Abstract

We study the eigenvalues distribution of a GUE matrix with a variance profile that is perturbed by an additive random matrix that may possess spikes. Our approach is guided by Voiculescu’s notion of freeness with amalgamation over the diagonal and by the notion of deterministic equivalent. This allows to derive a fixed point equation to approximate the spectral distribution of certain deformed GUE matrices with a variance profile and to characterize the location of potential outliers in such models in a non-asymptotic setting. We also consider the singular values distribution of a rectangular Gaussian random matrix with a variance profile in a similar setting of additive perturbation. We discuss the application of this approach to the study of low-rank matrix denoising models in the presence of heteroscedastic noise, that is when the amount of variance in the observed data matrix may change from entry to entry. Numerical experiments are used to illustrate our results.

Keywords: Deformed random matrix, Variance profile, Outlier detection, Free probability, Freeness with amalgamation, Operator-valued Stieltjes transform, Gaussian spiked model, Low-rank model.
We first introduce some questions related to the estimation of the eigenvalues distribution of the sum of a GUE matrix with a variance profile and a deterministic matrix that may
possess spikes. Then, we present the main theoretical contributions of the paper and its organization.

1.1 Aim of the article

1.1.1 Deformed Hermitian random matrices

We recall that for any integer $N \geq 1$, a $N \times N$ GUE matrix is a Hermitian random matrix $(\frac{x_{i,j}}{\sqrt{N}})_{i,j=1,\ldots,N}$, such that the sub-diagonal entries are independent, the variables $x_{i,i}$, $i = 1, \ldots, N$, are centered real Gaussian variables with variance one, and the non-diagonal entries $x_{i,j}$ (with $i \neq j$) are centered complex Gaussian variables with variance one. Let $\Gamma_N = (\gamma_N(i,j))_{i,j=1,\ldots,N}$ be a symmetric matrix with non negative entries. Then, the random matrix

$$X_N := \left( \gamma_N(i,j) \frac{x_{i,j}}{\sqrt{N}} \right)_{i,j=1,\ldots,N} \quad (1.1)$$

is called a *GUE matrix of size $N$ with variance profile $\Gamma_N$. Furthermore, we consider a deterministic Hermitian matrix $Y_N$. The random matrix

$$H_N := X_N + Y_N \quad (1.2)$$

is then an (additively) *deformed random matrix*. We shall also study the situation where $Y_N$ has spikes that may generate a finite number of eigenvalues (called outliers) that detach from the rest of the spectrum.

There are two dual ways to interpret model (1.2) in RMT. From the point of view of mathematical physics, the random matrix $X_N$ models the Hamiltonian of a system, and the deformation $Y_N$ is a perturbation coming from an external source. A variance profile for $X_N$ can model impurities of the system. From the signal processing point of view, $Y_N$ models a signal and $X_N$ an additive noise which deforms the data.

In this paper, we present results and heuristics to approximate the eigenvalues distribution of a deformed random matrix by a deterministic function, called classically a *deterministic equivalent*. Our approach involves tools in RMT and free probability, some of them having been created quite recently. They may have applications for the numerical analysis of the spectral distribution of data organized in a matrix form such as noisy images.

The substance of our mathematical arguments is a legacy of the investigations of Haagerup and Thorbjørnsen who developed a strategy in [37] for the study of block GUE matrices, involving classical techniques of Gaussian calculus and complex analysis. In their seminal paper, such a strategy served as an ingredient to reinforce the link between RMT and the abstract theory of $C^*$-algebras of free groups. The analysis in [37] and in our work are both based on Voiculescu’s equations of subordination with amalgamation. Adapting Haagerup and Thorbjørnsen’s method we want to spread their method to more applied contexts that we believe to be of interest in high-dimensional statistics and signal processing. We notify the reader that the aim of this work is not to obtain sharp estimations (that is with optimal rates of convergence) of the spectral distribution of deformed random matrix with a variance profile.
1.1.2 Information plus noise model

If model (1.2) is quite natural from the mathematical point of view, it is more interesting for application in statistics to study rectangular matrices with no symmetry. We define a standard Gaussian matrix as a \( N \times M \) rectangular random matrix \( x_{i,j} \) with \( x_{i,j} \) independent centered complex Gaussian entries with variance one for all \( i,j \). Let \( \Gamma_{N,M} = (\gamma_{N,M}^2(i,j))_{i,j} \) be a \( N \times M \) matrix with non negative entries. Then, the random matrix

\[
X_{N,M} = \left( \gamma_{N,M}(i,j) \frac{x_{i,j}}{\sqrt{M}} \right)_{i=1,...,N, j=1,...,M}
\]

is called a Gaussian matrix with variance profile \( \Gamma_{N,M} \). Let \( Y_{N,M} \) be a deterministic matrix of size \( N \times M \). The random matrix

\[
H_{N,M} = X_{N,M} + Y_{N,M}
\]

is called an information plus noise model with a variance profile.

When \( Y_{N,M} \) is a finite rank matrix, then the rectangular information plus noise model (1.3) corresponds to the low-rank matrix denoising problem which arises in various applications, where it is of interest to estimate a \( N \) by \( M \) signal matrix \( Y_{N,M} \) from noisy data. When the variance profile \( \Gamma_{N,M} \) is a matrix with equal entries, the problem of estimating the low-rank matrix \( Y_{N,M} \) has been extensively studied in statistics and machine learning [19, 26, 45, 53] using spectral estimators constructed from the singular value decomposition of \( H_{N,M} \). These works build upon well understood results of the asymptotic behavior (as \( \min(N, M) \to \infty \)) of the singular values of \( H_{N,M} \) in the Gaussian spiked population model [14, 27].

Hence, low-rank matrix estimation is well understood when the additive noise is Gaussian with homoscedastic variance (that is a constant variance profile). However, in many applications the noise can be highly heteroscedastic meaning that the amount of variance in the observed data matrix may significantly change from entry to entry. Examples can be found in photon imaging [51], network traffic analysis [11] or genomics for microbiome studies [20]. In such applications, the observations are count data that are modeled by Poisson or multinomial distributions which leads to heteroscedasticity. The literature on statistical inference from high-dimension matrices with heteroscedastic noise has thus recently been growing [16, 32, 42, 56, 60].

As discussed in [60], the Gaussian model (1.3) with a variance profile can serve as a prototype for various applications involving low-rank matrix denoising in the presence of heteroscedasticity. The analysis of the outliers in model (1.3) with a non-constant variance profile is also of interest in neural networks to model synaptic matrix [50] and in signal processing for radio communications [24]. Nevertheless, to the best of our knowledge, the characterization of potential outliers in model (1.3) when the noise matrix \( X_{N,M} \) has a variance profile that is not constant has not received very much attention so far.

We now address the two questions on deformed random matrices we investigate.
1.1.3 Global behavior of the eigenvalues of an additive perturbation

The first issue is the global behavior of the eigenvalues of $H_N$. In particular, under general assumptions on $\Gamma_N$ and $Y_N$, we shall consider the question of how approximating the empirical spectral distribution (e.s.d.) of $H_N$:

$$\mu_{H_N} := \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i(H_N)},$$

where $\delta$ is the Dirac mass and $\lambda_i(H_N)$ the $i$-th eigenvalue of $H_N$. As commonly done in random matrix theory (RMT), we use the Stieltjes transform $g_{H_N}$ of the e.s.d. of $H_N$, namely the map defined by the trace of the resolvent $(\lambda I_N - H_N)^{-1}$ of $H_N$ that is

$$g_{H_N}(\lambda) := \frac{1}{N} \text{Tr}[(\lambda I_N - H_N)^{-1}].$$

In above formula, $\lambda$ belongs to the set $\mathbb{C}^+$ of complex number with positive imaginary part. When the context is clear, we write $\lambda - H_N$ for the matrix $\lambda I_N - H_N$. This transform has numerous properties and applications, we refer to Benaych-Georges and Knowles lecture notes [13, Section 2.2] for an introduction. Note that for $\lambda = t + i\eta$ (with $t \in \mathbb{R}$ and $\eta > 0$) then $\frac{1}{\pi} \Im m g_{H_N}(\lambda) = (\mu_{H_N} * \rho_\eta)(t)$ where $*$ denotes the convolution product and $\rho_\eta(t) = \frac{\eta}{\pi^2 + \eta^2}$ is an approximate delta function (Cauchy kernel) as $\eta \to 0^+$. Hence for $\eta$ chosen sufficiently small, the function

$$t \mapsto \frac{1}{\pi} \Im m (g_{H_N}(t + i\eta))$$

is a good approximation of $\mu_{H_N}$ over $\mathbb{R}$ by a random analytic function. In what follows, the smooth density defined by (1.5) will be referred to as the inverse Stieltjes transform of $g_{H_N}$.

Then, we first address the question of the construction of a deterministic analytic function $g_{H_N}^* : \mathbb{C}^+ \to \mathbb{C}^-$, which only depends on $N$, on the variance profile $\Gamma_N$ and on $Y_N$, and that approximates $g_{H_N}$ with high probability.

1.1.4 Outliers location in finite rank perturbation models

The second issue is the effect of a low rank additive perturbation. For notational convenience, we introduce another deterministic Hermitian matrix $Z_N$ of finite rank $k$ (not depending on $N$). We can thus write $Z_N = U_{N,k} \Theta_k U_{N,k}^*$, where $\Theta_k$ is a $k \times k$ diagonal matrix with nonzero diagonal entries and $U_{N,k}$ is a $N \times k$ matrix whose columns are orthonormal vectors. The eigenvalues of $Z_N$ are commonly called the spikes. The matrix

$$H_N^* = X_N + Y_N + Z_N = H_N + U_{N,k} \Theta_k U_{N,k}^*$$

is a finite rank deformation of $H_N$. For such models, we consider the problem of how locating in the spectrum of $H_N^*$ the eigenvalues coming from $Z_N$ that detach from the spectrum of $H_N$. 

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To formulate more precisely this problem, we say that, for a given level of precision \( \varepsilon > 0 \), an outlier is a real eigenvalue \( t \) of \( H'_{N} \) which is not in an \( \varepsilon \)-neighbourhood of the spectrum of \( H_{N} \). Following [14], let us first factorize the spectrum of \( H_{N} \) in \( H'_{N} \), that is, for any complex \( \lambda \) which is not in the spectrum of \( H_{N} \), we consider the decomposition

\[
(\lambda - H_{N}) = (\lambda - H_{N}) \times \alpha_{N}(\lambda),
\]

where

\[
\alpha_{N}(\lambda) = I_{N} - (\lambda - H_{N})^{-1}Z_{N}.
\]

An outlier is then a real number \( t \in \mathbb{R} \) away from the spectrum of \( H_{N} \) such that \( \det(\alpha_{N}(t)) = 0 \). Therefore, noticing that the determinant of \( \alpha_{N}(\lambda) \) is equal to the determinant of the \( k \times k \) matrix

\[
\beta_{k}(\lambda) := I_{k} - U_{N,k}^{*}(\lambda - X_{N} - Y_{N})^{-1}U_{N,k}\Theta_{k},
\]

it follows that, to localize the outliers of \( H'_{N} \), it is sufficient to compute the real numbers \( t \in \mathbb{R} \) such that

\[
\det(\beta_{k}(t)) = 0.
\]

Then, the second question that we address in this paper is the construction of a deterministic matrix-valued function \( \beta_{k}^{\circ} \), which depends on \( N \), on the variance profile \( \Gamma_{N} \), and on the matrices \( Y_{N}, Z_{N} \), and that approximates \( \beta_{k} \). In this manner, the zeros of \( \beta_{k}^{\circ} \) that are away from the spectrum of \( H_{N} \) shall indeed be close to the outliers of \( H'_{N} \).

1.2 Main statements

We now provide the formal statements of our main results and the method to answer the two questions raised above. We let \( D_{N}(\mathbb{C}^{+}) \) (resp. \( D_{N}(\mathbb{C}^{-}) \)) denotes the set of diagonal matrices \( \Lambda = (\Lambda(i, j))_{i,j} \) of size \( N \) with diagonal entries having positive imaginary (resp. negative) parts. For any matrix \( A_{N} = (a_{i,j})_{i,j} \), we denote by \( \Delta(A_{N}) \) the diagonal matrix whose diagonal entries are those of \( A_{N} \). The operator-valued Stieltjes transform \( G_{A_{N}} \) of a Hermitian matrix \( A_{N} \) is defined as the map

\[
G_{A_{N}} : D_{N}(\mathbb{C}^{+}) \rightarrow D_{N}(\mathbb{C}^{-}), \quad \Lambda \mapsto \Delta[(\Lambda - A_{N})^{-1}].
\]

For any \( \Lambda \in D_{N}(\mathbb{C}^{+}) \), we also introduce the mapping

\[
\mathcal{R}_{N}(\Lambda) = \text{deg}(\frac{\Gamma_{N}}{N}\Lambda) = \mathbb{E}[X_{N} \Lambda X_{N}],
\]

that is a key tool in our analysis. The map \( \mathcal{R}_{N} \) may also be written as

\[
\mathcal{R}_{N}(\Lambda) = \text{deg}(\frac{\Gamma_{N}}{N}\Lambda) = \mathbb{E}[X_{N} \Lambda X_{N}],
\]
where $\text{deg}(A)$ for a matrix $A$ is the diagonal matrix whose $k$-diagonal element is the sum of the entries of the $k$-row of $A$. We now state our main result, on the construction of a deterministic equivalent of the operator-valued Stieltjes transform of deformed random matrices and its finite sample properties of approximation.

**Theorem 1.1.** There exists a unique function $G_{H_N}^0 : D_N(\mathbb{C})^+ \to D_N(\mathbb{C})^-$, analytic in each variable, solution of the following fixed point equation

$$G_{H_N}^0(\Lambda) = \Delta \left( \Lambda - R_N(G_{H_N}^0(\Lambda)) - Y_N \right)^{-1},$$

(1.12)

which holds for any $\Lambda \in D_N(\mathbb{C})^+$. Let $\gamma_{\text{max}}^2 = \max_{i,j} \gamma^2_N(i,j)$, denote $\eta = 1$ in the general case, and $\eta = 3/2$ if $Y_N$ is a diagonal matrix. Moreover, let $0 < \delta < 1$, and consider $\Lambda \in D_N(\mathbb{C})^+$ satisfying

$$\Im \Lambda \geq \frac{\gamma_{\text{max}} N^{-\eta/6}}{(1 - \delta)^{1/6}}.$$  

(1.13)

Let $d > 1$ and $N \geq 1$. Then, letting

$$\varepsilon_N(d) := 2\gamma_{\text{max}} \sqrt{\frac{d \log(N)}{\| \Im \Lambda \|^2}} N^{-1/2} + 4 \gamma_{\text{max}} \left( 1 + \frac{\gamma_{\text{max}}^2}{\delta} \| \Im \Lambda \|^2 \right) \| \Im \Lambda \|^{5N^{-\eta}},$$

we have

$$\mathbb{P} \left( \| G_{H_N}(\Lambda) - G_{H_N}^0(\Lambda) \| \geq \varepsilon_N(d) \right) \leq 4N^{1-d},$$

(1.14)

where $\| \cdot \|$ denotes the operator norm of a matrix.

The solution $G_{H_N}^0$ of the fixed point equation is referred to as the *deterministic equivalent* of the operator-valued Stieltjes transform of $H_N$. It has another description given in Section 6.2.2, as the limit of a functional on large random matrices. One may remark that the results of Theorem 1.1 hold without any assumption on the Hermitian matrix $Y_N$ and the variance profile $\Gamma_N$. In particular no bound from below for the entries $\Gamma_N$ is involved to derive the concentration inequality (1.14). If we are given sequences of matrices with growing dimension $N$, then this estimate is also meaningful when $\gamma_{\text{max}}$ grows slowly with $N$.

The proof of Theorem 1.1 is divided into three steps that are detailed in Section 5, Section 6 and Section 7. We then deduce the following methods to answer the two issues raised in Section 1.1.3 and Section 1.1.4.

**Approximation of the global behavior of the e.s.d. of $H_N$ by a smooth density.**

As a corollary of Theorem 1.1 we obtain the following approximation for the Stieltjes transform of $H_N$ (recall that $\gamma_{\text{max}}^2 = \max_{i,j} \gamma^2_N(i,j)$).

**Corollary 1.2.** Let $\eta = 1$ in the general case (and $\eta = 3/2$ if $Y_N$ is a diagonal matrix) and $0 < \delta < 1$. Then, for any $\lambda \in \mathbb{C}^+$ such that

$$\Im \lambda \geq \frac{\gamma_{\text{max}} N^{-\eta/6}}{(1 - \delta)^{1/6}},$$

(1.15)
if one denotes for some $d > 0$

$$
\tilde{\varepsilon}_N(d) := 2\sqrt{2} \frac{\gamma_{\max}}{\sqrt{d \log(N)}} \left( N^{-1} + \gamma_{\max}^4 \left( 1 + \frac{\gamma_{\max}^2}{\delta} |3m\lambda|^2 \right) |3m\lambda|^{-5} N^{-\eta} \right),
$$

and $g_{H_N}(\lambda) := \frac{1}{N} \text{Tr} G_{H_N}^0 (\lambda \mathbb{I}_N)$, it follows that

$$
\mathbb{P}\left( \left| g_{H_N}(\lambda) - g_{H_N}^0(\lambda) \right| \geq \tilde{\varepsilon}_N(d) \right) \leq N^{-d}, \tag{1.16}
$$

where $G_{H_N}^0 (\lambda \mathbb{I}_N)$ is the unique solution of (1.12) for the scalar matrix $\Lambda = \lambda \mathbb{I}_N$.

The concentration inequality (1.16) means that the deterministic equivalent $g_{H_N}^0(\lambda) = \frac{1}{N} \text{Tr} G_{H_N}^0 (\lambda \mathbb{I}_N)$ is a good approximation of $g_{H_N}(\lambda)$ when the imaginary part of $\lambda$ is allowed to decay to zero as the dimension $N$ grows at a rate given by the right hand side of Inequality (1.15). Hence, by the inverse Stieltjes transform formula (1.5), the map $t \mapsto \frac{1}{N} \text{Tr} G_{H_N}^0 (\lambda \mathbb{I}_N) \left( g_{H_N}^0(\lambda + i\eta) \right)$ is a good approximation for the e.s.d. of $H_N$ when $\eta$ is small. However, compared to existing results in the RMT literature, the lower bound (1.15) on $\Im \lambda$ is not optimal, and thus Corollary 1.2 can only be interpreted as a weak local law.

More generally, Condition (1.13) in Theorem 1.1 means that all diagonal entries of $\Lambda$ must have imaginary part larger than $\gamma_{\max}^4 \left( \frac{1}{\delta} |3m\lambda|^2 \right)^{-\frac{5}{4}}$. Therefore, Theorem 1.1 may be interpreted as an operator-valued weak local law for the operator-valued Stieltjes transform of $H_N$. In Section 2.4, we discuss more precisely the connections between our work and existing results on local laws for random matrices with a variance profile.

**Outliers localization in the case where $Y_N$ is diagonal.** From its definition, the natural way to construct a deterministic equivalent for $\beta_k(\lambda)$ is to replace $(\lambda \mathbb{I}_N - X_N - Y_N)^{-1}$ in expression (1.7) by an appropriate deterministic estimate. When $Y_N$ is diagonal, then for any $\Lambda \in \mathbb{D}_N(\mathbb{C})^+$, the matrix $\mathbb{E}\left[(\Lambda - X_N - Y_N)^{-1}\right]$ is also diagonal (thanks to Corollary 7.2). Using a concentration inequality, one has that $(\Lambda - X_N - Y_N)^{-1}$ is thus close to a diagonal matrix, and so we can replace this generalized resolvent with the deterministic equivalent $G_{X_N + Y_N}^0 (\Lambda)$ of the operator-valued Stieltjes transform which is by definition a diagonal matrix. Hence, introducing the matrix valued-function

$$
\beta_k^0(\lambda) = \mathbb{I}_k - U_{N,k}^* G_{H_N}^0 (\lambda \mathbb{I}_N) U_{N,k} \Theta_k, \quad \text{for } \lambda \in \mathbb{C}^+, \tag{1.17}
$$

and the deterministic equivalent $\beta_k^0(\lambda)$ is close to $\beta_k(\lambda)$, in the sense that

$$
\mathbb{P}\left( \| \beta_k(\lambda) - \beta_k^0(\lambda) \| \geq \varepsilon'_N(d) \right) \leq 4k^2 N^{-d}, \tag{1.18}
$$

where, for any $d > 0$,

$$
\varepsilon'_N(d) := 2k \gamma_{\max} \sqrt{d \log(N)} N^{-1/2} + \gamma_{\max}^4 \left( 1 + \frac{\gamma_{\max}^2}{\delta} |3m\lambda|^2 \right) |3m\lambda|^{-5} N^{-\eta}.
$$

The proof of Inequality (1.18) can be found in Section 7.
**Outliers localization, general case.** In general, if $Y_N$ is not diagonal then the expectation of the generalized resolvent, that is $\mathbb{E}[(\Lambda-X_N-Y_N)^{-1}]$, is no longer a diagonal matrix. Hence it is no longer correct to approximate $\beta_k^{\ast}(\lambda)$ by replacing $(\Lambda-X_N-Y_N)^{-1}$ with $G^0_{H_N}(\lambda I_N)$ in Equation (1.7). Yet, assuming that the resolvent $(\Lambda-X_N-Y_N)^{-1}$ is close to its expectation, we can use the so-called Master equality (4.1), that is presented in Section 4, which provides the following approximation
\begin{align*}
(\Lambda-X_N-Y_N)^{-1} &\approx G_{Y_N}(\Omega_{X_N,Y_N}(\Lambda)) = (\Omega_{X_N,Y_N}(\Lambda) - Y_N)^{-1},
\end{align*}
(1.19)
with $\Omega_{X_N,Y_N}(\Lambda) := \Lambda - \mathcal{R}_N(\mathbb{E}[G_{X_N+Y_N}(\Lambda)])$. It appears that the matrix $\Omega_{X_N,Y_N}(\Lambda)$ can be interpreted as an approximate operator-valued subordination function, and we refer to Section 2.2 for further details and discussion on this heuristic. Hence, in the general case, we suggest to define the following deterministic equivalent
\begin{align*}
\tilde{\beta}_k^{\ast}(\lambda) &= I_k - U_{N,k}^*(\Omega^0_{H_N}(\lambda I_N) - Y_N)^{-1}U_{N,k}\Theta_k, \quad \text{for } \lambda \in \mathbb{C}^+, \quad (1.20)
\end{align*}
where $\Omega^0_{H_N}(\Lambda) := \Lambda - \mathcal{R}_N(G^0_{H_N}(\Lambda))$, with $G^0_{H_N}(\Lambda)$ solution of the fixed point Equation (1.12).

**Remark 1.3.** The definition of the estimate (1.20) for $\beta_k^{\ast}(\lambda)$ is driven by heuristic considerations that are detailed in Section 2.2 and Section 2.3. Indeed, the arguments developed to obtain the proof of Theorem 1.14 do not allow to derive a concentration inequality in operator norm between $(\Lambda-X_N-Y_N)^{-1}$ and its approximation by $(\Omega^0_{H_N}(\lambda I_N) - Y_N)^{-1}$. Nevertheless, we use numerical experiments to illustrate the benefits of using $\tilde{\beta}_k^{\ast}(\lambda)$ instead of $\beta_k^{\ast}(\lambda)$ as a deterministic equivalent when $Y_N$ is not diagonal.

**Remark 1.4.** In general, even when $Y_N = 0$ the locations of outliers possibly depend on the eigenvectors of $Z_N$, and there is no longer a canonical way to associate an outlier to a specific spike as it is the case when the variance profile has constant entries. We shall discuss this specific case in the literature review proposed in Section 2, and this property will be illustrated by numerical experiments in Section 3.

**1.3 Organization of the paper**

In Section 2 we relate the approach followed in this paper to the existing literature in RMT and free probability on deformed models and outlier detection. Then, we report the results of numerical experiments in Section 3 to shed some light on the benefits of our approach to localize potential outliers in information plus noise models with a variance profile. We present the strategy of the proof of Theorem 1.1 and its organization in Section 4. The details of the main steps of the proof are then gathered in Section 5, Section 6 and Section 7.

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2 Related literature and methods

2.1 Additive perturbations and the standard spiked population model

We first discuss the case when all entries of the variance profile equal one, that is \( X_N \) is a standard GUE matrix. The celebrated Wigner’s Theorem [59] states that the empirical spectral distribution of \( X_N \) converges to the semicircular distribution \( \mu_x : = \frac{(2\pi)^{-1/2} (1 - \rho^2) \mathbb{1}_{[-2,2]} dt}{\sqrt{4 - \rho^2}} \). Pastur studies in [47] the global behaviour of a GUE matrix with additive perturbation. Assuming that the e.s.d. \( \mu_Y \) of \( Y_N \) converges to a measure \( \mu_y \), then almost surely the e.s.d. of \( X_N \) `\( Y_N \) converges to a measure denoted \( \mu_x \) `\( Y_N \). In general, there is no explicit description of \( \mu_x \) `\( Y_N \), but the limiting Stieltjes transform \( g_x \) `\( Y_N \) satisfies the so-called Pastur’s equation, which is expressed in terms of the Stieltjes transform \( g_y \) of \( \mu_y \) as follows: for all \( \lambda \) in \( \mathbb{C} \),

\[
g_x + y(\lambda) = g_y(\lambda - g_x + y(\lambda)).
\]

(2.1)

Note that \( Y_N = 0 \) implies \( \mu_y = \delta_0 \) and the above equation reads \( g_x(\lambda) = (\lambda - g_x(\lambda))^{-1} \).

In practical applications, we are given a matrix \( Y_N \) of fixed size and not a sequence whose e.s.d. converges to a certain distribution \( \mu_y \). The deterministic equivalent method, for Pastur’s equation (2.1), consists in replacing \( \mu_y \) by the true e.s.d. \( \mu_Y \). In this setting, there is a unique analytic map \( g_{X_N + Y_N}^\circ : \mathbb{C}^+ \rightarrow \mathbb{C}^- \), solution of the fixed-point equation

\[
g(\lambda) = g_Y(\lambda - g(\lambda)), \quad \forall \lambda \in \mathbb{C}^+,
\]

(2.2)

for a map \( g : \mathbb{C}^+ \rightarrow \mathbb{C}^- \). Note that there is a convenient abuse of notation in the sense that \( g_{X_N + Y_N}^{\circ} \) depends only on \( Y_N \). We say that \( g_{X_N + Y_N}^{\circ} \) is a deterministic equivalent of \( g_{X_N + Y_N} \). The interest is that \( g_{X_N + Y_N}^{\circ} \) is a good approximation of \( g_{X_N + Y_N} \) of the Stieltjes transform that we can approximate numerically thanks to the fixed-point equation (2.2).

A fundamental result of phase transition for finite rank deformed random matrices was discovered by Ben Arous, Baik and Péché [10] in the slightly different model of Gaussian matrices without symmetry. Called in short BBP-transition, the analogue result for GUE matrices [49] states that, in the large \( N \) limit, a spike \( \theta \) of \( Z_N \) will create an outlier \( \sigma \in X_N + Z_N \) only if \( \theta > 1 \) in which case \( \sigma = \theta + \theta^{-1} \). For related results, we also refer to [22,23,33,34].

Remark 2.1. When all entries of the variance profile equal one, we emphasis that the existence and the position of an outlier is independent of the eigenvector of \( Z_N \) associated to the spike \( \theta \).

2.2 The free probability approach

More generally, the issues discussed above have also been considered when \( X_N \) is a unitary invariant random matrix. In this context, Voiculescu’s notion of asymptotic
freeness [58] implies a generalization of Pastur’s equation. Assume that \( \mu_{X_N} \) and \( \mu_{Y_N} \) have limiting e.s.d. \( \mu_x \) and \( \mu_y \) respectively. Recall that, denoting \( g_x \) the Stieltjes transform of \( \mu_x \), the \( R \)-transform \( R_x \) of \( x \) is the analytic map satisfying

\[
g_x(\lambda) = \left( \lambda - R_x(g_x(\lambda)) \right)^{-1}. \tag{2.3}
\]

Then \( \mu_{X_N} + Y_N \) converges to a measure \( \mu_{x+y} = \mu_x \boxplus \mu_y \), called the free convolution of \( \mu_x \) and \( \mu_y \). This limit is characterized by the so-called subordination property: \( \forall \lambda \in \mathbb{C}^+ \),

\[
g_{x+y}(\lambda) = g_y(\lambda - R_x(g_{x+y}(\lambda))). \tag{2.4}
\]

The map \( R_x \) is linear if and only if \( \mu_x \) is a centered semicircular distribution. The method of deterministic equivalent can be extended to this case using Voiculescu’s equation (2.4) instead of Pastur’s one. The difficulty in general is to compute the \( R \)-transform \( R_x \), or to replace it with a good approximation.

For finite rank deformation, an important discovery was made in the early decade by Capitaine [21]. We recall the heuristic presented in [12] and refer to this paper for the mathematical arguments, without defining the notions of free probability. In the context of Voiculescu’s problem, the limit \( x \) of \( X_N \) and \( y \) of \( Y_N \) are modelled in the free von Neumann algebra generated by two self-adjoint variables \( x \) and \( y \) with distribution \( \mu_x \) and \( \mu_y \) respectively. Let \( E_y \) be the projection on the von Neumann algebra generated by \( y \). Then Biane proved [15] that there exists an analytic map \( \omega_{x,y} \) defined outside the spectrum of \( x+y \), called the subordination function, such that

\[
E_y[(\lambda - x - y)^{-1}] = (\omega_{x,y}(\lambda) - y)^{-1}, \forall \lambda \in \mathbb{C}^+. \tag{2.5}
\]

The above equality means that the projection of the resolvant of \( x+y \) equals the resolvant of \( y \) evaluated at the subordination function. Taking the trace in the identity yields the relation for Stieltjes transforms

\[
g_{x+y} = g_y(\omega_{x,y}(\lambda)), \forall \lambda \in \mathbb{C}^+.
\]

The reader not familiar with free probability language can still translate this result into a heuristic for unitary invariant matrices by replacing the condition expectation \( E_y \) by the classical expectation \( E \) and the variables by the matrices. One method to introduce an approximate subordination function consists in setting

\[
\omega_{X_N,Y_N}(\lambda) := (E[ (\lambda - X_N - Y_N)^{-1} ])^{-1} + Y_N,
\]

so that we have \( E[ (\lambda - X_N - Y_N)^{-1} ] = (\omega_{X_N,Y_N}(\lambda) - Y_N)^{-1} \), similar to (2.5).

The matrix \( \omega_{X_N,Y_N} \) is not a scalar, but this property is true for the compression involved in the outlier detection problem. If \( X_N \) is unitarily invariant, then \( E[ (\lambda - X_N - Y_N)^{-1} ] \) belongs to the unital algebra generated by \( Y_N \). Let now assume that \( Z_N = U_{N,k} \Theta U_{N,k}^* \), whose eigenvectors are orthogonal to those of \( Y_N \). This property ensures that the spikes are eigenvalues of \( Y_N + Z_N \). It implies that for any matrix \( A \) in
the algebra generated by $Y_N$, its compression $A \mapsto U_{N,k}^*AU_{N,k}$ is a scalar matrix. Hence in particular

$$U_{N,k}^*\omega_{X_N,Y_N}(\lambda)U_{N,k} = (U_{N,k}^*\mathbb{E}[(\lambda - X_N - Y_N)^{-1}]U_{N,k})^{-1} + 0,$$

is a scalar matrix, and it is actually a good approximation for $\omega_{x,y}(\lambda)\|k$.

Concentration properties of unitarily invariant matrices implies that the function $(\beta_k)$ defined in (1.7) is close to its expectation, and so $\det(\beta_k(\lambda))$ is well approximated by

$$\prod_{i=1}^k \left( 1 - \frac{\theta_i}{\omega_{x,y}(\lambda)} \right),$$

where $\theta_1, \ldots, \theta_k$ denote the eigenvalues of $Z_N$. Hence, we retrieve the fundamental Capitaine’s relation [21] between spikes and outliers, namely in the large $N$ limit, the locations of the outliers belong to the pre-image by $\omega_{x,y}$ of the spikes.

**Remark 2.2.** When $X_N$ is unitary invariant random matrix, we remark that the locations of outliers in $X_N + Y_N + Z_N$ depend individually on the eigenvalues of $Z_N$. They do not depend on its eigenvectors, and the outliers generated by a spike $\theta$ do not depend on the other spikes.

### 2.3 Random matrices with a variance profile in free probability

The asymptotic of GUE matrix with a variance profile is characterized by Shlyakhtenko in [54] in the multi-matrix setting of operator-valued free probability over the diagonal. Assuming the variance profiles are of the form $\gamma^2_N(i,j) = \gamma^2 \left( \frac{i}{N}, \frac{j}{N} \right)$, for some bounded real-valued function $\gamma : [0,1]^2 \rightarrow \mathbb{R}^+$, Shlyakhtenko proved that independent GUE matrices with variance profiles are asymptotically free with amalgamation over the diagonal.

In particular, this implies that the e.s.d. of $X_N$ converges almost surely, and that the limit is characterized by an integral operator with kernel $\gamma^2$. The approach is in the lineage of previous works on band matrices, see [17, 40, 46], for which it was observed that to derive the limiting e.s.d. of a random matrix one can derive a system of linear equations for the diagonal of the resolvent $(\lambda I_N - X_N)^{-1}$ of the matrix. An interest in Shlyakhtenko’s approach is the use of the notion of operator-valued free probability, which (in particular) nicely generalizes Pastur’s equation.

More formally, Shlyakhtenko considers in [54] the operator-valued Stieltjes transform $G_{X_N}$ of $X_N$, which the a map between sets of diagonal matrices defined by (1.9). For any bounded function $\Lambda : [0,1] \rightarrow \mathbb{C}^+$, let $\Lambda_N \in D_N(\mathbb{C})^+$ defined by $\Lambda_N(i,i) = \Lambda \left( \frac{i}{N} \right)$. Then, for any $\Lambda : [0,1] \rightarrow \mathbb{C}^+$, the diagonal matrix $G_{X_N}(\Lambda_N)$, seen as a piece-wise constant function on $[0,1]$, converges to a function $G_x(\Lambda)$ in $L^\infty([0,1],\mathbb{C}^-)$. The functional map $G_x$ is characterized the identity

$$G_x(\Lambda) = \left( \Lambda - \mathcal{R}_x(G_x(\Lambda)) \right)^{-1},$$

where for any $\Lambda$ in $L^\infty([0,1],\mathbb{C})$,

$$\mathcal{R}_x(\Lambda) := \int_0^1 \gamma(\cdot, y)\Lambda(y)dy.$$
Since $R_x$ is linear, we say that the abstract limit $x$ of $X_N$ (which lives in a von Neumann algebra) is a semicircular variable with amalgamation over the diagonal. Note that the mapping $R_N$ defined by (1.10) is a discretization of the functional map $R_x$.

Recently, the traffic method yields to the observation that asymptotic freeness over the diagonal was a generic rule for large permutation invariant random matrices with a variance profile [7]. We mention briefly a consequence of this result, referring to [44] for definitions. Let $X_N = X'_N \circ \Gamma_N$ be the entry-wise product of a permutation invariant random matrix $X'_N$ that converges in traffic distribution and of a matrix $\Gamma_N$ that converges in graphons topology [44, see second item in Corollary 2.19]. Let $Y_N$ be a matrix bounded in operator norm that converge in traffic distribution. Then, under the additional assumption that $\Gamma_N$ and $Y_N$ are permutation invariants, $X_N$ and $Y_N$ are asymptotically free over the diagonal. They converge to elements $x$ and $y$ of a von Neumann algebra endowed with a conditional expectation $\Delta$, whose operator-valued Stieltjes transforms satisfy

$$G_{x+y}(\Lambda) = G_y\left(\Lambda - R_x(G_{x+y}(\Lambda))\right),$$  

(2.8)

where $R_x$ is such that the above equation is valid for $y = 0$. For more details about the operator-valued subordination property, we refer to [57].

A motivation of our work is to specify this statement, in a comprehensive way, when $X_N$ is a GUE matrix with a variance profile and to give an estimate for its operator-valued Stieltjes transform. Then, we explicit the associated deterministic equivalent and we show how to adapt Capitaine’s approach for outlier detection.

### 2.4 The Dyson equations and deterministic equivalents in RMT

In the RMT literature, Hermitian random matrices with centered entries but non-equal distribution are referred to as generalized Wigner matrices for which many asymptotic properties are now well understood in a precise sense. For example, under the assumption that the variance profile is bi-stochastic (that is its rows and columns elements sum up to one), bulk universality at optimal spectral resolution for local spectral statistics have been established in [31] and they are shown to converge to those of the GUE. The case of a Wigner matrix with a variance profile that is not necessarily bi-stochastic has been studied in [2], and non-hermitian random matrices with a variance profile have been considered in [25,38,39] using the notion of deterministic equivalent.

Under mild assumptions on the variance profile $\Gamma_N$, the e.s.d. of a generalized Wigner matrix converges to a limiting spectral measure for which there is generally no explicit formula. Currently, a classical method to approximate an asymptotic spectral measure is to solve a nonlinear system of deterministic equations that are referred to as the Dyson equation [2–5]. For each fixed dimension $N$, the solution of this equation yields a deterministic equivalent of the resolvent of $H_N = X_N + Y_N$. These equations are also equivalent to the operator-valued equation of the subordination functions where the parameter is scalar, instead of being functional. For example, the vector Dyson equation studied in details in [2] corresponds to Equation (1.12) with $\Lambda = \lambda I_N$ and

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$Y_N$ a diagonal matrix. In our setting, the matrix Dyson equation, introduced in [3] to study Hermitian random matrices with correlated entries and nonzero expectation, is the following nonlinear matrix equation formulated for an unknown matrix-valued function $A_N: \mathbb{C}^+ \rightarrow \mathbb{C}^{N \times N}$

\[ \mathbb{I}_N - (\lambda \mathbb{I}_N - \mathcal{S}_N(A_N(\lambda)) - Y_N) A_N(\lambda) = 0 \]  

(2.9)

for $\lambda \in \mathbb{C}^+$, where $\mathcal{S}_N$ is the mapping (see Equation (1.3) in [3])

\[ \mathcal{S}_N(A) = \mathbb{E}[X_N A X_N] = \mathcal{R}_N(\Delta(A)), \text{ for } A \in \mathbb{C}^{N \times N}. \]  

(2.10)

Thus, provided that $A_N(\lambda)$ is invertible, Equation (2.9) may be written as

\[ A_N(\lambda) = (\lambda \mathbb{I}_N - \mathcal{R}_N(\Delta(A_N(\lambda)) - Y_N)^{-1}. \]

Hence, applying the operator $\Delta$ of both sides of the above equality yields the fixed point Equation (1.12) with scalar parameter $\Lambda = \lambda \mathbb{I}_N$ and $Y_N$ an arbitrary Hermitian matrix.

The existence and stability of the solutions of the vector and matrix Dyson equation are studied in details in [2] and [3] respectively. These deterministic vector or matrix valued functions (parametrized by $\lambda \in \mathbb{C}^+$) are used to prove local laws for the resolvent of $H_N$. In RMT, the derivation of local laws refers to results controlling the difference between the Stieltjes transform $g_{H_N}(\lambda)$ (or the resolvent $(\lambda \mathbb{I}_N - H_N)^{-1}$) and a deterministic function when $\Im \lambda$ is allowed to decay to zero at a rate depending on $N$. The notion of local semicircular law for Wigner matrices, which constituted the central open question known as Wigner-Dyson-Mehta conjecture, was solved in 2011 independently by [28] and [55]. We refer to [13] for a presentation of this notion. For a recent overview on the derivation of optimal local laws for Hermitian random matrices using the deterministic solution of a Dyson equation, we refer to Erdős’s lecture notes [29]. In [2,3], the authors make several assumptions to derive local laws, in particular they suppose that the entries of the variance profile $\Gamma_N$ are bounded away from below. In [30] this assumption is relaxed: a weak local law is obtained for the model $H_N = X_N + Y_N$ without lower bound assumptions on the variance profile. The generalized Wigner matrix $X_N$ can possibly have correlated entries that are not necessarily Gaussian random variables (see Theorem 2.1 in [30] as well as the preliminary result [6, Lemma B.1]).

It would be interesting to have a generalization of optimal local laws as in [2,3,30] but in an operator-valued sense, that is for $(\Lambda - H_N)^{-1}$, when $\Lambda$ is not a scalar matrix as we consider in the current paper.

## 3 Applications and numerical illustrations

In this section, we report results on numerical experiments on the localization of outliers in the rectangular information plus noise model (1.3) for various variance profiles and additive perturbations that may posses spikes generating outliers.
3.1 Information plus noise model with a variance profile

To include the setting where the additive perturbation may possess spikes, we introduce, for notational convenience, the model

\[ H_{N,M}' = X_{N,M} + Y_{N,M} + Z_{N,M}. \]  

(3.1)

where \( Z_{N,M} = U_{N,k} \Theta_k V_{M,k}^* \) is a low-rank matrix \( N \times M \) with \( k \) spikes that are equal to the singular values of \( Z_{N,M} \), where \( \Theta_k \) is a \( k \times k \) diagonal matrix with positive diagonal entries and \( U_{N,k} \) (resp. \( V_{M,k} \)) is the matrix whose columns are the left (resp. right) singular vectors of \( Z_{N,M} \).

The question of locating potential outliers in model (3.1) can be answered using the approach developed in this paper for the Hermitian setting. To this end, we use the principle of Hermitian dilation [48] which corresponds to embed any rectangular matrix \( A_{N,M} \) (with complex entries) of size \( N \times M \) within a larger Hermitian block matrix by defining

\[ D(A_{N,M}) = \begin{bmatrix} 0 & A_{N,M} \\ A_{N,M}^* & 0 \end{bmatrix}. \]  

(3.2)

Note that if one denotes by \( \sigma_1 \geq \ldots \geq \sigma_r > 0 \) the singular values of \( A_{N,M} \) assumed to be of rank \( r \), then the spectrum of the Hermitian matrix \( D(A_{N,M}) \) is

\[ \{-\sigma_1 \leq \ldots \leq -\sigma_r \leq 0 \leq \sigma_r \leq \ldots \leq \sigma_1\} \]

where the eigenvalue 0 is of multiplicity \( M + N - 2r \). By applying Hermitian dilation to Equation (1.3), we obtain that

\[ D(H_{N,M}') = D(X_{N,M}) + D(Y_{N,M}) + D(Z_{N,M}), \]

which is a finite rank deformation of the GUE model \( D(H_{N,M}) = D(X_{N,M}) + D(Y_{N,M}). \) For \( i \leq N \) and \( j \geq N + 1 \) the entry \((i,j)\) of \( D(X_{N,M}) \) is a centered complex Gaussian variable with variance \( \frac{\gamma_{N,M}(i,j)}{M} \) satisfying \( x_{ij} = \sqrt{\frac{N+M}{M}} \left( \frac{x_{ij}}{\sqrt{M}} \right) \). Therefore, \( D(X_{N,M}) \) is a GUE matrix of size \( N + M \) with variance profile \( \frac{N+M}{M} D(\Gamma_{N,M}) \), where the zero entries of \( D(X_{N,M}) \) are considered as centered Gaussian variables with variance equal to zero. The additive perturbation \( D(Z_{N,M}) \) is a matrix of rank \( 2k \) that can be written as

\[ D(Z_{N,M}) = W_{N+M,2k} \begin{bmatrix} \Theta_k & 0 \\ 0 & -\Theta_k \end{bmatrix} W_{N+M,2k}^* \]

where

\[ W_{N+M,2k} = \frac{1}{\sqrt{2}} \begin{bmatrix} -U_{N,k} & U_{N,k} \\ -V_{M,k} & -V_{M,k} \end{bmatrix} \]

is a \((N + M) \times 2k\) matrix whose columns are orthonormal vectors. After introducing the Hermitian dilation of model (1.3), one may thus consider the function \( G_{D(H)}^{\text{herm}} \):
D_{N+M}(\mathbb{C})^+ \rightarrow D_{N+M}(\mathbb{C})^-, \text{ analytic in each variable, that is the unique solution of the fixed point equation}

\[ G_{D(H)}^o(\Lambda) = \Delta \left( \Lambda - R_{N+M} \left( G_{D(H)}^o(\Lambda) - D(Y_{N,M}) \right)^{-1} \right). \]  

which holds for any \( \Lambda \in D_{N+M}(\mathbb{C})^+ \). Following Equation (1.11), \( R_{N+M} \) is the map defined on \( D_{N+M}(\mathbb{C})^+ \) by

\[ R_{N+M}(\Lambda) = \text{deg} \left( \frac{1}{N+M} \times \frac{N+M}{M} \Delta(\Gamma) \Lambda \right) = \text{deg} \left( \frac{D(\Gamma)}{M} \Lambda \right). \]  

Hence, after Hermitian dilation, one may follow the approach described in Section 1.2 to approximate of the global behavior of the empirical distribution of the singular values of \( H_{N,M} \) and to localize potential outliers generated by the spikes of \( Z_{N,M} \).

**Remark 3.1.** In the RMT literature, the Hermitian dilation (3.2) of a rectangular random matrices is classically referred to as Girko’s Hermitization trick [35,36]. As our study of a GUE matrix with variance profile allows the setting where large blocks of \( D(\Gamma_{N,M}) \) are equal to zero, treating the setting of the information plus noise model using Girko’s Hermitization is an immediate application of our results in the Hermitian case. As explained e.g. in [4,5] this Hermitization trick may also be used beyond the Gaussian case. However, a direct application of the results in [2,3] on the vector and matrix Dyson equation for the study of generalized Wigner matrices is not possible as a key assumption in these papers is that the entries of the variance profile must be bounded from below. Hence, beyond the Gaussian case, the use of Girko’s Hermitization requires a specific treatment in [4,5] and the introduction of a system of quadratic vector equations extensively studied in [1] that relates the resolvent of \( X_{N,M} X_{N,M}^* \) to the resolvent of \( D(X_{N,M}) \) via the equality \( (\lambda^2 I_N - X_N X_N^*)^{-1} = A_{1,1}(\lambda)/\lambda \), where \( A_{1,1}(\lambda) \) denotes the upper left \( N \times N \) block of \( (\lambda I_{N+M} - D(X_{N,M}))^{-1} \).

### 3.2 Additive deformation of rank one

Let us first consider the rectangular model (3.1) under the simplified setting \( Y_{N,M} = 0 \) and the low rank denoising model with an additive deformation of rank \( k = 1 \), that is

\[ H_{N,M}^k = X_{N,M} + Z_{N,M}, \quad \text{where} \quad Z_{N,M} = \theta u_N v_M^*, \]  

where \( u_N \in \mathbb{R}^N, v_M \in \mathbb{R}^M \) are unit vectors, \( \theta > 0 \) is a spike, and \( X_{N,M} \) is a rectangular Gaussian matrix with a variance profile \( \Gamma_{N,M} = (\gamma_{N,M}^2(i,j))_{i,j} \). We assume that the variance profile satisfies

\[ \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{M} \frac{\gamma_{N,M}^2(i,j)}{M} = 1, \]  

which ensures the normalization condition \( \mathbb{E} \left[ \frac{1}{N} \text{Tr} X_{N,M} X_{N,M}^* \right] = 1 \). In all the numerical experiments we took \( N = 360 \) and \( M = 400 \). Following the discussion in Section 1.2 and
the principle of Hermitian dilation described in Section 3.1, a potential outlier in model 
(3.5) may be found by searching for a positive real \( \lambda \) such that

\[
\det \left( \beta_2^\alpha(\lambda) \right) = 0, \tag{3.7}
\]

where

\[
\beta_2^\alpha(\lambda) = \mathbb{I}_2 - \frac{1}{2} \begin{bmatrix}
-\frac{u_N^*}{u_N} & -\frac{v_M^*}{v_M} \\
\frac{u_N}{u_N} & \frac{v_M}{v_M}
\end{bmatrix} G_{\mathcal{D}(\mathcal{H})}^0(\lambda) \mathbb{I}_{N+M} \begin{bmatrix}
-\frac{u_N}{u_N} & \frac{v_M}{v_M} \\
\frac{v_M^*}{v_M} & -\frac{u_N^*}{u_N}
\end{bmatrix} \begin{bmatrix}
\theta & 0 \\
0 & -\theta
\end{bmatrix}, \tag{3.8}
\]

and \( G_{\mathcal{D}(\mathcal{H})}^0(\lambda) \) is the solution of the fixed point equation (3.3), with \( \mathcal{D}(\mathcal{Y}_{N,M}) = 0 \).

Remark 3.2. We choose to directly search for potential outliers by minimizing \( \beta_2^\alpha \) over the set of reals \( \lambda \in \mathbb{R} \) rather than over the set of complex values \( (\lambda = t + i\eta)_{t \in \mathbb{R}} \) with a small and fixed value \( \eta > 0 \) as both approaches lead to the same numerical results.

A numerical approximation (for a given value of \( \lambda \)) is easily obtained by the following iterative procedure

\[
G_{n+1}^0(\lambda) = \Delta \left( \lambda \mathbb{I}_{N+M} - \mathcal{R}_{N+M}(G_n^0(\lambda)) \right)^{-1}, \tag{3.9}
\]

that is stopped for \( n \) sufficiently large or when the difference between two successive iterations is sufficiently small. Since \( \lambda \mathbb{I}_{N+M} - \mathcal{R}_{N+M}(G_n^0(\lambda)) \) is a diagonal matrix, the fixed point iteration corresponds to the numerical evaluation of the vector Dyson equation [2], and it simplifies to the vector equation

\[
G_{n+1}^0(\lambda) = \frac{1}{\lambda \mathbb{I}_{N+M} - \mathcal{D}(\mathcal{Y}_{N,M})} G_n^0(\lambda). \tag{3.10}
\]

In practice, to find a potential solution to equation (3.7), we use a numerical optimization procedure to obtain a minimizer \( \lambda = \hat{\lambda}(\theta) \) of the function \( \lambda \rightarrow \det (\beta_2^\alpha(\lambda)) \) over \( \mathbb{R}_+ \). To this end, we have used Matlab’s command \texttt{fminsearch} which is based on the Nelder-Mead simplex method. Then, if the value \( \det (\beta_2^\alpha(\hat{\lambda}(\theta))) \) is sufficiently close to zero, we conclude that \( \hat{\lambda}(\theta) \) is an outlier.

Finally, a smooth approximation of the singular values distribution (s.v.d.) of \( \mathcal{H}_{N,M} = X_{N,M} \) at location \( t \in \mathbb{R} \) may be obtained from the inverse Stieltjes transform formular (1.5). Letting \( g^\alpha_N(\lambda) = \frac{1}{N+M} \text{Tr} G^\alpha_N(\lambda) \), we define \( f_n(t) = \frac{2}{\pi} \Im (g^\alpha_N(t + i\eta)) \) for \( \eta > 0 \) small enough. Now, recall that \( g^\alpha_N(\lambda) \) is an approximation of the dilation matrix \( \mathcal{D}(\mathcal{H}_{N,M}) \) whose eigenvalue values are 0 (with multiplicity \( M - N \)) and \( \{-\sigma_1, \ldots, -\sigma_N, \sigma_N, \ldots, \sigma_1\} \) where \( \sigma_N \leq \ldots \leq \sigma_1 \) are the singular values of \( \mathcal{H}_{N,M} \), and that the inverse Stieltjes transform (1.5) amounts to approximate a measure by a convolution with the Cauchy kernel \( t \mapsto \frac{\eta}{t^2 + \eta^2} \). Therefore, an approximation of the s.v.d. of \( \mathcal{H}_{N,M} \) is given by the density

\[
\hat{f}_n(t) = \frac{2}{1 - (M - N)/(M + N)} \left( f_n(t) - \frac{M - N}{M + N} \frac{\eta}{t^2 + \eta^2} \right), \quad t \geq 0. \tag{3.11}
\]
3.2.1 Constant variance profile

We propose to validate this way of localizing outliers by first considering the standard case where the variance profile \( \Gamma_{N,M} \) has constant entries equal to one. This setting corresponds the so-called Gaussian spike population model for which the asymptotic behavior (as \( \min(N,M) \to +\infty \)) of the singular values of \( H_{N,M}' \) is well understood \([14, 27, 43, 53]\) when the rank \( k \) of the additive deformation \( Z_{N,M} \) is held fixed. In the asymptotic framework where the sequence \( M = M_N \geq N \) is such that \( \lim_{N \to +\infty} \frac{N}{M} = c \) with \( 0 < c \leq 1 \), it is well known \([9]\) that the empirical distribution of the singular values of \( X_{N,M} \) converges, as \( N \to +\infty \), to the quarter circle distribution if \( c = 1 \), or to its generalized version if \( c < 1 \), called the Marchenko-Pastur distribution, which is supported on the compact interval \( [c_-, c_+] \) with \( c_\pm = 1 \pm \sqrt{c} \) where \( c_+ \) is the so-called bulk (right) edge. Now, if one denotes by \( \sigma_1 \geq \ldots \geq \sigma_N > 0 \) (recall that \( N \leq M \)) the singular values of \( H_{N,M}' \), then the following result holds (see e.g. Theorem 2.8 in \([14]\) and Proposition 9 in \([53]\)). Almost surely, one has that

\[
\lim_{N \to +\infty} \sigma_1 = \left\{ \begin{array}{ll} 
\lambda_c(\theta) & \text{if } \theta > c^{1/4}, \\
 c_+ & \text{otherwise},
\end{array} \right.
\]

and \( \lim_{N \to +\infty} \sigma_N = c_- \), where

\[
\lambda_c(\theta) = \sqrt{\frac{(1 + \theta^2)(c + \theta^2)}{\theta^2}} \text{ for any } \theta > c^{1/4}.
\] (3.12)

The interpretation of this result, called the BBP transition after \([10]\), is as follows. If the spike \( \theta \) in model (3.5) is larger than \( c^{1/4} \) then an outlier exists and it is asymptotically located at \( \lambda_c(\theta) > c_+ \). To the contrary, if \( \theta \leq c^{1/4} \) then there exists no outlier as the largest singular value \( H_{N,M}' \) converges to the bulk edge \( c_+ \).

Let us now compute the solution \( G^c_{D(H)}(\lambda I_{N+M}) \) of the fixed point equation (3.3) for a constant variance profile, namely \( \Gamma_{N,M}(i,j) = 1 \) for any \( i, j \). Using the particular structure of the variance profile \( D(\Gamma_{N,M}) \) and the expression (3.4) of \( R_{N+M} \), one obtains by simple calculations that \( G^c_{D(H)}(\lambda I_{N+M}) = \begin{bmatrix} g_N^c(\lambda)I_N & 0 \\
0 & g_M^c(\lambda)I_M \end{bmatrix} \) where \( g_N^c, g_M^c \) are complex-valued functions satisfying \( g_N^c(\lambda) = (\lambda - g_M^c(\lambda))^{-1} \) and

\[
g_M^c(\lambda) = \frac{1 + \lambda^2 - \frac{N}{M} - \sqrt{(1 + \lambda^2 - \frac{N}{M})^2 - 4\lambda^2}}{2\lambda}.
\]

Inserting this expression for \( G^c_{D(H)}(\lambda I_{N+M}) \) into (3.8), one obtains that

\[
\det(\beta_2(\lambda)) = 1 - \theta^2 g_N^c(\lambda) g_M^c(\lambda).
\]

It can then be shown that the equation \( \det(\beta_2(\lambda)) = 0 \) admits a solution given by

\[
\lambda_{N/M}(\theta) = \sqrt{\frac{(1 + \theta^2)(\frac{N}{M} + \theta^2)}{\theta^2}} \text{ provided that } \theta > \left( \frac{N}{M} \right)^{1/4}.
\] (3.13)

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Figure 1: Constant variance profile. (a) Histogram of the singular values of one realization of $H'_{N,M}$ for $\theta = 2$. The black curve is the smooth approximation by $\tilde{f}_n$ of the singular values distribution of $X_{N,M}$. The red vertical line denotes the value $\lambda_{N/M}(2) \approx 2.48$ which is the approximation of the location of the outlier, while the blue vertical dashed line denotes the location of the singular value of $H'_{N,M}$ which is the closet to $\lambda_{N/M}(2)$. (b) The red line is the curve $\theta \mapsto \max \left( 1 + \sqrt{\frac{N}{M}}, \lambda_{N/M}(\theta) \right)$, and the blue dots are the points $(\theta, \bar{\lambda}(\theta))$ where $\bar{\lambda}(\theta)$ is found by numerical minimization of $\lambda \mapsto \det \left( \beta^2_\lambda(\lambda) \right)$ for $\theta$ ranging in a grid of 50 regularly spaced values in $[0,3]$. The black vertical line is located at $(\frac{N}{M})^{1/4}$ and its height is $1 + \sqrt{\frac{N}{M}}$. (c) The blue dashed line is the curve $\theta \mapsto \det \left( \beta^2_\lambda(\bar{\lambda}(\theta)) \right)$.

Note that the condition $\theta > (\frac{N}{M})^{1/4}$ guarantees that $\lambda_{N/M}(\theta) > 1 + \sqrt{\frac{N}{M}}$. Hence, we retrieve the expression (3.12) of the asymptotic location of an outlier in the Gaussian spike population model where the asymptotic ratio $c = \lim_{N \to +\infty} \frac{N}{M}$ is replaced with its non-asymptotic approximation $\lambda_{N/M}$. As expected, we also remark that the localization $\lambda_{N/M}(\theta)$ of an outlier does not depend on the singular vectors $u_N$ and $v_N$ of the additive
perturbation $Z_{N,M}$.

In Figure 1(a), we display the histogram of the singular values of one realization of $H'_{N,M} = X_{N,M} + \theta u_N v_M^*$ with $\theta = 2$, where $u_N$ and $v_M$ are chosen to be unit vectors with constant entries. There is clearly an outlier outside the interval $[1 - \sqrt{N/M}, 1 + \sqrt{N/M}] \approx [0.05, 1.95]$. In Figure 1(a), we also plot the curve $x \mapsto \tilde{f}_n(x)$ which shows that the density defined by (3.11) is a very satisfactory approximation the distribution of the singular values of $X_{N,M}$. In Figure 1(b), we plot the curve $\theta \mapsto \max \left(1 + \sqrt{N/M}, \lambda_{N/M}(\theta)\right)$ for $\theta \in [0, 3]$, which gives the location of outliers for any $\theta > (N/M)^{1/4} \approx 0.974$.

For a set of regularly spaced values of $\theta$ on $[0, 3]$, we also report the results of the numerical procedure that we use to compute a minimizer $\lambda(\theta)$ of the function $\lambda \mapsto \det \left(\beta_2^2(\lambda)\right)$ over $\mathbb{R}_+$. In Figure 1(c), we display the curve $\theta \mapsto \lambda(\theta)$ found by numerical minimization which coincides with $\theta \mapsto \lambda_{N/M}(\theta)$ for $\theta > (N/M)^{1/4}$.

Interestingly, for all values of $\theta$ smaller than $(N/M)^{1/4}$ it appears that $\lambda(\theta) = 1 + \sqrt{N/M}$, suggesting that $\lambda \mapsto \det \left(\beta_2^2(\lambda)\right)$ admits a minimizer at the bulk edge.

### 3.2.2 Piecewise constant variance profile

We now consider the following example of a piecewise constant variance profile

$$
\Gamma_N = \begin{bmatrix}
\gamma_1 \mathbb{1}_{N/4}^* \mathbb{1}_{M/4}^* & \gamma_2 \mathbb{1}_{N/4}^* \mathbb{1}_{3M/4}^* \\
\gamma_2 \mathbb{1}_{3N/4}^* \mathbb{1}_{M/4}^* & \gamma_1 \mathbb{1}_{3N/4}^* \mathbb{1}_{3M/4}^* \mathbb{1}_M
\end{bmatrix},
$$

(3.14)

where $\mathbb{1}_q$ denotes the vector of length $q$ with all entries equal to one, and $\gamma_1, \gamma_2$ are positive constant such that $\gamma_2 = 200 \times \gamma_1$. Then, we compare two settings where either $u_N$ and $v_M$ are unit vectors with constant entries, or $u_N$ (resp. $v_M$) is equal to the first vector $e_1^N$ (resp. $e_1^M$) of the canonical basis of $\mathbb{R}^N$ (resp. $\mathbb{R}^M$).

In Figure 2, we display the histogram of the singular values of one realization of $H'_{N,M} = X_{N,M} + \theta u_N v_M^*$ for different values of $\theta \leq 1$. When $u_N = \frac{1}{\sqrt{N}} \mathbb{1}_N$ and $v_M = \frac{1}{\sqrt{M}} \mathbb{1}_M$ are unit vectors with constant entries, a spike $\theta \in [0.28, 1]$ clearly generates an outlier at $\lambda \in [0.2, 0.5]$, while in the case $u_N = e_1^N$ and $v_M = e_1^M$ there is no outlier for such values of the spike. For each setting, we also display in Figure 3 the curves $\theta \mapsto \lambda(\theta)$ and $\theta \mapsto \det \left(\beta_2^2(\lambda(\theta))\right)$ where $\lambda(\theta)$ is found by numerical minimization of $\lambda \mapsto \det \left(\beta_2^2(\lambda)\right)$ over $\mathbb{R}_+$. In the case where $u_N$ and $v_M$ have constant entries, the value of $\det \left(\beta_2^2(\lambda(\theta))\right)$ is close to zero when $\theta \in [0.28, 1]$ which confirms the existence of outliers for such values of the spike. When $u_N = e_1^N$ and $v_M = e_1^M$, one has clearly that $\det \left(\beta_2^2(\lambda(\theta))\right) \neq 0$ when $\theta \in [0.28, 1]$ and thus, for such spikes, there is no outlier. In
Figure 2: Piecewise constant variance profile. (c-f) Histograms of the singular values of one realization of $H'_{N,M} = X_{N,M} + \theta u_N v_M^*$ and smooth approximation of the singular values distribution of $X_{N,M}$ (black curve). The red (resp. green) vertical line denotes the value $\bar{\lambda}_p(\theta)$ when $u_N = \frac{1}{\sqrt{N}} 1_N$ and $v_M = \frac{1}{\sqrt{M}} 1_M$ (resp. $u_N = e_1^N$ and $v_M = e_1^M$), while the blue vertical dashed line denotes the location of the singular value of $H'_{N,M}$ which is the closest to $\bar{\lambda}(\theta)$.

both settings, when $\theta$ is sufficiently large (e.g. $\theta \geq 1$), there exists an outlier $\bar{\lambda}(\theta) > 1.5$ but with a location depending on the values of $u_N$ and $v_M$.

3.2.3 Bernoulli variance profile

We now consider variance profiles whose entries may be equal to zero and are chosen randomly (and independently) as follows. Each entry of $\Gamma_{N,M}$ takes either the value zero with probability $1 - p$ (for some $0 < p < 1$) or a fixed positive value $\gamma^2 > 0$ with probability $p$. After randomly fixing the entries of $\Gamma_{N,M}$ in this way, the value of $\gamma^2$ is chosen such that the normalisation condition (3.6) is satisfied. In Figure 4, we display the histogram of the singular values of one realization of $H'_{N,M} = X_{N,M} + \theta u_N v_M^*$ for
Figure 3: Piecewise constant variance profile. (a) The dashed black line is the curve $	heta \mapsto \max \left( 1 + \sqrt{\frac{N}{M}}, \lambda_{N/M}(\theta) \right)$, and the red (resp. green) dots are the points $(\theta, \tilde{\lambda}(\theta))$ when $u_N = \frac{1}{\sqrt{N}} 1_N$ and $v_M = \frac{1}{\sqrt{M}} 1_M$ (resp. $u_N = e_1^N$ and $v_M = e_1^M$). When $u_N$ and $v_M$ are unit vectors with constant entries, outliers are generated within the interval $[0.2, 0.5]$ for spikes $\theta \in [0.28, 1]$. (b) The dashed lines are the curves $\theta \mapsto \det \left( \beta_2^2(\tilde{\lambda}(\theta)) \right)$ depending on the choice of $(u_N, v_M)$.

\[ \theta = 2, \quad \text{with} \quad u_N = \frac{1}{\sqrt{N}} 1_N, v_M = \frac{1}{\sqrt{M}} 1_M, \quad \text{and for various Bernoulli variance profiles by letting the sampling probability} \quad p \quad \text{ranging from} \quad \frac{5}{M} \quad \text{to} \quad \frac{40}{M}. \quad \text{For values of} \quad p \quad \text{larger than} \quad \frac{5}{M}, \quad \text{the value} \quad \lambda(2) \quad \text{is an accurate approximation of the location of an outlier in the singular values distribution of} \quad H_{N,M}'. \quad \text{The shape of the smooth approximation by} \quad \tilde{f}_n \quad \text{of the singular values distribution of} \quad X_{N,M} \quad \text{clearly depends on the value of} \quad p. \]

### 3.2.4 Doubly stochastic variance profile

We finally assume that $N = M = 400$, and we consider the setting where $\Gamma_{N,M}$ is a doubly stochastic matrix. Under such an assumption, it can be easily shown that the solution to (3.3) is a scalar matrix

\[ G_{D(H)}^0(\lambda I_{N+M}) = g_N^0(\lambda) I_{N+M}, \]

where $g_N^0$ is complex-valued function satisfying $g_N^0(\lambda) = (\lambda - g_N^0(\lambda))^{-1}$ for $\lambda \in \mathbb{C}^+$. Therefore, one has that $g_N^0(\lambda) = \frac{\lambda - \sqrt{\lambda^2 - 4}}{2}$ which is the Stieltjes transform of the semi-circular law.

Hence, using exactly the same calculations than those made for a constant variance profile, one obtains that, for any unit vectors $u_N$ and $v_N$, the equation

\[ \det \left( \beta_2^2(\lambda) \right) = 1 - \theta^2 (g_N^0(\lambda))^2 = 0 \]
Figure 4: Bernoulli variance profile. Histogram of the singular values of one realization of $H'_{N,M}$ for $\theta = 2$ and with a Bernoulli variance profile for different values of the sampling probability $p$. In each figure, the black curve is the smooth approximation by $\tilde{f}_n$ of the singular values distribution of $X_{N,M}$, the red vertical line denotes the value $\bar{\lambda}p^2$ which is the approximation of the location of an outlier, while the blue vertical dashed line denotes the location of the singular value of $H'_{N,M}$ which is the closest to $\bar{\lambda}(2)$.

admits a solution given by

$$\lambda_1(\theta) = \frac{(1 + \theta^2)}{\theta} = \theta^{-1} + \theta \quad \text{provided that } \theta > 1,$$

(3.15)

Interestingly, for a doubly stochastic variance profile, we obtain exactly the expression (3.12) of the asymptotic location of an outlier in the standard Gaussian spike population model for the ratio $c = 1$. In Figure 5, we display the histogram of the singular values of one realization of $H'_{N,M} = X_{N,M} + \theta u_N v_M^*$ with $\theta = 2$ and $u_N$ and $v_M$ chosen to be unit vectors with constant entries. The normalized variance profile $\frac{\Gamma_{N,M}}{M}$ of $X_{N,M}$ is chosen as follows

$$\frac{\Gamma_{N,M}}{M} = \frac{1}{K} \sum_{k=1}^{M} P_k,$$

(3.16)
where $P_1, \ldots, P_K$ are permutation matrices (obtained by random permutations of the columns of the identity matrix $I_N$). For small values of $K$, such variance profiles have many entries equal to zero. In Figure 5, we display the histogram of the singular values of one realization of $H_{N,M}'$ for $\theta = 2$ and with a variance profile given by (3.16) for different values of $K$. In each figure, the black curve is the smooth approximation by $\tilde{f}_n$ of the singular values distribution of $X_{N,M}$, the red vertical line denotes the value $\lambda_1(2) = 2.5$ which is the approximation of the location of an outlier, while the blue vertical dashed line denotes the location of the singular value of $H_{N,M}'$ which is the closet to $\lambda_1(2)$.

For $K \geq 4$, there is clearly an outlier located approximately at $\lambda_1(\theta) = 2.5$. The quality of the smooth approximation by $\tilde{f}_n$ of the singular values distribution of $X_{N,M}$ also clearly depends on the value of $\gamma_{\max}^2$ (maximum value of the entries of the variance profile $\Gamma_{N,M}$) which is consistent with our theoretical results in Theorem 1.1 on the control of the deviation between the deterministic equivalent $\tilde{g}_{H_N}$ and the Stieltjes transform $g_{H_N}$.
3.3 General deformed models

Let us now consider the general setting of the rectangular information plus noise model (3.1) with $Y_{N,M} \neq 0$.

3.3.1 Simulated model

Taking again $N = 360$ and $M = 400$ we generate one realization from the model $H'_{N,M} = X_{N,M} + Y_{N,M} + Z_{N,M}$ as follows. The matrix $X_{N,M}$ is a Gaussian matrix with piecewise constant variance profile given by (3.14). Then, we generate a $N \times M$ matrix $W_{N,M}$ with i.i.d. real entries sampled from a Gaussian distribution with zero mean and variance $\tau^2 = \frac{1}{12M}$, that we write using singular value decomposition (SVD) as $W_{N,M} = U \Sigma V^*$. Denoting $\sigma_j$ the left singular vectors associated to the three largest singular values $\sigma_j$ of $W_{N,M}$, we finally define

$$Y_{N,M} = W_{N,M} - \sum_{j=1}^{3} \sigma_j U_j V_j^*$$

and

$$Z_{N,M} = \sum_{j=1}^{3} \theta_j U_j V_j^*$$

with $\theta_1 = 4$, $\theta_2 = 3$, $\theta_3 = 2$.

The histograms of the s.v.d. of $X_{N,M}$, $Y_{N,M}$ and $H'_{N,M}$ are displayed in Figure 6. It can be seen that the $k = 3$ spikes of $Z_{N,M}$ clearly generate 3 outliers.

In Figure 6(c), we also display the smooth approximation of the s.v.d. of the random matrix $H_{N,M} = X_{N,M} + Y_{N,M}$ using the inverse Stieltjes transform (1.5) of $g_n(\lambda) = \frac{1}{N+M} \text{Tr} G_n(\lambda)$. Since $Y_{N,M}$ is not a diagonal matrix, the deterministic equivalent $G_n(\lambda)$ of the operator-valued Stieltjes transform of $H_{N,M}$ is obtained by iterating the following matrix equation

$$G_{n+1}(\lambda) = \Delta \left[ \lambda I_{N+M} - R_{N+M}(G_n(\lambda)) - Y_{N,M} \right]^{-1}. \tag{3.17}$$

In this way, for the problem of localizing the outliers in the s.v.d. of $H'_{N,M}$, we are now able to compare the performances of two deterministic equivalents $\beta_k^\ast(\lambda)$ defined by (1.17) and $\tilde{\beta}_k^\ast(\lambda)$ defined by (1.20). To this end, the curves of the mapping $\lambda \mapsto \log(\beta_k^\ast(\lambda))$ and $\lambda \mapsto \log(\tilde{\beta}_k^\ast(\lambda))$ are displayed in Figure 6(d). It can be seen that, there exist 3 distinct values $\lambda_3 < \lambda_2 < \lambda_1$ (resp. $\tilde{\lambda}_3 < \tilde{\lambda}_2 < \tilde{\lambda}_1$) where the function $\beta_k^\ast$ (resp. $\tilde{\beta}_k^\ast$ ) is equal to zero. However, one has that $\lambda_j \neq \tilde{\lambda}_j$ for all $1 \leq j \leq 3$, and it appears from Figure 6(c) that the $\tilde{\lambda}_j$'s yield more accurate predictions of the locations the true outliers in the s.v.d. of $H'_{N,M}$. This suggests that it is preferable to use $\tilde{\beta}_k^\ast(\lambda)$ instead of $\beta_k^\ast(\lambda)$ to localize potential outliers in a more precise way.

3.3.2 A model of noisy images with heteroscedasticity

To conclude this section on numerical experiments, we study an example inspired by the problem of low-rank matrix denoising in image processing in the presence of Poisson noise. In this setting, one observe a $N \times M$ data matrix such that each $(i,j)$-th entry
Figure 6: Deformed model with a piecewise constant variance profile. Histogram of the s.v.d. of (a) $X_{N,M}$ and (b) $Y_{N,M}$. (c) Histogram of the singular values of $H'_{N,M} = X_{N,M} + Y_{N,M} + Z_{N,M}$ with $k = 3$ spikes. The black curve is the smooth approximation of the s.v.d. of $H_{N,M} = X_{N,M} + Y_{N,M}$. The red (resp. blue) vertical lines denote the values $(\tilde{\lambda}_j)_{1 \leq j \leq 3}$ (resp. $(\lambda_j)_{1 \leq j \leq 3}$) which are the approximation of the locations of outliers given by the zeros of $\tilde{\beta}_{2k}$ (resp. $\beta_{2k}$), while the blue vertical dashed lines denote the locations of the true outliers in the s.v.d. of $H'_{N,M}$. (d) Graph of $\lambda \mapsto \log(\det(\beta_{2k}(\lambda)))$ (blue curves) and $\lambda \mapsto \log(\det(\tilde{\beta}_{2k}(\lambda)))$ (red curves).

is independently sampled from a Poisson distribution with parameter $\kappa_{i,j} > 0$. Under such an assumption, the expectation and variance of each entry are thus equal to $\kappa_{i,j}$.

Therefore, the following rectangular information plus noise model

$$H'_{N,M} = X_{N,M} + \frac{\Gamma_{N,M}}{M},$$

(3.18)

where $X_{N,M}$ is a rectangular Gaussian matrix with a variance profile $\Gamma_{N,M} = (\gamma_{N,M}^2(i,j))_{i,j}$, may be viewed as a prototype for studying low-rank matrix denoising in the presence
Figure 7: Gaussian setting with equal mean and variance. (a) Image of the values (in grayscale) of the entries of the $N \times M$ normalized variance profile $\frac{\Gamma_{N,M}}{M^2}$ with $N = 340$ and $M = 510$ (b) Image of the modulus of the entries of the matrix $H'_{N,M}$. (c) Histogram of the singular values of the matrix $\Gamma_{N,M}$.

of Poisson noise (with $\kappa_{i,j} = \gamma_{n,m}^2(i,j)/M$), as considered e.g. in [60]. We shall refer to model (3.18) as the Gaussian setting with equal mean and variance. In Figure 7, we display the image made of the entries of the normalized variance profile $\frac{\Gamma_{N,M}}{M^2}$ that is considered in these numerical experiments, as well as the histogram of the singular values of this matrix which is scaled so that it satisfies the normalization condition:

$$\frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{M} \frac{\gamma_{N,M}^2(i,j)}{M} = 15. \quad (3.19)$$

Now, let us consider the SVD of the normalized variance profile $\frac{\Gamma_{N,M}}{M^2} = U\Theta V^*$. For any
\[1 \leq k \leq \min(N, M),\] model (3.18) can be written as
\[
H_{N,M}^{'\prime} = X_{N,M} + Y_{N,M}^{(k)} + Z_{N,M}^{(k)}, \quad \text{with} \quad Z_{N,M}^{(k)} = U_{N,k} \Theta_{k} V_{M,k}^{*},
\]
where \(\Theta_{k}\) is a diagonal matrix whose elements are the \(k\) largest singular values of \(\frac{1}{M} \Gamma_{N,M}\) and \(U_{N,k}\) (resp. \(V_{M,k}\)) is the matrix made of the associated left (resp. right) singular vectors, and
\[
Y_{N,M}^{(k)} = \frac{1}{M} \Gamma_{N,M} - U_{N,k} \Theta_{k} V_{M,k}^{*}.
\]
is the matrix obtained by keeping only the remaining smallest singular values in the SVD of the normalized variance profile.

In Figure 8, we display the histogram of the s.v.d. of \(H_{N,M}^{'\prime}\) sampled from model (3.18). We also report the smooth approximation of the singular value distributions of the random matrices \(X_{N,M}\) and \(H_{N,M}^{(k)} := X_{N,M} + Y_{N,M}^{(k)}\) for \(k = 1, 2, 3\) using the inverse Stieltjes transform (1.5) of \(\beta_{k}^{(n)}(\lambda) = \frac{1}{N+M} \text{Tr} G_{n}^{*}(\lambda).\) As described previously, for the matrix \(X_{N,M},\) the deterministic equivalent \(G_{n}^{*}(\lambda)\) of its operator-valued Stieltjes transform is obtained by iterating the vector Dyson equation (3.10). For the matrix \(H_{N,M}^{(k)},\) such a deterministic equivalent is obtained by iterating the matrix equation (3.17).

In Figure 8, we also report, for \(k = 1, 2, 3,\) the values of the mapping \(\lambda \mapsto \det(\tilde{\beta}_{2k}(\lambda))\) and \(\lambda \mapsto \det(\tilde{\beta}_{2k}(\lambda))\) for \(8 \leq \lambda \leq 15.\) We observe that the values of \(\lambda\) such that \(\det(\tilde{\beta}_{2k}(\lambda)) \approx 0\) give the best prediction of the locations of the outliers. Again, this illustrates the benefits of using \(\tilde{\beta}_{2k}\) instead of \(\beta_{2k}\) for outliers detection. For \(k = 1\) or \(k = 2\) spikes, one predicts exactly \(k\) outliers, while for \(k = 3\) spikes one has only 2 outliers.

### 4 Organization of the proofs

We first describe the mains steps to derive the proof of Theorem 1.1 using the notation of Section 1.2. In Section 5, following the method and terminology of Haagerup and Thorbjørnsen in [37], we start by proving a Master equality (see Lemma 5.7) involving the expectation of the generalized resolvent \((\Lambda - X_{N} - Y_{N})^{-1}\) that can be decomposed as follows
\[
\mathbb{E}[(\Lambda - X_{N} - Y_{N})^{-1}] = (\Omega_{X_{N},Y_{N}}(\Lambda) - Y_{N})^{-1} + F_{N}(\Lambda),
\]
with
\[
\Omega_{X_{N},Y_{N}}(\Lambda) = \Lambda - \mathcal{R}_{N}\left(\mathbb{E}[G_{X_{N}+Y_{N}}(\Lambda)]\right),
\]
and \(F_{N}(\Lambda) = \left(\Omega_{H_{N}^{\prime}}(\Lambda) - Y_{N}\right)^{-1} E_{N},\) where \(E_{N}\) is the matrix of covariance between \((\Lambda - X_{N} - Y_{N})^{-1}\) and \(\mathcal{R}_{N}\left(\mathbb{E}[G_{X_{N}+Y_{N}}(\Lambda)]\right)\) given explicitly in (5.11). We deduce this result from an identity on the resolvent \((\Lambda - H_{N})^{-1}\) that is a consequence of the Gaussian integration by part formula (see Lemma 5.5 and Lemma 5.6 below).
Figure 8: Gaussian setting with equal mean and variance. First row: histogram of the s.v.d. of $H_{N,M} = X_{N,M} + Y_{N,M}^{(k)} + Z_{N,M}^{(k)}$ for $k$ spikes with $k = 1, 2, 3$. The black curves are the smooth approximations of the s.v.d. of $H_{N,M}^{(k)} = X_{N,M} + Y_{N,M}^{(k)}$ for different values of $k$, while the green curve is the smooth approximation of the s.v.d. of $X_{N,M}$. The red (resp. blue) vertical lines denote the locations which are the approximation of the locations of outliers given by the zeros of $\tilde{\beta}_{2k}$ (resp. $\beta_{2k}^{(0)}$), while the blue vertical dashed lines denote the locations of the true outliers in the s.v.d. of $H_{N,M}$. Second row: graph of $\lambda \mapsto \log (\det (\beta_{2k}^{(0)}(\lambda)))$ (blue curves) and $\lambda \mapsto \log (\det (\beta_{2k}^{(0)}(\lambda)))$ (red curves) for different values of $k$.

Following the heuristic of Section 2.2, in particular (2.5), the matrix $\Omega_{X_N,Y_N}(\Lambda)$ is a good candidate to approximate an operator-valued subordination function. Applying the operator $\Delta$ on both sides of equality (4.1) provides the following approximate equation for operator-valued Stieltjes transforms:

$$\mathbb{E}[G_{X_N+Y_N}(\Lambda)] = G_{Y_N}(\Lambda - \mathcal{R}_N(\mathbb{E}[G_{X_N+Y_N}(\Lambda)]) + \Theta_N(\Lambda).$$

(4.2)

with $\Theta_N(\Lambda) := \Delta[F_N(\Lambda)]$. Equation (4.2) tells that the expectation of $G_{X_N+Y_N}$ satisfies, up to additive error term, the fixed point equation (1.12) that defines its deterministic equivalent. Then, using Gaussian Poincaré inequality (see Lemma 5.9), we obtain an upper bound on the error term $\max_{(k,k')\in[N]\times[N]}|F_N(\Lambda)(k,k')|$ that we call Master inequality (see Lemma 5.8 below), following again the terminology of Haagerup and
We recall if $M$ of matrices with complex entries that we repeatedly use in the proof. For any matrix $N$ we let

$$\|\Theta_N(A)\| \leq \gamma_{max}^4 N^{-\eta} \times \|3m \Lambda^{-1}\|^5$$

with $\eta = 1$, and if $Y_N$ is diagonal then this estimate on $\|\Theta_N(A)\|$ holds with $\eta = \frac{3}{2}$.

In Section 6 we prove the existence of the deterministic equivalent $G_{H_N}^0$, solution of equation (4.2) when $\Theta_N(A) = 0$. We also prove that the upper bound (4.3) for $\Theta_N(A)$ implies a bound for the difference $\|E[G_{H_N}(A)] - G_{H_N}^0(A)\|$, thanks to an analysis of regularity of the fixed point problem (1.12). More precisely, $G_{H_N}^0$ is a good approximation of $G_{H_N}$ out of a small strip around the real axe as stated below.

**Lemma 4.1.** For any $0 < \delta < 1$ and any $\Lambda$ such that $\gamma_{max}^6 N^{-\eta} \|3m \Lambda^{-1}\|^6 \leq 1 - \delta$, we have

$$\|E[G_{H_N}(A)] - G_{H_N}^0(A)\| \leq \gamma_{max}^4 (1 + \gamma_{max}^2/\delta \|3m \Lambda^{-1}\|^2) \|3m \Lambda^{-1}\|^5 \times N^{-\eta},$$

where $\eta = 1$ in general and $\eta = \frac{3}{2}$ if $Y_N$ is diagonal.

The rest of the proof of Theorem 1.1 is finally based on the control of the difference between the generalized resolvent $(\Lambda - H_N)^{-1}$ and the expectation of $G_{H_N}(A)$. As explained in Section 7, the comparison between the random quantity $G_{H_N}(A)$ and the deterministic equivalent $G_{H_N}^0(A)$ follows from the combination of Lemma 4.1 and Gaussian concentration inequality of Lipschitz functions allowing to show that the entries of the generalized resolvent $(\Lambda - H_N)^{-1}$ are close to their expectation with high probability. Finally, Section 7 ends with a mathematical justification of the convergence of the numerical method used to approximate the solution of the fixed point equation (1.12).

## 5 The approximate subordination property

### 5.1 Notation and preliminaries

We let $[N]$ be the set of integers between 1 and $N$. Then, we recall some basic properties of matrices with complex entries that we repeatedly use in the proof. For any matrix $A$ in $M_N(\mathbb{C})$ we denote by $\|A\|$ its operator norm, namely

$$\|A\| = \sup_{x \in \mathbb{C}^N, \|x\|=1} \|Ax\|_2, \quad \text{where} \quad \|x\|_2 = (\langle x, x \rangle_{\mathbb{C}^N})^{\frac{1}{2}}, \quad \langle x, y \rangle_{\mathbb{C}^N} = \sum_i \bar{x}_i y_i.$$

We recall if $A$ is a Hermitian matrix then $\|A\|$ is the spectral radius of $A$, and in general it is the square-root of the spectral radius of $AA^*$. In particular it satisfies the $C^*$-norm condition $\|A\|^2 = \|A^*\|^2 = \|AA^*\|$. We denote by $\Re A$ and $\Im A$ the real and imaginary parts of $A$, which are Hermitian matrices defined by

$$\Re A = \frac{1}{2}(A + A^*), \quad \Im A = \frac{i}{2}(A - A^*), \quad i^2 = 1.$$

We write $A \geq 0$ (resp. $A > 0$) whenever the matrix $A$ is Hermitian and positive (resp. positive definite), and $A \leq 0$ (resp. $A < 0$) if $-A$ satisfies this property. For a Hermitian matrix $A$, we denote its eigenvalues in decreasing order by $\sigma_1(A) \geq \ldots \geq \sigma_n(A)$.
Lemma 5.1. Let $A$ in $M_N(\mathbb{C})$ such that $\Im A > 0$. Then $A$ is invertible and
\[ \|A^{-1}\| \leq \| (\Im A)^{-1}\|. \]  
\hfill (5.1)

Proof. We follow the proof of Lemma 3.1 in [37]. For any unit vector $x$ in $\mathbb{C}^N$, since $\langle \Re(A)x, x \rangle$ and $\langle \Im(A)x, x \rangle$ are real, we have
\[
\|Ax\|_2 = \|Ax\|_2 \|x\|_2 \geq |\langle Ax, x \rangle| = |\langle \Re(A)x, x \rangle + k(\Im A)x, x \rangle| 
\geq |\langle \Im(A)x, x \rangle| \geq \| (\Im A)^{-1}\|^{-1} \|x\|_2 = \| (\Im A)^{-1}\|^{-1}.
\]
Hence $A$ is injective, so it is invertible. Since $1 = \|x\|_2 = \|A^{-1}Ax\|_2 \leq \|A^{-1}\| \times \|Ax\|_2$, we get from the previous lower bound that $\|A^{-1}\| \leq \| (\Im A)^{-1}\|$. \qed

For a diagonal matrix $\Lambda$, note that $\|\Lambda\| = \max_{i \in [N]} |\Lambda(i, i)|$ and $\Lambda \geq 0$ whenever $\Lambda(i, i) \geq 0$ for any $i = 1, \ldots, N$. Recall that we defined $\Delta(A) = \text{diag}(A(i, i))$ for any matrix $A$.

Lemma 5.2. For any $A$ in $M_N(\mathbb{C})$, one has
\[ \|\Delta(A)\| \leq \|A\|. \]  
\hfill (5.2)

Moreover, $A \leq 0$ implies $\Delta(A) \leq 0$ and $A < 0$ implies $\Delta(A) < 0$.

Proof. We have $\|\Delta(A)\| = \max_{i \in [N]} |A(i, i)| = \max_{i \in [N]} |\langle e_i, Ae_i \rangle| \leq \|A\|$, where $(e_i)_{i \in [N]}$ denotes the canonical basis of $\mathbb{C}^N$. Moreover, if $A \leq 0$ then $A = -BB^*$ for some Hermitian matrix $B$. Hence for any $i = 1, \ldots, N$, $\Delta(A)(i, i) = -\sum_{j=1}^N |B(i, j)|^2 < 0$, and so we get $\Delta(A) \leq 0$. If moreover $A < 0$ then $A$ is invertible so the $i$-th line of $B$ is nonzero for any $i = 1, \ldots, N$ and hence $\Delta(A) < 0$. \qed

If $\Lambda$ is an invertible diagonal matrix, note also that $\|\Lambda^{-1}\| = \left( \min_{i \in [N]} |\Lambda(i, i)| \right)^{-1}$. Moreover, we have $3m(\Lambda^{-1}) = (3m \Lambda)^{-1}$ and so we can drop the parentheses and write $3m \Lambda^{-1}$ without confusion. We use several time the following direct consequence of Lemmas 5.1 and 5.2.

Lemma 5.3. For any Hermitian matrix $A$ and any diagonal matrix $\Lambda$ such that $\Im \Lambda > 0$, the matrix $(\Lambda - A)$ is invertible and
\[ \| (\Lambda - A)^{-1} \| \leq \| 3m \Lambda^{-1} \|. \]  
\hfill (5.3)

Hence the map $G_A(\Lambda) = \Delta[(\Lambda - A)^{-1}]$ is indeed well defined and $\| G_A(\Lambda) \| \leq \| 3m \Lambda^{-1} \|$. Moreover, we have $3m G_A(\Lambda) < 0$.

Recall that we defined $\mathcal{R}_N : G \mapsto \text{diag}\left( \sum_{j=1}^N \frac{\gamma_j^2}{N} G_{i,j} \right)$ for diagonal matrices $G$. Throughout the paper, we denote $\gamma_{\max} = \max_{i,j} \gamma_{i,j}$. 31
Lemma 5.4. For any diagonal matrix $G$ such that $\Im G < 0$, one has $\mathcal{R}_N(G) \preceq 0$. Moreover, for any diagonal matrix $G$ in $M_N(\mathbb{C})$, the following operator norm bound holds

$$\|\mathcal{R}_N(G)\| \leq \gamma_{\text{max}}^2 \|G\|.$$  

Proof. For any $G$ such that $\Im G < 0$, since the matrix $\Gamma_N$ has nonnegative entries, we have

$$\Im(\mathcal{R}_N(G)) = \sum_{i \in [N]} \frac{1}{2i} \left[ \sum_{j=1}^N \frac{\gamma_{i,j}^2}{N} G_{j,j} E_{i,i} - \left( \sum_{j=1}^N \frac{\gamma_{i,j}^2}{N} G_{j,j} E_{i,i} \right)^* \right]$$

$$= \sum_{i \in [N]} \sum_{j=1}^N \frac{\gamma_{i,j}^2}{N} \Im(G_{j,j}) E_{i,i},$$

where $E_{i,i} = e_i \otimes e_i$ denotes the matrix with entries equal to zero except the $(i,i)$-th one which is equal to one. Hence the diagonal entries of $\mathcal{R}_N(G)$ are indeed nonpositive. Moreover, for any diagonal matrix $G$ we have

$$\|\mathcal{R}_N(G)\| = \max_{i \in [N]} |\mathcal{R}_N(G)(i,i)| \leq \gamma_{\text{max}}^2 \frac{1}{N} \sum_{j=1}^N |G(j,j)|.$$

Since $|G(j,j)| \leq \|G\|$, we obtain the expected inequality. \hfill \Box

5.2 The Master equality

The notation are as in Section 1.2, in particular we denote $H_N = X_N + Y_N$. We start with the following observation.

Lemma 5.5. For any $\Lambda \in D_N(\mathbb{C}^+)$, diagonal matrix with positive imaginary part, recalling that $G_{H_N}(\Lambda) = \Delta[(\Lambda - H_N)^{-1}]$, we have the equality between the $N \times N$ matrices

$$\mathbb{E}[X_N(\Lambda - H_N)^{-1}] = \mathbb{E}[\mathcal{R}_N(G_{H_N}(\Lambda))(\Lambda - H_N)^{-1}].$$  \hfill (5.4)

Lemma 5.5 is a consequence of the well-known Gaussian integration by part formula (see e.g. Lemma 3.3 in [37]) that we recall below.

Lemma 5.6. Let $f : \mathbb{R}^q \to \mathbb{C}$ be a continuously differentiable function, and $X_1, \ldots, X_q$ a sequence of independent centered real Gaussian variables with possibly different variances $\text{Var}(X_k) = \gamma_k^2$ for $1 \leq k \leq q$. Then, under the conditions that $f$ and its first order derivatives $\frac{\partial}{\partial x_1} f, \ldots, \frac{\partial}{\partial x_q} f$ are polynomially bounded, one has that

$$\mathbb{E}[X_k f(X_1, \ldots, X_q)] = \gamma_k^2 \mathbb{E} \left[ \frac{\partial}{\partial x_k} f(X_1, \ldots, X_q) \right].$$  \hfill (5.5)
Proof. In order to prove Lemma 5.5, let us write the random matrix $X_N$ in an appropriate orthonormal basis to make its dependency on only real Gaussian variables more explicit. Let $E_{i,j} = e_i \otimes e_j$ be the canonical basis of $N \times N$ matrices, and define

$$F_{i,j} = \begin{cases} E_{i,j} & \text{if } i = j \\ \frac{1}{\sqrt{2}} (E_{i,j} + E_{j,i}) & \text{if } i > j \\ \frac{1}{\sqrt{2}} (E_{i,j} - E_{j,i}) & \text{if } i < j \end{cases}$$ (5.6)

Then, $X_N$ can be decomposed as

$$X_N = \sum_{i,j=1}^{N} x'_{i,j} F_{i,j},$$ (5.7)

where the $x'_{i,j}$ are i.i.d. real Gaussian random variables, centered and such that $x_{i,j}$ has variance $\frac{\gamma^2_{i,j}}{N}$. This implies that

$$\mathbb{E} \left[ X_N (\Lambda - H_N)^{-1} \right] = \sum_{i,j=1}^{N} F_{i,j} \mathbb{E} \left[ x'_{i,j} (\Lambda - H_N)^{-1} \right].$$

By the Gaussian integration by part (5.5), we have

$$\mathbb{E} \left[ x'_{i,j} (\Lambda - H_N)^{-1} \right] = \frac{\gamma^2_{i,j}}{N} \mathbb{E} \left[ \frac{d}{d \epsilon} \big|_{\epsilon=0} (\Lambda - H_N - \epsilon F_{i,j})^{-1} \right].$$

Now, recalling that $H_N = X_N + Y_N$, we can compute

$$\frac{d}{d \epsilon} \big|_{\epsilon=0} (\Lambda - H_N - \epsilon A)^{-1} = (\Lambda - H_N)^{-1} A (\Lambda - H_N)^{-1}$$ (5.8)

for any direction $A \in \mathbb{C}^{N \times N}$, and get the following relation

$$\mathbb{E} \left[ X_N (\Lambda - H_N)^{-1} \right] = \sum_{i,j=1}^{N} \frac{\gamma^2_{i,j}}{N} F_{i,j} \mathbb{E} \left[ (\Lambda - H_N)^{-1} F_{i,j} (\Lambda - H_N)^{-1} \right].$$ (5.9)

Note that (5.3) ensures that the function and its derivatives are bounded, so we can correctly apply (5.5). Moreover, since $\gamma_{ij} = \gamma_{ji}$, we have for any matrix $A$

$$\sum_{i,j=1}^{N} \frac{\gamma^2_{i,j}}{N} F_{i,j} A F_{i,j} = \sum_{i=1}^{N} \frac{\gamma^2_{i,i}}{N} E_{i,i} A E_{i,i} + \frac{1}{2} \sum_{i>j} \frac{\gamma^2_{i,j}}{N} (E_{i,j} + E_{j,i}) A (E_{i,j} + E_{j,i})$$

$$- \frac{1}{2} \sum_{i<j} \frac{\gamma^2_{i,j}}{N} (E_{i,j} - E_{j,i}) A (E_{i,j} - E_{j,i})$$

$$= \sum_{i=1}^{N} \frac{\gamma^2_{i,i}}{N} E_{i,i} A E_{i,i} + \sum_{i>j} \frac{\gamma^2_{i,j}}{N} (E_{i,j} A E_{j,i} + E_{j,i} A E_{i,j})$$

$$= \sum_{i,j=1}^{N} \frac{\gamma^2_{i,j}}{N} A_{jj} E_{ii} = \mathcal{R}_N (\Delta(A)).$$ (5.10)
Combined with (5.9), the equality implies the expected result
\[ \mathbb{E}\left[ X_N(\Lambda - H_N)^{-1} \right] = \mathbb{E}\left[ R_N(\Delta(\Lambda - H_N)^{-1})(\Lambda - H_N)^{-1} \right]. \]
\[ \square \]

Let us now introduce \( E_N = E_N(\Lambda) \) defined as the covariance between the matrices \( R_N(G_{H_N}(\Lambda)) \) and \( (\Lambda - H_N)^{-1} \), namely
\[ E_N = \mathbb{E}\left[ R_N(G_{H_N}(\Lambda))(\Lambda - H_N)^{-1} \right] - \mathbb{E}\left[ R_N(G_{H_N}(\Lambda)) \right] \times \mathbb{E}\left[ (\Lambda - H_N)^{-1} \right]. \quad (5.11) \]
We can now state and prove the so-called Master equality introduced in Equation (4.1).

**Lemma 5.7 (Master equality).** For any \( \Lambda \in D_N(\mathbb{C})^+ \), we define the the diagonal matrix
\[ \Omega_{H_N}(\Lambda) = \Lambda - R_N\left( \mathbb{E}\left[ G_{H_N}(\Lambda) \right] \right). \quad (5.12) \]
Then, we have \( \Im m\Omega_{H_N}(\Lambda) > \Im m\Lambda \), and the following equality holds for all \( \Lambda \in D_N(\mathbb{C})^+ \):
\[ \mathbb{E}\left[ (\Lambda - X_N - Y_N)^{-1} \right] = (\Omega_{H_N}(\Lambda) - Y_N)^{-1} \times (I_N + E_N). \quad (5.13) \]

**Proof.** Starting with the left hand side \( X_N(\Lambda - H_N)^{-1} \) in (5.4), we want to obtain an expression involving only generalized resolvent. Recall that \( H_N = X_N + Y_N \). If we where solely considering the random matrix \( X_N \) and assume \( Y_N = 0 \), we should write
\[ X_N(\Lambda - X_N)^{-1} = \left( - (\Lambda - X_N) + \Lambda \right)(\Lambda - X_N)^{-1} = -I_N + \Lambda(\Lambda - H_N)^{-1}. \]

When \( Y_N \) is non zero, we first fix a deterministic matrix \( \Omega \in D_N(\mathbb{C}^+) \) whose choice is determined later on, and multiplying on the left by \( (\Omega - Y_N)^{-1} \) our expression under consideration: we have
\[ (\Omega - Y_N)^{-1}X_N(\Lambda - H_N)^{-1} = (\Omega - Y_N)^{-1}\left[ - (\Lambda - X_N - Y_N) + (\Omega - Y_N) + (\Lambda - \Omega) \right](\Lambda - H_N)^{-1} \]
\[ = -(\Omega - Y_N)^{-1} + (\Lambda - H_N)^{-1} + (\Omega - Y_N)^{-1}(\Lambda - \Omega)(\Lambda - H_N)^{-1}. \]

Moreover, since \( Y_N \) is deterministic, (5.4) implies
\[ \mathbb{E}\left[ (\Omega - Y_N)^{-1}X_N(\Lambda - H_N)^{-1} \right] = \mathbb{E}\left[ (\Omega - Y_N)^{-1}R_N(G_{H_N}(\Lambda))(\Lambda - H_N)^{-1} \right]. \]
Introducing the map \( f : A \mapsto \mathbb{E}\left[ (\Omega - Y_N)^{-1}A(\Lambda - H_N)^{-1} \right] \) for any random matrix \( A \), we obtain
\[ \mathbb{E}\left[ (\Lambda - X_N - Y_N)^{-1} \right] = (\Omega - Y_N)^{-1} + f\left( R_N(G_{H_N}(\Lambda)) \right) - f(\Lambda - \Omega). \]
Since $f$ is linear, we have
\[
\begin{align*}
  f\left(\mathcal{R}_N(G_{H_N}(\Lambda))\right) - f(\Lambda - \Omega) \\
  &= f\left(\mathbb{E}[\mathcal{R}_N(G_{H_N}(\Lambda)) - \Lambda + \Omega]\right) + f\left(\mathcal{R}_N(G_{H_N}(\Lambda)) - \mathbb{E}[\mathcal{R}_N(G_{H_N}(\Lambda))]\right).
\end{align*}
\]
We set $\Omega = \Omega_{H_N}(\Lambda) = \Lambda - \mathcal{R}_N(\mathbb{E}[G_{H_N}(\Lambda)])$, so that the first term in the above equality vanishes. By Lemma 5.4, we have that $\Im(\mathcal{R}_N(G_{H_N}(\Lambda))) \leq 0$ and so $\Omega_{H_N}$ belongs to $D_N(\mathbb{C}^\dagger)$, and is such that $\Im(\Omega_{H_N}(\Lambda)) > \Im(\Lambda)$. Hence we obtain the following expression
\[
\mathbb{E}\left[ (\Lambda - X_N - Y_N)^{-1} \right] = (\Omega_{H_N}(\Lambda) - Y_N)^{-1} + f\left(\mathcal{R}_N(G_{H_N}(\Lambda)) - \mathbb{E}[\mathcal{R}_N(G_{H_N}(\Lambda))]\right).
\]
Since $Y_N$ is deterministic, the above identity completes the proof of Lemma 5.7. $\square$

5.3 The Master inequality

5.3.1 Statement and use of Poincaré inequality

We prove the following estimate on $E_N$ defined in (5.11).

**Lemma 5.8** (Master inequality). Recall that we denote $\gamma_{\max} = \max_{i,j} \gamma_{i,j}$. For any $\Lambda, \Omega$ belonging to $D_N(\mathbb{C}^\dagger)$ and any $k, k'$ in $[N]$, we have
\[
\left| \left[ (\Omega - Y_N)^{-1} E_N(\Lambda) \right](k,k') \right| \leq C_N \times \| \Im \Lambda^{-1} \| \times \| \Im \Omega^{-1} \|,
\]
where $C_N = \gamma_{\max}^4 N^{-1}$ in general, and $C_N = \gamma_{\max}^4 N^{-\frac{3}{2}}$ if $Y_N$ is a diagonal matrix.

By Lemma 5.7 one has that $\| \Im(\Omega_{H_N}(\Lambda)) \| \leq \| \Im \Lambda^{-1} \|$. Hence, by Lemma 5.8 the operator-valued subordination property (4.2) is approximatively holds up to an error term $\Delta[F_N(\Lambda)] = \Delta[(\Omega_{H_N} - Y_N)^{-1} E_N(\Lambda)]$ satisfying
\[
\left\| \Delta[F_N(\Lambda)] \right\| = \max_{k \in [N]} \left| \left[ (\Omega_{H_N}(\Lambda) - Y_N)^{-1} E_N(\Lambda) \right](k,k) \right| \leq C_N \| \Im \Lambda^{-1} \|^5.
\]

The rest of the section is devoted to the proof of Lemma 5.8. To abbreviate the notation, we define the random matrices
\[
\begin{align*}
  A_N &= (\Lambda - H_N)^{-1}, \\
  \hat{A}_N &= A_N - \mathbb{E}[A_N], \\
  B_N &= (\Omega - Y_N)^{-1}, \\
  D_N &= \mathcal{R}_N(\Delta(A_N)), \\
  \hat{D}_N &= D_N - \mathbb{E}[D_N],
\end{align*}
\]
so that we can write $(\Omega - Y_N)^{-1} E_N(\Lambda) = \mathbb{E}[B_N \hat{D}_N \hat{A}_N]$. Note that $B_N$ is deterministic, $D_N$ is diagonal. For any integers $k, k'$ in $[N]$:
\[
\left| \mathbb{E}\left[ (B_N \hat{D}_N \hat{A}_N)(k,k') \right] \right| = \left| \sum_{\ell=1}^{N} \mathbb{E}\left[ B_N(k,\ell) \hat{D}_N(\ell,\ell) \times \hat{A}_N(\ell,k') \right] \right| \\ 
\leq \left( \sum_{\ell=1}^{N} \mathbb{E}\left[ |B_N(k,\ell)\hat{D}_N(\ell,\ell)|^2 \right] \right) \times \sum_{\ell=1}^{N} \mathbb{E}\left[ |\hat{A}_N(\ell,k')|^2 \right] \frac{1}{2}.
\]
}

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where we have used Cauchy-Schwarz inequality with the scalar product $\langle x, y \rangle_{\mathbb{C}^N} = \sum_{\ell} \mathbb{E}[\bar{x}_\ell y_\ell]$. Hence, since $B_N$ is deterministic we have

$$
\left| \left( (\Omega - Y_N)^{-1} E_N(\Lambda) \right)(k, k') \right| \leq \left( \sum_{\ell=1}^{N} |B_N(k, \ell)|^2 \text{Var}(D_N(\ell, \ell)) \times \sum_{\ell=1}^{N} \text{Var}(A_N(\ell, k')) \right)^{\frac{1}{2}}.
$$

(5.15)

If moreover $Y_N$ is diagonal, then so is $B_N$ and we can simply write

$$
\left| \mathbb{E} \left[ (B_N \hat{D}_N \hat{A}_N)(k, k') \right] \right| = |B_N(k, k)| \times \mathbb{E} \left[ (\hat{D}_N \hat{A}_N)(k, k') \right]
$$

and so applying (5.3) to $B_N$ and the Cauchy-Schwarz inequality we get

$$
\left| \left( (\Omega - Y_N)^{-1} E_N(\Lambda) \right)(k, k') \right| \leq \| \Im \Omega^{-1} \| \times \left( \text{Var}(D_N(k, k)) \text{Var}(A_N(k, k')) \right)^{\frac{1}{2}}.
$$

(5.16)

We now estimate the variances that appear in inequalities (5.15) and (5.16). For this task, we recall the statement of Gaussian Poincaré inequality (see e.g. Proposition 4.1 in [37]).

**Proposition 5.9** (Gaussian Poincaré inequality). Let $f : \mathbb{R}^q \to \mathbb{C}$ be a continuously differentiable function, and $X_1, \ldots, X_q$ a sequence of independent centered real Gaussian variables with possibly different variances $\text{Var}(X_k) = \gamma_k^2$ for $1 \leq k \leq q$. Then, under the condition that $f$ and its first order derivatives are polynomially bounded, one has that

$$
\text{Var}(f(X_1, \ldots, X_q)) \leq \mathbb{E} \left( \| \Gamma^{1/2} \nabla f(X_1, \ldots, X_q) \|_2^2 \right)
$$

where $\Gamma = \text{diag}(\gamma_1^2, \ldots, \gamma_q^2)$, $\nabla f$ is the gradient of $f$, and $\| \cdot \|_2$ is the standard Euclidean norm of a vector with complex entries.

We write the matrices $A_N$ and $D_N$ as functions of the independent real Gaussian random variables defined in (5.7), namely: for all $\ell, k$ in $[N]$, we define for any real matrix $A = (a_{i,j})_{i,j}$,

$$
f_{1, \ell, k}(A) = \left( \Lambda - \sum_{i,j} a_{i,j} F_{i,j} - Y_N \right)^{-1}(\ell, k),
$$

$$
f_{2, \ell}(A) = \mathcal{R}_N \left( \Delta \left( \Lambda - \sum_{i,j} a_{i,j} F_{i,j} - Y_N \right)^{-1} \right)(\ell, \ell),
$$

where $(F_{i,j})_{i,j}$ is the basis of Hermitian matrices defined by (5.6), so that we have

$$
A_N(\ell, k) = \left( \Lambda - H_N \right)^{-1}(\ell, k) = f_{1, \ell, k}(x_{i,j}^{(i,j)}),
$$

$$
D_N(\ell, \ell) = \mathcal{R}_N \left( \Delta \left( \Lambda - H_N \right)^{-1} \right)(\ell, \ell) = f_{2, \ell}(x_{i,j}^{(i,j)}),
$$

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for the coordinates \((x'_{i,j})_{i,j}\) of \(X_N\) in the basis \((F_{i,j})_{i,j}\), see (5.7). By the same computation as for the derivative in (5.8), and using by the linearity of the maps \(\mathcal{R}_N\) and \(\Delta\), we obtain

\[
\frac{\partial}{\partial x_{i,j}'} f_1,\ell,k((x_{i,j}')_{i,j}) = \left[(\Lambda - H_N)^{-1} F_{i,j} (\Lambda - H_N)^{-1}\right](\ell, k) = (A_N f_{i,j} A_N)(\ell, k),
\]

\[
\frac{\partial}{\partial x_{i,j}'} f_2,\ell((x_{i,j}')_{i,j}) = \mathcal{R}_N \left[\Delta (\Lambda - H_N)^{-1} F_{i,j} (\Lambda - H_N)^{-1}\right] (\ell, \ell) = \mathcal{R}_N \left[\Delta [A_N F_{i,j} A_N]\right] (\ell, \ell),
\]

Thanks to Proposition 5.9 (Gaussian Poincaré inequality), we hence have for any \(k \in [N]\)

\[
\mathbb{V} \text{ar}(A_N(\ell, k)) \leq \mathbb{E} \left[ \sum_{i,j} \frac{\gamma_{i,j}^2}{N} \left| A_N F_{i,j} A_N \right|(\ell, k) \right]^2 \leq \frac{\gamma_{\text{max}}^2}{N} \mathbb{E} \left[ \sum_{i,j} \left| A_N F_{i,j} A_N \right|(\ell, k) \right]^2, \tag{5.17}
\]

\[
\mathbb{V} \text{ar}(D_N(\ell, \ell)) \leq \mathbb{E} \left[ \sum_{i,j} \frac{\gamma_{i,j}^2}{N} \mathcal{R}_N \left[\Delta [A_N F_{i,j} A_N]\right] (\ell, \ell) \right]^2 \leq \frac{\gamma_{\text{max}}^2}{N} \mathbb{E} \left[ \sum_{i,j} \mathcal{R}_N \left[\Delta [A_N F_{i,j} A_N]\right] (\ell, \ell) \right]^2. \tag{5.18}
\]

### 5.3.2 Estimation of the terms given by Poincaré inequality

Hence in order to obtain the required estimates for (5.15) and (5.16), one should find upper bounds for (5.17) and (5.18). This is the purpose of this section, where we derive below the three estimates (5.20) (5.21) and (5.22), which allow to complete the proof of Lemma 5.8.

We first write in a different way the term \(\sum_{i,j} \left| A_N F_{i,j} A_N \right|(\ell, k)\). Recall that \((e_k)_{k \in [N]}\) denotes the canonical basis of \(\mathbb{C}^N\). Recalling that the entry \((\ell, k)\) of a matrix \(M\) is equal to \(e^*_k M e_\ell\) and the elementary matrix \(E_{\ell,k}\) equals \(e_k e^*_\ell\), and since \(F_{ij}\) is a Hermitian matrix, we have

\[
\left| A_N F_{i,j} A_N \right|(\ell, k) = A_N F_{i,j} A_N(\ell, k) \times A_N^* F_{i,j} A_N^*(k, \ell) = e^*_k A_N F_{i,j} A_N e_k \times e^*_\ell A_N^* F_{i,j} A_N^* e_\ell.
\]

Note also that (5.10) implies the following equality, valid for any matrix \(M\):

\[
\sum_{i,j=1}^N F_{ij} M F_{ij} = \sum_{i,j=1}^N M_{ij} E_{ii} = \sum_{i,j=1}^N e_i e_j^* M e_j e_i^* \tag{5.19}
\]
Using (5.19) for $M = A_N e_k e_k^* A_N^*$, we get

$$\sum_{i,j} \left| [A_N F_{i,j} A_N] (\ell, k) \right|^2 = e_{\ell}^* A_N \sum_{i,j} (F_{i,j} M F_{i,j}) A_N^* e_\ell = e_{\ell}^* A_N \sum_{i,j} (e_i e_j^* M e_i^* e_j^*) A_N^* e_\ell = \sum_{i,j} e_{\ell}^* A_N e_i e_j^* A_N e_k e_k^* A_N^* e_\ell = \sum_{i,j} |A_N (\ell, i)|^2 |A_N (j, k)|^2.$$

Hence the application of Poincaré inequality (5.17) yields

$$\sum_{\ell=1}^N \text{Var}(A_N (\ell, k)) \leq \gamma_{\max}^2 N^{-1} \mathbb{E} \left[ \sum_{i,\ell} |A_N (\ell, i)|^2 \sum_j |A_N (j, k)|^2 \right] = \gamma_{\max}^2 \mathbb{E} \left[ \left( \frac{1}{N} \text{Tr}[A_N A_N^*] \right) \times \left( [A_N A_N^*] (k, k) \right) \right].$$

Recalling that $\|A_N\| = \| (A - H_N)^{-1} \| \leq \| m A^{-1} \|$ by (5.3), we get a first estimate uniform in $k \in [N]$:  

$$\sum_{\ell=1}^N \text{Var}(A_N (\ell, k)) \leq \gamma_{\max}^2 \mathbb{E} \|A_N\|^4 \leq \gamma_{\max}^2 \| m A^{-1} \|^4. \quad (5.20)$$

Moreover, specifying Poincaré inequality for $\ell = k$, we have

$$\text{Var}(A_N (k, k)) \leq \gamma_{\max}^2 N^{-1} \mathbb{E} \left[ \sum_{i,j} |A_N (k, i)|^2 |A_N (j, k)|^2 \right] \leq \gamma_{\max}^2 N^{-1} \mathbb{E} \left[ [A_N A_N^*] (k, k)^2 \right] \leq \gamma_{\max}^2 N^{-1} \mathbb{E} \|A_N\|^4 \leq \gamma_{\max}^2 N^{-1} \| m A^{-1} \|^4. \quad (5.21)$$

Similarly, we re-write the term $\sum_{i,j} |\mathcal{R}_N (\Delta [A_N F_{i,j} A_N]) (\ell, \ell)|^2$. Since for any matrix $M$,

$$\mathcal{R}_N (\Delta M) (\ell, \ell) = \sum_{e} \frac{\gamma_{\ell e}^2}{N} e_{\ell} e_{e}^* M e_{e},$$

we obtain

$$\left| \mathcal{R}_N (\Delta [A_N F_{i,j} A_N]) (\ell, \ell) \right|^2 = \mathcal{R}_N (\Delta [A_N F_{i,j} A_N]) (\ell, \ell) \mathcal{R}_N (\Delta [A_N^* F_{i,j} A_N^*]) (\ell, \ell) = \sum_{e, e'} \gamma_{\ell e}^2 \gamma_{\ell' e'}^2 N^{-2} e_{\ell e}^* A_N (F_{i,j} A_N e_{e} e_{e'} A_N^* F_{i,j}) A_N^* e_{e'}. \quad (5.30)$$
Using (5.19) for $M = A_N e_\ell e_{\ell'}^* A_N^*$, we get the expressions and estimates

$$
\sum_{i,j} \left| R_N \left( \Delta \left[ A_N F_{i,j} A_N \right] \right)(\ell, \ell) \right|^2
= \sum_{i,j,\ell,\ell'} \gamma_{\ell,\ell'}^2 \gamma_{\ell,\ell'}^2 N^{-2} e_\ell e_{\ell'}^* A_N e_i e_j^* A_N e_\ell e_{\ell'} A_N^* e_{\ell'} e_i A_N e_{\ell'} e_i A_N^* e_\ell e_{\ell'}
= \sum_{i,j,\ell,\ell'} \gamma_{\ell,\ell'}^2 \gamma_{\ell,\ell'}^2 N^{-2} A_N(\ell', i) A_N(j, \ell') A_N^*(\ell'', j) A_N^*(i, \ell''')
\leq \gamma_{\max}^4 N^{-2} \sum_{\ell,\ell'} \left| \sum_i A_N(\ell', i) A_N(j, \ell') A_N^*(\ell'', j) A_N^*(i, \ell''') \right|
= \gamma_{\max}^4 N^{-2} \sum_{\ell,\ell'} \left| (A_N A_N^*)(\ell', \ell'') \times (A^* A)(\ell'', \ell') \right|.
$$

The Cauchy-Schwarz inequality for $\sum_{\ell,\ell'}$ implies

$$
\sum_{i,j} \left| R_N \left( \Delta \left[ A_N F_{i,j} A_N \right] \right)(\ell, \ell) \right|^2
\leq \gamma_{\max}^4 N^{-2} \times \left( \sum_{\ell,\ell'} \left| (A_N A_N^*)(\ell', \ell'') \right|^2 \right)^{\frac{1}{2}} \times \left( \sum_{\ell,\ell'} \left| (A^* A)(\ell'', \ell') \right|^2 \right)^{\frac{1}{2}}
= \gamma_{\max}^4 N^{-2} \times \text{Tr} \left[ (A_N A_N^*)^2 \right] \leq \gamma_{\max}^4 N^{-1} \| \Im \Lambda \|^4.
$$

Hence with (5.18), the above inequality gives

$$
\sum_{\ell=1}^N \left| B_N(k, \ell) \right|^2 \text{Var} \left( D_N(\ell, \ell) \right) \leq \left( \sum_{\ell=1}^N \left| B_N(k, \ell) \right|^2 \right) \times \gamma_{\max}^6 N^{-2} \| \Im \Lambda \|^4
= (B_N B_N^*)(k, k) \times \gamma_{\max}^6 N^{-2} \| \Im \Lambda \|^4
\leq \gamma_{\max}^6 N^{-2} \| \Im \Omega^{-1} \|^2 \times \| \Im \Lambda \|^4. \quad (5.22)
$$

Inserting the estimates (5.20) and (5.22) in (5.15) gives that

$$
\left| \left[ (\Omega - Y_N)^{-1} E_N(\Lambda) \right](k, k') \right| \leq \left( \gamma_{\max}^6 N^{-2} \| \Im \Omega^{-1} \|^2 \| \Im \Lambda \|^4 \times \gamma_{\max}^2 N^{-1} \| \Im \Lambda \|^4 \right)^{\frac{1}{2}}
= \gamma_{\max}^4 N^{-1} \| \Im \Lambda \|^4 \| \Im \Omega^{-1} \|.
$$

Moreover, if $Y_N$ is diagonal, then using (5.16), (5.21) and (5.15) implies

$$
\left| \left[ (\Omega - Y_N)^{-1} E_N(\Lambda) \right](k, k') \right| \leq \left( \gamma_{\max}^6 N^{-2} \| \Im \Omega^{-1} \|^2 \| \Im \Lambda \|^4 \times \gamma_{\max}^2 N^{-1} \| \Im \Lambda \|^4 \right)^{\frac{1}{2}}
= \gamma_{\max}^4 N^{-\frac{3}{2}} \| \Im \Lambda \|^4 \| \Im \Omega^{-1} \|,
$$

which proves Inequality (5.14) and complete the proof of Lemma 5.8.
6 The fixed point equation

In this section, we prove the existence and uniqueness of the deterministic equivalent $G^c_{H_N}(\Delta)$, and we derive an estimate for the difference $G^c_{H_N} - \mathbb{E}[G_{H_N}]$.

6.1 Fixed point formulation

Recall that for any $\Lambda$ in $D_N(\mathbb{C}^+)$, we denote $G_Y(\Lambda) = \Delta[(\Lambda - Y)^{-1}]$. For any $\Lambda$ in $D_N(\mathbb{C}^+)$ we consider the function

$$
\psi_\Lambda : D_N(\mathbb{C}^{-}) \to D_N(\mathbb{C}^{-})
$$

and so

$$
G \mapsto G_Y(\Lambda - \mathcal{R}_N(G)).
$$

So $G^c_{H_N}(\Lambda)$ is solution of Equation (1.12) of Theorem 1.1 if and only if $G^c_{H_N}(\Lambda)$ is a fixed point of $\psi_\Lambda$ for any $\Lambda$. Note that by Lemma 5.4, when $G \in D_N(\mathbb{C}^{-})$ then $\mathcal{R}_N(G) \leq 0$ and so $\Lambda - \mathcal{R}_N(G) \in D_N(\mathbb{C}^+)$. Hence we can correctly evaluate the function $G_Y$ in this diagonal matrix and the expression defining $\psi_\Lambda$ makes sense. Moreover, by the last statement of Lemma 5.3 we indeed have $\psi_\Lambda(G) \in D_N(\mathbb{C}^{-})$. We shall use the next statement later.

**Lemma 6.1.** For any $\Lambda \in D_N(\mathbb{C}^+)$ the function $\psi_\Lambda$ is bounded and Lipschitz for the operator norm: for any $G, G' \in D_N(\mathbb{C}^{-})$,

$$
\|\psi_\Lambda(G)\| \leq \|3m \Lambda^{-1}\|,
$$

$$
\|\psi_\Lambda(G) - \psi_\Lambda(G')\| \leq \gamma^2_{\text{max}}\|3m \Lambda^{-1}\|^2 \times \|G - G'\|.
$$

**Proof.** We have by Lemmas 5.3 and 5.4

$$
\|\psi_\Lambda(G)\| = \|G_Y((\Lambda - \mathcal{R}_N(G))\| \leq \|3m((\Lambda - \mathcal{R}_N(G))^{-1}\| \leq \|3m\Lambda^{-1}\|
$$

Moreover, by Lemma 5.2 and the fact that $\|\cdot\|$ is an algebra norm, for any $G, G'$ in $D_N(\mathbb{C}^{-})$ we have

$$
\|\psi_\Lambda(G) - \psi_\Lambda(G')\| = \|\Delta[(\Lambda - \mathcal{R}_N(G) - Y)^{-1}][\mathcal{R}_N(G - G')](\Lambda - \mathcal{R}_N(G') - Y)^{-1}]\|
\leq \|[(\Lambda - \mathcal{R}_N(G) - Y)]^{-1}\| \times \|\mathcal{R}_N(G - G')\| \times \|((\Lambda - \mathcal{R}_N(G') - Y)^{-1}\|
$$

By (5.3) we have $\|((\Lambda - \mathcal{R}_N(G) - Y)^{-1}\| \leq \|3m((\Lambda - \mathcal{R}_N(G))^{-1}\|$. But by Lemma 5.4 we have $3m(\Lambda - \mathcal{R}_N(G)) \geq 3m \Lambda$ and so $\|((\Lambda - \mathcal{R}_N(G) - Y)^{-1}\| \leq \|3m \Lambda^{-1}\|$. The same inequality holds for $G'$ instead of $G$. These inequalities together with the estimate $\|\mathcal{R}_N(G - G')\| \leq \gamma^2_{\text{max}}\|G - G'\|$ of Lemma 5.4 yield

$$
\|\psi_\Lambda(G) - \psi_\Lambda(G')\| \leq \gamma^2_{\text{max}}\|3m \Lambda^{-1}\|^2 \times \|G - G'\|.
$$

By Banach fixed-point theorem and Lemma 6.1, we get the existence and uniqueness a priori of the deterministic equivalent on a region on $D_N(\mathbb{C}^+)$.

**Corollary 6.2.** For any $\Lambda$ such that $\|3m \Lambda^{-1}\| < \gamma^{-1}_{\text{max}}$ there exists a unique deterministic diagonal matrix $G^c_{H_N}(\Lambda) \in D_N(\mathbb{C}^{-})$ such that $G^c_{H_N}(\Lambda) = \psi_\Lambda(G^c_{H_N}(\Lambda))$. 

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6.2 The deterministic equivalent

6.2.1 Setting of the problem

We want to extend the previous corollary for any \( \Lambda \) in \( D_N(\mathbb{C})^+ \). The difficulty is that \( \psi_\Lambda \) is not contractive in general. Fortunately, it will be enough in our problem to consider uniqueness in the class of analytic function in several variables. Recall from [52] that for any open set \( \Omega \) of \( D_N(\mathbb{C}) \), we say that a function \( G : \Omega \to M_N(\mathbb{C}) \) is analytic on \( \Omega \) whenever for any \( k, \ell = 1, \ldots, N \) the function
\[
(\lambda_1, \ldots, \lambda_N) \mapsto G(\text{diag}(\lambda_1, \ldots, \lambda_N))(k,\ell)
\]
are analytic in each variable \( \lambda_i \).

Lemma 6.3. There exists a unique deterministic analytic map \( G^0_{H_N} : D_N(\mathbb{C})^+ \to D_N(\mathbb{C})^- \) such that \( G^0_{H_N}(\Lambda) = \psi_\Lambda(G^0_{H_N}(\Lambda)) \) for any \( \Lambda \in D_N(\mathbb{C})^+ \). Moreover, for any \( \Lambda, \Lambda' \in D_N(\mathbb{C})^+ \),
\[
\|G^0_{H_N}(\Lambda) - G^0_{H_N}(\Lambda')\| \leq \|\Im m \Lambda^{-1}\| \|\Im m \Lambda'^{-1}\| \times \|\Lambda - \Lambda'\|. \tag{6.1}
\]

By [52, Conclusion 1.2.1.2], the analytic continuation principle holds for analytic maps in several variables. Hence, by Corollary 6.2 we know that there exists at most one analytic map \( G \) solution of the fixed point problem on \( D_N(\mathbb{C})^+ \), since all solutions must coincide in \( \Omega \) \( \in \) \( D_N(\mathbb{C}) \).

In Section 6.2.2 we prove the existence of such function \( G^0_{H_N} \) and that it satisfies the estimate (6.1). Then, Section 6.2.3 will be dedicated to the proof of analyticity.

6.2.2 Large random matrix model

Given \( N, \Gamma_N \) and \( Y_N \) fixed we consider an intermediate sequence of Hermitian random matrices \( H_{N,M} \) of size \( NM \) by \( NM \). Seeing a generic element \( A \) of \( M_{NM}(\mathbb{C}) \) as an element of \( M_{N,M}(\mathbb{C}) \), we denote
\[
A = \left( A_{i,i',j,j'} \right)_{i,j \in [N], i',j' \in [M]} = \sum_{i,j \in [N], i',j' \in [M]} A_{i,i',j,j'} E_{ij} \otimes E_{i',j'}.
\]

Then, we consider two new random and deterministic matrices.

- Let \( X_{N,M} \) be a G.U.E. matrix with variance profile \( \Gamma_{N,M} = \Gamma_N \otimes \mathbb{1}_{M,M} \), where \( \mathbb{1}_{M,M} \) is the matrix whose all entries are one. Hence the variance profile is constant on blocks of size \( M \times M \) and we can write
\[
X_{N,M} = \sum_{i,j \in [N], i',j' \in [M]} \gamma_{i,j} N^{-\frac{1}{2}} M^{-\frac{1}{2}} x_{i,j,i',j'} E_{ij} \otimes E_{i',j'},
\]
where the \( x_{i,j,i',j'} \) are complex Gaussian random variables, i.i.d. up to the Hermitian symmetry, centered and such that \( x_{i,j,i',j'} \) has variance 1.
- We denote by $Y_{N,M} = Y_N \otimes \mathbb{1}_{M,M}$ the deterministic constant on blocks of size $M \times M$, so that $Y_{N,M} = \sum_{i,j,i',j'} Y_N(i,j) E_{i,j} \otimes E_{i',j'}$.

We set $H_{N,M} = X_{N,M} + Y_{N,M}$. To avoid ambiguity, we denote by $\Lambda_N$ a generic element of $D_N(\mathbb{C})^+$ and $\Lambda_{M,N}$ a generic element of $D_{NM}(\mathbb{C})^+$. We consider the diagonal of the generalized resolvent

$$G_{H_{N,M}} : D_{NM}(\mathbb{C})^+ \to D_{NM}(\mathbb{C})^-$$

so that $\Lambda_{N,M} \mapsto \Delta\left(\Lambda_{N,M} - H_{N,M}\right)^{-1}$, and the deterministic function

$$G_{H_N}^{c_{i,j}} : D_N(\mathbb{C})^+ \to D_N(\mathbb{C})^-$$

where $\Lambda_N \mapsto \left(i d \otimes \frac{1}{M} \text{Tr}\left[\mathbb{E}\left[G_{H_{N,M}}(\Lambda_N \otimes \mathbb{1}_{M \times M})\right]\right]\right)$. (6.2)

Note first that since $\|G_{H_N}(\Lambda_N)\| \leq \|3 m \Lambda_N^{-1}\|$, we know that up to a subsequence $G_{H_N}^{c_{i,j}}(\Lambda_N)$ has a limit $G_{H_N}^{c_{i,j}}(\Lambda_N')$ as $N$ goes to infinity for any $\Lambda_N \in D_N(\mathbb{C})^-$. Moreover, the same computation as Lemma 6.1 yields

$$\|G_{H_N}^{c_{i,j}}(\Lambda_N) - G_{H_N}^{c_{i,j}}(\Lambda_N')\| \leq \|3 m \Lambda_N^{-1}\|\|3 m \Lambda_N'^{-1}\| \times \|\Lambda_N - \Lambda_N'\|.$$ (6.3)

Letting $M$ going to infinity along a subsequence, this implies that the estimate (6.1) is valid for any accumulation point $G_{H_N}(\Lambda_N)$ of $G_{H_N}^{c_{i,j}}(\Lambda_N)$.

We shall now prove that $G_{H_N}^{c_{i,j}}$ converges when $M$ goes to infinity to a solution of the fixed point problem. Thanks to Lemma 5.8, we may now apply Equality (4.2) to the random matrix $H_{N,M}$: for any $\Lambda_{N,M} \in D_{NM}(\mathbb{C})$, we have

$$\mathbb{E}\left[G_{H_{N,M}}(\Lambda_{N,M})\right] = \psi_{\Lambda_{N,M}}\left(\mathbb{E}\left[G_{H_{N,M}}(\Lambda_{N,M})\right]\right) + \Theta_{M,N},$$ (6.4)

where $\|\Theta_{M,N}\| \leq \gamma_6^{\max} N^{-\eta} M^{-\eta} \|3 m \Lambda_{M,N}^{-1}\|^5$ by Inequality (4.3) and

$$\psi_{\Lambda_{M,N}}(G) = G_{M,N}(\Lambda_{M,N} - \mathcal{R}_{N,M}(G))^{-1},$$

for any $G \in D_{NM}(\mathbb{C})$. Note that $\gamma_6^{\max}$ is indeed the maximum of the variances in the profile $\Gamma_{N,M}$. Moreover, because of the definition of $\Gamma_{N,M}$ the map $\mathcal{R}_{N,M}$ is given by: for any $G \in D_{NM}(\mathbb{C})$, for any $i \in [N], i' \in [M]$

$$\mathcal{R}_{N,M}(G)(i, i', i) = \sum_{j,j'} \left(\Gamma_N(i,j) \otimes \mathbb{1}_{M,M}(i',j')\right) \times G(j, j', j, j') = \sum_{j \in [N]} \left(\frac{\Gamma_N(i,j)}{N}\right) \frac{1}{M} \sum_{j' \in [M]} G(j, j', j, j') = \mathcal{R}_N\left((id \otimes \frac{1}{M} \text{Tr}(G)) \otimes \mathbb{1}_M\right).$$

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Moreover note that if \( \Lambda_{N,M} = \Lambda_N \otimes 1_M \) then \( G_{Y_N,M}(\Lambda_{N,M}) = G_{Y_N}(\Lambda_N) \otimes 1_M \). Hence we get for any \( G \in D_{N,M}(\mathbb{C}) \):
\[
\psi_{\Lambda_N \otimes 1_M}(G) = \psi_{\Lambda_N} \left( (id \otimes \frac{1}{M} \text{Tr})(G) \right) \otimes 1_M.
\]
Hence, applying \( id \otimes \frac{1}{M} \text{Tr} \) in (6.4) yields the following formula:
\[
G_{H_N}^{o,M}(\Lambda_N) = \psi_{\Lambda_N} \left( G_{H_N}^{o,M}(\Lambda_N) \right) + (id \otimes \frac{1}{M} \text{Tr})(\Theta_{M,N}).
\]
Since \( \|\Theta_{M,N}\| \) goes to zero as \( M \) goes to infinity and by the continuity of \( \psi_{\Lambda_N} \), letting \( M \) go to infinity proves that any accumulation point \( G_{H_N}^{o,M} \) of \( G_{H_N}^{o,M} \) is solution of the fixed point problem that satisfies (6.1) thanks to Inequality (6.3).

6.2.3 Analyticity

Let us justify that the quantities under consideration up to now are analytic functions.

**Lemma 6.4.** For any Hermitian random matrix \( M \), the function \( \Lambda \mapsto \mathbb{E}\left[(\Lambda - M)^{-1}\right] \) is analytic on \( D_N(\mathbb{C})^+ \).

**Proof.** Let us first assume that the matrices are deterministic. We prove the lemma by induction of the size \( N \) of the matrices. Since for any \( m \in \mathbb{R} \) the map \( \lambda \mapsto (\lambda - m)^{-1} \) is analytic on \( \mathbb{C}^+ \), the lemma is true for \( N = 1 \). From now we fix \( N \geq 2 \) and we assume the lemma is true for all deterministic Hermitian matrices of size \( N - 1 \).

We write \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N) \) and recall that we denote \( A_N = (\Lambda - M)^{-1} \). For any \( k \in [N] \), let \( A_{N-1}^{(k)} \) be the inverse of the \( N-1 \) by \( N-1 \) matrix obtained from \( (\Lambda - M) \) by removing the \( k \)-th line and column. Let \( m^{(k)} \) be the vector of size \( N - 1 \) obtained from the \( k \)-th column of \( M \) by removing the \( k \)-th entry. Recall the Schur complement formula [8, Appendix A.1.4]:
\[
(\Lambda - M)^{-1}(k, k) = A_N(k, k) = \left( \lambda_k - M(k, k) - m^{(k)*} A_{N-1}^{(k)} m^{(k)} \right)^{-1}, \quad \forall k \in [N]
\]
\[
(\Lambda - M)^{-1}(k, \ell) = -(A_{N-1}^{(k)*} m^{(k)}) (\ell) \times A_N(k, k), \quad \forall k > \ell \in [N]
\]
\[
(\Lambda - M)^{-1}(k, \ell) = -(m^{(k)} A_{N-1}^{(k)*} (\ell)) \times A_N(k, k), \quad \forall k < \ell \in [N].
\]

By induction hypothesis, \( A_N^{(k)} \) is analytic on \( D_{N-1}(\mathbb{C})^+ \). By Lemma 5.3 we have
\[
\Im m \left( \lambda_k - M(k, k) - m^{(k)*} A_{N-1}^{(k)} m^{(k)} \right) = \Im m \lambda_k - m^{(k)*} (\Im m A_{N-1}^{(k)}) m^{(k)} \geq \Im m \lambda_k
\]
Hence the maps \( \Lambda \mapsto (\Lambda - M)^{-1}(k, k) \) are analytic in each variable for each \( k = 1, \ldots, N \), and hence so are the maps \( \Lambda \mapsto (\Lambda - M)^{-1}(k, \ell) \) for any \( k, \ell \).

Let now assume that \( M \) is random. Each realization \( \Lambda \mapsto (\Lambda - M)^{-1} \) is analytic and the map is bounded. Hence \( \Lambda \mapsto \mathbb{E}\left[(\Lambda - M)^{-1}\right] \) is also analytic. \( \square \)

Hence the map \( G_{H_N}^{o,M} \) defined by (6.2) is indeed analytic. Since it is Lipschitz by Inequality (6.3), it follows that every accumulation point \( G_{H_N}^{o,M} \) of the sequence is also analytic. This finishes the proof of Lemma 6.3.
6.3 Stability of the fixed point equation and proof of Lemma 4.1

Let \( G^0_{H_N} : D_N(\mathbb{C}^+) \to D_N(\mathbb{C})^- \) be the deterministic equivalent, unique analytic solution of the fixed point problem

\[
G^0_{H_N}(\Lambda) = G_Y N\left(\Lambda - \mathcal{R}_N\left(\mathbb{E}\left[G_{H_N}(\Lambda)\right]\right)\right).
\]

For reading convenience, we recall that \( \mathbb{E}\left[G_{H_N}(\Lambda)\right] = \mathbb{E}\left[\Delta[(\Lambda - H_N)^{-1}]\right] \) satisfies the approximate subordination property, namely

\[
\mathbb{E}\left[G_{H_N}(\Lambda)\right] = G_Y N\left(\Lambda - \mathcal{R}_N\left(\mathbb{E}\left[G_{H_N}(\Lambda)\right]\right)\right) + \Theta_N(\Lambda), \tag{6.5}
\]

where \( \Theta_N(\Lambda) = \Delta[(\Omega_{H_N}(\Lambda) - Y_N)^{-1}E_N(\Lambda)] \), whose operator norm satisfies

\[
\|\Theta_N(\Lambda)\| \leq \gamma^4_{\text{max}}N^{-\eta}\|\Im m\Lambda^{-1}\|^5, \tag{6.6}
\]

by Lemma 5.8 (with \( \eta = 1 \) in general and \( \eta = \frac{3}{2} \) if \( Y_N \) is diagonal). The purpose of this section is to prove Lemma 4.1, giving an estimate for the norm of the difference \( \|G^0_{H_N}(\Lambda) - \mathbb{E}\left[G_{H_N}(\Lambda)\right]\| \).

We define two diagonal matrices by

\[
\tilde{\Lambda} = \Lambda - \mathcal{R}_N(\Theta_N(\Lambda)) = \Lambda - \mathcal{R}_N(\mathbb{E}\left[G_{H_N}(\Lambda)\right] - \tilde{G}_N(\Lambda)).
\]

Provided we can justify that \( \tilde{\Lambda} \) belongs to \( D_N(\mathbb{C})^+ \), we have

\[
\tilde{G}_N(\Lambda) = G_Y N\left(\Lambda - \mathcal{R}_N\left(\mathbb{E}\left[G_{H_N}(\Lambda)\right]\right)\right) = G_Y N\left(\tilde{\Lambda} - \mathcal{R}_N(\tilde{G}_N(\Lambda))\right) = \psi_{\tilde{\Lambda}}(\tilde{G}_N(\Lambda)). \tag{6.7}
\]

With \( \mathbb{I}_N \) denoting the identity matrix and using Lemma 5.4 and the bound (6.6) for \( \|\Theta_N(\Lambda)\| \), we have

\[
\Im m\tilde{\Lambda} = \Im m\Lambda - \Im m\mathcal{R}_N(\Theta_N(\Lambda)) \geq \Im m\Lambda - \|\mathcal{R}_N(\Theta_N(\Lambda))\| \times \mathbb{I}_N
\]

\[
\geq \Im m\Lambda - \gamma^6_{\text{max}}N^{-\eta}\|\Im m\Lambda^{-1}\|^5 \times \mathbb{I}_N. \tag{6.8}
\]

Assuming that

\[
\gamma^6_{\text{max}}N^{-\eta}\|\Im m\Lambda^{-1}\|^6 < 1, \tag{6.9}
\]

we indeed have \( \tilde{\Lambda} \in D_N(\mathbb{C})^+ \) and so \( \tilde{G}_N(\Lambda) \) is solution of the fixed point problem for \( \psi_{\tilde{\Lambda}} \).

Hence, by Lemma 6.3 we obtain the equality \( \tilde{G}_N(\Lambda) = G^0_{H_N}(\tilde{\Lambda}) \) and so, by Equalities (6.5) and (6.7), we obtain

\[
\mathbb{E}\left[G_{H_N}(\Lambda)\right] = G^0_{H_N}(\tilde{\Lambda}) + \Theta_N(\Lambda).
\]

By Lemma 6.4, the map \( \Lambda \mapsto \mathbb{E}\left[G_{H_N}(\Lambda)\right] \) is analytic on \( D_N(\mathbb{C})^+ \). Recall that \( \Theta_N(\Lambda) = \Delta[(\Omega_{H_N}(\Lambda) - Y_N)^{-1}E_N] \) where \( E_N \) is defined in (5.11) and \( \Omega_{H_N}(\Lambda) \) is defined in (5.12).
One checks easily that the map $\Lambda \mapsto \Theta_N(\Lambda)$ is analytic on $D_N(\mathbb{C})^+$, which implies that so are $\Lambda \mapsto \tilde{G}_N(\Lambda)$ and $\Lambda \mapsto \tilde{\Lambda}$.

Hence the equality extends by analyticity for all $\Lambda > 0$ such that $\tilde{\Lambda} > 0$. Hence we get
\[
\| \mathbb{E}[G_{H_N}(\Lambda)] - G_{H_N}^0(\Lambda) \| \leq \| G_{H_N}^0(\tilde{\Lambda}) - G_{H_N}^0(\Lambda) \| + \| \Theta_N(\Lambda) \|. \tag{6.10}
\]
Moreover, with the same proof as for (6.1), we have the estimate
\[
\| G_{H_N}^0(\tilde{\Lambda}) - G_{H_N}^0(\Lambda) \| \leq \| \Im \Lambda \| \| \Im \tilde{\Lambda}^{-1} \| \| \Lambda - \tilde{\Lambda} \|. \tag{6.11}
\]
We have by Lemma 5.4, $\| \Lambda - \tilde{\Lambda} \| = \| \mathcal{R}_N(\Theta_N(\Lambda)) \| \leq \gamma_{\max}^2 \| \Theta_N(\Lambda) \|$. Moreover, under the assumption
\[
\gamma_{\max}^6 N^{-\eta} \| \Im \Lambda \| \| \Im \tilde{\Lambda}^{-1} \| \| \Lambda - \tilde{\Lambda} \| \leq 1 - \delta, \text{ for some } 0 < \delta < 1, \tag{6.12}
\]
we obtain by (6.8) that
\[
\| \Im \tilde{\Lambda}^{-1} \| \geq \| \Im \Lambda \| \gamma_{\max}^6 \| \Im \Lambda \|^{-5} N^{-\eta} \| \Lambda - \tilde{\Lambda} \| \geq \frac{\delta}{\| \Im \Lambda \| \| \Lambda - \tilde{\Lambda} \|}.
\]
Hence provided that Condition (6.12) holds, we have $\| \Im \tilde{\Lambda}^{-1} \| / \delta$. Hence combining (6.10) with (6.11) and (6.6), the previous estimates on $\| \Lambda - \tilde{\Lambda} \|$ and $\| \Im \tilde{\Lambda}^{-1} \|$ give
\[
\| \mathbb{E}[G_{H_N}(\Lambda)] - G_{H_N}^0(\Lambda) \| \leq \| \Im \Lambda \|^{-2} \gamma_{\max}^2 \| \Theta_N(\Lambda) \| + \| \Theta_N(\Lambda) \|
\leq \gamma_{\max}^4 (1 + \gamma_{\max}^2 / \delta) \| \Im \Lambda \|^{-5} N^{-\eta},
\]
which completes the proof of Lemma 4.1.

7 Analyse of the resolvent

We now have all the ingredients to control the difference between the resolvent $(\lambda \mathbb{I}_N - H_N)^{-1}$ and the deterministic equivalent $G_{H_N}^0(\lambda \mathbb{I}_N)$ which will finally complete the proof of Theorem 1.1.

7.1 Expectation out of the diagonal

We first establish results allowing to show that the expectation of the resolvent is a diagonal matrix. Recall that a random matrix $A$ is unitarily invariant whenever $UAU^*$ has the same law as $A$ for any unitary matrix $U$.

Lemma 7.1. Let $A$ be a $N$ by $N$ unitarily invariant random matrix whose entries have finite moment of any orders and let $\Sigma \in \mathcal{M}_N(\mathbb{C})$ Then for any $n \geq 1$, the matrix $\mathbb{E}[(\Sigma \circ A)^n]$ is diagonal.
Proof. For any \( i_1, i_{n+1} \in [N] \) we have
\[
E[(\Sigma \circ A)^n](i_1, i_{n+1}) = \sum_{i_2, \ldots, i_n=1}^{N} \left( \prod_{k=1}^{n} \sigma(i_k, i_{k+1}) \right) \times E\left[ \prod_{k=1}^{n} A(i_k, i_{k+1}) \right],
\]
where \( \sigma(i_k, i_{k+1}) \) denotes the \((i_k, i_{k+1})\)-th entry of the matrix \( \Sigma \).

We shall prove that for any \( i_2, \ldots, i_n \in [N] \) then \( E\left[ \prod_{k=1}^{n} A(i_k, i_{k+1}) \right] = 0 \) when \( i_1 \neq i_{n+1} \). For this purpose we introduce a matrix function \( A_t \) that depends on an implicit parameter \( G \): for any anti-Hermitian matrix \( G \), i.e. such that \( G^* = -G \), and for any \( t \in \mathbb{R} \), we denote \( A_t = e^{tG}Ae^{-tG} \). Note that \( A_0 = A \) and the derivative of \( A_t \) with respect to \( t \) is \( \partial_t A_t = GA_t - A_tG \). Moreover, the unitary invariance of \( A \) implies that \( A \) and \( A_t \) have the same law. In particular for any \( i_1, \ldots, i_{n+1} \in [N] \) and any \( t \in \mathbb{R} \) we have
\[
E\left[ \prod_{k=1}^{n} A(i_k, i_{k+1}) \right] = E\left[ \prod_{k=1}^{n} A_t(i_k, i_{k+1}) \right].
\]

We now differentiate the above equality with respect to \( t \) and take \( t = 0 \): using Leibniz formula,
\[
0 = E\left[ \partial_t \left( \prod_{k=1}^{n} A_t(i_k, i_{k+1}) \right) \right]_{t=0}
\]
\[
= \sum_{k=1}^{n} E\left[ A(i_1, i_2) \cdots A(i_{k-1}, i_k)(GA - AG)(i_k, i_{k+1})A(i_{k+1}, i_{k+2}) \cdots A(i_n, i_{n+1}) \right].
\]

Recall that the above equality is a priori valid under the assumption that \( G \) is anti-Hermitian. But the relation is linear in \( G \), and the set of anti-Hermitian matrices spans \( \mathbb{M}_N(\mathbb{C}) \) as a vector space (any matrix \( A \) can be written as a linear combination of Hermitian matrices \( A = \Re \cdot A + \Im \cdot A \) and a matrix \( G \) is Hermitian whenever \( iG \) is anti-Hermitian). We can hence specify the equality for the elementary matrix \( G = E_{i_1, i_1} \).

Note that for any \( k \in [N] \), we have
\[
(GA - AG)(i_k, i_{k+1}) = (\delta_{i_k, i_k} - \delta_{i_k, i_{k+1}})A(i_k, i_{k+1}).
\]

Hence we obtain from Equation (7.2) a telescopic sum
\[
0 = E\left[ A(i_1, i_2) \cdots A(i_n, i_{n+1}) \right] \times \sum_{k=1}^{n} (\delta_{i_k, i_k} - \delta_{i_k, i_{k+1}})
\]
\[
= E\left[ A(i_1, i_2) \cdots A(i_n, i_{n+1}) \right] (1 - \delta_{i_1, i_{n+1}}).
\]
If \( i_1 \neq i_{n+1} \) then we get \( E\left[ \prod_{k=1}^{n} A(i_k, i_{k+1}) \right] = 0 \) for any \( i_2, \ldots, i_n \) and hence by Equation (7.1) we deduce that \( E[(\Sigma \circ A)^n] \) is a diagonal matrix.

\[\Box\]

**Corollary 7.2.** Let \( A \) and \( \Sigma \) be as in Lemma 7.1. Assume moreover that \( \Sigma \circ A \) is Hermitian. Then for any \( \Lambda \in \mathbb{D}_N(\mathbb{C}) \), the expectation of the generalized resolvent \( E[(\Lambda - \Sigma \circ A)^{-1}] \) is a diagonal matrix.
Proof. We first assume that there is a constant $B > 0$ such that almost surely one has $\|A\| \leq B$. For any matrix $M$ we denote by $\|M\|_F$ its Frobenius norm $\|M\|_F = (\text{Tr}[M^*M])^{\frac{1}{2}}$ and we recall that $\|M\| \leq \|M\|_F \leq \sqrt{N}\|M\|$. Hence we have for $M = \Sigma \circ A$,

$$\|\Sigma \circ A\| \leq \left( \text{Tr}\left[ (\Sigma \circ A)^*(\Sigma \circ A) \right] \right)^{\frac{1}{2}} = \left( \sum_{i,j} |\sigma(i,j)|^2 |A(i,j)|^2 \right)^{\frac{1}{2}} \leq \sigma_{\max} \sqrt{N} \|A\|,$$

where $\sigma(i,j)$ denotes the $(i,j)$-th entry of the matrix $\Sigma$, and $\sigma_{\max} = \max_{i,j} |\sigma(i,j)|$. For any $\Lambda \in \text{D}_N(\mathbb{C})^+$ such that $\Im \Lambda > \sigma_{\max} \sqrt{N} B I_N$, we have that $\| (\Sigma \circ A)\Lambda^{-1} \| < 1$, and thus the following identity holds

$$(\Lambda - \Sigma \circ A)^{-1} = \sum_{n \geq 0} \Lambda^{-1} ((\Sigma \circ A)\Lambda^{-1})^n,$$

where the convergence of the sum is normal. In particular we can interchange summation and expectation, namely

$$\mathbb{E}[ (\Lambda - \Sigma \circ A)^{-1} ] = \sum_{n \geq 0} \Lambda^{-1} \mathbb{E} \left[ ((\Sigma \circ A)\Lambda^{-1})^n \right].$$

Moreover, we have the equality $(\Sigma \circ A)\Lambda^{-1} = \Sigma' \circ A$ where $\Sigma' = \Sigma \Lambda^{-1}$. Hence by Lemma 7.1 for any $\Lambda \in \text{D}_N(\mathbb{C})^+$ and any $n \geq 1$ the matrix $\mathbb{E} \left[ ((\Sigma \circ A)\Lambda^{-1})^n \right]$ is diagonal. So for any $\Lambda$ such that $\Im \Lambda > \sigma_{\max} \sqrt{N} B I_N$, the matrix $\mathbb{E}[G_X(\Lambda)]$ is also diagonal. This fact extends for any $\Lambda \in \text{D}_N(\mathbb{C})^+$ by analytic continuation thanks to Lemma 6.4.

To treat the general case, we use a classical spectral truncation argument. Let us denote by $\lambda_1, \ldots, \lambda_N$ and $u_1, \ldots, u_N$ the eigenvalues and the associated eigenvectors of $A$, so that we have $A = \sum_{i=1}^N \lambda_i u_i^* u_i$. For any $B > 0$, we denote by $A^{(B)}$ the matrix $A^{(B)} = \sum_{i=1}^N \lambda_i 1(|\lambda_i| \leq B) u_i^* u_i$, where $1$ denote the indicator function. Hence $A^{(B)}$ is uniformly bounded in operator norm by $B$. By the previous case, the matrix $\mathbb{E}[ (\Lambda - \Sigma \circ A^{(B)})^{-1} ]$ is diagonal. Moreover, we have

$$\left\| (\Lambda - \Sigma \circ A)^{-1} - (\Lambda - \Sigma \circ A^{(B)})^{-1} \right\| \leq \|\Im \Lambda^{-1}\| \left\| \Sigma \circ (A^{(B)} - A) \right\| \leq \|\Im \Lambda^{-1}\| \sigma_{\max} \sqrt{N} \|A^{(B)} - A\|.$$
7.2 Concentration argument and proof of the main results

We now complete the proof of Theorem 1.1 by combining Lemma 4.1 with a concentration argument for the resolvent \((\Lambda - X_N - Y_N)^{-1}\) towards its expectation \(\mathbb{E}[(\Lambda - H_N)^{-1}]\).

**Lemma 7.3.** Let \(\Lambda \in D_N(\mathbb{C})^+\). Then, for any pair of unit vectors \(v, w\) (that is \(\|v\|_2 = \|w\|_2 = 1\)), one has that, for all \(t > 0\),

\[
\mathbb{P}\left(\|v^*((\Lambda - H_N)^{-1} - \mathbb{E}[(\Lambda - H_N)^{-1}])w\| > t\right) \leq 4 \exp\left(-N^t \frac{\|3m\Lambda\|_2^4}{4\gamma_{\max}^2}\right), \quad (7.3)
\]

where \(\gamma_{\max}^2\) is the maximum of the variances in the profile \(\Gamma_N\). Moreover, for any \(\lambda \in \mathbb{C}^+\) and all \(t > 0\),

\[
\mathbb{P}\left(\|g_{H_N}(\lambda) - \mathbb{E}[g_{H_N}(\lambda)]\| > t\right) \leq 4 \exp\left(-N^t \frac{\|3m\lambda\|_2^4}{8\gamma_{\max}^2}\right), \quad (7.4)
\]

where \(g_{H_N}\) is the Stieltjes transform of \(H_N\).

To prove this result, we use the following result of Gaussian concentration inequality for Lipschitz functions (see e.g. [18, Theorem 5.6]).

**Theorem 7.4.** Let \(X = (X_1, \ldots, X_n)\) be a vector of \(n\) independent standard normal random variables. Let \(f : \mathbb{R}^n \to \mathbb{R}\) be a \(L\)-Lipschitz function for the Euclidean norm of \(\mathbb{R}^n\). Then for all \(t > 0\) we have

\[
\mathbb{P}\left(f(X) - \mathbb{E}[f(X)] > t\right) \leq e^{-t^2/(2L)^2}. \quad (7.5)
\]

**Proof.** We let \(H_N(\mathbb{C}) \subset M_N(\mathbb{C})\) be the subset of Hermitian matrices. For two unit vectors \(v, w\) and any \(\Lambda\) in \(D_N(\mathbb{C}^+)\) we consider the function

\[
\phi_{\Lambda, v, w} : H_N(\mathbb{C}) \to \mathbb{C}, \quad A \mapsto v^*(\Lambda - A - Y_N)^{-1}w.
\]

In order to use Theorem 7.4 for the real and the imaginary parts of \(\phi_{\Lambda, v, w}\), we shall estimate its Lipschitz constant. We denote by \(\|A\|_F = \left(\text{Tr}[A^*A]\right)^{\frac{1}{2}}\) the Frobenius norm of a matrix \(A\). We use the isomorphism between \(M_N(\mathbb{C})\) endowed with \(\|\cdot\|_F\) and \(\mathbb{R}^{N^2}\) endowed as the Euclidean norm. Using Cauchy-Schwarz’s inequality and Lemma 5.3, it follows that, for any \(A, A'\) in \(H_N(\mathbb{C})\),

\[
|\phi_{\Lambda, v, w}(A) - \phi_{\Lambda, v, w}(A')| = \|v^*(\Lambda - A - Y_N)^{-1}(A - A')(\Lambda - A' - Y_N)^{-1}w\| \\
\leq \|v^*(\Lambda - A - Y_N)^{-1}\|_2\|(A - A')(\Lambda - A' - Y_N)^{-1}w\|_2 \\
\leq \|(\Lambda - A - Y_N)^{-1}\|\|A - A'\|\|(\Lambda - A' - Y_N)^{-1}\| \\
\leq \|3m\Lambda^{-1}\|\|A - A'\| \leq \|3m\Lambda^{-1}\|\|A - A'\|_F. \quad (7.6)
\]
Since $A$ can be decomposed in the basis (5.6) of Hermitian matrices as $A = \sum_{i,j=1}^{N} \tilde{a}_{i,j} F_{i,j}$, where the $\tilde{a}_{i,j}$ are reals, we have that $\phi_A^{w,w}$ is a $L$-Lipschitz function (with Lipschitz constant $L = \| \Im m \Lambda^{-1} \|^2$) of the $N \times N$ matrix $\tilde{A} = (\tilde{a}_{i,j})$ having real entries. Now, recall the decomposition (5.7) of $X_N$ into the basis (5.6)

$$X_N = \sum_{i,j=1}^{N} \tilde{\gamma}_{i,j} \frac{1}{\sqrt{N}} \tilde{x}_{i,j} F_{i,j},$$

where the $\tilde{x}_{i,j}$ are i.i.d. real Gaussian random variables that are centered and of variance one. By Inequality (7.6) and after a change of variable, we get that the function $(\tilde{x}_{i,j})_{i,j} \rightarrow \phi_A^{w,w}(X_N)$ is a $L$-Lipschitz function (that is complex-valued) with Lipschitz constant

$$L = \| \Im m \Lambda^{-1} \|^2 \gamma_{\max} \sqrt{N}.$$

Hence (7.2) follows by Gaussian concentration inequality, namely Equation (7.5). Now, let $\lambda \in \mathbb{C}^+$ and recall that the Stieltjes transform $H_N = X_N + Y_N$ is the map

$$g_{H_N}(\lambda) = \frac{1}{N} \text{Tr} \left[ G_{H_N}(\lambda \mathbb{I}_N) \right] = \frac{1}{N} \text{Tr} \left[ (\lambda \mathbb{I}_N - H_N)^{-1} \right],$$

and let us consider the mapping $A \rightarrow \phi_\lambda(A) := \text{Tr} \left[ (\lambda \mathbb{I}_N - A - Y_N)^{-1} \right]$ for $A \in H_N(\mathbb{C})$. For any $A, A' \in H_N(\mathbb{C})$, we remark that

$$|\phi_\lambda(A) - \phi_\lambda(A')| = |\text{Tr} \left[ (\lambda \mathbb{I}_N - A - Y_N)^{-1} (A - A') (\lambda \mathbb{I}_N - A' - Y_N)^{-1} \right]|
\geq |\text{Tr} \left[ (\lambda \mathbb{I}_N - A - Y_N)^{-1} (\lambda \mathbb{I}_N - A' - Y_N)^{-1} (A - A') \right]| \quad (7.8)$$

To obtain an appropriate upper bound for (7.8), we use the following inequalities (see e.g. Lemma II.2 in [41]): denoting by $\sigma_1(A) \leq \cdots \leq \sigma_N(A)$ the singular values of $A$,

$$\sum_{i=1}^{N} \sigma_{n-i+1}(\Re B) \sigma_i(C) \leq \Re \left( \text{Tr} [BC] \right) \leq \sum_{i=1}^{N} \sigma_i(\Re B) \sigma_i(C) \quad (7.9)$$

which hold for any $B \in M_N(\mathbb{C})$ and $C \in H_N(\mathbb{C})$. For two such matrices, Inequality (7.9) combined with Cauchy-Schwarz’s inequality implies that

$$|\Re (\text{Tr} [BC])| \leq |\sigma_1(\Re B)| \sum_{i=1}^{N} |\sigma_i(C)| \leq \| \Re B \| \| \text{Tr} [BC] \| \sqrt{N} \sum_{i=1}^{N} |\sigma_i(C)|^2.$$ 

Since $\|C\|_F^2 = \sum_{i=1}^{N} |\sigma_i(C)|^2$ and $\| \Re B \| \leq \| B \|$ (by the same argument as for the imaginary part in the proof of Lemma 5.1), one finally obtains that

$$|\Re (\text{Tr} [BC])| \leq \sqrt{N} \| B \| \| C \|_F.$$ 

Using the fact $\Im m A = -\Re (\mathbf{1}^T A)$, one obtains by similar arguments that

$$|\Im m (\text{Tr} [BC])| \leq \sqrt{N} \| B \| \| C \|_F.$$
which finally yields
\[ |\text{Tr}[BC]| \leq |\Re(\text{Tr}[BC])| + |\Im(\text{Tr}[BC])| \leq 2\sqrt{N}\|B\|\|C\|_F \]

Hence, combining the above inequality with (7.8) and Lemma 5.3, it follows that
\[ |\phi_\lambda(A) - \phi_\lambda(A')| \leq 2\sqrt{N}\|\Im\lambda\|^{-2}\|A-A'|_F. \]

Therefore, thanks to the decomposition (7.7) for \( X_N \), the mapping \( \tilde{X}_N \rightarrow \frac{1}{N}\text{Tr}[\lambda I_N - X_N - Y_N]^{-1} = \frac{1}{N}\phi_\lambda(X_N) \) is a \( L \)-Lipschitz function with Lipschitz constant
\[ L = 2|\Im\lambda|^{-2}\gamma_{\text{max}}N^{-\eta}. \]

Therefore, using again Gaussian concentration for Lipschitz functions, one obtains Inequality (7.4), which completes the proof of Lemma 7.3.

Now, let us fix \( 0 < \delta < 1 \), and consider \( \Lambda \in D_N(\mathbb{C})^+ \) satisfying Condition (1.13) so that \( \|\mathbb{E}(G_{H_N}(\Lambda)) - G_{H_N}^0(\Lambda)\| \leq t_N^{(1)} \), by Lemma 4.1, where
\[ t_N^{(1)} := \gamma_{\text{max}}^4 \left( 1 + \frac{\gamma_{\text{max}}}{\delta}\|\Im\Lambda^{-1}\|_2^2 \right) \|\Im\Lambda^{-1}\| \|\Lambda\|^{-5}N^{-\eta}. \]

Note that, for \( v = e_i \) and \( w = e_j \), Inequality (7.2) yields a concentration result of each entry \((\Lambda - H_N)^{-1}[i,j]\) of the resolvent towards its expectation. Since \( G_{H_N}(\Lambda) \) and its expectation are diagonal matrices one has that the operator norm of their difference satisfies
\[ \|G_{H_N}(\Lambda) - \mathbb{E}[G_{H_N}(\Lambda)]\| = \max_{1 \leq i \leq N} \left| (\Lambda - H_N)^{-1}[i,i] - \mathbb{E}[(\Lambda - H_N)^{-1}[i,i]] \right|. \]

Thus, combining Inequality (7.2) with a union bound yields the following concentration inequality: for all \( t > 0 \),
\[ \mathbb{P}(\|G_{H_N}(\Lambda) - \mathbb{E}[G_{H_N}(\Lambda)]\| \geq t) \leq 4N \exp \left( -N \frac{t^4\|\Im\Lambda\|^4}{4\gamma_{\text{max}}^2} \right), \]

Hence, taking \( t = t_N^{(2)} := \sqrt{\frac{4\gamma_{\text{max}}^2\log(N)}{\|\Im\Lambda\|^4}} N^{-1/2} \) (for some \( d > 1 \)), one finally obtains that
\[ \mathbb{P}(\|G_{H_N}(\Lambda) - G_{H_N}^0(\Lambda)(\lambda)\| \geq t_N^{(2)} + t_N^{(1)} ) \leq 4N^{1-d}, \]

which proves Inequality (1.14), and completes the proof of Theorem 1.1.

Then, to derive the proof of Corollary 1.2, one has to obtain a concentration inequality for the Stieltjes transform \( g_{H_N}(\lambda) \). Since \( g_{H_N}(\lambda) = \frac{1}{N}\text{Tr}[G_{H_N}(\lambda I_N)] \) and given that \( \left| \frac{1}{N}\text{Tr}[A] \right| \leq \|A\| \) for any diagonal matrix \( A \in M_N(\mathbb{C}) \), we obtain from Lemma 4.1 that, for \( \Lambda = \lambda I_N \) satisfying Condition (1.13),
\[ |\mathbb{E}[g_{H_N}(\lambda)] - g_{H_N}(\lambda)| \leq \tilde{t}_N^{(1)}, \quad \text{where} \quad \tilde{t}_N^{(1)} := \frac{\gamma_{\text{max}}^2}{\delta} \left( 1 + \frac{\gamma_{\text{max}}}{\delta}\|\Im\lambda\|^{-2}\|\Im\lambda\|^{-5}N^{-\eta} \right). \]
where \( \eta = 1 \) in general (and \( \eta = \frac{3}{2} \) if \( Y_N \) is diagonal). Therefore, taking \( t = \tilde{r}_N^{(2)} := \sqrt{\frac{8d^c}{3m \lambda^4}} N^{-1} \) (for some \( d > 0 \)) one obtains, by combining Inequalities (7.11), (7.12) and (7.13), that

\[
P\left( |g_{H_N}(\lambda) - g_{\hat{H}_N}(\lambda)| \geq \tilde{r}_N^{(2)} + \tilde{r}_N^{(1)} \right) \leq N^{-d},
\]

which proves Inequality (1.16), and completes the proof of Corollary 1.2.

Finally, to complete the proof of the main results stated in Section 1.2, we now assume that \( Y_N \) is a diagonal matrix and we take \( \Lambda = \lambda \mathbb{I}_N \) satisfying Condition (1.13). Since \( Y_N \) is supposed to be Hermitian, it is a diagonal matrix with real entries. Therefore, by Corollary 7.2, one has that \( \mathbb{E}[\mathbb{I}(\Lambda \mathbb{I}_N - H_N)^{-1}] \) is diagonal. Then, we remark that

\[
\|\beta_k(\lambda) - \beta_k^a(\lambda)\| \leq \|U_{N,k}^a((\Lambda \mathbb{I}_N - H_N)^{-1} - G_{\hat{H}_N}(\Lambda \mathbb{I}_N))U_{N,k}\| \|\Theta_k\| \quad (7.11)
\]

Now, one obviously has that

\[
\|U_{N,k}^a \mathbb{E}[G_{H_N}(\Lambda \mathbb{I}_N)] - G_{\hat{H}_N}(\Lambda \mathbb{I}_N)\| U_{N,k} \leq \|\mathbb{E}[G_{H_N}(\Lambda \mathbb{I}_N)] - G_{\hat{H}_N}(\Lambda \mathbb{I}_N)\| \leq \tilde{r}_N^{(1)}. \quad (7.12)
\]

Then, if we denote by \( u_1, \ldots, u_k \) the columns of the matrix \( U_{N,k} \), we remark that

\[
\|U_{N,k}^a((\Lambda \mathbb{I}_N - H_N)^{-1} - \mathbb{E}[(\Lambda \mathbb{I}_N - H_N)^{-1}]) U_{N,k}\| \leq k \max_{1 \leq i, \ell \leq k} u_i^a ((\Lambda \mathbb{I}_N - H_N)^{-1} - \mathbb{E}[(\Lambda \mathbb{I}_N - H_N)^{-1}]) u_\ell^a.
\]

Thus, by combining Inequality (7.2) with the fact that \( \mathbb{E}[(\Lambda \mathbb{I}_N - H_N)^{-1}] = \mathbb{E}[G_{H_N}(\Lambda \mathbb{I}_N)] \), it follows by a union bound argument that, for all \( t > 0 \),

\[
P \left( \frac{1}{k} \|U_{N,k}^a((\Lambda \mathbb{I}_N - H_N)^{-1} - \mathbb{E}[G_{H_N}(\Lambda \mathbb{I}_N)]) U_{N,k}\| \geq t \right) \leq 4k^2 \exp \left( -N \frac{t^2|3m \lambda|^4}{4 \gamma_{\max}^2} \right). \quad (7.13)
\]

Therefore, combining Inequalities (7.11), (7.12) and (7.13) we obtain that

\[
P \left( \|\beta_k(\lambda) - \beta_k^a(\lambda)\| \geq \|\Theta_k\|(kt_{N}^{(2)} + \tilde{r}_N^{(1)}) \right) \leq 4k^2 N^{-d},
\]

with \( t_{N}^{(2)} := \sqrt{\frac{4 \gamma_{\max} d \log(N)}{|3m \lambda|^4}} N^{-1/2} \), which finally yields Inequality (1.18), and completes the proofs of the results stated in Section 1.2.

### 7.3 Convergence of the fixed point algorithm

A key step in these numerical experiments in the numerical approximation of the solution of the fixed point equation (1.12) through an iterative algorithm whose convergence is first briefly discussed.

**Lemma 7.5.** For any diagonal function \( G_N^{(0)} : \mathbb{D}_N(\mathbb{C})^+ \rightarrow \mathbb{D}_N(\mathbb{C})^- \), we consider the sequence of diagonal functions \( (G_N^{(n)})_{n \geq 0} \) given by \( G_N^{(n+1)}(\Lambda) = \psi_\Lambda(G_N^{(n)}(\Lambda)) \) for any \( \Lambda \in \mathbb{D}_N(\mathbb{C})^+ \). Then, as \( n \) goes to infinity, the sequence \( G_N^{(n)}(\Lambda) \) converges to \( G_{\hat{H}_N}(\Lambda) \), where \( G_{\hat{H}_N} \) denotes the deterministic equivalent characterized in Lemma 6.3.

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Proof. Recall that by Lemma 6.1 the sequence \( (G_N^{(n)}(\Lambda))_{n \geq 0} \) is bounded, and so up to a subsequence it converges to some diagonal matrix \( G_N(\Lambda) \). If \( \Im \Lambda > 2 \max \), by Corollary 6.2 then \( G_N(\Lambda) = G_{H_N}^{(\circ)}(\Lambda) \) by contractivity of the fixed point problem. Moreover, for each \( n \), the function \( G^{(n)} \) is an analytic function of the variable \( \Lambda \), uniformly bounded for \( \Im \Lambda > \varepsilon \) for any \( \varepsilon \). By dominated convergence, the limit \( G_N \) up to any subsequence is analytic. By analytic continuation, all limit coincide and are equal to \( G_{H_N}^{(\circ)} \). \( \square \)

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