SOME NEW PERSPECTIVES ON \textit{d}-ORTHOGONAL POLYNOMIALS

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Abstract. The aim of this paper are two folds. The first part is concerned with the associated and the so-called co-polynomials, i.e., new sequences obtained when finite perturbations of the recurrence coefficients are considered. Moreover, the second part deals with Darboux factorization of Jacobi matrices. Here the respective co-polynomials solutions are explicitly expressed in terms of the fundamental solutions of a \((d+2)\)-term recurrence relation. New identities and formulas related to determinants with co-polynomials entries are obtained. Accordingly, further determinants bring out partial generalizations of Christoffel Darboux formula. Some of new sequences proved useful for determining the entries of matrices in LU and UL decomposition of Jacobi matrix. The last one gives rise of a \(d\)-analogue of kernel polynomials with quite a few properties, and further a new characterization of the \(d\)-quasi-orthogonality. Kernel polynomials also appear in the \((d+1)\)-decomposition of a \(d\)-symmetric sequence. Exploiting properties of \(d\)-symmetric sequences, reveal a simple proof of Darboux factorizations. It turns out that Jacobi matrix for \(d\)-OPS is a product of \(d\) lower bidiagonal matrices and upper bidiagonal matrix and that each lower bidiagonal matrix is in fact a closed connection between two adjacent components for some \(d\)-(symmetric)OPS. Furthermore, we pointed out that if the first component is Hahn classical \(d\)-OPS then the corresponding \(d\)-symmetric sequence as well as all the components are Hahn classical \(d\)-OPS as well. Oscillation matrices assert that zeros of \(d\)-OPS are positive and simple whenever the recurrence coefficients are strict positive. Further interlacing properties are justified by the same approach.

1. Introduction

This paper deals with the theory of \(d\)-orthogonal polynomials appeared at first time in the thesis of Van Iseghem [69] on the study of vectorial Padé approximations and this was the starting point where it was remarked that these polynomials satisfy a \((d+2)\)-term recurrence relation.

Two years later, the theory gives rise with a paper due to Maroni [55] tightly describes this new theory. Maroni with new algebraic approach figured out many interesting characterizations of the \(d\)-orthogonality rely almost all on the orthogonality’s vectorial form. Moreover, he introduced the \(d\)-quasi-orthogonality’s notion in the same paper. Since then, some attempts are given in order to improve as well as to understand this new theory. The challenge tackled by Douak and Maroni was Hahn’s property. They succeed one time by characterizing the class of \(d\)-orthogonal polynomials with \(d\)-orthogonal derivatives in terms of Pearson equation. The latter class (orthogonal polynomials sequence whose sequence of derivatives is also orthogonal) is referred to as Hahn classical \(d\)-orthogonal polynomials according to Hahn’s work on the characterization of classical orthogonal polynomials from...
algebraic point of view [42]. Other researchers have also analyzed some characterization’s problems which led to construct many d-analogue of classical families of polynomials as well as to discovery some new ones.

Classical orthogonal polynomials sequences (OPS in short) constitute very important class of special functions with wide range of their applications mainly in numerical analysis, probability and statistics, stochastic processes, combinatorics, number theory, potential theory, scattering theory, physics, biology, automatic control ... and more. Starting from classical OPS, there were many ways to formulate as well as to introduce analogous problems in d-orthogonality’s context and the construction of new d-OPS families is then quite fruitful. In this way, several examples of d-OPS families which reduce to the classical cases are given in literature. In contrast to the classical OPS case, to the best of our knowledge, Pearson equation is the only characterization extended to Hahn classical d-OPS. Although, Maroni has pointed out another one in the case d=2 [59, Prop. 6.2]. In this paper, thanks to new characterizations of the d-quasi-orthogonality in terms of linear combinations (see proposition 2.5 and theorem 7.2), we shall show out two new characterizations of Hahn classical d-OPS in terms of linear combinations as well. We would like to emphasize that one of these latter gives us ideas to construct new polynomial families which possess Hahn’s property. We shall present some of them in forthcoming papers.

Shohat has used Pearson equation in his study of classical OPS which led later to discovery the semi-classical OPS, he also coined the quasi-orthogonality’s notion were first introduced by Riesz in 1923. Later on, Al-Salam and Chihara remarked in [3] that classical OPS could be further, characterized in terms of linear combination of OPS, i.e., with the aid of quasi-orthogonality. The latter result, led to significant development since the late of 1980’s. Among the generalizations of classical OPS, an important problem raised by Askey (see [3, p.69]) to characterize sequences of OPS whose derivatives are quasi-orthogonal. Many authors interested to study Askey’s problem and used different approaches as well as gave many attempts to resolve the problem. One of the most successful of these has been done by Maroni who presented in [58] an algebraic theory of semi-classical OPS including very interesting characterizations. Ever since, quasi-orthogonality has been aptly motivated. Indeed, quasi-orthogonality and linear combinations of OPS as well as modifications of orthogonality’s measures are, however, a powerful tool in constructing new families of OPS.

Riesz and Shohat discussed further the zeros of linear combinations of OPS. For a positive definite linear form, it is well known that zeros of the corresponding OPS are real and simple. Notice that the interlacing property is by no means sufficient to ensure orthogonality. Nevertheless, a necessary and sufficient conditions on the interlacing of zeros of two and three arbitrary polynomials that can be embedded in an OPS are discussed respectively by Wendroff [71] and very recently in [10]. Moreover, many recent results derive sufficient conditions for a linear combination of OPS to have simple zeros might be found. For the multiple as well as the d-OPS, it seems no longer true at the time!

We are interested in the zeros of OPS because of their important role mainly in interpolation and approximation theory, Gauss-Jacobi quadrature, spectral theory, and in image analysis and pattern recognition. The first real investigation of the zeros of multiple OPS was not performed till 2011, when Haneczok with Van Assche gave sufficient conditions for
zeros of the latter class to be real and distinct [43], a result we shall build upon and generalize to the d-orthogonality with the aid of oscillation matrices which involves further, some interlacing properties.

However, the general theory is far from being complete, and many natural questions remain unanswered or have only partial explanations, intuitively this is why the theory is not so deeply understood yet. This study aims to significantly answer on some open questions mainly for Hahn’s property and the multiplicity as well as the nature of zeros. We desperately hope that our results could give some helps to find then new applications as for usual OPS which are ubiquitous in several areas. Our approach to study Hahn’s property is very simple and based on a new characterization. Although, this new characterization shows that the sequence of its normalized derivative possesses again Hahn’s property. Thus, without doubt, this is an affirmative answer to a first open question.

The layout of the paper is as follows. After recalling the definition and some characterizations of the d-orthogonality, section 3 provides a deep study of the so-called co-polynomials reached by some modifications and perturbations in the recurrence coefficients. In other words, we discuss the associated as well as anti-associated polynomials, co-recursive polynomials, co-dilated polynomials and finally the co-modified polynomials where the solutions of these new families are explicitly singled out in terms of the fundamental set of solutions of a \((d+2)\)-term recurrence relation. Some properties of the formal Stieltjes functions are also presented. In section 4 we throw a shadow on Casorati determinants. In other words, we consider some Casorati determinants whose entries are d-OPS. Many identities satisfied by these type of determinants with d-OPS and co-polynomials entries are given. Almost all our proofs in that section are based on one or two \((d+2)\)-recurrence relation connects \(d+2\) consecutive polynomials in level of association. In section 5, we deduce that some Casorati determinants also provide partial generalizations of Christoffel-Darboux type formulas for d-OPS.

In section 6 we investigate Darboux transformations. We started by looking for the existence as well as at sequences generated by LU and UL factorization of Jacobi matrix associated with a d-OPS. Fortunately, our study has been motivated by the fact that d-OPS generated by the first factorization is d-quasi-orthogonal of order one with respect to that generated by the second one. This is what we shall show in section 7. In fact, we have discovered a d-analogue of structure relation termed out by Maroni in his study of semi-classical polynomials (for usual orthogonality) [58]. Moreover, some properties have emerged in this section characterize kernel polynomials in the case \(d = 1\) from the quasi-orthogonality point of view. For this end, we have also looked at Uvarov transformation in order to find out further properties of kernel polynomials, since they were the crucial point in characterizing such transformations. However, for \(d > 1\), this seems no longer true. Nevertheless, in section 8, a deep investigation on the \((d+1)\)-decomposition of a d-symmetric sequence, reveals that the latter fact could be a starting point for many studies mainly for zeros and to construct new families of polynomials using products of bi-diagonal matrices. Roughly speaking, we have explicitly characterized Hahn classical d-OPS and we have showed further that the derivatives sequence of any order of Hahn classical d-OPS always possesses Hahn property. However our approach to investigate zeros, in section 9, is to appeal the theory of totally nonnegative (TN) matrices. We started in a first time (see section 8) by considering higher order three term recurrence relation (all recurrence coefficients are zero except the last one),
i.e., $d$-symmetric case (8.3), and then regarding for all nonsymmetric sequences in which their recurrence coefficients are expressed in terms of the nonzero parameter in $d$-symmetric sequence (8.3) above. Although this higher order recurrence relation has been investigated many times, see for instance [4, 5, 37], our idea to find out some zero’s properties is based on previous reasonings and techniques, but technically more detailed, extremely intricate and somehow different of the approaches used in the aforesaid references. Indeed, it is well known in this case, that there are $(d + 1)$ nonsymmetric $d$-OPS families called components of the $d$-symmetric sequence defined by this higher order three term recurrence relation (8.3) [33] (see also [37] for a quite different construction). As a result, the recurrence coefficients of the above components are in fact the symmetric functions. Which inevitably leads to think about TN matrices. Moreover, Jacobi matrices of the components are product of $d$ lower bidiagonal TN matrices and one upper bidiagonal TN matrix whenever the coefficient of the oscillation is that the recurrence coefficients should be strict positive. Consequently, this is a much simpler proof than that presented in [51]. Next, we have only showed that Jacobi matrices are oscillation matrices whenever all the recurrence coefficients are strictly positive. Unfortunately, these conditions are sufficient but not necessary as $d$-Laguerre polynomials show. The outcome of the interlacing properties could be deduced by appealing oscillation matrix’s tools.

2. Basic background

Let $\{P_n\}_{n \geq 0}$ be a monic sequence in the space of polynomials $P$ with $\deg P_n = n$, $n \geq 0$. By the euclidean division, there always exist complex sequences $\{\beta_n\}_{n \geq 0}$, $\langle \chi_{n,v} \rangle$, $0 \leq v \leq n$ such that

$$P_0 (x) = 1, \quad P_1 (x) = x - \beta_0,$$

$$P_{n+2} (x) = (x - \beta_{n+1}) P_n (x) - \sum_{v=0}^n \chi_{n,v} P_v (x), \quad n \geq 0.$$  

The dual sequence $\{u_n\}_{n \geq 0}$, $u_n \in P'$ of $\{P_n\}_{n \geq 0}$ is defined by the duality bracket denoted throughout as $\langle u_n, P_m \rangle := \delta_{n,m}$, $n, m \geq 0$. The latter equality is sometimes called a bi-orthogonality between two sequences. In particular, we denote by $\langle u_r \rangle_n = \langle u_r, x^n \rangle$, $n \geq 0$, the moments of $u_r$. Using the definition of dual sequence, it is easy seen that we have

$$\beta_n = \langle u_n, x P_n (x) \rangle, \quad n \geq 0,$$

$$\chi_{n,v} = \langle u_v, x P_{n+1} (x) \rangle, \quad 0 \leq v \leq n.$$  

(2.1)

For a linear form $u$, let $S (u)$ be its Stieltjes function defined by

$$S (u) (z) = - \sum_{n \geq 0} \frac{\langle u \rangle_n}{z^{n+1}} = - \frac{1}{z} \left\langle u, \sum_{n \geq 0} \frac{x^n}{z} \right\rangle = \left\langle u, \frac{1}{x - z} \right\rangle,$$

of course the functional (linear form) $u$ acts on the variable $x$. In particular, we have $S (\delta) (z) = -1/z$. Stieltjes function, called also Stieltjes transform, play an important role in the determination of the orthogonality’s measure. It is worthwhile to notice that if we know specifically a generating function $F (x) = \sum_{n} (u)_n x^n$ of the moments sequence corresponding to the form $u$, then we could determine Stieltjes transform explicitly as $zS (z) = F (1/z)$. We prove, in the next section, some new algebraic identities satisfied by Stieltjes function. For this end, some properties are needed
Lemma 2.1. [56] For any \( p \in \mathcal{P} \) and any \( u, v \in \mathcal{P}' \), we have

(a) \( S(x^{-1}u)(z) = S(u)/z \),
(b) \( S(uv)(z) = -zS(u)(z)S(v)(z) \),
(c) \( S(pu)(z) = p(z)S(u)(z) + (u\theta_0 p)(z) \).

Now for any polynomial \( \pi \) and any \( c \in \mathbb{C} \), we can define the following forms \( Du = u' \), \( \pi u \) and \( \delta_c \) by

\[
\langle u', p \rangle := -\langle u, p' \rangle, \quad \langle \pi u, p \rangle := \langle u, \pi p \rangle, \quad \langle \delta_c, p \rangle := p(c), \quad p \in \mathcal{P},
\]

and for each \( \lambda \in \mathbb{C} \) and \( s \in \mathbb{N} \), we consider the operators \( \theta_\lambda \) and \( \sigma \) defined respectively as

\[
(\theta_\lambda p)(x) = \frac{p(x) - p(\lambda)}{x - \lambda}, \quad (\sigma_s p)(x) := p(x^s), \quad p \in \mathcal{P}.
\]

Before we dive into the \( d \)-orthogonality, let us briefly recall the standard orthogonality in formal essence. The sequence \( \{P_n\}_{n \geq 0} \) is said to be orthogonal with respect to some linear form (called also moment functional) \( u \), if

\[
\langle u, P_n P_m \rangle := r_n \delta_{n,m}, \quad n, m \geq 0, \quad r_n \neq 0, \quad n \geq 0.
\]

In this case, necessarily, \( u = \lambda u_0 \), \( \lambda \neq 0 \). Further, we have \( u_n = (\langle u_0, P_n^2 \rangle)^{-1} P_n u_0 \), \( n \geq 0 \) [56]. In terms of (2.1), \( \{P_n\}_{n \geq 0} \) is orthogonal if and only if

\[
\chi_{n,v} = 0, \quad 0 \leq v \leq n - 1, \quad n \geq 1 \quad \text{and} \quad \chi_{n,n} \neq 0, \quad n \geq 0.
\]

For a generalization of the above standard orthogonality we will deal with the concept of \( d \)-orthogonality (it can be also regarded as the multiple OPS of type II at the step line). Let us recall the definition and some characterizations which will be needed in the sequel. Throughout this paper all the sequence of polynomials are supposed to be monic.

Definition 2.2. [55, 57] A sequence of monic polynomials \( \{P_n\}_{n \geq 0} \) is said to be a \( d \)-orthogonal polynomial sequence, in short a \( d \)-OPS, with respect to the \( d \)-dimensional vector of linear forms \( U = (u_0, \ldots, u_{d-1})^T \) if

\[
\begin{align*}
\langle u_r, x^m P_n(x) \rangle &= 0, \quad n \geq md + r + 1, \quad m \geq 0, \\
\langle u_r, x^m P_{md+r}(x) \rangle &\neq 0, \quad m \geq 0,
\end{align*}
\]

for each \( 0 \leq r \leq d - 1 \).

The first and second conditions of (2.2) are called respectively the \( d \)-orthogonality conditions and the \( d \)-regularity conditions. In this case, the \( d \)-dimensional form \( U \) is called regular. Notice further that if \( d = 1 \), then we meet again the notion of ordinary orthogonality.

Let us now recall the following characterizations which is the analogue of Favard’s theorem.

Theorem 2.3. [55] Let \( \{P_n\}_{n \geq 0} \) be a monic sequence of polynomials, then the following statements are equivalent.

(a) The sequence \( \{P_n\}_{n \geq 0} \) is \( d \)-OPS with respect to \( U = (u_0, \ldots, u_{d-1}) \).
(b) The sequence \( \{P_n\}_{n \geq 0} \) satisfies a \( (d + 2) \)-term recurrence relation

\[
P_{m+d+1}(x) = (x - \beta_{m+d}) P_{m+d}(x) - \sum_{\nu=0}^{d-1} \gamma_{m+d-\nu} P_{m+d-1-\nu}(x), \quad m \geq 0,
\]
Section UL to that generated by polynomials generated by the matrix Lu. Jacobi matrix, reveals a glimpse on the kernel polynomials. Accordingly, the sequence of for every 0 \leq \nu \leq d - 1, there exist d polynomials \phi_{\nu, \mu}, 0 \leq \mu \leq d - 1 such that \[ u_{nd+\nu} = \sum_{\mu=0}^{d-1} \phi_{\nu, \mu} u_{\mu}, \quad n \geq 0, \quad 0 \leq \nu \leq d - 1, \]
and verifying
\[ \begin{align*}
\deg \phi_{\nu, \mu} &= n, & 0 \leq \nu \leq d - 1, & \text{and if } d \geq 1, \\
\deg \phi_{\nu, \mu} &\leq n, & 0 \leq \mu \leq \nu - 1, & \text{if } 1 \leq \nu \leq d - 1, \\
\deg \phi_{\nu, \mu} &\leq n - 1, & \nu + 1 \leq \mu \leq d - 1, & \text{if } 0 \leq \nu \leq d - 2.
\end{align*} \]

Now, if we multiply the recurrence of \( P_{(n+1)d+r} \) by \( x^n \), we get under the action of \( u_r \)
\[ \langle u_r, x^{n+1} P_{(n+1)d+r} \rangle = \gamma_{nd+r+1}^0 \langle u_r, x^n P_{nd+r} \rangle, \]
and then
\[ \prod_{\nu=0}^{n} \gamma_{\nu d+r+1}^0 = \frac{\langle u_r, x^{n+1} P_{(n+1)d+r} \rangle}{\langle u_r, P_r \rangle}, \quad 0 \leq r \leq d - 1. \]

When \( r = d - 1 \) in (2.5), and if we set \( \langle u_{d-1}, P_{d-1} \rangle = \gamma_{0}^0 \), we obtain [55]
\[ \prod_{\nu=0}^{n} \gamma_{\nu d}^0 = \langle u_{d-1}, x^n P_{(n+1)d-1} \rangle. \]

Furthermore, and in a similar way we have [55]
\[ \beta_{\nu} = \langle u_{\nu}, x P_{\nu} \rangle, \quad 0 \leq \nu \leq d - 1, \]
\[ \gamma_{\nu+1}^{\nu+1} = \langle u_{\nu+1}, x P_{\nu+1} \rangle, \quad 1 \leq \nu \leq d - 1 - r, \quad 0 \leq r \leq d - 2, \]
\[ \gamma_{n+1}^{\nu} = \langle u_{n+\nu}, x P_{n+d} \rangle, \quad 0 \leq \nu \leq d - 1, \quad n \geq 0. \]

Later on we will give a new characterization of the d-quasi-orthogonality which is closely related to our previous one and further, it could be also regarded as the d-analogue of Al-Salam-Chihara’s characterization for classical OPS. The inspection of UL decomposition of Jacobi matrix, reveals a glimpse on the kernel polynomials. Accordingly, the sequence of polynomials generated by the matrix \( LU \) is d-quasi-orthogonal of order one with respect to that generated by \( UL \). We defer further details concerning Darboux transformation to Section 6, where we prove some amazing results and we recall here the definition and the only characterization exists in the literature at the time of writing.

**Definition 2.4.** [55] A sequence \( \{ P_n \}_{n \geq 0} \) is said \( d \)-quasi-orthogonal of order \( s \) with respect to the form \( U = (u_0, ..., u_{d-1})^T \), if for every \( 0 \leq r \leq d - 1 \), there exist \( s_r \geq 0 \) and \( \sigma_r \geq s_r \), integers such that
\[ \begin{align*}
\langle u_r, P_m P_n \rangle &= 0, & n &\geq (m + s_r) d + r + 1, & m &\geq 0, \\
\langle u_r, P_{\sigma_r} P_{(\sigma_r + s_r)d+r} \rangle &\neq 0, & m &\geq 0,
\end{align*} \]
for every \( 0 \leq r \leq d - 1 \). We put \( s = \max_{0 \leq r \leq d - 1} s_r. \)
Unfortunately, some characterizations of the d-quasi-orthogonality are proved by Maroni [55] subject to some relations only between regular forms. In the usual orthogonality, we emphasize that an OPS is also quasi-orthogonal of order $l$ with respect to another regular form, if and only if it is a linear combination of $l$ terms of the corresponding sequence of the second form. The latter has been generalized to the d-orthogonality as follows

**Proposition 2.5.** [65] For any two d-OPS’s $\{P_n\}_{n \geq 0}$ and $\{Q_n\}_{n \geq 0}$ relative to $\mathcal{U}$ and $\mathcal{V}$, respectively, the following are equivalent

1. There exist $d^2$ polynomials $\phi^r_s$, $0 \leq s, r \leq d - 1$ such that
   \[ v_r = \sum_{s=0}^{d-1} \phi^s_r u_s, \quad 0 \leq r \leq d - 1, \]
   where
   \[ \deg \phi^r_s = l, \]
   \[ \deg \phi^r_s \leq l, \quad 0 \leq s \leq r - 1, \quad \text{if} \quad 1 \leq r \leq d - 1, \]
   \[ \deg \phi^r_s \leq l - 1, \quad r + 1 \leq s \leq d - 1, \quad \text{if} \quad 0 \leq r \leq d - 2. \]

2. There exists non negative integer $l$ such that
   \[ P_n (x) = Q_n (x) + \sum_{i=1}^{dl} a_{n,i} Q_{n-i} (x), \quad n \geq dl, \]
   with $a_{n,dl} \neq 0$.

The above characterization (2.9) reduces to [51, p. 294] first proved for classical OPS (d=1). The determination of the matrix polynomials, i.e., the link between the two vector forms, is based on the following useful characterization theorem

**Theorem 2.6.** [55] For each sequence $\{P_n\}_{n \geq 0}$ d-OPS with respect to $\mathcal{U}$, then the following statements are equivalent.

(i) There exist $\mathcal{L} \in \mathcal{P}'$ and a nonnegative integer number $s$ such that
   \[ \langle \mathcal{L}, P_n \rangle = 0, \quad n \geq s + 1 \quad \text{and} \quad \langle \mathcal{L}, P_s \rangle \neq 0. \]

(ii) There exist $\mathcal{L} \in \mathcal{P}'$, a nonnegative integer number $s$, and $d$ polynomials $\phi^\alpha$, $0 \leq \alpha \leq d - 1$, such that $\mathcal{L} = \sum_{\alpha=0}^{d-1} \phi^\alpha u_\alpha$ with the following properties: if $s = qd + r$, $0 \leq r \leq d - 1$, we have
   \[ \deg \phi^r = q, \quad 0 \leq r \leq d - 1, \quad \text{and} \quad d \geq 2, \]
   \[ \deg \phi^\alpha \leq q, \quad 0 \leq \alpha \leq r - 1, \quad \text{if} \quad 1 \leq r \leq d - 1, \]
   \[ \deg \phi^\alpha \leq q - 1, \quad r + 1 \leq \alpha \leq d - 1, \quad \text{if} \quad 0 \leq r \leq d - 2. \]

The most notable moment functionals are those in the positive definite case. For instance, in this case, the zero of the corresponding sequence of orthogonal polynomials exhibit special features. We will back to this context later in section 9, and we show that the definition of positive definite moment functional, in the sense of Chihara [26], can be extended in a natural way into the d-orthogonality and that the positive definiteness may be characterized by the recurrence coefficients.

Actually, an OPS can be seen as the characteristic polynomial of a certain tridiagonal matrix. So, it is not surprising that quite a few results on OPS can be verified with tools from matrix theory.
In our case, it is well known that we can express (2.3) in terms of matrices as $x^\mathbb{P} = J_d \mathbb{P} := J \mathbb{P}$ (see (2.11) below) where $J_d = (a_{i,j})_{i,j=0}^\infty$ is a $(d+2)$ -banded lower Hessenberg matrix, i.e., that is to say

$$
\begin{align*}
    a_{i,i+1} &= 1, \quad i \geq 0 \\
    a_{i,i} &= \beta_i, \quad i \geq 0 \\
    a_{i+r,i} &= \gamma_{i+1}^{d-r}, \quad i \geq 0, \quad 1 \leq r \leq d.
\end{align*}
$$

Matrix (2.10) is called the monic Jacobi matrix of the monic d-OPS $\{P_n\}_{n \geq 0}$.

To describe our results we have introduced some notation which will be kept throughout. For the sake of uniformity, we will often use the following notations

$$
\begin{align*}
    A_k^r &= \left( A_k^{(r)}, A_{k+1}^{(r)}, \ldots, A_{k+d}^{(r)} \right)^T, \\
    A_{k,-l}^r &= \left( A_k^{(r)}, A_{k-1}^{(r)}, \ldots, A_{k+l}^{(r)} \right), \\
    A_k^r &= \left( A_0^{(r)}, A_1^{(r)}, \ldots \right)^T, \\
    A_k^{r,0} &= \left( A_k^{(r)}, A_{k+1}^{(r)}, \ldots \right).
\end{align*}
$$

3. Modification of the recurrence coefficients

Some modifications of the recurrence coefficients in equations (2.3)-(2.4), lead to new families of d-OPS such as the associated and the co-recursive polynomials as well as to some interesting Jacobi matrices. Indeed, by deleting the first $r$ rows and columns from the Jacobi matrix, the corresponding OPS are the associated polynomials of order $r$, denoted by $P_n^{(r)}$. Instead of deleting rows and columns, if we add $r$ new rows and columns at the beginning of the Jacobi matrix, then the corresponding new OPS are called anti-associated of order $r$ denoted by $P_n^{(-r)}$ [63].

The purpose of this section is to discuss the associated as well as the anti-associated polynomials in a greater generality manner and further, to introduce particular perturbation of the coefficients. These families were initialized for $d \geq 2$ in [57, 64], but we repeat some main results here for the sake of uniformity of treatment and for completeness.

3.1. The associated sequence. The associated sequence of $\{P_n\}_{n \geq 0}$ (with respect to $u_0$), is the sequence $\{P_n^{(1)}\}_{n \geq 0}$ defined by

$$
P_n^{(1)}(x) = \left\langle u_0, \frac{P_{n+1}(x) - P_{n+1}(\xi)}{x - \xi} \right\rangle, \quad n \geq 0.
$$

$P_n^{(1)}$ is a monic polynomial of degree $n$. Let us denote by $\{u_n^{(1)}\}_{n \geq 0}$ the dual sequence of $\{P_n^{(1)}\}_{n \geq 0}$. Then, it results straightforwardly from the left product of a form by polynomial that [57, 64]

**Proposition 3.1.** When $\{P_n\}_{n \geq 0}$ is d-OPS with respect to $\mathcal{U} = (u_0, \ldots, u_{d-1})^T$, then $\{P_n^{(1)}\}_{n \geq 0}$ is d-OPS with respect to $\mathcal{U}^{(1)} = (u_0^{(1)}, \ldots, u_{d-1}^{(1)})^T$ with the following properties

$$
P_n^{(1)}(x) = (u_0 \theta_0 P_{n+1})(x), \quad n \geq 0,
$$

$$
\begin{align*}
    u_\nu^{(1)} &= x(u_{\nu+1} u_0^{(1)}), \quad 0 \leq \nu \leq d - 2, \quad d \geq 2, \\
    \gamma_1 u_{d-1}^{(1)} &= -x^2 u_0^{(1)} - \sum_{\nu=0}^{d-2} x^{d-1-\nu} x (u_{\nu+1} u_0^{(1)}).
\end{align*}
$$
Accordingly, the successive associated sequences are defined recursively [57]

\[ P_n^{(r+1)}(x) = (P_n^{(r)}(x))^{(1)} \]

and \( u_n^{(r+1)} = (u_n^{(r)})^{(1)} \), \( n, r \geq 0 \),

with \( P_n^{(0)} = P_n \) and \( u_0^{(0)} = u_0 \). That is to say

\[ P_n^{(r+1)}(x) = (u_0^{(r)} \theta_0 P_n^{(r+1)}(x)) ; \quad u_n^{(r+1)} = (x u_{n+1}^{(r)}) (u_0^{(r)})^{-1}, \quad n, r \geq 0. \]

When \( \{P_n\}_{n \geq 0} \) is \( d \)-OPS with respect to \( \mathcal{U} = (u_0, ..., u_{d-1}) \), it verifies a recurrence relation of type (2.3), we deduce immediately that the associated sequence \( \{P_n^{(r)}\}_{n \geq 0} \) satisfies the following recurrence relation

\[ P_m^{(r)}(x) = (x - \beta_m + d + r) P_m^{(r)}(x) - \sum_{\nu=0}^{d-1} \theta_{m+d+r-\nu} P_{m+d-1-\nu}^{(r)}(x) \quad \text{for} \quad m \geq 0, \]

with the initial conditions

\[ P_0^{(r)}(x) = 1, \quad P_n^{(r)}(x) = x - \beta_r, \]

\[ P_m^{(r)}(x) = (x - \beta_m + r) P_m^{(r)}(x) - \sum_{\nu=0}^{m-2} \theta_{m+r-\nu} P_{m-2-\nu}^{(r)}(x), \quad 2 \leq m \leq d. \]

Furthermore, the sequence of polynomials and its corresponding associated sequence are also connected through

\[ P_n^{(r+1)}(x) = (u_r \theta_0 P_{n+r+1}^{(r+1)})(x), \quad n, r \geq 0. \]

**Proposition 3.2.** When \( \{P_n\}_{n \geq 0} \) is \( d \)-OPS with respect to \( \mathcal{U} = (u_0, ..., u_{d-1}) \), then \( \{P_n^{(r)}\}_{n \geq 0} \) is \( d \)-OPS with respect to \( \mathcal{U}^{(r)} = (u_0^{(r)}, ..., u_{d-1}^{(r)}) \) defined by (3.3). In addition \( u_n^{(r+1)} \) satisfies the following relation

\[ u_n^{(r+1)} = x^{r+1} (u_n^{(r+1)})(u_0^{(0)} u_0^{(1)} u_0^{(2)} ... u_0^{(r)})^{-1} = (x u_{n+r+1}) u_r^{-1}, \quad n, r \geq 0. \]

**Proof.** The first equality in the most left of (3.6) has been proven in [64]. Then, from this last equality we deduce when \( n = 0 \), that

\[ u_0^{(0)} u_0^{(1)} u_0^{(2)} ... u_0^{(r)} = x^r u_r, \]

hence,

\[ u_n^{(r)} (x^{r-1} u_{r-1}) = x^r u_{n+r}. \]

Now the left product of a form by polynomial gives the second equality. \( \square \)

When \( r = 0 \), our results (both formulas in (3.6)) reduce again to the classical result of Maroni, i.e., \( u_n^{(1)} = (x u_{n+1}) u_0^{-1} \) [55, (1.9)]. In addition, the proof of some formulas in [64] become quite easily. For example, we can apply Stieltjes function to the first and last term of (3.6), and using properties listed in lemma 2.1, to obtain

\[ S(u_{r-1})(z) S(u_n^{(r)})(z) = -S (u_{n+r})(z). \]

If \( n = 0 \), the last formula reduces to the case \( d = 1 \) (called Markov theorem) [68, 72].

When \( r > d \), we can express the element of the sequence \( \{P_n^{(r)}\}_{n \geq 0} \) in terms of the original polynomials and their first \( d \) consecutive polynomials in association as follows

\[ P_n^{(r)} = a_1 (x) P_{n+r} + a_2 (x) P_{n+r-1}^{(1)} + ... + a_{d+1} (x) P_{n+r-d}^{(d)}. \]
Indeed, the polynomials \( \{ P_n, P_{n-1}^{(1)}, \ldots, P_{n-d}^{(d)} \} \) are the basic solutions of the linear recurrence (2.3) by definition. They are linearly independent see Proposition 4.3 below. To compute the coefficients \( \{ a_i (x) \}_{i=1}^d \) we use the initial conditions \( P_0^{(r)} = 1 \) and \( P_{-n}^{(r)} = 0 \) if \( 1 \leq n \leq d \) (see also [68]).

For the anti-associated \( d \)-OPS, the corresponding Jacobi matrix, denoted \( J_d^{(-r)} \) [63], contains \( (d + 1) r \) new parameters and they satisfy
\[
\left( P_{n+r}^{(-r)} (x) \right)^{(k)} = P_{n+r}^{(k-r)} (x).
\]

For \( r = 1 \) we have
\[
J_d^{(-1)} = \begin{pmatrix} \beta_{-1} & 1 \\ \Gamma & J_d \end{pmatrix}, \quad \Gamma^T = (\gamma_0^d, \ldots, \gamma_0^0).
\]

The associated sequence as well as the anti-associated sequence can be both defined by (3.1). Indeed, the anti-associated polynomials are defined by means of (3.1) as follows
\[
P_n^{(-r)} (x) = \left\{ u_0, \frac{P_{n-r} (x) - P_{n-r} (\xi)}{x - \xi} \right\}, \quad n \geq 0.
\]

The latter one allows us to verify that the family of \( P_n^{(-r)} \) satisfies the recurrence (3.4) by shifting \( \beta_{m+d+r} \) and \( \gamma_{m+d+r-\nu}^{d-1-\nu} \) to \( \beta_{m+d-r} \) and \( \gamma_{m+d-r-\nu}^{d-1-\nu} \) respectively. Moreover, if we denote the corresponding dual sequence of the anti-associated polynomials by \( U^{(-r)} = (u_0^{(-r)}, \ldots, u_{d-1}^{(-r)}) \), then following the same idea used in [55], we recover a closed connection between \( P_n^{(-1)} \) and \( u_0^{(-1)} \) to the original sequences in easy way
\[
P_n^{(-1)} (x) = (u_0 \theta_0 P_{n-1}) (x) \quad \text{and} \quad u_0^{(-1)} = (x u_1 - u_0^{-1}) \quad n \geq 1.
\]

Again, since the anti-associated polynomials are also \( d \)-OPS, then we could expand them as a linear combination of the basic solutions of their \((d+2)\)-term recurrence relation as follows
\[
P_n^{(-r)} = b_1 (x) P_{n+r} + b_2 (x) P_{n+r-1}^{(1)} + \cdots + b_{d+1} (x) P_{n+r-d}^{(d)}.
\]

The coefficients \( \{ b_i (x) \}_{i=1}^d \) can be determined using the initial conditions \( P_0^{(-r)} = 1 \) and \( P_{-n}^{(-r)} = 0 \) if \( 1 \leq n \leq d \).

### 3.2. Finite modifications.

The general modification consists in perturbing some terms of the sequences \( \{ \beta_n \}_{n \geq 0} \) and \( \{ \gamma_{n, \nu} \leq \nu \leq d - 1 \}_{n \geq 0} \) of the recurrence by adding or multiplying by some complex numbers.

To start with, let us explain a little bit the construction of co-recursive sequences. Given a \( d \)-dimensional vector \( \mu = (\mu_0, \mu_1, \ldots, \mu_{d-1}) \) and an array \( \eta = \{ \eta_{\nu+1}^\nu \}_{0 \leq \nu \leq d-1} \), the co-recursive sequence \( \{ P_n \}_{n \geq 0} \) where \( P_n (x) := P_n (x, \mu, \eta) \), is defined by modifying the initial values of the sequences \( \{ \beta_n \}_{n \geq 0} \) and \( \{ \gamma_{n, \nu} \leq \nu \leq d - 1 \}_{n \geq 0} \) as follows [57, 64, 53]
\[
P_0^c (x) = 1, \quad P_1^c (x) = x - \alpha_0,
\]
\[
P_m^c (x) = (x - \alpha_{m-1}) P_{m-1}^c (x) - \sum_{\nu=\nu_0}^{m-2} \xi \nu_{m-1-\nu} P_{m-\nu}^c (x), \quad 2 \leq m \leq d,
\]
\[
P_{m+d}^c (x) = (x - \beta_{m+d}) P_{m+d}^c (x) - \sum_{\nu=\nu_0}^{d-1} \gamma \nu_{m+d-\nu} P_{m+d-\nu}^c (x), \quad m \geq 0,
\]

where \( \xi \nu_{m-1-\nu} \) and \( \gamma \nu_{m+d-\nu} \) are some fixed polynomials.
where \( \xi_n^0 = \gamma_n^0, \forall n \geq 1 \) and
\[
\alpha_n = \beta_n + \mu_n, \quad \text{for } 0 \leq n \leq d - 1,
\]
\[
\xi_n^\nu = \gamma_n^\nu + \eta_n^\nu, \quad \text{for } 1 \leq n \leq \nu, \quad 1 \leq \nu \leq d - 1.
\]

Further generalizations of this perturbation can be done by translating the perturbation at level \( k \geq 0 \). That is to say, we define a generalized co-recursive polynomials by the following recurrence
\[
P_m^c(x) = P_m(x), \quad m \leq k, \quad \text{with} \quad P_m^c(x) \equiv 0, \quad m < 0,
\]
\[
P_m^c(x) = (x - \alpha_{m-1}) P_{m-1}^c(x) - \sum_{\nu=0}^{d-1-\nu} \xi_{m-1-\nu}^{d-1-\nu} P_{m-\nu}^c(x), \quad k + 1 \leq m \leq d + k,
\]
\[
P_{m+d+1}^c(x) = (x - \beta_{m+d}) P_{m+d}^c(x) - \sum_{\nu=0}^{d-1} \gamma_{m+d-\nu}^{d-1-\nu} P_{m+d-\nu}^c(x), \quad m \geq k.
\]

**Corollary 3.3.** The general solution of the recurrence (3.9)-(3.10), can be written as
\[
P_n^c(x) = P_n(x) - \sum_{i=1}^{d} A_i(x) P_{n-k-i}^c(x), \quad d \geq 1, \quad n, k \geq 0,
\]
\[
P_n^c(x) = P_n(x), \quad n \leq k.
\]

Indeed, using the initial conditions (3.7)-(3.8) to determine the coefficient \( \{A_k\}_{k=1}^d \) explicitly. In fact, for \( m = k + 1 \), we get
\[
P_{k+1}^c = (x - \beta_k - \mu_k) P_k^c - (\gamma_k^{d-1} + \eta_k^{d-1}) P_{k-1}^c - \ldots - (\gamma_1^{d-k} + \eta_1^{d-k}) P_0^c
\]
\[
= P_{k+1} - \mu_k P_k - \eta_k^{d-1} P_{k-1} - \ldots - \eta_1^{d-k} P_0
\]
\[
= P_{k+1} - A_1
\]
i.e.,
\[
A_1 = \mu_k P_k + \eta_k^{d-1} P_{k-1} + \ldots + \eta_1^{d-k} P_0.
\]

By induction on \( m \), we obtain for \( m = k + d \),
\[
P_{d+k}^c = (x - \beta_{d+k-1} - \mu_{d+k-1}) P_{d+k-1}^c - (\gamma_{d+k-1}^{d-1} + \eta_{d+k-1}^{d-1}) P_{d+k-2}^c - \ldots
\]
\[
- (\gamma_{k+1}^1 + \eta_{k+1}^1) P_k^c - \gamma_k^0 P_{k-1}^c,
\]
then using the expansion (3.11), we get
\[
P_{d+k}^c = (x - \beta_{d+k-1} - \mu_{d+k-1}) \left[ P_{d+k-1} - A_1 P_{d-k-2}^c - \ldots - A_{d-1} P_0^{(d+k-1)} \right]
\]
\[
- (\gamma_{d+k-1}^{d-1} + \eta_{d+k-1}^{d-1}) \left[ P_{d+k-2} - A_1 P_{d-k-3}^c - \ldots - A_{d-2} P_0^{(d+k-2)} \right] - \ldots
\]
\[
- (\gamma_{k+2}^2 + \eta_{k+2}^2) \left[ P_{k+1} - A_1 P_0^{(k+1)} \right] - (\gamma_{k+1}^1 + \eta_{k+1}^1) P_k - \gamma_k^0 P_{k-1}
\]
\[
= P_{d+k} - A_1 P_{d-k}^c - \ldots - A_{d-1} P_0^{(d+k-1)} - A_d P_0^{(d+k)}
\]
whence, finally
\[
A_d = \mu_{d+k-1} \left[ P_{d+k-1} - A_1 P_{d-k-2}^c - \ldots - A_{d-1} P_0^{(d+k-1)} \right]
\]
\[
+ \eta_{d+k-1}^{d-1} \left[ P_{d+k-2} - A_1 P_{d-k-3}^c - \ldots - A_{d-2} P_0^{(d+k-2)} \right]
\]
\[
+ \ldots + \eta_{k+2}^2 \left[ P_{k+1} - A_1 P_0^{(k+1)} \right] + \eta_{k+1}^1 P_k.
\]
Notice that when \( k = 0 \), we find the definition of the co-recursive polynomials \([57, 64, 53]\), and when \( d = 1 \) the above results reduce to those analyzed in \([25, 30, 39, 40, 52, 62, 67]\), among others.

Now as it was remarked in \([64]\), we again have the following

\[
(P_n^{(d+k)})(x) = P_n^{(d+k)}(x), \quad d \geq 1, \quad n, k \geq 0.
\]

That is, if we denote the dual sequence of the co-recursive sequence by \( \{L_n\}_{n \geq 0} \), then there exist constants \( \theta \) such that

\[
L^{(d+k)} = \theta u^{(d+k)}
\]

In order to determine the parameters \( \theta \), we first prove the following lemma.

**Lemma 3.4.** We have

\[
\langle u^{(1)}_{\nu}, P_{\nu+2+i} \rangle = \gamma_{\nu+2}^{d-1-i}, \quad 0 \leq \nu, i \leq d - 1,
\]

\[
\langle u^{(r)}_{\nu}, P_{r(d+1)+\nu} \rangle = \prod_{i=1}^{r} \gamma_{d(r-\nu)+\nu+r+1}^{0}, \quad 0 \leq \nu \leq d - 1, \quad r \geq 1.
\]

**Proof.** From (3.6) we have

\[
\langle u^{(1)}_{\nu}, P_{\nu+2+i} \rangle = \langle u_{\nu+1}u_{0}^{-1}, xP_{\nu+2+i} \rangle = \langle u_{\nu+1}, xP_{\nu+2+i} \rangle.
\]

Next, relations (2.5) show the first equality. For the second equality, one can use the first part of proposition 3.2. Indeed, by taking \( m + d + 1 = r(d + 1) + \nu \) in the recurrence (2.3), we get using also the definition of dual sequence that

\[
\langle u^{(r)}_{\nu}, P_{dr+r+\nu} \rangle = \langle u_{\nu+r}, x^{r}P_{dr+r+\nu} \rangle = \gamma_{d(r-1)+\nu+r+1}^{0} \langle u_{\nu+r}, x^{r-1}P_{d(r-1)+r+\nu} \rangle.
\]

Therefore, this lemma shows that \( \theta = 1, \forall \nu \geq 0 \) because of the following

\[
\langle L^{(d+k)}_{\nu}, P_{(d+k)(d+1)+\nu} \rangle = \theta \nu \langle u_{\nu+d+k}, x^{d+k}P_{(d+k)(d+1)+\nu} \rangle.
\]

It thus follows by proposition 3.2 that

\[
(xL_{\nu})L_{d+k-1}^{-1} = (xu_{\nu}) u_{d+k-1}^{-1}, \quad \nu \geq d, \quad k \geq 0.
\]

A little more generally, we study now further finite modifications of all the recurrence coefficients and we start with the so-called co-dilated and later on with the co-modified polynomials \([30, 52, 62]\). The main idea consists in modifying the recurrence coefficients and keeping unchanged the regularity conditions in order to preserve the orthogonality according to Favard’s theorem. In doing so, next we shall multiply the last terms \( (\gamma_n^0) \) in the recurrence by a non-zero complex parameter \( \lambda \). We emphasize that in the previous modification, the regularity is well satisfied.

As is customary, the co-dilated of a d-OPS \( \{P_n\}_n \) denoted by \( \{\tilde{P}_n\}_n \), is the family of polynomials generated by the recurrence formula (2.3) in which \( \gamma_n^0 \) is replaced by \( \lambda \gamma_n^0 \). Hence, by regarding the initial conditions, this is equivalent to the following recurrence

\[
\tilde{P}_n = P_n, \quad n \leq d,
\]

\[
\tilde{P}_{d+1} = (x - \beta_d) \tilde{P}_d - \sum_{\nu=0}^{d-2} \gamma_{d-\nu}^{d-1-\nu} \tilde{P}_{d-1-\nu} - \lambda \gamma_1^0,
\]

\[\text{(3.12)}\]
\[ \hat{P}_{n+d+1} = (x - \beta_{n+d}) \hat{P}_{n+d} - \sum_{\nu=0}^{d-1} \gamma_{n+d-\nu}^{d-1} \hat{P}_{n+d-\nu}, \quad n \geq 1. \]

Using the initial conditions (3.12), the general solution of this recurrence can be written as

\[ (3.13) \quad \hat{P}_n (x) = P_n (x) + \gamma_0^0 (1 - \lambda) P_{n-(d+1)} (x), \quad d \geq 1, \quad n \geq 0. \]

Remark that we have again \( \hat{P}_n^{(1)} = P_n^{(1)}, n \geq 0. \) Hence, if we denote by \( \tilde{U} = (\tilde{u}_0, \ldots, \tilde{u}_{d-1}) \) the corresponding vector linear form of co-dilated sequence, then there exist \( \omega, \) constants such that

\[ (3.14) \quad (\tilde{u}_\nu)^{(1)} := \tilde{u}_\nu^{(1)} = \omega_\nu u^{(1)}_\nu. \]

Since the co-dilated are obtained under the modification of only one parameter, then from lemma 3.4 we see that

\[ (3.15) \quad \omega_\nu = 1 \quad \text{for} \quad 0 \leq \nu \leq d - 1. \]

At first sight, we have the following result

**Proposition 3.5.** When \( \{P_n\}_{n \geq 0} \) is \( d \)-OPS with respect to \( U = (u_0, \ldots, u_{d-1}), \) then the co-dilated sequence \( \{\hat{P}_n\}_{n \geq 0} \) is \( d \)-OPS with respect to \( \tilde{U} = (\tilde{u}_0, \ldots, \tilde{u}_{d-1}) \) satisfying

\[ (3.16) \quad \tilde{u}_\nu = u_\nu \left[ \lambda \delta + \lambda (1 - \beta_0 x^{-1}) u_0 - \lambda \sum_{\nu=0}^{d-2} \gamma_{1}^{d-1-\nu} x^{-\nu} \right]^{-1} \]

for \( 0 \leq \nu \leq d - 1, \quad d \geq 1 \) where \( \lambda + \tilde{\lambda} = 1. \)

**Proof.** From (3.14)-(3.15), we deduce by using the second equality in (3.2), that

\[ (3.17) \quad \tilde{u}_{\nu+1} \tilde{u}_0^{-1} = u_{\nu+1} u_0^{-1}, \quad 0 \leq \nu \leq d - 2, \quad d \geq 2, \]

together with the third equality in (3.2), we obtain

\[ x^2 [\tilde{u}_0^{-1} - \lambda u_0^{-1}] + \tilde{\lambda} \sum_{\nu=0}^{d-2} \gamma_{1}^{d-1-\nu} x (u_{\nu+1} u_0^{-1}) = 0. \]

The left product of a regular form by polynomial and [64, lem. 24] give

\[ \tilde{u}_0 = u_0 \left[ \lambda \delta + \lambda (1 - \beta_0 x^{-1}) u_0 - \lambda \sum_{\nu=0}^{d-2} \gamma_{1}^{d-1-\nu} x^{-\nu} \right]^{-1}. \]

whence, using the fact that \( \delta' u = -x^{-1} u \) [56, (1.16)]

\[ (3.18) \quad \tilde{u}_0 = u_0 \left[ \lambda \delta + \lambda (1 - \beta_0 x^{-1}) u_0 - \lambda \sum_{\nu=0}^{d-2} \gamma_{1}^{d-1-\nu} x^{-\nu} \right]^{-1}. \]

Replace (3.18) in (3.17) to obtain the desired result. \( \Box \)

In terms of the Stieltjes function, by lemma 2.1 we obtain straightwardly

**Corollary 3.6.** The Stieltjes function of the co-dilated sequence satisfies

\[ S (\tilde{u}_\nu) (z) = \frac{S (u_\nu) (z)}{\lambda - \lambda P_1 (z) S (u_0) (z) + \tilde{\lambda} \sum_{\nu=0}^{d-2} \gamma_{1}^{d-1-\nu} S (u_{\nu+1}) (z)}, \]

for \( 0 \leq \nu \leq d - 1, \quad d \geq 1. \)
See [30] for a similar result in the case \( d = 1 \). The extension of co-dilated at level \( k \geq 1 \) was introduced in [52] for \( d = 1 \). In case of the \( d \)-orthogonality, we can multiply the constant \( \gamma_k^0 \) by a nonzero complex number \( \lambda \). In this case, new family is defined by the following recurrence

\[
\begin{align*}
P_n &= P_n, \quad n \leq d + k - 1, \\
\tilde{P}_{d+k} &= (x - \beta_{d+k-1}) P_{d+k-1} - \sum_{\nu=0}^{d-2} \gamma_{d+k-1-\nu}^0 \tilde{P}_{d+k-2-\nu} - \lambda \gamma_k^0 \tilde{P}_{k-1}, \\
\tilde{P}_{n+d+1} &= (x - \beta_{n+d}) \tilde{P}_{n+d} - \sum_{\nu=0}^{d-1} \gamma_{n+d+1-\nu} \tilde{P}_{n+d-1-\nu}, \quad n \geq k + 1.
\end{align*}
\]

Using the initial conditions (3.19), the general solution of the latter above recurrence could be written as

\[
\tilde{P}_n (x) = P_n (x) + \gamma_0^\lambda \tilde{P}_{k-1} (x) P_{n-(d+k)} (x), \quad d \geq 1, \quad n \geq 0.
\]

Furthermore, we have \( \tilde{P}_n^{(k)} = P_n^{(k)} \) for \( n \geq 0, k \geq 1 \), and then again by lemma 3.4

\[
(\tilde{u}_n^{(k)}) := \tilde{u}_n^{(k)}, \quad 0 \leq \nu \leq d - 1.
\]

Hence, by proposition 3.2 and the left product of a form by polynomial, we deduce

\[
\tilde{u}_{k-1} \tilde{u}_{k+\nu}^{-1} = u_{k-1} u_{k+\nu}^{-1}, \quad 0 \leq \nu \leq d - 1.
\]

Now, combining the results of co-recursive and co-dilated, a new family of polynomials might be generated by modifying the recurrence coefficients all together [30, 39, 40, 52, 62]. The new family obtained, denoted by \( \{ \tilde{P}_n \}_{n \geq 0} \), called co-modified sequence, and it is generated by the following recurrence relation

\[
\begin{align*}
\tilde{P}_0 (x) &= 1, \\
\tilde{P}_1 (x) &= x - \beta_0 - \mu_0, \\
\tilde{P}_m (x) &= (x - \alpha_{m-1}) \tilde{P}_{m-1} (x) - \sum_{\nu=0}^{m-2} \xi_{m-1-\nu} \tilde{P}_{m-2-\nu} (x), \quad 2 \leq m \leq d, \\
\tilde{P}_{d+1} (x) &= (x - \beta_d) \tilde{P}_d (x) - \sum_{\nu=0}^{d-2} \gamma_{d-1-\nu} \tilde{P}_{d-1-\nu} (x) - \lambda \gamma_k^0, \\
\tilde{P}_{m+d+1} (x) &= (x - \beta_{m+d}) \tilde{P}_{m+d} (x) - \sum_{\nu=0}^{d-1} \gamma_{m+d+1-\nu} \tilde{P}_{m+d-1-\nu} (x), \quad m \geq 1,
\end{align*}
\]

where \( \alpha_n \) and \( \xi_n, 1 \leq \nu \leq d - 1 \) are given by (3.8).

In general framework, for \( k \geq 0 \) we could define the co-modified sequence by \( \tilde{P}_m (x) = P_m (x) \), for \( m \leq k \) and for \( m > k \) by the following

\[
\begin{align*}
\tilde{P}_m (x) &= (x - \alpha_{m-1}) \tilde{P}_{m-1} (x) - \sum_{\nu=0}^{m-2} \xi_{m-1-\nu} \tilde{P}_{m-2-\nu} (x), \\
\tilde{P}_{d+1} (x) &= (x - \beta_d) \tilde{P}_d (x) - \sum_{\nu=0}^{d-2} \gamma_{d-1-\nu} \tilde{P}_{d-1-\nu} (x) - \lambda \gamma_k^0, \\
\tilde{P}_{m+d+1} (x) &= (x - \beta_{m+d}) \tilde{P}_{m+d} (x) - \sum_{\nu=0}^{d-1} \gamma_{m+d+1-\nu} \tilde{P}_{m+d-1-\nu} (x), \quad m \geq 1.
\end{align*}
\]

From the previous results, the general solution of this recurrence, connects all the above modified sequences through

\[
\begin{align*}
\tilde{P}_n (x) &= P_n (x) - \sum_{i=1}^{d} A_i (x) P_{n-k-i}^{(k+i)} (x) + \gamma_k^0 \tilde{P}_k (x) P_{n-(d+k+1)}^{(d+k+1)} (x), \quad d \geq 1, \quad n, k \geq 0, \\
\tilde{P}_n (x) &= P_n (x), \quad n \leq k.
\end{align*}
\]

And also we have

\[
\tilde{P}_n^{(d+k)} (x) = P_n^{(d+k)} (x), \quad d \geq 1, \quad n, k \geq 0.
\]
That is, if we denote the dual sequence of co-modified polynomials by \( \{ \tilde{u}_n \}_{n \geq 0} \), then lemma 3.2 provides that necessarily we have
\[
\tilde{u}^{(d+k)}_\nu = u^{(d+k)}_\nu.
\]

In other words, proposition 3.2 shows that
\[
(x \tilde{u}_\nu) \tilde{u}^{-1}_{d+k-1} = (xu_\nu) u^{-1}_{d+k-1}, \quad \nu \geq d.
\]

We could give formally further connections between \( \{ \tilde{u}_n \}_{\nu \geq 0} \) and \( \{ u_{\nu} \}_{\nu \geq 0} \) similar to that in proposition 3.5 and its corollary (see \([30]\) for the usual orthogonality, i.e., \( d=1 \)).

4. DETERMINANTS WITH CO-POLYNOMIALS ENTRIES

In this section we shed new light on the theory of determinants whose entries are \( d \)-OPS, and we give mild generalization as well as few identities that characterize some Casorati determinants related to co-polynomials discussed in section 3.

To begin with, we try to give a 4-analogue of some well known properties related to associated polynomials in the usual orthogonality. First, we have the following formula which leads to formula (4.3). The latter one plays in turn, a pivotal role in proving almost all the results of this section.

**Proposition 4.1.** We have for any \( d \)-OPS \( \{ P_n \} \) the following expansion
\[
P^{(r)}_{n+m} = P^{(n+r)}_m P^{(r)}_n - \left( \sum_{i=1}^d \gamma^{d-i}_{n+r} P^{(n+r+i)}_{m-i} \right) P^{(r)}_{n-1} - \left( \sum_{i=1}^{d-1} \gamma^{d-i-1}_{n+r-1} P^{(n+r+i)}_{m-i} \right) P^{(r)}_{n-2} - \ldots - \gamma^0_{n+r-d+1} P^{(n+r+1)}_{m-1} P^{(r)}_{n-d}.
\]

**Proof.** The above identity can be easily proved by mathematical induction. Indeed, the equality is satisfied for \( m = 0 \) and \( m = 1 \) (\( \forall n, r \geq 0 \)). Assume that it is true up to a fixed \( m \). Then, by replacing (4.1) in the recurrence of \( P^{(r)}_{n+m+1} \), and after getting \( P^{(r)}_{n-i} \), \( 0 \leq i \leq d \), as a factor in the obtained expression, the result follows.

By interchanging the role of \( n \) and \( m \) we find
\[
P^{(r)}_{n+m} = P^{(m+r)}_n P^{(r)}_m - \left( \sum_{i=1}^d \gamma^{d-i}_{m+r} P^{(m+r+i)}_{n-i} \right) P^{(r)}_{m-1} - \left( \sum_{i=1}^{d-1} \gamma^{d-i-1}_{m+r-1} P^{(m+r+i)}_{n-i} \right) P^{(r)}_{m-2} - \ldots - \gamma^0_{n+r-d+1} P^{(m+r+1)}_{n-1} P^{(r)}_{m-d}.
\]

The latter relation gives a link between a polynomials of \( d + 2 \) levels of association. By setting \( m = 1 \) we get
\[
P^{(r)}_{n+1}(x) = (x - \beta_r) P^{(r+1)}_n(x) - \sum_{i=1}^d \gamma^{d-i}_{r+1} P^{(r+1+i)}_{n-i}(x),
\]
which is a dual formula of (3.4). When \( d = 1 \), we obtain the result of Belmehdi and Van Assche (see also \([8, 9, 29]\)).

As a consequence of the relation (4.3) is the following. Take \( P^{(k)}_{n-k} \) in the place of \( P^{(r)}_{n+1} \) and expand the polynomials \( P^{(k+1)}_{n-(k+1)} \) by means of (4.3) to get

\[
P^{(k)}_{n-k} = P^{(k)}_2 P^{(k+2)}_{n-(k-2)} - \left[ \gamma^{d-1}_{k+2} P^{(k)}_{1} + \gamma^{d-2}_{k+1} P^{(k+3)}_{n-(k-3)} - \ldots - \left[ \gamma^1_{k+2} P^{(k)}_{1} + \gamma^0_{k+1} \right] P^{(k+d+1)}_{n-(k+d+1)} - \gamma^0_{k+2} P^{(k)}_{1} P^{(k+d+1)}_{n-(k+d+1)}.\]
Proceeding in the same way $r$ times we obtain the following expression
\[(4.4) \quad P_{n-k}^{(k)} = P_r^{(k)} P_{n-(k+r)}^{(k+r)} - q_{1,r-1} P_{n-(k+r+1)}^{(k+r+1)} - \ldots - q_{d,r-1} P_{n-(k+r+d)}^{(k+r+d)},\]
where $q_{1,r-1} \ldots q_{d,r-1}$ are polynomials on $x$ of degree $r - 1$.

Formula (4.3) constitutes the key ingredient in our approach of the present section. On the other hand, since each of the perturbed sequence discussed in section 3 satisfies the same recurrence as $\{P_n\}$ from certain level $k$, we could give an analogue of the expansion (4.1) as well as of (4.2) for any perturbed sequence aforementioned by following a similar approach. For example, $\{P_n\}$ and its corresponding $d$-co-recursive sequence $\{Q_n\}$ satisfy the same recurrence relation for $n \geq d + 1$, for this end, by regarding the result in the proposition 4.1, we can easily obtain an analogous expression for the co-recursive polynomials and the result is just replacing $P_n^{(r)}$ by $Q_n$ in (4.1) for $n \geq d + 1$
\[
Q_{n+m} = P_m^{(n)} Q_n - \left( \sum_{i=1}^{d} \gamma_{n-i} P_{m-i}^{(n+1)} \right) Q_{n-1} - \left( \sum_{i=1}^{d-1} \gamma_{n-i} P_{m-i}^{(n+1)} \right) Q_{n-2} - \ldots - \gamma_0 P_{m-1}^{(n+1)} Q_{n-d}.
\]

In what follows, we present some Casorati determinants according to our notation (2.11). First, let us consider, for $n, r \geq 0$ and $d \geq 1$, the following determinant
\[(4.5) \quad B_n^{(r)} = \left| P_{n-1}^{r} \ P_{n+1-r}^{r} \ldots P_{n+d-1-r}^{r} \right|^T.\]

Now express each of the polynomials in the first column in the determinant (4.5) by means of the recurrence relation (4.3), and then use the linearity of the determinant with respect to its first column, it is not difficult to check that

**Proposition 4.2.** The determinant (4.5) satisfies the following linear recurrence
\[
B_n^{(r)} = (-1)^d \gamma_1 B_{n-1}^{(r+1)} - (-1)^{2(d-1)} \gamma_{r+1} \gamma_{r+2} B_{n-2}^{(r+2)} - \ldots
\]
\[
- (-1)^{(d-1)(d-1)} \gamma_{r+2} \ldots \gamma_{d-2} \gamma_{d-1} B_{n-(d-1)}^{(r+d-1)}
\]
\[
+ (-1)^{d(d-1)} (x - \beta_{r+d-1}) \gamma_{r+1} \ldots \gamma_{r+d-1} B_{n-d}^{(r+d)}
\]
\[
+ (-1)^{d+1} \gamma_{r+d} B_{n-(d+1)}.\]

Notice that if one expresses each of the polynomials in the first column in the determinant (4.5) according to the recurrence relation (3.4) instead of the dual recurrence (4.3), one gets a recurrence of order $(d + 1)$ but for the same level of association $r$ in each row. When $d = 2$, the recurrence (4.6) reduces to that of de Bruin [28, Lem.1, p.372].

Now if we add, for instance, the next row in bottom and the next column at rightmost in the determinant $B_n$, we obtain a constant. This shows that those families are linearly independent (see proposition 4.3 below).

Let us now consider the following Casorati determinants
\[(4.7) \quad \Delta_n^{(r)} := \left| P_n^r \ P_{n+1}^r \ldots P_{n-d}^r \right| = \left| P_n^r \ P_{n+1}^r \ldots P_{n+d}^r \right|^T,
\]
\[
\nabla_n^{(r)} := \left| Q_n \ P_n^r \ldots P_{n-r-d+1}^r \right|.
\]

Now we are able to prove the following identity.
**Proposition 4.3.** The determinant $\Delta_n^{(r)}$ satisfies the following identity

$$\Delta_n^{(r)} = (-1)^{(d+1)n} \prod_{i=1}^{n} \gamma_{i+r}^{0} \quad \text{with} \quad \Delta_0^{(r)} = 1$$

**Proof.** The proof of (4.8) follows easily from the dual recurrence relation (4.3). Indeed, we express each of the polynomials in the first column in the determinant $\Delta_n^{(r)}$ by means of (4.3) and using the linearity of the determinant with respect to its first column, we obtain

$$\Delta_n^{(r)} = (-1)^{d+1} \gamma_{r+1}^{0} \Delta_{n-1}^{(r+1)}.$$ 

Then, the result follows by induction on $n$. \qed

More generally, we have the following result

**Theorem 4.4.** For any integers $m_1, ..., m_d > n \geq 0$, and $r \geq 0$, we have

$$F_n^{(r)} = \begin{vmatrix} P_n^{r} & P_{m_1}^{r} & \cdots & P_{m_d}^{r} \end{vmatrix}_T = \Delta_n^{(r)} \begin{vmatrix} P_{m_1-n-1,1}^{r+n+1} & P_{m_2-n-1,1}^{r+n+1} & \cdots & P_{m_d-n-d,1}^{r+n+1} \end{vmatrix}_T$$

**Proof.** The equality (4.9) can be obtained in a similar manner as in the previous proposition, we express each of the polynomials in the first column using the dual recurrence (4.3), to get

$$F_n^{(r)} = (-1)^{d+1} \gamma_{r+1}^{0} F_{n-1}^{(r+1)}.$$ 

Proceeding in a similar way $n$ times we have

$$F_n^{(r)} = \Delta_n^{(r)} F_0^{(r+n)},$$

which implies the required result. \qed

Having in mind that the first $d$ polynomials in association, i.e., the polynomials $\{P_n^{(r)}\}_{0 \leq r \leq d}$ are linearly independent, then any determinant of type (4.7) of dimension $m \times m$, with $m \geq d + 1$, is identically zero.

The above results can be proved in another way by using the companion matrix. That is to say, the recurrence relation (2.3), can be presented in terms of another matrix $C_n$ called companion or transfer matrix as follows $\mathbb{P}_{n+1}^i = C_n^{(i)} \mathbb{P}_{n}^i$ where $C_n^{(i)}$ is the matrix

$$C_n^{(i)} = \begin{pmatrix} 0 & I_d \\ -V & x - \beta_{n+d+i} \end{pmatrix}, \quad V = (\gamma_{n+i+1}^{0}, \gamma_{n+i+2}^{1}, \ldots, \gamma_{n+d+i}^{d-1}).$$

By virtue of our notation (2.11), let us introduce the following Casorati determinant

$$D_{n-i}^{(i)} = \begin{vmatrix} \mathbb{P}_{n-i}^i & \mathbb{P}_{n-r}^i & \cdots & \mathbb{P}_{n-r-d+1}^i \end{vmatrix}, \quad 0 \leq i \leq r - 1, \quad r \geq 1.$$ 

In this case, since $\det(C_n^{(i)}) = (-1)^{d+1} \gamma_{n+i+1}^{0}$, then we have

$$D_{n-i}^{(i)} = (-1)^{d+1} \gamma_{n}^{0} D_{n-i-1}^{(i)}.$$ 

Accordingly, we get by recurrence that

**Proposition 4.5.** For any $n \geq 0$, $r \geq 1$, and $0 \leq i \leq r - 1$, we have

$$D_{n-i}^{(i)} = (-1)^{(d+1)(n-r+1)} \prod_{k=r}^{n} \gamma_{k}^{0} D_{r-i-1}^{(i)}(x) = \Delta_{n-r+1}^{(r-1)} P_{r-i-1}^{(i)}(x),$$
which presents another proof of the proposition 4.3 (see also [46]).
Of course this can also be checked by replacing each polynomials in the first column of $D_n$ by the corresponding dual recurrence (4.4). Similar identity for co-recursive polynomials can be obtained by replacing the polynomial $P_m$ by $Q_m$.

However, this idea is not good enough to work with if one uses different sequences because the vector $V$ is not the same. Nevertheless, it is more convenient sometimes, to use the expansion (4.3) instead of the recurrence (2.3). For example, we could express and calculate Casorati determinant (4.9) using transfer matrix. Indeed, we have
\[
P^r_{m_i} = \tilde{C}_r P^{r+1}_{m_i-1},
\]
where the matrix $\tilde{C}$ is obtained from the matrix $C$ by replacing the vector $V$ by the vector $\tilde{V} = (\gamma^0_{r+1}, \gamma^1_{r+1}, \ldots, \gamma^{d-1}_{r+1})$. Accordingly, we get
\[
P^{(r)}_n = (-1)^{d+1} \gamma^0_{r+1} F^{(r+1)}_{n-1}.
\]
Thus our next task is to show analogous results for sequences obtained by a finite modification in the recurrence coefficients. First, taking into account the initial conditions (3.9)-(3.10) and the corollary 3.3, we are able to prove the following identities satisfied by the determinant $\nabla^{(r)}_n$.

**Proposition 4.6.** The determinant $\nabla^{(r)}_n$ satisfies $\nabla^{(0)}_n = (-1)^{d+1} A_d \Delta^{(0)}_n$ and for $r \geq 1$ the following identities
\[
\nabla^{(r)}_n = Q_{r-1} \Delta^{(r-1)}_{n-r+1} = \begin{cases}
\begin{bmatrix}
P_{r-1} - A_1 P_{r-2}^{(1)} & \ldots & - A_{r-1} P_0^{(r-1)}
\end{bmatrix} \Delta^{(r-1)}_{n-r+1} & , \text{if } r \leq d \\
P_{r-1} - A_1 P_{r-2}^{(1)} & \ldots & - A_d P_{d-r-1}^{(d)}
\end{bmatrix} \Delta^{(r-1)}_{n-r+1} & , \text{if } r > d.
\end{cases}
\]

**Proof.** We express each of the polynomials in the first column of the determinant $\nabla^{(r)}_n$ according to the expansion (3.11), we get
\[
\nabla^{(r)}_n = D^{(0)}_n - A_1 D^{(1)}_{n-1} - \ldots - A_{r-1} \Delta^{(r-1)}_{n-r+1} - \ldots - A_d D^{(d)}_{n-d},
\]
now proposition 4.5 completes the proof.

Analogous formulas for co-dilated as well as for co-modified are the following. If we replace the vector $Q_n$ in $\nabla^{(r)}_n$ at first time by $\tilde{P}_n$ and by $\tilde{P}_n$ in a second time, then the resulting determinants are denoted respectively by $\tilde{\nabla}^{(r)}_n$ and $\tilde{\nabla}^{(r)}_n$. i.e.,
\[
\tilde{\nabla}^{(r)}_n = \left| \tilde{P}_n \tilde{P}_{n-r} \ldots \tilde{P}_{n-r-d+1} \right|
\]
and as above, we have the following results.

**Proposition 4.7.** Casorati determinants corresponding to the co-modified polynomials satisfy
\[
\tilde{\nabla}^{(r)}_n = \begin{cases}
(-1)^{d+1} \left[ A_d + \tilde{\lambda} \gamma_1 \right] \Delta^{(0)}_n & , \text{if } r = 0, \\
\lambda \Delta^{(0)}_n & , \text{if } r = 1,
\lambda \Delta^{(1)}_{n-1} & , \text{if } r = 2,
\end{cases}
\]
\[
\begin{cases}
[Q_{r-1} + \tilde{\lambda} \gamma_1 P_{r-d-1}^{(d+1)}] \Delta^{(r-1)}_{n-r+1} & , \text{if } r > 3.
\end{cases}
\]
The determinants \( \tilde{\nabla}^{(r)}_n \) could be obtained as a particular case from \( \tilde{\nabla}^{(0)}_n \) by taking \( A_i \equiv 0 \).

Proof. Using the expression (3.13), then the determinant \( \tilde{\nabla}^{(r)}_n \) reads, when \( r = 0 \)
\[
\tilde{\nabla}^{(0)}_n = (-1)^d \overline{\lambda} \gamma_1^0 \left| P_n \right| P_{n-1} \cdots P_{n-d+1} P_{n-d-1}^{-1}.
\]

Once again, express the polynomials in the first column by means of the dual recurrence (4.3) and use the fact that \( \Delta_n^{(0)} = (-1)^{r+1} \gamma_1^0 \Delta_{n-1}^{(1)} \) to get the value of \( \tilde{\nabla}^{(0)}_n \). Then induction gives the result.

For the determinant \( \tilde{\nabla}^{(r)}_n \), it suffices to remark that we have the following relation between all the perturbed families and the starting one

\[
\tilde{P}_n = Q_n + \tilde{P}_n - P_n,
\]

that is, in view of the above notation and results,
\[
\tilde{\nabla}^{(r)}_n = \nabla^{(r)}_n + \tilde{\nabla}^{(r)}_n - P_{r-1} \Delta_{n-r+1}^{(r-1)}.
\]

This finishes the proof of the proposition. \( \square \)

We can use the results of this section to give analogous results for the modified sequences. Indeed, combine the proposition 4.5 and 4.6, it is not difficult to see that we again have

**Theorem 4.8.** For any integers \( m_1, \ldots, m_d > n \geq 0 \), and \( r \geq 1 \), we have

\[
\tilde{R}^{(r)}_n = \left| \begin{array}{cccc}
Q_n & Q_{m_1} & \cdots & Q_{m_d} \\
\tilde{P}_n^r & \tilde{P}_{m_1}^r & \cdots & \tilde{P}_{m_d}^r \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{P}_{n-r-1}^r & \tilde{P}_{m_1-r-1}^r & \cdots & \tilde{P}_{m_d-r-1}^r \\
\end{array} \right|^T = \left\{ \begin{array}{l}
(-1)^{d+1} A_d F_n^{(0)}, \quad \text{if } r = 0, \\
Q_{r-1} F_{n-r+1}^{(r-1)}, \quad \text{if } r \geq 1.
\end{array} \right.
\]

By analogy, let us denote the first column of the determinant \( R^{(r)}_n \) by \( \tilde{\mathcal{R}} \). If we replace this last vector by \( \tilde{\mathcal{R}} \), and denote the resulting determinant by \( \tilde{R}^{(r)}_n \), we then obtain the following

**Corollary 4.9.** The Casorati determinants corresponding to the co-dilated and the co-modified polynomials satisfy

\[
\tilde{R}^{(r)}_n = \left\{ \begin{array}{l}
(-1)^{d+1} [A_d + \overline{\lambda} \gamma_1^0] F_n^{(0)}, \quad \text{if } r = 0, \\
\lambda F_n^{(0)}, \quad \text{if } r = 1, \\
Q_1 F_{n-1}^{(1)}, \quad \text{if } r = 2, \\
[Q_{r-1} + \overline{\lambda} \gamma_1^0 P_{r-d-1}^{(d+1)}] F_{n-r+1}^{(r-1)}, \quad \text{if } r \geq 3.
\end{array} \right.
\]

The determinants \( \tilde{R}^{(r)}_n \) are obtained from \( \tilde{R}^{(r)}_n \) by taking \( A_i \equiv 0 \).

Further generalization of \( F_n^{(r)} \) are the following determinants

**Corollary 4.10.** The determinants \( G_n \) satisfy the following recurrence

\[
G_d(n) = \left| \begin{array}{c}
P_{n-s_0}^{(s_0)} \quad \cdots \quad P_{n-s_d}^{(s_d)} \end{array} \right|^T = (-1)^{d+1} \gamma_{n-d}^0 G_d(n-1).
\]
4.1. A characterization of d-orthogonality. Next we give a generalization of the characterization of orthogonality pointed out by Al-Salam [44]. From the general theory, for any linear recurrence relations of d+1 terms, there are d linearly independent solutions, i.e., the Wronskian of these d solutions is different from zero. It follows then, that every solutions is a linear combination of d linear independent solutions.

We now set d + 1 polynomials $S_n^{(i)}$, $1 \leq i \leq d + 1$ defined by the initial conditions $S_{k-1}^k \neq 0$ and $S_n^k = 0$ when $n < k - 1$ for $1 \leq k \leq d + 1$. This construction allows us to assert that the set $\{S_n^i, 1 \leq i \leq d + 1\}$, forms a basic solution of a $d + 2$-term linear recurrence relation. Furthermore, we have the following results [44]

Lemma 4.11. A necessary and sufficient condition that there exists a relation

$$|S_n^{(1)} \ldots S_n^{(d+1)}| = \Delta_n \neq 0$$

is that the sequence of polynomials $\{S_n^{(i)}, 1 \leq i \leq d + 1\}$ are d-OPS.

Now suppose that $\{f_n\}$ is d-OPS satisfying the following recurrence relation

$$f_{n+d+1}(x) = (A_{n+d}x + B_{n+d}) f_{n+d}(x) + \gamma_{n+d}^{d-1} f_{n+d-1}(x) + \ldots + \gamma_{n+1}^0 f_n(x).$$

It is evident from (4.4), that for each integer $p \geq 1$, we have

$$f_{n+d+p}(x) = T_p^{(d+1)}(x) f_{n+d}(x) + \ldots + T_p^{(d+1)}(x) f_n(x),$$

where $T_p^{(i)}(x)$ are polynomials on $x$ of degree $p$ and $p - 1$ for $i = 0$ and $2 \leq i \leq d + 1$ respectively and where

$$\begin{align*}
T_0^{(1)} &= 1, \quad T_0^{(i)}(x) = 0, \quad 1 \leq i \leq d + 1, \\
T_1^{(1)}(x) &= A_n x + B_n, \quad T_1^{(i)}(x) = \gamma_{n+d-1}^{d+1-i}, \quad 2 \leq i \leq d + 1, \\
T_2^{(1)}(x) &= (A_{n+1} x + B_{n+1}) (A_n x + B_n) + \gamma_{n+d}^{d-1}, \\
T_2^{(i)}(x) &= (A_{n+1} x + B_{n+1}) \gamma_{n+d+2-i}^{d+1-i} + \gamma_{n+d+3-i}^{d-i}, \quad 2 \leq i \leq d + 1, \\
&\vdots
\end{align*}$$

Then we can prove the following

Theorem 4.12. The polynomials $T_p^{(i)}(x)$, $1 \leq i \leq d + 1$, appeared in (4.11) are also d-OPS. Moreover, they satisfy the following identity

$$|T_p^{(1)} T_p^{(2)} \ldots T_p^{(d+1)}|^T = \Delta_n^{-1} \Delta_{n+p} \neq 0.$$  

Proof. It follows from (4.11), that

$$\begin{align*}
|S_n^{(1)} \ldots S_n^{(d+1)}|^T = |S_n^{(1)} \ldots S_n^{(d+1)}|^T T_p^{(1)} T_p^{(2)} \ldots T_p^{(d+1)}|^T.
\end{align*}$$

Hence

$$\Delta_n^{-1} |T_p^{(1)} T_p^{(2)} \ldots T_p^{(d+1)}|^T = \Delta_{n+p}.$$  

Then, lemma 4.11 completes the proof. □
Notice that for \( d = 1 \), a sequence of quasi-orthogonal polynomials satisfies a three term recurrence relation with polynomial coefficients [24]. Furthermore, any linear combination in \( l \) term \((l > d)\) of a d-OPS with constant coefficients, could be expressed as a linear combination in terms of only \( d + 1 \) term with polynomial coefficients. This attempt treated by Joulak in \([45]\). For a given d-OPS \( \{P_n\}_{n \geq 0} \) defined by the recurrence \((2.3)\), let us consider the following linear combination

\[
Q_n(x) = P_n(x) + a^{(1)}_n P_{n-1}(x) + \ldots + a^{(r)}_n P_{n-r}(x), \quad n \geq 1.
\]

If \( r \) is multiple of \( d \), the sequence \( \{Q_n\} \) is called d-quasi-orthogonal.

**Proposition 4.13.** For \( r > d \), the polynomials sequence \( \{Q_n\} \) defined by \((4.13)\) might be given in the following form

\[
Q_n(x) = U_{r-1} P_{n-r+1} + \left[ a^{(r)}_n - \sum_{i=0}^{d-1} \gamma^{d-1-i} U_{r-2-i} \right] P_{n-r} - \left[ \sum_{i=0}^{d-2} \gamma^{d-2-i} U_{r-2-i} \right] P_{n-r-1} - \left[ \sum_{i=0}^{d-3} \gamma^{d-3-i} U_{r-2-i} \right] P_{n-r-2} - \ldots - \gamma^0_{n-r-d+2} U_{r-2} P_{n-r-d+1},
\]

where

\[
U_r - a^{(r)}_n = (x - \beta_{n-r}) U_{r-1} - \sum_{\nu=0}^{d-1} \gamma^{d-1-\nu} U_{r-2-i},
\]

with \( U_0 = 1 \) and \( U_s \equiv 0 \) for \( s \geq 1 \).

In the above expansion the polynomials \( \{U_n\} \) are d-OPS iff \( a^{(i)}_n \equiv 0 \), \( 1 \leq i \leq r \). In this case, \( \{P_n = Q_n\} \) is also d-OPS (quasi-orthogonal of order zero).

5. Christoffel-Darboux type formulas

Our next wishes are to give some formulas of Christoffel-Darboux type. Let us first notice that, for the determinant \( F^{(r)}_n \) above \((4.9)\), when \( m_i = n + i \) for \( 1 \leq i \leq d - 1 \), and if we replace \( m_{d} \) by \( m - i \), we readily get the following identities

\[
\left| P^r_{n} \ldots P^r_{n+d-1} P^r_{m-i} \right|^T = \Delta^{(r)}_n P^{(n+r+d)}_{m-n-i-d},
\]

and in a similar way, also using the dual recurrence relation \((4.3)\), we have

\[
\left| P^r_{n} \ldots P^r_{n+d-2} P^r_{n+d} P^r_{m-i} \right|^T = \Delta^{(r)}_n \left[ P^{(n+r+d-1)}_{m-n-i-d} P^{(n+r+d)}_{m-n-i-d} - P^{(n+r+d-1)}_{m-n-i-d+1} \right] = \Delta^{(r)}_n \left[ \sum_{j=1}^{d} \gamma^{d-j}_{r+n+d} P^{(n+r+d+j)}_{m-n-d-i-j} \right].
\]

Now we want to give a motivation of these latter identities. Especially when \( d = 2 \) we can give further new type of Christoffel-Darboux formula. Indeed, the next results, given by theorem 5.1 and corollary 5.2, are established for \( d = 2 \). In this particular case, we are able to prove the following formula.
**Theorem 5.1.** For any integers \( k > m > n \geq 0 \) and \( r \geq 0 \), we have

\[
H_m := |P_n^r P_m^r P_k^r| = \begin{vmatrix} P^{(r)}_n & P^{(r+1)}_{n-1} & P^{(r+2)}_{n-2} \\ P^{(r)}_m & P^{(r+1)}_{m-1} & P^{(r+2)}_{m-2} \\ P^{(r)}_k & P^{(r+1)}_{k-1} & P^{(r+2)}_{k-2} \end{vmatrix}
\]

(5.1)

\[
= \Delta_n^{(r)} \begin{bmatrix} P^{(m+r+1)}_{k-m-1} \\ P^{(m+r+1)}_{m-n} \\ P^{(m+r+1)}_{m-n} \\ P^{(m+r+1)}_{m-n} \\ \end{bmatrix} + \gamma^0_{m+r} P^{(m+r+2)}_{m-n-1} + \gamma^0_{m+r} P^{(m+r+2)}_{m-n-1} + \gamma^0_{m+r} P^{(m+r+2)}_{m-n-1} + \gamma^0_{m+r} P^{(m+r+2)}_{m-n-1} + \gamma^0_{m+r} P^{(m+r+2)}_{m-n-1}
\]

Proof. When \( d = 2 \), since \( k > m \), then expanding \( P^{(r)}_k, P^{(r+1)}_{k-1} \) and \( P^{(r+2)}_{k-2} \) by means of proposition 4.1 in the following forms

\[
P^{(r+i)}_{k-i} = P^{(m+r)}_{k-m} P^{(r+i)}_{m-i} - \gamma^0_{m+r} P^{(m+r+1)}_{k-m-1} P^{(r+i)}_{m-i-1} - \gamma^1_{m+r} P^{(m+r+1)}_{k-m-1} P^{(r+i)}_{m-i-1},
\]

(5.2)

for \( i = 0, 1, 2 \). Then replacing each polynomials in the bottom row of \( H_m \) by the corresponding recurrence from (5.2), and by theorem 4.4, we get

\[
H_m = \left( \gamma^1_{m+r} P^{(m+r+1)}_{k-m-1} + \gamma^0_{m+r} P^{(m+r+2)}_{k-m-2} \right) \Delta_n^{(r)} \begin{bmatrix} P^{(r+n+1)}_{m-n-3} & P^{(r+n+2)}_{m-n-4} \\ P^{(r+n+1)}_{m-n-3} & P^{(r+n+2)}_{m-n-4} \end{bmatrix}.
\]

Next the following formula completes the proof

\[
\begin{bmatrix} P^{(r+n+1)}_{m-n-1} & P^{(r+n+2)}_{m-n} \\ P^{(r+n+1)}_{m-n} & P^{(r+n+2)}_{m-n} \end{bmatrix} = \gamma^1_{m+r} \begin{bmatrix} P^{(r+n+1)}_{m-n-2} & P^{(r+n+2)}_{m-n-3} \\ P^{(r+n+1)}_{m-n-2} & P^{(r+n+2)}_{m-n-3} \end{bmatrix} + \gamma^0_{m+r} \begin{bmatrix} P^{(r+n+1)}_{m-n-1} & P^{(r+n+2)}_{m-n} \\ P^{(r+n+1)}_{m-n} & P^{(r+n+2)}_{m-n} \end{bmatrix}.
\]

\[\square\]

We are still working with the case \( d = 2 \), now we shall give a first Christoffel-Darboux type formula. By setting

\[
J_m = \left( -1 \right)^m \prod_{i=1}^{m} \gamma^0_{i+m} \left( -1 \right) \begin{bmatrix} P^{(r+n+1)}_{m-n-1} & P^{(r+n+2)}_{m-n-2} \\ P^{(r+n+1)}_{m-n} & P^{(r+n+2)}_{m-n-1} \end{bmatrix},
\]

we get

\[
\left( -1 \right)^m \prod_{i=1}^{m} \gamma^0_{i+m} H_m = \left( -1 \right)^n \prod_{i=1}^{n} \gamma^0_{i+r} \begin{bmatrix} P^{(m+r+1)}_{k-m-1} J_m - P^{(m+r+2)}_{k-m-2} J_{m-1} \end{bmatrix},
\]

with \( J_n = 0 \). In addition, since \( H_n = 0 \), we conclude that
Corollary 5.2. The following relation holds true for any integers \( k > m > n \geq 0 \) and \( r \geq 0 \)

\[
\sum_{v=n+1}^{m} \left( -1 \right)^{v} \prod_{l=1}^{v} \gamma_{l+v} \left( -1 \right)^{-1} \left| \begin{array}{c}
\sum_{r=0}^{(r+n+1)} \left| \begin{array}{c}
P_{m-n-1}^{(m+r+1)} \left( x_{d} \right) \end{array} \right|
\end{array} \right|
\]

We believe that there exist generalizations of (5.1) as well as of (5.3) for \( d \geq 3 \). Of course, it may be difficult to explicitly compute them for any \( d \geq 3 \) in this direction. Whereas, it seems that the above formula might be affords an alternative way of understood of the connection between the polynomials \( \{K_{n}\} \) and \( \{L_{n}(.;c)\} \) appeared in section 7.

We end this section by a somewhat more generalization of Christoffel-Darboux formula. From the following recurrences

\[
\begin{align*}
\text{(S)} & \\
x_{1}P_{n+d-1}^{(r)}(x_{1}) & = P_{n+d}^{(r)}(x_{1}) + \beta_{n+r+d-1}P_{n+d-1}^{(r)}(x_{1}) + \ldots + \gamma_{n+r}^{0}P_{n-1}^{(r)}(x_{1}) , \\
x_{2}P_{n+d-2}^{(r+1)}(x_{2}) & = P_{n+d-1}^{(r+1)}(x_{2}) + \beta_{n+r+d-1}P_{n+d-2}^{(r+1)}(x_{2}) + \ldots + \gamma_{n+r}^{0}P_{n-2}^{(r+1)}(x_{2}) , \\
& \vdots \\
x_{d+1}P_{n-1}^{(r+d)}(x_{d+1}) & = P_{n}^{(r+d)}(x_{d+1}) + \beta_{n+r+d-1}P_{n-1}^{(r+d)}(x_{d+1}) + \ldots + \gamma_{n+r}^{0}P_{n-d-1}^{(r+d)}(x_{d+1})
\end{align*}
\]

we get

\[
\left( \Delta_{n}^{(r)} \right)^{-1} \left| \begin{array}{cccc}
\mathbb{P}_{n-1}^{(r)}(x_{1}) & \mathbb{P}_{n-1}^{(r+1)}(x_{2}) & \ldots & \mathbb{P}_{n-d-1}^{(r+d)}(x_{d+1}) \\
x_{1}P_{n+d-1}^{(r)}(x_{1}) & x_{2}P_{n+d-2}^{(r+1)}(x_{2}) & \ldots & x_{d+1}P_{n-1}^{(r+d)}(x_{d+1}) \\
\end{array} \right|^T = I_{n} - I_{n-1},
\]

where

\[
I_{n} = \left( \Delta_{n}^{(r)} \right)^{-1} \left| \begin{array}{cccc}
\mathbb{P}_{n}^{(r)}(x_{1}) & \mathbb{P}_{n-1}^{(r+1)}(x_{2}) & \ldots & \mathbb{P}_{n-d}^{(r+d)}(x_{d+1}) \\
\end{array} \right|^T ,
\]

since \( I_{0} = 1 \), we have the following generalized Christoffel-Darboux type formula

\[
\sum_{v=1}^{n} \Delta_{n}^{(r)} \left| \begin{array}{cccc}
\mathbb{P}_{v-1}^{(r)}(x_{1}) & \mathbb{P}_{v-1}^{(r+1)}(x_{2}) & \ldots & \mathbb{P}_{v-d-1}^{(r+d)}(x_{d+1}) \\
x_{1}P_{v+d-1}^{(r)}(x_{1}) & x_{2}P_{v+d-2}^{(r+1)}(x_{2}) & \ldots & x_{d+1}P_{v-1}^{(r+d)}(x_{d+1}) \\
\end{array} \right|^T
\]

This is a generalization of the formula given in [54] as well as of the formula in [11, Prop. 2.10]. Notice also that when \( x_{1} = x_{2} = \ldots = x_{d+1} \), we find the identity (4.8).

In the remainder of this section we use the notation \( X^{[n]} \) to indicate the \( n^{th} \) derivative of \( X \).

Now, replace in the system (S) the recurrence of \( P_{n+d-i}^{(r)}(x_{i+1}) \) by that of \( P_{n+d}^{(r)}(x_{i+1}) \) for \( 0 \leq i \leq d \) respectively. Next taking the \( (i-1)^{th} \) derivative of the \( i^{th} \) equation, and eliminate
the coefficients $\beta_n$ and $\gamma_n$. Then after dividing by $\Delta_n^{(r)}$ and taking sum, we obtain

$$\sum_{v=1}^{n} \frac{\Delta_n^{(r)}}{\Delta_v^{(r)}} \begin{bmatrix} \mathbb{P}_{v-1}^{(r)}(x_1) & [\mathbb{P}_{v-1}^{(r)}(x_2)]' & \cdots & [\mathbb{P}_{v-1}^{(r)}(x_d)]' \\ \vdots & \vdots & \ddots & \vdots \\ x_1 P_{v+d-1}^{(r)}(x_1) & x_2 P_{v+d-1}^{(r)}(x_2) & \cdots & x_{d+1} P_{v+d-1}^{(r)}(x_{d+1}) \end{bmatrix}^T = \left| \begin{array}{c} \mathbb{P}_n^{(r)}(x_1) \\ \vdots \\ \mathbb{P}_n^{(r)}(x_d) \end{array} \right|^T - \frac{d}{k=1} k! \Delta_n^{(r)}.$$

and when $x_1 = \ldots = x_{d+1}$ we get

$$\sum_{v=1}^{n} \frac{\Delta_n^{(r)}}{\Delta_v^{(r)}} \begin{bmatrix} \mathbb{P}_v^{(r)}(x_1) & [\mathbb{P}_v^{(r)}(x_2)]' & \cdots & [\mathbb{P}_v^{(r)}(x_d)]' \\ \vdots & \vdots & \ddots & \vdots \\ 0 & P_{v+d-1}^{(r)} & \cdots & P_{v+d-1}^{(r)} \end{bmatrix}^T = \left| \begin{array}{c} \mathbb{P}_n^{(r)}(x_1) \\ \vdots \\ \mathbb{P}_n^{(r)}(x_d) \end{array} \right|^T - \frac{d}{k=1} k! \Delta_n^{(r)}.$$

Similarly by taking in the system $(S)$, the $(i-1)^{th}$ derivative of the $i^{th}$ equation we obtain, using the similar approach above, the following

$$\sum_{v=1}^{n} \frac{\Delta_n^{(r)}}{\Delta_v^{(r)}} \begin{bmatrix} \mathbb{P}_v^{(r)}(x_1) & [\mathbb{P}_v^{(r+1)}(x_2)]' & \cdots & [\mathbb{P}_v^{(r+d)}(x_{d+1})]' \\ \vdots & \vdots & \ddots & \vdots \\ x_1 P_{v+d-1}^{(r)}(x_1) & x_2 P_{v+d-1}^{(r+1)}(x_2) & \cdots & x_{d+1} P_{v+d-1}^{(r+d)}(x_{d+1}) \end{bmatrix}^T = \left| \begin{array}{c} \mathbb{P}_n^{(r+1)}(x_1) \\ \vdots \\ \mathbb{P}_n^{(r+d)}(x_d) \end{array} \right|^T.$$

and when $x_1 = \ldots = x_{d+1}$ we infer that

$$\sum_{v=1}^{n} \frac{\Delta_n^{(r)}}{\Delta_v^{(r)}} \begin{bmatrix} \mathbb{P}_v^{(r+1)}(x_1) & [\mathbb{P}_v^{(r+1)}(x_2)]' & \cdots & [\mathbb{P}_v^{(r+d)}(x_{d+1})]' \\ \vdots & \vdots & \ddots & \vdots \\ 0 & P_{v+d-2}^{(r+1)} & \cdots & P_{v+d-1}^{(r+d)} \end{bmatrix}^T = \left| \begin{array}{c} \mathbb{P}_n^{(r+1)}(x_1) \\ \vdots \\ \mathbb{P}_n^{(r+d)}(x_d) \end{array} \right|^T.$$

6. Darboux transformations

This section deals with $LU$ as well as $UL$ decomposition of the Jacobi matrix $J_d$. A motivation of this decomposition comes out in the study of Kostant-Toda lattice [6] where operators in the commutator are banded matrices. The authors show further that the matrix $L$ could be written as a product of $d$ bi-diagonal matrices $L = L_1 L_2 \ldots L_d$ with a full description in case $d = 2$. Moreover, two of the authors considered latter $d$ Darboux transformations of $J_d$ and defined $d$ new matrices through

$$J_d^{(0)} = J_d, \quad J_d^{(i)} = L_{i+1} \ldots L_d U L_1 \ldots L_i + \lambda I, \text{ for } i = 1, 2, \ldots, d.$$

Therein, they have shown that the above transformations generate $d$ solutions denoted $\{P_n^{(i)}\}$ of $(d + 2)$-term recurrence relation, i.e. that is to say, any circular permutation between the matrices $L_i$ for $1 \leq i \leq d$ and the matrix $U$ brings forth another solution of the recurrence
have obtained the following connection
\[ L_{j+1} L_{j+2} \ldots L_j v^{(i)}(z) = v^{(j)}(z), \quad 0 \leq j < i \leq d, \]
\[(6.2)\]
where \( l^{(i)}_{(d+1)m+i+2} \) are the entries at position \((m + 1, m)\) of the matrix \( L_i \). We would like to point out that the latter transformations have also been investigated in [12].

The \( d \)-orthogonality of the above polynomials gives evidence of the following question: what kind are these polynomials? We back to the above recursion many times in the next two sections, we establish that the above polynomials are in fact the \((d+1)\)-decomposition of some \( d \)-symmetric sequence.

Let us denote the matrices \( U \) and \( L \) as follows
\[(6.3)\]
\[
U = \begin{pmatrix} m_1 & 1 \\ m_2 & 1 \\ \vdots & \vdots \\ \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 1 \\ \vdots & \vdots \\ l_{d1} & \cdots & l_{dd} & 1 \\ 0 & \cdots & \end{pmatrix}
\]

First let us express the matrix \( J_d \) as the product of \( U \) times \( L \). We have the following results generalizing those in [22]

**Proposition 6.1.** Let \( \{P_n(x)\}_{n \geq 0} \) be \( d \)-OPS defined by the \( J_d \) given in (2.10). Assume that \( P_n(0) \neq 0, \ n \geq 1 \). Then, for the LU decomposition of the matrix \( J_d \), we have
\[(6.4)\]
\[
m_1 = \beta_0, \\
m_n = \beta_{n-1} - l_{n-1,n-1}, \quad \text{for } n \geq 2, \\
l_{n+d-1,n} = \gamma_0^d / m_n, \quad \text{for } n \geq 1, \\
l_{n+i,n} + l_{n+i,n+1} m_{n+1} = \gamma_0^{d-i}, \quad \text{for } 1 \leq i \leq d,
\]
where the elements \( l_{n,k} \) can be computed recursively in the following manner
\[(6.5)\]
\[
l_{i,1} m_1 = \gamma_1^{d-i}, \quad \text{for } 1 \leq i \leq d, \\
l_{n+i,n+1} = \gamma_0^{d-i} / m_{n+1} - l_{n+i,n} / m_{n+1}, \quad \text{for } 1 \leq i \leq d.
\]

Moreover, we have
\[(6.6)\]
\[
m_n = -P_n(0) / P_{n-1}(0).
\]

**Proof.** The product of \( L \) times \( U \) gives
\[
\beta_0 = m_1, \\
\gamma_1^{d-i} = l_{i,1} m_1, \\
\gamma_0^{d-i} = l_{n+i,n} + l_{n+1,n+1} m_{n+1},
\]
whence the recursions (6.4)-(6.5).

The equality (6.6) can be checked by induction on \( n \). Since \( P_1(0) = -\beta_0 \), then
\[
m_1 = \beta_0 = -P_1(0) / P_0(0).
\]
Assume that \( m_k = -P_k(0)/P_{k-1}(0) \) for \( k \leq n \). Then from the recurrence relation (2.3) we get
\[
P_{n+1}(0) = -\beta_n P_n(0) - \gamma_{n-1}^d P_{n-1}(0) - \ldots - \gamma_{n-d+1}^0 P_{n-d}(0),
\]
hence
\[
-P_{n+1}(0)/P_n(0) = -\beta_n + \frac{\gamma_{n-1}^d}{P_n(0)/P_{n-1}(0)} + \ldots + \frac{\gamma_{n-d+1}^0}{[P_n(0)/P_{n-1}(0)] \ldots [P_{n-d+1}(0)/P_{n-d}(0)]},
\]
using the induction hypothesis as well as (6.5), we infer that
\[
-P_{n+1}(0)/P_n(0) = -\beta_n - \frac{\gamma_{n-1}^d}{m_n} + \ldots + (-1)^{n+d-1} \frac{\gamma_{n-d+1}^0}{m_n m_{n-1} \ldots m_{n-d+1}}.
\]
Now, from the first equality in (6.5), we remark that we can write the last two terms as
\[
(-1)^{n+d-2} \left[ \frac{\gamma_{n-d+2}^1 - \gamma_{n-d+1}^0}{m_n m_{n-1} \ldots m_{n-d+2}} \right] = (-1)^{n+d-2} \frac{\gamma_{n-d+2}^1 - \gamma_{n-d+1}^0}{m_n m_{n-1} \ldots m_{n-d+2}} = (-1)^{n+d-2} \frac{\gamma_{n-d+2}^1 - \gamma_{n-d+1}^0}{m_n m_{n-1} \ldots m_{n-d+2}} = (-1)^{n+d-2} \frac{\gamma_{n-d+2}^1 - \gamma_{n-d+1}^0}{m_n m_{n-1} \ldots m_{n-d+2}}.
\]
By induction we get at end
\[
m_{n+1} = -\frac{P_{n+1}(0)}{P_n(0)} = -\beta_n - \left[ \frac{\gamma_{n-1}^d}{m_n} - \frac{l_{n,n-1}}{m_n} \right] = \beta_n - l_{n,n}.
\]

Now, for the UL decomposition we have

\textbf{Proposition 6.2.} Assume that \( J_d = UL \) denotes the UL factorization of the Jacobi matrix \( J_d \). We have for \( 1 \leq j \leq d \), the following initial conditions
\[
(6.7) \quad l_{jj} = \beta_{j-1} - \mu_{j-1},
\]
\[
l_{ji} = \gamma_i^{d-j+i} - \eta_i^{d-j+i}, \quad \text{for} \quad 1 \leq i \leq j - 1,
\]
where
\[
(6.8) \quad \mu_{j-1} = m_j, \quad \text{for} \quad j \geq 1,
\]
\[
\eta_i^{d-j+i} = m_j l_{j-1,i}, \quad \text{for} \quad 1 \leq i \leq j - 1 \leq d - 1,
\]
are free parameters, and for \( n \geq d + 1 \), the following
\[
(6.9) \quad m_{d+n} = \gamma_i^0 / l_{d+n-1,n}, \quad \text{for} \quad n \geq 1,
\]
\[
l_{d+n,i} = \gamma_i^{l_n - n} - m_{d+n} l_{d+n-1,i}, \quad \text{for} \quad n + 1 \leq i \leq n - 2.
\]
In addition, the free parameters \( \mu_i \) and \( \eta_i \) define a new sequence of co-recursive polynomials which can be used to determine \( l_{ij} \). Furthermore, for \( 1 \leq n \leq d \) we have the following
\[
(6.10) \quad -l_{nn} = Q_1^{(n-1)}(0) = \mu_{n-1} - \beta_{n-1},
\]
\[
-\frac{l_{n,n-1}}{l_{n,n}} = Q_2^{(n-2)}(0) + l_{nn} Q_1^{(n-2)}(0),
\]
\[
\vdots
\]
\[
-\frac{l_{n,n+1-d}}{l_{n,n}} = Q_d^{(n-d)}(0) + l_{nn} Q_{d-1}^{(n-d)}(0) + \ldots + l_{n,n+2-d} Q_1^{(n-d)}(0),
\]
and the recursion (6.9) for \( n \geq d + 1 \).

**Proof.** The product of \( U \) times \( L \) gives, for \( 1 \leq j \leq d \)

\[
\beta_{j-1} = l_{jj} + m_j,
\]

\[
\gamma_{j}^{d-j+i} = l_{ji} + m_j l_{j-1,i}, \text{ with } 1 \leq i \leq j - 1,
\]

which shows that there are exactly \( d \) free parameters \( \{m_j\}_{1 \leq j \leq d-1} \), and when \( n = d + 1 \), we get the following

\[
m_{d+1} = \gamma_1^0/\ell_d, \quad l_{d+1,i} = \gamma_i^{d-1} - m_{d+1}\ell_i, \text{ for } 2 \leq i \leq d.
\]

For \( n \geq d + 1 \), we get (6.9).

The proof of (6.10) follows readily by combining the co-recursive’s recurrence relation and the associated polynomials. Notice that this is just a simple idea on how to compute the coefficients \( l_{n,k} \). Indeed, let us denote the sequence of co-recursive polynomials generated by perturbing the recurrence of \( \{P_n\} \) through the free parameters \( \mu_i \) and \( \eta_i \) by \( \{Q_n\} \). In this case, we have for \( 1 \leq n \leq d \)

\[
-Q_n(0) = l_{nn}Q_{n-1}(0) + l_{n,n-1}Q_{n-2}(0) + ... + l_{n,n+1-d}Q_{n-d}(0) + + \gamma_{n-d}^{0}Q_{n-d-1}(0).
\]

Now we determine the coefficients \( l_{n,i} \). Remark first that

\[
-Q_1^{(n-1)}(0) = \beta_{n-1} - \mu_{n-1} = l_{n,n},
\]

and also

\[
-Q_2^{(n-2)}(0) = l_{n,n}Q_1^{(n-2)}(0) + l_{n,n-1}Q_0^{(n-2)}(0),
\]

hence the proof follows by induction on \( n \). \( \square \)

### 7. Kernel polynomials and quasi-orthogonality

The following question indicates just how little we know about kernel polynomials.

Darboux transformation allows us to deduce a new family of polynomials for which our sequence \( \{P_n\} \) is \( d \)-quasi-orthogonal of order one [65]. For the usual orthogonality, the expressions given in the following proposition define Kernel polynomials from the quasi-orthogonality’s point of view. Furthermore, we have the following result

**Proposition 7.1.** Let \( \{P_n\} \) be a \( d \)-OPS and \( J_d \) the corresponding Jacobi matrix, and \( \{K_n\} \) denotes the sequence of polynomials generated by \( J_d^T = UL \). Then

\[
P_n = K_n + l_{n,n}K_{n-1} + l_{n,n-1}K_{n-2} + ... + l_{n,n-d+1}K_{n-d}, \quad n \geq 0
\]

and

\[
x K_n(x) = P_{n+1}(x) - \frac{P_{n+1}(0)}{P_n(0)} P_n(x), \quad n \geq 0.
\]

**Proof.** Define a monic polynomials sequence \( \{R_n\} \) by

\[
R_{n+1} = P_{n+1} + m_{n+1}P_n.
\]

Then,

\[
x \mathbb{P} = J_d \mathbb{P} = UL \mathbb{P} = L (R_1, R_2, ...)^T,
\]

that is

\[
x P_n = R_{n+1} + l_{n,n}R_n + l_{n,n-1}R_{n-1} + ... + l_{n,n-d+1}R_{n-d+1}.
\]
Remark that $R_n (0) = 0$ because at least $l_{n,n-d+1} \neq 0$. That is, $R_n (x) = xS_{n-1} (x)$ whence (7.1). On the other hand, we get
\[xS = U \mathbb{P} = UL \mathbb{S}\]
which means that the sequence $\{S_n\}$ is d-OPS corresponding to the Darboux transformation $J^d = UL$, i.e., $\{S_n\} = \{K_n\}$. \[\square\]

The expression (7.1) means that $\{P_n\}$ is d-quasi-orthogonal of order one with respect to the corresponding vector linear form of $\{K_n\}$ (2.9) [65]. In the usual orthogonality, recurrences (7.1) and (7.2) reduce respectively to the formulas (9.5) and (9.4) in [26, p.45] (see also exercise 9.6 p.49). Then these define kernel polynomials for the d-orthogonality sense.

Notice also that the polynomials generated by Darboux transformations $J_d$ and $J^d$ can be related through the matrix of change of basis $L$ in the form $\mathbb{P} = L \mathbb{K}$.

On the other hand, it is obvious that the recurrence of kernel polynomials $\{K_n\}$ as well as of $\{P_n\}$ could be extremely determined using only the two recurrences (7.1)-(7.2). Indeed, suppose that $J_d = UL$ and define $\mathbb{P} = L \mathbb{K}$, i.e., that is by (7.1). Then
\[x \mathbb{K} = UL \mathbb{K} = U \mathbb{P}\]
hence
\[L U \mathbb{P} = x L \mathbb{K} = x \mathbb{P}\]
which means that $\{P_n\}$ is d-OPS generated by $LU$. Using once again the recurrence (7.1)-(7.2) we get
\[x P_n = P_{n+1} + (l_{nn} + m_{n+1}) P_n + \sum_{i=0}^{d-2} (l_{n,n-1-i} + l_{n,n-i}m_{n-i}) P_{n-i-1} + l_{n,n-d+1}m_{n-d+1}P_{n-d}\]
and
\[x K_n = K_{n+1} + (l_{nn+1} + m_{n+1}) K_n + \sum_{i=0}^{d-2} (l_{n,n-i+1} + l_{n,n-i+1}m_{n+1}) K_{n-i-1} + l_{n,n-d+1}m_{n+1}K_{n-d}\]
then according to propositions 6.1 and 6.2 we have respectively the recurrence of $\{P_n\}$ as well as that of $\{K_n\}$.

To determine the dual sequence of $\{K_n\}$ which we denote by $\mathcal{V} = (v_0, ..., v_{d-1})^T$, we use the d-quasi-orthogonality as it was already pointed out in [65]. Indeed, since
\[\langle v_r, P_n \rangle = \langle v_r, k_n \rangle + l_{n,n} \langle v_r, K_{n-1} \rangle + ... + l_{n,n-d+1} \langle v_r, K_{n-d} \rangle = 0, \quad n \geq r + d + 1,\]
\[\langle v_r, P_{r+d} \rangle = l_{r+d,r+1} \langle v_r, K_r \rangle \neq 0,\]
then, there exists $r \leq t_r \leq r + d$ such that
\[\langle v_r, P_n \rangle = 0, \quad n \geq t_r + 1,\]
\[\langle v_r, P_{t_r} \rangle \neq 0.\]

According to theorem 2.6, there exist $d$ polynomials $\phi_r^\mu$, $0 \leq r, \mu \leq d - 1$, such that
\[v_r = \sum_{\mu=0}^{d-1} \phi_r^\mu u_\mu.\]
Set \( t_r = q_r d + p_r, \) \( 0 \leq p_r \leq d - 1 \). Since \( r \leq t_r \leq r + d \) and \( p_r \leq d - 1 \), then \( q_r \leq 1 \). Further, if \( q_r = 1 \), then \( p_r \leq r \). Hence, the above expression of \( v_r \) takes the following form

\[
(7.3) \quad v_r = \sum_{i=0}^{r} (a^i_r x - b^i_r) u_i - \sum_{j=r+1}^{d-1} b^j_r u_j.
\]

Now, applying both sides of \( v_r \) recursively on the polynomials \( P_0, ..., P_{d-1} \), and making use of (2.7), we obtain the following expressions

\[
(7.4) \quad b^0_r = a^0_r \beta_0 - 1, \quad b^0_r = a^0_r \beta_0 + a^1_r, \quad \text{for } 1 \leq r \leq d - 1,
\]

\[
\begin{align*}
& b^r_r = a^r_0 \gamma^d_{d-r} + a^1_r \gamma^d_{d-r+1} + \ldots + a^{r-1}_r \gamma^d_{d-1} + a^r_r \beta_r - 1, \quad \text{for } 1 \leq r \leq d - 1, \\
& b^r_r = a^r_0 \gamma^d_{d-i} + a^1_r \gamma^d_{d-i+1} + \ldots + a^{r-1}_r \gamma^d_{d-1} + a^r_r \beta_i + a^i_r, \quad \text{for } 1 \leq i < r \leq d - 1, \\
& b^r_r = a^r_0 \gamma^d_{d-i} + a^1_r \gamma^d_{d-i+1} + \ldots + a^r_r \gamma^d_{d-i+r} - l_i, \quad \text{for } 1 \leq r < i \leq d - 1.
\end{align*}
\]

The \( a_i \)'s are easily obtained from the following

\[
\langle v_r, P_{d+i} \rangle = l_{d+i, r+1} = a^0_r \gamma^d_{d-i+1} + a^1_r \gamma^d_{d-i+2} + \ldots + a^r_r \gamma^d_{d-i+1}, \quad \text{for } 0 \leq i \leq r.
\]

It may be worthwhile to consider analogous problems of the usual orthogonality in which kernel polynomials appeared as particular or as a solution of the whole problem. It could be then of interest to study Uvarov modification of the measure since the regularity as well as the recurrence coefficients corresponding to the new sequence are expressed in terms of kernel polynomials. For this end, let us consider the following problem.

Given a \( d \)-OPS \( \{ P_n \} \) with respect to some regular \( U = (u_0, \ldots, u_{d-1})^T \), and define a new vector form \( V = (v_0, \ldots, v_{d-1})^T \) as

\[
(7.5) \quad v_r = u_r + \lambda \delta_r, \quad 0 \leq r \leq d - 1.
\]

When \( V \) is regular, we denote the corresponding \( d \)-OPS by \( \{ Q_n \} \).

Now suppose that \( V \) is regular and write

\[
Q_m = P_m + \sum_{i=0}^{m-1} a_{m,i} P_i.
\]

Then, by taking \( m = dn + k \) with \( 0 \leq k \leq d - 1 \), we get

\[
\langle v_r, Q_{dn+r} \rangle = \begin{cases} 
\langle v_r, P_{dn+r} \rangle = \prod_{j=1}^{n} \gamma_{d(j-1)+r+1}^0, & \text{for } j = n, \\
\prod_{j=1}^{n} \gamma_{d(j-1)+r+1}, & \text{for } 0 \leq j \leq n - 1,
\end{cases}
\]

and by (7.5) we also have

\[
\langle u_r, Q_{dn+r} \rangle = \begin{cases} 
\langle v_r, Q_{dn+r} \rangle - \lambda Q_{dn+r}(c) P_n(c), & \text{for } j = n, \\
-\lambda Q_{dn+r}(c) P_j(c), & \text{for } 0 \leq j \leq n - 1.
\end{cases}
\]

Accordingly, we obtain

\[
(7.6) \quad Q_{dn+k}(x) = P_{dn+k}(x) - \lambda Q_{dn+k}(c) L_{dn+k-1}(x; c),
\]
where $L_{n}(x;.)$ is the polynomial defined as

\begin{equation}
L_{dn+k}(x; c) = \sum_{j=0}^{n-1} P_{j}(c) \left\{ \sum_{r=0}^{d-1} \frac{P_{j+r}(x)}{\langle u_r, P_{d+r} \rangle} \right\} + P_{n}(c) \left\{ \sum_{r=0}^{k} \frac{P_{dn+r}(x)}{\langle u_r, P_{dn+r} \rangle} \right\}.
\end{equation}

Set $x = c$ in (7.6) to get

\[ Q_{dn+k}(c) \left[ 1 + \lambda L_{dn+k-1}(c; c) \right] = P_{dn+k}(c), \]

where necessarily $1 + \lambda L_{dn+k-1}(c; c) \neq 0$, otherwise we also have $P_{dn+k}(c) = 0$. It now follows by induction that $c$ should be a common zero for more than $d$ consecutive polynomials of the sequence $\{P_n\}$ which is impossible (see corollary 9.5 below). Hence

\[ Q_{dn+k}(c) = P_{dn+k}(c) \left[ 1 + \lambda L_{dn+k-1}(c; c) \right]^{-1} \]

and then

\begin{equation}
Q_{m}(x) = P_{m}(x) - \frac{P_{m}(c)}{1 + \lambda L_{m-1}(c; c)} L_{m-1}(x; c).
\end{equation}

While it is obviously seen that the polynomials $L_{n}(x; c)$ given by the formula (7.7) are exactly kernel polynomials in the case $d = 1$, it is not yet so clear how this sequence is connected to the polynomials $K_{n}$ appeared in proposition 7.1. Moreover, the correct expression of kernel polynomials as a sum in terms of $\{P_n\}$ that provides such more general properties is not yet come over. In the usual orthogonality ($d = 1$), the polynomials $L_{n}(x; c)$ are extremely used to characterize the regularity of the corresponding linear form $\mathcal{V}$, they were never the main heroes here, but we leave regularity questions of the Uvarov transform outside of the scope of this paper.

Let us now focus on the d-quasi-orthogonality appeared in the proposition 7.1. In fact, the above results (on the d-quasi-orthogonality), allow us to extract another characterization of the d-quasi-orthogonality. Actually, the d-quasi-orthogonality between two sequences of d-OPS subject to some conditions [65].

**Theorem 7.2.** Let $\{P_{n}\}$ and $\{Q_{n}\}$ be two d-OPS with respect to $\mathcal{U}$ and $\mathcal{V}$ respectively. Then $\{P_{n}\}$ is d-quasi-orthogonal of order $l$ with respect to $\mathcal{V}$ if and only if there exist polynomial $\pi_{l}$ of degree $l$ and complex numbers $b_{n,k}$ such that the following formula holds true

\begin{equation}
\pi_{l}(x)Q_{n}(x) = b_{n+l,0}P_{n+l}(x) + b_{n+l,1}P_{n+l-1}(x) + \ldots + b_{n+l,t}P_{n}(x).
\end{equation}

**Proof.** Suppose that we have the formula (7.9), then from the fact that $\{Q_{n}\}$ is d-OPS with respect to $\mathcal{V}$, we get

\[ \langle v_{r}, \pi_{l}Q_{n} \rangle = b_{n+l,0} \langle v_{r}, P_{n+l} \rangle + \ldots + b_{n+l,t} \langle v_{r}, P_{n} \rangle = 0, \quad n \geq d+1, \]

whence $\{P_{n}\}$ is d-quasi-orthogonal of order $l$ with respect to $\mathcal{V}$ by definition (2.8).

Conversely, suppose that $\{P_{n}\}$ is d-quasi-orthogonal of order $l$ with respect to $\mathcal{V}$, and expand $\pi_{l}Q_{n}$ as

\[ \pi_{l}Q_{n} = \sum_{i=0}^{k} b_{n+l,i}P_{n+l-i}, \quad k \geq 1, \]

then

\[ \langle v_{r}, \pi_{l}Q_{n} \rangle = \sum_{i=0}^{k-1} b_{n+l,i} \langle v_{r}, P_{n+l-i} \rangle + b_{n+l,k} \langle v_{r}, P_{n-l-k} \rangle = 0, \]
for \( n + l - k \geq dl + r + 1 \), then \( l \leq k \). Whence necessarily \( k = l \) [55, lem. 3.2, p. 117].

The latter result enables us to give an analogue of the structure relation (as a new characterization) of semi-classical \( d \)-OPS were first proved in the usual orthogonality by Maroni as an attempt to generalizing the Al-Salam-Chihara’s characterization [3]. In fact, \( \{ P_n \} \) is Hahn classical (semi-classical of order \( s=0 \)) \( d \)-OPS, means that the sequence \( \{ P_n^{[i]} = P_{n+1}^{'}/(n + 1) \} \) is also \( d \)-orthogonal. In this case, the sequence \( \{ P_n \} \) itself is \( d \)-quasi-orthogonal of order two at most with respect to the vector linear form of \( \{ P_n^{[i]} \} \) [66]. Consequently, we immediately have the following result

**Corollary 7.3.** Let \( \{ P_n \} \) be a \( d \)-OPS with respect to regular \( \mathcal{U} \). The following properties are equivalent:

1. \( \{ P_n \} \) is Hahn classical.
2. there exist a polynomial \( \pi \in \mathcal{P}_2 \) and a complex parameters \( a_n, b_n \) and \( c_n \), with \( c_n \neq 0 \), such that
   \[
   \pi(x)P_n'(x) = a_nP_{n+1}(x) + b_nP_n(x) + c_nP_{n-1}(x), \; \forall n \geq 0.
   \]
3. there exist complex numbers \( \lambda_{n,\nu} \) not all zero, such that [53]
   \[
   P_n(x) = \sum_{i=0}^{d+1} \lambda_{n,\nu} P_{n-\nu}^{[i]}(x), \; \forall n \geq 0.
   \]

Notice that the equivalence between (1) and (3) in the latter corollary is obvious. Indeed, by taking derivative of the recurrence satisfied by \( \{ P_n \} \), it is easily seen that (7.11) is sufficient to prove the \( d \)-orthogonality of the derivatives sequence. Conversely, suppose that \( \{ P_n \} \) is Hahn classical \( d \)-OPS, then by differentiating its recurrence relation and replace \( xP_n^{[1]}(x) \) from the \( (d + 2) \)-term recurrence relation satisfied by \( \{ P_n^{[i]} \} \) where we have denoted their recurrence coefficients by \( \xi_n \) and \( \gamma_n^{[i]} \), we obtain

\[
\begin{align*}
  P_n &= P_n^{[1]} + n(\beta_n - \xi_{n-1}) P_{n-1}^{[1]} + (n-1) \gamma_n^{d-1} - n \gamma_{n-1}^{[d-1]} + \frac{1}{2} \left[ (n-2) \gamma_n^{d-2} - n \gamma_{n-1}^{[d-2]} \right] P_{n-3}^{[1]} + \ldots + \left( (n-d) \gamma_{n+1-d}^{0} - n \gamma_{n-d}^{[0]} \right) P_{n-d-1}^{[1]}.
\end{align*}
\]

which is (7.11) written down explicitly.

In the usual orthogonality \( (d=1) \), it is well known that if a sequence of OPS \( \{ P_n \} \) is classical, then their derivative sequences of any order are again classical whereas this is not direct conclusion if \( d \geq 2 \). In fact, the derivative sequence \( \{ P_n^{[i]} \} \) is \( d \)-orthogonal with respect to \( \mathcal{V} = \Phi \mathcal{U} \) [34, 66]. Suppose that there exists matrix \( \Upsilon \) such that \( \Upsilon \Phi \Psi = \Psi \Upsilon \Phi \), then \( (\Phi_1 \mathcal{V})' = \Psi_1 \mathcal{V} \) with \( \Phi_1 = \Upsilon \Phi \) and \( \Psi_1 = \Phi_1 + \Psi \Upsilon \). Thus, the \( d \)-orthogonality of second derivatives sequence is judged according to the matrix \( \Upsilon \).

For this end, at the beginning we thought that we should distinguish whether the derivatives sequence is \( d \)-OPS, then we proposed to call a \( d \)-OPS families for which their derivatives of any order are still \( d \)-OPS, a very classical \( d \)-OPS. Although, we could give a first characterization of the very classical \( d \)-OPS families (see [53, cor.13]).
Corollary 7.4. A d-OPS \( \{P_n\} \) is very classical if and only if there exist complex numbers \( \lambda_{n,\nu} \) not all zero such that

\[
P_n^{[m]}(x) = \sum_{i=0}^{d+1} \lambda_{n,\nu} P_{n-\nu}^{[m+1]}(x), \quad \forall n, m \geq 0.
\]  

Furthermore, by differentiating (7.11) \( m \) times we find (7.13) which characterizes the very classical d-OPS. Accordingly, all Hahn classical d-OPS are in fact very classical d-OPS, and then there is no need to introduce this appellation.

Example 7.5. First of all, the structure relation (7.10) is satisfied by the d-Appell (d-Hermite) [31] with \( \pi(x) = 1, a_n = b_n \equiv 0 \), hence (7.13) is also satisfied with \( \lambda_{n,\nu} = 0, \nu \neq n \). The d-symmetric d-OPS are Hahn classical [13, cor. 4.7]. This means that there exists a structure relation of type (7.10) much simpler than that [36, eq. (3.1)]. The d-Laguerre polynomials satisfy (7.10) with \( \pi(x) = x \) [16, eq. (6.3)] (See also [32, prop. 3.1]). The d-analogue of q-Meixner and big q-Laguerre [50, prop. 3.4] as well as little q-Laguerre [19, prop.3.2] verify (7.10) with \( a_n = b_n \equiv 0 \), i.e, are Hahn classical d-OPS.

Another interesting example is the d-analogue of Laguerre type \( P_n(x) := P_n^\alpha(x;d) \) presented in [70]. From equations (13)-(15) and making use also of (16) in that paper we get

\[
xP_n^\prime(x) = nP_n(x) + \sum_{k=2}^{d+1} a_{n,k}P_{n-k+1}(x),
\]

which shows that this family is not Hahn classical at all.

The question now is: can we determine all the Hahn classical d-OPS families (for fixed \( d \)) as in the work of Al-Salam-Chihara [3]? The next section provides further information about Hahn classical d-OPS’s.

8. \((d+1)\)-DECOMPOSITION AND d-SYMMEtRIZATION

Starting from the classical problem of symmetrization [26], Douak and Maroni introduced a natural generalization, i.e., the d-symmetrization as well as the \((d+1)\)-decomposition of a sequence of polynomials.

A sequence of polynomials \( \{B_n\} \) is called d-symmetric if it fulfills \( B_n(x_k) = x_k^n B_n(x) \), for each \( 0 \leq k \leq d \) and \( n \geq 0 \) where \( x_k = \exp\{2ik\pi/(d+1)\} \). Notice that when \( d=1 \), we get the definition of a symmetric sequence \( B_n(-x) = (-1)^n B_n(x) \).

Besides, a vector form \( V = (v_0, \ldots, v_{d-1})^T \) is called d-symmetric if for each \( 0 \leq r \leq d-1 \), the moments of the linear form \( v_r \) satisfy

\[(v_r)_{(d+1)\neq s} = 0 \] whenever \( r \neq s, \quad 0 \leq s \leq d, \quad n \geq 0.\]

As a result, \( \{B_n\} \) is d-symmetric if and only if it could be written in the following form

\[
B_{(d+1)n+s} = x^n \sum_{p=0}^{d} a_{(d+1)n+s,d+1,p+s} x^{(d+1)p}, \quad 0 \leq s \leq d, \quad n \geq 0.
\]

The components of the sequence \( \{B_n\} \) are \( d+1 \) sequences denoted by \( \{B_n^i\} \), for \( 0 \leq i \leq d \), and defined as follows

\[
B_{(d+1)n+s}(x) = x^n B_n^i(x^{d+1}), \quad 0 \leq s \leq d, \quad n \geq 0.
\]
In the same paper, they have also proven that a sequence of polynomials is $d$-symmetric if and only if the corresponding vector linear form is $d$-symmetric. Furthermore, a $d$-symmetric sequence of polynomials \( \{B_n\} \) is $d$-OPS if and only if it satisfies a \((d+1)\)-order linear recurrence relation of the form

\[
\begin{aligned}
B_{n+d+1}(x) &= xB_{n+d}(x) - \rho_{n+1}B_n(x), \quad n \geq 0, \\
B_n(x) &= x^n, \quad 0 \leq n \leq d.
\end{aligned}
\]  

As it was pointed out by Douak and Maroni [33, p.85-86] that, first, each component of a $d$-symmetric $d$-OPS is again $d$-OPS and, second, that there exist some links between the components. Besides that links, we can also give a connection between any two components either consecutive or not adjacent. Further, the last component is connected to the first one which presents, in fact, kernel polynomials as it was shown by Chihara [26, p.45], i.e., the second component in the quadratic decomposition of a symmetric OPS defines kernel polynomials.

Indeed, replace first \( n + d + 1 \) in (8.3) by \((d+1)n\) we get

\[
B_{(d+1)n}(x) = xB_{(d+1)(n-1)+d}(x) - \rho_{(d+1)(n-1)+1}B_{(d+1)(n-1)}(x),
\]

and according to (8.2), we find

\[
xB_n^d(x) = B_{n+1}^0(x) + \rho_{n+1}B_n^0(x).
\]  

Furthermore, since the components are not $d$-symmetric (see (8.9) below), then \( B_n^0(0) \neq 0 \), \( \forall n \geq 0 \), we thus obtain

\[
B_{(d+1)(n+1)}(0) = B_{n+1}^0(0) = -\rho_{n+1}B_{n+1}^0(0),
\]

i.e.,

\[
\rho_{n+1} = -B_{n+1}^0(0)/B_n^0(0) = m_{n+1}.
\]

This shows that the \((d+1)\)th and the first component are respectively the kernel polynomials \( \{K_n\} \) and the original sequence \( \{P_n\} \) (see Chihara’s book for a comparative results).

Next, by replacing \( n + d + 1 \) in (8.3) by \((d+1)n + s\) for \( s = 1 \) to \( d \) recursively, and making use of (8.2) once again, we get an analogue result of that expounded in (6.2)

\[
B_{n+1}^i = B_{n+1}^{i+1} + \rho_{(d+1)n+i+2}B_{n+1}^{i+1}, \quad 0 \leq i \leq d - 1.
\]  

The latter connection shows that we can express the component \( B_n^i \) as a linear combination in terms of any other component \( B_n^k \). Indeed, replace recursively the system obtained above, we get the following recursion

\[
B_{n+1}^i = B_{n+1}^{i+k} + \sum_{1 \leq j_1 < j_2 < \cdots < j_l \leq k+1} \rho_{(d+1)n+i+j_1} \rho_{(d+1)(n-1)+i+j_2} \cdots \rho_{(d+1)(n-l+1)+i+j_l} B_{n+1}^{i+k},
\]

which shows, by taking \( i = 0 \) and \( k = d \) therein, that the coefficients \( l_{n,k} \) in (6.10) (or equivalently in the expansion (7.1)) are given explicitly in terms of \( \rho_n \), and if we replace (8.6) with \( i = 0 \) in (8.4) we find the recursion of Douak and Maroni which we prefer to write
in the following form
(8.7)
\[ x^d B_n^i = B_{n+1}^i + \sum_{t=0}^{i} \sum_{1 \leq j_0 < j_1 < \cdots < j_t \leq i}^{i} \rho_{d+1)(n+1)+j_0} \rho_{(d+1)(n-1)+j_1} \cdots \rho_{(d+1)(n-t)+j_t} B_{n-t}^i, \quad 0 \leq i \leq d. \]

In other words, recursions (8.5)-(8.7) all together, could define kernel polynomials as well as the first component as a linear combination in (i+1)-term of the component \( \{B_n^i\} \).

As a matter of fact, the system (8.6) is also quite useful to determine the coefficients of the recurrence relation for each component [21, thm. 2.3] (see also [33, p. 88]). First off, the recurrence relation of kernel polynomials is already obtained from (8.7) with \( i = d \), whilst the recurrence of any other component can be obtained by combining (8.6) and (8.5). Instead of following this long way, the recurrence of the ongoing sequences are already obtained in the proof of [21, thm. 2.3] implicitly. In what follows, we need the explicit form. Hence as a much deeper, with extra computation we explicitly have

(8.8)
\[ x^r B_n = B_{n+r} + \sum_{k=1}^{\lceil\frac{n}{d}\rceil} \rho_{d+1}(n+i_1+1+2d) \cdots \rho_{d+1}(n+s) B_{n+r-k(d+1)}. \]

Now, with \( r = d + 1 \) and \( n \rightarrow n(d + 1) + s \), we get using (8.2)

(8.9)
\[ x^d B_n^s = B_{n+1}^s + \sum_{k=1}^{d+1} \rho_{d+1}(n+1)+s \cdots \rho_{d+1}(n+k+s) B_{n+1-k}^s, \]

when \( d = 2 \) the recursion (8.9) reduces to that given in [33, p.88].

It is easily concluded from the preceding that the above recurrence of components (8.9) could be used to give an analogous of Chihara’s theorem [26, thm.9.1, p.46]. Actually, we have already pursued this viewpoint here to show out the recursion (8.9). Accordingly, the relation (8.1) is satisfied whenever (8.9) is satisfied and vice-versa.

We back now to the section 6 especially to show that sequences generated by Darboux transformations (6.1) are in fact our \( \{B_n^i\} \). To this end, we shall formulate sufficient condition in terms of the coefficients which makes them easy to check.

According to (8.5), the components are defined recursively as \( B^i = R_i^t B^{i+1} \) where \( R_i^t \) are bidiagonal matrices with 1 in the main diagonal and \( \rho_{m(d+1)+i+2} \) at the position \( (m+2, m+1) \) for \( m \geq 0 \). Since the polynomials are monic, it suffice then to show that \( L_{i+1} = R_i \) where \( L_i \) are the matrices given in (6.1)-(6.2). We proceed by induction on \( i \) and we shall show that \( J_d^i \) is the corresponding Jacobi matrix of \( \{B_n^i\} \). In fact, on one hand, since \( J_d^1 \) is the Jacobi matrix for the first component, it follows then that \( x R_0 B^1 = J_d^1 R_0 B^1 \), that is \( J_d^1 = R_0^{-1} J_d^0 R_0 \). On the other hand, from \( L_1 J_d^1 = J_d^0 L_1 \) in (6.2), it follows by comparing the entries in both main diagonal using the recursion (8.9) that \( L_1 = R_0 \). From which the result follows readily by induction.

Consequently, this is another easy way to prove that the lower matrix in (6.2) is the product \( L = R_0 R_1 \cdots R_{d-1} \).

A word about the measure of orthogonalities. Let us first denote the corresponding orthogonality’s vector form of \( \{B_n^i\} \) by \( U^i = (u_0^i, \ldots , u_{d-1}^i)^T \). It has been already shown in [53,
thm. 3] that \( \mathcal{U}^{i+1} = \Phi \mathcal{U}^i \) where \( \Phi_i = \{\phi_i^\mu(i)\}_{\mu,v=0}^{d-1} \) is a \( d \times d \) matrix polynomial with entries

\[
\begin{align*}
\phi_{r+1}^i(i) &= \rho_{(d+1)r+i+2}, \quad &\phi_r^i(i) &= 1, \text{ for } 0 \leq r \leq d - 2, \\
\phi_r^\mu &= 0, \text{ for } r + 2 \leq \mu \leq d - 1 \text{ and } 0 \leq r \leq d - 3, \\
\phi_r^0 &= 0, \quad &\phi_r^d - 1(i) &= 1 - (\rho_{d^2 + i + 1} \gamma_1^0(i)) / \gamma_1^0(i), \text{ for } 1 \leq \mu \leq d - 2, \\
\phi_r^{d-1} = -\left(\rho_{d^2 + i + 1} \gamma_1^d - \mu(i) / \gamma_1^d(i), \text{ for } 1 \leq \mu \leq d - 2,
\end{align*}
\]

where \( \beta_n(i) \) and \( \gamma_n^\mu(i) \) are the coefficients of the \((d+2)\)-term recurrence relation of \( \{B_n^i\} \) which could be all expressed in terms of \( \rho_n \) from (8.9). For instance, we have

\[
\gamma_n^0(i) = \prod_{\nu=n}^{d+n} \rho_{(\nu-1)d+n+i}, \quad 0 \leq i \leq d - 1, \quad n \geq 1,
\]

this shows further that the brackets (2.5) could be expressed in terms of \( \rho_n \) as follows

\[
\langle u_i^i, P_{dn+r}^i P_n^i \rangle = \prod_{\nu=1}^{n-1} \prod_{j=1}^{n+1} \rho_{d(j+\nu-2)d+r+i+1}, \quad 0 \leq i \leq d - 1, \quad n \geq 1.
\]

It thus follows, that all \( \mathcal{U}^i, 0 \leq i \leq d \) could be determined whenever one of them is explicitly known, and according to our notation in the previous section, we infer that

\[
\mathcal{V} = \mathcal{U}^d = \Phi_{d-1} \Phi_{d-2} \cdots \Phi_0 \mathcal{U}^0 = \Phi \mathcal{U},
\]

which is another expression and of course another way to determine the matrix \( \Phi \) given in (7.3)-(7.4). It is well known that when \( d = 1 \), the orthogonality’s measure of kernel polynomials is just the corresponding measure of \( B_n^0 \) multiplied by \( x \) [26, p. 35], or more generally times \( x - c \) if one consider the transformation \( UL + cI \) instead (in our case \( c = 0 \)).

Remark further that the corresponding matrix Pearson equation of a Hahn classical d-OPS [34] is very nice in the d-symmetric case. Indeed, if we denote the vector form of the sequence \( \{B_n\} \) by \( \mathcal{W} = (w_0, \ldots, w_{d-1})^T \) and by adding tilde to the corresponding vector form as well as to the recurrence coefficients for its derivative, then the connection between the two forms might be written as

\[
\begin{align*}
\tilde{w}_{d-1} &= a_{d-1}^d x^2 w_0 + b_{d-1}^d w_{d-1}, \\
\tilde{w}_r &= b_r^r w_r + a_r^{r+1} x w_{r+1}, \quad &\text{for } 0 \leq r \leq d - 2, \\
b_r^r &= \frac{(d + r + 1)\rho_{r+1}^0 - r^2 \rho_{r+2}^0}{\rho_{r+1}^0 + \rho_{r+2}^0}, \quad &b_r^r + a_r^{r+1} = 1, \quad &\text{for } 0 \leq r \leq d - 1.
\end{align*}
\]

It should be noted that from the connection \( u_i^i = \sigma_{d+1} \left( x^i w_{r(d+1)+i} \right) \), for \( 0 \leq i \leq d \), the dual sequence of the components of the derivatives sequence of \( \{B_n\} \) [33, (5.4)] could be also expressed explicitly in terms of \( w_r \) using (8.10).

Let us now move on to Hahn’ property. When the sequence \( \{B_n\} \) is Hahn classical, then, first from [13, cor. 4.7] we know that its derivative is again d-symmetric and Hahn classical d-OPS (which can be obtained differently from (7.11)), and second from the paper of Blel [21, thm 2.4], that all the components are also Hahn classical d-OPS. In other words, these results mean that the derivative of any order of a d-symmetric Hahn classical d-OPS
and its components are again d-symmetric Hahn classical d-OPS and Hahn classical d-OPS respectively.

Hence an interesting question to think about is the converse, i.e., is a d-symmetric sequence possesses Hahn’s property if their components are all of Hahn type (Hahn classical), or at least one of them is needed to be of Hahn type? Further, does the kernel sequence of Hahn classical one also of Hahn type?

It is not at all difficult to see why the inverse situation is also true. Indeed, given a d-symmetric sequence \( \{B_n\} \) and denote its components by \( \{B_s n\} \). Then from [33, thm. 5.3] we know that the sequence \( \{A_n = (n + 1)^{-1}B_{n+1}'\} \) is also d-symmetric, we denote its components as \( \{A_s n\} \). Now, we assume that the components \( \{B_s n\} \) are Hahn classical d-OPS, and we try to prove that \( \{B_n\} \) is Hahn classical d-OPS, namely that \( \{B_n\} \) and \( \{A_n\} \) are both d-OPS. Actually, it suffice to show that if the components \( \{B_s n\} \) are d-OPS, then \( \{B_n\} \) should be d-OPS too. In fact, from (8.9) with \( s = 0 \), we get a recurrence of the type (8.8)

\[
x^{d+1}B_n = B_{n+d+1} + \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq d-k+2} \rho_{n-(d+1)+i_1+1} \cdots \rho_{n-k(d+1)+i_k+k} B_{n+(1-k)(d+1)}. 
\]

From this, by taking \( n = d(d+1) + r \), and from the definition of dual sequences it follows that

\[
\langle w_r, x^{d+1}B_{d(d+1)+r} \rangle = \prod_{\nu=1}^{d+1} \rho_{(d-\nu)(d+1)+\nu+r+1} \langle w_r, B_r \rangle \neq 0,
\]

whence the desired result.

Remark that we only needed to suppose that the first component is d-OPS. It is sufficient enough to prove the d-orthogonality of its derivative to suppose only that the first component is Hahn classical. Indeed, it is well understood that \( (n + 1)^{-1} (B_{n+1}^0)' = A_n^1 \) [33, thm. 5.3], hence the result follows in the same easy way by taking care now of \( A_n^1 \) as above.

Consequently, this answers affirmatively to the above two questions. In other words, it results from the above discussion that if the first component is Hahn classical d-OPS, then all the components as well as their corresponding d-symmetric sequence are also Hahn classical d-OPS as well.

As a conclusion of this section, it results that under the umbrella of each d-symmetric Hahn classical d-OPS there are \( (d+1) \) Hahn classical (nonsymmetric) d-OPS families. And since there are \( 2^d \) d-symmetric Hahn classical d-OPS, then the d-symmetric sequences and their components constitute a set of \( (d+1)2^d \) families of Hahn classical d-OPS.

This gives evidence to think about the following question: Can we distribute all Hahn classical d-OPS onto \( d \) sets? Namely, is it possible to consider \( d \) sets of specific d-OPS families in which the recurrence coefficients are all zero except one parameter (d-symmetric case), two parameter, ..., \( d \) parameter and repeat the above study in order to generate the maximum number possible of Hahn classical d-OPS into which their recurrence coefficients are determined by only one parameter (d-symmetric case), two parameter, ..., \( d \) parameter? Second: is there any d-OPS of Hahn type which could not be a component of any of the above \( d \) sets, i.e., under no umbrella?

**Example 8.1.** A good example for a d-symmetric sequence and its components, is presented in [14]. The authors showed that there are only two d-symmetric d-OPS families of Brenke
type. Furthermore, the corresponding generating functions of the components are all explicitly determined. See also the components of d-Hermite polynomial [31, p. 287-288] and [15, sec. 5] for 2-Laguerre.

Let us consider d-Chebyshev polynomials of second kind [35] with \( \rho_n = \rho \). Accordingly, from (8.9) it is readily seen that the recurrence coefficients differ only for the initial conditions, i.e., \( \beta_n(i) = \beta_n(j) \) and \( \gamma^k_n(i) = \gamma^k_n(j) \) for \( n \geq 1 \).

For instance, to check (7.2) we need only to look at the initial conditions. Remark that \( \beta_0(i) = (i + 1)\rho \) and \( \gamma^k_1(i) = (i + 1)(d - (i/2))\rho^2 \). Hence, \( B^d_1(x) := K_1(x) = x - (d + 1)\gamma \) and \( B^0_2(x) := P_2(x) = (x - \beta_1)(x - \rho) - d\rho^2 \). Consequently, we obtain

\[
P_2(x) - (P_2(0)/P_1(0)) P_1(x) = (x - \beta_1)(x - \rho) - d\rho^2 + (\beta_1 - d\rho)(x - \rho) = xK_1.
\]

We desperately hope that this study provides a spotlight to pursue an analytic approach in order to explore for instance the zeros and to find out some applications since the recurrence coefficients are expressed in terms of only one parameter. In fact, a bonus following from this is by supposing that the parameter \( \rho_n > 0, \forall n \geq 1 \), we remark by inspection for small \( n \), that the zeros are positive, equidistributed according to \( \beta_n(i) \) and satisfy some interlacing properties. But we are not yet able to prove and to comment these remarks. However, sufficient conditions for zeros to be real and simple are presented in the next section.

9. SOME PROPERTIES OF ZEROS

Facing now the set of zeros. And recall that a zero of a polynomial \( \pi(x) \) at an interior point of \([a, b]\) is said to be nodal or nonnodal according as \( \pi(x) \) changes or does not change sign in the neighborhood of the zero.

Let \( \{P_n\}_{n \geq 0} \) be d-OPS with respect to \( U = (u_0, ..., u_{d-1})^T \). The following theorem given by Maroni [54] in the sense of \( 1/p \) orthogonality

**Proposition 9.1.** Suppose that \( \gamma^0_{m+1} > 0, m \geq 0 \). Then each polynomial \( P_{dn+q}, 1 \leq q \leq d \) has at least \( n + 1 \) distinct nodal zeros.

The previous proposition stated without proof, but it is readily proved using only the recurrence relation (see [69, p. 56] for a such proof). Recall now the following definition

**Definition 9.2.** [26] A moment functional \( u \) is called positive definite if \( \langle u, \pi(x) \rangle > 0 \) for every polynomial \( \pi(x) \) that is not identically zero and is non-negative for all real \( x \).

Since the moment of linear form may be expressed in terms of the recurrence coefficients of the corresponding OPS, then it is straightforward that the respective OPS as well as the recurrence coefficients should be real in the positive definite case.

Let \( \{x_i\}_{i=1}^k \) be all the nodal zeros of \( P_m \) and set

\[
\pi_k(x) = (x - x_1) \cdots (x - x_k),
\]

then \( \pi_k(x) P_{dk+r}(x) \geq 0 \). In addition, from (2.5) and the definition of the \( d \)-orthogonality, we have

\[
\langle u_r, \pi_k(x) P_{dk+r} \rangle = \langle u_r, x^k P_{dk+r} \rangle = \prod_{\nu=0}^{k-1} \gamma^0_{\nu d+r+1},
\]
which gives explicitly the determinants $H_{md}$ in [55, eq. (2.7), (2.9)], whereas [65, p.878]

\[
\langle u_r, x^k P_{dk+r-i} \rangle = \sum_{j=0}^{i} \gamma_{d(k-1)+r+1-j} \langle u_r, x^{k-1} P_{d(k-1)+r-j} \rangle, \quad 1 \leq i \leq d-1,
\]

\[
\langle u_r, x^k P_{dk+r-d} \rangle = \beta_{d(k-1)+r} \langle u_r, x^{k-1} P_{d(k-1)+r} \rangle + \sum_{j=0}^{d-1} \gamma_{d(k-1)+r+1-j} \langle u_r, x^{k-1} P_{d(k-1)+r-j} \rangle,
\]

\[
\langle u_r, x^k P_{dk+r-l} \rangle = \langle u_r, x^{k-1} P_{dk+r-l+1} \rangle + \beta_{dk+r-l} \langle u_r, x^{k-1} P_{dk+r-l} \rangle + \sum_{j=0}^{d-1} \gamma_{dk+r-l-j} \langle u_r, x^{k-1} P_{dk+r-l-j} \rangle, \quad d+1 \leq l \leq dk + r.
\]

Hence, on account of [55, eq. (2.6)-(2.9)] and [65, p.876-878] we have the following result

**Proposition 9.3.** $u_r$ is positive definite if and only if $\beta_{\nu}, \gamma_{\nu+1}^s, 1 \leq s \leq d - 1$, are real and $\gamma_{\nu+1}^0 > 0, \forall \nu \geq 0$.

Further, suppose that there are $s$ nonnodal zeros and let

\[
\phi_s(x) = (x - x_1) \ldots (x - x_s),
\]

then $\phi_s P_{dk+r}$ has only nodal zeros. Hence

\[
\langle u_r, \phi_s P_{dn+q} \rangle = 0, \quad dn + q \geq ds + r + 1,
\]

this means that $s \leq n - 1$. We have then proven the following consequence of the previous proposition

**Corollary 9.4.** If there exist nonnodal zeros for the polynomial $P_{dn+q}, 1 \leq q \leq d$, then there are $n - 1$ distinct zeros at most.

Accordingly, from formulae (5.4) as well as (4.8) which shows that $\Delta_n^{(r)} \neq 0$, we readily deduce that

**Corollary 9.5.** The multiplicity of zeros of any $d$-OPS is at most $d$. Moreover, any $d+1$ consecutive polynomials as well as any $d+1$ consecutive polynomials from the $r$-associated sequence $\{P_n^{(r)}\}$, have no common zero. And for any $r \geq 0$, the polynomials $P_n^{(r)}, P_n^{(r+1)}, \ldots, P_n^{(r+d)}$ have no common zero.

Corollaries 9.1 and 9.5 show again that the zeros are simple when $d = 1$.

### 9.1. Chebyshev systems.

Zeros of OPS interlace as a consequence generally from the recurrence relation.

A system of real functions $\{\mu_i\}_{i=0}^d$ defined on an abstract set $E$ is called a Chebyshev system (T-system) of order $d$ on $E$ if any polynomial (any linear combination)

\[
P(t) = \sum_{i=0}^{d} c_i \mu_i(t), \quad \text{with} \quad \sum_{i=0}^{d} c_i^2 \neq 0,
\]

has at most $d$ zeros on $E$ [48].
It is readily seen that \( \{\mu_i\}_{i=0}^d \) is a T-system on \( E \) if and only if the determinant
\[
\det (\mathbb{P}_0(t_0) \mathbb{P}_0(t_1) \ldots \mathbb{P}_0(t_d))
\]
does not vanish for any pairwise distinct \( t_0, \ldots, t_d \in E \). This follows at once by considering a system of \( n + 1 \) homogeneous equations
\[
\sum_{i=0}^d c_i \mu_i(t_j) = 0, \quad j = 0, 1, \ldots, d,
\]
in \( c_1, \ldots, c_d \).

The interlacing property for the zeros of polynomials orthogonal with respect to a Markov system proved by Kershaw with respect to Lebesgue measure in [47], and under a weak condition, with respect to the Borel measure in [38]. The same argument used in [43] to prove the interlacing property for the type II multiple OPS with respect to measures that form an AT system.

Recall that a system of measures \( (\mu_1, \ldots, \mu_r) \) forms an AT system for the set of integers \( (n_1, \ldots, n_r) \) on \( [a, b] \) if the measures \( \mu_j \) are absolutely continuous with respect to a measure \( \mu \) on \( [a, b] \), with \( d\mu_j(x) = \omega_j(x) d\mu(x) \) and
\[
\{\omega_1, x\omega_1, \ldots, x^{n_1-1}\omega_1, \omega_2, \ldots, x^{n_r-1}\omega_r\}
\]
is a Chebyshev system on \( [a, b] \) of order \( n = n_1 + \ldots + n_r - 1 \) [48, 60].

In view of all the above results, we are able to announce and prove the following result

**Proposition 9.6.** Let \( \{P_n\} \) be an \( d \)-OPS with respect to an AT system \( \mathcal{U} = (\mu_0, \ldots, \mu_{d-1}) \). Then the zeros of \( P_{dn+r} \) and \( P_{dn+r+1} \) interlace for \( n \geq 0 \) and \( 0 \leq r \leq d - 1 \).

**Proof.** The proof is analogue of that used to prove [43, theorem 2.1]. Replace the polynomials \( P_n^\gamma \) and \( P_n^\gamma + \tau \) by \( P_{dn+r} \) and \( P_{dn+r+1} \) respectively and use the following determinant
\[
W_{dn+r}(x_1, \ldots, x_{dn+r+1}) = 
\begin{vmatrix}
\omega_1(x_1) & \omega_2(x_1) & \ldots & x_1^{n-1}\omega_r(x_1) & x_1^{n-1}\omega_d(x_1) \\
\omega_1(x_2) & \omega_2(x_2) & \ldots & x_2^{n-1}\omega_r(x_2) & x_2^{n-1}\omega_d(x_2) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\omega_1(x_{dn+r}) & \omega_2(x_{dn+r}) & \ldots & x_{dn+r}^{n-1}\omega_r(x_{dn+r}) & x_{dn+r}^{n-1}\omega_d(x_{dn+r})
\end{vmatrix}
\]
for the point \( x_1, \ldots, x_{dn+r+1} \) on \( [a, b] \) instead.

Following the same resonance in [43, theorem 2.1] we conclude that the zeros \( x_k \) and \( y_i \) of \( P_{dn+r} \) and \( P_{dn+r+1} \) respectively, are in the following situation
\[
y_i < x_i < y_{i+1} \quad \text{for} \quad i = 1, \ldots, dn + r \quad \text{and} \quad n \geq 0.
\]

**Remarks.** First of all, we are not working onto empty set. The zeros of \( d \)-Laguerre OP [16] are real, positive and simple for \( \alpha_i + 1 > 0, \quad i = 0, \ldots, d \) [23]. This gives evidence to think whether is it possible to avoid such condition, i.e., that is to say, to look if there are extra informations on the zeros that could be find out from the recurrence coefficients. Such sufficient conditions are termed out for type II multiple OPS [43, thm. 2.2]. For the \( d \)-orthogonality, we believe that we could give an analogous of the latter condition using the
recurrence (8.9) (resp. and maybe some analogue of it) which provides conditions only on one parameter \( \rho_n \) (resp. on few parameters). Although next section shows, with the aid of totally nonnegative matrices, that such sufficient conditions on the recurrence coefficients are available.

9.2. **Totally positive matrix.** In this section we provide new approach based on totally positive matrices to show that zeros of \( d \)-OPS could be real and simple. Let us first recall some terminologies and definitions.

**Definition 9.7.** A \( n \times m \) matrix \( A \) is said to be:

1. totally nonnegative (TN) if all its minors are nonnegative.
2. totally positive (TP) if all its minors are strictly positive.
3. an oscillation matrix if \( A \) is TN and some power of \( A \) are TP.

An interesting class between TP and TN matrices was introduced by Gantmakher and Krein which share the spectral properties of TP matrices. It is more convenient to consider this class of non-symmetric matrices with the oscillatory properties. Actually, there are relatively simple criteria for determining if a TN matrix is an oscillation matrix.

**Theorem 9.8.** [41] A \( n \times n \) matrix \( A = (a_{i,j})_{i,j=0}^n \) is an oscillation matrix if and only if \( A \) is TN, nonsingular, and \( a_{i,i+1}, a_{i+1,i} > 0 \), \( i = 1, ..., n - 1 \). Furthermore, if \( A \) is oscillation matrix, then \( A^{n-1} \) is TP.

If a matrix \( A \) is TP (resp. TN), then \( A^T \) (transpose of \( A \)) is TP (resp. TN) as well as every submatrix of \( A \) and \( A^T \) is TP (resp. TN). Furthermore, since the product of TN matrices is TN matrix, then the following proposition is with important interest in our study of zeros. It could be also proved readily using planar network.

**Proposition 9.9.** [61, p.155] A bi-diagonal lower triangular matrix is TN if and only if all its elements are nonnegative.

The eigenvalues of oscillation matrices are simple and positive. Although, the following theorem shows that the eigenvalues of the two principal submatrices obtained form \( A \) by deleting either the first row and column, or the last row and column, strictly interlace the eigenvalues of \( A \)

**Proposition 9.10.** [61, p.136] Let \( A \) be an \( n \times n \) TP. Then its eigenvalues are positive and simple. In addition, if these eigenvalues are denoted by \( \lambda_1 > ... > \lambda_n > 0 \), and \( \mu_1^{(k)} > ... > \mu_{n-1}^{(k)} > 0 \) are the eigenvalues of the principal submatrix of \( A \) obtained by deleting its \( k \)-th row and column, then

\[
\lambda_j > \mu_j^{(k)} > \mu_{j+1}^{(k)}, \quad j = 1, ..., n - 1,
\]

for \( k = 1 \) and \( k = n \).

On the other hand, TP matrices are dense in the class of TN matrices [61, th. 2.6], i.e., for a \( n \times m \) TN matrices \( A \) there exists a sequence of \( n \times m \) TP matrices \( \{A_k\}_{k \geq 1} \) such that \( \lim_{k \to \infty} A_k = A \). The latter fact allows us to assert that the eigenvalues of TN matrices are both real and nonnegative.

It results now from the above discussion the following conclusion. For a \( d \)-symmetric \( d \)-OPS \( \{B_n\} \), if we assume that \( \{\rho_n\} \) is a sequence of positive numbers, then it is readily
seen that the recurrence coefficients in (8.10) are all positive. Furthermore, according to the
factorization \( J_d^n = R_{d+1}...R_{d-1}UR_0...R_1 \), Theorem 9.8 shows that the \( n \times n \) leading Jacobi
submatrices of the components are all oscillation matrices, and then their eigenvalues are positive and simple. This result already proved for Faber polynomials in [37] and for 4 term
recurrence relation in [27].

Now we want to show that the zeros of any \( d \)-OPS are positive and simple whenever
the recurrence coefficients are strictly positive. Notice first that, in this case, according to
Theorem 9.8 it is enough to show that \( n \times n \) Jacobi matrices are TN for any integer \( n \). That
is to say, it is always possible to write the matrix \( L \) as a product of \( d \) bi-diagonal lower TN
matrices.

From [7, thm.2], we construct recursively our matrices \( L_i \) such that \( L = L_1...L_d \) where \( L_i \)
are bidiagonal matrices with 1 in the main diagonal and \( l_{m+1}^i > 0 \) at the position \((m+2,m+1)\)
for \( m \geq 0 \). Let us begin with constructing \( L_1 \). In other words, we look for two matrices \( L_1 \)
and \( T_1 \) with strict positive entries such that \( L = L_1T_1 \) where

\[
T_1 = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
t_{11} & 1 & \cdots & 0 \\
& \ddots & \ddots & \ddots \\
t_{d-1,1} & \cdots & t_{d-1,d-1} & 1 \\
0 & \ddots & \ddots & \ddots \\
\end{pmatrix}.
\]

(9.1)

Now by equating both sides (\( L \) with the product \( L_1T_1 \)), we get from the first line \( l_{11}^1 = \)
\( t_{11} + l_1^1 > 0 \). We choose the entries of the matrix \( L_1 \) recursively. Suppose that \( l_{m}^1 > 0 \)
are chosen up to some integer \( k - 1 \). Then, the entries of \( L \) at line \( k+1 \) show that \( l_{k}^1 \) could be
chosen strict positive and satisfies the following inequalities

\[
\begin{align*}
t_{k1} &= l_{k1} - l_1^1 (l_{k2} - l_2^1 (l_{k3} - \cdots (l_{kk-1} - l_{k-1}^1 (l_{kk} - l_{k}^1)))) > 0, \\
t_{kk} &= l_{kk} - l_{k}^1 > 0, \\
t_{ki} &= l_{ki} - l_{i}^1 l_{k,i+1} > 0, \quad 1 \leq i \leq k - 1,
\end{align*}
\]

for \( 1 \leq k \leq d - 1 \).

For \( k \geq d \), the entries at line \( d + i, \ i \geq 0 \), show that \( l_{i}^1 \) and \( t_{i,j} \) could be chosen to be strict
positive in the following manner

\[
\begin{align*}
l_{d-1+i,i} &= l_{i}^1 t_{d-1+i,i+1}, \quad i \geq 1, \\
l_{d-1+i,j} &= t_{d-1+i,j} + l_{j}^1 t_{d-1+i,j+1}, \quad i + 1 \leq j \leq i + d - 1.
\end{align*}
\]

(9.3)

Repeatedly, we construct \( L_2, ..., L_d \) with strict positive entries \( l_{m}^i \) and, then, the leading
submatrix of \( L \) is TN.

According to Theorem 9.8 it terms out that our Jacobi matrix is oscillation matrix. Now
Proposition 9.10 asserts that the eigenvalues of Jacobi matrix are positive and simple whenever
its entries \( \{\beta_n\}_n \) and \( \{\gamma_n^i\}_n \) as well as the entries \( \{m_n\}_n \) and \( \{l_{n,m}\}_{n,m} \)
of the matrices \( U \) and \( L \) respectively page 24 are strict positive (see proposition 6.1).

Denoting by \((J)_n \) the leading principal submatrix of \( J_d \) (see 2.10) of size \( n \times n \), and by
\((P)_n = (P_0(x), ..., P_n(x))^T \), we get

\[
x(P)_{n-1} = (J)_n(P)_{n-1}
\]

(9.4)
if and only if $x$ is a zero of $P_n$. This identifies the zeros of $P_n$ as eigenvalues of the matrix $(J)_n$. This can also be seen by expanding the determinant $\det (xI)_n - (J)_n$ along the last row to get that this determinant is $P_n(x)$. In the same way, it is readily seen that the zeros of $P_{n-1}^{(1)}$ (resp. $P_{n-1}$) are the eigenvalues of the principal submatrix of $(J)_n$ obtained by deleting its last (resp. first) row and column. Hence, according to Proposition 9.10, the zeros of $P_n$ and $P_{n-1}$ as well as that of $P_n$ and $P_{n-1}^{(1)}$ interlace.

However, this condition is too strong (see examples below) and one needs to look for weaker condition that ensures zero’s simplicity and interlacing.

### 9.3. Examples

Let us look at zeros of some $d$-OPS families. First off, in [23] the authors tell us that zeros of $d$-Laguerre polynomials are positive and simple, whereas the recurrence coefficients are not all positive (see [16, p.597] for $d=2$). Accordingly, in account of this result, the strict positivity of the recurrence coefficients is sufficient but not necessary.

For $q$-Appell OPS ($d=1$), Al-Salam [2] gives explicitly the recurrence coefficients. We can mimic him to get the recurrence coefficients for $d > 2$ as follows

$$
\beta_n = q^n \beta_0, \quad \gamma^i_{n+1} = \left[ \begin{array}{c} n + d - i \\ d - i \end{array} \right] q^n \gamma^i_1, \quad 0 \leq i \leq d - 1, \quad \forall n \geq 0.
$$

Then, zero’s interlacing as well as simplicity are guaranteed for $d$-analogue of $q$-Appell whenever $\beta_0$ and $\gamma^i_1$ are strict positive. When $q \to 1$ we find $d$-analogue of Appell ($d$-Hermite) studied by Douak [31] with the same conclusion. $d$-Charlier polynomials [20] are also of Appell type (known as $\Delta_w$-Appell or discrete Appell), defined by their recurrence coefficients $\beta_n = wn - \beta_0$ and $\gamma^i_{n+1} = -\beta_i(n+1)_{d-1}$ for $n \geq 0$. Accordingly, by $\beta_i < 0$ for $0 \leq i \leq d - 1$, the interlacing property is satisfied. The same conclusion for Dunkl-Appell $d$-OPS studied in [17] where the recurrence relation is

$$
xP_n(x) = P_{n+1}(x) - \sum_{k=1}^{d} \beta_k \frac{\gamma_{\mu}(n)}{\gamma_{\mu}(n-k)} P_{n-k}(x).
$$

Hence, a sufficient condition for simplicity of zeros is $\beta_k < 0$, $k \geq 0$.

Humbert polynomials defined by the following generating function given in terms of hypergeometric function

$$
(1 - xt + t^{d+1})^{-\alpha} = (1 + t^{d+1})^\alpha \, _1F_0 \left( \alpha, -, \frac{xt}{1 + t^{d+1}} \right) = \sum_{n \geq 0} H^\alpha_n(x) t^n,
$$

are $d$-symmetric. Their components denoted by $\left\{ \frac{\alpha+r}{\alpha} B^a+r_n(x, (\theta_r)) \right\}$, are explicitly given by [49]

$$
B^a+r_n(x, (\theta_r)) = \frac{(-1)^n (\alpha + r)_n}{n!} \, _{d+1}F_d \left( \begin{array}{c} -n, \Delta(d, n + \alpha + r), \\ (\theta_r) \end{array} \right)(q, z),
$$

where $(\theta_r)$ designates the set $\left\{ \frac{\alpha+1+i}{d+1}; i = 0, \ldots, d \text{ and } i \neq d - \alpha \right\}$ and $\Delta(p, a)$ abbreviates the array of $p$ parameters $(a + i - 1)/p$, for $i = 1, \ldots, p$.

Monic Humbert polynomials satisfy the recurrence (8.3) with

$$
\rho_{n+1} = \frac{(n + 1)d+1}{(\alpha + n)d+1} \left( \frac{(d+1)(\alpha-1)}{n + d + 1} + 1 \right).
$$
Then, in order that the recurrence coefficients in (8.9) be strict positive it suffices to take $\alpha > 0$. Hence the interlacing properties are satisfied for components of Humbert polynomials. The same conclusion for component’s zeros of $d$-symmetric Dunkl $d$-OPS [18, p.213] since $\rho_n > 0$ [18, p.201].

Let us consider the classical $d$-OPS generated by

$$\exp \left\{ \frac{xt}{1-at} + \sum_{k=0}^{d-1} b_k \frac{t^k}{k!} \right\} := \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!}.$$  

Notice that when $a = 0$ the above generating function reduces to Appell ones [31]. Now, denote by $Q_r(x) = P_{r+1}(x)/(r+1)$. In this case, upon writing $b_i \equiv 0$ if $i \geq d$, we have

$$P_{n+1}(x) = (x + 2an + b_1) P_n(x) - n [a^2(n - 1) + 2ab_1 - b_2] P_{n-1}(x)$$

$$+ \sum_{k=2}^{d} \binom{n}{k} (b_{k+1} - 2akb_k + a^2k(k - 1)b_{k-1}) P_{n-k}(x).$$

and

$$Q_{n+1}(x) = (x + a(2n + 1) + b_1) Q_n(x) - n [a^2n + 2ab_1 - b_2] Q_{n-1}(x)$$

$$+ \sum_{k=2}^{d} \binom{n}{k} (b_{k+1} - 2akb_k + a^2k(k - 1)b_{k-1}) Q_{n-k}(x).$$

Accordingly, the following conditions $a < 0$ and $b_i < 0$, $i = 1, \ldots, d$, are sufficient for the zeros to be positive and distinct for both of the latter sequences.

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SOME NEW PERSPECTIVES ON $d$-ORTHOGONAL POLYNOMIALS

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