The Abraham-Lorentz force and electrodynamics at the classical electron radius

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The Abraham-Lorentz force is a finite remnant of the UV singular structure of the self interaction of a point charge with its own field. The satisfactory description of such interaction needs a relativistic regulator. This turns out to be a problematic point because the energy of regulated relativistic cutoff theories is unbounded from below. However one can construct point splitting regulators which keep the Abraham-Lorentz force stable. The classical language can be reconciled with QED by pointing out that the effective quantum theory for the electric charge supports a saddle point producing the classical radiation reaction forces.

I. INTRODUCTION

The radiation reaction problem, the intrinsic instability of the interaction of a point charge with its own field, has been clearly stated since more than a century, however the discovery of quantum mechanics somehow deflected the interest of the majority of the physics community. Nevertheless number of methods have been developed to tackle the problem and several solution have been proposed in the meantime. The present work is based on the point of view that a field theory of point particles displays singular short distance dynamics and needs a cutoff, a minimal distance down to which the theory is applicable, both in the classical and the quantum domain. The UV divergences of quantum field theories are a well known problem but one should not loose sight of the singular classical Coulomb field of point charges, one of the tumbling stones of the radiation reaction problem. The main point of this work is to show that electrodynamics supports stable self interaction for point charges when supplied an appropriate regulator.

Let us approach the radiation reaction problem, the last open chapter of classical elec-
trodynamics, in three steps. First concerns the origin of the phenomenon. It is known that an accelerated charge looses part of its energy by radiation hence there is a radiation reaction force. However the Lagrangian do not contain higher than first order time derivatives hence can not give account the energy, lost to the radiation. The solution of the problem in the jargon of field theory of today is that the radiation reaction is an effective force and it appears only in the closed equation of motion for the charge. This equation can be obtained by solving the Maxwell equation for a given particle world line and inserting the solution back into the mechanical equation of motion. This is the construction of the effective theory for the charge and the effective dynamics, generated by the feedback of the electromagnetic field contains the radiation reaction force. The calculation of the force of a point charge is hindered by the divergence of the near field and the Abraham-Lorentz force was found by checking the energy-momentum balance equation for the radiating charge [1, 2].

The second step is about the attempts of reconciliation of an unacceptable feature of the Abraham-Lorentz force, namely it is an \( O(\dddot{x}) \) dissipative force which generates self-accelerating, runaway trajectories. Such a state of affairs questions the applicability of field theory to point particles and a wide range of remedies have been proposed. (i) The problem is related to the third auxiliary condition, needed to integrate the equation of motion. By imposing an additional final condition one trades the instability into acausality [3], an idea which can be realized by assuming a complete absorption of the electromagnetic radiation at the final state [4]. (ii) One can expand the solution in the retardation and the Abraham-Lorentz force can be approximated by making an iterative step, using the second order equations. The result is an \( O(\dddot{x}) \) equation which remains stable as long as the resolution is worse than \( r_0 \) [5, 6]. The same equation can be recovered by restricting the trajectories of the Abraham-Lorentz force to the stable manifold [7]. (iii) A more flexible family of modifications results from giving up the local nature of the reaction force in time. One way to achieve this is to retain a memory term in the reaction force and the resulting integro-differential equation offers important improvements [8]. Another possibility is to assume and extended charge distribution [9] of non-electromagnetic origin [10, 11] or the presence of a polarizable medium [12]. Yet another approach is to assume a suitable chosen form factor [13]. A particular non-locality has been evoked by assuming a discrete structure in time [14]. (iv) Finally, one can step back and seek a change of the effective equation of motion, based on physical intuition [15], on magnetic moment charge [16], or on quantum
effects [17]. So far no generally accepted solution has been found to cure the instability. The point of view, developed below, is based on a non-local dynamics hence is closest to the group (iii) with the difference of being minimalistic, namely we stay within QED in an imaginary world without other particles and interactions. The non-locality arises during the unavoidable, conventional regularization of the theory and is treated in the usual manner, known from the renormalization of quantum field theories.

The third step is to embed the radiation reaction problem into QED by considering it as saddle point physics. The Abraham-Lorentz force can be identified in the $\hbar \to 0$ limit of the shift of the position of a wave packet [18] and the Landau-Lifshitz form of the equation [6] follows from considering the shift of the momentum of the particle [19]. The more direct derivation which goes parallel with the classical arguments is based on the retarded Green’s function. It has already been considered in the presence of a cutoff [20] but the non-locality prevented a systematic, analytical exploitation of the equation of motion.

We are accustomed to UV finite classical theories because the iterative solution of the classical equations of motion can be presented in the form of series, involving $\mathcal{O}(\hbar^0)$, tree-level graphs. But this holds for elementary, closed theories only. If there are unobserved degrees of freedom, an environment, and we are looking for a closed system of equations for the observed quantities, then the elimination of the unobserved coordinates generates loop diagrams. The iterative solution of the classical equation of motion of a charge, moving on the background of a fixed electromagnetic field, can be represented by an infinite series of tree-level graphs where the electromagnetic field is attached to the world line as an external leg. However the elimination of the electromagnetic field by the help of the Maxwell equations, using the world line as a source, couples the pairs of the external legs and forms loops. The loop integral is $\mathcal{O}(\hbar^0)$, a power of $\hbar$ is lost compared to the usual counting in QED because the line, representing the charge is classical, $\mathcal{O}(\hbar^0)$. The loop integrals of classical effective theories have already been spotted as the on-shell contributions to the loop integrals of the full quantum theory [21]. The self-interaction of a point charge with its own field is UV singular due to the $\mathcal{O}(r^{-1})$ near field, requiring the usual regularization and renormalization procedure of quantum field theories, applied already in the classical domain.

The traditional strategy to derive the effective equation of motion without touching the issue of divergences is to exploit the energy-momentum conservation for the energy-
momentum flux, calculated for a tube around the point particle world line [3]. However energy-momentum is modified by the regulator, making the conserved energy unbounded from below in relativistic theories, thereby removing a sufficient condition of stability [22]. The regulator, used in this work is the point splitting of the interaction, consisting of the use of a smeared electromagnetic field in the interaction term. The smearing is chosen in such a manner that superluminal charges do not interact. The resulting effective equation of motion provides stable dynamics since a runaway charge must acquire velocities beyond the speed of light. The appearance of the cutoff in the classical effective equation of motion makes certain concepts of the renormalization group method important for classical field theories.

First we consider the linearized effective equation of motion [23] in section II A which describes an stable dynamics for low cutoff. If the resolution in the space-time is better than the classical electron radius then the usual $\mathcal{O}(\vec{x})$ Abraham-Lorentz force appears and induces self accelerating, unstable particle trajectories. Next we turn to the full, non-linear, integro-differential equation of motion in section II B. Its numerical solution displays two distinct scale regimes, separated by the classical electron radius. The usual linear Abraham-Lorentz force is displayed in the IR domain and the non-linear effects prevent the runaway and keep the motion stable in the UV regime. The issue of embedding this scenario in QED is taken up briefly in section III where it is argued that the radiation reaction problem is a tree-level saddle point effect governed by an $\mathcal{O}(\hbar^0)$ effective equation of motion and hiding deeply within the quantum regime of the theory.

II. CLASSICAL EFFECTIVE DYNAMICS OF A POINT CHARGE

The effective dynamics of a point charge is found by solving Maxwell’s equation for the electromagnetic field, $A_\mu(x)$, induced by a point charge which follows a given world line, $x^\mu(s)$, and substituting the result into the equation of motion of the charge. The electromagnetic field,

$$A_\mu(x) = e \int ds D_{M\mu}(x - x(s)) \dddot{x}^\mu(s),$$

(1)

is given by the retarded Green’s function, $D_{M\mu}(x) = -4\pi(g_{\mu\nu} - \partial_\mu \partial_\nu / \Box)D_0(x)$, where $D_0(x) = -\Theta(x^0)\delta(x^2)/2\pi$ is the free, massless Green’s function, and $\dddot{x}(s) = d\dot{x}(s)/ds$. The
mechanical equation of motion,

\[ m_B \ddot{x} = \frac{e}{c} [\partial^{\mu} A_{\nu}(x) - \partial_{\nu} A^{\mu}(x)] \dot{x}_\nu - k^\mu, \tag{2} \]

contains the bare mass, \( m_B \), and an external source, \( k^\mu(s) \), to diagnose the dynamics. We avoid the UV singularities of the near field self-interaction by introducing a cutoff, realized by the smearing of the Dirac-delta, \( \delta(x^2) \to \delta_B(x^2) \), in the retarded Green’s function.

The regulated Dirac-delta should satisfy three conditions: (i) To preserve the flux of the radiated field we require the normalization condition

\[ \int_{-\infty}^{\infty} dz \delta_B(z) = 1. \tag{3} \]

(ii) The Lorentz symmetry prevents us to smear the factor \( \Theta(x^0) \) of the retarded Green’s function, we impose \( \delta_B(0) = 0 \) to separate the singular points in the product of two generalized functions in the Green’s function. (iii) It will be argued below that the stability of the dynamics requires the suppression of the interactions for charges moving faster than the light, expressed by the condition \( \delta_B(z) = 0 \) for \( z < 0 \). The simplest possibility is to displace its retarded Green’s function slightly off the light-cone,

\[ \delta_B(x^2) = \delta(x^2 - \ell^2), \tag{4} \]

It leads to oscillations in the momentum space which can be avoided by smearing the singularity,

\[ \delta_B(x^2) = \frac{\Theta(x^2)}{12\ell^4} x^2 e^{-\sqrt{\frac{x^2}{\ell^2}}}. \tag{5} \]

The effective equation of motion with the regulated self interaction is

\[ \ddot{x} = 4r_{0B} \int_{-\infty}^{s} ds' \delta_B((x - x')^2)((x - x')(\dot{x} x') - [\dot{x}(x - x')]\dot{x'}], \tag{6} \]

where \( x = x(s) \) and \( x' = x(s') \), and \( r_{0B} = e^2/m_B c^2 \) stands for the bare classical electron radius and \( \delta'(z) = d\delta(z)/dz \). The characteristic scale of the radiation reaction problem is provided by the only dimensional constant of this equation, the classical electron radius, playing the role of the coupling constant.

A. Linearized equation of motion

The linearized equation of motion,

\[ \ddot{x} = 4r_{0B} \int_{-\infty}^{0} du \delta'(u^2)(x - x' + u\dot{x'}), \tag{7} \]
contains a regular and a singular force on the right hand side,

\[ F_r = 2r_0B \int_{-\infty}^{0} \frac{du}{u^2} \delta(u^2) \left[ x - x' + u\dot{x}' - u^2 \left( \ddot{x}' - \frac{1}{2} x' \right) + \frac{2u^3}{3} \right], \]

\[ F_s = -r_0B \int_{0}^{\infty} du \delta(u^2) \left( \ddot{x} + \frac{4u}{3} \right), \tag{8} \]

where the expression in the square bracket of the integrand of \( F_r \) contains the regular, \( \mathcal{O}(u^4) \) terms making up a uniformly convergent integral. The rest, \( F_s \), can be written as

\[ F_s = -\frac{r_0B}{2} \ddot{x} \int_{0}^{\infty} \frac{dz}{\sqrt{z}} \delta_B(z) + \frac{2}{3} r_0B \ddot{x}, \tag{9} \]

giving rise to a mass renormalization, \( m = m_B + \delta m \), with

\[ \delta m = \frac{e^2}{2e^2} \int_{0}^{\infty} \frac{dz}{\sqrt{z}} \delta_B(z). \tag{10} \]

and to the Abraham-Lorentz force.

It is illuminating to check the order of magnitude of the two forces as the cutoff is removed. We assume that \( \Lambda = 1/\ell \) is large enough to approximate the square bracket in the integrand of the uniformly convergent part by \( u^2/\ell^3 \) where \( \ell \) is the length scale of the world line and find

\[ F_r \approx 2r_0B \ell \int d\tilde{u} \tilde{\delta}_B(\tilde{u}^2) \tilde{u}^2, \tag{11} \]

in terms of the dimensionless variable \( \tilde{u} = u/\ell \) and the regular function \( \tilde{\delta}_B(\tilde{u}^2) = \ell^2 \delta_B(u^2) \). The regular force is suppressed during the renormalization owing to the smallness of the important integration region, \( u = \mathcal{O}(\ell) \) of an \( \delta(u^2)u^2 = \mathcal{O}(\ell^0) \) integrand. The important integration region is \( u = \mathcal{O}(\ell) \) in the non-uniformly convergent part, too. The linearly divergent mass renormalization comes from an \( \mathcal{O}(\ell^{-2}) \) integrand and the reduced \( \mathcal{O}(\ell^{-1}) \) divergence of the integrand in the Abraham-Lorentz force yields a cutoff independent result. Such a cutoff-independent cutoff-scale contribution owes its existence to the non-uniform convergence and is the hallmark of anomalies. In fact, the coefficient of \( \ddot{x} \) is an “accidentally finite” loop-integral as in the case of the chiral anomaly for massless fermions.

The retarded world line Green’s function, \( F^r \), is defined by

\[ \dot{x}^\mu(s) = \int_{-\infty}^{\infty} ds' F^r(s - s') k^\mu(s'), \tag{12} \]

and its Fourier transform,

\[ F^r_\omega = \int_{-\infty}^{\infty} ds e^{i\omega s} F^r(s), \tag{13} \]
can be written as \( F^r_\omega = 1/[(\omega + i\epsilon)^2 \chi^r_\omega] \), in terms of the susceptibility

\[
\chi^r_\omega = 1 + r_0 \left[ \frac{2}{3} i\omega \right. \\
- \frac{2}{\omega^2} \int_{-\infty}^{0} du \delta_B(u^2) \left( 1 + i\omega u - u^2 \omega^2 \right) e^{-i\omega \ell} - 1 + \frac{1}{2} u^2 \omega^2 - i \frac{2}{3} u^3 \omega^3 \bigg] \]

(14)

and the long memory tail when the regulator (5) is used requires \( \text{Im} \omega > 0 \). The harmonic effective dynamics is stable and causal if the susceptibility is analytic and pole-free on the upper part of the complex \( \omega \) plane.

The rational function, multiplying the Dirac-delta in the integrand is \( \frac{3}{8} \omega^4 u^2 (1 + \mathcal{O}(\omega u)) \)

hence

\[
\chi^r_\omega = 1 + r_0 \omega \left[ \frac{2}{3} i + \mathcal{O}(\omega \ell) \right].
\]

(15)

As the cutoff is removed with a fixed frequency the Abraham-Lorentz force is left behind and the infamous self-acceleration is recovered. The unstable dynamics generates high frequency components which make the particular details of the regulator influence the motion. The numerical monitoring of the \( \ell \)-dependence shows that both regulators lead to a susceptibility which develops a pole with positive imaginary part, destroying the stability when the cutoff is comparable or larger than the classical electron radius, as predicted by the cutoff-independent acausal pole of the susceptibility (15).

One would think that the appearance of the instability at \( \ell \sim r_0 \) implies that the source of the instability is related to the dynamics around the scale \( r_0 \). But the dynamics is shaped both by the \( \mathcal{O}(\omega^2) \) kinetic energy and the self interaction. The \( \mathcal{O}(\omega) \) friction term is forbidden by Lorentz symmetry, the impossibility of measuring absolute velocity, thus the radiation energy loss is represented by the \( \mathcal{O}(\omega^3) \) dissipative force. This latter which is weak for slow motions turns out however to be dominant at high frequencies, the crossover being around \( \omega \sim 1/\ell \) and makes the linearized dynamics unstable in the UV regime.

B. Full effective dynamics

The regulator was introduced in section II in such a manner that the interaction becomes suppressed if the velocity of the particle exceed the speed of light, making the instability, arising from the self interaction, unable to drive the particle to arbitrarily high velocity. Therefore it is natural to inquire whether the non-linear terms of the effective equation of
motion (6) can stabilize the dynamics. The traditional derivation of the effective equation of motion, based on the energy-momentum conservation, leads to an equation where the only non-linearity arises from the projection of the reaction force onto the linear subspace, orthogonal to the four-velocity [24]. The cutoff-dependence of the instability suggests that if the non-linearities stabilize the dynamics they should come from another source than this cutoff-independent projection operator.

The shifted Dirac-delta, (4), produces the equation of motion with a finite delay,

\[
\ddot{x} = r_0 \frac{m}{m_B} \frac{1}{\dot{x}'(x' - x)} \left[ \frac{\ddot{x}'(x' - x) + 1}{\dot{x}'(x - x')} \left\{ (x - x')(\dot{x}'\dot{x}' - [\dot{x}(x - x')\dot{x}']) + (x - x'(\dot{x}'\dot{x}' + [\dot{x}(x' - x)]\dot{x}')] \right\} \right],
\]

where the retarded source point, \( x' \), is found by the condition \( \ell^2 = (x - x')^2 \). The equation of motion, found by the help of the smeared Dirac-delta, (5), has infinitely long memory and contains the velocities only in the right hand side,

\[
\ddot{x} = r_0 \frac{m}{3\ell^4 m_B} \int_{-\infty}^{0} du \left( 1 - \sqrt{\frac{(x - x')^2}{2\ell}} \right) e^{-\sqrt{\frac{(x - x')^2}{\ell}}} \times \{ (x - x')(\dot{x}'\dot{x}' + [\dot{x}(x' - x)]\dot{x}')] \}.
\]

We impose the initial condition that the charge is at rest, \( x(t) = (t, 0) \), for \( t < t_0 \) and the charge follows a prescribed trajectory, \( \mathbf{x}_i(t) \) for \( t_0 < t < t_0 + t_i \). A certain external source, \( k_i(s) \), is supposed to generate this motion which is turned off after this initial phase and the invariant length, \( s \), of the world line is measured from the time \( t_0 + t_i \). The numerical solution of the equation of motion becomes straightforward with such initial conditions: One introduces a small but finite \( \Delta t \) step size and writes eqs. (16) or (17) as differential equation and finds the retarded time, \( u \), or calculates the integral of the memory term numerically, respectively at each step, \( t \to t + \Delta t \).

The equation of motion has two free, adjustable parameters, the cutoff, \( \ell \), and the bare mass, \( m_B \). Hence we need a renormalization condition to fix the theory, it is chosen to be

\[
\chi_\omega^r = 1 + \frac{2}{3} i r_0 \omega,
\]

cf. eq. (15).

The numerical solution of the equation of motion indicates stable dynamics for sufficiently weak force, ie. small \( |m/m_B| \). The acceleration changes in a monotonous, exponential
FIG. 1: A component of the spatial acceleration, \( a_{r_0} = |\ddot{x}|_{r_0} \), plotted against the proper time, \( s/r_0 \) for the smeared Dirac-delta regularization, \( r_0/\ell = 3 \), (a): \( m/m_B = 1.95 \) (dashed line), \( m/m_B = 1.98 \) (solid line) and \( m/m_B = 2 \) (dotted line) and (b): \( m/m_B = -3.8 \) (dashed line), \( m/m_B = -3.91 \) (solid line) and \( m/m_B = -4.1 \) (dotted line).

manner after some transient period, depending on the initial conditions, if \( m_B > 0 \) as shown in Fig. 1 (a) and the renormalization condition, (18), can be satisfied by monitoring the relaxation for large \( s \). When \( m_B < 0 \) then the acceleration is oscillatory with exponentially exploding or decreasing envelope, cf. Fig. 1 (b). The relaxation of the envelope is used to find the physical theory, obeying eq. (18) in this case. The precise value of \( m/m_B \) at the stability edge is found to be slightly dependent on the initial, prescribed trajectory. This might come from the finite \( \Delta s \) resolution of the finite difference equation, solved numerically because the unstable, runaway trajectories support no fixed, finite \( \Delta s \). The existence of stable regions suggests that despite the unboundedness of the energy in regulated electrodynamics there are energy barriers which stabilize the charge.

The phase structure of the effective theory is shown in Fig. 2. The stability region narrows as the cutoff is removed, \( \ell \to 0 \), since the regulator subjects the trajectory to some deformation within the length scale \( \Delta s \sim \ell \), inducing a larger value of the loop integral in the effective equation of motion, (6), and requiring smaller coefficient, \( r_{0B} \). The non-monotonous behavior of the renormalized trajectory indicates the presence of an IR and an UV scaling regime, separated by the intrinsic length scale, \( r_0 \). There are two solutions of the renormalization condition, one with \( m_B > 0 \) and another with \( m_B < 0 \). The latter is qualitatively consistent with the linearized equation of motion and displays “Zitterbewegung”, fast oscillations. The non-linear terms of the equation of motion play an important
FIG. 2: The phase structure of the effective theory with (a): shifted and (b): smeared regulated Dirac-delta on the plane \((r_0/\ell, m/m_B)\). The dynamics is stable within the shaded region and the solid lines indicates the solution of the renormalization condition, (18). The dotted line belongs to the linearized theory, fixed by the counterterm (10).

role at any value of the cutoff since the linearized theory is unstable despite having bare parameters within the stability region of the full equation. There is no numerical evidence of a Landau-pole, an obstruction of the limit \(\ell \to 0\).

The quality of satisfying the renormalization conditions is shown in Fig. 3 (a). While the oscillatory motion of \(\ddot{x}\) generates strongly localized minima in \(|\ddot{x}(s)|\) in a periodic manner the envelope follows the prediction of the Abraham-Lorentz force with a remarkable precision despite the non-linearity of the equation of motion. The zoom into Fig. 3 (a), shown in Fig. 3 (b), supports the expectation that the length of an oscillation scales with the cutoff. Similar behavior can be found for \(m_B > 0\) where the monotonous trajectory shows a single exponential relaxation.

III. QED

The physics around the classical electron radius is deeply within the quantum domain and we turn to the question of placing the classical considerations, presented above, into the context of QED.
FIG. 3: (a): The spatial acceleration, $|\ddot{x}|_0$, plotted against the proper time, $s/r_0$ for the smeared Dirac-delta regularization at $r_0/\ell = 3$ (fat line) and $r_0/\ell = 15$ (thin line) with $m_B < 0$, plotted together with the prediction of the renormalization condition (dotted line). (b): The zoom is into a more restricted scale region.

A. Scale hierarchy and subclassical physics

Weakly coupled quantum field theories have an intrinsic hierarchy of scales, assured by the smallness of the dimensionless strength of interaction. In the case of QED with electrons four of the scales $r_n = \alpha^n r_0$, $\alpha = e^2/\hbar c$, have already been identified, the Bohr radius, $r_{-2} = a_0 = \hbar^2/mc^2$, the Compton wavelength, $r_{-1} = \lambda_C = \alpha a_0 = \hbar/mc$, the classical electron radius, $r_0 = \alpha^2 a_0 = e^2/mc^2$ and finally the Lamb shift scale, $r_1 = \alpha^3 a_0 = e^4/\hbar mc^2$.

The scale dependence of the dynamics is driven by different elementary processes at different scale regimes. This can easily be seen at the first three scales by identifying the fundamental constant which is missing from the expression of the scales. In fact, the driving force comes from the non-relativistic quantum mechanics at the Bohr radius (absence of $c$), from pair creation at the Compton wavelength, independently of the specific nature of the underlying interactions (absence of $e$) and from classical electrodynamics at the classical electron radius (absence of $\hbar$). The physics of the Lamb shift at $r_1$ is driven by involved vacuum polarization effects. The scales with $n \geq 2$ and $n \leq -3$ are covered by the Electro-Weak theory and the collective phenomena in many-body systems or chemistry, respectively and it is not easy to identify them.

The classical electron radius is well below the quantum-classical transition scale and the dominance of the scaling laws by classical physics is to be taken with a grain of salt. The best is to look at this issue within the context of the saddle point expansion, based on
dimensionless small parameter, $\lambda$, appearing in QED when the rescaling, $\hbar \to \lambda \hbar$ which shifts the quantum-classical transition scales, is performed. This induces the change $\alpha \to \alpha / \lambda$, showing that the saddle point and the usual weak coupling expansion represent two opposite extrema. In fact, we have $r_n \to \lambda^{-2n} r_n$, the gradual turning on of the quantum fluctuations by moving $\lambda$ from 0 to 1 reshuffles the scale hierarchy: The imaginary world with weak quantum fluctuations and large fine structure constant, $a_0 < \lambda_C < r_0 < r_n$, $n > 0$ and our world, $r_n < r_0 < \lambda_C < a_0$, are separated by a strongly coupled regime without scale separation, $r_n \sim r_0 \sim \lambda_C \sim a_0$ at $\lambda \sim 1/137$. Since the separation of scales is an important ingredient of asymptotic expansions one expects complications in extrapolating from the weak to the physical quantum fluctuations. However it is fair to say that the strong field of a point charge creates an $O(\hbar^0)$ saddle point contribution around $r_0$, embedded deeply within the quantum domain.

But the classical limit of a quantum system is more than the recovery of some classical equations of motion. It is instructive in this respect to consider an extension of this problem, the expectation value of local operators in quantum field theory. These expectation values define space-time dependent functions which satisfy integro-differential equation of motion and thus can superficially be viewed as classical fields of a classical effective dynamics as a reminiscence of Ehrenfest’s theorem. However the local field variable is classical only if its reduced density matrix is strongly peaked on the diagonal matrix elements. The off-diagonal values characterize the importance of the linear superposition in the averages and must be negligible in the classical domain. Actually it is better to call the expectation values of local operators subclassical fields [25], the difference between them and the classical fields being the worse space-time resolution, i.e., lower UV cutoff, and the strong decoherence in the latter case.

B. Effective saddle point dynamics

The cutoff theory supports an open dynamics owing to the unobserved UV degrees of freedom hence the regularization of quantum field theories is to be performed in a framework designed for open systems [26], namely within the Closed Time Path (CTP) formalism [27]. This is a CQCO scheme, it handles classical, quantum, closed and open systems on equal footing and treats initial rather than boundary value problems. The redoubling of the degrees
of freedom, the distinguishing feature of this scheme, allows the extension of the variational principle of classical mechanics for dissipative forces in open systems [28] and the quantum effects arise as an $O(\sqrt{\hbar})$ separation of the two coordinates, describing the same degree of freedom. Furthermore we obviously have to rely on initial rather than boundary conditions in problems, related to the radiation.

There is yet another reason to use the CTP scheme: The radiation reaction provides an effective force which appears only by eliminating the electromagnetic field, by considering it as the unobserved environment of the charge. To pick up the effects of the outgoing radiation, the friction forces, the electromagnetic field must be allowed to occupy any excited final state. This condition requires the CTP formalism.

The motion of a charged particle can be reconstructed from the expectation value of the charge density, to be extended in a relativistic treatment to the expectation value of the electric current. Hence we seek the effective theory for the electric current. The effective action is constructed from the generator functional for the connected Green’s functions of the electric current [25],

$$e^{i\hat{W}[\sigma]} = \text{Tr} \left[ U[a^+, \eta^+, \eta^-] |0\rangle \langle 0| U^+[a^-, \eta^-, \eta^-] \right]$$  \hspace{1cm} (19)

where the source $a_\mu$, coupled linearly to $A_\mu$ generates the Green’s functions and $\eta$, and $\bar{\eta}$ are coupled linearly to $\psi$ and $\bar{\psi}$, respectively to produce a coherent initial state, describing an electron. In the path integral representation of this functional,

$$e^{i\hat{W}[\sigma]} = \int D[\bar{\psi}] D[\psi] D[\hat{\bar{A}}] \exp \frac{i}{\hbar} \left[ S_M[\hat{\bar{A}}] + S_D[\bar{\psi}, \psi] + S_i[\bar{\psi}_B, \psi_B, \hat{\bar{A}}_B + \hat{\bar{a}}] + \int dx [\bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)] \right]$$  \hspace{1cm} (20)

the integration is over the CTP doublet fields, $\hat{\bar{A}} = (A^+, A^-)$, $\bar{\psi} = (\psi^+, \psi^-)$ and $\hat{\psi} = (\bar{\psi}^+, \bar{\psi}^-)$. One uses a similar notation for the sources, $\hat{\bar{a}} = (a^+, a^-)$, $\hat{\eta} = (\eta^+, \eta^-)$ and $\hat{\bar{\eta}} = (\bar{\eta}^+, \bar{\eta}^-)$, as well. The first two contribution to the action are the Maxwell action,

$$S_M[\hat{\bar{A}}] = \frac{1}{2c} \int dx dy \hat{\bar{A}}(x) \hat{D}_{\text{CI}}^{-1}(x,y) \hat{\bar{A}}(y)$$  \hspace{1cm} (21)

with a relativistic gauge fixing term,

$$\hat{D}_{\text{CI}}^{-1\mu\nu} = -4\pi(T^{\mu\nu} + \xi L^{\mu\nu})\hat{D}_0$$  \hspace{1cm} (22)
with \( L^{\mu\nu} = \partial^\mu \partial^\nu / \Box \) and \( T^{\mu\nu} = g^{\mu\nu} - L^{\mu\nu} \),

\[
\hat{D}_m(p) = \left( \begin{array}{cc}
D^n(p) + iD^i(p) & -D_f(p) + iD^i(p) \\
D_f(p) + iD^i(p) & -D^n(p) + iD^i(p)
\end{array} \right) \\
= \left( \begin{array}{cc}
\frac{1}{p^2 - m^2 + i\epsilon} & -2\pi i\delta(p^2 - m^2)\Theta(-p^0) \\
-2\pi i\delta(p^2 - m^2)\Theta(p^0) & -\frac{1}{p^2 - m^2 - i\epsilon}
\end{array} \right)
\]

(23)

standing for the propagator of a scalar particle of mass \( m \). The CTP propagator contains the Feynman propagator, \( D^{++} \), and the retarded and advanced Green’s functions, \( D^\pm = D^n \pm D^f \). The free Dirac action,

\[
S_D[\hat{\psi}, \hat{\bar{\psi}}] = \frac{1}{c} \int dx dy \hat{\bar{\psi}}(x) \hat{G}^{-1}(x, y) \hat{\psi}(y),
\]

(24)

contains the inverse of the electron propagator \( \hat{G}_m(p) = (\not{p} + m) \hat{D}_m(p) \). The interaction is described by the action

\[
S_i[\hat{\psi}, \hat{\bar{\psi}}, \hat{A}] = e \sum_\sigma \int dx \hat{\bar{\psi}}^\sigma(x) \hat{A}^{\sigma}(x) \hat{\psi}^\sigma(x).
\]

(25)

The CTP symmetry of the action,

\[
S[\phi^+, \phi^-] = -S^*[\phi^-, \phi^+],
\]

(26)

\( \phi^\pm \) denoting the CTP doublet pair of a generic field variable, follows from the definition of the generator functional.

The point splitting has already been used for gauge theories [29], and the particular regularization, implemented here is the replacement of the local fields with smeared ones, \( \hat{A}_B = \hat{\kappa} \hat{\sigma} \hat{A}, \hat{\psi}_B = \hat{\chi} \hat{A} \hat{\bar{\sigma}} \hat{\psi} \) and \( \hat{\psi}_B = \hat{\psi} \hat{\bar{\sigma}} \hat{\chi}^{-1}[\hat{A}] \) with \( \hat{\chi} = \gamma^0 \chi^1 \gamma^0 \), in the interaction where \( \hat{\sigma} = \text{Diag}(1, -1) \) denotes the simplectic metric tensor of the CTP scheme [30]. The smeared photon field contains the transverse component only, \( \hat{\kappa}^{\mu\nu} = T^{\mu\nu} \hat{\kappa}_T \), the longitudinal components being suppressed in the interactions. The action is kept invariant under gauge transformations, \( A \rightarrow A + \partial \alpha \) and \( \psi \rightarrow e^{-ie\alpha} \psi \), by applying the replacement, \( \partial_\mu \rightarrow \partial_\mu + ieLA_\mu \), within the smearing function, \( \hat{\chi} \). The physical, gauge invariant components of the electromagnetic field do not appear in the smearing function of the charged field and the potentially dangerous regulator vertices of the higher order derivative scheme [31] are avoided. It is advantageous to perform the change of integral variable, \( \hat{\psi}_B \rightarrow \hat{\psi}, \hat{\psi}_B \rightarrow \hat{\psi}, \hat{A}_B \rightarrow \hat{A}, \)
\[ \tilde{\eta} \chi^{-1} \to \tilde{\eta} \text{ and } \tilde{\chi}^{-1} \hat{\eta} \to \hat{\eta}, \]
in the generator functional, (20), which amounts to the replacement \( \hat{D}_{CI} \to \hat{D}_{CIB} = \hat{\kappa} \hat{D}_{CI} \hat{\kappa} \) and \( \hat{G} \to \hat{G}_B = \chi \hat{G}_B \tilde{\chi} \) of the propagators. The action, expressed in terms of the smeared, bare fields, displays local interaction and modified free dispersion relations [22].

The regularization of the retarded Green’s function, used in section II, can be extended to the whole CTP Green’s function. Owing to the positivity of the energy of the excitations the support of the spectral function, \( iD^+ \), is over positive negative energy, \( iD^-(q) \text{sign}(q^0)D^+(q) = \text{sign}(q^0)i\text{Im}D^+(q) \), resulting the Feynman propagator \( D^{++}(q^0, q) = D^r(|q^0|, q) \). For instance, the retarded Green’s function, defined by eq. (4),

\[ D^r_B(q) = -\frac{1}{|q|} \int_0^\infty dr \frac{r}{r^\ell} e^{ir(q^0 + i\epsilon)} \sin |q|r, \quad (27) \]

where \( r^\ell = \sqrt{\ell^2 + r^2} \), yields

\[ D^r_B(0, q) = -\frac{1}{q^2} \int_0^\infty du \frac{u}{\sqrt{q^2\ell^2 + u^2}} e^{-\frac{u}{\ell} \sqrt{q^2\ell^2 + u^2}} \sin u, \]

\[ D^r_B(q^0, 0) = -\frac{1}{q^0 \ell^2} \int_0^\infty du \frac{u^2}{\sqrt{q^0 \ell^2 + u^2}} e^{(i - \frac{i}{q^0}) \sqrt{q^0 \ell^2 + u^2}}. \quad (28) \]

Thus the free Green’s functions is \( \mathcal{O}(|p^2|^{-\frac{3}{2}}) \) in the UV regime and renders the Feynman graphs with internal photon lines finite. The CTP matrix, \( \hat{\kappa} \), is assumed to have the block structure of the propagator, (23), with \( \kappa^{++}(q) = \sqrt{D^{++}_B(q)}D^{++}(q) \) and \( \kappa^r(q) = \sqrt{D^r_B(q)}D^r(q) \). The choice \( \hat{\eta} = \sqrt{-\Lambda \hat{G}_\Lambda} \), with \( \Lambda = 1/\ell \) and the replacement \( \partial_{\mu} \to \partial_{\mu} + icLA_{\mu} \), preserves causality [22].

The perturbation series is generated by the formal expression

\[ e^{\hat{\pi}W[\hat{a}]} = e^{i\frac{\hbar c^2}{2\epsilon} \int dx dy \gamma^\delta(x) \hat{D}_{CI} \gamma^\delta(y) e^{\hat{\pi}W_0[\hat{a}]}, \quad (29) \]

where the free generator functional is given by

\[ e^{\hat{\pi}W_0[\hat{a}]} = \int D[\hat{\psi}] D[\hat{\bar{\psi}}] e^{\frac{i}{\pi c} \int dx \hat{D}[\hat{\psi}] e^{\hat{\pi}G^{-1} + \hat{\delta} \hat{\phi}} - \hat{\bar{\psi}} \hat{\delta} \hat{\phi} + \hat{\bar{\psi}} \hat{\delta} \hat{\phi} + \hat{\bar{\psi}} \hat{\delta} \hat{\phi}. \quad (30) \]

The Gaussian integration leads to

\[ W_0[\hat{a}] = -\hat{\bar{\psi}} \frac{1}{c(G^{-1} - \hat{\delta} \hat{\phi})} \hat{\bar{\psi}} - i\hbar \text{Tr ln}[\hat{G}^{-1} - \hat{\delta} \hat{\phi}], \quad (31) \]

the sum of a tree-level and a quantum fluctuation contribution. If one uses the powers of \( \hbar \) to trace the weight of the quantum fluctuations then the Compton wavelength of the electron
$mc/h$, the mass term in $\hat{G}$ must be considered as a fixed, $h$-independent number. The resulting $h$-independence of the first term is due to the charge conservation. The effective action for the electric current is given by the Legendre transformation,

$$\Gamma[\hat{\mathcal{J}}] = W[\hat{\mathcal{A}}] - \hat{\mathcal{A}} \hat{\mathcal{J}}, \quad \hat{\mathcal{J}} = \frac{\delta W[\hat{\mathcal{A}}]}{\delta \hat{\mathcal{A}}}$$

(32)

and the Euler-Lagrange equation,

$$\frac{\delta \Gamma[\hat{\mathcal{J}}]}{\delta \hat{\mathcal{J}}} = \hat{\mathcal{A}}$$

(33)

is satisfied by the subclassical fields.

A Gaussian integral can be reproduced by solving the saddle point equation for the variable,

$$\int_{-\infty}^{\infty} d\mathcal{A} e^{i \frac{d^2}{2} \mathcal{A}^2 + i J \mathcal{A}} = e^{-i \frac{J^2}{2} \int_{-\infty}^{\infty} d\mathcal{A} e^{i \frac{d^2}{2} \mathcal{A}^2}},$$

(34)

the linear equation of motion can be used as an exact operator equation. Thus there is tree-level saddle point contribution to the generator function of the connected Green’s functions,

$$W[\hat{\mathcal{A}}] = W_0[\hat{\mathcal{A}}] - e^2 \frac{c^2}{2} \int dx dy \hat{j}(x) \hat{D}_{\mathcal{A}}(x - y) \hat{j}(y) + \mathcal{O}(h),$$

(35)

where the quantum corrections correspond to the free Dirac see. The effective action for the current is therefore

$$\Gamma[\hat{\mathcal{J}}] = \Gamma_0[\hat{\mathcal{J}}] - e^2 \frac{c^2}{2} \int dx dy \hat{j}(x) \hat{D}_{\mathcal{A}}(x - y) \hat{j}(y) + \mathcal{O}(h),$$

(36)

where

$$\Gamma_0[\hat{\mathcal{J}}] = W_0[\hat{\mathcal{A}}] - \hat{\mathcal{A}} \hat{\mathcal{J}}$$

(37)

stands for the effective action in the free Dirac-see.

The free effective action is highly involved, displaying a non-local, non-polynomial structure without a small parameter to organize an expansion [32]. Rather than seeking an approximate solution we assume that it approaches the form,

$$\Gamma_0[\hat{\mathcal{J}}] = -m_B c \int ds (\sqrt{\dot{x}^2} - \sqrt{\dot{x}_-^2})$$

(38)

in the point-like limit,

$$\hat{j}^{\sigma \mu}(x) = \frac{\delta W_0[\hat{\mathcal{A}}]}{\delta \hat{\mathcal{A}}(x)} \to \int ds \delta(x - x^{\sigma}(s)) \dot{x}^{\sigma \mu}(s),$$

(39)
in the absence of pair creation, \( \dot{x}^0 > 0 \). The action (36) is the CTP extension of the action-at-a-distance theory [4, 33], including the retarded radiation field [23]. The tree-level effective action,

\[
\Gamma[\hat{x}] = -m_B c \int ds (\sqrt{\dot{x}^2} - \sqrt{\dot{x}'^2}) - \frac{e^2}{2c} \sum_{\sigma\sigma'} \int ds ds' [\dot{x}_\mu(s) \dot{x}'_\mu(s')] D_{\sigma\sigma'}^{\mu\nu} (x^\sigma(s) - x'^\sigma(s')),
\]

contains the one-loop electron self interaction. This latter is of \( \mathcal{O}(\hbar^0) \) because the electron line of the corresponding Feynman graph describes a coherent state and is \( \mathcal{O}(\hbar^0) \).

The form \( \Gamma[\hat{x}] = \Gamma_1[x^+] - \Gamma_1[x^-] + \Gamma_2[\hat{x}] \), of the effective action with real \( \Gamma_1 \) and \( \Gamma_2[x^+, x^-] = -\Gamma_2^*[x^-, x^+] \) is consistent with the symmetry (26). The unitarity of the time evolution in the full QED implies \( W[a, a] = 0 \) in eq. (19) and consequently \( \Gamma[x, x] = \Gamma_2[x, x] = 0 \). The expectation value \( \langle x \rangle \) is identical when calculated by the help of \( U \) or \( U^\dagger \) in using the generator functional and the solution of the equations of motion produces \( x^+ = x^- \). The equation of motion for \( x^+ \), if \( x^+ = x^- \),

\[
0 = \frac{\delta \Gamma_1[x]}{\delta x} + \frac{\delta \Gamma_2[x, x]}{\delta x} \bigg|_{x'=x},
\]

is identical of eq. (6) and shows the dissipative nature of the radiation reaction force [34].

**IV. CONCLUSIONS**

The scaling laws of an electron around the classical electron radius, \( r_0 = 2.8 fm \), deeply within the quantum regime, are governed by the classical equation of motion, the dominant force arises from the interaction of the electron with its own field and the standard regularization procedure must be employed. The effective dynamics of a point charge is derived in this work by the help of a point splitting regularization which smears the electromagnetic field over the invariant distance \( ds^2 = \ell^2 \). The linearized equation of motion describes a stable, causal dynamics for \( \ell \gg r_0 \) and cutoff-dependent instability arises if \( \ell \ll r_0 \). However the non-linear terms of the equation of motion owing their existence to the cutoff stabilize the dynamics. Two different renormalized trajectories are found and one of them fits qualitatively to the dynamics, described by the unstable, linearized equation. The removal of the cutoff seems to be numerically possible, there is no evidence of a Landau pole within the tree-level renormalization. The radiation reaction can be fit into QED by realizing it as a tree-level saddle point effect.
The historical name of the scale $r_0$ expresses expectations that the classical electrodynamics of a point charge is ill-defined at shorter distances [6]. One should at this point distinguish two different inquiries. Are we looking into the physics of an imaginary world with classical physics only, $\hbar = 0$, or into a physical phenomenon of our world where $\hbar \neq 0$?

The latter scenarios is followed here by adopting the point of view that classical physics is supposed to be derived from the quantum level and classical electrodynamics should join smoothly to QED at the quantum-classical crossover. Regarding the radiation reaction problem from this point of view one encounters two remarkable features of the Abraham-Lorentz force which are prone to lead to misunderstanding, namely its $\hbar$- and cutoff-independence.

While the radiation reaction can be identified in classical electrodynamics and is therefore a purely classical phenomenon however there are three considerations indicating that it is not a typical classical physics problem. First, the radiation reaction force originates from a scale region which is deeply quantum and the correspondence principle, a guiding rule of our intuition, is strongly violated. Second, the tree-level effective equation of motion applies to the expectation value of the world line only, leaving a necessary condition of the classical limit, the decoherence, an open issue. The decoherence, being an IR effect [34], is not generated at the scale $r_0$ and the charge maintains its coherent quantum state at this scale, in other words, the Abraham-Lorentz force is a subclassical effect [25]. The third point concerns the origin and the features of the radiation reaction which bear the fingerprint of quantum field theory, namely being generated by a loop-integral. This integral is divergent and needs a regulator, implying the techniques and concepts of the renormalization group, developed in quantum field theory.

The radiation reaction force of a point particle is obviously an UV cutoff-effect, the velocity of a massive particle is bounded by the speed of light hence the world line of a point particle can not cut through the light cones of its own radiation field. Among the several cutoff-dependent terms in the effective equation of motion the Abraham-Lorentz force is distinguished by being cutoff-independent. It is generated by the cutoff but its strength is independent of the cutoff scale. This is a well known phenomenon in quantum field theory, has the somehow unfortunate name of anomaly, and reflects the non-uniform convergence of the loop-integrals of the perturbative solutions when the cutoff is removed [34].

The puzzle of the radiation reaction force, the apparent instability of the Abraham-Lorentz force, can be resolved by bearing in mind that the effective classical dynamics
contains a one-loop integrals which needs regularization. The introduction of the cutoff
makes the parameters of the equation of motion non-physical and forces us to follow the
painsstaking limit $\ell \to 0$ and to construct the renormalized trajectory. The result is a scale-
dependent dynamics where it is too naive to expect a simple local differential equation be
valid globally, at all scales.

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