THE MORSE MINIMAL SYSTEM IS NEARLY CONTINUOUSLY KAKUTANI EQUIVALENT TO THE BINARY ODOMETER

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Abstract. Ergodic homeomorphisms $T$ and $S$ of Polish probability spaces $X$ and $Y$ are evenly Kakutani equivalent if there is an orbit equivalence $\phi : X_0 \to Y_0$ between full measure subsets of $X$ and $Y$ such that, for some $A \subset X_0$ of positive measure, $\phi$ restricts to a measurable isomorphism of the induced systems $T_A$ and $S_{\phi(A)}$. The study of even Kakutani equivalence dates back to the seventies, and it is well known that any two zero-entropy loosely Bernoulli systems are evenly Kakutani equivalent. But even Kakutani equivalence is a purely measurable relation, while systems such as the Morse minimal system are both measurable and topological.

Recently del Junco, Rudolph and Weiss studied a new relation called nearly continuous Kakutani equivalence. A nearly continuous Kakutani equivalence is an even Kakutani equivalence where also $X_0$ and $Y_0$ are invariant $G_\delta$ sets, $A$ is within measure zero of both open and closed, and $\phi$ is a homeomorphism from $X_0$ to $Y_0$. It is known that nearly continuous Kakutani equivalence is strictly stronger than even Kakutani equivalence, and nearly continuous Kakutani equivalence is the natural strengthening of even Kakutani equivalence to the nearly continuous category—the category where maps are continuous after sets of measure zero are removed. In this paper we show that the Morse minimal substitution system is nearly continuously Kakutani equivalent to the binary odometer.

1. Introduction

Even Kakutani equivalence is one of the most natural examples in the theory of restricted orbit equivalence of ergodic and finite measure preserving dynamical systems. In this paper we study even Kakutani equivalence in the nearly continuous category. A nearly continuous dynamical system is given by a triple $(X,\mu,T)$, where $X$ is a Polish space, $\mu$ is a Borel probability measure on $X$, and $T : X \to X$ is an ergodic measure preserving homeomorphism. Recall that a measurable orbit equivalence between two such systems $(X,\mu,T)$ and $(Y,\nu,S)$ is an invertible, bi-measurable, and measure preserving map $\phi : X \to Y$ that sends orbits to orbits. A measurable orbit equivalence $\phi : X \to Y$ is a nearly continuous orbit equivalence if there exist invariant and $G_\delta$ subsets $X_0 \subset X$ and $Y_0 \subset Y$ of full measure so that $\phi : X_0 \to Y_0$ is a homeomorphism.

The first result in this category is the celebrated theorem of Keane and Smorodinsky [9] that any two Bernoulli shifts of equal entropy are finitarily isomorphic, namely, that the isomorphism between them can be made a homeomorphism almost everywhere. In a later paper, Denker and Keane [3] established a general framework for studying measure

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preserving systems that also preserve a topological structure. We refer the reader to a paper by del Junco, Rudolph, and Weiss [1] for a more complete history of the area. We only mention here that interest in the orbit equivalence theory for this category was more recently revived by the work of Hamachi and Keane in [6] where they proved that the binary and ternary odometers are nearly continuously orbit equivalent. Their work inspired similar results for other pairs of examples (see [7], [8], [12], [13], [14], and [15]). These examples were later subsumed as special cases of a Dye’s Theorem in this category proved by del Junco and Şahin [2].

Around the same time as a nearly continuous Dye’s Theorem was established, del Junco, Rudolph, and Weiss proved in [1] that if one does not impose the condition that the invariant sets of full measure on which the orbit equivalence is a homeomorphism are $G_\delta$ sets, then any restricted orbit equivalence classification is exactly the same as in the measure theoretic case. In particular, they showed that any orbit equivalence can be regularized to be a homeomorphism on a set of full measure, but could not prove that the set of full measure had any topological structure.

The importance of the topological structure in the theory is even more striking for the study of even Kakutani equivalence. Recall that in the measurable category two ergodic and finite measure preserving systems $(X,\mu,T)$ and $(Y,\nu,S)$ are even Kakutani equivalent if there exists a measurable orbit equivalence $\phi : X \to Y$, and measurable sets $A \subset X, B \subset Y$ with $\mu A = \nu B > 0$ with the property that $\phi : A \to B$ is a measurable isomorphism of the induced transformations $T_A$ and $S_B$. We call the orbit equivalence $\phi$ an even Kakutani equivalence between $T$ and $S$. It follows from [1] that any even Kakutani equivalence can be made to be a homeomorphism on a set of full measure. In the same paper they show that if one imposes the additional condition that the sets $A$ and $B$ be nearly clopen, meaning within a set of measure zero of an open set and also of a closed set, then there is a new invariant for even Kakutani equivalence of nearly continuous dynamical systems called near unique ergodicity. They use this new invariant to show that nearly continuous even Kakutani equivalence is stronger than measure theoretic even Kakutani equivalence. The example they construct is, in some sense, not natural, and begs the question whether there are any natural examples of nearly continuous systems that are measurably evenly Kakutani equivalent but not nearly continuously so.

Rudolph began looking for examples in the family of zero entropy Loosely Bernoulli systems. Recall that any two zero entropy Loosely Bernoulli transformations are measurably even Kakutani equivalent. Furthermore, many natural examples of nearly continuous systems including rotations, all adding machines, and in fact all finite rank transformations, are Loosely Bernoulli. In [16], Roychowdhury and Rudolph proved that any two adding machines are nearly continuously even Kakutani equivalent. Shortly after, Dykstra and Rudolph showed in [5] that all irrational rotations are nearly continuously Kakutani equivalent to the binary odometer.

In [16], new machinery, called templates, was introduced to construct the nearly continuous Kakutani equivalence. There, templates were defined using the natural topological tower structure present in adding machines. The construction in [5] showed that the template machinery can be adapted to the case where the underlying system does not have
a canonical symbolic structure. More recently, Springer [18] expanded on their ideas and adapted templates further to prove that all minimal isometries of compact metric spaces are nearly continuously Kakutani equivalent to the binary odometer. Salvi [17] adapted templates to the setting of $\mathbb{R}$ actions and used the machinery to prove Rudolph’s Two-Step Coding Theorem in the nearly continuous category.

Each result mentioned above has required more sophisticated and technically intricate incarnations of templates. On the other hand each proof has also established the usefulness and flexibility of the machinery. In this paper we adapt the template machinery even further to show our main result:

**Theorem 1.1.** The Morse minimal system is nearly continuously even Kakutani equivalent to the binary odometer.

The version of the template machinery in this paper is designed to address the new complication of the additional tower present in the rank two Morse system. We believe the generalization we give here is the appropriate starting place to prove more generally that finite rank nearly continuous systems are all nearly continuously Kakutani equivalent to the binary odometer.

Finally we note that this manuscript is a culmination of work that the first author began in 2009 while he was a post-doctoral fellow working with Daniel Rudolph at Colorado State University. The initial architecture of the constructions and the main ideas were all established collaboratively by Dykstra and Rudolph. The second author joined the project after the untimely death of Rudolph in 2010, and the manuscript was completed in 2014.

2. Template Machinery

In this section, deferring some formal definitions until later, we give an overview of the construction and introduce templates. Let $(X, T, \mu)$ denote the Morse minimal system and $(Y, S, \nu)$ the binary odometer. Recall that each system has a canonical refining, generating sequence of clopen partitions that are given by the finite rank structure of each system. The Morse system is rank two, so at each stage the partition is defined by a pair of towers. The odometer is rank one, so the sequence of partitions is defined by a sequence of single towers. The construction of the orbit equivalence uses an inductive “back and forth” procedure. Intuitively, at each stage we need to construct a set map from the levels of the tower of one system to the levels of the tower of the other system, switching the domain and range of the set maps at each stage. The orbit equivalence will be defined on the set of points for which our procedure will converge. In order for this set to be a $G_\delta$ set, and the map to be a homeomorphism, for each point where we have convergence we need to have the procedure stabilize after a finite number of steps. In other words, once we have defined the set map at a particular stage $n$, we cannot modify its domain at any successive stage.

This introduces an obvious complication in the construction. At a particular stage $n$, we have to know that our choice of the set map on stage $n$ towers will be consistent with the choices that we will make for all stages after. To address this complication, informally speaking, we do not actually choose a particular set map at any stage. Instead, at each
stage we construct a collection of set maps that are possible extensions of previous stage set maps, and that all agree on a set we call the good set. The convergence then depends on us being able to provide enough choices at each stage $n$ so that it is possible to construct sufficiently many choices of maps at stage $n+1$ that extend the $n$-stage maps.

Templates are a combinatorial tool that have been designed to facilitate the intensive book keeping required to describe such a procedure. Formally, template is an ordered multiset. For example, the multiset $\{a, a, b, b, c\}$, together with the ordering $a < a < b < c < b$, gives a template $\tau$, which we write as

$$\tau : a < a < b < c < b.$$  

The elements of a template (with multiplicity) are called levels. In our work, each level of a template will correspond to a clopen set that is a level of a tower. In particular, the towers themselves can be thought of as templates where each level appears exactly once and the ordering on the levels is exactly the ordering on the sets that is imposed by the underlying dynamics.

Notice that a set map from one tower to another can be thought of as a re-ordering of the levels of the domain tower according to the levels that they are being mapped to in the image tower. We replace the notion of a set map with maps between templates, where the re-ordering is not described by the map, but rather the ordering given by the image template.

More formally, given templates $\tau$ and $\tau'$, written

$$\tau : c_0 \prec c_1 \prec \cdots \prec c_{n-1} \quad \text{and} \quad \tau' : d_0 \prec d_1 \prec \cdots \prec d_{m-1},$$

we represent $\tau$ and $\tau'$ with the intervals

$$I = [0,1,\ldots,n-1] \subset \mathbb{Z} \quad \text{and} \quad J = [0,1,\ldots,m-1] \subset \mathbb{Z}$$

via the correspondences $c_i \leftrightarrow i$ and $d_i \leftrightarrow i$. A partial interval bijection is an ordered quintuple $\hat{f} = [I,J,A,B,f]$, where $A \subset I$, $B \subset J$, and $f : A \to B$ is some bijection.

This perspective allows for an explicit combinatorial understanding of how maps need to be defined at each stage in order to construct maps at later stages, on larger domains. It also allows for an explicit combinatorial description of how maps from one stage are constructed from maps of a previous stage, so it is easy to prove that the construction has indeed stabilized for points on a $G_\delta$ set of full measure.

2.1. The Induction. In our proof we will construct an increasing sequence $(k_n)$, templates for each system and partial interval bijections between template sets that constitute the “back and forth” diagram given in Figure 1 below.

The objects in the diagram are template sets. The template sets on the left $(\mathcal{P}_{k_n}, \tilde{\mathcal{P}}_{k_n}, \mathcal{T}_{k_n}, \text{and } \tilde{T}_{k_n})$ belong to the Morse minimal system, while those on the right $(\mathcal{Q}_{k_n}, \tilde{\mathcal{Q}}_{k_n}, \mathcal{\Omega}_{k_n}, \text{and } \tilde{\Omega}_{k_n})$ belong to the binary odometer. The maps $\phi_*$ are partial interval bijections. A key ingredient of the diagram is its almost commutative nature, as introduced by Roychowdhury and Rudolph in [16]. Interpreting the levels of templates as levels of towers that form a refining sequence of partitions, we see that every level at stage $n$ of a tower is a subset of a level from a previous stage template, and the maps $\zeta$ and $\pi$ are the natural inclusion
maps. The consistency of set maps from one level to another is achieved by requiring that on the good set, all partial interval bijections agree when composed with $\zeta$ and $\pi$.

There are two key differences our work here and that of [16] or [5] in how we use templates from a particular stage to construct later stage templates. In particular, we cannot use the notion of concatenation as was defined in the earlier papers, instead we define overlapping concatenations. In addition, to accommodate the combinatoric structure of the towers of the Morse minimal system we introduce a new family of partial interval bijections called reordering maps. Once the diagram is built up, however, the argument that it produces a well-defined nearly continuous Kakutani equivalence is nearly identical to the arguments in both [16] and [5]. We include it here for completeness (see Sections 16 and 17).

It is our hope that one day we might discover a more general machine that could characterize broad classes of systems, perhaps even all zero entropy loosely Bernoulli systems. But at the moment it is not clear how such a machine, if one exists, could be sufficiently general to account for the differences between systems.

2.2. The Organization of the Paper. The paper is organized as follows:

Section 3. We define the tools for constructing templates and partial interval bijections that we will use throughout the construction.
Sections 4 - 5: We give preliminary definitions of the Morse minimal system and the binary odometer. In particular, we define the template sets $P_k$, $\tilde{P}_k$, $Q_k$, and $\tilde{Q}_k$, for $k \geq 0$. These template sets are given by the towers in the respective system.

Section 6: We define the template sets $\Omega_k$ and $\tilde{\Omega}_k$, for $k \geq 0$. These are templates in the odometer system that are rearrangements of the tower templates, reflecting set maps that map Morse towers to odometer towers.

Section 7: We construct stage $n = 2$ of the diagram by explicitly defining $k_2$, along with the partial interval bijections $\phi_\omega : P_{k_2} \cup \tilde{P}_{k_2} \to \Omega_{k_2} \cup \tilde{\Omega}_{k_2}$.

Section 8: We define the template sets $T_k$ and $\tilde{T}_k$, for $k \geq 0$. These are new templates in the Morse system, describing how odometer towers will be mapped to Morse towers.

Section 9: We define the sequence $\left( k_n \right)$ recursively.

Sections 10 - 15: We assume the diagram has been built down to stage $n$, where $n$ is even, and show how to build it down to stage $n + 2$. Because the construction depends on whether $n \equiv 2$ or $n \equiv 0 \mod 4$, we proceed as follows:

- In Sections 10 - 11.3, we introduce notation and machinery that is used for every even $n$.
- In Sections 12 - 13.3.3, we illustrate the $n \equiv 2 \mod 4$ case by building the diagram down to stage 4.
- In Sections 14.1 - 14.3, we illustrate the $n \equiv 0 \mod 4$ case by building the diagram down to stage 6.
- In Section 15, we indicate how the induction looks in stages $n \geq 8$.

Sections 16 - 17: We use the properties of the diagram to prove that our procedure produces a nearly continuous Kakutani equivalence between the two systems.

### 3. Partial Interval Bijections and Concatenations

In this section we introduce the terminology and tools necessary to build and extend partial interval bijections.

**Definition 3.1.** Given templates $\tau$ and $\tau'$, and a partial interval bijection $\hat{f} = [I, J, A, B, f]$ from $\tau$ to $\tau'$, the **domain** of $\hat{f}$ consists of those levels in $\tau$ that are represented by $A$. The **range** of $\hat{f}$ consists of those levels in $\tau'$ that are represented by $B$.

For the next definition, suppose $\tau'' : e_0 \prec e_1 \prec \cdots \prec e_m$ is another template which, like $\tau'$, is represented by $J \subset \mathbb{Z}$. Suppose there is a partial interval bijection $\hat{g} = [I, J, A, C, g]$ from $\tau$ to $\tau''$. Note that $\hat{f}$ and $\hat{g}$ agree in their first three components ($I, J$, and $A$).

**Definition 3.2.** Suppose $\hat{f} : \tau \to \tau'$ and $\hat{g} : \tau \to \tau''$ are two partial interval bijections, given by $\hat{f} = [I, J, A, B, f]$ and $\hat{g} = [I, J, A, C, g]$. Then $\hat{f}$ and $\hat{g}$ **match** if, for each integer $i \in A$, the level in $\tau'$ that is represented by $f(i)$ is identical to the level in $\tau''$ that is represented by $g(i)$.
Definition 3.3. Two partial interval bijections are *equivalent* if one is a translate of the other. More precisely, \([I, J, A, B, f] \sim [I', J', A', B', f']\) if there exist \(t, s \in \mathbb{Z}\) and \(I' = I + t, J' = J + s, A' = A + t, B' = B + s\), and \(f'(i + t) = f(t + s)\).

Definition 3.4. Given a partial interval bijection \(\hat{f} = [I, J, A, B, f]\), the *inverse* of \(\hat{f}\) is the partial interval bijection \(\hat{f}^{-1} = [J, I, B, A, f^{-1}]\).

3.1. Simple Concatenations. Suppose \(\hat{f}_i = [I_i, J_i, A_i, B_i, f_i]\) for \(i = 1, 2\) are two partial interval bijections, and assume each \(I_i\) and \(J_i\) begin at 0. Let \(t = \#I_1\) and \(s = \#J_1\). Define the *simple concatenation* of \(\hat{f}_1\) and \(\hat{f}_2\) by

\[
\hat{f}_1 \ast \hat{f}_2 = [I_1 \cup (I_2 + t), J_1 \cup (J_2 + s), A_1 \cup (A_2 + t), B_1 \cup (B_2 + s), f],
\]

where

\[
f(i) = \begin{cases} 
  f_1(i) & \text{if } i \in A_1; \\
  f_2(i - t) + s & \text{if } i \in A_2 + t
\end{cases}
\]

Note that this is an associative semigroup action on the space of all partial interval bijections. Also note that, if \(\hat{f}_1\) and \(\hat{f}_2\) are partial interval bijections, then so is \(\hat{f}_1 \ast \hat{f}_2\).

3.2. Sticky Notes. For our construction, some partial interval bijections will need to be decomposed into a form

\[
\hat{f} = \hat{f}(1) \ast \hat{f}(2) \ast \hat{f}(3),
\]

where \(\hat{f}(1)\) and \(\hat{f}(3)\) make up a very small portion of the overall map. We will refer to \(\hat{f}(1)\) and \(\hat{f}(3)\) as the *bottom* and *top sticky notes* of \(\hat{f}\). The *body* is \(\hat{f}(2)\).

3.3. Overlapping Concatenations. We will often want to “glue” partial interval bijections together. Top and bottom sticky notes will be our means of doing this via the following definition.

Definition 3.6. Suppose \(\hat{f}_i = [I_i, J_i, A_i, B_i, f_i]\) for \(i = 1, 2\) are two partial interval bijections that are decomposed via \([3.5]\) as

\[
\hat{f}_i = \hat{f}_i(1) \ast \hat{f}_i(2) \ast \hat{f}_i(3).
\]

If \(\hat{f}_1(3) \sim \hat{f}_2(1)\), then define the *overlapping concatenation* of \(\hat{f}_1\) and \(\hat{f}_2\), denoted \(\hat{f}_1 \ast \hat{f}_2\), by

\[
\hat{f}_1 \ast \hat{f}_2 = \hat{f}_1(1) \ast \hat{f}_1(2) \ast \hat{f}_1(3) \ast \hat{f}_2(2) \ast \hat{f}_2(3) = \hat{f}_1(1) \ast \hat{f}_1(2) \ast \hat{f}_2(2) \ast \hat{f}_2(3)
\]

3.4. Generalized Sticky Notes. The sticky notes described in Section \[3.2\] will be used only in the early stages of the construction. From then on, top and bottom sticky notes will *overlap* with the body, so that a typical partial interval bijection will need to be decomposed into the form

\[
\hat{f} = \hat{f}(1) \ast \hat{f}(2) \ast \hat{f}(3),
\]
where again \( \hat{f}(1) \) and \( \hat{f}(3) \) make up a very small portion of the overall map. Here again, we will refer to \( \hat{f}(1) \) and \( \hat{f}(3) \) as the bottom and top sticky notes, and \( \hat{f}(2) \) as the body, of \( \hat{f} \).

3.5. Generalized Overlapping Concatenations. Once we are far enough along in the construction that sticky note decompositions take the form (3.7), we will no longer be able to use Definition 3.6 to glue partial interval bijections together. We will instead use the following.

**Definition 3.8.** Suppose \( \hat{f}_i = [I_i, J_i, A_i, B_i, f_i] \) for \( i = 1, 2 \) are two partial interval bijections that are decomposed via (3.7) as \( \hat{f}_i = \hat{f}_i(1) \hat{f}_i(2) \hat{f}_i(3) \).

If \( \hat{f}_1(3) \sim \hat{f}_2(1) \), then define the overlapping concatenation of \( \hat{f}_1 \) and \( \hat{f}_2 \), denoted \( \hat{f}_1 \hat{f}_2 \), by

\[
\hat{f}_1 \hat{f}_2 = \hat{f}_1(1) \hat{f}_1(2) \hat{f}_1(3) \hat{f}_2(2) \hat{f}_2(3) = \hat{f}_1(1) \hat{f}_1(2) \hat{f}_2(1) \hat{f}_2(2) \hat{f}_2(3)
\]

3.6. Reordering Maps. A reordering map is a partial interval bijection of the form \( \hat{p} = [J, J, J, J, p] \). Unlike arbitrary partial interval bijections, in a reordering map each of the first four components is the same interval. Therefore it is possible to compose two reordering maps, as follows. If \( p_1 : J \to J \) and \( p_2 : J \to J \) are two bijections, then, as a usual composition of functions, \( p_2 \circ p_1 : J \to J \) is a bijection. Therefore we can define \( \hat{p}_2 \) composed with \( \hat{p}_1 \) to be the reordering map \( \hat{p}_2 \circ \hat{p}_1 = [J, J, J, p_2 \circ p_1] \).

Reordering maps will be used to move certain levels in the “bottom part” of a template up to the “top part,” while shifting all levels in the “middle part” down. For example, consider the template

\[
\tau : c_0 \prec c_1 \prec c_2 \prec c_3 \prec c_4 \prec c_5 \prec c_6 \prec c_7 \prec c_8 \prec c_9 \prec c_{10} \prec c_{11} \prec c_{12} \prec c_{13},
\]

and think of \( c_7 \prec c_8 \prec c_9 \) as the “middle part.” Suppose we wish to shift this middle part down by two positions. We could accomplish this, for example, by moving \( c_2 \) and \( c_6 \) from the bottom part to the top part, as follows. Let \( J = [0, 1, \ldots, 13] \) and define \( p_1 : J \to J \) by

\[
p_1(j) = \begin{cases} 
9 & \text{if } j = 2 \\
 j - 1 & \text{if } 3 \leq j \leq 9 \\
 j & \text{if } 0 \leq j \leq 1 \text{ or } 10 \leq j \leq 13 
\end{cases}
\]

and \( p_2 : J \to J \) by

\[
p_2(j) = \begin{cases} 
11 & \text{if } j = 5 \\
 j - 1 & \text{if } 6 \leq j \leq 11 \\
 j & \text{if } 0 \leq j \leq 4 \text{ or } 12 \leq j \leq 13
\end{cases}
\]

Notice that \( \hat{p}_1(\tau) \) is the (new) template

\[
c_0 \prec c_1 \prec c_3 \prec c_4 \prec c_5 \prec c_6 \prec c_7 \prec c_8 \prec c_9 \prec c_2 \prec c_{10} \prec c_{11} \prec c_{12} \prec c_{13},
\]
and $\tilde{p}_2 \circ \tilde{p}_1(\tau)$ is the (new) template
\[
c_0 < c_1 < c_3 < c_4 < c_5 \leq c_7 < c_8 < c_9 \leq c_2 < c_{10} < c_{11} < c_6 < c_{12} < c_{13}.
\]

We can also compose a reordering map $\tilde{p} = [J, J, J, p]$ with an arbitrary partial interval bijection $\hat{f} = [I, J, A, B, \hat{f}]$ by defining $\tilde{p} \circ \hat{f} = [I, J, A, \hat{p}(B), p|_{B} \circ \hat{f}]$, where $p|_{B}$ is the restriction of $p$ to $B$.

4. Morse Minimal System Preliminaries

Given a binary word $B = b_1 b_2 \cdots b_k \subset \{0, 1\}^k$ of length $k$, the flip of $B$ is the word $\overline{B} = \overline{b}_1 \overline{b}_2 \cdots \overline{b}_k$, where $\overline{b}_i = 0$ if $b_i = 1$ and $\overline{b}_i = 1$ if $b_i = 0$.

Let $\sigma$ be the substitution rule on the symbols 0 and 1 given by $\sigma(0) = 01$ and $\sigma(1) = 10$. Iterating $\sigma$ on the symbol 0 determines, for each $k \in \mathbb{N}$, a word $u_k := \sigma^k(0)$ of length $2^k$:
\[
\begin{align*}
  u_1 &= \sigma(0) = 01 \\
  u_2 &= \sigma^2(0) = 0110 \\
  u_3 &= \sigma^3(0) = 01101001 \\
  &\vdots
\end{align*}
\]

Observe that $\overline{u}_k = \sigma^k(1)$ and that $u_{k+1} = u_k \overline{u}_k$. The Morse sequence is the sequence in $\{0, 1\}^\mathbb{N}$ whose first $2^k$ symbols are the word $u_k$.

Let $X$ denote the set of all doubly infinite sequences $x = (x_i)_{i \in \mathbb{Z}}$ in $\{0, 1\}^\mathbb{Z}$ such that every finite subword of $x$ occurs as a subword of the Morse sequence. Given $x$ and $x'$ in $X$, let $\rho(x, x') = 1$ if $x_0 \neq x'_0$; otherwise, define $\rho(x, x') = \frac{1}{2^n}$, where $n$ is maximal such that $x_i = x'_i$ for all $i \in \{-n, \ldots, n\}$. Then $\rho$ is a metric on $X$ that determines a Borel sigma algebra $B$. The Morse minimal system is then the system $(X, T, B, \mu)$, where $T : X \rightarrow X$ is the left shift, and $\mu$ is the unique complete ergodic Borel probability measure.

The following results are proved in [15].

**Proposition 4.1 ([15]).** For each $x \in X$ and $k \in \mathbb{N}$, there exists a unique partition of $\mathbb{Z}$ into intervals of length $2^{k+1}$ so that the subword of $x$ on each interval of this partition is either $u_k \overline{u}_k$ or $\overline{u}_k u_k$.

**Corollary 4.2 ([15]).** For each $x \in X$ and $k \in \mathbb{N}$, there exists a unique partition of $\mathbb{Z}$ into intervals of length $2^k$ so that the subword of $x$ on each interval of this partition is either $u_k$ or $\overline{u}_k$.

**Proposition 4.3 ([15]).** For $x \in X$ and $k \in \mathbb{N}$, partition $\mathbb{Z}$ into intervals as in Corollary 4.2 and let $t_k(x)$ be the position occupied by 0 in its interval, $0 \leq t_k(x) < 2^k$. The functions $t_k$ are continuous.

4.1. Canonical Templates. Given $x \in X$ and $k \in \mathbb{N}$, let $t_k(x)$ be as defined in Proposition 4.3. For $0 \leq i < 2^k$, let $u_{k,i}$ denote the $i$th symbol in $u_k$, and let $\overline{u}_{k,i}$ denote the $i$th symbol in $\overline{u}_k$. Define sets $u_k(i) = \{x \in X \mid t_k(x) = i \text{ and } x_0 = u_{k,i}\}$
and 

\[ \overline{u}_k(i) = \{ x \in X \mid t_k(x) = i \text{ and } x_0 = \overline{u}_{k,i} \}. \]

Then \( u_k(i) \) and \( \overline{u}_k(i) \) are obtained by taking the clopen set where \( t_k(x) = i \) and splitting it according to whether the symbol at the origin is 0 or 1. Therefore \( u_k(i) \) and \( \overline{u}_k(i) \) are clopen. These \( 2^{k+1} \) sets, all of equal measure, are called the \( k \)-canonical cylinders in \( X \).

We will often refer to \( k \)-canonical cylinders as \textit{levels}.

Define the \textit{k-canonical templates} in \( X \) by:

\[ \mathcal{P}_k(0) : u_k(0) \prec u_k(1) \prec \cdots \prec u_k(2^k - 1) \]

and 

\[ \mathcal{P}_k(1) : \overline{u}_k(0) \prec \overline{u}_k(1) \prec \cdots \prec \overline{u}_k(2^k - 1). \]

Note that each \( k \)-canonical template has height \( 2^k \). The order on the \( k \)-canonical templates is simply the order given by the action of \( T \). The set of levels in the \( k \)-canonical templates gives a partition of \( X \). Let \( \mathcal{P}_k = \{ \mathcal{P}_k(0), \mathcal{P}_k(1) \} \).

4.2. Maps Between Canonical Templates. Given \( 0 \leq k' < k \), each level \( c \in \mathcal{P}_k \) is a subset of a unique level \( c' \in \mathcal{P}_{k'} \). Sending \( c \mapsto c' \) then gives a map \( \pi : \mathcal{P}_k \to \mathcal{P}_{k'} \). Observe that \( \pi \) is measure preserving in the sense that the measure of the pull back of a set in \( \mathcal{P}_{k'} \) is the same as its measure.

4.3. Modified Canonical Templates. Define three modified versions of each \( k \)-canonical template: one in which the bottom level is removed, a second in which there is an extra copy of \( u_k(0) \) tacked on to the top, and a third in which there is an extra copy of \( \overline{u}_k(0) \) tacked on to the top. The superscripts \( m, e(0), \) and \( e(1) \) will denote “missing bottom level”, “extra copy of \( u_k(0) \)”, and “extra copy of \( \overline{u}_k(0) \)”, respectively:

\[ \mathcal{P}_k(0) : u_k(0) \prec u_k(1) \prec \cdots \prec u_k(2^k - 1) \]

\[ \mathcal{P}_k^m(0) : u_k(1) \prec u_k(2) \prec \cdots \prec u_k(2^k - 1) \]

\[ \mathcal{P}_k^{e(0)}(0) : u_k(0) \prec u_k(1) \prec \cdots \prec u_k(2^k - 1) \prec u_k(0) \]

\[ \mathcal{P}_k^{e(1)}(0) : u_k(0) \prec u_k(1) \prec \cdots \prec u_k(2^k - 1) \prec \overline{u}_k(0) \]

Define \( \mathcal{P}_k^m(1), \mathcal{P}_k^{e(0)}(1), \) and \( \mathcal{P}_k^{e(1)}(1) \) by analogy. Let

\[ \mathcal{P}_k = \{ \mathcal{P}_k^m(0), \mathcal{P}_k^{e(0)}(0), \mathcal{P}_k^{e(1)}(0), \mathcal{P}_k^m(1), \mathcal{P}_k^{e(0)}(1), \mathcal{P}_k^{e(1)}(1) \} \]

be the set of all modified canonical templates at stage \( k \).

5. Binary Odometer Preliminaries

Let \( Y = \{0, 1\}^\mathbb{N} \). Then \( Y \) is compact and metrizable; the metric \( \rho(y, y') = 2^{-\ell} \), where \( \ell = \min\{|i| : y_i \neq y'_i\} \), induces the topology and determines a Borel sigma algebra \( \mathcal{F} \). Define \( S : Y \to Y \) by \( S(y) = y + 1 \), where \( 1 = (1, 0, 0, \ldots) \) and the addition is coordinate-wise mod 2 with right carry. The \textit{binary odometer} is the system \( (Y, S, \mathcal{F}, \nu) \), where \( \nu \) is the unique complete ergodic Borel probability measure.
5.1. Canonical Templates. A $k$-canonical cylinder is a set \( \{ y \in Y \mid y \text{ begins with } d \} \), where \( d \in \{0, 1\}^k \) is a binary word of length \( k \). These \( k \)-canonical cylinders are the levels of templates for the binary odometer.

The \( k \)-canonical template in \( Y \), denoted \( Q_k(0) \), is the set of \( k \)-canonical cylinders together with the order \( \prec \) inherited by the action of \( S \), where \( 0 = (0, 0, 0, \ldots) \) is an element of the first cylinder. For example, the 3-canonical template is

\[
(5.1) \quad Q_3(0) : 000 \prec 100 \prec 010 \prec 110 \prec 001 \prec 101 \prec 011 \prec 111.
\]

In general, let \( v_k(i) \) denote the \( i \)-th level in \( Q_k(0) \), so that

\[
Q_k(0) : v_k(0) \prec v_k(1) \prec \cdots \prec v_k(2^k - 1).
\]

Note that \( Q_k(0) \) has height \( 2^k \), and the set of levels in \( Q_k(0) \) gives a partition of \( Y \). Let \( Q_k = \{ Q_k(0) \} \).

5.2. Maps Between Canonical Templates. Given \( 0 \leq k' < k \), each level \( d \in Q_k \) is a subset of a unique level \( d' \in Q_{k'} \). Sending \( d \mapsto d' \) then gives a map \( \pi : Q_k \to Q_{k'} \). Observe that \( \pi \) is measure preserving in the sense that the measure of the pull back of a set in \( Q_{k'} \) is the same as its measure.

5.3. Modified Canonical Templates. Define two modified versions of the \( k \)-canonical template \( Q_k(0) \): one in which the level \( v_k(0) \) is removed, and another in which there is an extra copy of \( v_k(0) \) tacked on to the top. The superscripts \( m \) and \( e \) will denote “missing \( v_k(0) \)” and “extra copy of \( v_k(0) \)”, respectively:

\[
\begin{align*}
Q_k(0) & : v_k(0) \prec v_k(1) \prec \cdots \prec v_k(2^k - 1) \\
Q_k^m(0) & : v_k(1) \prec v_k(2) \prec \cdots \prec v_k(2^k - 1) \\
Q_k^e(0) & : v_k(0) \prec v_k(1) \prec \cdots \prec v_k(2^k - 1) \prec v_k(0)
\end{align*}
\]

Let

\[
\tilde{Q}_k = \{ Q_k^m(0), Q_k^e(0) \}
\]

be the set of all modified canonical templates at stage \( k \). Define \( k_0 = k_1 = 0 \), so that both \( Q_{k_0} \) and \( Q_{k_1} \) are the trivial canonical templates for the binary odometer (each consisting of just one level) as indicated in Figure 1.

6. Binary Odometer Template Sets \( \Omega_k \) and \( \tilde{\Omega}_k \)

The template sets \( \Omega_k \) and \( \tilde{\Omega}_k \), defined in Section 6.7, consist of templates of the following types: basic, diminished, augmented, missing and extra. We begin by describing these types of templates.
6.1. Basic Templates. A basic template at stage \( k \) is any template that satisfies all of the following conditions: it has height \( 2^k \); its elements are \( k \)-canonical cylinders; and its ordering is “allowed” by the action of \( S \).

For the ordering of a template \( \omega \) to be “allowed” by the action of \( S \), \( \omega \) must have one of the following forms. Either it is a special basic template \( \omega = \mathcal{Q}_k(0) \) that we will refer to as the zero-template, or for some (unique) \( i \in \{1, 2, \ldots, 2^k - 1\} \), \( \omega \) is:

\[
\omega : v_k(i) < v_k(i + 1) < \cdots < v_k(2^k - 1) < v_k(0) < v_k(1) < \cdots < v_k(i - 1).
\]

Let \( \mathcal{B}_k(Y) \) be the set of all basic templates for the binary odometer at stage \( k \).

In each basic template, the level \( v_k(0) \) occurs once and only once. Call this level the global cut.

6.2. Predecessor and Successor Templates. If \( \omega \in \mathcal{B}_k(Y) \) is a basic template, define the predecessor template for \( \tau \), denoted \( \omega_p \), to be the basic template whose global cut is one position higher (mod \( 2^k \)), and define the successor template for \( \omega \), denoted \( \omega_s \), to be the basic template whose global cut is one position lower (mod \( 2^k \)). For example, if \( \omega \in \mathcal{B}_k(Y) \) is:

\[
\omega : v_k(i) < v_k(i + 1) < \cdots < v_k(2^k - 1) < v_k(0) < v_k(1) < \cdots < v_k(i - 1),
\]

then \( \omega_p \) is

\[
\omega_p : v_k(i - 1) < v_k(i) < \cdots < v_k(2^k - 1) < v_k(0) < v_k(1) < \cdots < v_k(i - 2),
\]

and \( \omega_s \) is

\[
\omega_s : v_k(i + 1) < v_k(i + 2) < \cdots < v_k(2^k - 1) < v_k(0) < v_k(1) < \cdots < v_k(i).
\]

6.3. Diminished Templates. Given a basic template \( \omega \in \mathcal{B}_k(Y) \), define two additional, diminished templates \( \omega^-(d) \) and \( \omega^-(u) \) as follows:

- \( \omega^-(d) \) is \( \omega \) with the global cut removed, and with an extra level tacked on at the bottom (the level that would naturally precede the bottom level in \( \omega \), namely the pre-image of the bottom level of \( \omega \) under the map \( S \)).
- \( \omega^-(u) \) is \( \omega \) with the global cut removed, and with an extra level tacked on at the top (the level that would naturally follow the top level in \( \omega \), namely the image of the top level in \( \omega \) under the map \( S \)).

For example, if \( \omega \in \mathcal{B}_k(Y) \) is given by

\[
\omega : v_k(i) < v_k(i + 1) < \cdots < v_k(2^k - 1) < v_k(0) < v_k(1) < \cdots < v_k(i - 1),
\]

then \( \omega^-(d) \) is given by

\[
\omega^-(d) : v_k(i - 1) < v_k(i) < \cdots < v_k(2^k - 1) < v_k(0) < v_k(1) < \cdots < v_k(i - 1).
\]

The letters “d” and “u” refer to “down” and “up”, respectively, for reasons that will be made clear later.

Let \( \mathcal{D}_k(Y) \) be the set of all diminished templates for the binary odometer at stage \( k \). Note that all diminished templates in \( \mathcal{D}_k(Y) \) have height \( 2^k \).
6.4. **Augmented Templates.** Given a basic template $\omega \in B_k(Y)$, define two additional, *augmented* templates as follows:

- $\omega^+(d)$ is $\omega$ with an extra copy of the global cut, $v_k(0)$, inserted right next to the actual global cut, and with the bottom level deleted.
- $\omega^+(u)$ is $\omega$ with an extra copy of the global cut, $v_k(0)$, inserted right next to the actual global cut, and with the top level deleted.

For example, if $\omega \in B_k(Y)$ is given by

$$
\omega : v_k(i) \prec v_k(i + 1) \prec \cdots \prec v_k(2^k - 1) \prec v_k(0) \prec v_k(1) \prec \cdots \prec v_k(i - 1),
$$

then $\omega^+(d)$ is given by

$$
\omega^+(d) : v_k(i + 1) \prec \cdots \prec v_k(2^k - 1) \prec v_k(0) \prec v_k(1) \prec \cdots \prec v_k(i - 1).
$$

Let $O_k(Y)$ be the set of all augmented templates for the binary odometer at stage $k$. Note that all augmented templates in $O_k(Y)$ have height $2^k$.

6.5. **Missing Templates.** Given a basic template $\omega \in B_k(Y)$, define one additional, *missing* template, denoted $\omega^m$, to be $\omega$ with its bottom level removed. For example, if $\omega$ is the zero-template, then

$$
\omega^m = v_k(1) \prec v_k(2) \prec \cdots \prec v_k(2^k - 1).
$$

Let $M_k(Y)$ be the set of all missing templates for the binary odometer at stage $k$. Note that all missing templates have height $2^k - 1$.

6.6. **Extra Templates.** Given a basic template $\omega \in B_k(Y)$, define one additional, *extra* template, denoted $\omega^e$, to be $\omega$ with an one extra level tacked on at the top (the level that would naturally follow the top level). For example, if $\omega \in B_k(Y)$ is given by

$$
\omega : v_k(i) \prec v_k(i + 1) \prec \cdots \prec v_k(2^k - 1) \prec v_k(0) \prec v_k(1) \prec \cdots \prec v_k(i - 1),
$$

then $\omega^e$ is given by

$$
\omega^e : v_k(i) \prec v_k(i + 1) \prec \cdots \prec v_k(2^k - 1) \prec v_k(0) \prec v_k(1) \prec \cdots \prec v_k(i - 1) \prec v_k(i).
$$

Let $E_k(Y)$ be the set of all extra templates for the binary odometer at stage $k$. Note that all extra templates in $E_k(Y)$ have height $2^k + 1$.

6.7. **Definition of $\Omega_k$ and $\tilde{\Omega}_k$.** Define

$$
(6.1) \quad \Omega_k = B_k(Y) \cup D_k(Y) \cup A_k(Y) \text{ and } \tilde{\Omega}_k = M_k(Y) \cup E_k(Y).
$$

7. **Second stage of the induction**

Define $k_2 = 2$, so that the canonical templates in $P_{k_2}$ have height 4. For each canonical template $P_{k_2}(i) \in P_{k_2}$ for the Morse system, and for each template $\omega \in \Omega_{k_2}$ in the binary odometer, we will define a partial interval bijection $\phi_{\omega} : P_{k_2}(i) \to \omega$ from a subset of the levels in $P_{k_2}(i)$ to a subset of the levels in $\omega$. Only after we have done this will we define partial interval bijections from the modified canonical templates in $\tilde{P}_{k_2}$ to the modified templates in $\tilde{\Omega}_{k_2}$. 
7.1. Maps to Basic Templates. There are exactly four basic templates in $B_{k_2}(Y)$:

- $\omega_1 : v_{k_2}(0) \prec v_{k_2}(1) \prec v_{k_2}(2) \prec v_{k_2}(3)$,
- $\omega_2 : v_{k_2}(3) \prec v_{k_2}(0) \prec v_{k_2}(1) \prec v_{k_2}(2)$,
- $\omega_3 : v_{k_2}(2) \prec v_{k_2}(3) \prec v_{k_2}(0) \prec v_{k_2}(1)$, and
- $\omega_4 : v_{k_2}(1) \prec v_{k_2}(2) \prec v_{k_2}(3) \prec v_{k_2}(0)$.

For $0 \leq i \leq 1$ and $1 \leq j \leq 4$, define $\phi_{\omega_j} : P_{k_2}(i) \to \omega_j$ to be the partial interval bijection $\phi_{\omega_j} = [I, I, A, B_j, f_j]$, where:

- $I = [0, 1, 2, 3] \subset \mathbb{Z}$,
- $A = \{2\}$,
- $B_1 = B_2 = \{2\}$,
- $B_3 = B_4 = \{1\}$, and
- Each $f_j$ is the obvious bijection $f_j : A \to B_j$.

Note that each $\phi_{\omega_j}(0)$ is equivalent to $\phi_{\omega_1}(1)$ as a formal map between intervals in $\mathbb{Z}$. But of course $\phi_{\omega_j}(0)$ and $\phi_{\omega_j}(1)$ are different as set maps because the levels represented by $A$ in $P_{k_2}(0)$ and $P_{k_2}(1)$ are different. Now recall Definition 3.1, where the domain and range of a partial interval bijection are defined. The following propositions are obvious.

**Proposition 7.1.** For each canonical template $P_{k_2}(i) \in P_{k_2}$, neither the bottom level nor the top level is in the domain of any $\phi_{\omega}$.

**Proposition 7.2.** Given $\omega \in B_{k_2}(Y)$, the global cut in $\omega$ is not in the range of $\phi_{\omega}$.

We now define the “good set” at stage 2, and establish its most important property (Proposition 7.4), which is obvious at this stage because $Q_{k_0}$ is the trivial canonical template.

**Definition 7.3.** Let $G_2 = \{u_{k_2}(2), \pi_{k_2}(2)\}$.

Recall the vertical map $\pi : Q_{k_1} \to Q_{k_0}$ which was defined in Section 4.2. We similarly define $\zeta : \Omega_{k_2} \to Q_{k_1}$. The following is immediate since $Q_{k_0} = Q_{k_1} = Y$.

**Proposition 7.4.** If $c \in G_2$ and $\omega, \omega' \in B_{k_2}(Y)$, then $c$ is in the domain of both $\phi_{\omega}$ and $\phi_{\omega'}$, and $\pi \circ \zeta \circ \phi_{\omega}(c) = \pi \circ \zeta \circ \phi_{\omega'}(c)$.

7.2. Maps to Diminished Templates. Let $\omega$ be a basic template in $B_{k_2}(Y)$. Define $\phi_{\omega^-(u)}$ to match with $\phi_{\omega}$, and define $\phi_{\omega^-(d)}$ to match with $\phi_{\omega^p}$, as in Definition 3.2. That these bijections are well defined follows from Proposition 7.2. For example, suppose $\omega$ is given by

$$\omega : v_{k_2}(3) \prec v_{k_2}(0) \prec \boxed{v_{k_2}(1)} \prec v_{k_2}(2),$$

where the box indicates the level that is in the range of $\phi_{\omega}$. Then, using this same “box” notation to indicate the levels in the ranges of the corresponding partial interval bijections, we have

$$\omega^-(u) : v_{k_2}(3) \prec \boxed{v_{k_2}(1)} \prec v_{k_2}(2) \prec v_{k_2}(3),$$

$$\omega^-(d) : v_{k_2}(2) \prec \boxed{v_{k_2}(3)} \prec v_{k_2}(1) \prec v_{k_2}(2),$$

for $\omega$.
Lemma 7.6. Given any basic template \( \omega \in \mathcal{B}_{k_2}(Y) \), \( \phi_{\omega} \) matches with \( \phi_{\omega_p} \).

Proof. Both \( \phi_{\omega} \) and \( \phi_{\omega_p} \) are defined to match with \( \phi_{\omega_p} \).

\[ \omega_p : v_{k_2}(2) \prec v_{k_2}(3) \prec v_{k_2}(0) \prec v_{k_2}(1). \]

Now observe that \( \omega_p^-(u) \), and its corresponding partial interval bijection, are:

\[ \omega_p^-(u) : v_{k_2}(2) \prec v_{k_2}(3) \prec v_{k_2}(1) \prec v_{k_2}(2). \]

That \( \phi_{\omega_p^-}(d) \) matches with \( \phi_{\omega^-(d)} \) is not a coincidence:

Lemma 7.5. Given any basic template \( \omega \in \mathcal{B}_{k_2}(Y) \), \( \phi_{\omega} \) matches with \( \phi_{\omega_p^-}(u) \).

Proof. Both \( \phi_{\omega^-(u)} \) and \( \phi_{\omega_p}(u) \) are defined to match with \( \phi_{\omega_p} \).

\[ \omega : \omega_{k_2}, A, B, f \]

7.3. Maps to Augmented Templates. Similarly, given \( \omega \in \mathcal{B}_{k_2}(Y) \), define \( \phi_{\omega^+(d)} \) to match with \( \phi_{\omega} \), and define \( \phi_{\omega^+(u)} \) to match with \( \phi_{\omega_p^+} \). That these definitions are possible again follows from Proposition 7.2.

Lemma 7.6. Given any basic template \( \omega \in \mathcal{B}_{k_2}(Y) \), \( \phi_{\omega^+(u)} \) matches with \( \phi_{\omega_p^+(d)} \).

Proof. Both \( \phi_{\omega^+(u)} \) and \( \phi_{\omega_p^+}(d) \) are defined to match with \( \phi_{\omega_p} \).

7.4. Maps to Missing and Extra Templates. Recall the modified template sets \( \mathcal{P}_{k_2} \) and \( \mathcal{B}_{k_2} \), defined in Section 8.1 and 8.1. Given \( i \in \{0, 1\} \) and \( \omega \in \mathcal{M}_{k_2}(Y) \), we know from Propositions 7.1 and 7.2 that the bottom level of \( \mathcal{P}_{k_2}(i) \) is not in the domain of \( \phi_{\omega} : \mathcal{P}_{k_2}(i) \rightarrow \omega \), nor is the bottom level of \( \omega \) in the range. Therefore, if \( \phi_{\omega} = [I, I, A, B, f] \), where \( I = [0, 1, \ldots, 2^{k_2} - 1] \), then we may define \( \phi_{\omega^m} : \mathcal{P}_{k_2}(i) \rightarrow \omega^m \) by \( \phi_{\omega^m} = [I', I', A, B, f] \), where \( I' = [1, 2, \ldots, 2^{k_2} - 1] \). In other words, \( \phi_{\omega^m} : \mathcal{P}_{k_2}(i) \rightarrow \omega^m \) is identical to \( \phi_{\omega} : \mathcal{P}_{k_2}(i) \rightarrow \omega \) except that the bottom level of \( \omega \) is technically missing.

Similarly, given \( i, j \in \{0, 1\} \) and \( \omega \in \mathcal{E}_{k_2}(Y) \), if again \( \phi_{\omega} = [I, I, A, B, f] \), then we may define \( \phi_{\omega^e} : \mathcal{P}_{k_2}(i) \rightarrow \omega^e \) by \( \phi_{\omega^e} = [I'', I'', A, B, f] \), where \( I'' = [0, 1, \ldots, 2^{k_2}] \). In other words, \( \phi_{\omega^e} : \mathcal{P}_{k_2}(i) \rightarrow \omega^e \) is identical to \( \phi_{\omega} : \mathcal{P}_{k_2}(i) \rightarrow \omega \) except that there is technically an extra level at the top.

8. Morse system template sets \( \mathcal{T}_{k} \) and \( \breve{T}_{k} \)

Similar to \( \Omega_k \) and \( \breve{\Omega}_k \), the template sets \( \mathcal{T}_{k} \) and \( \breve{T}_{k} \), defined in Section 8.7 below, consist of templates of the following types: basic, diminished, augmented, missing, and extra. We define these types here.

8.1. Basic Templates. A basic template at stage \( k \) is any template that satisfies the all of the following conditions: it has height \( 2^k \), its elements are \( k \)-canonical cylinders, and its ordering is “allowed” by the action of \( T \).

For the ordering of a template \( \tau \) to be “allowed” by the action of \( T \), \( \tau \) must have one of the following forms:

1. \( \tau = \mathcal{P}_{k}(0) \). This is a special basic template we call the zero-template.
(2) \( \tau = \mathcal{P}_k(1) \). This is a special basic template we call the one-template.

(3) For some (unique) \( i \in \{1, 2, \ldots, 2^k - 1\} \), \( \tau \) is one of the following four templates:

\[ \tau : u_k(i) < u_k(i + 1) < \cdots < u_k(2^k - 1) < u_k(0) < u_k(1) < \cdots < u_k(i - 1) \]
\[ \tau : u_k(i) < u_k(i + 1) < \cdots < u_k(2^k - 1) < \overline{u}_k(0) < \overline{u}_k(1) < \cdots < \overline{u}_k(i - 1) \]
\[ \tau : \overline{u}_k(i) < \overline{u}_k(i + 1) < \cdots < \overline{u}_k(2^k - 1) < u_k(0) < u_k(1) < \cdots < u_k(i - 1) \]
\[ \tau : \overline{u}_k(i) < \overline{u}_k(i + 1) < \cdots < \overline{u}_k(2^k - 1) < \overline{u}_k(0) < \overline{u}_k(1) < \cdots < \overline{u}_k(i - 1) \]

Let \( \mathcal{B}_k(X) \) be the set of all basic templates for the Morse system at stage \( k \).

In each basic template, exactly one of the levels is either \( u_k(0) \) or \( \overline{u}_k(0) \). Call that level the global cut.

8.2. Predecessor and Successor Templates. If \( \tau \in \mathcal{B}_k(X) \) is a basic template whose global cut is neither the bottom level nor the top level, then \( \tau \) must be one of the four templates listed in item 3 of Section 8.1 above. In this case, define the predecessor template for \( \tau \), denoted \( \tau_p \), to be the basic template of that same form whose global cut is one position lower. For example, if \( \tau \in \mathcal{B}_k(X) \) is the template

\[ \tau : u_k(i) < u_k(i + 1) < \cdots < u_k(2^k - 1) < \overline{u}_k(0) < \overline{u}_k(1) < \cdots < \overline{u}_k(i - 1), \]

then \( \tau_p \in \mathcal{B}_k(X) \) is

\[ \tau_p : u_k(i - 1) < u_k(i + 1) < \cdots < u_k(2^k - 1) < \overline{u}_k(0) < \overline{u}_k(1) < \cdots < \overline{u}_k(i - 2), \]

and \( \tau_s \in \mathcal{B}_k(X) \) is

\[ \tau_s : u_k(i + 2) < u_k(i + 1) < \cdots < u_k(2^k - 1) < \overline{u}_k(0) < \overline{u}_k(1) < \cdots < \overline{u}_k(i). \]

If \( \tau \in \mathcal{B}_k(X) \) is a basic template whose global cut is the bottom level, then \( \tau \) is either the zero-template or the one-template. In this case, define two predecessor templates for \( \tau \), denoted \( \tau_{p(0)} \) and \( \tau_{p(1)} \), as follows:

- \( \tau_{p(0)} \) is \( \tau \) with the top level removed and \( u_k(2^k - 1) \) tacked on at the bottom.
- \( \tau_{p(1)} \) is \( \tau \) with the top level removed and \( \overline{u}_k(2^k - 1) \) tacked on at the bottom.

Also in this case, define two successor templates for \( \tau \), denoted \( \tau_{s(0)} \) and \( \tau_{s(1)} \), as follows:

- \( \tau_{s(0)} \) is \( \tau \) with the bottom level removed and \( u_k(0) \) tacked on at the top.
- \( \tau_{s(1)} \) is \( \tau \) with the bottom level removed and \( \overline{u}_k(0) \) tacked on at the top.

If \( \tau \in \mathcal{B}_k(X) \) is a basic template whose global cut is the top level (there are exactly two such basic templates), then define two predecessor templates for \( \tau \), denoted \( \tau_{p(0)} \) and \( \tau_{p(1)} \), as follows:

- \( \tau_{p(0)} \) is \( \tau \) with the top level (the global cut) removed and \( u_k(0) \) tacked on at the bottom.
- \( \tau_{p(1)} \) is \( \tau \) with the top level (the global cut) removed and \( \overline{u}_k(1) \) tacked on at the bottom.
Also in this case, define a single successor template for $\tau$, denoted $\tau_s$, to be the basic template that is $\tau$ with its bottom level removed, and with the level that would naturally follow the top level tacked on at the top. For example, if $\tau \in B_k(X)$ is
\[
\tau : u_k(1) \prec u_k(2) \prec \cdots \prec u_k(2^k - 1) \prec \overline{u}_k(0),
\]
then $\tau_s$ is
\[
\tau_s : u_k(2) \prec u_k(3) \prec \cdots \prec u_k(2^k - 1) \prec \overline{u}_k(0) \prec \overline{u}_k(1).
\]
Because certain templates have multiple predecessor/successor templates, while others have only one, the following definition will be useful.

**Definition 8.1.** If $\tau \in B_k(X)$ is a basic template, then a predecessor template for $\tau$ is any basic template of the form $\tau_p$, $\tau_p(0)$, or $\tau_p(1)$, as defined above. A successor template for $\tau$ is any basic template of the form $\tau_s$, $\tau_s(0)$, or $\tau_s(1)$, as defined above.

**Lemma 8.2.** If $\tau \in B_k(X)$ is a basic template, then all predecessor templates for $\tau$ agree in every level except possibly the bottom. Also, all successor templates for $\tau$ agree in every level except possibly the top.

### 8.3. Diminished Templates.

Given a basic template $\tau \in B_k(X)$ that is neither the zero-template nor the one-template, define two additional, diminished templates $\tau^-(d)$ and $\tau^-(u)$ as follows:

- $\tau^-(d)$ is $\tau$ with the global cut removed, and with an extra level tacked on at the bottom (the level that would naturally precede the bottom level).
- $\tau^-(u)$ is $\tau$ with the global cut removed, and with an extra level tacked on at the top (the level that would naturally follow the top level).

For example, suppose $\tau \in B_k(X)$ is given by
\[
\tau : u_k(i) \prec u_k(i + 1) \prec \cdots \prec u_k(2^k - 1) \prec \overline{u}_k(0) \prec \overline{u}_k(1) \prec \cdots \prec \overline{u}_k(i - 1).
\]
Then the global cut is $\overline{u}_k(0)$, and the bottom level is $u_k(i)$. The level that would naturally precede this bottom level is $u_k(i - 1)$. Therefore the diminished template $\tau^-(d)$ is given by
\[
\tau^-(d) : u_k(i - 1) \prec u_k(i) \prec \cdots \prec u_k(2^k - 1) \prec \overline{u}_k(1) \prec \cdots \prec \overline{u}_k(i - 1)
\]

Now suppose $\tau \in B_k(X)$ is either the zero-template or the one-template. Then there are two levels that could naturally precede the bottom level (either $u_k(2^k - 1)$ or $\overline{u}_k(2^k - 1)$), as well as two levels that could naturally follow the top level (either $u_k(0)$ or $\overline{u}_k(0)$). For this reason, we define four diminished templates, as follows:

- $\tau^-(d,0)$ is $\tau$ with the global cut (which, in this case, is also the bottom level) removed, and with the level $u_k(2^k - 1)$ tacked on in its place.
- $\tau^-(d,1)$ is $\tau$ with the global cut (which, in this case, is also the bottom level) removed, and with the level $\overline{u}_k(2^k - 1)$ tacked on in its place.
- $\tau^-(u,0)$ is $\tau$ with the global cut (which, in this case, is also the bottom level) removed, and with the level $u_k(0)$ tacked on at the top.
- $\tau^-(u,1)$ is $\tau$ with the global cut (which, in this case, is also the bottom level) removed, and with the level $\overline{u}_k(0)$ tacked on at the top.
For example, if $\tau$ is the zero template, then $\tau^{-}(d, 1)$ is
\[
\tau^{-}(d, 1) = \overline{\nu}_{k}(2^{k} - 1) \prec u_{k}(1) \prec u_{k}(2) \prec \cdots \prec u_{k}(2^{k} - 1).
\]
Let $\mathcal{D}_{k}(X)$ be the set of all diminished templates for the Morse system at stage $k$. Note that all diminished templates in $\mathcal{D}_{k}(X)$ have height $2^{k}$.

8.4. **Augmented Templates.** Given a basic template $\tau \in \mathcal{B}_{k}(X)$, define four additional, augmented templates as follows:

- $\tau^{+}(d, 0)$ is $\tau$ with an extra copy of $u_{k}(0)$ inserted directly before the global cut, and with the bottom level deleted.
- $\tau^{+}(d, 1)$ is $\tau$ with an extra copy of $\overline{\nu}_{k}(0)$ inserted directly before the global cut, and with the bottom level deleted.
- $\tau^{+}(u, 0)$ is $\tau$ with an extra copy of $u_{k}(0)$ inserted directly before the global cut, and with the top level deleted.
- $\tau^{+}(u, 1)$ is $\tau$ with an extra copy of $\overline{\nu}_{k}(0)$ inserted directly before the global cut, and with the top level deleted.

For example, if $\tau \in \mathcal{B}_{k}(X)$ is given by
\[
\tau : u_{k}(i) \prec \cdots \prec u_{k}(2^{k} - 1) \prec \overline{\nu}_{k}(0) \prec \overline{\nu}_{k}(1) \prec \cdots \prec \overline{\nu}_{k}(i - 1),
\]
then $\tau^{+}(d, 0)$ is given by
\[
\tau^{+}(d, 0) : u_{k}(i + 1) \prec u_{k}(i + 2) \prec \cdots \prec u_{k}(2^{k} - 1) \prec u_{k}(0) \prec \overline{\nu}_{k}(0) \prec \cdots \prec \overline{\nu}_{k}(i - 1).
\]
Let $\mathcal{A}_{k}(X)$ be the set of all augmented templates for the Morse system at stage $k$. Note that all augmented templates in $\mathcal{A}_{k}(X)$ have height $2^{k}$.

8.5. **Missing Templates.** Given a basic template $\tau \in \mathcal{B}_{k}(X)$, define one additional, missing template, denoted $\tau^{m}$, to be $\tau$ with its bottom level removed. For example, if $\tau$ is the zero-template, then
\[
\tau^{m} = u_{k}(1) \prec u_{k}(2) \prec \cdots \prec u_{k}(2^{k} - 1).
\]
Let $\mathcal{M}_{k}(X)$ be the set of all missing templates for the Morse system at stage $k$. Note that all missing templates have height $2^{k} - 1$.

8.6. **Extra Templates.** Given a basic template $\tau \in \mathcal{B}_{k}(X)$ that is neither the zero-template nor the one-template, define one additional, extra template, denoted $\tau^{e}$, to be $\tau$ with one extra level tacked on at the top (the level that would naturally follow the top level). For example, if $\tau \in \mathcal{B}_{k}(X)$ is given by
\[
\tau : u_{k}(i) \prec \cdots \prec u_{k}(2^{k} - 1) \prec \overline{\nu}_{k}(0) \prec \overline{\nu}_{k}(1) \prec \cdots \prec \overline{\nu}_{k}(i - 1),
\]
then $\tau^{e}$ is given by
\[
\tau^{e} : u_{k}(i) \prec \cdots \prec u_{k}(2^{k} - 1) \prec \overline{\nu}_{k}(0) \prec \overline{\nu}_{k}(1) \prec \cdots \prec \overline{\nu}_{k}(i - 1) \prec \overline{\nu}_{k}(i).
\]
If $\tau$ is the zero-template or the one-template, then define two extra templates, as follows:

- $\tau^{e}(0)$ is $\tau$ with the level $u_{k}(0)$ tacked on at the top.
- $\tau^{e}(1)$ is $\tau$ with the level $\overline{\nu}_{k}(0)$ tacked on at the top.
Let $\mathcal{E}_k(X)$ be the set of all extra templates for the Morse system at stage $k$. Note that all extra templates in $\mathcal{E}_k(X)$ have height $2^k + 1$.

8.7. **The Template Sets** $\mathcal{T}_k$ and $\tilde{\mathcal{T}}_k$. Define 

$$
\mathcal{T}_k = \mathcal{B}_k(X) \cup \mathcal{D}_k(X) \cup \mathcal{A}_k(X) \quad \text{and} \quad \tilde{\mathcal{T}}_k = \mathcal{M}_k(X) \cup \mathcal{E}_k(X).
$$

9. **The Sequences** $(\varepsilon_n)$ and $(k_n)$

Let $(\varepsilon_n)$ be a summable sequence. Recall that we defined $k_0 = k_1 = 0$ and $k_2 = 2$. Now define $k_n$, $n > 2$, by the following two-step recursion. Given $k_n$, for $n$ even, define $k_{n+1}$ and $k_{n+2}$ as follows.

First choose $m_1 \in \mathbb{N}$ large enough so that

$$
\frac{2(1 + 2^{k_n})}{2(1 + 2^{k_n}) + m_1} < \varepsilon_n,
$$

and pick $k_{n+1} \in \mathbb{N}$ large enough so that

$$
2^{k_{n+1}} \geq m_1 \cdot 2^{k_n} + 2 \cdot (2^{k_n} + 2^{k_n} \cdot 2^{k_n}).
$$

Then choose $m_2 \in \mathbb{N}$ large enough so that

$$
\frac{2(1 + 2^{k_{n+1}})}{2(1 + 2^{k_{n+1}}) + m_2} < \varepsilon_{n+1},
$$

and pick $k_{n+2} \in \mathbb{N}$ large enough so that

$$
2^{k_{n+2}} \geq m_2 \cdot 2^{k_{n+1}} + 2 \cdot (2^{k_{n+1}} + 2^{k_{n+1}} \cdot 2^{k_{n+1}}).
$$

Note that inequalities (9.3) and (9.4) are the same as (9.1) and (9.2), only with $k_n$, $k_{n+1}$, $m_1$, and $\varepsilon_n$ replaced with $k_{n+1}$, $k_{n+2}$, $m_2$, and $\varepsilon_{n+1}$.

10. **From Stage $n$ to $n + 2$: Frequently Used Notation**

Assume the diagram in Figure 1 has been built down to stage $n$, where $n$ is even. Using our choices of $k_{n+1}$ and $k_{n+2}$ from Section 9 we then build the diagram down to stage $n + 2$. The construction is largely the same whether $n \equiv 2$ or $n \equiv 0 \mod 4$; Sections 10 - 11.3 apply in either case.

Define $J, J' \subset \mathbb{Z}$ by

$$
J = [0, 1, \ldots, 2^{k_{n+2}} - 1] \quad \text{and} \quad J' = [0, 1, \ldots, 2^{k_{n+1}} - 1].
$$

Define the bottom and top global safe zones in $J$ to be the subintervals

$$
[0, \ldots, 2^{k_{n+1}} + 2^{k_{n+1}} \cdot 2^{k_{n+1}} - 1] \quad \text{and} \quad [2^{k_{n+2}} - (2^{k_{n+1}} + 2^{k_{n+1}} \cdot 2^{k_{n+1}}), \ldots, 2^{k_{n+2}} - 1]
$$

of $J$, respectively. By (9.4) the global safe zones are well-defined, and by (9.3) the fraction of $J$ in the global safe zones is less than $\varepsilon_{n+1}$.

Similarly, define the bottom and top intermediate safe zones in $J'$ to be the subintervals

$$
[0, \ldots, 2^{k_{n+1}} + 2^{k_{n}} \cdot 2^{k_{n}} - 1] \quad \text{and} \quad [2^{k_{n+1}} - (2^{k_{n}} + 2^{k_{n}} \cdot 2^{k_{n}}), \ldots, 2^{k_{n+1}} - 1]
$$
of $J'$, respectively. By (9.2) the intermediate safe zones are well-defined, and by (9.1) the fraction of $J'$ in the intermediate safe zones is less than $\varepsilon_n$.

Let

$$B_{k,n+2} = \begin{cases} B_{k,n+2}(X) & \text{if } n \equiv 2 \pmod{4} \\ B_{k,n+2}(Y) & \text{if } n \equiv 0 \pmod{4} \end{cases}$$

be the set of basic templates at stage $n+2$. Let $L = 2^{k_{n+2}} / 2^{k_{n+1}}$ and $L' = 2^{k_{n+1}} / 2^{k_n}$. Fix a basic template $\tau \in B_{k,n+2}$, represent $\tau$ with the interval $J$, and let $g = g(\tau)$ be the position in $J$ where the global cut occurs. Let $a \in \{0, 1, \ldots, 2^{k_{n+1}} - 1\}$ and $b \in \{0, 1, \ldots, 2^{k_n} - 1\}$ be such that $g \equiv a \pmod{2^{k_{n+1}}}$ and $g \equiv b \pmod{2^{k_n}}$. Let $c = \left\lfloor \frac{a}{2^{k_n}} \right\rfloor$.

10.1. Intermediate and Local Block Partitions of Basic Templates at Stage $n+2$.

In this section we define two partitions of $\tau$, which we call the intermediate and local block partitions of $\tau$. We also define the intermediate and local cuts in $\tau$.

**Definition 10.1 (Intermediate Blocks in $J$).** Depending on $a$, the intermediate block partition of $J$ is either (10.2) or (10.3) below:

If $a = 0$, then

$$J = J(0) \cup J(1) \cup \cdots \cup J(L - 1),$$

where $J(m) = [m \cdot 2^{k_{n+1}}, \ldots, (m + 1) \cdot 2^{k_{n+1}} - 1]$ for $0 \leq m \leq L - 1$.

If $a \neq 0$, then

$$J = J(0) \cup J(1) \cup \cdots \cup J(L),$$

where:

- $J(0) = [0, \ldots, a - 1]$
- $J(m) = [a + (m - 1) \cdot 2^{k_{n+1}}, \ldots, a + m \cdot 2^{k_{n+1}} - 1]$ for $1 \leq m \leq L - 1$
- $J(L) = [a + (L - 1) \cdot 2^{k_{n+1}}, \ldots, 2^{k_{n+2}} - 1]$.

Whether the intermediate block partition of $J$ takes the form (10.2) or (10.3), the sub-intervals $J(m) \subset J$ are called the intermediate blocks in $J$.

**Definition 10.4 (Intermediate Blocks in $\tau$).** The intermediate block partition of $\tau$ is either

$$\tau = \tau(0) \cup \tau(1) \cup \cdots \cup \tau(L - 1)$$

or

$$\tau = \tau(0) \cup \tau(1) \cup \cdots \cup \tau(L),$$

depending on whether the intermediate block partition of $J$ takes the form (10.2) or (10.3), respectively. Either way, the sub-templates $\tau(m) \subset \tau$ consist of those levels in $\tau$ that occur in positions from $J(m)$, and are called the intermediate blocks in $\tau$.

**Definition 10.7 (Intermediate Cuts in $\tau$).** The first level in an intermediate block $\tau(m)$ is called an intermediate cut in $\tau$. 


**Definition 10.8** (Local Blocks in $J'$). The *local block partition of* $J'$ is
\[(10.9)\]
\[J' = J'(0) \cup J'(1) \cup \cdots \cup J'(L' - 1),\]
where $J'(i) = [i \cdot 2^{k_n}, \ldots, (i + 1) \cdot 2^{k_n} - 1]$ for $0 \leq i \leq L' - 1$. The sub-intervals $J'(m) \subset J'$, each of which has length $2^{k_n}$, are called the *local blocks in* $J'$.

**Definition 10.10** (Local Blocks within Intermediate Blocks of height $2^{k_{n+1}}$). If $\tau(m) \subset \tau$ is an intermediate block of height $2^{k_{n+1}}$, then the *local block partition of* $\tau(m)$ is
\[(10.11)\]
\[\tau(m) = \tau(m, 0) \cup \tau(m, 1) \cup \cdots \cup \tau(m, L' - 1),\]
where the sub-templates $\tau(m, i)$ consist of those levels in $\tau(m)$ that occur in positions from $J'(i)$, and are called the *local blocks in* $\tau(m)$.

**Definition 10.12.** Given an interval $I = [i, i + 1, \ldots, i + \ell - 1] \subset \mathbb{Z}$ of length $\ell$ and $d \in \{0, \ldots, \ell - 1\}$, denote the subinterval of $I$ consisting of the last $d$ integers in $I$ by $[I]^d$ and the subinterval of $I$ consisting of the first $\ell - d$ integers by $[I]_d$. Namely, $[I]_d = I \setminus [I]^d$.

**Definition 10.13** (Local Blocks in $[J]_a$ and $[J]^a$). The *local block partitions of* $[J]_a$ and $[J]^a$ are
\[(10.14)\]
\[[J]_a = J'(0) \cup J'(1) \cup \cdots \cup J'(L' - c - 2) \cup [J'(L' - c - 1)]_b,\]
and
\[(10.15)\]
\[[J]^a = \begin{cases} [J'(L' - c - 1)]_b \cup J'(L' - c) \cup \cdots \cup J'(L' - 1) & \text{if } a \neq 0 \\ \emptyset & \text{if } a = 0. \end{cases}\]
The sub-intervals in the partitions (10.14) and (10.15) are called the *local blocks in* $[J]_a$ and $[J]^a$.

**Remark 10.16.** If $a = 0$, then $b = c = 0$, so the local block partition (10.14) of $[J]'_0$ is identical to the local block partition (10.9) of $J'$. Also $J' = [J]'_0$. Therefore Definition 10.8 is just a special case of Definition 10.13.

**Definition 10.17** (Local Blocks within Intermediate Blocks of height $< 2^{k_{n+1}}$). Intermediate blocks of height $< 2^{k_{n+1}}$ can only exist if $a \neq 0$; in this case, $\tau(L)$ and $\tau(0)$ are the only two such. Represent $\tau(L)$ with $[J]'_a$ and $\tau(0)$ with $[J]^a$. Then the *local block partitions of* $\tau(L)$ and $\tau(0)$ are
\[(10.18)\]
\[\tau(L) = \tau(L, 0) \cup \tau(L, 1) \cup \cdots \cup \tau(L, L' - c - 2) \cup [\tau(L, L' - c - 1)]_b,\]
and
\[(10.19)\]
\[\tau(0) = [\tau(0, L' - c - 1)]_b \cup \tau(0, L' - c) \cup \cdots \cup \tau(0, L' - 1),\]
where the sub-templates in the partitions (10.18) and (10.19) consist of those levels in $\tau(L)$ and $\tau(0)$ that occur in positions from the corresponding local blocks in $[J]'_a$ and $[J]^a$, and are called the *local blocks in* $\tau(L)$ and $\tau(0)$.

**Definition 10.20** (Local Blocks and Local Cuts in $\tau$). A *local block* in $\tau$ is any local block from Definitions 10.10 or 10.17. The *local block partition of* $\tau$ is the partition of $\tau$ into its local blocks. The first level in a local block is called a *local cut in* $\tau$. 

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10.2. **Block Partitions of Non-Basic Templates at Stage** $n+2$. Recall that each basic template $\tau \in B_{k,n+2}$ has four variation types: diminished, augmented, missing, and extra. And within a given variation type, there may be multiple templates. But any such template is constructed by applying one or both of the following operations to $\tau$:

1. Remove one level from the bottom of a local block in $\tau$.
2. Insert one new level at the top (or bottom) of a local block in $\tau$.

Therefore, if $\tau'$ is a diminished, augmented, missing, or extra version of $\tau$, then the intermediate and local block partitions of $\tau$ determine intermediate and local block partitions of $\tau'$, as follows:

1. Suppose a level, $d$, is removed from the bottom of a local block $B$. Let $C$ be the intermediate block that contains $B$. Then, in the intermediate and local block partitions of $\tau'$, replace $B$ and $C$ with $B \setminus \{d\}$ and $C \setminus \{d\}$. Leave all other intermediate and local blocks alone.
2. Suppose a level, $d$, is inserted at the top (resp., bottom) of a local block $B$. Let $C$ be the intermediate block that contains $B$. Then, in the intermediate and local block partitions of $\tau'$, replace $B$ and $C$ with $B \cup \{d\}$ and $C \cup \{d\}$, where, in the new order, $d$ is the top (resp., bottom) level in $B \cup \{d\}$. Leave all other intermediate and local blocks alone.

11. **The Reordering Maps** $\hat{p}_1 \hat{p}_2$

Recall the definition of a reordering map in Section 3.6. We will employ two types of reordering maps: *global* and *intermediate*. Roughly speaking, the *global* reordering map takes a small number of levels in $\tau$ that occur in positions from the bottom global safe zone in $J$ and moves them, one by one, up to the top global safe zone. This has the effect of sliding all levels in $\tau$ that do not occur in the global safe zones (those in the “middle part”) down. Here is the formal definition:

**Definition 11.1 (Global Reordering).** The *global reordering map* for $\tau$ is a map $\hat{p}_1 = [J,J,J,J,p_1]$ where

1. If $a = 0$, then $p_1$ and therefore $\hat{p}_1$ is the identity.
2. If $a \neq 0$ then $p_1$ is defined so that $\hat{p}_1$ takes the bottom levels in $\tau(1), \tau(2), \ldots, \tau(a)$ and inserts them, in order, directly after the top levels in $\tau(L - a), \tau(L - a + 1), \ldots, \tau(L - 1)$. Observe that this means $\hat{p}_1$ shifts all levels in $\tau$ that do not occur in the global safe zones down by exactly $a$ positions.

More formally when $a \neq 0$, for each intermediate block $J(m)$ in $J$, let $s_m$ and $\ell_m$ denote the smallest and largest integers in $J(m)$, respectively. Then $p_1 = q_a \circ q_{a-1} \circ \cdots \circ q_1$, where, for $1 \leq m \leq a$,

$$q_m(j) = \begin{cases} 
\ell_{L-a+m-1} & \text{if } j = s_m - m + 1 \\
\ell_j & \text{if } s_m - m + 1 < j \leq \ell_{L-a+m-1} \\
\ell_j & \text{if } j < s_m - m + 1 \text{ or } j > \ell_{L-a+m-1}.
\end{cases}$$
Proposition 11.2. The intermediate cuts in \( \hat{p}_1(\tau) \) that do not occur in the global safe zones occur in positions in \( J \) that are congruent to 0 mod \( 2^{k_n+1} \).

This follows immediately from the definitions and implies that the intermediate cuts in \( \hat{p}_1(\tau) \) that do not occur in the global safe zones “line up” with intermediate cuts in the zero template \( \tau^* = \mathcal{P}_{k_n+2}(0) \). This in turn implies that the intermediate blocks in \( \hat{p}_1(\tau) \) that do not occur in the global safe zones also line up with intermediate blocks in the zero template.

11.1. Block Partitions of \( \hat{p}_1(\tau) \). If \( a = 0 \), then the intermediate block partition of \( \tau \) is given by (10.5) and \( \hat{p}_1(\tau) = \tau \). In this case we define the intermediate block partition of \( \hat{p}_1(\tau) \) to be identical to the intermediate block partition of \( \tau \). Formally

\[
\hat{p}_1(\tau) = [\hat{p}_1(\tau)](0) \cup [\hat{p}_1(\tau)](1) \cup \cdots \cup [\hat{p}_1(\tau)](L-1),
\]

where each \([\hat{p}_1(\tau)](m) = \tau(m) \). In this \( a = 0 \) case we define the the local block partition of \( \hat{p}_1(\tau) \) to be identical to the local block partition of \( \tau \) (see (10.11)). Formally, for \( 0 \leq m \leq L-1 \),

\[
[\hat{p}_1(\tau)](m) = [\hat{p}_1(\tau)](m,0) \cup [\hat{p}_1(\tau)](m,1) \cup \cdots \cup [\hat{p}_1(\tau)](m,L'-1)
\]

where each \([\hat{p}_1(\tau)](m,i) = \tau(m,i) \).

Now suppose \( a \neq 0 \), so that the intermediate block partition of \( \tau \) is given by (10.6). Then \( \hat{p}_1 \) takes the bottom levels in \( \tau(1), \tau(2), \ldots, \tau(a) \) and inserts them, in order, directly after the top levels in \( \tau(L-a), \tau(L-a+1), \ldots, \tau(L-1) \). Then we define the intermediate block partition of \( \hat{p}_1(\tau) \) by

\[
\hat{p}_1(\tau) = [\hat{p}_1(\tau)](0) \cup [\hat{p}_1(\tau)](1) \cup \cdots \cup [\hat{p}_1(\tau)](L)
\]

where:

• For \( m \notin \{1, \ldots, a\} \cup \{L-a, \ldots, L-1\} \), \([\hat{p}_1(\tau)](m) = \tau(m) \), and

• For \( m \in \{1, \ldots, a\} \), \([\hat{p}_1(\tau)](m) \) is \( \tau(m) \) with its bottom level removed, and \([\hat{p}_1(\tau)](L-a + m - 1) \) is \( \tau(L-a + m - 1) \) with the bottom level of \( \tau(m) \) inserted at the top.

If \( a \neq 0 \) and \( m \notin \{1, \ldots, a\} \cup \{L-a, \ldots, L-1\} \), then define the local block partition of \([\hat{p}_1(\tau)](m) \) to be identical to the local block partition of \( \tau \). Formally, if \( m \notin \{0,1, \ldots, a\} \cup \{L-a, \ldots, L\} \), then the local block partition of \([\hat{p}_1(\tau)](m) \) is given by (11.4). The local block partitions of \([\hat{p}_1(\tau)](L) \) and \([\hat{p}_1(\tau)](0) \) are

\[
[\hat{p}_1(\tau)](L) = [\hat{p}_1(\tau)](L,0) \cup \cdots \cup [\hat{p}_1(\tau)](L,L'-c-2) \cup [\hat{p}_1(\tau)](L,L'-c-1)
\]

and

\[
[\hat{p}_1(\tau)](0) = [[\hat{p}_1(\tau)](0,L'-c-1)]_b \cup [\hat{p}_1(\tau)](0,L'-c) \cup \cdots \cup [\hat{p}_1(\tau)](0,L')
\]

where \([\hat{p}_1(\tau)](L,i) = \tau(L,i) \) for each \( i \).

Finally, if \( a \neq 0 \) and \( m \in \{1, \ldots, a\} \), then define the local block partitions of \([\hat{p}_1(\tau)](m) \) and \([\hat{p}_1(\tau)](L-a + m - 1) \) by

\[
[\hat{p}_1(\tau)](m) = [\hat{p}_1(\tau)](m,0) \cup [\hat{p}_1(\tau)](m,1) \cup \cdots \cup [\hat{p}_1(\tau)](m,L'-1)
\]
and

\[(11.9) \quad [\hat{p}_1(\tau)](L - a + m - 1) = [\hat{p}_1(\tau)](L - a + m - 1, 0) \cup \cdots \cup [\hat{p}_1(\tau)](L - a + m - 1, L' - 1)\]

where:

- For \(i \neq 0\), \(\hat{p}_1(\tau)(m, i) = \tau(m, i)\),
- For \(i \neq L' - 1\), \(\hat{p}_1(\tau)(L - a + m - 1, i) = \tau(L - a + m - 1, i)\), and
- \(\hat{p}_1(\tau)(m, 0)\) is \(\tau(m, 0)\) with the bottom level removed, and \(\hat{p}_1(\tau)(L - a + m - 1, L' - 1)\) is \(\tau(L - a + m - 1, L' - 1)\) with the bottom level of \(\tau(m, 0)\) inserted at the top.

**Proposition 11.10.** Let \(\hat{p}_1(\tau)(m)\) be an intermediate block in \(\hat{p}_1(\tau)\) that does not occur in a global safe zone. Then the levels in \(\hat{p}_1(\tau)(m)\) occur in the same positions as those in the intermediate block \(\tau^*(\ell)\) in the zero template where

\[
\ell = \begin{cases} 
  m & \text{if } a = 0 \\
  m - 1 & \text{if } a \neq 0.
\end{cases}
\]

Therefore if \(\tau\) is an odometer template then \(\zeta([\hat{p}_1(\tau)](m)) = \zeta(\tau^*(\ell))\). If \(\tau\) is a Morse template then either \(\zeta([\hat{p}_1(\tau)](m)) = \zeta(\tau^*(\ell))\) or \(\zeta([\hat{p}_1(\tau)](m)) = \zeta(\tau^*(\ell))\).

**Proof.** Follows from Proposition \[11.2\]. \(\square\)

### 11.2. The Intermediate Reordering Map \(\hat{p}_2\)

In this section we define a reordering map \(\hat{r}_m\) for each intermediate block \([\hat{p}_1(\tau)](m)\) in \(\hat{p}_1(\tau)\). We then define the intermediate reordering map to be the concatenation, denoted \(\hat{p}_2\), of the maps \(\hat{r}_m\) (either \(\hat{p}_2 = \hat{r}_0 \cdots \hat{r}_{L-1}\) or \(\hat{p}_2 = \hat{r}_0 \cdots \hat{r}_L\)). Here is the formal definition.

**Definition 11.11** (The Intermediate Reordering Map). Depending on whether \(a = 0\) or \(a \neq 0\), define the intermediate reordering map for \(\tau\) to be the concatenation \(\hat{p}_2 = \hat{r}_0 \cdots \hat{r}_{L-1}\) or \(\hat{p}_2 = \hat{r}_0 \cdots \hat{r}_L\), respectively, where each \(\hat{r}_m = [J', J', J', J', r_m]\) is defined as follows. If \([\hat{p}_1(\tau)](m)\) is an intermediate block in \(\hat{p}_1(\tau)\) that occurs in a global safe zone, then \(r_m = \text{identity}\). If \([\hat{p}_1(\tau)](m)\) is an intermediate block in \(\hat{p}_1(\tau)\) that does not occur in a global safe zone then, letting

\[
x = \begin{cases} 
  m & \text{if } a = 0 \\
  m - 1 & \text{if } a \neq 0,
\end{cases}
\]

**Proposition 11.10** determines two cases:

1. If \(\zeta([\hat{p}_1(\tau)](m)) = \zeta(\tau^*(x))\), then \(r_m\) is the identity.
2. If \(\zeta([\hat{p}_1(\tau)](m)) = \zeta(\tau^*(x))\), then \(r_m\) is defined so that \(\hat{r}_m\) takes the bottom levels in \([\hat{p}_1(\tau)](m, 1), \ldots, [\hat{p}_1(\tau)](m, 2^k)\) and inserts them, in order, directly after the top levels in \([\hat{p}_1(\tau)](m, L' - 2^k), \ldots, [\hat{p}_1(\tau)](m, L' - 1)\). This shifts all other local blocks down by exactly \(2^k\) positions. More formally, for \(1 \leq t \leq L'\), let \(s_t\) and
\( \ell_t \) denote the smallest and largest integers in \( J'(t) \), respectively. Then let \( r_m = q_{2^kn} \circ q_{2^{kn-1}} \circ \cdots \circ q_1 \) where, for \( 1 \leq t \leq 2^kn \),

\[
q_t(j) = \begin{cases} 
\ell_{L'-2^kn+t-1} & \text{if } j = s_t - t + 1 \\
1 & \text{if } s_t - t + j \leq \ell_{L-2^kn+t-2} \\
1 & \text{if } j > s_t - t + 1 \text{ or } j > \ell_{L-2^kn+t-2}.
\end{cases}
\]

### 11.3. Block Partitions of \( \hat{\tau} = \hat{p}_2 \circ \hat{p}_1(\tau) \)

Let \( \hat{\tau} = \hat{p}_2 \circ \hat{p}_1(\tau) \) and, depending on whether \( a = 0 \) or \( a \neq 0 \), define the intermediate block partition of \( \hat{\tau} \) to be either

\[
\hat{\tau} = \hat{\tau}(0) \cup \hat{\tau}(1) \cup \cdots \cup \hat{\tau}(L-1) \quad \text{or} \quad \hat{\tau} = \hat{\tau}(0) \cup \hat{\tau}(1) \cup \cdots \cup \hat{\tau}(L)
\]

respectively, where, for each \( m \), \( \hat{\tau}(m) = \hat{r}_m([\hat{p}_1(\tau)](m)) \).

Given an intermediate block \( \hat{\tau}(m) \) in \( \hat{\tau} \), if \( r_m = \text{identity} \), then define the local block partition of \( \hat{\tau}(m) \) to be identical to the local block partition of \([\hat{p}_1(\tau)](m)\). Denote the local blocks in \( \hat{\tau}(m) \) by \( \hat{\tau}(m,i) \), \( [\hat{\tau}(m,i)]_b \), or \( [\hat{\tau}(m,i)]^b \) depending on the form that \( \hat{\tau}(m) = [\hat{p}_1(\tau)](m) \) takes (the various possible forms are described in Section [11.1]).

If \( r_m \neq \text{identity} \), then \( r_m = q_{2^kn} \circ \cdots \circ q_1 \), as in Definition [11.1]. In this case, define the local block partition of \( \hat{\tau}(m) \) to be

\[
\hat{\tau}(m) = \hat{\tau}(m,0) \cup \hat{\tau}(m,1) \cup \cdots \cup \hat{\tau}(m,L'-1)
\]

where:

- \( \hat{\tau}(m,0) \) consists of the \( 2^kn \) consecutive levels in \( \hat{\tau}(m) \) that occur in positions from \( [0, \ldots, 2^kn - 1] \) in \( J' \);
- For \( 1 \leq i \leq 2^kn \), \( \hat{\tau}(m,i) \) consists of the \( 2^kn - 1 \) consecutive levels in \( \hat{\tau}(m) \) that occur in positions from \( [i \cdot 2^kn - i + 1, \ldots, (i + 1) \cdot 2^kn - i - 1] \) in \( J' \);
- For \( 2^kn + 1 \leq i \leq L' - 2^kn - 2 \), \( \hat{\tau}(m,i) \) consists of the \( 2^kn \) consecutive levels in \( \hat{\tau}(m) \) that occur in positions from \( [(i - 1) \cdot 2^kn, \ldots, i \cdot 2^kn - 1] \) in \( J' \);
- For \( L' - 2^kn - 1 \leq i \leq L' - 2 \), \( \hat{\tau}(m,i) \) consists of the \( 2^kn + 1 \) consecutive levels in \( \hat{\tau}(m) \) that occur in positions from \( [(i - 1) \cdot 2^kn + i - (L' - 2^kn - 1), \ldots, i \cdot 2^kn + i - (L' - 2^kn - 1)] \) in \( J' \); and
- \( \hat{\tau}(m,L'-1) \) consists of the \( 2^kn \) consecutive levels in \( \hat{\tau}(m) \) that occur in positions from \( [(L'-1)2^kn, \ldots, 2^{kn+1} - 1] \) in \( J' \).

### 12. The Good Set

All the machinery we have defined up to this point will be used to construct the partial interval bijections from Figure [1]. The global and intermediate reordering maps in particular are defined to guarantee the existence of “good sets”. Recall from the introduction that these are subsets of \( X \) and \( Y \) defined for each stage of the construction on which all partial interval bijections match.
Define the good set in $J'$ to be the subset $G' \subset J'$ given by

$$G' = \bigcup_{t=0}^{L/2 - \frac{2^{kn}}{2} - 1} J'(2^{kn} + 2t).$$

Then $G'$ consists of every other local block in $J'$ that does not occur in the intermediate safe zones. For $0 \leq s \leq L - 1$, define $G(s) = G' + s \cdot 2^{kn+1}$, and

$$G_{n+2} = \bigcup_{s=2^{kn+1}+1}^{L-(2^{kn+1}+2^{kn+1})-1} G(s).$$

Note that $G_{n+2} \subset J$; we call $G_{n+2}$ the good set within $J$ at stage $n + 2$.

For $n$ congruent to 0 mod 4 (2 mod 4) we set $G_{n+2} \subset Q_{kn+2}$ ($H_{n+2} \subset P_{kn+2}$) to consist of those levels in $Q_{kn+2}$ ($P_{kn+2}$) that occur in positions from $G_{n+2} \subset J$. We call $G_{n+2}$ or $H_{n+2}$ the good set at stage $n + 2$. Note that, because the global and intermediate safe zones make up a very small proportion of $J$, $G_{n+2}$ consists of roughly half of $J$. Therefore $\nu(G_{n+2})$ and $\mu(H_{n+2})$ are both approximately $\frac{1}{2}$.

**Proposition 12.1.** Let $\tau_1$ and $\tau_2$ be basic templates in $B_{n+2}(X)$ ($B_{n+2}(Y)$), and let $g \in G_{n+2}$ ($g \in H_{n+2}$). Let $c_1 \in \hat{\tau}_1$ and $c_2 \in \hat{\tau}_2$ be the levels that occur in position $g$ within $\hat{\tau}_1$ and $\hat{\tau}_2$. Then $\pi \circ \zeta(c_1) = \pi \circ \zeta(c_2)$.

**Proof.** The global reordering map was defined so that, after global reordering, the intermediate block structures of all templates line up outside of the global safe zone (see Proposition 11.10). For the odometer system, then, after global reordering, every local block that is not in the global safe zones matches the corresponding local block in the zero template, in the sense that it has the same image under $\pi \circ \zeta$.

For the Morse system, the intermediate reordering map was defined so that, after both global and intermediate reordering, every other local block that is neither in the global nor intermediate safe zones moved down by exactly one complete local block. As a consequence of the combinatoric structure of the Morse system, now every other such local block matches the corresponding local block in the zero template. To see why this is the case, consider the Morse sequence and its flip:

Morse sequence = \[\begin{array}{cccccccc}
0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
\end{array}\]

Flip = \[\begin{array}{cccccccc}
.1 & 0 & 1 & 1 & 0 & 1 & 1 \\
\end{array}\]

Notice that, if we shift the flip to the left by one coordinate, then it matches the Morse sequence in every other coordinate. The same would be true for sequences of substitution blocks (just replace 0’s and 1’s with blocks $\theta^n(0)$ and $\theta^n(1)$).

**Proposition 12.2.** Let $n < m$ be natural numbers congruent to 0 mod 4 (2 mod 4). Then $G_n$ and $G_m$ ($H_n$ and $H_m$) are independent events in $Y$ ($X$).

**Proof.** We only give the proof in the case where $n$ and $m$ are congruent to 0 mod 4. The other case follows similarly. Since $n < m$, we have $2^{kn} = p \cdot 2^{kn}$, where $p = 2^{kn}$. Then
\( \mathcal{Q}_{kn} \) is partitioned into \( p \) “copies” of \( \mathcal{Q}_{kn} \):

\[
\mathcal{Q}_{km} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \cdots \cup \mathcal{L}_p,
\]

where each \( \mathcal{L}_i \) is a union of exactly \( 2^k \) consecutive levels in \( \mathcal{Q}_{kn} \), and \( \pi(\mathcal{L}_i) = \mathcal{Q}_{kn} \) for each \( i \). Because \( \mathcal{G}_m \) is a union of local blocks from stage \( m - 2 \) and \( n < m - 2 \), we can write \( \mathcal{G}_m \) as a disjoint union

\[
\mathcal{G}_m = \bigcup_{i \in I} \mathcal{L}_i
\]

for some \( I \subset \{1, \ldots, p\} \). Because \((12.3)\) is obtained by cutting and stacking \( \mathcal{Q}_{kn} \) \( p \) times, we have \( \nu(\mathcal{G}_m) = \nu(\mathcal{G}_n) \) and therefore \( \nu(\mathcal{G}_n \cap \mathcal{L}_i) = \nu(\mathcal{L}_i) \nu(\mathcal{G}_n) \) for each \( i \). Since \((12.4)\) is a disjoint union,

\[
\mu(\mathcal{G}_n|\mathcal{G}_m) = \frac{\nu(\mathcal{G}_n \cap \mathcal{G}_m)}{\nu(\mathcal{G}_m)} = \frac{1}{\nu(\mathcal{G}_m)} \sum_{i \in I} \nu(\mathcal{G}_n \cap \mathcal{L}_i) = \frac{1}{\nu(\mathcal{G}_m)} \sum_{i \in I} \nu(\mathcal{L}_i) \nu(\mathcal{G}_n) = \nu(\mathcal{G}_n).
\]

Therefore \( \mathcal{G}_m \) and \( \mathcal{G}_n \) are independent.

\[\square\]

13. Stage 4 of the Induction

13.1. Heads and Tails. Suppose \( \tau \in B_{k4}(X) \) and \( b \neq 0 \). Then the local block partitions of \( \hat{\tau}(0) \) and \( \hat{\tau}(L) \) are

\[
\hat{\tau}(0) = \hat{\tau}(0, L' - c - 1)b \cup \hat{\tau}(0, L' - c) \cup \cdots \hat{\tau}(0, L' - 1)
\]

and

\[
\hat{\tau}(L) = \hat{\tau}(L, 0) \cup \cdots \hat{\tau}(L, L' - c - 2) \cup [\hat{\tau}(L, L' - c - 1)]_b,
\]

as described Section 11.3, and in \((11.7)\) and \((11.6)\). Notice that the very first local block in \( \hat{\tau} \) has height \( b \neq 2^k \), i.e., it is a partial block. Correspondingly, the bottom \( b \) levels in \( \mathcal{Q}_{k4} \) can be thought of as the top \( b \) levels in a collection of templates \( \omega \in \Omega_{k2} \). Define

\[
\tau_{\text{tail}} = \{ \omega \in \Omega_{k2} : \text{the top } b \text{ levels in } \omega \text{ are the bottom } b \text{ levels in } \mathcal{Q}_{k4} \}.
\]

Similarly, define

\[
\tau_{\text{head}} = \{ \omega \in \Omega_{k2} : \text{the bottom } 2^k - b \text{ levels in } \omega \text{ are top } 2^k - b \text{ levels in } \mathcal{Q}_{k4} \}.
\]

If \( b = 0 \), then define \( \tau_{\text{tail}} = \tau_{\text{head}} = \emptyset \).

13.2. Maps to Basic Templates. Given \( \tau \in B_{k4}(X) \), in this section we define a collection of partial interval bijections of the form

\[
\phi_\tau = \phi_{\tau}^{\text{tail}} \ast \phi_{\tau}^{\text{body}} \ast \phi_{\tau}^{\text{head}}.
\]

There will be one such partial interval bijection for each pair \( (\omega_1, \omega_2) \in \tau_{\text{tail}} \times \tau_{\text{head}} \). We call \( \phi_{\tau}^{\text{tail}} \) and \( \phi_{\tau}^{\text{head}} \) the bottom and top sticky notes of \( \phi_\tau \). If \( b = 0 \), then each \( \phi_\tau \) is defined on an interval of length \( 2^k \). However, if \( b \neq 0 \), then each \( \phi_\tau \) is defined on an interval of length approximately \( 2^k + 2^k \).
13.2.1. **Sticky Notes** $\phi^\text{tail}_\tau$ and $\phi^\text{head}_\tau$. If $b = 0$, then define just one top sticky note and just one bottom sticky note, namely, the trivial partial interval bijection $[I, J, A, B, f]$ where $I = J = \emptyset$.

If $b \neq 0$, then, for each $\omega \in \tau_{\text{tail}}$, define $\phi^\text{tail}_\tau = \phi^-_\omega$. To conserve notation, since the particular choice of $\omega$ will not matter for our construction, the expression $\phi^\text{tail}_\tau$ does not indicate dependence on $\omega$.

Similarly, for each $\omega \in \tau_{\text{head}}$, define $\phi^\text{head}_\tau = \phi^-_\omega$.

13.2.2. **The Body Map** $\phi^\text{body}_\tau$. The local block partition of $\tau$ determines a partition of $J = [0, 1, \ldots, 2^{k_4} - 1]$ into subintervals, which in turn determines a partition of $Q_{k_4}$, which we call the local block partition of $Q_{k_4}$. The local block partition of $Q_{k_4}$ has the form

$$
\omega_1 \cup \omega_2 \cup \cdots \cup \omega_{LL'} \quad \text{if } b = 0
$$

$$
\omega_1 \cup \omega_2 \cup \cdots \cup \omega_{LL'+1} \quad \text{if } b \neq 0,
$$

where the $\omega_i$ are consecutively occurring sets of levels in $Q_{k_4}$. If $b = 0$, then, for each $i$, $\pi(\omega_i)$ is a template from $\Omega_{k_2} \cup \Omega_{k_2}$. In this case, define

$$
\phi^\text{body}_\tau = \phi^-_{\omega_1} \ast \phi^-_{\omega_2} \ast \cdots \ast \phi^-_{\omega_{LL'}}.
$$

If $b \neq 0$, then for $2 \leq i \leq LL'$, $\pi(\omega_i)$ is a template from $\Omega_{k_2} \cup \Omega_{k_2}$. In this case, define

$$
\phi^\text{body}_\tau = \phi^-_{\omega_2} \ast \phi^-_{\omega_3} \ast \cdots \ast \phi^-_{\omega_{LL'}}.
$$

**Lemma 13.2.** Let $c \in G_4$ be a level in $Q_{k_4}$ from the good set at stage 4. Then, given $\tau_1$ and $\tau_2$ in $B_{k_4}(X)$, either both $\phi^\text{body}_{\tau_1}$ and $\phi^\text{body}_{\tau_2}$ are undefined on $c$, or else $\pi \circ \zeta \circ \phi^\text{body}_{\tau_1}(c) = \pi \circ \zeta \circ \phi^\text{body}_{\tau_2}(c)$.

**Proof.** By construction, together with Proposition 12.1.

It follows from Lemma 13.2 that it is possible to extend the domain of definition of $\phi^\text{body}_\tau$ to a partial interval bijection, call it $\phi^\text{body}_\tau$, between the set of all levels in $G_4$ and those levels in $\tau$ that occur in positions from $G_4$. Moreover, the same bijection can be used for all reordered templates $\tau$ so that, given $\tau_1$ and $\tau_2$ in $B_{k_4}(X)$ and $c \in G_4$, $\pi \circ \zeta \circ \phi^\text{body}_{\tau_1}(c) = \pi \circ \zeta \circ \phi^\text{body}_{\tau_2}(c)$.

Finally, define

$$
\phi^\text{body}_\tau = \begin{cases} (p_2 \circ p_1)^{-1} \circ \phi^\text{body}_{\tau} & \text{if } b = 0 \\ (p_2 \circ p_1)^{-1} \circ \phi^\text{body}_{\tau} & \text{if } b \neq 0, \end{cases}
$$

where $p_2 \circ p_1$ denotes the restriction of $p_2 \circ p_1$ to the subinterval of $J \subset J$ corresponding to $\omega_2 \cup \cdots \cup \omega_{LL'}$. (Note that $p_2 \circ p_1$ is well-defined because $p_2 \circ p_1$ is the identity outside of $J$.)'
Proposition 13.3. Neither the bottom level nor the top level of $Q_{k4}$ is in the domain of any $\phi_\tau$.

Proof. Given $\phi_\tau$ of the form $\phi_\tau = \phi_{\tau}^{\text{tail}} \circ \phi_{\tau}^{\text{body}} \circ \phi_{\tau}^{\text{head}}$ determined by $(\omega_1, \omega_2) \in \tau_{\text{tail}} \times \tau_{\text{head}}$, the bottom level in $Q_{k4}$ occurs as the cut in $\omega_1$. But by Proposition 13.2, this level is not in the range of the stage 2 map $\phi_{\omega_1}$. Therefore it is not in the domain of $\phi_{\tau}^{\text{tail}}$.

Proposition 13.4. Given $\tau \in B_{k4}(X)$, the global cut in $\tau$ is not in the range of any $\phi_\tau$.

Proof. Follows from Proposition 7.3.

Proposition 13.5. Let $c \in G_4$. Then, given $\tau_1$ and $\tau_2$ in $B_{k4}(X)$ and partial interval bijections $\phi_{\tau_1}$ and $\phi_{\tau_2}$ of the form (13.1), both $\phi_{\tau_1}$ and $\phi_{\tau_2}$ are defined on $c$, and $\pi \circ \zeta \circ \phi_{\tau_1}(c) = \pi \circ \phi_{\tau_2}(c)$.

Proof. Follows from Lemma 13.2 together with the definition of the partial interval bijections $\phi_\tau$.

13.3. Partial Interval Bijections. We now define maps to diminished, augmented, missing, and extra templates, and establish analogues of Lemmas 7.3 and 7.6, which are needed to establish analogues of Propositions 14.3 and 14.4 in stage 8.

13.3.1. Maps to Diminished Templates. Given $\tau \in B_{k4}(X)$ that is neither the zero-template nor the one-template, define $\phi_{\tau}^{-(u,i)}$ to match with $\phi_\tau$, and define $\phi_{\tau}^{-(d,i)}$ to match with $\phi_{\tau_\rho}$.

If $\tau \in B_{k4}(X)$ is either the zero-template or the one-template, then for $i \in \{0, 1\}$, define

$\phi_{\tau}^{-(u,i)}$ to match with $\phi_\tau$, and define $\phi_{\tau}^{-(d,i)}$ to match with $\phi_{\tau_\rho}$.

Lemma 13.6. If $\tau \in B_{k4}(X)$ is neither the zero-template nor the one-template, then $\phi_{\tau}^{-(d,i)}$ matches with $\phi_{\tau_\rho}^{-(u,i)}$. If $\tau \in B_{k4}(X)$ is either the zero-template or the one-template, then for $i \in \{0, 1\}$, $\phi_{\tau}^{-(d,i)}$ matches with $\phi_{\tau_\rho}^{-(i)(u)}$.

Proof. By construction.

13.3.2. Maps to Augmented Templates. Given $\tau \in B_{k4}(X)$ and $i \in \{0, 1\}$, define $\phi_{\tau}^{+(d,i)}$ to match with $\phi_\tau$. If $\tau$ is neither the zero-template nor the one-template, then define $\phi_{\tau}^{+(u,i)}$ to match with $\phi_{\tau_\rho}$. If $\tau$ is the zero-template or the one-template, then define $\phi_{\tau}^{+(u,i)}$ to match with $\phi_{\tau_\rho(i)}$.

Lemma 13.7. Given $i \in \{0, 1\}$, and $\tau \in B_{k4}(X)$, if $\tau$ is neither the zero-template nor the one-template, then $\phi_{\tau}^{+(u,i)}$ matches with $\phi_{\tau_\rho}^{+(d,i)}$. If $\tau$ is the zero-template or the one-template, then $\phi_{\tau}^{+(u,i)}$ matches with $\phi_{\tau_\rho(i)}^{+(d,i)}$.

Proof. By construction.
13.3.3. Maps to Missing and Extra Templates. Given $\tau^m \in \mathcal{M}_{k_4}(Y)$, define $\tau^m_{\text{head}} = \tau_{\text{head}}$. Note that, if $\omega \in \tau^m_{\text{head}}$, then $\omega^-(u) \in \tau^m_{\text{head}}$. Define

$$
\tau^m_{\text{tail}} = \{\omega \in \Omega_{k_2} : \text{the top } b - 1 \text{ levels in } \omega \text{ are the bottom } b - 1 \text{ levels in } Q^m_{k_4}\}.
$$

Note that $\tau_{\text{tail}} \subseteq \tau^m_{\text{tail}}$, but $\tau_{\text{tail}} \neq \tau^m_{\text{tail}}$ since, in particular, if $\omega \in \tau_{\text{tail}}$, then $\omega^-(d) \in \tau^m_{\text{tail}}$, but $\omega^-(d) \notin \tau_{\text{tail}}$.

Define $\phi_{\tau^m} = \phi_{\tau_{\text{tail}}} \circ \phi_{\tau_{\text{body}}} \circ \phi_{\tau_{\text{head}}}$ as follows. For each $\omega \in \tau^m_{\text{head}}$, define $\phi_{\tau_{\text{head}}} = \phi^{-1}$, and, for each $\omega \in \tau^m_{\text{tail}}$, define $\phi_{\tau_{\text{tail}}} = \phi^{-1}$. Define $\phi_{\tau^m} = \phi_{\tau_{\text{body}}}$. Similarly, given $\tau \in \mathcal{B}_{k_4}(X)$ that is neither the zero-template nor the one-template, define $\tau_{\text{tail}} = \tau_{\text{tail}}$. Note that, if $\omega \in \tau_{\text{tail}}$, then $\omega^+(d) \in \tau_{\text{tail}}$. Define

$$
\tau^e_{\text{head}} = \{\omega \in \Omega_{k_2} : \text{the bottom } 2^{k_2} - b + 1 \text{ levels in } \omega \text{ are the top } 2^{k_2} - b + 1 \text{ levels in } Q^e_{k_4}\}.
$$

Note that, if $\omega \in \tau_{\text{head}}$, then $\omega^+(u) \in \tau^e_{\text{head}}$.

14. Stage 6 of the Induction

14.1. Reordering Maps, Block Partitions, the Good Set, and Heads and Tails. Stage 6 is analogous to stage 4, but with one new layer of complexity: Whereas the maps in stage 4 were defined as concatenations of maps from stage 2, the maps in stage 6 will be overlapping concatenations of the maps from stage 4. The top and bottom sticky notes defined in stage 4 are used to glue these overlapping concatenations together. We need to verify that there are enough sticky notes defined in stage 4 to choose from so that these overlapping concatenations are well-defined.

The global reordering map in stage 6, denoted $\hat{p}_1$, is the same as it was in stage 4 except, of course, now $J = [0, 1, \ldots, 2^{k_6} - 1]$. In stage 4, an intermediate reordering map was also used to ensure that every other local block that does not occur in a safe zone has the same image under $\pi \circ \zeta$ as the corresponding local block in the zero-template (see Proposition 12.1). But an intermediate reordering map is not needed in stage 6 because the odometer has just one canonical tower at each stage—not two. So in stage 6, let the intermediate reordering map $\hat{p}_2$ be simply the identity. Given $\omega \in \mathcal{B}_{k_6}(Y)$, let $\hat{\omega} = \hat{p}_2 \circ \hat{p}_1(\omega)$.

Define the intermediate and local block partitions of $\hat{p}_1(\omega)$, as well as the intermediate block partition of $\hat{\omega}$, in exactly the same way that they were defined in stage 4 (see Sections 11.1 and 11.3) except, of course, that $\tau$ is replaced with $\omega$. Given an intermediate block $\hat{\omega}(m)$ in $\omega$, because the intermediate reordering map $\hat{p}_2$ is simply the identity, just like in stage 4, define the local block partition of $\hat{\omega}(m)$ to be identical to the local block partition of $[\hat{p}_1(\omega)](m)$.

The good set within $J$ at stage 6, $G_6$, is defined in Section 12. Let $G_0$, called the good set at stage 6, consist of those levels in $P_{k_6}$ that occur in positions from $G_6$. Note that Proposition 12.1 holds in stage 6.

Given $\omega \in \mathcal{B}_{k_6}(Y)$, the sets $\omega_{\text{tail}}$ and $\omega_{\text{head}}$ are defined analogously to the definitions in Section 13.1 (just replace $\tau$, $\omega$, $k_2$, and $k_4$ with $\omega$, $\tau$, $k_4$, and $k_6$, respectively). However,
there is an added layer of complexity in stage 6: If \( \tau \in \omega_{head} \) (or \( \tau \in \omega_{tail} \)), then \( \tau \) has head and tail sets of its own, \( \tau_{head} \) and \( \tau_{tail} \).

14.2. Maps to Basic Templates. Much like in stage 6, given \( \omega \in B_{k_4}(Y) \), the partial interval bijections \( \phi_\omega \) take the form

\[
(14.1) \quad \phi_\omega = \phi_\omega^{head} \ast \phi_\omega^{body} \ast \phi_\omega^{tail},
\]

where \( \ast \) represents overlapping concatenation as defined in Definition 3.6. There is one such partial interval bijection for each pair \((\tau_1, \tau_2) \in \omega_{tail} \times \omega_{head}\). The maps \( \phi_\omega^{tail} \), \( \phi_\omega^{body} \), and \( \phi_\omega^{head} \) are defined analogously to the definitions in Sections 13.2.1 and 13.2.2 except that within the body map \( \phi_\omega^{body} \) (Section 13.2.2), concatenations \((\ast)\) are replaced with overlapping concatenations \((\tilde{\ast})\). We now show that the overlapping concatenations within \( \phi_\omega^{body} \) can be glued together. Suppose \( b \neq 0 \) (the case that \( b = 0 \) is nearly identical). Then, following what was done in Section 13.2.2, we define \( \phi_\omega^{body} \) to take the form

\[
(14.2) \quad \tilde{\phi}_\omega^{body} = \phi_{\tau_2}^{-1} \tilde{\ast} \phi_{\tau_4}^{-1} \tilde{\ast} \cdots \tilde{\ast} \phi_{\tau_L}^{-1}.
\]

Proposition 14.3. With the templates \( \tau_i \) from (14.2) listed in order of overlapping concatenation \((\tau_2 \prec \tau_3 \prec \cdots \prec \tau_{LL'})\), the following six types of successive pairs can occur:

1. \( \tau \prec \tau' \) where \( \tau, \tau' \in \{P_{k_4}(0), P_{k_4}(1)\} \),
2. \( \tau \prec \tau \) where \( \tau \in B_{k_4}(X) \setminus \{P_{k_4}(0), P_{k_4}(1)\} \),
3. \( \tau^{(i)} \prec \tau \), where \( i \in \{0, 1\} \) and \( \tau = P_{k_4}(i) \),
4. \( \tau^{(i)} \prec \tau^{(m)} \), where \( \tau \in B_{k_4}(X) \setminus \{P_{k_4}(0), P_{k_4}(1)\} \),
5. \( \tau \prec \tau^{(i)} \), where \( i, j \in \{0, 1\} \) and \( \tau = P_{k_4}(i) \), and
6. \( \tau^e \prec \tau^{(i)} \), where \( \tau \in B_{k_4}(X) \setminus \{P_{k_4}(0), P_{k_4}(1)\} \).

Proof. By construction. \( \square \)

Proposition 14.4. In any of the six cases of Proposition 14.3, top and bottom sticky notes can be chosen so that the overlapping concatenation of the corresponding partial interval bijections \( \phi^{-1}_\tau \) are well-defined.

Proof. Case 1 is trivial because partial interval bijections to the zero- and one-templates do not have sticky notes (there is no overlap to worry about). For case 2, observe that if \( \omega \in B_{k_4}(Y) \) is a basic template from stage 2 such that \( \omega \in \tau_{tail} \), then \( \omega \in \tau_{head} \) as well, so we can glue \( \phi^{-1}_\omega \) together with itself by picking such \( \phi^{-1}_\omega \) on the overlap. For case 3, observe that there exists \( \omega \in B_{k_4}(Y) \) such that \( \phi^{-1}_\omega(d) \in \phi^{-1}_\omega^{tail} \) and \( \phi^{-1}_\omega(u) \in \phi^{-1}_\omega^{head} \). Lemma 7.5 then guarantees that there is a bottom sticky note on \( \phi_\tau \) that matches with a top sticky note on \( \phi^{-1}_\tau \). Cases 4-6 are similar. \( \square \)

The following shows that (14.1) is well defined.

Proposition 14.5. Top and bottom sticky notes can be chosen so that the overlapping concatenations \( \phi_\omega^{tail} \ast \phi_\omega^{body} \) and \( \phi_\omega^{body} \ast \phi_\omega^{head} \) are well-defined.
Proof. Similar to the proof of Proposition 14.4. □

The following propositions are analogous to propositions from stage 4.

Proposition 14.6. Neither the bottom level nor the top level of $P_{k_6}(0)$ or $P_{k_6}(1)$ is in the domain of any $\phi_\omega$.

Proof. Follows from Proposition 13.4. □

Proposition 14.7. Given $\omega \in B_{k_6}(Y)$, the global cut in $\omega$ is not in the range of any $\phi_\omega$.

Proof. Follows from Proposition 13.3. □

Proposition 14.8. Let $g \in G_6$ and let $c \in P_{k_6}(i)$ be a level that occurs in position $g$, where $i \in \{0, 1\}$. Then, given $\omega_1$ and $\omega_2$ in $B_{k_6}(Y)$ and partial interval bijections $\phi_{\omega_1}$ and $\phi_{\omega_2}$ of the form (14.2), both $\phi_{\omega_1}$ and $\phi_{\omega_2}$ are defined on $c$, and $\pi \circ \zeta \circ \phi_{\omega_1}(c) = \pi \circ \zeta \circ \phi_{\omega_2}(c)$.

Proof. Proposition 12.1 can be used to show that Lemma 13.2 holds in stage 6 (with appropriate notational modifications). The proposition follows. □

14.3. Maps to Diminished, Augmented, Missing, and Extra Templates. The definitions of these modified maps are analogous to those in stage 4. Maps to missing and extra templates are needed to define the maps at stage 8 in places where individual levels have been deleted or inserted. Analogues of Lemmas 13.6 and 13.7 hold in stage 6, and are used to glue sticky notes together in stage 10.

15. Completing the induction

The essential components of the induction have now been established. Stages 8, 12, 16, ... are analogous to stage 4, while stages 10, 14, 18, ... are analogous to stage 6. In each stage $n \geq 8$, concatenations of the (inverses of) the partial interval bijections at stage $n - 2$ are glued together using Definition 3.8.

16. Convergence of Partial Interval Bijections

Given a level $c \in G_4$, Proposition 13.5 guarantees that $\pi \circ \zeta \circ \phi_{\omega}(c)$ is a level in $P_{k_2}$ that does not depend on which $\tau \in B_{k_4}$ is used. Moreover, Proposition 13.5 holds in every stage $n \equiv 0 \mod 4$, which permits us to define $\phi^\nu : Q_{k_n} \to P_{k_{n-2}}$ to be the restriction of $\pi \circ \zeta \circ \phi_{\omega}$ to $G_{k_n}$. Similarly, for $n \equiv 2 \mod 4$, we can define $\phi^\eta : P_{k_n} \to Q_{k_{n-2}}$ to be the restriction of $\pi \circ \zeta \circ \phi_{\omega}$ to $G_{k_n}$.

Given $n \equiv 0 \mod 4$ and $y \in Y$, let $d_{k_n}(y)$ denote the unique $k_n$-canonical cylinder in $Y$ that contains $y$. Let $H \subset Y$ be the set of $y \in Y$ such that $d_{k_n}(y) \in G_n$ for infinitely many $n$.

Similarly, given $n \equiv 2 \mod 4$ and $x \in X$, let $c_{k_n}(x)$ denote the unique $k_n$-canonical cylinder in $X$ that contains $x$. Let $G \subset X$ be the set of $x \in X$ such that $c_{k_n}(x) \in G_n$ for infinitely many $n$.

Theorem 16.1. The sets $G$ and $H$ are $G_\delta$ sets of full measure.
Proof. By Proposition 12.2, the sets $G_n$ for $n \equiv 0 \mod 4$ are independent with respect to $\nu$. Therefore $\nu(G) = 1$ by the Borel-Cantelli Lemma. Moreover, the sets $G_n$ are open, so $G$ is a $G_\delta$ subset of $Y$. The argument for $H$ is similar. \hfill \square

**Lemma 16.2.** Given $t > 0$, if $n \equiv 0 \mod 4$ and $d \in G_n$ and $d' \in G_{n+4t}$ are levels such that $d' \subset d$, then $\phi^{n+4t}(d') \subset \phi^n(d)$. Also, if $n \equiv 2 \mod 4$, $c \in G_n$, $c' \in G_{n+4t}$, and $c' \subset c$, then $\phi^{n+4t}(c') \subset \phi^n(c)$.

Proof. Given $\tau \in T_{k_{n+4t}} \cup \tilde{T}_{k_{n+4t}}$, the map $\phi_\tau$ at stage $n + 4t$ is an extension of a concatenation of the maps $\phi_\tau$ at stage $n$. It follows that

$$\pi \circ \phi^{n+4t}(d') = \phi^n \circ \pi(d') = \phi^n(d),$$

where, depending on the context, $\pi$ refers either to the map $\pi : P_{k_{n+4t+2}} \to P_{k_{n-2}}$ or to the map $\pi : Q_{k_{n+4t}} \to Q_{kn}$. The argument when $n \equiv 2 \mod 4$ is nearly identical. \hfill \square

Given $y \in H$, let $(n(i))_{i \in \mathbb{N}}$ be the increasing sequence of indices, each congruent to 0 mod 4, such that each $d_{k_{n(i)}}(y) \in G_{k_{n(i)}}$. Then, by Lemma 16.2, the levels $(\phi^{n(i)}(d_{k_{n(i)}}(y)))_{i \in \mathbb{N}}$ form a nested sequence. It follows that there is a unique point in the intersection $\bigcap_{i \in \mathbb{N}} \phi^{n(i)}(d_{k_{n(i)}}(y))$.

Similarly, if $(m(i))_{i \in \mathbb{N}}$ is the analogous sequence of indices for $x \in G$, then there is a unique point in the intersection $\bigcap_{i \in \mathbb{N}} \phi^{m(i)}(c_{k_{n(i)}}(x))$. This permits the following definition.

**Definition 16.3.** Given $x \in G$, let $\phi(x)$ be the unique point in the intersection $\bigcap_{i \in \mathbb{N}} \phi^{m(i)}(c_{k_{n(i)}}(x))$.

Given $y \in H$, let $\psi(y)$ be the unique point in the intersection $\bigcap_{i \in \mathbb{N}} \phi^{n(i)}(d_{k_{n(i)}}(y))$.

**Theorem 16.4.** The maps $\phi$ and $\psi$ are continuous in the relative topologies on $G$ and $H$.

Proof. Let $x \in G$ and $\varepsilon > 0$. Choose $n \equiv 2 \mod 4$ large enough so that $1/2^{k_{n-2}} < \varepsilon$ and such that $x \in G_{k_n}$. Choose $\delta > 0$ small enough so that $\rho(x, x') < \delta$ implies $c_{k_n}(x) = c_{k_n}(x')$. In particular, $\rho(x, x') < \delta$ implies $x' \in G_{k_n}$. It then follows from Lemma 16.2 that, for all $t > 0$ such that $c_{k_n+4t}(x) \in G_{k_{n+4t}}$,

$$\phi^{n+4t}(c_{k_n+4t}(x)) \subset \phi^n(c_{k_n}(x))$$

and, for all $i > 0$ such that $c_{k_{n+4t}}(x') \in G_{k_{n+4t}}$,

$$\phi^{n+4t}(c_{k_{n+4t}}(x')) \subset \phi^n(c_{k_n}(x')) = \phi^n(c_{k_n}(x)).$$

Therefore

$$\rho(\phi(x), \phi(x')) < \frac{1}{2^{k_{n-2}}} < \varepsilon.$$

The continuity of $\psi$ is proved similarly. \hfill \square

**Theorem 16.5.** The maps $\phi$ and $\psi$ are measure preserving.
Proof. Fix a natural number \( m \equiv 0 \mod 4 \) and a level \( d \) in \( \mathcal{Q}_{km} \). We wish to show that \( \mu(\phi^{-1}(d)) = \nu(d) = 1/2^{km} \). Given \( n > 0 \), let

\[
J_n(d) = \{ \text{levels } c \in \mathcal{P}_{m+4n-2} : \zeta \circ \phi_\omega(c) = d \ \forall \omega \in \Omega_{m+4n-2} \cup \tilde{\Omega}_{m+4n-2} \},
\]

where \( \zeta \) is the map \( \Omega_{m+4n-2} \cup \tilde{\Omega}_{m+4n-2} \to \mathcal{Q}_{km} \).

Let \( D_n = \mathcal{G}_{m+4n-2} \) and \( E_n = D_n \cap J_n(d) \). Observe that

\[
|E_n| \geq 2^{km+4n-2}/3 + 2^{km+4n-2}/3 = 2 \cdot 2^{km+4n-2}/3.
\]

(16.7)

For \( 1 \leq a \leq n - 1 \), let \( \tilde{D}_{n-a} \) denote the set of levels in \( \mathcal{P}_{km+4(n-a)-2} \) that are contained in levels from \( \mathcal{G}_{km+4(n-a)-2} \), and recursively define \( D_{n-a} = \tilde{D}_{n-a} \setminus \bigcup_{a' = 0}^{a-1} D_{n-a'} \) and \( E_{n-a} = D_{n-a} \cap J_n(d) \). Observe that

\[
|E_{n-a}| = 2^{km}
\]

and

(16.8)

(16.9)

(Again, 1/3 is a conservative lower bound—it is actually closer to 1/2). It follows from (16.7) and (16.9) that

\[
\sum_{a = 0}^{n-1} |D_{n-a}| \geq 2 \cdot 2^{km+4n-2} \left( 1 - \left( \frac{2}{3} \right)^n \right).
\]

It now follows from (16.6) and (16.8) that

\[
|J_n(d)| \geq \sum_{a = 0}^{n-1} |E_{n-a}| = 2^{km} \sum_{a = 0}^{n-1} |D_{n-a}| \geq 2 \cdot 2^{km+4n-2} \cdot \frac{1 - (\frac{2}{3})^n}{2^{km}}.
\]

This implies that

\[
\mu(J_n(d)) \geq \frac{1}{2^{km}} - \left( \frac{1}{2^{km}} \right) \cdot \left( \frac{2}{3} \right)^n.
\]

Letting \( n \to \infty \) gives \( \mu(\phi^{-1}(d)) \geq \nu(d) \). This being true for each level \( d \in \mathcal{Q}_{km} \) trivially implies that \( \mu(\phi^{-1}(d)) = \nu(d) \).

\[ \square \]
Theorem 16.10. If \( x \in \mathcal{G} \) and \( \phi(x) \in \mathcal{H} \), then \( \psi(\phi(x)) = x \). Similarly, if \( y \in \mathcal{H} \) and \( \psi(y) \in \mathcal{G} \), then \( \phi(\psi(y)) = y \).

Proof. Let \( \varepsilon > 0 \) and let \( n \equiv 2 \mod 4 \) be such that \( \phi(x) \in \mathcal{H}_{k_n+2} \) and \( 1/2^{k_n} < \varepsilon \). Let \( t \geq 1 \) be such that \( x \in \mathcal{G}_{k_n+4t} \). Then

\[
\phi^{n+2} \circ \pi \circ \phi^{n+4t}(c_{k_n+4t}(x)) = \pi(c_{k_n+4t}(x)) = c_{k_n}(x)
\]

where, depending on the context, \( \pi \) refers either to the map \( \pi : Q_{k_n+4t-2} \to Q_{k_n+2} \) or to the map \( \pi : P_{k_n+4t} \to P_{k_n} \). Since \( \phi(x) \in \phi^{n+4t}(c_{k_n+4t}(x)) \), we have \( \phi(x) \in \pi \circ \phi^{n+4t}(c_{k_n+4t}(x)) \), which implies \( \psi(\phi(x)) \in c_{k_n}(x) \). Therefore \( \rho(x, \psi(\phi(x))) < 1/2^{k_n} < \varepsilon \). The second statement is proved similarly. \( \square \)

17. Kakutani Equivalence

Let

\[
X_1 = \bigcap_{i \in \mathbb{Z}} T^i(\mathcal{G}) \quad \text{and} \quad Y_1 = \bigcap_{i \in \mathbb{Z}} S^i(\mathcal{H}).
\]

Then \( X_1 \) and \( Y_1 \) are invariant \( G_\delta \) subsets of full measure. Let \( X_0 = X_1 \cap \phi^{-1}(Y_1) \) and \( Y_0 = \phi(X_0) \). Then \( X_0 \) and \( Y_0 \) are full measure subsets because \( \phi \) is measure preserving. And \( X_0 \) and \( Y_0 \) are \( G_\delta \) subsets because \( \phi \) is continuous in the relative topology on \( X_1 \).

In this section we show that \( X_0 \) and \( Y_0 \) are invariant and that \( \phi : X_0 \to Y_0 \) is an orbit equivalence that is a conjugacy when restricted to \( \mathcal{G}_2 \), the good set at stage 2.

Lemma 17.1. For \( x \in X_1 \), \( \phi \) maps the \( T \)-orbit of \( x \) into the \( S \)-orbit of \( \phi(x) \). And for \( y \in Y_1 \), \( \psi \) maps the \( S \)-orbit of \( y \) into the \( T \)-orbit of \( \psi(y) \).

Proof. Let \( x = x_1 \in X_1 \) and \( x_2 = T^{-r}(x_1) \) for some \( r > 0 \). Recall that the bottom global safe zone at stage \( n \) has height \( h(n) := 2^{k_{n-1}} + 2^{k_{n-1}} \cdot 2^{k_{n-1}} \). Pick \( n_0 \equiv 2 \mod 4 \) such that \( h(n_0) > r \) and \( x_1 \in \mathcal{G}_{k_{n_0}} \). Then \( x_1 \) and \( x_2 \) are in the same tower in \( P_{k_{n_0}} \) and \( c_{k_{n_0}}(x_2) = T^{-r}(c_{k_{n_0}}(x_1)) \). Moreover, since each \( \mathcal{G}_{k_n} \) consists of complete local blocks from stage \( n-2 \), for each \( n > n_0 \) \((n \equiv 2 \mod 4)\) such that \( x_1 \in \mathcal{G}_{k_n} \), we have \( x_2 \in \mathcal{G}_{k_n} \) and \( c_{k_n}(x_2) = T^{-r}(c_{k_n}(x_1)) \). Fix such \( n \). For \( i \in \{1, 2\} \), let \( d_i \in Q_{k_{n-2}} \) be such that \( \phi^i(c_{k_n}(x_1)) = d_i \). Let \( t \in \mathbb{Z} \) be such that \( |t| < 2^{k_{n-2}} \) and \( d_1 = S^t(d_2) \).

Let \( m > n \) \((m \equiv 2 \mod 4)\) such that \( c_{k_m}(x_1) \in \mathcal{G}_{k_m} \). For \( i \in \{1, 2\} \), let \( e_i \in Q_{k_{m-2}} \) be levels such that \( \phi^m(c_{k_m}(x_1)) = e_i \). Then because the partial interval bijections \( \phi_\omega \) at stage \( m \) are extensions of concatenations of those at stage \( n \), we have \( e_1 = S^t(e_2) \). Since \( m \) was arbitrary, it follows that \( \phi(x_1) = S^t(\phi(x_2)) \). Therefore \( \phi \) maps the backward \( T \)-orbit of \( x \) into the \( S \)-orbit of \( \phi(x) \). By similar argument, \( \phi \) maps the forward \( T \)-orbit of \( x \) into the \( S \)-orbit of \( \phi(x) \). The argument for \( \psi \) is also similar. \( \square \)

Theorem 17.2. The sets \( X_0 \subset X \) and \( Y_0 \subset Y \) are invariant \( G_\delta \) subsets of full measure, and \( \phi : X_0 \to Y_0 \) carries \( T \)-orbits bijectively to \( S \)-orbits.
Proof. We have already seen that \(X_0\) and \(Y_0\) are \(G_δ\) subsets of full measure. Let \(x \in X_0\). Then \(x \in X_1\) and \(φ(x) ∈ Y_1\) by definition. Let \(x’ = T^r(x)\) for some \(r ∈ Z\). Then \(x’ \in X_1\) because \(X_1\) is \(T\)-invariant. And \(φ(x’) ∈ Y_1\) by Lemma 17.1 (and because \(φ(x) ∈ Y_1\)). Therefore \(x’ \in X_0\), so \(X_0\) is \(T\)-invariant. Similarly, \(Y_0\) is \(S\)-invariant.

Now suppose \(y\) is a point in the \(S\)-orbit of \(φ(x)\). Then \(y ∈ Y_1 \subset H\), so \(ψ(y)\) is in the orbit of \(x\). Hence \(ψ(y) ∈ G\). Therefore \(φ\) carries the orbit of \(x\) onto the orbit of \(φ(x)\).

\[\Box\]

Theorem 17.3. The map \(φ\) is a conjugacy between the two induced maps \(T_{G_2}\) and \(S_{φ(G_2)}\). Both \(G_2\) and \(φ(G_2)\) are nearly clopen. Therefore \(φ\) is a nearly continuous Kakutani equivalence of \(T\) and \(S\).

Proof. Let \(x_1\) and \(x_2 = T^r(x_1)\) be two points in \(G_2 \cap X_0\). Then, by Theorem 17.2, \(φ(x_2) = S^k(φ(x_1)\) for some \(k ∈ Z\). We wish to show that \(r\) and \(k\) have the same sign. This will imply that \(φ\) restricted to \(G_2 \cap X_0\) is order preserving on orbits, and hence a conjugacy between the induced maps.

As we saw in the proof of Lemma 17.1, \(c_{k_0}(x_2) = T^r(c_{k_0}(x_1))\) for all sufficiently large \(n \equiv 2 \mod 4\). Let \(n\) be minimal among such \(n\). If \(n = 2\), then \(r\) and \(k\) automatically have the same sign because each partial interval bijection \(φ_{c_n}\) at stage 2 maps the levels in \(G_2\) in an order-preserving way, and this is then carried through the diagram via concatenations.

If \(n > 2\), then because \(n\) is minimal, \(c_{k_0}(x_2)\) and \(c_{k_0}(x_1)\) must lie in different towers in \(P_{k_0-4}\). And the partial interval bijections at stage \(n\) are extensions of concatenations of those at stage \(n - 4\). So if \(r > 0\), then the stage-\((n - 4)\) partial interval bijection that acts on \(x_1\) in stage \(n\) comes before the stage-\((n - 4)\) partial interval bijection that acts on \(x_2\) in stage \(n\). This order-preservation at stage \(n\) is then carried through the diagram via concatenations, so \(k > 0\). Similarly, \(r < 0\) implies \(k < 0\).

\[\Box\]

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