ALGEBRAIC DREAMS

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ABSTRACT

Nature’s attraction to unique mathematical structures provides powerful hints for unraveling her mysteries. None is at present as intriguing as eleven-dimensional M-theory. The search for exceptional structures specific to eleven dimensions leads us to exceptional groups in the description of space-time. One specific connection, through the coset $F_4/\text{SO}(9)$, may provide a generalization of eleven-dimensional supergravity. Since this coset happens to be the projective space of the Exceptional Jordan Algebra, its charge space may be linked to the fundamental degrees of freedom underlying M-theory.
1 Introduction

Nature relishes unique mathematical structures. A prime example is the consistency of superstring theory which seems, time and again, to rely on miracles often traced to algebraic “coincidences” that naturally appear in special algebraic structures. This provides us mortals with a strong guiding principle to unravel Her mysteries.

Nature shows that space-time symmetries with dynamics associated with gravity, and internal symmetries with their dynamics described by Yang-Mills theories, can coexist peacefully. How does She do it? M-theory and Superstring theories \[1\] are the only examples of theories where this union appears possible, but there remain important unanswered questions. While theory is formulated in terms of local operators in space-time, space-time as we know it, is probably not as fundamental as we think, and only a solution of a more general theory. Does this solution contain clues as to the nature of the underlying theory? In Quantum Mechanics, time is treated very differently from space, and space-time symmetries do not seem natural, leading some to speculate \[2\] that only compact symmetries reflect those of the underlying theory. This line of reasoning, applied to supersymmetry, leads to all types of questions: is it clear that M-theory is manifestly supersymmetric in eleven dimensions? Could it be supersymmetric only in the local limit or when compactified to lower dimensions?

2 Is Space-Time Exceptional?

The Exceptional Algebras are most unique and beautiful among Lie Algebras, and no one should be surprised if Nature uses them. To that effect, we present some mathematical and physical factoids which may suggest new lines of exceptional inquiry:

- Patterns of the quantum numbers of the elementary particles point to their embedding in Exceptional Groups \[3\]. The sequence
\[ E_8 \supset E_7 \supset E_6 \supset SO(10) \supset SU(5) \supset SU(3) \times SU(2) , \]
obtained by chopping off one dot from these algebras’ Dynkin diagrams leads to the non-Abelian symmetries of the Standard Model. This sequence is realized in the Heterotic string [4] where the gauge algebra is \( E_8 \times E_8 \).

- Exceptional Groups contain orthogonal groups capable of describing space-time symmetries. Some compact embeddings are

\[ E_8 \supset SO(16) , \quad E_7 \supset SO(12) \times SO(3) , \quad E_6 \supset SO(10) \times SO(2) . \]

This may occur along non-compact groups as well, for instance along the sequence from \( E_8(-24) \) to \( SO(9) \):

\[ E_8(-24) \supset E_7(-25) \times SU(1,1) , \quad E_7(-25) \supset E_6(-26) \times SO(1,1) , \quad E_6(-26) \supset F_4(-20) \supset SO(9) . \]

- The consistency of superstring [5] theories in \( 9 + 1 \) dimensions relies on the triality of the light-cone little group \( SO(8) \), which links its tensor and spinor representations via a \( Z_3 \) symmetry. The exceptional group \( F_4 \) is the smallest which realizes this triality explicitly. It was surprising to find another consistent theory in one more space dimension since the \( SO(9) \) little group has very different spinor and tensor representations. A possible hint for fermion-boson confusion is the anomalous embedding of \( SO(9) \) into an orthogonal group in which the vector representation of the bigger group is identified with the spinor of the smaller group

\[ SO(16) \supset SO(9) , \quad 16 = 16 . \]
• The use of exceptional groups to describe space-time symmetries has not been as fruitful. One obstacle has been that exceptional algebras relate tensor and spinor representations of their orthogonal subgroups, while Spin-Statistics requires them to be treated differently. Yet there are some mathematical curiosities worth noting. For one, the anomalous Dynkin embedding of $F_4$ inside $SO(26)$

$$SO(26) \supset F_4 , \quad 26 = 26,$$

or its non-compact variety

$$SO(25,1) \supset F_4(-20),$$

together with the embedding

$$F_4 \supset SO(9) , \quad 26 = 1 \oplus 9 \oplus 16,$$

or its non-compact form $F_4(-20) \supset SO(9)$, might point to a (M)-heterotic construction from the bosonic string to M-theory.

• A formulation of finite-dimensional Hilbert spaces in terms of the algebra of observables, proposed by P. Jordan [6], has not yet proven fruitful in Physics, in spite of many attempts. In all but one case, it is akin to rewriting the familiar Dirac ket description in terms of density matrices, but it also unearthed a unique structure on which Quantum Mechanics can be implemented, even though it cannot be described by kets in Hilbert space. Our interest lies in the fact that its automorphism group is $F_4$ and its natural description lies in the sixteen-dimensional (Cayley) projective space $F_4/SO(9)$.

• There is a whiff of the exceptional group $F_4$ in the supergravity supermultiplet in eleven dimensions, as we now proceed to show.
3 Supergravity in Eleven Dimensions

Eleven dimensional $N = 1$ Supergravity is the ultimate field theory that includes gravity, but it is not renormalizable, and does not stand on its own as a physical theory. Its counterpart, the ultimate field theory without gravity, is the finite $N = 4$ Super Yang-Mills theory in four dimensions. Recently, the eleven-dimensional theory has been revived as the limit of M-theory which, like characters on the walls of Plato’s cave, has revealed itself through its compactified version onto lower-dimensional manifolds. In the absence of a definitive description of M-theory, it behooves us to scrutinize what is known about 11-d Supergravity theory.

3.1 SuperAlgebra

$N = 1$ supergravity in eleven dimension is a local field theory that contains three massless fields, the familiar symmetric second-rank tensor, $h_{\mu\nu}$ which represents gravity, a three-form field $A_{\mu\nu\rho}$, and the Rarita-Schwinger spinor $\Psi_{\mu \alpha}$. From its Lagrangian, one can derive the expression for the super Poincaré algebra, which in the unitary transverse gauge assumes the particularly simple form in terms of the nine $(16 \times 16)$ $\gamma_i$ matrices which form the Clifford algebra

$$\{ \gamma^i, \gamma^j \} = 2\delta^{ij}, \quad i, j = 1, \ldots, 9.$$  

Supersymmetry is generated by the sixteen real supercharges

$$Q^a_\pm = Q^{a*}_\mp,$$

which satisfy

$$\{ Q^a_+, Q^b_+ \} = \sqrt{2} p^+ \delta^{ab}, \quad \{ Q^a_-, Q^b_- \} = \frac{\vec{p} \cdot \vec{p}}{\sqrt{2} p^+} \delta^{ab}, \quad \{ Q^a_+, Q^b_- \} = - (\gamma_i)^{ab} p^i,$$

and transform as Lorentz spinors

$$[M^{ij}, Q^a_\pm] = \frac{i}{2} (\gamma^{ij} Q_\pm)^a, \quad [M^{+-}, Q^a_\pm] = \pm \frac{i}{2} Q^a_\pm.$$
\[ [M^{\pm i}, Q_a^\pm] = 0, \quad [M^{\pm i}, Q_a^a] = \pm \frac{i}{\sqrt{2}} (\gamma^i Q^a) \pm i, \]

A very simple representation of the 11-dimensional super-Poincaré generators can be constructed, in terms of sixteen anticommuting real \( \chi \)'s and their derivatives, which transform as the spinor of \( SO(9) \), as

\[ Q_a^+ = \partial \chi^a + \frac{1}{\sqrt{2}} p^i \chi^a, \quad Q_a^- = -\frac{p^i}{p^+} (\gamma^i Q_+)^a, \]

\[ M^{ij} = x^j p^i - x^i p^j - \frac{i}{2} \chi \gamma^{ij} \partial \chi, \quad M^{+-} = -x^- p^+ - \frac{i}{2} \chi \partial \chi, \quad M^{+i} = -x^i p^+, \quad M^{-i} = x^- p^i - \frac{1}{2} \{ x^i, P^- \} + \frac{ip^i}{2p^+} \chi \gamma^i \gamma^j \partial \chi. \]

The light-cone little group transformations are generated by

\[ S^{ij} = -\frac{i}{2} \chi \gamma^{ij} \partial \chi, \]

which satisfy the \( SO(9) \) Lie algebra. In order to examine the spectrum, we rewrite the supercharges in terms of eight complex Grassmann variables

\[ \theta^a \equiv \frac{1}{\sqrt{2}} (\chi^a + i\chi^{a+8}), \quad \bar{\theta}^a \equiv \frac{1}{\sqrt{2}} (\chi^a - i\chi^{a+8}), \]

and

\[ \frac{\partial}{\partial \theta^a} \equiv \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial \chi^a} - i \frac{\partial}{\partial \chi^{a+8}} \right), \quad \frac{\partial}{\partial \bar{\theta}^a} \equiv \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial \chi^a} + i \frac{\partial}{\partial \chi^{a+8}} \right), \]

where \( \alpha = 1, 2, \ldots, 8 \). The eight complex \( \theta \) transform as the \( (4, 2) \), and \( \bar{\theta} \) as the \( (\overline{4}, 2) \) of the \( SU(4) \times SU(2) \) subgroup of \( SO(9) \). The eight complex supercharges
\[
\begin{align*}
Q_+^\alpha & \equiv \frac{1}{\sqrt{2}} \left( Q_+^\alpha + i Q_+^{\alpha+8} \right) = \frac{\partial}{\partial \theta^\alpha} + \frac{1}{\sqrt{2}} p^+ \theta^\alpha, \\
Q_+^{\alpha\dagger} & \equiv \frac{1}{\sqrt{2}} \left( Q_+^\alpha - i Q_+^{\alpha+8} \right) = \frac{\partial}{\partial \theta^\alpha} + \frac{1}{\sqrt{2}} p^+ \theta^\alpha,
\end{align*}
\]

satisfy
\[
\{ Q_+^\alpha, Q_+^{\beta\dagger} \} = \sqrt{2} p^+ \delta^{\alpha\beta}.
\]

They act irreducibly on chiral superfields which are annihilated by the covariant derivatives
\[
\left( \frac{\partial}{\partial \theta^\alpha} - \frac{1}{\sqrt{2}} p^+ \theta^\alpha \right) \Phi(y^-, \theta) = 0,
\]

where
\[
y^- = x^- - \frac{i \theta \theta}{\sqrt{2}}.
\]

Expansion of the superfield in powers of the eight complex \(\theta\)'s yields 256 components, with the following \(SU(4) \times SU(2)\) properties
\[
\begin{align*}
1 & \sim (1, 1), \\
\theta & \sim (4, 2), \\
\theta \theta & \sim (6, 3) \oplus (10, 1), \\
\theta \theta \theta & \sim (\bar{20}, 2) \oplus (\bar{4}, 4), \\
\theta \theta \theta \theta & \sim (15, 3) \oplus (1, 5) \oplus (20', 1),
\end{align*}
\]

and the higher powers yield the conjugate representations by duality. These make up the three representations of \(N = 1\) supergravity
\[
\begin{align*}
44 & = (1, 5) \oplus (6, 3) \oplus (20', 1) \oplus (1, 1), \\
84 & = (15, 3) \oplus (10, 1) \oplus (6, 3) \oplus (1, 1), \\
128 & = (20, 2) \oplus (\bar{20}, 2) \oplus (4, 4) \oplus (\bar{4}, 4) \oplus (4, 2) \oplus (\bar{4}, 2).
\end{align*}
\]
For future reference we note the $SU(4) \times SU(2)$ weights of the $\theta$s, using the notation $(a_1, a_2, a_3; a)$,

\[
\begin{align*}
\theta^1 &\sim (1, 0, 0; 1) , & \theta^5 &\sim (1, 0, 0; -1) , \\
\theta^4 &\sim (-1, 1, 0; 1) , & \theta^6 &\sim (-1, 1, 0; -1) , \\
\theta^7 &\sim (0, -1, 1; 1) , & \theta^8 &\sim (0, -1, 1; -1) , \\
\theta^7 &\sim (0, 0, -1; 1) , & \theta^3 &\sim (0, 0, -1; -1) ,
\end{align*}
\]

which enables us to find the highest weights of the supergravity representations

\[
\begin{align*}
44 & : \theta^1 \theta^4 \theta^5 \theta^8 = (0, 2, 0; 0) \sim (20', 1) \\
84 & : \theta^1 \theta^8 = (2, 0, 0; 0) \sim (10, 1) \\
128 & : \theta^1 \theta^4 \theta^5 \theta^8 = (1, 1, 0; 1) \sim (20, 2)
\end{align*}
\]

together with their $SU(4) \times SU(2)$ properties. All other states are generated by acting on these highest weight states with the lowering operators. The highest weight chiral superfield that describes $N = 1$ supergravity in eleven dimensions is simply

\[
\Phi = \theta^1 \theta^8 h(y^-, \vec{x}) + \theta^1 \theta^4 \theta^8 \psi(y^-, \vec{x}) + \theta^1 \theta^4 \theta^5 \theta^8 \Lambda(y^-, \vec{x}),
\]

which summarizes the spectrum of the super-Poincaré algebra in eleven dimensions of either a free field theory or a free superparticle.

Since the little group generators act on a 256-dimensional space, we can express them in terms of sixteen $(256 \times 256)$ matrices, $\Gamma^a$, which satisfy the Dirac algebra
\( \{ \Gamma^a, \Gamma^b \} = 2\delta^{ab}. \)

This leads to an elegant representation of the \( SO(9) \) generators

\[
S^{ij} = -\frac{i}{4}(\gamma^{ij})^{ab} \Gamma^a \Gamma^b \equiv \frac{i}{2} f^{ij ab} \Gamma^a \Gamma^b.
\]

The coefficients

\[
f^{ij ab} \equiv \frac{1}{2}(\gamma^{ij})^{ab},
\]

naturally appear in the commutator between the generators of \( SO(9) \) and any spinor operator \( T^a \), as

\[
[T^{ij}, T^a] = \frac{i}{2}(\gamma^{ij} T)^a = if^{ij ab} T^b.
\]

But there is more to it, the \( (\gamma^{ij})^{ab} \) can also be viewed as structure constants of a Lie algebra. Manifestly antisymmetric under \( a \leftrightarrow b \), they can appear in the commutator of two spinors into the \( SO(9) \) generators

\[
[T^a, T^b] = \frac{i}{2}(\gamma^{ij})^{ab} T^{ij} = f^{a b i j} T^{ij},
\]

and one easily checks that they satisfy the Jacobi identities. Remarkably, the 52 operators \( T^{ij} \) and \( T^a \) generate the exceptional Lie algebra \( F_4 \), showing explicitly how an exceptional Lie algebra appears in the light-cone formulation of supergravity in eleven dimensions!

### 3.2 Character Formula

The degrees of freedom are labelled by the light-cone little group \( SO(9) \) acting on the transverse vector indices, as \( h_{(ij)} \sim (2000), A_{[ijkl]} \sim (0010), \Psi_{i\alpha} \sim (1001) \), with their little group representations in Dynkin’s notation.

Their group-theoretical properties are summarized in the following table
where $D$ is the dimension of the representation, and $I_n$ are the Dynkin indices of the representations, related to the four Casimir operators of $SO(9)$. We note that the dimension and Dynkin indices of the fermion is the sum over those of the bosons, except for $I_8$, indicating that these three representations have much in common.

Amazingly, the supergravity fields are the first of an infinite number of triplets \[8\] of $SO(9)$ representations which display the same group-theoretical relations: equality of dimension and all Dynkin indices except $I_8$ between one representation and the sum of the other two. Quantum theories of these Euler triplets may have very interesting divergence properties, as these numbers typically occur in higher loop calculations, and such equalities usually increase the degree of divergence, and the failure of the equality for $I_8$ is probably related to the lack of renormalizability of the theory \[9\].

This mathematical fact has been traced to a character formula \[10\] related to the three equivalent embeddings of $SO(9)$ into $F_4$. The character formula is given by

$$V_\lambda \otimes S^+ - V_\lambda \otimes S^- = \sum_c \text{sgn}(c) U_{c\lambda}.$$ 

On the left-hand side, $V_\lambda$ is a representation of $F_4$ written in terms of its $SO(9)$ subgroup, $S^\pm$ are the two spinor representations of $SO(16)$ written in terms of its anomalously embedded subgroup $SO(9)$, $\otimes$ denotes the normal Kronecker product of representations, and the $-$ denotes the naive substraction of representations. On the right-hand side, the sum is over $c$, the elements of the Weyl group which map the Weyl chamber of $F_4$ into that of
$SO(9)$. In this case there are three elements, the ratio of the orders of the Weyl groups (it is also the Euler number of the coset manifold), and $U_{c \cdot \lambda}$ denotes the $SO(9)$ representation with highest weight $c \cdot \lambda$, where

$$c \cdot \lambda = c(\lambda + \rho_{F_4}) - \rho_{SO(9)},$$

and the $\rho$'s are the sum of the fundamental weights for each group, and $\text{sgn}(c)$ is the index of $c$. Thus to each $F_4$ representation corresponds a triplet, called Euler triplet. The supergravity case is rather trivial as

$$SO(16) \supset SO(9), \quad S^+ \sim 128 = 128, \quad S^- \sim 128' = 44 + 84,$$

and the character formula reduces to the truism

$$128 - 44 - 84 = 128 - 44 - 84.$$

This construction yields the general form of the Euler triplets: the Euler triplet corresponding to the $F_4$ representation $[a_1 a_2 a_3 a_4]$ is made up of the following three $SO(9)$ representations listed in order of increasing dimensions:

$$\quad (2 + a_2 + a_3 + a_4, a_1, a_2, a_3), \quad (a_2, a_1, 1 + a_2 + a_3, a_4), \quad (1 + a_2 + a_3, a_1, a_2, 1 + a_3 + a_4)$$

The spinor representations appear with odd entries in the fourth place. Euler triplets with the largest spinor and two bosons must have both $a_3$ and $a_4$ even or zero.

Since the Dynkin indices of the product of two representations satisfy the composition law

$$I^{(n)}[\lambda \otimes \mu] = d_\lambda I^{(n)}[\mu] + d_\mu I^{(n)}[\lambda],$$

it follows that the deficit in $I^{(8)}$ is always proportional to the dimension of the $F_4$ representation that generates it.
3.3 The Kostant Operator

This character formula can be viewed as the index formula of a Dirac-like operator formed over the coset $F_4/SO(9)$. This coset is the sixteen-dimensional Cayley projective plane, over which we introduce the previously considered Clifford algebra

$$\{ \Gamma^a, \Gamma^b \} = 2 \delta^{ab}, \quad a, b = 1, 2, \ldots, 16,$$

generated by $(256 \times 256)$ matrices. The Kostant equation is defined as

$$\mathcal{K} \Psi = \sum_{a=1}^{16} \Gamma^a T^a \Psi = 0,$$

where $T^a$ are $F_4$ generators not in $SO(9)$, with commutation relations

$$[T^a, T^b] = i f^{abij} T^{ij}.$$

Although it is taken over a compact manifold, it has non-trivial solutions. To see this, we rewrite its square as the difference of positive definite quantities,

$$\mathcal{K} \mathcal{K} = C^2_{F_4} - C^2_{SO(9)} + 72,$$

where

$$C^2_{F_4} = \frac{1}{2} T^{ij} T^{ij} + T^a T^a,$$

is the $F_4$ quadratic Casimir operator, and

$$C^2_{SO(9)} = \frac{1}{2} \left( T^{ij} - i f^{abij} \tilde{\Gamma}^{ab} \right)^2,$$

is the quadratic Casimir for the sum

$$L^{ij} \equiv T^{ij} + S^{ij},$$
where $S^{ij}$ is the previously defined $SO(9)$ generator which acts on the supergravity fields. We have also used the quadratic Casimir on the spinor representation

$$\frac{1}{2} S^{ij} S^{ij} = 72.$$ 

Kostant’s operator commutes with the sum of the generators,

$$[\mathcal{K}, L^{ij}] = 0,$$

allowing its solutions to be labelled by $SO(9)$ quantum numbers.

The same construction of Kostant’s operator applies to all equal rank embeddings, and its trivial solutions display supersymmetry $[10, 12, 13, 14]$. In particular we note the cases $E_6/SO(10) \times SO(2)$, with Euler number 27, $E_7/SO(12) \times SO(3)$ with Euler number 63, and $E_8/SO(16)$, where the Euler triplets contain 135 representations $[8]$. These cosets with dimensions 32, 64, and 128 could be viewed as complex, quaternionic and octonionic Cayley plane $[15]$.

### 3.4 Oscillator Representation of $F_4$

Schwinger’s celebrated representation of $SU(2)$ generators of in terms of one doublet of harmonic oscillators can be extended to other Lie algebras $[10]$. The generalization involves several sets of harmonic oscillators, each spanning the fundamental representation. For example, $SU(3)$ is generated by two sets of triplet harmonic oscillators, $SU(4)$ by two quartets. In the same way, all representations of the exceptional group $F_4$ are generated by three sets of oscillators transforming as $26$. We label each copy of 26 oscillators as $A_0^{[\kappa]}$, $A_i^{[\kappa]}$, $i = 1, \ldots, 9$, $B_a^{[\kappa]}$, $a = 1, \ldots, 16$, and their hermitian conjugates, and where $\kappa = 1, 2, 3$. Under $SO(9)$, the $A_i^{[\kappa]}$ transform as $9$, $B_a^{[\kappa]}$ transform as $16$, and $A_0^{[\kappa]}$ is a scalar. They satisfy the commutation relations of ordinary harmonic oscillators
\[
[A^{[\kappa]}_i, A^{[\kappa']\dagger}_j] = \delta_{ij} \delta^{[\kappa][\kappa']} , \quad [A^{[\kappa]}_0, A^{[\kappa']\dagger}_0] = \delta^{[\kappa][\kappa']} .
\]

Note that the \(SO(9)\) spinor operators satisfy Bose-like commutation relations

\[
[B^{[\kappa]}_a, B^{[\kappa']\dagger}_b] = \delta_{ab} \delta^{[\kappa][\kappa']} .
\]

The generators \(T_{ij}\) and \(T_a\)

\[
T_{ij} = -i \sum_{\kappa=1}^4 \left\{ \left( A^{[\kappa]\dagger}_i A^{[\kappa]}_j - A^{[\kappa']\dagger}_j A^{[\kappa]}_i \right) + \frac{1}{2} B^{[\kappa]\dagger} \gamma_{ij} B^{[\kappa]} \right\} ,
\]

\[
T_a = -i \sum_{\kappa=1}^4 \left\{ (\gamma_i)^{ab} \left( A^{[\kappa]\dagger}_i B^{[\kappa]}_b - B^{[\kappa']\dagger}_b A^{[\kappa]}_i \right) - \sqrt{3} \left( B^{[\kappa]\dagger}_a A^{[\kappa]}_0 - A^{[\kappa]\dagger}_0 B^{[\kappa]}_a \right) \right\} ,
\]
satisfy the \(F_4\) algebra,

\[
[T_{ij}, T_{kl}] = -i (\delta_{jk} T_{il} + \delta_{il} T_{jk} - \delta_{ik} T_{jl} - \delta_{jl} T_{ik}) ,
\]

\[
[T_{ij}, T_a] = \frac{i}{2} (\gamma_{ij})_{ab} T_b ,
\]

\[
[T_a, T_b] = \frac{i}{2} (\gamma_{ij})_{ab} T_{ij} ,
\]

so that the structure constants are given by

\[
f_{ijab} = f_{abij} = \frac{1}{2} (\gamma_{ij})_{ab} .
\]

The last commutator requires the Fierz-derived identity

\[
\frac{1}{4} \theta \gamma^{ij} \theta \chi \gamma^{ij} \chi = 3 \theta \chi \theta + \theta \gamma^i \chi \gamma^j \theta ,
\]

from which we deduce

\[
3 \delta^{ac} \delta^{db} + (\gamma^i)^{ac} (\gamma^i)^{db} - (a \leftrightarrow b) = \frac{1}{4} (\gamma^{ij})^{ab} (\gamma^{ij})^{cd} .
\]

To satisfy these commutation relations, we have required both \(A_0\) and \(B_a\) to obey Bose commutation relations (Curiously, if both are anticommuting,
the $F_4$ algebra is still satisfied). One can just as easily use a coordinate representation of the oscillators by introducing real coordinates $u_i$ which transform as transverse space vectors, $u_0$ as scalars, and $\zeta_a$ as space spinors which satisfy Bose commutation rules

$$
A_i = \frac{1}{\sqrt{2}}(u_i + \partial u_i), \quad A_i^\dagger = \frac{1}{\sqrt{2}}(u_i - \partial u_i),
$$

$$
B_a = \frac{1}{\sqrt{2}}(\zeta_a + \partial \zeta_a), \quad B^\dagger_a = \frac{1}{\sqrt{2}}(\zeta_a - \partial \zeta_a),
$$

$$
A_0 = \frac{1}{\sqrt{2}}(u_0 + \partial u_0), \quad A_0^\dagger = \frac{1}{\sqrt{2}}(u_0 - \partial u_0).
$$

Using square brackets $[\cdots]$ to represent the Dynkin label of $F_4$, and round brackets $(\cdots)$ to represent those of $SO(9)$, we list some of the combinations which will be used for investigating the solutions of Kostant’s equation

$$
\begin{align*}
&u_1 + iu_2 \sim [0 0 0 1] \sim (1 0 0 0), \\
u_3 + iu_4 \sim [1 0 0 -1] \sim (-1 1 0 0), \\
&\zeta_1 + i\zeta_9 \sim [0 0 1 -1] \sim (0 0 0 1), \\
&\zeta_8 + i\zeta_{16} \sim [0 1 -1 0] \sim (0 0 1 -1), \\
&\zeta_3 - i\zeta_{11} \sim [1 -1 1 0] \sim (0 1 -1 1), \\
&\zeta_6 - i\zeta_{14} \sim [1 0 -1 1] \sim (0 1 0 -1).
\end{align*}
$$

Hence $u_1 + iu_2$ and $\zeta_1 + i\zeta_9$ are the highest weights of the $SO(9)$ representations $9$, and $16$, respectively.

### 3.5 Solutions of Kostant’s Equation

For every representation of $F_4$, $[a_1, a_2, a_3, a_4]$, there is one $SO(9)$ Euler triplet solution of Kostant’s equation

$$
(2 + a_2 + a_3 + a_4, a_1, a_2, a_3), \quad (a_2, a_1, 1 + a_2 + a_3, a_4), \quad (1 + a_2 + a_3, a_1, a_2, 1 + a_3 + a_4)
$$

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The trivial solution with \(a_1 = a_2 = a_3 = a_4 = 0\), yields the \(N = 1\) supergravity multiplet in eleven dimensions, \((2000) \oplus (0010) \oplus (1001)\). We have seen that the highest weight solution are \(\theta^1\theta^4\theta^5\theta^8\), \(\theta^1\theta^8\), and \(\theta^1\theta^4\theta^8\), described by the chiral superfield

\[
\Phi_{0000} = \theta^1\theta^8 h_{0000}(y^- , \bar{x}) + \theta^1\theta^4\theta^8 \psi_{0000}(y^- , \bar{x}) + \theta^1\theta^4\theta^5\theta^8 A_{0000}(y^- , \bar{x}) .
\]

In general, the highest weight solutions appear in the form of \(f(u_i, \zeta_a) \Theta(\theta)\), where both \(f(u_i, \zeta_a)\) and \(\Theta(\theta)\) are the highest weights \(SO(9)\) states with respect to the earlier defined \(T_{ij}\) and \(S_{ij}\). The solutions have the quantum numbers of their sum \(L_{ij} = S_{ij} + T_{ij}\), which commutes with Kostant’s operator. \(\Theta(\theta)\) is one of the three polynomials above, \(\theta^1\theta^4\theta^5\theta^8\), \(\theta^1\theta^8\), or \(\theta^1\theta^4\theta^8\).

The highest weight solutions corresponding to each fundamental representation of \(F_4\) are \([17]\)

1. \(a_1 = a_2 = a_3 = 0, a_4 \geq 1\)

   These representations are built with only one copy. The highest weight solutions with \(a_4 = 1\) are uniquely given by

   \[
   \theta^1\theta^4\theta^5\theta^8(u_1 + i u_2), \\
   \theta^1\theta^8(\zeta_1 + i \zeta_9), \\
   \theta^1\theta^4\theta^8(\zeta_1 + i \zeta_9),
   \]

   where \((u_1 + i u_2)\) and \((\zeta_1 + i \zeta_9)\) are the highest weights of \(SO(9)\) representations \((1000)\) and \((0001)\), respectively.

2. \(a_1 = a_2 = a_4 = 0, a_3 \geq 1\)

   In this case we need two copies \((\kappa = 1, 2)\). For \(a_3 = 1\), the highest weight solutions are
\[
\theta^1 \theta^4 \theta^5 \theta^8 \left[ u_1 + iu_2 , \zeta_1 + i\zeta_9 \right], \\
\theta^1 \theta^8 \left[ \zeta_1 + i\zeta_9 , \zeta_8 + i\zeta_{16} \right], \\
\theta^1 \theta^4 \theta^8 \left[ u_1 + iu_2 , \zeta_1 + i\zeta_9 \right],
\]

where

\[
[a, b] \equiv a^{[1]} b^{[2]} - a^{[2]} b^{[1]},
\]

is the determinant of 2 copies of \( a \) and \( b \) states. Note that \([ u_1 + iu_2 , \zeta_1 + i\zeta_9 ]\) and \([ \zeta_1 + i\zeta_9 , \zeta_8 + i\zeta_{16} ]\) are the highest weights of the \( SO(9) \) representations (1001) and (0010) respectively.

3. \( a_1 = a_3 = a_4 = 0, a_2 \geq 1 \)

Here three copies, \( \kappa = 1, 2, 3 \) are needed. The \( a_2 = 1 \) highest weight solutions are

\[
\theta^1 \theta^4 \theta^5 \theta^8 \left[ u_1 + iu_2 , \zeta_1 + i\zeta_9 , \zeta_8 + i\zeta_{16} \right], \\
\theta^1 \theta^8 \left[ u_1 + iu_2 , \zeta_1 + i\zeta_9 , \zeta_8 + i\zeta_{16} \right], \\
\theta^1 \theta^4 \theta^8 \left[ u_1 + iu_2 , \zeta_1 + i\zeta_9 , \zeta_8 + i\zeta_{16} \right],
\]

where \([ u_1 + iu_2 , \zeta_1 + i\zeta_9 , \zeta_8 + i\zeta_{16} ]\) is the highest weight of the \( SO(9) \) representation (1010), and \([ a, b, c ]\) is the determinant (antisymmetric product) of 3 copies of \( a, b \) and \( c \) states.

4. \( a_2 = a_3 = a_4 = 0, a_1 = 1 \)

The \( F_4 \) states are represented by antisymmetric products of \( \kappa = 2 \) copies of 26 states. The highest weight solutions are

\[
\theta^1 \theta^4 \theta^5 \theta^8 \left( [ u_1 + iu_2 , u_3 + iu_4 ] + [ \zeta_1 + i\zeta_9 , \zeta_6 - i\zeta_{14} ] + [ \zeta_8 + i\zeta_{16} , \zeta_3 - i\zeta_{11} ] \right), \\
\theta^1 \theta^8 \left( [ u_1 + iu_2 , u_3 + iu_4 ] + [ \zeta_1 + i\zeta_9 , \zeta_6 - i\zeta_{14} ] + [ \zeta_8 + i\zeta_{16} , \zeta_3 - i\zeta_{11} ] \right), \\
\theta^1 \theta^4 \theta^8 \left( [ u_1 + iu_2 , u_3 + iu_4 ] + [ \zeta_1 + i\zeta_9 , \zeta_6 - i\zeta_{14} ] + [ \zeta_8 + i\zeta_{16} , \zeta_3 - i\zeta_{11} ] \right),
\]

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where \((u_1 + iu_2, u_3 + iu_4] + [\zeta_1 + i\zeta_9, \zeta_6 - i\zeta_{14}] + [\zeta_8 + i\zeta_{16}, \zeta_3 - i\zeta_{11}]\)

is the highest weight of the \(SO(9)\) representation \((0100)\).

This last case implies that only three copies of 26 oscillators suffice to generate all \(F_4\) representations. It is not possible to construct the \([1000]\) state out of four copies of states in the 26. Hence all representations of \(F_4\) can be obtained by three copies of harmonic oscillators.

Since the \(T_{ij}\) do not alter the degree of homogeneity of polynomials in \(u_i\) and \(\zeta_a\), all solutions are given by the functions \(f(u_i, \zeta_a)\) as homogeneous polynomials of their variables. The general highest weight solutions are then

\[
\begin{align*}
\theta^1\theta^4\theta^5\theta^8 w_1^{a_1} w_2^{a_2} w_3^{a_3} w_4^{a_4}, \\
\theta^1\theta^8 w_1^{a_1} w_2^{a_2} v_3^{a_3} v_4^{a_4}, \\
\theta^1\theta^4\theta^8 w_1^{a_1} w_2^{a_2} w_3^{a_3} v_4^{a_4},
\end{align*}
\]

where

\[
\begin{align*}
w_1 &= [u_1 + iu_2, u_3 + iu_4] + [\zeta_1 + i\zeta_9, \zeta_6 - i\zeta_{14}] + [\zeta_8 + i\zeta_{16}, \zeta_3 - i\zeta_{11}], \\
w_2 &= [u_1 + iu_2, \zeta_1 + i\zeta_9, \zeta_8 + i\zeta_{16}], \\
w_3 &= [u_1 + iu_2, \zeta_1 + i\zeta_9], \quad v_3 = [\zeta_1 + i\zeta_9, \zeta_8 + i\zeta_{16}], \\
w_4 &= (u_1 + iu_2), \quad v_4 = (\zeta_1 + i\zeta_9).
\end{align*}
\]

All other states are generated by application of the four \(SO(9)\) lowering operators.

### 3.6 Super Euler Triplets (SET)

We have just displayed the solutions to Kostant’s equation as products of a \(\theta\) part and an internal part that depends polynomially on new variables. Since the \(\theta\) parts describe a superparticle in eleven dimensions, it is tempting
to interpret states in the other Euler triplets as superparticles dressed with fields described by these new variables, vector coordinates $u^{[κ]}_i$ and twistor coordinates $ζ^{[κ]}_a$. Should we think of these as coordinates (vector and spinor) in the $SO(9)$ space of some other particle? To satisfy the spin-statistics connection, the twistors $ζ_a$ can only appear quadratically, since odd powers would generate fermions ($SO(9)$ spinors) with Bose properties. This is true of the triplets for which $a_3$ and $a_4$ are even, with no restrictions on $a_1$ and $a_2$.

The lowest Euler triplet which describes supergravity is supersymmetric with an equal number of fermions and bosons. None of the other Euler triplets display space-time supersymmetry, although a subclass does contain equal number of fermions ($SO(9)$ spinors) and bosons ($SO(9)$ tensors), those for which the Dynkin indices $a_3$ and $a_4$ are even, with no restrictions on $a_1$ and $a_2$. Curiously, spin-statistics requires those super-like Euler triplets (SETs) that share this one feature of supersymmetry, although they are not space-time supersymmetric by themselves.

There are four different families of SETs:

- The simplest set has $a_4$ even, and $a_3 = a_2 = a_1 = 0$. Their superfields depend on the symmetric products of two vectors $u_i u_j$, and two spinors, $ζ_a ζ_b$. The simplest highest weight solutions ($a_4 = 2$) are

$$θ^1 θ^4 θ^5 θ^8 (u_1 + iu_2)^2, \quad θ^1 θ^8 (ζ_1 + iζ_9)^2, \quad θ^4 θ^8 (ζ_1 + iζ_9)^2,$$

where $(u_1 + iu_2)^2$ and $(ζ_1 + iζ_9)^2$ are the highest weights of $SO(9)$ representations $(2000)$ and $(0002)$, respectively, described by one set of vector and twistor coordinates, which enter as a symmetric second rank tensor represented by $(2000)$ coupled to gravity, and a four-form $(0002)$, coupled to the three-form and Rarita-Schwinger fields of supergravity. The highest weight component of its superfield is given by
\[
\Phi_{(0001)} = \phi_{(0001)} \theta^1 \theta^4 \theta^5 \theta^8 (u_1 + iu_2)^2 + \\
+ A_{(0001)} \theta^1 \theta^8 (\zeta_1 + i\zeta_9)^2 + \psi_{(0001)} \theta^1 \theta^4 \theta^8 (\zeta_1 + i\zeta_9)^2 ,
\]

where the fields \( \phi, A, \psi \) depend also on the center of mass variables \( \zeta^- \) and the transverse vector \( \vec{u} \). This family requires one new set of vector and twistor coordinates.

- The case \( a_3 \neq 0 \) even, and \( a_1 = a_2 = a_4 = 0 \) requires two sets of extra coordinates, \( \{ u_i^{[\kappa]} \}, \zeta_a^{[\kappa]} \), \( \kappa = 1, 2 \), as its highest weight superfield is

\[
\Phi_{(0010)} = \phi_{(0010)} \theta^1 \theta^4 \theta^5 \theta^8 [u_1 + iu_2, \zeta_1 + i\zeta_9]^2 + \\
+ A_{(0010)} \theta^1 \theta^8 [\zeta_1 + i\zeta_9, \zeta_8 + i\zeta_{16}]^2 + \psi_{(0010)} \theta^1 \theta^4 \theta^8 [u_1 + iu_2, \zeta_1 + i\zeta_9]^2 .
\]

The two sets of internal coordinates arrange themselves in the symmetric product of two three forms that couple to the supergravity three-form, and two RS spinors that couples to gravity and the original RS fields of supergravity.

- When only \( a_2 \neq 0 \), all three supergravity multiplets are dressed by the same \((1010)\) representation, described by triple products of vector and two spinors. The highest weight superfield is now

\[
\Phi_{(0100)} = \left( \phi_{(0100)} \theta^1 \theta^4 \theta^5 \theta^8 + A_{(0100)} \theta^1 \theta^8 + \psi_{(0100)} \theta^1 \theta^4 \theta^8 \right) \times \\
\times [u_1 + iu_2, \zeta_1 + i\zeta_9, \zeta_8 + i\zeta_{16}]^{a_2} ,
\]

which requires three sets of coordinates \( \{ u_i^{[\kappa]} \}, \zeta_a^{[\kappa]} \), \( \kappa = 1, 2, 3 \)

- Finally, if \( a_1 \neq 0 \) only, the three supergravity states are dressed the same way by something with the quantum number of a 2-form, \((0100)\),
made up of two vectors and two spinors. Although it is most complicated in terms of the underlying coordinates, it is simplest in terms of representations.

$$\Phi_{(1000)} = \left( \phi_{(1000)} \theta^1 \theta^4 \theta^5 \theta^8 + A_{(1000)} \theta^1 \theta^8 + \psi_{(1000)} \theta^1 \theta^4 \theta^8 \right) \times$$

$$\times \left( [u_1 + i u_2, u_3 + i u_4] + [\zeta_1 + i \zeta_9, \zeta_6 - i \zeta_{14}] + [\zeta_8 + i \zeta_{16}, \zeta_3 - i \zeta_{11}] \right).$$

These SETs require only two copies. This doubling may indicate the presence of $E_6$, the complex extension of $F_4$.

We note that none of the solutions depend on the singlet variable $u_0$, which can be traced to the equation

$$\Gamma^a \frac{\partial}{\partial \zeta_a} = 0.$$ 

We may be tempted to think of these solutions as products of the supergravity ground state and excitations from new objects (at least two in all but one case) with both transverse space vector and spinor (twistor) coordinates. Since vector and spinor coordinates obey Bose statistics, their description is not space-time supersymmetric, although it could be relativistic.

Let us assume that the Euler triplets describe states on which the Poincaré symmetries can be implemented. Then one of two possibilities arises:

- If these states are massless, we already have the necessary ingredients, the $SO(9)$ generators given by the sum $L^{ij} = T^{ij} + S^{ij}$ and there is no further addition to the light-cone Hamiltonian $P^-$. But these states represent higher spin particles and if they are massless, one can expect grave difficulties in implementing their interactions [18, 19], although there is an infinite number of triplets [20].

- If the excited Euler triplets describe massive states, we must be able to produce a non-linear realization of the generators of the massive little
group $SO(10)$. This entails the construction of a transverse vector $L^i$, with the commutation relations

$$\left[ L^i, L^j \right] = iM^2 L^{ij},$$

where $M^2$ is the mass squared operator which commutes with $L^{ij}$. Then adding $M^2$ to $P^-$, and including $L^i$ in the light-cone boosts satisfies the Poincaré algebra. For strings and superstrings, the $L^i$ are cubic in the oscillators, and the commutation relation works only in the right number of dimensions.

A necessary condition to realize the massive Euler triplets is to assemble the triplets in $SO(10)$ representations, the massive little group in eleven dimensions, but this is not possible with Euler triplets alone. To see this, consider the $SO(8)$ fermion representations, with Dynkin labels $(1 + a_2 + a_3, a_1, a_2, 1 + a_3 + a_4)$ with $a_3, a_4$ even. They can be expressed in terms of fields with one spinor index and a tensor structure given by the partition

$$\{ 1 + a_1 + a_2 + a_3 + \frac{a_3 + a_4}{2}, a_1 + a_2 + \frac{a_3 + a_4}{2}, a_2 + \frac{a_3 + a_4}{2}, a_3 + a_4 \},$$

and we see that the first row is always larger than the second row. The corresponding $SO(10)$ tensor will also have more in the first row than in its second, but it will also have a partition where the excess in the first row are identified with the tenth direction. This produces an $SO(8)$ tensor with equal number in its first two rows, which is not a triplet spinor. We conclude, in analogy with the Higgs mechanism, that new degrees of freedom are needed to make the Euler triplets massive. If successfull this would describe an object with a massless supersymmetric sector and massive non-supersymmetric states with equal number of fermions and bosons, similar in spirit to Witten’s model in $2 + 1$ dimensions.
Either way, our approach still lacks an organizing principle for the inclusion and exclusion of Euler triplets. We have already seen that the spin-statistics connection limits the multiplets \((a_3, a_4 \text{ even})\), but nothing so far tells us how many Euler triplets should participate. For instance, an infinite number would suggest the inclusion of \(F_4\) inside a non-compact group.

We see that the Euler triplets generate the spectrum of a Poincaré covariant object which has a ground state with supersymmetry! This object would be described by its center of mass coordinates \(x^-\) and \(x_i\), and internal coordinates \(u^{(\kappa)}_i, \zeta^{(\kappa)}_a\), where \(\kappa\) may run over three values at most. Contrast this with the superstring which is also described by its center of mass coordinates \(x^-, x_m, m = 1, \ldots, 8\) and an infinite number of internal variables \(x^{(n)}_m\), and anticommuting spinor variables \(\zeta^{(n)}_\alpha, n = 1, 2, \ldots \infty, \alpha = 1, \ldots 8\).

In this language, the Euler triplets emerge as much simpler than superstrings, since they have a finite number of internal variables, although their internal spinor variables satisfy Bose commutations.

Could these label the end points of an open string in the zero tension limit?

Could this new internal space be generated by the degrees of freedom of the Exceptional Jordan Algebras?

4 Exceptional Jordan Algebra

We have built Kostant’s operator from the coset \(F_4/\text{SO}(9)\). This coset is a very special projective plane, called the Cayley or Moufang plane, since its projective geometry is the only one not to satisfy Desargues’ theorem. Its points can be identified with the projection operators associated with the quantum mechanical states of the Exceptional Jordan Algebra.

Jordan algebras \([21]\) are an alternate description of finite-dimensional Hilbert space in terms of its observables. One defines the symmetric Jordan product.
\[ J_a \circ J_b = J_b \circ J_a , \quad \text{(I)} \]

which maps observables into observables. It is the symmetric matrix product, but since matrices do not commute, the Jordan associator

\[ (J_a , J_b , J_c ) \equiv J_a \circ (J_b \circ J_c ) - (J_a \circ J_b ) \circ J_c . \]

is not necessarily zero, although it satisfies the Jordan identity

\[ (J_a , J_b , J_a ^2 ) = 0 . \quad \text{(II)} \]

Equations (I) and (II) are the postulates for the commutative but non-associative Jordan Algebras, unlike the matrix multiplication in Hilbert space which is non-commutative but associative.

To any quantum-mechanical state described by the Hilbert space ket \( |\alpha > \), corresponds the observable idempotent (density) matrix

\[ P_\alpha = \frac{|\alpha >\langle \alpha |}{\langle \alpha | \alpha >} , \quad P_\alpha \circ P_\alpha = P_\alpha . \]

All familiar quantum mechanics of finite Hilbert space can be expressed in this language. For instance, linear dependence among three states translates into the Jordan statement among their associated projection operators

\[ \text{Tr} [(P_\alpha \times P_\beta ) \circ P_\gamma ] = 0 , \]

using the Freudenthal product

\[ J_a \times J_b \equiv J_a \circ J_b - \frac{1}{2} J_a \text{Tr}(J_b) - \frac{1}{2} J_b \text{Tr}(J_a) - \frac{1}{2} \text{Tr}(J_a \circ J_b ) + \frac{1}{2} \text{Tr}(J_a) \text{Tr}(J_b) . \]

Unitary maps in Hilbert space are written in terms of Jordan operations

\[ \delta P_\alpha \equiv \mathcal{D}_{B,C} P_\alpha = (B , P_\alpha , C ) , \]
which reduces to the commutator relation,

$$\delta P_\alpha = \frac{1}{4} [[B, C], P_\alpha] ,$$
in complex Hilbert space. In Jordanese, two observables represented by $A$ and $B$, are compatible if their associator $(A, J, B)$ with any element $J$ of the Jordan algebra vanishes.

A particular example is time (or light-cone time) evolution generated by

$$i \hbar \frac{\partial J}{\partial t} = (A, J, B) ,$$
where the Hamiltonian in given by

$$H = \frac{i}{4} [A, B] .$$

Jordan, Von Neumann and Wigner [22] found that the Jordan axioms were also realized by $3 \times 3$ hermitian matrices over octonions. Octonions, sometimes called Cayley numbers, are non-associative generalizations of real, complex and quaternionic numbers. This Exceptional Jordan Algebra (EJA), has intrigued many people [21, 23, 24], but no compelling case for its use in physics has ever been made. The non-associativity forbids its interpretation in terms of 3-dimensional Hilbert space with kets representing physical states. Its elements are of the form

$$J(\alpha_i, \omega_i) = \begin{pmatrix} \alpha_1 & \omega_3 & \omega_2 \\ \omega_3 & \alpha_2 & \omega_1 \\ \omega_2 & \omega_1 & \alpha_3 \end{pmatrix} = \sum_{i=1}^{3} \alpha_i E_i + (\omega_3)_{12} + (\omega_2)_{31} + (\omega_1)_{23} ,$$
where $\alpha_i$ are real numbers, and the $\omega_i$ are three octonions. Octonions are written in terms of eight real numbers as

$$\omega = a_0 + \sum_{i=1}^{7} a_\alpha e_\alpha ; \quad \overline{\omega} = a_0 - \sum_{i=1}^{7} a_\alpha e_\alpha ,$$

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where \(e_\alpha\) are the seven imaginary octonion units. They satisfy the relations
\[
e_\alpha e_\beta = -\delta_{\alpha\beta} + \Psi_{\alpha\beta\gamma} e_\gamma ,
\]
\(\Psi_{\alpha\beta\gamma}\) are the totally antisymmetric octonion structure functions, with only non-zero elements
\[
\Psi_{123} = \Psi_{246} = \Psi_{435} = \Psi_{572} = \Psi_{714} = \Psi_{367} = 1 .
\]
The Cayley algebra is non-associative, but alternative: the associator of three octonions
\[
[\omega_1, \omega_2, \omega_3] \equiv (\omega_1 \omega_2) \omega_3 - \omega_1 (\omega_2 \omega_3) ,
\]
is completely antisymmetric
\[
[e_\alpha, e_\beta, e_\gamma] = 2\bar{\Psi}_{\alpha\beta\gamma\delta} e_\delta ,
\]
where
\[
\bar{\Psi}_{\alpha_1\alpha_2\alpha_3\alpha_4} = \frac{1}{3!} \epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7} \Psi_{\alpha_5\alpha_6\alpha_7} ,
\]
is the dual of the octonion structure functions.

Unitary maps on the EJA are replaced by its group of automorphisms, the 52-parameter exceptional group \(F_4\), under which
\[
F_4 : \quad \delta J = \mathcal{D}_{h_1, h_2} J = (h_1, J, h_2) ,
\]
where \(h_1, h_2\) are \((3 \times 3)\) traceless octonionic hermitian matrices, each labelled by 26 real parameters. Because of the non-associativity of the matrix elements, the Jordan associator does not reduce to a commutator. The traceless Jordan matrices span the 26 representations of \(F_4\). One can supplement the \(F_4\) transformation by an additional 26 parameters, and define
\[
\mathcal{D}_X J \equiv X \circ J ,
\]
leading to a group with 78 parameters. These extra transformations are non-compact, and close on the \(F_4\) transformations, leading to the exceptional
group $E_{6(-26)}$. The subscript in parenthesis denotes the number of non-compact minus the number of compact generators.

An important subgroup is $SO(9)$, the automorphism of the $(2 \times 2)$ Jordan matrices over octonions, generated by transformations that leave an idempotent, say $E_3$, invariant

$$D_{(\omega)_{12}(\tau)_{12}}, D_{(\omega)_{12},E_1-E_2},$$

where $\omega$ and $\tau$ are octonions. The first, antisymmetric under $\omega \leftrightarrow \tau$, represents the 28 transformations of $SO(8)$. Traceless $(2 \times 2)$ matrices transform as the nine components of the vector representation of $SO(9)$.

It follows that the EJA dynamic evolution is generated by $F_4$ transformations, which can be catalogued in terms of unbroken symmetries. Could the $SO(9)$ subgroup of EJA automorphisms be identified with the light-cone little group in eleven space-time dimensions?  

EJA states are represented by points in the projective geometry over $F_4/SO(9)$, which can be written in the form

$$P = \Omega \frac{1}{\sqrt{\Omega^\dagger \Omega}} \Omega^\dagger,$$

where $\Omega^T = (\omega_1, \omega_2, \omega_3)$ is an octonionic vector. With seven redundant phases and one normalization condition, one of the three octonions can be set equal to one, leaving us with 16 real parameters to determine a point, the labels of the Moufang projective space. It is also the coset space $F_4/SO(9)$, since $F_4$ acts on the points, while its $SO(9)$ subgroup leaves any one point invariant, and the transformations in $F_4/SO(9)$ map this point into other points. As mentioned earlier, this projective geometry is unique as it does not satisfy Desargues’ theorem.

For those who do not remember that theorem: take three lines $p, q, r$ that meet at a point. Take any two points on each of these lines, call them $p_1, p_2, q_1, q_2$ and $r_1, r_2$. The lines connecting $p_1q_1$ and $p_2q_2$ meet at the point $A_{pq}$, while $p_1r_1$ and $p_2r_2$ meet at $A_{pr}$. Finally the lines $q_1r_1$ and $q_2r_2$ meet at $A_{qr}$.
Desargues theorem states that the three points \(A_{pq}, A_{pr}, A_{qr}\) lie on the same line.

To conclude, we have investigated several schemes which involve exceptional groups in the description of space(-time); especially interesting is the connection between the light-cone little group in eleven dimensions and \(F_4\). In the process we have shown how the superparticle could be generalized to include Euler triplets. Also we have established some curious mathematical links between EJA quantum mechanics [25] and Euler triplets. It behooves us to continue to investigate [17] these relations and endow them with physical significance.

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