A simple model for the macroscopic fluctuations of temporal networks

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Many real-world social networks constantly change their global properties over time, such as the number of edges, size and density. While temporal and local properties of social networks have been extensively studied, the origin of their global fluctuations is not yet well understood. A network may grow or shrink if a) the total population of nodes, including resting ones, changes and/or b) the chance of two nodes being connected varies over time. Here, we develop a method that classifies the source of global fluctuations of temporal networks according to these two mechanisms. We propose a dynamic hidden-variable model to formally define the two dynamical classes, with which we show that the global fluctuations in a real-world dynamical system can be explained by either of the two simple mechanisms. Our findings will contribute to a better understanding of the origin of the time-varying nature of complex networks.

I. INTRODUCTION

Along with the increasing availability of high-resolution data sets, dynamics of human social communication have been extensively studied over the past decades [1–6]. Many of these studies are based on data sets of online interactions, such as emails [7], text messages [8, 9] and mobile phones [2, 3, 10, 11], but the recent development of sensor devices has also enabled us to collect time-stamped data from face-to-face interactions in physical space [1, 12–15]. Those data therefore cover a wide range of social contexts in which dynamic interactions among individuals form temporal social networks [6, 16].

These real-world social networks exhibit very often non-stationarity: their structure constantly changes over time not only in shape but also in size. For example, face-to-face networks in a school tend to be denser during breaks than during class time [11–13]. Similarly, the overall activity of social agents (i.e., individuals, financial institutions, etc) has intrinsic diurnal and/or daily rhythms [19–23].

Generally, these dynamics are present because the studied system is not closed: it is in fact a common property of real-world social and economic networks that agents are free to enter and exit. For online communication tools, such as email [7], text messages [9] and phone calls [2, 3, 10], the number of nodes that are ready to interact with others can vary according to external factors such as work schedule and time differences between cities, etc. In social networks like Twitter and Facebook, anyone can basically join or quit the existing communication networks at any time. In financial markets, a bank becomes a part of an interbank network if it borrows from or lends to other banks and exits the network when the loan is repaid [24, 25]. Another non-conservative aspect of real-world networks is the fact that even if the population is constant, the networking activity might vary due to external factors. Diurnal, weekly and monthly rhythms drive traffic in mobility networks, both the flows between cities [19, 26] and within [23]. At a smaller scale, schedules and organisation can for instance hinder or facilitate the creation of links (coffee breaks in a conference [13], pauses between classes in a school [5, 17], lunch breaks in a company [15, 27], etc).

While fluctuations of the structure at the macroscopic scale are a common phenomenon in temporal social networks, the mechanisms responsible for them remain unknown. In the present work, we focus on the evolution of two quantities that condition the macroscopic network structure: the numbers of active nodes and edges. In principle, we can reduce the mechanisms to two factors that would lead to global fluctuations: number of nodes \(N\) and the number of edges \(M\) will fluctuate if a) the size of population (i.e., potential number of nodes) changes and/or b) the chance of two nodes being connected varies over time. Here, we develop a method that classifies the source of global fluctuations of temporal networks according to these two mechanisms. We propose a dynamic hidden-variable model to formally define the two dynamical classes, with which we show that the global fluctuations in a real-world dynamical system can be explained by either of the two simple mechanisms. Our findings will contribute to a better understanding of the origin of the time-varying nature of complex networks.

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tween the numbers of active nodes and edges. To model the behaviour of nodes, we use a dynamical version of a hidden-variable model in which the temporal probability of two nodes being connected is given by a product of “fitness” parameters \[28, 29\]. The fitness parameters are considered to be intrinsic and constant features of the nodes, the time-evolving aspect arises from two supplementary mechanisms. First, we introduce a parameter that modulates the average activity level of nodes. This modulation parameter allows the size of generated networks to vary over time while keeping the total number of nodes in the system, including resting nodes, constant. Second, we allow the population size to vary in time. In the original fitness model \[28, 29\], there is no distinction between population and the number of active nodes, because the population size is assumed to be large enough so that virtually all nodes in the network are active \[25\]. However, if the population of a network is not sufficiently large, a certain fraction of existing nodes may not be active \[25\], and thereby a change in the population size affects the rate at which the number of edges grows with the number of active nodes.

In the following, we first expose the empirical evidence for the existence of simple mechanisms governing the fluctuations of global activity in temporal networks. We then present a dynamical hidden-variable model with which we investigate the emergence of these temporal patterns. From this model we extract two theoretical equations that connect \(N\) and \(M\) under different specifications on the activity rhythm of agents and the population size. The proposed method allows us to estimate the actual activity rhythm and the (unobservable) population size. From this we are then able to identify for each empirical case which key factor drives the observed global fluctuations of the temporal network. We also briefly mention a variation of the model for cases where the population is fixed and known, allowing us to fit the empirical distribution of node fitnesses to a beta distribution. We conclude by a discussion of our results and the limitations of the model.

II. RESULTS

A. Evidence from empirical data

We consider six data sets of social and economic interest, taken from contexts of very different nature (see IV A in Methods for a full description of the data sets):

- Interbank (bilateral transactions in the online interbank market in Italy);
- Enron (email communication network from the Enron Corporation \[21, 30]\);
- CollegeMsg (online social network at the University of California, Irvine \[8, 9]\);
- RealityMining (phone call data from the Reality Commons project \[2\]);
- LondonBike (bike trips from the London Bicycle Sharing Scheme \[31]\);
- Highschool (face-to-face contacts network in a French high school \[17\]).

We investigate in this empirical data the dynamical relationship between the number of active nodes \(N\) and the corresponding number of edges \(M\) present in each snapshot of the temporal networks. Fig. 1 shows scatter plots of \(M\) against \(N\) for each social context. Two important features appear. First and foremost, there is a strong positive correlation between \(N\) and \(M\) in all the data sets we examine. In particular, we observe superlinear scaling, i.e., the rate at which \(M\) rises with \(N\) is larger than that expected by a linear growth, as is occasionally reported for many real-world systems \[10, 32, 33\].

Second, there are two different patterns as to how \(M\) grows with \(N\). One is the superlinear scaling we mentioned, in which the exponent of scaling is constant (>1), showing as a straight line on a log-log scale plot. In Fig. 1 Interbank, Enron, CollegeMsg and RealityMining appear showing as a straight line on a log-log scale plot. In Fig. 1, Interbank, Enron, CollegeMsg and RealityMining appear showing as a straight line on a log-log scale plot. In Fig. 1, Interbank, Enron, CollegeMsg and RealityMining appear showing as a straight line on a log-log scale plot. In Fig. 1, Interbank, Enron, CollegeMsg and RealityMining appear showing as a straight line on a log-log scale plot. In Fig. 1, Interbank, Enron, CollegeMsg and RealityMining appear showing as a straight line on a log-log scale plot. In Fig. 1, Interbank, Enron, CollegeMsg and RealityMining appear showing as a straight line on a log-log scale plot. In Fig. 1, Interbank, Enron, CollegeMsg and RealityMining appear showing as a straight line on a log-log scale plot.

Both behaviours are striking, as they suggest the existence of simple mechanisms for the dynamics of global activity in temporal networks. However, the empirical dynamical relationship we observe in Fig. 1 cannot be reproduced by a class of common growing network models in which a new node joins the network with \(m\) edges \[34–36\]. While these models are intended to explain the emergence of scaling in empirical degree distributions \[34\], the number of edges in the network has a linear correlation with the number of active nodes asymptotically, i.e., \(M \propto N\), which is not consistent with our finding. In the following, we present a model which explains how these different types of global behaviours can emerge from temporal social interactions.

B. A dynamic hidden-variable model

We consider a dynamical version of the hidden variable model in which the probability of two nodes being connected at time interval \(t\) is given by:

\[
p_{ij,t} = \kappa_t a_i a_j, \quad i, j = 1, \ldots, N_{p,t}, \quad t = 1, \ldots, T. \tag{1}
\]

where \(a_i\) is the “fitness” of node \(i\) that represents the activity level of the node \[28, 29, 37\]. In the baseline model we assume that \(a_i\) is uniformly distributed on \([0, 1]\) because in general we do not have any prior information about the distribution of activity levels. We will also consider a beta distribution as an alternative case in section \[11D\].
There are two time-varying parameters in the model. One is $N_{p,t}$ which represents the potential number of active nodes in the system at time $t$, i.e., the total of active and inactive nodes that are in the system at time $t$. The number of active nodes having at least one edge at time $t$ is denoted by $N_t$. We note that the number of active nodes $N_t$ is always observable, but the potential number of nodes $N_{p,t}$ is not. In social networks, for instance, we do not usually know how many people are ready to interact with other people and what fraction of them actually created at least one edge. In most cases, what we can observe from data is the number of active nodes that appear in the record of interaction history, while there is no record of nodes without interactions. Since the observed active nodes may account for only a fraction of the potential nodes, it is generally written as $N_t = (1 - q_{0,t})N_{p,t}$, where $q_{0,t}$ denotes the share of resting nodes having no edge. To take an example of social networks, changes in $N_p$ may represent a situation in which the number of students in the classroom changes over time according to the class schedule, leading to a variation in the maximum possible size of face-to-face contact networks. The potential number of nodes that are ready to interact with others is the first key parameter of the model, as it physically constrains the size of networks to be observed.

The second time-varying parameter of the model is $\kappa_t > 0$, which modulates the global activity level of nodes. In the financial system, for instance, the chance that two banks trade during the lunch time would be intrinsically lower than that in the morning [25], in which case the banks’ activity levels may have a certain diurnal pattern. In social networks where individuals communicate with each other, $\kappa$ would vary according to the time-schedule of the school, workplace, academic conferences, or the circadian rhythm of humans [4, 20, 21, 38].

With this specification, the observed network size $N$ and the number of edges $M$ co-evolve as either $N_p$ or $\kappa$ or both change over time. One can see a change in $N$ due to a shift in $N_p$ represents an extensive margin effect, while a shift in $\kappa$ leads to an intensive margin effect. Parameters $\kappa$ and $N_p$ can thus explain two different origins of the time-varying nature of networks.

1. Analytical solution for $N$ and $M$

One can derive the analytical forms for $N$ and $M$ for given parameters $(\kappa, N_p)$ from the model (see Appendix for derivation):

$$N = N_p \left[1 - \frac{2}{\kappa N_p} \left(1 - \left(1 - \frac{\kappa}{2}\right) N_p\right)\right],$$

$$M = \frac{1}{8} \kappa N_p (N_p - 1),$$

FIG. 1. Relationship between the number of active nodes $N$ and the number of edges $M$ in each snapshot for different social contexts. For Interbank, Enron, CollegeMsg and RealityMining data, each dot represents the realisation of ($N, M$) in a particular time window (annotated in the top) of a day. For LondonBike and Highschool data, each dot represents the realisation of ($N, M$) in a 10-minutes time window of a day (0:00–24:00). Black dotted and dashed lines denote the theoretical upper ($M = N(N - 1)/2$) and lower ($M = N/2$) bounds, respectively.
where we drop the time subscript $t$ for brevity. Eq. 2 can be rewritten as follows to link the numbers of active nodes $N$ to the population $N_p$:

$$N = (1 - q_0(\kappa, N_p)) N_p, \quad (4)$$

$$q_0(\kappa, N_p) = \frac{2}{\kappa N_p} \left[ 1 - \left( 1 - \frac{\kappa}{2} \right)^{N_p} \right], \quad (5)$$

This leads to interesting limit behaviours: if $|1 - \kappa/2| < 1$ and $N_p$ is sufficiently large, then $q_0(\kappa, N_p) \simeq 0$ and thereby $N \simeq N_p$ and $M \propto N^2$ (Eq. 3), as is shown in the study of the static fitness model [25, 29, 37]. In contrast, if $N_p$ is not large enough, then $q_0(\kappa, N_p) > 0$ and $N < N_p$, in which case $M$ is not of order $N^2$ and the scaling exponent will take a value between 1 and 2 [25]. Note that $\kappa$ is not per se a probability, and that its value does not have any a priori upper bound (as it depends on the activity distribution). Clearly, the larger the population $N_p$ and the overall activity $\kappa$, the lower the share of resting nodes in the population (Fig. S1). In practice, the value of $\kappa$ for empirical data sets is very small ($< 0.1$) as is shown below.

2. Effects of $\kappa$ and $N_p$ on the emergence of scaling

![Graph showing the relationship between $N_p$ and $M$](image)

**FIG. 2.** Pairs of $(N, M)$ indicated by the dynamic hidden-variable model. Each colour represents a particular value of $N_p$, while different symbols denote different values of $\kappa$. Black dotted and dashed lines denote the theoretical upper ($M = N(N - 1)/2$) and lower ($M = N/2$) bounds, respectively.

Using Eqs. 2 and 3, we are able to analyse numerically how $M$ scales with $N$ for a given parameter pair $(\kappa, N_p)$. First we observe that if the value of $\kappa$ is kept constant while $N_p$ varies, the dynamical relationship between $N$ and $M$ is close to a straight line in a log-log plot, as seen in some empirical data (Fig. 2). If $\kappa$ is close to 0, the scaling is close to linear. However, as $\kappa$ increases, the scaling becomes more and more superlinear, which can be seen in Fig. 2 by following the same symbol in different colours.

By contrast, if we vary $\kappa$ for a given value of $N_p$, the slope will bend upward. This can be seen in Fig. 2 by following different symbols in the same colour. This reproduces the accelerating growth behaviour observed in the empirical data. Although the scaling relationships appear to be quite regular, it proves to be very difficult (if not impossible) to extract from Eqs. 2 and 3 an analytical expression for them, because of the complicated dependencies of $N$ on $N_p$.

C. Identifying the source of network dynamics in empirical networks

Given that the model appears to be able to reproduce the two types of dynamical behaviours, we propose methods to estimate $\kappa$ and $N_p$ from the empirical data. The two parameters may be estimated in two ways. One is to directly solve the two nonlinear equations Eqs. 2 and 3 with respect to $(\kappa, N_p)$ for a given observation of $(N, M)$. This direct calculation gives us a one-to-one mapping of $(N, M)$ to $(\kappa^*, N_p^*)$, where asterisk denotes the solution of the system of two equations. However, such a method proves to be unable to correctly estimate the parameters (see IV B 3).

Another method is to use the dynamical relationship between $N$ and $M$ (see IV B in Methods for a detailed description). We consider two different representations of the model for empirical fitting: Model I, which assumes that overall activity $\kappa$ is constant and the evolution of the potential number of nodes $N_{p,t}$ is endogenous; Model II, which assumes that the population size $N_p$ is constant and the evolution of overall activity $\kappa_t$ is endogenous. We fit both on each data set and then select the model which returns the best fit.

Fig. 3 shows the results for the CollegeMsg and LondonBike data sets (see Figs. S2 and S3 in SI for the other data sets). Our results illustrate the fact that global fluctuations in social and economic temporal networks may be alternatively driven by the two previously described factors. For Interbank, Enron, CollegeMsg and RealityMining, Model I is selected, which means the time-varying nature of the global network properties comes from shifts in the potential number of nodes, i.e. the population in the system changes over time. On the other hand, for LondonBike and Highschool, Model II is selected, which means the population remains almost unchanged, and the changes in the numbers of edges and active nodes are due to time-varying connecting probabilities. Since all we need for model classification is a variety of combinations of $(N, M)$, one can implement the method for any timescale. For instance, if we see daily activity in the Interbank data set, the macroscopic behaviours on a vast majority of days are still better modelled as Model I (Fig. S4). For LondonBike dataset,
the global fluctuations across different days are identified as being driven by a time-varying \( \kappa \) for each time interval (Fig. S5), again indicating that the population (i.e., the number of bike stations) is essentially fixed throughout the data period.

For the data sets for which Model I is selected, we note that the estimated values of \( \kappa \) are fairly small, ranging from 0.011 (RealityMining) to 0.078 (Interbank). \( \hat{\kappa} \) is time-varying for LondonBike and Highschool, but \( \hat{\kappa} \) is still small with the maximum value being no larger than 0.02. This suggests that the direct calculation discussed in section IV B 1 would not work well for empirical networks (Fig. 6a).

We compare the theoretical and the empirical network density in Fig. 3 (right panels) for CollegeMsg and LondonBike (see Fig. S6 for the other data sets) (see IV C in Methods for how we estimate the density). For the data sets for which Model I is selected (Interbank, Enron, CollegeMsg and RealityMining), the density monotonically decreases as \( N \) increases, approaching the asymptotic value \( \hat{\kappa}/4 \) (dashed line). For the other data sets (LondonBike and Highschool) on the other hand, there exists a threshold value for \( N \) above which the empirical and theoretical density increases with \( N \). This is a sort of finite size effect: in this region \( N \) gets close to \( \hat{N}_p \), thus the positive impact of \( \kappa \) on density becomes dominant, as the dilution effect through an increase in \( N \) vanishes.

### D. Activity distribution

The estimation methods we propose assume that activity parameters \( \{a_i\} \) are distributed uniformly, because in many real-world systems we have no prior knowledge about the activity level of (unobservable) resting nodes. Nevertheless, if we could have further information about the system (in addition to \( N \) and \( M \)), we could also obtain an estimate of the empirical activity distribution that covers the entire set of nodes. See IV D in Methods for the description of the activity distribution estimation method.

We focus on the systems in which \( \hat{N}_p \) is considered to be constant (i.e., for which Model II is selected), namely LondonBike and Highschool, and assume that \( \hat{N}_p \) is given by the total number of active nodes of a day. The estimation results suggest that the activity distribution is skewed to the left in both data sets, and the generalised regression equation still well fits the empirical \( N-M \) curve (Fig. 4). We note that while the goodness of fit generally improves due to the introduction of addi-
In left panels, red line shows the theoretical equation (Eq. 11) with optimised parameters \((\alpha^*, \beta^*)\). Inset: Estimated activity distribution \(p(\alpha)\). In right panels, the number of active nodes \(N\) (dotted blue), estimated \(\hat{N}_p\) (black solid) and the empirical counterpart of \(N_p\), denoted by \(N_p^{\text{max}}\) (red circle), are shown. 5% confidence interval for \(\hat{N}_p\) is depicted by black dotted lines.

FIG. 4. Estimates based on generalised regression equations. (a) LondonBike and (b) Highschool

III. DISCUSSION

We proposed a method to identify the source of global fluctuations in temporal networks, namely the fluctuations of the numbers of active nodes and edges. Building on a model including both population and activity dynamics, we showed that these two mechanisms are sufficient to explain these global fluctuations. The estimating method we developed enables us to compute the parameters for the activity rhythm \(\kappa\) and the population size \(N_p\) (and thereby the number of resting nodes \(N_p - N\)). While an observation of \((N, M)\) in a particular snapshot is not sufficient to identify the source of global fluctuations, a sequence of \(N\) and \(M\) allows for such an estimation. We apply the method to six empirical data sets, and identify for each the main driving factor for global fluctuations.

Going further in the analysis, we found that in social systems for which the population size is fixed, as the number of active nodes increases the density decreases. Indeed, as the activity increases, new links are more likely to recruit nodes from the potential population, which dilutes the network. This effect however stops at a certain threshold, as the number of active nodes \(N\) approaches its upper bound \(N_p\). After this point any increase in the activity makes the network denser, as it adds links in a population where all nodes are already active. In contrast, for systems in which the activity is constant and the fluctuations are driven by the dynamics of the population, network density always decreases as \(N\) increases. This difference in dynamics might thus be used to identify the source of aggregate fluctuations.

While our baseline model assumed a uniform activity distribution, the model can be extended to a more general case in which activity parameters follow a beta distribution. Given the assumption that the total number of unique node IDs is a proxy of \(N_p\), we showed that the estimated activity distributions are skewed to the left, compared to the uniform distribution.

While our framework is useful for understanding the evolution of temporal networks in any contexts, there remain some issues that need to be addressed in future research. First, our method assumes that there are two types of systems, which are described as Model I (i.e., activity rhythm \(\kappa\) is constant and population size \(N_p\) is time-varying) and Model II (i.e., population size \(N_p\) is constant and activity rhythm \(\kappa\) is time-varying). In real-world systems, there may exist an intermediate state in which both the activity rhythm and the population size are evolving with similar time scales. To study those systems, one needs to include additional information other than \(N\) and \(M\) to inform the model, in order to be able to separate the effect of both mechanisms. Second, one key parameter of the model is the distribution of node fitnesses. Currently, we specified this distribution to be either a uniform or a beta distribution, which gives satisfactory estimates of the dynamical parameters. The method would of course yield more accurate estimates if we could incorporate an empirical distribution of fitnesses. However, measuring those is a complicated task: to do so, one needs to observe the activity levels of totally inactive nodes (i.e., nodes without edges), which is paradoxical. The fitness of a node in the model is indeed a rather abstract property, which integrates many realistic characteristics that depend on the context. Such characteristics can also be time dependent.

IV. METHODS

A. Data sets

All data sets are converted to temporal networks with undirected and unweighted edges. Bidirectional edges (i.e., edges in both directions) are regarded as undirected edges with weight 1. We use the following six datasets.

The Interbank data set is constructed from bilateral transactions in the online interbank market in Italy between September 4, 2000 and December 31, 2015 (i.e.,
3,922 business days). The data is commercially available from e-MID SIM S.p.A. based in Milan, Italy (http://www.e-mid.it). From the data we build a temporal network where nodes are users, with one snapshot per day. For each day, two users are connected by an edge if at least one e-mail has been sent from one user to the other between 11:00 and 16:00.

The Enron data set is an email-based communication network from the Enron Corporation [7, 30] collected from May 11, 1999 to June 21, 2002. From the data we build a temporal network where nodes are employees, with one snapshot per day. For each day, two employees are connected by an edge if at least one e-mail has been sent from one employee to the other between 11:00 and 12:00.

The LondonBike data set describes the trips taken by customers of London Bicycle Sharing Scheme [31] collected on January 12, 2016. From the data we build a temporal network where nodes are users, with one snapshot per day. For each day, two users are connected if there has been a phone call between them or a voicemail has been left, during the 8:00–12:00 time window.

The Highschool data set is a face-to-face contacts network recorded in a high school in France on December 6, 2013, using wearable sensors by the SocioPatterns collaboration [11, 17]. As in LondonBike, from the data we build a temporal network where nodes are individuals, with snapshots constructed every 20 seconds with a 10 minutes sliding time window. For each 10-minutes time interval, two stations are connected if there has been at least one trip between them.

The CollegeMsg data set is an online social network at the University of California, Irvine collected from March 23, 2004 to October 26, 2004 [8, 9]. From the data we build a temporal network where nodes are individuals, with one snapshot per day. For each day, two individuals are connected if there has been a phone call between them or a voicemail has been left, during the 8:00–12:00 time window.

The RealityMining data set is built from the call data from the Reality Commons project [2] collected from September 24, 2004 to January 7, 2005. From the data we build a temporal network where nodes are individuals, with one snapshot per day. For each day, two individuals are connected if there has been a phone call between them or a voicemail has been left, during the 8:00–12:00 time window.

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2. Exploiting dynamical relationship: Distinguishing two classes of systems

This method is based on the idea that the estimation bias due to the overlap of \((N, M)\) for multiple combinations of \((\kappa, N_p)\) could be avoided if we exploit the dynamical relationship between \(N\) and \(M\) rather than a particular realisation of \((N, M)\) in a given snapshot. In this method, we fit the empirical \(N\)-\(M\) relationship to the theoretical equations, which will give us nonlinear least squares estimators of \(\kappa\) and \(N_p\).

Since the observed variables \(N\) and \(M\) appear separately in Eqs. (2) and (3), respectively, we formulate a regression equation by relating \(N\) with \(M\) through the substitution of \(\kappa\) or \(N_p\). By doing so, we essentially categorise the empirical dynamic networks into two classes. In the regression equation for the first class, we express \(N\) as a function of \(M\) and parameter \(\kappa\) to endogenise the time-variation of \(N_p\). Hereafter we call this type of formulation “Model I”. This corresponds to a situation in which bilateral connection probabilities between nodes are constant while the potential size of networks is time-varying.

In “Model II”, on the other hand, we specify \(N\) as a function of \(M\) and parameter \(N_p\) to endogenise the time-variation of \(\kappa\). This type of model would be appropriate when the set of nodes is fixed while bilateral connecting probabilities are affected by diurnal or circadian rhythms.

The regression equations in the two models are respectively given as follows:

**Model I**: \(N_p\) is time-varying and \(\kappa\) is constant.

\[ N = G(M; \kappa) \\equiv N_p(M, \kappa) \left[ 1 - \frac{2}{\kappa N_p(M, \kappa)} \left( 1 - \left( 1 - \frac{\kappa}{2} \right)^{N_p(M, \kappa)} \right) \right] \tag{6} \]

where \(N_p\) is expressed as a function of \(M\) and \(\kappa\):

\[ N_p(M, \kappa) \equiv \frac{1 + \sqrt{1 + 32M/\kappa}}{2} \] (see Eq. A16). We obtain the estimator of \(\kappa\), denoted by \(\tilde{\kappa}\), by regressing \(N\) on \(M\) using a method of nonlinear least squares, where \(N = G(M; \tilde{\kappa}) + \varepsilon_1\), and \(\varepsilon_1\) denotes the \(T \times 1\) vector of residuals. Estimates of time-varying \(N_p\) are then given by:

\[ \tilde{N}_{p,t} = \frac{1 + \sqrt{1 + 32M_t/\kappa}}{2}. \tag{7} \]

**Model II**: \(N_p\) is constant and \(\kappa\) is time-varying.

\[ N = F(M; N_p) \equiv N_p \left[ 1 - \frac{2}{\kappa(M, N_p)} \left( 1 - \left( 1 - \frac{\kappa(M, N_p)}{2} \right)^{N_p} \right) \right] \tag{8} \]

where \(\kappa\) is expressed as a function of \(M\) and \(N_p\):

\[ \kappa(M, N_p) \equiv \frac{2M}{N_p(N_p - 1)} \] (see, Eq. A16). We estimate
κ ranging from 0.001 to 0.99) for a given parameter combination (κ, Np) may yield different observations of (N, M). Using a particular pair of (N, M) is therefore not sufficient to infer the true model parameters. Indeed, the solution of Eqs. (2) and (3) leads to a biased estimate of Np especially when the true values of κ and Np are small (Fig. 6b). This is expected from Fig. 2 in which there is a large amount of data overlap in the lower left area of the corn. In fact, κ tends to take small values (e.g., < 0.1) in real-world networks, in which case the biased estimation can become a serious problem.

Fig. 5 and c shows the error bars of the estimated parameters for the second method over 1,000 runs. The estimated values of Np and κ nicely match the true values even when the network size is fairly small and thereby multiple combinations of (κ, Np) can yield the same (N, M). This is an advantage of this method with which we do not rely on a particular realisation of (N, M), but rather we exploit the whole dynamical relationship. Furthermore, in the case where Np is fixed and κ varies in time, Model II also gives a better estimate than the direct calculation.

It should be noted that the realised N can be much lower than its potential value Np, which suggests that the potential number of active nodes cannot necessarily be inferred directly from the observed number of nodes. This is particularly true when κ is so small that the network is fairly sparse (Fig. S7 in SI).

C. Calculating the network density

From the estimates of κ and Np we can write the theoretical network density as

\[ \frac{2M}{N(N-1)} = \frac{\hat{\kappa}}{8} \left( \frac{1}{1 - q_0(\hat{\kappa}, \hat{N}_p)} \right)^2 \left( 1 + \frac{q_0(\hat{\kappa}, \hat{N}_p)}{N - 1} \right). \]

The parameter q0 approaches 0 and thereby N ≈ Np as Np becomes sufficiently large. This implies that in Model I in which κ is constant, the theory suggests that the density converges to κ/4 as Np (and N) grows.

In Model II where Np is constant, the density can be regarded as a function of κ. A shift in κ has two effects on the density. First, an increase in κ leads the network to be denser because it has a positive impact on the probability of two nodes being connected. Second, an increase in κ causes an increase of the number of active nodes N, which has a negative impact on the density. Since there is a finite fraction of inactive nodes when the network is not large enough (i.e., q0 > 0), the number of active nodes can increase in accordance with a rise in κ. This increases the denominator of the density by definition, which would lead to a reduction in the theoretical density. We find that there exists a threshold of N above which the former effect dominates the latter (Fig. 6b, right).

3. Validation

We check the accuracy of the proposed estimation method by using synthetic networks. For the estimation of Model I (resp. Model II), we generate 500 synthetic networks under various Np ranging from 20 to 300 (resp. κ ranging from 0.001 to 0.99) for a given κ (resp. Np).

While solving the system of two nonlinear equations is straightforward in principle, the question is whether the obtained solution matches the true values of κ and Np. Obviously, the network generating mechanism is in reality not deterministic but stochastic, which means the model selection is illustrated in Fig. [5]. Note that the criterion of model selection is effectively the same as that of the Akaike information criterion (AIC) and Bayesian information criterion (BIC) because we have only one parameter in both models.
D. Estimating the activity distribution

We propose a method to estimate activity distribution when the total number of potentially active nodes in the system (i.e., \(N_p\)) is known. Since in general we do not know how many nodes are ready to be active, we use the total number of active nodes of a day as a proxy for \(N_p\). The implicit assumption here is that nodes that are ready to be active would have at least one temporal edge during a day.

We choose a beta distribution, \(\rho(a) = f(a; \alpha, \beta) \equiv \frac{a^\alpha(1-a)^{\beta-1}}{B(\alpha, \beta)}\) for \(a \in [0,1]\), as a general form for the activity distribution. Parameters \(\alpha\) and \(\beta\) are estimated so that the estimated \(N_p\) matches the empirical counterpart.

A generalised version of the nonlinear regression equation (Eq. 8) is given by (see, Eq. A12 in Appendix)

\[
N = F(M; N_p, \alpha, \beta) = N_p \left[1 - \int da f(a, \alpha, \beta) \times \left(1 - \left(\frac{\alpha + \beta}{\alpha}\right) \frac{2Ma}{N_p(N_p - 1)}\right)^{(N_p - 1)}\right]. \tag{11}
\]

Note that endogenous variable \(\kappa\) is now expressed as a function of \(M\), taking parameters \(N_p\), \(\alpha\) and \(\beta\) as given:

\[
\kappa(M; N_p, \alpha, \beta) \equiv \left(\frac{\alpha + \beta}{\alpha}\right)^2 \frac{2M}{N_p(N_p - 1)}.
\]

The estimation procedure under a generalised activity distribution is then given by the following four steps:

1. For a given combination of \((\alpha, \beta)\), obtain the estimate of \(N_p\), denoted by \(\hat{N}_p(\alpha, \beta)\), by implementing the non-linear least squares on Eq. 11.

2. Repeat step 1 for various combinations of \((\alpha, \beta)\).

3. Find a combination of \((\alpha^*, \beta^*)\) such that \(\hat{N}_p(\alpha^*, \beta^*) = \arg \min_{\alpha, \beta} |\hat{N}_p(\alpha, \beta) - N_p^{\text{max}}|\), where \(N_p^{\text{max}}\) denotes the empirical counterpart of the total number of nodes in the system including temporally resting nodes.

4. The estimator of \(N_p\) is given by \(\hat{N}_p(\alpha^*, \beta^*)\).

APPENDIX: ANALYTICAL EXPRESSION FOR \(N\) AND \(M\)

In this appendix we show an analytical solution of the dynamic hidden-variable model when the network size is finite. The following derivation is based on \[25\].

Node \(i\) \((1 \leq i \leq N_p)\) is assigned activity or “fitness” \(\alpha_i \in [0,1]\) which is drawn from density \(\rho(\alpha)\) \[28\]. Let \(u(a_i, a_j)\) be the probability of nodes \(i\) and \(j\) being connected. The numbers of active nodes \(N\) and edges \(M\) can be expressed as functions of parameters \(\kappa\) and \(N_p\):

\[
\begin{align*}
N &= (1 - q_0(\kappa, N_p))N_p, \\
M &= \frac{1}{2} (\kappa N_p)^2 N_p,
\end{align*}
\]  \tag{A1}
where \( q_0(\kappa, N_p) \) is the probability of a randomly chosen node being isolated (i.e., no edges attached) and \( \bar{\kappa}(\kappa, N_p) \) denotes the average degree over all the existing nodes including isolated ones. To obtain the functional forms of \( N \) and \( M \), we need to find the functional forms of \( q_0(\kappa, N_p) \) and \( \bar{\kappa}(\kappa, N_p) \).

Given the vector of each node’s activity \( \vec{a} = (a_1, a_2, \ldots, a_{N_p}) \), the probability that node \( i \) has degree \( k_i \) is decomposed as:

\[
g(k_i|\vec{a}) = \sum_{c_i} \left[ \prod_{j \neq i} a(ai, aj)^{c_{ij}}(1 - u(ai, aj))^{1-c_{ij}} \right] \times \delta \left( \sum_{j \neq i} c_{ij}, k_i \right), \tag{A2}\]

where \( c_{ij} \in \{0, 1\} \) is the \((i, j)\)-element of the \( N_p \times N_p \) adjacency matrix, whose \( i \)-th column is given by \( \vec{c}_i = (c_{i1}, c_{i2}, \ldots, c_{iN_p})^T \), and function \( \delta(x, y) \) denotes the Kronecker delta.

Let us redefine a product term in the square bracket of \( \text{[A2]} \) as:

\[
f_j(c_{ij}; ai, aj) \equiv u(ai, aj)^{c_{ij}}(1 - u(ai, aj))^{1-c_{ij}}. \tag{A3}\]

Since \( g(k_i|\vec{a}) \) is the convolution of \( \{f_j(c_{ij}; ai, aj)\} \), its generating function:

\[
\hat{g}_i(z|\vec{a}) \equiv \sum_{k_i} z^{k_i} g(k_i|\vec{a}) \tag{A4}\]

is decomposed as:

\[
\hat{g}_i(z|\vec{a}) = \prod_{j \neq i} \hat{f}_j(z; ai, aj), \tag{A5}\]

where \( \hat{f}_j \) is the generating function of \( f_j(c_{ij}; ai, aj) \), given by:

\[
\hat{f}_j(z; ai, aj) \equiv \sum_{aij} z^{aij} f_j(c_{ij}; ai, aj). \tag{A6}\]

Degree distribution \( p(k_i; \kappa, N_p) \) is defined by the probability that node \( i \) has degree \( k_i \) and is related to \( g(k_i|\vec{a}) \) so that:

\[
p(k_i; \kappa, N_p) = \int g(k_i|\vec{a}) \rho(\vec{a}) d\vec{a}, \tag{A7}\]

where we define \( \rho(\vec{a}) \equiv \prod_i \rho(ai) \) and \( d\vec{a} \equiv \prod_i da_i \). Therefore, differentiation of \( \hat{g}_i(z|\vec{a}) \) with respect to \( z \) gives the average degree \( \bar{\kappa}(\kappa, N_p) \):

\[
\bar{\kappa}(\kappa, N_p) = \sum_{k_i} k_ip(k_i; N_p) = \sum_{k_i} k_i \int g(k_i|\vec{a}) \rho(\vec{a}) d\vec{a} = \frac{d}{dz} \int \hat{g}_i(z|\vec{a}) \rho(\vec{a}) d\vec{a} \bigg|_{z=1} = \frac{d}{dz} \int \rho(\vec{a}) da_i \prod_{j \neq i} \int \hat{f}_j(z; ai, aj) \rho(aj) da_j \bigg|_{z=1} = \int \rho(\vec{a}) da_i \frac{d}{dz} \left[ \int \hat{f}(z; ai, h) \rho(h) dh \right]^{N_p-1} \bigg|_{z=1} = (N_p - 1) \int \rho(\vec{a}) da_i \left[ \int da_\rho(\vec{a}) \hat{f}(z; ai, a) \right]^{N_p-2} \bigg|_{z=1}. \tag{A8}\]

From Eqs. \( \text{[A3]} \) and \( \text{[A6]} \), we have \( \hat{f}(z; ai, a) = \sum_{c_{ij}} z^{c_{ij}} f(c_{ij}; ai, a) = (z - 1)u(ai, a) + 1 \). It follows that:

\[
\int da_\rho(\vec{a}) \hat{f}(z; ai, a) = (z - 1) \int da_\rho(\vec{a}) u(ai, a) + 1, \tag{A9}\]

\[
\int da_\rho(\vec{a}) \frac{d}{dz} \hat{f}(z; ai, a) = \int da_\rho(\vec{a}) u(ai, a). \tag{A10}\]

Substituting these into Eq. \( \text{[A8]} \) leads to:

\[
\bar{\kappa}(\kappa, N_p) = (N_p - 1) \int da_\rho(\vec{a}) \hat{f}(z; ai, a) = (z - 1) \int da_\rho(\vec{a}) u(ai, a). \tag{A11}\]

It should be noted that \( \text{[A11]} \) is equivalent to Eq. \( \text{[21]} \) of Ref. \( \text{[29]} \) if \( N_p - 1 \) is replaced with \( N \).

From \( \text{[A7]} \), the probability of a node being isolated, \( q_0(\kappa, N_p) \equiv p(k_i = 0; \kappa, N_p) \), is given by:

\[
q_0(\kappa, N_p) = \int g(k_i = 0|\vec{a}) \rho(\vec{a}) d\vec{a} = \int da_\rho(\vec{a}) \left[ 1 - \int da_i \rho(ai) da_i \right]^{N_p-1}. \tag{A12}\]

Then, substituting \( \rho(ai) = 1 \) (i.e., uniform distribution on \([0, 1]\)) and \( u(ai, a') = \kappa aai' \) into Eq. \( \text{[A11]} \) gives:

\[
\bar{\kappa}(\kappa, N_p) = \frac{\kappa}{4} (N_p - 1). \tag{A13}\]

Similarly, substituting the same conditions into Eq. \( \text{[A12]} \) gives:

\[
q_0(\kappa, N_p) = \int \left( 1 - \frac{\kappa a_i}{2} \right)^{N_p-1} da_i. \tag{A14}\]
By changing the integration variable to \( x = 1 - \frac{\kappa N}{2} \), we have:

\[
q_0(\kappa, N_p) = \frac{2}{\kappa} \int_{1-\frac{\kappa N}{2}}^{1} x^{N_p-1} dx = \frac{2}{\kappa N_p} \left[ 1 - \left(1 - \frac{\kappa}{2}\right)^{N_p} \right]. \tag{A15}
\]

Note that \( q_0(\kappa, 1) = 1 \) and \( \lim_{N_p \to \infty} q_0(\kappa, N_p) = 0 \). Combining these results with Eq. (A1), we have:

\[
\begin{align*}
N &= N_p \left[ 1 - \frac{2}{\kappa N_p} \left(1 - \frac{\kappa}{2}\right)^{N_p} \right], \\
M &= \frac{1}{8\kappa} N_p (N_p - 1).
\end{align*}
\tag{A16}
\]

If \( N_p \) is sufficiently large, then \( q_0(\kappa, N_p) \simeq 0 \) and thereby \( N \sim N_p \) and \( M \propto N^2 \).

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AUTHOR CONTRIBUTIONS

T.K. conceived and directed the study. T.K. and M.G. defined the model. T.K. performed the analytical calculations and the data analyses. T.K. and M.G. drafted the final manuscript.

ADDITIONAL INFORMATION

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Supporting Information:

“A simple model for the macroscopic fluctuations of temporal networks”
Teruyoshi Kobayashi and Mathieu Génois

FIG. S1. Fraction of inactive nodes in the population. The larger the population $N_p$ and the overall activity $\kappa$, the lower the fraction of resting nodes $q_0$. 

FIG. S2. Fitted $M$-$N$ curve. Activity is uniformly distributed: $a \in [0, 1]$. 
FIG. S3. Estimated time-varying parameters. Activity is uniformly distributed: $a \in [0,1]$. Model I: Interbank, Enron and RealityMining. Model II: Highschool.
FIG. S4. Model fit to the intra-day networks in the Interbank data set. Red line denotes theoretical values indicated by the selected model. For each day, each dot represents a snapshot network of a 20-minute time window. Snapshots are created every 5 minutes. Model I (Model II) is selected for panels showing estimated $\kappa (N_p)$. Model II is selected for September 13, 14 and 15 while Model I is selected for the other 12 days.
FIG. S5. Model fit for the LondonBike dataset in each time bin. The data period ranges from January 10, 2016 to August 31, 2016 (235 days). Red line denotes theoretical values indicated by the selected model. Model II is selected for all time bins, indicating that the number of existing nodes is constant.
FIG. S6. Empirical and theoretical density. Activity is uniformly distributed: $a \in [0, 1]$. Model I: Interbank, Enron and RealityMining. Model II: Highschool.
FIG. S7. Validation of the proposed estimation method based on synthetic networks. Color denotes the average value over 1,000 runs (color bar in left panel).