SEPARATED SETS AND THE FALCONER CONJECTURE FOR POLYGONAL NORMS

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Abstract. The Falconer conjecture [F86] asserts that if \( E \) is a planar set with Hausdorff dimension strictly greater than 1, then its Euclidean distance set \( \Delta(E) \) has positive one-dimensional Lebesgue measure. We discuss the analogous question with the Euclidean distance replaced by non-Euclidean norms \( \| \cdot \|_X \) in which the unit ball is a polygon with \( 2K \) sides. We prove that for any such norm, and for any \( \alpha > K/(K-1) \), there is a set of Hausdorff dimension \( \alpha \) whose distance set has Lebesgue measure 0.

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\( \S 0. \) INTRODUCTION

A conjecture of Falconer [F86] asserts that if a set \( E \subset \mathbb{R}^2 \) has Hausdorff dimension strictly greater than 1, then its Euclidean distance set

\[ \Delta(E) = \Delta_{l^2}(E) = \left\{ \| x - x' \|_{l^2} : x, x' \in E \right\} \]

has positive one-dimensional Lebesgue measure. The current best result in this direction is due to Wolff [W99], who proved that the conclusion is true if \( E \) has Hausdorff dimension greater than 4/3. Erdogan [Er03] extended this result to higher dimensions, proving that the same conclusion holds for subsets of \( \mathbb{R}^d \) with Hausdorff dimension greater than \( d(d+2)/2(d+1) \). This improves on the earlier results of Falconer [F86], Mattila [M87], and Bourgain [B94].

A similar question can be posed for more general two-dimensional normed spaces. More precisely, if \( X \) is such a space and \( E \subset X \), then we define the \( X \)-distance set of \( A \) as

\[ \Delta_X(E) = \{ \| x - x' \|_X : x, x' \in E \} \]

and ask how the size of \( \Delta_X(E) \) depends on the dimension of \( E \) as well as on the properties of the norm \( \| \cdot \|_X \). Simple examples show that Falconer’s conjecture as stated above, but with \( \Delta(E) \) replaced by \( \Delta_X(E) \), cannot hold for all normed spaces \( X \). For instance, let

\[ \| x \|_{l^2_{\infty}} = \max(|x_1|, |x_2|) \]

and let \( E = F \times F \), where \( F \) is a subset of \([0,1]\) with Hausdorff dimension 1 such that \( F - F := \{ x - x' : x, x' \in F \} \) has measure 0. (It is an easy exercise to modify the Cantor set construction to produce such a set.) Then \( E \) has Hausdorff dimension 2, but its \( l^2_{\infty} \)-distance set \( F - F \) has measure 0.

Here and below, we use \( \dim(E) \) to denote the Hausdorff dimension of \( E \), \( |F|_d \) to denote the \( d \)-dimensional Lebesgue measure of \( F \), and \( |A| \) to denote the cardinality of a finite set \( A \).

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**Definition 0.1.** Let \( 0 < \alpha < 2 \). We will say that the \( \alpha \)-Falconer conjecture holds in \( X \) if for any set \( E \subset X \) with \( \dim(E) > \alpha \) we have \( |\Delta_X(E)|_1 > 0 \).

Iosevich and the second author [IL04] proved that the \( 3/2 \)-Falconer conjecture holds if the unit ball in \( X \),

\[
BX = \{ x \in \mathbb{R}^2 : \| x \|_X \leq 1 \},
\]
is strictly convex and its boundary \( \partial BX \) has everywhere nonvanishing curvature, in the sense that the diameter of the chord

\[
\{ x \in BX : x \cdot v \geq \max_{y \in BX} (y \cdot v) - \epsilon \},
\]

where \( v \) is a unit vector and \( \epsilon > 0 \), is bounded by \( C\sqrt{\epsilon} \) uniformly for all \( v \) and \( \epsilon \).

We do not know of any counterexamples to the 1-Falconer conjecture in normed spaces with \( BX \) strictly convex.

On the other hand, if \( BX \) is a polygon, then the above example shows that the \( \alpha \)-Falconer conjecture may fail for all \( \alpha < 2 \). The purpose of this paper is to examine this situation in more detail.

**Theorem 1.** Let \( BX \) be a symmetric convex polygon with \( 2K \) sides. Then there is a set \( E \subset [0,1]^2 \) with Hausdorff dimension \( \geq K/(K-1) \) such that \( |\Delta_X(E)|_1 = 0 \).

If we assume that there is a coordinate system in which the slopes of all sides of \( K \) are algebraic, then a stronger result is known [KL04].

**Corollary 2.** [KL04] If \( BX \) is a polygon with finitely many sides, and if there is a coordinate system in which all sides of \( BX \) have algebraic slopes, then there is a compact \( E \subset X \) such that the Hausdorff dimension of \( E \) is 2 and the Lebesgue measure of \( \Delta_X(E) \) is 0.

In particular, Corollary 2 can be applied to all polygons \( BX \) with 4 or 6 sides. We do not know if the same assertion is true for all polygonal norms. However, using recent results on Diophantine approximations, one can prove it for almost all polygons \( BX \). Fixing a coordinate system, we can define, for any non-degenerate segment \( I \subset X \), its slope \( Sl(I) \): if the line containing \( I \) is given by an equation \( u_1x_1 + u_2x_2 + u_3 = 0 \), then we set \( Sl(I) = -u_1/u_2 \). We write \( Sl(I) = \infty \) if \( u_2 = 0 \).

**Theorem 3.** For any integer \( K \geq 2 \) and for almost all \( \gamma_1, \ldots, \gamma_K \) the following is true. If \( BX \) is a symmetric convex polygon with \( 2K \) sides, and the slopes of non-parallel sides are equal to \( \gamma_1, \ldots, \gamma_K \), then there is a compact \( E \subset X \) such that the Hausdorff dimension of \( E \) is 2 and the Lebesgue measure of \( \Delta_X(E) \) is 0.

Actually, we will prove the stronger result: if the slopes of 3 non-parallel sides of \( BX \) are fixed, then for almost all choices of slopes of other \( K-3 \) non-parallel sides the required compact \( A \) exists (recall that for \( K \leq 3 \) Theorem 3 follows from Corollary 2).
§1. PROOF OF THEOREM 1

We may assume that $K \geq 4$, since otherwise Corollary 2 applies. We use $B(x, r)$ to denote the closed Euclidean ball with center at $x$ and with radius $r$. We also denote $A - A = \{a - a' : a, a' \in A\}$ and $A \cdot v = \{a \cdot v : a \in A\}$.

Let $b_1, \ldots, b_K$ be vectors such that

$$BX = \bigcap_{k=1}^K \{x : |x \cdot b_k| \leq 1\}.$$

Then for any $x \in X$,

$$\|x\|_X = \max_{1 \leq k \leq K} |x \cdot b_k|.$$

Let also $a_1, \ldots, a_K$ be unit vectors parallel to the $K$ sides of $BX$, so that

$$a_j \cdot b_j = 0, \quad j = 1, \ldots, K.$$

**Lemma 1.1.** Assume that $K \geq 4$. Then there are arbitrarily large integers $n$ for which we may choose sets $A = A(n) \subset B(0, 1/2)$ such that $|A| = n$ and

$$|(A - A) \cdot b_k| \ll n^{1-1/K}, \quad k = 1, 2, \ldots, K,$$

(in particular, $|\Delta_X(A)| \ll n^{1-1/K}$), and

$$\|x - x'\|_X \gg n^{-1/2}, \quad x, x' \in A, \quad x \neq x',$$

with the implicit constants independent of $n$.

**Proof.** Fix a large integer $N$, and let $u_1, \ldots, u_K$ be numbers in $[1, 2]$, to be determined later. Define

$$S = \left\{ \sum_{k=1}^K \frac{j_k}{N} u_k a_k : j_k \in \{1, \ldots, N\} \right\}.$$

We claim that the set

$$U = \{(u_1, \ldots, u_K) \in \mathbb{R}^K : |S| < N^K \}$$

has $K$-dimensional measure 0. Indeed, if $|S| < N^K$, then we must have

$$\sum_{k=1}^K \frac{j_k}{N} u_k a_k = 0$$

for some $j_1, \ldots, j_K \in \{1 - N, \ldots, N - 1\}$, not all zero. Fix such $j_1, \ldots, j_K$. Then the $2 \times K$ matrix with columns $\frac{j_k}{N} u_k a_k$, $k = 1, \ldots, K$, has rank at least 1, hence its nullspace has dimension at most $K - 1$. It follows that $U$ is a union of a finite number of hyperplanes of dimension at most $K - 1$, therefore has $K$-dimensional measure 0 as claimed.
We will assume henceforth that \((u_1, \ldots, u_K) \notin U\). Then \(|S| = N^K\) and \(S \subset B(0,2K)\). Our goal is to obtain (1.1), (1.2) for \(n = N^K\) and \(A = (4K)^{-1}S\).

We first prove that (1.1) holds, i.e.

\[(S - S) \cdot b_k \ll N^{K-1} \ll n^{-1/K}, \ k = 1, 2, \ldots, K.\]  

Indeed, let \(x \in S - S\), then \(x = \sum_{k=1}^{K} \frac{1}{N} u_k a_k\) for some \(j_1, \ldots, j_K \in \{1-N, \ldots, N-1\}\). Fix \(k_0 \in \{1, \ldots, k\}\), then

\[x \cdot b_{k_0} = \sum_{k=1}^{K} \frac{j_k}{N} u_k a_k \cdot b_{k_0} = \sum_{k \neq k_0} \frac{j_k}{N} u_k a_k \cdot b_{k_0},\]

where we also used (1.4). The last sum can take at most \((2N)^{K-1}\) possible values, which proves (1.5).

It remains to verify that there is a choice of \(u_1, \ldots, u_K\) for which (1.2) also holds. We will do so by proving that if \(t\) is a sufficiently small constant, depending only on \(K\) and on the angles between the non-parallel sides of \(BX\), then the set

\[(u_1, \ldots, u_K) \in [1, 2]^K: \ |x|_X \leq tN^{-K/2} \text{ for some } x \in S - S\]

has \(K\)-dimensional Lebesgue measure strictly less than 1.

Let \(x \in S - S\), then \(x = \sum_{k=1}^{K} \frac{1}{N} u_k a_k\) for some \(j_k \in \{1-N, \ldots, N-1\}\). Suppose that \(x \neq 0\) and

\[|x|_X \leq tN^{-K/2}.\]

Assume that \(|j_{k_1}| \geq |j_{k_2}| \geq \cdots \geq |j_{k_K}|\), and that \(|j_{k_1}| \in [2^s, 2^{s+1})\) for some integer \(s\) such that \(1 \leq 2^s \leq N\). If we had \(|j_{k_2}| < 2^{s-2}/K\), then we would also have

\[|x|_X \geq \left| \frac{j_{k_1}}{N} u_{k_1} a_{k_1} \right|_X - \sum_{k \neq k_1} \left| \frac{j_k}{N} u_k a_k \right|_X \geq \frac{2^s}{N} - K \cdot \frac{2^{s-2}}{KN} = \frac{2^{s-1}}{N} \geq \frac{1}{2N}.\]

But if \(K \geq 4\), then (1.7) implies that \(|x|_X \leq tN^{-2}\), which contradicts the last inequality if \(t \leq 1\) and \(N > 2\). It follows that

\[|j_{k_1}| \geq 2^{s}, \ |j_{k_2}| \geq 2^{s-2}/K.\]

Fix \(j_{k_1}, j_{k_2}\) as in (1.8). Fix also \(y = \sum_{k \neq k_1, k_2} \frac{1}{N} u_k a_k\), and consider the set of \((u_{k_1}, u_{k_2}) \in \mathbb{R}^2\) such that (1.7) holds, i.e.

\[\left| \frac{j_{k_1}}{N} u_{k_1} a_{k_1} + \frac{j_{k_2}}{N} u_{k_2} a_{k_2} + y \right|_X \leq tN^{-K/2}.\]

By (1.8), this set has 2-dimensional measure

\[\leq c_1(tN^{-K/2})^2 \cdot \frac{N}{2^s} \cdot \frac{NK}{2^{s-2}} = 4c_1K \cdot t^2N^{2-K}/2^{2s}.\]
Here and through the rest of the proof of the lemma, $c_1, c_2, c_3$ denote constants which may depend on $K$ and on the angles between the non-parallel sides of $BX$, but are independent of $t$ and $N$.

Integrating over $u_k$, $k \neq k_1, k_2$, we see that the set
\[ \{(u_1, \ldots, u_K) \in [1, 2]^K : \| \sum_{k=1}^{K} \frac{jk}{N} u_k \|_X \leq tN^{-K/2} \}, \]
with fixed $j_1, \ldots, j_K$ such that
\[ 2^s \leq \max_{k=1, \ldots, K} |j_k| < 2^{s+1}, \]
has $K$-dimensional measure $\leq 4c_1 K \cdot t^2 N^{2-K} / 2^{2s}$.

The number of $K$-tuples $j_1, \ldots, j_K$ satisfying (1.9) is $\leq (2^{s+2})^K$, hence summing over all such $K$-tuples we get a set of measure
\[ \leq c_2 t^2 N^{2-K} 2^{(K-2)s}. \]
Now sum over all $s$ with $2^s \leq N$. We find that the measure of the set in (1.6) is
\[ \leq c_2 \sum_{s=1}^{\infty} t^2 N^{2-K} 2^{(K-2)s} \leq c_3 t^2 N^{2-K} 2^{K-2} = c_4 t^2. \]
This is less than 1 if $t < \sqrt{c_3}$, as claimed.

Proof of Theorem 1. We construct $E$ as follows. Take a small positive number $c$ which will be specified later. Let $A_j = A(n_j)$ be as in Lemma 1.1, where a nondecreasing sequence $\{n_j\}$ and a sequence $\{N_j\}$ are such that
\[ N_j = \prod_{\nu=1}^{j} n_\nu, \quad n_j \to \infty \quad (j \to \infty), \quad \log n_{j+1} / \log N_j \to 0 \quad (j \to \infty). \]
(We consider that the empty product for $j = 0$ is equal to 1.) Also, fix $s = (K-1)/K > 1/2$. Let also $c$ be small enough so that for any $j$ the discs $B(x, cn_j^{-s})$, $x \in A_j$, are mutually disjoint and contained in $B(0,1)$; this is possible by (1.2). Denote
\[ \delta_j = cn_j^{-s}, \quad \Delta_j = \prod_{\nu=1}^{j} \delta_j = c^j N_j^{-s}. \]
Let $E_1 = \bigcup_{x \in A_j} B(x, \delta_1)$. We then define $E_2, E_3, \ldots$ by induction. Namely, suppose that we have constructed $E_j$ which is a union of $N_j$ disjoint closed discs $B_i$ of radius $\Delta_j$ each. Then $E_{j+1}$ is obtained from $E_j$ by replacing each $B_i$ by the image of $\bigcup_{x \in A_{j+1}} B(x, \delta_{j+1})$ under the unique affine mapping which takes $B(0,1)$ to $B_i$ and preserves direction of vectors. We then let $E = \bigcap_{j=1}^{\infty} E_j$.

We will first prove that $E$ has Hausdorff dimension at least $1/s$. The calculation follows closely that in [F85], pp. 16–18.

Let $B_j$ be the family of all discs of radius $\Delta_j$ used in the construction of $E_j$, and let $B = \bigcup_{j=0}^{\infty} B_j$, where we set $B_0 = \{B(0,1)\}$. We then define
\[ \mu(F) = \inf \left\{ \sum_{i=1}^{\infty} N_j^{-1} : F \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), B(x_i, r_i) \in B_j(i) \right\}, \]
\[ 5 \]
for all $F \subset E$. Clearly, $\mu$ is an outer measure on subsets of $E$. Observe that if $B = B(x, \Delta_j) \in B_j$, then

\[
N_j^{-1} = n_{j+1} \cdot N_{j+1}^{-1} = \sum_{B' \in B_{j+1} : B' \subset B} (N_{j+1})^{-1},
\]

hence the sum in (1.11) does not change if we replace a disc $B \in B_j$ by all its subdiscs from the next iteration $B_{j+1}$. In particular, we may assume that all the discs in the covering of $F$ in (1.11) have radius less than $\delta$ for any $\delta > 0$.

We first claim that if $B_0 = B_0(x_0, r_0) \in B_j$ then

\[
\mu(E \cap B_0) = N_j^{-1}.
\]

The inequality $\mu(E \cap B_0) \leq N_j^{-1}$ is obvious, by taking a covering of $E \cap B_0$ by the single ball $B_0$. Let now $E \cap B_0 \subset \bigcup_i B_i$, where $B_i \in B$ has radius $r_i = \Delta_j(i)$. We need to prove that

\[
\sum r_i^{1/s} \geq r_0^{1/s}.
\]

Since $E$ is compact and $B_i$ are open relative to $E$, we may assume that the covering is finite. We may also assume that all $B_i$ are disjoint, since otherwise we may simply remove any discs contained in any other disc of the covering. If the covering consists of the single disc $B_0$, we are done. Otherwise, let $B_I$ be one of the covering discs with smallest $r_i$, say $B_I \in B_j$, and let $\tilde{B}_I \in B_{j-1}$ be such that $B_I \subset \tilde{B}_I$. Then $\tilde{B}_I \subset B_0$, hence all discs in $B_j$ contained in $\tilde{B}_I$ are also contained in $B_0$. By the minimality of $r_I$, these discs belong to the covering $\{B_i\}$. We then replace all these discs by the single disc $\tilde{B}_I$; by (1.12), the sum on the left side of (1.14) does not change. Iterating this procedure, we eventually arrive at a covering consisting only of $B_0$, which proves (1.14).

Next, we prove that for any $s' > s$

\[
\mu(E \cap B) \ll r^{1/s'}
\]

for any disc $B = B(x, r)$, not necessarily in $B$, where the constant in $\ll$ may depend on $s'$. We may assume that $r \leq 1$, since otherwise we have from (1.13) with $B_0 = B(0, 1)$

\[
\mu(E \cap B) \leq \mu(E) = 1 \leq r^{1/s'},
\]

which proves (1.15). Let $j \geq 0$ be such that $r \in (\Delta_{j+1}, \Delta_j)$, and consider all discs in $B_j$ which intersect $E \cap B$. They are closed, mutually disjoint discs which intersect $B$ and have radius no less than $r$; hence there are at most 6 such discs. Applying (1.13) to each of these discs and summing up, we have

\[
\mu(E \cap B) \leq 6 N_j^{-1}.
\]

Moreover,

\[
r > \Delta_{j+1} = N_j^{-s} n_{j+1}^{-s} e^{-j-1},
\]

and we get (1.15) using (1.10).
Thus, if $s' > s$ and $\{B_i\}_{i=1}^\infty$ is a covering of $E$ by discs of radii $r_i$, then from (1.15) we have

$$\sum_{i=1}^\infty r_i^{1/s'} \gg \sum_{i=1}^\infty \mu(E \cap B_i) \geq \mu(E).$$

Taking the infimum over all such coverings, we see that

$$H_{1/s}(E) > 0.$$ 

Since $s' > s$ is arbitrary, we conclude that the Hausdorff dimension of $E$ is at least $K/(K-1)$.

It remains to prove that $|\Delta_X(E)|_1 = 0$. From (1.1) we have

(1.16) \[ |(A - A) \cdot b_k| \leq CN^{1-1/K}, \quad k = 1, 2, \ldots, K, \]

with $C$ independent of $n$. We choose $c$ small enough so that

(1.17) \[ cC < 1/2. \]

Let $D_j$ be the set of the centers of the discs in $B_j$. We claim that

(1.18) \[ |(D_j - D_j) \cdot b_k| \leq C^jN_j^s, \quad k = 1, 2, \ldots, K. \]

Indeed, for $j = 1$ this is (1.16). Assuming (1.18) for $j$, we now prove it for $j + 1$. Let $x, x' \in D_{j+1}$. Then $x \in B(y, \Delta_j)$, $x' \in B(y', \Delta_j)$, $y, y' \in D_j$. We write

(1.19) \[ (x - x') \cdot b_k = (y - y') \cdot b_k + ((x - y) - (x' - y')) \cdot b_k. \]

The first term on the right is in $(D_j - D_j) \cdot b_k$, hence has at most $C^jN_j^s$ possible values. Also, by construction $x - y, x' - y'$ are in $\Delta_j A_{j+1}$, hence the second term is in $\Delta_j (A_{j+1} - A_{j+1}) \cdot b_k$ and has at most $CN_{j+1}^s$ possible values, by (1.16). This gives at most $C^{j+1}N_{j+1}^s$ possible values for (1.19), as required.

By (1.18), (1.3) and the triangle inequality, $\Delta_X(E_j)$ can be covered by at most $KC^jN_j^s$ intervals of length $2c_0\Delta_j = 2c_0c^jN_j^-s$, where $c_0$ is the $X$-diameter of $B(0, 1)$. It follows that

$$|\Delta_X(E_j)|_1 \leq 2Kc_0(cC)^j \leq 2Kc_0(1/2)^j,$$

by (1.17). The last quantity goes to 0 as $j \to \infty$. Since $\Delta_X(E) \subset \Delta_X(E_j)$, this proves our claim that $|\Delta_X(E)|_1 = 0$. The proof of the theorem is complete.

Remark. It is easy to check that the set constructed in the proof of Theorem 1 has the Hausdorff dimension exactly $K/(K-1)$.

§2. PROOF OF THEOREM 3

The case $K \leq 3$ is covered by Corollary 2. We consider that $K > 3$ and denote $d = K - 3$. Denote

$$\mathcal{I} = \{l_1, \ldots, l_d\} \in \mathbb{Z}_+^d,$$

$$\mathcal{L}(L) = \{\mathcal{I} : 0 \leq l_k < L (k = 1, \ldots, d)\}.$$ 

For a real vector $\mathcal{P} = (\gamma_1, \ldots, \gamma_d)$ we write $\mathcal{P} \in (KM)$ if for any positive integer $L$ and for any $\varepsilon > 0$

$$\inf_{\mathcal{I} \in \mathcal{L}(L)} \sum_{|l_1|^{\varepsilon} \ldots |l_d|^{(1+\varepsilon)L^d}} \left( \max_{\mathcal{I} \in \mathcal{L}(L)} |n_{\mathcal{P}}(\mathcal{I})|^{(1+\varepsilon)L^d} \right) > 0,$$

where infimum is taken over all nonzero integral vectors $\{n_{\mathcal{P}} : \mathcal{I} \in \mathcal{L}\}$. The following theorem easily follows from the results of Kleinbock and Margulis [KM98].
Theorem A. For almost all $\gamma \in \mathbb{R}^d$ we have $\gamma \in (KM)$.

The results of [KM98] have been refined in [BKM01], [Be02], [BBKM02].

Now we formulate the main result of this section.

Theorem 4. Let $\gamma \in (KM)$, $K = d + 3$, and let $BX$ be a symmetric convex polygon with $2K$ sides, and the slopes of non-parallel sides are equal to $\gamma_1, \ldots, \gamma_d, 0, 1, \infty$, then there is a compact $E \subset X$ such that the Hausdorff dimension of $E$ is $2$ and the Lebesgue measure of $\Delta_X(E)$ is $0$.

Formally, Theorem 4 deals with polygons $BX$ of special kind, but it is easy to see that for any polygon we can make slopes of three sides of it equal to $0, 1, \infty$ by a choice of a coordinate system. Indeed, if $I_1, I_2, I_3$ are 3 non-parallel sides of $BX$, then, taking the $x_1$-coordinate axis and the $x_2$-coordinate axis of a new coordinate system parallel to $I_1$ and $I_3$ respectively, we get $\text{Sl}(I_1) = 0, \text{Sl}(I_3) = \infty$; moreover, the slope of $I_2$ can be made equal to $1$ by scaling and, if necessary, reflecting, the $x_2$-coordinate axis. Thus, combining Theorem A and Theorem 4 we get Theorem 3 (and also its stronger version mentioned in the end of §0).

We use notation introduced in the beginning of §1. To prove Theorem 4, we need a lemma similar to Lemma 1.1.

Lemma 2.1. Assume that $K, d, \gamma, BX$ satisfy the conditions of Theorem 4. Then for any $\varepsilon > 0$ there are arbitrarily large integers $n$ for which we may choose sets $A = A(n) \subset B(0, 1/2)$ such that $|A| = n$ and

\begin{equation}
(A - A) \cdot b_k \ll n^{(1/2)+\varepsilon}, \ k = 1, 2, \ldots, K,
\end{equation}

(in particular, $|\Delta_X(A)| \ll n^{(1/2)+\varepsilon}$), and

\begin{equation}
\|x - x'\|_X \gg n^{-1/2-\varepsilon}, \ x, x' \in A, \ x \neq x',
\end{equation}

where the implicit constants may depend on $\varepsilon$ but are independent of $n$.

Proof. Fix a positive integer $L > 1/\varepsilon$. Next, fix a large integer $N$. Define

\begin{equation}
S_0 = \left\{ \sum_{T \in \mathcal{L}(L)} \frac{\tilde{\gamma}}{N} \gamma_1^{l_1} \cdots \gamma_d^{l_d} : \tilde{\gamma} \in \{1, \ldots, N\} \right\}.
\end{equation}

and $S = S_0 \times S_0$, that is

$S = \{(x_1, x_2) : x_1, x_2 \in S_0\}$.

For any $x \in S_0$ we have

\[|x| \leq \sum_{T \in \mathcal{L}(L)} |\gamma_1|^{l_1} \cdots |\gamma_d|^{l_d} = \sum_{l=0}^{L-1} |\gamma_1|^l \cdots \sum_{l=0}^{L-1} |\gamma_d|^l \leq \gamma^{dL},\]

where

$\gamma = \max(|\gamma_1|, \ldots, |\gamma_d|) + 1.$
Therefore, \( S \subset B(0, 2\gamma dL) \). Our goal is to check that \(|S| = n\) and to obtain (2.1), (2.2) for \( n = N^{2Ld}\) and \( A = (4\gamma dL)^{-1}S\).

We consider that \( a_k \) \((k = 1, \ldots, d)\) are parallel to the sides with slopes \( \gamma_1, \ldots, \gamma_d \) respectively and \( a_{d+1}, a_{d+2}, a_{d+3} \) are parallel to the sides with slopes \( 0, 1, \infty \) respectively. Thus, we can take \( b_k = (-\gamma_k, 1) \) for \( k = 1, \ldots, d \), \( b_{d+1} = (0, 1), b_{d+2} = (-1, 1), b_{d+3} = (1, 0) \).

We first prove (2.1) for \( k = 1, \ldots, d \), i.e.

\[
(2.4) \quad |(S - S) \cdot b_k| \ll n^{\frac{1}{2} + \epsilon}.
\]

Indeed, for \( x \in (S - S) \cdot b_k, k_0 = 1, 2, \ldots, d \), we have a representation

\[
x \cdot b_k = -\gamma_k \sum_{T \in L(L)} \frac{j_T}{N} \gamma_1^{l_1} \cdots \gamma_d^{l_d} + \sum_{T \in L(L)} \frac{j'_T}{N} \gamma_1^{l_1} \cdots \gamma_d^{l_d},
\]

where \( j_T, j'_T \in \{1 - N, \ldots, N - 1\} \) \((T \in L(L))\).

Denote

\[
L(L, k_0) = \{ T : 0 \leq l_k < L \ (k = 1, \ldots, d; k \neq k_0), \ 0 \leq l_{k_0} \leq L \}.
\]

Then we have

\[
x \cdot b_{k_0} = \sum_{T \in L(L, k_0)} \frac{j_T}{N} \gamma_1^{l_1} \cdots \gamma_d^{l_d}
\]

with

\[
j_T \in \{2 - 2N, \ldots, 2N - 2\} \quad (T \in L(L, k_0)).
\]

Hence,

\[
|(S - S) \cdot b_{k_0}| \ll (4N)^{Ld + Ld - 1}.
\]

By the choice of \( L \) we have \( L^d + L^{d-1} \ll (1 + \epsilon)L^d \), and we get (2.4). for \( k = 1, \ldots, d \).

Next, (2.4) holds for \( k = d+1, d+2, d+3 \) because for those \( k \) and for \( x \in (S - S) \cdot b_k \) we have a representation

\[
x \cdot b_k = \sum_{T \in L(L)} \frac{j_T}{N} \gamma_1^{l_1} \cdots \gamma_d^{l_d}
\]

with

\[
j_T \in \{2 - 2N, \ldots, 2N - 2\} \quad (T \in L(L)).
\]

Hence,

\[
|(S - S) \cdot b_k| \leq (4N)^{Ld},
\]

and we again get (4.2) for sufficiently large \( N \). So, (2.1) is proved.

Now observe that the supposition \( T \in (KM) \) implies that elements of \( S_0 \) with different representations (2.3) are distinct. This gives \(|S_0| = N^{L^d}\) and thus \(|S| = |S_0|^2 = n\) as required. Moreover, since for any \( x, x' \in S_0 \) there is a representation

\[
x - x' = \sum_{T \in L(L, k_0)} \frac{j_T}{N} \gamma_1^{l_1} \cdots \gamma_d^{l_d}
\]
with \( j \in \{1 - N, \ldots, N - 1\} \) \( (l \in L(L, k_0)) \).

we conclude from the supposition \( \gamma \in (KM) \) that for \( x \neq x' \)

\[(2.5) |x - x'| \gg (2N)^{-1+0.1\varepsilon}L^d - 1.\]

By the choice of \( L \), we have \((1 + 0.1\varepsilon)L^d + 1 \leq (1 + 1.1\varepsilon)L^d \), and from (2.5) we get for sufficiently large \( N \) and distinct \( y, y' \in A \)

\[\|y - y'\|_X \gg (4\gamma^dL)^{-1} (2N)^{-(1+1.1\varepsilon)}L^d \gg N^{-(1+2\varepsilon)L^d} = n^{-1/2-\varepsilon}.\]

This completes the proof of Lemma 2.1.

**Proof of Theorem 4.** We construct \( E \) as follows. Let \( A_j = A(n_j) \) be as in Lemma 2.1 with \( \varepsilon = \varepsilon_j \), where a nondecreasing sequence \( \{n_j\} \), a sequence \( \{N_j\} \), and a sequence \( \{\varepsilon_j\} \) are such that

\[N_j = \prod_{\nu=1}^j n_{\nu}, \quad n_j \to \infty (j \to \infty), \quad \log n_{j+1}/\log N_j \to 0, \quad \varepsilon_j \to 0 (j \to \infty).\]

(We consider that the empty product for \( j = 0 \) is equal to 1.) Let also all \( n_j \) be large enough so that for any \( j \) the discs \( B(x, n_j^{-1/2-2\varepsilon_j}) \), \( x \in A_j \), are mutually disjoint and contained in \( B(0, 1) \); this is possible by (2.2). Denote

\[\delta_j = n_j^{-1/2-2\varepsilon_j}, \quad \Delta_j = \prod_{\nu=1}^j \delta_j.\]

Let \( E_1 = \bigcup_{x \in A_1} B(x, \delta_1) \). We then define \( E_2, E_3, \ldots \) by induction. Namely, suppose that we have constructed \( E_j \) which is a union of \( N_j \) disjoint closed discs \( B \) of radius \( \Delta_j \) each. Then \( E_{j+1} \) is obtained from \( E_j \) by replacing each \( B_i \) by the image of \( \bigcup_{x \in A_{j+1}} B(x, \delta_{j+1}) \) under the unique affine mapping which takes \( B(0, 1) \) to \( B_i \) and preserves direction of vectors. We then let \( E = \bigcap_{j=1}^\infty E_j \). The verification of properties \( \text{dim}(E) = 2 \) and \( |\Delta_X(E)| = 0 \) is exactly as in the proof of Theorem 1.

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