The Formal Development of a Relative Measurement

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Abstract. Based on Krechmer K 2018 Relative Measurement Theory, Measurement 116 pp. 77-82, this paper presents a formal development of a relative measurement which includes accuracy and precision. Accuracy and precision, usually considered artifacts of experimental equipment and systems, are inherent in relative measurements. When accuracy and precision are included, experimental and theoretical measurements of the same magnitude property are shown to be equal.

1. Introduction
Prior to any measurement, only the observable property exists, e.g., distance, mass/energy, or time. How observable properties are identified is not a subject of this paper. However an observable property is always quantifiable in units of measure (units). The relative measurement of an observable property includes three processes:

1. The projection of stipulated units onto the observable property shown in Fig. 2.
2. The conversion of each unit to a measuring apparatus interval (MAI) shown in Fig. 3.
3. The comparison of each MAI to a reference (calibration) also shown in Fig. 3.

The first process determines a magnitude in units and is often described as a measurement. L. Euler [1] identifies that all measurement results are relative to each other. Without the next two processes a measurement is not relative. Without process 2 the measurement magnitude is a count of equal units, not a sum of MAI. Without process 3 any variation in each MAI is assumed to cancel. Assuming no variation of the MAI is valid when counting the teeth on a gear. Each physical gear tooth statistically varies and the sum of the MAI variation cancels. This assumption is not valid when making a measurement relative to a metre stick. In this case, the variation of each MAI due to calibration ($\Delta u_t$) tends not to cancel, as each $\Delta u_t$ is likely to be close to equal, see Fig. 3.

Fig. 1. Relative measurement system

In Fig. 1 the minimum single axis relative measurement system includes three entities: observable property, measuring apparatus with finite intervals, and a reference. The inner product (vertical arrows) of each observable's property $A$ and $B$ and each measuring apparatus establishes the $A$ and the $B$ vector magnitudes in intervals of $a_i$ and $b_j$. The reference $u$ is tightly correlated (relative) by calibration (diagonal arrows) to each experimental measuring apparatus interval (MAI). Fig. 1 does not include the variation of each MAI due to precision, i.e., the smallest resolution of the measuring apparatus.

When two calibration functions occur (which correlate each MAI of each measuring apparatus to one reference), measurement result $\Sigma a_i$ and measurement result $\Sigma b_j$ and the reference become relative to each
other (thin dash-dot line) and can be compared via a common factor of \( u \) (common reference). Relative describes the now correlated relation of the measurement result’s intervals, e.g., \( 2a_i = u = b_j \), as well as the now correlated relation of the measurement result’s relative magnitudes (\( A = 6 \) and \( B = 3 \)) in \( u_i \), or \( 6a_i = 3b_j \).

2. Formal Measurement
In an experimental measurement, the MAI and their coordinate axes are defined by the SI (International System of Units) [2]. The SI is the experimental vector space which describes experimental results. An MAI is correlated to the appropriate SI standard(s) using metrology. But each MAI is not often exactly equal to the others from the same measuring apparatus or exactly equal to the appropriate SI standard(s).

In this paper, the magnitudes of observables and their intervals are formalized without consideration for interactions with the measuring apparatus (i.e., observer effects) or any external effects such as noise. In carefully designed measurement and calibration experiments, these observer and external effects may be minimized or canceled and are not considered inherent.

3. Inner Product Function
An observable’s property (e.g., \( A \) or \( B \)) exists prior to any relative relation. Such an unmeasured magnitude can only be a norm or \( n(1/n)=1 \). Norms (bold) are self-relative and represent all the magnitude possibilities of that observable property.

The inner product function (Fig. 2) converts an observable's property's norm to a magnitude \( n \) of interval norms, \( u \). A measure (scalar magnitude in \( u \)) is calculated when each interval of the measuring apparatus (\( u \)) which projects (upward arrows) on the observable's property, is counted.

![Fig. 2. Inner product function](image)

In a normed vector space, (1) formalizes Fig. 2 as a sum of inner products \( \langle .. \rangle \) where \( i \in \{1,2,...,n\} \).

A single axis measurement reduces the inner product to a dot product where the \( \cos \theta = 1 \).

\[
\sum_{i=1}^{n} \langle \frac{V}{n}, u_i \rangle = nu_i \cos \theta = nu_i \hspace{1cm} (1)
\]

The inner product (1) correlates the observable and the measuring apparatus. That is, when they are separated, their relative absolute magnitude relationship continues. Fig. 1 shows that two observables, each correlated with different, but calibrated, measuring apparatuses, are also correlated with each other.

An example of measurement correlation in classical mechanics: A blindfolded carpenter takes two dowels, \( A \) and \( B \), aligns by feel one end of the two dowels and makes one saw cut across both dowels. The alignment (zero setting) and saw cut (projection) is an inner product which correlates the dowels. Each dowel's length magnitude, before it is correlated, is "one dowel", a norm or all the possible dowel lengths. The correlated dowels, when separated to anywhere in space-time, have an equal (with equal variation), previously unknown, length magnitude, \( |A| = |B| \), relative to a stipulated \( u \).
The linear result of an inner product is a scalar measure which is not relative to a coordinate axis. A non-linear calibration function in an SI vector space is required to transform a scalar measure into a relative measurement.

4. Calibration Function
Equation (1) and Fig. 2 present each $\mathbf{u}_i$ as normalized and equal. Exactly equal $\mathbf{u}_i$ are not possible in any experimental measuring apparatus [3]. Therefore a relative measurement is in theory $\Sigma u_i$ where $i=1$ to $n$, and in experiments the $\Sigma$MAI, not the count of $n \mathbf{u}_i$ interval norms. To convert Fig. 2 to an SI measurement result, a separate calibration function correlates $u$ to $\mathbf{u}_i$, creating the applied references $u_i \equiv MAI$, where $\equiv$ indicates equal with variation, as an exact (in theory) measurement result of each MAI is not possible (experimentally) due to the measuring apparatus precision $\pm s$ shown in Fig. 3.

Fig. 3, the calibration function ($cf$), projects each $u$ to transform the $i^{th}$ $\mathbf{u}_i$ to the $i^{th}$ $u_i$ with a total variation $\Delta u_i \pm s$:

![Fig. 3. The calibration function](image)

By stipulation, $u$ may be a primary standard, e.g., a metre, kilogram or second, or correlated to a primary standard, or $u$ may be a common reference, where $u$ is referenced by two or more calibration functions to establish relative relations. From Fig. 3 the difference $\Delta u_i = u_i - u$ (accuracy) is calculated from: $\sqrt{1-(cf/u)^2} = u_i$ and:

$$u_i \rightarrow u_i = u + \Delta u_i \pm s \quad (2)$$

Precision, $\pm s$ in (2), describes the smallest magnitude ($s$, a sample, in Fig. 3) that the perfect measuring apparatus can reliably detect. In a perfect measuring apparatus (which cannot create perfect measurement results), when $s$ of the measuring apparatus projects on the observable, it is $s$. When $s$ does not project on the observable, it is 0. When the magnitude of one measuring apparatus's $s$ projection is $>0$ and $<s$, that $s$ will be uncertain, 0 or $s$. Each of the two possible $s$ increments of $u_i$ shown in Fig. 3 may be uncertain due to zero setting or projection.

$\pm s$ has three possible magnitudes: $+s$, 0 (i.e., $<|s|$), -$s$. The zero magnitude is not a state (does not occur in theory), but is a transition between the $+s$ and -$s$ states, which has a low experimental probability, and is usually treated as an experimental error. The standard deviation of such a three magnitude distribution is just $<s$.

5. Relative Measurement Function
Replacing $\mathbf{u}_i$ in (1) with $u_i$ from (2) formalizes the relative measurement function:

$$\Sigma u_i = \sum_{i=1}^{n} \left(\frac{1}{n} \left(u + \Delta u_i \pm s\right)\right) \quad (3)$$

Equation (3) includes both accuracy (4), which tends to be large when $n$ is large, and precision (5), which tends to be small when $n$ is large:

$$\sum_{i=1}^{n} \Delta u_i = \text{accuracy} \quad (4) \quad \sum_{i=1}^{n} \pm s = \text{precision} \quad (5)$$
6. Experimental Variation
Consider a four digit voltmeter where 00.01 is the voltmeter's display of an MAI. Measuring a fixed voltage source multiple times produces a distribution of measurement results. A theoretical measurement calculation example: maintaining \(\pm u_i=0.01\) requires \(\leq \pm 0.005\) volt (v) precision (s) based upon sampling theory [4], i.e., each \(u_i\) is between 0.005 and 0.015 v. The smallest precision (magnitude of one sample) of a measuring apparatus is one cause of the variation of measurement results, and is never zero. The precision variation has three possible magnitudes: +0.005, 0.000 (<|0.005|), -0.005. A 1.000 ±0.001 v fixed voltage source is applied. ±0.001 v is the independently determined maximum variation (accuracy plus precision) of the 1.000 v fixed source relative to a primary standard. Applying (2), there are nine possible magnitudes of \(u_i \geq 0.001\) when \(u = 0.010\):

| \(\pm \Delta u_i \pm s\) | \(u_i\) | \(\pm \Delta u_i + s\) | \(u_i\) | \(\pm \Delta u_i + s\) | \(u_i\) |
|--------------------------|-------|------------------------|-------|------------------------|-------|
| -0.001 -0.005            | 0.004 | -0.001 +0.000          | 0.009 | -0.001 +0.005          | 0.014 |
| +0.000 -0.005            | 0.005 | +0.000 +0.000          | 0.010 | +0.000 +0.005          | 0.015 |
| +0.001 -0.005            | 0.006 | +0.001 +0.000          | 0.011 | +0.001 +0.005          | 0.016 |

Table 1. Nine magnitudes of \(u_i\)

Summing the nine possible magnitudes of \(u_i\) over \(n = 100\) of the \(u_i\) (of the 1.000 v fixed source) identifies 9100 combinations of sums which will, as the number of different voltmeter measurements of the same observable increase, converge to a normal distribution as described by the central limit theorem, i.e., 100 times 0.004 to 0.016 = 0.400 to 1.600 v.

An experimental calibration example: The four digit display of the voltmeter is adjusted to 00.00 (zero setting). The fixed voltage source (1.000v) is applied and a second voltmeter adjustment (reference setting) changes the display to 1.00. These calibration adjustments remove most of the variation, \(\sum(\Delta u_i \pm s)\), and establish 100(MAI) = 1.00 ±0.006 v. The theoretical measurement is 1.00 ± 0.01 v. This change from calculated distribution to calibrated measurement result distribution identifies why calibration must be included in a relative measurement.

7. Conclusion
The value of (3) is that this description of accuracy and precision is equal to the quantum mechanical description of measurement uncertainty and accuracy is also equal to the Lorentz transform in special relativity [5]. Equation (3) is thus the formal description of all experimental and theoretical measurements.

References
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