PATTERN FORMATION OF A DIFFUSIVE
ECO-EPIDEMIOLOGICAL MODEL WITH PREDATOR-PREY
INTERACTION

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Abstract. We consider a predator-prey system with a ratio-dependent functional response when a prey population is infected. First, we examine the global attractor and persistence properties of the time-dependent system. The existence of nonconstant positive steady-states are studied under Neumann boundary conditions in terms of the diffusion effect; namely, pattern formations, arising from diffusion-driven instability, are investigated. A comparison principle for the parabolic problem and the Leray-Schauder index theory are employed for analysis.

1. Introduction. Over the last three decades, predator-prey models have been studied extensively by many researchers. In particular, ratio-dependent predator-prey models, in which the per capita predator growth rate depends on a function of the ratio of prey to predator abundance, have been proposed by Arditi and Ginzburg [1]. Ratio-dependent models have been mathematically studied for both the spatially homogeneous case [13, 14, 15] and spatially inhomogeneous case [6, 20, 24]. The actual evidence and justification of these models have also been studied [2, 3, 9, 11].

On the other hand, because of Kermack-McKendrick’s model, epidemic models have also received a lot of attention. We are interested in the study of ecological systems with the influence of epidemiological factors. A reasonable number of studies have been done on the spatially homogeneous case [5, 7, 8, 12, 26, 27].

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Arino et al. [4] suggested the following non-dimensionalized model, which is a non-spatial version:

\[
\begin{align*}
\frac{dS}{dt} &= rS\left(1 - \frac{S}{K}\right) - \frac{SP}{k_1P + S + I} - \lambda SI, \quad S(0) > 0 \\
\frac{dI}{dt} &= \lambda SI - \frac{IP}{k_1P + S + I} - \delta_I I, \quad I(0) > 0 \\
\frac{dP}{dt} &= \frac{k(\alpha S + \alpha I)P}{k_1P + S + I} - \delta_P P, \quad P(0) > 0,
\end{align*}
\]

where \(S, I,\) and \(P\) denote the population density of the susceptible prey, infected prey, and predator, respectively. The initial conditions are \(R^3 = \{(S, I, P) \in \mathbb{R}^3 : S \geq 0, I \geq 0, P \geq 0\}\). The authors studied the behavior of the system near the equilibria, and obtained the criteria of the persistence of the system.

In this paper, with this motivation, we consider a diffusive eco-epidemiological model with infection in the prey and ratio-dependent functional responses:

\[
\begin{align*}
\frac{du}{dt} - d\Delta u &= u[r - \frac{r}{K}u - \frac{\alpha w}{m_0 + u + v} - bv] \\
\frac{dv}{dt} - d\Delta v &= v[\beta w - (\delta_1 + \frac{c\beta v}{m_0 + u + v})] \\
\frac{dw}{dt} - D\Delta w &= w[-d_2 + \frac{m_0 + u + v}{m_0 + u + v} + \frac{c\beta v}{m_0 + u + v}] \\
\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = \frac{\partial w}{\partial \eta} &= 0 \quad \text{on } (0, \infty) \times \partial \Omega, \\
u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad w(0,x) = w_0(x) \quad \text{in } \Omega,
\end{align*}
\]

where \(\Omega \subseteq \mathbb{R}^N\) is a bounded region with smooth boundary \(\partial \Omega\), and \(r, m, K, b, d_1, D, c, \alpha,\) and \(\beta\) are positive constants. The initial functions \(u_0, v_0,\) and \(w_0\) are not identically zero in \(\Omega\); \(u, v,\) and \(w\) represent the densities of the susceptible prey, infected prey, and predator, respectively, and \(\eta\) is the outward directional derivative normal to \(\partial \Omega\). Furthermore, \(\alpha\) and \(\beta\) are the searching efficiency constants of the predation rate for the susceptible and infective prey, respectively. \(\frac{\alpha}{m}\) and \(\frac{\beta}{m}\) are the maximum per capita capturing rates of the predator for the susceptible prey and infected prey, respectively. \(m\) is the predation rate for the susceptible prey and infected prey. Finally, \(b\) is the force of infection, \(d_1\) and \(d_2\) are the death rates of the infected prey and predator, respectively, and \(c\) is a conversion rate. The homogeneous Neumann boundary condition describes an environment with no flux at the boundary of the region.

Model (1) is based on the following assumptions:

- In the absence of disease, the prey population grows according to logistic law with carrying capacity \(K > 0\) and an intrinsic growth rate \(r > 0\).
- In the presence of disease, the prey consists of two classes: susceptible prey and infected prey.
- Only susceptible prey can reproduce themselves logistically and contribute to its carrying capacity. Infected prey do not grow, recover, or reproduce.
- Disease can only be spread among the prey, and it is not inherited. Disease transmission follows the simple law of mass-action.

For additional background information pertaining to (1), we refer to [4] and the references therein.
The goal of this study is to determine the coexistence states by investigating the non-constant positive solutions of the following time-independent system:

\[
\begin{align*}
-\Delta u &= u \left[ r - \frac{r}{K} u - \frac{\alpha w}{mw + u + v} - bv \right] \\
-\Delta v &= v \left[ bu - d_1 - \frac{\beta w}{cw + u + v} \right] \\
-D\Delta w &= w \left[ -d_2 + \frac{u + v}{cw + u + v} + \frac{c\beta v}{cw + u + v} \right] \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = \frac{\partial w}{\partial \eta} &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

Note that the given growth rates in (2) are not defined at \((u, v, w) = (0, 0, 0)\). However, since

\[
\lim_{(u, v, w) \to (0, 0, 0)} \frac{uw}{mw + u + v} = \lim_{(u, v, w) \to (0, 0, 0)} \frac{vw}{mw + u + v} = 0,
\]

the domain of \(\frac{uw}{mw + u + v}\) and \(\frac{vw}{mw + u + v}\) may be extended to \(\{(u, v, w) : u \geq 0, v \geq 0, w \geq 0\}\) so that \((0, 0, 0)\) becomes a trivial solution to (2) [15]. Furthermore, note that the growth rates of the given system are not quasimonotone nondecreasing.

The remainder of this paper is organized as follows. In Section 2, the global attractor and persistence properties are given for solutions to the time-dependent system in (1). In Section 3, we estimate the bounds of positive solutions to system (2). Section 4 provides the existence and non-existence of non-constant positive solutions to (2). Finally, the results obtained are analyzed briefly in terms of biological interpretations in Section 5.

2. Global attractor and persistence properties of time-dependent system.

In this section, the global attractor and persistence properties are investigated for solutions to the time-dependent system in (1).

For convenience, we denote the growth rate terms as follows:

\[
\begin{align*}
f_1(u, v, w) &:= r - \frac{r}{K} u - \frac{\alpha w}{mw + u + v} - bv, \\
f_2(u, v, w) &:= bu - d_1 - \frac{\beta w}{cw + u + v}, \\
f_3(u, v, w) &:= -d_2 + \frac{c\alpha u}{cw + u + v} + \frac{c\beta v}{cw + u + v}.
\end{align*}
\]

Using the uniform bound of \(u, v\) and \(w\), one can show that \((uf_1, vf_2, wf_3)\) satisfies the Lipschitz condition. Using the upper and lower solution method in [22], it can also be shown that (1) has a nonnegative solution.

The next theorem states that the solution to (1) is uniformly bounded.

**Theorem 2.1.** The solution \((u, v, w)\) of (1) is uniformly bounded; specifically,

\[
0 \leq u(t, x) \leq B_1, \quad 0 \leq v(t, x) \leq B_2, \quad 0 \leq w(t, x) \leq B_3,
\]

where \(B_i\) is defined by

\[
\begin{align*}
B_1 &:= \max \{ K, \|u_0\|_\infty \}, \\
B_2 &:= \max \left\{ \frac{1}{d_1} \frac{K}{r} \left( \frac{r + d_1}{2} \right)^2, \|u_0\|_\infty + \|v_0\|_\infty \right\}, \\
B_3 &:= \max \left\{ \|w_0\|_\infty, \frac{c(\alpha + \beta) - d_2}{d_2m} B_2 \right\}.
\end{align*}
\]
Proof. First, note that from the strong maximum principle and the Hopf Boundary Lemma for parabolic problems, \( u(t, x), v(t, x), \) and \( w(t, x) \) are nonnegative.

Next, we show that the solution to (1) is bounded above on \([0, \infty) \times \Omega\). Since \( u_t - d\Delta u = uf_1(u, v, w) \leq u[r - \frac{r}{K}u] \), by a comparison argument for the parabolic equation, \( 0 \leq u(t, x) \leq B_1 \) holds.

Adding the first equation to the second equation in (1) yields,

\[
(u + v)_t - d\Delta (u + v) = uf_1(u, v, w) + vf_2(u, v, w) \leq u(r + d_1 - \frac{r}{K}u) = d_1(u + v) \leq K \frac{r + d_1}{r} \frac{2}{2} - d_1(u + v).
\]

The last inequality follows from the fact that the maximum value of \( u(r + d_1 - \frac{r}{K}u) \) is \( \frac{K}{r} \frac{(r + d_1)^2}{2} \).

Now, let \( Z := u + v \). Consider \( Z_t - d\Delta Z = \frac{K}{r} \frac{(r + d_1)^2}{2} - d_1 Z \) in \((0, \infty) \times \Omega\) and \( \frac{\partial Z}{\partial \eta} = 0 \) on \((0, \infty) \times \partial \Omega\). Then by the maximum principle, \( Z \leq B_2 \). Since \( v \leq Z \), the desired result is obtained.

Finally, the upper bound \( B_3 \) of \( w \) follows from the maximum principle, the fact that \( u \leq Z \leq B_2 \), and the following inequality:

\[
w_t - D\Delta w = wf_3(u, v, w) \leq w \left[ c\alpha \frac{u}{mw + u} + \frac{c\beta v}{mw + v} - d_2 \right] \leq w \left[ \frac{c\alpha B_2}{mw + B_2} + \frac{c\beta B_2}{mw + B_2} - d_2 \right] = w \left[ \frac{c(\alpha + \beta) - d_2}{mw + B_2} \right].
\]

Next, we examine the dissipation and persistence of parabolic system (1).

**Theorem 2.2.** For a solution \( u = (u(t, x), v(t, x), w(t, x)) \) to the parabolic system (1),

\[
\limsup_{t \to \infty} u \leq \left( K, \frac{1}{d_1} \frac{K}{r} \frac{(r + d_1)^2}{2}, \frac{c(\alpha + \beta) - d_2}{d_2m} \right),
\]

if \( c(\alpha + \beta) > d_2 \).

**Proof.** First, \( \limsup_{t \to \infty} u(t, x) \leq K \) follows from a comparison argument for parabolic problems, since \( f_1(u, v, w) \leq (r - \frac{r}{K}u) \).

To obtain the inequality involving \( v \), we use the same argument used in the proof of Theorem 2.1. Consider the following parabolic problem:

\[
\begin{cases}
Z_t - d\Delta Z = \frac{K}{r} \frac{(r + d_1)^2}{2} - d_1 Z & \text{in } (0, \infty) \times \Omega, \\
\frac{\partial Z}{\partial \eta} = 0 & \text{on } (0, \infty) \times \partial \Omega, \\
Z(0, x) = u(0, x) + v(0, x) & \text{on } \Omega.
\end{cases}
\]

Clearly \( Z(t, x) = Z(0, x)e^{-d_1t} + \frac{1}{d_1} \frac{K}{r} \frac{(r + d_1)^2}{2} \left( 1 - e^{-d_1t} \right) \) is the solution to this problem; thus, \( \limsup_{t \to \infty} v(t, x) \leq \frac{1}{d_1} \frac{K}{r} \frac{(r + d_1)^2}{2} \). The same inequality also holds for \( u \). Thus, there exists a \( T \) such that \( u(t, x), v(t, x), w(t, x) \leq \frac{1}{d_1} \frac{K}{r} \frac{(r + d_1)^2}{2} + \varepsilon \) on \( \Omega \) for time \( t \geq T \).
Finally, using the same method used in Theorem 2.1 to find the uniform bound of $w$, we obtain $\limsup_{t \to \infty} w(t, x) \leq \frac{\alpha \beta}{\alpha + \beta} - d_2 \left( \frac{1}{d_1 T} \left( \frac{r + d_1}{2} \right)^2 + \varepsilon \right)$. Therefore, since $\varepsilon$ is arbitrary, the desired result is achieved. \(\Box\)

The following theorem states the persistence property of system (1).

**Theorem 2.3.** Assume that $\beta \geq \alpha > \frac{d_2}{c}$, $r > \min \{ \frac{b K (r + d_1)}{a_1 r} \left( \frac{r + d_1}{2} \right)^2 + \frac{\alpha}{m}, \frac{1}{b} \left( d_1 + \frac{\beta}{m} \right) + \frac{\alpha}{m} \}$. Then

$$\liminf_{t \to \infty} u \geq \left( \Theta_1, \Theta_2, \Theta_3 \right),$$

where $\Theta_1 := \left( r - \frac{b K (r + d_1)}{a_1 r} \left( \frac{r + d_1}{2} \right)^2 - \frac{\alpha}{m} \right) K$, $\Theta_2 := \frac{1}{b} \left( r - \frac{1}{b} \left( d_1 + \frac{\beta}{m} \right) - \frac{\alpha}{m} \right)$, and $\Theta_3 := \frac{c \alpha - d_2}{d_2 m} \Theta_1$ for $\frac{a}{m} \Theta_3 \leq b$.

**Proof.** Note from Theorem 2.2 that for an arbitrary $\varepsilon > 0$, there exists $T_1$ such that $v \leq \frac{1}{d_1} \left( \frac{r + d_1}{2} \right)^2 + \varepsilon$ on $\overline{\mathcal{I}}$ for $t \geq T_1$. In light of this fact, it follows that

$$u_t - d \Delta u = uf_1(u, v, w) \geq u \left[ r - \frac{b K (r + d_1)}{a_1 r} \left( \frac{r + d_1}{2} \right)^2 - \frac{\alpha}{m} - \varepsilon - \frac{r}{K} u \right].$$

Thus, there exists $T_2 \geq T_1$ such that $u \geq \Theta_1 - \varepsilon$ on $\overline{\mathcal{I}}$ for $t \geq T_2$. Next, find $\Theta_3$, where $\Theta_3$ is the lower bound of $w$. Note that since $\beta > \alpha$, the term $\frac{c \alpha u + c \beta v}{m w + u + v}$ in $f_3$ is monotone increasing for $v \geq 0$. Since

$$w_t - d \Delta w = w f_3(u, v, w) \geq w \left[ \frac{c \alpha (\Theta_1 - \varepsilon)}{m w + \Theta_1 - \varepsilon} - d_2 \right] \geq w \left[ \frac{(c \alpha - d_2)(\Theta_1 - \varepsilon) - d_2 m w}{m w + \Theta_1 - \varepsilon} \right],$$

there exists $T_3 \geq T_2$ such that $w \geq \Theta_3 - \varepsilon$ on $\overline{\mathcal{I}}$ for $t \geq T_3$.

Finally, we show that $\liminf_{t \to \infty} v \geq \Theta_2$. Notice that $(f_1)_v \leq 0$ if $\frac{\alpha}{m^2 \Theta_3} \leq b$. For a nonnegative solution $w(t, x) > 0$ in $[T_3, \infty) \times \overline{\mathcal{I}}$, consider the following parabolic system:

$$\left\{ \begin{array}{ll}
    u_t - d \Delta u = u \left[ r - \frac{b K (r + d_1)}{a_1 r} \left( \frac{r + d_1}{2} \right)^2 + \frac{\alpha}{m} \right] - bv \\
    v_t - d \Delta v = v \left[ b u - d_1 - \frac{\beta w}{m w + u + v} \right] & \text{in } (T_3, \infty) \times \Omega, \\
    \left. \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} \right| = 0 & \text{on } (T_3, \infty) \times \partial \Omega. 
\end{array} \right.$$  

(3)

Since the given growth rates are not quasimonotone nondecreasing, we are not able to directly apply the comparison theorem for a reaction diffusion system. Thus,
we extend system (3) to four equations (see [23]) :
\[
\begin{cases}
    u_t - d\Delta u = u[r - \frac{r}{K} u - \frac{\alpha w}{mw + u + M - Q} - b(M - Q)] \\
    v_t - d\Delta v = v[bu - d_1 - \frac{\beta w}{mw + u + v}] \\
    P_t - d\Delta P = (M - P)[-r + \frac{r}{K}(M - P) + \frac{\alpha w}{mw + M - P + v} + bv] \\
    Q_t - d\Delta Q = (M - Q)[-b(M - P) + d_1 + \frac{\alpha w}{mw + M - P + M - Q}]
\end{cases}
\] (4)

where \( M := \max\{B_i : i = 1, 2, 3\} \), and \( B_i \) is defined in Theorem 2.1. Observe that (4) is quasimonotone nondecreasing. Furthermore, \((u, v)\) is a solution to (3) if and only if \((u, v, P, Q) = (u, v, M - u, M - v)\) is a solution of (4) (see Lemma 3.1 in [23]).

Finally, we derive \( \limsup_{t \to \infty} Q \leq \frac{\alpha}{m} = \frac{\beta}{m} \) for a sufficiently small positive constant \( \delta \ll 1 \); by the comparison principle, \( P \leq P^* \) and \( Q \leq Q^* \), where \((P^*, Q^*)\) satisfies the following parabolic system:
\[
\begin{cases}
    \hat{P}_t - d\Delta \hat{P} = (M - \hat{P})[-r + \frac{r}{K}(M - \hat{P}) + \frac{\alpha}{m} + bM - b(1 - \delta)\hat{Q}] \\
    \hat{Q}_t - d\Delta \hat{Q} = (M - \hat{Q})[d_1 - bM + b\hat{P} + \frac{\beta}{m}] \text{ in } (0, \infty) \times \Omega, \\
    \hat{P}(0, x) = P(T_3, x), \quad \hat{Q}(0, x) = Q(T_3, x) \quad \text{ in } \Omega.
\end{cases}
\] (5)

The equilibrium point \((\hat{P}^*, \hat{Q}^*) = (M - \frac{1}{b}(d_1 + \frac{\beta}{m}), \frac{M}{1 - \delta} - \frac{\Theta_1}{1 - \beta})\) is globally asymptotically stable. This can be verified by considering the Lyapunov function
\[
E(t) = \int_{\Omega} \left[ A_1 \left( M - \hat{P} - (M - \hat{P}^*) - (M - \hat{P}^*) \log \frac{M - \hat{P}}{M - \hat{P}^*} \right) \\
+ A_2 \left( M - \hat{Q} - (M - \hat{Q}^*) - (M - \hat{Q}^*) \log \frac{M - \hat{Q}}{M - \hat{Q}^*} \right) \right]
\]
for a suitable positive constant \( A_1 \). Thus, since \( \delta \) is arbitrary and \( M - \hat{Q} \leq v \),
\( \liminf_{t \to \infty} v(t, x) \geq \Theta_2 \). \( \square \)

3. Bound estimates of positive steady-states. First, we estimate the bounds of the positive solutions to the elliptic system (2). To estimate the \( a\)-priori bounds for positive solutions to (2), the Maximum Principle [16] and Harnack inequality, introduced by Lin et al. [17], are used.

Note that a positive solution to (2) is contained in \( [C^2(\Omega)]^3 \) by the standard regularity theorem for elliptic equations \([10, 25]\); hence the Harnack inequality can be applied to (2).

**Theorem 3.1.** If \( c \in \{\alpha, \beta\} \geq d_2 \), then positive solutions \((u, v, w)\) of (2) satisfy
\[
u(x) \leq K, \quad v(x) \leq \frac{K(r + d_1)^2}{4d_1r}, \quad w(x) \leq \frac{\alpha d_2 - d_2 K(r + d_1)^2}{d_2m + 4d_1r} \quad \text{on } \Omega. \quad (6)
\]
Proof. Applying the Maximum Principle to the first equation in (2) yields the first assertion.

To establish the second assertion, we add the first and second equations in (2), and then we derive

\[-d\Delta (u + v) \leq u\left(r + d_1 - \frac{r}{K} u\right) - d_1 (u + v) \leq K\left(r + d_1\right)^2 - d_1 (u + v)
\]
in \(\Omega\). Thus from the Maximum principle, we obtain

\[u(x) + v(x) \leq \frac{K\left(r + d_1\right)^2}{4d_1 r}\]
on \(\bar{\Omega}\), which gives the second assertion.

Finally, using the maximum principle again in the third equation in (2), we have the third assertion. \(\square\)

We now estimate a positive lower bound for classical positive solutions to (2). For simplicity, we let \(\Gamma\) denote \((r, K, \alpha, \beta, b, m, d_i)\).

**Theorem 3.2.** Assume \(d \in [d^*, \infty)\) and \(D \in [d_*, D^*]\) for a fixed \(d^*, d_*\) and \(D^* > 0\). Then there exists a positive constant \(C_4(N, \Omega, d^*, d_*, D^*, \Gamma)\) such that a positive solution \((u, v, w)\) of (2) satisfies

\[
\min_{\bar{\Omega}} u(x), \min_{\bar{\Omega}} v(x), \min_{\bar{\Omega}} w(x) > C_4,
\]

if

\[
\begin{align*}
& c \min\{\alpha, \beta\} > d_2, \\
& b\left(r - \frac{\alpha}{m}\right) \frac{K}{r} > \frac{\beta}{m} + d_1.
\end{align*}
\]

Proof. It is easy to verify that \(\frac{f_1}{d}, \frac{f_2}{d}, \frac{f_3}{d} \in C(\bar{\Omega})\) for \(d \geq d^*, D \geq d_*\). By the Harnack inequality, there exists a positive constant \(C_4(N, \Omega, d^*, \Gamma)\) such that

\[
\max_{\bar{\Omega}} u \leq C_4 \min_{\bar{\Omega}} u, \quad \max_{\bar{\Omega}} v \leq C_4 \min_{\bar{\Omega}} v, \quad \max_{\bar{\Omega}} w \leq C_4 \min_{\bar{\Omega}} w
\]

are satisfied.

Suppose to the contrary that (7) is not satisfied. Then by (9), there exist sequences \(\{d_n, D_n\}\) and a corresponding positive solution \((u_n, v_n, w_n)\) to (2) such that \(d_n \geq d^*\) and \(D_n \in [d_*, D^*]\) for \(n \in \mathbb{N}\); furthermore, \(\max_{\bar{\Omega}} u_n \to 0\), \(\max_{\bar{\Omega}} v_n \to 0\), or \(\max_{\bar{\Omega}} w_n \to 0\) as \(n \to \infty\).

By the regularity theory of elliptic equations, there exists a subsequence of \(\{(u_n, v_n, w_n)\}\) and nonnegative functions \(\tilde{u}, \tilde{v}, \tilde{w} \in C^2(\bar{\Omega})\) such that \(u_n, v_n, w_n \to (\tilde{u}, \tilde{v}, \tilde{w})\) as \(n \to \infty\). Moreover, we assume that \((d_n, D_n) \to (\tilde{d}, \tilde{D}) \in [d^*, \infty) \times [d_*, D^*]\) as \(n \to \infty\). Since \(\max_{\bar{\Omega}} u_n \to 0\), \(\max_{\bar{\Omega}} v_n \to 0\), or \(\max_{\bar{\Omega}} w \to 0\) as \(n \to \infty\), \(\tilde{u} \equiv 0\), \(\tilde{v} \equiv 0\), or \(\tilde{w} \equiv 0\). If we only consider the cases that satisfy the last sentence, we are reduced to the following:

(i) \(\tilde{u} \equiv 0\), \(\tilde{v} \not\equiv 0\), \(\tilde{w} \not\equiv 0\) or \(\tilde{u} \equiv 0\), \(\tilde{v} \not\equiv 0\), \(\tilde{w} \equiv 0\).

(ii) \(\tilde{u} \equiv 0\), \(\tilde{v} \equiv 0\), \(\tilde{w} \not\equiv 0\).

(iii) \(\tilde{u} \not\equiv 0\), \(\tilde{v} \equiv 0\), \(\tilde{w} \not\equiv 0\).

(iv) \(\tilde{u} \not\equiv 0\), \(\tilde{v} \not\equiv 0\), \(\tilde{w} \equiv 0\).

(v) \(\tilde{u} \not\equiv 0\), \(\tilde{v} \equiv 0\), \(\tilde{w} \equiv 0\).

(vi) \(\tilde{u} \equiv 0\), \(\tilde{v} \equiv 0\), \(\tilde{w} \equiv 0\).
The following integral equations are obtained by integrating by parts:

\[
\begin{aligned}
\int_{\Omega} u_n f_1(u_n, v_n, w_n) &= 0, \\
\int_{\Omega} v_n f_2(u_n, v_n, w_n) &= 0, \\
\int_{\Omega} w_n f_3(u_n, v_n, w_n) &= 0.
\end{aligned}
\] (10)

(i): Note that \(v_n, w_n > 0\) hold by (9), and \(\bar{u}, \bar{v}, \text{and} \bar{w}\) satisfy (9) since \(u_n, v_n, \text{and} w_n\) satisfy (9). Thus, \(\bar{v} > 0\), since \(\bar{v} \neq 0\). Furthermore, since \(v_n \rightarrow \bar{v} > 0\) and \(u_n \rightarrow 0\) as \(n \rightarrow \infty\), \(v_n f_2(u_n, v_n, w_n) < 0\) for a sufficiently large \(n\). This is a contradiction to the second equation in (10).

(ii): Similarly, for a sufficiently large \(n, w_n f_3(u_n, v_n, w_n) < 0\) since \(w_n > 0, v_n \rightarrow \bar{v} \equiv 0, \text{and} u_n \rightarrow \bar{u} \equiv 0.\) This is a contradiction to the third equation in (10).

(iii): Since \(v_n \rightarrow \bar{v} \equiv 0\) as \(n \rightarrow \infty\), \((\bar{u}, \bar{w})\) is a solution to the \(2 \times 2\) predator-prey model with diffusion \((\bar{d}, \bar{D})\). Furthermore, \(\bar{u} \geq \frac{K}{r}(r - \frac{\alpha}{m})\).

Consider the second equation in (10):

\[
\begin{aligned}
bw_n - d_1 - \frac{\beta w_n}{m w_n + u_n + v_n} &\rightarrow b\bar{u} - d_1 - \frac{\beta \bar{w}}{m \bar{w} + \bar{u}} \geq b \frac{K}{r} \left(r - \frac{\alpha}{m}\right) - d_1 - \frac{\beta}{m} > 0,
\end{aligned}
\]

as \(n \rightarrow \infty\) by (8). This, together with the fact \(v_n > 0\), violates the second integral equation of (10).

(iv): We know that \(u_n, v_n > 0\) on \(\Omega\) for a sufficiently large \(n\); consequently, \(\bar{u}\) and \(\bar{v} > 0\) as well. So as \(n \rightarrow \infty, \frac{\alpha u_n + c \beta v_n}{u_n + v_n} - d_2 \rightarrow \frac{\alpha \bar{u} + c \beta \bar{v}}{\bar{u} + \bar{v}} - d_2 \geq c \min\{\alpha, \beta\} - d_2 > 0\) uniformly on \(\Omega\) by (9). This contradicts the third integral equation of (10).

(v): Note that \(\bar{u} > 0\). Since \(u_n \rightarrow \bar{u}\) and \(v_n, w_n \rightarrow 0\) as \(n \rightarrow \infty, f_3(u_n, v_n, w_n) \rightarrow c \alpha - d_2 > 0\) by (8). This contradicts the third integral equation in (10).

(vi): Consider the following elliptic system under homogeneous Neumann boundary conditions:

\[
\begin{aligned}
-d_n \Delta U_n &= U_n [r - \frac{r}{K} u_n - \frac{\alpha W_n}{m w_n + u_n + v_n} - bw_n], \\
-d_n \Delta V_n &= V_n [bw_n - d_1 - \frac{m w_n + u_n + v_n}{m w_n + u_n + v_n} - d_1 - \frac{\beta}{m} W_n], \\
-d_n \Delta W_n &= W_n f_3(U_n, V_n, W_n) \quad \text{in} \quad \Omega,
\end{aligned}
\] (11)

where \(U_n = \frac{u_n}{\|u_n\|_{\infty} + \|v_n\|_{\infty} + \|w_n\|_{\infty}}\), \(V_n = \frac{v_n}{\|u_n\|_{\infty} + \|v_n\|_{\infty} + \|w_n\|_{\infty}}\), and \(W_n = \frac{w_n}{\|u_n\|_{\infty} + \|v_n\|_{\infty} + \|w_n\|_{\infty}}.\) The following integral identities hold for \(n \geq 1:\)

\[
\begin{aligned}
\int_{\Omega} U_n [r - \frac{r}{K} u_n - \frac{\alpha W_n}{m w_n + u_n + v_n} - bw_n] &= 0, \\
\int_{\Omega} V_n [bw_n - d_1 - \frac{m w_n + u_n + v_n}{m w_n + u_n + v_n} - d_1 - \frac{\beta}{m} W_n] &= 0, \\
\int_{\Omega} W_n f_3(U_n, V_n, W_n) &= 0.
\end{aligned}
\] (12)

Similar to the first part of this proof, there exists a subsequence \((U_n, V_n, W_n)\) that converges to \((\bar{U}, \bar{V}, \bar{W})\) for some nonnegative functions \(\bar{U}, \bar{V}, \text{and} \bar{W} \in C^2(\Omega).\) These nonnegative functions satisfy \(\|\bar{U}\|_{\infty} + \|\bar{V}\|_{\infty} + \|\bar{W}\|_{\infty} = 1\) since \(\|U_n\|_{\infty} + \|V_n\|_{\infty} + \|W_n\|_{\infty} = 1.\) Also \(\bar{U} + \bar{V} + \bar{W} > 0\) on \(\Omega\) since \(\|\bar{U}\|_{\infty} + \|\bar{V}\|_{\infty} + \|\bar{W}\|_{\infty} = 1.\) These nonnegative pairs satisfy the Harnack inequality as well.
If necessary, we consider a subsequence and assume $d_n \to d \in [d^*, \infty]$ and $D_n \to D \in [d^*, D^*]$. Consider the case when $\hat{d} < \infty$. If $\hat{W} \equiv 0$, then $\hat{U} + \hat{V} > 0$ on $\Omega$; moreover, if $\hat{U} \equiv 0$, $\hat{V} > 0$ holds on $\Omega$. It is also true that $\int_{\Omega} V(bu - d_1) = \int_{\Omega} -d_1 \hat{V} < 0$; however, this contradicts the second equation in (12). Conversely, if $\hat{V} \equiv 0$, then $\hat{U} > 0$ holds on $\Omega$ and $\int_{\Omega} \hat{U}(r - \frac{\alpha}{2} \hat{u} - b \hat{v}) = \int_{\Omega} r \hat{U} > 0$. Again this contradicts the first equation in (12). Thus, $\hat{W} > 0$ on $\Omega$.

Therefore, $bu_n - d_1 - \frac{\beta \hat{w}_n}{m \hat{w}_n + u_n + v_n} < 0$ uniformly on $\Omega$ as $n \to \infty$. This contradicts the second integral equation of (12).

Now consider the case when $\hat{d} = \infty$. In this case, $\hat{U} = A$ and $\hat{V} = B$, where $A$ and $B$ are nonnegative constants. Similar to the case when $\hat{d} < \infty$, one can show that $\hat{W} > 0$ on $\Omega$. However, this violates the second integral equation in (12), since $u_n \to \hat{u} \equiv 0$ as $n \to \infty$. □

4. Nonconstant positive solutions. In this section, we obtain the results for the nonexistence and existence of a nonconstant positive solution to system (2).

First, we observe that if the following conditions are satisfied:

$$AS^2 + BS + C < 0 \text{ and } d_2 < c\beta, \quad (13)$$

where

$$A = cmr\beta b,$$
$$B = -c(r\beta(\beta + md_1) + mk(r\beta + \alpha d_1)b) - d_2(-r\beta + Kb(\beta - \alpha)),$$
$$C = K(r\beta + \alpha d_1)[c(\beta + md_1) - d_2],$$
$$S = \frac{(r\beta + \alpha d_1)K}{r\beta + \alpha bK},$$

then there exists a unique positive equilibrium point $u_\ast = (u_\ast, v_\ast, w_\ast)$, where

$$u_\ast = \frac{-B - \sqrt{B^2 - 4AC}}{2A},$$
$$v_\ast = -\left(\frac{r}{bK} + \frac{\alpha}{\beta}\right) u_\ast + \left(\frac{r}{b} + \frac{\alpha d_1}{b\beta}\right),$$
$$w_\ast = \frac{(c\alpha - d_2)u_\ast + (c\beta - d_2)v_\ast}{d_2m}.$$

4.1. Nonexistence of nonconstant positive solutions. In this subsection, we investigate a condition that implies the nonexistence of nonconstant positive solutions to (2). This result is also used in the index approach to show the existence of nontrivial positive steady states, which is the main result of this section.

Before developing our argument, we define the following notation, which is similar to the notation defined in [18, 21].

Notation.

(i) $\mu_i$: Eigenvalue of $-\Delta$ on $\Omega$ under Neumann boundary condition.
(ii) $E(\mu_i)$: The eigenspace corresponding to $\mu_i$.
(iii) $\{\varphi_{ij}: j = 1, \ldots, \dim E(\mu_i)\}$: An orthonormal basis of $E(\mu_i)$.
(iv) $X_{ij} = \{c \cdot \varphi_{ij} | c \in \mathbb{R}^3\}$
(v) $X = \{u = (u, v, w) \in [C^1(\Omega)]^3 | \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0 \text{ on } \partial \Omega\}$. 
The eigenvalues in (i) satisfy \( 0 = \mu_1 < \mu_2 < \mu_3 < \cdots \to \infty \). Also, \( X = \bigoplus_{i=1}^{\infty} X_i \), where \( X_i = \bigoplus_{j=1}^{\dim E_i} X_{ij} \).

**Theorem 4.1.** Assume that \( D\mu_2 > 2c\alpha + 2c\beta - d_2 \). There exists a positive constant \( \bar{d}(N, \Omega, D, \Gamma) \) such that for any \( d > \bar{d} \), (2) has no non-constant positive solution.

**Proof.** First, multiply the first, second, and third equations in (2) by \((u - \overline{u})\), \((v - \overline{v})\), and \((w - \overline{w})\), respectively. Here, \( \overline{\varphi} = \frac{1}{\text{measure} \{\Omega\}} \int_{\Omega} \varphi \). This yields the following:

\[
\begin{align*}
-\alpha(u - \overline{u}) \Delta u & = u(u - \overline{u}) f_1(u, v, w), \\
-\alpha(v - \overline{v}) \Delta v & = v(v - \overline{v}) f_2(u, v, w), \\
-\alpha(w - \overline{w}) \Delta w & = w(w - \overline{w}) f_3(u, v, w).
\end{align*}
\]

By integrating (14) over \( \Omega \) and applying Green’s first identity, the left hand sides of (14) become \( \int_{\Omega} d|\nabla u|^2 + d|\nabla v|^2 + d|\nabla w|^2 \), respectively. By summing the equations in (14), we find that

\[
\int_{\Omega} [((u - \overline{u})(uf_1(u, v, w) - \overline{u}f_1(\overline{u}, \overline{v}, \overline{w}))+ (v - \overline{v})(vf_2(u, v, w) - \overline{v}f_2(\overline{u}, \overline{v}, \overline{w}))+ (w - \overline{w})(wf_3(u, v, w) - \overline{w}f_3(\overline{u}, \overline{v}, \overline{w})))
= \int_{\Omega} \left[ (u - \overline{u})^2 \left( r - \frac{r}{K} (u + \overline{u}) - b\overline{v} - \frac{\alpha \overline{uv}}{\overline{\varphi}} - \frac{\alpha mw\overline{w}}{\overline{\varphi}} \right) \right. \\
+ (v - \overline{v})^2 \left( \frac{\beta mw\overline{w}}{\overline{\varphi}} - \frac{\beta \overline{uw}}{\overline{\varphi}} \right) \\
+ (w - \overline{w})^2 \left. \left( \frac{\beta \overline{vw}}{\overline{\varphi}} - \frac{\beta \overline{uw}}{\overline{\varphi}} - \frac{c\alpha \overline{vw}}{\overline{\varphi}} + \frac{c\beta \overline{uv}}{\overline{\varphi}} \right) \right]
\]

where \( \varphi = (mv + u + v)(mv\overline{w} + u\overline{v} + \overline{v}) \).

The integral equation in (15) is less than or equal to the following:

\[
\int_{\Omega} \left[ (u - \overline{u})^2 \left( \frac{c\alpha \overline{vw}}{\overline{\varphi}} + \frac{c\beta \overline{uv}}{\overline{\varphi}} \right) \right. \\
+ (v - \overline{v})^2 \left( \frac{c\alpha \overline{vw}}{\overline{\varphi}} + \frac{c\beta \overline{uv}}{\overline{\varphi}} \right) \\
+ (w - \overline{w})^2 \left( \frac{c\alpha \overline{vw}}{\overline{\varphi}} + \frac{c\beta \overline{uv}}{\overline{\varphi}} \right) \left( \frac{c\alpha \overline{vw}}{\overline{\varphi}} + \frac{c\beta \overline{uv}}{\overline{\varphi}} - \frac{c\alpha \overline{vw}}{\overline{\varphi}} - \frac{c\beta \overline{uv}}{\overline{\varphi}} \right)
\]

where \( \overline{\varphi} = (mv + u + v)(mv\overline{w} + u\overline{v} + \overline{v}) \).

By the Poincaré inequality, the following is satisfied:

\[
\int_{\Omega} d|\nabla u|^2 + d|\nabla v|^2 + D|\nabla w|^2 \geq \int_{\Omega} d\mu_2 (u - \overline{u})^2 + d\mu_2 (v - \overline{v})^2 + D\mu_2 (w - \overline{w})^2.
\]
For a sufficiently small \( \varepsilon_0 \), \( D \mu_2 > -d_2 + 2\alpha + 2\beta + \frac{2\beta + 2\alpha + 2\varepsilon_0}{2} \varepsilon_0 + \frac{2\alpha + 2\varepsilon_0 + 2\alpha}{2} \varepsilon_0 \) since \( D \mu_2 > -d_2 + 2\alpha + 2\beta \). Now, choose \( \bar{d} = \frac{1}{\mu_2} \max\{r + \frac{bK+\mu+\mu\rho+\rho}{2}, b\bar{K} - d_1 + \frac{bK+\mu+\mu\rho+\rho}{2} + \frac{2\beta + 2\alpha + 2\varepsilon_0}{2 \varepsilon_0} \} \}. Therefore, we conclude that

**Remark 4.2.** Note that increasing values of \( D \) have little effect on \( \bar{d} \); the value of \( \varepsilon_0 \) is flexible if \( D \) is a large constant.

### 4.2. Existence of nonconstant positive solutions.

In this subsection, we study the existence of nonconstant positive solutions using the Leray-Schauder Theorem. For convenience, set \( u = (u(x), v(x), w(x))^T \) and

\[
\mathcal{F}(u) = \begin{pmatrix}
(I - d\Delta)^{-1}[u(f_1(u, v, w) + 1)] \\
(I - d\Delta)^{-1}[v(f_2(u, v, w) + 1)] \\
(I - D\Delta)^{-1}[w(f_3(u, v, w) + 1)]
\end{pmatrix}.
\]

Then the reaction diffusion system in (2) is equivalent to \( (I - \mathcal{F})u = 0 \). Notice that \( \mathcal{F} : \mathbb{X} \to \mathbb{X} \) is a compact operator since the equation expressed in the bracket is bounded and the operator \( (I - \rho\Delta)^{-1} : C^1(\Omega) \to C^1(\Omega) \) is compact for a finite constant \( \rho > 0 \). In particular, the operator \( (I - \rho\Delta)^{-1} \) is defined as follows: for \( \varphi \in C^1(\Omega) \), \( (I - \rho\Delta)^{-1}\varphi = \psi \) if \( \psi \) is the unique solution to the Neumann boundary value problem \(-\rho\Delta\psi + \psi = \varphi \) in \( \Omega \).

To apply index theory, we investigate the eigenvalues of the following problem:

\[
-(I - \mathcal{F}(u_\ast))\Psi = \lambda\Psi, \quad \Psi \neq 0,
\]

where \( \Psi = (\psi_1, \psi_2, \psi_3) \), and \( u_\ast = (u_\ast, v_\ast, w_\ast) \) is the unique positive equilibrium point of (2). Then by the Leray-Schauder Theorem (Theorem 2.8.1 in [19]),

\[
\text{index}(I - \mathcal{F}, u_\ast) = (-1)^\gamma, \quad \gamma = \sum_{\lambda > 0} n_\lambda,
\]

where \( n_\lambda \) is the multiplicity of the positive eigenvalue \( \lambda \).

Let \( \mathbf{F} = (uf_1, vf_2, wf_3) \). Then

\[
\mathbf{F}(u_\ast) = \begin{pmatrix}
u_\ast - \frac{r}{K} + \frac{\alpha w_\ast}{(m_\ast + u_\ast + v_\ast)^2} & \frac{\alpha w_\ast}{(m_\ast + u_\ast + v_\ast)^2} - b & -\alpha u_\ast \frac{u_\ast + v_\ast}{(m_\ast + u_\ast + v_\ast)^2} \\
v_\ast \left( \frac{\beta w_\ast}{(m_\ast + u_\ast + v_\ast)^2} + b \right) & \frac{\beta v_\ast}{(m_\ast + u_\ast + v_\ast)^2} - \beta v_\ast & -\beta v_\ast \frac{u_\ast + v_\ast}{(m_\ast + u_\ast + v_\ast)^2} \\
w_\ast \left( \frac{\epsilon \alpha m_\ast + \epsilon \beta w_\ast}{(m_\ast + u_\ast + v_\ast)^2} \right) & \frac{\epsilon \beta m_\ast + \epsilon \alpha v_\ast}{(m_\ast + u_\ast + v_\ast)^2} & -m w_\ast \frac{u_\ast + v_\ast}{(m_\ast + u_\ast + v_\ast)^2}
\end{pmatrix}
\]

For simplicity, denote

\[
\mathbf{F}(u_\ast) := \begin{pmatrix}
L_{11} & L_{12} & L_{13} \\
L_{21} & L_{22} & L_{23} \\
L_{31} & L_{32} & L_{33}
\end{pmatrix}.
\]
Then system (17) can be written as the following elliptic system:

\[
\begin{cases}
-d(\lambda + 1)\Delta \psi_1 + (\lambda - L_{11})\psi_1 - L_{12}\psi_2 - L_{13}\psi_3 = 0 \\
-d(\lambda + 1)\Delta \psi_2 - L_{21}\psi_1 + (\lambda - L_{22})\psi_2 - L_{23}\psi_3 = 0 \\
-D(\lambda + 1)\Delta \psi_3 - L_{31}\psi_1 - L_{32}\psi_2 + (\lambda - L_{33})\psi_3 = 0 \quad \text{in } \Omega, \\
\frac{\partial \psi_1}{\partial \eta} = \frac{\partial \psi_2}{\partial \eta} = \frac{\partial \psi_3}{\partial \eta} = 0 \quad \text{on } \partial \Omega,
\end{cases}
\]

(19)

Thus, (19) is equivalent to finding the positive roots of the characteristic equation $B_k(\lambda) = 0$ for $k \geq 1$, where

\[
B_k(\lambda) = \det \begin{pmatrix}
\lambda + \frac{d\mu_k - L_{11}}{1 + d\mu_k} & -\frac{L_{12}}{1 + d\mu_k} & -\frac{L_{13}}{1 + d\mu_k} \\
-\frac{L_{21}}{1 + d\mu_k} & \lambda + \frac{d\mu_k - L_{22}}{1 + d\mu_k} & -\frac{L_{23}}{1 + d\mu_k} \\
-\frac{L_{31}}{1 + d\mu_k} & -\frac{L_{32}}{1 + d\mu_k} & \lambda + \frac{d\mu_k - L_{33}}{1 + d\mu_k}
\end{pmatrix}
\]

for $k \geq 1$. Hence, the above Leray-Schauder Theorem can be rewritten as follows:

\[\text{index}(I - \mathcal{F}, u_*) = (-1)^\gamma, \quad \gamma = \sum_{k \geq 1} \sum_{\lambda_k > 0} n_{\lambda_k},\]

where $n_{\lambda_k} = m_{\lambda_k} \dim E(\mu_k)$, $m_{\lambda_k}$ is the multiplicity of $\lambda_k$ as a positive root of $B_k(\lambda) = 0$, and $E(\mu_k)$ is defined in Notation 4.1.

From Theorem 4.1, we know that there does not exist a nonconstant positive solution to (2) if $d > \bar{d}$ for a sufficiently large $\bar{d}$ when $D\mu_2 > 2c\alpha + 2c\beta - d_2$. Thus, for the index approach, we investigate the index value at $u_*$ when $d$ is a sufficiently large.

**Lemma 4.3.** Assume that (13) and (8) hold. Suppose $mc \geq 1$ and $\alpha = \beta$. If there exists a positive constant $\bar{d} = \bar{d}(N, \Omega, \Gamma, D)$ such that $d \geq \bar{d}$, then $\text{index}(I - \mathcal{F}, u_*) = 1$.

**Proof.** Consider the case when $k = 1$, i.e., $\mu_1 = 0$. Then

\[
B_1(\lambda) = \lambda^3 + (L_{11} - L_{22} - L_{33})\lambda^2 + (L_{11}L_{22} + L_{22}L_{33} + L_{11}L_{33} - L_{13}L_{31} \\
- L_{32}L_{23} - L_{21}L_{12})\lambda - L_{11}L_{22}L_{33} - L_{21}L_{13}L_{32} - L_{31}L_{12}L_{23} + L_{13}L_{31}L_{22} + L_{32}L_{23}L_{11} + L_{21}L_{12}L_{33}.
\]

With some careful calculation, we can show that the given coefficients were all positive. Thus, $B_1(\lambda) > 0$ for all $\lambda \geq 0$.

Now, consider the case when $k \geq 2$, i.e., $\mu_k > 0$. The polynomial $B_k(\lambda)$ can be written as follows:

\[
B_k(\lambda) = (\lambda + 1)^2 \left( \lambda + \frac{D\mu_k - L_{33}}{1 + D\mu_k} \right) + \mathcal{O}\left( \frac{1}{\bar{d}} \right).
\]

Thus, there exists a large positive constant $\bar{d}$ depending on $\Gamma$, $N$, $\Omega$, and $D$ such that $B_k(\lambda) > 0$ for all $d \geq \bar{d}$ and $\lambda \geq 0$. Therefore, $B_k(\lambda) > 0$ for all $\lambda \geq 0$, $k \geq 1$ and $d \geq \bar{d}$, implying $\gamma = \sum_{k \geq 1} \sum_{\lambda_k > 0} n_{\lambda_k} = 0$. \qed

**Remark 4.4.** If $D$ is a large positive constant and all the claims of Lemma 4.3 are satisfied, the same index value is obtained.

Now, we focus on the positive coexistence of (2) when $D$ is a large constant.
Lemma 4.5. Assume that (13) and (8) hold. Suppose $mc \geq 1$ and $\alpha = \beta > \frac{d_2}{r}$. Furthermore, if for some $k_0 \geq 2$, $\bar{\mu} \in (\mu_{k_0}, \mu_{k_0+1})$, where
\[
\bar{\mu} = \frac{1}{22} \left[ L_{11} + L_{22} + \sqrt{(L_{11} + L_{22})^2 - 4(L_{11}L_{22} - L_{21}L_{12})} \right],
\]
and
\[
\frac{r}{K} \geq \frac{boc^2m(r + d_1)}{(co - d_2)d_2},
\]
then there exists a positive constant $\bar{D}(N, \Omega, \Gamma, d)$ such that if $D \geq \bar{D}$, the polynomial $B_k(\lambda) = 0$ has one simple positive root for $2 \leq k \leq k_0$.

Proof. If $k = 1$, the proof follows using the argument in the first part of Lemma 4.3. Thus, $B_1(\lambda) > 0$ for all $\lambda \geq 0$.

If $k \geq 2$, the polynomial $B_k(\lambda)$ can be written as
\[
B_k(\lambda) = (\lambda + 1)(\lambda^2 + p(\mu_k)\lambda + q(\mu_k)) + O\left(\frac{1}{D}\right),
\]
where $p(\mu_k) = \frac{2d_{\mu_k} - L_{11} - L_{22}}{1 + d_{\mu_k}}$ and $q(\mu_k) = \frac{(d_{\mu_k} - L_{11})(d_{\mu_k} - L_{22}) + L_{11}L_{22} - L_{21}L_{12}}{1 + d_{\mu_k}}$.

Note that (21) implies the negativity of $L_{11}L_{22} - L_{21}L_{12}$; specifically, In fact,
\[
L_{11}L_{22} - L_{21}L_{12} = u_*v_*\left( -\alpha r \frac{\alpha \alpha - d_2}{K} \frac{d_2 m}{u_* + v_*}(m w_* + u_* + v_*)^2 + b^2 \right) 
\]
\[
= u_*v_*\left( -\alpha r \frac{\alpha \alpha - d_2}{K} \frac{d_2 m}{(\alpha\alpha)^2(u_* + v_*)} + b^2 \right) 
\]
\[
= u_*v_*b\left( -\frac{r}{K} \frac{\alpha \alpha - d_2}{C^2 \alpha m} \frac{d_2}{r + \frac{K}{2} u_* + d_1} + b \right) 
\]
\[
< u_*v_*b\left( -\frac{r}{K} \frac{\alpha \alpha - d_2}{C^2 \alpha m} \frac{d_2}{r + d_1} + b \right) \leq 0.
\]

We now investigate the roots of $r_k(\lambda) := \lambda^2 + p(\mu_k)\lambda + q(\mu_k)$. First, $r_k(\lambda)$ has two real roots if $(p(\mu_k))^2 - 4q(\mu_k) > 0$. In fact, $(p(\mu_k))^2 - 4q(\mu_k) = \frac{1}{(1 + d_{\mu_k})^2}[(G + H)^2 - 4(G \cdot H - L_{11}L_{21})]$, where $G = d_{\mu_k} - L_{11}$ and $H = d_{\mu_k} - L_{22}$. Thus, $(G - H)^2 + 4L_{21}L_{12} = (L_{11} + L_{22})^2 - 4(L_{11}L_{22} - L_{21}L_{12}) > 0$ is satisfied. Hence, $r_k(\lambda)$ has two real roots.

Next, we determine the sign of $q(\mu_k) = \frac{1}{(1 + d_{\mu_k})^2} \bar{q}(\mu_k)$, where $\bar{q}(\mu_k) = d_2^2 \mu_k^2 - (L_{11} + L_{22})d_{\mu_k} + L_{11}L_{22} - L_{21}L_{12}$. Note that the polynomial $\bar{q}(\mu_k)$ has two real roots. Moreover, these two roots have different signs. Since $\mu_k > 0$ for $k \geq 2$, we consider only the positive root, i.e., $\bar{\mu}$ defined in (20). Thus, since $\bar{\mu} \in (\mu_{k_0}, \mu_{k_0+1})$, the sign of $q(\mu_k)$ is negative for $2 \leq k \leq k_0$.

Therefore, $r_k(\lambda)$ has one negative real root and one positive real root for $2 \leq k \leq k_0$. If $k \geq k_0 + 1$, $q(\mu_k)$ and $p(\mu_k)$ are positive since $\bar{\mu} \in (\mu_{k_0}, \mu_{k_0+1})$. Thus, $r_k(\lambda)$ has two negative real roots.

Hence, as $D \to \infty$, the coefficients of $B_k(\lambda)$ converge to the coefficients of $(\lambda + 1)(\lambda^2 + p(\mu_k)\lambda + q(\mu_k))$. If $D \geq \bar{D}$ for a positive constant $\bar{D}$, the desired result is achieved.

Finally, we prove the main result of this section under certain conditions.

Theorem 4.6. Assume that (13), (8), and (21) hold. Suppose that $mc \geq 1$, $\alpha = \beta$, and $\bar{\mu} \in (\mu_{k_0}, \mu_{k_0+1})$ for some $k_0 \geq 2$. If $\sum_{k=2}^{k_0} \text{dim} E(\mu_k)$ is odd, then there exists a positive constant $\bar{D}$, depending only on $\Gamma$, $N$, $\Omega$, and $d$, such that (2) has at least one nontrivial positive solution provided that $D \geq \bar{D}$. \qed
Proof. For $\theta \in [0, 1]$, define

$$F_{\theta}(u) = \begin{pmatrix}
(I - \theta d) - (1 - \theta) d \Delta)^{-1}[u(f_1(u,v,w) + 1)] \\
(I - \theta d) - (1 - \theta) d \Delta)^{-1}[v(f_2(u,v,w) + 1)] \\
(I - \theta D) - (1 - \theta)(\frac{2c\alpha + 2c\beta - d}{\mu_2} + 1) \Delta)^{-1}[w(f_3(u,v,w) + 1)]
\end{pmatrix}.$$ 

Here, $d \geq \bar{d}$, where $\bar{d}$ is defined in Lemma 4.3, and $\bar{d}$ is defined in Theorem 4.1.

By Theorems 3.1 and 3.2, there exist positive constants $C_2(\Gamma, \bar{d}, \bar{D}, N, \Omega)$ and $C^\alpha(\Gamma)$ such that for all $\theta \in [0, 1]$, the positive solutions to $F_{\theta}(u) = 0$ are contained in $\Lambda = \{u \in X \mid c_2 < u, v, w < c_3\}$. Then for all $u \in \partial \Lambda$, $F_{\theta}(u) \neq 0$. Thus, the degree $\text{deg}(I - F_{\theta}(u), \Lambda, 0)$ is well-defined since $F_{\theta}(u) : \Lambda \times [0, 1] \to X$ is compact. Moreover, by applying the homotopy invariance of the Leray-Schauder degree, $\text{deg}(I - F_0(u), \Lambda, 0) = \text{deg}(I - F_1(u), \Lambda, 0)$. If $\theta = 0$, Theorem 4.1 implies that $F_0(u) = 0$ has no nonconstant positive solution since $\bar{d}$ is greater than or equal to $d$. Hence, $\text{deg}(I - F_0(u), \Lambda, 0) = \text{index}(I - F_0, u_\ast)$. Furthermore, Lemma 4.3 implies $\text{index}(I - F_0, u_\ast) = 1$.

On the other hand, $\text{deg}(I - F_1(u), \Lambda, 0) = \text{index}(I - F_1, u_\ast) = (-1)^{\sum_{k=2}^{m} \text{dim} E(\mu_k)} = -1$ by Leray-Schauder Theorem. This contradiction gives the desired result. \qed

5. Concluding remarks. A diffusive predator-prey model with a ratio-dependent functional response and infected prey population was investigated under homogeneous Neumann boundary conditions. First, the global attractor and persistence properties of the given time-dependent system were examined. We showed that there is no nonconstant positive solution (i.e., no pattern formation occurs) under a suitable diffusion condition. In particular, if the diffusion of the predator is not too small, no pattern occurs when the diffusion of the susceptible and infected prey is sufficiently large. On the other hand, when there is no nonconstant positive solution, a Turing pattern is induced by the predator’s large diffusion value. Therefore, diffusion of the predator creates a spatially nonconstant positive solution from Turing instabilities under suitable conditions.

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REFERENCES

[1] R. Arditi and L. R. Ginzburg, Coupling in predator-prey dynamics: ratio dependence, J. Theor. Biol., 139 (1989), 311–326.
[2] R. Arditi, L. R. Ginzburg and H. R. Akcakaya, Variation in plankton densities among lakes: a case for ratio-dependent models, American Naturalist, 138 (1991), 1287–1296.
[3] R. Arditi and H. Saiah, Empirical evidence of the role of heterogeneity in ratio-dependent consumption, Ecology, 73 (1992), 1544–1551.
[4] O. Arino, A. El. abdillahou, J. Mikram and J. Chattopadhyay, Infection in prey population may act as a biological control in raito-dependent predator-prey models, Nonlinearity, 17 (2004), 1101–1116.
[5] E. Beltrami and T. Carroll, Modelling the role of viral disease in recurrent phytoplankton blooms, J. Math. Biol., 32 (1994), 857–863.
[6] R. S. Cantrell and C. Cosner, On the dynamics of predator-prey models with the Beddington-DeAngelis functional response, J. Math. Anal. Appl., 257 (2001), 206–222.
[7] J. Chattopadhyay and O. Arino, A predator-prey model with disease in the prey, *Nonlinear Anal.*, 36 (1999), 747–766.
[8] J. Chattopadhyay and S. Pal, Viral infection of phytoplankton-zooplankton system-a mathematical modeling, *Ecol. Modelling.*, 151 (2002), 15–28.
[9] C. Cosner, D. L. DeAngelis, J. S. Ault and D. B. Olson, Effects of spatial grouping on the functional response of predators, *Theoret. Population Biol.*, 56 (1999), 65–75.
[10] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, New York, 1983.
[11] A. P. Gutierrez, The physiological basis of ratio-dependent predator-prey theory: a metabolic pool model of Nicholson's blowflies as an example, *Ecology*, 73 (1992), 1552–1563.
[12] K. Hadeler and H. Freedman, Predator-prey population with parasite infection, *J. Math. Biol.*, 27 (1989), 609–631.
[13] S. B. Hsu, T. W. Hwang and Y. Kuang, Rich dynamics of a ratio-dependent one-prey two-predators model, *J. Math. Biol.*, 43 (2001), 377–396.
[14] S. B. Hsu, T. W. Hwang and Y. Kuang, A ratio-dependent food chain model and its applications to biological control, *Math. Biosci.*, 181 (2003), 55–83.
[15] Y. Kuang and E. Beretta, Global qualitative analysis of a ratio-dependent predator-prey system, *J. Math. Biol.*, 36 (1998), 389–406.
[16] Y. Lou and W. M. Ni, Diffusion, self-diffusion and cross-diffusion, *J. Differential Equations*, 131 (1996), 79–131.
[17] C. S. Lin, W. M. Ni and I. Takagi, Large amplitude stationary solutions to a chemotaxis system, *J. Differential Equations*, 72 (1988), 1–27.
[18] Z. Lin and M. Pedersen, Stability in a diffusive food-chain model with Michaelis-Menten functional response, *Nonlinear Anal.*, 57 (2004), 421–433.
[19] L. Nirenberg, *Topics in Nonlinear Functional Analysis*, Courant Institute of Mathematical Science, New York, 1974.
[20] P. Y. H. Pang and M. X. Wang, Qualitative analysis of a ratio-dependent predator-prey system with diffusion, *Proc. Roy. Soc. Edinburgh Sect. A*, 33 (2003), 919–942.
[21] P. Y. H. Pang and M. X. Wang, Strategy and stationary pattern in a three-species predator-prey model, *J. Differential Equations*, 200 (2004), 245–273.
[22] C. V. Pao, *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, New York, 1992.
[23] C. V. Pao, Quasisolutions and global attractor of reaction-diffusion systems, *Nonlinear Anal.*, 26 (1996), 1889–1903.
[24] K. Ryu and I. Ahn, Positive solutions to ratio-dependent predator-prey interacting systems, *J. Differential Equations*, 218 (2005), 117–135.
[25] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, 2nd edition, Springer-Verlag, New York, 1994.
[26] Y. Xiao and L. Chen, A ratio-dependent predator-prey model with disease in the prey, *Appl. Maths. Comp.*, 131 (2002), 397–414.
[27] Y. Xiao and L. Chen, Analysis of a three species eco-epidemiological model, *J. Math. Anal. Appl.*, 258 (2001) 733–754.

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