Recent Progress in Mathematical Analysis of Vortex Sheets

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Abstract

We consider the motion of the interface separating two domains of the same fluid that moves with different velocity along the tangential direction of the interface. We assume that the fluids occupying the two domains are of constant densities that are equal, are inviscid, incompressible and irrotational, and that the surface tension is zero. We discuss results on the existence and uniqueness of solutions for given data, the regularity of solutions, singularity formation and the nature of solutions after the singularity formation time.

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1. Introduction

Vortex dynamics is of fundamental importance for a wide variety of concrete physical problems, such as lift of airfoils, mixing of fluids, separation of boundary layers, and generation of sounds. In mathematical analysis, one often neglects surface tension and viscosity, when they are small in the real physical problem. This necessitates justifying such simplifications.

In this paper, we consider the motion of the interface separating two domains of the same fluid in $R^2$ that moves with different velocity along the tangential direction of the interface. We assume that the fluids occupying the two domains separated by the interface are of constant densities that are equal, are inviscid, incompressible and irrotational. We also assume that the surface tension is zero, and there is no external forces. The interface in the aforementioned fluid motion is a so called vortex sheet. We want to study the following problem:

Given a vortex sheet initial data, is there a unique solution to this problem?

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In general, there are two approaches to the aforementioned problem. One is to solve the initial value problem of the incompressible Euler equation in $\mathbb{R}^2$:

$$\begin{align*}
\begin{cases}
  v_t + v \cdot \nabla v + \nabla p &= 0, \\
  \text{div } v &= 0, \\
  v(x, y, 0) &= v_0(x, y) \quad (x, y) \in \mathbb{R}^2, \quad t \geq 0
\end{cases}
\end{align*} \tag{1}$$

where the initial incompressible velocity $v_0 \in L^2_{\text{loc}}(\mathbb{R}^2)$, in which the vorticity $\omega_0 = \text{curl } v_0$ is a finite Radon measure. Here $v$ is the fluid velocity, $p$ is the pressure, and the density of the fluid is assumed to be one. Notice that a vortex sheet gives a measure valued vorticity supported on the interface. This approach was posed by DiPerna and Majda in 1987 [9]. In 1991, J.M. Delort [8] proved the existence of weak solutions global in time of the 2-D incompressible Euler equation (1) for measure-valued initial vorticity in $H^{-1/2}_{\text{loc}}(\mathbb{R}^2)$ that has a distinguished sign. However the problem of uniqueness of the weak solution is still unresolved. In 1963, Yudovich [33] obtained the existence and uniqueness of weak solutions of the 2-D incompressible Euler equation (1) for initially bounded vorticity. The best results on uniqueness upto date are given by Yudovich [34] and Vishik [31] for weak solutions with vorticity in a class slightly larger than $L^\infty$. This does not include vortex sheets, which admit measure-valued vorticity. Examples of weak solutions with the velocity field $v \in L^2(\mathbb{R}^2 \times (-T, T))$ that is compactly supported in spacet ime was constructed by V. Scheffer [29] and later by A. Shnirelman [30]. This gives non-uniqueness of weak solutions in $L^2(\mathbb{R}^2 \times (-T, T))$. However non-uniqueness in the physically relevant class of conserved energy $v \in L^\infty([0, \infty), L^2_{\text{loc}}(\mathbb{R}^2))$ remains open. Numerical evidences of non-uniqueness of weak solutions for vortex sheet data can be found in [25], [18].

Furthermore, weak solutions give little information of the specific nature of the vortex sheet evolution. For instance, does the vorticity remain supported on a curve for a later time given that the initial vorticity is supported on a curve in $\mathbb{R}^2$? Assume further that the free interface between the two fluid domains remains a curve in $\mathbb{R}^2$ at a later time, equation (1) can be reduced to an evolutionary differential-integral equation along the interface. This is the Birkhoff-Rott equation, written explicitly by Birkhoff in [2] and implied in the work of Rott [26]. The second approach uses the Birkhoff-Rott equation as a model for the evolution of the vortex sheet.

2. The Birkhoff-Rott equation

For convenience, we use complex variable $z = x + iy$ to denote a point in $\mathbb{R}^2$. $\overline{z} = x - iy$ denotes the complex conjugate and $f_x = \partial_x f$ is the partial derivative of the function $f$. $H^s$ indicates Sobolev spaces.

In search of the equation for the evolution of the vortex sheet, we suppose that at time $t \geq 0$ the vorticity is a measure supported on the curve $\Gamma(t)$ given by the complex position $\xi = \xi(s, t)$ in the arclength $s$, in which $\xi(0, t)$ is the particle path of a reference particle; and on this curve the vorticity density is $\gamma = \gamma(s, t)$. That
is the vorticity at time \( t \) is \( \omega(x, y, t) \) satisfying

\[
\int \int \phi(x, y) \omega(x, y, t) \, dx \, dy = \int \phi(\xi(s, t)) \gamma(s, t) \, ds,
\]
for any \( \phi \in C_0^\infty(R^2) \).

From the Biot-Savart law, the velocity field \( v \) induced by the vorticity is given by

\[
v(z, t) = \frac{1}{2\pi i} \int \frac{\gamma(s', t)}{z - \xi(s', t)} \, ds',
\]
for \( z \notin \Gamma(t) \).

Notice that the velocity is discontinuous just on \( \Gamma(t) \). We define the velocity on the sheet as the average of the velocities at the two sides of the sheet, that is given by the principle value integral:

\[
\overline{v}(\xi(s, t), t) = \frac{1}{2\pi i} \text{p.v.} \int \frac{\gamma(s', t)}{\xi(s, t) - \xi(s', t)} \, ds'.
\]  \hspace{1cm} (2)

As suggested by the properties of the Euler equation, we assume that the vortex sheet is convected by the average velocity (2), and the vorticity is conserved along the particle path. We arrive at the evolution equation of the vortex sheet:

\[
\begin{align*}
\xi_t(s, t) + a(s, t) \xi_s(s, t) &= v(\xi(s, t), t) \\
\gamma_t(s, t) + \partial_s(a(s, t) \gamma(s, t)) &= 0
\end{align*}
\]  \hspace{1cm} (3)

where \( a(s, t) \) is a real valued function satisfying \( a(0, t) = 0 \).

A rigorous justification of the equivalence between equation (3) and equation (1) for smooth graphs \( \xi(s, t) \) and smooth vortex strength \( \gamma(s, t) \) can be found in [19]. It is not hard to extend it to all smooth curves.

Assume \( \alpha(s, t) = \int_0^s \gamma(s', t) \, ds' \) defines an increasing function of \( s \), we make a change of variables: \( z(\alpha, t) = \xi(s(\alpha, t), t) \), in which \( s(\alpha, t) \) is the inverse of \( \alpha(s, t) \): \( \alpha(s(\alpha, t), t) = \alpha \). We get from equation (3) the Birkhoff-Rott equation

\[
\partial_t \varphi(\alpha, t) = \frac{1}{2\pi i} \text{p.v.} \int \frac{1}{z(\alpha, t) - z(\beta, t)} \, d\beta.
\]  \hspace{1cm} (4)

Notice that \( z = z(\alpha, t) \) is a parameterization of the vortex sheet in the circulation variable \( \alpha \), \( 1/|z_\alpha| = \gamma \) is the vortex strength. A steady solution of (4) is the flat sheet \( z = \alpha \).

Equation (4) has been under active investigations over the last four decades. A well-known property of (4) is that perturbations of the flat sheet grow due to the Kelvin-Helmholtz instability, following from a linearization of equation (4) about the flat sheet. For given analytic data, Sulem, Sulem, Bardos and Frisch [28] established the short time existence and uniqueness of solutions in analytic class for 2-D and 3-D vortex sheet evolution. Duchon and Robert [10] obtained the global existence of solutions of equation (4) for a special class of initial data that is close to the flat sheet. However, numerous results show that a vortex sheet can develop a curvature...
singularity in finite time from analytic data. D.W. Moore [21] was the first to provide analytical evidence that predicts the occurrence and time of singularity formation, which was verified numerically by Meiron, Baker and Orszag [20] and by Krasny [13]. Caflisch and Orellana [3] proved existence almost up to the time of expected singularity formation for analytic data that is close to the flat sheet. Duchon and Robert [10] and Caflisch and Orellana [4] constructed specific examples of solutions of equation (4) where a curvature singularity develops in finite time from analytic data. The example of Caflisch and Orellana [4] has the form

\[ z(\alpha, t) = z(\alpha) + S(\alpha, t) + r(\alpha, t), \]

where

\[ S(\alpha, t) = \epsilon(1 - i)\{(1 - e^{-t/2+ia})^{1+\mu} - (1 - e^{-t/2-ia})^{1+\mu}\} \]

is a solution of the linearized equation in which \( \epsilon \) is small, \( \mu > 0 \); \( r(\alpha, t) \) is the correction term that is negligible relative to \( S(\alpha, t) \) in the sense that \( S(\alpha, t) + r(\alpha, t) \) exhibits the same kind of behavior as \( S(\alpha, t) \) [4]. Notice that \( S(\alpha, t) \) is an analytic function for \( t > 0 \), but \( S(\alpha, 0) \) has an infinite second derivative at \( \alpha = 0 \) for \( \mu \in (0, 1) \). In fact, the \( (1 + \nu) \)th derivative of \( S(\alpha, 0) \) for \( \nu > \mu \) becomes infinite at \( \alpha = 0 \). Now inverting time gives an example \( \hat{z}(\alpha, t) \) that is analytic at \( t_0 < 0 \), but has an infinite second derivative at \( \alpha = 0, t = 0 \). At the singularity formation time \( t = 0 \), the vortex strength \( \frac{1}{|\hat{z}_\alpha|} \) of this example satisfies

\[ 0 < c \leq \frac{1}{|\hat{z}_\alpha|} \leq C < \infty \]

for some constants \( c \) and \( C \); and \( \hat{z}(\alpha, t) \in C^{1+\rho}(R \times [t_0, 0]) \) for \( 0 < \rho < \mu \).

These examples also show that the initial value problem of the Birkhoff-Rott equation (4) is ill-posed in \( C^{1+\nu}(R) \), \( \nu > 0 \), and in Sobolev spaces \( H^s(R) \), \( s > 3/2 \) in the Hadamard sense [4], [10]. Ill-posedness was also proved by Ebin [11] using a different approach. However the existence of solutions in spaces less regular than \( C^{1+\nu}(R) \) or \( H^s(R) \), and the nature of the vortex sheet at and beyond the singularity time remained unknown analytically in general.

This suggests that we look for solutions of the Birkhoff-Rott equation in the largest possible spaces where the equation makes sense. For the purpose of this paper, we consider functions \( z(\alpha, t) \) so that for each fixed time \( t \), both sides of the equation (4) are functions locally in \( L^2 \), and on which the \( L^2 \)-analysis is available. This leads us to consider chord-arc curves, thanks to the work of G. David [7].

Another reason that chord-arc curves are to be considered is due to the numerical calculation of Krasny [14] [15]. Krasny studied the evolution of the vortex sheet beyond singularity using the vortex blob method. He found that the approximating solutions have the form of a spiral beyond singularity. Convergence of the approximating sequence to a weak solution of the Euler equation (1) was proved by J-G. Liu and Z-P. Xin [17], under the assumption that the initial vorticity has a distinguished sign. A special example of chord-arc curves is a logarithmic spiral.
3. Chord-arc curves and some recent results

Let \( \Gamma \) be a rectifiable Jordan curve in \( \mathbb{R}^2 \) given by \( \xi = \xi(s) \) in the arclength \( s \). We say \( \Gamma \) is a chord-arc curve, if there is a constant \( M \geq 1 \), such that

\[
|s_1 - s_2| \leq M|\xi(s_1) - \xi(s_2)|, \quad \text{for all } s_1, s_2.
\]

The infimum of all such constants \( M \) is called the chord-arc constant.

For a chord-arc curve \( \xi = \xi(s) \), \( s \) the arclength, it is proved in [6] that \( \xi'(s) \) exists almost everywhere, and there is a choice of the argument function \( b \in BMO \), such that \( \xi'(s) = e^{ib(s)} \). In particular, if the chord-arc constant is close to 1, there is a choice of \( b \in BMO \), such that \( \|b\|_{BMO} \) is close to 0. Moreover the subset of all those functions \( b \) is an open subset of \( BMO \).

Examples of chord-arc curves include Lipschitz curves and logarithmic spirals \( r = \pm e^{\theta} \), \( \theta \in \mathbb{R} \), where \((r, \theta)\) is the polar coordinates.

A Theorem of G. David [7] states that

**Theorem (G. David [7]).** For all chord-arc curves \( \Gamma : \xi = \xi(s) \), \( s \) the arclength, the corresponding Cauchy integral operator \( \Gamma f(s) = p.v. \int \frac{f(s')}{\xi(s) - \xi(s')} d\xi(s') \),

is bounded from \( L^2(ds) \) to \( L^2(ds) \).

In fact, the result of G. David [7] is stronger than stated above. He proved that the Cauchy integral operator \( \Gamma \) is bounded from \( L^2(ds) \) to \( L^2(ds) \) if and only if \( \Gamma \) is a regular curve. A rectifiable curve \( \Gamma \) is said to be regular if there is a constant \( M \) such that for every \( r > 0 \) and every disc \( D \) with radius \( r \), the length of \( \Gamma \cap D \) does not exceed \( Mr \). A chord-arc curve is regular but not vice versa.

Now we go back to the Birkhoff-Rott equation. Notice that the Biot-Savart integral representing an incompressible velocity field \( v \) in terms of the vorticity \( \omega \) may be divergent if \( \omega \) does not vanish fast enough at infinity, even if the velocity field \( v \) is well defined, we extend the definition of the Birkhoff-Rott equation (4) by considering the differences of the velocities between any two points:

\[
\begin{align*}
\frac{z_t(\alpha,t) - z_t(\alpha',t)}{z(\alpha,t) - z(\alpha',t)} &= \frac{1}{2\pi i} p.v. \int_{|\beta| \leq N} \left\{ \frac{1}{z(\alpha,t) - z(\beta,t)} - \frac{1}{z(\alpha',t) - z(\beta,t)} \right\} d\beta \\
&\phantom{=} + \frac{1}{2\pi i} p.v. \int_{|\beta| > N} \frac{z(\alpha',t) - z(\alpha,t)}{(z(\alpha,t) - z(\beta,t))(z(\alpha',t) - z(\beta,t))} d\beta,
\end{align*}
\]

for all \((\alpha,t), (\alpha',t)\), and some \( N > |\alpha| + |\alpha'| + 1 \). This admits a larger class of solutions. In particular, the integral on the right hand side of equation (6) is convergent for those similarity solutions considered in [12], [23]-[25], which otherwise
give divergent Cauchy integrals in (4) due to the divergent contributions from the vorticities at infinity. It follows from the Theorem of G. David that the integral on the right hand side of the equation (6) is convergent for a.e. \((\alpha, t), (\alpha', t)\) and is in \(L^\infty([0, T], L^2_{\text{loc}}(\alpha) \times L^2_{\text{loc}}(\alpha'))\) for the solutions considered in Theorem 1 in the following.

Roughly speaking, if a function \(z = z(\alpha, t)\) satisfies equation (4), it will also satisfy equation (6). On the other hand, if \(z = z(\alpha, t)\) satisfies (6), and the Cauchy integral

\[
\frac{1}{2\pi i} \text{p.v.} \int \frac{1}{z(\alpha, t) - z(\beta, t)} \, d\beta
\]

is convergent, then there is a function \(c = c(t)\), such that \(z(\alpha, t) + c(t)\) satisfies equation (4).

For a local integrable function \(f = f(\alpha)\) defined on \((a, b)\), we say \(f\) is of bounded local mean oscillation on \((a, b)\) if there exists \(\delta_0 > 0\) such that

\[
\|f\|_{\text{BMO}(a, b), \delta_0} = \sup_{I \subset (a, b), |I| \leq \delta_0} \frac{1}{|I|} \int_I |f(\alpha) - f_I| \, d\alpha < \infty,
\]

here \(f_I = \frac{1}{|I|} \int_I f(\alpha) \, d\alpha\), \(I\) is an interval. We say \(f\) is analytic on \((a, b)\) if \(f \in C^{\infty}(a, b)\), and for any compact subset \(K\) of \((a, b)\), there is a constant \(\rho > 0\), such that

\[
\sum_{m=0}^{\infty} \frac{\rho^m}{m!} \int_K |\partial^m_{\alpha} f(\alpha)|^2 \, d\alpha < \infty.
\]

Notice that \(\ln z\) is multi-valued for complex number \(z\). In the following, \(\ln z\), refers to one choice of the multi-values. We have the following results concerning the solutions of the Birkhoff-Rott equation (6) or (4).

**Theorem 1** [32]. Assume that \(z \in H^1([0, T], L^2_{\text{loc}}(R)) \cap L^2([0, T], H^1_{\text{loc}}(R))\) is a solution of the Birkhoff-Rott equation (6) for \(0 \leq t \leq T\), satisfying that

1. There are constants \(m > 0, M > 0\), independent of \(t\), such that

\[
m|\alpha - \beta| \leq |z(\alpha, t) - z(\beta, t)| \leq M|\alpha - \beta| \quad \text{for all } \alpha, \beta, 0 \leq t \leq T. \tag{7}
\]

2. For the interval \((a, b)\), there exists \(\delta_0 > 0\), independent of \(t\), such that for all \(0 \leq t \leq T\),

\[
\|\ln z_{\alpha}(\cdot, t)\|_{\text{BMO}(a, b), \delta_0} \leq C(m, M), \tag{8}
\]

here \(C(m, M)\) is a universal constant depending on \(m\) and \(M\). Assume further that \(\ln z_{\alpha} \in L^2([0, T], L^2_{\text{loc}}(R))\). Then \(z_{\alpha} \in C((a, b) \times (0, T))\), and for each \(t_0 \in (0, T)\), \(z_{\alpha}(\cdot, t_0)\) is analytic on \((a, b)\).

**Remark** Notice that assumption 2. is satisfied if \(\ln z_{\alpha} \in C([a, b] \times [0, T])\). Assumption 1. is equivalent to assuming that the vortex strength \(\gamma = 1/|z_{\alpha}|\) is bounded away from 0 and \(\infty\), with bounds independent of \(t\), and \(z = z(\cdot, t)\) defines a chord-arc curve for each fixed \(t \in [0, T]\), with the chord-arc constant independent of \(t\).
Assumption 1. can be relaxed by requiring that $1/|z_\alpha(\alpha, t)|$ is bounded away from 0 and $\infty$ for $\alpha \in (a, b)$ only, and by assuming some weaker conditions for $\alpha \not\in (a, b)$.

A version of Theorem 1 under assumptions that the solution of the Birkhoff-Rott equation (4): $z = z(\alpha, t) \in C^{1+\rho_0}$, for some $\rho_0 > 0$, the vortex strength $c_0 < \gamma(\alpha, t) < C_0$ for some constants $c_0 > 0, C_0 > 0$, and the vortex sheets $\Gamma(t)$ are closed Jordan curves is also obtained by Lebeau [16] using an independent approach.

As a consequence of Theorem 1, the example constructed by Caflisch and Orellana [4] will fail to satisfy properties 1 and 2 as stated in Theorem 1 near the singularity after the singularity formation time.

Regarding the existence of solutions, we have the following

**Theorem 2** [32]. For any real valued function $w_0 \in H^{\frac{2}{3}}(R)$, there exists $T = T(\|w_0\|_{H^{\frac{2}{3}}}) > 0$, such that the Birkhoff-Rott equation (6) has a solution $z = z(\alpha, t)$ for $0 \leq t \leq T$, satisfying $\ln z_\alpha \in L^\infty([0, T], H^{\frac{2}{3}}(R)) \cap \text{Lip}([0, T], H^{\frac{2}{3}}(R))$ and $\text{Im}((1+i) \ln z_\alpha(\alpha, 0)) = w_0(\alpha)$, with the properties that there exist constants $m > 0, M > 0$ independent of $t$, such that

$$m|\alpha - \beta| \leq |z(\alpha, t) - z(\beta, t)| \leq M|\alpha - \beta|, \quad \text{for all } \alpha, \beta, 0 \leq t \leq T;$$

and there exists $\delta_0 > 0$, independent of $t$, such that

$$\|\ln z_\alpha(\cdot, t)\|_{\text{BMO}(R), \delta_0} \leq C(m, M), \quad \text{for } 0 \leq t \leq T,$$

here $C(m, M)$ is the universal constant as in Theorem 1.

Theorem 2 states that if only half of the data $z_\alpha(\alpha, 0)$ is given, there is a solution of the Birkhoff-Rott equation (6) for a finite time period. Theorem 2 is a generalization of the existence result of Duchon and Robert [10] to general data.

The following result implies that Theorem 2 is optimal, in the sense that in general, there is no solution of the Birkhoff-Rott equation satisfying properties 1. and 2. as stated in Theorem 1 beyond the initial time $t = 0$ for arbitrarily given data.

**Theorem 3** [32]. Assume that $z \in H^1([0, T], L^2_{\text{loc}}(R)) \cap L^2([0, T], H^1_{\text{loc}}(R))$ is a solution of the Birkhoff-Rott equation (6) for $0 \leq t \leq T, T > 0$, satisfying the property 1. and property 2. on some interval $(a, b)$ as stated in Theorem 1. Assume further that $\ln z_\alpha \in L^2([0, T], L^2_{\text{loc}}(R))$, and $w_0 = \text{Im}\{(1+i)\ln z_\alpha(\cdot, 0)\}$ is analytic on $(a, b)$. Then $z_\alpha \in C((a, b) \times [0, T])$ and $\text{Re}(1+i)\ln z_\alpha(\cdot, 0)$ is also analytic on $(a, b)$.

4. Open questions

The reason that Theorem 1-3 holds is that under the assumptions 1. and 2. in Theorem 1, the Birkhoff-Rott equation (6) is of “elliptic” type on $(a, b) \times [0, T]$. 
It would be interesting to see whether the assumption 2 that requires small local mean oscillation on \( \ln z_\alpha(\cdot, t) \) can be removed, or to see whether one can construct a non-smooth solution of equation (6) that violates (8). A good place to start is to construct similarity solutions of the Birkhoff-Rott equation. Similarity solutions are studied numerically in Pullen [23] [25], Pullen and Phillips [24], and analytically in Kambe [12]. However, the similarity solutions in [12], [23]-[25] violate (7). It would also be interesting to relax the assumption 1. by considering vortex strength that is not necessarily bounded away from 0 and infinity on the interval \((a, b)\).

Theorem 1-3 implies that a solution of the Birkhoff-Rott equation is either analytic, or doesn’t satisfy properties 1. and 2. in Theorem 1. And there are in general no solutions of reasonable regularity (properties 1. and 2.) beyond the initial time for an arbitrarily given data. Are there solutions of the Birkhoff-Rott equation in spaces of even less regularity? Notice that we can define the solutions of the Birkhoff-Rott equation in the distribution sense by

\[
\partial_t \left( \int z(\alpha, t) \eta(\alpha) \, d\alpha \right) = \frac{1}{4\pi} \iint \frac{\eta(\alpha) - \eta(\beta)}{z(\alpha, t) - z(\beta, t)} \, d\alpha \, d\beta,
\]

for all \( \eta \in C^\infty_0 \).

This admits some even larger classes of solutions \( z = z(\alpha, t) \). Is there a solution in the distribution sense for the Birkhoff-Rott equation for any given data? How is it related to the weak solution of the Euler equation?

Theorem 1-3 might as well suggest that the vortex sheet in general fails to be a curve beyond the initial time for general data. Therefore it becomes interesting to study the vortex layers or considering the effects of viscosity. Numerical analysis of the vortex layers can be found in Baker and Shelley [1] etc.

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