Positive solutions to some systems of coupled nonlinear Schrödinger equations

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Abstract
We study the existence of nontrivial bound state solutions to the following system
of coupled nonlinear time-independent Schrödinger equations

\[-\Delta u_j + \lambda_j u_j = \mu_j u_j^3 + \sum_{k=1, k\neq j}^{N} \beta_{jk} u_j u_k^2, \quad u_j \in W^{1,2}(\mathbb{R}^n); \quad j = 1, \ldots, N\]

where \( n = 1, 2, 3; \lambda_j, \mu_j > 0 \) for \( j = 1, \ldots, N \), the coupling parameters \( \beta_{jk} = \beta_{kj} \in \mathbb{R} \) for \( j, k = 1, \ldots, N, j \neq k \). Precisely, we prove the existence of nonnegative bound state solutions for suitable conditions on the coupling factors. Additionally, with more restrictive conditions on the coupled parameters, we show that the bound states founded are positive.

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1 Introduction

Problems on coupled Nonlinear Schrödinger Equations (NLSE for short) have been widely investigated in the last years. They arise naturally in nonlinear Optics, and in the Hartree-Fock theory for Bose-Einstein condensates, among other physical problems. For example, a planar light beam propagating in the z direction in a non-linear medium can be described by a NLSE of the form

\[ iE_z + E_{xx} + \theta |E|^2 E = 0, \]

where \( i, E(x,z) \) denote the imaginary unit and the complex envelope of an Electric field respectively, and \( \theta > 0 \) is a normalization constant, corresponding to the fact that the medium is self-focusing.

Here we consider the following more general system of \( N \)-coupled NLSE

\[
\begin{cases}
-i \frac{\partial}{\partial t} E_j - \Delta E_j = \mu_j |E_j|^2 E_j + \sum_{k \neq j, k=1}^{N} \beta_{jk} |E_k|^2 E_j, & x \in \mathbb{R}^n, \ t > 0 \\
E_j = E_j(x,t) \in \mathbb{C}, \ E_j(x,t) \to 0 \text{ as } |x| \to \infty
\end{cases}
\]

(1.1)

for \( j = 1, \ldots, N \), the coupled parameters \( \beta_{jk} = \beta_{kj} \in \mathbb{R} \) for \( j, k = 1, \ldots, N, j \neq k; \lambda_j > 0 \) and \( \mu_j > 0 \) is for self-focusing in the \( j \)-th component of the beam. The solution \( E_j \) represents the \( j \)-th component of the beam. The coupling constant \( \beta_{jk} \) means the interaction between the \( j \)-th and \( k \)-th component of the beam. If they are positive, the interaction is attractive while if they are negative we have a repulsive interaction. The mixed case involves different sign on the coupling parameters \( \beta_{jk} \). Here, we study the attractive and mixed interactions.

When one looks for solitary wave solutions of the form \( E_j(x,t) = e^{i\lambda_j t} u_j(x) \), with \( \lambda_j > 0, u_j \) are the real valued functions called standing wave solutions which solve the following system of \( N \)-coupled nonlinear time-independent Schrödinger equations

\[
-\Delta u_j + \lambda_j u_j = \mu_j u_j^3 + \sum_{k \neq j, k=1}^{N} \beta_{jk} u_k^2 u_j, \quad u(x) \to 0 \text{ as } |x| \to \infty,
\]

(1.2)

for \( j = 1, \ldots, N \). Solutions \( u_j \) belong to the Sobolev space \( W^{1,2} (\mathbb{R}^n) \) (\( n = 1, 2, 3 \)) for all \( j = 1, \ldots, N \).

In the last years there has been a very active investigation on coupled systems of NLSE, from the physical point of view; see for instance [12, 19] or the book [1] and the references therein, as well as many works of mathematicians, from a more theoretical point of view, dealing with the existence, multiplicity, uniqueness and qualitative properties of bound and ground state solutions; see the earlier works [3, 4, 5, 7, 15, 18, 20, 21, 24], and the more recent list (far
from complete) [8, 11, 13, 17, 22, 23]. This kind of systems have also been recently studied in the framework of the fractional and bi-harmonic Schrödinger operators in [10] and [2] respectively.

Most of the existing works have been developed for systems with two equations, i.e., $N = 2$. For a general $N$-system, in [15] Lin-Wei studied the attractive interaction case $(\beta_{ij} > 0)$ in the dimensional case $n = 1, 2, 3$ and small coupling factors, i.e., when the ground state could coincide with a semitrivial solution, which corresponds to the ground state of a single NLS equation in one component and zero the others. More recently, in [17] Liu-Wang gave a sufficient condition on large coupling coefficients for the existence of a nontrivial ground state solution in a system of $N$-system (i.e., $N$-system for short) NLSE for the dimensional case $n = 2, 3$.

The existence of positive bound states was firstly studied by Ambrosetti-C. in [3, 4], where for a 2-system in the dimensional case $n = 2, 3$ was proved, among other results, the existence of positive bound states for positive non-large coupling factors; see also [10] for $N = 2, n = 1$. Even more, that existence result was also proved in [4] for $N \geq 3$ in the attractive interaction case, provided the coupling factors are sufficiently small. Later in [13] with $N = 2$, Ikoma-Tanaka shown that bound states obtained by Ambrosetti-C. in [3, 4] are indeed least energy (positive) solutions in some range of the small coupling factor. That result was recently improved by Chen-Zou in [8] for a better suitable range of the parameters.

Here, we improve the only result (up to our knowledge) of the existence of bound state solutions for a system of coupled NLSE such us (1.2) with $N \geq 3$. Precisely, we demonstrate the existence of new positive and non-trivial bound state solutions to (1.2) in dimensions $n = 1, 2, 3$ different from all previously studied. We want to emphasize that the interaction between the components of the beam is analyzed here in the attractive case and the mixed one.

The paper is organized as follows. In Section 2 we introduce the functional setting, the notation and give some definitions. Section 3 contains some previous known results, while the last one, Section 4 is devoted to prove the main results of the work.

## 2 Functional setting and notation

In this section, as we just mentioned above, we will establish the corresponding functional framework, the notation and give some definitions. The functional setting comes from the variational structure of (1.2), which can be derived from

$$-\Delta u_j + \lambda_j u_j = f_{u_j}(u_1, \ldots, u_n)$$

where

$$f(u_1, \ldots, u_n) = \sum_{j,k=1}^{N} \alpha_{jk} u_j^2 u_k^2,$$
with \( \alpha_{jj} = \frac{1}{4} \mu_j \) for \( j = 1, \ldots, N \); the coupling parameters \( \alpha_{jk} = \frac{1}{4} \beta_{jk} \) are symmetric, \( \beta_{jk} = \beta_{kj} \), both for \( k, j = 1, \ldots, N \) with \( k \neq j \). We will work on the Sobolev functional space \( W^{1,2}(\mathbb{R}^n) \), i.e., we will assume that \( u_j \in W^{1,2}(\mathbb{R}^n) \) for \( j = 1, \ldots, N \). We remember that the Sobolev space \( E_1 = W^{1,2}(\mathbb{R}^n) \) can be defined as the completion of \( C_0^1(\mathbb{R}^n) \) with the norm given by

\[
\| u \|_E = \left( \int_{\mathbb{R}^n} (|\nabla u|^2 + u^2) \, dx \right)^{\frac{1}{2}},
\]

which comes from the scalar product

\[
\langle u | v \rangle_E = \int_{\mathbb{R}^n} (\nabla u \cdot \nabla v + uv) \, dx.
\]

We will denote the following equivalent norms in \( E \),

\[
\| u \| = \| u \|_j = \left( \int_{\mathbb{R}^n} (|\nabla u|^2 + \lambda_j u^2) \, dx \right)^{\frac{1}{2}},
\]

with the corresponding scalar products

\[
(u | v) = (u | v)_j = \int_{\mathbb{R}^n} (\nabla u \cdot \nabla v + \lambda_j uv) \, dx; \quad j = 1, \ldots, N,
\]

dropping the sub-index \( j \) in (2.1)-(2.2) for short. In this manner, solutions of (1.2) are the critical points \( u = (u_1, \ldots, u_N) \in E = E \times \cdots \times E \) of the corresponding energy functional defined by

\[
\Phi(u) = \frac{1}{2} \| u \|^2 - F(u),
\]

where

\[
F(u) = \int_{\mathbb{R}^n} f(u) \, dx \quad \text{and} \quad \| u \|^2 = \sum_{j=1}^{N} \| u_j \|^2.
\]

Also we define

\[
I_j(u) = \frac{1}{2} \int_{\mathbb{R}^n} (|\nabla u|^2 + \lambda_j u^2) \, dx - \frac{1}{4} \mu_j \int_{\mathbb{R}^n} u_j^4 \, dx \quad \text{for} \quad j = 1, \ldots, N
\]

With respect to the coupling factors, we assume \( m \in \mathbb{N} \), moreover \( 2 \leq 2m \leq N \), and define \( \beta_k = \beta_{2k-1,2k} \) and \( \lambda_{2k-1} = \lambda_{2k} \) for \( 1 \leq k \leq m \). If \( 2m < N \), we define \( \beta^{(2k)}_{m+\ell} = \beta_{2m+\ell,2k} \) for any \( 1 \leq \ell \leq N - 2m \) and \( k \neq 2m + \ell \). We also suppose that \( \beta^{(2k)}_{m+\ell} = \epsilon \beta^{(2k)}_{m+\ell} \), \( \beta^{(2k-1)}_{m+\ell} = \epsilon \beta^{(2k-1)}_{m+\ell} \).

Let us denote \( \Phi_{\epsilon} = \Phi \) to emphasize its dependence on \( \epsilon \), then we can rewrite the functional \( \Phi_{\epsilon} \) as

\[
\Phi_{\epsilon}(u) = \Phi_0 - \epsilon \tilde{F}(u)
\]
where

\[
\Phi_0(u) = \sum_{j=1}^{N} I_j(u_j) - \frac{1}{2} \sum_{k=1}^{m} \beta_k \int_{\mathbb{R}^n} u_{2k-1}^2 \, dx
\]

and

\[
\tilde{F}(u) = \frac{1}{2} \sum_{\ell=1}^{N-2m} \sum_{k=1}^{m} \int_{\mathbb{R}^n} \left( \tilde{\beta}_{k+\ell}^{(2k-1)} u_{2k-1}^2 + \tilde{\beta}_{k+\ell}^{(2k)} u_{2k}^2 \right) u_{2m+\ell}^2 \, dx.
\]

For a function of two components we also set

\[
\Phi_k(u, v) = I_{2k-1} + I_{2k} - \frac{1}{2} \beta_k \int_{\mathbb{R}^n} u^2 v^2 \, dx \quad 1 \leq k \leq m.
\] (2.3)

We denote \(0 = (0, \ldots, 0)\), then we say that a vector function (or a function to simplify) \(u \in \mathbb{E}\) is positive (non-negative), namely \(u > 0\) (\(u \geq 0\)) if every component of \(u\) is positive (non-negative), i.e., \(u_j > 0\) (\(u_j \geq 0\)) for every \(j = 1, \ldots, N\).

**Definition 2.1** A bound state \(u \in \mathbb{E}\) of (1.2) is a critical point of \(\Phi\). A non-trivial bound state \(u = (u_1, \ldots, u_N)\) is a bound state with all the components \(u_j \neq 0, j = 1, \ldots, N\). Moreover, a positive bound state \(u \geq 0\) such that its energy is minimal among all the non-trivial bound states, namely

\[
\Phi(u) = \min\{\Phi(v) : v \in \mathbb{E} \setminus \{0\}, \Phi'(v) = 0\},
\] (2.4)

is called a ground state of (1.2).

Note that a ground state can have some of the components equal zero, thus in some works the definition of a ground state corresponds to a positive ground state here.

We will focus on the existence of (non-trivial) non-negative and also on positive bound states.

Let us denote \(2^* = \frac{2n}{n-2}\) if \(n \geq 3\) and \(2^* = \infty\) if \(n = 1, 2\), and remark that the functional make sense because \(E \hookrightarrow L^q(\mathbb{R}^n)\) for \(n = 1, 2, 3\). Concerning the Palais-Smale (PS) condition, one cannot expects a compact embedding neither in \(E\) nor in the restriction on the even functions of \(E\) into \(L^q(\mathbb{R})\) for any \(1 < q < \infty\). If \(n = 2, 3\), the PS condition is easy to obtain because the space of radially symmetric functions of the Sobolev space \(E\), namely \(H = E_{\text{rad}}\), is compactly embedded into \(L^q(\mathbb{R}^n)\) for \(q < 2^*\) (in particular for \(q = 4\), see for instance [16]). Nevertheless, as we will see, the lack of compactness will not be a problem provided we will work on \(H\) for all \(n = 1, 2, 3\). The reason is that we can obtain strong convergence in \(\mathbb{H} = H \times \cdots \times H\) thanks to the Local Inversion Theorem by a perturbation argument involving some non-degenerate critical points.
3 Previous results

In order to prove the main result of the paper we introduce some known results we are going to use.

**Remark 3.1** Let $U$ be the unique positive solution in $H$ of the equation $-\Delta u + u = u^3$ (see the paper of Kwong [14]) then the scaled function

$$U_j(x) = \sqrt[\lambda_j \mu_j] U(\sqrt[\lambda_j] x), \quad j = 1, \ldots, N \quad (3.1)$$

is the unique positive solution in $H$ of $-\Delta u + \lambda_j u = \mu_j u^3$ for $j = 1, \ldots, N$. We also recall that $U_j$ is the ground state, as well as a non-degenerate critical point of the corresponding functional $I_j$, $j = 1, \ldots, N$.

Let us consider the following 2-system of NLSE

$$(S_2) \equiv \begin{cases} -\Delta u + \lambda u = \mu_1 u^3 + \beta v^2 u \\ -\Delta v + \lambda u = \mu_2 v^3 + \beta u^2 v. \end{cases} \quad (3.2)$$

Throughout this subsection we will maintain the notation of Section 2 for a 2-system or a 3-system of NLSE with the natural meaning.

Let us define

$$a_k \beta = \sqrt[\mu_k \mu_j] \frac{\mu_k(\mu_j - \beta)}{\mu_k \mu_j - \beta^2}, \quad k \neq j, \quad k = 1, 2.$$ 

If $\lambda_1 = \lambda_2 = \lambda$, we have the following explicit and positive solution of $(S_2)$, given by

$$(u^0, v^0) = (a_{1\beta} U_1, a_{2\beta} U_2) \quad (3.3)$$

with $U_1, U_2$ defined in (3.1), see also [4, Remarks 5.8-(b)].

**Remark 3.2** In [11, Lemma 2.2 and Theorem 3.1] Dancer-Wei proved that $(u^0, v^0)$ is a non-degenerate critical point to the corresponding energy functional of $(S_2)$. ■

Even more, Wei-Yao proved in [23] the following property of the critical point $(u^0, v^0)$.

**Theorem A.** Assume that $\beta > \max\{\mu_1, \mu_2\}$, then $(u^0, v^0)$ is the unique positive solution of $(S_2)$.

**Remark 3.3** According to [4, Remark 5.8-(b)] and Theorem A one has that $(u^0, v^0)$ is a ground state of $(S_2)$, i.e.,

$$\Phi(u^0, v^0) = \min\{\Phi(v) : v \in \mathbb{E} \setminus \{0\}, \Phi'(v) = 0\},$$

which also corresponds to

$$\Phi(u^0, v^0) = \min_{u \in \mathbb{N}} \Phi(u)$$

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where \( \mathcal{N} \) is the corresponding Nehari manifold, defined as
\[
\mathcal{N} = \{ \mathbf{u} \in \mathbb{E} \setminus \{0\} : (\Phi'(\mathbf{u})|\mathbf{u}) = 0 \},
\]
that is a natural constraint for \( \Phi \); see [4] for more details. Furthermore,
\[
\| (u^0, v^0) \| = \inf_{\mathbf{u} \in \mathbb{E} \setminus \{0\}} \frac{\| (u_1, u_2) \|^2}{\left( \mu_1 \int_{\mathbb{R}^n} u_1^4 \, dx + \mu_2 \int_{\mathbb{R}^n} u_2^4 \, dx + 2\beta \int_{\mathbb{R}^n} u_1^2 u_2^2 \, dx \right)^{\frac{1}{2}}}
\]
\[
= \inf_{\mathbf{u} \in \mathbb{E} \setminus \{0\}} \frac{\| (u_1, u_2) \|^2}{\left( \mu_1 \int_{\mathbb{R}^n} u_1^4 \, dx + \mu_2 \int_{\mathbb{R}^n} u_2^4 \, dx + 2\beta \int_{\mathbb{R}^n} u_1^2 u_2^2 \, dx \right)^{\frac{1}{2}}}.\]

\[\tag{3.4}\]

\section{Existence of bound states}

In order to prove the existence of a bound state solution to System \((1.2)\), first we demonstrate a reduced result to both, clarify and also simplify the proof of the main result in Theorem 4.4. To do so we consider the following 3-system of NLSE
\[
(S_3) \equiv \begin{cases} 
-\Delta u_1 + \lambda_1 u_1 & = \mu_1 u_1^4 + \beta_{12} u_1^2 u_2^2 + \beta_{13} u_3^2 u_1^2 \\
-\Delta u_2 + \lambda_2 u_2 & = \mu_2 u_2^4 + \beta_{21} u_2^2 u_1^2 + \beta_{23} u_3^2 u_2^2 \\
-\Delta u_3 + \lambda_3 u_3 & = \mu_3 u_3^4 + \beta_{31} u_3^2 u_1^2 + \beta_{32} u_2^2 u_3^2 
\end{cases}
\]
with \( \lambda_j, \mu_j > 0, u_j \in H, j = 1, 2, 3 \) and \( \beta_{jk} = \beta_{kj} \) for all \( j, k = 1, 2, 3; j \neq k \).

\begin{remark}
Let
\[
\sigma^2_\lambda = \inf_{\varphi \in \mathbb{E} \setminus \{0\}} \frac{\| \varphi \|^2_{L^4}}{\| \varphi \|_{L^4}},
\]
denotes the best Sobolev constant in the embedding of \((E, \| \cdot \|)\) into \(L^4(\mathbb{R}^n)\), where
\[
\| u \|^2_{\Lambda} = \int_{\mathbb{R}^n} (|\nabla u|^2 + \lambda u^2) \, dx.
\]
Easily one obtains that \( \sigma_\lambda \) is attained at
\[
v_\lambda(x) = \sigma^{-1}_\lambda \sqrt{\lambda} U(\sqrt{\lambda}x),
\]
and one has
\[
\sigma^4_\lambda = \lambda^2 \int_{\mathbb{R}^n} U^4(\sqrt{\lambda}x) \, dx = \lambda^2 - \frac{5}{2} \int_{\mathbb{R}^n} U^4(x) \, dx.
\]
\end{remark}
Let us set $u_0 = (u^0, v^0, U_3)$, where $(u^0, v^0), U_3$ are given by (3,3), (3,1) respectively. Then we have the following.

**Theorem 4.2** If $\lambda_1 = \lambda_2 = \lambda, \beta_{12} > \max\{\mu_1, \mu_2\}, \text{ and } \beta_{j3} = \varepsilon\beta_{j3}, j = 1, 2$, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, (S3) has a radial bound state $u_\varepsilon \geq 0$ with $u_\varepsilon \to u_0$ as $\varepsilon \to 0$. Furthermore, if both $\beta_{13}, \beta_{23} \geq 0$ then $u_\varepsilon > 0$. Notice that this result corresponds to Remark 4.3

Note that hypothesis $\beta_{12} > \max\{\mu_1, \mu_2\}$ comes from Theorem A, on the contrary (S2) have no positive solutions. Multiplying the first (third) equation of (S3) by $u_3$ and integrating on $\mathbb{R}^n$ we get

\[ (\mu_1 - \beta_{13}) \int_{\mathbb{R}^n} u_1^3 u_3 \, dx + (\beta_{12} - \beta_{23}) \int_{\mathbb{R}^n} u_1 u_2^2 u_3 \, dx + (\beta_{13} - \mu_3) \int_{\mathbb{R}^n} u_1 u_3^3 \, dx = 0. \] (4.1)

Multiplying the second (third) equation of (S3) by $u_3$ and integrating again on $\mathbb{R}^n$ we find

\[ \int_{\mathbb{R}^n} u_2^3 u_3 \, dx + (\beta_{12} - \beta_{13}) \int_{\mathbb{R}^n} u_1^2 u_2 u_3 \, dx + (\beta_{23} - \mu_3) \int_{\mathbb{R}^n} u_2 u_3^3 \, dx = 0. \] (4.2)

As a consequence, if for instance $\mu_3 < \beta_{13}$, thanks to (4.1), system (S3) has no positive solutions. Similarly, if $\mu_3 < \beta_{23}$, due to (4.1), system (S3) has no positive solutions. But $\mu_3 > \max\{\beta_{13}, \beta_{23}\}$ provided $\varepsilon_0$ is sufficiently small.

**Proof of Theorem 4.2** Taking $\varepsilon_0 > 0$ sufficiently small, we can suppose that $\mu_3 > \max\{\beta_{13}, \beta_{23}\}$ and the compatibility conditions to have non-negative solutions given in (4.1), (4.2) are satisfied. According to the notation of Section 2 with $N = 3, m = 1, \ell = 1$, we have

\[ \Phi_\varepsilon(u) = \Phi_0 - \varepsilon\tilde{F}(u) \]

where

\[ \Phi_0(u) = \sum_{j=1}^3 I_j(u_j) - \frac{1}{2} \beta \int_{\mathbb{R}^n} u_1^2 u_2^2 \, dx \]

and

\[ \tilde{F}(u) = \frac{1}{2} \int_{\mathbb{R}^n} \left( \beta_{13} u_1^2 u_3^2 + \beta_{23} u_2^2 u_3^2 \right) \, dx. \]

Let us consider the critical point $u_0$ of the unperturbed functional $\Phi_0$. We note that $U_3$ is a non-degenerate critical point of $I_3$ on $H$; see [14] and Remark 8.3. Also, as we pointed out in Remark 3.2, $(u^0, v^0)$ is a non-degenerate critical point to the corresponding energy functional (of (S2) with $\lambda_1 = \lambda_2 = \lambda$ and $\beta_{12} = \beta$) acting on $\mathbb{H}$. As an immediate consequence, $u_0$ is a non-degenerate critical point of $\Phi_0$ on $\mathbb{H}$, then a straightforward application of the Local Inversion Theorem yields the existence of a critical point $u_\varepsilon$ of $\Phi_\varepsilon$ for any $0 < \varepsilon < \varepsilon_0$ with $\varepsilon_0$ sufficiently small; see [6] for more details. Moreover, $u_\varepsilon \to u_0$ on $\mathbb{H}$ as $\varepsilon \to 0$. Notice that this result corresponds to N = 3, m = 1, ℓ = 1.
To complete the proof it remains to show that \( u_\epsilon \geq 0 \), and furthermore, in case \( \beta_{13}, \beta_{23} \geq 0 \) then \( u_\epsilon > 0 \). We follow an argument of [9] with suitable modifications by separating the positive and negative parts and using energy type estimates.

Let us denote the positive part \( u^+_\epsilon = (u^+_{1\epsilon}, u^+_{2\epsilon}, u^+_{3\epsilon}) \) and the negative part \( u^-_\epsilon = (u^-_{1\epsilon}, u^-_{2\epsilon}, u^-_{3\epsilon}) \). Notice that \((u^0, v^0)\) satisfies the identity (3.4) and \( U_3 \) satisfies the following identity

\[
\|U_3\| = \inf_{u \in H \setminus \{0\}} \left( \mu_3 \int_{\mathbb{R}^n} u^4 \, dx \right)^{1/2}.
\] (4.3)

As a consequence, from (3.4), resp. (4.3), it follows that

\[
\left( \mu_1 \int_{\mathbb{R}^n} (u^+_{1\epsilon})^4 \, dx + \mu_2 \int_{\mathbb{R}^n} (u^+_{2\epsilon})^4 \, dx + 2 \beta \int_{\mathbb{R}^n} (u^+_{1\epsilon})^2 (u^+_{2\epsilon})^2 \, dx \right)^{1/2} \geq \|(u^0, v^0)\|,
\] (4.4)

resp.

\[
\left( \mu_3 \int_{\mathbb{R}^n} (u^+_{3\epsilon})^4 \, dx \right)^{1/2} \geq \|U_3\|.
\] (4.5)

Multiplying the third equation of \((S_3)\) by \( u^+_{3\epsilon} \) and integrating on \( \mathbb{R}^n \) one infers

\[
\|u^+_{3\epsilon}\|^2 = \mu_3 \int_{\mathbb{R}^n} (u^+_{3\epsilon})^4 \, dx + \epsilon \int_{\mathbb{R}^n} \left[ (u^+_{3\epsilon})^2 (\beta_{13} u^2_{1\epsilon} + \beta_{23} u^2_{2\epsilon}) \right] \, dx
\]

\[
\leq \mu_3 \int_{\mathbb{R}^n} (u^+_{3\epsilon})^4 \, dx
\]

\[
+ \epsilon \left( \int_{\mathbb{R}^n} (u^+_{3\epsilon})^4 \, dx \right)^{1/2} \left[ \beta_{13} \left( \int_{\mathbb{R}^n} u^4_{1\epsilon} \right)^{1/2} + \beta_{23} \left( \int_{\mathbb{R}^n} u^4_{2\epsilon} \right)^{1/2} \right].
\]

This, jointly with (4.5), yields

\[
\|u^+_{3\epsilon}\|^2 \leq \frac{\|u^+_{3\epsilon}\|^4}{\|U_3\|^2} + \epsilon \vartheta_\epsilon \frac{\|u^+_{3\epsilon}\|^2}{\|U_3\|},
\]

where

\[
\vartheta_\epsilon = \mu_3^{-1/2} \left[ \beta_{13} \left( \int_{\mathbb{R}^n} u^4_{1\epsilon} \right)^{1/2} + \beta_{23} \left( \int_{\mathbb{R}^n} u^4_{2\epsilon} \right)^{1/2} \right].
\]

Since \( u_\epsilon \to u_0 \), clearly \((u_{1\epsilon}, u_{2\epsilon}) \to (u^0, v^0)\), then one has \( \epsilon \vartheta_\epsilon \to 0 \) as \( \epsilon \to 0 \). Hence, if \( \|u^+_{3\epsilon}\| > 0 \), one obtains

\[
\|u^+_{3\epsilon}\|^2 \geq \|U_3\|^2 + o(1),
\] (4.6)
where \( o(1) = o_\varepsilon(1) \to 0 \) as \( \varepsilon \to 0 \). Using again \( u_\varepsilon \to u_0 \), then \( u_{3\varepsilon} \to U_3 > 0 \) and as a consequence, for \( \varepsilon \) small enough, \( \|u_{3\varepsilon}^+\| > 0 \). Thus (4.6) gives

\[
\|u_{3\varepsilon}^+\|^2 \geq \|U_3\|^2 + o(1).
\]  

(4.7)

Multiplying now the first, resp. the second equation of (\( S_3 \)) by \( u_{1\varepsilon}^+ \), resp. \( u_{2\varepsilon}^+ \) and integrating on \( \mathbb{R}^n \) one infers

\[
\| (u_{1\varepsilon}^+, u_{2\varepsilon}^+) \|^2 = \mu_1 \int_{\mathbb{R}^n} (u_{1\varepsilon}^+)^4 \, dx + \mu_2 \int_{\mathbb{R}^n} (u_{2\varepsilon}^+)^4 \, dx
\]

\[
+ \beta \int_{\mathbb{R}^n} [(u_{1\varepsilon}^+)^2 u_{2\varepsilon}^2 + (u_{2\varepsilon}^+)^2 u_{1\varepsilon}^2] \, dx
\]

\[
+ \varepsilon \int_{\mathbb{R}^n} u_{3\varepsilon}^2 \left[ (\beta_{13}(u_{1\varepsilon}^2) + \beta_{23}(u_{2\varepsilon}^2) \right] \, dx
\]

\[
\leq \mu_1 \int_{\mathbb{R}^n} (u_{1\varepsilon}^+)^4 \, dx + \mu_2 \int_{\mathbb{R}^n} (u_{2\varepsilon}^+)^4 \, dx + 2\beta \int_{\mathbb{R}^n} (u_{1\varepsilon}^+)^2 (u_{2\varepsilon}^+)^2 \, dx
\]

\[
+ \beta \int_{\mathbb{R}^n} [(u_{1\varepsilon}^+)^2 u_{2\varepsilon}^2 + (u_{2\varepsilon}^+)^2 u_{1\varepsilon}^2] \, dx
\]

\[
+ \varepsilon \left( \int_{\mathbb{R}^n} u_{3\varepsilon}^4 \right)^{1/2} \left[ \beta_{13} \left( \int_{\mathbb{R}^n} (u_{1\varepsilon}^+)^4 \right)^{1/2} + \beta_{23} \left( \int_{\mathbb{R}^n} (u_{2\varepsilon}^+)^4 \right)^{1/2} \right] \, dx.
\]

This, jointly with \( (4.4) \), yields

\[
\| (u_{1\varepsilon}^+, u_{2\varepsilon}^+) \|^2 \leq \frac{\| (u_{1\varepsilon}^+, u_{2\varepsilon}^+) \|^4}{\| (u_0^+, v_0^-) \|^2} + \beta \int_{\mathbb{R}^n} [(u_{1\varepsilon}^+)^2 (u_{2\varepsilon}^+)^2 + (u_{2\varepsilon}^+)^2 (u_{1\varepsilon}^+)^2] \, dx
\]

\[
+ \varepsilon \psi \| (u_{1\varepsilon}^+, u_{2\varepsilon}^+) \|^2,
\]

\[
(4.8)
\]

where

\[
\psi = C \left( \int_{\mathbb{R}^n} u_{3\varepsilon}^4 \right)^{1/2} \quad \text{for some constant} \quad C = C(\mu_1, \mu_2, \lambda, \beta) > 0.
\]

Using that \( u_{3\varepsilon} \to U_3 \) we obtain

\[
\varepsilon \psi \to 0 \quad \text{as} \quad \varepsilon \to 0,
\]

\[
(4.9)
\]

and, from \( (u_{1\varepsilon}, u_{2\varepsilon}) \to (u^0, v^0) > 0 \), we get \( (u_{1\varepsilon}^+, u_{2\varepsilon}^-) \to 0 \) which implies

\[
\beta \int_{\mathbb{R}^n} [(u_{1\varepsilon}^+)^2 (u_{2\varepsilon}^-)^2 + (u_{2\varepsilon}^+)^2 (u_{1\varepsilon}^-)^2] \, dx \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

\[
(4.10)
\]

Since \( (u_{1\varepsilon}^+, u_{2\varepsilon}^+) \to (u^0, v^0) > 0 \), we have \( \| (u_{1\varepsilon}^+, u_{2\varepsilon}^+) \| > 0 \) for \( \varepsilon \) sufficiently small, thus \( (4.8)-(4.10) \) gives

\[
\| (u_{1\varepsilon}^+, u_{2\varepsilon}^+) \|^2 \geq \| (u^0, v^0) \|^2 + o(1).
\]

\[
(4.11)
\]
If \( \|(u_{1e}^{-}, u_{2e}^{-})\| > 0 \), it is not so easy to obtain a similar estimate like (4.10) with \( \|(u_{1e}^{-}, u_{2e}^{-})\|^2 \) replaced by \( \|(u_{1e}^{-}, u_{2e}^{-})\|^2 \). To do so, we need to estimate more carefully the mixed term by (4.10) since it has the same order of decay as \( \|(u_{1e}^{-}, u_{2e}^{-})\|^2 \).

By the Cauchy-Schwarz inequality, the convergence \( u_e \to u_0 \) and Remark 4.1 we have

\[
\beta \int_{\mathbb{R}^n} [(u_{1e}^-)^2 (u_{2e}^+)^2 + (u_{1e}^+)^2 (u_{2e}^-)^2] \, dx \leq \beta \left[ \left( \int_{\mathbb{R}^n} (u_{1e}^-)^4 \right)^{1/2} \left( \int_{\mathbb{R}^n} (u_{2e}^+)^4 \right)^{1/2} + \left( \int_{\mathbb{R}^n} (u_{1e}^+)^4 \right)^{1/2} \left( \int_{\mathbb{R}^n} (u_{2e}^-)^4 \right)^{1/2} \right] 
\]

\[ \leq \beta \frac{1}{\sigma^2 \lambda^2} \|u_{1e}^-\|^2 \left( \lambda^{1-\frac{n}{4}} \left( \int_{\mathbb{R}^n} U^4 \, dx \right)^{\frac{1}{2}} + o(1) \right) \frac{\beta - \mu_1}{\beta^2 - \mu_1 \mu_2} 
\]

\[ + \beta \frac{1}{\sigma^2 \lambda^2} \|u_{2e}^-\|^2 \left( \lambda^{1-\frac{n}{4}} \left( \int_{\mathbb{R}^n} U^4 \, dx \right)^{\frac{1}{2}} + o(1) \right) \frac{\beta - \mu_2}{\beta^2 - \mu_1 \mu_2} \]

\[ \leq (\rho + o(1)) \|(u_{1e}^-, u_{2e}^-)\|^2, \quad (4.12) \]

where

\[ \rho = \max \left\{ \frac{\beta - \mu_1}{\beta^2 - \mu_1 \mu_2}, \frac{\beta - \mu_2}{\beta^2 - \mu_1 \mu_2} \right\}. \quad (4.13) \]

Even more, taking into account that \( \beta > \max\{\mu_1, \mu_2\} \), then \( 0 < \rho < 1 \). As a consequence, if \( \|(u_{1e}^{-}, u_{2e}^{-})\| > 0 \), by (4.8), (4.9) and (4.12) we have the following estimate,

\[ \|(u_{1e}^{-}, u_{2e}^{-})\|^2 \geq (1 - \rho) \|(u_0, v_0)\|^2 + o(1). \quad (4.14) \]

Now, suppose by contradiction that there exists \( k \in \{1, 2, 3\} \) such that \( \|u_{3e}^-\| > 0 \). Then we have two possibilities:

- If \( k = 3 \), as in (4.6) one obtains \( \|u_{3e}^-\|^2 \geq \|U_3\|^2 + o(1) \), hence

\[ \|u_e^-\|^2 = \|(u_{1e}^{-}, u_{2e}^{-})\|^2 + \|u_{3e}^-\|^2 \geq \|U_3\|^2 + o(1). \quad (4.15) \]

Next, we evaluate the functional \( \Phi(u_e) = \frac{1}{2} \|u_e\|^2 = \frac{1}{4} \left[ \|u_e^+\|^2 + \|u_e^-\|^2 \right] \).

On one hand, using (4.7), (4.11) and (4.15) we infer

\[ \Phi(u_e) \geq \frac{1}{4} \|u_0\|^2 + \frac{1}{4} \|U_3\|^2 + o(1). \quad (4.17) \]

On the other hand, since \( u_e \to u_0 \) we also find

\[ \Phi(u_e) = \frac{1}{4} \|u_e\|^2 \to \frac{1}{4} \|u_0\|^2, \quad (4.18) \]

which is in contradiction with (4.17), proving that \( u_{3e} \geq 0 \).
• If $k \in \{1, 2\}$, by (4.14) we have
\[
\|u_{\varepsilon}^k\|^2 = \|(u_{1\varepsilon}, u_{2\varepsilon})\|^2 + \|u_{3\varepsilon}^k\|^2 \geq (1 - \rho)\|(u^0, v^0)\|^2 + o(1).
\] (4.19)

Using (4.11), (4.14), (4.16) and (4.19), we get
\[
\Phi(u_{\varepsilon}) \geq \frac{1}{2}\|u_0\|^2 + \frac{1}{2}(1 - \rho)\|(u^0, v^0)\|^2 + o(1).
\]

This is a contradiction with (4.18), proving that $(u_{1\varepsilon}, u_{2\varepsilon}) \geq 0$.

In conclusion, we have proved that $u_{\varepsilon} \geq 0$. Finally, if $\beta_{12}, \beta_{23} \geq 0$, using once more that $u_{\varepsilon} \to u_0$ and applying the maximum principle it follows that $u_{\varepsilon} > 0$.

Let us set
\[
z = ((u_1^0, v_1^0), \ldots, (u_m^0, v_m^0), U_{2m+1}, \ldots, U_N),
\] (4.20)
where $(u_k^0, v_k^0)$ is given by (3.2) with the parameters $\lambda = \lambda_{2k}$, $\mu_1 = \mu_{2k-1}$, $\mu_2 = \mu_{2k}$, $\beta = \beta_k$, and $1 \leq k \leq m$; see Theorem A. And
\[
U_{2m+\ell}(x) = \sqrt{\frac{\lambda_{2m+\ell}}{\mu_{2m+\ell}}} I (\sqrt{\lambda_{2m+\ell}} x)
\]
where $U$ is given by (3.1); see Remark 3.1.

The main result of the paper is the following.

**Theorem 4.4** If $\beta_k > \max\{\mu_{2k-1}, \mu_{2k}\}$ for $1 \leq k \leq m$, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, (1.2) has a radial bound state $u_{\varepsilon} \geq 0$ with $u_{\varepsilon} \to z$ as $\varepsilon \to 0$. Furthermore, if $\beta_{m+\ell} \geq 0$ for all $1 \leq \ell \leq N - 2m$ then $u_{\varepsilon} > 0$.

**Remark 4.5** Notice that a condition on the coupling factors like in Remark 4.3 holds provided $\varepsilon_0$ is small enough.

**Proof of Theorem 4.4** We follow, with suitable modifications, the arguments in the proof of Theorem 4.2.

Let us consider the critical point $z$ (defined in (4.20)) of the unperturbed functional $\Phi_0$. We note that $U_{2m+\ell}$ is a non-degenerate critical point of $Z_{2m+\ell}$ on $H$ for each $\ell$; see [14] and Remark 3.1. Also, $(u_k^0, v_k^0)$ is a non-degenerate critical point of the corresponding functional $\Phi_k$ described in (4.20) for each $1 \leq k \leq m$; see Remark 3.2. As an immediate consequence, $u_0$ is a non-degenerate critical point of $\Phi_0$ on $H$, then the Local Inversion Theorem provides us with the existence of a critical point $u_{\varepsilon}$ of $\Phi_{\varepsilon}$ for any $\varepsilon < \varepsilon_0$ with $\varepsilon_0$ sufficiently small. Furthermore, $u_{\varepsilon} \to u_0$ on $H$ as $\varepsilon \to 0$. To complete the proof it remains to show that $z \geq 0$ and if $\beta_{m+\ell} \geq 0$ for all $1 \leq \ell \leq N - 2m$, then $u_{\varepsilon} > 0$.

Following the notation of $z$ by (4.20), let us denote
\[
u_{\varepsilon} = ((u_1^\varepsilon, v_1^\varepsilon), \ldots, (u_m^\varepsilon, v_m^\varepsilon), u_{2m+1}^\varepsilon, \ldots, u_N^\varepsilon).
\]
Using that $u_\varepsilon \to z > 0$, as in (4.7) and (4.11) it follows
\[ \|(u_k^+, v_k^+)\|^2 \geq \|(u_k^0, v_k^0)\|^2 + o(1) \quad \text{for any } 1 \leq k \leq m, \]
(4.21)
and
\[ \|(u_{2m+\ell}^+)\|^2 \geq \|U_{2m+\ell}\|^2 + o(1) \quad \text{for any } 1 \leq \ell \leq N - 2m. \]
(4.22)
Suppose by contradiction that there exists either $1 \leq k_0 \leq m$, or $1 \leq \ell_0 \leq N - 2m$ such that either
\[ \|(u_{k_0}^-, v_{k_0}^-)\| > 0, \]
(4.23)
or
\[ \|(u_{2m+\ell_0}^-)\| > 0. \]
(4.24)
• If (4.23) holds, by (4.14)
\[ \|(u_{k_0}^-, v_{k_0}^-)\|^2 \geq (1 - \rho_{k_0})\|(u_{k_0}^0, v_{k_0}^0)\|^2 + o(1), \]
(4.25)
for some $0 < \rho_{k_0} < 1$ defined by (4.13) with the parameters $\lambda = \lambda_{2k_0}$, $\mu_1 = \mu_{2k_0-1}$, $\mu_2 = \mu_{2k_0}$, $\beta = \beta_{k_0}$. Then we obtain
\[ \|u_{k_0}^-\|^2 \geq (1 - \rho_{k_0})\|(u_{k_0}^0, v_{k_0}^0)\|^2 + o(1), \]
(4.26)
and by (4.21), (4.25) we have the following inequality
\[ \Phi(u_\varepsilon) \geq \frac{1}{4}\|z\|^2 + \frac{1}{4}(1 - \rho_{k_0})\|(u_{k_0}^0, v_{k_0}^0)\|^2 + o(1). \]
(4.27)
• If (4.24) holds, by (4.7)
\[ \|(u_{2m+\ell_0}^-)\|^2 \geq \|U_{2m+\ell_0}\|^2 + o(1), \]
(4.28)
which implies
\[ \|u_{\ell_0}^-\|^2 \geq \|U_{2m+\ell_0}\|^2 + o(1), \]
(4.29)
and by (4.22) and (4.28) the following inequality holds
\[ \Phi(u_\varepsilon) \geq \frac{1}{4}\|z\|^2 + \frac{1}{4}\|U_{2m+\ell_0}\|^2 + o(1). \]
(4.30)
Since $u_\varepsilon \to z$ we have
\[ \Phi(u_\varepsilon) = \frac{1}{4}\|u_\varepsilon\|^2 \to \frac{1}{4}\|z\|^2. \]
This is a contradiction with (4.27) and also with (4.30), proving that $u_\varepsilon \geq 0$.

To finish, if $\beta_{m+\ell} \geq 0$ for all $1 \leq \ell \leq N - 2m$, using once more that $u_\varepsilon \to z$ and applying the maximum principle it follows that $u_\varepsilon > 0$. □

If $2m = N$, or equivalently $\ell = 0$, we set $z_1 = ((u_0^0, v_0^0), \ldots, (u_m^0, v_m^0))$, and if $m = 0$, or equivalently $\ell = N$, we set $z_2 = (U_1, \ldots, U_N)$. Then as a consequence of Theorem 4.4 we have the following.
Corollary 4.1 For $\varepsilon$ small enough,

(i) if $2m = N \iff \ell = 0$, then $[\text{(1.2)}]$ has a radial bound state $u^{(1)}_{\varepsilon} > 0$ and $u^{(1)}_{\varepsilon} \to z_1$ as $\varepsilon \to 0$.

(ii) if $m = 0 \iff \ell = N$, then $[\text{(1.2)}]$ has a radial bound state $u^{(2)}_{\varepsilon} \geq 0$ and $u^{(2)}_{\varepsilon} \to z_2$ as $\varepsilon \to 0$. Furthermore, if $\beta_\ell \geq 0$ for all $1 \leq \ell \leq N$ then $u^{(2)}_{\varepsilon} > 0$.

We observe that Corollary 4.1-(ii) is nothing but [4, Theorem 6.4].

Remark 4.6 Note that in Theorem 4.4 some of the coupling factors $\beta_k$, $1 \leq k \leq m$ are big while the others $\beta_{k+\ell}$, $1 \leq \ell \leq N-2m$ are small, and some of these can be negative giving rise to a non-negative new non-trivial bound states. Similar remark can be done for Corollary 4.1(ii). $lacksquare$

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