A novel quantum key distribution scheme with orthogonal product states

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The general conditions for the orthogonal product states of the multi-state systems to be used in quantum key distribution (QKD) are proposed, and a novel QKD scheme with orthogonal product states in the $3 \times 3$ Hilbert space is presented. We show that this protocol has many distinct features such as great capacity, high efficiency. The generalization to $n \times n$ systems is also discussed and a fancy limitation for the eavesdropper’s success probability is reached.

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I. INTRODUCTION

Cryptography is created to satisfy the people’s desire of transmitting secret messages. With the development of the quantum computation, especially the proposal of Shor’s algorithm [1], the base of the most important classic cryptographic scheme was shocked. But at the same time, the principles of quantum mechanics have also shed new light on the field of cryptography as these fundamental laws guarantee the secrecy of quantum cryptosystems. Any intervention of an eavesdropper, Eve, must leave some trace which can be detected by the legal users of the communication channel. All kinds of quantum key distribution (QKD) schemes, such as BB84 protocol [2], B92 protocol [3], and the EPR scheme [4] have been proposed. Recently, quantum cryptography with 3-state systems was also introduced [5]. Experimental research on QKD is also progressing fast, for instance, the optical-fiber experiment of BB84 and B92 protocols have been realized up to 48 km [6], and QKD in free space for B92 scheme has been achieved over 1 km distance [7].

In paper [8], Lior and Lev presented a quantum cryptography based on orthogonal states firstly. Then there is quantum cryptographic scheme involving truly two orthogonal states [8]. The basic technic is to split the transfer of one-bit information into two steps, ensuring that only a fraction of the bit information is transmitted at a time. Then the non-cloning theorem of orthogonal states [11] guarantee its security. Based on the impossibility of cloning nonorthogonal mixed states, the no-cloning theorem of orthogonal states says that the two (or more) orthogonal states $\rho_i(AB)$ of the system composed of $A$ and $B$ cannot be cloned if the reduced density matrices of the subsystem which is available first (say $A$) $\rho_i(A) = Tr_B[\rho_i(AB)]$ are nonorthogonal and nonidentical, and if the reduced density matrices of the second subsystem are nonorthogonal. It is a very surprising result since it means that entanglement is not vital for preventing cloning of orthogonal states. In the case of a composite system made of two subsystems, if the subsystem are only available one after the other, then there are various cases that orthogonal states cannot be cloned.

For the multi-state systems, Bennett et al. have shown that there are orthogonal product pure states in the $3 \times 3$ Hilbert space and proved that these states may have some degree of nonlocality without entanglement [11]. There was also experimental demonstration of three mutually orthogonal polarization states [12], where biphotons are used as multi-state systems.

We propose the general conditions for the orthogonal product states of the multi-state composite systems to be used in QKD, then present a QKD scheme with the orthogonal product states of $3 \times 3$ system which has several distinct features, such as high efficiency and great capacity. The generalization to the $n$-state systems, and eavesdropping is analyzed where a peculiar limitation, $1/2$, for the success probability of an efficient eavesdropping strategy is found as $n$ becomes large enough.

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II. THE QKD SCHEME WITH ORTHOGONAL PRODUCT STATES

In the present QKD scheme with orthogonal product states in the $n \times n$ Hilbert space, the transmission processing is same as the QKD scheme with common orthogonal states [3]. The information is encoded in the holistic state of the two particles, and these two particles are sent separately to ensure that any eavesdropper cannot hold both particles at the same time. Since only orthogonal product state are employed, operations on one subsystem have no effect to the other. There are some basic conditions for any set of orthogonal product states in the $n$-state composite systems to be used in the present QKD scheme: for any density matrix of any subsystem, $\rho_i (P)$ , there must be at least one $\rho_j (P)$ which is both nonidentical and nonorthogonal to $\rho_i (P)$ . ($P$ represents subsystem $A$ or $B$. i and j represent different states of the set.) Then from the point of any subsystem [10], the standard non-cloning theorem [13] is satisfied, this guarantees the security of the protocol. What’s more, we can transmit $2 \log_2 n$ bits information, double the value of existing QKD protocol with usual orthogonal states [8,9]. It is evident that this is the maximal information that can be transmitted by the $n \times n$ system.

For $2 \times 2$ system, there is obviously no such orthogonal product states that satisfy the orthogonal states cryptography conditions. The reason is that if $\rho_0 (P)$ are nonidentical and nonorthogonal to $\rho_1 (P)$, then $\rho_0 (A) \otimes \rho_0 (B)$ can not be orthogonal to $\rho_1 (A) \otimes \rho_1 (B)$.

Next, we consider the $3 \times 3$ system. A general set of orthogonal product states in this Hilbert space is as follows

$$
\Psi_1 = |1\rangle_A (a |1\rangle_B + b |0\rangle_B),
\Psi_2 = |1\rangle_A (b^* |1\rangle_B - a^* |0\rangle_B),
\Psi_3 = |1\rangle_A + d (|0\rangle_A |2\rangle_B),
\Psi_4 = |d^* |1\rangle_A - e^* |0\rangle_A |2\rangle_B),
\Psi_5 = |2\rangle_A (e |0\rangle_B + f |2\rangle_B),
\Psi_6 = |2\rangle_A (f^* |0\rangle_B - e^* |2\rangle_B),
\Psi_7 = (g |0\rangle_A + h |2\rangle_A |1\rangle_B),
\Psi_8 = (h^* |0\rangle_A - g^* |2\rangle_A |1\rangle_B),
\Psi_9 = |0\rangle_A |0\rangle_B,
$$

(1)

where $a, b, c, d, e, f, g, h$ are complex number and $|a|^2 + |b|^2 = |c|^2 + |d|^2 = |e|^2 + |f|^2 = |g|^2 + |h|^2 = 1$.

This set of states has been proved to have some degree of nonlocality without entanglement, when $a = b = c = d = e = f = g = h = 1/\sqrt{2}$ [11]. For the general case, no satisfying proof for the existing of the nonlocality has been found yet. But if they satisfy the conditions mentioned above, they can still be used in this QKD scheme.

The process of this QKD scheme is as follows: 1. Alice prepares two particles $A$ and $B$ randomly in one of the nine orthogonal product states shown above and sends particle $A$ to Bob, when Bob receives it, he informs Alice through open classical channel. Then Alice sends out particle $B$. When particle $A$ and $B$ are both in the hand of Bob, he makes a collective orthogonal measurement under the basis of Eqs. (1) to determine which state of the existing nonlocality has been found yet. But if they satisfy the conditions mentioned above, they can still be used in this QKD scheme.

In order to find possible eavesdropping, Alice and Bob randomly compare some bits to verify whether the correlations have been destroyed. If it is true with as high probability as they require, it can be believed that there is no eavesdropper and all of the rest results can be used as cryptographic keys. Otherwise, all the keys are discarded and they must be redistributed.

What is vital to this scheme is that Alice sends the second particle only when the first one reaches Bob to eliminate the possibility of any eavesdropper to possess the two particles at the same time. This protocol has some distinct features. As all raw keys except a small portion chosen for checking eavesdroppers is usable, it is very efficient (nearly 100%), and it has large capacity since $\log_2 9$ bits information is transmitted by a $3 \times 3$ system.

III. EA VESDROPPING AND THE GENERALIZATION TO THE $N$-STATE SYSTEM

We first consider one efficient eavesdropping strategy. In this strategy, Eve measures the first particle from Alice and sends it to Bob. She measures the second particle corresponding to the measurement result of the first one and sends it to Bob. It has been found that with this strategy, Eve has the success eavesdropping probability as high as 0.94 for QKD with orthogonal states in the $2 \times 2$ Hilbert space [14].

The eavesdropping is as following, Eve intercepts particle $A$ and makes an orthogonal measurement in the basis $\{|0\rangle, |1\rangle, |2\rangle\}$. Suppose particle $A$ is found in state $|1\rangle_A$, Eve knows that the two-particle states of $A$ and $B$ is $\Psi_1, \Psi_2$ with probability $1/9$ respectively, or $\Psi_3, \Psi_4$ with probability $|c|^2/9$ and $|d|^2/9$, respectively. Then she sends it to Bob.
When particle $B$ comes, she intercepts it too and measures it in the basis $\{ |2\rangle_B, a |1\rangle_B + b |0\rangle_B, b |1\rangle_B - a |0\rangle_B \}$ and then sends it to Bob. If Eve sees that particle $B$ is in $a |1\rangle_B + b |0\rangle_B$ or $b |1\rangle_B - a |0\rangle_B$, then obviously she knows that the two-particle state is $Ψ_1$ or $Ψ_2$. In this case she is lucky enough to conceal from Alice and Bob for the two-particle state is not disturbed. And if Eve finds particle $B$ in state $|2\rangle_B$, then the two-particle state is $Ψ_3$ or $Ψ_4$, which has collapsed to $|1\rangle_A |2\rangle_B$, then Bob has the partial probability of $|c|^2$ or $|d|^2$ to find the two-particle state in $Ψ_3$ or $Ψ_4$, respectively. So it is clear that for the case Eve measures the particle $A$ in the state $|1\rangle_A$, the probability for he to eavesdrop without being detected is $(2 + |c|^2 + |d|^2)/9$. Analyzed in the same way, the total probability that Eve eavesdrops the key information without being detected is

$$P_3 = 5/9 + 2(|c|^4 + |d|^4 + |h|^4 + |g|^4)/9. \tag{2}$$

Then it is evident that $P_3$ gets the minimal value of 7/9 when $|c|^2 = |d|^2 = |h|^2 = |g|^2 = 1/2$.

We depict this set of states in the $3 \times 3$ Hilbert space in a visual graphical way as Figure 1 shows. The four dominoes represent the four pairs of states that involve superposition of the basis states $|0\rangle$, $|1\rangle$ or $|2\rangle$. It is obvious that this figure is 4-fold rotation symmetric, and we will show later that this symmetry is one of the basic requirements for there may be two symmetrical eavesdropping strategies. From this figure we can obviously see that all the states included in one row, where particle $A$ is in basis states, can be eavesdropped without being detected in this strategy. But for all other states, there is only certain probability, the quartic of probability amplitude of the superposition states of particle $A$ in the basis states, for Eve’s success eavesdropping. Then the total probability is $P_3$. Now let’s consider the case of $n \times n$ system. Since we only utilize orthogonal product states, and any superposition of $n$ basis states that covers all the grids of any row (any column) in the graphic depiction will surely be distinguished by Eve which will have no use in the present QKD scheme. In other words, any set of states including such superposition state will violate the above conditions, then the possible superposition states can just be the ones of less than $n$ basis states which cover no more than $n - 1$ grids of the figures. The $4 \times 4$ system can be depicted in Figure 2. We can see that the only difference between the figures of the $3 \times 3$ and $4 \times 4$ systems is that there are four states in the center of the $4 \times 4$ system, which can be eavesdropped without being detected. Then generalized straightforwardly, the $2n \times 2n$ system can be analyzed in the similar way as the $(2n - 1) \times (2n - 1)$ system. Thus here we take the $(2n - 1) \times (2n - 1)$ system for example. ($n = 2, 3, 4, \ldots$).

The figure 3 depicts a set of orthogonal product states of the $5 \times 5$ system, which is generalized straightforwardly from $3 \times 3$ system. For any complete set of orthogonal product states, the success probability for Eve to eavesdropping without being detected is

$$P_2 = \left[ 9P_3 + 8 + \left( \sum_{k,i,J} |\alpha_{J_k} |3\rangle_B \right)^4 + \sum_{k,i,J} |\alpha_{J_k} |4\rangle_B \right] / 25, \tag{3}$$

where $|\alpha_{J_k} |l\rangle_B$ is the probability amplitude of the superposition states of particle $A$ in the basis states and $\sum_J \alpha_{J_k}^2 |l\rangle_B = 1$ for any $|l\rangle_B$. $k$ and $i$, and $|l\rangle_B$ means that particle $B$ is in the basis state $|l\rangle_B$, $k$ denotes dominoes in line $|l\rangle_B$, $i_k$ denotes the superposition states in domino $k$, and $J_k$ denotes the basis states that superposition state $i_k$ involves, $P_3$ is Eve’s success probability for the $3 \times 3$ system and 8/25 is the probability for states in row $|3\rangle_A$ and $|4\rangle_A$. We know from the graph that the dominoes in column $|3\rangle_B$ and $|4\rangle_B$ cover 4 grids in total (If 5 grids are covered, all the states in this column can be eavesdropped without being detected.), and we have shown that $J_k \leq 4$. It can be proved easily that $P_3$ reaches its minimal value 17/25 when $P_3$ gets its minimal value and only one domino is in column $|3\rangle_B$ or $|4\rangle_B$ and $|\alpha_{J_k} |l\rangle_B|^2 = 1/4$. This corresponds to the set of states depicted in Figure 3. Then for the $(2n - 1) \times (2n - 1)$ system, the success probability is

$$P_{2n+1} = \left[ (2n - 1)^2 \times P_{2n-1} + 4n + \left( \sum_{k,i,J} |\alpha_{J_k} |2n-1\rangle_B \right)^4 + \sum_{k,i,J} |\alpha_{J_k} |2n\rangle_B \right] / (2n + 1)^2. \tag{4}$$

The minimal probability can be obtained similarly:

$$\min \{P_{2n+1} \} = \left[ (2n - 1)^2 \times \min \{P_{2n-1} \} + 4n + 2 \right] / (2n + 1)^2 = 1/2 + (1 + 4n) / 2(2n + 1)^2, \tag{5}$$

where $n \geq 1$. We can deduce immediately that this value approaches 1/2 when $n$ gets large enough.
Of course, there are other ways of plotting the graph symmetrically. But this set of states is of the most secure. In fact, the value of \( \min\{P_{2n+1}\} \) can be deduced straightforwardly from the symmetry of the plot. For any vertical domino contributes \( 1/(2n+1)^2 \) to this value, any state in the horizontal dominoes and the state in the center each contributes \( 1/(2n+1)^2 \). Due to the symmetry, there are at least \( 2n \) vertical or horizontal dominoes and \( [(2n+1)^2 - 1]/2 \) states in the horizontal dominoes.

For the \( 2n \times 2n \) system, the same result can be reached. That is to say there is a limitation in the probability of the success eavesdropping when the Hilbert space becomes large enough. And it is evident that in this strategy only particle \( A \) may be demolished, and particle \( B \) is not infected at all. The function of the operation to \( B \) which is depended on the result of the measurement on \( A \) is just to extract more information.

Eve may adopt the complementary eavesdropping strategy, in which Eve try to eavesdrop some information by intercepting and operating only on the second particle \( B \), which may cause demolition to it. Then for the set of states in \( n \times n \) systems, whose graphic depictions are 4-fold rotation symmetric, the probability to eavesdrop some information without being detected is equal to that of the first strategy, i.e., \( P_n \). But for those states without such symmetry, it can be verified that one of the success probabilities for the complementary strategies is larger than \( P_n \). So we employ the symmetric states in the present scheme.

Of course, there are other strategies, for example, she can hold up the first particle \( A \) and send out a substitute particle \( C \) to Bob. When \( B \) comes, she makes a collective measurement under the two-particle orthogonal basis, then sends out a particle \( D \) in the state of \( B \). In this strategy, Eve can eavesdrop the information entirely, but the probability for she to pass the checking process is only \( 1/n \), which tends to zero, for the state of particle \( C \) is randomly chosen from a \( n \) Hilbert space.

**IV. CONCLUSION**

We have proposed the general conditions for the orthogonal product states to be used in QKD, then presented a QKD scheme with the orthogonal product states of \( 3 \times 3 \) system which has several distinct features, such as high efficiency and great capacity. The generalization to the \( n \)-state systems, and eavesdropping is analyzed where a peculiar limitation, \( 1/2 \), for the success probability of an efficient eavesdropping strategy is found as \( n \) becomes large enough.

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**Figure Captions:**
Figure 1: The graphical depiction of the set of orthogonal product states in the \( 3 \times 3 \) Hilbert space.
Figure 2: The graphical depiction of the set of orthogonal product states in the \( 4 \times 4 \) Hilbert space.
Figure 3: The graphical depiction of the set of orthogonal product states in the \( 5 \times 5 \) Hilbert space.
