On the physical contents of q-deformed Minkowski spaces

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Abstract

Some physical aspects of q-deformed spacetimes are discussed. It is pointed out that, under certain standard assumptions relating deformation and quantization, the classical limit (Poisson bracket description) of the dynamics is bound to contain unusual features. At the same time, it is argued that the formulation of an associated q-deformed field theory is fraught with serious difficulties.

1 Introduction

Since the early days of quantum theory there have been attempts to quantize the spacetime manifold, the latest one being probably related to superstring theory. This question is now being actively discussed using different arguments and from different backgrounds (see, e.g., [1] and references therein).

A recent generalization of Lie groups and algebras, the quantum groups (or deformations of the algebra of functions on Lie groups and of their enveloping algebras) provides another framework to construct non-commutative spacetime coordinates. This is achieved by deforming the kinematical Lie groups and in particular the Poincaré and Lorentz groups [2, 3]. In this scheme, a quantum Minkowski spacetime $\mathcal{M}_q$ may be introduced by extending to the quantum ($q \neq 1$) case the ‘classical’ or undeformed ($q = 1$) Lie group construction, by which spacetime (a four-vector) is constructed out of two (dotted and undotted) spinors.

It has been pointed out [4] that the $R$-matrix [5] and the reflection equations (RE) (see [6, 7] and references therein as well as [8, 9, 10] in the context

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of braided quantum groups) constitute a suitable formalism to describe in an unified manner certain $q$-deformations of the Minkowski spacetime. The physical contents of these deformations is, however, up to now unclear. This letter is an attempt to discuss some general features of these deformations to see whether there are grounds to favour some of them and to check the consistency of the interpretations. In fact, some mathematical and physical (see, in particular, (14), (15), (16)) properties of deformed Poincaré algebras have been recently discussed but it is fair to say that a complete picture of a $q$-deformed physical relativistic system (not to say of a quantum field theory) is still missing.

Such a picture should include:

1. a suitable spacetime non-commutative coordinate algebra $\mathcal{M}_q$
2. a deformed Poincaré group and its coaction on the spacetime coordinates
3. a deformed Poincaré algebra, part of it generated by the deformed momenta
4. an appropriate definition of phase space from the $q$-spacetime coordinates and $q$-momenta, as well as the associated algebra of observables.

Apart from these considerations, a precise relation between $q$-deformation ($q \neq 1$) and quantization ($\hbar \neq 0$) has to be postulated if $q$ and $\hbar$ are not independent constants (see below). In fact, once commutation relations among the $q$-coordinates and $q$-momenta covariant under the action of the $q$-Poincaré group are determined, it is possible to interpret them as:

a) the algebra of quantum observables, and then study its irreducible representations (see (13), (2), (7))
b) as a ‘classical’ algebra of ‘$q$-numbers’ with a possible further quantization to introduce Planck’s constant (here, usually a $q$-path integral formalism is invoked, see e.g. (18)).

The second interpretation is in line with the quantization of mechanics with Grassmann variables and supersymmetry. Here we shall take the ‘standard’ point of view, so that non-commuting quantities will already be considered as operators. As a result, the elements of the deformed Poincaré group(s) must be treated also as operators. This confers the group parameters a certain dynamical character absent in the undeformed case, in which they are real numbers. Thus, the symmetry and the transformation properties of a relativistic system with respect to such a quantum group are different from those studied in (19), (20), where the ground state and the Hamiltonian are invariant under the coaction of the quantum group, which plays the rôle of an internal symmetry and is not the quantum kinematical group itself.

Since the one-particle case already provides the basic ground to analyze the above problems, we shall not consider here the question of multiparticle systems, which lead to the braided Hopf algebra structure of quantum Minkowski spacetime (8), (3), (10), (21).
2 Possible $q$-Minkowski spaces

To introduce non-commutative ‘coordinates’ for spacetime it is natural to assume that this non-commutativity is determined by their transformation properties under the corresponding quantum Lorentz group. The definitions for this group to be considered here will all be based on the well established $SL_q(2, C)$ quantum group. This is defined through an ‘RTT’ relation \[5\] for the $2 \times 2$ matrix $M$ of the $SL_q(2, C)$ generator

$$R_{12} M_1 M_2 = M_2 M_1 R_{12} \quad ,$$

(1)

where $M_1 = M \otimes 1$, $M_2 = 1 \otimes M$ and $R_{12}$ is the $SL_q(2, C)$ $4 \times 4$ $R$-matrix.

Thinking of the classical homomorphism $SL(2, C) \to L$, the spacetime ‘coordinates’ are defined as the entries of a $2 \times 2$ matrix $K$ (the analog of $\sigma_{\mu} x^\mu$) transforming as

$$K \mapsto K' = K M K^\dagger \quad , \quad K = K^\dagger \quad ,$$

(2)

where in the quantum case $(M^\dagger)^{-1}$ is an independent copy of the $SL_q(2, C)$ generators matrix also satisfying $R_{12} (M_1^\dagger)^{-1} (M_2^\dagger)^{-1} = (M_2^\dagger)^{-1} (M_1^\dagger)^{-1} R_{12}$ with $q$ real and $\mathcal{P} R_{12} \mathcal{P} = R_{21} = R_{12}^{-1}$ ($\mathcal{P}$ is the permutation operator in $C^2 \otimes C^2$). The commutation properties of the generators of a $q$-Minkowski algebra defined by means of $K$ may be expressed by a RE as \[4\]

$$R_{12} (1) K_1 R_{21} (2) K_2 = K_2 R_{32} (3) K_1 R_{41} (4) \quad .$$

(3)

The covariance condition now defining the $q$-Minkowski algebra is expressed by saying that the coaction (2) preserves (3). This gives

$$R_{12} (1) M_1 M_2 = M_2 M_1 R_{12} (1) \quad , \quad M_1^\dagger R_{21} (2) M_2 = M_2 R_{21} (2) M_1^\dagger \quad ,$$

(4)

plus $R_{12} = R_{41}^\dagger$ and $R_{32} = \mathcal{P} R_{21} (2) \mathcal{P}^{-1} (R_{12} (3) = R_{21} (2))$. Thus, the coaction (2) plus the preservation of the algebra (3) does not lead to a unique $q$-Minkowski algebra $\mathcal{M}_q$ or $q$-Lorentz group (4), since there are many consistent solutions for the $4 \times 4$ matrices $R_{ij} (i = 1, ..., 4)$ in (3). Let us point out some of them

1) $R_{12} (1) = R_{12}$, $R_{21} (2) = R_{21}$. This defines the $q$-Minkowski space $\mathcal{M}_q (1)$ \[2, 3\].

2) $R_{12} (1) = R_{12}$, $R_{21} (2) = I_4$. This possibility leads to a $\mathcal{M}_q (2)$ algebra \[2, 17, 22\] isomorphic to the $GL_q(2)$ quantum group.

3) $R_{12} (1) = I_4$, $R_{21} (2) = V \equiv diag(q^2, 1, 1, q^2)$. This defines a ‘twisted’ Minkowski space $\mathcal{M}_q (3)$ (we shall use $q$ although this case \[23\] is not a proper deformation).

Once we have commutation properties preserved under a certain Lorentz quantum group defined by (4) for a specific choice of $R_{ij}$, the corresponding $q$-Lorentz algebra which acts on coordinates may be introduced by duality between them as dual Hopf algebras \[1, 24\]. Also, the $q$-momenta to be introduced below (the translation part of the $q$-Poincaré algebra) may be related
to $q$-coordinates by duality in a general setting [21]. However, as mentioned in the introduction we shall restrict ourselves to the algebraic aspects of $\mathcal{M}_q^{(i)}$.

The translation part of a $q$-Poincaré algebra ($q$-momenta) is introduced in this scheme by means of the non-commuting entries of a $2 \times 2$ matrix $Y$ which satisfies

$$R^{(1)} Y_1 R^{(3)^{-1}} Y_2 = Y_2 R^{(2)^{-1}} Y_1 R^{(4)} .$$

(5)

The matrices $R^{(i)}$ for which (5) is invariant under the coaction

$$Y \mapsto Y' = (M^\dagger)^{-1} Y M^{-1} ,$$

(6)
determine through (5) the corresponding $q$-derivative algebras $\mathcal{D}^{(i)}_q$. Because of the transformation properties (2) and (6), we may call $Y$ 'covariant' if $K$ is 'contravariant'. The $q$-Lorentz invariant scalar product is given [4] by the $q$-trace [5, 25]. For instance, a $q$-analogue of the dilatation operator $s = x^\mu \partial_\mu$ is given by

$$s = tr_q(KY) = tr(DKY) ,$$

(7)

where $D \propto tr(2) (P((R^{(1)}_1)^{-1})^t_1)$ (the trace is in the second space and the transposition in the first one; the proportionality factor is fixed by convenience being 1 for $q=1$). Mixed commutation relations are defined by an invariant inhomogeneous RE [4]

$$Y_2 R^{(1)} K_1 R^{(2)} = R^{(3)} K_1 R^{(4)^{-1}} Y_2 + J$$

(8)

which extends to the $q$-case the familiar relation $\partial_j x^i = x^i \partial_j + \delta^i_j$. A complete $q$-differential calculus (see [3] for $\mathcal{M}_q^{(1)}$) may be developed within this scheme [4, 17].

In order to extract some physical consequences of the $q$-deformation it is natural to consider the simplest $\mathcal{M}_q^{(3)}$ case [23] since, as it has been pointed out [24], it corresponds to a twisted algebra [24, 27] situation (i.e., not a proper $q$-deformation) and thus it must be simpler as the diagonal form of $V$ above already suggests. The defining properties of the coordinate and derivative algebras (3), (5), (8) may be expressed as

$$Z_1 V Z_2 = Z_2 V Z_1 , \quad D_1 V^{-1} D_2 = D_2 V^{-1} D_1 ,$$

$$D_1 Z_2 = V Z_2 D_1 V^{-1} + \mathcal{P} .$$

(9)

where we have relabelled $K=K^{(3)}=Z$ in (3), and in (5), $Y=Y^{(3)}=D$. Defining

$$Z = \begin{pmatrix} z^1 \\ z^2 \\ z^4 \\ z^3 \end{pmatrix} , \quad Z' = M Z M^\dagger , \quad D = \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} , \quad D' = (M^\dagger)^{-1} D M^{-1}$$

(10)

($Z=Z^\dagger$, $D=-D^\dagger$; we denote by $\delta_i$ the non-commuting derivatives and reserve $\partial_i$ for the ordinary commuting ones) the commuting properties of the $z^i$, $\delta_i$ are easily found from eqs. (3). As an example, the first eq. in (3) gives e.g. [23]

$$z^1 z^2 = q^2 z^2 z^1 , \quad z^1 z^3 = z^3 z^1 , \quad z^4 z^1 = q^2 z^1 z^4 ,$$

$$z^2 z^3 = q^2 z^3 z^2 , \quad z^2 z^4 = z^4 z^2 , \quad z^3 z^4 = q^2 z^4 z^3 .$$

(11)
Given the matrix elements of an algebra of one type, contravariant or covariant (cf. eqs. (2), (9)), we may use the $4 \times 4$ matrix \( \hat{V}^{\epsilon} = \epsilon_2 \mathcal{P} \epsilon_2 \) \((\epsilon_2 = 1 \otimes \epsilon, \epsilon = i\sigma_2, \epsilon \mathcal{M}^{-1} = (\mathcal{M}^{-1})^t)\),

\[
\hat{V}^{\epsilon} = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & -q^2 & 0 & 0 \\
0 & 0 & -q^2 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix},
\]

(12)

to obtain coordinates of the other type since

\[
\hat{V}^{\epsilon}(M \otimes M^*)\hat{V}^{\epsilon-1} = M^{-1 \dagger} \otimes M^{-1 t} .
\]

(13)

This change of type will be denoted by adding (or removing) an overbar to the original matrix. For instance, \( \bar{Z}^{ij} = \hat{V}^{\epsilon}_{ij}Z^{ij} \), \( \bar{Z}^{ij} = (M^\dagger - 1)\bar{Z}M^{-1} \), is covariant (if we write \( \bar{Z}^{ij} = \hat{V}^{\epsilon}_{ij}Z^{ij} \)). Similarly, \( \bar{D}^{ij} = \hat{V}^{\epsilon-1}D^{ij} \) is contravariant \((\bar{D}' = \mathcal{M} \bar{D} \mathcal{M}^\dagger)\) because of (13)). As a result, \( \text{tr}(\bar{Z}Z) = \text{tr}(Z\bar{Z}) \) is an invariant and defines the scalar product

\[
(Z, Z) \equiv \frac{1}{2} \text{tr}(\bar{Z}Z) = z^1z^3 - q^2z^2z^4 \quad (\bar{Z} = \begin{pmatrix}
z^3 & -q^2z^4 \\
-q^2z^2 & z_1^1
\end{pmatrix}).
\]

(14)

In fact, \( (Z, Z) = \frac{1}{2} \bar{Z}_{ij}g_{ij,kl}Z_{kl} \) defines the q-Lorentz invariant metric for \( \mathcal{M}_q^{(3)} \),

\[
g = \hat{V}^{\epsilon}\mathcal{P} = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -q^{-2} & 0 \\
0 & -q^{-2} & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}.
\]

(15)

If we write \( (Z, Z) = \frac{1}{2} g_{ij}z^i z^j \) with \( i, j = 1, 2, 3, 4 \), then \( g_{13} = g_{31} = 1 \) and \( g_{24} = g_{42} = -q^{-2} \). A similar procedure \[17\] may be used to introduce the metric in the other cases; for the present \( \mathcal{M}_q^{(3)} \) case the trace in (14) is the ordinary trace, cf. (1).

An important consequence of the simple (twisted) nature of the (diagonal) \( R \)-matrices defining \( \mathcal{M}_q^{(3)} \) is that its non-commuting properties may be accounted for by introducing two operators \( u, v \) such that

\[
vu = q^2uv \quad , \quad u^{-1} = u^\dagger , \quad v = v^\dagger .
\]

(16)

If we now define \( Z \) in (10) as

\[
Z \equiv \begin{pmatrix}
vx^1 \\
u^{-1}x^4
\end{pmatrix} \quad , \quad X = \begin{pmatrix}
x^1 & qx^4 \\
qx^2 & x^3
\end{pmatrix}
\]

(17)

it is simple to see that the commutation properties (14) and \( Z_1VZ_2 = Z_2VZ_1 \), imply that the \( x \) components are commuting, \( X_1X_2 = X_2X_1 \). This reduction
to the commutative case also applies to the commutation relations that two vectors \( Z \) and \( Z' \) have to satisfy so that their linear combination \( \alpha Z + \beta Z' \) is isomorphic to \( Z \) (cf. [23]) i.e., that the braided coaddition (cf. [10]) is defined. In the present case, the braiding is given by \( Z_1 V Z'_2 = Z'_2 V Z_1 \) and the same identification ([17]) leads to the trivial relation \( X_1 X'_2 = X'_2 X_1 \).

For the derivatives we may similarly introduce the realization
\[
D = \begin{pmatrix}
v^{-1} \partial_1 & u^{-1} \partial_2 \\
u \partial_1 & v \partial_3
\end{pmatrix}, \quad \bar{D} = \hat{V}^{-1} D = \begin{pmatrix}
v \partial_3 & -q^{-2} u^{-1} \partial_2 \\
-q^{-2} u \partial_4 & v^{-1} \partial_1
\end{pmatrix}, \quad (18)
\]
(\text{cf. (17)}) and find \( \partial_1 \partial_2 = \partial_2 \partial_1 \). Thus, the matrices \( X \) and \( \partial \) correspond to the ordinary (commuting) coordinates and derivatives. Indeed, it is not difficult to show that the mixed equation
\[
\bar{D}_1 V Z_2 = Z_2 V \bar{D}_1 + 2 P_- , \quad P_- = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \quad (19)
\]
is invariant, and that the above ‘undressing’ of \( Z \) and \( \bar{D} \) reduces it to
\[
\bar{\partial}_1 X_2 = X_2 \bar{\partial}_1 + 2 P_-
\]
which is just a rewriting of \( \partial_i x^j = x^i \partial_j + \delta^j_i \). This shows that, in spite of the initial non-commutativity of the entries of \( Z \) and \( D \), the use of two non-commuting variables \( (u, v) \) that commute with \( x \) and \( \partial \) allows us to ‘undress’ \( M_q^{(3)} \) and \( D_q^{(3)} \). This possibility differs from that used in the second reference in [23], where the relation between the commutative and the non-commutative coordinates is given in terms of a ‘\( q \)-bein’, hence requiring 16 additional non-commutative quantities.

At the algebra level one may use [24] only one operator \( q^L \) (\( L \) being an element of the classical Lorentz algebra) but, unlike \( u \) and \( v \), it does not commute with the coordinates. In fact, the twisting [24] of a Hopf algebra \( A \) does not change its algebra structure (the multiplication and hence commutation rules), while the coproduct is changed by the similarity transformation \( \Delta_F (\cdot) = F \Delta (\cdot) F^{-1} \) where \( F \) is an element of \( A \otimes A \) which satisfies certain requirements [24, 27]. To preserve the *-structure of the Hopf algebra \( A \) the element \( F \) must be unitary, \( F^\dagger = F^{-1} \). It was shown [26] that the \( M_q^{(3)} \) case is related to the classical Poincaré algebra through twisting. In order to respect the *-structure \textit{i.e.}, the reality condition, an \( F \) different from that in [26] must be used. Let us find this \( F \) explicitly. The commutators of the Poincaré algebra are given by
\[
[L^{mn}, L^{pk}] = g^{mk} L^{np} + g^{np} L^{mk} - g^{np} L^{nk} - g^{nk} L^{mp} ,
\]

\[
[\partial^k, L^{mn}] = g^{km} \partial^m - g^{kn} \partial^m , \quad [\partial^k, \partial^l] = 0 \quad ,
\]

\[
[\partial^k, \partial^l] = 0 \quad .
\]
where the non-zero elements of the metric \((g_{\mu\nu} = g^{\mu\nu})\) are \(g_{13} = 1 = g_{31}\) and \(g_{24} = -1 = g_{42}\) (see below eq. (15) for \(q = 1\)); \(\partial_1^\dagger = -\partial_1, \partial_3^\dagger = -\partial_3, \partial_2^\dagger = -\partial_4\).

With \(L_{12} = -L_3, L_{13} = L_5, L_{14} = -L_4, L_{23} = L_1, L_{24} = -L_6\) and \(L_{34} = -L_2\) (to compare with \[23\]) we have

\[
L_1^\dagger = -L_2, \quad L_3^\dagger = -L_4, \quad L_5^\dagger = -L_6, \quad L_6^\dagger = L_6. \tag{22}
\]

All these Poincaré Lie algebra generators are ‘primitive elements’ for the co-product \(\Delta(X) = X \otimes 1 + 1 \otimes X\). To preserve the \(*\)-structure of the Poincaré algebra in the twisting process we introduce

\[
F = q^\alpha L_5 \otimes L_6, \quad \alpha \in R, \quad F^\dagger = F^{-1}. \tag{23}
\]

This twisting operator, together with the non-linear transformations

\[
L_i = (L_1, ..., L_6) \mapsto (q^\alpha L_5 L_1, L_2 q^\alpha L_6, q^{-\alpha} L_6 L_3, L_4 q^{-\alpha} L_6, L_5, L_6) \equiv \tilde{L}_i, \quad \tag{24}
\]

\[
\left( \begin{array}{cc}
\partial_1 & \partial_2 \\
\partial_4 & \partial_3 \\
\end{array} \right) \mapsto \left( \begin{array}{cc}
\partial_1 q^\alpha L_6 & \partial_2 \\
\partial_4 & \partial_3 q^{-\alpha} L_6 \\
\end{array} \right) \equiv \tilde{\partial}, \tag{25}
\]

gives (for \(\alpha = -2\)) the ‘q-deformed’ Lorentz algebra and the non-commuting translations (the change in the form of the commutators is due to the redefinitions (24) and (23)) as well as the coproduct introduced in (23) for the \(\tilde{L}\)’s and the \(\tilde{\partial}\)’s; it also preserves the hermiticity relations (22). We see here that the matrix \(\tilde{\partial}\) provides another possibility for the ‘dressing’ of the \(\partial\)’s now in terms of one operator (cf. (18)), since its entries satisfy the same commutation relations as those of \(D\) (see \[3\]) and \(\tilde{\partial}\) is also antihermitian.

### 3 Physical considerations

Let us now discuss some physical problems of the \(q\)-deformation of spacetime, and specially of the Minkowski space \(\mathcal{M}_q^{(3)}\) as the simplest example. Due to its twisted character, we have seen that there is a basis for this deformation which coincides with the basis of the standard Poincaré algebra. This might lead us to believe that there is nothing new in this case. However, the global transformations in the deformed case introduce new non-commutative quantities (operators) and this will result in new features as it will be shown below.

Let us first discuss the quasiclassical limit \(i.e.,\), the transition from the commutation relations to the corresponding Poisson brackets; this limit is performed similarly in all cases \(\mathcal{M}_q^{(i)}\) from the appropriate eq. (3). The limit \(q = e^{\gamma h} \to 1, h \to 0\), is governed by the correspondence \(1/i h [\hat{A}, \hat{B}] \to \{A, B\}\) from the operators \(\hat{A}, \hat{B}\) to the commuting quantities \(A, B\) provided that the \(R\)-matrices are normalized as

\[
R^{(i)} = 1 + \gamma h r^{(i)} + \mathcal{O}(h^2), \tag{26}
\]
\( r^{(i)} \) being their quasiclassical counterparts. Two novel features appear as a result of the quasiclassical limit, irrespective of the specific \( q \)-deformed spacetime \( \mathcal{M}_q^{(i)} \) considered. First, the Poisson brackets (PB) of \( x^i \) and \( p_i \) (given by the limit of the commutation rules established by the RE for coordinates and derivatives) which now are \( c \)-numbers, become non-canonical, the departure from the canonical ones being governed by the new constant \( \gamma \). Secondly, although \( K' = MKM^\dagger \) becomes the usual Lorentz transformation of spacetime (for \( K = \sigma_{\mu\nu}x^\mu \)), the elements of the usual Lorentz group now have non-trivial PB in general. For instance, it follows from (4)

\[
\{M_1, M_2^\dagger\} = i\gamma(M_1 r^{(2)} M_2 - M_2^\dagger r^{(2)} M_1) .
\]

For the case of \( \mathcal{M}_q^{(3)} \), to which we shall restrict from now on, \( r^{(1)} = r^{(4)} = 0 \) and \( r^{(2)} = r^{(3)} = r = \text{diag}(1, 0, 0, 1) \).

The PB for coordinates \( Z \) and momenta \( P \) (which transforms as the derivative matrix \( D \)) for \( \mathcal{M}_q^{(3)} \) are obtained from the quasiclassical limits of (4) where \( D \) is replaced by \( P/(-i\hbar) \) and \( \delta_i \) by \( p_i/(-i\hbar) \) (cf. (13)) \( (P = P^\dagger) \). In this way, for \( q = 1 \) (4) reduces to \([\hat{x}^i, \hat{p}_j] = i\hbar\delta^i_j \). The quasiclassical limits now give

\[
\begin{align*}
\{Z_1, Z_2\} &= i\gamma[Z_1 r Z_1, \mathcal{P}] \\
\{P_1, P_2\} &= -i\gamma[P_1 r P_1, \mathcal{P}] \\
\{P_1, Z_2\} &= i\gamma[P_1 Z_2, r] - \mathcal{P} .
\end{align*}
\]

These PB are invariant under the usual Lorentz transformations provided that the entries of \( M \) and \( M^\dagger \) have zero Poisson brackets with those of \( Z \) and \( P \) (which has to be the case since they already commuted in the \( q \)-case) and (27) is valid. But as eqs. (27) and (28) (and similar ones for the other \( q \)-spacetimes \( \mathcal{M}_q^{(i)} \)) show, even at the classical level a deformed Minkowski phase space cannot be reduced to the classical one (as a dynamical or invariant relativistic system). In particular, the parameters of the Lorentz group have non-trivial Poisson brackets (27) among themselves; we obtain a Lie-Poisson group [24, 28] rather than a Lie group and its homogeneous Poisson space.

To discuss the Hamiltonian dynamics of a particle we shall assume the Dirac constraint formalism. We shall take the mass-shell constraint \( \varphi = p^2 - m^2 = (p_1 p_3 - q^{-2} p_2 p_4) - m^2 \) as usual, and introduce the additional one \( \varphi' = (z^1 - \tau) \) to eliminate the unwanted degree of freedom and separate an evolution parameter. This means that we use light-cone-like variables as suggested by the real elements of \( Z \); we shall take \( p_1 = (m^2 + |p_2|^2)/p_3 \) as energy in the quasiclassical limit. In this picture, the \( 2 \times 2 \) PB matrix \( C \) of the constraints is specially simple, \( C = \{\varphi, \varphi'\}(i\sigma_2) = -ip_3\sigma_2 \). The new (Dirac) PBs are obtained from

\[
\{A, B\}^* = \{A, B\} - (\{A, \varphi\}, \{A, \varphi'\})C^{-1}(\{\varphi, B\}, \{\varphi', B\})^t .
\]

Since \( \{P, \varphi\} = 0 \) (the d’Alembertian was already central), the PB of momenta do not change. Then, from the standard Hamiltonian equations \( \dot{A} = \{H, A\} \).
we obtain with $p_1$ as Hamiltonian
\[
\begin{pmatrix}
0 & v^4 \\
v^2 & v^3 \\
\end{pmatrix}^* = \begin{pmatrix}
0 & 0 & -p_2/p_3 - i\gamma p_1 z^4 \\
-p_4/p_3 + i\gamma p_1 z^2 & p_1/p_3 \\
\end{pmatrix} .
\]

(30)

Thus, even in this simple case, we obtain rather surprising expressions for the velocities $v^{2,4}$ which for $\gamma \neq 0$ depend on the coordinates. For the momenta we get
\[
\dot{P} = \begin{pmatrix}
\dot{p}_1 \\
\dot{p}_2 \\
\dot{p}_4 \\
\dot{p}_3 \\
\end{pmatrix} = \{p_1, P\} = i\gamma \begin{pmatrix}
0 & -p_1 p_2 \\
p_1 p_4 & 0 \\
\end{pmatrix} .
\]

(31)

Hence,
\[
p_1(\tau) = \text{const} , \quad p_3(\tau) = \text{const} ,
\]

\[
p_2(\tau) = \exp(-i\gamma p_1 \tau) p_2(0) = (p_4(\tau))^* 
\]

($p_2 = (p_4)^*$ is not conserved) and
\[
z^3(\tau) = \frac{p_1}{p_3} \tau + z^3(0) ,
\]

\[
z^2(\tau) = (z^2(0) - \frac{p_4(0)}{p_3} \tau) \exp(i\gamma p_1 \tau) = (z^4(\tau))^* .
\]

(32)

(33)

Although the limit $\gamma = 0$ reproduces the standard constant momenta and linear evolution of coordinates with respect to the parameter $\tau$, the $\gamma$-deformed behaviour is, even in this simple case, strongly oscillating in the $(x, y)$ plane and may relate points separated by spacelike intervals. Hence the dynamical trajectories do not coincide with the trajectories of the Lie group action (straight lines for translations). This is because the group action vector fields are not Hamiltonian ones due to the Lie-Poisson group nature of the problem. We stress that the PB for the other $q$-spacetimes are similar to (27), (28) and that our treatment is general. However, the equations of the motion are less transparent than those for $\mathcal{M}_{q}^{(3)}$. It is worth recalling that the crucial point in the previous analysis was the assumption $q \simeq 1 + \gamma \hbar$ between the deformation and Planck constants; in the absence of a definite ‘correspondence principle’ intertwining deformation and quantization, other assumptions could be possible. For instance, if the dependence of $\hbar$ is of higher order, $(q - 1)/\gamma \hbar \rightarrow 0$ for $\hbar \rightarrow 0$, no trace of the deformation survives in the classical theory. If, on the other hand, $q$ does not depend on $\hbar$ (i.e., the deformation is completely unrelated to quantization), then one needs an analogue of the classical mechanics for non-commuting $q$-numbers.

To complement the description of the dynamics for the simple $\mathcal{M}_{q}^{(3)}$ case, we now look at the first step towards a field theory, the free wave equation (for a discussion of $q$-wave equations see [23, 24]). In our framework, such an equation must translate the constraint $\varphi$ into a condition on the wavefunctions $\Phi(Z)$,
\[
(\Box_q + m^2)\Phi(z^1, ..., z^4) = 0 ,
\]

(34)
where \( \Box_q \) is the deformed d’Alembertian, \( \Box_q = 1/2 \, tr(\bar{D}D) = det_q D \). Let now \( \bar{P} \) be the contravariant 2×2 matrix of the eigenvalues of \( \bar{D} \), with commuting properties given by

\[
\bar{P}_1 \bar{V} \bar{P}_2 = \bar{P}_2 \bar{V} \bar{P}_1 \quad , \quad \bar{P}_1 \bar{V} \bar{Z}_2 = \bar{Z}_2 \bar{V} \bar{P}_1 \quad , \quad D_1 \bar{P}_2 = \bar{V} \bar{P}_2 D_1 \bar{V}^{-1}
\]

(i.e., the same as those for \( Z \) and \( D \) but without the inhomogeneous term in the third eq. in (3)). The scalar product \((P, Z) = (Z, P)\) between the momenta \( P \) and the coordinates \( Z \) is given (cf. (14)) by

\[
(P, Z) \equiv \frac{1}{2} tr(PZ) = \frac{1}{2} tr(\bar{Z} \bar{P}) = \frac{1}{2} (p_1 z^3 + p_3 z^1 - q^2 (p_2 z^4 + p_4 z^2)) , \tag{36}
\]

where again the covariant \( P \) is given by

\[
P = \begin{pmatrix} p_1 & p_4 \\ p_2 & p_3 \end{pmatrix} \quad , \quad \bar{P} = \begin{pmatrix} p_1 & -q^2 p_4 \\ p_2 & p_3 \end{pmatrix} . \tag{37}
\]

The scalar product (36) is invariant and central, \( [(P, Z), Z] = 0 = [(P, Z), P] \).

Since

\[
\Box_q (P, Z)^n = (P, Z)^n \Box_q + 2n(P, Z)^{n-1} (P, D) + n(n-1)(P, Z)^{n-2} (P, P) \quad , \tag{38}
\]

acting on the unity at the right \((D.1 = 0)\) this implies that

\[
\Box_q \sum_n \frac{(iP, Z)^n}{n!} = - \sum_n \frac{(iP, Z)^{n-2} (P, P)}{(n-2)!} . \tag{39}
\]

This means that we may also define here the equivalent of the Klein-Gordon plane wave since

\[
(\Box_q + m^2) \exp i(P, Z) = (m^2 - (P, P)) \exp i(P, Z) = 0 \quad \tag{40}
\]

when the mass-shell constraint \((P, P) = m^2\) is fulfilled. It is interesting to point out that the commutation relations of \( Z \) and \( D \) with the dilatation operator (7) for \( \mathcal{M}_q^{(1, 2)} \), \( s = tr(ZD) \), are the same as the undeformed ones \((sZ = Z(s+1), \quad sD = D(s-1))\) whereas for \( \mathcal{M}_q^{(3)} \) they include the deformation parameter \( q \).

The fact that the non-commuting factors \( u, v \) drop from the scalar (invariant) products and from the product of the four \( dp^i \) allows us to write

\[
\Phi_q (Z) = \int d^4 p \, \delta(p^2 - m^2)[a_q(p)e^{i(P, Z)} + h.c.] \quad , \tag{41}
\]

where \( d^4 p \) and \( \delta(p^2 - m^2) \) are the ordinary integral measure and mass shell delta function respectively. This \( q \)-scalar field thus depends in practice on commuting coordinates and momenta due to the cancellation of the \( (u, v) \) factors in the ‘undressing’ process; its only difference with respect to the standard Klein-Gordon field is in the presence of the twisting parameter \( q \) in the scalar...
products. This produces a trivial deformation of the pole structure of the naïve Green functions of the theory associated with the re-definitions $p^{2,4} \rightarrow qp^{2,4}$ of the transverse momenta and coordinates but, apart from this, it looks like the ordinary free theory. However, a closer inspection of this and the other $\mathcal{M}_q^{(i)}$ cases reveals that a proper definition of the Green functions is lacking. Moreover, due to the peculiarities of the quasiclassical limit, a possible path integral derivation of the Green functions appears to be fraught with great difficulties related to the canonical measure and action functional.

4 Conclusions

We have considered in an unified way several deformed spacetime algebras $\mathcal{M}_q^{(i)}$, the different possibilities being related to the two $R$-matrices in (4). Although $\mathcal{M}_q^{(3)}$, for instance, is very close to the standard case, and $\mathcal{P}_q^{(3)}$ has the same irreducible unitary representations, on the whole the physical picture is different from the usual one. The assumption of the standard Dirac bracket formalism and the corresponding Hamiltonian formulation gives rise to quadratic Poisson brackets of coordinates and momenta reflecting the Lie-Poisson nature of the situation. This leads to trajectories which coincide with the classical ones only when the new parameter $\gamma$ is set to zero; similar features appear in other cases with a more complicated $R$-matrix structure. In fact, the situation for the $\mathcal{M}_q^{(1,2)}$ cases is worse: for instance, it is not possible to define simultaneously $\mathcal{M}_q^{(1,2)}$ and $\mathcal{D}_q^{(1,2)}$ with the usual hermiticity properties under the star operation [3, 4, 7], which leads to rather unusual momenta (or coordinates). As for the solutions of the $q$-deformed Klein-Gordon equation as given by the $q$-d’Alembertian operator, the wave expansion requires the introduction of $q$-number parameters and the corresponding $q$-integration. This is not a problem for the simple $\mathcal{M}_q^{(3)}$ case where $q$-plane waves may be easily constructed, but the situation is much more complicated for other $q$-spacetimes.

In fact, it is not clear what should be the specific physical criteria (beyond the general ones in Sec.1) that would select the appropriate physical $q$-spacetime. As already mentioned, different $q$-Lorentz groups (see [11]) exist, and a mathematical classification of quantum Poincaré groups and their corresponding $q$-Minkowski spaces has been recently given in [32]. Although this classification includes the example [23] specially discussed here and others such as the $\kappa$-Poincaré group [11], it does not incorporate the case of [3] although, for instance, both the $q$-Minkowski spaces of [3] and [23] appear as special ones in our framework. Moreover, a detailed analysis of the physical contents of the different deformed spacetimes is lacking despite the fact that they may give rise to unusual properties such as non-commutative time (see [21] for the case of $\kappa$-Minkowski space). Our discussion indicates that a solution to these problems requires a better understanding of the possible relation between $q$-deformation and quantization, and that more work is needed to investigate
whether a $q$-deformed field theory based in this approach is feasible.

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