The Free Quon Gas Suffers Gibbs’ Paradox

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I. INTRODUCTION

In series of papers [1, 3] O.W. Greenberg and collaborators have suggested a new way of interpolating between Bose and Fermi statistics, which would presumably allow small deviations from the standard description of identical particles. Their proposal is to consider “Quon” creation and annihilation operators satisfying the relations

\[ a(f)a^\dagger(g) - q a^\dagger(g)a(f) = \langle f, g \rangle \mathbb{1}, \]

where \( f, g \) are test functions, i.e., elements of the one-particle space with inner product \( \langle f, g \rangle \), \( a(f) \) is a Hilbert space operator with adjoint \( a^\dagger(f) \) depending linearly on \( f \). The deformation parameter \( q \) must be real, and we take it in the interval \(-1 < q < 1\). At the endpoints of this interval the relations become the canonical commutation relations \((q = 1)\), and the canonical anticommutation relations \((q = -1)\), respectively. The Quons with \( q = 0 \) were the first example of this type considered by Greenberg [1]. The relations \((q = 0)\) also play a central rôle in the theory of freely independent random variables [3, 4, 5] initiated by Voiculescu [7]. Such systems have been used as a driving noise [3] in quantum stochastic differential equations [8]. The extension of this work to other values of \( q \) has led to an independent proposal of the relations \((q = 0)\). The relations also arise as the commutation relations of collective degrees of freedom in a system of many components obeying anomalous statistics [10].

\[ \mathbb{1} \]

The Quon system with a single degree of freedom, for which the test function space is one-dimensional, and only one relation \( qa^\dagger - aq^\dagger \mathbb{1} = \mathbb{1} \) remains, is called a q-oscillator. It naturally arises in the theory of quantum SU\(_2\) [11] and has also been introduced in this context by Biedenharn [12] and MacFarlane [13]. The theory of this system can be made to look very much like the theory of the ordinary oscillator by means of “q-analysis” [4, 14]. A “gas” of q-oscillators can be defined and studied using standard quantum mechanical procedures, as soon as one decides whether these systems should be Bosons or Fermions. The q-commutation relations then only enter as a way of defining a special one-particle Hamiltonian (see e.g. [17]).

In this note we consider the rather less trivial “Quon gas” according to Greenberg’s original proposal, i.e. we demand that different quonic degrees of freedom, corresponding to orthogonal test functions \( f, g \), also satisfy the relations \((q = 0)\). In particular, we address the question of the \( q \)-dependence of the partition function of a free Quon gas. We begin by establishing the construction of Quon second quantization. We then derive the formulas for computing partition functions and expectation values of second quantized observables, and show that these formulas do not contain the parameter \( q \). Finally, we show that the Statistical Mechanics of non-relativistic Quons exhibits the Gibbs’ Paradox known from Classical Statistical Mechanics. The main intention of this paper is to state this Paradox as simply as possible, observing accepted standard procedures of theoretical physics. However, at the end we discuss some possible variations of the approach described in the paper.

II. QUON SECOND QUANTIZATION

The free Quon gas is a system described in a Fock Hilbert space generated from a vacuum vector \( \Omega_q \) with \( a(f)\Omega_q = 0 \) by successive application of creation operators \( a^\dagger(f) \). The scalar product of the vectors \( a^\dagger(f_1) \cdots a^\dagger(f_n)\Omega_q \), which by definition generate the q-Fock space, can be computed using only the commutation relations \((q = 0)\) and the condition \( a(f)\Omega_q = 0 \), and gives [3]

\[
\langle a^\dagger(f_1) \cdots a^\dagger(f_n)\Omega_q, a^\dagger(g_1) \cdots a^\dagger(g_m)\Omega_q \rangle
\]

\[
= \begin{cases} 
0 & \text{if } n \neq m \\
\sum_{\pi \in S_n} q^{\text{I}(\pi)} \langle f_1, g_{\pi(1)} \rangle \cdots \langle f_n, g_{\pi(n)} \rangle & \text{otherwise.} 
\end{cases}
\]
Here the sum is over all permutations $\pi$ of $n$ elements, and $I(\pi)$ denotes the number of inversions of the permutation $\pi$, i.e. the number of pairs $(i, j)$ such that $i < j$ and $\pi(i) > \pi(j)$. The positivity of the scalar product follows from the positive definiteness of $q^I(\pi)$ as a function of $\pi$, shown in [9] (compare [14,18]). Completing with respect to this scalar product we obtain the Hilbert space of a second quantized Quon system. For $q = 0$ this space is exactly the “full Fock space”, i.e. the direct sum of the unsymmetrized $n$-fold tensor products of the one-particle space. In equation (2) we have followed the notational convention that the $q$-dependence is attached to the vacuum vector $\Omega_q$, and the creation and annihilation operators are denoted by the same symbols for all $q$.

The connection between $q$-Fock space and the full Fock space at $q = 0$ is then given by an operator $S_q$ with

$$S_q a^\dagger(f_1) \cdots a^\dagger(f_n) \Omega_q = a^\dagger(f_1) \cdots a^\dagger(f_n) \Omega_0 \ .$$

(3)

It was shown in [10,12,14] that, restricted to the $n$-particle space, this operator is boundedly invertible for all $-1 < q < 1$.

What makes a Quon system “free” is that its Hamiltonian and time evolution are canonically given in terms of the corresponding objects on the one-particle Hilbert space: if $u_t f = e^{-it\hbar} f$ describes the time evolution of a single Quon, the free time evolution on Fock space is given by

$$U_t = e^{-i\mathcal{H}t}$$

with

$$U_t a^\dagger(f) U_t^\dagger = a^\dagger(u_t f) \ .$$

We will use the notations $U_t = \Gamma_q(u_t)$ and $H = d\Gamma_q(h)$ for this “functor” of second quantization. In particular, $d\Gamma_q(1) = N$ is the number operator in Fock space.

We will now collect a few properties of $\Gamma_q$, which will be useful later on. These hold for all $q$ including $q = \pm 1$.

We first extend the definition of $\Gamma_q$ and $d\Gamma_q$ from unitary and self-adjoint operators to more general operators on one-particle space by setting

$$\Gamma_q(R) a^\dagger(f_1) \cdots a^\dagger(f_n) \Omega_q = a^\dagger(R f_1) \cdots a^\dagger(R f_n) \Omega_q \ .$$

(5a)

$$d\Gamma_q(h) a^\dagger(f_1) \cdots a^\dagger(f_n) \Omega_q$$

$$= a^\dagger(h f_1) a^\dagger(f_2) \cdots a^\dagger(f_n) \Omega_q$$

$$+ a^\dagger(f_1) a^\dagger(h f_2) \cdots a^\dagger(f_n) \Omega_q$$

$$+ \cdots + a^\dagger(f_1) a^\dagger(f_2) \cdots a^\dagger(h f_n) \Omega_q \ .$$

(5b)

Then it is elementary to check the following relations:

$$\Gamma_q(R^*) = \Gamma_q(R)^*$$

(6a)

$$\Gamma_q(R) \Gamma_q(S) = \Gamma_q(R S)$$

(6b)

$$d\Gamma_q(\lambda h + \mu k) = \lambda d\Gamma_q(h) + \mu d\Gamma_q(k) \ .$$

(6c)

$$e^{d\Gamma_q(h)} = \Gamma_q(e^h) \ .$$

(6d)

From the definition (2) of the operator $S_q$ it is clear that, for $-1 < q < 1$,

$$\Gamma_q(R)= S_q^{-1} \Gamma_0(R) S_q$$

(7)

and

$$d\Gamma_q(h) = S_q^{-1} d\Gamma_0(h) S_q \ .$$

Hence $\Gamma_q(R)$ and $\Gamma_0(R)$ are similar. Using the results of [10,14] one can even define a unitary operator $S'_q$ satisfying equation (2), but we will not need this fact.

### III. QUON STATISTICAL MECHANICS

The density matrix of the grand canonical ensemble of a free Quon gas with one-particle Hamiltonian $h$ is proportional to

$$e^{-\beta(d\Gamma_q(h))} = \Gamma_q \left( e^{-\beta h + \beta \mu} \right)$$

and the grand canonical partition function is the trace of this expression. Hence the fundamental computation on which the theories Bose, Fermi, or Quon gases can be built is the evaluation of $\text{tr} \, \Gamma_q(R)$. In the Bose and Fermi cases the answer is in every textbook (though maybe not quite in this form):

$$\text{tr} \, \Gamma_{+1}(R) = \exp \text{tr} \ln \frac{1}{1 - R}$$

(8a)

$$\text{tr} \, \Gamma_{-1}(R) = \exp \text{tr} \ln (1 + R) \ .$$

(8b)

With $R = \exp(-\beta h + \beta \mu)$ these formulas give back the standard partition functions for quantum gases. Since the scalar products (2) depend continuously on $q$ it is natural to expect for $\text{tr} \, \Gamma_q(R)$ a formula interpolating between the two expressions above.

For computing this expression we make use of the operator $S_q$ introduced in equation (2). Let $P_n$ denote the projection on the $n$-particle space in either full Fock space or $q$-Fock space, i.e. the projection onto the eigenspace of $\Gamma_q(1)$ or $\Gamma_0(1)$ with eigenvalue $n$. Then $S_q P_n = P_n S_q$, and we have

$$\text{tr} \left(P_n \Gamma_q(R)\right) = \text{tr} \left(P_n S_q^{-1} \Gamma_0(R) S_q\right)$$

$$= \text{tr} \left(P_n S_q S_q^{-1} \Gamma_0(R)\right)$$

$$= \text{tr} \left(P_n \Gamma_0(R)\right) \ .$$

(9)

This computation shows that $\text{tr} \, \Gamma_q(R)$ is independent of $q$. Moreover, $P_n \Gamma_0(R)$ is just the $n$-fold tensor product of $R$ with itself. Since the trace of a tensor product of operators is the product of the traces, we obtain

$$\text{tr} \, P_n \Gamma_q(R) = (\text{tr} \, R)^n$$

(10a)
\[
\text{tr} \Gamma_q(R) = \frac{1}{1 - \text{tr} R} .
\]

Thus we have found the analog of equations \([\text{3}]\). However, in spite of the continuity of \([\text{3}]\) in \(q\), the right hand side of \([\text{10b}]\) does not interpolate between \([\text{3a}]\) and \([\text{31}]\), since for \(-1 < q < 1\) it does not even depend on \(q\). Consequently, the grand canonical partition function

\[
\mathcal{Q}(h, \beta, \mu) = \text{tr} e^{-\beta(d\Gamma_q(h) - \mu N)} = \text{tr} \Gamma_q \left( e^{-\beta h + \beta \mu} \right) = \frac{1}{1 - e^{\beta \mu} \text{tr} e^{-\beta h}}
\]

and the canonical partition function

\[
\mathcal{Z}(h, \beta, N) = \text{tr} P_N \Gamma_q(e^{-\beta h}) = \left( \text{tr} e^{-\beta h} \right)^N
\]

of the free Quon gas do not depend on \(q\).

Moreover, the probability distributions of second quantized one-particle operators \(d\Gamma_q(k)\) in the corresponding ensembles are independent of \(q\). In the classical case this would follow from the independence of the partition functions, since all moments of such probability distributions can be obtained by differentiating the partition function with respect to suitable parameters in \(h\). In Quantum Statistical Mechanics this works only for the first moment, and fails in general since \(\text{tr} \exp(A + B) \neq \text{tr} (\exp A \exp B)\). For quasi-free states, i.e. for the equilibrium states of free evolutions, however, the conclusion is still valid: the Fourier transform of the probability distribution of \(d\Gamma_q(k)\) is

\[
\lambda \mapsto \frac{\text{tr} e^{i\lambda d\Gamma_q(k)} \Gamma_q(R)}{\text{tr} \Gamma_q(R)} = \frac{\text{tr} \Gamma_q(e^{i\lambda k}) \Gamma_q(R)}{\text{tr} \Gamma_q(R)} = \frac{\text{tr} \Gamma_q(e^{i\lambda k} R)}{\text{tr} \Gamma_q(R)},
\]

and, by \([\text{10b}]\), the right hand side of this equation does not depend on \(q\).

There is an algebraic way of looking at these structures. For \(|q| < 1\) it follows from the relations \([\text{1}]\) that \(a^\dagger(f)\) is a bounded operator, and we may look at the norm closed algebra (C*-algebra) generated by these operators in any particular realization. The C*-algebras are especially important when one wishes to consider the relations \([\text{1}]\) without necessarily assuming the existence of a vacuum vector \(\Omega_q\). It turns out that the C*-algebra generated by the \(a^\dagger(f)\) is essentially independent of the value of \(q\), so that the insensitivity of Quon Statistical Mechanics to the value of \(q\) has its counterpart in the insensitivity of these algebras. More precisely, for \(|q| < \sqrt{2} - 1 \approx 0.41\) it was shown in \([\text{2}]\) that the C*-algebra generated by the Fock representation contains the full algebraic information, i.e. that every other realization of the relations \([\text{1}]\) either generates an algebra isomorphic to the Fock representation, or to the quotient of the Fock representation modulo the compact operators in Fock space. Moreover, the C*-algebras of the Fock representations for different \(q\) are all isomorphic in the range \(|q| < \sqrt{2} - 1\). For \(q = 0\) the algebra obtained in Fock space is known as the Cuntz-Toeplitz algebra, and its quotient by the compact operators as the Cuntz algebra \([\text{4}]\), and is a well-studied mathematical structure. In the results \([\text{19}]\) the restriction on \(q\) is expected to be only technical, and they are conjectured to be true for all \(|q| < 1\) \([\text{22}]\). Somewhat sharper results can be obtained, when one considers only the Fock representations to begin with \([\text{14}]\).

These algebraic results imply that for \(-1 < q < 1\) the only requirement we may impose in addition to equation \([\text{11}]\) without making the system contradictory is to demand the strict positivity of \(\sum_i a^\dagger(e_i)a(e_i)\), singling out the Cuntz, rather than the Cuntz-Toeplitz algebra. This is in stark contrast to the situation at \(q = \pm 1\): there the operators in Fock space satisfy the additional relation

\[
a(f)a(g) - q a(g)a(f) = 0 ,
\]

which is thus seen to be consistent with \([\text{11}]\) for \(q = \pm 1\). There are, of course, other realizations of the relations \([\text{1}]\) at \(q = \pm 1\) not satisfying \([\text{14}]\). The reason \([\text{14}]\) is satisfied in Boson/Fermion Fock space is the same as the reason for the difference between Boson/Fermion and Quon Statistical Mechanics, namely the degeneracy of the \(q\)-dependent scalar product \([\text{3}]\) at the points \(q = \pm 1\).

## IV. GIBBS’ PARADOX

The partition function \([\text{11}]\) has a strange property: let us assume for definiteness that the single Quon is a non-relativistic spinless particle of mass \(m\) in a \(d\)-dimensional region of volume \(V\). The Hamiltonian is then the Laplacian \(h = -(1/2m)\Delta\) with Dirichlet boundary conditions. Then, provided the volume is large enough, and has a decent boundary, Weyl’s formula for the asymptotic behaviour of the eigenvalues of the Laplacian (e.g. Theorem XII.78 in \([\text{24}]\)) gives (for large volumes):

\[
\text{tr} e^{-\beta(h - \mu)\mathbb{1}} \approx e^{\beta \mu} \lambda^{-dV} ,
\]

where \(\lambda = (2\pi \beta \hbar^2 / m)^{1/2}\) is the thermal de Broglie wavelength. For the expectation value of the particle number we get

\[
\langle N \rangle = \frac{1}{\beta} \frac{d}{d\mu} \ln \mathcal{Q}(h, \beta, \mu) = \frac{V}{e^{-\beta \mu} \lambda^d - V} .
\]

Thus \(\langle N \rangle\) is not proportional to the volume. Moreover, at the volume \(e^{-\beta \mu} \lambda^d\) the expected particle number, and indeed the partition function itself diverges. This is in flat contradiction to the assumptions made in the interpretation of the grand canonical ensemble as the density matrix of a small subsystem in interaction with a large
system serving as a reservoir for heat and particles. If there is an upper limit to the size of Quon gas containers, the idea of a “particle bath” breaks down.

It is therefore necessary to go back to the canonical ensemble, i.e. to consider systems with fixed particle number. From (13) and the canonical partition function (12) we get the Helmholtz free energy

$$F(\beta, V, N) = -\frac{N}{\beta} \ln \left( \lambda^{-d} V \right),$$

(17)

but we have not got rid of the paradox, since now the free energy at fixed density $\rho = N/V$ contains a term growing like $N \ln N$. Finally, we could go back to the microcanonical ensemble, where we would find that the entropy fails to be an extensive quantity.

In the Statistical Mechanics of large classical systems this problem is well known as Gibbs’ Paradox [24,25]. In that context it can be resolved by fixing the overall statistical weight of the $N$-particle phase space by a suitable convention. The factor $(h^{3N} N!)^{-1}$ used to do this is ultimately to be determined from the classical limit of Quantum Statistical Mechanics. The classical limits of the Fermi and Bose gases coincide in this limit. However, the classical limit of a Quon gas would revive Gibbs’ old problem, for it does not give back the factor $1/N!$. Since Classical Statistical Mechanics by itself offers no canonical choice for the normalization of phase space measure, Gibbs’ Paradox in the classical theory can be put aside as a weakness which may be (and was, in fact) overcome by a more powerful theory. The situation is more serious for the Quon gas: here the weight is determined unambiguously by Quantum Statistical Mechanics, and it comes out wrong.

Whichever way we put it, the Quon gas does not behave like some “material” of which one can have a large, homogeneous sample. The thermodynamic limit, which implicitly or explicitly is the basis of equilibrium Statistical Mechanics does not make sense for this system. Things get worse if we consider problems like stability of matter. It is well-known that a world of only Bosons would collapse under its electromagnetic interaction. Quons have an even stronger tendency to cling together, and this is true even for nearly Fermionic values $q \approx -1$. So perhaps the most powerful test supporting the standard assumption $q = \pm 1$ is the stability of matter itself.

V. DISCUSSION

The second quantization of Quons described above is based on the postulate that symmetries of the one-particle system must become symmetries of the many-Quon system. It is therefore insensitive to modifications of the creation operators by factors which commute with all these symmetries. Let $\alpha$ be real, and consider

$$b(f) = a(f) \Gamma_q(q^{\alpha} \mathbb{1}) = q^{\alpha} \Gamma_q(q^{\alpha} \mathbb{1}) a(f),$$

(18)

with $\Gamma_q(q^{\alpha} \mathbb{1}) = \exp(\alpha N \ln q) = q^{\alpha N}$. Then we get

$$b(f)b^\dagger(g) = q^{2\alpha(N+1)} \langle f, g \rangle + q^{2\alpha+1} b^\dagger(g)b(f).$$

(19)

Then for $\alpha = -1/4$ (and a single degree of freedom) we obtain the $q$-oscillator as introduced by Biedenharn [22], and independently (with a different parameter $\tilde{q} = q^{-1/2}$) by [13]. On the other hand, we can use the “Woronowicz normalization”, named after his version of the one-dimensional relations, which appears as a subalgebra of quantum SU(2) [24], i.e. operators $w(f) = (1-q)^{1/2}a(f)$, satisfying $w(f)w^\dagger(g) = (1 - q)\langle f, g \rangle + q w^\dagger(g)w(f)$. In both of these cases the second quantized operators $\Gamma_q(R)$ and $d\Gamma_q(h)$ are simply the same as before, and all our results carry over. The point here is that the second quantized observables are defined by the second quantization of symmetries, and not by explicit expressions in the creation operators. Although there are such expressions, they are not very useful, since they are of infinite order in $a(f)$ and $a^\dagger(f)$ [22,27]. Indeed, formulas such as

$$H = \sum_{ij} a^\dagger(f_i)(h_{f_j})a(f_j),$$

(20)

familiar from the second quantization of Bosons or Fermions, make little sense for Quons. For example, for $q = 0$ the above operator $H$ acts only on the “first” particle. One could think of a suitable average over permutations, but this fails, because there is no natural unitary action of the permutations group on the Quon Fock space. Clearly, for some Hamiltonians defined by fixed algebraic expressions in creation and annihilation operators the partition functions will depend on $q$. We leave open the question whether there is any sensible second quantization scheme of this type.

By applying the rules of Statistical Mechanics we have restricted consideration to Quons in equilibrium. Of course, one can avoid our conclusions if one assumes, firstly, that in the early universe for some reason there were no Quons, and, secondly, that for all Quons $q$ is so close to $\pm 1$ that the interaction on the cosmic time scale is not sufficient to bring the Quonic degrees of freedom into equilibrium. Such arguments have been suggested by Greenberg and Mohapatra [28]. The cosmological questions raised by this way out of the dilemma of Gibbs’ Paradox are beyond the scope of the present paper.

Discussing the Gibbs’ Paradox Greenberg [1] compares Quon Statistical Mechanics ($q = 0$) to the Statistical Mechanics of para-Bosons and para-Fermions [29]. It is well-known that para-Bosons of finite order can be considered as Bosons with a finite number of internal degrees of freedom (and similarly for para-Fermions). In this analogy the Quons, as particles with infinite statistics, might be expected to behave like particles with infinitely many internal degrees of freedom. This would then reduce Quon Statistical Mechanics to the Statistical Mechanics of ordinary systems. Greenberg does not make this connection explicit, and it can be expected that an attempt to do
this would lead into difficulties. The crucial problem is the dynamical behaviour of the internal degrees of freedom. Even in the para-Boson case this is a problem, when one assumes that there is absolutely no interaction between the internal and the external degrees of freedom. The internal degrees of freedom then never go into equilibrium, and the equilibrium of the external degrees of freedom depends on the prior history of the system. For example, when initially all particles are in the same internal state, there is no difference between this system and ordinary Bosons. A possible way out is the assumption that the internal degrees of freedom are always completely randomized. The partition functions are then the same as for a one-particle Hamiltonian all of whose eigenvalue multiplicities are multiplied with the same factor. Clearly, the Statistical Mechanics of such a system is not qualitatively different from that of the original system, and does not exhibit Gibbs’ Paradox. However, this way out is impossible for infinitely many internal degrees of freedom. Infinitely many degrees of freedom cannot be completely randomized, because the unit operator cannot be normalized as a density matrix. If one makes an arbitrary choice, like e.g. an equilibrium state for a suitable Hamiltonian for the internal degrees of freedom, the system will effectively behave like one with finitely many degrees of freedom, and will not exhibit Gibbs’ Paradox. Consider, for example, Fermions in a spacetime with additional compactified space dimensions, as suggested in [27]. If we include a kinetic energy term associated with the additional dimensions, the internal degrees of freedom have an equilibrium state due to the compactness of the added dimensions. This system is well-behaved from the point of view of Statistical Mechanics, and does not exhibit Gibbs’ Paradox. On the other hand, if we do not include the additional kinetic energy term, there will be no equilibrium states at all. Thus Quon Statistical Mechanics appears to be qualitatively different from the Statistical Mechanics of Bosons or Fermions with (finitely or infinitely many) internal degrees of freedom.

The Gibbs’ Paradox concerns the composition of different Quon systems. This problem also arises in Quon field theory, where one has to specify how the observable algebras of two subregions combine into the observable algebra for the whole region. Certainly, for discussing Quons in the early universe, this problem of Quon field theory cannot be avoided. It is one of the fundamental results in general (algebraic) quantum field theory that (massive) particles with infinite statistics, such as Quons, cannot occur [29,74]. Since this theory proceeds axiomatically this objection is even valid for interacting Quon systems, and is essentially independent of the way the Quon observable algebras are defined in terms of creation and annihilation operators. As an illustration consider a relativistic field theory of Quons, set up by taking as one-particle space an irreducible representation space of the Poincaré group, and applying the second quantization scheme introduced above. If \( m \geq 0 \) for the one-particle representation the second quantized system will also obey the spectral (positive energy) condition. In the Fermi-or Bose case the local algebra associated with a bounded open region in space-time is defined in terms of the real-linear subspace of the one-particle space, consisting of the Fourier transforms of the real test functions supported in the given region [31]. The Boson local field algebra is then generated by the operators \( a(f) + a^\dagger(f) \) with \( f \) in the associated subspace. It is easy to see from the relations (6) that the corresponding prescription fails to produce algebras commuting at spacelike separation. Even restricting to the gauge invariant subalgebra of the field algebra (as is done in the Fermi case) will not make these algebras commute. On the other hand, for commuting operators \( R, S \), the operators \( \Gamma_q(R) \) and \( \Gamma_q(S) \) also commute. Therefore, if we could associate with each space-time region a complex-linear subspace such that spacelike regions would correspond to orthogonal subspaces, we could generate commuting local algebras by the operators \( \Gamma_q(R) \) with \( R \) supported by the appropriate subspace. It is well-known that such a net of orthogonal subspaces can be found, as soon as one replaces the irreducible representation by a reducible one, containing both negative and positive frequencies. In that case, however, the field theory would no longer obey the spectral condition.

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