CAUCHY TRANSFORM AND POISSON’S EQUATION

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Abstract. Let $u \in W^{2,p}_0$, $1 \leq p \leq \infty$ be a solution of the Poisson equation $\Delta u = h$, $h \in L^p$, in the unit disk. It is proved that $\|\nabla u\|_{L^p} \leq a_p \|h\|_{L^p}$ with sharp constant $a_p$ for $p = 1$ and $p = \infty$ and that $\|\partial u\|_{L^p} \leq b_p \|h\|_{L^p}$ with sharp constant $b_p$ for $p = 1$, $p = 2$ and $p = \infty$. In addition is proved that for $p > 2$ $||\partial u||_{L^\infty} \leq c_p \|h\|_{L^p}$, and $||\nabla u||_{L^\infty} \leq C_p \|h\|_{L^p}$, with sharp constants $c_p$ and $C_p$. An extension to smooth Jordan domains is given. These problems are equivalent to determining the precise value of $L^p$ norm of Cauchy transform of Dirichlet’s problem.

Contents

1. Introduction 1
2. Some lemmas 7
3. $L^\infty$ norm of gradient 13
4. $L^p$ norm of Cauchy transform 16
5. The Hilbert norm of Cauchy transform 21
6. Refinement of $L^p$ norm 28
References 30

1. INTRODUCTION

1.1. Notation. By $\Omega$ is denoted the unit disk in the complex plane and by $\mathcal{T}$ its boundary. By $\Omega$ is denoted a bounded domain in complex plane. By $dA(z) = dx dy$, $z = x + iy$,

is denoted the Lebesgue area measure in the unit disk and by

$$d\mu(z) = \frac{1}{\pi} dx dy$$

is denoted normalized area measure. Here $W^{k,p}(\Omega)$ is the Banach space of $k$-times weak differentiable $p$–integrable functions. The norm in $W^{k,p}(\Omega)$ is

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defined by
\[\|u\|_{W^{k,p}} := \left(\int_\Omega \sum_{|\alpha| \leq k} |D^\alpha u|^p d\mu\right)^{1/p},\]
where \(\alpha \in \mathbb{N}_0^2\). If \(k = 0\), then \(W^{k,p} = L^p\) and instead of \(\|u\|_{L^p}\) we sometimes write \(\|u\|_p\). Another Banach space \(W^{k,p}_0(\Omega)\) arises by taking the closure of \(C^{k,0}_0(\Omega)\) in \(W^{k,p}(\Omega)\) (here \(C^{k,p}_0(\Omega)\) is the space of \(k\) times continuously differentiable functions with compact support in \(\Omega\), [9, p. 153-154]).

The main subject of this paper is a weak solution of Dirichlet problem
\[
\begin{cases}
u z = g(z), & z \in \Omega \\
u \in W^{1,p}_0(\Omega)
\end{cases}
\]
where \(4u z = \Delta u\) is the Laplacian. This equation is the Poisson’s equation. A weak differentiable function \(u\) defined in a domain \(\Omega\) with \(u \in W^{2,p}(\Omega)\) is a weak solution of Poisson’s equation if \(D_1 u\) and \(D_2 u\) are locally integrable in \(\Omega\) and
\[
\int_{\Omega} (D_1 u D_1 v + D_2 u D_2 v + 4 g v) d\mu(z) = 0,
\]
for all \(v \in C^{1,0}(\Omega)\).

It is well known that for \(g \in L^p(\Omega), \ p > 1\), the weak solution \(u\) of Poisson’s equation is given explicitly as the sum of Newtonian potential
\[
N(g) = \frac{2}{\pi} \int_{\Omega} \log |z - w| g(w) dudv, \quad w = u + iv
\]
and a harmonic function \(h\) such that \(h|_{\partial \Omega} + N(g)|_{\partial \Omega} \equiv u|_{\partial \Omega}\). In particular if \(\Omega = U\), then the function
\[
u(z) = \frac{2}{\pi} \int_{U} \log \left|\frac{z - w}{1 - \overline{z}w}\right| g(w) dA(w)
\]
is the explicit solution of \((1.1)\).

The function \(G\) given by
\[
G(z, w) = \frac{2}{\pi} \log \left|\frac{z - w}{1 - z\overline{w}}\right|, \quad z, w \in U,
\]
is called the Green function of the unit disk \(U \subset \mathbb{C}\) w.r. to Laplace operator. For \(g \in L^p(U), \ p > 1\), and
\[
u(z) = \frac{2}{\pi} \int_{U} \log \left|\frac{z - \omega}{1 - z\overline{\omega}}\right| g(\omega) dA(\omega),
\]
the Cauchy transform and conjugate Cauchy transform for Dirichlet’s problem (see [5, p. 155]) of \(g\) are defined by
\[
\mathcal{C}_U[g](z) = \frac{\partial u}{\partial z} = \frac{1}{\pi} \int_{U} \frac{1 - |\omega|^2}{(\omega - z)(\overline{\omega}z - 1)} g(\omega) dA(\omega)
\]
and
\[(1.5) \quad \tilde{C}_U[g](z) = \frac{\partial u}{\partial \bar{z}} = \frac{1}{\pi} \int_U \frac{1 - |\omega|^2}{(\bar{\omega} - \bar{z})(\omega \bar{z} - 1)} g(\omega) dA(\omega).\]

Here we use the notation
\[\frac{\partial}{\partial z} := \frac{1}{2} \left( \frac{\partial}{\partial x} + \frac{1}{i} \frac{\partial}{\partial y} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left( \frac{\partial}{\partial x} - \frac{1}{i} \frac{\partial}{\partial y} \right).\]

It is well-known that for \(p > 1\) Cauchy transforms \(C_U, \bar{C}_U : L^p(U) \to L^p(U)\) and \(\tilde{C}_U, \tilde{\bar{C}}_U : L^p(U) \to L^p(U)\) are bounded operators. Recall that the norm of an operator \(T : X \to Y\) between normed spaces \(X\) and \(Y\) is defined by
\[\|T\|_{X \to Y} = \sup\{\|Tx\| : \|x\| = 1\}.\]

The Jacobian matrix of a mapping \(u : \mathbb{C} \to \mathbb{C}\) is defined by
\[\nabla u = \begin{pmatrix} D_{11}u & D_{21}u \\ D_{12}u & D_{22}u \end{pmatrix}.\]

The matrix \(\nabla u\) is given by
\[(1.6) \quad \nabla u(z)h = \frac{2}{\pi} \int_U \left\langle \frac{(1 - |\omega|^2)}{(\omega - z)(\bar{\omega} \bar{z} - 1)}, h \right\rangle g(\omega) dA(\omega), \quad h \in \mathbb{C}.\]

Here \(\langle \cdot, \cdot \rangle\) denotes the scalar product. The equation \((1.6)\) defines the differential operator of Dirichlet’s problem
\[D_U : L^p(U, \mathbb{C}) \to L^p(U, \mathcal{M}_{2,2}), \quad D_U[g] = \nabla u.\]

Here \(\mathcal{M}_{2,2}\) is the space of square \(2 \times 2\) matrices \(A\) by the induced norm:
\[|A| = \max\{|Ah| : |h| = 1\}.\]

With respect to the induced norm there holds
\[(1.7) \quad |\nabla u| = |\partial u| + |\bar{\partial} u|,\]

and this implies that
\[(1.8) \quad |D_U[g]| = |C_U[g]| + |\tilde{C}_U[g]|.\]

1.2. Background. The starting point of this paper is the celebrated Calderon-Zygmund Inequality which states that. Let \(g \in L^p(\Omega), 1 < p < \infty\), and let \(w\) be the Newtonian potential of \(g\). Then \(u \in W^{2,p}(\Omega), \Delta u = g\) a.e. and
\[(1.9) \quad \|D^2u\|_p \leq C\|g\|_p\]

where \(D^2u\) is the weak Hessian matrix of \(u\) and \(C\) depends only on \(n\) and \(p\). Calderon-Zygmund Inequality is one of the main tools in establishing the a priori bound of \(W^{2,p}\) norm of \(u\) in terms of the function \(g\) and boundary condition (see [9] Theorem 9.13 or classical paper by Agmon, Douglis and Nirenberg [1]). It follows from these a priori bounds that for \(p > 1\) there exists a constant \(C_p\), such that
\[(1.10) \quad \|\nabla u\|_p \leq C_p\|g\|_p, \quad \text{for} \quad u \in W^{1,p}_0(U).\]
We refer to [9, Problem 4.10, p. 72] for some related estimates that are not sharp for the case $u \in C_0^2(\mathbb{B}^n)$, where $\mathbb{B}^n$ is the unit ball in $\mathbb{R}^n$.

Suppose now that $g$ is in $L^2(\Omega)$, where $\Omega$ is a bounded domain in the complex plane, and that $g = 0$ outside $\Omega$. The Cauchy transform $\mathcal{C}[g]$ of $g$, is defined by

$$\mathcal{C}[g] = \frac{1}{\pi} \int_{\Omega} \frac{g(w)}{w - z} dA(w).$$

Similarly is defined the Cauchy transform with respect to some positive Radon measure $\nu$. The operator $\mathcal{C}$ is a bounded operator from $L^2(\Omega)$ into itself. We want to point out the following result of Anderson and Hinkkanen [3]. If $\Omega = \mathbb{U}$, the Cauchy transform $\mathcal{C}[g]$ restricted to $\mathbb{U}$, satisfies

$$\|\mathcal{C}[g]\|_2 \leq \frac{2}{\alpha} \|g\|_2,$$

where $\alpha \approx 2.4048$ is the smallest positive zero of the Bessel function $J_0$:

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!^2} \left( \frac{x}{2} \right)^{2k}.$$

This inequality is sharp. This result has been extended by Dostanić to the smooth domains with sharp constants [13], and for $p \neq 2$ with some constants that are asymptotically sharp when $p$ is close to 1 or 2, see [12].

Associated to this Cauchy transform is the Beurling transform (called also Ahlfors-Beurling transform or Hilbert transform)

$$\mathfrak{B}[g](z) = \partial_z \mathcal{C}[g](z) = \text{pv} \int_{\mathbb{U}} \frac{g(w)}{(w - z)^2} d\mu(w),$$

where ”pv” indicates the standard principal value interpretation of the integral.

On the other hand, associated to Cauchy transform of Dirichlet’s problem is the Beurling transform of Dirichlet’s problem [5]

$$\mathcal{S}[g](z) = \partial_z \mathcal{C}_\mathbb{U}[g](z) = \text{pv} \int_{\mathbb{U}} \left( \frac{1}{(w - z)^2} + \frac{\mu^2}{(1 - wz)^2} \right) g(w) d\mu(w).$$

The Beurling transform and Beurling transform of Dirichlet’s problem are bounded operator in $L^p$, $1 < p < \infty$. This follows from Calderon-Zygmund Inequality. However, determining the precise value of the $L^p$-norm for $p \neq 2$ of Beurling transform is a well-known and long-standing open problem. On the other for $p = 2$, both Beurling transforms are the isometries of Hilbert space $L^2(\mathbb{U})$, and therefore have the norms equal to 1, see [5, Theorem 4.8.3] and [2, p. 87-111]. Beurling transforms are important in connection with nonlinear elliptic system in the plane and Beltrami equation (see [6], [2, Chapter V], [5, Chapter IV]). Cauchy transform and Cauchy transform of Dirichlet’s problem are connected by

$$\mathcal{C}_\mathbb{U}[g](z) = (\mathcal{C} - \mathcal{J}_\mathbb{U}^* )[g](z),$$
where
\[ J_0^*[g](z) = \frac{1}{\pi} \int_{\Omega} \frac{\omega}{1 - \bar{z}\omega} g(\omega) d\omega, \]
which satisfies
\[ J_0^* = B\mathcal{C}. \]
Thus
\[ (1.12) \quad C_U = \mathcal{C} - B\mathcal{C}. \]

The same can be repeated for conjugate Cauchy transform for Dirichlet’s problem and conjugate Beurling transform for Dirichlet’s problem. See [11] for this topic. Unlike the Beurling transform, the Cauchy transform is not a bounded operator considered as a mapping from \( L^2(\mathbb{C}) \) into itself. The reason is that the Lebesgue measure \( d\omega \) of the complex plane do not satisfies linear growth condition. Let \( \nu \) be a continuous positive Radon measure on \( \mathbb{C} \) without atoms. According to the result of Tosla [19], the Cauchy integral of the measure \( \nu \) is bounded on \( L^2(\mathbb{C},\nu) \) if and only if \( \nu \) has linear growth and satisfies the local curvature condition.

One of primary aims of this paper is to give an explicit constant \( C_p \) of inequality (1.10), and to generalize the inequality (1.11) for Cauchy transform of Dirichlet’s problem, which is equivalent with the problem of estimation of the following norms \( \|C_U\|_{L^p_{\Omega} \to L^p_{\Omega}} \), \( \|\bar{C}_U\|_{L^p_{\Omega} \to L^p_{\Omega}} \), and \( \|D_U\|_{L^p_{\Omega} \to L^p_{\Omega}} \). It follows from (1.8) and (1.10) that these norms are finite and that they can be estimated in terms of \( p \). In this paper we deal with the exact values of these norms.

The first main result of this paper is

**Theorem A** Let \( \alpha \approx 2.4048 \) be the smallest positive zero of the Bessel function \( J_0 \). For \( 1 \leq p \leq 2 \) we have

\[ (1.13) \quad \|C_U\|_{L^p_{\Omega} \to L^p_{\Omega}} \leq \frac{2}{\alpha^{2-2/p}} \text{ and } \|\bar{C}_U\|_{L^p_{\Omega} \to L^p_{\Omega}} \leq \frac{2}{\alpha^{2-2/p}}, \]

and for \( 2 \leq p \leq \infty \) we have

\[ (1.14) \quad \|C_U\|_{L^p_{\Omega} \to L^p_{\Omega}} \leq \frac{4}{3} \left( \frac{3}{2\alpha} \right)^{2/p} \text{ and } \|\bar{C}_U\|_{L^p_{\Omega} \to L^p_{\Omega}} \leq \frac{4}{3} \left( \frac{3}{2\alpha} \right)^{2/p}. \]

The equality is attained in all inequalities in (1.13) and (1.14) for \( p = 1, p = 2 \) and \( p = \infty \). Moreover for \( 1 \leq p \leq 2 \) there holds the inequality

\[ (1.15) \quad \|D_U\|_{L^p_{\Omega} \to L^p_{\Omega}} \leq 4\alpha^{2/p-2}, \]

and if \( 2 \leq p \leq \infty \)

\[ (1.16) \quad \|D_U\|_{L^p_{\Omega} \to L^p_{\Omega}} \leq \frac{16}{3\pi} \left( \frac{3\pi}{4\alpha} \right)^{2/p}. \]

The equality is attained in (1.15) and (1.16) for \( p = 1 \) and \( p = \infty \).
Notice that both Cauchy transforms $C_{U}$ and $C_{U}$ have the same Hilbert norm (cf. inequalities (1.11) and (1.13) for $p = 2$). It remains an open problem the precise determining of $L^p$ norm of $C_{U}$ for $1 < p < 2$ and $2 < p < \infty$ and of $D_{U}$ for $1 < p < \infty$.

For $q > 2$, and $g \in L^q$ the solution $u$ of Poisson equation (1.1) is in $C^{1,\alpha}(\Omega)$ for some $\alpha > 0$ (see for example [14]) which in particular implies that if $K \subset \Omega$ is a compact set, then there exists a constant $C_K$ such that $|\nabla u(z)| \leq C_K$, $z \in K$. The condition $q > 2$ is the best possible (see Example 3.2 below). We will show that for the unit disk, or more generally for smooth domains, the gradient of solution is globally bounded on the domain, see Corollary 3.5.

The second main result of the paper is precise estimation of $L^\infty$ norm of gradient which can be written in terms of operator norms as follows.

**Theorem B** For $q > 2$, and $p : \frac{1}{p} + \frac{1}{q} = 1$, there hold the following relations

\[(1.17) \quad \|C_{U}\|_{L^q \to L^\infty} = c_p, \]

\[(1.18) \quad \|\overline{C}_{U}\|_{L^q \to L^\infty} = c_p, \]

and

\[(1.19) \quad \|D_{U}\|_{L^q \to L^\infty} = C_p, \]

where

\[c_p = B(1 + p, 1 - p/2), \]

$B$ is the beta function, and

\[C_p = \frac{2^{2-p}\Gamma[(1 + p)/2]}{\sqrt{\pi}\Gamma[1 + p/2]}c_p. \]

The condition $q > 2$ is the best possible.

Together with this section, the paper contains five other sections. Section 2 contains some important formulas and sharp inequalities for potential type integrals. One of the main tools for the proving of these results are Möbius transformations of the unit disk and the Gauss hypergeometric function. Section 3 contains the proof of Theorem B together with an extension to smooth Jordan domain. Section 4 contains the proof of a weak form of Theorem A with exact constants for $p = 1$ and for $p = \infty$. Section 5 contains the proof of Theorem A for the Hilbert case, namely for $p = 2$. The proof is based on Boyd theorem ([7, Theorem 1, p. 368]), and involves the zeros of Bessel function. By making use of Riesz-Thorin interpolation theorem, in Section 6 we complete the proof of Theorem A.
2. SOME LEMMAS

We recall the classical definition of the Gauss hypergeometric function:

\[ _2F_1(a, b; c; z) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n, \]

where \((d)_n = d(d+1) \cdots (d+n-1)\) is the Pochhammer symbol. The series converges at least for complex \(z\in \mathbb{U}\) and for \(z\in \mathbb{T}\), if \(c > a + b\). We begin with the lemma which will be used in two our main inequalities.

**Lemma 2.1** (The main technical lemma). If \(1 \leq p < 2\), \(0 \leq \rho < 1\) and

\[ I_p = 2(1 - \rho^2)^{2-p} \int_0^1 r^{1-p}(1 - r^2)^p \frac{1 + r^2 \rho^2}{(1 - r^2 \rho^2)^3} dr, \]

then

\[ I_p = \frac{\Gamma[1 + p] \Gamma[1 - \rho^2]}{\Gamma[2 + p]} _2F_1\left(\frac{p}{2} - 1; p; \frac{p}{2} + 2; \rho^2\right), \]

where \(_2F_1\) is Gauss hypergeometric function. Moreover \(I_p\) is decreasing in \([0, 1]\) and there hold

\[ I_p(0) = \frac{\Gamma[1 + p] \Gamma[1 - \rho^2]}{\Gamma[2 + p]} \]

and

\[ I_p(1) := \lim_{\rho \to 1^{-0}} I_p(\rho) = \frac{\Gamma[1 + p] \Gamma[1 - \rho^2] \Gamma[3 - p]}{2 \Gamma[2 - \frac{p}{2}]} . \]

**Proof.** By applying partial integration we obtain

\[ I_p = \int_0^1 \frac{p(1 - \rho^2)^{2-p}(1 - r)^p r^{-p/2}(-4r + p(1 + r)^2)}{2(1 - r)^2(1 - \rho^2 r)} dr. \]

Since

\[ \frac{p(1 - \rho^2)^{2-p}(-4r + p(1 + r)^2)}{2(1 - r)^2(1 - \rho^2 r)} = \frac{2(p - p^2)}{(1 - \rho^2)^p(1 - r)} + \frac{2(p - p^2)(1 - \rho^2)}{(1 - \rho^2)^p(1 - r)^2} + \frac{p(-4\rho^2 + p(1 + \rho^2)^2)}{2(1 - \rho^2)^p(1 - r \rho^2)}, \]

by using the well known formulas

\[ _2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b) \Gamma(c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1} dt, \]

\[ _2F_1(a, b; c; 1) = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)}, \]

we obtain that
\[ I_p = \frac{\Gamma[1 + p] \Gamma[1 - \frac{p}{2}]}{4 \Gamma[2 + \frac{p}{2}]} L_p, \]

where

\[ L_p = \frac{(4 - p^2 - (2p + p^2)\rho^2 + p(-4\rho^2 + p(1 + \rho^2)^2)\phantom{2} {}_2F_1(1; 1 - \frac{p}{2}; \frac{4 + p}{2}; \rho^2))}{(1 - \rho^2)^p}. \]

By using the formula

\[ {}_2F_1(a, b; c; z) = (1 - z)^{c - a - b} \phantom{2} {}_2F_1(c - a, c - b; c; z) \]

we obtain

\[ (1 - \rho^2)^{-p} {}_2F_1(1; 1 - \frac{p}{2}; \frac{4 + p}{2}; \rho^2) = {}_2F_1(1 + \frac{p}{2}; 1 + p; \frac{4 + p}{2}; \rho^2). \]

Having in mind the fact

\[ (1 - \rho^2)^{-p} = \sum_{n=0}^{\infty} (-1)^n \left( \frac{-p}{n} \right) \rho^{2n} = \sum_{n=0}^{\infty} \frac{(p)_n}{n!} \rho^{2n}, \]

by calculating the Taylor coefficients we obtain

\[ L_p = 4 + \sum_{n=1}^{\infty} \frac{4p(p^2 - 4)(p)_n}{(2(n - 1) + p)(2n + p)(2(n + 1) + p)n!} \rho^{2n}, \]

where

\[ (p)_n := \prod_{k=0}^{n-1} (p + k). \]

It follows that

\[ L_p = 4 \sum_{n=0}^{\infty} \frac{(\frac{p}{2} - 1)_n (p)_n}{(\frac{p}{2} + 2)_n} \rho^{2n} \]

\[ = 4 \phantom{2} {}_2F_1(\frac{p}{2} - 1; p; \frac{p}{2} + 2; \rho^2). \]

From

\[ L_p = 4 \phantom{2} {}_2F_1(\frac{p}{2} - 1; p; \frac{p}{2} + 2; \rho^2), \]

because \( \frac{p}{2} + 2 > \frac{p}{2} - 1 + p \), we obtain

\[ \lim_{\rho \to 1} I_p(\rho) = \frac{\Gamma[1 + p] \Gamma[1 - \frac{p}{2}]}{\Gamma[2 + \frac{p}{2}]} \phantom{2} {}_2F_1(\frac{p}{2} - 1; p; \frac{p}{2} + 2; 1) \]

\[ = \frac{\Gamma[1 + p] \Gamma[1 - \frac{p}{2}] \Gamma[3 - p]}{2 \Gamma[2 - \frac{p}{2}]} \].
Lemma 2.2. For
\[ I_p(z) := \int_{U} \left( \frac{1 - |\omega|^2}{|z - \omega| \cdot |1 - \bar{z}\omega|} \right)^p d\mu(\omega), \quad 1 \leq p < 2 \]
there holds the sharp inequality
\[ I_p(z) \leq I_p(0) = B(1 + p, 1 - p/2), \]
where B is the beta function. Moreover
\[ I_1(z) = \frac{4_2F_1(-1/2; 1; 5/2; |z|^2)}{3} \leq \frac{4}{3}. \]
The case \( p = 1 \) of (2.4) has been already established in [15, Lemma 2.3].

Proof. For a fixed \( z \), we introduce the change of variables
\[ \frac{z - \omega}{1 - \bar{z}\omega} = a, \]
or, what is the same,
\[ \omega = \frac{z - a}{1 - \bar{z}a}. \]
Then
\[ I_p = \int_{U} \left( \frac{1 - |\omega|^2}{|z - \omega| \cdot |1 - \bar{z}\omega|} \right)^p d\mu(\omega) \]
\[ = \int_{U} \left( \frac{1 - |\omega|^2}{|a| \cdot |1 - \bar{z}\omega|^2} \right)^p d\mu(\omega) \]
\[ = \int_{U} \frac{1 - |\omega|^2}{|a|^p \cdot |1 - \bar{z}\omega|^2} \frac{(1 - |z|^2)^2}{|1 - \bar{z}a|^4} d\mu(a) \]
\[ = \int_{U} \frac{(1 - |a|^2)^p(1 - |z|^2)^2}{|a|^p \cdot |1 - \bar{z}a|^{4+2p} |1 - \bar{z}\omega|^{2p}} d\mu(a). \]
Since
\[ 1 - \bar{z}\omega = 1 - \bar{z} \frac{z - a}{1 - \bar{z}a} = \frac{1 - |z|^2}{1 - \bar{z}a}, \]
by using polar coordinates, we see that
\[ I_p = (1 - |z|^2)^{2-p} \int_{U} \frac{1 - |a|^2}{|a|^p |1 - \bar{z}a|^4} d\mu(a) \]
\[ = \frac{1}{\pi} (1 - |z|^2)^{2-p} \int_0^1 \rho^{1-p} |1 - \rho^2|^{2-p} d\rho \int_0^{2\pi} |1 - \bar{z}\rho e^{i\varphi}|^{-4} d\varphi. \]
By Parseval’s formula (see [17, Theorem 10.22]), we get
\[ \frac{1}{\pi} \int_0^{2\pi} \frac{dt}{|1 - ze^{it}|^4} = \frac{1}{\pi} \int_0^{2\pi} \frac{dt}{|(1 - ze^{it})^2|^2} \]
\[ = \frac{1}{\pi} \int_0^{2\pi} \left| \sum_{n=0}^{\infty} (n+1) \bar{z} \rho^n e^{nit} \right|^2 dt \]
\[ = 2 \sum_{n=0}^{\infty} (n+1)^2 |z|^{2n} \rho^{2n}. \]

Thus
\[ I_p = 2(1 - |z|^2)^{2-p} \sum_{n=0}^{\infty} (n+1)^2 |z|^{2n} \int_0^1 \rho^{1-p}(1 - \rho^2)^p \rho^{2n} d\rho, \]
which can be written in closed form as
\[ I_p = 2(1 - |z|^2)^{2-p} \int_0^1 \rho^{1-p}(1 - r^2)^p \frac{1 + r^2 \rho^2}{(1 - r^2 \rho^2)^3} dr. \]

From Lemma 2.1 it follows that
\[ I_p(z) \leq I_p(0) = \frac{\Gamma[1 + p] \Gamma(1 - p/2) \Gamma[1 + p/2] \Gamma[3 - p]}{\sqrt{\pi} \Gamma[1 + \frac{p}{2}] \Gamma[2 + \frac{p}{2}]}, \]
where B is the beta function.

**Lemma 2.3.** For \( \varphi \in [0, 2\pi] \), \( z = e^{i\varphi} \rho \in U, \ 1 \leq p < 2 \) and
\[ (2.6) \quad I_p := \frac{2}{\pi} \int_U \left| \text{Re} \frac{e^{-i\varphi}(1 - |\omega|^2)}{(\omega - z)(z\bar{\omega} - 1)} \right|^p dA(\omega) \]
there holds the equality
\[ (2.7) \quad I_p = 2 \frac{\Gamma[1 + p] \Gamma[1 - \frac{p}{2}] \Gamma[1 + \frac{p}{2}]}{\sqrt{\pi} \Gamma[1 + \frac{p}{2}] \Gamma[2 + \frac{p}{2}]} \text{F}_1\left(\frac{p}{2} - 1; p; \frac{p}{2} + 2; \rho^2\right). \]

Moreover
\[ \frac{\Gamma[1 + p] \Gamma[1 - \frac{p}{2}] \Gamma[1 + \frac{p}{2}]}{\sqrt{\pi} \Gamma[2 - \frac{p}{2}] \Gamma[1 + \frac{p}{2}]} \leq I_p \leq 2 \frac{\Gamma[1 + p] \Gamma[1 - \frac{p}{2}] \Gamma[1 + \frac{p}{2}]}{\sqrt{\pi} \Gamma[1 + \frac{p}{2}] \Gamma[2 + \frac{p}{2}]} - \frac{2}{\sqrt{\pi} \Gamma[1 + \frac{p}{2}] \Gamma[2 + \frac{p}{2}]}. \]

In particular
\[ (2.8) \quad I_1 = 16 \text{F}_1(-1/2; 1; 5/2; \rho^2) \leq \frac{16}{3\pi}. \]

**Proof.** For \( \varphi \in [0, 2\pi] \) let \( \xi = e^{i\varphi} \). Then by introducing the change \( w = e^{i\varphi} \omega \) we obtain
\[ \frac{2}{\pi} \int_U \left| \text{Re} \frac{1 - |\omega|^2}{(e^{i\varphi} \omega - \zeta)(\zeta e^{i\varphi} \omega - 1)} \right|^p dA(\omega) \]
\[ = \frac{2}{\pi} \int_U \left| \text{Re} \frac{1 - |\omega|^2}{(w - \zeta)(\zeta w - 1)} \right|^p dA(w). \]
It follows in particular that

$$I_p = \frac{2}{\pi} \int_U \left| \text{Re} \frac{e^{i\alpha}(1 - |\omega|^2)}{(\omega - z)(\bar{\omega} - 1)} \right|^p dA(\omega).$$

As in Lemma 2.2 for a fixed $z$, we introduce the change of variables

$$\omega = z - a.$$ 

Then

$$I_p = 2 \int_U \left| \text{Re} \frac{e^{i\alpha}(1 - |\omega|^2)}{a \cdot (1 - \bar{\omega} a)^2} \right|^p d\mu(\omega)$$

$$= 2 \int_U \left| \text{Re} \frac{e^{i\alpha}(1 - |\omega|^2)}{a \cdot (1 - \bar{\omega} a)^2} \right|^p \frac{(1 - |z|^2)^2}{|1 - \bar{\omega} a|^4} d\mu(a)$$

$$= 2 \int_U \left| \text{Re} \frac{e^{i\alpha}}{a \cdot (1 - \bar{\omega} a)^2} \right|^p \frac{(1 - |a|^2)^p(1 - |z|^2)^2 + p}{|1 - \bar{\omega} a|^{4 + 2p}} d\mu(a).$$

Since

$$1 - \bar{\omega} a = 1 - \bar{\omega} \frac{z - a}{1 - \bar{\omega} a} = \frac{1 - |z|^2}{1 - \bar{\omega} a},$$

as in the proof of Lemma 2.2, we obtain

$$I_p = 2(1 - |\omega|^2)^{2 - p} \int_U (1 - |a|^2)^p \text{Re} \left( \frac{e^{i\alpha}}{a} \right)^2 \frac{1}{|1 - \bar{\omega} a|^{4 + 2p}} d\mu(a).$$

Introducing polar coordinates $a = re^{ix}$ we have that

$$I_p = \frac{2}{\pi} (1 - \rho^2)^{2 - p} \int_0^1 \int_0^{2\pi} (1 - r^2)^p |\cos x + r^2 \rho^2 \cos x - 2r \rho|^p \frac{d\rho}{(1 + r^2 \rho^2)^{2 + p}} dx dr.$$ 

Let $\tau = \frac{2r \rho}{1 + r^2 \rho^2}$. It is clear that $0 \leq \tau \leq 1$. Then

$$\int_0^{2\pi} \frac{|\cos x + r^2 \rho^2 \cos x - 2r \rho|^p}{(1 + r^2 \rho^2)^{2 + p}} dx$$

$$= \frac{1}{(1 + r^2 \rho^2)^2} \int_{-\pi}^\pi \left| \frac{\tau - \cos x}{1 - \tau \cos x} \right|^p \frac{1}{(1 - \tau \cos x)^2} dx.$$

$$= \frac{2}{(1 + r^2 \rho^2)^2} \int_0^\pi \left| \frac{\tau - \cos x}{1 - \tau \cos x} \right|^p \frac{1}{(1 - \tau \cos x)^2} dx.$$
Introducing the change
t = \frac{\tau - \cos x}{1 - \tau \cos x},
or what is the same
\cos x = \frac{\tau - t}{1 - \tau t},
we obtain
\int_0^\pi \left| \frac{\tau - \cos x}{1 - \tau \cos x} \right|^p \frac{1}{(1 - \tau \cos x)^2} dx = \int_{-1}^1 |t|^p \frac{(1 - \tau t)(1 - \tau^2)^{-1}}{(1 - t^2)^{1/2}(1 - \tau^2)^{1/2}} dt
= (1 - \tau^2)^{-3/2} \int_{-1}^1 \frac{|t|^p}{(1 - t^2)^{1/2}} dt
= (1 - \tau^2)^{-3/2} \frac{\sqrt{\pi} \Gamma((1 + p)/2)}{\Gamma[1 + p/2]}.

Therefore
(2.9) \quad I_p = \frac{4}{\sqrt{\pi}} \frac{\Gamma[(1 + p)/2]}{\Gamma[1 + p/2]} \frac{(1 - \rho^2)^{2-p}}{r^{2-p}(1 - r^2)^p} \frac{1 + r^2 \rho^2}{(1 - r^2 \rho^2)^3} dr
i.e.
(2.10) \quad I_p = \frac{2}{\sqrt{\pi}} \frac{\Gamma[(1 + p)/2]}{\Gamma[1 + p/2]} I_p,
where
I_p = 2(1 - \rho^2)^{2-p} \int_0^1 r^{1-p}(1 - r^2)^p \frac{1 + r^2 \rho^2}{(1 - r^2 \rho^2)^3} dr.

Now (2.7) follows from (2.1).

From (2.10) and (2.1) we obtain
\[ I_p(0) = \frac{2 \Gamma[1 + p] \Gamma[1 - \frac{p}{2}] \Gamma[\frac{1 + p}{2}]}{\sqrt{\pi} \Gamma[1 + \frac{p}{2}] \Gamma[2 + \frac{p}{2}]}, \]
and
\[ \lim_{\rho \to 1} I_p(\rho) = \frac{\Gamma[1 + p] \Gamma[1 - \frac{p}{2}] \Gamma[\frac{1 + p}{2}] \Gamma[3 - p]}{\sqrt{\pi} \Gamma[2 - \frac{p}{2}] \Gamma[1 + \frac{p}{2}]} \]

\[ \square \]

Corollary 2.4. For \( z \in \mathbb{U} \) and \( 1 \leq p < 2 \) there hold the equalities
\[ \int_{\mathbb{U}} \left| \operatorname{Re} \frac{(1 - |\omega|^2)}{(\omega - z)(z\bar{\omega} - 1)} \right|^p dA(\omega) \]
(2.11)
\[ = \int_{\mathbb{U}} \left| \operatorname{Im} \frac{(1 - |\omega|^2)}{(\omega - z)(z\bar{\omega} - 1)} \right|^p dA(\omega) \]
\[ = \frac{\sqrt{\pi} \Gamma[1 + p/2]}{2 \Gamma[(1 + p)/2]} \int_{\mathbb{U}} \left| \frac{(1 - |\omega|^2)}{(\omega - z)(z\bar{\omega} - 1)} \right|^p dA(\omega). \]
Notice that for $1 \leq p < 2$

$$\frac{\pi}{4} \leq \frac{\sqrt{\pi\Gamma[1+p/2]}}{2\Gamma[(1+p)/2]} < \frac{\sqrt{\pi\Gamma[1+2/2]}}{2\Gamma[(1+2)/2]} = 1.$$ 

3. $L^\infty$ norm of gradient

**Theorem 3.1.** If $u \in W^{2,p}_0$ is a solution, in the sense of distributions, of Dirichlet’s problem $u_{z\bar{z}} = g(z), g \in L^q(U), q > 2, 1/p + 1/q = 1$, then for

$$(3.1) \quad c_p = B(1 + p, 1 - p/2),$$

where $B$ is the beta function, and

$$(3.2) \quad C_p = \frac{2^{2-p}\Gamma[(1+p)/2]}{\sqrt{\pi}\Gamma[1+p/2]} c_p$$

there hold the following sharp inequalities

$$(3.3) \quad |\partial u(z)| \leq c_p\|g\|_q, \quad z \in U,$$

$$(3.4) \quad |\bar{\partial} u(z)| \leq c_p\|g\|_q, \quad z \in U,$$

$$(3.5) \quad |\nabla u(z)| \leq C_p\|g\|_q, \quad z \in U.$$ 

The condition $q > 2$ is the best possible. From $3.3$–$3.5$ we have the following relations

$$(3.6) \quad \|C_\mathbf{U}\|_{L^n \to L^\infty} = c_p,$$

$$(3.7) \quad \|\bar{\partial} C_\mathbf{U}\|_{L^n \to L^\infty} = c_p,$$

and

$$(3.8) \quad \|\nabla C_\mathbf{U}\|_{L^n \to L^\infty} = C_p.$$ 

In the following example it is shown that the condition $q > 2$ i.e. $p < 2$ in Theorem 3.1 is the best possible.

**Example 3.2.** For $z \in U \setminus \{0\}$ define $g$ by

$$g(z) = \frac{z}{|z| \log |z|} \left( \frac{1 - |z|^2}{|z|^2} \right).$$

It is easy to verify that $g(z) \in L^2(U)$. On the other hand for the solution $u \in W^{1,2}_0$ of Poisson equation $\Delta u = 4g$, $\nabla u(0)$ do not exist and $\nabla u(z)$ is unbounded in every neighborhood of 0.

Since $C_1 = \frac{16}{3\pi}$ and $c_1 = \frac{4}{3}$ we obtain

**Corollary 3.3.** Under the condition of Theorem 3.1 for $p = 1$, i.e. $q = \infty$ we have the following sharp inequalities

$$\|\nabla u\|_\infty \leq \frac{16}{3\pi}\|g\|_\infty,$$

$$\|\partial u\|_\infty \leq \frac{4}{3}\|g\|_\infty,$$
\[ \|\partial u\|_\infty \leq \frac{4}{3}\|g\|_\infty. \]

For a positive nondecreasing continuous function \( \omega : [0, l] \to \mathbb{R}, \omega(0) = 0 \) we will say that is Dini’s continuous if it satisfies the condition
\[ \int_0^l \frac{\omega(t)}{t} dt < \infty. \]

A smooth Jordan curve \( \gamma \) with the length \( l = |\gamma| \), is said to be Dini’s smooth if the derivative of its natural parametrization \( g \) has the modulus of continuity \( \omega \) which is Dini’s continuous.

**Proposition 3.4** (Kellogg). (See [16] and [20]) Let \( \gamma \) be a Dini’s smooth Jordan curve and let \( \Omega = \text{Int}(\gamma) \). If \( \varphi \) is a conformal mapping of \( U \) onto \( \varphi \), then \( \varphi' \) and \( \log \varphi' \) are continuous on \( U \).

For a conformal mapping \( \varphi \) there holds
\[ \Delta(u \circ \varphi)(z) = |\varphi'(z)|^2 \Delta u(\varphi(z)), \]
and
\[ |\nabla(u \circ \varphi)(z)| = |\varphi'(z)||\nabla u(\varphi(z))|. \]

By using Theorem 3.1 and Proposition 3.4 and relations (3.10) and (3.11), we obtain

**Corollary 3.5.** Let \( \Omega \) be a Jordan domain bounded by a Dini’s smooth Jordan curve \( \gamma \). If \( u \in W^{2,p}_0(\Omega) \) is a solution, in the sense of distributions, of Dirichlet’s problem \( u|_{\gamma} = g(z), g \in L^q(\Omega), q > 2, \frac{1}{p} + \frac{1}{q} = 1 \), then
\[ |\partial u(z)| \leq c_p C_\Omega \|g\|_q, \quad z \in \Omega, \]
\[ |\bar{\partial} u(z)| \leq c_p C_\Omega \|g\|_q, \quad z \in \Omega, \]
\[ |\nabla u(z)| \leq C_p C_\Omega \|g\|_q, \quad z \in \Omega, \]
with the constant \( c_p \) and \( C_p \) defined in (3.1) and (3.2) and
\[ C_\Omega = \inf \left\{ \max\left\{ \frac{|\varphi'(z)|^2}{\min\{|\varphi'(z)| : z \in T\}} : z \in T \right\} \right\}, \]
where \( \varphi \) ranges over all conformal mappings of the unit disk onto \( \Omega \).

**Proof of Theorem 3.1.** We start by the formula (1.4) to obtain
\[ |\partial u(z)| \leq \int_U \frac{1 - |\omega|^2}{|\omega - z| \cdot |\omega z - 1|} |g(\omega)|d\mu(\omega). \]

According to Hölder’s inequality it follows that
\[ |\partial u(z)| \leq \left( \int_U \left( \frac{1 - |\omega|^2}{|z - \omega| \cdot |1 - \bar{\omega}z|} \right)^p d\mu(\omega) \right)^{1/p} \left( \int_U |g(\omega)|^q d\mu(\omega) \right)^{1/q}. \]
By using Lemma 2.2 we obtain
\[ |\nabla u(z)|^p \leq I_p(0)\|g\|_q^p = c_p\|g\|_q^p. \]

The inequality (3.3) easily follows. To show that the inequality is sharp take
\[ g(z) = -\frac{z}{|z|} \left( \frac{1 - |z|^2}{|z|} \right)^{p-1}. \]

Then
\[
\partial u(0) = \frac{1}{\pi} \int_{B^2} \frac{1 - |\omega|^2}{-w} \cdot \left( -\frac{\omega}{|\omega|} \left( \frac{1 - |\omega|^2}{|w|} \right)^{p-1} \right) dA(\omega) \\
= \frac{1}{\pi} \int_{B^2} \left( \frac{1 - |\omega|^2}{|\omega|} \right)^p dA(\omega) \\
= \frac{1}{\pi} \int_0^1 \int_{2\pi}^r r^{1-p} (1 - r^2)^p dr dt \\
= 2 \int_0^1 r^{1-p} (1 - r^2)^p dr \\
= B(1 + p, 1 - p/2) \\
= I_p(0) \\
= I_p(0)^{1/p} I_p(0)^{1/q} \\
= c_p\|g\|_q.
\]

To prove the inequality (3.5), we begin by the equality
\[ \nabla u(z) h = 2 \int_{U} \left\langle \frac{(1 - |\omega|^2)}{(\omega - z)(\bar{\omega} - 1)} , h \right\rangle g(\omega) d\mu(\omega), \quad h = e^{i\varphi} \in T. \]

It follows that
\[ |\nabla u(z)| \leq 2 \left( \int_{U} \left| \Re \frac{e^{-i\varphi} (1 - |\omega|^2)}{(\omega - z)(\bar{\omega} - 1)} \right|^p d\mu(\omega) \right)^{1/p} \|g\|_q. \]

Lemma 2.3 implies
\[ |\nabla u(z)| \leq 2 \left( \frac{\Gamma(1 + p)\Gamma[1 - \frac{p}{2}]\Gamma[1 + \frac{p}{2}]}{\sqrt{\pi} \Gamma[1 + \frac{p}{2}]\Gamma[2 + \frac{p}{2}]} \right)^{1/p} \|g\|_q. \]

The equality is achieved by the following function
\[ g(re^{it}) = \left| \Re \frac{1 - r^2}{re^{it}} \right| \frac{\varphi}{\varphi} \sign(\cos t). \]
Namely
\[
\|g\|_q^q = \frac{1}{\pi} \int_0^{2\pi} |\cos t|^p dt \int_0^1 r \left( \frac{1-r^2}{r} \right)^p dr
\]
\[
= \frac{2}{\sqrt{\pi}} \frac{\Gamma((1 + p)/2)}{\Gamma(1 + p/2)} \int_0^1 r^{1-p}(1 - r^2)^p dr
\]
\[
= 2^{p-2} C_p^p
\]
and
\[
\nabla u(0)(1,0) = \frac{1}{2\pi} \int_U \text{Re} \left( \frac{1-|\omega|^2}{-\omega} \right) g(\omega) dA(\omega)
\]
\[
= \frac{1}{2\pi} \int_0^{2\pi} |\cos t|^{1+p} dt \int_0^1 r^{1-p}(1 - r^2)^p dr
\]
\[
= \frac{1}{\sqrt{\pi}} \frac{\Gamma((1 + p)/2)}{\Gamma(1 + p/2)} \int_0^1 r^{1-p}(1 - r^2)^p dr
\]
\[
= 2^{p-3} C_p^p = C_p \|g\|_q.
\]

Remark 3.6. The solution to the Poisson’s equation with homogeneous boundary condition over the unit disk with \( g \) defined in (3.15) for \( p = 1 \), i.e. for \( g(z) = -z/|z| \) is

\[
u(z) = \frac{4}{3} e^{-it}(r - r^2) = \frac{4}{3} \bar{z}(1 - |z|).
\]

It would be of interest to find the solution for arbitrary \( 1 \leq p < 2 \).

4. \( L^p \) norm of Cauchy transform

In this section we consider the situation \( 1 \leq p < \infty \).

Theorem 4.1. If \( u \) is a solution, in the sense of distributions, of Dirichlet’s problem \( u_{\zeta\zeta} = g(z) \), \( u \in W^{1,p}_0(U) \), \( z \in U \), \( g \in L^p(U) \), \( 1 < p < \infty \), then

\[
\|\nabla u\|_p \leq 4 \left( \frac{4}{3\pi} \right)^{1-1/p} \|g\|_p
\]
(4.1)

\[
\|\partial u\|_p \leq 2 \left( \frac{2}{3} \right)^{1-1/p} \|g\|_p
\]
(4.2)

and

\[
\|\bar{\partial} u\|_p \leq 2 \left( \frac{2}{3} \right)^{1-1/p} \|g\|_p
\]
(4.3)
Remark 4.2. The inequalities (4.1) and (4.2) are asymptotically sharp as $p$ approaches 1 or $\infty$. We will treat the Hilbert case $p = 2$ separately in order to obtain the sharp constant for the case $p = 2$ (see Section 5). Making use of this fact and by using Riesz-Thorin interpolation theorem, we will improve inequalities (4.1) and (4.2) (see Section 6). It remains to find sharp inequalities for $0 < p < 2$ and $2 < p < \infty$.

**Corollary 4.3.** Under the conditions of Theorem 4.1 there hold the following sharp inequalities

(4.4) \[ \| \nabla u \|_1 \leq 4 \| g \|_1, \]
(4.5) \[ \| \partial u \|_1 \leq 2 \| g \|_1, \]
and
(4.6) \[ \| \bar{\partial} u \|_1 \leq 2 \| g \|_1. \]

**Proof of Corollary 4.3.** Let $u_n$ be a solution of $\Delta u = 4g_n$ for $g_n = n^2 \chi_{_{1/n}}^{U}$. By using (1.4) and polar coordinates, we obtain

\[
\frac{\partial u_n}{\partial z} = \frac{1}{\pi} \int_{U} \frac{1 - |\omega|^2}{(\omega - z)(\omega z - 1)} g_n(\omega) dA(\omega)
\]
\[
= \frac{n^2}{\pi} \int_{1/nU} \frac{1 - |\omega|^2}{(\omega - z)(\omega z - 1)} dA(\omega)
\]
\[
= \frac{n^2}{\pi} \int_{0}^{1/n} r(1 - r^2) dr \int_{0}^{2\pi} \frac{1}{(re^{it} - z)(re^{-it} - 1)} dt
\]
\[
= \frac{n^2}{\pi} \int_{0}^{1/n} r \int_{|\zeta| = 1} \frac{(1 - r^2) d\zeta}{i(r\zeta - z)(rz - \zeta)}
\]

Let
\[
\lambda_z(r) = \int_{|\zeta| = 1} \frac{d\zeta}{i(r\zeta - z)(rz - \zeta)}.
\]

Then by Cauchy residue theorem, for almost every $r$\[
\lambda_z(r) = \text{Ind}_T \left( \frac{z}{r} \right) \text{Res}_{\zeta = \frac{z}{r}} \frac{2\pi i(1 - r^2)}{i(r\zeta - z)(rz - \zeta)} + \text{Res}_{\zeta = zr} \frac{2\pi i(1 - r^2)}{i(r\zeta - z)(rz - \zeta)}\]

Therefore
\[
\lambda_z(r) = \begin{cases} 0, & \text{if } |\frac{z}{r}| < 1; \\ \frac{2\pi}{r}, & \text{if } |\frac{z}{r}| > 1. \end{cases}
\]

Thus
\[
\frac{\partial u_n}{\partial z} = \begin{cases} 2n^2 \int_{|z|}^{1/n} \frac{r}{r} dr, & \text{if } |z| < \frac{1}{n}; \\ 2n^2 \int_{0}^{1/n} \frac{r}{r} dr, & \text{if } \frac{1}{n} \leq |z| \leq 1. \end{cases}
\]

It follows that
\[
\frac{\partial u_n}{\partial z} = \begin{cases} \frac{|z|^2 n^2}{r}, & \text{if } |z| < 1/n; \\ \frac{1}{z}, & \text{if } 1/n \leq |z| < 1. \end{cases}
\]
Then
\[ a_n := \int_U |\frac{\partial u_n}{\partial z}| d\mu = 2 \left(1 - \frac{2}{3n}\right). \]
On the other hand
\[ b_n := \int_U |g_n(z)| d\mu(z) = 1. \]
Since
\[ \lim_{n \to \infty} \frac{a_n}{b_n} = 2, \]
the inequality (4.5) is sharp. On the other hand since \( u_n \) is a real function, it follows that
\[ |\nabla u_n(z)| = 2 \left|\frac{\partial u_n}{\partial z}\right| \]
and this shows that (4.4) is sharp.

Observe that the sequence \( g_n \) converges to Dirac delta function \( \delta(0) = \infty, \delta(z) = 0, z \neq 0, \int \delta(z)d\mu = 1. \)

The solution \( u_n \in W^{2,p}_0 \) of equation \( \Delta u = g_n \) converges to \( u(z) = \frac{1}{z} \) which is a solution of \( \Delta u = \delta. \)\]

In order to prove Theorem 4.1 we need the following lemmas.

**Lemma 4.4** (Young’s inequality for convolution). [22, pp. 54-55; 8, Theorem 20.18] If \( h \in L^p(\mathbb{R}), \) and \( \rho \in C_c^\infty(\mathbb{R}), \) such that \( \int \rho dt = 1 \), then
\[ \|h \ast \rho\|_p \leq \|h\|_p. \]

We make use of the following immediate corollary of [9, Lemma 9.17]:

**Lemma 4.5** (Stability Lemma). Let \( v \) be a weak solution of \( \Delta v = h \) with \( v \in W^{1,p}_0, 1 < p < \infty. \) Then for some \( C = C(p) \)
\[ \|\nabla v\|_{L^p} \leq C\|h\|_p. \]

**Lemma 4.6.** For \( \omega \in U \) the function defined by
\[ J(\omega) := \frac{1}{2} \int_U \frac{1 - |\omega|^2}{|z - \omega| \cdot |1 - \bar{z}\omega|} d\mu(z), \]
is equal to
\[ J(\omega) = \frac{1 - |\omega|^2}{2|\omega|} \log \frac{1 + |\omega|}{1 - |\omega|}. \]

**Proof.** For a fixed \( \omega \), we introduce the change of variables
\[ a(\omega) = \frac{\omega - z}{1 - z\omega}, \]
and recall that
\[ a'(\omega) = -\frac{1 - |\omega|^2}{(1 - z\omega)^2}. \]
We obtain
\[
J(\omega) = \frac{1}{2} \int_U \frac{1 - |\omega|^2}{|1 - a\omega|^2} \frac{1}{|a'(z)|^2} d\mu(a)
= \frac{1}{2} \int_U \frac{|1 - a\omega|^{-2}}{(1 - |\omega|^2)^{-1}} |a|^{-1} d\mu(a)
= \frac{1}{2} (1 - |\omega|^2) \int_U |a|^{-1} |1 - a\omega|^{-2} d\mu(a)
= \frac{1}{2} (1 - |\omega|^2) \int_0^1 dt \int_0^{2\pi} |(1 - \omega e^{it})^{-1}|^2 dt.
\]

By using Parseval’s formula to the function
\[
f(a) = (1 - \omega a)^{-1} = \sum_{k=0}^\infty \bar{\omega}^k a^k,
\]
we obtain
\[
J(\omega) = (1 - |\omega|^2) \sum_{k=0}^\infty \frac{|\omega|^{2k}}{2k + 1}
= \frac{1 - |\omega|^2}{2|\omega|} \log \frac{1 + |\omega|}{1 - |\omega|}.
\]

\[\square\]

**Proof of Theorem 4.1.** Let \( I_1(z) \) be the function defined by (2.8). We introduce appropriate mollifiers: Fix a smooth function \( \rho : \mathbb{R} \to [0,1] \) which is compactly supported in the interval \((-1,1)\) and satisfies \( \int \rho = 1 \). For \( \varepsilon > 0 \) consider the mollifier
\[
(4.10) \quad \rho_\varepsilon(t) := \frac{1}{\varepsilon} \rho \left( \frac{t}{\varepsilon} \right).
\]
It is compactly supported in the interval \((-\varepsilon, \varepsilon)\) and satisfies \( \int \rho_\varepsilon = 1 \). For \( \varepsilon > 0 \) define
\[
g_\varepsilon(x) = \int_\mathbb{R} g(y) \frac{1}{\varepsilon} \rho \left( \frac{x-y}{\varepsilon} \right) dy = \int_\mathbb{R} g(x - \varepsilon z) \rho(z) dz.
\]
Then \( g_\varepsilon \) converges to \( g \) as \( \varepsilon \to 0 \) in \( L^p \) norm. Let \( u_\varepsilon(z) \in C^\infty_0(U) \) be a solution to \( u_{zz} = g_\varepsilon \). Then by using the Jensen’s inequality, and having in mind the fact that the measure
\[
d\nu_\varepsilon(\omega) := 2 \text{Re} \left( \frac{e^{-i\varphi} (1 - |\omega|^2)}{(\omega - z)(\omega z - 1)} \right) d\mu(\omega)
\]
is a probability measure in the unit disk, for \( h = e^{i\varphi} \), by using (1.4), we obtain
$|\nabla u_\varepsilon(z)|^p = \left| \int_U 2 \left\langle \frac{1 - |\omega|^2}{(\omega - z)(\overline{\omega}z - 1)}, h \right\rangle g_\varepsilon(\omega) d\mu(\omega) \right|^p$

\leq \left( \int_U 2 |\text{Re} e^{-i\varphi} (1 - |\omega|^2) g_\varepsilon(\omega)| d\mu(\omega) \right)^p

= \mathcal{I}_1^p(z) \left( \int_U |g_\varepsilon(\omega)| \text{Re} \frac{2e^{-i\varphi} (1 - |\omega|^2)}{(\omega - z)(\overline{\omega}z - 1)} d\mu(\omega) \right)^p

\leq \mathcal{I}_1^{p-1}(z) \int_U |g_\varepsilon(\omega)|^p d\mu(\omega).

Thus

(4.11) $|\nabla u_\varepsilon(z)|^p \leq \mathcal{I}_1^{p-1}(z) \int_U |\frac{2(1 - |\omega|^2)}{(\omega - z)(\overline{\omega}z - 1)}||g_\varepsilon(\omega)||^p d\mu(\omega).

Integrating (4.11) over the unit disk $U$, by using Fubini’s theorem it follows that

$\int_U |\nabla u_\varepsilon(z)|^p d\mu(z) \leq \int_U \mathcal{I}_1^{p-1}(z) \int_U |\frac{2(1 - |\omega|^2)}{(\omega - z)(\overline{\omega}z - 1)}||g_\varepsilon(\omega)||^p d\mu(\omega) d\mu(z)

= \int_U |g_\varepsilon(\omega)|^p d\mu(\omega) \int_U \mathcal{I}_1^{p-1}(z) |\frac{2(1 - |\omega|^2)}{(\omega - z)(\overline{\omega}z - 1)}| d\mu(z).

Since

$\mathcal{I}_1 = \frac{162F_1(-1/2; 1; 5/2; |z|^2)}{3\pi} \leq \frac{16}{3\pi},$

we obtain first that

$\mathcal{I}_1^{p-1} \leq \left( \frac{16}{3\pi} \right)^{p-1}.$

On the other hand by (4.9),

$\int_U \frac{2(1 - |\omega|^2)}{|z - \omega| \cdot |1 - \overline{z}\omega|} d\mu(z) \leq 4.$

It follows that

$\|\nabla u_\varepsilon\|_p^p \leq 4 \left( \frac{16}{3\pi} \right)^{p-1} \|g_\varepsilon\|_p^p,$

i.e.

$\|\nabla u_\varepsilon\|_p \leq 4 \left( \frac{4}{3\pi} \right)^{1-1/p} \|g_\varepsilon\|_p.$

Take $h = g - g_\varepsilon$. Then by (4.7) we obtain

$\|\nabla u_\varepsilon - \nabla u\|_p \leq C\|g - g_\varepsilon\|_p.$

This implies that

$\lim_{\varepsilon \to 0} \|\nabla u_\varepsilon - \nabla u\|_p = \lim_{\varepsilon \to 0} C\|g - g_\varepsilon\|_p = 0$.
and therefore
\[(4.12) \lim_{\varepsilon \to 0} \|\nabla u_\varepsilon\|_p^p = \|\nabla u\|_p^p,\]
On the other hand by Lemma 4.4 we obtain
\[(4.13) \|g_\varepsilon\|_p^p \leq \|g\|_p^p.\]
The conclusion is
\[\|\nabla u\|_p \leq 4 \left(\frac{4}{3\pi}\right)^{1/p} \|g\|_p^p,\]
as desired. An analogous proof yields the inequalities (4.2) and (4.3). In this case we make use of the following probability measure
\[d\mu_z(\omega) := \frac{e^{-i\varphi(1 - |\omega|^2)}}{(\omega - z)(\omega \bar{z} - 1)} I_1(z),\]
and the relation
\[I_1 = \frac{4_2F_1(-1/2; 1; 5/2; |z|^2)}{3} \leq \frac{4}{3},\]
which coincides with (2.5).

5. The Hilbert norm of Cauchy transform

In this section we determine the precise value of the operator norm \(C_U\) when considered as an operator from the Hilbert space \(L^2(U)\) into itself. It follows from our proof that the Hilbert norm of \(C_U\) coincides with the Hilbert norm of \(\mathcal{C} : L^2(U) \rightarrow L^2(U)\), which has been determined by Anderson and Hinkkanen in [3]. For \(k \in \mathbb{Z}\), we denote by \(\alpha_k\) the smallest positive zero of the Bessel function
\[J_k(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! (m + k)!} \left(\frac{x}{2}\right)^{2m+k},\]
the smallest positive zero of the Bessel function \(J_0\) satisfies \(\alpha := \alpha_0 \approx 2.4048256\) by [21, p. 748], and hence \(\frac{2}{\alpha_0} \approx 0.83166\).

Concerning the zeros of Bessel functions there hold the following

Lemma 5.1. [21, section 15.22, p. 479] The sequence \(\alpha_k\) is increasing for \(k \geq 0\).

The main theorem of this section is the theorem:

Theorem 5.2. The norm of the operator \(C_U : L^2(U) \rightarrow L^2(U)\) is
\[\|C_U\| = \frac{2}{\alpha}.\]
In other words
\[(5.1) \|C_U[g]\|_2 \leq \frac{2}{\alpha} \|g\|_2, \quad \text{for} \quad g \in L^2(U).\]
The equality holds in (5.1) if and only if \(g(z) = c|z|J_0(\alpha|z|), \text{ for a.e. } z \in U\), where \(c\) is a complex constant.
To prove Theorem 5.2 it suffices to show that
\[
\|C_U[P]\| \leq \frac{2}{\alpha}\|P\|_2
\]
whenever
\[
P(z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn} z^n \bar{z}^m
\]
is a polynomial in \(z\) and \(\bar{z}\), since such functions are dense in \(L^2(U)\) and \(\frac{2}{\alpha}\) is the best constant. In this case only finitely many of the complex numbers \(a_{mn}\) are nonzero. It is evident that there exist radial functions \(f_d, d \in \mathbb{Z}\) such that
\[
P(z) = \sum_{d=-\infty}^{\infty} g_d(z),
\]
where \(g_d(z) = f_d(r)e^{idt}, d = m - n\). Observe that \(g_{d_1}\) and \(g_{d_2}\) are orthogonal for \(d_1 \neq d_2\) in Hilbert space \(L^2(U)\). In the following proof we will show that \(C_{U}[g_{d_1}]\) and \(C_{U}[g_{d_2}]\) are orthogonal in Hilbert space \(L^2(U)\).

Thus
\[
\|C_{U}[P]\|_2 \leq \frac{2}{\alpha}\|P\|_2
\]
if and only if
\[
\sum_{d=-\infty}^{\infty} \|C_{U}[g_d]\|_2 \leq \frac{2}{\alpha}\sum_{d=-\infty}^{\infty} \|g_d\|_2.
\]
We will show a bit more, we will prove the following lemma, which is the main ingredient of the proof of Theorem 5.2.

**Lemma 5.3.** For \(d \in \mathbb{Z}\) there holds the following sharp inequality
\[
\|C_{U}[g_d]\|_2 \leq \frac{2}{\alpha|d|}\|g_d\|_2,
\]
where \(g_d(z) = f_d(r)e^{idt} \in L^2(U), z = re^{it}\).

Before proving Lemma 5.3 we need some preparation. By using (1.4) and polar coordinates, we obtain
\[
C_{U}[g_d] = \frac{1}{\pi} \int_U \frac{1 - |\omega|^2}{(\omega - z)(\bar{\omega}z - 1)} g_d(\omega) dA(\omega)
\]
\[
= \frac{1}{\pi} \int_U \frac{(1 - |\omega|^2)f_d(r)e^{idt}}{(\omega - z)(\bar{\omega}z - 1)} dA(\omega)
\]
\[
= \frac{1}{\pi} \int_0^1 r f_d(r) dr \int_0^{2\pi} \frac{(1 - r^2)e^{idt}}{(re^{it} - z)(re^{-it}z - 1)} dt
\]
\[
= \frac{1}{\pi} \int_0^1 r f_d(r) \int_{|\zeta|=1} \frac{(1 - r^2)\zeta^d}{i(r\zeta - z)(r\zeta - z)} d\zeta.
\]
Let
\[ \lambda_z(r) = \int_{|\zeta|=1} \frac{\zeta^d(1-r^2)d\zeta}{i(r\zeta-z)(rz-\zeta)}. \]

Then by Cauchy residue theorem, for every \( r \neq |z| \)
\[ \lambda_z(r) = \text{Ind}_T \left( \frac{z}{r} \right) \text{Res}_{\zeta=\bar{z}} \frac{2\pi i(1-r^2)\zeta^d}{i(r\zeta-z)(rz-\zeta)} + \text{Res}_{\zeta=0} \frac{2\pi i(1-r^2)\zeta^d}{i(r\zeta-z)(rz-\zeta)}. \]

Thus
\[ \lambda_z(r) = \begin{cases} 
-\text{Ind}_T \left( \frac{z}{r} \right) \frac{2\pi z^d-1}{r^d} + \frac{2\pi z^d-1}{r^d}, & \text{if } d \leq 0; \\
-\text{Ind}_T \left( \frac{z}{r} \right) \frac{2\pi z^d-1}{r^d} + 2\pi z^{d-1}r^d, & \text{if } d > 0.
\end{cases} \]

Therefore for \( d \leq 0 \)
\[ \lambda_z(r) = \begin{cases} 
0, & \text{if } r > |z|; \\
\frac{2\pi z^d-1}{r^d}, & \text{if } r < |z|,
\end{cases} \]

and hence
\[ (5.6) \quad C_U[g_d] = 2z^{d-1} \int_0^{|z|} f_d(r)r^{-d+1}dr. \]

For \( d > 0 \)
\[ \lambda_z(r) = \begin{cases} 
-\left( \frac{z}{r} \right) \frac{2\pi z^d-1}{r^d} + 2\pi z^{d-1}r^d, & \text{if } r > |z|; \\
2\pi z^{d-1}r^d, & \text{if } r < |z|,
\end{cases} \]

and therefore
\[ (5.7) \quad C_U[g_d] = 2z^{d-1} \left( \int_0^1 f_d(r)r^{d+1}dr - \int_{|z|}^1 f_d(r)r^{-d+1}dr \right). \]

First of all, it follows from (5.6) and (5.7) that for \( d_1 \neq d_2, C_U[g_{d_1}] \) and \( C_U[g_{d_2}] \) are orthogonal.

In particular if \( f_d(r) = r^{n+m} \), i.e. \( g_d(z) = z^m\bar{z}^n \), and \( d = m-n \leq 0 \), then from (5.6) we have that
\[ (5.8) \quad C_U[z^m\bar{z}^n] = \frac{1}{n+1} z^m\bar{z}^{n+1}. \]

If \( d = m-n > 0 \), from (5.7) we obtain
\[ (5.9) \quad C_U[z^m\bar{z}^n] = \frac{1}{n+1} \left( z^m\bar{z}^{n+1} - \frac{m-n}{m+1} z^m\bar{z}^{n-1} \right). \]

\textit{Proof of Lemma 5.3} We divide the proof into two cases.
5.1. Case \( d = m - n \leq 0 \). From (5.6) we deduce that

\[
L_d := \int_U |C_U|^2 d\mu(z) = 8 \int_0^1 r^{2d-2} \left| \int_0^r f_d(s) s^{-d+1} ds \right|^2 dr.
\]

On the other hand let

\[
R_d := \int_0^1 r |f_d(r)|^2 dr.
\]

We should find the best constant \( A_d \) such that

\[
L_d \leq A_d R_d,
\]

i.e. the best constant \( B_d = \frac{1}{A_d} \) such that

\[
\int_0^1 r^{2d-2} \left| \int_0^r f_d(s) s^{-d+1} ds \right|^2 dr \leq B_d \int_0^1 r |f_d(r)|^2 dr.
\]

Without loss of generality, assume that \( f_d \) is real and positive in \([0, 1]\). By setting \( h(s) = f_d(s) s^{-d+1} \) we obtain the following inequality

\[
\int_0^1 x^{2d-1} \left( \int_0^r h(s) ds \right)^2 dx \leq B_d \int_0^1 r^{2d-1} (h(r))^2 dr.
\]

This problem, which involves an inequality of Hardy type, can be solved by appealing to a more general result of Boyd ([7, Theorem 1, p. 368]) (Proposition 5.4).

To formulate the result of Boyd we need some definitions and facts. For \( \omega, m \in C^1(a, b) \), we assume that \( w(x) > 0 \) and \( m(x) > 0 \) for \( a < x < b \). By Proposition 5.4.

\[T_1 f(x) = \omega(x)^{1/p} m(x)^{-1/p} \int_a^x f(t) dt.\]

Let

\[L_m = \left\{ f : \|f\|_s := \left( \int_0^1 |f|^s m(x) dx \right)^{1/s} < \infty \right\} .\]

A simple sufficient condition for \( T \) to be compact from \( L_m \to L_{m'} \) (here \( s' \) is conjugate of \( s \) \( (s' = s/(s-1)) \)), is that the function \( k \) defined by

\[k(x, t) := \omega(x)^{1/p} m(x)^{-1/p} m(t)^{-1} \chi_{[a,x]}(t)\]

have finite \( (r', s) \)-double norm

\[
(5.10) \quad \|T\| = \left\{ \int_a^b \left[ \int_a^b k(x, t) r' m(t) dt \right] \frac{1}{m(x)} dx \right\}^{1/s} < \infty, \quad s > 1, \quad r > 1.
\]

Here \( r' = r/(r-1) \). For this argument we refer to [23, p. 319].

**Proposition 5.4.** [7] Suppose that \( \omega, m \in C^1(a, b) \), that \( w(x) > 0 \) and \( m(x) > 0 \) for \( a < x < b \), that \( p > 0, \ r > 1, \ 0 < q < r \), and that the operator \( T_1 \) is compact from \( L_{m'} \to L_{m'}(s = pr/(r-q)).\)
Then, the following eigenvalue problem (P) has solutions \((y, \lambda), y \in C^2(a,b)\) with \(y(x) > 0, y'(x) > 0\) in \((a,b)\).

\[
\frac{d}{dx}(r\lambda y^{-1}y' - qy^{q-1}\omega) + py^{p-1}y'^{q\omega} = 0
\]

\[
\lim_{x \to a^+} y(x) = 0, \quad \lim_{x \to b^-} (r\lambda y^{-1}y' - qy^{q-1}\omega(x)) = 0
\]

\[
\|y'\|_r = 1.
\]

There is a largest value \(\lambda\) such that (5.11) has a solution and if \(\lambda^*\) denotes this value, then for any \(f \in L^r_m\),

\[
\int_a^b \left| \int_a^x f \right| q\omega(x)dx \leq \frac{r\lambda^*}{p+q} \left\{ \int_a^b |f|^r m(x)dx \right\}^{(p+q)/r}.
\]

Equality holds in (5.12) if and only if \(f = cy\) a.e. where \(y\) is a solution of (5.11) corresponding to \(\lambda = \lambda^*\), and \(c\) is any constant.

In our case we have \(r = 2, p = 2, q = 0, m(x) = x^{2d-1}\) and \(\omega(x) = x^{2d-1}\), \(s = pr/(r-q) = 2\) and \(s' = s/(s-1) = 2\). The corresponding differential equality is equivalent to

\[
x^2y'' + (2d-1)xy' + \frac{x^2}{\lambda}y = 0.
\]

The additional compactness condition required is proved by proving that the operator

\[T_1f(x) = \int_0^x f(t)dt\]

is compact from \(L^2_m \to L^2_m\). Namely by (5.10) we have to show that \(\|T_1\| < \infty\). In this case

\[k(x, t) = t^{1-2d}1_{[0,x]}(t).
\]

Thus

\[\|T_1\| = \left\{ \int_0^1 \int_0^x t^{2-4d}t^{2d-1}dt.x^{2d-1}dx \right\}^{1/2} = \frac{1}{2\sqrt{1-d}} < \infty.
\]

The positive solution of (5.13) is

\[
y(x) = x^{-d+1}J_{-d+1}(\frac{x}{\sqrt{\lambda}}),
\]

where \(J_{-d+1}\) is the Bessel function.

Then by [21] Section 3.2, p. 45

\[y'(x) = \frac{x^{-d+1}}{\sqrt{\lambda}}J_{-d}(\frac{x}{\sqrt{\lambda}})
\]

and therefore

\[y'(1) = \frac{1}{\sqrt{\lambda}}J_{-d}(\frac{1}{\sqrt{\lambda}}).\]
Thus the largest possible value of $\lambda$ is $B_d = \lambda_d$ where
$$
\frac{1}{\sqrt{\lambda_d}} = \alpha_d,
$$
and $\alpha_d$ is the smallest positive zero of $J_{-d}$. We note that then also $y(x) > 0$ for $0 < x < 1$ since the smallest positive zero of $J_{-d+1}$ is larger than that of $J_{-d}$. Finally
\begin{equation}
(5.15)
A_d = \frac{4}{\alpha_d^2 |d|}
\end{equation}
as desired.

5.2. The case $d > 0$. Let $b_n = a_{n+d,n}$, where $a_{mn}$ are the coefficients of expression (5.3). Then
$$
g_d(z) = \sum_{n=0}^{\infty} b_n z^{n+d}z^n.
$$
Thus
\begin{align*}
\|g_d\|_2^2 &= \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \left| \sum_{n=0}^{\infty} b_n r^{n+d}e^{idt} \right|^2 r dr dt \\
&= 2 \sum_{n,l \geq 0} \frac{b_n b_l}{n + l + 2d + 2}.
\end{align*}
On the other hand, by using (5.9), we obtain
\begin{align*}
\|C[U[g_d]]\|_2^2 &= \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \left| \sum_{n=0}^{\infty} \frac{b_n}{n+1} \left[ w^{n+d}w^{n+1} - \left( \frac{d}{m+1} \right) w^{d-1} \right] \right|^2 r dr dt \\
&= \sum_{n,l \geq 0} \frac{b_n b_l}{(n+1)(l+1)} \left[ \frac{1}{n + l + d + 2} - \frac{d}{(l + d + 1)(n + d + 1)} \right] \\
&= \sum_{n,l \geq 0} \frac{b_n b_l}{(1 + l + d)(1 + n + d)(2 + l + n + d)}.
\end{align*}
So we seek the best constant $A_d$ such that for all choices of $b_n \in \mathbb{C}$ (since $P$ is a polinom, only finitely many $b_n$ are nonzero, but the proof works as well, without this assumption), there holds the inequality
\begin{equation}
(5.16) \sum_{l,n \geq 0} \frac{b_n b_l}{(1 + l + d)(1 + n + d)(2 + l + n + d)} \leq A_d \sum_{l,n \geq 0} \frac{b_n b_l}{n + l + d + 1}.
\end{equation}
Let
$$
\psi(t) = \sum_{n=0}^{\infty} b_n t^n.
$$
Then the previous inequality is equivalent with
(5.17) \[ \int_0^1 \frac{1}{r^{d+1}} \left( \int_0^r \psi(s)s^d ds \right)^2 dr \leq A_d \int_0^1 r^d |\psi(r)|^2 dr. \]

Since the quadratic forms in (5.16) are symmetric with real coefficients \((b_{n1}b_n + b_{b_n} is a real number), it suffices to consider the inequality (5.17) for arbitrary real-valued continuous functions on \([0, 1]\). Setting \(h(r) = \psi(r)s^d\), the inequality is equivalent to

\[ \int_0^1 \frac{1}{r^{d+1}} \left( \int_0^r h(s) ds \right)^2 dr \leq A_d \int_0^1 \frac{1}{r^d} h^2(r) dr. \]

Here we again make use of Proposition [5.4]. In this case we have \(r = 2, p = 2, q = 0, s = s' = 2, m(x) = \frac{1}{x^d} \) and \(\omega(x) = \frac{1}{x^d}\). The additional compactness condition required is easily proved by observing that

\[ T_1f(x) = \frac{1}{\sqrt{x}} \int_0^x f(t) dt \]

and applying (5.10) to show that it is compact from \(L^2_m \rightarrow L^2_m\). Namely

\[ k(x, t) = x^{-1/2} t^d \chi_{[0, x]}(t), \]

and therefore

\[ \|T_1\| = \left( \int_0^1 \int_0^x \frac{t^2 d t}{x^d x^d} dr \right)^{1/2} = (d + 1)^{-1/2} < \infty. \]

The corresponding differential equality is equivalent to

\[ xy'' - dy' + \frac{x}{\lambda} y = 0, \]

which can be transformed by making use of the change \(x = \frac{\lambda}{4} z\) to the equality

\[ zy'' - dy' + \frac{y}{4} = 0. \]

The solution of the last inequality, by [21] Formula (7) in section 4.31, p. 97, is given by

\[ y = z^{\frac{1}{2}(d+1)} J_{d+1}(\sqrt{z}) = \left(2\sqrt{x/\lambda}\right)^{1+d} J_{1+d} \left(2\sqrt{x/\lambda}\right). \]

Then by [21] Section 3.2, p. 45

\[ y'(x) = \frac{1}{\sqrt{\lambda x}} \left(2\sqrt{x/\lambda}\right)^{1+d} J_d \left(2\sqrt{x/\lambda}\right) \]

and therefore

\[ \lambda^{\frac{2+d}{2} - 2 - d} y'(1) = J_d \left(2\sqrt{1/\lambda}\right). \]

Thus the largest permissible value of \(\lambda\) is

\[ \lambda^* = \frac{4}{\alpha_d^2} \]
where \( \alpha_d \) is the smallest positive zero of \( J_d \). Then as in the case \( d \leq 0 \), \( y(x) > 0 \) for \( 0 < x < 1 \) since, by Lemma 5.3, the smallest positive zero of \( J_{d+1} \) is larger than that of \( J_d \). Finally we obtain

\[
A_d = \frac{4}{\alpha_d^2}.
\]

This finishes the proof of Lemma 5.3.

Proof of Theorem 5.2. In view of comments after the statement of Theorem 5.4, the inequality follows from Lemma 5.3. The equality statement follows from the fact that \( \alpha_0 < \alpha_1 < \cdots < \alpha_d < \cdots \), Lemma 5.3, relation (5.14) and Proposition 5.4.

6. Refinement of \( L^p \) norm

We make use of the following interpolation theorem.

Proposition 6.1. \[18]\ Let \( T \) be a linear operator defined on a family \( F \) of functions that is dense in both \( L^{p_1} \) and \( L^{p_2} \) (for example, the family of all simple functions). And assume that \( Tf \) is in both \( L^{p_1} \) and \( L^{p_2} \) for any \( f \) in \( F \), and that \( T \) is bounded in both norms. Then for any \( p \) between \( p_1 \) and \( p_2 \) we have that \( F \) is dense in \( L^p \), that \( Tf \) is in \( L^p \) for any \( f \) in \( F \) and that \( T \) is bounded in the \( L^p \) norm. These three ensure that \( T \) can be extended to an operator from \( L^{p_1} \) to \( L^{p_2} \).

In addition an inequality for the norms holds, namely for \( t \in (0, 1) \) such that

\[
\frac{1}{p} = \frac{1-t}{p_1} + \frac{t}{p_2}
\]

there holds

\[
\|T\|_{L^{p_1}\to L^{p_2}} \leq \|T\|_{L^{p_1}\to L^{p_1}}^{1-t} \cdot \|T\|_{L^{p_2}\to L^{p_2}}^t.
\]

Theorem 6.2. For \( 1 \leq p \leq 2 \) we have

(6.1) \[
\|C_U\|_{L^p\to L^p} \leq \frac{2}{\alpha^{2-2/p}} \quad \text{and} \quad \|\bar{C}_U\|_{L^p\to L^p} \leq \frac{2}{\alpha^{2-2/p}};
\]

and for \( 2 \leq p \leq \infty \) we have

(6.2) \[
\|C_U\|_{L^p\to L^p} \leq \frac{4}{3} \left( \frac{3}{2\alpha} \right)^{2/p} \quad \text{and} \quad \|\bar{C}_U\|_{L^p\to L^p} \leq \frac{4}{3} \left( \frac{3}{2\alpha} \right)^{2/p}.
\]

There holds the equality in all inequalities in (6.1) and (6.2) for \( p = 1 \), \( p = 2 \) and \( p = \infty \). Moreover for \( 1 \leq p \leq 2 \) there holds the inequality

(6.3) \[
\|D_U\|_{L^p\to L^p} \leq 4\alpha^{2/p-2};
\]

and if \( 2 \leq p \leq \infty \)

(6.4) \[
\|D_U\|_{L^p\to L^p} \leq \frac{16}{3\alpha} \left( \frac{3\pi}{4\alpha} \right)^{2/p}.
\]
Proof. Let $T$ be a linear operator defined by $T = C_U : L^p(U, \mathbb{C}) \to L^p(U, \mathbb{C})$. The inequalities follow by applying the Riesz-Thorin theorem, Theorem 3.1, Theorem 4.1 and Theorem 5.2 by taking $t = 2 - \frac{2}{p}$, for $1 = p_1 \leq p \leq p_2 = 2$, and $t = 1 - \frac{2}{p}$ for $2 = p_1 \leq p \leq p_2 = \infty$ to the operator $T$ and observing that

$$2^{-1+\frac{2}{p}} \left( \frac{2}{\alpha} \right)^{2-2/p} = \frac{2}{\alpha^{2-2/p}}, \quad 1 \leq p \leq 2$$

and

$$\left( \frac{2}{\alpha} \right)^{1-(1-2/p)} \cdot \left( \frac{4}{3} \right)^{1-2/p} = \frac{4}{3} \cdot \left( \frac{3}{2\alpha} \right)^{2/p}, \quad 2 \leq p \leq \infty.$$

To prove (6.3) and (6.4) we take the operator $T = D_U : L^p(U, \mathbb{C}) \to L^p(U, \mathbb{M}_{2,2})$.

The inequalities (6.3) and (6.4) follow from (4.4), $\|D_U\|_p \leq \|\bar{C}_U\|_p + \|C_U\|_p$ and the relations

$$4^{-1+2/p} \left( \frac{4}{\alpha} \right)^{2-2/p} = 4 \alpha^{2/p-2}, \quad 1 \leq p \leq 2$$

$$\left( \frac{4}{\alpha} \right)^{2/p} \left( \frac{16}{3\pi} \right)^{1-2/p} = \frac{16}{3\pi} \left( \frac{3\pi}{4\alpha} \right)^{2/p}, \quad 2 \leq p \leq \infty.$$

By Riesz-Thorin theorem, the function $[0, 1] \ni s \to \log \|C_U\|_{L^{1/s}} \to L^{1/s}$ is convex and therefore continuous. This, together with Theorem 6.2 imply the fact

**Corollary 6.3.** There are exactly two absolute constants $1 < p_1 < 2$ and $2 < p_2 < \infty$ such that

$$\|C_U\|_{L^{p_1} \to L^{p_1}} = \|C_U\|_{L^{p_2} \to L^{p_2}} = 1.$$

As $\Delta u = 4u_{zz}$ we obtain the following result.

**Corollary 6.4.** Let $u \in W_0^{2,p}(U)$, $1 \leq p \leq \infty$ be a solution of the Poisson equation $\Delta u = h$, $h \in L^p$. Then

$$\|\partial u\|_{L^p} \leq \frac{\alpha^{2/p-2}}{2} \|h\|_{L^p}, \quad \text{and} \quad \|\bar{\partial} u\|_{L^p} \leq \frac{\alpha^{2/p-2}}{2} \|h\|_{L^p}, \quad 1 \leq p \leq 2$$

and

$$\|\partial u\|_{L^p} \leq \frac{1}{3} \left( \frac{3}{2\alpha} \right)^{2/p} \|h\|_{L^p}, \quad \text{and} \quad \|\bar{\partial} u\|_{L^p} \leq \frac{1}{3} \left( \frac{3}{2\alpha} \right)^{2/p} \|h\|_{L^p}, \quad 2 \leq p \leq \infty.$$

The inequalities are sharp for $p \in \{1, 2, \infty\}$. Moreover

$$\|\nabla u\|_{L^p} \leq \frac{\alpha^{2/p-2}}{2} \|h\|_{L^p}, \quad 1 \leq p \leq 2$$

and
\[ \| \nabla u \|_{L^p} \leq \frac{4}{3\pi} \left( \frac{3\pi}{4\alpha} \right)^{2/p} \| h \|_{L^p}, \quad 2 \leq p \leq \infty, \]

with sharp constants for \( p = 1 \) and \( p = \infty \).

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