Class field theory for curves over $p$-adic fields

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Abstract

We develop class field theory of curves over $p$-adic fields which extends the unramified theory of S. Saito [4]. The class groups which approximate abelian étale fundamental groups of such curves are introduced in the terms of algebraic $K$-groups by imitating G. Wiesend’s class group for curves over finite fields [6].

1 Introduction

Let $X$ be a regular curve over a finite field $k$ with function field $K$, $\overline{X}$ the regular compactification of $X$, that is the regular and proper curve which contains $X$ as an open subvariety, and $X_{\infty}$ the finite set of closed points in the boundary $\overline{X} \setminus X$ of $X$. Class field theory describes the abelian étale fundamental group $\pi_1(X)^{ab}$ of $X$ by a topological abelian group $\mathcal{C}_X$ which is called the class group. In terms of (Milnor) $K$-groups, the group $\mathcal{C}_X$ is the cokernel of the map

$$K_1(K) \to \bigoplus_{x \in X_0} K_0(k(x)) \oplus \bigoplus_{x \in X_\infty} K_1(K_x)$$

induced by the inclusion $K \hookrightarrow K_x$ and the boundary map $K_1(K_x) \to K_0(k(x))$, where $k(x)$ is the residue field at $x$, $K_x$ is the completion of $K$ at $x$ and $X_0$ is the set of closed points in $X$ (cf. [6]). The reciprocity map $\rho_X : \mathcal{C}_X \to \pi_1(X)^{ab}$ is defined by class field theory of finite fields, local class field theory and the reciprocity law. It has dense image and the kernel is the connected component of 0 in $\mathcal{C}_X$.

The aim of this note is to develop class field theory for curves over local fields. Here, a local field means a complete discrete valuation field with

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finite residue field. Let $X$ be a regular curve over a local field $k$ with function field $K$. The class group $C_X$ of $X$ is defined to be the cokernel of the homomorphism

$$K_2(K) \to \bigoplus_{x \in X_0} K_1(k(x)) \oplus \bigoplus_{x \in X_\infty} K_2(K_x),$$

induced by the inclusion $K \hookrightarrow K_x$ and the boundary map $K_2(K_x) \to K_1(k(x))$ (see Def. 2 for the precise definition). Note that the residue field $k(x)$ at $x \in X_0$ is a local field, and $K_x$ is a 2-dimensional local field in the sense of K. Kato, that is a complete discrete valuation field whose residue field is a local field. Next, a canonical continuous homomorphism $\sigma_X : C_X \to \pi_1(X)^{\text{ab}}$ shall be defined by local class field theory, 2-dimensional local class field theory [1] and the reciprocity law due to S. Saito [4]. Our main result is the following determination of its kernel and cokernel when the characteristic of $k$ is 0.

**Theorem 1.** Let $X$ be a regular and geometrically connected curve over a finite extension $k$ of $\mathbb{Q}_p$.

(i) The kernel of $\sigma_X$ is the maximal divisible subgroup of $C_X$.

(ii) The quotient of $\pi_1(X)^{\text{ab}}$ by the topological closure $\overline{\text{Im}(\sigma_X)}$ of the image of $\sigma_X$ is isomorphic to $\hat{\mathbb{Z}}^r$ with some $r \geq 0$.

Further assume that the variety $X$ is proper. In this case, the class group $C_X$ is nothing other than $SK_1(X)$. By using this, S. Saito [4] showed the above theorem and it plays an important role in higher dimensional class field theory of K. Kato and S. Saito. The invariant $r$ in the above theorem is called the rank of the compactification $\overline{X}$ of $X$ (op. cit., Def. 2.5). It depends on the type of the reduction of $\overline{X}$. In particular, we have $r = 0$ if it has potentially good reduction.

**Remark.** As in op. cit., for a local field $k$ with characteristic $p > 0$, the theorem above can be proved with restriction to “the prime-to-$p$ part” in the assertion (i).

After introducing the class group of $X$ and the reciprocity map in Section 2 we shall prove Theorem 1 in Section 3.

Throughout this paper, a curve over a field is an integral separated scheme of finite type over the field of dimension 1. For an abelian group $A$, we denote by $A/n$ the cokernel of the map $n : A \to A$ defined by $x \mapsto nx$ for any positive integer $n$.

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2 Class Groups

Let $X$ be a regular curve over a local field $k$, $\overline{X}$ the regular compactification of $X$, and $X_\infty$ the finite set of closed points in the boundary $\overline{X} \setminus X$ of $X$. We define a group $\mathcal{I}_X$ by

$$\mathcal{I}_X = \bigoplus_{x \in X_0} K_1(k(x)) \oplus \bigoplus_{x \in X_\infty} K_2(K_x).$$

The topology of $K_2(K_x)$ is defined in [1] (cf. op. cit., I, Sect. 7). In particular, if the characteristic of $k$ is 0, we take the discrete topologies of $K_x^\times$ and $K_2(K_x)$. The group $\mathcal{I}_X$ is endowed with the direct sum topology, that is, a subset is open if the intersection with each finite partial sum is open.

**Definition.** Define the class group $C_X$ associated with $X$ by the cokernel of the natural map $K_2(K) \to \mathcal{I}_X$ which is defined by the boundary map $K_2(K_x) \to K_1(k(x))$ and the inclusion $K \hookrightarrow K_x$. The quotient topology makes this an abelian topological group.

The reciprocity map $\sigma_X : C_X \to \pi_1(X)^{ab}$ is defined as follows: For $x \in X_0$, the reciprocity map of local class field theory $K_1(k(x)) \to \pi_1(X)^{ab}$ and the natural map $\pi_1(x)^{ab} \to \pi_1(X)^{ab}$ give $K_1(k(x)) \to \pi_1(X)^{ab}$. For any $x \in X_\infty$, the reciprocity map of 2-dimensional local class field theory $K_2(K_x) \to \pi_1(Spec(K_x))^{ab}$ and the natural map $\pi_1(Spec(K_x))^{ab} \to \pi_1(X)^{ab}$ define a map $K_2(K_x) \to \pi_1(X)^{ab}$. Thus, we have $\mathcal{I}_X \to \pi_1(X)^{ab}$. Finally, the reciprocity law of $K$ ([1], Chap. II, Prop. 1.2) and 2-dimensional local class field theory (op. cit., Chap. II, Th. 3.1) show that the homomorphism $\mathcal{I}_X \to \pi_1(X)^{ab}$ defined above factors through $C_X$. Thus the required homomorphism $\sigma_X : C_X \to \pi_1(X)^{ab}$ is obtained.

The structure map $X \to Spec(k)$ induces a map $N : C_X \to k^\times$ which is defined by norms over $k$ and one denotes the kernel of this map by $V(X)$. It makes the following diagram commutative:

\[
\begin{array}{ccc}
0 & \longrightarrow & V(X) & \longrightarrow & C_X & \longrightarrow & k^\times \\
& & \uparrow{\tau_X} & & \downarrow{\sigma_X} & & \downarrow{\rho_k} \\
0 & \longrightarrow & \pi_1(X)^{ab,geo} & \longrightarrow & \pi_1(X)^{ab} & \longrightarrow & \pi_1(Spec(k))^{ab}.
\end{array}
\]

Here, the group $\pi_1(X)^{ab,geo}$ is defined by the exactness of the lower horizontal row.
Remark. As in [6], we can define a class group and a reciprocity map for a regular variety over the local field \( k \). More generally, for a regular variety over a higher dimensional local field, a class group may be defined as an abstract group by using Milnor \( K \)-groups of higher degree. However, there is no appropriate topology in the \( K \)-groups for degree \( > 2 \).

3 Proof of the Theorem

In this section, we shall prove Theorem [1]. We denote by \( \pi_1(X)_{\text{cs}}^{\text{ab}} \) the quotient of \( \pi_1(X)_{\text{cs}}^{\text{ab}} \) which classifies the abelian covers of \( X \) which are completely split. The assertion (ii) is reduced to the unramified case ([4], Chap. II, Prop. 2.2, Th. 2.4) as follows:

\[
\pi_1(X)^{\text{ab}}/\text{Im}(\sigma_X) \simeq \pi_1(X)_{\text{cs}}^{\text{ab}} = \pi_1(X)_{\text{cs}}^{\text{ab}} \simeq \hat{\mathbb{Z}}^r,
\]

where \( r \) is the rank of \( \overline{X} \).

The lemma below is used in the proof of the assertion (i) in an auxiliary role.

Lemma 2. (i) The image of \( \tau_X \) is finite.
(ii) The cokernel of \( \tau_X \) is isomorphic to \( \hat{\mathbb{Z}}^r \).

Proof. Since the map \( N : C_X \to k^x \) is induced by norms over \( k \), its image is finite index in \( k^x \). Thus, the commutative diagram (1) implies \( \pi_1(X)^{\text{ab}}/\text{Im}(\sigma_X) \simeq \pi_1(X)_{\text{cs}}^{\text{ab},\text{geo}}/\text{Im}(\tau_X) \). There is an exact sequence of étale cohomology groups

\[
0 \to H^1(\overline{X}, \mathbb{Q}/\mathbb{Z}) \to H^1(X, \mathbb{Q}/\mathbb{Z}) \to \bigoplus_{x \in X_{\infty}} H^2_x(\overline{X}, \mathbb{Q}/\mathbb{Z})
\]

and an isomorphism \( H^2_x(\overline{X}, \mathbb{Q}/\mathbb{Z}) \simeq H^0(k(x), \mathbb{Q}/\mathbb{Z}(-1)) \) of finite groups. The abelian étale fundamental group has the description \( (\pi_1(X)^{\text{ab}})^* \simeq H^1(X, \mathbb{Q}/\mathbb{Z}) \), where the superscript “*” denotes the Pontrjagin dual. Thus, Theorem 1 in [7] and the above exact sequence imply the following description of \( \pi_1(X)^{\text{ab},\text{geo}} \):

\[
0 \to \pi_1(X)^{\text{ab},\text{geo}}_{\text{tor}} \to \pi_1(X)^{\text{ab},\text{geo}} \to \hat{\mathbb{Z}}^r \to 0,
\]

where the torsion subgroup \( \pi_1(X)^{\text{ab},\text{geo}}_{\text{tor}} \) of \( \pi_1(X)^{\text{ab},\text{geo}} \) is finite (Note that, the rank of \( \overline{X} \) is the rank of the special fiber of the Néron model of the Jacobian variety of \( X \), cf. [4], Chap. II, Th. 6.2). Since the quotient group \( \pi_1(X)^{\text{ab},\text{geo}}/\text{Im}(\tau_X) \) and \( \pi_1(X)^{\text{ab},\text{geo}} \) are \( \hat{\mathbb{Z}} \)-modules of rank \( r \), the image of \( \tau_X \) is finite. The assertions (i) and (ii) follows from it. \( \square \)
If we assume the following lemma, then the rest of the proof of the assertion (i) in Theorem 1 is essentially the same as in the proof of Theorem 5.1 in Chapter II of [4] (by using Lem. 2).

**Lemma 3.** Let \( n \) be a positive integer. Then the map \( \sigma_X : C_X \to \pi_1(X)^{ab} \) induces the injection \( C_X/n \hookrightarrow \pi_1(X)^{ab}/n \).

**Proof.** (Compare with the proof of [4], Chap. II, Lem. 5.3.) By the duality theorem of étale cohomology groups with compact support, we have

\[
\pi_1(X)^{ab}/n = H^1(X, \mathbb{Z}/n) \cong H^3(X, \mathbb{Z}/n(2)) \cong H^3(\mathbb{X}, j_*\mathbb{Z}/n(2)),
\]

where \( j : X \hookrightarrow \mathbb{X} \) is the open immersion. Let us consider the following diagram:

\[
\begin{array}{ccccccccc}
K_2(K)/n & \to & \bigoplus_{x \in X_0} K_1(k(x))/n & \oplus & \bigoplus_{x \in X_\infty} K_2(K_x)/n & \to & C_X/n & \to & 0 \\
\downarrow h_2 & & \downarrow h & & \downarrow h & & \downarrow & & \\
H^2(K, \mathbb{Z}/n(2)) & \to & \bigoplus_{x \in X_0} H^3_x(\mathbb{X}, j_*\mathbb{Z}/n(2)) & \to & H^3(\mathbb{X}, j_*\mathbb{Z}/n(2)).
\end{array}
\]

Here, the horizontal sequences are exact, and the left vertical map \( h_n^2 \) is the isomorphism by the Merkur’ev-Suslin theorem [2]. The vertical map \( h \) is an isomorphism defined as follows: For \( x \in X_0 \), by excision and the purity theorem we have

\[
H^3_x(\mathbb{X}, j_*\mathbb{Z}/n(2)) \cong H^3_x(X, j_*\mathbb{Z}/n(2)) \cong H^1(k(x), \mathbb{Z}/n(1)).
\]

Thus, Kummer theory gives an isomorphism

\[
(3) \quad K_1(k(x))/n \cong H^1(k(x), \mathbb{Z}/n(1)) \cong H^3(\mathbb{X}, j_*\mathbb{Z}/n(2)).
\]

For \( x \in X_\infty \), let \( O^{hx}_{X,x} \) be the henselization of \( O_{X,x} \), \( K^h_x \) the field of fractions of \( O^{hx}_{X,x} \), and \( j_x : \text{Spec}(K^h_x) \hookrightarrow \text{Spec}(O^{hx}_{X,x}) \) the inclusion. By excision and Proposition 1.1 in [3], we have

\[
H^3_x(\mathbb{X}, j_*\mathbb{Z}/n(2)) \cong H^3_x(\text{Spec}(O^{hx}_{X,x}), j_*\mathbb{Z}/n(2)) \cong H^2(K^h_x, \mathbb{Z}/n(2)).
\]

The Merkur’ev-Suslin theorem gives an isomorphism

\[
K_2(K_x)/n \cong K_2(K^h_x)/n \cong H^2(K^h_x, \mathbb{Z}/n(2)) \cong H^3_x(\mathbb{X}, j_*\mathbb{Z}/n(2)).
\]

By composing this and (3), the isomorphism \( h \) is defined. From the above diagram and (2), we obtain an injection

\[
C_X/n \hookrightarrow H^3(\mathbb{X}, j_*\mathbb{Z}/n(2)) \cong \pi_1(X)^{ab}/n
\]

which is nothing other than the map \( \sigma_X \) modulo \( n \). \( \square \)
Remark. Q. Tian \cite{5} established a similar result by using relative $K$-groups $SK_1(\overline{X}, D)$, where $D := \overline{X} \setminus X$ is the reduced Weil divisor on $\overline{X}$. However, it seems that the theorem (op. cit., Th. 3.11) corresponding to the lemma above is not proved completely.

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