On scale-free and poly-scale behaviors of random hierarchical networks

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Abstract. In this paper the question of statistical properties of block-hierarchical random matrices is raised for the first time in connection with structural characteristics of random hierarchical networks obtained by a ‘mipmapping’ procedure. In particular, we compute numerically the spectral density of large random adjacency matrices defined via a hierarchy of the Bernoulli distributions \(\{q_1, q_2, \ldots\}\) on matrix elements, where \(q_\gamma\) depends on the hierarchy level \(\gamma\) as \(q_\gamma = p^{-\mu \gamma}\) (\(\mu > 0\)). For the spectral density we clearly see scale-free behavior. We show also that for the Gaussian distributions on matrix elements with zero mean and variances \(\sigma_\gamma = p^{-\nu \gamma}\), the tail of the spectral density, \(\rho_G(\lambda)\), behaves as \(\rho_G(\lambda) \sim |\lambda|^{-(2-\nu)/(1-\nu)}\) for \(|\lambda| \to \infty\) and \(0 < \nu < 1\), while for \(\nu \geq 1\) the power-law behavior is terminated. We also find that the vertex degree distribution of such hierarchical networks has a poly-scale fractal behavior extended over a very broad range of scales.

Keywords: random graphs, networks
1. Introduction

Determinations of the structural organization of a complex system with statistical disorder often refer to a ‘network paradigm’, implying the investigation of topological and statistical characteristics of the contact map between the system elements. Analytical approaches to this task commonly use the well-defined relationship between the spectral properties of the adjacency (connectivity) matrix, which codes the contacts, and a particular topological structure of the network. In the case of the uncorrelated random networks (graphs) known as Erdős–Rényi (ER) graphs [1], there exist a group of methods for analytical study of spectra of the adjacency matrix—from mathematically rigorous to less rigorous. The latter are often borrowed from the statistical mechanics of disordered systems. Among the most used are the replica [2] and the Flory mean-field [3] approaches.

In the last decade, an explosive growth of computational power has allowed us to accumulate a large body of data on statistical and topological characteristics of real networks of diverse nature. The ‘network paradigm’ is currently spread from protein folding, intermolecular contacts in biopolymers, genetic maps and cell metabolism, to natural networks (including the World Wide Web, various ecological, social, financial and economic entities, etc), statistical data analysis, and even to Bose–Einstein condensation. Many of these topics are reviewed in [4]. In the majority of cases it turned out that the statistical characteristics of real network topology, as well as the spectral properties of the adjacency matrices, essentially differ from the characteristics of random ER graphs. For example, it has been found that the probability distributions of typical topological characteristics (e.g. the vertex degree distribution, or the clustering coefficient) have power-law tails for many real networks, in contrast to the exponential tails for ER graphs. Thus, the networks of such topology stood out as a class of ‘scale-free’ networks just because of the power-law behavior of some observables. In practice, the network is often referred to as belonging to the ‘scale-free’ class if the vertex degree distribution (i.e. the distribution of nearest-neighboring contacts over the network vertices) has a power-law tail. However in a general setting, dealing with the analysis of the spectral density of the adjacency matrix, we shall understand as included in ‘scale-free’ behavior the existence of a power-law tail in the spectral density. The topological characteristics of these networks,
such as the vertex degree, have extremely wide distributions exceeding the corresponding distributions for ER graphs by orders of magnitude. Such anomalously wide distributions we shall call ‘poly-scaled’ to distinguish them from ones with ‘scale-free’ behavior.

The main idea of our work consists in the construction of networks with block-hierarchical adjacency matrices resulting from an appropriate randomization of the standard Parisi matrix—one of the key objects in the theory of spin glasses (see [5] for example). In other words, we study below the random block-hierarchical (RBH) networks called in [6] locally constant Parisi (LCP) networks. By the behavior of the spectral density of adjacency matrices, the RBH networks fall into the ‘scale-free’ class, but in the majority of cases, they have ‘poly-scaled’ distribution of the vertex degree.

It would not be an exaggeration to say that the block-hierarchical ‘ordering’ is typical rather than exceptional for many complex systems which are both random and multi-scaled. Examples of such networks can be easily found in different areas of mathematics, physics and biology: from chaotic maps in Hamiltonian systems [7, 8] to condensed (globular) structures of polyelectrolyte chains [9] and hierarchical organization of biopolymers [10]. We anticipate our consideration of spectral properties of the RBH adjacency matrices with a generic example of possible realization of a block-hierarchical contact map in a globular phase of unknotted ring polymer molecules with topological interactions.

It is known that the ‘non-phantomness’ of a polymer chain causes two kinds of interactions: (i) volume interactions vanishing for infinitely thin chains, and (ii) topological interactions, which present even for chains of zero thickness. For sufficiently high temperatures, a polymer molecule strongly fluctuates without a reliable thermodynamic state called a coil state. However for temperatures below some critical value, $\theta$, the polymeric chain exhibits a dense weakly fluctuating globular (drop-like) structure [11, 12]. In classical works [11, 12] devoted to the coil-to-globule phase transition without topological constraints, it has been shown that, for $T < \theta$, the globular state can be described by using just two- and three-body interaction constants: $B = b(T - \theta)/\theta < 0$ and $C = \text{const} > 0$ (see [13, 14]). The approach developed in [11, 12] is regarded as the basic one in modern statistical theory of collapsed polymeric states.

The topological constraints in the globular phase of an unknotted macromolecule act the part of repulsive interactions. For the temperatures below the $\theta$-point (i.e. in a poor solvent), against the temperature and the energy of volume interactions there exists a certain scale of the chain length, $g^*$, such that the chains longer than $g^*$ collapse. Taking a long enough chain, we can define these $g^*$-unit parts as new ‘block monomers’ (or ‘folds of minimal scale’). In figure 1(a) they are denoted as the first-level folds. Sufficiently long parts of the chain with several folds of minimal scale should again ‘collapse on themselves’, i.e. they should form the second-level folds, if other chain parts do not interfere with this. The chain of such new sub-blocks of the second-level folds collapses again forming third-level folds, and so on. This block-hierarchical folding is completed when the initial chain units are united into one fold of the largest scale. The first three consecutive steps of such a process are shown in figure 1(a). Note that the line representing the chain folded in this way resembles the 3D analogue of the well known self-similar Peano curve. The specific feature of the crumpled globule consists in the fact that different chain parts are not entangled with each other, completely fill the allowed volume of space and are ‘collapsed in on themselves’ starting from the characteristic scale $g^*$.
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Figure 1. (a) Three subsequent stages of the construction of a hierarchical contact map for the ‘crumpled globule’ (see the text for details). (b) Block-hierarchical $p$-adic Parisi matrix $T$ ($p = 2$).

The scale of the fold, shown by hues of gray in figure 1(a), can be considered as a cutoff for the interaction distance between $g^*$-block monomers in the current fold. The values $t_i^{(n)}$ can be thought of as constants of interaction between these $g^*$-block monomers in the $\gamma$-level fold and, thus, they constitute the contact map $T$ depicted in figure 1(b).

In [15] and later, more rigorously, in [16] it has been argued that the absence of knots in a densely packed polymer ring causes a very peculiar fractal structure of the chain trajectory, strongly affecting all thermodynamic properties of the macromolecule in the globular phase. The corresponding structure of a collapsed unknotted polymer ring was called a crumpled globule. The chain trajectory in the crumpled globule densely fills the volume, such that all parts of the chain become segregated from each other over a broad region of scales. This model was used later (see, for instance, [17,18]) to describe self-similar hierarchical organization in some biopolymers, like DNA and chromatin.

Below we discuss some topological properties of block-hierarchical random networks. On the basis of the results obtained we propose a new way of building hierarchical networks with scale-free and poly-scaled properties in a memoryless locally uniform way. This method does not demand the control of the current state of the network.

2. ‘Heavy tails’ in the spectral density of random block-hierarchical adjacency matrices

Let us start with a description of a generic procedure of the RBH network construction. Taking $N$ points as potential vertices of our forthcoming network, we raise a hierarchical network by connecting the vertices by edges in a specific way. At the outset, we introduce an ensemble of the $N \times N$ adjacency matrices, $T$, each of them encoding the edges between
connected vertices in a network realization. Namely, the element $T_{i,j}$ of $T$ is 1 if the vertices $i$ and $j$ are directly connected, and otherwise $T_{i,j} = 0$. We consider the adjacency matrix in the very peculiar form of a $p$-adic translation-noninvariant Parisi matrix. This matrix is shown in figure 1(b) for $p = 2$. Obviously, $T_{i,j} = T_{j,i}$ and $T_{i,i} = 0$. All matrix elements, $t^{(n)}_{\gamma}$, are Bernoulli-distributed random variables:

$$
t^{(n)}_{\gamma} = \begin{cases} 
1 & \text{with the probability } q_{\gamma}, \\
0 & \text{with the probability } 1 - q_{\gamma},
\end{cases}
$$

(1)

where $\gamma$ counts the hierarchy levels ($1 \leq \gamma \leq \gamma_{\text{max}} \equiv \Gamma$) and $n$ enumerates different blocks corresponding to a given hierarchy level $\gamma$ (see figure 1(b)). Note that the probability $q_{\gamma}$ does not depend on $n$. The full ensemble of $N \times N$ matrices $T$, where $N = p^{\Gamma}$, is completely determined by the set of probabilities, $\{Q\} = \{q_1, q_2, \ldots, q_{\Gamma}\}$. Thus, the elements $T_{i,j}$, being random variables, are hierarchically organized in probability. Below we consider the set of probabilities, $\{Q\}$, with $q_{\gamma} = p^{-\mu\gamma}$ ($\mu > 0$).

The systematic study of statistical properties of ensembles of random graphs (networks) deals with the investigation of the spectral properties of a graph adjacency matrix [19,20]. Let $\lambda_i$ ($1 \leq i \leq N$) be the eigenvalue of the adjacency matrix. The spectral density of the ensemble of random symmetric adjacency matrices is defined in the standard way,

$$\rho(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \langle \delta(\lambda - \lambda_i) \rangle_{\{q_1, \ldots, q_{\Gamma}\}},$$

(2)

where $\langle \cdot \rangle_{\{q_1, q_2, \ldots, q_{\Gamma}\}}$ denotes the averaging over the distributions of the matrix elements, $t^{(n)}_{\gamma}$.

Computing numerically the spectral density, $\rho(\lambda)$, of networks with block-hierarchical adjacency matrices, we found that the tails of the spectral density $\rho(\lambda)$ follow a power-law asymptotic behavior $\rho(\lambda) \sim |\lambda|^{-\chi}$ with the exponent $\chi = \chi(\mu)$. The sample plots of the spectral density $\rho(\lambda)$ are shown for $N = 256, 2048$ and $\mu = 0.2$ in figure 2(a) in semi-log coordinates. The corresponding log–log plot of the left- and right-hand tails of the spectral density for $N = 256$ is drawn in figure 2(b). It is interesting to note that the right-hand tail of $\rho(\lambda)$, while demonstrating the same behavior, shows poorer averaging.

Some analytic arguments supporting the power-law behavior found for the spectral statistics of the RBH graphs can be introduced by means of the spectral density, $\rho_G(\lambda)$, of the Gaussian ensemble of the Parisi matrices $T$, where the Bernoulli distributions on matrix elements of $T$ are replaced by the Gaussian distributions with zero means and a set of variances $\{S\} = \{\sigma_1, \sigma_2, \ldots, \sigma_{\Gamma}\}$, $\sigma_{\gamma} = p^{-\nu\gamma}$ ($\nu > 0$). To get the spectral density $\rho_G(\lambda)$, note the eigenvalues of the standard (i.e. translation-invariant) Parisi matrix with $T_{i,i} = 0$ can be expressed in terms of matrix elements $t^{(n)}_{\gamma}$ as follows ([21]):

$$\lambda_{\gamma} = p^\gamma t_{\gamma} - (1 - p^{-1}) \sum_{\gamma' = 1}^{\gamma} p^{\gamma'} t_{\gamma'}, \quad (\gamma = 1, \ldots, \Gamma).$$

(3)

The eigenvalue $\lambda_{\gamma}$ is $p^{\Gamma-\gamma}$ times degenerate ($\gamma = 1, \ldots, \Gamma$). In addition there is one extra eigenvalue $\lambda_0 = -\sum_{\gamma=1}^{\Gamma} p^{\Gamma-\gamma} \lambda_{\gamma}$. For the translation-noninvariant Parisi matrix
we generalize (3) in a way similar to that used for block-hierarchical kinetic matrices (see [22]). Recall that for eigenvalues of kinetic matrix, one has

$$\lambda_{\gamma,n} = -p^n t_{\gamma}^{(n)} - (1 - p^{-1}) \sum_{\gamma' = \gamma+1}^{\Gamma} p^{\gamma'} t_{\gamma'}^{(n')},$$

i.e. the eigenvalue $\lambda_{\gamma,n}$ can be expressed via a linear combination ($\Sigma$ in (4)) of weighted matrix elements $t_{\gamma'}^{(n')}$ coming from the vertices $(\gamma', n')$ along a unique path on the $p$-adic Cayley tree from the vertex $(\gamma = \gamma, n' = n)$ towards the root vertex $(\gamma = \Gamma, n' = 1)$—see [6] for more details. In the framework of the same geometrical interpretation, the eigenvalue $\lambda_{\gamma,n}$ of the block-hierarchical matrix with $T_{i,i} = 0$ (shown in figure 1(b)) seems to be a linear combination of weighted matrix elements $t_{\gamma'}^{(n')}$ along a path on the $p$-adic Cayley tree from the bottom level $\gamma' = 1$ to the vertex $(\gamma' = \gamma, n' = n)$. However this construction has ambiguity since the corresponding path is not uniquely defined by the pair $(\gamma, n)$. So, rigorously speaking, the eigenvalues of a particular realization of the random block-hierarchical adjacency matrix cannot be parameterized by the pairs $(\gamma, n)$, and we cannot write an exact expression for the eigenvalues in the form of expression (3). However for computation of the spectral density, $\rho_G(\lambda)$, we use below a posteriori self-averaging arguments, which make our consideration self-consistent. Moreover, the extensive numerical simulations confirm our analytic prediction of $\rho_G(\lambda)$ for $|\lambda| \gg 1$ in the interval $0 < \nu < 1$. Thus, formally extending (3) to the case of $\lambda_{\gamma,n}$ for the block-hierarchical matrix $T$, we replace in (3) the first term by $t_{\gamma}^{(n)}$ and the sum by $\sum_{\gamma' = 1}^{\Gamma} p^{\gamma'} t_{\gamma'}^{(n')}$, where the summation runs now along the paths on the Cayley tree from the hierarchical level $\gamma' = 1$ to the vertex $(\gamma, n)$ located on the hierarchical level $\gamma' = \gamma$ (compare to (4)).
Supposing the distribution of the matrix elements $t^{(n)}_{\gamma}$ to be Gaussian,

$$P(t^{(n)}_{\gamma}) = \frac{1}{\sqrt{\pi} \sigma^{2}_{\gamma}} \exp \left( -\frac{(t^{(n)}_{\gamma})^{2}}{\sigma^{2}_{\gamma}} \right),$$  \hspace{1cm} (5)$$

and using for $\lambda_{\gamma,n}$ the guessed expression as a linear combination of matrix elements, we end up with the following equation for the spectral density, $\rho_{G}(\lambda)$:

$$\rho_{G}(\lambda) = p^{-1} \sum_{\gamma,n} (\delta(\lambda - \lambda_{\gamma,n})) P(t^{(n)}_{\gamma}) = \frac{1}{\sqrt{\pi}} \sum_{\gamma=1}^{\Gamma} p^{-\gamma} \frac{1}{\sqrt{u^{2}_{\gamma}}} \exp \left( -\frac{\lambda^{2}}{u^{2}_{\gamma}} \right),$$ \hspace{1cm} (6)

where for $\sigma_{\gamma} = p^{-\nu_{\gamma}}$ we have

$$u^{2}_{\gamma} = p^{2(\nu-1)}\sigma^{2}_{\gamma} + (1 - p^{-1}) \sum_{\gamma' = 1}^{\gamma-1} p^{2\nu_{\gamma'}}\sigma^{2}_{\gamma'} = \frac{p - 2}{p} p^{2(1-\nu)\gamma} + \frac{(p - 1)^{2}}{p - p^{\nu}} (p^{2(1-\nu)\gamma} - 1).$$ \hspace{1cm} (7)

For $p = 2$ we can rewrite $\rho_{G}(\lambda)$ in (6) for $\Gamma \rightarrow \infty$ as follows:

$$\rho_{G}(\lambda) \simeq \frac{1}{\sqrt{\pi}} \sum_{\gamma=1}^{\infty} 2^{-(2-\nu)\gamma} \exp \left[ -\lambda^{2} \frac{4 - 4^{\nu}}{2^{2(1-\nu)\gamma} - 1} \right].$$ \hspace{1cm} (8)

Taking into account that

$$\sum_{\gamma=1}^{\infty} p^{c_{1}\gamma} e^{-t^{2}p^{-c_{2}\gamma}} \simeq t^{-c_{1}/c_{2}} \quad (t \gg 1),$$ \hspace{1cm} (9)

and substituting $c_{1} = 2 - \nu$ and $c_{2} = 1 - \nu$, we arrive at the following asymptotic form for the spectral density at $|\lambda| \gg 1$:

$$\rho_{G}(\lambda) \simeq |\lambda|^{-\xi(\nu)} \quad (0 < \nu < 1),$$ \hspace{1cm} (10)

where

$$\xi(\nu) = \frac{2 - \nu}{1 - \nu}. $$ \hspace{1cm} (11)

The arguments supporting our derivation of the expression (7) for the spectral density $\rho_{G}(\lambda)$ are as follows. First of all, note that (6) and (7) become exact if we skip the dependence on $n$ in the matrix elements $t^{(n)}_{\gamma}$ and, hence, restore the translational invariance in the block-hierarchical matrix $T$. Secondly, we found in the extensive numerical simulations summarized in figure 3 that the conjectured behavior (10) and (11) does indeed actually hold for translation-noninvariant Parisi matrices. In figure 3 we have plotted the tails of the spectral density $\rho_{G}(\lambda)$ for the Gaussian ensemble of block-hierarchical matrices for $N = 256$. The solid and dot–dashed lines have the slopes $-\xi(\nu)$, where $\xi(\nu = 0) = 2$ (solid line) and $\xi(\nu = 0.8) = 6$ (dashed line). The scatter graphs from top to bottom correspond to $\nu = 0.0, 0.2, 0.4, 0.6, 0.8, 1.0$.

The fact that (10) gives the right asymptotic behavior occurs apparently due to an effective self-averaging of the sum of matrix elements along each particular path on a Cayley tree for the distribution $\{S\} = \{\sigma_{1}, \sigma_{2}, \ldots\}$. One sees that in the sum in (7) for $\sigma_{\gamma} = p^{-\nu_{\gamma}}$ the lower limit of the summation can be shifted from $\gamma' = 1$ to $\gamma' \rightarrow -\infty$. 

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Figure 3. Spectral density for symmetric Gaussian translationally noninvariant block-hierarchical matrices. The solid and dot-dashed lines have the slopes $\xi(\nu = 0) = 2$ and $\xi(\nu = 0.8) = 6$; scatter graphs: $\nu = 0$ (+), $\nu = 0.2$ ($\Phi$), $\nu = 0.4$ ($\nabla$), $\nu = 0.6$ ($\Delta$), $\nu = 0.8$ ($\bullet$), $\nu = 1.0$ ($\blacksquare$).

Asymptotically the result for $u_2^2$ will remain unchanged if $0 < \nu < 1$. Such an extension of the summation means that the computation of the spectral density (and, in particular, of $\lambda_{\gamma,n}$) involves the summation along the infinite paths running from $-\infty$ to the hierarchical level $\gamma$. We expect that for $\sigma_\gamma = p^{-\nu\gamma}$ ($0 < \nu < 1$) due to the convergence of the sum $\sum_{\gamma'=\infty} p^{2\gamma'}\sigma_{\gamma'}^2$, the eigenvalue $\lambda_{\gamma,n}$ does not depend on each particular path on a Cayley tree and, hence, the eigenvalue does not depend on the index $n$. Once this point of view is accepted, we return to an effective translation-invariant block-hierarchical matrix for which (8) is exact.

One can see from (10), (11) and figure 3 that for $\nu \geq 1$ the power-law behavior of the spectral density $\rho_G(\lambda)$ terminates. This termination deserves special attention. Indeed, for $\nu \geq 1$ we cannot extend the lower limit of summation over $\gamma'$ to $-\infty$ since the corresponding sum diverges. Hence, the contributions to the eigenvalues (and, therefore, to the spectral density) strongly depend on the particular configuration of the path. In this case the self-averaging, suggested in our computation, is invalid now and we cannot say anything about the behavior of the spectral density in the region $\nu \geq 1$. The opposite case $\nu < 0$ deserves special attention because formally the sum (7) converges for any $\nu < 0$ and our arguments concerning self-averaging seem to work. However the details of the analysis of this case are beyond the scope of the current work and will be discussed elsewhere.

Our approach to the construction of the networks with the scale-free behavior in the spectral density is not unique. As is shown in the recent work [23], the procedure of dividing Gaussian matrices by a random variable, as well as the same procedure applied to random graphs, leads to the spectral density interpolating between the Erdős–Rényi and the scale-free models.
3. Hierarchical ‘growing’ of poly-scaled networks

One can roughly distinguish two methods of constructing the scale-free networks. The first method has been developed mainly for illustrative purposes and deals with the hierarchical construction [24] of deterministic scale-free graphs with predetermined fractal properties such as, say, vertex degree distribution. Since in this case the graphs obtained are deterministic, it is senseless to talk about any statistics of their spectra. The second method deals with the variants of iterative ‘preferential attachment’ construction [25], where new nodes are added to a vertex of the network with the probability depending on the already existing vertex degree (the number of one-step connections to other vertices). Almost all known statistical characteristics of scale-free networks, including the spectral density of the adjacency matrix, are obtained for the networks constructed using this method. Typically, the spectral density of the ensemble of scale-free networks designed by the preferential attachment method has a triangle-like shape in the ‘bulk’ part with power-law tails.

Note that the construction of scale-free networks by the preferential attachment method is based on locally nonuniform incremental growth with unlimited evolutionary memory. In contrast to this, our proposal below is another physically motivated approach to the construction of the random networks with power-law spectral density by a ‘parallel’ (i.e. non-stepwise) and uniform procedure.

The peculiarity of our construction consists in the following. We build clusters of edges with hierarchically organized probabilities, while the typical procedure consists in hierarchical grouping of vertices. Our procedure does not impose any additional metric structure on the graph and leaves the graph (network) purely topological. On the contrary, grouping of vertices imposes a metric structure on the graph (network) since such a grouping operates usually with the notion of ‘close’ (or ‘distant’) vertices.

To shed light on spectral properties of random hierarchical graphs we exploit the link between the random matrix theory (RMT) and the graph theory (GT). It is known that the spectral density $\rho(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \langle \delta(\lambda - \lambda_i) \rangle_{\{q_1,q_2,\ldots,q_r\}}$ of the ensemble of adjacency matrices is directly related to the topological structure of the corresponding network since the value

$$M_k = \frac{1}{N} \int \lambda^k \rho(\lambda) \, d\lambda = \frac{1}{N} \sum_{i=1}^{N} \lambda_i^k$$

(12)

defines (up to the factor $N$) the average number of $k$-step loops in the network (see, for example, [19]).

In particular, in [26] it has been shown that in the thermodynamic limit the spectral properties of random Erdős–Rényi graphs [27] coincide with the spectral properties of random real symmetric matrices. This result is one of the benchmarks of our consideration. The elements of the adjacency matrix, $A_{i,j}$, of the Erdős–Rényi graph are Bernoulli-distributed random variables: $A_{i,j} = 1$ or $0$ with probabilities $q$ and $1-q$ correspondingly. For the ensemble of ER random graphs the density, $\rho_A(\lambda)$, of eigenvalues of adjacency matrices $A$ can be analytically computed in the thermodynamic limit $N \to \infty$ and has for some $q$ a celebrated Wigner–Dyson semicircle law known for ensembles of Gaussian matrices [28]. Namely, the following statement is proved [26]. Let $B$ be a real symmetric $N \times N$ matrix with independently distributed entries $B_{i,j}$ from, say, the Gaussian
Figure 4. Comparison of spectral densities of the random block-hierarchical matrix and Erdős–Rényi (ER) random graphs: (a) in semi-logarithmic coordinates in the region $-0.5 < \lambda/N < 0.5$; and (b) in linear coordinates in the region $-0.05 < \lambda/N < 0.05$.

distribution $P(B_{i,j})$ with $\langle B_{i,j} \rangle = 0$ and $\langle B_{i,j} \rangle = \sigma^2$. Then the spectral density, $\rho_B(\lambda)$, of the ensemble of matrices $B$ converges in the limit $N \to \infty$ to the semicircle distribution

$$
\rho_B(\lambda) = \begin{cases} 
\frac{1}{2 \pi \sigma^2} \sqrt{4N\sigma^2 - \lambda^2} & \text{if } |\lambda| < \sqrt{4N\sigma^2}, \\
0 & \text{if } |\lambda| > \sqrt{4N\sigma^2}.
\end{cases}
$$

(13)

If $\sigma^2 = q(1-q)$, then $\rho_A(\lambda) = \rho_B(\lambda)$ for $N \to \infty$, i.e. the spectral densities of ensembles of random ER graphs and of Gaussian symmetric matrices coincide in the thermodynamic limit. Nevertheless, such a coincidence of spectral densities for random graphs and random Gaussian matrices should not be understood in a literal sense: some spectral properties of random ER graphs and random matrices are different [19, 29]. For example, since for the adjacency matrix $A$ of random ER graphs one has $\langle A_{i,j} \rangle = q$, then the corresponding largest eigenvalue, $\lambda_1$, grows linearly with the system size, $N$, i.e. $\lambda_1 = Nq$, meaning that the semicircular distribution for random graphs is valid only for the matrix $A - \langle A \rangle$. Also, the tails of spectral distributions near the spectrum edges are different for random graphs and random matrices. Nevertheless, as a first approximation, the random Gaussian matrices could serve as a very natural benchmark for the corresponding statistical analysis.

The comparison of spectral properties of hierarchical and random Erdős–Rényi graphs is demonstrated in figure 4. Figure 4(a) shows the semi-log plot of the spectral densities for: (i) random hierarchical graphs with LCP adjacency matrices for $N = 256$ and $\mu = 0.2$ (solid line), and (ii) random Erdős–Rényi graphs for $N = 256$ and $p = 0.2$ (dotted line), for $N = 256$ and $p = 0.02$ (dashed line). In figure 4(b) we have redrawn the central part of figure 4(a) on a linear scale.

The numerical results for the probability distributions $P(M_k)$ ($k = 2, 3$) of the number of $k$-step loops in random hierarchical graphs for $N = 256$ and $\mu = 0.2$ are compared in figure 5 with the distributions of loops on Erdős–Rényi random graphs for $N = 256$ and $p = 0.2$. 

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Recall that $P(M_2)$ defines the probability of having in the finite graph the average connection degree equal to $M_2$.

One sees that the distribution functions $P(M_2)$ and $P(M_3)$ for our hierarchical random graphs are much broader than the corresponding distributions for Erdős–Rényi graphs with the same number of vertices. Hence, the topological structure of random hierarchical graphs is much more ‘flexible’ than that of random ER graphs. This is consistent with the behavior found for the spectral density: the distribution function $\rho(\lambda)$ has ‘heavy’ tails and decays much more slowly than that for random ER graphs. According to the behavior of the distribution functions $P(M_2)$ and $P(M_3)$, it is natural to call our random block-hierarchical graph ‘poly-scaled’.

The method of generating functions allows us to compute easily the vertex degree distribution in the ensemble of random block-hierarchical graphs directly from their adjacency matrices. To do that, let us consider any (for example, the first) row in the adjacency matrix $T$ (see figure 1(b)). The total number of links of the first graph vertex to other vertices is defined by the number of matrix elements, $t^{(1)}_{\gamma}$ ($\gamma = 1, \ldots, \Gamma$), having in the first row a nonzero value (i.e. taking the value ‘1’). Thus, the distribution of the number of connections (i.e. vertex degree distribution), $\mathcal{P}(m)$, is the probability of the sum of matrix elements in the first row being exactly equal to $m$ under the condition that the matrix elements are grouped in hierarchical blocks and have the binomial distributions $\{q_1, q_2, \ldots, q_\Gamma\}$ as defined in (1). Finally we arrive at the following expression for the degree distribution $\mathcal{P}(m)$ (for the sake of simplicity we have used the notation $q_\gamma(t^{(1)}_{\gamma}) \equiv q_\gamma(t_\gamma)$):

$$
\mathcal{P}(m) = \sum_{\{t_1, \ldots, t_\Gamma\}} \left[ \prod_{\gamma=1}^{\Gamma} q_\gamma(t_\gamma) \right] \Delta \left( \sum_{\gamma=0}^{\Gamma} p^{\gamma} t_{\gamma+1} - m \right),
$$

where the binomial distributions $q_\gamma(t_\gamma)$ have the form

$$
q_\gamma(t_\gamma) = p^{-\mu \gamma} \delta_{t_\gamma,1} + (1 - p^{-\mu \gamma}) \delta_{t_\gamma,0}
$$
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Figure 6. (a) The family of distributions $\mathcal{P}(m)$ for $\Gamma = 16$ at $\mu = 0.1, 1.0$; (b) distribution $\mathcal{P}(m)$ on a double-logarithmic scale for $\mu = 0.1$ and $\Gamma = 16$; (c) comparison of $\mathcal{P}(m)$ for the random block-hierarchical graph ($\mu = 1.0, \Gamma = 16$) and $\mathcal{P}_{ER}(m)$ for the Erdös–Rényi graph ($q = 0.1, 0.5, N = 2^{16}$).

and $\Delta(\cdot \cdot \cdot)$ is the Kronecker symbol:

$$\Delta(x) = \frac{1}{2\pi i} \oint dz z^{-x-1} = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

Substituting (15) and (16) in (14), after elementary transformations we get

$$\mathcal{P}(m) = \frac{1}{2\pi i} \oint dz z^{-(m+1)} \prod_{\gamma=1}^{\Gamma} W(z, \gamma),$$

where

$$W(z, \gamma) = p^{-\mu} z^{p^\gamma} + 1 - p^{-\mu}. \quad (18)$$

For not too large values of $\Gamma$, the distribution $\mathcal{P}(m)$ can be analyzed numerically. Using the fact that the function $W(z, \gamma)$ is a polynomial of $z$, let us represent $\mathcal{P}(m)$ as

$$\mathcal{P}(m) = \frac{1}{(m+1)!} \frac{d^{m+1} [\prod_{\gamma=1}^{\Gamma} W(z, \gamma)]}{dz^{m+1}} \bigg|_{z=0}.$$ 

In figure 6(a) we have depicted the family of curves $\mathcal{P}(m)$ for $\Gamma = 16$ and $\mu = 0.1, 1.0$. For comparison, in figure 6(c) we have plotted the distribution $\mathcal{P}(m)$ for $\mu = 1.0, \Gamma = 16$ as well as the binomial distribution $\mathcal{P}_{ER}(m) = C_{N}^{m} q^{m}(1-q)^{N-m}$ for the standard Erdös–Rényi graph with the number of vertices $N = 2^{16} = 2^{16}$ and for two values $q = 0.1, 0.5$. One can see in figure 6(c) how much wider the distribution $\mathcal{P}(m)$ is with respect of the corresponding distribution $\mathcal{P}_{ER}(m)$ for ER graphs.

The fractal structure of the distribution $\mathcal{P}(m)$ for random hierarchical networks shown in figures 6(a) and (b) is apparently deeply linked to the invariant multifractal measures appearing in chaotic Hamiltonian systems in connection with the problems of 'number-theoretic chaos'—see, for example, [30]. Actually, the condition $\Delta(\sum_{\gamma=1}^{\Gamma} p^\gamma t_{\gamma} - m)$ in equation (14) for $\Gamma \to \infty$ is nothing but the binary expansion of the number $m$: $m = t_{1} 2^{0} + t_{2} 2^{1} + t_{3} 2^{2} + \cdots + t_{\gamma+1} 2^{\gamma} + \cdots$, where the coefficients $t_{\gamma}$ take values 1 or 0 with corresponding probabilities $q_{t_{\gamma}}$ defined in (15). Let us note that a similar expansion of the form $\sum_{k=1}^{\infty} \varepsilon_{k} u^{-k}$, where $u > 1$ and $\varepsilon_{k} = \pm 1$ (with equal probabilities...
independent of \( k \), is known in the literature as the ‘singular Erdös measure’ \([31]–[33]\). The observed fractal structure of the distribution function \( P(m) \) appears due to the effects of the incommensurability of number-theoretic origin: some binary expansions (of the number \( m \)) with random coefficients have relatively high probabilities of appearing in the adjacency matrix, while other binary expansions have much less possibility.

4. Conclusion

First of all, let us note that our consideration of spectral properties of random hierarchical graphs is far from being complete and many other properties are of interest (for example, eigenvectors, inverse participation ratio, etc—see, for instance \([20]\)). However even in this preliminary investigation we would like to emphasize the crucial difference between random hierarchical and random Erdös–Rényi graphs.

Roughly speaking, there are two generic ways of achieving scale-free network construction. The similarities and differences of these two approaches we would like to emphasize below, again.

- The first way, widely discussed in the literature, deals basically with the ‘preferential attachment’ procedure, where the network is raised by the essentially non-Markovian (in the increments) evolution process with unlimited memory. An evolutionary process of such a kind can be viewed, to some extent, as a ‘generalized Brownian motion’. The realization of such a procedure demands the monitoring of the whole network structure at each step, because the appearance of new links depends on the current degree of graph vertices. From this point of view the sequential construction of the corresponding network can be tentatively denoted as a ‘nonlinear evolution’.

- The second way, discussed in the present work, exploits an essentially different mechanism of scale-free network formation. The hierarchical organization of \( \text{probabilities} \) of links in the topological network constitutes the basic idea of our construction. Specifically, we construct networks with the scale-free eigenvalue distribution of the adjacency matrices, where the latter originates from an appropriate randomization of the standard Parisi matrix—one of the key objects in the theory of spin glasses. Our method allows one to build a hierarchical network in a memoryless locally uniform way. The application of the elements of the \( p \)-adic analysis \([34]\) permits us to analyze the basic spectral properties of ensembles of randomized Parisi-type adjacency matrices.

To finish, let us emphasize that there are two important features of hierarchical networks constructed in our paper. First of all, any sub-graph belonging to a particular hierarchy level, \( \gamma \), is just an independent graph because the formation of clusters of bonds on each hierarchy level is entirely uncorrelated. Secondly, the random sub-graphs, associated with different hierarchy levels of the network, can be different, so the network as a whole can be essentially nonuniform. Nevertheless as we have seen, the ‘mipmapping’ construction with different sets of parameters (i.e. the hierarchical embeddings of sub-graphs corresponding to the different hierarchical levels) leads under some conditions to scale-free behavior in the spectral density of the adjacency matrix and to poly-scale (and even to fractal) behavior in various topological characteristics of the graph. This
observation is rather unexpected since in our case the scale-free behavior is reached by essentially Markovian and memoryless procedures.

In physics, the random graphs of such a hierarchical genesis can be encountered among the scale-free networks whose natural origins are associated with random events with low correlation, occurring with short evolutionary memory. In particular, the networks of hierarchical genesis may by interesting as regards prebiology or the earliest biology.

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