Abstract

In this paper we propose a reduction scheme for polydifferential operators phrased in terms of $L_\infty$-morphisms. The desired reduction $L_\infty$-morphism has been obtained by applying an explicit version of the homotopy transfer theorem. Finally, we prove that the reduced star product induced by this reduction $L_\infty$-morphism and the reduced star product obtained via the formal Koszul complex are equivalent.
1 Introduction

This paper aims to propose a reduction scheme for equivariant polydifferential operators that is phrased in terms of $L_\infty$-morphisms, generalizing the results from [14], obtained for polyvector fields. Our main motivation comes from formal deformation quantization: Deformation quantization has been introduced by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer in [1] and it relies on the idea that the quantization of a phase space described by a Poisson manifold $M$ is described by a formal deformation, so-called star product, of the commutative algebra of smooth complex-valued functions $C_\infty(M)$ in a formal parameter $\hbar$. The existence and classification of star products on Poisson manifolds has been provided by Kontsevich’s formality theorem [22], whereas the invariant setting of Lie group actions has been treated by Dolgushev, see [10, 11]. More explicitly, the formality theorem provides an $L_\infty$-quasi-isomorphism between the differential graded Lie algebra (DGLA) of polyvector fields $T_{poly}(M)$ and the polydifferential operators $D_{poly}(M)$ resp. the invariant versions. As such, it maps in particular Maurer-Cartan elements in the DGLA of polyvector fields, i.e. (formal) Poisson structures, to Maurer-Cartan elements in the DGLA of polydifferential operators, which correspond to star products.

One open question and our main motivation is to investigate the compatibility of deformation quantization and phase space reduction in the Poisson setting, and in this present paper we propose a way to describe the reduction on the quantum side by an $L_\infty$-morphism. Given a Lie group $G$ acting on a manifold $M$, we aim to reduce equivariant star products $(\star, H)$, i.e. pairs consisting of an invariant star product $\star$ and a quantum momentum map $H = \sum_{r=0}^{\infty} \hbar^r J_r : g \rightarrow \mathcal{C}_\infty(M)[[\hbar]]$, where $g$ is the Lie algebra of $G$. In this case, $J_0$ is a classical momentum map for the Poisson structure induced by $\star$. Interpreting it as smooth map $J_0 : M \rightarrow g^*$ and assuming that $0 \in g^*$ is a value and regular value, it follows that $C = J^{-1}(\{0\})$ is a closed embedded submanifold of $M$ and by the Poisson version of the Marsden-Weinstein reduction [25] we know that under suitable assumptions the reduced manifold $M_{red} = C/G$ is again a Poisson manifold if the action on $C$ is proper and free. In this setting, there is a well-known BRST-like reduction procedure [3][13] of equivariant star products on $M$ to star products on $M_{red}$.

In order to describe this reduction by an $L_\infty$-morphism, we have to fix at first the DGLA controlling Hamiltonian actions in the quantum setting, i.e. a DGLA whose Maurer-Cartan elements correspond to equivariant star products. We denote it by

$$(D_g(M)[[\hbar]], \hbar \lambda, \partial^\hbar = [J_0, \cdot]_g, [\cdot, \cdot]_g),$$
where $\lambda = e^t \otimes (e_i)_M$ is given by the fundamental vector fields of the $G$-action in terms of a basis $e_1, \ldots, e_n$ of $\mathfrak{g}$ with dual basis $e^1, \ldots, e^n$ of $\mathfrak{g}^*$. It is called the DGLA of equivariant polydifferential operators.

The construction of the desired $L_\infty$-morphism to $(D_{\text{poly}}(M_{\text{red}}), \partial, [\cdot, \cdot]_G)$ is then based on the following steps:

- Assuming for simplicity $M = C \times \mathfrak{g}^*$, which always holds locally in suitable situations, we can perform a Taylor expansion around $C$ and end up with a DGLA $D_{\text{poly}}(C \times \mathfrak{g}^*)$. Using a 'partial homotopy', we find a deformation retract to a DGLA structure on the space $(\prod_{i=0}^{\infty} (S^i \mathfrak{g} \otimes D_{\text{poly}}(C)))^G$, i.e. we get rid of differentiations in $\mathfrak{g}^*$-direction.

- For the polyvector fields in [14] we used the canonical linear Poisson structure $\pi_{\text{Kosz}}$ on the dual of the action Lie algebroid $C \times \mathfrak{g}$ for the reduction. The analogue structure in our quantum setting is the product on the quantized universal enveloping algebra $U_h(C \times \mathfrak{g})$ of the action Lie algebroid. We use this product to perturb the deformation retract from the last point. This is more complicated as the polyvector field case since we have to use now the homological perturbation lemma to perturb the involved chain maps, and the deformed maps are no longer compatible with the Lie brackets.

- We use the homotopy transfer theorem to construct the $L_\infty$-projection from the Taylor expansion to $(\prod_{i=0}^{\infty} (S^i \mathfrak{g} \otimes D_{\text{poly}}(C)))^G$ with transferred $L_\infty$-structure. Notice that in the polyvector field case it was not necessary to transfer the DGLA structure.

- We check in Proposition 3.10 that the transferred $L_\infty$-structure is just a DGLA structure, and in Proposition 3.11 that the transferred Lie bracket is compatible with the projection to $D_{\text{poly}}(M_{\text{red}})[[\hbar]]$. Thus we get the reduction $L_\infty$-morphism from the Taylor expansion to the polydifferential operators on $M_{\text{red}}$. Twisting it by the product on the universal enveloping algebra ensures that we start in the right curved DGLA structure.

Finally, the morphism can be globalized to general smooth manifolds $M$ with sufficiently nice Lie group actions and we get the following result (Theorem 3.15):

**Theorem** There exists an $L_\infty$-morphism

$$D_{\text{red}} : (D_{\theta}(M)[[\hbar]], \hbar \lambda, \partial - [J_0, \cdot]_G, [\cdot, \cdot]_G) \longrightarrow (D_{\text{poly}}(M_{\text{red}})[[\hbar]], 0, \partial, [\cdot, \cdot]_G), \quad (1.1)$$

called reduction $L_\infty$-morphism.

Finally, we compare the reduction of equivariant star products via $D_{\text{red}}$ to a slightly modified version of the BRST reduction from [14][15], see Theorem 3.14:

**Theorem** Let $(\ast, H)$ be an equivariant star product on $M$. Then the reduced star product induced by $D_{\text{red}}$ from [14] and the reduced star product via the formal Koszul complex are equivalent.

Note that together with [14] Theorem 5.1 we have now the diagram:

$$\begin{align*}
(T^*_{\theta}(M)[[\hbar]], \hbar \lambda, [-J_0, \cdot]_G, [\cdot, \cdot]_G) & \xrightarrow{T_{\text{red}}} (D^*_{\theta}(M)[[\hbar]], \hbar \lambda, \partial - [J_0, \cdot]_G, [\cdot, \cdot]_G) \\
(T_{\text{poly}}(M_{\text{red}})[[\hbar]], 0, 0, [\cdot, \cdot]_S) & \xrightarrow{F_{\text{red}}} (D_{\text{poly}}(M_{\text{red}})[[\hbar]], 0, \partial, [\cdot, \cdot]_G),
\end{align*}$$

where $F_{\text{red}}$ is the standard Dolgushev formality with respect to a torsion-free covariant derivative on $M_{\text{red}}$. Moreover, in [16] we show that the Dolgushev formality is compatible with $\lambda$ under suitable flatness assumptions. In these flat cases it induces an $L_\infty$-morphism

$$F^G : (T^*_{\theta}(M)[[\hbar]], \hbar \lambda, [-J_0, \cdot]_G, [\cdot, \cdot]_G) \longrightarrow (D^*_{\theta}(M)[[\hbar]], \hbar \lambda, \partial - [J_0, \cdot]_G, [\cdot, \cdot]_G),$$

which gives the forth arrow in the above diagram, and we plan to investigate its commutativity (up to homotopy) in future work.
Example 2.1 (Curved DGLA)

A basic example of a (curved) differential graded Lie algebra \((g,R,d,[\cdot,\cdot])\) by setting \(Q_0(1) = -R\), \(Q_1 = -d\), \(Q_2(\gamma \lor \mu) = -(-1)^{[\gamma][\mu]}[\gamma,\mu]\) and \(Q_i = 0\) for all \(i \geq 3\). Note that we denoted by \(|\cdot|\) the degree in \(g[1]\).
Let us consider two $L_\infty$-algebras $(\mathfrak{L}, Q)$ and $(\tilde{\mathfrak{L}}, \tilde{Q})$. A degree 0 counital coalgebra morphism

$$F : S^c(\mathfrak{L}) \to S^c(\tilde{\mathfrak{L}})$$

such that $FQ = \tilde{Q}F$ is said to be an $L_\infty$-morphism. A coalgebra morphism $F$ from $S^c(\mathfrak{L})$ to $S^c(\tilde{\mathfrak{L}})$ such that $F(1) = 1$ is uniquely determined by its components (also called Taylor coefficients)

$$F_n : S^n(\mathfrak{L}[1]) \to \tilde{\mathfrak{L}}[1],$$

where $n \geq 1$. Namely, we set $F(1) = 1$ and use the formula

$$F(\gamma_1 \vee \ldots \vee \gamma_n) = \sum_{p \geq 1} \sum_{k_1, \ldots, k_p \geq 1} \sum_{\sigma \in \text{Sh}(k_1, \ldots, k_p)} \frac{e(\sigma)}{p!} F_{k_1}(\gamma_{\sigma(1)} \vee \ldots \vee \gamma_{\sigma(k_1)}) \vee \ldots \vee F_{k_p}(\gamma_{\sigma(n-k_p+1)} \vee \ldots \vee \gamma_{\sigma(n)}),$$

where $\text{Sh}(k_1, \ldots, k_p)$ denotes the set of $(k_1, \ldots, k_p)$-shuffles in $S_n$ (again we set $\text{Sh}(n) = \{\text{id}\}$). We also write $F_k = F^k$ and similarly to (2.3) we get coefficients $F^k_n : S^n \mathfrak{L}[1] \to S^k \tilde{\mathfrak{L}}[1]$ of $F$ by taking the corresponding terms in [17 Equation (2.15)]. Note that $F^k_n$ only depends on $F^1_k = F_k$ for $k \leq n - j + 1$. Given an $L_\infty$-morphism $F$ of (non-curved) $L_\infty$-algebras $(\mathfrak{L}, Q)$ and $(\tilde{\mathfrak{L}}, \tilde{Q})$, we obtain the map of complexes

$$F_1 : (\mathfrak{L}, Q_1) \to (\tilde{\mathfrak{L}}, \tilde{Q}_1).$$

In this case the $L_\infty$-morphism $F$ is called an $L_\infty$-quasi-isomorphism if $F_1$ is a quasi-isomorphism of complexes. Given a DGLA $(\mathfrak{g}, d, [\cdot, \cdot])$ and an element $\pi \in \mathfrak{g}[1]^0$ we can obtain a curved Lie algebra by defining a new differential $d + [\pi, \cdot]$ and considering the curvature $R^\pi = d\pi + \frac{1}{2}[\pi, \pi]$. In fact the same procedure can be applied to a curved Lie algebra $(\mathfrak{g}, R, d, [\cdot, \cdot], [\cdot, \cdot])$ to obtain the twisted curved Lie algebra $(\mathfrak{L}, R^\pi, d + [\pi, \cdot], [\cdot, \cdot])$, where

$$R^\pi := R + d\pi + \frac{1}{2}[\pi, \pi].$$

The element $\pi$ is called a Maurer–Cartan element if it satisfies the equation

$$R + d\pi + \frac{1}{2}[\pi, \pi] = 0.$$  

Finally, it is important to recall that given a DGLA morphism, or more generally an $L_\infty$-morphism, $F : \mathfrak{g} \to \mathfrak{g}'$ between two DGLAs, one may associate to any (curved) Maurer–Cartan element $\pi \in \mathfrak{g}[1]^0$ a (curved) Maurer–Cartan element

$$\pi_F := \sum_{n \geq 1} \frac{1}{n!} F_n(\pi \vee \ldots \vee \pi) \in \mathfrak{g}'[1]^0.$$  

In order to make sense of these infinite sums we consider DGLAs with complete descending filtrations

$$\cdots \supseteq \mathcal{F}^{-2}\mathfrak{g} \supseteq \mathcal{F}^{-1}\mathfrak{g} \supseteq \mathcal{F}^0\mathfrak{g} \supseteq \mathcal{F}^1\mathfrak{g} \supseteq \cdots, \quad \mathfrak{g} \cong \varprojlim \mathfrak{g}/\mathcal{F}^n\mathfrak{g}$$

and

$$d(\mathcal{F}^k\mathfrak{g}) \subseteq \mathcal{F}^{k+1}\mathfrak{g} \quad \text{and} \quad [\mathcal{F}^k\mathfrak{g}, \mathcal{F}^l\mathfrak{g}] \subseteq \mathcal{F}^{k+l}\mathfrak{g}.$$  

In particular, $\mathcal{F}^1\mathfrak{g}$ is a projective limit of nilpotent DGLAs. In most cases the filtration is bounded below, i.e. bounded from the left with $\mathfrak{g} = \mathcal{F}^k\mathfrak{g}$ for some $k \in \mathbb{Z}$. If the filtration is unbounded, then we assume always that it is in addition exhaustive, i.e. that

$$\mathfrak{g} = \bigcup_n \mathcal{F}^n\mathfrak{g},$$

even if we do not mention it explicitly. Moreover, we assume that the DGLA morphisms are compatible with the filtrations. Considering only Maurer–Cartan elements in $\mathcal{F}^1\mathfrak{g}^1$ ensures the
well-definedness of \((2.6)\). Mainly, the filtration is induced by formal power series in a formal parameter \(h\). Starting with a DGLA \((\mathfrak{g}, d, [\cdot, \cdot])\), its \(h\)-linear extension to formal power series \(\mathcal{E} = \mathfrak{g}[[h]]\) of a DGLA \(\mathfrak{g}\), has the complete descending filtration \(\mathfrak{F}^k \mathcal{E} = h^k \mathcal{E}\).

One can not only twist the DGLAs resp. \(L_\infty\)-algebras, but also the \(L_\infty\)-morphisms between them. Below we need the following result, see \([9, \text{Prop. 2}]\) and \([11, \text{Prop. 1}]\).

**Proposition 2.2** Let \(F: (\mathfrak{g}, Q) \to (\mathfrak{g}', Q')\) be an \(L_\infty\)-morphism of DGLAs, \(\pi \in \mathfrak{F}^1 \mathfrak{g}'\) a Maurer-Cartan element and \(S = F^\ast(\exp(\pi)) \in \mathfrak{F}^1 \mathfrak{g}'\).

i.) The map

\[
F^\ast = \exp(-S \triangledown) F \exp(\pi \triangledown): \mathcal{F}(\mathfrak{g}[1]) \to \mathcal{F}(\mathfrak{g}'[1])
\]

defines an \(L_\infty\)-morphism between the DGLAs \((\mathfrak{g}, d + [\pi, \cdot])\) and \((\mathfrak{g}', d + [S, \cdot])\).

ii.) The structure maps of \(F^\ast\) are given by

\[
F^\ast_n(x_1, \ldots, x_n) = \sum_{k=0}^n \frac{1}{k!} F_{n+k}(\pi, \ldots, \pi, x_1, \ldots, x_n).
\]

iii.) Let \(F\) be an \(L_\infty\)-quasi-isomorphism such that \(F^1\) is not only a quasi-isomorphism of filtered complexes \(L \to L'\) but even induces a quasi-isomorphism

\[
F^1: \mathfrak{F}^k L \to \mathfrak{F}^k L'
\]

for each \(k\). Then \(F^\ast\) is an \(L_\infty\)-quasi-isomorphism.

### 2.2 Equivariant Polydifferential Operators

In the following we present some basic results concerning equivariant polydifferential operators, which are basically folklore knowledge and are based on \([33]\).

Let us consider the DGLA of *polydifferential operators* on a smooth manifold \(M\)

\[
(D^\ast_{\text{poly}}(M), \partial = [\mu, \cdot]_G, [\cdot, \cdot]_G)
\]

(2.11)

Here

\[
D^\ast_{\text{poly}}(M) = \bigoplus_{n=1}^\infty D^n_{\text{poly}}(M)
\]

where \(D^n_{\text{poly}}(M) = \text{Hom}_{\text{diff}}(\mathcal{C}^\infty(M)^{\otimes n+1}, \mathcal{C}^\infty(M))\) are the differentiable Hochschild cochains vanishing on constants. We use the sign convention from \([5]\) for the Gerstenhaber bracket \([\cdot, \cdot]\), not the original one from \([17]\).

Explicitly

\[
[D, E]_G = (-1)^{|E||D|} \left( D \circ E - (-1)^{|D||E|} E \circ D \right)
\]

(2.12)

with

\[
D \circ E(a_0, \ldots, a_{d+e}) = \sum_{i=0}^{\lfloor D \rfloor} (-1)^{|E|D} D(a_0, \ldots, a_i-1, E(a_i, \ldots, a_{i+e}), a_{i+e+1}, \ldots, a_{d+e})
\]

(2.13)

for homogeneous \(D, E \in D^\ast_{\text{poly}}(M)\) and \(a_0, \ldots, a_{d+e} \in \mathcal{C}^\infty(M)\). Moreover, \(\mu\) denotes the commutative pointwise product on \(\mathcal{C}^\infty(M)[[h]]\) and \(\partial\) is the usual Hochschild differential.

We are interested in the case of group actions where we always consider a (left) action \(\Phi: G \times M \to M\) of a connected Lie group \(G\). Let \(M\) be now equipped with a \(G\)-invariant star product \(*\), i.e. an associative product \(* = \mu + \sum_{r=1}^\infty h^r C_r = \mu_0 + h m_* \in (D^1_{\text{poly}}(M))^{G[[h]]}\). Recall that a linear map \(H: \mathfrak{g} \to \mathcal{C}^\infty(M)[[h]]\) is called a quantum momentum map if

\[
\mathcal{L}_{\xi_M} = -\frac{1}{h} [H(\xi), \cdot]_* \quad \text{and} \quad \frac{1}{h} [H(\xi), H(\eta)]_* = H([\xi, \eta]),
\]

where \(\xi_M\) denotes the fundamental vector field corresponding to the action \(\Phi\).

A pair \((\ast, H)\) consisting of an invariant star product \(* = \mu + h m_*\) and a quantum momentum map \(H\) is also called *equivariant star product*. They are useful since they allow for a BRST like reduction scheme, compare Appendix \([\Delta]\) We introduce now the DGLA that contains the data of Hamiltonian actions, i.e. of equivariant star products. Here we follow \([33]\).
Definition 2.3 (Equivariant polydifferential operators) The DGLA of equivariant polydifferential operators \( (D^*_g(M), \partial^g, [\cdot, \cdot]_g) \) is defined by

\[
D^*_g(M) = \bigoplus_{2i+j+k} (S^i g^* \otimes D^j_{\text{poly}}(M))^G
\]

with bracket

\[
[a \otimes D_1, \beta \otimes D_2]_g = a \triangledown \beta \otimes [D_1, D_2]_G
\]

and differential

\[
\partial^g(a \otimes D_1) = a \otimes \partial D_1 = a \otimes [\mu, D_1]_G
\]

for \( a \otimes D_1, \beta \otimes D_2 \in D^*_g(M) \). Here we denote by \( \partial \) and \([\cdot, \cdot]_G\) the usual Hochschild differential and Gerstenhaber bracket on the polydifferential operators and by \( \mu \) the pointwise multiplication of \( \mathcal{C}^\infty(M) \).

Notice that invariance with respect to the group action means invariance under the transformations \( \text{Ad}_g^* \otimes \Phi_g^* \) for all \( g \in G \), and that the equivariant polydifferential operators can be interpreted as equivariant polynomial maps \( g \mapsto D_{\text{poly}}(M) \). We introduce the canonical linear map

\[
\lambda: g \ni \xi \mapsto \mathcal{L}_{\xi M} \in D^0_{\text{poly}}(M),
\]

and see that \( \lambda \in D^2_g(M) \) is central and moreover \( \partial^g \lambda = 0 \). This implies that we can see \( D^*_g(M) \) either as a flat DGLA with the above structures or as a curved DGLA with the above structures and curvature \( \lambda \). In the case of formal power series we rescale the curvature again by \( h^2 \) and obtain the following characterization of Maurer-Cartan elements:

Lemma 2.4 A curved formal Maurer-Cartan element \( \Pi \in hD^1_g(M)[[h]] \), i.e. an element \( \Pi \) satisfying

\[
h^2 \lambda + \partial^g \Pi + \frac{1}{2}[\Pi, \Pi]_g = 0,
\]

is equivalent to a pair \((m_*, H)\), where \( m_* \in D^1_{\text{poly}}(M)^G[[h]] \) defines a \( G \)-invariant star product via \( \ast = \mu + h m_* \) with quantum momentum map \( \hat{H}: g \to \mathcal{C}^\infty(M)[[h]] \). In other words, \((\ast, H)\) is an equivariant star product.

Proof: Let us decompose \( \Pi = h m_* - h H \in h(D^1_{\text{poly}}(M))^G \oplus (g^* \otimes D^{-1}_{\text{poly}}(M))^G[[h]] \). Then the curved Maurer-Cartan equation applied to an element \( \xi \in g \) reads

\[
-h^2 \mathcal{L}_{\xi M} = -h^2 \lambda(\xi) = \partial^g \Pi(\xi) + \frac{1}{2}[\Pi, \Pi]_g(\xi)
\]

\[
= h[\mu, m_*]_G + h^2 [m_*, m_*]_G - h^2 [m_*, H(\xi)]_G.
\]

This is equivalent to the fact that \( h m_* \) is Maurer-Cartan in the flat setting and that \( \mathcal{L}_{\xi M} = -\frac{1}{h}[H(\xi), \cdot]_G \), since \( h[\mu, m_*]_G(f) = -[H(\xi), f]_G \), for \( f \in \mathcal{C}^\infty(M) \). Then the invariance of both elements implies that \( \ast = \mu + h m_* \) is a \( G \)-invariant star product with quantum momentum map \( H \). \( \square \)

Two equivariant star products \( h(m_* - H) \) and \( h(m_*' - H') \) are called equivariantly equivalent if they are gauge equivalent, i.e. if there exists an \( hT \in hD^0_{\text{poly}}(M)^G[[h]] \subset D^0_g(M) \) such that

\[
h(m_*' - H') = \exp(h[T, \cdot]_g) \ast h(m_* - H) = \exp(h[T, \cdot]_g)(\mu + h(m_* - H)) - \mu.
\]

This means that \( S = \exp(hT) \) satisfies for all \( f, g \in \mathcal{C}^\infty(M)[[h]] \)

\[
S(f \ast g) = Sf \ast' Sg \quad \text{and} \quad SH = H'.
\]
3 Reduction of the Equivariant Polydifferential Operators

Now we aim to describe a reduction scheme for general equivariant polydifferential operators via an $L_\infty$-morphism denoted by $D_{\text{red}}$, generalizing the results for the polyvector fields from [14].

Let $M$ be a smooth manifold with action $\Phi: G \times M \to M$ of a connected Lie group and let $(\star, H = J + hJ')$ be an equivariant star product, i.e. a curved formal Maurer-Cartan element in the equivariant polydifferential operators, see Lemma [23]. Here the component $J: M \to \mathfrak{g}^*$ of the quantum momentum map $H$ in $h$-order zero is a classical momentum map with respect to the Poisson structure induced by the skew-symmetrization of the $h^1$-part of $\star$. We assume from now on that $0 \in \mathfrak{g}^*$ is a value and a regular zero of $J$ and set $C = J^{-1}\{0\}$. In addition, we require the action to be proper around $C$ and free on $C$. Then $M_{\text{red}} = C/G$ is a smooth manifold and we denote by $\iota: C \to M$ the inclusion and by $\text{pr}_\iota: C \to M_{\text{red}}$ the projection on the quotient. Moreover, the properness around $C$ implies that there exists an $G$-invariant open neighbourhood $M_{\text{nice}} \subseteq M$ of $C$ and a $G$-equivariant diffeomorphism $\Psi: M_{\text{nice}} \to U_{\text{nice}} \subseteq C \times \mathfrak{g}^*$, where $U_{\text{nice}}$ is an open neighbourhood of $C \times \{0\}$ in $C \times \mathfrak{g}^*$. Here the Lie group $G$ acts on $C \times \mathfrak{g}^*$ as $\Phi_g = \Phi_0^g \times \text{Ad}_{\mathfrak{g}^*}^{-1}$, where $\Phi_C$ is the induced action on $C$, and the momentum map on $U_{\text{nice}}$ is the projection to $\mathfrak{g}^*$ (see [2] Lemma 3, [13]).

From now on we assume $M = M_{\text{nice}}$. Then we can define an equivariant prolongation map by

$$\text{prol}: \mathcal{C}^\infty(C) \ni \phi \longmapsto (pr_1 \circ \Psi)^* \phi \in \mathcal{C}^\infty(M_{\text{nice}})$$

and we directly get $e^* \text{prol} = \text{id}_{\mathcal{C}^\infty(C)}$.

Consider the Taylor expansion around $C$ in $\mathfrak{g}^*$-direction as in [14] Section 4.1], which is a map

$$D_{\mathfrak{g}^*}: D^k_{\text{poly}}(C \times \mathfrak{g}^*) \longmapsto \prod_{i=0}^{\infty} (S_i \mathfrak{g} \otimes T^{k+1}(S\mathfrak{g}^*) \otimes D^k_{\text{poly}}(C)),$$

where $T^*(S\mathfrak{g}^*)$ denotes the tensor module of $S\mathfrak{g}^*$. Note that we are only interested in a subspace since we consider polydifferential operators vanishing on constants. Slightly abusing the notation, the Taylor expansion of the equivariant polydifferential operators takes then the following form

$$D_{\text{poly}}(C \times \mathfrak{g}^*) = \left(S\mathfrak{g}^* \otimes \prod_{i=0}^{\infty} (S_i \mathfrak{g} \otimes T(S\mathfrak{g}^*) \otimes D_{\text{poly}}(C)) \right)^G$$

and one easily checks that this yields an equivariant DGLA morphism

$$D_{\mathfrak{g}^*}: (D_{\mathfrak{g}}(M), \lambda, \partial, \cdot, \cdot, \cdot^g) \longrightarrow (D_{\text{poly}}(C \times \mathfrak{g}^*), \lambda, \partial, \cdot, \cdot, \cdot^g).$$

Our goal consists in finding a reduction morphism from

$$D_{\text{red}}: (D_{\text{poly}}(C \times \mathfrak{g}^*)[[h]], h\lambda, \partial + [-J, \cdot, \cdot, \cdot]_g) \longrightarrow (D_{\text{poly}}(M_{\text{red}})[[h]], \partial, \cdot, \cdot, \cdot)_G.$$ (3.3)

Following a similar strategy as in [14], we construct $L_\infty$-morphisms

$$D_{\text{poly}}(C \times \mathfrak{g}^*)[[h]] \longrightarrow \left(\prod_{i=0}^{\infty} (S_i \mathfrak{g} \otimes D_{\text{poly}}(C)) \right)^G[[h]] \longrightarrow D_{\text{poly}}(M_{\text{red}})[[h]]$$

with suitable $L_\infty$-structures on the three spaces, where $(\prod_{i=0}^{\infty} (S_i \mathfrak{g} \otimes D_{\text{poly}}(C)))^G[[h]]$ is a candidate for a Cartan model.

3.1 A 'partial' Homotopy for the Hochschild Differential

In order to find a suitable analogue of the Cartan model for the polydifferential operators, we need to understand the cohomology of

$$(D_{\mathfrak{g}}(M), \partial^g - [J, \cdot, \cdot, \cdot]_g)$$
and in particular the role of the differential \([-J, \cdot]\). To this end we construct a 'partial' homotopy for \(\partial^g - [J, \cdot]_g\). Here we use the results concerning the homotopy for the Hochschild differential from \([8]\). In particular, we restrict ourselves to the subspace of normalized differential Hochschild cochains, i.e. polydifferential operators vanishing on constants. One can show that they are quasi-isomorphic to the differential ones. Recall the maps

\[
\Phi: D^a_{\text{poly}}(M) \longrightarrow D^{a-1}_{\text{poly}}(M)
\]

\[
\Phi(A)(f_0, \ldots, f_{a-1}) = \sum_{t=1}^n \sum_{i \leq j < a} (-1)^j A\left(f_0, \ldots, f_{i-1}, f_t, \ldots, f_{j-1}, f_{j+1}, \ldots, f_{a-1}\right)
\]

for \(f_1, \ldots, f_{a-1} \in \mathcal{E}^\infty(M)\) and

\[
\Psi: D^a_{\text{poly}}(M) \ni A \longmapsto (-1)^a [x^1, A]_G \cup \frac{\partial}{\partial x^1} = (-1)^{a+1} \sum_{i=1}^n (A \circ e^i) \cup \frac{\partial}{\partial e^i} \in D^a_{\text{poly}}(M)
\]

for local coordinates \((x^1, \ldots, x^n)\) of \(M\). They satisfy, by \([8\) Proposition 4.1], the following condition:

\[
\Phi \circ \partial + \partial \circ \Phi = -(\text{deg}_D \cdot \text{id} + \Psi),
\]

where \(\text{deg}_D\) is the order of the differential operator.

We assume from now on for simplicity \(M = C \times g^*\) and \(J = \text{pr}_+\) and we want to find a suitable Cartan model for the polydifferential operators. Similarly to \([14\) Definition 4.14] for the polyvector field case, we want to obtain a DGLA structure on

\[
\left(\prod_{i=0}^\infty (S^g \otimes D_{\text{poly}}(C))\right)^G.
\]

Hence we adapt the maps \(\Phi\) and \(\Psi\) in such a way that they only include coordinates \(J_i = \alpha_i = e_i\) on \(g^*\) with \(i = 1, \ldots, n:\)

\[
\Phi(A)(f_0, \ldots, f_{a-1}) = \sum_{t=1}^n \sum_{i \leq j < a} (-1)^j A\left(f_0, \ldots, f_{i-1}, e_t, \ldots, f_{j-1}, f_{j+1}, \ldots, f_{a-1}\right),
\]

\[
\Psi(A) = (-1)^{a+1} \sum_{i=1}^n (A \circ e_i) \cup \frac{\partial}{\partial e_i},
\]

where \(A \in D^a_{\text{poly}}(C \times g^*)\) and \(f_0, \ldots, f_{a-1} \in \mathcal{E}^\infty(C \times g^*)\).

**Proposition 3.1** One has on \(D_{\text{poly}}(C \times g^*)\)

\[
\Phi \circ \partial + \partial \circ \Phi = -(\text{deg}_g \cdot \text{id} + \Psi),
\]

where \(\text{deg}_g\) is the order of differentiations in direction of \(g^*-coordinates).

**Proof:** The proof follows the same lines as in \([8\) Proposition 4.1]. It is proven by induction on the degree of \(a\) of \(A \in D^a_{\text{poly}}(C \times g^*)\). For \(a = 0\) and \(a \in D^a_{\text{poly}}(C \times g^*)\) as well as \(f \in \mathcal{E}^\infty(C \times g^*)\) we get

\[
((\Phi \circ \partial + \partial \circ \Phi)(A))(f) = (\partial A)\left(e_i \frac{\partial}{\partial e_i} f\right) = e_i A\left(\frac{\partial}{\partial e_i} f\right) - A\left(e_i \frac{\partial}{\partial e_i} f\right) + A(e_i) \frac{\partial}{\partial e_i} f
\]

\[
= (-\text{deg}_g(A) - \Psi(A))(f).
\]

Note that \(\Psi\) has the following compatibility with the \(\cup\)-product:

\[
\Psi(A \cup B) = (\Psi A) \cup B + A \cup (\Psi B) + (-1)^a(A \circ e_i) \cup \left(\frac{\partial}{\partial e_i} \cup B + (-1)^b B \cup \frac{\partial}{\partial e_i}\right).
\]
From (3.7) we know deg

Proposition 3.2 polydifferential operators, where we can show:

on these generators with Φ fields. Finally, note that ˆferential operators, where we restrict ourselves again to polydifferential operators vanishing on □ Thus the proposition is shown.

As in [14], we define a homotopy on the equivariant polydifferential operators

The operators (i(ei) ◦ ∂ + ∂ ◦ i(ei)) and (∂ ◦ i(∂ ei) − i (∂ ei) ◦ ∂) are graded commutators of derivations of the ∪-product and therefore themselves graded derivations. Thus they are determined by their action on D−1poly(C × g∗) and D0poly(C × g∗). The first one obviously vanishes. The second coincides on these generators with

A ↦→ −(∂ ei A + (−1)a A ∪ ∂ ei)

and the proposition is shown. □

As in [14], we define a homotopy on the equivariant polydifferential operators

The fact that ˆh maps invariant elements to invariant ones follows as in the case of polyvector fields. Finally, note that Φ and Ψ are equivariant, whence they can be extended to the equivariant polydifferential operators, where we can show:

Proposition 3.2 One has on (Sg∗ ⊗ Dpoly(C × g∗))G

where deg g is again the order of differentiations in direction of g∗-coordinates.

Proof: From (3.7) we know [Φ, ∂θ] = −(deg g · id + Ψ). In addition, one has for homogeneous P ⊗ D

Moreover, since we consider only differential operators vanishing on constants one checks easily that also [Φ, [−J, · ]g] = 0. Finally,

Thus the proposition is shown. □

The above contructions work also for the Taylor series expansion of the equivariant polydifferential operators, where we restrict ourselves again to polydifferential operators vanishing on
constants. We slightly abuse the notation and denote them again by $D_{\tau_0}(C \times g^*)$, compare (5.1). Writing
\[
 h = \begin{cases} 
 \frac{1}{\omega_{S^2 g^*} + \deg g} (\hat{h} - \Phi) & \text{if } \deg_{S^2 g^*} + \deg g \neq 0, \\
 0 & \text{else,}
\end{cases}
\tag{3.9}
\]
we get the following result:

**Proposition 3.3** One has a deformation retract
\[
\left( (\prod_{i=0}^{\infty} (S^i g \otimes D_{poly}(C)))^g[[h]], \partial \right) \xrightarrow{\pi} (D_{\tau_0 y}(C \times g^*)[[h]], \partial + [-J, \cdot]), \xrightarrow{\iota} h \tag{3.10}
\]
where $\pi$ and $\iota$ denote the obvious projection and inclusion. This means that one has $\pi \iota = \text{id}$ and $\text{id} - ip = [h, \partial + [-J, \cdot]]$. Moreover, the identities $hi = 0 = ph$ hold.

**Remark 3.4** Note that one has $h^2 \neq 0$, i.e. the above retract is no special deformation retract. However, by the results of [21, Remark 2.1] we know that this could also be achieved.

The reduction works now in two steps: At first, we use the homological perturbation lemma from Lemma [A.1] to deform the differential on $D_{\tau_0 y}(C \times g^*)[[h]]$, and in a second step we use the homotopy transfer theorem, see Theorem [B.2] to extend the deformed projection to an $L_\infty$-morphism. This will possibly give us higher brackets on $(\prod_{i=0}^{\infty} (S^i g \otimes D_{poly}(C)))^g[[h]]$ that we have to discuss.

### 3.2 Application of the Homological Perturbation Lemma

In our setting, the bundle $C \times g \to C$ can be equipped with the structure of a Lie algebroid since $g$ acts on $C$ by the fundamental vector fields. The bracket of this action Lie algebroid is given by
\[
[\xi, \eta]_{C \times g}(p) = [\xi(p), \eta(p)] - (L_{\xi_C} \eta)(p) + (L_{\eta_C} \xi)(p) \tag{3.11}
\]
for $\xi, \eta \in \mathcal{C}^\infty(C \times g)$. The anchor is given by $\rho(p, \xi) = -\xi_C p |_p$. In particular, one can check that $\pi_{\text{ker}}$ is the negative of the linear Poisson structure on its dual $C \times g^*$ in the convention of [30].

For Lie algebroids there is a well-known construction of universal enveloping algebras [29,30,33]. It turns out that in our special case we get a simpler description of the universal enveloping algebra:

**Proposition 3.5** The universal enveloping algebra $U(C \times g)$ of the action Lie algebroid $C \times g$ is isomorphic to $\mathcal{C}^\infty(C) \rtimes U(g)$ with product
\[
(f, x) \cdot (g, y) = \sum (f L(x_{(1)})(g), x_{(2)}y). \tag{3.12}
\]
Here $y_{(1)} \otimes y_{(2)} = \Delta(y)$ denotes the coproduct on $U(g)$ induced by extending $\Delta(\xi) = 1 \otimes \xi + \xi \otimes 1$ as an algebra morphism. Moreover, $L: U(g) \to \text{DiffOp}(\mathcal{C}^\infty(C))$ is the extension of the anchor of the action algebroid, i.e. of the negative fundamental vector fields, to the universal enveloping algebra. The same holds also in the formal setting of $U_h(g)$ with bracket rescaled by $h$. Note that in this case one has to rescale $L$ by powers of $h$, i.e. $L_\xi = -h L_{\xi_C}$ for $\xi \in g$.

**Proof:** Note that the product is associative since
\[
((f, x) \cdot (g, y)) \cdot (h, z) = \sum (f L(x_{(1)})(g), x_{(2)}y) \cdot (h, z) = \sum (f L(x_{(1)})(g) L(x_{(2)})(y_{(1)}h), x_{(3)}y_{(2)}z)
\]
\[
= \sum (f, x) \cdot (g L(y_{(1)}h), y_{(2)}z) = (f, x) \cdot ((g, y) \cdot (h, z)),
\]
where the penultimate identity follows with the coassociativity of $\Delta$ and the identity $L(x)(fg) = L_{x_{(1)}}(f)L_{x_{(2)}}(g)$. Note that the inclusions $\kappa_C: \mathcal{C}^\infty(C) \to \mathcal{C}^\infty(C) \rtimes U(g)$ and $\kappa: \mathcal{C}^\infty(C) \otimes g \to \mathcal{C}^\infty(C) \rtimes U(g)$ satisfy
\[
[\kappa(s), \kappa_C(f)] = \kappa(\rho(s)f), \quad \text{and} \quad \kappa_C(f)\kappa(s) = \kappa(fs).
\]
Thus the universal property gives the desired morphism $U(C \times g) \to \mathcal{C}_\infty(C) \otimes U(g)$. Recursively we can show that the right hand side is generated by $u \in \mathcal{C}_\infty(C)$ and $\xi \in \mathcal{C}_\infty(C) \otimes g$ which gives the surjectivity of the morphism. Concerning injectivity, suppose $(f^{i_1}, e_{i_1}) \cdots (f^{i_n}, e_{i_n}) = 0$ in $\mathcal{C}_\infty(C) \times U(g)$. We have to show that also $(f^{i_1}e_{i_1}) \cdots (f^{i_n}e_{i_n}) = 0$ in $U(C \times g)$. But this follows from a direct comparison of the terms in the corresponding associated graded algebras. □

Recall that by the Poincaré-Birkhoff-Witt Theorem the map

$$S(g) \ni x_1 \vee \ldots \vee x_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(n)} \in U(g)$$

is a coalgebra isomorphism with respect to the usual coalgebra structures induced by extending $\Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi$ for $\xi \in g$, see e.g. [2, 19]. This statement holds also in the case of formal power series in $h$ whence we can transfer the product on the universal enveloping algebra as in Proposition 3.5 to an associative product $\ast$.

Proposition 3.7 to an associative product $\ast$

$\text{power series in } \hbar$ we have with the above lemma with $B$

we can show that the right hand side is generated by $\tilde{\Phi}$ is the combination of $\Phi$ with the degree-counting coefficient from $h$ from (3.9). We want to take a closer look at the induced differential:

$$\text{Lemma 3.6 The Gutt product } \ast_G \text{ on } \mathcal{C}_\infty(C) \otimes S(g)[[h]] \text{ is } G\text{-invariant and } J = \text{pr}_g : M = C \times g^* \to g^* \text{ is a momentum map, i.e.}$$

$$-\mathcal{L}_{\xi_M} = \frac{1}{\hbar} \text{ad}_{\ast_G}(J(\xi)). \quad (3.13)$$

Proof: Both statements follow directly from the explicit formula in Proposition 3.5. □

We deform the differential $\partial + [-J, \cdot]$ by $[hm_G, \cdot]$, i.e. exactly by the higher orders of this product. The perturbed differential $\partial^\ast + [hm_G - J, \cdot] = [\ast_G - J, \cdot]$ squares indeed to zero since we have with the above lemma

$$[\ast_G - J, \cdot]^2 = \frac{1}{2} [\ast_G - J, \ast_G - J], \cdot] = [-h\lambda, \cdot] = 0,$$

where again $\lambda = e^t \otimes (e_i)_M$. By the homological perturbation lemma as formulated in Section 3.1 this yields a homotopy retract

$$\begin{array}{c}
\left((\prod_{i=0}^\infty (S^i g \otimes D_{\text{poly}}(C)))^G \right)[[h]], \partial_h \rightleftharpoons \left((D_{\text{poly}}(C \times g^*)[[h]], [\ast_G - J, \cdot]), \rightleftharpoons h_n
\end{array}$$

with $B = [hm_G, \cdot]$ and

$$A = (\text{id} + Bh)^{-1}B, \quad \partial_h = \partial + pAi, \quad i_h = i - hAi,$$

$$p_h = p - pAh, \quad h_h = h - hAh, \quad (3.15)$$

compare Lemma A.1. More explicitly, we have

$$i_h = \sum_{k=0}^\infty (\tilde{\Phi} \circ B)^k \circ i, \quad \text{ and } \quad h_h = h \circ \sum_{k=0}^\infty (-Bh)^k, \quad (3.16)$$

where $\tilde{\Phi}$ is the the combination of $\Phi$ with the degree-counting coefficient from $h$ from (3.9). We want to take a closer look at the induced differential:

Proposition 3.7 One has

$$p_h = p \quad \text{and} \quad \partial_h = \partial + \delta$$

with

$$\delta(P \otimes D) = (-1)^d P(1) \otimes D \cup \mathcal{L}_{P(0)} - (-1)^d P \otimes D \cup \text{id}$$

for homogeneous $P \otimes D \in Sg \otimes D\text{poly}(C)$. 

\[12\]
Proof: The fact that \( p_h = p \) follows since \( B_h \) always adds differentials in \( g \)-direction. For the deformed differential we compute for homogeneous \( P \otimes D \in S g \otimes D^d_{poly}(C) \) and \( f_i \in \mathcal{C}^\infty(C) \)

\[
(\delta(P \otimes D))(f_0, f_1, \ldots, f_{d+1}) = (p \circ \sum_{k=0} \phi_k B \circ i(P \otimes D))(f_0, f_1, \ldots, f_{d+1})
\]

\[
= p(B(P \otimes D)(f_0, f_1, \ldots, f_{d+1}))
\]

\[
= (-1)^d p(hm_C(P \otimes D(f_0, \ldots, f_d), f_{d+1})
\]

\[
= (-1)^d P(1) \otimes D(f_0, \ldots, f_d) \cdot \mathcal{L}_{P(2)} f_{d+1}
\]

for all \( P(2) \neq 1 \). Here we used the explicit form of the Gutt product as in Proposition 3.8 and the fact that \( S(g)[[\hbar]] \) and \( U_h(\mathfrak{g}) \) are isomorphic coalgebras. \( \square \)

Since the classical homotopy equivalence data \( (3.10) \) is no special deformation retract, the perturbed one is also no special one. But it still has some nice properties.

Proposition 3.8 One has

\[
p_h \circ h_h = 0 = h_h \circ i_h \quad \text{and} \quad p_h \circ i_h = \text{id}.
\]

Proof: The properties follow from \( p \circ h = 0 = h \circ i, p \circ i = \text{id} \) and \( \tilde{\Phi}^2 = 0 \). \( \square \)

Thus the deformation retract \( (3.14) \) satisfies all properties of a special deformation retract except for \( h_h \circ h_h = 0 \), and we can still apply the homotopy transfer theorem.

3.3 Application of the Homotopy Transfer Theorem

We use the homotopy transfer theorem to extend \( p_h \) to an \( L_\infty \)-morphism. We denote the \( L_\infty \)-structure on the Taylor expansion by \( Q \) and the extension of \( h_h \) to the symmetric algebra as in \( (3.2) \) by \( H \). Then applying the homotopy transfer theorem in the form of Theorem 3.2 to the deformation retract \( (3.14) \) induces higher brackets \( (Q_C)_k^1 \) on \( (\prod_{i=0}^\infty (S^i g \otimes D_{poly}(C)))^G[[\hbar]][1] \):

Proposition 3.9 The maps

\[
(Q_C)_1^1 = - \partial_h,
\]

\[
(Q_C)_{k+1}^1 = P_k^1 \circ Q_{k+1}^k \circ i_h^{\vee(k+1)},
\]

where

\[
P_1^1 = p_h = p,
\]

\[
P_{k+1}^1 = \sum_{\ell=2}^{k+1} Q_{\ell} \circ P_{k+1}^1 - P_k^1 \circ Q_{k+1}^k \big) \circ H_{k+1} \quad \text{for} \quad k \geq 1,
\]

induce a codifferential \( Q_C \) on the symmetric coalgebra of \( (\prod_{i=0}^\infty (S^i g \otimes D_{poly}(C)))^G[[\hbar]][1] \) and an \( L_\infty \)-quasi-isomorphism \( P \)

\[
P: (D_{\text{poly}}(C \times g^*))[\hbar], [x_G - J, \cdot], [..., \cdot] \longrightarrow \left( \prod_{i=0}^\infty (S^i g \otimes D_{poly}(C))^G[\hbar], Q_C \right).
\]

Proof: The proposition follows directly from homotopy transfer theorem as in Theorem 3.2. Note that we do indeed not need \( h_h \circ h_h = 0 \), only the other properties of a special deformation retract from Proposition 3.8. \( \square \)

Let us take a closer look at the higher brackets \( Q_C \) induced by the homotopy transfer theorem. One can check that they vanish:

Proposition 3.10 One has

\[
(Q_C)_{k+1}^1 = 0 \quad \forall k \geq 2.
\]
Proof: In the higher brackets with $k \geq 2$ one has

$$H_k \circ Q^k_{k+1} \circ i^\nu(k+1),$$

where in $H_k$ one component consists of the application of $\Phi$, i.e., contains an insertion of a linear coordinate function $e_i$. We claim that it has to vanish. At first, it is clear that the image of $i$ vanishes if one argument is $e_i$. Let us now show that $i_h$ satisfies the same property, which directly gives the proposition since then also the bracket vanishes if one inserts a $g^*$-coordinate.

We can compute for homogeneous $D \in D^a_{red}(C \times g^*)$ and $f_0, \ldots, f_d \in \prod_i (S'[g \otimes \mathcal{C}^\infty(C))]$

$$\Phi \circ B(D)(f_0, \ldots, f_d) = \sum_{i=1}^n \sum_{i \leq j < d+1} \left( -1 \right)^i h m_G(f_0, D(f_1, \ldots, f_{i-1}, e_i, \ldots, \frac{\partial}{\partial e_t} f_j, \ldots, f_d) )$$

$$= \sum_{i=1}^n \sum_{i \leq j < d+1} \left( -1 \right)^i (h m_G(f_0, D(f_1, \ldots, f_{i-1}, e_i, \ldots, \frac{\partial}{\partial e_t} f_j, \ldots, f_d) )$$

$$- D(h m_G(f_0, f_1), \ldots, f_{i-1}, e_i, \ldots, \frac{\partial}{\partial e_t} f_j, \ldots, f_d) + \ldots$$

$$+ (-1)^d h m_G(D(f_0, \ldots, f_{i-1}, e_i, \ldots, \frac{\partial}{\partial e_t} f_j, \ldots, f_{d-1}), f_d) ).$$

If $D$ vanishes if one of the arguments is a $g^*$-coordinate, then this simplifies to

$$\Phi \circ B(D)(f_0, \ldots, f_d) = \sum_{j=0}^d h m_G(e_t, D(f_0, \ldots, f_{i-1}, \ldots, \frac{\partial}{\partial e_t} f_j, \ldots, f_d) )$$

$$- D(h m_G(e_t, f_0), \ldots, f_{i-1}, \ldots, \frac{\partial}{\partial e_t} f_j, \ldots, f_d) + \sum_{j=1}^d D(h m_G(f_0, e_t), \ldots, f_{i-1}, \ldots, \frac{\partial}{\partial e_t} f_j, \ldots, f_d) + \ldots,$$

where $e_i$ is always an argument of $h m_G$. In particular, we know $h m_G(e_i, e_j) = \frac{h}{2}[e_i, e_j]$ and we see that the above sum vanishes if one of the functions $f_i$ is a $g^*$-coordinate, i.e., $\Phi \circ B(D)$ has the same vanishing property as $D$. The same holds for $\Phi \circ B(D)$ and this shows by induction that the image of $i_h$ has the same property and the proposition is shown.

Considering $(Q_C)_1^2$, we can simplify (3.19) to

$$(Q_C)_1^2 = \sum_{k=1}^\infty p \circ Q^k_1 \circ ( (\Phi \circ B)^k \circ i \lor i + i \lor (\Phi \circ B)^k \circ i ) + p \circ Q^k_2 \circ ( i \lor i ),$$

where the last term is the usual Gerstenhaber bracket. This is clear since $\Phi$ adds a differential in $g^*$-direction and the bracket can only eliminate it on one argument. Recall that we also have the canonical projection $\text{pr}: (\prod_{i=0}^\infty (S'[g \otimes D_{poly}(C)^G])) \rightarrow D_{poly}(M_{red})$ which projects first to symmetric degree zero and then restricts to $c\mathcal{C}^\infty(C)^G \cong c\mathcal{C}^\infty(M_{red})$. It is a DGLA morphism with respect to classical structures, i.e., Hochschild differentials and Gerstenhaber brackets. We extend it $h$-linearly and can show that it is also a DLGA morphism with respect to the deformed DGLA structure $Q_C$:

**Proposition 3.11** The projection induces a DGLA morphism

$$\text{pr}: \left( \prod_{i=0}^\infty (S'[g \otimes D_{poly}(C)^G])^G \right)^{[[h]], Q_C} \rightarrow (D_{poly}(M_{red})[[h]], \partial, [\cdot, \cdot]_G).$$

(3.22)
Proof: By the explicit form of the differential \((Q_C)^1\) we know that \(\pr \circ Q = \pr \circ \partial = \partial \circ \pr\). Thus it only remains to show that we have \(\pr \circ (Q_C)^2 = Q_2^1 \circ \pr \circ \partial\), which is equivalent to showing

\[
\pr \circ \sum_{k=1}^{\infty} p \circ Q_2^k \circ \left( (\tilde{\Phi} \circ B)^k \circ i \lor i \lor (\tilde{\Phi} \circ B)^k \circ i \right) = 0. \tag{*}
\]

In the proof of Proposition 3.10 we computed \(\Phi \circ B(D)\) of some \(D \in D^{d}_{	ext{top}}(C \times \mathfrak{g}^*)\) and we saw that the image of \(i\) vanishes if one inserts a \(\mathfrak{g}^*\)-coordinate and that \(\Phi \circ B\) preserves this property. Therefore, we get for such a \(D\) that vanishes if one of the arguments is \(e_i\)

\[
\Phi \circ B(D)(f_0, \ldots, f_d) = \sum_{j=0}^{d} (hm_G(e_t, D(f_0, \ldots, f_{i-1}, \partial D_j, f_j, \ldots, f_d))
- D(hm_G(e_t, f_0), \ldots, f_{i-1}, \partial D_j, f_j, \ldots, f_d))
+ \sum_{j=1}^{d} D(hm_G(e_t, f_0, e_t), \ldots, f_{i-1}, \partial D_j, f_j, \ldots, f_d) - \ldots
- D(f_0, \ldots, f_{i-1}, \partial D_j, f_j, f_d)
\]

where \(f_0, \ldots, f_d \in \prod (S^t \mathfrak{g} \otimes \mathfrak{c}^{\infty} (C))\). Let us consider now applied to homogeneous \(P \otimes D \lor Q \otimes D'\), where \(P, Q \in S^1 \mathfrak{g}\) and \(D, D' \in D^{d}_{\text{poly}}(C)[[h]]\). At first we note that this is zero if both \(P \neq 1 \neq Q\) since the Gerstenhaber bracket can cancel at most one term. Similarly, it is zero if both \(P = 1 = Q\). Thus we consider w.l.o.g. \(D, Q \otimes D'\) with \(Q \neq 1\) and \(D \in (D^{d}_{\text{poly}}(C))^G[[h]]\), where the only possible contributions are

\[
\pr \circ \pr \circ Q_2^k \left( (\tilde{\Phi} \circ B)^k \circ (Q \otimes D') \right) = (-1)^{d+(dd')} \pr \circ \pr \circ \left( (\tilde{\Phi} \circ B)^k \circ (Q \otimes D') \right)
\]

for all \(k \geq 1\). Note that, up to a sign, this is \((\tilde{\Phi} \circ B)^k \circ (Q \otimes D')\) applied to invariant functions \(\mathfrak{c}^{\infty} (C)G[[h]]\) and then projected to \(S^1 \mathfrak{g}\). But on invariant functions the vertical vector fields and the differentials in \(\mathfrak{g}^*\)-direction vanish, and we have only one slot where they can give a non-trivial contribution, namely \(Q \otimes D'\). We fix the symmetric degree \(Q \in S^1 \mathfrak{g}\) and get

\[
\pr \circ \pr \circ Q_2^k \left( (\tilde{\Phi} \circ B)^k \circ (Q \otimes D') \right) = \left( \frac{(-1)^{d+(dd')}}{i} \right) \pr \circ \pr \circ \left( (\tilde{\Phi} \circ B)^k \circ (Q \otimes D') \right)
\]

Here \((B(\tilde{\Phi}B)^{k-1}D)\) denotes the component of \(B(\tilde{\Phi}B)^{k-1}D\) with \(i\) differentiations in \(\mathfrak{g}^*\)-direction.

The 1/i comes from the degree of the homotopy \([3,13]\) since we have no \(S^1 \mathfrak{g}^*\)-degree and since the only term that can be non-trivial is the one with \(i\) differentiations in \(\mathfrak{g}^*\)-direction applied to \(Q\).

We compute with

\[
\pr \circ \pr \circ Q_2^1 \left( (\tilde{\Phi} \circ B)^k \circ (Q \otimes D') \right) = \left( \frac{(-1)^{d+(dd')}}{i} \right) \pr \circ \pr \circ \left( (\tilde{\Phi} \circ B)^k \circ (Q \otimes D') \right)
\]

\[
= \left( \frac{(-1)^{d+(dd')}}{i} \right) \pr \circ \pr \circ \left( (-hL_{(e_t)}C \circ \pr |_{Sp \mathfrak{g}} \tilde{\Phi} \circ B)^{k-1}D) \circ \pr |_{Sp \mathfrak{g}} \right) \circ (Q \otimes D')
\]

\[
- \left( \pr |_{Sp \mathfrak{g}} \tilde{\Phi} \circ B)^{k-1}D \circ \pr |_{Sp \mathfrak{g}} \right) \circ (Q \otimes D')
\]

\[
= \left( \frac{(-1)^{d+(dd')}}{i} \right) \pr \circ \pr \circ \left( (-hL_{(e_t)}C \circ \pr |_{Sp \mathfrak{g}} \tilde{\Phi} \circ B)^{k-1}D) \circ \pr |_{Sp \mathfrak{g}} \right) \circ (Q \otimes D')
\]

\[
= \left( \frac{(-1)^{d+(dd')}}{i} \right) \pr \circ \pr \circ \left( ((hm_G(e_t, \partial |_{Sp \mathfrak{g}}) \tilde{\Phi} \circ B)^{k-1}D) \circ (Q \otimes D') \right).
\]
But we know $hm_G(e_i, \cdot) = -hL(e_i)_C + hm_g(e_i, \cdot)$, where $hm_g$ denotes the higher components of the Gutt product on $\mathfrak{g}^*$. Moreover, we have by the invariance

$$-[\mathcal{L}(e_i)_C, \text{pr} |_{S^0\mathfrak{g}} (\tilde{\Phi} \circ B)^k D]_G = \left[ -f_{ik} e_j \frac{\partial}{\partial e_k}, \text{pr} |_{S^0\mathfrak{g}} (\tilde{\Phi} \circ B)^k D \right]_G$$

and thus

$$h \text{ pr} \circ \left( \left[ -\mathcal{L}(e_i)_C, \text{pr} |_{S^0\mathfrak{g}} (\tilde{\Phi} \circ B)^k D \right]_G \circ (\frac{\partial}{\partial e_I} Q \otimes D') \right) = h \text{ pr} \circ \left( \text{pr} |_{S^0\mathfrak{g}} (\tilde{\Phi} \circ B)^k D \circ \left( f_{ik} e_j \frac{\partial}{\partial e_k} (\text{pr} |_{S^0\mathfrak{g}} (\tilde{\Phi} \circ B)^k D \circ (\frac{\partial}{\partial e_I} Q \otimes D')) \right) \right) = 0.$$

The only remaining terms are

$$\text{pr} \circ \left( (\tilde{\Phi} \circ B)^k D \circ (Q \otimes D') \right) = (1)^{d+(d'd')} \text{pr} \circ \left( \left( \text{pr} |_{S^0\mathfrak{g}} (\tilde{\Phi} \circ B)^k D \circ (\frac{\partial}{\partial e_I} Q \otimes D') \right) \right).$$

We know that $hm_g(e_i, \frac{\partial}{\partial e_I} Q)$ is either zero or in $S^0\mathfrak{g}$ and the statement follows by induction. □

In particular, we can compose this projection $\text{pr}$ with the $L_\infty$-projection from Proposition 3.10 that we constructed with the homotopy transfer theorem. Summarizing, we have shown:

**Theorem 3.12** There exists an $L_\infty$-morphism

$$D_{\text{red}} = \text{pr} \circ \text{pr}: (D_{tv}(\mathbb{C} \times \mathfrak{g}^*)[[h]], \ast_G, J, \cdot, [\cdot, \cdot], \partial) \longrightarrow (D_{\text{poly}}(M_{\text{red}})[[h]], \partial, [\cdot, \cdot]) \quad (3.23)$$

Finally, as in the polyvector field case in [14], we can twist the above morphism to obtain an $L_\infty$-morphism from the curved equivariant polydifferential operators into the Cartan model and therefore also into the polydifferential operators on $M_{\text{red}}$, see Proposition 3.13 for the basics of the twisting procedure.

**Proposition 3.13** Twisting the reduction $L_\infty$-morphism $D_{\text{red}}$ from [3.28] with $-hm_G$ yields an $L_\infty$-morphism

$$D_{\text{red}}^{-hm_G}: (D_{tv}(\mathbb{C} \times \mathfrak{g}^*)[[h]], h\lambda, \partial + [-J, \cdot], [\cdot, \cdot]) \longrightarrow (D_{\text{poly}}(M_{\text{red}})[[h]], \partial, [\cdot, \cdot]) \quad (3.24)$$

where $\lambda = e^i \otimes (e_i)_M$ denotes the curvature.

**Proof:** At first we check that the curvature is indeed given by

$$e^i \otimes [-e_i, -hm_G]_G = e^i \otimes (-[e_i, \cdot])_G = e^i \otimes (hL(e_i)_C - h \text{ad}(e_i)) = h\lambda,$$

compare Lemma 3.6. The only thing left to show is that the DGLA structure on $M_{\text{red}}$ is not changed, which is equivalent to

$$\sum_{k=1}^{\infty} \frac{(-h)^k}{k!} (D_{\text{red}})_k (m_G \lor \cdots \lor m_G) = 0.$$ 

But using the explicit form of $P$ from Proposition 3.9 we see inductively that $P$ vanishes if every argument has a differential in $\mathfrak{g}^*$-direction and the statement is shown. □

**Remark 3.14** In the polyvector field case from [14] Proposition 4.29] we saw that the structure maps of the twisted morphism coincide with the structure maps of the original one. In our case it is not clear, i.e. one might indeed have $D_{\text{red}}^{-hm_G} \neq D_{\text{red}}$. 

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This reduction morphism can be used to obtain a reduction morphism of the equivariant polydifferential operators $D^*_g(M)$ of more general manifolds $M \neq C \times g^*$. More explicitly, assuming that the action is proper around $C$ and free on $C$, we can restrict at first to $M_{\text{n}} \cong U_{\text{n}} \subset C \times g^*$, i.e. we have

$$\psi \circ U_{\text{n}}^*: (D^*_g(M)[[h]], h\lambda, \partial\theta - [J, \cdot]_g, [\cdot, \cdot]_g) \longrightarrow (D^*_g(U_{\text{n}})[[h]], h\lambda|_{U_{\text{n}}}, \partial\theta - [J|_{U_{\text{n}}}, \cdot]_g, [\cdot, \cdot]_g).$$

But on $U_{\text{n}}$, we can perform the Taylor expansion that is a morphism of curved DGLAs

$$D^*_g: (D^*_g(U_{\text{n}})[[h]], h\lambda|_{U_{\text{n}}}, \partial\theta - [J|_{U_{\text{n}}}, \cdot]_g, [\cdot, \cdot]_g) \rightarrow (D^*_{\text{Tay}}(C \times g^*)[[h]], h\lambda, \partial - [J, \cdot], [\cdot, \cdot]).$$

Finally, we can compose it with $D^*_{\text{red}}$ and obtain the following statement:

**Theorem 3.15** The composition of the above morphisms results in an $L_\infty$-morphism

$$D_{\text{red}}: (D^*_g(M)[[h]], h\lambda, \partial\theta - [J, \cdot]_g, [\cdot, \cdot]_g) \longrightarrow (D^*_{\text{poly}(M_{\text{red}})}[[h]], 0, \partial, [\cdot, \cdot]|_C),$$

called reduction $L_\infty$-morphism.

**Remark 3.16 (Choices)** Note that the only non-canonical choice we made is a open neighbourhood of $C$ in $M$ which is diffeomorphic to a star shaped open neighbourhood of $C$ in $C \times g^*$. Recall that the choice of this neighbourhood works as follows: Take an arbitrary $G$-equivariant tubular neighbourhood embedding $\psi: \nu(C) \rightarrow U \subset M$, where $\nu(C)$ denotes the normal bundle. Then define

$$\phi: \nu(C) \ni [v_p] \mapsto (p, J(\psi([v_p]))) \in C \times g^*$$

which is close to $C$ a diffeomorphism. After some suitable restriction we obtain the identification. Nevertheless, we had to choose a $G$-equivariant tubular neighbourhood and any two choices differ by a $G$-equivariant local diffeomorphism around $C$

$$A: C \times g^* \longrightarrow C \times g^*$$

which is, restricted to $C$ the identity. One can show that one has in the Taylor expansion

$$D^*_g(A^*f) = e^X D^*_g(f)$$

for a vector field $X \in \prod_{i \geq 1} (S^i g \otimes \mathfrak{X}(C))^G \subset D^*_{\text{Tay}}(C \times g^*)$. Since any vector field is closed, $X$ does not derive in $g^*$-direction and $\lambda$ is central, we obtain an inner automorphism

$$e^{[X, \cdot]}: (D^*_{\text{Tay}}(C \times g^*)[[h]], h\lambda, \partial - [J, \cdot], [\cdot, \cdot]) \longrightarrow (D^*_{\text{Tay}}(C \times g^*)[[h]], h\lambda, \partial - [J, \cdot], [\cdot, \cdot])$$

of curved Lie algebras which acts trivially on the level of equivalence classes of Maurer-Cartan elements. We are moreover certain, that the two reduction $L_\infty$-morphisms are homotopic in a suitable curved setting, which, to our knowledge, is not developed yet.

As a last remark of this section, we want to mention a very interesting observation, which is not directly connected to the rest of this paper and/or the results of it. Nevertheless, we felt that it can be interesting from many other different perspectives.

**Remark 3.17 (Cartan model)** One can show that DGLA structure $Q_C$ from Proposition 3.9 on $\prod_{i \geq 0} (S^i g \otimes D^*_{\text{poly}(C)})^G[[h]]$ restricts to $(S^i g \otimes D^*_{\text{poly}(C)})^G[[h]]$ and hence can be evaluated at $h = 1$. Moreover, we still have the DGLA map

$$\text{pr}: (S^i g \otimes D^*_{\text{poly}(C)})^G \longrightarrow D^*_{\text{poly}(M_{\text{red}})}.$$
Picking a $G$-invariant covariant derivative (not necessarily torsion-free) for which the fundamental vector fields are flat in fiber direction one can, using the PBW-ismorphism for Lie algebroids (see [25] and [31]) prove that there is an equivariant cochain map $K: D_{poly}(C) \to T_{poly}(C)$ and an equivariant homotopy $h: D_{poly}^\bullet(C) \to D_{poly}^{\bullet-1}(C)$, such that

$$T_{poly}(C) \xrightarrow{\text{hkr}} (D_{poly}(C), \partial) \xleftarrow{\text{hkr}} h$$

is a special deformation retract. Additionally, one can show that

$$K(D_1 \cup D_2) = K(D_1) \wedge K(D_2) \quad \text{and} \quad K(\mathcal{L}_P) = \begin{cases} -P_C, & \text{for } P \in \mathfrak{g} \subset S\mathfrak{g} \\ 0, & \text{else} \end{cases}$$

for $D_1, D_2 \in D_{poly}(C)$ and $P \in S\mathfrak{g}$. We extend now (3.26) to

$$((S\mathfrak{g} \otimes T_{poly}(C))^G, 0) \xrightarrow{\text{hkr}} ((S\mathfrak{g} \otimes D_{poly}(C))^G, \partial) \xleftarrow{\text{hkr}} h$$

to obtain a special deformation retract. Now we include $\delta$ as in Proposition 3.7 and see it as a perturbation of $\partial$. One can show that the perturbation is small in the sense of the homological perturbation lemma as in [7], and we obtain

$$((S\mathfrak{g} \otimes T_{poly}(C))^G, \delta) \xrightarrow{\text{hkr}} ((S\mathfrak{g} \otimes D_{poly}(C))^G, \partial + \delta) \xleftarrow{\text{hkr}} h$$

where $\delta$ is the differential

$$\delta(P \otimes X) = i(e^i)P \otimes (e_i)_C \wedge X$$

obtained in [14] Definition 4.14] on $(S\mathfrak{g} \otimes T_{poly}(C))^G$. Finally, one can show that

$$((S\mathfrak{g} \otimes T_{poly}(C))^G, \delta) \xrightarrow{\text{hkr}} ((S\mathfrak{g} \otimes D_{poly}(C))^G, \partial + \delta)$$

$$T_{poly}(M_{\text{red}}, 0) \xrightarrow{\text{hkr}} (D_{poly}(M_{\text{red}}), \partial)$$

commutes and both of the horizontal maps are quasi-isomorphisms as well as the left vertical one which implies the claim.

4 Comparison of the Reduction Procedures

At the level of Maurer-Cartan elements, we know that the $L_\infty$-morphism $D_{\text{red}}$ from Theorem 3.15 induces a map from equivariant star products $(\ast, H)$ with quantum momentum map $H = J + O(h)$ on $M$ to star products $\ast_{\text{red}}$ on the reduced manifold $M_{\text{red}}$. We conclude with a comparison of this reduction procedure with the reduction of formal Poisson structures via the quantized Koszul complex as in [4, 18], see also our adapted version in Appendix A.

We assume for simplicity $M = C \times \mathfrak{g}^*$ and work in the Taylor expansion of the equivariant polydifferential operators. Moreover, we identify $\mathcal{C}^\infty(C)$ with $\text{pro}l\mathcal{C}^\infty(C) \subset \mathcal{C}^\infty(C \times \mathfrak{g}^*)$. Let us start with an equivariant star product $(\ast, H = J + hH')$ on $C \times \mathfrak{g}^*$, which means that $h\pi_\ast - hH' = \ast - \ast_G - (H - J)$ is Maurer-Cartan element in

$$\{D_{\text{ray}}(C \times \mathfrak{g}^*)[[h]], [\ast_G - J, \cdot], [\cdot, \cdot]]\}.$$
Proposition 4.1 Defining $I^1_1 = i_h$ and $I^2_k = h^{-1} \circ Q^2_k \circ I^2_{k+1}$ gives an $L_\infty$-morphism

$$I: \left( \prod_{j=0}^\infty (S^j \mathfrak{g} \otimes D_{\text{poly}}(C)) \right)^G \to (D_{\text{tor}}(C \times \mathfrak{g}^*)[[h]], [\ast_G - J, \cdot], [\cdot, \cdot]).$$ (4.1)

Moreover, one $I$ is a quasi-inverse of the $L_\infty$-projection $P$ from Proposition 3.7 and one has $P \circ I = \text{id}$.

Proof: Note that we have in general $h^2 \neq 0$, but the only part of the homotopy that appears in the above recursions is $\tilde{\Phi}$, where we know $\tilde{\Phi} \circ \tilde{\Phi} = 0$. Therefore, the statement follows from Proposition 3.7. \qed

We get with Corollary 3.5

Corollary 4.2 The $L_\infty$-morphism $I$ is compatible with the filtration induced by $h$ and

$$h \tilde{\pi}_* = (I \circ P)^1(\exp(h\pi_* - h'H')) \in (D_{\text{tor}}(C \times \mathfrak{g}^*)[[h]], [\ast_G - J, \cdot], [\cdot, \cdot]).$$ (4.2)

is a well-defined Maurer-Cartan element that is equivalent to $h\pi_* - h'H'$. In particular, $(\tilde{\pi} = *_G + h\tilde{\pi}_*, J)$ is a strongly invariant star product, i.e., an equivariant star product s.t. the quantum momentum map is just the classical momentum map, and it is equivariantly equivalent to $(\ast, H)$.

The reduction of $(\tilde{\pi}, J)$ via the reduction $L_\infty$-morphism $D_{\text{red}}$ is now easy:

Lemma 4.3 The reduction $L_\infty$-morphism

$$D_{\text{red}} = \text{pr} \circ P: (D_{\text{tor}}(C \times \mathfrak{g}^*)[[h]], [\ast_G - J, \cdot], [\cdot, \cdot]) \to (D_{\text{poly}}(M_{\text{red}})[[h]], \partial, [\cdot, \cdot])_G$$ (4.3)

from Theorem 3.7 maps $h\tilde{\pi}_*$ to a Maurer-Cartan element $\text{hm}_{\text{red}} = \text{pr} \circ P^1(\exp h\tilde{\pi}_*)$ in the poly-differential operators on $M_{\text{red}}$. The corresponding star product $\tilde{\pi}_{\text{red}} = \mu + \text{hm}_{\text{red}}$ is given by

$$\text{pr}^* (u_1 \tilde{\pi}_{\text{red}} u_2) = \iota^*(\text{pr}^* u_1 \tilde{\pi}_{\text{red}} \text{pr}^* u_2))$$ (4.4)

for all $u_1, u_2 \in \mathcal{O}_G(M_{\text{red}})[[h]]$.

Proof: By definition of $h\tilde{\pi}_*$ we know $h\tilde{\pi}_* h\tilde{\pi}_* = \tilde{\Phi}(h\tilde{\pi}_*) = 0$, and thus

$$\text{hm}_{\text{red}} = \text{pr} \circ P^1(\exp h\tilde{\pi}_*) = \text{pr} \circ P(h\tilde{\pi}_*).$$

Equation (4.4) follows since $\text{hm}_G(\text{pr}^* u_1 \tilde{\pi}_{\text{red}} \text{pr}^* u_2) = 0$. \qed

Moreover, we know by Lemma A.5 that the BRST reduction of $\mu + \mu_G + h\tilde{\pi}_*$ coincides with $\tilde{\pi}_{\text{red}}$, and we have shown:

Theorem 4.4 Let $(\ast, H)$ be an equivariant star product on $M$. Then the reduced star product induced by $D_{\text{red}}$ from Theorem 3.7 and the reduced star product via the formal Koszul complex A.14 are equivalent.

Proof: We know that both reduction procedures map equivalent equivariant star products to equivalent reduced star products. Moreover, we saw above that both reduction procedures coincide on $(\tilde{\pi} = \ast_G + h\tilde{\pi}_*, J)$ which is equivariantly equivalent to $(\ast, H)$. \qed

A BRST Reduction of Equivariant Star Products

We recall a slightly modified version of the reduction of equivariant star products as introduced in [18], see also [16] for a discussion of this reduction scheme in the context of Hermitian star products. It relies on the quantized Koszul complex and the homological perturbation lemma.
A.1 Homological Perturbation Lemma

At first we recall from [7, Theorem 2.4] and [32, Chapter 2.4] a version of the homological perturbation lemma that is adapted to our setting. Let

\[(C, d_C) \xrightarrow{i} (D, d_D) \xrightarrow{h}\]

be a homotopy retract (also called homotopy equivalence data), i.e. let \((C, d_C)\) and \((D, d_D)\) be two chain complexes together with two quasi-isomorphisms

\[i: C \rightarrow D \quad \text{and} \quad p: D \rightarrow C\]  

(A.1)

and a chain homotopy

\[h: D \rightarrow D \quad \text{with} \quad \text{id}_D - ip = d_D h + h d_D\]  

(A.2)

between \(\text{id}_D\) and \(ip\). Then we say that a graded map

\[B: D_\bullet \rightarrow D_{\bullet - 1}\]

with \((d_D + B)^2 = 0\) is a perturbation of the homotopy retract. The perturbation is called small if \(\text{id}_D + Bh\) is invertible, and the homological perturbation lemma states that in this case the perturbed homotopy retract is again a homotopy retract, see [7, Theorem 2.4] for a proof.

**Proposition A.1 (Homological perturbation lemma)** Let

\[(C, d_C) \xrightarrow{i} (D, d_D) \xrightarrow{h}\]

be a homotopy retract and let \(B\) be small perturbation of \(d_D\), then the perturbed data

\[(C, \hat{d}_C) \xrightarrow{j} (D, \hat{d}_D) \xrightarrow{H}\]  

with

\[A = (\text{id}_D + Bh)^{-1} B, \quad \hat{d}_D = d_D + B, \quad \hat{d}_C = d_C + pAi, \]

\[I = i - hAi, \quad P = p - pAh, \quad H = h - hAh\]  

(A.4)

is again a homotopy retract.

**Remark A.2** In [7] it is shown that perturbations of special deformation retracts are again special deformation retracts, which is in general not true for deformation retracts, see Section [3] for the different notions.

We are interested in even simpler complexes of the following form:

\[
\begin{array}{cccccccc}
0 & \xleftarrow{} & D_0 & \xleftarrow{d_{D,1}} & D_1 & \xleftarrow{d_{D,2}} & \cdots \\
& & p & \downarrow & h_0 & & & \\
0 & \xleftarrow{} & C_0 & \xleftarrow{} & 0
\end{array}
\]  

(A.5)

In this case, the perturbed homotopy retract corresponding to a small perturbation \(B\) according to (A.4) is given by

\[I = i, \quad P = p - p(\text{id}_D + B_1 h_0)^{-1} B_1 h_0, \quad H = h - h(\text{id}_D + Bh)^{-1} Bh\]

and, using the geometric power series, this can be simplified to

\[I = i, \quad P = p(\text{id}_D + B_1 h_0)^{-1}, \quad H = h(\text{id}_D + Bh)^{-1}\]  

(A.6)

Here we denote by \(B_1: D_1 \rightarrow D_0\) the degree one component of \(B\), analogously for \(h\). By Remark [A.2] we know that deformation retracts are in general not preserved under perturbations. However, in this case we see that, starting with a deformation retract, the additional condition \(h_0 i = 0\) suffices to guarantee

\[PI = p(\text{id}_D + B_1 h_0)^{-1} i = pi = \text{id}_{C_0}.\]
A.2 Quantized Koszul Complex

Let now $(M, \{\cdot, \cdot\})$ be a smooth Poisson manifold with a left action of the Lie group $G$. Moreover, let $J: M \to \mathfrak{g}^*$ be a classical (equivariant) momentum map. As usual, we assume that $0 \in \mathfrak{g}^*$ is a value and a regular value of $J$ and set $C = J^{-1}\{0\}$. In addition, we require the action to be proper on $M$ (or at least around $C$) and free on $C$, which implies that $M_{red} = C/G$ is a smooth manifold. The reduction via the classical Koszul complex $\Lambda^\bullet \mathfrak{g} \otimes \mathcal{C}^\infty(M)$ is one way to show that $M_{red}$ is even a Poisson manifold, but we need the quantum version to show that we have an induced star product on $M_{red}$. The Koszul differential $\partial$ is given by

$$\partial: \Lambda^0 \mathfrak{g} \otimes \mathcal{C}^\infty(M) \to \Lambda^1 \mathfrak{g} \otimes \mathcal{C}^\infty(M), \quad a \mapsto i(J)a = J_1 a(e^1) a,$$

(A.7)

where $i$ denotes the left insertion and $J = J_1 e^1$ the decomposition of $J$ with respect to a basis $e^1, \ldots, e^n$ of $\mathfrak{g}^*$. Then $\partial^2 = 0$ follows immediately with the commutativity of the pointwise product in $\mathcal{C}^\infty(M)$. The differential $\partial$ is also a derivation with respect to associative and super-commutative product on the Koszul complex, consisting of the $\wedge$-product on $\Lambda^\bullet \mathfrak{g}$ tensored with the pointwise product on the functions. Moreover, it is invariant with respect to the induced $\mathfrak{g}$-representation

$$\mathfrak{g} \ni \xi \mapsto \rho(\xi) = \text{ad}(\xi) \otimes \text{id} - \text{id} \otimes \mathcal{L}_\xi \in \text{End}(\Lambda^\bullet \mathfrak{g} \otimes \mathcal{C}^\infty(M))$$

(A.8)

as we have

$$\partial \rho(e_a)(x \otimes f) = f^{b}a_{j}e_k \wedge i(e^{j}) \wedge i(e^{k}) x \otimes J_{0,i} f + f^{a}_{j} i(e^j) x \otimes J_{0,i} f$$

$$+ i(e^i)x \otimes J_{0,i} (J_{0,a}, f)_0$$

$$= \rho(e_a) \partial(x \otimes f)$$

for all $x \in \Lambda^\bullet \mathfrak{g}$ and $f \in \mathcal{C}^\infty(M)$.

One can show that the Koszul complex is acyclic in positive degree with homology $\mathcal{C}^\infty(C)$ in order zero, and that one has a $G$-equivariant homotopy

$$h: \Lambda^\bullet \mathfrak{g} \otimes \mathcal{C}^\infty(M) \to \Lambda^{\bullet+1} \mathfrak{g} \otimes \mathcal{C}^\infty(M),$$

(A.9)

see [4, Lemma 6] and [18]. In other words, this means that

$$\text{prol}: (\mathcal{C}^\infty(C), 0) \cong (\Lambda^\bullet \mathfrak{g} \otimes \mathcal{C}^\infty(M), \partial): \iota^* h$$

is a HE data of the special type of [18], i.e. we have the following diagram:

$$0 \leftarrow \mathcal{C}^\infty(M) \xrightarrow{\iota^*} \Lambda^0 \mathfrak{g} \otimes \mathcal{C}^\infty(M) \xrightarrow{\partial_G} \Lambda^1 \mathfrak{g} \otimes \mathcal{C}^\infty(M) \xrightarrow{\partial_G} \cdots$$

$$0 \leftarrow \mathcal{C}^\infty(C) \xrightarrow{\text{prol}}$$

For the reduction of equivariant star products, we need to deform it to the quantized Koszul complex. The quantized Koszul differential $\theta: \Lambda^\bullet \mathfrak{g} \otimes \mathcal{C}^\infty(M_{\text{nice}})[[h]] \to \Lambda^{\bullet+1} \mathfrak{g} \otimes \mathcal{C}^\infty(M_{\text{nice}})[[h]]$ is defined by

$$\theta^{(\kappa)}(x \otimes f) = \iota(e^a)x \otimes H_a \ast f - \frac{h}{2} f_{ab} c_c \wedge i(e^a) i(e^b) x \otimes f + h \kappa f_{ab} i(e^a)(x \otimes f)$$

(A.10)

for $\kappa \in \mathbb{C}[[h]], \ x \in \Lambda^\bullet \mathfrak{g}[[h]]$ and $f \in \mathcal{C}^\infty(M_{\text{nice}})[[h]]$, where $\Delta = f_{ab} e^a$ is the modular one-form of $\mathfrak{g}$.

Remark A.3 Note that in the literature [4,18] a different convention is used:

$$\theta^{(\kappa)}(x \otimes f) = \iota(e^a)x \otimes f \ast H_a + \frac{h}{2} f_{ab} c_c \wedge i(e^a) i(e^b) x \otimes f + h \kappa i(\Delta)(x \otimes f)$$

for $\kappa \in \mathbb{C}[[h]]$. In particular, $\theta^{(\kappa)}$ is left $\ast$-linear. However, in order to simplify the comparison of the BRST reduction with the reduction via $D_{\text{red}}$ in Section 4, we want the quantized Koszul differential to be right $\ast$-linear, which leads to our convention in (A.10).
The reduction of the star product in our convention works analogously to [3,13] since $\delta^{(\kappa)}$ satisfies all the desired properties:

**Lemma A.4** Let $(\ast, H)$ be an equivariant star product and $\kappa \in \mathbb{C}[[h]]$.

i.) One has $\delta^{(0)} \circ i(\Delta) + i(\Delta) \circ \delta^{(0)} = 0$.

ii.) $\delta^{(\kappa)}$ is right $\ast$-linear.

iii.) $\delta^{(\kappa)} = \partial + O(h)$.

iv.) $\delta^{(\kappa)}$ is $G$-equivariant.

v.) One has $\delta^{(\kappa)} \circ \delta^{(\kappa)} = 0$.

**Proof:** The proof is analogue to [13] Lemma 3.4.

Assume that we have chosen a value $\kappa \in \mathbb{C}[[h]]$ and write $\delta = \delta^{(\kappa)}$. Then by the homological perturbation lemma one gets a perturbed homotopy retract

\[
\begin{array}{c}
0 \leftarrow \mathcal{E}^\infty(M_{\text{nice}})[[h]] \xleftarrow{\partial_1} \Lambda^1 g \otimes \mathcal{E}^\infty(M_{\text{nice}})[[h]] \xrightarrow{\partial_2} \cdots \\
\uparrow \text{prol} \uparrow \\
0 \leftarrow \mathcal{E}^\infty(C)[[h]] \leftarrow 0,
\end{array}
\]

where

\[
\text{prol} = \text{prol}, \quad \ast = \ast \circ (\text{id} + B_1 h_0)^{-1}, \quad h = h \circ (\text{id} + B h)^{-1},
\]

and where $\delta - \partial = B$, see (A.3). One can show that the deformed restriction map $\ast$ is given by

\[
\ast = \ast \circ S = \sum_{r=0} \hbar^r \ast_r : \mathcal{E}^\infty(M_{\text{nice}})[[h]] \longrightarrow \mathcal{E}^\infty(C)[[h]]
\]

(A.12)

with a $G$-equivariant formal series of differential operators $S = \text{id} + \sum_{r=1} \hbar^r S_r$ on $\mathcal{E}^\infty(M_{\text{nice}})$ and with $S_r$ vanishing on constants. Moreover, it is uniquely determined by the properties

\[
\ast_0 = \ast, \quad \ast \partial_1 = 0 \quad \text{and} \quad \ast \text{prol} = \text{id}_{\mathcal{E}^\infty(C)[[h]]}.
\]

(A.13)

The reduced star product $\ast_{\text{red}}$ on $M_{\text{red}} = C/G$ is then given by

\[
\text{pr}^\ast (u_1 \ast_{\text{red}} u_2) = \ast (\text{prol}(\text{pr}^\ast u_1) \ast \text{prol}(\text{pr}^\ast u_2))
\]

(A.14)

for all $u_1, u_2 \in \mathcal{E}^\infty(M_{\text{red}})[[h]]$, compare [3] Theorem 32. In [32] Lemma 4.3.1 it has been shown that equivariantly equivalent star products reduce to equivalent star products on $M_{\text{red}}$.

For the comparison of the reduction procedures in Section 4 we need the following observation:

**Lemma A.5** Let $(\ast = \mu + h \pi_\ast + h m_G, J)$ be an equivariant star product on $C \times g^\ast$, and choose $\kappa = -1$ for the quantized Koszul differential. If one has $\tilde{\Phi}(h \pi_\ast) = 0 = \Phi(h \pi_\ast)$, then it follows for all $u_1, u_2 \in \mathcal{E}^\infty(M_{\text{red}})[[h]]$

\[
\text{pr}^\ast (u_1 \ast_{\text{red}} u_2) = \ast (\text{prol}(\text{pr}^\ast u_1) \ast \text{prol}(\text{pr}^\ast u_2)) = \ast (\text{prol}(\text{pr}^\ast u_1) \ast \text{prol}(\text{pr}^\ast u_2)).
\]

(A.15)

**Proof:** We have for a polynomial function $f = P \otimes \phi \in S^g \otimes \mathcal{E}^\infty(C) \subset \mathcal{E}^\infty(C \times g^\ast)$

\[
(\partial - \partial)h_0(P \otimes \phi) = \frac{1}{j} (h(\pi_\ast + m_G)(e_i, i(e^j) P \otimes \phi) + h \kappa f_{ib}^j i(e^j) P \otimes \phi)
\]

\[
= \frac{1}{j} (\Phi(h \pi_\ast + h m_G)(P \otimes \phi) + h \kappa f_{ib}^j i(e^j) P \otimes \phi)
\]

\[
= \frac{1}{j} (h m_G(e_i, i(e^j) P \otimes \phi) + h \kappa f_{ib}^j i(e^j) P \otimes \phi)
\]

\[
= \frac{1}{j} (h m_G(e_i, i(e^j) P \otimes \phi) - i(e^j) P \otimes h \mathcal{L}_{(e_i)} \phi + h \kappa f_{ib}^j i(e^j) P \otimes \phi),
\]

where
where $hm_{\mathfrak{g}}$ denotes the non-trivial part of the Gutt product on $\mathfrak{g}^*$. We know that $\text{im}(hm_{\mathfrak{g}}(e_i, \cdot)) \subset S^0\mathfrak{g}[h]$, hence it follows

$$
\iota^* \circ (\partial - \partial) h_0 (P \otimes \phi) = \frac{1}{2} \iota^* ( - i(e^i) P \otimes h\mathcal{L}(e_i)_c \phi + h\kappa f^i_{ij} i(e^j) P \otimes \phi).
$$

(*)

On an invariant polynomial $P \otimes \phi \in (S^i \mathfrak{g} \otimes \mathcal{L}^\infty(C))^G$ we have

$$
- i(e^i) P \otimes h\mathcal{L}(e_i)_c \phi = - h i(e^i) \text{ad}(e_i) P \otimes \phi = - h f^i_{ij} i(e^j) P \otimes \phi,
$$

hence $\square$ vanishes for $\kappa = -1$. Thus we have in this case

$$
\text{pr}^*(u_1 \ast_{\text{red}} u_2) = \iota^* (\text{prol} \text{pr}^* u_1) \ast \text{prol} \text{pr}^* u_2 = \iota^* (\text{prol} \text{pr}^* u_1) \ast \text{prol} \text{pr}^* u_2)
$$

and the statement is shown. $\Box$

**B Explicit Formulas for the Homotopy Transfer Theorem**

In is well-known that $L_\infty$-quasi-isomorphisms always admit $L_\infty$-quasi-inverses. Moreover, it is well-known that given a homotopy retract one can transfer $L_\infty$-structures, see e.g. [26, Section 10.3]. Explicitly, a homotopy retract (also called homotopy equivalence data) consists of two cochain complexes $(A,d_A)$ and $(B,d_B)$ with chain maps $i,p$ and homotopy $h$ such that

$$(A,d_A) \xrightarrow{i} (B,d_B) \xleftarrow{p} h$$

with $h \circ d_B + d_B \circ h = \text{id} - i \circ p$, and such that $i$ and $p$ are quasi-isomorphisms. Then the homotopy transfer theorem states that if there exists a flat $L_\infty$-structure on $B$, then one can transfer it to $A$ in such a way that $i$ extends to an $L_\infty$-quasi-isomorphism. By the invertibility of $L_\infty$-quasi-isomorphisms there also exists an $L_\infty$-quasi-isomorphism into $A$ denoted by $P$, see e.g. [26, Proposition 10.3.9].

In this section we state a version of this statement adapted to our applications. For simplicity, we assume that we have a deformation retract satisfying

$$p \circ i = \text{id}_A.$$

By [21, Remark 2.1] we can assume that we have even a special deformation retract, also called *contraction*, where

$$h^2 = 0, \quad h \circ i = 0 \quad \text{and} \quad p \circ h = 0.$$

Assume now that $(B,Q_B)$ is an $L_\infty$-algebra with $(Q_B)_1 = - d_B$. In the following we give a more explicit description of the transferred $L_\infty$-structure $Q_A$ on $A$ and of the $L_\infty$-projection $P: (B,Q_B) \rightarrow (A,Q_A)$ inspired by the symmetric tensor trick [3][20][21][27]. The map $h$ extends to a homotopy $H_n: S^n(B[1]) \rightarrow S^n(B[1])[-1]$ with respect to $Q_{B,n}^n: S^n(B[1]) \rightarrow S^n(B[1])[1]$, see e.g. [26, p. 383] for the construction on the tensor algebra, which adapted to our setting works as follows: we define the operator

$$K_n: S^n(B[1]) \rightarrow S^n(B[1])$$

by

$$K_n(x_1 \vee \cdots \vee x_n) = \frac{1}{n!} \sum_{i=0}^{n-1} \sum_{\sigma \in S_n} \epsilon(\sigma) \frac{1}{n-i} iP X_\sigma(1) \vee \cdots \vee iP X_\sigma(i) \vee X_\sigma(i+1) \vee X_\sigma(n).$$

Note that here we sum over the whole symmetric group and not the shuffles, since in this case the formulas are easier. We extend $-h$ to a coderivation to $S(B[1])$, i.e.

$$\hat{H}_n(x_1 \vee \cdots \vee x_n) := - \sum_{\sigma \in \text{Sh}(1,n-1)} \epsilon(\sigma) hx_\sigma(1) \vee x_\sigma(2) \vee \cdots \vee x_\sigma(n)$$

and the statement is shown.
and define
\[ H_n = K_n \circ \bar{H}_n = \bar{H}_n \circ K_n. \]  
(B.2)

Since \(i\) and \(p\) are chain maps, we have \(K_n \circ Q^n_{B,n} = Q^n_{B,n} \circ K_n\), where \(Q^n_{B,n}\) is the extension of the differential \(Q^n_{B,1} = -d_B\) to \(S^n(B[1])\) as a coderivation. Hence we have
\[ Q^n_{B,n}H_n + H_nQ^n_{B,n} = (n \cdot \text{id} - ip) \circ K_n, \]
where \(ip\) is extended as a coderivation to \(S(B[1])\). A combinatorial and not very enlightening computation shows that finally
\[ Q^n_{B,n}H_n + H_nQ^n_{B,n} = \text{id} - (ip)^{\vee n}. \]  
(B.3)

Now assume that we have a codifferential \(Q_A\) and a morphism of coalgebras \(P\) with structure maps \(P^i : S^i(B[1]) \to A[1]\) such that \(P\) is an \(L_\infty\)-morphism up to order \(k\), i.e.
\[ \sum_{\ell=1}^m P^\ell_1 \circ Q^\ell_{B,m} = \sum_{\ell=1}^m Q^\ell_{A,\ell} \circ P_1^\ell, \]
for all \(m \leq k\). Then we have the following statement, whose proof can be found in [13].

**Lemma B.1** Let \(P : S(B[1]) \to S(A[1])\) be an \(L_\infty\)-morphism up to order \(k \geq 1\). Then
\[ L_\infty,k+1 = \sum_{\ell=2}^{k+1} Q^\ell_{A,\ell} \circ P_1^{k+1} - \sum_{\ell=1}^{k} P^\ell_1 \circ Q^\ell_{B,k+1} \]  
(B.4)
satisfies
\[ L_\infty,k+1 \circ Q_{B,k+1}^{k+1} = -Q_{A,1}^1 \circ L_\infty,k+1. \]  
(B.5)

This allows us to prove one version of the homotopy transfer theorem.

**Theorem B.2 (Homotopy transfer theorem)** Let \((B,Q_B)\) be a flat \(L_\infty\)-algebra with differential \(Q_B^1 = -d_B\) and contraction
\[ (A,d_A) \xleftarrow{i} (B,d_B). \]  
(B.6)

Then
\[ (Q_A)^1 = -d_A, \quad (Q_A)^1_{k+1} = \sum_{i=1}^{k} P^i_1 \circ (Q_B)^1_{k+1} \circ i^{\vee(k+1)}, \]  
(B.7)

\[ P^1_{k+1} = P^i_{k+1} = L_\infty,k+1 \circ H_{k+1} \quad \text{for } k \geq 1 \]
turns \((A,Q_A)\) into an \(L_\infty\)-algebra with \(L_\infty\)-quasi-isomorphism \(P : (B,Q_B) \to (A,Q_A)\). Moreover, one has \(P^1_k \circ i^{\vee k} = 0\) for \(k \neq 1\).

**Proof:** We observe \(P^1_{k+1}(ix_1 \vee \cdots \vee ix_{k+1}) = 0\) for all \(k \geq 1\) and \(x_i \in A\), which directly follows from \(h \circ i = 0\) and thus \(H_{k+1} \circ i^{\vee(k+1)} = 0\). Suppose that \(Q_A\) is a codifferential up to order \(k \geq 1\), i.e. \(\sum_{\ell=1}^{m} (Q_A)^1_{\ell}(Q_A)^m_{\ell} = 0\) for all \(m \leq k\), and that \(P\) is an \(L_\infty\)-morphism up to order \(k \geq 1\). We know that these conditions are satisfied for \(k = 1\) and we show that they hold for \(k+1\). Starting with \(Q_A\) we compute
\[
(QAQ_A)_{k+1}^1 = (QAQ_A)_{k+1}^1 \circ P^1_{k+1} \circ i^{\vee(k+1)} = \sum_{\ell=1}^{k+1} (QAQ_A)^1_{\ell} P^\ell_{k+1} i^{\vee(k+1)} = (QAQAP)_{k+1}^1 i^{\vee(k+1)} \\
= \sum_{\ell=2}^{k+1} (Q_A)^1_{\ell} (QAP)^1_{k+1} i^{\vee(k+1)} + (Q_A)^1_{k+1} (QAP)^1_{k+1} i^{\vee(k+1)}
\]
\[
\begin{align*}
&= \sum_{\ell = 2}^{k+1} (QA)^1_\ell (PQ_B)_{k+1}^\ell v^{(k+1)} + (QA)^1_{k+1} \cr
&= (QPQ_B)_{k+1}^k v^{(k+1)} - (QA)^1_{k+1} + (QA)^1_{k+1} \cr
&= \sum_{\ell = 1}^k (QA)^1_\ell (QB)_{k+1}^\ell v^{(k+1)} + (QA)_{k+1}^1 (PB)_{k+1} v^{(k+1)} \cr
&= \sum_{\ell = 1}^k (QB)_{k+1}^\ell (QB)_{k+1}^k v^{(k+1)} + (QA)_{k+1}^1 (PB)_{k+1} v^{(k+1)} \cr
&= -(QPQ_B)_{k+1}^k v^{(k+1)} (QA)_{k+1} + (QA)_{k+1}^1 (PB)_{k+1} v^{(k+1)} (QA)_{k+1} \cr
&= -(QA)_{k+1}^1 (QA)_{k+1} + (QA)_{k+1}^1 (QA)_{k+1} \cr
&= 0.
\end{align*}
\]

By the same computation as in Lemma \[B.1\] where one in fact only needs that $Q_A$ is a codifferential up to order $k+1$, it follows that

\[
L_{\infty,k+1} \circ Q_B^{k+1}_{k+1} = - Q_A^1 \circ L_{\infty,k+1}.
\]

It remains to show that $P$ is an $L_\infty$-morphism up to order $k+1$. We have

\[
P_{k+1}^1 \circ (Q_B)_{k+1}^k = L_{\infty,k+1} \circ H_{k+1} \circ (Q_B)_{k+1}^k \cr
&= L_{\infty,k+1} - L_{\infty,k+1} \circ (Q_B)_{k+1}^k \circ H_{k+1} - L_{\infty,k+1} \circ (i \circ p) v^{(k+1)} \cr
&= L_{\infty,k+1} + (QA)^1_{k+1} \circ P_{k+1}^1
\]

since

\[
L_{\infty,k+1} \circ (i \circ p) v^{(k+1)} = \left( \sum_{\ell = 2}^{k+1} Q_A^1_\ell \circ P_{k+1}^\ell - \sum_{\ell = 1}^k P_{k+1}^\ell \circ Q_A^1_{k+1} \right) \circ (i \circ p) v^{(k+1)} \cr
= (QA)^1_{k+1} \circ p v^{(k+1)} - (QA)^1_{k+1} \circ p v^{(k+1)} = 0.
\]

Therefore

\[
P_{k+1}^1 \circ (Q_B)_{k+1}^k - (QA)^1_{k+1} \circ P_{k+1}^1 = L_{\infty,k+1},
\]

i.e. $P$ is an $L_\infty$-morphism up to order $k+1$, and the statement follows inductively.

Note that a special case of the above theorem, for $i$ being a DGLA morphism, has been proven in \[14\] Proposition 3.2. We also want to give an explicit formula for a $L_\infty$-quasi-inverse of $P$, generalizing \[14\] Proposition 3.3.

**Proposition B.3** The coalgebra map $I : S^\bullet(A[1]) \to S^\bullet(B[1])$ recursively defined by the maps $I_1^1 = i$ and $I_1^k = h \circ L_{\infty,k+1}$ for $k \geq 1$ is an $L_\infty$-quasi inverse of $P$. Since $h^2 = 0 = h \circ i$, one even has $I_{k+1}^1 = h \circ \sum_{\ell = 2}^{k+1} Q_B^1_\ell \circ I_{k+1}^\ell$ and $P \circ I = \text{id}_A$.

**Proof:** We proceed by induction: assume that $I$ is an $L_\infty$-morphism up to order $k$, then we have

\[
I_{k+1}^1 Q_{A,k+1}^{k+1} - Q_{B,1}^1 I_{k+1}^k = -Q_{B,1}^1 \circ h \circ L_{\infty,k+1} + h \circ L_{\infty,k+1} \circ Q_{A,k+1}^{k+1} \cr
= -Q_{B,1}^1 \circ h \circ L_{\infty,k+1} - h \circ Q_{B,1}^1 \circ L_{\infty,k+1} \cr
= (id \circ i \circ p) L_{\infty,k+1}.
\]

We used that $Q_{B,1}^1 = -d_B$ and the homotopy equation of $h$. Moreover, we get with $p \circ h = 0$

\[
p \circ L_{\infty,k+1} = p \circ \left( \sum_{\ell = 2}^{k+1} Q_B^1_\ell \circ I_{k+1}^\ell - \sum_{\ell = 1}^k I_{k+1}^\ell \circ Q_A^1_{k+1} \right) \cr
= \sum_{\ell = 2}^{k+1} (P \circ Q_B^1_\ell \circ I_{k+1}^\ell - \sum_{\ell = 1}^k P_{k+1}^\ell \circ Q_B^1_\ell \circ I_{k+1}^\ell - Q_A^1_{k+1} \cr
\]
\[
\sum_{\ell=2}^{k+1} (Q_A \circ P)_\ell \circ I_{k+1}^\ell - \sum_{i=2}^{k+1} \sum_{\ell=i}^{k+1} P_i^1 \circ Q_i^\ell \circ I_{k+1}^\ell - Q_A^{1,k+1} = 0,
\]

and therefore \( I \) is an \( L_\infty \)-morphism.

**Remark B.4** Note that in the homotopy transfer theorem the property \( h^2 = 0 \) is not needed, and that one can also adapt the above construction of \( I \) to this more general case.

Note that there exists a homotopy equivalence relation \( \sim \) between \( L_\infty \)-morphisms, see e.g. [12] such that equivalent \( L_\infty \)-morphisms map Maurer-Cartan elements to equivalent Maurer-Cartan elements, see e.g. [5, Lemma B.5] for the case of DGLAs and [23, Proposition 1.4.6] for the case of flat \( L_\infty \)-algebras. Then we get:

**Corollary B.5** In the above setting one has \( P \circ I = id_A \) and \( I \circ P \sim id_B \). In particular, assume that one has complete descending filtrations on \( A, B \) such that all the maps are compatible. Then every Maurer-Cartan element \( \pi \in \mathcal{F}^{1}B \) is equivalent to \((I \circ P)^{1}(\exp(\pi))\).

**Proof:** By [23, Proposition 3.8] \( P \) admits a quasi-inverse \( I' \) such that \( P \circ I' \sim id_A \) and \( I' \circ P \sim id_B \), which implies

\[ I \circ P = id_B \circ I \circ P \sim I' \circ P \circ I \circ P = I' \circ P \sim id_B, \]

the rest of the statement is then clear. \( \square \)

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