Qualitons at High Density

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Abstract

In the color-flavor-locked (CFL) phase of the QCD superconductor we show that baryons behave as qualitons (called “superqualitons”) with quantum numbers \( B = (1 \text{ mod } 2)/3, \ S = 1/2 \) and \( Y = B \). An intriguing possibility implied by this identification is that light baryonic modes in the form of superqualitons could be excited below the (color) superconducting gap in the CFL phase, a novel phenomenon foreign to normal superconductors.

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1. Dense hadronic matter poses still a challenging problem to theoretical physicists despite years of vested efforts. With the advent of dedicated fixed target accelerators (SIS) and heavy-ion colliders (SPS, RHIC, LHC) the theoretical issues have become more imperative as experiments may shed new light and spur new interest in the problem.

At subnuclear matter densities, hadronic matter behaves as a free gas of nucleons and a good understanding of this phase can be reached by using virial type analyses [1]. A liquid-gas or Ising transition has been proposed [2] and recently observed [3]. At a few times nuclear matter densities, phenomenological analyses guided by nuclear structure data and constrained by neutron star observations have revealed a variety of phenomena including the possibility of a kaon-condensed phase that would be at the origin of proton-rich stars [4]. At densities larger than nuclear matter densities it has been recently suggested that a variety of diquark-rich and superconducting phases may take place with potentially large gaps [5, 6, 7, 8, 9]. Present neutron star cooling scenarios however appear to disfavor large gaps at moderate densities, although this conclusion is by no means definitive [10].

At asymptotic nuclear matter densities, quarks interact weakly [11] due to asymptotic freedom [12]. The ensuing metallic phase is supposedly screened except for possible infrared problems. It was originally suggested by Bailin and Love [13] and subsequently restressed by others [5] that in the asymptotic regime quarks pair at the Fermi surface resulting in small energy gaps. The perturbative arguments were recently revisited in light of the fact that the magnetic modes undergo Landau damping instead of screening, with the possibility of large gaps [6, 7, 8, 9]. Here we just note that perturbation theory may still fail at asymptotic densities due to the nearness of the Gribov horizon in gauge theories [14].

Assuming that the superconducting phase is favored at asymptotic densities, it was recently observed that for three massless flavors, quarks may pair into diquarks with color and flavor locked forming the CFL (color-flavor-locked) phase [15]. It was further suggested that the CFL phase exhibits a spectrum with matching quantum numbers to that of the zero density phase [16], a situation reminiscent of the strongly coupled standard model [17] and more intriguingly of the Cheshire-Cat phenomenon in low-energy hadronic structure (see [18] and references therein). The CFL phase appears to be energetically favored by effective theory calculations [8].

In this letter we show that in the superconducting CFL phase, baryons behave as qualitons (called in short “superqualitons”), realizing Kaplan’s scenario in the high density phase [19, 20]. They carry quantum numbers $B = Y = (1 \text{ mod } 2)/3$ and $S = 1/2$, and may even appear below the superconducting gap, a phenomenon unseen in normal superconductors. In section 2, we will give arguments for the effective action in the CFL phase. In section 3, we show that superqualitons are topologically stable and identify their quantum numbers. In section 4, we give two complementary estimates for their mass and size. In section 5, we discuss their quantized spectrum. Our conclusions are in section 6.

We are suggesting that the notion of quark soliton or qualiton is more appropriate at high density than at zero density at which the qualiton picture has met with little success [21].
2. In the CFL superconducting phase, quarks with opposite Fermi momenta pair with color and flavor locked. Model calculations [16] and effective theory calculations [8] have shown that this phase is favored energetically when the quark masses are light for a reasonable choice of parameters. The pertinent condensate will be taken in the $(\bar{3}, \bar{3})$ color-flavor representation as

$$\langle q^a_L q^b_L \rangle = -\langle \bar{q}^a_R \bar{q}^b_R \rangle = \kappa \epsilon^{ij} \epsilon^{\alpha\beta} \epsilon_{\alpha\beta i}$$

(1)

where $\kappa$ is some constant, $i, j$ are $SL(2, C)$ indices, $a, b$ are color indices, and $\alpha, \beta$ are flavor indices. By allowing for an arbitrary relative phase between the left- and right-condensate, parity could be spontaneously broken at high densities [7]. For finite $\kappa$, both global color $SU(3)_c$ and flavor $SU(3)_{fL,R}$ symmetries are broken. The flavor-color locking in (1) implies spontaneous breaking through the color-flavor diagonal. Specifically: $SU(3)_c \times (SU(3)_L \times SU(3)_R) \to SU(3)_{c+L+R}$, with the emergence of 8 pseudoscalar Nambu-Goldstone bosons together with 8 scalar Nambu-Goldstone bosons eaten up by gluons through the Higgs mechanism. There is an extra Nambu-Goldstone boson associated with $U(1)_B \to Z_2$ with no relevance to our discussion.

To describe the low-energy dynamics of the color-flavor locking phase, we introduce a field $U_L(x)$ which maps space-time to the coset space, $M_L = SU(3)_c \times SU(3)_L/SU(3)_{c+L}$. Specifically, the left-handed field is

$$U_{Laa}(x) \equiv \lim_{y \to x} \frac{|x-y|^{\gamma_m}}{\kappa} \epsilon^{ij} \epsilon_{\alpha\beta} \epsilon_{\alpha\beta i} q^a_L(\tilde{v}_F, x) q^b_L(\tilde{v}_F, y),$$

(2)

where $\gamma_m$ is the anomalous dimension of the diquark field of order $\alpha_s$ and $q(\tilde{v}_F, x)$ denotes the quark field with momentum close to a Fermi momentum $\mu \tilde{v}_F$. The pairing involves quarks near the opposite edge of the Fermi surface. Similarly, we introduce a right-handed field $U_R(x)$, also a map from space-time to $M_R = SU(3)_c \times SU(3)_R/SU(3)_{c+R}$, to describe the excitations of the right-handed diquark condensate. Under an $SU(3)_c \times SU(3)_L \times SU(3)_R$ transformation by unitary matrices $(g_c, g_L, g_R)$, $U_L$ transforms as $U_L \to g_c^* U_L g_R$ and $U_R$ transforms as $U_R \to g_c^* U_R g_R$. In the ground state of the CFL superconductor, $U_L$ and $U_R$ take the same constant value. QCD symmetries imply

$$\langle U_{Laa} \rangle = -\langle U_{Raa} \rangle = \kappa \delta_{aa}.$$ 

(3)

The Nambu-Goldstone bosons are the low-lying excitations of the condensate, given as unitary matrices $U_L(x) = g_c^T(x) g_L(x)$ and $U_R(x) = g_c^T(x) g_R(x)$. For the present decomposition, we note the extra invariance under the (hidden) local transformation $g_{c+L+R}(x)$ within the diagonal $SU(3)_{c+L+R}$ through

$$g_c^T(x) \to g_{c+L+R}(x) g_c^T(x) \quad g_{L,R}(x) \to g_{c+L+R}(x) g_{L,R}(x)$$

(4)

The breaking of global color is of course a misdemeanor that we will not address here. See [13] and Langfeld and Rho in [8] for discussions on this point.
Hence, the spontaneous breaking of $SU(3)_c \times (SU(3)_L \times SU(3)_R) \rightarrow SU(3)_{c+L+R}$ can be realized non-linearly through the use of $U_{L,R}(x)$ or linearly through the use of $g_{c,L,R}(x)$ with the addition of an octet vector gauge field transforming inhomogeneously under local $g_{c+L+R}(x)$. This is the hidden local symmetry approach \[21\], in which the ensuing vector gauge field is composite and Higgsed, which is to be contrasted with a recent suggestion \[22\]. Clearly the composites carry color-flavor in the unbroken subgroup, with a mass of the order of the superconducting gap. Under some general conditions (e.g. vector dominance), the linear and nonlinear representations are the same \[23\]. Some of these points will be further discussed elsewhere.

The current associated with $\det U_L \det U_R$ is the left (right)-handed $U(1)_L \cdot U(1)_R$ baryon number current. Because of the $U(1)$ axial anomaly, the field $\det U_L / \det U_R$ is massive due to instantons \[^3\]. We decouple the massive field from the low energy effective action by imposing $\det U_L / \det U_R = 1$. The Nambu-Goldstone boson associated with spontaneously broken $U(1)_B$ symmetry is described by $\det U_L \cdot \det U_R$ and responsible for baryon superfluidity. But since it is not directly relevant for our problem, we further choose $\det U_L \cdot \det U_R = 1$ to isolate the Nambu-Goldstone bosons resulting from the spontaneous breaking of chiral symmetry, from the massive ones eaten up by the gluons. We now parameterize the unitary matrices $U_{L,R}$ as

$$U_L(x) = \exp \left( 2i \Pi_A^A T^A / F \right), \quad U_R(x) = \exp \left( 2i \Pi_R^A T^A / F \right),$$

where $T^A$ are $SU(3)$ generators, normalized as $\text{Tr} T^A T^B = \delta^{AB} / 2$. The Nambu-Goldstone bosons $\Pi^A$ transform nonlinearly under $SU(3)_c \times SU(3)_L \times SU(3)_R$ but linearly under the unbroken symmetry group $SU(3)_{c+L+R}$.

The effective Lagrangian for the CFL phase at asymptotic densities follows by integrating out the ‘hard’ quark modes at the edge of the Fermi surface as suggested by Polchinski \[24\] and Weinberg \[25\]. This program has been carried out partly by one of us \[8\]. The effective Lagrangian for $U_{L,R}$ is a standard non-linear sigma model in $d=4$ dimensions and should include the interaction of Nambu-Goldstone bosons with colored but “screened” gluons $G^{\mu \nu}$ \[^4\]. The $SU(3)_c$ current of Nambu-Goldstone bosons in the CFL phase consists of two pieces; one is from the Noether color-current and the other one is from the WZW term, given as

$$J_{cL}^{A\mu} = \frac{i}{2} F^{\mu \nu} \text{Tr} U_L^{-1} T^A \partial_\nu U_L + \frac{1}{24\pi^2} \epsilon^{\mu \rho \sigma \tau} \text{Tr} T^A U_L^{-1} \partial_\rho U_L U_L^{-1} \partial_\sigma U_L U_L^{-1} \partial_\tau U_L,$$

where the first term is the Noether current and the second one is from the WZW term. Expanding in powers of derivative, the effective Lagrangian for the (colored) Nambu-Goldstone

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\[^3\]The axial anomaly persists even in perturbation theory, and does not fully require the presence of instantons. The instantons of course persist even at asymptotic densities, albeit in a screened form.

\[^4\]The magnetic gluons are Landau-damped in perturbation theory, but this may not be valid at asymptotic densities as suggested in \[14\], if $m_\ast \sim (m^2_\Delta)^{1/3}$. 

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bosons is then
\[ \mathcal{L} = + \frac{F_T^2}{4} \text{Tr}(\partial_0 U_L \partial_0 U_L^\dagger) - \frac{F_S^2}{4} \text{Tr}(\partial_i U_L \partial_i U_L^\dagger) + g_s G \cdot J_{cL} + n_L \mathcal{L}_{WZW} \]
\[ + (L \to R) + \mathcal{O} \left( \frac{\partial^4}{(4\pi\Delta)^4} \right). \] (7)

In the presence of a chemical potential, only $O(3)$ is enforced in space. In carrying the derivative expansion, we assume the superconducting scale $4\pi\Delta$ to be the large scale. (In effective theories the emergence of $4\pi\Delta$ is purely a loop effect in $d=4$). The Wess-Zumino-Witten \[26, 27\] (WZW) term is nonlocal in $d=4$. Its normalization will be discussed below. The temporal and spatial decay constants $F_{T,S}$ are fixed by the ‘hard’ modes at the Fermi surface. Their exact values are determined by the dynamics at the Fermi surface. For simplicity, we will assume $F_S \sim F_T \sim F \sim \Delta$ from now on.

We note that (7) only accounts for the gluon interaction to leading order in $g_s$. To order $g_s^2$ the gluon mediated interaction yields $L,R$ mixing in a parity-invariant way. The breaking of chiral symmetry in the CFL phase via composite operators, e.g. $\langle (\bar{q}q)^{2n} \rangle \sim | \langle \bar{q}q \rangle |^{2n}$, does not imply additional order parameters. No additional Goldstone modes are needed as all expectation values follow through simple Fierz rearrangements.

The effective Lagrangian (7) in the CFL phase bears much in common with Skyrme-type Lagrangians \[28, 29, 18\], with the screened color mediated interaction analogous to the exchange of massive vector mesons. Indeed in the CFL phase the ‘screened’ gluons and the ‘Higgsed’ gauge composites are the analogue of the massive vector mesons in the low density phase \[16\]. In the CFL phase the WZW term is needed to enforce the correct flavor anomaly structure. Its structure is best described in 5-dimensional space as the homotopy of the Nambu-Goldstone manifold $M$, $\Pi_3(M) = Z$. Through the normalization $S_{WZW} = 2m\pi$ with $m \in Z$, its coefficient $n_L$ and $n_R$ must be an integer to have a consistent quantum theory.

3. Much like the effective Lagrangian for QCD at low density (giving rise to skyrmions), the low-energy effective Lagrangian in the CFL phase admits a stable (static) soliton solution, with a winding number given by the homotopy $\Pi_3(M) = Z$. It is stable by the balance between the kinetic energy (attractive force) and the Coulomb energy (repulsive force) \[30, 31\]. Since the soliton quantum numbers are determined by the WZW term upon quantization, we need to determine the coefficient $n$. For skyrmions, Witten \[26\] has shown that $n_L$ and $n_R$ are fixed by the underlying flavor anomalies in QCD. Here we proceed to show that $n_L$ and $n_R$ are fixed by analogous color-flavor anomalies in the CFL phase.

First, consider the anomalous $SU(3)_L$ current, the pertinent $U(1)$ subgroup of which

\[9^5\]There can be additional higher derivative terms that may significantly contribute to the stabilization of the qualiton which we shall not address here. Differently to the qualiton in the vacuum, one would also have to consider terms that are specific to the Fermi surface.
is usually at the origin of the pion decay into two photons (i.e., ABJ triangle-anomaly). Since under the $U(1)_{em}$ electro-magnetic transformation the quark transforms as $q \rightarrow e^{i\epsilon Q} q$ with $Q = \text{diag} (2/3, -1/3, -1/3)$, the $U(1)_{em}$ transformation of the CFL condensate can be undone by a $U(1)_Y$ hyper-charge transformation of $SU(3)_c$ with $Y = -Q$. Therefore, the unbroken $U(1)_{\tilde{Q}}$ gauge boson (or modified photon) is a linear combination of the original photon and gluon

$$\tilde{A}_\mu = A_\mu \cos \theta + G^Y_\mu \sin \theta$$

(8)

where $\cos \theta = g_s/\sqrt{e^2 + g_s^2}$ and $G^Y_\mu$ is the gluon field for $U(1)_Y \subset SU(3)_c$. Since the quark now couples to the modified photon with strength $\tilde{e} = e g_s/\sqrt{e^2 + g_s^2}$, the unit charge of $U(1)_{\tilde{Q}}$ is $\tilde{e}$.

A $U(1)_{\tilde{Q}}$ transformation $\exp(i\epsilon\tilde{Q})$ on $U$ is nothing but a simultaneous transformation of $g_c = \exp(i\epsilon Y)$ and $g = \exp (i\epsilon Q)$:

$$U \rightarrow e^{i\epsilon Q} U e^{-i\epsilon Q},$$

(9)

where we have used $Y = -Q$ for the left multiplication. We see that the $U(1)_{\tilde{Q}}$ transformation Eq. (1) is identical to the $U(1)_{em}$ transformation of the Nambu-Goldstone bosons associated with chiral symmetry breaking $SU(3)_L \times SU(3)_R \rightarrow SU(3)_V$ in the QCD vacuum, except for the fact that the unit charge is now $\tilde{e} = e g_s/\sqrt{e^2 + g_s^2}$ instead of $e$.

Therefore, all the modified electric charges of the Nambu-Goldstone bosons in the CFL phase are integral multiple of $\tilde{e}$, the modified electric charge of the electron. Note that all the quarks and gluons have integer (modified) electric charges too, since quarks and gluons transform as Eq. (9) under $U(1)_{\tilde{Q}}$. Specifically,

$$G_\mu \rightarrow e^{i\tilde{Q}} G_\mu e^{-i\tilde{Q}} = e^{iY} G_\mu e^{-iY}$$

(10)

$$q \rightarrow e^{i\tilde{Q}} q e^{i\tilde{Q}} = e^{iY} q e^{iY}$$

(11)

The $SU(3)_L$ anomaly is now given as

$$\partial^\mu J^A_{L\mu} = \frac{e^2}{32\pi^2} \text{Tr} \left(T^A \tilde{Q}^2\right) \epsilon_{\mu\nu\rho\sigma} \tilde{F}^{\mu\nu} F^{\rho\sigma},$$

(12)

where $\tilde{F}^{\mu\nu}$ is the field strength tensor of the modified photon associated with $U(1)_{\tilde{Q}}$. Note that there is no color or flavor factor in the coefficient because neither color nor flavor is running around the triangle diagram responsible for the $SU(3)_L$ anomaly. In the CFL phase color and flavor are locked, and the modified pion and photon carry color. In the CFL phase,

$$\tilde{Q}_{\alpha\alpha} = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

(13)

and only three quarks contribute to the anomaly in the $J^3_{L\mu}$ current: $\text{Tr} \left(T^3 \tilde{Q}^2\right) = 1/2 \cdot (1^2 + 1^2) + (-1/2) (-1)^2 = 1/2$. 
To fix the coefficient \( n \) of the WZW term from the \( SU(3)_L \) anomaly, we introduce the modified electro-magnetic interaction by gauging the \( U(1)_{\tilde{Q}} \) symmetry in the effective action (7) through

\[
\partial^\mu U \mapsto D^\mu U = \partial^\mu U - i\tilde{e} \tilde{A}_\mu (Y_{ab} U_{b\alpha} + U_{a\beta} Q_{\beta\alpha}).
\]

(14)

As shown by Witten [26], the \( U(1)_{\tilde{Q}} \) invariant WZW term then contains a term

\[
S_{WZW}(U, \tilde{A}_\mu) \ni n_L \tilde{e}^2 \frac{F}{64\pi^2} \epsilon^{\mu\nu\alpha\beta} \tilde{F}_{\mu\nu} \tilde{F}_{\alpha\beta},
\]

(15)

which agrees with (12) if \( n_L = 1 \). Similarly, one can also show that \( n_R = 1 \).

On the other hand, the topologically conserved current for the soliton in (7) is

\[
J^\mu_L = \frac{1}{24\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{Tr} \; U^{-1}_L \partial_\nu U_L U^{-1}_L \partial_\rho U_L U^{-1}_L \partial_\sigma U_L.
\]

(16)

Since the sigma model description of the \( SU(3)_L \) quark current \( J^A_{L\mu} = \bar{q}_L T^A \gamma_\mu q_L \) contains an anomalous piece from the WZW term

\[
J^A_{L\mu} \ni \frac{1}{24\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{Tr} \; T^A U^{-1}_L \partial_\nu U_L U^{-1}_L \partial_\rho U_L U^{-1}_L \partial_\sigma U_L
\]

(17)

and the topological current Eq. (16) corresponds to the anomalous piece of the \( U(1)_L \) current in the \( \sigma \) model description [26, 32, 33], we find the soliton of unit winding number has a quark number 1 or baryon number \( B = \frac{1}{3} \). This is Kaplan’s qualiton [19] in the CFL phase. We are referring to it as the superqualiton to distinguish it from Kaplan’s qualiton in the vacuum. Since in the CFL phase baryon number is spontaneously broken \( U_B(1) \rightarrow Z_2 \), it is clear that \( B = \frac{1}{3} \) is the same as \( B = 1 \), hence \( B = (1 \text{ mod } 2)/3 \). The fact that \( n_L = 1 \) (or \( n_R = 1 \)) implies that the superqualiton is a fermion. The superqualiton carries positive parity, since the CFL condensate is parity even. Further quantum numbers along with the excited spectrum of the superqualiton will be discussed below.

4. Some important questions regarding superqualitons concern their mass \( M_S \) and their interactions. Indeed, if \( M_S \) happens to be less than \( \Delta \) (where \( \Delta \) is the superconductivity gap), then superqualitons may exist inside the superconducting CFL gap with ordinary baryon quantum numbers, a phenomenon unseen in conventional superconductors. Attractive interactions among light superqualitons may even trigger a total rearrangement inside the QCD superconductor.

A simple estimate of the mass follows from scale (virial-like) arguments that are in general not specific to the detailed form of (7). Indeed, for stable static configurations a typical gradient in (7) is \( \partial \sim \Delta \). Since \( F \sim \Delta \) and the size of the superqualiton \( R_S \sim 2\pi/(4\pi\Delta) \), then

\[
M_S \sim \left( \frac{1}{2} F^2 \Delta^2 \right) \left( \frac{4}{3} \pi R_S^3 \right) \sim \frac{1}{4} \Delta
\]

(18)
which is less than \( \Delta/2 \). Both the mass and the size are fixed by \( \Delta \). To create 2 superqualitons costs less than to excite 2 bare quarks from the CFL phase. Such superqualitons may either condense (crystallize) into a new superconductor with a reduced gap, or exhibit a liquid or gas phase with either normal or superconducting properties. In order to address these issues, the specific dynamics of the many-superquliton interaction is required. Unfortunately we have presently no good understanding of this aspect of the problem.

A more conservative estimate of the mass of the superqualiton can be made by relying on the effective Lagrangian (7). Following Skyrme [28] and others [18] (and references therein) we seek a static configuration for the field \( U_L \) in \( SU(3) \) by embedding an \( SU(2) \) hedgehog in color-flavor in \( SU(3) \), with

\[
U_{Lc}(x) = e^{i \vec{r} \cdot \hat{r} \theta(r)}, \quad U_R = 0, \quad G_Y^0 = \omega(r), \quad \text{all other } G_{\mu}^A = 0, \quad \tag{19}
\]

where \( \tau \)'s are Pauli matrices. The radial function \( \theta(r) \) is monotonous and satisfies

\[
\theta(0) = \pi, \quad \theta(\infty) = 0 \quad \tag{20}
\]

for a soliton of winding number one. (Note that we can also look for a right-handed soliton by switching off the \( U_L \) field. The solution should be identical because (7) is invariant under parity.) This configuration has only non-vanishing color charge in the \( Y \) direction

\[
J_Y^0 = \frac{\sin^2 \theta \theta'}{2 \pi r^2} \quad \tag{21}
\]

and all others are zero. As shown in [19], the energy of the configuration (19) is given as

\[
E[\omega, \theta] = \int 4\pi r^2 dr \left[ -\frac{1}{2} \omega'^2 + F^2 \left( \theta'^2 + 2 \frac{\sin \theta^2}{r^2} \right) + \frac{g_s}{2\pi^2} \frac{\omega}{r^2} \sin^2 \theta \theta' \right]. \quad \tag{22}
\]

The total charge within a radius \( r \) is

\[
Q_Y^0(r) = g_s \int_0^r \text{Tr} \, Y J_Y^0(r') 4\pi r'^2 dr' = -g_s \left( \frac{\theta(r) - \sin \theta(r) \cos \theta(r) - \pi}{\pi} \right). \quad \tag{23}
\]

Using Gauss law with a screened charge density, we can trade \( \omega \) in terms of \( \theta(r) \),

\[
\omega' = \frac{Q_Y^0(r)}{4\pi r^2} e^{-m_E r}, \quad \tag{24}
\]

where \( m_E = \sqrt{6\alpha_s/\pi \mu} \) is the electric screening mass for the gluons (note that the magnetic gluons are not needed for the static configuration). Hence, the energy functional simplifies to

\[
E[\theta] = \int_0^\infty \mathcal{E}(r) dr = \int_0^\infty 4\pi dr \left[ F^2 \left( r^2 \theta'^2 + 2 \sin^2 \theta \right) + \frac{\alpha_s}{2\pi} \left( \frac{\theta - \sin \theta \cos \theta - \pi}{2r} \right)^2 e^{-2m_E r} \right], \quad \tag{25}
\]
where \( \alpha_s = g_s^2/(4\pi) \). The squared size of the superqualiton is \( R_S^2 = \langle r^2 \rangle \) where the averaging is made using the (weight) density \( \mathcal{E}(r) \). The equation of motion for the superqualiton profile \( \theta(r) \) is

\[
\left( r^2 \theta' \right)' = \sin 2\theta + \frac{\alpha_s e^{-2m_E r}}{4\pi \langle F r \rangle^2} \sin^2 \theta \left( \pi - \theta + \frac{1}{2} \sin 2\theta \right)
\]

(26)

subject to the boundary conditions (20).

We have solved (26) numerically for several values of \( m_E \) and \( \alpha_s \). Given a fixed \( \alpha_s = 1 \), we find the soliton mass \( M_S = 1.53, 1.06, 0.75 \times 4\pi F \) and the soliton radius \( R_S = 0.29, 0.20, 0.15 \) \( F^{-1} \) for \( m_E/F = 2, 5, 10 \), respectively. For various values of \( \alpha_s = 1.0, 0.5, 0.2 \) with a given \( m_E/F = 2 \), we get \( M_S = 1.53, 1.21, 0.87 \times 4\pi F \) and \( R_S = 0.29, 0.24, 0.17 \) \( F^{-1} \). We see that both the mass and the size of the soliton decrease as the repulsive Coulomb force decreases, showing that the soliton gets smaller for the weaker coupling, since the color-electric force, which balances the kinetic energy of the soliton, is less repulsive \([30]\). For the high density region, where the QCD coupling is small, we expect the higher order derivative terms to become more important in stabilizing the soliton, since the Coulomb forces become negligible following screening (an alternative would be through the use of gauge composites). This trend is expected from the screened character of the interaction, and seems to differ from the one we suggested earlier by our simple scaling arguments. Which scenario (among the various alternatives) is favored will depend on the content of the effective Lagrangian implied by first principles of QCD which may not be reliably embodied in \([7]\).

5. To access the quantum numbers and the spectrum of the superqualiton, we note as usual that for any static solution to the equations of motion, one can generate another solution by a rigid \( SU(3) \) rotation,

\[
U(x) \rightarrow AU(x)A^{-1}, \quad A \in SU(3).
\]

(27)

The matrix \( A \) corresponds to the zero modes of the superqualiton. Note that two \( SU(3) \) matrices are equivalent if they differ by a matrix \( h \in U(1) \subset SU(3) \) that commutes with \( SU(2) \) generated by \( \vec{\tau} \otimes I \). The Lagrangian for the zero modes is given by substituting \( U(\vec{x}, t) = A(t)U_c(\vec{x})A(t)^{-1} \) \([34]\). Hence,

\[
L[A] = -M_S + \frac{1}{2} I_{\alpha\beta} \text{Tr} T^\alpha A^{-1} \dot{A} \text{Tr} T^\beta A^{-1} \dot{A} - \frac{i}{2} \text{Tr} Y A^{-1} \dot{A},
\]

(28)

where \( I_{\alpha\beta} \) is an invariant tensor on \( \mathcal{M} = SU(3)/U(1) \) and the hypercharge \( Y \) is

\[
Y = \frac{1}{3} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -2
\end{pmatrix}.
\]

(29)
Under the transformation $A(t) \mapsto A(t)h(t)$ with $h \in U(1)_Y$,

$$L \mapsto L - \frac{i}{2} \text{Tr} \ Y h^{-1} \dot{h}. \quad (30)$$

Therefore, if we rotate adiabatically the soliton by $\theta$ in the hypercharge space in $SU(3)$, $h = \exp(iY\theta)$, for time $T \to \infty$, then the wave function of the soliton changes by a phase in the semiclassical limit:

$$\psi(T) \sim e^{i \int dt L} \psi(0) = e^{i\theta/3} \psi(0), \quad (31)$$

where we neglected the irrelevant phase $-M_S T$ due to the rest mass energy. The simplest and lowest energy configuration that satisfies Eq. (31) is the fundamental representation of $SU(3)$. Similarly, under a spatial (adiabatic) rotation by $\theta$ around the $z$ axis, $h = \exp(i\tau^3\theta)$, the phase of the wave function changes by $\theta/2$. Therefore, the ground state of the soliton is a spin-half particle transforming under the fundamental representation of both the flavor and the color group, which leads us to conclude that the soliton is a massive left-handed (or right-handed) quark in the CFL phase.

It is interesting to note that the phase continuity between high-density quark matter and low-density nuclear matter, conjectured by Schäfer and Wilczek [16], carries to the Skyrme picture of baryons. One way to address the specifics of the continuity besides a spectrum analysis, is via anomaly-matching. This issue will be addressed elsewhere.

6. In the CFL phase of the QCD superconductors, baryons emerge as qualitons (superqualitons). We have shown that the quantum numbers of the superqualitons are dictated by a WZW term suitably normalized by the CFL triangle-anomaly. We have presented general arguments for the effective Lagrangian in the CFL phase. Superqualitons can be stabilized by screened colored electric interactions for a variety of chemical potentials, exhibiting a rich and colorful spectrum. Our estimate of the mass is on the higher side of the superconducting gap, although simple scaling arguments show it otherwise. Our calculations do not exhaust all parameter space for effective model calculations in QCD superconductors, and point at the necessity of a first principle assessment of the effective Lagrangian (7). One fascinating possibility suggested by the superqualiton description of the hadron-quark continuity is that light fermionic modes with $M_s < \Delta$ get excited within the CFL superconducting gap, a phenomenon unseen in normal superconductors. On general grounds, the interaction between light superqualitons may cause a rearrangement in the superconducting phase with important implications for the bulk properties such as the specific heat. The latter is paramount to cooling curves of neutron stars.

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Note added

After posting our paper, [35] appeared in which some of the present issues were discussed. We note that the linear interpretation of the symmetry breaking pattern adopted by the authors in [35] differs from ours and that our nonlinear description does contain $L, R$ mixing to order $g_s^2$ contrary to their statement.
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