Abstract. We calculate the expectation values of the stress-energy tensor for both a massless minimally-coupled and dilaton-coupled 2D field propagating on an extremal Reissner-Nordström black hole, showing its regularity on the horizon in contrast with previous claims in the literature.

1. Introduction

Hawking discovery of particles production by black holes [1], gave rise to a great investigation of quantum effects in strong gravitational fields.

As we do not possess a full quantum theory of gravitation, the framework in which we work is the so-called semiclassical approximation, in which the gravitational field is treated at the classical level, whereas the matter fields are quantized. The classical energy-momentum tensor in the r.h.s of Einstein’s equations is replaced by its quantum expectation values in a suitable quantum state:

\[
G_{\mu\nu}(g_{\alpha\beta}) = \frac{8\pi}{\hbar c G} \langle T_{\mu\nu}(g_{\alpha\beta}) \rangle
\]

(in units \(\hbar = c = G = 1\)). The main problem is just to find a covariant expression for the quantum stress-energy tensor in a general curved spacetime.

The exact form of \(\langle T_{\mu\nu}(g_{\alpha\beta}) \rangle\) is in general unknown in four dimensions and therefore, with the hypothesis of spherical symmetry in 4D for the metric and selecting the \(s\)-wave component of the matter field, we work in a two-dimensional spacetime.

We study the energy-momentum tensor of a quantum scalar field propagating on an extremal Reissner-Nordström black hole. The interest in such black holes finds its justification in the fact that, being characterized by zero Hawking temperature, they are supposed not to emit particles and so they can be considered as the final state of the evaporation of nonextremal ones.

In the literature it is shown that, in two dimensions, the \(\langle T_{\mu\nu}(g_{\alpha\beta}) \rangle\) of a scalar field is divergent on the horizon of extremal black holes [2, 3]. Such divergence is physically unacceptable because, as we said, the quantum stress-energy tensor is supposed to be the source term of the semiclassical Einstein’s equations. The question we shall try to answer now is if this divergence is indeed realized in a physically realistic situation.

2. Scalar field in \(s\)-wave approximation

Let us consider a four dimensional massless minimally-coupled scalar field, which has the following classical action

\[
S^{(4)} = -\frac{1}{8\pi} \int d^4 x \sqrt{-g^{(4)}}(\nabla f)^2.
\]
A spherically symmetric four-dimensional metric can be written as

$$ds^2_{(4)} = ds^2_{(2)} + r_0^2 e^{-2\phi} d\Omega^2,$$

(2)

where we have parameterized the radius of the two-sphere \( r^2 = r_0^2 e^{-2\phi} \) through a dilaton field \( \phi \) and \( r_0 \) is an arbitrary scale factor. In a spherically symmetric background any field can be expanded in spherical harmonics, of which we pick up only the \( s \)-wave component: \( f = f(t, r) \). With these hypothesis, the 4D action can be integrated with respect to the angular coordinates, obtaining so a two-dimensional action

$$S^{(2)} = -\frac{1}{2} \int d^2 x \sqrt{-g^{(2)}} e^{-2\phi} (\nabla f)^2,$$

(3)

where \( g^{(2)} \) represents the radial sector of the metric. We remark that the scalar field acquires now a non-trivial coupling with the dilaton field besides the usual one to the metric.

3. Near horizon approximation

As we are interested in the behavior of the stress-energy tensor on the horizon of a black hole, we first consider the near horizon approximation. Fixing the radius of the two-sphere equal to the horizon’s one, \( r_0 = r_H, \phi = 0 \), we obtain just the action of a 2D scalar field

$$S^{(2)} = -\frac{1}{2} \int d^2 x \sqrt{-g^{(2)}} (\nabla f)^2.$$

(4)

It is useful to work in conformal coordinates: \( ds^2_{(2)} = -e^{2\rho(x)} dx^+ dx^- \). The classical action satisfies two invariance laws: general covariance, to which it corresponds a conserved stress-energy tensor \( (\nabla \mu T^{\mu\nu} = 0) \), and Weyl-invariance, which has as consequence a traceless energy-momentum tensor.

An intuitive way to derive the quantum stress-energy tensor is by the use of Virasoro anomaly and the equivalence principle (for details see [4]). One obtains the following expression for the expectation value of the stress tensor in a generic state \( |\Psi\rangle \)

$$\langle \Psi | T_{\pm\pm}(x^\pm) | \Psi \rangle = \langle \Psi | T_{\pm\pm}(x^\pm) : | \Psi \rangle - \frac{1}{12\pi} (\partial_{\pm\pm} \partial_{\pm\pm} \rho - \partial_{\pm}^2 \rho).$$

(5)

At the quantum level it is impossible to preserve the two invariances. Therefore, preserving general covariance and imposing the stress-energy tensor conservation, the Weyl-invariance is broken and an anomalous trace appears

$$\langle T \rangle = \frac{1}{24\pi} R.$$

(6)

Alternatively, one can obtain the quantum stress tensor by variation of the non local Polyakov action: \( S_P = -1/(96\pi) \int d^2 x \sqrt{-g^{(2)}} R \Box^{-1} R \).

From the expression of the quantum stress-energy tensor (Eq. (5)), one can obtain the relation between its expectation values in two different vacuum states \( |x^\pm\rangle \) and \( |\tilde{x}^\pm\rangle \):

$$\langle \tilde{x}^\pm | T_{\pm\pm}(x^\pm) | \tilde{x}^\pm \rangle = \langle x^\pm | T_{\pm\pm}(x^\pm) | x^\pm \rangle - \frac{1}{24\pi} \{ \tilde{x}^\pm, x^\pm \},$$

(7)

where \( |x^\pm\rangle \) \( (|\tilde{x}^\pm\rangle) \) represents the vacuum state determined by the expansion of the field in the modes which are positive frequency with respect to the time \( (x^+ + x^-)/2 \) \( ((\tilde{x}^+ + \tilde{x}^-)/2) \) and

$$\{ \tilde{x}^\pm, x^\pm \} = \frac{d^2 \tilde{x}^\pm}{dx^\pm dx^-} - \frac{2}{3} \left( \frac{d^2 \tilde{x}^\pm}{dx^\pm dx^-} \right)^2$$

is the Schwarzian derivative associated to the conformal transformation \( \{ x^\pm \} \to \{ \tilde{x}^\pm \} \).
4. Nonextremal Reissner-Nordström black holes

We shall now apply the formalism developed above to the case of the propagation of a scalar field in the background of a charged black hole, which is characterized by the following line element

$$ds^2 = -f(r)du dv = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)du dv$$

and possesses two horizons: $r_\pm = M \pm \sqrt{M^2 - Q^2}$ (with $M$ and $Q$ respectively the mass and the charge of the black hole). We shall analyze the behavior of the energy-momentum tensor on the outer horizon $r_+$. In the study of black holes, one usually consider three vacuum states.

- **Boulware state**

It is constructed by expanding the field in modes which are positive frequencies with respect to the asymptotically Minkowskian time coordinate: $(4\pi\omega)^{-1/2}e^{-i\omega u}$, $(4\pi\omega)^{-1/2}e^{-i\omega u}$. These modes reduce at infinity to plane waves, so the Boulware state reproduces, in the limit $r \to \infty$, the notion of vacuum in flat spacetime. This state is supposed to describe the vacuum polarization outside a static star, whose radius is bigger than the horizon’s one.

With the use of Eq. (5), with $\rho = \frac{1}{2}\ln f$, we can evaluate the energy-momentum tensor in this state

$$\langle B|T_{uu}|B \rangle = \langle B|T_{vv}|B \rangle = \frac{1}{24\pi r^3} \left(-M + \frac{3M^2 + Q^2}{2r} - \frac{3MQ^2}{r^2} + \frac{Q^4}{r^4}\right).$$

As it is plausible to expect, the stress-energy tensor in this states goes to zero at infinity. To check the regularity on the horizon, we have to express its components in a coordinate system which is regular there. To this end, we introduces the Kruskal coordinates

$$U = -\frac{1}{\kappa_+}e^{-\kappa_+ u}, \quad V = \frac{1}{\kappa_+}e^{\kappa_+ v},$$

where $\kappa_+ = \frac{\partial f}{\partial r}|_{r_+} = \sqrt{M^2 - Q^2/r_+^2}$ is the surface gravity on the outer horizon. With these coordinates, the regularity conditions on the future horizon become [5]

$$f^{-2}(T_{uu}) < \infty, \quad (T_{vv}) < \infty, \quad f^{-2}(T_{uv}) < \infty$$

and the same with just $u$ and $v$ interchanged on the past horizon. It is easy to check that the Boulware state is divergent on both the future and the past horizons.

- **Hartle-Hawking state**

This state is constructed by expanding the field in ingoing (outgoing) modes which are positive frequencies with respect to the affine parameter on the future (past) horizon: $(4\pi\omega)^{-1/2}e^{-i\omega V}$, $(4\pi\omega)^{-1/2}e^{-i\omega U}$. One can evaluate the quantum stress-energy tensor in this state, by means of relation (7)

$$\langle H|T_{uu}|H \rangle = \langle B|T_{uu}|B \rangle - \frac{1}{24\pi}\{U, u\} = \langle B|T_{uu}|B \rangle + \frac{1}{48\pi}\kappa_+^2 = \langle H|T_{vv}|H \rangle.$$  

(12)

The difference between the two vacuum states is induced by the Schwarzian derivative, which, by virtue of the exponential relation between Eddington-Finkelstein and Kruskal coordinates, turns out to be proportional to the square of the surface gravity. The Hartle-Hawking state satisfies the regularity conditions (11), but at infinity it is different from zero: the asymptotic limit represents thermal radiation at the Hawking temperature $T_H$

$$\langle H|T_{uu}|H \rangle = \langle H|T_{vv}|H \rangle \sim \frac{1}{12}\kappa_+^2 = \frac{\pi}{12}T_H^4.$$  

(13)

The physical interpretation of this state is that of thermal equilibrium of the black hole with its own radiation.
- Unruh state

It is constructed by choosing ingoing modes positive frequency with respect to the asymptotic Minkowski time, and outgoing modes positive frequency with respect to the affine parameter on the past horizon: $(4\pi\omega)^{-1/2}e^{-i\omega v}$, $(4\pi\omega)^{-1/2}e^{-i\omega U}$. This state represents the vacuum of particles incoming from $I^-$. Incoming modes propagating on the geometry of a collapsing body, after being reflected at the origin, emerge, just before the horizon formation, exponentially redshifted as positive frequency with respect to $U$. This "hybrid" construction describes the late-time behavior of a quantum field in the spacetime of a collapsing body forming a black hole. The stress-energy tensor turns out to be

$$\langle U|T_{uu}|U \rangle = \langle H|T_{uu}|H \rangle = \langle B|T_{uu}|B \rangle + \frac{1}{48\pi}\kappa^2_+,$$

$$\langle U|T_{vv}|U \rangle = \langle B|T_{vv}|B \rangle.$$  

(14)

(15)

In the case of a realistic collapse one should consider a different vacuum state, the so-called in vacuum, containing extra terms depending on the details of the collapse, which drop exponentially to zero at late time, leaving just the constant contribution of the Schwarzian derivative:

$$\langle in|T_{\mu\nu}|in \rangle \xrightarrow{r \to \infty} \langle U|T_{\mu\nu}|U \rangle + O(e^{-\kappa_+u}).$$

The Unruh state is regular on the future horizon. At future infinity it gives a constant outgoing flux which represents the asymptotical value of Hawking radiation

$$\langle U|T_{vv}|U \rangle \xrightarrow{t\to\infty} \frac{1}{48\pi}\kappa^2_+.$$  

(17)

5. Extremal Reissner-Nordström black holes

Let us consider now the more peculiar case of the extreme charged black holes:

$$ds^2 = -f(r)du dv = -\left(1 - \frac{M}{r}\right)^2 du dv.$$  

(18)

The two horizons now coincide at the value $r_H = M$. The surface gravity is null and, being zero the Schwarzian derivatives $\{U, u\}$ and $\{V, v\}$, it could seem that the three vacuum states all coincide

$$\langle T_{uu} \rangle = \langle T_{vv} \rangle = -\frac{1}{24\pi r^3}\left(1 - \frac{M}{r}\right)^3 \equiv H^{extr}(r).$$

(19)

This stress-energy tensor does not satisfy the regularity conditions and we have the problem of the divergence on the horizon. However, we have to note that the Kruskal coordinates we have used to construct a regular state on the horizon (see (10)), are not defined in the limit $\kappa_+ \to 0$. To analyze the problem, we have to understand if the quantum state whose energy-momentum tensor in given by Eq. (19) is realizable by means of a physical process. To this end, let us consider the process of formation of an extremal black hole, starting by a flat spacetime, by means of the collapse of an ingoing charged null shell located at $v = v_0$ (see Fig.1)[6]. The metric will be the following

$$v < v_0 \quad ds^2 = -du_in dv_in \quad \text{Minkowski spacetime},$$

$$v > v_0 \quad ds^2 = -\left(1 - \frac{M}{r}\right)^2 du dv \quad \text{Extremal black hole}.$$ 

Matching the two metrics across the shell, we obtain the relation $u(u_{in})$

$$u = u_{in} - 4M\left[\ln\left(\frac{v_0 - u_{in}}{2M} - 1\right) - \frac{1}{2\left(\frac{v_0 - u_{in}}{2M} - 1\right)}\right].$$

(20)
Figure 1. Penrose diagram describing the formation of an extreme Reissner-Nordström black hole.

A positive frequency \( v_{in} \) mode at \( I^- \), after being reflected at \( r = 0 \), becomes at late retarded time \((u \to +\infty)\) a positive frequency \( U_{\text{mode}} \), where

\[
 u = -4M \left[ \ln \left( -\frac{U}{M} \right) + \frac{M}{2U} \right] \quad \text{(with} \ u_{in} \to 2U - 2M + v_0). \tag{21}
\]

With this new Kruskal-type coordinate, the \( in \) vacuum state turns out to be

\[
 \langle in | T_{uu} | in \rangle = H_{\text{extr}}(r), \tag{22}
\]
\[
 \langle in | T_{vv} | in \rangle = H_{\text{extr}}(r). \tag{23}
\]

The stress-energy tensor is now regular on the future horizon \((U \sim -(r - M))\):

\[
 \lim_{r \to M} \left( \frac{du}{dU} \right)^2 \langle in | T_{uu} | in \rangle = -\lim_{r \to M} \frac{1}{24\pi} \left[ \frac{M}{r^2} \frac{1}{r - M} - \frac{1}{M(r - M)} + \text{finite} \right] = -\frac{1}{24\pi} \frac{3}{2M^2} < \infty. \tag{24}
\]

The divergence coming from the vacuum polarization part is canceled by the divergent term induced by the Schwarzian derivative. Let us note that in this case the transient terms in the stress-energy tensor for the \( in \) vacuum do not drop to zero exponentially, but with power law; however, unlike the nonextremal case, one cannot simply discard this contribution to get the late-time behavior, since this procedure would lead to the incorrect result we have previously seen. Also the vacuum polarization vanishes at late retarded time and it is necessary to consider both terms to have regularity on the future horizon.

The result has also been generalized to the case of the collapse of a timelike shell \([8]\). In this case the relation \( u(U) \) is

\[
 u = -4M \left[ \ln \left( -\frac{U}{cM} \right) + \frac{cM}{2U} \right], \tag{25}
\]

where \( c \) is a constant related to the details of the collapse which does not affect the regularity of the stress-energy tensor on the horizon.
6. Dilatonic theory

The result obtained in the previous section can be extended to the most general case of a dilaton-coupled scalar field, whose classical action is given by (3). Exactly as we did for the Polyakov theory, we can derive the expression for the quantum expectation values of the energy-momentum tensor by means of the generalized Virasoro anomaly and the equivalence principle [9]

\[ \langle \Psi | T_{\pm\pm}(x^+, x^-) | \Psi \rangle = \langle \Psi | :T_{\pm\pm}(x^+, x^-): | \Psi \rangle - \frac{1}{12\pi} (\partial_+ \rho \partial_+ \rho - \partial_2^2 \rho) + \frac{1}{2\pi} [\partial_+ \rho \partial_\phi + \rho (\partial_\phi)^2]. \]

We note that new local terms depending on the dilaton field appear. From the expression of the quantum stress-energy tensor we obtain the following relation between the expectation values of the same tensor in two different vacuum states

\[ \langle \tilde{x}^+ | T_{\pm\pm}(x^+, x^-) | \tilde{x}^+ \rangle = \langle x^+ | T_{\pm\pm}(x^+, x^-) | x^+ \rangle - \frac{1}{24\pi} \{ \tilde{x}^+, x^+ \} - \frac{1}{4\pi} \ln \left( \frac{dx^- dx^+}{d\tilde{x}^- d\tilde{x}^+} \right) (\partial_\phi)^2 + \frac{1}{4\pi} d^2x^\pm (d\tilde{x}^\pm)^2 \partial_\phi. \]  

(26)

Evaluating the \( \langle in | T_{UU} | in \rangle \) component of the stress-energy tensor for the extremal Reissner-Nordström geometry, we can see that new divergent terms, both polynomial and logarithmic, appear in addition to those already present in the Polyakov case. However it is easy to show (see [8] for more details) that the polynomial terms exactly cancel each other on the horizon and the logarithmic ones too sum up to give a finite value. Therefore the stress-energy tensor turn out to be finite on the future horizon in this more involved case too.

7. Conclusions

Unlike previous claims in the literature, we have shown the regularity of the energy-momentum tensor at the horizon of an extremal Reissner-Nordström black hole formed by gravitational collapse in the two cases of a massless minimally-coupled 2D scalar field, and a massless dilaton-coupled 2D scalar field. Besides we have shown that the regularity is independent on the details of the collapse.

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