COUNTING SQUARE DISCRIMINANTS

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Abstract. Hee Oh and Nimish Shah prove in [OS] that the number of integral binary quadratic forms whose coefficients are bounded by a quantity $X$, and with discriminant a fixed square integer $d$, is $cX \log X + O(X(\log X)^{3/4})$. This result was obtained by the use of ergodic methods. Here we use the method of shifted convolution sums of Fourier coefficients of certain automorphic forms to obtain a sharpened result of a related asymptotic, obtaining a second main term and an error of $O(X^{1/2})$.

1. Introduction

The aim of this paper is to answer an elementary question concerning the asymptotics of the number of solutions of an indefinite quadratic ternary form. We fix the form to be the discriminant $b^2 - 4ac$, and look at the number of integer triples $(a, b, c)$ for which $0 < a, |b|, c \leq X$ and $b^2 - 4ac = h > 0$. The main term of the asymptotics depends on whether $h$ is a square. This question was answered in [OS] as a part of a solution of counting lattice points on hyperbolic surfaces. In particular, they prove the following.

Theorem 1.1 (Oh-Shah). For any non-zero square $h \in \mathbb{Z}$ there exists $c > 0$ such that

$$
\# \{ Q(x, y) = ax^2 + bxy + cy^2 : a, b, c \in \mathbb{Z}, \text{disc}(Q) = h, a^2 + b^2 + c^2 \leq X^2 \} = cX \log X + O(X(\log X)^{3/4}).
$$

Our approach to this problem uses the analytical properties of the Dirichlet series

$$
\sum_{a,c=1}^{\infty} \frac{\tau(4ac + h)}{a^s c^w}
$$

where $\tau$ is a square indicator function. This Dirichlet series can be obtained from the Petersson inner product

$$
\langle P_{h, Y}^{-}(\ast, s, \delta), \text{Im}(\ast)^{\frac{1}{2}} \theta E^{\ast} \left( 4\ast, \frac{1+s}{2} \right) \rangle
$$

taken over the domain $\Gamma_0(4) \backslash \mathbb{H}$. The function $\theta(z) E^{\ast} \left( 4z, \frac{1+s}{2} \right)$ will be denoted $V_v(z)$ for brevity. The function $P_{h, Y}^{-}(z, s, \delta)$ is a Poincaré series, often just

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denoted \(P\), and is derived from a heuristic form given by

\[
P_h^{-\frac{1}{2}}(z, s) = \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma_0} \Im(\gamma z)^4 e^{-2\pi i h \gamma z} \frac{j(\gamma, z)}{|j(\gamma, z)|}
\]

(3)

The letters \(Y\) and \(\delta\) appearing in \(P_{h,Y}^{-\frac{1}{2}}(z, s, \delta)\) are auxiliary variables that will ensure square-integrability and good spectral behavior.

The function \(\tilde{V}_v\) is an automorphic form of half integral weight on the space \(\Gamma_0(4) \setminus \mathbb{H}\). In order to understand the analytic behavior of (1), we use the spectral decomposition of the Poincaré series.

\[
\langle P, \tilde{V}_v \rangle = \sum_j \langle P, u_j \rangle \langle u_j, \tilde{V}_v \rangle + \langle P, \Im(*)^{\frac{1}{2} \delta} \Im(*)_{\delta}^{\frac{1}{2}} \tilde{V}_v \rangle \\
+ \sum_{a \text{ cusps}} \frac{\nu}{4\pi} \int_{-\infty}^{\infty} \langle P, E(\ast 1/2 + it) \rangle \langle E_a(\ast, 1/2 + it), \tilde{V}_v \rangle dt.
\]

(4)

Here \(\nu\) is the volume of the hyperbolic surface \(\Gamma_0(4) \setminus \mathbb{H}\). By considering each term in the above expansion separately, we find the locations of the poles and the residues of the meromorphic continuation of (1).

The difficulties are threefold. Two of these were overcome in [HH], where a similar Poincaré series of weight 0 was defined and studied. The case of weight \(-\frac{1}{2}\), which is what we are interested in, is not significantly different. First of all \(P_h^{-\frac{1}{2}}\), as given in (3), is not an \(L^2\) function. In fact, it has exponential growth in \(y\). Therefore, the spectral expansion does not exist, and we must work with a truncated function \(P_{h,Y}(z, s)\) supported on a compact domain. After taking the inner product, we can take the limit

\[
\lim_{Y \to \infty} \langle P_{h,Y}(\ast, s), \tilde{V}_v \rangle
\]

(5)
to recover the Dirichlet series.

The second difficulty is that, although \(P_{h,Y}(z, s)\) can be given a spectral decomposition, quantities such as \(\langle P_{h,Y}(\ast, s), u_j \rangle\) do not make sense as \(Y \to \infty\). That difficulty is overcome by introducing another auxiliary variable, \(\delta > 0\). The final form of the Poincaré series is given below in (13). Convergence of the Dirichlet series occurs for \(s\) in some right half-plane and in this region, \(\delta\) can be meaningfully sent to zero. On the spectral side the expansion (4) converges in some left half-plane as \(\delta \to 0\).

The third difficulty does not have to do with the analytical incompatibilities of the Poincaré series, but is due to the fact that \(\tilde{V}_v\) is of moderate growth and thus not square-integrable. We overcome this by subtracting a modular form of the same weight, character and growth as \(\tilde{V}_v\). More specifically we subtract linear combinations of Eisenstein series at various cusps, evaluated at specific values of the holomorphic variable and the resulting difference is square-integrable.
In later sections of the paper, we use inverse Mellin transforms to get from the Dirichlet series to the truncated sum, and then move the lines of integration to obtain the desired asymptotic results. Our main result is the following.

**Theorem 1.2.** Let $h > 0$ be an integer and $X > 0$ a large real number. Then we have the asymptotic expansion

$$\sum_{a,c=1}^{\infty} \tau(h + 4ac) e^{- \frac{(h + 4ac)X}{2}} = c_1(h) X \log X + c_2(h) X + O_h(X^{\frac{1}{2}})$$

(6)

where $c_1(h) = 0$ if $h$ is a non-square.

**Remark 1.** The sum on the left-hand side of (6) is a smoothed version of the sum

$$\sum_{a,c=1}^{X} \tau(h + 4ac)$$

(7)

which counts the number of points in the set

$$\{(a, b, c) \in \mathbb{Z}_+^3 : a, b, c \leq X \text{ and } b^2 - 4ac = h\}$$

as $X \to \infty$, which will be demonstrated in the next section. The authors would like to note the techniques in this paper can be similarly used to derive an asymptotic expansion of (7) with the same main terms as (6) and an error term of $O_h(X^{1-\delta})$ for some $\delta > 0$. In fact we can choose $\delta = 1/12$ without much effort. This is achieved by replacing the inverse Mellin transforms in Section 4 with the finite version of Perron’s integral formula.

**Remark 2.** The question for negative $h$ happens to be surprisingly simpler, owing to the fact that the Poincaré series involved is the ordinary one as defined in (3) rather than the one of exponential growth as defined in [HH]. Therefore we will only concentrate on the case where $h$ is strictly positive.

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2. **The Automorphic Function $V(z)$**

For notational convenience let $\Gamma = \Gamma_0(4)$, the set of matrices in $\text{SL}(2, \mathbb{Z})$ which are upper triangular when reduced modulo four. Modular forms of weight $k$, for $k$ a half-integer, are functions $f$ on the upper half plane satisfying

$$f(\gamma z) = j(\gamma, z)^{2k} f(z)$$
where \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \) and

\[
j(\gamma, z) = \varepsilon_d^{-1} \left( \frac{c}{d} \right) (cz + d)^{1/2}
\]
is the weight 1/2 cocycle, as in [Shi73].

Given a cusp \( a \in \mathbb{Q} \cup \{\infty\} \), we call the stabilizer of the cusp

\[
\Gamma_a := \{ \gamma \in \Gamma : \gamma a = a \}.
\]

For the group \( \Gamma \), any cusp is equivalent to one of the three inequivalent cusps \( 0, \frac{1}{2}, \infty \).

Call \( E(z, w) \) to be the standard real-analytic Eisenstein series on \( SL(2, \mathbb{Z}) \), and

\[
E^*(z, w) := E(z, w)\zeta^*(2w)
\]
where

\[
\zeta^*(w) = \pi^{-\frac{w}{2}} \Gamma\left(\frac{w}{2}\right)\zeta(w).
\]

Furthermore, for \( k \) a half integer, let \( E_k^a(z, w) \) be the weight \( k \) Eisenstein series at the cusp \( a \) as in [GH85]. Our notation differs somewhat from the notation of [GH85], as we shift the position of the complex variable by replacing the \( w \) with \( w - \frac{k}{2} \). This amounts to using the normalized cocycle \( j(\gamma, z)/|j(\gamma, z)| \) in the definition of the Eisenstein series instead of \( j(\gamma, z) \) alone. Also, note that in [GH85] the weight is denoted by \( \frac{k}{2} \) but throughout this work we use \( k \).

We use the Fourier coefficients of the classical theta series

\[
\theta(z) = \sum_{n \in \mathbb{Z}} e^{2\pi in^2z} = \sum_{n=0}^{\infty} \tau(n)e^{2\pi inz}
\]
as a square indicator function. Here

\[
\tau(n) = \begin{cases} 
1 & \text{if } n = 0 \\
2 & \text{if } n \text{ is a nonzero square} \\
0 & \text{if } n \text{ is not a square}.
\end{cases}
\]

The theta series is a modular form of weight 1/2. Finally we define the Gauss sum,

\[
g_n(c) = \sum_{d \mid c \text{ odd}} \varepsilon_d \left( \frac{c}{d} \right) e^{2\pi i c/d}.
\]

With this background in mind, we begin by fixing a positive integer \( h \). We can approach the problem of counting the number of ways of writing \( h \) as a discriminant by counting the number of integer triples \( (a, b, c) \) in a box of size \( X \) satisfying the equality \( h = b^2 - 4ac \). If \( \tau \) is a square indicator function, then we have that

\[
\sum_{a, c=1}^{X} \tau(4ac + h) = \# \{(a, b, c) \in \mathbb{Z}_{>0}^3 : a, c \leq X \text{ and } h + 4ac = b^2 \}.
\]
If $a, c \leq X$ and $h \leq 4X$, then $b$ will be bounded by $2X$ also, so we have the relation

$$
\sum_{a,c=1}^{X} \tau(4ac + h) = \# \{(a, b, c) \in \mathbb{Z}^3_{>0} : a, b, c \leq X \text{ and } b^2 - 4ac = h\}. \tag{10}
$$

for fixed $h$ as $X \to \infty$. The Dirichlet series,

$$
\sum_{a,c=1}^{\infty} \frac{\tau(4ac + h)}{a^{s+v} c^{s}} \tag{11}
$$

which is obtained via an inverse Mellin transform, will aid us in estimating the desired asymptotics of (10). Letting $m = ac$, the above Dirichlet series becomes

$$
\sum_{m=1}^{\infty} \frac{\tau(4m + h)}{m^{s}} \sum_{a|m} 1 \frac{1}{a^{v}} = \sum_{m=1}^{\infty} \frac{\tau(4m + h)\sigma_{-v}(m)}{m^{s}}.
$$

We will use the Fourier coefficients of Eisenstein series of level 1 and weight 0 on $\text{SL}(2, \mathbb{Z}) \backslash \mathbb{H}$ as a source for the divisor function. Then what we have is a shifted convolution sum of the Fourier coefficients of theta function and those of the Eisenstein series. This is obtained from the function

$$
\overline{V_v}(z) = y^{4}(\theta(z)E^{*}(4z, \frac{1+v}{2}) - 4^{1+v} \zeta^{*}(1+v)E^{*}_{\infty}(z, \frac{3}{2} + \frac{v}{2}) - \zeta^{*}(1+v)E^{*}_{0}(z, \frac{3}{2} + \frac{v}{2}) \tag{12}
$$

upon taking an inner product with the Poincaré series. The Eisenstein series which were subtracted ensure that $V_v(z)$ is in $L^2(\Gamma \backslash \mathbb{H})$.

We can obtain this Dirichlet series naturally from an inner product of $V_v$ and a weight $-1/2$ automorphic function on $\mathbb{H}$ of level 4 and a new type of Poincaré series, as in [HH]. Let

$$
P_{h,Y}^{\frac{1}{2}}(z, s; \delta) = \sum_{\gamma \in \Gamma_{\infty} \backslash \Gamma} \text{Im}(\gamma z)^{s} \Psi_Y(\text{Im} \gamma z) e^{-2\pi i h \text{Re}(\gamma z)} e^{2\pi h \text{Im}(\gamma z)(1-\delta)} \frac{j(\gamma, z)}{j(\gamma, z)} \tag{13}
$$

be the Poincaré series where $\Psi_Y$ is the indicator function of the interval $[Y^{-1}, Y]$. Again note for brevity that we will often denote this as simply $P$. This Poincaré series has been constructed to pick out the $h^{th}$ term from the Fourier expansion of an automorphic function of weight $1/2$. The factor of $j$’s in the sum make $P$ itself, as a function of $z$, an automorphic function of weight $-1/2$. Without the factor of the indicator function in the sum, the series as a function of $z$ would be growing exponentially in the imaginary part $y$ of $z$; this would have made it difficult to exploit the spectral decomposition of the function $P$. Right now, it is compactly supported, and hence in $L^2(\Gamma \backslash \mathbb{H}, -\frac{1}{4})$. 
3. The Dirichlet Series

In this section we will demonstrate that the inner product of \( P \) and first term of \( V_\nu \) produces a Dirichlet series of the form we desire. As in the Rankin-Selberg convolution, we begin by unfolding the inner product given in (5):

\[
\langle P, \text{Im}(\ast) \frac{\tau}{v} E^\ast(4s, \frac{1+v}{2}) \rangle = \int \int_{Y^{-1} 0} y^{s - \frac{3}{4}} e^{-2\pi i h x} e^{2\pi i y (1 - \delta)} \theta(z) E^\ast(4z, \frac{1+v}{2}) \frac{dxdy}{y}.
\]

Substituting the Fourier expansions for \( \theta(z) \) and the Eisenstein series the integral, the inner product becomes

\[
\int \int_{Y^{-1} 0} y^{s - \frac{3}{4}} e^{-2\pi i h x} e^{2\pi i y (1 - \delta)} \left( \sum_{n=0}^\infty \tau(n) e^{2\pi in z} \right) \left( \sum_{n=0}^\infty a_v(n, 4y) e^{-2\pi in(4x)} \right) \frac{dxdy}{y},
\]

where

\[
a_v(n, y) = \begin{cases} 
\zeta^\ast(1 + v) y^{\frac{v}{2} + \frac{\delta}{2}} + \zeta^\ast(v) y^{\frac{v}{2} - \frac{\delta}{2}} & \text{if } n = 0 \\
2 y^{\frac{v}{2}} \sigma_v(n) |n|^{\frac{v}{2}} K_\frac{v}{2}(2\pi |n| y) & \text{if } n \neq 0.
\end{cases}
\]

Therefore we obtain the expansion

\[
\langle P, \text{Im}(\ast) \frac{\tau}{v} E^\ast(4s, \frac{1+v}{2}) \rangle = \tau(h) 4^{\frac{s}{2} - s} \int_{Y^{-1}} e^{-2\pi y \delta h (4y)^{s - \frac{3}{4}}} \left( \zeta^\ast(1 + v)(4y)^{\frac{v}{2}} + \zeta^\ast(v)(4y)^{-\frac{v}{2}} \right) \frac{dy}{y} \quad (14a)
\]

\[
+ 4 \sum_{m=1}^{\lfloor \frac{h}{4} \rfloor} \tau(h - 4m) \sigma_v(m) |m|^{\frac{v}{2}} \int_{Y^{-1}} y^{s - \frac{3}{4}} e^{2\pi y (4m - \delta h)} K_\frac{v}{2}(2\pi |4m| y) \frac{dy}{y} \quad (14b)
\]

\[
+ 4 \sum_{m=1}^\infty \tau(h + 4m) \sigma_v(m) |m|^{\frac{v}{2}} \int_{Y^{-1}} y^{s - \frac{3}{4}} e^{-2\pi y (4m + \delta h)} K_\frac{v}{2}(2\pi |4m| y) \frac{dy}{y}. \quad (14c)
\]

We see that by taking the respective limits of \( \delta \) and \( Y \), when Re(\( s \)) is sufficiently large, (14c) becomes

\[
\frac{(16\pi)^{\frac{3}{4} - s}}{\Gamma(s + \frac{1}{4})} \Gamma(s - \frac{1}{4} + \frac{\delta}{2}) \Gamma(s - \frac{1}{4} - \frac{\delta}{2}) \sum_{a,c=1}^\infty \frac{\tau(h + 4ac)}{a^{s + \frac{\delta}{2} - \frac{\delta}{4}} c^{s - \frac{\delta}{2} - \frac{\delta}{4}}} \quad (15)
\]

which we see is the Dirichlet series we want to study.

Now, when Re \( s > \frac{1}{4} + |\text{Re}(\frac{\delta}{2})| \) we can take the limit as \( Y \) goes to infinity and (14a) becomes

\[
\tau(h) 4^{\frac{s}{2} - s} \left( (\delta h \pi)^{\frac{3}{4} - s - \frac{\delta}{2}} \zeta^\ast(1 + v) \Gamma(s - \frac{1}{4} + \frac{\delta}{2}) + (\delta h \pi)^{\frac{3}{4} - s + \frac{\delta}{2}} \zeta^\ast(v) \Gamma(s - \frac{1}{4} - \frac{\delta}{2}) \right),
\]

\[
\text{(Counting Square Discriminants)} \quad 6
\]
which has a meromorphic continuation to all \( s \in \mathbb{C} \). Similarly, when \( \text{Re} \, s > \frac{1}{4} + |\text{Re}(\frac{3}{2})| \), (14b) becomes

\[
4 \sum_{m=1}^{\left\lfloor \frac{1}{2} \right\rfloor} \frac{\tau(h - 4m)\sigma_m(m)}{(8\pi m)^{s - \frac{1}{2}}} \int_0^\infty y^{s - \frac{1}{2}} e^{y(1 - \frac{4h}{y})} K_{\frac{s}{2}}(y) \frac{dy}{y} = 4 \sum_{m=1}^{\left\lfloor \frac{1}{2} \right\rfloor} \frac{\tau(h - 4m)\sigma_m(m)}{(8\pi m)^{s - \frac{1}{2}}} \left( \frac{\pi}{2} \right)^{\frac{s}{2}} M_0(s + \frac{1}{4}, \frac{v}{2\pi}, \delta h, \frac{3m}{2}).
\]

as \( Y \to \infty \). The function \( M_k(s, z/i, \delta) \) has a meromorphic continuation to all \((s, z) \in \mathbb{C}^2 \) for fixed \( k \in \mathbb{R} \); its general definition and other relevant properties are due to [Hul13] and are summarized in Proposition A.2 in the appendix. Both (14a) and (14b) contribute poles at \( s = \frac{1}{4} \pm \frac{i}{2} \).

In order to understand the inner product of \( P \) with \( V_c \), we also need to examine the inner product of \( P \) with the half-integral weight Eisenstein series at the infinity cusp,

\[
\lim_{Y \to \infty} \langle P, \zeta^*(1 + v)E_{\frac{1}{2}}^\sharp(s, \frac{3}{4} + \frac{v}{2}) \rangle
= \int_0^\infty \int_0^1 y^{s-1}e^{2\pi i h y(1-\delta)} e^{-2\pi i h x} \zeta^*(1 + v)E_{\frac{1}{2}}^\sharp(z, \frac{3}{4} + \frac{v}{2}) \frac{dx \, dy}{y}
= i^\frac{1}{2} \frac{\pi^{\frac{1}{2} + \frac{v}{2} h} \zeta^*(1 + v)D_{\infty}(\frac{3}{4} + \frac{v}{2}; h)(2\pi h)^{1-s} M_\frac{1}{2}(s, \frac{1}{4} + \frac{v}{2}, \delta)}{\Gamma(1 + \frac{v}{2})} \tag{16}
\]

and at the zero cusp,

\[
\lim_{Y \to \infty} \langle P, \zeta^*(1 + v)E_0^\sharp(s, \frac{3}{4} + \frac{v}{2}) \rangle
= \int_0^\infty \int_0^1 y^{s-1}e^{2\pi i h y(1-\delta)} e^{-2\pi i h x} \zeta^*(1 + v)E_0^\sharp(z, \frac{3}{4} + \frac{v}{2}) \frac{dx \, dy}{y}
= i^\frac{1}{2} \frac{\pi^{\frac{1}{2} + \frac{v}{2} h} \zeta^*(1 + v)D_0(\frac{3}{4} + \frac{v}{2}; h)(2\pi h)^{1-s} M_\frac{1}{2}(s, \frac{1}{4} + \frac{v}{2}, \delta)}{\Gamma(1 + \frac{v}{2})}. \tag{17}
\]

Here

\[
D_{\infty}(s; h) = \sum_{n=1}^{\infty} \frac{g_n(4n)}{(4n)^s}
\]

is the \( h \)th Fourier coefficient of \( E_{\frac{1}{2}}^\sharp(z, s) \). Similarly \( D_0(s; h) \) is the \( h \)th Fourier coefficient of \( E_0^\sharp(z, s) \) and only differs from \( D_{\infty}(s; h) \) in the 2-place. This is described more fully in [GH85]. In particular, in Corollary (1.3), the authors show that

\[
D_a(s; h) = L^*(2s - \frac{1}{2}, (\frac{4h}{\pi}) F_a(s; h)
\]
where \( F_a(s; h) \) is a function which is analytic for \( \text{Re}(s) \geq \frac{1}{2} \). To be concrete, for positive \( h \),

\[
F_\infty(s; h) = \frac{1}{\zeta(4u - 1)} \frac{h^{s + \frac{1}{4} - \frac{3}{2}} \cos \left( -\frac{h}{2} + \frac{1}{4} \right)}{(24s - 1)!} \cdot \cos \left( \left( \frac{h}{2} + \frac{1}{4} \right) \right) \]

\[
\times \prod_{p \mid h, p \neq 2} \left( \sum_{i=0}^{\lfloor \frac{h}{p} \rfloor} \frac{1}{p^{4(s - \frac{1}{2})}} + \frac{1}{\zeta(2)} \left( 1 - \left( \frac{h/p}{p^{2s - \frac{1}{2}}} \right) \right) \right)
\]

So to summarize, we have proven the following proposition.

**Proposition 3.1.** The function

\[
\mathcal{D}(s; h, \delta) := \frac{4}{(8\pi)^{s-\frac{1}{4}}} \sum_{a,c=1}^{\infty} \frac{\tau(4ac + h)}{a^{s+\frac{1}{4}} e^{s+\frac{1}{4}}} \int_{0}^{\infty} y^{s-\frac{1}{4}} e^{-y - y(1 + \frac{3}{2})} \frac{dy}{y} \tag{19}
\]

is absolutely convergent for \( \text{Re}(s) > \frac{5}{4} + |\text{Re}(\frac{a}{2})| \) and has the expansion

\[
\mathcal{D}(s; h, \delta) = \lim_{Y \to \infty} \langle P, V_e \rangle \tag{20a}
\]

\[
= -4 \sum_{m=1}^{\infty} \frac{\tau(h - 4m)\zeta_v(m)\pi}{(8\pi m)^{s-\frac{1}{4}}} \sqrt{\pi} M_0(s + \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) \tag{20b}
\]

\[
- \frac{\tau(h)(\delta h \pi)^{s-\frac{1}{4}}}{4^{s-\frac{1}{4}}}(\zeta(1+v)\Gamma(s-\frac{1}{4} + \frac{1}{2}) + (\delta h \pi)^s \zeta(1+v)(s-\frac{1}{4} - \frac{1}{2})) \tag{20c}
\]

in this region.

Since (19) converges to (15) as \( \delta \to 0 \), in the next section we will use this proposition and the inverse Mellin transform of (20b) to relate the asymptotic expansion of the smoothed sum in Theorem 1.2 to the analytic properties of \( \langle P, V_e \rangle \).

4. **The inverse Mellin integrals**

To get an asymptotic estimate of

\[
\sum_{a,c=1}^{\infty} \tau(h + 4ac) e^{-\frac{\tau(h + 4ac)}{2 \pi}}, \tag{21}
\]

we take an inverse Mellin transform of the equation in Proposition 3.1. In particular, we will reduce our problem to a better computation of

\[
I := \lim_{\delta \to 0} \left( \frac{1}{2\pi i} \right)^2 \int_{(\frac{1}{2})}\int_{(\frac{1}{2})} \lim_{Y \to \infty} \langle P, V_e \rangle \Gamma(s + \frac{1}{4}) (16\pi)^{s-\frac{1}{4}} X^{2s-\frac{1}{4}} ds \, dv, \tag{22}
\]
which we will do in the next section.

**Proposition 4.1.** For the quantity $I$ defined above, one has the decomposition

$$
\sum_{a,c=1}^{\infty} \tau(h + 4ac)e^{-\frac{a+c}{h}} = I - \tau(h)cX \log X - dX + O(X^{\frac{1}{2}})
$$

(23)

for some constants $c \neq 0$ and $d$ depending on $h$.

**Proof.** From (15) and a simple change of variables we have that

$$
\lim_{\delta \to 0} \left( \frac{1}{2\pi i} \right)^2 \int \int D(s; h, \delta) \frac{\Gamma(s + \frac{1}{2})}{(16\pi)^{s/2}} X^{2s-\frac{1}{2}} ds \, dv = \sum_{a,c=1}^{\infty} \tau(h + 4ac)e^{-\frac{a+c}{h}}
$$

(24)

Using the expansion of $D(s; h, \delta)$ given in Proposition 3.1 we have that

$$
\sum_{a,c=1}^{\infty} \tau(h + 4ac)e^{-\frac{a+c}{h}} = I - T_1 - T_3; \infty + T_3; 0
$$

(25)

where

$$
T_1 := \left( \frac{1}{2\pi i} \right)^2 \int \int \frac{\tau(h)(4\pi)^{s-\frac{1}{2}}}{(\delta h\pi)^{s/2}} \zeta^*(1 + v) \Gamma(s - \frac{1}{4} + \frac{v}{2}) \Gamma(s + \frac{1}{2}) X^{2s-\frac{1}{2}} ds \, dv
$$

and

$$
T_2 := \left( \frac{1}{2\pi i} \right)^2 \int \int \frac{\tau(h)(4\pi)^{s-\frac{1}{2}}}{(\delta h\pi)^{s/2}} \zeta^*(v) \Gamma(s - \frac{1}{4} - \frac{v}{2}) \Gamma(s + \frac{1}{2}) X^{2s-\frac{1}{2}} ds \, dv
$$

is due to (20d),

$$
T_3 := \left( \frac{1}{2\pi i} \right)^2 \int \int \frac{\tau(h)(4\pi)^{s-\frac{1}{2}}}{(\delta h\pi)^{s/2}} \zeta^*(1 + v) \Gamma(s - \frac{1}{4} + \frac{v}{2}) \Gamma(s + \frac{1}{2}) X^{2s-\frac{1}{2}} ds \, dv
$$

is due to (20c), and

$$
T_{3, a} := \left( \frac{1}{2\pi i} \right)^2 \int \int \frac{\tau(h)(4\pi)^{s-\frac{1}{2}}}{(\delta h\pi)^{s/2}} \zeta^*(1 + v) \Gamma(s - \frac{1}{4} + \frac{v}{2}) \Gamma(s + \frac{1}{2}) X^{2s-\frac{1}{2}} ds \, dv
$$

is due to (20e) and (20f).

For $T_1$, we will deal with the two terms separately. For the first integral of $T_1$, we will shift the line of integration to $\text{Re}(s) = -\frac{1}{2} - \varepsilon$, picking up poles at $s = \frac{1}{4} - \frac{v}{2}$ and $s = -\frac{1}{4}$. The residue at $s = -\frac{1}{4}$ is

$$
\left( \frac{1}{2\pi i} \right)^2 \int \frac{\tau(h)(4\pi)^{-1}(\delta h\pi)^{\frac{1}{2}-\frac{v}{2}}}{(\delta h\pi)^{\frac{1}{2}}} \zeta^*(1 + v) \Gamma(\frac{\varepsilon-1}{2}) X^{-\frac{1}{2}} ds
$$

and we can shift the line of integration for $v$ left to $\text{Re}(v) = 1 - \varepsilon$. We can then take $\delta$ to 0. The residue at $s = \frac{1}{4} - \frac{v}{2}$ is independent of $\delta$ and has growth $O(X^{-\frac{1}{2}})$. 


The resulting residual term is

\[
\frac{1}{2\pi i} \int_{(\frac{1}{2})} \tau(h)(4\pi)^{\frac{1}{2}} \zeta^*(v) \Gamma\left(\frac{v+1}{2}\right) X^v \, dv = c_1 \tau(h)X + O(X^{\frac{3}{2}})
\]

where \( c_1 \) is a computable constant. We see that this residue is independent of \( \delta \).

For \( T_2 \), we will move the line of integration for \( v \) left to \( \text{Re} v = \frac{1}{2} + \varepsilon \). Then we shift the line of integration for \( s \) left to \( \text{Re} s = -\varepsilon \), picking up poles at \( s = \frac{1}{2} + \frac{v}{2} \) and \( s = \frac{1}{2} - \frac{v}{2} \). We can take \( \delta \to 0 \) for the shifted integral, as \( M_0 \) converges to a ratio of gamma functions for \( s \) in that range; the shifted integral has growth \( O(X^{-\frac{1}{2} - 2\varepsilon}) \).

At \( s = \frac{1}{2} + \frac{v}{2} \), we have residue (after taking \( \delta \) to 0)

\[
\frac{1}{2\pi i} \int_{(\frac{1}{2} + \varepsilon)} \frac{1}{2} \zeta^*(1 + v)D_\delta\left(\frac{3}{4} + \frac{v}{2}, h\right) \frac{2\pi^{-v} \Gamma\left(\frac{1}{2} + v\right)}{\Gamma\left(\frac{1}{2} + \frac{v}{2}\right)} X^{1+v} \, dv
\]

which we see is \( O(X^\varepsilon) \) by shifting the line of integration for \( v \) left to \( \text{Re} v = \varepsilon \). The residue at \( s = \frac{1}{4} - \frac{v}{2} \) is treated similarly.

For \( T_{3,a} \) we move the line of integration for \( v \) to \( \text{Re} v = \frac{1}{2} + \varepsilon \). We then shift \( s \) left to \( \text{Re} s = -\varepsilon \), picking up poles at \( s = \frac{3}{4} + \frac{v}{2} \), \( s = \frac{1}{4} - \frac{v}{2} \) and \( s = \frac{5}{4} - \frac{v}{2} \). The residue at \( s = \frac{3}{4} + \frac{v}{2} \), after taking \( \delta \to 0 \), is

\[
\frac{1}{2\pi i} \int_{(1+\varepsilon)} i \frac{\zeta^*(1+v)}{2} D_\delta\left(\frac{3}{4} + \frac{v}{2}, h\right) \frac{2\pi^{-v} \Gamma\left(\frac{1}{2} + v\right)}{\Gamma\left(\frac{1}{2} + \frac{v}{2}\right)} X^{1+v} \, dv.
\]

Now we move \( \text{Re}(v) \) left to \(-\frac{1}{2} - \varepsilon \), picking up poles at \( v = 0 \) and \( v = -\frac{1}{2} \). If \( h \) is not a square, both poles are simple and we obtain that

\[
\lim_{\delta \to 0} (T_3 + T_4) = c_2 \log X + O(X^{\frac{3}{2}})
\]

If \( h \) is a square, the pole at \( v = 0 \) is a double pole instead of a simple one, and we obtain:

\[
\lim_{\delta \to 0} (T_3 + T_4) = c_3 X \log X + c_4 X + O(X^{\frac{3}{2}})
\]

where in the above, \( c_2, c_3 \) and \( c_4 \) are computable constants. The residues at \( s = \frac{1}{4} - \frac{v}{2} \) and \( s = \frac{5}{4} - \frac{1}{4} \) can be dealt with similarly. \( \square \)

5. Spectral Expansion

The function \( P_{h,Y}(z, s; \delta) \) is in \( L^2(\Gamma_0(4) \backslash \mathbb{H}, -\frac{1}{2}) \); in fact it is compactly supported, so we can expand this function in the spectrum of the Laplacian \( \Delta_{-\frac{1}{2}} \). Let the \( u_j \) be an orthonormal basis of Maass forms i.e. eigenfunctions of the Laplacian which vanish at the cusps. We parametrize the eigenvalues of these Maass forms as \( 1/4 + t_j^2 \). Each Maass form has the Fourier expansion

\[
u_j(z) = \sum_{n \neq 0} \rho_j(n) n^{-\frac{1}{2}} W_{-\frac{1}{2}, n} \left( 4\pi |n| y \right) e^{2\pi i n x}.
\]
The continuous spectrum of $L^2(\Gamma_0(4) \backslash \mathbb{H}, -\frac{1}{2})$ is spanned by the Eisenstein series at various cusps, $E_{\theta}^{-\frac{1}{2}}(z, \frac{1}{2} + it)$ defined for $\text{Re} u > 1$ by

$$E_{\theta}^{-\frac{1}{2}}(z, u) = \sum_{\Gamma_u \setminus \Gamma_0(4)} (\text{Im} \sigma_{\theta}^{-1} \gamma z)^u \frac{j'(\gamma, z)}{|j(\gamma, z)|}.$$  

This Eisenstein series has a meromorphic continuation to the whole complex plane with a pole at $u = \frac{3}{4}$, and

$$\text{Res}_{u=\frac{3}{4}} E_{\theta}^{-\frac{1}{2}}(z, u) = c_{\theta} \theta(z) y^{\frac{1}{4}}$$

is the residual spectrum. Here $c_{\theta}$ is some constant. Then we can decompose the function $P$:

$$P_{h, Y}^{-\frac{1}{2}}(z, s, \delta) = \sum_j (P, u_j) u_j(z) + \langle P, \text{Im}(\cdot)^{-\frac{1}{4}} \rangle y^{\frac{1}{4}} \theta(z)$$

$$+ \frac{Y}{4\pi i} \sum_{n=\infty, 0} \int_{\pm\frac{1}{Y}} \langle P, E_{\theta}^{-\frac{1}{2}}(\cdot, u) \rangle E_{\theta}^{-\frac{1}{2}}(z, u) du.$$  

(26)

Now we see what each of these inner products is

$$\langle P, u_j \rangle = \int_{\Gamma_0(4) \setminus \mathbb{H}} P_{h, Y}^{-\frac{1}{2}}(z, s, \delta) u_j(z) \frac{dx dy}{y^2}$$

(27)

$$= \frac{\rho_j(-h)}{h^{\frac{s}{2}}} \frac{1}{(2\pi h)^{s-1}} \int_{\mathbb{H} \setminus Y^{-1}} \gamma^{-1} e^{\gamma(1-\delta)} W_{\frac{1}{4}, it_j}(2y) \frac{dy}{y}$$

$$= (2\pi)^{-(s-1)} \frac{\rho_j(-h)}{h^{\frac{s}{2}}} M_{Y, h, \frac{1}{2}}(s, t_j, \delta),$$

where the function $M_{Y, h, \frac{1}{2}}(s, t_j, \delta)$ is just defined as the integral on the second line of (27) above and is described in more detail in the appendix and in [Hul13]. When $\text{Re} s > \frac{1}{2} + \max_{t_j} |\text{Im}(t_j)|$, Proposition A.1 lets us take the limit as $Y \to \infty$ through the sum over $u_j$ in (26), so that the $M_{Y, h, \frac{1}{2}}(s, t_j, \delta)$ become $M_{\frac{1}{2}}(s, t_j, \delta)$.

We similarly compute

$$\langle P_{h, Y}^{-\frac{1}{2}}(\cdot, s, \delta), \text{Im}(\cdot)^{-\frac{1}{4}} \rangle = \int_{\Gamma_0(4) \setminus \mathbb{H}} P_{h, Y}^{-\frac{1}{2}}(z, s, \delta) \theta(z) y^{\frac{1}{4}} \frac{dx dy}{y^2}$$

(28)

$$= \int_{Y^{-1}}^{\infty} \int_0^1 y^{s-\frac{3}{4}} e^{-2\pi hy} e^{2\pi h(1-\delta)} \left( \sum_{n=0}^{\infty} \tau(n)e^{2\pi inz} \right) \frac{dy}{y}$$

$$= \frac{\tau(h)}{(2\pi \delta h)^{s-\frac{3}{4}}} \int_{2\pi h \delta Y^{-1}}^{2\pi h \delta Y} y^{s-\frac{3}{4}} e^{-y} \frac{dy}{y}$$

which, for $\text{Re}(s) > 3/4$, uniformly converges as $Y \to \infty$ to

$$\frac{\tau(h)}{(2\pi \delta h)^{s-\frac{3}{4}}} \Gamma(s - \frac{3}{4}).$$
We use the Fourier coefficients of Eisenstein series to compute the inner product of \( P \) and the Eisenstein series. By untiling we get that
\[
\langle P_{h,V}(\ast,s;\delta), E_a^{-\frac{1}{2}}(\ast,u) \rangle = \int_{Y^{-1}}^{Y} \int_{0}^{1} y^s e^{-2\pi h x} e^{2\pi h y (1-\delta)} E_a^{-\frac{1}{2}}(z,u) \frac{dx dy}{y^2}.
\]
Thus when \( \text{Re } s > \frac{1}{2} + |\text{Re } u - \frac{1}{2}| \), we have that
\[
\lim_{Y \to \infty} \langle P_{h,V}(\ast,s;\delta), E_a^{-\frac{1}{2}}(\ast,u) \rangle = i^{\frac{1}{2}} \frac{\pi u h^{u-1}}{\Gamma(u + \frac{3}{4})} D_a(u;h)(2\pi h)^{1-s} M_{\frac{1}{2}}(s, \frac{1}{12}(u - \frac{1}{2}), \delta),
\]
and Proposition A.1 allows us to take this limit through the integral in \( u \) in (26).

The spectral expansion of the Poincaré series given in (26), can now be used to complete our computation of the asymptotic expansion given in (23) by allowing us to compute
\[
I := \lim_{\delta \to 0} \left( \frac{1}{2\pi i} \right)^2 \int \int \lim_{Y \to \infty} \langle P, V_v \rangle \Gamma(\frac{s}{4}) (16\pi)^{s - \frac{1}{4}} X^{2s - \frac{1}{2}} ds dv.
\]
Indeed, following from (26) we have
\[
\langle P, V_v \rangle = \sum_{j} \langle P, u_j \rangle \langle u_j, V_v \rangle + \langle P, \text{Im}(\ast) \frac{1}{2}\theta \rangle \langle \text{Im}(\ast) \frac{1}{2}\theta, V_v \rangle + \frac{Y}{4\pi} \sum_{a = \infty, 0, \frac{1}{2}} \int_{-\infty}^{\infty} \langle P, E_a(\ast, \frac{1}{2} + it) \rangle \langle E_a(\ast, \frac{1}{2} + it), V_v \rangle dt.
\]
One useful fact to note is that we have as much polynomial decay in \( v \) for \( \langle u_j, V_v \rangle \) as is needed for convergence. This can be easily seen by writing down the Fourier expansion of \( V_v(z) \) and noting that each term has a factor of either \( \zeta^*(v) \), \( \zeta^*(1+v) \), \( K_{\frac{1}{2}}(y) \) or \( W_{\pm \frac{1}{2}, \frac{1}{4} - \frac{1}{2}}(y) \).

Now we can combine all the computations for the inner products of \( P \) with various eigenfunctions. Summarizing the results, we have
\[
\lim_{Y \to \infty} \langle P_{h,Y}(\ast,s;\delta), u_j \rangle = (2\pi)^{-(s-1)} \frac{\rho_j(-h)}{h^{s-\frac{1}{2}}} M_{\frac{3}{4}}(s,t_j,\delta)
\]
\[
\lim_{Y \to \infty} \langle P_{h,Y}(\ast,s;\delta), \text{Im}(\ast) \frac{1}{2}\theta \rangle = \frac{\tau(h)}{(2\pi \delta h)^{-\frac{3}{4}}} \Gamma(s - \frac{3}{4})
\]
\[
\lim_{Y \to \infty} \langle P_{h,Y}(\ast,s;\delta), E_a(\ast,u) \rangle = i^{\frac{1}{2}} \frac{\pi u h^{u-1}}{\Gamma(u + \frac{3}{4})} D_a(u;h)(2\pi h)^{1-s} M_{\frac{1}{2}}(s, \frac{1}{12}(u - \frac{1}{2}), \delta).
\]
Note that the \( s \) dependence in (30) is due to these terms. As was done with the Dirichlet series in the previous section, we substitute the spectral expansion for \( \langle P, V_v \rangle \) into \( I \) as given in (22).
Proposition 5.1. For the $I$ defined as in (22)

$$I = \tau(h)cX + O(X^{\frac{3}{2}}).$$

for some computable constant $c$ which is independent of $h$.

As a corollary we will get Theorem 1.2.

Proof. We begin by dealing with each component of the spectral expansion separately. By substituting (31) into the cuspidal part of (30) and inserting this into the definition of $I$ as given in (22), we get the integral

$$I_{\text{cusp}} := \lim_{\delta \to 0} \sum_j \left( \frac{1}{2\pi i} \right)^2 \int_Y \int_\mathbb{R} \lim_{Y \to \infty} \langle P_{h,Y}(s,\delta), u_j \rangle \langle u_j, V_v \rangle \times \Gamma(s + \frac{1}{4})(16\pi)^{s-\frac{3}{4}} X^{2s-\frac{3}{2}} ds dv$$

$$= \lim_{\delta \to 0} \sum_j \left( \frac{1}{2\pi i} \right)^2 \int_{(\frac{3}{2})(2)} \pi^s \rho_j(-h) 2^{2s-3h-\frac{3}{2}} M_{\frac{1}{2}}(s,t_j,\delta) \langle u_j, V_v \rangle \Gamma(s + \frac{1}{4}) X^{2s-\frac{3}{2}} ds dv.$$

Note that the poles in $s$ occur at $s = \frac{1}{2} \pm it_j$ where $\frac{1}{4} + t_j^2$ are eigenvalues of the Laplacian. So when we move the line of integration in $s$ left to $\text{Re}(s) = \frac{3}{8}$, picking up all the residues at $s = \frac{1}{2} \pm it_j$, we obtain

$$I_{\text{cusp}} = \sum_{\pm t_j} \frac{1}{2\pi i} \int_{(\frac{3}{2})} 4^{it_j} \pi^s \rho_j(-h) \Gamma(2it_j) \langle u_j, V_v \rangle \Gamma(\frac{3}{4} + it_j) X^{\frac{1}{2} + 2it_j} dv + O(X^{\frac{3}{2}}),$$

by using proposition A.2 for the residue calculation of $M_{\frac{1}{2}}(s,t_j,\delta)$ and letting $\delta \to 0$. Since the Eisenstein series is a component of $V_v$, we can compute $\langle u_j, V_v \rangle$ by general unfolding techniques to get that

$$\langle u_j, V_v \rangle \ll_{Re(v)} [(1 + |t_j + \text{Im}(\frac{v}{2})|)(1 + |t_j - \text{Im}(\frac{v}{2})|)]^{c_{Re(v)}} e^{-\frac{v}{2}||t_j|-|\text{Im}(\frac{v}{2})||},$$

where $c_{Re(v)}$ is dependent only on $\text{Re}(v)$, which gives us the sum and integral in $I_{\text{cusp}}$ are simultaneously absolutely convergent. We note that $I_{\text{cusp}}$ is $O(X^{\frac{3}{2}})$ as $t_j > 0$ for all $t_j$, which is to say that none of the $t_j$s are exceptional. Indeed, the Shimura correspondence for Maass forms, as investigated by Sarnak and Goldfeld in [Sar84], gives that each weight-$1/2$ Maass form on $\Gamma_0(4)$ and eigenvalue $t_j$ lifts to a weight-zero Maass form on $\Gamma_0(4)$ with eigenvalue $2t_j$. Since there are no exceptional eigenvalues for weight zero and level two, as is computed explicitly in [LMF], there are no exceptional eigenvalues for weight $1/2$ and level four.
Now we consider the residual spectrum.

\[ I_{\text{res}} := \lim_{\delta \to 0} \frac{1}{2\pi i} \int_{(\frac{1}{2})} \langle P_{h,Y}(z, s, \delta), \theta(z) y^{\frac{1}{4}}, V_\psi \rangle \times \Gamma \left( s + \frac{1}{4} \right) (16\pi)^{s-\frac{3}{4}} X^{2s-\frac{7}{4}} ds \, dv \]

\[ = \frac{1}{2\pi i} \int_{(\frac{1}{2})} \langle \theta(z) y^{\frac{1}{4}}, V_\psi \rangle (16\pi)^{\frac{3}{4}} X \, dv \]

\[ + \lim_{\delta \to 0} \frac{1}{2\pi i} \int_{(\frac{1}{2})} \frac{\tau(h)}{(\delta h)^{s-\frac{3}{4}}} \Gamma(s-\frac{3}{4}) \langle \theta(z) y^{\frac{1}{4}}, V_\psi \rangle \Gamma \left( s + \frac{1}{4} \right) (16\pi)^{s-\frac{3}{4}} X^{2s-\frac{7}{4}} ds \, dv \]

\[ = \tau(h) cX \]

for some constant \( c \). In fact we have the exact formula

\[ c = 2\pi \int_{(\frac{1}{2})} \langle \theta(z) y^{\frac{1}{4}}, V_\psi \rangle \, dv. \]

The residue of the Eisenstein series at \( s = 1 \) will not contribute anything, because the \( \delta \)th Fourier coefficient of \( I \), constant function is 0.

Finally we show that the continuous spectrum also adds a term of \( O(X^{\frac{1}{2}}) \) to \( I \), as defined in (22).

\[ I_{\text{cts}} := \lim_{\delta \to 0} \frac{\nu}{4\pi} \sum_{a=\infty,0} \int_{(\frac{1}{2})} \left( \frac{1}{2\pi i} \right)^2 \int_{(\frac{1}{2})} \langle P_{h,Y}(\ast, s, \delta), E_\alpha(\ast, u) \rangle \langle E_\alpha(\ast, u), V_\psi \rangle \times \Gamma \left( s + \frac{1}{4} \right) (16\pi)^{s-\frac{3}{4}} X^{2s-\frac{7}{4}} ds \, dv \, du \]

\[ = \frac{\nu}{4\pi} \sum_{a=\infty,0} \int_{(\frac{1}{2})} \left( \frac{1}{2\pi i} \right)^2 \int_{(\frac{1}{2})} \delta^{u \frac{3}{4} - \frac{1}{2}} D_a(u; h) (2\pi h)^{1-s} M_4(s, \frac{1}{4}(u-\frac{1}{2}), \delta) \times \langle E_\alpha(\ast, u), V_\psi \rangle \Gamma \left( s + \frac{1}{4} \right) (16\pi)^{s-\frac{3}{4}} X^{2s-\frac{7}{4}} ds \, dv \, du \]

Now we move the line of integration in \( u \) to \( \text{Re}(u) = 5/8 \). This separates the poles at \( s = u \) and \( s = 1 - u \). Then move the \( s \) line of integration down to \( \text{Re}(s) = 7/16 \). During the process we will pick up a pole from the \( M \) function at \( s = u \).

\[ I_{\text{cts}} = \lim_{\delta \to 0} \frac{\nu}{4\pi} \sum_{a=\infty,0} \int_{(\frac{1}{2})} \left( \frac{1}{2\pi i} \right)^2 \int_{(\frac{1}{2})} \delta^{u \frac{3}{4} - \frac{1}{2}} D_a(u; h) \left( \frac{\Gamma(2u-1)}{\Gamma(u-\frac{1}{4})} + O(\delta^{\frac{1}{4}}) \right) \times \langle E_\alpha(\ast, u), V_\psi \rangle 16^{u-\frac{3}{4}} X^{2u-\frac{7}{4}} dv \, du \]
\[ + \frac{V}{4\pi} \sum_{a=0}^{\infty} \int_{\frac{1}{2}}^{\infty} \frac{1}{2\pi i} \int \frac{\Gamma(u + \frac{1}{4})}{\Gamma(u + \frac{1}{2})} D_a(u; h) \frac{\Gamma(2u - 1)}{\Gamma(u - \frac{1}{4})} \times \langle E_a(s, s), V \rangle \pi \sum_{y} \frac{1}{2} ds \ dv \ du \]
\[ = \frac{V}{4\pi} \sum_{a=0}^{\infty} \int_{\frac{1}{2}}^{\infty} \frac{1}{2\pi i} \int \frac{\Gamma(u + \frac{1}{4})}{\Gamma(u + \frac{1}{2})} D_a(u; h) \frac{\Gamma(2u - 1)}{\Gamma(u - \frac{1}{4})} \times \langle E_a(s, s), V \rangle \pi \sum_{y} \frac{1}{2} ds \ dv \ du \]
\[ + \lim_{\delta \to 0} O(X^{1/2 + \delta}) + O(X^{1/2}) \]
\[ = O(X^{1/2}). \]

**Appendix A.**

Consider the functions

\[ M_{Y, h, k}(s, z/i, \delta) := \int_{Y - 12\pi h}^{Y + 12\pi h} y^{s-1} e^{y(1-\delta)} W_{\frac{Y}{2}, \delta} (2y) \ dy. \]  

and

\[ M_k(s, z/i, \delta) := \int_{0}^{\infty} y^{s-1} e^{y(1-\delta)} W_{\frac{Y}{2}, \delta} (2y) \ dy \]

for \( s, z \in \mathbb{C}, k \in \mathbb{R}, Y \gg 1, h \in \mathbb{Z}_{\geq 1} \) and small \( \delta > 0 \). These functions have been thoroughly investigated in [Hul13], further generalizing a construction first studied in [HH]. The following two propositions summarize the relevant properties of these functions used in this work, and their proofs can be found in [Hul13].

**Proposition A.1.** Let

\[ M_{Y, h, k}(s, z/i, \delta) := \int_{Y - 12\pi h}^{Y + 12\pi h} y^{s-1} e^{y(1-\delta)} W_{\frac{Y}{2}, \delta} (2y) \ dy. \]  

For fixed \( \varepsilon > 0, Y \gg 1, 1 > \delta > 0, \) and \( A \in \mathbb{Z}_{\geq 0}, \) we have that for \( \Re(s) > \frac{1}{2} + |\Re(z)| + \varepsilon \)

\[ |M_{Y, h, k}(s, z/i, \delta) - M_k(s, z/i, \delta)| \]

\[ \ll e^{-Y2\pi h \delta} (Yh)^{\Re(s) + \frac{1}{2} + A + \varepsilon - 2} \left( \frac{1 + |\Im(z)|}{1 + |\Im(z)|} \right)^{A} \]

where the implied constant is dependent on \( A, k, \Re(s), \Re(z) \) and \( \varepsilon \).

**Proposition A.2.** Fix small \( \varepsilon > 0 \) and \( \delta > 0 \), and let \( k \in \mathbb{R} \). Furthermore let \( s = \sigma + ir \) where \( \sigma, r \in \mathbb{R} \) and \( \Im(z) = t \). The function \( M_k(s, z/i, \delta) \) has a meromorphic continuation to all \( (s, z) \in \mathbb{C}^2 \) with simple polar lines at the points
and are as described above. The Laurent series around these double poles are of the form

$\frac{(-1)^{\ell} 2^\ell \Gamma(\frac{1}{2} + \ell \mp z) \Gamma(\frac{1}{2} + \ell - z)}{\ell! \Gamma(\frac{1}{2} - \frac{\ell}{2} + z) \Gamma(\frac{1}{2} - \frac{\ell}{2} - z)} + O_\ell, \Re z \left( (1 + |t|)^{-\ell \pm \frac{1}{2} - \Re z} e^{-r|t|} \delta^\ell \right) \tag{38}$

where $\ell \in \mathbb{Z}_{\geq 0}$ and $b < \min(-1, \frac{1}{2} - \sigma - \Re(z), -2 \Re(z))$. If $\ell \pm 2z \in \mathbb{Z}_{\geq 0}$ then $M_k(s, z/i, \delta)$ has a double pole at $s = \frac{1}{2} - \ell \mp z$. Otherwise the poles are simple and as described above. The Laurent series around these double poles are of the form

$M(s, z/i, \delta) = \frac{c_2^\pm(\ell, z, k) + O_{\ell, z, k}(\delta)}{(s - \frac{1}{2} + \ell \mp z)^2} + \frac{c_1^\pm(\ell, z, k) + O_{\ell, z, k}(\delta)}{(s - \frac{1}{2} + \ell \mp z)} + O_{\ell, z, k}(1) + O_{\ell, z, k}(\delta^{1-\varepsilon}). \tag{39}$

If we restrict $z$ to the region $0 < |\Re(z)| < \varepsilon$, we also have poles in $z$ of the form

$\frac{(-1)^{s-m} \Gamma(2s - m + 1) \Gamma(1 - s - \frac{k}{2})}{m! \Gamma(1 - s - m - \frac{k}{2}) \Gamma(s + m - \frac{k}{2})} + O_{\sigma, m, k, b} \left( \frac{\Gamma(2s + m - 1)}{\Gamma(s + m - \frac{k}{2})} (1 + |r|)^{1 - 2\sigma - b} \delta \right). \tag{40}$

when $s$ is near the line $\sigma = \frac{1}{2} - m$. These residues have a meromorphic continuation to $\sigma < \frac{1}{2} - m + \varepsilon$ that agrees with the representation above.

For $s$ and $z$ at least a distance of $\varepsilon > 0$ from the poles, there exists $A \in \mathbb{R}$, independent of $\delta$, $r$, and $t$, such that $A > 1 + |\sigma| + |\Re(z)| + |\frac{k}{2}|$ and

$M_k(s, z/i, \delta) \ll A_{\sigma, \varepsilon} \delta^{\frac{1}{4}} (1 + |t|)^{2\sigma - 2 - 2A + k}(1 + |r|)^{2A - 2b} e^{-\frac{\pi}{2}|r|}. \tag{41}$

For $\sigma < \frac{1}{2} - \varepsilon_0$ and $s$ at least a distance of $\varepsilon$ away from the poles of $M_k(s, z/i, \delta)$ and $\delta(1 + |t|)^2 \leq 1$ we have

$M_k(s, z/i, \delta) = \frac{2^{1-s} \Gamma(s - \frac{1}{2} - z) \Gamma(s - \frac{1}{2} + z) \Gamma(1 - s - \frac{1}{2})}{\Gamma(\frac{1}{2} - \frac{k}{2} + z) \Gamma(\frac{1}{2} - \frac{k}{2} - z)} + O_{A, b, \varepsilon_0} \left( (1 + |t|)^{2\sigma - 2 + k} (1 + |r|)^{2A - 2b} e^{-\frac{\pi}{2}|r|} \delta^{\varepsilon_0} \right). \tag{42}$

while for $\delta(1 + |t|)^2 > 1$ we have

$M_k(s, z/i, \delta) \ll A_{\sigma, \varepsilon} (1 + |t|)^{2\sigma - 2 + k}(1 + |r|)^{2A - 2b} e^{-\frac{\pi}{2}|r|}. \tag{43}$

When $\Re(z) = 0$ and $|t|, |r| \gg 1$, $|s \pm z - \frac{1}{2} - m| = \varepsilon > 0$, for $\varepsilon$ small, we have

$M_k(s, z/i, \delta) \ll_{A, b} \epsilon^{-1} \delta^{\frac{1}{4}} (1 + |r|)^{2A - 2b} e^{-\frac{\pi}{2}|r|}. \tag{44}$
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