Internal Josephson phenomena in a tuned linear-nonlinear two-state Bose-Einstein condensate

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We discuss the coherent oscillations between two coupled quantum states of a Bose-Einstein condensate in two-dimensional space at zero temperature. In the system we consider, weak interparticle repulsive interactions occur between the particles in state one only, while the state two particles remain non-interacting. The system is described by the coupled Gross-Pitaevskii (non-linear) and Schrödinger (linear) equations for macroscopic wave functions of the two states. Analytical as well as numerical solution reveals a variety of regimes of dynamics of the relative quantum phase and population imbalance between the two states of the condensate, such as harmonic and anharmonic Rabi oscillations, internal a.c. Josephson effect, and macroscopic quantum self-trapping. We show that there is a value of energy detuning between the states at which oscillations are fully suppressed.

PACS numbers: 03.75.Kk, 67.85.Fg, 74.50.+r, 71.36.+c

I. INTRODUCTION

Following the original discovery by Josephson in 1962 [1], the so-called external Josephson effect was extensively studied in superconducting [2–4], superfluid helium [5] and atomic bosonic weak links [6–13], both theoretically and experimentally. It has also been argued for a system of superfluid 3He-A [14, 15] that Josephson-like effects may occur for two condensates which are not spatially separated but occupy different quantum states. Later, experimental observation of internal Josephson effect in a two-state BEC of 87Rb atoms was also reported [16].

Recently, a new ground for the Josephson physics investigation appeared: quantum-optical strongly correlated photonic systems in single-cavity [17, 18], multi-cavity [19, 20], and extended photonic media (see the recent review [21] and references therein). Within the field of cavity electrodynamics, great theoretical and experimental activity was aimed to study Josephson phenomena in exciton-polariton BECs coupled via a double-well potential [22–28]. Bosons participating in the phenomena are composite half-light half-matter quasiparticles interacting via their excitonic component [28], while the photon component allows them to condense at high critical temperatures [30]. Despite the fact that different dynamical regimes which result from the interplay between the tunneling coupling strength of the two condensates and the interparticle interactions depend on the specific system under consideration [22, 27, 28], the oscillations discussed in context of polaritons always imply that the two polaritonic BECs are spatially separated by a tunnel barrier. However, since polaritons itself are composite and result from strong coupling between the photon and quantum well exciton modes inside a planar optical cavity, they may be considered as a mixture of the two Bose-condensed components performing mutual resonant transformations. The internal dynamics of exciton-polariton gas was kept behind the scenes until the year of 2014, when observation of the relaxation oscillations in a polariton system was reported [31]. This experiment, as well as following theoretical works [32, 33], demonstrate that there is a rich variety of dynamical behaviors in the system of exciton-polaritons still to be explored.

The present paper is dedicated to the study of various regimes of internal oscillations in the system of coupled linear (non-interacting) and non-linear (interacting) BECs occupying the same two-dimensional space. The theoretical problem in consideration is general, although we keep the polariton background in mind for the sake of realistic simulation parameters, which we extract from the recent microcavity experiments. Whereas for a pure quantum two-level system with no interactions the dynamics is described by single-particle harmonic Rabi oscillations, we show that in a more sophisticated system such as BEC with interactions and non-zero energy detuning, various scenarios of collective oscillations become possible, including anharmonic Rabi oscillations, the analog of internal a.c. Josephson effect, and the regime of macroscopic quantum self-trapping of populations [3] and full suppression of oscillations.

The paper is organized as follows. In Section II we introduce all notations and the theoretical model, analyze the time-dependent coupled equations describing temporal dynamics of the system and discuss the analogies with internal Josephson effect in related systems. In Section III we solve the evolution equations, both analytically and numerically, and discuss various dynamical regimes of the system which can be realized, depending on the parameters. Section IV summarizes our results.
II. TWO-MODE GROSS-PITAEVSKII MODEL

Within the mean-field approximation, ground-state unit area energy functional for weakly nonhomogeneous two-component Bose condensate with weak repulsive interactions in one component can be introduced in the following form [34]:

\[
\mathcal{E}[\psi_1, \psi_2] = -\frac{\hbar^2}{2m_1} \psi_1^* \nabla \psi_1 + E_1^0|\psi_1|^2 + \frac{g}{2}|\psi_1|^4
\]

\[
-\frac{\hbar^2}{2m_2} \psi_2^* \nabla \psi_2 + E_2^0|\psi_2|^2 + \frac{\hbar \Omega_R}{2}(\psi_1^* \psi_2 + \psi_2^* \psi_1),
\]

with \(\psi_{1,2}\) the two macroscopic wave functions of the condensate components, \(E_{1,2}^0\) the bottoms of energy dispersions and \(m_{1,2}\) the effective masses of the two types of particles. \(\tilde{g} > 0\) is the constant of repulsive interactions, \(\hbar \Omega_R\) the Rabi splitting energy. Externally trapping potential and spin degree of freedom are omitted here for simplicity. The corresponding terms can be included in the theoretical [11] whenever needed. We assume \(T = 0\) and take no account of particles gain or dissipation. Those non-equilibrium effects are out of scope of the present discussion.

Using variational principle \(i\hbar \partial_t \psi_{1,2} = \delta \mathcal{E}/\delta \psi_{1,2}^*\), one gets a set of two coupled differential equations of the Gross-Pitaevskii type,

\[
i \hbar \partial_t \psi_1 = \left[A_1^0 - \frac{\hbar^2 \nabla^2}{2m_1} + \tilde{g}|\psi_1|^2\right] \psi_1 + \frac{\hbar \Omega_R}{2} \psi_2,
\]

\[
i \hbar \partial_t \psi_2 = \left[A_2^0 - \frac{\hbar^2 \nabla^2}{2m_2}\right] \psi_2 + \frac{\hbar \Omega_R}{2} \psi_1.
\]

In the absence of interactions, initial particles dispersions split due to degeneracy into a pair of mixed eigenmodes. The positive sign chosen in Eqs. (2), (3) in front of the coupling term \(\hbar \Omega_R/2\) imposes that the antisymmetric mode \((\psi_1 - \psi_2)/\sqrt{2}\) with the relative quantum phase \(\pi\) is the lower energy level while the symmetric mode \((\psi_1 + \psi_2)/\sqrt{2}\) with zero quantum phase is the upper one. An initial state of the two-component BEC, being some linear combination of these two modes, results in density oscillations between the two particle subsystems. In the presence of interactions (\(\tilde{g} > 0\)), the effective lower energy level is blueshifted (while the upper energy level appears redshifted), and the eigenmodes are no longer the antisymmetric and the symmetric ones. Still, the relative phase oscillations which will be discussed below go around the time-average values \(\pi\) (\(\pi\)-phase modes) or \(0\) (zero-phase modes).

Since we are interested in the temporal evolution of the system, we restrict our consideration to the homogeneous case when the wave functions profiles are spatially uniform. Therefore we omit all the spatial derivatives in Eqs. (2) and (3). Further in this paper, we will rescale lengths and energies in terms of harmonic-oscillator units \(\sqrt{\hbar/m_1 \Omega_R}\) and \(\hbar \Omega_R\), respectively. Additionally rescaling time as \(\tilde{t} = \sqrt{\hbar/m_1 \Omega_R} \to t\) and the wavefunctions as \(\psi_{1,2}/\sqrt{\hbar/m_1 \Omega_R} \to \psi_{1,2}\), we get the equations

\[
i \partial_t \psi_1 = \left[\epsilon_1^0 + g|\psi_1|^2\right] \psi_1 + \frac{1}{2} \psi_2, \quad (4)
\]

\[
i \partial_t \psi_2 = \epsilon_2^0 \psi_2 + \frac{1}{2} \psi_1
\]

with the definitions \(\epsilon_1^0 = E_{1,2}^0/\hbar \Omega_R\) and \(g = \tilde{g} m_1/\hbar^2\).

After the Madelung transformation

\[
\psi_{1,2}(t) = \sqrt{n_{1,2}(t)} e^{i S_{1,2}(t)}
\]

we get four non-linear dynamical equations for the condensate densities \(n_{1,2}(t)\) and quantum phases \(S_{1,2}(t)\):

\[
\begin{aligned}
\dot{n}_1 &= -\sqrt{n_1 n_2} \sin (S_1 - S_2), \\
\dot{S}_1 &= -\epsilon_1^0 - g n_1 - \frac{1}{2} \sqrt{\frac{n_2}{n_1}} \cos (S_1 - S_2), \\
\dot{n}_2 &= \sqrt{n_1 n_2} \sin (S_1 - S_2), \\
\dot{S}_2 &= -\epsilon_2^0 - \frac{1}{2} \sqrt{\frac{n_1}{n_2}} \cos (S_1 - S_2). \\
\end{aligned}
\]

In order to investigate the dynamics, it is convenient to introduce a new set of variables: population imbalance between the two subsystems \(\rho(t) = n_1(t) - n_2(t)\) and the relative quantum phase \(S(t) = S_1(t) - S_2(t)\). These variables \(\rho\) and \(S\) obey the coupled evolution equations

\[
\dot{\rho} = -\sqrt{n^2 - \rho^2} \sin S, \quad (8)
\]

\[
\dot{S} = -\Delta E - \frac{\rho}{n} + \frac{\rho}{\sqrt{n^2 - \rho^2}} \cos S, \quad (9)
\]

where \(n = n_1(t) + n_2(t)\) is total number of particles in the condensate. The dimensionless effective detuning

\[
\Delta E = \delta + \frac{gn_1}{2}, \quad \delta = \epsilon_1^0 - \epsilon_2^0, \quad (10)
\]

and the dimensionless blueshift value

\[
\Lambda = \frac{gn_1}{2}, \quad (11)
\]

are the parameters which determine different regimes of the system behavior. It is worth noting that for a closed conservative system that we consider the equations (5), (9) have Hamiltonian form for canonically conjugate variables: \(\dot{\rho} = \partial H/\partial S, \dot{S} = -\partial H/\partial \rho\). The conserved energy is

\[
H(S, \rho) = \Delta E \rho + \Lambda \frac{\rho^2}{2n} + \sqrt{n^2 - \rho^2} \cos S. \quad (12)
\]
where total population $n$ remains constant. The Hamiltonian (12) is analogous to that of classical nonrigid pendulum with length dependent on its angular momentum $\rho$ (although, due to the choice of sign in front of the coupling term $\sim \hbar \Omega R / 2$ in (1), the pendulum’s “restoring force” is driving the tilt angle $S$ to $\pi$ instead of zero). Equations (8) and (9) admit analytical solution in terms of quadratures,

$$\cos S = \frac{H - \Lambda \rho^2 - \Delta E \rho}{\sqrt{n^2 - \rho^2}},$$  \quad (13)

$$t = \mp \int \frac{d\rho}{\sqrt{n^2 - \rho^2 - (H - \Lambda \rho^2 - \Delta E \rho)^2}}. \quad (14)$$

A system providing a physical analogy to the two-component condensate considered here is a double atomic BEC [16] with two condensate states separated by a hyperfine splitting produced by an external magnetic field. In this case, the model equations for the two wave functions both contain nonlinearities brought into the system by intra-species $g_{1,2}$ and inter-species $g_{12} = g_{21}$ interactions, and the equations are symmetric with respect to $1 \leftrightarrow 2$ replacement. Mathematically, such a system is described by the evolution equations equivalent to those for bosonic Josephson junction [8]. In the notation used here, the Hamiltonian for the atomic double-BEC reads:

$$H' = \Delta E' \rho + \Lambda' \frac{\rho^2}{2n} - \sqrt{n^2 - \rho^2} \cos S \quad (15)$$

with the parameters $\Delta E' = \delta + (g_1 - g_2)n/2$ and $\Lambda' = (g_1 + g_2)n/2$.

III. RESULTS AND DISCUSSION

A. Symmetric case $\Delta E = 0$

For the starting analysis we consider the system with no effective detuning: $\Delta E = 0$ (i.e. bare detuning $\delta$ compensates the blueshift $g n/2$). In this case, Eqs. (5) and (9) may be transformed to the following:

$$\dot{\rho} + \rho \left(1 + \Lambda \sqrt{1 - \left(\frac{\rho}{n}\right)^2} \cos S\right) = 0, \quad (16)$$

$$\dot{S} + \frac{1}{2} \frac{n^2 + \rho^2}{n^2 - \rho^2} \sin(2S) - \frac{\Lambda}{\sqrt{1 - \left(\frac{\rho}{n}\right)^2}} \sin S = 0. \quad (17)$$

If interactions are negligible ($\Lambda \to 0$), Eq. (16) reduces to $\dot{\rho} + \rho = 0$ and describes harmonic Rabi oscillations between the “state 1” and “state 2” with natural frequency $\Omega_R$ (which corresponds to $\omega = 1$ in unscaled units) and time-average value $\langle \rho \rangle = 0$. Eq. (17) in general case is referred to as the Hill equation and describes a so-called parametric oscillator. The explicit solutions for relative phase and population imbalance for $\Lambda = 0$ read

$$\rho(t) = \mp \sqrt{n^2 - H_0^2} \sin(t \mp t_0), \quad (18)$$

$$\cos S(t) = \frac{H_0}{\sqrt{n^2 \cos^2(t \mp t_0) + H_0^2 \sin^2(t \mp t_0)}}. \quad (19)$$

where $H_0 = H[S(0), \rho(0)] = \sqrt{n^2 - \rho^2(0)} \cos S(0)$ and initial phase of oscillations $t_0 = \arcsin[\rho(0)/\sqrt{n^2 - H_0^2}]$. The upper sign (‘-') corresponds to the case when $\pi/2 < S(0) < \pi$, the lower sign (‘+') corresponds to the case when $\pi < S(0) < 3\pi/2$ (for $\pi$-phase modes with $\langle S \rangle = \pi$). The tunneling to zero-phase modes with $\langle S \rangle = 0$ in the conservative system (which is characterized by initial phase difference in the vicinity of $\pi$) cannot be observed.

The shape of $S(t)$ temporal profile is strongly dependent on initial conditions. In case of small-amplitude oscillations around the equilibrium point $\pi$, $S(t)$ is periodic with a modulated “period” $T(t) = 2\pi/\sqrt{(n^2 + \rho^2(t))/\left(n^2 - \rho^2(t)\right)} (0 < T < 2\pi)$. Furthermore, in the limit $\rho \ll n$, Eq. (17) reduces to $\dot{S} + S = \pi$, and hence the relative phase oscillates with the same frequency $\Omega_R$ as does the population imbalance $\rho(t)$. If $|S(0) - \pi|$ is comparable to $\pi/2$ or $\rho(0) \to n$, $S(t)$ becomes strongly unharmonic while $\rho(t)$ oscillates with large amplitude $\sqrt{n^2 \sin^2 S(0) + \rho^2(0) \cos^2 S(0)}$. Fig. 1 shows population imbalance and relative phase...
against time for the case of small-amplitude oscillations ($\rho(0) = 0.1n$) and for large-amplitude oscillations ($\rho(0) = 0.99n$). Here and below we will always imply the initial phase difference $S(0) = \pi$ unless stated otherwise.

In the context of exciton-polaritons, the unscaled interaction constant $g$ is of the order of $10^{-3}$ (estimated from $g = 0.015$ meV·Å)$^2$ [30]). Thus, the type of behavior described above changes only for large values of the total density $n$ of the polaron condensate. Numerical solution of Eqs. (16) and (17) which takes into account interactions ($\Lambda \neq 0$) starts to noticeably differ from the analytical solution given by (18) and (19) only when the parameter $\Lambda$ becomes of the order of $10^{-1}$ and larger. For small-amplitude oscillations, this difference appears as a shift of the oscillation frequency

$$\omega = \Omega_R \sqrt{1 + \Lambda}.$$  \hfill (20)

Mathematically, as the parameter $\Lambda$ increases, the evolution becomes an interplay between the Rabi dynamics and an analog of internal Josephson effect with a variety of regimes. From this point of view, for small-amplitude oscillations discussed above one may say that the shift of natural frequency corresponds to Josephson “plasma frequency” $\omega_{pl} = \sqrt{\Lambda \Omega_R}$. For large-amplitude oscillations, when one considers large values of $\rho$ comparable to $n$, the transition from the Rabi to the Josephson dynamics becomes more dramatic. When $\Lambda = 0$, it is easy to see that (12) always yields oscillations of $\rho(t)$ around zero with the Rabi frequency $\Omega_R$ (see Eq. (16)). By contrast, when $\Lambda \neq 0$, oscillations frequency becomes dependent on the initial values $\rho(0)$ and $S(0)$. The regimes of oscillations at different values of $\Lambda$ are not shown here as they are completely analogous to those appearing in the bosonic Josephson junction described by the Hamiltonian (15) and thoroughly discussed in Ref. [8], with the only difference that the stable equilibrium of the Hamiltonian (15) corresponds to the point $S = \pi$ while for the Hamiltonian (12) it is $S = 0$. For $\Lambda > 1$, the sinusoidal oscillations in $\rho(t)$ shown in Fig. 1 acquire higher harmonics. As $\Lambda$ approaches some critical value $\Lambda_c$, the oscillations display a critical slowing down with logarithmic divergence. When $\Lambda$ exceeds $\Lambda_c$, the oscillations of population imbalance average to a nonzero number $\langle \rho \rangle \neq 0$ while relative phase $S(t)$ runs without bound (in the classical mechanics analogy, it is equivalent to pendulum undergoing complete turns). The direction in which $\langle \rho \rangle$ shifts with respect to zero (or, the direction of the pendulum rotation) is determined by the sign of $\rho(0)$. This regime is called macroscopic quantum self-trapping of the populations [7], and it is an effect which arises from the nonlinear interaction term $\sim gn^2\rho^2$ of the Hamiltonian (12). Critical value $\Lambda_c$ depends on $\rho(0)$ and $S(0)$. For instance, for the initial conditions $S(0) = \pi$ and $\rho(0) = 0.99n$ the critical number is $\Lambda_c = 2.69$, for $\rho(0) = 0.9n$ $\Lambda_c = 3.54$, and so on: $\Lambda_c \to \infty$ as $\rho(0) \to 0$, i.e. in the small-amplitude limit ($\rho \ll n$) there are always oscillations of zero average and frequency given by (20). For $g \sim 10^{-3}$, however, reaching even $\Lambda \sim 1$ would require condensate densities as large as $n \sim 10^3$. N.B., for realistic polariton densities, the unscaled $n$ is of the order of unity (estimated from $10^{10}$ cm$^{-2}$ [30]), hence for the closed conservative system with $n = \text{const}$ this regime is practically not realized, and the effect of interactions on the pure Rabi dynamics stays negligible.

A phase-plane portrait of the conjugate variables $\rho$ and $S$, summarizing the dynamical behavior of the system in the case $\Delta E = 0$, is displayed in Fig. 2. The gray dotted lines show the evolution of the condensates for vanishing value of the parameter $\Lambda \sim 10^{-3}$, with the initial population imbalance $\rho(0) = 0.01, 0.1, 0.2, 0.3, 0.5, 0.75$ and $0.9n$. These trajectories correspond to the oscillations shown in Fig. 1 where the population imbalance $\rho$ oscillates around 0 and the relative phase $S$ around $\pi$. The black solid lines mark the trajectories calculated for different $\Lambda/\Lambda_c$ at constant $\rho(0) = 0.9n$. The red solid line marks the separatrix (trajectory at the critical value of the parameter $\Lambda = \Lambda_c$ when the oscillations lose harmonicity and slow down). Black dashed lines mark the trajectories for the values of $\Lambda$ exceeding the critical value $\Lambda_c$, when the system is in the macroscopic quantum self-trapping regime.

![FIG. 2. (Color online) Phase-plane portrait of the conjugate variables $\rho$ (in units of $n$) and $S$ (in units of $\pi$).](image-url)

(a) $\Delta E = 0$: grey dotted trajectories for $\Lambda = 3 \cdot 10^{-4}\Lambda_c$ and $\rho(0)$ values as marked, dark (solid and dashed) trajectories for $\rho(0) = 0.9n$ and $\Lambda/\Lambda_c$ values as marked.

(b) $\Delta E \neq 0$: all trajectories for $\rho(0) = 0.1n$ and $\Delta E$ values as marked. Other physical values same as in Fig. 1.
B. Asymmetric case $\Delta E \neq 0$

Let us now focus at the main goal of the present research: the case of non-zero effective detuning, in which the system behavior is influenced by the additional term $(\delta + \Lambda)\rho$ in the Hamiltonian (12). From mathematical point of view, the resulting dynamics should be governed by a competition of the terms $\Delta \rho/n$ and $\Delta E$ in Eq. (9). Physically, the detuning between the modes is assumed to be less than or comparable to the Rabi splitting energy $\hbar \Omega_R$, which in unscaled units used in this paper means that absolute value of detuning ranges from zero to a number of the order one. Therefore, since the maximum ratio $\rho/n$ equals 1, for almost all values of detuning (except a narrow region $-2\Lambda < \delta < 0$, which is equivalent to $\Lambda > |\Delta E| \neq 0$) the term containing $\Delta E$ wins this competition. (As discussed in the previous section, expectable values of the parameter $\Lambda$ for the polariton system are of the order of $10^{-3}$.)

Whereas for the case $\Delta E = 0$ and small values of $\Lambda < \Lambda_c$ the time-average population imbalance was always zero, for $\Delta E \neq 0$ the oscillations of $\rho(t)$ always average to a non-zero number dependent on $\Lambda$, $\Delta E$ and the initial conditions. For vanishing $\Lambda$,

$$\langle \rho \rangle \to \Delta E \frac{\Delta E \rho(0) + \sqrt{n^2 - \rho^2(0)} \cos S(0)}{1 + (\Delta E)^2}. \quad (21)$$

Thus, for a given $\rho(0)$, the shift of $\langle \rho \rangle$ is determined by $\Delta E$ (although, since $\cos S(0)$ is negative, the initial shift direction is always opposite to the sign of $\Delta E$). This is analogous to an external magnetic field applied to a mixture of spin-up and spin-down atoms with populations performing Josephson-like oscillations. As can be seen from (21), the time-average $\langle \rho \rangle$ can be much larger or have the opposite sign than the initial $\rho(0)$ (in contrast to the regime of self-trapping, when $\langle \rho \rangle$ always has the same sign as $\rho(0)$ and never exceeds its value). Fig. 3(a) shows $\langle \rho \rangle$ against $\Delta E$ for $\rho(0) = 0.1n$, 0.5n, 0.99n.

Frequency of oscillations in $\rho(t)$ for $\Lambda \to 0$ is equal to $\Omega_R \sqrt{1 + (\Delta E)^2}$. Here, the interplay between the Rabi and the “a.c. Josephson” dynamics becomes apparent: for vanishing values of $\delta$ and $\Lambda$, when $|\Delta E| \ll 1$, we arrive at the Rabi regime discussed in the previous section. In the case of larger detunings which result in $|\Delta E|$ comparable to or larger than 1, the oscillations between the two condensates are entirely analogous to the internal a.c. Josephson effect. For the intermediate values of $|\Delta E|$, there are several regimes possible, which differ in the behavior of the relative phase $S(t)$. These regimes of oscillations are shown in Fig. 3.

For positive detunings and $\rho(0) > 0$ (or, equivalently, for negative detunings and $\rho(0) < 0$), as $|\Delta E|$ increases, harmonic Rabi oscillations in $\rho(t)$ shift from $\langle \rho \rangle = 0$ according to (21) and grow in amplitude. This is shown in Fig. 3(a)–(c). Oscillations of the relative phase $S(t)$ lose harmonicity and become of the shape of smoothed sawtooth with the amplitude growing up to $\pi/2$ when $\Delta E$ approaches its critical value $\Delta E^c_+$. This critical value $\Delta E^c_+$ is dependent on the initial values $\rho(0)$ and $S(0)$. More precisely, for $\Lambda \to 0$ we have

$$\Delta E^c_+ = \frac{\rho(0) - \sqrt{\rho^2(0) \sin^2 S(0) + n^2 \cos^2 S(0)}}{\sqrt{n^2 - \rho^2(0) \cos S(0)}}. \quad (22)$$

As follows from (22), for $\rho(0) \to 0$ the critical $\Delta E^c_+ \to 1$ (which means positive detuning $\to \hbar \Omega_R$), and for $\rho \to n$ it tends to zero. The dependence (22) is plotted in the inset of Fig. 3(b) for $S(0) = \pi$.

When $\Delta E$ exceeds $\Delta E^c_+$, the relative phase becomes monotonously decreasing (or increasing, depending on the sign of the detuning) in time while the population imbalance time-average starts to shift in the opposite direction (see (21) and Fig. 3(a)) and the amplitude...
of oscillations begins to decay (amplitude against $\Delta E$ is shown in Fig. 3(b)). This latter regime of running phase is shown in Fig. 4(a)–(c) and 4(e)–(f). The red solid lines mark the trajectories at the critical values $\Delta E^{\pm}_{c}$ (oscillations corresponding to these trajectories are shown in Fig. 4(c) and 4(f)). Black dashed lines correspond to the a.c. Josephson regime of oscillations (see Fig. 4(d) and 4(h)), when the values of $\Delta E$ exceed the critical values $\Delta E^{\pm}_{c}$.

**IV. CONCLUSION**

We have investigated the dynamical regimes of internal oscillations in a tuned two-component Bose condensate with interaction only in one of the components. Introducing the set of coupled temporal Gross-Pitaevskii and Schrödinger equations, we described the evolution of population imbalance and relative quantum phase between the subsystems for different values of energy detunings, being guided by the physical system of exciton-polaritons in an optical microcavity. For the symmetric case of zero effective detunings, we described the behavior of $n$ and $S(t)$ (in units of $\pi$) as functions of time, with the initial conditions $\rho(0) = 0.1n$, $S(0) = \pi$.

Red/dark dotted lines represent the normalized time-averaged $\langle \rho \rangle/n$ given by (21). Values of the effective detuning $\Delta E$: (a) $\Delta E = 0.1\Delta E^{+}_{c}$, (b) and (f) $\Delta E = 0.55\Delta E^{+}_{c}$, (c) and (g) $\Delta E = \Delta E^{+}_{c}$, (d) and (h) $\Delta E = 1.006\Delta E^{+}_{c}$. Other physical values same as in Fig. 1.

Upon reaching the critical value

$$\Delta E^{+}_{c} = \frac{\rho(0) + \sqrt{(\rho^{2}(0) \sin^{2}S(0) + n^{2} \cos^{2}S(0)})}{\sqrt{n^{2} - \rho^{2}(0) \cos S(0)}},$$

the behavior is once more changed to the a.c. Josephson regime characterized by a running phase. Another difference with the previous case is that when $\rho(0) \to n$, the critical value $\Delta E^{+}_{c}$ goes to infinity (see the inset of Fig. 3(b)), which means for large $\rho(0)$, in the case $\rho(0) = -\pi(\Delta E)$, the crossover to the Josephson regime doesn’t occur, and $\langle \rho \rangle$ shifts in the direction of the initial value $\rho(0)$.

Phase-plane portrait $S(\rho)$ for the case $\Delta E \neq 0$ is shown in Fig. 2(b). All trajectories are calculated for different values of $\Delta E$ (as marked) with $\rho(0)$ kept constant at 0.1$n$. Black solid lines correspond to the oscillations shown in Fig. 2(a)–(b) and 2(e)–(f). The red solid lines mark the trajectories at the critical values $\Delta E^{\pm}_{c}$. Other physical values same as in Fig. 1.

For the case when $\Delta E$ and $\rho(0)$ have opposite signs, the dynamics possesses certain differences. A peculiar feature is that when $|\Delta E|$ starts increasing, amplitude of oscillations in $S(t)$ and $\rho(t)$ decreases while the time-average population imbalance shifts from $\langle \rho \rangle = 0$ in the direction opposite to the sign of $\Delta E$ in agreement with (21). If $S(0) = \pi$, the amplitudes drop down to zero at $\Delta E = \Delta E^{+} = (\rho(0) \cos S(0) + n \sin S(0))/\sqrt{n^{2} - \rho^{2}(0)}$. The oscillations appear fully suppressed, and the values of population imbalance and the relative phase are fixed at $\rho = \rho(0)$ and $S = S(0) = \pi$. In case $S(0) \neq \pi$, the amplitudes reach local minima with values dependent on $S(0)$ and $\rho(0)$ (see Fig. 3(b)). When $\Delta E > \Delta E^{+}$, the amplitudes start to grow and the relative phase acquires the sawtooth-like temporal profile, analogous to the case discussed above.
V. ACKNOWLEDGEMENTS

Fruitful discussions with I.Carussotto are acknowledged. N.S.V is grateful for the conversations with the participants of the ESF POLATOM Network Conference in Cambridge, 2012. This work is partly supported by Russian Foundation for Basic Research (RFBR). Y.E.L. is supported by Program of Basic Research of HSE. The work of N.S.V. is partly supported by NRNU MEPhI “Young teacher” grants.

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