On the Polynomial Solution of Divided-Difference Equations of the Hypergeometric Type on Nonuniform Lattices

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Received: 31 January 2019; Accepted: 10 April 2019; Published: 21 April 2019

Abstract: In this paper, we provide a formal proof of the existence of a polynomial solution of fixed degree for a second-order divided-difference equation of the hypergeometric type on non-uniform lattices, generalizing therefore previous work proving existence of the polynomial solution for second-order differential, difference or \(q\)-difference equation of hypergeometric type. This is achieved by studying the properties of the mean operator and the divided-difference operator as well as by defining explicitly, the right and the “left” inverse for the second operator. The method constructed to provide this formal proof is likely to play an important role in the characterization of orthogonal polynomials on non-uniform lattices and might also be used to provide hypergeometric representation (when it does exist) of the second solution—non polynomial solution—of a second-order divided-difference equation of hypergeometric type.

Keywords: second-order differential/difference/\(q\)-difference equation of hypergeometric type; non-uniform lattices; divided-difference equations; polynomial solution

1. Introduction

Classical orthogonal polynomials of a continuous variable \((P_n)\) are known to satisfy a second-order differential equation of hypergeometric type

\[
\sigma(x) y''(x) + \tau(x) y'(x) + \lambda y(x) = 0,
\]

where \(\sigma\) is a polynomial of degree at most 2, \(\tau\) is a first degree polynomial and \(\lambda\) is a constant with respect to \(x\).

In [1,2], it is shown that Equation (1) has a polynomial solution of exactly \(n\) degree for a specific given constant \(\lambda = \lambda_n\). This is achieved mainly by showing that:

- the \(n\)th derivative \(y^{(n)}\) of any solution \(y\) of (1) satisfies an equation of the same type (hypergeometric aspect), that is, an equation of the form

\[
\sigma(x) y''(x) + \tau_n(x) y'(x) + \lambda_n y(x) = 0,
\]
where $\tau_n$ is a first degree polynomial and $\lambda_n$ is a constant given by

$$
\tau_n(x) = \tau + n\sigma'(x), \quad \lambda_n = \lambda + n\tau' + \frac{n(n-1)}{2}\sigma''.
$$

Any solution of (2) can be written as the $n$th derivative of a solution of (1), provided that $\lambda_j \neq 0$, $j = 1,...,n-1$.

The fact that the constant solution of (2) when $\lambda_n = 0$ is the $n$th derivative of a solution of (1) leads to the existence of a polynomial solution of (1), of exactly $n$ degree, when

$$
\lambda = \lambda_n = -n\tau' - \frac{n(n-1)}{2}\sigma''.
$$

This result proves not only the existence of a polynomial solution for Equation (1) but also allows for establishing the Rodrigues formula expressing the polynomial solution in the term of the $n$th derivative:

$$
P_n(x) = \frac{B_n}{\rho(x)}\left[\sigma^n(x)\rho(x)\right]^{(n)},
$$

where $B_n$ is a constant and $\rho$ is the weight function satisfying the Pearson equation

$$
(\sigma(x)\rho(x))' = \tau(x)\rho(x).
$$

It is worth mentioning that Hermite, Laguerre, Jacobi and Bessel polynomials are the polynomial eigenfunctions of the second-order linear differential operation given in (1).

Using the same approach, similar results have been established in [3] (See also [4]) for the classical orthogonal polynomials of a discrete variable satisfying instead a second-order difference equation of hypergeometric type

$$
\sigma(x)\Delta\nabla y(x) + \tau(x)\Delta y(x) + \lambda_n y(x) = 0,
$$

where $\Delta$ and $\nabla$ are the forward and the backward operators defined by

$$
\Delta f(s) = f(s+1) - f(s), \quad \nabla f(s) = f(s) - f(s-1).
$$

Furthermore, it should be noticed that Charlier, Krawtchuk, Meixner and Hahn polynomials are the polynomial eigenfunctions of the second-order linear difference operation given in (4).

The same result can be established in the same way to for the classical orthogonal polynomials of a $q$-discrete variable satisfying a second-order $q$-difference equation of hypergeometric type [5] (See also [6,7])

$$
\sigma(x)D^2_q y(x) + \tau(x)D_q y(x) + \lambda_n y(x) = 0,
$$

where $D_q$ is the Hahn operator [8] defined by

$$
D_q(f(x)) = \frac{f(qx) - f(x)}{(q-1)x}, \quad x \neq 0, \quad D_q f(0) := f'(0),
$$

provided that $f'(0)$ exists. Orthogonal polynomials which are eigenfunctions of the second-order $q$-difference operator given defined by (5) are [5]: Big $q$-Jacobi, Big $q$-Laguerre, Little $q$-Jacobi, Little $q$-Laguerre (Wall), q-Laguerre, Alternative q-Charlier, Al-Salam–Carlitz I, Al-Salam–Carlitz II, Stieltjes–Wigert, Discrete q–Hermite I, Discrete q–Hermite II, q-Hahn, q-Meixner, Quantum q-Krawtchouk, q-Krawtchouk, Affine q-Krawtchouk, the q-Charlier and the q-Charlier II polynomials.
Classical orthogonal polynomials on non-uniform lattices (including but not limited to Askey–Wilson polynomials, Racah and \(q\)-Racah polynomials), are known to satisfy a second-order divided-difference equation of the form\(^{[9,10]}\) (see also\(^{[11]}\))

\[ \frac{\Delta}{\Delta x(s - \frac{1}{2})} \left[ \frac{\nabla y(x(s))}{\nabla x(s)} \right] + \frac{\psi(x(s))}{2} \left[ \frac{\nabla y(x(s))}{\Delta x(s)} + \frac{\nabla y(x(s))}{\nabla x(s)} \right] + \lambda_n y(x(s)) = 0, \tag{6} \]

where \(\psi\) and \(\phi\) are polynomials of degree 1 and at most 2, respectively; \(\lambda_n\) is a constant depending on \(n\) and on the leading coefficients of \(\phi\) and \(\psi\). The lattice \(x(s)\) is defined by\(^{[9,10]}\)

\[ x(s) = \begin{cases} 
    c_1 q^{-s} + c_2 q^s + c_3 & \text{if } q \neq 1, \\
    c_4 s^2 + c_5 s + c_6 & \text{if } q = 1,
\end{cases} \tag{7} \]

is known as non-uniform lattice and fulfills various important properties.

Equation (6) can be transformed into equation\(^{[12]}\)

\[ \phi(x(s)) \frac{\Delta}{\Delta x(s)} y(x(s)) + \psi(x(s)) \frac{\nabla}{\nabla x(s)} y(x(s)) + \lambda_n y(x(s)) = 0, \tag{8} \]

called divided-difference equation of the hypergeometric type by means of the two companion operators \(\mathbb{D}_x\) (called divided-difference operator) and \(\mathbb{S}_x\) (called mean operator) defined as\(^{[9,10,12,13]}\)

\[ \mathbb{D}_x f(x(s)) = \frac{f(x(s + \frac{1}{2})) - f(x(s - \frac{1}{2}))}{x(s + \frac{1}{2}) - x(s - \frac{1}{2})}, \quad \mathbb{S}_x f(x(s)) = \frac{f(x(s + \frac{1}{2})) + f(x(s - \frac{1}{2}))}{2}. \tag{9} \]

Using appropriate bases, computer algebra software has been used to solve divided-difference Equation (8) for specific families of classical orthogonal polynomials on a non-uniform lattice. For some special values of the parameter for the specific case of Askey–Wilson polynomials, non-polynomial solution has been recovered together with the polynomial one\(^{[14]}\) (see page 15, Equations (62) and (63)). In addition, the operators \(\mathbb{D}_x\) and \(\mathbb{S}_x\) have played a decisive role not only for establishing the functional approach of the characterization theorem of classical orthogonal polynomials on non-uniform lattices, but also for providing algorithmic solution to linear homogeneous divided-difference equations with polynomial coefficients, allowing to solve explicitly\(^{[13]}\) the first-order divided-difference equations satisfied by the basic exponential function

\[ \mathbb{D}_x y(x(s)) = \frac{2wq^1}{1 - q} y(x(s)), \]

and the second-order divided-difference equation satisfied by the basic trigonometric functions

\[ \mathbb{D}_x^2 y(x(s)) = -\left( \frac{2wq^1}{1 - q} \right)^2 y(x(s)), \]

where \(w\) is a given constant.

The aim of this work is:

1. to define the right and the “left” inverses of the operator \(\mathbb{D}_x\);
2. to provide a formal proof of the existence of a polynomial solution of a preassigned degree of the divided-difference equation of hypergeometric type (8), extending and generalising therefore—by means of specialisation and limiting situations on the lattice \(x(s)\)—similar results obtained for second-order differential, difference or \(q\)-difference equation of hypergeometric type.
2. Preliminary Results: Known and New Properties

Since the main result of this paper is based on the operators $\mathbb{D}_x$ and $\mathbb{S}_x$ which are defined by using the lattice $x(s)$, we will provide in this section some known and basic properties of $x(s)$, $\mathbb{D}_x$ and $\mathbb{S}_x$. We will also derive new properties such as the right and the “left” inverses of the operator $\mathbb{D}_x$, required in the next section.

2.1. Known Properties of the Lattice $x(s)$

Taking into account the notation

$$x_\mu(s) = x(s + \frac{\mu}{2}),$$

the non-uniform lattice $x(s)$ defined by Equation (7) satisfies

$$\begin{align*}
\frac{x(s + k) - x(s)}{2} &= \gamma_k \nabla x_{k+1}(s), \\
\frac{x(s + k) + x(s)}{2} &= a_k x_k(s) + \beta_k,
\end{align*}$$

for $k = 0, 1, \ldots$, with

$$\alpha_0 = 1, \alpha_1 = \alpha, \beta_0 = 0, \beta_1 = \beta, \gamma_0 = 0, \gamma_1 = 1,$$

where the sequences $(\alpha_k)$, $(\beta_k)$, $(\gamma_k)$ satisfy the following relations:

$$\begin{align*}
\alpha_{k+1} - 2 \alpha \alpha_k + \alpha_{k-1} &= 0, \\
\beta_{k+1} - 2 \beta \beta_k + \beta_{k-1} &= 2 \beta \alpha_k, \\
\gamma_{k+1} - \gamma_{k-1} &= 2 \alpha_k,
\end{align*}$$

and are given explicitly by $[9,10]$

$$a_n = 1, \beta_n = \beta n^2, \gamma_n = n, \text{ for } n = 1,$$

and

$$a_n = \frac{q \alpha^2 + q^{-2}}{2}, \beta_n = \frac{\beta(1 - \alpha_n)}{1 - \alpha}, \gamma_n = \frac{q^2 - q^{-2}}{q^\frac{2}{\alpha^2} - q^{-\frac{2}{\alpha^2}}}, \text{ for } \alpha = \frac{q^\frac{2}{\alpha^2} + q^{-\frac{2}{\alpha^2}}}{2}.\quad (15)$$

2.2. Known Properties of the Operators $\mathbb{D}_x$ and $\mathbb{S}_x$

The operators $\mathbb{S}_x$ and $\mathbb{D}_x$ fulfil the so-called Product rules I $[13,14]$: 

$$\begin{align*}
\mathbb{D}_x (f(x(s))g(x(s))) &= \mathbb{S}_x f(x(s)) \mathbb{D}_x g(x(s)) + \mathbb{D}_x f(x(s)) \mathbb{S}_x g(x(s)), \\
\mathbb{S}_x (f(x(s))g(x(s))) &= U_2(x(s)) \mathbb{D}_x f(x(s)) \mathbb{D}_x g(x(s)) + \mathbb{S}_x f(x(s)) \mathbb{S}_x g(x(s)),
\end{align*}$$

where $U_2$ is a polynomial of degree 2

$$U_2(x(s)) = (\alpha^2 - 1) x^2(s) + 2 \beta (\alpha + 1) x(s) + \eta_x,$$

and $\eta_x$ is a constant given by $[14]$

$$\eta_x = \frac{x^2(0) + x^2(1) - (2\alpha^2 - 1) x(0) x(1) - \beta (\alpha + 1) \alpha^2 (x(0) + x(1)) + \beta^2 (\alpha + 1)^2}{2\alpha^2}. \quad (19)$$

The operators $\mathbb{D}_x$ and $\mathbb{S}_x$ also satisfy the so-called Product Rules II $[13,14]$: 

$$\begin{align*}
\mathbb{D}_x \mathbb{S}_x &= a \mathbb{S}_x \mathbb{D}_x + U_1(x(s)) \mathbb{D}_x^2, \\
\mathbb{S}_x^2 &= U_1(x(s)) \mathbb{S}_x \mathbb{D}_x + a U_2(x(s)) \mathbb{D}_x^2 + I,
\end{align*}$$

$$a \mathbb{S}_x \mathbb{D}_x + U_1(x(s)) \mathbb{D}_x^2, \quad a U_2(x(s)) \mathbb{D}_x^2 + I.$$
where $I$ is the identity operator $I f(x) = f(x)$, and

$$U_1(x(s)) = (a^2 - 1) x(s) + \beta (a + 1).$$

2.3. An Appropriate Basis for the Operators $D_x$ and $S_x$

Searching for a polynomial basis on which the action of the companion operators will give a linear combination of at most two elements of the basis, Foupouagnigni et al. proved in [14] that the polynomial $F_n$ defined by

$$F_n(x(s)) = F_n(x(s), x(\epsilon)),$$  \hspace{1cm} (21)

where $\epsilon$ is the unique solution (provided that the lattice $x(s)$ is quadratic or $q$-quadratic: i.e., the constants $c_j$ in (7) satisfy $c_1 c_2 \neq 0$ or $c_4 \neq 0$) in the variable $t$ of the equation $x_1(t) = x(t)$, is the right basis for the operators $S_x$ and $D_x$ because it satisfies the following properties

$$D_x F_n(x(s)) = \gamma_n F_{n-1}(x(s)),$$  \hspace{1cm} (22)

$$S_x F_n(x(s)) = a_n F_n(x(s)) + \frac{\gamma_n}{2} \nabla x_{n+1}(z_x) F_{n-1}(x(s)),$$  \hspace{1cm} (23)

where the constants $a_n$ and $\gamma_n$ are given in (14) and (15).

After reviewing some properties of the operators $D_x$ and $S_x$, we now state and prove the following proposition providing the left and right inverse of the operator $D_x$, to be used in the next section to complete the proof of the main theorem of this paper.

Proposition 1.

Let $F_x$ be a linear operator defined on the basis $(F_n)_n$ by

$$F_x F_n = \frac{F_{n+1}}{\gamma_{n+1}}, \ n \geq 0, \ F_x 0 := 0.$$  \hspace{1cm} (24)

Then, $F_x$ satisfies the following relations:

$$D_x F_x = I, \ F_x D_x = I - \delta_{x(e)},$$  \hspace{1cm} (25)

where $I$ is the identity operator and $\delta_{x(e)}$ is the Dirac delta distribution defined by

$$\langle \delta_{x(e)}, P \rangle = P(x(\epsilon)), \ \forall P,$$

with $\epsilon$ is defined in (21).

Proof. For all positive integer $n$, $F_n$ defined by (21) is a polynomial of degree exactly $n$. $(F_n)$ is therefore a basis of $C[x]$. Letting $f \in C_n[x]$, there exist $f_0, \ldots, f_n \in C[x]$ such that

$$f(x(s)) = \sum_{j=0}^{n} f_j F_j(x(s)).$$

We have

$$\langle \delta_{x(e)}, f \rangle = f(x(\epsilon)) = \sum_{j=1}^{n} f_j F_j(x(\epsilon)) + f_0 = f_0.$$  \hspace{1cm} (26)
\[
\mathcal{D}_x \mathcal{F}_x f(x(s)) = \sum_{j=0}^{n} f_j \mathcal{D}_x \mathcal{F}_j (x(s))
\]
\[
= \sum_{j=0}^{n} f_j \mathcal{D}_x \mathcal{F}_{j+1} (x(s)) / \gamma_{j+1}
\]
\[
= \sum_{j=0}^{n} f_j \mathcal{F}_j (x(s)) = f(x(s)).
\]

Hence, the first part of Relation (24) holds. Using (26), we have
\[
\mathcal{F}_x \mathcal{D}_x f(x(s)) + \langle \delta, f \rangle = \mathcal{F}_x \mathcal{D}_x f(x(s)) + f_0
\]
\[
= \sum_{j=1}^{n} f_j \mathcal{F}_j (x(s)) + f_0
\]
\[
= \sum_{j=1}^{n} f_j \mathcal{F}_j (x(s)) + f_0
\]
\[
= f(x(s)).
\]

The second part of (25) is therefore satisfied. \(\square\)

3. Existence of the Polynomial Solution of the Divided-Difference Equation of the Hypergeometric Type

Having stated and proved required properties of the operators \(\mathcal{D}_x\) and \(\mathcal{S}_x\), we will now state and prove the main theorem of this paper.

**Theorem 1.** Let \(n\) be a nonnegative integer, \(\psi\) and \(\phi\) be two polynomials of degree 1 and at most 2, respectively, such that
\[
\forall k \in \mathbb{N}, \eta_k := \phi_2 \gamma_k + \psi_1 \alpha_k \neq 0. \tag{27}
\]
Then, the divided-difference equation
\[
\phi(x(s)) \mathcal{D}_x^2 y(x(s)) + \psi(x(s)) \mathcal{S}_x \mathcal{D}_x y(x(s)) + \lambda_{n,0} y(x(s)) = 0, \tag{28}
\]
with
\[
\lambda_{n,0} = -\gamma_n (\phi_2 \gamma_{n-1} + \psi_1 \alpha_{n-1}) = -\gamma_n \eta_{n-1}, \tag{29}
\]
where \(\psi_1\) and \(\phi_2\) are leading coefficients of polynomials \(\psi\) and \(\phi\) respectively, has a polynomial solution of exactly \(n\) degree.

The proof of Theorem 1 will be organized as follows: we split the proof in five lemmas which we first state, prove, and then put these lemmas together in combination with Proposition 1 to deduce the proof of this theorem.

**Lemma 1.** If the function \(y_0\) is a solution of (28), then the function \(y_1 = \mathcal{D}_x y_0\) satisfies
\[
\phi^{[1]}(x(s)) \mathcal{D}_x^2 y(x(s)) + \psi^{[1]}(x(s)) \mathcal{S}_x \mathcal{D}_x y(x(s)) + \lambda_{n,1} y(x(s)) = 0, \tag{30}
\]
where
\[
\begin{aligned}
\phi^{[1]}(x(s)) &= S_x \phi(x(s)) + U_2(x(s)) D_x \psi + U_1(x(s)) S_x \psi(x(s)), \\
\psi^{[1]}(x(s)) &= D_x \phi(x(s)) + U_1(x(s)) D_x \psi + \alpha S_x \psi(x(s)), \\
\lambda_{n,1} &= \lambda_{n,0} + D_x \psi.
\end{aligned}
\] (31)

**Proof.** Assume that \( y_0 \) satisfies (28). Applying the operator \( D_x \) to (28) in which \( y \) is replaced by \( y_0 \) and using the product rule I in (16) and (17), we obtain
\[
D_x \phi(x(s)) S_x D_x^2 y_0(x(s)) + S_x \phi(x(s)) D_x^3 y_0(x(s)) + D_x \psi(x(s)) S_x^2 D_x y_0(x(s)) + S_x \psi(x(s)) D_x S_x D_x y_0(x(s)) + \lambda_{n,0} D_x y_0(x(s)) = 0.
\]
Using the product rules II in (20) to replace \( S_x^2 \) and \( D_x S_x \) in the previous equation, we have
\[
\phi^{[1]}(x(s)) D_x^3 y_0(x(s)) + \psi^{[1]}(x(s)) S_x D_x y(x(s)) + \lambda_{n,1} D_x y_0(x(s)) = 0,
\]
where \( \phi^{[1]}(x(s)), \psi^{[1]}(x(s)) \) and \( \lambda_{n,1} \) are defined by (31). Therefore, \( y_1 = D_x y_0 \) is a solution of Equation (30).

**Lemma 2.**
If the function \( y_0 \) is a solution of (28), then the function \( y_k = D_x^k y_0 \) is a solution of the equation
\[
\phi^{[k]}(x(s)) D_x^2 y(x(s)) + \psi^{[k]}(x(s)) S_x D_x y(x(s)) + \lambda_{n,k} y(x(s)) = 0,
\]
(32)
where the polynomials \( \phi^{[k]} \), \( \psi^{[k]} \) and the constant \( \lambda_{n,k} \) satisfy
\[
\begin{aligned}
\phi^{[k+1]}(x(s)) &= S_x \phi^{[k]}(x(s)) + U_2(x(s)) D_x \psi^{[k]} + U_1(x(s)) S_x \psi^{[k]}(x(s)), \\
\psi^{[k+1]}(x(s)) &= D_x \phi^{[k]}(x(s)) + U_1(x(s)) D_x \psi^{[k]} + \alpha S_x \psi^{[k]}(x(s)), \\
\lambda_{n,k+1} &= \lambda_{n,k} + D_x \psi^{[k]},
\end{aligned}
\] (33)
with the following initial values: \( \phi^{[0]} := \phi, \psi^{[0]} := \psi \).

**Proof.** Lemma 1 assures the validity of the result for \( k = 1 \).
Let \( k \) be a positive integer. Assume that \( y_k \) is solution of Equation (32). Applying the operator \( D_x \) to (32) in which \( y \) is replaced by \( y_k \) and using the Product Rules I, we obtain
\[
D_x \phi^{[k]}(x(s)) S_x D_x^2 y_k(x(s)) + S_x \phi^{[k]}(x(s)) D_x^3 y_k(x(s)) + D_x \psi^{[k]}(x(s)) S_x^2 D_x y_k(x(s)) + S_x \psi^{[k]}(x(s)) D_x S_x D_x y_k(x(s)) + \lambda_{n,k} D_x y_k(x(s)) = 0.
\]
Using the products rule II to replace \( S_x^2 \) and \( D_x S_x \) in the previous equation, we have
\[
\phi^{[k+1]}(x(s)) D_x^3 y_k(x(s)) + \psi^{[k+1]}(x(s)) S_x D_x y_k(x(s)) + \lambda_{n,k+1} D_x y_k(x(s)) = 0,
\]
where \( \phi^{[k+1]}(x(s)), \psi^{[k+1]}(x(s)) \) and \( \lambda_{n,k+1} \) are defined by (33). Thus, \( y_{k+1} = D_x y_k \) satisfies
\[
\phi^{[k+1]}(x(s)) D_x^2 y_{k+1}(x(s)) + \psi^{[k+1]}(x(s)) S_x D_x y_{k+1}(x(s)) + \lambda_{n,k+1} y_{k+1}(x(s)) = 0.
\]

**Lemma 3.**
If a given function \( y_1 \) satisfies (30) with \( \lambda_{n,0} \neq 0 \), then there exists a function \( y_0 \) satisfying (28) such that
\[
y_1 = D_x y_0.
\] (34)
Therefore, for any positive integer $n$, the coefficients from (39).

Solving this system of recurrence equations with the initial values $y_1 = D_x v_0$, then from (28) we can express $v_0$ as:

$$v_0(x(s)) = -\frac{1}{\lambda_{n,0}} \left[ \phi(x(s)) D_x y_1(x(s)) + \psi(x(s)) S_x y_1(x(s)) \right].$$

(35)

Now, it remains to verify that the function $v_0$ defined in terms of $y_1$ by (35) satisfies Equation (28) with $D_x v_0 = y_1$.

By applying $D_x$ to (35) and using product rules I, II and the fact that $y_1$ is solution of (30), we get

$$-\lambda_{n,0} D_x v_0(x(s)) = D_x [\phi(x(s)) D_x y_1(x(s)) + \psi(x(s)) S_x y_1(x(s))]$$

$$= \phi^{(1)}(x(s)) D_x^2 y_1(x(s)) + \psi^{(1)}(x(s)) S_x D_x y_1(x(s)) + (\lambda_{n,1} - \lambda_{n,0}) y_1(x(s))$$

$$= -\lambda_{n,0} y_1(x(s)).$$

Therefore, $D_x v_0 = y_1$ since $\lambda_{n,0} \neq 0$.

We prove that $v_0$ is solution of (28) by replacing $y_1$ in the Equation (35) by $D_x v_0$.

**Lemma 4.**

For any positive integer $n$, the coefficients $\lambda_{n,k}$ defined by relation (33) satisfy

$$\lambda_{n,k} = \lambda_{n,0} - \lambda_{k,0}, \quad 0 \leq k \leq n,$$

$$\lambda_{n,k} \neq 0, \quad \text{for } 0 \leq k \leq n - 1, \text{ and } (n, k) \neq (0, 0),$$

where

$$\lambda_{n,0} = -\gamma_n (\phi_2 \gamma_{n-1} + \psi_1 \alpha_{n-1}).$$

**Proof.** If we denote by $\phi^{(k)}(x(s)) = \phi_2^{(k)} F_2(x(s)) + \phi_1^{(k)} F_1(x(s)) + \phi_0^{(k)}$ and $\psi^{(k)}(x(s)) = \psi_1^{(k)} F_1(x(s)) + \psi_0^{(k)}$, then from (33), we have the following system of recurrence equation

$$\left\{ \begin{align*}
\phi_2^{(k+1)} &= a_2 \phi_2^{(k)} + \alpha \gamma_1 (\alpha^2 - 1) \psi_1^{(k)} + a_1 (\alpha^2 - 1) \psi_1^{(k)}, \\
\psi_1^{(k+1)} &= \gamma_2 \phi_2^{(k)} + (\alpha^2 - 1) \gamma_1 \psi_1^{(k)} + \alpha \psi_1^{(k)}, \\
\lambda_{n,k+1} &= \lambda_{n,k} + \psi_1^{(k)}. 
\end{align*} \right.$$

Using relations

$$a_2 = 2 \alpha^2 - 1, \quad \gamma_2 = 2 \alpha,$$

derived from Equations (12) and (13), the previous system of equations becomes

$$\left\{ \begin{align*}
\phi_2^{(k+1)} &= (2 \alpha^2 - 1) \phi_2^{(k)} + 2 \alpha (\alpha^2 - 1) \psi_1^{(k)}, \\
\psi_1^{(k+1)} &= 2 \alpha \phi_2^{(k)} + (2 \alpha^2 - 1) \psi_1^{(k)}, \\
\lambda_{n,k+1} &= \lambda_{n,k} + \psi_1^{(k)}. 
\end{align*} \right.$$

Solving this system of recurrence equations with the initial values $\phi_2^{(0)} = \phi_2, \psi_1^{(0)} = \psi_1$, we obtain for the $q$-quadratic lattice

$$\lambda_{n,k} = \frac{(q^k - q^n)}{(q - 1)^2 q^k q^n} \left[ \sqrt{q} (q^k q^n - q) \phi_2 + \frac{1}{2} (q - 1)(q + q^k q^n) \psi_1 \right].$$

(39)

Using the definition of $\lambda_{n,0}$ which of course coincides with the one of $\lambda_{n,k}$ for $k = 0$, we derive (37) from (39).
Solving the following the equation
\[ \lambda_{n,k} = 0, \]
in terms of the unknown \( k \) keeping in mind (27), gives a unique solution \( k = n \). Thus, relation (38) is satisfied. It can easily be proved in the same way that relation (38) is satisfied for the quadratic lattice. □

**Lemma 5.**
Let \( n \) be a fixed positive integer and let \( k \) be an integer such that \( 0 \leq k \leq n \). Then, if \( y_k \) is a solution of Equation (32), then there exists \( y_0 \) solution of Equation (28) such that
\[ y_k = D_x^k y_0. \]

**Proof.** Let \( k \) be a nonnegative integer with \( k \leq n \). Assume that \( y_k \) satisfies (32). Then, we obtain that there exists a function \( y_{k-1} \) solution of the equation obtained by replacing \( k \) in (32) by \( k - 1 \), namely,
\[ \phi^{(k-1)}(x(s)) D_x^2 y(x(s)) + \psi^{(k-1)}(x(s)) S_x D_x y(x(s)) + \lambda_{n,k-1} y(x(s)) = 0, \]
(40)
such that
\[ y_k = D_x y_{k-1}. \]
This is achieved using the fact that \( \lambda_{n,k-1} \neq 0 \) thanks to Lemma 4, and also using Lemma 3 but with the functions \( y_0 \) and \( y_1 \) replaced, respectively, by the functions \( y_{k-1} \) and \( y_k \) while Equation (28) is replaced by Equation (40). In addition, Equation (30) is replaced by the Equation (32). □

The proof is completed by repeating the same process for \( y_{k-1}, y_{k-2}, \ldots, y_1 \) and using Lemmas 3 and 4.

**Proof of Theorem 1.** Since, for \( k = n \), \( \lambda_{n,k} = \lambda_{n,n} = 0 \) thanks to (37), Equation (32) admits a constant solution, namely \( F_0(x(s)) = 1 \). We therefore deduce from Lemma 5 that there exists a function \( v_0 \) solution of (28) such that
\[ F_0(x(s)) = D_x^n v_0(x(s)). \]
(41)
Next, we apply the operator \( F_x \) on both members of the previous equation and deduce by applying the second relation of Equation (25) of Proposition 1 that
\[ F_x F_0(x(s)) = F_x D_x D_x^{n-1} v_0(x(s)) = D_x^{n-1} v_0(x(s)) - D_x^{n-1} v_0(x(s)) |_{s=\varepsilon}. \]

Hence,
\[ D_x^{n-1} v_0(x(s)) = F_x F_0(x(s)) + C_{n-1} F_0(x(s)), \]
where \( C_{n-1} = \mathbb{D}_x^{n-1} v_0(x(s)) |_{s=\varepsilon} \).

By applying again the operator \( F_x \) on both members of the previous equation and using the second relation of Equation (25), we get
\[ D_x^{n-2} v_0(x(s)) = F_x^2 F_0(x(s)) + C_{n-1} F_x F_0(x(s)) + C_{n-2} F_0(x(s)), \]
where \( C_{n-2} = \mathbb{D}_x^{n-2} v_0(x(s)) |_{s=\varepsilon} \). Repeating the same process, we express \( v_0 \) as
\[ v_0(x(s)) = F_x^n F_0(x(s)) + \sum_{j=0}^{n-1} C_j F_x^j F_0(x(s)) = \frac{F_n(x(s))}{\prod_{l=1}^{n} \gamma_l} + \sum_{j=0}^{n-1} C_j F_x^j F_0(x(s)), \]
where \( C_j = \mathbb{D}_x^j v_0(x(s)) |_{s=\varepsilon} \). Therefore, \( v_0(x(s)) \) is a polynomial of degree exactly \( n \) in \( x(s) \). □
4. Conclusions and Perspectives

In this work, we have derived the right and the “left” inverse of the operator $D_x$ and used the properties of the inverse operators, as well as those of the operators $D_x$ and $S_x$, to provide a formal proof that the divided-difference equation of hypergeometric type (28) has a polynomial solution of degree exactly $n$.

The novelty of our work is the formal proof of the existence of this polynomial solution, confirming therefore the fact that, in [14], by solving divided-difference (8) on a case by case basis and using most appropriate polynomial basis for each case, we have obtained for each family of classical orthogonal polynomials on non-uniform lattice, a hypergeometric or $q$-hypergeometric solution which happens to be a polynomial because of the form of one of the upper parameters obtained in the hypergeometric (or $q$-hypergeometric) representation of the obtained solution.

Finding hypergeometric representation of the non polynomial solution of (8) is not obvious and this was obtained unexpectedly for the Askey–Wilson polynomials when the parameters fulfill $b = a q^q$, $d = a q^{q-1}$ [14] (see page 15, Equations (62) and (63)). The method developed here might help to understand when and why such a hypergeometric representation exists for non-polynomial solutions.

As an additional potential application of our paper, the right and the “left” inverse of the operator $D_x$ are likely to play important role in the study of the properties of orthogonal polynomials on the non-uniform latices, and on the search of the solutions of divided-difference equations on non-uniform lattices, as well as on the hypergeometric representation (when they exist) of the second-solution—non polynomial solution—of Equation (28).

Author Contributions: These authors contributed equally to this work, in terms of investigation, conceptualization and methodology.

Funding: The Research of the first author was partially supported by the AIMS-Cameroon Research Allowance 2018–2019.

Acknowledgments: We would like to thank Gabriel Nguetseng from the Mathematics Department of the University of Yaounde I, for asking the right question whose answer led to this result. In addition, we would like to thank the anonymous reviewers whose comments and suggestions helped to improve the quality of our work.

Conflicts of Interest: The authors declare no conflict of interest.

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