ON THE UNIVALENCE OF POLY-ANALYTIC FUNCTIONS

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Abstract. A continuous complex-valued function $F$ in a domain $D \subseteq \mathbb{C}$ is Poly-analytic of order $\alpha$ if it satisfies $\partial^\alpha_z F = 0$. One can show that $F$ has the form $F(z) = \sum_{k=0}^{\alpha-1} z^k A_k(z)$, where each $A_k$ is an analytic function. In this paper, we prove the existence of a Landau constant for Poly-analytic functions and the special Bi-analytic case. We also establish the Bohr’s inequality for poly-analytic and bi-analytic functions which map $U$ into $U$. In addition, we give an estimate for the arclength over the class of poly-analytic mappings and consider the problem of minimizing moments of order $p$.

1. Introduction

There exist non-analytic functions with significant structure and with properties reminiscent of those satisfied by analytic functions. Such nice non-analytic functions are called Poly-analytic functions. A continuous complex-valued function $F$ defined in a domain $D \subseteq \mathbb{C}$ is Poly-analytic of order $\alpha$ if it satisfies the generalized Cauchy-Riemann equations $\partial^\alpha_z F = 0$. An iteration argument shows that $F$ has the form

$$F(z) = \sum_{k=0}^{\alpha-1} z^k A_k(z),$$

where $A_k$ are analytic functions for $k = 1, \ldots, \alpha - 1$. In particular, a continuous complex-valued function $F = u + iv$ in a domain $D \subseteq \mathbb{C}$ is said to be Bi-analytic if $\frac{\partial}{\partial \bar{z}} \left( \frac{\partial F}{\partial z} \right) = 0$. It is a poly-analytic function of order 2. Note that $F_\bar{z}$ is analytic in $D$. In any simply connected domain $D$, it can be shown that $F$ has the form

$$F(z) = \bar{z} A(z) + B(z),$$

where $A$ and $B$ are analytic function (and obviously $A = F_\bar{z}$).

Poly-analytic functions were first introduced by the Russian mathematician G.V. Kolossov in connection with his research in the mathematical theory of elasticity who studied in particular bi-analytic mappings (See [19]). The bi-analytic functions have been introduced to study physical fields with divergence or rotation at the present time and their theories and applications have been studied by many authors (see [26], [31]). Useful applications of this idea in mechanics are widely known from the remarkable investigations by Kolossoff, his student Muskhelishvili and their followers. The applications of poly-analytic functions to problems in elasticity are
well documented in his book [23]. Most important applications of the theory of functions of a complex variables were obtained in the plane theory of elasticity (see [19]-[23]). Poly-analytic function theory has been investigated intensively, notably by the Russian school led by Balk [7]. A new characterization of poly-analytic functions has been obtained by Agranovsky [1]. We note that a Poly-analytic complex function is poly-harmonic, but the converse is not true. If \( A_k(z) = 1 \) for all \( k \), then \( F(z) \) is a harmonic function (see [7]). The composition of poly-analytic function with conformal mapping from both sides, in general is not poly-analytic. The properties of poly-analytic functions can be different from those enjoyed by analytic functions, for example they can vanish on closed curves without vanishing identically. One such example the function \( F(z) = 1 - z^2 \). Still, many properties of analytic functions have found an extension to poly-analytic functions, often in a nontrivial form. We consider in Section 2 Landau’s Theorem and we give two different versions one for poly-analytic and the other for bi-analytic functions. In section 3 we show that the Bohr’s radius holds for poly-analytic functions and in section 4, we consider the case when the poly-analytic functions have starlike analytic counterparts, we find an upper bound for the arclength of poly-analytic functions and solve the problem of minimizing moments of order \( p \) for bi-analytic functions.

2. Landau’s Theorem for Poly-analytic functions

The classical Landau Theorem for bounded analytic functions states that if \( f \) is analytic in the unit disk \( U \) with \( f(0) = 0 \), \( f'(0) = 1 \) and \(|f(z)| < M\) for \( z \in U \), then \( f \) is univalent in the disk \( U_{\rho_0} = \{ z : |z| < \rho_0 \} \) with

\[
\rho_0 = \frac{1}{M + \sqrt{M^2 - 1}}
\]

and \( f(U_{\rho_0}) \) contains a disk \( U_{R_0} \) with \( R_0 = M\rho_0^2 \). This result is sharp. Chen, Gauthier and Hengartner [9] obtained a version of the Landau theorem for bounded harmonic mappings of the unit disk. Unfortunately their result is not sharp. Better estimates were given later in [10]. In specific, it was shown in [11] that if \( f \) is harmonic in the unit disk \( U \) with \( f(0) = 0 \), \( J_f(0) = 1 \) and \(|f(z)| < M\) for \( z \in U \), then \( f \) is univalent in the disk \( U_{\rho_1} = \{ z : |z| < \rho_1 \} \) with

\[
\rho_1 = 1 - \frac{2\sqrt{2}M}{\sqrt{\pi + 8M^2}}
\]

and \( f(U_{\rho_1}) \) contains a disk \( U_{R_1} \) with \( R_1 = \frac{\pi}{4M} - 2M\rho_1^2 \rho_1^2 \). This result is the best known but not sharp. Landau’s Theorem has been also proven for the classes of Biharmonic mappings by several authors and extended to the class of Polyharmonic mappings (see [2], [10]-[12]).

We first give a version of Landau’s theorem for Poly-analytic functions.

**Theorem 1.** Let \( F(z) = \sum_{k=0}^{\alpha-1} A_z^k A_k(z) \) be a poly-analytic function of order \( \alpha \) on \( U \), where \( A_k \) are analytic such that \( A_k(0) = 0 \), \( A_k'(0) = 1 \) and \(|A_k| \leq M \) for all \( k \).
Then there is a constant $0 < \rho_1 < 1$ so that $F$ is univalent in $|z| < \rho_1$. In specific, $\rho_1$ satisfies

$$1 - M \left( \frac{\rho_1(2 - \rho_1)}{(1 - \rho_1)^2} + \sum_{k=1}^{\alpha-1} \rho_1^k \frac{(1 + k - \rho_1)}{(1 - \rho_1)^2} \right) = 0,$$

and $F(U_{\rho_1})$ contains a disk $U_{R_1}$, where $R_1 = \rho_1 - \rho_1^2 (\frac{1 - \alpha^{-1}}{1 - \rho_1}) - M \sum_{k=0}^{\alpha-2} \rho_1^k$.

**Proof.** Fix $0 < \rho < 1$ and choose $z_1, z_2$ with $z_1 \neq z_2$, $|z_1| < \rho$ and $|z_2| < \rho$. We first note that $F_z = \sum_{k=0}^{\alpha-1} \bar{\alpha}' A_k$, $F_{\bar{\alpha}}(z) = \sum_{k=0}^{\alpha-1} k \bar{\alpha} A_k$, which implies $F_z(0) = A_0(0) = 1$, $F_{\bar{\alpha}}(0) = A_1(0) = 0$. We write each $A_k(z) = \sum_{n=1}^{\infty} a_{n,k} z^n$. It follows that on the line segment $[z_1, z_2]$ we have

$$|F(z_1) - F(z_2)| = \int_{[z_1, z_2]} F_z(z) dz + F_{\bar{\alpha}}(z) d\bar{\alpha}$$

$$= \int_{[z_1, z_2]} (F_z(0) dz + F_{\bar{\alpha}}(0) d\bar{\alpha}) + \int_{[z_1, z_2]} (F_z(z) - F_z(0)) dz + (F_{\bar{\alpha}}(z) - F_{\bar{\alpha}}(0)) d\bar{\alpha}.$$
For $|z| = \rho_1$,

$$|F(z)| = \left| \sum_{k=0}^{\alpha-1} z^k A_k(z) \right| = \left| a_{1,0} z + \sum_{k=1}^{\alpha-1} a_{1,k} z^{k-1} + \sum_{k=0}^{\alpha-1} \sum_{n=0}^{\infty} a_{n,k} z^n \right| \geq \rho_1 - \sum_{k=2}^{\alpha-1} \sum_{n=0}^{\rho_1^{n+k}} - M \sum_{k=0}^{\alpha-1} \sum_{n=0}^{\infty} \left| \frac{1 - \rho_1^{n-k}}{1 - \rho_1} \right| - M \sum_{k=0}^{\alpha-1} \sum_{n=0}^{\rho_1^{n+k}} \left| \frac{1 - \rho_1^{n-k}}{1 - \rho_1} \right| = \rho_1 - \rho_1^2 \left( \frac{1 - \rho_1^{-1}}{1 - \rho_1} \right) - M \sum_{k=0}^{\alpha-1} \sum_{n=0}^{\rho_1^{n+k}} \left| \frac{1 - \rho_1^{n-k}}{1 - \rho_1} \right| \geq \rho_1 - \rho_1^2 \left( \frac{1 - \rho_1^{-1}}{1 - \rho_1} \right).
$$

□

As a corollary, we conclude a version of Landau’s theorem for bi-analytic functions:

**Corollary 1.** Let $F(z) = \overline{z}A(z) + B(z)$ be a Bi-analytic function of $U$, where $A$ and $B$ are analytic such that $A(0) = B(0) = 0$, $A'(0) = B'(0) = 1$ and $|A|$ and $|B|$ are both bounded by $M$. Then there is a constant $0 < \rho_1 < 1$ so that $F$ is univalent in $|z| < \rho_1$. In specific, $\rho_1$ satisfies

$$1 - 2M \left( \frac{2\rho_1 - \rho_1^2}{(1 - \rho_1)^2} \right) = 0,$$

that is

$$\rho_1 = \frac{2M}{2M + 1} \left( 1 + \sqrt{\frac{2M + 1}{2M}} + \frac{1}{2M} \right).$$

Moreover, $F(U_{\rho_1})$ contains a disk $U_{R_1}$, where

$$R_1 = \rho_1 - \rho_1^2 - M \rho_1^3 + \rho_1^2 \frac{1}{1 - \rho_1}.$$

### 3. Bohr’s theorem for Poly-analytic functions

Bohr’s inequality says that if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in the unit disc $U$ and $|f(z)| < 1$ for all $z$ in $U$, then $\sum_{n=0}^{\infty} |a_n z^n| \leq 1$ for all $z \in U$ with $|z| \leq \frac{1}{2}$. This inequality was discovered by Bohr in 1914 (see[8]). Bohr actually obtained the inequality for $|z| \leq \frac{1}{2}$. Later Wiener, Riesz and Schur independently, established the inequality for $|z| \leq \frac{1}{2}$ and showed that $\frac{1}{2}$ is sharp (See [25], [28], [29]). More recently, the Bohr’s inequality has been of interest for several authors. We refer to [5] for a historical overview.

Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic function in $U$ and let $M(A) := \sum_{n=0}^{\infty} |a_n| r^n$ be the associated majorant series for $A$. If $A$ and $B$ are analytic functions, straight forward calculations we deduce the following:

(3.1) $$M(A + B) \leq M(A) + M(B)$$

(3.2) $$M(AB) \leq M(A)M(B).$$
In the next result, we establish a Bohr’s type inequality for Poly-analytic functions of order \( \alpha \) given by \( F(z) = \sum_{k=0}^{\alpha-1} z^k A_k(z) \). For this family of functions, we define the majorant series \( M(F, r) \) by

\[
M(F, r) = \sum_{n=0}^{\alpha-1} \sum_{k=0}^{n} |a_{n,k}| r^{k+n}.
\]

**Theorem 2.** Let \( F(z) = \sum_{k=0}^{\alpha-1} z^k A_k(z) \) be a Poly-analytic function of order \( \alpha \), where \( A_k \) are analytic mappings for \( k = 0, 1, ... \alpha - 1 \), such that \( f_k(z) = A_0(z) + A_k(z) \) preserves orientation for each \( k \). Suppose that \( A_0 \) is univalent and normalized by \( A_0(0) = 0, A_0'(0) = 1 \) and \( F(U) \subset U \). Then

\[
M(F, r) < 1 \quad \text{if} \quad |z| < r_0,
\]

where \( r_0 \) is the root of the polynomial \( r^\alpha + r^{\alpha-1} + ... + r^3 + 3r - 1 = 0 \). We can take \( r_0 \approx 0.318 \).

**Proof.** We first note that

\[
M(F, r) = M\left( \sum_{k=0}^{\alpha-1} z^k A_k(z) \right) \leq \sum_{k=0}^{\alpha-1} r^k M(A_k).
\]

Now since \( f_k(z) = A_0(z) + A_k(z) \) preserves the orientation for each \( k \), it follows that \( A_k'(z) = a_k(z) A_0'(z) \) for \( a_k \in H(U) \) and \( |a_k(z)| < 1 \) for all \( z \in U \), which gives that

\[
M(A_k) \leq \int_0^r M(A_k')ds = \int_0^r M(a_k A_0')ds \leq \int_0^r M(A_0')ds = M(A_0).
\]

Hence

\[
(3.3)\quad M(F, r) \leq \sum_{k=0}^{\alpha-1} r^k M(A_0) = M(A_0) \frac{1 - r^\alpha}{1 - r}.
\]

We next find a bound for \( M(A_0) \). We have

\[
M(A_0) = \int_0^r M(A_0')ds \leq \int_0^\infty |na_{n,0} z^{n-1}|ds
= \int_0^\infty n|a_{n,0}| s^{n-1} ds \leq \int_0^\infty nr^n = \frac{r}{(1-r)^2}.
\]

Therefore

\[
M(F, r) \leq \frac{r(1 - r^\alpha)}{(1 - r)^3}.
\]

It follows that for all integer values of \( \alpha \), we have \( M(F, r) \leq \frac{r(1 - r^\alpha)}{(1 - r)^3} < 1 \) for \( |z| < r_0 \), where \( r_0 \) is the root of the polynomial \( r^\alpha + r^{\alpha-1} + ... + r^3 + 3r - 1 = 0 \).

Below is a table that indicates the value of the Bohr’s radius depending on the value of \( \alpha \). This tables shows that we can take \( r_0 \approx 0.318 \).
As a corollary, and as seen from the above table we obtain a Bohr’s type inequality for Bi-analytic functions. We note here that if \( F(z) = zA(z) + B(z) \) is a Bi-analytic function we take

\[
M(F, r) = \sum_{n=0}^{\infty} |a_n| r + |b_n|.
\]

**Corollary 2.** Let \( F(z) = zA(z) + B(z) \) be a Bi-analytic function such that \( f(z) = A(z) + B(z) \) preserves the orientation. Suppose that \( B \) is univalent and normalized by \( B(0) = 0, B'(0) = 1 \) and \( F(U) \subset U \). Then

\[
M(F, r) < 1 \quad \text{if} \quad |z| < \frac{1}{3}.
\]

The bound is sharp and is attained by suitable rotation of the Koebe function \( A(z) = \frac{z}{1-z^2} \).

Recently, Abu Muhanna has shown the following lemma for analytic functions \cite{3}, which gives the radius under which the majorant function is bounded by the distance to the boundary of the image of an analytic function.

**Lemma 1.** Let \( A(z) = \sum_{n=0}^{\infty} a_n z^n \) be analytic on \( U \) and suppose that \( A(U) \) misses at least two points then

\[
M(A) \leq \text{dist}(A(0), \partial A(U)),
\]

for \( |z| \leq e^{-\pi} = 4.3214 \times 10^{-2} \).

We generalize the above lemma to poly-analytic functions under certain conditions on \( A_k \).

**Theorem 3.** Let \( F(z) = \sum_{k=0}^{\alpha-1} z^k A_k(z) \) be a Poly-analytic function of order \( \alpha \), where \( A_k \) are analytic mappings for \( k = 0, 1, ..., \alpha - 1 \), such that \( f_k(z) = A_0(z) + A_k(z) \) preserves orientation for each \( k \). Suppose that \( A_0(z) \) misses at least two points and \( A_0(0) = a_0, A_k(0) = 0 \), for \( k = 1, ..., \alpha - 1 \). Then if \( |z| < e^{-\pi} \),

\[
M(F, r) \leq \frac{1 - r^\alpha}{1 - r} \text{dist}(a_0, \partial A_0(U)).
\]

**Proof.** The proof is a simple application of equation (3.3) and Lemma 1. \( \square \)
Corollary 3. Let $F(z) = \tau A(z) + B(z)$ be a Bi-analytic function such that $f(z) = B(z) + A(z)$ preserves the orientation, where $B(0) = b_0$, $A(0) = 0$, and suppose that $B(z)$ misses at least two points. Then if $|z| < e^{-\pi}$, we have
\[ M(F, r) \leq (1 + r)d(b_0, \partial B(U)). \]

4. Poly-analytic functions with starlike counterparts

We first recall that an analytic univalent mapping $A(z)$ is said to be starlike analytic if it satisfies
\[ \frac{\partial \arg A(re^{i\theta})}{\partial \theta} = \Re \frac{zA'(z)}{A(z)} > 0 \]
for all $z \in U$.

The next result establishes an upper estimate for the arclength of Poly-analytic functions with starlike analytic counterparts.

Theorem 4. Let $F(z) = \sum_{k=1}^{\alpha-1} \tau^k A_k(z)$ be a Poly-analytic functions such that each $A_k(z)$ is a starlike analytic function and $|A_k(z)| \leq M(r)$, for each $k = 1, \ldots, \alpha - 1$, $0 < r < 1$. Let $L(r)$ denote the arclength of the curve $C_r$, where $C_r$ denote the image of the circle $|z| = r < 1$ under the function $w = F(z)$. Then
\[ L(r) \leq \frac{2\pi M(r)r}{1 - r} \left[ (1 + r) \left( (\alpha - 1)r^\alpha - \alpha r^{\alpha-1} + 1 \right) \right] - \frac{(1 - r)}{(1 - r)^2}. \]

Proof. We have
\[
L(r) = \int_{C_r} |dF| = \int_0^{2\pi} \left| zF'_r - \tau zF'_\tau \right| d\theta \\
= \int_0^{2\pi} \left| \sum_{k=1}^{\alpha-1} \tau^k zA'_k(z) - \tau k z^{\alpha-1} A_k(z) \right| d\theta \\
\leq \int_0^{2\pi} \sum_{k=1}^{\alpha-1} |zA'_k(z)| \left| \frac{zA'_k(z)}{A_k(z)} - k \right| d\theta \\
\leq M(r) \int_0^{2\pi} \sum_{k=1}^{\alpha-1} r^k \int_0^{2\pi} \left| \frac{zA'_k(z)}{A_k(z)} - k \right| d\theta.
\]

Since $A_k(z)$ is a starlike analytic function, we have $\Re \left( \frac{zA'_k(z)}{A_k(z)} \right) > 0$ and it follows that $\frac{zA'_k(z)}{A_k(z)} - k$ is subordinate to $\frac{1 + z}{1 - z} - k = \frac{1}{1 - z} - k = \frac{1}{1 - z} + (1 + k) \frac{z}{1 - z}$. Therefore,
\[ L(r) \leq M(r) \sum_{k=1}^{\alpha-1} r^k \int_0^{2\pi} \left| \frac{1}{1 - z} + (1 + k) \frac{z}{1 - z} \right| d\theta. \]

We note that
\[ \left| \frac{1}{1 - z} \right| = \sum_{n=0}^{\infty} z^n \leq \sum_{n=0}^{\infty} r^n = \frac{1}{1 - r}, \]
and
\[ \left| \frac{1+z}{1-z} \right| \leq 1 + 2 \sum_{n=1}^{\infty} r^n = \frac{1 + r}{1 - r}. \]

Hence,
\[ L(r) \leq \frac{2 \pi M(r)}{1 - r} \sum_{k=1}^{\alpha-1} r^k (k - 1 + r(k + 1)) \]
\[ \leq \frac{2 \pi M(r)}{1 - r} \left[ (1 + r) \sum_{k=1}^{\alpha-1} kr^k - (1 - r) \sum_{k=1}^{\alpha-1} r^k \right] \]
\[ \leq \frac{2 \pi M(r)}{1 - r} \left[ (1 + r) \left( (\alpha - 1)r^\alpha - \alpha r^{\alpha-1} - 1 + r^{\alpha-1} \right) \right]. \]

□

In the special case of bi-analytic functions, we let $\alpha = 2$ and obtain by a simple calculation the following corollary:

**Corollary 4.** Let $F(z) = \overline{z}A(z)$ be a Bi-analytic functions such that $A(z)$ is a starlike analytic function. Suppose that $|A(z)| \leq M(r), 0 < r < 1$. Suppose that $|A_k(z)| \leq M(r), 0 < r < 1$. Let $L(r)$ denote the arclength of the curve $C_r$, where $C_r$ denote the image of the circle $|z| = r < 1$ under the function $w = F(z)$. Then
\[ L(r) \leq 4 \pi M(r) \frac{r^2}{1 - r}. \]

The function $F(z) = \frac{\overline{z}}{(1 - z)^2}$ shows the result is best possible.

We next consider the problem of minimizing the moments of order $p$ over a subclass of the class bi-analytic functions defined over the unit disc $U$.

**Theorem 5.** Let $f(z) = \overline{z}A(z)$ be a bi-analytic function defined on the unit disk $U$ such that $A(z)/z$ is starlike univalent analytic. Let $M_p(r, f)$ denote the moment of order $p, p \geq 0$. Then,
\[ M_p(r, f) \geq 2 \pi \frac{r^{3p+6}}{3p + 6}. \]

Equality holds if and only if $f(z) = \overline{z}z^2$.

**Proof.** Let $f(z) = \overline{z}A(z)$ be a bi-analytic function defined on the unit disk $U$ where $A(z) = z\phi(z), \phi$ is starlike univalent analytic. Let $M_p(r, f)$ denote the moment of order $p, p \geq 0$. Then,
\[ M_p(r, f) = \int_0^r \int_0^{2\pi} |f|^p \left( |f_z|^2 - |f_z|^2 \right) \rho d\theta d\rho \]
\[ = \int_0^r \int_0^{2\pi} r^{3p} \left| \frac{\phi(z)}{z} \right|^p \left( \left| \overline{z}\phi(z) + |z|^2 \phi'(z) \right|^2 - |z\phi(z)|^2 \right) \rho d\theta d\rho \]
\[ = \int_0^r \int_0^{2\pi} r^{3p} \left| \frac{\phi(z)}{z} \right|^p \left( \rho \left| \phi'(z) \right|^2 + 2\rho \left| \phi(z) \right|^2 \Re \frac{z\phi'(z)}{\phi(z)} \right) \rho d\theta d\rho. \]
Since \( \phi(z) \) is starlike, it follows that \( \Re \frac{z\phi'(z)}{\phi(z)} > 0 \).

Hence,

\[
M_p(r, f) \geq \int_0^r \int_0^{2\pi} \rho^{3p+5} \left| \frac{\phi(z)}{z} \right|^p \left| \phi'(z) \right|^2 \, d\theta \, d\rho.
\]

Writing

\[
\left( \frac{\phi(z)}{z} \right)^{p/2} \phi'(z) = 1 + \sum_{k=1}^{\infty} c_k z^k,
\]

we have

\[
\int_0^{2\pi} \rho^{3p+5} \left| \frac{\phi(z)}{z} \right|^p \left| \phi'(z) \right|^2 \, d\theta = 2\pi \rho^{3p+5} \left( 1 + \sum_{k=1}^{\infty} |c_k|^2 |z|^k \right).
\]

Therefore,

\[
M_p(r, f) \geq 2\pi \int_0^r \rho^{3p+5} \, d\rho = 2\pi \rho^{3p+6}.
\]

Equality holds if and only if \( \left( \frac{\phi(z)}{z} \right)^{p/2} \phi'(z) = 1 \) which gives \( \phi(z) = z \) and then \( f(z) = z^2 \).

**Remark 1.** If \( p = 0 \) in Theorem 5 then we have the problem of minimizing the area. Moreover, if \( p = 2 \), then we obtain the minimum of the moment of inertia.

We can in a similar way obtain the minimal area for a more general bi-analytic function.

**Theorem 6.** Let \( f(z) = zA(z) + B(z) \) be a bi-analytic function defined on the unit disk \( U \) such that \( A(z)/z \) is starlike univalent analytic and \( \Re(zA'B') \geq 0 \). Then, the minimal area is given by

\[
\int \int_{U_r} J_F \, dA \geq \frac{\pi r^6}{3},
\]

where \( U_r = \{ z : |z| \leq r \} \).

**Proof.** Proceeding as in the proof of the previous theorem we get

\[
\int \int_{U_r} J_F \, dA = \int_0^r \int_0^{2\pi} \left( |f_z|^2 - |f_z|^2 \right) \rho \, d\theta \, d\rho
\]

\[
= \int_0^r \int_0^{2\pi} (|zA'|^2 + |B'|^2 + 2\Re(zA'B') - |A|^2) \rho \, d\theta \, d\rho
\]

\[
\geq \int_0^r \int_0^{2\pi} \left( \rho^4 |\phi'(z)|^2 + 2\rho^2 |\phi(z)|^2 \Re \frac{z\phi'(z)}{\phi(z)} \right) \rho \, d\theta \, d\rho
\]

\[
\geq \int_0^r \int_0^{2\pi} \rho^5 |\phi'(z)|^2 \, d\theta \, d\rho,
\]
since $\Re \frac{z\phi'(z)}{\phi(z)} < 0$. Now, $\phi'(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$, hence $\int_{0}^{2\pi} |\phi'(z)|^2 d\theta \geq 1$. It follows that
\[
\int \int_{U_r} dA \geq \frac{\pi r^6}{3}.
\]

Finally in the next theorem, we establish a linkage between starlike analytic functions and Bi-analytic functions.

**Theorem 7.** Let $F(z) = \overline{z} A(z)$ be a Bi-analytic function, where $A \in H(U)$, with $A(0) = 0$ and $A'(0) = 1$. $A$ is starlike if and only if $\Phi(z) = z F(z)$ is starlike.

**Proof.** Suppose $A$ is starlike, then by direct calculations we have
\[
J_\Phi = |\Phi_z|^2 - |\Phi_{\overline{z}}|^2 = |F + z F_z|^2 - |z F_z|^2
\]
\[
= |F|^2 + |z F_z|^2 + 2 \Re z F_z \overline{F} - |z F_z|^2
\]
\[
= |z F_z|^2 + 2 |F|^2 \Re \frac{z F_z}{F}
\]
\[
= |z A'|^2 + 2 |F|^2 \Re \frac{z A'}{A}
\]
\[
> 0
\]
if $z \neq 0$. Moreover,
\[
\Re \frac{z \Phi_z - \overline{\Phi}_{\overline{z}}}{{\Phi}} = \Re \left( 1 + \frac{z F_z}{F} - \frac{\overline{z} F_{\overline{z}}}{F} \right) = \Re \left( \frac{z F_z}{F} \right) = \Re \left( \frac{z A'}{A} \right) > 0,
\]
since $\frac{z}{F} = 1$. The converse follows in a similar fashion.

\[\square\]

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