Solution of constant Yang–Baxter system in the dimension two

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Abstract

Complete solution of the Yang–Baxter system (3) – (6) in the dimension 2, more precisely, all invertible $4 \times 4$ matrices $R, Q$ that solve the system are given.

1 Introduction – origin of the Yang–Baxter system

The quantised braided groups were introduced recently in [1] combining Majid’s concept of braided groups [2] and the FRT formulation of quantum supergroups [3]. The generators of quantised braided groups $T = T^i_j, i, j \in \{1, \ldots, d = \dim V\}$ satisfy the algebraic and braid relations

$$Q_{12} R^{-1}_{12} T_1 R_{12} T_2 = R_{21}^{-1} T_2 R_{21} T_1 Q_{12}$$

(1)

$$\psi(T_1 \otimes R_{12} T_2) = R_{12} T_2 \otimes R_{12}^{-1} T_1 R_{12}$$

(2)

where the numerical matrices $Q, R$ satisfy the system of Yang–Baxter–type equations

$$Q_{12} Q_{13} Q_{23} = Q_{23} Q_{13} Q_{12},$$

(3)

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12},$$

(4)

$$Q_{12} R_{13} R_{23} = R_{23} R_{13} Q_{12},$$

(5)

$$R_{12} R_{13} Q_{23} = Q_{23} R_{13} R_{12}.$$  

(6)

The special cases of the quantised braided groups are the quantum supergroups [4], quantum anyonic groups [4] and braided $GL_q$ groups [3].
For obtaining all possible quantized braided groups we have to find all solutions of this Yang–Baxter system. (3) – (6). Besides that solution of this system give special solution of a more complicated system, see e.g. [6]

\begin{align*}
Q_{12}Q_{13}Q_{23} &= Q_{23}Q_{13}Q_{12}, \quad Q_{12}Q_{13}Q_{23} = Q_{23}Q_{13}Q_{12}, \\
R_{12}R_{13}R_{23} &= R_{23}R_{13}R_{12}, \quad R_{12}R_{13}Q_{23} = Q_{23}R_{13}R_{12}. 
\end{align*}

Indeed, if \((Q, R)\) is a solution of the system of Yang–Baxter type equations (3 – 6) then \((Q, R, Q = P R P R^{-1})\) is a solution of the system (7,8). The latter system is, on the other hand, a special case of Yang–Baxter type equations that appeared in [7] as a consistency conditions for quantization of nonultralocal models (see also [8]).

There are several simple solutions of the system (3–6). One of them is \(R = 1, Q\) - any solution of the Yang–Baxter equation (YBE). This gives the algebras that correspond to the ordinary (unbraided) quantum groups in the FRT formulation. Rather trivial solution is \(R, Q = P\) where P is the permutation matrix and \(R\) any solution of the YBE. This solution does not yield any algebraic relation because (1) is satisfied identically in this case.

Other simple solutions are \((R, Q = R)\) or \((R, Q = PR^{-1}P)\), \(R\) being any solution of the YBE. For \(R\) that is both invertible and has the so called second inversion \((R^{(2)})^{-1}\) they correspond to the (unquantised) braided groups [10].

In the paper [1] several nontrivial solutions of (3) – (6) were presented but in general there is very little experience with the Yang–Baxter systems to the best knowledge of the author.

The goal of this paper is to give the complete solution of the Yang–Baxter system (3) – (6) in the dimension 2, more precisely, find all invertible 4 × 4 matrices \(R, Q\) that solve (3) – (6). The starting point for this is the Hietarinta’s classification of 4 × 4 solutions of the YBE [11, 12].

2 Solution of the system

Before we start solving the system (3) – (6) we shall discuss its symmetries. It is well known that each of the YBE (3) and (4) have both continuous and discrete groups of symmetries. However, the group of symmetries of the system (3) – (6) is not the product of the groups of symmetries (3) and (4). On the other hand it is easy to check that the system (3) – (6) is invariant under

\begin{align*}
Q' &= \lambda(S \otimes S)Q(S \otimes S)^{-1}, \quad R' = \kappa(S \otimes S)R(S \otimes S)^{-1}, \quad \lambda, \kappa \in \mathbb{C}, \quad S \in SL(2, \mathbb{C}) \quad (9)
\end{align*}

and

\begin{align*}
Q'' &= Q^+ = PQP, \quad R'' = R^+ = PRP. 
\end{align*}

They are the symmetries we are going to use.

The procedure we adopt for solving the system (3) – (6) is the following. From the Hietarinta’s classification [11, 12] we know that up to the symmetries (9,10) there are just eleven invertible, i.e. rank 4, solutions of (3) (see also [13, 14]). For each of
them it is rather easy to solve the linear homogenous equations (3, 4) for \( Q \). Inserting
the solution of the linear equations into (3) we get equations for the coefficients of
linear combinations of matrices that solve (3, 4). Solving them we get the solution
of the whole system.

As mentioned in the introduction, it is clear that among others we must obtain
solutions \( Q = P, Q = R, Q = PR^{-1}P \) (not necessarily different) for each \( R \). The
interesting cases are those when some other solutions appear. As we shall see they
are rather rare.

### 2.1 Generic cases

To demonstrate the method sketched above and show a typical and simple example
we choose

\[
R = \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 1 & -1 & 0 \\
-1 & 0 & 0 & 1
\end{pmatrix}.
\]

This is the YBE solution \( R_{H0.2} \) in the classification [11, 12] or \( R_1 \) in [13]. The
solution of the linear equations (3)-\( (3, 4) \) in this case is

\[
Q = \alpha \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} + \beta \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix},
\]

and from the YBE (3) we get cubic equation for the coefficients \( \alpha \) and \( \beta \)

\[
\beta(\beta^2 - \alpha^2) = 0.
\]

It is easy to check that three solutions \( \beta = 0, \pm \alpha \) of this equation correspond just to
the expected solutions \( Q = \alpha P, Q = \alpha R, Q = \alpha PR^{-1}P \) and no other exists. These
solution, where \( R \) have the second inversion, yield (unquantized) braided group or
free algebras (for \( Q = P \)).

The same situation, namely that the only solutions of (3)-(4) are \((R, \alpha P), (R, \alpha R), (R, \alpha PR^{-1}P)\), happens for all but four invertible YBE solutions \( R \) of the
Hietarinta’s list.

The exceptional cases are investigated in the next section.

### 2.2 Special cases

The simplest solution of (3) that admits a wider set of solutions of the Yang–Baxter
system (3, 4) than just \( Q \propto P, R, PR^{-1}P \) is

\[
R = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & -1
\end{pmatrix}.
\]

3
This is a special case of the YBE solution $R_{H1.2}$ in the classification \[11, 12\] (or $R_3$ in \[13\]). The solutions of the linear equations (5,6) in this case form four-dimensional space

$$Q = \begin{pmatrix} 
\alpha + \beta & 0 & 0 & 0 \\
0 & \alpha - \delta & \beta + \delta & 0 \\
0 & \alpha & \beta & 0 \\
\gamma & 0 & 0 & \delta 
\end{pmatrix}$$  \(14\)

and from the YBE (3) we get three simple cubic equations for the coefficients $\alpha$, $\beta$, $\gamma$ and $\delta$

$$\alpha \gamma (\beta + \delta) = 0, \quad \alpha \beta (\beta + \delta) = 0, \quad \alpha (\alpha - \delta) (\beta + \delta) = 0.$$  \(15\)

The invertible solutions of these equations up to the transformation are $Q = \alpha P$ or

$$Q = \beta \begin{pmatrix} 
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
r & 0 & 0 & -t 
\end{pmatrix}.$$  \(16\)

Note that in general, the matrix $Q$ is proportional neither to $R$ nor $PR^{-1}P$.

The second type of nontrivial solutions to the system (3) – (6) is obtained when

$$R = \begin{pmatrix} 
0 & 0 & 0 & 1 \\
0 & 0 & t & 0 \\
0 & t & 0 & 0 \\
1 & 0 & 0 & 0 
\end{pmatrix}.$$  \(17\)

This is the solution $R_{H1.4}$ in the Hieterinta’s classification \[11, 12\] or $R_9$ in \[13\]. The solution space of the linear equations (5,6) in this case is three-dimensional

$$Q = \begin{pmatrix} 
\alpha & 0 & 0 & \gamma \\
0 & \beta & 0 & 0 \\
0 & \beta & 0 & 0 \\
\gamma & 0 & 0 & \alpha 
\end{pmatrix}$$  \(18\)

and two cubic equations for the coefficients $\alpha$, $\beta$ and $\gamma$ are obtained from the YBE (3)

$$\alpha^2 \gamma = 0, \quad \alpha (\gamma^2 + \alpha \beta - \beta^2) = 0.$$  \(19\)

They yield $Q = \alpha P$,

$$Q = \gamma \begin{pmatrix} 
0 & 0 & 0 & 1 \\
0 & 0 & t' & 0 \\
0 & t' & 0 & 0 \\
1 & 0 & 0 & 0 
\end{pmatrix}.$$  \(20\)

or noninvertible matrices. Once again the matrix $Q$ given by (20) is proportional neither to $R$ nor $PR^{-1}P$, in general.
The third class of nontrivial solutions of the Yang–Baxter system (3) – (6) come from

\[
R = \begin{pmatrix}
1 & 0 & 0 & 0 \\
x & 1 & 0 & 0 \\
y & 0 & 1 & 0 \\
z & y & x & 1
\end{pmatrix}.
\] (21)

This is the YBE solution \(R_{H23}\) in the classification \([11, 12]\) or \(R_{10}\) in \([14]\). The solutions of the linear equations (5,6) in this case form six-dimensional space

\[
Q = \begin{pmatrix}
\alpha_1 & 0 & 0 & 0 \\
\beta_1 & \alpha_2 & \alpha_1 - \alpha_2 & 0 \\
\beta_2 & \alpha_1 - \alpha_2 & \alpha_2 & 0 \\
\gamma & \delta_1 & \beta_1 + \beta_2 - \delta_1 & \alpha_1
\end{pmatrix}
\] (22)

and solutions of the set of cubic equations for the coefficients \(\alpha, \ldots, \delta_1\) that we get from the YBE (3) give two noninvertible matrices \(Q\).

\[
Q = \begin{pmatrix}
1 & 0 & 0 & 0 \\
 a & 1 & 0 & 0 \\
b & 0 & 1 & 0 \\
c & b & a & 1
\end{pmatrix}
\] (23)

and

\[
Q = \begin{pmatrix}
1 & 0 & 0 & 0 \\
-g & 1 & 0 & 0 \\
g & 0 & 1 & 0 \\
-gh & h & -h & 1
\end{pmatrix}.
\] (24)

These solutions were found in \([1]\) and the corresponding quantized braided groups were presented.

The last special case is given by diagonal \(R\). This possibility was investigated in the paper \([1]\) and the following lemma was proved.

**Lemma:** If \(R\) is a diagonal matrix, \(R = diag(x, u, v, y)\), \(xuvy \neq 0\), then there are three types of solutions of the system (3) – (6):

1) \(R \propto I\) i.e. \(x = u = v = y\) and \(Q\) is an arbitrary YBE solution.
2) \(R\) is proportional to sign–diagonal, i.e. \(x^2 = u^2 = v^2 = y^2\), and \(Q\) is an arbitrary YBE solution of the eight–or–less–vertex form

\[
Q = \begin{pmatrix}
q & 0 & 0 & a \\
0 & r & b & 0 \\
c & s & 0 & 0 \\
d & 0 & 0 & t
\end{pmatrix}
\] (25)

3) \(R\) is general diagonal matrix and \(Q\) is an arbitrary six–or–less–vertex YBE solution i.e. of the form (25) where \(a = d = 0\).

When classifying solutions following from these three possibilities we must remember that in general we do not have at our disposal the symmetry transformation

\[
Q' = (S \otimes S)Q(S \otimes S)^{-1}, \ S \in SL(2, C)
\] (26)
used in the classification of the YBE solutions because now it must be accompanied by
\[ R' = (S \otimes S)R(S \otimes S)^{-1} \] (27)
that might have been used for bringing \( R \) to the diagonal form. It means that we can use only such transformation of \( Q \) that keep \( R \) diagonal.

For scalar \( R \) i.e. \( R = \lambda \mathbf{1} \) this is no restriction so that we get as many nonequivalent solutions of the system (8) – (6) as there are solutions of the YBE [11, 12].

For the other types of diagonal \( R \), it rather easy to see that we can use only transformations (26, 27) generated either by diagonal or antidiagonal matrix \( S \). It means that we must classify the corresponding type of the YBE solutions up to this special type of symmetries.

Classification of the six–or–less vertex solutions up to this restricted type of transformations is not different from the usual one [13]. There are just four invertible solutions of the YBE, namely
\[
\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q - t & qt & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \quad \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q - t & qt & 0 \\ 0 & 0 & 0 & -t \end{pmatrix}, \quad \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix},
\]
and the permutation matrix.

On the other hand, classification of the invertible eight–or–less vertex YBE solutions (25) up to the restricted transformations is different from the usual one because besides the solutions presented in the classification [13], there are YBE solutions
\[ Q = \begin{pmatrix} a & 0 & 0 & x \\ 0 & \pm a & x & 0 \\ 0 & x & \pm a & 0 \\ x & 0 & 0 & a \end{pmatrix}, \quad x, a \neq 0 \]
(29)
that can be transformed solution to those in the paper [13] only by non-(anti)diagonal transformations (26, 27).

3 Conclusions

We have classified all invertible solutions of the Yang–Baxter system (3) – (6) up to the transformations (9, 10). They are given by pairs \( (R, Q) \) where \( R \) belong to the Hietarinta’s list [11, 12].

For all but four types of invertible solutions from the Hietarinta’s list, the only solutions of the Yang–Baxter system (3) – (6) are pairs \( (R, \lambda R) \) or \( (R, \lambda PR^{-1}P) \) or \( (R, \lambda P) \) where \( P \) is the permutation matrix.

The list of exceptional solutions of the Yang–Baxter system (3) – (6) up to the transformations (9, 10) is
\[ R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 - t & t & 0 \\ r & 0 & 0 & -t \end{pmatrix}, \]
(30)
\[
R = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & t & 0 \\
0 & t & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix},
Q = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & a & 0 \\
0 & a & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix},
\]
\text{(31)}

\[
R = \begin{pmatrix}
1 & 0 & 0 & 0 \\
x & 1 & 0 & 0 \\
y & 0 & 1 & 0 \\
z & y & x & 1
\end{pmatrix},
Q = \begin{pmatrix}
1 & 0 & 0 & 0 \\
a & 1 & 0 & 0 \\
b & 0 & 1 & 0 \\
c & b & a & 1
\end{pmatrix},
\]
\text{(32)}

\[
R = \begin{pmatrix}
1 & 0 & 0 & 0 \\
x & 1 & 0 & 0 \\
y & 0 & 1 & 0 \\
z & y & x & 1
\end{pmatrix},
Q = \begin{pmatrix}
1 & 0 & 0 & 0 \\
-g & 1 & 0 & 0 \\
g & 0 & 1 & 0 \\
-g h & h & -h & 1
\end{pmatrix},
\]
\text{(33)}

\[
R = \begin{pmatrix}
1 & 0 & 0 & 0 \\
x & 0 & 0 & 0 \\
y & 0 & y & 0 \\
z & 0 & 0 & z
\end{pmatrix},
Q = \begin{pmatrix}
q & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & q - t & qt & 0 \\
0 & 0 & 0 & q
\end{pmatrix},
\]
\text{(34)}

\[
R = \begin{pmatrix}
1 & 0 & 0 & 0 \\
x & 0 & 0 & 0 \\
y & 0 & y & 0 \\
z & 0 & 0 & z
\end{pmatrix},
Q = \begin{pmatrix}
q & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & q - t & qt & 0 \\
0 & 0 & 0 & -t
\end{pmatrix},
\]
\text{(35)}

\[
R = \begin{pmatrix}
1 & 0 & 0 & 0 \\
x & 0 & 0 & 0 \\
y & 0 & y & 0 \\
z & 0 & 0 & z
\end{pmatrix},
Q = \begin{pmatrix}
1 & 0 & 0 & 0 \\
a & 0 & 0 & 0 \\
b & 0 & 0 & c \\
0 & 0 & 0 & c
\end{pmatrix},
\]
\text{(36)}

and pairs \((R, Q)\) where

\[
R = \begin{pmatrix}
1 & 0 & 0 & 0 \\
x & 1 & 0 & 0 \\
y & 0 & 1 & 0 \\
0 & 0 & z & 1
\end{pmatrix},
x^2 = y^2 = z^2 = 1
\]
\text{(37)}

and \(Q\) is any solution of the following list

\[
Q = \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 1 & -1 & 0 \\
-1 & 0 & 0 & 1
\end{pmatrix},
Q = \begin{pmatrix}
1 + t & 0 & 0 & 1 \\
0 & s & 1 & 0 \\
0 & 1 & s & 0 \\
1 & 0 & 0 & 1 - t
\end{pmatrix},
s^2 = 1 + t^2,
\]
\text{(38)}

\[
Q = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 - t & t & 0 \\
1 & 0 & 0 & - t
\end{pmatrix},
Q = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}
\]
\text{(39)}
\[
Q = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & t & 0 \\
0 & t & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}, \quad Q = \begin{pmatrix}
a & 0 & 0 & 1 \\
0 & b & 1 & 0 \\
0 & 1 & b & 0 \\
1 & 0 & 0 & a
\end{pmatrix}, \quad b^2 = a^2.
\] (40)

This list is exhaustive i.e. any other invertible solution of the system (3) – (6) can be obtained by the transformations \((9, 10)\).

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