Fusion Rules of Modular Invariants*

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Abstract

Modular invariants satisfy remarkable fusion rules. Let $Z$ be a modular invariant associated to a braided subfactor $N \subset M$. The decomposition of the non-normalized modular invariants $ZZ^*$ and $Z^*Z$ into sums of normalized modular invariants is related to the decomposition of the full induced $M - M$ system of sectors.

Contents

1 Introduction 2
2 Preliminaries 4
3 A closer look at the $M-N$ system 6
  3.1 Varying the $\iota$-vertex on the $M-N$ graphs 7
  3.2 A curious identity 8
  3.3 Towards a general formula for $[\theta]$ 8
4 A closer look at the $M-M$ system 10
  4.1 Some remarks on products of modular invariants 11
  4.2 On the geometry of the $M-M$ system 11
5 Examples 13
  5.1 $SU(2)$-invariants 13
  5.2 $SU(3)$-invariants 15
  5.3 Towards a pattern 17
  5.4 Interesting invariants of $SU(n)_k$ 19

*This contribution is dedicated to Huzihiro Araki on the occasion of his seventieth birthday
1 Introduction

Suppose $N \subset M$ is a braided type III subfactor; i.e. say the type III factor $N$ possesses a non-degenerate system $N\mathcal{X}_N$ of braided endomorphisms with the inclusion generated by certain sectors of the system. Then we know by [4 2] that the system $N\mathcal{X}_N$ generates a representation of the modular group $SL(2; \mathbb{Z})$, with generators $S = \{S_{\lambda,\mu}; \lambda, \mu \in N\mathcal{X}_N \}$, $T = \{T_{\lambda,\mu}; \lambda, \mu \in N\mathcal{X}_N \}$. Moreover [7, 11, 18] the inclusion generates a modular invariant $Z$ through the process of $\alpha$-induction from sectors of $N$ to sectors of $M$:

$$Z_{\lambda,\mu} = \langle \alpha^+_\lambda, \alpha^-_\mu \rangle, \quad \lambda, \mu \in N\mathcal{X}_N. \quad (1)$$

The right hand side is interpreted as multiplicities of common sectors in the two inductions, which is clearly thus a matrix with positive integer entries. It commutes with both $S$ and $T$ matrices or the representation of $SL(2; \mathbb{Z})$. In particular this covers the case of all A-D-E $SU(2)$ modular invariants, and much more besides. We say that a modular invariant is sufferable if if can be realised from a subfactor in this way from $\alpha$-induction on a braided system of endomorphisms.

Moreover, we can generate from $\alpha$-induction the sectors $M\mathcal{X}_M^\pm$ which in turn generate the full system $M\mathcal{X}_M$ of $M$. The full system $M\mathcal{X}_M$ has $\sum_{\lambda,\mu} Z^2_{\lambda,\mu}$ irreducible sectors and clearly $\alpha$-induction gives representations of the original $N$-$N$ fusion rules on $M\mathcal{X}_M$. However it is the natural action of the $N$-$N$ sectors on the corresponding $N$-$M$ sectors $N\mathcal{X}_M$ is what gives the A-D-E classification and its generalizations. In particular the trace of $Z$, $\text{tr}Z = \sum_\lambda Z_{\lambda,\lambda}$, gives the number of $N$-$M$ sectors in $N\mathcal{X}_M$. Now $\sum_{\lambda,\mu} Z^2_{\lambda,\mu} = \text{tr}ZZ^*$. The matrix $ZZ^*$ is a modular invariant, in that it has positive integral entries, commutes with representation of the modular group $SL(2; \mathbb{Z})$, but in general will not be physical in not having the vacuum entry normalized to be one. It is therefore tempting to ask whether we can understand the full $M$-$M$ system in terms of an analysis of the modular invariant $ZZ^*$, and an inclusion $N \subset M_1$ with the full $M$-$M$ system being related to the chiral $N$-$M_1$ system, just as we understand the $N$-$M$ system from the modular invariant $Z$. This was the original motivation in [3, 10] to write the numerical count $\sum_{\lambda,\mu} Z^2_{\lambda,\mu}$ as $\text{tr}ZZ^*$, the trace of a modular invariant.

For example for $SU(2)$, where moreover always $Z = Z^*$, we have the following simple commutative fusion rules for the three modular invariants at level 16 labelled by the three Dynkin diagrams with Coxeter number 18:

$$Z^2_{D_{10}} = 2Z_{D_{10}}, \quad Z_{D_{10}}Z_{E_7} = Z_{E_7}Z_{D_{10}} = 2Z_{E_2}, \quad Z^2_{E_7} = Z_{D_{10}} + Z_{E_7}.$$

Consider the subfactor $N \subset M$ describing the $E_7$ modular invariant. Indeed the fusion graph of $\alpha^+_1$ for the $E_7$ example has two connected components, $D_{10}$ and $E_7$ with the decomposition $\text{tr}Z^2_{E_7} = \text{tr}Z_{D_{10}} + \text{tr}Z_{E_7}$ reflecting this decomposition of the $M$-$M$ graph. The aim of this paper is to begin to understand better the decomposition of the full $M$-$M$ system into its components via the decomposition of the matrix $ZZ^*$ into normalized modular invariants.
If a modular invariant $Z$ is associated to an inclusion $N \subset M$, then where would we try to understand the doubled or modular invariant $ZZ^*$? This will be our driving principle: For a subfactor $N \subset M$, there is a natural squaring or iteration procedure $N \subset M \subset M_1$ of the basic construction. Indeed, if the decomposition Eq. (1) formula for a type I invariant $Z = B^*B$ in matrix form is related to an inclusion $N \subset M$, with dual canonical endomorphism $\tau$, then it is natural to try to understand the iteration $ZZ^* = B^*BB^*B$, with the basic construction $N \subset M \subset M_1$ which has dual canonical endomorphism $\tau\tau$.

In the next section we outline our framework of preliminaries in more detail. In Section 3 we complete some analysis begun in [8] regarding changing the modular invariants, in particular $ZZ^*$ endomorphisms. We then in Section 4 look at the structure of the products of modular invariants can be realised in canonical ways with natural dual canonical endomorphisms are. Nevertheless there is a simple expression for the sum of all possible dual canonical endomorphisms in Subsection 3.2. This will be used for example in [20] for answering the question of which modular invariants are realisable in concrete situations with given modular data. Many subactors can give rise to the same modular invariant. However, in Subsection 3.3 we consider whether sufferable modular invariants can be realised in canonical ways with natural dual canonical endomorphisms. We then in Section 4 look at the structure of the products of modular invariants, in particular $ZZ^*$ and $Z^*Z$, and how their decomposition into normalised modular invariants is related to the geometry of the related $M$-$M$ system, its decomposition into $M^\vee_M^\vee$ orbits, and the decomposition of $M^\vee_M^\vee$ into $(ZZ^*)_{0,0}$ and $(Z^*Z)_{0,0}$ $M^\vee_M^\vee$ orbits respectively. Subsection 3.1 and Subsection 3.2 contain a discussion of concrete examples from $SU(2)$ and $SU(3)$ respectively. In particular the curious example of the full $M$-$M$ system for the conformal embedding modular invariant $SU(3)_9 \subset (E_6)_1$ where there are six $M^\vee_M^\vee$ orbits in $M^\vee_M^\vee$ yet the full system contains besides three copies of $E_1^{(12)}$, also three copies of the isospectral graph $E_2^{(12)}$.

We can write a sufferable modular invariant $Z$ in terms of rectangular branching matrices as $Z = B_+^*B_-$ so that $ZZ^* = B_+^*B_-B_+^*B_-$ and $Z^*Z = B_+^*B_-B_+^*B_-$. We look in Subsection 3.3 at the sandwiched $B_\pm B_\pm^*$. This is a modular invariant for the extended system which is in general not normalized but its decomposition into normalized modular invariants (usually permutations) and its relationship to the decomposition of the full system $M^\vee_M^\vee$ into $M^\vee_M^\vee$ orbits is discussed. Finally in Subsection 5.4 we discuss some interesting invariants of $SU(n)_n$. In the conclusions of [8] we speculated about modular invariants which look like type I or type II but really come from heterotic extensions, i.e. for which we have different intermediate local subfactors. We provide examples, actually making use of the heterotic $SO(16\ell)_1$ modular invariants ($\ell = 1, 2, 3, ...$) treated in [8], and conformal inclusions $SU(n)_n \subset SO(n^2-1)_1$. The simplest case is $SU(7)_7 \subset SO(48)_1$ and by pulling back the heterotic situation on $SO(48)_1$ we obtain our strange heterotic modular invariant on $SU(7)_7$ – which of course must be symmetric.
2 Preliminaries

We cite [19] as a general reference for operator algebras and subfactors, and recall the sector setting of [34]. Let $A$ and $B$ be type III von Neumann factors. A unital $\ast$-homomorphism $\rho : A \to B$ is called a $B$-$A$ morphism. The positive number $d_\rho = [B : \rho(A)]^{1/2}$ is called the statistical dimension of $\rho$; here $[B : \rho(A)]$ is the minimal Jones index [33] of the subfactor $\rho(A) \subset B$. If $\rho$ and $\sigma$ are $B$-$A$ morphisms with finite statistical dimensions, then the vector space of intertwiners

$$\text{Hom}(\rho, \sigma) = \{ t \in B : t\rho(a) = \sigma(a)t, \ a \in A \}$$

is finite-dimensional, and we denote its dimension by $\langle \rho, \sigma \rangle$. Indeed we will only consider morphisms of finite statistical dimension. To any $B$-$A$ morphism $\rho$ is assigned a conjugate $A$-$B$ morphism $\overline{\rho}$ so that the map $[\rho] \mapsto [\overline{\rho}]$ is additive, antimultiplicative and idempotent – generalizing the notion of inversion and conjugate representation in a group or group dual respectively.

We work with the setting of [11], i.e. we are working with a type III subfactor and finite system $\mathcal{N} \chi_\mathcal{N} \subset \text{End}(N)$ of (possibly degenerately) braided morphisms which is compatible with the inclusion $N \subset M$. Then the inclusion is in particular forced to have finite Jones index and also finite depth (see e.g. [19]). More precisely, we make the following

**Assumption 2.1** We assume that we have a type III subfactor $N \subset M$ together with a finite system of endomorphisms $\mathcal{N} \chi_\mathcal{N} \subset \text{End}(N)$ in the sense of [11, Def. 2.1] which is braided in the sense of [11, Def. 2.2] and such that $\theta = \overline{\tau} \in \Sigma(\mathcal{N} \chi_\mathcal{N})$ for the injection $M$-$N$ morphism $\iota : N \hookrightarrow M$ and a conjugate $N$-$M$ morphism $\overline{\iota}$.

With the braiding $\varepsilon$ on $\mathcal{N} \chi_\mathcal{N}$ and its extension to $\Sigma(\mathcal{N} \chi_\mathcal{N})$ (the set of finite sums of morphisms in $\mathcal{N} \chi_\mathcal{N}$) as in [11], one can define the $\alpha$-induced morphisms $\alpha^\pm_\lambda \in \text{End}(M)$ for $\lambda \in \Sigma(\mathcal{N} \chi_\mathcal{N})$ by the Longo-Rehren formula [37], namely by putting

$$\alpha^\pm_\lambda = \overline{\tau}^{-1} \circ \text{Ad}(\varepsilon^\pm(\lambda, \theta)) \circ \lambda \circ \tau,$$

where $\tau$ denotes a conjugate morphism of the injection map $\iota : N \hookrightarrow M$. Then $\alpha^+_\lambda$ and $\alpha^-_\lambda$ extend $\lambda$, i.e. $\alpha^\pm_\lambda \circ \iota = \iota \circ \lambda$, which in turn implies $d^\pm_\lambda = d_\lambda$ by the multiplicativity of the minimal index [35]. Moreover, we have $\alpha^\pm_\lambda \alpha^\pm_\mu = \alpha^\pm_\lambda \alpha^\pm_\mu$ if also $\mu \in \Sigma(\mathcal{N} \chi_\mathcal{N})$, and clearly $\alpha^-_\text{id}_M = \text{id}_M$. The morphism $\alpha^\pm_\lambda$ is a conjugate for $\alpha^\mp_\lambda$. Let $\gamma = \overline{\iota} \tau$ denote Longo’s canonical endomorphism from $\overline{M}$ into $N$.

We will assume that braiding on the system $\mathcal{N} \chi_\mathcal{N}$ is non-degenerate. In this case there is a natural representation of the modular group $SL(2; \mathbb{Z})$ where the $S$ and $T$ matrices are basically given by the Hopf link and twist respectively. More precisely, recall that the statistics phase of $\omega_\lambda$ for $\lambda \in \mathcal{N} \chi_\mathcal{N}$ is given as $d_\lambda \phi_\lambda(\varepsilon^+(\lambda, \lambda)) = \omega_\lambda \text{id}$, where the state $\phi_\lambda$ is the left inverse of $\lambda$. We set $z = \sum_{\lambda \in \mathcal{N} \chi_\mathcal{N}} d^2_\lambda \omega_\lambda$ if $z \neq 0$ we put $c = 4 \text{arg}(z)/\pi$, which is the central charge defined modulo 8. The $S$-matrix is defined by

$$S_{\lambda,\mu} = \frac{1}{|z|} \sum_{\rho \in \mathcal{N} \chi_\mathcal{N}} \frac{\omega_\lambda \omega_\mu}{\omega_\rho} N^\rho_{\lambda,\mu} d_\rho, \quad \lambda, \mu \in \mathcal{N} \chi_\mathcal{N},$$
with \( N^0_{\lambda,\mu} = \langle \rho, \lambda \mu \rangle \) denoting the fusion coefficients. [12, 22, 21]. (As usual, the label 0 refers to the identity morphism id \( \in \mathcal{N} \mathcal{X}_N \).) Let \( T \) be the diagonal matrix with entries \( T_{\lambda,\mu} = e^{i \pi \epsilon / 2 \lambda \mu} \). Then this pair of \( S \) and \( T \) matrices satisfy \( T S T S T S T = S \) and give a unitary representation of the modular group \( SL(2; Z) \). [12, 45]. Putting \( Z_{\lambda,\mu} = \langle \alpha^+, \alpha^- \rangle \) defines a matrix with positive integral entries normalized at the vacuum, \( Z_{0,0} = 1 \), commuting with \( S \) and \( T \). Consequently, \( Z \) gives a modular invariant [11, 13].

Let \( M \mathcal{X}_M \subset \text{End}(M) \) denote a system of endomorphisms consisting of a choice of representative endomorphisms of each irreducible subssector of sectors of the form \( [\lambda, \gamma] \), \( \lambda \in N \mathcal{X}_N \). We choose \( \text{id} \in \text{End}(M) \) representing the trivial sector in \( M \mathcal{X}_M \). Then we define similarly the chiral systems \( M \mathcal{X}^\pm_M \) and the \( \alpha \)-system \( M \mathcal{X}_\alpha^\pm_M \) to be the subsystems of endomorphisms \( \beta \in M \mathcal{X}_M \) such that \( \beta \) is a subsector of \( [\alpha_+^\pm \alpha^-] \) and of \( [\alpha^- \alpha_+] \), respectively, for some \( \lambda, \mu \in N \mathcal{X}_N \). The neutral system is defined as the intersection \( M \mathcal{X}^0_M = M \mathcal{X}^+_M \cap M \mathcal{X}^-_M \), so that \( M \mathcal{X}^0_M \subset M \mathcal{X}^+_M \subset M \mathcal{X}_\alpha^0_M \subset M \mathcal{X}^-_M \).

Suppose that we have two subfactors, \( N \subset M_a \) and \( N \subset M_b \) where the irreducible components of both dual canonical endomorphisms lie in the braided non degenerate system \( N \mathcal{X}_N \) with modular invariants \( Z^a \) and \( Z^b \) respectively. Let \( M_a \mathcal{X}_M b \) denote the irreducible subssectors of \( \iota_a \lambda \mathcal{X}_b \) where \( \iota_a \), \( \iota_b \) are the corresponding embeddings of \( N \) in \( M_a \) and \( M_b \) respectively. We can then by an extension of the ideas of [12] show that the complexification of the bimodule \( M_a \mathcal{X}_M b \) under the left action of \( M_a \mathcal{X}_M a \) and the right action of \( M_b \mathcal{X}_M b \) is isomorphic to

\[
\bigoplus_{\lambda,\mu \in N \mathcal{X}_N} H^c_{\lambda,\mu} \otimes \overline{H}^\perp_{\lambda,\mu},
\]

where

\[
H^c_{\lambda,\mu} = \bigoplus_{x \in N \mathcal{X}_M c} \text{Hom}(\lambda \mathcal{X}_c, x \mathcal{X}), \lambda, \mu \in N \mathcal{X}_N.
\]

is the Hilbert space of intertwiners of dimension \( Z^c_{\lambda,\mu}, \ c = a, b \). In particular the decomposition in Eq. (2) is compatible in the natural way as a bimodule with the complexification of the fusion rule algebra of \( M_a \mathcal{X}_M a \) as

\[
\bigoplus_{\lambda,\mu \in N \mathcal{X}_N} B(H^c_{\lambda,\mu}).
\]

A dimension counts shows that the number of irreducible \( M_a \cdot M_b \) sectors of \( M_a \mathcal{X}_M b \) is \( \text{tr}(Z^{a*} Z^b) \). If \( M_a = M_b = M \), then \( #M \mathcal{X}_M = \text{tr} Z^* Z \), and if \( M_a = N \) and \( M_b = M \), then \( #N \mathcal{X}_M = \text{tr} Z \). The action of \( N \mathcal{X}_N \times N \mathcal{X}_N \) on \( M_a \mathcal{X}_M b \) via \( \alpha \)-induction namely \( \nu, \rho \rightarrow \alpha^\nu_\gamma \alpha^\rho_\gamma \), on either the left via the induction \( N \subset M_a \) or on the right via \( N \subset M_b \), gives a doubled nimrep \((\nu, \rho) \rightarrow \Gamma_{\nu,\rho}\) whose spectrum is \( S_{\lambda,\mu} \in S_{\mu,\nu} S_0 S_{\nu,0} \) with multiplicity \( Z^a_{\lambda,\mu} Z^b_{\lambda,\mu} \). This reduces to parts 1 and 2 respectively of [12, Thm. 4.16] when \( M_a = M_b, M_a = N \) respectively. Applications of the existence of such \( Z^a \cdot Z^b \) nimreps for sufferable invariants and the question of the decomposition of the products \( Z^{a*} Z^b \) into normalised modular invariants will appear elsewhere.
We are particularly concerned here with modular invariants arising in WZW or loop group settings. The modular data (S, and T matrices etc) can be constructed from representation theory of unitary integrable highest weight modules over affine Lie algebras or in exponentiated form from the positive energy representations of loop groups. The subfactor machinery is invoked as follows. Let \( LG \) be a loop group (associated to a simple, simply connected loop group \( G \)). Let \( L_I G \) denote the subgroup of loops which are trivial off some proper interval \( I \subset S^1 \). Then in each level \( k \) vacuum representation \( \pi_0 \) of \( LG \), we naturally obtain a net of type III factors \( \{ N(I) \} \) indexed by proper intervals \( I \subset S^1 \) by taking \( N(I) = \pi_0(L_I G)'' \) (see \([15, 23, 1]\)). Since the Doplicher-Haag-Roberts DHR selection criterion (cf. \([27]\)) is met in the (level \( k \)) positive energy representations \( \pi_\lambda \), there are DHR endomorphisms \( \lambda \) naturally associated with them. (By some abuse of notation we use the same symbols for labels of positive energy representations and endomorphisms.)

The rational conformal field theory RCFT modular data matches that in the subfactor setting – in particular the RCFT Verlinde fusion coincides with the (DHR superselection) sector fusion, i.e. that \( N_{\lambda, \mu} = \langle \lambda \mu, \nu \rangle \). The statistics S- and T-matrices are identical with the Kac-Peterson S- and T-modular matrices which perform the conformal character transformations.

Antony Wassermann has informed us that he has extended his results for \( SU(n)_k \) fusion \([16]\) to all simple, simply connected loop groups; and with Toledano-Laredo all but \( E_8 \) using a variant of the Dotsenko-Fateev differential equation considered in his thesis \([32]\), see also \([10, 33, 32, 3, 4]\).

Two subfactor cases are of particular interest in this context, that of conformal embeddings \([17, 3, 6]\) and simple current or orbifold constructions \([8]\). For a conformal embedding \( G_k \subset H_1 \) we have subfactors \( N = \pi_0(L_I G)'' \subset \pi_0(L_I H)'' = M \), with \( \pi_0 \) denoting the level 1 vacuum representation of \( LH \). Here, the subfactor comes equipped with non-degenerately braided systems of endomorphisms on \( N \) and \( M \) isomorphic to the level \( k \) representations of \( G \) and level 1 representations of \( H \) respectively, and is relevant for the role of studying conformal embedding modular invariants. The centre \( Z_n \) of \( SU(n) \) acts on the algebra \( N = \pi_0(L_I SU(n))'' \), for say the vacuum level 1 representation. We can form the crossed product subfactor \( N(I) \subset N(I) \rtimes Z_n \), which will recover the orbifold modular invariants, but this extended system is only local if and only if \( k \in 2n\mathbb{N} \) if \( n \) is even and \( k \in n\mathbb{N} \) if \( n \) is odd \([6]\).

### 3 A closer look at the M-N system

For a (non-degenerately) braided subfactor it is the M-N (or N-M) system which is relevant for the diagonal part of the modular invariant. Therefore it is in particular the key to understand the role of (Coxeter) exponents.
3.1 Varying the \( \iota \)-vertex on the \( M \)-\( N \) graphs

We assume that we are dealing with a braided (type III) subfactor \( N \subset M \). For \( a \in N \mathcal{X}_M \) consider the (irreducible) subfactor \( a(M) \subset N \) and let

\[
a(M) \subset N \subset L
\]

be its basic extension. Note that then \( \theta_L = a \overline{a} \) has a Q-system \([36]\) for \( a(M) \subset N \) so that it is a canonical endomorphism, i.e. \( \theta_a \) is the dual canonical endomorphism of \( N \subset L \). Thus \( \theta_a = a \overline{a} = \tau_L \iota_L \) for \( \iota_L : N \hookrightarrow L \) the injection homomorphism and \( \tau_L \in \text{Mor}(L, N) \) a conjugate so that \( \tau_L(L) = a(M) \). We conclude that \( \tau_L^{-1} \circ a \) is an isomorphism in \( \text{Mor}(M, L) \) with conjugate (i.e. inverse) \( a^{-1} \circ \tau_L \in \text{Mor}(L, M) \). For any \( b \in \text{Mor}(M, N) \) we now associate \( x_b \in \text{Mor}(L, N) \) by putting

\[
x_b = b \circ a^{-1} \circ \tau_L.
\]

Note that \( x_b \) is irreducible if and only if \( b \) is and that \( x_a = \tau_L \).

**Lemma 3.1** Varying \( b \in N \mathcal{X}_M \), the \( x_b \)'s yield all the \( N \)-\( L \) sectors, and this provides a canonical bijection between \( N \mathcal{X}_M \) and \( N \mathcal{X}_L \).

**Proof.** Note that for any \( b \in N \mathcal{X}_M \) there is some \( \lambda \in N \mathcal{X}_N \) such that \( \langle b \overline{a}, \lambda \rangle \neq 0 \) as \( b \overline{a} \in \Sigma(N \mathcal{X}_N) \). Thus

\[
\langle x_b, \lambda \overline{a}_L \rangle = \langle b \overline{a}^{-1} \tau_{L,L}, \lambda \rangle = \langle b \overline{a}^{-1} \overline{a}, \lambda \rangle \neq 0,
\]

implying that \([x_b]\) is one of the \( N \)-\( L \) sectors. Conversely, assume that there is some \( x \in N \mathcal{X}_L \) such that \( \langle x, x_b \rangle = 0 \), i.e. \( \langle x, b \overline{a}^{-1} \tau_L \rangle = 0 \) for all \( b \in N \mathcal{X}_M \). This implies \( \langle x, \lambda \overline{a}^{-1} \tau_L \rangle = \langle x, \lambda \overline{a}_L \rangle \neq 0 \) for all \( \lambda \in N \mathcal{X}_N \), in contradiction to \( x \in N \mathcal{X}_L \). \( \square \)

**Lemma 3.2** For \( b, c \in N \mathcal{X}_M \) we have

\[
\langle x_b, \nu x_c \rangle = \langle b, \nu c \rangle,
\]

i.e. the (graphs describing the) multiplication rules of \( N \mathcal{X}_N \) on \( N \mathcal{X}_M \) and \( N \mathcal{X}_L \) are the same.

**Proof.** This is just

\[
\langle x_b, \nu x_c \rangle = \langle b \overline{a}^{-1} \tau_L, \nu c \overline{a}^{-1} \overline{a} \tau_L \rangle = \langle b, \nu c \overline{a}^{-1} \tau_L \overline{a}^{-1} \rangle = \langle b, \nu c \rangle,
\]

using that \( \iota^{-1}_L \) is a conjugate morphism of \( a^{-1} \tau_L \). \( \square \)

Note that the lemma implies in particular that at least the diagonal part of the coupling matrices produced from \( N \subset M \) and \( N \subset L \) are the same. That in fact the full coupling matrix (and not only the diagonal part) remains invariant under this change of the \( \iota \)-vertex has been shown in \([10]\).
3.2 A curious identity

We here assume that we are dealing with a non-degenerately braided (type III) subfactor \( N \subset M \). We have seen that for a given braided subfactor \( N \subset M \), realizing a coupling matrix \( Z \) and a category of morphisms, we obtain irreducible subfactors with dual canonical endomorphisms \( \theta_a = a\pi, a \in N\mathcal{X}_M \), realizing the same \( Z \) [1].  

It seems likely to be true that this way we in fact exhaust all irreducible subfactors producing equivalent categories.

Given a modular invariant matrix \( Z \), it is usually not easy to decide whether it can be realized from a subfactor or not, and, if yes, how the possible dual canonical endomorphisms might look like. In the latter case, i.e. if there is some \( N \subset M \) realizing \( Z \), at least a statement on the sum of all these endomorphisms can be made in the following

**Proposition 3.3** If the braiding on \( N\mathcal{X}_N \) is non-degenerate we have the identity

\[
\bigoplus_{a \in N\mathcal{X}_M} [a\pi] = \bigoplus_{\lambda, \mu \in N\mathcal{X}_N} Z_{\lambda, \mu}[\lambda\pi]. \tag{6}
\]

**Proof.** The multiplicity of \([\nu]\) on the left-hand side is for all \( \nu \in N\mathcal{X}_N \)

\[
\sum_{a} \langle a\pi, \nu \rangle = \sum_{a} \langle a, \nu a \rangle = \text{tr}(G_{\nu}) = \sum_{\rho} Z_{\rho, \rho} S_{\rho, \nu} S_{\rho, 0} ,
\]

where we used [12, Thm. 4.16]. The multiplicity of \([\nu]\) on the right-hand side is for all \( \nu \in N\mathcal{X}_N \)

\[
\sum_{\lambda, \mu} Z_{\lambda, \mu} \langle \lambda\pi, \nu \rangle = \sum_{\lambda, \mu} Z_{\lambda, \mu} N_{\lambda, \mu} = \sum_{\lambda, \mu, \rho} Z_{\lambda, \mu} S_{\rho, \mu} S_{\rho, 0} S_{\rho, \nu} S_{\rho, 0} \tag{5}
\]

where we used the Verlinde formula and modular invariance. \( \square \)

3.3 Towards a general formula for \([\theta]\)

Looking at a couple of examples, it seems that a “physical invariant” \( Z \), which can be realized from some subfactor, can in fact be realized with a dual canonical endomorphism given by something like

\[
[\theta] = \bigoplus_{\lambda} Z_{\lambda, \lambda}[\lambda] . \tag{7}
\]

In general summing over a subset of \( N\mathcal{X}_N \) related to Frobenius-Schur indicators and conformal dimensions. Let us consider some examples.

For \( \mathbb{Z}_n \) conformal field theories with \( n \) odd, this works perfectly. In this situation, there are \( n \) sectors, labelled by \( \lambda = 0, 1, 2, \ldots, n - 1 \) (mod \( n \)), obeying \( \mathbb{Z}_n \) fusion rules, and conformal dimensions of the form \( h_\lambda = a\lambda^2/2n \) (mod \( 1 \)), where \( a \) is an integer.
mod $2n$, $a$ and $n$ coprime and $a$ is even whenever $n$ is odd. The modular invariants of such models have been classified [13]. They are labelled (with notation as in [9, 14]) by the divisors $\delta$ of $\tilde{n}$, where $\tilde{n} = n$ if $n$ is odd and $\tilde{n} = n/2$ if $n$ is even. Let us take $n$ odd. Then it is not hard to show that $Z^{(\delta)}_{\lambda, \lambda} = Z^{(n/\delta)}_{\lambda, \lambda}$. Thus by Eq. (8.1) we find $Z^{(\delta)}_{\lambda, \lambda} = 1$ for $\lambda = 0 \mod \delta$ and $Z^{(\delta)}_{\lambda, \lambda} = 0$ otherwise, and in fact $\theta = \bigoplus_{j=0}^{n/\delta-1} [\rho_j \delta]$ realizes $Z^{(\delta)}$, see [9]. (By the way: Since $\langle \alpha_j^+ \alpha^-_j, \gamma \rangle = \langle \alpha_j^+ \alpha^-_j, \gamma \rangle = 1$ it is easy to see that $[\gamma] = \sum_{j=0}^{\tilde{n}/\delta-1} [\alpha_j^+ \alpha^-_j]$ for all $\mathbb{Z}_n$ theories, no matter whether $n$ is even or odd.)

Note that for the conjugation invariant $C$ we would usually insert all morphisms in the $[\theta]$. This does not work for the $\mathbb{Z}_n$ CFT’s with $n$ even because we must use the even labels only [9]. So for some reasons the odd labels have to be ruled out. (Moreover, if $n$ is a multiple of 4 we do not want to see the self-conjugate even label $n/2$ in the dual canonical endomorphism realizing the trivial invariant.) A similar thing happens for $SU(2)_k$. Here $Z_{\lambda, \lambda} = Z_{\lambda, \lambda}$, but if we restrict the sum to even spins then we can in fact realize each A-D-E invariant by Eq. (7).

Let us start with the subfactors used to produce the A-D-E modular invariants in [9, 11, 12], i.e. the corresponding dual canonical endomorphisms $[\theta]$ are given by:

$$
\begin{align*}
A_{\ell}, & \quad \ell = k + 1 : \quad [\lambda_0] \\
D_{\ell}, & \quad k = 2\ell - 4 : \quad [\lambda_0] \oplus [\lambda_k] \\
E_6, & \quad k = 10 : \quad [\lambda_0] \oplus [\lambda_6] \\
E_7, & \quad k = 16 : \quad [\lambda_0] \oplus [\lambda_8] \oplus [\lambda_{16}] \\
E_8, & \quad k = 28 : \quad [\lambda_0] \oplus [\lambda_{10}] \oplus [\lambda_{18}] \oplus [\lambda_{28}] 
\end{align*}
$$

Now we choose the following $M$-$N$ morphisms $[\alpha]$: For $A_\ell$ (where $\ell$ is trivial) we choose $\lambda_{[\ell/2]} \equiv \lambda_{[\ell/2]}$. Here $[x]$ denotes the greatest possible integer less than or equal to $x$. For $D_\ell$ we choose $\lambda_{[\ell/2] - 1}$. For $E_6$ we choose $\sigma$ with $\sigma$ the marked vertex with statistical dimension $\sqrt{2}$. For $E_7$ we choose the morphism denoted by $\tau$ in [14, Fig. 41]. For $E_8$ we choose $\alpha_6^{(1)}$ with $\alpha_6^{(1)}$ the neutral or marked vertex as in [14, Fig. 8]. It is now straightforward to compute the sectors $[\sigma \alpha \tau]$ which will be our new $[\theta]$’s. For example, for $D_\ell$ we compute $[\lambda_{[\ell/2] - 1} \lambda_{[\ell/2] - 1}] = [\lambda_{[\ell/2] - 1}]^2 ([\lambda_0] \oplus [\lambda_k])$. For $E_6$ we compute $[\tau \sigma \alpha \ell] = [\tau] ([\alpha_0] \oplus [\alpha_{10}]) [\ell] = ([\lambda_0] \oplus [\lambda_{10}]) ([\lambda_0] \oplus [\lambda_6])$. Only for $E_7$ we need to sit down a bit, using $[\tau] = [\tau \lambda_2] \oplus [\tau \lambda_4] \oplus [\tau \lambda_6]$. This gives:

$$
\begin{align*}
A_{\ell}, & \quad \ell = k + 1 : \quad [\lambda_0] \oplus [\lambda_2] \oplus [\lambda_4] \oplus \ldots \oplus [\lambda_{2k/2}] \\
D_{2q}, & \quad k = 4q - 4 : \quad [\lambda_0] \oplus [\lambda_2] \oplus \ldots \oplus [\lambda_{2q-4}] \oplus [\lambda_{2q-2}] \oplus [\lambda_{2q}] \oplus \ldots \oplus [\lambda_k] \\
D_{2q+1}, & \quad k = 4q - 2 : \quad [\lambda_0] \oplus [\lambda_2] \oplus [\lambda_4] \oplus \ldots \oplus [\lambda_k] \\
E_6, & \quad k = 10 : \quad [\lambda_0] \oplus [\lambda_4] \oplus [\lambda_6] \oplus [\lambda_{10}] \\
E_7, & \quad k = 16 : \quad [\lambda_0] \oplus [\lambda_4] \oplus [\lambda_6] \oplus [\lambda_{8}] \oplus [\lambda_{10}] \oplus [\lambda_{12}] \oplus [\lambda_{16}] \\
E_8, & \quad k = 28 : \quad [\lambda_0] \oplus [\lambda_6] \oplus [\lambda_{10}] \oplus [\lambda_{12}] \oplus [\lambda_{16}] \oplus [\lambda_{18}] \oplus [\lambda_{22}] \oplus [\lambda_{28}].
\end{align*}
$$

So here we indeed find exactly the even spins of the diagonal. (Note that the $[\theta]$’s for $A$ and $D_{odd}$ are the same (at levels $k = 6, 10, 14,...$). Thus these are examples for
subfactors producing different $Z$'s but having the same dual canonical endomorphism sector.

It is interesting to note what the canonical endomorphism looks like in these possibly natural subfactors:

\begin{align*}
A_\ell, \quad \ell = k + 1 : & \quad [a_0] \oplus [a_2] \oplus [a_4] \oplus \ldots \oplus [a_{2[k/2]}] \\
D_{2q}, \quad k = 4q - 4 : & \quad [a_0] \oplus [a_2] \oplus \ldots \oplus [a_{2q-4}] \oplus [a_{2q-2}] \oplus [a_{2q-2}] \oplus [\epsilon] \oplus \oplus [\beta_2] \oplus [\delta_4] \oplus \ldots \oplus [\beta_{2q-4}] \oplus [\eta] \oplus [\eta'] \\
D_{2q+1}, \quad k = 4q - 2 : & \quad [a_0] \oplus [a_2] \oplus [a_4] \oplus \ldots \oplus [a_k] \\
E_6, \quad k = 10 : & \quad [a_0] \oplus [a_{10}] \oplus [\delta] \oplus [\delta'] \\
E_7, \quad k = 16 : & \quad [a_0] \oplus [\eta] \oplus [\delta] \oplus [\alpha^+_6] \oplus [\eta'] \oplus [\delta] \oplus [\alpha^+_4] \oplus [\alpha^+_3] \oplus [\eta_s] \oplus [\eta'] \\
E_8, \quad k = 28 : & \quad [a_0] \oplus [a^{(1)}_6] \oplus [\delta] \oplus [\chi] \oplus [\omega] \oplus [\pi] \oplus [\eta] \oplus [\eta'].
\end{align*}

For $D_{2q}, D_{2q+1}, E_6, E_7, E_8$, we have used the notation of \cite{7, Fig. 9, 12, Fig. 40}, \cite{4, Fig. 2}, \cite{12, Fig. 42}, \cite{4, Fig. 5} respectively.

At least in this $SU(2)$ setting, there is a fusion rule symmetry on $M_X^\pm$ obtained by interchanging $\alpha^+_\lambda$ with $\alpha^-_\lambda$ taking $M_X^+ \to M_X^-$. In terms of the above figures for $D_{2q}, D_{2q+1}, E_6, E_7, E_8$, this is the flip around the vertical through the vacuum. (When we change in the above examples the subfactor $N \subset M$ but retain the same modular invariant the systems of sectors $M_X^+, M_X^0, M_X^-, M_X$ remain isomorphic to the old ones, so we retain the same figures). Again for $E_7$ we need to do some work, e.g. $\langle b', \alpha^+_i \alpha^-_j \rangle = \langle \iota([\lambda_2] \oplus [\lambda_4] \oplus [\lambda_6]2\tau, \alpha^+_i \alpha^-_j) \rangle = \langle ([\lambda_2] \oplus [\lambda_4] \oplus [\lambda_6])^2, \tau \alpha^+_i \alpha^-_j \rangle = \langle ([\lambda_2] \oplus [\lambda_4] \oplus [\lambda_6])^2, [\lambda_1][\lambda_2](\lambda_0 \oplus [\lambda_8] \oplus [\lambda_{16}]) \rangle$. Then the the "real" part of the full system are the sectors fixed under the flip, i.e. the sectors lying on the vertical through the vacuum. Then the canonical endomorphism is the even part of the "real" part of the full system - presumably these are the ones of Frobenius-Schur indicator one.

For $SU(3)_k$ we have checked for $k = 1$ and $k = 2$ that $Z = C$ is indeed realized by Eq. \cite{7}, the sum taken over all $SU(3)_k$ weights. Assuming that the subfactor exists for $k = 3$ it is easy to check it for this case as well. Similarly it is easy to check for $k \leq 3$ that that $[\theta]$ given as sum over all selfconjugate sectors indeed produces $Z = 1$ since the $[\theta]$ is just the square of the only non-trivial self-conjugate sector $[\lambda_{2,1}]$. Squaring larger $[\lambda]$'s instead, this procedure should also work at any higher levels $k$.

4 A closer look at the $M-M$ system

Here we discuss the structure of the entire $M-M$ system. We will only consider proper modular invariants here, i.e. we assume the $N$-$N$ system is non-degenerate. First some observations.

4.1 Some remarks on products of modular invariants

A modular invariant from a subfactor is of the form

\[ Z_{\lambda,\mu} = \sum_{\tau \in M^{0}_{M}} b_{\tau,\lambda}^{+} b_{\tau,\mu}^{-}. \]

Now let consider the fusion graph of \( \alpha_{\lambda}^{+} \) in the entire system \( M^{0}_{M} \). (We will here consider the non-degenerate case only.) We know since [12] that the multiplicity of the eigenvalue \( S_{\lambda,\rho} / S_{0,\rho} \) is given by \( \sum_{\mu} Z_{\rho,\mu}^{2} = (ZZ^{*})_{\rho,\rho} \), and that this exhausts the spectrum. Since this graph contains the chiral graph as a subgraph, we must have \( (ZZ^{*})_{\rho,\rho} \geq Z_{\rho,\rho}^{+} \), where \( Z^{+} \) denotes the type I parent, \( Z_{\lambda,\mu}^{+} = \sum_{\tau} b_{\tau,\lambda}^{+} b_{\tau,\mu}^{+} \). And indeed, we can compute quite generally

\[ (ZZ^{*})_{\lambda,\mu} = \sum_{\nu} Z_{\lambda,\nu} Z_{\mu,\nu} = \sum_{\nu} \sum_{\tau,\tau'} b_{\tau,\nu}^{+} b_{\tau',\nu}^{+} b_{\tau,\mu}^{-} b_{\tau',\mu}^{-} \geq \sum_{\nu} \sum_{\tau} b_{\tau,\nu}^{+} b_{\tau,\mu}^{-} b_{\tau,\mu}^{-} \geq \sum_{\tau} b_{\tau,\mu}^{+} = Z_{\lambda,\mu}^{+}, \]

where we used that for each \( \tau \in M^{0}_{M} \) there is a \( \nu \) such that \( b_{\tau,\nu}^{+} \geq 1 \). (And of course we obtain similarly \( (Z^{*}Z)_{\lambda,\mu} \geq Z_{\lambda,\mu}^{-}, \) etc. etc.) Note that \( ZZ^{*} - Z^{+} \) must be modular invariant, and looking at the above calculation we see that it is even non-negative. So what about normalization? We distinguish the two cases: (1) \( Z \) is a pure permutation. Then in fact \( ZZ^{*} = Z^{+} = 1 \) (2) otherwise there is a \( \lambda \) with \( Z_{0,\lambda} \neq 0 \), and consequently \( (ZZ^{*})_{0,0} = \sum_{\lambda} Z_{0,\lambda}^{2} > 1 \). If this gives exactly 2 then we know that \( ZZ^{*} - Z^{+} \) is another normalized integral modular invariant, but if it is larger then it is not clear whether \( ZZ^{*} - Z^{+} \) can always written as a positive integer linear combination of normalized integral modular invariants.

It is clear that if we always obtain a positive integer linear combination of normalized integral modular invariants, then the number of such invariants (counting multiplicities) will be \( (ZZ^{*})_{0,0} = \sum_{\lambda} Z_{0,\lambda}^{2} = \langle \theta_{+}, \theta_{+} \rangle \). Each invariant is expected to correspond to a component of the full fusion graph, so we expect \( \sum_{\lambda} Z_{0,\lambda}^{2} \) components. (By components we mean here a connected component of the fusion graph of a generator \( \alpha_{\lambda}^{+} \). Equivalently one can decompose the sum of the full fusion matrices of all chiral sectors into irreducible components.) That at least this numbering for the connected components is indeed correct is shown in the following subsection.

4.2 On the geometry of the \( M-M \) system

Let us recall that the \( M-M \) system has subsystems \( M^{0}_{M} \supset M^{+}_{M} \supset M^{0}_{M} \). Under the action (fusion) of a chiral system, say \( M^{+}_{M} \), the \( M^{+}_{M} \) system decomposes into \( M^{+}_{M} \) orbits. These correspond to the connected components of the fusion graph of a generator of \( M^{+}_{M} \) in \( M^{0}_{M} \). We may draw such a graph using straight lines, and the graph arising from the corresponding generator of \( M^{+}_{M} \) using dotted lines as in [4, Figs. 2,5,8,9] or [12, Figs. 40,42,43]. For the \( E_{8} \) example we find 4 \( M^{+}_{M} \) orbits which are precisely the 4 straight-lined \( E_{8} \) “layers” in [4, Fig. 5]. How many such layers do we usually have? A first answer is this:
Lemma 4.1 The number of $M\mathcal{X}_M^\pm$ orbits in $M\mathcal{X}_M$ is equal to the number of $M\mathcal{X}_M^0$ orbits in $M\mathcal{X}_M^\mp$. In fact, all $M\mathcal{X}_M^\pm$ orbits in $M\mathcal{X}_M$ intersect with the subset $M\mathcal{X}_M^\mp \subset M\mathcal{X}_M$, and the intersections are precisely the $M\mathcal{X}_M^0$ orbits in $M\mathcal{X}_M^\mp$.

Proof. Consider the identity component $\Gamma_{(0)}^+$ of the fusion graph of $M\mathcal{X}_M^+$ in $M\mathcal{X}_M$ (which is essentially $M\mathcal{X}_M^+$ itself). Since $M\mathcal{X}_M^+$ and $M\mathcal{X}_M^-$ generate $M\mathcal{X}_M$, each connected component $\Gamma_{(j)}^+$ of the fusion graph of $M\mathcal{X}_M^+$ in $M\mathcal{X}_M$ must touch $\Gamma_{(0)}^+$ somewhere. (E.g. the identity component $\Gamma_{(0)}^+$ meets $\Gamma_{(0)}^-$ exactly on the ambichiral vertices.) Hence the number of $M\mathcal{X}_M^-$ orbits in $M\mathcal{X}_M$ is equal to the number of groups of vertices on $\Gamma_{(0)}^+$ lying on the same component $\Gamma_{(j)}^-$. Two vertices on $\Gamma_{(0)}^+$ corresponding to sectors $\beta_1, \beta_2 \in M\mathcal{X}_M^+$ lie on the same component $\Gamma_{(j)}^-$ if and only if there is a $\beta \in M\mathcal{X}_M^-$ such that $\langle \beta_1, \beta_2 \rangle \neq 0$. But $\langle \beta, \beta_1 \beta_2 \rangle \neq 0$ if means that $\beta$ is ambichiral. Hence two vertices on $\Gamma_{(0)}^+$ corresponding to sectors $\beta_1, \beta_2 \in M\mathcal{X}_M^+$ lie on the same component $\Gamma_{(j)}^-$ if and only if they are in the same ambichiral orbit. The proof is completed by exchanging + and − signs.

A more concrete answer is now obtained in the following

Lemma 4.2 The number of $M\mathcal{X}_M^0$ orbits in $M\mathcal{X}_M^\pm$ is given by $\sum_\lambda (b_{0,\lambda}^\pm)^2$.

Proof. Let $\Gamma_{\tau,0}^\pm$ be the fusion matrix of $\tau \in M\mathcal{X}_M^0$ in $M\mathcal{X}_M^\pm$, as in [12 Sect. 4]. The sum matrix $Q = \sum_\tau \Gamma_{\tau,0}^\pm$ will not be irreducible as long as we have more than one $M\mathcal{X}_M^0$ fusion orbit (i.e. as long as $M\mathcal{X}_M^0 \neq M\mathcal{X}_M^\pm$). In fact $Q$ must decompose into a number of irreducible blocks which is exactly the number of fusion orbits. Nevertheless the vector $\tilde{d}$ with entries $d_{\beta}, \beta \in M\mathcal{X}_M^\pm$ is an eigenvector of $Q$ with eigenvalue $\sum_\tau d_{\tau}$. Since all the entries are strictly positive, it must be the direct sum of the Perron-Frobenius eigenvalues of each irreducible block (up to a scaling by a positive factor for each block). Thanks to the Perron-Frobenius theorem, the number $\sum_\tau d_{\tau}$ is thus the (non-degenerate) Perron-Frobenius eigenvalue of each irreducible block. It follows that the number of irreducible components is given by the multiplicity of the eigenvalue $\sum_\tau d_{\tau}$, i.e. by the multiplicity of $\chi_{\tau}^{\text{ext}}(\tau)$ in $\Gamma_{\tau,0}^\pm$. By the diagonalization of the $\Gamma_{\tau,0}^\pm$'s derived in [12, Thm. 4.16], we know that this multiplicity is exactly $\sum_\lambda (b_{0,\lambda}^\pm)^2$.

Note that $\sum_\lambda (b_{0,\lambda}^+)^2 = \sum_\lambda Z_{\lambda,0}^2$ and $\sum_\lambda (b_{0,\lambda}^-)^2 = \sum_\lambda Z_{0,\lambda}^2$. In fact, the consideration of the $M\mathcal{X}_M^0$ in $M\mathcal{X}_M^\pm$ was instructive, but not really necessary to get the number of $M\mathcal{X}_M^0$ orbits in $M\mathcal{X}_M$. Thanks to the generating property, we could also have determined the number of $M\mathcal{X}_N$ orbits in $M\mathcal{X}_M$ via the induced $[\alpha^+]$ and $[\alpha^-]$. Then the statement of [12, Thm. 4.14] would similarly determine the multiplicity of the Perron-Frobenius eigenvalues $\chi_0(\lambda)$ as $\sum_\mu Z_{\lambda,0,\mu}^2$ and $\sum_\mu Z_{0,\mu,\lambda}^2$, respectively.
5 Examples

Suppose $N \subset M$ is a braided subfactor with $\iota : N \hookrightarrow M$ being the injection map and basic construction $N \subset M \subset M_1$. Thus if $\iota_1 : M \hookrightarrow M_1$ is the corresponding inclusion, then by naturality the sector $[\mathcal{I}_1 \iota_1]$ of the dual canonical endomorphism for $M \subset M_1$ is identified with the sector of the canonical endomorphism for $N \subset M$, i.e. $\mathcal{I}$. Hence the sector of the dual canonical endomorphism $\theta_1$ for $N \subset M_1$ is $[\mathcal{I}_1 \iota_1 \iota] = [\mathcal{I} \iota] = [\theta^2]$, which lies in $\Sigma(N \mathcal{X}_N)$ as $\theta$ does. In particular, if $N \mathcal{X}_N$ is braided, we can certainly apply $\alpha$-induction to the inclusion $N \subset M_1$. Note that in this context, that the inclusion, $N \subset M_1$ rarely satisfies chiral locality by Corollary 3.6 [5]. We have the naturality equations for $\alpha$-induced morphisms

$$x \varepsilon^\pm (\rho, \lambda) = \varepsilon^\pm (\rho, \mu) \alpha^\pm_\rho (x)$$

whenever $x \in \text{Hom}(\iota \lambda, \iota \mu)$ and $\rho \in \Sigma(N \mathcal{X}_N)$, see e.g. [3, Eq. (9)]. In particular, inducing from $N$ to $M_1$, we have taking $\lambda = \mu = id$, and $x \in \text{Hom}(\iota_1 \iota, \iota_1 \iota)$, that $\alpha_\rho(x) = x$ on $N' \cap M_1$, for all $\rho$.

We will look again at the $SU(3)$ and $SU(2)$ situations in detail in this basic construction.

5.1 $SU(2)$-invariants

By the A-D-E classification [14], we know that there are at most three invariants for each level labelled by Dynkin diagrams. They satisfy the following fusion rules:

$$Z^2_{D_{2g}} = 2Z_{D_{2g}}, \quad Z^2_{D_{2g+1}} = Z_{A_{4g-1}}, \quad Z^2_{E_6} = 2Z_{E_6}, \quad Z^2_{E_7} = Z_{D_{10}} + Z_{E_7}, \quad Z^2_{E_8} = 4Z_{E_8}.$$

(i) Example: $D_j$

We start with $SU(2)$ at even level $k$ and the simple current or orbifold invariants. Here there is a $Z_2$ extension: $N \subset N \rtimes Z_2$, with $N$ as $\pi^0(L SU(2))''$ in the vacuum representation at level $k$, and dual canonical endomorphism $[\lambda_0] \oplus [\lambda_\ell]$. If $k = 4l - 4$, then the extension is local, the corresponding modular invariant is $D_{2l}$, and the canonical endomorphism is $\gamma = [id] \oplus [\alpha_1^\pm]$. If $k = 4l - 2$, then the extension is not local, the corresponding modular invariant is $D_{2l+1}$ and the canonical endomorphism is $\gamma = [id] \oplus [\epsilon]$, where $\epsilon$ is an irreducible subsector of $[\alpha_1^+ \alpha_1^-]$. In either case, the basic construction is by Takesaki duality:

$$N \subset N \rtimes Z_2 \subset N \rtimes Z_2 \rtimes \hat{Z}_2 = N \otimes \text{Mat}_2.$$

Thus by the above naturality, $\alpha_\lambda = \lambda \otimes id$, as here $N' \cap M_1 = \text{Mat}_2$, the $2 \times 2$ complex matrices. Thus $N \mathcal{X}_N$ is identified with $M_1 \mathcal{X}_{M_1}$, and we do not appear to have anything interesting. To see the finer structure, we need to look closer at the dual canonical endomorphism $[\theta_1]$, which decomposes in the local case $k = 4l - 4$, into $[\mathcal{I}]$ and $[\mathcal{I} \alpha_1^\pm]$. Both are dual canonical endomorphisms in their own right. The first can be thought of as giving the sheet of $D_j$ in the full $M_1 \mathcal{X}_M$ system starting at
[id_M] and the second sector as giving the other sheet in the full \( M \mathcal{X}_M \) system. All this becomes clearer in the type I conformal embedding modular invariants.

(ii) Example: \( E_6, SU(2)_{10} \subset SO(5)_1 \)

We now consider the \( E_6 \) modular invariant for \( SU(2) \):

\[
Z_{E_6} = |\chi_0 + \chi_6|^2 + |\chi_4 + \chi_{10}|^2 + |\chi_3 + \chi_7|^2.
\]

This is exhibited by the conformal embedding \( SU(2)_{10} \subset SO(5)_1 \). Here the dual canonical endomorphism \( \theta \) is given by the vacuum sector \( [\theta] = [\lambda_0] \oplus [\lambda_6] \), and the corresponding canonical endomorphism was computed in [7] as:

\[
[\gamma] = [id] \oplus [\alpha^+_1 \alpha^-_1].
\]

Then for the corresponding basic construction \( \mathcal{N} \subset M \subset M_1 \) we have

\[
[\theta_1] = [\lambda_1 \alpha_1] = [\bar{\eta}] \oplus [\bar{\tau} \alpha^+_1 \alpha^-_1].
\]

This time the dual canonical endomorphism \( [\bar{\eta}] \) gives the first sheet of the full \( M \mathcal{X}_M \) system, whilst the second term gives the second sheet of the full system where the sector \( [\alpha^+_1 \alpha^-_1] \) in the full system is identified with the \( N-M \) sector \( [\alpha^-_1 \lambda] \) using the changing the \( \lambda \) vertex argument.

(iii) Example: \( E_8, SU(2)_{28} \subset (G_2)_1 \)

Next let us revisit the \( E_8 \) modular invariant at level \( k = 28 \):

\[
Z_{E_8} = |\chi_0 + \chi_{10} + \chi_{18} + \chi_{28}|^2 + |\chi_6 + \chi_{12} + \chi_{16} + \chi_{22}|^2.
\]

This is exhibited by the conformal embedding \( SU(2)_{28} \subset (G_2)_1 \). The dual canonical endomorphism is again given by the vacuum sector

\[
[\theta] = [\lambda_0] \oplus [\lambda_{10}] \oplus [\lambda_{18}] \oplus [\lambda_{28}].
\]

The canonical endomorphism was computed in [7] as:

\[
[\gamma] = [id_M] \oplus [\alpha^+_1 \alpha^-_1] \oplus [\alpha^+_2 \alpha^-_2] \oplus [\eta],
\]

where \( [\eta] \) was described as an irreducible subsector of \( [\alpha^+_3 \alpha^-_3] \). However by comparing Fig. 8 of [7] with Fig. 5 of [7], and with the above experience for \( E_6 \), we suspect that \( [\eta] \) can be identified with \( [\alpha^+_5 \alpha^-_5] \), where \( [\alpha^+_5] = [\alpha^+_5(1)] \oplus [\alpha^+_5(2)] \) and \( [\alpha^-_5] = [\alpha^{-}_5(1)] \oplus [\alpha^{-}_5(2)] \). We can compute

\[
\langle \gamma, \alpha^+_5(2) \alpha^-_5(2) \rangle = \langle \bar{\tau}, \alpha^+_5(2) \alpha^-_5(2) \rangle = \langle id, \bar{\tau} \alpha^+_5(2) \alpha^-_5(2) \rangle
\]

\[
= \langle id, \bar{\tau} [\alpha^+_5(1) \oplus \alpha^+_3(2) \oplus \alpha^-_5(2)] [\alpha^-_5(1) \oplus \alpha^-_3(2) \oplus \alpha^-_5(2) \rangle
\]

\[
= \langle id, \bar{\tau} [\lambda_5] \oplus [\lambda_3] \oplus [\lambda_7] \rangle^2
\]

\[
= 1
\]

using the Verlinde fusion rules for \( SU(2) \) at level 28. We can similarly show that \( [\alpha^+_5(2) \alpha^-_5(2)] \) is irreducible and disjoint from \( [id_M] \), \( [\alpha^+_1 \alpha^-_1] \) and \( [\alpha^+_2 \alpha^-_2] \). Hence \( [\eta] = [\alpha^+_5(2) \alpha^-_5(2)] \).

There are four sheets in the full system \( M \mathcal{X}_M \), all copies of \( E_8 \). The four terms in \( \gamma \) give rise to the four sheets in the full system with vertices \( [id_M], [\alpha^+_1 \alpha^-_1], [\alpha^+_2 \alpha^-_2], [\alpha^+_5 \alpha^-_5] \) identified with base points \( \iota, \alpha^-_1 \iota, \alpha^-_2 \iota \) and \( \alpha^-_5 \iota \) on the \( N-M \) graph \( E_8 \) using again the argument of changing the \( \iota \) vertex.
5.2 $SU(3)$-invariants

We now move on the the case of $SU(3)$ and its modular invariants.

(i) Example: $E^{(8)}$, $SU(3)_5 \subset SU(6)_1$.

The first conformal embedding invariant is at level 8:

$$Z_{E^{(8)}} = |\chi_{0,0} + \chi_{4,2}|^2 + |\chi_{2,0} + \chi_{5,3}|^2 + |\chi_{2,2} + \chi_{5,5}|^2 + |\chi_{3,0} + \chi_{3,3}|^2 + |\chi_{3,1} + \chi_{5,5}|^2 + |\chi_{3,2} + \chi_{5,0}|^2$$

This can be obtained from the conformal inclusion $SU(3)_5 \subset SU(6)_1$ with dual canonical endomorphism given by the vacuum sector $[\theta] = [\lambda_{0,0}] + [\lambda_{4,2}]$ with the canonical endomorphism computed in [7] as $[\gamma] = [id] \oplus \alpha^-_{1,0} \alpha^-_{1,1}$.

(ii) Example: $E^{(12)}$, $SU(3)_9 \subset (E_6)_1$.

This modular invariant is at level 12:

$$Z_{E^{(12)}} = |\chi_{0,0} + \chi_{9,0} + \chi_{9,9} + \chi_{4,1} + \chi_{1,4} + \chi_{4,4}|^2 + 2|\chi_{2,2} + \chi_{5,2} + \chi_{2,5}|^2.$$

It is obtained from the conformal embedding $SU(3)_9 \subset (E_6)_1$, with dual canonical endomorphism given by the vacuum sector:

$$[\theta] = [\lambda_{0,0}] + [\lambda_{0,0}] + [\lambda_{0,0}] + [\lambda_{4,1}] + [\lambda_{1,4}] + [\lambda_{4,4}], \quad (8)$$

This modular invariant can also be realized from the dual canonical endomorphism

$$\oplus \lambda Z_{\chi_{\lambda}}[\lambda] = [\lambda_{0,0}] + [\lambda_{0,0}] + [\lambda_{0,0}] + [\lambda_{4,1}] + [\lambda_{1,4}] + [\lambda_{4,4}] + 2[\lambda_{2,2}] + 2[\lambda_{5,2}] + 2[\lambda_{2,5}],$$

where the sum is over all sectors in $\mathcal{N}_\mathcal{X}_\mathcal{N}$ using [10] the extension $N \subset M \times \mathbb{Z}_3$, as $E_6$ at level 1 has $\mathbb{Z}_3$ fusion rules.

Now the canonical endomorphism corresponding to Eq. (8) was computed in [7] as

$$[\gamma] = [id] \oplus \alpha^-_{1,0} \alpha^-_{1,1} \oplus \alpha^+_{1,1} \alpha^-_{1,0} \oplus \alpha^+_{2,0} \alpha^-_{2,2} \oplus \alpha^+_{2,2} \alpha^-_{2,0} \oplus \alpha^+_{2,1} \alpha^-_{2,1}.$$

So we expect six sheets in the full $M$-$M$ system, but this is where a surprise appears. We do not get six copies of the $N$-$N$ graph $E^{(12)}_1$. We only get three copies of $E^{(12)}_1$, located at the three sectors $[id]$, $[\alpha^+_{1,0} \alpha^-_{1,1}]$, $[\alpha^+_{1,1} \alpha^-_{1,0}]$ in the $M\mathcal{X}_M$ graph and three copies of the isospectral graph $E^{(12)}_2$ located at the three sectors $[\alpha^+_{2,0} \alpha^-_{2,2}]$, $[\alpha^+_{2,2} \alpha^-_{2,0}]$, $[\alpha^+_{2,1} \alpha^-_{2,1}]$ in $M\mathcal{X}_M$.

Here we show that for the conformal inclusion $SU(3)_9 \subset (E_6)_1$, for which we have six $M\mathcal{X}_M^+$ orbits in $M\mathcal{X}_M$, we find three copies of the graph $E^{(12)}_1$ and three copies of $E^{(12)}_2$.

Let us draw the fusion graph of the generator $[\alpha^+_{(1,0)}]$ in $M\mathcal{X}_M$ in blue. (We use the labelling as in [7, Fig. 12].) The vacuum column forces its identity component, i.e. the chiral fusion graph of $[\alpha^+_{(1,0)}]$, to be $E^{(12)}_1$, see [7]. Now let us think of the fusion graph of $[\alpha^-_{(1,0)}]$ in $M\mathcal{X}_M$ as being red. We now use the fact that $E^{(12)}_j$, $j = 1, 2, 3$, exhaust the list of isospectral graphs. The connected components of the red graph will
correspond to nimreps and hence must be $\mathcal{E}^{(12)}_i$, $j = 1, 2, 3$. (Note that the modular invariant obeys $Z^*Z = 6Z$, hence we must have six layers.) Which one of the three graphs can touch the vertices of the blue $\mathcal{E}_1^{(12)}$? At the identity vertex this is clearly the other chiral graph, determined by the vacuum row to be (a red) $\mathcal{E}_1^{(12)}$. These two (blue and red) $\mathcal{E}_1^{(12)}$’s intersect exactly on the marked (ambichiral) vertices. The other red “coset” graphs will connect the other $M\mathcal{A}^0_M$ fusion orbits in $M\mathcal{A}^+_M$. Now the $M\mathcal{A}^0_M$ fusion orbits are just the $\mathbb{Z}_3$ symmetry orbits of $\mathcal{E}_1^{(12)}$. Thus we will have six red layers: The first is the already determined $\mathcal{E}_1^{(12)}$ corresponding to the $M\mathcal{A}^0_M$ orbit of id. Then there will be one layer connecting $[\alpha^{+(1)}_{(1,0)}]$, $[\alpha^{+(1)}_{(3,1)}]$ and $[\alpha^{+(2)}_{(3,1)}]$, similarly one layer connecting $[\alpha^{+(1)}_{(1,1)}]$, $[\alpha^{+(1)}_{(3,2)}]$ and $[\alpha^{+(2)}_{(3,2)}]$, and finally each $M\mathcal{A}^0_M$ fixed point $[\alpha^{+(2,0)}_1]$, $[\alpha^{+(2,1)}_1]$ and $[\alpha^{+(2,2)}_1]$ are connected to one red layer. To determine the red layer which touches $[\alpha^{+(1)}_{(1,0)}]$, we compute

$$\langle \alpha^{+(1,0)}_1\alpha^{-(1,0)}_1, \alpha^{+(1,0)}_1\alpha^{-(1,0)}_1 \rangle = \langle \alpha^{+(1,0)}_1\alpha^{+(1,1)}_1, \alpha^{-(1,0)}_1\alpha^{-(1,1)}_1 \rangle = 1.$$  

Thus $[\alpha^{+(1)}_{(1,0)}]$ has only one target vertex on the red graph. Hence we must have here either one of the three extremal vertices of $\mathcal{E}_1^{(12)}$ or the unique isolated extremal vertex of $\mathcal{E}_2^{(12)}$. Since $\mathcal{E}_3^{(12)}$ does not have such a vertex, this one is ruled out here. Now note that the target vertices of these extremal vertices have itself two and four target vertices for $\mathcal{E}_1^{(12)}$ and $\mathcal{E}_2^{(12)}$, respectively. But since

$$\langle \alpha^{+(1,0)}_1\alpha^{-(1,0)}_1, \alpha^{+(1,0)}_1\alpha^{-(1,0)}_1 \rangle = 2,$$

we conclude that a red $\mathcal{E}_1^{(12)}$ touches $[\alpha^{+(1)}_{(1,0)}]$. The same is checked for $[\alpha^{+(1)}_{(1,1)}]$, and it cannot lie on the same red $\mathcal{E}_1^{(12)}$ as $[\alpha^{+(1)}_{(1,0)}]$ since this would mean that one is the fusion product of the other by an ambichiral sector. Next we check what red graph touches $[\alpha^{+(2,0)}_1]$. Since

$$\langle \alpha^{+(2,0)}_1\alpha^{-(1,0)}_1, \alpha^{+(2,0)}_1\alpha^{-(1,0)}_1 \rangle = 1,$$

we must again locate an extremal vertex of $\mathcal{E}_1^{(12)}$ or $\mathcal{E}_2^{(12)}$ here. But now

$$\langle \alpha^{+(2,0)}_1\alpha^{-(1,0)}_1, \alpha^{+(2,0)}_1\alpha^{-(1,0)}_1 \rangle = 4,$$

forces us to select $\mathcal{E}_2^{(12)}$. (We use $[\alpha^{+(2,0)}_1][\alpha^{+(2,2)}_1] = [\text{id}] \oplus [\alpha^{+(2,1)}_1] \oplus [\alpha^{+(2,2)}_1]$.) A similar argument applies to $[\alpha^{+(2,1)}_1]$ and $[\alpha^{+(2,2)}_1]$. Thus we have indeed found three layers of $\mathcal{E}_1^{(12)}$ and three layers of $\mathcal{E}_2^{(12)}$.

Di Francesco and Zuber actually produced three isospectral graphs $\mathcal{E}_i^{(12)}$, $i = 1, 2, 3$, whose spectrum reproduced the diagonal part of the modular invariant $\mathcal{E}^{(12)}(SU(3)_9 \subset (E_6)_1)$, and we realized two of those graphs $\mathcal{E}_1^{(12)}$ and $\mathcal{E}_2^{(12)}$ in $\mathbb{H}$. The third was apparently eliminated by $\mathbb{E}$. We certainly know that $\mathcal{E}_3^{(12)}$ does not appear
in a “natural” way in the sense that we have some subfactor \( N \subset M \) producing \( \mathcal{E}_3^{(12)} \) as \( M-N \) graph in the following sense: We know that such a subfactor would have intermediate subfactors \( N \subset M_+ = M_- \) producing the same invariant \( Z_{\mathcal{E}_3^{(12)}} \) and with \( M_+-N \) graph \( \mathcal{E}_1^{(12)} \). This subfactor could not have the “natural” property that the dual canonical endomorphism of \( M_+ \subset M \) decomposes exclusively into ambichiral sectors. This is because we know that the only irreducible braided extensions (relative to the ambichiral system) are the trivial one \( M_+ \subset M = M_+ \rtimes \mathbb{Z}_3 \) where in turn \( N \subset M \) produces \( \mathcal{E}_1^{(12)} \) and \( \mathcal{E}_2^{(12)} \), respectively \([3, 10]\).

(iii) Example: \( \mathcal{E}^{(24)} SU(3)_{21} \subset (\mathbb{E}_7)_1 \):

The corresponding modular invariant reads

\[
Z_{\mathcal{E}^{(24)}} = |\chi_{0,0} + \chi_{21,0} + \chi_{21,21} + \chi_{8,4} + \chi_{17,4} + \chi_{17,13} + \chi_{11,1} + \chi_{11,10} + \chi_{20,10} + \chi_{12,6} + \chi_{15,6} + \chi_{15,9}|^2 + |\chi_{6,0} + \chi_{21,6} + \chi_{15,15} + \chi_{15,0} + \chi_{21,15} + \chi_{6,6} + \chi_{11,4} + \chi_{17,7} + \chi_{14,10} + \chi_{11,7} + \chi_{14,4} + \chi_{17,10}|^2,
\]

therefore

\[
[\theta] = [\lambda_{0,0}] \oplus [\lambda_{21,0}] \oplus [\lambda_{21,21}] \oplus [\lambda_{8,4}] \oplus [\lambda_{17,4}] \oplus [\lambda_{17,13}] \\
[\lambda_{11,1}] \oplus [\lambda_{11,10}] \oplus [\lambda_{20,10}] \oplus [\lambda_{12,6}] \oplus [\lambda_{15,6}] \oplus [\lambda_{15,9}].
\]

Taking the extension \( N \subset M \rtimes \mathbb{Z}_2 \), as the extended system \( E_7 \) at level 1 has \( \mathbb{Z}_2 \) fusion rules, the modular invariant can also be realised from \( \oplus \lambda Z_{\lambda,\lambda}^*[\lambda] \) where the sum is over all sectors in \( N \mathcal{X}_N \).

### 5.3 Towards a pattern

We have seen that we have exactly \( \sum_{\lambda} Z_{\lambda,0}^2 \) (respectively \( \sum_{\lambda} Z_{\lambda,0}^2 \)) \( \mathcal{X}_M^- \) (respectively \( \mathcal{X}_M^+ \)) orbits in \( \mathcal{X}_M \). These intersect with \( \mathcal{X}_M^- \) (respectively \( \mathcal{X}_M^+ \)), i.e. with the \( \mathcal{X}_M^- \) (respectively \( \mathcal{X}_M^+ \)) orbit containing [id], precisely on its \( \mathcal{X}_M^0 \) orbits. We are interested in the particular shape of the \( \mathcal{X}_M^+ \) or \( \mathcal{X}_M^- \) orbits in the full system \( \mathcal{X}_M \). For all examples we know, the products \( ZZ^* \) and \( Z^*Z \) are integral linear combinations of physical invariants, and the linear combination corresponds precisely to the decomposition of the full system in chiral orbits. Note that each \( \mathcal{X}_M^+ \) orbit must be a nimrep. As long as we have a one-to-one correspondence between irreducible\(^1\) nimreps and diagonals of modular invariants we find that at least the diagonal part of \( ZZ^* \) and \( Z^*Z \) can be written as a positive integral linear combination of diagonal parts of modular invariants. Since there are no distinct\(^2\) modular invariants known sharing the same diagonal part, this is a strong indication that there is indeed a general rule.

\(^1\)We do not mean irreducibility in the usual sense for representations here — this would mean “one-dimensional” since our braided systems \( \mathcal{X}_N \) are commutative. Here we rather mean irreducibility in the sense that the sum of the representation matrices is irreducible (in the sense of \([23]\)).

\(^2\)Here we do not worry about the distinction between a sufferable modular invariant and its transpose, which can be obtained from the same subfactor by reversing the braiding.
We can write $Z$ in terms of rectangular branching matrices as $Z = B_+^* B_-$ so that $ZZ^* = B_+^* B_- B_+^* B_-$ and $Z^* Z = B_+^* B_+ B_+^* B_-$. Let us look at the sandwiched $B_\pm B_\pm^*$ which must be invariant under the extended $S$- and $T$-matrices thanks to the intertwining rules of [8, Thm. 6.5]. The extended $S$- and $T$-matrices have at most permutation invariants. If these invariants in fact span the entire commutant of $S$ and $T$ (may well be in general) then $B_\pm B_\pm^*$ must be a linear combination of these permutations. Unfortunately, it is not clear whether this is always an integral linear combination.

It is very instructive to look at some examples. Even type I invariants are interesting here, i.e. when we have $B_+ = B_-$. For instance for the $D_{10}$ invariant of $SU(2)_{16}$ we have

$$B_+ B_+^* = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} = 2 \cdot 1_4 \oplus \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 1_6 + t_0,$$

where $t_0$ is the transposition matrix which exchanges the two marked vertices $[\alpha_8^{(j)}]$, $j = 1, 2$, on the short legs of $D_{10}$. For the $E_7$ invariant we have $B_- = \Pi B_+$, where the permutation $\Pi$ is either $t_j$, $j = 1, 2$, the transpositions exchanging $[\alpha_8^{(j)}]$ with the marked vertex $[\alpha_2]$, or one of the two cyclic permutations $c_1, c_2$. For example, if $\Pi = t_1$, then $B_- B_-^* = 1 + t_2$. Next let us consider the $D^{(12)}$ invariant of $SU(3)_{9}$. Here we find

$$B_+ B_+^* = 3 \cdot 1_6 \oplus \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = 1_9 + c_1 + c_2,$$

where here $c_1, c_2$ denote the two non-trivial cyclic permutations of the three fixed points $[\alpha_{(j)}^{(6,3)}]$, $j = 1, 2, 3$. For simple current invariants with a single full fixed point we probably have a sum over all cyclic permutations of the fixed point constituents in general.

For the conformal inclusion invariant $\mathcal{E}^{(12)}$ we find

$$B_+ B_+^* = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & 3 & 3 \end{pmatrix} = 3 \cdot 1_3 + 3 \cdot C,$$

with the $Z_3$ charge conjugation $C$ exchanging the two non-trivial marked vertices $[\eta_1]$ and $[\eta_2]$. These numbers reflect exactly the appearance of three times $\mathcal{E}_1^{(12)}$ which corresponds to $1$ and three times $\mathcal{E}_2^{(12)}$ which corresponds to $C$. The very special property of this example is that the orbifold corresponding to charge conjugation changes the graph non-trivially, $\mathcal{E}_2^{(12)}$ is the $Z_3$ orbifold of $\mathcal{E}_1^{(12)}$ whereas the modular invariant is self-conjugate, $Z = CZ$. We do not know any other example where this
happens. Other examples for self-conjugate modular invariants which are non-self-conjugate on the extended level are $D_{4\ell}$ for $SU(2)$. But for $D_4$, the conjugation of the extended conjugation is obtained by a $Z_3$ orbifold and $D_4$ is its own $Z_3$ orbifold. For $\ell > 1$, the conjugation will no longer be obtained as an orbifold since we do not have a simple current group as extended theory, but apparently the $D_{4\ell}$ graphs are in general identical with there non-group-like orbifolds. Another example is the conformal inclusion $SU(4)_2 \subset SU(6)_1$, for which the extended conjugation is also obtained by a $Z_3$ orbifold, however, the graphs are their own $Z_3$ orbifolds.

Another strange but different case is the conformal inclusion $SU(4)_6 \subset SU(10)_1$ invariant for which

$$B_+B_+^* = \left( \begin{array}{cc} 4 & 0 \\ 0 & 4 \end{array} \right) \oplus \left( \begin{array}{cc} 3 & 1 \\ 1 & 3 \end{array} \right) \otimes \mathbf{1}_4 = 3 \cdot \mathbf{1}_{10} + C,$$

with the $\mathbb{Z}_{10}$ charge conjugation $C$. An in fact, here we expect three layers of the chiral graph and one layer of the conjugation graph in the entire $M$-$M$ system. (See subsection below.)

### 5.4 Interesting invariants of $SU(n)_n$

In the conclusions of [8] we speculated about modular invariants which look like type I or type II but really come from heterotic extensions, i.e. for which we have different intermediate local subfactors $M_+ \neq M_-$. By the results of [8, Sect. 4] this means that at least for one $\lambda$ we have $\text{Hom}(\text{id}, \alpha^\lambda_+) \neq \text{Hom}(\text{id}, \alpha^\lambda_-)$ in spite of $Z_{\lambda,0} = Z_{0,\lambda}$. Since $\text{Hom}(\text{id}, \alpha^\lambda_+) \subset \text{Hom}(\ell, \ell \lambda)$ this will necessarily require $(\theta, \lambda) \geq 2$ for such $\lambda$. In [8] we pointed out that such a case may be possible but also that did not know of an example.

Here are examples, actually making use of the heterotic $SO(16\ell)_1$ modular invariants ($\ell = 1, 2, 3, \ldots$) treated in [8]. For this we consider once more the series of conformal inclusions $SU(n)_n \subset SO(n^2 - 1)_1$. Note that for $n = 7, 9, 15, 17, 23, \ldots$, i.e. for $n = 8r \pm 1$, $r = 1, 2, 3, \ldots$, the number $n^2 - 1$ is a multiple of 16, so that the ambient algebra has a heterotic extension. (The simplest case is therefore $SU(17)_7 \subset SO(48)_1$.) Using the standard labelling for the sectors of $SO(16\ell)_1$, the two heterotic invariants can be written as

$$Z = \chi^0(\chi^0)^* + \chi^s(\chi^0)^* + \chi^0(\chi^c)^* + \chi^s(\chi^c)^*$$

and $Z^*$ (their coupling matrices are denoted by $Q$ and $tQ$ in [8]). Now let $N \subset M$ denote the conformal inclusion subfactor of $SU(n)_n \subset SO(n^2 - 1)_1$ for $n = 8r \pm 1$, $r = 1, 2, 3, \ldots$. As we know from [8, Sect. 7], there is a crossed product extension by all $SO(64r^2 \pm 16r)_1$ sectors $v$, $s$, $c$ (and 0) $\tilde{M} \subset M = \tilde{M} \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ producing $Z$ and $Z^*$ (using braiding and its opposite). The local intermediate extensions are different, namely the $\mathbb{Z}_2$ extensions corresponding to $s$ and $c$ separately. So let us consider the subfactor $N \subset M$. Then its maximal local intermediate extensions will therefore be $M_+ = \tilde{M} \rtimes_s \mathbb{Z}_2$ and $M_- = \tilde{M} \rtimes_c \mathbb{Z}_2$. Nevertheless the $SU(8r \pm 1)_{8r \pm 1}$ invariant arising from $N \subset M$ does not seem to have non-symmetric vacuum coupling
— all known $SU(n)$ invariants are symmetric. Therefore we expect that the sectors $s$ and $c$ of $SO(64r^2 \pm 16r)_{1}$ will have the same branching rules, i.e. have the same restriction to $SU(8r \pm 1)_{8r \pm 1}.$ (This is quite natural due to the similarity of the sectors $s$ and $c$ of $SO(n)_{1}.$ For $n$ an odd multiple of 8 this similarity even covers the sector $v$ and e.g. for the conformal inclusion $SU(3)_{3} \subset SO(8)_{1}$ all three sectors $v, s, c$ have the same $SU(3)$ restriction.) Anyway, then here we have a heterotic extension, but the identical branching rules of $s$ and $c$ would imply that $Z,$ when written in $SU(8r \pm 1)_{8r \pm 1}$ characters, has symmetric coupling matrix and looks and in particular does not look heterotic anymore. In fact, upon restriction to $SU(8r \pm 1)_{8r \pm 1},$ both $Z$ and $Z^{*}$ will be identical with the invariants $|\chi^{0} + \chi^{s}|^{2}$ and $|\chi^{0} + \chi^{c}|^{2}.$ Hence there will be a 4-fold degeneracy. Due to the permutation $s \leftrightarrow c,$ the original conformal inclusion invariant $|\chi^{0}|^{2} + |\chi^{v}|^{2} + |\chi^{s}|^{2} + |\chi^{c}|^{2}$ will be two-fold degenerate.

Now let $\lambda$ be an $SU(8r \pm 1)_{8r \pm 1}$ sector appearing in the restriction of $s$ and hence of $c.$ Since $Z$ contains $\chi^{s}(\chi^{0})^{*}$ and $\chi^{0}(\chi^{c})^{*}$ it follows that $Z_{\lambda,0} = Z_{0,\lambda}$ is non-zero. On the other hand, since the dual canonical endomorphism sector $[\theta]$ of the full subfactor $N \subset M$ is the $\sigma$-restriction of $[id] \oplus [v] \oplus [s] \oplus [c]$ it follows that $\langle \theta, \lambda \rangle \geq 2$ — as it must be.

Now let us concentrate on the simplest example of this series, the conformal embedding $SU(7)_{7} \subset SO(48)_{1},$ which already seems to produce quite interesting $SU(7)_{7}$ modular invariants. The center of the Weyl alcove, the simple current fixed point $(1, 1, 1, 1, 1, 1)$ (or $[6, 5, 4, 3, 2, 1]$ as a Young frame) appears in the restriction of $s$ and $c,$ with a multiplicity 4. Indeed the branching rules are $[24, \text{Eq. (5.30)}]:$

$$0 \quad \rightarrow \quad \left( (0, 0, 0, 0, 0, 0) \oplus (1, 0, 0, 2, 1, 0) \oplus (0, 1, 0, 0, 0, 2) \right) \times Z_{7} \oplus \left((1, 1, 1, 1, 1, 1) \right)$$

$$v \quad \rightarrow \quad \left( (1, 0, 0, 0, 0, 1) \oplus (3, 0, 0, 1, 0, 0) \oplus (1, 2, 0, 1, 1, 0) \oplus (1, 1, 0, 0, 1, 1) \right) \times Z_{7}$$

$$s, c \quad \rightarrow \quad 4 \cdot (1, 1, 1, 1, 1, 1),$$

where “$\times Z_{7}$” means that the entire $Z_{7}$ simple current orbit has to be taken. Note that then indeed the modular invariants 1 or $W$ of $SO(48)_{1}$ (in the notation of $[3, \text{Sect. 7}]$) restrict to the same $SU(7)_{7}$ invariant, let us call it $Z_{1},$ and similarly a different $SU(7)_{7}$ invariant, let us call it $Z_{s},$ is obtained from either $X_{s}, X_{c}, Q$ or $^{4}Q$ of $SO(48)_{1}$ (i.e. the latter is the specialization of the above $Z$ or $Z^{*}.$) Also note that $Z_{1},$ as it arises from the diagonal invariant $|\chi^{0}|^{2} + |\chi^{v}|^{2} + |\chi^{s}|^{2} + |\chi^{c}|^{2},$ has only two (large) non-zero matrix blocks, because the identical $s$ and $c$ blocks intersect with the vacuum block — this is actually the first modular invariant with this property we have encountered so far. The first block, including the vacuum, is a $36 \times 36$ block, containing 1’s everywhere except a single $33 = 1 + 2 \cdot 4^{2}$ on the corner corresponding to the label $(1, 1, 1, 1, 1, 1).$ Then there is a $28 \times 28$ block of 1’s coming from $v.$ The other invariant $Z_{s},$ arising from $|\chi^{0} + \chi^{s}|^{2}$ (either from $X_{s}, X_{c}, Q$ or $^{4}Q$ in the notation of $[3, \text{Sect. 7}]$) has only a single $36 \times 36$ block, containing a $35 \times 35$ block of 1’s being cornered by a row and a column of 35 entries 5, and they meet with a 25 on the corner corresponding to the label $(1, 1, 1, 1, 1, 1).$ Anyway, these seem to
be interesting modular invariants. First note that we have the curious multiplication rules $Z_1 \cdot Z_1 = 28Z_1 + 8Z_8$ and $Z_6 \cdot Z_8 = 60Z_6$. (Clearly both $Z$’s are selfconjugate in both senses, i.e. $Z = \overline{Z}$ and $Z = Z^*$.) Since $\text{tr}Z_1 = 96$ and $\text{tr}Z_8 = 60$ we will have $\#_M\lambda_N = 96 = \#_M\lambda_M^\pm$, $\#_M\lambda_M = 3168$ for (the two-fold degenerate) $Z_1$, and $\#_M\lambda_N = 60 = \#_M\lambda_M^\mp$, $\#_M\lambda_M = 3600$ for (the 4-fold degenerate) $Z_8$, and that for $Z_1$ the full system will decompose into 28 copies of its chiral graph plus 8 copies of the chiral graph for $Z_8$ whereas we expect for $Z_8$ itself that the full system will decompose into 60 layers of its own chiral graph. (Note that $SU(7)_7$ has 1716 primaries.)

It is tempting to conjecture that for type I invariants, this fusion graph will always consist exclusively of copies of the chiral graph. This is however not the case, as for instance for the $E^{(12)}$ modular invariant of $SU(3)_{9}$ the full system contains besides 3 copies of $E^{(12)}_1$ also 3 copies of the isospectral graph $E^{(12)}_2$, see below. Moreover, even for type I invariance the product $ZZ^*$ is not necessarily a multiple of $Z$. For instance the modular invariant arising from the conformal inclusion $SU(4)_6 \subset SU(10)_1$ fulfills $ZZ^* = 3Z + CZ$, see [10].

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