AROUND GENERAL NERON DESINGULARIZATION

DORIN POPESCU

ABSTRACT. It gives some new forms of General Neron Desingularization and new applications.

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INTRODUCTION

Using the proof of the Artin approximation property of the convergent power series rings over \( \mathbb{C} \) (see [1]), Ploski [14] succeeds to prove a theorem in the idea of Neron desingularization [13]. Ploski’s result was the starting point of our General Neron Desingularization (see here Theorem 2). Later we found other types of such theorems one conjectured by Artin [4] and solved positively in [7] (see here Theorem 6) using an idea of [5, Theorem 2.3] and another one reminding of the Artin strong approximation property (see [19] and here Theorem 7) which has an interesting consequence (see Corollary 10) studied in a special case in Section 2 mainly in the arc frame. The proof of Theorem 12 is inspired by [15] where we had to consider General Neron Desingularization given along the maps defined by formal power series not in an explicit way in order to use Computer Algebra Systems. The paper ends with an application of the General Neron Desingularization to an extension of the Bass-Quillen Conjecture in the frame of factors of polynomial algebras by monomial ideals (see Theorems 24, 25).

1. A VERSION OF GENERAL NERON DESINGULARIZATION IN THE IDEA OF PLOSKI

A ring morphism \( u : A \to A' \) has regular fibers if for all prime ideals \( P \in \text{Spec } A \) the ring \( A'/PA' \) is a regular ring, i.e. its localizations are regular local rings. It has geometrically regular fibers if for all prime ideals \( P \in \text{Spec } A \) and all finite field extensions \( K \) of the fraction field of \( A/P \) the ring \( K \otimes_{A/P} A'/PA' \) is regular. If \( A \supset \mathbb{Q} \) then regular fibers of \( u \) are geometrically regular. We call \( u \) regular if it is flat and its fibers are geometrically regular. A Henselian local ring \( (A, m) \) is excellent if it is Noetherian and the completion map \( A \to \hat{A} \) is regular. A regular morphism is smooth if it is finitely presented and it is essentially smooth if it is a localization of a finitely presented morphism.
Theorem 1. (Ploski [14]) Let \( C[x], x = (x_1, \ldots, x_n), f = (f_1, \ldots, f_s) \) be some convergent power series from \( C[x, Y], Y = (Y_1, \ldots, Y_N) \) and \( \hat{y} \in C[[x]]^N \) with \( \hat{y}(0) = 0 \) be a solution of \( f = 0 \). Then the map \( v : B = C[x, Y]/(f) \to C[[x]] \) given by \( y \to y \) factors through an \( A \)-algebra of type \( B' = C[x, Z] \) for some variables \( Z = (Z_1, \ldots, Z_s) \), that is \( v \) is a composite map \( B \to B' \to C[[x]] \).

This result shows a particular case of the following theorem in Krull dimension > 1, the case when \( A, A' \) are DVR being given by the so called Neron p-desingularization [13, 2].

Theorem 2. (General Neron Desingularization, Popescu [16, 17], Andre [6], Swan [22, Spivakovsky [21]) Let \( v : A \to A' \) be a regular morphism of Noetherian rings and \( B \) a finite type \( A \)-algebra. Then any \( A \)-morphism \( v : B \to A' \) factors through a smooth \( A \)-algebra \( C \), that is \( v \) is a composite \( A \)-morphism \( B \to C \to A' \).

The following theorem is a consequence of Theorem 2 in the idea of Ploski.

Theorem 3. Let \( (A, m) \) be an excellent Henselian local ring, \( \hat{A} \) its completion, \( B \) a finite type \( A \)-algebra and \( v : B \to \hat{A} \) an \( A \)-morphism. Then \( v \) factors through an \( A \)-algebra of type \( A < Z \) if some variables \( Z = (Z_1, \ldots, Z_s) \), that is \( A < Z \) is the Henselization of \( A[Z](m, Z) \).

Proof. By Theorem 2 we see that \( v \) factors through a smooth \( A \)-algebra \( B' \), let us say \( v' \). Using the local structure of smooth algebras given by Grothendieck (see [22, Theorem 2.5]) we may assume that \( B'_{\psi^{-1}(m, A)} \) is a localization of a smooth \( A \)-algebra of type \( (A[Z, T]/(g))_{g'h} \), where \( Z = (Z_1, \ldots, Z_s) \), \( g' = \partial g/\partial T \). Choose \( h \) such that \( v' \) factors through \( C' = (A[Z, T]/(g))_{g'h} \) let us say \( v' \) is the composite map \( B' \to C' \xrightarrow{w} \hat{A} \).

Suppose that \( w \) is given by \( (Z, T) \to (\hat{Z}, \hat{T}) \in \hat{A} \). We claim that we may reduce to the case when \( \hat{Z}(0) = 0, \hat{T}(0) = 0 \). Indeed, choose \( z_0 \in A^N, t_0 \in A \), such that \((z_0, t_0) \equiv (\hat{Z}, \hat{T}) \) modulo \( m \hat{A} \) and set \( \hat{z}' = \hat{z} - z_0 \in m \hat{A}, \hat{t}' = \hat{t} - t_0 \in m \hat{A} \). Changing \( (Z, T) \) by \( (z_0 + Z', t_0 + T') \), \( Z' = (Z_1', \ldots, Z_N') \) in \( C \) and correspondingly \( (\hat{Z}, \hat{T}) \) by \( (z_0 + \hat{z}', t_0 + \hat{t}) \) we get our claim fulfilled.

Clearly \( w \) extends to a map \( w' : C' = (A < Z > [T]/(g))_{g'h} \to \hat{A} \) and we have \( C' \cong A < Z > \) since \( C' \) is an etale neighborhood of \( A < Z > \).

The following consequence of Theorem 2 was noticed and hinted by N. Radu to M. Andre. This was the origin of Andre’s interest to read our theorem and to write later [6].

Corollary 4. Let \( v : A \to A' \) be a regular morphism of Noetherian rings and \( B \) a finite type \( A \)-algebra. Then the differential module \( \Omega_{A'/A} \) is flat.

For the proof note that by Theorem 2 it follows that \( A' \) is a filtered inductive limit of some smooth \( A \)-algebras \( C \) and so \( \Omega_{A'/A} \) is a filtered inductive limit of \( A' \otimes_C \Omega_{C/A} \), the last modules being free modules.

A Noetherian local ring \((A, m)\) has the property of Artin approximation if the solutions in \( A \) of every system of polynomial equations \( f \) over \( A \) is dense with
respective of the $m$-adic topology in the set of solutions of $f$ in the completion $\hat{A}$ of $A$. In fact $(A, m)$ has the property of Artin approximation if and only if every system of polynomial equations $f$ over $A$ has a solution in $\hat{A}$ if it has one in $\hat{A}$. The following theorem was conjectured by M. Artin in [3] and it is a consequence of Theorem 2.

**Theorem 5.** (Popescu [16], [20]) An excellent Henselian local ring has the property of Artin approximation.

This theorem follows easily from Theorem 2 using the Implicit Function Theorem. But it is much easier to apply Theorem 3. Indeed, let $(A, m)$ be an excellent Henselian local ring, $f = (f_1, \ldots, f_r)$ some polynomials from $A[Y], Y = (Y_1, \ldots, Y_N)$ and $\hat{y} \in \hat{A}^N$ a solution of $f = 0$. We will show that $f$ has a solution in $A$. Set $B = A[Y]/(f)$ and $v : B \to \hat{A}$ the map given by $Y \to \hat{y}$. By Theorem 3 $v$ factors through $A < Z >$ for some $Z = (Z_1, \ldots, Z_s)$, that is $v$ is the composite map $B \hat{\to} A < Z > \to \hat{A}$. Set $y = t(Y + (f))$. Then $y(0)$ is a solution of $f$ in $A$.

A special form of Theorem 2 is the following theorem which is a positive answer to a conjecture of M. Artin [4].

**Theorem 6.** (Cipu-Popescu [7]) Let $u : A \to A'$ be a regular morphism of Noetherian rings, $B$ a finite type $A$-algebra, $v : B \to A'$ an $A$-morphism and $D \subset \text{Spec } B$ the open smooth locus of $B$ over $A$. Then there exist a smooth $A$-algebra $C$ and two $A$-morphisms $t : B \to C$, $w : C \to A'$ such that $v = wt$ and $C$ is smooth over $B$ at $t'^{-1}(D), t^* : \text{Spec } C \to \text{Spec } B$ being induced by $t$.

Another type of General Neron Desingularization is the following theorem.

**Theorem 7.** (Popescu [19]) Let $(A, m)$ be a Noetherian local ring with the completion map $A \to \hat{A}$ regular. Then for every finite type $A$-algebra $B$ there exists a function $\lambda : N \to N$ such that for every positive integer $c$ and every morphism $v : B \to A/m^{\lambda(c)}$ there exists a smooth $A$-algebra $C$ and two $A$-algebra morphisms $t : B \to C$, $w : C \to A/m^c$ such that $wt$ is the composite map $B \hat{\to} A/m^{\lambda(c)} \to A/m^c$.

**Remark 8.** Can be $\lambda$ computed? The proof from [19] is not constructive. Perhaps $\lambda$ defined by $B = A[Y]/(f), Y = (Y_1, \ldots, Y_n), f = (f_1, \ldots, f_r) \in A[Y]^r$ can be related with the Artin function $\nu$ associated to $f$ considered over $\hat{A}$ since $\hat{A}$ has the so called the Artin strong approximation property (see [2], [9], [16], [20]).

**Corollary 9.** Let $(A, m)$ be a Noetherian local ring with the completion map $A \to \hat{A}$ regular, $c \in N, B$ a finite type $A$-algebra and $\lambda$ its function defined by Theorem 7. Let $v : B \to \hat{A}/m^{\lambda(c)}\hat{A}$ be an $A$-morphism. Then there exists a $B$-algebra $C$ which is smooth over $A$ and an $A$-morphism $w : C \to \hat{A}$ such that the composite map $B \to C \hat{\to} \hat{A}$ coincides with $v$ modulo $m^c\hat{A}$. In particular, $\lambda \geq \nu$, the last map being the Artin function over $\hat{A}$ associated to the system of polynomials $f$ defining $B$.

**Proof.** Let $\lambda$ be given by Theorem 7. Then there exists a $B$-algebra $C$ which is smooth over $A$ and an $A$-morphism $\hat{w} : C \to A/m^c$ such that the composite map $B \to C \hat{\to} \hat{A}/m^c\hat{A}$ coincides with $v$ modulo $m^c$. By the Implicit Function Theorem
\( \tilde{w} \) can be lifted to an \( A \)-morphism \( w : C \to \hat{A} \). Therefore, \( v \) coincides with the composite map \( B \to C \xrightarrow{w} \hat{A} \) modulo \( m\hat{A} \).

**Corollary 10.** Let \((A, m)\) be a Noetherian local ring with the completion map \( A \to \hat{A} \) regular, \( c \in \mathbb{N}, B \) a finite type \( A \)-algebra and \( \lambda \) its function defined by Theorem 7. Let \( g : B \to \hat{A}/m^{\lambda} \hat{A} \) be an \( A \)-morphism. Then there exists a \( B \)-algebra \( C \) which is smooth over \( A \) with the property that for any \( A \)-morphism \( v : B \to \hat{A} \) congruent with \( g \) modulo \( m^{\lambda} \hat{A} \), there exists an \( A \)-morphism \( w : C \to \hat{A} \) such that the composite map \( B \to C \xrightarrow{w} \hat{A} \) coincides with \( v \) modulo \( m\hat{A} \).

For the proof apply Corollary 9 for \( g \).

**Remark 11.** It will be nice to get above that \( v \) coincides with the composite map \( B \to C \xrightarrow{w} A' \) because then \( C \) does not depend on \( v \) but only on \( g \) and \( \lambda(c) \). This is possible in a special case (see Theorem 12).

2. A special case of Corollary 10

Let \( k \) be a field, \( A = k[[x]], A' = k[[x]], B = A[Y]/I, Y = (Y_1, \ldots, Y_n) \) a finite type \( A \)-algebra, If \( f = (f_1, \ldots, f_r) \), \( r \leq n \) is a system of polynomials from \( I \) then we can define the ideal \( \Delta_f \) generated by all \( r \times r \)-minors of the Jacobian matrix \( \left( \frac{\partial f_i}{\partial X_j} \right) \). After Elkik [8], let \( H_{B/A} \) be the radical of the ideal

\[
\sum_{r=1}^{n} \sum_{f} ((f) : I) \Delta_f B,
\]

where the second sum is taken over all systems of polynomials \( f = (f_1, \ldots, f_r) \) from \( I \). Then \( B_P \) is essentially smooth over \( A \) if and only if \( P \not\cong H_{B/A} \) by a theorem of Elkik [8]. Thus the set of prime ideals \( V(H_{B/A}) \) is the non smooth locus of \( B \) over \( A \). Let \( c \in \mathbb{N} \) and \( v : B \to A'/(x^{4c+1}) \) be an \( A \)-morphism such that \( v(H_{B/A}) \not\subseteq (x)^c/(x^{4c+1}) \), let us say there exist \( f = (f_1, \ldots, f_r) \) in \( I \), \( N \in ((f) : I) \) and \( M \) an \( r \times r \)-minor of the Jacobian matrix \( \left( \frac{\partial f_i}{\partial Y_j} \right) \) such that \( v(NM) \not\subseteq (x)^c/(x^{4c+1}) \), where for simplicity we write \( v(NM) \) instead \( v(NM + I) \). We may assume that \( M = \det((\frac{\partial f_i}{\partial Y_j})_{i,j\in[r]}) \).

**Theorem 12.** Then there exists a \( B \)-algebra \( C \) which is smooth over \( A \) such that every \( A \)-morphism \( v' : B \to A' \) with \( v' \equiv v \) modulo \( x^{4c+1} \) (that is \( v'(Y) \equiv v(Y) \) modulo \( x^{4c+1} \)) factors through \( C \).

**Proof.** Set \( P = NM, B_1 = B[Z]/(x^c - PZ) \). Then each \( A \)-morphism \( v' : B \to A' \) with \( v' \equiv v \) modulo \( x^{4c+1} \) satisfies \( v'(P) \not\subseteq (x)^c \) and so \( v'(P) \not\subseteq (x)^c \) and therefore \( v' \) factors through \( B_1 \), that is \( v' \) is the composite map \( B \to B_1 \xrightarrow{w'_1} A' \), where \( w'_1 \) is given by \( Z \to x^c/v'(P) \in A' \). Set \( I_1 = (I, f_{r+1}) \), where \( f_{r+1} = x^c - PZ \). Changing \( B \) by \( B_1 \) and \( v \) by a certain \( v_1 \) we may assume that \( x^{2c} \equiv P \) modulo \( I \) for some new \( P = NM \) with \( N \in ((f) : I) \) and \( M \) a \((r + 1) \times (r + 1)\)-minor of the corresponding Jacobian matrix. In fact the new \( M \) is the old minor multiplied with \( P \) and the new \( N \) is the old one multiplied with \( Z^2 \).
Since $A/(x^{4c+1}) \cong A’/(x^{4c+1})$ we may choose $y' \in A'$, such that $v(Y + I) = y' + (x^{4c+1})$ (recall that now $Y$ denotes the former $(Y, Z)$ and the new $r$ is the old $r + 1$). Thus $I(y') \equiv 0$ modulo $x^{4c+1}$. Then we have $P(y') \equiv x^{2c}$ modulo $x^{4c+1}$ and so $x^{2c}s = P(y')$ for some $s \in 1 + x^{2c+1}A$, that is invertible since $A$ is local.

Let $H$ be the $n \times n$-matrix obtained adding down to $(\partial f/\partial Y)$ as a border the block $(0|\text{Id}_{n-r})$ (we assume as above that $M$ is given on the first $r$ columns of the Jacobian matrix). Let $G'$ be the adjoint matrix of $H$ and $G = s^{-1}NG'$. We have

$$GH = HG = s^{-1}NM\text{Id}_n = s^{-1}P\text{Id}_n$$

and so

$$x^{2c}\text{Id}_n = s^{-1}P(y')\text{Id}_n = G(y')H(y').$$

Let

$$h = Y - y' - x^{2c}G(y')T,$$

where $T = (T_1, \ldots, T_n)$ are new variables. Since

$$Y - y' \equiv x^{2c}G(y')T \mod h$$

and

$$f(Y) - f(y') \equiv \sum_j \partial f/\partial Y_j(y')(Y_j - y'_j)$$

modulo higher order terms in $Y_j - y'_j$ by Taylor’s formula we see that we have

$$f(Y) - f(y') \equiv \sum_j x^{2c}\partial f/\partial Y_j(y')G_j(y')T_j + x^{4c}Q =$$

$$x^{2c}s^{-1}P(y')T + x^{4c}Q = x^{4c}(T + Q) \mod h,$$

where $Q \in T^2A[T]^{r}$. This is because $(\partial f/\partial Y)G = (s^{-1}P\text{Id}_n|0)$. We have $f(y') = x^{4c}a$ for some $a \in xA'$. Set $g_i = a_i + T_i + Q_i$, $i \in [r]$ and $E = A[Y, T]/(I, g, h)$. Clearly, it holds $x^{4c}g \subset (f, h)$ and $(f) \subset (h, g)$. Note that $U = (A[T]/(g))_{s'}$ is smooth for some $s' \in 1 + (T)$ because the $r \times r$-minor of the Jacobian matrix $(\partial g/\partial T)$ given by the first $r$ columns has the form $\det(\text{Id}_r + (\partial Q_i/\partial T_j)_{i,j\in[r]})$. In particular, $U$ is flat over $A$ and so $x$ is regular in $U$. We claim that $E_{s',s''} \cong U_{s''}$ for some $s'' \in 1 + (x, T)$. It will be enough to show that $I \subset (h, g)A[Y, T]_{s',s''}$. We have $s^{-1}PI \subset (f) \subset (h, g)$ and so $s^{-1}P(y' + x^{2c}L)I \subset (h, g)$ where $L = G(y')T$. But $s^{-1}P(y' + x^{2c}L)$ has the form $x^{2c}s''$ for some $s'' \in 1 + (x, T)$. It follows that $s''I \subset (h, g)A[Y, T]_{s',s''}$ since $x$ is regular in $U$ and so $I \subset (h, g)A[Y, T]_{s',s''}$. Thus $C = E_{s',s''}$ is a $B$-algebra smooth over $A$.

Remains to see that an arbitrary $A$-morphism $v' : B \to A'$ with $v' \equiv v$ modulo $x^{4c+1}$ factors through $C$. We have $v'(Y + I) \equiv v(Y + I) \equiv y'$ modulo $x^{4c+1}$ and so there exists $\varepsilon \in xA^m$ such that $v'(Y) - y' = x^{4c}\varepsilon$. Then $t := H(y')\varepsilon \in xA^m$ satisfies

$$G(y')t = s^{-1}P(y')\varepsilon = x^{2c}\varepsilon$$

and so

$$v'(Y) - y' = x^{2c}G(y')t,$$

that is $h(v'(Y), t) = 0$. Note that $x^{4c}g(t) \in (h(v(Y)), t), f(v(Y))) = (0)$ and it follows that $g(t) = 0$ since $A'$ is a domain. Thus $v'$ factors through $E$, that is $v'$ is a composite map $B \to E \xrightarrow{\alpha} A'$, where $\alpha$ is a $B$-morphism given by $T \to t$. As
\(\alpha(s') \equiv 1 \mod x\) and \(\alpha(s'') \equiv 1 \mod (x, t)\), \(t \in xA'\) we see that \(\alpha(s'), \alpha(s'')\) are invertible because \(A'\) is local and so \(\alpha\) (thus \(v'\)) factors through the standard smooth \(A\)-algebra \(C\).

**Corollary 13.** In the assumptions and notations of the above theorem, let \(\rho : B \to C\) be the structural algebra map. Then \(\rho\) induces a bijection \(\rho^*\) between \(\{w \in \text{Hom}_A(C, A') : w\rho \equiv v \mod x^{4c+1}\}\) and \(\{v' \in \text{Hom}_B(B, A') : v' \equiv v \mod x^{4c+1}\}\) given by \(\rho^*(w) = w\rho\).

**Proof.** By the above theorem \(\rho^*\) is surjective. Let \(w, w' \in \text{Hom}_A(C, A')\) be such that \(w\rho = w'\rho \equiv v \mod x^{4c+1}\). Since \(H(y')(Y - y') \equiv x^{4c}T\) modulo \(h\) by construction of \(E\) we see that \(x^{4c}E \subset \text{Im}\ \rho\) and so \(w|_E = w'|_E\) because \(A'\) is a domain. It follows \(w = w'\). \(\square\)

**Corollary 14.** In the assumptions and notations of the above corollary, suppose that there exists an \(A\)-morphism \(\tilde{\nu} : B \to A'\) with \(\tilde{\nu} \equiv v \mod x^{4c+1}\). Then the following statements hold:

1. there exists an unique \(A\)-morphism \(\tilde{w} : C \to A'\) such that \(\tilde{w}\rho = \tilde{\nu}\),
2. there exists a bijection \(\gamma : \{\alpha \in \text{Hom}_A(C, A') : \text{Im } \alpha \subset (x)\} \to \{w' \in \text{Hom}_A(C, A') : w' \equiv \tilde{w} \mod x^{4c+1}\}\) given by \(\alpha \to \tilde{w} + x^{4c+1}\alpha\),
3. there exists a bijection \(\{\alpha \in \text{Hom}_A(C, A') : \text{Im } \alpha \subset (x)\} \xrightarrow{\rho^*\gamma} \{v' \in \text{Hom}_B(B, A') : v' \equiv v \mod x^{4c+1}\}\).

**Proof.** By Corollary 13 we take \(\tilde{w} = \rho^*(\tilde{\nu})\). If \(\gamma(\alpha) = \gamma(\alpha')\) then \(x^{4c+1}(\alpha - \alpha') = 0\) and so \(\alpha = \alpha'\) since \(A'\) is a domain. Conversely, if \(\text{Im}(w' - \tilde{w}) \subset x^{4c+1}A'\) for some \(w' \in \text{Hom}_A(C, A')\) then \(a \to (w' - \tilde{w})(a)/x^{4c+1}\) defines \(\alpha\) with \(\gamma(\alpha) = w'\). Clearly, we have \(\text{Im } \alpha \subset (x)\). \(\square\)

By a Grothendieck’s theorem [22] Theorem 2.5, \(C\) can be chosen of the form \((A[Z, V]/(g))_{g''n}\), where \(Z = (Z_1, \ldots, Z_s)\) and \(g' = \partial g/\partial V\).

**Corollary 15.** In the assumptions and notations of the above corollary, the map \(\{\alpha \in \text{Hom}_A(C, A') : \text{Im } \alpha \subset (x)\} \to x \text{Hom}_A(A[Z], A') \cong A^s\) induced by the restriction given by \(A[Z] \to C\) is a bijection. In particular, there exists a canonical bijection \(A^s \cong x \text{Hom}_A(A[Z], A') \to \{v' \in \text{Hom}_B(B, A') : v' \equiv v \mod x^{4c+1}\}\).

**Proof.** If \(\alpha \in \text{Hom}_A(C, A')\) satisfies \(\text{Im } \alpha \subset (x)\) we get \(\text{Im } \alpha|_{A[Z]} \subset (x)\). Since \((Z, V) \to 0\) defines an \(A\)-morphism \(C \to A'\) corresponding to \(\tilde{w}\) we see that \(g(0, 0) = 0, g'(0, 0), h(0, 0) \not\in (x)\). It follows that \(g(\alpha(Z), 0) \in (x)\) and \(g'(\alpha(Z), 0), h(\alpha(Z), 0) \not\in (x)\). Then there exists an unique solution \(\alpha(V)\) of \(g(\alpha(Z), V) = 0\) in \(A'\) by the Implicit Function Theorem. It follows that the above restriction is bijective. \(\square\)

Let \(k\) be a field and \(F\) a finite type \(k\)-algebra, let us say \(F = k[U]/J, U = (U_1, \ldots, U_r)\). An arc \(\text{Spec } k[[x]] \to \text{Spec } F\) (see [12]) is given by a \(k\)-morphism
$g : F \to A' = k[[x]]$. Assume that $H_{F/k} \neq 0$ (this happens for example when $F$ is reduced and $k$ is perfect). Moreover suppose that $g(H_{F/k}) \neq 0$. Let $A = k[x]$, $B = A \otimes_k F$ and $v : B \to A'$ be the $A$-morphism induced by $g$. Note that $A \otimes_k -$ induces a bijection $\text{Hom}_k(F, A') \to \text{Hom}_A(B, A')$, the injectivity follows by flatness of $A$ over $k$ and an $A$-morphism $B \to A'$ is induced by a solution of $J$ in $A'$, that is by a $k$-morphism $F \to A'$.

**Corollary 16.** The set \{ \text{g}' \in \text{Hom}_k(F, A') : \text{g}' \equiv g \text{ modulo x}^{4c+1} \} is in bijection with an affine space $A'^s$ over $A'$ for some $s \in \mathbb{N}$.

### 3. An application of General Neron Desingularization to an extension of Bass-Quillen Conjecture

Let $R[T], T = (T_1, \ldots, T_n)$ be a polynomial algebra in $T$ over a Noetherian ring $R$. A finitely generated projective $R[T]$-module $M$ is extended from $R$ if there exists a finitely generated projective $R$-module $M'$ such that $M \cong R[T] \otimes_R M'$. An extension of Serre’s Problem proved by Quillen and Suslin is the following

**Conjecture 17.** (Bass-Quillen) If $R$ is a regular ring then every finitely generated projective module over $R[T]$ is extended from $R$.

If $(R, m)$ is regular local ring then the above conjecture says that every finitely generated projective module over $R[T]$ is free. For a possible proof it is enough to consider only this case using the Local-Global Principle.

**Theorem 18.** (Lindel) The Bass-Quillen Conjecture holds if $R$ is essentially of finite type over a field.

Swan’s notes on Lindel’s paper contain two interesting remarks.

1) Lindel’s proof goes also when $(R, m)$ is regular local essentially of finite type over a DVR $A$ such that its local parameter $p \notin m^2$.

2) The Bass-Quillen Conjecture holds if $(R, m)$ is a regular local ring containing a field, or $p = \text{char } R/m \notin m^2$ providing the following question has a positive answer.

**Question 19.** (Swan) Is a regular local ring a filtered inductive limit of regular local rings essentially of finite type over $\mathbb{Z}$?

Indeed, suppose for example that $R$ contains a field and Swan’s question has a positive answer. Then $R$ is a filtered inductive limit of regular local rings $R_i$ essentially of finite type over a prime field $P$. A finitely generated projective $R[T]$-module $M$ is an extension of a finitely generated projective $R_i[T]$-module $M_i$ for some $i$, that is $M \cong R[T] \otimes_{R_i[T]} M_i$. By Theorem 18 we get $M_i$ free and so $M$ is free too.

**Theorem 20.** (Popescu [18]) Swan’s Question 19 holds for regular local rings $(R, m, k)$ which are in one of the following cases:

1) $R$ contains a field,
2) the characteristic $p$ of $k$ is not in $m^2$,
3) $R$ is excellent Henselian.
Corollary 21. (Popescu [18]) The Bass-Quillen Conjecture holds if $R$ is a regular local ring in one of the cases (i), (ii) of the above theorem.

Remark 22. Theorem 20 does not solve completely the Question 19. A complete solution is hard to do it. However, (iii) from Theorem 20 says that a positive answer is expected in general. Unfortunately, there exist no result similar to Lindel’s saying that the Bass-Quillen Conjecture holds for all regular local rings essentially of finite type over $\mathbb{Z}$ even Theorem 20 was published a lot of time ago.

An interesting problem is to replace in the Bass-Quillen Conjecture the polynomial algebra $R[T]$ by other $R$-algebras. A useful result in this idea is the following theorem which we found too late to use it in [18].

Theorem 23. (Vorst [23]) Let $A$ be a ring, $A[X]$, $X = (X_1, \ldots, X_r)$ a polynomial algebra, $I \subset A[X]$ a monomial ideal and $B = A[X]/I$. Then every finitely generated projective $B$-module $M$ is extended from a finitely generated projective $A$-module if for all $n \in \mathbb{N}$ every finitely generated projective $A[T]$-module, $T = (T_1, \ldots, T_n)$ is extended from a finitely generated projective $A$-module.

Using the above theorem and Corollary 21 we get the following theorem.

Theorem 24. Let $R$ be a regular local ring in one of the cases (i), (ii) of Theorem 20, $I \subset R[X]$, $X = (X_1, \ldots, X_r)$ be a monomial ideal and $B = R[X]/I$. Then any finitely generated projective $B$-module is free.

The Bass-Quillen Conjecture could also hold when $R$ is not regular as shows the following theorem.

Theorem 25. Let $R$ be a regular local ring in one of the cases (i), (ii) of Theorem 20, $I \subset R[X]$, $X = (X_1, \ldots, X_r)$ be a monomial ideal and $A = R[X]/I$. Then every finitely generated $A[T]$-module, $T = (T_1, \ldots, T_n)$ is free.

For the proof note that $A[T]$ is a factor of $R[X, T]$ by the monomial ideal $IR[X, T]$.

Remark 26. If $I$ is not monomial above then the Bass-Quillen Conjecture may fail for $A$. Indeed, if $A = R[X_1, X_2]/(X_1^2 - X_2^3)$ then there exist finitely generated projective $A[T]$-modules of rank one which are not free (see [10, (5.10)]).

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DORIN POPESCU, SIMION STOILOW INSTITUTE OF MATHEMATICS, RESEARCH UNIT 5, UNIVERSITY OF BUCHAREST, P.O.Box 1-764, BUCHAREST 014700, ROMANIA
E-mail address: dorin.popescu@imar.ro