BV YANG-MILLS AS A HOMOTOpy CHERN-SIMONS VIA SFT

Anton M. Zeitlin

Department of Mathematics, Yale University,
442 Dunham Lab, 10 Hillhouse Ave, New Haven, CT 06511,
anton.zeitlin@yale.edu

Received Day Month Year
Revised Day Month Year

We show explicitly how BV Yang-Mills action emerges as a homotopy generalization of Chern-Simons theory from the algebraic constructions arising from String Field Theory.

Keywords: Yang-Mills, Batalin-Vilkovisky, String Field Theory

PACS numbers: 11.25.Hf, 11.25.Sq, 11.30.Na

1. Introduction: Chern-Simons vs Yang-Mills

Chern-Simons-like theories have played the important role in both Quantum Field Theory and String Theory for a long time. The interest to such theories started from original 3d Chern-Simons theory\(^1\)\(^2\) with the action functional

\[
S_{CS} = \int_{M^3} \text{Tr}\left(\frac{1}{2} A \wedge dA + \frac{1}{3!} [A \wedge A] \wedge A\right),
\]

which was one of the first considered Topological Field Theories. Soon after that, the generalized Chern-Simons theories with the action

\[
S^{\text{gen}}_{CS} = \frac{1}{2} \langle \Psi, Q\Psi \rangle + \frac{1}{3!} \{\Psi, \Psi, \Psi\},
\]

where \(Q\) is some nilpotent operator, \(\langle \cdot, \cdot \rangle\) is some pairing and \(\{\cdot, \cdot, \cdot\}\) is some graded (anti)symmetric 3-linear operation, drew a lot of attention, e.g. in relation to open String Field Theory\(^3\).

The equations of motion for the theories of type \(^2\) are generalized Maurer-Cartan equations

\[
Q\Psi + \frac{1}{2} [\Psi, \Psi] = 0,
\]

where \([\cdot, \cdot]\) is some graded (anti)symmetric bilinear operation, which in the case of the usual 3d Chern-Simons theory is simply the zero curvature equation.

\(^1\)http://math.yale.edu/~az84, \(^2\)http://www.ipme.ru/zam.html
In paper Ref. 4, motivated by the well-known fact from open SFT\cite{5-7}, that the Maxwell equations are given by the linear equation
\[ Q\phi_A = 0, \quad \phi_A = (-i\gamma_1 A_\mu(x) a_{\mu}^{\dagger} - c_0 \partial^\mu A_\mu(x))|0\rangle, \] (4)
where \( Q \) is BRST operator of open string\cite{8}, we gave homological meaning to general Yang-Mills equations. Namely, we considered the subcomplex \( \mathcal{F} \) of the BRST complex of open SFT and constructed the graded (w.r.t. the ghost number) operations \([\cdot,\cdot]_h, [\cdot,\cdot,\cdot]_h\) on \( \mathcal{F}_g = \mathcal{F} \otimes g \) (\( g \) is some reductive Lie algebra), which together with BRST operator formed a homotopy Lie algebra. The Yang-Mills equations
\[ \partial_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}] = 0 \] (5)
appeared to be the generalized Maurer-Cartan equations associated with this homotopy Lie algebra:
\[ Q\phi_A + 1/2[\phi_A, \phi_A]_h + 1/3! [\phi_A, \phi_A, \phi_A]_h = 0. \] (6)
In this paper, we write the action for the Yang-Mills theory and moreover its Batalin-Vilkovisky (BV) version\cite{9}
\[ S_{BV}^{YM} = \int d^D x (1/2 (F_{\mu\nu}(x) F^{\mu\nu}(x))_K + 2(D_\mu \omega(x), A^\mu(x))_K - \langle [\omega(x), \omega(x)], \omega^*(x)\rangle_K), \] \[ S_{HCS} = -1/2 \langle \Phi, Q \Phi \rangle - 1/3! \{\Phi, \Phi, \Phi\}_h - 1/4! \{\Phi, \Phi, \Phi, \Phi\}_h, \] (8)
where \( \langle \cdot,\cdot \rangle \) is appropriately defined pairing, and \( \{\cdot,\cdot,\cdot\}_h = \langle [\cdot,\cdot,\cdot]_h, \cdot \rangle \) are the graded antisymmetric n-linear operations (n=3,4) on \( \mathcal{F}_g \). In the language of open SFT, \( \Phi \in \mathcal{H}_g \), which leads to the BV Yang-Mills action, has the following form:
\[ \Phi(\omega, \omega^*, A, A^*) = \langle \omega(x) - i\gamma_1 A_\mu(x) a^\mu_1 - c_0 \partial_\mu A^\mu(x) + ic_1 c_0 A^\mu_1(x) a^\mu_1 - c_1 c_0 c_1 \omega^*(x)\rangle |0\rangle, \] (9)
where \( \omega \) is the ghost field and \( \omega^*, A^* \) are antifields of ghost and gauge field. We note here that in Refs. 10 and 11 the nonabelian versions of Yang-Mills actions were obtained from the effective actions of canonical open SFT\cite{9} and WZW-like superSFT\cite{12} correspondingly. In Ref. 13 the BV Yang-Mills action up to 3-point terms was obtained from open SFT. It would be interesting to find out how our constructions are related to the algebraic structure of open SFT.
Continuing our comparison of Yang-Mills and Chern-Simons theories, we remind that the construction of BV quantization of usual 3d Chern-Simons theory leads to the action (see e.g. Refs. 14 and 15):
\[ S_{CS}^{BV} = \int_{M^3} Tr (1/2 \Psi d\Psi + 1/3 \Psi^3), \] (10)
where

$$\Psi = \omega + A_\mu dx^\mu + \frac{1}{4} \epsilon^{\mu \nu \rho} A_\nu^* dx^\rho \wedge dx^\nu + \frac{1}{24} \epsilon^{\mu \nu \rho} \omega^* dx^\mu \wedge dx^\nu \wedge dx^\rho. \quad (11)$$

One can see the evident similarity between the expressions (9) and (11).

Finally, we note that the action (8) reminds the action for the closed SFT, constructed by Zwiebach\(\^{16}\), which in comparison to the action (8) contains infinite number of terms.

The outline of the paper is as follows. In section 2, we firstly remind some constructions we considered in Ref.\(\^{4}\). We give the definition of Yang-Mills chain complex\(\mathcal{F}_g\) and show that the BRST operator\(\mathcal{Q}\) can be reduced to the\(sl(2, \mathbb{R})\) Chevalley operator. We also give a new realization of this complex which allows us to construct the graded (w.r.t the ghost number) symmetric pairing. Then, we define the 2-linear and 3-linear graded antisymmetric operations which satisfy, together with operator\(\mathcal{Q}\), the homotopy Lie algebra\(\^{17}\) relations. To formulate the homotopy Chern-Simons action, we introduce the multilinear operations \{\cdot, \ldots, \cdot\}_h\) which we already mentioned above. After that, we show that the pure Yang-Mills action can be reformulated in the form (8). In the last part of section 2, we demonstrate how we arrived to the multilinear operations generating the homotopy Lie algebra, namely, we show how it emerges from the OPE in the boundary CFT of open string on the upper half-plane\(\^{18}\).

The main result of section 3 is the formulation of the BV Yang-Mills action (7) in the form (8). To do this, first of all we consider a tensor product of the complex\(\mathcal{F}_g\) with some Grassmann algebra\(A\) to introduce the degrees of freedom of fermion statistics. Then, it is reasonable to change the grading in the complex and consider the total ghost number (taking into account\(\mathbb{Z}\)-gradation in\(A\)) making the resulting complex\(\mathcal{H}_g = \mathcal{F}_g \otimes A\) to be infinite. After that, we appropriately redefine all algebraic structures we constructed for\(\mathcal{F}_g\) in the case of\(\mathcal{H}_g\). In the end of section 3, we find the correspondence between (7) and (8).

In section 4, we give a quick review of BV formalism\(\^{9, 19, 20}\) and consider the gauging of BV Yang-Mills action in the homotopy Chern-Simons form. The paper ends with Final Remarks Section.

It should be noted, that the local version (when all the fields are constant) of\(L_\infty\) algebras, corresponding to Yang-Mills theory was considered in Ref.\(\^{21}\). One should also note that since the original Ref.\(\^{14}\) where the abstract statement about the relation between BV formalism and\(L_\infty\) algebras was given, there was a lack of consideration of explicit field theory examples (except for Ref.\(\^{21}\) the explicit examples were different versions of Chern-Simons theory). We hope that here we partly fill this gap.

In this paper, our main goal is to point out the relation between the structures coming from open SFT and BV formalism of pure Yang-Mills theory. In separate paper, Ref.\(\^{22}\) we shall consider the extension of our results to the case of scalar and fermion fields coupled to gauge theory.
Notation and Conventions

**BRST operator in open String theory.** In the case of open string theory in dimension $D$, one has $D$ scalar fields $X^\mu(z)$ such that the mode expansion is

$$X^\mu(z) = x^\mu - 2p^\mu \log |z|^2 + i \sum_{n=-\infty, n \neq 0}^{n=+\infty} \frac{a_n}{n} (z^{-n} + \bar{z}^{-n}),$$

(12)

where $\eta^\mu\nu$ is the constant metric in the flat $D$-dimensional space either of Euclidean or Minkowski signature. One can define the Virasoro generators

$$L_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} :a^\mu_{n-m}a_{\mu m} :,$$

(13)

such that $a^\mu_0 = 2p^\mu$, and associate to them the so-called BRST operator $Q$,

$$Q = \sum_{n=-\infty}^{\infty} c_n L_{-n} + \sum_{m,n=-\infty}^{\infty} \frac{(m-n)}{2} :c_m c_n b_{-m-n} : - c_0,$$

(14)

where $\{c_n, b_m\} = \delta_{m,n}$ and :: stand for Fock normal ordering. It is well known that $Q_B$ is nilpotent, when $D = 26$. We note that we put usual $\alpha'$ parameter equal to 2.

The so-called conformal vacuum $|0\rangle$, which is $sl(2,\mathbb{R})$-invariant (under the action of $L_0, L_{\pm 1}$), satisfies the following conditions under the action of the corresponding modes:

$$a^\mu_n |0\rangle = 0, \quad n \geq 0,$$

$$b_n |0\rangle = 0, \quad n \geq -1,$$

$$c_n |0\rangle = 0, \quad n > 1.$$

(15)

This leads to the relation $Q_B |0\rangle = 0$. We define the ghost number operator $N_g$ by

$$N_g = \frac{3}{2} + \frac{1}{2}(c_0 b_0 - b_0 c_0) + \sum_{n=1}^{\infty} (c_{-n} b_n - b_{-n} c_n).$$

(16)

The constant shift $(+3/2)$ is included to make the ghost number of $|0\rangle$ be equal to 0.

**Bilinear operation and Lie brackets.** In this paper, we will meet two bilinear operations $[\cdot, \cdot], \{\cdot, \cdot\}_h$. The first one, without the subscript, denotes the Lie bracket in the given finite-dimensional Lie algebra $\mathfrak{g}$ and the second one, with subscript $h$, denotes the graded antisymmetric bilinear operation in the homotopy Lie superalgebra.

**Operations on differential forms.** We will use three types of operators acting on differential forms with values in some finite-dimensional reductive Lie algebra $\mathfrak{g}$. The first one is the de Rham operator $d$. The second one is the Maxwell operator $m$, which maps 1-forms to 1-forms. Say, if $A = A_\mu dx^\mu$ is a 1-form, then

$$mA = (\partial_\nu \partial^\mu A_\nu - \partial^\mu \partial_\nu A_\mu)dx^\nu,$$

(17)
where indices are raised and lowered w.r.t. the metric $\eta^\mu\nu$. The third operator maps 1-forms to 0-forms, this is the operator of divergence $\text{div}$. For a given 1-form $A$,
\[
\text{div}A = \partial_\mu A^\mu.
\]
(19)

For $g$-valued 1-forms, one can also define the following (anti)symmetric bilinear and 3-linear operations:
\[
(A, B) = (A_\mu, B^\mu)_K,
\]
\[
\{A, B\} = ([A_\mu, \partial^\mu B_\nu] + [B_\mu, \partial^\mu A_\nu] + [\partial_\nu A_\mu, B^\mu] + [\partial_\nu B_\mu, A^\mu] + \partial^\mu [A_\mu, B_\nu] + \partial^\mu [B_\mu, A_\nu])dx^\nu,
\]
\[
A \cdot W = [A^\mu, W_\mu],
\]
\[
\{A, B, C\} = ([A_\mu, [B^\mu, C_\nu]] + [B_\mu, [A^\mu, C_\nu]] + [C_\mu, [B^\mu, A_\nu]] + [B_\mu, [C^\mu, A_\nu]] + [A_\mu, [C^\mu, B_\nu]] + [C_\mu, [A^\mu, B_\nu]])dx^\nu,
\]
(20)

where $(\cdot, \cdot)_K$ is the canonical invariant form on the Lie algebra $g$.

2. Yang-Mills Chain Complex and Homotopy Lie Algebra

2.1. Chain Complex

Let's consider some finite-dimensional reductive Lie algebra $g$ and the following states of open SFT
\[
\rho_u = u(x)|0\rangle, \quad \phi_A = (-ic_1 A_\mu(x)a^\mu_{-1} - c_0 \partial_\mu A^\mu(x))|0\rangle,
\]
\[
\psi_W = -ic_1 c_0 W_\mu(x)a^\mu_{-1}|0\rangle, \quad \chi_\alpha = 2c_1 c_0 c_{-1} a(x)|0\rangle
\]
(22)

associated with $g$-valued functions $u(x)$, $a(x)$ and 1-forms $A = A_\mu(x)dx^\mu$, $W = W_\mu(x)dx^\mu$. It is easy to check that the resulting space, spanned by the states like (22), is invariant under the action of the BRST operator, moreover, the following proposition holds.

**Proposition 2.1.** Let the space $\mathcal{F}_g$ be spanned by all possible states of the form (22). Then we have a chain complex:
\[
0 \rightarrow g \mathcal{F}_g^0 \xrightarrow{Q} \mathcal{F}_g^1 \xrightarrow{Q} \mathcal{F}_g^2 \xrightarrow{Q} \mathcal{F}_g^3 \rightarrow 0,
\]
(23)

where $\mathcal{F}_g^i$ $$(i=0,1,2,3)$$ is a subspace of $\mathcal{F}$ corresponding to the ghost number $i$ and $Q$ is the BRST operator (15).

**Proof.** Really, it is easy to see that we have the following formulas:
\[
Q\rho_u = 2\phi_{du}, \quad Q\phi_A = 2\psi_{mA}, \quad Q\psi_W = -\chi_{\text{div}W}, \quad Q\chi_\alpha = 0.
\]
(24)

Then the statement can be easily obtained.

**Remark 2.1.** From (24), one can see that in the case if $g$ is abelian, the first cohomology module $H_1^g(\mathcal{F})$ can be identified with the space of abelian gauge fields, satisfying the Maxwell equations modulo gauge transformations.
Now, we construct another realization of this chain complex which appears to be quite useful. Namely, we notice that acting on $F_g$, we use only $sl(2, \mathbb{R})$ part of the operator $Q$, so one can reduce $Q$ to the usual Chevalley differential for the $sl(2, \mathbb{R})$ algebra generated by $L_{\pm 1}, L_0$, that is, one can reduce $Q$ to the following operator:

$$Q = \sum_{n=-1}^{1} c_n L_{-n} - c_0 c_1 b_1 + c_0 c_{-1} b_{-1} - 2c_{-1} c_1 b_0.$$  \hspace{1cm} (25)

Moreover, one can easily see that $Q$ can be reduced further, to be presented as a differential operator

$$Q = \sum_{n=-1}^{1} c_n s_n - 2c_{-1} c_1 \frac{\partial}{\partial c_0},$$  \hspace{1cm} (26)

where

$$s_0 = -2 \frac{\partial^2}{\partial x^\mu \partial x^\mu}, \quad s_1 = -i 2 \frac{\partial^2}{\partial x^\mu \partial q^\mu}, \quad s_{-1} = -i 2 q^\mu \frac{\partial}{\partial x^\mu}.$$  \hspace{1cm} (27)

using the usual differential operator representation for the Heisenberg algebra:

$$a_0^\mu = 2 p^\mu \rightarrow -2i \frac{\partial}{\partial x^\mu}, \quad a_{-1}^\mu \rightarrow q^\mu, \quad a_1^\mu \rightarrow \frac{\partial}{\partial q^\mu}, \quad b_n \rightarrow \frac{\partial}{\partial c_n}.$$  \hspace{1cm} (28)

and eliminating all terms from $Q$, which act as zero. Therefore, it is reasonable to write $\rho_u, \phi_A, \psi_W, \chi_a$, which span the space of our chain complex, as the following Lie algebra-valued functions of $c_n, x^\mu$, and $q^\nu$:

$$\rho_u = u(x), \quad \phi_A = -i c_1 A_\mu(x) q^\mu - c_0 \partial_\mu A^\mu(x),$$
$$\psi_W = -i c_1 c_0 W_\mu(x) q^\mu, \quad \chi_a = 2 c_1 c_0 c_{-1} a(x).$$  \hspace{1cm} (29)

We note that the ghost number operator can be reduced to

$$N_g = \sum_{n=-1}^{n=1} c_n \frac{\partial}{\partial c_n}.$$  \hspace{1cm} (30)

Our next task is to define an inner product on the chain complex $F_g$ which behaves in a reasonable way under the action of the differential. First of all, for any vector $\Psi \in F_g$, which has the explicit form:

$$\Psi = \rho_u + \phi_A + \psi_W + \chi_a$$  \hspace{1cm} (31)

for some $u, A, W, a$, we define the “conjugate” differential operator:

$$\Psi^* = u(x) - i c_{-1} A_\mu(x) \frac{\partial}{\partial q^\mu} + c_0 \partial_\mu A^\mu(x) -$$
$$i c_{-1} c_0 W_\mu(x) \frac{\partial}{\partial q^\mu} - 2c_{-1} c_0 c_1 a(x).$$  \hspace{1cm} (32)

\hspace{1cm} ^aThe reader with experience in CFT can notice that it is nothing but BPZ conjugate state.
Now, we are ready to define the pairing.

**Definition 2.1.** Consider two elements $\Phi, \Psi$ of $\mathcal{F}_g$. Their pairing $\langle \Psi, \Phi \rangle$ is defined by the following formula:

$$
\langle \Psi, \Phi \rangle = \int d^Dx \int dc_1 dc_0 dc_1 (\Psi^*, \Phi) K(x, c_i),
$$

(33)

where $(\cdot, \cdot)_K$ is the canonical invariant form on $g$ and the integral over $c_{-1}, c_0, c_1$ is the standard Berezin integral.

In the case when $\Psi = \rho_u + \phi_A + \psi_U + \chi_a$ and $\Phi = \rho_v + \phi_B + \psi_V + \chi_b$, the pairing is given by:

$$
\langle \Psi, \Phi \rangle = \int d^Dx ((A, V) + (U, B))(x) - 2(u(x), b(x)) K - 2(a(x), v(x)) K.
$$

(34)

Now, we formulate as a proposition, how this inner product behaves under the action of the differential $Q$.

**Proposition 2.2.** Let $\Phi, \Psi \in \mathcal{F}_g$ be of ghost numbers $n_\Phi, n_\Psi$. Then, the following relation holds:

$$
\langle Q\Phi, \Psi \rangle = -(-1)^{n_\Phi n_\Psi} \langle Q\Psi, \Phi \rangle.
$$

(35)

**Proof.** It is easy to see that it is enough to show, that the following relations hold:

$$
\langle Q\psi W, \rho_u \rangle = -(Q\rho_u, \psi W), \quad \langle Q\phi A, \phi B \rangle = \langle Q\phi B, \phi A \rangle.
$$

(36)

The proof is straightforward:

$$
\langle Q\psi W, \rho_u \rangle = \int 2(\partial_{\mu} W^\mu, u) K = - \int 2(W^\mu, \partial_{\mu} u) K = -2(\rho_{\mu}, \psi W) = -(Q\rho_u, \psi W),
$$

$$
\langle Q\phi A, \phi B \rangle = \int 2(\partial_{\mu} \partial^{\nu} A^\nu, B_{\nu}) K = \int 2(\partial_{\mu} \partial^{\nu} B_{\nu}, A^{\nu}) K = \langle Q\phi B, \phi A \rangle.
$$

(37)

2.2. **Homotopy Lie Algebra and Yang-Mills action**

Now, we construct the graded bilinear and 3-linear operations on the space of our chain complex.

**Definition 2.2.** We define the bilinear operation

$$
[\cdot, \cdot]_h : \mathcal{F}^i_g \otimes \mathcal{F}^j_g \rightarrow \mathcal{F}^{i+j}_g,
$$

(38)
which is graded (w.r.t. to the ghost number) antisymmetric bilinear operation by the following relations on the elements of \( \mathcal{F}_g \):

\[
[p_{u}, p_{v}]_{h} = 2p_{[u,v]}, \quad [p_{u}, \phi_{A}]_{h} = 2\phi_{[u,A]}, \quad [p_{u}, \psi_{W}]_{h} = 2\phi_{[u,W]}, \\
[p_{u}, \chi_{a}]_{h} = 2\chi_{[u,a]}, \quad [\phi_{A}, \phi_{B}]_{h} = 2\phi_{(A,B)}, \quad [\phi_{A}, \psi_{W}]_{h} = -\chi_{A.W},
\]

where \( p_{u}, \phi_{v} \in \mathcal{F}^0_g, \phi_{A}, \phi_{B} \in \mathcal{F}^1_g, \psi_{W} \in \mathcal{F}^2_g, \chi_{A} \in \mathcal{F}^3_g \).

**Definition 2.3.** The operation

\[
\{\cdot, \cdot, \cdot\}_h : \mathcal{F}^i_g \otimes \mathcal{F}^j_g \otimes \mathcal{F}^k_g \rightarrow \mathcal{F}^{i+j+k-1}_g
\]

is defined to be nonzero only when all arguments lie in \( \mathcal{F}^1_g \). For \( \phi_{A}, \phi_{B}, \phi_{C} \in \mathcal{F}^1_g \), we have:

\[
[\phi_{A}, \phi_{B}, \phi_{C}]_{h} = 2\psi_{(A,B,C)}.
\]

**Remark 2.2.** Here, we note that the bilinear operation, defined in this subsection, corresponds to the lowest orders in \( \alpha' \) of that introduced in Ref. [18] (we will return to this in subsection 2.3.).

We claim that these graded antisymmetric multilinear operations satisfy the relations of a homotopy Lie algebra. Namely, the following proposition holds.

**Proposition 2.3.** Let \( a_1, a_2, a_3, b, c \in \mathcal{F}_g \) be of ghost numbers \( n_{a_1}, n_{a_2}, n_{a_3}, n_b, n_c \) correspondingly. Then the following relations hold:

\[
Q[a_1, a_2]_{h} = [Qa_1, a_2]_{h} + (-1)^{n_{a_1}}[a_1, Qa_2]_{h}, \\
Q[a_1, a_2, a_3]_{h} + [Qa_1, a_2, a_3]_{h} + (-1)^{n_{a_1}}[a_1, Qa_2, a_3]_{h} + \\
(-1)^{n_{a_1}+n_{a_2}}[a_1, a_2, Qa_3]_{h} + [a_1, [a_2, a_3]]_{h} - [[a_1, a_2], a_3]_{h} - \\
(-1)^{n_{a_1}+n_{a_2}}[a_2, [a_1, a_3]]_{h} = 0, \\
[b, [a_1, a_2, a_3]]_{h} + (-1)^{n_{a_1}+n_{a_2}+n_{a_3}}[a_1, [a_2, a_3, b]]_{h} + \\
(-1)^{n_{a_2}+n_{a_1}}[a_2, [b, a_1, a_3]]_{h} - (-1)^{n_{a_3}+n_{a_2}+n_{a_1}}[a_3, [b, a_1, a_2]]_{h} + \\
(-1)^{n_{a_1}+n_{a_2}}[a_3, [b, a_1, a_2]]_{h} = 0.
\]

The proof is given in Appendix.

**Remark 2.3.** Denoting \( d_0 = Q, d_1 = [\cdot, \cdot]_{h}, d_2 = [\cdot, \cdot, \cdot]_{h}, \) the relations [42], together with condition \( Q^2 = 0 \), can be summarized in the following way:

\[
D^2 = 0,
\]

where \( D = d_0 + \theta d_1 + \theta^2 d_2 \). Here, \( \theta \) is some formal parameter anticommuting with \( d_0 \) and \( d_2 \). We remind that \( d_0 \) raises ghost number by 1, \( d_1 \) leaves it unchanged.
while $d_2$ lowers ghost number by 1. Therefore, $d_0$, $d_2$ are odd elements as well as the parameter $\theta$, but $d_1$ is even. Hence, (43) gives the following relations:

$$
\begin{align*}
&d_0^2 = 0, \quad d_0 d_1 - d_1 d_0 = 0, \quad d_1 d_1 + d_0 d_2 + d_2 d_0 = 0, \\
&d_1 d_2 - d_2 d_1 = 0, \quad d_2^2 = 0,
\end{align*}
$$

(44)

which are in agreement with (42).

Now, we will define the following multilinear forms on the complex $F_g$ which appear to be very useful for the construction of the Yang-Mills action.

**Definition 2.4.** For any $a_1, a_2, a_3, a_4 \in F_g$, one can define the following n-linear forms ($n = 2, 3, 4$)

$$
\{\cdot, \ldots, \cdot\}_h : F_g \otimes \ldots \otimes F_g \to C
$$

in the following way:

$$
\begin{align*}
\{a_1, a_2\}_h &= \langle Q a_1, a_2 \rangle, \\
\{a_1, a_2, a_3\}_h &= \langle [a_1, a_2]_h, a_3 \rangle, \\
\{a_1, a_2, a_3, a_4\}_h &= \langle [a_1, a_2, a_3]_h, a_4 \rangle.
\end{align*}
$$

(45)

These bilinear operations satisfy the following remarkable property.

**Proposition 2.4.** The multilinear forms, introduced in Definition 2.5., are graded antisymmetric, i.e.

$$
\{a_1, \ldots, a_i, a_{i+1}, \ldots, a_n\}_h = (-1)^{n_{a_i} n_{a_{i+1}}} \{a_1, \ldots, a_{i+1}, a_i, \ldots, a_n\}_h,
$$

(46)

where $n_{a_i}$ denotes the ghost number of $a_i$.

The proof is given in Appendix.

Now, we are ready to formulate the Yang-Mills action as a homotopy Chern-Simons theory.

**Proposition 2.5.** The Yang-Mills action

$$
S_{YM} = 1/2 \int d^D x (F_{\mu \nu}(x), F^{\mu \nu}(x))_K, \quad F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]
$$

(48)

can be written as follows:

$$
\begin{align*}
S_{YM} &= - \sum_{n=2}^{4} \frac{1}{n!} \{\phi^n_A\}_h = \\
&= -\frac{1}{2} \langle Q \phi_A, \phi_A \rangle - \frac{1}{6} \{\phi_A, \phi_A, \phi_A\}_h - \frac{1}{24} \{\phi_A, \phi_A, \phi_A, \phi_A\}_h.
\end{align*}
$$

(49)

**Proof.** Really, from the definition of the brackets, one can see that:

$$
\begin{align*}
Q \phi_A &= 2\psi_{W_1}, \quad W_{1\mu} = \partial_\nu \partial^\nu A_\mu - \partial_\mu \partial^\nu A_\nu, \\
[\phi_A, \phi_A]_h &= 4\psi_{W_2}, \quad W_{3\mu} = [\partial_\nu A^\nu, A_\mu] + 2[A^\nu, \partial_\nu A_\mu] - [A^\nu, \partial_\mu A_\nu], \\
[\phi_A, \phi_A, \phi_A]_h &= 12\psi_{W_3}, \quad W_{5\mu} = [A_\nu, [A^\nu, A_\mu]].
\end{align*}
$$

(50)
Therefore,
\[ \frac{1}{2} \langle Q\phi_A, \phi_A \rangle = \langle \int d^D x (A^\nu(x), \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x))_K \rangle = \]
\[ -\frac{1}{2} \int d^D x (\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x), \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x))_K, \]
\[ \frac{1}{6} \{ \phi_A, \phi_A, \phi_A \}_h = -\int d^D x (\partial^\nu A^\mu(x) - \partial^\mu A^\nu(x))[A_\nu(x), A_\mu(x)], \]
\[ \frac{1}{24} \{ \phi_A, \phi_A, \phi_A, \phi_A \}_h = -\frac{1}{2} \int d^D x ([A_\mu(x), A_\nu(x)], [A^\mu(x), A^\nu(x)])_K. \]  

Therefore, we see that
\[ \frac{1}{2} \langle Q\phi_A, \phi_A \rangle + \frac{1}{6} \{ \phi_A, \phi_A, \phi_A \}_h + \]
\[ \frac{1}{24} \{ \phi_A, \phi_A, \phi_A, \phi_A \}_h = -\frac{1}{2} \int d^D x (F^\mu\nu(x), F^\mu\nu(x))_K. \]  

This finishes the proof. \( \square \)

Varying the action \( \text{(49)} \) with respect to \( \phi_A \), one obtains that the resulting equations of motion are the generalized Maurer-Cartan equations, which obviously coincide with Yang-Mills equations. Moreover, it appears that the gauge transformations coincide with the transformations which preserve the Maurer-Cartan equations. This can be summarized in the following proposition.

**Proposition 2.6.** \( \square \) Let \( \phi_A \) be the element of \( \mathcal{F}_0^1 \) associated with 1-form \( A = A_\mu dx^\mu \) and \( \rho_u \) be the element of \( \mathcal{F}_0^0 \) associated with Lie algebra-valued function \( u(x) \). Then, the Yang-Mills equations for \( A \) and its infinitesimal gauge transformations:
\[ \partial_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}] = 0, \quad A_\mu \rightarrow A_\mu + \epsilon (\partial_\mu u + [A_\mu, u]) \]  

can be rewritten as follows:
\[ Q\phi_A + \frac{1}{2!} \{ \phi_A, \phi_A \}_h + \frac{1}{3!} \{ \phi_A, \phi_A, \phi_A \}_h = 0, \]
\[ \phi_A \rightarrow \phi_A + \frac{\epsilon}{2} (Q\rho_u + [\phi_A, \rho_u]_h). \]

### 2.3. OPE origin of the bilinear operation \([\cdot, \cdot]_h\)

In this part, we show how we found the bilinear operation \([\cdot, \cdot]_h\).

We consider the open string theory on the disk, conformally mapped to the upper half-plane. The operator products between the coordinate fields are the following ones (we include the usual \( \alpha' \)-coefficient in the operator product):
\[ X^\mu(z_1)X^\nu(z_2) \sim -\frac{\alpha'}{2} \eta^{\mu\nu} \log |z_1 - z_2|^2 - \frac{\alpha'}{2} \eta^{\mu\nu} \log |z_1 - \bar{z}_2|^2. \]  

\[ \text{(56)} \]
The states (22) correspond to the following operators:

\[ u(X), \ cA_\mu(X)\partial X^\mu - \frac{\alpha'}{2}\partial c\partial_\mu A^\mu(X), \ c\partial cW_\mu(X)\partial X^\mu, \ c\partial c\partial^2 c a(X). \]  

Let \( A(z), B(z) \) be any two operators. Let’s consider the expression

\[ [A(t + \epsilon), B(t)], \]  

where \( t \) lies on the real axis and \([·, ·]\) means the commutator in Lie algebra \( g \). Due to (56), this object is the series in \( \epsilon \) and \( \log(\epsilon/\mu) \), therefore, this allows us to define the following operation.

**Definition 2.5.** For any two operators \( A(z), B(z) \) we define a bilinear operation:

\[ R(A, B)(t) = \mathcal{P}[A(t + \epsilon), B(t)] - (-1)^{n_A n_B} \mathcal{P}[B(t + \epsilon), A(t)], \]  

where \( \mathcal{P} \) is the projection on the \( \epsilon^0(\log(\epsilon/\mu))^0 \) term, \( t \) lies on the real axis and \( n_A, n_B \) are the ghost numbers of \( A, B \) correspondingly.

By means of straightforward calculation, substituting the operators (57) in (59), one obtains that the operation \( R \) at the lowest orders in \( \alpha' \) reproduces the bilinear operation \([·, ·]_h\). Moreover, due to the following proposition, one obtains the first nontrivial relation of the homotopy algebra, namely, the commutation relation between the operator \( Q \) and \([·, ·]_h\).

**Proposition 2.7.** Let \( A(t), B(t) \) be some operators of ghost numbers \( n_A, n_B \). Then

\[ [Q, R(A, B)] = R([Q, A], B) + (-1)^{n_A} R(A, [Q, B]), \]  

where \( Q \) is BRST operator.

For the further information on the subject see Ref. [18].

3. Fields, Antifields and BV Yang-Mills

3.1. Fermionic degrees of freedom and total ghost number

In order to introduce ghosts, antifields, i.e. the fermion degrees of freedom, we consider the tensor product of our chain complex \((F_g, Q)\) with some Grassmann algebra \( A \). We assume that \( A \) is \( \mathbb{Z} \)-graded: \( A = \oplus_{i \in \mathbb{Z}} A^i \) and if \( \lambda^i \in A^i \) and \( \lambda^j \in A^j \), then \( \lambda^i \lambda^j = (-1)^{ij} \lambda^j \lambda^i \).

Moreover, we introduce the following notation: if \( \lambda \in A^i \), we will say that \( \lambda \) is of target space ghost number \( i \). Therefore, it is reasonable to introduce the gradation w.r.t. the total ghost number, which is equal to the sum of the worldsheet ghost number, generated by the operator \( N_g \) and this target space ghost number on the space \( \mathcal{H}_g = F_g \otimes A \). Hence, if the element \( \Phi \in \mathcal{H}_g^n \) (i.e. of total ghost number \( n \)), which is written in the form \( \sum_s \Phi_s \otimes \xi_s \) such that \( \Phi_s \in F_g \) of ghost number \( n^w_s \) and \( \xi_s \in A \) of ghost number \( n^t_s \), then \( n^w_s + n^t_s = n \) for all \( s \). In the following, to simplify
the notation, we will refer to the total ghost number simply as the ghost number. In such a way, one can consider a new chain complex, which is now infinite
\[ Q^{-1} \mathcal{H}_g^{-1} Q^{-1} \mathcal{H}_g^0 Q^{-1} \mathcal{H}_g^1 Q^{-1} \mathcal{H}_g^2 Q^{-1} \ldots, \]
where \( Q = Q \otimes 1 \). The space \( \mathcal{H}_g \) is therefore spanned by the elements of the form \( \rho_u, \phi_A, \psi_W, \chi_\alpha \), where they are associated with the functions and 1-forms, which take values in \( g \otimes A \).

### 3.2. Algebraic operations and multilinear forms on the complex \( \mathcal{H}_g \)

Now, we can construct the algebraic structures, defined for the complex \( F_g \) in section 2, in the case of complex \( \mathcal{H}_g \).

**Definition 3.1.** Let \( \Phi_i \in \mathcal{H}_g \) (\( i=1,2,3 \)) such that \( \Phi_i = \sum s \Phi^s_i \otimes \xi^s_i \), where \( \Phi^s_i \in F_g \) and \( \xi^s_i \in A \). Then, one can define the following multilinear algebraic operations:

\[
\langle \cdot, \cdot \rangle : \mathcal{H}_g \otimes \mathcal{H}_g \rightarrow A, \\
[\cdot, \cdot]_h : \mathcal{H}_g \otimes \mathcal{H}_g \rightarrow \mathcal{H}_g, \\
[\cdot, \cdot, \cdot]_h : \mathcal{H}_g \otimes \mathcal{H}_g \otimes \mathcal{H}_g \rightarrow \mathcal{H}_g
\]

by means of the following expressions:

\[
\langle \Phi_1, \Phi_2 \rangle = \sum_{s,s'} \langle \Phi^s_1, \Phi^{s'}_2 \rangle \otimes \xi^s_1 \xi^{s'}_2 (-1)^{n_{\xi^s_1} n_{\xi^{s'}_2}}, \\
[\Phi_1, \Phi_2]_h = \sum_{s,s'} [\Phi^s_1, \Phi^{s'}_2]_h \otimes \xi^s_1 \xi^{s'}_2 (-1)^{n_{\xi^s_1} n_{\xi^{s'}_2}}, \\
[\Phi_1, \Phi_2, \Phi_3]_h = \sum_{s,s',s''} [\Phi^s_1, \Phi^{s'}_2, \Phi^{s''}_3]_h \otimes \xi^s_1 \xi^{s'}_2 \xi^{s''}_3 (-1)^{n_{\xi^s_1} (n_{\xi^{s'}_2} + n_{\xi^{s''}_3})} (-1)^{n_{\xi^{s'}_2} n_{\xi^{s''}_3}},
\]

where \( n \) denotes the ghost number.

It is easy to see that the operation \( \langle \cdot, \cdot \rangle \) is graded symmetric on the space \( \mathcal{H}_g \) with respect to the ghost number, i.e. for \( \Phi, \Psi \in \mathcal{H}_g \)

\[
\langle \Phi, \Psi \rangle = (-1)^{n_{\Phi} n_{\Psi}} \langle \Psi, \Phi \rangle.
\]

Similarly, one can show that on the space \( \mathcal{H}_g \), \( [\cdot, \cdot]_h \) and \( [\cdot, \cdot, \cdot]_h \) are graded anti-symmetric w.r.t. the ghost number and satisfy, together with \( Q \), the relations of the homotopy Lie algebra \([42]\). Now, we give a definition which is analogous to Definition 2.4.

**Definition 3.2.** For any \( a_1, a_2, a_3, a_4 \in \mathcal{H}_g \) one can define the following n-linear forms (\( n=2,3,4 \))

\[
\{\cdot, \cdot, \cdot\}_h : \mathcal{H}_g \otimes \ldots \otimes \mathcal{H}_g \rightarrow A
\]
in the following way:

\[
\{a_1, a_2\}_h = \{QA_1, a_2\}, \quad \{a_1, a_2, a_3\}_h = \{[a_1, a_2]_h, a_3\}, \\
\{a_1, a_2, a_3, a_4\}_h = \{[a_1, a_2, a_3]_h, a_4\}.
\] (66)

Using Proposition 2.4. and Definition 3.1., we find that the operations \{\cdot, \cdot\}_h are graded antisymmetric on \(H_\Phi\).

Finally, we mention that the properties of antisymmetric multilinear operations are related to the action of the element of Grassmann algebra \(A\) on them. First of all, we have the natural right action of an element \(a \in A\) on \(\Phi \in H_\Phi\), namely, \(\Phi \cdot a = \Phi(1 \otimes a)\). Due to Definitions 3.1. and 3.2., we have:

\[
\begin{align*}
\{\Phi_1, \ldots, \Phi_1 \cdot a + \Phi'_1, \ldots, \Phi_k\}_h &= \{\Phi_1, \ldots, \Phi_1, \ldots, \Phi_k\}_h \cdot a(-1)^{n(\chi, \chi_1 + \ldots + n_k)} \\
\{\Phi_1, \ldots, \Phi_1 \cdot a + \Phi'_1, \ldots, \Phi'_k\}_h &= \{\Phi_1, \ldots, \Phi_1 \cdot a + \Phi'_1, \ldots, \Phi'_k\}_h \cdot a(-1)^{n(\chi, \chi_1 + \ldots + n_k)}
\end{align*}
\] (67)

where \(k = 2, 3, m = 2, 3, 4\) and \(\Phi_r \in H_\Phi\) for all \(r\).

### 3.3. Ghosts, antifields and BV Yang-Mills

One of the main characters throughout this subsection is the element \(\Phi \in H_\Phi\) of ghost number 1. Using the realization of \(F_\Phi\), we have constructed in subsection 2.1., one can write the expression for such element as follows:

\[
\Phi(\omega, \omega^\ast, A, A^\ast) = \rho_\omega + \Phi_A - \psi_A \cdot -1/2 \chi_\omega^\ast = \\
\omega(x) - ic_1 A_\mu(x)q^\mu - c_0 \partial_\mu A^\mu(x) + ic_1 c_0 A^\ast_\mu(x)q^\mu - c_1 c_0 c_1 \omega^\ast(x),
\] (68)

where the target space ghost numbers of \(\omega, \omega^\ast, A, A^\ast\) are 1, -2, 0, -1 correspondingly. In this case, the analogue of Proposition 2.5. looks as follows.

**Proposition 3.1.** Consider \(\Phi = \Phi(\omega, \omega^\ast, A, A^\ast) \in H_\Phi^1\) with the notation as in (68). Then, the Homotopy Chern-Simons action

\[
S_{HC\text{S}} = -\sum_{n=2}^{4} \frac{1}{n!} \{\Phi^n\}_h = \\
-\frac{1}{2} \{Q\Phi, \Phi\} - \frac{1}{6} \{\Phi, \Phi, \Phi\}_h - \frac{1}{24} \{\Phi, \Phi, \Phi, \Phi\}_h
\] (69)

coincides with the BV Yang-Mills action

\[
S_{B\text{VYM}} = \\
S_{YM}[A] + 2 \int d^Dx(D_\mu \omega(x), A^\ast_\mu(x))_{K} - ([\omega(x), \omega(x)], \omega^\ast(x))_{K},
\] (70)

where, as usual, \(D_\mu \omega = \partial_\mu \omega + [A_\mu, \omega]\).
Proof. Really, from the definition of the brackets, one can see that:

\[ Q\Phi = Q\phi_A + 2\phi_{i\mu} - 2\chi_{\text{div}}A^*, \]
\[ [\Phi, \Phi]_h = [\phi_A, \phi_A]_h + 2\rho_{[i\mu, j\nu]} - 4\phi_[i\mu, A^] - 4\psi_{[i\mu, A^*]} + 2\chi_{[i\mu, j\nu]} + 2\chi_A A^*, \]
\[ [\Phi, \Phi, \Phi]_h = [\phi_A, \phi_A, \phi_A]_h. \]

(71)

Therefore,

\[ \frac{1}{2}\langle Q\Phi, \Phi \rangle = \frac{1}{2}\langle Q\phi_A, \phi_A \rangle + \int d^Dx (A^+ + \partial^\mu \omega)_K + \]
\[ \int d^Dx (A^, \omega) = \frac{1}{2}\langle Q\phi_A, \phi_A \rangle - 2 \int d^Dx (\partial^\mu \omega(x), A^+_K) - \]
\[ \frac{1}{6}[\Phi, \Phi, \Phi]_h = \frac{1}{6}[\phi_A, \phi_A, \phi_A] - \int d^Dx (4(\omega(x), A^+_K) + 4(A^+_K, [\omega(x), A^+_K]) - 2(\omega^*, [\omega, \omega])_K - \]
\[ \int d^Dx ([A^+, \omega(x)], A^+(x))_K \]
\[ \frac{1}{24}[\Phi, \Phi, \Phi, \Phi]_h = \frac{1}{24}[\phi_A, \phi_A, \phi_A, \phi_A]. \]

(72)

Summing all these terms, we see that \( S_{\text{HCS}} = S_{\text{BVYM}} \). This finishes the proof. \( \square \)

4. BV Formalism, Gauge Conditions and Quantum Theory.

First of all, we shortly review the basic elements of BV formalism. Consider the set of all fields \( \psi_n(x) \) in some classical field theory in \( D \) dimensions. Let’s assign to each field \( \psi_n(x) \) the antifield \( \psi^*_n(x) \), which is of opposite statistics to \( \psi_n(x) \). Then, one can define the odd Laplace operator, sometimes called BV Laplacian, and the BV bracket:

\[ \Delta_{\text{BV}} = \int d^Dx \frac{\delta}{\delta \psi_n(x)} \frac{\delta}{\delta \psi^*_n(x)}, \]
\[ (F, G)_{\text{BV}} = \int d^Dx \left( \frac{\delta R}{\delta \psi_n(x)} \frac{\delta G}{\delta \psi^*_n(x)} - \frac{\delta G}{\delta \psi^*_n(x)} \frac{\delta R}{\delta \psi_n(x)} \right), \]

(73)

where \( F, G \) are some functionals on the space of fields and antifields which can be either even or odd. The classical BV action \( S_{\text{cl}}^{\text{BV}} \) is a bosonic functional, which satisfies the classical BV Master equation:

\[ (S_{\text{cl}}^{\text{BV}}, S_{\text{cl}}^{\text{BV}})_{\text{BV}} = 0. \]

(74)

For the gauge theories, in particular for Yang Mills with the gauge-invariant action \( S[A] \), where \( A \) is a gauge field, it is easy to construct an action satisfying equation (74):

\[ S_{\text{cl}}^{\text{BV}}[A, \omega, A^*, \omega^*] = \]
\[ S[A] + 2 \int d^Dx (D_\mu \omega(x), A^{*\mu}(x))_K - ([\omega(x), \omega(x)], \omega^*(x))_K. \]

(75)
Here, $\omega$ is a so-called ghost field, $\omega^*$ is its antifield, and $A^\ast$ is the antifield for $A$. As we have seen in the previous section for Yang-Mills gauge theory, this action can be written as a homotopy Chern-Simons theory.

The action (75) satisfies the so-called quantum BV Master equation:

$$\Delta_{BV}(e^{-\frac{1}{\hbar}S[\psi,\psi^*]}) = 0.$$  

(76)

The quantum BV theory is defined by means of a path integral over some Lagrangian submanifold $L$ which is defined with respect to the odd symplectic form $\sum_n \delta \psi_n \wedge \delta \psi_n^*$, i.e. $\sum_n \delta \psi_n \wedge \delta \psi_n^*\big|_F = 0$. One of the great advantages of this formalism is that the corresponding effective action, which we find after integrating over the appropriately chosen parts of fields and antifields, will also satisfy the quantum BV Master equation.

One of the most popular Lagrangian submanifolds for the action (75) is the following one:

$$\omega^* = 0, \quad F(A) = 0, \quad A^\ast = -\frac{\delta F}{\delta A},$$  

(77)

where $F$ is some Lie algebra-valued constraint on gauge fields, $\bar{\omega}$ is what is left of $A^\ast$ degree of freedom and is usually called antighost field, and $F^\bar{\omega} = \int d^Dx(\bar{\omega}(x), F(x))_K$.

Really, substituting the values for antifields in the action, we get

$$S_{gf}^{YM} = S_{YM} - 2 \int d^Dx(D_\mu \omega(x), \frac{\delta F}{\delta A^\mu(x)})_K \bigg|_{\partial_\mu A^\mu = 0},$$  

(78)

which coincides with the usual gauge fixed action for quantum Yang-Mills theory.

For example, in the case of Lorentz gauge, when $F = \partial_\mu A^\mu$, the action reads as follows:

$$S_{gf}^{L} = S_{YM} - 2 \int d^Dx(\bar{\omega}(x), \partial_\mu D_\mu \omega(x))_K \bigg|_{\partial_\mu A^\mu = 0}$$.  

(79)

Remember that in our formalism, the condition $b_0 \Phi(A, \omega, A^\ast, \omega^*) = 0$, which in open SFT is known as Siegel gauge, is equivalent to the following conditions on fields and antifields: $\partial_\mu A^\mu = 0, \omega^* = 0, A^\ast = 0$. This is not precisely what we are willing to obtain. However, one can consider the object $R_{\bar{\omega}} = 1/2(\chi_\omega, b_0 \Phi)$. Then, one can define

$$S_{HCS}^{gL}(\Phi) = S_{HCS}(\Phi) + (S_{HCS}(\Phi), R_{\bar{\omega}}(\Phi))_{BV},$$  

(80)

which coincides with $S_{gf}^{gL}$.

5. Final Remarks

In this paper, we have reformulated BV Yang-Mills theory in the form of Homotopy Chern-Simons theory. From the very beginning, we were motivated by the constructions of open SFT. In the case of closed SFT, one can get the action for linearized gravity in the abelian Chern-Simons-like form.
We begin from the case when Proposition 2.3. Appendix A.

Acknowledgements
I would like to thank D. Borisov, I.B. Frenkel, M.M. Kapranov, M. Mvsof, T. Panet, M. Rocek, H. Sati, J. Stasheff, D. Sullivan, M.A. Vasiliev and G. Zuckerman for numerous discussions on the subject and also I.B. Frenkel and N.Yu. Reshetikhin for their permanent encouragement and support.

Appendix A.

Proposition 2.3. Let \(a_1, a_2, a_3, b, c \in \mathcal{F}\) be of ghost numbers \(n_{a_1}, n_{a_2}, n_{a_3}, n_b, n_c\) correspondingly. Then the following relations hold:

\[
Q[a_1, a_2)_h = [Qa_1, a_2)_h + (-1)^{n_{a_1}} [a_1, Qa_2)_h,
\]
\[
Q[a_1, a_2, a_3)_h + [Qa_1, a_2, a_3)_h + (-1)^{n_{a_1}} [a_1, Qa_2, a_3)_h +
\]
\[
(-1)^{n_{a_1}+n_{a_2}} [a_1, a_2, Qa_3)_h + [a_1, [a_2, a_3)_h] - [a_1, a_2)_h, a_3)_h -
\]
\[
(-1)^{n_{a_1}+n_{a_2}} [a_2, [a_1, a_3)_h] = 0,
\]
\[
[b, [a_1, a_2, a_3)_h] - (-1)^{n_{a_1}+n_{a_2}+n_{a_3}} [a_1, [a_2, a_3, b)_h] +
\]
\[
(-1)^{n_{a_2}+n_{a_3}} [a_2, [b, a_1, a_3)_h - (-1)^{n_{a_3}+n_{a_2}+n_\phi} [a_3, [b, a_1, a_2)_h]
\]
\[
= [[b, a_1)_h, a_2, a_3)_h + (-1)^{n_{a_1}+n_\phi} [a_1, [b, a_2)_h, a_3)_h +
\]
\[
(-1)^{n_{a_2}+n_{a_3}} [a_2, [b, a_3)_h] + [a_1, a_2, b, a_3)_h]_h,
\]
\[
[[a_1, a_2, a_3)_h, b, c)_h = 0. \tag{A.1}
\]

Proof. Let’s start from the first relation:

\[
Q[a_1, a_2)_h = [Qa_1, a_2)_h + (-1)^{n_{a_1}} [a_1, Qa_2)_h. \tag{A.2}
\]

We begin from the case when \(a_1 = \rho_u \in \mathcal{F}_0^0\). Then, for \(a_2 = \rho_\phi \in \mathcal{F}_0^0\) we have:

\[
Q[\rho_u, \rho_\phi)_h = 4\phi_{[u,v]} = [\rho_u, 2\phi_{[u,v]}_h + [2\phi_{[u,v]}_h, \rho_\phi)_h = [Q\rho_u, \rho_\phi)_h + [\rho_u, Q\rho_\phi)_h. \tag{A.3}
\]

Let \(a_2 = \phi_A \in \mathcal{F}_0^1\). Then

\[
Q[\rho_u, \phi_A]_h = 4\phi_{[u,A]}, \tag{A.4}
\]
We find that
\[ m[u, A] = (\partial_\mu \partial_\nu [u, A_\nu] - \partial_\mu \partial_\nu [u, A_\mu]) dx^\nu. \] (A.5)

At the same time
\[ [Q \rho_u, \phi_A]_h = 2[\phi_{\partial_\mu u}, \phi_A]_h = 4\psi_Y, \] (A.6)

where
\[ Y_\nu = 2[\partial_\mu u, \partial_\nu A_\nu] + 2[A_\mu, \partial_\nu \partial_\nu A_\nu] + [\partial_\mu A_\nu, \partial_\nu u] + [\partial_\nu A_\mu, \partial_\mu u] + [\partial_\mu \partial_\nu u, A_\nu] + [\partial_\nu \partial_\mu u, A_\nu] + [\partial_\mu A_\nu, \partial_\nu u] \] (A.7)

and
\[ [\rho_u, Q \phi_A]_h = 4\psi_{u,mA}. \] (A.8)

Summing (A.6) and (A.8) and, therefore, the relation (A.2) also holds in this case. The last nontrivial case with \( a_1 = \rho_u \) is that when \( a_2 = \psi_W \). We see that
\[ Q[\rho_u, \psi_W]_h = -2\psi_{\text{div} u, W} = -2\psi_{\text{div} u - W} + 2\psi_{u, \text{div} W} = [Q \rho_u, \psi_W]_h + [\rho_u, Q \psi_W]_h. \] (A.9)

Let’s put \( a_1 = \phi_A \in \mathcal{F}_g^1 \). Then for \( a_2 = \phi_B \in \mathcal{F}_g^1 \), we get
\[ Q[\phi_A, \phi_B]_h = -2\chi_{\text{div}(A,B)}. \] (A.10)

We find that
\[ \text{div} \{A, B\} = [\partial_\nu A_\mu, \partial_\mu B_\nu] + [A_\mu, \partial_\nu \partial_\nu B_\nu] + [\partial_\nu B_\mu, \partial_\nu A_\mu] + + [\partial_\nu \partial_\nu B_\mu, A_\mu] + [\partial_\nu B_\mu, \partial_\nu A_\nu] + [\partial_\nu B_\nu, \partial_\nu A_\mu] = \] (mA) \cdot B + (mB) \cdot A. \] (A.11)

This leads to the relation:
\[ -2\chi_{\text{div} \{A, B\}} = -2\chi_{(mA) \cdot B} - 2\chi_{(mB) \cdot A} = [Q \phi_A, \phi_B] - [\phi_A, Q \phi_B]. \] (A.12)

Therefore, (A.2) holds in this case.

It is easy to see that the relation (A.2), for the other values of \( a_1 \) and \( a_2 \), reduces to trivial one \( 0 = 0 \). Thus, we proved (A.2).

Let’s switch to the proof of the second relation, including the graded antisymmetric 3-linear operation:
\[ Q[a_1, a_2, a_3]_h + [Q a_1, a_2, a_3]_h + (-1)^{n_{\alpha_1}} [a_1, Q a_2, a_3]_h + \] (A.13)

\[ (-1)^{n_{\alpha_1} + n_{\alpha_2}} [a_1, a_2, Q a_3]_h + [a_1, [a_2, a_3]]_h - [[a_1, a_2]_h, a_3]_h = \]
\[ (-1)^{n_{\alpha_1} + n_{\alpha_2}} [a_2, [a_1, a_3]]_h = 0. \] (A.13)

It is easy to see that (A.13) is worth proving in the cases, when \( a_1 \in \mathcal{F}_g^0 \), \( a_2 \in \mathcal{F}_g^1 \), \( a_3 \in \mathcal{F}_g^1 \) and \( a_1 \in \mathcal{F}_g^1 \), \( a_2 \in \mathcal{F}_g^1 \), \( a_3 \in \mathcal{F}_g^1 \). For the other possible values of \( a_1, a_2, a_3, \)
the relation (A.13) reduces to the permutations of the above two cases or simple consequences of Jacobi identity for the Lie algebra g.

So, let’s consider \( a_1 = \rho_u \in \mathcal{F}_g^0 \), \( a_2 = \phi_A \in \mathcal{F}_g^1 \), \( a_3 = \phi_B \in \mathcal{F}_g^1 \). In this case, (A.13) reduces to

\[
[Q\rho_u, \phi_A, \phi_B]_h + [\rho_u, [\phi_A, \phi_B]_h]_h - [[\rho_u, \phi_A]_h, \phi_B]_h = 0. \tag{A.14}
\]

or, rewriting it by means of the expressions for appropriate operations, we get:

\[
\psi_{\{u, \{A, B\}\}} + \psi_{\{u, [A, B]\}} - \psi_{\{B, [u, A]\}} = 0. \tag{A.15}
\]

Therefore, to establish (A.14), one needs to prove:

\[
\{du, A, B\} + [u, \{A, B\}] - \{A, [u, B]\} - \{B, [u, A]\} = 0. \tag{A.16}
\]

Really,

\[
\{A, [u, B]\} = (2[A_\mu, \partial^\mu u, B_\nu] + 2[A_\mu, [u, \partial^\mu B_\nu]] - 2[\partial_\mu A_\nu, [u, B_\mu]] + [\partial_\nu A_\lambda, [u, B_\mu]] - [\partial_\nu B_\lambda, A_\mu] - [A_\mu, [\partial^\mu u, B_\nu]] - [A_\mu, [u, \partial^\mu B_\nu]] + [\partial^\mu A_\mu, [u, B_\nu]])dx^\nu. \tag{A.17}
\]

Rearranging the terms and using Jacobi identity, we find that

\[
\{A, [u, B]\} = \{[u, [A_\mu, \partial^\nu B_\nu]] + [B_\mu, \partial^\mu A_\nu] + [A_\mu, [\partial^\nu u, B_\nu]] + [B_\mu, [\partial^\nu u, A_\nu]] + [A_\mu, [B_\mu, \partial^\nu u]] + [\partial^\nu u, [B_\mu, A_\nu]] + [B_\mu, [A_\nu, \partial^\nu u]] + [\partial^\nu u, [A_\nu, B_\mu]]\}dx^\nu = \{du, A, B\} + [u, \{A, B\}] = 0. \tag{A.18}
\]

In such a way we proved (A.14).

Let’s consider the case, when \( a_1 = \phi_A \in \mathcal{F}_g^1 \), \( a_2 = \phi_B \in \mathcal{F}_g^1 \), \( a_3 = \phi_C \in \mathcal{F}_g^1 \). For this choice of variables, (A.13) has the following form:

\[
Q[\phi_A, \phi_B, \phi_C]_h + [\phi_A, [\phi_B, \phi_C]_h]_h - [[\phi_A, \phi_B]_h, \phi_C]_h = 0. \tag{A.19}
\]

or, on the level of differential forms,

\[
div\{A, B, C\} + A \cdot \{B, C\} + C \cdot \{A, B\} + B \cdot \{C, A\} = 0. \tag{A.20}
\]

To prove (A.20), we write the expression for \( C \cdot \{A, B\} \):

\[
C \cdot \{A, B\} = 2[C^\nu, [A_\mu, \partial^\nu B_\nu]] - 2[C^\nu, [\partial_\mu A_\nu, B^\nu]] + 2[C^\nu, [\partial_\nu A_\mu, B_\nu]] + [C^\nu, [\partial_\nu B_\mu, A^\nu]] - [C^\nu, [A_\nu, \partial^\nu B_\mu]] + [C^\nu, [\partial^\nu A_\mu, B_\nu]] - [C^\nu, [\partial_\nu A_\mu, B_\nu]] + [C^\nu, [\partial_\nu B_\mu, A^\nu]] + [C^\nu, [\partial_\nu B_\mu, A^\nu]] + [C^\nu, [\partial_\nu B_\mu, A^\nu]]. \tag{A.21}
\]
In order to obtain the last equality, we have used Jacobi identity from \( \mathfrak{g} \). Now, we observe that adding to (A.21) its cyclic permutations, that is \( \mathbf{A} \cdot \{ \mathbf{B}, \mathbf{C} \} \) and \( \mathbf{B} \cdot \{ \mathbf{A}, \mathbf{C} \} \), one obtains that the sum of cyclic permutations of terms in circle brackets (see last equality of (A.21)) gives \( \text{div} \{ \mathbf{A}, \mathbf{B}, \mathbf{C} \} \) while all other terms cancel. This proves relation (A.20) and, therefore, (A.19). Hence we proved (A.13).

The relations left are:

\[
[b, [a_1, a_2, a_3]_h]_h - (-1)^{n_1(n_2 + n_3)} [a_1, [a_2, a_3, b]_h]_h +
(-1)^{n_2(n_3 + n_1)} [a_2, [b, a_1, a_3]_h]_h - (-1)^{n_3(n_1 + n_2 + n_3)} [a_3, [b, a_1, a_2]_h]_h
= [[b, a_1]_h, a_2, a_3]_h + (-1)^{n_1 n_3} [a_1, [b, a_2]_h, a_3]_h +
(-1)^{n_3 n_1 + n_2} [a_1, [a_2, b]_h, a_3]_h,
\]

\[
[[a_1, a_2, a_3]_h, [b, c]_h] = 0. \tag{A.22}
\]

However to prove the first one, it is easy to see, that this relation is nontrivial only, when \( b \in \mathcal{F}_0^1 \) or \( b \in \mathcal{F}_0^1 \) and \( a_i \in \mathcal{F}_0^1 \). Therefore, it becomes a consequence of Jacobi identity from \( \mathfrak{g} \). The second one is trivial since the 3-linear operation takes values in \( \mathcal{F}_0^2 \) and it is zero for any argument lying in \( \mathcal{F}_0^2 \).

Proposition 2.3. is proved.

\[\Box\]

Proposition 2.4. The multilinear products introduced in Definition 2.5. are graded antisymmetric, i.e.

\[
\{ a_1, ..., a_i, a_{i+1}, ..., a_n \} \cdot h = -(-1)^{n_i n_{i+1}} \{ a_1, ..., a_{i+1}, a_i, ..., a_n \} \cdot h, \tag{A.23}
\]

where \( n_{a_i} \) denotes the ghost number of \( a_i \).

**Proof.** For the bilinear form \( \{ \cdot, \cdot \} \), the statement is the direct consequence of Proposition 2.2. To prove (A.23) of the 3-linear form \( \{ \cdot, \cdot, \cdot \} \), we need to check four relations:

\[
\langle [\rho_a, \rho_b]_h, \chi_a \rangle = \langle [\chi_a, \rho_a]_h, \rho_b \rangle, \quad \langle [\phi_{\mathbf{A}}, \phi_{\mathbf{B}}]_h, \phi_{\mathbf{C}} \rangle = \langle [\phi_{\mathbf{C}}, \phi_{\mathbf{B}}|h, \phi_{\mathbf{A}}]_h \rangle,
\]

\[
\langle [\rho_a, \phi_{\mathbf{A}}]_h, \psi_{\mathbf{W}} \rangle = \langle [\psi_{\mathbf{W}}, \rho_a]_h, \phi_{\mathbf{A}} \rangle = \langle [\phi_{\mathbf{A}}, \psi_{\mathbf{W}}]_h, \rho_a \rangle. \tag{A.24}
\]

Almost all of them are the simple consequence of the basic property of the invariant form:

\[
(X, [Y, Z])_K = ([X, Y])_K, \tag{A.25}
\]

where \( X, Y, Z \in \mathfrak{g} \). The only nontrivial one is

\[
\langle [\phi_{\mathbf{A}}, \phi_{\mathbf{B}}]_h, \phi_{\mathbf{C}} \rangle = \langle [\phi_{\mathbf{A}}, \phi_{\mathbf{C}}]_h, \phi_{\mathbf{B}} \rangle. \tag{A.26}
\]

Let’s prove it. From the definition of the inner product and bilinear operation in the homotopy Lie superalgebra we see, that it is equivalent to the following statement:

\[
\int d^D x (\{ \mathbf{A}, \mathbf{B} \}, \mathbf{C})(x) = \int d^D x (\{ \mathbf{A}, \mathbf{C} \}, \mathbf{B})(x) \tag{A.27}
\]
Let’s write it explicitly:

\[ \int d^D x (\{A, B\}, C)(x) = \int d^D x (C')(x), 2[A_\mu, \partial^\mu B_\nu](x) + 2[B_\mu, \partial^\mu A_\nu](x) + [\partial_\nu A_\mu, B^\mu](x) + [\partial_\nu B_\mu, A^\mu](x) = \int d^D x (B'(x), \partial^\mu [A_\mu, C_\nu](x) - 2[C_\mu, \partial_\nu A_\mu](x) - [\partial_\mu A_\nu, C''](x) + \partial^\nu [C_\mu, A_\nu](x) + [C_\nu, \partial_\mu A^\mu](x) + \partial^\nu [C^\mu, A^\mu](x) + [\partial_\nu A_\mu, C''](x) + [\partial_\nu C_\mu, A''](x) + [\partial^\nu A_\mu, C''](x) + [\partial^\nu C_\mu, A''](x)] ) = \int d^D x (\{A, C\}, B)(x). \tag{A.28} \]

Thus, we have proved the proposition for 3-linear form. So, to finish the proof, we need to check the relation (A.23) for the 4-linear product. In other words, we need to show that

\[ \{ \phi_A, \phi_B, \phi_C, \phi_D \} = \{ \phi_A, \phi_B, \phi_D, \phi_C \}. \tag{A.29} \]

By definition, this is equivalent to the relation

\[ \int d^D x (\{A, B, C\}, D)(x) = \int d^D x (\{A, B, D\}, C)(x), \tag{A.30} \]

which can be easily shown to be true by the iterated use of (A.25).

Thus, Proposition 2.4. is proved. \(\square\)

References

1. A.S. Schwarz, *New Topological Invariants Arising in the Theory of Quantized Fields*, Baku International Topological Conference, Abstracts (Part 2), Baku, 1987.
2. E. Witten, *Quantum Field Theory and Jones Polynomial*, Comm. Math. Phys. **121** (1989) 351.
3. E. Witten, *Noncommutative Geometry and String Field Theory*, Nucl. Phys. **B268** (1986) 253.
4. A. M. Zeitlin, *Homotopy Lie Superalgebra in Yang-Mills Theory*, JHEP09 (2007) 068, arXiv:0708.1773.
5. W. Siegel, *Introduction to String Field Theory*, World Scientific, 1988.
6. C. Thorn, *String Field Theory*, Phys. Rep. **175** (1989) 1.
7. W. Taylor, *String Field Theory*, hep-th/0605202.
8. M. Kato, K. Ogawa, *Covariant Quantization of String Based on BRS Invariance*, Nucl. Phys. **B212** (1983) 443;
   D. Friedan, E.J. Martinec, S.H. Shenker, *Covariant Quantization of Superstrings*, Phys. Lett. **B160** (1985) 55;
   N. Ohta, *Covariant Quantization Of Superstrings Based On BRS Invariance*, Phys. Rev. **D33** (1986) 1681;
   N. Ohta, *BRST cohomology in superstring theories*, Phys. Lett. **B179** (1986) 347;
   K. Furuchi, N. Ohta, *On the no-ghost theorem in string theory*, Prog. Theor. Phys. **116** (2006) 601, hep-th/0607105.
9. I.A. Batalin, G.A. Vilkovisky, *Gauge Algebra and Quantization*, Phys. Lett. **B102** (1981) 27. *Quantization Of Gauge Theories with Linearly Dependent Generators*, Phys. Rev. **D28** (1983) 2567.

10. E. Coletti, I. Sigalov, W. Taylor, *Abelian and Nonabelian Vector Field Effective Actions from String Field Theory*, [hep-th/0306041](https://arxiv.org/abs/hep-th/0306041).

11. N. Berkovits, M. Schnabl, *Yang-Mills Action from Open Superstring Field Theory*, [hep-th/0307019](https://arxiv.org/abs/hep-th/0307019).

12. N. Berkovits, *Review of Open Superstring Field Theory*, [hep-th/0105230](https://arxiv.org/abs/hep-th/0105230).

13. H. Feng, W. Siegel, *Yang-Mills Gauge Conditions from Witten’s Open String Field Theory*, [hep-th/0611307](https://arxiv.org/abs/hep-th/0611307).

14. M. Alexandrov, M. Kontsevich, A. Schwarz, O. Zaboronsky, *The Geometry of Master Equation and Topological Quantum Field Theory*, [hep-th/9502010](https://arxiv.org/abs/hep-th/9502010).

15. A.S. Schwarz, *Topological Quantum Field Theories*, [hep-th/0011260](https://arxiv.org/abs/hep-th/0011260).

16. B. Zwiebach, *Closed String Field Theory: Quantum Action and the B-V Master equation*, Nucl. Phys. **B300** (1993) 33.

17. T. Lada, J. Stasheff, *Introduction to SH Lie Algebras for Physicists*, [hep-th/9209099](https://arxiv.org/abs/hep-th/9209099).

18. A.M. Zeitlin, *Formal Maurer-Cartan Structures: from CFT to Classical Field Equations*, JHEP12(2007)098, [arXiv:0708.0955](https://arxiv.org/abs/0708.0955).

19. A. Schwarz, *Geometry of Batalin-Vilkovisky Quantization*, Comm. Math. Phys. **155** (1993) 249.

20. J. Stasheff, *The (Secret?) Homological Algebra of the Batalin-Vilkovisky approach*, Secondary Calculus and Cohomological Physics (Moscow, 1997) 195-210, Contemp. Math. **219** Amer. Math. Soc., Providence, RI, 1998.

21. M. Movshev, A. Schwarz, *On Maximally Supersymmetric Yang-Mills Theories*, [hep-th/0311132](https://arxiv.org/abs/hep-th/0311132); *Algebraic Structure of Yang-Mills Theory*, [hep-th/0404183](https://arxiv.org/abs/hep-th/0404183).

22. A.M. Zeitlin, *SFT-inspired Algebraic Structures in Gauge Theories*, [arXiv:0711.3843](https://arxiv.org/abs/0711.3843).

23. J. Polchinski, *String Theory*, Volume 1, CUP, 1998.

24. S. Weinberg, *The Quantum Theory of Fields*, Volume 2, CUP, 1996.

25. A.S. Losev, A. Marshakov, A.M. Zeitlin, *On the First Order Formalism in String Theory*, Phys. Lett. **B633** (2006) 375; [hep-th/0510065](https://arxiv.org/abs/hep-th/0510065).

26. A.M. Zeitlin, *BRST, Generalized Maurer-Cartan Equations and CFT*, Nucl. Phys. **B759** (2006) 370; [hep-th/0610208](https://arxiv.org/abs/hep-th/0610208).

27. A.M. Zeitlin, *Perturbed Beta-Gamma Systems and Complex Geometry*, Nucl. Phys. **B794** (2008) 381, [arXiv:0708.0682](https://arxiv.org/abs/0708.0682).