A NEW LOGISTIC-TYPE MODEL FOR PRICING EUROPEAN OPTIONS

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ABSTRACT. We propose a family of models for the evolution of the price process $S(t)$ of a financial market. We prove that this family of models is well defined in the sense that the SDEs that define each model have unique non-explosive solutions; we prove that each model from this family is free of arbitrage opportunities and it is (state) complete. We calibrate a model from the proposed family and Heston model (Heston [10]) with option prices written over the S&P500 obtained from the CBOE from December 2007 to December 2008. The empirical results obtained in both models show similarities for periods of low volatility, but the model studied shows a better performance on the period of higher volatility.

1. INTRODUCTION

We define a family of models where the evolution of the price process $S(t)$ is given by the system of stochastic differential equations

\[
\begin{align*}
    dS(t) &= (\sigma \hat{S} \theta - \delta + r) S(t) dt + \sigma \hat{S} S(t) dW(t) \quad S(0) = s_0 \\
    d\hat{S}(t) &= -\delta \hat{S}(t) dt + (\sigma \hat{S} - \theta) \hat{S} dW(t) \quad \hat{S}(0) = s_0.
\end{align*}
\]

where $\sigma$ is a twice continuous differentiable function, and $\delta$, $r$ and $\theta$ are constants for the dividend rate, the short interest rate and the market price of risk. In this paper we prove that as long as $\sigma(\cdot)$ satisfies a behavior defined by equations (11) below, the given system of differential equations defines a global solution (with non explosion). We also prove that the market with stock whose price evolution is given by $S(t)$ and short interest rate $r$ is free of (state) arbitrage opportunities; in addition as long as $d\sigma(\cdot)/dx$ satisfies the non-singularity condition given by equation (12) the market defined above is (state) complete. The definitions of (state) completeness and (state) arbitrage opportunities are new and developed by the author ([19]). We also analyze the empirical behavior whenever $\sigma(x) = n_2(P - x)$, and $\theta = n_1/n_2$ where $n_1$, $n_2$, $P$ are constants; this family has a simple economic interpretation (see remarks made after equation (4)). For the sake of clarity, we develop the theory for the particular model and later we extend the results to the general family of models.

The particular model whose empirical properties are analyzed in this paper is better than others proposed in the literature, because the evolution of all variables has a clear economic meaning and the market model is a (state) complete market. Moreover the model is extremely simple, it is easy to calibrate and has reduced run times because it depends on very few parameters; it captures most stylized facts observed on the market and it seems to behave as good as Heston’s model in terms of RMSE on regular times but better than the Heston’s Model in times with higher volatility and uncertainty (for example during the peak of the crisis of

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2008). Also, the proposed model showed better dynamics of the volatility surface, showed evidence of auto-correlation in squared log-returns, and there exists evidence of negative correlation between the volatility process and the level of prices. The model’s empirical properties analyzed are reviewed in section 4.

The simplest model for equity prices is a geometric Brownian motion (see Black and Scholes [2]). Desirable properties of this model include market completeness along with absence of arbitrage opportunities. Nevertheless, the Black-Scholes model has widely documented problems. There exists evidence of time, price and strike value affecting the volatility process of financial assets; those findings have been materialized on two concepts: the volatility smile and the cluster effect, where volatility clustering is the empirical observation that there appear to be high volatility and low volatility time periods. These effects have been empirically documented by Aït-Sahalia and Lo [11], Jackwerth and Rubinstein [15], Bollerslev et al. [3], Derman [7], Rebonato [24], Derman [7].

In order to overcome Black-Scholes’ shortcomings, research have extended this market model to allow for richer dynamics of financial asset prices. Some of the extended models are proposed by Merton [21], Derman and Kani [8], Dupire [9], Hobson and Rogers [11], Hull and White [12], Heston [10].

In some cases those models violate market completeness; as a consequence, unique prices under absence of arbitrage will not be obtained (See Londoño [19], Rebonato [24], Broadie and Detemple [4]).

The family of models studied in this paper is an example of models where a pricing theory can be obtained using the results of Londoño [20, 19]. In general, the theory developed in Londoño [20] does not impose conditions on the eigenvalues of the volatility matrix. Traditional models usually require that the eigenvalues of the volatility matrix remain away from 0 (see Karatzas and Shreve [17]). The methodology developed in this paper give sufficient conditions for the existence of non-explosive solutions and market completeness for the family of models proposed (see remark 1), that in general do not impose conditions on the eigenvalues of volatility. The main results in this paper are complementary to results in Londoño [20], since it gives us concrete examples beyond standard models of financial markets.

In Londoño [20] a general theoretical framework for valuation and arbitrage is developed, but in order to provide models beyond the standard literature models it is needed to guarantee existence of non-explosive solutions for the stochastic differential equations than define the models, and it is necessary to give conditions to guarantee market completeness (that according to Corollary 2 are equivalent to the existence of density functions of the proposed models).

The contents of the paper are as follows: In Section 2 we develop a model for the evolution of the price of stocks and prove that an existence of non explosive solutions for the stochastic differential equations which defines the model; we note that the model developed is free of (state) arbitrage opportunities. Moreover we prove that market defined by the model proposed is (state) complete and we introduce the numerical methodology used to approximate asset prices as solutions to SDEs (stochastic differential equations); we also generalize the results obtained for a family of models that includes the model proposed. In Section 3 we present the calibration results on the model presented and compare to results obtained from the Heston model. In Section 4 we describe some empirical characteristics of the model.
2. DEFINITION AND CHARACTERIZATION OF NEW LOGISTIC-TYPE MODELS

A first approximation to equilibrium theory is to assume short periods of time where the price of a stock is essentially constant, modulo arbitrage considerations. Namely if we assume an equilibrium price at time $0$ the short time evolution of the price of the stock should take into account non-arbitrage considerations and should account for changes in the interest rate and alike. If we believe that there is a short term equilibrium price, and the observed market price is a noisy proxy the equilibrium price, then it is natural to have mean reversion in the volatility, but this reversion should be with respect to a changing equilibrium price (see discussion on equation (4) below).

From the above arguments, one of the possible one dimensional stochastic extensions derived from the deterministic logistic equation under Londoño [20] framework is:

\[
\begin{align*}
\frac{dS(t)}{S(t)} &= (r - \delta + n_1(P - \hat{S}(t)))S(t)dt + n_2(P - \hat{S}(t))S(t)dW(t) \quad S(0) = s_0 \\
\frac{dH_0(t)}{H_0(t)} &= -H_0(t)(r dt + \theta dW(t)), \quad H_0(0) = 1
\end{align*}
\]

where $r, \delta, P$, and $n_1, n_2 \neq 0$ are real constants and $\hat{S}(t) = S(t)H_0(t)$ is the financial asset price discounted by the state price density process.

We point out that the volatility can be instantaneously 0, but as a consequence of proposition 2 the set of time values where this occurs has measure 0. It follows that

\[
\begin{align*}
b(t) + \delta - r &= n_1(P - \hat{S}(t)) \\
\sigma(t) &= n_2(P - \hat{S}(t)).
\end{align*}
\]

where $b(t)$, and $\sigma(t)$ are the return and volatility processes. We first notice that the system of equations define a global (non explosive solution); see subsection 2.2. Throughout this paper we shall call the market defined by equations (1), and (2) as the linear model.

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It follows that

\[
b(t) - r + \delta = \sigma(t)\theta(t).
\]

Therefore there are not state tame arbitrage opportunities (see Londoño [19]). However, we notice that we allow singular volatilities. We have

\[
\sigma(t) = n_2H_0(t)(PH_0^{-1}(t) - S(t))
\]

where the volatility is mean reverting against the process $PH_0^{-1}(t)$. If we think that $P$ is the short term equilibrium price at time $0$ and that $S$ is the market price, then there is mean reversion towards the “short equilibrium price” $P$ at time $t$.

Finally as a consequence of corollary 2 below the proposed model is complete. Market completeness together with absence of arbitrage will provide consistent and unique prices for European contingent claims. Models belonging to a logistic category have not been widely developed on modern research. Onyango [23] proposed an extension of the logistic equations which
governs population growth to model the behavior of an asset price for one dimensional worlds. In Onyango [23], asset prices are assumed to obey the following SDE (adapted to use the previous notation),

$$dS(t) = n_1(P - S(t))S(t)dt + n_2(P - S(t))S(t)dW(t).$$

The above framework is a direct extension of a deterministic logistic equation of the form,

$$dS = n_1(P - S)dt.$$ 

However Onyango [23] shows several sources of weakness. The most serious drawback on Onyango [23] is the absence of a theoretical framework sustained by a complete and arbitrage-free market.

2.1. Numerical approximation method. Due to complexity of explicit solutions for some SDE systems, this work will use a numerical approximation procedure. Among all the methods in the current literature, we have chosen Wong-Zakai type approximations to be used due to its simplicity of implementation. Wong-Zakai (1965) demonstrated that, if the solution to an equation of the form,

$$dS(t) = \tilde{b}(t, S(t))dt + \tilde{\sigma}(t, S(t))dW(t), \quad S(0) = s_0$$

wants to be approximated using a partition \(0 = t_0 < t_1 < t_2 \cdots < t_k = T\) of the time interval \([0, T]\); it is possible to accomplish it through the solution of the following ODE for each interval \([t_{i-1}, t_i], i = 1 \cdots k\),

$$\frac{dS^*}{dt} = \tilde{b}(t, S^*) + \tilde{\sigma}(t, S^*) (W_{t_i} - W_{t_{i-1}})$$

with, \(S^*(t_i) = s_i, \quad W_{t_i} - W_{t_{i-1}} \sim N(0, t_i - t_{i-1})\).

The last differential equation is in the sense of Stratonovich and \(\tilde{b}(\cdot)\) and \(\tilde{\sigma}(\cdot)\) are twice continuous differentiable functions on the spatial variable and continuously differentiable in the time variable. Convergence of Wong-Zakai solution is defined on an almost sure sense. See Wong and Zakai [26] for more details.

The ODE system defined over the interval \(0 = t_0 < t_1 < t_2 \cdots < t_k = T\), which approximates the solution of the system of equations 1 and 2 on each interval \([t_{i-1}, t_i], i = 1 \cdots k\) is recursively solved using Matlab® ODE solvers and random number generators. The main objective is to obtain approximations of \(S(T)\) to calculate European contingent claim prices with expiration \(T\).

2.2. Some Basic Properties. Next we develop the model proposed and prove that the system of proposed SDE’s defines a non-explosive solution. We should emphasize that this is not straightforward since the coefficients that define the SDE are not Lipchitz continuous.

Moreover we prove that the model satisfies a condition given by Proposition 2 below, that implies as a consequence of Londoño [19, Theorem 4.1] that the model defined is (state) complete. It should be emphasize that the results on this section are the core of this contribution and they can be summarized as mathematical results that give sufficient conditions for the theory of Londoño [19] to be applied. We also point out that the results in this section are generalized to any model given by the system of SDE’s defined by equation (10) as long as the conditions given by equation (11) and (12) are met. Although this paper proposes a huge collection of models, we just review in detail a model to illustrate properties, and to give an overview of empirical properties that the model proposed has.
We first observe the linear model defined by equations (1) and (2) is equivalent to
\begin{align}
\dot{S}(t) &= (r - \delta + n_1(P - \hat{S}(t))) S(t) dt + n_2(P - \hat{S}(t)) S(t) dW(t) \quad S(0) = s_0 \\
\dot{\hat{S}}(t) &= -\delta \hat{S}(t) dt + (n_2 P - \theta - n_2 \hat{S}(t)) \hat{S} dW(t) \quad \hat{S}(0) = s_0.
\end{align}
(7)
The solutions are equivalent in the sense that they produce the same solution for $S$ and $\hat{S}$. Since the system of equations (7) is locally Lipschitz continuous there exist a unique local solution to the system of differential equations. In order to prove global existence it is just sufficient to prove that there is not explosion on positive time. This is a consequence of proposition 1 below. First, using Itô’s rule we obtain the following lemma:

**Lemma 1.** Assume the unique local solution of the differential equation
\[ d\hat{S}(t) = \alpha(\hat{S}) \hat{S} dt + \beta(\hat{S}) \hat{S} dW(t) \quad \hat{S}(0) = s_0 > 0 \]
for $t \in [0, \tau)$, where $\tau$ is the explosion time for the differential equation on $\mathbb{R}^+$ (the positive real numbers), and where $\alpha(\cdot)$, $\beta(\cdot)$ are differentiable functions defined on $\mathbb{R}^+$. Then $\hat{Y} = 1/\hat{S}$, $t \in [0, \tau)$ is the maximal local solution of the differential equation
\[ d\hat{Y}(t) = \left[ \beta^2 \left( \frac{1}{\hat{Y}} \right) - \alpha \left( \frac{1}{\hat{Y}} \right) \right] \hat{Y} dt - \beta \left( \frac{1}{\hat{Y}} \right) \hat{Y} dW(t) \quad \hat{Y}(0) = 1/s_0. \]
(8)
As a consequence of the previous lemma we obtain the following proposition:

**Proposition 1.** There exist a unique non-explosive solution of the stochastic differential equation in $\mathbb{R}^+ = (0, \infty)$
\[ d\hat{S}(t) = \beta \hat{S} dt + (\alpha - \kappa \hat{S}(t)) \hat{S} dW(t) \quad \hat{S}(0) = s_0 > 0 \]
for any $\beta, \alpha, \kappa \in \mathbb{R}$.

**Proof** Since the coefficients of the previous SDE are differentiable and therefore locally Lipschitz continuous it follows that there exist a unique local maximal solution for the SDE for an random time interval $[0, \tau)$. Therefore, It is sufficient to prove that for any $T > 0$ there is not explosion in $[0, \tau \wedge T)$ for the solution of the stochastic differential equation in $(0, \infty)$. Then, it is sufficient to prove that for any $T > 0$
\[ \limsup_{t \in [0, \tau \wedge T)} \hat{S}(t) < \infty \quad \text{and,} \quad \liminf_{t \in [0, \tau \wedge T)} \hat{S}(t) > 0 \]
almost everywhere. Let $T > 0$ be given; we first prove that $\limsup_{t \in [0, \tau \wedge T)} \hat{S}(t) < \infty$. For this we notice that there exist a global solution (in the interval $[0, T]$) for the stochastic differential equation
\[ dS_1(t) = (\alpha - \kappa S_1(t)) \left( \frac{\alpha}{2} - \kappa S_1(t) \right) S_1(t) dt + (\alpha - \kappa S_1(t)) S_1 dW(t) \quad \hat{S}(0) = s_0 \]
In fact there exist a closed form solution of a SDE whose drift and diffusion agrees with the coefficients of the previous differential equations outside a ball large enough. (see Kloeden and Platen [18]). Since for $x$ large enough
\[ \beta x \leq (\alpha - \kappa x)(\alpha/2 - \kappa x)x \]
it follows, using a localization argument and stochastic inequalities (Karatzas and Shreve [16 Proposition 2.18]), that $\limsup_{t \in [0, \tau \wedge T)} \hat{S}(t) < \infty$.
Finally, in order to prove that $\liminf_{t \in [0, \tau \wedge T)} \hat{S}(t) > 0$ for any $T > 0$ it is sufficient to prove that $\lim sup_{t \in [0, \tau \wedge T)} \hat{Y}(t) < \infty$, where $\hat{Y}(t) = 1/\hat{S}(t)$. Since $\hat{Y}(t)$ is the
maximal local solution of the stochastic differential equation given by (8), \( \hat{Y}(t) \) satisfies

\[
d\hat{Y}(t) = \left( (\alpha^2 - \beta)\hat{Y} - 2k\alpha + \frac{k^2}{\hat{Y}} \right) dt - \left( \alpha\hat{Y} - k \right) dW(t) \quad \hat{Y}(0) = 1/s_0.
\]

It follows by a localization argument and stochastic inequalities (Karatzas and Shreve [16, Proposition 2.18]) that any maximal solution of \( \hat{Y} \) must be dominated from above by \( \max(1, Y_u(t)) \) where \( Y_u(t) \) the global solution of the following stochastic differential equation

\[
dY_u = \left( (\alpha^2 - \beta)Y_u - 2k\alpha + k^2 \right) dt - \left( \alpha Y_u - k \right) dW(t), \quad Y_u(0) = 1/s_0.
\]

We notice that the coefficients of the previous stochastic differential equation are Lipschitz continuous and therefore there exist a non explosive global solution in any interval \([0,T]\). It follows that \( \sup_{t \in [0,T]} \hat{Y}(t) < \infty \) as required where \( \tau \) is the explosion time of \( \hat{Y} \) exiting \((0,\infty)\).

**Corollary 1.** There exist a unique global non explosive solution to the system of stochastic differential equations of (7).

**Proof** By proposition 1 there exists a unique non-explosive solution \( \hat{S}(t) \) to the second equation of the system of equations (7). It follows using Itô’s rule’s that there exist a closed form (non-explosive) solution of \( S(t) \) in terms of \( \hat{S}(t) \) (see for instance Karatzas and Shreve [17, equation 1.9]).

Finally we prove that the model proposed by the system of equations (7) is a (state) complete market. For this it is necessary the following proposition:

**Proposition 2.** The unique non explosive solution \( \hat{S}(t) \) of the stochastic differential equation (9) has a density for any \( \beta, \alpha, \kappa \in \mathbb{R} \setminus \{0\} \) and any initial condition \( s_0 > 0 \).

**Proof** We notice that \( \hat{S}(t) \) is also the solution in the sense of Stratonovich of the following SDE

\[
d\hat{S} = \left( \beta - (\alpha - \kappa\hat{S})\frac{\alpha}{2} - \kappa\hat{S} \right) \hat{S} dt + (\alpha - \kappa\hat{S}(t))\hat{S} dW(t) \quad \hat{S}(0) = s_0 > 0.
\]

It follows by Itô’s rule that for any twice continuous differentiable function \( f \in C^2 \)

\[
df(\hat{S}_t) = A_0 f(\hat{S}_t) \ dt + A_1 f(\hat{S}_t) \circ dW_t
\]

where \( A_0 \) is the differential operator

\[
A_0 = \left( \beta - (\alpha - \kappa x)\frac{\alpha}{2} - \kappa x \right) x \frac{d}{dx}
\]

and \( A_1 \) is the differential operator

\[
A_1 = (\alpha - \kappa x)x \frac{d}{dx}.
\]

Then, the Lie bracket \([A_0, A_1]\) evaluated at \( \alpha/\kappa \) is

\[
[A_0, A_1]_{\alpha/\kappa} = -\beta \frac{\alpha^2}{\kappa} \frac{d}{dx}
\]

and therefore the result follows as a consequence of the Hörmander condition (Hörmander [13] and Ikeda and Watanabe [14]).
Corollary 2. The model given by the system of stochastic differential equations of equation (7) where it is assumed that $\delta, n_1, n_2, P \in \mathbb{R} \setminus \{0\}$ is free of (state) arbitrage opportunities and it is (state) complete.

Proof By construction, equation (5) holds and it follows by Londoño [19, Theorem 3.1] that the market is free of state arbitrage opportunities. In order to prove that the proposed model is a (state) complete market we observe that the volatility for $S(t)$ is $n_2(P - \hat{S}(t))$; it follows by Londoño [19, Theorem 4.1] that it is sufficient to prove that $\hat{S}(t)$ has a density, but this is the conclusion of Proposition 2. □

Remark 1. If we assume a constant dynamic for the interest rate, the market price of risk and the dividend process ($r$, and $\theta$, $\delta$), then the new models can be constructed in the following way. Let $\sigma(\hat{S}) = f(\hat{S})$ where $f$ is a function with two continuous derivatives. Define $b(\hat{S}) = f(\hat{S}) \theta - \delta + r$; if the system of stochastic differential equations
\begin{align*}
\frac{dS(t)}{S(t)} &= b(\hat{S}) dt + \sigma(\hat{S}) dW(t) \quad S(0) = s_0 \\
\frac{d\hat{S}(t)}{\hat{S}(t)} &= -\delta \hat{S}(t) dt + (\sigma(\hat{S}) - \theta) \hat{S} dW(t) \quad \hat{S}(0) = s_0.
\end{align*}
(10)
where $\hat{S}(t) = H_0(t)S(t)$, is a system of non-explosive differential equations then Londoño [19, Theorem 3.1] implies that the system defines a market free of state arbitrage opportunities. Using a similar argument to the one used in proposition 1 it can be shown that if
\begin{equation}
\limsup_{x \to \infty} \left| \sigma \left( \frac{1}{x} \right) \right| < \infty
\end{equation}
(11)
then the system of equations is a non-explosive system. Using a similar argument to the one used in Proposition 2 it can be proved that as long as
\begin{equation}
d\sigma(x)/dx \neq \theta \quad \text{for any } x > 0 \quad \text{with } \sigma(x) = \theta
\end{equation}
(12)
then the market defined by the system of equations (10) is a (state) complete market. More models can be constructed assuming a stochastic evolution for the interest rate and the market price of risk.

Remark 2. We notice that the standard theory of valuation and arbitrage is not well suited for the model proposed in this paper. The difficulty of the proposed model arises in the fact that the volatility of the price process $S(t)$ given by equation (7) is allowed to take singular values. To overcome this difficulty we use the theory of arbitrage and valuation proposed in Londoño [19] and Londoño [20].

3. Model Calibration
Following standard procedures, the model is calibrated minimizing an error function. Shouten et al. [25] considered absolute option price differences (AP). Other alternative is relative prices (RP) as in Mikhailov and Nögel [22]. In this paper, relative implied volatility (RV) differences (Shouten et al. [25]) are used to implement the calibration procedure.
We define the error functions (average relative percentage error) as,

\[ RV := \sqrt{\frac{1}{n} \sum_{i=1}^{n} \omega_i \left( \frac{IV_{i,\text{mod}} - IV_{i,\text{mar}}}{IV_{i,\text{mar}}} \right)^2} \]

where \( IV_{i,\text{mod}} \) and \( IV_{i,\text{mar}} \) refers to the implied volatility given by the model output and by the market value of the corresponding option, \( n \) is the number of options considered on any given date and

\[ \omega_i = \frac{1}{n_{\text{mar}} n_{\text{str}}^i} \]

where \( n_{\text{mat}} \) and \( n_{\text{str}}^i \) will be the number of maturities and the number of strikes with the same maturity as observation \( i \) respectively.

Since we do not know an analytical expression for the values of European contingent claims for the model proposed by the system of equations (7), Monte Carlo simulation techniques will be implemented. In this document the error function is minimized using a heuristic method of direct search. Specifically, we follow the Generalized Pattern Search (GPS) algorithm implemented by the function patternsearch on Matlab.

As benchmark model we use the model proposed in Heston [10]. The Heston(93) model has been chosen because it has analytical closed solutions (see Carr and Madan [6]). Even though the Heston(93) model violates the market completeness assumption, market completeness is overcome by assuming a specific functional form for the volatility market price of risk.

3.1. Volatility surface calibration results.

3.1.1. Data. The implied volatility surface is recovered from closing mid-prices of plain vanilla calls and puts written over the S&P500 every Tuesday from December 5th 2007 until December 3rd 2008, using the Heston model (93) and the linear model. The database was filtered in order to leave options showing a daily trading volume greater than 1,000 transactions per day and expiration between 0.1 and 0.8 years. The latter will avoid illiquid option which will distort the calibration results. The year 2008 represents a challenge to any model that intends to recover implied volatility from market option prices. Due to the liquidity crunch generated by the sub-prime crisis, market volatility peaked to one of the highest levels ever. For example, the VIX index reached 80 on October 27th and November 20th 2008. This increase in volatility was accompanied by a decrease of 58.64% in the S&P500 index.

On average, there are 4 different maturities and 54 options per day which fulfill the maturity and volume criteria. Because the S&P 500 is a stock index, the calibration procedure assumed a fixed annual dividend yield of 1.89%, the 2007 average dividend yield according to Standard & Poor’s.

Because these models assume constant interest rates, they have been calibrated as follows. For the Heston(93) model, each option price was obtained using the corresponding spot interest rate. However, the lineal model used the spot interest rate only for the first maturity. The linear model has been calibrated using a numerical approximation, and asset price paths simulated for a certain option maturity were
also used to simulate asset prices on subsequent maturities and for these we used forward rates. In order to avoid discrepancies due to methodology implementation, calibration results using only closest maturity options are also reported. The spot interest rate curve was estimated from treasury bill yields provided by the FED.

3.1.2. Results. The error function based on relative implied volatility differences using all the option maturities was minimized for 52 days. Figure 1 (left) depicts the error evolution for the calibration period. Statistics on the error function are provided on Table 1. Figure 2 shows the fit of models to market implied volatility on February 6th 2008.

As observed in Table 1, the mean valued of the error function for the Heston model (93) and the linear model are similar. The evolution of the error function is similar even if we only used just the closest maturity options for calibration. Table 1 also shows statistics of calibration results including only closing maturity calls and puts (see Figure 1 on right side for evolution of errors using only options with the closest maturity).

Table 1. Error statistics for the calibration procedure on the average relative percentage error for the linear model (equation (7)) and the Heston model.

| Complete set       | Closest maturity options |
|--------------------|--------------------------|
|                    | Linear Model  | Heston Model | Linear Model | Heston Model |
| **Mean**           | 7.70%         | 7.81%        | 7.76%        | 7.65%        |
| **Stand. dev.**    | 2.39%         | 3.43%        | 2.81%        | 4.20%        |

Though the means of errors are similar as can be seen from Table 1, the standard deviation of the linear model error was lower than the one estimated using the
Figure 2. Implied volatility on February 6th 2008 using market data, the model proposed on equation (7), and the Heston Model for different maturities. Mat: 0.12 (upper/left), 0.2 (upper/right), 0.37 (bottom/left) and 0.62 (bottom/right) years.

The evolution for the calibrated parameters for both models are shown in figure 3. One particular observation is the behavior of the parameter $P$ on the linear model. On average, $P$ was located $7.04\%$ above the initial index value ranging from $1.4\%$ up to $14.70\%$. This finding strengthens the idea of asset price level dependency of $P$. Also, figure 3 shows the volatility and drift of the price processes $S(t)$ of equation (7). As shown, the volatility and mean processes remain stable until September 17th 2008. Afterwards, they almost tripled. This increment is mainly explained by the disruption of the interbank credit market motivated by the bankruptcy or sell of several investment or commercial banks.

4. SOME EMPIRICAL FACTS

First, we assume a particular set of parameters for the linear model: $T = 0.5, S(0) = 987.5, P = 1075.1, n_2 = -0.003, r = 1.57\%, n_1 = 0.0013, \delta = 1.89\%$. The dividend has been estimated using dividend yield reported by Standard & Poor’s. These parameters followed from calibration results in section 3 over the S&P500 using...
relative volatility minimization on data from October 20th 2008. In figure 5 a random realization is chosen and its associated volatility process have been reported.

4.1. Implied volatility smiles and parameter analysis. Logistic features of the proposed model produce a high volatility regime when the discounted price is far from \( P \) and viceversa. Results in figure 5 also suggest that the model reproduce a world where the volatility process is negatively related to the stock price level. We notice that this property does not follow from the structure of the driving SDE since the volatility is a function of the price process discounted the the state price density and it is not a function of the price process alone. This feature is a common effect found in real data due to leverage reasons. In a leveraged company, if the stock price decreases, the company Debt/Equity ratio will increase though the level of debt is unchanged. An increasing leverage normally causes a high volatility level. See for example Campbell and Hentschel [5] for more details on the leverage effect evidence.

4.2. Floating smile and volatility surface. As discussed before a desirable model feature is the ability to reproduce floating smiles. Figure 6 (left) shows a simulated path of underlying price using \( P = 1075.1, n_2 = 0.005, r = 1.57\%, n_1 = 0.0013 \) and \( \delta = 1.89\% \) and (right) several implied volatility curves calculated using the linear model at 5 selected underlying prices. As shown, the volatility surfaces obtained are able to recover implied volatility steepness specially on short maturity options. This specific feature is persistently
Figure 4. Volatility and Mean process of the model given by the system of equation (7) for the calibration period. The mean process has been estimated using the 1M risk free interest rate.

observed on volatility surfaces extracted from option market data over the S&P500 from December 5th 2007 to December 3rd 2008.

4.2.1. Cluster effect in the volatility process. One of the common features found in price processes is volatility clustering. Volatility clustering is the empirical observation that there appear to be high volatility and low volatility time periods. A Ljung-Box Q-test was done over squared returns on the simulated prices using the parameters obtained from calibration of the linear model from S&P500 option prices each Tuesday of each of the 52 weeks starting at Dec-5-2007. The null hypothesis was that the series of squared returns exhibits no auto-correlation for a fixed number of lags $L = 20$, against the alternative that some auto-correlation coefficient $\rho(k), k = 1, ..., L$, is nonzero. The test statistic is

$$Q = n(n + 2) \sum_{k=1}^{L} \frac{\rho^2(k)}{n-k}$$

where $n$ is the sample size, $L$ is the number of auto-correlation lags, and $\rho(k)$ is the sample auto-correlation at lag $k$. Under the null hypothesis, the asymptotic distribution of $Q$ is chi-square with $L$ degrees of freedom. The test was done 10,000 times with $\alpha = 5\%$. Results on the simulations are found on figure 8.

Simulations suggest that there exist a relation between correlation over squared returns and $P/S_0$, where $P$ is the parameter that represent the “equilibrium price” (as discussed after equation (4)) and $S_0$ is the current price.
**Figure 5.** Price and volatility process for a randomly selected realization. \( T = 0.5, S(0) = 987.5, P = 1075.1, n_2 = -0.005, r = 1.57\%, n_1 = 0.0013, \delta = 1.89\%, \) and a partition of 500 points of the time interval.

**Figure 6.** Evolution of implied volatility of the linear model for a fixed maturity time interval of \( T = 1 \) year. Left: Simulated path of underlying price using \( P = 1075.1, n_2 = 0.005, r = 1.57\%, n_1 = 0.0013 \) and \( \delta = 1.89\%. \) Right: Implied volatility curves for 1-year maturity calculated using the linear model at 5 selected underlying prices.
Finally as a consequence of persistence, a standard procedure from the GARCH family can be used to implement volatility forecasting. Figure 9 shows a simulated volatility process and the corresponding calibrated GARCH(1,1) fitted over the same selected stock price realization obtained from the linear model calibrated on October 20, 2008.

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Figure 8. Percentage of simulated price process that showed significant auto-correlations different from zero using a 95% confidence interval with a fixed number of lags $L = 20$. Each point represents a different set of parameters for the linear model calibrated from the S&P500 option prices at every Tuesday of each of the 52 weeks starting at Dec-5-2007.

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Figure 9. Volatility process and corresponding one-period ahead Garch(1,1) forecasting for a randomly selected index value path from the model given by the system of equations \([7]\).

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