Research Article

On Modifications of the Gamma Function by Using Mittag-Leffler Function

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Received 25 March 2021; Revised 16 April 2021; Accepted 30 April 2021; Published 24 June 2021

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Mittag-Leffler function is a natural generalization of the exponential function. Recent applications of Mittag-Leffler function have reshaped the scientific literature due to its fractional effects that cannot be obtained by using exponential function. Present motivation is to define a new special function by modification in the original gamma function with Mittag-Leffler function. Properties of this modified function are discussed by investigating a new series representation involving delta function. Hence, the results are also validated with the earlier obtained results for gamma function as special cases. Furthermore, the new function is used to generate a probability density function, and its statistical properties are explored. Similar properties of existing distributions can be deduced.

1. Introduction and Preliminaries

1.1. Motivation. Use of fractional operators has become a remarkable tool to study new technology and researches [1–17]. One attractive aspect of this approach is to model complex systems by taking into account the effects of filtration and memory. How the fractional operators are defined? It might be summarized as follows: “each fractional operator is the action of a particular special function on a general function, which usually represents the solution of a well-defined problem by using a suitable integral transform.” See for example [18–22]. Hence, a basic motivation and a significant contribution of this research are to study a special function containing the Mittag-Leffler function (also known as “queen of fractional calculus” [23]), which is represented in terms of complex delta function. Hence, its action on a suitably chosen function over a specific domain is easily obtainable in view of the standard properties of the delta function. This gives insights for the emergence of a new fractional operator as one of the natural consequences of this research.

The review of literature [1–23] demonstrates that many of the new researches depend on the involvement of Mittag-Leffler function in place of exponential function. More recently, Mainardi and Masina [24] have presented modified exponential integral by using Mittag-Leffler function. Mainardi et al. [25] have explored a new model which depends on \( \nu \in [0, 1] \). This model represents the Maxwell body at one end point, namely, \( \nu = 0 \), and the Becker body at other end point, namely \( \nu = 1 \). Referring to these phenomena, creep rule is generalized in the sense that Becker model involves Mittag-Leffler function of order \( \nu \) in place of exponential function. Following them, Paris [26] has discussed asymptotic behavior of the modified exponential integral and studied its more general form by using two-parameter Mittag-Leffler functions. Taking motivation from these researches [24–26], the aim of the present paper is to study the modified gamma function by using Mittag-Leffler function. Before going on to our main results, we describe necessary background and preliminaries in the subsequent paragraphs.
1.2. Preliminaries. Mittag-Leffler [27] proposed a function that appears a natural substitute of exponential function in the case of fractional order. It is known as Mittag-Leffler function defined by

\[ E_\alpha(s) = \sum_{r=0}^{\infty} \frac{s^r}{\Gamma(\alpha r + 1)} \quad \alpha \in \mathbb{C}, \Re(\alpha) > 0. \]  

(1)

For \( \alpha = 1 \), one-parameter Mittag-Leffler function reduces to the exponential function \( E_1(s) = \sum_{r=0}^{\infty} \frac{s^r}{r!} = e^s \).

There are many generalizations of Mittag-Leffler function. For example, two- and three-parameter Mittag-Leffler functions are defined by

\[ E_{\alpha,\beta}(s) = \sum_{r=0}^{\infty} \frac{s^r}{\Gamma(\alpha r + \beta)} \alpha, \beta \in \mathbb{C}, \Re(\alpha) > 0, \]  

\[ E_{\alpha,\beta}^\prime(s) = \sum_{r=0}^{\infty} \frac{s^r}{\Gamma(\alpha r + \beta)} \alpha, \beta, \gamma \in \mathbb{C}, \Re(\alpha) > 0, \]  

respectively. Here, \( \mathbb{C} \) denotes the set of complex numbers and \( \Re \) denotes the real part of any complex number. Three-parameter Mittag-Leffler function is also known as Prabhakar function [20]. The decades of this century are mainly devoted to the analysis and applications of this function, see for example [1–24], and related references therein. For our purposes, we would be using the following form of Mittag-Leffler function [28]:

\[ E_\varepsilon(t^\varepsilon) = \sum_{r=0}^{\infty} \frac{t^{r\varepsilon}}{\Gamma(r\varepsilon + 1)} \quad (t \in \mathbb{R}; 0 < \varepsilon < 1). \]  

(3)

Here, \( \Re \) denotes the set of real numbers. Similarly, the definition of fractional circular functions is [28, 29]

\[ \sin_\varepsilon(t) = \sum_{r=0}^{\infty} (-1)^r \frac{t^{(2r+1)}}{\Gamma(2r+1)!} \]  

(4)

\[ \cos_\varepsilon(t) = \frac{\sum_{r=0}^{\infty} (-1)^r t^{2r+1}}{\Gamma(2r+1)} \]  

(5)

which are obviously related to the Mittag-Leffler function defined in equation (3). Similarly, the hyperbolic functions can be defined as [28, 29]

\[ \sinh_\varepsilon(t) = \frac{E_\varepsilon(t^\varepsilon) - E_\varepsilon(-t^\varepsilon)}{2} = \sum_{r=0}^{\infty} \frac{t^{(2r+1)}}{\Gamma(2r+1)!}, \]  

(6)

\[ \cosh_\varepsilon(t) = \frac{E_\varepsilon(t^\varepsilon) + E_\varepsilon(-t^\varepsilon)}{2} = \sum_{r=0}^{\infty} \frac{t^{2r+1}}{\Gamma(2r+1)} \]  

The focus point of the current investigation is to study a modified (generalized) form of gamma function [30], defined by

\[ \Gamma(u) = \int_{0}^{\infty} t^{u-1} e^{-t} dt \quad (u \in \mathbb{C}; \Re(u) > 0). \]  

(7)

It is a basic special function so that Pochhammer symbols \((u)_\kappa\) have the following relationship with the gamma function:

\[ (u)_\kappa = \frac{\Gamma(u + \kappa)}{\Gamma(u)} = \begin{cases} 1, & (\kappa = 0), \\ u(u + 1) \ldots (u + n - 1) & (\kappa \in \mathbb{C}\setminus\{0\}; \kappa = n \in \mathbb{N}; u \in \mathbb{C}). \end{cases} \]  

(8)

Here, \( \mathbb{N} \) denotes the set of natural numbers or positive integers. Further detailed analysis of the gamma function can be found in [31–34] and related bibliography therein. It is important to mention that this contribution remained unachievable without reviewing [35–46]. As we will be using delta function for our results and therefore we briefly describe about the spaces of test functions and distributions in the subsequent paragraphs.

A dual space also known as space of generalized functions (distributions) corresponds to a space of test functions. Such phenomenon involves the representation of singular functions and hence the certain functions can be taken into account and various calculus operations can be applied in this way. Let us begin with the following simple identities:

\[ \langle \delta(y - \sigma), u(y) \rangle = u(\sigma), \quad \sigma \in \mathbb{R}, \]  

(9)

and for a nonvanishing parameter \( \sigma \), \( \delta(-y) = \delta(y) \); \( \delta(\sigma y) = \delta(y)/|\sigma| \).

Gelfand and Shilov [47] have provided a comprehensive discussion and explanation of distributions (or generalized functions). According to them, [47], the commonly used test functions that have compact support are the elements of a space of test functions denoted by \( D \) and another space denoted by \( S \) is of infinitely differentiable as well as fast decaying functions. These spaces \( D \) and \( S \) have their dual spaces denoted by \( D' \) and \( S' \), respectively. When we talk about the closeness of such spaces under Fourier transform, then these spaces behave differently. Hence, the spaces \( S \) and \( S' \) are closed but \( D \) and \( D' \) are not closed under Fourier transform. As a result, the elements of \( D' \) have Fourier transforms that form distributions for entire functions spaces \( Z \) whose Fourier transforms belong to \( D \) [48]. In addition to it, since the entire function is nonzero only for a specific interval \( m_1 < y < m_2 \), so the following relation among these spaces is valid

\[ Z \subset S \subset S' \subset Z'; Z \cap D \equiv 0; D \subset S \subset S' \subset D'. \]  

(10)

For the benefit of this investigation, space \( Z \) comprises of entire and analytic functions sustaining the subsequent criteria
where $q$ and $C_q$ are numbers that depend on function $u$. The subsequent equations ([46], Vol 1, p. 169, equation (8); [47], p. 159, equation (4); and [48], p. 201, equation (10)) are vital to follow the steps of the proofs in this research

\[ \mathcal{F}[e^{xt}; \xi] = 2\pi\delta(\xi - iy). \] (12)

Hence, $\mathcal{F}[e^{xt}; \xi] \in \mathcal{Z}$ and for each $u \in \mathcal{Z}$,

\[ u(y + \theta) = \sum_{j=0}^{\infty} \theta_j(y) \frac{\theta^j}{j!}; \] (13)

\[ \delta(y + \theta) = \sum_{j=0}^{\infty} \delta_j(y) \frac{\theta^j}{j!}; \] (14)

\[ \delta(\sigma_1 - y) \delta(\sigma_2 - y) = \delta(\sigma_1 - \sigma_2). \] (15)

In addition to this, the other functions whose Fourier transform involves delta function are $\sin(\theta)$, $\cos(\theta)$, $\sinh\theta$, and $\cosh\theta$. Further detailed study about similar type of functions is given in [47–50].

Plan of the remaining paper is given as follows: Section 2.1 deals with the definition of the new function by investigating a series form of modified gamma function in terms of complex delta function. Section 2.2 discusses the conditions of persistence along with possible uses of the new series. Authentication of such results is discussed in Section 2.3. Advance properties of the new function as a distribution are analyzed in Section 2.4. Further examples by using new representation are discussed in Section 2.5. Generalized gamma distribution is discussed in Section 2.6. This work is concluded in Section 3. The results presented here are validated with the earlier obtained results in [35–37].

2. Results

2.1. Modified Form of the Gamma Function and Its Series Representation. By following the methodology of [24], we introduce

\[ \Gamma_\varepsilon(u) = \int_0^\infty t^{\varepsilon-1} E_\varepsilon(-t^\varepsilon) dt, \quad (u \in \mathbb{C}; 0 < \Re(u) < 1; 0 < \varepsilon < 1), \] (16)

and for $\varepsilon = 1$, we get the original gamma function (6).

According to the particular asymptotic behavior of Mittag-Leffler function ([2], p. 286, equations 2.8-2.9), this integral converges for $0 < \Re(u) < 1$ when $0 < \varepsilon < 1$. Next, by following Lebedev [30], we rewrite (16) as

\[ \int_0^\infty t^{\varepsilon-1} E_\varepsilon(-t^\varepsilon) dt = \int_0^1 t^{\varepsilon-1} E_\varepsilon(-t^\varepsilon) dt + \int_1^\infty t^{\varepsilon-1} E_\varepsilon(-t^\varepsilon) dt = P_\varepsilon(u) + Q_\varepsilon(u). \] (17)

Here, first integral $P_\varepsilon(u)$ is an analytic function of $u$ and second integral $Q_\varepsilon(u)$ is an entire function. Now, we replace $E_\varepsilon(-t^\varepsilon)$ by its power series such as

\[ P_\varepsilon(u) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)} \int_0^1 t^n u^{-1} dt \]

\[ = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)(u+ne)}, \quad (u \in \mathbb{C}; \Re(u) > 0). \] (18)

Uniform convergence of the involved integral allows one to reverse the order of integration and summation in equation (18). It shows that $\Gamma_\varepsilon(u)$ has singularities at $u = ne$.

Continuing in the same way, we investigate a new series representation of $\Gamma_\varepsilon(u)$ in the form of the following theorem.

**Theorem 1.** The modified gamma function $\Gamma_\varepsilon(u)$ has the following representation:

\[ \Gamma_\varepsilon(u) = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)} \delta(\theta - i(v + ne)). \] (19)

**Proof.** Let us first replace $t = e^x$ and $u = v + i\theta$ in equation (16), then we get

\[ \Gamma_\varepsilon(u) = \int_{-\infty}^{\infty} e^{i(v+\theta)x} E_\varepsilon(-e^{20}) dx. \] (20)

Next, we replace Mittag-Leffler function by its series form as given in (1),

\[ E_\varepsilon(-e^{20}) = \sum_{n=0}^{\infty} \frac{(-e^{20})^n}{\Gamma(n+1)}, \] (21)

and then by rearranging the involved exponential terms

\[ \Gamma_\varepsilon(u) = \int_{-\infty}^{\infty} e^{x(v+i\theta)} \sum_{n=0}^{\infty} \frac{(-e^{20})^n}{\Gamma(n+1)} dx, \] (22)

we get

\[ \Gamma_\varepsilon(u) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)} \int_{-\infty}^{\infty} e^{ixd(\varepsilon+ne)x} dx. \] (23)

Next, the uniform convergence of the involved integral allows us to interchange the operations of integration and summation. Hence, by using equation (12), we can write

\[ \int_{-\infty}^{\infty} e^{i\theta x} e^{(\varepsilon+ne)x} dx = \mathcal{F}[e^{(\varepsilon+ne)x} ; \theta] = 2\pi\delta(\theta - i(v + ne)). \] (24)

The above expressions (23) and (24) lead to the result (19) as mentioned above.

**Corollary 1.** The modified gamma function $\Gamma_\varepsilon(u)$ has the subsequent demonstration
\[
\Gamma_\varepsilon(u) = 2\pi \sum_{n,r=0}^{\infty} \frac{(-1)^n (-i(\varepsilon + ne))^r}{\Gamma(ne + 1)r!}\delta^{(r)}(\theta).
\] (25)

**Proof.** This can be obtained by using equations (14) and (19) as follows:

\[
\delta(\theta - i(\varepsilon + ne)) = \sum_{r=0}^{\infty} \frac{(-i(\varepsilon + ne))^r}{r!}\delta^{(r)}(\theta).
\] (26)

Next, by using this relation in (19), we obtain the desired result. \[\Box\]

**Corollary 2.** The modified gamma function \(\Gamma_\varepsilon(u)\) has the subsequent demonstration

\[
\Gamma_\varepsilon(u) = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(ne + 1)} \delta(u + ne).
\] (27)

**Proof.** This result can be obtained by a small alteration in equation (19) as follows:

\[
\int_{-\infty}^{\infty} e^{ix\theta} e^{(\varepsilon + ne)x} dx = \mathcal{F} \left[ e^{(\varepsilon + ne)x} ; \theta \right] = 2\pi \delta(\theta - i(\varepsilon + ne))
\]

\[
= 2\pi \delta \left( \frac{1}{\varepsilon} (i\theta + (\varepsilon + ne)) \right)
\]

\[
= 2\pi \delta(\varepsilon + i\theta + ne) = 2\pi \delta(u + ne),
\] (28)

which leads to the required result in view of (19). \[\Box\]

**Corollary 3.** The modified Gamma function \(\Gamma_\varepsilon(u)\) has the subsequent demonstration:

\[
\Gamma_\varepsilon(u) = 2\pi \sum_{n,r=0}^{\infty} \frac{(-1)^n (ne)^r}{\Gamma(ne + 1)r!}\delta^{(r)}(u).
\] (29)

**Proof.** This can be deduced in view of the relation (14) as follows:

\[
\delta(u + ne) = \sum_{r=0}^{\infty} \frac{(ne)^r}{r!}\delta^{(r)}(u).
\] (30)

Next, by making use of this result in (27), we obtain the required form. \[\Box\]

**Corollary 4.** Gamma function has the subsequent representation ([36], p. 2092, equation (2.7))

\[
\Gamma(u) = \Gamma_1(u) = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (\theta - i(\varepsilon + n)).
\] (31)

**Proof.** This can be deduced by fixing \(\varepsilon = 1\) in equation (19). \[\Box\]

**Corollary 5.** Gamma function has the subsequent series representation [37]

\[
\Gamma(u) = 2\pi \sum_{n,r=0}^{\infty} \frac{(-1)^n (-i(\varepsilon + n))^r}{n!r!}\delta^{(r)}(\theta).
\] (32)

**Proof.** This can be deduced by fixing \(\varepsilon = 1\) in equation (25). \[\Box\]

**Corollary 6.** Gamma function has the following representation [37]:

\[
\Gamma(u) = 2\pi \sum_{n=0}^{\infty} (-1)^n \delta(u + n).
\] (33)

**Proof.** This can be deduced by fixing \(\varepsilon = 1\) in equation (27). \[\Box\]

**Corollary 7.** Gamma function has the subsequent representation [37]:

\[
\Gamma(u) = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n n!}{\Gamma(n)\Gamma(u + n)}\delta^{(n)}(u).
\] (34)

**Proof.** This can be deduced by fixing \(\varepsilon = 1\) in equation (29). \[\Box\]

**Remark 1.** It is to be remarked that the newly obtained series representation is only meaningful if defined as a distribution acting over a space of test functions. We can observe it by simply multiplying the both sides of equation (27) with \(1/\Gamma_\varepsilon(u)\),

\[
1 = \frac{2\pi \sum_{n=0}^{\infty} (-1)^n n! \delta(u + ne)}{\Gamma_\varepsilon(u)},
\] (35)

which produces the following:

\[
1 = \frac{2\pi \sum_{n=0}^{\infty} (-1)^n n!}{\Gamma_\varepsilon(-ne)}.
\] (36)

As proved above that \(\Gamma_\varepsilon(u)\) has poles at \(u = -ne\), here, the right hand side sum will disappear because \(\Gamma_\varepsilon(-ne)\) is in the denominator

\[
\Rightarrow 1 = 0,
\] (37)

which is obviously false. At the same time, if we consider the following inner product,

\[
\left\langle \frac{\Gamma_\varepsilon(u)}{\Gamma_\varepsilon(u)}, \frac{1}{\Gamma_\varepsilon(u)} \right\rangle = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n n!}{\Gamma_\varepsilon(-ne)}
\] (38)
we obtain
\[
\int_{\mathbb{C}} 1 \, du = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \Gamma_{\varepsilon}(-ne).
\] (39)

Now, the right-hand side will vanish due to the fact that \( \Gamma_{\varepsilon}(u) \) has poles at \( u = -ne \), hence
\[
\int_{\mathbb{C}} 1 \, du = 0, \quad \Rightarrow \varepsilon \to 0 = 0,
\] (40)

which is false. This discussion demonstrates that we have to be very careful and rigorous in selecting the set of functions for a meaningful definition of this new representation. This is a part of Section 2.2.

2.2. Existence and Applications of New Representation.

This exemplification of the generalized gamma function \( \Gamma_{\varepsilon}(u) \) is an infinite summation over delta function, which is certain only if defined in the way as distributions (generalized functions). Therefore, it is curious to show that the obtained series of delta function is a distribution (generalized function) on a particular space of test functions as proved in the subsequent theorem.

**Theorem 2.** Prove that the modified gamma function \( \Gamma_{\varepsilon}(u) \) acts as a distribution over the space of slowly increasing complex test functions denoted by \( Z \).

**Proof.** Let us first take the following expression for any \( f^*(u), f^{**}(u) \in Z \), and \( c^*, c^{**} \in \mathbb{C} \):

\[
\langle \Gamma_{\varepsilon}(u), c^* f^*(u) + c^{**} f^{**}(u) \rangle = \left\langle 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)} \delta(u + en), c^* f^*(u) + c^{**} f^{**}(u) \right\rangle,
\]
(41)

\[
\Rightarrow \langle \Gamma_{\varepsilon}(u), c^* f^*(u) + c^{**} f^{**}(u) \rangle = c^* \langle \Gamma_{\varepsilon}(u), f^*(u) \rangle + c^{**} \langle \Gamma_{\varepsilon}(u), f^{**}(u) \rangle.
\]

It shows that \( \Gamma_{\varepsilon}(u) \) is a linear function over \( Z \). Now, we take an arbitrary sequence \( Z_{\langle u_{\rho} \rangle_{\rho=1}^{\infty}} \) and use the fact that \( \delta(u) \) acts as continuous function over \( Z \),

\[
\Rightarrow \left\langle \Gamma_{\varepsilon}(u), f_{\rho}(u) \right\rangle_{\rho=1}^{\infty} = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)} \left\langle \delta(u + en), f_{\rho}(u) \right\rangle_{\rho=1}^{\infty}.
\] (42)

Then, we can show that \( \left\langle \delta(u + en), f_{\rho}(u) \right\rangle_{\rho=1}^{\infty} \to 0 \) (tends to zero). Hence, the modified gamma function \( \Gamma_{\varepsilon}(u) \) is a distribution (continuous linear functional) over \( Z \) because of the convergence of (27) given by

\[
\langle \Gamma_{\varepsilon}(u), f(u) \rangle = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)} \langle \delta(u + en), f(u) \rangle \quad (\forall f(u) \in Z),
\]
(43)

\[
= 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)} f(-ne). \quad (44)
\]

In the above equation, we have applied the following property for delta function,

\[
\langle \delta(u + en), f(u) \rangle = f(-ne).
\] (45)

In this way, the attained functions \( f(-ne) \in Z \) are test functions of slow growth (bounded by a polynomial); also, we can observe that summation of the coefficients in (44) is

\[
\text{sum over the coefficients} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)} = E_\varepsilon(-1). \quad (46)
\]

Hence, we get a finite and definite answer, which shows that \( \langle \Gamma_{\varepsilon}(u), f(u) \rangle \) is convergent, for \( \forall f(u) \in Z \). Here, we have used the well-known fact that “the product of a rapid decay and slow growth functions is convergent.” However, it is also obvious from the famous Abel’s theorem. Hence, the same is true for its special cases which are given in (25)–(29).

Let us move to a larger class of functions and consider \( f(u) = u^\varepsilon \). In view of equation (27) and shifting characteristic of delta functions, the product \( f(u), \Gamma_{\varepsilon}(u) \) is computed as follows:

\[
\int_{\mathbb{C}} u^\varepsilon \Gamma_{\varepsilon}(u) \, du = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)} \varepsilon t^{-n} \exp(-t) \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)} \varepsilon t^{-n} \exp(-t) = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)} \varepsilon t^{-n} \exp(-t).
\] (47)

By putting \( \varepsilon = 1 \) in (47), we can get the following known result for gamma function:

\[
\int_{\mathbb{C}} u^\varepsilon \Gamma(u) \, du = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \varepsilon t^{-n} \exp(-t) = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \varepsilon t^{-n} \exp(-t).
\] (48)

**Remark 2.** Particular engineering problems involve sum of delta functions. It is remarked that when one considers the product of \( 2\pi E_\varepsilon(-1) \) with sequence of delta functions \( \{\delta(u + en)\}_{n=0}^{\infty} \), then it yields the modified form \( \Gamma_{\varepsilon}(u) \). One can validate this result for gamma function \( \Gamma(u) \) at \( \varepsilon = 1 \).

The above discussion explores the existence of further results. To illustrate, we take \( \varepsilon = e^{-1} \) in equation (47), which
then leads to Laplace transform of $\Gamma_\varepsilon(u)$. Hence, we consider the validation of such outcomes in the subsequent part.

2.3. Validation of the Results Obtained by New Representation.
In this section, we will testify our results with the existing results. Let us take $t = e^x$ and $u = y + iT^2$ in (16), then the modified gamma function $\Gamma_\varepsilon(u)$ is expressible as
\[
\Gamma_\varepsilon(v + i\theta) = \sqrt{2\pi} \mathcal{F} [e^{\varepsilon x} E_i(-e^{\varepsilon x})] \xi, \tag{49}
\]
and putting $\varepsilon = 1$, in the above equation, we obtain
\[
\Gamma(v + i\theta) = \sqrt{2\pi} \mathcal{F} [e^{\varepsilon x} \exp(-e^{\varepsilon x})] \theta. \tag{50}
\]

We know that Fourier transform satisfies the following dual property for any function $u(t)$.
\[
\mathcal{F} [\sqrt{2\pi} \mathcal{F} [u(t); \theta]] = 2\pi \mu(-\xi). \tag{51}
\]

Hence, by applying this property for equations (49)–(50), we get the following result:
\[
\mathcal{F} [\Gamma_\varepsilon(u)(v + i\theta)] \xi = \mathcal{F} [\sqrt{2\pi} \mathcal{F} [e^{\varepsilon x} E_i(-e^{\varepsilon x})]] \xi \nonumber = f(-\xi) = 2\pi e^{-\varepsilon x} E_i(-e^{-\varepsilon x}), \tag{52}
\]
or
\[
\int_{-\infty}^{\infty} e^{\varepsilon i\tau} \Gamma_\varepsilon(u)(v + i\theta) d\tau = 2\pi e^{-\varepsilon x} E_i(-e^{-\varepsilon x}). \tag{53}
\]

This can also be accessible as a particular case of the main result (47) when one puts $\tau = e; u = v + i\theta$. So, the results obtained by using the new series representation are accurate when compared with well-known results. Furthermore, when one takes $\xi = 0$ in (49), then the following identity can be obtained:
\[
\int_{-\infty}^{\infty} \Gamma_\varepsilon(u)(v + i\theta) d\theta = 2\pi E_i(-1), \tag{54}
\]
which is also a special case of (53). Hence, it is testified that the new representation of $\Gamma_\varepsilon(u)$ creates new results that are unachievable in view of known representations but the special cases of these novel results are reliable with the existing methods. Continuing in this way, by taking $\varepsilon = 1$ in (53) and (54), we obtain
\[
\int_{-\infty}^{\infty} e^{\varepsilon i\tau} \Gamma_\varepsilon(v + i\theta) d\tau = \int_{-\infty}^{\infty} e^{\varepsilon i\tau} \Gamma(v + i\theta) d\tau = 2\pi e^{-\varepsilon x} \exp(-e^{-\varepsilon x}), \tag{55}
\]
and $\xi = 0$ in (55) yields the following well-known result:
\[
\int_{-\infty}^{\infty} \Gamma(v + i\theta) d\theta = \frac{2\pi}{e}. \tag{56}
\]

2.4. Advanced Distributional Properties of Modified Gamma Function. Here, by taking motivation from ([48], Chapter 7), we state and prove a list of basic properties of the modified gamma function.

**Theorem 3.** For an arbitrary test function $f(u)\in Z$, $\Gamma_\varepsilon(u)$ do satisfy the subsequent properties as a distribution over the space of test functions:
\[
\begin{align*}
(1) \quad & \langle \Gamma_\varepsilon(u), f(u) \rangle = \langle \Gamma_\varepsilon(u), f^*(u) \rangle + \langle \Gamma_\varepsilon(u), f^{**}(u) \rangle, \\
(2) \quad & \langle \varepsilon^* \Gamma_\varepsilon(u), f(u) \rangle = \langle \Gamma_\varepsilon(u), c^* f(u) \rangle, \\
(3) \quad & \langle \Gamma_\varepsilon(u - \gamma), f(u) \rangle = \langle \Gamma_\varepsilon(u), f(u + \gamma) \rangle, \\
(4) \quad & \langle \varepsilon^* \Gamma_\varepsilon(u), f(u) \rangle = \langle \Gamma_\varepsilon(u), 1/c^* f(u/c^*) \rangle, \\
(5) \quad & \langle \varepsilon^*(u - \gamma), f(u) \rangle = \langle \Gamma_\varepsilon(u), 1/c^* f(u/c^* + t\gamma) \rangle, \\
(6) \quad & \text{For any regular distribution } \psi(u), \psi(u)\varepsilon^*(u)\in Z. \quad \text{is a distribution over } Z.
\end{align*}
\]

Proof. First six results (1)–(6) can be proved by using the basic properties of delta function and by following the procedure of Theorem 2. Here, we will prove the next result (7). Hence, by taking nth derivative of representation (27) w.r.t $u$ and using (14), one can obtain
\[
\langle \Gamma_\varepsilon^{(m)}(u), f(u) \rangle = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} f^{(m)}(-en), \tag{57}
\]
which is a well-defined and finite product. Then, one can prove the next result as follows:
\[
\langle \Gamma_\varepsilon(u + e), f(u) \rangle = \langle \Gamma_\varepsilon(u), f(u + e) \rangle, \tag{58}
\]
Again, by using the property of delta function, one can get the next result (9).
\[
\langle \Gamma_{\varepsilon}(\omega_1 - u)\Gamma_{\varepsilon}(u - \omega_2), f(u) \rangle = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)} \langle \delta(\omega_1 - \omega_2), f(u) \rangle
\]

\[
= (2\pi\varepsilon(-1)^2\langle \delta(\omega_1 - \omega_2), f(u) \rangle.
\]

(59)

Similarly, the results (10)–(15) are obvious to hold in view of Fourier transform and the properties of complex delta function. Let us start proving the next identity (10) from the above list,

\[
\langle \mathcal{F}[\Gamma_{\varepsilon}(u)], f(u) \rangle = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)} \langle \mathcal{F}[(\delta(u + \varepsilon n)], f(u) \rangle
\]

\[
= 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)} \langle (\delta(u + \varepsilon n), \mathcal{F}[f(u)] \rangle = \langle \Gamma_{\varepsilon}(u), \mathcal{F}[f(u)] \rangle.
\]

(60)

Next, we prove identity (11) from the above list,

\[
\langle \mathcal{F}[\Gamma_{\varepsilon}(u)], \mathcal{F}[f(u)] \rangle = 2\pi \langle \mathcal{F}[\Gamma_{\varepsilon}(u)], \mathcal{F}[f(u)] \rangle
\]

\[
= 2\pi \langle \Gamma_{\varepsilon}(t), f(-t) \rangle,
\]

and identity (12),

\[
\langle \mathcal{F}[\Gamma_{\varepsilon}(u)], \mathcal{F}[f(u)] \rangle = 2\pi \langle \mathcal{F}[\Gamma_{\varepsilon}(u)], \mathcal{F}[f(u)] \rangle
\]

\[
= 2\pi \langle \Gamma_{\varepsilon}(t), f(-t) \rangle = 2\pi \langle \Gamma_{\varepsilon}(t), f^t(t) \rangle,
\]

(62)

where \( f^t \) represents transpose of the involved test function \( f \). Hence, the above identity (13) can be observed by the following:

\[
\langle \mathcal{F}[\Gamma_{\varepsilon}^{(1)}(u)], f(u) \rangle = \langle \mathcal{F}[\Gamma_{\varepsilon}(u)], f^{(1)}(u) \rangle = \langle \Gamma_{\varepsilon}(u), \mathcal{F}[f^{(1)}(u)] \rangle,
\]

\[
\Rightarrow \langle \mathcal{F}[\Gamma_{\varepsilon}^{(1)}(u)], f(u) \rangle = \langle \Gamma_{\varepsilon}(u), (-it)f(u)e^{-\varepsilon ut} \rangle,
\]

\[
\Rightarrow \langle \mathcal{F}[\Gamma_{\varepsilon}^{(1)}(u)], f(u) \rangle = \langle (-it)\mathcal{F}[\Gamma_{\varepsilon}(u)], f(u) \rangle.
\]

(65)

and so on, we get

\[
\langle \mathcal{F}[\Gamma_{\varepsilon}^{(m)}(u)], f(u) \rangle = \langle (-it)^m \mathcal{F}[\Gamma_{\varepsilon}(u)], f(u) \rangle.
\]

(66)

Continuing in this way, one can prove the last identity (16) by making use of the result ([48], p. 201),

\[
\langle \Gamma_{\varepsilon}(u + c^*), f(u) \rangle = \langle \Gamma_{\varepsilon}(u), f(u - c^*) \rangle = \lim_{y \to \infty} \langle \Gamma_{\varepsilon}(u), \sum_{n=0}^{\infty} \frac{(-c^*)^n}{n!} f^{(n)}(u) \rangle
\]

\[
= \lim_{y \to \infty} \sum_{n=0}^{\infty} \frac{c^n}{n!} \Gamma_{\varepsilon}^{(n)}(u), f(u) \rangle,
\]

(68)

as required.

\[\square\]
Let $\delta(t)$ be a Dirac delta function. It is a linear mapping which can map each function to its value at zero. By taking into account this fact, we consider the following examples. Let $u = t$, to real numbers, then we have

$$\Gamma_t (t) = 2\pi \sum_{n,r=0}^{\infty} \frac{(-1)^n (ue)^r}{(ne + 1)r!} \delta^{(r)}(t).$$

Hence,

$$\langle \Gamma_t (t), f(t) \rangle = 2\pi \sum_{n,r=0}^{\infty} \frac{(-1)^n (ue)^r}{(ne + 1)r!} \delta^{(r)}(t) = 2\pi \sum_{n,r=0}^{\infty} \frac{(-1)^n (ue)^r}{(ne + 1)r!} (-1)^r f^{(r)}(0).$$

Example 3. Let $f(t) = 1/(1-t)$, then $f^{(r)}(0) = r!$,

$$\langle \Gamma_t (t), \frac{1}{1-t} \rangle = 2\pi \sum_{n,r=0}^{\infty} \frac{(-1)^n (ue)^r}{(ne + 1)r!} (-1)^r r!$$

$$= 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{(ne + 1)(1 + ne)} = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{(ne + 2)} = 2\pi E_{z,2} (-1).$$

Example 4. Let $f(t) = \ln(1+t)$, then $f^{(r)}(0) = (-1)^{r+1} (r-1)!$,

$$\langle \Gamma_t (t), \ln(1+t) \rangle = 2\pi \sum_{n,r=0}^{\infty} \frac{(-1)^n (ue)^r}{(ne + 1)r!} (-1)^r (-1)^{r+1} (r-1)!$$

$$= 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n \ln(1-ne)}{(ne + 1)}.$$

Here, we restrict the value of $\epsilon$ very small such that $1 - ne > 0$. These examples show that our new representation of the modified gamma function is meaningful for all those functions that have derivatives of all orders at point 0. Similar results hold for its special case gamma function given by equation (34).

Next, we consider Laplace transform of delta function that yields the Laplace transform of the modified gamma and the original gamma function itself. We know that

$$L[f^{(r)}(u); s] = s^r.$$  

Therefore, we get
It can also be verified that
\[
\int_0^\infty \Gamma_\epsilon(X)dX = \int_0^\infty \frac{\mu_{\epsilon-1}E_\epsilon(-t^2)}{\Gamma_\epsilon(u)}dt = 1. 
\]

Next, we discuss the usual properties of a random variable \(X\) which is distributed according to the probability density function \(\Gamma_\epsilon(X)\) given in (81).

The general moment of order \(n\) for the random variable \(X\) is obtained as
\[
E[X^n] = \int_0^\infty t^n \frac{\mu_{\epsilon-1}E_\epsilon(-t^2)}{\Gamma_\epsilon(u)}dt = \frac{\Gamma_\epsilon(u+n)}{\Gamma_\epsilon(u)},
\]
where \(E\) denotes the mathematical expectation.

Mean and expected value for the probability distribution function is the first moment of the variable \(X\) and obtained as a special case of the above for \(n = 1\),
\[
\mu_X = E[X] = \int_0^\infty t \frac{\mu_{\epsilon-1}E_\epsilon(-t^2)}{\Gamma_\epsilon(u)}dt = \frac{\Gamma_\epsilon(u+1)}{\Gamma_\epsilon(u)},
\]
and for \(n = 2\),
\[
E[X^2] = \int_0^\infty t^2 \frac{\mu_{\epsilon-1}E_\epsilon(-t^2)}{\Gamma_\epsilon(u)}dt = \frac{\Gamma_\epsilon(u+2)}{\Gamma_\epsilon(u)}.
\]

Hence, the variance of the random variable \(X\) is computed as
\[
\sigma_X^2 = E[X^2] - (E[X])^2 = \frac{\Gamma_\epsilon(u+2)}{\Gamma_\epsilon(u)} - \left(\frac{\Gamma_\epsilon(u+1)}{\Gamma_\epsilon(u)}\right)^2.
\]

By using (84), the characteristic function of the random variable \(X\) is given as follows:
\[
E[e^{itX}] = \sum_{n=0}^\infty \frac{(it)^n}{n!} E[X^n] = \sum_{n=0}^\infty \frac{(it)^n}{n!} \frac{\Gamma_\epsilon(u+n)}{\Gamma_\epsilon(u)}.
\]

Similarly, the generalized moment generating function of the random variable \(X\) by using Mittag-Leffler function instead of exponential function is given as
\[
E[E_{\epsilon}(tX)] = \sum_{n=0}^\infty \frac{(tX)^n}{\Gamma(n+1)} = \sum_{n=0}^\infty \frac{t^n}{n!} \frac{\Gamma_\epsilon(u+n)}{\Gamma_\epsilon(u)}
\]
and the usual moment generating function of the random variable \(X\) is obtained as a special case of (89),
\[
M_X(t) = E[e^{tX}] = \sum_{n=0}^\infty t^n \frac{\Gamma_\epsilon(u+n)}{\Gamma_\epsilon(u)}.
\]

The cumulative distribution function of the random variable \(X\) for \(X > 0\) is computed as
\[
L(\Gamma_\epsilon(u); s) = L\left(2\pi \sum_{\nu=0}^\infty \frac{(-1)^\nu (ne)^\nu}{\Gamma((ne+1)r)} \delta^{(r)}(u); s \right)
= 2\pi \sum_{\nu=0}^\infty \frac{(-1)^\nu (ne)^\nu}{\Gamma((ne+1)r)} L\left(\delta^{(r)}(u); s \right)
= 2\pi \sum_{\nu=0}^\infty \frac{(-1)^\nu (ne)^\nu}{\Gamma((ne+1)r)} \pi^{\frac{s}{r}} \sum_{\nu=0}^\infty \frac{(-1)^\nu}{\Gamma((ne+1))} (e^{\nu s})^r
= 2\pi e_c (-e^{\nu c}).
\]

Putting \(\epsilon = 1\) in the above equation gives
\[
L(\Gamma_\epsilon(u); s) = 2\pi \exp(-e^s),
\]
which yields further
\[
L(\Gamma_\epsilon(u-c); s) = 2\pi e^{-sc} \exp(-e^s),
\]
and
\[
L(\Gamma_\epsilon(u-c); s) = 2\pi e^{-sc} \exp(-e^s).
\]

A special case taking \(s = 0\) in (78) leads to the well-known result given above in (56). Hence, it validates that the results obtained by new representation are consistent with the known results.

**Remark 4.** It is to be remarked that the success of the application of this new representation lies in the fact that the sum over the coefficients in the new representation is finite and well-defined as given in (46).

### 2.6. Generalized Gamma Distribution.

Using a probability density function, Dirac delta function is frequently used to represent certain probability distributions. For example, considering the probability density function \(u(x)\) of a discrete distribution with points \(x = \{x_1, \ldots, x_i, \ldots\\}\) and corresponding probabilities \(p_i\), it can be written as
\[
u(x) = \sum_{i=0} p_i \delta(x-x_i).
\]

Moreover, gamma distribution and its generalizations are of interest for a large audience (see [51] and the cited bibliography therein). Let us consider the following new probability density function of a statistical distribution for a random variable \(X\) as follows:
\[
\Gamma_\epsilon(X) = \begin{cases} 
\chi^{\nu-1}E_\epsilon(-\chi^2) & u > 0, \\
0 & \text{elsewhere},
\end{cases}
\]
where \(\Gamma_\epsilon(u)\) is defined in (16) and it is suitably assumed that for the involved parameters,
\[
\Gamma_\epsilon(X) > 0.
\]
\[
\mathcal{F}_\varepsilon(X) = \int_0^X t^{\mu-1} E_\varepsilon(-t^\mu) \, dt = \frac{1}{\Gamma_\varepsilon(u)} \int_0^X t^{\mu-1} E_\varepsilon(-t^\mu) \, dt = \frac{1}{\Gamma_\varepsilon(u)} \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)} \int_0^X t^{\mu n-1} \, dt = \frac{1}{\Gamma_\varepsilon(u)} \sum_{n=0}^{\infty} \frac{(-1)^n X^{\mu n}}{\Gamma(n+1)(u+n)} \ (u \in \mathbb{C}; \ \Re(u) > 0).
\]  
(91)

The Lorenz curve as introduced by Lorenz [52] is commonly used to represent the disparity of money and earnings in economics. For our new probability function, it is computed for a random variable \( X \) as follows:

\[
L(X) = \frac{\mathcal{F}_\varepsilon(X)}{\mu_X} = \frac{1}{\Gamma_\varepsilon(u+1)} \sum_{n=0}^{\infty} \frac{(-1)^n X^{\mu n}}{\Gamma(n+1)(u+n)} \ (u \in \mathbb{C}; \ \Re(u) > 0).
\]  
(92)

Hence, it is to be remarked that the results for the other probability functions, for example,

\[
f(X) = \begin{cases} 
\frac{X^{\mu-1} e^{-t}}{\Gamma(u)}; & u > 0, \\
0, & \text{elsewhere},
\end{cases}
\]  
(93)
can be obtained as special case of the above statistical measures by taking \( \varepsilon = 1 \).

3. Conclusion

During this study, we explored a new special function as a modified form of gamma function so that the gamma function can be obtained as a special case of it. We discussed some basic properties of this function. The combination of distribution theory with different integral transforms is well explored for the analysis of partial differential equations (PDE). Numerous practical questions are impossible to be answered by applying the known techniques but became possible by using this combination. Hence, a new fractional operator naturally emerges as a consequence of this research which can act on a suitably chosen function \( f(x) \) as follows:

\[
\Gamma_\varepsilon(f(x)) = \int_{\mathbb{C}} \Gamma_\varepsilon(s) f(s) \, ds = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)} f(-ne).
\]  
(94)

A new probability density function involving the new function is introduced, and its properties are studied. We end this discussion by proposing a generalized representation of the familiar beta function,

\[
B(x; y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)} \ (x, y \in \mathbb{C}; \ \Re(x); \Re(y) > 0),
\]  
(95)
in terms of the new function as follows:

\[
B_\varepsilon(x; y) = \frac{\Gamma_\varepsilon(x) \Gamma_\varepsilon(y)}{\Gamma_\varepsilon(x+y)} \ (x, y \in \mathbb{C}; \ \Re(x); \Re(y) > 0).
\]  
(96)

It is customary to consider Mittag-Leffler function as given in equation (3) for certain applications. However, it can be noticed that if we consider (1) instead of (3) in (16) for

Data Availability

No data were generated or analyzed for this submission.

Conflicts of Interest

The authors declare no conflicts of interest.

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