Supplementary Material for “Study of entanglement via a multi-agent dynamical quantum game”

Proof of Theorem 1

Let us write explicitly the correlation matrix for \( n \) species, including a special, \((n + 1)\)th, species that employs two observables. Thus,

\[
C = \begin{bmatrix} M(n,p,\rho) & \Gamma \\ \Gamma^T & D \end{bmatrix}
\]

(1)

Here, the \( n \times n \) sub-matrix, \( M(n,p,\rho) = I_{n \times n} + \rho A(n,p) \), where \( A(n,p) \) is the adjacency matrix of \( G(n,p) \). In addition,

\[
\Gamma^T = \begin{bmatrix} \rho^0_{1,n+1} & \cdots & \rho^0_{n,n+1} \\ \rho^1_{1,n+1} & \cdots & \rho^1_{n,n+1} \end{bmatrix}
\]

(2)

and

\[
D = \begin{bmatrix} 1 & \eta \\ \eta & 1 \end{bmatrix}
\]

(3)

where it is assumed, without loss of generality, that \( \eta \) is real. In what follows it is assumed that \( \rho \) is non-negative.

As \( C \) is a correlation matrix it must be positive semidefinite. The block decomposition of \( C \) allows for an equivalent requirement,

\[
M(n,p,\rho) > 0, \quad D - \Gamma^T M(n,p,\rho)^{-1} \Gamma \succeq 0
\]

(4)

known as Schur complement condition for positive semidefiniteness.

As \( C \) is a random matrix there should be some distribution of the largest possible \( \rho \) admitting the conditions (4). A known result in random matrix theory, based on concentration of measure arguments, has it that the distribution of the eigenvalues of \( A(n,p)/\sqrt{np} \) approaches the Wigner semicircle distribution with radius 2 in the \( n \to \infty \) limit (see Theorem 2.8.1 in [1], and Exercise 3.1 in [2]),

\[
\text{Prob} \left( \lambda \left( \frac{A(n,p)}{\sqrt{np}} \right) \leq x \right) \longrightarrow \text{Wsc}(x)
\]

Therefore, for sufficiently large \( n \) it follows that

\[
\lambda \left( M(n,p,\rho)^{-1} \right) = (1 + \gamma \rho \sqrt{np})^{-1}
\]

(5)

where \( \gamma \sim \text{Wsc}(\gamma) \). Furthermore, that \( M(n,p,\rho) > 0 \) means \( \rho < -1/(\gamma \sqrt{np}) = O(1/\sqrt{np}) \), for \( \gamma < 0 \) and sufficiently large \( n \).

The additional condition in (4) further restricts the bound on \( \rho \). To find out how we shall express the distribution of \( \rho^* \), the maximum value of \( \rho \), such that (4) hold.
From (4) and (5) it follows that for large $n$,

$$D - \frac{\Gamma T}{1 + \gamma \rho \sqrt{np}} = D - \frac{np\rho^2 (1 + \delta)}{1 + \gamma \rho \sqrt{np}} \left[ \begin{array}{c} 1 \\ p \end{array} \right] \succeq 0$$

(6)

where $\delta$ is a discrete random variable satisfying $\delta \leq (n - \lfloor np \rfloor)/\lfloor np \rfloor \simeq (1 - p)/p$. The expression in the numerator on the right follows from known concentration inequalities concerning sums of Bernoulli random variables; thus, for large $n$, the sum $\sum_{i=1}^n (\rho_{i,n+1}^I)^2$ centers around $nE[(\rho_{i,n+1}^I)^2] = np\rho^2$, and, similarly, $\sum_{i=1}^n (\rho_{i,n+1}^0)^2 nE[\rho_{i,n+1}^0 \rho_{i,n+1}^1]$ centers around $np^2\rho^2$. The random variable $\delta$ accounts for the uncertainty in these estimates.

The right hand side in (6) is equivalent to non-negativity of the trace and determinant of the underlying matrix difference. This leads to,

$$np\rho^2 (1 + \delta) \left( \frac{1 + \gamma}{\gamma \rho \sqrt{np}} \right) \leq g(\eta, p)$$

(7)

where $g(\eta, p)$ the same one defined in the theorem. In other words,

$$(\rho \sqrt{np})^2 - |\gamma| g'(\eta, p, \delta)(\rho \sqrt{np}) - g'(\eta, p, \delta) \leq 0$$

(8)

where $g'(\eta, p, \delta) = g(\eta, p)/(1 + \delta)$. Solving for $\rho \sqrt{np}$ yields,

$$\rho \sqrt{np} \leq \frac{1}{2} \left( |\gamma| g'(\eta, p, \delta) + \sqrt{\gamma^2 g'(\eta, p, \delta)^2 + 4g'(\eta, p, \delta)} \right) \leq |\gamma| g'(\eta, p, \delta) + 2 \sqrt{g'(\eta, p, \delta)}$$

(9)

We thus may identify the right hand side above as the random variable $\rho^* \sqrt{np}$. Therefore,

$$\frac{\rho^* \sqrt{np}}{g'(\eta, p, \delta)} = |\gamma| + \frac{2 \sqrt{g'(\eta, p, \delta)}}{\sqrt{g'(\eta, p, \delta)}}$$

(10)

which holds for sufficiently large $n$.

Because $\gamma$ is distributed according to Wsc($\gamma$) in the $n \to \infty$ limit, and similarly $|\gamma|$ is distributed according to $\text{Wsc}_+(|\gamma|)$, it follows from (10) that

$$\text{Prob} \left( \frac{\rho^* \sqrt{np}}{g(\eta, p)} \leq \frac{\nu}{1 + \delta} + 2 \sqrt{\frac{1 + \delta}{g(\eta, p)}} \left| \delta \right. \right) =$$

$$\text{Prob} \left( \frac{\rho^* \sqrt{np}}{g'(\eta, p, \delta)} \leq \nu + 2 \left| \frac{\delta}{\sqrt{g'(\eta, p, \delta)}} \right. \right) \to \text{Wsc}_+ (\nu)$$

(11)

where $\nu \in [0, 2]$. That is,

$$\text{Prob} \left( \frac{\rho^* \sqrt{np}}{g(\eta, p)} \leq \zeta \left| \delta \right. \right) \to \text{Wsc}_+ \left( (1 + \delta) \zeta - 2 \frac{(1 + \delta)^{3/2}}{\sqrt{g(\eta, p)}} \right)$$

(12)
Note that by the very definitions of $\zeta$ and $g(\eta, p)$, the product $\zeta g(\eta, p) = O(\sqrt{1-\eta})$. Finally, in the $n \to \infty$ limit,
\[
\text{Prob} \left( \frac{\rho^* \sqrt{np}}{g(\eta, p)} \leq \zeta \right) = \mathbb{E}_\delta \left[ \text{Prob} \left( \frac{\rho^* \sqrt{np}}{g(\eta, p)} \leq \zeta \mid \delta \right) \right] \to \mathbb{E}_\delta \left[ Wsc_+ \left( (1+\delta)\zeta - 2 \frac{(1+\delta)^{3/2}}{\sqrt{g(\eta, p)}} \right) \right]
\]
from which the theorem follows.

**Nonlocality and dynamics**

**Linearization**

Locally, the system may be described by its Lyapunov exponents, the eigenvalues of $\exp(\mathbf{J}t)$ with the Jacobian $\mathbf{J}$, where
\[
\mathbf{J} = \begin{bmatrix}
\gamma - \sum_{i=1}^n B_i v_i & -B_1 c & \cdots & -B_n c \\
B_1 v_1 & -\zeta_1 + B_1 c & 0 \\
\vdots & \ddots & \ddots \\
B_n v_n & 0 & -\zeta_n + B_n c
\end{bmatrix}.
\]

Let us assume that $\forall i \in [n], \zeta_i = \zeta, v_i = v$; and also $\forall i \in \{2, \ldots, n\}, B_i = B$. Note that $B_1$ may be different than the others! Rewriting the Jacobian:
\[
\mathbf{J} = \begin{bmatrix}
\gamma - B_1 v - (n-1)Bv & -B_1 c & -Bc & \cdots & -Bc \\
B_1 v & -\zeta + B_1 c & 0 & \cdots & 0 \\
Bv & 0 & -\zeta + Bc & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
Bv & 0 & \cdots & 0 & -\zeta + Bc
\end{bmatrix}
\]

The characteristic polynomial of $\mathbf{J}$:
\[
p_J(\lambda) = (\gamma - [(n-1)B + B_1]v - \lambda) (-\zeta + Bc - \lambda)^{n-1} (-\zeta + B_1 c - \lambda) + \]
\[
+ B_1^2 cv (-\zeta + Bc - \lambda)^{n-1} + (n-1)B^2 cv (-\zeta + B_1 c - \lambda) (-\zeta + Bc - \lambda)^{n-2} =
\]
\[
= (-\zeta + Bc - \lambda)^{n-2} (\gamma - [(n-1)B + B_1]v - \lambda) (-\zeta + Bc - \lambda) (-\zeta + B_1 c - \lambda) + 
\]
\[
+ (-\zeta + Bc - \lambda)^{n-2} B_1^2 cv (-\zeta + Bc - \lambda) + (-\zeta + Bc - \lambda)^{n-2} (n-1)B^2 cv (-\zeta + B_1 c - \lambda)
\]

Which implies that $\lambda_{mul} = Bc - \zeta$ appears as an eigenvalue with algebraic multiplicity of (at least) $n-2$. $\lambda_{mul}$ corresponds to dynamical modes for which $c, v_1$ and $\sum_{k=2}^n v_k$ are constant; i.e., the only dynamics are amongst the $n-1$ “homogeneous” virus species, and therefore not particularly interesting for our purposes.
The other three eigenvalues are the solutions of the cubic equation:

\[(\gamma - [(n - 1)B + B_1] v - \lambda)(-\zeta + Bc - \lambda)(-\zeta + B_1c - \lambda) + B_1^2cv(-\zeta + Bc - \lambda) + (n - 1)B^2cv(-\zeta + B_1c - \lambda) = 0.\] (17)

We shall examine the dynamics of the system near a point where the normalized populations are all 1. Moreover, we are only interested in cases where the viruses all have positive population payoffs, i.e. \(B_1, B > 0\). First, let us rewrite (17) using the variable \(\tilde{\lambda} := \lambda + \zeta\):

\[\left(\gamma + \zeta - [(n - 1)B + B_1] v - \tilde{\lambda}\right)\left(Bc - \tilde{\lambda}\right)\left(B_1c - \tilde{\lambda}\right) + B_1^2cv\left(Bc - \tilde{\lambda}\right) + (n - 1)B^2cv\left(B_1c - \tilde{\lambda}\right) = 0.\] (18)

Now, we substitute \(B_1 = \frac{\beta}{\sqrt{2}} \cos \theta, B = \frac{\beta}{\sqrt{2(n-1)}} \sin \theta, c = v = 1, and also define \(\delta := \gamma + \zeta\):

\[\left(\delta - \sqrt{\frac{n - 1}{2}} \beta \sin \theta - \frac{\beta}{\sqrt{2}} \cos \theta - \tilde{\lambda}\right)\left(\frac{\beta}{\sqrt{2(n-1)}} \sin \theta - \tilde{\lambda}\right)\left(\frac{\beta}{\sqrt{2}} \cos \theta - \tilde{\lambda}\right) + \frac{\beta^2}{2} \cos^2 \theta \left(\frac{\beta}{\sqrt{2(n-1)}} \sin \theta - \tilde{\lambda}\right) + \frac{\beta^2}{2} \sin^2 \theta \left(\frac{\beta}{\sqrt{2}} \cos \theta - \tilde{\lambda}\right) = 0.\] (19)

Generally, given parameters \(\beta, \gamma, \zeta, \theta\), one may find some value \(n_c = n(\beta, \gamma, \zeta, \theta)\), such that for any \(n \leq n_c\) the system would admit at least one non-negative Lyapunov exponent (note for some range of the parameters we would have \(n_c = 0\)). This function \(n(\beta, \gamma, \zeta, \theta)\) is given implicitly by considering \(\Re(\tilde{\lambda}_{\text{max}}) = \zeta\), where \(\tilde{\lambda}_{\text{max}}\) is the solution to (19) having the largest real part.

**Non-entangled case**

In the non-entangled (equally-correlated) case, we assume \(\mathcal{R}_{CV_k} = 2\sqrt{2/n} =: \mathcal{R}\) for all \(k\), implying \(B_1 = B = \beta/\sqrt{2n}\). We may define a new dynamic variable: \(\tilde{v} \triangleq (\sum_{i=1}^n v_i)/n\), and replace our system of \(n + 1\) equations with only two:

\[
\begin{align*}
\dot{c} &= \gamma c - Bcn\tilde{v} \\
\dot{\tilde{v}} &= -\zeta \tilde{v} + Bc\tilde{v}.
\end{align*}
\] (20)

This system admits the following equilibrium point:

\[
\tilde{v} = \frac{\sqrt{2}\gamma}{\sqrt{n}\beta}, \quad c = \frac{\sqrt{2n}\zeta}{\beta}
\] (21)

and the following Jacobian:

\[
J_{NE} = \begin{bmatrix}
\gamma - Bn\tilde{v} & -Bcn \\
B\tilde{v} & -\zeta + Bc
\end{bmatrix}
\] (22)
Let us find its eigenvalues. The characteristic polynomial is:

\[
\det(J_{NE} - \lambda I) = (\gamma - Bn\bar{v} - \lambda)(-\zeta + Bc - \lambda) + B^2cn\bar{v} = \\
\lambda^2 + (\gamma + Bn\bar{v} + \zeta - Bc)\lambda - \gamma\zeta + \gamma Bc + \zeta Bn\bar{v}.
\] (23)

Let us use the notation \(\delta = \gamma + \zeta\) and assume \(c \approx 1, v_{tot} \approx 1\):

\[
\lambda_{\pm} = \frac{1}{2} \left[ \gamma - \zeta - \frac{\beta}{\sqrt{2n}}(n-1) \right] \pm \frac{1}{2} \left( \left[ \gamma - \zeta - \frac{\beta}{\sqrt{2n}}(n-1) \right]^2 - 4(-\gamma\zeta + \gamma Bc + \zeta Bn\bar{v}) \right)^{1/2} \\
\approx \frac{1}{2} \left[ \gamma - \zeta - \frac{\beta}{\sqrt{2n}}(n-1) \right] \pm \frac{1}{2} \left( \delta^2 - \frac{2\gamma\beta}{\sqrt{2n}}(n+1) - \frac{2\zeta\beta}{\sqrt{2n}} + \frac{\beta^2}{2n}(n-1)^2 \right)^{1/2}.
\] (24)

The discriminant is:

\[
\Delta = \delta^2 - 2B(n+1)\delta + B^2(n-1)^2 = \delta^2 - \sqrt{2}\beta\delta \left( \sqrt{n} + \frac{1}{\sqrt{n}} \right) + \frac{\beta^2}{2} \left( n - 2 + \frac{1}{n} \right),
\] (25)

which vanishes for \(\delta_{\pm} = \frac{\beta(n+1)}{\sqrt{2n}} \pm \sqrt{2}\beta\). For large enough values of \(n\) we have:

\[
\Delta \approx \frac{\beta^2n}{2} \left[ 1 - 2\sqrt{\frac{2}{n}} \frac{\delta}{n} + \frac{2}{\beta^2n} \right].
\]

Thus, it follows that \(\lambda_+ \approx -\zeta + \frac{\delta^2}{2\sqrt{2n}\beta}\) and \(\lambda_- \approx \gamma - \beta\sqrt{\frac{n}{2}} + \frac{2\beta^2 - \delta^2}{2\sqrt{2n}\beta} \approx \gamma - \beta/\beta\). Now we shall find the eigenvectors:

\[
J_{NE} - \lambda_+ I = \left[ \delta - \beta\frac{\sqrt{2}}{\sqrt{2n}} - \frac{\delta^2}{2\sqrt{2n}\beta} - \frac{\beta\sqrt{2}}{2\sqrt{2n}\beta} \right] + O \left( \frac{1}{n} \right),
\]

\[
J_{NE} - \lambda_- I = \left[ \frac{\delta^2 - 2\beta^2}{2\sqrt{2n}\beta} \right] \pm \frac{\beta}{\sqrt{2n}} + \frac{\delta^2}{2\sqrt{2n}\beta} + O \left( \frac{1}{n} \right).
\]

Thus we have:

\[
u_+ \approx \left[ \frac{\delta^2 - 2\beta^2}{2\beta^2n} \right], \quad u_- \approx \left[ \frac{1}{\delta^2 - 2\beta^2} \right].
\] (27)
Maximally-entangled case

In the maximally-entangled case, \( B_{CV_1} = 2\sqrt{2} \) and for any \( k > 1 \), \( B_{CV_k} = 0 \). Thus, \( B_1 = \beta/\sqrt{2} \) and \( B = 0 \). The system admits the following equilibrium point:
\[
c = \sqrt{2}\zeta/\beta, \quad v_1 = \sqrt{2}\gamma/\beta, \quad v_k = 0 \quad \forall k > 1.
\] (28)

The eigenvalues are \( \lambda_{mul} = -\zeta \) with algebraic multiplicity \( n - 2 \), and the other three are solutions of the following cubic equation:
\[
(\gamma - B_1 v - \lambda) (-\zeta - \lambda) (-\zeta + B_1 v - \lambda) + B_1^2 cv (-\zeta - \lambda) = 0
\]

\[
\iff (\lambda + \zeta) \left[ (\gamma - B_1 v - \lambda) (-\zeta + B_1 v - \lambda) + B_1^2 cv \right] = 0.
\] (29)

So we see that \( \lambda = -\zeta \) actually has algebraic multiplicity \( n - 1 \), and the remaining two eigenvectors are the roots of:
\[
(\gamma - B_1 v - \lambda) (-\zeta + B_1 v - \lambda) + B_1^2 cv = 0
\]

\[
\iff \lambda^2 - \lambda (\gamma - \zeta) - \gamma B_1 v + \zeta B_1 v - B_1^2 v^2 + B_1^2 cv = 0
\]

\[
\iff \lambda_{\pm} = \frac{1}{2} (\gamma - \zeta) \pm \frac{1}{2} \left[ (\gamma - \zeta)^2 - 4 (-\gamma + \gamma B_1 v + \zeta B_1 v - B_1^2 v^2 + B_1^2 cv) \right]^{1/2} =
\]

\[
= \frac{1}{2} (\gamma - \zeta) \pm \frac{1}{2} \left[ (\gamma - \zeta)^2 - 4 \left( -\gamma v/\sqrt{2} + \beta v^2 + \beta^2 v/2 \right) \right]^{1/2}.
\] (30)

Let us assume that \( v \approx c \approx 1 \). Using these approximations, the Jacobian has the form:
\[
J = \begin{bmatrix}
\gamma - B_1 & -B_1 & 0 & \cdots & 0 \\
B_1 & -\zeta + B_1 & 0 & \cdots & 0 \\
0 & 0 & -\zeta & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
0 & 0 & \cdots & 0 & -\zeta
\end{bmatrix}
\] (31)

and the remaining two eigenvalues are:
\[
\lambda_{\pm} = \frac{1}{2} (\gamma - \zeta) \pm \frac{1}{2} \left[ (\gamma - \zeta)^2 - 4 \left( -\gamma + \gamma B_1 v + \zeta B_1 v - B_1^2 v^2 + B_1^2 cv \right) \right]^{1/2} =
\]

\[
= \frac{1}{2} (\gamma - \zeta) \pm \frac{1}{2} \left[ (\gamma + \zeta)^2 - 4 \right. \left( -\gamma + \gamma B_1 v + \zeta B_1 v - B_1^2 v^2 + B_1^2 cv \right) \right]^{1/2} =
\]

\[
= \frac{1}{2} (\gamma - \zeta) \pm \frac{1}{2} \left[ \delta \left( \delta - 2\sqrt{2} \beta \right) \right]^{1/2}
\] (32)
where we have used the notation $\delta = \gamma + \zeta$. The corresponding eigenvectors are:

$$u_{\pm} = \begin{bmatrix} \delta - 2B_1 \mp \sqrt{\delta (\delta - 2\sqrt{2}\beta)} \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{33}$$

And for $\lambda_{\text{mul}}$ we have the eigenvectors $\{e_j\}_{j=2}^{n}$, where $e_j$ is the vector with $j$th entry 1 and all other entries 0 (note the indices start from 0).

We must consider three cases: $\delta = 4B_1$, $\delta < 4B_1$ and $\delta > 4B_1$ (recall that $B_1 = \beta / \sqrt{2}$). We start with the former.

1. $\gamma + \zeta = 4B_1 = 2\sqrt{2}\beta$: we obtain the eigenvalue $\lambda_2 = (\gamma - \zeta) / 2$ with algebraic multiplicity 2. Let us express it with $\beta$ rather than $\zeta$:

$$\lambda_2 = \frac{\gamma - \zeta}{2} = \frac{\gamma + \gamma - 2\sqrt{2}\beta}{2} = \gamma - \sqrt{2}\beta = \gamma - 4\beta / \mathcal{R}_{CV_1}, \tag{34}$$

while the eigenvalue $\lambda_{\text{mul}}$, with multiplicity $n - 1$, obtains the form:

$$\lambda_{\text{mul}} = -\zeta = \gamma - 2\sqrt{2}\beta = \gamma - 8\beta / \mathcal{R}_{CV_1}. \tag{35}$$

Compare with the Lyapunov exponents of the non-entangled case: $\lambda_- = \gamma - \beta \sqrt{n/2} + O(1/\sqrt{n}) \approx \gamma - 2\beta / \mathcal{B}$ and $\lambda_+ = -\zeta + O(1/\sqrt{n}) \approx \gamma - 2\sqrt{2}\beta$. Interestingly, for $n = 4$ the cells have the same Lyapunov exponents in the non-entangled and maximally-entangled states, up to terms in the order of $1/\sqrt{n}$.

2. $\gamma + \zeta < 4B_1$: Let us denote $\Delta \triangleq \sqrt{(\gamma + \zeta) (4B_1 - \gamma - \zeta)}$, thus, we obtain $\lambda_{\pm} = \frac{1}{2} (\gamma - \zeta \pm i\Delta)$.

3. $\gamma + \zeta > 4B_1$: For the case of $\gamma + \zeta > 4B_1$, we denote $\Delta \triangleq \sqrt{(\gamma + \zeta) (\gamma + \zeta - 4B_1)}$, thus, we obtain $\lambda_{\pm} = \frac{1}{2} (\gamma - \zeta \pm \Delta)$.

[1] T. Tao, *Topics in random matrix theory*. American Mathematical Society, 2012.

[2] C. Bordenave, “Lecture notes on random matrix theory,” 2019.