Are there contact transformations for discrete equations?

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Abstract

We define infinitesimal contact transformations for ordinary difference schemes as transformations that depend on \(K + 1\) lattice points (\(K \geq 1\)) and can be integrated to form a local or global Lie group. We then prove that such contact transformations do not exist.

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1. Introduction

Let us consider a system of ordinary or partial differential equations involving the independent variables \(x \in \mathcal{R}^p\) and the dependent variables \(y \in \mathcal{R}^q\). Three types of transformation of variables play important roles in the theory of differential equations.

The first are point transformations, in which the new variables (\(\tilde{x}, \tilde{y}\)) depend only on the old variables (\(x, y\)). The methods of Lie point symmetry groups have been adapted to make them applicable to difference equations and differential-difference equations [2, 3, 8, 10, 11].

The second are generalized transformations for which (\(\tilde{x}, \tilde{y}\)) depend not only on \(x\) and \(y\) but also on derivatives of \(y\) of all orders. These are important in the theory of integrable partial differential equations. They too have been adapted to differential-difference equations, difference equations and other equations involving discrete variables (or variables on lattices) [7–9, 11–13, 17].

Intermediate between point and generalized transformations are contact ones. As symmetry transformations they provide nontrivial results only for scalar equations [1, 6]. For them we have

\[
\tilde{y} = \Omega(x, y, y_x), \quad \tilde{x} = \Lambda(x, y, y_x) \tag{1}
\]
and a contact condition ensures that we have
\[ \tilde{y}_x = \Phi(x, y, y_x) \] (2)
(no second derivatives). The contact condition also guarantees that the \( n \)th derivative \( \tilde{y}_{nx} \)
depends only on the derivatives \( \tilde{y}_{kx} \), \( 1 \leq k \leq n \).

The Lie algebra of contact transformations is closed under commutation, i.e. it does not introduce higher derivatives. This makes it possible to integrate the Lie algebra to a Lie group of transformations of the forms (1) and (2). The inverse transformations have the same form. For some differential equations contact transformations provide analytical solutions or at least a reduction of the equation, not obtainable by using point transformations [5, 15, 18]. Contact transformations for ordinary differential equations (ODEs) are necessarily of first order (i.e. involve first derivatives only) [14].

Contact transformations have, so far, not been adapted to difference equations. In this paper, we will just consider the case of ordinary difference equations. The purpose of this paper is to analyze the situation in this case and show that contact transformations for ordinary difference equations do not exist.

We give a precise definition of what we mean by contact transformations for difference equations in section 4. Roughly speaking, these are transformations that depend on \( n \) points on the lattice with \( 2 \leq n \leq \infty \). They take solutions of the difference equation into solutions. In the continuous limit, they reduce to contact transformations of the forms (1) and (2) for the ODE obtained (in this limit).

In section 2, we will review the theory of contact transformations in the case of ordinary differential equations while in section 3, we review the theory of Lie point symmetries for difference schemes. Section 4 is then dedicated to proving that contact transformations for difference equations do not exist.

2. Point and contact transformations for ordinary differential equations

Let us consider an ODE of order \( n \):
\[ y^{(n)} = F(x, y, y', \ldots, y^{(n-1)}) \] (3)
and the \( n \)th prolongation of an element of its symmetry algebra
\[ X = \xi \partial_x + \phi \partial_y, \quad pr^{(n)}X = \xi \partial_x + \phi \partial_y + \phi^{(1)} \partial_{y'} + \cdots + \phi^{(n)} \partial_{y^{(n)}}. \] (4)
The coefficients in the prolongation are calculated recursively:
\[ \phi^{(n)} = D_x \phi^{(n-1)} - y^{(n)} D_x \xi, \quad \phi^{(0)} = \phi, \] (5)
where \( D_x \) is the total derivative operator. The symmetry condition is
\[ pr^{(n)}X(y^{(n)} - F)|_{y^{(n)}=F} = 0. \] (6)
For point transformations, we have (by definition)
\[ \xi = \xi(x, t) \quad \phi = \phi(x, y) \] (7)
and hence equation (5) implies that \( \phi^{(n)} = \phi^{(n)}(x, y, y', \ldots, y^{(n)}) \). A Lie algebra of point transformations can be integrated to a Lie group. Indeed we can integrate the vector field \( X \):
\[ \frac{d\hat{x}}{d\lambda} = \xi(\hat{x}, \hat{y}), \quad \frac{d\hat{y}}{d\lambda} = \phi(\hat{x}, \hat{y}) \]
\[ \hat{x}|_{\lambda=0} = x, \quad \hat{y}|_{\lambda=0} = y. \] (8)
The transformation formulas for the derivatives \( \hat{y}_{k}\) can be obtained (for any \( k \)) either by the chain rule, once \( \hat{y}(\hat{x}) \) is known, or by integrating \( pr^{(k)}X \).
Now let us consider generalized symmetries containing higher order derivatives up to order $n$. In this case, instead of (7), we postulate
\[
\xi = \xi(x, y, y', \ldots, y^{(n)}), \quad \phi = \phi(x, y, y', \ldots, y^{(n)}).
\] (9)

To be able to integrate the Lie algebra of such transformations to a Lie group, we need the coefficients $\phi(k)$ in the $n$th prolongation (for $k = 1, 2, \ldots, n$) to depend only on $x, y, y', \ldots, y^{(n)}$ but not on $y^{(n+j)}$, $j \geq 1$. Stephani [14] provides an elegant proof of the fact that this condition cannot be satisfied for $n > 1$.

For $n = 1$, contact transformations do exist if
\[
\phi_y = y' \xi_y
\] (10)
and an ODE can be invariant under a group of contact transformations. The elements of Lie algebra have the form
\[
X = \xi(x, y, y') \partial_x + \phi(x, y, y') \partial_y + \phi^{(1)}(x, y, y') \partial_{y'}.
\] (11)

The contact condition (10) eliminates $y''$ from $\phi^{(1)}$ and assures that $\phi(k)$ for $k \geq 1$ is given by (5). Thus, first order contact transformations of the form (11) form a Lie algebra that can be integrated to give a Lie group of contact transformations.

3. Ordinary difference schemes and their point transformation

A difference equation on a fixed nontransforming lattice has very few continuous symmetries. One way of making a Lie symmetry approach to difference equations fruitful is to consider difference equations on flexible lattices that themselves transform under the transformations [2, 3, 11]. Thus, instead of difference equations we will deal with difference schemes.

Let us consider the case of one independent variable $x$ and one dependent one $y$. The independent variable $x$ is sampled at several points $x_k$ and $y(x)$ is evaluated at the same points $y_k = y_k(x_k)$. An ordinary difference scheme ($O\Delta S$) consists of two relations between the $K$ points $x_k, y_k$:
\[
E_a([x_k], [y_k], k = n + M, n + M + 1, \ldots, n + N) = 0
\] (12)
\[a = 1, 2 \quad K = N + M + 1, \quad n, M, N \in \mathbb{Z}, \quad N > M.
\]

In the continuous limit $x_j - x_{j+1} = h_j \to 0$, one of the two equations (12) should go into an ODE of order $K - 1$ (or less), and the other into an identity (like $0 = 0$). If the values $x_k, y_k$ are given in $K - 1$ neighboring points then we must be able to calculate their values in the next point. The corresponding independence condition is
\[
\frac{\partial (E_1, E_2)}{\partial (x_{n+K}, y_{n+K})} \neq 0, \quad \forall n.
\] (13)

Thus, the $O\Delta S$ (12) determines both the lattice and the solution of the difference equation. The general solution of (12) can be written as
\[
x_n = x(n, c_1, c_2, \ldots, c_{2(K-1)})
\]
\[
y_n = y(n, c_1, c_2, \ldots, c_{2(K-1)}) = Y(x_n, c_1, c_2, \ldots, c_{2(K-1)}).
\] (14)

Point symmetries of the $O\Delta S$ (12) can be determined algorithmically. We require that the elements of the Lie point symmetry algebra of (12) have the form
\[
X = \xi_n \partial_{x_n} + \phi_n \partial_{y_n}
\] (15)
with \( \xi_n = \xi(x_n, y_n) \), \( \phi_n = \phi(x_n, y_n) \) and the prolongation of the vector field (15) should have the form

\[
pr^{(K)}X = \sum_{k=n+M}^{n+N} (\xi_k \partial_{x_k} + \phi_k \partial_{y_k}),
\]

(16)

where the sum is over all points figuring in the OΔS (12). The determining equations are obtained from the conditions

\[
pr^{(K)}X E_a|_{E_1=0, E_2=0} = 0, \quad a = 1, 2.
\]

(17)

Since each of the functions \( \xi_k \) and \( \phi_k \) (for point transformations) depends only on one point \((x_k, y_k)\), equation (17) actually splits into a set of several functional equations for the functions \( \xi_k \) and \( \phi_k \).

If we fix the value of \( n \), then the \( K \) points in (12) form a 'stencil' [3]. Within a stencil, we can choose different coordinates, more appropriate for taking a continuous limit. Let us consider a stencil with the points \( \{x_{n+k-1}, y_{n+k-1}, 1 \leq k \leq K\} \) for some fixed \( n \). An alternative set of coordinates on the stencil is [16]

\[
\{x_n, y_n, p(1)_{n+1}, p(2)_{n+2}, p(3)_{n+3}, \ldots, p(K)_{n+K-1}, h_{n+1}, h_{n+2}, \ldots, h_{n+K-1}\}
\]

(18)

with

\[
h_{nk} = x_{nk} - x_{nk-1},
\]

\[
p^{(1)}_{n+1} = \frac{y_{n+1} - y_n}{x_{n+1} - x_n},
\]

\[
p^{(2)}_{n+2} = \frac{2p^{(1)}_{n+2} - p^{(1)}_{n+1}}{x_{n+2} - x_n},
\]

\[
p^{(K-1)}_{n+K-1} = (K-1) \frac{p^{(K-2)}_{n+K-1} - p^{(K-2)}_{n+K-2}}{x_{n+K-1} - x_n}.
\]

(19)

In the continuous limit, we have

\[
h_{nk} \to 0, \quad p^{(1)}_{n+1} \to y', \quad p^{(2)}_{n+2} \to y'', \ldots, p^{(N)}_{n+K-1} \to y^{(K-1)}.
\]

(20)

If we transform the ‘discrete’ prolongation (16) of the vector field (15) to the new variables (18), we obtain

\[
pr X = \xi_n \partial_{x_n} + \phi_n \partial_{y_n} + \sum_{k=1}^{K} \lambda^{(k)} \partial_{h_{nk}} + \sum_{k=1}^{K} \phi^{(k)} \partial_{p^{(k)}_{nk}}.
\]

(21)

The general formulas for the coefficients \( \lambda^{(k)} \) and \( \phi^{(k)} \) are

\[
\lambda^{(k)} = \xi_{nk} - \xi_{nk-1},
\]

\[
\phi^{(k)} = \sum_{j=1}^{k} \frac{h_{nk} \Delta^{T} \phi^{(k-1)}_{nk-1} - p^{(k)}_{nk}}{h_{nk}} \sum_{j=1}^{k} h_{nj} \Delta^{T} \xi_{nk-j-1}, \quad k = 1, 2, \ldots, K,
\]

(22)

where the \( \Delta^{T} \) is the total difference operator

\[
\Delta^{T} F(x_n, y_n, p^{(1)}_{n+1}, p^{(2)}_{n+2}, \ldots) = \frac{1}{h_{n+1}} \left\{ F(x_{n+1}, y_{n+1}, p^{(1)}_{n+2}, p^{(2)}_{n+3}, \ldots) - F(x_n, y_n, p^{(1)}_{n+1}, p^{(2)}_{n+2}, \ldots) \right\}.
\]

(23)
4. Do contact transformations for OΔS exist?

Let us again consider the OΔS (12). The question that we pose is the following. Is it possible to define contact transformations that leave the solution set of the OΔS (12) invariant? We shall impose the following restrictions to define contact transformations in the discrete case.

Definition 4.1. A contact transformation for the OΔS (12) satisfies the following.

1. The vector fields forming the Lie algebra $L_c$ of contact and point transformations should have the form (15) with at least one of the coefficients $\xi_n$ and $\phi_n$ in one of the fields $X$ depending on $K + 1$ points:

   \[ \xi_n = \xi_n(x_n, y_n, x_{n+1}, y_{n+1}, \ldots, x_{n+K}, y_{n+K}), K \in \mathbb{Z}^+ \]

   \[ \phi_n = \phi_n(x_n, y_n, x_{n+1}, y_{n+1}, \ldots, x_{n+K}, y_{n+K}). \]  

   (24)

2. The Lie algebra should be integrable to a Lie group. This implies that all coefficients (22) in the $K$th prolongation of $X$ (21) should depend only on the points $(x_n, y_n, \ldots, x_{n+K}, y_{n+K})$ and not on any further ones.

3. In the continuous limit, the algebra and the group of contact transformations of the OΔS (12) should reduce to the corresponding Lie algebra and Lie group of contact transformations of the corresponding ODE.

Definition 4.2. A contact transformation for a difference scheme is said to be of order $\ell$ if it involves $\ell + 1$ points of the lattice.

Let us now prove the following theorem.

Theorem 4.1. Contact transformations of order 1 of OΔS do not exist.

Proof.

Let us consider the first prolongation of a vector field depending on two points:

\[ pr^{(1)}X = \xi_n \partial x_n + \phi_n \partial y_n + \lambda^{(1)} \partial h_{n+1} + \phi^{(1)}_{n+1} \partial p^{(1)}_{n+1}. \]  

(25)

We have

\[ \xi_n = \xi_n(x_n, y_n, x_{n+1}, y_{n+1}) = \xi_n(x_n, y_n, x_n + h_{n+1}, y_n + h_{n+1} P^{(1)}_{n+1}) \]

\[ \phi_n = \phi_n(x_n, y_n, x_{n+1}, y_{n+1}) = \phi_n(x_n, y_n, x_n + h_{n+1}, y_n + h_{n+1} P^{(1)}_{n+1}) \]  

(26)

\[ \lambda^{(1)} = \xi_{n+1} - \xi_n \]

\[ \phi^{(1)}_{n+1} = \frac{\phi_{n+1} - \phi_n}{h_{n+1}} \]  

(27)

where

\[ \xi_{n+1} = \xi_{n+1}(x_{n+1}, y_{n+1}, x_{n+2}, y_{n+2}) \]

\[ = \xi_{n+1}(x_n + h_{n+1}, y_n + h_{n+1} P^{(1)}_{n+1}, x_n + h_{n+1}, y_n) \]

\[ + (h_{n+1} + h_{n+2}) \left[ \frac{P^{(1)}_{n+1}}{2} P^{(2)}_{n+2} \right] \]

\[ \phi_{n+1} = \phi_{n+1}(x_{n+1}, y_{n+1}, x_{n+2}, y_{n+2}) \]

\[ = \phi_{n+1}(x_n + h_{n+1}, y_n + h_{n+1} P^{(1)}_{n+1}, x_n + h_{n+1}, y_n) \]

\[ + (h_{n+1} + h_{n+2}) \left[ \frac{P^{(1)}_{n+1}}{2} P^{(2)}_{n+2} \right]. \]  

(28)
As one can see in (28), both $\lambda^{(1)}$ and $\phi_{n+1}$ depend on $(h_{n+2}, p_{n+2})$. The question is whether this dependence can cancel out by proper choice of the symmetry generator form. We have

$$\frac{\partial \phi_{n+1}}{\partial p_{n+2}} = \frac{h_{n+2}(h_{n+1} + h_{n+2})}{h_{n+1}} \left[ \frac{\partial \phi_{n+1}}{\partial y_{n+2}} - p_{n+1} \frac{\partial \xi_{n+1}}{\partial y_{n+2}} \right] = 0,$$

which provide a contact condition similar to the continuous one (10). However, in this case we have to require the further condition

$$\frac{\partial \lambda^{(1)}}{\partial p_{n+2}} = \frac{h_{n+2}(h_{n+1} + h_{n+2})}{2} \frac{\partial \xi_{n+1}}{\partial y_{n+2}} = 0$$

and hence $\partial \xi_{n+1}/\partial y_{n+2} = 0$ and consequently from (29) $\partial \phi_{n+1}/\partial y_{n+2} = 0$. In turn, these conditions imply

$$\frac{\partial \xi_{n}}{\partial y_{n+1}} = \frac{\partial \phi_{n}}{\partial y_{n+1}} = 0.$$ (31)

Similarly $\partial \lambda^{(1)}/\partial h_{n+2} = 0$ and $\partial \phi_{n+1}/\partial y_{n+2} = 0$ imply

$$\frac{\partial \xi_{n}}{\partial x_{n+1}} = \frac{\partial \phi_{n}}{\partial x_{n+1}} = 0,$$ (32)

but (31) and (32) mean that (25) corresponds to a point transformation, rather than a contact one. □

By a similar calculation one can prove that if $\xi$ and $\phi$ depend on $K$ points then the $K$th prolongation will depend on one more point and hence one cannot obtain an integrable (closed) system of contact transformations for an OΔS. To sum up we state

**Theorem 4.2.** Contact transformation of any order $K \geq 1$ for OΔS does not exist.

If we consider discrete equations on a fixed (nontransforming) lattice then most, or all, point symmetries are absent. In this case, the variable $x_n = h n + x_0$ with $h$ and $x_0$ fixed numbers and thus the infinitesimal generator is just of evolutionary form, i.e.

$$X = \phi_n(y_n, y_{n+1}) \partial_{y_n} + \phi_{n+1}(y_{n+1}, y_{n+2}) \partial_{y_{n+1}} = \phi_n \partial_{y_n} + \phi_{n+1} \partial_{y_{n+1}}.$$ (33)

Then (33) is a special case of (21) with $\xi_n$ and $\lambda^{(k)}$ absent. Thus, the nontransforming lattice is a special case of the previous theorem.

**5. Conclusions**

Theorem 4.2 amounts to a ‘no-go theorem’. It states that for ordinary difference schemes contact transformations, as defined in definition 4.1, do not exist. This leaves open the possibility that generalized symmetries might exist that in the continuous limit go into contact symmetries of an ODE. For comparison we recall that when considering integrable differential-difference equations on fixed (nontransforming) lattices, the following situation occurs. An infinite Lie algebra of generalized symmetries exists of which a small subset ‘contracts’ to point symmetries in the continuous limit [4]. Lie point symmetries of the corresponding differential equation are recovered in this manner.
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