Harmonic mappings of an annulus, Nitsche conjecture and its generalizations

Tadeusz Iwaniec, Leonid V. Kovalev, Jani Onninen

American Journal of Mathematics, Volume 132, Number 5, October 2010, pp. 1397-1428 (Article)

Published by Johns Hopkins University Press
DOI: https://doi.org/10.1353/ajm.2010.0000

For additional information about this article
https://muse.jhu.edu/article/396804/summary
HARMONIC MAPPINGS OF AN ANNULUS,
NITSCHÈ CONJECTURE AND ITS GENERALIZATIONS

By Tadeusz Iwaniec, Leonid V. Kovalev, and Jani Onninen

Abstract. As long ago as 1962 Nitsche conjectured that a harmonic homeomorphism $h: A(r, R) \rightarrow A(r^*, R^*)$ between planar annuli exists if and only if $\frac{R_*}{r_*} \geq \frac{1}{2} \left( \frac{R}{r} + \frac{r}{R} \right)$. We prove this conjecture when the domain annulus is not too wide; explicitly, when $R \leq e^{3/2} r$. We also treat the general annuli $A(r, R)$, $0 < r < R < \infty$, and obtain the sharp Nitsche bound under additional assumption that either $h$ or its normal derivative have vanishing average along the inner circle of $A(r, R)$. We consider the family of Jordan curves in $A(r, R)$ obtained as images under $h$ of concentric circles in $A(r, R)$. We refer to such family of Jordan curves as harmonic evolution of the inner boundary of $A(r, R)$. In the borderline case $\frac{R_*}{r_*} = \frac{1}{2} \left( \frac{R}{r} + \frac{r}{R} \right)$ the evolution begins with zero speed. It will be shown, as a generalization of the Nitsche Conjecture, that harmonic evolution with positive initial speed results in greater ratio $\frac{R_*}{r_*}$ in the deformed (target) annulus. To every initial speed there corresponds an underlying differential operator which yields sharp lower bounds of $\frac{R_*}{r_*}$ in our generalization of the Nitsche Conjecture.

Contents.
1. Introduction and overview.
   1.1. The Nitsche bound.
   1.2. Extremal mappings.
2. Background.
3. Circular means of a harmonic evolution.
4. Extremal mappings.
5. Convexity operators.
6. Variance is a subsolution to all $\Lambda^\gamma$.
7. Proof of Theorems 1.5(i) and 1.7.
8. Proof of Theorem 1.4.
   8.1. The case $1 < R \leq e$.
   8.2. The case $e < R \leq e^{3/2}$.
9. Proof of Theorem 1.5(ii).
10. Proof of Theorem 1.6.
References.

Manuscript received March 15, 2009.
Research of the first author supported by NSF grant DMS-0800416 and the Academy of Finland grant 1128331; research of the second author supported by NSF grants DMS-0913474 and DMS-0968756; research of the third author supported by NSF grants DMS-0701059 and DMS-1001620.

American Journal of Mathematics 132 (2010), 1397–1428. © 2010 by The Johns Hopkins University Press.
1. Introduction and overview. The Riemann Mapping Theorem tells us that planar simply connected domains (different from \( \mathbb{C} \)) are conformally equivalent. Annuli are the first place one meets obstructions to the existence of conformal mappings. The famous theorem due to Schottky (1877) \([14]\), asserts that an annulus

\[ A = A(r, R) = \{ z \in \mathbb{C} : r < |z| < R \}, \quad 0 < r < R \tag{1.1} \]

can be mapped conformally onto the annulus

\[ A^* = A(r^*, R^*) = \{ w \in \mathbb{C} : r^* < |w| < R^* \}, \quad 0 < r^* < R^* \tag{1.2} \]

if and only if \( \text{Mod} A := \log \frac{R}{r} = \log \frac{R^*}{r^*} =: \text{Mod} A^* \); that is,

\[ \frac{R}{r} = \frac{R^*}{r^*}. \tag{1.3} \]

Moreover, modulo rotation, every conformal mapping \( h: A \to A^* \) takes the form

\[ h(z) = \frac{r^*}{r} z \quad \text{or} \quad h(z) = \frac{r^* R}{z}. \tag{1.4} \]

Note that the latter map, though orientation preserving, reverses the order of the boundary circles. Such a mapping problem becomes more flexible if we admit harmonic mappings \( h: A \onto A^* \) in which the real and imaginary parts need not be harmonic conjugate. A univalent (one-to-one) complex-valued harmonic function will be referred to as harmonic homeomorphism. We denote by \( \mathcal{H}(A, A^*) \) the class of orientation preserving harmonic homeomorphisms \( h: A \onto A^* \) which preserve the order of the boundary circles, see Section 2 for a brief discussion of annuli and homeomorphisms between them. For a recent account of the theory of harmonic mappings we refer to the book by P. Duren \([4]\).

J. C. C. Nitsche \([12]\) showed that the annulus \( A \) cannot be mapped by a harmonic homeomorphism onto \( A^* \) if the target is conformally too thin compared to \( A \). Let us devote a few lines to a simple proof of this fact via normal family arguments. Suppose, to the contrary, that we are given harmonic homeomorphisms \( h_j: A \onto A(1, r_j) \), where \( r_j \searrow 1 \). There is a subsequence, still denoted by \( \{ h_j \} \), which converges to a harmonic function \( h: A \to \mathbb{C} \) uniformly together with its derivatives on compact subsets of \( A \). We have \( |h(z)|^2 \equiv 1 \) on \( A \). Hence

\[ 0 = \Delta |h|^2 = 4 \frac{\partial^2}{\partial z \bar{z}} (h\bar{h}) = |h_z|^2 + |h_{\bar{z}}|^2. \]

This implies that \( h \) is constant. On the other hand, since each \( h_j \) is a sense preserving homeomorphism its winding number over any circle

\[ \mathbb{T}_\rho := \{ z : |z| = \rho \} \quad \text{where} \ r < \rho < R \]
equals 1. Precisely, we have

$$\frac{1}{2\pi i} \oint_{\partial B} \frac{1}{h_j} \frac{\partial h_j}{\partial \theta} = 1.$$  

Passing to the limit as $r_j \downarrow 1$, this equation remains valid for the constant map $h$, which gives the desired contradiction.

Note that the above proof does not provide us with any lower bound for $\text{Mod} A^*$. In 1962, when studying doubly connected minimal surfaces J.C.C. Nitsche conjectured the following (also see [13, §878], [4, p. 138], [1, Conj. 21.3.2]).

**Conjecture 1.1.** (Nitsche [12]) An annulus $A = A(r, R)$ can be mapped by a harmonic homeomorphism onto the annulus $A^* = A(r_*, R_*)$ if and only if

$$(1.5) \quad \frac{R_*}{r_*} \geq \frac{1}{2} \left( \frac{R}{r} + \frac{r}{R} \right).$$

We call it the *Nitsche bound*.

Once this condition is satisfied, the following complex harmonic function

$$h(z) = az + b\bar{z}^{-1}, \quad a = \frac{R_* R - r_* r}{R^2 - r^2}, \quad b = \frac{(Rr_* - R_* r) rR}{R^2 - r^2}$$

gives an example of a homeomorphism between such annuli. Such functions will hereafter be referred to as the *Nitsche mappings*.

Explicit lower bounds of $\frac{R_*}{r_*}$ have been obtained by A. Lyzzaik [10] (whose estimate exhibits the linear growth of $R_*/r_*$ as $R/r \to \infty$), by A. Weitsman [15]:

$$\frac{R_*}{r_*} \geq 1 + \frac{1}{2} \frac{r^2}{R^2} \log^2 \frac{R}{r}$$

and by D. Kalaj [9]:

$$\frac{R_*}{r_*} \geq 1 + \frac{1}{2} \log^2 \frac{R}{r}.$$  

**1.1. The Nitsche bound** However, none of these lower bounds reach the critical number $\frac{1}{2} \left( \frac{R}{r} + \frac{r}{R} \right)$. In the present paper we prove (1.5) when the domain annulus is conformally not too thick.

**Theorem 1.2.** An annulus $A = A(r, R)$ whose modulus $\text{Mod} A = \log \frac{R}{r} \leq 3/2$ can be mapped harmonically onto the annulus $A^* = A(r_*, R_*)$ if and only if

$$\frac{R_*}{r_*} \geq \frac{1}{2} \left( \frac{R}{r} + \frac{r}{R} \right).$$
We also show that, regardless of the modulus of $\mathbb{A}$, the Nitsche bound holds for a class of harmonic homeomorphisms $h: \mathbb{A} \mapsto \mathbb{A}^*$ having vanishing average on the inner circle of $\mathbb{A}$; that is,

$$\lim_{\rho \searrow r} \int_{\mathbb{T}_\rho} h = 0.$$ 

Let us denote by $\mathcal{H}_D(\mathbb{A}, \mathbb{A}^*) \subset \mathcal{H}(\mathbb{A}, \mathbb{A}^*)$ the subclass of such mappings. Another class, denoted by $\mathcal{H}_N(\mathbb{A}, \mathbb{A}^*) \subset \mathcal{H}(\mathbb{A}, \mathbb{A}^*)$, consists of harmonic homeomorphisms with vanishing average normal derivative along Jordan curves separating the boundary components; that is,

$$\int_{\mathbb{T}_\rho} \frac{\partial h}{\partial n} = 0, \quad \text{for some (equivalently for all) } r < \rho < R.$$

**Theorem 1.3.** Suppose $h: \mathbb{A} \rightarrow \mathbb{A}^*$ belongs to $\mathcal{H}_D(\mathbb{A}, \mathbb{A}^*)$ or $\mathcal{H}_N(\mathbb{A}, \mathbb{A}^*)$. Then

$$\frac{R_*}{r_*} \geq \frac{1}{2} \left( \frac{R}{r} + \frac{r}{R} \right).$$

It turns out that the round shape of the outer boundary of the target annulus is not essential. To formulate exact statements we simplify matters by normalizing $\mathbb{A}$ so that its inner boundary is the unit circle $\mathbb{T} = \{ z : |z| = 1 \}$; that is, $\mathbb{A} = A(1, R)$. Moreover, the target $\mathcal{A} := h(\mathbb{A})$ will be a half round annulus. Precisely $\mathcal{A}$ will be a doubly connected domain whose inner boundary is the unit circle. The outer boundary of $\mathcal{A}$, however, can be arbitrary. The mean outer radius of the image of $\mathbb{A}$ under the mapping $h \in \mathcal{H}(\mathbb{A}, \mathcal{A})$ is defined by

$$R_*(h) = \lim_{\rho \searrow R} \left( \int_{\mathbb{T}_\rho} |h|^2 \right)^{1/2}.$$ 

Now Theorems 1.2 and 1.3 are special cases of the following results.

**Theorem 1.4.** Let $h \in \mathcal{H}(\mathbb{A}, \mathcal{A})$ with $\text{Mod} \mathbb{A} \leq 3/2$. Then

$$R_*(h) \geq \frac{1}{2} \left( \frac{R + \frac{1}{R}}{r} \right).$$

No restriction on the size of the annulus $\mathbb{A}$ will be imposed if the mapping has vanishing average on the inner circle, or vanishing average of the normal derivative.

**Theorem 1.5.** Suppose $h$ belongs to one of the following

(i) $\mathcal{H}_D(\mathbb{A}, \mathcal{A})$

(ii) $\mathcal{H}_N(\mathbb{A}, \mathcal{A})$. 
Then
(1.8) \[ R_*(h) \geq \frac{1}{2} \left( R + \frac{1}{R} \right). \]

Further related generalizations demand a few preliminary remarks.

### 1.2. Extremal mappings.
Let us first look at the Nitsche mappings \( h^\lambda: A(1,R) \rightarrow A(1,R_*) \).

(1.9) \[ h^\lambda(z) = \frac{1}{1+\lambda} \left( z + \frac{\lambda}{z} \right): \mathbb{A}^* \rightarrow \mathbb{A}, \quad -1 < \lambda \leq 1. \]

The above restriction on the parameter \( \lambda \) amounts to saying that \( R_* \geq \frac{1}{2} (R + \frac{1}{R}) \). Later it will be of interest to look at the Nitsche mappings with \( \lambda \geq 1 \) as well. The outer radius \( R_* = \frac{R^2 + \lambda R}{R + \lambda R} \) is smaller than \( R \) if \( \lambda > 0 \) and greater than \( R \) if \( -1 < \lambda < 0 \), and \( h^1 \) is the identity mapping. These mappings minimize the Dirichlet energy, see Section 4 for details. They have vanishing average over the inner boundary, actually over any circle \( T_\rho = \{ z: |z| = \rho \} \subset \mathbb{A} \). Except for the critical case, corresponding to \( \lambda = 1 \), each \( h^\lambda: \mathbb{A} \rightarrow \mathbb{A}^* \) has positive Jacobian determinant in the closure of \( \mathbb{A} \). That is why for \( \lambda < 1 \) any small perturbation of the boundary homeomorphisms \( h^1: T \rightarrow T \) and \( h^1: T_R \rightarrow T_{R_*} \), upon harmonic extension inside \( \mathbb{A} \), gives again a homeomorphism of \( \mathbb{A} \) onto \( \mathbb{A}^* \). Thus, if \( \frac{R_*}{R_\text{se}} \) exceeds the Nitsche lower bound there are many harmonic homeomorphisms \( h: A(r, R) \rightarrow A(r_*, R_*) \). The situation is dramatically different if \( R_* = \frac{1}{2} (R + \frac{1}{R}) \).

The critical Nitsche mapping
(1.9) \[ h^1(z) = \frac{1}{2} \left( z + \frac{1}{z} \right) = \frac{1}{2} \left( \rho + \frac{1}{\rho} \right) e^{i\theta}, \quad z \in \rho e^{i\theta}, \]

as opposed to other extremal mappings in (1.9), has Jacobian determinant

\[ J(z, h^1) = |h^1_z|^2 - |h^1_{\bar{z}}|^2 = \frac{|z|^4 - 1}{4 |z|^4} \]

vanishing on the inner boundary of \( \mathbb{A} \). Consequently, slightly perturbing the boundary data \( h^1: T \rightarrow T \) and \( h^1: T_R \rightarrow T_{R_*} \) we lose the injectivity inside \( \mathbb{A} \). In fact, we have the following uniqueness statement.

**Theorem 1.6.** (Uniqueness up to rotation) Let \( (\mathbb{A}, \mathbb{A}^*) \) be a pair of annuli \( \mathbb{A} = A(1, R) \) and \( \mathbb{A}^* = A(1, R_*) \) with \( R_* = \frac{1}{2} (R + \frac{1}{R}) \). Then,

\[ \mathcal{H}_D(\mathbb{A}, \mathbb{A}^*) = \mathcal{H}_N(\mathbb{A}, \mathbb{A}^*) = \{ \alpha h^1: |\alpha| = 1 \}. \]

Moreover, \( \mathcal{H}(\mathbb{A}, \mathbb{A}^*) = \{ \alpha h^1: |\alpha| = 1 \} \) when \( \text{Mod } \mathbb{A} \leq \frac{3}{2} \).
A natural generalization of the Nitsche bound in Theorems 1.2-1.5 comes upon observation that the extremal mapping \( h^1 \) represents so-called free harmonic evolutions of the unit circle. To be precise, we regard \( h \in \mathcal{H}(A, A) \) as a function of concentric circles \( T_\rho = \{ z : |z| = \rho \}, 1 < \rho < R \), into Jordan curves \( \mathcal{T}_* = h(T_\rho) \) in \( A \). We refer to
\[
\nu_\circ(h^\lambda) = \lim_{\rho \searrow 1} \frac{d}{d\rho} \left( \frac{\int_{T_\rho} |h|^2}{\rho^2} \right)^{\frac{1}{2}}, \quad \nu_\circ(h^\lambda) \geq 0.
\]
as the initial speed of the evolution of circles. Free evolution begins with zero initial speed. As might be expected positive initial speed, or simply forced harmonic evolution, results in larger ratio \( R_* / r_* \). The extremal mappings \( h^\lambda, -1 < \lambda < 1 \), are representatives of the forced evolutions of circles with positive initial speed. The speed is given by:
\[
(1.10) \quad \nu_\circ(h^\lambda) = \lim_{\rho \searrow 1} \frac{d}{d\rho} \left( \frac{\int_{T_\rho} |h^\lambda|^2}{\rho^2} \right)^{\frac{1}{2}} = \frac{1 - \lambda}{1 + \lambda}, \quad 0 < \nu_\circ(h^\lambda) < \infty
\]

**Theorem 1.7.** Suppose \( h \in \mathcal{H}(A, A) \) has the initial speed
\[
\lim_{\rho \searrow 1} \frac{d}{d\rho} \left( \frac{\int_{T_\rho} |h|^2}{\rho^2} \right)^{\frac{1}{2}} = \frac{1 - \lambda}{1 + \lambda}, \quad -1 < \lambda \leq 1.
\]

Then for \( 1 < \rho < R \) we have
\[
\max_{|z| = \rho} |h(z)| \geq \left( \frac{\int_{T_\rho} |h|^2}{\rho^2} \right)^{1/2} \geq \left( \frac{\int_{T_\rho} |h^\lambda|^2}{\rho^2} \right)^{1/2} = \frac{\rho^2 + \lambda}{(1 + \lambda) \rho} \geq \frac{1}{2} \left( \rho + \frac{1}{\rho} \right)
\]

Equality occurs if and only if \( h = \alpha h^\lambda \) for some \( |\alpha| = 1 \).

*Added in revision.* After this paper was submitted, we succeeded in proving Theorems 1.2, 1.4 and 1.6 without the assumption \( \text{Mod} A \leq 3/2 \), thus completing the solution of the Nitsche Conjecture. See [5]. The connection between the Nitsche Conjecture and minimal surfaces is further explored in [6].

**2. Background.** We shall work with the open annulus \( A = A(1, R) = \{ z : 1 < |z| < R \} \) whose inner boundary is the unit circle \( T = \{ z : |z| = 1 \} \). The basic complex harmonic functions in \( A \) are the integer powers \( z^n, \bar{z}^n, n = 0, \pm 1, \pm 2, \ldots \), and the logarithm \( \log |z| \). If we combine these functions suitably in pairs, we obtain an orthogonal basis for harmonic functions. Precisely, every harmonic function \( h : A \to \mathbb{C} \) can be decomposed into an infinite sum of so-called
Fourier components

\[ h(z) = \sum_{n \in \mathbb{Z}} h_n(z), \]

where \( h_n(z) = a_n z^n + b_n \bar{z}^{-n} \) for \( n \neq 0 \) and \( h_0(z) = a_0 \log |z| + b_0 \). For every circle \( T_\rho = \{ z : |z| = \rho \}, \ 1 \leq \rho \leq R, \) we have

\[ \int_{T_\rho} h_n \bar{h}_m = \begin{cases} 0 & \text{if } n \neq m \\ |a_n \rho^n + b_n \rho^{-n}|^2 & \text{if } n = m \neq 0 \\ |a_0 \log \rho + b_0|^2 & \text{if } n = m = 0. \end{cases} \]

It should be pointed out that \( h \) need not be defined on the boundary of \( \mathbb{A} \), while its Fourier components \( h_n \) are well defined in the entire punctured complex plane \( \mathbb{C} \setminus \{0\} \). The orthogonality of \( h_n \), as well as their derivatives, will prove useful in the subsequent computations.

Given the rotational invariance of the annulus \( \mathbb{A} \) and the radial symmetry of the extremal harmonic mappings, we shall express the complex variable \( z = \rho e^{i\theta} \) as a function of the polar coordinates \( 1 < \rho < R, \ 0 \leq \theta < 2\pi \). Then the Cauchy-Riemann derivatives are

\[ h_z = \frac{\partial h}{\partial z} = \frac{1}{2} e^{-i\theta} \left( h_\rho - \frac{i}{\rho} h_\theta \right) \]
\[ h_{\bar{z}} = \frac{\partial h}{\partial \bar{z}} = \frac{1}{2} e^{i\theta} \left( h_\rho + \frac{i}{\rho} h_\theta \right). \]

Hence the Laplacian

\[ \Delta h = 4 \frac{\partial^2 h}{\partial z \partial \bar{z}} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial h}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 h}{\partial \theta^2} = h_{\rho \rho} + \frac{1}{\rho} h_\rho + \frac{1}{\rho^2} h_{\theta \theta}. \]

We will use the Hilbert-Schmidt norm of the Jacobian matrix of \( h \), which is equal to

\[ \|Dh\|^2 = 2 \left( |h_z|^2 + |h_{\bar{z}}|^2 \right) = |h_\rho|^2 + \rho^{-2} |h_\theta|^2. \]

The Jacobian determinant is expressed in polar coordinates as

\[ J(z, h) = |h_z|^2 - |h_{\bar{z}}|^2 = \frac{1}{\rho} \Im (\bar{h}_\rho h_\theta). \]

We shall now fix a round annulus

\[ \mathbb{A} = A(1, R) = \{ z : 1 < |z| < R \}, \ 1 < R < \infty. \]
and consider homeomorphisms \( h: \mathbb{A} \rightarrow \mathbb{A} \). Here \( \mathcal{A} \subset \mathbb{C} \) is a topological annulus, i.e., an open connected set whose complement \( \hat{\mathbb{C}} \setminus \mathcal{A} \) consists of two disjoint nonempty closed sets. One of these sets called the outer complement, denoted by \( G_O \), contains \( \infty \). The other set, called the inner complement, is the closed unit disk, denoted by \( G_I \).

In general \( h \) does not extend continuously to the closure of \( \mathbb{A} \), yet it “takes” the boundary circles

\[
\mathbb{T} = \{ z: |z| = 1 \} \quad \text{and} \quad \mathbb{T}_R = \{ z: |z| = R \}
\]

into two different components of \( \partial \mathcal{A} \) in the sense of cluster sets [11, p. 156]. We write

\[
h(\mathbb{T}) = \{ \lim_{z \to \mathbb{T}} h(z): z \in \mathbb{A} \} = \partial G_I
\]

and

\[
h(\mathbb{T}_R) = \{ \lim_{z \to \mathbb{T}_R} h(z): z \in \mathbb{A} \} = \partial G_O.
\]

There are four homotopy classes of homeomorphisms \( h: \mathbb{A} \rightarrow \mathbb{A} \), each determined by the orientation of \( h \) and the order of the boundary components \( h(\mathbb{T}) \) and \( h(\mathbb{T}_R) \) in \( \partial \mathcal{A} \). Without loosing any generality, we restrict our considerations to the following class.

**Definition 2.1.** The class \( \mathscr{H}(\mathbb{A}, \mathcal{A}) \) consists of all orientation preserving homeomorphisms \( h: \mathbb{A} \rightarrow \mathbb{A} \) such that \( h(\mathbb{T}) \) and \( h(\mathbb{T}_R) \) are the inner and outer boundary of \( \mathcal{A} \), respectively. The subclass of harmonic mappings in \( \mathscr{H}(\mathbb{A}, \mathcal{A}) \) will be denoted by \( \mathcal{H}(\mathbb{A}, \mathcal{A}) \).

Recall from introduction that \( h \in \mathscr{H}(\mathbb{A}, \mathcal{A}) \) is viewed as a function of circles \( \mathbb{T}_\rho = \{ \rho e^{i \theta}: 0 \leq \theta < 2\pi \} \) into Jordan curve \( \mathbb{T}^* = h(\mathbb{T}_\rho) \), \( 1 < \rho < R \). This function is called the *evolution of circles*. For the generalization of the Nitsche conjecture it will be essential to assume that the inner boundary of \( \mathcal{A} \) is the unit circle \( \mathbb{T} \) and that \( h(\mathbb{T}) = \mathbb{T} \). We then note that for \( h \in \mathscr{H}(\mathbb{A}, \mathcal{A}) \) the function \( z \to |h(z)| \) extends continuously to the inner circle of \( \mathbb{A} \), with value 1 on \( \mathbb{T} \).

3. **Circular means of a harmonic evolution.** We shall introduce a number of integral means over the circles \( \mathbb{T}_\rho \), \( 1 < \rho < R \), first defined for harmonic functions \( h: \mathbb{A} \rightarrow \mathbb{C} \) and then restricted to the mappings in \( \mathcal{H}(\mathbb{A}, \mathcal{A}) \). Our aim is to indicate in some detail how to understand these integral means on the inner circle \( \mathbb{T} \), as \( h \) need not be even defined on \( \mathbb{T} \). The orthogonal decomposition (2.1)
of $h$ on $\mathbb{A}$ comes handful. Accordingly, the circular means

$$\int_{\mathbb{T}_\rho} h = \frac{1}{2\pi \rho} \int_{\mathbb{T}_\rho} h = a_0 \log \rho + b_0, \quad 1 < \rho < R, \quad (3.1)$$

extend continuously to the closed interval $[1, R],$

$$\lim_{\rho \downarrow 1} \int_{\mathbb{T}_\rho} h = b_0. \quad (3.2)$$

Their $\rho$-derivatives can also be given a meaning at the boundary circles,

$$\lim_{\rho \downarrow 1} \frac{d}{d\rho} \int_{\mathbb{T}_\rho} h = \lim_{\rho \downarrow 1} \int_{\mathbb{T}_\rho} \rho h = a_0. \quad (3.3)$$

Hereafter, passing with differentiation inside the integral is justified by a general commutation rule, which in symbols reads as

$$\frac{d}{d\rho} \int_{\mathbb{T}_\rho} h = \int_{\mathbb{T}_\rho} \frac{d}{d\rho}, \quad \text{for } 1 < \rho < R. \quad (3.4)$$

Next we introduce

$$U = U(\rho) = U(\rho, h) = \int_{\mathbb{T}_\rho} |h|^2, \quad 1 < \rho < R \quad (3.5)$$

and call $U(\rho)$ the quadratic mean of $h$ over the circle $\mathbb{T}_\rho$. Using the Green’s formula $\int_{\Omega} (u\Delta v - v\Delta u) = \int_{\partial\Omega} (u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n})$, with $u = 1$ and $v = |h|^2$, this leads to

$$\int_{\mathbb{T}_\rho} |h|^2 - \int_{\mathbb{T}_r} |h|^2 = \int \int_{r \leq |z| \leq \rho} \Delta |h|^2 = 4 \int \int_{r \leq |z| \leq \rho} \frac{\partial^2 |h|^2}{\partial z \partial \bar{z}}$$

$$= 4 \int \int_{r \leq |z| \leq \rho} (|h_z|^2 + |h_{\bar{z}}|^2) = 2 \int \int_{r \leq |z| \leq \rho} \| Dh \|^2. \quad (3.6)$$

Here $1 < r \leq \rho < R$. By virtue of the commutation rule (3.4) we obtain

$$\rho \dot{U}(\rho) - r \dot{U}(r) = 2 \int \int_{r \leq |z| \leq \rho} \| Dh \|^2, \quad 1 < r \leq \rho < R \quad (3.7)$$
where, as usual, dot over \( U \) stands for the \( \rho \)-derivative of \( U \). Differentiation of (3.7) with respect to \( \rho \) yields

\[
\frac{1}{\rho} \frac{d}{d\rho} \left[ \rho \frac{d}{d\rho} U \right] = 4\pi \int_{\pi \rho}^{\|Dh\|^2 > 0, 1 < \rho < R.}
\]

It should be said, therefore, that \( U \) is a subsolution to the differential operator

\[
L = \frac{1}{\rho} \frac{d}{d\rho} \left[ \rho \frac{d}{d\rho} \right], \quad L[U] \geq 0 \; \text{in} \; (1, R).
\]

In Section 5, we shall see many more second order differential operators for which various integral means are subsolutions. As regards the extension of \( \bar{U}(\rho) \) and its derivative \( \dot{\bar{U}}(\rho) \) to \([1, R]\), we restrict ourselves to the mappings \( h \in H(A, A) \), so that the function \( z \rightarrow |h(z)| \) is continuous up to the inner circle of \( A \). In particular, setting \( \bar{U}(1) = 1 \) gives the desired continuous extension of \( \bar{U}(\rho) \) to all radii \( 1 \leq \rho < R \). Then formula (3.8) will allow us to extend \( \bar{U}(\rho) \) to \([1, R] \). To this end, we first infer from (3.8) that \( \rho \bar{U}(\rho) \) is strictly increasing. Hence

\[
\dot{\bar{U}}(\rho) = \frac{1}{\rho \log \rho} \int_{1}^{\rho} \bar{U}(\rho) \frac{dt}{t} > \frac{1}{\rho \log \rho} \int_{1}^{\rho} \dot{\bar{U}}(t) \frac{dt}{t} = \frac{\bar{U}(\rho) - \bar{U}(1)}{\rho \log \rho} > 0.
\]

We then conclude that the following limit exists, and is nonnegative:

\[
\lim_{\rho \searrow 1} \dot{\bar{U}}(\rho) = \lim_{\rho \searrow 1} \rho \dot{\bar{U}}(\rho) =: \dot{\bar{U}}(1) \geq 0.
\]

Now the function \( \dot{\bar{U}}(\rho) \), so defined at \( \rho = 1 \), is clearly continuous on \([1, R] \). Moreover \( \dot{\bar{U}}(1) \) agrees with the usual definition of the derivative,

\[
\frac{\bar{U}(\rho) - 1}{\rho - 1} = \frac{1}{\rho - 1} \int_{1}^{\rho} \dot{U}(t) dt \rightarrow \dot{U}(1) \quad \text{as} \; \rho \searrow 1.
\]

Formula (3.7) now remains valid for \( r = 1 \),

\[
2 \int_{1 \leq |z| \leq \rho} \|Dh\|^2 = \rho \dot{U}(\rho) - \dot{U}(1) \leq \rho \dot{U}(\rho).
\]

Hence

\[
\dot{U}(\rho) \geq \frac{2}{\rho} \int_{1 \leq |z| \leq \rho} \|Dh\|^2 > 0 \quad \text{for} \; 1 < \rho < R
\]

and, as a consequence, we infer that:
Proposition 3.1. Suppose \( h \in \mathcal{H}(\mathbb{A}, \mathcal{A}) \). Then the quadratic means \( U(\rho) = \frac{1}{2\pi} \int_{\mathbb{T}} |h|^2 \) are strictly increasing and the integral of \( \|Dh\|^2 \) over any annulus \( A(1, \rho) \) is finite for \( 1 < \rho < R \).

It is natural to define

\[
U(R) := \lim_{\rho \to R} U(r) = (R_*(h))^2 \in [0, +\infty].
\]

Remark 3.2. When \( \rho \) approaches the outer radius of \( \mathbb{A} \), the energy integral \( \int \|Dh\|^2 \) may grow to infinity. A short sketch proof of this fact runs somewhat as follows:

Let \( f: \mathbb{T} \to \mathbb{T} \) be a homeomorphism of the unit circle onto itself. We consider the Poisson integral extension of \( f \) into the unit disk \( \mathbb{D} \), still denoted by \( f \). This extension is a homeomorphism of \( \mathbb{D} \) onto itself by the Radó-Kneser-Choquet Theorem, and \( C^\infty \)-smooth diffeomorphism in the open disk by Lewy’s Theorem, see [4]. In general, the Dirichlet energy of \( f \) need not be finite. The best that one can guarantee is that \( \|Df\| \) lies in the Marcinkiewicz space \( L^2_{\text{weak}}(\mathbb{D}) \), see [7].

Having such a harmonic homeomorphism \( f: \mathbb{D} \to \mathbb{D} \) with infinite energy, we look at the inverse image of an annulus \( A = A(r, 1) \subset \mathbb{D}, 0 < r < 1 \), to observe that \( f^{-1}(A) \) is a doubly connected domain with smooth boundaries (real analytic). There exists a conformal mapping \( \varphi: \mathbb{A} \to f^{-1}(A) \) of a round annulus \( \mathbb{A} = A(1, R) \) onto \( f^{-1}(A) \). This mapping is a diffeomorphism up to the boundary of \( \mathbb{A} \), even in a neighborhood of \( \overline{\mathbb{A}} \). In this way we arrive at the harmonic homeomorphism with infinite energy

\[
h = \frac{1}{R} (f \circ \varphi): A(1, R) \to A(1, R_*), \quad R_* = \frac{1}{r}.
\]

We turn next to the variance of \( h \). Observe that the circular means \( \frac{1}{2\pi} \int_{\mathbb{T}} h = a_0 \log |z| + b_0 \) form a harmonic function, so is the function \( H = h - \frac{1}{2\pi} \int_{\mathbb{T}} h \) whose quadratic average is the variance of \( h \), namely

\[
V = V(\rho) = V(\rho, h) = \frac{1}{2\pi} \left| \int_{\mathbb{T}_\rho} h - \int_{\mathbb{T}_\rho} \frac{1}{2\pi} \int_{\mathbb{T}} h \right|^2 = \frac{1}{2\pi} \left| \int_{\mathbb{T}_\rho} h \right|^2 - \frac{1}{2\pi} \left| \int_{\mathbb{T}_\rho} \frac{1}{2\pi} \int_{\mathbb{T}} h \right|^2 = U(\rho, H).
\]

The orthogonal decomposition (2.1) gives rise to a decomposition of the circular means,

\[
U = U(\rho, h) = \sum_{n \in \mathbb{Z}} U_n(\rho), \quad U_\rho(\rho) = U(\rho, h_\rho)
\]

\[
V = V(\rho, h) = \sum_{n \neq 0} U_n(\rho).
\]

We shall explore these formulas throughout this paper.
At this stage we take advantage of the orthogonal decomposition to deduce that \( V(\rho) \) is convex in \( \rho \):

\[
V(\rho) = \sum_{n \neq 0} |a_n\rho^n + b_n\rho^{-n}|^2 = \sum_{n \neq 0} \left( |a_n|^2 \rho^{2n} + |b_n|^2 \rho^{-2n} + 2 \Re a_n\bar{b}_n \right).
\]

The second derivative is indeed positive

\[
\ddot{V}(\rho) = 2 \rho^2 \sum_{n \in \mathbb{Z}} \left[ n(2n - 1) |a_n|^2 \rho^{2n} + n(2n + 1) |b_n|^2 \rho^{-2n} \right] > 0. \tag{3.15}
\]

Thus \( V \) is a subsolution of the operator \( \frac{d^2}{d\rho^2} \). However, this operator is not good enough to conclude with the Nitsche conjecture. For, the critical Nitsche mapping fails to be a solution of this operator.

4. Extremal mappings. Let us look more closely at the minimizers of the Dirichlet integral

\[
\mathcal{E}[h] = \iint_A \|Dh\|^2
\]

subject to all homeomorphisms \( h \in \mathcal{H}(\mathcal{A}, \mathcal{A}^*) \) between round annuli \( \mathcal{A} = A(1, R) \) and \( \mathcal{A}^* = A(1, R^*) \). We invoke the results in [2], [8]. Within the Nitsche range (1.5) for the annuli \( \mathcal{A} \) and \( \mathcal{A}^* \) the minimum is obtained (uniquely up to rotation) by the harmonic mapping

\[
h_\lambda(z) = \frac{1}{1 + \lambda} \left( z + \frac{\lambda}{z} \right) \quad \text{where} \quad \lambda = \frac{R^2 - RR^*}{RR^* - 1} \in (-1, 1]. \tag{4.2}
\]

Outside the Nitsche range (1.5) for the annuli \( \mathcal{A} \) and \( \mathcal{A}^* \) the infimum in \( \mathcal{H}(\mathcal{A}, \mathcal{A}^*) \) is not attained [2], [8]. It is a general fact concerning mappings between domains in \( \mathbb{C} \); any minimizer of the Dirichlet energy is harmonic outside the branch set. The reader may wish to know that the nonharmonic mapping \( h(z) = z/|z| \) of \( \mathcal{A} \) onto the unit circle is a minimizer of the Dirichlet integral [8]. We then see that the Nitsche conjecture implies nonexistence of minimizers outside of the Nitsche bound, but not vice versa.

If one thinks of \( h_\lambda \) as evolution of the inner boundary of in \( A(1, R) \), then the parameter \( \lambda, -1 < \lambda \leq 1 \), tells us about the initial speed of the evolution. Let us denote by \( r_*(\rho, h_\lambda) \) the radii of the circles \( h_\lambda(T_\rho) \). It is easily seen that

\[
\dot{r}_*(1, h_\lambda) = \frac{d}{d\rho} r_*(\rho, h_\lambda) \bigg|_{\rho=1} = \frac{1 - \lambda}{1 + \lambda} \in [0, \infty).
\]

This observation will be useful in Section 7 where we show that \( h_\lambda \) also serves
as extremal among all harmonic evolutions \( h \in \mathcal{H}_D(\Lambda, \mathcal{A}) \) of the given initial speed.

5. Convexity operators. The idea is to generalize the usual convexity operator \( \frac{d^2}{d\rho^2} \) in such a way that variance \( V = V(\rho) = V(\rho, h) \) of all complex harmonic functions satisfy the inequality

\[
\mathcal{L}[V] := \dot{V} + A(\rho)V + B(\rho)V \geq 0
\]

with equality occurring exactly for the Nitsche mapping

\[
h^\lambda = \frac{1}{1 + \lambda} \left( \frac{z + \lambda}{\bar{z}} \right),
\]

\(-1 < \lambda \leq 1\). This idea is carried through the following operator

\[
\mathcal{L}^\lambda = \frac{d^2}{d\rho^2} + \frac{3\lambda - \rho^2}{\rho(\rho^2 + \lambda)} \frac{d}{d\rho} - \frac{8\lambda}{(\rho^2 + \lambda)^2}
\]

or, in divergence form

\[
\mathcal{L}^\lambda[V] = \frac{\rho^2 + \lambda}{\rho^3} \frac{d}{d\rho} \left[ \rho^3 \frac{d}{d\rho} \left( \frac{V}{\rho^2 + \lambda} \right) \right]
\]

which is defined for any \( V \in C^2(1, R) \). Before passing to subsolutions of \( \mathcal{L}^\lambda \), see Proposition 6.1, two other formulas for \( \mathcal{L}^\lambda \) are in order.

**Lemma 5.1.** Let \( h \) be a complex harmonic function on \( \Lambda = A(1, R) \) and \( U = U(\rho) = \int_{\Gamma_\rho} |h|^2 \). Then

\[
\mathcal{L}^\lambda[U] = 2 \int_{\Gamma_\rho} \left[ \|Dh\|^2 - \frac{1}{\rho} \frac{d}{d\rho} \left( \frac{\rho^2 - \lambda}{\rho^2 + \lambda} |h|^2 \right) \right]
\]

\[
\mathcal{L}^\lambda[U] = \frac{2|\rho|}{\rho^2} \int_{\Gamma_\rho} \left[ |h_\rho|^2 - |h|^2 + (\rho^2 + \lambda)^2 \frac{d}{d\rho} \left( \frac{\rho}{\rho^2 + \lambda} \right) |h|^2 \right].
\]

**Proof:** In view of the commutation rule (3.4) we find that differentiation yields

\[
\dot{U}(\rho) = \int_{\Gamma_\rho} \left| h^2 \right|_\rho = 2 \int_{\Gamma_\rho} \text{Re} \left( \bar{h} h_\rho \right)
\]

and

\[
\ddot{U}(\rho) = 2 \int_{\Gamma_\rho} \left[ |h_\rho|^2 + \text{Re} \left( \bar{h} h_{\rho\rho} \right) \right].
\]
Since \( h \) is harmonic, we have the Laplace equation
\[
h_{\rho\rho} = -h_{\rho}/\rho - h_{\theta\theta}/\rho^2.
\]
Integrating \( hh_{\theta\theta} \) by parts over the circle \( \mathbb{T}_\rho \) yields
\[
\dot{U}(\rho) = \int_{\mathbb{T}_\rho} \left[ 2|h_{\rho}|^2 - \frac{1}{\rho}|h^2|_{\rho} + \frac{2}{\rho^2}|h_{\theta}|^2 \right].
\]
Substitute these values of \( U, \dot{U} \) and \( \ddot{U} \) into (5.1) to obtain
\[
\mathcal{L}_\lambda[U] = \int_{\mathbb{T}_\rho} \left[ 2|h_{\rho}|^2 + \frac{2}{\rho^2}|h_{\theta}|^2 - \frac{\rho^2 - \lambda}{\rho(\rho^2 + \lambda)} |h^2|_{\rho} - \frac{8\lambda}{(\rho^2 + \lambda)^2} |h^2| \right].
\]
This yields formula (5.3) since in polar coordinates \( ||Dh||^2 = |h_{\rho}|^2 + \rho^{-2} |h_{\theta}|^2 \).

For formula (5.4), however, we proceed in the following way
\[
\mathcal{L}_\lambda[U] = \frac{2}{\rho^2} \int_{\mathbb{T}_\rho} \left\{ |h_{\theta}|^2 + \rho^2 |h_{\rho}|^2 - \frac{\rho(\rho^2 - \lambda)}{(\rho^2 + \lambda)} |h^2|_{\rho} - \frac{4\lambda \rho^2}{(\rho^2 + \lambda)^2} |h^2| \right\}
\]
\[
= \frac{2}{\rho^2} \int_{\mathbb{T}_\rho} \left\{ |h_{\theta}|^2 - |h|^2 + \left[ \frac{(\rho^2 - \lambda)^2}{(\rho^2 + \lambda)^2} |h|^2 + \rho^2 |h_{\rho}|^2 - \frac{\rho(\rho^2 - \lambda)}{\rho^2 + \lambda} |h_{\theta}|^2 \right]\right\}
\]
It only remains to verify that the expression in the rectangular parentheses coincides with the term
\[
(\rho^2 + \lambda)^2 \left| \frac{d}{d\rho} \left( \frac{\rho h}{\rho^2 + \lambda} \right) \right|^2.
\]
Indeed,
\[
(\rho^2 + \lambda)^2 \left| \frac{d}{d\rho} \left( \frac{\rho h}{\rho^2 + \lambda} \right) \right|^2
\]
\[
= (\rho^2 + \lambda)^2 \left| \frac{\rho h_{\theta}}{\rho^2 + \lambda} - \frac{\rho^2 - \lambda}{(\rho^2 + \lambda)^2} h \right|^2
\]
\[
= (\rho^2 + \lambda)^2 \left[ \frac{\rho^2 |h_{\rho}|^2}{(\rho^2 + \lambda)^2} + \frac{4\lambda \rho^2}{(\rho^2 + \lambda)^2} |h|^2 - 2\frac{\rho(\rho^2 - \lambda)}{(\rho^2 + \lambda)^3} \Re \bar{h} h_{\rho} \right]
\]
\[
= (\rho^2 - \lambda)^2 \frac{\rho^2 + \lambda}{(\rho^2 + \lambda)^2} + \rho^2 |h_{\rho}|^2 - \frac{\rho(\rho^2 - \lambda)}{\rho^2 + \lambda} |h^2|_{\rho}
\]
as desired. \[\square\]
Two special cases of the parameter $\lambda$ are worth noting. First, the operator $\mathcal{L}^1$ for the critical Nitsche mapping $h^1 = \frac{1}{2} \left( z + \frac{1}{\bar{z}} \right)$ takes the form

$$
\mathcal{L}^1 = \frac{d^2}{d\rho^2} + \frac{3 - \rho^2}{\rho(\rho^2 + 1)} \frac{1}{d\rho} - \frac{8}{(\rho^2 + 1)^2}, \quad \mathcal{L}^1[U] = 0 \quad \text{with} \quad U = \frac{(\rho^2 + 1)^2}{4\rho^2}.
$$

On the other hand, letting $\lambda = 0$, we obtain the operator associated with the identity mapping $h_0(z) = z$,

$$
\mathcal{L}^0 = \frac{d^2}{d\rho^2} - \frac{1}{\rho} \frac{d}{d\rho}, \quad \mathcal{L}^0[U] = 0 \quad \text{with} \quad U = \rho^2.
$$

For $\lambda = 1$ the critical Nitsche mapping $h^1$ represents harmonic mapping $h \in \mathcal{H}(A, \mathcal{A})$ with vanishing normal derivative on the inner circle. Such mappings extend harmonically beyond the unit circle by reflection

$$
h(z) = h \left( \frac{1}{\bar{z}} \right) \quad \text{for} \quad z \in A(1/R, 1).
$$

The extended mapping is a double cover of $\mathcal{A}$. For $\lambda = 0$, on the other hand, the identity mapping $h^0(z) = z$ represents all conformal evolutions of the unit circle.

**Lemma 5.2.** Every evolution of the inner circle in $\mathcal{A}$ that is generated by a conformal mapping $h \in \mathcal{H}(\mathcal{A}, \mathcal{A})$ begins with the unit speed

$$
\lim_{\rho \searrow 1} \frac{d}{d\rho} \left( \int_{\mathcal{P}} |h|^2 \right)^{1/2} = 1.
$$

**Proof.** First, we extend $h$ conformally to the annulus $A(1/R, R)$, by reflection

$$
h(z) = [h(1/\bar{z})]^{-1}, \quad \text{for} \quad \frac{1}{R} < |z| < 1.
$$

The Cauchy-Riemann system $h_\nu = 0$ reads in polar coordinates as $h_\rho = \frac{\nu}{\rho} h_\theta$. Moreover, the winding number of $h$ equals 1. Hence,

$$
\int_{\mathcal{P}} \frac{-ih_\theta}{h} = 1, \quad \text{for every} \quad \frac{1}{R} < \rho < R.
$$

Now the computation of the initial speed proceeds as follows.

$$
\lim_{\rho \searrow 1} \frac{d}{d\rho} \left( \int_{\mathcal{P}} |h|^2 \right)^{1/2} = \frac{1}{2} \int_{\mathcal{P}} |h^2|_\rho = \frac{1}{2} \int_{\mathcal{P}} (h_\rho \overline{h} + \overline{h_\rho} h)
$$

$$
= \frac{1}{2} \int_{\mathcal{P}} \frac{-ih_\theta}{h} + \frac{1}{2} \int_{\mathcal{P}} \frac{-ih_\theta}{h} = 1. \quad \Box
$$
6. Variance is a subsolution to all $\mathcal{L}^\lambda$. The quadratic means of a harmonic function $H(\rho e^{i\theta}) = h - \frac{1}{f_{2\pi}} h$ are none other than the variance of $h$ which can be computed by using orthogonal decomposition (2.1):

$$V(\rho, h) = U(\rho, H) = \sum_{n \neq 0} U(\rho, h_n).$$

We employ formula (5.4) to estimate $\mathcal{L}^\lambda[V]$. Neglecting the nonnegative term

$$\frac{d}{d\rho} \left( \frac{\rho}{\rho^2 + \lambda} H \right) \geq 0$$

yields the desired inequality

$$\mathcal{L}^\lambda[V] \geq \frac{2}{\rho^2} \int_{\mathbb{T}_\rho} (|H_\theta|^2 - |H|^2)$$

$$= \frac{2}{\rho^2} \sum_{n \neq 0} \int_{\mathbb{T}_\rho} \left( \frac{d}{d\theta} |h_n|^2 - |h_n|^2 \right)$$

$$= \frac{2}{\rho^2} \sum_{n \neq 0} (n^2 - 1) U(\rho, h_n) \geq 0.$$

Of particular interest to us is the case $\mathcal{L}^\lambda[V] = 0$. For this, we must have equality at (6.2), which yields

$$H(\rho e^{i\theta}) = \frac{1}{1 + \lambda} \left( \rho + \frac{\lambda}{\rho} \right) C(\theta),$$

where $C = C(\theta)$ is a $2\pi$-periodic function in $0 \leq \theta \leq 2\pi$. As $H$ is harmonic, using the Laplace equation in polar coordinates, we find that

$$0 = \Delta H = H_{\rho\rho} + \frac{1}{\rho} H_{\rho} + \frac{1}{\rho^2} H_{\theta\theta}$$

$$= \frac{\rho^2 + \lambda}{(1 + \lambda)\rho^2} [C(\theta) + \dot{C}(\theta)].$$

The general solution to this ODE is $C(\theta) = \alpha e^{i\theta} + \beta e^{-i\theta}$, so

$$H(z) = \frac{\alpha}{1 + \lambda} \left( z + \frac{\lambda}{z} \right) + \frac{\beta}{1 + \lambda} \left( \bar{z} + \frac{\lambda}{z} \right) = \alpha h^\lambda(z) + \beta \overline{h^\lambda(z)}, \quad \alpha, \beta \in \mathbb{C}.$$ 

Hence

$$h(z) = \frac{\alpha}{1 + \lambda} \left( z + \frac{\lambda}{z} \right) + \frac{\beta}{1 + \lambda} \left( \bar{z} + \frac{\lambda}{z} \right) + a_0 \log |z| + b_0.$$
Of additional interest to us is the case when \( h \in H(A, A) \). This \( h \), given by (6.4), has modulus 1 on \( \mathbb{T} \). In polar coordinates,

\[
(6.5) \quad h(e^{i\theta}) = \alpha e^{i\theta} + \beta e^{-i\theta} + b_0.
\]

It is easily seen that the condition \( |h(e^{i\theta})|^2 \equiv 1 \) yields

\[
(6.6) \quad |\alpha|^2 + |\beta|^2 + |b_0|^2 = 1, \quad \beta\bar{\alpha} = 0, \quad b_0\beta + \bar{b}_0\alpha = 0.
\]

The possibility \( \alpha = \beta = 0 \) is ruled out because \( h(e^{i\theta}) \not\equiv \text{const.} \) This leaves two cases:

**Case 1.** \( \alpha = 0 \) and \( \beta \neq 0 \). The equations (6.6) reduce to \( |\beta| = 1 \) and \( b_0 = 0 \), so we obtain

\[
(6.7) \quad h(z) = \frac{\beta}{1 + \lambda} \left( \frac{\bar{z} + \frac{\lambda}{z}}{1 + \lambda} \right) + a_0 \log |z|.
\]

Next we look at how this mapping stands up to the circular mean of the Jacobian determinant \( J(z, h) = |h_z|^2 - |h_{\bar{z}}|^2 \), where

\[
h_z = -\frac{\beta\lambda}{(1 + \lambda)z^2} + \frac{a_0}{2z} \quad \text{and} \quad h_{\bar{z}} = \frac{\beta}{1 + \lambda} + \frac{a_0}{2\bar{z}}.
\]

Using orthogonality of the power functions on circles, we obtain a contradiction

\[
0 \leq \oint_{\mathbb{T}_\rho} J(z, h) = \frac{|\beta|^2}{(1 + \lambda)^2} \left( \frac{\lambda^2}{\rho^4} - 1 \right) < 0 \quad \text{for } \rho > 1.
\]

Therefore, the only possibility is:

**Case 2.** \( \beta = 0 \) and \( \alpha \neq 0 \). As before, the mapping \( h \) takes the form

\[
h(z) = \frac{\alpha}{1 + \lambda} \left( z + \frac{\lambda}{\bar{z}} \right) + a_0 \log |z|.
\]

We just proved the following:

**PROPOSITION 6.1.** Variance \( V = V(\rho) \) of a harmonic function \( h: A(1,R) \to \mathbb{C} \) is a subsolution to all operators \( \mathcal{L}^\lambda \) with \(-1 < \lambda < \infty\); that is, \( \mathcal{L}^\lambda[V] \geq 0 \). Equality \( \mathcal{L}^\lambda[V] \equiv 0 \), for some \( \lambda \), occurs if and only if

\[
(6.7) \quad h(z) = b_0 + a_0 \log |z| + \frac{\alpha}{1 + \lambda} \left( z + \frac{\lambda}{\bar{z}} \right) + \frac{\beta}{1 + \lambda} \left( \bar{z} + \frac{\lambda}{z} \right)
\]
where \( a_0, b_0, \alpha, \beta \) are arbitrary complex coefficients. If, moreover, \( h \in \mathcal{H}(\mathcal{A}, \mathcal{A}) \), then

\[
h(z) = a_0 \log |z| + \frac{\alpha}{1 + \lambda} \left( z + \frac{\lambda}{\bar{z}} \right), \quad \text{with } |\alpha| = 1.
\]

Here \( 1 < \lambda \leq 1 \) and the coefficient \( a_0 \) must be small enough to ensure that \( h \) is injective in the entire annulus \( \mathcal{A} \).

The reader may wish to notice that the equality \( \mathcal{L}_\lambda[V] = 0 \) for \( h \in \mathcal{H}(\mathcal{A}, \mathcal{A}) \) implies \( \lim_{\rho \downarrow 1} \int_{T_\rho} h = 0 \). There is yet further reduction of formula (6.8) for mappings between round annuli, meaning that \( |h(z)| = R_* \) for \(|z| = R\). In this case equality \( \mathcal{L}_\lambda[V] \equiv 0 \) occurs only for

\[
h(z) = \frac{\alpha}{1 + \lambda} \left( z + \frac{\lambda}{\bar{z}} \right) = \alpha h^\lambda(z), \quad |\alpha| = 1,
\]

which is, up to rotation, the extremal mapping for the Dirichlet energy \( \iint_{\mathcal{A}} \|Dh\|^2 \).

We have in this case \( \int_{T_\rho} h = 0 \) for every \( 1 < \rho < R \).

## 7. Proof of Theorems 1.5(i) and 1.7.

Theorem 1.7 is a special case of the following result of independent interest in the study of harmonic functions.

**Proposition 7.1.** Consider an arbitrary harmonic function \( h: A(1, R) \to \mathbb{C} \). For notational simplicity we normalize \( h \) at the inner circle by the conditions

\[
\lim_{\rho \downarrow 1} \int_{T_\rho} h = 0 \quad \text{and} \quad \lim_{\rho \downarrow 1} \int_{T_\rho} |h|^2 = 1.
\]

Suppose that the evolution of circles under \( h \) begins with initial speed

\[
\lim_{\rho \downarrow 1} \frac{d}{d\rho} \left( \int_{T_\rho} |h|^2 \right)^{1/2} = \frac{1 - \lambda}{1 + \lambda}, \quad -1 < \lambda \leq \infty.
\]

Then for \( 1 < s < R \),

\[
\left( \int_{T_s} |h|^2 \right)^{1/2} \geq \frac{s^2 + \lambda}{(1 + \lambda)s}.
\]

The inequality in (7.3) turns into equality if and only if \( h = \alpha h^\lambda = \frac{\alpha}{1 + \lambda} \left( z + \frac{\lambda}{\bar{z}} \right) \) with \(|\alpha| = 1\).

Before passing to the proof of Proposition 7.1 let us note that Theorem 1.7 follows as a corollary:
Proof of Theorem 1.7. For a harmonic homeomorphism $h \in H_{D}(A, A)$ the initial speed is nonnegative, so $-1 < \lambda \leq 1$. This yields
\[
\max_{|z| = \rho} |h(z)| \geq \left( \int_{\mathbb{T}_\rho} |h|^2 \right)^{\frac{1}{2}} \geq \frac{\rho^2 + \lambda}{(1 + \lambda)\rho} \geq \frac{1}{2} \left( \rho + \frac{1}{\rho} \right).
\]

Proof of Proposition 7.1. We shall examine the variance
\[
V = V(\rho) = \int_{\mathbb{T}_\rho} |h|^2 - \left| \int_{\mathbb{T}_\rho} h \right|^2.
\]

A key step in obtaining (7.3) is the inequality from Proposition 6.1,
\[
0 \leq \mathcal{L}^\lambda[V] = \frac{\rho^2 + \lambda}{\rho^3} \frac{d}{d\rho} \left[ \rho^3 \frac{d}{d\rho} \left( \frac{V}{\rho^2 + \lambda} \right) \right]
\]
which tells us that the function $\rho \mapsto \rho^3 \frac{d}{d\rho} \left( \frac{V}{\rho^2 + \lambda} \right)$ is nondecreasing. An obvious consequence of it is that
\[
\frac{d}{d\rho} \left( \frac{V}{\rho^2 + \lambda} \right) \geq \frac{C}{\rho^2}, \quad 1 \leq \rho < R,
\]
where the constant $C$ is given by
\[
C = \left. \frac{d}{d\rho} \left( \frac{V}{\rho^2 + \lambda} \right) \right|_{\rho = 1} = \frac{\dot{V}(1)}{1 + \lambda} - \frac{2V(1)}{(1 + \lambda)^2}.
\]

We express $C$ in terms of the initial speed. Using the normalization conditions in (7.1) gives $V(1) = 1$. More generally,
\[
V(\rho) = \int_{\mathbb{T}_\rho} |h|^2 - \left| \int_{\mathbb{T}_\rho} h \right|^2 = \int_{\mathbb{T}_\rho} |h|^2 - |a_0|^2 \log^2 \rho,
\]
where we employed the orthogonal decomposition (2.1) with $b_0 = \int_{\mathbb{T}} h = 0$. Differentiation yields
\[
\dot{V}(1) = \dot{U}(1) = 2 \frac{1 - \lambda}{1 + \lambda}.
\]

On substituting these values of $V(1)$ and $\dot{V}(1)$ into (7.6) we find the constant $C$
\[
C = -\frac{2\lambda}{(1 + \lambda)^2}.
\]
Then inequality (7.5) takes the explicit form

\[
\frac{d}{d\rho} \left( \frac{V}{\rho^2 + \lambda} \right) \geq -\frac{2\lambda}{(1 + \lambda)^3} \frac{1}{\rho^3}.
\]  

(7.8)

We integrate it over the interval \(1 < \rho < s\) to obtain

\[
V(s) \geq \frac{(s^2 + \lambda)^2}{(1 + \lambda)^2 s^2}.
\]

(7.9)

which yields the desired inequality,

\[
\int_{T_s}^{s} |h|^2 = V(s) + \left| \int_{T_s}^{s} h \right|^2 \geq \left[ \frac{s^2 + \lambda}{(1 + \lambda)s} \right]^2.
\]

(7.10)

Now suppose that the equality occurs in (7.10). Then \(\mathcal{L}^\lambda[V] \equiv 0\) and \(\int_{T_s} h = 0\). The latter, together with (7.1), yields \(a_0 = b_0 = 0\). The equality case of Proposition 6.1 implies \(h = \alpha h^\lambda\) where \(|\alpha| = 1\).

Proof of Theorem 1.5(i). Choose \(\lambda \in (-1, 1]\) so that \(\dot{U}(1) = \frac{1-\lambda}{1+\lambda}\), where \(\dot{U}(1)\) is the initial speed of harmonic evolution, defined by (3.10). Theorem 1.7 yields

\[
R^* (h) \geq \frac{R^2 + \lambda}{(1 + \lambda)R} \geq \frac{1}{2} \left( R + \frac{1}{R} \right).
\]

The latter inequality turns into identity when \(\lambda = 1\).

Remark 7.2. Suppose \(h \in \mathcal{H}_D(\hat{\lambda}, \hat{\lambda}^*)\) with \(R^* = \frac{1}{2} \left( R + \frac{1}{R} \right)\). Then

\[
U(R) = R^2 = \left( \frac{R^2 + 1}{2R} \right)^2.
\]

In view of (7.10) this is only possible if \(\lambda = 1\) and equality holds in (7.8) for all \(\rho \in (1, R)\). Therefore, (7.4) also turns into identity for all \(\rho \in (1, R)\). As in the proof of Theorem 1.7, the equality case of Proposition 6.1 implies

\[
\mathcal{H}_D(\hat{\lambda}, \hat{\lambda}^*) = \{ \alpha h^1: |\alpha| = 1 \}.
\]

(7.11)

The case \(\lambda = 0\) of Theorem 1.7 also gains in interest if we combine it with Lemma 5.2. We obtain a refinement of Schottky’s theorem. Burckel and Poggi-Corradini [3] provided us with a different proof of Corollary 7.3.
**Corollary 7.3.** Let \( h: A(1, R) \to A \) be a conformal mapping such that

\[
\lim_{\rho \searrow 1} \int \psi_{\rho} h = 0.
\]

Then

\[
R_*(h) := \lim_{\rho \nearrow R} \left( \int \psi_{\rho} |h|^2 \right)^{1/2} \geq R.
\]

Also, the area of the target annulus \( A \) is not smaller than that of the domain \( \mathbb{A} = A(1, R) \).

**Proof.** Inequality (7.13) is obtained from Proposition 7.1 by setting \( \lambda = 0 \). To estimate the area of \( A \), we consider the Laurent expansion of \( h \) around zero,

\[
h(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad \text{where} \quad a_0 = \lim_{\rho \searrow 1} \int \psi_{\rho} h = 0.
\]

We explore the orthogonality of the powers of \( z \) to compute

\[
R_*(h) - 1 = \lim_{\rho \nearrow R} \int \psi_{\rho} |h|^2 - \lim_{\rho \searrow 1} \int \psi_{\rho} |h|^2 = \sum_{n \neq 0} |a_n|^2 (R^{2n} - 1),
\]

where the sum is taken to be \( \infty \) when the series diverges. Hence, by (7.13),

\[
\sum_{n \neq 0} |a_n|^2 (R^{2n} - 1) \geq R^2 - 1.
\]

On the other hand, the area of \( A \) is equal to

\[
\int_{\mathbb{A}} |h'(z)|^2 = \pi \sum_{n \neq 0} n |a_n|^2 (R^{2n} - 1)
\]

\[
\geq \pi \sum_{n \neq 0} |a_n|^2 (R^{2n} - 1),
\]

which is greater than or equal to the area of \( \mathbb{A} \), by virtue of (7.14). Here we used the inequality \( n(R^{2n} - 1) \geq (R^{2n} - 1) \), which is valid for all integers.

**Remark 7.4.** Consider a polar mapping \( h: \mathbb{A} \to \mathbb{A}^* \); that is, \( |h(z)| = \text{const} \) on every circle \( \mathbb{T}_\rho \). The observant reader may notice that not only for the variance but also for the quadratic mean \( U = U(\rho, h) \) we have \( L^\lambda(U) \geq 0 \). Indeed, (5.4)
and Hölder’s inequality imply

\[ L^\lambda[U] \geq \frac{2}{\rho^2} \int_{T_\rho} (|h_\theta|^2 - |h|^2) \geq \frac{2}{\rho^2} \left[ \left( \int_{T_\rho} |h_\theta| \right)^2 - \int_{T_\rho} |h|^2 \right] = 0. \]

Here \( \frac{1}{\rho} \int_{T_\rho} |h_\theta| \) represents the length of the curve \( C_\rho^* = h(T_\rho) \) which equals \( 2\pi h(\rho) \). Therefore, the Nitsche bound (1.5) holds for polar mappings.

8. Proof of Theorem 1.4.

8.1. The case \( 1 < R \leq e \). We will prove the following, more precise statement: if for some \(-1 < \lambda \leq 1\)

\[ \dot{U}(1) \geq 2 \frac{1 - \lambda}{1 + \lambda} \tag{8.1} \]

and

\[ R^2 - 1 - (R^2 - \lambda) \log R \geq 0, \tag{8.2} \]

then

\[ R_*(h) \geq \frac{R^2 + \lambda}{(1 + \lambda)R}. \tag{8.3} \]

Theorem 1.4 for \( 1 < R \leq e \) follows by choosing \( \lambda = 1 \) above. We may assume that \( R_*(h) < \infty \), otherwise (8.3) is vacuous.

In this proof we examine \( L^\lambda[U] \) for the quadratic mean \( U(\rho) = \int_{T_\rho} |h|^2 \). Examples show that for a general harmonic mapping \( h \in \mathcal{H}(A, \mathcal{A}) \), the corresponding value \( L^\lambda[U] \) need not be nonnegative pointwise in the entire interval \((1, R)\). But it does in an average sense, when the domain annulus \( A = A(1, R) \) is not too wide. We shall integrate \( L^\lambda[U] \) against a carefully adopted weight in the interval \((1, R)\). It is important that such a weighted average of \( L^\lambda[U] \) will depend only on \( U(1), \dot{U}(1) \) and \( U(R) \), which we defined in Section 3. The following integral is the key ingredient,

\[ \mathcal{K}^\lambda[U] := \int_1^R \rho(R^2 - \rho^2) \frac{L^\lambda[U]}{\rho^2 + \lambda} d\rho. \tag{8.4} \]

This, possibly improper, integral has a well-defined meaning in \((-\infty, +\infty]\). Indeed,

\[ L^\lambda[U] = L^\lambda[V] + L^\lambda[U_0] \]

where \( L^\lambda[V] \geq 0 \) by Proposition 6.1 and \( L^\lambda[U_0] \) is bounded on \([1, R]\). We
integrate (8.4) by parts using the divergence form (5.2) of $L^\lambda[U]$ to obtain the identity

\begin{equation}
K^\lambda[U] = \frac{R^2}{R^2 + \lambda} U(R) - \frac{2\lambda R^2 + 1}{(1 + \lambda)^2} U(1) - \frac{R^2 - 1}{1 + \lambda} \dot{U}(1).
\end{equation}

Since $U(1) = 1$, (8.5) takes the form

\begin{equation}
K^\lambda[U] = \frac{2R^2}{R^2 + \lambda} \left\{ U(R) - \left[ \frac{R^2 + \lambda}{(1 + \lambda)R} \right]^2 \right\} - \frac{R^2 - 1}{1 + \lambda} \left( \dot{U}(1) - 2 \frac{1 - \lambda}{1 + \lambda} \right).
\end{equation}

The desired bound (8.3) will follow once we know that

\begin{equation}
K^\lambda[U] \geq 0.
\end{equation}

Before proving (8.7), observe that equality occurs for the mapping

\begin{equation}
h^\lambda(z) = \frac{1}{1 + \lambda} \left( z + \frac{\lambda}{\bar{z}} \right) = \left( \frac{\rho}{1 + \lambda} + \frac{\lambda}{(1 + \lambda)\rho} \right) e^{i\theta}
\end{equation}

because $L^\lambda[U(\rho, h^\lambda)] \equiv 0$, see Section 5. This suggests writing our mapping $h$ in the form

\begin{equation}
h(\rho e^{i\theta}) = h^\lambda(\rho e^{i\theta}) g(\rho e^{i\theta}), \quad 1 < \rho < R.
\end{equation}

We need a lemma.

**Lemma 8.1.** Let $h \in \mathcal{H}(\mathbb{A}, \mathcal{A})$, then

\begin{equation}
\lim_{\rho \to 1} \int_{\mathbb{T}_1} \text{Im} \bar{h} h_{\theta} = 1,
\end{equation}

hence

\begin{equation}
\lim_{\rho \to 1} \int_{\mathbb{T}_1} \text{Im} \bar{g} g_{\theta} = 0.
\end{equation}
Proof. To prove (8.10) we compute the area of a domain $\Omega_{T}$ inside the Jordan curve $T_{\rho}^{*} = h(T_{\rho})$ as follows.

\[
|\Omega_{T}| = \frac{1}{2i} \int_{T_{\rho}} \bar{h} dh = \frac{1}{2i} \int_{0}^{2\pi} \bar{h} h_{\theta} d\theta = \pi \int_{T_{\rho}} \text{Im} \bar{h} h_{\theta}.
\]

On the other hand $|\Omega_{T}| - \pi$ is the area of the region enclosed between $T$ and $T_{\rho}^{*}$, which converges to 0 as $\rho \searrow 1$.

To prove (8.11) we differentiate the mapping $g = h/h_{\lambda}$ and find that

\[
\tilde{g} g_{\theta} = \left| h_{\lambda} \right|^{-2} \left[ \bar{h} h_{\theta} - \frac{1}{2} \frac{h_{\lambda}}{h_{\lambda}} \right].
\]

This implies

\[
\lim_{\rho \searrow 1} \frac{1}{T_{\rho}} \int_{T_{\rho}} \text{Im} \tilde{g} g_{\theta} = \lim_{\rho \searrow 1} \frac{(1 + \lambda)^{2} \rho^{2}}{(\rho^{2} + \lambda)^{2}} \left[ \int_{T_{\rho}} \text{Im} \bar{h} h_{\theta} - \int_{T_{\rho}} |h|^{2} \text{Im} \frac{h_{\lambda}}{h_{\lambda}} \right]
\]

\[
= \lim_{\rho \searrow 1} \frac{1}{T_{\rho}} \int_{T_{\rho}} \text{Im} \bar{h} h_{\theta} - \int_{T_{\rho}} \text{Im} \frac{h_{\lambda}}{h_{\lambda}} = 1 - 1 = 0.
\]

The same circular means, $\int_{T_{\rho}} \tilde{g} g_{\theta}$, link us with the Jacobian determinant

\[
J(z, g) = \frac{1}{\rho} \text{Im} \tilde{g} \rho g_{\theta} = |g_{\lambda}|^{2} - |g_{\lambda}|^{2}.
\]

Indeed, we differentiate with respect to $\rho$ and integrate by parts along the circle $T_{\rho}$ to obtain

\[
\frac{d}{d\rho} \frac{1}{T_{\rho}} \int_{T_{\rho}} \tilde{g} g_{\theta} = \int_{T_{\rho}} (\tilde{g} \rho g_{\theta} + \tilde{g} g_{\theta}) = \int_{T_{\rho}} (\tilde{g} \rho g_{\theta} - \tilde{g} \rho g_{\theta})
\]

\[
= 2i \int_{T_{\rho}} \text{Im} \tilde{g} \rho g_{\theta}.
\]

Hence the formula

\[
(8.12) \quad \frac{d}{d\rho} \frac{1}{T_{\rho}} \int_{T_{\rho}} \text{Im} \tilde{g} g_{\theta} = \frac{1}{\pi} \int_{T_{\rho}} (|g_{\lambda}|^{2} - |g_{\lambda}|^{2}).
\]

We now take advantage of formula (5.4) for the operator $L^{\lambda}$,

\[
L^{\lambda}[U] = \frac{2}{\rho^{2}} \int_{T_{\rho}} \left| h_{\theta} \right|^{2} - |h| + (\rho^{2} + \lambda)^{2} \left| \frac{d}{d\rho} \left( \frac{\rho h}{\rho^{2} + \lambda} \right) \right|^{2}.
\]

\[
(8.13) \quad L^{\lambda}[U] = \frac{2}{\rho^{2}} \int_{T_{\rho}} \left| h_{\theta} \right|^{2} - |h| + (\rho^{2} + \lambda)^{2} \left| \frac{d}{d\rho} \left( \frac{\rho h}{\rho^{2} + \lambda} \right) \right|^{2}.
\]
In order to express \( L^\lambda[U] \) by means of \( g \) we compute the terms under the integral sign,

\[
|h|^2 = \frac{(\rho^2 + \lambda)^2}{(1 + \lambda)^2 \rho^2} |g|^2 ;
\]
\[
|h_\theta|^2 = \frac{(\rho^2 + \lambda)^2}{(1 + \lambda)^2 \rho^2} |g_\theta + ig|^2 ;
\]
\[
\frac{d}{d\rho} \left( \frac{\rho h}{\rho^2 + \lambda} \right) = e^{i\theta} \frac{g_\rho}{1 + \lambda} g_\rho.
\]

Therefore,

\[
L^\lambda[U] = \frac{2}{\rho^4} \frac{(\rho^2 + \lambda)^2}{(1 + \lambda)^2} \int_{T_\rho} \left( |g_\theta + ig|^2 - |g|^2 + \rho^2 |g_\rho|^2 \right)
\]
\[
= \frac{2}{(1 + \lambda)^2 \rho^2} \int_{T_\rho} \left( |g_\rho|^2 + \rho^{-2} |g_\theta|^2 + 2\rho^{-2} \text{Im} (\bar{g} g_\theta) \right)
\]
\[
= \frac{4}{(1 + \lambda)^2 \rho^2} \int_{T_\rho} \left( |g_z|^2 + |g_\theta|^2 + \rho^{-2} \text{Im} (\bar{g} g_\theta) \right).
\]

Substitute this into (7.1) to obtain

\[
(8.14) \quad K^\lambda[U] = I + II,
\]

where

\[
I = \frac{4}{(1 + \lambda)^2} \int_1^R \frac{(R^2 - \rho^2)(\rho^2 + \lambda)}{\rho} \int_{T_\rho} \left( |g_z|^2 + |g_\theta|^2 \right)
\]
\[
II = \frac{4}{(1 + \lambda)^2} \int_1^R \frac{(R^2 - \rho^2)(\rho^2 + \lambda)}{\rho^3} \int_{T_\rho} \text{Im} (\bar{g} g_\theta).
\]

By Fubini’s theorem \( I \) takes the form of a double integral

\[
I = \frac{4}{(1 + \lambda)^2 \pi} \iint_{\mathbb{H}} \frac{(R^2 - \rho^2)(\rho^2 + \lambda)}{2\rho^2} \left( |g_z|^2 + |g_\theta|^2 \right).
\]

Before converting \( II \) into double integral we shall first integrate by parts. For this we express the factor in front of the circular mean as

\[
(8.15) \quad \frac{(R^2 - \rho^2)(\rho^2 + \lambda)}{\rho^3} = -\frac{d}{d\rho} \left( \frac{R^2 - \lambda \log R}{\rho} - \frac{(R^2 - \rho^2)(\rho^2 - \lambda)}{2\rho^2} \right).
\]
The expression in the square brackets vanishes at the endpoint \( \rho = R \), whereas \( \lim_{\rho \searrow 1} \int_{\pi} \Im \bar{g}g = 0 \) by (8.11). Therefore, integration by parts will not produce the endpoint terms. In view of formula (8.12) we obtain

\[
II = 4 \left(1 + \frac{\lambda}{1 + \lambda^2} \right) \pi \int_1^R \left[ (R^2 - \lambda) \log \frac{R}{\rho} - \frac{(R^2 - \rho^2)(\rho^2 - \lambda)}{2\rho^2} \right] \int_{\pi} \left( |g_z|^2 - |g_{\bar{z}}|^2 \right).
\]

Adding up \( I \) and \( II \) we arrive at the formula

\[
K^\lambda[U] = 4 \left(1 + \frac{\lambda}{1 + \lambda^2} \right) \pi \int_1^R \left[ (R^2 - \lambda) \log \frac{R}{\rho} + \frac{(R^2 - \rho^2)\lambda}{\rho^2} \right] |g_z|^2
\]

\[
+ 4 \left(1 + \frac{\lambda}{1 + \lambda^2} \right) \pi \int_1^R \left[ (R^2 - \rho^2) - (R^2 - \lambda) \log \frac{R}{\rho} \right] |g_{\bar{z}}|^2.
\]

We leave to the reader a routine task of verifying that the factor in from of \( |g_z|^2 \) is nonnegative; that is,

\[
(R^2 - \lambda) \log \frac{R}{\rho} + \frac{(R^2 - \rho^2)\lambda}{\rho^2} \geq 0,
\]

whenever \( 1 \leq \rho \leq R \) and \(-1 < \lambda \leq 1\).

To establish the inequality (8.7) it suffices to ensure that

\[
(R^2 - \rho^2) - (R^2 - \lambda) \log \frac{R}{\rho} \geq 0.
\]

This expression, regarded as a function in \( 1 \leq \rho \leq R \), is concave, vanishes at \( \rho = R \), and is nonnegative at \( \rho = 1 \) by virtue of (8.2). This completes the proof of Theorem 1.4 in case \( 1 < R \leq e \).

**Remark 8.2.** As \( \lambda \) decreases from 1 to \(-1\) the condition (8.2) becomes more restrictive but it still holds for \( R \) sufficiently close to 1. For example, if \( \lambda = 0 \), then (8.2) certainly holds whenever \( 1 < R \leq 2 \).

**Remark 8.3.** For the critical case, suppose \( 1 < R \leq e \) and \( h \in \mathcal{H}(\mathbb{A}, \mathbb{A}^*) \) with \( R_* = \frac{1}{2} \left( R + \frac{1}{R} \right) \). Since \( U(R) = R_* \), we must have \( \lambda = 1 \) in (8.3). Also, \( K^1[U] \leq 0 \), because of (8.6). By (8.16) we have \( g = \text{const.} \) Thus

\[
\mathcal{H}(\mathbb{A}, \mathbb{A}^*) = \{ \alpha h^1 : |\alpha| = 1 \} \quad \text{where} \quad 1 < R \leq e.
\]

**8.2. The case e < R \leq e^{3/2}.** In this case we rely heavily on the orthogonal decomposition (2.1). The operator \( L^\lambda \) and associated integral \( K^\lambda \) from the previous subsection will be used here only with \( \lambda = 1 \) and denoted simply as \( L \) and
Let us state here the relevant versions of identities (8.4) and (8.5), namely

\[ K[U] := \int_1^R \frac{\rho(R^2 - \rho^2)}{\rho^2 + 1} \mathcal{L}[U] d\rho \]  

and

\[ K[U] = \frac{2R^2}{R^2 + 1} U(R) - \frac{R^2 + 1}{2} U(1) - \frac{R^2 - 1}{2} U(1). \]  

We require the following lemma, whose proof is postponed to the end of the section.

**Lemma 8.4.** Suppose that \( R > e \) and \( h \in \mathcal{H}(A, A) \). Then

\[ K[V] \geq (R^2 - 1) |b_0|^2. \]  

**Proof of Theorem 1.4 for \( e < R \leq e^{3/2} \).** Inequality (8.21) yields

\[ K[U] = K[V] + K[U_0] \geq (R^2 - 1) |b_0|^2 + K[U_0]. \]  

From (8.20) we have

\[ K[U_0] = \frac{2R^2}{R^2 + 1} |a_0 \log R + b_0|^2 - \frac{R^2 + 1}{2} |b_0|^2 - 2 \text{Re} (a_0 \bar{b}_0) \frac{R^2 - 1}{2}, \]

hence

\[ K[U] \geq \frac{2R^2 \log^2 R}{R^2 + 1} |a_0|^2 + \frac{R^4 + 2R^2 - 3}{2(R^2 + 1)} |b_0|^2 \]

\[ + 2 \text{Re} (a_0 \bar{b}_0) \left( \frac{2R^2 \log R}{R^2 + 1} - \frac{R^2 - 1}{2} \right). \]

Let us record for future use that (8.23) is valid whenever \( R > e \), as the condition \( R \leq e^{3/2} \) was not used yet.

The quadratic form with respect to \( a_0 \) and \( b_0 \) in the righthand side of (8.23) is positive definite, provided that the quantity (minus discriminant)

\[ \left( \frac{2R^2 \log^2 R}{R^2 + 1} \right) \left( \frac{R^4 + 2R^2 - 3}{2(R^2 + 1)} \right) - \left( \frac{2R^2 \log R}{R^2 + 1} - \frac{R^2 - 1}{2} \right)^2 \]

is positive. Multiplying (8.24) by \( 4(R^2 + 1) \), we arrive at the function

\[ \phi(R) := 4R^2(R^2 - 3) \log^2 R + 8R^2(R^2 - 1) \log R - (R^2 - 1)(R^4 - 1). \]
It remains to prove that $\phi(R) > 0$ for $e \leq R \leq e^{3/2}$. First compute

$$
\phi(e) = 13e^4 - e^6 - 19e^2 - 1 > 0 \quad \text{and} \quad \phi(e^{3/2}) = 22e^6 - e^9 - 38e^3 - 1 > 0.
$$

Since the second derivative

$$
d^2dR^2 (R^{-4} \phi(R)) = \frac{2}{R^5} \{4R^4 \log R + 36R^2(\log^2 R - \log R) + 2R^2(R^4 - 1) + 10\}
$$

is negative for $R \geq e$, it follows that $\phi(R) > 0$ for $e \leq R \leq e^{3/2}$.

Thus, $K[U] \geq 0$, which by (8.20) yields

$$
\frac{2R^2}{R^2 + 1} U(R) \geq \frac{R^2 + 1}{2} U(1) + \frac{R^2 - 1}{2} \dot{U}(1) \geq \frac{R^2 + 1}{2},
$$

as required.

\[ \square \]

**Remark 8.5.** For the extremal case $R_* = \frac{1}{2} \left( R + \frac{1}{R} \right)$ suppose $e < R \leq e^{3/2}$ and $h \in \mathcal{H}(\Lambda, \Lambda^*)$. Since $U(R) = R_*^2$, we have $K[U] \leq 0$ because of (8.20). On the other hand, the quadratic form in (8.23) is strictly positive unless $a_0 = b_0 = 0$. Invoking Remark 7.2, we arrive at

$$
\mathcal{H}(\Lambda, \Lambda^*) = \mathcal{H}_D(\Lambda, \Lambda^*) = \{ \alpha h^1 : |\alpha| = 1 \} \quad \text{where} \quad e < R \leq e^{3/2}.
$$

**Proof of Lemma 8.4.** Let us assume for now that $h$ is continuously differentiable up to the inner circle $T$; this assumption will be removed later. It is easy to see that

$$
\frac{1}{i} \int_T h \bar{h}_{\theta} - \int_T |h|^2 + \left| \int_T h \right|^2 = \sum_{n \neq 0} (n - 1) |a_n + b_n|^2.
$$

We claim that

$$
\int_1^R \frac{\rho(R^2 - \rho^2)}{\rho^2 + 1} \mathcal{L}[U_n] d\rho \geq (R^2 - 1)(n - 1) |a_n + b_n|^2, \quad n \neq 0.
$$

Indeed, in Section 6 we found that the lefthand side of (8.28) is nonnegative for $n \neq 0$. Thus, we only need to prove (8.28) for $n \geq 2$. Using the identity (8.20), we find

$$
\int_1^R \frac{\rho(R^2 - \rho^2)}{\rho^2 + 1} \mathcal{L}[U_n] d\rho = (R^2 - 1)(n - 1) |a_n + b_n|^2

= \frac{1}{2(R^2 + 1)} \left\{ A_n |a_n|^2 + B_n |b_n|^2 + 2C_n \text{Re}(a_n \bar{b}_n) \right\},
$$
where

\begin{align}
A_n &= 4R^{2n+2} + (R^2 - 3)(R^2 + 1) - 4n(R^4 + 1); \\
B_n &= 4R^{2-2n} + (R^2 - 3)(R^2 + 1); \\
C_n &= -(R^2 - 1)(2n(R^2 + 1) - R^2 - 3).
\end{align}

Our goal is to show that the quadratic form in (8.29) is positive definite as long as \( n \geq 2 \) and \( R \geq e \). To this end, we can replace the coefficient \( B_n \) with the smaller quantity \( \tilde{B}_n = (R^2 - 3)(R^2 + 1) \). Since \( R^2 \geq e^2 > 3 \), we have \( \tilde{B}_n > 0 \).

Therefore, it remains to prove that

\begin{align}
D(n, R) := A_n \tilde{B}_n - C_n^2 > 0 \quad \text{for } R \geq e, \ n \geq 2.
\end{align}

After a simplification,

\begin{align*}
D(n, R) &= 4\{R^{2n+2}(R^4 - 2R^2 - 3) - n^2 R^8 + (4n - 2)R^6 \\
&\quad + 2n^2 R^4 + (6 - 4n)R^2 - n^2\}.
\end{align*}

First consider the case \( n = 2 \):

\begin{align*}
D(2, R) &= 4(R^2 - 1)(R^8 - 5R^6 - 2R^4 + 6R^2 + 4) > 0
\end{align*}

because \( R^8 \geq e^2 R^6 > 7R^6 \). We will show that \( D(n, R) \) is convex and increasing with respect to \( n \geq 2 \) for each \( R \geq e \). Indeed

\[ \frac{\partial D(n, R)}{\partial n} = 8\{R^{2n+2}(R^4 - 2R^2 - 3) \log R - nR^8 + 2R^6 + 2nR^4 - 2R^2 - n\}. \]

This derivative is positive for \( n = 2 \), as it simplifies to

\[ 8\left\{R^6(R^4 - 2R^2 - 3) \log R - 2R^8 + 2R^6 + 4R^4 - 2R^2 - 2\right\} \]

\[ \geq 8\left\{R^6(R^4 - 2R^2 - 3) - 2R^8 + 2R^6 + 4R^4 - 2R^2 - 2\right\} \]

\[ = 8(R^2 + 1)(R^8 - 5R^6 + 4R^4 - 2) > 0. \]

This leads us to consider the second derivative

\[ \frac{\partial^2 D(n, R)}{\partial n^2} = 16R^{2n+2}(R^4 - 2R^2 - 3) \log^2 R - 8(R^4 - 1)^2. \]

Since \( R^4 - 2R^2 - 3 = (R^2 - 3)(R^2 + 1) > 0 \), the second derivative is increasing
with \( n \). For \( n = 2 \) it is equal to

\[
16R^6(R^4 - 2R^2 - 3)\log^2 R - 8(R^4 - 1)^2 = 16(R^{10} - 2R^8 - 3R^6) - 8(R^4 - 1)^2
\]

\[
= 8(R^2 + 1)(2R^8 - 7R^6 + R^4 + R^2 - 1)
\]

which is positive since \( R^8 > 7R^6 \). Thus, \( D(n, R) \) is convex and increasing with respect to \( n \geq 2 \). This completes the proof of (8.31) and therefore of (8.28).

Summing (8.28) over \( n \neq 0 \) and using (8.27), we obtain

\[
\int_1^R \frac{\rho(R^2 - \rho^2)}{\rho^2 + 1} \mathcal{L}[V] \, d\rho \geq (R^2 - 1) \left\{ \frac{1}{r} \int_{\mathbb{T}} h \, dh_\theta - \int_{\mathbb{T}} |h|^2 + \int_{\mathbb{T}} h^2 \right\}.
\]

By Lemma 8.1 the righthand side of (8.32) is equal to \((R^2 - 1)|b_0|^2\).

Finally, we remove the assumption that \( h \) is smooth up to \( \mathbb{T} \). For \( r \in (1, R/e) \) we can apply (8.32) to the mapping \( f: A(1, R/r) \to \mathbb{C} \) defined by \( f(z) = h(rz) \).

Using Lemma 8.1, we conclude that

\[
\left( \frac{1}{r} \int_{\mathbb{T}} f h_\theta - \int_{\mathbb{T}} |f|^2 + \int_{\mathbb{T}} h^2 \right) \to \int_{\mathbb{T}} h^2 = |b_0|^2
\]

as \( r \searrow 1 \). Also, substitution \( \rho = t/r \) yields

\[
\int_1^R \frac{\rho(R^2 - \rho^2)}{\rho^2 + 1} \mathcal{L}[V(\rho, f)] \, d\rho = \frac{1}{r^2} \int_r^R \frac{t(R^2 - t^2)}{t^2 + r^2} \mathcal{L}[V(t, h)] \, dt.
\]

Recall that \( \mathcal{L}[V] \geq 0 \) by Proposition 6.1. Using the monotone convergence theorem, we conclude that

\[
\int_{\mathbb{T}} \frac{t(R^2 - t^2)}{t^2 + r^2} \mathcal{L}[V(t, h)] \, dt \to \int_1^R \frac{t(R^2 - t^2)}{t^2 + 1} \mathcal{L}[V(t, h)] \, dt.
\]

as \( r \searrow 1 \). Thus, inequality (8.32) remains true without the smoothness assumption on \( h \).

9. Proof of Theorem 1.5(ii). The case \( R \leq e^{3/2} \) was already covered by Theorem 1.4. Thus we may and we do assume that \( R > e \). In this case (8.23) is known to be true. Since \( h \in \mathcal{H}_N(A, A) \), we have \( a_0 = \int_{\mathbb{T}} h_\rho = 0 \). Inequality (8.23) takes the form

\[
K[U] \geq \frac{R^4 + 2R^2 - 3}{2(R^2 + 1)} |b_0|^2 \geq 0.
\]
It follows from (8.20) that
\[
\frac{2R^2}{R^2 + 1} U(R) = \frac{R^2 + 1}{2} U(1) + \frac{R^2 - 1}{2} U(1) + K^1(U) \geq \frac{R^2 + 1}{2},
\]
hence
\[
U(R) \geq \left( \frac{R^2 + 1}{2R} \right)^2
\]
as required. \(\square\)

**Remark 9.1.** For the critical case, suppose \( h \in \mathcal{H}_N(\mathbb{A}, \mathbb{A}^*) \) with \( R_* = \frac{1}{2} \left( R + \frac{1}{R} \right) \). Since \( U(R) = R^2 \), we have \( K[U] \leq 0 \) because of (8.20). Contrasting this with (9.1), we are led to the conclusion \( b_0 = 0 \). Then, by Remark 7.2, we conclude with

(9.2) \[ \mathcal{H}_N(\mathbb{A}, \mathbb{A}^*) = \mathcal{H}_D(\mathbb{A}, \mathbb{A}^*) = \{ c h^1 : |c| = 1 \}. \]

**10. Proof of Theorem 1.6.** Combining (7.11), (8.18), (8.26) and (9.2), Theorem 1.6 follows. \(\square\)

DEPARTMENT OF MATHEMATICS, SYRACUSE UNIVERSITY, SYRACUSE, NY 13244

E-mail: tiwaniec@syr.edu

E-mail: lvkovale@syr.edu

E-mail: jkonnine@syr.edu

REFERENCES

[1] K. Astala, T. Iwaniec, and G. J. Martin, *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane*, Princeton University Press, 2009.

[2] Astala, Iwaniec, and Martin, Deformations of smallest mean distortion, *Arch. Ration. Mech. Anal.* 195 (2010), no. 3, 899–921.

[3] R. B. Burckel and P. Poggi-Corradini, Some remarks about analytic functions defined on an annulus, unpublished.

[4] P. Duren, *Harmonic Mappings in the Plane*, Cambridge Tracts in Math., vol. 156, Cambridge University Press, Cambridge, 2004.

[5] T. Iwaniec, L. V. Kovalev, and J. Onninen, The Nitsche conjecture, preprint, arXiv:0908.1253.

[6] Iwaniec, Doubly connected minimal surfaces and extremal harmonic mappings, preprint, arXiv:0912.3542.

[7] T. Iwaniec, G. Martin, and C. Sbordone, \( L^p \)-integrability & weak type \( L^2 \)-estimates for the gradient of harmonic mappings of \( D \), *Discrete Contin. Dyn. Syst. Ser. B* 11 (2009), no. 1, 145–152.

[8] T. Iwaniec and J. Onninen, \( n \)-Harmonic mappings between annuli, preprint.

[9] D. Kalaj, On the Nitsche conjecture for harmonic mappings in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \), *Israel J. Math.* 150 (2005), 241–251.
[10] A. Lyzzaik, The modulus of the image annuli under univalent harmonic mappings and a conjecture of J.C.C. Nitsche, *J. London Math. Soc.* **64** (2001), 369–384.

[11] M. H. A. Newman, *Elements of the Topology of Plane Sets of Points*, 2nd ed., Cambridge University Press, 1951.

[12] J. C. C. Nitsche, On the modulus of doubly connected regions under harmonic mappings, *Amer. Math. Monthly* **69** (1962), 781–782.

[13] ———, *Vorlesungen über Minimalflächen*, Springer-Verlag, Berlin-New York, 1975.

[14] F. H. Schottky, Über konforme Abbildung von mehrfach zusammenhängenden Fläche, *J. für Math.* **83** (1877).

[15] A. Weitsman, Univalent harmonic mappings of annuli and a conjecture of J.C.C. Nitsche, *Israel J. Math.* **124** (2001), 327–331.