GEOMETRIC PROPERTIES OF
THE KAZHDAN-LUSZTIG SCHUBERT BASIS

CRISTIAN LENART, CHANGJIAN SU, KIRILL ZAINOULLINE, AND CHANGLONG ZHONG

Abstract. We study classes determined by the Kazhdan-Lusztig basis of the Hecke algebra in the $K$-theory and hyperbolic cohomology theory of flag varieties. We first show that, in $K$-theory, the two different choices of Kazhdan-Lusztig bases produce dual bases, one of which can be interpreted as characteristic classes of the intersection homology mixed Hodge modules. In equivariant hyperbolic cohomology, we show that if the Schubert variety is smooth, then the class it determines coincides with the class of the Kazhdan-Lusztig basis; this was known as the Smoothness Conjecture. For Grassmannians, we prove that the classes of the Kazhdan-Lusztig basis coincide with the classes determined by Zelevinsky’s small resolutions. These properties of the so-called KL-Schubert basis show that it is the closest existing analogue to the Schubert basis for hyperbolic cohomology; the latter is a very useful testbed for more general elliptic cohomologies.

Contents

1. Introduction 1
2. Formal affine Demazure algebra and its dual 3
3. Hecke algebra, motivic Chern class, and the smoothness criterion 6
4. Dual bases in $K$-theory and characteristic classes of mixed Hodge modules 8
5. The smoothness conjecture for hyperbolic cohomology 13
6. KL-Schubert classes and small resolutions 15
References 22

1. Introduction

Let $G$ be a split semi-simple linear algebraic group with a fixed Borel subgroup $B$ and a maximal torus $T \subset B$. Let $P$ be a parabolic subgroup containing the Borel subgroup $B$. The varieties $G/P$ and $G/B$ are called flag varieties, and they are among the most concrete objects in algebraic geometry, because of the Bruhat decompositions. For instance, the equivariant cohomology (Chow group) of flag varieties is freely spanned by the classes of Schubert varieties $X(w)$. Similarly, the equivariant $K$-theory of flag varieties is spanned by the structure sheaves of Schubert varieties. The field of studying intersection theory of these classes is called Schubert calculus, and is related to combinatorics, representation theory, and enumerative geometry.

Due to the failure of Schubert varieties being smooth, the present paper deals with two different directions in generalizing classical Schubert calculus. The first one is concerned with the Chern classes. Although the classical Chern class theory does not work for the singular Schubert varieties, there are generalizations to this case, which are called Chern-Schwartz-MacPherson (CSM) classes in homology and motivic Chern (MC) classes in $K$-theory [M74, S65a, S65b, BSY10, AMSS19, FRW18]. These generalized Chern classes of Schubert cells are closely related to the corresponding...
stable bases of the cotangent bundle $T^*G/B$, defined by Maulik and Okounkov in their study of quantum cohomology/$K$-theory of Nakajima quiver varieties [MO19, O17]. These classes are permuted by various Demazure-Lusztig operators [AMI16, Su17, SZZ17, AMSS19, MNS20], and are related to unramified principal series representations of the Langlands dual group over a non-archimedean local field [SZZ17, AMSS19].

We focus on the Kazhdan-Lusztig bases of the Hecke algebra, which are related to the intersection cohomology of Schubert varieties. Classically, there are two choices of Kazhdan-Lusztig bases. In this paper, we consider the $K$-theory classes determined by these two collections of Kazhdan-Lusztig bases. The cohomology case is considered in [MS20]. As our first main result, we show that they are dual to each other in Theorem 10 and 18. These dualities are closely related to the characteristic classes of mixed Hodge modules, studied by Schürmann and his collaborators [S11, S17, BSY10]. Moreover, we interpret one collection of these classes as the motivic Hodge Chern classes of the intersection homology mixed Hodge modules of the Schubert varieties, which immediately implies that they are invariant under the Serre-Grothendieck duality, see Proposition 14 and Corollary 15.

The other direction is to look at more general cohomology theories, namely the equivariant oriented cohomology theories of Levine-Morel. They are those contravariant functors $h_T$ from the category of smooth (quasi)-projective varieties to the category of commutative rings, such that for any proper map of varieties, a push-forward of the cohomology groups is defined. One can then define Chern classes, where the first Chern class of the tensor product of line bundles determines a one-dimensional commutative formal group law. The structure of the equivariant oriented cohomology of flag varieties is studied in [CZZ12, CZZ13, CZZ14, LZZ19]. Roughly speaking, there is an algebra generated by push-pull operators between $h_T(G/B)$ and $h_T(G/P)$, called the formal affine Demazure algebra $D_F$, whose dual $D_F^*$ is isomorphic to $h_T(G/B)$.

To resolve the singularities of a Schubert variety $X(w)$, one often uses the Bott-Samelson resolution, which is defined by fixing a reduced decomposition of the Weyl group element $w$. For oriented cohomology beyond singular cohomology/$K$-theory, the classes determined by such resolutions depend on the choice of the reduced decomposition. This corresponds to the fact that, for general $h_T$, the push-pull operators do not satisfy the braid relations. Because of this fact, there are no canonically defined Schubert classes.

Aiming for the definition of Schubert classes, in [LZ17, LZZ19], the authors consider the so-called hyperbolic cohomology, denoted by $\mathfrak{h}$. A Riemann-Roch type map is defined from $K$-theory to the hyperbolic cohomology theory, which induces an action of the Hecke algebra (considered on the $K$-theory side) on the hyperbolic cohomology of $G/B$. In this way, the action of the Kazhdan-Lusztig basis defines classes $KL_w$ in $h_T(G/B)$, called KL-Schubert classes. In [LZ17, LZZ19], there is a conjecture stating that, if the Schubert variety $X(w)$ is smooth, then its fundamental class coincides with the class $KL_w$. It is proved in some special cases in [LZ17, LZZ19]. The second main result of this paper is to prove this conjecture in full generality, see Theorem 23.

The idea of the proof is as follows: if $X(w)$ is smooth, then all the Kazhdan-Lusztig polynomials $P_{y,w}$ for any $y \leq w$ are equal to 1, so the Kazhdan-Lusztig basis for $w$ is the sum of the Demazure-Lusztig operators. As mentioned above, the MC classes of Schubert cells in $K$-theory are permuted by the Demazure-Lusztig operators. So the MC class of $X(w)$ coincides with the KL class in $K$-theory, and the restriction formula for the former is obtained in [AMSS19] by generalizing a result of Kumar [K96]. Translating this formula to the hyperbolic side, we prove the Smoothness Conjecture (Theorem 23). For partial flag varieties, a similar property is also proved.

Restricting to type $A$ Grassmannians, we prove more geometric and combinatorial properties. For example, Zelevinsky constructed small resolutions of all Schubert varieties [Z83]. We prove that the classes determined by these resolutions coincide with the KL-Schubert classes, which is the third main result of this paper, see Theorem 38. By the uniqueness of the Kazhdan-Lusztig
basis, it follows that all small resolution classes are the same. The proof of this result can be summarized as follows: Zelevinsky’s small resolutions are similar to the Bott-Samelson resolutions, except that, instead of using minimal parabolic subgroups, one considers more general parabolic subgroups. So the small resolution classes can be computed by using relative push-pull operators between \( G/P \) and \( G/Q \). These operators were studied in [CZZ13]. On the other hand, in [KL00], a factorization of the Kazhdan-Lusztig basis elements for Grassmannians is exhibited. By carefully transforming this factorization, one can write the Kazhdan-Lusztig basis elements as products of “relative” Kazhdan-Lusztig elements. Finally, by identifying the latter with the relative push-pull operators, one proves Theorem 38.

There have been important developments in Schubert calculus for general cohomology theories. More specifically, for elliptic cohomology, a stable basis in the cotangent bundle \( T^*G/B \) was defined (see [AO16, O20], which generalizes stable bases for cohomology and \( K \)-theory), and canonical classes were associated with Bott-Samelson resolutions of Schubert varieties [RW19, KRW20]. The elliptic cohomology used in the latter papers can be considered as the oriented cohomology theory associated with a certain elliptic formal group law determined by the Jacobi theta functions; meanwhile, the mentioned cohomology classes are elliptic analogues of the CSM classes in ordinary cohomology and the MC classes in \( K \)-theory. On the other hand, the hyperbolic formal group law we consider here comes from a generic singular Weierstrass curve, see [BB10]. The properties of the KL-Schubert basis proved in this paper (namely, the Smoothness Conjecture and the interpretation in terms of the Zelevinsky small resolutions) show that this basis is the closest existing analogue to the Schubert basis for hyperbolic cohomology. Furthermore, the latter is a very useful testbed for more general elliptic cohomologies.

The paper is organized as follows. In Section 2, we recall the algebraic construction of the equivariant oriented cohomology of flag varieties. In Section 3, we recall basic facts about the Hecke algebra, MC classes, and the smoothness criterion. In Section 4, we use Kazhdan-Lusztig bases to define the two collections of KL classes in \( K_T(G/B) \) and \( K_T(G/P) \), and show that they are dual to each other. We also give a geometric interpretation for one of them using mixed Hodge modules. In Section 5, we recall the definition of KL-Schubert classes in hyperbolic cohomology, and prove the Smoothness Conjecture. In Section 6, we prove Theorem 38, which connects small resolutions for Grassmannians with the corresponding KL-Schubert classes.

**Acknowledgments:** We would like to thank Samuel Evens for helpful conversations. C. L. gratefully acknowledges the partial support from the NSF grants DMS-1362627 and DMS-1855592. K. Z. acknowledges the partial support from the NSERC Discovery grant RGPIN-2015-04469, Canada. C. S. thanks J. Schürmann for useful discussions, and further to P. Aluffi, L. Mihalcea, H. Naruse and G. Zhao for related collaborations.

## 2. Formal affine Demazure algebra and its dual

We recall the definition of the formal affine Demazure algebra and its relation with equivariant generalized (oriented) cohomology of flag varieties following [HMSZ, CZZ12, CZZ13] and especially the paper [CZZ14].

**Notation.** Let \( G \) be a semisimple simply connected linear algebraic group over \( \mathbb{C} \), and fix \( B \) a Borel subgroup with a maximal torus \( T \subset B \). Let \( X^*(T) \) denote the character lattice of \( T \). Let \( W = N_G(T)/T \) be the Weyl group.

Let \( \Sigma \) denote the set of associated roots and let \( \Sigma^+ \) denote the subset of roots in \( B \). For any root \( \alpha \), let \( \alpha > 0 \) (resp. \( \alpha < 0 \)) denote \( \alpha \in \Sigma^+ \) (resp. \( -\alpha \in \Sigma^+ \)).

Let \( \Pi = \{ \alpha_1, \ldots, \alpha_n \} \) denote the set of simple roots. Let \( \ell: W \to \mathbb{Z} \) denote the length function. For any \( J \subset \Pi \), denote by \( W_J \) the parabolic subgroup corresponding to \( J \), by \( w_J \) its longest
element, and by $W^J$ the set of minimal length representatives of right cosets $W/W_J$. Specifically, $w_0 := w_{\Pi} \in W$ is the longest element. More generally, if $J' \subseteq J \subseteq \Pi$, denote $w_{J,J'} := w_Jw_{J'} \in W^{J'}$, that is, $w_{J,J'}$ is the maximal element (in terms of the Bruhat order) in the set $W_J \cap W^{J'}$. Denote $\Sigma_J := \{ \alpha \in \Sigma | s_\alpha \in W_J \}$, and $\Sigma_J^+ := \Sigma_J \cap \Sigma^\pm$.

**Formal group algebra.** Let $F$ be a one dimensional formal group law over a commutative unital ring $R$. The formal group algebra $R[[X^*(T)]]_F$ is defined to be the quotient of the completion $R[[x_\lambda | \lambda \in X^*(T)]]/J_F$, where $J_F$ is the closure of the ideal generated by $\langle x_0, F(x_\lambda, x_\mu) - x_{\lambda + \mu} \mid \lambda, \mu \in X^*(T) \rangle$. For simplicity it will be denoted by $S$. It can be shown that if $\{\omega_1, \ldots, \omega_n\}$ is a basis of $X^*(T)$, then $S$ is (non-canonically) isomorphic to $R[[\omega_1, \ldots, \omega_n]]$.

**Localized twisted group ring.** Let $Q = S \left[ \frac{1}{x_\alpha} | \alpha > 0 \right]$, and $Q_W = Q \otimes R[[W]]$. Denote the canonical left $Q$-basis of $Q_W$ by $\delta_w, w \in W$, and define a product on $Q_W$ by

$$(p\delta_w) \cdot (p'\delta_{w'}) := pw(p')\delta_{ww'}, \quad p, p' \in Q, w, w' \in W.$$  

In particular, we have $\delta_w = v(p)\delta_v, p \in Q$.

**Push-pull elements.** For each root $\alpha$, define the formal push-pull element

$$Y_\alpha := (1 + \delta_{s_\alpha}) \frac{1}{x_\alpha} \in Q_W.$$

For any reduced word $w = s_{i_1} \cdots s_{i_k}$, where $s_i$ is the simple reflection corresponding to the $i$th simple root in $\Pi$, define $I_w = (i_1, \ldots, i_k)$, and $Y_{I_w} = Y_{\alpha_{i_1}} \cdots Y_{\alpha_{i_k}}$. The product $Y_{I_w}$ depends on the choice of the reduced sequence, unless the formal group law $F$ is of the form $x + y + \beta xy$ with $\beta \in R$. For simplicity, denote $\delta_i := \delta_{s_i}, Y_i := Y_{\alpha_i}$ and $x_{\pm i} := x_{\pm \alpha_i}$.

**Formal affine Demazure algebra.** Let $D_F$ be the subring of $Q_W$ generated by elements of $S$ and push-pull elements $Y_i, i = 1, \ldots, n$. This is called the formal affine Demazure algebra. It is proved in [CZZ12] that $D_F$ is a free left $S$-module with basis $\{Y_{I_w} | w \in W\}$.

**Example 1.** If $R = \mathbb{Z}$ and $F_m(x, y) = x + y - xy$ (multiplicative formal group law), then

$$S \cong \mathbb{Z}[X^*(T)]^\wedge, \quad x_\alpha \mapsto 1 - e^{-\alpha},$$

where the completion is taken with respect to the kernel of the trace map. The ring $D_F$ is then isomorphic to the (completed) affine 0-Hecke algebra.

For $J' \subseteq J \subseteq \Pi$, denote

$$x_{J,J'} := \prod_{\alpha \in \Sigma_J \setminus \Sigma_J^+} x_\alpha, \quad x_J := x_{J,\emptyset}.$$

Fixing a set of left coset representatives $W_{J,J'}$ of $W_J/W_{J'}$, we define a push-pull element

$$Y_{J,J'} := \left( \sum_{w \in W_{J,J'}} \delta_w \right) \frac{1}{x_{J,J'}}, \quad Y_J := Y_{J,\emptyset} = \left( \sum_{w \in W_J} \delta_w \right) \frac{1}{x_J}.$$  

Note that the definition of $Y_{J,J'}$ does not depend on the choice of $W_{J,J'}$. If $J = \Pi$, $x_{\Pi}$ and $Y_{\Pi}$ are correspondingly defined. For instance, if $J = \{i\}$, then $Y_{\{i\}} = Y_{\alpha_i}$. Note that in general $Y_{J,J'} \in Q_W$, but $Y_J \in D_F$. We have

$$Y_{J,J'}Y_{J'} = Y_J.$$
There is an anti-involution $\iota$ of $D_F$, defined by
\begin{equation}
\iota(p\delta_v) := \delta_v^{-1} p \frac{v(x_{\Pi})}{x_{\Pi}} = v^{-1}(p) \frac{x_{\Pi}}{v^{-1}(x_{\Pi})} \delta_v, \quad p \in Q, \, v \in W.
\end{equation}
By definition, if $I^{-1}$ is the sequence obtained from $I$ by reversing the order, then
\begin{equation}
\iota(Y_I) = Y_{I^{-1}}.
\end{equation}

**Dual of the Demazure algebra.** Let $D_F^*$ denote the $S$-linear dual $\text{Hom}_S(D_F, S)$ with the dual basis $Y_{I^w}, \, w \in W$. One can also consider the $Q$-linear dual $Q_W^* = \text{Hom}_Q(Q_W, Q)$, which is isomorphic to the set-theoretic $\text{Hom}(W, Q)$. There is the dual basis $f_w, w \in W$ of $Q_W^*$ such that $f_w(\delta_v) = \delta_{w,v}^K$ and $f_w \cdot f_v = \delta_{w,v}^K f_w$. It turns $Q_W^*$ into a commutative ring with identity $1 = \sum_w f_w$. By definition, we have $D_F^* \subset Q_W^*$ (where the former is a $S$-module, and the latter is considered as a $Q$-module), and the product on $Q_W^*$ restricts to the product on $D_F^*$.

**Two actions on the dual.** There are actions denoted ‘$\cdot$’ and ‘$\odot$’ of the ring $Q_W$ on its $Q$-linear dual $Q_W^*$ defined as:
\begin{equation}
(p\delta_v) \cdot (qf_w) := qww^{-1}(p)f_{w^{-1}} \quad \text{and} \quad (p\delta_v) \odot (qf_w) := pv(q)f_{vw}, \quad v, w \in W, \, p, q \in Q.
\end{equation}
It follows from [LZZ19, §3] that the $\cdot$-action is $Q$-linear, while the $\odot$-action is not, and the two actions commute. We also have $z \cdot pt_e = \iota(z) \odot pt_e$. Moreover, the two actions induce (via the embeddings $D_F \subset Q_W$ and $D_F^* \subset Q_W^*$) corresponding actions of $D_F$ on $D_F^*$. For homology and $K$-theory, the $\cdot$ and $\odot$ actions correspond to the right and left actions considered in [MNS20].

**The class of a point.** For each $w \in W$ define the element
\[ pt_w := x_{\Pi} \cdot f_w = w(x_{\Pi})f_w, \]
and call it the class of a point. From the definition, we have $z \cdot pt_e = \iota(z) \odot pt_e, \, z \in Q_W$, where $e \in W$ denotes the identity element.

**Generalized (oriented) cohomology.** Given a formal group law $F$ over $R$, let $h$ be the corresponding free algebraic generalized (oriented) cohomology theory obtained from the algebraic cobordism $\Omega$ of Levine-Morel [LM07] by tensoring with $F$, i.e.
\[ h(-) := \Omega(-) \otimes_{\Omega(pt)} R. \]
Here $\Omega(pt)$ is the Lazard ring, the coefficient ring of universal formal group law, and $\Omega(pt) \to R$ is the evaluation map defining $F$. We refer to [LM07] for all the properties of such theories.

In particular, for the additive formal group law $F_a(x, y) = x + y$ one obtains the Chow ring and for the multiplicative group law $F_m$ one gets the usual $K$-theory.

**Equivariant generalized cohomology.** Let $h_T$ be the respective $T$-equivariant generalized (oriented) cohomology theory of [CZZ14, §2]. Replacing $h_T$ if necessary by its characteristic completion (see [CZZ14, §3]), the main result of [CZZ14] says that the formal affine Demazure algebra $D_F$ and its dual $D_F^*$ are related to generalized cohomology $h_T(G/B)$ and $h_T(G/P_J)$ as follows:

1. There is an isomorphism $D_F^* \cong h_T(G/B)$, which maps the element $Y_{I^w} \cdot pt_e = Y_{I^w} \odot pt_e$ to the Bott-Samelson class determined by the sequence $I^w$.
2. Via the above isomorphism, the map $Y_{\Pi} \cdot \_ : D_F^* \to (D_F^*)^W \cong S$ coincides with the map $h_T(G/B) \to h_T(\text{Spec}(k))$. 

(3) The group $W$ acts on $D_p^*$ by restriction of the $\bullet$-action via the embedding $W \subset D_p$. For any subset $J \subset \Pi$, one has an isomorphism $(D_p^*)^W \cong h_T(G/P_J)$, and the map $Y_J : D_p^* \to (D_p^*)^W$ coincides with the push-forward map $h_T(G/B) \to h_T(G/P_J)$. More generally, the map $Y_{J,J} : D_p^* \to (D_p^*)^{W_J}$ restricts to a map $(D_p^*)^{W_J} \to (D_p^*)^{W_J'}$, which corresponds to $h_T(G/P_{J'}) \to h_T(G/P_J)$.

(4) The embedding $D_p^* \to Q^*_W$ coincides with the restriction to $T$-fixed points map $h_T(G/B) \to Q \otimes_S h_T(W)$, and the element $pt_w$ is mapped to the class $e_w$ of $T$-fixed point of $G/B$.

Remark 2. Observe that the localization axiom [CZZ14, A3] used to prove the above properties can be replaced by a weaker CD-property of [NPSZ, Def. 3.3] which holds for any $h_T$ defined using the Borel construction (see [NPSZ, Example 3.6]).

3. Hecke algebra, motivic Chern class, and the smoothness criterion

In this section, we recall the definition of the Kazhdan-Lusztig Schubert (KL-Schubert) classes, following [LZZ19].

The multiplicative case. Set $R = \mathbb{Z}[t,t^{-1},(t + t^{-1})^{-1}]$, where $t$ is a parameter. Definitions of section 2 applied to the multiplicative formal group law $F_m$ over $R$ give the respective formal group algebra and its localization:

$$S_m := R[[X^*(T)]|_{F_m}], \quad Q_m := S_m\left[\frac{1}{h_\gamma}\mid \gamma > 0\right];$$

the localized twisted group algebra and the formal affine Demazure algebra:

$$Q_{m,W} := Q_m \otimes_R R[W], \quad D_m := \langle S_m,Y_1,...,Y_n \rangle \subset Q_{m,W}.$$

The Demazure-Lusztig elements. Define the Demazure-Lusztig elements in $Q_{m,W}$ as

$$\tau_i := Y_{i}^m(t - t^{-1}e^{\alpha_i}) - t = \frac{t^m - t}{1 - e^{\alpha_i}} + \frac{t - t^{-1}e^{-\alpha_i}}{1 - e^{-\alpha_i}} \delta_i^m.$$

It can be shown that $\tau_i \in D_m, i = 1,...,n$ satisfy the standard quadratic relation $\tau_i^2 = (t^2 - t)\tau_i + 1$, and the braid relations. So they generate the Hecke algebra $H$ over $R$.

Remark 3. Let $y = -t^{-2}$. As operators on $D_m^* \cong K_T(G/B)$, then $t^{-1}\tau_i \circ -$ agrees with $T_i^L$, and $t^{-1}\tau_i \bullet -$ agrees with $T_i^{R,R}$, respectively, where the latter are notions from [MNS20, Section 5.3].

The Kazhdan-Lusztig basis. Consider the involution of the Hecke algebra $H \to H, z \mapsto \overline{z}$ such that

$$\mathcal{T} = t^{-1}, \quad \overline{T_i} = T_i^{-1}.$$

There is a basis of $H$ over $R$ denoted by $\{\gamma_w\}_{w \in W}$ and called the Kazhdan-Lusztig basis. It is invariant under this involution and satisfies

$$\gamma_w \in \tau_w + \sum_{v < w} t\mathbb{Z}[t]T_v.$$

We set $t_w = t^{\ell(w)}$ and

$$\gamma_w = \sum_{v \leq w} t_w^{t_v^{-1}}P_{v,w}(t^2)T_v,$$

where $P_{v,w}$ are the Kazhdan-Lusztig polynomials. In addition to this, there is another canonical basis defined by (see [KL79])

$$\overline{\gamma}_w := \sum_{v \in W} e_w e_v t_w^{-1} t_v P_{v,w}(t^2)T_v \in \tau_w + \sum_{v < w} t^{-1}\mathbb{Z}[t^{-1}]T_v.$$
More generally, for \( J' \subset J \subseteq \Pi \), denote
\[
\gamma_J := \gamma_{w_J} = \sum_{v \leq w_J} t_{w_J} t_v^{-1} \tau_v, \quad \gamma_{J/J'} := \sum_{v \in W_J \cap W_{J'}} t_{w_J} t_v^{-1} \tau_v.
\]
(7)

It is not difficult to see that
\[
\gamma_J = \gamma_{J/J'} \gamma_{J'}. \quad \text{(8)}
\]

If \( Q \subset P \) are the parabolic subgroups corresponding to \( J' \subset J \), respectively, denote \( \gamma_{P/Q} = \gamma_{J/J'} \).

It will be used in considering KL-Schubert classes in hyperbolic cohomology of partial flag varieties below.

**Motivic Chern classes.** We recall the definition of the motivic Chern classes, following [BSY10, PRW18, AMSS19]. Let \( X \) be a quasi-projective, non-singular, complex algebraic variety with an action of the torus \( T \). Let \( G^T_0(\text{var}/X) \) be the (relative) Grothendieck group of varieties over \( X \). By definition, it is the free abelian group generated by isomorphism classes \([f : Z \to X]\) where \( Z \) is a quasi-projective \( T \)-variety and \( f \) is a \( T \)-equivariant morphism modulo the usual additivity relations
\[
[f : Z \to X] = [f : U \to X] + [f : Z \setminus U \to X],
\]
for any \( T \)-invariant open subvariety \( U \subset Z \).

**Theorem 4.** There exists a unique natural transformation \( \text{MC}_{-t^{-2}} : G^T_0(\text{var}/X) \to K_T(X)[t^{-2}] \) satisfying the following properties:

1. It is functorial with respect to \( T \)-equivariant proper morphisms of non-singular, quasi-projective varieties.
2. It satisfies the normalization condition
\[
\text{MC}_{-t^{-2}}[\text{id}_X : X \to X] = \sum (-1)^j t^{-2j}[\wedge^j T^*_X] =: \lambda_{-t^{-2}}(T^*_X) \in K_T(X)[t^{-2}].
\]

The non-equivariant case is proved in [BSY10], and the equivariant case is shown in [AMSS19, PRW18].

Let
\[
D(-) := (-1)^{\text{dim} X} \text{RHom}_{O_X}(-, \omega_X)
\]
be the Serre-Grothendieck duality functor on \( K_T(X) \), where \( \omega_X := \wedge^{\text{dim} X} T^*_X \) is the canonical bundle of \( X \). Extend it to \( K_T(X)[t^{\pm 1}] \) by setting \( D(t^i) = t^{-i} \).

**Definition 5.** Let \( Z \subset X \) be a \( T \)-invariant subvariety.

1. Define the motivic Chern class of \( Z \) to be
\[
\text{MC}_{-t^{-2}}(Z) := \text{MC}_{-t^{-2}}([Z \hookrightarrow X]).
\]
2. Further assume that \( Z \) is pure-dimensional. Define the Segre motivic Chern classes of \( Z \) as follows (see [MNS20, Definition 6.2]),
\[
\text{SMC}_{-t^{-2}}(Z) := t^{-2 \text{dim} Z} \frac{D(\text{MC}_{-t^{-2}}(Z))}{\lambda_{-t^{-2}}(T^*_X)}.
\]

**Smoothness of Schubert varieties.** Consider the variety of complete flags \( G/B \). Let \( X(w)^o := BwB/B \) and \( Y(w)^o := B^{-w}B/B \) be the Schubert cells. The closures \( X(w) := \overline{X(w)^o} \) and \( Y(w) := \overline{Y(w)^o} \) are the Schubert varieties. Observe that \( u \leq v \) with respect to the Bruhat order if and only if \( X(u) \subset X(v) \). Let \( pt^m_w = w(x_{11})f^m_w \) denote the class of a point in \( Q^*_m, \) and let \( c_w \) denote the respective \( T \)-fixed point in \( G/B \).

The key property of the motivic Chern classes of the Schubert cells that we need are listed below.
Theorem 6. (1) [MNS20, Theorem 7.6] For any \( w \in W \), we have
\[
\text{MC}_{-t-2}(X(w)^0) = t_w^{-1} \tau_w \circ pt_e^m.
\]

(2) [AMSS19, Theorem 9.1] For any \( u \leq w \in W \), the Schubert variety \( X(w) \) is smooth at \( \mathfrak{e}_u \) if and only if
\[
\text{MC}_{-t-2}(X(w))|_u = \prod_{\alpha > 0, u_s \alpha \leq w} (1 - e^{u_\alpha}) \prod_{\alpha > 0, u_s \alpha \leq w} (1 - t^{-2} e^{u_\alpha}).
\]

Remark 7. This theorem is used to prove the Bump, Nakasuji and Naruse’s conjectures about Casselman basis in unramified principal series representations, see [BN11, BN19, NT14, AMSS19, Su19].

Proof. The first part follows from the reference mentioned. The second one follows from the fact \( w_0 \circ (\text{MC}_{-t-2}(Y(w))) = \text{MC}_{-t-2}(X(w_0 w)) \).

Given \( w \in W \), define the coefficients \( a_{w,u} \in \mathbb{Q}_m \) by the following formulas:
\[
\Gamma_w := \sum_{v \leq w} t_v^{-1} \tau_v = \sum_{u \leq w} a_{w,u} \delta_u^m \in \mathbb{Q}_m W.
\]

Note that if the Schubert variety \( X(w) \) is smooth, then \( P_{v,w} = 1 \) for all \( v \leq w \), so \( \Gamma_w = t_w^{-1} \gamma_w \). It is immediate to get the following corollary from Theorem 6.

Corollary 8. For any \( u \leq w \in W \), the Schubert variety \( X(w) \) is smooth at the fixed point \( \mathfrak{e}_u \) if and only if
\[
a_{w,u} = \prod_{\alpha > 0, u_s \alpha \leq w} \frac{1 - t^{-2} e^{u_\alpha}}{1 - e^{u_\alpha}}.
\]

Proof. By Theorem 6 (1) and (9), we have
\[
\text{MC}_{-t-2}(X(w)) = \sum_{v \leq w} \text{MC}_{-t-2}(X(v)^0) = \sum_{v \leq w} t_v^{-1} \tau_v \circ pt_e^m
\]
\[
= \sum_{v \leq w} a_{w,v} \delta_v^m \circ pt_e^m = \sum_{v \leq w} a_{w,v} \prod_{\alpha > 0} (1 - e^{v_\alpha}) f_v.
\]
Thus, we have
\[
\text{MC}_{-t-2}(X(w))|_u = a_{w,u} \prod_{\alpha > 0} (1 - e^{u_\alpha}).
\]

The corollary follows from this and Theorem 6 (2).

4. Dual bases in \( K \)-theory and characteristic classes of mixed Hodge modules

In this section, we use the two Kazhdan-Lusztig bases of the Hecke algebra to define two collections of classes in \( K \)-theory, and show that they are actually dual to each other. We also give a geometric interpretation of one of these collections using the intersection homology mixed Hodge modules. These are also generalized to the partial flag variety case.
Therefore, since \( q(t) = \sum_{m=0}^{\infty} q_m t^m \) is a \( \infty \)-periodic function, we have
\[
q(t) = \sum_{m=0}^{\infty} q_m (t^{m+1} + t^{-m-1}).
\]

On the other hand, since \( c_n(t) = \sum_{m=0}^{\infty} c_{n,m} t^m \) is the \( \infty \)-periodic function, we have
\[
c_n(t) = \sum_{m=0}^{\infty} c_{n,m} (t^{m+1} + t^{-m-1}).
\]

Therefore, we have
\[
q(t) = c_n(t).
\]

By definition, \( \gamma_n(t) = \sum_{m=0}^{\infty} \gamma_{n,m} t^m \) is the \( \infty \)-periodic function, we have
\[
\gamma_n(t) = \sum_{m=0}^{\infty} \gamma_{n,m} t^m.
\]

Therefore, we have
\[
\gamma_n(t) = \lambda_n(t).
\]

By definition, \( \gamma_n(t) = \sum_{m=0}^{\infty} \gamma_{n,m} t^m \) is the \( \infty \)-periodic function, we have
\[
\gamma_n(t) = \lambda_n(t).
\]

Therefore, we have
\[
\gamma_n(t) = \lambda_n(t).
\]

By definition, \( \gamma_n(t) = \sum_{m=0}^{\infty} \gamma_{n,m} t^m \) is the \( \infty \)-periodic function, we have
\[
\gamma_n(t) = \lambda_n(t).
\]

Therefore, we have
\[
\gamma_n(t) = \lambda_n(t).
\]
Thus,
\[
\widetilde{C}_w = \tilde{\gamma}_{w^{-1}w_0} \cdot pt^m_{w_0} = \sum_{v \geq w} \epsilon_w \epsilon_v t_{w^{-1}w_0} t_{v^{-1}w_0}^{-1} P_{v^{-1}w_0, w^{-1}w_0} (t^{-2}) \tau_{w_0}^{-1} \cdot pt^m_{w_0}
\]
\[
= \prod_{\alpha > 0} (1 - t^{-2} e^{-\alpha}) \sum_{v \geq w} \epsilon_w \epsilon_v t_{w^{-1}w_0} P_{v^{-1}w_0, w^{-1}w_0} (t^{-2}) \text{SMC}_{-t^{-2}}(Y(v)^\circ),
\]
where the last step follows from Lemma 11 (1).

Therefore, we have
\[
\langle C_w, \widetilde{C}_y \rangle = \prod_{\alpha > 0} (1 - t^{-2} e^{-\alpha}) t_w t_y^{-1} w_0 \sum_u P_{u, w} \sum_v \epsilon_v \epsilon_y P_{v^{-1}w_0, y^{-1}w_0} \delta_{w,v}^{K_r}
\]
\[
= \prod_{\alpha > 0} (1 - t^{-2} e^{-\alpha}) t_w t_y^{-1} w_0 \sum_u P_{u, w} \epsilon_y P_{w_0 u, w_0 y}
\]
\[
= \prod_{\alpha > 0} (1 - t^{-1} e^{-\alpha}) \delta_{w,y}^{K_r},
\]
where the first equality follows from Lemma 11 (2), the second follows from \(P_{u, v} = P_{u^{-1}, v^{-1}}\), and the third one follows from (10).

An immediate corollary of the proof is the following.

**Corollary 13.** If the Schubert variety \(X(w)\) is smooth,
\[
C_w = \sum_{u \leq w} t_w \text{MC}_{-t^{-2}}(X(u)^\circ) = t_w \text{MC}_{-t^{-2}}(X(w)) \in K_T(G/B)[t^{\pm 1}].
\]

**Proof.** It follows directly from (11) and the fact \(P_{u, w} = 1\) for all \(u \leq w\). \qed

**Characteristic classes of mixed Hodge modules.** For any parabolic subgroup \(P_J\), let \(K^0(\text{MHM}(G/P_J, B))\) denote its Grothendieck group of \(B\)-equivariant mixed Hodge modules. Recall there is a motivic Hodge Chern transformation (see [ST1, Definition 5.3 and Remark 5.5])
\[
\text{MHC}_{-t^{-2}} : K^0(\text{MHM}(G/P_J, B)) \rightarrow K_B(G/P_J)[t^{\pm 1}] \cong K_T(G/P_J)[t^{\pm 1}],
\]
such that for any \([f : Z \rightarrow G/P_J] \in G^B_0(\text{var}/(G/P_J))\),
\[
\text{MC}_{-t^{-2}}([f : Z \rightarrow G/P_J]) = \text{MHC}_{-t^{-2}}([f!Q^H_Z]),
\]
where \([Q^H_Z] := [k_* Q^H_{pt}] \in K^0(\text{MHM}(Z, B))\) and \(k : Z \rightarrow \text{pt}\) is the structure morphism. The construction also works for \(B^-\)-equivariant mixed Hodge modules, where \(B^-\) is the opposite Borel subgroup. The natural transformation \(\text{MC}_{-t^{-2}}\) commutes with the Serre-Grothendieck dual as follows, see Corollary 5.19 of loc. cit.,
\[
\text{MHC}_{-t^{-2}} \circ \mathcal{D} = \mathcal{D} \circ \text{MHC}_{-t^{-2}},
\]
where the \(\mathcal{D}\) on the left hand side is the dual of the mixed Hodge modules, while the other one is the Serre-Grothendieck dual. Here both are denoted by \(\mathcal{D}\), if no confusion is possible.

For any \(u \in W\), let \(i_u : X(u)^\circ \hookrightarrow G/B\) and \(j_u : Y(u)^\circ \hookrightarrow G/B\) be the inclusions. Then by (13)
\[
\text{MC}_{-t^{-2}}(X(u)^\circ) = \text{MHC}_{-t^{-2}}([i_u! Q^H_{X(u)^\circ}]),
\]
Since \(\mathcal{D} \circ j_{u!} = j_{u*} \circ \mathcal{D}\), and
\[
\mathcal{D}(Q^H_{Y(v)^\circ}) = Q^H_{Y(v)^\circ} [2 \dim Y(v)^\circ](\dim Y(v)^\circ),
\]
where $[2 \dim Y(v)\circ]$ means shift by $2 \dim Y(v)\circ$ and $(\dim Y(v)\circ)$ denotes the twist by the Tate Hodge module $Q^H(1) \otimes \dim Y(v)\circ$, Equation (14) gives

$$\text{SMC}_{t-2}(Y(v)\circ) = \frac{\text{MHC}_{t-2}(\lambda_{t-2}(T^{*}_{G/B}))}{\lambda_{t-2}(T^{*}_{G/B})}.$$ 

Using these, Lemma 11(2) can also be proved using mixed Hodge modules by J. Schürmann. For the analogue in equivariant homology, see [S17, Theorem 1.2].

For any Schubert variety $X(w)$, let $[IC_{H}^{X(w)\circ}] \in K^0(\text{MHM}(G/B))$ denote the intersection homology Hodge module on $X(w)$. Then it is well known that (see [KL80, T87, KT02]),

$$[IC_{H}^{X(w)\circ}] = \sum_{u \leq w} \epsilon_{u}P_{u,w}(t^{-2})[\epsilon_{u}Q_{H}^{X(u)\circ}].$$

Thus,

$$\text{MHC}_{t-2}([IC_{H}^{X(w)\circ}]) = \sum_{u \leq w} \epsilon_{w}P_{u,w}(t^{-2})\text{MC}_{t-2}(X(u)\circ).$$

Comparing with (11), we get the following geometric interpretation of the KL classes $C_{w}$ in Definition 9.

**Proposition 14.** For any $w \in W$,

$$C_{w} = t_{w} \epsilon_{w} \text{MC}_{t-2}([IC_{H}^{X(w)\circ}]) \in K_{T}(G/B)[t^{\pm 1}].$$

An immediate Corollary is the following.

**Corollary 15.** The canonical basis $C_{w}$ is invariant under the Serre-Grothendieck duality, i.e.,

$$D(C_{w}) = C_{w} \in K_{T}(G/B)[t^{\pm 1}].$$

**Proof.** Since

$$D([IC_{H}^{X(w)\circ}]) = [IC_{H}^{X(w)\circ}(\dim X(w))],$$

Equation (14) and Proposition 14 give

$$D(C_{w}) = D(t_{w} \epsilon_{w} \text{MC}_{t-2}([IC_{H}^{X(w)\circ}])) = t_{w}^{-1} \epsilon_{w} \text{MC}_{t-2}(D([IC_{H}^{X(w)\circ}])) = C_{w}.$$ 

**Parabolic case.** In this subsection, we generalize the above results to the parabolic case. Let $J \subset \Pi$ be a subset of simple roots, with corresponding parabolic subgroup $P_{J}$. Schubert cells and varieties and opposite Schubert cells and varieties of $G/P_{J}$ are indicated by subscripts $J$. Recall there exist parabolic Kazhdan-Lusztig polynomials (see [D79, KT02]), denoted by $P_{v,w}^{J} \in \mathbb{Z}[t^{-2}]$, where $v, w \in W^{J}$. Here our $P_{v,w}^{J}$ is the $u = -1$ parabolic KL polynomials in [D79], which is also denoted by $P_{v,w}^{J}$ in [KT02, Remark 2.1]. We have the following property, which generalizes [D79, Proposition 3.4].

**Lemma 16.** [LZZ19, Proposition 5.19] For any $v, w \in W^{J}$ and $u \in W_{J}$,

$$P_{u,v,v,w}^{J} = P_{u,v,w}^{J}.\,$$

Let $Q_{u,v} = P_{w_{0}w,w_{0}u}$ denote the usual inverse KL polynomials, which satisfy

$$\sum_{u} \epsilon_{u} \epsilon_{w} Q_{u,w} P_{w,v} = \delta_{u,v}^{K_{F}}.$$
For any $u, w \in W^J$, let $Q^J_{u,w} \in \mathbb{Z}[t^{-2}]$ denote the inverse parabolic KL polynomial (see [KT02]). Then
\begin{equation}
\sum_{w \in W^J} \epsilon_u \epsilon_w Q^J_{u,w} P^J_{w,v} = \delta^J_{u,v}.
\end{equation}

Moreover, it is related to the usual $Q_{u,w}$ as follows, see [KT02, Proposition 2.6] or [S97]:
\begin{equation}
Q^J_{u,w} = \sum_{v \in W^J} \epsilon_u \epsilon_w Q_{u,w,v}.
\end{equation}

Following Equations (11) and (12), we define the parabolic canonical bases in $K_T(G/P_J)[t, t^{-1}]$ as follows.

**Definition 17.** For any $w \in W^J$, let
\begin{equation}
C^J_w := \sum_{u \in W^J, w \leq u} t_u P^J_{u,w} (t^{-2}) \MC_{-t^{-2}}(X(u)_J^0),
\end{equation}
and
\begin{equation}
\tilde{C}^J_w := \prod_{\alpha \in \Sigma^+ \setminus \Sigma^+_J} (1 - t^{-2} e^{-\alpha}) \sum_{v \in W^J, v \geq w} \epsilon_v \epsilon_w t_{w,v} Q^J_{v,u,v} (t^{-2}) \SMC_{-t^{-2}}(Y(v)_J^0).
\end{equation}

Then if $J = \emptyset$, then $C^0_w = C_w$, and $\tilde{C}^0_w = \tilde{C}_w$, as defined before.

Let $\langle -, - \rangle_J$ denote the non-degenerate tensor product pairing on $K_T(G/P_J)$. The parabolic analog of Lemma 12 also holds (see [MNS20, Theorem 7.2]): for any $u, v \in W^J$,
\begin{equation}
\langle \MC_{-t^{-2}}(X(u)_J^0), \SMC_{-t^{-2}}(Y(v)_J^0) \rangle_J = \delta^J_{u,v}.
\end{equation}
Combining this with (15), we immediately get the following generalization of Theorem 10.

**Theorem 18.** For any $u, w \in W^J$,
\begin{equation}
\langle C^J_w, \tilde{C}^J_w \rangle_J = \delta^J_{u,w} \prod_{\alpha \in \Sigma^+ \setminus \Sigma^+_J} (t - t^{-1} e^{-\alpha})
\end{equation}

We now investigate the relation between KL classes of $G/B$ and $G/P_J$. For any $w \in W^J$, let us still use $i_u$ denote the inclusion $X(u)_J^0 \to G/P_J$. Then the following identity holds in $K^0(MHM(G/P_J, B))$ (see [KT02, Corollary 5.1]),
\begin{equation}
[I^J_{X(u)_J^0}] = \sum_{u \in W^J, u \leq w} \epsilon_u P^J_{u,w} [i_u ! I^J_{X(u)_J^0}].
\end{equation}

Thus, we get the following parabolic analog of Proposition 14 and Corollary 15.

**Proposition 19.** For any $w \in W^J$,
\begin{equation}
C^J_w = t_w \epsilon_w \MHM_{-t^{-2}}(I^J_{X(u)_J^0}).
\end{equation}
Moreover, let $D_J$ denote the Serre-Grothendieck duality functor on $G/P_J$. Then
\begin{equation}
D_J(C^J_w) = C^J_w.
\end{equation}

Recall $\pi_J : G/B \to G/P_J$ denotes the natural projection. The relation between $C_w$ and $C^J_w$ is given by the following proposition.

**Proposition 20.** Let $P_J(t) = \sum_{v \in W_J} t_v$ be the Poincaré polynomial of $W_J$. then for any $w \in W^J$,
\begin{equation}
\pi_J^* (C^J_{wwJ}) = t_w^{-1} P_J(t^2) C^J_w \in K_T(G/P_J)[t, t^{-1}].
\end{equation}

\footnote{Our $Q^J_{u,w}$ is denoted by $Q^J_{u,w}^{\text{proj}}$ in [KT02].}
Proof. By \cite{AMSS19} Remark 5.5, for any \( u \in W^J \) and \( v \in W_J \),
\[
\pi_{J*}(MC_{t-2})(X(u)v) = t_v^{-2}MC_{t-2}(X(u)^0),
\]
which also follows directly from the identity about mixed Hodge modules
\[
\pi_{J*}(i_{uv}^!Q^H_{X(u)v}) = Q^H_{X(u)^0}[-2\ell(v)](-\ell(v)).
\]
Thus,
\[
\pi_{J*}(C_{uw,J}) = \sum_{u \in W^J, \alpha_u \leq w} \sum_{v \in W_J} t_w t_{w,J} P_{uv,ww,J} \pi_{J*} MC_{t-2}(X(u)v)
\]
\[= \sum_{u \in W^J, \alpha_u \leq w} t_w t_{w,J} P_{uv,ww,J} MC_{t-2}(X(u)^0) \sum_{v \in W_J} t_v^{-2}
\]
\[= C_w^J \sum_{v \in W_J} t_v^{-2} t_{w,J} = C_w^J \sum_{v \in W_J} t_{w,J} t_v^{-2} t_{w,J} = C_w^J \sum_{v \in W_J} t_v^{-2} t_{w,J} = C_w^J t_v^{-2} P_J(t^2),
\]
where the second equality follows from Lemma \ref{lem:mc}. \(\square\)

5. The Smoothness Conjecture for Hyperbolic Cohomology

In this section, we use the smoothness criterion to prove the Smoothness Conjecture. Since we will be working with multiplicative and hyperbolic formal group laws in the same time, we add superscripts or subscripts \( m \) (resp. \( t \)) in the multiplicative case (resp. hyperbolic case).

The hyperbolic case. Set \( \mu = t + t^{-1} \). Consider the hyperbolic formal group law over \( R \)
\[
F_t(x, y) := \frac{x + y - xy}{1 - \mu - 2xy}.
\]
The definitions of Section \ref{sec:smoothness} applied to \( F_t \) give the respective rings
\[
S_t, Q_t, Q_{t,W}, D_t.
\]
Consider a map of formal group laws
\[
g: F_t \to F_m, \quad g(x) = \frac{(1-t^2)x}{x-(t^2+1)},
\]
so that \( F_m(g(x), g(y)) = g(F_t(x, y)) \). It induces ring embeddings
\[
\psi: S_m \hookrightarrow S_t, \quad \psi(f(x)) = f(g(x)), \quad f(x) \in R[[x]],
\]
and
\begin{equation}
(16)
\psi: Q_m \hookrightarrow R \left[ \frac{1}{1-t^2} \right] \otimes_Q Q_t.
\end{equation}
Consequently, we have a ring embedding
\[
\psi: Q_{m,W} \to R \left[ \frac{1}{1-t^2} \right] \otimes_R Q_{t,W}, \quad \psi(p\delta^m_w) = \psi(p)\delta^t_w, \quad p \in Q_m, w \in W.
\]
It can be shown that
\begin{equation}
(17)
\psi(\tau_i) = \mu Y_i^t - t \in D_t \subset Q_{t,W}.
\end{equation}
Note that in (16), for the target, we have to invert \( t^2 - 1 \), but for the one in (17), it is not necessary.

One of the most interesting properties of \( \psi \) is the following (see \cite{LZZ19} Corollary 5.5 (2)):
\begin{equation}
(18)
\mu^{-\ell(w,J')} \psi(\gamma_{J'/J}) Y_{J'}^t = Y_J^t.
\end{equation}
In other words, \( \psi(\gamma_{J'/J}) \) behaves like a replacement of \( Y_{J'/J} \); see \cite{LZZ19} Remark 5.6. In particular, letting \( J' = \emptyset \), one then has
\[
\mu^{-\ell(w,J)} \psi(\gamma_{w,J}) = Y_J^t.
\]
Let $h$ denote the respective oriented cohomology theory for the hyperbolic formal group law $F_t$.

**Definition 21.** Define the KL-Schubert class for $w \in W^J$ to be

$$ KL_w^J := \mu^{-\ell(w)} \psi(\gamma_w) \odot pt_e^t \in (D_t)^{W_J} \cong h_T(G/P_J). $$

**Remark 22.** Following [LZZ19] one can define certain involution on some subset $N_J := \psi(H) \odot pt_e^t \subset D_t^*$ so that $KL_{N_J}$ is invariant under such involution, similar to the parabolic Kazhdan-Lusztig basis of Deodhar.

We now prove the Smoothness Conjecture [LZZ19, Conjecture 5.14]. Several special cases were proved in [LZ17, LZZ19], such as the case of $w = w_{J/J'}$ for $J' \subset J \subseteq \Pi$ (i.e., $w$ has ‘relative’ maximal length), and that of Schubert varieties in complex projective spaces.

**Theorem 23.** If $X(w)$ is smooth, then the class determined by $X(w)$ in $h_T(G/B)$ coincides with the KL-Schubert class $KL_w$.

**Proof.** Since $X(w)$ is smooth, $P_{v,w} = 1$ for any $v \leq w$, see [BL00, 6.1.19]. Therefore,

$$\gamma_w = \sum_{v \leq w} t_w t_v^{-1} \tau_v = t_w \sum_{v \leq w} t_v^{-1} \tau_v = t_w \Gamma_w = t_w \sum_{v \leq w} a_{w,v} y_v^m.$$ 

From the definition of $\psi$, it is easy to verify that

$$ \psi\left(\frac{1 - t^{-2} e^{\alpha}}{1 - e^{\alpha}}\right) = \frac{t^{-1} \mu}{x_{-\alpha}}. $$

Then for any $w \in W$, we have

$$ KL_w = \mu^{-\ell(w)} \psi(\gamma_w) \odot pt_e^t $$

$$ = \mu^{-\ell(w)} \psi\left(t_w \sum_{v \leq w} a_{w,v} y_v^m\right) \odot pt_e^t $$

$$ = \mu^{-\ell(w)} \sum_{v \leq w} \psi\left(\prod_{\alpha > 0, \forall s_{\alpha} \leq w} \frac{1 - t^{-2} e^{\alpha}}{1 - e^{\alpha}}\right) \delta_{v}^t \odot pt_e^t $$

$$ = \mu^{-\ell(w)} \sum_{v \leq w} \left(\prod_{\alpha > 0, \forall s_{\alpha} \leq w} \frac{t^{-1} \mu}{x_{-\alpha}}\right) \cdot v(x_{\Pi}^t) f_v^t $$

$$ = \sum_{v \leq w} \left(\prod_{\alpha < 0, \forall s_{\alpha} \leq w} x_{\alpha}\right) f_v^t $$

$$ = \sum_{v \leq w} \left(\prod_{\alpha > 0, \forall s_{\alpha} \leq w} x_{-\alpha}\right) f_v^t. $$

Here the fifth identity follows from the following well-known fact:

*for any $v \leq w \in W$, if $X(w)$ is smooth, $|\{\alpha > 0 \mid s_{\alpha} v \leq w\}| = \ell(w),*
and the last one is proved as follows: for any \( v \leq w \in W, \)
\[
\prod_{\alpha<0} x_{\alpha} = \prod_{\alpha>0, s_{\alpha} v < w} x_{\alpha} \cdot \prod_{\alpha>0, v < s_{\alpha} w} x_{-\alpha}
\]
Comparing with the restriction formula of \([X(w)]\) in \([LZZ19, (5.6)]\), we see that \(KL_w = [X(w)].\)

The proof is finished.

We now look at the case of partial flag varieties. Let \( P_J \) be the parabolic subgroup with the projection map \( \pi_J : G/B \to G/P_J. \) Let \( w_J \) be the longest element in the subgroup \( W_J \) of \( W \) determined by \( J, \) and \( W^J \subset W \) be the set of minimal length representatives of \( W/W_J. \) Recall \( X(w)_J \) denotes the Schubert variety of \( G/P_J \) determined by \( w \in W^J. \)

For \( G/P_J \), the definition of KL-Schubert class \( KL^J_w \) corresponding to \( w \in W^J \) is defined by using the so-called parabolic Kazhdan-Lusztig basis. According to the paragraph right after \([LZZ19, \text{Definition } 5.9]\), via the embedding \( \pi_J^*: h_T(G/P_J) \to h_T(G/B), \) we have
\[
\pi_J^*(KL^J_w) = KL_{ww_J}.
\]

**Corollary 24.** Conjecture 5.14 of \([LZZ19]\) holds for any partial flag variety \( G/P_J, \) that is, if the Schubert variety \( X(w)_J \) of \( G/P_J \) is smooth for \( w \in W_J, \) then the KL-Schubert class \( KL^J_w \) of \( w \)

coincides with the fundamental class \( [X(w)_J].\)

**Proof.** We have the following Cartesian diagram:
\[
\begin{array}{ccc}
\pi_J^{-1}(X(w)_J) & \xrightarrow{i'} & G/B \\
\downarrow{\pi_J} & & \downarrow{\pi_J} \\
X(w)_J & \xrightarrow{i} & G/P_J.
\end{array}
\]
Moreover, \( \pi_J^{-1}(X(w)_J) = X(ww_J). \) Since \( X(w)_J \) is smooth, \( X(ww_J) \) is also smooth. Thus, Theorem \([23]\) implies \([X(ww_J)] = KL_{ww_J}.\) On the other hand, we get the following by proper base change:
\[
\pi_J^*[X(w)_J] = \pi_J^*i_*[1_{X(w)_J}] = i'_*\pi_J^*[1_{X(w)_J}] = i'_*[X(ww_J)] = [X(ww_J)],
\]
where the third equality follows from the fact that the pull-back \( \pi_J^* \) preserves identity. Since \( \pi_J^*(KL^J_w) = KL_{ww_J} \) and \( \pi_J^* \) is injective, we get \( KL^J_w = [X(ww_J)] \in h_T(G/P_J). \)

6. KL-Schubert classes and small resolutions

In this section, we give a geometric interpretation of the KL-Schubert classes (for hyperbolic cohomology) in the case of type \( A \) Grassmannians.

For subsets \( J' \subset J \subset \Pi, \) for hyperbolic cohomology, we will use relative push-pull elements \( Y^t_{J/J'} \)
defined in \([I]\). For simplicity, we will skip the superscript \( t. \) Moreover, if \( Q \subset P \) are the parabolic subgroups corresponding to \( J' \subset J, \) respectively, we will denote \( Y_{P/Q} = Y_{J/J'}. \)

Consider the Grassmannian \( Gr_d(\mathbb{C}^{n-d}) = SL_n/P_J, \) where the set of simple roots \( \Pi \) is identified with \( \{1, \ldots, n-1\} \) and \( J := \Pi \setminus \{d\}. \) Fix a Schubert variety \( X(\lambda) \) of it, which is indexed by a partition \( \lambda = (\lambda_1 \geq \ldots \geq \lambda_l > 0) \) contained inside the \( d \times (n-d) \) rectangle; here we mean that \( \lambda \)
is identified with a Young diagram (in English notation), whose top left box is placed on the top left box of the mentioned rectangle.

Alternatively, the Schubert variety \( X(\lambda) \) is indexed by a \( d \)-subset \( I_\lambda \) of \([n] := \{1, \ldots, n\} \), which is constructed as follows. Place the above \( d \times (n - d) \) rectangle inside the first quadrant of the \( xy \)-plane, such that its southwest corner is the origin. Label each horizontal (resp. vertical) unit segment whose left (resp. bottom) endpoint is a lattice point \((x, y)\) by \( x + y + 1 \). Consider the lattice path from \((0, 0)\) to \((n - d, d)\) defining the southeast boundary of the Young diagram \( \lambda \) when embedded into the \( d \times (n - d) \) rectangle as stated above. Then \( I_\lambda \) consists of the labels on the vertical steps of this path.

Yet another indexing of the Schubert variety \( X(\lambda) \) is by a Grassmannian permutation \( w_\lambda \) in the symmetric group \( W = S_n \), which has its unique descent in position \( d \). Written in one-line notation, \( w_\lambda \) consists of the entries in \( I_\lambda \) followed by the entries in \([n] \setminus I_\lambda \), where both sets of entries are ordered increasingly. Thus, \( w_\lambda \) belongs to the set \( W^J \) of lowest coset representatives modulo the parabolic subgroup \( W_J \). Moreover, it has the following reduced decomposition:

\[
w_\lambda = \prod_{(i, j) \in \lambda} s_{d + j - i};
\]

here \((i, j)\) is the box of the Young diagram \( \lambda \) in row \( i \) and column \( j \), while in the product we scan the rows of \( \lambda \) from bottom to top, and each row from right to left.

**Example 25.** We use as a running example the same one as in [BL00, Example 9.1.11], namely \( n = 10, d = 5, \lambda = (5, 5, 3, 2, 2), I_\lambda = \{3, 4, 6, 9, 10\} \). In order to illustrate (20), we place the number \( d + j - i \) in the box \((i, j)\) of \( \lambda \), as follows:

\[
\begin{array}{cccccc}
5 & 6 & 7 & 8 & 9 \\
4 & 5 & 6 & 7 & 8 \\
3 & 4 & 5 \\
2 & 3 \\
1 & 2
\end{array}
\]

Thus, we have

\[
w_\lambda = [3, 4, 6, 9, 10, 1, 2, 5, 7, 8] = (s_2s_1)(s_3s_2)(s_5s_4s_3)(s_8s_7s_6s_5s_4)(s_9s_8s_7s_6s_5).
\]

In [BL00, Section 9.1], the permutation \( w_\lambda \) is identified with the \( d \)-subset \( I_\lambda \), and they are encoded into a \( 2 \times m \) matrix

\[
\begin{pmatrix}
k_1 & \ldots & k_m \\
a_1 & \ldots & a_m
\end{pmatrix},
\]

which can be read off from the above lattice path as follows. The entries \( 0 < k_1 < \ldots < k_m \leq n \) are the labels of the last steps in consecutive sequences of vertical (unit) steps. The entries \( a_1, \ldots, a_m \) are the lengths of these sequences. The numbers \( b_0, \ldots, b_{m - 1} \) calculated in [BL00] are the lengths of the sequences of horizontal steps, where we set \( b_0 := 0 \) if \( l < d \) (i.e., if the lattice path starts with a vertical step). Recall that we also set \( a_0 = b_m := \infty \).

Now recall that the Schubert variety \( X(\lambda) \) has small resolutions, which were defined by Zelevinsky [ZS3]. We briefly recall their construction following [BL00, Section 9.1]. This construction starts with the choice of an index \( i \), with \( 0 \leq i < m \), such that \( b_i \leq a_i \) and \( a_{i+1} \leq b_{i+1} \) (any such choice can be made). While it is clear that such an index always exists, we avoid the choice of \( i = 0 \) if \( l < d \). Then, a new permutation \( w^2 \) is obtained from \( w^1 := w_\lambda \) via a certain procedure, which can be rephrased as follows. Consider the \( i \)-th outer corner of \( \lambda \) (counting from 0), from southwest to northeast, where the origin is an outer corner if and only if \( l < d \). Consider the rectangle \( R_1 \) (inside \( \lambda \)) whose southeast vertex is the mentioned outer corner, and which is maximal such that
its removal from $\lambda$ still leaves a Young diagram. It is clear that the size of $R_1$ is $b_i \times a_{i+1}$. Then $w^3$ is the Grassmannian permutation corresponding to the Young diagram $\lambda \setminus R_1$.

The above procedure is then iterated. We thus tile the Young diagram $\lambda$ with rectangles $R_1, \ldots, R_r$. Let us denote by $p_i$ and $q_i$ the height and width of $R_i$, respectively. We also define the sequence of Grassmannian permutations $w^1, \ldots, w^r$, such that the Young diagram of $w^i$ is $\lambda^i := \lambda \setminus \rho^{i-1}$, where $\rho^j := R_1 \cup \ldots \cup R_j$. In particular, the Young diagram of $w^r$ is $R_r$, and the Schubert variety $X(w^r)$ is smooth. Note that $r = m$ if $l = d$, and $r = m - 1$ if $l < d$.

**Example 26.** We continue Example 25. The encoding of $w_\lambda$ by the $2 \times m$ matrix (23) and the successive choices of $w^1, w^2, w^3$ based on it are described in detail in [BL00]. In our setup, the tiling of $\lambda$ with the corresponding rectangles $R_1, R_2, R_3$ is illustrated below (the number in a box is the index of the rectangle to which that box belongs).

```
  3 3 2 2 2
  3 3 2 2 2
  3 3 3 3 3
```

In order to complete the construction of the Zelevinsky resolution, following [BL00] Section 9.1, we need the stabilizer $P_{w_\lambda}$ of the Schubert variety $X(\lambda) = X(w_\lambda)$. This is the parabolic subgroup corresponding to the subset $\Pi \setminus \{k_1, \ldots, k_m\}$, cf. (23). More generally, consider the stabilizers $P_i := P_{w^i}$, for $i = 1, \ldots, r$, and $P_{r+1} := P_J$; for simplicity, we use the same notation for the corresponding subsets of $\Pi$. Also let $Q_i := P_i \cap P_{i+1}$, for $i = 1, \ldots, r$, both as parabolic subgroups and subsets of $\Pi$. Then the Zelevinsky resolution of $X(w)$ is expressed as follows:

$$X(w) =: \widetilde{X}(w_\lambda) \rightarrow X(w_\lambda).$$

Therefore, the cohomology class of $\widetilde{X}(w_\lambda)$ is computed by the following composition of relative push-pull operators:

$$Y_{P_1/Q_1} \cdots Y_{P_r/Q_r} Y_J.$$

Here we recall the fact that $Y_{J/J'} \bullet : (D_t^W)_{J'}^W \rightarrow (D_t^W)_{J}$ coincides with the canonical map $\eta_T(G/P_J) \rightarrow \eta_T(G/P_J)$; see [CZZ14] Lemma 8.13 for more details.

**Example 27.** Continuing Example 26, the operator in (25) is written explicitly as follows:

$$Y_{(\Pi \setminus \{4, 5\})/(\Pi \setminus \{4, 5, 6\})} Y_{(\Pi \setminus \{5\})/(\Pi \setminus \{5, 7\})} Y_{(\Pi \setminus \{7\})/(\Pi \setminus \{5, 7\})} Y_{\Pi \setminus \{5\}}.$$

Indeed, the parabolic subsets $P_i$ for these examples were exhibited in [BL00], while they can also be read off from the Young diagram of $\lambda = (5, 5, 3, 2, 2)$ as indicated above.

We will now state the main technical result of this section, Theorem 29, which is interesting itself, and is needed to make the connection with the KL-Schubert classes for the Grassmannian, cf. [LZZ19]. To this end, we introduce more notation in the above setup. Given the rectangle $R_i$, with its embedding into the Young diagram of $\lambda$ and the first quadrant, let $C_i$ and $D_i$ be the sets of labels on its left vertical side and its top horizontal side, respectively. Let

$$c_i := \min C_i, \quad d_i := \max D_i = c_i + p_i + q_i - 1, \quad C'_i := C_i \setminus \{\max C_i\}, \quad D'_i := D_i \setminus \{d_i\}.$$

Finally, let $J_i := C_i \sqcup D'_i$ and $J'_i := C'_i \sqcup D'_i$.

We also need to define the subsets $K'_i \subsetneq K_i$ of $\Pi$, $i = 1, \ldots, r$. First recall that above we defined the shape $\rho^i$ as the union of the rectangles $R_1, \ldots, R_i$. It is not hard to see that $\rho^i$ is a union of completely disjoint Young diagrams (i.e., they do not share even a single point), aligned from southwest to northeast. Let $C_i$ be set of indices $j \in \{1, \ldots, i\}$ such that the left side of $R_j$ is
contained in the left boundary of a component of $\rho_i$. Similarly, let $D_i$ be set of indices $k \in \{1, \ldots, i\}$ such that the top side of $R_k$ is contained in the top boundary of a component of $\rho_i$. We now define

$$K'_i := \bigcup_{j \in C_i} C'_j \cup \bigcup_{k \in D_i} D'_k, \quad K_i := K'_i \cup \{\max C_i\}.$$ 

Note that $J_i \subseteq K_i$ and $J'_i \subseteq K'_i$.

**Example 28.** Continuing Example 27 we have

- $K'_1 = J'_1 = \emptyset \subseteq K_1 = J_1 = \{5\}$,
- $K'_2 = J'_2 = \{6, 8, 9\} \subseteq K_2 = J_2 = \{6, 7, 8, 9\}$,
- $J'_3 = \{1, 2, 3, 4, 6\} \subseteq J_3 = \{1, 2, 3, 4, 5, 6, 9\}$,
- $K'_3 = \{1, 2, 3, 4, 6, 8, 9\} \subseteq K_3 = \{1, 2, 3, 4, 5, 6, 8, 9\}$.

As indicated above, all this information is easily read off from the Young diagram of $\lambda = (5, 3, 2, 2)$.

**Theorem 29.** In $H \subset Q_{m,W}$, we have

$$(26) \quad \gamma_{w_\lambda} \gamma_J = \gamma_{J_1/J'_1} \cdots \gamma_{J_r/J'_r} \gamma_J = \gamma_{K_1/K'_1} \cdots \gamma_{K_r/K'_r} \gamma_J.$$ 

In order to prove Theorem 29 we start by recalling some results from [KL00], related to the factorization of Kazhdan-Lusztig elements for the Grassmannian. This paper introduces an element $Z_{w_\lambda}$ of the Hecke algebra, defined as a product of linear factors in the generators, which are associated with the boxes of the Young diagram $\lambda$. Instead of recalling the precise definition, which is not needed here, we will state a weaker form of the factorization, which turns out to be related to factorizations in (26). We will use notation introduced above.

The rectangle $R_i$ corresponds to the following Grassmannian permutation, cf. (20) and Example 25

$$v^i := (s_{c_i+q_i-1} \cdots s_{c_i}) (s_{c_i+q_i} \cdots s_{c_i+1}) \cdots (s_{c_i+q_i+q_i-2} \cdots s_{c_i+q_i-1}).$$

It is not hard to see that we have the following factorization of $w_\lambda$, which corresponds to a reduced decomposition of $w_\lambda$ obtained from (20) only by commuting simple reflections:

$$(27) \quad w_\lambda = v^1 \cdots v^r.$$ 

**Example 30.** In our running example, the reduced decomposition corresponding to (27) (to be compared with (22), cf. also (21)) is

$$w_\lambda = [3, 4, 6, 9, 10, 1, 2, 5, 7, 8] = (s_8) ((s_8 s_7 s_6)(s_9 s_8 s_7)) ((s_2 s_1)(s_3 s_2)(s_4 s_3)(s_5 s_4)(s_6 s_5)).$$

The factorization of $Z_{w_\lambda}$ needed here is the following one, which corresponds to the factorization (27) of $w_\lambda$:

$$(28) \quad Z_{w_\lambda} = Z_{v^1} Z_{v^2} = Z_{v^1} \cdots Z_{v^r}.$$ 

See the proof of [KL00] Theorem 3] for details.

The connection between the element $Z_{w_\lambda}$ and the corresponding parabolic Kazhdan-Lusztig basis element is made in [KL00] Theorem 3].

**Theorem 31.** [KL00] In $H \subset Q_{m,W}$, we have

$$Z_{w_\lambda} \gamma_J = \gamma_{w_\lambda w_J}.$$ 

The proof of Theorem 29 also relies on the following lemmas.

**Lemma 32.** Consider $J' \subset J \subseteq \Pi$, and assume that $J \subset [a, b]$ with $a, b \in \Pi$. If $A \subseteq \Pi \setminus [a-1, b+1]$, then we have

$$\gamma_{J/J'} = \gamma_{J \cup A/J' \cup A} \in Q_{m,W}, \quad Y_{J/J'} = Y_{J \cup A/J' \cup A} \in D_l.$$
The similar property for $Y$ is clear that $\rho$. Lemma 34. and $W$ are indicated below; the boxes of $R$ component(s) of the relevant Young diagram to its right (respectively at the bottom). It is also useful to observe that all unit segments with the same label form a northwest to southeast staircase shape, and the labels increase by 1 as we move northeast.

Let $B$ denote the set of labels on the boundary of the rectangle $R_i$. Using the above notation, in all three cases in (30), we have

$$B = C_i \sqcup D_i = \{c_i, \ldots, d_i\}, \quad K_i \setminus B = K_i' \setminus B, \quad K_i \cap B = C_i \sqcup D_i' = B \setminus \{d_i\} =: J_i.$$

On another hand, we have $d_i \not\in K_i'_{i-1}$; indeed, in the first and last case in (30), the label $d_i$ is on the left side of a rectangle $R_j$ with $j \in C_{i-1}$, but $d_i \not\in C_j'$, because it is the top label on the
mentioned side. We conclude that $K_{i-1}' \subseteq K_i$. In fact, the inclusion is strict because we also have $c_i + q_i - 1 \in (K_i \cap B) \setminus K_{i-1}'$.

For the second part, we note that, in addition to the above facts, we have $K_i' \cap B = C_i' \cup D_i' =: J_i'$ and $c_i - 1 \notin K_i$. For the latter part, note that, in the last two cases in (29), the label $c_i - 1$ is on the left side of a rectangle $R_j$ with $j \in C_i$ and $j \neq i$, but $c_i - 1 \notin C_i'$, because it is the top label on the mentioned side. The proof is concluded by applying Lemma 32.

\textbf{Proof of Theorem 29} Using the above setup, we have

\begin{equation}
\gamma_{J/K_1/K_1'} \cdots \gamma_{J/K_r/K_r'}\gamma J \\
\overset{\sharp_1}{=} \gamma_{J/K_1/K_1'} \cdots \gamma_{J/K_r/K_r'}\gamma J \\
\overset{\sharp_2}{=} \gamma_{J/K_1/K_1'} \cdots \gamma_{J/K_r/K_r'}\gamma J \\
\overset{\sharp_3}{=} \gamma_{J/K_1/K_1'} \cdots \gamma_{J/K_r/K_r'}\gamma J \\
\overset{\sharp_4}{=} \gamma_{J/K_1/K_1'} \cdots \gamma_{J/K_r/K_r'}\gamma J \\
\overset{\sharp_5}{=} \gamma_{J/K_1/K_1'} \cdots \gamma_{J/K_r/K_r'}\gamma J \\
\overset{\sharp_6}{=} \gamma_{J/K_1/K_1'} \cdots \gamma_{J/K_r/K_r'}\gamma J \\
\overset{\sharp_7}{=} \gamma_{J/K_1/K_1'} \cdots \gamma_{J/K_r/K_r'}\gamma J \\
\overset{\sharp_8}{=} \gamma_{J/K_1/K_1'} \cdots \gamma_{J/K_r/K_r'}\gamma J \\
\overset{\sharp_9}{=} \gamma_{J/K_1/K_1'} \cdots \gamma_{J/K_r/K_r'}\gamma J \\
\overset{\sharp_{10}}{=} \gamma_{J/K_1/K_1'} \cdots \gamma_{J/K_r/K_r'}\gamma J.
\end{equation}

Here $\sharp_1$, $\sharp_2$, $\sharp_3$, $\sharp_4$ are based on Lemma 34 (1) and Lemma 33 (2), $\sharp_5$ on (8), and $\sharp_6$ on the repeated use of an argument similar to $\sharp_2$ followed by $\sharp_3$.

We now prove the theorem using induction on $r$, with base case $r = 0$, which is trivial. We have

\begin{equation}
\gamma_{w\lambda w_j} \overset{\sharp_1}{=} Z_{w\lambda} \gamma J \\
\overset{\sharp_2}{=} Z_{w\lambda} \gamma J \\
\overset{\sharp_3}{=} Z_{w\lambda} \gamma J \\
\overset{\sharp_4}{=} Z_{w\lambda} \gamma J \\
\overset{\sharp_5}{=} Z_{w\lambda} \gamma J \\
\overset{\sharp_6}{=} Z_{w\lambda} \gamma J \\
\overset{\sharp_7}{=} Z_{w\lambda} \gamma J \\
\overset{\sharp_8}{=} Z_{w\lambda} \gamma J \\
\overset{\sharp_9}{=} Z_{w\lambda} \gamma J \\
\overset{\sharp_{10}}{=} Z_{w\lambda} \gamma J.
\end{equation}

Here $\sharp_1$, $\sharp_2$, $\sharp_3$, $\sharp_4$ are based on Theorem 31, $\sharp_5$ on (28), $\sharp_6$ on the induction hypothesis, $\sharp_7$, $\sharp_8$ on Lemma 34 (2), $\sharp_9$ on (31), and $\sharp_{10}$ on (30); additionally, in $\sharp_7$ we use the fact that $K_1 = J_1 = C_1' \cup D_1' = \{c_1, \ldots, d_1 - 1\}$, $K_1' = J_1' = C_1' \cup D_1' = K_1 \setminus \{\max C_1\}$, and thus we have $v^1w_{K_1}' = w_{K_1}$.

\textbf{Remark 35}. We could not have carried out the above proof by using only one of the pairs $(J_i, J_i')$ and $(K_i, K_i')$. Indeed, the first pair does not satisfy the property in Lemma 34 (1), which is crucial in the proof. On the other hand, the induction procedure cannot be applied based on the second pair because the respective sets for $\lambda^1 = \lambda$ and $\lambda^2$ (corresponding to $w^2$) are different.

In order to relate Theorem 29 to the Zelevinsky resolution, and more specifically to the operator (25), we need the following result.

\textbf{Lemma 36}. For every $i = 1, \ldots, r$, we have

\[ Y_{J_i/J_i'} = Y_{K_i/K_i'} = Y_{P_i/Q_i}. \]

\textbf{Proof}. By using Lemma 34 (2), it suffices to prove $Y_{J_i/J_i'} = Y_{P_i/Q_i}$. Moreover, it suffices to consider $i = 1$, as we can just replace the partition $\lambda^1 = \lambda$ with $\lambda^1$. Recall that $P_1$ is obtained by considering the lattice path from $(0, 0)$ to $(n - d, d)$ defining the southeast boundary of $\lambda^1$, and by excluding from $\Pi$ the last label in each sequence of vertical steps. Similarly, $P_2$ corresponds to $\lambda^2 := \lambda \setminus R_1$. 

\[ Y_{J_i/J_i'} = Y_{K_i/K_i'} = Y_{P_i/Q_i}. \]
Let $B$ denote the set of labels on the boundary of the rectangle $R_1$; see the diagram below, where the boxes of $R_1$ are marked with $\star$.

Using the above notation, we have $B = C_1 \cup D_1 = \{c_1, \ldots, d_1\}$. Based on the above interpretation of $P_1$ and $P_2$, we deduce

$$P_1 \cap B = C_1 \cup D_1' =: J_1 = B \setminus \{d_1\}, \quad P_2 \cap B = C_1' \cup D_1 =: J_1', \quad P_1 \setminus B \subseteq P_2 \setminus B \implies P_1 \setminus B = Q_1 \setminus B.$$ Moreover, we have $c_1 - 1 \notin P_1$ and $d_1 \notin P_1$. Thus, we are under the hypotheses of Lemma 32 so the conclusion follows.

We now rephrase Theorem 29 as follows, via the map $\psi$.

**Corollary 37.** We have

$$\mu^{-\ell(w_{\lambda}w_{\gamma})} (\gamma_{w_{\lambda}w_{\gamma}}) = Y_{P_i/Q_i} \cdots Y_{P_t/Q_t} Y_J \in D_t.$$ 

**Proof.** We start by observing the following:

$$w_{K_i/K_i'} = w_{J_i/J_i'} = v^i \implies \ell(w_{K_i/K_i'}) = p_iq_i = |R_i|,$$

where $|R_i|$ denotes the number of boxes of the rectangle $R_i$. Here the first equality is based on (29) and the fact that this result can be applied to the pairs $(J_i, J_i')$ and $(K_i, K_i')$, as discussed in the proof of Lemma 34; the second equality is clear by the definition of $v^i$.

We now apply $\mu^{-\ell(w_{\lambda}w_{\gamma})} (\cdot)$ to the first and last part of (26). After doing this, the latter can be written as follows:

$$\mu^{-\ell(w_{\lambda}w_{\gamma})} (\gamma_{K_i/K_i'}) \cdots (\gamma_{K_t/K_t'}) (\gamma_J)$$

$$\overset{\#1}{=} (\mu^{-\ell(w_{K_i/K_i'})} (\gamma_{K_1/K_1'}) \cdots (\mu^{-\ell(w_{K_t/K_t'})} (\gamma_{K_t/K_t'})) \mu^{-\ell(w_{\gamma})} (\gamma_J))$$

$$\overset{\#2}{=} (\mu^{-\ell(w_{K_i/K_i'})} (\gamma_{K_1/K_1'}) \cdots (\mu^{-\ell(w_{K_t/K_t'})} (\gamma_{K_t/K_t'})) Y_J)$$

$$\overset{\#3}{=} (\mu^{-\ell(w_{K_i/K_i'})} (\gamma_{K_1/K_1'}) \cdots (\mu^{-\ell(w_{K_t/K_t'})} (\gamma_{K_t/K_t'})) Y_{K_t} Y_{K_t} \ell(Y_{J/K_t'}))$$

$$\overset{\#4}{=} Y_{K_1} \ell(Y_{K_2/K_1'}) \cdots \ell(Y_{J/K_t'})$$

$$\overset{\#5}{=} Y_{K_1/K_t'} Y_{K_t' \ell(Y_{K_2/K_t'})} \cdots \ell(Y_{J/K_t'})$$

$$\overset{\#6}{=} Y_{K_1/K_t'} Y_{K_t'} Y_{K_2} \cdots \ell(Y_{J/K_t'})$$

$$= \ldots \overset{\#8}{=} Y_{K_1} Y_{K_2} \cdots Y_{K_t} Y_J Y_{P_1/Q_1} \cdots Y_{P_t/Q_t} Y_J.$$

Here $\#1$ is based on (33) and the fact that $\ell(w_{\lambda}) = \sum_i |R_i|$, $\#2$, $\#4$ are based on (18), $\#3$, $\#7$ on Lemma 33 (2), $\#5$ on the repeated use of an argument similar to $\#3$ followed by $\#4$, $\#6$ on the repeated use of an argument similar to $\#6$ followed by $\#7$, and $\#8$ on Lemma 36.

We now state the main result of this section.
Theorem 38. The KL-Schubert classes for the Grassmannian coincide with the hyperbolic cohomology classes of the corresponding Zelevinsky resolutions.

Proof. The result is now immediate by comparing the left- and right-hand sides of Definition 21 and (25), respectively.

□

Remark 39. Theorem 38 implies that all the Zelevinsky resolutions of a Schubert variety in the Grassmannian have the same class in hyperbolic cohomology (i.e., the corresponding KL-Schubert class). This agrees with a result of Totaro’s [T00], which says that the algebraic theories in a larger class (defined by Krichever [BB10]), which includes hyperbolic cohomology, are invariant under small resolutions.

References

[AO16] M. Aganagic, A. Okounkov. Elliptic stable envelopes, to appear in JAMS, arXiv:1604.00423, 2016.
[AM16] P. Aluffi and L. Mihalcea. Chern-Schwartz-MacPherson classes for Schubert cells in flag manifolds, Compositio Math., 152 (12):26032625, 2016.
[AMSS17] P. Aluffi, L. Mihalcea, J. Schürmann, and C. Su. Shadows of characteristic cycles, Verma modules, and positivity of Chern-Schwartz-MacPherson classes of Schubert cells, arXiv:1709.08697, 2017.
[AMSS19] P. Aluffi, L. Mihalcea, J. Schürmann, and C. Su. Motivic Chern classes of Schubert cells with applications to Casselman’s problem, arXiv:1902.10101, 2019.
[BL00] S. Billey and V. Lakshmibai. Singular Loci of Schubert Varieties, Progress in Mathematics, 182, Birkhäuser Boston Inc., Boston, MA, 2000.
[BSY10] J. Brasselet, J. Schürmann, and S. Yokura. Hirzebruch classes and motivic Chern classes for singular spaces, J. Topol. Anal., 2 (1):155, 2010.
[BB10] V. Buchstaber and E. Bunkova. Elliptic formal group laws, integral Hirzebruch genera and Krichever genera, arXiv:1010.0944, 2010.
[BN11] D. Bump and M. Nakasuji. Casselman’s basis of Iwahori vectors and the Bruhat order, Canad. J. Math., 63 (6): 12381253, 2011.
[BN19] D. Bump and M. Nakasuji. Casselman’s basis of Iwahori vectors and Kazhdan-Lusztig polynomials, Canad. J. Math., 71 (6):13511366, 2019.
[CZZ12] B. Calmès, K. Zainoulline, and C. Zhong. A coproduct structure on the formal affine Demazure algebra, Math. Zeitschrift, 282 (3):11911218, 2016.
[CZZ13] B. Calmès, K. Zainoulline, and C. Zhong. Push-pull operators on the formal affine Demazure algebra and its dual, Manuscripta Math., 160 (1-2):950, 2019.
[CZZ14] B. Calmès, K. Zainoulline, and C. Zhong. Equivariant oriented cohomology of flag varieties, Doc. Math., Extra Volume: Alexander S. Merkurjev’s Sixtieth Birthday (2015), 113144.
[CG09] N. Chriss and V. Ginzburg. Representation theory and complex geometry, Springer Science & Business Media, 2009.
[D79] V. Deodhar. On some geometric aspects of Bruhat orderings. II. The parabolic analogue of Kazhdan-Lusztig polynomials, J. Algebra, 111:483506, 1979.
[FRW18] L. Fehér, R. Rimányi, and A. Weber. Motivic Chern classes and $K$-theoretic stable envelopes, Proc. London Math. Soc., to appear. arXiv:1802.01503.
[HMSZ] A. Hoffnung, J. Malagon-Lopez, A. Savage, and K. Zainoulline. Formal Hecke algebras and algebraic oriented cohomology theories, Selecta Math., 20 (4):12131245, 2014.
[KT02] M. Kashiwara and T. Tanisaki. Parabolic Kazhdan-Lusztig polynomials and Schubert varieties, J. Algebra, 249 (2):306325,2002.
[KL79] D. Kazhdan and G. Lusztig. Representations of Coxeter groups and Hecke algebras, Invent. Math., 53 (2):165184, 1979.
[KL80] D. Kazhdan and G. Lusztig. Schubert varieties and Poincaré duality, Proc. Sympos. Pure Math., 36:185203, 1980.
[KL96] A. Kirillov and A. Lascoux. Factorization of Kazhdan-Lusztig elements for Grassmanians, Combinatorial Methods in Representation Theory, Advanced Studies in Pure Mathematics, 28:143514, 2000.
[KK86] B. Kostant and S. Kumar. The nil Hecke ring and cohomology of $G/P$ for a Kac-Moody group $G$, Adv. Math., 62 (3):187237,1986.
[KK90] B. Kostant and S. Kumar. $T$-equivariant $K$-theory of generalized flag varieties, J. Differential Geometry, 32 (2):549603, 1990.
[K96] S. Kumar. The nil Hecke ring and singularity of Schubert varieties, Invent. Math., 123 (3):471506, 1996.
S. Kumar, R. Rimányi, and A. Weber. Elliptic classes of Schubert varieties, *Math. Ann.*, to appear. [arXiv:1910.02313.](https://arxiv.org/abs/1910.02313)

C. Lenart and K. Zainoulline. A Schubert basis in equivariant elliptic cohomology, *New York J. Math.*, 23:711737, 2017.

C. Lenart, K. Zainoulline, and C. Zhong. Parabolic Kazhdan-Lusztig basis, Schubert classes and equivariant oriented cohomology. *J. Inst. Math. Jussieu*, to appear. DOI 10.1017/s1474748018000592.

M. Levine and F. Morel. *Algebraic cobordism*, Springer Monographs in Math., Springer, Berlin, 2007, xii+244 pp.

R. MacPherson. Chern classes for singular algebraic varieties, *Ann. Math. (2)*, 100:423432, 1974.

D. Maulik and A. Okounkov. Quantum groups and quantum cohomology, *Astérisque*, 408, 2019.

M. Levine and F. Morel. *Algebraic cobordism*, Springer Monographs in Math., Springer, Berlin, 2007, xii+244 pp.

H. Naruse. Schubert calculus and hook formula, Slides at 73rd Sém. Lothar. Combin., Strobl, Austria, 2014.

A. Neshitov, V. Petrov, N. Semenov, and K. Zainoulline. Motivic decompositions of twisted flag varieties and representations of Hecke-type algebras, *Adv. Math.*, 340:791871, 2018.

A. Okounkov. Lectures on K-theoretic computations in enumerative geometry, *Geometry of Moduli Spaces and Representation Theory*, volume 24 of IAS/Park City Mathematics Series, 251380, 2017.

A. Okounkov. Inductive construction of stable envelopes and applications, 1, [arXiv:2007.09094](https://arxiv.org/abs/2007.09094), 2020.

R. Rimányi and A. Weber. Elliptic classes of Schubert varieties via Bott-Samelson resolution, *J. Topology*, 13 (3):1139–1182, 2020.

C. Su. Restriction formula for stable basis of the Springer resolution. *Selecta Math. (N.S.)*, 23 (1):497518, 2017.

C. Su. Motivic Chern classes and Iwahori invariants of principal series, *Proceedings of International Congress of Chinese Mathematicians*, to appear, 2019.

T. Tanisaki. Hodge modules, equivariant K-theory and Hecke algebras, *Publications of the Research Institute for Mathematical Sciences*, 23:841870,1987.

B. Totaro. Chern numbers for singular varieties and elliptic homology, *Ann. of Math.*, 151 (2):757791, 2000.

A. V. Zelevinski. Small resolutions of singularities of Schubert varieties, *Functional Anal. Appl.*, 17:142144, 1983.
State University of New York at Albany, 1400 Washington Avenue, Albany, NY 12222
E-mail address: clenart@albany.edu

University of Toronto, 40 St. George St., Toronto, ON M5S 2E4, Canada
E-mail address: csu@math.toronto.edu

Department of Mathematics and Statistics, University of Ottawa, 150 Louis-Pasteur, Ottawa, ON, K1N 6N5, Canada
E-mail address: kirill@uottawa.ca

State University of New York at Albany, 1400 Washington Avenue, Albany, NY 12222
E-mail address: czhong@albany.edu