The radial supersymmetry of the
(d+1)-dimensional relativistic rotating
oscillators

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Abstract

We study the supersymmetry of the radial problems of the models of quantum relativistic rotating oscillators in arbitrary dimensions, defined as Klein-Gordon fields in backgrounds with deformed anti-de Sitter metrics. It is pointed out that the shape invariance of the supersymmetric partner radial potentials leads to simple operators forms of the Rodrigues formulas for the normalized radial wave functions.

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1 Introduction

The geometric quantum models in general relativity are devoted to the study of quantum modes of the free fields in given backgrounds. Of a special interest are the analytical solvable models since these can be analyzed in all details giving thus information about the relation between the geometry of the background and the properties of the specific quantum objects. In this way we can better understand how the geometry determines the mean features of the whole algebras of observables that include the dynamical algebras and the operators involved in supersymmetry.

One of the simplest geometric models is that of the relativistic quantum oscillator simulated by the scalar (Klein-Gordon) field in the anti-de Sitter background \([1]\). We generalized this model to families of (1+1)-dimensional relativistic oscillators \([2]\) or relativistic rotating oscillators in (3+1) \([3]\) or arbitrary dimensions \([4, 5]\). In general, the backgrounds of these models have local charts with deformed anti-de Sitter or de Sitter metrics. In the case of the deformed anti-de Sitter metrics the Klein-Gordon equation lead to standard Pöschl-Teller problems \([6]\) in (1+1) dimensions \([7]\) or to similar radial problems in higher dimensions \([4, 5]\), all of them having countable discrete energy spectra and square integrable energy eigenfunctions. This is the motive why the properties of the dynamical algebras as well as the supersymmetry of these models can be easily studied. Thus we have shown that all the (1+1) Pösch-Teller models have the same dynamical algebra, \(so(1, 2)\) \([8]\), and good properties of supersymmetry and shape invariance which allowed us to write down the Rodrigues formulas of the normalized energy eigenfunctions in operator closed form \([7]\).

In the present article we should like to continue this study with the supersymmetry of the radial problems of the Pöschl-Teller-type models in arbitrary (d+1) dimensions we have recently solved by using traditional methods \([5]\). Our main objective is to introduce the operators involved in supersymmetry and to derive the operator version of the Rodrigues formulas of the normalized radial functions which enter in the structure of the energy eigenfunctions.

In Sec.2 we briefly present the radial problems of the models of rotating oscillators in (d+1) dimensions showing that these are Pöschl-Teller-type problems with a very simple parameterization. The supersymmetry of these problems and the shape invariance of the supersymmetric partner (super-
partner) radial potentials is discussed in the next section where we obtain the operator form of the mentioned Rodrigues formulas. Sec.4 is devoted to a special model of this family which is very similar to the usual nonrelativistic harmonic oscillator. The usual expressions of the Rodrigues formulas for the radial functions of all these models are given in Appendix. We work in natural units with \( \hbar = c = 1 \).

2 Radial Pöschl-Teller problems

The \((d+1)\)-dimensional rotating oscillators are simple geometric models of scalar particles freely moving on backgrounds able to simulate oscillatory geodesic motions. These background are static and spherically symmetric (central) having static charts with generalized spherical coordinates, \( r, \theta_1, ..., \theta_{d-1} \), commonly related with the Cartesian ones \( x \equiv (x^1, x^2, ..., x^d) \). It is convenient to choose the radial coordinate such that \( g_{rr} = -g_{00} \) since then the radial scalar product is simpler. The metrics of our models are one-parameter deformations of the AdS metric given by the line elements

\[
\begin{align*}
 ds^2 &= \left(1 + \frac{1}{\epsilon^2} \tan^2 \omega r \right) (dt^2 - dr^2) - \frac{1}{\omega^2} \tan^2 \omega r \, d\theta^2 \tag{1}
\end{align*}
\]

where we denote \( \omega = \epsilon \omega, \epsilon \in [0, \infty) \), and

\[
 d\theta^2 = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \sin^2 \theta_1 \sin^2 \theta_2 d\theta_3^2 + \cdots + \sin^2 \theta_1 \sin^2 \theta_2 \cdots \sin^2 \theta_{d-2} d\theta_{d-1}^2 \tag{2}
\]

is the usual line element on the sphere \( S^{d-1} \). The deformation parameter \( \epsilon \) determines the geometry of the background while \( \omega \) remains fixed. It is clear that for \( \epsilon = 1 \) we obtain just the AdS metric (with the hyperboloid radius \( R = 1/\omega \)). An interesting case is the model with \( \epsilon = 0 \) called normal oscillator since its line element,

\[
 ds^2 = (1 + \omega^2 r^2)(dt^2 - dr^2) - r^2 d\theta^2 , \tag{3}
\]

defines a background where the relativistic quantum motion is similar to that of the nonrelativistic harmonic oscillator. In general, the radial domain of any RO is \( D_r = [0, \pi/2\hat{\omega}) \) which means that the whole space domain is \( D = D_r \times S^{d-1} \). For the models with \( \epsilon \neq 0 \) the time might satisfy the
condition $t \in [−\pi/\hat{\omega}, \pi/\hat{\omega})$ as in the AdS case but here we consider that $t \in (−\infty, \infty)$ which corresponds to the universal covering spacetimes of the hyperbolic original ones.

In these models the oscillating test particle is described by a scalar quantum field $\phi$ of mass $M$, minimally coupled with the gravitational field. Its quantum modes are given by the particular solutions of the Klein-Gordon equation

$$\frac{1}{\sqrt{g}} \partial_{\mu} (\sqrt{g} g^{\mu\nu} \partial_{\nu} \phi) + M^2 \phi = 0 \quad g = |\det(g_{\mu\nu})|,$$

which, in the case of our models, must be square integrable functions [5] on the domain $D$, orthonormalized with respect to the relativistic scalar product [9]

$$\langle \phi, \phi' \rangle = i \int_{D} d^d x \sqrt{g} g^{00} \phi^* \frac{\partial}{\partial x^0} \phi'.$$

The spherical variables can be separated by using generalized spherical harmonics, $Y_{l(l)}(x/r)$. These are normalized eigenfunctions of the angular Laplace operator [10],

$$-\Delta_S Y_{l(l)}^{d-1}(x/r) = l(l + d - 2) Y_{l(l)}^{d-1}(x/r),$$

corresponding to eigenvalues depending on the angular quantum number $l$ which takes only integer values, 0, 1, 2, ..., selected by the boundary conditions on the sphere $S^{d-1}$ [10]. The notation $(\lambda)$ stands for a collection of quantum numbers giving the multiplicity of these eigenvalues [10],

$$\gamma_l = (2l + d - 2) \frac{(l + d - 3)!}{l! (d - 2)!}.$$  

In general, the particular solutions of the Klein-Gordon equation of energy $E$ (and positive frequency) have the form [3]

$$\phi_{E,l(\lambda)}^{(+)}(t, x) = \frac{1}{\sqrt{2E}} (\hat{\omega} \cot \hat{\omega} r)^{d-1} \frac{d-1}{2} R(r) Y_{l(l)}^{d-1}(x/r) e^{-iEt},$$

involving the radial wave functions $R(r)$ defined such that the scalar product [3] leads to the simplest radial scalar product,

$$\langle R, R' \rangle = \int_{D_r} dr \; R^*(r) R'(r).$$
when the spherical harmonics are normalized to unity.

In Ref. [5] we have shown that, after the separation of spherical variables, the remaining radial Klein-Gordon equation can be treated as an independent Pöschl-Teller problem, with discrete energy spectra and a very simple parameterization given by the quantum number $l$ and the specific parameter

$$k = \sqrt{\frac{M^2}{\hat{\omega}^2} + \frac{d^2}{4} + \frac{d}{2}}$$

which concentrates all the other ones, playing thus the role of the main parameter of our models. Therefore, we can consider that the geometric parameters ($\omega$ and $\epsilon$) are fixed while the mass of the test particle is given by $M_k = \epsilon^2 \hat{\omega}^2 k(k - d)$. For this reason we denote the model by $[k]$ understanding that it generates the radial problems $(k, l)$, each one having its own sequence of radial wave functions, $R_{k,l,n_r}(r)$, labeled by the radial quantum number $n_r = 0, 1, 2, \ldots$ [5]. With these notations, the radial equation for a given pair $(k, l)$ can be put in the form

$$\left[ -\frac{1}{\hat{\omega}^2} \frac{d^2}{dr^2} + \frac{2s(2s - 1)}{\sin^2 \hat{\omega}r} + \frac{2p(2p - 1)}{\cos^2 \hat{\omega}r} \right] R_{k,l,n_r} = \nu^2 R_{k,l,n_r}$$

where the values of the parameters $s$ and $p$ corresponding to the regular modes are

$$2s = l + \delta, \quad 2p = k - \delta, \quad \delta = \frac{d - 1}{2}$$

while the quantization condition reads

$$\nu = 2(n_r + s + p).$$

Hereby it results that the discrete energy levels,

$$E_{k,n,l}^2 = \hat{\omega}^2(k + n)^2 + \hat{\omega}^2(\epsilon^2 - 1) \left[ k(k - d) - \frac{1}{\epsilon^2}l(l + d - 2) \right].$$

depend on the main quantum number, $n = 2n_r + l$. If $n$ is even then $l = 0, 2, 4, \ldots, n$ while for odd $n$ we have $l = 1, 3, 5, \ldots, n$. In both cases the degree of degeneracy of the level $E_{k,n,l}$ is given by [7].

In the following we use only the quantum number $n_r$ since this labels the sequences of radial functions $R_{k,l,n}$, which form the energy bases of the Pöschl-Teller radial problems $(k, l)$ in the Hilbert space $\mathcal{H}_r$, of the square integrable
functions with respect to the radial scalar product (\(\mathbf{9}\)). We specify that all our radial functions accomplish the boundary conditions

\[ R(0) = R(\pi/2\hat{\omega}) = 0 \]

such that the Hermitian conjugation of the operators on \(\mathcal{H}_r\) can be correctly defined with respect to this scalar product (e.g., \(\partial_r^\dagger = -\partial_r\)). Thus we obtain a familiar approach similar to that of the non-relativistic radial problems in the coordinate representation of the Schrödinger picture.

3 Radial supersymmetry

The supersymmetric formalism of the radial problems \((k, l)\) can be constructed like that of the relativistic (1+1)-dimensional Pöschl-Teller models \([7]\). To this end it is convenient to introduce the operator

\[ \{\Delta [V] R\}(r) = \left( -\frac{d^2}{dr^2} + V(r) \right) R(r), \] (15)

which should play the same role as the Hamiltonian of the one-dimensional nonrelativistic problems.

Starting with the normalized ground-state radial function

\[ R_{k,l,0}(r) = \sqrt{2\hat{\omega}} \left[ \frac{\Gamma(k + l + 1)}{\Gamma(l + \frac{d}{2})\Gamma(k + 1 - \frac{d}{2})} \right]^{\frac{1}{2}} \sin^{2s}\hat{\omega}r \cos^{2p}\hat{\omega}r \] (16)

we obtain the radial superpotential

\[ W(k, l, r) = -\frac{1}{R_{k,l,0}(r)} \frac{dR_{k,l,0}(r)}{dr} = \hat{\omega}[2p\tan\hat{\omega}r - 2s\cot\hat{\omega}r] \] (17)

that help us to find the superpartner radial potentials

\[ V_\pm(k, l, r) = \pm \frac{dW(k, l, r)}{dr} + W(k, l, r)^2 \]

\[ = \hat{\omega}^2 \left[ \frac{2s(2s \pm 1)}{\sin^2\hat{\omega}r} + \frac{2p(2p \pm 1)}{\cos^2\hat{\omega}r} - (2s + 2p)^2 \right]. \] (18)

Now Eq.(11) can be rewritten as

\[ \Delta[V_- (k, l)] R_{k,l,nr} = d_{k,l,nr} R_{k,l,nr} \] (19)
where
\[ d_{k,l,n_r} = \hat{\omega}^2 [\nu^2 - (\nu|_{n_r=0})^2] = 4\hat{\omega}^2 n_r (n_r + k + l) \] (20)
satisfies \( d_{k,l,0} = 0 \).

Furthermore, according to the standard procedure [12], we define the pair of adjoint operators, \( A_{k,l} \) and \( A_{k,l}^\dagger \), having the action
\[
(A_{k,l} R)(r) = \left( \frac{d}{dr} + W(k,l,r) \right) R(r), \quad (A_{k,l}^\dagger R)(r) = \left( -\frac{d}{dr} + W(k,l,r) \right) R(r)
\] (21) (22)
and allowing us to write
\[
\Delta [V_-(k,l)] = A_{k,l}^\dagger A_{k,l}, \quad \Delta [V_+(k,l)] = A_{k,l} A_{k,l}^\dagger. \] (23)

Let us observe now that the radial potentials \( V_-(k,l) \) and \( V_+(k,l) \) are shape invariant since
\[ V_+(k,l,r) = V_-(k+1,l+1,r) + 4\hat{\omega}^2(k+l+1). \] (24)
Consequently, we can verify that
\[
\Delta [V_+(k,l)] R_{k+1,l+1,n_r-1} = d_{k,l,n_r} R_{k+1,l+1,n_r-1}, \quad n_r = 1, 2, \ldots ,
\] (25)
which means that the spectrum of the operator \( \Delta [V_+(k,l)] \) coincides with that of \( \Delta [V_-(k,l)] \), apart from the lowest eigenvalue \( d_{k,l,0} = 0 \). From (23) combined with (19) it results that the normalized radial functions satisfy
\[
A_{k,l} R_{k,l,n_r} = \sqrt{d_{k,l,n_r}} R_{k+1,l+1,n_r-1}, \quad A_{k,l}^\dagger R_{k,l,n_r} = \sqrt{d_{k,l,n_r}} R_{k,l,n_r}. \] (26) (27)
Hence, we have obtained the desired relation between the energy bases of the radial problems \((k,l)\) and \((k+1,l+1)\) which have the superpartner radial potentials \( V_-(k,l) \) and \( V_+(k,l) \) respectively. In general, any normalized radial function of the basis \((k,l)\) of the model \([k]\) can be written as
\[
R_{k,l,n_r} = \frac{1}{(2\hat{\omega})^{n_r}} \left[ \frac{\Gamma(n_r + k + l)}{n_r! \Gamma(2n_r + k + l)} \right]^\frac{1}{2} A_{k,l}^\dagger A_{k+1,l+1}^\dagger \cdots \]
\[ \cdots A_{k+n_r-1,l+n_r-1} A_{k+n_r,l+n_r,0}. \] (28)

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where $R_{k+n_r,l+n_r,0}$ is the normalized ground-state radial function of the energy basis $(k + n_r, l + n_r)$ of the model $[k + n_r]$, given by Eq.(16). Thus we obtain the operator form of the Rodrigez formula of the radial functions as in the case of the (1+1)-dimensional relativistic Pöschl-Teller models [7].

4 The normal oscillator

A special case is that of $\epsilon \to 0$ when one obtains the normal oscillator which has similar radial functions as those of the nonrelativistic harmonic oscillator [3]. In this limit we have $\omega \to 0$, $k \to \infty$, but $\epsilon^2 k \to M/\omega$, such that the superpotential becomes

$$W(l, r) = \lim_{\epsilon \to 0} W(k, l, r) = \omega Mr - \frac{l + \delta}{r}$$

(29)

giving the superpartner radial potentials

$$V_{\pm}(l, r) = \lim_{\epsilon \to 0} V_{\pm}(k, l, r)$$

$$= \omega^2 M^2 r^2 + \frac{(l + \delta)(l + \delta \pm 1)}{r^2} - \omega M[2(l + \delta) \mp 1]$$

(30)

which are shape invariant since

$$V_{+}(l, r) = V_{-}(l + 1, r) + 4\omega M.$$  

(31)

Thus we see that the shape of the superpartner potentials is given only by the parameter $l$. Therefore, we can consider all the other parameters (including the mass) as being fixed and denote the radial problems of the normal oscillator by $(l)$.

Since $\lim_{\epsilon \to 0} d_{k,l,n_r} = 4\omega^2 M n_r$, it is convenient to introduce the new operators

$$a_l = \frac{1}{2\sqrt{\omega M}} \lim_{\epsilon \to 0} A_{k,l}$$

(32)

which have the action

$$(a_l R)(r) = \frac{1}{2\sqrt{\omega M}} \left( \frac{d}{dr} + \omega Mr - \frac{l + \delta}{r} \right),$$

(33)

$$(a_l^{\dagger} R)(r) = \frac{1}{2\sqrt{\omega M}} \left( -\frac{d}{dr} + \omega Mr - \frac{l + \delta}{r} \right),$$

(34)
and satisfy
\[ \Delta[V_-] = 4\omega M a_l^\dagger a_l, \quad \Delta[V_+] = 4\omega M a_l a_l^\dagger. \] (35)

As in the previous cases, from Eqs. (30) and (31) we deduce that these operators have the same spectrum (apart from the lowest eigenvalue) and, consequently, we can write
\[ a_l R_{l,n_r} = \sqrt{n_r} R_{l+1,n_r-1}, \quad a_l^\dagger R_{l+1,n_r-1} = \sqrt{n_r} R_{l,n_r}. \] (36)

Finally we obtain the operator form of the Rodrigues formula,
\[ R_{l,n_r} = \frac{1}{\sqrt{n_r!}} a_{l+1}^\dagger \cdots a_{l+n_r-1}^\dagger R_{l+n_r,0} \] (37)

which express any radial function of the problem (l) in terms of the ground-state radial function of the problem (l + n_r). Hereby it result the usual Rodrigues formula of the radial functions of the normal oscillator (as given in Appendix).

5 Concluding remarks

The radial problems of our rotating oscillators have two interesting properties. The first one refers to the relation among symmetries and supersymmetries. Thus we have seen that all of our models, apart from that with \( \epsilon = 0 \), have the same properties concerning the supersymmetry and shape invariance of the radial potentials, despite of the fact that their backgrounds have different spacetimes symmetries. Indeed, all the models with \( \epsilon \neq 1 \) have central backgrounds with the symmetry given by the group \( T(1) \otimes SO(d) \) of time translations and space rotations. On the other hand, the background of the model with \( \epsilon = 1 \) is just the \( (d+1) \) dimensional anti-de Sitter spacetime which is the homogeneous space of the group \( SO(d,2) \) that coincides with the isometry group of this background. Thus we see that problems with different spacetime symmetries have similar behaviors from the point of view of the supersymmetry and shape invariance.

The second property which is worth pointing out concerns the parameterization of these models. The radial problems studied here depend on two parameters of different nature, \( l \) which is the quantum number of the whole
angular momentum and the usual parameter $k$. What is interesting is that both these parameters have the same behavior from the point of view of the shape invariance. More precisely the problems with superpartner radial potentials have simultaneously $\Delta l = \pm 1$ and $\Delta k = \pm 1$ even though these parameters are of different origins. Thus we find similar properties with those studied in the simplest case of the (1+1) models of relativistic oscillators [7].

A Rodrigues formulas

Starting with the observation that, according to Eqs.(21) and (17), we have

$$\left( A_{k,l} R \right)(r) = (\sin \hat{\omega} r)^{2s}(\cos \hat{\omega} r)^{2p} \frac{d}{dr} \left( \sin \hat{\omega} r \right)^{-2s}(\cos \hat{\omega} r)^{-2p} R(r),$$

it is not difficult to show that Eq.(28) gives the Rodrigues formula of the normalized radial functions in the new variable $\hat{u} = \cos 2\hat{\omega}r$. This is

$$R_{k,l,n}(u) = \frac{(-1)^{n_r}}{2^{\frac{1}{2}+n_r}} \left[ \frac{2 \hat{\omega} (2n_r + k + l) \Gamma(n_r + k + l)}{n_r! \Gamma(n_r + l + \frac{d}{2}) \Gamma(n_r + k + 1 - \frac{d}{2})} \right]^{\frac{1}{2}} \times (1 - u)^{-\frac{k-\delta-1}{2}}(1 + u)^{-\frac{k-\delta+1}{2}} (1 + u)^{l+\delta+n_r - \frac{d}{2}} (1 - u)^{l+\delta+n_r}.$$  

(39)

In the case of the normal oscillator we have

$$\left( a_l R \right)(r) = \frac{1}{2\sqrt{\omega M}} r^{l+\delta} e^{\omega Mr^2/2} \frac{d}{dr} e^{-\omega Mr^2/2} R(r)$$

(40)

and

$$R_{l,0}(r) = \lim_{\epsilon \to 0} R_{k,l,0}(r) = \left[ \frac{2(\omega M)^{l+\frac{d}{2}}}{\Gamma(l + \frac{d}{2})} \right]^{\frac{1}{2}} e^{\omega Mr^2/2}$$

(41)

so that Eq.(37) gives the usual Rodrigues formula of the normalized radial functions of the normal oscillator,

$$R_{l,n}(z) = \frac{(4\omega M)^{\frac{1}{2}}}{\sqrt{n_r! \Gamma(n_r + l + \frac{d}{2})}} z^{-\frac{l+\delta+1}{2}} e^{z/2} \frac{d^n}{dz^n} z^{n_r + l + \delta - \frac{d}{2}} e^{-z},$$

(42)

where $z = \omega Mr^2$.

Thus the supersymmetry and shape invariance help us to easily recover our previous results obtained by using traditional methods [3].
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