On Mumford Orbifolds

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Abstract

Formulae for the number of branch points of one-dimensional orbifolds defined over a non-archimedean local field and uniformisable by discrete projective linear groups are given. They depend only on the uniformising group. The method of equivariant pasting reveals the possible relative position of the branch points.

1 Introduction

Non-archimedean orbifolds first appear in [1] in the context of uniformising differential equations for certain covers of the $p$-adic projective line. They are in particular useful for classifying triangle groups. Special types of $p$-adic triangle groups, appearing in characteristic zero only for the residue characteristics 2, 3 and 5 are Kato’s triangle groups of Mumford type [8] and [4]. They belong to one-dimensional orbifolds allowing three-point covers of the projective line by Mumford curves.

For his uniformisation purpose, André developed a theory of orbifold fundamental groups in characteristic zero which turned out to be very useful in classifying components of $p$-adic Hurwitz spaces for covers between Mumford curves. This is the theme of the author’s dissertation [3].

Kato’s extensive study of triangle groups of Mumford type yields them all. His quotient pasting method is applied in this paper to all finitely generated discrete projective linear groups over local non-archimedean fields for finding formulae for the number of branch points of covers defined as quotients by those groups, and to gain control of their relative position.

The setup for this paper is Berkovich’s theory of strictly $K$-analytic spaces [2], successfully giving rigid analytic spaces a topology suitable for studying covers of such spaces. For example, Berkovich paths on a one-dimensional manifold $X$ may be seen as continuous versions of their rigid analytic counterparts on the graphs associated to reductions of $X$ along pure affinoid coverings of $X$.

Until recently, the author was unaware of an unpublished article by van der Put and Voskuil [12] proving a special case of these results with entirely different methods.

A large portion of this paper can be found in the author’s dissertation [3] which may serve as an introduction to the theory of $K$-analytic spaces.


2 Orbifolds

Let $K$ be a non-archimedean local field. Its residue field will be denoted $k$. A manifold over $K$ or simply a $K$-manifold, is paracompact, strictly $K$-analytic space $X$, admitting locally an étale morphism to $A^\dim X_K$. A space fulfilling the latter property is called smooth. There is also an ad-hoc definition of manifold allowing some non-smooth spaces such as affinoid discs to be named manifolds [1].

Definition 1. An orbifold $\mathcal{X} = (X, (Z_i, G_i))$ over $K$ consists of a $K$-manifold $X$ together with a locally finite family $(Z_i)$ of irreducible divisors and finite groups $G_i$ with the property that for each $i$ there is an open $U_i \supseteq Z_i$ in $X$ and a ramified Galois cover $V_i \xrightarrow{G_i} U_i$ un-ramified outside $Z_i$.

If $\text{char} K = 0$, this is a special case of an orbifold in the sense of [1]: the groups $G_i$ may be replaced by cyclic groups. André then replaces these groups by their orders.

If $\text{char} K > 0$, there is a nice subclass of orbifolds with $G_i$ of the form $G_i \cong B(t, n) := E_t \rtimes C_n$, where $E_t := C_p^t$ and $n \mid p^t - 1$ (depending on $i$, of course). These orbifolds are called ordinary, and may be written as $\mathcal{X} = (X, (\xi_i; n_i, t_i))$ with $n_i \mid p^t_i - 1$.

We will always assume that all groups $G_i$ of the orbifold $\mathcal{X} = (X, (Z_i, G_i))$ are non-trivial.

Definition 2. A Mumford orbifold is a one-dimensional orbifold $\mathcal{X} = (X, (\xi_i, G_i))$ admitting a Galois cover $Y \to X$ by a Mumford curve $Y$, ramified exactly above the $\xi_i$ with decomposition group isomorphic to $G_i$.

The importance of ordinarity is

Lemma 3. In characteristic $p > 0$ any Mumford orbifold is ordinary.

Proof. [9, Lemma 4.4.2]

We also have

Lemma 4. If $\mathcal{X} = (X, (\xi_i, G_i))$ is a Mumford orbifold, then $X$ is a Mumford curve.

Proof. For example, the proof of [3, Lemma 4.2] exhibits this fact.

3 Kato trees

We will be subsequently working with topological graphs. Only the edges homeomorphic to the interval $[0, 1]$ will be called "edges". The ones homeomorphic to $[0, 1)$ are named cusps.

Let $\Omega \subseteq \mathbb{P}^1$ be a connected, open, strictly analytic subset. It is well-known that $\Omega$ is simply connected. In fact, there is exactly one injective path connecting two points on $\Omega$. Thus, for each $x \in \Omega$ we get a partial order $\leq_x$ on $\Omega$ by setting

$$y \leq_x z :\iff z \text{ is on the path } x \leadsto y.$$ 

It is easily seen that for any pair $y, z \in \Omega$ there is a unique $\sup(y, z) \in \Omega$ for $\leq_x$. This leads to an embedding of $\Omega$ into the space $\hat{\Omega}$ of all partial orders with suprema for pairs compatible with paths on $\Omega$ [2, 4.1.3].
Definition 5. The connected hull of \( \hat{\Omega} \setminus \Omega \) in \( \hat{\Omega} \) is called the skeleton of \( \Omega \).

Let \( X \) be a non-singular projective curve over \( K \). Replacing \( K \) by a finite extension, if necessary, guarantees the existence of a pure affinoid covering \( \mathcal{U} \) of \( X \) such that the reduction \( X_U \) along \( \mathcal{U} \) is semi-stable [1]. Let \( \pi_U : X \to X_U, x \mapsto \bar{x} \) be the corresponding Tate map. It is well known that

\[
\pi^{-1}_U(\bar{x}) \text{ is a(n)} \quad \begin{cases} 
\text{point}, & \text{if } \bar{x} \in \bar{X}_{\text{gen}} \\
\text{open disk}, & \text{if } \bar{x} \in \bar{X}_{\text{ns}} \\
\text{open annulus}, & \text{if } \bar{x} \in \bar{X}_{\text{sing}}
\end{cases}
\]

[2, 4.3.1]. Here \( \bar{X}_{\text{gen}}, \bar{X}_{\text{ns}} \) resp. \( \bar{X}_{\text{sing}} \) means the set of generic, non-singular resp. singular points of the connected curve \( \bar{X} \) defined over \( k \). The skeleton \( \Delta(\pi^{-1}_U(\bar{x})) \) is therefore either empty (\( \bar{x} \in \bar{X}_{\text{gen}} \) or \( \bar{X}_{\text{ns}} \)) or homeomorphic to the open unit interval. The graph \( \Delta_U(X) \) is defined as follows: its vertex set is \( \pi^{-1}_U(\bar{X}_{\text{gen}}) \), and the edges are the closures \( \pi^{-1}_U(\bar{x})_{\text{cl}} \) in \( X \) (and therefore homeomorphic either to \([0, 1]\) or \([0, 1] \)).

Definition 6. The graph \( \Delta_U(X) \) is called the analytic skeleton of \( X \) with respect to the pure affinoid covering \( \mathcal{U} \).

A discrete group \( G \) acting discontinuously on \( \Omega \) induces an action on the skeleton \( \Delta(\Omega) \), which in fact is a tree [2, 4.1.7]. If \( X = \Omega/G \) is a non-singular irreducible projective curve, then there is a pure affinoid covering \( \mathcal{U} \) of \( X \) such that

\[
\Delta(\Omega)/G \cong \Delta_U(X).
\]

The topological graph of groups \((\Delta_U(X), G_*)\) will be called a quotient skeleton.

Let now be given a finitely generated discrete group \( N \subseteq \text{PGL}_2(K) \). If we take \( \Omega^*(N) \) to be the complement of the set \( \mathcal{L}^*(N) \) of limit points of \( N \) and the fixed points of \( N \)'s elements of finite order, then its skeleton \( \Delta^*(N) := \Delta(\Omega^*) \) is a tree whose set of cusps is exactly \( \mathcal{L}^*(N) \).

Definition 7. The quotient skeleton \( \Gamma^*(N) := (\Delta^*(N)/N, N_*) \) is called the Kato graph of \( N \).

There is also \( \Omega(N) := \mathbb{P}^1 \setminus \mathcal{L}(N) \), the complement of the set \( \mathcal{L}(N) \) of limit points of \( N \) in \( \mathbb{P}^1 \). The corresponding quotient skeleton

\[
\Gamma(N) = (\Delta(\Omega(N))/N, N_*)
\]

is a finite graph of groups, and the map \( \Omega(N) \to \Omega(N)/N \) is an orbifold cover of the Mumford curve \( \Omega(N)/N \). The graph \( \Gamma(N) \) is obtained by a contraction of the Kato graph \( \Gamma^*(N) \): cusps are cut off, and edges \( e \) whose stabiliser equals either \( N_{\delta(e)} \) or \( N_{\tau(e)} \) are replaced by the extremal vertex \( v \) with the larger group, if the valency of \( v \) is less than 3.

Kato states in [3, Proposition 2.2]

Lemma 8. There is a one-to-one correspondence between the set of cusps of \( \Gamma^*(N) \) and the set of branch points of the cover \( \Omega(N) \to \Omega(N)/N \). The stabiliser of a cusp is the decomposition group of the corresponding branch point.

The relationship between Mumford orbifolds and Kato graphs is

Theorem 1. A one-dimensional orbifold is a Mumford orbifold if and only if there exists a global Galois orbifold chart whose quotient skeleton is a Kato graph.
Proof. Denote the orbifold with $X = (X, (\xi, e_i))$.

The implication $\Rightarrow$ is quite clear: for an orbifold chart $\varphi: C \xrightarrow{/[G]} X$ take $\omega: \Omega \to C$ to be the universal cover of the Mumford curve $C$. Then the composition $\varphi \circ \omega$ is the quotient by an action of a discrete group $N \subseteq \text{PGL}_2(K)$, and the quotient graph $(\Delta^*(N)/N, N_*)$ is a Kato graph.

For $\Leftarrow$: let $C \to X$ be a chart whose quotient skeleton is a Kato graph for $N$. Being a fundamental group of a graph whose vertex and edge stabilisers are finite subgroups of $\text{PGL}_2(K)$, $N$ contains a free normal subgroup $H$ of finite index $[5, \text{I.\S}3]$. Let $\Omega \subseteq \mathbb{P}^1$ be their domain of regularity. Then there is a commuting diagramme

\[
\begin{array}{ccc}
\Omega & \xrightarrow{/[H]} & Y \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
X & \xrightarrow{/[N]} & \mathbb{P}^1
\end{array}
\]

with a Mumford curve $Y$. Since the vertical arrow is an orbifold chart, we are done. \qed

In his construction of triangle groups, Kato first studies the Kato trees of the finite subgroups of $\text{PGL}_2(K)$ $[8]$.

**Definition 9.** The Kato trees $T^*(G)$ for the finite groups of $\text{PGL}_2(K)$ are called elementary Kato trees.

**Proposition 10.** The elementary Kato trees look like this, if $\text{char } K = 0$:

1. If $\text{char } k > 5$, then all elementary Kato trees have only one vertex:

\[
\begin{array}{c}
\text{\rotatebox{-90}{\hspace{1cm}}}
\end{array}
\]

2. If $\text{char } k \leq 5$, then the elementary Kato trees for the groups of order divisible by $p$ are also star-shaped as in the above case, but many of them are instable (after contracting the cusps).

**Proof.** $[8]$.

**Remark 11.** The existence of edges in $\text{char } k \leq 5$ turns out to be important for constructing $p$-adic triangle groups of Mumford type $[8]$.

If $\text{char } K > 0$, the Kato tree for $G = C'_p$, as defined above, is empty $[4, \text{I.4.4.(i)}]$. We will resolve the problem in the following manner: since all $\gamma \in C'_p \setminus \{1\}$ are parabolic, the fixed point set in $\mathbb{P}^1$ of $G$ is a one-cusped subtree of $\Delta(\mathbb{P}^1 \setminus \mathbb{P}^1(K))$, and may be contracted to a vertex with one cusp. We will call that tree, by abuse of language, the Kato tree for $C'_p$.

The geodesic in $\mathbb{P}^1$ between the two $K$-rational fixed points of an elliptic transformation $\gamma$ is called the mirror $M(\gamma)$ of $\gamma$. 


Let $P(2,p^t)$ denote either $\text{PGL}_2(\mathbb{F}_{p^t})$ or $\text{PSL}_2(\mathbb{F}_{p^t})$. In the former case, we will abbreviate $n_- := p^t - 1$ and $n_+ := p^t + 1$, whereas in the latter, $n_- := \frac{p^t - 1}{2}$ and $n_+ := \frac{p^t + 1}{2}$.

In the proof of the following proposition, we will take particular care whether (two-pointed) mirrors are folded, i.e. mapped onto a half-line, or not.

**Proposition 12.** If $\text{char } K = p > 0$, then any elementary Kato tree $T^*(G)$ for $G \subseteq \text{PGL}_2(K)$ viewed as a subgroup of $\text{PGL}_2(\mathbb{F}_{p^m})$ is one of the following:

1. $G = C_n$ for $(n, p) = 1$: $n \leftarrow C_n \rightarrow n$

2. $G = D_n$ for

   (a) $p \neq 2$ and $n \mid p^m \pm 1$

   (b) $p = 2$ and $(n, 2) = 1$

3. $G = B(t, n)$ for $t \leq m$ and

   (a) $n \mid p^m - 1$, $n \mid p^t - 1$, $n > 1$

   (b) $n = 1$

4. $G = P(2, p^t)$: $n_+ \rightarrow P(2, p^t) \rightarrow B(t, n_-)$

5. $G = T$ for $p \neq 2, 3$

6. $G = O$ for $p \neq 2, 3$

7. $G = I$ for $5 \mid p^{2m} - 1$ and

   (a) $p \neq 2, 3, 5$
(b) $p = 3$: \[5 \xrightarrow{I} B(1,2)\]

**Proof.** The list of groups is Dickson’s classification of finite subgroups of $\text{PGL}_2(K)$ [4, II.8.27], to be found also in [10, Theorem 2.9]. The trees of 1., 2.(a), 5., 6., 7.(a) have been constructed in [8, Appendix A] for char $K = 0$. Kato’s construction is also possible in our situation. 3.(b) is our convention from above.

2.(b): $D_n = C_n \rtimes C_2$, and $C_2$ interchanges the two $K$-rational fixed points of $C_n$, thus folding the mirror of a generator of $C_n$, while the generator of $C_2$ is parabolic.

3.(a): The tree is the intersection of the two mirrors $M(\sigma)$ and $M(\gamma)$ of an elliptic $\sigma$ of order $n > 1$ and $K$-rational fixed points, say, 0 and $\infty$, and a parabolic $\gamma$ with fixed point 0. So one of the cusps of $M(\sigma)$ is stabilised by all of $B(t, n)$.

4.: $P(2, q^t)$ contains the dihedral group $D_{n+}$. The element of order two in it folds the mirror of any other generator of order $n_+$. Since $T^*(P(2, q^t)$ has only two cusps, anyway, the other one is stabilised by $B(t, n_-)$. Since, according to [4, Lemma 4.3], every path emanating from the vertex $v$ with stabiliser $P(2, p^t)$ is locally stabilised by either $C_{n+}$ or $B(t, n_-)$, the Kato tree has the shape as drawn.

7.(b): As $I$ contains dihedral groups $D_5$ and $D_2$, the mirrors of elements of orders 2 and 5 are folded. But the former mirror has a cusp in common with the mirror of a parabolic transformation. \[\square\]

**Definition 13.** A tree of groups with non-trivial edge groups is called an irreducible tree. A maximal irreducible subtree of a graph $\Gamma$ of groups is called an irreducible component of $\Gamma$.

The following Proposition is decisive for the rest of this paper.

**Proposition 14.** Every irreducible Kato tree $T = (T^*(N), N_*)$ is obtained by glueing the elementary Kato trees $T^*(N_v)$ for the vertex groups of $T$ along the trees $T^*(N_e)$ for the edge groups:

$$T \cong \lim_{\rightarrow} T^*(N_e).$$

**Proof.** The proof in [8, Proposition 3.9] works in all characteristics. \[\square\]

4 The structure of Kato graphs

An important question in the analysis of Kato graphs is whether mirrors fold or not under the action of a given group.

4.1 The characteristic zero case

**Theorem 2.** Let char $K = 0$. If $C$ denotes the number of cyclic vertex groups, $c$ the number of cyclic edge groups, $D$ the number of non cyclic vertex groups, and $d$ the number of non cyclic edge groups of $\Gamma(N)$, then the number of cusps of $\Gamma^*(N)$ is given by

$$\# \partial \Gamma^*(N) = 3(D - d) + 2(C - c).$$
Proof. First let $\Gamma^*(N) = T$ be an irreducible Kato tree. Then, according to Proposition 14, $T \cong \lim \to T^*(N_e)$ is obtained by glueing the elementary Kato trees for the vertex groups along the trees for the edge groups. So, all we have to do is to examine the glueing process, and check the formula at each step. A detailed description of the glueing morphisms as quotients of the injective glueing morphisms $\Delta^*(N_e) \to \Delta^*(N_v)$ for $v$ an extremity of $e$ has already been done in [8].

If $\text{char} \ k > 5$, then only cyclic groups can occur as edge groups. [8] tells us that in this case segments are obtained somewhat like

$T_1:\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ T_2:\$

\[
\begin{align*}
\begin{array}{c}
\xymatrix{ & m \ar[ld]_{\cong R \geq 0} \ar[rd]^{m} \ar[rr]^{C_m} & & m \ar[ld]_{\cong R \leq 0} \ar[rd]^{m} \\
& T_0: & & T_0: & \end{array}
\end{align*}
\]

along $T_0$:

\[
\begin{array}{c}
\xymatrix{ & m \ar[ld]_{\cong R \leq 0} \ar[rd]^{m} \ar[rr]^{C_m} & & m \ar[ld]_{\cong R \geq 0} \ar[rd]^{m} \\
& T_0: & & T_0: & \end{array}
\]

with glueing morphisms

$$\varphi_1: T_0 \to T_1, \ t \mapsto |t|, \ \ \ \ \ \ \varphi_2: T_0 \to T_1, \ t \mapsto -|1 - t|,$$

resulting in

$T_1 \#_{T_0} T_2:\$

\[
\begin{array}{c}
\xymatrix{ & m \ar[ld]_{\cong R \geq 0} \ar[rd]^{m} \ar[rr]^{C_m} & & m \ar[ld]_{\cong R \leq 0} \ar[rd]^{m} \\
& T_0: & & T_0: & \end{array}
\]

This proves the formula for irreducible Kato trees, if $\text{char} \ k > 5$.

Now, for $\text{char} \ k \leq 5$, there are elementary Kato trees with edges (homeomorphic to $[0, 1]$). This makes it possible to glue along trees for non cyclic groups in order to obtain trees with edges. Kato has shown [8] that in those cases one of the following occurs:

- Both glueing morphisms are injective.
- One of the glueing morphisms folds an unfolded mirror of $T_0$.
- Both glueing morphisms fold an unfolded mirror of $T_0$.

Any of these possibilities reduces the total amount of ends by three, and the formula holds in this case.

Now that the formula has been proven for irreducible trees, we examine the case that $T = \Gamma^*(N)$ is a general Kato tree. Then $N$ is a free tree product of fundamental groups $N_i$ of irreducible Kato trees $T_i = \Gamma^*(N_i)$ which are connected within $T$ by edges with trivial stabiliser. Obviously, the number of cusps of $T$ equals the sum of the number of cusps belonging to the irreducible components $T_i$. The formula is therefore valid for all Kato trees.

Now, let $\Gamma = \Gamma^*(N)$ be a Kato graph of positive genus $g$. Since the fundamental group $N$ of $\Gamma$ contains a free normal subgroup of rank $g$, we may remove $g$ edges with trivial stabiliser in such a way as to obtain a tree of groups with the same number of vertices. This proves the formula in all genera of $\Gamma^*(N)$. \qed
Example 1. One of the simpler triangle groups is for \( \text{char} \ k = 5 \): \( T_0 = \Gamma^* (D_5), \ T_1 = \Gamma^* (A_5), \ T_2 = \Gamma^* (D_{10m}) \).

\[ \begin{array}{c}
T_1 \rightarrow \ T_1 \# T_0 \ T_2 \\
\begin{array}{c}
\text{fold in } v_0 \\
\text{fold in } v_1 \\
\text{x} \rightarrow |\text{x}|,
\end{array}
\end{array} \]

Note that the point \( v_0 \) in \( T_1 \) is a vertex, whereas in \( T_0 \) and \( T_2 \) it is not: it is a marked point on the cusp viewed as the half-open interval \([0, 1)\).

4.2 The general case

Let now \( K \) be of any characteristic. If \( \text{char} \ K > 0 \), then the formula of Theorem 2 has to be stated in more general terms, as Kato trees for non-cyclic groups may have different numbers of cusps. Let \( V \) denote the set of vertices and \( E \) the set of edges of a given graph.

Theorem 3. The number of cusps in a Kato graph \( \Gamma^*(N) \) is given by

\[ \# \partial \Gamma^*(N) = \sum_{v \in V} \# \partial T^*(N_v) - \sum_{e \in E} \# \partial T^*(N_e). \]

Proof. The proof reduces to considering \( \text{char} \ K > 0 \). As in the characteristic zero case, we need to treat irreducible Kato trees, only. In positive characteristic, the glueing procedure turns out to be of the following:

1. For edge groups \( N_e \) we have \( \# \partial T^*(N_e) \leq 2 \).

2. If \( \# \partial T^*(N_e) = 1 \), then both glueing morphisms \( T^*(N_e) \rightarrow T^*(N_{o(e)}) \) and \( T^*(N_e) \rightarrow T^*(N_{t(e)}) \) are injective.

3. If \( \# \partial T^*(N_e) = 2 \), then the glueing morphisms either both fold the line \( T^*(N_e) \) in two different points, or one of the glueing morphisms is an isomorphism of trees with injective morphisms between stabilisers.

The proof of these claims.
1. Edge stabilisers are Borel groups [4, Lemma 4.1] (The cited Lemma holds without the restrictions of [4, Section 4]).
2. Obvious.
3. We have $N_e = B(t, n)$ with $n > 1$. If $N_e$ is cyclic (of order necessarily prime to $p$), then glueing is as in the characteristic zero case.

Let now $N_e = B(t, n)$ with $t > 0$. There are two possibilities: either the stabiliser of the origin $o(e)$ is of Borel type or not. In the first case, $N_{o(e)} = B(t', n')$, then $n' = n$ because the prime-to-$p$ part of $N_e$ is a maximally cyclic subgroup of $N_{o(e)}$ [8, Lemma 1].

In the second case, we have $N_{o(e)} = P(2, p^t)$. Then, as we have seen in the proof of Proposition 12, $t = t$ and $n = n_\infty$. An element of order two in $P(2, p^t)$ folds the geodesic tree $T^*(B(t, n))$ in its vertex $v_0$. The glueing morphism maps $v_0$ somewhere onto the cusp of $T^*(N_{o(e)})$ with stabiliser $B(t, n)$. The pre-image of $e$ in $T^*(N_{o(e)})$ is the path from its vertex to $v_0'$, and its stabiliser is $B(t, n)$. As shown in the proof of 4, 4.6], the ends of any geodesic in $T^*(N)$ going through $e$ are stabilised by $C_{n_\infty}$ and by $B(t, n)$, respectively, and the stabilisers of points on the latter end form an increasing sequence of Borel groups, as one moves away from $t(e)$. Therefore, $N_{t(e)} = B(t', n)$ with $t | t'$, and the map $T^*(N_e) \to T^*(N_{t(e)})$ merely replaces $B(t, n)$ by $B(t', n)$.

These are all possibilities.

**Example 2.** A non-elementary Kato tree with two cusps may be constructed in this way: let $n = p^t - 1$ and $t | s$, then

$$\begin{align*}
\begin{array}{c}
p^t+1 \quad \bullet \quad B(t, n) \quad \bullet \quad B(t, n) \\
PGL_2(p^t) \quad \downarrow v_0 \quad \uparrow \quad B(t, n)
\end{array}
\end{align*}$$

$$\begin{align*}
\begin{array}{c}
p^t+1 \quad \bullet \quad B(s, n) \quad \bullet \quad B(s, n) \\
PGL_2(p^t) \quad \downarrow v_0 \quad \uparrow \quad B(s, n)
\end{array}
\end{align*}$$

is Cartesian. The stabiliser of the edge in the upper right tree is the intersection of the two vertex groups: $PGL_2(p^t) \cap B(x, n) = B(t, n)$.

From the proofs of the two preceding theorems we get

**Proposition 15.** Let $\Gamma$ be a Kato graph. Then

1. Each vertex has at most three cusps or edges with non trivial stabilisers going out of it.
2. The vertex stabilisers are generated by the outgoing edges’ or cusps’ groups.

### 4.3 Separating the branch points

The knowledge we have over the glueing process allows us to control the separation of branch points.
Theorem 4. If $\mathcal{X} = (X, (x_i, G_i))$ is a Mumford orbifold, then there is a pure affinoid covering of $X$ separating the points $x_i$ into triplets, pairs and singlets.

Proof. An orbifold chart $\varphi: Y \xrightarrow{\pi} X$ with a Mumford curve $Y$ gives us a Kato graph $\Gamma^*(N) = (\Gamma, G_\bullet)$ with fundamental group $N = \varprojlim G_\bullet$. As we have seen in the proof of Theorem 2, $\Gamma$ is obtained by glueing elementary Kato trees. Such trees have at most three cusps, and therefore, at most three cusps emanate from each vertex of $\Gamma$.

Now, the cusps themselves are Berkovich paths from the branch points $x_i$ to the nearest vertex, i.e. a generic point of a disc with radius in $|K^\times|$ containing all points $x_i$ to which the cusps emanating from that vertex lead. In the case of two or three cusps, that disc is the smallest one containing the corresponding points which are equidistantly positioned on the disc’s boundary. Any nearest vertex in $\Gamma$ with cusps is a generic point of a disc with radius in $|K^\times|$ containing the corresponding points $x'_i$ in its boundary. The shortest path to the first vertex gives the distance between those two generic points. All edges emanating from those two vertices define two affinoid subsets of $X$ separating the set $\{x_i\}$ from $\{x'_i\}$ and the rest of the marked points of $\mathcal{X}$. As the intersection of the two affinoid sets consists of boundary components (coming from edges connecting the two points), we see that it is a pure subset of both affinoid pieces. Thus, one obtains a pure affinoid covering fulfilling the requirements of the theorem.

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