Robustness of the geometric phase under parametric noise

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Received 31 March 2009
Accepted for publication 9 April 2009
Published 27 May 2009
Online at stacks.iop.org/PhysScr/79/065012

Abstract
We study the robustness of the geometric phase in the presence of parametric noise. For this purpose we consider a simple case study, namely a semiclassical particle that moves adiabatically along a closed loop in a static magnetic field acquiring the Dirac phase. Parametric noise comes from the interaction with a classical environment, which adds a Brownian component to the path followed by the particle. After defining a gauge-invariant Dirac phase, we discuss the first and second moments of the distribution of the Dirac phase angle coming from the noisy trajectory.

PACS numbers: 03.65.Vf, 02.50.Ey, 03.67.−a

1. Introduction

The first reference to the role played by geometric phases in physics dates back to the work of Pancharatnam \([1]\) in the context of interferometry of polarized beams of light. Later, the same phenomenon was described by Berry \([2]\) for quantum mechanical systems in the adiabatic limit. A mathematical insight into its origin was provided by Simon \([3]\), who recognized that Berry phases could be interpreted as holonomies on a fiber bundle. Subsequently, quantum geometric phases (i.e. quantum holonomies) have been predicted and observed in various physical systems, and several generalizations and extensions were proposed \([4–6]\) (see \([7]\) and the references therein). Generally speaking, we may say that geometric phases appear in correspondence with a cyclic evolution of a relevant Hilbert (sub)space. The dimension of the cyclic (sub)space determines the features of the corresponding geometric phase: Abelian, i.e. \(U(1)\)-valued, for Hilbert space of unit dimension; non-Abelian, i.e. \(U(N)\)-valued, in the case of \(N\)-dimensional cyclic Hilbert space.

A few years after the seminal papers by Berry and Simon, the scientific community discovered the potentialities of quantum mechanical systems in the context of information and communication technology \([8]\), leading to the birth of quantum information science \([9]\). Since information processing obeys physical laws, a reversible quantum algorithm is represented by a unitary transformation as long as information is encoded as vectors in a Hilbert space. It follows that quantum holonomies, being unitary transformations, can serve as quantum logical gates to implement quantum algorithms. This idea was first proposed and discussed in \([10]\), where the authors also proved that universal computation \([11]\) can be, in general, realized by means of solely (non-Abelian) geometric phases. A vast literature followed, which included both theoretical proposals \([12, 13]\) and experimental realizations \([14, 15]\) of simple holonomic quantum gates.

In view of the application for quantum information processing, one usually considers the case of the adiabatic geometric phase (also called the Berry phase). In this case the system Hamiltonian is supposed to be a smooth function of the coordinates on a suitable manifold, often called the ‘parameter manifold’. The geometric phase arises in the adiabatic approximation in correspondence with a closed loop in the parameter manifold.

It is worth noting, however, that a geometric approach to the computation can be rather demanding from a technological point of view. Nevertheless, its advantage with respect to the standard dynamical approaches relies on the fact that Berry phases are argued to be particularly robust with respect to noise. In particular, it is argued that holonomic quantum gates can be robust with respect to a certain kind of classical parametric noise \([16]\). This kind of noise can
be modeled as coming from the interaction with a classical environment.

The robustness of quantum logic gates is indeed a crucial issue because of the inherent fragility of quantum mechanical systems. Hence, although quantum error correction protocols exist [17], quantum gates that are a priori robust are welcome. It is worthwhile to mention another remarkable proposal for a fault-tolerant computation, namely topological computation [18]. This approach is based on non-Abelian Aharonov–Bohm topological phases.

As we have anticipated, the geometric phase is believed to be robust with respect to classical parametric noise. This argument has been the central issue of several investigations, in particular we mention the work by De Chiara and Palma [19], as well as the results presented in [20]. Several physical models have been taken into consideration to study the robustness of the geometric phase. Recently, the robustness of the geometric phase has been tested experimentally in trapped polarized ultra-cold neutrons [21]. However, it is worth noticing that this issue is largely independent of the details of the physical model under consideration. For this reason, our discussion will be devoted to the simplest settings in which geometric phases appear. In the following sections, we consider a charged semiclassical particle that is adiabatically and cyclically moved in a static magnetic field. The particle acquires the Dirac phase which is proportional to the magnetic flux enclosed by the particle trajectory. Two different settings will be described: in the first case, the semiclassical particle is in the presence of a homogeneous magnetic field; in the second example, the particle is subject to the field generated by a magnetic monopole.

2. Dirac phase under noise

In the following sections, we consider the effects of parametric noise on quantum holonomies in two simple but remarkable examples. We also consider two models for the noise, respectively, represented by a Wiener and by an Ornstein–Uhlenbeck process. The examples involve a simple physical system made of a charged semiclassical particle subjected to Brownian motion due to the interaction with a contribution to the phase factor. Let the particle now be in the internal state of the particle will be described by the vector

\[ \psi_1(t) \]

if the internal state of the particle is initially described by the vector

\[ \psi_1(0) \]

We are going to consider a noisy path of the form (2) with a trivial noiseless Dirac phase. For the sake of simplicity, we consider the motion of the semiclassical particle as confined in the plane \( x-y \), perpendicular to the magnetic field.

3. The particle in a homogenous magnetic field

In this section we consider the case of a static homogeneous magnetic field \( B \equiv (0, 0, B) \). We can write the vector potential in the asymmetric gauge as \( A \equiv (-yB, 0, 0) \), hence the Dirac phase is determined by the integral

\[ \phi_g = \oint_r A \cdot dr \]

(3)

where \( G \) indicates the straight line joining the final point \( r(T) \) to the initial point \( r(0) \).

3.1. Wiener process

As a first model for the noise component, we consider the case of a Wiener process. Hence we impose the following conditions on two-time correlation functions of the Brownian component \( r_n(t) \):}

\[ \langle x(t) x(t) \rangle = D_x \delta(s-t), \]

(5)

\[ \langle y(t) y(t) \rangle = D_y \delta(s-t), \]

(6)

\[ \langle x(t) y(t) \rangle = 0. \]

(7)

The noisy path has initial points \( x(0), y(0) \) and final points \( x(T), y(T) \). In order to ensure gauge invariance, we add one
The Dirac phase angle can be written as the mean square limit (see e.g. [22]) of the following quantity:

\[ \phi_{\delta} = \lim_{N \to \infty} \frac{1}{N} \langle S + \frac{1}{2} \Sigma \Delta \rangle, \]

(18)

where

\[ S = \sum_{i=0}^{N-1} y(t_i)(x(t_i + \delta t) - x(t_i)), \]

(19)

with \( \delta t = T/N \), and the proper term has been added to ensure gauge invariance.

The variance of the Dirac phase angle is given by

\[ \sigma_{\delta}^2 = \lim(\langle S + \frac{1}{2} \Sigma \Delta \rangle^2) = \lim(\langle S \Sigma \Delta \rangle + \frac{1}{2} \langle \Sigma \rangle^2 (\Delta^2)). \]

(20)

The first term on the right-hand side of equation (20) is the limit of

\[ S^2 = \sum_{i,j} \langle y(t_i)y(t_j) \rangle \langle x(t_i + \delta t) \rangle \langle x(t_j + \delta t) \rangle \]

(21)

where the average of \( y \) and \( x \) factorizes for the statistical independence of the processes. We have

\[ \Delta S^2 = \sum_{ij} \langle y(t_i)y(t_j) \rangle \left[ \langle (x(t_i+1)x(t_{j+1}) - \langle x(t_{i+1})x(t_j) \rangle \right. \]

\[ \left. + \langle x(t_i)x(t_j) \rangle - \langle x(t_{i+1})x(t_{j+1}) \rangle \right]. \]

(22)

Evaluating the two-time correlation functions, and putting \( D_x = D_y \) and \( \Gamma_x = \Gamma_y \), one obtains

\[ \Delta S^2 = \epsilon^2 \sum_{ij} \langle y(t_i)y(t_j) \rangle \left[ 2e^{-\Gamma|i-j|\delta t} \right. \]

\[ - e^{\Gamma|i-j+1|\delta t} - e^{-\Gamma|i-j+1|\delta t} \right]. \]

(23)

The term in curled brackets is

\[ 2(1 - e^{-\Gamma \delta t}) \simeq 2\Gamma \delta t \quad \text{for} \quad |i - j| = 0, \]

\[ -(1 - e^{-\Gamma \delta t})^2 \simeq -\Gamma^2(\delta t)^2 \quad \text{for} \quad |i - j| = 1, \]

\[ e^{-\Gamma|i-j|\delta t} (2 - e^{-\Gamma \delta t} - e^{-\Gamma \delta t}) \]

\( \simeq e^{-\Gamma|i-j|\delta t} \Gamma^2(\delta t)^2 \quad \text{for} \quad |i - j| > 1, \]

(24)

and \( \langle y(t_i)y(t_j) \rangle = e^2e^{-\Gamma|i-j|\delta t} \). Taking the limit \( \delta t \to 0 \), only the terms with \( |i - j| = 0 \) do not vanish, leading to

\[ \sigma_{\delta}^2 \simeq \epsilon^4 \sum_{j} 2\Gamma \delta t \simeq \epsilon^4 \int_0^T 2\Gamma dt = 2\epsilon^4 \Gamma T = 2\epsilon^4 N, \]

(25)

where \( N := \Gamma T \) is interpreted as the average number of statistically independent fluctuations.

The second term on the right-hand side of (20) is the limit of the quantity

\[ \sum_j \langle y(t_j)(x(t_j + \delta t) - x(t_j)) \rangle, \]

(26)

which equals

\[ \epsilon^4 \sum_j \left[ e^{-\Gamma t_j} + e^{-\Gamma(T-t_j)} \right] \left[ e^{-\Gamma(t_j+\delta t)} - e^{-\Gamma(T-t_j-\delta t)} \right. \]

\[ - e^{-\Gamma t_j} + e^{-\Gamma(T-t_j)} \right]. \]

(27)
and, in the limit of $\delta t \to 0$ reads

$$- \Gamma \epsilon^4 \sum_j \left[ e^{-\Gamma t_j} + e^{-\Gamma T} e^{\Gamma t_j} \right] \left[ e^{-\Gamma t_j} + e^{-\Gamma T} e^{\Gamma t_j} \right] \delta t$$

$$\simeq -4 \Gamma \epsilon^4 e^{-\Gamma T} \int_0^T \cosh \left[ \left( \frac{T}{2} - t \right) \right] \cosh \left[ \left( \frac{T}{2} - t \right) \right] dt. \quad (28)$$

The last integral reads

$$-2 \epsilon^4 \Gamma \left( 1 - e^{-2\Gamma T} + T \right). \quad (29)$$

Finally, the third term on the right-hand side of (20) is

$$\epsilon^4 (1 - e^{-\Gamma T})^2. \quad (30)$$

Summing all the contribution, and taking the limit $e^{-\Gamma T} \to 0$, we can write

$$\sigma^2 \simeq 2 \epsilon^4 (\Gamma T + 1). \quad (31)$$

From the last equation we see that, in contrast to the case of the Wiener process, for the Orstein–Uhlenbeck process the variance

$$\sigma \simeq \epsilon^2 \sqrt{2(\Gamma T + 1)} \quad (32)$$

grows with the square root of the operational time. This expression for the variance of the Dirac phase angle can be heuristically explained as the result of the sum of $N = \Gamma T$ statistically independent variables, each contributing with a variance of the order $\epsilon^2$.

4. The particle in the field of a magnetic monopole

In this section we consider another simple physical example, in which the semiclassical reference frame is subjected to the field of a magnetic monopole. If the magnetic monopole is sitting at the origin of the reference frame, we can write the corresponding vector potential, for $z/R \neq -1$, as follows:

$$\mathbf{A} \cdot d\mathbf{r} = \frac{1 - z/R}{x^2 + y^2} ( -y \, dx + x \, dy), \quad (33)$$

where $R^2 = x^2 + y^2 + z^2$, or simply in spherical coordinates, for $\theta \neq \pi$, as

$$\mathbf{A} \cdot d\mathbf{r} = \frac{1}{R} \tan \frac{\theta}{2} \, d\varphi, \quad (34)$$

where $\cos \theta = z/R$ and $\tan \varphi = y/x$. For sufficiently small amplitude of the noise and short operational time, we can take a linearized version of (33):

$$\mathbf{A} \cdot d\mathbf{r} \simeq \mathbf{A}_L \cdot d\mathbf{r} = - \left[ f_0 + f_x (x - x_0) + f_y (y - y_0) + f_z (z - z_0) \right] \, dx + \left[ g_0 + g_x (x - x_0) + g_y (y - y_0) + g_z (z - z_0) \right] \, dy. \quad (35)$$

In this approximation, we can write the following expression for the gauge-invariant Dirac phase angle:

$$\phi_0 = - \int \mathbf{A}_L \cdot d\mathbf{r} + \left[ f_0 - \frac{1}{2} (f_x \Delta_x + f_y \Delta_y + f_z \Delta_z) \right] \Delta_x$$

$$- \left[ g_0 - \frac{1}{2} (g_x \Delta_x + g_y \Delta_y + g_z \Delta_z) \right] \Delta_y, \quad (36)$$

where $\Delta_x = x(0) - x(T)$, $\Delta_y = y(0) - y(T)$ and $\Delta_z = z(0) - z(T)$.

4.1. Wiener process

In this section we consider a tri-dimensional Wiener process as the noise model. We consider a trivial noiseless loop to which the Wiener process is superimposed. Denoting $D_x$, $D_y$, and $D_z$ as the diffusion constants, we obtain the following expression for the mean value of (36):

$$\langle \phi_0 \rangle = \frac{f_x}{2} (\Delta_x^2) - \frac{g_x}{2} (\Delta_x^2) = \frac{1}{2} (g_x D_x - f_x D_x) T, \quad (37)$$

which does not vanish in general and grows linearly with the operational time.

Regarding the corresponding variance, we obtain

$$\sigma_x^2 = \langle \phi_0^2 \rangle - \langle \phi_0 \rangle^2 = \frac{1}{2} \Delta T^2. \quad (38)$$

Hence the variance

$$\sigma_x = \frac{1}{2} \sqrt{\Delta T}, \quad (39)$$

grows linearly with the operational time, and

$$\Delta = D_x (3 D_x f_y^2 + 2 D_y f_x^2 + 2 D_z f_z^2) + D_y (2 D_x g_y^2 + 3 D_y g_x^2 + 2 D_z g_z^2). \quad (40)$$

4.2. Ornstein–Uhlenbeck process

In the examples discussed above we computed the variance of the Dirac phase angle caused by a Brownian motion of the semiclassical particle. We explicitly considered the case of a trivial noiseless loop, in which the trajectory is purely Brownian. We obtained that the leading term in the variance is of the second order in the amplitude of the noise. Moreover, it increases linearly with the operational time for the Wiener process, and with the square root in the case of the Ornstein–Uhlenbeck process. This behavior can be compared with the results presented in [19], where the variance of the Berry phase was computed in the presence of a noise component modeled by an Ornstein–Uhlenbeck process. This noisy component is superimposed to a drift loop which is a precession around the $z$-axis. In that case it was shown that the leading term in the variance is of the first order in the amplitude of the noise. Moreover, it decreases linearly with the operational time, leading to negligible fluctuations in the Berry phase.

It is interesting to compare the variances of the Dirac phase obtained in the case of different drift loops. We have considered both the case of a trivial noiseless component $\mathbf{r}_0(t) = (\sin (\vartheta_0) \cos (\varphi_0), \sin (\vartheta_0) \sin (\varphi_0), \cos (\vartheta_0))$, and the case of a precession about the $z$ axis described by the loop

$$\mathbf{r}_0(t) = (\sin (\vartheta_0) \cos (\varphi_0 + 2\pi t), \sin (\vartheta_0) \times \sin (\varphi_0 + 2\pi t), \cos (\vartheta_0)). \quad (41)$$

We have numerically simulated (following [23]) an Ornstein–Uhlenbeck process affecting these loops and estimated the variance of the corresponding Dirac phase angle. The results are plotted in figure 1 for $\vartheta_0 = 1/\sqrt{3}$. 

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The plot shows the numerically estimated variance of the geometric phase angle for a particle in a monopole field. The variance is plotted as a function of the average number of noise fluctuations for a Ornstein–Uhlenbeck process modeling the noise component, with amplitude $\epsilon = 0.05$. The data represented by dots refer to a trivial drift loop (purely Brownian motion), they are well fitted by a square root law $\sigma_g = a\sqrt{N} + b$, with $a \approx 0.0025$ and $b = -0.00016$. The data represented by circles correspond to the drift loop in equation (41) (Brownian motion superimposed to precession).

We notice two different patterns of the variance of the Dirac phase angle as a function of the average number of fluctuations in the noisy component. In the case of trivial drift loop (purely Brownian motion) the variance always increases as the square root of the average number of fluctuations. On the other hand, for nontrivial noiseless loop (Brownian component superimposed to precession) a transient behavior is present in which the variance decreases with the number of fluctuations. This behavior is in agreement with what was found in [19] and is due to the contribution of the first order in the noise amplitude. By increasing the value of $N$, the first-order contributions become negligible and the second-order ones become predominant.

5. Conclusion

We have computed the mean value and the variance of the Dirac phase acquired by a semiclassical particle subjected to Brownian motion. If the trajectory is purely Brownian the variance of the Dirac phase angle always increases as a function of the operational time (or the average number of noise fluctuations). On the other hand, a transient behavior is observed if the Brownian motion is superimposed to a noiseless drift loop.

In the case of pure Brownian motion, we have obtained an expression for the variance which is of the second order in the amplitude of noise and increases with operational time. In particular, if the noise is modeled by an Ornstein–Uhlenbeck process, the variance grows with the square root of the operational time.

The case of the Dirac phase can be viewed as an instance of the geometric phase. Hence we can compare our results with others which refer to the Berry phase. In [19], it was shown that in the case of an adiabatic precession of a 1/2-spin the leading term in the variance of the Berry phase is of the first order in the amplitude of the noise. Moreover, these terms decrease linearly with the operational time. This behavior is in accordance with the transient behavior of the Dirac phase for nontrivial noiseless drift loop. Notice that the second-order effects become relevant for long enough operational time.

To the best of our knowledge, the presentation of the effects of second order in the variance of the Dirac phase introduces a new element in the study of the robustness of geometric phases. We argue that second-order effects can feasibly be observed in experimental settings as in [21]. Recently, the effects of nonadiabaticity in the noise component were studied in [24]. The pattern of the variance of the corresponding geometric phase as a function of the operational time is qualitative analogous to the results presented here, in the sense that the squared variance grows linearly in time. Quantitatively, this effect is of the first order in the noise amplitude in [24] while in our analysis, which assumes the adiabatic approximation, the effect is of the second order.

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