Plane Waves and Vacuum Interpolation

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Abstract

A $\frac{1}{2}$-BPS family of time dependent plane wave spacetimes which give rise to exactly solvable string backgrounds is presented. In particular a solution which interpolates between Minkowski spacetime and the maximally supersymmetric homogeneous plane wave along a timelike direction is analyzed. We work in $d = 4$, $N = 2$ supergravity, but the results can be easily extended to $d = 10, 11$. The conformal boundary of a particular class of solutions is studied.

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Introduction

Since the discovery of the maximally supersymmetric plane wave solution in type IIB theory [1], great progress has been achieved in the AdS/CFT context in going beyond the supergravity approximation [2]. It is known that pp-waves yield exact classical backgrounds for string theory since all curvature invariants vanish and therefore receive no $\alpha'$ corrections. Hence, pp-wave spacetimes correspond to exact conformal field theories. This fact has been known for some time [3], and after the work of [4] the interest in them has been revived. Plane waves in particular provide simple toy models for studying string propagation and singularity issues because it has been shown that they lead to exactly solvable models in the light cone gauge [5] (see also [6]).

After reviewing how to obtain the maximally supersymmetric solutions of $d = 4$, $N = 2$ supergravity in sections 1 and 2, we present the $\frac{1}{2}$-BPS time dependent pp-wave backgrounds in section 3. In section 4 we develop a general method to study the conformal boundary of these time dependent solutions and then for a particular class of metrics the boundary is analyzed. We end with some conclusions and directions for future research.

# 1 Constraints from Maximal Supersymmetry

We first review the spacetimes of $d = 4$, $N = 2$ supergravity with maximal supersymmetry, this means spacetimes with 8 Killing spinors. The Killing spinor equation is [7]

$$\hat{\nabla} \xi \equiv (d + \frac{1}{4} \omega - i \frac{1}{4} F \Gamma) \xi = 0,$$

(1)

where $d = \partial_\mu dx^\mu$, $\omega = \omega^{ab}_\mu \Gamma_{ab} dx^\mu$, $F = F^{ab} \Gamma_{ab}$ and $\Gamma = \Gamma^a e_a^\mu dx^\mu$. Maximal supersymmetry implies that the commutator of covariant derivatives as a spinor matrix must vanish

$$\left[ \hat{\nabla}_\mu, \hat{\nabla}_\nu \right] \xi = 0, \quad \forall \xi \Rightarrow \left[ \hat{\nabla}_\mu, \hat{\nabla}_\nu \right]_{\alpha\beta} = 0.$$  

(2)

Writing $\hat{\nabla}_\mu = D_\mu - i \frac{1}{4} F \Gamma_\mu$ where $D_\mu \equiv \partial_\mu + \frac{1}{4} \omega_\mu^{ab} \Gamma_{ab}$ the commutator (2) takes the form

$$\frac{1}{4} \Gamma_{ab} R^{ab}_{\mu \nu} - \frac{i}{4} D_\mu (F \Gamma_\nu) + \frac{i}{4} D_\nu (F \Gamma_\mu) - \frac{1}{16} [F \Gamma_\mu, F \Gamma_\nu] = 0.$$  

(3)

Expanding (3) in gamma matrices and setting to zero each independent coefficient we get for $\Gamma_{ab}, \Gamma^a, \Gamma^\alpha$ and $\Gamma^5$

$$R^{ab}_{\mu \nu} - e_{\mu \alpha} e_{\nu \beta} (F^{ac} F^{bd} - F^{bc} F^{ad}) = 0$$

(4)

$$D_\mu F_{\nu \alpha} - D_\nu F_{\mu \alpha} = 0,$$

(5)

$$D_\mu \tilde{F}_{\nu \alpha} - D_\nu \tilde{F}_{\mu \alpha} = 0,$$

(6)

$$\tilde{F}_{\mu \rho} F^\rho_{\nu} - \tilde{F}_{\nu \rho} F^\rho_{\mu} = 0,$$

(7)

where $\tilde{F}_{\mu \nu} \equiv \frac{1}{2} \varepsilon_{\mu \nu \alpha \beta} F^{\alpha \beta}$. The set of equations (4)-(7) are the constraints that a maximally supersymmetric spacetime of $d = 4$, $N = 2$ supergravity must satisfy [8].

Equation (7) doesn’t impose any constraint (cf. eqn.(8) of [8]), it is trivially satisfied due to the identity $F_{\mu \rho} \tilde{F}^\rho_{\nu} = \frac{1}{4} \delta^\nu_{\rho} F_{\rho \sigma}$.

\[\text{1 See appendix for conventions.}\]
Equations (5) and (6) imply, upon contracting with the covariant Levi-Civita tensor, the Bianchi identity and the equations of motion for $F_{\mu\nu}$. In particular, a strong constraint follows for the electromagnetic field, by substituting (5) into the Bianchi identity

$$D_\alpha F_{\mu\nu} + D_{\mu} F_{\nu\alpha} + D_{\nu} F_{\alpha\mu} = 0,$$

one finds that maximal supersymmetry requires that the electromagnetic field must be covariantly constant

$$D_\alpha F_{\mu\nu} = 0.$$  

This is a natural property for a maximally supersymmetric “vacuum” solution.

The $d = 4$ spaces admitting a covariantly constant 2-form were classified in [9], where it was shown that there exist two classes of solutions according to $F$ being either non-null (Robinson-Bertotti [10]) or null (pp-waves [12],[13]). One can check that the Robinson-Bertotti $AdS_2 \times S^2$ solution satisfies the constraint (4) as is well known [11]. So let’s move to the null electromagnetic field class of solutions.

### 2 Maximally Supersymmetric Plane Waves

To find the maximally supersymmetric plane wave solution [8], we start from the general pp-wave solution of the Einstein-Maxwell action [12]

$$ds^2 = du(dv + H(u, \zeta, \bar{\zeta})du) + d\zeta d\bar{\zeta},$$  

$$F = \frac{1}{2} du \wedge (\partial_\zeta \phi d\zeta + \partial_{\bar{\zeta}} \bar{\phi} d\bar{\zeta}),$$  

where

$$H(u, \zeta, \bar{\zeta}) = f(u, \zeta) + \bar{f}(u, \bar{\zeta}) - \phi(u, \zeta) \bar{\phi}(u, \bar{\zeta}), \quad F \equiv \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu.$$  

Here $f(u, \zeta)$ and $\phi(u, \zeta)$ are two arbitrary holomorphic complex functions with complex conjugates $\bar{f}(u, \bar{\zeta})$ and $\bar{\phi}(u, \bar{\zeta})$. Maximal supersymmetry constrains the functional form of $f(u, \zeta)$ and $\phi(u, \zeta)$.

Having a covariantly constant electromagnetic field (9) implies that

$$\phi(u, \zeta) = C(u) + \lambda \zeta.$$  

The function $C(u)$ can be absorbed in $f(u, \zeta)$, and $\lambda$ can always be made real by a rotation in $\zeta$, so we get

$$H(u, \zeta, \bar{\zeta}) = f(u, \zeta) + \bar{f}(u, \bar{\zeta}) - \lambda^2 \zeta \bar{\zeta},$$  

$$F = \frac{\lambda}{2} du \wedge (d\zeta + d\bar{\zeta}).$$  

The only independent non-zero components of the Riemann tensor for the spacetime (10) are

$$R_{\alpha\nu\mu\bar{\zeta}} = -\frac{1}{2} \partial_\alpha \partial_\nu H(u, \zeta, \bar{\zeta}),$$  

where $i, j = (\zeta, \bar{\zeta})$ refer to transverse space coordinates.

Equation (4) implies

$$\partial_\zeta \partial_{\bar{\zeta}} H = -\lambda^2,$$  

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The first equation is automatically satisfied because it is just the Einstein equation. Equations (18) imply that the space should be conformally flat and then

\[ f(u, \zeta) = A(u) + B(u)\zeta. \]  

Here \( A(u), B(u) \) are two arbitrary complex functions of the real variable \( u \), but \( f(u, \zeta) \) can be set to zero by a diffeomorphism. Eqn. (20) together with (13) imply that the maximally supersymmetric pp-wave solution belongs to the subclass of plane wave spacetimes. The maximally supersymmetric plane wave solution is then

\[ ds^2 = du(dv - \lambda^2 \zeta \bar{\zeta} du) + d\zeta d\bar{\zeta}, \quad F = \frac{\lambda}{2} du \wedge(d\zeta + d\bar{\zeta}). \]  

This is called the Brinkmann form of the plane wave [14]. Despite its appearance the spacetime (21) is completely homogeneous, having a seven dimensional group of isometries [15]. The isometry group of \((M, F)\) is six dimensional and this coincides with the dimension of the isometry group of the Robinson-Bertotti solution. This observation was at the root of the connection between the homogeneous plane waves and the \( AdS \times S \) via Penrose limits and group contractions [16][4][17].

We have seen that maximal supersymmetry for \( d = 4, N = 2 \) supergravity implies that the spacetime must be conformally flat. This statement extends to \( d = 6, 10 \) but not to \( d = 5, 11 \) [18],[19].

### 3 \( \frac{1}{2} \)-BPS Time Dependent Plane Wave Solutions

Now we are going to find some new time dependent \( \frac{1}{2} \)-BPS solutions which can be used in \( d = 10 \) as string backgrounds leading to gaussian models in the light-cone gauge.

The Killing spinor equations (1) for the background (10)-(11) take the form

\[ \left( \partial_\zeta - \frac{i}{8} \partial_\zeta \phi \Omega \right) \xi = 0, \]
\[ \left( \partial_\zeta - \frac{i}{8} \partial_\zeta \phi \Omega' \right) \xi = 0, \]
\[ \left( \partial_u + \frac{1}{4} \Upsilon \right) \xi = 0, \]
\[ \partial_\zeta \xi = 0, \]

where

\[ \Omega = \Gamma^1 \Gamma^2 \Gamma^- \]
\[ \Omega' = \Gamma^2 \Gamma^1 \Gamma^- \]
\[ \Upsilon = \Gamma^- (\partial_\zeta H \Gamma^1 + \partial_\bar{\zeta} H \Gamma^2) + \frac{i}{2} (\partial_\zeta \phi \Gamma^1 + \partial_\bar{\zeta} \bar{\phi} \Gamma^2) \Gamma^- \Gamma^+. \]

\(^2\)One can see that the contraction of the Riemann tensor gives Einstein equations

\[ R_{\mu\nu} = 2(F_{\mu\alpha} F_{\nu\beta} g^{\alpha\beta} - \frac{1}{4} g_{\mu\nu} F^2). \]  

\(^3\)The number comes as follows: to the usual \( 2d - 3 \) killing vectors that any plane wave spacetime has [18], in our case 5, one must add, from \( H(u, \zeta, \bar{\zeta}) \) being \( H(u, \zeta, \bar{\zeta}) = -\lambda^2 \zeta \bar{\zeta} \), translations in \( u \) and rotations in the transverse space \( i(\bar{\zeta} \partial_\zeta - \zeta \partial_\bar{\zeta}) \). The appearance of \( \partial_u \) as a Killing vector makes the spacetime homogeneous.
The spinor projection\footnote{The coordinate independent way of expressing (29) is \((l \cdot \Gamma) \xi = 0\) where \(l\) is the covariantly constant null vector of the pp-wave \cite{22}.}
\[ \Gamma^- \xi = 0, \quad (29) \]
was used in \cite{1},\cite{20}. We now further impose the constraint
\[ \phi(u, \zeta) = g(u) \zeta, \quad (30) \]
where \(g(u)\) is an arbitrary complex function and show that we have \(1 \over 2\)-supersymmetry. Using the constraint (29), the system (22)-(25) reduces to
\[ \partial_v \xi = \partial_{\zeta} \xi = \partial_{\bar{\zeta}} \xi = 0, \quad (31) \]
\[ \left( \partial_u + i \over 2 (\partial_{\zeta} \phi \Gamma^1 + \partial_{\bar{\zeta}} \bar{\phi} \Gamma^2) \right) \xi = 0, \quad (32) \]
and we see that a consistent \(1 \over 2\)-BPS solution is found if (30) is satisfied.

From now on we take \(g(u)\) to be real. The solution for the Killing spinors is
\[ \xi = e^{-i \over 2 (\Gamma^1 + \Gamma^2) G(u)} \xi_0, \quad (33) \]
where \(G(u) = \int_{u_0}^{u} du' g(u')\) and \(\xi_0\) satisfies (29). The exponential in (33) can be resummed using the parametrization for the gamma matrices given in the appendix, obtaining
\[ \xi = \left\{ \cos(G(u)) + \gamma^2 \sin(G(u)) \right\} \xi_0. \quad (34) \]
This shows that the Killing spinors obtained by the projection (29) always depend on the "time-like" \(u\) coordinate, are independent of the transverse coordinates and correspond to kinematical supersymmetries \cite{21}.

For the maximally supersymmetric solutions \(g(u)\) is constant, \(G(u)\) becomes linear and then half of the spinors are periodic in the \(u\) coordinate. Explicitly, the Killing spinors are given by
\[ \xi = (1 + i \lambda \over 8 (\Omega \zeta + \Omega \bar{\zeta})) e^{-i \lambda G(u) \Gamma^1 \Gamma^2} \xi_0 \quad \text{(35)} \]
Using the parametrization given in the appendix, the exponential can be resummed obtaining
\[ \xi = (1 + i \lambda \over 8 (\Omega \zeta + \Omega \bar{\zeta})) (A \cos \lambda u - iB \sin \lambda u) \xi_0, \quad (36) \]
(expressions for \(A\) and \(B\) are given in the appendix). Eqn. (36) shows the periodicity of the spinors in the \(u\) coordinate and from the eigenvalues of \(A\) and \(B\) one gets that only half of the spinors depend explicitly on \(u\).

\section{Conformal Boundary}

From the discussion of the last section we know that a \(1 \over 2\)-BPS family of solutions is obtained by the ansatz\footnote{The coordinate independent way of expressing (29) is \((l \cdot \Gamma) \xi = 0\) where \(l\) is the covariantly constant null vector of the pp-wave \cite{22}.}
\[ ds^2 = du dv - g^2(u) x_1^2 du^2 + dx_1^2, \quad (37) \]
\[ F = g(u) du \wedge dx_1, \quad (38) \]
here $i = 1, 2$ refers to the real transverse directions. Computing the curvature of (10), one notices that the only nontrivial pp-waves which are non-singular are precisely of the form (37) with smooth $g(u)$ [24]. In the $g(u) = \text{const}$ case, supersymmetry gets enhanced and we get the fully supersymmetric solution [21].

From the form (37) we see that the spacetime is foliated by null hypersurfaces $u = \text{const}$ and the vector $t = \partial_u$ connecting these is timelike everywhere except at the origin of the transverse space.

We will now analyse the conformal boundary of the solution (37) for particular choices of $g(u)$. The plan is, given that the solutions are conformally flat, to embed them into the Einstein Static Universe (ESU) and study where the conformal factor blows up. This is done by going from Brinkman (37) to Rosen coordinates [24] where the metric takes the form

$$ ds^2 = d\tilde{u}d\tilde{v} + p^2(u) d\tilde{x}_i^2, $$

where $p(u)$ is given by the solution of

$$ \ddot{p}(u) = -g^2(u) p(u). $$

Factorizing out $p^2(u)$ in (39) one obtains the conformally flat expression

$$ ds^2 = \frac{1}{\Pi^2(\tilde{u})} (d\tilde{u}d\tilde{v} + d\tilde{x}_i^2). $$

where $1/\Pi^2(\tilde{u}) = p^2(u(\tilde{u}))$ and $\tilde{u} = \int \frac{1}{p(u)} du$. Now using the standard coordinate transformations [27]

$$ \tilde{u} = \frac{\sin \psi + \sin \xi \cos \theta}{\cos \psi + \cos \xi}, \quad \tilde{v} = \frac{-\sin \psi + \sin \xi \cos \theta}{\cos \psi + \cos \xi}, \quad \tilde{x}_1 = \frac{\sin \xi \sin \theta \cos \phi}{\cos \psi + \cos \xi}, \quad \tilde{x}_2 = \frac{\sin \xi \sin \theta \sin \phi}{\cos \psi + \cos \xi}, $$

one can embed (41) into the ESU. The result is

$$ ds^2 = \frac{1}{(\cos \psi + \cos \xi)^2 \Pi^2(\tilde{u})} (-d\psi^2 + d\xi^2 + \sin^2 \xi (d\theta^2 + \sin^2 \theta d\phi^2)). $$

The conformal boundary is defined as the set of points where the conformal factor

$$ \Omega(x^\mu) = (\cos \psi + \cos \xi) \Pi(\tilde{u}), $$

vanishes. The normal to the hypersurfaces $\Omega(x^\mu) = \text{const}$ is given by $n_\mu = \partial_\mu \Omega$ and the nature of the boundary can be analysed by computing

$$ n^2 = \frac{1}{2} \Pi(\tilde{u}) \left( \Pi(\tilde{u}) (\cos 2\psi - \cos 2\xi) + 4\Pi(\tilde{u}) (\sin \psi \cos \xi - \cos \psi \sin \xi \cos \theta) \right). $$

and evaluating it at $\Omega = 0$.

i) Minkowski space: This very well known case corresponds to $g(u) = 0$ in (37). In Rosen coordinates we get $p(u) = A + Bu$, where $A, B$ are two arbitrary constants. By taking $B = 0$ we end up in ESU with the standard expression

$$ ds^2 = \frac{1}{(\cos \psi + \cos \xi)^2} (-d\psi^2 + d\xi^2 + \sin^2 \xi (d\theta^2 + \sin^2 \theta d\phi^2)). $$

The change of coordinates to go from (36) to (39) is $v = \tilde{v} - p(u)p(u) \tilde{x}_i^2$, $x_i = p(u) \tilde{x}_i$ where the index $i = 1, 2$ refers to the transverse space coordinates. $p(u)$ is a solution of (40) which can be interpreted as a harmonic oscillator with time dependent frequency.

5 The change of coordinates to go from (36) to (39) is $v = \tilde{v} - p(u)p(u) \tilde{x}_i^2$, $x_i = p(u) \tilde{x}_i$ where the index $i = 1, 2$ refers to the transverse space coordinates. $p(u)$ is a solution of (40) which can be interpreted as a harmonic oscillator with time dependent frequency.
Figure 1: (a) Minkowski spacetime eqn. (46). (b) Hpw spacetime eqn. (49). The base of the solid cylinders represents $S^3$, the center of it being $z_4 = -1$ and the boundary being $z_4 = 1$. The vertical direction corresponds to the time coordinate $\psi$. As usual, the boundary of the cylinder should be regarded as 1 dimensional. The thick lines in (a) and (b) correspond to the set of points where the conformal factors in (46) and (49) vanish. The time coordinate difference between the tips of the cones is $\Delta \psi = 2\pi$.

In this case $\Pi(\tilde{u}) = 1$ and computing (45) on $\Omega = 0$ we see that the boundary is a null co-dimension 1 hypersurface.

A non standard expression for the conformal compactification of Minkowski space is obtained by choosing $A = 0$ and $B = 1$, this is

$$ds^2 = \frac{1}{(\sin \psi + \sin \xi \cos \theta)^2}(-d\psi^2 + d\xi^2 + \sin^2 \xi (d\theta^2 + \sin^2 \theta \, d\phi^2)). \quad (47)$$

Here we get $\Pi(\tilde{u}) = \sqrt{1 + \tilde{u}^2}$ and of course we obtain again a co-dimension 1 null boundary

Picturing the $S^3$ parametrized by $(\xi, \theta, \phi)$ as $z_1^2 + z_2^2 + z_3^2 + z_4^2 = 1$ where

$$z_1 = \sin \xi \sin \theta \cos \phi, \quad z_2 = \sin \xi \sin \theta \sin \phi, \quad z_3 = \sin \xi \cos \theta, \quad z_4 = \cos \xi, \quad (48)$$

one sees that the conformal factors in (46) or (47) correspond to the slicing of the $S^3$ by $z_4 = -\cos \psi$ or $z_3 = -\sin \psi$ respectively. This gives rise to the pictorial representation of compactified Minkowski space as the interior of a cone followed by an inverted cone (see figure 1(a)).

ii) Maximally supersymmetric homogeneous plane wave (Hpw): The conformal boundary for this spacetime in the $d = 10$ case has been studied in [26] (see also [27]). Setting $g(u) = 1$ by an appropriate change of coordinates, we obtain

$$ds^2 = \frac{1}{(\cos \psi + \cos \xi)^2 + (\sin \psi + \sin \xi \cos \theta)^2}(-d\psi^2 + d\xi^2 + \sin^2 \xi (d\theta^2 + \sin^2 \theta \, d\phi^2)). \quad (49)$$

We get $\Pi(\tilde{u}) = \sqrt{1 + \tilde{u}^2}$ and by (45) we see again that the boundary is a null surface. In this example, however, the conformal boundary is 1-dimensional and this comes about because we must look for the intersection of the two null hypersurfaces $S_1(x^\mu) = \cos \psi + \cdots$.
cos \xi = 0 and \( S_2(x^\mu) = \sin \psi + \sin \xi \cos \theta = 0 \). Naïvely one would expect a co-dimension 2 surface (co-dimension \( D-2 \) in the general case), but in this particular case the intersection of the two hypersurfaces \( S_1(x^\mu) = \cos \psi + z_4 = 0 \) and \( S_2(x^\mu) = \sin \psi + z_3 = 0 \) occurs for any value of \( \psi \) at the north/south pole of the \( S_3^{D,\phi} \), where the fiber corresponding to the \( \phi \) coordinate in \( (43) \) collapses. In other words this means that at any given fixed time \( \psi \) the intersection of the two null hypersurfaces is just one point. In the higher dimensional cases this same phenomenon happens and the conformal null boundary of the maximally supersymmetric plane waves is again 1-dimensional (see figure \[1\](b)).

iii) Minkowski→Hpw→Minkowski plane wave (MHMpw): An analytically tractable example is obtained by choosing \( g(u) \) of the form

\[
g(u) = \frac{1}{1 + a^2 u^2}.
\]

(50)

For large absolute values values of \( u \) we asymptotically get Minkowski spacetime and the homogeneous plane-wave region appears for \( u \approx 0 \). In particular in the limit \( a \to \infty \) we get Minkowski space and for \( a \to 0 \) we get the homogeneous plane wave \( (49) \).

The solution to \( (40) \) is

\[
p(u) = A_1 \sqrt{1 + a^2 u^2} \cos \left( \frac{\sqrt{1 + a^2}}{a} \arctan(a u) + A_2 \right).
\]

(51)

Without loss of generality we can take \( A_1 = 1 \) and \( A_2 = 0 \), the expression for \( \Pi(\tilde{u}) \) is

\[
\Pi^2(\tilde{u}) = \frac{1 + \tilde{u}^2}{1 + \tan^2 \left( \frac{1}{\eta} \arctan \tilde{u} \right)},
\]

(52)

where \( \eta = \sqrt{1 + a^2} > 1 \). (Minkowski spacetime is obtained for \( \eta = 1 \) and the maximally supersymmetric Hpw for \( \eta = \infty \).) The embedding in ESU takes the form

\[
ds^2 = \frac{1 + \tan^2 \left( \frac{1}{\eta} \arctan \left( \frac{\sin \psi + \sin \xi \cos \theta}{\cos \psi + \cos \xi} \right) \right) \arctan(a u) + A_2}{(\cos \psi + \cos \xi)^2 + (\sin \psi + \sin \xi \cos \theta)^2} ds^2_{ESU}.
\]

(53)

From this expression it is clear that the conformal boundary will be given by the set of points corresponding to the 1-d “helix” described in the last subsection, where the denominator vanishes, plus the ones which make the numerator of \( (53) \) blow up, this means the argument of the tangent in the numerator of \( (53) \) being equal to \( \frac{\pi}{2} + n \pi \).

To understand the conformal boundary of \( (53) \) it is necessary to understand how the \( \tilde{u} = \text{const.} \) null hypersurfaces in \( (41) \) get compactified when we go go to the ESU (recall that the argument of \( \arctan \) is just \( \tilde{u} \) see eqn.\( (42) \)). It is important to have in mind that these null hypersurfaces which we will take to be \( \tilde{u} = t + \tilde{x}_3 \) correspond to plane fronts and should be distinguished from the ones in Penrose’s compactification which are spherical fronts \( \tilde{u} = t + r \). By looking at the hypersurfaces as \( \psi \)-dependent intersections of the \( S^3 \) parametrized by \( (\xi, \theta, \phi) \), from \( (42) \) and \( (48) \) we get that they correspond to the slicing

\[
z_3 = \tan \alpha z_4 + \frac{\sin(\alpha - \psi)}{\cos \alpha}
\]

(54)

\footnote{After an obvious rescaling in \( \tilde{u} \).}
where $\tilde{u} = \tan \alpha$. A visualization for the $\tilde{u} = 0$ case is given in figure 2. In the general case the null hypersurface consists of three parts: two 1-dimensional sectors represented in figure 2 by the lines ‘ab’ and ‘gh’ and one 3-dimensional between points ‘b’ and ‘g’. As $\tilde{u}$ becomes negative the points ‘b’ and ‘g’ approach ‘a’ and ‘f’ respectively, and the points ‘d’ and ‘e’ get closer to ‘f’. In the case of $\tilde{u}$ becoming positive all the points mentioned before move in opposite direction. With this picture in mind one concludes that the top and bottom cones in the figure 1.(a) should be identified with $\tilde{u} = \pm \infty$ respectively. Furthermore, from this viewpoint one recognizes the conformal boundary of the Hpw as nothing but the intersection of the boundary of Minkowski spacetime (figure 1.(a)) with the $\tilde{u} = \text{const} < \infty$ null hyperplane (cf. with the definitions in terms of $S_{1,2}$ in (ii)).

Returning to (53) one immediately realizes that for $\eta > 1$ there exists no value of $\tilde{u} \in (-\infty, \infty)$ making the numerator blow up. However, an important point is worth mentioning: when going from (37) to the conformally embedded expression (43) taking (50), one can see that the coordinate range $\tilde{u} \in (-\infty, \infty)$, used as an intermediate step to obtain (53), only covers the patch $|u| < \frac{1}{\eta} \tan\left(\frac{\sqrt{1+\alpha^2}}{2}\right)$ of the original Brinkman coordinate system. This fact is manifested mathematically in (53) by the presence of the arctan function which should then be analytically continued in order to cover the original geodesically complete spacetime. To be precise: for $\psi \in (2n\pi, (2n + 2)\pi)$ the value of arctan should be restricted to be within $((-1/2 + n)\pi, (1/2 + n)\pi)$. With this prescription it becomes clear that the conformal boundary of the spacetime (53) will consist of two null $\tilde{u} = \text{const}$ hypersurfaces (see figure 3) where the numerator of (53) will blow up, joined by the 1-dimensional null line depicted in figure 2(b), where the denominator in (53) vanishes. One may say that the two cones present in the conformal compactification of Minkowski space and intersecting at $i_0$ have now been separated and joined by the null 1-dimensional helix (see figure 3). The $\psi$ separation between the two null hypersur-
Conclusion and Discussion

We have obtained new $\frac{1}{2}$-BPS time dependent backgrounds which when extended to $d = 10$ will give gaussian models with “time dependent masses” in the light-cone gauge.

A general procedure for analyzing the conformal boundary of the time dependent spacetimes was developed and applied to a particular example.

It may perhaps be worth noticing that for the particular class of spacetimes (37) the only non-zero Ricci component is $R_{uu}$, this means that they can be supported by any p-form of the form $F = g(u) \, du \wedge \varphi$ where $\varphi$ is a constant (p-1)-form in the transverse flat space. The resulting background can then be choosen to be soported by Neveu-Schwarz or Ramond-Ramond field with constant dilaton field. For the particular case of $g(u) = \text{const}$ the background supported by the NS 3-form field strength was obtained long ago by a WZW construction over a generalized Heisenberg group [28].

The study of strings propagating on these backgrounds is under investigation.

N.B: While this work was being written up, the paper [29] appeared where similar issues are discussed.

Acknowledgments

It is pleasure to thank Pascal Bain, Sean Hartnoll, Venkata Suryanarayana Nemani, Carlos Núñez, Rubén Portugués and especially Gary Gibbons for many fruitful conversations. Research supported by CONICET and Fundación Antorchas, Argentina.
Appendix

General pp-wave solution

The covariantly constant null vector of (10) is \( l = l^\mu \partial_\mu = \partial/\partial v \) and we have used light-cone coordinates \((u,v)\) and complex coordinates \((\zeta, \bar{\zeta})\) related to the usual cartesian ones by \( \zeta = x_1 + ix_2 \). The Riemann tensor for the spacetime (10) can be written as \( R_{\mu
u\rho\sigma} = l^\mu_{[\rho} k_{\sigma]} \) where \( l^\mu \) are the covariant components of the null constant vector and the symmetric tensor \( k_{\mu\nu} \) is non-zero only for the transverse coordinates \( i,j = \zeta, \bar{\zeta} \) and has the form \( k_{ij} = \frac{1}{2} \partial_i \partial_j H \). The electromagnetic field (11) can be written as \( F_{\mu\nu} = l^\mu [\sigma \nu] \) where \( s^\mu = \sqrt{2} (\partial_\zeta \phi, \partial_{\bar{\zeta}} \bar{\phi}, 0, 0) \) in the \( x^\mu = (\zeta, \bar{\zeta}, u, v) \) coordinate system. Note that \( l \cdot s = 0 \).

The metric (21) can also be obtained from a Wess-Zumino-Witten construction over the Heisenberg-Cangemi-Jackiw (HCJ) group \([15]\) (the \( d = 6, 10 \) maximally supersymmetric plane waves can be obtained as well as the bi-invariant metrics over generalized HCJ-groups via a WZW construction \([28]\)). The expression found by Nappi-Witten is obtained by making in (21) the change of coordinates \( \zeta = e^{-i\lambda u} \omega \)

\[
ds^2 = dudv + i\lambda(\bar{\omega}d\omega - \omega d\bar{\omega}) du + d\omega d\bar{\omega}, \quad F = \frac{\lambda}{2} du \wedge (e^{-iCu} d\omega + e^{iCu} d\bar{\omega}).
\]

In cartesian coordinates \( \omega = y_1 + iy_2 \) this takes the form

\[
ds^2 = dudv + 2\lambda(y_2 dy_1 - y_1 dy_2) du + dy_1^2 + dy_2^2, \quad F = \lambda du \wedge (\cos(\lambda u) dy_1 + \sin(\lambda u) dy_2)
\]

Note that the maximally supersymmetric solution is supported by a 1-form gauge field instead of a 2-form gauge fields. The \( d = 10 \) case works along the same lines, with the metric supported by the self-dual RR 5-form and having constant dilaton.

Tetrads, spin connections and gamma matrices conventions

My conventions for \( \Gamma \) matrices are: \( \{\Gamma_a, \Gamma_b\} = 2\eta_{ab} \) with \( \eta_{ab} \) mostly plus, \( \Gamma_{ab} = \frac{1}{2} \epsilon_{abcd} \Gamma^c \Gamma^d \) and \( \epsilon_{0123} = -1. a,b \ldots \) are flat indices while \( \mu, \nu \ldots \) are spacetime indices.

For the general pp-wave solution (10) I choose the tetrads

\[
e^1 = d\zeta, \quad e^2 = d\bar{\zeta}, \quad e^- = du, \quad e^+ = dv + H(u, \zeta, \bar{\zeta}) du,
\]

which have the tangent space inverse metric

\[
\eta^{ab} = \begin{pmatrix}
0 & 2 & 0 & 0 \\
2 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 \\
0 & 0 & 2 & 0
\end{pmatrix}
\]

Here \( a,b = 1,2,-,+ \). The spin connections are given by

\[
\omega_{-1} = \frac{1}{2} \partial_\zeta H \ du, \quad \omega_{-2} = \frac{1}{2} \partial_{\bar{\zeta}} H \ du.
\]

A real representation of the gamma matrices satisfying \( \{\Gamma^a, \Gamma^b\} = 2\eta^{ab} \) is given by

\[
\Gamma^1 = i\gamma^2 + \gamma^3, \quad \Gamma^2 = i\gamma^2 - \gamma^3, \quad \Gamma^- = \gamma^0 + \gamma^1, \quad \Gamma^+ = \gamma^0 - \gamma^1,
\]
where $\gamma^\mu$ are the standard chiral representation for the gammas with mostly minus flat metric given by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix},$$

(60)

$$\sigma^\mu = (I_2, \sigma), \quad \bar{\sigma}^\mu = (I_2, -\bar{\sigma}).$$

The relation between curved and tangent space gamma matrices $\Gamma_\mu = e_{a\mu} \Gamma^a$ gives

$$\Gamma_\zeta = \frac{1}{2} \Gamma^2, \quad \Gamma_{\bar{\zeta}} = \frac{1}{2} \Gamma^1, \quad \Gamma_u = \frac{1}{2} (\Gamma^+ + H \Gamma^-), \quad \Gamma_v = \frac{1}{2} \Gamma^-$$

(61)

The matrices (26)-(27) take the explicit form

$$\Omega = \begin{pmatrix} 0 & 0 \\ 4(I_2 - \sigma^1) & 0 \end{pmatrix}, \quad \Omega' = \begin{pmatrix} 0 & 4(I_2 + \sigma^1) \\ 0 & 0 \end{pmatrix}.$$  

(62)

The matrices $A$ and $B$ appearing in (36) are

$$A = \frac{1}{2} \left( \begin{array}{cc} I_2 + \sigma^1 & 0 \\ 0 & I_2 - \sigma^1 \end{array} \right), \quad B = \frac{1}{2} \left( \begin{array}{cc} 0 & -\sigma^3 + i\sigma^2 \\ -\sigma^3 - i\sigma^2 & 0 \end{array} \right).$$

(63)

Both $A, B$ are hermitian and commute between themselves. $A$ has eigenvalues $\lambda_A = 0, 0, 1, 1$ whereas $B$ has $\lambda_B = 0, 0, 1, -1$. In other words, $A$ and $B$ have simultaneously 4 null eigenvectors.

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