Stochastic Population Dynamics Driven by Lévy Noise

Jianhai Bao and Chenggui Yuan*
Department of Mathematics,
Swansea University, Swansea SA2 8PP, UK

Abstract

This paper considers stochastic population dynamics driven by Lévy noise. The contributions of this paper lie in that (a) Using Khasminski-Mao theorem, we show that the stochastic differential equation associated with the model has a unique global positive solution; (b) Applying an exponential martingale inequality with jumps, we discuss the asymptotic pathwise estimation of such model.

Keywords: Brownian motion, Lévy noise; Exponential martingale inequality with jumps.
Mathematics Subject Classification (2000) 93D05, 60J60, 60J05.

1 Introduction

Stochastic population dynamics perturbed by Brownian motion has been studied extensively by many authors. There are a great amount of literature on this topic. In [6] Mao, Marion and Renshaw investigate stochastic $n$-dimensional Lotka-Volterra system

$$dX(t) = \text{diag}(X_1(t), \ldots, X_n(t)) \left((b + AX(t))dt + \sigma X(t)dW(t)\right), \quad (1.1)$$

where

$$X = (X_1, \ldots, X_n)^T, \quad b = (b_1, \ldots, b_n)^T, \quad A = (a_{ij})_{n \times n}, \quad \sigma = (\sigma_{ij})_{n \times n},$$

and reveal the important role that the environmental noise can suppress a potential population explosion; Mao, Sabanis and Renshaw [7] further discuss the asymptotic behaviour of population process determined by Eq. (1.1). Then the techniques developed in [6] [7] have been applied successfully to study stochastic delay population dynamics in [2] [9];
functional Kolmogorov-type systems \cite{11, 12}; hybrid competitive Lotka-Volterra models in \cite{8, 13, 14, 15}.

The population may suffer sudden environmental shocks, e.g., earthquakes, hurricanes, epidemics, etc. However, stochastic extension of population process described by Eq. (1.1) cannot explain the phenomena above. To explain these phenomena, introducing a jump process into underlying population dynamics is one of the important methods. To our knowledge there is few systematic work so far in which the noise source is a jump process. The work presented here is to take some steps in this direction, building extensively on the existing results mentioned above for the Brownian motion case, is also a sequel to that of \cite{3}, where competitive Lotka-Volterra population dynamics with jumps

\[
dY(t) = Y(t)\left[(a(t) - b(t)Y(t))dt + \sigma(t)dW(t) + \int_{\mathbb{Y}} \gamma(t, u)\tilde{N}(dt, du)\right]
\]

is investigated, and the explicit solution, sample Lyapunov exponent and invariant measure are also addressed.

We focus in this paper on stochastic population dynamics (1.1) that is further perturbed by Lévy noise, that is,

\[
dx(t) = \text{diag}(X_1(t), \cdots, X_n(t))\left[(b + AX(t))dt + \sigma X(t)dW(t) + \int_{\mathbb{Y}} H(X(t^-), u)\tilde{N}(dt, du)\right]. \tag{1.2}
\]

Here \(b, A, \sigma\) are defined as in model (1.1), \(W(t)\) is a scalar Brownian motion defined on the probability space \(\{\Omega, \mathcal{F}, P\}\) with the filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) satisfying the usual condition, \(X(t^-) := \lim_{s \uparrow t} X(s), N(dt, du)\) is a real-valued Poisson counting measure with characteristic measure \(\lambda\) on measurable subset \(\mathbb{Y}\) of \([0, \infty)\) with \(\lambda(\mathbb{Y}) < \infty\), \(\tilde{N}(dt, du) := N(dt, du) - \lambda(du)dt\), and \(H : \mathbb{R}^n \times \mathbb{Y} \to \mathbb{R}^n\). Throughout this paper, we further assume that \(W\) is independent of \(N\).

In reference to the existing results in the literature, our contributions are as follows:

- We use jump diffusion to model the evolutions of population dynamics when they suffer sudden environmental shocks;
- Using Khasminskii-Mao theorem, we show that the Stochastic Differential Equation (SDE) associated with the model has a unique global positive solution;
- Applying an exponential martingale inequality with jumps, together with the standard Borel-Cantelli lemma, we discuss the asymptotic pathwise estimation of such model.

2 Global Positive Solutions

Since \(X\) denotes the population sizes of the \(n\) interacting species, it is natural to require the solution of Eq. (1.2) not only to be positive but also not to explode in a finite time.
Therefore, in this section we intend to show that Eq. (1.2) has a unique global positive solution under some conditions. Since the coefficients don’t satisfy linear growth condition or weak coercivity condition, even they satisfy local Lipschitz condition, the solutions of Eq. (1.2) may explode in a finite time. Khasminskii [4, Theorem 4.1, p85] and Mao [5] gave the Lyapunov function argument, which is a powerful test for nonexplosion of solutions without linear growth condition and is referred as Khasminskii-Mao theorem. In what follows, we shall also apply Khasminskii-Mao approaches to show that Eq. (1.2) has a unique global positive solution \( X(t), t \geq 0 \). In this section we will show:

- Jump processes can suppress the explosion;
- Brownian motion can also suppress the explosion to our new model, which is similar to that of [6].

In what follows let \( K > 0 \) be a generic constant whose values may vary for its different appearances. To show the main result let us recall the following facts.

Consider 1-dimensional SDE with jumps

\[
dX(t) = F(X(t))dt + G(X(t))dW(t) + \int_{\mathbb{Y}} \Phi(X(t^{-}), u)\tilde{N}(dt, du), \quad t \geq 0
\]

with initial condition \( X(0) = x_0 \in \mathbb{R} \), where \( W(t) \) is a real-valued Brownian motion, \( F, G : \mathbb{R} \to \mathbb{R}, \Phi : \mathbb{R} \times \mathbb{Y} \to \mathbb{R} \).

The following conclusion is given by (see [10, Lemma 2].)

**Lemma 2.1.** Let \( F(0) = G(0) = \Phi(0, u) = 0 \) for \( u \in \mathbb{Y} \) and \( F, G, \Phi \) satisfy local Lipschitz condition. Set

\[
J(x) := \int_{\mathbb{Y}} \left( \ln \left| \frac{x + \Phi(x, u)}{|x|} \right| \right)^2 \lambda(du), \quad x \neq 0.
\]

Assume that

\[
\sup_{0 < |x| \leq m} J(x) < \infty \quad \text{for each } m \geq 1. \tag{2.2}
\]

Then for \( x \neq 0 \)

\[
P(X(t, x) \neq 0 \text{ and } X(t^{-}, x) \neq 0 \text{ for any } t \geq 0) = 1,
\]

where \( X(t, x) \) denotes the solution of Eq. (2.1) starting from \( x \) at time \( t = 0 \).

For convenience of reference, we recall some fundamental inequalities stated as a lemma.

**Lemma 2.2.**

\[
x^r \leq 1 + r(x - 1), \quad x \geq 0, \quad 1 \geq r \geq 0, \tag{2.3}
\]

\[
n^{(1 - \frac{p}{2}) \wedge 0}|x|^p \leq \sum_{i=1}^{n} x_i^p \leq n^{(1 - \frac{p}{2}) \wedge 0}|x|^p, \quad \forall p > 0, \quad x \in \mathbb{R}_+^n, \tag{2.4}
\]

where \( \mathbb{R}_+^n := \{ x \in \mathbb{R}^n : x_i > 0, 1 \leq i \leq n \} \), and

\[
\ln x \leq x - 1, \quad x > 0. \tag{2.5}
\]
2.1 Explosive Suppression by Jump Processes

For jump-diffusion coefficient we assume that

\[(H1)\] For any \(m \geq 1, x \in \mathbb{R}^n, u \in \mathcal{Y}\) and \(i = 1, \cdots, n\)

\[H_i(x, u) > -1, \quad H_i(0, u) = 0, \quad (2.6)\]

\[
\sup_{0 < |x| \leq m} \int_{\mathcal{Y}} (\ln |1 + H_i(x, u)|)^2 \lambda(du) < \infty, \quad (2.7)
\]

and for each \(k > 0\) there exists constant \(L_k > 0\) such that

\[
\int_{\mathcal{Y}} |H(x, u) - H(y, u)|^2 \lambda(du) \leq L_k |x - y|^2 \quad (2.8)
\]

whenever \(x, y \in \mathbb{R}^n\) with \(|x| \vee |y| \leq k\).

Theorem 2.1. Let assumption \((H1)\) hold. Assume further that for \(p \in (0, 1)\) there exist constants \(\delta > 0, \alpha > 2\) such that for \(x \in \mathbb{R}^n, i = 1, \cdots, n\),

\[J_i(x, p) := \int_{\mathcal{Y}} [(1 + H_i(x, u))^p - 1 - pH_i(x, u)] \lambda(du) \leq -\delta |x|^{\alpha} + o(|x|^{\alpha}), \quad (2.9)\]

where \(o(|x|^\alpha)/|x|^\alpha \to 0\) as \(|x| \to \infty\). Then, for any initial condition \(\bar{x} \in \mathbb{R}^n_+\), Eq. \((1.2)\) has a unique global solution \(X(t) \in \mathbb{R}^n_+\) for any \(t \geq 0\) almost surely.

Before we prove the theorem, we give an example such that condition \((2.9)\) holds.

Example 2.1. For \(i = 1, \cdots, n\), let \(\gamma_i > 0\) and \(\int_{\mathcal{Y}} (1 \vee \gamma_i(u)) \lambda(du) < \infty\). Assume that

\[H_i(x, u) := \gamma_i(u)H(|x|), \quad x \in \mathbb{R}^n, u \in \mathcal{Y},\]

where \(H(|x|)\) is a polynomial of degree \(\alpha > 2\) with positive leading coefficient. Then by a straightforward computation we have for some \(\delta > 0\)

\[J_i(x, p) = \int_{\mathcal{Y}} [(1 + \gamma_i(u)H(|x|))^p - 1 - p\gamma_i(u)H(|x|)] \lambda(du)
\leq -\delta |x|^{\alpha} + o(|x|^{\alpha}).\]

Therefore condition \((2.9)\) holds.

Proof of Theorem 2.1. By \((2.8)\), for arbitrary initial value \(\bar{x} \in \mathbb{R}^n_+\) there is a unique local solution \(X(t)\) for \(t \in [0, \tau_\varepsilon)\), where \(\tau_\varepsilon\) is the explosion time. By Eq. \((1.2)\) the \(i\)th component \(X_i(t)\) of \(X(t)\) admits the form

\[
dX_i(t) = X_i(t) \left[ \left( b_i + \sum_{j=1}^{n} a_{ij} X_j(t) \right) dt + \sum_{j=1}^{n} \sigma_{ij} X_j(t) dW(t) + \int_{\mathcal{Y}} H_i(X(t^-), u) \tilde{N}(dt, du) \right]. \quad (2.10)
\]
Note that for any \( t \in [0, \tau_e) \)

\[
X_i(t) = \bar{x}_i \exp \left\{ \int_0^t \left( b_i + \sum_{j=1}^n a_{ij} X_j(s) - \frac{1}{2} \left( \sum_{j=1}^n \sigma_{ij} X_j(s) \right)^2 \right. \right.
\]
\[\left. + \int_\mathbb{Y} (\ln(1 + H_i(X(s), u)) - H_i(X(s), u)) \lambda(du) \right) ds \]
\[\left. + \int_0^t \sum_{j=1}^n \sigma_{ij} X_j(s) dW(s) + \int_0^t \int_\mathbb{Y} \ln(1 + H_i(X(s^-, u))) \tilde{N}(ds, du) \right\}. \]

This, together with \( \bar{x} \in \mathbb{R}_+^n \), yields that \( X_i(t) \geq 0 \) for any \( t \in [0, \tau_e) \). On the other hand, due to (2.7), for Eq. (2.10) condition (2.2) holds. Then Lemma 2.1 gives \( X_i(t) > 0 \) for any \( t \in [0, \tau_e) \) since \( \bar{x} \in \mathbb{R}_+^n \). Next we show \( \tau_e = \infty \) a.s. Let \( k_0 > 0 \) be sufficiently large such that \( |\bar{x}| < k_0 \). For each \( k > k_0 \) define a stopping time \( \tau_k := \inf \{ t \in [0, \tau_e) : |X(t)| > k \} \). Clearly, \( \tau_k \) is increasing as \( k \to \infty \). Set \( \tau_\infty := \lim_{k \to \infty} \tau_k \), whence \( \tau_\infty \leq \tau_e \) a.s., then it is sufficient to check \( \tau_\infty = \infty \) a.s. Introduce a Lyapunov function for any \( p \in (0, 1) \)

\[
V(x) := \sum_{i=1}^n x_i^p, \ x \in \mathbb{R}_+^n. \tag{2.11}
\]

Let \( T > 0 \) be arbitrary. For any \( 0 \leq t \leq \tau_k \wedge T \), the Itô formula yields

\[
dV(X(t)) = \mathcal{L}V(X(t)) dt + p \sum_{i=1}^n X_i^p(t) \sum_{j=1}^n \sigma_{ij} X_j(t) dW(t)
\]
\[\left. + \sum_{i=1}^n \int_\mathbb{Y} [(1 + H_i(X(t^-), u))^p - 1] \tilde{N}(dt, du) X_i^p(t), \right\} \tag{2.12}
\]

where for \( x \in \mathbb{R}_+^n \)

\[
\mathcal{L}V(x) := p \sum_{i=1}^n \left[ b_i + \sum_{j=1}^n a_{ij} x_j - \frac{1-p}{2} \left( \sum_{j=1}^n \sigma_{ij} x_j \right)^2 \right] x_i^p
\]
\[\left. + \sum_{i=1}^n \int_\mathbb{Y} [(1 + H_i(x, u))^p - 1 - pH_i(x, u)] \lambda(du) x_i^p \right\} \tag{2.13}
\]

By inequality (2.4)

\[
K_1(x, p) \leq K|x|^{2+p} + o(|x|^{2+p}),
\]

and, thanks to (2.9)

\[
K_2(x, p) \leq -\delta |x|^\alpha + o(|x|^\alpha).
\]
Thus, for $\alpha > 2$

$$\mathcal{L}V(x) \leq K \quad \text{for any } x \in \mathbb{R}^n_+.$$  \hfill (2.14)

Define for each $u > 0$

$$\mu(u) := \inf\{V(x), |x| \geq u\}.$$  

Thanks to inequality (2.4), it is easy to see that

$$\lim_{u \to \infty} \mu(u) = \infty.$$  \hfill (2.15)

Then we obtain from (2.14) that for some constant $K > 0$

$$\mu(k)\mathbb{P}(\tau_k \leq T) \leq \mathbb{E}(V(X(\tau_k))I_{(\tau_k \leq T)}) \leq \mathbb{E}V(X(\tau_k \wedge T)) \leq K.$$  

Recalling (2.15) and letting $k \to \infty$ yields

$$\mathbb{P}(\tau_\infty \leq T) = 0.$$  

Since $T$ is arbitrary, we must have

$$\mathbb{P}(\tau_\infty = \infty) = 1$$

and Eq. (1.2) admits a unique global solution $X(t) \in \mathbb{R}^n_+$ on $t \geq 0$.

**Remark 2.1.** In [6], under the condition

$$(\text{H2}) \quad \sigma_{ii} > 0 \text{ if } 1 \leq i \leq n \text{ while } \sigma_{ij} \geq 0 \text{ if } i \neq j.$$  

Mao, Marion and Renshaw reveal the important fact that Brownian motion noise can suppress a potential population explosion. Theorem 2.1 shows that Lévy noise can also play the same role, without any conditions being imposed on the diffusion coefficient $\sigma$.

As for population dynamics, in general, the following Lyapunov function for $p \in (0, 1)$

$$U(x) := \sum_{i=1}^{n} [x_i^p - 1 - p \ln x_i], \quad x \in \mathbb{R}^n_+,$$  \hfill (2.16)

is constructed to show that the SDE associated to the model admits a unique global positive solution, see, e.g., [6, 9, 13, 14, 15]. In what follows, under suitable conditions we can also show that Eq. (1.2) has a unique global positive solution through the Lyapunov function defined by (2.16).

**Theorem 2.2.** Suppose that assumptions (2.6), (2.8) and (2.9) hold. Assume further that there exist constants $\beta \in (0, \alpha]$ and $\nu > 0$ such that

$$\int_{\mathbb{Y}} [H_i(x, u) - \ln(1 + H_i(x, u))] \lambda(du) \leq \nu |x|^\beta + o(|x|^\beta)$$  \hfill (2.17)

for $i = 1, \ldots, n$ and $x \in \mathbb{R}^n_+$. Then, for any initial condition $\bar{x} \in \mathbb{R}^n_+$, Eq. (1.2) has a unique global solution $X(t) \in \mathbb{R}^n_+$ for any $t \geq 0$ almost surely.
Proof. Since the argument is similar to that of [6, Theorem 2.1], we here only sketch the proof to point out the variation from the Brownian motion case. Let $k_0 \in \mathbb{N}$ be sufficiently large such that every component of $\bar{x}$ is contained in the interval $(\frac{1}{k_0}, k_0)$. For each $k > k_0$ define a stopping time

$$
\tau_k := \inf \left\{ t \in [0, \tau_e) : X_i(t) \notin \left(\frac{1}{k}, k\right) \text{ for some } i = 1, \ldots, n \right\},
$$

where $\tau_e$ is the explosion time. In the sequel, we show $\tau_\infty := \lim_{k \to \infty} \tau_k = \infty$ a.s. Let $T > 0$ be arbitrary. For any $0 \leq t \leq \tau_k \wedge T$, applying Itô’s formula, we obtain

$$
\mathcal{L}U(x) := p \sum_{i=1}^{n} \left[ b_i(x_i^p - 1) - (x_i^p - 1) \sum_{j=1}^{n} a_{ij} x_j + \left( \frac{p-1}{2} x_i^p + 1 \right) \left( \sum_{j=1}^{n} \sigma_{ij} x_j \right)^2 \right] + \sum_{i=1}^{n} \int_{Y} \left[ (1 + H_i(x, u))^p - 1 - pH_i(x, u) \right] \lambda(du) x_i^p \\
+ p \sum_{i=1}^{n} \int_{Y} \left[ H_i(x, u) - \ln(1 + H_i(x, u)) \right] \lambda(du)
:= I_1(x, p) + I_2(x, p) + I_3(x, p).
\tag{2.18}
$$

By (2.4), note that

$$
I_1(x, p) \leq K |x|^{2+p} + o(|x|^{2+p}).
$$

Also, due to (2.9) and (2.17)

$$
I_2(x, p) + I_3(x, p) \leq -\delta |x|^\alpha + np \nu |x|^\beta + o(|x|^{\alpha+p}).
$$

Then for $p \in (0, 1)$ and $\beta \in (0, \alpha]$

$$
\mathcal{L}U(x) \leq K, \quad x \in \mathbb{R}_+^n.
$$

Define for each $u > 1$

$$
\mu(u) := \inf \left\{ U(x) : x_i \geq u \text{ or } x_i \leq \frac{1}{u} \text{ for some } i = 1, \ldots, n \right\}.
$$

Due to the property of function $h(x) := x - 1 - \ln x, x > 0$, we see that

$$
\lim_{x \uparrow \infty} h(x) = \infty \quad \text{and} \quad \lim_{x \downarrow 0} h(x) = \infty
$$

and hence

$$
\lim_{u \to \infty} \mu(u) = \infty. \tag{2.19}
$$

The proof is then complete by repeating the procedure of [6, Theorem 2.1].
2.2 Explosive Suppression by Brownian Motion

In this subsection, we further show that Brownian motion can also suppress the explosion to our model under condition \((H2)\), but weaker conditions imposed on jump-diffusion coefficient.

**Theorem 2.3.** Assume that assumptions \((H1)\) and \((H2)\) hold. Then, for any initial condition \(\bar{x} \in \mathbb{R}^n_+\), Eq. \((1.2)\) has a unique global solution \(X(t) \in \mathbb{R}^n_+\) for any \(t \geq 0\) almost surely.

**Proof.** Since the proof is very similar to that of Theorem 2.1, we here only give an outline of the argument. In \((2.13)\), note from \((H2)\) and inequality \((2.4)\) that for \(p \in (0,1)\) and \(x \in \mathbb{R}^n_+\)

\[
K_1(x,p) \leq -\frac{p(1-p)n^2}{2} \min_{1 \leq i \leq n} \sigma_{ii} |x|^{2+p} + o(|x|^{2+p}). \tag{2.20}
\]

On the other hand, by inequality \((2.3)\), for any \(p \in (0,1)\) and \(x \in \mathbb{R}^n_+\)

\[
\int_Y [(1 + H_i(x,u))^p - 1 - pH_i(x,u)] \lambda(du) \leq 0,
\]

due to which we have

\[
K_2(x,p) \leq 0, \quad p \in (0,1) \quad \text{and} \quad x \in \mathbb{R}^n_+.
\]

Thus

\[
\mathcal{L}V(x) \leq K \quad \text{for any} \quad x \in \mathbb{R}^n_+.
\]

The conclusion then follows by carrying out the procedure of Theorem 2.1. \(\square\)

Applying Lyapunov function \(U(x)\) in \((2.16)\), under suitable conditions we can still guarantee that Eq. \((1.2)\) admits a unique global positive solution.

**Theorem 2.4.** Let conditions \((2.6), (2.8), (H2)\) hold and assume further that condition \((2.17)\) holds with \(\beta \in (0,2]\). Then, for any initial condition \(\bar{x} \in \mathbb{R}^n_+\), Eq. \((1.2)\) has a unique global solution \(X(t) \in \mathbb{R}^n_+\) for any \(t \geq 0\) almost surely.

**Proof.** The argument is similar to that of Theorem 2.1. In Eq. \((2.13)\), by inequality \((2.3)\)

\[
I_2(x,p) \leq 0, \quad p \in (0,1) \quad \text{and} \quad x \in \mathbb{R}^n_+.
\]

What’s more, condition \((2.17)\) leads to, for \(\beta \in (0,2]\) and \(x \in \mathbb{R}^n_+\),

\[
I_3(x,p) \leq np\nu |x|^\beta + o(|x|^\beta).
\]

Combining \((2.20)\), for \(p \in (0,1)\) and \(\beta \in (0,2]\), we can conclude that for some \(K > 0\)

\[
\mathcal{L}U(x) \leq K, \quad x \in \mathbb{R}^n_+.
\]

Then the proof can be done by carrying out the procedure of Theorem 2.1. \(\square\)
Remark 2.2. By constructing two different Lyapunov functions $V(x)$ and $U(x)$ defined by (2.11) and (2.16), respectively, under different conditions, namely (2.7) and (2.17), we show that Eq. (1.2) has a unique global positive solution. Comparing conditions (2.7) and (2.17), we see that condition (2.7) is a local one while (2.17) has a growth restriction on jump diffusion $H$ in whole space. Therefore, the argument developed in Theorem 2.3 is easier to verify than that of Theorem 2.4.

3 Asymptotic Moment Properties

In the last section, under suitable conditions we have shown that Eq. (1.2) admits a unique global positive solution. From the biological point of view, the nonexplosion property and positive solution in a population dynamical system are often not good enough while the moment properties are more desired. In this section we shall show that the $p$th moment with $p \in (0, 1)$ and the average in time of the moment of the solution to Eq. (1.2) are both bounded.

Theorem 3.1. Under conditions of Theorem 2.1 (or Theorem 2.2, or Theorem 2.3, or Theorem 2.4), for any $p \in (0, 1)$ and some $K > 0$

$$\limsup_{t \to \infty} \mathbb{E}|X(t)|^p \leq K. \quad (3.1)$$

Proof. We only give a sketch of proof under conditions of Theorem 2.1 due to the similarities of arguments. For any $p \in (0, 1)$ let $V$ be defined by (2.11). For any $|\bar{x}| < k$ define a stopping time

$$\sigma_k := \inf\{t \geq 0 : |X(t)| > k\}.$$

By the argument of Theorem 2.1 we have $\sigma_k < \infty$ and $\sigma_k \uparrow \infty$ a.s. as $k \to \infty$. Applying the Itô formula yields

$$\mathbb{E}(e^{t \wedge \sigma_k} V(X(t \wedge \sigma_k))) = V(\xi(0)) + \mathbb{E} \int_0^{t \wedge \sigma_k} e^s[V(X(s)) + \mathcal{L}V(X(s))] ds, \quad (3.2)$$

where $\mathcal{L}V$ is defined by (2.13). Since the leading terms of $\mathcal{L}V$ are $-\delta|x|^{\alpha+p}$ with $\alpha > 2$ we can deduce that there exists a constant $K > 0$ such that $V(x) + \mathcal{L}V(x) \leq K$. Hence

$$\mathbb{E}(e^{tV(X(t)))} \leq K(1 + e^t)$$

and the desired conclusion follows from inequality (2.4).

Going through the arguments of Theorem 2.1 (or Theorem 2.2 or Theorem 2.3 or Theorem 2.4), we can also derive that the average in time of the moment of the solution to Eq. (1.2) is bounded.

Theorem 3.2. (i). Under conditions of Theorem 2.1 or Theorem 2.2 for any $p \in (0, 1)$ and any initial value $\bar{x} \in \mathbb{R}_+^n$, there exists constant $K > 0$ such that

$$\limsup_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{E}|X(s)|^{p+2} ds \leq K.$$
(ii). Under conditions of Theorem 2.3 or Theorem 2.4, for any \( p \in (0, 1) \) and any initial value \( \tilde{x} \in \mathbb{R}_+^n \), there exists constant \( K > 0 \) such that

\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{E}|X(s)|^{p+2}ds \leq K.
\]

**Proof.** Since the proofs of (i) and (ii) are very similar, we here only prove (ii) under the conditions of Theorem 2.3. The argument is motivated by that of [7, Theorem 2]. By (2.12) and (2.13) we have

\[
V(X(t)) \leq V(X(0)) + \int_0^t \left( p \sum_{i=1}^n b_i + p \sum_{i=1}^n \sum_{j=1}^n X_i^p(s)X_j(s) - \frac{p(p-1)}{2} \sum_{i=1}^n \sigma_i^2 X_i^{2+p}(s) \right) ds
\]

\[+ M_1(t) + M_2(t),\]

where \( M_1(t), M_2(t) \) are two local martingales. Noting that the polynomial

\[
p \sum_{i=1}^n b_i + p \sum_{i=1}^n \sum_{j=1}^n x_i^p x_j - \frac{p(p-1)}{4} \sum_{i=1}^n \sigma_i^2 x_i^{2+p}
\]

has a upper bound \( K \) (dependent on \( p \)), therefore

\[
V(X(t)) + \frac{p(p-1)}{4} \sum_{i=1}^n \sigma_i^2 x_i^{2+p}(t) \leq V(X(0)) + Kt + M_1(t) + M_2(t).
\]

Taking the expectation, and then dividing by \( t \) on both sides, we obtain the result. \( \square \)

By taking another different Lyapunov function, we will have the following theorem.

**Theorem 3.3.** Under the conditions of Theorem 2.1 let \( p^T = (p_1, \ldots, p_n) \) be positive numbers such that \( p_1 + \ldots + p_n < 1 \), and assume there exist constants \( \beta_1 \) and \( \beta_2 \in (0, \alpha) \) such that

\[
\int_{\mathbb{Y}} \left[ \Pi_{i=1}^n (1 + H_i(x, u))^{p_i} - \sum_{i=1}^n (1 + H_i(x, u))^{p_i} \right] \lambda(du) \leq \beta_1 |x|^{\beta_2} + o(|x|^{\beta_2}),
\]

where constant \( \alpha > 2 \) was given in (2.9). Then

\[
\mathbb{E} (\Pi_{i=1}^n X_i^{p_i}(t)) < \infty, \text{ for all } t \geq 0.
\]

**Remark 3.1.** For \( n = 1 \), condition (3.3) must be true. Moreover, Example 2.1 also demonstrates that condition (3.3) holds in some cases.

**Proof of Theorem 3.3.** Define a \( C^2 \)-function \( V : \mathbb{R}_+^n \to \mathbb{R}_+ \) by

\[
V(x) := \Pi_{i=1}^n x_i^{p_i}.
\]
Compute
\[ \mathcal{L}V(x) = V(x)p^T(b + Ax) - \frac{1}{2}V(x)x^T \sigma^T[\text{diag}(p_1, \ldots, p_n) - pp^T] \sigma x \]
\[ + V(x) \int_Y \left[ \prod_{i=1}^n (1 + H_i(x,u))^p_i - 1 - \sum_{i=1}^n p_i H_i(x,u) \right] \lambda(du) \]
\[ = V(x)p^T(b + Ax) - \frac{1}{2}V(x)x^T \sigma^T[\text{diag}(p_1, \ldots, p_n) - pp^T] \sigma x \]
\[ + V(x) \sum_{i=1}^n \int_Y \left[ (1 + H_i(x,u))^p_i - 1 - p_i H_i(x,u) \right] \lambda(du) \]
\[ + V(x) \int_Y \left[ \prod_{i=1}^n (1 + H_i(x,u))^p_i - \sum_{i=1}^n (1 + H_i(x,u))^p_i + n - 1 \right] \lambda(du). \quad (3.4) \]

Noting conditions (2.9) and (3.3), we derive that there exist positive constants \( C_1 \) and \( C_2 \) such that
\[ \mathcal{L}V(x) \leq V(x)(C_1 - C_2|x|^\alpha). \]

For each \( k > |\bar{x}| \) define a stopping time
\[ \tau_k := \inf\{t \geq 0 : |X(t)| > k\}. \]

By Theorem 2.1 \( \tau_k < \infty \) and \( \tau_k \to \infty \) as \( k \to \infty \) almost surely. Using the Itô formula we obtain
\[ \mathbb{E}V(X(t \wedge \tau_k)) = V(X_0) + \mathbb{E} \int_0^{t \wedge \tau_k} \mathcal{L}V(X(s))ds \]
\[ \leq V(X_0) + C_1 \mathbb{E} \int_0^{t \wedge \tau_k} V(X(s))ds. \]

Hence applying the well-known Gronwall inequality and letting \( k \to \infty \) gives
\[ \mathbb{E}V(X(t)) \leq V(X_0)e^{C_1t}, \]
and the required assertion follows.

## 4 Asymptotic Pathwise Estimation

In the last section we have discussed how the solutions vary in \( \mathbb{R}^n_+ \) in probability or in moment. In this section we examine pathwise properties of the solutions. To discuss the pathwise properties of Eq. (1.2), we cite the following exponential martingale inequality with jumps, e.g., [5, Theorem 5.2.9, p291].

**Lemma 4.1.** Assume that \( g : [0, \infty) \to \mathbb{R} \) and \( h : [0, \infty) \times Y \to \mathbb{R} \) are both predictable \( F_t \)-adapted processes such that for any \( T > 0 \)
\[ \int_0^T |g(t)|^2 dt < \infty \quad \text{a.s. and} \quad \int_0^T \int_Y |h(t, u)|^2 \lambda(du)dt < \infty \quad \text{a.s.} \]
Then for any constants $\alpha, \beta > 0$

\[
\mathbb{P}\left\{ \sup_{0 \leq t \leq T} \left[ \int_0^t g(s) dW(s) - \frac{\alpha}{2} \int_0^t |g(s)|^2 ds + \int_0^t \int_\mathcal{Y} h(s, u) \tilde{N}(ds, du) \right] > 0 \right\} \leq e^{-\alpha\beta}.
\]

**Theorem 4.1.** Let conditions of Theorem 2.1 hold. Assume further that there exists constant $\theta \in (0, \alpha]$ such that

\[
\int_\mathcal{Y} [(\ln Q(x, u))^2 + Q(x, u)] \lambda(du) \leq K |x|^\theta + o(|x|^\theta),
\]

where, for $p \in (0, 1)$ and $x \in \mathbb{R}_+$

\[
Q(x, u) := \sum_{i=1}^n (1 + H_i(x, u))^p x_i^n / \sum_{i=1}^n x_i^n.
\]

There exists $K > 0$, independent of initial value $\bar{x} \in \mathbb{R}_n^+$, such that the solution $X(t), t \geq 0$, of Eq. (1.2) has the property

\[
\limsup_{t \to \infty} \frac{\ln(|X(t)|)}{\ln t} \leq K, \quad \text{a.s.}
\]

**Remark 4.1.** By inequality (2.4), it is easy to see that condition (4.1) holds for $H_i$ in Example 2.1.

In what follows we complete the argument of Theorem 4.1.

**Proof of Theorem 4.1** Note by Theorem 2.1 that Eq. (1.2) has a unique global positive solution for any initial value $\bar{x} \in \mathbb{R}_n^+$. Let $V(x), K_1(x, p)$ be defined by (2.11) and (2.13), respectively, and $Z(x) := \frac{1}{V(x)} \sum_{i=1}^n p x_i^n \sum_{j=1}^n \sigma_{ij} x_j$ for $x \in \mathbb{R}_n^+$ and $p \in (0, 1)$. Applying the Itô formula to $e^t \ln V(x)$ yields

\[
e^t \ln V(X(t)) = \ln V(\bar{x}) + \int_0^t e^s \left[ \ln V(X(s)) + \frac{1}{V(X(s))} K_1(X(s), p) - \frac{1}{2} Z^2(X(s)) \right] ds
\]

\[
+ \int_\mathcal{Y} \left( \ln Q(X(s), u) - \frac{p}{V(X(s))} \sum_{i=1}^n X_i^p(s) H_i(X(s), u) \lambda(du) \right) ds
\]

\[
+ \int_0^t e^s Z(X(s)) dW(s) + \int_0^t \int_\mathcal{Y} e^s \ln Q(X(s^-), u) \tilde{N}(ds, du).
\]

By virtue of Lemma 4.1 for any $\alpha, \beta, T > 0$ we have

\[
\mathbb{P}\left\{ \omega : \sup_{0 \leq t \leq T} \left[ \int_0^t e^s Z(X(s)) dW(s) - \frac{\alpha}{2} \int_0^t e^{2s} Z^2(X(s)) ds 
\right.
\]

\[
+ \int_0^t \int_\mathcal{Y} e^s \ln Q(X(s^-), u) \tilde{N}(ds, du)
\]

\[
- \frac{1}{\alpha} \int_0^t \int_\mathcal{Y} \left( Q^{ae^s}(X(s), u) - 1 - \alpha e^s \ln Q(X(s), u) \right) \lambda(du) ds \right] \geq \beta \right\} \leq e^{-\alpha\beta}.
\]
Choose $T = k, \alpha = \epsilon e^{-k}$ and $\beta = \frac{2e^k\ln k}{\epsilon}$, where $k \in \mathbb{N}, 0 < \epsilon < \frac{1}{2}$, in the above equation. Since $\sum_{k=1}^{\infty} k^{-2} < \infty$, we can deduce from the Borel-Cantalli lemma that there exists an $\Omega_0 \subseteq \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that for any $\omega \in \Omega_0$ we can find an integer $k_0(\omega) > 0$ such that

$$\int_0^t e^{s}Z(X(s))dW(s) + \int_0^t \int_{\mathbb{Y}} e^{s} \ln Q(X(s)^{-}, u)\tilde{N}(ds, du) \leq \frac{2e^k\ln k}{\epsilon} + \frac{e^{-k}}{2} \int_0^t e^{2s}Z^2(X(s))ds \quad (4.5)$$

whenever $0 \leq t \leq k$ and $k \geq k_0(\omega)$. Hence, for any $\omega \in \Omega_0$, $0 \leq t \leq k$ and $k \geq k_0(\omega)$

$$\ln V(X(t)) \leq e^{-t} \ln V(\bar{x}) + \frac{2e^{k-t} \ln k}{\epsilon}$$

$$+ \int_0^t e^{s-t} \left[ \ln V(X(s)) + \frac{1}{V(X(s))}K_1(X(s), p) - \frac{1}{2} \epsilon Z^2(X(s)) \right] ds$$

$$+ \int_0^t e^{s-t} \int_{\mathbb{Y}} \left( \ln Q(X(s), u) - \frac{p}{V(X(s))} \sum_{i=1}^{n} x_i p H_i(X(s), u) \right) \lambda(du) ds$$

$$+ \frac{1}{e^{t-k}} \int_0^t \int_{\mathbb{Y}} \left( Q^{\epsilon e^{s-k}}(X(s), u) - 1 - \epsilon e^{s-k} \ln Q(X(s), u) \right) \lambda(du) ds$$

$$:= J_1(t) + J_2(t) + J_3(t) + J_4(t).$$

For any $x \in \mathbb{R}_+^n$ and $u \in \mathbb{Y}$, compute that

$$\ln Q(x, u) - \frac{p}{V(x)} \sum_{i=1}^{n} x_i p H_i(x, u)$$

$$= \log Q(x, u) - Q(x, u) + 1 + Q(x, u) - \frac{p}{V(x)} \sum_{i=1}^{n} x_i p H_i(x, u) - 1$$

$$\leq Q(x, u) - \frac{p}{V(x)} \sum_{i=1}^{n} x_i p H_i(x, u) - 1$$

$$= \frac{1}{V(x)} \sum_{i=1}^{n} [(1 + H_i(x, u))^p - 1 - pH_i(x, u)] x_i^p,$$

where in the second step we used the inequality (2.5). In the light of a Taylor’s series expansion, for $\epsilon \in (0, \frac{1}{2}], x \in \mathbb{R}_+^n, u \in \mathbb{Y}$ and $s \leq k$

$$Q^{\epsilon e^{s-k}}(x, u) = 1 + \epsilon e^{s-k} \ln Q(x, u) + \frac{e^2 \epsilon^2 (s-k)}{2} (\ln Q(x, u))^2 Q^\xi(x, u),$$

where $\xi$ lies between 0 and $\epsilon$. Thus

$$J_4(t) = \int_0^t \int_{\mathbb{Y}} \frac{\epsilon e^{2s-t-k}}{2} (\ln Q(x, u))^2 Q^\xi(x, u) \lambda(du) ds.$$
Note that for any \( \omega \in \Omega_0, t \leq k \) and \( k \geq k_0(\omega) \)
\[
J_4(t) = \int_0^t \int_{0<Q(x,u)<1} \frac{\epsilon e^{2s-t-k}}{2} (\ln Q(x,u))^2 Q^\xi(x,u) \lambda(du) ds \\
+ \int_0^t \int_{Q(x,u) \geq 1} \frac{\epsilon e^{2s-t-k}}{2} (\ln Q(x,u))^2 Q^\xi(x,u) \lambda(du) ds \\
= : \Gamma_1(t) + \Gamma_2(t).
\]
For \( 0 < Q(x,u) < 1 \) and \( 0 \leq \xi \leq \epsilon \leq \frac{1}{2} \), we have \( Q^\xi(x,u) \leq 1 \). Hence
\[
\Gamma_1(t) \leq \int_0^t \int_{0<Q(x,u) \leq 1} \frac{\epsilon e^{2s-t-k}}{2} (\ln Q(x,u))^2 \lambda(du) ds \\
\leq \int_0^t \int_{\mathcal{Y}} \frac{\epsilon e^{2s-t-k}}{2} (\ln Q(x,u))^2 \lambda(du) ds.
\]
On the other hand, recalling the fundamental inequality
\[
\ln x \leq 4(x^{\frac{1}{4}} - 1) \text{ for } x \geq 1,
\]
and observing \( Q^\xi(x,u) \leq Q^\frac{1}{4}(x,u) \) for \( Q(x,u) \geq 1 \) and \( 0 \leq \xi \leq \epsilon \leq \frac{1}{2} \), we have
\[
\Gamma_2(t) \leq 16 \int_0^t \int_{Q(x,u) \geq 1} \frac{\epsilon e^{2s-t-k}}{2} Q(x,u) \lambda(du) ds \\
\leq 16 \int_0^t \int_{\mathcal{Y}} \frac{\epsilon e^{2s-t-k}}{2} Q(x,u) \lambda(du) ds.
\]
Consequently, for any \( \omega \in \Omega_0, t \leq k \) and \( k \geq k_0(\omega) \)
\[
J_4(t) \leq \frac{\epsilon}{2} \int_0^t \int_{\mathcal{Y}} e^{s-t} [\ln Q(x,u))^2 + 16Q(x,u)] \lambda(du) ds.
\]
Then, for any \( \omega \in \Omega_0, 0 \leq t \leq k \) and \( k \geq k_0(\omega) \), we have
\[
J_2(t) + J_3(t) + J_4(t) \leq \int_0^t e^{s-t} \left[ \ln V(X(s)) + \frac{1}{V(X(s))} K_1(X(s), p) - \frac{1}{2} Z^2(X(s)) \\
+ \int_{\mathcal{Y}} \sum_{i=1}^n \frac{X_i(s)^p}{V(X(s))} [(1 + H_i(X(s), u))^p - 1 - pH_i(X(s), u)] \lambda(du) \\
+ \frac{\epsilon}{2} \int_{\mathcal{Y}} [\ln Q(X(s), u))^2 + 16Q(X(s), u)] \lambda(du) \right] ds \\
:= \int_0^t e^{s-t} [M_1(X(s), p) + M_2(X(s), p) + M_3(X(s), p)] ds.
\]
By inequality \((2.4)\) and inequality \((2.5)\), for \( x \in \mathbb{R}_+^n \) and \( p \in (0, 1) \) we obtain
\[
M_1(x, p) \leq K|x|^2 + o(|x|^2).
\]

14
Moreover, again by inequality (2.4), together with (2.9) and (4.1), we can deduce that for \( x \in \mathbb{R}^n_+ \) and \( \alpha > 2 \)
\[
M_2(x, p) + M_3(x, p) \leq -K|x|^\alpha + o(|x|^\alpha).
\]
Noting that \( M_1(x, p) + M_2(x, p) + M_3(x, p) \) is bounded by a polynomial with the negative leading coefficient, we arrive at
\[
J_2(t) + J_3(t) + J_4(t) \leq K \int_0^t e^{s-t} ds = K(1 - e^{-t}).
\]
Thus, for any \( \omega \in \Omega_0, 0 \leq t \leq k \) and \( k \geq k_0(\omega) \)
\[
\ln V(X(t)) \leq e^{-t} \ln V(\bar{x}) + \frac{2e^{k-t} \ln k}{\epsilon} + K.
\]
In particular, for \( \omega \in \Omega_0, k - 1 \leq t \leq k \) and \( k \geq k_0(\omega) + 1 \), we have
\[
\frac{\ln(|X(t)|^p)}{\ln t} \leq \frac{\ln V(X(t))}{\ln t} \leq \frac{1}{\ln(k-1)} \left[ e^{-t} \ln V(\bar{x}) + K \right] + \frac{2e \ln k}{\epsilon \ln(k-1)}.
\]
This implies that
\[
\limsup_{t \to \infty} \frac{\ln(|X(t)|)}{\ln t} \leq \frac{2e}{p \epsilon}.
\]
The desired assertion then follows by letting \( \epsilon \uparrow \frac{1}{2} \).

Noting the limit \( \lim_{t \to \infty} \frac{\ln t}{t} = 0 \), we can easily deduce from Theorem 4.1 that the sample Lyapunov exponent of Eq. (1.2) is less or equal to zero, which is stated as the following corollary.

**Corollary 4.1.** Under conditions of Theorem 4.1
\[
\limsup_{t \to \infty} \frac{\ln(|X(t)|)}{t} \leq 0, \quad \text{a.s.}
\]

**Remark 4.2.** Our theories developed can also be applied to discuss stochastic functional Kolmogorov-type population dynamics with jumps and switching-diffusion ecosystems with jumps, respectively, which will be reported in separated papers.

**References**

[1] Applebaum, D., *Lévy Processes and Stochastics Calculus*, Cambridge University Press, 2nd Edition, 2009.

[2] Bahar, A. and Mao, X., Stochastic delay Lotka-Volterra model, *J. Math. Anal. Appl.*, 292 (2004), 364-380.

[3] Bao, J., Mao, X., Yin, G. and Yuan, C., Competitive Lotka-Volterra Population Dynamics with Jumps, *arXiv:1102.2163v1*. 

15
[4] Khasminskii, R., *Stochastic Stability of Differential Equations*, Alphen: Sijthoff and Noordhoff (translation of the Russian edition, Moscow: Nauka 1969), 1980.

[5] Mao, X., A note on the LaSalle-type theorems for stochastic differential delay equations, *J. Math. Anal. Appl.*, **268**, (2002), 125-142.

[6] Mao, X., Marion, G. and Renshaw, E., Environmental Brownian noise suppresses explosions in population dynamics, *Stochastic Process. Appl.*, **97** (2002), 95-110.

[7] Mao, X., Sabanis, S. and Renshaw, E., Asymptotic behaviour of the stochastic Lotka-Volterra model, *J. Math. Anal. Appl.*, **287** (2003) 141-156.

[8] Mao, X. and Yuan, C., *Stochastic Differential Equations with Markovian Switching*, Imperial College Press, 2006.

[9] Mao, X., Yuan, C. and Zou, J., Stochastic differential delay equations of population dynamics, *J. Math. Anal. Appl.*, **304** (2005), 296-320.

[10] Wee, I. S., Stability for multidimensional jump-diffusion processes, *Stoch. Proc. Appl.*, **80** (1999), 193-209.

[11] Wu, F. and Hu, S., Stochastic functional Kolmogorov-type population dynamics, *J. Math. Anal. Appl.*, **347** (2008), 534-549.

[12] Wu, F. and Hu, Y., Existence and uniqueness of global positive solutions to the stochastic functional Kolmogorov-type system, *IMA J. Appl. Math.*, **75** (2010), 317-332.

[13] Yuan, C., Mao, X. and Lygeros, J., Stochastic hybrid delay population dynamics: well-posed models and extinction, *J. Biol. Dyn.*, **3** (2009), 1-21.

[14] Zhu, C. and Yin, G., On hybrid competitive Lotka-Volterra ecosystems, *Nonlinear Anal.*, **71** (2009), e1370-e1379.

[15] Zhu, C. and Yin, G., On competitive Lotka-Volterra model in random environments, *J. Math. Anal. Appl.*, **357** (2009), 154-170.