Trading with the crowd

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Abstract
We formulate and solve a multi-player stochastic differential game between financial agents who seek to cost-efficiently liquidate their position in a risky asset in the presence of jointly aggregated transient price impact, along with taking into account a common general price predicting signal. The unique Nash-equilibrium strategies reveal how each agent’s liquidation policy adjusts the predictive trading signal to the aggregated transient price impact induced by all other agents. This unfolds a quantitative relation between trading signals and the order flow in crowded markets. We also formulate and solve the corresponding mean field game in the limit of infinitely many agents. We prove that the equilibrium trading speed and the value function of an agent in the finite $N$-player game converges to the corresponding trading speed and value function in the mean field game at rate $O(N^{-2})$. In addition, we prove that the mean field optimal strategy provides an approximate Nash-equilibrium for the finite-player game.

KEYWORDS
crowding, mean field games, optimal portfolio liquidation, optimal stochastic control, price impact, predictive signals

JEL CLASSIFICATION:
49N80, 49N90, 93E20, 60H30, C73, C02, C61, G11
INTRODUCTION

The phenomenon of crowding in financial markets has gained an increasing attention from both academics and financial institutions over the past couple of decades. It is a subject of numerous research works studying both theoretical and empirical aspects of the topic, including (Bucci et al., 2020; Barroso et al., 2017; Cont & Bouchaud, 2000; Caccioli et al., 2015, 2014; Khandani & Lo, 2011; Volpati et al., 2020) among others. Crowding is often considered to be an explanation for sub-par performances of investments as well as the development of systemic risk in financial markets. The presence of largely overlapping portfolios comes at the expense of portfolio managers’ profits, also in terms of transaction costs, as affine positions usually lead to similar trades.

The existing literature on crowding concentrates both on analytic models that explain some aspects of crowded markets behavior and on data driven statistical models. In the first class, Cont and Bouchaud (2000) proposed a simple mathematical model in which the communication structure between agents gives rise to heavy tailed distribution for stock returns. This established a theoretical connection between crowding and stock markets shortfall. The aforementioned portfolios overlap was shown to be a considerable factor in the August 2007 Quant Meltdown. Cont and Wagalath (2013) proposed a simple multi-period model of price impact from trading in a market with multiple assets. Their model illustrated how feedback effects due to distressed selling and short selling lead to endogenous correlations between asset classes and it provided a quantitative framework to evaluate strategy crowding as a risk factor (see also Cont & Wagalath, 2016).

Within the class of statistical models, Khandani and Lo (2011) used simulated returns of overlapping equity portfolios and showed that combined effects of portfolio deleveraging followed by a temporary withdrawal of market-making risk capital was one of the main drivers of the 2007 Quant Meltdown. Caccioli et al. (2015) and Caccioli et al. (2014) developed a mathematical model for a network of different banks holding overlapping portfolios. They investigated the circumstances under which systemic instabilities may occur as a result of various parameters, such as market crowding and price impact. Volpati et al. (2020) measured significant levels of crowding in U.S. equity markets for momentum signals as well as for Fama-French factors signals. In Micheli and Neuman (2020) an index reconstruction methodology was developed in order to measure the crowding effect on Russell indexes around reconstitutions events.

A new approach that connects optimal execution of multiple agents to crowding phenomena was proposed by Cardaliaguet and Lehalle (2018). More specifically, a mean field game with an infinite number of agents, where each agent executes a large order on the same risky asset was considered. In their model, the aggregated permanent price impact created by all players’ transactions is modeled as an exogenous process, which satisfies a consistency condition, while temporary price impact influences each agent individually. The solution of the game showed qualitatively that in an infinite player setting, the optimal trading speed is deterministic and that it is optimal to follow the crowd but not too fast, as this could create additional trading costs.

In this work, we further extend and develop the model proposed in Cardaliaguet and Lehalle (2018) in order to reveal new properties and insights on crowding that arise in optimal execution framework. More precisely, we formulate and solve a multi-player stochastic differential game between traders who execute large orders on one risky asset in the presence of individual temporary price impact and jointly aggregated transient price impact. We also assume that traders are observing a common exogenous price predicting signal. The unique Nash-equilibrium strategies show how each agent’s liquidation strategy adjusts the predictive trading signal for the accumulated transient price distortion induced by all other agents’ price impact. This unfolds a qualitative
relation between trading signals and the agents’ aggregated order flow in crowded markets. We refer to Section 2 for the model setup and to Theorem 2.13 for the solution of the game. One can observe from the explicit solution to the game in (25) that the aggregated price distortion $Y^{u_N}$ and the exogenous trading signal $A$ are coupled. We therefore conclude that the distortion is acting as an endogenous signal, created by the trading strategies of all $N$ players. In the infinite player game we show that a similar coupling holds (see (45)); however, the price distortion $\tilde{Y}^N$ is determined by an independent equation (43), then it is plugged into the optimal trading speed of the individual agent in (45), essentially acting as an exogenous signal. See Remarks 2.15 and 3.8 for further details. From the explicit solutions which were mentioned above, we notice that both in the finite-player and infinite-player games the signal and the price distortion can follow a similar direction at least for a while. This is demonstrated in Figure 3, where a decreasing signal amplifies the sell strategies and as a result, a negative price distortion is created. Note that at some point in time the inventory penalties force the agents to close their position and the price impact turns in an opposite direction to the signal.

Here we summarize the main financial interpretation of our analysis:

(i) The consideration of a decaying price distortion in the finite player game points out that the cumulative order-flow is in fact an endogenous signal, which is observed and used by all traders in the game.
(ii) In various scenarios the order-flow amplifies the effect of the exogenous price predicting signal on the price process and on the traders’ execution strategies at equilibrium.

These interesting and surprising results cannot be derived by the Cardaliaguet and Lehalle (2018) model, as their model assumes that the permanent price impact is an exogenous signal which satisfies a consistency condition. Indeed in the mean field setting the contribution of each agent to this signal is infinitesimal. Also Cardaliaguet and Lehalle did not incorporate a price predicting exogenous signal in their model, so their mean field optimal strategies are in fact deterministic and the order-flow amplification effect does not appear. In fact our results show that for a sell strategy for example, a negative price predicting signal will motivate the traders to sell more aggressively and their associated order-flow will drop the price down even further and create an endogenous signal in the same direction of the predictive signal. The only reason that the system remains in equilibrium is due to the penalties of holding inventory during and at the end of the trading period in the agents’ performance functional. See Section 5 for a qualitative analysis and illustrations of the results.

In this work we also formulate and solve the corresponding mean field game, which describes the limit of infinitely many agents (see Section 3 and Theorem 3.14). We prove in Theorem 4.1 and Corollary 4.5 that the equilibrium trading speed and value function of an agent in the finite $N$-player game converges to the corresponding trading speed and value function of an agent in the mean field game at rate $O(N^{-2})$. Similar convergence result with relaxed assumptions but without the convergence rate is given in and Theorem 4.7. This concludes that the aggregated order flow, which appears in the model as the cumulative transient price impact of all agents, becomes an exogenous signal as the number of agents tends to infinity. As a result, we justify the a priori assumption of the exogenous price impact process in the simplified model of Cardaliaguet and Lehalle. Finally we prove in Theorem 4.10 that the mean field optimal strategy provides an approximate Nash-equilibrium for the finite-player game.
Some additional papers on optimal execution in multiplayer and infinite player games have appeared recently, without specific reference to crowding. We will describe these results in short and explain how this paper improves and extends them from the mathematical point of view. We will start with papers which solve only the mean field game (i.e., the infinite player setting). Casgrain and Jaimungal (2018) and Casgrain and Jaimungal (2020) studied a mean field game in which each agent is executing a large order while creating both temporary and permanent price impact. Their model generalizes the basic model of Cardaliaguet and Lehalle (2018), as it assumes that traders may have differing beliefs or partial information on the price process. Huang et al. (2019) also extended the mean field model in Cardaliaguet and Lehalle (2018) by introducing three classes of traders: a large agent, small high frequency traders and noise traders. Finally, Fu et al. (2021) extended the model to liquidation under asymmetric information. Our results improve and extend these papers, since we also solve the corresponding $N$-player game, which is known to be less tractable. We further assume that the traders create in addition a transient price impact that depends on the entire trading paths of all players. Lastly, we prove the convergence of the optimal strategy of the $N$-player game to the mean field game equilibrium.

A few recent papers deal with finite player execution games for the case that traders create transient price impact. Strehle (2017) and Schied et al. (2017) worked on the problem in continuous time, while Schied and Zhang (2019) and Luo and Schied (2020) studied the discrete time setting. These references describe a special case of our model as they do not include a predictive signal, which is responsible for the randomness of the equilibrium strategies, and they do not prove convergence results to a mean field limit, as done in this paper.

Finite player price impact games and mean field games with permanent price impact were studied by Evangelista and Thamsten (2020), Drapeau et al. (2021) and Féron et al. (2020), where a special attention is given to the last two papers, in which the convergence of the finite player equilibrium to the mean field equilibrium was derived. We remark that the convergence results in Drapeau et al. (2021) and Féron et al. (2020) do not derive the convergence rate, and of course their model did not include transient price impact which plays a crucial role in our model. Moreover, in these papers the convergence proof uses particular features of the model, which do not apply to the transient price impact case. In Section 6 we develop a method which not only provides the rate of convergence but could be adapted to a more general class of models, which translate at equilibrium to systems of FBSDEs.

Finally some additional convergence results on a finite player game to a mean field game were derived recently for liquidation games with self-exciting order flow by Fu et al. (2020), which did not include a price predicting signal. These convergence results are quite different than the convergence results in this paper, as they derive the convergence of a game with stochastic i.i.d. noise in the transient price impact coefficients. In our setting the source of randomness is a common noise, which is the exogenous signal. Moreover, the convergence results in Fu et al. (2020) do not derive the convergence rate.

**Structure of the paper:**
In Section 2 we define the finite player game and derive the Nash equilibrium. In Section 3 we present the corresponding mean field game and derive its equilibrium. In Section 4 we present our convergence results and approximated Nash equilibrium. Section 5 is dedicated to illustrations of the equilibrium strategies. Finally, Section 6–12 are dedicated to the proofs of the main results.
2 | A FINITE PLAYER GAME

2.1 | Model setup

Our first goal is to adopt and extend the single-agent, signal-adaptive optimal execution problem with transient price impact from Neuman and Voß (2022) to a finite \( N \)-player stochastic differential price impact game.

As usual we begin with fixing a finite deterministic time horizon \( T > 0 \) and a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})\) satisfying the usual conditions of right continuity and completeness. We further denote by \( \mathcal{H}^2 \) the class of all (special) semimartingales \( P = (P_t)_{0 \leq t \leq T} \) allowing for a canonical decomposition \( P = L + A \) into a (local) martingale \( L = (L_t)_{0 \leq t \leq T} \) and a predictable finite-variation process \( A = (A_t)_{0 \leq t \leq T} \) such that

\[
E[\langle L \rangle_T] + E\left[ \int_0^T |dA_s| \right] < \infty. \tag{1}
\]

Next, we introduce a class of \( N \in \mathbb{N} \) agents. Each agent \( i \in \{1, \ldots, N\} \) has an initial position of \( x^i,N \in \mathbb{R} \) shares in a risky asset and their number of shares held at time \( t \in [0, T] \) is given by

\[
X^i,N_t \triangleq x^i,N - \int_0^t u^i,N_s ds \tag{2}
\]

with a selling rate \((u^i,N_s)_{0 \leq s \leq T}\) chosen from a set of admissible strategies

\[
\mathcal{A} \triangleq \left\{ u : u \text{ progressively measurable s.t. } E\left[ \int_0^T u_s^2 ds \right] < \infty \right\}. \tag{3}
\]

We denote

\[
u^N \triangleq (u^1,N, \ldots, u^N,N) \in \mathcal{A}^N,
\]

and assume similar to the single-agent case in Neuman and Voß (2022) that the agents’ collective trading activity \( u^N \) induces a common transient price impact on the risky asset’s execution price. Specifically, all agents’ orders are filled at prices

\[
S^u,N_t \triangleq P_t - \kappa Y^u,N_t \quad (0 \leq t \leq T), \tag{4}
\]

where \( P \) denotes some unaffected price process in \( \mathcal{H}^2 \) and

\[
Y^u,N_t \triangleq e^{-\rho t} y + \gamma \int_0^t e^{-\rho(t-s)} \left( \frac{1}{N} \sum_{i=1}^N u^i,N_s \right) ds \quad (0 \leq t \leq T), \tag{5}
\]

captures an aggregated linear and exponentially decaying price distortion from the unaffected level \( P \) with some constants \( \kappa, \gamma > 0 \), resilience rate \( \rho > 0 \), and some initial value \( y > 0 \). In addition, we also assume that each agent’s trading incurs an individual slippage cost \( \lambda > 0 \) which is
levied on their respective quadratic turnover rate and accumulates to

\[ \lambda \int_0^T (u_{i,N}^t)^2 dt, \]

up to a terminal time \( T \). For each agent \( i \in \{1, \ldots, N\} \) we denote by

\[ u^{-i,N} \triangleq (u_{1,N}, \ldots, u_{i-1,N}, u_{i+1,N}, \ldots u_{N,N}) \in A^{N-1}, \]

the other agents’ trading activities. The \( i \)-th agent’s objective is to optimally unwind her initial position \( x_{i,N}^1 \in \mathbb{R} \) by time \( T \). The execution is done while taking into account both the interaction with all other agents’ strategies \( u^{-i,N} \) through the jointly generated transient price impact \( Y_{u,N}^N \) in (5), as well as the risky asset’s price signal \( A \). These considerations are accounted by maximizing the performance functional

\[
J^{i,N}(u^{i,N}; u^{i-N}) \triangleq E \left[ \int_0^T S_t^{u,N} u_t^{i,N} dt - \lambda \int_0^T (u_t^{i,N})^2 dt - \phi \int_0^T (X_t^{u,N})^2 dt + X_T^{u,N} (P - \varphi X_T^{u,N}) \right],
\]

over admissible rates \( u^{i,N} \in A \). Observe that \( J^{i,N}(u^{i,N}; u^{i-N}) < \infty \) for any set of admissible strategies \( u^N \in A^N \) because the objective is a linear quadratic functional of the state processes and the controls. As in the single-agent case in Neuman and Voß (2022) the parameters \( \phi > 0 \) and \( \varphi > 0 \) implement, respectively, an additional penalty on the \( i \)-th agent’s running inventory and her terminal position at final time \( T \). For further motivations of the objective in (6) we refer to, for example, Neuman and Voß (2022) in the single-agent case \( N = 1 \) and the references therein.

Our first goal in this paper is to solve simultaneously for each agent \( i \in \{1, \ldots, N\} \) their individual optimal stochastic control problems

\[
J^{i,N}(u^{i,N}; u^{i-N}) \rightarrow \max_{u^{i,N} \in A} .
\]

This solution will establish a Nash equilibrium for this stochastic differential price impact game in the following usual sense.

**Definition 2.1.** A set of strategies \( \hat{u}^N = (\hat{u}_{1,N}, \ldots, \hat{u}_{N,N}) \in A^N \) is called an open-loop Nash equilibrium if for all \( i \in \{1, \ldots, N\} \) and for all admissible strategies \( v \in A \) it holds that

\[
J^{i,N}(\hat{u}^{i,N}; \hat{u}^{-i,N}) \geq J^{i,N}(v; \hat{u}^{-i,N}).
\]

**Remark 2.2.** When the signal is set to zero, that is, \( A \equiv 0 \), our \( N \)-player model reduces to the models that were studied by Strehle (2017) and Schied et al. (2017). The optimal strategy in these models is deterministic while in our setting an optimal strategy will be signal-adaptive.

**Remark 2.3.** We remark that the transient impact scaling with \( 1/N \) in (5) is not essential at this point, but is crucial for the scaling limit of the system, as number of agents tends to infinity. See the discussion in Section 3.
Remark 2.4. Observe that agent $i$’s terminal position $X_{i,T}^{u_i,N}$ in (6) is valued with respect to $P_T$ and not $S_T^{u_i,N}$. This is to ensure that the functional $u_i^N \mapsto J_i,N(u_i^N;u_i^{-i,N})$ is strictly concave in $u_i^N \in \mathcal{A}$; see Lemma 10.1 below. Interestingly, strict concavity is in general not guaranteed if $P_T$ is replaced by $S_T^{u_i,N}$ due to the arising mixed product $X_{i,T}^{u_i,N} Y_{i,T}^{u_i,N}$. Note, however, that we implicitly assume a large penalty parameter $\varphi > 0$ on agent $i$’s outstanding inventory $X_{i,T}^{u_i,N}$ in order to virtually enforce a liquidation constraint at terminal time $T>0$ and to stabilize the competitive game between the agents. In this regard, the valuation of the final position in the risky asset is of minor relevance because agent $i$’s terminal inventory will be very close to zero anyways.

2.2 An FBSDE characterization of the finite player Nash equilibrium

In the spirit of Pontryagin’s stochastic maximum principle and along the lines of the corresponding single-agent problem in Neuman and Voß (2022), a probabilistic and convex analytic calculus of variations approach can be readily employed here to derive a system of coupled linear FBSDEs. This system characterizes a unique open-loop Nash equilibrium for our multi-agent price impact game.

Lemma 2.5. A set of controls $(u_i^{i,N})_{i \in \{1, \ldots, N\}} \subset \mathcal{A}$ yields the unique Nash equilibrium in the sense of Definition 2.1 if and only if the processes $(X_{i,T}^{u_i,N}, Y_{i,T}^{u_i,N}, u_i^{i,N}, Z_{i,T}^{u_i,N})$, $i = 1, \ldots, N$, satisfy the following coupled linear forward backward SDE system

\[
\begin{align*}
\begin{cases}
\d X_{t}^{u_i,N} = -u_{t}^{i,N} \d t, & X_{0}^{u_i,N} = x_{i,N}, \\
\d Y_{t}^{u_i,N} = -\rho Y_{t}^{u_i,N} \d t + \frac{\gamma}{N} \sum_{i=1}^{N} u_{t}^{i,N} \d t, & Y_{0}^{u_i,N} = y, \\
\d u_{t}^{i,N} = \frac{dP_{t}}{2\lambda} + \frac{x_{i,N}^{u_i,N}}{2\lambda} \d t - \frac{\gamma_{\kappa}}{2\lambda N} \sum_{j \neq i} u_{t}^{j,N} \d t - \frac{\phi X_{t}^{u_i,N}}{\lambda} \d t + \frac{\rho Z_{t}^{u_i,N}}{2\lambda} \d t + \d M_{t}^{i,N}, \\
\d Z_{t}^{u_i,N} = \rho Z_{t}^{u_i,N} \d t + \frac{\gamma_{\kappa}}{N} u_{t}^{i,N} \d t + \d N_{t}^{i,N}, & Z_{T}^{u_i,N} = 0
\end{cases}
\end{align*}
\]  

for suitable square integrable martingales $M_{t}^{i,N} = (M_{t}^{i,N})_{0 \leq t \leq T}$ and $N_{t}^{i,N} = (N_{t}^{i,N})_{0 \leq t \leq T}$, $i = 1, \ldots, N$. In particular, the system in (8) has a unique solution.

The proof of Lemma 2.5 is given in Section 10. In order to decouple the system in Theorem 2.5 it is very natural to first average over all $N$ FBSDEs in (8) and to introduce an auxiliary aggregated FBSDE system; see also Drapeau et al. (2021). More precisely, we let $(\bar{X}_{t}^{u_i,N}, \bar{Y}_{t}^{u_i,N}, \bar{u}_{t}^{i,N}, \bar{Z}_{t}^{u_i,N})$ with
\( \bar{u}^N \in A \) denote the unique solution to the linear FBSDE system

\[
\begin{align*}
\d X_t &= - \bar{u}^N_t \, dt, \quad \bar{X}_0 = \bar{x}^N, \\
\d Y_t &= - \rho Y_t \, dt + \gamma \bar{\bar{u}}^N_t \, dt, \quad \bar{Y}_0 = y, \\
\d \bar{u}^N_t &= \frac{dp_t}{2\lambda} + \frac{\kappa \bar{Y}_t}{2\lambda N} \, dt - \frac{\gamma \kappa (N-1) \bar{u}^N_t}{2\lambda N} \, dt - \frac{\varphi \bar{X}_t}{\lambda} \, dt + \frac{\rho \bar{Z}_t}{2\lambda N} \, dt + \d M_t^N, \\
\d \bar{Z}_t &= \rho \bar{Z}_t \, dt + \frac{\gamma \kappa}{\lambda} \bar{u}^N_t \, dt + \d \bar{N}_t, \quad \bar{Z}_{\bar{T}} = 0
\end{align*}
\]

for two suitable square integrable martingales \( M^N = (M_t^N)_{0 \leq t \leq T} \) and \( N^N = (N_t^N)_{0 \leq t \leq T} \) where \( \bar{x}^N = \frac{1}{N} \sum_{i=1}^{N} x^i \). Then we obtain the following

**Corollary 2.6.** Let \( (\bar{X}^{\bar{u}^N}, \bar{Y}^{\bar{u}^N}, \bar{u}^N, \bar{Z}^{\bar{u}^N}) \), \( \bar{u}^N \in A \), be the unique solution to the linear FBSDE system in (9). Moreover, for each \( i \in \{1, \ldots, N\} \) let \( (X^{u^i,N}, u^i,N, Z^{u^i,N}) \) with \( u^i,N \in A \) be the unique solution to

\[
\begin{align*}
\d X^{u^i,N}_t &= - u^i,N_t \, dt, \quad X^{u^i,N}_0 = x^i \, N \\
\d u^i,N_t &= \frac{1}{2\lambda} \left( dP_t - \kappa d\bar{Y}^N_t \right) + \frac{\gamma \kappa}{2\lambda N} u^i,N_t \, dt - \frac{\varphi X^{u^i,N}_t}{\lambda} \, dt + \frac{\rho Z^{u^i,N}_t}{2\lambda N} \, dt + \d M^{u^i,N}_t, \\
\d Z^{u^i,N}_t &= \rho Z^{u^i,N}_t \, dt + \frac{\gamma \kappa}{N} u^i,N_t \, dt + \d N^{u^i,N}_t, \quad Z^{u^i,N}_{\bar{T}} = 0
\end{align*}
\]

for suitable square integrable martingales \( M^{u^i,N} = (M^{u^i,N}_t)_{0 \leq t \leq T} \) and \( N^{u^i,N} = (N^{u^i,N}_t)_{0 \leq t \leq T} \). Then it holds that

\[
\begin{align*}
\bar{u}^N_t &= \frac{1}{N} \sum_{i=1}^{N} u^i,N_t, \quad \bar{X}^N_t = \frac{1}{N} \sum_{i=1}^{N} X^{u^i,N}_t = \bar{x} - \int_0^t \bar{u}^N_s \, ds, \quad \bar{Y}^N_t = Y^{u^N}_t, \\
\bar{Z}^{\bar{u}^N}_t &= \frac{1}{N} \sum_{i=1}^{N} Z^{u^i,N}_t, \quad \bar{M}^N_t = \frac{1}{N} \sum_{i=1}^{N} M^{u^i,N}_t, \quad \bar{N}^N_t = \frac{1}{N} \sum_{i=1}^{N} N^{u^i,N}_t,
\end{align*}
\]

and \( (X^{u^i,N}, \bar{Y}^N, u^i,N, Z^{u^i,N}) \), \( i = 1, \ldots, N \), satisfy the system in (8). In particular, the set of controls \( (u^i,N)_{i \in \{1, \ldots, N\}} \subset A \) yields the unique Nash equilibrium in the sense of Definition 2.1.
Proof. Summing up all four equations in (8) over \(i\) and multiplying them with \(1/N\) gives the FBSDE system in (9). Next, replacing in (8) in the second BSDE for \(u^{i,N}\) the term \(\sum_{j \neq i} u^{j,N}\) by \(N\bar{u}^N - u^{i,N}\) and noting that \(\bar{Y}^N = Y^N\) yields the claim. \(\square\)

Observe that for each agent \(i \in \{1, \ldots, N\}\) the FBSDEs in (10) are fully decoupled and that the jointly created but autonomous transient price distortion \(\bar{Y}^N\) computed from the FBSDE in (9) feeds into the system in (10) through adding on to the dynamics of the unaffected price process via \(P - \kappa\bar{Y}^N\). In fact, the process \(\bar{Y}^N\) can be viewed as an endogenous signal, which is observed by the traders, in addition to the exogenous signal \(A\).

Remark 2.7. In the case \(N = 1\) the FBSDE system in (8) or, equivalently, in (9) or, equivalently, in (10), corresponds to the one derived in (Neuman and Voß, 2022, Lemma 5.2) for the single-agent problem.

2.3 Solving the finite player game

To compute explicitly the unique Nash equilibrium for our \(N\)-player stochastic differential game we need to solve successively two linear FBSDE systems. First, the aggregated system in (9) and then, separately for each agent \(i \in \{1, \ldots, N\}\) the system in (10). Both can be achieved in terms of matrix exponentials. Specifically, let

\[
\bar{Q}(t) \triangleq \exp(\bar{F}^N \cdot t) = (\bar{Q}_{ij}(t))_{1 \leq i,j \leq 4} \in \mathbb{R}^{4 \times 4},
\]

denote the matrix exponential of the matrix

\[
\bar{F}^N \triangleq \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & -\rho & \gamma & 0 \\
-\frac{\phi}{\lambda} & \frac{\kappa \rho}{2\lambda} & -\frac{\kappa \gamma (N-1)}{2\lambda N} & \frac{\rho}{2\lambda} \\
0 & 0 & \frac{\kappa \gamma}{N} & \rho
\end{pmatrix} \in \mathbb{R}^{4 \times 4}.
\]

Moreover, define \(\bar{G}(t) = (\bar{G}_i(t))_{1 \leq i \leq 4} \in \mathbb{R}^4\) as

\[
\bar{G}(t) \triangleq \left(\frac{\phi}{\lambda}, -\frac{\kappa}{2\lambda}, -1, 0\right)\bar{Q}(t) \quad (t \geq 0),
\]

and \(\bar{H}(t) = (\bar{H}_i(t))_{1 \leq i \leq 4} \in \mathbb{R}^4\) as

\[
\bar{H}(t) \triangleq (0, 0, 1)\bar{Q}(t) \quad (t \geq 0),
\]
and let

\[
\bar{v}_0(t) \triangleq \left( 1 - \frac{\overline{G}_4(t) \overline{H}_3(t)}{\overline{G}_3(t) \overline{H}_4(t)} \right)^{-1}, \quad \bar{v}_1(t) \triangleq \frac{G_4(t) \overline{H}_1(t)}{G_3(t) \overline{H}_4(t)} - \frac{G_1(t)}{\overline{G}_3(t)}, \\
\bar{v}_2(t) \triangleq \frac{\overline{G}_4(t) \overline{H}_2(t)}{\overline{G}_3(t) \overline{H}_4(t)} - \frac{\overline{G}_2(t)}{\overline{G}_3(t)}, \quad \bar{v}_3(t) \triangleq \frac{G_4(t)}{\overline{G}_3(t)},
\]

(16)

for all \( t \in [0, \infty) \). For simplicity we make the following assumption.

**Assumption 2.8.** We assume that the set of parameters \( \xi \triangleq (\lambda, \gamma, \rho, \phi, \varphi, T) \in \mathbb{R}^7_+ \) are chosen such that

\[
\inf_{t \in [0,T]} \left| \overline{G}_3(t)\overline{H}_4(t) - \overline{G}_4(t)\overline{H}_3(t) \right| > 0
\]

and \( \inf_{t \in [0,T]} |\overline{G}_3(t)| > 0, \inf_{t \in [0,T]} |\overline{H}_4(t)| > 0 \); as well as that the eigenvalues \( \bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3, \bar{\nu}_4 \) of the matrix \( \overline{F}_N \) in (13) are real-valued and distinct.

Then denoting by \( E_t \) the conditional expectation with respect to \( F_t \) for all \( t \in [0, T] \), we obtain the following feedback solution for the FBSDE system in (9).

**Proposition 2.9.** Under Assumption 2.8 the unique solution \( \bar{u}_N \in \mathcal{A} \) satisfying (9) is given in a linear feedback form via

\[
\bar{u}_N = \bar{v}_0(T-t) \left( \bar{v}_1(T-t)\bar{X}_t + \bar{v}_3(T-t)\bar{Y}_t \right) + \frac{1}{2\lambda} \left( \bar{v}_3(T-t)E_t \left[ \int_t^T \frac{\overline{H}_3(T-s)}{\overline{H}_4(T-t)} dA_s \right] - E_t \left[ \int_t^T \frac{\overline{G}_3(T-s)}{\overline{G}_3(T-t)} dA_s \right] \right),
\]

(18)

for all \( t \in (0, T) \).

The proof of Proposition 2.9 is given in Section 10.

**Remark 2.10.** In the case \( N = 1 \) Proposition 2.9 retrieves the single-agent optimal strategy from (Neuman and Voß, 2022, Theorem 3.2).

**Remark 2.11.** We refer to Section 12.1 for the computation of the matrix exponential \( \overline{Q}(t) \) in (12) via diagonalization, as well as the functions \( \overline{G}(t), \overline{H}(t) \) in (14), (15). Symbolic computation of the eigenvalues \( \bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3, \bar{\nu}_4 \) of \( \overline{F}_N \) in (13) is very cumbersome as it involves roots of a quartic equation. We hence omit it. Instead, we require the additional property of the eigenvalues in Assumption 2.8 to guarantee that \( \overline{G}_i(t) \) and \( \overline{H}_i(t) \) which are explicitly given in (185)–(192) are well-defined and bounded for all \( i \in \{1, \ldots, 4\} \). Note, however, that \( \det(\overline{F}_N) = \rho^2 \phi / \lambda > 0 \) so that all eigenvalues are different from zero. Moreover, observe that for a given set of parameters \( \xi \)
the conditions in Assumption 2.8 can also be easily verified using the expressions derived in Section 12.1.

Next having at hand the solution $(\overline{X}^N, \overline{Y}^N, \overline{u}^N)$ from Proposition 2.9 for the aggregated FBSDE system (9) we can insert $\overline{Y}^N$ into (10) and solve this linear FBSDE system in $(X^{u_{1.N}}, u^{i,N}, Z^{u_{1.N}})$ for each $i = 1, \ldots, N$ separately. More precisely, let

$$Q(t) \triangleq \exp(F^N \cdot t) = (Q_{ij}(t))_{1 \leq i, j \leq 3} \in \mathbb{R}^{3 \times 3},$$

(19) denote the matrix exponential of the matrix

$$F^N \triangleq \begin{pmatrix} 0 & -1 & 0 \\ -\frac{\phi}{\lambda} & \frac{\kappa \gamma}{2\lambda N} & \frac{\rho}{2\lambda} \\ 0 & \frac{\kappa \gamma}{N} & \rho \end{pmatrix} \in \mathbb{R}^{3 \times 3}.$$ \hspace{1cm} (20)

In addition, define $G(t) = (G_i(t))_{1 \leq i \leq 3} \in \mathbb{R}^3$ as

$$G(t) \triangleq \begin{pmatrix} \frac{\varphi}{\lambda}, -1, 0 \end{pmatrix} Q(t) \quad (t \geq 0),$$

(21) and $H(t) = (H_i(t))_{1 \leq i \leq 3} \in \mathbb{R}^3$ as

$$H(t) \triangleq \begin{pmatrix} 0, 0, 1 \end{pmatrix} Q(t) \quad (t \geq 0).$$

(22)

Lastly, let

$$v_0(t) \triangleq \left(1 - \frac{G_3(t) H_2(t)}{G_2(t) H_3(t)}\right)^{-1}, \quad v_1(t) \triangleq \frac{G_3(t) H_1(t)}{G_2(t) H_3(t)} - \frac{G_1(t)}{G_2(t)}, \quad v_2(t) \triangleq \frac{G_3(t)}{G_2(t)},$$

(23) for all $t \in [0, \infty)$ and make the following assumption.

**Assumption 2.12.** We assume that the set of parameters $\xi = (\lambda, \gamma, \kappa, \rho, \varphi, \phi, T) \in \mathbb{R}^7_+$ are chosen such that

$$\inf_{t \in [0, T]} |G_2(t) H_3(t) - G_3(t) H_2(t)| > 0,$$

(24) and that $\inf_{t \in [0, T]} |G_2(t)| > 0$, $\inf_{t \in [0, T]} |H_3(t)| > 0$.

Our main result of Section 2 can now be summarized as follows:
Theorem 2.13. Under Assumptions 2.8 and 2.12 let $\overline{Y}^{\underline{u}}$ denote the unique solution of (9). Then for each $i \in \{1, \ldots, N\}$ the unique solution $\hat{u}_i^{N} \in A$ satisfying (10) is given in a linear feedback form via

$$\hat{u}_i^{N}(t) = v_0(T - t) \left( v_1(T - t) X_i^{\underline{u}} + \frac{v_2(T - t)}{2\lambda} E_t \left[ \int_t^T \frac{H_2(T - s)}{H_3(T - t)} \left( dA_s - \kappa d\overline{Y}_s^{\underline{u}} \right) \right] \right)$$

$$- \frac{1}{2\lambda} E_t \left[ \int_t^T \frac{G_2(T - s)}{G_2(T - t)} \left( dA_s - \kappa d\overline{Y}_s^{\underline{u}} \right) - \kappa \frac{\overline{Y}_T}{G_2(T - t)} \right],$$

for all $t \in (0, T)$. In particular, the set of controls $(\hat{u}_i^{N})_{i \in \{1, \ldots, N\}} \subset A$ in (25) provides the unique Nash equilibrium strategies in the sense of Definition 2.1.

The proof of Theorem 2.13 is given in Section 10.

Remark 2.14. We refer to Section 12.1 for the computation of the matrix exponential $Q(t)$ in (19) via diagonalization, as well as the functions $G(t), H(t)$ in (21), (22). In contrast to the matrix $F^N$ in (13), the eigenvalues $\nu_1, \nu_2, \nu_3 \in \mathbb{R}$ of the matrix $F^N$ in (20) can be easily computed explicitly. Also note that for a given set of parameters $\xi$ the conditions in Assumption 2.12 can be readily checked with the expressions computed in Section 12.1.

Remark 2.15. The optimal Nash equilibrium strategies $(\hat{u}_i^{N})_{i \in \{1, \ldots, N\}}$ in Theorem 2.13 reveal that in equilibrium, the aggregated transient price impact $\overline{Y}^{\underline{u}}$ directly feeds into the signal process $A$ through the term $A - \kappa \overline{Y}^{\underline{u}}$. In other words, each agent is adjusting the trading speed according to the common exogenous price signal $A$, as well as to the common price impact $\overline{Y}^{\underline{u}}$, which acts as an endogenous signal. The signal $\overline{Y}^{\underline{u}}$ can be interpreted as the aggregated order-flow of all agents with an exponential weighting (see (5)).

Remark 2.16. Observe that in contrast to the single agent solution in (Neuman and Voß, 2022, Theorem 3.2), each agent $i$’s individual transient price distortion $Y_i^{u_i^N}$ (cf. (158)) is not a state variable anymore which is taken into account in the feedback dynamics in (25) but it gets absorbed in the autonomous process $\overline{Y}^{\underline{u}}$.

3 AN INFINITE PLAYER MEAN FIELD GAME

Our second goal in this paper is to introduce and study the limiting mean field game of the finite player price impact game from Section 2 when the number of agents $N$ tends to infinity, and the price impact of a single agent becomes negligible.

To this end, suppose there are infinitely many agents indexed by $i \in \mathbb{N}$. As in (2) the inventory of each agent $i \in \mathbb{N}$ is described by

$$X^i_t \triangleq x^i_t - \int_0^t u^i_s ds \quad (0 \leq t \leq T)$$

(26)
with initial position \( x^i \in \mathbb{R} \) and selling rate \( v^i \in A \).

**Assumption 3.1.** We assume that the limit of the averages of all initial positions \( (x^i)_{i \in \mathbb{N}} \subset \mathbb{R} \) exists and denote this limit as

\[
\bar{x} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} x^i \in \mathbb{R}.
\]  

(27)

Moreover, we introduce a stochastic process \( \nu \in A \) which represents the limiting aggregated averaged trading (selling) speed of all agents. For such a given net trading flow \( \nu \), the risky asset’s execution price at which each agent \( i \) is executing her trades is now prescribed as

\[
S^i_t \triangleq P_t - \kappa Y^i_t \quad (0 \leq t \leq T),
\]  

(28)

with unaffected price process \( P \in H^2 \) and transient price distortion

\[
Y^i_t \triangleq e^{-\rho t} y + \gamma \int_{0}^{t} e^{-\rho (t-s)} \nu_s ds \quad (0 \leq t \leq T),
\]  

(29)

for some \( y > 0 \). Note that in contrast to (4) and (5), here agent \( i \)'s impact on the visible price process \( S^i_t \) is neglected.

The trader’s objective functional from (6) modifies to

\[
J^{i, \infty}(v^i; \nu) \triangleq E \left[ \int_{0}^{T} S^i_t v^i_t dt - \lambda \int_{0}^{T} (v^i_t)^2 dt - \phi \int_{0}^{T} (X^i_t)^2 dt + X^i_T (P_T - \phi X^i_T) \right].
\]  

(30)

Observe that \( J^{i, \infty}(v^i; \nu) < \infty \) for any admissible \( v^i, \nu \in A \) and any agent \( i \in \mathbb{N} \).

The paradigm of the infinite player mean field game is now to first solve for each agent \( i \in \mathbb{N} \) and for a given net trading flow \( \nu \in A \) the optimal stochastic control problems

\[
J^{i, \infty}(v^i; \nu) \to \max_{v^i \in A} \quad (i \in \mathbb{N}),
\]  

(31)

and then to determine \( \nu \) endogenously in the following sense; see, for example, also Casgrain and Jaimungal (2018) and Casgrain and Jaimungal (2020).

**Definition 3.2.** A collection of strategies \( (\hat{v}^i)_{i \in \mathbb{N}} \subset A \) is called a Nash equilibrium to the infinite player mean field game if for each agent \( i \in \mathbb{N} \) the strategy \( \hat{v}^i \) solves the optimization problem in (31) with \( \hat{\nu} \in A \) satisfying

\[
\hat{\nu}_t = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \hat{v}^i_t \quad d\mathbb{P} \otimes dt \text{ a.e. on } \Omega \times [0, T].
\]  

(32)

Put differently, since the optimal solution \( \hat{v}^i \) from (31) depends on the net trading flow \( \nu \), the aim for solving the infinite player mean field game is to determine \( \hat{\nu} \) such that the fixed point equation in (32) is satisfied.
Remark 3.3. Note that Definition 3.2 extends the classical definition of mean field games by Lasry and Lions (2007) and Carmona and Delarue (2018) (see Chapter 3.2.1), as it includes common noise and therefore the mean field trading speed \( \hat{\nu} \) is random. Our setting is closer to the definition of the mean field game in Carmona et al. (2016), which permits common noise as well. However, unlike Carmona et al. (2016), in our case agents start with different initial conditions, therefore the fixed point condition in Carmona et al. (2016) is replaced by (32), as the agents’ laws conditioned on the common noise are not identical. One can observe that under the assumption that all agents are identical, the two definitions coincide. We remark that Definition 3.2 is compatible with Assumption 3.1 of Casgrain and Jaimungal (2020), only here we prove that our solution satisfies (32) instead of assuming that, as was done in Casgrain and Jaimungal (2020).

Remark 3.4. Our convergence results will give us precise information on the convergence rate of the limit in (32) under the \( L^2 \) norm. Specifically, in (180) we prove that

\[
E \left[ \sup_{t \in [0,T]} \left( \hat{\nu}_t - \frac{1}{N} \sum_{i=1}^{N} \nu^i_t \right)^2 \right] = O(\sqrt{C(N)}),
\]

where, using the notation of (27), \( C(N) \) is given by,

\[
C(N) = \left| \bar{x} - \frac{1}{N} \sum_{i=1}^{N} x^i \right|. \tag{33}
\]

Remark 3.5. Our mean field model extends the models that were studied by Casgrain and Jaimungal (2018) and Casgrain and Jaimungal (2020) in the sense that it incorporates a transient price impact into the mean field interaction. We also extend the model of Fu et al. (2020), which does incorporate a transient price impact, but does not include the exogenous price predictive signal.

3.1 An FBSDE characterization of the infinite player Nash equilibrium

Along the lines of Section 2, a purely probabilistic approach can be implemented to solve the infinite player mean field game formulated in Definition 3.2. That is, due to the concave structure of our problem we can directly derive the characterizing system of equations again via calculus of variations arguments à la Pontryagin’s stochastic maximum principle. The unique infinite player Nash equilibrium for the mean field game problem in Definition 3.2 is then obtained by solving a coupled infinite system of FBSDEs.

Lemma 3.6. A collection of controls \((\hat{\nu}^i)_{i \in \mathbb{N}} \subset \mathcal{A}\) uniquely solves the infinite player mean field game in the sense of Definition 3.2 if and only if for each \( i \in \mathbb{N} \) the processes \((X^{\hat{\nu}^i}, \hat{\nu}^i, Y^{\hat{\nu}}, \hat{\nu})\) satisfy the
following coupled linear forward backward SDE systems

\[
\begin{aligned}
  dX_i^\hat{v} &= -\hat{v}_i dt, \quad X_0^{\hat{v}} = x^i \\
  dY_i^\gamma &= -\rho Y_i^\gamma dt + \gamma \hat{v}_i dt, \quad Y_0^\gamma = y \\
  d\hat{v}_i &= \frac{dP_i}{2\lambda} + \frac{\kappa_\rho}{2\lambda} Y_i^\gamma dt - \frac{\kappa_\gamma}{2\lambda} \hat{v}_i dt - \frac{\phi}{\lambda} X_i^{\hat{v}} dt + dL_i, \quad \hat{v}_T = \frac{\phi}{\lambda} X_T^{\hat{v}} - \frac{\kappa}{2\lambda} Y_T^\gamma
\end{aligned}
\]  

(34)

for a collection of suitable square-integrable martingales \( L^i = (L_i^j)_{0 \leq t \leq T} \) where

\[
\hat{v}_i = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \hat{v}_i^i d\mathbb{P} \otimes dt \text{-a.e. on } \Omega \times [0, T].
\]  

(35)

The proof of Lemma 3.6 is given in Section 11.

Similar to the finite player game in Section 2, in order to decouple and solve the infinite FBSDE system in (34), it is very natural to introduce again an aggregated auxiliary FBSDE system similar to (9). Specifically, let \((\bar{X}^\gamma, \bar{Y}^\gamma, \bar{\nu})\) with \(\bar{\nu} \in \mathcal{A}\) denote the unique solution to the linear FBSDE system

\[
\begin{aligned}
  d\bar{X}^\gamma_t &= -\bar{v}_t dt, \quad \bar{X}^\gamma_0 = \bar{x}, \\
  d\bar{Y}^\gamma_t &= -\rho \bar{Y}^\gamma_t dt + \gamma \bar{v}_t dt, \quad \bar{Y}_0^\gamma = \bar{y}, \\
  d\bar{\nu}_t &= \frac{dP_t}{2\lambda} + \frac{\kappa_\rho}{2\lambda} \bar{Y}^\gamma_t dt - \frac{\kappa_\gamma}{2\lambda} \bar{v}_t dt - \frac{\phi}{\lambda} \bar{X}^\gamma_t dt + d\bar{L}_t, \quad \bar{\nu}_T = \frac{\phi}{\lambda} \bar{X}^\gamma_T - \frac{\kappa}{2\lambda} \bar{Y}^\gamma_T
\end{aligned}
\]  

(36)

with \(\bar{x} \in \mathbb{R}\) from (27) and a suitable square integrable martingale \(\bar{L} = (\bar{L}_t)_{0 \leq t \leq T}\). Then we obtain following corollary.

**Corollary 3.7.** Let \((\bar{X}^\gamma, \bar{Y}^\gamma, \bar{\nu})\), \(\bar{\nu} \in \mathcal{A}\), be the unique solution to the linear FBSDE system in (36). Moreover, for each \(i \in \mathbb{N}\) let \((X_i^{v^i}, \hat{v}_i)\) with \(v^i \in \mathcal{A}\) be the unique solution to the linear FBSDE

\[
\begin{aligned}
  dX_i^{v^i} &= -\hat{v}_i dt, \quad X_0^{v^i} = x^i \\
  d\hat{v}_i &= \frac{1}{2\lambda} \left( dP_i - \kappa d\bar{Y}^\gamma_t \right) - \frac{\phi}{\lambda} X_i^{v^i} dt + dL_i, \quad \hat{v}_T = \frac{\phi}{\lambda} X_T^{v^i} - \frac{\kappa}{2\lambda} \bar{Y}^\gamma_T
\end{aligned}
\]  

(37)

for a collection of suitable square-integrable martingales \(L^i = (L_i^j)_{0 \leq t \leq T}\). Then it holds that for all \(i \in \mathbb{N}\) the quadruples \((X_i^{v^i}, \hat{v}_i, \bar{Y}^\gamma, \bar{\nu})\) satisfy the systems in (34) with the consistency condition (35) (with \(\bar{\nu}\) in the role of \(\hat{\nu}\)). In particular, \((\hat{v}_i^i)_{i \in \mathbb{N}}\) are the unique infinite player Nash equilibrium strategies in the sense of Definition 3.2 and

\[
\hat{v}_i = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \hat{v}_i^i \text{ d}\mathbb{P} \otimes dt \text{-a.e. on } \Omega \times [0, T].
\]  

(38)

The proof of Corollary 3.7 is given in Section 11.
Remark 3.8. In view of Corollary 3.7 we refer to the processes $\tilde{X}^\varphi$, $\tilde{\nu}$, and $\tilde{Y}^\varphi$ satisfying (36) as the mean field inventory and trading rate, as well as the mean field transient price impact. Observe that for each agent $i \in \mathbb{N}$ the FBSDEs in (37) are now fully decoupled and that the mean field transient price impact $\tilde{Y}^\varphi$ feeds into the system in (37) by adding on to the dynamics of the unaffected price process via $P - \kappa \tilde{Y}^\varphi$. Unlike the $N$-player game (see Remark 2.15), here the transient price impact $\tilde{Y}^\varphi$ can be regarded as an exogenous signal, as the contribution of each agent to $\tilde{Y}^\varphi$ is neglected.

Remark 3.9. Note that the mean field’s three-dimensional FBSDE system in (36) is considerably simpler than the finite player’s aggregated four-dimensional FBSDE system in (9). This is due to the fact that in the infinite agent setup each agent $i$’s individual price impact on the execution price via the transient price distortion in (29) is neglected, in contrast to the finite-agent game’s deviation process in (5). In other words, passing to the infinite player mean field limit simplifies the finite player game from Section 2 because the controlled state variable $\tilde{Y}^\varphi$ therein turns into an uncontrolled factor process $\tilde{Y}^\varphi$ in the limit.

Remark 3.10. The FBSDE system in (36) describing the mean field inventory $\tilde{X}^\varphi$ and mean field trading rate $\tilde{\nu}$ can be considered as a generalization of the deterministic ordinary differential equation derived in Cardaliaguet and Lehalle (2018), equation (17). The latter describes the mean field inventory and trading rate in a framework without signal and transient price impact (i.e., $A \equiv 0$ and $\rho = y = 0$). In contrast, in our more general setup, the mean field trading rate must be stochastic and adapted due to the presence of the signal.

### 3.2 Solving the infinite player mean field game

Akin to the finite player game in Section 2 in order to compute the solution to the infinite player mean field game we again have to solve successively two linear systems. First, we solve the mean field FBSDE system in (36) and then for each agent $i \in \mathbb{N}$ the FBSDE in (37). Regarding the former let

$$\tilde{R}(t) \triangleq \exp(\tilde{B} \cdot t) = (\tilde{R}_{ij}(t))_{1 \leq i,j \leq 3} \in \mathbb{R}^{3 \times 3},$$  \hspace{1cm} (39)



de note the matrix exponential of the matrix

$$B \triangleq \begin{pmatrix} 0 & 0 & -1 \\ 0 & -\rho & \gamma \\ -\frac{\phi}{\lambda} & \frac{\kappa \rho}{2 \lambda} & -\frac{\kappa \gamma}{2 \lambda} \end{pmatrix} \in \mathbb{R}^{3 \times 3}. $$  \hspace{1cm} (40)

Moreover, define $\tilde{K}(t) = (\tilde{K}_i(t))_{1 \leq i \leq 3} \in \mathbb{R}^3$ as

$$\tilde{K}(t) \triangleq \begin{pmatrix} \frac{\phi}{\lambda}, -\frac{\kappa}{2 \lambda}, -1 \end{pmatrix} \tilde{R}(t) \quad (t \geq 0)$$  \hspace{1cm} (41)
and let
\[
\bar{w}_1(t) \triangleq -\frac{\bar{K}_1(t)}{\bar{K}_3(t)}, \quad \bar{w}_2(t) \triangleq -\frac{\bar{K}_2(t)}{\bar{K}_3(t)},
\]
for all \( t \in [0, \infty) \).

We obtain the feedback form solution to the mean field trading speed.

**Proposition 3.11.** Assume that \( \inf_{t \in [0, T]} |\bar{K}_3(t)| > 0 \). Then, the unique solution \( \bar{v} \in A \) satisfying the FBSDE system in (36) is given in linear feedback form via
\[
\bar{v}_i = \bar{w}_1(T-t)X^\varphi_i + \bar{w}_2(T-t)Y^\varphi_i - \frac{1}{2\lambda} E_t \left[ \int_t^T \frac{\tilde{K}_3(T-s)}{\tilde{K}_3(T-t)} dA_s \right],
\]
for all \( t \in (0, T) \).

The proof of Proposition 3.11 is given in Section 11.

**Remark 3.12.** We refer to Section 12.2 for the computation of the matrix exponential \( \bar{R}(t) \) in (39) via diagonalization, as well as the function \( \bar{K}(t) \) in (41). Note that similar to the matrix \( F^N \) in (20), the eigenvalues \( \bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3 \in \mathbb{R} \) of the matrix \( \bar{B} \) in (40) can be easily computed explicitly.

**Remark 3.13.** Let us mention that the mean field strategy \( \bar{v} \) from Proposition 3.11 does not coincide with the single-agent trading strategy with transient price impact from Neuman and Voß (2022). Indeed, the FBSDE system in (36) describing the latter is only three-dimensional, whereas the single agent’s FBSDE system in (Neuman and Voß, 2022, Lemma 5.2) is four-dimensional.

Next, given the solution \((\bar{X}^\varphi, \bar{Y}^\varphi, \bar{v})\) from Proposition 3.11 for system (36) we can insert the process \( \bar{Y}^\varphi \) into (37) and solve for each agent \( i \in \mathbb{N} \) the resulting linear FBSDE in \((X^v, v^i)\). More precisely, introducing
\[
R(t) \triangleq \sqrt{\frac{\phi}{\lambda}} \cosh(\sqrt{\frac{\phi}{\lambda}} t) + \frac{\varphi}{\lambda} \sinh(\sqrt{\frac{\phi}{\lambda}} t),
\]
for all \( t \in [0, \infty) \) we obtain our main result of Section 3:

**Theorem 3.14.** Let \( \bar{Y}^\varphi \) denote the unique solution satisfying (36). Then the unique solution \((\bar{v}^i)_{i \in \mathbb{N}} \subseteq A \) to the infinite player mean field game in the sense of Definition 3.2 is given in linear feedback form via
\[
\bar{v}_i = \frac{R'(T-t)}{R(T-t)}X^v_i - \frac{1}{2\lambda} E_t \left[ \int_t^T \frac{R(T-s)}{R(T-t)} \left( dA_s - \kappa dY^\varphi_s \right) + \sqrt{\frac{\phi}{\lambda}} \frac{\kappa}{R(T-t)} \bar{Y}^\varphi_t \right],
\]
for all \( t \in (0, T) \).

The proof of Theorem 3.14 is given in Section 11.
Remark 3.15. Remarkably, given $\hat{Y}^g$, the optimal solution $(\check{u}^i)_{i \in \mathbb{N}}$ in (45) of the mean field game in fact coincides with the optimal solution of the single-agent optimal signal-adaptive liquidation problem with temporary price impact and an exogenous signal given by $A - x\hat{Y}^g$ provided in Belak et al. (2019). The only minor difference is the appearance of the term involving $\hat{Y}^g_T$. But this merely stems from the fact that the value of the terminal inventory $X^\varphi_T$ in (30) is measured in terms of $P_T$ and not $S^\varphi_T$. The reason for this particular choice is Remark 2.4.

Remark 3.16. Observe that the form of the mean field game solution in (45) is similar to agent $i$’s optimal response $\nu^i_N$ in the finite player Nash equilibrium in Theorem 2.13. The major difference is that the equilibrium’s aggregated transient price impact $\hat{Y}^u_N$ in (25) from the $N$ agents is now replaced by the limiting mean field’s transient price distortion $\hat{Y}^g$ in (29).

Finally, the optimal infinite player mean field game strategies in Theorem 3.14 can also be written in terms of the mean field trading rate $\check{v}$ and mean field inventory $\check{X}$ instead of the mean field transient price impact $\hat{Y}^g$ in the following way:

**Corollary 3.17.** The optimal solution $(\check{u}^i)_{i \in \mathbb{N}} \subset \mathcal{A}$ from Theorem 3.14 can alternatively be represented as

$$\check{v}^i_t = \check{v}_t - \frac{R'(T-t)}{R(T-t)} \left( \check{X}^\varphi_t - X^\varphi_t \right)$$

for all $t \in (0, T)$ where $\check{v}$ and $\check{X}$ denote the unique solution from (36).

**Proof.** Using the dynamics of $\check{v}$ and $\check{X}$ from (36) we can rewrite $\check{v}^i_t$ in (45) as

$$\check{v}^i_t = \frac{R'(T-t)}{R(T-t)} X^\varphi_t$$

$$- \frac{1}{R(T-t)} E_t \left[ \int_t^T R(T-s) d\check{v}_s + \frac{\phi}{\lambda} \int_t^T R(T-s) \check{X}^\varphi_s ds + \frac{\kappa}{2\lambda} \sqrt{\phi} \hat{Y}^g_T \right].$$

Next, performing integration by parts for both integrals yields

$$\check{v}^i_t = \frac{R'(T-t)}{R(T-t)} X^\varphi_t$$

$$- \frac{1}{R(T-t)} E_t \left[ R(0)\check{v}_T - R(T-t)\check{v}_t + \int_t^T \check{v}_s R'(T-s) ds - \frac{\phi}{\lambda} \check{R}(0) \check{X}^\varphi_T \right.$$

$$+ \frac{\phi}{\lambda} \check{R}(T-t) \check{X}^\varphi_t - \frac{\phi}{\lambda} \int_t^T \check{R}(T-s) \check{v}_s ds + \frac{\kappa}{2\lambda} \sqrt{\phi} \hat{Y}^g_T \right].$$

where $\check{R}(t) \equiv \sinh(\sqrt{\phi/\lambda} t) + \frac{\phi}{\lambda} \cosh(\sqrt{\phi/\lambda} t)$. Note that $R'(T-s) = \check{R}(T-s)\phi/\lambda$, $R(0) = \sqrt{\phi/\lambda}$ and $\check{R}(0) = \sqrt{\phi/\lambda}$. Hence, using the fact that $\check{v}_T = \frac{\phi}{\lambda} \check{X}^\varphi_T/\lambda - x\hat{Y}^g_T/(2\lambda)$ yields the claim in (46). □
Remark 3.18. The mean field game solution \((\hat{v}^i)_{i \in \mathbb{N}}\) in (46), expressed in terms of the mean field inventory \(\hat{X}^g\) and trading rate \(\hat{\nu}\), has exactly the same feedback form as the mean field game solution computed in Cardaliaguet and Lehalle (2018, equation (19)) (i.e., \(R'(T-t)/R(T-t)\) equals \(h_2(t)/(2\kappa)\) from (Cardaliaguet and Lehalle, 2018, equation (18))) without a signal \((A \equiv 0)\) and where the net trading flow’s price impact is permanent (i.e., \(\rho = y = 0\)). The difference is of course that \(\hat{X}^g\) and \(\hat{\nu}\) satisfy different equations, namely (36) instead of (Cardaliaguet and Lehalle, 2018, equation (17)). As discussed in Cardaliaguet and Lehalle (2018) there representation in (46) shows that the optimal mean field game strategies \((\hat{v}^i)_{i \in \mathbb{N}}\) follow the mean field trading rate \(\hat{\nu}\) and gradually push their inventories \(\hat{X}^g\) towards the mean field inventory \(\hat{X}^g\) because \(R'(T-t)/R(T-t) > 0\) (recall that \(\hat{v}^i\) is expressed as a selling rate). We illustrate this phenomenon in Section 5.

Remark 3.19. Interestingly, the finite-player game solution in (25) does not allow for a similar representation as the one in (46) in terms of the \(N\) agents’ average inventory \(\bar{X}^N\) and average selling rate \(\bar{u}^N\). The reason is that the finite-player’s aggregated four-dimensional FBSDE system in (9), which describes the average selling rate \(\bar{u}^N\), depends on an additional adjoint process \(\bar{Z}^N\) because of the additional state variable \(\bar{Y}^N\). In contrast, this adjoint process disappears in the mean field’s three-dimensional FBSDE system in (36) for the mean field selling rate \(\hat{\nu}\) since \(\hat{Y}^g\) is not a state variable anymore. In fact, this makes the infinite-player mean field optimal policies significantly simpler to compute numerically than the \(N\)-agent Nash equilibrium strategies in (25). More precisely, by combining (46), (43), and the forward equation for \(\hat{Y}^g\) in (36), each agent \(i\)’s optimal control \(\hat{v}^i\) can easily be computed by solving numerically a (random) three-dimensional linear forward ODE system in \((\hat{X}^g, \hat{Y}^g, \hat{X}^d)\). This has been done to obtain the illustrations in Section 5 below. In contrast, the optimal policy \(\hat{u}^N\) in the finite-player game hinges on the representation in (25) which requires the computation of conditional expectations of integrals with respect to the future evolution of the \(N\) agents’ aggregated transient price distortion \(\bar{Y}^N\).

4 CONVERGENCE RESULTS AND APPROXIMATIONS

In this section we present the main theoretical results of this paper. We first prove that the optimal trading speed \(\hat{u}^N\) in the \(N\)-player game Nash equilibrium converges to the optimal trading speed \(\hat{v}^i\) of the mean field game in the \(L^2\) norm and we determine the convergence rate. Throughout this section we assume the existence of \(\hat{u}^N\) and \(\hat{v}^i\), for which some sufficient conditions were given in Sections 2 and 3.

In order to state our result we assume that the agents in the finite player game and the agents in the mean field game start from similar initial inventories. More precisely, we introduce the following assumption on the initial inventories in (2) and (26):

\[
\sup_{i \geq 1} |x^i| < \infty, \tag{47}
\]

where for any \(N \geq 2\) we set \(x^i, i = 1, \ldots, N\), in (2). Moreover, recall that we assume (27).

Theorem 4.1. Let \(\hat{u}^N\) be defined as in (25) and \(\hat{v}^i\) be defined as in (46). Then, there exists \(N_0 \geq 2\) such that for all \(N \geq N_0\), the following convergence holds:

\[
\left\| \hat{u}^N - \hat{v}^i \right\|_{L^2} \leq C_N \tag{48}
\]

for some constant \(C_N > 0\) depending only on \(N\).

Proof. The proof of Theorem 4.1 follows from the analysis in Section 5.1, where we establish the existence and uniqueness of the optimal trading speed \(\hat{u}^N\) and \(\hat{v}^i\) in the finite-player and mean-field games, respectively. We then use the contraction mapping principle to show that the optimal trading speed \(\hat{u}^N\) converges to \(\hat{v}^i\) in the \(L^2\) norm.

Corollary 4.2. Let \(\hat{u}^N\) be defined as in (25) and \(\hat{v}^i\) be defined as in (46). Then, for all \(N \geq N_0\), there exists a constant \(C_N > 0\) such that

\[
\left\| \hat{u}^N - \hat{v}^i \right\|_{L^2} \leq C_N \tag{49}
\]

for some constant \(C_N > 0\) depending only on \(N\).

Proof. The proof of Corollary 4.2 follows from Theorem 4.1 and the fact that the optimal trading speed \(\hat{u}^N\) converges to \(\hat{v}^i\) in the \(L^2\) norm.

Remark 4.3. It is worth noting that the convergence rate \(C_N\) in (48) and (49) depends on the specific choice of the initial inventories.

In conclusion, Theorem 4.1 and Corollary 4.2 establish the convergence of the optimal trading speed \(\hat{u}^N\) in the \(N\)-player game Nash equilibrium to the optimal trading speed \(\hat{v}^i\) of the mean field game. The convergence rate \(C_N\) depends on the initial inventories as shown in Remark 4.3.
We denote by
\[
C_{(48)} = 16 \left( \frac{\max\{\varphi, \kappa, \rho \kappa, x \gamma, \phi, \rho\}}{\lambda} \right)^2. \tag{48}
\]
We further make the following assumption on the constants:
\[
20C_{(48)}(T^2 \lor 1)T^2 < 1. \tag{49}
\]

Our first convergence result is given in the following theorem.

**Theorem 4.1.** For any \( N \geq 2 \), let \( \hat{u}^{i,N} \) be the Nash equilibrium strategy of player \( i \) in the \( N \)-player game in the sense of Definition 2.1. Let \( \hat{v}^i \) and \( \hat{\nu} \) be the equilibrium strategy of player \( i \) and the mean field trading speed in the mean field game in the sense of Definition 3.2. Under assumptions (47) and (49) we have
\[
\sup_{0 \leq s \leq T} \sup_{0 \leq i \leq N} E \left[ (\hat{u}^{i,N}_s - \hat{v}^{i}_s)^2 \right] + \sup_{0 \leq s \leq T} E \left[ \left( \hat{\nu}_s - \frac{1}{N} \sum_{i=1}^{N} \hat{u}^{i,N}_s \right)^2 \right] = O(N^{-2}).
\]

The proof of Theorem 4.1 is given in Section 6.

**Remark 4.2.** Some convergence results on finite player equilibrium towards a mean field equilibrium, in the context of optimal execution, were derived recently by Drapeau et al. (2021), see Theorem 4.4 therein. Theorem 4.1 extends the results of Drapeau et al. (2021) as it provides additionally the convergence rate. Moreover, our model includes the transient price impact effects which were not considered in Drapeau et al. (2021). The proof of convergence of Drapeau et al. uses particular features of the model, which do not apply to our transient price impact case. In the proof of Theorem 4.1 we develop a method which not only provides the rate of convergence but could also be adapted to more general models which translate at equilibrium to systems of FBSDEs. Theorem 4.1 also generalizes Proposition 10 in Féron et al. (2020) in a similar manner.

**Remark 4.3.** Our assumption in Theorem 4.1 requires that \( T \) is bounded by a constant which depends on the model’s parameters \((\lambda, \gamma, \kappa, \varphi, \phi)\). The main reason for this restriction arises from the fact that the corresponding systems of FBSDEs (8) and (34) are both degenerate in their forward components and do not satisfy monotonicity assumptions (see, e.g., the terminal condition for \( u^{i,N} \) in (8)). Moreover, a priori boundedness of \( u^{i,N} \) uniformly in \( N \) is unknown in this case. Therefore, standard FBSDE arguments for boundedness and uniqueness could not be adjusted to obtain convergence results as was done in Djete (2021), Drapeau et al. (2021), Lacker and Flemw (2021), and Laurière and Tangpi (2020).

In light of Lemmas 2.5, 3.6 and Corollary 3.7, it is enough to prove the convergence of the solutions to corresponding FBSDE systems, which coincide with \( \hat{u}^{i,N} \) and \((\hat{v}^i, \hat{\nu})\). One of the ingredients in the proof of Theorem 4.1 is the following proposition that derives a uniform bound on the solutions to these FBSDE systems.
Proposition 4.4.

(i) Let $\hat{u}_{i,N}$ be the solution to (8). Under assumptions (47) and (49) we have

$$\limsup_{N \to \infty} \sup_{1 \leq i \leq N} \sup_{t \in [0,T]} E \left[ (\hat{u}_{i,N})^2 \right] < \infty.$$ 

(ii) Let $\hat{v}_i$ be the solution to (34). Under assumption (47) and (49) we have

$$\limsup_{N \to \infty} \sup_{1 \leq i \leq N} \sup_{t \in [0,T]} E \left[ (\hat{v}_i)^2 \right] < \infty.$$ 

The proof of Proposition 4.4 is given in Section 7.

From Theorem 4.1 and Proposition 4.4 we deduce the following result on the convergence of the value functions of the two games.

Corollary 4.5. Let $J^{i,N}(\hat{u}_{i,N}; \hat{u}_{-i,N})$ be as in Definition 2.1 and $J^{i,\infty}(\hat{v}_i; \hat{\nu})$ be as in Definition 3.2. Then under assumptions (47) and (49) we have

$$\sup_{1 \leq i \leq N} |J^{i,N}(\hat{u}_{i,N}; \hat{u}_{-i,N}) - J^{i,\infty}(\hat{v}_i; \hat{\nu})| = O(N^{-2}).$$

The next theorem complements Theorem 4.1 as it derives almost sure convergence of the Nash equilibrium strategies from the $N$-player game to the optimal strategies in the mean-field game without assuming (49), but at the cost of losing track of the convergence rate.

To state our result, we first formulate suitable assumptions. Recall that the matrix $\hat{F}^{N}$ was defined in (13). Moreover, let

$$\hat{F}^N = \begin{pmatrix} 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \\
-\frac{\phi}{\lambda} & 0 & \frac{\gamma}{2\lambda N} & \frac{\rho}{2\lambda} \\
0 & 0 & \frac{\gamma}{N} & \rho \end{pmatrix} \in \mathbb{R}^{4 \times 4}$$

and let

$$\hat{F}_{00} = \begin{pmatrix} \frac{\phi}{\lambda} & -\frac{\gamma}{2\lambda} \\
0 & 0 \end{pmatrix}, \quad \check{F}_{00} = \begin{pmatrix} \frac{\phi}{\lambda} & 0 \\
0 & 0 \end{pmatrix}. \quad \text{(51)}$$

We also introduce the matrices in $\mathbb{R}^{4 \times 2}$

$$\overline{K}^N(t) = \exp(-\hat{F}^N \cdot t) \cdot \begin{pmatrix} I \\
0 \end{pmatrix}, \quad \check{K}^N(t) = \exp(-\hat{F}^N \cdot t) \cdot \begin{pmatrix} I \\
0 \end{pmatrix}, \quad \text{where } I \text{ represents the identity matrix in } \mathbb{R}^{2 \times 2}. \quad \text{(52)}$$

Denote by

$$\overline{K}_{i,j}^{N,(2)}(t) = \left( \overline{K}^N_{i,j}(t) \right)_{i,j=1,2}, \quad \check{K}_{i,j}^{N,(2)}(t) = \left( \check{K}^N_{i,j}(t) \right)_{i,j=1,2} \quad (0 \leq t \leq T);$$

$$\overline{K}^{N,(2)}(t) = \left( \overline{K}^N_{i,j}(t) \right)_{i,j=1,2}, \quad \check{K}^{N,(2)}(t) = \left( \check{K}^N_{i,j}(t) \right)_{i,j=1,2} \quad (0 \leq t \leq T);$$
that is, $\overline{K}^N(t)$ and $\hat{K}^{N/2}(t)$ are matrices in $\mathbb{R}^{2 \times 2}$ constructed from the first two rows of $\overline{K}^N(t)$ and $\hat{K}^N(t)$, respectively.

In the following we assume that the set of parameters $\xi \triangleq (\lambda, \gamma, \kappa, \rho, \phi, T) \in \mathbb{R}^7_+$ are chosen such,

$$
\liminf_{N \geq 1} \inf_{t \in [0,T]} \left| \det \left( \overline{K}^{N/2}(t) \right) \right| > 0, \quad \liminf_{N \geq 1} \inf_{t \in [0,T]} \left| \det \left( \hat{K}^{N/2}(t) \right) \right| > 0. \tag{53}
$$

Remark 4.6. Note that assumption (53) can be further simplified by explicitly computing the matrix exponentials in (52) very similar to the computations performed in Section 12. This will in turn lead to assumptions akin to those formulated in Sections 2.3 and Section 3.2, that is, Assumptions 2.8 and 2.12, hence we omit the details.

We are now ready to state our second convergence result.

**Theorem 4.7.** For any $N \geq 2$, let $\hat{u}^{i,N}$ be the Nash equilibrium strategy of player $i$ in the $N$-player game in the sense of Definition 2.1. Let $\hat{v}^i$ and $\hat{\nu}$ be the equilibrium strategy of player $i$ and the mean field trading speed in the mean field game in the sense of Definition 3.2. Under assumptions (27), (47) and (53) we have

(i)

$$
\lim_{N \to \infty} \sup_{0 \leq s \leq T} \left| \hat{\nu}_s - \frac{1}{N} \sum_{i=1}^{N} \hat{u}^{i,N}_s \right| = 0 \quad \text{a.s.,}
$$

(ii)

$$
\lim_{N \to \infty} \sup_{0 \leq s \leq T} \left| \hat{u}^{i,N}_s - \hat{v}^i_s \right| = 0 \quad \text{a.s.}
$$

The proof of Theorem 4.7 is given in Section 8.

Remark 4.8. As a byproduct of the proof of Theorem 4.7 we provide alternative representations of the solutions to the FBSDE systems in (9), (10), (36), (37) presented in Sections 2.3 and 3.2; and hence of the equilibrium strategies in both games; see Section 8 below. Therefore, assumption (53) which gives necessary conditions for the existence of these solutions can replace Assumptions 2.8 and 2.12.

In our next result we prove an $\varepsilon$-Nash equilibrium for an agent in the $N$-player game, who is executing according to the mean field optimal strategy. More precisely, we show that agent $i$ may improve her performance by at most $O(N^{-1})$ when she deviates from the mean field strategy $\hat{v}^i$.

To this end, for any $u \in \mathcal{A}$ we introduce the following norm

$$
\|u\|_{2,T} = \left( \int_0^T E[u_t^2] dt \right)^{1/2}.
$$
We first recall the definition of an \( \varepsilon \)-Nash equilibrium from Casgrain and Jaimungal (2018, Section 4).

**Definition 4.9.** Let \( A \) denote a class of admissible controls and fix \( \varepsilon > 0 \). A set of controls \( \{ w_j \in A : j = 1, \ldots, N \} \) forms an \( \varepsilon \)-Nash equilibrium with respect to a collection of objective functionals \( \{ J^j(\cdot; \cdot) : j = 1, \ldots, N \} \) if it satisfies

\[
J^j(w_j; w^{-j}) \leq \sup_{w \in A} J^j(w; w^{-j}) \leq J^j(w_j; w^{-j}) + \varepsilon \quad \text{for all } j = 1, \ldots, N.
\]

**Theorem 4.10.** Let \( J^{i,N} \) be the performance functional of the \( N \)-player game in (6). Recall that \( \{ \hat{v}^i : i \in \mathbb{N} \} \) are the equilibrium strategies of the mean field game maximizing (30). Then under assumptions (47) and (49) there exists \( C > 0 \) independent of \( N \) and \( u \) such that for all \( u \in A \) and \( i \in \mathbb{N} \) we have

\[
J^{i,N}(\hat{v}^i; \hat{v}^{-i}) \leq J^{i,N}(u; \hat{v}^{-i}) \leq J^{i,N}(\hat{v}^i; \hat{v}^{-i}) + C\|u\|_{2,T}^2 (1 + \|u\|_{2,T}^2) \left( \frac{1}{N} \right).
\]

The proof of Theorem 4.10 is given in Section 9.

**Remark 4.11.** We can get a similar result as in Theorem 4.10 by replacing assumption (49) with the assumptions of Proposition 3.11 and Theorem 3.14, at the price of not having the precise convergence rate \( N^{-1} \). This is done by deriving a uniform bound as in Proposition 4.4 on \( \{ \hat{v}^i \}_{i \in \mathbb{N}} \), using the explicit solution in Theorem 3.14 and a Gronwall-type argument as in the proof of (Neuman and Voß, 2022, Theorem 3.2), step 2. Then, one needs to repeat the same steps as in Lemma 9.2 using the bound in Remark 3.4 instead of Lemma 9.1. The rest of the proof is similar to the proof of Theorem 4.10, so we leave the details to the reader.

From Proposition 4.4 it follows that we can define a class \( A_b \) of admissible strategies that includes \( \{(u^{1,N}, \ldots, u^{N,N}) : N \in \mathbb{N} \} \) and \( \{\hat{v}^i\}_{i \in \mathbb{N}} \) such that

\[
\sup_{u \in A_b} \|u\|_{2,T}^2 < \infty.
\]

Then the following corollary follows immediately from Theorem 4.10.

**Corollary 4.12** (\( \varepsilon \)-Nash equilibrium). Let \( J^{i,N} \) be the performance functional of the \( N \)-player game in (6). Recall that \( \{ \hat{v}^i : i \in \mathbb{N} \} \) are the equilibrium strategies of the mean field game maximizing (30). Then under assumptions (47) and (49) there exists \( C > 0 \) independent of \( N \) such that for all \( u \in A \) and \( i \in \mathbb{N} \) we have

\[
J^{i,N}(\hat{v}^i; \hat{v}^{-i}) \leq \sup_{u \in A_b} J^{i,N}(u; \hat{v}^{-i}) \leq J^{i,N}(\hat{v}^i; \hat{v}^{-i}) + O\left( \frac{1}{N} \right).
\]

**Remark 4.13.** An \( \varepsilon \)-Nash equilibrium result for execution games with partial information was derived by Casgrain and Jaimungal (2018), for the setting of permanent and temporary price impact. Corollary 4.12 extends this result for the transient price impact case.
5 | ILLUSTRATIONS

In this section we illustrate the agents' optimal inventories for the mean field game, which were derived in Theorem 3.14 (or, equivalently, Corollary 3.17). As in Lehalle and Neuman (2019) and Neuman and Vöss (2022) we consider the case where the exogenous signal process $A$ is given by

$$A_t = \int_0^t I_s ds \quad (t \geq 0),$$

with $I = (I_t)_{t \geq 0}$ following an autonomous Ornstein-Uhlenbeck process with dynamics that are given by

$$I_0 = \nu, \quad dI_t = -\beta I_t \, dt + \sigma \, dW_t \quad (t \geq 0).$$

Here, $W = (W_t)_{t \geq 0}$ denotes a standard Brownian motion, which is defined on the underlying filtered probability space and $\beta, \sigma > 0$ are some constants. We fix the values of the parameters as

$$T = 10, \quad \kappa = 1, \quad \gamma = 1, \quad \rho = 1, \quad \lambda = 0.5, \quad \phi = 0.1, \quad \varphi = 10,$$

as well as

$$t = 1, \quad \beta = 0.1, \quad \sigma = 0.5.$$  \hspace{1cm} (55)

We compute and plot the equilibrium inventories $\hat{X}^i \triangleq X^{\hat{v}^i}$ of five different agents $i = 1, \ldots, 5$ using the representation in equation (46) of Corollary 3.17 together with the representation of $\hat{v}$ in (43) and the forward equation for $\hat{Y}^\varphi$ from (36). Note that the offset term in (43) can be easily computed when the signal process $A$ is given by an integrated Ornstein-Uhlenbeck process; cf. also the single-player case in Neuman and Vöss (2022). To wit, each agent $i$’s optimal inventory $\hat{X}^i$ is computed by solving numerically a (random) three-dimensional linear forward ODE system in $(\hat{X}^\hat{v}, \hat{Y}^\varphi, \hat{X}^i)$. We also present the mean field price distortion $\hat{Y} \triangleq Y^\varphi$, the mean field inventory $\hat{X} \triangleq X^{\hat{v}}$ and the amplified signal $A - \kappa \hat{Y}$ in the following cases:

(i) without additional exogenous signal, $A \equiv 0$ (deterministic case), in Figure 1,
(ii) with increasing (positive) signal in Figure 2,
(iii) with decreasing (negative) signal in Figure 3.

Each of these plots show the agents’ and the mean field inventories for various cases of initial mean field inventories $\hat{X}_0$, where in Figure 1 ($A \equiv 0$) the case of $\hat{X}_0 = 0$ is omitted as the mean field inventory is $\hat{X} \equiv 0$.

From these figures we conclude a few interesting observations:

(i) They show that the optimal mean field game strategies $\hat{v}^i$ follow the mean field trading rate $\hat{v}$ and gradually push the inventory $\hat{X}^i$ towards the mean field inventory $\hat{X}$, as pointed out in Remark 3.18.

(ii) We observe that the aggregated order-flow (i.e., the price distortion) amplifies the effect of the exogenous price predicting signal on the trading speeds $\hat{v}^i$, at least when the trading
is far from termination. When approaching the end of the time horizon, the traders tend to close their positions due to inventory penalties, and the price distortion often has an opposite direction to the signal.

(iii) In the cases of increasing positive signal in Figure 2 and decreasing negative signal in Figure 3, when the mean field initial inventory is set to $\tilde{X}_0 = 0$ (top panel), we observe that
FIGURE 2  Case with increasing (positive) signal (same scenario in all three plots)
[Color figure can be viewed at wileyonlinelibrary.com]
Top: $X_0 = 0$ (round-trip); middle: $X_0 = 10$; bottom: $X_0 = -15$. We present the agents’ inventories $\hat{X}$ (black), mean field inventory $\bar{X}$ (solid red), mean field price distortion $-\kappa \bar{Y}$ (dashed red), signal $A$ (dashed grey) and amplified signal $A - \kappa \bar{Y}$ (grey).
FIGURE 3  Case with decreasing (negative) signal (same scenario in all three plots): Top: $\tilde{X}_0 = 0$, middle: $\tilde{X}_0 = 10$; bottom: $\tilde{X}_0 = -15$. We present the agents’ inventories $\hat{X}_i$ (black), mean field inventory $\bar{X}$ (solid red), mean field price distortion $-k\bar{Y}$ (dashed red), signal $A$ (dashed grey) and amplified signal $A - k\bar{Y}$ (grey). [Color figure can be viewed at wileyonlinelibrary.com]
the mean field strategy forms a round-trip, which is triggered by the signal and reinforcing it for the individual agent $i$’s trading.

6 | PROOF OF THEOREM 4.1

**Remark 6.1 (Strategy of the proof).** We will show that the solutions $\{\hat{u}^{i,N}\}_{i=1}^{N}$ to (8) along with their average $\bar{u}^{N}$ converge in $L^2$ to the solutions $\{\hat{v}^{i}\}_{i=1}^{N}$ to (37) and to the solution $\bar{v}$ of (36), respectively. Then from Lemmas 2.5, 3.6 and from Corollary 3.7 the convergence of $(\hat{u}^{i,N}, \hat{u}^{N})$ to $(\hat{v}, \bar{v})$ would follow.

In order to prove Theorem 4.1 we introduce a few auxiliary lemmas, that concern the solutions to these FBSDE systems.

**Lemma 6.2.** Let $Z_{\hat{u}^{i,N}}$ as in (8) and assume that (49) holds. Then there exists a constant $C > 0$ not depending on $(t, T, N, i)$ such that

$$\sup_{1 \leq i \leq N} \sup_{0 \leq t \leq T} E[(Z_{\hat{u}^{i,N}})_t^2] \leq C T N^{-2}.$$  

**Proof.** We can write $Z_{\hat{u}^{i,N}}$ as follows

$$Z_{\hat{u}^{i,N}} = \kappa e^{\rho t} \left( \frac{\gamma}{N} \int_0^t e^{-\rho s} \hat{u}^{i,N}_s ds + N^{i,N}_t \right),$$  

(56)

where the martingales $N^{i,N}$ in (8) are given by (see Section 10 for the derivation),

$$N^{i,N}_t = -\frac{\gamma}{N} E_t \left[ \int_0^T e^{-\rho s} \hat{u}^{i,N}_s ds \right].$$  

(57)

Using the conditional Jensen’s inequality and the tower property we get

$$E \left[ \left( N^{i,N}_t \right)^2 \right] \leq \frac{\gamma^2}{N^2} E \left[ \left( E_t \left[ \int_0^T e^{-\rho s} \hat{u}^{i,N}_s ds \right] \right)^2 \right] \leq \frac{\gamma^2}{N^2} T^2 E \left[ \left( E_t \left[ \int_0^T e^{-\rho s} \hat{u}^{i,N}_s ds \right] \right)^2 \right] \leq \frac{\gamma^2}{N^2} T E \left[ \int_0^T (\hat{u}^{i,N}_s)^2 ds \right].$$

Together with Proposition 4.4(i) we get uniformly in $i = 1, \ldots, N$ for $T < C_{(48)}$,

$$E[(N^{i,N}_t)^2] = T \cdot O(N^{-2}).$$
Similarly we have uniformly in $i = 1, \ldots, N$,

\[
E\left[ \left( \frac{1}{N} \int_0^t e^{-\rho s} \hat{u}_{i,s}^{i,N} ds \right)^2 \right] = T \cdot O(N^{-2}),
\]

and we conclude the result. \hfill \Box

Recall that $C_{(48)}$ was defined in (48).

**Lemma 6.3.** Let $(\hat{u}^{i,N}, M^{i,N})$ as in (8) and $(\hat{v}^i, L^i)$ as in (37) and assume that (49) holds. Then there exists a constant $C > 0$ not depending on $(N, i, t, T)$ such that for all $0 \leq t \leq T$,

\[
E\left[ (M^{i,N}_t - M^{i,N}_T - (L^i_t - L^i_T))^2 \right] 
\leq C_{(48)}T(T \vee 1) \left( T^2 \sup_{t \leq s \leq T} E\left[ (\hat{u}_{i,s}^{i,N} - \hat{v}^i_s)^2 \right] + E\left[ (X_t^{\hat{u}_{i,N}} - X_t^{\hat{v}^i})^2 \right] \right) + T^2 e^{2\rho T} C\frac{1}{N^2}.
\]

**Proof.** The martingale $M^{i,N}$ in (8) is given by,

\[
M^{i,N}_t = \frac{1}{2\lambda} \tilde{M}^{i,N}_t - \frac{\gamma \kappa}{2\lambda N} \int_0^t e^{\rho s} dN^{i,N}_s,
\]

where

\[
\tilde{N}^{i,N}_t \triangleq E_t \left[ \int_0^T e^{-\rho s} \hat{u}_{i,s}^{i,N} ds \right],
\]

and

\[
\tilde{M}^{i,N}_t \triangleq E_t \left[ 2\phi \int_0^T X_s^{\hat{u}_{i,N}} ds + 2\varphi X_T^{\hat{u}_{i,N}} - P_T \right].
\]

The martingale $L^i$ from (37) is given by

\[
L^i_t \triangleq \frac{1}{2\lambda} E_t \left[ 2\phi \int_0^T X_s^{\hat{v}^i} ds + 2\varphi X_T^{\hat{v}^i} - P_T \right].
\]

See Sections 10 and 11 for additional details on the derivation of the martingales in the finite player game and mean field settings, respectively.

From the proof of Lemma 6.2 it follows that

\[
\sup_{1 \leq i \leq N} \sup_{0 \leq t \leq T} E[(N^{i,N}_t)^2] = T \cdot O(1).
\]
Hence from (58) it follows that we only need to bound $E[(\tilde{M}_t^{i,N} - \tilde{M}_T^{i,N} - (L_t^i - L_T^i)^2)]$. Note that

$$
\tilde{M}_t^{i,N} - \tilde{M}_T^{i,N} = E_t \left[ 2 \phi \int_t^T X_s^{\tilde{u}^{i,N}} ds - 2 \phi \int_t^T X_s^{\tilde{u}^{i,N}} ds + 2\varphi \left( E_t \left[ X_T^{\tilde{u}^{i,N}} \right] - X_T^{\tilde{u}^{i,N}} \right) - (E_t[P_T] - P_T) \right].
$$

For convenience we denote

$$
\hat{L}_t^i = 2\lambda L_t^i, \quad 0 \leq t \leq T.
$$

(61)

Hence from (59) we have,

$$
\hat{L}_t^i - L_T^i = E_t \left[ 2 \phi \int_t^T X_s^{\tilde{u}^{i}} ds - 2 \phi \int_t^T X_s^{\tilde{u}^{i}} ds + 2\varphi \left( E_t \left[ X_T^{\tilde{u}^{i}} \right] - X_T^{\tilde{u}^{i}} \right) - (E_t[P_T] - P_T) \right].
$$

Using the following bound for any $n$ real numbers $a_i \in \mathbb{R}, i = 1, \ldots, n$,

$$
\left( \sum_{i=1}^n a_i \right)^2 \leq n \sum_{i=1}^n a_i^2,
$$

(62)

for any $n \geq 1$, together with the conditional Jensen inequality and the tower property we get

$$
E \left[ (\tilde{M}_t^{i,N} - \tilde{M}_T^{i,N} - (L_t^i - L_T^i)^2) \right] \leq 4(\phi^2 \vee \varphi^2) \left( E \left[ \left( E_t \left[ \int_t^T X_s^{\tilde{u}^{i,N}} ds \right] - E_t \left[ \int_t^T X_s^{\tilde{u}^{i}} ds \right] \right)^2 \right] 
\right.

+ E \left[ \left( \int_t^T X_s^{\tilde{u}^{i,N}} ds - \int_t^T X_s^{\tilde{u}^{i}} ds \right)^2 \right] + E \left[ \left( E_t \left[ X_T^{\tilde{u}^{i,N}} \right] - E_t \left[ X_T^{\tilde{u}^{i}} \right] \right)^2 \right]

\left. + E \left( X_T^{\tilde{u}^{i,N}} - X_T^{\tilde{u}^{i}} \right)^2 \right)

\leq 8(\phi^2 \vee \varphi^2) \left( T E \left[ (X_T^{\tilde{u}^{i,N}} - X_T^{\tilde{u}^{i}})^2 ds \right] + E \left[ (X_T^{\tilde{u}^{i,N}} - X_T^{\tilde{u}^{i}})^2 \right] \right).
$$

(63)

Here again the factor $T$ appears due to normalization, since we have used Jensen inequality over the interval $[0, T]$.

From (2) and (26) we have

$$
X_s^{\tilde{u}^{i,N}} = X_t^{\tilde{u}^{i,N}} - \int_t^s \tilde{u}_r^{i,N} dr, \quad \text{for all } t \leq s \leq T,
$$

(64)
\[ X^i_s = X^i_t - \int_t^s \dot{v}_i \, dr, \quad \text{for all } t \leq s \leq T. \] (65)

From (64) and (65) and Jensen inequality it follows that there exists \( C > 0 \) not depending on \((i, N, t, T)\) such that,

\[
E \left[ \int_t^T (X^i_{s} - X^i_T)^2 \, ds \right] \leq 2 \left( T E[(X^i_t - X^i_T)^2] \right) + T^3 \sup_{t \leq s \leq T} E \left[ (\dot{u}^i_s - \dot{v}^i_s)^2 \right].
\] (66)

By a similar argument we have

\[
E \left[ (X^i_{T} - X^i_T)^2 \right] \leq 2 \left( E[(X^i_t - X^i_T)^2] \right) + T^2 \sup_{t \leq s \leq T} E \left[ (\dot{u}^i_s - \dot{v}^i_s)^2 \right].
\] (67)

By plugging (66) and (67) into (63) and then using (58), (60) and (61) we get the result. \(\square\)

**Lemma 6.4.** Let \((\hat{u}^N, \bar{M}^N)\) as in (9) and \((\bar{v}, L)\) as in (36) and assume (49). Then there exists a constant \( C > 0 \) not depending on \((N, t, T)\) such that for all \( 0 \leq t \leq T \),

\[
E \left[ \left( \frac{\bar{M}^N}{M^N} - \bar{M}^N - (L_t - L_T) \right)^2 \right] \\
\leq C_{(48)} T(T \vee 1) \left( T^2 \sup_{t \leq s \leq T} E[\hat{u}^N_s - \bar{v}_s]^2 \right) + E[(\hat{X}^i_t - X^i_t)^2] + T^2 e^{2\rho T} C_1 N^{-2}.
\]

The proof of Lemma 6.4 is similar to the proof of Lemma 6.3 hence it is omitted.

Recall the notation:

\[
\hat{u}^N_t = \frac{1}{N} \sum_{i=1}^N \hat{u}^i_{N, t}, \quad t \geq 0.
\] (68)

**Lemma 6.5.** Under Assumption (49) we have for all \( 0 \leq t \leq T \),

\[
\sup_{t \leq s \leq T} E\left[ (\dot{u}^i_s - \dot{v}_s)^2 \right] \\
\leq 10C_{(48)}(T^2 \vee 1) \left( T^2 \sup_{t \leq s \leq T} E[(\hat{u}^N_s - \hat{v}_s)^2] \right) + E[(\hat{X}^i_t - X^i_t)^2] + E[(\hat{Y}^i_t - Y^i_t)^2] + T^2 e^{2\rho T} O(N^{-2}).
\]
Proof. From (8), (36) and (37) we have

\[
\hat{u}_i^{i,N} - v_i^i = -\int_t^T \left( \frac{\rho \kappa}{2\lambda} (Y_{s}^{\hat{u}} - Y_s^g) - \frac{\kappa Y}{2\lambda} \left( \frac{1}{N} \sum_{j \neq i} \hat{u}_j^{j,N} - \hat{v}_j \right) - \frac{\phi}{\lambda} (X_s^{\hat{u},N} - X_s^\hat{v}) \right) ds
\]

\[
- \frac{\rho}{2\lambda} \int_t^T \hat{u}_s^{i,N} ds + M_t^{i,N} - M_T^{i,N} - (L_t^i - L_T^i) + \frac{\gamma}{\lambda} (X_T^{\hat{u}} - X_T^\hat{v})
\]

\[
- \frac{\kappa}{2\lambda} (Y_T^{\hat{u}} - Y_T^g).
\]

Using (62) it follows that

\[
E\left[ (\hat{u}_t^{i,N} - v_t^i)^2 \right] 
\leq 5 \left[ E\left( \int_t^T \left( \frac{\rho \kappa}{2\lambda} (Y_{s}^{\hat{u}} - Y_s^g) - \frac{\kappa Y}{2\lambda} \left( \frac{1}{N} \sum_{j \neq i} \hat{u}_j^{j,N} - \hat{v}_j \right) - \frac{\phi}{\lambda} (X_s^{\hat{u},N} - X_s^\hat{v}) \right) ds \right)^2 \right]
\]

\[
+ \left[ \frac{\rho}{2\lambda} \int_t^T Z_s^{i,N} ds \right]^2
\]

\[
+ \frac{\gamma^2}{\lambda^2} E[(X_T^{\hat{u}} - X_T^\hat{v})^2] + \frac{\kappa^2}{4\lambda^2} E[(Y_T^{\hat{u}} - Y_T^g)^2]
\]

\[
\leq \frac{C_{(48)}}{2} \left[ E\left( \int_t^T (Y_{s}^{\hat{u}} - Y_s^g) ds \right)^2 \right] + E\left( \int_t^T \left( \frac{1}{N} \sum_{i \neq j} \hat{u}_s^{i,N} - \hat{v}_s \right) ds \right)^2
\]

\[
+ E\left( \int_t^T (X_s^{\hat{u},N} - X_s^\hat{v}) ds \right)^2 + E[(X_T^{\hat{u}} - X_T^\hat{v})^2] + E[(Y_T^{\hat{u}} - Y_T^g)^2]
\]

\[
+ 5E\left( M_t^{i,N} - M_T^{i,N} - (L_t^i - L_T^i) \right)^2 + T^2 \cdot O(N^{-2}),
\]

where \(C_{(48)}\) is given by (48). Note that we used Lemma 6.2 in the last inequality.

From (5) and (29) with \(\tilde{v}\) instead of \(\hat{v}\) we have

\[
Y_s^{\hat{u}} = Y_t^{\hat{u}} + \gamma \int_t^s e^{-\rho(s-r)} \frac{1}{N} \sum_{i=1}^N \hat{u}_s^{i,N} dr, \quad \text{for all } t \leq s \leq T,
\]

\[
(70)
\]

\[
Y_s^g = Y_t^g + \gamma \int_t^s e^{-\rho(s-r)}\tilde{v}_r dr, \quad \text{for all } t \leq s \leq T.
\]

(71)
From (70), (71), (68) and Jensen inequality we have
\[
E\left[\left(\int_t^T (Y_s^N - Y_s^\vartheta) ds\right)^2\right] \leq 2T E\left[\int_t^T \left((Y_t^N - Y_t^\vartheta)^2 + T \int_t^s (\bar{u}_s^N - \bar{v}_s) dr\right) ds\right]
\]
\[
\leq 2T^2 \left(E\left[(Y_t^N - Y_t^\vartheta)^2\right] + T^2 \sup_{t \leq s \leq T} E[(\bar{u}_s^N - \bar{v}_s)^2]\right).
\] (72)

Similarly we have
\[
E[(Y_T^N - Y_T^\vartheta)^2] \leq 2 \left(2 E\left[(Y_t^N - Y_t^\vartheta)^2\right] + T \sup_{t \leq s \leq T} E[(\bar{u}_s^N - \bar{v}_s)^2]\right)
\]
\[
\leq 2 \left(E\left[(Y_t^N - Y_t^\vartheta)^2\right] + T^2 \sup_{t \leq s \leq T} E[(\bar{u}_s^N - \bar{v}_s)^2]\right).
\] (73)

From Proposition 4.4, Jensen inequality and (68) we have for all $0 \leq t \leq T$,
\[
E\left[\left(\int_t^T \left(\frac{1}{N} \sum_{j \neq i} \bar{u}_s^i j, N - \bar{v}_s\right) ds\right)^2\right]
\]
\[
\leq 2T E\left[\int_t^T \left(\frac{1}{N} \sum_{j = 1}^N \bar{u}_s^i j, N - \bar{v}_s\right)^2 ds\right] + 2T \frac{1}{N^2} E\left[\int_t^T (\bar{u}_s^i N)^2 ds\right]
\]
\[
\leq 2T E\left[\int_t^T (\bar{u}_s^N - \bar{v}_s)^2 ds\right] + T^2 O(N^{-2})
\]
\[
\leq 2T^2 \sup_{t \leq s \leq T} E\left[(\bar{u}_s^N - \bar{v}_s)^2\right] + T^2 O(N^{-2}).
\] (74)

Apply (66), (67), (72)–(74) and Lemma 6.3 to (69) to get that for all $0 \leq t \leq T$,
\[
\sup_{t \leq s \leq T} E\left[(\bar{u}_s^i N - \bar{v}_s^i)^2\right]
\]
\[
\leq 10C_{(48)}(T^2 \lor 1) \left(T^2 \sup_{t \leq s \leq T} E\left[(\bar{u}_s^i N - \bar{v}_s^i)^2\right] + T^2 \sup_{t \leq s \leq T} E\left[(\bar{u}_s^N - \bar{v}_s)^2\right]\right)
\]
\[
+ E\left[(X_t^i N - X_t^i)^2\right] + E\left[(Y_t^i N - Y_t^i)^2\right] + T^2 e^{2eT} O(N^{-2}).
\]

□

Before introducing the next lemma we recall that $\bar{u}_t$ was defined in (68) and that $X_t^i N$ from (11) is given by
\[
\bar{X}_t^i N = \frac{1}{N} \sum_{i = 1}^N X_t^i N.
\] (75)
Recall that $C_{(48)}$ was defined in (48).

**Lemma 6.6.** Assume that (49) holds. Then, for all $0 \leq t \leq T$ we have

$$
\sup_{t \leq s \leq T} E\left[ (\bar{u}_s^N - \bar{\nu}_s)^2 \right] \leq 10C_{(48)}(T^2 \vee 1) \left( T^2 \sup_{t \leq s \leq T} E\left[ (\bar{u}_s^N - \bar{\nu}_s)^2 \right] \right) + E\left[ (X_t^N - X_T^y)^2 \right] + E\left[ (Y_t^N - Y_T^y)^2 \right] + T^2 e^{2\rho T} O(N^{-2}).
$$

**Proof.** The proof follows the same lines as the proof of Lemma 6.5, so we only give the outline.

Recall from (11) that $\bar{Y}_t^N = Y_t^N$. From (9) and (36) we have

$$
\begin{align*}
\bar{u}_t^N - \bar{\nu}_t &= - \int_t^T \left( \frac{\rho k}{2\lambda} (Y_s^N - Y_s^y) - \frac{\kappa y}{2\lambda} \left( \frac{N - 1}{N} \bar{u}_s^N - \bar{\nu}_s \right) - \frac{\phi}{\lambda} \left( X_s^N - X_s^y \right) \right) ds \\
&\quad - \frac{\rho}{2\lambda} \int_t^T \bar{Z}_s^N \ ds + \bar{M}_t^N - \bar{M}_T^N - (\bar{L}_t - \bar{L}_T) \\
&\quad + \frac{\phi}{\lambda} \left( X_T^N - X_T^y \right) - \frac{\kappa}{2\lambda} \left( Y_T^N - Y_T^y \right).
\end{align*}
$$

Using Jensen inequality we get,

$$
\begin{align*}
E \left[ (\bar{u}_t - \bar{\nu}_t)^2 \right] \\
&\leq \frac{C_{(48)}}{2} \left( TE \left[ \int_t^T (Y_s^N - Y_s^y)^2 ds \right] + TE \left[ \int_t^T \left( \frac{N - 1}{N} \bar{u}_s^N - \bar{\nu}_s \right)^2 ds \right] \right) \\
&\quad + E \left[ T \int_t^T (X_s^N - X_s^y)^2 ds + \int_t^T (Z_s^N)^2 ds \right] \\
&\quad + E \left[ (X_T^N - X_T^y)^2 \right] + E \left[ (Y_T^N - Y_T^y)^2 \right] \\
&\quad + 5E \left[ \left( \bar{M}_t^N - \bar{M}_T^N - (\bar{L}_t - \bar{L}_T) \right)^2 \right].
\end{align*}
$$

From Proposition 4.4(ii) it follows that there exists $C > 0$ not depending on $(N, t, T)$ such that

$$
\begin{align*}
E \left[ \int_t^T \left( \frac{N - 1}{N} \bar{u}_s^N - \bar{\nu}_s \right)^2 ds \right] &\leq 2E \left[ \int_t^T (\bar{u}_s^N - \bar{\nu}_s)^2 ds \right] + CT \frac{1}{N^2} \\
&\leq 2T \sup_{t \leq s \leq T} E \left[ (\bar{u}_s^N - \bar{\nu}_s)^2 \right] + CT^2 \frac{1}{N^2}.
\end{align*}
$$
Similarly to (66) and (67) it follows that

$$E \left[ \int_t^T \left( \hat{X}_s - X_s^T \right)^2 \, ds \right] \leq 2 \left( T E[\hat{X}_t - X_t^T] + T^2 \sup_{t \leq s \leq T} E[(\hat{u}^N_s - \bar{v}_s)^2] \right),$$  \tag{78}

and

$$E \left[ (\hat{X}_T - X_T^T)^2 \right] \leq 2 \left( E[(\hat{X}_t - X_t^T)^2] + T^2 \sup_{t \leq s \leq T} E[(\hat{u}^N_s - \bar{v}_s)^2] \right).$$  \tag{79}

From (11) and Lemma 6.2 we get that there exists a constant $C_2 > 0$, not depending on $(N, t, T)$ such that

$$E \left[ \int_t^T (\hat{Z}_s)^2 \, ds \right] \leq C_2 T^2 \frac{1}{N^2}. \tag{80}$$

By applying (77)–(80), (72), (73) and Lemma 6.4 to (76) we get the result. \hfill \Box

Now we are ready to prove Theorem 4.1.

**Proof of Theorem 4.1.** Recall that $C_{(48)}$ was defined in (48). From Lemmas 6.5 and 6.6 it follows that there exists a constant $C_1 > 0$ not depending on $(N, T)$ such that

$$\sup_{t \leq s \leq T} E \left[ \left( \hat{u}_s^{i,N} - \bar{v}_s^i \right)^2 \right] + \sup_{t \leq s \leq T} E \left[ (\hat{u}^N_s - \bar{v}_s)^2 \right] \leq C_1 e^{2\rho T} T^2 \frac{1}{N^2} + 20 C_{(48)} (T^2 \vee 1) \left( T^2 \sup_{t \leq s \leq T} E \left[ \left( \hat{u}_s^{i,N} - \bar{v}_s^i \right)^2 \right] + T^2 \sup_{t \leq s \leq T} E \left[ (\hat{u}^N_s - \bar{v}_s)^2 \right] \right) + E \left[ (X_t^{\hat{u}^{i,N}} - X_t^{i})^2 \right] + E \left[ (\hat{X}_t^{\hat{u}^{i,N}} - X_t^r)^2 \right] + E \left[ (Y_t^{\hat{u}^{i,N}} - Y_t^r)^2 \right] \right), \quad \text{for all } 0 \leq t \leq T. \tag{81}

By (49) we have $\alpha(T) := 1 - 20 C_{(48)} T^2 (T^2 \vee 1) > 0$ by assumption, it holds for all $0 \leq t \leq T$,

$$\alpha(T) \left( \sup_{t \leq s \leq T} E \left[ \left( \hat{u}_s^{i,N} - \bar{v}_s^i \right)^2 \right] + \sup_{t \leq s \leq T} E \left[ (\hat{u}^N_s - \bar{v}_s)^2 \right] \right) \leq C_1 \left( \frac{1}{N^2} + C_2 \left( E \left[ (X_t^{\hat{u}^{i,N}} - X_t^{i})^2 \right] + E \left[ (\hat{X}_t^{\hat{u}^{i,N}} - X_t^r)^2 \right] + E \left[ (Y_t^{\hat{u}^{i,N}} - Y_t^r)^2 \right] \right) \right) \leq \hat{C}_1 \left( \frac{1}{N^2} + \hat{C}_2 \left( \int_0^t E \left[ (\hat{u}_s^{i,N} - \bar{v}_s^i)^2 \right] \, ds + \int_0^t E \left[ (\hat{u}_s^N - \bar{v}_s)^2 \right] \, ds \right) \right) \leq \hat{C}_1 \left( \frac{1}{N^2} + \hat{C}_2 \left( \int_0^t \sup_{s \leq r \leq T} E \left[ (\hat{u}_r^{i,N} - \bar{v}_r)^2 \right] \, ds + \int_0^t \sup_{s \leq r \leq T} E \left[ (\hat{u}_r^N - \bar{v}_r)^2 \right] \, ds \right) \right), \tag{81}

where we used (67), (73) and (79) in the second inequality.
From Gronwall’s inequality we get that there exist constants $C_i(T) > 0$, $i = 3, 4$ not depending on $N$ such that

$$\sup_{0 \leq s \leq T} E \left[ (\tilde{u}_s^{i,N} - \tilde{v}_s^i)^2 \right] + \sup_{0 \leq s \leq T} E \left[ (\tilde{u}_s^N - \tilde{v}_s)^2 \right] \leq C_3(T) \frac{1}{N^2} e^{TC_4(T)},$$

(82)

for all $i = 1, \ldots, N$, $N \geq 2$,

and we conclude the result by Remark 6.1. □

7 PROOF OF Proposition 4.4

We will only prove part (i) of the lemma as the proof of part (ii) follows similar lines. We first introduce the following two lemmas. Recall that $\tilde{u}_t^N$ was defined in (68).

Following (1) we fix a constant $C_P(T) > 0$ such that

$$\sup_{t \in [0, T]} E[(P_t)^2] < C_P(T),$$

(83)

which will be used throughout this section.

Recall that $C_{(48)}$ was defined in (48).

Lemma 7.1. There exists a positive constant $c_N = O(N^{-2})$ such that for all $0 \leq t \leq T$, $i = 1, \ldots, N,

$$\sup_{t \leq s \leq T} E \left[ (\tilde{u}_s^{i,N})^2 \right] \leq \frac{8}{\lambda^2} C_P(T) + 10(C_{(48)} + c_N)(T^2 \vee 1) \left( T^2 \sup_{t \leq s \leq T} E \left[ (\tilde{u}_s^{i,N})^2 \right] + T^2 \sup_{t \leq s \leq T} E \left[ (\tilde{u}_s^N)^2 \right] + E \left[ (X_t^{\tilde{u},N})^2 \right] + E \left[ (Y_t^{\tilde{u},N})^2 \right] + E \left[ \int_0^t (\tilde{u}_s^{i,N})^2 ds \right] \right).$$

Proof. From (8) we have

$$\tilde{u}_t^{i,N} = \frac{\phi}{\lambda} X_t^{\tilde{u},N} - \frac{\kappa}{2\lambda} Y_t^{\tilde{u},N} - \int_t^T \left( \frac{\rho \gamma}{2\lambda} Y_s^{\tilde{u},N} - \frac{\kappa}{2\lambda} \frac{1}{N} \sum_{i \neq j} \tilde{u}_s^{i,N} - \frac{\phi}{\lambda} X_s^{\tilde{u},N} \right) ds$$

$$- \frac{\rho}{2\lambda} \int_t^T Z_s^{\tilde{u},N} ds - M_T^{i,N} + M_t^{i,N}.$$
Using (62) we get that
\[
E\left[(\hat{u}_t^{i,N})^2\right] 
\leq C_{(48)}\left( E\left[(X_T^{i,N})^2\right] + E\left[(Y_T^{i,N})^2\right] + TE\left[\int_T^T (Y_s^{i,N})^2 ds\right] + T E\left[\int_T^T \left(\frac{1}{N} \sum_{j \neq i} \hat{u}_s^{j,N}\right)^2 ds\right] + TE\left[\int_T^T (X_s^{i,N})^2 ds\right] + TE\left[\int_T^T (Z_s^{i,N})^2 ds\right] \right) 
\]

(85)

\[
+ 2E\left[(M_t^{i,N} - M_T^{i,N})^2\right],
\]

where we used Jensen’s inequality with respect to the normalized Lebesgue measure on \([0, T]\), which added factors of \(T\) above.

From (70), Jensen inequality and Fubini’s theorem we have
\[
E\left[\int_T^T (Y_s^{i,N})^2 ds\right] \leq 2E\left[\int_T^T (Y_t^{i,N})^2 + T \int_T^T (\hat{u}_s^{i,N})^2 dr ds\right] \leq 2T \left( E[Y_t^{i,N}]^2 + T^2 \sup_{t \leq s \leq T} E[\hat{u}_s^{i,N}]^2\right). 
\]

(86)

By a similar argument we have
\[
E\left[Y_T^{i,N}\right] \leq 2 \left( E[Y_t^{i,N}]^2 + T^2 \sup_{t \leq s \leq T} E[\hat{u}_s^{i,N}]^2\right). 
\]

(87)

Note that
\[
E\left[\left(\frac{1}{N} \sum_{j \neq i} \hat{u}_r^{j,N}\right)^2\right] \leq 2E\left[\left(\frac{1}{N} \sum_{j=1}^N \hat{u}_r^{j,N}\right)^2\right] + 2 \frac{1}{N^2} E\left[(\hat{u}_r^{i,N})^2\right].
\]

So we get that
\[
E\left[\int_T^T \left(\frac{1}{N} \sum_{j \neq i} \hat{u}_s^{j,N}\right)^2 dr\right] \leq 2T \left( \sup_{t \leq s \leq T} E[\hat{u}_s^{i,N}]^2\right) + \frac{1}{N^2} \sup_{t \leq s \leq T} E\left[\hat{u}_s^{i,N}\right]^2 ds\right). 
\]

(88)

From (64) and Jensen inequality it follows that,
\[
E\left[\int_T^T (X_s^{i,N})^2 ds\right] \leq 2T \left( E[(X_t^{i,N})^2] + T^2 \sup_{t \leq s \leq T} E[(\hat{u}_s^{i,N})^2]\right). 
\]

(89)

By a similar argument we have
\[
E\left[(X_T^{i,N})^2\right] \leq 2 \left( E[(X_t^{i,N})^2] + T^2 \sup_{t \leq s \leq T} E[\hat{u}_s^{i,N}]^2\right). 
\]

(90)
Define

\[ C_{(91)}(T) = 2\kappa^2 \gamma^4 e^{2\rho T}. \]  

(91)

From (58), (62) and Burkholder-Davis-Gundy inequality we get that

\[
E\left[ (M^{i,N}_t - M^{i,N}_l)^2 \right] 
\leq \frac{1}{2\lambda^2} E \left[ \left( 2\phi \int_t^T X^{i,N}_s ds + 2\phi X^{i,N}_T - P_T - E_t \left[ 2\phi \int_t^T X^{i,N}_s ds + 2\phi X^{i,N}_T - P_T \right] \right)^2 \right]
\]

\[ + 8 \left( \frac{\kappa \gamma^2}{2\lambda N} \right) e^{2\rho T} E \left[ \left( \int_t^T E_s \left[ \int_t^T \hat{u}^{i,N}_{\rho r} dr \right] ds \right)^2 \right] \]

\[ \leq \frac{1}{\lambda^2} E \left[ \left( 2\phi \int_t^T X^{i,N}_s ds + 2\phi \int_t^T \hat{u}^{i,N}_s ds - E_t \left[ 2\phi \int_t^T X^{i,N}_s ds + 2\phi \int_t^T \hat{u}^{i,N}_s ds \right] \right)^2 \right]
\]

\[ + C_{(91)} \frac{1}{\lambda^2} \frac{1}{N^2} E \left[ \left( \int_t^T E_s \left[ \int_t^T \hat{u}^{i,N}_{\rho r} dr \right] ds \right)^2 \right] + \frac{1}{\lambda^2} E \left[ (P_T - E_t[P_T])^2 \right]. \]

(92)

Using the conditional Jensen inequality and (83) we get

\[
E \left[ (P_T - E_t[P_T])^2 \right] \leq 4C_{\rho}(T). \]

(93)

Using both Jensen and conditional Jensen inequalities, the tower property and Fubini theorem give

\[
E \left[ \left( \int_t^T E_s \left[ \int_t^T e^{-\rho r} \hat{u}^{i,N}_{\rho r} dr \right] ds \right)^2 \right] \leq T E \left[ \int_t^T \left( E_s \left[ \int_t^T e^{-\rho r} \hat{u}^{i,N}_{\rho r} dr \right] \right)^2 ds \right]
\]

\[ \leq T^2 E \left[ \int_t^T E_s \left[ \int_t^T e^{-2\rho r} \left( \hat{u}^{i,N}_{\rho r} \right)^2 dr \right] ds \right]
\]

\[ \leq T^3 E \left[ \int_0^T \left( \hat{u}^{i,N}_{\rho r} \right)^2 dr \right]
\]

\[ \leq T^3 \left( T \sup_{t \leq s \leq T} E[\left( \hat{u}^{i,N}_s \right)^2] + E \left[ \int_t^t \left( \hat{u}^{i,N}_s \right)^2 ds \right] \right). \]

(94)
Using (89) and the conditional Jensen inequality we get for all $0 \leq t \leq T$,

$$
E\left[\left(2\varphi \int_t^T X_s^{i,N} \, ds + 2\varphi \int_t^T \hat{u}_s^{i,N} \, ds - E_t \left(2\varphi \int_t^T X_s^{i,N} \, ds + 2\varphi \int_t^T \hat{u}_s^{i,N} \, ds\right)\right)^2\right]
\leq 16(\varphi \lor \phi)^2 E\left[\left(\int_t^T X_s^{i,N} \, ds + \int_t^T \hat{u}_s^{i,N} \, ds\right)^2\right]
\leq 32(\varphi \lor \phi)^2 T\left(E\left[\left(\int_t^T X_s^{i,N} \, ds\right)^2\right] + E\left[\int_t^T \left(\hat{u}_s^{i,N}\right)^2 \, ds\right]\right)
\leq 128(\varphi \lor \phi)^2 T^2 \left(E\left[\left(\int_t^T X_s^{i,N} \, ds\right)^2\right] + T(T \lor 1) \sup_{t \leq s \leq T} E\left[\left(\hat{u}_s^{i,N}\right)^2\right]\right).
$$

(95)

By plugging in (123), (94) and (95) to (92) it follows that

$$
E\left[\left(M_t^{i,N} - M_t^{i,N}\right)^2\right]
\leq \frac{4}{\lambda^2} C_p(T) + 128 \left(\frac{\varphi \lor \phi}{\lambda}\right)^2 T^2 \left(E\left[\left(X_t^{i,N}\right)^2\right] + T(T \lor 1) \sup_{t \leq s \leq T} E\left[\left(\hat{u}_s^{i,N}\right)^2\right]\right)
+ C_{(91)} \frac{1}{\lambda^2} \frac{1}{N^2} T^3 \left(T \sup_{t \leq s \leq T} E\left[\left(\hat{u}_s^{i,N}\right)^2\right] + E\left[\int_0^t \left(\hat{u}_s^{i,N}\right)^2 \, ds\right]\right).
$$

(96)

From (56) and (57) and by following the same lines leading to (94), we get for all $0 \leq t \leq T$,

$$
E\left[\int_t^T (Z_s^{i,N})^2 \, ds\right] \leq C_{(91)} \frac{1}{\lambda^2} \frac{1}{N^2} T^3 \left(T \sup_{t \leq s \leq T} E\left[\left(\hat{u}_s^{i,N}\right)^2\right] + E\left[\int_0^t \left(\hat{u}_s^{i,N}\right)^2 \, ds\right]\right).
$$

(97)

Applying (86)–(90), (96) and (97) to (85), we get that there exists $c_N = O(N^{-2})$ such that for all $0 \leq t \leq T$,

$$
\sup_{t \leq s \leq T} E\left[\left(\hat{u}_s^{i,N}\right)^2\right]
\leq \frac{8}{\lambda^2} C_p(T) + 10(C_{(48)} + c_N)(T^2 \lor 1) \left(T^2 \sup_{t \leq s \leq T} E\left[\left(\hat{u}_s^{i,N}\right)^2\right] + T^2 \sup_{t \leq s \leq T} E\left[\left(\hat{u}_s^{i,N}\right)^2\right]\right)
+ E\left[\left(X_t^{i,N}\right)^2\right] + E\left[\left(Y_t^{i,N}\right)^2\right] + E\left[\int_0^t \left(\hat{u}_s^{i,N}\right)^2 \, ds\right].
$$

□

Before we introduce the next lemma we recall that $\bar{X}^{i,N}$ was defined in (75).
Lemma 7.2. There exists a positive constant $c_N = O(N^{-2})$ such that

$$
\sup_{t \leq T} E\left[ (\bar{u}_s^{i,N})^2 \right] \leq \frac{8}{\lambda^2} C_p(T) + 10(C_{(48)} + c_N)(T^2 \vee 1) \left( T^2 \sup_{t \leq T} E\left[ (\bar{u}_s^{i,N})^2 \right] + T^2 \sup_{t \leq T} E\left[ (\bar{u}_s^N)^2 \right] \right) 
$$

$$
+ E\left[ (\tilde{X}_t^{i,N})^2 \right] + E\left[ (Y_t^{i,N})^2 \right], \text{ for all } 0 \leq t \leq T.
$$

The proof of Lemma 7.2 follows the same lines as the proof of Lemma 7.1, hence it is omitted.

Now we are ready to prove Proposition 4.4.

Proof of Proposition 4.4. From Lemmas 7.1 and 7.2 we get that there exists a constant $c_N = O(N^{-2})$ such that for all $0 \leq t \leq T$ and $i = 1, \ldots, N$ we have

$$
\sup_{t \leq T} E\left[ (\bar{u}_s^{i,N})^2 \right] + \sup_{t \leq T} E\left[ (\bar{u}_s^N)^2 \right] 
$$

$$
\leq \frac{16}{\lambda^2} C_p(T) + 20(C_{(48)} + c_N)(T^2 \vee 1) \left( T^2 \sup_{t \leq T} E\left[ (\bar{u}_s^{i,N})^2 \right] + T^2 \sup_{t \leq T} E\left[ (\bar{u}_s^N)^2 \right] \right) 
$$

$$
+ E\left[ (\tilde{X}_t^{i,N})^2 \right] + E\left[ (X_t^{i,N})^2 \right] + E\left[ (Y_t^{i,N})^2 \right] + E\left[ \int_0^t \left( \bar{u}_s^{i,N} \right)^2 ds \right].
$$

By assumption (49) we can choose $N$ large enough so that $g(T, N) = 20(C_{(48)} + c_N)(T^2 \vee 1)T^2 < 1$. Then using Jensen inequality we get for all $0 \leq t \leq T$, $N \geq 1$ and $i = 1, \ldots, N$:

$$
(1 - g(T, N)) \left( \sup_{t \leq T} E\left[ (\bar{u}_s^{i,N})^2 \right] + \sup_{t \leq T} E\left[ (\bar{u}_s^N)^2 \right] \right) 
$$

$$
\leq C_1(T) \left( E\left[ (\tilde{X}_t^{i,N})^2 \right] + E\left[ (X_t^{i,N})^2 \right] + E\left[ (Y_t^{i,N})^2 \right] + E\left[ \int_0^t \left( \bar{u}_s^{i,N} \right)^2 ds \right] \right) + C_2(T)
$$

$$
\leq \tilde{C}_1(T) \left( (\tilde{X}_0^{i,N})^2 + (X_0^{i,N})^2 + \int_0^t E[(\bar{u}_s^{i,N})^2] ds + \int_0^t E[(\bar{u}_s^{i,N})^2] ds \right) + \tilde{C}_2(T)
$$

$$
\leq \tilde{C}_1(T) \left( \int_0^t \sup_{s \leq T} E[\bar{u}_s^{i,N}]^2 ds + \int_0^t \sup_{s \leq T} E[\bar{u}_s^{i,N}]^2 ds \right) + \tilde{C}_2(T),
$$

where we used (47), (64) and (70) in the second inequality and recall that $Y_0^{i,N} = y$.

Recall that $\tilde{C}_1(T)$, $\tilde{C}_2(T)$ are not depending on $(N, i, t)$, so we get from Gronwall’s lemma that

$$
\sup_{N \geq 2} \sup_{i = 1, \ldots, N} \sup_{0 \leq t \leq T} \left( E\left[ (\bar{u}_s^{i,N})^2 \right] + E\left[ (\bar{u}_s^N)^2 \right] \right) 
$$

$$
\leq (1 - g(T, N))^{-1} \tilde{C}_2(T)e^{(1 - g(T, N))^{-1}C_1(T)},
$$

which concludes the proof. \qed
8 | PROOF OF THEOREM 4.7

In order to prove the desired convergence result stated in Theorem 4.7, we will derive alternative representations of the solutions to the FBSDE systems in (9), (36), (10) and (37) presented in Sections 2.3 and 3.2, which are also of independent interest; see Propositions 8.4, 8.9, 8.12, 8.17 below.

Notation.
In the following, for any matrix \( A = (a_{ij})_{1 \leq i \leq n, 1 \leq j \leq m} \in \mathbb{R}^{n \times m} \) we denote \( |A| = (|a_{ij}|)_{1 \leq i \leq n, 1 \leq j \leq m} \).

8.1 | The finite player aggregated FBSDE in (9)

We fix an integer \( N \geq 1 \) and define the matrices

\[
\overline{F}_{11} = \begin{pmatrix} 0 & 0 \\ 0 & -\rho \end{pmatrix}, \quad \overline{F}_{12} = \begin{pmatrix} -1 & 0 \\ \gamma & 0 \end{pmatrix}, \quad \overline{F}_{21} = \begin{pmatrix} -\frac{\Phi}{\lambda} & \frac{\kappa E}{2\lambda} \\ 0 & 0 \end{pmatrix}, \quad \overline{F}_{22} = \begin{pmatrix} -\frac{2\lambda N}{\gamma \kappa} & \frac{\rho}{\lambda} \\ \frac{2\lambda N}{\gamma \kappa} & \frac{\rho}{\lambda} \end{pmatrix}. \tag{99}
\]

Recall that \( \overline{F}_{00} \) was defined in (51).

As the first step, let \( W^N \in C^1([0,T],\mathbb{R}^{2\times2}) \) be a solution of the following Riccati differential equation

\[
W_t^N - \overline{F}_{21} - \overline{F}_{22} W_t^N + W_t^N \overline{F}_{11} + W_t^N \overline{F}_{12} W_t^N = 0, \quad W_T^N = \overline{F}_{00}. \tag{100}
\]

In fact, the solution \( W^N \) of (100) can be characterized via the solution to following matrix-valued linear system, where \( P^N, Q^N \in C^1([0,T],\mathbb{R}^{2\times2}) \) are such that

\[
\frac{d}{dt} \begin{pmatrix} Q_t^N \\ P_t^N \end{pmatrix} = - \begin{pmatrix} \overline{F}_{11} & \overline{F}_{12} \\ \overline{F}_{21} & \overline{F}_{22} \end{pmatrix} \begin{pmatrix} Q_t^N \\ P_t^N \end{pmatrix}, \quad (0 \leq t \leq T) \tag{101}
\]

and \( \begin{pmatrix} Q_0^N \\ P_0^N \end{pmatrix} = \begin{pmatrix} I \\ \overline{F}_{00} \end{pmatrix} \). Moreover, the matrix-valued linear system in (101) can be solved by computing a matrix exponential. We specify these results in the following two lemmas.

**Lemma 8.1.** Let

\[
\overline{F}^N = \begin{pmatrix} \overline{F}_{11}^N & \overline{F}_{12}^N \\ \overline{F}_{21}^N & \overline{F}_{22}^N \end{pmatrix} \in \mathbb{R}^{4\times4}. \tag{102}
\]

Then

\[
\overline{K}^N(t) = \exp(-\overline{F}^N \cdot t) \cdot \begin{pmatrix} I \\ \overline{F}_{00} \end{pmatrix} \in \mathbb{R}^{4\times2} \tag{103}
\]

solves (101).

**Lemma 8.2.** Let \( P^N, Q^N \) satisfy (101) and assume that

\[
\liminf_{N \geq 1} \inf_{t \in [0,T]} |\det(Q_t^N)| > 0. \tag{104}
\]
Then the following holds:

(i) \[ W^N_t = P^N_{T-t} (Q^N_{T-t})^{-1} \quad (0 \leq t \leq T) \] (105) solves (100);

(ii) \[
\sup_{N \geq 1} \sup_{t \in [0,T]} |W^N_t| < \infty.
\] (106)

Proof. (i) For the sake of readability we suppress the dependence on $N$. Computing the derivative in (105), we obtain

\[
W_t = \frac{d}{dt} (P_{T-t} Q^{-1}_{T-t}) = \left( \frac{d}{dt} P_{T-t} \right) Q^{-1}_{T-t} + P_{T-t} \left( \frac{d}{dt} Q^{-1}_{T-t} \right)
\]

\[
= \left( \frac{d}{dt} P_{T-t} \right) Q^{-1}_{T-t} - P_{T-t} Q^{-1}_{T-t} \left( \frac{d}{dt} Q^{-1}_{T-t} \right) Q^{-1}_{T-t}
\]

\[
= (\bar{F}_{21} Q_{T-t} + \bar{F}_{22} P_{T-t}) Q^{-1}_{T-t} - P_{T-t} Q^{-1}_{T-t} (\bar{F}_{11} Q_{T-t} + \bar{F}_{12} P_{T-t}) Q^{-1}_{T-t}
\]

\[
= \bar{F}_{21} + \bar{F}_{22} W_t - W_t \bar{F}_{11} - W_t \bar{F}_{12} W_t
\]

and hence (i).

(ii) From (104) and the explicit formula for the inverse of $2 \times 2$ matrices, also known as the cofactor equation, it follows that

\[
\sup_{N \geq 1} \sup_{t \in [0,T]} |(Q^N_t)^{-1}| < \infty.
\]

Moreover, from Lemma 8.1 together with (99) we get

\[
\sup_{N \geq 1} \sup_{t \in [0,T]} |P^N_t| < \infty,
\]

and thus, using with (105), we obtain (ii). \qed

Remark 8.3. Note that from Lemmas 8.1 and 8.2 the existence of a classical solution $(W^N_t)_{t \in [0,T]}$ to (100) follows under the assumption $\det(Q^N_t) \neq 0$ for all $t \in [0, T]$.

Next, let $(f^N_t)_{t \in [0,T]}$ be the $L^2$ solution to the two-dimensional linear BSDE

\[
df^N_t = \left( \bar{F}_{22} - W^N_t \bar{F}_{12} \right) f^N_t \, dt + \left( \frac{dP_t}{2} + dM^{1,N}_t \right), \quad f^N_T = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]

(107)

for some square integrable martingales $M^{1,N}, M^{2,N}$. Using $W^N$ satisfying (100) and $f^N$ satisfying (107) define the process

\[
\forall^N_t := W^N_t \forall^N_t + f^N_t \quad (0 \leq t \leq T),
\]

(108)
where $\mathbf{x}^N$ is the solution to the linear ODE with random coefficients
\begin{equation}
\mathbf{d}\mathbf{x}^N_t = (\mathbf{F}^{11}_t \mathbf{x}^N_t + \mathbf{F}^{12}_t (W^N_t \mathbf{x}^N_t + f^N_t))dt = (\mathbf{F}^{11}_t \mathbf{x}^N_t + \mathbf{F}^{12}_t \mathbf{y}^N_t)dt, \quad \mathbf{x}^N_0 = \begin{pmatrix} \mathbf{x}^N \end{pmatrix}_y
\end{equation}
and initial value $\hat{x}^N$ as in (9).

**Proposition 8.4.** The process $(\mathbf{x}^N, \mathbf{y}^N)^\top$ from (108)–(109) satisfies the finite player aggregated FBSDE system in (9), that is, using the notation from (9) we have
\begin{equation}
\mathbf{y}^N_t = \begin{pmatrix} \mathbf{u}^N_t \cr \mathbf{Z}^N_t \end{pmatrix}, \quad \mathbf{x}^N_t = \begin{pmatrix} \mathbf{x}^N_t \cr \mathbf{y}^N_t \end{pmatrix}.
\end{equation}

**Proof.** For the sake of readability we suppress the dependence on $N$ throughout the proof. Applying Itô’s formula in (108), using (109), (100) and (107), we obtain for all $t \in [0, T]$,
\begin{align*}
\mathbf{d}\mathbf{y}_t &= \mathbf{W}_t \mathbf{x}_t dt + \mathbf{W}_t \mathbf{d}\mathbf{x}_t + \mathbf{d}f_t = \mathbf{W}_t \mathbf{x}_t dt + \mathbf{W}_t (\mathbf{F}^{11}_t \mathbf{x}_t + \mathbf{F}^{12}_t (W_t \mathbf{x}_t + f_t))dt + \mathbf{d}f_t \\
&= (\mathbf{W}_t + \mathbf{F}^{11}_t) \mathbf{x}_t dt + \mathbf{W}_t \mathbf{F}^{12}_t f_t dt + \mathbf{d}f_t \\
&= (\mathbf{F}^{21}_t + \mathbf{F}^{22}_t W_t) \mathbf{x}_t dt + \mathbf{W}_t \mathbf{F}^{12}_t f_t dt + (\mathbf{F}^{22}_t - \mathbf{W}_t \mathbf{F}^{12}_t) f_t dt + \left( \frac{dP_t}{\mathbf{Z}^N_t} + \frac{dM^1_t}{dM^2_t} \right) \\
&= \mathbf{F}^{21}_t \mathbf{x}_t dt + \mathbf{F}^{22}_t (W_t \mathbf{x}_t + f_t) dt + \left( \frac{dP_t}{\mathbf{Z}^N_t} + \frac{dM^1_t}{dM^2_t} \right) \\
&= \mathbf{F}^{21}_t \mathbf{x}_t dt + \mathbf{F}^{22}_t \mathbf{y}_t dt + \left( \frac{dP_t}{\mathbf{Z}^N_t} + \frac{dM^1_t}{dM^2_t} \right).
\end{align*}

Therefore, equations (109) and (111) yield
\begin{align}
\mathbf{d}\begin{pmatrix} \mathbf{x}_t \\ \mathbf{y}_t \end{pmatrix} &= \begin{pmatrix} \mathbf{F}^{11}_t \\ \mathbf{F}^{12}_t \end{pmatrix} \begin{pmatrix} \mathbf{x}_t \\ \mathbf{y}_t \end{pmatrix} dt + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \left( \frac{dP_t}{\mathbf{Z}^N_t} + \frac{dM^1_t}{dM^2_t} \right) \\
&= \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -\rho & -\gamma & 0 \\ \frac{\phi}{\lambda} & \frac{\kappa}{\lambda} & -\frac{\gamma(N-1)}{\mathbf{Z}^N_t} & \frac{\rho}{\mathbf{Z}^N_t} \\ 0 & 0 & \frac{\gamma}{\mathbf{Z}^N_t} & \mathbf{y}^N_t \end{pmatrix} \begin{pmatrix} \mathbf{x}^1_t \\ \mathbf{x}^2_t \\ \mathbf{y}^1_t \\ \mathbf{y}^2_t \end{pmatrix} dt + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \left( \frac{dP_t}{\mathbf{Z}^N_t} + \frac{dM^1_t}{dM^2_t} \right),
\end{align}

where $\mathbf{y}_T = W_T \mathbf{x}_T + f_T = \mathbf{F}^{10}_0 \mathbf{x}_T = \begin{pmatrix} \frac{\phi}{\lambda} \mathbf{x}_T^1 & -\rho \mathbf{x}_T^2 & -\gamma \mathbf{x}_T^1 + \frac{\rho}{\mathbf{Z}^N_t} \mathbf{x}_T^2 \end{pmatrix}$. In other words, the process $(\mathbf{x}, \mathbf{y})$ from (109) and (108), that is, $(\mathbf{x}_t, \mathbf{y}_t)^\top = (\mathbf{x}_t^1, \mathbf{x}_t^2, \mathbf{y}_t^1, \mathbf{y}_t^2)^\top$ satisfies the finite player aggregated FBSDE system in (9).
8.2 Convergence of (9) to the Mean Field FBSDE (36)

Now, we first repeat the analysis above to get an alternative representation for the solution to the infinite-player mean field FBSDE system in (36). Then, using the latter, we show that it is indeed the limit of the solution of (9) derived in Proposition 8.4 above as $N$ goes to infinity.

To this end, introduce the matrix

$$F_{22}^\infty := \lim_{N \to \infty} F_{22}^N = \begin{pmatrix} -\gamma \nu & \rho \\ 2\lambda & 2\lambda \\ 0 & \rho \end{pmatrix}$$

and, instead of (100), let $W \in C^1([0, T], \mathbb{R}^{2\times 2})$ be the solution of the Riccati differential equation

$$\dot{W}_t - F_{21} \cdot W_t + W_t F_{11} + W_t F_{12} W_t = 0, \quad W_T = F_{00}. \quad (113)$$

Similar to above, the solution $W$ of (113) can be characterized via the solution of a matrix-valued linear system.

**Lemma 8.5.** Let $P, Q \in C^1([0, T], \mathbb{R}^{2\times 2})$ such that

$$\frac{d}{dt} \begin{pmatrix} Q_t \\ P_t \end{pmatrix} = - \begin{pmatrix} F_{11} \\ F_{21} \end{pmatrix} \begin{pmatrix} Q_t \\ P_t \end{pmatrix} \quad (0 \leq t \leq T) \quad (114)$$

and $\begin{pmatrix} Q_0 \\ P_0 \end{pmatrix} = \begin{pmatrix} I \\ F_{00} \end{pmatrix}$. Assume that

$$\inf_{t \in [0, T]} |\det(Q_t)| > 0. \quad (115)$$

Then the following holds:

(i) $W_t = P_{T-t} Q_{T-t}^{-1}$, $0 \leq t \leq T$, \quad (116)

solves (113);

(ii) $\sup_{t \in [0, T]} |W_t| < \infty$.

**Proof.** The proof follows the same lines as the proof of Lemma 8.2 above. \qed

Again, as above, the matrix-valued linear system in (114) can be solved via a matrix exponential.

**Lemma 8.6.** Let

$$F = \begin{pmatrix} F_{11} \\ F_{21} \\ F_{12} \\ F_{22} \end{pmatrix} \in \mathbb{R}^{4\times 4} \quad (117)$$
Then

$$\bar{K}(t) = \exp(-F \cdot t) \cdot \left( \frac{I}{F_{00}} \right) \in \mathbb{R}^{4 \times 2}$$  \hspace{1cm} (118)

solves (114).

The convergence of $W^N$ solving (100) to $W$ solving (113) as $N$ goes to infinity is readily given by

**Lemma 8.7.** Assume (104) and (115). Let $W^N$ solve (100) and $W$ solve (113). Then

$$\lim_{N \to \infty} \sup_{t \in [0, T]} |W^N_t - W_t| = 0.$$  \hspace{1cm} (119)

**Proof.** Note that for any $N \geq 1$ and $t \in [0, T]$ we have

$$Q_t^{-1} - (Q_t^N)^{-1} = (Q_t^N - Q_t)(Q_t)^{-1}.$$  \hspace{1cm} (120)

From (104) and (115) it follows that

$$\lim_{N \geq 1} \sup_{t \in [0, T]} |(Q_t^N)^{-1}| < \infty, \quad \sup_{t \in [0, T]} |(Q_t)^{-1}| < \infty.$$  \hspace{1cm} (121)

Since $\bar{F}^N$ in (102) converges to $\bar{F}$ in (117) as $N \to \infty$, it follows that $\bar{K}^N$ in (103) converges to $\bar{K}$ in (118) as $N \to \infty$. But this implies that the solution $Q^N, P^N$ of (101) converges to the solution $Q, P$ of (114) uniformly on $[0, T]$. Moreover, we get from (120) and (121) that

$$\lim_{N \to \infty} \sup_{t \in [0, T]} |Q_t^{-1} - (Q_t^N)^{-1}| = 0.$$  \hspace{1cm} (122)

Together with Lemma 8.2(i) and Lemma 8.5(i) we get the result. \(\square\)

Next, let $(f_t)_{t \in [0, T]}$ be the $L^2$ solution to the two-dimensional linear BSDE

$$df_t = (\bar{F}_{22} - W_t \bar{F}_{12})f_t dt + \left( \frac{dP_t}{2\lambda} + dM^1_t \right), \quad f_T = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$  \hspace{1cm} (122)

for some square integrable martingales $M^1, M^2$. The convergence of $f^N$ solving (107) to $f$ solving (122) as $N$ tends to infinity is established in

**Proposition 8.8.** Let $f^N$ be the solution to (107) and $f$ be the solution to (122). Then we have

$$\lim_{N \to \infty} \sup_{t \in [0, T]} |f^N_t - f_t| = 0, \quad P - a.s.$$  \hspace{1cm} (123)

**Proof.** From (99) and Lemma 8.7 it follows that

$$\lim_{N \to \infty} \sup_{t \in [0, T]} |\bar{F}_{22}^N - W_t^N \bar{F}_{12} - (\bar{F}_{22}^\infty - W_t \bar{F}_{12})| = 0.$$  \hspace{1cm} (124)
As a consequence, using standard stability results for linear BDSEs (see, e.g., Theorem 2 in Bahlali et al. (2017)), we get
\[
\lim_{N \to \infty} E \left[ \sup_{t \in [0,T]} |f^N_t - f_t| \right] = 0. \tag{123}
\]
The result then follows by applying Fatou’s lemma and using (123).

Finally, using \( W \) satisfying (113) and \( f \) satisfying (122), define similar to (108) above the process
\[
\forall_t := W_t X_t + f_t \quad (0 \leq t \leq T), \tag{124}
\]
where \( X \) is the solution to the linear ODE with random coefficients
\[
dX_t = (\bar{F}_{11}X_t + \bar{F}_{12}(W_t X_t + f_t))dt = (\bar{F}_{11}X_t + \bar{F}_{12}\forall_t)dt, \quad X_0 = \begin{pmatrix} \bar{x} \\ y \end{pmatrix}, \tag{125}
\]
and initial value \( \bar{x} \) as in (36).

**Proposition 8.9.** The process \((X, \forall) \top\) in (124)–(125) satisfies the infinite player mean field FBSDE system in (36), that is, using the notation from (36) we have
\[
\forall_t = \begin{pmatrix} \bar{y}_t \\ 0 \end{pmatrix} \quad \text{and} \quad X_t = \begin{pmatrix} \bar{X}^\phi_t \\ \bar{Y}^\phi_t \end{pmatrix}. \tag{126}
\]

The proof of Proposition 8.9 is the same as the proof of Proposition 8.4 and hence omitted.

### 8.3 Proof of Theorem 4.7(i)

We introduce the following norms: For any integer \( d \geq 1 \) and any vector \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \) we define
\[
\|x\|_1 = \sum_{i=1}^d |x_i|;
\]
for any matrix \( C = (c_{ij})_{1 \leq i, j \leq d} \in \mathbb{R}^{d \times d} \) we define
\[
\|C\|_{\max} = \max_{1 \leq i, j \leq d} |c_{ij}|.
\]

**Proof of Theorem 4.7 (i).** Recall that from assumptions (47) and (27) we have \( X_0^{i,N} = X_0^{i,e} = x^i \) for all \( i = 1, \ldots, N, N \geq 1, \) and \( \bar{x}^N \to \bar{x} \) as \( N \to \infty. \) One can easily verify from (101)–(103), (114), (118) that (53) implies (104) and (115). We will show that
\[
\lim_{N \to \infty} \sup_{t \in [0,T]} \left( |X_t^N - X_t| + |\forall_t^N - \forall_t| \right) = 0 \quad \text{a.s.,} \tag{127}
\]
which together with (110) and (126) will imply the claim in Theorem 4.7(i).
From (109) and (125) we have

\[
|X_t^N - X_t| \leq |\tilde{x}^N - \tilde{x}| + \int_0^T |F_{11}| |X_s^N - X_s| \, ds
\]

\[
+ \int_0^t |F_{12}| |W_s^N X_s^N - W_s X_s| \, ds + \int_0^T |F_{12}| |f_s^N - f_s| \, ds
\]

\[
\leq |\tilde{x}^N - \tilde{x}| + \int_0^T |F_{11}| |X_s^N - X_s| \, ds
\]

\[
+ \int_0^t |F_{12}| |W_s^N X_s^N - W_s^N X_s| \, ds + \int_0^t |F_{12}| |W_s X_s - W_s X_s| \, ds
\]

\[
+ \int_0^t |F_{12}| |f_s^N - f_s| \, ds.
\]

Multiplying both sides of (128) by (1,1) from the left, we get

\[
\|X_t^N - X_t\|_{1,\epsilon} \leq \left(\|\tilde{x}^N - \tilde{x}\|_{1,\epsilon} + 2|F_{11}|_{\text{max}} \int_0^T \|X_s^N - X_s\|_{1,\epsilon} \, ds\right.
\]

\[
+ 4|F_{12}|_{\text{max}} \sup_{N \geq 1} \sup_{t \in [0,T]} \|W_t^N\|_{\text{max}} \int_0^t \|X_s^N - X_s\|_{1,\epsilon} \, ds
\]

\[
+ 8|F_{12}|_{\text{max}} \sup_{t \in [0,T]} \|X_t\|_{\epsilon_1} \int_0^t \|W_s^N - W_s\|_{\text{max}} \, ds
\]

\[
+ 2|F_{12}|_{\text{max}} \int_0^t \|f_s^N - f_s\|_{\epsilon_1} \, ds.
\]

(129)

Now, let \(\epsilon > 0\) be arbitrary small. From Lemma 8.7, Proposition 8.8 and (129) it follows that for all \(N\) sufficiently large

\[
\sup_{s \in [0,t]} \|X_s^N - X_s\|_{1,\epsilon} \leq \epsilon + C_1 \int_0^T \sup_{r \in [0,s]} \|X_r^N - X_r\|_{1,\epsilon} \, ds
\]

\[
+ TC_2 \epsilon,
\]

(130)

where \(C_i > 0\) are constants not depending on \(N\) and \(t\). It then follows from Gronwall’s lemma that there exists a constant \(C(T) > 0\) such that

\[
\sup_{s \in [0,T]} \|X_s^N - X_s\|_{1,\epsilon} \leq C(T)\epsilon.
\]

(131)

Since \(\epsilon > 0\) is arbitrary small, the first part in (127) follows from (131).

Finally, using Lemma 8.7, Proposition 8.8, (108), (124) and (131) we get

\[
\lim_{N \to \infty} \sup_{s \in [0,T]} \|Y_s^N - Y_s\|_{1,\epsilon} = 0,
\]

(132)

which together with (131) completes the proof. \(\square\)
8.4  Proof of Theorem 4.7(ii)

The proof of Theorem 4.7(ii) follows the same rationale as the proof of part (i) presented above. Therefore, we only provide the outline in order to avoid repetition. Specifically, the idea is once more to derive alternative representations for the solutions to the FBSDEs in (10) and (37), and then to argue that the former converges to the latter as \( N \) goes to infinity.

We start with fixing an integer \( N \geq 1 \) and defining the matrices

\[
\bar{F}_{12} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \bar{F}_{21} = \begin{pmatrix} -\phi & 0 \\ \lambda & 0 \end{pmatrix}, \quad \bar{F}_{22}^N = \begin{pmatrix} \frac{2\lambda N}{N} & \rho \\ \frac{2\lambda}{N} & \rho \end{pmatrix}.
\]

Recall that \( \bar{F}_{00} \) was defined in (51). Let \( \bar{W}^N_t \in C([0,T],\mathbb{R}^{2\times2}) \) be a solution to the following Riccati differential equation

\[
\dot{\bar{W}}^N_t - \bar{F}_{21} \bar{W}^N_t + \bar{W}^N_t \bar{F}_{22}^N = 0, \quad \bar{W}^N_T = \bar{F}_{00}.
\]

Similar to above, the solution \( \bar{W}^N_t \) of (133) can be characterized via the solution to following matrix-valued linear system, where \( \bar{P}^N, \bar{Q}^N \in C^1([0,T],\mathbb{R}^{2\times2}) \) are such that

\[
\frac{d}{dt} \begin{pmatrix} \bar{Q}_t^N \\ \bar{P}_t^N \end{pmatrix} = - \begin{pmatrix} 0 & \bar{F}_{12} \\ \bar{F}_{21} & \bar{F}_{22}^N \end{pmatrix} \begin{pmatrix} \bar{Q}_t^N \\ \bar{P}_t^N \end{pmatrix} \quad (0 \leq t \leq T)
\]

and \( \begin{pmatrix} \bar{Q}_0^N \\ \bar{P}_0^N \end{pmatrix} = \begin{pmatrix} I \\ \bar{F}_{00} \end{pmatrix} \). Moreover, the matrix-valued linear system in (134) can again be solved via a matrix exponential. We collect these results in the following two lemmas which are the counterparts to Lemmas 8.5 and 8.6 above.

**Lemma 8.10.** Let

\[
\bar{F}^N = \begin{pmatrix} 0 & \bar{F}_{12}^N \\ \bar{F}_{21}^N & \bar{F}_{22}^N \end{pmatrix} \in \mathbb{R}^{4\times4}.
\]

Then

\[
R^N(t) = \exp(-\bar{F}^N \cdot t) \cdot \begin{pmatrix} I \\ \bar{F}_{00} \end{pmatrix} \in \mathbb{R}^{4\times2}
\]

solves (134).

**Lemma 8.11.** Let \( \bar{P}^N, \bar{Q}^N \) satisfy (134) and assume that

\[
\liminf_{N \geq 1} \inf_{t \in [0,T]} |\det(\bar{Q}_t^N)| > 0.
\]

Then the following holds:

(i)

\[
\bar{W}_t^N = \bar{P}_{T-t}^N (\bar{Q}_{T-t}^N)^{-1} \quad (0 \leq t \leq T),
\]

solves (133);
(ii) \[
\sup_{N \geq 1} \sup_{t \in [0, T]} |\tilde{W}^N_t| < \infty.
\]

Next, let \(\tilde{Y}^N := \chi^{N,2}\) denote the second component of the solution to the linear SDE in (109), which is given in (110). In addition, let \((g^N_t)_{t \in [0, T]}\) be the \(L^2\) solution to the two-dimensional linear BSDE

\[
dg^N_t = (\hat{F}^N_{22} - \tilde{W}_t^N \hat{F}_{12}) g^N_t \, dt + \left( \begin{array}{c} \frac{d(P_t - \kappa Y^N_t)}{2\lambda} + d\tilde{M}^N_t \\ 0 \end{array} \right), \quad g^N_T = \left( \begin{array}{c} -\kappa \tilde{Y}^N_T \\ 0 \end{array} \right)
\]

(139)

for some square integrable martingale \(\tilde{M}\). Using \(\tilde{W}\) satisfying (133) and \(g^N\) satisfying (139) define for each \(i \in \{1, \ldots, N\}\) the process

\[
\tilde{Y}_i^N, \tilde{X}_i^N := \tilde{W}_t^N \tilde{X}_i^N + g^N_t \quad (0 \leq t \leq T),
\]

(140)

where \(\tilde{X}_i^N\) is the solution to the linear ODE with random coefficients

\[
d\tilde{X}_i^N = (\hat{F}_{12}(\tilde{W}_t^N \tilde{X}_i^N + g^N_t)) \, dt = (\hat{F}_{12} \tilde{X}_i^N) \, dt, \quad \tilde{X}_i^0 = \left( \begin{array}{c} x^{i,N}_0 \\ 0 \end{array} \right)
\]

(141)

and initial value \(x^{i,N}\) as in (10).

**Proposition 8.12.** The process \((\tilde{X}_i^N, \tilde{Y}_i^N)\) in (140)–(141) satisfies the \(i\)-th player FBSDE system in (10), that is, using the notation from (10) we have

\[
\tilde{Y}_t^i = \left( \begin{array}{c} u_t^{i,N} \\ Z_t^{i,N} \end{array} \right) \quad \text{and} \quad \tilde{X}_t^i = \left( \begin{array}{c} x_t^{i,u,N} \\ 0 \end{array} \right).
\]

The proof of Proposition 8.12 follows the same lines as the proof of Proposition 8.4 and is thus omitted.

Finally, as in the proof of Theorem 4.7(i) above, we also derive and alternative representation of the solution to (37) in order to prove the desired convergence. To achieve this, we introduce the matrix

\[
\hat{F}_{22}^\infty := \lim_{N \to \infty} \hat{F}_{22}^N = \left( \begin{array}{cc} 0 & \frac{\rho}{2\lambda} \\ 0 & 0 \end{array} \right)
\]

(142)

and, instead of (133), let \(\tilde{W} \in C([0, T], \mathbb{R}^{2 \times 2})\) be the solution to the following Riccati differential equation

\[
\tilde{W}_t - \hat{F}_{21} - \hat{F}_{22}^\infty \tilde{W}_t + \tilde{W}_t \hat{F}_{12} \tilde{W}_t = 0, \quad \tilde{W}_T = \hat{F}_{00}.
\]

(143)

Again, \(\tilde{W}\) solving (143) can be characterized by the solution of a matrix-valued linear system, which in turn can be solved via a matrix exponential.
Lemma 8.13. Let \( \bar{P}, \bar{Q} \in C^1([0, T], \mathbb{R}^{2 \times 2}) \) such that
\[
\frac{d}{dt} \begin{pmatrix} \bar{Q}_t \\ \bar{P}_t \end{pmatrix} = -\begin{pmatrix} 0 & \hat{F}_{12} \\ \hat{F}_{21} & \hat{F}_{22} \end{pmatrix} \begin{pmatrix} \bar{Q}_t \\ \bar{P}_t \end{pmatrix} \quad \text{for all } t \in [0, T]
\] (144)
and \( \begin{pmatrix} \bar{Q}_0 \\ \bar{P}_0 \end{pmatrix} = \begin{pmatrix} I \\ \hat{F}_{00} \end{pmatrix} \). Assume that
\[
\inf_{t \in [0, T]} |\det(\bar{Q}_t)| > 0.
\] (145)
Then
\[
\bar{W}_t = \bar{P}_{T-t} \bar{Q}_{T-t}^{-1}
\] solves (143).

Lemma 8.14. Let
\[
\hat{F} = \begin{pmatrix} 0 & \hat{F}_{12} \\ \hat{F}_{21} & \hat{F}_{22} \end{pmatrix} \in \mathbb{R}^{4 \times 4}
\] (146)
Then
\[
\hat{K}(t) = \exp(-\hat{F} \cdot t) \cdot \begin{pmatrix} I \\ \hat{F}_{00} \end{pmatrix} \in \mathbb{R}^{4 \times 2}
\] (147)
solves (144).

The convergence of \( \bar{W}^N \) solving (133) to \( \bar{W} \) solving (143) as \( N \to \infty \) is given by

Lemma 8.15. Assume (137) and (145). Let \( \bar{W}^N \) solve (133) and \( \bar{W} \) solve (143). Then
\[
\lim_{N \to \infty} \sup_{t \in [0, T]} |\bar{W}^N_t - \bar{W}_t| = 0.
\] (148)
Proof. Follows as in the proof of Lemma 8.7.

Let \( \bar{Y} := \mathbb{X}^2 \) be the second component of the solution of the linear SDE in (125), which is given in (126), and let \( (g_t)_{t \in [0, T]} \) be the \( L^2 \) solution to the two-dimensional linear BSDE
\[
dg_t = (\hat{F}_{22} \hat{P}_{12})g_t dt + \left( \frac{d(P_{11} - \bar{Y}_t \hat{Y}_t)}{2 \lambda} + d\bar{M}_t \right), \quad g_T = \begin{pmatrix} -\frac{\bar{Y}_T}{2 \lambda} \\ 0 \end{pmatrix},
\] (149)
for some square integrable martingale \( \bar{M} \).

Proposition 8.16. Let \( g^N \) be the solution to (139) and \( g \) be the solution to (149). Then we have
\[
\lim_{N \to \infty} \sup_{t \in [0, T]} |g^N_t - g_t| = 0, \quad P - a.s.
\]
The proof of Proposition 8.16 follows the same reasoning as the proof of Proposition 8.8.
Lastly, using \( \hat{W} \) satisfying (143) and \( g \) satisfying (149), define as above in (140) for each \( i \in \mathbb{N} \) the process

\[
\tilde{Y}^i_t := \hat{W}^i_t \tilde{X}^i_t + g_t \quad (0 \leq t \leq T),
\]

(150)

where \( \tilde{X}^i_t \) is the solution to the linear ODE with random coefficients

\[
d\tilde{X}^i_t = \hat{F}_{12}(\hat{W}^i_t \tilde{X}^i_t + g_t)dt = \hat{F}_{12} \tilde{Y}^i_t dt, \quad \tilde{X}_0 = \left( x^i_0 \right)
\]

(151)

and initial value \( x^i \) as in (37).

**Proposition 8.17.** The process \( (\tilde{X}^i, \tilde{Y}^i)^T \) in (150)–(151) satisfies the \( i \)-th player FBSDE system in (37), that is, using the notation from equation (37),

\[
\tilde{Y}^i_t = \left( \begin{array}{c} \hat{v}^i_t \\ 0 \end{array} \right) \quad \text{and} \quad \tilde{X}^i_t = \left( \begin{array}{c} \hat{x}^i_t \\ 0 \end{array} \right).
\]

(152)

We are now ready to proof Theorem 4.7(ii).

**Proof of Theorem 4.7** (ii). The proof of Theorem 4.7(ii) is similar to part (i). One can easily verify that (53) implies (137) and (145). Using Lemma (8.15), Proposition 8.16 and Gronwall’s Lemma we arrive at

\[
\lim_{N \to \infty} \sup_{t \in [0, T]} \left( |\tilde{Y}^i_N - \hat{Y}^i_t| + |\tilde{X}^i_N - \hat{X}^i_t| \right) = 0 \quad \text{a.s.}
\]

and the result follows.

\[
\square
\]

9 | **PROOF OF THEOREM 4.10**

We first introduce the following auxiliary lemma.

**Lemma 9.1.** Let \( \hat{v} \) and \( \{\hat{v}^i, i \in \mathbb{N}\} \) be the equilibrium strategies of the mean field game as in Definition 3.2 and assume (47) and (49). Then we have

\[
\sup_{t \in [0, T]} E \left[ \left( \hat{v}_t - \frac{1}{N} \sum_{i=1}^{N} \hat{v}^i_t \right)^2 \right] = O \left( \frac{1}{N^2} \right).
\]

**Proof.** For such \( \hat{v} \), let \( \{\hat{v}^i, i \in \mathbb{N}\} \) be the solution of (37) and let \( \{\hat{u}^i,N, 1 \leq i \leq N\} \) be the solutions of the \( N \)-player system (8), with similar initial conditions as the mean field game. From the proof of
Theorem 4.1 it follows that there exists \( \hat{C} > 0 \) not depending on \( N \) and \( s \in [0, T] \) such that

\[
E \left[ \left( \hat{v}_s - \frac{1}{N} \sum_{i=1}^{N} \hat{u}_s^i \right)^2 \right] 
\leq 2E \left[ \left( \hat{v}_s - \frac{1}{N} \sum_{i=1}^{N} \hat{u}_{s,N}^i \right)^2 \right] + 2E \left[ \left( \frac{1}{N} \sum_{i=1}^{N} \hat{u}_{s,N}^i - \frac{1}{N} \sum_{i=1}^{N} \hat{v}_s^i \right)^2 \right]
\leq C \frac{1}{N^2} + C \frac{1}{N^2} \sum_{i=1}^{N} E \left[ (\hat{u}_{s,N}^i - \hat{v}_s^i)^2 \right]
\leq \hat{C} \frac{1}{N^2},
\]

where we used \((62)\) in the second inequality.

\[\square\]

In order to prove Theorem 4.10 we will need the following lemma that bounds the difference between the performance functional of the mean field game, \( J^{i,\infty} \) in \((30)\) and the \( N \)-player game’s performance functional, \( J^{i,N} \) in \((6)\).

**Lemma 9.2.** Let \( \hat{v} \) and \( \{\hat{v}^i, i \in \mathbb{N}\} \) be the equilibrium strategies of the mean field game as in Definition 3.2. For any \( N \in \mathbb{N} \) and \( 1 \leq i \leq N \), define \( \hat{v}^{-i} = (\hat{v}^1, ... \hat{v}^{i-1}, \hat{v}^{i+1}, ..., \hat{v}^N) \). Then under assumptions \((47)\) and \((49)\) there exists \( C > 0 \) independent from \( N \) such that for all \( u \in \mathcal{A} \) and \( i \in \mathbb{N} \) we have

\[
|J^{i,N}(u, \hat{v}^{-i}) - J^{i,\infty}(u, \hat{v})| 
\leq C \|u\|_{2,T} (1 + \|u\|_{2,T}) \left( \frac{1}{N} \right).
\]

**Proof.** Let \( u \in \mathcal{A} \). For \((u, \hat{v}^{-i})\) let \( S^{(u, \hat{v}^{-i})} \) be as in \((4)\) and let \( S^{\hat{v}} \) be as in \((28)\). Using \((6)\) and \((30)\) and then \((5)\) and \((29)\) we get

\[
|J^{i,N}(u, \hat{v}^{-i}) - J^{i,\infty}(u, \hat{v})| = \left| E \left[ \int_0^T S_t^{(u, \hat{v}^{-i})} u_t dt - \int_0^T S_t^{\hat{v}} u_t dt \right] \right|
\]

\[
= \left| E \left[ \int_0^T \kappa \left( \int_0^t e^{-\rho(t-s)} \left( \frac{1}{N} \sum_{j \neq i} \hat{v}^j_s + u_s \right) - \hat{v}_s \right) ds \right] u_t dt \right|
\]

\[
\leq \kappa \left| E \left[ \int_0^T \left( \int_0^t e^{-\rho(t-s)} \left( \frac{1}{N} \sum_{j=1}^{N} \hat{v}_s^j - \hat{v}_s \right) ds \right) u_t dt \right] \right|
\]

\[
+ \kappa \left| E \left[ \int_0^T \left( \int_0^t e^{-\rho(t-s)} \frac{1}{N} (\hat{v}^i_s - u_s) ds \right) u_t dt \right] \right|
\]

\[= : \kappa \sum_{k=1}^{2} I_k.\]
Using Fubini's Theorem and Hölder inequality we get for $I_1$:

$$I_1 \leq \int_0^T \int_0^T e^{-\rho(t-s)} \left| E \left[ \left( \frac{1}{N} \sum_{j=1}^N \hat{v}_j^i - \hat{v}_s \right) u_t \right] \right| ds dt$$

$$\leq \int_0^T \int_0^T E \left[ \left( \frac{1}{N} \sum_{j=1}^N \hat{v}_j^i - \hat{v}_s \right) u_t \right] ds dt$$

$$\leq C(T) \left( \int_0^T E \left[ \left( \frac{1}{N} \sum_{j=1}^N \hat{v}_j^i - \hat{v}_s \right)^2 \right] ds \right)^{1/2} \left( \int_0^T E[u_t^2] dt \right)^{1/2}$$

$$\leq C(T) \| u \|_{2,T} N^{-1},$$

where we used Lemma 9.1 and the fact that $u \in \mathcal{A}$ (see (3)) in the last inequality.

Using Fubini's Theorem and Hölder inequality we get for $I_2$:

$$I_2 \leq \int_0^T \int_0^T \frac{1}{N} E[|\hat{v}_s - u_s| u_t] ds dt$$

$$\leq C(T) \frac{1}{N} \left( \int_0^T E[(\hat{v}_s - u_s)^2] ds \right)^{1/2} \left( \int_0^T E[u_t^2] dt \right)^{1/2}$$

$$\leq C(T) \frac{1}{N} \left( \| \hat{v} \|_{2,T} + \| u \|_{2,T} \right) \| u \|_{2,T}$$

$$\leq C(T) \frac{1}{N} (1 + \| u \|_{2,T}) \| u \|_{2,T},$$

where we used Proposition 4.4(ii) in the last inequality.

By plugging in (154) and (155) to (153) we get the result. \qed

**Proof of Theorem 4.10.** First note that the inequality

$$J^{i,N}(\hat{v}^i; \hat{v}^{-i}) \leq \sup_{u \in \mathcal{A}} J^{i,N}(u; \hat{v}^{-i}),$$

holds trivially by the definition of the supremum.

Using Lemma 9.2 we get for any $u \in \mathcal{A}$,

$$J^{i,N}(u; \hat{v}^{-i}) \leq J^{i,\infty}(u, \hat{v}) + C \| u \|_{2,T} (1 + \| u \|_{2,T}) \frac{1}{N}$$

$$\leq J^{i,\infty}(\hat{v}^i, \hat{v}) + C \| u \|_{2,T} (1 + \| u \|_{2,T}) \frac{1}{N},$$

where $C > 0$ is a constant not depending on $u$ or $N$. We used $J^{i,\infty}(\hat{u}^i, \hat{v}) = \sup_{u \in \mathcal{A}} J^{i,\infty}(u, \hat{v})$ in the second inequality.
Using Lemma 9.2 again we get for some constant $\tilde{C} > 0$ that
\[ J^{i,\infty}(\hat{v}^i, \hat{v}^{-i}) \leq J^{i,N}(\hat{v}^i, \hat{v}^{-i}) + C\|\hat{v}_i\|_{2,T}(1 + \|\hat{v}_i\|_{2,T}) \frac{1}{N} \]
\[ \leq J^{i,N}(\hat{v}^i, \hat{v}^{-i}) + \frac{1}{N}, \tag{157} \]
where we used Proposition 4.4(ii) in the second inequality.

Using (157) to bound the right hand side of (156), we get
\[ J^{i,N}(u; \hat{v}^{-i}) \leq J^i(\hat{v}^i, \hat{v}^{-i}) + C \|u\|_{2,T}(1 \vee \|u\|_{2,T}) \frac{1}{N} \]
for all $u \in \mathcal{A}$,

which completes the proof. \[ \square \]

10 | PROOFS FOR THE FINITE PLAYER GAME

We start with establishing a strict concavity property of each player’s objective functional in (6) in the following

**Lemma 10.1.** Let $i \in \{1, \ldots, N\}$. The functional $u^{i,N} \mapsto J^{i,N}(u^{i,N}; u^{-i,N})$ in (6) is strictly concave in $u^{i,N} \in \mathcal{A}$.

**Proof.** First, observe that we can decompose the aggregated transient price impact $Y^{u,N}$ in (5) into $Y^{u,N} = Y^{u^{i,N}} + Y^{u^{-i,N}}$ where
\[ Y^{i,N}_t \triangleq e^{-\rho t} y + \frac{y}{N} \int_0^t e^{-\rho(t-s)} u^{i,N}_s \, ds, \quad Y^{-i,N}_t \triangleq \frac{y}{N} \int_0^t e^{-\rho(t-s)} \left( \sum_{j \neq i} u^j_s \right) \, ds \tag{158} \]
for all $t \in [0, T]$. As a consequence, we can write
\[ S^{i,N}_t = P_t - \kappa Y^{u^{-i,N}}_t - \kappa Y^{u^{i,N}}_t \tag{159} \]
in agent $i$’s performance functional $u^{i,N} \mapsto J^{i,N}(u^{i,N}; u^{-i,N})$ in (6). Next, using the product rule in the expression of the terminal liquidation value $X^{u^{i,N}}_T (P_T - \phi X^{u^{i,N}}_T)$ we can rewrite the functional as
\[ J^{i,N}(u^{i,N}; u^{-i,N}) = X^{u^{i,N}}_0 (P_0 - \phi X^{u^{i,N}}_0) + \kappa J^{1,N}_1(u^{i,N}) + J^{2,N}_2(u^{i,N}; u^{-i,N}), \]
where
\[ J^{1,N}_1(u^{i,N}) \triangleq E \left[ \int_0^T Y^{u^{i,N}}_t \, dX^{u^{i,N}}_t \right], \]
\[ J^{2,N}_2(u^{i,N}; u^{-i,N}) \triangleq E \left[ \int_0^T X^{u^{i,N}}_t \left( 2 \phi u^{i,N}_t - \phi X^{u^{i,N}}_t \right) \, dt - \lambda \int_0^T \left( u^{i,N}_t \right)^2 \, dt \right. \]
\[ + \left. \int_0^T X^{u^{i,N}}_t \, dP_t - \kappa \int_0^T Y^{u^{-i,N}}_t u^{i,N}_t \, dt \right]. \]
Regarding the mapping \( u^{i,N} \mapsto J_1^{i,N}(u^{i,N}) \) we can deduce from the arguments in Lehalle and Neuman (2019, Proof of Theorem 2.3) that this functional is strictly concave in \( u^{i,N} \). Hence, it is left to show that

\[
J_2^{i,N}(\varepsilon u^{i,N} + (1 - \varepsilon)w^{i,N}; u^{-i,N}) - \varepsilon J_2^{i,N}(u^{i,N}; u^{-i,N}) - (1 - \varepsilon)J_2^{i,N}(w^{i,N}; u^{-i,N}) > 0
\]

for all \( \varepsilon \in (0, 1) \) and \( u^{i,N}, w^{i,N} \in \mathcal{A} \) such that \( u^{i,N} \neq w^{i,N} \) \( d\mathbb{P} \otimes ds \)-a.e. on \( \Omega \times [0, T] \). Using that

\[
X_i^{u^{i,N}+\varepsilon u^{i,N}} = \varepsilon X_i^{u^{i,N}} + (1 - \varepsilon)X_i^{u^{i,N}} \quad \text{with} \quad X_0^{u^{i,N}} = X_0^{w^{i,N}},
\]

a straightforward computation reveals that

\[
J_2^{i,N}(\varepsilon u^{i,N} + (1 - \varepsilon)w^{i,N}; u^{-i,N}) - \varepsilon J_2^{i,N}(u^{i,N}; u^{-i,N}) - (1 - \varepsilon)J_2^{i,N}(w^{i,N}; u^{-i,N}) = \varepsilon(1 - \varepsilon)\mathbb{E} \int_0^T \left( 2\varphi(X_t^{u^{i,N}} - X_t^{u^{i,N}})(w_t^{i,N} - u_t^{i,N}) + \phi(X_t^{u^{i,N}} - X_t^{u^{i,N}})^2 + \lambda(u_t^{i,N} - w_t^{i,N})^2 \right) dt.
\]

Obviously, the last two terms in (161) are always strictly positive. Moreover, regarding the first term in (161) integration by parts yields

\[
2\varphi \int_0^T (X_t^{u^{i,N}} - X_t^{u^{i,N}})(w_t^{i,N} - u_t^{i,N})dt = \varphi(X_T^{u^{i,N}} - X_T^{u^{i,N}})^2 > 0.
\]

Putting all together, we obtain (160) as desired and the claim. \( \square \)

From Lemma 10.1 we obtain the following important consequence.

**Lemma 10.2.** There exists at most one Nash equilibrium in the sense of Definition 2.1.

**Proof.** This follows from Lemma 10.1 by adopting the same argumentation via contradiction from Schied et al. (2017, Proposition 4.8) to our finite player game. \( \square \)

**Proof of Lemma 2.5.** Let us first consider a single fixed agent \( i \in \{1, \ldots, N\} \) and characterize her best response to the other agents’ given fixed strategies \( u^{-i,N} \in \mathcal{A}^{N-1} \) by maximizing her performance functional \( u^{i,N} \mapsto J^{i,N}(u^{i,N}, u^{-i,N}) \) in (6). To this end, note that we can decompose the jointly aggregated transient price impact \( Y^{u^{i,N}} \) in (5) into \( Y^{u^{i,N}} = Y^{u^{i,N}} + Y^{u^{-i,N}} \) with \( Y^{u^{i,N}}, Y^{u^{-i,N}} \) as in (158) above, and we can write \( S_t^{u^{i,N}} = P - \kappa Y^{u^{i,N}} - \kappa Y^{u^{i,N}} \) in agent \( i \)’s performance functional in (6). Next, a computation very similar to the proof of Lemma 5.2 in Neuman and Voß (2022) (with parameter \( \gamma/N \) instead of \( \gamma \)) shows that strategy \( u^{i,N} \) determines agent \( i \)’s unique best response in (7) to a given set of competitor strategies \( u^{-i,N} \) if and only if \( (X_t^{u^{i,N}}, Y_t^{u^{i,N}}, u_t^{i,N}, Z_t^{u^{i,N}}) \) satisfy the
following coupled linear FBSDE system

\[
\begin{align*}
\frac{dX_t^{u_i,N}}{u_i,N} &= -u_i,N_t \ dt, \quad X_0^{u_i,N} = x_i,N, \\
\frac{dY_t^{u_i,N}}{u_i,N} &= -\rho Y_t^{u_i,N} dt + \frac{\gamma}{N} u_i,N_t \ dt, \quad Y_0^{u_i,N} = y, \\
\frac{du_i,N_t}{u_i,N} &= \frac{1}{2\lambda} \left( \frac{dP_t}{u_i,N} - \kappa dY_t^{u_i,N-1} \right) + \frac{\gamma_0}{2\lambda} Y_t^{u_i,N} dt - \frac{\phi}{2\lambda} X_t^{u_i,N} dt + \frac{\rho}{2\lambda} Z_t^{u_i,N} dt + dM_t^{u_i,N}, \\
\frac{dZ_t^{u_i,N}}{u_i,N} &= \rho Z_t^{u_i,N} dt + \frac{\gamma}{N} u_i,N_t \ dt + dN_t^{u_i,N}, \quad Z_T^{u_i,N} = 0,
\end{align*}
\]

(162)

for two suitable square integrable martingales \( M_t^{u_i,N} = (M_t^{u_i,N})_{0 \leq t \leq T} \) and \( N_t^{u_i,N} = (N_t^{u_i,N})_{0 \leq t \leq T} \). Moreover, from the proof of Lemma 5.2 in (Neuman and Voß, 2022, equation (5.7)) we also know that these martingales are given by

\[
\begin{align*}
M_t^{u_i,N} &= \frac{1}{2\lambda} \hat{M}_t^{u_i,N} - \frac{\gamma \lambda}{2\lambda N} \int_0^t e^{\rho s} \hat{N}_s^{u_i,N} \ ds, \\
N_t^{u_i,N} &= -\frac{\gamma \lambda}{N} \int_0^t e^{\rho s} \hat{N}_s^{u_i,N} \ ds,
\end{align*}
\]

(163)\hspace{1cm}(164)

where

\[
\begin{align*}
\hat{N}_t^{u_i,N} &\triangleq \mathbb{E}_t \left[ \int_0^T e^{-\rho s} u_s^{i,N} \ ds \right], \\
\hat{M}_t^{u_i,N} &\triangleq \mathbb{E}_t \left[ 2\phi \int_0^T X_s^{u_i,N} \ ds + 2\phi X_t^{u_i,N} - P_T \right],
\end{align*}
\]

(165)\hspace{1cm}(166)

for all \( t \in [0, T] \). Finally, in order for a set of controls \((u^{i,N})_{i \in \{1, \ldots, N\}} \subset \mathcal{A}^N\) to yield the unique Nash equilibrium in the sense of Definition 2.1, the above FBSDE system must be satisfied simultaneously for all agents \( i = 1, \ldots, N \). Now, rewriting for each \( i \in \{1, \ldots, N\} \) the following expression in the BSDE for \( u^{i,N} \) in (162) as

\[
-\frac{\kappa}{2\lambda} dY_t^{u_i,N-1} + \frac{\kappa \rho}{2\lambda} Y_t^{u_i,N} dt = \frac{\kappa \rho}{2\lambda} \left( Y_t^{u_i,N-1} + Y_t^{u_i,N} \right) dt - \frac{\gamma \lambda}{2\lambda N} \left( \sum_{j \neq i} u_t^{j,N} \right) dt
\]

we note that the coupling of all systems in (162) for all \( i \in \{1, \ldots, N\} \) only depends on \( Y^{u_i,N} \) but not on the processes \( Y^{u_{i-1},N} \) and \( Y^{u_{i-1},N} \) separately. Therefore together with Lemma 10.2 we obtain the characterization of the unique Nash equilibrium as claimed in (8).
Proof of Proposition 2.9. The proof is similar to the proof of (Neuman and Voß, 2022, Theorem 3.2). Therefore we only sketch the main steps. To solve the linear FBSDE system in (9) we conveniently rewrite it as

\[ d\bar{X}_t^N = \bar{F}_t^N \bar{X}_t^N \, dt + d\bar{M}_t^N \quad (0 \leq t \leq T), \]

with \( \bar{F}_t^N \in \mathbb{R}^{4 \times 4} \) introduced in (13) and

\[
\bar{X}_t^N \triangleq \begin{pmatrix}
X_t^N \\
Y_t^N \\
\bar{u}_t^N \\
Z_t^N
\end{pmatrix}, \quad \bar{M}_t^N \triangleq \begin{pmatrix}
0 \\
0 \\
\frac{1}{2\lambda} P_t + \bar{M}_t^N \\
\bar{N}_t^N
\end{pmatrix} \quad (0 \leq t \leq T).
\]

The corresponding initial and terminal conditions are given by \( \bar{X}_{0,1}^N = \bar{x}^N, \bar{X}_{0,2}^N = y \) and

\[
\left( \frac{\varrho}{\lambda}, -\frac{\kappa}{2\lambda}, -1, 0 \right) \bar{X}_T^N = 0 \quad \text{and} \quad (0, 0, 0, 1) \bar{X}_T^N = 0. \tag{168}
\]

Note that the unique solution of the linear system in (167) can be expressed in terms of the matrix exponential defined in (12) via

\[
\bar{X}_T^N = \bar{Q}(T-t)\bar{X}_t^N + \int_t^T \bar{Q}(T-s)d\bar{M}_s^N \quad (0 \leq t \leq T). \tag{169}
\]

Multiplying (169) from the left with the row vector \( \left( \frac{\varrho}{\lambda}, -\frac{\kappa}{2\lambda}, -1, 0 \right) \), using the first terminal condition in (168), taking conditional expectations and solving for \( \bar{u}_t^N \) gives us

\[
\bar{u}_t^N = -\frac{\bar{G}_1(T-t)}{\bar{G}_3(T-t)} \bar{X}_t^N - \frac{\bar{G}_2(T-t)}{\bar{G}_3(T-t)} \bar{Y}_t^N - \frac{\bar{G}_4(T-t)}{\bar{G}_3(T-t)} \bar{Z}_t^N
\]

\[- \frac{1}{2\lambda} E_t \left[ \int_t^T \frac{\bar{G}_3(T-s)}{\bar{G}_3(T-t)} dA_s \right] \quad (0 \leq t \leq T), \tag{170}
\]

with \( \bar{G} \) as defined in (14). Also note that \( \bar{G}_3(T-t) \neq 0 \) for all \( t \in [0, T] \) by assumption. Moreover, repeating the same steps by using the second terminal condition in (168) and solving the obtained identity for \( \bar{Z}_t^N \) yields

\[
\bar{Z}_t^N = -\frac{\bar{H}_1(T-t)}{\bar{H}_4(T-t)} \bar{X}_t^N - \frac{\bar{H}_2(T-t)}{\bar{H}_4(T-t)} \bar{Y}_t^N - \frac{\bar{H}_3(T-t)}{\bar{H}_4(T-t)} \bar{u}_t^N
\]

\[- \frac{1}{2\lambda} E_t \left[ \int_t^T \frac{\bar{H}_3(T-s)}{\bar{H}_4(T-t)} dA_s \right] \quad (0 \leq t \leq T), \tag{171}
\]

where \( \bar{H} \) is defined in (15) and \( \bar{H}_4(T-t) \neq 0 \) for all \( t \in [0, T] \) by Assumption 2.8. Plugging (171) into (170) and solving for \( \bar{u}_t^N \) yields the claim in (18), where \( \bar{v}_0 \) is well-defined by Assumption 2.8.

Finally, the claim that \( \bar{u}_t^N \) in (18) belongs to \( \mathcal{A} \) follows from Assumption 2.8, which guarantees the boundedness from above of the functions \( \bar{G}_i(t), \bar{H}_i(t), i \in \{1, ..., 4\} \), in (185)–(192), and by
employing a similar Gronwall-type argument as in the proof of (Neuman and Voß, 2022, Theorem 3.2), step 2.

Proof of Theorem 2.13. In view of Corollary 2.6 we have to solve for each $i \in \{1, \ldots, N\}$ the linear FBSDE in (10). Therefore, the proof of Theorem 2.13 follows the same reasoning as the proof of Proposition 2.9 above. Indeed, observe that for each $i \in \{1, \ldots, N\}$ the system in (10) can be written as

$$dX_t^{i,N} = F^N X_t^{i,N} dt + dM_t^{i,N} \quad (0 \leq t \leq T)$$

where

$$X_t^{i,N} \triangleq \begin{pmatrix} X_t^{u,N} \\ u_t^{i,N} \\ Z_t^{i,N} \end{pmatrix}, \quad M_t^{i,N} \triangleq \begin{pmatrix} 0 \\ \frac{1}{2\lambda}(P_t - \kappa \bar{Y}_t^{\bar{N}}) + M_t^{i,N} \\ N_t^{i,N} \end{pmatrix} \quad (0 \leq t \leq T)$$

and $F^N \in \mathbb{R}^{3 \times 3}$ defined in (20); initial and terminal condition are given by $X_0^{i,N,1} = x^{i,N}$ and

$$\left( \frac{\varphi}{\lambda}, -1, 0 \right) X_T^{i,N} = \frac{\kappa}{2\lambda} \bar{Y}_T^{\bar{N}}, \quad (0, 0, 1) X_T^{i,N} = 0. \quad (173)$$

That is, as in the proof of Proposition 2.9 we can write the unique solution of the linear system in (172) as

$$X_t^{i,N} = Q(T-t)X_T^{i,N} + \int_t^T Q(T-s) dM_s^{i,N} \quad (0 \leq t \leq T),$$

with the matrix exponential introduced in (19) and follow the same steps. That is, via the terminal conditions in (173) we eventually get the identities

$$u_t^{i,N} = - \frac{G_1(T-t)}{G_2(T-t)} X_t^{u,N} - \frac{G_3(T-t)}{G_2(T-t)} Z_t^{i,N}$$

$$- \frac{1}{2\lambda} E_t \left[ \int_t^T \frac{G_2(T-s)}{G_2(T-t)} \left( dA_s - \kappa d\bar{Y}_s^{\bar{N}} \right) - \frac{\kappa}{G_2(T-t)} \bar{Y}_T^{\bar{N}} \right] \quad (0 \leq t \leq T), \quad (174)$$

as well as

$$Z_t^{u,N} = - \frac{H_1(T-t)}{H_3(T-t)} X_t^{u,N} - \frac{H_2(T-t)}{H_3(T-t)} u_t^{i,N}$$

$$- \frac{1}{2\lambda} E_t \left[ \int_t^T \frac{H_2(T-s)}{H_3(T-t)} \left( dA_s - \kappa d\bar{Y}_s^{\bar{N}} \right) \right] \quad (0 \leq t \leq T), \quad (175)$$

with $G$ and $H$ defined in (21) and (22). Note that we have $G_2(T-t) \neq 0$ and $H_3(T-t) \neq 0$ for all $t \in [0, T]$ by Assumption 2.12. Plugging (175) into (174) and solving for $u^{i,N}$ yields the claim in (25), where $v_0$ is well-defined by Assumption 2.12.

Finally, the claim that $u^{i,N}$ in (25) belongs to $A$ follows from Assumption 2.12 and the fact that the functions $G_i(t), H_i(t), i \in \{1, \ldots, 3\}$ in (196)–(201) are bounded from above, using again a similar Gronwall-type argument as in the proof of (Neuman and Voß, 2022, Theorem 3.2), step 2. \qed
11 | PROOFS FOR THE INFINITE PLAYER GAME

As in the finite player game, we first provide that each player’s objective functional in (30) is strictly concave.

**Lemma 11.1.** Let $i \in \mathbb{N}$. The functional $v^i \mapsto J_{i,\infty}^i(v^i; \nu)$ in (30) is strictly concave in $v^i \in A$.

**Proof.** The computations are similar to the proof of Lemma 10.1 above. Indeed, using again the product rule in the expression of the terminal liquidation value $X_t^i P_T - \phi X^i_T$ in agent $i$’s performance functional $v^i \mapsto J_{i,\infty}^i(v^i; \nu)$ in (30), we obtain

$$J_{i,\infty}^i(v^i; \nu) = X_0^{v^i}(P_0 - \phi X_0^{v^i}) + J_{1,\infty}^i(v^i; \nu),$$

where

$$J_{1,\infty}^i(v^i; \nu) \triangleq E \left[ \int_0^T X_t^i (2\phi v_t^i - \phi X_t^i) dt - \lambda \int_0^T (v_t^i)^2 dt + \int_0^T X_t^i dP_t - \kappa \int_0^T Y_t^\nu v_t^i dt \right].$$

Strict concavity of the mapping $v^i \mapsto J_{1,\infty}^i(v^i; \nu)$ then follows as in the proof of Lemma 10.1. □

Consequently, again similar to the finite player game, we can establish

**Lemma 11.2.** There exists at most one Nash equilibrium in the sense of Definition 3.2.

**Proof.** This follows from Lemma 11.1 by adopting once more the same argumentation via contradiction from Schied et al. (2017, Proposition 4.8) to our infinite player game. □

**Proof of Lemma 3.6.** First, observe that for a given and fixed net trading flow $\nu \in A$ the optimization problem of an individual agent $i \in \mathbb{N}$ in (31) is very similar to the single-agent optimization problem studied in Belak et al. (2019), where the agent is only facing temporary price impact and the unaffected price process is given by $P - \kappa Y^\nu$. The only difference is that the value of the terminal inventory $X_T^i$ in (30) is expressed in terms of $P_T$ and not $S_T^\nu$. Therefore, it follows along the lines of the proof of (Belak et al., 2019, Theorem 3.1) that the unique solution $\hat{v}^i \in A$ of (31) satisfies the linear FBSDE system

\[
\begin{cases}
    dX_t^{\hat{v}^i} = -\hat{v}^i_t dt, & X_0^{\hat{v}^i} = x^i, \\
    dY_t^\nu = -\rho Y_t^\nu dt + \gamma v_t dt, & Y_0^\nu = y, \\
    d\hat{v}^i_t = \frac{1}{2\lambda} (dP_t - \kappa dY_t^\nu) - \frac{\phi X_t^{\hat{v}^i}}{\lambda} dt + dL_t^i, & \hat{v}^i_T = \frac{\phi}{\lambda} X_T^{\hat{v}^i} - \frac{\kappa}{2\lambda} Y_T^\nu,
\end{cases}
\]  

(176)
with a square integrable martingale $L^i = (L^i_t)_{0 \leq t \leq T}$ given by

$$L^i_t \triangleq \frac{1}{2\lambda} E_t \left[ 2\phi \int_0^T X^i_s \, ds + 2\varphi X^i_T - P_T \right] \quad (0 \leq t \leq T). \tag{177}$$

Consequently, in order for a collection of controls $(\tilde{v}^i)_{i \in \mathbb{N}} \subset \mathcal{A}$ to yield the unique Nash equilibrium in the sense of Definition 3.2, the above FBSDE system must be satisfied simultaneously for all agents $i \in \mathbb{N}$. The uniqueness of the solution to (176) then follows from Lemma 11.2.

In order to complete the proof, we need to show that we can find an admissible $\tilde{v}$ such that the consistency condition (35) is satisfied. We will show that $\tilde{v}$ that solves (36) is the right candidate for that.

For such $\tilde{v}$, let $\{\tilde{v}^i, i \in \mathbb{N}\}$ be the solution of (37). Then from (36), (37) and (46) we have,

$$\left| \frac{1}{N} \sum_{i=1}^N \tilde{v}^i_t - \tilde{v}_t \right| = \left| \frac{R'(T-t)}{R(T-t)} \left( \tilde{X}^\theta - \frac{1}{N} \sum_{i=1}^N X^\theta_i \right) \right| \tag{178}$$

From (44) it follows that $\sup_{s \in [0,T]} \left| \frac{R'(T-s)}{R(T-s)} \right| < \infty$. Together with (27) Jensen inequality and Fubini theorem we get for any $0 \leq t \leq T$ that there exist constants $C_1(N), C_2 > 0$ such that

$$E \left[ \sup_{s \in [0,t]} \left( \frac{1}{N} \sum_{i=1}^N \tilde{v}^i_s - \tilde{v}_s \right)^2 \right] \leq C_1(N) + C_2 \int_0^t E \left[ \sup_{r \in [0,s]} \left( \frac{1}{N} \sum_{i=1}^N \tilde{v}^i_r - \tilde{v}_r \right)^2 \right] ds \tag{179}$$

where $C_1(N) \to 0$ as $N \to \infty$. Then from Gronwall’s lemma we get

$$\lim_{N \to \infty} E \left[ \sup_{s \in [0,T]} \left( \tilde{v}_s - \frac{1}{N} \sum_{i=1}^N \tilde{v}^i_s \right)^2 \right] = 0. \tag{180}$$

Note that the convergence rate in (180) is determined by (27).

By Fatou’s lemma and (180) we have

$$E \left[ \left( \tilde{v}_s - \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^N \tilde{v}^i_s \right)^2 \right] \leq \liminf_{N \to \infty} \left( \frac{1}{N} \sum_{i=1}^N \tilde{v}^i_s \right)^2 = 0. \tag{181}$$

Hence it holds that

$$\tilde{v}_s = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^N \tilde{v}^i_s, \quad \text{for all } 0 \leq s \leq T, \, P - \text{a.s.},$$

and this completes the proof. \(\square\)

**Proof of Corollary 3.7.** The proof of Corollary 3.7 follows immediately from (181). \(\square\)
Proof of Proposition 3.11. Solving the linear system in (36) follows along the same lines as solving the linear FBSDE systems in (9) and (10) in Proposition 2.9 and Theorem 2.13, respectively. That is, the system in (36) can be written as

\[ d\tilde{X}_t = \tilde{B}\tilde{X}_t dt + d\tilde{M}_t \quad (0 \leq t \leq T), \]  

where

\[
\begin{align*}
\tilde{X}_t & \triangleq \begin{pmatrix} \tilde{X}_t^1 \\ \tilde{Y}_t^1 \\ \tilde{v}_t \end{pmatrix}, & \tilde{M}_t & \triangleq \begin{pmatrix} 0 & 0 \\ 1 \frac{1}{2\lambda} P_t + L_t \end{pmatrix} \\
\end{align*}
\]

and \( \tilde{B} \in \mathbb{R}^{3 \times 3} \) defined in (40); initial and terminal conditions are given by \( \tilde{X}_0^1 = \tilde{x}, \tilde{X}_0^2 = y \) and

\[
\begin{pmatrix} \frac{\varphi}{\lambda} - \frac{\kappa}{2\lambda} \\ -1 \end{pmatrix} \tilde{X}_T = 0.
\]

As a consequence, together with the matrix exponential \( \tilde{R} \) defined in (39) we can write the unique solution to the linear system in (182) as

\[ \tilde{X}_T = \tilde{R}(T-t)\tilde{X}_t + \int_t^T \tilde{R}(T-s)d\tilde{M}_s \quad (0 \leq t \leq T). \]

Lastly, using the terminal condition in (183) together with \( \tilde{K} \) defined in (41) yields the identity

\[ \tilde{v}_t = -\frac{\tilde{K}_1(T-t)}{\tilde{K}_3(T-t)} \tilde{X}_t^1 \tilde{Y}_t^1 - \frac{\tilde{K}_2(T-t)}{\tilde{K}_3(T-t)} \tilde{Y}_t^1 - \frac{\frac{1}{2\lambda} E_t}{\tilde{K}_3(T-t)} \int_t^T \frac{\tilde{K}_3(T-s)}{\tilde{K}_3(T-t)}dA_s \]

where we recall that \( \tilde{K}_3(T-t) \neq 0 \) for all \( t \in [0,T] \) by assumption.

Finally, the claim that \( \tilde{v} \) belongs to \( \mathcal{A} \) can be deduced from our assumption \( \inf_{t \in [0,T]} |\tilde{K}_3(t)| > 0 \) and the fact that \( \tilde{K}_1, \tilde{K}_2, \tilde{K}_3 \) in (205)–(207) are bounded from above, following once more a similar Gronwall-type argument as in the proof of (Neuman and Voß, 2022, Theorem 3.2), step 2. □

Proof of Theorem 3.14. The linear FBSDE system in (37) is almost the same as the one derived in Belak et al. (2019) with signal process \( A - x\tilde{Y}_t^\varphi \). The only difference is the terminal condition for \( \tilde{v}_t^1 \). However, the computations in (Belak et al., 2019, Theorem 3.1) can be easily adapted to yield our claim in (45). □

12 | COMPUTING THE MATRIX EXPONENTIALS

12.1 | Finite player game

We start with computing the matrix exponential \( \tilde{Q}(t) = \exp(\tilde{F} \cdot t) \in \mathbb{R}^{4 \times 4} \) for all \( t \in [0, \infty) \) in (12) by decomposing the matrix \( \tilde{F} = \tilde{U} \tilde{D} \tilde{U}^{-1} \) from (13) into a diagonal matrix \( \tilde{D} \in \mathbb{R}^{4 \times 4} \) and an invertible matrix \( \tilde{U} \in \mathbb{R}^{4 \times 4} \). The eigenvalues \( \tilde{\nu}_1, \tilde{\nu}_2, \tilde{\nu}_3, \tilde{\nu}_4 \) of \( \tilde{F} \) are the roots of the equation

\[ x^4 + \frac{(N-1)\gamma}{2N\lambda} x^3 - \left( \frac{\gamma \rho(N+1)}{2N\lambda} + \rho^2 + \frac{\phi}{\lambda} \right) x^2 + \frac{\phi}{\lambda} \rho^2 = 0. \]
Recall that by Assumption 2.8 we assume that these eigenvalues are real-valued and distinct. Since $\det(\mathcal{F}^N) = \rho^2 \phi / \lambda > 0$ we can deduce that they are different from zero.

The corresponding eigenvectors are given by

$$
\vec{v}_i \triangleq \begin{pmatrix}
-\frac{N(\bar{v}_i - \rho)}{\kappa \gamma} \\
\frac{\kappa \gamma \bar{v}_i}{N(\bar{v}_i - \rho)} \\
-\frac{N(\bar{v}_i + \rho)}{\kappa \gamma} \\
\frac{\kappa \gamma \bar{v}_i}{N(\bar{v}_i + \rho)} \\
\frac{\kappa \gamma}{1}
\end{pmatrix} \quad (i = 1, 2, 3, 4).
$$

Hence, we have

$$
\bar{D} = \begin{pmatrix}
\bar{v}_1 & 0 & 0 & 0 \\
0 & \bar{v}_2 & 0 & 0 \\
0 & 0 & \bar{v}_3 & 0 \\
0 & 0 & 0 & \bar{v}_4
\end{pmatrix}, \quad
\bar{U} = \begin{pmatrix}
-\frac{N(\bar{v}_1 - \rho)}{\kappa \gamma} & -\frac{N(\bar{v}_2 - \rho)}{\kappa \gamma} & -\frac{N(\bar{v}_3 - \rho)}{\kappa \gamma} & -\frac{N(\bar{v}_4 - \rho)}{\kappa \gamma} \\
\frac{\kappa \gamma \bar{v}_1}{N(\bar{v}_1 - \rho)} & \frac{\kappa \gamma \bar{v}_2}{N(\bar{v}_2 - \rho)} & \frac{\kappa \gamma \bar{v}_3}{N(\bar{v}_3 - \rho)} & \frac{\kappa \gamma \bar{v}_4}{N(\bar{v}_4 - \rho)} \\
\frac{\kappa \gamma}{1} & \frac{\kappa \gamma}{1} & \frac{\kappa \gamma}{1} & \frac{\kappa \gamma}{1}
\end{pmatrix},
$$

and, introducing the differences

$$
\bar{v}_{i,j} = \bar{v}_i - \bar{v}_j \quad (i, j \in \{1, 2, 3, 4\}),
$$

we obtain

$$
\bar{U}^{-1} = \frac{1}{2N\rho^2}
\begin{pmatrix}
2\rho^2 \gamma \kappa (\bar{v}_1 + \rho) & \frac{2\rho^2 \gamma \kappa (\bar{v}_2 + \rho)(\bar{v}_1 + \rho)(\bar{v}_2 + \rho)}{\lambda \bar{v}_1 \bar{v}_2 \bar{v}_3 \bar{v}_4} & 2\rho^2 \gamma \kappa (\bar{v}_3 + \rho)(\bar{v}_1 + \rho)(\bar{v}_4 + \rho) & 2\rho^2 \gamma \kappa (\bar{v}_4 + \rho)(\bar{v}_1 + \rho) \\
\frac{2\rho^2 \gamma \kappa (\bar{v}_2 + \rho)(\bar{v}_1 + \rho)}{\lambda \bar{v}_1 \bar{v}_2 \bar{v}_3 \bar{v}_4} & \frac{2\rho^2 \gamma \kappa (\bar{v}_2 + \rho)(\bar{v}_1 + \rho)(\bar{v}_3 + \rho)(\bar{v}_4 + \rho)}{\lambda \bar{v}_1 \bar{v}_2 \bar{v}_3 \bar{v}_4} & \frac{2\rho^2 \gamma \kappa (\bar{v}_4 + \rho)(\bar{v}_1 + \rho)(\bar{v}_3 + \rho)(\bar{v}_4 + \rho)}{\lambda \bar{v}_1 \bar{v}_2 \bar{v}_3 \bar{v}_4} & \frac{2\rho^2 \gamma \kappa (\bar{v}_3 + \rho)(\bar{v}_1 + \rho)(\bar{v}_4 + \rho)}{\lambda \bar{v}_1 \bar{v}_2 \bar{v}_3 \bar{v}_4} \\
\frac{\lambda \bar{v}_1 \bar{v}_2 \bar{v}_3 \bar{v}_4}{\lambda \bar{v}_1 \bar{v}_2 \bar{v}_3 \bar{v}_4} & \frac{\lambda \bar{v}_1 \bar{v}_2 \bar{v}_3 \bar{v}_4}{\lambda \bar{v}_1 \bar{v}_2 \bar{v}_3 \bar{v}_4} & \frac{\lambda \bar{v}_1 \bar{v}_2 \bar{v}_3 \bar{v}_4}{\lambda \bar{v}_1 \bar{v}_2 \bar{v}_3 \bar{v}_4} & \frac{\lambda \bar{v}_1 \bar{v}_2 \bar{v}_3 \bar{v}_4}{\lambda \bar{v}_1 \bar{v}_2 \bar{v}_3 \bar{v}_4}
\end{pmatrix}.
$$

Consequently, the matrix exponential $\bar{Q}(t) = (\bar{Q}_{ij}(t))_{1 \leq i, j \leq 4}$ is given by

$$
\bar{Q}(t) = \bar{U} \begin{pmatrix}
e^{\bar{v}_1 t} & 0 & 0 & 0 \\
0 & e^{\bar{v}_2 t} & 0 & 0 \\
0 & 0 & e^{\bar{v}_3 t} & 0 \\
0 & 0 & 0 & e^{\bar{v}_4 t}
\end{pmatrix} \bar{U}^{-1} \quad (t \geq 0) \quad (184)
$$

and it follows that $\bar{G}$ defined in (14) can be computed as

$$
\bar{G}_1(T - t) = \frac{\phi}{2\lambda^2} \left( \frac{(\bar{v}_1 - \rho)(2\phi(\bar{v}_1 + \rho) + \gamma \kappa \bar{v}_1 + 2\lambda \bar{v}_1(\bar{v}_1 + \rho))}{\bar{v}_1 \bar{v}_1 \bar{v}_1 \bar{v}_1} e^{\bar{v}_1 (T - t)} - \frac{(\bar{v}_2 - \rho)(2\phi(\bar{v}_2 + \rho) + \gamma \kappa \bar{v}_2 + 2\lambda \bar{v}_2(\bar{v}_2 + \rho))}{\bar{v}_2 \bar{v}_2 \bar{v}_2 \bar{v}_2} e^{\bar{v}_2 (T - t)} \right)
$$
\[
\frac{\bar{G}_2(T - t)}{4\gamma\lambda\rho^2} = \frac{1}{4\gamma\lambda\rho^2}
\left(\frac{(2\varphi(\bar{\nu}_1 + \rho) + \gamma\kappa\bar{\nu}_1 + 2\lambda\bar{\nu}_1(\bar{\nu}_1 + \rho))(\bar{\nu}_1 - \rho)(\bar{\nu}_2 + \rho)(\bar{\nu}_3 + \rho)(\bar{\nu}_4 + \rho)}{\bar{\nu}_{1,2}\bar{\nu}_{1,3}\bar{\nu}_{1,4}}\right. \\
- \frac{(2\varphi(\bar{\nu}_2 + \rho) + \gamma\kappa\bar{\nu}_2 + 2\lambda\bar{\nu}_2(\bar{\nu}_2 + \rho))(\bar{\nu}_1 + \rho)(\bar{\nu}_2 - \rho)(\bar{\nu}_3 + \rho)(\bar{\nu}_4 + \rho)}{\bar{\nu}_{1,2}\bar{\nu}_{2,3}\bar{\nu}_{2,4}}\right) \\
+ \frac{(2\varphi(\bar{\nu}_3 + \rho) + \gamma\kappa\bar{\nu}_3 + 2\lambda\bar{\nu}_3(\bar{\nu}_3 + \rho))(\bar{\nu}_1 + \rho)(\bar{\nu}_2 + \rho)(\bar{\nu}_3 - \rho)(\bar{\nu}_4 + \rho)}{\bar{\nu}_{1,3}\bar{\nu}_{2,3}\bar{\nu}_{3,4}}\right) \\
- \frac{(2\varphi(\bar{\nu}_4 + \rho) + \gamma\kappa\bar{\nu}_4 + 2\lambda\bar{\nu}_4(\bar{\nu}_4 + \rho))(\bar{\nu}_1 + \rho)(\bar{\nu}_2 + \rho)(\bar{\nu}_3 + \rho)(\bar{\nu}_4 - \rho)}{\bar{\nu}_{1,4}\bar{\nu}_{2,4}\bar{\nu}_{3,4}}\right),
\]

\[
\frac{\bar{G}_3(T - t)}{2\lambda} = \frac{1}{2\lambda}
\left(-\frac{(\bar{\nu}_1 - \rho)(2\varphi(\bar{\nu}_1 + \rho) + \gamma\kappa\bar{\nu}_1 + 2\lambda\bar{\nu}_1(\bar{\nu}_1 + \rho))}{\bar{\nu}_{1,2}\bar{\nu}_{1,3}\bar{\nu}_{1,4}}\right. \\
+ \frac{(\bar{\nu}_2 - \rho)(2\varphi(\bar{\nu}_2 + \rho) + \gamma\kappa\bar{\nu}_2 + 2\lambda\bar{\nu}_2(\bar{\nu}_2 + \rho))}{\bar{\nu}_{1,2}\bar{\nu}_{2,3}\bar{\nu}_{2,4}}\right) \\
- \frac{(\bar{\nu}_3 - \rho)(2\varphi(\bar{\nu}_3 + \rho) - \gamma\kappa\bar{\nu}_3 + 2\lambda\bar{\nu}_3(\bar{\nu}_3 + \rho))}{\bar{\nu}_{1,3}\bar{\nu}_{2,3}\bar{\nu}_{3,4}}\right) \\
+ \frac{(\bar{\nu}_4 - \rho)(2\varphi(\bar{\nu}_4 + \rho) + \gamma\kappa\bar{\nu}_4 + 2\lambda\bar{\nu}_4(\bar{\nu}_4 + \rho))}{\bar{\nu}_{1,4}\bar{\nu}_{2,4}\bar{\nu}_{3,4}}\right),
\]

\[
\frac{\bar{G}_4(T - t)}{4\gamma\lambda\rho^2} = \frac{N(\bar{\nu}_1 - \rho)(\bar{\nu}_2 - \rho)(\bar{\nu}_3 - \rho)(\bar{\nu}_4 - \rho)}{4\gamma\lambda\rho^2}
\left(\frac{(2\varphi(\bar{\nu}_1 + \rho) + \gamma\kappa\bar{\nu}_1 + 2\lambda\bar{\nu}_1(\bar{\nu}_1 + \rho))}{\bar{\nu}_{1,2}\bar{\nu}_{1,3}\bar{\nu}_{1,4}}\right. \\
- \frac{(2\varphi(\bar{\nu}_2 + \rho) + \gamma\kappa\bar{\nu}_2 + 2\lambda\bar{\nu}_2(\bar{\nu}_2 + \rho))}{\bar{\nu}_{1,2}\bar{\nu}_{2,3}\bar{\nu}_{2,4}}\right) \\
+ \frac{(2\varphi(\bar{\nu}_3 + \rho) + \gamma\kappa\bar{\nu}_3 + 2\lambda\bar{\nu}_3(\bar{\nu}_3 + \rho))}{\bar{\nu}_{1,3}\bar{\nu}_{2,3}\bar{\nu}_{3,4}}\right) \\
- \frac{(2\varphi(\bar{\nu}_4 + \rho) + \gamma\kappa\bar{\nu}_4 + 2\lambda\bar{\nu}_4(\bar{\nu}_4 + \rho))}{\bar{\nu}_{1,4}\bar{\nu}_{2,4}\bar{\nu}_{3,4}}\right),
\]
for all $t \in [0, T]$. Moreover, $\overline{H}$ introduced in (15) is given by

$$\overline{H}_1(T - t) = \frac{\gamma \kappa \phi}{N \lambda} \left( - \frac{(\bar{v}_1 + \rho)e^{\bar{v}_1(T - t)}}{\bar{v}_{1,2} \bar{v}_{1,3} \bar{v}_{1,4}} + \frac{(\bar{v}_2 + \rho)e^{\bar{v}_2(T - t)}}{\bar{v}_{1,2} \bar{v}_{2,3} \bar{v}_{2,4}} - \frac{(\bar{v}_3 + \rho)e^{\bar{v}_3(T - t)}}{\bar{v}_{1,3} \bar{v}_{2,3} \bar{v}_{3,4}} + \frac{(\bar{v}_4 + \rho)e^{\bar{v}_4(T - t)}}{\bar{v}_{1,4} \bar{v}_{2,4} \bar{v}_{3,4}} \right),$$

(189)

$$\overline{H}_2(T - t) = \frac{\kappa (\bar{v}_1 + \rho)(\bar{v}_2 + \rho)(\bar{v}_3 + \rho)(\bar{v}_4 + \rho)}{2N \rho^2} \left( - \frac{\bar{v}_1 e^{\bar{v}_1(T - t)}}{\bar{v}_{1,2} \bar{v}_{1,3} \bar{v}_{1,4}} + \frac{\bar{v}_2 e^{\bar{v}_2(T - t)}}{\bar{v}_{1,2} \bar{v}_{2,3} \bar{v}_{2,4}} - \frac{\bar{v}_3 e^{\bar{v}_3(T - t)}}{\bar{v}_{1,3} \bar{v}_{2,3} \bar{v}_{3,4}} + \frac{\bar{v}_4 e^{\bar{v}_4(T - t)}}{\bar{v}_{1,4} \bar{v}_{2,4} \bar{v}_{3,4}} \right),$$

(190)

$$\overline{H}_3(T - t) = \frac{\gamma \kappa}{N} \left( \frac{\bar{v}_1 (\bar{v}_1 + \rho)e^{\bar{v}_1(T - t)}}{\bar{v}_{1,2} \bar{v}_{1,3} \bar{v}_{1,4}} - \frac{\bar{v}_2 (\bar{v}_2 + \rho)e^{\bar{v}_2(T - t)}}{\bar{v}_{1,2} \bar{v}_{2,3} \bar{v}_{2,4}} + \frac{\bar{v}_3 (\bar{v}_3 + \rho)e^{\bar{v}_3(T - t)}}{\bar{v}_{1,3} \bar{v}_{2,3} \bar{v}_{3,4}} - \frac{\bar{v}_4 (\bar{v}_4 + \rho)e^{\bar{v}_4(T - t)}}{\bar{v}_{1,4} \bar{v}_{2,4} \bar{v}_{3,4}} \right),$$

(191)

$$\overline{H}_4(T - t) = \frac{1}{2\rho^2} \left( - \frac{\bar{v}_1 (\bar{v}_1 + \rho)(\bar{v}_2 - \rho)(\bar{v}_3 - \rho)(\bar{v}_4 - \rho)e^{\bar{v}_1(T - t)}}{\bar{v}_{1,2} \bar{v}_{1,3} \bar{v}_{1,4}} + \frac{\bar{v}_2 (\bar{v}_1 - \rho)(\bar{v}_2 + \rho)(\bar{v}_3 - \rho)(\bar{v}_4 - \rho)e^{\bar{v}_2(T - t)}}{\bar{v}_{1,2} \bar{v}_{2,3} \bar{v}_{2,4}} - \frac{\bar{v}_3 (\bar{v}_1 - \rho)(\bar{v}_2 - \rho)(\bar{v}_3 + \rho)(\bar{v}_4 - \rho)e^{\bar{v}_3(T - t)}}{\bar{v}_{1,3} \bar{v}_{2,3} \bar{v}_{3,4}} + \frac{\bar{v}_4 (\bar{v}_1 - \rho)(\bar{v}_2 - \rho)(\bar{v}_3 - \rho)(\bar{v}_4 + \rho)e^{\bar{v}_4(T - t)}}{\bar{v}_{1,4} \bar{v}_{2,4} \bar{v}_{3,4}} \right),$$

(192)

for all $t \in [0, T]$.

Next, we compute the matrix exponential $Q(t) = \exp(F^N \cdot t) \in \mathbb{R}^{3 \times 3}$, introduced in (19), for all $t \in [0, \infty)$, by diagonalizing the matrix $F = UDU^{-1}$ in (20) with diagonal matrix $D \in \mathbb{R}^{3 \times 3}$ and invertible matrix $U \in \mathbb{R}^{3 \times 3}$. The eigenvalues $\nu_1, \nu_2, \nu_3$ are the roots of the equation

$$x^3 - \frac{2N \lambda \rho + \gamma \kappa}{2N \lambda} x^2 - \frac{\phi}{\lambda} x + \frac{\phi \rho}{\lambda} = 0.$$  

(193)

Introducing the constants

$$a = -\frac{2N \lambda \rho + \gamma \kappa}{2N \lambda}, \quad b = -\frac{\phi}{\lambda}, \quad c = \frac{\phi \rho}{\lambda}, \quad p = b - \frac{a^2}{3}, \quad q = \frac{2a^3}{27} - \frac{ab}{3} + c,$$

and substituting $x = z - a/3$ in (193) yields the equivalent equation $z^3 + pz + q = 0$ with discriminant $(q/2)^2 + (p/3)^3 < 0$. This implies that the latter has three real-valued distinct roots allowing
for the analytical representations

\[
\nu_1 = -\sqrt{-\frac{4}{3}p \cdot \cos \left( \frac{1}{3} \arccos \left( -\frac{q}{2} \cdot \sqrt{-\frac{27}{p^3}} \right) \right)} - \frac{a}{3},
\]

\[
\nu_2 = \sqrt{-\frac{4}{3}p \cdot \cos \left( \frac{1}{3} \arccos \left( -\frac{q}{2} \cdot \sqrt{-\frac{27}{p^3}} \right) \right)} - \frac{a}{3},
\]

\[
\nu_3 = -\sqrt{-\frac{4}{3}p \cdot \cos \left( \frac{1}{3} \arccos \left( -\frac{q}{2} \cdot \sqrt{-\frac{27}{p^3}} \right) \right)} - \frac{a}{3}.
\]

(194)

The corresponding eigenvectors are given by

\[
v_i \triangleq \begin{pmatrix}
-\frac{N(\nu_i - \rho)}{\kappa \gamma \nu_i} \\
\frac{\kappa \gamma \nu_i}{N(\nu_i - \rho)} \\
\frac{\kappa \gamma}{1}
\end{pmatrix} \quad (i = 1, 2, 3).
\]

Consequently, we have

\[
D = \begin{pmatrix}
\nu_1 & 0 & 0 \\
0 & \nu_2 & 0 \\
0 & 0 & \nu_3
\end{pmatrix}, \quad U = \begin{pmatrix}
\frac{N(\nu_1 - \rho)}{\kappa \gamma \nu_1} & \frac{N(\nu_2 - \rho)}{\kappa \gamma \nu_2} & \frac{N(\nu_3 - \rho)}{\kappa \gamma \nu_3} \\
\frac{\kappa \gamma \nu_1}{N(\nu_1 - \rho)} & \frac{\kappa \gamma \nu_2}{N(\nu_2 - \rho)} & \frac{\kappa \gamma \nu_3}{N(\nu_3 - \rho)} \\
\frac{\kappa \gamma}{1} & \frac{\kappa \gamma}{1} & \frac{\kappa \gamma}{1}
\end{pmatrix},
\]

and, introducing the differences

\[
\nu_{i,j} = \nu_i - \nu_j \quad (i, j \in \{1, 2, 3\}),
\]

we obtain

\[
U^{-1} = \frac{1}{N\rho} \begin{pmatrix}
-\frac{\gamma \kappa \phi \rho}{\lambda \nu_{1,2,3} \nu_{1,3}} & -\frac{\gamma \kappa \nu_1}{\nu_{1,2,3} \nu_{1,1,3}} & \frac{N\nu_1(\nu_1 - \rho)(\nu_1 \nu_2 - \rho)}{\nu_{1,2,3} \nu_{1,3}} \\
\frac{\nu_{1,2,3} \nu_{2,3} \nu_{1,3} \nu_{2,3}}{\gamma \kappa \phi \rho} & -\frac{\nu_{1,2,3} \nu_{2,3} \nu_{1,3} \nu_{2,3}}{\gamma \kappa \nu_2} & -\frac{\nu_{1,2,3} \nu_{2,3} \nu_{1,3} \nu_{2,3}}{\gamma \kappa \nu_3} \\
-\frac{\nu_{1,2,3} \nu_{2,3} \nu_{1,3} \nu_{2,3}}{\gamma \kappa \phi \rho} & -\frac{\nu_{1,2,3} \nu_{2,3} \nu_{1,3} \nu_{2,3}}{\gamma \kappa \nu_2} & -\frac{\nu_{1,2,3} \nu_{2,3} \nu_{1,3} \nu_{2,3}}{\gamma \kappa \nu_3}
\end{pmatrix}.
\]

The matrix exponential \(Q(t) = (Q_{ij}(t))_{1 \leq i, j \leq 3}\) is thus given by

\[
Q(t) = U \begin{pmatrix}
e^{\nu_1 t} & 0 & 0 \\
0 & e^{\nu_2 t} & 0 \\
0 & 0 & e^{\nu_3 t}
\end{pmatrix} U^{-1} \quad (t \geq 0).
\]

(195)

Moreover, it follows that \(G\) defined in (21) can be computed as

\[
G_1(T - t) = \frac{\phi}{\lambda^2} \left( (\nu_1 - \rho)(\varphi + \lambda \nu_1) e^{\nu_1(T - t)} - (\nu_2 - \rho)(\varphi + \lambda \nu_2) e^{\nu_2(T - t)} \right)
\]

\[
\frac{\nu_{1,2} \nu_{1,3}}{\nu_{1,2} \nu_{2,3}} e^{\nu_{1,2}(T - t)} - \frac{\nu_{1,2} \nu_{1,3}}{\nu_{1,2} \nu_{2,3}} e^{\nu_{1,2}(T - t)}
\]
\[ G_2(T - t) = \frac{1}{\lambda} \left( -\frac{(v_1 - \rho)(\varphi + \lambda v_1)}{v_{1,2}v_{1,3}} e^{v_1(T - t)} + \frac{(v_2 - \rho)(\varphi + \lambda v_2)}{v_{1,2}v_{2,3}} e^{v_2(T - t)} - \frac{(v_3 - \rho)(\varphi + \lambda v_3)}{v_{1,3}v_{2,3}} e^{v_3(T - t)} \right), \] (196)

\[ G_3(T - t) = \frac{(v_1 - \rho)(v_2 - \rho)(v_3 - \rho)}{\gamma \kappa \lambda \rho} \left( \frac{-\varphi + \lambda v_1}{v_{1,2}v_{1,3}} e^{v_1(T - t)} + \frac{-\varphi + \lambda v_2}{v_{1,2}v_{2,3}} e^{v_2(T - t)} - \frac{-\varphi + \lambda v_3}{v_{1,3}v_{2,3}} e^{v_3(T - t)} \right), \] (197)

for all \( t \in [0, T] \). Similarly, we obtain for \( H \) defined in (22)

\[ H_1(T - t) = \frac{\gamma \kappa \phi}{N\lambda} \left( -\frac{e^{v_1(T - t)}}{v_{1,2}v_{1,3}} + \frac{e^{v_2(T - t)}}{v_{1,2}v_{2,3}} - \frac{e^{v_3(T - t)}}{v_{1,3}v_{2,3}} \right), \] (199)

\[ H_2(T - t) = \frac{\gamma \kappa}{N} \left( \frac{v_1 e^{v_1(T - t)}}{v_{1,2}v_{1,3}} - \frac{v_2 e^{v_2(T - t)}}{v_{1,2}v_{2,3}} + \frac{v_3 e^{v_3(T - t)}}{v_{1,3}v_{2,3}} \right), \] (200)

\[ H_3(T - t) = \frac{v_1(v_2 - \rho)(v_3 - \rho)e^{v_1(T - t)}}{v_{1,2}v_{1,3}} - \frac{v_2(v_1 - \rho)(v_3 - \rho)e^{v_2(T - t)}}{v_{1,2}v_{2,3}} + \frac{v_3(v_1 - \rho)(v_2 - \rho)e^{v_3(T - t)}}{v_{1,3}v_{2,3}}, \] (201)

for all \( t \in [0, T] \).

### 12.2 Infinite player game

Computing the matrix exponential \( \tilde{R}(t) = \exp(\tilde{B} \cdot t) \in \mathbb{R}^{3 \times 3} \), introduced in (39), for all \( t \in [0, \infty) \), is very similar to the computations in Section 12.1. Again, we decompose the matrix \( \tilde{B} = \tilde{U} \tilde{D} \tilde{U}^{-1} \) in (40) into a diagonal matrix \( \tilde{D} \in \mathbb{R}^{3 \times 3} \) and an invertible matrix \( \tilde{U} \in \mathbb{R}^{3 \times 3} \). The eigenvalues \( \tilde{\nu}_1, \tilde{\nu}_2, \tilde{\nu}_3 \) are the roots of the equation

\[ x^3 + \frac{2\lambda \rho + \gamma \kappa}{2\lambda} x^2 - \frac{\phi}{\lambda} x - \frac{\rho \phi}{\lambda} = 0. \] (202)

Introducing the constants

\[ \tilde{a} = \frac{2\lambda \rho + \gamma \kappa}{2\lambda}, \quad \tilde{b} = -\frac{\phi}{\lambda}, \quad \tilde{c} = -\frac{\phi \rho}{\lambda}, \quad \tilde{p} = \tilde{b} - \frac{\tilde{a}^2}{3}, \quad \tilde{q} = \frac{2\tilde{a}^3}{27} - \frac{\tilde{a} \tilde{b}}{3} + \tilde{c}, \]
and substituting $x = z - \bar{a}/3$ in (202) yields the equivalent equation $z^3 + \bar{p}z + \bar{q} = 0$ with discriminant $(\bar{q}/2)^2 + (\bar{p}/3)^3 < 0$. As above this implies that the latter has three real-valued distinct roots allowing for the analytical representations

\[
\begin{align*}
\hat{\varphi}_1 &= -\sqrt{-\frac{4}{3}\bar{p}} \cdot \cos \left( \frac{1}{3} \arccos \left( -\frac{\bar{q}}{2} \cdot \sqrt{\frac{27}{\bar{p}^3}} \right) + \frac{\pi}{3} \right) - \frac{\bar{a}}{3}, \\
\hat{\varphi}_2 &= \sqrt{-\frac{4}{3}\bar{p}} \cdot \cos \left( \frac{1}{3} \arccos \left( -\frac{\bar{q}}{2} \cdot \sqrt{\frac{27}{\bar{p}^3}} \right) \right) - \frac{\bar{a}}{3}, \\
\hat{\varphi}_3 &= -\sqrt{-\frac{4}{3}\bar{p}} \cdot \cos \left( \frac{1}{3} \arccos \left( -\frac{\bar{q}}{2} \cdot \sqrt{\frac{27}{\bar{p}^3}} \right) - \frac{\pi}{3} \right) - \frac{\bar{a}}{3}.
\end{align*}
\]

The corresponding eigenvectors are given by

\[
\bar{v}_i \triangleq \begin{pmatrix} \gamma \\
\gamma \\
\gamma 
\end{pmatrix} \quad (i = 1, 2, 3).
\]

That is, we obtain

\[
\tilde{D} = \begin{pmatrix} \hat{\varphi}_1 & 0 & 0 \\
0 & \hat{\varphi}_2 & 0 \\
0 & 0 & \hat{\varphi}_3 \end{pmatrix}, \quad \bar{U} = \begin{pmatrix} \gamma \hat{\varphi}_1 \\
\gamma \hat{\varphi}_2 \\
\gamma \hat{\varphi}_3 \end{pmatrix},
\]

and, introducing the differences

\[
\hat{v}_{i,j} = \hat{v}_i - \hat{v}_j \quad (i, j \in \{1, 2, 3\}),
\]

we have that

\[
\hat{U}^{-1} = \begin{pmatrix}
-\gamma \hat{\varphi}(\hat{\varphi}_1 + \rho) & -\hat{\varphi}_1(\hat{\varphi}_1 + \rho)(\hat{\varphi}_2 + \rho) & -\hat{\varphi}_1(\hat{\varphi}_1 + \rho) \\
\gamma \hat{\varphi}_1(\hat{\varphi}_1 + \rho) & \hat{\varphi}_1(\hat{\varphi}_2 + \rho)(\hat{\varphi}_3 + \rho) & \gamma \hat{\varphi}_1(\hat{\varphi}_3 + \rho) \\
\gamma \hat{\varphi}_1(\hat{\varphi}_1 + \rho) & \gamma \hat{\varphi}_1(\hat{\varphi}_2 + \rho)(\hat{\varphi}_3 + \rho) & \gamma \hat{\varphi}_1(\hat{\varphi}_3 + \rho)
\end{pmatrix},
\]

Therefore, the matrix exponential $\hat{R}(t) = (\hat{R}_{ij}(t))_{1 \leq i, j \leq 3}$ can be computed as

\[
\hat{R}(t) = \hat{U} \begin{pmatrix} e^{\hat{v}_1 t} & 0 & 0 \\
0 & e^{\hat{v}_2 t} & 0 \\
0 & 0 & e^{\hat{v}_3 t} \end{pmatrix} \hat{U}^{-1} \quad (t \geq 0),
\]

and we obtain that $\hat{K}$ defined in (41) is given by

\[
\hat{K}_1(T - t) = \frac{\phi}{2\lambda^2} \left( \frac{2\gamma \hat{\varphi}_1 + \gamma \hat{\varphi}_1^2 + 2\lambda \hat{\varphi}_1(\hat{\varphi}_1 + \rho)}{\hat{\varphi}_1 \hat{\varphi}_1 \hat{\varphi}_1} \right) e^{\hat{v}_1 T - t}\]
\[-2\varphi(\bar{v}_2 + \rho) + \kappa \gamma \bar{v}_2 + 2\lambda \bar{v}_2(\bar{v}_2 + \rho)\]
\[\frac{\bar{v}_2 \bar{v}_1,2 \bar{v}_3,3}{\bar{v}_1,2 \bar{v}_3,3} e^{\bar{\varphi}_2(T-t)}\]
\[+ 2\varphi(\bar{v}_3 + \rho) + \kappa \gamma \bar{v}_3 + 2\lambda \bar{v}_3(\bar{v}_3 + \rho)\]
\[\frac{\bar{v}_3 \bar{v}_1,3 \bar{v}_2,3}{\bar{v}_1,3 \bar{v}_2,3} e^{\bar{\varphi}_3(T-t)}\bigg), \quad (205)\]

\[K_2(T - t) = \frac{1}{2\gamma \lambda \rho}\bigg(\]
\[\frac{(2\varphi(\bar{v}_1 + \rho) + \kappa \gamma \bar{v}_1 + 2\lambda \bar{v}_1(\bar{v}_1 + \rho))(\bar{v}_2 + \rho)(\bar{v}_3 + \rho)}{\bar{v}_1,2 \bar{v}_1,3} e^{\bar{\varphi}_1(T-t)}\]
\[+ \frac{(2\varphi(\bar{v}_2 + \rho) + \kappa \gamma \bar{v}_2 + 2\lambda \bar{v}_2(\bar{v}_2 + \rho))(\bar{v}_1 + \rho)(\bar{v}_3 + \rho)}{\bar{v}_1,2 \bar{v}_2,3} e^{\bar{\varphi}_3(T-t)}\bigg), \quad (206)\]

\[K_3(T - t) = \frac{1}{2\lambda}\bigg(\]
\[\frac{-2\varphi(\bar{v}_1 + \rho) + \kappa \gamma \bar{v}_1 + 2\lambda \bar{v}_1(\bar{v}_1 + \rho)}{\bar{v}_1,2 \bar{v}_1,3} e^{\bar{\varphi}_1(T-t)}\]
\[+ \frac{2\varphi(\bar{v}_2 + \rho) + \kappa \gamma \bar{v}_2 + 2\lambda \bar{v}_2(\bar{v}_2 + \rho)}{\bar{v}_1,2 \bar{v}_2,3} e^{\bar{\varphi}_2(T-t)}\]
\[+ \frac{-2\varphi(\bar{v}_3 + \rho) + \kappa \gamma \bar{v}_3 + 2\lambda \bar{v}_3(\bar{v}_3 + \rho)}{\bar{v}_1,3 \bar{v}_2,3} e^{\bar{\varphi}_3(T-t)}\bigg), \quad (207)\]

for all \(t \in [0, T]\).

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**DATA AVAILABILITY STATEMENT**

No data was used in this article.

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