Regularization of the Reissner-Nordström black hole

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An inner de Sitter region is glued smoothly and consistently with an outer Reissner-Nordström (RN) spacetime on a spherical thin-shell. Mass and charge of the outer RN spacetime are defined by the de Sitter and shell parameters. Radius of the shell plays the role of a cut-off which by virtue of regular de Sitter inside removes the singularity at \( r = 0 \). The topology of inner de Sitter with the radius of the thin-shell becomes compact. For stability the perturbed shell is shown to satisfy a modified polytropic equation of state which has vanishing mass and pressure on the unperturbed shell as dictated by the junction conditions.

I. INTRODUCTION

Since the inception of the cut and paste technique following the seminal work of Israel’s junction conditions \cite{1} the topic of thin-shells has been popularized extensively. Application of thin-shells to wormholes \cite{2} in general relativity has been another major topic that found vast applications. In that construction (preferably) two asymptotically flat spacetimes are glued at a minimal radius that defines the throat of the wormhole \cite{3}. Through that throat an observer passes from one universe to the other easily. A wormhole may connect two black holes which may be interpreted in the language of modern physics as entanglement \cite{4}. It should also be reminded that the existence of a minimum radius lead Einstein and Rosen to interpret a wormhole as a geometrical model of a particle \cite{5}. Very special spacetimes satisfy Israel’s junction conditions \cite{1} to be glued smoothly \cite{6}. In \cite{6}, \cite{7} inner flat / Minkowski spacetime was glued to the outer extremal RN. However, Zaslavskii in \cite{8} has shown that Minkowski spacetime can not be glued smoothly to extremal RN but instead Bertotti-Robinson spacetime was successfully glued to extremal RN black hole at its horizon.

In this paper we glue an inner de Sitter with an outer Reissner-Nordström (RN) spacetime on a spherical shell that satisfies the junction conditions of smooth match. The reasons and advantages for such an option have already been explained in details by Lemos and Zanchin \cite{9} and Uchihata et. al \cite{10}. Our analysis is closely related to their works while the distinction from theirs will be justified below. Our choice, is made such that as in \cite{9} no energy momentum tensor exists on the interface hypersurface \cite{8} i.e., the mass = the pressure = 0 at equilibrium. In this sense our work is different from that of Frolov, et. al \cite{11}. (Frolov, in a more recent work, presented also a generic approach to the non-singular models of black holes in static spherically symmetric spacetime in four and higher dimensions \cite{12}.) We employ the matching conditions by replacing the singular inside of a RN black hole with a regular de Sitter spacetime with a compact topological structure. In the literature there are both regular black holes \cite{13} as well as regularization methods \cite{14} which involve a change in topology of the spacetime. Specifically, the stability analysis of the shell distinguishes our work in the present study from that of \cite{14}. The external RN spacetime which has parameters mass \( M \) and charge \( Q \) are determined from the satisfaction of the boundary conditions required for a smooth match. Stated otherwise, the mass and charge are defined ‘from geometry’ in accordance with Wheeler’s geometrodynamics \cite{15}. Regularization is to be understood in the sense that is reminiscent of some renormalization / regularization techniques that were used in field theory. The aim in those techniques was to eliminate divergences in field theory. In doing this, experimental values of particles, such as charge, mass, magnetic moment etc. were used as guidelines. Insertion of measured quantities into the theory played major role in choosing the cut-offs. As a result finite quantities emerged from the divergent ones, as a physical requirement. In this study we shall insert a thin-shell of radius \( R_0 \neq 0 \), as our cut-off to eliminate the singularity at the origin. The radius \( R_0 \) of the shell must be finely tuned since it will be related to the mass \( M \), charge \( Q \) and the cosmological constant \( \ell \).

In general relativity also singularities, i.e., diverging curvature invariants lie at the heart of gravitational theory. Most black holes admit singularities at their center which make invariants divergent. The worst of such singularities is the spacelike ones as encountered in the Schwarzschild black hole. Addition of electric charge (i.e., the RN solution) makes the central singularity timelike, which is the subject matter of the present article. By cutting the central singularity and pasting a regular de Sitter spacetime we get rid of the \( r = 0 \) singularity. In turn, the shell must satisfy certain conditions, especially upon perturbation for stability requirement a fluid energy-momentum arises naturally. This is in the form of a modified polytropic fluid whose energy density and transverse pressures satisfy the conservation law. We discuss briefly the physical properties of such a fluid. Being highly nonlinear we choose a particular case and confine the argument to the vicinity of the static shell. Before

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perturbation the shell may be taken as a false vacuum state as in the field theory in which the surface energy-momentum of the fluid vanishes i.e., energy density $\sigma$ and pressure $p$ are both zero. The total energy analysis after the perturbation suggests an energy zone that makes the shell and therefore our model, stable against linear radial perturbations.

Organization of the paper is as follows. In Section II we introduce our model of gluing inner de Sitter with the outer RN metrics. Energy-momentum and Maxwell equations on the shell are discussed in Section III. Section IV analyses the stability of the model. Our Conclusion and Discussion appears in Section V.

II. THE MODEL

In $3 + 1-$dimension, let’s consider the following static, spherically symmetric spacetimes

$$ds^2 = -f_i (r_i) dt^2 + \frac{dr^2_i}{f_i (r_i)} + r_i^2 (d\theta_i^2 + \sin^2 \theta_i d\phi_i^2)$$

(1)

for inside ($i = 1$) and outside ($i = 2$) of a timelike shell defined by $F := r - R_0 = 0$ where $R_0$ is the constant radius of the shell. Following the Israel junction formalism [1], the induced metric on the shell is found to be

$$ds^2 = -dr^2 + R_0^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

(2)

The energy momentum tensor components on the shell are from $S_{\mu}^{\nu} = diag (\sigma_0, p_0, p_0)$ such that

$$\sigma_0 = -\frac{1}{4\pi G} \left( \sqrt{f_2 (R_0)} - \sqrt{f_1 (R_0)} \right)$$

(3)

and

$$p_0 = \frac{1}{8\pi G} \left( \frac{f'_2 (R_0)}{2\sqrt{f_2 (R_0)}} - \frac{f'_1 (R_0)}{2\sqrt{f_1 (R_0)}} + \frac{\sqrt{f_2 (R_0)} - \sqrt{f_1 (R_0)}}{R_0} \right)$$

(4)

where a prime means $\frac{d}{dr}$ at $r = R_0$. Next, we set

$$f_1 = 1 - \frac{r^2}{\ell^2}$$

(5)

and

$$f_2 = 1 - \frac{2M}{r_2} + \frac{Q^2}{r_2^2}$$

(6)

as representatives of the inner ($f_1$) and the outer ($f_2$) spacetimes, respectively [9]. Our aim is to glue the two spacetimes smoothly such that $\sigma_0$ and $p_0$ are determined on the shell: interestingly both vanish. For this we impose $f_1 (R_0) = f_2 (R_0)$ and $f'_1 (R_0) = f'_2 (R_0)$ which leads to

$$M = \frac{2R_0^3}{\ell^2}$$

(7)

and

$$Q^2 = \frac{3R_0^4}{\ell^2}.$$  

(8)

Thus, geometrical conditions of continuity of the metric and its first derivative automatically determine these fine-tuning conditions that play crucial role in the problem. Let us add that in this identification the dimensions of $M$ and $Q$ are same as $R_0$ and $\ell$. For a double horizon case we must have the condition $R_0 > \sqrt{3}\ell$ satisfied. The choice $R_0 = \sqrt{3}\ell$ will obviously correspond to the extremal RN and $R_0 < \sqrt{3}\ell$ will give rise to no horizon case. It is observed that for a nontrivial matching the limit $R_0 \to 0$, must be excluded. In Fig. 1 we plot $f (r) = f_1 (r) \Theta (R_0 - r) + f_2 (r) \Theta (r - R_0)$ in which $\Theta (\cdot)$ stands for the Heaviside step function, for different values of $Q$ and $M$ (and consequently $R_0$ and $\ell^2$).

**FIG. 1:** The metric function $f (r) = f_1 (r) \Theta (R_0 - r) + f_2 (r) \Theta (r - R_0)$ versus $r$ for different values of $Q$ and $M$. $Q = 2.2, 2$ and $1.8$ (or $(R_0 = 1.61, \ell^2 = 4.20), (R_0 = 4.3, \ell^2 = 64/27)$ and $(R_0 = 1.08, \ell^2 = 1.26$). The vertical lines are the locations of the interface shell, i.e. $r = R_0$.

Our Fig. 1 may be compared with the Fig. 4 of Ref. [11] to see the difference with the de-Sitter-Schwarzschild matching. The thin-shell, or transition layer in the terminology of Frolov et. al. [11] has a total non-zero mass / energy whereas in our case by virtue of (7) and (8) and the definition of the surface energy-momentum tensor $S_{\mu}^{\nu} = diag (\sigma_0, p_0, p_0)$ we have $S_{\mu\nu} = 0$. Upon perturbation, as we shall show below we shall have $S_{\mu\nu} \neq 0$. Since $r < R_0$ for inside and $R_0$ can be chosen arbitrary
in terms of \( \ell \), say \( R_0 = \alpha \ell \), where \( \alpha \geq \sqrt{3} \). The topology of de Sitter is adjusted accordingly.

From 1 - \( \frac{2M}{\ell} + \frac{Q^2}{\ell^2} \) = 0 we have \( r_{\pm} = M \pm \sqrt{M^2 - Q^2} \). With the substitutions (7), (8) and \( R_0 = \alpha \ell \) we obtain \( r_{\pm} = R_0 (2x^2 \pm \alpha \sqrt{4x^2 - 3}) \). Let us discuss the following cases:

i) For \( \alpha^2 = 3/4 \) we have \( r_+ = r_- = \frac{3}{2} R_0 \) which implies that the horizon lies outside the shell. Upon substitution of \( r = \sin \psi < \frac{\sqrt{3}}{2} \) we obtain for the spatial part of de Sitter the line element

\[
 ds^2 = \ell^2 (d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\varphi^2))
\]

which is compact \( S^3 \) metric with \( -\sin^{-1} \frac{\sqrt{3}}{2} < \psi < \sin^{-1} \frac{\sqrt{3}}{2} \).

ii) \( \alpha = 1 \), yields \( r_- = R_0 \), \( r_+ = 3R_0 \) in which the inner horizon coincides with the shell. In this case we cast the spatial line element into \( S^3 \) with \( -\frac{\sqrt{3}}{2} < \psi < \frac{\sqrt{3}}{2} \). Note that in this case our shell lies at the inner (Cauchy) horizon \( r_- \) of RN so that it becomes null which we shall not elaborate on. Instability of the Cauchy horizon suggests [16] that we must exclude this choice.

iii) \( \alpha = \sqrt{3} \), gives \( r_- = 3R_0 (2 - \sqrt{3}) \) and \( r_+ = 3R_0 (2 + \sqrt{3}) \). For this case and in general for any \( \alpha > 1 \), we have \( r > 1 \) so that the coordinate change \( r > \cosh \tau \) and \( t = \ell \rho \) yields

\[
 \frac{ds^2}{\ell^2} = -d\tau^2 + \sinh^2 \tau d\rho^2 + \cosh^2 \tau (d\theta^2 + \sin^2 \theta d\varphi^2).
\]

This also gives a compact topology since \( \tau \) is bounded from both below and above as long as \( \alpha \) stands for a finite number. Thus our result is in conform with the Theorem of Borde [14], which says that regularity of a black hole demands that the topology changes. Let us also add that in order to avoid the Cauchy horizon and be on the safe side we must make the choice \( \alpha > \sqrt{3} \). Further, since the overcharging case of the RN doesn’t correspond to a black hole we ignore that discussion. In [9] they consider the case also with \( Q^2 > M^2 \).

III. THE ENERGY-MOMENTUM AND MAXWELL EQUATIONS ON THE SHELL

We shall explain in this section that although the energy-momentum \( T^{\mu}_{\nu} \) on the shell i.e., \( S^2 \) vanishes \( \Delta T^{\mu}_{\nu} = \lim_{r \to R_0^+} T^{\mu}_{\nu} - \lim_{r \to R_0^-} T^{\mu}_{\nu} \) is not necessarily zero (A detailed discussion on this issue is made by Bonnor and Vickers in [17]). This amounts to a jump in \( T^{\mu}_{\nu} \) across the shell. We have, for instance in the present problem that

\[
 \Delta T^\rho_\varphi = \Delta T^\varphi_\rho = \frac{6}{\ell^2} \tag{11}
\]

whereas \( \Delta T^t_t = \Delta T^r_r = 0 \). This is due to the fact that the Israel junction conditions involve up to first order derivative of the metric functions across the spherical shell while the angular energy momentum tensor involves the second order derivatives of the metric function. In other words, from the Israel junction condition we put constraint on the metric functions and their first derivatives on the shell while their second derivatives are free. Hence, while we set \( a_0 = p_0 = 0 \) on the shell we only had to impose continuity condition on \( f(r) \) and \( f'(r) \) across the shell but \( f''(r) \) is free on the shell as we find

\[
 \lim_{r \to R_0^-} \Delta f''(r) = \frac{12}{\ell^2} \tag{12}
\]

For the discussion of Maxwell equation on the shell we make use of the distributional potential 1-form given by

\[
 A = Q \left( \frac{1}{r} - \frac{1}{R_0} \right) \Theta (r - R_0) \, dt \tag{13}
\]

in which \( \Theta (r - R_0) \) represents the Heaviside step function and \( Q \) is the charge. Note that with this choice we assume that for \( r > R_0 \), up to a gauge transformation, we have the Coulomb potential \( A_0 = \frac{Q}{r} \). The electromagnetic field 2-form becomes

\[
 F = dA = \frac{Q}{r^2} \Theta (r - R_0) \, dt \wedge dr \tag{14}
\]

where the notation is such that \( d \) and \( \wedge \) stand for the exterior derivative and the wedge product, respectively. The dual 2-form of \( F \) is given accordingly by

\[
 ^* F = Q \sin \theta \Theta (r - R_0) \, d\theta \wedge d\varphi \tag{15}
\]

so that the Maxwell equation on the shell takes the form

\[
 d^* F = ^* J. \tag{16}
\]
Here \( j \) is interpreted as the charge density 3–form defined by
\[
\ast j = Q \delta (r - R_0) \sin \theta dr \wedge d\theta \wedge d\varphi
\]  
(17)
where \( \delta (r - R_0) \) is the Dirac delta function coming from the derivative of \( \Theta (r - R_0) \) in the sense of distributions. To show that Maxwell equation holds on the shell we check the integral of \( \ast j \) which amounts to
\[
Q = \frac{1}{4\pi} \int \ast j
\]  
(18)
and is manifestly satisfied.

To complete this section let’s add that having the Maxwell field non-zero on one side of the shell must not lead to a conclusion that the energy momentum on the shell can not be zero as the field is not extended to the other side of the shell. We recall that the energy momentum on the shell is made by the field from both sides of the shell. Hence, the presence of the cosmological constant inside the shell guaranties that \( p_0 = \sigma_0 = 0 \) provided (7) and (8) are satisfied. As a matter of fact Eq.s (7) and (8) may be interpreted as fine-tuning condition among parameters, mass, charge and the cosmological constant. We see therefore that without the cosmological constant inside the shell such a perfect match would not be possible, justifying the choice of de Sitter as the inner spacetime.

**IV. STABILITY OF THE MODEL**

Once we adopt that the two spacetimes are glued on the timelike shell \( F := r - R = 0 \) we investigate next its stability. Here we assume a radial perturbation of the shell which causes \( R \) changing with respect to the proper time \( \tau \). The standard calculation of the energy-momentum tensor of the shell when \( R = R(\tau) \) yields
\[
\sigma = -\frac{1}{4\pi G} \left( \sqrt{f_2(R) + R^2} - \sqrt{f_1(R) + R^2} \right), 
\]  
(19)
and
\[
p = \frac{1}{8\pi G} \left( \frac{2\dot{R}(\tau) + f_2'(R)}{2f_2(R) + R^2} - \frac{2\dot{R}(\tau) + f_1'(R)}{2f_1(R) + R^2} + \sqrt{f_2(R) + R^2} - \sqrt{f_1(R) + R^2} \right). 
\]  
(20)
in which \( f_1 \) and \( f_2 \) are given in (5) and (6) and a dot represents \( \frac{d}{d\tau} \). We note that the energy conservation equation imposes that \( \sigma \) and \( p \) given in (19) and (20) satisfy
\[
\frac{d\sigma}{dR} + \frac{2}{R} (p + \sigma) = 0. 
\]  
(21)
An equation of state in the form of \( p = p(\sigma) \) in this equation manifests the exact form of \( \sigma \) and \( p \) after the perturbation irrespective of the form of \( f_1 \) and \( f_2 \). The latter equation admits
\[
\int_0^\sigma \frac{d\sigma}{p(\sigma) + \sigma} = 2 \ln \left( \frac{R_0}{R} \right) 
\]  
(22)
which suggests that \( p(\sigma) \) can not be an arbitrary function as it must satisfy \( p(0) = 0 \). Note that the integration constant \( R_0 \) is identified as the unperturbed radius of the shell. For instance a linear gas with EoS \( p = \omega \sigma \) \((\omega = \text{const.})\) can not be a physical choice. We recall that a massive shell was chosen in [10] which is different from our choice. An equation of state of the form
\[
p = -\sigma + \omega \sigma^\nu 
\]  
(23)
in which \( 0 < \nu < 1 \) is a suitable candidate for the fluid presented on the surface of the shell after the perturbation. This is a modified version of a polytropic fluid [13]. We note that the non-zero \( p \) and \( \sigma \) after the perturbation can be attributed to the energy given during the perturbation. Obviously it is observed from (22) that \( p = 0 \) when \( \sigma = 0 \). Integrating (22) with (23) one finds
\[
\sigma (R) = \left( 2\omega (1 - \nu) \ln \left( \frac{R_0}{R} \right) \right)^{\frac{1}{1-\nu}}. 
\]  
(24)
Herein \( \omega \) is a constant which can be adjusted but as \( R \) gets values on both sides of \( R_0 \) one has to set \( \nu \) in such a way that the right-hand side remains real. For instance \( \nu = \frac{1}{2} \) leaves the expression real while \( \nu = \frac{1}{4} \) does not. The total energy on the shell can be obtained as
\[
E = \int \sigma (R) \delta (r - R) \sqrt{-g} d^4 x = 4\pi R^2 \left( 2\omega (1 - \nu) \ln \left( \frac{R_0}{R} \right) \right)^{\frac{1}{1-\nu}}. 
\]  
(25)
In Fig. 2 we plot \( E \) versus \( R \) for \( \omega = 1, \nu = \frac{1}{2} \) and for the three different \( R_0 \) values used in Fig. 1. In accordance with Fig. 2, for some extension, more deviation from \( R = R_0 \) requires more energy and physically this is an indication of stability. From Fig. 2 it is also seen that the minimum of energy formed at \( R = R_0 \) is strongly stable from right side. From the left, on the other hand, overcoming the energy barrier causes the shell to collapse leaving behind a flat spacetime in accordance with (7) and (8).

To justify the polytropic property, i.e., \( PV^n = \text{const.} \), (with \( n = \text{const.} \) of the equation of state (23) we choose a particular parameter, namely, \( \nu = \frac{3}{4} \) and make analysis in the vicinity of \( R = R_0 \). With \( \nu = \frac{3}{4} \) we have from (23)
\[
p = \omega^2 \ln \left( \frac{R_0}{R} \right) \left( 1 - \ln \left( \frac{R_0}{R} \right) \right). 
\]  
(26)
Now we take $R = R_0 + \epsilon$, where $|\epsilon| \ll 1$ and upon expansion we obtain

$$p \simeq -\frac{\epsilon \omega^2}{R_0}.$$  

(27)

Recalling that the volume, (in fact the area) $V \sim R_0^2$ for $S^2$ we have $PV^{1/2} \simeq -\epsilon \omega^2 = \text{const.}$ so that it corresponds to $n = \frac{1}{2}$ law for the polytropic gas on the shell.

V. CONCLUSION AND DISCUSSION

By applying the cut and paste technique via a thin-shell we regularize the inner part of the RN spacetime which removes its central singularity. Simply the patched regular de Sitter spacetime constitutes the inner part. This amounts to the choice of the distributional metric function $f(r) = \left(1 - \frac{r^2}{r^2_\ell}\right) \Theta(R - r) + \left(1 - \frac{2m}{r_\ell^2} + \frac{Q^2}{r_\ell^2}\right) \Theta(r - R)$, in which $R$ stands for the radius of the shell. At the static case $R = R_0$, application of the Israel junction conditions yields no source on the surface of the shell for a smooth match. This requires that the metric and its first derivative are continuous on the shell which makes the model feasible to certain curves of energy versus $R$ limit. These conditions emerge as a result of fine-tuning of parameters via (7) and (8). However, upon radial perturbation we can have for both $R > R_0$ and $R < R_0$ a source of modified polytropic fluid. In this sense the shell may be considered as a ‘false vacuum state’ for the environmental fluid described by the equation of state (23) whose limit $R \to R_0$ agrees with such a vacuum. Relying on the curves of energy versus $R$ we predict a restricted stability of the shell which makes the model feasible to certain extend. It should also be added that the boundary shell must be finely tuned to avoid the Cauchy horizon and its inherent instability. Finally, we must add that RN singularity is a time-like one (for $M > Q$) which may be considered weaker than the spacelike singularity of the Schwarzschild black hole. Although our method has no immediate answer for the removal of the Schwarzschild’s singularity what we have shown in this study is that in the case of RN it remarkably works through a change in topology. This is due to the fact that the de-Sitter geometry must have only a compact topology which confirms a theorem proved in [14]. We add that recent methods of removing the singularity of black holes are available in the literature which also are based on the topology change (see [19] and the references cited therein).

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