Cross-Linked Structure of Network Evolution

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We study the temporal covariance of network co-evolution via the cross-link structure of networks, for which we take advantage of the formalism of hypergraphs to map cross-link structures back to network nodes. We investigate two sets of temporal network data in detail. In a network of coupled nonlinear oscillators, hyperedges that consist of network edges with temporally co-varying weights uncover the driving co-evolution patterns of edge weight dynamics both within and between oscillator communities. In the human brain, networks that represent temporal changes in brain activity during learning exhibit early co-evolution that then settles down with practice, and subsequent decreases in hyperedge size are consistent with emergence of an autonomous subgraph whose dynamics no longer depends on other parts of the network. Our results on real and synthetic networks give a poignant demonstration of the ability of cross-link structure to uncover unexpected co-evolution attributes in both real and synthetic dynamical systems. This, in turn, illustrates the utility of analyzing cross-links for investigating the structure of temporal networks.

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Many complex systems can be represented as temporal networks, which consist of components (i.e., nodes) that are connected by time-dependent edges [1, 2]. The edges can appear, disappear, and change in strength over time. To obtain a deep understanding of real and model networked systems, it is critical to try to determine the underlying drivers of such edge dynamics. The formalism of temporal networks provides a means to study dynamic phenomena in biological [3–5], financial [6, 7], political [8–10], social [11–17] systems, and more.

Capturing salient properties of temporal edge dynamics is critical for characterizing, imitating, predicting, and manipulating system function. Let’s consider a system that consists of the same $N$ components for all time. One can parsimoniously represent such a temporal network as a collection of edge-weight time series. For undirected networks, we thus have a total of $N(N-1)/2$ time series, which are of length $T$. The time series can either be inherently discrete or they can be obtained from a discretization of continuous dynamics (e.g., from the output of a continuous dynamical system). In some cases, the edge weights that represent the connections are binary, but this is not true in general.

Several types of qualitative behavior can occur in time series that represent edge dynamics [18, 19]. For example, unvarying edge weights are indicative of a static system, and independently varying edge weights indicate that a system does not exhibit meaningfully correlated temporal dynamics. A much more interesting case, however, occurs when there are meaningful transient or long-memory dynamics. As we illustrate in this Letter, one can obtain interesting insights in such situations by examining network cross-links, which are defined via the temporal covariance in edge weights. Illuminating the structure of cross-links has the potential to enable predictability.

To gain intuition about the importance of analyzing cross-links, it is useful to draw an analogy from biology. The cellular cytoskeleton [20] is composed of actin filaments that form bridges (edges) between different parts (nodes) of a cell. Importantly, the bridges are themselves linked to one another via actin-binding proteins. Because the network edges in this system are not independent of each other, the structure of cross-links has important implications for the mechanical and transport properties of the cytoskeleton. Similarly, one can think of time-dependent relationships between edge weights as cross-links that might change the temporal landscape for dynamic phenomena like information processing, social adhesion, and systemic risk. Analyzing cross-links allows one to directly investigate time-dependent correlations in a system, and it thereby has the potential to yield important insights on the (time-dependent) structural integrity of a diverse variety of systems.

In this Letter, we develop a formalism for uncovering the structure in time-dependent networks by extracting groups of edges that share similar temporal dynamics.
We map these cross-linked groups of edges back to the nodes of the original network using hypergraphs [21]. We define a co-evolution hypergraph via a set of hyperedges that captures cross-links between network edges, where each hyperedge is given by the set of edges that exhibit statistically significant similarities to one another in the edge-weight time series (see Fig. 1). A single temporal network can contain multiple hyperedges, and each of these can capture a different temporal pattern of edge-weight variation.

We illustrate our approach using ensembles of time-dependent networks extracted from a nonlinear oscillator model and empirical neuroscience data.

**CROSS-LINK STRUCTURE**

To quantify network co-evolution, we extract sets of edges whose weights co-vary in time. For a temporal network $A_t$, where each $t$ indexes a discrete sequence of $N \times N$ adjacency matrices, we calculate the $E \times E$ adjacency matrix $A$, where the matrix element $A_{ab}$ is given by the Pearson correlation coefficient between the time series of weights for edge $a$ and that for edge $b$. Note that $E = N(N-1)/2$ is the total number of possible (undirected) edges per layer in a temporal network. The layers can come from several possible sources: data can be inherently discrete, so that each layer represents connections at a single point in time; the output of a continuous system can be discretized (e.g., via constructing time windows); etc. We identify the statistically significant elements of the edge-edge correlation matrix $A$ (see the Supplemental Material), and we retain these edges (with their original weights) in a new matrix $A'$. We set all other elements of $A'$ to 0.

We examine the structure of the edge-edge co-variation represented by the $E \times E$ matrix $A'$ by identifying sets of edges that are connected to one another by significant temporal correlations (i.e., by identifying cross-links; see Fig. 1). If $A'$ contains multiple connected components, then we study each component as a separate edge set. If $A'$ contains a single connected component, then we extract edge sets using community detection (see SM). We represent each edge set as a hyperedge, and we thereby construct a co-evolution hypergraph $H$. The nodes are the original $N$ nodes in the temporal network, and they are connected via a total of $\eta$ hyperedges that we identified from $A'$.

**Diagnostics.** To evaluate the structure of co-evolution hypergraphs, we compute several diagnostics. To quantify the extent of co-evolution, we define the strength of co-variation as the sum of all elements in the edge-edge correlation matrix: $\nu_{A_t} = \sum_{a,b} A'_{ab}$. To quantify the breadth of a single co-variation profile, we define the size of a hyperedge as the number of cross-links that comprise the hyperedge: $s(h) = \frac{1}{2} \sum_{a,b \in \lambda} a'_{ab} > 0 \lambda$, where the square brackets denote a binary indicator function (i.e., 1 if is true and 0 if it is false) and $\lambda$ indicates the set of edges that are present in the hyperedge $h$ of the matrix $A'$. To quantify the prevalence of hyperedges in a single node in the network, we define the hypergraph degree of a node $i$ to be equal to the number of hyperedges $\eta_i$ associated with node $i$.

**NETWORKS OF NONLINEAR OSCILLATORS**

Synchronization provides an example of network co-evolution, as the coherence (represented using edges) between many pairs of system components (nodes) can increase in magnitude over time [22, 23]. Pairs of edge-weight time series exhibit temporal covariance (i.e., nontrivial cross-links) because they experience such a trend. Perhaps less intuitively, nontrivial network co-evolution can also occur even without synchronization. To illustrate this phenomenon, we construct temporal networks from the time-series output generated by interacting Kuramoto oscillators [24], which are well-known dynamical systems that have been studied for their synchronization properties (both with and without a nontrivial underlying network structure) [22, 23, 25–30]. By coupling Kuramoto oscillators on a network with community structure [29], we can probe the co-evolution of edge weight time series both within and between synchronizing communities.

In Fig. 2A, we depict the block-matrix community structure in a network of 128 Kuramoto oscillators with 8 equally-sized communities. The phase $\theta_i(t)$ of the $i^{th}$ oscillator evolves in time according to

$$\frac{d\theta_i}{dt} = \omega_i + \sum_j \kappa C_{ij} \sin(\theta_j - \theta_i), \quad i \in \{1, \ldots, N\},$$

where $\omega_i$ is the natural frequency of oscillator $i$, the matrix $C$ gives the binary-valued (0 or 1) coupling between each pair of oscillators, and $\kappa$ (which we set to 0.2) is a positive real constant that indicates the strength of the
coupling. We draw the frequencies $\omega_i$ from a Gaussian distribution with mean 0 and standard deviation 1. Each node is connected to 13 other nodes (chosen uniformly at random) in its own community and to one node outside of its community. This external node is chosen uniformly at random from the set of all nodes from other communities. This external node is connected to 13 other nodes (chosen uniformly at random) in its own community and to one node outside of its community. This external node is chosen uniformly at random from the set of all nodes from other communities.

To quantify the temporal evolution of synchronization patterns, we define a set of temporal networks from the time-dependent correlations (which, following Ref. [29], we use to measure synchrony) between pairs of oscillators: $A_{ij}(t) = \langle |\cos(\theta_i(t) - \theta_j(t))| \rangle$, where the angular brackets indicate an average over 20 simulations. We perform simulations, each of which use a different realization of the coupling matrix C. (See the SM for details of the numerics.) Importantly, edge weights not only vary (see Fig. 2B) but they also co-vary with one another (see Fig. 2C) in time: the strength of network co-evolution, which we denote by $\nu_{A_h}$, is greater than that expected in a null-model network in which each edge-weight time series is independently permuted uniformly at random.

In this example, the cross-links given by the non-zero elements of $A'$ form a single connected component due to the extensive co-variance. One can distinguish cross-links according to their roles relative to the community structure in Fig. 2A [31]: (i) pairs of within-community edges, (ii) pairs of between-community edges, and (iii) pairs composed of one within-community edge and one between-community edge. Assortative pairings (i.e., cases (i) and (ii)) are significantly more represented than disassortative pairings (i.e., case (iii)) (see Fig. 2D). The assortative nature of cross-links might be driven by the underlying community structure in the block structure in Fig. 2A: within-community edges are directly connected to one another via shared nodes, whereas between-community edges are more distantly connected to one another via a common input (e.g., a sparse but frequently updating representation of the state of other oscillators).

Using community detection (see SM), we identified 5 distinct edge sets (i.e., hyperedges) in $A'$ with distinct temporal profiles (see Fig. 2E). The first hyperedge tends to connect within-community edges to each other. On average, they tend to synchronize early in our simulations. The second and third hyperedges tend to connect between-community edges to each other. The second hyperedge connects edges that tend to exhibit a late synchronization, and the third one connects edges that tend to exhibit an initial synchronization followed by a desynchronization. The fourth and fifth hyperedges are smaller in size (i.e., contain fewer edges) than the first three, and their constituent edges oscillate between regimes with high and low synchrony. The edges that constitute the fifth hyperedge oscillate at approximately one frequency, whereas those in the fourth hyperedge have multiple frequency components.

Together, our results demonstrate the presence of multiple co-evolution profiles: early synchronization, late synchronization, desynchronization, and oscillatory behavior [26]. Moreover, the assortative pairing of cross-links indicates that temporal information in this dynamic system is segregated not just within separate synchronizing communities but also in between-community edges.

**NETWORKS OF HUMAN BRAIN AREAS**

Our empirical data captures the changes in regional brain activity over time as experimental subjects learn a complex motor-sequencing task that is analogous to playing complex keyboard arpeggios. Twenty individuals practiced on a daily basis for 6 weeks, and we acquired MRI brain scans of blood oxygenated-level-dependent (BOLD) signal at four times during this period. We extracted time series of MRI signals from $N = 112$ parts of each individual’s brain [32]. Covariation in BOLD...
measurements between brain areas can indicate shared information processing, communication, or input; and changes in levels of coherence over time can reflect the network structure of skill learning. We summarize such functional connectivity [33] patterns using an \( N \times N \) coherence matrix [3, 4], which we calculate for each experimental block. We extract temporal networks, which each consist of 30 time points, for naive (experimental blocks corresponding to 0–50 trials practiced), early (60–230), middle (150–500), and late (690–2120) learning [32]. We hypothesize that learning should be reflected in changes of hypergraph properties over the very long time scales (6 weeks) associated with this experiment.

Temporal brain networks exhibit interesting dynamics: all four temporal networks have a non-zero variance in edge weights over time (see Fig. 3A). Importantly, edge weights not only vary but co-vary in time: the strength of network co-evolution \( \nu_A \) is greater in the 4 real temporal networks than expected in a random null-model network in which each edge-weight time series is independently permuted uniformly at random (see Fig. 3B). The magnitude of temporal covariance between functional connections is modulated by learning: it is smallest prior to learning and largest during early learning (i.e., amidst most performance gains). These results are consistent with the hypothesis that the adjustment of synaptic weights during learning alters the synchronization properties of neurophysiological signals [3], which could manifest as a steep gain in the co-evolution of synchronized activity of large-scale brain areas.

To uncover groups of co-evolving edges, we study the edge-edge correlation matrix \( \Lambda' \), whose density across the 4 temporal networks and the 20 study participants ranged from approximately 1% to approximately 95%. We found that the significant edges were already associated with multiple connected components, so we did not further partition the edge sets into communities. The distribution of component sizes \( s \) is heavy-tailed (see Fig. 3C), which perhaps reflects inherent variation in the communication patterns that are necessary to perform multiple functions required during learning [3]. With long-term training, hyperedges decrease in size (see Fig. 3C), which might reflect an emerging autonomy of sensorimotor regions that can support sequential motor behavior without relying on association cortex.

Hyperedges indicate temporal covariation of putative communication routes in the brain and can be distributed across different anatomical locations. The hypergraph node degree quantifies the number of hyperedges that are connected to each brain region. We observe that nodes with high hypergraph degree are located predominantly in brain regions known to be recruited in motor sequence learning [34]: the primary sensorimotor strip in superior cortex and the primary visual area in posterior cortex (see Fig. 3D).

FIG. 3. Co-evolution Properties of Brain Network Dynamics. (A) A histogram of the number of edges as a function of the standard deviation in edge weights over time for the 4 temporal networks. (B) The strength of network co-evolution \( \nu_A \), of 4 temporal networks and the respective null-model networks (gray). Error bars indicate standard deviation of the mean over study participants. (C) Probability distribution of the size \( s \) of hyperedges in the 4 learning hypergraphs. (D) Anatomical distribution of early-learning hypergraph node degree (averaged over the 20 participants). We obtain qualitatively similar results from the early, middle, late, and extended learning temporal networks. In panels (A–C), color and shape indicate the temporal network corresponding to naive (black circles), early (orange stars), middle (green diamonds), and late (blue squares) learning.

CONCLUSION

Networked systems are ubiquitous in technology, biology, physics, and culture. The development of conceptual frameworks and mathematical tools to uncover meaningful structure in network dynamics is critical for the determination and control of system function. We have demonstrated that the cross-link structure of network co-evolution, which can be represented parsimoniously using hypergraphs, can be used to identify unexpected temporal attributes in both real and simulated temporal dynamical systems. This, in turn, illustrates the utility of analyzing cross-links for investigating the structure of temporal networks.

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Supplemental Material for

“Cross-Linked Structure of Network Evolution”

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In this supplementary document, we include the following material to support the work described in the main text.

1. A detailed description of statistical corrections for edge-edge association matrices.

2. A description of the community-detection techniques that we applied to the edge-edge association matrix.

3. A characterization of the temporal profiles and final synchronization patterns of hyperedges in the network of Kuramoto oscillators.

4. Figure S1: Hyperedge Identification in a Network of Kuramoto Oscillators.

5. Figure S2: Final Synchronization Patterns and Temporal Profiles of Hyperedges.

6. Appendix on numerical implementation.
STATISTICAL CORRECTIONS FOR EDGE-EDGE ASSOCIATION MATRICES

In the main text, we describe a method for extracting cross-links from temporal networks. For a temporal network $A_t$, we calculate the $E \times E$ adjacency matrix $\Lambda$, where the matrix element $\Lambda_{ab}$ gives the Pearson correlation coefficient between the time series of weights for edge $a$ and the time series of weights for edge $b$. Note that $E = N(N - 1)/2$ is the total number of possible (undirected) edges per layer in a temporal network. (Each layer can come from a single point in time, aggregation over a given time window, etc.) For simplicity, we employ a correlation coefficient as a measure of statistical association to examine linear relationships in ensembles of edge-weight time series [1]. Because we seek to determine sets of edges that might have a common driver, we do not employ sparse network methods such as the graphical lasso [2] or Bayesian network [3] methods that attempt to estimate pairwise relationships between time series in a manner that is independent of other variables.

Given the very large number of statistical tests that the above procedure entails, we threshold the edge-edge correlation matrix $\Lambda$ to retain only statistically significant connections, which we determine by estimating the p-value associated with the Pearson coefficient $r$ for each edge-edge correlation. Using a false-positive correction for multiple comparisons, we threshold $\Lambda$ by identifying significant matrix elements as those whose associated p-value satisfies

$$p < \frac{1}{M} = \frac{2}{E(E - 1)},$$

where $M$ is the number of tests that were performed. We retain the original weights of significant matrix elements in a new matrix $\Lambda'$ and set nonsignificant matrix elements to 0.

The type of multiple comparisons correction that one uses to control for Type I errors (i.e., false positives) in correlation matrices derived from (both real and simulated) dynamical systems is itself interest [4, 5]. The false-positive correction of $p < 1/M$ that we applied is an increasingly common choice in the study of correlation matrices in the neuroscience literature [6–12]. It has been argued that alternative choices, such as the false-discovery rate [13, 14] and Bonferroni-correction methods [15–17], are too stringent for situations like correlation matrices in which variables are highly inter-dependent [5], and they can lead to an overly large number of Type II errors (i.e., false negatives) [18].

After performing the statistical correction to obtain the weighted thresholded matrix $\Lambda'$, we wish to extract cohesive sets of co-evolving edges. Two potential cases are apparent. The
simpler case occurs when \( \Lambda' \) is composed of disconnected components that each contain a set of co-evolving edges. We illustrate this scenario in the main manuscript using networks of brain regions. In a second case, \( \Lambda' \) contains a single large connected component — which can but need not include all of a network’s edges — from which one must further extract sets of co-evolving edges. We illustrate this scenario, which arises from extensive and broadly distributed temporal covariance, in the main manuscript using networks of Kuramoto oscillators.

To study the second scenario, we need to use a method for extracting sets of strongly cross-linked edges in \( \Lambda' \). One possible approach is to choose a more stringent statistical threshold for creating \( \Lambda' \) in the first place. For example, one could tune the threshold so that it fragments \( \Lambda \) into several disconnected components. However, such an approach requires the choice of an arbitrarily stringent threshold on the p-value \( p \) and entails the risk of Type II errors (i.e., false negatives) [5]. In this paper, we employ an alternative approach: we extract sets of strongly cross-linked edges using community detection [19, 20]. An advantage of this approach is that we can exploit the complete information housed in \( \Lambda' \) by using community-detection methods that account for cross-link weights and their signs [21].

COMMUNITY DETECTION ON EDGE-EDGE ASSOCIATION MATRICES

Methods for detecting communities in networks make it possible to algorithmically extract groups of nodes that are highly and mutually interconnected [19, 20, 22]. In this paper, we seek sets of edges that are strongly and densely cross-linked to one another[23]. We identify such “communities” (or “modules”) of edges by optimizing a modularity quality function that is suitable for signed matrices [21]:

\[
Q = \sum_{ab} \left[ \Lambda'_{ab} - \gamma^+ P^+_{ab} - \gamma^- P^-_{ab} \right] \delta(g_a, g_b),
\]

where \( \Lambda' = (\Lambda'_{ab}) \) is the \( E \times E \) thresholded and weighted correlation matrix, edge \( a \) is assigned to community \( g_a \), edge \( b \) is assigned to community \( g_b \), the Kronecker delta \( \delta(g_a, g_b) = 1 \) if \( g_a = g_b \) and it equals 0 otherwise, \( \gamma^+ \) and \( \gamma^- \) are resolution parameters, and \( P^+_{ab} \) and \( P^-_{ab} \) are the respective expected weights of the positive and negative cross-links that connect edge \( a \) and edge \( b \) via a specified null model. We employ a signed null model [21] with \( \gamma^+ = \gamma^- \)
(also see [24], who study the case $\gamma^+ = \gamma^- = 1$), so that

$$P_{ab}^+ = \frac{k_a^+ k_b^+}{\sum_{ab} k_a^+}, \quad P_{ab}^- = \frac{k_a^- k_b^-}{\sum_{ab} k_a^-},$$

(3)

where $k_a^\pm = \sum_b \Lambda'_{ab}^\pm$ is the strength of cross-link $a$ in the matrix $\Lambda'^\pm$. The matrix $\Lambda'^+$ retains all positively weighted elements of $\Lambda'_ab$ and sets all negatively weighted elements of $\Lambda'_ab$ to 0. The matrix $\Lambda'^-$ retains all negatively weighted elements of $\Lambda'_ab$ and sets all positively weighted elements of $\Lambda'_ab$ to 0.

Maximization of $Q$ yields a hard partition of the edge-edge network into communities such that the total cross-link weight inside of communities is as large as possible (relative to the null model and subject to the limitations of the employed computational heuristics, as optimizing $Q$ is NP-hard [19, 20, 25]). Given the near-degeneracy of the landscape of the modularity function $Q$ [26], we perform 100 optimizations of Eq. 2 and obtain consensus partitions over these optimizations via a comparison to an appropriate null model. (See Ref. [27] for a detailed description of the method.)

The structural resolution parameter $\gamma = \gamma^+ = \gamma^-$ is a tunable scalar that sets the size of the communities in the (near) optimal partition [27, 28]. Small values of $\gamma$ produce a few large communities, whereas large values of $\gamma$ produce many small communities. By tuning $\gamma$, one can therefore examine the community structure at different scales [29–34] of both real [35–39] and simulated [40, 41] dynamical systems.

For simplicity, we focus on a single resolution-parameter value for detailed investigation. We choose a value that provides insight into the relationship between the community structure of edges and the community structure of nodes, and later we discuss at length the procedure that we used to select this value. In Fig. 1A, we show a template block structure that summarizes the community structure of nodes in a network of Kuramoto oscillators. Each block contains edges that are located either (i) within communities (template blocks 1–8) or (ii) between communities (template blocks 9–36). We characterize the similarity between this template (which yields a network partition that we label by $\alpha$) and the community structure of edges at a given value of the structural resolution parameter $\gamma$ (which yields a partition that we label by $\beta$) using the z-score of the Rand coefficient [42]. We use $w_{11}$ to denote the count of edge pairs that are classified together in both partitions (e.g., $\alpha$ and $\beta$). We use $w_{10}$ to denote the count of edge pairs that are classified together in the first partition but classified separately in the second partition, and we define $w_{01}$ analogously.
as the count of edge pairs that are classified separately in the first partition but classified together in the second partition. We use \( w_{00} \) to denote the count of edge pairs that are classified separately in both partitions. The total number \( R \) of node pairs is then given by the sum of these quantities: \( R = w_{11} + w_{10} + w_{01} + w_{00} \). We calculate the Rand \( z \)-score in terms of the network’s total number of node pairs \( R \), the number of pairs \( R_\alpha \) classified the same way in partition \( \alpha \), the number of pairs \( R_\beta \) classified the same way in partition \( \beta \), and the number of node pairs \( w_{\alpha\beta} \) that are assigned to the same community both in partition \( \alpha \) and in partition \( \beta \). The \( z \)-score of the Rand coefficient comparing these two partitions is

\[
z_{\alpha\beta} = \frac{1}{\sigma_{w_{\alpha\beta}}} \left( w_{\alpha\beta} - \frac{R_\alpha R_\beta}{R} \right),
\]

where \( \sigma_{w_{\alpha\beta}} \) is the standard deviation of \( w_{\alpha\beta} \) (as in [42]).

In the resolution parameter range \( \gamma \in [0, 2, 4] \), the \( z \)-score of the Rand coefficient between the template and partitions into communities of edges appears to have two regimes (see Fig. 1B). For \( \gamma \leq 1.8 \), the \( z \)-score exhibits large variability over multiple optimizations of the modularity quality function in Eq. 2, which suggests that the optimization landscape of \( Q \) is replete with local maxima [26]. However, for \( \gamma \geq 1.8 \), the \( z \)-score has a much smaller variability over the multiple optimizations, which suggests that the partitions in this regime are relatively robust [27]. In this second regime, \( \gamma \geq 1.8 \), the mean \( z \)-score also decreases with increasing \( \gamma \), which indicates that partitions with a large number of small communities (i.e., for \( \gamma \) values closer to 4) exhibit less similarity to the template than partitions with a small number of large communities (i.e., \( \gamma \) values closer to 1.8).

We choose to examine the community structure in the edge-edge correlation matrix at the resolution parameter \( \gamma = 1.8 \) for two reasons: (i) at this resolution-parameter value, partitions are more robust (i.e., less variable) over multiple optimizations than they are at lower values of \( \gamma \); and (ii) this approximately maximizes the similarity, as measured by the Rand \( z \)-score, between the community structure of edges and the community structure of nodes. To visualize the cross-linked edge communities that are present at \( \gamma = 1.8 \), we construct a consensus partition [43] over the 100 optimizations using a method that corrects for statistical noise in sets of partitions defined in comparison to a null model [27]. The consensus partition assigns each edge to one of 5 communities of varying sizes (see Fig. 1C). Each community yields a hyperedge, and we note that the pattern of hyperedges in the network has an inherently different structure than the final synchronization pattern of the
network of Kuramoto oscillators (compare Figs. 1C and D) [44]. In the next section, we characterize the differences between these two structures in greater detail.

**TEMPORAL PROFILES AND FINAL SYNCHRONIZATION PATTERNS OF HYPEREDGES IN THE NETWORK OF KURAMOTO OSCILLATORS**

Each hyperedge that we identified in the network of Kuramoto oscillators consists of a set of edges with a different temporal weight profile (see Fig. 2A). Edges are cross-linked based on the similarity in their temporal weight profile, and community detection makes it possible to extract cohesive groups of edges with similar profiles. The first two hyperedges, whose dynamics we show in the left two panels of Fig. 2, tend to consist of between-community edges (see Fig. 2B) and exhibit either late increases in weight (which yields late synchronization) or decreases in weight (desynchronization) over time (see Fig. 2A). The hyperedge whose dynamics we show in the center panel of Fig. 2 includes the majority of the within-community edges and exhibits a strong increase in weight (and hence oscillator synchronization) early in the simulation. The final two hyperedges, whose dynamics we show in the right two panels of Fig. 2, consist of edges that exhibit high-frequency oscillatory behavior in their weights.

Our investigation of cross-links and subsequent hyperedge extraction identifies similarities between edges that are based on their temporal profiles and can therefore be different from their final synchronization values. For example, hyperedges 1–3 in Fig. 2 each include edges with a wide range of final synchronization values that range from very strong ($A_{ij} = 0.9$) to very weak ($A_{ij} = -0.2$). Each hyperedge instead captures a property of edge dynamics: the trajectory that that edge followed to attain a given final synchronization value.

**APPENDIX**

To simulate the dynamics of the network of Kuramoto oscillators, we solved the discrete-time equation

$$\theta_t = \theta_{t-1} + \tau \omega_i + \sum_j \kappa C_{ij} \sin(\theta_j - \theta_i),$$

(5)

where $\omega_i$ is the natural frequency of oscillator $i$, the matrix $C$ gives the binary-valued (0 or 1) coupling between each pair of oscillators, $\tau$ (which we set to 0.1) is a positive real constant that indicates the time step, and $\kappa$ (which we set to 0.2) is a positive real constant.
that indicates the strength of the coupling. We solve this equation for \( t \in \{1, \ldots, T\} \) for a maximum of \( T = 101 \) time points. The simulation mechanics are modeled after the implementation in Ref [45]. Each matrix in the temporal network \( A_t \) gives the time-dependent correlations between pairs of oscillators measured at time point \( t \).

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FIG. 1. Hyperedge Identification in a Network of Kuramoto Oscillators. (A) Template indicating the block structure of the community structure of nodes in a network of 128 Kuramoto oscillators. Blocks 1–8 contain within-community edges, and blocks 9–36 contain between-community edges. Color indicates the block number. (B) The z-score of the Rand coefficient between the upper triangle of the template in (A) and the partition of the thresholded and weighted edge-edge correlation matrix $\Lambda'$ into communities of edges. Box plots indicate quartiles and 95% confidence intervals over the 100 optimizations of the signed modularity quality function in Eq. 2. (C) Consensus over partitions obtained from 100 optimizations at $\gamma = 1.8$. Each community of edges constitutes a hyperedge, and color indicates hyperedge number. (D) The final synchronization pattern of the network of Kuramoto oscillators at the final time ($T = 100$), which is reminiscent of the community structure of the network (which we show in Fig.2A in the main manuscript). Color indicates time-dependent correlation between pairs of oscillators (which we use to indicate their level of synchrony, following [40]): $A_{ij}(t) = \langle |\cos[\theta_i(t) - \theta_j(t)]|\rangle$, where the angular brackets indicate an average over 20 simulations.
FIG. 2.  Temporal Profiles and Final Synchronization Patterns of Hyperedges. (A) The mean synchronization of edges as a function of time \[ S(t) = \sum_{(i,j) \in h} A_{ij}(t) \] and (B) the final synchronization weights of each edge. From left to right, we plot these for hyperedge 1 (in the left panel) to hyperedge 5 (right panel). Color indicates time-dependent correlation between pairs of oscillators: \[ A_{ij}(t) = \langle |\cos[\theta_i(t) - \theta_j(t)]| \rangle \], where the angular brackets indicate an average over 20 simulations. Matrix elements highlighted in gray indicate edges that are members of a hyperedge other than their own.