ParNes: A rapidly convergent algorithm for accurate recovery of sparse and approximately sparse signals

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Abstract In this article we propose an algorithm, NESTA-LASSO, for the LASSO problem (i.e., an underdetermined linear least-squares problem with a one-norm constraint on the solution) that exhibits linear convergence under the restricted isometry property (RIP) and some other reasonable assumptions. Inspired by the state-of-the-art sparse recovery method, NESTA, we rely on an accelerated proximal gradient method proposed by Nesterov in 1983 that takes $O(\sqrt{1/\varepsilon})$ iterations to come within $\varepsilon > 0$ of the optimal value. We introduce a modification to Nesterov’s method that regularly updates the prox-center in a provably optimal manner, resulting in the linear convergence of NESTA-LASSO under reasonable assumptions.

Our work is motivated by recent advances in solving the basis pursuit denoising (BPDN) problem (i.e., approximating the minimum one-norm solution to an underdetermined least squares problem). Thus, one of our goals is to show that NESTA-LASSO can be used to solve the BPDN problem. We use NESTA-LASSO to solve a subproblem within the Pareto root-finding method used by the state-of-the-art BPDN solver SPGL1. The resulting algorithm is called PARNES, and we show, experimentally, that it is comparable to currently available solvers.

Keywords basis pursuit · Newton’s method · Pareto curve · Nesterov’s method · compressed sensing · convex minimization · duality

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1 Introduction

We would like to find a solution to the sparsest recovery problem with noise

\[
\min \|x\|_0 \quad \text{s.t.} \quad \|Ax - b\|_2 \leq \sigma. \tag{1}
\]

Here, \( \sigma \) specifies the noise level, \( A \) is an \( m \)-by-\( n \) matrix with \( m \ll n \), and \( \|x\|_0 \) is the number of nonzero entries of \( x \). This problem comes up in fields such as image processing [29], seismics [22, 21], astronomy [6], and model selection in regression [15]. Since (1) is known to be ill-posed and NP-hard [18, 23], various convex, \( l_1 \)-relaxed formulations are often used.

Relaxing the zero-norm in (1) gives the basis pursuit denoising (BPBDN) problem

\[
\text{BP}(\sigma) \min \|x\|_1 \quad \text{s.t.} \quad \|Ax - b\|_2 \leq \sigma. \tag{2}
\]

The special case of \( \sigma = 0 \) is the basis pursuit problem [12]. Two other commonly used \( l_1 \)-relaxations are the LASSO problem [30]

\[
\text{LS}(\tau) \min \|Ax - b\|_2 \quad \text{s.t.} \quad \|x\|_1 \leq \tau, \tag{3}
\]

and the penalized least-squares problem

\[
\text{QP}(\lambda) \min \|Ax - b\|_2^2 + \lambda \|x\|_1, \tag{4}
\]

proposed by Chen, Donoho, and Saunders [12]. A large amount of work has been done to show that these formulations give an effective approximation of the solution to (1); see [13, 31, 10]. In particular, under certain conditions on the sparsity of the solution \( x \), \( x \) can be recovered exactly provided that \( A \) satisfies the restricted isometry property (RIP).

There are a wide variety of algorithms which solve the BP(\( \sigma \)), QP(\( \lambda \)), and LS(\( \tau \)) problems. Refer to Section 5 for descriptions of some of the current algorithms. Our work has been motivated by the accuracy and speed of the recent solvers NESTA and SPGL1. In [24], Nesterov presents an algorithm to minimize a smooth convex function over a convex set with an optimal convergence rate. An extension to the nonsmooth case is presented in [25]. NESTA solves the BP(\( \sigma \)) problem using the nonsmooth version of Nesterov’s work.

For appropriate parameter choices of \( \sigma, \lambda, \) and \( \tau \), the solutions of BP(\( \sigma \)), QP(\( \lambda \)), and LS(\( \tau \)) coincide [33]. Although the exact dependence is usually hard to compute [33], there are solution methods which exploit these relationships. The Matlab solver SPGL1 is based on the Pareto root-finding method [33] which solves BP(\( \sigma \)) by approximately solving a sequence of LS(\( \tau \)) problems. In SPGL1, the LS(\( \tau \)) problems are solved using a spectral projected-gradient (SPG) method.

While we are ultimately interested in solving the BPBDN problem in (2), our main result is an algorithm for solving the LASSO problem. Our algorithm, NESTA-LASSO (cf. Algorithm 2), effectively uses Nesterov’s work to solve the LASSO problem. Additionally, we propose a modification of Nesterov’s work which results in the local linear convergence of NESTA-LASSO under reasonable assumptions. Finally, we show that replacing the SPG method in the Pareto root-finding procedure, used in SPGL1, with our NESTA-LASSO method leads to an effective method for solving BP(\( \sigma \)). We call this modification PARNES and compare its efficacy with the state-of-the-art solvers presented in Section 5.
1.1 Organization of the paper

In Section 2, we present and describe the background of NESTA-LASSO. We show in Section 3 that, under some reasonable assumptions, NESTA-LASSO can exhibit linear convergence. In Section 4, we describe the Pareto root-finding procedure behind the BPDN solver SPGL1 and show how NESTA-LASSO can be used to solve a subproblem. Section 5 describes some of the available algorithms for solving BPDN and the equivalent QP(λ) problem. Lastly, in Section 6, we show in a series of numerical experiments that using NESTA-LASSO in SPGL1 to solve BPDN is comparable with current competitive solvers.

2 NESTA-LASSO

We present the main parts of our method to solve the LASSO problem. Our algorithm, NESTA-LASSO (cf. Algorithm 2), is an application of the accelerated proximal gradient algorithm of Nesterov [24] outlined in Section 2.1. Additionally, we have a prox-center update improving convergence which we describe in Section 3. In each iteration, we use the fast \(l^1\)-projector of Duchi, Shalev-Shwartz, Singer, and Chandra [14] given in Section 2.3.

2.1 Nesterov’s Algorithm

Let \(Q \subseteq \mathbb{R}^n\) be a convex closed set. Let \(f : Q \to \mathbb{R}\) be smooth, convex and Lipschitz differentiable with \(L\) as the Lipschitz constant of its gradient, i.e.

\[
\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|, \quad \text{for all } x, y \in Q.
\]

Nesterov’s accelerated proximal gradient algorithm iteratively defines a sequence \(x_k\) as a judiciously chosen convex combination of two other sequences \(y_k\) and \(z_k\), which are in turn solutions to two quadratic optimization problems on \(Q\). The sequence \(z_k\) involves a strongly convex prox-function, \(d(x)\), which satisfies

\[
d(x) \geq \alpha_2 \|x - c\|_2^2. \tag{5}
\]

For simplicity, we have chosen the right hand side of (5) with \(\alpha = 1\) as our prox-function throughout this paper. The \(c\) in the prox-function is called the prox-center. With this prox-function, we have:

\[
y_k = \arg\min_{y \in Q} \nabla f(x_k)^\top (y - x_k) + \frac{L}{2}\|y - x_k\|_2^2,
\]

\[
z_k = \arg\min_{z \in Q} \sum_{i=0}^{k} \frac{i + 1}{2} [f(x_i) + \nabla f(x_i)^\top (z - x_i)] + \frac{L}{2}\|z - c\|_2^2,
\]

\[
x_k = \frac{2}{k + 3} z_k + \frac{k + 1}{k + 3} y_k.
\]

Nesterov showed that if \(x^*\) is the optimal solution to

\[
\min_{x \in Q} f(x),
\]
then the iterates defined above satisfy

\[ f(y_k) - f(x^*) \leq \frac{L}{k(k+1)} \|x^* - c\|^2 = O \left( \frac{L}{k^2} \right). \]  

(6)

An implication is that the algorithm requires \( O(\sqrt{L}/\varepsilon) \) iterations to bring \( f(y_k) \) to within \( \varepsilon > 0 \) of the optimal value.

**Algorithm 1** Accelerated proximal gradient method for convex minimization

**Input:** function \( f \), gradient \( \nabla f \), Lipschitz constant \( L \), prox-center \( c \).

**Output:** \( x^* = \arg \min_{x \in Q} f(x) \)

1: initialize \( x_0 \);
2: for \( k = 0, 1, 2, \ldots \) do
3: compute \( f(x_k) \) and \( \nabla f(x_k) \);
4: \( y_k = \arg \min_{y \in Q} \nabla f(x_k)^\top (y - x_k) + \frac{L}{2} \|y - x_k\|^2; \)
5: \( z_k = \arg \min_{z \in Q} \sum_{i=0}^{k-1} \frac{L_i}{2} \|f(x_i) + \nabla f(x_i)^\top (z - x_i)\| + \frac{L}{2} \|z - c\|^2; \)
6: \( x_{k+1} = \frac{2}{k+1} y_k + \frac{k}{k+1} z_k + \frac{1}{k+1} x_k; \)
7: end for

In [25], Nesterov extends his work to minimize nonsmooth convex functions \( f \). Nesterov shows that one can obtain the minimum by applying his algorithm for smooth minimization to a smooth approximation \( f_\mu \) of \( f \). Since \( \nabla f_\mu \) is shown to have Lipschitz constant \( L_\mu = 1/\mu \), if \( \mu \) is chosen to be proportional to \( \varepsilon \), it takes \( O \left( \frac{1}{\varepsilon} \right) \) iterations to bring \( f(x_k) \) within \( \varepsilon \) of the optimal value.

The recent algorithm NESTA solves \( \text{BP}(\sigma) \) using Nesterov’s algorithm for nonsmooth minimization. Our algorithm, NESTA-LASSO, solves \( \text{LS}(\tau) \) using Nesterov’s smooth minimization algorithm. We are motivated by the accuracy and speed of NESTA, and the fact that smooth version of Nesterov’s algorithm has a faster convergence rate than the nonsmooth version.

2.2 NESTA-LASSO: An accelerated proximal gradient algorithm for LASSO

We apply Nesterov’s accelerated proximal gradient method in Algorithm 1 to the LASSO problem \( \text{LS}(\tau) \), and in each iteration, we use the fast \( l_1 \)-projector \( \text{proj}_1 \) described in the next section. The pseudocode for this is given in Algorithm 2. We make one slight improvement to Algorithm 1, namely, we update our prox-centers from time to time. In fact, we will see in Section 3 that the prox-centers may be updated in an optimal fashion and that this leads to linear convergence under a suitable application of RIP (see Corollary 1 for details).

In our case, \( f = \frac{1}{2} \|b - Ax\|^2_2 \), \( \nabla f = A^\top (Ax - b) \), and \( Q \) is the one-norm ball \( \|x\|_1 \leq \tau \). The initial point \( x_0 \) is used as the prox-center \( c \). To compute the iterate \( y_k \), we have

\[
\begin{align*}
y_k &= \arg \min_{\|y\|_1 \leq \tau} \nabla f(x_k)^\top (y - x_k) + \frac{L}{2} \|y - x_k\|^2 \\
&= \arg \min_{\|y\|_1 \leq \tau} y^\top y - 2(x_k - \nabla f(x_k)/L)^\top y \\
&= \arg \min_{\|y\|_1 \leq \tau} \|y - (x_k - \nabla f(x_k)/L)\|_2 \\
&= \text{proj}_1(x_k - \nabla f(x_k)/L, \tau)
\end{align*}
\]
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where \( \text{proj}_1(v, \tau) \) returns the projection of the vector \( v \) onto the one-norm ball of radius \( \tau \). By similar reasoning, computing \( z_k \) can be shown to be equivalent to computing

\[
z_k = \text{proj}_1 \left( c - \frac{1}{L} \sum_{i=0}^{k} i + \frac{1}{2} \nabla f(x_i), \tau \right).
\]

**Algorithm 2** NESTA-LASSO algorithm with prox-center updates

**Input:** initial point \( x_0 \), LASSO parameter \( \tau \), tolerance \( \eta \).

**Output:** \( x_\tau = \arg\min \{ \|b - Ax\|_2 \mid \|x\|_1 \leq \tau \} \).

1: for \( j = 0, \ldots, j_{\text{max}} \), do
2: \( c_j = x_0, h_0 = 0, r_0 = b - Ax_0, g_0 = -A^T r_0, \eta_0 = \|r_0\|_2 - (b^T r_0 - \tau \|g_0\|_\infty) / \|r_0\|_2; \)
3: for \( k = 0, \ldots, k_{\text{max}} \), do
4: if \( \eta_k \leq e^{-2\eta_0} \) then
5: return \( y_k, \eta_k \)
6: end if
7: \( y_k = \text{proj}_1 (x_k - g_k / L, \tau); \)
8: \( h_k = \frac{1}{k+1} g_k; \)
9: \( z_k = \text{proj}_1 (c_j - h_k / L, \tau); \)
10: \( x_k = h_k + \frac{1}{k+1} g_k; \)
11: \( r_k = b - Ax_k; \)
12: \( g_k = -A^T r_k; \)
13: \( \eta_k = \|r_k\|_2 - (b^T r_k - \tau \|g_k\|_\infty) / \|r_k\|_2; \)
14: end for
15: \( x_0 = y_k; \)
16: if \( \eta_k \leq \eta \) then
17: return \( x_\tau = y_k; \)
18: end if
19: end for

2.3 One Projector

The one-projector, \( \text{proj}_1 \), is used twice in each iteration of Algorithm 2. We briefly describe the algorithm of Duchi, Shalev-Schwartz, Singer, and Chandra [14] for fast projection to an \( l^1 \)-ball in high-dimension. A similar algorithm is presented in [33]. The algorithm costs \( O(n \log n) \) in the worst case, but it has been shown to cost much less experimentally [33]. Note that the two calls to one-projector in each iteration can be reduced to one call with results in Tseng’s paper [32]. However, we make no change to our algorithm due to the low cost of the one-projector.

Consider the projection of an \( n \)-vector \( c \) onto the one-norm ball \( \|x\|_1 \leq \tau \). This is given by the minimization problem

\[
\text{proj}_1(c, \tau) := \arg\min_x \| c - x \|_2 \quad \text{s.t.} \quad \|x\|_1 \leq \tau.
\]

Without loss of generality, the symmetry of the one-norm ball allows us to assume that \( c \) has all nonnegative entries. Assuming the coefficients of the vector \( c \) are ordered from largest to smallest, the solution \( x^*_i = \text{proj}_1(c, \tau) \) is given by

\[
x^*_i = \max \{ 0, c_i - \eta \} \quad \text{with} \quad \eta = \frac{\tau - (c_1 + \cdots + c_k)}{k}.
\]
where $k$ is the largest index such that $\eta \leq c_k$. Please refer to [33][14] for more details on the implementation.

3 Optimality

In NESTA-LASSO, we update the prox-center, $c$, in a provably optimal way which yields a local linear convergence rate. We analyze the case where the solution $x^*$ is $s$-sparse. The case where $x^*$ is approximately sparse has not been analyzed; we plan to do this for future work. Nesterov has shown that when his accelerated proximal gradient algorithm is applied to minimize the objective function $f$, the $k$th iterate $y_k$ and the minimizer $x^*$ satisfy

$$f(y_k) - f(x^*) \leq \frac{L}{k(k + 1)}\|x^* - c\|_2^2$$

(9)

where $L$ is the Lipschitz constant for $\nabla f$ and $c$ is the prox-center [24][25].

In our case, $f(x) = \|Ax - b\|_2^2$ where the underdetermined matrix $A$ is understood to represent a dictionary of atoms, usually sampled via some measurements. We will assume that $A$ satisfies the restricted isometry property (RIP) of order $2s$ as described in [33][9]. Namely, there exists a constant $\delta_{2s} \in (0, 1)$ such that

$$(1 - \delta_{2s})\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta_{2s})\|x\|_2^2$$

(10)

whenever $\|x\|_0 \leq 2s$. Since the RIP helps ensure that the solution to (1) is closely approximated by the solution to (2) [9], and we are ultimately interested in solving (2), this is a reasonable assumption.

Additionally, we assume the iterates $y_k$ are $s$-sparse when sufficiently close to $x^*$. This is reasonable since each iteration of NESTA-LASSO involves projecting the current iteration onto a 1-norm ball. Due to the geometry of the projection, there is a high likelihood that our assumption will hold. Under the RIP and the assumption that $x^*$ is $s$-sparse, it follows by Theorem 3.2 in [25] that the LASSO problem has a unique solution. Since the one-norm ball is compact, this implies that $y_k$ converges to $x^*$.

Let $\delta = 1 - \delta_{2s}$. We have

$$\|A(x^* - y_k)\|_2^2 + 2(y_k - x^*)^\top A^\top (Ax^* - b) = f(y_k) - f(x^*) \geq \|A(y_k - x^*)\|_2^2 \geq \delta\|y_k - x^*\|_2^2.$$

To see the first inequality, let $y = x^* + \tau (y_k - x^*)$ for $\tau \in [0, 1]$. Due to the convexity of the one-norm ball, $y$ is feasible. Since $x^*$ is the minimum, for any $\tau \in [0, 1],$

$$f(y) - f(x^*) = \tau^2 \|A(x^* - y_k)\|_2^2 + 2\tau(y_k - x^*)^\top A^\top (Ax^* - b) \geq 0.$$

Thus, $(y_k - x^*)^\top A^\top (Ax^* - b) \geq 0$. The second inequality follows from (10). Then from (9), we have

$$\delta\|y_k - x^*\|_2^2 \leq \frac{L}{k(k + 1)}\|x^* - c\|_2^2.$$

Putting everything together gives

$$\|y_k - x^*\|_2 \leq \sqrt{\frac{L}{k(k + 1)\delta}}\|x^* - c\|_2 \leq \frac{1}{k} \sqrt{\frac{L}{\delta}}\|c - x^*\|_2.$$

(11)
The above relation and (9) suggest that we can speed up Algorithm 1 by updating the prox-center, \( c \), with the current iterate \( y_k \) every \( k \) steps. With our assumptions, we prove in the following theorem that there is an optimal number of such steps. Allow the iterates to be represented by \( y_{jk} \) where \( j \) is the number of times the prox-center has been changed (the outer iteration) and \( k \) is number of iterations after the last prox-center change (the inner iteration).

**Theorem 1** Suppose \( A \) satisfies the restricted isometry property of order \( 2s \), the solution \( x^* \) is \( s \)-sparse, and the iterates \( y_{jk} \) are eventually \( s \)-sparse. Then for each \( j \),

\[
\|y_{jk_{\text{opt}}} - x^*\| \leq \frac{1}{e}\|y_{j1} - x^*\|
\]

where

\[
k_{\text{opt}} = e \sqrt{\frac{L}{\delta}}
\]

and \( e \) is the base of the natural logarithm. Moreover, the total number of iterations, \( j_{\text{tot}}k_{\text{opt}} \), to get \( \|y_{jk} - x^*\|_2 \leq \varepsilon \) is minimized with this choice of \( k_{\text{opt}} \).

**Proof** First observe that (11) implies

\[
\|y_{jk} - x^*\|_2 \leq \frac{1}{k} \sqrt{\frac{L}{\delta}} \|y_{j1} - x^*\|_2 \leq \left( \frac{1}{k} \sqrt{\frac{L}{\delta}} \right)^j \|y_{j1} - x^*\|_2 \leq \varepsilon \|y_{j1} - x^*\|_2
\]

when

\[
j \log \left( \frac{1}{k} \sqrt{\frac{L}{\delta}} \right) = \log \varepsilon.
\]

This relation allows us to choose \( k \) to minimize the product \( jk \). Since

\[
jk = \frac{k \log \varepsilon}{\log \sqrt{L/\delta} - \log k},
\]

taking derivative of the expression on the right shows that \( jk \) is minimized when

\[
k_{\text{opt}} = e \sqrt{\frac{L}{\delta}}
\]

where \( e \) is the base of the natural logarithm. Thus,

\[
j_{\text{tot}}k_{\text{opt}} = -e \sqrt{\frac{L}{\delta} \log \varepsilon}.
\]

The assumption that the iterates \( y_{jk} \) will eventually be \( s \)-sparse is reasonable precisely because we expect the LASSO relaxation (3) to be a good proxy for (1). In other words, we expect our solutions to (3) to be sparse. Of course this argument is merely meant to be a heuristic; a more rigorous justification along the lines of [13] may be possible and is in our future plans.

Let \( k_{\text{opt}} \) be as in (12). For each prox-center change, \( j \), perform \( k_{\text{opt}} \) inner iterations, and let \( p_j = y_{jk_{\text{opt}}} \) be the output to the \( j \)th outer iteration. An immediate consequence of Theorem 1 is that the relative decrease after \( k_{\text{opt}} \) steps of the inner iteration in Algorithm 2 is \( e^{-2} \), i.e.

\[
\|p_j - x^*\|_2^2 \leq e^{-2}\|p_{j-1} - x^*\|_2^2
\]

and in general, this is the best possible.
Corollary 1 If $A$ satisfies the restricted isometry property of order $2s$, the solution $x^*$ is $s$-sparse, and the iterates $p_j$ are eventually $s$-sparse, then Algorithm 2 is linearly convergent.

In our experiments, there are some cases where updating the prox-center will eventually cause the duality gap to jump to a higher value than the previous iteration. This can cause the algorithm to run for more iterations than necessary. A check is added to prevent the prox-center from being updated if it no longer helps.

In Tables 1 and 2, we give some results showing that updating the prox-center is effective when using NESTA-LASSO to solve the LASSO problem.

4 PARNES

In applications where the noise level of the problem is approximately known, it is preferable to solve $bp(\sigma)$. The Pareto root-finding method used by van den Berg and Friedlander interprets $bp(\sigma)$ as finding the root of a single-variable nonlinear equation whose graph is called the Pareto curve. Their implementation of this approach is called SPGL1. In SPGL1, an inexact version of Newton’s method is used to find the root, and at each iteration, an approximate solution to the LASSO problem, $ls(\tau)$, is found using an SPG approach. Refer to [33] for more information on the inexact Newton method. In Section 6, we show experimentally that using NESTA-LASSO in place of the SPG approach can lead to improved results. We call this version of the Pareto root-finding method, PARNES. The pseudocode of PARNES is given in Algorithm 3.

4.1 Pareto Curve

Suppose $A$ and $b$ are given, with $0 \neq b \in \text{range}(A)$. The points on the Pareto curve are given by $(\tau, \varphi(\tau))$ where $\varphi(\tau) = \|Ax_\tau - b\|_2$, $\tau = \|x_\tau\|_1$, and $x_\tau$ solves $ls(\tau)$. The Pareto curve gives the optimal trade-off between the 2-norm of the residual and 1-norm of the solution to $ls(\tau)$. It can also be shown that the Pareto curve also characterizes the optimal trade-off between the 2-norm of the residual and 1-norm of the solution to $bp(\sigma)$. Refer to [33] for a more detailed explanation of these properties of the Pareto curve.
Let $\tau_{bp}$ be the optimal objective value of BP(0). The Pareto curve is restricted to the interval $\tau \in [0, \tau_{bp}]$ with $\varphi(0) = \|b\|_2 > 0$ and $\varphi(\tau_{bp}) = 0$. The following theorem, proved in [33], shows that the Pareto curve is convex, strictly decreasing over the interval $\tau \in [0, \tau_{bp}]$, and continuously differentiable for $\tau \in (0, \tau_{bp})$.

**Theorem 2** The function $\varphi$ is

1. convex and nonincreasing.
2. continuously differentiable for $\tau \in (0, \tau_{bp})$ with $\varphi'(\tau) = -\lambda_{\tau}$ where $\lambda_{\tau} = \|A^T y_{\tau}\|_\infty$ is the optimal dual variable to LS($\tau$) and $y_{\tau} = r_{\tau}/\|r_{\tau}\|_2$ with $r_{\tau} = Ax_{\tau} - b$.
3. strictly decreasing and $\|x_{\tau}\|_1 = \tau$ for $\tau \in [0, \tau_{bp}]$.

### 4.2 Root Finding

Since the Pareto curve characterizes the optimal trade-off for both BP($\sigma$) and LS($\tau$), solving BP($\sigma$) for a fixed $\sigma$ can be interpreted as finding a root of the non-linear equation $\varphi(\tau) = \sigma$. The iterations consist of finding the solution to LS($\tau$) for a sequence of parameters $\tau_k \to \tau_{\sigma}$ where $\tau_{\sigma}$ is the optimal objective value of BP($\sigma$).

Applying Newton’s method to $\varphi$ gives

$$\tau_{k+1} = \tau_k + (\sigma - \varphi(\tau_k))/\varphi'(\tau_k).$$

Since $\varphi$ is convex, strictly decreasing and continuously differentiable, $\tau_k \to \tau_{\sigma}$ superlinearly for all initial values $\tau_0 \in (0, \tau_{bp})$ (see Proposition 1.4.1 in [4]). By Theorem 2, $\varphi(\tau_k)$ is the optimal value to LS($\tau_k$) and $\varphi'(\tau_k)$ is the dual solution to LS($\tau_k$). Since evaluating $\varphi(\tau_k)$ involves solving a potentially large optimization problem, an inexact Newton method is carried out with approximations of $\varphi(\tau_k)$ and $\varphi'(\tau_k)$. 

![Fig. 1 An example of a Pareto curve. The solid line is the Pareto curve; the dotted red lines give two iterations of Newton’s method.](image-url)
Let $\overline{y}_\tau$ and $\overline{\lambda}_\tau$ be the approximations of the $y_\tau$ and $\lambda_\tau$ defined in Theorem 2. The duality gap at each iteration is given by
\[
\eta_\tau = \|r_\tau\|_2 - (b^T y_\tau - \tau \lambda_\tau).
\]
The authors of [33] have proved the following convergence result.

**Theorem 3** Suppose $A$ has full rank, $\sigma \in (0, \|b\|_2)$, and the inexact Newton method generates a sequence $\tau_k \rightarrow \tau_\sigma$. If $\eta_k := \eta_{\tau_k} \rightarrow 0$ and $\tau_0$ is close enough to $\tau_\sigma$, we have
\[
|\tau_{k+1} - \tau_\sigma| = \gamma_1 \eta_k + \zeta_k |\tau_k - \tau_\sigma|,
\]
where $\zeta_k \rightarrow 0$ and $\gamma_1$ is a positive constant.

4.3 Solving the LASSO problem

Approximating $\varphi(\tau_k)$ and $\varphi'(\tau_k)$ require approximately minimizing $L_\sigma(\tau)$. The solver SPGL1 uses a spectral projected-gradient (SPG) algorithm. The method follows the algorithm by Birgin, Martínez, and Raydan [5] and is shown to be globally convergent. The costs include evaluating $Ax$, $A^T r$, and a projection onto the one-norm ball $\|x\|_1 \leq \tau$. In PARNES, we replace this SPG algorithm with our algorithm, NESTA-LASSO (cf. Algorithm 2).

**Algorithm 3** PARNES: Pareto curve method with NESTA-LASSO

**Input:** initial point $x_0$, BPDN parameter $\sigma$, tolerance $\eta$.

**Output:** $x_\sigma = \text{argmin}\{\|x\|_1 \mid \|Ax - b\|_2 \leq \sigma\}$

1: $\tau_0 = 0$, $\varphi_0 = \|b\|_2$, $\varphi'_0 = \|A^T b\|_\infty$;
2: for $k = 0, \ldots, k_{\max}$, do
3: $\tau_{k+1} = \tau_k + (\sigma - \varphi_k)/\varphi'_k$;
4: $x_{k+1} = \text{NESTA-LASSO}(x_k, \tau_{k+1}, \eta)$;
5: $r_{k+1} = b - Ax_{k+1}$;
6: $\varphi_{k+1} = \|r_{k+1}\|_2$;
7: $\varphi'_{k+1} = -\|A^T r_{k+1}\|_\infty/\|r_{k+1}\|_2$;
8: if $\|r_{k+1}\|_2 - \sigma \leq \eta \cdot \max\{1, \|r_{k+1}\|_2\}$ then
9: return $x_\sigma = x_{k+1}$;
10: end if
11: end for

5 Other solution techniques and tools

In the NESTA paper, [3], extensive experiments are carried out, comparing the effectiveness of state-of-the-art sparse reconstruction algorithms. The code used to run these experiments is available at [http://www.acm.caltech.edu/~nesta](http://www.acm.caltech.edu/~nesta). We have modified this NESTA experiment infrastructure to include PARNES in its tests and repeated some of the tests in [3] with the same experimental standards and parameters. Refer to the [3] for a detailed description of the experiments. The algorithms tested and their experimental details are described below. Note that the algorithms either solve BP($\sigma$) or QP($\lambda$).
5.1 NESTA [3]

NESTA is used to solve \( \text{bp}(\sigma) \). Its code is available at \( \text{http://www.acm.caltech.edu/~nesta} \). The parameters for NESTA are set to be
\[
x_0 = A^*b, \quad \mu = 0.02,
\]
where \( x_0 \) is the initial guess and \( \mu \) is the smoothing parameter for the one-norm function in \( \text{bp}(\sigma) \).

Continuation techniques are used to speed up NESTA in [3]. Such techniques are useful when it is observed that a problem involving some parameter \( \lambda \) is faster for large \( \lambda \), [27,19]. Thus, the idea of continuation is to solve a sequence of problems for decreasing values of \( \lambda \). In the case of NESTA, it is observed that convergence is faster for larger values of \( \mu \). When continuation is used in the experiments, there are four continuation steps with \( \mu_0 = \|x_0\|_\infty \) and \( \mu_t = (\mu/\mu_0)^{t/4} \mu_0 \) for \( t = 1, 2, 3, 4 \).

5.2 GPSR: Gradient Projection for Sparse Reconstruction [16]

GPSR is used to solve the penalized least-squares problem \( \text{qp}(\lambda) \). The code is available at \( \text{http://www.lx.it.pt/~mtf/GPSR} \). The problem is first recast as a bound-constrained quadratic program (BCQP) by using a standard change of variables on \( x \). Here, \( x = u_1 - u_2 \), and the variables are now given by \( [u_1, u_2] \) where the entries are positive. The new problem is then solved using a gradient projection (GP) algorithm. The parameters are set to the default values in the following experiments.

A version of GPSR with continuation is also tested. The number of continuation steps is 40, the variable TOLERANCEA is set to \( 10^{-3} \), and the variable MINITERA is set to 1. All other parameters are set to their default values.

5.3 SpaRSA: Sparse reconstruction by separable approximation [17]

SpaRSA is used to minimize functions of the form \( \phi(x) = f(x) + \lambda c(x) \) where \( f \) is smooth and \( c \) is non-smooth and non-convex. The \( \text{qp}(\lambda) \) problem is a special case of functions of this form. The code for SpaRSA is available at \( \text{http://www.lx.it.pt/~mtf/SpaRSA} \).

In a sense, SpaRSA is an iterative shrinkage/thresholding algorithm. Utilizing continuation and a Brazilai-Borwein heuristic [1] to find step sizes, the speed of the algorithm can be increased. The number of continuation steps is set to 40 and the variable MINITERA is set to 1. All remaining variables are set to their default values.

5.4 SPGL1 [33] and SPARCO [34]

SPGL1 is available at \( \text{http://www.cs.ubc.ca/labs/scl/spgl1} \). The parameters for our numerical experiments are set to their default values.

Due to the vast number of available and upcoming algorithms for sparse reconstruction, the authors of SPGL1 and others have created SPARCO [34]. In SPARCO, they provide a much needed testing framework for benchmarking algorithms. It consists of a large collection of imaging, compressed sensing, and geophysics problems. Moreover, it includes a library of standard operators which can be used to create new test problems. SPARCO is implemented in MATLAB and was originally created to test SPGL1. The toolbox is available at \( \text{http://www.cs.ubc.ca/labs/scl/sparco} \).
5.5 FISTA: Fast iterative soft-thresholding algorithm \[2\]

FISTA solves $\text{QP}(\lambda)$. It can be thought of as a simplified version of the Nesterov algorithm in Section \[2,\] since it involves two sequences of iterates instead of three. In Section 4.2 of \[3\], FISTA is shown to give very accurate solutions provided enough iterations are taken. Due to its ease of use and accuracy, FISTA is used to compute reference solutions in \[3\] and in this paper. The code for FISTA can be found in the NESTA experiments code at \url{http://www.acm.caltech.edu/~nesta}.

5.6 FPC: Fixed point continuation \[19,20\]

FPC solves the general problem $\min_x \|x\|_1 + \lambda f(x)$ where $f(x)$ is differentiable and convex. The special case with $f(x) = \frac{1}{2} \|Ax - b\|_2^2$ is the $\text{QP}(\lambda)$ problem. The algorithm is available at \url{http://www.caam.rice.edu/~optimization/L1/fpc}.

FPC is equivalent to iterative soft-thresholding. The approach is based on the observation that the solution solves a fixed-point equation $x = F(x)$ where the operator $F$ is a composition of a gradient descent-like operator and a shrinkage operator. It can be shown that the algorithm has $q$-linear convergence and also, finite-convergence for some components of the solution. Since the parameter $\lambda$ affects the speed of convergence, continuation techniques are used to slowly decrease $\lambda$ for faster convergence. A more recent version of FPC, FPC-BB, uses Brazilai-Borwein steps to speed up convergence. Both versions of FPC are tested with their default parameters.

5.7 FPC-AS: Fixed-point continuation and active set \[35\]

FPC-AS is an extension of FPC into a two-stage algorithm which solves $\text{QP}(\lambda)$. The code can be found at \url{http://www.caam.rice.edu/~optimization/L1/fpc}. It has been shown in \[19\] that applying the shrinkage operator a finite number of times yields the support and signs of the optimal solution. Thus, the first stage of FPC-AS involves applying the shrinkage operator until an active set is determined. In the second stage, the objective function is restricted to the active set and $\|x\|_1$ is replaced by $c^T x$ where $c$ is the vector of signs of the active set. The constraint $c_i \cdot x_i > 0$ is also added. Since the objective function is now smooth, many available methods can now be used to solve the problem. In the following tests, the solvers L-BFGS and conjugate gradients, CG (referred to as FPC-AS (CG)), are used. Continuation methods are used to decrease $\lambda$ to increase speed. For experiments involving approximately sparse signals, the parameter controlling the estimated number of nonzeros is set to $n$, and the maximum number of subspace iterations is set to 10. The other parameters are set to their default values. All other experiments were tested with the default parameters.

5.8 Bregman \[36\]

The Bregman Iterative algorithm consists of solving a sequence of $\text{QP}(\lambda)$ problems for a fixed $\lambda$ and updated observation vectors $b$. Each $\text{QP}(\lambda)$ is solved using the Brazilai-Borwein version of FPC. Typically, very few (around four) outer iterations are needed. Code for the Bregman algorithm can be found at \url{http://www.caam.rice.edu/~optimization/L1/2006/10/bregman-iterative-algorithms-for.html}. All parameters are set to their default values.
6 Numerical results

As mentioned above, some of the algorithms we test solve $\text{QP}(\lambda)$ and some solve $\text{BP}(\sigma)$. Comparing the algorithms thus requires a way of finding a $(\sigma, \lambda)$ pair which for which the solutions coincide. The tests in [3] use a two-step procedure. From the noise level, $\sigma$ is chosen, and then SPGL1 is used to solve $\text{BP}(\sigma)$. The SPGL1 dual solution gives an estimate of the corresponding $\lambda$, and then FISTA is used to compute a second $\sigma$ corresponding to this $\lambda$ with high accuracy.

In the NESTA experiments, FISTA is used to determine the accuracy of the computed solutions while NESTA is used to compute a solution that is used in the stopping criteria. Please refer to [3] for a complete description of the experimental details. Section 4.2 of [3], shows that FISTA gives very accurate solutions provided enough iterations are taken. For each test, FISTA is run twice. FISTA is first run, with no limit on the number of iterations, until the relative change in the function value is less than $10^{-14}$. This solution is used to determine the accuracy of the computed solutions. FISTA is run a second time with one of the following stopping criteria; the results of this run are recorded in the tables.

In [3], NESTA (with continuation), is used to compute a solution $x_{\text{NES}}$. In the tests, the following stopping criteria are used where $\hat{x}_k$ is the $k$th-iteration in the algorithm being tested.

$$
\| \hat{x}_k \|_{\ell_1} \leq \| x_{\text{NES}} \|_{\ell_1} \quad \text{and} \quad \| b - A\hat{x}_k \|_{\ell_2} \leq 1.05 \| b - Ax_{\text{NES}} \|_{\ell_2},
$$

or

$$
\lambda \| \hat{x}_k \|_{\ell_1} + \frac{1}{2} \| A\hat{x}_k - b \|_{\ell_2}^2 \leq \lambda \| x_{\text{NES}} \|_{\ell_1} + \frac{1}{2} \| Ax_{\text{NES}} - b \|_{\ell_2}^2.
$$

In other words, the algorithms are run until they achieve a solution at least as accurate as NESTA.

The rationale for having two stopping criteria is to reduce any potential bias arising from the fact that some algorithms solve $\text{QP}(\lambda)$, for which (14) is the most natural, while others solve $\text{BP}(\sigma)$, for which (13) is the most natural. It is evident from the tables below that there is not a significant difference whether (13) or (14) is used. The algorithms are recorded to not have converged (DNC) if the number of calls to $A$ or $A^*$ exceeds 20,000.

In Tables 4 and 5, we repeat the experiments done in Tables 5.1 and 5.2 of [3]. These experiments involve recovering an unknown signal that is exactly $s$-sparse with $n = 262144$, $m = n/8$, and $s = m/5$. The experiments are performed with increasing values of the dynamic range $d$ where $d = 20, 40, 60, 80, 100$ dB. For each run, the measurement operator is a randomly subsampled discrete cosine transform, and the noise level is set to 0.1.

The dynamic range, $d$, is a measure of the ratio between the largest and smallest magnitudes of the non-zero coefficients of the unknown signal. Problems with a high dynamic range occur often in applications. In these cases, high accuracy becomes important since one must be able to detect and recover lower-power signals with small amplitudes which may be obscured by high-power signals with large amplitudes.
Table 3 Comparison of accuracy using experiments from Table 4. Dynamic range 100 dB, $\sigma = 0.100$, $\mu = 0.020$, sparsity level $s = m/5$. Stopping rule is (13).

| Methods  | $N_A$ | $\|x\|_1$ | $\|Ax - b\|_2$ | $\frac{\|x - x^*\|_1}{\|x\|_1}$ | $\|x - x^*\|_\infty$ | $\|x - x^*\|_2$ |
|----------|-------|------------|----------------|---------------------|-------------------|----------------|
| PARNES   | 632   | 942197.606 | 2.692          | 0.000093            | 8.312             | 46.623         |
| NESTA    | 15227 | 942402.960 | 2.661          | 0.004124            | 45.753            | 255.778        |
| NESTA + CT| 787   | 942211.581 | 2.661          | 0.000812            | 9.317             | 52.729         |
| GPSR     | DNC   | DNC        | DNC            | DNC                 | DNC               | DNC            |
| GPSR + CT| 11737 | 942211.377 | 2.725          | 0.001420            | 15.646            | 90.493         |
| SPARS  | 693   | 942197.785 | 2.728          | 0.000783            | 9.094             | 51.839         |
| SPGL1    | 504   | 942211.520 | 2.628          | 0.001326            | 14.806            | 84.560         |
| FISTA    | 12462 | 942211.540 | 2.654          | 0.000363            | 4.358             | 26.014         |
| FPC-AS   | 287   | 942210.925 | 2.498          | 0.000672            | 9.374             | 45.071         |
| FPC-AS (CG) | 361   | 942210.512 | 2.508          | 0.000671            | 9.361             | 45.010         |
| FPC      | 96114 | 942211.540 | 2.719          | 0.001422            | 15.752            | 90.665         |
| FPC-BB   | 1082  | 942209.854 | 2.726          | 0.001378            | 15.271            | 87.963         |
| BREGMAN-BB | 1408  | 942286.656 | 1.326          | 0.000891            | 9.303             | 52.449         |

Table 4 Number of function calls where the sparsity level is $s = m/5$ and the stopping rule is (13).

| Method      | 20 dB | 40 dB | 60 dB | 80 dB | 100 dB |
|-------------|-------|-------|-------|-------|--------|
| PARNES      | 122   | 172   | 214   | 470   | 632    |
| NESTA       | 383   | 809   | 1639  | 4341  | 15227  |
| NESTA + CT  | 483   | 513   | 583   | 685   | 787    |
| GPSR        | 64    | 622   | 5030  | DNC   | DNC    |
| GPSR + CT   | 271   | 219   | 357   | 1219  | 11737  |
| SPARS       | 323   | 387   | 465   | 541   | 693    |
| SPGL1       | 58    | 102   | 191   | 374   | 504    |
| FISTA       | 69    | 267   | 1020  | 3465  | 12462  |
| FPC-AS      | 209   | 231   | 299   | 371   | 287    |
| FPC-AS (CG)| 253   | 289   | 375   | 481   | 361    |
| FPC         | 474   | 386   | 478   | 1068  | 9614   |
| FPC-BB      | 164   | 168   | 206   | 278   | 1082   |
| BREGMAN-BB  | 211   | 223   | 309   | 455   | 1408   |

The last two tables, Tables 6 and 7, replicate Tables 5.3 and 5.4 of [3]. There are five runs of each experiment. Each run involves an approximately sparse signals obtained from a permutation of the Haar wavelet coefficients of a $512 \times 512$ image. The measurement vector $b$ consists of $m = n/8 = 512^2/8 = 32,768$ random discrete cosine measurements, and the noise level is set to 0.1. For more specific details, refer to [3]. In applications, the signal to be recovered is often approximately sparse rather than exactly sparse. Again, high accuracy is important when solving these problems.

It can be seen that NESTA + CT, SPARS, SPGL1, PARNES, and both versions of FPC-AS perform well in the case of exactly sparse signals for all values of the dynamic range. However, in the case of approximately sparse signals, all versions of FPC and SPARS no longer converge in 20,000 function calls. PARNES still performs well, converging in under 2000 iterations for all runs. The accuracy of the various algorithms is compared in Table 3.
Table 5  Number of function calls where the sparsity level is $s = m/5$ and the stopping rule is (14).

| Method       | 20 dB | 40 dB | 60 dB | 80 dB | 100 dB |
|--------------|-------|-------|-------|-------|--------|
| PARNES       | 74    | 116   | 166   | 364   | 562    |
| NESTA        | 383   | 809   | 1639  | 4341  | 15227  |
| NESTA + CT   | 483   | 513   | 583   | 685   | 787    |
| GPSR         | 62    | 618   | 5026  | DNC   | DNC    |
| GPSR + CT    | 271   | 219   | 369   | 1237  | 11775  |
| SPARSIA      | 323   | 387   | 463   | 541   | 689    |
| SPGL1        | 45    | 99    | 185   | 365   | 488    |
| FISTA        | 72    | 261   | 1002  | 3477  | 12462  |
| FISTA + CT   | 115   | 167   | 159   | 371   | 281    |
| FPC-AS (CG)  | 142   | 210   | 198   | 481   | 355    |
| FPC          | 472   | 386   | 466   | 1144  | 9734   |
| FPC-BB       | 164   | 164   | 202   | 276   | 1092   |
| BREGMAN-BB   | 211   | 223   | 309   | 455   | 1408   |

Table 6  Recovery results of an approximately sparse signal (with Gaussian noise of variance 1 added) and with (14) as a stopping rule.

| Method       | Run 1 | Run 2 | Run 3 | Run 4 | Run 5 |
|--------------|-------|-------|-------|-------|-------|
| PARNES       | 838   | 810   | 1038  | 1098  | 654   |
| NESTA        | 8817  | 10867 | 9887  | 9903  | 11211 |
| NESTA + CT   | 3807  | 3045  | 3047  | 3225  | 2735  |
| GPSR         | DNC   | DNC   | DNC   | DNC   | DNC   |
| GPSR + CT    | DNC   | DNC   | DNC   | DNC   | DNC   |
| SPARSIA      | 2143  | 2353  | 1977  | 1613  | DNC   |
| SPGL1        | 916   | 892   | 1115  | 1437  | 938   |
| FISTA        | 3375  | 2940  | 2748  | 2538  | 3855  |
| FISTA + CT   | DNC   | DNC   | DNC   | DNC   | DNC   |
| FPC-AS (CG)  | DNC   | DNC   | DNC   | DNC   | DNC   |
| FPC          | DNC   | DNC   | DNC   | DNC   | DNC   |
| FPC-BB       | 5614  | 7906  | 5986  | 4652  | 6906  |
| BREGMAN-BB   | 3298  | 1281  | 1507  | 2892  | 3104  |

Table 7  Recovery results of an approximately sparse signal (with Gaussian noise of variance 0.1 added) and with (14) as a stopping rule.

| Method       | Run 1 | Run 2 | Run 3 | Run 4 | Run 5 |
|--------------|-------|-------|-------|-------|-------|
| PARNES       | 1420  | 1772  | 1246  | 1008  | 978   |
| NESTA        | 11573 | 10457 | 10705 | 8807  | 13795 |
| NESTA + CT   | 7543  | 13655 | 11515 | 3123  | 2777  |
| GPSR         | DNC   | DNC   | DNC   | DNC   | DNC   |
| GPSR + CT    | DNC   | DNC   | DNC   | DNC   | DNC   |
| SPARSIA      | 12509 | DNC   | DNC   | 3117  | DNC   |
| SPGL1        | 1652  | 1955  | 2151  | 1311  | 2365  |
| FISTA        | 10845 | 12165 | 10050 | 7647  | 11997 |
| FISTA + CT   | DNC   | DNC   | DNC   | DNC   | DNC   |
| FPC-AS (CG)  | DNC   | DNC   | DNC   | DNC   | DNC   |
| FPC          | DNC   | DNC   | DNC   | DNC   | DNC   |
| FPC-BB       | 5614  | 7906  | 5986  | 4652  | 6906  |
| BREGMAN-BB   | 3298  | 1281  | 1507  | 2892  | 3104  |
6.1 Choice of parameters

As Tseng observed, accelerated proximal gradient algorithms will converge so long as the condition given as equation (45) in [32] is satisfied. In our case this translates into

\[
\min_{x \in \mathbb{R}^n} \left\{ \nabla f(y_k)^T x + \frac{L}{2} \|x - x_k\|^2_2 + P(x) \right\} \geq \nabla f(y_k)^T y_k + P(y_k),
\]

upon setting \( \gamma_k = 1 \) and

\[
P(x) = \begin{cases} 0 & \text{if } \|x\|_1 \leq \tau, \\ \infty & \text{otherwise}, \end{cases}
\]

in (45) in [32]. In other words, the value of \( L \) need not necessarily be fixed at the Lipschitz constant of \( \nabla f \) but may be decreased and decreasing \( L \) has the same effect as increasing the stepsize. Tseng suggested to decrease \( L \) adaptively by a constant factor until (45) is violated then backtrack and repeat the iteration (cf. Note 6 in [32]). For simplicity, and very likely at the expense of speed, we do not change our \( L \) adaptively in \textsc{pares} and \textsc{nesta-lasso}. Instead, we choose a small fixed \( L \) by trying a few different values so that (15) is satisfied for all \( k \), and likewise for the tolerance \( \eta \) in Algorithm 2. However, even with this crude way of selecting \( L \) and \( \eta \), the results obtained are still rather encouraging.

7 Conclusions

As seen in the numerical results, \textsc{spgl1} and \textsc{nesta} are among some of the top performing solvers available for basis pursuit denoising problems. We have therefore made use of Nesterov’s accelerated proximal gradient method in our algorithm \textsc{nesta-lasso} and shown that updating the prox-center leads to improved results. Through our experiments, we have shown that using \textsc{nesta-lasso} in the Pareto root-finding method leads to results comparable to the results given by currently available state-of-the-art methods.

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