Boundary controllability for a degenerate and singular wave equation

BRAHIM ALLAL
Faculté des Sciences et Techniques
Université Hassan 1er
Laboratoire MISI, B.P. 577
Settat 26000, Morocco
e-mail: b.allal@uhp.ac.ma

ALHABIB MOUMNI
Moulay Ismail University of Meknes,
FST Errachidia, MAIS Laboratory, MAMCS Group,
P.O. Box 509, Boutalamine 52000, Errachidia, Morocco
email: alhabibmoumni2020@gmail.com

JAWAD SALHI
Moulay Ismail University of Meknes,
FST Errachidia, MAIS Laboratory, MAMCS Group,
P.O. Box 509, Boutalamine 52000, Errachidia, Morocco
email: sj.salhi@gmail.com

Abstract
In this paper, we deal with the boundary controllability of a one-dimensional degenerate and singular wave equation with degeneracy and singularity occurring at the boundary of the spatial domain. Exact boundary controllability is proved in the range of subcritical/critical potentials and for sufficiently large time, through a boundary controller acting away from the degenerate/singular point. By duality argument, we reduce the problem to an observability estimate for the corresponding adjoint system, which is proved by means of the multiplier method and a new special Hardy-type inequality.

1 Introduction

Controllability issues for nondegenerate hyperbolic equations have been a mainstream topic over the past several years, and numerous developments have been pursued (see for example [4, 7, 21, 23, 27, 32] and the references therein).

In the present paper, we consider the following degenerate/singular hyperbolic equation

\begin{equation}
\begin{aligned}
y_{tt} - (x^{\alpha} y_x)_x - \frac{\mu}{x^2} y &= 0, & (t, x) \in Q := (0, T) \times (0, 1), \\
y(1) &= f, & t \in (0, T), \\
\begin{cases}
y(t, 0) = 0, & \text{if } 0 \leq \alpha < 1, \\
(x^{\alpha} y_x)(t, 0) = 0, & \text{if } 1 \leq \alpha < 2, \\
y(0, x) = y_0(x), & y_t(0, x) = y_1(x), & x \in (0, 1),
\end{cases}
\end{aligned}
\end{equation}

where $\alpha$ and $\mu$ are two real parameters, $(y, y_t)$ is the state variable, $(y_0, y_1)$ is regarded as being the initial value, $T > 0$ stands for the length of the time-horizon and $f \in L^2(0, T)$ is
the control at $x = 1$ (that is, away from the degenerate and singular point). In particular, if $\alpha \in (0, 1)$ we say that the problem is weakly degenerate (WD), if $\alpha \in [1, 2)$ then it is strongly degenerate (SD).

In order to study problem (1.1), we assume that the parameters $\alpha$ and $\mu$ satisfy the following assumption:

$$\alpha \in [0, 2) \setminus \{1\} \text{ and } \mu \leq \mu(\alpha),$$

(1.2)

where

$$\mu(\alpha) := \frac{(1 - \alpha)^2}{4}$$

(1.3)

is the constant appearing in the following generalized Hardy inequality: for all $\alpha \in [0, 2)$,

$$\frac{(1 - \alpha)^2}{4} \int_0^1 \frac{u^2}{x^{2-\alpha}} dx \leq \int_0^1 x^{\alpha} u_x^2 dx,$$

(1.4)

for all $u \in C^\infty_c(0, 1)$ (the space of infinitely smooth functions compactly supported in $(0, 1)$). We refer for example to [12, chap 5.3].

We emphasise that (1.4) ensures that, if $\alpha \in [0, 2) \setminus \{1\}$ and if $u \in H^1_{\text{loc}}((0, 1))$ is such that $\frac{u}{x^{(2-\alpha)/2}}$ belongs to $L^2(0, 1)$. On the contrary, in the case $\alpha = 1$, (1.4) (which reduces to a trivial inequality) does not provide this information anymore. Hence, it is not surprising if with our techniques we cannot handle this latter special case and we refer to [14] and [24] where this issue is attacked in a different way for the heat equation.

Now, observe that when $\mu = 0$, the problem above is purely degenerate. In this case, controllability properties by means of a boundary control have been investigated in various papers. We refer the reader to the following pioneering contributions [1, 2, 13, 16, 18, 28]. We also refer to [29, 30] for other works on controllability problems by means of a locally distributed control.

On the other hand, when $\alpha = 0$, system (1.1) becomes purely singular with a singularity that takes the form of an inverse-square potential. As far as we know, [10] and [25] are the unique published works on this subject; they are concerned with the problem of exact controllability for the linear multidimensional wave equation with singular potentials.

We underline that this is the first paper to consider the exact boundary controllability for the system (1.1) that couples a degenerate variable coefficient in the principal part with a singular potential. More precisely, we will solve the following problem: Given a time-horizon $T > 0$, initial data $(y_0, y_1)$ and a target $(y^T_0, y^T_1)$, we ask whether there is a suitable control function $f$ such that the corresponding solution of the system (1.1) satisfies

$$(y, y_t)(T, \cdot) = (y^T_0, y^T_1)(\cdot).$$

(1.5)

In the view of the linearity and the time-reversibility of system (1.1), it can be shown that this system is exactly controllable through the boundary Dirichlet conditions at $x = 1$ if and only if it is null controllable (see [31, Proposition 2.3.1]). This guarantees that (1.1) is exactly controllable if and only if, for any $(y_0, y_1)$ there exists a control $f$ such that the corresponding solution $(y, y_t)$ of (1.1) satisfies

$$y(T, \cdot) = y_t(T, \cdot) = 0.$$

(1.6)

This result is actually equivalent to an observability inequality for the solutions of the adjoint
system
\[
\begin{aligned}
&u_{tt} - (x^\alpha u_x)_x - \frac{\mu}{x^{2-\alpha}} u = 0, \\
&u(t, 0) = 0, \quad \text{if } 0 \leq \alpha < 1, \\
&(x^\alpha u_x)(t, 0) = 0, \quad \text{if } 1 < \alpha < 2, \\
&u(1) = 0, \quad t \in (0, T), \\
&u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in (0, 1).
\end{aligned}
\] (1.7)

The main contribution of this paper consists precisely in proving this observability inequality. To be more precise, we show that, for \( T \) large enough and all \( \mu \leq \mu(\alpha) \), there exists \( C > 0 \) such that
\[
\int_0^1 \left\{ u_t^2(0, x) + x^\alpha u_x^2(0, x) - \frac{\mu}{x^{2-\alpha}} u^2(0, x) \right\} \, dx \leq C \int_0^T u_x^2(t, 1) \, dt.
\] (1.8)

We refer to Theorem 4.1 for a precise statement of this result. The proof of (1.8) relies on the multiplier method which has first been given in [17] and can be found in [19, 20, 22]. As a consequence of this inequality, it follows that system (1.1) is exactly controllable in time \( T \) by a control acting at \( x = 1 \). More precisely, our main result is the boundary exact controllability of (1.1) for \( T > 0 \) sufficiently large and independent of \( \mu \leq \mu(\alpha) \) (see Theorem 5.1 for a rigorous statement).

Let us notice that in the subcritical case \( \mu < \mu(\alpha) \), in view of Lemma 3.1, one could proceed as in the case of the purely degenerate wave equation (when \( \mu = 0 \)). However, this would produce a controllability time \( T_\mu^\alpha \) depending on \( \mu \) and such that \( T_\mu^\alpha \to +\infty \) as \( \mu \to \mu(\alpha) \). Hence the time of controllability \( T_\mu^\alpha \) would not be uniform with respect to the parameter \( \mu \). Moreover, by this method, no result could be expected in the critical case \( \mu = \mu(\alpha) \).

In order to overcome this issue, we prove the following new weighted Hardy-type inequality that generalizes [25, Theorem 1.1], which is crucial in order to get a uniform time of controllability and to also treat the critical case \( \mu = \mu(\alpha) \).

**Theorem 1.1.** Let \( \mu(\alpha) \) be as in (1.3). For all \( \alpha \in [0, 2) \) and for all \( u \in C^\infty_c(0, 1) \), we have
\[
\int_0^1 x^\alpha u_x^2 \, dx \leq \int_0^1 \left( x^\alpha u_x^2 - \mu(\alpha) \frac{u^2}{x^{2-\alpha}} \right) \, dx + \frac{(1-\alpha)(\alpha-3)}{4} \int_0^1 u^2 \, dx.
\] (1.9)

The proof of this theorem is given in an appendix at the end of the paper.

**Remark 1.** Actually, using multiplier method, the authors in [1] proved the null controllability of purely degenerate wave equations with a more general speed coefficient. In the pure power case (i.e. the system (1.1) with \( \mu = 0 \)), they provided an explicit expression for the controllability time given by
\[
\bar{T}_\alpha = \frac{1}{2-\alpha} \left( 4 + 2\alpha \min\{2, \frac{1}{\sqrt{2-\alpha}}\} \right).
\]

Different from [1], it is worth mentioning that the authors in [28], also proved the null controllability of (1.1) when \( \mu = 0 \) by using a spectral approach. In particular, they provided a sharp controllability time given by
\[
T_\alpha = \frac{4}{2 - \alpha}.
\]

Here, by means of the multiplier method and the special Hardy-type inequality (1.9), we retrieve the expected minimal time of controllability \( T_\alpha \), which coincides with the one that the spectral method gives for the purely degenerate wave equation.
2 Preliminary results

In this section, we state some lemmas that play an important role in the rest of the paper. First of all, we prove the following new weighted Hardy-Poincaré inequality which is crucial in every situation.

Lemma 2.1. Let $\mu(\alpha)$ be as in (1.3). Then, for all $\alpha \in [0, 2)$

$$
\int_0^1 u^2 \, dx \leq C_\alpha \int_0^1 \left( x^{\alpha} u_x^2 - \mu(\alpha) \frac{u^2}{x^{2-\alpha}} \right) \, dx \quad \forall u \in C_c^\infty(0, 1),
$$

where

$$
C_\alpha := \frac{16}{(2-\alpha)^2}.
$$

Proof. The result will be obtained by the expansion of the square method as in [6]. Assume that $u \in C_c^\infty(0, 1)$ and let $\lambda > 0$. Then we write

$$
0 \leq \int_0^1 \left( x^{\alpha/2} u_x - \frac{1-\alpha}{2} \frac{1}{x^{(2-\alpha)/2}} u + \lambda u \right)^2 \, dx.
$$

Expanding the above inequality, we obtain

$$
0 \leq \int_0^1 x^{\alpha} u_x^2 \, dx + \frac{(1-\alpha)^2}{4} \int_0^1 \frac{1}{x^{2-\alpha}} u^2 \, dx + \lambda^2 \int_0^1 u^2 \, dx + \frac{1-\alpha}{2} \int_0^1 \frac{1}{x^{1-\alpha}} (u^2)_x \, dx
$$

$$
+ \lambda \int_0^1 x^{\alpha/2} (u^2)_x \, dx - \lambda (1-\alpha) \int_0^1 \frac{1}{x^{(2-\alpha)/2}} u^2 \, dx.
$$

Then integrations by parts lead to

$$
0 \leq \int_0^1 x^{\alpha} u_x^2 \, dx + \frac{(1-\alpha)^2}{4} \int_0^1 \frac{1}{x^{2-\alpha}} u^2 \, dx + \lambda^2 \int_0^1 u^2 \, dx
$$

$$
- \frac{(1-\alpha)^2}{2} \int_0^1 \frac{1}{x^{2-\alpha}} u^2 \, dx - \lambda (1-\alpha) \int_0^1 \frac{1}{x^{(2-\alpha)/2}} u^2 \, dx
$$

$$
= \int_0^1 \left( x^{\alpha} u_x^2 - \mu(\alpha) \frac{u^2}{x^{2-\alpha}} \right) \, dx + \int_0^1 \left[ \lambda^2 - \lambda (1-\alpha) \frac{1}{x^{(2-\alpha)/2}} \right] u^2 \, dx.
$$

Using the fact that $\alpha \in [0, 2)$, we deduce that

$$
0 \leq \int_0^1 \left( x^{\alpha} u_x^2 - \mu(\alpha) \frac{u^2}{x^{2-\alpha}} \right) \, dx + \left[ \lambda^2 - \lambda (1-\alpha) \frac{1}{x^{(2-\alpha)/2}} \right] \int_0^1 u^2 \, dx.
$$

Now, observe that

$$
\lambda^2 - \lambda (1-\alpha) \geq \frac{-(2-\alpha)^2}{16},
$$

with equality if and only if $\lambda = \frac{2-\alpha}{4}$. By choosing this value, we get the result. \qed

By the definition of $\alpha$ and (1.9) combined with (2.1), we obtain the following inequalities which are needed for proving a refined trace result (see Lemma 3.4).

Lemma 2.2. Let $\mu(\alpha)$ be as in (1.3).

1. If $\alpha \in [0, 1)$, then

$$
\int_0^1 x^2 u_x^2 \, dx \leq \int_0^1 \left( x^{\alpha} u_x^2 - \mu(\alpha) \frac{u^2}{x^{2-\alpha}} \right) \, dx \quad \forall u \in C_c^\infty(0, 1). \tag{2.2}
$$
2. If $\alpha \in [1, 2)$, then
\[
\int_0^1 x^2 u_x^2 \, dx \leq C'_\alpha \int_0^1 \left( x^{\alpha} u_x^2 - \mu(\alpha) \frac{u^2}{x^{2-\alpha}} \right) \, dx \quad \forall u \in C_c^\infty(0, 1),
\]
where
\[
C'_\alpha := \left[ 1 + \frac{4(1-\alpha)(\alpha-3)}{(2-\alpha)^2} \right].
\]

Remark 2. In the weakly degenerate case, observe that owing to (1.9) and (2.1), we obtain two different bounds for $\|u\|_{L^2(0, 1)}$ in terms of $\int_0^1 \left( x^{\alpha} u_x^2 - \mu(\alpha) \frac{u^2}{x^{2-\alpha}} \right) \, dx$. By taking the minimum of the two corresponding constants, one can deduce the following sharp Hardy-Poincaré inequality: for all $\alpha \in [0, 1)$
\[
\int_0^1 u^2 \, dx \leq c_\alpha \int_0^1 \left( x^{\alpha} u_x^2 - \mu(\alpha) \frac{u^2}{x^{2-\alpha}} \right) \, dx \quad \forall u \in C_c^\infty(0, 1),
\]
where
\[
c_\alpha := \min \left( \frac{4}{(1-\alpha)(3-\alpha)}, \frac{16}{(2-\alpha)^2} \right).
\]

3 Basic properties for the degenerate and singular wave equation

Before considering controllability issues, we address the question of well-posedness of the degenerate and singular hyperbolic problem (1.1). To this end, we first need to state some basic properties of nonhomogeneous wave equations of the following type:

\[
\begin{cases}
  u_{tt} - (x^{\alpha} u_x)_x - \frac{\mu}{x^{2-\alpha}} u = h, & (t, x) \in Q, \\
  u(t, 0) = 0, & \text{if } 0 \leq \alpha < 1, \\
  (x^{\alpha} u_x)(t, 0) = 0, & \text{if } 1 < \alpha < 2, \\
  u(1) = 0, & t \in (0, T), \\
  u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), & x \in (0, 1),
\end{cases}
\]

(3.1)

where $h \in L^2(Q)$ is a given source term.

3.1 Finite energy solutions for homogeneous Dirichlet boundary conditions

The first step is to prove the existence of finite energy solutions for (3.1). In this purpose, we briefly recall some usual weighted Sobolev spaces that are naturally associated with degenerate problems (see [8]). For all $0 \leq \alpha < 2$, we consider the following weighted Hilbert space

\[
H^1_{\alpha,0}(0, 1) := \left\{ u \in L^2(0, 1) \cap H^1_{loc}((0, 1]) \mid x^{\alpha/2} u_x \in L^2(0, 1) \right\},
\]

with the norm
\[
\|u\|_{H^1_{\alpha,0}(0, 1)}^2 := \|u\|_{L^2(0, 1)}^2 + \|x^{\alpha/2} u_x\|_{L^2(0, 1)}^2.
\]

Obviously, for any $u \in H^1_{\alpha,0}(0, 1)$, the trace at $x = 1$ exists. On the other hand, the trace of $u$ at $x = 0$ only makes sense when $0 \leq \alpha < 1$ (see for example [9]). For this reason, we consider the following space $H^1_{\alpha,0}$ depending on the value of $\alpha$:
(i) For $0 \leq \alpha < 1$, we define
\[ H_{\alpha,0}^1(0,1) := \{ u \in H_{\alpha}^1(0,1) \ | \ u(0) = u(1) = 0 \}. \]

(ii) For $1 \leq \alpha < 2$, we define
\[ H_{\alpha,0}^1(0,1) := \{ u \in H_{\alpha}^1(0,1) \ | \ u(1) = 0 \}. \]

We recall that in both cases, $H_{\alpha,0}^1(0,1)$ is the closure of $C_c^\infty(0,1)$ for the norm $\| \cdot \|_{H_{\alpha,0}^1(0,1)}$, see for example [9]. Therefore one can deduce that (1.4) holds true for any $u \in H_{\alpha,0}^1(0,1)$. Then, one can show that
\[ \forall u \in H_{\alpha,0}^1(0,1), \quad \| u \|_{H_{\alpha,0}^1(0,1)} := \left( \int_0^1 x^\alpha u_x^2 dx \right)^{\frac{1}{2}}, \]
defines a norm on $H_{\alpha,0}^1(0,1)$ that is equivalent to $\| \cdot \|^2_{H^1(0,1)}$.

Next, we also set
\[ H^2(0,1) := \{ u \in H_{\alpha,0}^1(0,1) \ | \ x^\alpha u_x \in H^1(0,1) \}, \]
where $H^1(0,1)$ denotes the classical Sobolev space of all functions $u \in L^2(0,1)$ such that $u_x \in L^2(0,1)$.

Let us pass to introduce the functional setting associated to the degenerate/singular problems (see [5] or [24]). For any $\mu \leq \mu(\alpha)$, we consider the Hilbert space $H_{\alpha,\mu}^1(0,1)$ given by
\[ H_{\alpha,\mu}^1(0,1) := \left\{ u \in L^2(0,1) \cap H^1_{loc}(0,1) \ | \ \int_0^1 \left( x^\alpha u_x^2 - \frac{\mu}{x^{2-\alpha}} u^2 \right) dx < +\infty \right\} \]
edowed with the scalar product
\[ \langle u, v \rangle_{H_{\alpha,\mu}^1} := \int_0^1 uv + x^\alpha u_x v_x - \frac{\mu}{x^{2-\alpha}} uv dx. \]

According to [24], the trace at $x = 0$ of any $u \in H_{\alpha,\mu}^1(0,1)$ makes sense as soon as $\alpha < 1$. This leads us to introduce the next space:

(i) For $0 \leq \alpha < 1$, we define
\[ H_{\alpha,0}^{1,\mu}(0,1) := \{ u \in H_{\alpha,\mu}^1(0,1) \ | \ u(0) = u(1) = 0 \}. \]

(ii) For $1 < \alpha < 2$, we change the definition of $H_{\alpha,0}^{1,\mu}(0,1)$ in the following way
\[ H_{\alpha,0}^{1,\mu}(0,1) := \{ u \in H_{\alpha,\mu}^1(0,1) \ | \ u(1) = 0 \}. \]

Let us mention that in both cases, $H_{\alpha,0}^{1,\mu}(0,1)$ may be seen as the closure of $C_c^\infty(0,1)$ with respect to the norm induced by $\langle \cdot, \cdot \rangle_{H_{\alpha,\mu}^1}$ and thus (1.4), (1.9) and (2.1) also hold true in $H_{\alpha,0}^{1,\mu}(0,1)$. Therefore, thanks to (1.4), one can see that
\[ \| u \|_{H_{\alpha,0}^{1,\mu}(0,1)} := \left( \int_0^1 x^\alpha u_x^2 - \frac{\mu}{x^{2-\alpha}} u^2 dx \right)^{\frac{1}{2}} \]
defines a norm on $H_{\alpha,0}^{1,\mu}(0,1)$ which is equivalent to $\| \cdot \|_{H_{\alpha,0}^{1,\mu}(0,1)}$. Hence $H_{\alpha,0}^{1,\mu}(0,1)$ is a Hilbert space for the scalar product
\[ \langle u, v \rangle_{H_{\alpha,0}^{1,\mu}} := \int_0^1 x^\alpha u_x v_x - \frac{\mu}{x^{2-\alpha}} uv dx. \]
Moreover, we also remark that in the case of a sub-critical parameter $\mu < \mu(\alpha)$, thanks to (1.4), it is easy to see that $H^{1,\mu}_{\alpha,0}(0,1) = H^1_{\alpha,0}(0,1)$. On the contrary, for the critical value $\mu = \mu(\alpha)$, the space is enlarged (see [26] for this observation in the case $\alpha = 0$):

$$H^1_{\alpha,0}(0,1) \subsetneq H^{1,\mu}_{\alpha,0}(0,1).$$

To be more precise, in the subcritical case, one can prove the next result.

**Lemma 3.1.** Assume that $\alpha \in (0,2) \setminus \{1\}$ and $\mu < \mu(\alpha)$. Then there exist two constants $C^1_{\alpha,\mu} > 0$ and $C^2_{\alpha,\mu} > 0$ such that, for every $u \in H^1_{\alpha,0}(0,1)$

$$C^1_{\alpha,\mu} \|u\|_{H^1_{\alpha,0}(0,1)} \leq \|u\|_{H^{1,\mu}_{\alpha,0}(0,1)} \leq C^2_{\alpha,\mu} \|u\|_{H^1_{\alpha,0}(0,1)}.$$

More precisely,

$$C^1_{\alpha,\mu} = 1 - \frac{\max(0,\mu)}{\mu(\alpha)}, \quad C^2_{\alpha,\mu} = 1 - \frac{\min(0,\mu)}{\mu(\alpha)}.$$

Further, we define $H^{-1,\mu}_{\alpha,0}(0,1)$ the dual space of $H^{1,\mu}_{\alpha,0}(0,1)$ with respect to the pivot space $L^2(0,1)$, endowed with the natural norm

$$\|f\|_{H^{-1,\mu}_{\alpha,0}(0,1)} := \sup_{\|g\|_{H^{1,\mu}_{\alpha,0}(0,1)} = 1} \langle f, g \rangle_{H^{1,\mu}_{\alpha,0}(0,1),H^{-1,\mu}_{\alpha,0}(0,1)}.$$  

Besides, we also set

$$H^{2,\mu}_{\alpha}(0,1) := \left\{ u \in H^{1,\mu}_{\alpha,0}(0,1) \cap H^2_{\alpha,0}(0,1) \cap \{ (x^\alpha u_x)_x + \frac{\mu}{x^{2-\alpha}} u \in L^2(0,1) \} \right\}.$$

Finally, for all $\mu \leq \mu(\alpha)$, we define the operator

$$A^\mu_\alpha := (x^\alpha u_x)_x + \frac{\mu}{x^{2-\alpha}} u$$

with domain

$$D^\mu_\alpha := D(A^\mu_\alpha) = H^{1,\mu}_{\alpha,0}(0,1) \cap H^{2,\mu}_{\alpha}(0,1).$$

Now, we are ready for the well posedness result of (3.1).

**Theorem 3.1.** Assume that (1.2) holds.

(i) For every $(u_0, u_1) \in H^{1,\mu}_{\alpha,0}(0,1) \times L^2(0,1)$ and $h \in L^2(Q)$, there exists a unique mild (or weak) solution $u \in C([0,T];H^{1,\mu}_{\alpha,0}(0,1)) \cap C^1([0,T];L^2(0,1))$ of (3.1) satisfying the following estimate:

$$\|(u(t), u_1(t))\|_{H^{1,\mu}_{\alpha,0} \times L^2} \leq C\left( \|(u_0, u_1)\|_{H^{1,\mu}_{\alpha,0} \times L^2} + \|h\|_{L^1([0,T];L^2(0,1))} \right), \quad \forall t \in [0,T].$$

(ii) Moreover, if $(u_0, u_1) \in D^\mu_\alpha \times H^{1,\mu}_{\alpha,0}(0,1)$ and $h \in C^1([0,T];L^2(0,1))$, then problem (3.1) admits a unique classical solution $u \in C([0,T];D^\mu_\alpha) \cap C^1([0,T];H^{1,\mu}_{\alpha,0}(0,1)) \cap C^2([0,T];L^2(0,1))$ that satisfies

$$\|(u(t), u_1(t))\|_{D^\mu_\alpha \times L^2} \leq C\left( \|(u_0, u_1)\|_{D^\mu_\alpha \times L^2} + \|h\|_{L^1([0,T];L^2(0,1))} \right), \quad \forall t \in [0,T].$$

**Proof.** Observe that the evolution equation (3.1) is equivalent to

$$\begin{cases}
u(t) = v(t), & v(t) = A^\mu_\alpha u(t) + h(t), \\
w(0) = u_0, & v(0) = u_1.
\end{cases}$$

(3.2)
Now, we introduce the energy space
\[ \mathcal{H} := H^{1,\mu}_{\alpha,0}(0,1) \times L^2(0,1), \]
equipped with the inner product defined by
\[ \langle U_1, U_2 \rangle_{\mathcal{H}} := \int_0^1 x^\alpha u_{1,x} u_{2,x} - \frac{\mu}{x^{2-\alpha}} u_1 u_2 + v_1 v_2 \, dx, \]
where
\[ U_i = (u_i, v_i)^T \in \mathcal{H}, \quad i = 1, 2. \]
We know that \( \mathcal{H} \) is a Hilbert space equipped with the adequate scalar product \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \).

The differential system (3.2) can be rewritten in the abstract form
\[
(CP) \quad \begin{cases}
\frac{dU}{dt}(t) = A^\mu_\alpha U(t) + F(t), \\
U(0) = (u_0, v_0),
\end{cases}
\]
where
\[ \frac{dU}{dt}(t) = \left( \begin{array}{c}
u(t) \\
u(t) \end{array} \right) \quad \text{and} \quad F(t) = \left( \begin{array}{c}0 \\
h(t) \end{array} \right). \]
Here \( A^\mu_\alpha \) is the unbounded linear operator defined by
\[
A^\mu_\alpha \begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} v \\ A^\mu_\alpha u \end{pmatrix}, \quad \forall \begin{pmatrix} u \\ v \end{pmatrix} \in D(A^\mu_\alpha),
\]
with domain
\[ D(A^\mu_\alpha) = \{ (u, v) \in \mathcal{H} \mid u \in D(A^\mu_\alpha), v \in H^{1,\mu}_{\alpha,0}(0,1) \}. \]
We claim that \( A^\mu_\alpha \) is maximal dissipative on \( \mathcal{H} \). Indeed, for \( U = (u, v)^T \in D(A^\mu_\alpha) \), we have
\[
\langle A^\mu_\alpha U, U \rangle_{\mathcal{H}} = \langle \begin{pmatrix} v \\ A^\mu_\alpha u \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \rangle_{\mathcal{H}}
= \int_0^1 x^\alpha u_{x} v_{x} - \frac{\mu}{x^{2-\alpha}} u v \, dx + \int_0^1 \left( (x^\alpha u_{x})_{x} + \frac{\mu}{x^{2-\alpha}} u \right) v \, dx
= 0.
\]
Hence \( A^\mu_\alpha \) is dissipative.

Next, we are going to show that \( I - A^\mu_\alpha \) is surjective. Given a vector \( g = (g_1, g_2) \in \mathcal{H} \), we seek a solution \( (u, v) \in D(A^\mu_\alpha) \) of
\[ (I - A^\mu_\alpha)(u, v)^T = (g_1, g_2)^T. \]
This is equivalent to the following system:
\[
\begin{aligned}
u = u - A^\mu_\alpha u &= g_1 + g_2, \\
v &= u - g_1.
\end{aligned}
\]
Suppose that we have found \( u \) with the appropriate regularity, then we solve (3.4) and we obtain \( v \in H^{1,\mu}_{\alpha,0}(0,1) \).

Now we state the process on how to get \( u \). For this, we recall that \( H^{1,\mu}_{\alpha,0}(0,1) \) is a Hilbert space for the scalar product \( \langle \cdot, \cdot \rangle_{H^{1,\mu}_{\alpha,0}} \). Consequently, for all \( (g_1, g_2)^T \in \mathcal{H} \), there exists a unique \( u \in H^{1,\mu}_{\alpha,0}(0,1) \) such that
\[
\forall \psi \in H^{1,\mu}_{\alpha,0}(0,1), \quad \langle u, \psi \rangle_{H^{1,\mu}_{\alpha,0}} = \int_0^1 (g_1 + g_2) \psi \, dx.
\]
Since $C_c^\infty(0, 1) \subset H^{1,\mu}_{\alpha,0}(0, 1)$, we have
\[
\int_0^1 u\psi + x^\alpha u_x \psi_x - \frac{\mu}{x^{2-\alpha}} u\psi \, dx = \int_0^1 (g_1 + g_2) \psi \, dx \quad \forall \psi \in C_c^\infty(0, 1).
\]
By duality, this implies that
\[
u - (x^\alpha u_x)_x - \frac{\mu}{x^{2-\alpha}} u = g_1 + g_2,
\]
in the sense of distribution. Hence $u \in D(A^{\mu}_{\alpha})$ and
\[
u - A^{\mu}_{\alpha} u = g_1 + g_2 \quad \text{a.e. in } (0, 1),
\]
and the claim follows.

Applying the Hille-Yosida theorem (see [3, Theorem 4.5.1] or [11, Theorem A.7]), we conclude that if $F \in C^1([0, T]; \mathcal{H})$, i.e., $h \in C^1([0, T]; L^2(0, 1))$ and $(u_0, u_1) \in D(A^{\mu}_{\alpha})$, i.e., $u_0 \in H^{1,\mu}_{\alpha,0}(0, 1), A^{\mu}_{\alpha} u_0 \in L^2(0, 1), u_1 \in H^{1,\mu}_{\alpha,0}(0, 1)$, then system (3.2) (equivalently problem (3.1)) has a unique solution
\[
u \in C^1([0, T]; H^{1,\mu}_{\alpha,0}(0, 1)), \quad \frac{du}{dt} \in C^1([0, T]; L^2(0, 1)),
\]
\[A^{\mu}_{\alpha} u \in C([0, T]; L^2(0, 1)).
\]
If $u_0 \in H^{1,\mu}_{\alpha,0}(0, 1), u_1 \in L^2(0, 1)$ and $h \in L^2(0, 1)$ then the Cauchy problem (CP) has a mild solution
\[(u, u_t) \in C([0, T]; H^{1,\mu}_{\alpha,0}(0, 1)) \times C([0, T]; L^2(0, 1)).\]

\[\square\]

**Remark 3.** In order to prove our main controllability result we use the so-called multiplier method. The justification of the computations may sometimes be delicate since we work in non standard weighted spaces, specially in the critical case. For this reason, we make formal computations that may be justified by the regularization process described in [24, Remark 5]. More precisely, one can define a regularized operator
\[A^{\mu, n}_{\alpha} u := (x^\alpha u_x)_x + \frac{\mu}{(x + \frac{1}{n})^{2-\alpha}} u\]
whose domain is the same as in the purely degenerate case, that is to say $D^{\mu}_{\alpha} := H^{2}_{\alpha,0}(0, 1)$ (see [8]). Therefore the classical solutions $u^n$ of the regularized problem possess all the regularity required to justify the computations (see [1]). Passing to the limit as $n \to +\infty$, we recover the result for the solutions $u$ of (3.1). In what follows, we directly write the computations formally for the solutions $u$ of (3.1).

For all $\mu \leq \mu(\alpha)$, we define the generalized energy of a mild solution $u$ of (1.7) by:
\[
E_u(t) = \frac{1}{2} \int_0^1 \left\{ u^2_t(t, x) + x^\alpha u^2(t, x) - \frac{\mu}{x^{2-\alpha}} u^2(t, x) \right\} \, dx
\]
\[
= \frac{1}{2} \left[ \|u_t(t)\|_{L^2(0, 1)}^2 + \|u(t)\|_{H^{1,\mu}_{\alpha,0}}^2 \right], \quad \forall t \geq 0.
\] \[(3.5)\]

Classical computations show that the generalized energy $E_u$ of the solution is constant.

**Proposition 3.1.** Assume (1.2) holds and consider $(u_0, u_1) \in H^{1,\mu}_{\alpha,0}(0, 1) \times L^2(0, 1)$. Then the energy $t \mapsto E_u(t)$ of the mild solution $u$ of (1.7) is constant in time, i.e.
\[
E_u(t) = E_u(0), \quad \forall t \geq 0.
\] \[(3.6)\]
Proof. Suppose first that \( u \) is a classical solution of (1.7). Then, multiplying the equation by \( u_t \) and integrating by parts, we obtain

\[
0 = \int_0^1 u_t(t,x) \left\{ u_{tt}(t,x) - (x^\alpha u_x(t,x))_x - \frac{\mu}{x^{2-\alpha}} u(t,x) \right\} \, dx = \frac{d}{dt} E_u(t)
\]

\[
= \int_0^1 \left\{ u_t(t,x) u_{tt}(t,x) + x^\alpha u_x(t,x) u_{tx}(t,x) - \frac{\mu}{x^{2-\alpha}} u_t(t,x) u(t,x) \right\} \, dx
\]

\[- \left[ x^\alpha u_t(t,x) u_x(t,x) \right]_{x=0}^{x=1}.
\]

Using the boundary conditions, we see that the boundary terms vanish. We conclude that the energy of \( u \) is constant. The same conclusion can be extended to any mild solution by an approximation argument.

We give now some important lemmas that we shall need to handle boundary conditions in the proof of the multiplier identity (3.10).

**Lemma 3.2.** Let \( 0 \leq \alpha < 1 \) be given. Then, for all \( u \in H^2_\alpha(0,1) \),

\[
x^{\alpha-1}u^2(x) \to 0 \quad \text{as} \quad x \to 0^+.
\]

**Proof.** By the definition of \( H^2_\alpha(0,1) \) in the case \( 0 \leq \alpha < 1 \), we know that \( x^\alpha u_x \in H^1(0,1) \subset L^\infty(0,1) \) and \( u(0) = 0 \). Then,

\[
|u_x(x)| \leq c x^{-\alpha} \quad \text{and} \quad |u(x)| = \left| \int_0^x u_x(\sigma) \, d\sigma \right| \leq c x^{1-\alpha}.
\]

This implies \( |x^{\alpha-1}u^2(x)| \leq c x^{1-\alpha} \). Thus, \( \alpha < 1 \) yields the claim.

**Lemma 3.3.** Let \( 1 < \alpha < 2 \) be given. Then, for all \( u \in H^2_{\alpha,0}(0,1) \),

\[
x^{\alpha-1}u^2(x) \to 0 \quad \text{as} \quad x \to 0^+.
\]

**Proof.** Let \( u \) be given in \( H^2_{\alpha,0}(0,1) \). By the definition of \( H^2_{\alpha,0}(0,1) \), we know that \( u \in L^2(0,1) \) and \( x^{\alpha/2} u_x \in L^2(0,1) \). Then \( x^{\alpha-1} u^2 \in L^1(0,1) \). Moreover, we have:

\[
(x^{\alpha-1}u^2)_x = (\alpha-1)x^{\alpha-2}u^2 + 2x^{\alpha-1}u u_x.
\]

By (1.4), it is easy to see that \( x^{\alpha-2}u^2 \in L^1(0,1) \) and \( x^{\alpha-1}u u_x = (x^{\alpha/2}u) (x^{\alpha/2} u_x) \in L^1(0,1) \). Hence, \( x^{\alpha-1}u^2 \in W^{1,1}(0,1) \). Thus, \( x^{\alpha-1}u^2 \to 0 \) as \( x \to 0^+ \). Finally, \( L = 0 \) since \( L \neq 0 \) would imply \( x^{\alpha-2}u \notin L^2(0,1) \). This completes the proof.

### 3.2 Hidden regularity result

In this subsection, we prove a regularity result for (1.7) which is often called hidden regularity result.

**Lemma 3.4.** Assume that (1.2) holds. For any mild solution \( u \) of (1.7) we have that \( u_x(.,1) \in L^2(0,T) \) for every \( T > 0 \) and

\[
\int_0^T u_x^2(t,1) \, dt \leq (2T + 4) E_u(0), \quad \text{if} \quad 0 \leq \alpha < 1 \quad \text{and}
\]

\[
\int_0^T u_x^2(t,1) \, dt \leq (2T + 4C_\alpha') E_u(0), \quad \text{if} \quad 1 \leq \alpha < 2,
\]

(3.9)
where $C_\alpha$ is the constant appearing in (2.3).

Moreover,

$$
\int_0^T u_2^2(t,1) \, dt = \int_0^T \left\{ u_1^2(t,x) + (1 - \alpha) \left( x^\alpha u_2^2(t,x) - \frac{\mu}{x^2-\alpha} u^2(t,x) \right) \right\} \, dx \, dt \\
+ 2 \left[ \int_0^1 x u_x(t,x) u_t(x) \, dx \right]_{t=0}^{t=T}.
$$

(3.10)

**Proof.** The proof relies on the multiplier method combined with the new Hardy-type inequality (1.9). Suppose first that $(u_0, u_1) \in D_0^\alpha \times H^\mu_{\alpha,0}(0,1)$, so that $u$ is a classical solution of (1.7). Then, by multiplying (1.7) by $u_x$ and integrating over $(0,T) \times (0,1)$, we obtain

$$
0 = \int_0^T \int_0^1 x u_x(t,x) \left( u_{tt}(t,x) - (x^\alpha u_x(t,x))_x - \frac{\mu}{x^2-\alpha} u(t,x) \right) \, dx \, dt \\
= \left[ \int_0^T x u_x(t,x) u_t(t,x) \, dx \right]_{t=0}^{t=T} - \int_0^T \int_0^1 x u_x(t,x) u_t(t,x) \, dx \, dt \\
- \int_0^T \int_0^1 (\alpha x^\alpha u_2^2(t,x) + x^{\alpha+1} u_x(t,x) u_{xx}(t,x)) \, dx \, dt - \mu \int_0^T \int_0^1 x^{\alpha-1} u_x(t,x) u(t,x) \, dx \, dt \\
= \left[ \int_0^T x u_x(t,x) u_t(t,x) \, dx \right]_{t=0}^{t=T} - \int_0^T \int_0^1 \alpha x^\alpha u_2^2(t,x) \, dx \, dt \\
- \int_0^T \int_0^1 \left( x \left( \frac{u_1^2(t,x)}{2} \right)_x + x^{\alpha+1} \left( \frac{u_2^2(t,x)}{2} \right)_x + \mu x^{\alpha-1} \left( \frac{u_2^2(t,x)}{2} \right)_x \right) \, dx \, dt.
$$

(3.11)

Arguing as in the proof of [1, Lemma 3.2], integrations by parts lead to

$$
\int_0^T \int_0^1 x \left( \frac{u_1^2(t,x)}{2} \right)_x \, dx \, dt = -\frac{1}{2} \int_0^T \int_0^1 u_1^2(t,x) \, dx \, dt \\
\int_0^T \int_0^1 x^{\alpha+1} \left( \frac{u_2^2(t,x)}{2} \right)_x \, dx \, dt = \frac{1}{2} \int_0^T u_2^2(t,1) \, dt - \frac{(\alpha + 1)}{2} \int_0^T \int_0^1 x^\alpha u_2^2(t,x) \, dx \, dt.
$$

(3.12)

We proceed to integrate by parts the last term in the right hand side of (3.11). We obtain

$$
\int_0^T \int_0^1 x^{\alpha-1} \left( \frac{u_2^2(t,x)}{2} \right)_x \, dx \, dt = \frac{1}{2} \int_0^T \left[ x^{\alpha-1} u_2^2(t,x) \right]_{x=0}^{x=1} \, dt - \frac{(\alpha - 1)}{2} \int_0^T \int_0^1 \frac{u_2^2(t,x)}{x^{2-\alpha}} \, dx \, dt.
$$

Clearly $x^{\alpha-1} u_2^2 |_{x=1} = 0$. Now, we show also that

$$
x^{\alpha-1} u_2^2 |_{x=0} = 0.
$$

(3.13)

To this end, we recall that according to Remark 3, we consider regular solutions of the regularized problem. Thus $u$ takes its values in $D_0^\alpha = H^\mu_2(0,1)$. Hence, in the case $0 \leq \alpha < 1$, the claim follows by Lemma 3.2. Moreover, applying Lemma 3.3, the same conclusion still holds true if $1 < \alpha < 2$.

It follows that

$$
\int_0^T \int_0^1 x^{\alpha-1} \left( \frac{u_2^2(t,x)}{2} \right)_x \, dx \, dt = -\frac{(\alpha - 1)}{2} \int_0^T \int_0^1 \frac{u_2^2(t,x)}{x^{2-\alpha}} \, dx \, dt.
$$

(3.14)

Then the identity (3.10) follows by inserting (3.12) and (3.14) into (3.11).

Next, we proceed to prove (3.9). At this step, we need to distinguish two cases.
If $\alpha \in [0, 1)$. For any $\mu \leq \mu(\alpha)$, we have
\[
\left| \int_0^1 x u_x(t, x) u_t(t, x) \, dx \right| \leq \frac{1}{2} \int_0^1 \left\{ u_t^2(t, x) + x^2 u_x^2(t, x) \right\} \, dx
\]
(by (2.2))
\[
\leq \frac{1}{2} \int_0^1 \left\{ u_t^2(t, x) + x^\alpha u_x^2(t, x) - \frac{\mu}{x^{2-\alpha}} u^2(t, x) \right\} \, dx
= E_u(0) \quad \forall t \geq 0.
\] (3.15)

If $\alpha \in (1, 2)$. Similarly, by Young inequality and using (2.3), we also have for any $\mu \leq \mu(\alpha)$
\[
\left| \int_0^1 x u_x(t, x) u_t(t, x) \, dx \right| \leq \frac{C_\mu}{2} \int_0^1 \left\{ u_t^2(t, x) + x^\alpha u_x^2(t, x) - \frac{\mu}{x^{2-\alpha}} u^2(t, x) \right\} \, dx
= C_\mu E_u(0) \quad \forall t \geq 0.
\] (3.16)

Now, we deduce (3.9) from (3.10), (3.15), (3.16) and the constancy of the energy. The conclusion has thus been proved for classical solutions.

In order to extend (3.9) and (3.10) to the mild solution associated with the initial data $(u_0, u_1) \in H^{1,\mu}_{\alpha,0}(0, 1) \times L^2(0, 1)$, it suffices to approximate such data by $(u_0^n, u_1^n) \in D_\alpha \times H^{1,\mu}_{\alpha,0}(0, 1)$ and use (3.9) to show that the normal derivatives of the corresponding classical solutions give a Cauchy sequence in $L^2(0, T)$.

3.3 Solutions defined by transposition

Since we are dealing with boundary controls, we need to define the solution of (1.1) by the transposition method in the spirit of [20].

**Definition 3.1.** Let $f \in L^2(0, T)$ and let $(y_0, y_1) \in L^2(0, 1) \times H^{-1,\mu}_{\alpha,0}(0, 1)$ be fixed arbitrarily. We say that $y$ is a solution by transposition of (1.1) if
\[
y \in C^1([0, T]; H^{-1,\mu}_{\alpha,0}(0, 1)) \cap C([0, T]; L^2(0, 1))
\] satisfies, for all $T > 0$ and all, $(w_0^0, w_1^0) \in H^{1,\mu}_{\alpha,0}(0, 1) \times L^2(0, 1),$
\[
\left\langle y'(T), w_0^0 \right\rangle_{H^{-1,\mu}_{\alpha,0}(0, 1), H^{1,\mu}_{\alpha,0}(0, 1)} - \int_0^1 y(T) w_1^0 \, dx = \left\langle y_1, w(0) \right\rangle_{H^{-1,\mu}_{\alpha,0}(0, 1), H^{1,\mu}_{\alpha,0}(0, 1)} - \int_0^1 y_0 w'(0) \, dx - \int_0^T f(t) w_x(t, 1) \, dt.
\] (3.17)

where $w$ is the solution of the backward equation
\[
\begin{cases}
  w_{tt} - (x^\alpha w_x)_x - \frac{\mu}{x^{2-\alpha}} w = 0, & (t, x) \in Q, \\
  w(t, 1) = 0, & t \in (0, T), \\
  w(t, 0) = 0, & 0 \leq \alpha < 1, \\
  (x^\alpha w_x)(t, 0) = 0, & 1 < \alpha < 2, \\
  w(T, x) = w_0^0(x), & w_1(T, x) = w_1^0(x), & x \in (0, 1).
\end{cases}
\] (3.18)

Note that equation (3.18) can be reduced to (1.7) by changing $t$ in $T - t$. In fact, thanks to the change of variable $u(t, x) = w(T - t, x)$ and according to Theorem 3.1, the backward system (3.18) admits a unique solution $w \in C^1([0, T]; L^2(0, 1)) \cap C([0, T]; H^{1,\mu}_{\alpha,0}(0, 1))$ for each
For any mild solution $u$ of \((w_0^0, w_1^0) \in H^{-1,\mu}_{0,0}(0,1) \times L^2(0,1)\)
and the energy $E_w$ of $w$ is conserved through time.

Now let us define $L$ by

$$L(u,v) = \langle y_1, w(0) \rangle_{H^{-1,\mu}_{0,0}(0,1), H^{1,\mu}_{0,0}(0,1)} - \int_0^1 y_0 w'(0) \, dx - \int_0^T f(t) w(t,1) \, dt.$$ 

In view of the direct inequality (3.9), there exists a constant $D > 0$ such that

$$\int_0^T w^2(t,1) \, dt \leq DE_w(0) = DE_w(T),$$

where

$$E_w(T) = \frac{1}{2} \left[ \|w_1^0\|_{L^2}^2 + \|w_0^0\|_{H^{-1,\mu}_{0,0}}^2 \right].$$

Thus, the mapping $L$ is a continuous linear form with respect to $(w_0^0, w_1^0) \in H^{-1,\mu}_{0,0}(0,1) \times L^2(0,1)$. Therefore, from Riesz Theorem, there exist a unique couple $(y_0^0, y_1^0) \in L^2(0,1) \times H^{-1,\mu}_{0,0}(0,1)$ such that

$$\forall (w_0^0, w_1^0), \quad L(w_0^0, w_1^0) = \langle y_1^0, w_0^0 \rangle_{H^{-1,\mu}_{0,0}(0,1), H^{1,\mu}_{0,0}(0,1)} - \int_0^1 y_0^0 w_1^0 \, dx.$$ 

Hence, there is a unique solution by transposition $y \in C^1([0,T]; H^{-1,\mu}_{0,0}(0,1))^\ast \cap C([0,T]; L^2(0,1))$ of (1.1) (we refer to [20, Theorem 4.2, page 46–54] for the details).

4 Boundary observability

In this section the problem of boundary observability of the degenerate/singular wave equation (1.7) is studied. Our result guarantees the observability of system (1.7) under the condition (1.2). In order to prove such a result, we first prove the following identity.

**Lemma 4.1.** For any mild solution $u$ of (1.7) we have that, for each $T > 0$,

$$\int_0^T \left\{ \frac{\mu}{x^{2-\alpha}} u^2(t,x) - u^2(t,x) - u_1^2(t,x) \right\} \, dt \, dx + \left[ \int_0^T u(t,x) u_1(t,x) \, dx \right]_{t=0}^{t=T} = 0. \tag{4.1}$$

**Proof.** Suppose that $u$ is a classical solution of (1.7). Multiplying the wave equation (1.7) by $u$ and integrating over $(0,1) \times (0,T)$, we obtain

$$0 = \int_0^T \left. \int_0^1 u(t,x) \left( u_{tt}(t,x) - (x^\alpha u_x(t,x))_{x} - \frac{\mu}{x^{2-\alpha}} u(t,x) \right) \, dx \, dt \right|_{t=0}^{t=T}$$

$$= \left[ \int_0^1 u(t,x) u_1(t,x) \, dx \right]_{t=0}^{t=T} - \int_0^T \int_0^1 u_1^2(t,x) \, dx \, dt$$

$$- \int_0^T \int_0^1 x^\alpha u(t,x) u_x(t,x) \, dx \, dt + \int_0^T \int_0^1 x^\alpha u_x^2(t,x) \, dx \, dt - \int_0^T \int_0^1 \frac{\mu}{x^{2-\alpha}} u(t,x) u_x(t,x) \, dx \, dt.$$

The conclusion follows from the above identity because $x^\alpha u(t,x) u_x(t,x)$ vanishes at $x = 0, 1$, owing to [1, Proposition 2.5] and also Remark 3. An approximation argument allows us to extend the conclusion to mild solutions. \hfill \square

We are now ready to prove the following inverse or observability inequality.
**Theorem 4.1.** Assume (1.2) holds and let \( u \) be the mild solution of (1.7). Then, for every \( T > 0 \),

\[
\int_0^T u_x^2(t, 1) \, dt \geq \{ (2 - \alpha) T - 4 \} E_u(0). \tag{4.2}
\]

**Proof.** As usual, we suppose that \( u \) is a classical solution of (1.7), since the case of mild solutions can be recovered by approximation arguments. By adding to the right-hand side of (3.10) the left side of (4.1) multiplied by \( \frac{T}{2} \), we obtain

\[
\int_0^T u_x^2(t, 1) \, dt = (1 - \frac{\alpha}{2}) \int_0^T \int_0^1 \{ u_x^2(t, x) + x^\alpha u_x^2(t, x) - \frac{\mu}{x^2 - \alpha} u^2(t, x) \} \, dx \, dt
\]

\[
+ 2 \left[ \int_0^1 x u_x(t, x) u_t(t, x) \, dx \right]_{t=0}^{t=T} + \frac{\alpha}{2} \left[ \int_0^1 u(t, x) u_t(t, x) \, dx \right]_{t=0}^{t=T}.
\]

Using the definition of \( E_u \) and having in mind (3.6), this can be rewritten as

\[
\frac{1}{2} \int_0^T u_x^2(t, 1) \, dt = (1 - \frac{\alpha}{2}) T E_u(0) + \left[ \int_0^1 u_t(t, x) \left( x u_x(t, x) + \frac{\alpha}{4} u(t, x) \right) \, dx \right]_{t=0}^{t=T}. \tag{4.3}
\]

We now estimate the last term on the right-hand side of the inequality (4.3). First we write

\[
\left| \int_0^1 u_t(t, x) \left( x u_x(t, x) + \frac{\alpha}{4} u(t, x) \right) \, dx \right| \leq \frac{1}{2} ||u_t||_{L^2(0, 1)}^2 + \frac{1}{2} ||x u_x + \frac{\alpha}{4} u||_{L^2(0, 1)}^2. \tag{4.4}
\]

Next, we compute:

\[
||x u_x + \frac{\alpha}{4} u||_{L^2(0, 1)}^2 = \int_0^1 x^2 u_x^2 \, dx + \frac{\alpha}{4} \int_0^1 x (u^2)_x \, dx + \frac{\alpha^2}{16} \int_0^1 u^2 \, dx
\]

\[
= \int_0^1 x^2 u_x^2 \, dx + \left( \frac{\alpha^2}{16} - \frac{\alpha}{4} \right) \int_0^1 u^2 \, dx.
\]

Using (1.9), it follows that

\[
||x u_x + \frac{\alpha}{4} u||_{L^2(0, 1)}^2 \leq ||u||_{H^1_{\alpha, 0} \cap H^1_{\alpha, \alpha}}^2 + \frac{1}{4} + \frac{\alpha^2 - 4 \alpha}{16} \int_0^1 u^2 \, dx
\]

\[
\leq ||u||_{H^1_{\alpha, 0} \cap H^1_{\alpha, \alpha}}^2.
\]

Hence (4.4) becomes

\[
\left| \int_0^1 u_t(t, x) \left( x u_x(t, x) + \frac{\alpha}{4} u(t, x) \right) \, dx \right| \leq \frac{1}{2} ||u_t||_{L^2(0, 1)}^2 + \frac{1}{2} ||u||_{H^1_{\alpha, 0} \cap H^1_{\alpha, \alpha}}^2.
\]

Since \( || \cdot ||_{H^1_{\alpha, 0} \cap H^1_{\alpha, \alpha}} \leq || \cdot ||_{H^1_{\alpha, 0}} \) (\( \forall \mu \leq \mu(\alpha) \)), we get

\[
\left| \int_0^1 u_t(t, x) \left( x u_x(t, x) + \frac{\alpha}{4} u(t, x) \right) \, dx \right| \leq E_u(t) = E_u(0).
\]

Combining this last estimate together with (4.3), we obtain the observability inequality (4.2) with explicit constants.

We recall that (1.7) is said to be observable in time \( T > 0 \) if there exists a constant \( C > 0 \) such that for every \( (u_0, u_1) \in H^1_{\alpha, 0}(0, 1) \times L^2(0, 1) \), the mild solution of (1.7) satisfies

\[
\int_0^T u_x^2(t, 1) \, dt \geq CE_u(0).
\]
To make sure that the constant $C > 0$, we have to impose
\[ T > T_\alpha := \frac{4}{2 - \alpha}. \]

Summarizing, the main observability result is as follows:

**Corollary 4.1.** Assume (1.2). Then (1.7) is observable in time $T$, provided that
\[ T > T_\alpha. \]

**Remark 4.** Notice that when $\alpha = 0$, the system (1.1) is a non-degenerate linear wave equation perturbed by a singular inverse-square potential. By the known controllability results in [25] and [10], the observability time is $T_0 = 2$, which coincides with the classical observability time for the classical wave equation, see [4]. Letting $\alpha$ tend to zero, one can find that $\lim_{\alpha \to 0} T_\alpha = T_0$ and thus the above result complements those in [25] and [10] in the 1-dimensional case.

## 5 Boundary controllability

In this section, we study the boundary controllability of the degenerate and singular wave equation (1.1). Our main result guarantees the exact controllability of system (1.1) under the condition (1.2). More precisely, the main controllability result of this paper can be stated as follows.

**Theorem 5.1.** Assume (1.2) holds. Then, for every $T > T_\alpha$ and for any $(y_0, y_1) \in L^2(0, 1) \times H^{-1,0}_{\alpha,0} \mu(0, 1)$, there exists a control $f \in L^2(0, T)$ such that the solution of (1.1) (in the sense of transposition) satisfies $(y, y_t)(T, \cdot) = (0, 0)$.

**Proof.** The proof relies on the use of the Hilbert uniqueness method (HUM) introduced by J.-L. Lions in [20]. Let $(y_0, y_1) \in L^2(0, 1) \times H^{-1,0}_{\alpha,0} \mu(0, 1)$, $(w_0^0, w_1^0, v_0^0, v_1^0) \in H^{1,0}_{\alpha,0} \mu(0, 1) \times H^2(0, 1)$ be arbitrary pairs. Let $w$ and $v$ be the mild solutions of the backward problem (3.18) with final conditions $W^T := (w_0^T, w_1^T)$ and $V^T := (v_0^T, v_1^T)$, respectively. Let us consider the bilinear form $\Lambda$ on $H^{1,0}_{\alpha,0} \mu(0, 1) \times L^2(0, 1)$ defined by

\[ \Lambda(W^T, V^T) := \int_0^T w_x(t, 1)v_x(t, 1) \, dt \quad \forall W^T, V^T \in H^{1,0}_{\alpha,0}(0, 1) \times L^2(0, 1). \]

From the direct inequality (3.9), it is clear that $\Lambda$ is continuous on $H^{1,0}_{\alpha,0} \mu(0, 1) \times L^2(0, 1)$. On the other hand, thanks to the observability inequality (4.2), $\Lambda$ is coercive on $H^{1,0}_{\alpha,0} \mu(0, 1) \times L^2(0, 1)$ provided $T > T_\alpha$.

Next, we define the continuous linear map

\[ \ell(V^T) := \langle y_1, v(0) \rangle_{H^{-1,0}_{\alpha,0} \mu(0, 1), H^{1,0}_{\alpha,0} \mu(0, 1)} - \int_0^1 y_0 v'(0) \, dx, \quad \forall V^T \in H^{1,0}_{\alpha,0}(0, 1) \times L^2(0, 1). \]

Since $\Lambda$ is continuous and coercive on $H^{1,0}_{\alpha,0} \mu(0, 1) \times L^2(0, 1)$, and $\ell$ is continuous on the Hilbert space $H^{1,0}_{\alpha,0} \mu(0, 1) \times L^2(0, 1)$, by the Lax-Milgram Lemma, the variational problem

\[ \Lambda(W^T, V^T) = \ell(V^T) \quad \forall V^T \in H^{1,0}_{\alpha,0}(0, 1) \times L^2(0, 1) \]

has a unique solution $W_T \in H^{1,0}_{\alpha,0}(0, 1) \times L^2(0, 1)$. Then setting $f = w_x(t, 1)$ and $T > T_\alpha$, where $w \in C([0, T]; H^{1,0}_{\alpha,0}(0, 1)) \cap C^1([0, T]; L^2(0, 1))$ is the mild solution of the backward
problem (3.18) with \((w^0_T, w^1_T)\) as the final data, we see that
\[
\int_0^T f(t)v_x(t, 1) \, dt = \int_0^T w_x(t, 1)v_x(t, 1) \, dt = \Lambda (W^T, V^T)
\]
\[
= \langle y_1, v(0) \rangle_{H^{-1, \alpha}_{-0, 0}(0, 1), H^{1, \alpha}_{0, 0}(0, 1)} - \int_0^1 y_0v'(0) \, dx, \quad \forall V^T \in H^{1, \mu}_{0, 0}(0, 1) \times L^2(0, 1).
\]

On the other hand, if \(y\) is the solution by transposition of the problem (1.1), then equality (3.17) implies that, for all \(V^T \in H^{1, \mu}_{0, 0}(0, 1) \times L^2(0, 1)\), we have
\[
- \int_0^T f(t)v_x(t, 1) \, dt = \langle y'(T), v^0_T \rangle_{H^{-1, \alpha}_{-0, 0}(0, 1), H^{1, \alpha}_{0, 0}(0, 1)} - \int_0^1 y(T)v^1_T \, dx
\]
\[
- \langle y_1, v(0) \rangle_{H^{-1, \alpha}_{-0, 0}(0, 1), H^{1, \alpha}_{0, 0}(0, 1)} + \int_0^1 y_0v'(0) \, dx.
\]

Comparing the last relations (5.1)-(5.2), it follows that
\[
\langle y'(T), v^0_T \rangle_{H^{-1, \alpha}_{-0, 0}(0, 1), H^{1, \alpha}_{0, 0}(0, 1)} - \int_0^1 y(T)v^1_T \, dx = 0, \quad \forall \ (v^0_T, v^1_T) \in H^{1, \mu}_{0, 0}(0, 1) \times L^2(0, 1).
\]

From this we finally deduce that \((y(T), y'(T)) \equiv (0, 0)\), i.e. the system (1.1) is boundary null controllable in time \(T > T_\alpha\).

\section{Conclusions and open problems}

In this paper, we have analyzed the boundary controllability of the 1-D degenerate/singular wave equation. By means of the multiplier method, we have shown that the equation is observable. As a consequence, applying the Hilbert uniqueness method, we deduced the exact controllability result when the control acts on the nondegenerate/nonsingular boundary. Moreover, the optimal time of controllability was given.

We present hereafter a non-exhaustive list of comments and open problems related to our work.

1. As a first thing, we point out that combining our proofs with the ideas of [24], boundary exact controllability result can be obtained for the case of a degenerate/singular operator with \(\frac{\alpha}{2} = \beta \leq 2 - \alpha\) instead of \(\frac{\alpha}{2}\).

2. In [1], the authors treat the case of a degenerate operator \((a(x)u_x)_x\) with a general coefficient \(a(x)\) that vanishes at \(x = 0\). It would be interesting to consider a simultaneously degenerate and singular equation with a general degenerate inhomogeneous speed and general singular potential.

3. Inspired by the results in [18], it would also be interesting to study the wave equation with degeneracy and singularity at the interior of the space domain as done in [15] for the heat equation.

4. Finally, the study of null controllability properties of degenerate or singular coupled wave equations is still to be done and many further directions remain to be investigated.

\section{Appendix}

This section is devoted to the proof of Theorem 1.1 which is inspired by [25, Theorem 1.1]. The main point in the proof is the following change of variables
\[
U(x) = x^{\frac{\alpha}{2}-1} u(x), \quad i.e., \quad u(x) = x^{\frac{1}{2}-\frac{\alpha}{2}} U(x).
\]
Next we compute

\[
\begin{aligned}
\int_0^1 x^2 u_x^2 \, dx &= \int_0^1 x^2 \left( \frac{1-\alpha}{2} x^{-\frac{1}{2}} U + x^{-\frac{1}{2}} x^2 \right)^2 \, dx \\
&= \int_0^1 \left( \frac{1-\alpha}{2} \right)^2 x^{1-\alpha} U^2 + x^{3-\alpha} U_x^2 + (1-\alpha)x^2-\alpha \left( \frac{U_x^2}{2} \right) \, dx \\
&= \int_0^1 \left( \frac{1-\alpha}{2} \right)^2 x^{1-\alpha} U^2 + x^{3-\alpha} U_x^2 - \frac{(1-\alpha)(2-\alpha)}{2} x^{1-\alpha} U^2 \, dx.
\end{aligned}
\]

On the other hand, we have

\[
\begin{aligned}
\int_0^1 x^\alpha u_x^2 \, dx &= \int_0^1 x^\alpha \left( \frac{1-\alpha}{2} x^{-\frac{1}{2}} U + x^{-\frac{1}{2}} x^2 \right)^2 \, dx \\
&= \int_0^1 \left( \frac{1-\alpha}{2} \right)^2 x^{-1} U^2 + x U_x^2 + (1-\alpha) \left( \frac{U_x^2}{2} \right) \, dx \\
&= \int_0^1 \left( \frac{1-\alpha}{2} \right)^2 x^{-1} U^2 + x U_x^2 \, dx.
\end{aligned}
\]

Therefore,

\[
\begin{aligned}
\int_0^1 \left( x^\alpha u_x^2 - \mu(x) \frac{u^2}{x^{2-\alpha}} \right) \, dx &= \int_0^1 \left( \frac{1-\alpha}{2} \right)^2 x^{-1} U^2 + x U_x^2 - \left( \frac{1-\alpha}{2} \right)^2 x^{1-\alpha} U^2 \, dx \\
&= \int_0^1 x U_x^2 \, dx.
\end{aligned}
\]

Hence (1.9) may be rewritten exactly as follows:

\[
\int_0^1 x^{3-\alpha} U_x^2 \, dx \leq \int_0^1 x U_x^2 \, dx.
\]

And this inequality is trivially true by the definition of \( \alpha \).

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