Localization of fermions in rotating electromagnetic fields

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(Dated: March 5, 2022)

Parameters of localization are defined in the lab and rotating frame for solutions of the Dirac equation in the field of a traveling circularly polarized electromagnetic wave and constant magnetic field. The radius of localization is of the order of the electromagnetic wavelength and lesser.

PACS numbers: 03.65.Pm, 03.65.Ta, 31.30.jx, 06.20.Jr

INTRODUCTION

Recently 3D transformation for rotating frames was deduced using the ‘invariance of the Dirac equation under rotation with a constant frequency provided that the cylindrical radius is invariable’ [1]. The transformation was obtained from very general assumptions and may be used in variety applications. With help of the transformation the wave function of Dirac equation can be translated to a rotating frame.

In the paper we study localization of fermions on an example of exact solutions of Dirac’s equation in the field of a traveling circularly polarized electromagnetic wave and constant magnetic field.

We start from a brief description of the transformation and exact localized solutions of the Dirac equation.

THE TRANSFORMATION

We use the dimensionless units

$$t \rightarrow \frac{ct}{\lambda}, \quad (x, y, z) \rightarrow \left(\frac{x}{\Omega}, \frac{y}{\Omega}, \frac{z}{\Omega}, t\right),$$

where the normalized frequency \(\Omega\) is the Compton wavelength \(\lambda = \hbar/mc\).

The transformation of the cylindrical coordinates \(\varphi, z, t\) has the form [1]

$$\tilde{\varphi} = \varphi + z\Omega - \Omega t, \quad \tilde{z} = \frac{-r^2\Omega^2 \varphi}{\sqrt{1 - r^2\Omega^2}} + z\sqrt{1 - r^2\Omega^2} + \frac{r^2\Omega^2 t}{\sqrt{1 - r^2\Omega^2}}, \quad \tilde{t} = \frac{-r^2\Omega^2}{\sqrt{1 - r^2\Omega^2}} + \frac{t}{\sqrt{1 - r^2\Omega^2}},$$

where the normalized frequency \(\Omega \rightarrow \Omega\lambda/c\).

The determinant of this transformation equals 1.

This 3D transformation is one-parametrical, that is variables with and without the tilde coincide in absence of rotation.

A distinguish feature of Eqs. (3) is an upper boundary for radius \(r^2\Omega^2 \leq 1\). In the non-normalized units this inequality is equivalent to

$$r \leq \frac{\lambda}{2\pi},$$

where \(\lambda\) is the wavelength corresponding to the frequency \(\Omega\).

The invariant transformation of the Dirac equation is realized by multiplication of this equation by operator

$$P = \exp\left(\frac{1}{2}\alpha_3\Phi_1 + \frac{1}{2}\alpha_2\Phi\right).$$

For the transformation of spinor the operator \(\tilde{P} = \beta P\beta\) is used, where \(\alpha_2, \alpha_3, \beta\) are Dirac matrices [1].

SOLUTIONS OF DIRAC’ EQUATION IN ROTATION ELECTROMAGNETIC FIELD

Consider Dirac’s equation

$$\{ -i\frac{\partial}{\partial t} - i\alpha \frac{\partial}{\partial x} - \alpha A + \beta \} \Psi = 0.$$  \(9\)

in the electromagnetic field with the potential

$$A_x = -\frac{1}{2}H_y + \frac{1}{\Omega}H\cos(\Omega t - kz), \quad A_y = \frac{1}{2}H_x + \frac{1}{\Omega}H\sin(\Omega t - kz).$$ \(10\) (11)

This potential describes a traveling circularly polarized electromagnetic wave of the frequency \(\Omega\) propagating along constant magnetic field \(H_z\). In the normalized dimensionless units the propagation constant \(k = \Omega\). The normalized potential is defined as

$$A \rightarrow \frac{e\alpha}{c\hbar} A, \quad H \rightarrow \frac{e\alpha^2}{c\hbar} H.$$ \(12\)

The Dirac’s equation [1] has exact solutions localized in the cross section perpendicular to the propagation direction of the wave [2]. In the lab frame only non-stationary states are possible. In contrast to that stationary states exist in a rotating frame.
The solutions in the lab frame can be presented as follows
\[ \Psi = \exp[-iEt + ipz - \frac{1}{2}z_1z_2(\Omega t - \Omega z) + D], \] 
where
\[ D = -\frac{d}{2}x_1^2 - i d_2 x_1 + d_2 y_1, \] 
and
\[ d = -\frac{1}{2}H_z, \quad d_2 = \frac{\mathcal{E}_0h}{2(E - \mathcal{E}_0)}. \]  
(13)

In the normalized units $d_2 \to d_2 \lambda, d \to d \lambda^2 E \to E/mc^2, \ p \to p/mc.$ $E$ obeys the characteristic equation
\[ \mathcal{E}(E + 2p - \Omega) - 1 - \frac{\mathcal{E}h^2}{E - \mathcal{E}_0} = 0, \]  
(14)

\[ \mathcal{E} = E - p, \quad \mathcal{E}_0 = \frac{2d}{\Omega}, \quad h = \frac{1}{\Omega}H, \]  
(15)

the parameter $h$ usually is small.

The parameter $\mathcal{E}_0$ in the non-normalized units is defined as
\[ \mathcal{E}_0 = -\frac{2\mu H_z}{\Omega^2}, \]  
(16)

where $\mu = e\hbar/(2mc)$ is the Bohr magneton. Equating $2/\mathcal{E}_0$ to g-factor turns the definition (18) in the classical condition of the magnetic resonance. The charge, for definiteness, is assumed to be negative $e = -|e|$. The equality (18) is kept by the sign change of $e$, provided the constant magnetic field $H_z$ has the opposite direction or by the the opposite polarization of the electromagnetic wave.

Below, for definiteness, we assume that $\mathcal{E}_0$ is always positive.

The spinor $\psi$ depend of $x_1, y_1$. A constant spinor describes the ground state, a spinor polynomial corresponds to excited state.

The spinor $\psi$ corresponding to the ground state has shape
\[ \psi = N \begin{pmatrix} h\mathcal{E} \\ -(\mathcal{E} + 1)(\mathcal{E} - \mathcal{E}_0) \\ -(\mathcal{E} - 1)(\mathcal{E} - \mathcal{E}_0) \end{pmatrix}, \]  
(19)

$N$ is the normalization constant defined by the normalization integral $\int \Psi^*\Psi ds = 1$.

\[ N^2|\hbar^2\mathcal{E}^2 + (\mathcal{E}^2 + 1)(\mathcal{E} - \mathcal{E}_0)^2| \frac{\pi}{d} \exp(\frac{d_2^2}{d}) = 1 \]  
(20)

We restrict ourselves the consideration of the ground state.

Spinor (18) as well as spinors of excited states never can be presented in the form a small and large two-component spinors. It means that the spinor describes only relativistic fermion.

Typically $\mathcal{E}$ is expanded in power series in $\hbar^2$. However, there 'singular solutions' are possible. For such solutions $\mathcal{E}$ is expanded in power series in $h$
\[ \mathcal{E}_{1,2} = \mathcal{E}_0 + h\mathcal{E}_{1,2} + h^2 \mathcal{E}_2 + \ldots, \quad \mathcal{E}_{1,2} = \pm \frac{\mathcal{E}_0}{\sqrt{\mathcal{E}_0^2 + 1}}, \]  
(21)

This expansion corresponds to a pair solutions. Odd terms in (21) have positive and negative signs for one and other solution in the pair. In the first approximation
\[ d_2 \approx \pm \frac{\sqrt{\mathcal{E}_0^2 + 1}}{2}. \]  
(22)

The necessary condition for existence of the expansion is a certain momentum for both the states in the pair.
\[ p = \frac{1}{2}(\frac{1}{\mathcal{E}_0} - \mathcal{E}_0) + \frac{1}{2} \Omega. \]  
(23)

With this momentum the energy $E$ also coincide but with accuracy $\sim h$
\[ E = \frac{1}{2}(\frac{1}{\mathcal{E}_0} + \mathcal{E}_0) + \frac{1}{2} \Omega + \ldots \]  
(24)

Eq. (18) is of third power. The third root of this equation is
\[ \mathcal{E} = -\frac{1}{\mathcal{E}_0} - \frac{\mathcal{E}_0}{(1 + \mathcal{E}_0^2)}h^2 + \ldots \]  
(25)

In the classical magnetic resonance, usually, a linearly oscillating magnetic field is applied. Such a field is a combination of two circularly polarized fields with opposite rotation. Fermion itself 'chooses' the convenient polarization. The second polarization results in a weakly dependence of spin on time, which may be neglected. But in assumed experiments with the singular solutions the polarization should be taken into account.

**LOCALIZATION**

Localization in the lab frame

The parameter localization is defined from the integral
\[ \int_{-\infty}^{+\infty} \Psi^*r^2\Psi dx dy \]  
(26)

For the wave function (13) this integral can be easily calculated. In non-normalized units
\[ \sqrt{x^2 + y^2} = \lambda \sqrt{\frac{\mathcal{E}_0^2 + 1}{\pi \mathcal{E}_0}} \]  
(27)
Localization in the rotating frame

As an example, we consider here the singular solutions. This case allows to present parameter of localization in a simple form.

The parameter of localization in the rotating frame is defined by the ratio

\[
\frac{\int \tilde{\Psi}^* r^2 \tilde{\Psi} r dr d\varphi}{\int \tilde{\Psi}^* r^2 \tilde{\Psi} r dr d\varphi}.
\] (28)

The procedure of the wave function definition in the rotating frame consist of: multiplication of \(\Psi\) by \(\exp \frac{1}{2} \alpha_1 \alpha_2 \varphi\) and operator \(\hat{P}\) by transition to the cylindrical coordinates and the rotating frame. Further \(\Psi\) should be multiplied by

\[
\exp(-\frac{1}{2} \alpha_1 \alpha_2 \varphi) = \exp(-\tilde{\varphi}_{12})
\] (29)

for transition to the Cartesian coordinates in the rotating frame. As result the wave function takes form

\[
\tilde{\Psi} = \exp[-\tilde{\varphi}_{12} + \frac{1}{2} \alpha_2 \alpha_3 \Phi_1 - \frac{1}{2} \alpha_2 \Phi + \tilde{\varphi}_{12} + D] \psi
\] (30)

The limits of integration with respect to \(\varphi\) in the rotating frame are the same because of the condition (2), while the upper limit for \(r\) bounded by the condition (17). After a simplification the normalization integral \(\int \tilde{\Psi}^* \tilde{\Psi} ds\) takes the form

\[
2 \pi \int_{\varphi=0}^{1/\Omega} \psi^* I_0(u) + \alpha_1 r \Omega I_0 u Y(-w) \psi r dr d\varphi,
\] (31)

where \(I_0(u)\) is the modified Bessel functions of the first kind, \(Y(w) = \exp(w)\); \(u = 2d_2r\), \(w = -dr^2\).

Equate

\[
r \Omega = \sin \theta,
\] (32)

and use the parameter \(\varkappa\)

\[
\varkappa = \frac{c}{\Omega \lambda} = \frac{\lambda}{\lambda C}, \quad \lambda C = 2\pi \varkappa.
\] (33)

For the singular solutions

\[
\psi^* \psi = 4h^2 E_0^2, \quad \psi^* \alpha_1 \psi = \mp 4h^2 \frac{E_0^3}{\sqrt{1 + E_0^2}},
\] (34)

\[
u = \pm \varkappa \sqrt{E_0^2 + 1} \sin \theta,
\] (35)

\[
w = -\frac{\varkappa E_0}{2} \sin^2 \theta.
\] (36)

Denote

\[
\eta = \int_{\theta=0}^{\pi/2} I_0 Y \sin \theta d\theta,
\] (37)

\[
\zeta = \int_{\theta=0}^{\pi/2} I_{0,u} Y \sin^2 \theta d\theta,
\] (38)

\[
\xi = \int_{\theta=0}^{\pi/2} I_0 Y \sin \theta \cos^2 \theta d\theta.
\] (39)

In order to carry out the calculation of ratio (28) the required integrals in the numerator can be expressed as functions of parameters \(\eta, \zeta, \xi\)

\[
\int_{\theta=0}^{\pi/2} I_0 Y \sin^3 \theta d\theta = \eta - \xi,
\] (40)

\[
\int_{\theta=0}^{\pi/2} I_{0,u} Y \sin^4 \theta d\theta = \zeta + \frac{\sqrt{E_0^2 + 1}}{E_0} \xi + \frac{3}{2 \varkappa} \zeta,\xi.
\] (41)

These parameters cannot be calculated analytically. However, taken into account that \(|\varkappa|\) is very large \(\sim 10^9\), this problem can be solved by means of an asymptotic expansion of the parameters for large \(|\varkappa|\).

Differentiate \(\eta, \zeta, \xi\) in respect to \(\varkappa\) obtain after simplification the system of equation

\[
\zeta, \xi = \pm \sqrt{E_0^2 + 1} \eta + \frac{1}{2} \sqrt{E_0^2 + 1} \xi - \frac{3}{2} \eta, \xi, -E - \frac{3}{2} \xi, \xi.
\] (42)

\[
\xi - E = \frac{E_0}{2} \xi + \frac{1}{2} \sqrt{E_0^2 + 1} \eta - \frac{3}{2} \eta, \xi, -E - \frac{3}{2} \xi, \xi.
\] (43)

\[
\eta - E = \pm \sqrt{E_0^2 + 1} \xi - \frac{E_0}{2} \eta + \frac{E_0}{2} \xi = 0.
\] (44)

Neglecting terms \(\sim 1/\varkappa\) obtain the asymptotic expansion. With accuracy \(1/\varkappa\)

\[
\xi = C_1 \rho_1,
\] (45)

\[
\eta = -E_0 C_1 \rho_1 + C_2 \rho_2 - C_3 \rho_3,
\] (46)

\[
\zeta = \pm \sqrt{E_0^2 + 1} C_1 \rho_1 + C_2 \rho_2 - C_3 \rho_3,
\] (47)

where \(C_k\) is a constant, the parameters \(\rho_k\) is

\[
\rho_1 = \exp \frac{E_0^2 + 1}{2 E_0} \varkappa,
\] (48)

\[
\rho_2 = \exp(\pm \sqrt{E_0^2 + 1} - \frac{E_0}{2} \varkappa),
\] (49)

\[
\rho_3 = \exp(\mp \sqrt{E_0^2 + 1} - \frac{E_0}{2} \varkappa).
\] (50)

Using the expansions obtain

\[
\sqrt{\frac{\int \tilde{\Psi}^* \tilde{\Psi} ds}{\int \tilde{\Psi}^* \tilde{\Psi} ds}} = \frac{\lambda}{2\pi}
\] (51)

Conclusion

We have calculated the parameter of localization, that is root from the average radius squared. Surprisingly this parameter in the rotating frame is few times smaller than that in the lab frame (27).
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