On $\mathcal{R}^4$ terms and MHV amplitudes in $\mathcal{N} = 5, 6$ supergravity vacua of Type II superstrings

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Abstract

We compute one-loop threshold corrections to $\mathcal{R}^4$ terms in $\mathcal{N} = 5, 6$ supergravity vacua of Type II superstrings. We then discuss non-perturbative corrections generated by asymmetric D-brane instantons. Finally we derive generating functions for MHV amplitudes at tree level in $\mathcal{N} = 5, 6$ supergravities.
Introduction

$\mathcal{N} = 5,6$ supergravities in $D = 4$ enjoy many of the remarkable properties of $\mathcal{N} = 8$ supergravity. Their massless spectra are unique and consist solely of the supergravity multiplets. Their R-symmetries are not anomalous \cite{1}. Regular BH solutions can be found whereby the scalars are stabilized at the horizon by the attractor mechanism. It is thus tempting to conjecture that if pure $\mathcal{N} = 8$ supergravity turned out to be UV finite \cite{3, 4, 5, 6, 7} then $\mathcal{N} = 5,6$ supergravities should be so, too.

As shown in \cite{8, 9, 10}, Type II superstrings or M-theory accommodate $\mathcal{N} = 8$ supergravity in such a way as to include non-perturbative states that correspond to singular BH solutions in $D = 4$. The same is true for $\mathcal{N} = 5,6$ supergravities. While the embedding of $\mathcal{N} = 8$ supergravity corresponds to simple toroidal compactifications, the embedding of $\mathcal{N} = 5,6$ supergravities, pioneered by S. Ferrara and C. Kounnas in \cite{11} and recently reviewed in \cite{12}, requires asymmetric orbifolds \cite{13, 14} or free fermion constructions \cite{16, 15, 17, 18, 19}.

The inclusion of BPS states, whose possible singular behavior from a strict 4-d viewpoint is resolved from a higher dimensional perspective, generate higher derivative corrections to the low-energy effective action. In particular a celebrated $\mathcal{R}^4$ term appears that spoils the continuous non-compact symmetry of ‘classical’ supergravity. Absence of such a term has been recently shown for pure $\mathcal{N} = 8$ supergravity in \cite{20}. In superstring theory, the $\mathcal{R}^4$ term receives contribution at tree-level, one-loop and from non-perturbative effects associated to D-instantons \cite{21} and other wrapped branes \cite{22}. Proposals for the relevant modular form of the $E_7(\mathbb{Z})$ U-duality group have been recently put forward in \cite{23, 24, 25} that seem to satisfy all the checks.

In this note we consider one-loop threshold corrections to the same kind of terms in superstring models with $\mathcal{N} = 5,6$ supersymmetry in $D = 4$ and $\mathcal{N} = 6$ in $D = 5$. After excluding $\mathcal{R}^2$ terms, we will derive formulae for the ‘perturbative’ threshold corrections. In $D = 4$ we will also discuss other MHV amplitudes\cite{26} that can be obtained by orbifold techniques from the generating function of $\mathcal{N} = 8$ supergravity amplitudes \cite{27}.

Aim of the analysis is three-fold. First, we would like to show that $\mathcal{N} = 5,6$ supersymmetric models in $D = 4$ behave very much as their common $\mathcal{N} = 8$ supersymmetric parent. Second, (gauged) $\mathcal{N} = 5,6$ supergravities have played a crucial role in the recent understanding of M2-brane dynamics \cite{28, 29, 30, 31} and non-perturbative tests may be refined by considering the effects of world-sheet instantons in $CP^3$ \cite{32, 33, 34, 35} along the

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1For a recent review see e.g. \cite{2}.
2$\mathcal{R}^3$ terms cannot be supersymmetrized on shell when all particles are in the supergravity multiplet \cite{20}.
3For a recent review see e.g. \cite{26}.
lines of our present (un-gauged) analysis. Finally, in addition to world-sheet instantons, D-brane instantons corresponding to Euclidean bound states of ‘exotic’ D-branes should contribute that generalize ‘standard’ D-brane instanton calculus to Left-Right asymmetric backgrounds.

Plan of the note is as follows. In Section 1 we briefly review $\mathcal{N} = 5, 6$ supergravities in $D = 4, 5$ and their embedding in Type II superstrings. We then pass to consider in Section 2 a 4-graviton amplitude at one-loop which allows to derive the ‘perturbative’ threshold corrections to $\mathcal{R}^4$ terms, thus excluding $\mathcal{R}^2$ terms. For simplicity, we only give the explicit result for $\mathcal{N} = 6$ in $D = 5$ in Section 3 and sketch how to complete the non-perturbative analysis by including asymmetric D-brane instantons [12] in Section 4. Finally, in Section 5 we consider MHV amplitudes in $\mathcal{N} = 5, 6$ supergravities in $D = 4$ and show how they can be obtained at tree-level by orbifold techniques from the generating function for MHV amplitudes in $\mathcal{N} = 8$ supergravity [27]. Section 6 contains a summary of our results and directions for further investigation.

1 Type II Superstring models with $\mathcal{N} = 5, 6$ in $D = 4, 5$

Let us briefly recall how $\mathcal{N} = 5, 6$ supergravities can be embedded in String Theory. The highest dimension where classical $\mathcal{N} = 6$ supergravity with 24 supercharges can be defined is $D = 6$. However the resulting $\mathcal{N} = (2, 1)$ theory is anomalous and thus inconsistent at the quantum level [36]. So we are led to consider $D = 5$ and then reduce to $D = 4$. $\mathcal{N} = 5$ supergravity with 20 supercharges can only be defined is $D = 4$ and lower. Although we will only focus on $\mathcal{R}^4$ terms in $D = 4$ the parent $D = 5$ theory is instrumental to the identification of the relevant BPS instantons.

1.1 $\mathcal{N} = 6 = 2_L + 4_R$ supergravity in $D = 5$

The simplest way to embed $\mathcal{N} = 6$ in Type II superstrings, is to quotient a toroidal compactification $T^5 = T^4 \times S^1$ by a chiral $Z_2$ twist of the L-movers (‘T-duality’) on four internal directions

$$X^i_L \to -X^i_L, \quad \Psi^i_L \to -\Psi^i_L, \quad i = 6, 7, 8, 9$$

(1)

accompanied by an order two shift that make twisted states massive. As a result half of the supersymmetries in the L-moving sector are broken. The perturbative spectrum is coded in the one-loop torus partition function.

In the untwisted sector, one finds

$$T_u = \frac{1}{2} \{ (Q_o + Q_v)\overline{Q}\Lambda_{5,[0]} + (Q_o - Q_v)(X_o - X_v)\overline{Q}\Lambda_{1,5,[1]} \}$$

(2)
where $X_o - X_v = 4\eta^2/\theta_2^2$ (with $\eta$ denoting Dedekind’s function and $\theta_1, \theta_4$ denoting Jacobi’s elliptic functions) describes the effect of the $Z_2$ projection on four internal L-moving bosons, while

$$\Lambda_{l,r}^{[a]} = \sum_{p_L, p_R} e^{i\tau [a_L p_L - a_R p_R]} q^{b_L^2/2} \eta \left( q^{b_R^2/2} \right)^2$$

are (shifted) Lorentzian lattice sums of signature $(l, r)$ and $Q = V_8 - S_8$, $Q_o = V_4 O_4 - S_4 S_4$, $Q_v = O_4 V_4 - C_4 C_4$, with $O_n, V_n, S_n, C_n$ the characters of $SO(n)$ at level $\kappa = 1$.

At the massless level, in $D = 5$ notation with $SO(3)$ little group, one finds

$$(V_3 + O_3 - 2\Sigma_3) \times (V_3 + 5\bar{O}_3 - 4\Sigma_3) \rightarrow (g + b_2 + \phi)_{NS-NS} + 6A_{NS-NS} + 5\phi_{NS-NS} + 8A_{R-R} + 8\phi_{R-R} - Fermi$$

that form the $N = 6$ supergravity multiplet in $D = 5$

$$SG^{D=5}_{N=6} = \{ g_{\mu\nu}, 6\psi_{\mu}, 15A_{\mu}, 20\chi, 14\phi \}$$

The R-symmetry is $Sp(6)$ while the ‘hidden’ non-compact symmetry is $SU^*(6)$, of dimension 35 and rank 3 generated by 6 × 6 matrices of the form $Z = (Z_1, Z_2, -\bar{Z}_2, \bar{Z}_1)$ with $Tr(Z_1 + \bar{Z}_1) = 0$.

For later purposes, let us observe that the 128 massless states of $N = 6$ supergravity in $D = 5$ are given by the tensor product of the 8 massless states of $N = 2$ SYM (for the Left-movers) and the 16 massless states of $N = 4$ SYM (for the Right-movers) viz.

$$SG^{D=5}_{N=6} = SYM^{D=5}_{N=2} \otimes SYM^{D=5}_{N=4} = \{ A_{\mu}, 2\lambda, \phi \}_L \otimes \{ \tilde{A}_\mu, 4\tilde{\lambda}, 5\tilde{\phi} \}_R$$

After dualizing all massless 2-forms into vectors, the $15 = 7_{NS-NS} + 8_{R-R}$ vectors transform according to the antisymmetric tensor of $SU^*(6)$. The $14 = 1_{NS-NS} + 5_{NS-NS} + 8_{R-R}$ scalars parameterize the moduli space

$$\mathcal{M}^{D=5}_{N=6} = SU^*(6)/Sp(6)$$

By world-sheet modular transformations (first $S$ and then $T$) one finds the contribution of the twisted sector

$$\mathcal{T}_1 = \frac{1}{2} \left\{ (Q_s + Q_c)(X_s + X_c)\bar{Q}\Lambda_{1,5}^{[1]} + (Q_s - Q_c)(X_s - X_c)\bar{Q}\Lambda_{1,5}^{[1]} \right\}$$

where $X_s + X_c = 4\eta^2/\theta_2^2$, $X_s - X_c = 4\eta^2/\theta_3^2$, $Q_s = O_4 S_4 - C_4 O_4$ (‘massive’), $Q_c = V_4 C_4 - S_4 V_4$ (‘massive’). Due to the (L-R symmetric) $Z_2$ shift, the massless spectrum receives no contribution from the twisted sector. Non-perturbative states associated to L-R asymmetric bound states of D-branes were studied in [12]. There are several other ways to embed $N = 6$ supergravity in Type II superstrings, reviewed in [12].

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4For $n$ odd $S_n$ coincides with $C_n$ and will be denoted by $\Sigma_n$. 

1.2 $\mathcal{N} = 6$ supergravities in $D = 4$

Reducing on another circle with or without further shifts, yields $\mathcal{N} = 6$ supergravity in $D = 4$ [11].

The massless spectrum is given by

$$ (V_2 + 2O_2 - 2S_2 - 2C_2) \times (V_2 + 6\bar{O}_2 - 4\bar{S}_2 - 4\bar{C}_2) \rightarrow (9) $$

$$ (g + b + \phi)_{NS-NS} + 8A_{NS-NS} + 12\phi_{NS-NS} + 8A_{R-R} + 16\phi_{R-R} - \text{Fermi} $$

and gives rise to the $\mathcal{N} = 6$ supergravity multiplet in $D = 4$

$$ SG_{D=4}^{N=6} = \{g_{\mu\nu}, 6\psi_\mu, 16A_\mu, 26\chi, 30\phi \} $$

For later purposes, let us observe that the 128 massless states of $\mathcal{N} = 6$ supergravity in $D = 4$ are given by the tensor product of the 8 massless states of $\mathcal{N} = 2$ SYM (for the Left-movers) and the 16 massless states of $\mathcal{N} = 4$ SYM (for the Right-movers) viz. viz.

$$ SG_{D=4}^{N=6} = SYM_{D=4}^{N=2} \otimes SYM_{D=4}^{N=4} = \{A_\mu, 2\lambda, 2\phi \}_L \otimes \{\tilde{A}_\nu, 4\bar{\lambda}, 6\bar{\phi} \}_R $$

The hidden non-compact symmetry is $SO^\ast(12)$, of dimension 66 and rank 3 generated by $12 \times 12$ matrices of the form $Z = (Z_1, Z_2; -\bar{Z}_2, \bar{Z}_1)$ with $Z_1 = -Z_1^\dagger$ and $Z_2$ hermitean. They satisfy $L^\dagger J L = J$ with $J = -J^\dagger = -J^\dagger$ the symplectic metric in 12-d. After dualizing all massive 2-forms into axions, the $30 = 2_{NS-NS} + 12_{NS-NS} + 16_{R-R}$ scalar parameterize the moduli space

$$ \mathcal{M}_{D=4}^{N=6} = SO^\ast(12)/U(6) $$

The $16 = 8_{NS-NS} + 8_{R-R}$ vectors together with their magnetic duals transform according to the 32 dimensional chiral spinor representation of $SO^\ast(12)$.

Due to the (L-R symmetric) $Z_2$ shift, the massless spectrum receives no contribution from the twisted sector. Non-perturbative states associated to L-R asymmetric bound states of D-branes were studied in [12].

1.3 $\mathcal{N} = 5 = 1_L + 4_R$ supergravity in $D = 4$

The highest dimension where $\mathcal{N} = 5$ supergravity exists is $D = 4$. In $D = 5$ because one cannot impose a symplectic Majorana condition on an odd number of spinors. A simple way to realize $\mathcal{N} = 5 = 1_L + 4_R$ supergravity in $D = 4$ is to combine $Z_2^L \times Z_2^L$ twists, acting by T-duality along $T_{6789}^4$ and $T_{4389}^4$, with order two shifts, that eliminate massless twisted states. In [11], “minimal” $\mathcal{N} = 5$ superstring solutions of this kind have been
classified into four classes which correspond to different choices of the basis sets of free fermions or inequivalent choices of shifts in the orbifold language.

Due to the uniqueness of $\mathcal{N} = 5$ supergravity in $D = 4$, all models display the same massless spectrum

$$SG_{\mathcal{N}=5}^{D=4} = \{g_{\mu\nu}, 5\psi_\mu, 10A_\mu, 11\chi, 10\phi\}$$

For later purposes, let us observe that the 64 massless states of $\mathcal{N} = 5$ supergravity in $D = 4$ are given by the tensor product of the 4 massless states of $\mathcal{N} = 1$ SYM (for the Left-movers) and the 16 massless states of $\mathcal{N} = 4$ SYM (for the Right-movers) viz.

$$SG_{\mathcal{N}=5}^{D=4} = SYM_{\mathcal{N}=1}^{D=4} \otimes SYM_{\mathcal{N}=4}^{D=4} = \{A_\mu, \lambda\}_L \otimes \{\tilde{A}_\nu, 4\tilde{\lambda}, 6\tilde{\phi}\}_R$$

The massless scalars parameterize the moduli space

$$\mathcal{M}_{\mathcal{N}=5}^{D=4} = SU(5,1)/U(5)$$

The graviphotons together with their magnetic duals transform according to the 20 complex (3-index totally antisymmetric tensor) representation of $SU(5,1)$.

## 2 Four-graviton one-loop amplitude

Since $\mathcal{N} = 5,6$ supergravities can be obtained as asymmetric orbifolds of tori, tree-level scattering amplitudes of untwisted states such as gravitons are identical to the corresponding amplitudes in the parent $\mathcal{N} = 8$ theory. In particular, denoting by $f_{\mathcal{R}^4}^{N=5,6}(\varphi)$ the moduli dependent coefficient function of the $\mathcal{R}^4$ term, one has

$$f_{\mathcal{R}^4}^{N=5,6} = \frac{2}{n} \zeta(3) \frac{V(T^d)}{g_s^2 \ell_s^2} + \frac{\mathcal{T}^{N=8}_{d,d}}{n\ell_s^2} + ...$$

where $n$ is the order of the orbifold group, that reduces the volume of $T^d$ with $d = 5,6$ to the volume of the orbifold, $\ell_s^2 = \alpha'$ and ... stands for non-perturbative terms. The one-loop threshold integral is given by

$$\mathcal{T}^{N=8}_{d,d} = (2\pi)^d \int_F \frac{d^2\tau}{\tau_2^2} \left[ \tau_2^{d/2} \Gamma_{d,d}(G,B;\tau) - \tau_2^{d/2} \right] = 2\pi^{d-4} \Gamma \left( \frac{d}{2} - 1 \right) \mathcal{E}^{SO(d,d|Z)}_{v=2d,s=\frac{d}{4}-1}$$

where

$$\mathcal{E}^{SO(d,d|Z)}_{v=2d,s=\frac{d}{4}-1} = \sum_{\vec{m},\vec{n}:\vec{m} \cdot \vec{n} = 0} \left[(\vec{m} + B\vec{n})^i G^{-1}(\vec{m} + B\vec{n}) + \vec{n}^i G\vec{n}\right]^{-d+2}$$

is a constrained Epstein series that encodes the contribution of perturbative 1/2 BPS states i.e. those satisfying $\vec{m} \cdot \vec{n} = 0$. The subtraction eliminates IR divergences, i.e. the
terms with \( \vec{m} = \vec{n} = 0 \). For \( \mathcal{N} = 5, 6 \) the contribution of the \((r, s) = (0, 0)\) ‘un-twisted’ sector is up to a factor \( 1/n \) the same as in toroidal Type II compactifications with restricted metric \( G_{ij} \) and anti-symmetric tensor \( B_{ij} \).

In the following we will focus on the contribution of the ‘twisted’ sector\(^5\) with \((r, s) \neq (0, 0)\).

Recall that the partition function reads

\[
Z = \frac{1}{n} \sum_{r,s} \theta_1(u_{rs}) \prod_{I=1}^3 \bar{\theta}_I(u_{rs}) \eta^3 \Gamma^{[r]}
\]

where \( u_{rs} \) encode the effect of the Left-moving twist on the three complex internal directions, while \( \Gamma^{[r]} \) denote the twisted and shifted lattice sums.

Following the analysis in \([37]\) for one-loop scattering of vector bosons in unoriented D-brane worlds and exploiting the ‘factorization’ of world-sheet correlation functions one has

\[
\mathcal{A}_{4h} = \frac{1}{n} \sum_{r,s} \int \frac{d^2 \tau}{\tau_2^2} \Gamma^{[r]} \mathcal{C}_L^4 \mathcal{C}_R^4
\]

Since in both \( \mathcal{N} = 5, 6 \) cases the orbifold projection only acts by a shift of the lattice on the Left-movers, \textit{i.e.} preserves all four space-time supersymmetries, their contribution is simply

\[
\mathcal{C}_L^4 = \text{const}
\]

after summing over spin structures. In the terminology of \([37]\) only terms with 4 fermion pairs contribute. Recall that the graviton vertex in the \( q = 0 \) superghost picture reads

\[
V_h = h_{\mu\nu}(\partial X^\mu + ik\bar{\psi}\psi^\mu)(\bar{\partial} \bar{X}^\nu + ik\bar{\bar{\psi}}\psi^\nu)e^{ikX}
\]

and, for fixed graviton helicity\(^6\), one can exploit ‘factorization’ of the physical polarization tensor

\[
h^{(2\sigma)}_{\mu\nu} = a^{(\sigma)}_{\mu} a^{(\sigma)}_{\nu}
\]

in terms of photon polarization vectors.

In the R-moving sector however, the orbifold projection breaks \( 1/2 \) \((\mathcal{N} = 6)\) or \( 3/4 \) \((\mathcal{N} = 5)\) of the original four space-time supersymmetries. Correlation functions of two and three fermion bilinears will be non vanishing, too.

\(^5\)We write ‘twisted’ in quotes, since the terminology includes projections of the untwisted sector, \textit{i.e.} amplitudes with \( r = 0 \) and \( s = 1, \ldots, n - 1 \)

\(^6\)Henceforth we use \( D = 4 \) notation but the analysis is valid in \( D = 5 \) too.
For two fermion bilinears one has [37]
\[
\langle \partial X^{\mu_1} \partial X^{\mu_2} k_3 \psi \psi^{\mu_3} k_4 \psi \psi^{\mu_4} \rangle = [\eta^{\mu_1 \mu_2} \partial_1 \partial_2 G_{12} - \sum_{i \neq 1} k_i^{\mu_1} \partial_1 G_{1i} + \sum_{j \neq 2} k_j^{\mu_2} \partial_2 G_{1j}] [k_3 k_4 \eta^{\mu_3 \mu_4} - k_3^{\mu_3} k_4^{\mu_4}]
\]
(24)
where \( G_{ij} \) denotes the scalar propagator on the torus (with \( \alpha' = 2 \))
\[
G_{zw} = - \log \frac{|\theta_1(z-w)|}{|\theta_1'(0)|} - \pi \frac{Im(z-w)^2}{Im\tau}
\]
(25)
Similarly, for three fermion bilinears, one finds [37]
\[
\langle \partial X^{\mu_1} k_2 \psi \psi^{\mu_2} k_3 \psi \psi^{\mu_3} k_4 \psi \psi^{\mu_4} \rangle = \sum_{i \neq 1} k_i^{\mu_1} \partial_1 G_{1i} [k_2 k_3 k_4 \eta^{\mu_2 \mu_3 \mu_4} - \cdots] \omega_{234}
\]
(26)
with \( \omega_{234} = \partial \log \theta_1(z_{23}) + \partial \log \theta_1(z_{34}) + \partial \log \theta_1(z_{42}) \)

For four fermion bilinears, disconnected contractions yield [37]
\[
\langle k_1 \psi \psi^{\mu_1} k_2 \psi \psi^{\mu_2} k_3 \psi \psi^{\mu_3} k_4 \psi \psi^{\mu_4} \rangle_{\text{disc}} = \{ [k_1 k_2 \eta^{\mu_1 \mu_2} - k_1^{\mu_2} k_2^{\mu_1}] [k_3 k_4 \eta^{\mu_3 \mu_4} - k_3^{\mu_4} k_4^{\mu_3}] \} \times
\]
\[
(\varphi_{12} + \varphi_{34} - \Delta_{rs}) + \cdots
\]
(27)
where \( \varphi \) is Weierstrass function
\[
\varphi(z) = \frac{1}{z^2} + \sum'_{m,n} \frac{1}{(z + n + m\tau)^2} - \frac{1}{(n + m\tau)^2}
\]
\[
= -\partial_z^2 \log \theta_1(z) - 2\eta_1 = -2\partial_z^2 \tilde{G}(z, \bar{z}) - \frac{\pi^2}{3} E_2
\]
(28)
with \( \eta_1 = -\theta''_1 / 6\theta'_1 \)

Finally, connected contractions of four fermion bilinears yield [37]
\[
\langle k_1 \psi \psi^{\mu_1} k_2 \psi \psi^{\mu_2} k_3 \psi \psi^{\mu_3} k_4 \psi \psi^{\mu_4} \rangle_{\text{conn}} = [k_1^{\mu_1} k_2^{\mu_2} k_3^{\mu_3} k_4^{\mu_4}] \varphi_{12} \omega_{13} \omega_{143} + \Delta_{rs}
\]
(29)
where, for \( N = 6 \)
\[
\Delta_{rs} = \varphi(u_{rs})
\]
(30)
while, for \( N = 5 \)
\[
\Delta_{rs} = 3\eta_1 + \frac{1}{6} \frac{H''(u_{rs})}{H'(u_{rs})}
\]
(31)
with \( H'/H = \sum_i \partial \log \theta_1(u_i^T) \), which is clearly moduli independent, since no NS-NS moduli survive except for the axio-dilaton. Dependence on R-R moduli and the axio-dilaton is expected to be generated by L-R asymmetric bound-states of Euclidean D-branes and NS5-branes.
2.1 World-sheet Integrations

Worldsheet integrations can be performed with the help of \( \int d^2 z \partial_z G_{zw} = 0 = \int d^2 z \partial^2 \partial G_{zw} \) as well as of

\[
\int d^2 z d^2 w (\partial_z G_{zw})^2 = -\tau_2 \hat{E}_2 \frac{\pi^2}{3}
\]  

(32)

and

\[
\int d^2 z d^2 w [\eta^{\mu_1 \mu_2} \partial_1 \partial_2 G_{12} k_1 k_2 G_{12} - \sum_{i \neq 1} k_i^{\mu_1} \partial_1 G_{1i} \sum_{f \neq 2} k_f^{\mu_2} \partial_2 G_{1j}] = -\tau_2 \hat{E}_2 \frac{\pi^2}{3} [\eta^{\mu_1 \mu_2} k_1 k_2 - k_1^{\mu_1} k_2^{\mu_2}]
\]  

(33)

For \( \mathcal{N} = 6 = 4_L + 2_R \), setting \( f^{L/R}_{\mu \nu} = k_\mu a_\nu^{L/R} - k_\nu a_\mu^{L/R} \), one has

\[
\mathcal{L}_{\text{twist}}^{\text{eff}} = \frac{1}{n} \sum_{r,s}^\prime \int d^2 \tau \Gamma^{[3]} \langle f_1 f_2 f_3 f_4 \rangle_{L}^{\text{MHV}} \left\{ 4[(f_1 f_2)(f_3 f_4) + \ldots]_R \frac{\pi^2}{3} \hat{E}_2 + \right.
\]

\[
+ [(f_1 f_2)(f_3 f_4) + \ldots]_R \left( -2\frac{\pi^2}{3} \hat{E}_2 + \varphi(u_{rs}) \right) + [(f_1 f_2 f_3 f_4) + \ldots]_R \left( -2\frac{\pi^2}{3} \hat{E}_2 - \varphi(u_{rs}) \right) \right\}
\]

(34)

where, including all permutations,

\[
\langle f_1 f_2 f_3 f_4 \rangle^{\text{MHV}} = (f_1 f_2 f_3 f_4) + (f_1 f_3 f_4 f_2) + (f_1 f_4 f_2 f_3) - 2(f_1 f_2)(f_3 f_4) - 2(f_1 f_3)(f_4 f_2) - 2(f_1 f_4)(f_2 f_3)
\]

(35)

is the structure that appears in 4-pt vector boson amplitudes, that are necessarily MHV (Maximally Helicity Violating) in \( D = 4 \).

Combining the R-moving contributions one eventually finds

\[
\mathcal{L}_{\text{eff}}^{\text{twist}} = \langle \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 \mathcal{R}_4 \rangle_{\text{MHV}} \frac{1}{n} \sum_{r,s}^\prime \int d^2 \tau \Gamma^{[3]} \left( +2\frac{\pi^2}{3} \hat{E}_2 - \varphi(u_{rs}) \right)
\]

(36)

where \( \mathcal{R}_i \) denote the linearized Riemann tensors of the four gravitons and

\[
\langle \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 \mathcal{R}_4 \rangle_{\text{MHV}} = \langle f_1 f_2 f_3 f_4 \rangle_{L}^{\text{MHV}} \langle f_1 f_2 f_3 f_4 \rangle_{R}^{\text{MHV}}
\]

(37)

reproduces the expected \( \mathcal{R}^4 \) structure, which is MHV in \( D = 4 \), and no lower derivative \( \mathcal{R}^2 \) and/or \( \mathcal{R}^3 \) terms [20].

For \( \mathcal{N} = 5 = 4_L + 1_R \) in \( D = 4 \) one gets similar results with \( \mathcal{E}_{\mathcal{N}=2_R} = \Gamma^{[7]} \) replaced by \( \mathcal{E}_{\mathcal{N}=1_R} = \mathcal{T}_{ab} \mathcal{H}' / \mathcal{H}(\alpha' \tau_2)^{-2} \) which is moduli independent.

Henceforth we will focus on the \( \mathcal{N} = 6 = 4_L + 2_R \) case and explore NS-NS moduli dependence of the one-loop threshold in \( D = 5 \).

\[7\text{In } D = 5 \text{ there is more than one ‘helicity’, but the tensor structure has the same form [38 99].}\]
3 One-loop Threshold Integrals

One-loop threshold integrals for toroidal compactifications have been briefly reviewed above and shown to represent the contribution of the \((r, s) \neq (0, 0)\) un-twisted sector. For \((r, s) = (0, 0)\) the threshold integrals involve shifted lattice sums as in heterotic strings with Wilson lines \([46, 45, 47, 48, 49]\).

For simplicity let us discuss here the case of \(N = 6\) in \(D = 5\). For definiteness we consider \(n = 2\) \((\mathbb{Z}_2\) shift orbifold) and start at the special point in the moduli space where \(T^5 = T^4_{SO(8)} \times S^1\). Later on we will include off-diagonal moduli that effectively behave as Wilson lines.

In the ‘twisted’ \([0]_1\) sector, the relevant threshold integral is of the form

\[
\mathcal{I}_{1,5}^{N=6[0]} = (2\pi)^5 \int_{\tau} \frac{d^2\tau}{\tau^2} \Gamma_{1,1[1]}(0) \hat{O}_8 \left( \frac{2\pi^2}{3} \hat{E}_2 + \varphi(1/2) \right)
\]

\[
= (2\pi)^5 \int_{\tau} \frac{d^2\tau}{\tau^2} \frac{R}{\sqrt{\alpha'}} \sum_{m,n} e^{-|2m+(2n+1)\pi R^2/4\alpha'|\tau_2} \hat{O}_8 \left[ \frac{2\pi^2}{3} \hat{E}_2 + \varphi(1/2) \right]
\]

Setting \((2m, 2n + 1) = (2\ell + 1)(2m', 2n' + 1)\) and using invariance of \(\hat{O}_8\) under \(\tau \to \tau + 2\) allows to unfold the integral to the double strip

\[
(2\pi)^5 \frac{R}{\sqrt{\alpha'}} \int_{-1}^{+1} d\tau_1 \int_{0}^{\infty} d\tau_2 \sum_{\ell} e^{-|2\ell + 1\pi R^2/4\alpha'|\tau_2} \sum_{N,N} d_N \bar{q}_N c_N \bar{q}_N
\]

where \(\hat{O}_8 = \sum_{N=\sqrt{r^2/2}} d_N \bar{q}_N\) and \(\frac{2\pi^2}{3} \hat{E}_2 + \varphi(1/2) = \sum_{N} c_N \bar{q}_N\). Performing the trivial integral over \(\tau_1\) (level matching \(N = \bar{N}\)) and the less trivial integral over \(\tau_2\) by means of

\[
\int_{0}^{\infty} dy y^{\nu-1} e^{-cy-b/y} = (b/c)^{\nu/2} K_{\nu}(\sqrt{bc}),
\]

where \(K_\nu(z)\) is a Bessel function of second kind, finally yields

\[
\mathcal{I}_{1,5}^{N=6[0]}(R, A_i = 0) = (2\pi)^5 \left( \frac{R}{\sqrt{\alpha'}} \right)^{3/2} \sum_{\ell=0}^{\infty} \sum_{N=1}^{\infty} (2\ell+1) \sqrt{N} d_N \sigma_1(N) K_1 \left( 4\pi(2\ell + 1) \sqrt{\frac{NR}{\sqrt{\alpha'}}} \right)
\]

where

\[
\sigma_1(N) = \sum_{d|N} = \psi(N) - \psi(1) = c_N / N
\]

from the expansion of \(\hat{E}_2\) in powers of \(q\).

The result can be easily generalized to the other sectors of the \(Z_2\) orbifold under consideration as well as to different (orbifold) constructions that give rise to different
shifted lattice sums. Manifest $SO(1,5|Z)$ symmetry can be achieved turning on off-diagonal components of $B$ and $G$ (subject to restrictions). Denoting by $2A_i = G_{i5} + B_{i5}$ and observing that $G_{i5} - B_{i5} = 0$ by construction, one finds

$$I_{1,5}^{N=6}(R, A_i) = (2\pi)^5 \frac{R}{2\sqrt{\alpha'}} \sum_{\ell=0}^{\infty} \sum_{\vec{w} \in \Gamma_{[\ell]}} c_{i\ell} \int_{0}^{\infty} dy (2\ell + 1) e^{-(2\ell+1)\pi R^2/4\alpha' y} - 2\pi i \vec{w} \cdot \vec{w} + 2\pi i \vec{w} \cdot \vec{A}$$

$$= (2\pi)^5 \left( \frac{R}{\sqrt{\alpha'}} \right)^{3/2} \sum_{\ell=0}^{\infty} \sum_{\vec{w} \in \Gamma_{[\ell]}} \sigma_1 \left( \frac{\vec{w}^2}{2} \right) (2\ell + 1) \sqrt{\vec{w}^2} e^{2\pi i \vec{w} \cdot \vec{A}} K_1 \left( 4\pi (2\ell + 1) \sqrt{\frac{\vec{w}^2 R}{2\sqrt{\alpha'}}} \right)$$

(43)

Summing up the contributions of the various sectors, i.e. various shifted lattice sums, yields the complete one-loop threshold correction to the $R^4$ terms for $N = 6$ superstring vacua in $D = 5$. Clearly only NS-NS moduli (except the dilaton) appear that expose $SO(1,5)$ T-duality symmetry.

The analysis is rather more involved in $D = 4$ where one-loop threshold integrals receive contribution from trivial, degenerate and non-degenerate orbits [40, 41]. Alternative methods for unfolding the integrals over the fundamental domain have been proposed [42, 43].

Explicit computation is beyond the scope of the present investigation. It proceeds along the lines above and presents close analogy with threshold computations in $N = 2$ heterotic strings sectors in the present of Wilson lines [46, 45, 48] or, equivalently, $N = 4$ heterotic strings in $D = 8$ [44]. Rather than focussing on this interesting but rather technical aspect of the problem, let us turn our attention onto the non-perturbative dependence on the other R-R moduli as well as dilaton. This is brought about by the inclusion of asymmetric D-brane instantons.

4 Low-energy action and U-duality

In [12] the conserved charges coupling to the surviving R-R and NS-NS graviphotons, were identified as combinations of those appearing in toroidal compactifications. In the case of maximal $N = 8$ supergravity, the 12 NS-NS graviphotons couple to windings and KK momenta. Their magnetic duals to wrapped NS5-branes (H-monopoles) and KK monopoles. The 32 R-R graviphotons (including magnetic duals) couple to 6 D1-, 6 D5- and 20 D3-branes in Type IIB and to 1 D0-, 15 D2-, 15 D4- and 1 D6-branes in Type IIA.

An analogous statement applies to Euclidean branes inducing instanton effects. In toroidal compactifications with $N = 8$ supersymmetry, one has 15 kinds of worldsheet
instantons (EF1), 1 D(-1), 15 ED1, 15 ED3 and one each of EN5, ED5, EKK5 for Type IIB. For Type IIA superstrings one finds 6 ED0, 20 ED2, 6 ED4 and one each of EN5 and EKK5.

In a series of papers [23, 24], a natural proposal has been made for the non-perturbative completion of the modular form of \( E_{d+1}(Z) \) that represent the scalar dependence of the \( R^4 \) and higher derivative terms in \( \mathcal{N} = 8 \) superstring vacua. The explicit formulae are rather simple and elegant. In particular

\[
f_{R^4}^{\mathcal{N}=8}(\Phi) = \mathcal{E}^{E(d+1|Z)}_{[10^d],3/2}(\Phi) \tag{44}
\]

where \( \mathcal{E}^{E(d+1|Z)}_{[10^d],3/2}(\Phi) \) is an Einstein series of the relevant U-duality group. The above proposal satisfies a number of consistency checks including perturbative string limit i.e. small string coupling in which \( E(d + 1|Z) \to SO(d, d|Z) \) and \([10..0] \to 2d\), large radius limit in which \( E(d + 1|Z) \to E(d|Z) \) and \([10..0] \to [10..0] \) and M-theory limit in which \( E(d + 1|Z) \to SL(d + 1|Z) \) and \([10..0] \to [10..0]\)'. Moreover \( f_{R^4} \) only receives contribution from 1/2 BPS states as expected for a supersymmetric invariant that can be written as an integral over half of (on-shell) superspace.

An independent but not necessarily inequivalent proposal has been made in [25].

We expect similar results for \( R^4 \) terms in \( \mathcal{N} = 5, 6 \) superstring vacua with the following caveats. First, in \( \mathcal{N} = 5, 6 \) superspace \( R^4 \) terms are 1/5 and 1/3 BPS respectively, since they require integrations over 16 Grassman variables. Indeed we have explicitly seen that one-loop threshold correction involve the left-moving sector, in which supersymmetry is partially broken, in an essential way. Second, the U-duality group is not of maximal rank and the same applies to the T-duality subgroup, present in the \( \mathcal{N} = 6 \) case. Third, \( \mathcal{N} = 5, 6 \) only exist in \( D \leq 5 \) or \( D \leq 4 \). Some decompactification limits should produce \( \mathcal{N} = 8 \) vacua in \( D = 10 \).

Let us try and identify, the relevant 1/3 or 1/5 BPS Euclidean D-brane bound states.

### 4.1 \( \mathcal{N} = 6 \) ED-branes

In the Type IIB description, the chiral \( Z_2 \) projection (‘T-duality’) from \( \mathcal{N} = 8 \) to \( \mathcal{N} = 6 \) yields Euclidean D-brane bound states of the form

\[
D(-1) + ED3_{T^4} \quad ED1_{T^2} + ED5_{T^2 \times T^4} \quad ED1_{S^1 \times S^1} + ED3_{S^1 \times T^4} \tag{45}
\]

\[
ED1_{T^2} + ED1_{T^2} \quad ED3_{T^2 \times T^2} + ED3_{T^2 \times T^2} \tag{46}
\]

The above bound states of Euclidean D-branes are 1/3 BPS since they preserve 8 supercharges out of the 24 supercharges present in the background.
A similar analysis applies to world-sheet and ENS5 instantons.

There are several other superstring realizations of $\mathcal{N} = 6$ supergravity in $D = 4$. Given the uniqueness of the low-energy theory, they all share the same massless spectrum but the massive spectrum and the relevant (Euclidean) D-brane bound-states depend on the choice of model.

### 4.2 Non-perturbative threshold corrections

By analogy with $\mathcal{N} = 8$ one would expect $f_{R^4} = \Theta_G$ i.e. an automorphic form of the U-duality group $G$ i.e. $G = SO^*(12) (SU^*(6))$ for $\mathcal{N} = 6$ in $D = 4$ ($D = 5$) and $G = SU(5, 1)$ for $\mathcal{N} = 5$ in $D = 4$. The relevant ‘instantons’ should be associated to BPS particles in one higher dimension (when possible).

For $\mathcal{N} = 6$, in the decompactification limit the relevant decomposition under $SO^*(12) \to SU(5, 1) \times R^+$ is

$$66 \to 35_0 + 1_0 + 15_{+2} + 15'_{-2} \quad (47)$$

so that the 15 particle charges in $D = 6$ satisfy 15 1/3 BPS ‘purity’ conditions in $D = 5$

$$\frac{\partial I_3}{\partial Q^{ij}} = 0 \quad (48)$$

where $I_3^{\mathcal{N}=6,D=5} = \varepsilon_{ijklmn}Q^{[ij]}Q^{[kl]}Q^{[mn]}$. The moduli space decomposes according to

$$\frac{SO^*(12)}{U(6)} \supset \frac{SU(5, 1) \times R^{15} \times R^+}{Sp(6)} \quad (49)$$

More precisely the 15 charges decompose under $SO(1, 5)$ into a 15-dim irrep. The ‘purity’ conditions include $detQ = 0$, viewed as a $6 \times 6$ antisymmetric matrix.

For $\mathcal{N} = 6$, in the string theory limit the relevant decomposition under $SO^*(12) \to SO(2, 6) \times SL(2)_S$ is

$$32 \to (8_v, 2)_{NS-NS} + (8_s, 1)_{R-R} + (8_c, 1)_{R-R} \quad (50)$$

that yields

$$66 \to (28, 1) + (1, 3) + (8_s, 2) + (8_c, 2) + 3(1, 1) \quad (51)$$

The moduli space decomposes according to

$$\frac{SO^*(12)}{U(6)} \supset \frac{SO(6, 2)}{SO(6) \times SO(2)} \times \frac{SL(2)}{U(1)} \times R^{16} \quad (52)$$

Further decomposition under $SL(2)_T \times SL(2)_U \times SL(2)_S$ should allow to get the ‘non-Abelian’ part of the automorphic from from the ‘Abelian’ one by means of $SL(2)_{U=T} \equiv$
$SL(2)_B$. In particular the action for a (T-duality invariant) bound state of ED5 and three ED1’s into the action of EN5 and EF1’s. While the action of (T-duality invariant) bound state of ED(-1) and three ED3’s is invariant (singlet). Clearly further detailed analysis is necessary.

4.3 $N = 5$ ED-branes

In the Type IIB description, the two chiral $Z_2$ projections (‘T-duality’ on $T^4_{1234}$ and $T^4_{3456}$) from $N = 8$ to $N = 5$ yield Euclidean D-brane bound states of the form

$$D(-1) + ED3_{T^4_{1234}} + ED3_{T^4_{3456}} + ED3_{T^4_{1256}}$$ (53)
$$ED(-1)_{12} + ED5_{123456} + ED1_{34} + ED1_{56}$$ (54)
$$ED1_{i_1i_2} + ED3_{i_1j_2k_3l_4} + ED3_{j_1j_2k_3l_4} + ED1_{j_1j_2}$$ (55)
$$ED1_{i_1j_3} + ED3_{i_1j_2k_3l_4} + ED3_{j_1j_2k_3l_4} + ED1_{j_1k_3}$$ (56)
$$ED1_{i_2i_3} + ED1_{j_2j_3} + ED3_{i_1j_1j_2i_3} + ED3_{i_1j_1i_2j_3}$$ (57)

Bound states of Euclidean D-branes carrying the above charges are 1/5 BPS since they preserve 4 supercharges out of the 20 supercharges present in the background.

As in the $N = 6$ case, a different analysis applies to BPS states carrying KK momenta or windings or their magnetic duals. However, at variant with the $N = 6$, the three massive gravitini cannot form a single complex 2/5 BPS multiplet. One of them, together with its superpartners, should combined with string states which are degenerate in mass at the special rational point in the moduli space where the chiral $Z_2 \times Z_2$ projection is allowed.

5 Generating MHV amplitudes in $N = 5, 6$ SG’s

Very much like, tree-level amplitudes in $N = 8$ supergravity in $D = 4$ can be identified with ‘squares’ of tree-level amplitudes in $N = 4$ SYM theory [3, 4], tree-level amplitudes in $N = 5, 6$ supergravity in $D = 4$ can be identified with ‘products’ of tree-level amplitudes in $N = 4$ and $N = 1, 2$ SYM theory.

As previously observed, a first step in this direction is to show that the spectra of $N = 5, 6$ supergravity are simply the tensor products of the spectra of $N = 4$ and $N = 1, 2$ SYM theory.

The second step is to work in the helicity basis and focus on MHV amplitudes\(^8\). In $N = 4$ SYM the generating function for (colour-ordered) $n$-point MHV amplitudes is

\(^8\)For a recent review see e.g. [26].
given by \[50\]
\[\mathcal{F}_{MHV}^{\mathcal{N}=4\text{SYM}}(\eta_i^a, u_i^\alpha) = \frac{\delta^8(\sum_i \eta_i^a u_i^\alpha)}{\langle u_1 u_2 \rangle \langle u_2 u_3 \rangle \cdots \langle u_n u_1 \rangle}\] (58)
where \(\eta_i^a\) with \(i = 1, \ldots n\) and \(a = 1, \ldots 4\) are auxiliary Grassmann variables and \(u_i\) are commuting left-handed spinors, such that \(p_i = u_i \bar{u}_i\).

Individual amplitudes obtain by taking derivatives wrt the Grassman variables \(\eta\)'s according to the rules
\[A^+ \rightarrow 1 \quad \lambda_a^+ \rightarrow \frac{\partial}{\partial \eta^a} \quad \ldots \quad A^- \rightarrow \frac{1}{4!} \epsilon^{abcd} \frac{\partial^4}{\partial \eta^a \ldots \partial \eta^d}\] (59)
The intermediate derivatives representing scalars (\(\varphi \sim \partial^2 / \partial \eta^2\)) and right-handed gaugini (\(\lambda^- \sim \partial^3 / \partial \eta^3\)).

One can reconstruct all tree-level amplitudes, be they MHV or not, from MHV amplitudes using factorization, recursion relations or otherwise, see e.g. \[26\].

One can easily derive (super)gravity MHV amplitudes by simply taking the product of the generating functions for SYM amplitudes
\[G_{MHV}^{\mathcal{N}=8\text{SG}}(\eta_i^A, u_i^\alpha) = \frac{\mathcal{C}(u_i) \delta^{16}(\sum_i \eta_i^A u_i^\alpha)}{\langle u_1 u_2 \rangle \langle u_2 u_3 \rangle \cdots \langle u_n u_1 \rangle^2} = \mathcal{C}(u_i) \mathcal{F}_{MHV,L}^{\mathcal{N}=4\text{SYM}}(\eta_i^{aL}, u_i^\alpha) \mathcal{F}_{MHV,R}^{\mathcal{N}=4\text{SYM}}(\eta_i^{aR}, u_i^\alpha)\] (60)
where \(\eta^A = (\eta_i^{aL}, \eta_i^{aR})\) with \(A = 1, \ldots 8\) and the correction factor \(\mathcal{C}(u_i)\) is only a function of the spinors \(u_i\), actually of the massless momenta \(p_i = u_i \bar{u}_i\) \[27\].

The relevant dictionary would read
\[h^+ \rightarrow 1 \quad \psi_A^+ \rightarrow \frac{\partial}{\partial \eta_A} \quad \ldots \quad h^- \rightarrow \frac{1}{\mathcal{N}!} \frac{\partial^8}{\partial \eta^8}\] (61)

In principle one can reconstruct all tree-level amplitudes, be they MHV or not, from MHV amplitudes using factorization, recursion relations or otherwise, see e.g. \[26\]. Unitary methods allow to extend the analysis beyond tree-level. If all \(\mathcal{N} = 8\) supergravity amplitudes were expressible in terms of squares of \(\mathcal{N} = 4\) SYM amplitudes, UV finiteness of the latter would imply UV finiteness of the former. Although support to this conjecture at the level of 4-graviton amplitudes, which are necessarily MHV, seems to exclude the presence of \(\mathcal{R}^4\) corrections, which are 1/2 BPS saturated, it would be crucial to explicitly test the absence \(D^8\mathcal{R}^4\) corrections, the first that are not BPS saturated.

Going back to the problem of expressing MHV amplitudes in \(\mathcal{N} = 5, 6\) supergravities in terms of SYM amplitudes, one has to resort to ‘orbifold’ techniques.

In the \(\mathcal{N} = 6\) case, half of the 4 \(\eta\)'s (say \(\eta_2^L\) and \(\eta_4^L\)) of the ‘left’ \(\mathcal{N} = 4\) SYM factor are to be projected out i.e. ‘odd’ under a \(Z_2\) involution. As a result the generating function
is the same as in $\mathcal{N} = 8$ supergravity but the dictionary gets reduced to

$$
\begin{align*}
& h^+ \rightarrow 1 \quad \psi^+_A \rightarrow \frac{\partial}{\partial \eta^A}, \quad A_0^+ = \frac{\partial^2}{\partial \eta^3 L \partial \eta^4 L}, \quad A^+_{A'B'} = \frac{\partial^2}{\partial \eta^A \partial \eta^{B'}} \quad \cdots

& \quad h^- = \frac{1}{6!} \varepsilon^{A'_1 \ldots A'_6} \frac{\partial^{2+6}}{\partial \eta^3 L \partial \eta^4 L \partial \eta^{A'_1} \ldots \partial \eta^{A'_6}} \tag{62}
\end{align*}
$$

where $A' = 1, \ldots 6$.

Further reduction is necessary for $\mathcal{N} = 5$ case, 3 of the 4 $\eta$’s of the ‘left’ $\mathcal{N} = 4$ SYM factor are to be projected out. For instance they may acquire a phase $\omega = \exp(i2\pi/3)$ under a $Z_3$ projection.

The same projections should be implemented on the intermediate states flowing around the loops. Although tree-level amplitudes in $\mathcal{N} = 5, 6$ supergravity are simply a subset of the ones in $\mathcal{N} = 8$ supergravity, naive extension of the argument at loop order does not immediately work [52, 51, 53]. Several cancellations are not expected to take place despite the residual supersymmetry of the left SYM factor. However, in view of the recent observations on the factorization of $\mathcal{N} = 4$ SYM into a kinematical part and a group theory part, where the latter satisfies identities similar to the former [54, 55, 56] and can thus be consistently replaced with the former giving rise to consistent and UV finite $\mathcal{N} = 8$ SG amplitudes, it may well be the case that a similar decomposition can be used to produce, possibly UV finite, $\mathcal{N} = 5, 6$ SG amplitudes. Our results on $R^4$ lend some support to this viewpoint.

## 6 Conclusions

Let us summarize our results. We have shown that the first higher derivative corrections to the low-energy effective action around superstring vacua with $\mathcal{N} = 5, 6$ supersymmetry are $R^4$ terms as in $\mathcal{N} = 8$. Contrary to $\mathcal{N} \leq 4$, no $R^2$ terms appear. Relying on previous results on vector boson scattering at one-loop in unoriented D-brane worlds [37], we have studied four graviton scattering amplitudes and derived explicit formulae for the one-loop threshold corrections in asymmetric orbifolds that realize the above vacua. In addition to a term $1/n f^{\mathcal{N}=8}_{cr4}$, coming from the $(0, 0)$ sector, contributions from non-trivial sectors of the orbifold to $f^{\mathcal{N}=5,6}_{cr4}$ display a close similarity with Heterotic threshold corrections in the presence of Wilson lines [46, 45, 48]. For illustrative purposes, we have computed the relevant integrals for $\mathcal{N} = 6$ in $D = 5$ exposing the expected $SO(1,5)$ T-duality symmetry. The analysis in $D = 4$ is technically more involved and will be performed elsewhere. We have also identified the relevant 1/3 or 1/5 BPS bound states of Euclidean D-branes that contribute to the non-perturbative dependence of the thresholds on R-R scalars and on the axio-dilaton. By analogy with $\mathcal{N} = 8$ it is natural to conjecture
the possible structure of the automorphic form of the relevant U-duality group. A more
detailed analysis of this issue is however necessary. Finally, in view of the potential UV
finiteness of $\mathcal{N} = 5,6$ supergravities, we have discussed how to compute tree-level MHV
amplitudes using generating function and orbifolds techniques [27]. All other tree-level
amplitudes should follow from factorization and in fact should coincide with $\mathcal{N} = 8$
amplitudes involving only $\mathcal{N} = 5$ or $\mathcal{N} = 6$ supergravity states in the external legs. Loop
amplitudes require a separate investigation. In particular no generalization of the KLT
relations is known beyond tree level [57].

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References

[1] N. Marcus, Phys. Lett. B 157 (1985) 383.

[2] A. Ceresole and S. Ferrara, arXiv:1009.4175 [hep-th].

[3] Z. Bern, L. J. Dixon and R. Roiban, Phys. Lett. B 644, 265 (2007)
arXiv:hep-th/0611086).

[4] Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, Phys. Rev. Lett. 103, 081301 (2009) arXiv:0905.2326 [hep-th]]).

[5] K. S. Stelle, Fortsch. Phys. 57, 446 (2009).

[6] N. Beisert, H. Elvang, D. Z. Freedman, M. Kiermaier, A. Morales and S. Stieberger,
arXiv:1009.1643 [hep-th].

[7] R. Kallosh, arXiv:1009.1135 [hep-th].

[8] M. B. Green, H. Ooguri and J. H. Schwarz, Phys. Rev. Lett. 99, 041601 (2007)
arXiv:0704.0777 [hep-th]].
[9] M. Bianchi, S. Ferrara and R. Kallosh, Phys. Lett. B 690, 328 (2010) [arXiv:0910.3674 [hep-th]].

[10] M. Bianchi, S. Ferrara and R. Kallosh, JHEP 1003, 081 (2010) [arXiv:0912.0057 [hep-th]].

[11] S. Ferrara and C. Kounnas, Nucl. Phys. B 328, 406 (1989).

[12] M. Bianchi, Nucl. Phys. B 805, 168 (2008) [arXiv:0805.3276 [hep-th]].

[13] K. S. Narain, M. H. Sarmadi and C. Vafa, Nucl. Phys. B 288, 551 (1987).

[14] A. Dabholkar and J. A. Harvey, JHEP 9902, 006 (1999) [arXiv:hep-th/9809122].

[15] I. Antoniadis and C. Bachas, Nucl. Phys. B 298, 586 (1988).

[16] I. Antoniadis, C. P. Bachas and C. Kounnas, Nucl. Phys. B 289, 87 (1987).

[17] H. Kawai, D. C. Lewellen and S. H. H. Tye, Nucl. Phys. B 288, 1 (1987).

[18] H. Kawai, D. C. Lewellen and S. H. H. Tye, Phys. Rev. D 34, 3794 (1986).

[19] H. Kawai, D. C. Lewellen and S. H. H. Tye, Phys. Rev. Lett. 57, 1832 (1986) [Erratum-ibid. 58, 429 (1987)].

[20] J. Broedel and L. J. Dixon, JHEP 1005, 003 (2010) [arXiv:0911.5704 [hep-th]].

[21] M. B. Green and M. Gutperle, Nucl. Phys. B 498, 195 (1997) [arXiv:hep-th/9701093].

[22] K. Becker, M. Becker and A. Strominger, Nucl. Phys. B 456, 130 (1995) [arXiv:hep-th/9507158].

[23] M. B. Green, J. G. Russo and P. Vanhove, Phys. Rev. D 81, 086008 (2010) [arXiv:1001.2535 [hep-th]].

[24] M. B. Green, S. D. Miller, J. G. Russo and P. Vanhove, [arXiv:1004.0163 [hep-th]].

[25] B. Pioline, JHEP 1003, 116 (2010) [arXiv:1001.3647 [hep-th]].

[26] see e.g. J. M. Drummond, [arXiv:1010.2418 [hep-th]].

[27] M. Bianchi, H. Elvang and D. Z. Freedman, JHEP 0809, 063 (2008) [arXiv:0805.0757 [hep-th]].

[28] J. Bagger and N. Lambert, Phys. Rev. D 75, 045020 (2007) [arXiv:hep-th/0611108].

[29] A. Gustavsson, Nucl. Phys. B 811, 66 (2009) [arXiv:0709.1260 [hep-th]].
[30] O. Aharony, O. Bergman and D. L. Jafferis, JHEP 0811, 043 (2008) [arXiv:0807.4924 [hep-th]].

[31] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, JHEP 0810, 091 (2008) [arXiv:0806.1218 [hep-th]].

[32] K. Hosomichi, K. M. Lee, S. Lee, S. Lee, J. Park and P. Yi, JHEP 0811, 058 (2008) [arXiv:0809.1771 [hep-th]].

[33] A. Cagnazzo, D. Sorokin and L. Wulff, JHEP 1005, 009 (2010) [arXiv:0911.5228 [hep-th]].

[34] M. Bianchi, R. Poghossian and M. Samsonyan, JHEP 1010, 021 (2010) [arXiv:1005.5307 [hep-th]].

[35] N. Drukker, M. Marino and P. Putrov, [arXiv:1007.3837] [hep-th].

[36] R. D’Auria, S. Ferrara and C. Kounnas, Phys. Lett. B 420, 289 (1998) [arXiv:hep-th/9711048].

[37] M. Bianchi and A. V. Santini, JHEP 0612, 010 (2006) [arXiv:hep-th/0607224].

[38] M. B. Green, J. H. Schwarz and E. Witten, Cambridge, Uk: Univ. Pr. (1987) 596 P. (Cambridge Monographs On Mathematical Physics)

[39] M. B. Green, J. H. Schwarz and E. Witten, Cambridge, Uk: Univ. Pr. (1987) 469 P. (Cambridge Monographs On Mathematical Physics)

[40] L. J. Dixon, V. Kaplunovsky and J. Louis, Nucl. Phys. B 355, 649 (1991).

[41] P. Mayr, H. P. Nilles and S. Stieberger, Phys. Lett. B 317, 53 (1993) [arXiv:hep-th/9307171].

[42] M. Trapletti, JHEP 0302, 012 (2003) [arXiv:hep-th/0211281].

[43] M. Cardella, JHEP 0905, 010 (2009) [arXiv:0812.1549 [hep-th]].

[44] W. Lerche, S. Stieberger and N. P. Warner, Adv. Theor. Math. Phys. 3, 1575 (1999) [arXiv:hep-th/9811228].

[45] E. Kiritsis and N. A. Obers, JHEP 9710, 004 (1997) [arXiv:hep-th/9709058].

[46] C. Bachas, C. Fabre, E. Kiritsis, N. A. Obers and P. Vanhove, Nucl. Phys. B 509, 33 (1998) [arXiv:hep-th/9707126].
[47] M. Bianchi, E. Gava, F. Morales and K. S. Narain, Nucl. Phys. B 547, 96 (1999) [arXiv:hep-th/9811013].

[48] N. A. Obers and B. Pioline, Fortsch. Phys. 49 (2001) 359 [arXiv:hep-th/0101122].

[49] M. Bianchi and J. F. Morales, JHEP 0802, 073 (2008) [arXiv:0712.1895 [hep-th]].

[50] V. P. Nair, Phys. Lett. B 214 (1988) 215.

[51] Z. Bern, J. J. M. Carrasco and H. Johansson, arXiv:1007.4297 [hep-th].

[52] P. Katsaroumpas, B. Spence and G. Travaglini, JHEP 0908, 096 (2009) [arXiv:0906.0521 [hep-th]].

[53] J. M. Drummond, P. J. Heslop and P. S. Howe, arXiv:1008.4939 [hep-th].

[54] H. Tye and Y. Zhang, arXiv:1007.0597 [hep-th].

[55] S. H. Henry Tye and Y. Zhang, JHEP 1006, 071 (2010) [arXiv:1003.1732 [hep-th]].

[56] Z. Bern, T. Dennen, Y. t. Huang and M. Kiermaier, Phys. Rev. D 82 (2010) 065003 [arXiv:1004.0693 [hep-th]].

[57] H. Kawai, D. C. Lewellen and S. H. H. Tye, Nucl. Phys. B 269, 1 (1986).