LOCALIZATIONS OF ONE-SIDED EXACT CATEGORIES

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ABSTRACT. One-sided exact categories were introduced by S. Bazzoni and S. Crivei by weakening the axioms of Quillen exact categories. In this paper, we consider quotients of one-sided exact categories by percolating subcategories. This generalizes the quotient of an abelian category by a Serre subcategory. When applied to exact categories, this framework extends earlier localization theories. As an application, we show that the compact and the discrete abelian groups form percolating subcategories of the locally compact abelian groups.

These quotients induce a Verdier localization on the level of the bounded derived categories. This will be investigated in a follow-up paper.

1. Introduction

Abelian categories were introduced as a framework for homological theories [15, 16]. Shortly after their introduction, the need for more general frameworks became apparent. The category of topological vector spaces and the category of filtered objects in an abelian category are not themselves abelian categories. For these examples, the notion of Quillen exact categories provides a natural framework.
A Quillen exact category is an additive category endowed with a set of chosen kernel-cokernel pairs, called conflations, satisfying some additional axioms. We refer to the kernel morphism of a conflation as an inflation and to the cokernel morphism as a deflation.

The original axioms of a Quillen exact category, as given in [24], can be partitioned into two dual sets: one set only referring to inflations, and one set only referring to deflations. In a one-sided exact category, we require only one of these sets (see [1, 28]). These categories still enjoy many useful homological properties.

Similar one-sided exact structures have occurred in several guises throughout the literature. The main source of examples is based on left or right almost abelian categories (see [26]). The axioms of a one-sided exact category are closely related to those of a Grothendieck pretopology (see [25]), to homological categories (see [3]), and to categories with fibrations (or cofibrations) and Waldhausen categories (see [34]). The latter allows for a $K$-space to be associated to a one-sided exact category.

Furthermore, the notion of one-sided exact categories has helped to understand two-sided exact categories, for example to find maximal exact structures on additive categories (see [12, 28], see also [32]).

Being a framework for homological algebra and $K$-theory, one can expect quotients of one-sided categories to play an important role. Such quotients are the focus of this paper.

Quotients of abelian categories occur naturally as localizations in algebra and geometry, and are well understood (see [14]): given an abelian category $C$ and a Serre subcategory $A$, the quotient category $C/A$ can be constructed using a calculus of fractions, namely $C/A = S^{-1}_A C$ where $S_A$ is the set of morphisms with kernels and cokernels in $A$.

When the category $C$ is not abelian, but merely exact, similar quotient constructions were given in [11, 30]. Here, the Serre subcategory $A$ is replaced by a subcategory that localizes $C$ (see [11, 4.0.35] or see [21, Appendix A]); in our terminology, such subcategories are called two-sided abelian percolating subcategories or by a left or right special filtering subcategory (see [30]). In both cases, the quotient categories satisfy the expected $K$-theoretic properties.

However, these frameworks do not accommodate some natural examples. It was observed in [5, example 4] that, in the category LCA of locally compact abelian groups, neither the subcategory LCA$_C$ of compact abelian groups nor the subcategory LCA$_D$ of discrete abelian groups satisfy the $s$-filtering condition used in [30], nor do these examples satisfy the conditions in [11]. This indicates a need for a more general localization theory.

To provide a localization theory that encompasses the previous example, we extend these quotients to the one-sided exact setting. It is worth noting that even if the initial category $C$ is Quillen exact instead of merely one-sided exact, our results are not recovered by the aforementioned localization theories. In this way, theorem 1.1 below is a complement to the localization theories in [11, 30].

Thus, let $C$ be a deflation-exact category. Our replacement of the Serre subcategory $A \subseteq C$ is a right percolating subcategory (see definition 3.1 for a precise formulation). First, consider any conflation $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $C$. If the quotient functor $Q: C \rightarrow C/A$ is to be exact (thus, mapping conflations of $C$ to kernel-cokernel pairs in $C/A$), it is clear that the following conditions need to hold: if $X \in A$, then $Q(f)$ is invertible, and if $Z \in A$, then $Q(g)$ is invertible. Thus, let $S_A$ be the set of morphisms (closed under composition) containing the inflations with cokernels in $A$ and the deflations with kernels in $A$. We refer to the morphisms in $S_A$ as weak $A^{-1}$-isomorphisms or just weak isomorphisms if there is no fear of confusion. The following theorem summarizes our main result (see theorem 4.7 in the text).

**Theorem 1.1.** Let $C$ be a deflation-exact category and let $A \subseteq C$ be a right percolating subcategory. There is an exact functor $Q: C \rightarrow C/A$ between deflation-exact categories such that $Q(A) = 0$ and which is universal with respect to this property in the following sense: any exact functor $F: C \rightarrow E$ between deflation-exact categories with $F(A) = 0$ factors through $Q$.

Furthermore, the corresponding set of weak isomorphisms $S_A$ is a right multiplicative system and the category $C/A$ is equivalent to the corresponding localization $S^{-1}_A C$.

For a left or right percolating subcategory $A$ of an exact category $C$, theorem 1.1 shows that the quotient $C/A$ is one-sided exact, but in general not two-sided exact. An explicit example is given in 7.6, based on the theory of gliders ([9, 10]). Hence, theorem 1.1 can be used to construct new one-sided exact categories.

It is, however, possible to stay within the framework of exact categories. We provide the following adjustment of theorem 1.1 (see corollary 4.9 in the text).
Corollary 1.2. Let $A$ be a right percolating subcategory of an exact category $C$. There is an exact functor $Q: C \to \overline{C/A}$ between exact categories such that $Q(A) = 0$ and which satisfies the following universal property: any exact functor $F: C \to E$ between exact categories with $F(A) = 0$ factors uniquely through $Q$.

Note that the category $\overline{C/A}$ need not be given by a calculus of fractions as in theorem 1.1. The relation between the usual quotient $C/A$ and the quotient $\overline{C/A}$ is given in [25]: the latter is the exact hull of the former.

In some examples of interest, the percolating subcategory $A \subseteq C$ is an abelian category. This is the case, for example, for the localizations considered in [11] and the aforementioned subcategories $\text{LCA}_C$ and $\text{LCA}_D$ of the category of locally compact abelian groups $\text{LCA}$. Under this additional assumption, the axioms of a percolating subcategory simplify and the set of weak isomorphisms satisfies some additional properties such as saturation (proposition 5.15) and the 2-out-of-3 property (proposition 5.13). Furthermore, all weak isomorphisms are then admissible (proposition 5.10).

So far we have mentioned quotients of exact categories by four different types of subcategories: two-sided abelian percolating subcategories [11], special filtering subcategories [30], and (abelian) percolating subcategories in one-sided exact categories introduced in this paper. The relations between these concepts are given in figure 1: we draw an arrow $a \to b$ if a subcategory of type $a$ is automatically a subcategory of type $b$. The only non-obvious statement is that every special filtering subcategory is percolating; this is proven in proposition 7.4.

In upcoming work, we consider the derived category of a one-sided exact category and show that the localization sequence $A \to C \to \overline{C/A}$ given by a percolating subcategory in a one-sided exact category, yields a Verdier localization $D^b(C) \to D^b(\overline{C/A})$ as in [30]. Moreover, we will provide an alternative construction of the exact hull $\overline{C}$ of a one-sided exact category. The derived categories $D^b(C)$ and $D^b(\overline{C})$ are triangle-equivalent so that both categories contain the same homological information.

Finally, the framework of derived categories allows us to use [17] to show that the quotients $\text{LCA} / \text{LCA}_D$ and $\text{LCA} / \text{LCA}_C$ are two-sided exact.

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2. Preliminaries

Throughout this paper, we will assume that all categories are small.

2.1 Properties of pullbacks and pushouts. For easy reference, it will be convenient to collect some properties of pullbacks and pushouts. We start by recalling the Pullback Lemma.

**Proposition 2.1 (Pullback lemma).** Consider the following commutative diagram in any category:

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X' & \longrightarrow & Y'
\end{array}
\begin{array}{ccc}
\longrightarrow & & \longrightarrow \\
& & \\
& & \longrightarrow \\
Z & \longrightarrow & Z'
\end{array}
$$
Assume that the right square is a pullback. The left square is a pullback if and only if the outer rectangle is a pullback.

The following statement is [22, proposition I.13.2] together with its dual.

**Proposition 2.2.** Let $C$ be any pointed category.

1. Consider a diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow{h} & & \downarrow{g} \\
X & \xrightarrow{f} & Y \\
\end{array}
\]

where $f$ is the kernel of $g$. The left-hand side can be completed to a pullback square if and only if $f'$ is the kernel of $gh$.

2. Consider a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{h} & & \downarrow{g} \\
Y' & \xrightarrow{g'} & Z' \\
\end{array}
\]

where $g$ is the cokernel of $f$. The right-hand side can be completed to a pushout square if and only if $g'$ is the cokernel of $hf$.

2.2. **Right exact categories.** We now recall the notion of a one-sided exact category as introduced by [1, 25, 27]. In the remainder of the text we follow the conventions of Rosenberg [25], that is, one-sided exact categories containing all axioms referring to the deflation-side are called right exact categories. This convention is opposite to the terminology used by [1].

**Definition 2.3.** Let $C$ be an additive category. A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ in $C$ where $f = \ker g$ and $g = \coker f$ is called a kernel-cokernel pair.

A *conflation category* $C$ is an additive category $C$ together with a chosen class of kernel-cokernel pairs, closed under isomorphisms, called confluences. A map that occurs as the kernel (or the cokernel) in a confluence is called an inflation (or a deflation). Inflations will often be denoted by $\rightarrowtail$ and deflations by $\rightarrowhead$.

A map $f : X \rightarrow Y$ is called an admissible morphism if it admits a deflation-inflation factorization, i.e. $f$ factors as $X \rightarrow Z \rightarrowtail Y$. The set of admissible morphisms in $C$ is denoted by $\text{Adm}(C)$.

Let $C$ and $D$ be conflation categories. An additive functor $F : C \rightarrow D$ is called exact if confluences in $C$ are mapped to confluences in $D$.

**Definition 2.4.** A right exact category or a deflation-exact category $C$ is a conflation category satisfying the following axioms:

- **R0** The identity morphism $1_0 : 0 \rightarrow 0$ is a deflation.
- **R1** The composition of two deflations is again a deflation.
- **R2** The pullback of a deflation along any morphism exists and is again a deflation, i.e. $\begin{array}{ccc} X & \xrightarrow{f} & Y \\
\downarrow{h} & & \downarrow{g} \\
Z & \xrightarrow{f'} & W \end{array}$

Dually, we call an additive category $C$ left exact or inflation-exact if the opposite category $C^{\text{op}}$ is right exact. Explicitly, a left exact category is a conflation category such that the inflations satisfy the following axioms:

- **L0** The identity morphism $1_0 : 0 \rightarrow 0$ is an inflation.
- **L1** The composition of two inflations is again an inflation.
- **L2** The pushout of an inflation along any morphism exists and is again an inflation, i.e. $\begin{array}{ccc} X & \xrightarrow{f} & Y \\
\downarrow{h} & & \downarrow{g} \\
Z & \xrightarrow{g'} & W \end{array}$

**Definition 2.5.** Let $C$ be a conflation category. In addition to the properties listed in definition 2.4, we will also consider the following axioms:
R0* For any $A \in \text{Ob}(C)$, $A \to 0$ is a deflation.
R3 If $i: A \to B$ and $p: B \to C$ are morphisms in $C$ such that $p$ has a kernel and $pi$ is a deflation, then $p$ is a deflation.
L0* For any $A \in \text{Ob}(C)$, $0 \to A$ is an inflation.
L3 If $i: A \to B$ and $p: B \to C$ are morphisms in $C$ such that $i$ has a cokernel and $pi$ is an inflation, then $i$ is an inflation.

A right exact category satisfying R3 is called strongly right exact or strongly deflation-exact. Dually, a left exact category satisfying L3 is called strongly left exact or strongly inflation-exact.

Remark 2.6.
(1) An exact category in the sense of Quillen (see [24]) is a conflation category $C$ satisfying axioms R0–R3, L0–L3. In [19, Appendix A], Keller shows that axioms R0, R1, R2, and L2 suffice to define an exact category.
(2) Axioms R3 and L3 are sometimes referred to as Quillen’s obscure axioms (see [8, 33]).
(3) In [28], the notions of one-sided exact categories includes the obscure.

Remark 2.7.
(1) In a weakly idempotent complete right exact category, axiom R3 is equivalent to the following statement: if $gf$ is a deflation, then $g$ is a deflation. A proof of this fact can be found in [8, proposition 7.6]. (Compare also to [1, proposition 6.4].)
(2) A right exact category $C$ satisfies axiom R0* if and only if every split kernel-cokernel pair is a conflation.

Lemma 2.8. Let $C$ be a right exact category. Then:
(1) Every isomorphism is a deflation.
(2) If $C$ is strongly right exact, then $C$ satisfies R0*.
(3) Every inflation is a monomorphism. An inflation which is an epimorphism is an isomorphism.
(4) Every deflation is an epimorphism. A deflation which is a monomorphism is an isomorphism.

Proof. (1) Let $f: X \to Y$ be an isomorphism. One easily checks that

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

is a pullback diagram. By R0 and R2, we know that $f: X \to Y$ is a deflation.

(2) Since $1_A: A \to A$ is the kernel of $p: A \to 0$ and the composition of $0 \xrightarrow{i} A \xrightarrow{p} 0$ is a deflation by R0, it follows that $p$ is a deflation. This establishes R0*.

(3) Every inflation is a kernel and kernels are monic. If an inflation is an epimorphism, then the cokernel is zero. As an inflation is the kernel of its cokernel, we infer that the inflation is an isomorphism.

(4) Similar. □

Proposition 2.9. Let $C$ be a right exact category.
(1) For a commutative square

$$
\begin{array}{ccc}
A & \xrightarrow{i} & B \\
\downarrow{f} & & \downarrow{g} \\
A' & \xrightarrow{i'} & B'
\end{array}
$$

where the horizontal arrows are inflations, the following statements are equivalent:
(a) the square is a pushout,
(b) the square can be extended to a diagram

$$
\begin{array}{ccc}
A & \xrightarrow{i} & B & \xrightarrow{p} & C \\
\downarrow{f} & & \downarrow{g} & & \downarrow{h} \\
A' & \xrightarrow{i'} & B' & \xrightarrow{p'} & C
\end{array}
$$

where the rows are conflations.
In this case, the commutative square is also a pullback.

(2) For a commutative square

\[
\begin{array}{ccc}
B & \xrightarrow{p} & C \\
\downarrow{f} & & \downarrow{g} \\
B' & \xrightarrow{p'} & C'
\end{array}
\]

where the horizontal arrows are deflations, the following statements are equivalent:

(a) the square is a pullback,
(b) the square can be extended to a diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B & \xrightarrow{p} & C \\
\downarrow{A'} & & \downarrow{f} & & \downarrow{g} \\
B' & \xrightarrow{p'} & C'
\end{array}
\]

where the rows are conflations.

In this case, the commutative square is also a pushout.

Proof. (1) The implication (1a) ⇒ (1b) is straightforward to prove. For the reverse implication, one can verify that the proof of [8, proposition 2.12] still holds. If (1b) holds, then proposition 2.2 shows that the given commutative square is a pullback.

(2) The first equivalence is [1, proposition 5.4]. The last statement again follows from proposition 2.2. □

Proposition 2.10. Let \( \mathcal{C} \) be a right exact category. Every morphism \((f, g, h)\) between conflations \(X \xrightarrow{i} Y \rightarrow Z\) and \(X' \xrightarrow{i'} Y' \rightarrow Z'\) factors through some conflation \(X' \rightarrow P \rightarrow Z\):

\[
\begin{array}{ccc}
X & \xrightarrow{i} & Y & \xrightarrow{p} & Z \\
\downarrow{f} & & \downarrow{g} & & \downarrow{h} \\
X' & \xrightarrow{i'} & Y' & \xrightarrow{p'} & Z'
\end{array}
\]

such that the upper-left and lower-right squares are both pullbacks and pushouts.

Proof. The factorization property is [1, proposition 5.2]. The statements about the pushouts and pullbacks follow from proposition 2.9. □

Lemma 2.11. Let \( \mathcal{C} \) be a right exact category. The pullback of an inflation \( f \) along a deflation is an inflation \( f' \).

Proof. Let \( f: X \rightarrow Z \) be an inflation and \( g: Y \rightarrow Z \) be a deflation. Consider the commutative diagram

\[
\begin{array}{ccc}
P & \xrightarrow{f'} & Y \\
\downarrow{g} & & \downarrow{g} \\
X & \xrightarrow{f} & Z & \rightarrow \text{coker}(f)
\end{array}
\]

where the square is a pullback diagram and the bottom row is a conflation. It follows from proposition 2.2(1) that \( f' \) is the kernel of the composition \( Y \rightarrow Z \rightarrow \text{coker}(f) \) and hence an inflation by axiom R1. □

2.3. Localizations and right calculus of fractions. The material of this section is based on [13, 18].

Definition 2.12. Let \( \mathcal{C} \) be any category and let \( S \subseteq \text{Mor} \mathcal{C} \) be any subset of morphisms of \( \mathcal{C} \). The localization of \( \mathcal{C} \) with respect to \( S \) is a universal functor \( Q: \mathcal{C} \rightarrow S^{-1} \mathcal{C} \) such that \( Q(s) \) is invertible, for all \( s \in S \).
Remark 2.13. By universality, we mean that any functor $F: \mathcal{C} \to \mathcal{D}$ such that every morphism in $S$ becomes invertible in $\mathcal{D}$ factors uniquely through $Q: \mathcal{C} \to S^{-1}\mathcal{C}$. Put differently, for every category $\mathcal{D}$, the functor $(Q \circ -): \text{Fun}(S^{-1}\mathcal{C}, \mathcal{D}) \to \text{Fun}(\mathcal{C}, \mathcal{D})$ induces an isomorphism between $\text{Fun}(S^{-1}\mathcal{C}, \mathcal{D})$ and the full subcategory of $\text{Fun}(\mathcal{C}, \mathcal{D})$ consisting of those functors $F: \mathcal{C} \to \mathcal{D}$ which make every $s \in S$ invertible.

Remark 2.14. Since all the categories in this paper are small, localizations always exist.

In this paper, we often consider localizations with respect to so-called right multiplicative systems.

Definition 2.15. Let $\mathcal{C}$ be a category and let $S$ be a set of arrows. Then $S$ is called a right multiplicative system if it has the following properties:

RMS1 For every object $A$ of $\mathcal{C}$ the identity $1_A$ is contained in $S$. Composition of composable arrows in $S$ is again in $S$.

RMS2 Every solid diagram

\[
\begin{array}{ccc}
X & \xrightarrow{g} & Y \\
\downarrow{t} & & \downarrow{s} \\
Z & \xrightarrow{f} & W
\end{array}
\]

with $s \in S$ can be completed to a commutative square with $t \in S$.

RMS3 For every pair of morphisms $f, g: X \to Y$ and $s \in S$ with source $Y$ such that $s \circ f = s \circ g$ there exists a $t \in S$ with target $X$ such that $f \circ t = g \circ t$.

Often arrows in $S$ will be endowed with $\sim$.

For localizations with respect to a right multiplicative system, we have the following description of the localization.

Construction 2.16. Let $\mathcal{C}$ be a category and $S$ a right multiplicative system in $\mathcal{C}$. We define a category $S^{-1}\mathcal{C}$ as follows:

1. We set $\text{Ob}(S^{-1}\mathcal{C}) = \text{Ob}(\mathcal{C})$.
2. Let $f_1: X_1 \to Y, s_1: X_1 \to X, f_2: X_2 \to Y, s_2: X_2 \to X$ be morphisms in $\mathcal{C}$ with $s_1, s_2 \in S$.
   We call the pairs $(f_1, s_1), (f_2, s_2) \in (\text{Mor}\mathcal{C}) \times S$ equivalent (denoted by $(f_1, s_1) \sim (f_2, s_2)$) if there exists a third pair $(f_3: X_3 \to Y, s_3: X_3 \to X) \in (\text{Mor}\mathcal{C}) \times S$ and morphisms $u: X_3 \to X_1, v: X_3 \to X_2$ such that

\[
\begin{array}{ccc}
X & \xrightarrow{s_1} & Y \\
\downarrow{u} & & \downarrow{f_1} \\
X_1 & & \\
\downarrow{s_2} & & \downarrow{f_2} \\
X_2 & \xrightarrow{f_3} & Y
\end{array}
\]

is a commutative diagram.

3. $\text{hom}_{S^{-1}\mathcal{C}}(X, Y) = \{ (f, s) \mid f \in \text{hom}_\mathcal{C}(X', Y), s: X' \to X \text{ with } s \in S \} / \sim$

4. The composition of $(f: X' \to Y, s: X' \to X)$ and $(g: Y' \to Z, t: Y' \to Y)$ is given by $(g \circ h: X'' \to Z, s \circ u: X'' \to X)$ where $h$ and $u$ are chosen to fit in a commutative diagram

\[
\begin{array}{ccc}
X'' & \xrightarrow{h} & Y' \\
\downarrow{u} & & \downarrow{t} \\
X' & \xrightarrow{f} & Y
\end{array}
\]

which exists by RMS2.

Proposition 2.17. Let $\mathcal{C}$ be a category and $S$ a right multiplicative system in $\mathcal{C}$.

1. The assignment $X \mapsto X$ and $(f: X \to Y) \mapsto (f: X \to Y, 1_X: X \to X)$ defines a functor $Q: \mathcal{C} \to S^{-1}\mathcal{C}$ called the localization functor. The functor $Q$ is a localization of $\mathcal{C}$ with respect to the set $S$ as in definition 2.12.

2. For any $s \in S$, the map $Q(s)$ is an isomorphism.

3. The localization functor commutes with finite limits.
If $C$ is an additive category, then $S^{-1}C$ is an additive category and the localization functor $Q$ is an additive functor.

Remark 2.18. It follows that if $C$ is an additive category and $S$ a right multiplicative system the functor $Q$ preserves kernels and pullbacks.

Definition 2.19. Let $C$ be any category and let $S \subseteq \text{Mor} C$ be any subset.

1. We say that $S$ satisfies the 2-out-of-3 property if, for any two composable morphisms $f, g \in \text{Mor} C$, we have that if two of $f, g, fg$ are in $S$, then so is the third.

2. Let $Q: C \to S^{-1}C$ be the localization of $C$ with respect to $S$. We say that $S$ is saturated if $S = \{ f \in \text{Mor} C \mid Q(f) \text{ is invertible} \}$.

3. Percolating subcategories

Let $C$ be a right exact category. In this section, we define the notion of a percolating subcategory of $C$. To place this notion in context: if the category $C$ is abelian, a subcategory $A \subseteq C$ is percolating if and only if it is a Serre subcategory; if $C$ is an exact category, then the notion of a percolating subcategory is weaker than the notion of a right s-filtering subcategory in [30].

3.1. Definitions and basic properties. We start by defining percolating subcategories. As this definition does not refer to the right exact structure of $C$, we formulate the definition for a more general conflation category.

Definition 3.1. Let $C$ be a conflation category. A non-empty full subcategory $A$ of $C$ is called a right percolating subcategory of $C$ if the following axioms are satisfied:

- **P1** $A$ is a Serre subcategory, meaning:
  - If $A' \hookrightarrow A \twoheadrightarrow A''$ is a conflation in $C$, then $A \in \text{Ob}(A)$ if and only if $A', A'' \in \text{Ob}(A)$.
  - **P2** For all morphisms $C \rightarrow A$ with $C \in \text{Ob}(C)$ and $A \in \text{Ob}(A)$, there exists a commutative diagram
    \[
    \begin{array}{ccc}
    A' & \rightarrow & A \\
    \downarrow & & \downarrow \\
    C & \rightarrow & A
    \end{array}
    \]
  - with $A' \in \text{Ob}(A)$ and where $C \twoheadrightarrow A'$ is a deflation.

- **P3** If $a: C \twoheadrightarrow D$ is an inflation and $b: C \rightarrow A$ is a deflation with $A \in \text{Ob}(A)$, then the pushout of $a$ along $b$ exists and yields an inflation and a deflation, i.e.
  \[
  \begin{array}{ccc}
  C & \twoheadrightarrow & D \\
  \downarrow & & \downarrow \\
  A & \rightarrow & P
  \end{array}
  \]

- **P4** For all inflations $i: A \rightarrow X$ and all deflations $p: X \rightarrow B$ with $A, B \in \text{Ob}(A)$, there exists objects $A', B' \in A$ such that there exists a commutative diagram as below:
  \[
  \begin{array}{ccc}
  A' & \rightarrow & X \\
  \downarrow & & \downarrow \\
  A & \rightarrow & B
  \end{array}
  \]

By dualizing the above axioms one obtains a similar notion of a left percolating subcategory.

Definition 3.2.

1. Following the conventions by [30], a non-empty full subcategory $A$ of a conflation category $C$ satisfying axioms P1 and P2 is called right filtering.

2. If $A$ is a right filtering subcategory of $C$ such that the map $A' \rightarrow A$ in axiom P2 can be chosen as a monic map, we will call $A$ a strongly right filtering subcategory.

3. A right percolating subcategory which is also strongly right filtering will be abbreviated to a strongly right percolating subcategory.

The notions of a left filtering, strongly left filtering, and strongly left percolating subcategory are defined dually.

Remark 3.3.
(1) If $\mathcal{C}$ is an exact category, then a right s-filtering subcategory $\mathcal{A}$ of $\mathcal{C}$ is right percolating in $\mathcal{C}$ (see proposition 7.4 below).

(2) If $\mathcal{C}$ is an exact category, then $\mathbf{P_3}$ automatically holds (see for example the dual of [8, proposition 2.15]).

(3) If $\mathcal{A}$ is a right percolating subcategory of a right exact category $\mathcal{C}$, then $\mathcal{A}$ is a right exact category (with the conflations induced by the conflations in $\mathcal{C}$).

We start by showing that an apparent strengthening of axiom $\mathbf{P_4}$ holds.

**Proposition 3.4.** Let $\mathcal{C}$ be a right exact category and $\mathcal{A}$ a right percolating subcategory. For all inflations $i: X \rightarrowtail Y$ and all deflations $p: Y \twoheadrightarrow B$ with $B \in \mathcal{A}$, there exists a commutative diagram:

$$
\begin{array}{c}
X \\
\downarrow k \\
A \\
\end{array} 
\begin{array}{c}
Y \\
\downarrow p \\
B
\end{array} 
\begin{array}{c}
\downarrow l \\
B'' \\
\end{array}
$$

with $A, B'' \in \mathcal{A}$.

**Proof.** Choose $i$ and $p$ as above. By axiom $\mathbf{P_2}$ there exists a commutative diagram:

$$
\begin{array}{c}
X \\
\downarrow p' \\
A \\
\end{array} 
\begin{array}{c}
Y \\
\downarrow p \\
B
\end{array}
$$

with $A \in \mathcal{A}$. By axiom $\mathbf{P_3}$, we obtain a commutative diagram (where the square is a pushout):

$$
\begin{array}{c}
X \\
\downarrow p' \\
A \\
\end{array} 
\begin{array}{c}
Y \\
\downarrow p \\
\alpha \\
\downarrow \beta \\
Q \\
\downarrow \gamma \\
B
\end{array}
$$

Applying axiom $\mathbf{P_2}$ to $\gamma: Q \rightarrow B$ yields:

$$
\begin{array}{c}
X \\
\downarrow p' \\
A \\
\end{array} 
\begin{array}{c}
Y \\
\downarrow p \\
\alpha \\
\downarrow \beta \\
Q \\
\downarrow \delta \\
B' \\
\downarrow \gamma \\
B
\end{array}
$$

where $\gamma = \varepsilon \circ \delta$ and $B' \in \mathcal{A}$. Applying axiom $\mathbf{P_4}$ to $\delta \circ \beta$, we find that there exists a commutative diagram:

$$
\begin{array}{c}
X \\
\downarrow p' \\
A \\
\end{array} 
\begin{array}{c}
Y \\
\downarrow p \\
\alpha \\
\downarrow \beta \\
Q \\
\downarrow \delta \\
A' \\
\end{array} 
\begin{array}{c}
B' \\
\downarrow \varepsilon \\
B
\end{array}
$$

such that $A', B'' \in \mathcal{A}$. The desired result now follows from axiom $\mathbf{R_1}$. \qed
3.2. **Homological consequences of axiom P3.** Throughout this subsection, let $\mathcal{C}$ be a right exact category and $\mathcal{A}$ a non-empty full subcategory of $\mathcal{C}$ satisfying axiom P3. We show that the existence of such a subcategory yields a weak version of axiom R3 (see proposition 3.5 below), and we show that a weak version of the $3 \times 3$-lemma holds (see proposition 3.6 below). If $\mathcal{C}$ is a strongly right exact category, these two properties are automatically satisfied (see [1]).

**Proposition 3.5.** Let $g: Y \to Z$ be a map such that $g$ has a kernel belonging to $\mathcal{A}$ and such that there exists a deflation $f: X \twoheadrightarrow Y$ such that $gf$ is also a deflation. Then $g$ is a deflation.

**Proof.** Proposition 2.2 yields the following commutative diagram

\[
\begin{array}{c}
\ker(gf) \xrightarrow{k'} X \xrightarrow{f} Z \\
\downarrow f' \downarrow \downarrow \\
\ker(g) \xrightarrow{k} Y \xrightarrow{g} Z
\end{array}
\]

where the left-hand square is a pullback. Axiom R2 implies that $f'$ is a deflation. Proposition 2.9 yields the existence of the following commutative diagram

\[
\begin{array}{c}
K \xrightarrow{i} K \\
\downarrow l \downarrow \downarrow \\
\ker(gf) \xrightarrow{k'} X \\
\downarrow f' \downarrow \downarrow \\
\ker(g) \xrightarrow{k} Y
\end{array}
\]

where the columns are conflations. By proposition 2.2(2), we know that the lower square is a pushout. Since $\ker(g) \in \text{Ob}(\mathcal{A})$, axiom P3 implies that $k: \ker(g) \twoheadrightarrow Y$ is an inflation. Proposition 2.9 implies that $g$ is the cokernel of the inflation $k$, and hence $g$ is a deflation. \qed

The next proposition is a weak version of the $3 \times 3$-lemma.

**Proposition 3.6.** Consider a commutative diagram

\[
\begin{array}{c}
X \xrightarrow{f} Y \xrightarrow{g} Z \\
\downarrow \downarrow \downarrow \\
X' \xrightarrow{f'} Y' \xrightarrow{g'} Z'
\end{array}
\]

where the rows are conflations and the vertical arrows are deflations. If $X' \in \text{Ob}(\mathcal{A})$, then the above diagram can be completed to a commutative diagram

\[
\begin{array}{c}
X'' \xrightarrow{f''} Y'' \xrightarrow{g''} Z'' \\
\downarrow \downarrow \downarrow \\
X' \xrightarrow{f'} Y' \xrightarrow{g'} Z'
\end{array}
\]

where the rows and the columns are conflations. Moreover, the upper left square is a pullback and the lower right square is a pushout.

**Proof.** By proposition 2.10, the diagram can be extended to a commutative diagram

\[
\begin{array}{c}
X \xrightarrow{i} Y \xrightarrow{p} Z \\
\downarrow f \downarrow \downarrow B \\
X' \xrightarrow{p'} Z'
\end{array}
\]

where $i$ is a monomorphism, $p$ is a epimorphism, and $i'$ is a monomorphism.
such that the square $D$ is a pullback and square $A$ is both a pullback and a pushout. By axioms $P3$ and $R2$ the maps $Y \to P$ and $P \to Y'$ are deflations. Applying proposition 2.9 yields the following commutative diagrams:

\[
\begin{array}{c}
X'' \xrightarrow{E} Y \xrightarrow{A} P \xrightarrow{F} Z''
\end{array}
\]

where the rows and columns are conflations. Starting from the conflations $X'' \rightarrow Y \rightarrow P$ and $Z'' \rightarrow P \rightarrow Y'$, we construct the commutative diagram

\[
\begin{array}{c}
X'' \xrightarrow{G} Y' \xrightarrow{I} P \xrightarrow{H} Z''
\end{array}
\]

where the rows and columns are conflations. Here, the dotted morphism $Y'' \longrightarrow Z''$ is chosen such that the square $H$ is a pullback (see proposition 2.2, the chosen morphism is automatically a deflation by $R2$). The square $G$ is given by proposition 2.9.

Putting the commutative squares together, we obtain the commutative diagram:

\[
\begin{array}{c}
X'' \xrightarrow{G} Y' \xrightarrow{I} P \xrightarrow{H} Z''
\end{array}
\]

where the right-most column composes to the morphism $Z'' \rightarrow Z$ by the square $F$. As $X'' \rightarrow X$, $Y'' \rightarrow Y$, and $Z'' \rightarrow Z$ have been constructed as kernels of $X \rightarrow X'$, $Y \rightarrow Y'$, and $Z \rightarrow Z'$, respectively, we obtain the $3 \times 3$-diagram in the statement of the proposition.

Finally, consider the $3 \times 3$-diagram in the statement of the proposition. It follows from proposition 2.2 that the upper left square is a pullback and the lower right square is a pushout. □

**Lemma 3.7.** Let $C$ be a right exact category and let $A$ be a full subcategory satisfying axiom $P3$. Consider a commutative diagram

\[
\begin{array}{c}
X \xrightarrow{i} Y \xrightarrow{p} Z
\end{array}
\]

where the rows are conflations. If $A, B, C \in A$, then $\beta$ is a deflation.

**Proof.** Proposition 2.10 gives the following diagram:

\[
\begin{array}{c}
X \xrightarrow{i} Y \xrightarrow{p} Z
\end{array}
\]

where the right-most column composes to the morphism $Z'' \rightarrow Z$ by the square $F$. As $X'' \rightarrow X$, $Y'' \rightarrow Y$, and $Z'' \rightarrow Z$ have been constructed as kernels of $X \rightarrow X'$, $Y \rightarrow Y'$, and $Z \rightarrow Z'$, respectively, we obtain the $3 \times 3$-diagram in the statement of the proposition.

Finally, consider the $3 \times 3$-diagram in the statement of the proposition. It follows from proposition 2.2 that the upper left square is a pullback and the lower right square is a pushout. □
where the rows are conflations, and the upper-left and the lower-right squares are both pullbacks and pushouts. The middle column composes to $\beta: Y \to B$. It follows from P3 that $Y \to Q$ is a deflation and from R2 that $Q \to B$ is a deflation. By axiom R1, $\beta$ is a deflation. □

3.3. Weak isomorphisms form a right multiplicative system. Let $F: \mathcal{C} \to \mathcal{D}$ be an exact functor between conflation categories. Let $\mathcal{A} \subseteq \mathcal{C}$ be a full subcategory and assume that $F(\mathcal{A}) = 0$. It is clear that, for any conflation $X \xrightarrow{f} Y \xrightarrow{g} Z$, we have that $X \in \mathcal{A}$ implies that $F(g)$ is an isomorphism. Likewise, $Z \in \mathcal{A}$ implies that $F(f)$ is an isomorphism. This observation motivates the following definition (the terminology is based on [11, 30]).

Definition 3.8. Let $\mathcal{C}$ be a conflation category and let $\mathcal{A}$ be a non-empty full subcategory of $\mathcal{C}$.

1. An inflation $f: X \to Y$ in $\mathcal{C}$ is called an $A^{-1}$-inflation if its cokernel belongs to $\mathcal{A}$.
2. A deflation $f: X \to Y$ in $\mathcal{C}$ is called a $A^{-1}$-deflation if its kernel belongs to $\mathcal{A}$.
3. A morphism $f: X \to Y$ is called a weak $A^{-1}$-isomorphism (or simply a weak isomorphism if $\mathcal{A}$ is implied) if it is a finite composition of $A^{-1}$-inflations and $A^{-1}$-deflations. We often endow weak isomorphisms with “$\sim$”.

The set of weak isomorphisms is denoted by $S_\mathcal{A}$. Given a weak isomorphism $f$, the composition length of $f$ is defined as the smallest natural number $n$ such that $f$ can be written as a composition of $n$ $A^{-1}$-inflations or $A^{-1}$-deflations.

The following propositions show that, under mild assumptions, $A^{-1}$-inflations and $A^{-1}$-deflations compose to $A^{-1}$-inflations and $A^{-1}$-deflations, respectively. Under these conditions, a weak isomorphism can be written as the composition of a finite alternating sequence of $A^{-1}$-inflations and $A^{-1}$-deflations.

Proposition 3.9. Let $\mathcal{C}$ be a right exact category and let $\mathcal{A}$ be a non-empty full subcategory satisfying axiom P3. Consider the composable inflations $X \overset{f}{\to} Y \overset{g}{\to} Z$. If $f$ is an $A^{-1}$-inflation, then $gf$ is an inflation.

Proof. Let $q: Y \to \text{coker}(f)$ be the cokernel of $f$. By axiom P3 we can take the pushout of $q$ along $g$:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
Y & \xrightarrow{g} & Z \\
\downarrow q & & \downarrow q' \\
\text{coker}(k) & \xrightarrow{q'} & P
\end{array}
\]

which is also a pullback square by proposition 2.9. Proposition 2.2 shows that $gf: X \to Z$ is the kernel of the deflation $q': Z \to P$, and hence an inflation. □

Proposition 3.10. Let $\mathcal{C}$ be a right exact category and let $\mathcal{A}$ be a non-empty full subcategory.

1. If $\mathcal{A}$ satisfies axiom P1, then the composition of $A^{-1}$-deflations is again an $A^{-1}$-deflation,
2. If $\mathcal{A}$ satisfies axiom P1 and P3, then the composition of $A^{-1}$-inflations is again an $A^{-1}$-inflation.

Proof. (1) Let $U \overset{a}{\to} V \overset{b}{\to} W$ be $A^{-1}$-deflations. Axiom R1 shows that $ba: U \to W$ is a deflation. Propositions 2.2 and 2.9 now yield the following commutative diagram:

\[
\begin{array}{ccc}
\ker(a') & \xrightarrow{k_a'} & \ker(a) \\
\downarrow k'_a & & \downarrow k_a \\
P & \xrightarrow{k_{ab}} & U \\
\downarrow q' & & \downarrow b \\
\ker(b) & \xrightarrow{k_b} & V \\
\downarrow & & \downarrow b \\
& & W
\end{array}
\]

where the rows and columns are conflations, and the lower-left square is a pullback. As $\ker(a)$, $\ker(b) \in \mathcal{A}$, axiom P1 implies that $P \in \mathcal{A}$. Proposition 2.2(1) implies that $P = \ker(ba)$. It follows that $ba \in S_\mathcal{A}$, as required.
(2) Let $a$ and $b$ be composable $A^{-1}$-inflations. It follows from proposition 3.9 that the composition of $A^{-1}$-inflations is again an inflation. By propositions 2.2 and 2.9, we obtain the following diagram

\[
\begin{array}{ccc}
U & \xrightarrow{a} & V \\
\downarrow & & \downarrow \xrightarrow{c_a} \coker(a) \\
W & \xrightarrow{b} & P \\
\downarrow \xrightarrow{c_b} & & \downarrow \xrightarrow{c_b'} \\
\coker(b) & \xrightarrow{coker(b')} & \coker(b')
\end{array}
\]

where the rows and columns are conflations, and the upper-right square is a pushout. As $\coker(a)$, $\coker(b) \in A$, axiom $\textbf{P1}$ implies that $P \in A$. Hence, $ba$ is an $A^{-1}$-inflation. \hfill \Box

The following proposition is a straightforward strengthening of [30, lemma 1.13]. We omit the proof.

**Proposition 3.11.** Let $C$ be a right exact category and let $A$ be a right filtering subcategory. The set $S_A$ of weak isomorphisms is a right multiplicative system. Moreover, every solid diagram

\[
\begin{array}{ccc}
X & \xrightarrow{s} & Y \\
\downarrow & & \downarrow \\
Z & \xrightarrow{f} & W
\end{array}
\]

with $s \in S_A$ can be completed to a commutative square such that $t \in S_A$ and the length of $t$ is at most the length of $s$.

If $A$ is a strongly right filtering subcategory, then the square in axiom $\textbf{RMS2}$ can be chosen as a pullback-square.

**Proposition 3.12.** Let $f: X \twoheadrightarrow Y$ be a deflation. For any weak isomorphism $s: Z \simrightarrow Y$, the pullback along $f$ is a weak isomorphism.

**Proof.** This follows from proposition 2.9, lemma 2.11 and the pullback lemma. \hfill \Box

3.4. **Further properties of weak isomorphisms.** We establish further properties of the set of weak isomorphisms related to a right percolating subcategory that we use later in this paper. In particular, these properties allow us to perform certain operations with weak isomorphisms without increasing the composition length (see definition 3.8) of the weak isomorphisms involved.

**Proposition 3.13.** Let $C$ be a right exact category and let $A$ be a right percolating subcategory. Any diagram

\[
\begin{array}{ccc}
U & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \xrightarrow{g} \downarrow \\
B & \xrightarrow{P} & Z
\end{array}
\]

whose row and column are conflations with $B \in A$ can be completed to a commutative diagram
where the rows and columns are conflations, and where \( A, B', C \in \mathcal{A} \).

**Proof.** Applying proposition 3.4 to the inflation \( i \) and deflation \( g \), we obtain a commutative diagram

\[
\begin{array}{c}
X \\ \downarrow i \\
Y \\ \downarrow p \\
Z \\
\end{array}
\begin{array}{c}
A \\ \downarrow g'' \\
B'' \\ \downarrow C' \\
\end{array}
\]

where the rows are conflations, the bottom row belongs to \( \mathcal{A} \), and \( g': Y \to B \) factors as \( Y \xrightarrow{g''} B'' \to B \). By axiom P2, the induced map \( Z \to C' \) factors as \( Z \to C \to C' \). By axiom R2, taking the pullback of \( C \to C' \) along \( B'' \to C' \) we obtain the commutative diagram

\[
\begin{array}{c}
X \\ \downarrow i \\
Y \\ \downarrow p \\
Z \\
\end{array}
\begin{array}{c}
A \\ \downarrow g' \\
B' \\ \downarrow C \\
\end{array}
\begin{array}{c}
A \\ \downarrow \\
B'' \\ \downarrow C' \\
\end{array}
\]

such that the composition \( Y \xrightarrow{g'} B' \to B'' \) equals \( g'' \). By axiom P1, \( B' \) belongs to \( \mathcal{A} \). By lemma 3.7, the map \( g' \) is a deflation. Let \( f': U' \to Y \) be the kernel of \( g': Y \to B' \). Since \( A \in \mathcal{A} \), proposition 3.6 allows us to use the \( 3 \times 3 \)-lemma on the upper half of the diagram; this yields most of the required diagram. Furthermore, since \( g': Y \to B \) factors as \( Y \to B' \to B \), we know, using the universal properties of kernels, that \( f': U' \to Y \) factors as \( f: U' \to U \to Y \).

\[ \square \]

**Corollary 3.14.** Let \( \mathcal{C} \) be a right exact category and let \( \mathcal{A} \) be a right percolating subcategory. Let \( s: X \to Y \) and \( t: Y \to Z \) be an \( \mathcal{A}^{-1} \)-inflation and \( \mathcal{A}^{-1} \)-deflation, respectively. There exists a map \( u: X' \to X \) yielding a commutative diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{u} & X \\
\downarrow s & & \downarrow \sim \\
Y & \xrightarrow{t} & Z
\end{array}
\]

where all arrows, except \( u \), are \( \mathcal{A}^{-1} \)-inflations or \( \mathcal{A}^{-1} \)-deflations.

**Proof.** Applying proposition 3.13 to the diagram

\[
\begin{array}{cccc}
X & \xrightarrow{i} & Y & \xrightarrow{p} Z \\
\downarrow \ker(t) & & \downarrow \sim & \\
\ker(s) & & \sim & \\
& & \downarrow \coker(s) & \\
& & & \coker(s)
\end{array}
\]

yields the desired result.

\[ \square \]

**Corollary 3.15.** Let \( \mathcal{C} \) be a right exact category and let \( \mathcal{A} \) be a right percolating subcategory. Let \( f: Y \to T \) be any map in \( \mathcal{C} \). If there is a weak isomorphism \( s: X \to Y \) such that \( f \circ s = 0 \), then there is an \( \mathcal{A}^{-1} \)-inflation \( t: X' \to Y \) such that \( f \circ t = 0 \).

**Proof.** If \( f = 0 \), then the statement is easy. So, we assume that \( f \neq 0 \). Let \( t: X \to Y \) be a weak isomorphism with minimal length \( l \), satisfying \( f \circ t = 0 \). We wish to show that \( l = 1 \). Seeking a contradiction, assume that \( l > 1 \). We write \( t = t_1 \circ \ldots \circ t_2 \circ t_1 \) where each \( t_i \) is either an \( \mathcal{A}^{-1} \)-inflation or an \( \mathcal{A}^{-1} \)-deflation (and not an isomorphism). As the length of \( t \) is minimal, we know that \( t_1 \) is not an epimorphism. Hence, \( t_1 \) is an \( \mathcal{A}^{-1} \)-inflation. We then apply corollary 3.14 to find that a morphism \( u: X' \to X \) such that \( t \circ u = t_1 \circ \ldots \circ t_2 \circ t_1 \) where \( t_2 \) is an \( \mathcal{A}^{-1} \)-deflation. Note that \( (f \circ t) \circ u = 0 \), so that we infer that \( t' = t_1 \circ \ldots \circ t_2 \) is an \( \mathcal{A}^{-1} \)-isomorphism such that \( f \circ t' = 0 \) and the length of \( t' \) is
at most $l - 1$. This contradicts the minimality of $l$. Hence, $l = 1$ and $t$ is either an $A^{-1}$-inflation or an $A^{-1}$-deflation. As $f \neq 0$, we may conclude that $t$ is an $A^{-1}$-inflation, as required. \hfill \Box

The next proposition characterizes zero maps in the localization $S_A^{-1}C$. Recall from proposition 3.11 that the set $S_A$ of weak isomorphisms is a right multiplicative set. We will use notation from §2.3.

Proposition 3.16. Let $\mathcal{C}$ be a right exact category and let $A \subseteq \mathcal{C}$ be a right percolating subcategory. Let $(f: X' \to Y, s: X' \to X)$ be a morphism in $S_A^{-1}C$. The following are equivalent:

1. $(f, s) = 0$ in $S_A^{-1}C$,
2. there is an $A^{-1}$-inflation $t: X \sim X'$ such that $f \circ t = 0$,
3. $f$ factors through an object of $A$.

Proof. Assume that (1) holds. Then $f \circ s^{-1} = 0$ in $S_A^{-1}C$ and $s^{-1}$ is invertible in $S_A^{-1}C$. It follows that $Q(f) = 0$ in $S_A^{-1}$ by definition of $S_A^{-1}C$ there exists a weak isomorphism $u: M \to X'$ such that $f \circ u = 0$ in $C$. By corollary 3.15, there exists an $A^{-1}$-inflation $t: X \sim X'$ such that $f \circ t = 0$.

Assume that (2) holds. Since $f \circ t = 0$, we know that $f$ factors through coker$(t)$. Moreover, since $t$ is an $A^{-1}$-inflation, coker$(t) \in A$. Hence, $f$ factors through an object of $A$.

Assume that (3) holds. By assumption, there exists an object $A \in \mathcal{A}$ such that $f = f_2 \circ f_1$ where $f_1: X' \to A$ and $f_2: A \to Y$. By axiom P2, the map $f_1$ factors as $X' \to B \to A$ where $B \in A$. Write $k: K \to X'$ for the kernel of $X' \to B$. Note that $k \in S_A$ and $f \circ k = 0$. Hence $Q(f) = 0 = Q(k)$ is invertible. It follows that $(s, f) = 0$ in $S_A^{-1}C$. \hfill \Box

Proposition 3.17. Let $\mathcal{C}$ be a right exact category and let $A \subseteq \mathcal{C}$ be a full subcategory satisfying P3. Given a diagram

\[
\begin{array}{ccc}
X' & \longrightarrow & Y \\
\downarrow f & & \downarrow g \\
X & \longrightarrow & Y \\
\end{array}
\]

where the rows are conflations. If $f: X' \to X$ is an $A^{-1}$-inflation, then $g: Z' \to Z$ is an $A^{-1}$-deflation.

Proof. We use proposition 2.10 to obtain the commutative diagram

\[
\begin{array}{ccc}
X' & \longrightarrow & Y \\
\downarrow f & & \downarrow g \\
X & \longrightarrow & P \\
\downarrow b & & \downarrow P \\
X & \longrightarrow & Y \\
\end{array}
\]

where the rows are conflations, the upper-left square and the bottom-right square are both pullbacks and pushouts, and $b \circ a = 1_Y$. As the upper-left square is a pushout, we infer that coker$(a) \cong$ coker$(f) \in A$. As $a$ is a section, we find that $P \cong Y \oplus$ coker$(f)$.

It follows from $b \circ a = 1_Y$ that ker$(b) \cong$ coker$(a)$. As the lower-right square is a pullback, we find that ker$(g) \cong$ ker$(b)$, hence ker$(g) \in A$. Applying proposition 3.5 to the composition $Y \to Z' \xrightarrow{g} Z$ shows that $g$ is a deflation with kernel in $A$. \hfill \Box

4. QUOTIENTS OF ONE-SIDED EXACT CATEGORIES VIA LOCALIZATIONS

Throughout this section, let $\mathcal{C}$ denote a right exact category and $A$ a right percolating subcategory. We write $S_A$ for the corresponding set of weak isomorphisms (see definition 3.8). The aim of this section is to show that $S_A^{-1}C$ has a canonical right exact structure such that the localization functor $Q: \mathcal{C} \to S_A^{-1}C$ is exact. Moreover, we show that $S_A^{-1}C$ is universal in the sense of the following definition.

Definition 4.1. Let $\mathcal{C}$ be a right exact category and $A$ a full right exact subcategory. We define the quotient of $\mathcal{C}$ by $A$ as a right exact category $\mathcal{C}/A$ together with an exact quotient functor $Q: \mathcal{C} \to \mathcal{C}/A$ satisfying the following universal property: for any exact functor $F: \mathcal{C} \to \mathcal{D}$ of right exact categories such that $F(A) \cong 0$ for all $A \in \text{Ob}(A)$ there exists a unique exact functor $G: \mathcal{C}/A \to \mathcal{D}$ such that the
following diagram commutes:

\[
\begin{array}{c}
\bullet A \\
\downarrow 0 \\
\bullet C \\
\downarrow F \\
\bullet D \\
\downarrow G \\
\bullet C/A
\end{array}
\]

**Remark 4.2.** The next two observations motivate the definitions of axiom **P1** and weak isomorphisms.

1. Let \( A \to X \to Y \) be a conflation in \( C \) with \( A \in \text{Ob}(A) \). Then \( 0 \to Q(X) \to Q(Y) \) is a conflation in \( C/A \). It follows \( Q(X) \to Q(Y) \) is invertible in \( C/A \). Similarly, if \( X \to Y \to A \) is a conflation in \( C \) with \( A \in \text{Ob}(A) \), then \( Q(X) \to Q(Y) \) is invertible. In particular, all weak isomorphisms become isomorphisms under \( Q \).

2. The kernel of any exact functor \( F: C \to D \) is a Serre subcategory of \( C \), i.e., it satisfies **P1**.

Let \( A \) be a right percolating subcategory of a right exact category \( C \). The main theorem (Theorem 4.7 below) states that the localization functor \( C \to S^{-1}_A C \) is a quotient functor. The proof consists of two major steps: in the first step we endow \( S^{-1}_A C \) with the structure of a conflation category such that \( Q: C \to S^{-1}_A C \) is exact; in the second step, we show that the conflation category \( S^{-1}_A C \) is a right exact category.

### 4.1. The category \( S^{-1}_A C \) is a conflation category.

**Lemma 4.3.** Let \( X \to Y \to Z \) be a conflation in \( C \). For every weak isomorphism \( s: Y_2 \sim Y \), there is a weak isomorphism \( s': Y' \sim Y_2 \) and a commutative diagram

\[
\begin{array}{c}
X' \ar{r}{i'} \ar{d}{?} & Y' \ar{r}{p'} \ar{d}{?} & Z' \\
X \ar{r}{i} \ar{d}{?} & Y \ar{r}{p} \ar{d}{?} & Z
\end{array}
\]

where the rows are conflations and the vertical maps are weak isomorphisms.

**Proof.** Recall that \( s \) is a composition of \( A^{-1} \)-inflations and \( A^{-1} \)-deflations. We consider the following two cases separately.

Case I: Assume that \( s \) factors as \( Y_2 \overset{s_2}{\to} Y_1 \overset{s_1}{\to} Y \). Taking the pullback of \( i \) along \( s_1 \) yields the following commutative diagram

\[
\begin{array}{c}
Y_2 \ar{d}{s_2} \\
X_1 \ar{r}{i_1} \ar{d}{?} & Y_1 \ar{r}{p_1} \ar{d}{?} & Z \\
X \ar{r}{i} \ar{d}{?} & Y \ar{r}{p} \ar{d}{?} & Z
\end{array}
\]

(see Lemma 2.11). Clearly, the length of \( s_2 \) is strictly smaller than the length of \( s \).

Case II: Assume that \( s \) factors as \( Y_2 \overset{s_2}{\to} Y_1 \overset{s_1}{\to} Y \). By Proposition 3.13, we obtain the solid part of the following commutative diagram in \( C \):

\[
\begin{array}{c}
X_1 \ar{r}{i_1} \ar{d}{?} & U_1 \ar{r}{s_2} \ar{d}{?} & P_1 \ar{d}{t_2} \\
X \ar{r}{i} \ar{d}{?} & Y \ar{r}{p} \ar{d}{?} & Z
\end{array}
\]

(see Lemma 2.11). Clearly, the length of \( s_2 \) is strictly smaller than the length of \( s \).
Here, the dotted arrows are obtained by axiom RMS2. Moreover, by proposition 3.11, the length of \(s_2'\) is bounded above by the length of \(s_2\) and, thus, is strictly smaller than the length of \(s\). We obtain the commutative diagram

\[
\begin{array}{ccc}
P_1 & \cong & s_2' \\
\downarrow & & \downarrow \\
X_1 & \xrightarrow{i_1} & U_1 \\
\downarrow & & \downarrow \\
X & \xrightarrow{i} & Y \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow \\
X_1/U_1 & \xrightarrow{p_1} & X_1 \\
\downarrow & & \downarrow \\
\rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow \\
Z & \xrightarrow{\varphi} & Z
\end{array}
\]

where the rows are conflations and the length of \(s_2'\) is less than the length of \(s_2\).

Repeatedly applying these two cases yields the required commutative diagram.

\[\square\]

**Proposition 4.4.** Let \(\mathcal{C}\) be a right exact category and let \(\mathcal{A}\) be a right percolating subcategory. The localization functor \(Q: \mathcal{C} \to S_{\mathcal{A}}^{-1}\mathcal{C}\) maps conflations to kernel-cokernel pairs.

**Proof.** Let \(X \xrightarrow{i} Y \xrightarrow{p} Z\) be a conflation in \(\mathcal{C}\). As \(S_{\mathcal{A}}\) is a right multiplicative system (see proposition 3.11), we know that \(Q(i)\) is the kernel of \(Q(p)\). We only need to show that \(Q(p)\) is the cokernel of \(Q(i)\).

For this, we consider the following diagram:

\[
\begin{array}{ccc}
T & \xrightarrow{f''} & Y'' \\
\downarrow & & \downarrow \\
X' & \xrightarrow{i'} & Y' \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow \\
X & \xrightarrow{i} & Y \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow \\
Z & \xrightarrow{p} & Z
\end{array}
\]

where the composition \((f, s)\circ(i, 1)\) is zero in \(S_{\mathcal{A}}^{-1}\mathcal{C}\). We will show that \((p, 1)\) is the cokernel of \((i, 1)\) by showing that, in \(S_{\mathcal{A}}^{-1}\mathcal{C}\), the morphism \((f, s)\) factors through \((p, 1)\). Using lemma 4.3, we find the following diagram

\[
\begin{array}{ccc}
T & \xrightarrow{f'} & Y' \\
\downarrow & & \downarrow \\
X' & \xrightarrow{i'} & Y' \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow \\
X & \xrightarrow{i} & Y \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow \\
Z & \xrightarrow{p'} & Z
\end{array}
\]

where the rows are conflations. As \((f, s)\circ(i, 1)\) is zero in \(S_{\mathcal{A}}^{-1}\mathcal{C}\), we infer that \(Q(f'i') = 0\). By corollary 3.15, there exists an \(\mathcal{A}^{-1}\)-inflation \(t\) such that \((f'i')t = 0\). By proposition 3.9, the composition \(i't\) is an inflation. Proposition 3.17 now yields a commutative diagram

\[
\begin{array}{ccc}
T & \xrightarrow{f''} & Y'' \\
\downarrow & & \downarrow \\
X'' & \xrightarrow{i''} & Y'' \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow \\
X' & \xrightarrow{i'} & Y' \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow \\
Z & \xrightarrow{p''} & Z''
\end{array}
\]

with exact rows where \(f' \circ i'' = 0\). As \(p'' = \text{coker}(i'')\), we know that \(f'\) factors through \(p''\). Careful examination of the above diagrams shows that \((f, s)\) factors through \((p, 1)\) in \(S_{\mathcal{A}}^{-1}\mathcal{C}\), as required. This establishes that \(Q(p)\) is a weak cokernel of \(Q(i)\).

To show that \(Q(p)\) is the cokernel of \(Q(i)\), it suffices to show that \(Q(p): Q(Y) \to Q(Z)\) is an epimorphism. Let \((g, t)\) be any morphism in \(S_{\mathcal{A}}^{-1}\mathcal{C}\) such that \((g, t)\circ Q(p) = 0\). By proposition 3.12, we find the
following diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{p} & Z \\
\downarrow{t} & & \downarrow{g} \\
Y' & \xrightarrow{p'} & Z' \\
\end{array}
\]

where the square is a pullback and the vertical arrows are weak isomorphisms. As \(Q(g \circ p') = 0\), proposition 3.16 shows that there is an \(A^{-1}\)-inflation \(s: W \rightarrow Y\) such that \(g \circ p' \circ s = 0\) in \(C\). It now follows from lemma 4.3 that there is a commutative diagram

\[
\begin{array}{ccc}
U & \xrightarrow{p''} & Z'' \\
\downarrow{v} & & \downarrow{u} \\
W & \xrightarrow{\iota} & Z \\
\downarrow{\iota} & & \downarrow{\iota} \\
Y' & \xrightarrow{p'} & Z' \\
\end{array}
\]

As the composition \(g \circ p' \circ s \circ u = g \circ v \circ p''\) from \(U\) to \(T\) is zero in \(C\), we see that \(g \circ v = 0\). As \(v\) is a weak isomorphism, this shows that \(Q(g) = 0\). We conclude that \(Q(p)\) is an epimorphism. Hence, \(Q(p)\) is the cokernel of \(Q(t)\).

\[\square\]

**Definition 4.5.** Let \(A\) be a right percolating subcategory in a right exact category \(C\). We say that a sequence \(X \rightarrow Y \rightarrow Z\) is a conflation in \(S^{-1}_A C\) if there is a conflation \(\bar{X} \rightarrow \bar{Y} \rightarrow \bar{Z}\) and a commutative diagram

\[
\begin{array}{ccc}
Q(\bar{X}) & \xrightarrow{Q(\bar{Y})} & Q(\bar{Z}) \\
\downarrow{Q(t)} & & \downarrow{Q(u)} \\
X & \xrightarrow{Y} & Z \\
\end{array}
\]

in \(S^{-1}_A C\) where the vertical arrows are isomorphisms.

**Remark 4.6.**
1. Thus, \(X \rightarrow Y \rightarrow Z\) is a conflation in \(S^{-1}_A C\) if and only if it is isomorphic to the image of a conflation in \(C\) under \(Q\).
2. It follows from proposition 4.4 that definition 4.5 endows \(S^{-1}_A C\) with a conflation structure.
3. With this choice, the localization functor \(Q: C \rightarrow S^{-1}_A C\) is exact.

**4.2. The category \(S^{-1}_A C\) is a right exact category.** We are now in a position to prove the main theorem, namely that the conflation category \(S^{-1}_A C\) from definition 4.5 is right exact.

**Theorem 4.7.** Let \(C\) be a right exact category and let \(A\) be a right percolating subcategory. The conflation category \(S^{-1}_A C\) (see definition 4.5) is a right exact category. The localization functor \(Q: C \rightarrow S^{-1}_A C\) is exact and satisfies the universal property of the quotient \(C \rightarrow C/A\). Moreover, if \(C\) satisfies axiom \(R0^*\), so does \(C/A\).

**Proof.** It is easy to see that axiom \(R0\) (respectively axiom \(R0^*\)) descends to \(S^{-1}_A C\). We now check that \(S^{-1}_A C\) satisfies axioms \(R1\) and \(R2\).

\(R1\) We consider two deflations \(X \rightarrow Y\) and \(Y \rightarrow Z\) in \(S^{-1}_A C\). By definition 4.5, this means that there are deflations \(\bar{X} \rightarrow \bar{Y}\) and \(\bar{Y} \rightarrow \bar{Z}\) and a diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{p} & \bar{Z} \\
\downarrow{\iota} & & \downarrow{\iota} \\
X & \xrightarrow{\bar{p}} & \bar{Y} \\
\downarrow{\iota} & & \downarrow{\iota} \\
\bar{X} & \xrightarrow{\bar{p}} & \bar{Y} \\
\end{array}
\]
which descends to a commutative diagram in $S^{-1}_A \mathcal{C}$. Here, we chose the direction of the isomorphisms in $S^{-1}_A \mathcal{C}$ in such a way to get the particular arrangement of arrows in $S_A$. The first step of the proof is to find a better representation in $\mathcal{C}$ of this composition of deflations in $S^{-1}_A \mathcal{C}$.

By proposition 3.11, we can obtain the dotted arrows by axiom RMS2. Note that the induced map $P \to Y''$ descends to an isomorphism in $S^{-1}_A \mathcal{C}$. It follows that we can represent the outer edge by the solid part of the following commutative diagram:

\[
\begin{array}{ccc}
Z & \rightarrow & P \\
\downarrow & & \downarrow \\
X & \rightarrow & Y
\end{array}
\]

By axiom RMS1, the composition $P \xrightarrow{\sim} Y'' \xrightarrow{\sim} Y$ belongs to $S_A$. Axiom R2 yields the pullback square $RPVX$ in $\mathcal{C}$. As $P \to Y$ descends to an isomorphism and $Q$ commutes with pullbacks (see remark 2.18), the map $R \to X$ descends to an isomorphism as well. It follows that the original composition of deflations in $S^{-1}_A \mathcal{C}$ can be represented by the composition $R \to P \xrightarrow{\sim} Y \xrightarrow{\sim} Z$.

Relabeling, we now see that is suffices to show that $X \xrightarrow{p_1} Y \xrightarrow{\sim} U \xrightarrow{p_2} Z$ descends to a deflation in $S^{-1}_A$. We consider the following two cases.

Case I: Assume that $s$ factors as $Y \xrightarrow{\sim} V \xrightarrow{\sim} U$. By axiom R1 in $\mathcal{C}$, the composition $p_2s_1$ is a deflation. This reduces the problem as the length of $s_2$ is strictly smaller than the length of $s$.

Case II: Assume that $s$ factors as $Y \xrightarrow{\sim} V \xrightarrow{\sim} U$. By proposition 3.13, there exists a commutative diagram given by

\[
\begin{array}{ccc}
W'' & \xrightarrow{\sim} & V' \\
\downarrow & & \downarrow \\
W & \xrightarrow{\sim} & U
\end{array}
\]

where the dotted arrows are induced by axiom RMS2 and axiom R2 in $\mathcal{C}$. Write $p'_2$ for the cokernel of $i'_2$. By proposition 3.13, the maps $p_2$ and $p'_2$ descend to isomorphic maps in $S^{-1}_A \mathcal{C}$. As the maps $t_1$, $t_2$ and $t_3$ descend to isomorphisms under $Q$, it follows that the composition $Q(p'_2 \circ s'_2 \circ p'_1)$ is isomorphic to the composition $Q(p_2 \circ s \circ p_1)$. Moreover, by proposition 3.11, the decomposition length of $s'_2$ is strictly smaller than that of $s$.

Iteratively applying these two cases yields the desired result. This proves that axiom R1 is satisfied.

R2 We now show that the pullback along a deflation exists and yields a deflation in $S^{-1}_A \mathcal{C}$. For this, consider a co-span $X \rightarrow Y \leftarrow Z$ in $S^{-1}_A \mathcal{C}$. The co-span can be represented by the following diagram in $\mathcal{C}$:

\[
\begin{array}{ccc}
X & \xrightarrow{\sim} & X'' \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\sim} & Y'' & \xrightarrow{\sim} & Z
\end{array}
\]

\[
\begin{array}{ccc}
X & \leftarrow & X'' \\
\downarrow & & \downarrow \\
Y & \leftarrow & Y'' & \leftarrow & Z
\end{array}
\]
Let Corollary 4.9. between exact categories satisfying the following universal property: any exact functor $\gamma: C \to E$ factors uniquely through $\gamma$. The category $\overline{C}$ is called the exact hull of $C$.

The next corollary is an immediate application of the previous proposition and theorem 4.7.

Corollary 4.9. Let $C$ be an exact category and let $A$ be a right percolating subcategory of $C$. The composition of the exact quotient functor $Q: C \to C/A$ and the embedding $\gamma: C/A \to \overline{C/A}$ is an exact functor between exact categories satisfying the following universal property: any exact functor $F: C \to E$ where $E$ is an exact category such that $F(A) \cong 0$ for all $A \in \text{Ob}(A)$ factors uniquely through $\gamma \circ Q$.

By remark 3.3, any subcategory $A$ of an exact category $C$ satisfies axiom $P3$. If $C$ is only right exact, axiom $P3$ is no longer automatically satisfied by any subcategory. We end this section by showing that theorem 4.7 requires axiom $P3$. The example uses terminology of section 5.

Example 4.10. Let $U$ be the category of finite-dimensional representations of the quiver

$$A_4 : 1 \leftarrow 2 \leftarrow 3 \leftarrow 4.$$ 

The category $U$ can be visualized by its Auslander-Reiten quiver

$$
\begin{array}{c}
P_1 \searrow \quad P_3 \\
\downarrow \quad P_2 \\
S_1 \quad S_2 \\
\downarrow \quad I_2 \\
S_3 \quad S_4 \\
\downarrow \quad I_3 \\
\end{array}
$$

where $\tau$ is the Auslander-Reiten translate.

Let $C$ be the full additive subcategory of $U$ generated by the objects $S_1, P_2, P_3, P_4, S_2, S_3$ and $I_2$. We claim that $C$ is a right exact category whose deflations are given by epimorphisms. Let $D$ be the full additive subcategory of $U$ generated by $C$ and $\tau^{-1}P_2$. As $D$ is an extension-closed subcategory of an abelian category, it is exact. It is clear that $C$ satisfies axioms $R0^r$ and $R1$. We now show that $C$ satisfies axiom $R2$. Consider a pullback diagram

$$
\begin{array}{c}
P \longrightarrow Z \\
\downarrow \quad \downarrow \\
X \longrightarrow Y \\
\end{array}
$$

in $D$ where $X, Y, Z \in \text{Ob}(C)$. One verifies that $\tau^{-1}P_2$ is not a summand of $P$, for all choices of $X, Y, Z \in \text{Ob}(C)$. It follows that the above diagram is a pullback diagram in $C$. It follows that $C$ is a right exact category.

Let $A$ be the full additive subcategory of $U$ generated by $S_2$. Clearly, $A$ is an abelian subcategory satisfying axioms $A1$ and $A2$ (and hence axiom $P4$ as well). On the other hand, one can check that the
consider the commutative diagram

\[
\begin{array}{ccc}
P_2 & \rightarrow & P_3 \\
\downarrow & & \downarrow \\
S_2 & \rightarrow & S_3 \\
\end{array}
\]

in \(C\). If \(C/A\) is a conflation category and the localization functor \(Q: C \rightarrow C/A\) is exact, \(Q(S_3)\) is the cokernel of \(Q(i)\). The above diagram yields an induced map \(Q(S_3) \rightarrow Q(I_2)\) in \(C/A\). On the other hand, one can verify explicitly that such a morphism cannot be obtained by localizing with respect to the weak isomorphisms. It follows that \(C/A\) does not satisfy theorem 4.7.

**Remark 4.11.** Although example 4.10 does not satisfy theorem 4.7, the localization \(C/A\) exists and satisfies the universal property of definition 4.1. Consider the obvious functor \(F: C \rightarrow D/A\) and endow \(D/A\) with the weakest right exact structure such that the functor \(F\) is exact. A tedious verification shows that \(F: C \rightarrow D/A\) satisfies the universal property of \(C/A\).

5. ABELIAN PERCOLATING SUBCATEGORIES

Let \(\mathcal{C}\) be a right exact category. In the previous section, we considered quotients of \(\mathcal{C}\) by right percolating subcategories. In this section, we consider the special case were the right percolating subcategory is abelian. Under this additional assumption, the set of weak isomorphisms is better behaved.

5.1. Basic definitions and results. We start with the definition of an abelian right percolating subcategory.

**Definition 5.1.** Let \(\mathcal{C}\) be a conflation category and let \(\mathcal{A}\) be a non-empty full subcategory of \(\mathcal{C}\). We call \(\mathcal{A}\) an *abelian right percolating subcategory* of \(\mathcal{C}\) if the following three properties are satisfied:

- **A1** \(\mathcal{A}\) is a Serre subcategory, meaning:
  
  If \(A' \rightarrow A \rightarrow A''\) is a conflation in \(\mathcal{C}\), then \(A \in \text{Ob}(\mathcal{A})\) if and only if \(A', A'' \in \text{Ob}(\mathcal{A})\).

- **A2** For all morphisms \(C \rightarrow A\) with \(C \in \text{Ob}(\mathcal{C})\) and \(A \in \text{Ob}(\mathcal{A})\), there exists a commutative diagram

  \[
  \begin{array}{ccc}
  C & \rightarrow & A \\
  \downarrow & & \downarrow \\
  A' & \rightarrow & A \\
  \end{array}
  \]

  with \(A' \in \text{Ob}(\mathcal{A})\), and where \(C \rightarrow A'\) is a deflation and \(A \rightarrow A'\) is an inflation.

- **A3** If \(a: C \rightarrow D\) is an inflation and \(b: C \rightarrow A\) is a deflation with \(A \in \text{Ob}(\mathcal{A})\), then the pushout of \(a\) along \(b\) exists and yields an inflation and a deflation, i.e.

  \[
  \begin{array}{ccc}
  C & \rightarrow & D \\
  \downarrow & & \downarrow \\
  A & \rightarrow & A \\
  \end{array}
  \]

**Remark 5.2.** It will be shown in proposition 5.5 below that an abelian right percolating subcategory \(\mathcal{A}\) in a right exact category \(\mathcal{C}\) is a right percolating subcategory \(\mathcal{A}\) that is additionally abelian. This explains the terminology.

**Remark 5.3.**

1. A conflation category with an abelian right percolating subcategory satisfies \(\text{R}0^*\). Indeed, it follows from **A1** that \(0 \in \mathcal{A}\) and from **A2** that any morphism \(X \rightarrow 0\) is a deflation.

2. Conditions **A1** and **A2** are also required by [11, definition 4.0.35].
(3) Given a right exact category $C$ and an abelian right percolating subcategory, axiom $A_2$ implies that $A$ is strongly right filtering. By proposition 3.11, pullbacks along weak isomorphisms exist and weak isomorphisms are stable under pullbacks.

**Lemma 5.4.** Let $A$ be an abelian right percolating subcategory of a conflation category $C$. Let $f : C \to A$ be a morphism in $C$ with $A \in A$. If $f$ is a monomorphism (epimorphism), then $f$ is an inflation (deflation). In particular, a morphism $X \to 0$ is a deflation.

**Proof.** This is an immediate application of axioms $A_1$ and $A_2$. □

**Proposition 5.5.** Let $C$ be a right exact category. A full subcategory $A \subseteq C$ is an abelian right percolating subcategory if and only if the following conditions are satisfied:

1. $A$ is a right filtering subcategory of $C$ (thus, it satisfies $P_1$ and $P_2$),
2. $A$ satisfies axiom $P_3$,
3. $A$ is abelian and the embedding $A \to C$ is exact.

Moreover, an abelian right percolating subcategory is a right percolating subcategory.

**Proof.** Assume that $A$ is an abelian right percolating subcategory of $C$. Axiom $P_1$ is identical to axiom $A_1$, axiom $P_2$ is implied by axiom $A_2$. Hence $A$ is right filtering in $C$. Axiom $P_3$ coincides with axiom $A_3$.

We now show that $A$ is an abelian category. By axiom $A_2$, any map in $A$ has a kernel and cokernel in $C$. Let $\alpha : A \to B$ be a morphism in $A$. Assume first that $\alpha$ is an epimorphism. It follows from lemma 5.4 that $\alpha$ is a deflation, and hence has a kernel in $C$, which lies in $A$ by $A_1$. Moreover, $\alpha = \text{coker}(\ker \alpha)$. Likewise, one shows that a monomorphism $\beta : A \to B$ satisfies $\text{ker}(\text{coker} \alpha)$. From [22, theorem 20.1], it follows that $A$ is abelian. The embedding $A \to C$ is exact by lemma 5.4.

Conversely, assume that $A$ is a right filtering subcategory of $C$ that is abelian, satisfies axiom $P_3$ and such that the canonical embedding $A \to C$ is exact. We only need to show that the $A$ satisfies axiom $A_2$.

Let $f : C \to A$ be a map in $C$ such that $A \in A$. By axiom $P_2$, $f$ factors as $C \xrightarrow{g} A' \xrightarrow{h} A$. Since $A$ is abelian and the map $h : A' \to A$ has an epi-mono factorization, $h = h_2 \circ h_1$. Since the embedding $A \to C$ is exact, $h_1$ is a deflation in $C$ and $h_2$ is an inflation in $C$. Hence $f = h_2 \circ (h_1 \circ g)$ where $h_1 \circ g$ is a deflation by axiom $R_1$ and $h_2$ is an inflation. This shows axiom $A_2$. Hence $A$ is an abelian right percolating subcategory of $C$.

Lastly, assume that $A$ is an abelian right percolating subcategory of $C$. By the above, we only need to show that $A$ satisfies axiom $P_4$ to show that $A$ is right percolating. It is straightforward to see that axiom $P_4$ follows from axiom $A_2$. □

5.2 Weak isomorphisms are admissible. Throughout this section, $C$ denotes a right exact category and $A$ denotes an abelian right percolating subcategory. Consider the set $\hat{S}_A := S_A \cap \text{Adm}(C)$ of admissible weak isomorphisms. The aim of this section is to show that $\hat{S}_A = S_A$.

**Remark 5.6.**

1. A morphism $f : X \to Y$ in $C$ belongs to $\hat{S}_A$ if and only if $f$ is admissible and $\ker(f), \text{coker}(f) \in A$.
2. For any admissible morphism $f$, one automatically has that $\text{coim}(f) \cong \text{im}(f)$ and $f$ factors as deflation-inflation through $\text{im}(f)$.
3. Admissible weak isomorphisms are called $A^{-1}$-isomorphisms in [11, definition 4.0.36].
4. Any morphism $\alpha : A \to B$ in an abelian right percolating subcategory $A \subseteq C$ belongs to $\hat{S}_A$.

Indeed, this is an immediate corollary of proposition 5.5.

We show two additional homological properties which are consequences of axiom $A_2$. The first is a strengthening of proposition 3.13.

**Corollary 5.7.** Let $X \to Y \to Z$ be a conflation and $f : Y \to B$ be a deflation. If $B \in \text{Ob}(A)$, then there is a commutative diagram

\[
\begin{array}{ccc}
X'' & \to & Y'' \\
\downarrow & & \downarrow \\
X & \to & Y \\
\downarrow & & \downarrow \\
A & \to & B \\
\downarrow & & \downarrow \\
A & \to & C
\end{array}
\]
where the rows and the columns are conflations, and where the bottom row lies in \( \mathcal{A} \). Moreover, the upper left square is a pullback and the lower right square is a pushout.

**Proof.** It follows from \( \mathbf{A2} \) that the composition \( X \to Y \to B \) factors as \( X \to A \to B \) with \( A \in \text{Ob}(\mathcal{A}) \). This gives the following commutative diagram:

\[
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow f \\
A & \longrightarrow & B
\end{array}
\]

with exact rows (the bottom rows lies in \( \mathcal{A} \) by \( \mathbf{A1} \)). The dotted arrow is induced by the universal property of the cokernel \( Y \to Z \). One easily verifies that the dotted arrow is an epimorphism and, thus, by lemma 5.4, a deflation. The statement now follows from proposition 3.6.

**Proposition 5.8.** Let \( f : X \to Y \) belong to \( \widehat{\mathcal{S}}_{\mathcal{A}} \) and let \( g : X \to A \) be any morphism. If \( A \in \text{Ob}(\mathcal{A}) \), then the pushout of \( f \) along \( g \) exists and the induced map belongs to \( \widehat{\mathcal{S}}_{\mathcal{A}} \).

**Proof.** By definition, \( f \) is an admissible map. By axiom \( \mathbf{A2} \), \( g \) is admissible as well. Since \( A' \in \mathcal{A} \), corollary 5.7 yields a commutative diagram

\[
\begin{array}{ccc}
Y & \longrightarrow & Q \\
\downarrow & & \downarrow R \\
X' & \longrightarrow & R
\end{array}
\]

with \( P \in \mathcal{A} \) and such that the square \( X'PAX \) is a pushout. Applying axiom \( \mathbf{A3} \) twice yields a commutative diagram

\[
\begin{array}{ccc}
Y & \longrightarrow & S \\
\downarrow & & \downarrow S \\
X' & \longrightarrow & S
\end{array}
\]

where all squares are pushout squares. It follows from the pushout lemma that the square \( YSAX \) is a pushout square as well. Since the map \( A \to S \) belongs to \( \mathcal{A} \), remark 5.6 yields that it is an admissible weak isomorphism.

The next lemma is crucial in showing that \( \mathcal{S}_{\mathcal{A}} = \mathcal{S}_{\mathcal{A}} \), i.e. that the weak isomorphisms are automatically admissible.

**Lemma 5.9.** Let \( a : U \sim \to V \) and \( b : V \sim \to W \) be an \( \mathcal{A}^{-1} \)-inflation and \( \mathcal{A}^{-1} \)-deflation, respectively. The composition \( b \circ a \) is an admissible weak isomorphism.
Proof. Using corollary 5.7, we find the commutative diagram

\[
\begin{array}{c}
\ker(c_a k_b) \xrightarrow{k'_a} U \xrightarrow{i'_a} \ker(c'_a) \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
\ker(b) \xrightarrow{k_b} V \xrightarrow{b} W \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
im(c_a k_b) \xrightarrow{\epsilon'_{a}} \coker(a) \xrightarrow{\epsilon'_{a}} \coker(c a k_b)
\end{array}
\]

such that the rows and columns are conflations. By axiom A1, the left column and lower row belong to \(A\). The upper-right square shows that \(ba = k'_a c'_b\) and thus \(ba\) is admissible. Clearly, \(\ker(ba) = \ker(c_a k_b)\) and \(\coker(ba) = \coker(c a k_b)\), both belonging to \(A\).  

\[\square\]

**Proposition 5.10.** Let \(C\) be a right exact category. If \(A \subseteq C\) an abelian right percolating subcategory, then \(S_A = \hat{S}_A\), in particular all weak isomorphisms are admissible. Moreover, \(S_A\) is a right multiplicative system such that the square in axiom RMS2 can be chosen as a pullback square; in particular, one can take pullbacks along weak isomorphisms.

Proof. The proof is a straightforward application of propositions 3.11 and 3.10 and lemma 5.9.  

\[\square\]

5.3. **The 2-out-of-3 property.** Throughout this section \(C\) is a right exact category and \(A\) is an abelian right percolating subcategory. We now show that the right multiplicative system \(S_A\) of admissible weak isomorphisms satisfies the 2-out-of-3 property. We first need some five-lemma type results.

**Lemma 5.11.** Consider a commutative diagram

\[
\begin{array}{c}
X \xrightarrow{f} Y \xrightarrow{g} Z \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
X' \xrightarrow{f'} Y' \xrightarrow{g'} Z
\end{array}
\]

with exact rows.

1. If \(f\) is an \(A^{-1}\)-inflation, then \(g\) is an \(A^{-1}\)-inflation.
2. If \(f\) is an \(A^{-1}\)-deflation, then \(g\) is an \(A^{-1}\)-deflation.

Proof. (1) From proposition 2.9, we obtain the following commutative diagram with exact rows and columns (where the left square is a pushout):

\[
\begin{array}{c}
X \xrightarrow{?} Y \xrightarrow{g} Z \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
X' \xrightarrow{f'} Y' \xrightarrow{g'} Z \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
\coker(f) \xrightarrow{\epsilon_g} \coker(g)
\end{array}
\]

As \(c_g: Y' \to \coker(g)\) is an epimorphism and \(\coker(f) \in \text{Ob}(A)\), lemma 5.4 yields that \(c_g\) is a deflation. Denote the kernel of \(c_g\) by \(K \to Y'\). By corollary 5.7 we obtain a commutative diagram:

\[
\begin{array}{c}
X \xrightarrow{?} Y \xrightarrow{g} Z \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
X' \xrightarrow{f'} Y' \xrightarrow{g'} Z \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
\coker(f) \xrightarrow{\epsilon_g} \coker(g) \xrightarrow{\epsilon_g} 0
\end{array}
\]
The dotted arrow is obtained by factoring \( g \) through the kernel of its cokernel. By the short five lemma ([1, lemma 5.3]), the induced map \( Y \to K \) is an isomorphism. It follows that \( g \) is an inflation. Since \( \text{coker}(g) \in A \), we find that \( g \in S_A \).

(2) By proposition 2.9, we know that the left square is a pushout and a pullback, and we obtain the following commutative diagram (where the columns are conflations)

\[
\begin{array}{ccc}
\text{ker}(f) & \rightarrow & \text{ker}(g) \\
| & k | & | \\
X & \rightarrow & Y \\
| & i | & | \\
X' & \rightarrow & Y'
\end{array}
\]

with \( \text{ker}(f) \in \text{Ob}(A) \) and such that \( ik \) is the kernel of \( g \).

By [1, proposition 5.5] (with \( d, d' \) the identity maps), the map \( (i' \ g) : X' \oplus Y \to Y' \) is a deflation. By [1, lemma 5.1], the map \( (f \ 0) : X \oplus Y \to X' \oplus Y \) is a deflation as well. Axiom \textbf{R1} yields that

\[
(i' \ g)(f \ 0) = (i' f \ 0) = (i' f \ g) = g (i' \ 1)
\]

is a deflation. Notice that we have the commutative diagram:

\[
\begin{array}{ccc}
& X & \rightarrow & Y \\
\downarrow & (1 \ 0) & & (0 \ 1) & \downarrow \\
& X & \rightarrow & Y \\
\downarrow & \left( \begin{smallmatrix} 1 \\ -1 \end{smallmatrix} \right) & & \left( \begin{smallmatrix} 1 \\ -1 \end{smallmatrix} \right) & \downarrow \\
& X' & \rightarrow & Y'
\end{array}
\]

It follows that the lower row is isomorphic to a conflation and thus \( (i' \ 1) \) is a deflation. By proposition 3.5 we conclude that \( g \) is a deflation. \( \square \)

**Proposition 5.12.**

(1) Consider the following commutative diagram in \( C \)

\[
\begin{array}{ccc}
X & \rightarrow & Y & \rightarrow & Z \\
\downarrow f & & \downarrow g & & \downarrow h \\
X' & \rightarrow & Y' & \rightarrow & Z'
\end{array}
\]

where the rows are conflations.

(a) If \( f \) is an \( A^{-1} \)-inflation, then \( g \) is an inflation.

(b) If additionally \( h \) is an \( A^{-1} \)-inflation, then \( g \) is an \( A^{-1} \)-inflation.

(2) Consider the following commutative diagram in \( C \)

\[
\begin{array}{ccc}
X & \rightarrow & Y & \rightarrow & Z \\
\downarrow f & & \downarrow g & & \downarrow h \\
X' & \rightarrow & Y' & \rightarrow & Z'
\end{array}
\]

where the rows are conflations.

(a) If \( f \) is an \( A^{-1} \)-deflation, then \( g \) is a deflation.

(b) If additionally \( h \) is an \( A^{-1} \)-deflation, then \( g \) is an \( A^{-1} \)-deflation.
Proof. Following proposition 2.10, we consider the following factorization of both diagrams in the statement of the proposition:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \mathbf{g} \\
X' & \xrightarrow{g_2} & Z \\
\downarrow & & \downarrow h \\
Y' & \xrightarrow{g_1} & Z' \\
\end{array}
\]

(1) If \( f \) is an inflation, then lemma 5.11 yields that \( g_1 \) is an \( \mathcal{A}^{-1} \)-inflation. As \( D \) is a pullback square and \( h \) an inflation, lemma 2.11 yields that \( g_2 \) is an inflation. It now follows from proposition 3.9 that \( g = g_2 \circ g_1 \) is an inflation. Moreover, if \( h \) is an \( \mathcal{A}^{-1} \)-inflation, its cokernel belongs to \( \mathcal{A} \). As \( D \) is also a pushout square, \( \text{coker}(g_2) = \text{coker}(h) \). Hence \( g_2 \) is an \( \mathcal{A}^{-1} \)-inflation. By proposition 3.10 we conclude that \( g = g_2 \circ g_1 \) is an \( \mathcal{A}^{-1} \)-inflation.

(2) If \( f \) is a deflation, lemma 5.11 yields that \( g_1 \) is an \( \mathcal{A}^{-1} \)-deflation. As \( D \) is a pullback square and \( h \) a deflation, axiom \( \text{R2} \) yields that \( g_2 \) is a deflation. By axiom \( \text{R1} \), \( g = g_2 \circ g_1 \) is a deflation. Moreover, if \( h \) is an \( \mathcal{A}^{-1} \)-deflation, proposition 3.10 yields that \( g_2 \) is an \( \mathcal{A}^{-1} \)-deflation and that \( g = g_2 \circ g_1 \) is an \( \mathcal{A}^{-1} \)-deflation. \( \square \)

Proposition 5.13. The two-out-of-three property holds, i.e. if \( f, g \) are composable morphisms, then if two of the three maps \( f, g, \) and \( gf \) belong to \( S_\mathcal{A} \), so does the third.

Proof. As we already showed that \( S_\mathcal{A} \) is a right multiplicative set, we know that \( f, g \in S_\mathcal{A} \) implies that \( gf \in S_\mathcal{A} \). We will first show that \( g, gf \in S_\mathcal{A} \) implies that \( f \in S_\mathcal{A} \).

Step 1: We now show that if \( g, gf \in S_\mathcal{A} \) and \( g \) is an inflation, then \( f \in S_\mathcal{A} \). Notice that \( \ker(gf) = \ker(f) \in \text{Ob}(\mathcal{A}) \) since \( g \) is monic and \( gf \in S_\mathcal{A} \). It follows that \( \text{coim}(gf) \cong \text{coim}(f) \). Hence we obtain the diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow \mathbf{g} & & \downarrow \mathbf{g} \\
\text{coim}(f) & \xrightarrow{\sim} & Z \\
\downarrow \mathbf{g} & & \downarrow \mathbf{g} \\
\text{coim}(gf) & & \end{array}
\]

Clearly the left-hand side of the diagram is commutative. Since \( X \twoheadrightarrow \text{coim}(f) \) is epic, the right side is commutative as well. Since \( gf \in S_\mathcal{A} \), we have that \( \text{coim}(gf) \cong \text{im}(gf) \). Since the right-hand side commutes, the map \( \text{im}(gf) \twoheadrightarrow Z \twoheadrightarrow \text{coker}(g) \) is zero. By corollary 5.7, we obtain a commutative diagram:

\[
\begin{array}{ccc}
\text{coim}(f) & \xrightarrow{\sim} & \text{im}(gf) & \xrightarrow{0} \\
\downarrow & & \downarrow \mathbf{g} & \downarrow \mathbf{g} & \downarrow \mathbf{g} \\
Y & \xrightarrow{\sim} & Z & \xrightarrow{\text{coker}(g)} \\
\downarrow \mathbf{g} & & \downarrow \mathbf{g} & & \downarrow \mathbf{g} \\
\text{coim}(gf) & \xrightarrow{\sim} & \text{coker}(g) & \xrightarrow{\text{coker}(g)}
\end{array}
\]

Using that \( g \) is monic, one readily verifies that \( \text{coim}(f) \twoheadrightarrow Y \) coincides with the dotted arrow from the previous diagram. It follows that \( f \) is admissible and \( \text{coker}(f) = K \in \mathcal{A} \). We conclude that \( f \in S_\mathcal{A} \).
Step 2: We now show that if \( g, gf \in S_A \) and \( g \) is a deflation, then \( f \in S_A \). Consider the commutative diagram:

\[
\begin{array}{ccc}
\ker(gf) & \rightarrow & X \\
\downarrow \phi & & \downarrow f \\
\ker(g) & \rightarrow & Y \\
\end{array}
\]

As \( \phi \) is a map in \( A \), remark 5.6 yields that \( \phi \) factors as a deflation-inflation through its image \( \text{im}(\phi) \). Moreover, proposition 2.2 implies that the left square is a pullback. As pullbacks preserve kernels, \( \ker(f) = \ker(\phi) \in \text{Ob}(A) \).

Axiom A3 yields the following commutative diagram:

\[
\begin{array}{ccc}
\ker(gf) & \rightarrow & X \\
\downarrow \phi & & \downarrow f \\
\ker(g) & \rightarrow & Y \\
\end{array}
\]

Indeed, the upper-left square is a pushout square constructed by axiom A3 and the lower-right square commutes as \( X \rightarrow P \) is epic. By proposition 5.12 we conclude that \( P \rightarrow Y \) is an \( A^{-1} \)-inflation. It follows that \( f \in S_A \). This concludes step 2.

Next, we will show that if \( f, gf \in S_A \) then \( g \in S_A \). Since \( f \) has a deflation-inflation factorization, it suffices to prove the statement separately assuming that \( f \) is a deflation and assuming \( f \) is an inflation. This will be done in step 1' and step 2'.

Step 1': Assume that \( f \) is a deflation. Then \( \text{coker}(gf) = \text{coker}(g) \in \text{Ob}(A) \). Hence we get the diagram

\[
\begin{array}{ccc}
Y & \rightarrow & Z \\
\downarrow h & & \downarrow g \\
X & \rightarrow & \text{im}(gf) \\
\end{array}
\]

Using that \( \text{im}(g) \rightarrow Z \) is monic, we see that this diagram is commutative.

As the composition \( \ker(f) \rightarrow X \rightarrow \text{im}(gf) \) is zero, one easily obtains the following commutative diagram:

\[
\begin{array}{ccc}
\ker(f) & \rightarrow & \ker(gf) \\
\downarrow & & \downarrow \\
\ker(f) & \rightarrow & C \\
\end{array}
\]

Here the induced map \( \ker(f) \rightarrow \ker(gf) \) is monic and belongs to \( A \) as \( A \) is abelian it has a cokernel \( C \). By proposition 2.2 the upper-right square is a pushout, by axiom A3 the map \( C \rightarrow Y \) is an inflation and by proposition 2.9, the map \( Y \rightarrow \text{im}(gf) \) is a deflation. As \( f \) is epic one sees that \( h = h' \). It follows that \( g \) has a deflation-inflation factorization. Since \( \ker(h) = C \in \text{Ob}(A) \) and \( \ker(h) = \ker(g) \) we conclude that \( g \) is an admissible weak isomorphism.
Step 2': Let $f$ be an inflation. We obtain a commutative diagram:

\[
\begin{array}{cccc}
X & \xrightarrow{f} & Y & \xrightarrow{\text{coker}(f)} \\
\downarrow & & \downarrow & \\
im(gf) & \xrightarrow{\phi} & Z & \xrightarrow{\text{coker}(gf)} \\
\downarrow & & \downarrow & \\
coker(\phi) & & \text{coker}(\phi)
\end{array}
\]

Here we used that the induced map $\phi$ belongs to $A$ and hence has a cokernel, the map $Z \twoheadrightarrow \text{coker}(\phi)$ is a deflation by axiom $\textbf{R1}$. The upper-right square is a pushout by proposition 2.2. It follows that $Z \twoheadrightarrow \text{coker}(\phi)$ is the cokernel of $g$.

By remark 5.6, $\phi$ factors as $\text{coker}(f) \twoheadrightarrow \text{im}(\phi) \twoheadrightarrow \text{coker}(gf)$. Taking the pullback of $\text{im}(\phi) \twoheadrightarrow \text{coker}(gf)$ along $Z \twoheadrightarrow \text{coker}(gf)$ we obtain the commutative diagram:

\[
\begin{array}{cccc}
X & \xrightarrow{f} & Y & \xrightarrow{\text{coker}(f)} \\
\downarrow & & \downarrow & \\
im(gf) & \xrightarrow{\phi} & Z & \xrightarrow{\text{coker}(gf)} \\
\downarrow & & \downarrow & \\
im(gf) & \xrightarrow{\text{im}(\phi)} & P & \xrightarrow{\text{im}(\phi)} \\
\downarrow & & \downarrow & \\
im(gf) & \xrightarrow{\text{im}(\phi)} & Z & \xrightarrow{\text{im}(\phi)} \\
\end{array}
\]

By lemma 2.11, $P \twoheadrightarrow Z$ is an inflation. By proposition 5.12, the map $Y \to P$ is a deflation whose kernel belongs to $A$. It follows that $g$ is an admissible weak isomorphism. □

### 5.4. Saturation

The 2-out-of-3-property yields saturation of the right multiplicative system $S_{-1}A\mathcal{C}$.

**Lemma 5.14.** Let $\mathcal{C}$ be a right exact category and let $A$ be an idempotent complete right percolating subcategory. Let $e: X \to X$ be any idempotent in $\mathcal{C}$. If $Q(e) = 0$, then $X \cong \ker(e) \oplus A$ for some $A \in A$.

**Proof.** This follows from the first part of the proof of [30, lemma 1.17.6] and corollary 3.15. □

**Proposition 5.15.** Let $\mathcal{C}$ be a right exact category and let $A$ be an abelian right percolating subcategory. The set $S_A$ of weak isomorphisms is saturated.

**Proof.** Let $f: Y \to Z$ be map that descends to an isomorphism in $S_{-1}A\mathcal{C}$. By definition, there exists a map $(g, s) \in \text{Mor}(S_{-1}A\mathcal{C})$ such $(f, 1) \circ (g, s) \sim (1, 1)$, i.e. there exists a commutative diagram:

\[
\begin{array}{ccc}
Z' & \xrightarrow{fg} & Z \\
\downarrow & & \downarrow \\
Z & \xrightarrow{\sim} & M & \xrightarrow{\sim} & Z \\
\downarrow & & \downarrow & & \downarrow \\
Z & \xrightarrow{s} & M & \xrightarrow{b} & Z
\end{array}
\]

Since $fg \in S_A$, we can take the pullback of $f$ along $fg$ (see proposition 5.10). We obtain the following commutative diagram

\[
\begin{array}{cccc}
Y & \xrightarrow{f} & Z & \\
\downarrow & & \downarrow & \\
P & \xrightarrow{\alpha} & M & \\
\downarrow & & \downarrow & \\
M & \xrightarrow{\beta} & Z
\end{array}
\]
where the square is a pullback and $\gamma: M \to P$ is induced by the pullback property. Clearly, $\alpha$ is a retraction and $\gamma$ is a section. Since $Q(f)$ is invertible in $S_A^{-1}C$ and the localization functor commutes with pullbacks (see proposition 2.17), we know that $Q(\alpha)$ is invertible in $S_A^{-1}C$ and that $Q(\alpha)^{-1} = Q(\gamma)$. It follows that the kernel of $\alpha$ is zero in $S_A^{-1}C$. This implies that $Q(1_P - \gamma \circ \alpha) = 0$. Lemma 5.14 shows that $Q(\alpha)^{-1} = Q(\gamma)$.

It follows that the kernel of $\alpha$ is zero in $S_A^{-1}C$. This implies that $Q(1_P - \gamma \circ \alpha) = 0$. Lemma 5.14 shows that $P \sim K \oplus A$ with $A \in A$. We infer that $\ker(\gamma \circ \alpha) = \ker(\alpha) = A$. As $C$ satisfies $R0^*$ (see remark 5.3) and $\alpha$ is a retraction, we may infer that $\alpha$ is a deflation in $C$. It follows that $A \in S_A$. From the 2-out-of-3 property, it follows that $f \in S_A$. \qed

6. Quillen’s obscure axiom under localizations

Let $A$ be a right percolating subcategory of a right exact category $C$. In this section, we show that if $C$ is weakly idempotent complete and satisfies $R3$, then the same holds for $C/A$, whenever $A$ is a strongly right percolating subcategory (see definition 3.2). Recall from remark 2.7 that for a weakly idempotent complete category $C$, the condition $R3$ is equivalent to: if $gf$ is a deflation, then $g$ is a deflation.

**Theorem 6.1.** Let $C$ be a weakly idempotent complete strongly right exact category and let $A$ be a strongly right percolating subcategory. The localization $C/A$ is also a weakly idempotent complete strongly right exact category.

**Proof.** By theorem 4.7, we know that $C/A = S_A^{-1}C$ is a right exact category obtained as the localization of $C$ with respect to the right multiplicative system $S$ of weak isomorphisms. We now show that if $f: X \to Y$ and $g: Y \to Z$ are two maps in $S_A^{-1}C$ such that $gf$ is a deflation, then $g$ is a deflation. Consider a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
Y & \xrightarrow{g} & Z
\end{array}
\]

in $S_A^{-1}C$. The diagram lifts to a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\uparrow & & \uparrow \\
X' & \xrightarrow{g'} & X'' \\
\downarrow & & \downarrow \\
Y & \xrightarrow{g''} & Z \\
\end{array}
\]

in $C$. We claim that we can choose a lift

\[
\begin{array}{ccc}
\tilde{X} & \sim & X \\
\downarrow & & \downarrow \\
\tilde{Y} & \sim & Z
\end{array}
\]

in $C$ such that this diagram descends to $gf$ under the localization functor $Q$. Indeed, applying axiom $RMS2$ four times we obtain the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X'' \\
\uparrow & & \uparrow \\
X' & \xrightarrow{g'} & X'' \\
\downarrow & & \downarrow \\
Y & \xrightarrow{g''} & Z \\
\end{array}
\]

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X'' \\
\uparrow & & \uparrow \\
X_1 & \xrightarrow{g'} & X_2 \\
\downarrow & & \downarrow \\
Y & \xrightarrow{g''} & Z \\
\end{array}
\]

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X'' \\
\uparrow & & \uparrow \\
X_1 & \xrightarrow{g'} & X_2 \\
\downarrow & & \downarrow \\
Y & \xrightarrow{g''} & Z \\
\end{array}
\]
which descends to a commutative diagram in $S^1_{\mathcal{A}}\mathcal{C}$. Rearranging the diagram and applying axiom RMS2 twice we obtain the diagram:

\[
\begin{array}{c}
X'' \xrightarrow{i} X_2 \\
\downarrow \quad \downarrow \\
X_1 \xrightarrow{\sim} X_3 \\
\downarrow \quad \downarrow \\
Y \xrightarrow{\sim} Y' \xrightarrow{\sim} Z \xrightarrow{\sim} Z' \xrightarrow{\sim} Z
\end{array}
\]

Applying axiom RMS2 to $X_3 \rightarrow Y$ along $\tilde{Y} \xrightarrow{\sim} Y$ we obtain the desired lift:

\[
\begin{array}{c}
\tilde{X} \quad \sim \quad X \\
\downarrow f' \quad \downarrow h \\
\tilde{Y} \quad \sim \quad g' \quad \rightarrow \quad Z
\end{array}
\]

This shows the claim. Hence, it suffices to show that, given a commutative diagram in $\mathcal{C}$:

\[
\begin{array}{c}
X \xrightarrow{\sim} \sim \quad X' \\
\downarrow f \quad \downarrow h \\
Y \rightarrow \quad g \quad \rightarrow \quad Z
\end{array}
\]

the morphism $g$ descends to a deflation in $S^1_{\mathcal{A}}\mathcal{C}$. We consider two cases:

**Case I** Assume that $s$ factor as $X \xrightarrow{\sim} X_1 \xrightarrow{\sim} X'$. By axiom R1 the composition $h \circ s_1$ is a deflation.

Moreover, $h$ and $h \circ s_1$ are isomorphic maps in $S^1_{\mathcal{A}}\mathcal{C}$. We have reduced the problem to a similar diagram where the length of $s$ strictly smaller.

**Case II** Assume that $s$ factor as $X \xrightarrow{\sim} X_1 \xrightarrow{\sim} X'$. By proposition 3.13 (considering only the right upper square), we obtain a commutative diagram:

\[
\begin{array}{c}
U \xrightarrow{h'} \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \right
Here the dotted arrow is obtained since $Y'$ is a pullback. As $g \circ f$ and $g' \circ f'$ are isomorphic maps in $\mathcal{C}/A$, and the length of $s_2'$ is strictly smaller than the length of $s$, we have again the problem.

Iterating both cases we find a replacement for $g \circ f$ such that $g \circ f$ equals a deflation in $S_{A}^{-1}\mathcal{C}$. Since $\mathcal{C}$ is weakly idempotent complete and satisfies axiom $\textbf{R3}$, remark 2.7 yields that $g$ descends to a deflation.

Lastly, we show that $S_{A}^{-1}\mathcal{C}$ is weakly idempotent complete. Let $r: B \to C$ be a retraction in $S_{A}^{-1}\mathcal{C}$ and $s: C \to B$ the corresponding section, then $rs = 1_{C}$ is a deflation. By the above, we conclude that $r$ is a deflation and hence has a kernel. This concludes the proof. □

7. Examples and applications

In this section, we give examples of the localizations studied in this paper. We start with a comparison with [11, 30]. Next, we show that a right exact category with $\textbf{R0}^{\ast}$ is a category with fibrations (thus, in particular, a coWaldhausen category). In this way, we have a natural definition of the $K$-theory of a right exact category. We show that the quotient behaves as expected on the level of the Grothendieck groups.

We then proceed by considering some more specific examples of percolating subcategories. In §7.3, we consider torsion theories in exact categories and give sufficient conditions for the torsion-free part to be a right percolating subcategory or a right special filtering subcategory. In §7.4, we consider the case of a quasi-abelian category (also called an almost abelian category) and show that axiom $\textbf{P2}$ simplifies in this setting.

Finally, we consider two more explicit examples. The first example (§7.5) concerns the category $R - \text{LC}$ locally compact modules over a discrete ring $R$. It was shown in [5] that the subcategory $R - \text{LC}_{D}$ of discrete modules is, in general, neither a left nor a right special filtering subcategory. We show that $R - \text{LC}_{D}$ is a right percolating subcategory so that we can consider the quotient category $R - \text{LC}/R - \text{LC}_{D}$.

In the second example (§7.6), we give an example coming from the theory of glider representations. Here, we show explicitly that the quotient is not left exact.

7.1. Comparison to localization theories of exact categories. Localizations of exact categories have been considered with an eye on $K$-theoretic applications in [11, 30]. We now compare these notions with the notions introduced in this paper. We refer to figure 1 for an overview.

7.1.1. Cardenas’ localization theory. The localization theory of exact categories developed by Cardenas in [11] is recovered completely by the framework of localizations with respect to percolating subcategories. The main theorem of the localization theory developed in [11] is the following:

**Theorem 7.1.** Let $\mathcal{C}$ be an exact category and $A$ be a full subcategory satisfying axioms $\textbf{A1}$, $\textbf{A2}$ and the dual of $\textbf{A2}$. There exists an exact category $\mathcal{C}/A$ and an exact functor $Q: \mathcal{C} \to \mathcal{C}/A$ satisfying the following universal property: for any exact category $\mathcal{E}$ and exact functor $F: \mathcal{C} \to \mathcal{E}$ such that $F(A) \cong 0$ for all $A \in A$, there exists a unique exact functor $G: \mathcal{C}/A \to \mathcal{E}$ such that $F = G \circ Q$.

**Proof.** Since $\mathcal{C}$ is exact, the subcategory $A$ automatically satisfies axiom $\textbf{A3}$. Hence $A$ is both an abelian right percolating subcategory and an abelian left percolating subcategory. By proposition 5.10 and its dual, the set $S_{A}$ is a multiplicative system. By theorem 4.7 and its dual, the category $S_{A}^{-1}\mathcal{C}$ is both left and right exact and the canonical localization functor $Q: \mathcal{C} \to S_{A}^{-1}\mathcal{C}$ is exact, moreover, $S_{A}^{-1}\mathcal{C} \cong \mathcal{C}/A$ and $Q$ satisfies the desired universal property. □

7.1.2. Schlichting’s localization theory. We recall the notion of a s-filtering subcategory of an exact category introduced by Schlichting (see [30, definition 1.5]). We use the reformulation given in [7, definition 2.12 and proposition A.2].

**Definition 7.2.** Let $\mathcal{C}$ be an exact category and let $A$ be a full subcategory. The subcategory $A$ is called right special if for every inflation $A \twoheadrightarrow X$ with $A \in A$ there exists a commutative diagram

$$
\begin{array}{ccc}
A & \twoheadrightarrow & B \\
\downarrow & & \downarrow \\
\uparrow & & \uparrow \\
A & \twoheadrightarrow & C
\end{array}
$$

such that the rows are conflations in $\mathcal{C}$ and the lower row belongs to $A$. Dually, $A$ is called left special if $A^{\text{op}}$ is right special in $\mathcal{C}^{\text{op}}$.

The subcategory $A$ is called right s-filtering if it is both right filtering, i.e. satisfies axioms $\textbf{P1}$ and $\textbf{P2}$, and right special in $\mathcal{C}$.
The main results of Schlichting's localization theory can be summarized as follows (see [30, propositions 1.16 and 2.6]):

**Theorem 7.3.** Let $C$ be an exact category and let $A$ be a right s-filtering subcategory. The localization functor $Q: C \rightarrow S^+_A C$ endows $S^+_A C$ with the structure of an exact category. The functor $Q$ is universal among exact functors from $C$ to exact categories that vanish on $A$, i.e. $C/A \cong S^+_A C$.

Moreover, if $A$ is idempotent complete, the sequence

$$D^b(A) \rightarrow D^b(C) \rightarrow D^b(C/A)$$

is a Verdier localization sequence.

The localization theory developed in [30] is compatible with the localization theory with respect to percolating subcategories in the following sense.

**Proposition 7.4.** Let $C$ be an exact category and $A \subseteq C$ a full subcategory. If $A$ is a right s-filtering subcategory, then $A$ is a right percolating subcategory.

**Proof.** We only need to verify that $A$ satisfies axiom P4. To that end, let $a: A \rightarrow X$ be an inflation and let $b: X \rightarrow B$ be a deflation with $A, B \in \text{Ob}(A)$. Write $k_b: Z \rightarrow X$ for the kernel of $b$ and $c_a: X \rightarrow Y$ for the cokernel of $a$. By definition, $c_a \circ k_b \in S_A$. By [30, lemma 1.17(3)], there exists an $A^{-1}$-inflation $t: U \rightarrow Z$ such that the composition $c_a \circ k_b \circ t$ is an $A^{-1}$-inflation. It follows that we obtain the following commutative diagram:

\[
\begin{array}{ccc}
0 & \rightarrow & U \\
\uparrow & & \uparrow \\
A & \rightarrow & X \\
\downarrow & & \downarrow \\
& \rightarrow & Y
\end{array}
\]

Here we used proposition 3.10 to see that $k_b \circ t$ is an $A^{-1}$-inflation.

Since $C$ is exact, we can apply the $3 \times 3$-lemma to the above diagram. In particular we obtain the commutative square

\[
\begin{array}{ccc}
A & \rightarrow & X \\
\downarrow & & \downarrow \\
& \rightarrow & Y \\
\downarrow & & \downarrow \\
& \rightarrow & B'
\end{array}
\]

where $b'$ is the cokernel of the composition $k_b \circ t: \sim$. It follows that $b$ factors through $b'$. This completes the proof. \hfill $\square$

Let $C$ be an exact category and let $A$ be a right s-filtering subcategory. By theorem 7.3, the category $S^+_A C$ is an exact category. Using the above proposition and theorem 4.7, we may only conclude that $S^+_A C$ is a right exact category. In section 7.6 we will show that the localization of an exact category with respect to a one-sided percolating subcategory need not be exact. The next example indicates more differences between the localization theories.

**Example 7.5.** Let $Q$ be the quiver $3 \rightarrow 2 \rightarrow 1$ with relation $\beta \circ \alpha = 0$. Let $U$ be the category of finite-dimensional representations of $Q$. The category $U$ can be visualized by its Auslander-Reiten quiver:

\[
\begin{array}{ccc}
P_2 & \rightarrow & I_2 \\
\downarrow & & \downarrow \\
S_1 & \rightarrow & S_3 \\
\downarrow & & \downarrow \\
S_2 & \rightarrow & S_3
\end{array}
\]

Let $A$ be the full additive subcategory generated by the simple representations $S_1$ and $S_3$. Note that $A$ is a Serre subcategory of the abelian category $U$. It follows that $A$ is an abelian right and left percolating subcategory of $U$. On the other hand, it is straightforward to see that $A$ is neither left nor right special in $U$. Moreover, $D^b(A) \rightarrow D^b(U) \rightarrow D^b(U/A)$ is not a Verdier localization sequence.

**Remark 7.6.** (1) Given an abelian category $U$ and a Serre subcategory $A$, the quotient $U/A$ is an abelian category. Clearly $A$ is both left and right percolating in $U$. Hence the localization theory with respect to left or right percolating subcategories generalizes the abelian setting. On the other hand, example 7.5 shows that $A$ need not be left or right s-filtering.
(2) In subsequent work, we show that, given a right exact category \( C \) and a right percolating subcategory \( A \), one still obtains a Verdier localization sequence

\[
D^b_A(C) \to D^b(C) \to D^b(C/A).
\]

Here \( D^b_A(C) \) denotes the thick triangulated subcategory of \( D^b(C) \) generated by \( A \).

(3) In example 7.5, one can verify explicitly that \( D^b(A) \not\cong D^b(U) \).

### 7.2. Waldhausen categories and the Grothendieck group

Given a right exact category \( C \) and an abelian right percolating subcategory \( A \), one can encode the localization \( C/A \) into a coWaldhausen category. In this way, one can study the K-theory of \( C/A \). In particular one obtains an immediate description of the Grothendieck group of \( C/A \) (see proposition 7.11). We refer the reader to [34] for more details.

**Definition 7.7.** Let \( C \) be a category and let \( \text{cofib}(C) \) be a set of morphisms in \( C \) called cofibrations (indicated by arrows \( \rightarrow \)). The pair \((C, \text{cofib}(C))\) is called a category with cofibrations if the following axioms are satisfied:

- **W0** Every isomorphism is a cofibration and cofibrations are closed under composition.
- **W1** The category \( C \) has a zero object 0 and for each \( X \in C \) the unique map \( 0 \to X \) is a cofibration.
- **W2** Pushouts along cofibrations exist and cofibrations are stable under pushouts.

Axioms W1 and W2 yield the existence of cokernels of cofibrations, thus for every cofibration \( X \to Y \) there is a canonical cofibration sequence \( X \to Y \to Z \). A category with fibrations is defined dually. A fibration is depicted by \( \twoheadrightarrow \) and the set of fibrations is denoted by \( \text{fib}(C) \).

**Remark 7.8.** A left exact category with \( \text{L}0^\ast \) is a category with cofibrations and, dually, a right exact category with \( \text{R}0^\ast \) is a category with fibrations.

**Definition 7.9.** Let \((C, \text{cofib}(C))\) be a category with fibrations and let \( wC \) be a set of morphisms in \( C \) called weak equivalences (indicated by arrows endowed with \( \sim \)). The triple \((C, \text{cofib}(C), wC)\) is called a Waldhausen category if \( wC \) contains all isomorphisms and is closed under composition and the following axiom (called the gluing axiom) is satisfied:

- **W3** For any commutative diagram of the form

\[
\begin{array}{ccc}
Z & \to & X \\
\downarrow & & \downarrow \\
Z' & \leftarrow & X'
\end{array}
\]

the induced map \( Z \cup_X Y \to Z' \cup_X Y' \) is a weak equivalence.

A coWaldhausen category is defined dually.

**Definition 7.10.** Let \((C, \text{cofib}(C), wC)\) be a Waldhausen category. The Grothendieck group \( K_0(C) \) (often denoted as \( K_0(wC) \)) is defined as the free abelian group generated by the isomorphism classes of objects in \( C \) modulo the relations:

1. \( [X] = [Y] \) if there is a weak equivalence \( X \sim Y \).
2. \( [Z] = [X] + [Y] \) for every cofibration sequence \( X \to Z \to Y \).

The Grothendieck group of a coWaldhausen category is defined dually.

**Proposition 7.11.** Let \( C \) be a right exact category satisfying axiom \( \text{R}0^\ast \) and let \( A \) be a right percolating subcategory. Let \( \text{fib}(C) \) be the set of deflations in \( C \) and let \( wC \) be the saturated closure of the set of weak isomorphisms with respect to the subcategory \( A \). The triple \((C, \text{fib}(C), wC)\) is a coWaldhausen category. Moreover, \( K_0(C/A) \cong K_0(wC) \), where \( K_0(C/A) \) is defined in the usual manner.

**Proof.** By assumption the category \( C \) satisfies axiom \( \text{R}0^\ast \). By remark 7.8, the pair \((C, \text{fib}(C))\) is a category with fibrations. We now show that \( wC \) satisfies the gluing axiom. Consider a commutative diagram:

\[
\begin{array}{ccc}
Z & \to & X \\
\downarrow & & \downarrow \\
Z' & \leftarrow & X'
\end{array}
\]

Here, arrows endowed with \( \sim \) are weak equivalences.
By axiom R0* and the dual of [1, proposition 5.7] we obtain a commutative diagram:

\[
\begin{array}{ccc}
Z \cap_X Y & \longrightarrow & Z \oplus Y \\
\downarrow & & \downarrow \\
Z' \cap_X Y' & \longrightarrow & Z' \oplus Y'
\end{array}
\]

As the localization commutes with kernels, the induced map \( Z \cap_X Y \rightarrow Z' \cap_X Y' \) descends to an isomorphism. It follows that the triple \((C, \text{fib}(C), wC)\) is a coWaldhausen category.

By theorem 4.7, the quotient category \( C/\mathcal{A} \) is a right exact category. Note that \( \text{Ob}(C) = \text{Ob}(C/\mathcal{A}) \).

One readily verifies that the map \( f: K_0(wC) \rightarrow K_0(C/\mathcal{A}) \), which is the identity on objects, is a group morphism. Let \( g: K_0(C/\mathcal{A}) \rightarrow K_0(wC) \) be the identity on objects. Let \( X \twoheadrightarrow Y \twoheadrightarrow Z \) be a conflation in \( C/\mathcal{A} \). This conflation can be represented by a diagram

\[
\begin{array}{ccc}
X & \twoheadrightarrow & Y \\
\downarrow & & \downarrow \\
Z
\end{array}
\]

which descends to a commutative diagram in \( C/\mathcal{A} \) and such that the vertical arrows descend to isomorphisms. Hence \([X] = [X], [Y] = [Y]\) and \([Z] = [Z]\) in \( K_0(wC) \). As \( X \twoheadrightarrow Y \twoheadrightarrow Z \) is a conflation in \( C \), we obtain \([Y] = [X] + [Z]\) in \( K_0(wC) \). It follows that \( g \) defines a group morphism as well. Clearly \( f \) and \( g \) are inverse to each other. We conclude that \( K_0(C/\mathcal{A}) \cong K_0(wC) \).

\[\square\]

7.3. Torsion theory in exact categories. In [4], a definition of a torsion theory is given in for general homological categories. We restrict ourselves to the context of exact categories. We relate torsion pairs in exact categories to percolating subcategories.

**Definition 7.12.** Let \( \mathcal{E} \) be a one-sided exact category. A torsion theory in \( \mathcal{E} \) is a pair of full subcategories \((\mathcal{T}, \mathcal{F})\) such that

1. \( \text{Hom}(T, F) = 0 \) for all \( T \in \mathcal{T} \) and all \( F \in \mathcal{F} \),
2. for any object \( M \in \mathcal{E} \) there exists a conflation

\[
T \twoheadrightarrow M \twoheadrightarrow F
\]

in \( \mathcal{E} \) with \( T \in \mathcal{T} \) and \( F \in \mathcal{F} \).

A torsion theory \((\mathcal{T}, \mathcal{F})\) is called hereditary if \( \mathcal{T} \) is a Serre subcategory of \( \mathcal{E} \) and cohereditary if \( \mathcal{F} \) is a Serre subcategory.

The following lemma is [4, lemma 4.2].

**Lemma 7.13.** Let \( \mathcal{E} \) be a one-sided exact category and let \((\mathcal{T}, \mathcal{F})\) be a torsion theory. For any object \( M \in \mathcal{E} \) a conflation

\[
M_T \twoheadrightarrow M \twoheadrightarrow M_F
\]

with \( M_T \in \mathcal{T} \) and \( M_F \in \mathcal{F} \) is unique up to isomorphism.

**Corollary 7.14.** The inclusion functor \( i: \mathcal{T} \rightarrow \mathcal{E} \) has a right adjoint \( t \) and the inclusion functor \( j: \mathcal{F} \rightarrow \mathcal{E} \) has a left adjoint \( f \).

The following is called a sequentially right exact functor in [23, definition 3.1].

**Definition 7.15.** Let \( \mathcal{E} \) and \( \mathcal{D} \) be one-sided exact categories and let \( F: \mathcal{E} \rightarrow \mathcal{D} \) be a functor. The functor \( F \) is called right exact if any conflation \( X \twoheadrightarrow Y \twoheadrightarrow Z \) is mapped to a sequence

\[
TX \rightarrow TY \rightarrow TZ
\]

where \( TX \rightarrow TY \) is admissible and \( TY \rightarrow TZ \) is its cokernel.

**Proposition 7.16.** Let \( \mathcal{E} \) be an exact category and let \((\mathcal{T}, \mathcal{F})\) be a cohereditary torsion theory.

1. The category \( \mathcal{F} \) satisfies axioms \( P1, P2, \) and \( P3 \).
2. If \( jf: \mathcal{E} \rightarrow \mathcal{E} \) is right exact, then \( \mathcal{F} \) is a right percolating subcategory of \( \mathcal{E} \).
3. If \( jf: \mathcal{E} \rightarrow \mathcal{E} \) is exact, then \( \mathcal{F} \) is a right s-filtering subcategory of \( \mathcal{E} \).
Proof. (1) Axiom P1 and P3 are automatic. We show that axiom P2 holds. Let \( \alpha : X \to F \) be a morphism with \( F \in \mathcal{F} \). By lemma 7.13 we obtain the commutative diagram:

\[
\begin{array}{ccc}
X_T & \xrightarrow{\alpha} & F \\
\downarrow & & \downarrow \\
X & \xrightarrow{\alpha} & F \\
\downarrow & & \downarrow \\
X_F & \xrightarrow{\alpha} & F \\
\end{array}
\]

The composition \( X_T \Rightarrow X \Rightarrow F \) is zero since \( \text{Hom}(\mathcal{T}, \mathcal{F}) = 0 \). It follows that \( f \) factors through the cokernel \( X \to X_F \). This shows axiom P2.

(2) Let \( \alpha : F \to X \) be an inflation with \( F \in \mathcal{F} \) and let \( \beta : X \to F' \) be a deflation with \( F' \in \mathcal{F} \). Consider the commutative diagram:

\[
\begin{array}{ccc}
F & \xrightarrow{\alpha} & X \\
\gamma L F & \xrightarrow{(jf)(\alpha)} & \gamma L X \\
\downarrow & & \downarrow \\
F & \xrightarrow{\beta} & F' \\
\end{array}
\]

By assumption \( (jf)(\alpha) \) is an admissible map and the dotted arrow is obtained as in the proof of axiom P2 above. Clearly \( \text{im}(jf)(\alpha)) \in \mathcal{F} \), this shows axiom P4.

(3) This is a straightforward adaptation of the above argument. \( \square \)

Example 7.17. Let \( \mathcal{U} \) be the category of finite-dimensional representations of the quiver \( A_4 \). The category \( \mathcal{U} \) can be visualized by its Auslander-Reiten quiver:

\[
\begin{array}{ccc}
P_4 & \xrightarrow{P_3} & I_2 \\
\downarrow & & \downarrow \\
P_2 & \xrightarrow{P_3} & X \\
\downarrow & & \downarrow \\
S_1 & \xrightarrow{S_2} & S_3 \\
\downarrow & & \downarrow \\
& \xrightarrow{S_4} & \\
\end{array}
\]

Let \( \mathcal{E} \) be the full additive subcategory of \( \mathcal{U} \) generated by \( S_1, P_2, P_3, P_4, S_2, X, I_2 \) and \( S_4 \). Clearly, \( \mathcal{E} \) is exact as it is an extension-closed subcategory of \( \mathcal{U} \). Let \( \mathcal{T} \) be the full additive subcategory of \( \mathcal{E} \) generated by \( S_1, P_4, I_2 \) and \( S_4 \) and let \( \mathcal{F} \) be the full additive subcategory of \( \mathcal{C} \) generated by \( S_2 \) and \( X \). One readily verifies that \( (\mathcal{T}, \mathcal{F}) \) is a cohereditary torsion pair in \( \mathcal{E} \). The functor \( jf \) is right exact and thus \( \mathcal{F} \) is a right percolating subcategory. On the other hand, \( \mathcal{F} \) is not right special in \( \mathcal{E} \).

Note that \( K_0(\mathcal{E}) \cong \mathbb{Z}^4 \) is the free abelian group generated by the indecomposable projectives \( S_1, P_2, P_3 \) and \( P_4 \). By proposition 7.11, the Grothendieck group \( K_0(\mathcal{E}/\mathcal{F}) \cong \mathbb{Z}^2 \) as \( S_1, P_2 \) and \( P_3 \) are weak isomorphic.

7.4. Quasi-abelian categories. Many interesting examples of localizations with respect to percolating subcategories arise in the context of quasi-abelian categories. We recall the following definition from [31]:

Definition 7.18. An additive category \( \mathcal{C} \) is called \textit{pre-abelian} if every morphism \( f : A \to B \) in \( \mathcal{C} \) has a kernel and cokernel. In particular, in a pre-abelian category \( \mathcal{C} \), the morphism \( f \) admits a factorization

\[
A \xrightarrow{\text{coin}(f)} \xrightarrow{\tilde{f}} \text{im}(f) \xrightarrow{\text{coker}(f)} B
\]

where \( \text{coin}(f) = \text{coker}(\text{ker}(f)) \) and \( \text{im}(f) = \text{ker}(\text{coker}(f)) \). A morphism \( f \) is called \textit{regular} if \( \tilde{f} \) is an isomorphism.

A pre-abelian category \( \mathcal{C} \) is called \textit{left quasi-abelian} if cokernels are stable under pullbacks and it is called \textit{right quasi-abelian} if kernels are stable under pushouts. A pre-abelian category is called quasi-abelian if it is both left and right quasi-abelian.

Remark 7.19.

(1) Left quasi-abelian categories have a natural deflation-exact structure and right quasi-abelian categories have natural inflation-exact structure. See also [1, section 4] for useful results on one-sided quasi-abelian categories. Quasi-abelian categories inherit a natural exact structure (see also [31]).
Proposition 7.23. and only if it is an open surjection.

Proof. Assume that \( A \) satisfies axiom \( P_2 \). Let \( f : X \hookrightarrow A \) be a monomorphism such that \( A \in A \). By axiom \( P_2 \), \( f \) factors as

\[
\begin{array}{ccc}
X & \rightarrow & B \\
\downarrow & & \downarrow \\
& & A
\end{array}
\]

with \( B \in A \). Since \( f \) is monic, the deflation \( X \twoheadrightarrow B \) is an isomorphism. Hence \( X \cong B \in A \).

Conversely, assume that \( A \) is closed under subobjects. Let \( f : X \rightarrow A \) be a morphism in \( C \) with \( A \in A \). Since \( C \) is left quasi-abelian, \( f \) factors as

\[
\begin{array}{ccc}
X & \rightarrow & \text{coim}(f) \\
\downarrow & & \downarrow \\
& & A
\end{array}
\]

By remark 7.19, the map \( \text{coim}(f) \rightarrow A \) is monic. Since \( A \) is closed under subobjects, \( \text{coim}(f) \in A \). Hence \( f \) has the desired factorization and axiom \( P_2 \) holds. \( \square \)

Corollary 7.21. Given a quasi-abelian category \( C \) and a full subcategory \( A \). The category \( A \) is a (strongly) right percolating subcategory if and only if \( A \) is a Serre subcategory, closed under subobjects, satisfying axiom \( P_4 \).

7.5. Locally compact modules. Let \( \text{LCA} \) be the category of locally compact (and Hausdorff) abelian groups. It is shown in [17, proposition 1.2] that \( \text{LCA} \) is a quasi-abelian category.

Let \( R \) be a unital ring, endowed with the discrete topology. We write \( R - \text{LC} \) for the category of locally compact (and Hausdorff) \( R \)-modules. We furthermore write \( R - \text{LC}_C \) or \( R - \text{LC}_D \) for the full subcategories given by those \( R \)-modules whose topology is compact or discrete, respectively.

Proposition 7.22. Let \( R \) be a unital ring.

1. The categories \( R - \text{LC} \) and \( \text{LC} - R \) are quasi-abelian.
2. There are quasi-inverse contravariant functors:

\[ \mathbb{D} : R - \text{LC} \rightarrow \text{LC} - R \quad \text{and} \quad \mathbb{D}' : \text{LC} - R \rightarrow R - \text{LC} \]

which interchange compact and discrete \( R \)-modules.

Proof. The first part follows from [17, proposition 1.2] (see also [5, proposition 2.2]).

The contravariant functors in the second statement are induced by the standard Pontryagin duality \( \text{LCA} \rightarrow \text{LCA} \) (see [20, theorem 1] or [5, theorem 2.3]). \( \square \)

It follows from [17] that the canonical exact structure on \( R - \text{LC} \) is described as follows: a morphism \( f : X \rightarrow Y \) is an injection if and only if it is a closed injection; a morphism \( f : X \rightarrow Y \) is a deflation if and only if it is an open surjection.

Proposition 7.23. (1) The category \( R - \text{LC}_D \) is an abelian right percolating subcategory of \( R - \text{LC} \).

The set \( S_{R - \text{LC}_D} \) of admissible weak isomorphisms is saturated.

(2) The category \( R - \text{LC}_C \) is an abelian left percolating subcategory of \( R - \text{LC} \). The set \( S_{(R - \text{LC}_C)} \) of admissible weak isomorphisms is saturated.

Proof. We first show that \( R - \text{LC}_D \) satisfies axiom \( A_1 \). Let \( A \rightarrow B \rightarrow C \) be a conflation in \( R - \text{LC} \). It is straightforward to show that if \( B \) is discrete, then so are \( A \) and \( C \). Conversely, assume that \( A \) and \( C \) are discrete. Since the singleton \( \{0_B\} \) is open in \( A \) and \( A \) has the subspace topology of \( B \), there exists an open \( U \subseteq B \) such that \( \{0_B\} = U \cap A \). Since the singleton \( \{0_C\} \) is open in \( C \), \( g^{-1}(\{0_C\}) = \ker(g) = A \) is open in \( B \). Hence \( \{0_B\} \) is open. It follows that \( B \) has the discrete topology.

Axiom \( A_2 \) follows from the observation that any map \( f : X \rightarrow A \) with \( A \) discrete induces an open surjective map \( X \rightarrow \text{im}(f) \) in \( R - \text{LC} \). Axiom \( A_3 \) is automatic since \( R - \text{LC} \) is an exact category. It follows from proposition 5.15 that the set \( S_{(R - \text{LC}_C)} \) is saturated and it follows from proposition 5.10 that weak isomorphisms are admissible. Pontryagin duality then implies the corresponding statements about \( R - \text{LC}_C \). \( \square \)
Example 4 of [5] shows that \( \text{LCA}_C \) is not left (or right) s-filtering in LCA in the sense of [30] (see definition 7.2). On the other hand, putting \( R = \mathbb{Z} \), the previous proposition implies that the category \( \text{LCA}_C \) is an abelian left percolating subcategory of LCA. It follows that \( \text{LCA} / \text{LCA}_C \) can be described as a localization with respect to the saturated left multiplicative system given by the weak \( \text{LCA}_C^{-1} \)-isomorphisms and the localization carries a natural left exact structure (see theorem 4.7).

The category \( \text{LCA}_D \) is not a left percolating subcategory of LCA. Indeed, the map \( 1_{\mathbb{R}}: (\mathbb{R}, \tau_{\text{discrete}}) \to (\mathbb{R}, \tau_{\text{trivial}}) \) is not admissible. Dually, the category \( \text{LCA}_C \) is not a right abelian percolating subcategory of LCA.

Following [5, 6], we write \( R - \text{LC}_{\mathbb{R}C} \) for the full subcategory of \( R - \text{LC} \) whose objects have a direct sum decomposition \( \mathbb{R}^n \oplus C \) (as topological groups) where \( C \) is compact. It is shown in [6, corollary 9.4] that \( R - \text{LC}_{\mathbb{R}C} \) is an idempotent complete fully exact subcategory of \( R - \text{LC} \). We write \( R - \text{LC}_{\mathbb{R}} \) for those objects of \( R - \text{LC} \) which are isomorphic to \( \mathbb{R}^n \) (with the standard topology).

As an application of proposition 7.16, we show that \( R - \text{LC}_C \) is left s-filtering in \( R - \text{LC}_{\mathbb{R}C} \). In this way, we recover [6, proposition 9.8].

**Proposition 7.24.**

1. The pair \((R - \text{LC}_C, R - \text{LC}_{\mathbb{R}C})\) is a torsion pair in \( R - \text{LC}_{\mathbb{R}C} \).

2. \( R - \text{LC}_C \) is left s-filtering in \( R - \text{LC}_{\mathbb{R}C} \).

3. \( R - \text{LC}_{\mathbb{R}} \) is right s-filtering in \( R - \text{LC}_{\mathbb{R}C} \).

**Proof.** It is clear that \((R - \text{LC}_C, R - \text{LC}_{\mathbb{R}C})\) is a torsion pair in \( R - \text{LC}_{\mathbb{R}C} \): the torsion of an object \( \mathbb{R}^n \oplus C \) is given by \( f(\mathbb{R}^n \oplus C) = \mathbb{R}^n \).

As \( R - \text{LC}_{\mathbb{R}C} \) is a fully exact subcategory of \( R - \text{LC} \) and \( R - \text{LC}_C \), it follows that \( R - \text{LC}_C \) satisfies \( \text{P1} \) in \( R - \text{LC} \). Moreover, as \( \text{R} \) is injective in LCA, it is clear that \( R - \text{LC}_{\mathbb{R}} \) is closed under extensions. Hence, we find that \( R - \text{LC}_{\mathbb{R}} \) also satisfies \( \text{P1} \) in \( R - \text{LC}_{\mathbb{R}C} \).

Lastly, given any conflation \( \mathbb{R}^{n_1} \oplus C_1 \rightarrow \mathbb{R}^{n_2} \oplus C_2 \rightarrow \mathbb{R}^{n_3} \oplus C_3 \) in \( R - \text{LC}_{\mathbb{R}C} \), we find, by applying the functor \( f: R - \text{LC}_{\mathbb{R}C} \rightarrow R - \text{LC}_{\mathbb{R}} \), the conflation \( \mathbb{R}^{n_1} \rightarrow \mathbb{R}^{n_2} \rightarrow \mathbb{R}^{n_3} \). The 3 \( \times 3 \)-lemma shows that the torsion part \( C_1 \rightarrow C_2 \rightarrow C_3 \) is also a conflation. \( \square \)

### 7.6. An example from representation theory.

In this section, we construct an example of an exact category \( \mathcal{E} \) and an abelian right percolating subcategory \( \mathcal{A} \) such that the localization \( \mathcal{E} / \mathcal{A} \) is right exact but not left exact. This shows that in general one cannot expect that localizing with respect to an (abelian) percolating subcategory preserves exactness. This example is based on the theory of *glider representations* (see for example [9, 10]). The example also fits into the framework of [29].

Let \( k \) be a field and let \( R \) be the matrix ring

\[
R = \begin{pmatrix}
  k & 0 & 0 \\
  k[t] & k & 0 \\
  k[t] & k[t] & k[t]
\end{pmatrix}.
\]

We write \( E_{i,j} \) for the \( 3 \times 3 \)-matrix defined by \((E_{i,j})_{k,l} = \delta_{i,k} \delta_{j,l}\), where \( \delta_{i,k} \) is the Kronecker delta. We write \( e_1, e_2, e_3 \) for the primitive orthogonal idempotents, i.e. \( e_i = E_{i,i} \). Let \( \mathcal{C} \) be the abelian category of left \( R \)-modules. Given an \( R \)-module \( M \), we have that \( M \cong e_1 M + e_2 M + e_3 M \) as a \( k \)-vector space. Note that \( e_3 M \) is a \( k[t] \)-module. Let \( \mathcal{E} \) be the full subcategory of \( \mathcal{C} \) of all left \( R \)-modules \( M \) such that the maps

\[
i_1: e_1 M \leftrightarrow e_2 M : m \mapsto E_{2,1}m \\
i_2: e_2 M \leftrightarrow e_3 M : m \mapsto E_{3,2}m
\]

are injective. For simplicity, we write an object of \( \mathcal{E} \) as \( e_1 M \leftrightarrow e_2 M \leftrightarrow e_3 M \). One readily verifies that \( \mathcal{E} \) is extension closed in \( \mathcal{C} \) and therefore inherits a natural exact structure. Using [2, proposition B.3] we see that \( \mathcal{E} \) is in fact a quasi-abelian category. Indeed, one can verify that \( \mathcal{E} \) is closed under subobjects and contains all projective \( R \)-modules. It follows that \( \mathcal{E} \) arises as the the torsion-free part of a cotilting torsion pair.

Let \( \mathcal{A} \) be the full subcategory of \( \mathcal{E} \) consisting of all \( R \)-modules such that \( e_1 M = 0 \) \( e_2 M \). Clearly \( \mathcal{A} \) is equivalent to the abelian category of \( k[t] \)-modules. Consider the map \( \phi \) in \( \mathcal{E} \) given by the following commutative diagram:

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\uparrow & & \uparrow \\
k & \rightarrow & k
\end{array}
\]

One readily verifies that \( \ker(\phi) = 0 = \operatorname{coker}(\phi) \) in \( \mathcal{E} \). It follows that if \( \phi \) is admissible, it is an isomorphism. However, \( \phi \) does not admit a right inverse. It follows that \( \mathcal{A} \) is not left percolating in \( \mathcal{E} \). On the other
hand, it is easy to see that \( \mathcal{A} \) is abelian right percolating in \( \mathcal{E} \). Hence, we can describe the localization \( \mathcal{E}/\mathcal{A} \) using theorem 4.7. Moreover, proposition 5.15 implies that the right multiplicative system of \( \mathcal{A}^{-1} \)-isomorphisms is saturated.

**Lemma 7.25.** Let \( M, N \in \text{Ob}(\mathcal{E}/\mathcal{A}) \) such that \( M \cong N \). Then \( e_1M \cong e_1N \) and \( e_2M \cong e_2N \) as \( k \)-vector spaces.

**Proof.** Let \( f : X \to Y \) be an \( \mathcal{A}^{-1} \)-isomorphism in \( \mathcal{E}/\mathcal{A} \) and write \( f_i \) for the induced map \( e_iX \to e_iY \). Since \( \ker(f_j), \coker(f_j) \in \mathcal{A} \), we have that \( \ker(f_j) = 0 = \coker(f_j) \) for \( j = 1 \) or 2. It follows that \( f_1 \) and \( f_2 \) are isomorphisms of \( k \)-vector spaces.

Assume that \( M \cong N \) in \( \mathcal{E}/\mathcal{A} \) and let \((g; L \to N, s; L \to M) \in \text{Hom}_{\mathcal{E}/\mathcal{A}}(M, N)\) be an isomorphism in \( \mathcal{E}/\mathcal{A} \). Since \( Q(g) \) is also an isomorphism in \( \mathcal{E}/\mathcal{A} \) and \( S_\mathcal{A} \) is saturated, \( g \) is an \( \mathcal{A}^{-1} \)-isomorphism. It follows that \( g_1, g_2, s_1 \) and \( s_2 \) are isomorphisms and hence \( e_1M \cong e_1N \) and \( e_2M \cong e_2N \) as \( k \)-vector spaces. □

We now show that the localization \( \mathcal{E}/\mathcal{A} \) is not left exact by explicitly showing the failure of axiom L1. Consider the following commutative diagram in \( \mathcal{E} \):

\[
\begin{array}{ccc}
\text{Re}_2 & \xrightarrow{f} & tk[t] \\
\text{Re}_2 \oplus \text{Re}_2 & \xrightarrow{g} & tk[t] \\
\text{Re}_1 & \xrightarrow{h} & kt \oplus kt^2 \\
\end{array}
\]

One can verify that \( f : \text{Re}_2 \to \text{Re}_2 \oplus \text{Re}_2 \) is an inflation, \( g \) is an \( \mathcal{A}^{-1} \)-isomorphism, and \( h : M \to \text{Re}_1 \) is an inflation. It follows that the composition \( \text{Re}_2 \xrightarrow{f} \text{Re}_2 \oplus \text{Re}_2 \xrightarrow{g} M \) descends to an inflation in \( \mathcal{E}/\mathcal{A} \). The cokernel of \( hgf \) in \( \mathcal{E}/\mathcal{A} \) is given by \( k \leftrightarrow k \leftrightarrow k \). A direct computation shows that \( \ker(\text{coker}(hgf)) \) is given by \( 0 \leftrightarrow kt \leftrightarrow kt^2 \leftrightarrow tk[t] \). By lemma 7.25, \( \ker(\text{coker}(hgf)) \not\cong \text{Re}_2 \) in \( \mathcal{E}/\mathcal{A} \). It follows that \( hgf \) is not an inflation in \( \mathcal{E}/\mathcal{A} \). This shows that axiom L1 is not satisfied.

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