Resilience in Collaborative Optimization:
Redundant and Independent Cost Functions

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Abstract

This report considers the problem of Byzantine fault-tolerance in multi-agent collaborative optimization. In this problem, each agent has a local cost function. The goal of a collaborative optimization algorithm is to compute a minimum of the aggregate of the agents’ cost functions. We consider the case when a certain number of agents may be Byzantine faulty. Such faulty agents may not follow a prescribed algorithm, and they may send arbitrary or incorrect information regarding their local cost functions. A reasonable goal in presence of such faulty agents is to minimize the aggregate cost of the non-faulty agents. In this report, we show that this goal can be achieved if and only if the cost functions of the non-faulty agents have a minimal redundancy property. We present different algorithms that achieve such tolerance against faulty agents, and demonstrate a trade-off between the complexity of an algorithm and the properties of the agents’ cost functions.

Further, we also consider the case when the cost functions are independent or do not satisfy the minimal redundancy property. In that case, we quantify the tolerance against faulty agents by introducing a metric called weak resilience. We present an algorithm that attains weak resilience when the faulty agents are in the minority and the cost functions are non-negative.

1 Introduction

The problem of collaborative optimization in multi-agent systems has gained significant attention in recent years [6, 18, 12, 21, 22]. In this problem, each agent knows its own local objective (or cost) function. In the fault-free setting, all the agents are non-faulty (or honest), and the goal is to design a distributed (or collaborative) algorithm to compute a minimum of the aggregate of their local cost functions. We refer to this problem as collaborative optimization. Specifically, we consider a system of $n$ agents where each agent $i$ has a local real-valued cost function $f_i(x)$ that maps a point $x$ in $d$-dimensional real-valued vector space (i.e. $\mathbb{R}^d$) to a real value. Unless otherwise stated, the cost functions are assumed to be convex\footnote{As noted later in Section 5, some of our results are valid even when the cost functions are non-convex.}. The goal of collaborative optimization is to determine a global minimum $x^*$, such that

$$x^* \in \arg \min_{x \in \mathbb{R}^d} \sum_{i=1}^n f_i(x).$$

(1)
Throughout the report, we use the shorthand ‘min’ for ‘\(\min_{x \in \mathbb{R}^d}\)’, unless otherwise mentioned.

As a simple example, \(f_i(x)\) may denote the cost for an agent \(i\) (which may be a robot or a person) to travel to location \(x\) from its current location. In this case, \(x^*\) is a location that minimizes the total cost for all the agents. Such multi-agent collaborative optimization is of interest in many practical applications, including collaborative machine learning [5, 6, 14], swarm robotics [22], and collaborative sensing [21]. Most of the prior work assumes all the agents to be non-faulty. Non-faulty agents follow a specified algorithm correctly. In our work we consider a scenario wherein some of the agents may be faulty and may behave incorrectly.

Su and Vaidya [26] introduced the problem of collaborative optimization in the presence of a Byzantine faulty agents. A Byzantine faulty agent may behave arbitrarily [15]. In particular, the faulty agents may send incorrect and inconsistent information in order to bias the output of a collaborative optimization algorithm, and the faulty agents may also collaborate with each other. For example, consider an application of multi-agent collaborative optimization to the case of collaborative sensing where the agents (or sensors) are observing a common object in order to collectively identify the object. However, the faulty agents may send arbitrary observations concocted to prevent the non-faulty agents from making the correct identification [9, 11, 20]. Similarly, in the case of collaborative learning, which is another application of multi-agent collaborative optimization, the faulty agents may send incorrect information based on mislabelled or arbitrary concocted data points to prevent the non-faulty agents from learning a good classifier [1, 2, 4, 8, 10, 30].

1.1 System architecture

The contributions of this paper apply to two different system architectures illustrated in Figure 1. In the server-based architecture, the server is assumed to be trustworthy, but up to \(t\) agents may be Byzantine faulty. The trusted server helps solve the distributed optimization problem in coordination with the agents. In the peer-to-peer architecture, the agents are connected to each other by a complete network, and up to \(t\) of these agents may be Byzantine faulty. Provided that \(t < \frac{n}{3}\), any algorithm for the server-based architecture can be simulated in the peer-to-peer system using the well-known Byzantine broadcast primitive [17].

For the simplicity of presentation, the rest of this report assumes the server-based architecture.

1.2 Resilience in collaborative optimization

As stated above, we will assume the server-based architecture in the rest of our discussion. We assume that up to \(t\) of the \(n\) agents may be Byzantine faulty, such that \(n > 2t\).
We assume that each agent \( i \) has a “true” cost function. Unless otherwise noted, each such cost function is assumed to be convex.

- If an agent \( i \) is non-faulty, then its behavior is consistent with its true cost function, say \( g_i(x) \). For instance, if agent \( i \) is required to send to the server the value of its cost function at some point \( x^\dagger \), then a non-faulty agent \( i \) will indeed send \( g_i(x^\dagger) \).

- If an agent \( i \) is faulty, then its behavior can be arbitrary, and not necessarily consistent with its true cost function, say \( f_i(x) \). For instance, if agent \( i \) is required to send to the server the value of its cost function at some point \( x^\dagger \), then a faulty agent \( i \) may send an arbitrary value instead of \( f_i(x^\dagger) \).

Clearly, when an agent is faulty, it may not share with the server correct information about its true cost function. However, it is convenient to define its true cost function as above, which is the cost function it would use in the absence of its failure.

Throughout this report, we assume the existence of a finite minimum for the aggregate of the true cost functions of the agents. Otherwise, the objective of collaborative optimization is vacuous. Specifically, we make following technical assumption.

**Assumption 1.** Suppose that the true cost function of each agent \( i \) is \( f_i(x) \). Then, for every non-empty set of agents \( T \), we assume that there exists a finite \( x^* \) such that \( x^* \in \arg \min_{x \in \mathbb{R}^d} \sum_{i \in T} f_i(x) \).

Suppose that the true cost function of agent \( i \) is \( f_i(x) \). Then, ideally, the goal of collaborative optimization is to compute a minimum of the aggregate of the true cost functions of all the \( n \) agents, \( \sum_{i=1}^{n} f_i(x) \), even if some of the agents are Byzantine faulty. In general, this may not feasible since the Byzantine faulty agents can behave arbitrarily. To understand the feasibility of achieving some degree of resilience to Byzantine faults, we consider two cases.

- **Independent functions:** A set of cost functions are independent if information about some of the functions in the set does not help learn any information about the remaining functions in the set. In other words, the cost functions do not contain any redundancy.

- **Redundant functions:** Intuitively speaking, a set of cost functions includes redundancy when knowing some of the cost functions helps to learn some
information about the remaining cost functions. As a trivial example, consider
the special case when it is known that there exists some function \( g(x) \) such
that \( g(x) \) is the true cost function of every agent. In this case, knowing the
true cost function of any agent suffices to learn the true cost functions of all
the agents. Also, any \( x \) value that minimize an individual agent’s true cost
function also minimizes the total true cost over all the agents.

Su and Vaidya [26] defined the goal of fault-tolerant collaborative optimization
as minimizing the aggregate of cost functions of just the non-faulty agents. Specifi-
cally, if \( f_i(x) \) is the true cost function of agent \( i \), and \( S \) denotes the set of non-faulty
agents in a given execution, then they defined the goal of fault-tolerant optimization
to be to output a point in

\[
\arg \min_{x \in \mathbb{R}^d} \sum_{i \in S} f_i(x).
\]

(2)

We refer to the above goal as \( t \)-resilience, formally defined below.

**Definition 1 (t-resilience).** A collaborative optimization algorithm is said to be \( t \)-resilient if it outputs a minimum of the aggregate of the true cost functions of
the non-faulty agents despite up to \( t \) agents being Byzantine faulty.

In general, Su and Vaidya [26] showed that, because the identity of the faulty
agents is a priori unknown, a \( t \)-resilient algorithm may not necessarily exist. In this
report, we provide an exact characterization of the condition under which \( t \)-resilience
is achievable. In particular, we show that \( t \)-resilience is achievable if and only if the
agents satisfy a property named \( 2t \)-redundancy, defined next.\(^2\) The definitions below
are vacuous if \( n \leq 2t \). Henceforth, we assume that the maximum number of faulty
agents \( t \) are in the minority, i.e., \( n > 2t \).

**Definition 2 (2t-redundancy).** Let \( f_i(x) \) denote the true cost function of agent \( i \).
The \( n \) agents are said to satisfy \( 2t \)-redundancy if the following holds for every two
subsets \( S_1 \) and \( S_2 \) each containing \( n - 2t \) agents.

\[
\emptyset \neq \bigcap_{i \in S_1} \arg \min_{x \in \mathbb{R}^d} f_i(x) = \bigcap_{i \in S_2} \arg \min_{x \in \mathbb{R}^d} f_i(x)
\]

(3)

The above definition of \( 2t \)-redundancy is equivalent to the definition below, as
shown in Appendix [3].

**Definition 3 (2t-redundancy).** Let \( f_i(x) \) denote the true cost function of agent \( i \).
The \( n \) agents are said to satisfy \( 2t \)-redundancy if the following holds for any sets of
agents \( \hat{S} \) and \( S \) such that \( |S| \geq n - t \), \(|\hat{S}| \geq n - 2t \), and \( \hat{S} \subseteq S \).

\[
\bigcap_{i \in \hat{S}} \arg \min_{x \in \mathbb{R}^d} f_i(x) = \arg \min_{x \in \mathbb{R}^d} \sum_{i \in S} f_i(x)
\]

(4)

\(^2\) The notion of \( 2t \)-redundancy can be extended to \( k \)-redundancy by replacing \( n - 2t \) in Definitions
[2] and [3] by \( n - k \).
Note that the \( t \)-resilience property pertains the point in \( \mathbb{R}^d \) that is the output of a collaborative optimization algorithm. \( t \)-resilience property does not explicitly impose any constraints on the function value. The notion of \((u, t)\)-weak resilience stated below relates to function values.

**Definition 4** \((u, t)\)-weak resilience. Let \( f_i(x) \) denote the true cost function of agent \( i \). Let \( S \) denote the set of all non-faulty agents. For \( 0 \leq u \leq |S| \), a collaborative optimization algorithm is said to be \((u, t)\)-weak resilient if it outputs a point \( \hat{x} \) for which there exists a subset \( \hat{S} \) of \( S \) such that \(|\hat{S}| \geq |S| - u\), and

\[
\sum_{i \in \hat{S}} f_i(\hat{x}) \leq \min_{x \in \mathbb{R}^d} \sum_{i \in S} f_i(x). \tag{5}
\]

It can be shown easily that \((0, t)\)-weak resilience implies \( t \)-resilience. The proof is deferred to Section 3. In many applications of multi-agent collaborative optimization, such as distributed machine learning, distributed sensing or hypothesis testing and swarm robotics, the cost functions are non-negative \([5, 6, 14, 21, 22]\). We constructively show that if the true cost functions of the agents are non-negative then \((u, t)\)-weak resilience for \( u \geq t \) can be achieved even if the cost functions are independent.

### 1.3 Prior Work

The prior work on resilience in collaborative multi-agent optimization by Su and Vaidya, 2016 \([26]\), and Sundaram and Gharesifard, 2018 \([28]\), only consider the special class of univariate cost functions, i.e., dimension \( d \) equals one. On the other hand, we consider the general class of multivariate cost functions, i.e., \( d \) can be greater than one. Specifically, they have proposed algorithms that output a minimum of the non-uniformly weighted aggregate of the non-faulty agents’ cost functions when \( d = 1 \). However, their proposed algorithms do not extend easily for the case when \( d > 1 \). On the other hand, the algorithms and the fault-tolerance results presented in this report are valid regardless of the value of the dimension \( d \) as long as it is finite.

Su and Vaidya have also considered a special case where the true cost functions of the agents are convex combinations of a finite number of basis convex functions in \([27]\). They have shown that if the basis functions have a common minimum then a minimum point (as in \([2]\)) can be computed accurately. This property of redundancy in the minimum of the basis functions, we note, is a special case of the \( 2t \)-redundancy property that we prove necessary and sufficient for \( t \)-resilience in this report. Other prior work related to the \( 2t \)-redundancy property is discussed in Section 2.2.

Yang and Bajwa, 2017 \([31]\) consider a very special case of collaborative optimization problem. They assume that the multivariate cost functions that can be split into independent univariate strictly convex functions. For this special, they have extended the fault-tolerance algorithm of Su and Vaidya, 2016 \([26]\) for approximate resilience. In general, however, the agents’ cost functions do not satisfy such specific
properties. In this report, we do not make such assumptions about the agents’ cost functions. We only assume the cost functions to be convex, differentiable and that the minimum of their sum is finite (i.e., Assumption 1). Note that these assumptions are fairly standard in the optimization literature, and are also assumed in all of the aforementioned prior work.

Outline of the report: The rest of the report is organized as follows. In Section 2 we present the case when the cost functions have redundancy. In Section 3 we present the case when the cost functions are independent. In Section 4 we summarize a gradient-based algorithm for $t$-resilience, which was proposed in our prior work [13]. In Section 5 we discuss direct extension of our results to the case when the cost functions are non-differentiable and non-convex. In the same section, we also present a summary of our results.

2 The Case of Redundant Cost Functions

This section presents the key result of this report for the case when the cost functions are redundant. Unless otherwise mentioned, in the rest of the report, the cost functions are assumed to be differentiable, i.e., their gradients exist at all the points in $\mathbb{R}^d$. Indeed, the cost functions are differentiable for most aforementioned applications of collaborative optimization [5, 6, 21, 22]. Nevertheless, as elaborated in Section 5 some of our results are also applicable for non-differentiable cost functions.

Before we present Theorem 1 below which states the key result of this section, in Lemma 2 we present an alternate, and perhaps more natural, equivalent condition of the $2t$-redundancy property for the specific case when the agents’ cost functions are differentiable. The proof of Lemma 2 uses Lemma 1 stated below.

Lemma 1. Suppose that Assumption 1 holds true, and $n > 2t$. For a non-empty set $T$, consider a set of functions $g_i(x)$, $i \in T$, such that

$$\bigcap_{i \in T} \arg \min_x g_i(x) \neq \emptyset.$$  

Then

$$\bigcap_{i \in T} \arg \min_x g_i(x) = \arg \min_x \sum_{i \in T} g_i(x).$$

Appendix A presents the proof of the above lemma.

Lemma 2. Suppose that Assumption 1 holds true, and $n > 2t$. When the true cost functions of the agents are convex and differentiable then the $2t$-redundancy property stated in Definition 2 or Definition 3 is equivalent to the following condition:

A point is a minimum of the sum of true cost functions of the non-faulty agents if and only if that point is a minimum of the sum of the true cost functions of any $n - 2t$ non-faulty agents.

Proof. Let the true cost function of each agent $i$ be denoted by $f_i(x)$. Recall that there can be at most $t$ Byzantine faulty agents. Let $S$ with $|S| \geq n - t$ be the set of
the non-faulty agents.

**Part I:** We first show that the condition stated in the lemma implies that in Definition 2. Recall that the conditions in Definitions 2 and 3 are equivalent.

The condition stated in the lemma is equivalent to saying that for every subset \( \hat{S} \) of \( S \) of size \( n - 2t \),

\[
\arg\min_{i \in \hat{S}} \sum_{i \in \hat{S}} f_i(x) = \arg\min_{i \in S} \sum_{i \in S} f_i(x). \tag{6}
\]

We show below that (6) together with Assumption 1 imply that for every subset \( \hat{S} \) of \( S \) of size \( n - 2t \),

\[
\bigcap_{i \in \hat{S}} \arg\min_{x} f_i(x) \neq \emptyset. \tag{7}
\]

Consider two arbitrary agents \( i, j \) in \( S \), and then consider two size \((n - 2t)\) subsets \( S_i \) and \( S_j \) of \( S \) such that \( i \in S_i \), \( j \in S_j \), and

\[
S_i \setminus \{i\} = S_j \setminus \{j\}. \tag{8}
\]

By Assumption 1 there exists a point \( x^* \in \arg\min_{x} \sum_{i \in S} f_i(x) \). Now, (6) implies that

\[
\nabla \sum_{i \in S_i} f_i(x^*) = \nabla \sum_{i \in S_j} f_i(x^*) = 0.
\]

The above equality and (8) imply that

\[
\nabla f_i(x^*) = \nabla f_j(x^*)
\]

This equality can be proven for any \( i, j \in S \). As the true cost functions \( f_1, \ldots, f_n \) are assumed convex, from above we obtain,

\[
x^* \in \arg\min_{x} f_i(x), \quad \forall i \in S.
\]

Therefore, for every subset \( \hat{S} \) of \( S \) of size \( n - 2t \),

\[
x^* \in \bigcap_{i \in \hat{S}} \arg\min_{x} f_i(x) \neq \emptyset.
\]

The above implies that for every subset \( \hat{S} \) of \( S \) of size \( n - 2t \),

\[
\arg\min_{i \in \hat{S}} \sum_{i \in \hat{S}} f_i(x) = \bigcap_{i \in \hat{S}} \arg\min_{i \in S} f_i(x).
\]

The above together with (6) implies the condition in Definition 2 i.e.,

\[
\bigcap_{i \in S_1} \arg\min_{i \in S_1} f_i(x) = \bigcap_{i \in S_2} \arg\min_{i \in S_2} f_i(x), \quad \forall S_1, S_2 \subset S, \quad |S_1| = |S_2| = n - 2t.
\]
Part II: We now show that the condition in Definition 3 implies the condition stated in the lemma. Now, \( \arg \min \sum_{i \in S} f_i(x) \) (i.e., the right side of (4)) is a non-empty set due to Assumption 1. This and (4) imply that for every subset \( \hat{S} \subset S \) of size \( n - 2t \),

\[
\bigcap_{i \in \hat{S}} \arg \min f_i(x) \neq \emptyset.
\]

Therefore, by Lemma 1,

\[
\bigcap_{i \in \hat{S}} \arg \min f_i(x) = \arg \min \sum_{i \in \hat{S}} f_i(x).
\]

Substituting the above in (4) implies (6) which is equivalent to the condition stated in the lemma.

The following theorem presents the main result of this section.

**Theorem 1.** Suppose that Assumption 7 holds true, and \( n > 2t \). When the true cost functions of the agents are convex and differentiable then \( t \)-resilience can be achieved if and only if the agents satisfy the \( 2t \)-redundancy property.

*Proof.* The case of \( t=0 \) is trivial, since there are no faulty agents. In the rest of the proof, we assume that \( t \geq 1 \).

**Sufficiency of \( 2t \)-redundancy:** Sufficiency of \( 2t \)-redundancy is proved constructively using the algorithm presented in Section 2.1. In particular, the algorithm is proved to achieve \( t \)-resilience if \( 2t \)-redundancy holds.

**Necessity of \( 2t \)-redundancy:** We consider the worst-case scenario where \( t \) arbitrary agents are faulty. Suppose that \( t \)-resilience can be achieved using an algorithm named \( \Pi \). Consider an execution \( E_S \) of \( \Pi \) in which set \( S \) with \( |S| = n - t \) is the actual set of non-faulty agents. All the remaining agents in the set \( C = \{1, \ldots, n\}\setminus S \) are the actual faulty agents. Suppose that the true cost function of agent \( i \) in execution \( E_S \) is \( g_i(x) \). We assume that the functions \( g_1, \ldots, g_n \) are differentiable and convex.

In any \( t \)-resilient algorithm for collaborative optimization, the server can communicate with the agents and learn some information about their local cost functions. The most information the server can learn about the cost function of an agent \( i \) is the complete description of its local cost function. To prove the necessity of \( 2t \)-redundancy, we assume that the server knows a cost function reported by each non-faulty agent \( i \).

Now consider the following executions.

- In execution \( E_0 \), all the agents are non-faulty. Let \( S_0 \) denote the set of all agents, which happen to be non-faulty in execution \( E_0 \). Thus, \( S_0 = \{1, 2, \ldots, n\} \). The true cost function of agent \( i \) is \( g_i(x) \), identical to its true cost function in execution \( E_S \).
• In execution $E_i$, where $1 \leq i \leq n$, agent $i$ is Byzantine faulty, and all the remaining $n - 1$ agents are non-faulty. Let $S_i = S_0 \setminus \{i\}$ denote the set of agents that happen to be non-faulty in execution $E_i$. In execution $E_i$, the true cost function of each non-faulty agent $i$ is $g_i(x)$, which is identical to its true cost function in execution $E_{S_i}$. Let the true cost function of faulty agent $i$ in execution $E_i$ be a differentiable and convex function $h_i(x)$. Assume that the functions $g_j(x), \forall j$, and $h_i(x)$ are independent. In execution $E_i$, suppose that the behavior of faulty agent $i$ from the viewpoint of the server is consistent with the cost function $g_i(x)$ (which equals the true cost function of agent $i$ in execution $E_0$).

Fix a particular $i$, $1 \leq i \leq n$. From the viewpoint of the server, execution $E_0$ and execution $E_i$ are indistinguishable. Thus, the $t$-resilient algorithm $\Pi$ will produce an identical output in these executions; suppose that this output is $x_\Pi$. As $\Pi$ is assumed to be $t$-resilient, we have by Definition 1 and Assumption 1,

$$x_\Pi \in \arg \min_x \sum_{j \in S_0} g_j(x), \quad \text{and}$$

$$x_\Pi \in \arg \min_x \sum_{j \in S_i} g_j(x).$$

For a differentiable cost function $g : \mathbb{R}^d \to \mathbb{R}$, we denote its gradient at a point $x$ by $\nabla g(x)$. Let $\mathbf{0}$ denote the zero-vector of dimension $d$. If $x^* \in \arg \min_x g(x)$ then

$$\nabla g(x^*) = \mathbf{0}.$$  

From (9) and (10) we obtain,

$$\nabla \sum_{j \in S_0} g_j(x_\Pi) = \sum_{j \in S_0} \nabla g_j(x_\Pi) = \mathbf{0}, \quad \text{and}$$

$$\nabla \sum_{j \in S_i} g_j(x_\Pi) = \sum_{j \in S_i} \nabla g_j(x_\Pi) = \mathbf{0}. \quad (11)$$

Recall that $S_0 = S_i \cup \{i\}$. Therefore,

$$\nabla g_i(x_\Pi) + \nabla \sum_{j \in S_i} g_j(x_\Pi) = \nabla \sum_{j \in S_0} g_j(x_\Pi). \quad (12)$$

From (11) and (12) we obtain,

$$\nabla g_i(x_\Pi) = \mathbf{0}\quad (13)$$

As the cost functions are assumed to be convex, the above implies that,

$$x_\Pi \in \arg \min_x g_i(x)$$

By repeating the above argument for each $i \in \{1, \ldots, n\}$, we have

$$x_\Pi \in \arg \min_x g_i(x), \quad \forall i \in \{1, \ldots, n\}. \quad (14)$$
Therefore,
\[ x_\Pi \in \bigcap_{i=1}^{n} \arg \min_x g_i(x) \neq \emptyset. \] (15)

Similarly, for every non-empty set of agents \( T \)
\[ x_\Pi \in \bigcap_{i \in T} \arg \min_x g_i(x) \neq \emptyset. \] (16)

Thus, \( \bigcap_{i \in T} \arg \min g_i(x) \neq \emptyset \). Then, Lemma 1 implies that
\[ \arg \min \sum_{i \in T} g_i(x) = \bigcap_{i \in T} \arg \min g_i(x) \neq \emptyset, \quad \forall \ \text{non-empty} \ T \subseteq \{1, \ldots, n\}. \] (17)

Now we consider execution \( E_S \) (defined earlier) in which the nodes in set \( S \) are non-faulty. Using the results derived in the proof so far, we will show that, for any \( \widehat{S} \subseteq S \) subject to \( |\widehat{S}| = n - 2t \),
\[ \arg \min_x \sum_{i \in S} g_i(x) = \bigcap_{i \in \widehat{S}} \arg \min_x g_i(x). \]

The proof concludes once we have shown the above equality.

Consider an arbitrary subset \( \widehat{S} \subseteq S \) subject to \( |\widehat{S}| = n - 2t \). It is trivially true that
\[ \bigcap_{i \in S} \arg \min_x g_i(x) \subseteq \bigcap_{i \in \widehat{S}} \arg \min_x g_i(x). \] (18)

So it remains to show that \( \bigcap_{i \in S} \arg \min g_i(x) \) is not a strict subset of \( \bigcap_{i \in \widehat{S}} \arg \min g_i(x) \). The proof below is by contradiction.

Suppose that
\[ \bigcap_{i \in S} \arg \min g_i(x) \subset \bigcap_{i \in \widehat{S}} \arg \min g_i(x). \] (19)

This implies that there exists a point
\[ x^* \in \bigcap_{i \in \widehat{S}} \arg \min g_i(x), \] (20)

such that
\[ x^* \notin \bigcap_{i \in S} \arg \min_x g_i(x). \] (21)

Footnote 2 noted that the notion of 2t-redundancy can be extended to \( k \)-redundancy. The proof so far has relied only on 1-redundancy, which is weaker than 2t-redundancy. The latter part of this proof makes use of 2t-redundancy.
Therefore, there exists an $i^\dagger \in S$ such that
\[ x^\dagger \notin \arg \min g_i(x). \tag{22} \]

Let $C = \{1, \cdots , n\} \setminus S$ and $F = S \setminus \widehat{S}$. Then $|C| = |F| = t$. Now we define executions $E_C$ and $E_F$.

- **Execution $E_C$:** In execution $E_C$ the $t$ agents in set $C$ are faulty, and the $n - t$ agents in set $S$ are non-faulty. In execution $E_C$, the behavior of each agent $i \in S$ is consistent with its true cost function being $g_i(x)$, which is identical to its true cost function in execution $E_S$. However, each faulty agent $j \in C$ behaves consistent with a differentiable and convex true cost function $h_j(x)$ that has a unique minimum at $x^\dagger$.

- **Execution $E_F$:** In execution $E_F$ the $t$ agents in set $F$ are faulty, and the remaining $n - t$ agents in $\widehat{S} \cup C$ are non-faulty. In execution $E_F$, the behavior of each agent $i \in S$ (including the faulty agents in $F$) is consistent with the cost function $g_i(x)$. Each non-faulty agent $j \in C$ behaves consistent with its true cost function being $h_j(x)$, which is defined in execution $E_C$. Recall that each $h_j(x)$ has a unique minimum at $x^\dagger$.

Observe that the server cannot distinguish between executions $E_C$ and $E_F$.

Now, (21) implies that $h_j(x)$ does not minimize at any point in $\bigcap_{i \in S} \arg \min_x g_i(x)$. That is, for every agent $j \in C$,
\[ \{x^\dagger\} = \arg \min h_j(x), \quad \text{and} \quad \left( \bigcap_{i \in S} \arg \min g_i(x) \right) \cap \arg \min h_j(x) = \emptyset \tag{23} \]

As $\Pi$ is $t$-resilient, in execution $E_F$, algorithm $\Pi$ must produce an output in
\[ \arg \min \left( \sum_{i \in \widehat{S}} g_i(x) + \sum_{j \in C} h_j(x) \right) \tag{24} \]
(Recall that the agents in $\widehat{S} \cup C$ are non-faulty in execution $E_F$.)

(20) and (23) together imply that
\[ \left( \bigcap_{i \in \widehat{S}} \arg \min g_i(x) \right) \cap \left( \bigcap_{j \in C} h_j(x) \right) = \{x^\dagger\} \]

That is, the above set contains only $x^\dagger$. This, in turn, by Lemma 1 implies that the set in (24) only contains the point $x^\dagger$, and thus, algorithm $\Pi$ must output $x^\dagger$ in execution $E_F$. 

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Now, since algorithm Π cannot distinguish between executions \( E_F \) and \( E_C \), it must also output \( x^\dagger \) in execution \( E_C \) as well. However, from (17) and (21), respectively, we know that

\[
\bigcap_{i \in S} \arg \min g_i(x) = \arg \min \sum_{i \in S} g_i(x)
\]

and

\[
x^\dagger \not\in \bigcap_{i \in S} \arg \min g_i(x).
\]

The above two equations imply that \( x^\dagger \not\in \arg \min \sum_{i \in S} g_i(x) \), and Π cannot output \( x^\dagger \) in execution \( E_C \) (otherwise Π cannot be \( t \)-resilient). This is a contradiction.

Therefore, we have proved that \( \bigcap_{i \in S} \arg \min g_i(x) \) is not a strict subset of \( \bigcap_{i \in \hat{S}} \arg \min g_i(x) \).

Above result together with (18) implies that

\[
\bigcap_{i \in S} \arg \min g_i(x) = \bigcap_{i \in \hat{S}} \arg \min g_i(x).
\]

Recall that \( \hat{S} \) is an arbitrary subset of \( S \) with \( |\hat{S}| = n - 2t \). Therefore, the above implies that for every subset \( \hat{S} \) of \( S \) with \( |\hat{S}| \geq n - 2t \),

\[
\bigcap_{i \in S} \arg \min g_i(x) = \bigcap_{i \in \hat{S}} \arg \min g_i(x).
\]

This together with (17) implies that

\[
\arg \min \sum_{i \in S} g_i(x) = \bigcap_{i \in \hat{S}} \arg \min g_i(x), \quad \forall \text{ non-empty } \hat{S} \subseteq S, \quad |\hat{S}| \geq n - 2t.
\]

Thus, if Π is \( t \)-resilient then the true cost functions of the agents satisfy the 2\( t \)-redundancy property as stated in Definition 3. Hence, proving the necessity of 2\( t \)-redundancy property for \( t \)-resilience.

The following collaborative optimization algorithm proves the sufficiency of 2\( t \)-redundancy for \( t \)-resilience.

### 2.1 A \( t \)-resilient algorithm

We present an algorithm and prove that it is \( t \)-resilient if the agents satisfy the 2\( t \)-redundancy property stated in Definition 2 or 3. We will suppose that Assumption 1 holds true and \( n > 2t \). We only consider the case when \( t > 0 \), since the case of \( t = 0 \) is trivial.
The server collects full description of the cost function of each agent. Suppose that the server obtains cost function \( h_j(x) \) from each agent \( j \in \{1, \cdots, n\} \). For each non-faulty agent \( i \), \( h_i(x) \) is the agent’s true objective function.

The proposed algorithm outputs a point \( x^\ast \) such that there exists a set \( A \) of \( n-t \) agents such that for any \( \hat{A} \subset A \) with \( |\hat{A}| = n-2t \),

\[
x^\ast \in \arg \min_{i \in \hat{A}} \sum_{i \in \hat{A}} h_i(x)
\]

If there are multiple candidate points that satisfy the condition above, then any one such point is chosen as the output.

Now we prove the correctness of the above algorithm if \( 2t \)-redundancy holds.

**Proof.** Assume that the \( 2t \)-redundancy property holds. First we observe that the algorithm will always be able to output a point if \( 2t \)-redundancy is satisfied. Let \( S \) denote the set of all non-faulty agents. Recall that \( |S| \geq n-t \). In particular, consider a set \( A \) that consists of any \( n-t \) non-faulty agents, that is, \( A \subseteq S \). For any \( \hat{A} \subset A \) where \( |\hat{A}| = n-2t \), due to \( 2t \)-redundancy (Definition 3) and Assumption 1, we have

\[
\bigcap_{i \in \hat{A}} \arg \min_{x \in \mathbb{R}^d} h_i(x) = \arg \min_{x \in \mathbb{R}^d} \sum_{i \in S} h_i(x) \tag{25}
\]

This implies that every point in \( \arg \min \sum_{i \in S} h_i(x) \) is a candidate for the output of the algorithm. Additionally, due to Assumption 1, \( \arg \min_{x \in \mathbb{R}^d} \sum_{i \in S} h_i(x) \) is guaranteed to be non-empty. Thus, the algorithm will always produce an output.

Next we show that the algorithm achieves \( t \)-resilience. Consider any set \( A \) for which the condition in the algorithm is true. The algorithm outputs \( x^\ast \). From the algorithm, we know that for any \( \hat{A} \subset A \) with \( |\hat{A}| = n-2t \),

\[
x^\ast \in \arg \min_{i \in \hat{A}} \sum_{i \in \hat{A}} h_i(x)
\]

Now, since at most \( t \) agents are faulty, there exists at least one set \( \hat{S} \) containing \( n-2t \) non-faulty agents such that \( \hat{S} \subseteq A \) (and also \( \hat{S} \subseteq S \) ). Thus,

\[
x^\ast \in \arg \min_{i \in \hat{S}} \sum_{i \in \hat{S}} h_i(x) \tag{26}
\]

Also, since \( \hat{S} \subseteq S \), due to \( 2t \)-redundancy (Definition 3), we have

\[
\bigcap_{i \in \hat{S}} \arg \min_{x \in \mathbb{R}^d} h_i(x) = \arg \min_{x \in \mathbb{R}^d} \sum_{i \in \hat{S}} h_i(x) \tag{27}
\]

Since \( \arg \min \sum_{i \in \hat{S}} h_i(x) \) is non-empty, the last equality implies that \( \bigcap_{i \in \hat{S}} \arg \min h_i(x) \) is non-empty. This, in turn, by Lemma 1 implies that

\[
\arg \min_{i \in \hat{S}} h_i(x) = \bigcap_{i \in \hat{S}} \arg \min h_i(x)
\]
The last equality, (26) and (27) together imply that

\[ x^* \in \arg \min_{x \in \mathcal{S}} \sum_{i \in \mathcal{S}} h_i(x). \]

Thus, the above algorithm achieves \( t \)-resilience.

It should be noted that the correctness of the \( t \)-resilient algorithm presented above does not require differentiability or convexity of the agents’ true cost functions. Therefore, the \( 2t \)-redundancy is a sufficient condition for \( t \)-resilience even when the agents’ cost functions are non-differentiable and non-convex.

**Alternate \( t \)-resilient algorithms:** There exist other, and more practical, algorithms to achieve \( t \)-resilience when \( 2t \)-redundancy holds. However, there is a trade-off between algorithm complexity and additional properties assumed for the cost functions.

- We present an alternate, computationally simpler, \( t \)-resilient algorithm in Section 3.1 for the case when the minimum values of each true cost function is zero.

- In our prior work [13], we proposed a gradient-descent based distributed algorithm that is \( t \)-resilient if the cost functions have certain additional properties presented in Section 4. The algorithm uses a computationally simple “comparative gradient clipping” mechanism to tolerate Byzantine faults.

### 2.2 Prior work on redundancy

To the best of our knowledge, there is no prior work on the tightness of \( 2t \)-redundancy property for \( t \)-resilience in collaborative optimization. Nevertheless, it is worthwhile to note that conditions with some similarity to \( 2t \)-redundancy are known to be necessary and sufficient for fault-tolerance in other systems, such as information coding and collaborative multi-sensing (or sensor fusion), discussed below. We note that collaborative multi-sensing can be viewed as a special case of the collaborative optimization problem presented in this report.

**Redundancy for error-correction coding:** Digital machines store or communicate information using a finite length sequence of symbols. However, these symbols are may become erroneous due to faults in the system or during communication. A way to recover the information despite such error is to use an error-correction code. An error-correction code transforms (or encodes) the original sequence of symbols into another sequence of symbols called a codeword. It is well-known that a code that generates codewords of length \( n \) can correct (or tolerate) up to \( t \) symbols errors if and only if the Hamming distance between any two codewords of the code is at least \( 2t + 1 \) [16, 29]. There exist codes (e.g., Reed-Solomon codes) such that the sequence of symbols encoded in a codeword can be uniquely determined using any \( n - 2t \) correct symbols of the codeword.
Redundancy for fault-tolerant state estimation: The problem of collaborative optimization finds direct application in distributed sensing [21]. In this problem, the system comprises multiple sensors, and each sensor makes partial observations about the state of the system. The goal of the sensors is to collectively compute the complete state of the system. However, if a sensor is faulty then it may share incorrect observations. The problem of fault-tolerance in collaborative sensing for the special case wherein the sensors’ observations are linear in the system state has gained significant attention in recent years [3, 11, 19, 20, 23, 24, 25]. Chong et al., 2015 [11] and Pajic et al., 2015 [20] showed that the system state can be accurately computed when up to \( t \) (out of \( n \)) sensors are faulty if and only if the system is \( 2t \)-sparse observable, i.e., the state can be computed uniquely using observations of only \( n - 2t \) non-faulty sensors. We note that the property of \( 2t \)-sparse observability is a special instance of the more general \( 2t \)-redundancy property presented in this report. Moreover, the necessity and sufficiency of the \( 2t \)-redundancy property proved in this report implies the necessity and sufficiency of \( 2t \)-sparse observability for fault-tolerant state estimation for a more general setting wherein the sensor observations may be non-linear; however, the converse is not true.

Next, we consider the case when the cost functions are independent, and may not satisfy the \( 2t \)-redundancy property.

3 The case of Independent Cost Functions

In this section, we present the case when the true cost functions of the agents are independent. Throughout this section we assume that \( t > 0 \), otherwise the problem of resilience is trivial.

We show below by construction that when the true cost functions are non-negative then \((u, t)\)-weak resilience can be achieved for \( u \geq t \) even if the true cost functions are independent. Note that, by Definition 4, when the true cost functions of the agents are non-negative then \((u, t)\)-weak resilience trivially implies \((u^\dagger, t)\)-weak resilience where \( u^\dagger \geq u \). Therefore, achievability of \((t, t)\)-weak resilience implies the achievability of \((u, t)\)-weak resilience for all \( u \geq t \).

In the subsequent subsection we present a collaborative optimization algorithm that guarantees \((t, t)\)-weak resilience when the true cost functions are non-negative and \( n > 2t \). In Section 3.3 we show that the algorithm below also achieves \( t \)-resilience under certain conditions.

3.1 Algorithm for \((t, t)\)-Weak Resilience

In the proposed algorithm, the server obtains a full description of the agents’ cost functions. We denote the function obtained by the server from agent \( i \) as \( h_i(x) \). Let the true cost function of each agent \( i \) be denoted \( f_i(x) \). Then for each non-faulty agent \( i \), \( h_i(x) = f_i(x), \ \forall \ x \). On the other hand, for each faulty agent \( i \), \( h_i(x) \) may not necessarily equal \( f_i(x) \).

The algorithm comprises three steps:
• **Pre-processing Step:** For any agent $j$, if $h_j(x)$ is not non-negative for some $x$ or $\min h_j(x)$ is not finite (or does not exist), then $j$ must be faulty. Remove $j$ from the system. Decrement $t$ and $n$ each by 1 for each agent thus removed. In other words, the cost functions of the remaining agents are non-negative. Also, it is easy to see that if the faulty agents are in the minority then $n > 2t$ after pre-processing for the updated values of $n$ and $t$.

• **Step 1:** For each set $A$ of agents such that $|A| = n - t$, compute

$$v_A = \min_{x \in \mathbb{R}^d} \sum_{i \in A} h_i(x).$$

• **Step 2:** Determine a subset $\hat{A}$ of size $n - t$ such that

$$v_{\hat{A}} = \min \{v_A | A \subseteq \{1, \ldots, n\}, |A| = n - t\} \tag{28}$$

Output a point $\hat{x} \in \arg \min_{x} \sum_{i \in \hat{A}} h_i(x)$.

Now we prove that the algorithm is $(t, t)$-weak resilient. It should be noted that the $(t, t)$-weak resilience property of the algorithm holds true despite the true cost function being non-convex and non-differentiable.

**Theorem 2.** Suppose that Assumption[1] holds, and $n > 2t$. If the true cost functions are non-negative then the above algorithm is $(t, t)$-weak resilient.

**Proof.** Suppose that, before the pre-processing step $n - t = a$ and $n - 2t = b$. In the proof, we consider the set of agents, and the values of $n$ and $t$ after the pre-processing step of the algorithm. In the worst-case for the algorithm, all faulty agents will send non-negative functions, thus, no faulty agents are removed in the pre-processing step. Also observe that, in general, for the updated values of $n$ and $t$ after the pre-processing step, (i) $n - t = a$ (i.e., $n - t$ remains unchanged), and (ii) $n - 2t \geq b$, and (iii) $n > 2t$.

For an execution of the proposed algorithm, let $F$ denote the set of up to $t$ faulty agents, and let $S$ denote the set of non-faulty agents. Thus, $|S| + |F| = n$.

Recall the definition of $\hat{A}$ in the algorithm above. Let

$$S_1 = S \cap \hat{A} \tag{29}$$

$$F_1 = F \cap \hat{A} \tag{30}$$

Thus, $\hat{A} = S_1 \cup F_1$. Since $|\hat{A}| = n - t$ and $|F| \leq t < n/2$, we have that $|S_1| \geq |S| - t$ and $|F_1| \leq t$.

---

4A worst-case adversary may ensure that $h_i(x)$ for faulty agent $i$ is non-negative, so that no faulty agents will be eliminated in the pre-processing step.
First, note that owing to the pre-processing step and Assumption \[1\] for every set of \(n - t\) agents \(A\), 
\[v_A = \min \sum_{i \in A} h_i(x)\] exists and is finite.

Now, note that
\[v_{\hat{A}} = \min \sum_{i \in \hat{A}} h_i(x) = \sum_{i \in S_1} h_i(\hat{x}) = \sum_{j \in F_1} h_j(\hat{x}).\]

From (28), \(v_{\hat{A}} \leq v_A\) for all sets \(A\) of size \(n - t\). Therefore, there exists a subset \(S' \subseteq S\) with \(|S'| = n - t\) such that
\[v_{\hat{A}} \leq v_{S'}.\]

From above we obtain,
\[\sum_{i \in S_1} h_i(\hat{x}) + \sum_{j \in F_1} h_j(\hat{x}) \leq v_{S'} = \min \sum_{i \in S'} h_i(x).\]

Recall that \(h_i(x) = f_i(x)\) for all \(i \in S\). As \(S_1\) and \(S'\) are subsets of \(S\), the above implies that,
\[\sum_{i \in S_1} f_i(\hat{x}) + \sum_{j \in F_1} h_j(\hat{x}) \leq v_{S'} = \min \sum_{i \in S'} f_i(x).\] (31)

Each \(h_j(\hat{x})\) is a non-negative function (due to the pre-processing step). Therefore, \(h_j(\hat{x}) \geq 0\) for all \(j \in F_1\). Substituting this in (31) implies,
\[\sum_{i \in S_1} f_i(\hat{x}) \leq \min \sum_{i \in S'} f_i(x).\] (32)

As \(S' \subseteq S\), non-negativity of cost functions implies that,
\[\min \sum_{i \in S'} f_i(x) \leq \min \sum_{i \in S} f_i(x).\]

Substituting the above in (32) implies,
\[\sum_{i \in S_1} f_i(\hat{x}) \leq \min \sum_{i \in S} f_i(x).\] (33)

Recall that \(|S_1| \geq |S| - t\). Recalling that the set of non-faulty agents is not affected by the pre-processing step, the above implies that the proposed algorithm is \((t, t)\)-weak resilient.

The algorithm above is \((t, t)\)-weak resilient for the case when each true cost function is non-negative. However, in general, there may exist collaborative optimization algorithms that are \((t, t)\)-weak resilient only for the case when each true cost function has minimum value 0. We present below a normalization technique for generalizing the weak resilience of such algorithms. Specifically, given a collaborative optimization algorithm that is \((u, t)\)-weak resilient for the case when each true
cost function has minimum value 0, the presented normalization technique general-
izes the algorithm to the case when the true cost functions are non-negative.

Later, we will see that the normalization technique renders a collaborative op-
timization algorithm that is \((t, t)\)-\textit{weak resilient} for the case when the true cost functions are non-negative, such as the one presented above, \textit{t-resilient} if the true cost functions satisfy the 2t-\textit{redundancy} property.

### 3.2 Normalized Implementation of \((u, t)\)-weak resilient algorithm

We denote the function obtained by the server from agent \(i\) as \(h_i(x)\). Let the true cost function of each agent \(i\) be denoted \(f_i(x)\). Then for each non-faulty agent \(i\), \(h_i(x) = f_i(x), \forall x\). For each faulty agent \(i\), \(h_i(x)\) may not necessarily equal \(f_i(x)\).

Consider an arbitrary algorithm \(\Pi\) that achieves \((u, t)\)-\textit{weak resilience} when each true cost function has minimum value 0. With \(\Pi\) as a building block, we design an algorithm \(\Pi^+\) using the two-step normalization procedure below. We will refer to \(\Pi^+\) as the \textit{normalized implementation} of \(\Pi\).

- **Step 1**: For each agent \(i\), compute \(\min h_i(x)\). If \(\min h_i(x)\) does not exist or is infinite then remove agent \(i\) from the system. Decrement \(n\) and \(t\) each by 1 for each agent thus removed. Otherwise, define an alternate \textit{effective cost function} \(h_i^\dagger\) such that

\[
h_i^\dagger(x) = h_i(x) - \min_{x \in \mathbb{R}^d} h_i(x), \quad \forall x \in \mathbb{R}^d.
\]  

\(34\)

It is easy to see that if the faulty agents are in the minority (i.e., \(n > 2t\) prior to the normalization step) then \(n > 2t\) upon completion of the normalization step for the updated values of \(n\) and \(t\).

The agents that remain after the above step are numbered 1 through \(n\), without loss of generality.

- **Step 2**: Execute \(\Pi\) on the \textit{effective cost functions} \(h_1^\dagger(x), \ldots, h_n^\dagger(x)\).

The resilience property of algorithm \(\Pi^+\) is stated below.

**Lemma 3.** Suppose that Assumption 2 holds true. If algorithm \(\Pi\) is \((u, t)\)-weak resilient when the true cost function of each agent has minimum value equal to zero then \(\Pi^+\), the normalized implementation of \(\Pi\), is \((u, t)\)-weak resilient when the true cost functions are non-negative.

**Proof.** In the proof, we consider the set of agents, and the values of \(n\) and \(t\) after the step 1 of the normalization procedure. Note that, due to Assumption 3 the set of non-faulty agents is not affected by step 1. Let the true cost function of each agent \(i\) be denoted \(f_i\). The true cost functions are assumed non-negative.

\(5\)A worst-case adversary may ensure that \(h_i(x)\) for faulty agent \(i\) is non-negative, so that no faulty agents will be eliminated in the normalization step.
Suppose that algorithm II is \((u, t)\)-weak resilient when each true cost function has minimum value equals zero. For an execution of the algorithm \(\Pi^+\), let set \(S\) with \(|S| \geq n - t\) denote the set of non-faulty agents. Let the output of \(\Pi^+\) be denoted by \(\hat{x}\).

Due to Assumption \([\text{I}]\) for each agent \(i\), \(\min_{x \in \mathbb{R}^d} f_i(x)\) exists and is finite. For each agent \(i\), let \(f^\dagger_i\) denote a function such that
\[
f^\dagger_i(x) = f_i(x) - \min_{y \in \mathbb{R}^d} f_i(y), \quad \forall x \in \mathbb{R}^d.
\]

Therefore, for each agent \(i\),
\[
\min_{x \in \mathbb{R}^d} f^\dagger_i(x) = \min_{x \in \mathbb{R}^d} \left( f_i(x) - \min_{y \in \mathbb{R}^d} f_i(y) \right) = \min_{x \in \mathbb{R}^d} f_i(x) - \min_{y \in \mathbb{R}^d} f_i(y) = 0. \tag{36}
\]

Note that from (34) in the step 1, if an agent \(i\) is non-faulty then \(h^\dagger_i = f^\dagger_i\). Therefore, the true cost functions in step 2, i.e., during the execution of algorithm \(\Pi\), are \(f^\dagger_1, \ldots, f^\dagger_n\). This together with (36) implies that each true cost function has minimum value equal to 0 during the execution of \(\Pi\). As \(\Pi\) is assumed \((u, t)\)-weak resilient for the case when each true cost function has minimum value equal to zero, by Definition \([\text{IV}]\) there exists \(\hat{S} \subseteq S\) with \(|\hat{S}| \geq |S| - u\) such that
\[
\sum_{i \in \hat{S}} f^\dagger_i(\hat{x}) \leq \min_{x \in \mathbb{R}^d} \sum_{i \in \hat{S}} f^\dagger_i(x).
\]
Substituting from (35) above we obtain,
\[
\sum_{i \in \hat{S}} \left( f_i(\hat{x}) - \min_{y \in \mathbb{R}^d} f_i(y) \right) \leq \min_{x \in \mathbb{R}^d} \sum_{i \in S} \left( f_i(x) - \min_{y \in \mathbb{R}^d} f_i(y) \right).
\]

Trivially, for each \(i\),
\[
\min_{x \in \mathbb{R}^d} \left( \min_{y \in \mathbb{R}^d} f_i(y) \right) = \min_{y \in \mathbb{R}^d} f_i(y).
\]

Therefore, from above we obtain,
\[
\sum_{i \in \hat{S}} f_i(\hat{x}) - \sum_{i \in S} \min_{y \in \mathbb{R}^d} f_i(y) \leq \min_{x \in \mathbb{R}^d} \sum_{i \in S} f_i(x) - \sum_{i \in S} \min_{y \in \mathbb{R}^d} f_i(y).
\]
Upon rearranging the terms we obtain,
\[
\sum_{i \in \hat{S}} f_i(\hat{x}) \leq \min_{x \in \mathbb{R}^d} \sum_{i \in S} f_i(x) - \sum_{i \in S \setminus \hat{S}} \min_{y \in \mathbb{R}^d} f_i(y). \tag{37}
\]
As the true cost functions \(f_1, \ldots, f_n\) are assumed non-negative, i.e., \(f_i(x) \geq 0\) for all \(x\) and \(i\), then \(\min_{y \in \mathbb{R}^d} f_i(y) \geq 0\) for all \(i\). From substituting this in (37) we obtain,
\[
\sum_{i \in \hat{S}} f_i(\hat{x}) \leq \min_{x \in \mathbb{R}^d} \sum_{i \in S} f_i(x).
\]
Thus, by Definition \([\text{IV}]\) the \textit{normalize implementation} of algorithm II, i.e., \(\Pi^+\), is \((u, t)\)-weak resilient when the true cost functions are non-negative. \[\blacksquare\]
3.3 $t$-Resilience Property

In this section, we show that if the true cost functions are non-negative, and satisfy the $2t$-redundancy property, then the normalized implementation of a $(t, t)$-weak resilient algorithm, such as the one presented above, is also $t$-resilient. First, let us consider the special case wherein each true cost function has minimum value equal to zero.

**Lemma 4.** Suppose that Assumption 1 holds true, and $n > 2t$. If the true cost functions of the agents satisfy the $2t$-redundancy property, and each true cost function has minimum value equal to zero, then a $(t, t)$-weak resilient algorithm is also $t$-resilient.

**Proof.** Let $\Pi$ be a $(t, t)$-weak resilient collaborative optimization algorithm. Consider an execution of $\Pi$, named $E_{\mathcal{F}}$, where $\mathcal{F}$ denotes the set of faulty agents with $|\mathcal{F}| \leq t$. The remaining agents in $S = \{1, \ldots, n\} \setminus \mathcal{F}$ are non-faulty. Suppose that the true cost function of each agent $i$ in execution $E_{\mathcal{F}}$ is $f_i$.

As $E_{\mathcal{F}}$ is an arbitrary execution, to prove the lemma it suffices to show that the output of $\Pi$ in execution $E_{\mathcal{F}}$ is a minimum of the sum of the true cost functions of all the non-faulty agents $S$.

We have assumed that the minimum values of the functions $f_1(x), \ldots, f_n(x)$ are zero, i.e.,

$$\min_{x \in \mathbb{R}^d} f_i(x) = 0, \quad 1 \leq i \leq n. \quad (38)$$

In the rest of the proof, the notation ‘$\min_{x \in \mathbb{R}^d}$’ is simply written as ‘$\min$’ unless otherwise noted.

By applying the condition in Definition 3 of $2t$-redundancy property for all possible $\hat{S} \subseteq S$ (where $|\hat{S}| \geq n - 2t$) we can conclude that the set $\arg \min \sum_{i \in \hat{S}} f_i(x)$ is contained in the set $\arg \min f_i(x)$ for each $i \in S$. This, and the fact that each individual cost function has minimum value 0, implies that

$$\min \sum_{i \in \hat{S}} f_i(x) = \sum_{i \in \hat{S}} \min f_i(x).$$

Substituting from (38) above implies that

$$\min_{i \in \hat{S}} f_i(x) = 0. \quad (39)$$

Let $x_{\Pi}$ denote the output of $\Pi$. As $\Pi$ is $(t, t)$-weak resilient, there exists a subset $\hat{S}$ of $S$ of size $|S| - t$ such that

$$\sum_{i \in \hat{S}} f_i(x_{\Pi}) \leq \min \sum_{i \in S} f_i(x).$$

Substituting from (39) above implies that

$$\sum_{i \in \hat{S}} f_i(x_{\Pi}) \leq 0.$$
From (38), \( f_i(x_{\Pi}) \geq 0, \forall i \). The above implies that
\[
f_i(x_{\Pi}) = 0, \quad \forall i \in \hat{S}.
\]

Alternately,
\[
x_{\Pi} \in \bigcap_{i \in \hat{S}} \arg \min f_i(x).
\]

As \( |\hat{S}| = |S| - t \geq n - 2t \), the 2t-redundancy property implies that
\[
\bigcap_{i \in \hat{S}} \arg \min f_i(x) = \arg \min \sum_{i \in S} f_i(x).
\]

From substituting the above in (40) we obtain,
\[
x_{\Pi} \in \arg \min \sum_{i \in S} f_i(x).
\]

Thus, algorithm \( \Pi \) achieves t-resilience.

Utilizing the Lemma 4 we show that the normalized implementation of a \((t, t)\)-weak resilience is t-resilience when the true cost functions are non-negative, and satisfy the 2t-redundancy property. Specifically, we have the following theorem.

**Theorem 3.** Suppose that Assumption 1 holds true, \( n > 2t \), and we are given an algorithm \( \Pi \) that is \((t, t)\)-weak resilient when each true cost function has minimum value 0. Then the algorithm \( \Pi^+ \) obtained as the normalized implementation of \( \Pi \) is t-resilient when the true cost functions are non-negative, and satisfy the 2t-redundancy property.

**Proof.** Let the true cost functions of each agent \( i \) be denoted by \( f_i \). The true cost functions are assumed to be non-negative, i.e,
\[
f_i(x) \geq 0, \quad \forall x \in \mathbb{R}^d, \quad i \in \{1, \ldots, n\}.
\]

The true cost functions \( f_1, \ldots, f_n \) are also assumed to satisfy the 2t-redundancy property, i.e., the condition stated in Definition 2 holds true.

Suppose that algorithm \( \Pi \) is a \((t, t)\)-weak resilient. Consider the normalized implementation of algorithm \( \Pi \) presented in Section 3.2. It is easy to see that if \( n > 2t \) a priori then \( n > 2t \) upon completion of the step 1 for the updated values of \( n \) and \( t \). Also, due to Assumption 1 the set of non-faulty agents are not affected by the step 1. For the rest of the proof, we consider the set of agents, and the values of \( n \) and \( t \) after the step 1\(^6\)

---

\(^6\)In the worst-case for the algorithm, all faulty agents will send functions that have finite minimum values, thus, no faulty agents are removed in the execution step.
Recall that due to Assumption 1, for each $i$, $\min_{y \in \mathbb{R}^d} f_i(y)$ exists and finite. Note that, due to (34) in step 1, the true cost function of each agent $i$ during the execution of $\Pi$ in Step 2, denoted by $f_i^\dagger$, satisfies the following:

$$f_i^\dagger(x) = f_i(x) - \min_{y \in \mathbb{R}^d} f_i(y), \quad \forall x \in \mathbb{R}^d. \quad (41)$$

As

$$\min_{x \in \mathbb{R}^d} \left( \min_{y \in \mathbb{R}^d} f_i(y) \right) = \min_{y \in \mathbb{R}^d} f_i(y),$$

for each $i$, $\min_{x \in \mathbb{R}^d} f_i^\dagger(x) = 0$ and

$$\arg \min_{x \in \mathbb{R}^d} f_i^\dagger(x) = \arg \min_{x \in \mathbb{R}^d} \left( f_i(x) - \min_{y \in \mathbb{R}^d} f_i(y) \right) = \arg \min_{x \in \mathbb{R}^d} f_i(x). \quad (42)$$

Now, consider two arbitrary sets of agents $S_1$ and $S_2$ each of size $n - 2t$. As the true cost functions $f_i$'s are assumed to satisfy the 2t-redundancy property, by Definition 2 and Assumption 1,

$$\emptyset \neq \bigcap_{i \in S_1} \arg \min_{x \in \mathbb{R}^d} f_i(x) = \bigcap_{i \in S_2} \arg \min_{x \in \mathbb{R}^d} f_i(x).$$

Substituting from (42) above we obtain,

$$\emptyset \neq \bigcap_{i \in S_1} \arg \min_{x \in \mathbb{R}^d} f_i^\dagger(x) = \bigcap_{i \in S_2} \arg \min_{x \in \mathbb{R}^d} f_i^\dagger(x).$$

As the above holds for any two such subsets $S_1$ and $S_2$, the cost functions $f_1^\dagger, \ldots, f_n^\dagger$ satisfy the 2t-redundancy property.

The above together with Lemma 4 implies that $\Pi$, which is executed in the step 2, is $t$-resilient when the true cost function of each agent $i$ is $f_i^\dagger$. Now, consider an execution of $\Pi^+$, the algorithm obtained as the normalized implementation of $\Pi$, where $S$ denotes the set of non-faulty agents. Let, $\hat{x}$ denote the output of this execution. Then,

$$\hat{x} \in \arg \min_{x \in \mathbb{R}^d} \sum_{i \in S} f_i^\dagger(x). \quad (43)$$

From (41),

$$\sum_{i \in S} f_i^\dagger(x) = \sum_{i \in S} \left( f_i(x) - \min_{y \in \mathbb{R}^d} f_i(y) \right) = \sum_{i \in S} f_i(x) - \sum_{i \in S} \min_{y \in \mathbb{R}^d} f_i(y).$$

This implies that

$$\arg \min_{x \in \mathbb{R}^d} \sum_{i \in S} f_i^\dagger(x) = \arg \min_{x \in \mathbb{R}^d} \sum_{i \in S} f_i(x).$$

Substituting this in (43) we obtain,

$$\hat{x} \in \arg \min_{x \in \mathbb{R}^d} \sum_{i \in S} f_i(x).$$

The above argument holds for every execution of $\Pi^+$. Hence, by Definition 1 the normalized implementation of $\Pi$ is $t$-resilient. ■
We have the following corollary of Theorem 2 and Theorem 3.

**Corollary 1.** If the true cost functions of the agents satisfy the $2t$-redundancy property, and are non-negative, then the normalized implementation of the proposed $(t, t)$-weak resilient algorithm in Section 3.1 is $t$-resilient.

Note that the algorithm presented in this section is computationally much simpler than the $t$-resilient algorithm previously presented in Section 2.1. However, the algorithm in this section relies on an additional assumption that the true cost function of each non-faulty agent is non-negative. In general, there is a trade-off between complexity of the algorithm, and the assumptions made regarding the true cost functions, as the discussion below also illustrates.

### 4 Gradient-Descent Based Algorithm

In certain application of collaborative optimization, the algorithms only use information about the gradients of the agents’ cost functions. Collaborative learning is one such application [5]. Due to its practical importance, fault-tolerance in collaborative learning has gained significant attention in recent years [1, 2, 4, 10, 30].

In this section, we briefly summarize a gradient-descent based distributed collaborative optimization algorithm wherein the agents only send gradients of their cost functions to the server, instead of sending their entire cost functions. The algorithm was proposed in our prior work [13], where we proved $t$-resilience of the algorithm when the true cost functions satisfy the $2t$-redundancy and certain additional properties.

The proposed algorithm is iterative. For an execution of the algorithm, let $S$ denote the set of non-faulty agents and suppose that the true cost functions of the agents are $f_1(x), \ldots, f_n(x)$. The server maintains an estimate of the minimum point, which is updated in each iteration of the algorithm. The initial estimate, named $x^0$, is chosen arbitrarily by the server from $\mathbb{R}^d$. In iteration $s \in \{0, 1, \ldots\}$, the server computes estimate $x^{s+1}$ in steps S1 and S2 as described below.

In Step S1, the server obtains from the agents the gradients of their local cost functions at $x^s$. A faulty agent may send an arbitrary $d$-dimensional vector for its gradient. Each non-faulty agent $i \in S$ sends the gradient of its true cost function at $x^s$, i.e., $\nabla f_i(x^s)$. In Step S2, to mitigate the detrimental impact of such incorrect gradients, the algorithm uses a filter to “robustify” the gradient aggregation step. In particular, the gradients with the largest $t$ norms are “clipped” so that their norm equals the norm of the $(t+1)$-th largest gradient (or, equivalently, the $(n-t)$-th smallest gradient). The remaining gradients remain unchanged. The resulting gradients are then accumulated to obtain the update direction, which is then used to compute $x^{s+1}$. We refer to the method used in Step S2 for clipping the largest $t$ gradients as “Comparative Gradient Clipping” (CGC), since the largest $t$ gradients are clipped to a norm that is “comparable” to the next largest gradient.

Detailed description of the algorithm and its resilience guarantee can be found in our prior work [13]. The above algorithm performs correctly despite the use of
a simple filter on the gradients, which only takes into account the gradient norms, not the direction of the gradient vectors. This simplification is possible due to the assumptions made on the cost functions [13]. Weaker assumptions will often necessitate more complex algorithms.

5 Summary of the Results

We have made the following key contributions in this report.

- **In case of redundant cost functions:** We proved the necessary and sufficient condition of $2t$-redundancy for $t$-resilience in collaborative optimization. We have presented $t$-resilient collaborative optimization algorithms to demonstrate the trade-off between the complexity of a $t$-resilient algorithm, and the properties of the agents’ cost functions.

- **In case of independent cost functions:** We introduced the metric of $(u, t)$-weak resilience to quantify the notion of resilience in case when the agents’ cost functions are independent. We have presented an algorithm that obtains $(u, t)$-weak resilience for all $u \geq t$ when the cost functions are non-negative and $n > 2t$.

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A Proof of Lemma 1

Lemma 1. For a non-empty set $T$, consider a set of functions $g_i(x)$, $i \in T$, such that

$$\bigcap_{i \in T} \arg \min_x g_i(x) \neq \emptyset.$$ 

Then

$$\bigcap_{i \in T} \arg \min g_i(x) = \arg \min_x \sum_{i \in T} g_i(x).$$

Proof. Consider any non-empty set $T$, and functions $g_i(x)$, $i \in T$, such that

$$\bigcap_{i \in T} \arg \min_x g_i(x) \neq \emptyset.$$ 

**Part I:** Consider any $x^0 \in \bigcap_{i \in T} \arg \min_x g_i(x)$. Since each cost function $g_i(x)$, $i \in T$, is minimized at $x^0$, it follows that $\sum_{i \in T} g_i(x)$ is also minimized at $x^0$. In other words, it is trivially true that

$$x^0 \in \bigcap_{i \in T} \arg \min_x g_i(x) \subseteq \arg \min_x \sum_{i \in T} g_i(x). \quad (44)$$

**Part II:** Let $x^-$ be a point such that

$$x^- \in \bigcap_{i \in T} \arg \min_x g_i(x).$$

Then

$$x^- \in \arg \min_x g_i(x), \quad \forall i \in T \quad (45)$$

From (44), $\arg \min_x \sum_{i \in T} g_i(x) \neq \emptyset$. Now we show that $\arg \min_x \sum_{i \in T} g_i(x) \subseteq \bigcap_{i \in T} \arg \min_x g_i(x)$. The proof is by contradiction.

Suppose that there exists a point $x^\dagger$ such that

$$x^\dagger \in \arg \min_x \sum_{i \in T} g_i(x), \quad (46)$$

and

$$x^\dagger \not\in \bigcap_{i \in T} \arg \min_x g_i(x). \quad (47)$$

This and (45) implies that there exists $i^\dagger \in T$ such that

$$g_{i^\dagger}(x^\dagger) > g_{i^\dagger}(x^-).$$

Also, from (45), for each $i \in T \setminus \{i^\dagger\}$,

$$g_i(x^\dagger) \geq g_i(x^-).$$
The above two inequalities together imply that,
\[
\sum_{i \in T} g_i(x^\dagger) = \sum_{i \in T \setminus \{i^\dagger\}} g_i(x^\dagger) + g_{i^\dagger}(x^\dagger) > \sum_{i \in T} g_i(x^-).
\]
The above is a contradiction of (46). Therefore, \(x^\dagger \in \bigcap_{i \in T} \arg \min_x g_i(x)\). This implies,
\[
\arg \min_{i \in T} \sum_{i \in T} g_i(x) \subseteq \bigcap_{i \in T} \arg \min g_i(x).
\]

From (44) and (48),
\[
\arg \min_{i \in T} \sum_{i \in T} g_i(x) = \bigcap_{i \in T} \arg \min g_i(x) \tag{49}
\]

**B Definitions 2 and 3 are Equivalent**

The lemma below shows that the two definitions of 2t-redundancy, namely Definition 2 and Definition 3, stated in Section 1.2 are equivalent.

**Lemma 5.** Suppose that Assumption 7 holds true, and \(n > 2t\). Then, conditions in Definition 2 and Definition 3 are equivalent.

**Proof.** Let the true cost functions of each agent \(i\) be denoted by \(f_i(x)\).

**Part I:** We first show that the condition in Definition 2 implies that in Definition 3. Suppose that the condition stated in Definition 2 holds true. Consider two arbitrary sets of agents \(S\) and \(\hat{S}\) with \(|S| \geq n - t\), \(|\hat{S}| \geq n - 2t\), and \(\hat{S} \subseteq S\). We need to show that (4) in Definition 3 holds true.

Note, (3) in Definition 2 implies that there exists a point \(x^*\) such that
\[
x^* \in \bigcap_{i \in \hat{S}^\dagger} \arg \min_{x \in \mathbb{R}^d} f_i(x), \quad \forall \hat{S}^\dagger \subseteq S, \quad |\hat{S}^\dagger| = n - 2t.
\]
Therefore,
\[
x^* \in \bigcap_{i \in S} \arg \min f_i(x) \neq \emptyset.
\]

Thus, from Lemma 1,
\[
\bigcap_{i \in S} \arg \min f_i(x) = \arg \min \sum_{i \in S} f_i(x) \tag{50}
\]

Now, consider an arbitrary subset \(S_1 \subseteq \hat{S}\) with \(|S_1| = n - 2t\). Then,
\[
\bigcap_{i \in S} \arg \min f_i(x) \subseteq \bigcap_{i \in \hat{S}} \arg \min f_i(x) \subseteq \bigcap_{i \in S_1} \arg \min f_i(x). \tag{51}
\]
We now show that when the condition in Definition 2 holds true then \( \bigcap_{i \in S} \arg\min f_i(x) = \bigcap_{i \in S_1} \arg\min f_i(x) \). The proof is by contradiction.

Suppose that

\[
\bigcap_{i \in S} \arg\min f_i(x) \subset \bigcap_{i \in S_1} \arg\min f_i(x). 
\]  

(52)

This implies that there exists a point \( x^\dagger \) in \( \bigcap_{i \in S_1} \arg\min f_i(x) \) such that

\[
x^\dagger \notin \bigcap_{i \in S} \arg\min f_i(x).
\]

This implies that there exists \( i^\dagger \in S \) such that \( x^\dagger \notin \arg\min f_{i^\dagger}(x) \). Now, consider a subset \( S_2 \subseteq S \) with \( |S_2| = n - 2t \) and \( i^\dagger \in S_2 \). Then,

\[
x^\dagger \notin \bigcap_{i \in S_2} \arg\min f_i(x).
\]

Since \( x^\dagger \in \bigcap_{i \in S_1} \arg\min f_i(x) \), the above implies that

\[
\bigcap_{i \in S_1} \arg\min f_i(x) \neq \bigcap_{i \in S_2} \arg\min f_i(x)
\]

which contradicts (3) in Definition 2. Therefore, (52) cannot hold, and so,

\[
\bigcap_{i \in S} \arg\min f_i(x) = \bigcap_{i \in S_1} \arg\min f_i(x). 
\]  

(53)

Substituting the above in (51) implies that

\[
\bigcap_{i \in S} \arg\min f_i(x) = \bigcap_{i \in \hat{S}} \arg\min f_i(x)
\]

The above together with (50) imply that

\[
\bigcap_{i \in \hat{S}} \arg\min f_i(x) = \arg\min \sum_{i \in S} f_i(x). 
\]  

(54)

Note that the above argument holds true for all pairs of sets \( \hat{S}, S \) with \( |\hat{S}| \geq n - 2t \), \( |S| \geq n - t \), and \( \hat{S} \subseteq S \). Therefore, the above implies that the condition stated in Definition 3 is true.

**Part II:** We now show that the condition in Definition 3 implies the condition in Definition 2. Suppose that the condition stated in Definition 3 holds true.

From Assumption 1

\[
\arg\min \sum_{i=1}^{n} f_i(x) \neq \emptyset.
\]
From substituting $S = \{1, \ldots, n\}$ in the equation (4) in Definition 3, we trivially obtain the following for every two subsets of agents $S_1$ and $S_2$ each containing $n - 2t$ agents.

$$\emptyset \neq \bigcap_{i \in S_1} \arg\min f_i(x) = \bigcap_{i \in S_2} \arg\min f_i(x).$$

Hence, the condition in Definition 3 implies the condition in Definition 2. ■