A General Framework for Solving Singular SPDEs with Applications to Fluid Models Driven by Pseudo-differential Noise

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Abstract

In this paper we focus on nonlinear SPDEs with singularities included in both drift and noise coefficients, for which the Gelfand-triple argument developed for (local) monotone SPDEs turns out to be invalid. We propose a general framework of proper regularization to solve such singular SPDEs. As applications, the (local and global) existence is presented for a broad class of fluid models driven by pseudo-differential noise of arbitrary order, which include the stochastic magnetohydrodynamics (hence Navier-Stokes/Euler) equations, stochastic Camassa-Holm type equations, stochastic aggregation-diffusion equation and stochastic surface quasi-geostrophic equation. Thus, some recent results derived in the literature are considerably extended in a unified way.

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Let $X$ be a progressively measurable process on $H$ of Hilbert-Schmidt operators from a larger Hilbert space $M$ such that $H$ is densely embedded into $M$. Hence weakly continuous in $M$, the variables $M$ representing larger Hilbert space $H$, $M$ and $H$ are described by how larger $M$ is than $H$. The study of stochastic partial differential equations (SPDEs), in particular for singular nonlinear models arising from fluid mechanics, is very rich and active. There exist a huge number of references in the literature, see, for instance, [4, 22, 30, 43, 45, 51] and monographs [11, 26, 27, 42]. In this paper, we intend to build up a general framework for nonlinear SPDEs with singularity in both drift and noise, such that a large class of models can be solved in a unified way.

1 Introduction

The study of stochastic partial differential equations (SPDEs), in particular for singular nonlinear models arising from fluid mechanics, is very rich and active. There exist a huge number of references in the literature, see, for instance, [4, 22, 30, 43, 45, 51] and monographs [11, 26, 27, 42]. In this paper, we intend to build up a general framework for nonlinear SPDEs with singularity in both drift and noise, such that a large class of models can be solved in a unified way.

1.1 Singular evolution equation in Hilbert space

We will study a stochastic system on a separable Hilbert space $H$, with coefficients taking values in a larger Hilbert space $M$ such that

$$H \hookrightarrow M,$$

that is, $H$ is densely embedded into $M$ with $\| \cdot \|_M \lesssim \| \cdot \|_H$. Here and in the sequel, for two nonnegative variables $A$ and $B$, $A \lesssim B$ means that there exists a constant $c > 0$ such that $A \leq c B$. The level of singularity is described by how larger $M$ is than $H$.

The system will be driven by the cylindrical Brownian motion $W$ on another separable Hilbert space $U$:

$$W(t) := \sum_{k \geq 1} W_k(t)e_k, \quad t \geq 0,$$

where $\{e_k\}_{k \geq 1}$ is a complete orthonormal basis of $U$, and $\{W_k\}_{k \geq 1}$ is a sequence of independent 1-D Brownian motions on a right-continuous complete filtration probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$. Let $L_2(U; M)$ be the space of Hilbert-Schmidt operators from $U$ to $M$.

We now consider the following stochastic equation for unknown process $X = (X(t))_{t \geq 0}$ on $H$:

$$dX(t) = \{b(t, X(t)) + g(t, X(t))\} dt + h(t, X(t))dW(t), \quad t \geq 0,$$

where

$$b : [0, \infty) \times H \to H, \quad g : [0, \infty) \times H \to M, \quad h : [0, \infty) \times H \to L_2(U; M)$$

are measurable. In applications, $b$ refers to the regular part of drift, while the drift term $g$ and the noise term $h(t, X(t))dW(t)$ are singular in the sense that they take values in $M$, which is larger than the state space $H$.

**Definition 1.1.** Let $\tau$ be a stopping time satisfying $\mathbb{P}(\tau > 0) = 1$, and let $(X, \tau) := (X_t)_{t \in [0, \tau)}$ be a progressively measurable process on $H$.

1. **(Local solution).** We call $(X, \tau)$ a local solution to (1.1), if $\mathbb{P}$-a.s. we have $t \mapsto X(t)$ is continuous in $M$ (hence weakly continuous in $H$),

$$\sup_{s \in [0, t]} \| X(s) \|_H < \infty, \quad t \in [0, \tau),$$
and the following equation holds on the space $\mathbb{M}$:

$$X(t) - X(0) = \int_0^t \{b(s, X(s)) + g(s, X(s))\} \, ds + \int_0^t h(s, X(s)) \, dW(s), \quad t \in [0, \tau).$$

(1.3)

(2) (Maximal solution). A local solution $(X, \tau)$ is called maximal, if

$$\limsup_{t \uparrow \tau} \|X(t)\|_\mathbb{M} = \infty \quad \text{a.s. on} \quad \{\tau < \infty\}.$$

Particularly, if $\mathbb{P}(\tau = \infty) = 1$, the solution is called global or non-explosive.

The first main result in this paper presents the following properties for solutions to (1.1) with coefficients given in (1.2).

**Main Result (I)** (see Theorem 2.1 below). *The existence, uniqueness, continuity in $\mathbb{H}$, blow-up and non-explosion criterion.*

### 1.2 SPDE with pseudo-differential noise

To see that (1.1) includes a large class of singular SPDEs as special situations, we introduce below a general type SPDE on $\mathbb{K}^d$ for some $d \in \mathbb{N}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$. We refer to Section 3.1 for the precise definitions of related notions.

Let $\{W_k, W_k\}_{k \geq 1}$ be independent 1-D Brownian motions, and let $dW_k(t)$ be the Stratonovich stochastic differential. Let $\mathcal{F}$ be the Fourier transform on $\mathbb{K}$ and $i = \sqrt{-1}$ be the imaginary unit. For $s > 0$ and $m \in \mathbb{N}$, we denote by $H^s(\mathbb{K}^d; \mathbb{R}^m)$ the standard Sobolev spaces of order $s$ for $\mathbb{R}^m$-valued functions on $\mathbb{K}^d$. Let $\text{OPS}^*$ be the class of pseudo-differential operators with symbols in $\mathbb{S}^*$, and let $\text{OPS}_{0}^*$ be subset of $\text{OPS}^*$ with symbols independent of $x$ (see (3.3) and (3.4)).

Let $\tilde{\mathbb{K}} := \mathbb{R}$ if $\mathbb{K} = \mathbb{R}$, and $\tilde{\mathbb{K}} := \mathbb{Z}$ for $\mathbb{K} = \mathbb{T}$. Assume that $\tilde{\Pi} : L^2(\mathbb{K}^d; \mathbb{R}^m) \to L^2(\mathbb{K}^d; \mathbb{R}^m)$ is a projection operator satisfying

$$[\mathcal{F}(\tilde{\Pi}f)](\xi) = \tilde{\pi}(\xi)(\mathcal{F} f)(\xi), \quad \langle \tilde{\Pi} f, g \rangle_{L^2} = \langle f, \tilde{\Pi} g \rangle_{L^2}, \quad \|\tilde{\Pi} f\|_{H^s} \leq \|f\|_{H^s},$$

for all $s \geq 0$, $\xi \in \tilde{\mathbb{K}}^d$, $f, g \in L^2$, and some measurable $\tilde{\pi} : \tilde{\mathbb{K}}^d \to C^{m \times m}$ such that

$$\tilde{\pi}(-\xi) = \overline{\tilde{\pi}(\xi)} := \text{Re}[\tilde{\pi}(\xi)] - \text{Im}[\tilde{\pi}(\xi)] i, \quad \xi \in \tilde{\mathbb{K}}^d.$$

Typical examples of $\tilde{\Pi}$ in the theory of PDEs modeling fluid dynamics include

$$\tilde{\Pi} = \begin{cases} I : \text{identity mapping}, \\ \Pi_d : \text{Leray projection on } \mathbb{K}^d, \text{ see (3.9)}, \\ \Pi_0 : \text{zero-average projection on } \mathbb{T}^d, \text{ see (3.12)}. \end{cases}$$

Particularly, if $\tilde{\Pi} = \Pi_d$, we assume $m = d$ and we will consider the equations in $H^s_{\text{div}}(\mathbb{K}^d; \mathbb{R}^d)$ (see (3.8)).

Consider the following nonlinear PDE in $\mathbb{H} := \tilde{\Pi}H^s(\mathbb{K}^d; \mathbb{R}^m)$:

$$\frac{d}{dt} X(t) = \tilde{\Pi} \mathcal{E} X(t) + \tilde{b}(X(t)) + \tilde{g}(X(t)), \quad t \geq 0,$$

(1.6)

where for some constants $p_0, q_0 > 0$,

$$\text{OPS}_{0}^{2p_0} \ni \mathcal{E} : H^s(\mathbb{K}^d; \mathbb{R}^m) \to H^{s-2p_0}(\mathbb{K}^d; \mathbb{R}^m)$$

is a negative semi-definite operator (see Section 3.1 and (E) below for more details),

$$\tilde{b} : \tilde{\Pi}H^s(\mathbb{K}^d; \mathbb{R}^m) \to \tilde{\Pi}H^s(\mathbb{K}^d; \mathbb{R}^m)$$
is the regular part, and
\[ \tilde{g} : \tilde{\Pi}H^s(\mathbb{K}^d; \mathbb{R}^m) \to \tilde{\Pi}H^{s-q_0}(\mathbb{K}^d; \mathbb{R}^m) \]
is the singular part losing regularities of order \( q_0 \) (see (F) for the statement). The parameters \( s, q_0, p_0 \) are to be determined in different examples (see Sections 1.4 and 4 for concrete \((\tilde{\mathcal{E}}, \tilde{h}, \tilde{g})\) and \((s, q_0, p_0)\)).

In the existing literature, SPDEs with transport noise has been intensively investigated in recent years, where the transport noise is given by

\[
(1.7) \quad \left( c_k X + \left( \sum_{i=1}^{d} d_{k,i} \partial x_i \right) X \right) \circ dW_k(t), \quad \partial_i := \frac{\partial}{\partial x_i},
\]

where \( c_k, d_{k,i} \) are nice \( \mathbb{R}^{m \times m} \)-valued and \( \mathbb{R} \)-valued functions, respectively, see for examples [2, 4, 5, 7, 22, 23, 28–30, 32, 35]. However, as far as we know, there is no result for the case that \( \partial_i \) in (1.7) is replaced by more general differential operators, for which the system is allowed to have non-local higher order singular noise coefficients.

Therefore, the second main result in this paper is to study (1.6) with pseudo-differential noise. We will consider the following SPDE for unknown process \( X(t) \) on \( \mathbb{H} = \tilde{\Pi}H^s(\mathbb{K}^d; \mathbb{R}^m) \):

\[
\begin{align*}
dX(t) &= \left\{ (\tilde{\Pi} \mathcal{E})X(t) + \tilde{b}(X(t)) + \tilde{g}(X(t)) \right\} dt \\
&\quad + \sum_{k=1}^{\infty} \left\{ (\tilde{\Pi} \mathcal{A}_k)X(t) \circ dW_k(t) + \tilde{\Pi} h_k(t, X(t)) d\tilde{W}_k(t) \right\}, \quad t \geq 0,
\end{align*}
\]

where \( \{\mathcal{A}_k\}_{k \geq 1} \subset \text{OPS}^{\infty} \) is a sequence of pseudo-differential operators not far away from anti-symmetric (see (D) for the precise assumptions) and \( h_k(t, \cdot) : H^s(\mathbb{K}^d; \mathbb{R}^m) \to H^s(\mathbb{K}^d; \mathbb{R}^m) \) \((k \geq 1, \ t \geq 0)\) are locally Lipschitz continuous (we refer to (F) for the details).

Then the second main result in this paper focuses on

**Main Result (II)** (see Theorem 3.1 below). **Well-posedness of (1.8), including existence, uniqueness, time-continuity, blow-up criterion and global existence.**

### 1.3 Comparison, motivation and remarks

To begin with, we give some comments on Definition 1.1. Due to the singularities of \( g \) and \( h \), the corresponding integrals in (1.3) are only defined in the larger space \( \mathbb{M} \), but their sum has to take value in the state space \( \mathbb{H} \) since \( X(t) \) is a process on \( \mathbb{H} \). In applications, the singular coefficients \( g \) and \( h \) may take values in \( \mathbb{M}_1 \) and \( L_2(U; \mathbb{M}_2) \) for some different Hilbert spaces \( \mathbb{M}_1 \) and \( \mathbb{M}_2 \). In this case, we take a larger Hilbert space \( \mathbb{M} \) with \( \mathbb{M}_i \hookrightarrow \mathbb{M} \) \((i = 1, 2)\).

#### 1.3.1 Comparing Main Result (I) with existing results

1. For the present model with singular noise coefficient, the Gelfand-triple argument developed for (local) monotone SPDEs turns out to be invalid. To see this, we consider the triplet of embedded Hilbert Spaces \( \mathbb{V} \hookrightarrow \mathbb{H} \hookrightarrow \mathbb{M} \) such that \( \mathbb{V} \hookrightarrow \mathbb{H} \) is dense and \( \langle X, Y \rangle_{\mathbb{M} \times \mathbb{V}} = \langle X, Y \rangle_{\mathbb{H}} \). Then we consider the following SDE:

\[
\Psi(t) = \Psi(0) + \int_0^t A dt' + \int_0^t B dW(t'), \quad A \in \mathbb{M}, \quad B \in L_2(U; \mathbb{H}), \quad \Psi(0) \in \mathbb{H}.
\]

In order to apply Itô’s formula to the above SDE, the condition that \( \Psi(t) \in \mathbb{V} \) (see [32, 48, 63]) is necessary because the dual product \( \langle A, \Psi \rangle_{\mathbb{M} \times \mathbb{V}} \) needs to be well-defined. But this means that \( \Psi(t) \in \mathbb{V} \hookrightarrow \mathbb{H} \ni \Psi(0) \), that is, \( \Psi(t) \) is more regular than its initial data \( \Psi(0) \), which is inconsistent for ideal fluid motion without viscosity. For example, for the following inviscid Burgers’ equation with the choice \( \mathbb{H} := H^s, \mathbb{V} := H^{s+1}, \mathbb{M} := H^{s-1} \) and \( s > 3/2 \) (see Sections 1.4 and 3.1 for more notations)

\[
\frac{d}{dt}X + X \partial X = 0, \quad X(0) \in \mathbb{H},
\]
one can only know that \( X \in \mathbb{H} \) and hence \( X \partial X \in \mathcal{M} \). Then, neither the inner product \( \langle X \partial X, X \rangle_{\mathbb{H}} \) nor the dual product \( \langle X \partial X, X \rangle_{\mathcal{M} \times \mathcal{V}} \) make no sense. Alternatively to the Galerkin approximation used in the monotone situation, we will propose the proper regularization (see Definition 2.1) to solve (1.1).

(2) A blow-up criterion of solutions in \( \mathbb{H} \) is described by the \( \mathcal{M} \)-norm, see (2.2) below. Since \( || \cdot ||_{\mathcal{M}} \lesssim || \cdot ||_{\mathbb{H}} \), the blow up in \( || \cdot ||_{\mathcal{M}} \) is stronger than that in \( || \cdot ||_{\mathbb{H}} \), but they are indeed equivalent under the present framework. Due to the singularities, in general, a solution \( \tilde{X}(t) \) to (1.1) may be only continuous in \( \mathcal{M} \) rather than \( \mathbb{H} \), so a criterion on the continuity in \( \mathbb{H} \) is provided in Theorem 2.1 (ii). The non-explosion is included in Theorem 2.1 (iii) for strong enough noise in the sense of condition (C), see also [12, 40, 49] for non-explosion results in similar spirits.

(3) In Theorem 2.1, \( X(0) \) is only assumed to be an \( \mathcal{F}_0 \)-measurable \( \mathbb{H} \)-valued variable without any moment condition. In this case the conditional expectation \( \mathbb{E}[\cdot | \mathcal{F}_0] \) will be used to replace the expectation \( \mathbb{E} \) in the construction of solutions, see Lemma 2.2. It seems that conditional expectation has been rarely used in the literature of SPDEs. Besides, it is worthwhile mentioning that our framework does not require any compactness of the embedding \( \mathbb{H} \rightarrow \mathcal{M} \), so that the main result applies not only to SPDEs on compact spaces, where the compactness is needed to apply Prokhorov’s Theorem and Skorokhod’s Theorem, but also to SPDEs on unbounded domain like \( \mathbb{R}^d \), as shown by Theorems 3.1 and examples in Section 4.

1.3.2 Motivation and remarks on Main Result (II)

Motivation. Pseudo-differential operators offer a non-local extension to classical differential operators. Exploring pseudo-differential noise can provide a versatile framework to model intricate random phenomena involving non-local random interactions. This can be particularly useful in turbulence models, where the behavior of fluid at one point is influenced by the behavior of fluid at distant points. To gain more insight into the non-locality arising from pseudo-differential noise, which classical transport noise cannot capture, we examine the stochastic Burgers’ equation as a simple yet intriguing example. Let \( W(t) \) be a standard 1-D Brownian motion. We first consider the case of classical derivative, i.e.,

\[
(1.9) \quad dX + X \partial X \, dt = \sqrt{2\mu} \partial X \circ dW(t), \quad \mu > 0.
\]

By utilizing the following relation for a semi-martingale \( \Theta(t) \):

\[
(1.10) \quad \Theta(t) \circ dW(t) = \Theta(t) \, dW(t) + \frac{1}{2} \left( \Theta, W \right) (t) \text{ with } \left\langle \cdot, \cdot \right\rangle \text{ being the quadratic variation},
\]

we can reformulate (1.9) as:

\[
dX + X \partial X \, dt = \sqrt{2\mu} \partial X \, dW(t) + \mu \partial^2 X \, dt.
\]

Let \( \Xi(t) := \exp \left\{ -\sqrt{\frac{2}{\mu}} W(t) \right\} \) be an operator-valued process. Then, in the sense of Fourier multiplier, we have the following SDE:

\[
d\Xi(t) = -\sqrt{\frac{2}{\mu}} \partial \Xi(t) \, dW(t) + \mu \partial^2 \Xi(t) \, dt.
\]

Therefore, for \( Y(t) := [\Xi X](t) \), we can derive

\[
(1.11) \quad dY = [d\Xi](X) + \Xi(dX) - 2\mu \partial^2 \Xi X \, dt = -\Xi(\Xi^{-1} Y \cdot \partial \Xi^{-1} Y) \, dt.
\]

Next, we consider the case of pseudo-differential noise, i.e.,

\[
dX + X \partial X \, dt = \sqrt{2\mu} (-\partial^2)^\alpha X \circ dW(t), \quad \mu > 0, \quad \alpha \in (0, 1/2].
\]

A similar argument yields

\[
\frac{d}{dt} Y + \Xi_{\alpha}(\Xi_{\alpha}^{-1} Y \cdot \partial \Xi_{\alpha}^{-1} Y) = 0, \quad Y(t) := [\Xi_{\alpha} X](t), \quad \Xi_{\alpha}(t) := \exp \left\{ -\sqrt{\frac{2}{\mu}} W(t)(-\partial^2)^\alpha \right\}.
\]

Comparing the two cases above, we observe that the kernels of \( \Xi \) and \( \Xi^{-1} \) are delta functions, which indicates their local-in-\( x \) nature. In the case of pseudo-differential noise, the non-locality arising from \( \sqrt{\frac{2}{\mu}} (-\partial^2)^\alpha X \circ dW(t) \) is characterized by the term \( \Xi_{\alpha}(\Xi_{\alpha}^{-1} Y \cdot \partial \Xi_{\alpha}^{-1} Y) \) at the level of \( Y \). However, it seems that the kernels of \( \Xi_{\alpha} \) and \( \Xi_{\alpha}^{-1} \) cannot be explicitly determined.
Projection $\Pi$. For simplicity, in (1.8) we assume that $\bar{b}$ and $\bar{g}$ already take values in projected space, as they come from deterministic PDEs. However, we keep $\Pi$ in $\Pi_{d}, \Pi_{d}A_{k}$ and $\Pi\tilde{h}_{k}$. For example, if $\Pi = \Pi_{d}$ and $\mathcal{E} = \Delta$, $\Pi_{d}\mathcal{E}$ is known as Stokes operator. As for $A_{k}$ and $\tilde{h}_{k}$, the projection is also necessary to make solutions stay in the projected space. However, in calculations it is non-trivial to deal with the case $\Pi \neq \mathbb{I}$ (see (D8) and Lemma 3.7) since $\Pi$ may not be a pseudo-differential operator. For instance, the Fourier multiplier of $\Pi = \Pi_{d}$ is singular at 0 (see (3.9) below).

Order of pseudo-differential operators. We mainly consider the case that the operator $A_{k}$ in (1.8) contains two parts: $x$-dependent part with order $r_{1} \in [0, 1]$ and $x$-independent part with order $r_{2} \geq r_{1}$ (see (D)). Each of them extends the classical transport type noise structure (1.7). In particular, $r_{2}$ can be arbitrary large. Accordingly, Theorem 3.1 enlarges many results derived in the literature by allowing highly non-local and singular noise (see examples in Section 1.4). We refer to [4, 5, 7, 22, 28, 32, 36] and the references therein for the known results with transport type noise given by (1.7).

1.4 Results on specific models

As (1.8) covers a broad class of SPDEs with different choices of $(\mathcal{E}, b, g)$, we apply Theorem 3.1 to some important models from fluid mechanics. Since there is an enormous literature on these equations, we do not try to provide a complete account but only mention a few results.

Recall that $\mathbb{K} = \mathbb{R}$ or $\mathbb{T}$ and $\partial_{i}$ stands for the $i$-th partial derivative on $\mathbb{K}^{d}$. For a function $f = (f_{i})_{1 \leq i \leq d} : \mathbb{K}^{d} \rightarrow \mathbb{R}^{d}$, we let

$$(f \cdot \nabla) := \sum_{i=1}^{d} f_{i} \partial_{i}, \quad \nabla \cdot f := \sum_{i=1}^{d} \partial_{i} f_{i}.$$  

In particular, when $d = 1$, we simply denote by $\partial = \partial_{1}$ the derivative in $\mathbb{K}$ and let (see (3.7) for precise definition)

$$\Lambda := (-\Delta)^{\frac{1}{2}}, \quad D := (I - \Delta)^{\frac{1}{2}}.$$  

Let $\text{diag}(\cdot, \cdots, \cdot)$ be the diagonal operator. For more notations, we refer to Section 3.1.

(1) Magnetohydrodynamics (MHD) equation. Let $d \geq 2$. Consider the following incompressible generalized magnetohydrodynamics equation with fractional kinematic dissipation and magnetic diffusion on $\mathbb{K}^{d}$, cf. [64],

$$\begin{aligned}
&\frac{d}{dt}V(t) + \mu_{1}\Lambda^{2\alpha_{1}}V(t) + \Pi_{d}(V(t) \cdot \nabla)V(t) - \Pi_{d}(M(t) \cdot \nabla)M(t) = 0, \\
&\frac{d}{dt}M(t) + \mu_{2}\Lambda^{2\alpha_{2}}M(t) + (V(t) \cdot \nabla)M(t) - (M(t) \cdot \nabla)V(t) = 0, \\
&\nabla \cdot V(t) = \nabla \cdot M(t) = 0,
\end{aligned}  \tag{1.12}$$

where $V : [0, \infty) \times \mathbb{K}^{d} \rightarrow \mathbb{R}^{d}$ is the velocity field, $M : [0, \infty) \times \mathbb{K}^{d} \rightarrow \mathbb{R}^{d}$ is magnetic field, $\Pi_{d}$ is the Leray projection, $\alpha_{1}, \alpha_{2} \in [0, 1]$ are the fractional powers and $\mu_{1}, \mu_{2} \geq 0$ stand for the kinematic viscosity and magnetic diffusivity constants, respectively. Letting $X = (V, M)^{T}$, (1.12) can be reformulated as

$$\frac{d}{dt}X(t) = \mathcal{E}_{\text{mhd}}X(t) + g^{\text{mhd}}(X(t))$$

with

$$\begin{aligned}
\mathcal{E}_{\text{mhd}} &:= -\text{diag}(\mu_{1}\Lambda^{2\alpha_{1}}, \mu_{2}\Lambda^{2\alpha_{2}}), \\
g^{\text{mhd}}(X) &:= \left(\Pi(M \cdot \nabla)M - \Pi(V \cdot \nabla)V, (M \cdot \nabla)V - (V \cdot \nabla)M\right)^{T}.
\end{aligned}  \tag{1.13}$$

Obviously, when $M \equiv 0$ and $\alpha_{1} = 1$, the equation for $V(t)$ covers the incompressible Navier-Stokes equation ($\mu_{1} > 0$) and the Euler equation ($\mu_{1} = 0$).

(2) Camassa-Holm (CH) type equation. The following equation for $X : [0, \infty) \times \mathbb{K} \rightarrow \mathbb{R}$

$$\begin{aligned}
&\frac{d}{dt}(X - \partial^{2}X)(t) + 3X(t)\partial X(t) = 2\partial X(t)\partial^{2}X(t) + X(t)\partial^{3}X(t)
\end{aligned}  \tag{1.14}$$
was first introduced by Fokas & Fuchssteiner [31] to study completely integrable generalizations of KdV equation with bi-Hamiltonian structure. In [15], Camassa & Holm proved that (1.14) can be connected to the unidirectional propagation of shallow water waves over a flat bottom (X represents the free surface of water), and now (1.14) is usually called the Camassa-Holm equation. In order to include some closely related equations (such as the Degasperis-Procesi equation [24], b-family equations [37], recently derived rotation Camassa-Holm equation [33], and even different drift terms in stochastic cases [2, 23]), we consider the following CH type equations:

\[
\frac{d}{dt}X(t) = b^{ch}(X(t)) + g^{ch}(X(t))
\]

by taking

\[
b^{ch}(X) := -\partial D^{-2}(\sum_{i=1}^{4} a_i X^i + a|\partial X|^2), \quad g^{ch}(X) := -X \partial X,
\]

where \(\{a, a_i\}_{1 \leq i \leq 4}\) are constants. We refer to [57] for the link of (1.15) to the above mentioned Camassa-Holm-family equations.

(3) Aggregation-diffusion (AD) equation. Let \(d \geq 2\). Consider the following aggregation-diffusion type models on \(\mathbb{K}^d\):

\[
\frac{d}{dt}X(t) + \nu \Lambda^{2\beta} X(t) + \gamma \nabla \cdot \{X(t) \nabla [\Phi \ast X(t)]\} = 0, \quad t \geq 0,
\]

where \(X : [0, \infty) \times \mathbb{K}^d \to \mathbb{R}\) represents the density of species (cells), \(\nu \geq 0\) and \(\gamma \in \mathbb{R} \setminus \{0\}\) are the diffusion and aggregation coefficients respectively, \(\beta \in [0, 1]\) is the diffusion order, \(\Phi\) is an interaction kernel and \(\ast\) stands for the convolution. The equation (1.16) has a range of applications arising in physics and biology with specific choices of \(\Phi\), see [39, 47] for self-organization of chemotactic movement, see [14] for biological swarm, and see the survey [16] for some other choices. In this paper we assume that \(\Phi \in H^\infty(\mathbb{K}^d, \mathbb{R}) := \cap_{s \geq 0} H^s(\mathbb{K}^d, \mathbb{R})\) such that \((\mathcal{F} \Phi)(\xi) \in S_0^{-2}\), where \(S_0^s\) on \(\mathbb{K}\) with \(s \in \mathbb{R}\) is defined in Section 3.1 below.

A typical example of \(\Phi\) is the Bessel kernel for which \(\{\Phi \ast\} = D^{-2} \in \text{OPS}^{-2}_{0}\) (see (3.7) and (3.4)), and in this case (1.16) reduces to the well-known parabolic-elliptic Keller-Segel system (see for example [21]):

\[
\frac{d}{dt}X(t) + \nu \Lambda^{2\beta} X(t) + \gamma \nabla \cdot \{X(t) \nabla Y(t)\} = 0, \quad Y(t) - \Delta Y(t) = X(t).
\]

Then we rewrite (1.16) as

\[
\frac{d}{dt}X(t) = \mathcal{E}^{\text{ad}} X(t) + g^{\text{ad}}(X(t)),
\]

where

\[
\mathcal{E}^{\text{ad}} := -\nu(-\Delta)^\beta, \quad g^{\text{ad}}(X) := -\gamma \nabla \cdot (XBX), \quad B := \nabla [\Phi \ast].
\]

(4) Surface quasi-geostrophic (SQG) equation. We consider the following equation on \(\mathbb{K}^2\):

\[
\frac{d}{dt}X(t) = g^{\text{seg}}(X(t)), \quad g^{\text{seg}}(X) := -((\mathcal{R} \perp X) \cdot \nabla X,
\]

where \(X : [0, \infty) \times \mathbb{K}^2 \to \mathbb{R}\), \(\mathcal{R} = (\mathcal{R}_j)_{j=1,2} = (\partial \lambda \Lambda^{-1})_{j=1,2} = (\partial (-\Delta)^{-\frac{1}{2}})_{j=1,2}\) is the Riesz transform on \(\mathbb{K}^2\), and

\[
\mathcal{R} \perp := (\mathcal{R}_2, \mathcal{R}_1).
\]

Equation (1.18) is an important model in geophysical fluid dynamics. Actually, it is a special case of the general quasi-geostrophic approximations for atmospheric and oceanic fluid flow with small Rossby and Ekman numbers. Besides, (1.18) is also an important example of a 2-D active scalar with a specific structure most closely related to the incompressible Euler equations. We refer to the recent monograph [17] for more details.

Considering stochastic variants of the above PDEs with both \(\mathbb{K} = \mathbb{R}\) and \(\mathbb{K} = \mathbb{T}\), the final main result of this paper is to establish
Main Result (III). Novel results on the well-posedness of the SPDE

\[ dX(t) = \left\{ (\tilde{\Pi}E)X(t) + \tilde{b}(X(t)) + \tilde{g}(X(t)) \right\} dt \]

\[ + \sum_{k=1}^{\infty} \left\{ (\tilde{\Pi}L_{A_k})X(t) \circ dW_k(t) + \tilde{\Pi}h_k(t, X(t)) d\tilde{W}_k(t) \right\}, \quad t \geq 0, \]

on \( \mathbb{H} := \tilde{\Pi}H^s(\mathbb{K}^d; \mathbb{R}^m) \) with the following choices:

- **Stochastic MHD equations** (cf. Theorem 4.1): \( d \geq 2, \ m = 2d, \mathbb{K} = \mathbb{R} \) or \( \mathbb{T}, \tilde{b} \equiv 0, \) \( (\mathcal{E}, \tilde{g}) = (\mathcal{E}^{\text{mhd}}, g^{\text{mhd}}) \) is given in (1.13), and \( \tilde{\Pi} = \text{diag}(\Pi_d, \Pi_d) \) if \( \mathbb{K} = \mathbb{T}, \) while \( \tilde{\Pi} = \text{diag}(\Pi_d \Pi_0, \Pi_d \Pi_0) \) if \( \mathbb{K} = \mathbb{T}, \) where \( \Pi_d \) and \( \Pi_0 \) are defined in (3.9) and (3.12), respectively;

- **Stochastic CH equation** (see Theorem 4.2): \( d = m = 1, \mathbb{K} = \mathbb{R} \) or \( \mathbb{T}, \tilde{\Pi} = \mathbb{I}, \mathcal{E} \equiv 0, \) \( (\tilde{b}, \tilde{g}) = (b^{\text{ch}}, g^{\text{ch}}) \) is given in (1.15);

- **Stochastic AD equation** (see Theorem 4.3): \( d \geq 2, \ m = 1, \mathbb{K} = \mathbb{R} \) or \( \mathbb{T}, \tilde{\Pi} = \mathbb{I}, \tilde{b} \equiv 0, \) \( (\mathcal{E}, \tilde{g}) = (\mathcal{E}^{\text{ad}}, g^{\text{ad}}) \) is given in (1.17);

- **Stochastic SQG equation** (cf. Theorem 4.4): \( d = 2, \ m = 1, \mathbb{K} = \mathbb{R} \) or \( \mathbb{T}, \tilde{b} \equiv 0, \mathcal{E} \equiv 0 \) and \( \tilde{g} = g^{\text{sqg}} \) are given in (1.18), \( \tilde{\Pi} = \mathbb{I} \) if \( \mathbb{K} = \mathbb{R} \) and \( \tilde{\Pi} = \Pi_0 \) if \( \mathbb{K} = \mathbb{T}, \) where \( \Pi_0 \) is given in (3.9).

To conclude this section, we briefly recall some references on nonlinear SPDEs. The stochastic MHD equation has been studied in [9, 38, 43] and the references therein, see [18–20, 35, 36, 49, 51, 52, 57] for the study of stochastic CH type equations, [28, 34, 46, 53, 56] for the stochastic AD equation, and [6, 13, 50, 65] for SQG type equations. Results presented in these references are now extended to the case with highly non-local and singular noise.

The remainder of the paper is organized as follows. In Section 2, we present a general framework (see Theorem 2.1) on the existence, uniqueness, blow-up and non-explosion criteria for (1.1). In Section 3, we apply our general framework to derive a well-posedness result (see Theorem 3.1) on SPDEs with pseudo-differential noises in Sobolev spaces. This result will be used in Section 4 to obtain novel results (see Theorems 4.1-4.4) on above mentioned stochastic models (MHD, CH, AD and SQG), see also Section 4.5 for two more examples of SPDEs.

## 2 A general framework

In this section, we establish a general framework to solve (1.1). Section 2.1 includes assumptions on the coefficients \( (b, g, h) \) and the main result Theorem 2.1 on the local existence, uniqueness, blow-up criterion and global existence. Complete proof of Theorem 2.1 is addressed in Section 2.2, and a further improved blow-up criterion is given in Section 2.3.

### 2.1 Assumptions and main results

We first introduce the notion of **proper regularization** for the singular coefficients \( g \) and \( h, \) which covers concrete regularizations used in [44, 60] for the deterministic case, and in [7, 43, 49, 57] for the stochastic case. For two topological spaces \( E_1 \) and \( E_2, \) let \( \mathcal{B}(E_1; E_2) \) be the class of all measurable maps from \( E_1 \) to \( E_2, \) while \( C(E_1; E_2) \) consists of all continuous maps from \( E_1 \) to \( E_2. \) When \( E_2 \) is a metric space, let \( \mathcal{B}_b(E_1; E_2) \) be the set of bounded elements in \( \mathcal{B}(E_1; E_2). \) Let \( \mathcal{K} \subset \mathcal{B}([0, \infty) \times [0, \infty); (0, \infty)) \) such that

\begin{equation}
\mathcal{K} := \left\{ K(x, y) \text{ is increasing in } y \text{ and locally integrable in } x \right\}.
\end{equation}

**Definition 2.1** (Proper Regularization). \( \{ (g_n, h_n) \}_{n \geq 1} \) is called a proper regularization of \( (g, h), \) if \( g_n : [0, \infty) \times \mathbb{H} \to \mathbb{H}, \quad h_n : [0, \infty) \times \mathbb{H} \to \mathcal{L}_2(U; \mathbb{H}), \quad n \geq 1 \)

are measurable such that the following conditions hold for some \( K \in \mathcal{K} \) and a dense subset \( \mathbb{M}_0 \subset \mathbb{M}: \)
(R₁) For any \( t \geq 0 \) and \( X \in \mathbb{H} \),
\[
\sup_{n \geq 1} \{ \| g_n(t, X) \|_M + \| h_n(t, X) \|_{L^2(U; M)} \} \leq K(t, \| X \|_\mathbb{H}),
\]
\[
\lim_{n \to \infty} \{ \| g_n(t, X) - g(t, X) \|_M + \| h_n(t, X) - h(t, X) \|_{L^2(U; M)} \} = 0.
\]

(R₂) For any \( n, N \geq 1 \),
\[
\sup_{t \in [0,T]} \left\{ \| g_n(t, X) \|_H + \| h_n(t, X) \|_{L^2(U; H)} \right\} < \infty,
\]
\[
\sup_{t \in [0,T]} \| g_n(t, X) - g_n(t, Y) \|_H + \| h_n(t, X) - h_n(t, Y) \|_{L^2(U; H)} < \infty.
\]

(R₃) For any \( Y \in M_0 \), \( T > 0 \) and \( \{ X_n, X \}_{n \geq 1} \subset \mathcal{B}_0([0, T]; \mathbb{H}) \cap C([0, T]; M) \) with \( X_n \to X \) in \( C([0, T]; M) \) as \( n \to \infty \),
\[
\lim_{n \to \infty} \int_0^T \left\{ \left| \langle g_n(t, X_n(t)) - g(t, X(t)), Y \rangle_H \right| + \sum_{k \geq 1} \left| \langle \{ h_n(t, X_n(t)) - h(t, X(t)) \} e_k, Y \rangle_M \right|^2 \right\} dt = 0.
\]

(R₄) (Cancellation of singularities) For any \( t \geq 0 \) and \( X \in \mathbb{H} \),
\[
\sup_{n \geq 1} \sum_{k \geq 1} \| h_n(t, X)e_k, X \|_H^2 \leq K(t, \| X \|_M) (1 + \| X \|_H^2),
\]
\[
\sup_{n \geq 1} \left\{ 2 \langle g_n(t, X), X \rangle_H + \| h_n(t, X) \|_{L^2(U; H)}^2 \right\} \leq K(t, \| X \|_M) (1 + \| X \|_H^2).
\]

With a proper regularization of \((g, h)\), we will prove the existence and uniqueness of maximal solution to (1.1) under the following assumption.

**Assumption (A).** There exists a proper regularization \( \{ (g_n, h_n) \}_{n \geq 1} \) of \((g, h)\) such that the following conditions hold for some \( K \in \mathcal{K} \) in (2.1) and some function \( \lambda : \mathbb{N} \times \mathbb{N} \to [0, \infty) \) with \( \lim_{n,l \to \infty} \lambda_{n,l} = 0 :

(A₁) (Regular drift) For any \( X \in \mathbb{H} \) and \( t, N \geq 0 \),
\[
\sup_{|X|_H, |Y|_H \leq N} \left\{ \frac{\| b(t, X) - b(t, Y) \|_H}{\| X - Y \|_H} + \frac{\| b(t, X) - b(t, Y) \|_M}{\| X - Y \|_M} \right\} \leq K(t, N).
\]

(A₂) (Asymptotic quasi monotonicity) For any \( n, l \geq 1 \), \( t \geq 0 \) and \( X, Y \in \mathbb{H} \),
\[
\sum_{k \geq 1} \left| \langle h_n(t, X) - h_l(t, Y) \rangle e_k, X - Y \rangle_M \right|^2
\]
\[
\leq K(t, \| X \|_H + \| Y \|_H) \| X - Y \|_M^2 \left( \lambda_{n,l} + \| X - Y \|_M^2 \right),
\]
\[
2 \langle g_n(t, X) - g_l(t, Y), X - Y \rangle_M + \| h_n(t, X) - h_l(t, Y) \|_{L^2(U; M)}^2
\]
\[
\leq K(t, \| X \|_H + \| Y \|_H) \left( \lambda_{n,l} + \| X - Y \|_M^2 \right).
\]
We call \( \textbf{(A}_2 \textbf{)} \) an asymptotic quasi monotonicity condition in \( \mathbb{M} \), since it becomes a monotonicity condition with coefficient \( K(t, \|X\|_H + \|Y\|_H) \) depending on the larger \( \mathbb{H} \)-norm when \( n, l \to \infty \).

Due to the singularity of \((g, h)\), Itô’s formula does not directly apply to \( \|X(t)\|^2_2 \) for a solution to (1.1), which is crucial to deduce the continuity in \( \mathbb{H} \) from the weak continuity imposed by definition. To this end, we make the following assumption, where the regularization operator \( T_n \) makes Itô’s formula available for \( \|T_nX(t)\|^2_{\mathbb{H}} \) instead of \( \|X(t)\|^2_{\mathbb{H}} \).

**Assumption (B).** There exist \( K \in \mathcal{K} \) in (2.1) and \( \{T_n\}_{n \geq 1} \subset \mathcal{L}(\mathbb{M}; \mathbb{H}) \) such that
\[
\lim_{n \to \infty} \|T_nX - X\|_{\mathbb{H}} = 0, \quad X \in \mathbb{H},
\]
and for all \( t \geq 0 \) and \( N \geq 1 \),
\[
\sup_{n \geq 1, \|X\|_{\mathbb{H}} \leq N} \left\{ \sum_{i=1}^{\infty} \langle T_nh(t, X)e_i, T_nX \rangle_{\mathbb{H}}^2, \right. \\
\left. 2 \langle T_ng(t, X), T_nX \rangle_{\mathbb{H}} + \|T_nh(t, X)\|^2_{\mathbb{H}^2(U;\mathbb{H})} \right\} \leq K(t, N).
\]

Finally, to prove the non-explosion, we assume the following Lyapunov type condition. When \( V'' < 0 \), a fast enough growth of the noise coefficient will kill the growth of other terms, such that the non-explosion is ensured.

**Assumption (C).** There exists a function \( 1 \leq V \in C^2([0, \infty)) \) satisfying
\[
V'(r) > 0, \quad V''(r) \leq 0 \quad \text{and} \quad \lim_{r \to \infty} V(r) = \infty,
\]
and a positive function \( F(\cdot) \in L^1_{\text{loc}}([0, \infty)) \) such that for all \((t, X) \in [0, \infty) \times \mathbb{H},
\[
V'(\|X\|_{\mathbb{H}}^2) \left\{ 2\langle b(t, X), X \rangle_{\mathbb{H}} + \|h(t, X)\|^2_{\mathbb{H}^2(U;\mathbb{H})} \right\} \\
+ 2V''(\|X\|_{\mathbb{H}}^2) \sum_{k=1}^{\infty} \langle h(t, X)e_k, X \rangle_{\mathbb{H}}^2 \leq F(t) V(\|X\|_{\mathbb{H}}^2).
\]

The main result is stated as follows.

**Theorem 2.1.** Assume \( \textbf{(A)} \) and let \( X(0) \) be an \( \mathcal{F}_0 \)-measurable \( \mathbb{H} \)-valued random variable. Then the following assertions hold.

(i) (1.1) has a unique maximal solution \((X, \tau^*)\), and the solution satisfies \( \mathbb{P} \)-a.s.
\[
\lim_{t \to \tau^*} \|X(t)\|_{\mathbb{M}} = \infty \quad \text{on} \quad \{\tau^* < \infty\}.
\]

(ii) Under \( \textbf{(B)} \) the solution is continuous in \( \mathbb{H} \), i.e., \( \mathbb{P}(X \in C([0, \tau^*]; \mathbb{H})) = 1 \).

(iii) Under \( \textbf{(C)} \) the maximal solution \((X, \tau^*)\) is non-explosive, i.e., \( \mathbb{P}(\tau^* = \infty) = 1 \).

### 2.2 Proof of Theorem 2.1

To construct the solution, we first localize the coefficients and then make regularity approximations. For any \( R > 1 \), we take a cut-off function \( \chi_R \in C^\infty([0, \infty); [0, 1]) \) such that \( \chi_R(r) = 1 \) for \( |r| \leq R \), and \( \chi_R(r) = 0 \) for \( r > 2R \). Consider the following localization of (1.1):

\[
\begin{cases}
  dX^{(R)}(t) = \chi_R^2(\|X^{(R)}(t) - X(0)\|_{\mathbb{M}}) \left[ b(t, X^{(R)}(t)) + g(t, X^{(R)}(t)) \right] dt \\
  + \chi_R(\|X^{(R)}(t) - X(0)\|_{\mathbb{M}}) h(t, X^{(R)}(t)) dW(t),
\end{cases}
\]

\( X^{(R)}(0) = X(0) \).
To solve this problem, we consider its regularization equation for every $n \geq 1$:

$$
\begin{align*}
\left\{ \begin{array}{l}
\frac{dX_n^{(R)}(t)}{dt} = \chi_R^2(\|X_n^{(R)}(t) - X(0)\|_{\mathbb{M}}) \left[ b(t, X_n^{(R)}(t)) + g_n(t, X_n^{(R)}(t)) \right] dt \\
\quad + \chi_R(\|X_n^{(R)}(t) - X(0)\|_{\mathbb{M}})h_n(t, X_n^{(R)}(t))dW(t),
\end{array} \right.
\end{align*}
$$

(2.4)

\[ X_n^{(R)}(0) = X(0). \]

**Lemma 2.1.** Assume (A) and let $X(0)$ be an $\mathcal{F}_0$-measurable $\mathbb{H}$-valued random variable.

1. For any $R, n \geq 1$, (2.4) has a unique global solution, which is continuous process on $\mathbb{H}$. Moreover, there exists a function $F : [0, \infty) \times [0, \infty) \to (0, \infty)$ increasing in both variables such that for any $R > 1$ and $T > 0$,

$$
\sup_{n \geq 1} \mathbb{E} \left[ \sup_{t \in [0, T]} \|X_n^{(R)}(t)\|^2_{\mathbb{H}} | \mathcal{F}_0 \right] \leq F(T, 2R + \|X(0)\|_{\mathbb{M}})(1 + \|X(0)\|^2_{\mathbb{H}}).
$$

(2.5)

2. For any $n, l \geq 1$ and $N > 0$, let

$$
\tau_n^{n,l}(R) := N \land \inf \left\{ t \geq 0 : \|X_n^{(R)}(t)\|_{\mathbb{H}} \vee \|X_l^{(R)}(t)\|_{\mathbb{H}} \geq N \right\}.
$$

Then $\mathbb{P}$-a.s.

$$
\lim_{n \to \infty} \sup_{l \geq n} \mathbb{E} \left[ \sup_{t \in [0, \tau_n^{n,l}(R)]} \|X_n^{(R)}(t) - X_l^{(R)}(t)\|^2_{\mathbb{H}} | \mathcal{F}_0 \right] = 0, \quad N > 0.
$$

(2.6)

**Proof.** (1) By (A$_1$) and (R$_2$), the drift and noise coefficients are locally Lipschitz continuous in $X \in \mathbb{H}$ locally uniform in $t$. Hence for any deterministic initial data, (2.4) has a unique solution, which is continuous in $\mathbb{H}$. See for instance [48, 63]. Since $\mathcal{F}_0$ is independent of the equation, (2.4) also admits a unique solution $X_n^{(R)}(t)$ for any $\mathcal{F}_0$-measurable $\mathbb{H}$-valued random variable $X(0)$, and $X_n^{(R)}(t)$ is continuous in $\mathbb{H}$ up to its lifespan $\tau_n(R)$ defined as

$$
\tau_n(R) := \lim_{N \to \infty} \tau_{n,N}(R), \quad \tau_{n,N}(R) := \inf \left\{ t \geq 0 : \|X_n^{(R)}(t)\|_{\mathbb{H}} \geq N \right\}, \quad N \geq 1.
$$

It remains to prove $\tau_n(R) = \infty$ and the estimate (2.5). Let

$$
\tilde{K}_{t, X(0)} := K(t, \|X(0)\|_{\mathbb{M}} + 2R), \quad t \geq 0.
$$

By (R$_4$) and Itô’s formula, we obtain

$$
\begin{align*}
d\|X_n^{(R)}(t)\|^2_{\mathbb{H}} - dM_n(t) \\
= \chi_R^2(\|X_n^{(R)}(t) - X(0)\|_{\mathbb{M}}) \left\{ \left\| h(t, X_n^{(R)}(t)) \right\|^2_{L^2(\mathbb{H})} + 2 \left\langle b(t, X_n^{(R)}(t)) + g_n(t, X_n^{(R)}(t)), X_n^{(R)}(t) \right\rangle_{\mathbb{H}} \right\} dt \\
\leq 2 \tilde{K}_{t, X(0)}(1 + \|X_n^{(R)}(t)\|^2_{\mathbb{H}}) dt,
\end{align*}
$$

where

$$
dM_n(t) := \chi_R(\|X_n^{(R)}(t) - X(0)\|_{\mathbb{M}}) \left\langle X_n^{(R)}(t), h_n(t, X_n^{(R)}(t)) dW(t) \right\rangle_{\mathbb{H}}
$$

satisfies

$$
d\langle M_n \rangle(t) \leq 4 \tilde{K}_{t, X(0)}(1 + \|X_n^{(R)}(t)\|^2_{\mathbb{H}}) dt.
$$
So, by BDG’s inequality, for any $T > 0$, we find constants $c_1, c_2 > 0$ such that for any $s \in [0, T]$ and $N \geq 1$,

$$
\mathbb{E}
\left[
\sup_{t \in [0, s \wedge \tau_{n,N}(R)]} \|X_n^{(R)}(t)\|_H^2 \bigg| \mathcal{F}_0
\right] - \|X(0)\|_H^2
\leq c_1 \mathbb{E}
\left[
\int_0^{s \wedge \tau_{n,N}(R)} \tilde{K}_{t,X(0)} \left(1 + \|X_n^{(R)}(t)\|_H^4\right) dt \bigg| \mathcal{F}_0
\right]
\leq \frac{1}{2} \mathbb{E}
\left[
\sup_{t \in [0, s \wedge \tau_{n,N}(R)]} \|X_n^{(R)}(t)\|_H^2 \bigg| \mathcal{F}_0
\right]
+ c_1 \mathbb{E}
\left[
\int_0^{s \wedge \tau_{n,N}(R)} \tilde{K}_{t,X(0)} \left(1 + \|X_n^{(R)}(t)\|_H^2\right) dt \bigg| \mathcal{F}_0
\right]
+ c_2 + c_2 \int_0^s \tilde{K}_{t,X(0)} \mathbb{E}
\left[
\sup_{r \in [0, t \wedge \tau_{n,N}(R)]} \|X_n^{(R)}(r)\|_H^2 \bigg| \mathcal{F}_0
\right] dt.
$$

By Grönwall’s inequality, there exists a function $F : [0, \infty) \times [0, \infty) \to (0, \infty)$ increasing in both variables such that for all $n, N \geq 1$,

$$
(2.7)
\mathbb{E}
\left[
\sup_{t \in [0, T \wedge \tau_{n,N}(R)]} \|X_n^{(R)}(t)\|_H^2 \bigg| \mathcal{F}_0
\right] \leq F(T, 2R + \|X(0)\|_H)(1 + \|X(0)\|_H^2).
$$

This implies that for all $n, N \geq 1$,

$$
\mathbb{P}(\tau_{n,N}(R) < T | \mathcal{F}_0) \leq \mathbb{P}
\left(
\sup_{t \in [0, T \wedge \tau_{n,N}(R)]} \|X_n^{(R)}(t)\|_H \geq N \bigg| \mathcal{F}_0
\right)
\leq \frac{F(T, 2R + \|X(0)\|_H)(1 + \|X(0)\|_H^2)}{N^2},
$$

so that $\tau_n(R) := \lim_{N \to \infty} \tau_{n,N}(R)$ satisfies

$$
\mathbb{P}(\tau_n(R) < T | \mathcal{F}_0) \leq \lim_{N \to \infty} \mathbb{P}(\tau_{n,N}(R) < T | \mathcal{F}_0) = 0.
$$

Hence, $\mathbb{P}(\tau_n(R) \geq T) = \mathbb{E}[\mathbb{P}(\tau_n(R) \geq T | \mathcal{F}_0)] = 1$ for all $T > 0$, which means $\mathbb{P}(\tau_n(R) = \infty) = 1$. By letting $N \to \infty$ in (2.7), we prove (2.5).

(2) Let $Z_{n,l}^{(R)} = X_n^{(R)} - X_l^{(R)}$ for $n, l \geq 1$. We have that

$$
(2.8)
dZ_{n,l}^{(R)}(t) = \sum_{i=1}^2 A_i^{n,l}(t) dt + \sum_{i=1}^2 B_i^{n,l}(t) dW(t), \quad Z_{n,l}^{(R)}(0) = 0,
$$

where

$$
A_i^{n,l}(t) := \left[\chi_R^2 \left(\|X_n^{(R)}(t) - X(0)\|_M\right) - \chi_R^2 \left(\|X_l^{(R)}(t) - X(0)\|_M\right)\right] b(t, X_n^{(R)}(t)),
$$

$$
A_2^{n,l}(t) := \chi_R^2 \left(\|X_l^{(R)}(t) - X(0)\|_M\right) \left[b(t, X_n^{(R)}(t)) - b(t, X_l^{(R)}(t))\right],
$$

$$
A_3^{n,l}(t) := \left[\chi_R^2 \left(\|X_n^{(R)}(t) - X(0)\|_M\right) - \chi_R^2 \left(\|X_l^{(R)}(t) - X(0)\|_M\right)\right] g_n(t, X_n^{(R)}(t)),
$$

$$
A_4^{n,l}(t) := \chi_R^2 \left(\|X_l^{(R)}(t) - X(0)\|_M\right) \left[g_n(t, X_n^{(R)}(t)) - g_l(t, X_l^{(R)}(t))\right],
$$

and

$$
B_1^{n,l}(t) := \chi_R \left(\|X_n^{(R)}(t) - X(0)\|_M\right) - \chi_R \left(\|X_l^{(R)}(t) - X(0)\|_M\right) h_n(t, X_n^{(R)}(t)),
$$

$$
B_2^{n,l}(t) := \chi_R \left(\|X_l^{(R)}(t) - X(0)\|_M\right) \left[h_n(t, X_n^{(R)}(t)) - h_l(t, X_l^{(R)}(t))\right].
$$
By the Itô formula, we obtain
\[
    d \left\| Z_{n,l}^{(R)}(t) \right\|_M^2 = 2 \sum_{i=1}^{2} \left\langle Z_{n,l}^{(R)}(t), B_{n,l}^{i}(t) \right\rangle d\mathcal{W}(t) + \left\{ \sum_{i=1}^{2} \left\| B_{n,l}^{i}(t) \right\|_{L^2(U;M)}^2 + 2 \sum_{i=1}^{4} \left\langle A_{n,l}^{i}(t), Z_{n,l}^{(R)}(t) \right\rangle \right\} dt.
\]
By the Lipschitz continuity of \( \chi_R \), (R₃), (A₁) and (A₂), we find a constant \( C_N > 0 \) such that for all \( n, l \geq 1 \) and \( t \in [0, T_N^{n,l}] \),
\[
    \sum_{i=1}^{2} \sum_{k=1}^{\infty} \left\langle Z_{n,l}^{(R)}(t), B_{n,l}^{i}(t) e_k \right\rangle^2 \leq C_N K(t, 2N) \left\| Z_{n,l}^{(R)}(t) \right\|_M^2 \left\{ \lambda_{n,l} + \left\| Z_{n,l}^{(R)}(t) \right\|_M^2 \right\},
\]
\[
    \sum_{i=1}^{2} \left\| B_{n,l}^{i}(t) \right\|_{L^2(U;M)}^2 + 2 \sum_{i=1}^{4} \left\langle A_{n,l}^{i}(t), Z_{n,l}^{(R)}(t) \right\rangle \leq C_N K(t, 2N) \left\{ \lambda_{n,l} + \left\| Z_{n,l}^{(R)}(t) \right\|_M^2 \right\}.
\]
Therefore, by BDG’s inequality to (2.8), we find constants \( a_1, a_2 > 0 \) depending on \( N \) such that for all \( n, l \geq 1 \) and \( s \in [0, N] \),
\[
    \mathbb{E} \left[ \sup_{t \in [0, s]\cap T_N^{n,l}(R)} \left\| Z_{n,l}^{(R)}(t) \right\|_M^2 \right| \mathcal{F}_0] \leq a_1 \mathbb{E} \left[ \int_0^s K(t, 2N) \left\{ \lambda_{n,l} + \left\| Z_{n,l}^{(R)}(t) \right\|_M^2 \right\} dt \right| \mathcal{F}_0] + a_2 \mathbb{E} \left[ \sup_{r \in [0, s]\cap T_N^{n,l}(R)} \left\| Z_{n,l}^{(R)}(r) \right\|_M^2 \right] \mathcal{F}_0]
\]
By Grönwall’s inequality and noticing \( \lambda_{n,l} \to 0 \) as \( n, l \to \infty \), we prove (2.6).

Next, we prove that up to a subsequence, \( X_{n}^{(R)} \) converges to a process on \( \mathbb{H} \).

**Lemma 2.2.** There exists an \( \mathcal{F}_1 \)-progressive measurable \( \mathbb{H} \)-valued process \( X^{(R)} = (X^{(R)}(t))_{t \geq 0} \) such that
\[
    \mathbb{E} \left[ \sup_{t \in [0, T]} \left\| X^{(R)}(t) \right\|_H^2 \right| \mathcal{F}_0] \leq F(T, 2R + \left\| X(0) \right\|_M) (1 + \left\| X(0) \right\|_H^2), \quad T > 0.
\]
Moreover, \( \{X_{n}^{(R)}\} \) has a subsequence (still labeled as \( \{X_{n}^{(R)}\} \) for simplicity) such that \( \mathbb{P} \)-a.s.,
\[
    X_{n}^{(R)} \xrightarrow{n \to \infty} X^{(R)} \text{ in } C([0, \infty); M).
\]
**Proof.** We first fix a time \( T > 0 \). For any \( N \geq T \) and \( \epsilon > 0 \), by using (1) in Lemma 2.1 and Chebyshev’s inequality, we obtain
\[
    P(\tau_{N}^{n,l}(R) < T \mid \mathcal{F}_0] \leq P \left( \sup_{t \in [0, T]} \left\| X_{n}^{(R)}(t) \right\|_H \geq N \right| \mathcal{F}_0] + P \left( \sup_{t \in [0, T]} \left\| X_{l}^{(R)}(t) \right\|_H \geq N \right) \]
\[
    \leq 2F(T, 2R + \left\| X(0) \right\|_M) (1 + \left\| X(0) \right\|_H^2) \leq \frac{N^2}{2}.\]
Then for any $N > T, n, l \geq 1$,
\[
\mathbb{P}\left( \sup_{t \in [0, T]} \| X_n^{(R)}(t) - X_l^{(R)}(t) \|_M > \epsilon \bigg| \mathcal{F}_0 \right) \\
\leq \mathbb{P}(\tau_N^+(R) < T \big| \mathcal{F}_0) + \mathbb{P}\left( \sup_{t \in [0, \tau_N^+(R)]} \| X_n^{(R)}(t) - X_l^{(R)}(t) \|_M > \epsilon \bigg| \mathcal{F}_0 \right) \\
\leq 2F(T, 2R + \| X(0) \|_M)(1 + \| X(0) \|_M^2) \\
+ \mathbb{P}\left( \sup_{t \in [0, \tau_N^+(R)]} \| X_n^{(R)}(t) - X_l^{(R)}(t) \|_M > \epsilon \bigg| \mathcal{F}_0 \right).
\]

According to (2) in Lemma 2.1, by first letting $n, l \to \infty$ and then $N \to \infty$, we obtain
\[
\lim_{n, l \to \infty} \mathbb{P}\left( \sup_{t \in [0, T]} \| X_n^{(R)}(t) - X_l^{(R)}(t) \|_M > \epsilon \bigg| \mathcal{F}_0 \right) = 0, \quad \epsilon, T > 0.
\]

By Fatou’s lemma, this implies
\[
\limsup_{n, l \to \infty} \mathbb{P}\left( \sup_{t \in [0, T]} \| X_n^{(R)}(t) - X_l^{(R)}(t) \|_M > \epsilon \right) \\
= \limsup_{n, l \to \infty} \mathbb{E}\left[ \mathbb{P}\left( \sup_{t \in [0, T]} \| X_n^{(R)}(t) - X_l^{(R)}(t) \|_M > \epsilon \bigg| \mathcal{F}_0 \right) \right] \\
\leq \mathbb{E}\left[ \limsup_{n, l \to \infty} \mathbb{P}\left( \sup_{t \in [0, T]} \| X_n^{(R)}(t) - X_l^{(R)}(t) \|_M > \epsilon \bigg| \mathcal{F}_0 \right) \right] = 0, \quad \epsilon, T > 0.
\]

Therefore, up to a subsequence, (2.10) holds for some progressively measurable process $X^{(R)}$ on $\mathbb{H}$, which together with (2.5) implies (2.9).

**Lemma 2.3.** Assume (A) and let $X(0)$ be an $\mathbb{H}$-valued $\mathcal{F}_0$-measurable random variable. Then for any $R \geq 1$, $X^{(R)}$ given by Lemma 2.2 is the unique global solution to (2.3) and
\[
\text{for any } R \geq 1, \quad \mathbb{E}\left[ \sup_{t \in [0, T]} \| X^{(R)}(t) \|^2_M \bigg| \mathcal{F}_0 \right] < \infty, \quad T > 0.
\]

**Proof.** Obviously, (2.11) follows from (2.9).

To prove that $X^{(R)}$ solves (2.3) in the sense of Definition 1.1, by (R3) we only need to show that $\mathbb{P}$-a.s.,
\[
\langle X^{(R)}(t) - X(0), Y \rangle_M = \int_0^t \langle A(s), Y \rangle_M ds + \int_0^t \langle B(s) \, dW(s), Y \rangle_M, \quad Y \in \mathcal{M}_0, \quad t > 0,
\]
where
\[
A(s) := \chi_R^2(\| X^{(R)}(s) - X(0) \|_M) \left\{ b(s, X^{(R)}(s)) + g(s, X^{(R)}(s)) \right\},
\]
\[
B(s) := \chi_R(\| X^{(R)}(s) - X(0) \|_M) h(s, X^{(R)}(s)).
\]

Note that this formula holds for $(X_n^{(R)}, g_n, h_n)$ replacing $(X^{(R)}, g, h)$, i.e., for any $n \geq 1$,
\[
\left\{ \begin{array}{l}
\langle X_n^{(R)}(t) - X(0), Y \rangle_M = \int_0^t \langle A_n(s), Y \rangle_M ds + \int_0^t \langle B_n(s) \, dW(s), Y \rangle_M, \\
A_n(s) := \chi_R^2(\| X_n^{(R)}(s) - X(0) \|_M) \left\{ b(s, X_n^{(R)}(s)) + g_n(s, X_n^{(R)}(s)) \right\}, \\
B_n(s) := \chi_R(\| X_n^{(R)}(s) - X(0) \|_M) h(s, X_n^{(R)}(s)).
\end{array} \right.
\]
By (R3), (A1) and (2.10), we have
\[
\lim_{n \to \infty} \left\{ \left| \langle A_n(t) - A(t), Y \rangle_M \right| + \sum_{k \geq 1} \left| \langle B_n(t) - B(t) \rangle e_k, Y \rangle_M^2 \right| \right\} = 0, \quad t \geq 0 \quad \mathbb{P}\text{-a.s.}
\]

By (2.5) and the fact that \( \chi_R(r) = 0 \) for \( r \geq 2R \), we find random variables \( C_1 = C_1(t, 2R + \|X_0\|_M) > 1 \) and \( C_2 = C_2(t, 2R + \|X_0\|_M) > 1 \) such that
\[
\mathbb{E} \left[ \sup_{s \in [0,t]} \left\{ \left| \langle A_n(s), Y \rangle_M^2 + \sum_{k \geq 1} \langle B_n(s) e_k, Y \rangle_M^2 \right| \right\} \mathcal{F}_0 \right] 
\leq C_1 + C_1 \mathbb{E} \left[ \sup_{s \in [0,t]} \|X_n^{(R)}(s)\|_H^2 \mathcal{F}_0 \right] \leq C_2, \quad n \geq 1.
\]

So, by the dominated convergence theorem we derive
\[
\lim_{n \to \infty} \mathbb{E} \left\{ \int_0^t \left| \langle A_n(s) - A(s), Y \rangle_M \right| + \left( \sum_{k \geq 1} \left| \langle B_n(s) - B(s) \rangle e_k, Y \rangle_M^2 \right| \right)_M^2 \right\} \mathcal{F}_0 = 0,
\]
and in the conditional probability \( \mathbb{P}(\cdot | \mathcal{F}_0) \) (equivalently, in \( \mathbb{P} \)),
\[
\lim_{n \to \infty} \int_0^t \sum_{k \geq 1} \left| \langle B_n(s) - B(s) \rangle e_k, Y \rangle_M^2 \right| \mathcal{F}_0 = 0.
\]

From this, (2.14), BDG’s inequality and the dominated convergence theorem, we find a constant \( c > 1 \) such that
\[
\lim_{n \to \infty} \mathbb{E} \left[ \sup_{s \in [0,t]} \left( \int_0^s \left| \langle B_n(r) - B(r) \rangle dW(r), Y \rangle_M \right| \mathcal{F}_0 \right] \right] \leq c \lim_{n \to \infty} \mathbb{E} \left[ \left( \int_0^t \sum_{k \geq 1} \left| \langle B_n(s) - B(s) \rangle e_k, Y \rangle_M^2 \right| \mathcal{F}_0 \right] \right] = 0.
\]

Combining this, (2.10) and (2.15), we prove (2.12) by letting \( n \to \infty \) in (2.13).

To prove the uniqueness, by (R1) and (A2), we may take \( n = m \to \infty \) to derive that
\[
2 \left| \langle g(t, X) - g(t, Y), X - Y \rangle_M + \|h(t, X) - h(t, Y)\|_{L_2(\mathbb{H})}^2 \right| \leq K(t, \|X\|_M + \|Y\|_M)\|X - Y\|_2^2, \quad t \geq 0, \quad X, Y \in \mathbb{H}.
\]

If \( \tilde{X} \) is another solution to (2.3) with \( \tilde{X}(0) = X(0) \) such that (2.11) holds for \( \tilde{X} \) replacing \( X \), by (2.16) and Itô’s formula for \( \|X^{(R)}(t) - \tilde{X}(t)\|_M^2 \), we prove \( \tilde{X}(t) = X^{(R)}(t) \) for all \( t \geq 0 \).

We are now ready to prove the main result in this section.

**Proof of Theorem 2.1.** \( \text{(i)} \) By Lemma 2.3, for any \( R \geq 1 \), (2.3) with initial value \( X(0) \) has a unique global solution \( X^{(R)} \) satisfying (2.11). Let
\[
\tau(R) := \inf \left\{ t \geq 0 : \|X^{(R)}(t) - X(0)\|_M \geq R \right\}.
\]

By the continuity of \( X^{(R)}(t) \) in \( M \), we have \( \mathbb{P}(\tau(R) > 0) = 1 \) for any \( R > 0 \). Since \( \chi_R(\|X^{(R)}(t) - X(0)\|_M) = 1 \) for \( t \leq \tau(R) \), (1.1) coincides with (2.3) up to time \( \tau(R) \). This together with (2.11) implies that \( (X^{(R)}, \tau(R)) \) is a local solution to (1.1). By the uniqueness of (2.3), we see that \( \tau(R) \) is increasing in \( R \), and
\[
X^{(R)}(t) = X^{(R+1)}(t), \quad t \leq \tau(R), \quad R \geq 1 \quad \mathbb{P}\text{-a.s.}
\]
Let $\tau^*: = \lim_{R \to \infty} \tau(R), \tau(0): = 0$ and we define

$$X(t) := \sum_{R=1}^{\infty} 1_{(\tau(R-1), \tau(R))}(t)X(R)(t), \quad t \in [0, \tau^*).$$

Then one can conclude that $(X, \tau^*)$ is a local solution to (2.3). Moreover, according to the definitions of $\tau^*$ and $\tau(R)$, we have $P$-a.s.,

$$\limsup_{t \to \tau^*} \|X(t)\|_{M} = \infty \text{ on } \{\tau^* < \infty\},$$

so that it is actually a maximal solution. Uniqueness follows from (2.16) and Itô’s formula for $\|X(t) - \bar{X}(t)\|^{2}_{M}$ if $(\bar{X}, \tau^*)$ is another maximal solution with initial value $X(0)$.

(ii) Let $(X, \tau^*)$ be the unique maximal solution as above. Since $X \in C([0, \tau^*]; \mathbb{M})$ and $\mathbb{H} \hookrightarrow \mathbb{M}$ is dense, $X$ is weakly continuous in $\mathbb{H}$ (cf. [62, page 263, Lemma 1.4]), so that $\|X(\cdot)\|_{\mathbb{H}}$ is lower semi-continuous. Thus,

$$\tau_N := N \wedge \inf \{t \geq 0 : \|X(t)\|_{\mathbb{H}} \geq N\}, \quad N \geq 1$$

are stopping times and $\tau^*: = \lim_{N \to \infty} \tau_N$. By the weak continuity of $X$ in $\mathbb{H}$, it suffices to prove

$$\|X(\cdot)\|_{\mathbb{H}} \in C([0, \tau_N]), \quad N \geq 1.$$  

Let $K_N = K(N, N)$ for $K$ in (B). By Itô’s formula, for any $n \geq 1$ we find a martingale $M^{(n)}_t$ such that for all $n \geq 1$,

$$\left\{ \begin{array}{ll}
-\kappa_N dt & \leq \|T_nX(\cdot)\|_{\mathbb{H}}^2 + dM^{(n)}(t) \leq K_N dt, & \quad t \in [0, \tau_N].
\end{array} \right.$$  

Then there exists a constant $C_N > 0$ such that

$$\mathbb{E}\left[\|T_nX(t \wedge \tau_N)\|_{\mathbb{H}}^2 - \|T_nX(s \wedge \tau_N)\|_{\mathbb{H}}^2\right] \leq C_N |t - s|^2, \quad t, s \geq 0, \quad n \geq 1.$$  

By (B) and Fatou’s lemma with $n \to \infty$, we derive

$$\mathbb{E}\left[\|X(t \wedge \tau_N)\|_{\mathbb{H}}^2 - \|X(s \wedge \tau_N)\|_{\mathbb{H}}^2\right] \leq C_N |t - s|^2, \quad t, s \geq 0.$$  

This together with Kolmogorov’s continuity theorem proves (2.20).

(iii) By Itô’s formula for $\|X(t)\|_{\mathbb{M}}^2$, we have

$$d\|X(t)\|_{\mathbb{M}}^2 = 2 \left(h(t, X(t))dW(t), X(t)\right)_{\mathbb{M}} + 2 \left(g(t, X(t)) + b(t, X(t)), X(t)\right)_{\mathbb{M}} dt + \left|X(t, X(t))\right|_{L_{2}(\mathbb{M}; \mathbb{M})}^{2} dt, \quad t \in [0, \tau^*).$$  

By (C) and Itô’s formula, this implies

$$dV(\|X(t)\|_{\mathbb{M}}^2) \leq F(t)V(\|X(t)\|_{\mathbb{M}}^2) dt + dM_t, \quad t \in [0, \tau^*),$$  

where $M_t$ is a martingale up to $\bar{\tau}_N$ (for any $N \geq 1$) defined as

$$\bar{\tau}_N := N \wedge \inf \{t \geq 0 : \|u(t)\|_{\mathbb{M}} \geq N\}, \quad N \geq 1.$$  

Thus,

$$\mathbb{E}\left[V(\|X(t \wedge \bar{\tau}_N)\|_{\mathbb{M}}^2)\right] \leq V(\|X(0)\|_{\mathbb{M}}^2)e^{\int_0^t F(s) ds}, \quad t \geq 0.$$  

Accordingly, by the continuity in $\mathbb{M}$ of $X(t)$ (see (2.10)), we derive that $\tilde{\tau}^*: = \lim_{N \to \infty} \bar{\tau}_N$ satisfies

$$\mathbb{P}(\tilde{\tau}^* < t | \mathcal{F}_0) \leq \mathbb{P}(\bar{\tau}_N < t | \mathcal{F}_0) \leq \frac{\mathbb{E}\left[V(\|X(t \wedge \bar{\tau}_N)\|_{\mathbb{M}}^2)\right]}{V(N^2)} \leq \frac{V(\|X(0)\|_{\mathbb{M}}^2)e^{\int_0^t F(s) ds}}{V(N^2)}, \quad N \geq 1, \quad t > 0.$$  

Letting $N \uparrow \infty$ then $t \uparrow \infty$ yields $\mathbb{P}(\tilde{\tau}^* < \infty | \mathcal{F}_0) = 0$. However, we infer from (2.2) that $\tau^* = \tilde{\tau}^*$ $P$-a.s., hence we obtain $\mathbb{P}(\tau^* < \infty) = 0$. $\square$
2.3 Improved blow-up criterion

According to Definition 1.1, for the maximal solution \((X, \tau^*)\) we have \(\limsup_{t \uparrow \tau^*} \|X(t)\|_M = \infty\) on \(\{\tau^* < \infty\}\). Since \(\| \cdot \|_M \lesssim \| \cdot \|_B\), (2.2) is a criterion of blow-up, i.e., \(X\) blows up in \(\mathbb{H}\) if and only if it blows up in \(M\), which has been used in the proof for (iii). The following result further improves this criterion. Particularly, if in the following \(B(\cdot) = \| \cdot \|_B\) with \(M \hookrightarrow \mathbb{B}\) for some Banach space \(\mathbb{B}\), then \(X\) blows up in \(\mathbb{H}\) if and only if it blows up in \(\mathbb{B}\).

**Proposition 2.1.** Assume (A) and let \(B(\cdot) \in C(\mathbb{M}; [0, \infty))\) be a subadditive function such that

\[
B(X) \lesssim \|X\|_M.
\]

If the growth factor \(K(t, \|X\|_M)\) in (R4) and (A1) is replaced by \(K(t, B(X))\), then \(\mathbb{P}\text{-a.s.}\) we have

\[
(2.22) \quad \limsup_{t \uparrow \tau^*} B(X) = \infty \text{ on } \{\tau^* < \infty\}.
\]

**Proof.** For any \(R \geq 1\), let

\[
\tau_B(R) := \inf \{t \geq 0 : B(X) \geq R\}.
\]

Now we consider the cut-off problem (2.3) with \(B(X(t) - X(0))\) replacing \(\|X(t) - X(0)\|_M\). Notice that the subadditivity of \(B(\cdot)\) implies \(B(Y) \leq B(Y - Z) + B(Z)\). Using this, (R4) and (A1) with \(K(t, B(X))\) being replaced by \(K(t, \|X\|_M)\) in the proof of (2.5), we find

\[
\sup_{n \geq 1} \mathbb{E} \left[ \sup_{t \in [0, T \wedge \tau_B(R)]} \|X^{(R)}(t)\|_{\mathbb{H}}^2 \right] F_0 \leq F(T, 2R + B(X(0)))(1 + \|X(0)\|_{\mathbb{H}}^2), \quad T, R \geq 1.
\]

By the subadditivity of \(B(\cdot)\) again, \(B(X) \lesssim \|X\|_M\) and the construction of \(X(t)\) (cf. Lemma 2.1 (2), (2.10) and (2.11)), we derive

\[
\mathbb{E} \left[ \sup_{t \in [0, T \wedge \tau_B(R)]} \|X(t)\|_{\mathbb{H}}^2 \right] F_0 \leq F(T, 2R + B(X(0)))(1 + \|X(0)\|_{\mathbb{H}}^2), \quad T, R \geq 1.
\]

Since \(\limsup_{t \uparrow \tau^*} \|X(t)\|_H = \infty\) on \(\{\tau^* < \infty\}\), this implies \(\tau_B(R) \leq \tau^*\) for any \(R \geq 1\), so that by the definition of \(\tau_B(R)\), we obtain (2.22). \(\Box\)

3 SPDE with pseudo-differential noise

In this part, we apply Theorem 2.1 to nonlinear SPDE (1.8) with pseudo-differential noise on \(\mathbb{H}^s(\mathbb{K}^d; \mathbb{R}^m)\) for some \(s > \frac{d}{2} + 1\). To this end, we first recall some notions and preliminary results on pseudo-differential operators, then state the main results and finally present a proof.

3.1 Notations and preliminary results

3.1.1 Pseudo-differential operators

We mainly focus on two cases: \(\mathbb{K} = \mathbb{R}\) and \(\mathbb{K} = \mathbb{T}\).

**Case of \(\mathbb{K} = \mathbb{R}\).** Let \(d, m \in \mathbb{N}\). Recall the Fourier and inverse Fourier transforms:

\[
(\mathcal{F} f)(\xi) := \int_{\mathbb{R}^d} f(x)e^{-i(x, \xi)}dx, \quad (\mathcal{F}^{-1} f)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(\xi)e^{i(x, \xi)}d\xi, \quad x, \xi \in \mathbb{R}^d,
\]

where \(i = \sqrt{-1}\) as before, and \((x, \xi) := \sum_{i=1}^m x_i\xi_i\). Let \(\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}_0^d := (\mathbb{N} \cup \{0\})^d\) be a multi-index and recall that \(\partial_k\) is the \(k\)-th partial derivate in \(\mathbb{R}^d\). We define

\[
|\alpha|_1 := \sum_{k=1}^d \alpha_k \quad \partial^\alpha := \prod_{k=1}^d \partial_k^{\alpha_k}.
\]
When both space variable $x$ and frequency variable $\xi$ appear, we use $\partial_x$ and $\partial_\xi$ to denote $\partial^\alpha$ in $x$ and $\xi$, respectively. Next, we will introduce the class of nice functions called symbols, from which we can define pseudo-differential operators. For any $s \in \mathbb{R}$ and $d, m \geq 1$, we define the class of symbols as

$$S^s(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{C}^{m \times m}) := \left\{ \varphi \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{C}^{m \times m}) : \sup_{(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d} \frac{|\partial_x^\beta \partial_\xi^\alpha \varphi(x, \xi)|_{m \times m}}{(1 + |\xi|)^{s-|\alpha|_1}} < \infty, \ \beta, \alpha \in \mathbb{N}_0^d \right\},$$

(3.1)

where, and in the sequel, $|\cdot|_{m \times m}$ and $|\cdot|$ are usual norms in $\mathbb{C}^{m \times m}$ and $\mathbb{R}^d$, respectively. For any $\varphi \in S^s(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{C}^{m \times m})$, the pseudo-differential operator $\text{OP}(\varphi)$ with symbol $\varphi$ is defined as

$$|\text{OP}(\varphi)|_s(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \varphi(x, \xi) (\mathcal{F} f)(\xi)e^{i(x-x')\xi} d\xi, \ \ x \in \mathbb{R}^d.$$

Throughout this paper, we focus on real-valued operators, i.e., $|\text{OP}(\varphi)|_s$ is real for real function $f$. Equivalently, as in (1.5), we require

$$\varphi(x, -\xi) = \varphi(x, \xi), \ \ (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d.$$

To highlight the difference between symbols depending on $(x, \xi)$ and symbols depending only on $\xi$, we use $\mathbb{R}^d_x$ and $\mathbb{R}^d_\xi$ to distinguish the space $\mathbb{R}^d$ for $x$ and $\xi$, respectively, and then $S^s(\mathbb{R}^d_x \times \mathbb{R}^d_\xi; \mathbb{C}^{m \times m})$ defined in (3.1) is relabeled as $S^s(\mathbb{R}^d_x \times \mathbb{R}^d_\xi; \mathbb{C}^{m \times m})$. Then we define

$$S(\mathbb{R}^d_x; \mathbb{C}^{m \times m}) := \left\{ \varphi \in S^s(\mathbb{R}^d_x \times \mathbb{R}^d_\xi; \mathbb{C}^{m \times m}) : \varphi(x, \xi) = \varphi(\xi) \right\}.$$

To simplify notations, when $d, m$ are clear in the context and no confusion can arise, we will write

$$S^s := \left\{ \varphi \in S^s(\mathbb{R}^d_x \times \mathbb{R}^d_\xi; \mathbb{C}^{m \times m}) : (3.2) \text{ holds} \right\},$$

$$S^s_0 := \left\{ \varphi \in S^s(\mathbb{R}^d_x \times \mathbb{R}^d_\xi; \mathbb{C}^{m \times m}) : (3.2) \text{ holds} \right\}.$$

Then, for any $s \in \mathbb{R}$, we define

$$\text{OPS}^s := \left\{ \text{OP}(\varphi) : \varphi \in S^s \right\}, \ \ \text{OPS}^s_0 := \left\{ \text{OP}(\varphi) : \varphi \in S^s_0 \right\}.$$  

It is well-known that $S^s$ is a Fréchet space equipped with the topology generated by seminorms $\{|:\mathbb{R}^d_x \times \mathbb{R}^d_\xi^{\beta, \alpha} s\}$, where

$$|\varphi|_{\beta, \alpha; s} := \sup_{(x, \xi) \in \mathbb{R}^d_x \times \mathbb{R}^d_\xi} \frac{|\partial_x^\beta \partial_\xi^\alpha \varphi(x, \xi)|_{m \times m}}{(1 + |\xi|)^{s-|\alpha|_1}}.$$

A set $S \subset S^s$ is called bounded, if $\sup_{\varphi \in S} |\varphi|_{\beta, \alpha; s} < \infty, \ \beta, \alpha \in \mathbb{N}_0^d$.

Let $L(E_1; E_2)$ be the space of bounded linear operators between two Banach spaces $E_1$ and $E_2$. Although $\text{OPS}^s$ can be measured by operator norm $L(\mathcal{H}; H^{q-s})$ (see Lemma 3.2), it is convenient to consider the boundedness for subsets of $\text{OPS}^s$ in terms of symbols, see for instance Lemmas 3.2, 3.4 and 3.5 below. Indeed, since $\varphi \mapsto \text{OP}(\varphi)$ is one-to-one, $\text{OPS}^s$ is also a Fréchet space with seminorms $\{|\text{OP}(\varphi)|_{\beta, \alpha; s}^{\mathbb{R}^d_x \times \mathbb{R}^d_\xi} := |\varphi|_{\beta, \alpha; s}^{\mathbb{R}^d_x \times \mathbb{R}^d_\xi}, \ \beta, \alpha \in \mathbb{N}_0^d$.

To emphasize the scalar symbols (i.e., $m = 1$ in (3.1)), as in (3.3), we simply write

$$S^s := \left\{ \varphi \in S^s(\mathbb{R}^d_x \times \mathbb{R}^d_\xi; \mathbb{C}) : (3.2) \text{ holds} \right\}, \ \ S^s_0 := \left\{ \varphi \in S^s(\mathbb{R}^d_x \times \mathbb{R}^d_\xi) : (3.2) \text{ holds} \right\}.$$

Then, as in (3.4), we denote by $\text{OPS}^s$ and $\text{OPS}^s_0$ the pseudo-differential operators with scalar symbols in $S^s$ and $S^s_0$, respectively.
We write $\xi$ we denote these classes as $S$ stand for the corresponding real pseudo-differential operator $s$ and $(3.5)$

Conversely, every symbol (see also [25, Corollary 2.11]) we see that if, $\varphi \in S^s(\mathbb{T}^d \times \mathbb{R}^d; \mathbb{C})$ satisfies

then $\tilde{\varphi} = \varphi|_{\mathbb{T}^d \times \mathbb{Z}^n}$ of a symbol $\varphi \in S^s(\mathbb{T}^d \times \mathbb{R}^d; \mathbb{C})$, where $\varphi$ satisfies (3.5). Therefore we see that $\text{OPS}_{\mathbb{T}^d \times \mathbb{R}^d; \mathbb{C}} = \text{OPS}_{\mathbb{T}^d \times \mathbb{R}^d; \mathbb{C}}$ and any bounded set in $\text{OPS}_{\mathbb{T}^d \times \mathbb{R}^d; \mathbb{C}}$ coincides with the restriction of a bounded set in $\text{OPS}_{\mathbb{T}^d \times \mathbb{R}^d; \mathbb{C}}$ (see also [25, Theorem 2.10 and Corollary 2.11]). By considering each element in a matrix, this also holds true for matrix-valued symbol, i.e., $\text{OPS}^{\mathbb{T}^d}$. Moreover, the following introduced Lemmas 3.2-3.5, which are known for pseudo-differential operators on $\mathbb{R}^d$, also hold for those on $\mathbb{T}^d$.
3.1.2 Function spaces and related estimates

Remember that $\mathbb{K} = \mathbb{R}$ or $\mathbb{T}$. Let $k \geq 1$. The space $W^{k,\infty}(\mathbb{K}^d; \mathbb{R}^m)$ is the set of weakly differential functions $f : \mathbb{K}^d \to \mathbb{R}^m$ such that

$$
\|f\|_{W^{k,\infty}} := \sum_{j=1}^{m} \sum_{0 \leq |\alpha| \leq k} \|\partial^\alpha f\|_{L^\infty} < \infty.
$$

Here, and in the following, the weak derivatives $\partial^\alpha f$ is understood in the sense of distribution:

$$
\langle \partial^\alpha f, g \rangle_{L^2} = (-1)^{|\alpha|} \langle f, \partial^\alpha g \rangle_{L^2}, \quad g \in C^\infty_0(\mathbb{R}^d) \quad \text{if} \quad \mathbb{K} = \mathbb{R} \quad \text{and} \quad g \in C^\infty(\mathbb{T}^d) \quad \text{if} \quad \mathbb{K} = \mathbb{T}.
$$

As before we denote by $\mathbb{I}$ the identity mapping. Then for any $s \in \mathbb{R}$, we define

$$
(3.7) \quad \mathcal{D}^s = (I - \Delta)^{\frac{s}{2}} := \mathcal{F}^{-1} \left((1 + |\cdot|^2)^{\frac{s}{2}} \mathbb{I}\right) \mathcal{F} \quad \text{and} \quad \Lambda^s = (-\Delta)^{\frac{s}{2}} := \mathcal{F}^{-1} \left(|\cdot|^s \mathbb{I}\right) \mathcal{F},
$$

respectively, where the function $|\cdot|$ is defined on $\mathbb{R}^d$ if $\mathbb{K} = \mathbb{R}$, and on $\mathbb{Z}^d$ if $\mathbb{K} = \mathbb{T}$. These two operators are self-adjoint and pseudo-differential operators if $s \geq 0$. For $s \geq 0$, $d, m \geq 1$, the Sobolev space $H^s(\mathbb{R}^d; \mathbb{R}^m)$ and $H^s(\mathbb{T}^d; \mathbb{R}^m)$ are defined as the completion of $C^\infty_0(\mathbb{R}^d; \mathbb{R}^m)$ and $C^\infty(\mathbb{T}^d; \mathbb{R}^m)$, respectively, under the inner product

$$
\langle X, Y \rangle_{H^s} := \langle \mathcal{D}^s X, \mathcal{D}^s Y \rangle_{L^2} = \sum_{j=1}^{m} \int_{\mathbb{R}^d} \left[\langle \mathcal{D}^s X_j, \mathcal{D}^s Y_j \rangle \right](x) \, dx.
$$

For $s \geq 0$ and $d \geq 2$, we denote by $H^s_{\text{div}}(\mathbb{R}^d; \mathbb{R}^d)$ a subset of $H^s(\mathbb{R}^d; \mathbb{R}^d)$ such that

$$
(3.8) \quad \begin{cases}
H^s_{\text{div}}(\mathbb{R}^d; \mathbb{R}^d) := \{ f : \nabla \cdot f = 0 \}, \\
H^s_{\text{div}}(\mathbb{T}^d; \mathbb{R}^d) := \{ f : \nabla \cdot f = 0, \quad \int_{\mathbb{R}^d} f(x) \, dx = 0 \},
\end{cases}
$$

where $\nabla \cdot f$ is also understood in the sense of distribution as mentioned above. Notice that $H^s_{\text{div}}(\mathbb{R}^d; \mathbb{R}^d)$ is a closed subspace of $H^s(\mathbb{R}^d; \mathbb{R}^d)$. The Leray projection $\Pi_d$ on $\mathbb{K}^d$ is defined by matrix-valued multiplier $\Pi$:

$$
(3.9) \quad \Pi_d := \mathcal{F}^{-1} \Pi \mathcal{F}, \quad \Pi := \Pi_{i,j} \mathbb{I}, \quad \Pi_{i,j} = \delta_{i,j} - \frac{\xi_i \xi_j}{|\xi|^2}, \quad 1 \leq i, j \leq d,
$$

where $\xi \in \mathbb{R}^d$ if $\mathbb{K} = \mathbb{R}$, $\xi \in \mathbb{Z}^d$ if $\mathbb{K} = \mathbb{T}$, and $\delta_{i,j}$ is the Kronecker delta.

When $\mathbb{K} = \mathbb{R}$, we have

$$
(3.10) \quad \Pi_d \in \mathcal{L}(H^s(\mathbb{R}^d; \mathbb{R}^d); H^s_{\text{div}}(\mathbb{R}^d; \mathbb{R}^d)).
$$

When $\mathbb{K} = \mathbb{T}$, the Laplacian has a spectral gap. To make it invertible we restrict to the zero-average sub-space $H^0_0(\mathbb{T}^d; \mathbb{R}^d)$ of $H^s(\mathbb{T}^d; \mathbb{R}^d)$, where in general,

$$
H^0_0(\mathbb{T}^d; \mathbb{R}^m) := \left\{ f \in H^s(\mathbb{T}^d; \mathbb{R}^m) : \int_{\mathbb{T}^d} f(x) \, dx = 0 \right\}, \quad d, m \in \mathbb{N}.
$$

For any $m \geq 1$, the natural zero-average projection on $L^2(\mathbb{T}^d; \mathbb{R}^m)$ is

$$
(3.11) \quad \Pi_0 f(x) := f(x) - \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(y) \, dy, \quad x \in \mathbb{T}^d, \quad f \in L^2(\mathbb{T}^d; \mathbb{R}^m).
$$

It is easy to see that $\Pi_d$ defined in (3.9) leaves $H^0_0(\mathbb{T}^d; \mathbb{R}^d)$ invariant, so that

$$
(3.13) \quad \Pi_d \Pi_0 \in \mathcal{L}(H^s(\mathbb{T}^d; \mathbb{R}^d); H^s_{\text{div}}(\mathbb{T}^d; \mathbb{R}^d)),
$$

where $H^s_{\text{div}}(\mathbb{T}^d; \mathbb{R}^d)$ is in (3.8).

When $d, m \in \mathbb{N}$ are fixed in the context, for $s \in \mathbb{R}$, $p \in [1, \infty]$, we will simply write

$$
H^s = H^s(\mathbb{K}^d; \mathbb{R}^m), \quad H^s_{\text{div}} = H^s_{\text{div}}(\mathbb{K}^d; \mathbb{R}^d), \quad W^{1,\infty} = W^{1,\infty}(\mathbb{K}^d; \mathbb{R}^m), \quad L^p = L^p(\mathbb{K}^d; \mathbb{R}^m).
$$

Recall that for two nonnegative variables $A$ and $B$, $A \lesssim B$ means that, for some constant $c > 0$, $A \leq cB$ holds. We have the following estimates on product of functions in $H^s$. 

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Lemma 3.1 ([8]). For any \( s \geq 0 \),
\[
\|fg\|_{H^s} \lesssim \|f\|_{L^\infty} \|g\|_{H^s} + \|g\|_{L^\infty} \|f\|_{H^s}, \quad f, g \in H^s \cap W^{1,\infty}.
\]

For an operator \( \mathcal{A} \), \( \mathcal{A}^* \) stands for the adjoint operator of \( \mathcal{A} \) in \( L^2 \). The following two lemmas with single \( \mathcal{Q} \) and a pair \( (\mathcal{Q}_1, \mathcal{Q}_2) \) on \( \mathbb{R}^d \) are well known in the literature, and they can be easily extended to bounded sets in \( \text{OPS}^s \), and to the case on \( \mathbb{T}^d \) (see Remark 3.1).

Lemma 3.2 (see [1, 3, 60] for \( \text{OPS}^s \) and [10] for \( \text{OPS}^r \)). Let \( r \in \mathbb{R} \) and \( \mathcal{Q} \in \text{OPS}^s \). Then \( \mathcal{Q}^* \in \text{OPS}^r \) and \( \mathcal{Q} \in \mathcal{L}(H^s; H^{s-r}) \) for any \( s \in \mathbb{R} \). Furthermore, for any bounded set \( \mathcal{O} \subset \text{OPS}^s \),
\[
\sup_{\mathcal{Q} \in \mathcal{O}} \|\mathcal{Q}\|_{\mathcal{L}(H^s, H^{s-r})} < \infty, \quad s \in \mathbb{R}.
\]

Lemma 3.3 (see [1, 3, 60] for \( \text{OPS}^r \) and [10, 58] for \( \text{OPS}^r \)). Let \( r_i \in \mathbb{R} \) \( (i = 1, 2) \), and let \( \mathcal{O}_i \subset \text{OPS}^{r_i} \) be bounded. Then
\[
\{\mathcal{Q}_1 \mathcal{Q}_2 : \mathcal{Q}_i \in \mathcal{O}_i, i = 1, 2\} \subset \text{OPS}^{r_1 + r_2}
\]
is bounded as well. If \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) have commuting matrices, then
\[
\{[\mathcal{Q}_1, \mathcal{Q}_2] : \mathcal{Q}_i \in \mathcal{O}_i, i = 1, 2\} \subset \text{OPS}^{r_1 + r_2 - 1}
\]
is also bounded.

For two operators \( \mathcal{A} \) and \( \mathcal{B} \), \( [\mathcal{A}, \mathcal{B}] := \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A} \). The original statements of the following two lemmas are for a single operator \( \mathcal{Q} \in \text{OPS}^s \) and for functions on \( \mathbb{R}^d \), where the constant \( C \) depends on seminorms of \( \mathcal{Q} \). Thus, it can be shown that these estimates hold uniformly for \( \mathcal{Q} \) in a bounded set of \( \text{OPS}^s \). Again, by Remark 3.1, they also hold true for functions on \( \mathbb{T}^d \).

Lemma 3.4 (Proposition 3.6.A in [60]). Let \( s > 0 \) and \( \mathcal{O} \subset \text{OPS}^s \) be bounded. Then for any \( \sigma \geq 0 \), \( g \in H^{s+\sigma} \cap W^{1,\infty} \), \( u \in H^{s+1+\sigma} \cap L^\infty \),
\[
\sup_{\mathcal{Q} \in \mathcal{O}} \|\mathcal{Q}(gI)u\|_{H^s} \lesssim \|g\|_{W^{1,\infty}} \|u\|_{H^{s+1+\sigma}} + \|g\|_{H^{s+\sigma}} \|u\|_{L^\infty}.
\]

Lemma 3.5 (Proposition 4.2 in [61]). Let \( s \geq 0 \) and \( \mathcal{O} \subset \text{OPS}^s \) be bounded. Then for any \( \sigma > 1 + \frac{d}{2} \) and \( q \in [0, \sigma - s] \),
\[
\sup_{\mathcal{Q} \in \mathcal{O}} \|\mathcal{Q}(gI)u\|_{H^q} \lesssim \|g\|_{H^s} \|u\|_{H^{q+s-1}}, \quad g \in H^s, \quad u \in H^{q+s-1}.
\]

We also recall the Friedrichs mollifier \( J_n \) on \( \mathbb{K}^d \). Let \( \phi \in \mathcal{S}(\mathbb{K}^d; \mathbb{R}) \) (the Schwarz space of rapidly decreasing \( C^\infty \) functions on \( \mathbb{R}^d \)) satisfy \( 0 \leq \phi(y) \leq 1 \) for all \( y \in \mathbb{R}^d \) and \( \phi(y) = 1 \) for any \( |y| \leq 1 \), and let
\[
\phi_n(\xi) := \phi_n(\xi/n), \quad n \geq 1
\]
for \( \xi \in \mathbb{R}^d \) when \( \mathbb{K} = \mathbb{R} \), and \( \xi \in \mathbb{Z}^d \) when \( \mathbb{K} = \mathbb{T} \). Then the Friedrichs mollifier \( \{J_n\}_{n \geq 1} \) is defined as
\[
J_n = \text{OP}(\phi_n), \quad n \geq 1.
\]

Of course, one may choose different \( J_n \) in different situations, but this is a simple and uniform choice for SPDEs considered in the paper.

Finally, to conclude this part, in the following lemma we summarize some frequently used properties regarding the above operators (cf. [43, 57, 59]):

Lemma 3.6. Let \( \mathcal{P}, (\mathcal{D}^s, \Lambda^s) \) and \( J_n \) be given in (1.4), (3.7) and (3.14), respectively. Then for any \( 1 \leq j \leq d, \) the following properties hold.

1. For all \( \sigma \geq 0 \) and \( f \in H^\sigma \), \( \sup_{n \geq 1} \|J_n\|_{\mathcal{L}(L^\infty; L^\infty)} < \infty, \|J_n\|_{\mathcal{L}(H^\sigma; H^\sigma)} = 1 \) \( (n \geq 1) \) and \( \lim_{n \to \infty} \|J_n f - f\|_{H^\sigma} = 0 \).
2. For any \( s_1, s_2 \geq 0 \),
\[
\|J_n\|_{\mathcal{L}(H^{s_1}; H^{s_2})} \lesssim n^{(s_2-s_1)^+}, \quad \|J_l - J_n\|_{\mathcal{L}(H^{s_1}; H^{s_2})} \lesssim (l \wedge n)^{-(s_1-s_2)^+}, \quad l, n \geq 1.
\]
(2) For all \( s, \sigma \in \mathbb{R} \) and \( n \geq 1 \), \( \tilde{\Pi}, \mathcal{D}^s, \Lambda^\sigma \) and \( J_n \) are self-adjoint in \( L^2 \) and
\[
\{T, \tilde{T}\} = [T, \partial_j] = 0, \quad T, \tilde{T} \in \{\tilde{\Pi}, \mathcal{D}^s, \Lambda^\sigma, J_n\}_{s, \sigma \in \mathbb{R}; n \geq 1}, \quad 1 \leq j \leq d.
\]
(3) For all \( g \in W^{1,\infty} \) and \( f \in L^2 \),
\[
\sup_{n \geq 1} \left\| [\tilde{J}_n, gI] \nabla f \right\|_{L^2} \lesssim \|g\|_{W^{1,\infty}} \|f\|_{L^2}, \quad g \in W^{1,\infty}.
\]

### 3.2 Assumptions and main results

Recall that \( A^* \) is the \( L^2 \)-adjoint operator of a linear operator \( A \). For \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{T} \) and \( d, m \geq 1 \), we simplify notation by letting
\[
\tilde{H}^s := \tilde{\Pi}H^s, \quad H^s = H^s(\mathbb{K}^d; \mathbb{R}^m), \quad s \geq 0.
\]
Because \( \tilde{\Pi} \) is a projection, for \( X, Y \in \tilde{H}^s \) with \( s \geq 0 \), we have
\[
\langle X, Y \rangle_{\tilde{H}^s} := \langle \tilde{\Pi}X, \tilde{\Pi}Y \rangle_{H^s} = \langle X, Y \rangle_{H^s}, \quad \|X\|_{\tilde{H}^s}^2 := \langle X, X \rangle_{\tilde{H}^s} = \|X\|_{H^s}^2.
\]

We first introduce the assumption on \( \{A_k\}_{k \geq 1} \) included in the noise coefficients of (1.8). As extension of skew-adjoint operators \( \{\tilde{\partial}_j\}_{1 \leq j \leq d} \) in the transport noise (1.7), each \( A_k \) is not far away from anti-symmetric, and contains a higher order operator with symbol independent of \( x \) and a lower order operator with symbol depending on \( x \).

**Assumption (D).** Let \( \tilde{\Pi} \) be a projection operator in \( L^2 \) given in (1.4) and (1.5). Let \( a_k, q_k \in \mathbb{R}, r_1 \in [0, 1], r_2 \geq r_1 \). For \( k \geq 1 \), we let
\[
A_k = a_k J_k + q_k K_k, \quad J_k := \text{diag}(J_{k,1}, \cdots, J_{k,m}), \quad K_k := \text{diag}(K_{k,1}, \cdots, K_{k,m}),
\]
where \( \{J_{k,i}\}_{k \geq 1} \in \text{OPS}^0 \) and \( \{K_{k,i}\}_{k \geq 1} \in \text{OPS}^1 \) \((1 \leq i \leq m)\) are bounded, respectively. We assume that the following conditions hold:

**D1** \( \{a_k\}_{k \geq 1}, \{q_k\}_{k \geq 1} \in l^2 \) and \( a_kq_k = 0 \) for \( k \geq 1 \).

**D2** There exist operators \( \mathcal{T}_k := \text{diag}(\mathcal{T}_{k,1}, \cdots, \mathcal{T}_{k,m}) \) and \( \mathcal{Q}_k := \text{diag}(\mathcal{Q}_{k,1}, \cdots, \mathcal{Q}_{k,m}) \) such that \( \{\mathcal{T}_{k,i}\}_{k \geq 1} \subset \text{OPS}^0 \) and \( \{\mathcal{Q}_{k,i}\}_{k \geq 1} \subset \text{OPS}^0 \) \((1 \leq i \leq m)\) are bounded, respectively, and
\[
\mathcal{J}_{k}^* = \mathcal{T}_k - \mathcal{J}_k, \quad \mathcal{K}_{k}^* = \mathcal{Q}_k - \mathcal{K}_k, \quad 1 \leq i \leq m, \quad k \geq 1.
\]

**D3** Either \( \tilde{\Pi} = \text{diag}(\tilde{\Pi}_1, \cdots, \tilde{\Pi}_m) \) with \( \tilde{\Pi}_i \in \text{OPS}^0 \) \((1 \leq i \leq m)\), or
\[
(3.18) \quad \tilde{\Pi}U|\tilde{H}^s = U|\tilde{H}^s, \quad s \geq r_0 := \max \left\{ r_2 1_{\{a_k \geq 0\}}, \quad r_1 1_{\{q_k \geq 0\}} \right\}
\]
holds for any \( U \in \{\mathcal{T}_k, \mathcal{T}_k, \mathcal{K}_k, \mathcal{Q}_k : k \geq 1\} \).

Then we assume that the operator \( -\mathcal{E} \) is a positive semi-definite operator satisfying the following assumption:

**Assumption (E).** Assume that there exists \( \mathcal{G}_j \in \text{OPS}^p \) with \( p_j \geq 0 \) and \( 1 \leq j \leq m \) such that for \( p_0 := \max \{p_1, \cdots, p_m\} \),
\[
\mathcal{E} \in \text{OPS}^{2p_0}, \quad -\langle \mathcal{E}X, X \rangle_{L^2} = \|\mathcal{G}X\|_{L^2}^2, \quad X \in H^{2p_0}, \quad \mathcal{G} = \text{diag}(\mathcal{G}_1, \cdots, \mathcal{G}_m).
\]

Usually, if \( \mathcal{G}_j \neq 0 \) with \( 1 \leq j \leq m \), then \( -\mathcal{E} \) is positive definite operator. In this work, we allow \( \mathcal{G}_j = 0 \) for some \( j \) to cover degenerate cases. Hence the case of degenerated diffusion in fluid models can be covered. For instance, degenerated diffusion appears in boundary layer equations.

Finally, we introduce the following assumptions on the regular noise coefficients \( \mathcal{h}_k : [0, \infty) \times H^s \to H^s \) and the drifts \( (\mathcal{b}, \mathcal{g}) : H^s \to \tilde{H}^s \) given by nonlinear PDEs, where \( \tilde{H}^s \) is given in (3.15).
**Assumption (F).** Assume that the following conditions hold for some

\[(3.19)\quad \theta_1, \theta_2 \geq 0, \; q_0 > 0, \; s_0 > \theta_0 + q_0 \text{ with } \theta_0 := \max\{\theta_1, \theta_2\},\]

some function \(K \in \mathcal{K}\) in (2.1) and an increasing function \(\tilde{K} : [0, \infty) \to (0, \infty)\).

(F_h) Let \(\sigma \in \{\theta_0, s_0\}. \; \tilde{h}_k : [0, \infty) \times H^{s_0} \to H^{s_0}\) satisfies that for all \(t \geq 0\) and \(X, Y \in H^{s_0}\),

\[
\sum_{k \geq 1} \|\tilde{h}_k(t, X)\|^2_{H^{s_0}} \leq K(t, \|X\|_{H^{s_0}})(1 + \|X\|^2_{H^{s_0}}),
\]

\[
\sum_{k=1}^{\infty} \|\tilde{h}_k(t, X) - \tilde{h}_k(t, Y)\|^2_{H^{s_0}} \leq K(t, \|X\|_{H^{s_0}} + \|Y\|_{H^{s_0}})\|X - Y\|^2_{H^{s_0}}.
\]

(F_b) Let \(\sigma \in \{\theta_0, s_0\}. \; \tilde{b} : H^{s_0} \to H^{s_0}\) satisfies that for all \(X, Y \in H^{s_0}\) and \(N > 0\),

\[
\|\tilde{b}(X)\|_{H^{s_0}} \leq \tilde{K}(\|X\|_{H^{s_0}})\|X\|_{H^{s_0}},
\]

\[
\sup_{\|X\|_{H^{s_0}}, \|Y\|_{H^{s_0}} \leq N} \mathbf{1}_{(X \neq Y)} \frac{\|\tilde{b}(X) - \tilde{b}(Y)\|_{H^{s_0}}}{\|X - Y\|_{H^{s_0}}} \leq \tilde{K}(N).
\]

(F_g) \(\tilde{g} : H^{s_0} \to H^{s_0 - q_0}\) satisfies for all \(X, Y \in H^{s_0}\) and \(N > 0\) that

\[
\|\tilde{g}(X)\|_{H^{s_0 - q_0}} \leq \tilde{K}(\|X\|_{H^{s_0}}),
\]

\[
\sup_{\|X\|_{H^{s_0}}, \|Y\|_{H^{s_0}} \leq N} \mathbf{1}_{(X \neq Y)} \frac{\|J_n\tilde{g}(J_nX) - J_n\tilde{g}(J_nY)\|_{H^{s_0}}}{\|X - Y\|_{H^{s_0}}} < \infty,
\]

\[
\sup_{n \geq 1} \left\{ \|\tilde{g}(J_nX), J_nX\|_{H^{s_0}} + \|\tilde{g}(J_nX), J_nX\|_{H^{s_0}} \right\} \leq \tilde{K}(\|X\|_{H^{s_0}})\|X\|^2_{H^{s_0}}.
\]

Moreover, there exists a function \(\lambda : \mathbb{N} \times \mathbb{N} \to (0, \infty)\) with \(\lambda_{n,l} \to 0\) as \(n, l \to \infty\) such that for all \(X, Y \in H^{s_0}\),

\[
\sup_{n \geq 1} (J_n\tilde{g}(J_nX) - J_n\tilde{g}(J_nY), X - Y)_{H^{s_0}} \leq \tilde{K}(\|X\|_{H^{s_0}} + \|Y\|_{H^{s_0}})(\lambda_{n,l} + \|X - Y\|^2_{H^{s_0}}), \; n, l \geq 1.
\]

Recall \(\tilde{H}^s\) in (3.15), \(r_0\) in (3.18), \(p_0\) in (E) and \(\theta_0\) in (F). Now we are in the position to state our main result for (1.8) as

**Theorem 3.1.** Assume (D) and (E). Let (F) hold with \(s_0 > \theta_0 + \max\{q_0, 2r_0, 2p_0\}\). Let \(X(0)\) be an \(\mathcal{F}_0\)-measurable \(H^{s_0}\)-valued random variable.

(a) (1.8) has a unique maximal solution \((X, \tau^*)\) in the sense of Definition 1.1 for the choice \(\mathbb{H} := \tilde{H}^{s_0}\) and \(\mathbb{M} := \tilde{H}^{\theta_0}\). Besides, \((X, \tau^*)\) satisfies \(P(X \in C([0, \tau^*]; H^{s_0})) = 1\) and \(\lim_{t \to \tau^*} \|X\|_{H^{s_0}} = \infty\) on \(\{\tau^* < \infty\}\).

(b) Let \(B(\cdot) \in C(\tilde{H}^{\theta_0}; [0, \infty))\) be a subadditive function satisfying \(B(X) \subseteq \|X\|_{H^{s_0}}\) for \(X \in \tilde{H}^{\theta_0}\). If in (F), \(K(t, \|X\|_{H^{s_0}})\) and \(\tilde{K}(\|X\|_{H^{s_0}})\) are replaced by \(K(t, B(X))\) and \(\tilde{K}(B(X))\), respectively, then

\[
\lim_{t \to \tau^*} \sup_{t \to \tau^*} B(X) = \infty \text{ on } \{\tau^* < \infty\}.
\]
(c) $P(\tau^*=\infty) = 1$ provided that

$$\limsup_{\|X\|_{H^{\theta_0}} \to \infty} \frac{\Psi(T, X, \theta_0)}{K(\|X\|_{H^{\theta_2}}\|X\|^2_{H^{\theta_0}})} < -1, \quad T \in (0, \infty),$$

where we define

$$\Psi(T, X, \kappa) := \sup_{t \in (0, T)} \sum_{k=1}^{\infty} \left( \left\| \tilde{h}_k(t, X) \right\|^2_{H^\kappa} - \frac{2(\tilde{h}_k(t, X), X)^2_{H^\kappa}}{e + \|X\|^2_{H^{\theta_0}}} \right), \quad \kappa > 0.$$  

**Remark 3.2.** Now we give some technical remarks on (F) and Theorem 3.1.

1. We assume the growth factor for noise coefficients $h_k$ in (F_k) involves $\|X\|_{\theta_1}$. Since $(\tilde{b}, \tilde{g})$ comes from nonlinear PDEs, we assume that the growth factor in (F_b) and (F_g) involves $\|X\|_{\theta_2}$. The parameters $(\theta_1, \theta_2)$ will be determined by $(\tilde{b}, \tilde{g})$ in specific models. In many cases we require $H^{\theta_2} \hookrightarrow W^{\kappa, \infty}$, for which we may take $\theta_2 = (\frac{d}{2} + k, \infty)$ by the Sobolev embedding theorem.

2. Assume (D) and (E). Theorem 3.1 means that if (F) holds for certain $\theta_1, \theta_2 \geq 0$ and some $s_0 > \theta_1 + \max\{q_0, 2r_0, 2p_0\}$, then $X(0) \in \tilde{H}^{s_0}$ implies that (1.8) has a unique solution $X \in C([0, \tau^*); \tilde{H}^{s_0})$. Here the parameter $s_0$ has freedom in the interval $(\theta_0 + \max\{q_0, 2r_0, 2p_0\}, \infty)$, so that the result implies stronger continuity of the solution when $s_0$ is replaced by larger parameters. For instance, if $X(0) \in \tilde{H}^{s_0} := \cap_{k \geq 1} \tilde{H}^k$ and (F) holds for any $s_0 \in (\theta_0 + \max\{q_0, 2r_0, 2p_0\}, \infty)$, then Theorem 3.1 implies that the unique solution is in $C([0, \tau^*); \tilde{H}^{s_0})$, where the lifespan is determined by $\|X\|_{\tilde{M}}$ (cf. (2.17) and (2.18)), and for any $s > \theta_0$, $\limsup_{t \to \tau^*} \|X(t)\|_{H^{s_0}} = \infty$ is equivalent to $\limsup_{t \to \tau^*} \|X(t)\|_{H^s} = \infty$. See Sections 4 for more examples.

### 3.3 Proof of Theorem 3.1

We denote

$$\tilde{\mathcal{U}} := \tilde{\Pi}\mathcal{U} \text{ for } \mathcal{U} \in \{\mathcal{E}, \mathcal{A}_k, \mathcal{J}_k, \mathcal{K}_k\}, \quad k \geq 1.$$  

Since $a_kq_k = 0$ in (D_1), for $\mathcal{A}_k$ in (3.17), we may write

$$\{\tilde{\mathcal{A}}_kX(t)\} \circ dW_k(t) = \{a_k\tilde{\mathcal{J}}_kX(t)\} \circ d\tilde{W}_k(t) + \{q_k\tilde{\mathcal{K}}_kX(t)\} \circ d\tilde{W}_k(t), \quad k \geq 1$$

for independent 1-D Brownian motions $\{\tilde{W}_k(t), \tilde{W}_k(t)\}$, which are also independent of $\{W_k(t)\}$. Thus, by (1.10), we rewrite (1.8) as

$$dX(t) = \left\{ (\tilde{E}X)(t) + \tilde{b}(X(t)) + \tilde{g}(X(t)) + \frac{1}{2} \sum_{k=1}^{\infty} \left[ (a_k\tilde{\mathcal{J}}_k)^2 X + (q_k\tilde{\mathcal{K}}_k)^2 X \right] (t) \right\} dt$$

$$+ \sum_{k=1}^{\infty} \left\{ (a_k\tilde{\mathcal{J}}_kX)(t) dW_k(t) + (q_k\tilde{\mathcal{K}}_kX)(t) d\tilde{W}_k(t) \right\}$$

$$+ \sum_{k=1}^{\infty} \tilde{h}_k(t, X(t)) d\tilde{W}_k(t), \quad t \geq 0.$$  

Before we prove Theorem 3.1, we state the following cancellation properties regarding on $\mathcal{A}_k$, which is required to verify (R_4).

**Lemma 3.7.** Assume (D). For any $\sigma \geq 0$, there exists a constant $C > 0$ such that $\mathcal{Y}_k \in \{a_k\tilde{\mathcal{J}}_k, q_k\tilde{\mathcal{K}}_k\}$ satisfies

$$\sup_{n \geq 1} \sum_{k=1}^{\infty} \langle J_n\mathcal{Y}_kX, J_nX \rangle_{H^\sigma}^2 + \sup_{n \geq 1} \sum_{k=1}^{\infty} \left\| J_n\mathcal{Y}_kX, X \right\|_{H^{\sigma+r_0}}^2 \leq C\|X\|_{H^\sigma}^4, \quad X \in \tilde{H}^{\sigma+r_0},$$

$$\sup_{n \geq 1} \sum_{k=1}^{\infty} \left\| J_n\mathcal{Y}_kX, J_nX \right\|_{H^\sigma}^2 + \sup_{n \geq 1} \sum_{k=1}^{\infty} \left\| J_n\mathcal{Y}_kX \right\|_{H^\sigma}^2 \leq C\|X\|_{H^\sigma}^2, \quad X \in \tilde{H}^{\sigma+2r_0},$$

(3.25)
\[ (3.26) \quad \sup_{n \geq 1} \sum_{k=1}^{\infty} \left\langle J_n^2 Y_n^2 J_n X, X \right\rangle_{H^*} + \left\| J_n Y_n J_n X \right\|^2_{H^*} \leq C \left\| X \right\|^2_{H^*}, \quad X \in \tilde{H}^{\sigma+2r_0}. \]

Proof. Verify (3.24) for \( Y_k = q_k \tilde{K}_k \). Remember that in this case \( r_0 = r_1 \in [0,1] \) (cf. (3.18)). Since \( J_n \) and \( D^\sigma \) have scalar symbols and \( K_k^* = Q_k - K_k \), we have for \( \{ P_n := D^\sigma J_n \} \subset \text{OPS}^\sigma \),

\[ \left\langle J_n \tilde{K}_k X, J_n X \right\rangle_{H^*} = \left\langle P_n K_k X, P_n X \right\rangle_{L^2} \]

\[ = \left( \left\langle P_n K_k X, P_n X \right\rangle_{L^2} + \left\langle P_n X, Q_k P_n X \right\rangle_{L^2} - \left\langle P_n X, K_k P_n X \right\rangle_{L^2} \right) \]

\[ = 2 \left( \left\langle P_n K_k X, P_n X \right\rangle_{L^2} + \left\langle P_n X, Q_k P_n X \right\rangle_{L^2} - \left\langle P_n X, K_k X \right\rangle_{L^2} \right), \]

which means

\[ (3.27) \quad \left\langle J_n \tilde{K}_k X, J_n X \right\rangle_{H^*} = \left\langle \{ P_n K_k X, P_n X \} \right\rangle_{L^2} + \frac{1}{2} \left\langle P_n X, Q_k P_n X \right\rangle_{L^2}. \]

It follows from (D), the boundedness of \( \{ P_n \}_{n \geq 1} \subset \text{OPS}^\sigma \), \( \{ K_k \}_{k \geq 1} \subset \text{OPS}^{r_1} \) and \( \{ Q_k \}_{k \geq 1} \subset \text{OPS}^0 \), Lemmas 3.2 and 3.3 that

\[ \sup_{n \geq 1} \sum_{k=1}^{\infty} \left\langle J_n(q_k \tilde{K}_k) X, J_n X \right\rangle^2_{H^*} \leq C \sup_{n \geq 1} \sum_{k=1}^{\infty} q_k^2 \left\| X \right\|^4_{H^*} \leq C \left\| X \right\|^4_{H^*}, \quad X \in \tilde{H}^{\sigma+r_0}. \]

Similarly, by Lemma 3.6, repeating the above process with \( P := D^\sigma \) leads to

\[ \sup_{n \geq 1} \sum_{k=1}^{\infty} \left\langle J_n(q_k \tilde{K}_k) X, J_n X \right\rangle^2_{H^*} \leq C \left\| X \right\|^4_{H^*}, \quad X \in \tilde{H}^{\sigma+r_0}. \]

Combining the above two estimates gives rise to (3.24) for the case \( Y_k = q_k \tilde{K}_k \).

Verify (3.25) for \( Y_k = a_k \tilde{J}_k \). Similar to (3.27), in this case we have \( [\tilde{J}_k, P_n] = 0 \) and hence we have

\[ \left\langle J_n \tilde{J}_k X, J_n X \right\rangle_{H^*} = \frac{1}{2} \left\langle P_n X, K_k P_n X \right\rangle_{L^2}. \]

By the same reason leading to (3.24) for the case \( A_k = q_k \tilde{K}_k \), we have

\[ \sup_{n \geq 1} \sum_{k=1}^{\infty} \left\langle J_n a_k \tilde{J}_k X, J_n X \right\rangle^2_{H^*} \leq C \sup_{n \geq 1} \sum_{k=1}^{\infty} a_k^2 \left\| X \right\|^4_{H^*} \leq C \left\| X \right\|^4_{H^*}, \quad X \in \tilde{H}^{\sigma+r_0}. \]

Similarly, one can obtain the desired bound for the other term. Hence (3.24) also holds true for the case \( Y_k = a_k \tilde{J}_k \).

Verify (3.25) for \( Y_k = q_k \tilde{K}_k \). Because \( \tilde{\Pi} \) is self-adjoint, we have

\[ (\tilde{K}_k)^* = K_k^* \tilde{\Pi} = Q_k \tilde{\Pi} - K_k \tilde{\Pi}. \]

We will frequently use the above facts as well as the following properties without further notice:

\[ \tilde{K}_k = \tilde{\Pi} K_k = \tilde{\Pi} \tilde{K}_k, \quad [P_n, \tilde{\Pi}] = 0, \quad \tilde{\Pi} = \tilde{\Pi}^*, \quad Q_k = Q_k^* \quad \text{and} \quad P_n X = \tilde{\Pi} P_n X \quad \text{for} \quad X \in \tilde{H}^{\sigma+2r_0}. \]

For \( k, n \in \mathbb{N} \), we let

\[ P_n := D^\sigma J_n, \quad \tilde{R}_{1,k} := [\tilde{K}_k, \tilde{\Pi} Q_k], \quad \tilde{R}_{2,k,n} := [P_n, \tilde{K}_k], \quad \tilde{R}_{3,k,n} := [\tilde{\Pi} R_{2,k,n}, \tilde{K}_k]. \]

We observe that for all \( X \in \tilde{H}^{\sigma+2r_0} \),

\[ \left\langle J_n \tilde{K}_k X, J_n X \right\rangle_{H^*} \]

\[ = \left\langle P_n \tilde{K}_k X, (\tilde{K}_k)^* P_n X \right\rangle_{L^2} + \left\langle \tilde{R}_{2,k,n} \tilde{K}_k X, P_n X \right\rangle_{L^2} \]

\[ = - \left\langle P_n \tilde{K}_k X, \tilde{K}_k P_n X \right\rangle_{L^2} + \left\langle P_n \tilde{K}_k X, Q_k P_n X \right\rangle_{L^2} + \left\langle \tilde{R}_{2,k,n} \tilde{K}_k X, P_n X \right\rangle_{L^2} \]

\[ = - \left\langle P_n \tilde{K}_k X, P_n \tilde{K}_k X \right\rangle_{L^2} + \left\langle P_n \tilde{K}_k X, \tilde{R}_{2,k,n} X \right\rangle_{L^2} \]

\[ + \left\langle P_n \tilde{K}_k X, Q_k P_n X \right\rangle_{L^2} + \left\langle \tilde{R}_{2,k,n} \tilde{K}_k X, P_n X \right\rangle_{L^2}. \]
which means

\[
\left< J_n \tilde{k}^2 X, J_n X \right>_{H^s} + \| J_n \tilde{k} X \|^2_{H^s} = \left< \mathcal{P}_n \tilde{k} X, \tilde{R}_{2,k,n} X \right>_{L^2} + \left< \mathcal{P}_n \tilde{k} X, \mathcal{Q}_k \mathcal{P}_n X \right>_{L^2} + \left< \tilde{R}_{2,k,n} \tilde{k} X, \mathcal{P}_n X \right>_{L^2}.
\]

Then, we observe that

\[
\begin{align*}
\left< J_n \tilde{k}^2 X, J_n X \right>_{H^s} + \| J_n \tilde{k} X \|^2_{H^s} &= \left< \tilde{k} \mathcal{P}_n X, \tilde{R}_{2,k,n} X \right>_{L^2} + \left< \tilde{R}_{2,k,n} \tilde{k} X, \tilde{R}_{2,k,n} X \right>_{L^2} \\
&\quad + \left< \tilde{R}_{2,k,n} \tilde{k} X, \mathcal{P}_n X \right>_{L^2} + \left< \mathcal{P}_n \tilde{k} X, \mathcal{Q}_k \mathcal{P}_n X \right>_{L^2} \\
&= - \left< \mathcal{P}_n X, \tilde{k} \tilde{R}_{2,k,n} \tilde{k} X \right>_{L^2} + \left< \mathcal{P}_n X, \mathcal{Q}_k \tilde{R}_{2,k,n} \tilde{k} X \right>_{L^2} \\
&\quad + \left< \tilde{R}_{2,k,n} X, \tilde{R}_{2,k,n} X \right>_{L^2} + \left< \mathcal{P}_n X, \mathcal{Q}_k \tilde{R}_{2,k,n} \tilde{k} X \right>_{L^2} + \left< \tilde{R}_{2,k,n} \tilde{k} X, \mathcal{P}_n X \right>_{L^2} \\
&= \left< \tilde{R}_{3,k,n} X, \mathcal{P}_n X \right>_{L^2} + \left< \mathcal{P}_n X, \mathcal{Q}_k \tilde{R}_{2,k,n} \tilde{k} X \right>_{L^2} \\
&\quad + \left< \tilde{R}_{2,k,n} X, \tilde{R}_{2,k,n} X \right>_{L^2} + \left< \mathcal{P}_n X, \mathcal{Q}_k \tilde{R}_{2,k,n} \tilde{k} X \right>_{L^2}.
\end{align*}
\]

Hence

\[
- \left< \mathcal{P}_n X, \tilde{k} \tilde{R}_{2,k,n} \tilde{k} X \right>_{L^2} + \left< \tilde{R}_{3,k,n} X, \mathcal{P}_n X \right>_{L^2} = \left< \tilde{R}_{3,k,n} X, \mathcal{P}_n X \right>_{L^2}.
\]

Once again, to deal with \( \mathcal{P}_n \tilde{k} \), we have

\[
\begin{align*}
\left< \mathcal{P}_n \tilde{k} X, \tilde{Q}_k \mathcal{P}_n X \right>_{L^2} &= - \left< \mathcal{P}_n X, \tilde{Q}_k \tilde{k} \mathcal{P}_n X \right>_{L^2} - \left< \mathcal{P}_n X, \tilde{R}_{1,k} \mathcal{P}_n X \right>_{L^2} \\
&\quad + \left< \mathcal{P}_n X, \mathcal{Q}_k \tilde{R}_{1,k} \mathcal{P}_n X \right>_{L^2} + \left< \tilde{R}_{2,k,n} X, \tilde{Q}_k \mathcal{P}_n X \right>_{L^2} \\
&= - \left< \tilde{Q}_k \mathcal{P}_n X, \tilde{k} \mathcal{P}_n X \right>_{L^2} - \left< \mathcal{P}_n X, \tilde{R}_{1,k} \mathcal{P}_n X \right>_{L^2} \\
&\quad + \left< \mathcal{P}_n X, \mathcal{Q}_k \tilde{R}_{1,k} \mathcal{P}_n X \right>_{L^2} + \left< \tilde{R}_{2,k,n} X, \tilde{Q}_k \mathcal{P}_n X \right>_{L^2}.
\end{align*}
\]

Accordingly, adding \( \left< \mathcal{P}_n \tilde{k} X, \tilde{Q}_k \mathcal{P}_n X \right>_{L^2} \) to both sides of the above equation and then using \( \tilde{R}_{2,k,n} \) yield

\[
\begin{align*}
2 \left< \mathcal{P}_n \tilde{k} X, \tilde{Q}_k \mathcal{P}_n X \right>_{L^2} &= - \left< \mathcal{P}_n X, \tilde{R}_{1,k} \mathcal{P}_n X \right>_{L^2} + \left< \mathcal{P}_n X, \mathcal{Q}_k \tilde{R}_{2,k,n} \mathcal{P}_n X \right>_{L^2} + 2 \left< \tilde{R}_{2,k,n} X, \tilde{Q}_k \mathcal{P}_n X \right>_{L^2}.
\end{align*}
\]

Combining the above estimates gives

\[
(3.28) \quad \left| \left< J_n \tilde{k}^2 X, J_n X \right>_{H^s} + \| J_n \tilde{k} X \|^2_{H^s} \right| \leq \sum_{i=1}^{6} |I_i|
\]

with

\[
\begin{align*}
I_1 &= \left< \tilde{R}_{3,k,n} X, \mathcal{P}_n X \right>_{L^2}, \quad I_2 = \left< \tilde{R}_{2,k,n} X, \tilde{R}_{2,k,n} X \right>_{L^2}, \\
I_3 &= \left< \mathcal{P}_n X, \tilde{Q}_k \tilde{R}_{2,k,n} X \right>_{L^2}, \quad I_4 = -\frac{1}{2} \left< \mathcal{P}_n X, \tilde{R}_{1,k} \mathcal{P}_n X \right>_{L^2}, \\
I_5 &= \frac{1}{2} \left< \mathcal{P}_n X, \mathcal{Q}_k \tilde{R}_{2,k,n} \mathcal{P}_n X \right>_{L^2}, \quad I_6 = \left< \tilde{R}_{2,k,n} X, \tilde{Q}_k \mathcal{P}_n X \right>_{L^2}.
\end{align*}
\]
Now we argue in two cases. We first consider $\bar{\Pi} = \{\bar{\Pi}_1, \cdots, \bar{\Pi}_m\}$ with $\bar{\Pi}_i \in \text{OPS}_i^0$ ($1 \leq i \leq m$) in (D). In this case, $\{K_k\}_{k \geq 1} \subset \text{OPS}_3^1$ and $\{\bar{\Pi}Q_k\}_{k \geq 1} \subset \text{OPS}_3^1$ are bounded, so that Lemmas 3.2 and 3.3 imply
\[
\sup_{n,k \geq 1} \left\{ \left\| \bar{\mathcal{R}}_{1,k} \right\|_{L^2(L^2;L^2)} + \left\| \bar{\mathcal{R}}_{2,k,n} \right\|_{L(H^s;L^2)} + \left\| \bar{\mathcal{R}}_{3,k,n} \right\|_{L(H^s;L^2)} \right\} < \infty.
\]
Hence (3.25) holds true for $\mathcal{Y}_k = q_k\bar{\Pi}_k$.

Next, we assume that (3.18) in (D) holds. In this case we let
\[
\mathcal{R}_{1,k} := \{K_k, Q_k\}, \quad \mathcal{R}_{2,k,n} := \{\bar{\Pi}, K_k\}, \quad \mathcal{R}_{3,k,n} := \{\bar{\Pi}Q_k, K_k\}.
\]
Then (3.18) and the fact that $\{\bar{\Pi}, \bar{\Pi}Q_k, K_k\}$ are bounded, so that Lemmas 3.2 and 3.3 imply
\[
\sup_{n,k \geq 1} \left\{ \left\| \mathcal{R}_{1,k} \right\|_{L^2(L^2;L^2)} + \left\| \mathcal{R}_{2,k,n} \right\|_{L(H^s;L^2)} + \left\| \mathcal{R}_{3,k,n} \right\|_{L(H^s;L^2)} \right\} < \infty.
\]
Consequently, for $X \in \bar{H}^{s+r_0}$, we have
\[
\sup_{n \geq 1} \sum_{k=1}^{\infty} \left\| J_n \left( q_k\bar{\Pi}_k \right)^2 X, J_n X \right\|_{H^s} + \left\| J_n(q_k\bar{\Pi}_k)X \right\|_{H^s}^2 \leq \sum_{k=1}^{\infty} q_k \left( \sum_{i=1}^{6} |I_i| \right) \leq C \|X\|_{H^s}^2,
\]
which is the desired result.

**Verify (3.25) for $\mathcal{Y}_k = a_k\bar{\Pi}_k$.** In this case, by the same argument leading to (3.28), we derive the same estimate for $(J_k, T_k)$ replacing $(\bar{\Pi}_k, Q_k)$. By (D), in each situations of (D) we have $\mathcal{R}_{1,k} = \mathcal{R}_{2,k,n} = \mathcal{R}_{3,k,n} = 0$ for $k, n \geq 1$, so that
\[
\sup_{n \geq 1} \sum_{k=1}^{\infty} \left\| J_n \left( a_k\bar{\Pi}_k \right)^2 X, J_n X \right\|_{H^s} + \left\| J_n(a_k\bar{\Pi}_k)X \right\|_{H^s}^2 \leq \sum_{k=1}^{\infty} a_k \left\| J_n T_k\bar{\Pi}_k T_k P_n X \right\|_{L^2} \leq C \|X\|_{H^s}^2.
\]

**Verify (3.26).** For the case that $\mathcal{Y}_k = q_k\bar{\Pi}_k$, going along the lines of the above proof of (3.25) with replacing $X$ by $J_nX$ gives the desired upper bound. The case of $\mathcal{Y}_k = a_k\bar{\Pi}_k$ is also similar. We omit the details for brevity.

With Lemma 3.7 at hand, now we are in the position to prove Theorem 3.1.

**Proof of Theorem 3.1.** Recall that (1.8) is equivalent to (3.23):
\[
dX(t) = \left\{ \left( \bar{\mathcal{E}}X \right)(t) + \bar{\mathcal{E}}(X(t)) + \bar{g}(X(t)) + \frac{1}{2} \sum_{k=1}^{\infty} \left( a_k\bar{\Pi}_k \right)^2 X + \left( q_k\bar{\Pi}_k \right)^2 X \right\} dt
\]
\[
+ \sum_{k=1}^{\infty} \left\{ a_k\bar{\Pi}_k X \right\} (t) dW_k(t) + \left( q_k\bar{\Pi}_k X \right) (t) d\bar{W}_k(t)
\]
\[
+ \sum_{k=1}^{\infty} \bar{h}_k(t, X(t)) d\bar{W}_k(t), \quad t \geq 0.
\]
(a) (Existence and uniqueness) The above equation is embedded into (1.1) with

\[
\begin{aligned}
b(t, X) := \tilde{b}(X), \\
g(t, X) = g(X) := \tilde{c}X + \tilde{g}(X) + \sum_{k=1}^{\infty} \left[ (a_k \tilde{f}_k)^2 X + (q_k \tilde{e}_k)^2 X \right], \\
h(t, X)e_{3k-2} := a_k \tilde{f}_k X, \quad k \geq 1, \\
h(t, X)e_{3k-1} := q_k \tilde{e}_k X, \quad k \geq 1, \\
h(t, X)e_{3k} := \tilde{h}_k(t, X), \quad k \geq 1, \\
\mathcal{W}(t) := \sum_{k=1}^{\infty} \left( \tilde{W}_k(t)e_{3k-2} + \tilde{W}_k(t)e_{3k-1} + \tilde{W}_k(t)e_{3k} \right).
\end{aligned}
\]  

(3.30)

Let \( \{J_n\}_{n \geq 1} \) be defined in (3.14). For any \( n \in \mathbb{N} \) and \( X \in \tilde{H}^{s_0} \), define

\[
\begin{aligned}
g_n(X) := J_n \tilde{c}J_n X + J_n \tilde{g}(J_n X) \\
&+ \frac{1}{2} \sum_{k=1}^{\infty} J_n^3(a_k \tilde{f}_k)^2 J_n X + \frac{1}{2} \sum_{k=1}^{\infty} J_n^3(q_k \tilde{e}_k)^2 J_n X, \quad k \geq 1, \\
h_n(t, X)e_{3k+1} := J_n(a_k \tilde{f}_k)J_n X, \quad k \geq 1, \\
h_n(t, X)e_{3k+2} := J_n(q_k \tilde{e}_k)J_n X, \quad k \geq 1, \\
h_n(t, X)e_{3k} := \tilde{h}_k(t, X), \quad k \geq 1.
\end{aligned}
\]  

(3.31)

In order to prove Theorem 3.1, by Theorem 2.1 and Proposition 2.1, we will show that \( (g_n, h_n) \) is the desired proper regularization satisfying (R1)-(R4) and assumptions (A) with the choice (cf. (3.15)):

\[
\mathbb{H} := \tilde{H}^{s_0}, \quad \mathcal{M} = \tilde{H}_{\theta_0}^{\theta_0},
\]  

(3.32)

where \( \theta_0 \) is given in (3.19). It is easy to see that Lemma 3.6, (D), (E) and (F) imply (R1), (R2) and (R3). It remains to verify (R4) and (A).

Verify (R4). By (F_k) and (3.24) (with \( \sigma = s_0 \)), we find some function \( \tilde{K} : [0, \infty) \times [0, \infty) \to (0, \infty) \) increasing in both variables such that for all \( X \in \mathbb{H} \) and \( t \geq 0 \),

\[
\sum_{k=1}^{\infty} \langle h_n(t, X)e_k, X \rangle_{\mathbb{H}}^2 \leq \tilde{K}(t, \|X\|_{\mathcal{M}})(1 + \|X\|_{\mathbb{H}}^2).
\]  

(3.33)

On the other hand, keeping in mind that \( \tilde{\Pi} X = X \) for \( X \in \mathbb{H} \) and \([\mathcal{D}^{s_0}, \mathcal{E}] = 0\), we infer from (E), (1.4) and Lemma 3.6 that

\[
-\langle J_n \tilde{c}J_n X, X \rangle_{H^{s_0}} = -\langle \mathcal{E} J_n \mathcal{D}^{s_0} X, J_n \mathcal{D}^{s_0} X \rangle_{L^2} = \|\mathcal{G} J_n \mathcal{D}^{s_0} X\|_{L^2, 2}^2, \quad X \in \mathbb{H}.
\]

Using this, (F_k), Lemma 3.6, (F_g) and (3.26) (with \( \sigma = s_0 \)), we derive for all \( X \in \tilde{H}^{s_0} \) and \( t \geq 0 \) that

\[
\begin{aligned}
2\langle g_n(X), X \rangle_{\mathbb{H}} + \|h_n(X)\|_{L^2, (0, t)}^2 \\
= 2\langle \tilde{c}J_n X, J_n X \rangle_{H^{s_0}} + 2\langle \tilde{g}(J_n X), J_n X \rangle_{H^{s_0}} \\
+ \sum_{k=1}^{\infty} \left\{ J_n^3(a_k \tilde{f}_k)^2 J_n X + J_n^3(q_k \tilde{e}_k)^2 J_n X, X \right\}_{H^{s_0}} \\
+ \sum_{k=1}^{\infty} \left\{ \|J_n(a_k \tilde{f}_k)J_n X\|_{H^{s_0}}^2 + \|J_n(q_k \tilde{e}_k)J_n X\|_{H^{s_0}}^2 + \|\tilde{h}_k(t, X)\|_{H^{s_0}}^2 \right\}
\end{aligned}
\]  

(3.34)

\[
\leq \tilde{K}(t, \|X\|_{\mathcal{M}})(1 + \|X\|_{\mathbb{H}}^2).
\]
Thus, (R4) holds.

**Verify (A).** Obviously, (A1) is a consequence of (Fb). It suffices to prove (A2). Remember that (cf. (3.22))

\[ \widetilde{U} := \Pi U, \quad U \in \{ E, \mathcal{A}_k, \mathcal{J}_k, \mathcal{K}_k \}, \quad k \geq 1. \]

By the definition of \( h_n \) in (3.31), Lemma 3.6, (Fb) and (3.24) (with \( \sigma = \theta_0 \) and \( X - Y \) replacing \( X \)), we can find a constant \( C > 0 \) and a function \( \widetilde{K} : [0, \infty) \times [0, \infty) \to (0, \infty) \) increasing in both variables, such that for any \( \epsilon \in (0, s_0 - \theta_0 - r_0) \),

\[
\sum_{k=1}^{\infty} \left\{ \left\langle h_n(t, X) - h_l(t, Y) \right\rangle \right\}_{\mathcal{M}}^2
\leq 2 \sum_{k=1}^{\infty} \left\{ \left\langle J_n \mathcal{A}_k J_n X - J_l \mathcal{A}_k J_l Y, X - Y \right\rangle_{H^{\theta_0}}^2
+ \| X - Y \|_{H^{\theta_0}}^2 \right\}
\leq 6 \sum_{k=1}^{\infty} \left\{ \left\langle J_n J_l (X - Y), J_l (X - Y) \right\rangle_{H^{\theta_0}}^2
+ 2K(t, \| X \|_{H^{\theta_0}} + \| Y \|_{H^{\theta_0}}) \| X - Y \|_{H^{\theta_0}}^2
\leq C(l \wedge n)^{-1} (s_0 - \theta_0 - \epsilon) \| X \|_{H^2} \| X - Y \|_{H^2}^2
\right\}
\]

Thus, which implies the first condition in (A2). To verify the second condition in (A2), we observe from (3.31) that

\[
2 \left\langle g_n(X) - g_l(Y), X - Y \right\rangle_{\mathcal{M}} + \| h_n(t, X) - h_l(t, Y) \|^2_{L^2(U; \mathcal{M})} = \Theta_1 + \Theta_2 + \sum_{k=1}^{\infty} \sum_{i=3}^{7} \Theta_{i,k},
\]

where

\[
\Theta_1 := 2 \left\langle J_n \mathcal{E} J_n X - J_l \mathcal{E} J_l Y, X - Y \right\rangle_{H^{\theta_0}}^2;
\]

\[
\Theta_2 := 2 \left\langle J_n \mathcal{G} J_n X - J_l \mathcal{G} J_l Y, X - Y \right\rangle_{H^{\theta_0}}^2
\]

\[
\Theta_{3,k} := \left\langle J_n^1 (ak \mathcal{K}_k)^2 J_n X - J_l^1 (ak \mathcal{K}_k)^2 J_l Y, X - Y \right\rangle_{H^{\theta_0}}^2;
\]

\[
\Theta_{4,k} := \left\langle J_n^1 (q_k \mathcal{K}_k)^2 J_n X - J_l^1 (q_k \mathcal{K}_k)^2 J_l Y, X - Y \right\rangle_{H^{\theta_0}}^2;
\]

\[
\Theta_{5,k} := \| h_n(t, X)e_{3k-2} - h_l(t, Y)e_{3k-2} \|^2_{H^{\theta_0}};
\]

\[
\Theta_{6,k} := \| h_n(t, X)e_{3k-1} - h_l(t, Y)e_{3k-1} \|^2_{H^{\theta_0}};
\]

\[
\Theta_{7,k} := \| \mathcal{H}_k(t, X) - \mathcal{H}_k(t, Y) \|^2_{H^{\theta_0}}.
\]

Firstly, by (Fb) and (Fh) we find an increasing function \( \widetilde{K} : [0, \infty) \to (0, \infty) \) and a map \( \lambda : \mathbb{N} \times \mathbb{N} \to (0, \infty) \) with \( \lambda_n, l \to 0 \) as \( n, l \to \infty \), such that

\[
\Theta_2 + \sum_{k=1}^{\infty} \Theta_{7,k} \leq \widetilde{K} (\| X \|_{\mathcal{M}} + \| Y \|_{\mathcal{M}}) (\lambda_n, l + \| X - Y \|_{H^2}^2), \quad n, l \geq 1, X, Y \in \mathbb{H}.
\]

Next, by (E) and Lemma 3.3, we have \( \mathcal{E} \in \text{OPS}^2_{p_0} \), where \( p_0 \) is given in (E). Then it follows from (E) and
Lemmas 3.6 and 3.2 that for any $\epsilon \in (0, s_0 - 2p_0 - \theta_0)$,

$$\Theta_1 = \left\langle (J_n - J_t)\tilde{E} J_n X, X - Y \right\rangle_{H^0, 0} \nonumber \nonumber$$

$$+ \left\langle J_t \tilde{E} (J_n - J_t) X, X - Y \right\rangle_{H^0, 0} \nonumber \nonumber$$

$$+ \left\langle J_t \tilde{E} J_t (X - Y), X - Y \right\rangle_{H^0, 0} \nonumber \nonumber$$

$$\lesssim (l \wedge n)^{-(s_0 - \theta_0 - 2p_0 - \epsilon)} \|X\|_{H^0} \|X - Y\|_{H^0} \nonumber \nonumber$$

(3.37)$$\lesssim \|X\|_{\mathbb{H}} \left( (l \wedge n)^{-(s_0 - \theta_0 - 2p_0 - \epsilon)} + \|X - Y\|_{\mathbb{H}}^2 \right), \quad X, Y \in \mathbb{H}, \quad n, l \geq 1. \nonumber \nonumber$$

Moreover, to estimate $\sum_{k=1}^{\infty} \{\Theta_{3,k} + \Theta_{5,k}\}$, we find

$$\Theta_{3,k} = \sum_{j=1}^{3} \Theta_{3,k,j}, \quad \Theta_{5,k} = \sum_{i,j=1}^{3} \left\langle \Theta_{5,k,i}, \Theta_{5,k,j} \right\rangle_{H^0, 0}, \nonumber \nonumber$$

where

$$\Theta_{3,k,1} := \left\langle (J_n^3 - J_t^3)(a_k \tilde{J}_k) J_n X, X - Y \right\rangle_{H^0, 0}, \quad \Theta_{5,k,1} := (J_n - J_t)(a_k \tilde{J}_k) J_n X, \nonumber \nonumber$$

$$\Theta_{3,k,2} := \left\langle J_t^3(a_k \tilde{J}_k)^2 (J_n - J_t) X, X - Y \right\rangle_{H^0, 0}, \quad \Theta_{5,k,2} := J_t(a_k \tilde{J}_k)(J_n - J_t) X, \nonumber \nonumber$$

$$\Theta_{3,k,3} := \left\langle J_t^3(a_k \tilde{J}_k)^2 J_t (X - Y), X - Y \right\rangle_{H^0, 0}, \quad \Theta_{5,k,3} := J_t(a_k \tilde{J}_k) J_t (X - Y). \nonumber \nonumber$$

Analogous to the analysis in (3.35), we have for any $\epsilon \in (0, s_0 - 2p_0 - \theta_0)$ and $X, Y \in \mathbb{H}$,

$$\sum_{k=1}^{\infty} \Theta_{3,k,1}, \sum_{k=1}^{\infty} \Theta_{3,k,2} \lesssim (l \wedge n)^{-(s_0 - 2p_0 - \theta_0 - \epsilon)} \|X\|_{\mathbb{H}} \|X - Y\|_{\mathbb{H}}, \nonumber \nonumber$$

$$\sum_{k=1}^{\infty} \sum_{i,j \in \{1,2\}} \left\langle \Theta_{5,k,i}, \Theta_{5,k,j} \right\rangle_{H^0, 0} \lesssim (l \wedge n)^{-(s_0 - 2p_0 - \theta_0 - \epsilon)} \|X\|_{\mathbb{H}}^2, \nonumber \nonumber$$

$$\sum_{k=1}^{\infty} \sum_{i \in \{1,2\}} \left\langle \Theta_{5,k,i}, \Theta_{5,k,3} \right\rangle_{H^0, 0} \lesssim (l \wedge n)^{-(s_0 - 2p_0 - \theta_0 - \epsilon)} \|X\|_{\mathbb{H}} \|X - Y\|_{\mathbb{H}} \nonumber \nonumber$$

Then we apply (3.26) (with $\sigma = \theta_0$ and $X - Y$ replacing $X$) to find

$$\sum_{k=1}^{\infty} \{\Theta_{3,k,3} + \left\langle \Theta_{5,k,3}, \Theta_{5,k,3} \right\rangle_{H^0, 0}\} \lesssim \|X - Y\|_{\mathbb{H}}^2, \quad X, Y \in \mathbb{H}. \nonumber \nonumber$$

In conclusion, we derive that for all $X, Y \in \mathbb{H}$ and $n, l \geq 1$,

$$\sum_{k=1}^{\infty} \{\Theta_{3,k} + \Theta_{5,k}\} \lesssim (1 + \|X\|_{\mathbb{H}}^2 + \|Y\|_{\mathbb{H}}^2) \left( (l \wedge n)^{-(s_0 - 2p_0 - \theta_0 - \epsilon)} + \|X - Y\|_{\mathbb{H}}^2 \right). \nonumber \nonumber$$

Similarly, the same estimate holds for $\sum_{k=1}^{\infty} \{\Theta_{4,k} + \Theta_{6,k}\}$. Combining these with (3.36) and (3.37), we verify the second condition in (A2). Therefore, Theorem 2.1 (i) implies that (3.23) has a unique maximal solution $(X, \tau^*)$ with

$$\lim_{t \to \tau^*} \|X\|_{\mathbb{H}} = \lim_{t \to \tau^*} \|X\|_{H^0, 0} = \infty \text{ on } \{\tau^* < \infty\}. \nonumber \nonumber$$

(a) (Time continuity) As explained in the proof of Theorem 2.1(ii), it suffices to prove the continuity of $[0, \tau^*) \ni t \mapsto \|X(t)\|_{\mathbb{H}}$. To this end, we recall (3.30), (3.32), (3.16), and reformulate (3.23) as

$$dX(t) - (\tilde{E}X)(t) dt = \{b(X) + \tilde{g}(X(t)) \} dt + h(t, X(t)) dW(t), \quad t \geq 0, \nonumber \nonumber$$

$$\tilde{g}(X(t)) := g(X(t)) - (\tilde{E}X)(t), \quad t \geq 0. \nonumber \nonumber$$

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By (E), (1.4), (1.5) and \([J_n, \mathcal{E}] = 0\), we arrive at

\[-\langle J_n \mathcal{E} X, J_n X \rangle_{H^\theta} = \|\mathcal{G}J_n D^\theta X\|^2_{L^2}, \quad X \in \mathbb{H}.\]

This, Lemmas 3.6 and 3.7, (F_b) and (F_g) yield that (B) holds true for \(\tilde{g}\) replacing \(g\) and \(T_n = J_n\), so that as in (2.21), for all \(n, N \geq 1\), there is a constant \(K_N > 0\) such that

\[
\begin{cases}
-\langle J_n \mathcal{E} X, J_n X \rangle_{H^\theta} \\ K_N dt \leq \|\mathcal{G}J_n D^\theta X\|^2_{L^2} dt + dM^{(n)}(t) \\ K_N dt \leq d\|J_n X(t)\|^2_{\mathbb{H}} + 2\|\mathcal{G}J_n X(t)\|^2_{\mathbb{H}} dt + dM^{(n)}(t), \quad t \in [0, \tau_N],
\end{cases}
\]

for some martingales \(M^{(n)}\). This implies

\[
\sup_{n \geq 1} \mathbb{E} \left[ \sup_{t \in [0, \tau_N]} \|J_n X(t)\|^2_{\mathbb{H}} + 2 \int_0^{\tau_N} \|\mathcal{G}J_n X(t)\|^2_{\mathbb{H}} dt \right] < \infty, \quad N \geq 1.
\]

By Fatou’s lemma

\[
\mathbb{E} \left[ \sup_{t \in [0, \tau_N]} \|X(t)\|^2_{\mathbb{H}} + 2 \int_0^{\tau_N} \|\mathcal{G}X(t)\|^2_{\mathbb{H}} dt \right] < \infty, \quad N \geq 1.
\]

This implies that the stopping times

\[\bar{\tau}_N := N \wedge \inf \left\{ t \geq 0 : \|X(t)\|_{\mathbb{H}} + \int_0^t \|\mathcal{G}X(s)\|^2_{\mathbb{H}} ds \geq N \right\}, \quad N \geq 1\]

satisfies \(\bar{\tau}_N \leq \tau_N\) and \(\mathbb{P}(\lim_{N \to \infty} \bar{\tau}_N = \tau^*) = 1\). Let \(\eta(t) := \|X(t)\|^2_{\mathbb{H}} + 2 \int_0^t \|\mathcal{G}X(s)\|^2_{\mathbb{H}} ds\). Then by the argument leading to (2.20), (3.38) implies

\[
\mathbb{E} \left[ |\eta(t \wedge \bar{\tau}_N) - \eta(s \wedge \bar{\tau}_N)|^4 \right] \leq G(N)|t - s|^2, \quad t, s \geq 0, \quad N \geq 1
\]

for some map \(G : \mathbb{N} \to (0, \infty)\). By Kolmogorov’s continuity theorem, \(\mathbb{P}(\eta(\cdot) \in C([0, \tau^*)]) = 1\). On the other hand, (3.39) implies that \(\mathbb{P}\)-a.s., \([0, \tau^*) \ni t \mapsto 2 \int_0^t \|\mathcal{G}X(s)\|^2_{\mathbb{H}} ds\) is continuous. Therefore, as desired, \(\|X(\cdot)\|^2_{\mathbb{H}} \in C([0, \tau^*)) \mathbb{P}\)-a.s.

(b) By (F_b), (3.33) and (3.34), we have

\[
\|\mathbb{b}(X)\|_{\mathbb{H}} \leq \tilde{K}(\mathbb{B}(X)) \|X\|_{\mathbb{H}},
\]

\[
\sum_{k=1}^{\infty} \langle h_n(t, X)e_k, X \rangle_{\mathbb{H}}^2 \leq \bar{K}(t, \mathbb{B}(X)) (1 + \|X\|_{\mathbb{H}}),
\]

\[
2 \langle g_n(t, X), X \rangle_{\mathbb{H}} + \|h_n(t, X)\|_{L^2(\mathbb{H}; \mathbb{H})}^2 \leq \bar{K}(t, \mathbb{B}(X)) (1 + \|X\|_{\mathbb{H}}^2).
\]

So, Proposition 2.1 implies Theorem 3.1 (b).

(c) Recall the coefficients \((b, g, h)\) in (3.30) and the definitions of \(\mathbb{H}\) and \(\mathbb{M}\) in (3.32). Let

\[
\mathcal{J}(t, X) := 2 \langle \mathbb{b}(X) + \mathbb{g}(X), X \rangle_{H^\theta_0} + \mathcal{L}(t, X),
\]

\[
\mathcal{L}(t, X) := \sum_{k=1}^{\infty} \left( \|\mathbb{I}h_k(t, X)\|_{H^\theta_0}^2 - \frac{2 \langle \mathbb{I}h_k(t, X), X \rangle_{H^\theta_0}^2}{\epsilon + \|X\|_{H^\theta_0}^2} \right).
\]

Taking \(\sigma = \theta_0\) and letting \(n \to \infty\) in (3.25) with noticing that \(\mathbb{H}\) is a self-adjoint projection, we obtain

\[
\sum_{k=1}^{\infty} \left| \langle a_k \mathbb{J}_k \rangle^2 X, X \rangle_{H^\theta_0} \right| + \|a_k \mathbb{J}_k X\|_{H^\theta_0}^2
\]

\[
+ \sum_{k=1}^{\infty} \left| \langle q_k \mathbb{K}_k \rangle^2 X, X \rangle_{H^\theta_0} \right| + \| q_k \mathbb{K}_k X\|_{H^\theta_0}^2 \lesssim \|X\|_{H^\theta_0}^2, \quad X \in \mathbb{H}^\theta_0.
\]
Notice that, by (E), \( \langle \tilde{E}X, X \rangle_{H^{s_0}} \leq 0, \) \( X \in \tilde{H}^{s_0} \). From these estimates and (F), we have

\[
J(t, X) \leq K(\|X\|_{H^{s_2}})\|X\|_{H^{s_0}}^2 \left( 1 + \frac{1}{K(\|X\|_{H^{s_2}})} \sup_{t \in [0, T]} L(t, X) \right).
\]

Therefore (3.20) implies

\[
\limsup_{\|X\|_{H^{s_0}} \to \infty} \sup_{t \in [0, T]} \frac{J(t, X(t))}{(e + \|X\|_{H^{s_0}}^2)(e + \|X\|_{H^{s_0}}^2)} < \infty, \quad T \in (0, \infty).
\]

Hence (C) holds for \( V(x) := \log(e + x) \), so that the proof is finished by Theorem 2.1 (iii). \( \square \)

4 Application to specific models with pseudo-differential noise

In this part, we apply Theorem 3.1 to specific models mentioned in Section 1.4. We state our results for each model in separate sections. As before, for \( d, m \geq 1 \) and \( s \geq 0 \), we write

\[
H^s = H^s(\mathbb{K}^d, \mathbb{R}^m), \quad \mathbb{K} = \mathbb{R} \text{ or } \mathbb{T}.
\]

As mentioned in Remark 3.2, \( (\tilde{b}, \tilde{g}) \) in following examples turns out to satisfy (F\(_b\)) and (F\(_g\)) with \( s_0 \) being replaced by \( s \) such that \( s \) is arbitrary in some range (see Lemmas 4.1, 4.2, 4.3 and 4.4). Therefore we also state the following assumption stronger than (F\(_h\)):

**Assumption (F\(_h\)'\).** There exists \( l \geq 1 \) such for all \( s > \frac{d}{2} + l, \tilde{h}_k : [0, \infty) \times H^s \to H^s \) satisfies

\[
\sum_{k \geq 1} \|\tilde{h}_k(t, X)\|_{H^s}^2 \leq K(t, \|X\|_{W^{s, \infty}})(1 + \|X\|_{H^s}^2), \quad t \geq 0, \quad X \in H^s,
\]

\[
\sum_{k=1}^{\infty} \|\tilde{h}_k(t, X) - \tilde{h}_k(t, Y)\|_{H^s}^2 \leq K(t, \|X\|_{H^s} + \|Y\|_{H^s})\|X - Y\|_{H^s}^2, \quad t \geq 0, \quad X, Y \in H^s.
\]

Obviously, since \( H^s \hookrightarrow W^{l, \infty} \) with \( s > \frac{d}{2} + l \), (F\(_h\)') implies (F\(_h\)) with \( \theta_1 \in (\frac{d}{2} + l, \infty) \) and \( s_0 = s \).

4.1 Stochastic MHD equations

Consider the stochastic MHD equations

\[
dX(t) = \left\{ (\tilde{\Pi}E^{mhd}X)(t) + gmhd(X(t)) \right\} dt
\]

\[
+ \sum_{k=1}^{\infty} \left\{ (\tilde{\Pi}A_kX)(t) \circ dW_k(t) + \tilde{\Pi}b_k(t, X(t))d\tilde{W}_k(t) \right\}, \quad t \geq 0,
\]

which is a special case of (1.8) with

\[
d \geq 1, \quad m = 2d, \quad X = (V, M)^T, \quad \tilde{b} \equiv 0, \quad E = E^{mhd}, \quad \tilde{g} = g^{mhd},
\]

\[
\tilde{\Pi} = \text{diag}(\Pi_d, \Pi_d) \quad \text{if } \mathbb{K} = \mathbb{R} \quad \text{and} \quad \tilde{\Pi} = \text{diag}(\Pi_0, \Pi_d, \Pi_0) \quad \text{if } \mathbb{K} = \mathbb{T}.
\]

Here we recall that \( E^{mhd} \) and \( g^{mhd} \) are given in (1.13) as

\[
E^{mhd} := -\text{diag}(\mu_1\Lambda^{2\alpha_1}, \mu_2\Lambda^{2\alpha_2}),
\]

\[
g^{mhd}(X) := \left( \Pi(M \cdot \nabla)M - \Pi(V \cdot \nabla)V, (M \cdot \nabla)V - (V \cdot \nabla)V \right)^T.
\]

and \( \Pi_d, \Pi_0 \) are defined in (3.9) and (3.12), respectively. Recalling \( H^s_{\text{div}}(\mathbb{K}^d; \mathbb{R}^d) \) in (3.8), and then using \( m = 2d, (3.10) \) and (3.13), we have

\[
\tilde{H}^s := \tilde{\Pi}H^s(\mathbb{K}^d; \mathbb{R}^m) = H^s_{\text{div}}(\mathbb{K}^d; \mathbb{R}^d) \times H^s_{\text{div}}(\mathbb{K}^d; \mathbb{R}^d).
\]
Lemma 4.1. Let \( n \) \( \in \mathbb{N} \), \( n \geq 1 \). Then for all \( \sigma > \frac{d}{2} \), \( g_{\text{mhd}} : \tilde{H}^{\sigma} \to \tilde{H}^{\sigma-1} \) and for all \( X, Y \in \tilde{H}^{\sigma} \),

\[
\sup_{n \geq 1} \|g_{\text{mhd}}^{(n)}(X)\|_{\tilde{H}^{\sigma-1}} + \|g_{\text{mhd}}^{(n)}(Y)\|_{\tilde{H}^{\sigma-1}} \lesssim \|X\|_{W^{1,\infty}} \|X\|_{\tilde{H}^{\sigma}},
\]

\[
\left\langle g_{\text{mhd}}^{(n)}(X) - g_{\text{mhd}}^{(n)}(Y), X - Y \right\rangle_{\tilde{H}^{\sigma-1}} \lesssim \|X\|_{\tilde{H}^{\sigma}} + \|Y\|_{\tilde{H}^{\sigma}} \|X - Y\|_{\tilde{H}^{\sigma-1}},
\]

\[
\sup_{n \geq 1} \left\{ \left| \left\langle g_{\text{mhd}}^{(n)}(X), X \right\rangle_{\tilde{H}^{\sigma-1}} \right| + \left| \left\langle J_n g_{\text{mhd}}^{(n)}(X), J_n X \right\rangle_{\tilde{H}^{\sigma-1}} \right| \right\} \lesssim \|X\|_{W^{1,\infty}} \|X\|_{\tilde{H}^{\sigma-1}}.
\]

Moreover, for any \( \zeta \in (\frac{d}{2}, \sigma - 1) \), there exists \( \lambda : \mathbb{N} \times \mathbb{N} \to (0, \infty) \) with \( \lambda_n \to 0 \) as \( n \to \infty \) such that for all \( X, Y \in \tilde{H}^{\sigma} \) and \( n, l \geq 1 \),

\[
\left| \left\langle g_{\text{mhd}}^{(n)}(X) - g_{\text{mhd}}^{(l)}(Y), X - Y \right\rangle_{\tilde{H}^{\sigma}} \right| \lesssim (1 + \|X\|_{\tilde{H}^{\sigma}} + \|Y\|_{\tilde{H}^{\sigma}}) (\lambda_n + \|X - Y\|_{\tilde{H}^{\sigma}}).
\]

Proof. Due to the divergence free condition, \( g_{\text{mhd}} : \tilde{H}^{\sigma} \to \tilde{H}^{\sigma-1} \). Moreover, by Lemmas 3.6 and 3.1, (4.2) holds true.

To prove (4.3), let \( X = (X_1, X_2), Y = (Y_1, Y_2) \). For simplicity, we let \( F = X_1 - Y_1 \) and \( H = X_2 - Y_2 \). By (1.13), we have

\[
- \left\langle g_{\text{mhd}}^{(n)}(Y), X - Y \right\rangle_{\tilde{H}^{\sigma-1}} = \left\langle (X_1 \cdot \nabla)X_1 - (Y_1 \cdot \nabla)Y_1, F \right\rangle_{\tilde{H}^{\sigma-1}} + \left\langle (Y_2 \cdot \nabla)Y_2 - (X_2 \cdot \nabla)X_2, F \right\rangle_{\tilde{H}^{\sigma-1}}
\]

\[
+ \left\langle (X_1 \cdot \nabla)X_2 - (Y_1 \cdot \nabla)Y_2, H \right\rangle_{\tilde{H}^{\sigma-1}} + \left\langle (Y_2 \cdot \nabla)Y_1 - (X_2 \cdot \nabla)X_1, H \right\rangle_{\tilde{H}^{\sigma-1}}.
\]

\[
:= \sum_{i=1}^{4} I_i,
\]

Since \( \sigma - 1 > \frac{d}{2} \), we have \( \tilde{H}^{\sigma-1} \to L^{\infty} \). Combining this with Lemmas 3.1 and 3.5 (with \( Q = D^{\sigma-1} \) and \( q = 0 \)) and the divergence free condition, we obtain

\[
|I_1| \lesssim \|X_1\|_{\tilde{H}^{\sigma}} + \|Y_1\|_{\tilde{H}^{\sigma}} \|F\|_{\tilde{H}^{\sigma-1}}^2,
\]

\[
|I_2| \lesssim \|X_2\|_{\tilde{H}^{\sigma}} \|F\|_{\tilde{H}^{\sigma-1}} \|H\|_{\tilde{H}^{\sigma-1}} + \|Y_1\|_{\tilde{H}^{\sigma}} \|H\|_{\tilde{H}^{\sigma-1}}^2.
\]

On the other hand, by using the divergence free conditions and integration by parts, we have

\[
-I_2 - I_4 = \left\langle D^{\sigma-1}((H \cdot \nabla)X_2), D^{\sigma-1}F \right\rangle_{L^2} + \left\langle \left[D^{\sigma-1}, (Y_2 \cdot \nabla)\right]H, D^{\sigma-1}F \right\rangle_{L^2}
\]

\[
+ \left\langle D^{\sigma-1}((H \cdot \nabla)X_1), D^{\sigma-1}H \right\rangle_{L^2} + \left\langle \left[D^{\sigma-1}, (Y_2 \cdot \nabla)\right]F, D^{\sigma-1}H \right\rangle_{L^2}.
\]

By \( \tilde{H}^{\sigma-1} \to L^{\infty} \) for \( s > \frac{d}{2} \), Lemmas 3.1 and 3.5, we arrive at

\[
|I_2 + I_4| \lesssim \|X_2\|_{\tilde{H}^{\sigma}} \|H\|_{\tilde{H}^{\sigma-1}} \|F\|_{\tilde{H}^{\sigma-1}} + \|Y_2\|_{\tilde{H}^{\sigma}} \|F\|_{\tilde{H}^{\sigma-1}} \|H\|_{\tilde{H}^{\sigma-1}} + \|X_1\|_{\tilde{H}^{\sigma}} \|H\|_{\tilde{H}^{\sigma-1}}^2.
\]

Collecting the above estimates, we obtain (4.3).

Concerning (4.4), we only prove the estimate on \( \left\langle J_n g_{\text{mhd}}^{(n)}(X), J_n X \right\rangle_{\tilde{H}^{\sigma}} \), since the other one can be derived similarly (even simpler because \( J_n \) is self-adjoint). We write

\[
\left\langle J_n g_{\text{mhd}}^{(n)}(X), J_n X \right\rangle_{\tilde{H}^{\sigma-1}}
\]

\[
= -\left\langle D^{\sigma-1}, (X_1 \cdot \nabla)X_1, D^{\sigma-1}J_n^2 X_1 \right\rangle_{L^2} - \left\langle (X_1 \cdot \nabla)D^{\sigma-1}X_1, D^{\sigma-1}J_n^2 X_1 \right\rangle_{L^2}
\]

\[
+ \left\langle (X_1 \cdot \nabla)X_2, D^{\sigma-1}J_n^2 X_1 \right\rangle_{L^2} + \left\langle (X_1 \cdot \nabla)D^{\sigma-1}X_2, D^{\sigma-1}J_n^2 X_1 \right\rangle_{L^2}
\]

\[
- \left\langle D^{\sigma-1}, (X_1 \cdot \nabla)X_2, D^{\sigma-1}J_n^2 X_2 \right\rangle_{L^2} - \left\langle (X_1 \cdot \nabla)D^{\sigma-1}X_2, D^{\sigma-1}J_n^2 X_2 \right\rangle_{L^2}
\]

\[
+ \left\langle (X_1 \cdot \nabla)X_1, D^{\sigma-1}J_n^2 X_2 \right\rangle_{L^2} + \left\langle (X_1 \cdot \nabla)D^{\sigma-1}X_1, D^{\sigma-1}J_n^2 X_2 \right\rangle_{L^2}
\]

\[
:= \sum_{i=1}^{8} N_i.
\]
Using Lemma 3.4 (with $Q = D^{\sigma - 1}$), we have
\[ |N_1|, |N_3|, |N_5|, |N_7| \lesssim (\|X_1\|_{W^{1, \infty}} + \|X_2\|_{W^{1, \infty}})(\|X_2\|_{H^{\sigma - 1}}^2 + \|X_1\|_{H^{\sigma - 1}}^2). \]
For $N_2$, we use the divergence free conditions and Lemma 3.6 to obtain,
\[ |N_2| = \left| \left\langle [J_n, (X \cdot \nabla)] D^{\sigma - 1}X_1, D^{\sigma - 1}J_nX_1 \right\rangle_{L^2} \right| \lesssim \|X_1\|_{W^{1, \infty}} \|X_1\|_{H^{\sigma - 1}}^2. \]
Similarly,
\[ |N_0| = \left| \left\langle [J_n, (X \cdot \nabla)] D^{\sigma - 1}X_2, D^{\sigma - 1}J_nX_2 \right\rangle_{L^2} \right| \lesssim \|X_1\|_{W^{1, \infty}} \|X_2\|_{H^{\sigma - 1}}^2. \]
Again, by the divergence free conditions,
\[ N_4 + N_8 = \left( \left\langle [J_n, (X_2 \cdot \nabla)] D^{\sigma - 1}X_2, D^{\sigma - 1}J_nX_1 \right\rangle_{L^2} + \left\langle (X_2 \cdot \nabla)D^{\sigma - 1}J_nX_2, D^{\sigma - 1}J_nX_1 \right\rangle_{L^2} \right) \]
\[ + \left( \left\langle [J_n, (X_2 \cdot \nabla)] D^{\sigma - 1}X_1, D^{\sigma - 1}J_nX_2 \right\rangle_{L^2} + \left\langle (X_2 \cdot \nabla)D^{\sigma - 1}J_nX_1, D^{\sigma - 1}J_nX_2 \right\rangle_{L^2} \right) \]
\[ = \left( \left\langle [J_n, (X_2 \cdot \nabla)] D^{\sigma - 1}X_2, D^{\sigma - 1}J_nX_1 \right\rangle_{L^2} + \left\langle [J_n, (X_2 \cdot \nabla)] D^{\sigma - 1}X_1, D^{\sigma - 1}J_nX_2 \right\rangle_{L^2} \right). \]
Therefore, we use Lemma 3.6 to find
\[ |N_4 + N_8| \lesssim \|X_2\|_{W^{1, \infty}} \|X_1\|_{H^{\sigma - 1}} \|X_2\|_{H^{\sigma - 1}}. \]
These imply the desired upper bound for
\[ \left| \left\langle J_n g^{mhd}(X), J_nX \right\rangle_{H^{\sigma - 1}} \right|. \]
Now we prove (4.5). We have
\[ J_n g^{mhd}(J_nX) - J_n g^{mhd}(J_Y) \]
\[ = (J_n - J_Y) g^{mhd}(J_nX) + J_n g^{mhd}(J_Y) \]
\[ := \sum_{i=1}^{3} Q_i. \]
To estimate $Q_2$, we write
\[ g^{mhd}(J_nX) - g^{mhd}(J_Y) = (\Pi_{p_{2,1,1}} - \Pi_{p_{2,1,2}}, p_{2,2,1} - p_{2,2,2})^T, \]
where
\[ p_{2,1,1} := ((J_n - J_Y)X_2 \cdot \nabla)J_nX_2 + (J_nX_2 \cdot \nabla)(J_n - J_Y)X_2, \]
\[ p_{2,1,2} := ((J_n - J_Y)X_1 \cdot \nabla)J_nX_1 + (J_nX_1 \cdot \nabla)(J_n - J_Y)X_1, \]
\[ p_{2,2,1} := (J_n - J_Y)X_2 \cdot \nabla)J_nX_1 + (J_nX_2 \cdot \nabla)(J_n - J_Y)X_1, \]
\[ p_{2,2,2} := (J_n - J_Y)X_1 \cdot \nabla)J_nX_2 + (J_nX_1 \cdot \nabla)(J_n - J_Y)X_2. \]
We infer from Lemmas 3.6 and 3.1, $H^\zeta \hookrightarrow L^\infty$ and divergence free conditions to find for $\epsilon \in (0, \sigma - 1 - \zeta),$
\[ \left| \langle J_n \Pi_{p_{2,1,1}}, X_1 - Y_1 \rangle_{H^\zeta} \right| + \left| \langle J_n \Pi_{p_{2,1,2}}, X_1 - Y_1 \rangle_{H^\zeta} \right| \lesssim (l \land n)^{-2\epsilon} \|X\|_{H^{\sigma}}^4 + \|X - Y\|_{H^\zeta}^2, \]
\[ \left| \langle J_n \Pi_{p_{2,2,2}}, X_2 - Y_2 \rangle_{H^\zeta} \right| + \left| \langle J_n \Pi_{p_{2,2,2}}, X_2 - Y_2 \rangle_{H^\zeta} \right| \lesssim (l \land n)^{-2\epsilon} \|X\|_{H^{\sigma}}^4 + \|X - Y\|_{H^\zeta}^2. \]
These estimates yield
\[ \left| \langle Q_2, X - Y \rangle_{H^\zeta} \right| \lesssim (l \land n)^{-2\epsilon} \|X\|_{H^{\sigma}}^4 + \|X - Y\|_{H^\zeta}^2. \]
Similarly, for $Q_1$, we have
\[ \|Q_1, X - Y \|_{H^\zeta} \leq \| (J_n - J_Y) g^{mhd}(J_nX) \|_{H^\zeta} \|X - Y\|_{H^\zeta} \]
\[ \lesssim (l \land n)^{-\epsilon} \|X\|_{H^{\sigma}}^2 \|X - Y\|_{H^\zeta} \lesssim (l \land n)^{-2\epsilon} \|X\|_{H^{\sigma}}^4 + \|X - Y\|_{H^\zeta}^2. \]
Finally, as $\zeta > d/2$, we use (4.3) to find
\[ \left| \langle Q_3, X - Y \rangle_{H^\zeta} \right| \lesssim \left( \|X\|_{H^{\sigma}} + \|Y\|_{H^{\sigma}} \right) \|X - Y\|_{H^\zeta}^2. \]
Collecting the above estimates, we obtain (4.5) and finish the proof.
Recall the $\alpha_1, \alpha_2 \in [0, 1]$ and $\mu_1, \mu_2 \geq 0$ come from the dissipation term $\mu_1 \Lambda^{2\alpha_1}$ and $\mu_2 \Lambda^{2\alpha_2}$. Let

$$a_0 := \max \{ \alpha_1 1_{(\mu_1 > 0)}, \alpha_2 1_{(\mu_2 > 0)} \}.$$ 

We also recall $r_0$ given in (3.18) and $H^\theta_{\text{div}}(K^d; \mathbb{R}^d)$ defined in (3.8). Then we have the following results on stochastic MHD equations (4.1):

**Theorem 4.1.** Let $d \geq 2$, $m = 2d$ and $K = \mathbb{R}$ or $\mathbb{T}$. Assume (D) with (3.18) holding for $\tilde{\Pi} = \text{diag}(\Pi_d, \Pi_d)$ if $K = \mathbb{R}$ and $\tilde{\Pi} = \text{diag}(\Pi_d \Pi_0, \Pi_d \Pi_0)$ if $K = \mathbb{T}$. Assume (F) for some $l \geq 1$. Then, for all $s > l + \frac{d}{2} + \max\{1, 2r_0, 2\alpha_0 \}$ and any $\mathcal{F}_0$-measurable $H^\theta_{\text{div}}(K^d; \mathbb{R}^d) \times H^\theta_{\text{div}}(K^d; \mathbb{R}^d)$-valued random variable $X(0)$, the following assertions hold.

1. (4.1) has a unique maximal solution $(X, \tau^*)$ such that Definition 1.1 is fulfilled with

$$H = H^\theta := H^\theta_{\text{div}}(K^d; \mathbb{R}^d) \times H^\theta_{\text{div}}(K^d; \mathbb{R}^d),$$

$$M = M^\theta := H^\theta_{\text{div}}(K^d; \mathbb{R}^d) \times H^\theta_{\text{div}}(K^d; \mathbb{R}^d), \quad \theta \in \left( l + \frac{d}{2}, s - \max\{1, 2r_0, 2\alpha_0 \} \right).$$

Moreover, $(X, \tau^*)$ defines a map $H^\theta \ni X(0) \rightarrow X(t) \in C([0, \tau^*); \mathbb{R}^d)$ $\mathbb{P}$-a.s., where $\tau^*$ does not depend on $s$, and

$$\limsup_{t \rightarrow \tau^*} \|X(t)\|_{W^{1, \infty}} = \infty \quad \text{on } \{ \tau^* < \infty \}.$$ 

2. The solution is non-explosive, if $\Psi(T, X, \theta)$ defined in (3.21) satisfies

$$\limsup_{\|X\|_{H^\theta} \rightarrow \infty} \frac{\Psi(T, X, \theta)}{\|X\|_{W^{1, \infty}}^2} < -1, \quad T \in (0, \infty).$$

**Proof.** It is easy to check that (E) holds for $E = \mathcal{E}^{mhd}$ with $m = 2d$, $p_0 = \alpha_0$ and

$$G_j := \begin{cases} \sqrt{n_1} \Lambda^{\alpha_1}, & \text{if } \alpha_1 < 1, 1 \leq j \leq d, \\ \sqrt{n_1} \partial_j, & \text{if } \alpha_1 = 1, 1 \leq j \leq d, \\ \sqrt{n_2} \Lambda^{\alpha_2}, & \text{if } \alpha_2 < 1, d + 1 \leq j \leq 2d, \\ \sqrt{n_2} \partial_j - \Lambda, & \text{if } \alpha_2 = 1, d + 1 \leq j \leq 2d. \end{cases}$$

Obviously, (F) holds since $\tilde{b} \equiv 0$. Since $H^\theta \hookrightarrow W^{1, \infty}$, we can infer from Lemmas 3.6 and 4.1 that (F) holds with $\tilde{g} = g^{mhd}$, $q_0 = 1$, $m = 2d$, $\theta_2 = \theta$, $K(\|X\|_{H^\theta})$ being replaced by $C\|X\|_{W^{1, \infty}}$ for some $C > 0$, $\tilde{\Pi} = \text{diag}(\Pi_d, \Pi_d)$ if $K = \mathbb{R}$ and $\tilde{\Pi} = \text{diag}(\Pi_d \Pi_0, \Pi_d \Pi_0)$ if $K = \mathbb{T}$.

Therefore, the statement of Theorem 4.1 follows from Theorem 3.1 (a), (b) with $B(X) = \|X\|_{W^{1, \infty}}$, and (c) respectively. The fact that $\tau^*$ is independent of $s$ has already been pointed out in Remark 3.2. \qed

### 4.2 Stochastic CH type equations

We consider the stochastic CH equation given by (1.8) with $d = m = 1$, $\mathcal{E} \equiv 0$, $\tilde{\Pi} = I$, $\tilde{b} = b^{ch}$ and $\tilde{g} = g^{ch}$, that is,

$$dX(t) = \left\{ b^{ch}(X(t)) + g^{ch}(X(t)) \right\} dt$$

$$+ \sum_{k=1}^{\infty} \left\{ (A_k X)(t) \circ dW_k(t) + \tilde{h}_k(t, X(t)) d\tilde{W}_k(t) \right\}, \quad t \geq 0,$$

where $b^{ch}$ and $g^{ch}$ are given in (1.15) as

$$b^{ch}(X) := -\partial (I - \partial^2)^{-1} \left( \sum_{i=1}^{4} a_i X^i + a \|\partial X\|^2 \right), \quad g^{ch}(X) := -\partial X.$$
Obviously, the solution space in this case is $\mathcal{H}^s(\mathbb{K}; \mathbb{R}) = H^s(\mathbb{K}; \mathbb{R})$. According to [57], we have

$$
\begin{align*}
&\|\delta^{ch}(X)\|_{H^s} \lesssim \psi(\|X\|_{W^{1,\infty}})\|X\|_{H^s}, \quad s > \frac{3}{2}, \quad X \in H^s; \\
&\|\delta^{ch}(X) - \delta^{ch}(Y)\|_{H^s} \lesssim \psi(\|X\|_{H^s} + \|Y\|_{H^s})\|X - Y\|_{H^s}, \quad s > \frac{3}{2}, \quad X, Y \in H^s,
\end{align*}
$$

where

$$
\psi(x) = |a_1| + (1 + |a_2| + |a|)x + |a_3|x^2 + |a_4|x^3.
$$

**Lemma 4.2.** Let $g_n^{ch}(X) := J_n g^{ch}(J_n X), n \geq 1$. Then for any $\sigma > 1 + \frac{1}{2}$ and $X \in H^s$, we have

$$
\sup_{n \geq 1} \|g_n^{ch}(X)\|_{H^{s-1}} + \|g^{ch}(X)\|_{H^{s-1}} \lesssim \|X\|_{W^{1,\infty}}\|X\|_{H^s},
$$

$$
\sup_{n \geq 1} \left\{ \left| \langle g_n^{ch}(X), X \rangle_{H^{s-1}} + \langle J_n g^{ch}(X), J_n X \rangle_{H^{s-1}} \right| \right\} \lesssim \|X\|_{W^{1,\infty}}\|X\|_{H^s}^2.
$$

Besides, for any $\zeta \in (\frac{1}{2}, \sigma - 1)$, there is $\lambda : \mathbb{N} \times \mathbb{N} \to (0, \infty)$ with $\lambda_{n,l} \to 0$ as $n, l \to \infty$ such that for any $n, l \geq 1$ and $X, Y \in H^s$,

$$
\left| \langle g_n^{ch}(X) - g_l^{ch}(Y), X - Y \rangle_{H^s} \right| \leq (1 + \|X\|_{H^s} + \|Y\|_{H^s}) \left( \lambda_{n,l} + \|X - Y\|_{H^s}^2 \right).
$$

**Proof.** One can verify the desired estimates as in the proof of Lemma 4.1, so we skip the details. □

Since for the present model we have $\overline{\Pi} = \Pi$ so that (D3) is trivial, in the following theorem we only assume (D) without this condition.

**Theorem 4.2.** Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{T}$. Let $d = m = 1$. Assume (D) without (D3), and assume (F') for some $l \geq 1$. For any $s > \frac{1}{2} + l + \max\{1, 2r_0\}$, where $r_0$ is given in (3.18), and any $\mathcal{F}_0$-measurable $H^s(\mathbb{K}; \mathbb{R})$-valued random variable $X(0)$, the following assertions hold.

1. (4.6) has a unique maximal solution $(X, \tau^*)$ in the sense of Definition 1.1 for $\mathbb{H} = \mathbb{H}^s := H^s(\mathbb{K}; \mathbb{R})$ and $\mathbb{M} = \mathbb{M}^d := H^0(\mathbb{K}; \mathbb{R})$ with $\theta \in (\frac{1}{2} + l, s - \max\{1, 2r_0\})$. Moreover, $(X, \tau^*)$ defines a map $\mathbb{H}^s \ni X(0) \mapsto X(t) \in C([0, \tau^*); \mathbb{H}^s)$ $\mathbb{P}$-a.s., where $\tau^*$ is independent of $s$, and

$$
\lim_{t \to \tau^*} \sup_{t \leq \tau^*} \|X(t)\|_{W^{1,\infty}} = \infty \quad \text{on} \quad \{\tau^* < \infty\}.
$$

2. $\mathbb{P}(\tau^* = \infty) = 1$ provided that

$$
\lim_{\|X\|_{H^s} \to \infty} \frac{\Psi(T, X, \theta)}{\psi(\|X\|_{W^{1,\infty}})\|X\|_{H^s}^2} < -1,
$$

where $\Psi(T, X, \theta)$ and $\psi(\cdot)$ are given in (3.21) and (4.8), respectively.

**Proof.** In this case $\mathcal{E} \equiv 0$ and $\overline{b} = \overline{b}^{ch}$. With (4.7) at hand, one can check (F) holds with $s_0 = s$ and $\theta_0 = \theta$. The details are similar to the proof of Theorem 4.1 and we skip them for brevity. □

### 4.3 Stochastic AD equation

Consider the stochastic AD equation embedded in (1.8) with $d \geq 2, m = 1, \overline{\Pi} = \Pi, \bar{b} \equiv 0, \mathcal{E} = \mathcal{E}^{ad}$ and $\bar{g} = \bar{g}^{ad}$, where $\mathcal{E}^{ad}$ and $\bar{g}^{ad}$ are given in (1.17). That is,

$$
\begin{align*}
dX(t) &= \left\{ (\mathcal{E}^{ad}X)(t) + \bar{g}^{ad}(X(t)) \right\} dt \\
&\quad + \sum_{k=1}^{\infty} \{ (A_k(t) \circ dW_k(t) + \overline{h}_k(t, X(t)) \circ d\overline{W}_k(t) \}, \quad t \geq 0,
\end{align*}
$$

where

$$
\mathcal{E}^{ad} := -\nu(-\Delta)^d, \quad \bar{g}^{ad}(X) := -\gamma \nabla \cdot (X B X), \quad B = (B_i)_{1 \leq i \leq d}, \quad \overline{h}_k(t, X(t)) := \left( \partial_{i} [\Phi^{*}] \right)_{1 \leq i \leq d}.
$$

Then the working space for (4.9) is $\mathcal{H}^s = \overline{\Pi} H(\mathbb{K}; \mathbb{R}) = H^s(\mathbb{K}; \mathbb{R})$ and we have the following
Lemma 4.3. Let $\Phi$ in (1.16) satisfy that $\Phi \in H^\infty$ such that $(\mathcal{F}\Phi)(\xi) \in S_0^{-2}$. Let $g_n^{ad}(X) := J_n g^{ad}(J_n X)$. Then for any $\sigma \geq \eta > \frac{d}{2} + 1$ and $X \in H^\sigma$,

\[
\|g_n^{ad}(X)\|_{H^{\sigma - 1}} + \sup_{n \geq 1} \|g_n^{ad}(X)\|_{H^{\sigma - 1}} \lesssim (\|X\|_{W^{1,\infty}} + \|BX\|_{W^{1,\infty}}) \|X\|_{H^\sigma},
\]

\[
\sup_{n \geq 1} \left\{ \|g_n^{ad}(X) - g_n^{ad}(Y)\|_{H^{\sigma - 1}} + \|J_n g^{ad}(X) - J_n g^{ad}(Y)\|_{H^{\sigma - 1}} \right\}
\lesssim (\|X\|_{W^{1,\infty}} + \|BX\|_{W^{1,\infty}}) \|X\|_{H^\sigma},
\]

and for any $\zeta \in (\frac{1}{2}, \sigma - 1)$, there is $\lambda : \mathbb{N} \times \mathbb{N} \to (0, \infty)$ with $\lambda_{n,l} \to 0$ as $n, l \to \infty$ such that for any $n, l \geq 1$,

\[
\left\| \left\langle g_n^{ad}(X) - g_n^{ad}(Y), X - Y \right\rangle_{H^{\sigma}} \right\|_{H^\lambda}
\leq \left(1 + \|X\|_{H^\sigma}^2 + \|Y\|_{H^\sigma}^2\right) \left(\lambda_{n,l} + \|X - Y\|_{H^\sigma}^2\right), \quad X, Y \in H^\sigma.
\]

Proof. By Lemma 3.2, $B_i = \partial_i [\Phi*] \in OПS_0^{-1}$, $1 \leq i \leq d$ provided $(\mathcal{F}\Phi)(\xi) \in S_0^{-2}$. Then, (4.10) comes from Lemma 3.1 and the fact $H^{\eta - 1} \to L^\infty$.

Concerning (4.11), only prove the upper bound estimate for $\langle J_n g^{ad}(X), J_n X \rangle_{H^\sigma}$, since the estimate for $\langle g_n^{ad}(X), X \rangle_{H^\sigma}$ can be derived similarly. Recalling that $H^s \to W^{1,\infty}$ for $s > 1 + \frac{d}{2}$, by Lemmas 3.1, 3.2, 3.4, 3.6, and integrating by parts, we obtain

\[
\left\| \left\langle D^\sigma J_n g^{ad}(X), D^\sigma J_n X \right\rangle_{L^2} \right\| \lesssim \left\| \left\langle D^\sigma (BX \cdot \nabla) X, D^\sigma J_n X \right\rangle_{L^2} \right\| + \left\| \left\langle J_n (BX \cdot \nabla), D^\sigma J_n X \right\rangle_{L^2} \right\|
\leq \|BX\|_{H^\sigma} \|\nabla X\|_{L^\infty} \|X\|_{H^\sigma} + \|BX\|_{W^{1,\infty}} \|X\|_{H^\sigma}^2 + \|X\|_{L^\infty} \|\nabla \cdot BX\|_{L^\infty} \|X\|_{H^\sigma}
\lesssim \|\nabla X\|_{L^\infty} \|\nabla \cdot BX\|_{H^\sigma} \|X\|_{H^\sigma} \|X\|_{H^\sigma}^2 \lesssim \|X\|_{H^\sigma} \|X\|_{H^\sigma}^2,
\]

so that the desired estimate holds.

Let $Z = X - Y$. By $H^\sigma \to W^{1,\infty}$, integration by parts, Lemmas 3.2 and 3.5, we derive

\[
\langle g_n^{ad}(X) - g_n^{ad}(Y), X - Y \rangle_{H^{\sigma - 1}} \lesssim (\|X\|_{H^\sigma} - \|Y\|_{H^\sigma}) \|X - Y\|_{H^{\sigma - 1}}.
\]

With this at hand, as in the proof of (4.5), we can verify (4.12) with using Lemmas 3.6 and 3.2. The details are omitted for brevity.

Recall the dissipation term $\nu \Lambda^{2\beta}$ in (1.16), and let $\beta_0 := \beta 1_{\nu > 0}$. Recall that $r_0$ is given in (3.18). As in Theorem 4.2, in the present case condition (D3) is trivial.

**Theorem 4.3.** Let $d \geq 2$, $m = 1$ and $K = \mathbb{R}$ or $\mathbb{T}$. Suppose that $\Phi$ in (1.16) satisfies $\Phi \in H^\infty(K^d; \mathbb{R})$ and $(\mathcal{F}\Phi)(\xi) \in S_0^{-2}$. Assume (D) without (D3), and assume (F) for some $l \geq 1$. Then the following assertions hold for any $s > \frac{d}{2} + l + \max\{1, 2r_0, 2\beta_0\}$ and $\mathcal{F}_0$-measurable $H^s$-valued random variable $X(0)$.
\(4.9\) admits a unique maximal solution \((X, \tau^*)\) in the sense of Definition 1.1 for \(\mathbb{H} = \mathbb{H}^s := H^s(\mathbb{K}; \mathbb{R})\) and \(\mathcal{M} = \mathcal{M}^\sigma := H^\sigma(\mathbb{K}; \mathbb{R})\), where \(\theta \in \left(\frac{1}{2} + l, s - \max\{1, 2r_0, 2\lambda_0\}\right)\). Furthermore, \((X, \tau^*)\) induces a map \(\mathbb{H}^s \ni X(0) \mapsto X(t) \in C([0, \tau^*); \mathbb{H}^\sigma) \mathcal{F}\text{-a.s.}, \) where \(\tau^*\) is independent of \(s\), and
\[
\limsup_{t \to \tau^*} \|X(t)\|_{W^{1, \infty}} + \|BX\|_{W^{1, \infty}} = \infty \text{ on } \{\tau^* < \infty\}.
\]

(2) The solution is non-explosive, if \(\Psi(T, X, \theta)\) given by \((3.21)\) enjoys
\[
\limsup_{\|X\|_{H^\theta} \to \infty} \frac{\Psi(T, X, \theta)}{\|X\|_{W^{1, \infty}} + \|BX\|_{W^{1, \infty}}} < -1, \quad T \in (0, \infty).
\]

Proof. In the same way as we prove Theorem 4.1, with Lemma 4.3 at hand, we can verify (E) and (F). We only remark that in this case, we take \(B(X) = \|X\|_{W^{1, \infty}} + \|BX\|_{W^{1, \infty}}\) and \(\mathcal{E} = \mathcal{E}^\text{ad} = \nu \Lambda^{2\beta}\).

### 4.4 Stochastic SQG equation

Let \(d = m = 1 \text{ and } s > 2\). Recall the operator \(\Pi_0\) defined by \((3.12)\). As before,
\[
\tilde{H}^s := \tilde{\Pi} H^s(\mathbb{K}; \mathbb{R}), \quad \tilde{\Pi} = \Pi \text{ if } \mathbb{K} = \mathbb{R} \text{ and } \tilde{\Pi} = \Pi_0 \text{ if } \mathbb{K} = \mathbb{T}.
\]

When \(\mathbb{K} = \mathbb{T}\), we recall \((3.11)\) and notice that
\[
(4.13) \quad (D^s \Pi_0 X, D^s \Pi_0 Y)_{L^s} \simeq (\Lambda^s \Pi_0 X, \Lambda^s \Pi_0 Y)_{L^s}, \quad X, Y \in \tilde{H}^s(\mathbb{T}^2, \mathbb{R}) = H_0^s(\mathbb{T}^2, \mathbb{R}),
\]
where \(D^s\) and \(\Lambda^s\) are defined in \((3.7)\). By \((3.16)\) and \((4.13)\), we have
\[
(X, Y)_{H_0^s} = (X, Y)_{H^s} \simeq (\Lambda^s X, \Lambda^s Y)_{L^s}, \quad X, Y \in H_0^s(\mathbb{T}^2, \mathbb{R}),
\]
and recall \((1.18)\), i.e., \(g_{\text{sqg}}(X) := - (\mathcal{R}^\perp X) \cdot \nabla X\). Then we consider the following stochastic SQG equation with \(t \geq 0\):
\[
(4.14) \quad \text{d}X(t) = g_{\text{sqg}}(X(t)) \text{d}t + \sum_{k=1}^{\infty} \left\{ (\Pi_0 A_k X)(t) \circ \text{d}W_k(t) + \tilde{\Pi}_0 k(t, X(t)) \text{d}\tilde{W}_k(t) \right\}.
\]

Lemma 4.4. Let \(g_{\text{sqg}}^n(X) := J_n g_{\text{sqg}}(J_n X)\). Then for any \(\sigma \geq \eta > 2\) and \(X \in \tilde{H}^s\), \(g_{\text{sqg}}(X) \in \tilde{H}^{\sigma-1}\) and
\[
(4.15) \quad \|g_{\text{sqg}}(X)\|_{H^{\sigma-1}} + \sup_{n \geq 1} \|g_{\text{sqg}}^n(X)\|_{H^{\sigma-1}} \lesssim (\|X\|_{W^{1, \infty}} + \|RX\|_{W^{1, \infty}}) \|X\|_{H^s},
\]
\[
\sup_{n \geq 1} \left\{ \left| \langle g_{\text{sqg}}^n(X), X \rangle_{H^s} \right| + \left| \langle J_n g_{\text{sqg}}(X), J_n X \rangle_{H^s} \right| \right\}
\]
\[
\lesssim (\|X\|_{W^{1, \infty}} + \|RX\|_{W^{1, \infty}}) \|X\|^2_{H^s}.
\]
Furthermore, for any \(\zeta \in (2, \sigma - 1)\), there is \(\lambda : \mathbb{N} \times \mathbb{N} \to (0, \infty)\) with \(\lambda_{n,l} \to 0\) as \(n, l \to \infty\) such that for any \(n, l \geq 1\),
\[
(4.17) \quad \left| \left\langle g_{\text{sqg}}^n(X) - g_{\text{sqg}}(v), X - Y \right\rangle_{H^s} \right| \lesssim (1 + \|X\|^2_{H^s} + \|Y\|^2_{H^s}) \left(\lambda_{n,l} + \|X - Y\|^2_{H^s}\right), \quad X, Y \in \tilde{H}^\sigma.
\]

Proof. By Lemma 3.1, the boundedness of \(\mathcal{R}\) in \(L^2\), \([J_n, \mathcal{R}^\perp] = 0\) and the fact \(g_{\text{sqg}}(X), g_{\text{sqg}}^n(X)\) have zero average on \(\mathbb{K} = \mathbb{T}\), we obtain \((4.15)\).

By Lemma 3.1 with \(s > \sigma > 2\) and the boundedness of \(\mathcal{R}\) in \(L^2\), we see that for any \(\sigma > 2\),
\[
\left| \left\langle g_{\text{sqg}}^n(X), X \right\rangle_{H^s} \right| = \left| \left\langle J_n [J_n (\mathcal{R}^\perp X) \cdot \nabla J_n X], X \right\rangle_{H^s} \right| \lesssim (\|X\|_{W^{1, \infty}} + \|RX\|_{W^{1, \infty}}) \|X\|^2_{H^s}, \quad X \in \tilde{H}^\sigma, \quad n \geq 1.
\]
Similarly, by Lemmas 3.4 and 3.6, integration by parts and $\nabla \cdot (R^\perp X) = 0$, we have that for $K = \mathbb{T}$,

$$\|\langle J_n g^{\text{ms}}(X), J_n X \rangle_{H^s} \| \lesssim \left| \langle [\Lambda^{\sigma}, (R X \cdot \nabla)]X, J_n^{2} \Lambda^{\sigma} X \rangle_{L^2} \right| + \left| \langle [J_n, (R^\perp X \cdot \nabla)] \Lambda^{\sigma} X, J_n \Lambda^{\sigma} X \rangle_{L^2} \right|$$

$$+ \left| \langle (R^\perp X \cdot \nabla) J_n \Lambda^{\sigma} X, J_n \Lambda^{\sigma} X \rangle_{L^2} \right|$$

$$\lesssim \left( \|X\|_{W^{1, \infty}} + \|\mathcal{R} X\|_{W^{1, \infty}} \right) \|X\|_{H^s}^2, \quad X \in \mathcal{H}^s, \; n \geq 1. \quad \text{(4.16)}$$

Hence (4.16) holds true for $\langle J_n g^{\text{ms}}(X), J_n X \rangle_{H^s}$. The estimate for $\langle g^{\text{ms}}(X), X \rangle_{H^s}$, and the estimate for the case $K = \mathbb{R}$ can be obtained in the same way. Finally, proceeding as in the proof of (4.5), we can verify (4.17). We omit the details.

Since $m = 1$ and $\mathcal{H} \in \text{OP}^0$, where $\mathcal{H} = \mathcal{I}$ for $K = \mathbb{R}$ and $\mathcal{H} = \Pi_0$ for $K = \mathbb{T}$, the condition (D3) is trivial.

**Theorem 4.4.** Let $d = 1$, $m = 1$ and $K = \mathbb{R}$ or $\mathbb{T}$. Assume (D) without (D3), and assume (F) for some $l \geq 1$. Then for any $s > 1 + l + \max\{1, 2r_0\}$, where $r_0$ is given in (3.18), and any $\mathcal{F}_0$-measurable $\mathbb{H}$-valued random variable $X(0)$, we have the following assertions.

1. (4.14) has a unique maximal solution $(X, \tau^*)$ such that Definition 1.1 is fulfilled with

$$\mathbb{H} = \mathbb{H}^s := \begin{cases} H^s(\mathbb{R}^2; \mathbb{R}) & \text{if } K = \mathbb{R}, \\ H^s(\mathbb{T}^2; \mathbb{R}) & \text{if } K = \mathbb{T}, \end{cases} \quad M = M^s := \begin{cases} H^0(\mathbb{R}^2; \mathbb{R}) & \text{if } K = \mathbb{R}, \\ H^0(\mathbb{T}^2; \mathbb{R}) & \text{if } K = \mathbb{T}, \end{cases}$$

where $H^s(\mathbb{T}^2; \mathbb{R})$ is defined in (3.11) and $\theta \in (l + \frac{d}{2}, s - \max\{1, 2r_0\})$. Besides, $(X, \tau^*)$ defines a map $\mathbb{H}^s \ni X(0) \mapsto X(t) \in C([0, \tau^*); \mathbb{H}^s)$ $\mathbb{P}$-a.s., where $\tau^*$ does not depend on $s$, and

$$\limsup_{t \to \tau^*} \|X(t)\|_{W^{1, \infty}} + \|\mathcal{R} X\|_{W^{1, \infty}} = \infty \text{ on } \{\tau^* < \infty\}.$$ 

2. The solution is non-explosive, if $\Psi(T, X, \theta)$ defined in (3.21) satisfies

$$\limsup_{\|X\|_{H^s} \to \infty} \left( \frac{\Psi(T, X, \theta)}{\|X\|_{W^{1, \infty}} + \|\mathcal{R} X\|_{W^{1, \infty}}} \right) < -1, \quad T \in (0, \infty).$$

**Proof.** With Lemma 4.4, one can prove this theorem in a way analogous to Theorem 4.1 with noticing that $B(X) = \|X\|_{W^{1, \infty}} + \|\mathcal{R} X\|_{W^{1, \infty}}$ in the current situation.

### 4.5 Further examples

In the above cases, we take $q_0 = 1$ and $\theta > \frac{d}{2} + l$. However, our general result Theorem 3.1 (with stronger assumption (F') replacing (F)) applies to many other models. Below we present two more examples where $\theta$ and $q_0$ have to be larger.

**Modified Camassa-Holm (MCH) equation.** Consider the modified CH equation

$$\begin{cases} \frac{d}{dt} Y + 2(\partial X) Y + X \partial Y := 0, \quad X = X(t, x) : [0, \infty) \times K \to \mathbb{R}, \\ Y := (I - \partial^2)^p X, \quad p \in \mathbb{N}, \end{cases} \tag{4.18}$$

which can be viewed as the Euler-Poincaré differential system on the Bott-Virasoro group with respect to the $H^k$-metric. For simplicity, we consider $p = 2$ and we refer to [55] and the references therein for the background of this equation. Then (4.18) can be reformulated as

$$\frac{d}{dt} X = g^{\text{mch}}(X(t)) + b^{\text{mch}}(X(t)), \quad \text{where} \quad g^{\text{mch}}(X(t)) := \frac{1}{2} \int_X \langle X, \partial \rangle X, \quad b^{\text{mch}}(X(t)) := \frac{1}{2} \int_X \langle X, \partial \rangle X.$$
where \( g^{\text{mch}}(X) = -X \partial X \) and
\[
\dot{b}^{\text{mch}}(X) := -\partial D^{-4}(X^2) - 2\partial D^{-4}[(\partial X)^2] + \frac{7}{2} \partial D^{-4}[(\partial^2 X)^2] + 3\partial D^{-4}(\partial[X \partial^3 X]).
\]

Now we consider the stochastic \textbf{MCH} equation:
\[
dX(t) = \left( g^{\text{mch}}(X(t)) + \dot{b}^{\text{mch}}(X) \right) dt + \sum_{k=1}^{\infty} \left\{ (A_k X(t) \circ dW_k(t) + \tilde{h}_k(t, X(t)) d\tilde{W}_k(t) \right\}, \quad t \geq 0.
\]

When \( s > 7/2 \), we have for \( X, Y \in H^s \) that, cf. [55],
\[
\| \dot{b}^{\text{mch}}(X) \|_{H^s} \lesssim \| X \|_{W^{3, \infty}} \| X \|_{H^s}, \quad s > 7/2,
\]
\[
\| \dot{b}^{\text{mch}}(X) - \dot{b}^{\text{mch}}(Y) \|_{H^s} \lesssim (\| X \|_{H^s} + \| Y \|_{H^s}) \| X - Y \|_{H^s}, \quad s > 7/2.
\]

The above estimates mean that \( \dot{b}^{\text{mch}} \) is locally Lipschitz in \( H^s \) for \( s > 7/2 \). Notice that \( g^{\text{mch}}(X(t)) = \dot{g}^{\text{ch}}(X(t)) \). Then the corresponding results for (4.19) can be stated as Theorem 4.2 for all \( s > \frac{1}{2} + \max\{3, l\} + \max\{1, 2r_0\} \) \( (\theta \in (\frac{1}{2} + \max\{3, l\}, s - \max\{1, 2r_0\}) \) in this case) with blow-up criterion
\[
\limsup_{t \to \tau^*} \| X(t) \|_{W^{\max\{3, l\}, \infty}} = \infty \quad \text{on} \quad \{ \tau^* < \infty \}.
\]

Global regularity criterion now reduces to
\[
\limsup_{\| X \|_{H^s} \to \infty} \frac{\Psi(T, X, \theta)}{\| X \|_{W^{3, \infty}} \| X \|_{H^s}^2} < -1,
\]
where \( \Psi(T, X, \theta) \) is in (3.21).

\textbf{Korteweg-De Vries (\textbf{KdV}) equation:} We consider the following Korteweg-De Vries equation for \( X = X(t, x) : [0, \infty) \times \mathbb{R} \to \mathbb{R}: \)
\[
\frac{d}{dt} X(t) = g^{\text{kdv}}(X(t)), \quad g^{\text{kdv}}(X) := -X \partial X - \partial^3 X = 0, \quad t \geq 0.
\]
This equation was introduced by Korteweg-de and Vries [41] to model the motion of long, unidirectional, weakly nonlinear water waves on a channel. Then we consider the stochastic \textbf{KdV} equation with \( t \geq 0: \)
\[
(4.19) \quad dX(t) = g^{\text{kdv}}(X(t)) dt + \sum_{k=1}^{\infty} \left\{ (A_k X(t) \circ dW_k(t) + \tilde{h}_k(t, X(t)) d\tilde{W}_k(t) \right\}.
\]

Notice that \( g^{\text{kdv}}(X) = \dot{g}^{\text{ch}}(X) - \partial_x^3 X \) and
\[
\langle \partial^3 f, f \rangle_{H^s} = -\langle \partial_x^3 \partial f, \partial_x^3 \partial f \rangle_{L^2} = -\frac{1}{2} \int_{\mathbb{R}} \partial(D^3 \partial f(x))^2 dx = 0, \quad f \in H^{s+3}.
\]
Therefore, one can also apply Theorem 3.1 to (4.19) and in this case \( q_0 = 3 \). The main result on \( K = \mathbb{R} \) can be stated as in Theorem 4.2 for some \( s > \frac{1}{2} + l + \max\{3, 2r_0\} \) \( (\text{in this case} \theta \in (\frac{1}{2} + l, s - \max\{3, 2r_0\}) \) with the global regularity criterion
\[
\limsup_{\| X \|_{H^s} \to \infty} \frac{\Psi(T, X, \theta)}{\| X \|_{W^{3, \infty}} \| X \|_{H^s}^2} < -1, \quad \psi(T, X, \theta) \text{ is given in (3.21).}
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