Hamiltonian paths on the Sierpinski gasket

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Abstract

We derive exactly the number of Hamiltonian paths $H(n)$ on the two dimensional Sierpinski gasket $SG(n)$ at stage $n$, whose asymptotic behavior is given by $\sqrt{3\left(2\sqrt{3}\right)^{3^{n-1}}} \times \left(\frac{52 \times 72 \times 172}{2^{12} \times 3^3 \times 13}\right)16^n$. We also obtain the number of Hamiltonian paths with one end at a certain outmost vertex of $SG(n)$, with asymptotic behavior $\sqrt{3\left(2\sqrt{3}\right)^{3^{n-1}}} \times \left(\frac{7 \times 17}{2^2 \times 3^5}\right)4^n$. The distribution of Hamiltonian paths on $SG(n)$ with one end at a certain outmost vertex and the other end at an arbitrary vertex of $SG(n)$ is investigated. We rigorously prove that the exponent for the mean $\ell$ displacement between the two end vertices of such Hamiltonian paths on $SG(n)$ is $\ell \log 2 / \log 3$ for $\ell > 0$.

Keywords: Hamiltonian paths, Sierpinski gasket, distribution, mean $\ell$ displacement, exact solutions

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I. INTRODUCTION

The origin of the name, Hamiltonian cycle or Hamiltonian circuit, is invented in 1857 by Sir William Rowan Hamilton when he asked for the construction of a cycle containing all the vertices of a dodecahedron. In general, a Hamiltonian cycle, or a closed Hamiltonian walk, is a cycle in a graph which visits each vertex exactly once and also returns to the starting vertex. A Hamiltonian path, or an open Hamiltonian walk, is a path in a graph which visits each vertex of the graph exactly once [1, 2]. Determining whether such paths and cycles exist in graphs is difficult in general [3]. It is one of the oldest problems in graph theory, and is closely related to the traveling salesman problem. A Hamiltonian path corresponds to a complete self-avoiding walk, that can be considered to represent one of the possible configurations of a close-packed, unbranched polymer in a solution [4, 5]. It has been also used in the study of protein folding [6]. The enumeration of Hamiltonian cycles or paths is a fundamental problem in physics [7, 8] and computer science [9, 10].

It is of interest to consider Hamiltonian paths and cycles on self-similar fractal lattices which have scaling invariance rather than translational invariance. Fractals are geometric structures of non-integer Hausdorff dimension realized by repeated construction of an elementary shape on progressively larger length scales [11, 12]. For a lattice Λ with $v(\Lambda)$ vertices, the number of Hamiltonian paths, $H_\Lambda$, grows as $\omega_v^{v(\Lambda)}$ for large $v(\Lambda)$, where the connectivity constant $\omega_\Lambda$ is defined as

$$\ln \omega_\Lambda = \lim_{v(\Lambda) \to \infty} \frac{\ln H_\Lambda}{v(\Lambda)}.$$  

(1.1)

It is known that $\omega_\Lambda$ is the same for Hamiltonian cycles and Hamiltonian paths [13, 14]. Some recent studies on the enumeration of closed and open Hamiltonian walks on various fractal lattices, including $n$-simplex, Sierpinski gasket and its generalization, were carried out in Refs. [15, 16, 17], where the connectivity constants and scaling forms of $H_\Lambda$ were determined. However, except the 3-simplex lattice, the exact expressions of $H_\Lambda$ on the fractal lattices were never obtained. The purpose of this paper is to derive the number of Hamiltonian paths on the two-dimensional Sierpinski gasket exactly in Sections III and IV that complements the previous asymptotic studies. Furthermore, we should consider the Hamiltonian paths with one end at a certain outmost vertex as defined in Section V and investigate the probability distribution of the location of the other end in Sections VI-VIII.
Especially, we show the upper and lower bounds for the mean $\ell$ displacement for $\ell > 0$ between these two end vertices and obtain its exponent in Section VIII.

II. PRELIMINARIES

We first recall some relevant definitions for graphs and the Sierpinski gasket in this section. A connected graph (without loops) $G = (V, E)$ is defined by its vertex (site) and edge (bond) sets $V$ and $E$ [1, 18]. Let $v(G) = |V|$ be the number of vertices and $e(G) = |E|$ the number of edges in $G$. The degree or coordination number $k_i$ of a vertex $v_i \in V$ is the number of edges attached to it. A $k$-regular graph is a graph with the property that each of its vertices has the same degree $k$.

The construction of the two-dimensional Sierpinski gasket $SG(n)$ at stage $n$ is shown in Fig. 1. At stage $n = 0$, it is an equilateral triangle; while stage $(n + 1)$ is obtained by the juxtaposition of three $n$-stage structures. The numbers of edges and vertices for $SG(n)$ are given by

$$e(SG(n)) = 3^{n+1}, \quad v(SG(n)) = \frac{3}{2}(3^n + 1).$$

(2.1)

Except the three outmost vertices which have degree two, all other vertices of $SG(n)$ have degree four, such that $SG(n)$ is 4-regular in the large $n$ limit.

![Fig. 1: The first four stages $n = 0, 1, 2, 3$ of the two-dimensional Sierpinski gasket $SG(n)$.](image)

Use the notation $o$ (origin), $a_n$ and $b_n$ for the leftmost, rightmost and topmost vertices of $SG(n)$, respectively, and the notation for general vertices will be given in Section V. The total number of Hamiltonian paths on $SG(n)$ will be denoted as $H(n)$, while its subset, the number of Hamiltonian paths with one end at $o$, is denoted as $H_0(n)$. The number of
Hamiltonian cycles on $SG(n)$ will be denoted as $HC(n)$. Let us define the quantities to be used as follows.

**Definition II.1** Consider the two-dimensional Sierpinski gasket $SG(n)$ at stage $n$. (i) Define $f_1(n)$ as the number of Hamiltonian paths whose end vertices are located at two certain outmost vertices, e.g. $o$ and $a_n$. (ii) Define $f_2(n)$ as the number of Hamiltonian paths whose one end vertex is located at a certain outmost vertex, e.g., $o$, while the other end vertex is not at the other two outmost vertices. (iii) Define $f_3(n)$ as the number of Hamiltonian paths whose both end vertices are not located at any of the outmost vertices.

It follows that

$$H_0(n) = 2f_1(n) + f_2(n), \quad (2.2)$$
$$H(n) = 3f_1(n) + 3f_2(n) + f_3(n) \quad (2.3)$$

for the numbers of Hamiltonian paths and

$$HC(n) = f_1(n - 1)^3 \quad (2.4)$$

for the number of Hamiltonian cycles. To calculate $f_1(n)$, $f_2(n)$ and $f_3(n)$, we need the following definitions.

**Definition II.2** Consider the two-dimensional Sierpinski gasket without one certain outmost vertices, e.g. $SG(n) \setminus \{b_n\}$. (i) Define $g_1(n)$ as the number of Hamiltonian paths whose end vertices are located at the remaining two outmost vertices, e.g. $o$ and $a_n$. (ii) Define $g_2(n)$ as the number of Hamiltonian paths whose one end vertex is located at one certain remaining outmost vertices, e.g., $o$, while the other end vertex is not at the other outmost vertex.

**Definition II.3** Divide the vertices of the two-dimensional Sierpinski gasket into two subsets $V_1(n)$ and $V_2(n)$. One of them contains two outmost vertices while the other contains only one outmost vertex, e.g. $o, a_n \in V_1(n)$, $b_n \in V_2(n)$. (i) Define $g_3(n)$ as the number of two Hamiltonian paths separately visiting all the vertices of $V_1(n)$ and $V_2(n)$, such that one of them has end vertices located at the two outmost vertices in $V_1(n)$, e.g. $o$ and $a_n$, while the other one has one end vertex located at the other outmost vertex in $V_2(n)$, e.g. $b_n$. 
(ii) Define $g_4(n)$ as the number of two Hamiltonian paths separately visiting all the vertices $V_1(n)$ and $V_2(n)$, such that one of them has only one end vertex located at a certain outmost vertex in $V_1(n)$, e.g. $o$, while the other one has one end vertex located at the outmost vertex in $V_2(n)$, e.g. $b_n$.

**Definition II.4** Consider the two-dimensional Sierpinski gasket without two certain outmost vertices, e.g. $SG(n) \setminus \{o, a_n\}$, and define $t_1(n)$ as the number of Hamiltonian paths on it.

**Definition II.5** Consider the two-dimensional Sierpinski gasket without one certain outmost vertices, e.g. $SG(n) \setminus \{b_n\}$, and divide its vertices into two subsets $V'_1(n)$ and $V'_2(n)$. Both of these subsets contain one outmost vertex, e.g. $o \in V'_1$, $a_n \in V'_2(n)$. Define $t_2(n)$ as the number of two Hamiltonian paths separately visiting all the vertices of $V'_1(n)$ and $V'_2(n)$, such that both of them have one end vertex located at an outmost vertex, e.g. $o$ and $a_n$.

The quantities defined above are illustrated in Fig. 2. Note that there are two equivalent ways to draw $f_2(n)$. When $n = 0$, only $f_1(0)$ and $g_1(0)$ are equal to one, and $f_2(0) = f_3(0) = g_2(0) = g_3(0) = g_4(0) = t_1(0) = t_2(0) = 0$.

![Illustration](image)

**FIG. 2:** Illustration for the quantities $f_1(n)$, $f_2(n)$, $f_3(n)$, $g_1(n)$, $g_2(n)$, $g_3(n)$, $g_4(n)$, $t_1(n)$ and $t_2(n)$. Two outmost vertices connected by a solid line belong to the same path. The outmost vertices are denoted by the following symbols: (i) an open circle corresponds to an end of a path; (ii) a solid circle corresponds to a middle point of a path; (iii) a cross is not passed by any paths; (iv) a symbol $\oplus$ corresponds to one end of a path but it is not connected with the other two outmost vertices. See text for details.
III. NUMBER OF HAMILTONIAN CYCLES AND NUMBER OF HAMILTONIAN PATHS WITH ONE END AT ORIGIN

In this section, we first rederive the number of Hamiltonian cycles, then enumerate the number of Hamiltonian paths with one end at origin. By Eqs. (2.2) and (2.4), it is sufficient to evaluate $f_1(n)$ and $f_2(n)$. We shall only consider $n ≥ 1$ throughout this paper as the case $n = 0$ is trivial.

A. Number of Hamiltonian cycles

Consider the quantities $f_1(n)$ and $g_1(n)$. The initial values are $f_1(1) = 2$, $g_1(1) = 3$ as illustrated in Figs. 3, 4. For $n ≥ 1$, we have the simple relations

$$f_1(n + 1) = 2f_1(n)^2g_1(n), \quad g_1(n + 1) = 2g_1(n)^2f_1(n),$$

such that

$$\frac{f_1(n + 1)}{g_1(n + 1)} = \frac{f_1(n)}{g_1(n)} = \frac{f_1(1)}{g_1(1)} = \frac{2}{3}. \quad (3.2)$$

![FIG. 3: Illustration for the expression of $f_1(n + 1)$.](image)

![FIG. 4: Illustration for the expression of $g_1(n + 1)$. The last drawing only applies for $n = 0$.](image)

Hence, we have

$$f_1(n) = 3f_1(n - 1)^3 = \cdots = 3^{2^n - 3}f_1(1)^{3n - 1} = 3^{2^n - 3}(2^{3n - 1}) = \frac{\sqrt{3}(2\sqrt{3})^{3n - 1}}{3} \quad (3.3)$$

and

$$g_1(n) = \frac{3f_1(n)}{2} = \frac{\sqrt{3}(2\sqrt{3})^{3n - 1}}{2} \quad (3.4)$$

for $n ≥ 1$. 

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By the definition of Hamiltonian cycles, its number is given by $HC(1) = 1$ and $HC(n) = f_1(n - 1)^3 = f_1(n)/3$ for $n \geq 2$ as obtained by Bradley [19].

**Theorem III.1 (Bradley, 1986)** The number of Hamiltonian cycles $HC(n)$ on the Sierpinski gasket $SG(n)$ is one for $n = 1$ and

$$HC(n) = \frac{\sqrt{3}(2\sqrt{3})^{3n-1}}{9}$$

for $n \geq 2$, such that the connectivity constant $\omega_{SG} = 12^{1/9}$.

**B. Number of Hamiltonian paths with one end at origin**

Consider the quantities $f_2(n), g_2(n), g_3(n)$ and $t_1(n)$ for $n \geq 1$, we have the following recursion equations as illustrated in Figs. 5-8

$$\begin{align*}
  f_2(n + 1) &= 2f_1(n)^2[g_1(n) + g_2(n) + g_3(n)] + 2f_1(n)f_2(n)g_1(n) , \\
  g_2(n + 1) &= 2f_1(n)g_1(n)[g_1(n) + g_2(n) + g_3(n)] + f_2(n)g_1(n)^2 + f_1(n)^2t_1(n) , \\
  g_3(n + 1) &= 2f_1(n)g_1(n)[2g_1(n) + g_2(n) + 3g_3(n)] + f_2(n)g_1(n)^2 + f_1(n)^2t_1(n) , \\
  t_1(n + 1) &= 2g_1(n)^2[g_1(n) + g_2(n) + g_3(n)] + 2f_1(n)g_1(n)t_1(n) .
\end{align*}$$

If we set $n = 0$ in Eq. (3.6), only the terms contain $f_1(0)$ and $g_1(0)$ as factors on the right-hand-sides give non-zero contribution. In addition, there are special drawings shown in Figs. 5-8 such that the initial values are $f_2(1) = 2$, $g_2(1) = 3$, $g_3(1) = 4$, $t_1(1) = 4$.

![FIG. 5: Illustration for the expression of $f_2(n + 1)$. The multiplication of two on the right-hand-side corresponds to the reflection symmetry with respect to the central vertical axis.](image)

Using $g_1(n) = 3f_1(n)/2$ for $n \geq 1$ from Eq. (3.4) to eliminate $g_1(n)$, Eq. (3.6) becomes

$$\begin{align*}
  f_2(n + 1) &= 3f_1(n)^3 + 2f_1(n)^2[g_2(n) + g_3(n)] + 3f_1(n)^2f_2(n) , \\
  g_2(n + 1) &= 3f_1(n)^2[\frac{3}{2}f_1(n) + g_2(n) + g_3(n)] + \frac{3}{4}f_1(n)^2f_2(n) + f_1(n)^2t_1(n) , \\
  g_3(n + 1) &= 3f_1(n)^2[3f_1(n) + g_2(n) + 3g_3(n)] + \frac{9}{4}f_1(n)^2f_2(n) + f_1(n)^2t_1(n) , \\
  t_1(n + 1) &= \frac{9f_1(n)^2}{2}[\frac{3}{2}f_1(n) + g_2(n) + g_3(n)] + 3f_1(n)^2t_1(n) .
\end{align*}$$
Define the ratios
\[ \alpha_2(n) = \frac{f_2(n)}{f_1(n)}, \quad \beta_2(n) = \frac{g_2(n)}{f_1(n)}, \quad \beta_3(n) = \frac{g_3(n)}{f_1(n)}, \quad \gamma_1(n) = \frac{t_1(n)}{f_1(n)}, \]
then they satisfy the recursion relations
\[
\begin{align*}
\alpha_2(n+1) &= 1 + \alpha_2(n) + \frac{2}{3}\beta_2(n) + \frac{2}{3}\beta_3(n), \\
\beta_2(n+1) &= \frac{3}{2} + \frac{3}{4}\alpha_2(n) + \beta_2(n) + \beta_3(n) + \frac{1}{3}\gamma_1(n), \\
\beta_3(n+1) &= 3 + \frac{3}{4}\alpha_2(n) + 3\beta_2(n) + 3\beta_3(n) + \frac{1}{3}\gamma_1(n), \\
\gamma_1(n+1) &= \frac{9}{4} + \frac{3}{2}\beta_2(n) + \frac{3}{2}\beta_3(n) + \gamma_1(n),
\end{align*}
\] (3.9)
where the relation \( f_1(n+1) = 3f_1(n)^3 \) in Eq. (3.3) is used. Define the vectors \( b = (1, \frac{3}{2}, 3, \frac{9}{4})^t \),
\[
\begin{array}{c}
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\times \times \times \times \times \times \times \times \times \time...
\[X(n) = (\alpha_2(n), \beta_2(n), \beta_3(n), \gamma_1(n))^t,\] and construct the matrix

\[
A = \begin{pmatrix}
1 & \frac{2}{3} & \frac{2}{3} & 0 \\
\frac{3}{4} & 1 & 1 & \frac{1}{3} \\
\frac{3}{4} & 1 & 3 & \frac{1}{3} \\
0 & \frac{3}{2} & \frac{3}{2} & 1
\end{pmatrix},
\]

such that \(X(1) = (1, 3/2, 2, 2)^t\) and for general \(n \geq 1,

\[
X(n) = AX(n-1) + b = A^2 X(n-2) + Ab + b = \cdots = A^{n-1} X(1) + \sum_{j=0}^{n-2} A^j b
\]

\[
= \begin{pmatrix}
(\frac{7 \times 17}{2^4 \times 3^2})^4n - \frac{2}{3^3} - (\frac{1}{2^2 \times 3^2})\delta_{n1} \\
(\frac{7 \times 17}{2^4 \times 3^2})^4n - \frac{7}{2^2 \times 3^2} - (\frac{1}{2^2 \times 3^2})\delta_{n1} \\
(\frac{7 \times 17}{2^4 \times 3^2})^4n - \frac{47}{2^2 \times 3^2} - (\frac{1}{2^2 \times 3^2})\delta_{n1} \\
(\frac{7 \times 17}{2^4 \times 3^2})^4n - \frac{5}{2^2 \times 3} - (\frac{1}{2^2 \times 3^2})\delta_{n1}
\end{pmatrix},
\]

(3.11)

where \(\delta_{ij}\) is the Kronecker delta function. Since \(H_0(n) = 2f_1(n) + f_2(n) = f_1(n)(2 + \alpha_2(n))\), we have the following theorem:

**Theorem III.2** The number of Hamiltonian paths on the Sierpinski gasket \(SG(n)\) with one end at origin for \(n \geq 1\) is given by

\[
H_0(n) = \sqrt{3}(2\sqrt{3})^{3^n-1} \left\{ (\frac{7 \times 17}{2^4 \times 3^2})^4n + \frac{2^2 \times 13}{3^3} - (\frac{1}{2^2 \times 3^2})\delta_{n1} \right\}. 
\]

(3.12)

**IV. NUMBER OF HAMILTONIAN PATHS**

In this section, we enumerate the total number of Hamiltonian paths. As we have obtained \(f_1(n)\) and \(f_2(n)\) in the previous section, we need to evaluate \(f_3(n)\) for \(n \geq 1\) before Eq. (2.3) is applied. Consider the quantities \(f_3(n), g_4(n),\) and \(t_2(n)\) for \(n \geq 1\), we have the following recursion equations as illustrated in Figs. 9-11.
\[
\begin{align*}
\begin{cases}
f_3(n+1) &= 6f_1(n)[f_2(n)g_2(n) + f_1(n)g_2(n) + f_1(n)g_4(n)], \\
g_4(n+1) &= f_1(n)[g_1(n)^2 + g_2(n)^2] + 4f_1(n)g_1(n)[g_2(n) + g_3(n) + g_4(n)] \\
&\quad + f_1(n)g_3(n)[2g_2(n) + 3g_3(n)] + f_1(n)^2[t_1(n) + t_2(n)] \\
&\quad + f_2(n)g_1(n)[g_1(n) + 2g_2(n) + 2g_3(n)] + f_1(n)f_2(n)t_1(n), \\
t_2(n+1) &= 2g_1^2(n)[g_1(n) + 2g_2(n) + 4g_3(n) + 2g_4(n)] \\
&\quad + 2g_1(n)g_3(n)[2g_2(n) + 3g_3(n)] + 2g_1(n)[g_2(n)^2 + f_2(n)t_1(n)] \\
&\quad + 4f_1(n)t_1(n)[2g_1(n) + g_2(n) + g_3(n)] + 4f_1(n)g_1(n)t_2(n).
\end{cases}
\end{align*}
\]

If we set \( n = 0 \) in Eq. (4.1), only the terms contain \( f_1(0) \) and \( g_1(0) \) as factors on the right-hand-sides give non-zero contribution. In addition, there are special drawings shown in Fig. [11] such that the initial values are \( f_3(1) = 0, g_4(1) = 1 \) and \( t_2(1) = 6 \).

\[
\begin{align*}
\text{FIG. 9: Illustration for the expression of } f_3(n+1). \text{ The multiplication of three on the right-hand-side corresponds to the three possible orientations of } SG(n+1).
\end{align*}
\]

Using \( g_1(n) = 3f_1(n)/2 \) for \( n \geq 1 \) from Eq. (3.4) to eliminate \( g_1(n) \), Eq. (4.1) becomes

\[
\begin{align*}
\begin{cases}
f_3(n+1) &= 6f_1(n)[f_2(n)g_2(n) + f_1(n)g_2(n) + f_1(n)g_4(n)], \\
g_4(n+1) &= \frac{9}{4}f_1(n)^3 + f_1(n)g_1(n)^2 + 6f_1(n)^2[g_2(n) + g_3(n) + g_4(n)] \\
&\quad + f_1(n)g_3(n)[2g_2(n) + 3g_3(n)] + f_1(n)^2[t_1(n) + t_2(n)] \\
&\quad + \frac{3}{2}f_2(n)g_1(n)[\frac{3}{2}f_1(n) + 2g_2(n) + 2g_3(n)] + f_1(n)f_2(n)t_1(n), \\
t_2(n+1) &= \frac{9}{2}f_1^2(n)[\frac{3}{2}f_1(n) + 2g_2(n) + 4g_3(n) + 2g_4(n)] \\
&\quad + 3f_1(n)g_3(n)[2g_2(n) + 3g_3(n)] + 3f_1(n)[g_2(n)^2 + f_2(n)t_1(n)] \\
&\quad + 4f_1(n)t_1(n)[3f_1(n) + g_2(n) + g_3(n)] + 6f_1(n)^2t_2(n).
\end{cases}
\end{align*}
\]

Define the ratios

\[
\begin{align*}
\alpha_3(n) = \frac{f_3(n)}{f_1(n)}, \quad \beta_4(n) = \frac{g_4(n)}{f_1(n)}, \quad \gamma_2(n) = \frac{t_2(n)}{f_1(n)},
\end{align*}
\]
then they satisfy the recursion relations

\[
\begin{align*}
\alpha_3(n+1) &= 2[\alpha_2(n)\beta_2(n) + \beta_2(n) + \beta_4(n)] , \\
\beta_4(n+1) &= \frac{3}{4} + \beta_2(n)[2 + \frac{\beta_3(n)}{3} + \beta_3(n)][2 + \frac{2}{3}\beta_2(n) + \beta_3(n)] \\
&\quad + \alpha_2(n)[\frac{3}{4} + \beta_2(n) + \beta_3(n) + \frac{\gamma_1(n)}{3}] + 2\beta_4(n) + \frac{1}{3} \gamma_1(n) + \gamma_2(n)] , \\
\gamma_2(n+1) &= \frac{2}{3} + \beta_2(n)[3 + \beta_2(n)] + \beta_3(n)[6 + 2\beta_2(n) + 3\beta_3(n)] \\
&\quad + \gamma_1(n)[4 + \alpha_2(n) + \frac{1}{3}\beta_2(n) + \frac{4}{3}\beta_3(n)] + 3\beta_4(n) + 2\gamma_2(n) .
\end{align*}
\]

Define vectors \( Y(n) = (\alpha_3(n), \beta_4(n), \gamma_2(n))^t \) and \( K(n) = (K_1(n), K_2(n), K_3(n))^t \) where

\[ K_1(n) = \alpha_3(n + 1) - 2\beta_4(n), \quad K_2(n) = \beta_4(n + 1) - 2\beta_4(n) - \gamma_2(n)/3, \quad K_3(n) = \gamma_2(n + 1) - 3\beta_4(n) - 2\gamma_2(n), \]

and the matrix

\[
B = \begin{pmatrix}
0 & 2 & 0 \\
0 & 2 & \frac{1}{3} \\
0 & 3 & 2
\end{pmatrix} \equiv PDP^{-1} , \quad (4.5)
\]

FIG. 10: Illustration for the expression of \( g_4(n+1) \).
where
\[ D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} , \quad P = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 0 & -3 & 9 \end{pmatrix} , \quad P^{-1} = \begin{pmatrix} 1 & -\frac{4}{3} & \frac{2}{9} \\ \frac{1}{2} & -\frac{1}{6} & -\frac{1}{18} \\ \frac{1}{6} & \frac{1}{18} & -\frac{1}{18} \end{pmatrix} , \quad (4.6) \]
such that \( Y(n+1) = K(n) + BY(n) \) for \( n \geq 1 \). We know \( Y(1) = (0, 1/2, 3)^t \), and for \( n \geq 2 \),
\[
Y(n) = \sum_{j=1}^{n-1} B^{n-1-j} K(j) + B^{n-1} Y(1)
\]
\[
= P \sum_{j=1}^{n-2} D^{n-1-j} P^{-1} K(j) + K(n-1) + PD^{n-1} P^{-1} Y(1)
\]
\[
= \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 0 & -3 & 9 \end{pmatrix} \left\{ \sum_{j=1}^{n-2} \frac{1}{6} \left( 3K_2(j) - K_3(j) \right) \right\} + \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{4} \end{pmatrix} + \begin{pmatrix} K_1(n-1) \\ K_2(n-1) \\ K_3(n-1) \end{pmatrix}.
\]
(4.7)
From the expressions of $\alpha_2(n), \beta_2(n), \beta_3(n), \gamma_1(n)$ given in (3.11), $K_1(n), K_2(n), K_3(n)$ for $n \geq 1$ can be solved as

$$K_1(n) = \left( \frac{7^2 \times 17^2}{2^5 \times 3^5} \right)(16)^n + \left( \frac{7 \times 17 \times 43}{2^5 \times 3^5} \right)4^n - \frac{5^2 \times 7}{2 \times 3^5} + \left( \frac{37}{2^4 \times 3^3} \right)\delta_{n1},$$

$$K_2(n) = \left( \frac{7^2 \times 17^2}{2^7 \times 3^3} \right)(16)^n - \left( \frac{5 \times 7 \times 17}{2^5 \times 3^3} \right)4^n - \frac{283}{2^3 \times 3^4} + \left( \frac{19}{2^4 \times 3^3} \right)\delta_{n1},$$

$$K_3(n) = \left( \frac{7^2 \times 17^2}{2^7 \times 3^3} \right)(16)^n - \left( \frac{7^2 \times 17}{2^6 \times 3^2} \right)4^n - \frac{283}{2^3 \times 3^3} - \left( \frac{19}{2^4 \times 3^2} \right)\delta_{n1}. \quad (4.8)$$

Hence, we have

$$\alpha_3(n) = \sum_{j=1}^{n-2} \left[ (1 + 3^{n-2-j})K_2(j) + \frac{1}{3}(3^{n-2-j} - 1)K_3(j) \right] + \frac{3^{n-1} - 1}{2} + K_1(n - 1)$$

$$= \left( \frac{5^2 \times 7^2 \times 17^2}{2^{12} \times 3^3 \times 13} \right)(16)^n - \left( \frac{7 \times 17}{2^4 \times 3^5} \right)4^n + \left( \frac{11 \times 257}{3^5 \times 2^3 \times 13} \right)3^n - \frac{1009}{2^3 \times 3^5} - \left( \frac{1}{2^4 \times 3^3} \right)\delta_{n2} \quad (4.9)$$

for $n \geq 2$. Since $H(n) = 3f_1(n) + 3f_2(n) + f_3(n) = f_1(n)(3 + 3\alpha_2(n) + \alpha_3(n))$, we have the following theorem:

**Theorem IV.1** The number of Hamiltonian paths on the Sierpinski gasket $SG(n)$ is twelve for $n = 1$ and

$$H(n) = \sqrt{3}(2\sqrt{3})^{3^{n-1}} \left\{ \left( \frac{5^2 \times 7^2 \times 17^2}{2^{12} \times 3^3 \times 13} \right)(16)^n + \left( \frac{7 \times 13 \times 17}{2^3 \times 3^5} \right)4^n + \left( \frac{11 \times 257}{3^5 \times 2^3 \times 13} \right)3^n \right\} + \frac{4391}{2^3 \times 3^5} - \left( \frac{1}{2^4 \times 3^3} \right)\delta_{n2} \quad (4.10)$$

for $n \geq 2$.

To consider the asymptotic behavior when $n$ is large, let us use the symbol $A(n) \sim B(n)$ to denote $\lim_{n \to \infty} A(n)/B(n) = 1$.

**Corollary IV.1** The asymptotic behavior for the number of Hamiltonian paths on the Sierpinski gasket $SG(n)$ when $n$ is large is given by

$$H(n) \sim \sqrt{3}(2\sqrt{3})^{3^{n-1}} \left( \frac{5^2 \times 7^2 \times 17^2}{2^{12} \times 3^3 \times 13} \right)(16)^n. \quad (4.11)$$
The ratio of the number of Hamiltonian paths on the Sierpinski gasket $SG(n)$ and that with one end at origin when $n$ is large is given by
\[
\frac{H(n)}{H_0(n)} \sim \left(\frac{5^2 \times 7 \times 17}{2^8 \times 3^2 \times 13}\right)^n.
\] (4.12)

Analogous to the consideration of self-avoiding paths in [20], consider $Z_n(\beta) = \sum_{w \in H(n)} e^{-L(w)\beta}$ with $\beta \in (0, \infty)$ and the length of $w$, denoted as $L(w)$, is equal to $\frac{3^{n+1}+1}{2}$ for every $w \in H(n)$. By Theorem IV.1, we have the following corollary.

**Corollary IV.2** There exists a critical $\beta_c = \ln \omega_{SG} = \frac{\ln(12)}{9}$ such that
\[
\lim_{n \to \infty} Z_n(\beta) = \begin{cases} 
0 & \text{if } \beta > \beta_c, \\
\infty & \text{if } \beta \leq \beta_c.
\end{cases}
\] (4.13)

Moreover, let $\beta_n = \beta_c + (\ln A_n)/3^n$ for some sequence $A_n > 0$. There is a critical $A_n^c = 2^{8n/3}$ such that
\[
\lim_{n \to \infty} Z_n(\beta_n) = \begin{cases} 
0 & \text{if } \lim_{n \to \infty} \frac{A_n^c}{A_n} = 0, \\
\infty & \text{if } \lim_{n \to \infty} \frac{A_n^c}{A_n} = \infty, \\
\sqrt[3]{\frac{5^2 \times 7^2 \times 17^2}{2^2 \times 3^2 \times 13}} & \text{if } A_n \sim A_n^c.
\end{cases}
\] (4.14)

**V. DEFINITION FOR THE DISTRIBUTION OF THE HAMILTONIAN PATHS ON $SG(n)$ WITH ONE END AT ORIGIN**

Consider the Hamiltonian paths on $SG(n)$ with one end at origin. We will study the distribution of the location of the other end. For that purpose, let us define the notation for the vertices of $SG(n)$, that is given progressively with increasing number of digits in the subscript as follows. First of all, fix the origin $o$ as the leftmost vertex, and all the Hamiltonian paths considered in the following sections have one end at $o$. Consider $SG(m)$ with $0 \leq m < n$ always has $o$ as its leftmost vertex, and denote $a_m$ and $b_m$ as its rightmost and topmost vertices, respectively. $c_m$ is defined such that the vertices $a_m$, $b_m$ and $c_m$ demarcate the largest lacunary triangle of $SG(m+1)$. We then define the vertex in the middle of the line connecting $a_m$ and $a_{m+1}$ with $m \geq 1$ as $a_{m,1}$. Similarly, $a_{m,1}$ and the associated $b_{m,1}$ and $c_{m,1}$ demarcate a lacunary triangle with $b_{m,1}$ on the left and $c_{m,1}$ on
the right. Next for \( m \geq 2 \), we append the subscript \((m,1,0)\) for the vertices of the largest lacunary inside the triangle with outmost vertices \( a_m, a_{m,1}, b_{m,1} \); the subscript \((m,1,1)\) for the vertices of the largest lacunary inside the triangle with outmost vertices \( a_{m,1}, a_{m+1}, c_{m,1} \); the subscript \((m,1,2)\) for the vertices of the largest lacunary inside the triangle with outmost vertices \( b_{m,1}, c_{m,1}, c_m \), etc. In general for the vertices of \( SG(n) \), we use the notation \( x_{\vec{\gamma}} \) where \( x \in \{a,b,c\} \) and the subscript \( \vec{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_s) \) has \(|\vec{\gamma}| = s\) components with \( 1 \leq s \leq n, 1 \leq \gamma_1 < n \) and \( \gamma_k \in \{0,1,2\} \) for \( k \in \{2,3,\ldots,s\} \). For the vertices above the line connecting \( o \) and \( c_{n-1} \) on \( SG(n) \), we will also use the notation \( \bar{x}_{\vec{\gamma}} \) such that it is the reflection of the vertex \( x_{\vec{\gamma}} \) with respect to this line. Similarly, for the vertices above the line connecting \( a_n \) and \( b_{n-1} \) on \( SG(n) \), we use the notation \( \hat{x}_{\vec{\gamma}} \) such that it is the reflection of the vertex \( x_{\vec{\gamma}} \) with respect to this line. For examples, \( a_{22} = b_{21} \), \( b_{221} = \bar{a}_{212} \), \( c_{222} = \bar{c}_{211} \), and \( b_{21} = \bar{b}_{21} \), \( c_{210} = \bar{a}_{212} \), \( a_{211} = \bar{c}_{211} \) on \( SG(3) \). There may be more than one ways to denote a vertex. For example, \( b_0 \) can be written as \( b_{10} \) on \( SG(2) \), or \( b_{200} \) on \( SG(3) \), etc.

The advantage of such vertex notation is that the quantities to be studied for the vertices \( x_{\gamma_1,\ldots,\gamma_s} \) with \( s \geq 2 \) components in the subscript on \( SG(n) \) can be expressed in terms of the quantities for the vertices with \( s-1 \) components in the subscript on \( SG(n-1) \) as discussed below.

Denote the number of Hamiltonian paths with one end at \( o \) and the other end at vertex \( x \) on \( SG(n) \) as \( f(n,x) \), and the distribution ratio \( F(n,x) = f(n,x)/f_1(n) \). Define the probability measure

\[
p(n,x) = \frac{f(n,x)}{H_0(n)},
\]

where \( H_0(n) \) is given in Eq. \((5.12)\). Denote the factor

\[
R(n) = (\frac{7 \times 17}{2^4 \times 3^3})4^n + \frac{2^2 \times 13}{3} - (\frac{1}{2^2 \times 3^2})\delta_{n1}
\]

such that \( H_0(n) = f_1(n)R(n) \), then

\[
p(n,x) = \frac{F(n,x)}{R(n)}.
\]

It is clear that the summation of \( p(n,x) \) over all the vertices of \( SG(n) \) should give one. The purpose of the following sections is to analyze \( F(n,x) \), whose expression is simpler than \( p(n,x) \) without the factor \( R(n) \). We need the following definitions: (i) \( g_2(n,x) \) denotes the number of \( g_2(n) \) such that the end points of the Hamiltonian path are located at \( o \) and \( x \), and \( G_2(n,x) = g_2(n,x)/g_1(n) \); (ii) \( g_3(n,x) \) denotes the number of \( g_3(n) \) such that one
FIG. 12: The notation for the vertices of the Sierpinski gasket $SG(4)$.

Hamiltonian path has end points located at $o$ and $a_n$, while the other Hamiltonian path has end points located at $b_n$ and $x$. Furthermore, define $u(n, x) = g_3(n, x) + g_3(n, \tilde{x})$ and $U(n, x) = u(n, x)/g_1(n)$; (iii) $t_1(n, x)$ denotes the number of $t_1(n)$ such that the end points of the Hamiltonian path are located at $b_n$ and $x$, and $T_1(n, x) = t_1(n, x)/g_1(n)$.

VI. DISTRIBUTION $F(n, x_m)$ WITH $x \in \{a, b, c\}$, $0 \leq m \leq n$ AND MEAN $\ell$ DISPLACEMENT FOR THESE VERTICES

Let us consider the vertex $x_m$ on $SG(n)$ with $x \in \{a, b, c\}$ and $0 \leq m \leq n$ in this section. It is easy to see that $f(n, a_m) = f(n, b_m)$ for all $m \leq n$ by symmetry, and $f(n, a_n) =
\[ f(n, b_n) = f_1(n) \] by definition. For \( m = n - 1 \geq 0 \), we have
\[
\begin{align*}
  f(n, a_{n-1}) &= f(n, b_{n-1}) = f_1(n-1)^2 g_1(n-1) = \frac{1}{2} f_1(n), \\
  f(n, c_{n-1}) &= 0, \\
  g_3(n, b_{n-1}) &= g_3(n, c_{n-1}) = \begin{cases} 2 f_1(n-1) g_1(n-1)^2 = g_1(n) & \text{for } n > 1, \\ 2 = \frac{2}{3} g_1(1) & \text{for } n = 1, \\ 0 & \text{for } n = 1 \end{cases}, \\
  g_3(n, a_{n-1}) &= 0,
\end{align*}
\]

where the expression for \( g_3(n, b_{n-1}) \) is illustrated in Fig. 13 and
\[
\begin{align*}
  g_2(n, a_{n-1}) &= g_2(n, b_{n-1}) = \begin{cases} f_1(n-1) g_1(n-1)^2 = \frac{1}{2} g_1(n) & \text{for } n > 1, \\ 1 = \frac{1}{3} g_1(1) & \text{for } n = 1, \\ \end{cases}, \\
  g_2(n, c_{n-1}) &= \begin{cases} 0 & \text{for } n > 1, \\ 1 = \frac{1}{3} g_1(1) & \text{for } n = 1. \\ \end{cases}
\end{align*}
\]

For \( 0 \leq m \leq n - 2 \), we have
\[
\begin{align*}
  f(n, x_m) &= f_1(n-1)^2 u(n-1, x_m), \\
  g_3(n, x_m) &= 2 f_1(n-1) g_1(n-1) g_3(n-1, x_m)
\end{align*}
\]

for \( x \in \{a, b, c\} \) as illustrated in Figs. 14 and 15 respectively.

\[
\begin{align*}
  \quad + \\
  \quad +
\end{align*}
\]

FIG. 13: Illustration for the expression of \( g_3(n, b_{n-1}) \).

FIG. 14: Illustration for the expression of \( f(n, x_m) \) with \( 0 \leq m \leq n - 2 \).

From Eqs. (3.1) and (6.3), \( g_3(n, x_m) \) for \( 0 \leq m \leq n - 2 \) can be written in general as
\[
  g_3(n, x_m) = r(n) g_3(n-1, x_m), \quad \text{where} \quad r(n) = \frac{g_1(n)}{g_1(n-1)},
\]

such that \( g_3(n, a_m) = 0 \) and
\[
  g_3(n, b_m) = g_3(n, c_m) = g_3(m+1, b_m) \prod_{j=m+2}^{n} r(j)
\]
FIG. 15: Illustration for the expression of $g_3(n, x_m)$ with $0 \leq m \leq n - 2$.

\[ g_1(m + 1) \prod_{j=m+2}^n r(j) = g_1(n) \quad (0 < m < n - 1) , \]
\[ \frac{2}{3} g_1(1) \prod_{j=2}^n r(j) = \frac{2}{3} g_1(n) \quad (0 = m < n - 1) , \]

where Eq. (6.1) is used. Combining Eqs. (6.1) and (6.5), we obtain

\[ U(n, a_m) = U(n, b_m) = \begin{cases} 1 & (0 < m < n) , \\ \frac{2}{3} & (0 = m < n) , \end{cases} \]
\[ U(n, c_m) = \begin{cases} \frac{1}{2} & (0 < m < n) , \\ \frac{1}{3} & (0 = m < n) , \end{cases} \]

as Eqs. (6.2) and (6.5) are combined. From Eq. (6.3), $f(n, x_m)$ for $m \leq n - 2$ is given as

\[ f(n, a_m) = f(n, b_m) = f_1(n - 1)^2 g_3(n - 1, b_m) = \begin{cases} \frac{1}{2} f_1(n) & (0 < m < n - 1) , \\ \frac{1}{3} f_1(n) & (0 = m < n - 1) , \end{cases} \]
\[ f(n, c_m) = 2 f_1(n - 1)^2 g_3(n - 1, c_m) = \begin{cases} f_1(n) & (0 < m < n - 1) , \\ \frac{2}{3} f_1(n) & (0 = m < n - 1) , \end{cases} \]

where Eqs. (6.1), (6.5) are used. Combining Eqs. (6.1) and (6.8), we have the following theorem:

**Theorem VI.1** On the Sierpinski gasket $SG(n)$, the distribution ratio $F(n, x_m)$ with $x \in$
\{a, b, c\} for integer 0 \leq m \leq n is given by

\[
F(n, a_m) = F(n, b_m) = \begin{cases} 
1 & (0 \leq m = n), \\
\frac{1}{2} & (0 < m < n), \\
\frac{1}{2} & (m = 0, n = 1), \\
\frac{1}{3} & (m = 0, n > 1), 
\end{cases}
\]

\[
F(n, c_m) = \begin{cases} 
0 & (0 \leq m = n - 1), \\
1 & (0 < m < n - 1), \\
\frac{2}{3} & (m = 0, n > 1).
\end{cases}
\] (6.9)

For any vertex \(x \in v(SG(n))\), denote the distance between \(o\) and \(x\) as \(|x|\), and set the distance between the vertices \(o\) and \(a_0\) equal to one. For any positive \(\ell > 0\), define the mean \(\ell\) displacement for the vertices \(x_m\) with \(x \in \{a, b, c\}\) and 0 \(\leq m \leq n\) as

\[
\xi'(n, \ell) = \frac{\sum_{x_m:j=0,1,2,\ldots,n;x=a,b,c} |x|^{\ell} p(n, x_j)}{\sum_{x_m:j=0,1,2,\ldots,n;x=a,b,c} p(n, x_j)} = \frac{\sum_{x_m:j=0,1,2,\ldots,n;x=a,b,c} |x|^{\ell} F(n, x_j)}{\sum_{x_m:j=0,1,2,\ldots,n;x=a,b,c} F(n, x_j)},
\] (6.10)

where Eq. (5.3) is used and \(p(n, c_n) = 0\) is assumed. From Theorem VI.1 for any integer \(n > 1\), we have

\[
\xi'(n, \ell) = \frac{2 \times \left(\frac{1}{3} + \frac{1}{2} \times \sum_{j=1}^{n-1} 2^{j\ell} + 1 \times 2^{n\ell} \right) + \frac{2}{3} \times \sqrt{3} + 1 \times \sum_{j=1}^{n-2} (2^{j} \times \sqrt{3})^{\ell}}{2 \times \left(\frac{1}{3} + \frac{1}{2} \times (n - 1) + 1\right) + \frac{2}{3} + 1 \times (n - 2)}
\]

\[
= \frac{2^{n\ell + 1} + \frac{2}{3}(1 + \sqrt{3}) + \frac{2^{n\ell} (1 + (\frac{\sqrt{3}}{2})^{\ell}) - 2^{\ell - (2\sqrt{3})^{\ell}}}{2^{\ell - 1}}}{2n + \frac{1}{3}}.
\] (6.11)

**Theorem VI.2** When \(n\) is large, we have the asymptotic expression

\[
\xi'(n, \ell) \sim \left(1 + \frac{1}{2} \left(1 + \left(\frac{\sqrt{3}}{2}\right)^{\ell}\right)\right) \times \frac{2^{n\ell}}{n}.
\] (6.12)

**Corollary VI.1** Consider only the vertices \(x_j\) with \(x \in \{a, b, c\}\) and \(j = 1, 2, \ldots, n\) on \(SG(n)\), and define the variable \(p'(n, x_j) = p(n, x_j)/\sum_{x_m:j=0,1,2,\ldots,n;x=a,b,c} p(n, x_j)\). The mean distance and the corresponding variance of it when \(n\) is large are given as

\[
\mu = \xi'(n, 1) \sim \frac{2^n (\frac{3}{2} + \sqrt{3})}{n},
\]

\[
\sigma^2 = \xi'(n, 2) - (\xi'(n, 1))^2 \sim \frac{31}{3} \times \frac{2^{n-3}}{n}.
\] (6.13)
VII. DISTRIBUTION \( F(n, x_{m,2}) \) WITH \( x \in \{a, b, c\} \) AND \( 1 \leq m < n \)

Let us consider the vertex \( x_{m,2} \) on \( SG(n) \) with \( x \in \{a, b, c\} \) and \( 1 \leq m < n \) in this section. The expressions for \( x_{m,1} \) can be obtained from those for \( x_{m,2} \) by symmetry. For \( m = n - 1 \), we have

\[
f(n, x_{n-1,2}) = f_1(n-1)g_1(n-1)f(n-1, \hat{x}_{n-2}) + f_1(n-1)^2g_2(n-1, \hat{x}_{n-2}) ,
\]

which is illustrated in Fig. 16, and \( F(n, x_{n-1,2}) \) can be obtained using Eqs. (6.1) and (6.2) as given below in Theorem VII.1

\[
\text{FIG. 16: Illustration for the expression of } f(n, x_{n-1,2}).
\]

Next consider \( m \leq n - 2 \), we have

\[
f(n, x_{m,2}) = f_1(n-1)^2u(n-1, x_{m,2}) ,
\]

similar to the first line of Eq. (6.3). It is then necessary to derive \( u(n-1, x_{m,2}) \) using Eq. (6.4) so that

\[
u(n-1, x_{m,2}) = u(m + 1, x_{m,2}) \prod_{j=m+2}^{n-1} r(j) ,
\]

where the product is set to one when \( n = m + 2 \) and

\[
u(m + 1, x_{m,2}) = f_1(m)^2t_1(m, x_{m-1}) + g_1(m)^2f(m, \hat{x}_{m-1}) + f_1(m)g_1(m)(g_2(m, \hat{x}_{m-1}) + g_2(m, \hat{x}_{m-1}) + 2u(m, x_{m-1}))
\]

as illustrated in Fig. 17.

The term \( t_1(m, x_{m-1}) \) is given by

\[
t_1(m, b_{m-1}) = t_1(m, c_{m-1}) = \begin{cases} 
  g_1(m-1)^3 = \frac{3}{2}f_1(m-1)g_1(m-1)^2 = \frac{3}{4}g_1(m) & (m > 1) \\
  1 = \frac{1}{3}g_1(1) & (m = 1) 
\end{cases}
\]

\[
t_1(m, a_{m-1}) = \begin{cases} 
  0 & (m > 1) \\
  2 = \frac{2}{3}g_1(1) & (m = 1)
\end{cases}
\]

(7.5)
Substituting the results in Eqs. (6.1), (6.2), (6.6) and (7.5) into Eq. (7.4), we have

\[
\begin{align*}
\text{Eq. (7.6)}
\end{align*}
\]

where \(g_1(2) = 36\), and

\[
\begin{align*}
\text{Eq. (7.7)}
\end{align*}
\]

\[
\begin{align*}
\text{Eq. (7.8)}
\end{align*}
\]

It follows that Eq. (7.3) becomes

\[
\begin{align*}
\text{Eq. (7.9)}
\end{align*}
\]

for \(m \leq n - 2\). Combining above results, we obtain the following theorem:
Theorem VII.1 On the Sierpinski gasket $SG(n)$ with $n \geq 2$, the distribution ratio $F(n, x_{n-1, \gamma_2})$ with $x \in \{a, b, c\}$ and $\gamma_2 \in \{1, 2\}$ is given by

\[
F(n, a_{n-1,2}) = F(n, c_{n-1,2}) = F(n, b_{n-1,1}) = F(n, c_{n-1,1}) = \begin{cases} 
\frac{1}{2} & (n > 2), \\
\frac{5}{12} & (n = 2), 
\end{cases}
\]

\[
F(n, b_{n-1,2}) = F(n, a_{n-1,1}) = \begin{cases} 
0 & (n > 2), \\
\frac{1}{6} & (n = 2), 
\end{cases}
\]  \hspace{1cm} (7.10)

and the distribution ratio $F(n, x_{m, \gamma_2})$ with integer $m \leq n - 2$, $x \in \{a, b, c\}$ and $\gamma_2 \in \{1, 2\}$ is given by

\[
F(n, a_{m,2}) = F(n, b_{m,1}) = \begin{cases} 
\frac{1}{2} & (m > 1), \\
\frac{11}{18} & (m = 1), 
\end{cases} F(n, b_{m,2}) = F(n, a_{m,1}) = \begin{cases} 
1 & (m > 1), \\
\frac{49}{72} & (m = 1), 
\end{cases}
\]

\[
F(n, c_{m,2}) = F(n, c_{m,1}) = \begin{cases} 
\frac{3}{2} & (m > 1), \\
\frac{73}{72} & (m = 1). 
\end{cases}
\]  \hspace{1cm} (7.11)

For $m = n - 1$, we also have

\[
g_2(n, x_{n-1,2}) = f_1(n-1)g_1(n-1)g_2(n-1, \tilde{x}_{n-2}) + f_1(n-1)^2t_1(n-1, \tilde{x}_{n-2}),
\]  \hspace{1cm} (7.12)

which is illustrated in Fig. 18 and $G_2(n, x_{n-1,2})$ can be obtained using Eqs. (6.2) and (7.5) as given below.

\[
\begin{array}{c}
\text{FIG. 18: Illustration for the expression of } g_2(n, x_{n-1,2}).
\end{array}
\]

Lemma VII.1 On the Sierpinski gasket $SG(n)$ with $n \geq 2$, the distribution ratio $G_2(n, x_{n-1,2})$ with $x \in \{a, b, c\}$ is given by

\[
G_2(n, a_{n-1,2}) = G_2(n, c_{n-1,2}) = \begin{cases} 
\frac{1}{2} & (n > 2), \\
\frac{5}{18} & (n = 2), 
\end{cases} G_2(n, b_{n-1,2}) = G_2(n, c_{n-1,1}) = \begin{cases} 
0 & (n > 2), \\
\frac{7}{18} & (n = 2). 
\end{cases}
\]  \hspace{1cm} (7.13)

The values for $G_2(n, x_{n-1,1})$ and $G_2(n, x_{m, \gamma_2})$ with $m \leq n - 2$, $x \in \{a, b, c\}$ and $\gamma_2 \in \{1, 2\}$ and equal to $F(n, x_{n-1,1})$ and $F(n, x_{m, \gamma_2})$, respectively, as discussed in the following section. The distribution ratios $F(n, x)$, $G_2(n, x)$ and $U(n, x)$ for $n = 1, 2$ are shown in Figs. 19 and 20 respectively.
VIII. DISTRIBUTION $F(n, x_{\vec{\gamma}})$ WITH $|\vec{\gamma}| \geq 3$ AND MEAN $\ell$ DISPLACEMENT FOR ALL VERTICES ON $SG(n)$

Using the vertex notation given in Section VI the subscript of a vertex $x_{\vec{\gamma}}$ on $SG(n)$ is given as $\vec{\gamma} = (\gamma_1, \gamma_2, \gamma_3, \cdots, \gamma_s)$ where $1 \leq \gamma_1 \leq n - 1$ and $\gamma_k \in \{0, 1, 2\}$ for $k = 2, 3, \ldots, s$. As the vertices $x_m$ and $x_{m,1, x_{m,2}}$, i.e. $s = |\vec{\gamma}| = 1, 2$, on $SG(n)$ with $x \in \{a, b, c\}$ and $m \leq n$ have been given in the previous two sections, the purpose of this section is to consider the other vertex $x_{\vec{\gamma}}$ with $3 \leq |\vec{\gamma}| \leq n$ and $x \in \{a, b, c\}$. We have to derive $F(n, x_{\vec{\gamma}})$ for the vertices with $\gamma_1 = n - 1$, $\gamma_2 = 2$ on $SG(n)$ and the vertices with $\gamma_1 < n - 1$. That is, we should consider the vertices within the upper $SG(n - 1)$ and the lower-left $SG(n - 1)$ that constitute $SG(n)$. For the vertices with $\gamma_1 = n - 1$, $\gamma_2 = 1$, i.e. the vertices within the lower-right $SG(n - 1)$, we use $F(n, x_{\vec{\gamma}}) = F(n, x_{\vec{\gamma}'})$ by symmetry. A quantity like $F(n, x_{\vec{\gamma}})$ on $SG(n)$ will be expressed in terms of the quantities on $SG(n - 1)$ as the recursion relation.

Let us denote the subscript $\vec{\gamma}'$ for the corresponding vertex $x_{\vec{\gamma}'}$ on $SG(n - 1)$. When the first component of $\vec{\gamma}'$, i.e. $\gamma_1$, is equal to $n - 1$ and $\gamma_2 \in \{1, 2\}$, we have $\vec{\gamma}' = (n - 2, \gamma_3, \cdots, \gamma_s)$ with $|\vec{\gamma}'| = |\vec{\gamma}| - 1$; while the first component of $\vec{\gamma}'$ is smaller than $n - 1$, we have $\vec{\gamma}' = \vec{\gamma}$.  

---

**FIG. 19:** The distribution ratios $F(1, x)$, $G_2(1, x)$, and $U(1, x)$ for $SG(1)$.

**FIG. 20:** The distribution ratios $F(2, x)$, $G_2(2, x)$, and $U(2, x)$ for $SG(2)$. 

\[
\begin{array}{ccc}
\frac{1}{2} & 0 & 1 \\
1 & 1/2 & 1 \\
1/2 & 1/2 & 1/2
\end{array}
\quad
\begin{array}{ccc}
1/3 & 1/3 & 1 \\
2/3 & 1/3 & 1 \\
1/3 & 2/3 & 1/3
\end{array}
\quad
\begin{array}{ccc}
\frac{4}{3} & 2/3 & 2/3 \\
4/3 & 1/3 & 1/3 \\
2/3 & 1/3 & 1/3
\end{array}
\quad
\begin{array}{ccc}
1 & 1/2 & 1/6 \\
5/12 & 5/12 & 5/12 \\
7/18 & 7/18 & 7/18
\end{array}
\quad
\begin{array}{ccc}
7/18 & 5/18 & 5/18 \\
11/9 & 11/9 & 11/9 \\
49/36 & 49/36 & 49/36
\end{array}
\quad
\begin{array}{ccc}
49/36 & 73/36 & 73/36 \\
49/36 & 73/36 & 73/36 \\
73/36 & 73/36 & 73/36
\end{array}
\]
By the argument of Eqs. (7.1) and (7.2), we have

\[
\begin{cases}
  f(n, x_{\vec{\gamma}}) = f_1(n-1)g_1(n-1)f(n-1, \hat{x}_{\vec{\gamma}}) + f_1(n-1)^2g_2(n-1, \hat{x}_{\vec{\gamma}}) & \text{if } \gamma_1 = n-1, \gamma_2 = 2, \\
  f(n, x_{\vec{\gamma}}) = f_1(n-1)^2u(n-1, x_{\vec{\gamma}}) & \text{if } \gamma_1 < n-1,
\end{cases}
\]

(8.1)

to obtain the following lemma:

**Lemma VIII.1** On the Sierpinski gasket SG(n), the distribution ratio \( F(n, x_{\vec{\gamma}}) \) with \( 3 \leq |\vec{\gamma}| \leq n \) is given by

\[
\begin{cases}
  F(n, x_{\vec{\gamma}}) = \frac{1}{2}F(n-1, \hat{x}_{\vec{\gamma}}) + \frac{1}{2}G_2(n-1, \hat{x}_{\vec{\gamma}}) & \text{if } \gamma_1 = n-1, \gamma_2 = 2, \\
  F(n, x_{\vec{\gamma}}) = \frac{1}{2}U(n-1, x_{\vec{\gamma}}) & \text{if } \gamma_1 < n-1.
\end{cases}
\]

(8.2)

To evaluate (8.2), we need the following lemmas.

**Lemma VIII.2** On the Sierpinski gasket SG(n), the distribution ratio \( G_2(n, x_{\vec{\gamma}}) \) is given as follows: (i) When \( \gamma_1 = n-1 \) and \( \gamma_2 = 1 \) with \( |\vec{\gamma}| \geq 2 \), we have

\[
G_2(n, x_{\vec{\gamma}}) = F(n, x_{\vec{\gamma}}).
\]

(8.3)

(ii) When \( \gamma_1 < n-1 \), we have

\[
G_2(n, x_{\vec{\gamma}}) = F(n, x_{\vec{\gamma}}) = \frac{1}{2}U(n-1, x_{\vec{\gamma}}), \quad U(n, x_{\vec{\gamma}}) = U(n-1, x_{\vec{\gamma}}).
\]

(8.4)

(iii) When \( \gamma_1 = n-1 \) and \( \gamma_2 = 2 \) with \( |\vec{\gamma}| \geq 3 \), we have

\[
G_2(n, x_{\vec{\gamma}}) = \begin{cases}
  G_2(n-1, \hat{x}_{\vec{\gamma}}) & \text{if } \gamma_3 = 1, 2, \\
  \frac{1}{2}(G_2(n-1, \hat{x}_{\vec{\gamma}}) + G_2(n-1, \tilde{x}_{\vec{\gamma}})) & \text{if } \gamma_3 = 0.
\end{cases}
\]

(8.5)

**Lemma VIII.3** On the Sierpinski gasket SG(n), the distribution ratio \( U(n, x_{\vec{\gamma}}) \) for \( \gamma_1 = n-1 \) and \( \gamma_2 = 2 \) with \( |\vec{\gamma}| \geq 3 \) is given by

\[
U(n, x_{\vec{\gamma}}) = \begin{cases}
  U(n-1, \hat{x}_{\vec{\gamma}}) + U(n-1, x_{\vec{\gamma}}) & \text{if } \gamma_3 = 2, \\
  G_2(n-1, \hat{x}_{\vec{\gamma}}) + G_2(n-1, \tilde{x}_{\vec{\gamma}}) + U(n-1, x_{\vec{\gamma}}) & \text{if } \gamma_3 = 0, 1.
\end{cases}
\]

(8.6)
Proof of Lemma VIII.3: Consider the vertex with \( \gamma_1 = n - 1 \) and \( \gamma_2 = 1 \) first. We have
\[
\begin{aligned}
    f(n, x_\vec{\gamma}) &= f_1(n-1)g_1(n-1)f(n-1, \hat{x}_\vec{\gamma}) + \frac{1}{2}f_2(n-1, \hat{x}_\vec{\gamma}), \\
    g_2(n, x_\vec{\gamma}) &= g_1(n-1)^2f(n-1, \hat{x}_\vec{\gamma}) + f_1(n-1)g_1(n-1)g_2(n-1, \hat{x}_\vec{\gamma}),
\end{aligned}
\]
(8.7)
which leads to (8.3).

For the vertex with \( \gamma_1 < n - 1 \), we have
\[
\begin{aligned}
    f(n, x_\vec{\gamma}) &= f_1(n-1)^2u(n-1, x_\vec{\gamma}), \\
    g_2(n, x_\vec{\gamma}) &= f_1(n-1)g_1(n-1)u(n-1, x_\vec{\gamma}), \\
    g_3(n, x_\vec{\gamma}) &= 2f_1(n-1)g_1(n-1)g_3(n-1, x_\vec{\gamma}),
\end{aligned}
\]
(8.8)
similar to Eq. (6.3), such that Eq. (8.4) is proved.

For Eq. (8.5) with \( \gamma_1 = n - 1 \) and \( \gamma_2 = 2 \), we have
\[
\begin{aligned}
    g_2(n, x_\vec{\gamma}) &= f_1(n-1)g_1(n-1)g_2(n-1, \hat{x}_\vec{\gamma}) + f_1(n-1)^2t_1(n-1, \tilde{x}_\vec{\gamma}),
\end{aligned}
\]
(8.9)
similar to Eq. (7.12), so that
\[
G_2(n, x_\vec{\gamma}) = \frac{1}{2}G_2(n-1, \hat{x}_\vec{\gamma}) + \frac{1}{3}T_1(n-1, \tilde{x}_\vec{\gamma}).
\]
(8.10)
The recursion relations for \( t_1(n, x_\vec{\gamma}) \) can be derived as
\[
\begin{aligned}
    t_1(n, x_\vec{\gamma}) &= g_1(n-1)^2g_2(n-1, \hat{x}_\vec{\gamma}) + f_1(n-1)g_1(n-1)t_1(n-1, \hat{x}_\vec{\gamma}) \quad \text{if} \quad \gamma_1 < n - 1, \\
    t_1(n, x_\vec{\gamma}) &= g_1(n-1)^2g_2(n-1, \tilde{x}_\vec{\gamma}) + f_1(n-1)g_1(n-1)t_1(n-1, \tilde{x}_\vec{\gamma}) \quad \text{if} \quad \gamma_1 = n - 1, \gamma_2 = 1, \\
    t_1(n, x_\vec{\gamma}) &= g_1(n-1)^2u(n-1, x_\vec{\gamma}) \quad \text{if} \quad \gamma_1 = n - 1, \gamma_2 = 2,
\end{aligned}
\]
(8.11)
or equivalently,
\[
\begin{aligned}
    T_1(n, x_\vec{\gamma}) &= \frac{3}{4}G_2(n-1, \hat{x}_\vec{\gamma}) + \frac{1}{2}T_1(n-1, \hat{x}_\vec{\gamma}) \quad \text{if} \quad \gamma_1 < n - 1, \\
    T_1(n, x_\vec{\gamma}) &= \frac{3}{4}G_2(n-1, \tilde{x}_\vec{\gamma}) + \frac{1}{2}T_1(n-1, \tilde{x}_\vec{\gamma}) \quad \text{if} \quad \gamma_1 = n - 1, \gamma_2 = 1, \\
    T_1(n, x_\vec{\gamma}) &= \frac{3}{4}U(n-1, \tilde{x}_\vec{\gamma}) \quad \text{if} \quad \gamma_1 = n - 1, \gamma_2 = 2.
\end{aligned}
\]
(8.12)
For the vertex \( x_\vec{\gamma} \) with \( \gamma_1 < n - 1 \), the corresponding \( \hat{x}_\vec{\gamma} \) is located in the upper \( SG(n-1) \), so that Eq. (8.10) can be applied to give
\[
\frac{3}{2}G_2(n, \hat{x}_\vec{\gamma}) = \frac{3}{4}G_2(n-1, \hat{x}_\vec{\gamma}) + \frac{1}{2}T_1(n-1, \hat{x}_\vec{\gamma}) = T_1(n, x_\vec{\gamma}),
\]
(8.13)
where the last equality follows from the first line of (8.12). For the vertex \( x_\gamma \) with \( \gamma_1 = n - 1 \) and \( \gamma_2 = 1 \), the corresponding \( \tilde{x}_\gamma \) is located in the upper \( SG(n - 1) \), so that Eq. (8.10) can be applied to give

\[
\frac{3}{2} G_2(n, \tilde{x}_\gamma) = \frac{3}{4} G_2(n - 1, \tilde{x}_\gamma) + \frac{1}{2} T_1(n - 1, \tilde{x}_\gamma) = \frac{3}{4} G_2(n - 1, \tilde{x}_\gamma) + \frac{1}{2} T_1(n - 1, x_\gamma) = T_1(n, \hat{x}_\gamma),
\]

(8.14)

where the last equality follows from the second line of (8.12), and the relations \( \hat{x}_\gamma = \tilde{x}_\gamma \) and \( T_1(n - 1, x_\gamma) = T_1(n - 1, \tilde{x}_\gamma) \) are used. For the vertex \( x_\gamma \) with \( \gamma_1 = n - 1 \) and \( \gamma_2 = 2 \), the corresponding \( \hat{x}_\gamma \) is located in the lower-left \( SG(n - 1) \), so that Eq. (8.8) can be applied to give

\[
\frac{3}{2} G_2(n, \hat{x}_\gamma) = \frac{3}{4} U(n, \hat{x}_\gamma) = T_1(n, x_\gamma),
\]

(8.15)

where the last equality follows from the third line of (8.12). Combining Eqs. (8.13)-(8.15) for the vertex \( x_\gamma \) with \( |\gamma| \geq 2 \), we obtain the following relation:

\[
T_1(n, x_\gamma) = \begin{cases} 
\frac{3}{2} G_2(n, \hat{x}_\gamma) & \text{if } \gamma_1 < n - 1 \text{ or } \gamma_1 = n - 1, \gamma_2 = 2, \\
\frac{3}{4} G_2(n, \tilde{x}_\gamma) & \text{if } \gamma_1 = n - 1, \gamma_2 = 1.
\end{cases}
\]

(8.16)

For the vertex \( x_\gamma \) with \( \gamma_1 < n - 1 \), or \( \gamma_1 = n - 1, \gamma_2 = 1 \), or \( \gamma_1 = n - 1, \gamma_2 = 2 \), the corresponding \( \tilde{x}_\gamma \) is located in the lower-right \( SG(n - 1) \), or upper \( SG(n - 1) \), or lower-left \( SG(n - 1) \), respectively. Therefore, the last term in Eq. (8.10) becomes

\[
\frac{1}{3} T_1(n - 1, \tilde{x}_\gamma) = \begin{cases} 
\frac{1}{2} G_2(n - 1, \tilde{x}_\gamma) & \text{if } \gamma_3 = 1, 2, \\
\frac{1}{2} G_2(n - 1, \tilde{x}_\gamma) = \frac{1}{2} G_2(n - 1, \hat{x}_\gamma) & \text{if } \gamma_3 = 0,
\end{cases}
\]

(8.17)

such that Eq. (8.5) is proved.

\[\text{Proof of Lemma VIII.3:}\] As \( U(n, x_\gamma) \) for \( \gamma_1 < n - 1 \) has been given in Eq. (8.4) and \( U(n, x_\gamma) = U(n, \hat{x}_\gamma) \) by definition, we only consider the vertices with \( \gamma_1 = n - 1 \) and \( \gamma_2 = 2 \) in this lemma. By the same argument of (7.4), we have

\[
u(n, x_\gamma) = f_1(n - 1)^2 t_1(n - 1, x_\gamma) + g_1(n - 1)^2 f(n - 1, \tilde{x}_\gamma) + f_1(n - 1) g_1(n - 1) (g_2(n - 1, \tilde{x}_\gamma) + g_2(n - 1, \hat{x}_\gamma) + 2 u(n - 1, x_\gamma))
\]

(8.18)
or equivalently,

\[
U(n, x) = \frac{1}{3}T_1(n-1, x) + \frac{1}{2}F(n-1, \hat{x}) + \frac{1}{2}G_2(n-1, \hat{x}) + \frac{1}{2}G_2(n-1, \tilde{x}) + U(n-1, x).
\]

(8.19)

Similar to the definition of \( \gamma' \), let us denote the subscript \( \gamma'' \) for the vertex on \( SG(n-2) \). When the first component of \( \gamma' \), i.e. \( \gamma_1 \), is equal to \( n-2 \) and \( \gamma_3 \in \{1, 2\} \), we have \( \gamma'' = (n-3, \gamma_4, \ldots, \gamma_s) \) with \( |\gamma''| = |\gamma'| - 1 \); while \( \gamma_3 = 0 \), we have \( \gamma'' = \gamma' \). Consider the vertex \( x \) with \( \gamma_3 = 2 \) first. The corresponding \( \hat{x} \) and \( \tilde{x} \) are both located in the lower-left \( SG(n-2) \). Using the third line of Eq. (8.11) and the first two lines of Eq. (8.8), we have

\[
\begin{align*}
t_1(n-1, x) &= g_1(n-2)^2 u(n-2, \hat{x}) = \frac{2}{3} g_1(n-1) U(n-2, \hat{x}) , \\
f(n-1, \hat{x}) &= f_1(n-2)^2 u(n-2, \hat{x}) = \frac{1}{2} f_1(n-1) U(n-2, \hat{x}) , \\
g_2(n-1, \hat{x}) &= g_2(n-1-2) u(n-2, \hat{x}) = f_1(n-2) g_1(n-2) u(n-2, \hat{x}) \\
&= \frac{1}{2} g_1(n-1) U(n-2, \hat{x}).
\end{align*}
\]

(8.20)

where \( U(n-2, \hat{x}) = U(n-1, \hat{x}) \) as given in Eq. (8.4). Substituting Eq. (8.20) into Eq. (8.19), the first line of (8.6) for \( \gamma_3 = 2 \) is proved.

Next consider the vertex \( x \) with \( \gamma_3 = 0 \). The corresponding \( \hat{x} \) and \( \tilde{x} \) are located in the upper \( SG(n-2) \) and lower-right \( SG(n-2) \), respectively. Using the first line of Eq. (8.11), the first line of Eq. (8.11), Eq. (8.9), and the second line of Eq. (8.7), we have

\[
\begin{align*}
t_1(n-1, x) &= g_1(n-2)^2 g_2(n-2, \hat{x}) + f_1(n-2) g_1(n-2) t_1(n-2, \hat{x}) , \\
f(n-1, \hat{x}) &= f_1(n-2) g_1(n-2) f(n-2, \hat{x}) + g_1(n-2) g_2(n-2) \hat{x} + f_1(n-2)^2 t_1(n-2, \hat{x}) , \\
g_2(n-1, \hat{x}) &= f_1(n-2) g_1(n-2) g_2(n-2, \hat{x}) + f_1(n-2)^2 t_1(n-2, \hat{x}) , \\
g_2(n-1, \tilde{x}) &= g_1(n-2)^2 f(n-2, \tilde{x}) + f_1(n-2) g_1(n-2) g_2(n-2, \tilde{x}) ,
\end{align*}
\]

(8.21)

or equivalently,

\[
\begin{align*}
T_1(n-1, x) &= \frac{2}{3} G_2(n-2, \hat{x}) + \frac{1}{2} T_1(n-2, \hat{x}) , \\
F(n-1, \hat{x}) &= \frac{1}{2} F(n-2, \hat{x}) + \frac{1}{2} G_2(n-2, \hat{x}) , \\
G_2(n-1, \hat{x}) &= \frac{1}{2} G_2(n-2, \hat{x}) + \frac{1}{2} T_1(n-2, \hat{x}) , \\
G_2(n-1, \tilde{x}) &= \frac{1}{2} F(n-2, \tilde{x}) + \frac{1}{2} G_2(n-2, \tilde{x}).
\end{align*}
\]

(8.22)

It follows that \( T_1(n-1, x) = \frac{2}{3} G_2(n-1, \hat{x}) \), \( F(n-1, \hat{x}) = G_2(n-1, \tilde{x}) \), and Eq. (8.19) becomes the second line of Eq. (8.6) for \( \gamma_3 = 0 \).
Finally, consider the vertex $x_{\gamma'}$ with $\gamma_3 = 1$. The corresponding $\hat{x}_{\gamma'}$ and $\tilde{x}_{\gamma'}$ are located in the lower-right $SG(n-2)$ and upper $SG(n-2)$, respectively. Using the second line of Eq. (8.11) and Eqs. (8.7), (8.9), we have

$$
\begin{aligned}
t_1(n-1, x_{\gamma'}) &= g_1(n-2)^2 g_2(n-2, \tilde{x}_{\gamma'}) + f_1(n-2) g_1(n-2) t_1(n-2, \hat{x}_{\gamma'}) , \\
f(n-1, x_{\gamma'}) &= f_1(n-2) g_1(n-2) f(n-2, x_{\gamma'}) + f_1(n-2)^2 g_2(n-2, x_{\gamma'}) , \\
g_2(n-1, x_{\gamma'}) &= g_1(n-2)^2 f(n-2, x_{\gamma'}) + f_1(n-2) g_1(n-2) g_2(n-2, x_{\gamma'}) , \\
g_2(n-1, \tilde{x}_{\gamma'}) &= f_1(n-2) g_1(n-2) g_2(n-2, \tilde{x}_{\gamma'}) + f_1(n-2)^2 t_1(n-2, \hat{x}_{\gamma'}) .
\end{aligned}
$$

(8.23)

Because $t_1(n-2, \hat{x}_{\gamma'}) = t_1(n-2, \hat{x}_{\gamma'})$, it follows that $T_1(n-1, x_{\gamma'}) = \frac{3}{2} G_2(n-1, \tilde{x}_{\gamma'})$, $F(n-1, \hat{x}_{\gamma'}) = G_2(n-1, \tilde{x}_{\gamma'})$, and Eq. (8.19) becomes the second line of Eq. (8.6) for $\gamma_3 = 1$.

The distributions $F(n, x), G_2(n, x)$ and $U(n, x)$ for $n = 3$ are shown in Fig. 21.

From Lemmas VIII.1 to VIII.3, we have the following proposition:

**Proposition VIII.1** For almost all the vertex $x_{\tilde{\gamma}}$ on the Sierpinski gasket $SG(n)$ with $n \geq 2$, we have $F(n, x_{\gamma}) = G_2(n, x_{\gamma})$, except the outmost vertices $a_n$, $b_n$, and the three vertices $x_{\tilde{\gamma}^0}$ with $\tilde{\gamma}^0 = (n-1, 2, 2, ..., 2)$ and $x \in \{a, b, c\}$. The values for the vertices $x_{\tilde{\gamma}^0}$ are given by

$$
F(n, a_{\tilde{\gamma}^0}) = \frac{5}{12} ,
$$

$$
F(n, b_{\tilde{\gamma}^0}) = \begin{cases} 
\frac{5}{12} & \text{for odd } n , \\
\frac{1}{6} & \text{for even } n ,
\end{cases} \quad F(n, c_{\tilde{\gamma}^0}) = \begin{cases} 
\frac{1}{6} & \text{for odd } n , \\
\frac{5}{12} & \text{for even } n ,
\end{cases}
$$

(8.24)

and

$$
G_2(n, a_{\tilde{\gamma}^0}) = \frac{5}{18} ,
$$

$$
G_2(n, b_{\tilde{\gamma}^0}) = \begin{cases} 
\frac{5}{18} & \text{for odd } n , \\
\frac{7}{18} & \text{for even } n ,
\end{cases} \quad G_2(n, c_{\tilde{\gamma}^0}) = \begin{cases} 
\frac{7}{18} & \text{for odd } n , \\
\frac{5}{18} & \text{for even } n .
\end{cases}
$$

(8.25)

**Proof:** For the outmost vertices $a_n$, $b_n$, we know $F(n, a_n) = F(n, b_n) = 1$, while there are no values assigned for $G_2(n, a_n)$ and $G_2(n, b_n)$ by definition. From Lemma VIII.2 we already know that $G_2(n, x_{\tilde{\gamma}}) = F(n, x_{\tilde{\gamma}})$ for the vertices with $\gamma_1 < n - 1$ and the vertices with
\[ \gamma_1 = n - 1 \quad \text{and} \quad \gamma_2 = 1. \] We should only consider the vertices with \( \gamma_1 = n - 1 \) and \( \gamma_2 = 2 \) in this proof.

Let us verify the values for the three special vertices \( x_{\tilde{\gamma}} \) with \( \gamma_1 = n - 1 \) and \( \gamma_2 = \gamma_3 = \cdots = 2 \) first. Eqs. (8.24) and (8.25) are correct for \( n = 2, 3 \) as shown in Figs. 20 and 21. It is easy to obtain Eq. (8.25) using the upper line of Eq. (8.5). Eq. (8.24) can also be verified by the upper line of Eq. (8.2) because \( \hat{x}_{\tilde{\gamma}} \) corresponds to the vertices with subscript \( \langle n - 1, 1, 1, \cdots \rangle \) such that \( G_2(n, \hat{x}_{\tilde{\gamma}}) = F(n, \hat{x}_{\tilde{\gamma}}) = F(n, \hat{x}_{\tilde{\gamma}}). \)

Now consider the vertices with \( \gamma_1 = n - 1, \gamma_2 = 2 \) other than the three special vertices discussed above. For the vertices with \( \gamma_1 = n - 1, \gamma_2 = 2, \gamma_3 = 0, F(n, x_\gamma) = G_2(n, x_\gamma) \) can be verified by comparing the lower line of Eq. (8.5) with the upper line of Eq. (8.2).
Both of them contain the term $\frac{1}{2}G_2(n-1, \hat{x}_\gamma)$, while $F(n-1, \hat{x}_\gamma) = G_2(n-1, \hat{x}_\gamma)$ for such vertices according to Eq. (8.3). For the vertices with $\gamma_1 = n-1$, $\gamma_2 = 2$, $\gamma_3 = 1$, compare the upper lines of Eqs. (8.5) and (8.2). We know $F(n-1, \hat{x}_\gamma) = G_2(n-1, \hat{x}_\gamma)$ for such vertices according to Eq. (8.4), while $G_2(n-1, \hat{x}_\gamma) = G_2(n-1, \hat{x}_\gamma) = G_2(n-1, \hat{x}_\gamma)$ for these vertices. Finally for the vertices with $\gamma_1 = n-1$, $\gamma_2 = 2$ and $\gamma_3 = 2$ other than the three special vertices discussed above, again compare the upper lines of Eqs. (8.5) and (8.2). We have $F(n-1, \hat{x}_\gamma) = G_2(n-1, \hat{x}_\gamma)$ for such vertices by induction, while $G_2(n-1, \hat{x}_\gamma) = F(n-1, \hat{x}_\gamma) = F(n-1, \hat{x}_\gamma) = F(n-1, \hat{x}_\gamma)$ for these vertices. \[\Box\]

For any positive $\ell > 0$, define the mean $\ell$ displacement $\xi(n, \ell)$ as
\[
\xi(n, \ell) = \sum_{x \in v(SG(n))} |x|^\ell p(n, x) , \quad \psi(n, \ell) = \sum_{x \in v(SG(n))} |x|^\ell F(n, x) , \quad (8.26)
\]
where $\xi(n, \ell) = \psi(n, \ell)/R(n)$ according to Eq. (8.3). We would like to study the behavior of $\xi(n, \ell)$ when $n$ is large. For that purpose, let us also define
\[
\phi(n, \ell) = \sum_{x \in V(SG(n))} |x|^\ell U(n, x) , \quad U(n) = \sum_{x \in v(SG(n)), |\gamma|>2} U(n, x) . \quad (8.27)
\]
We shall first derive upper and lower bounds for $\psi(n, \ell)$ and $\phi(n, \ell)$ as follows. By the first line of Eq. (8.2) and Proposition VIII.1, for every $n \geq 3$, the summation of the value of $F(n, x_\gamma)$ over most of the vertices in the upper $SG(n-1)$ is given as
\[
\sum_{x_\gamma \in v(SG(n)), |\gamma|>2, \gamma_1=n-1, \gamma_2=2} F(n, x_\gamma) = \sum_{x_\gamma \in v(SG(n-1))} F(n-1, x_\gamma) + O(1) = R(n-1) + O(1) , \quad (8.28)
\]
where from Eq. (8.2) $R(n) = \left(\frac{7n^3}{2^x x^3}\right)4^n + O(1)$ when $n$ is large, and the number of neglected vertices with $|\gamma| = 1, 2$ in the upper $SG(n-1)$ has order one. By Eq. (8.2), we have
\[
\sum_{x_\gamma \in v(SG(n))} F(n, x_\gamma) = R(n) = \frac{1}{2}U(n-1) + 2R(n-1) + O(n) , \quad (8.29)
\]
because the number of neglected vertices with $|\gamma| = 1, 2$ has order $n$, so that when $n$ is large
\[
U(n-1) = 2\left(R(n) - 2R(n-1)\right) + O(n) = R(n) + O(n) . \quad (8.30)
\]
For all the vertices $x_\gamma \in v(SG(n))$ with $|\gamma| > 2, \gamma_1 = n-1, \gamma_2 = 2$, the distance between $x_\gamma$ and $o$, i.e. $|x_\gamma|$, is larger than $2^{n-1}$ and less than $2^n$. Using the inequality: $2^{(n-1)\ell} \leq |x_\gamma|^{\ell} \leq 2^n$,
$2^{n\ell}$ for $\ell > 0$, we have the upper and lower bounds for $\psi(n, \ell)$ as

$$
\frac{1}{2} \phi(n-1, \ell) + 2 \{ 2^{(n-1)\ell} [R(n-1) + O(n)] \} \leq \psi(n, \ell) \leq \frac{1}{2} \phi(n-1, \ell) + 2 \{ 2^{n\ell} [R(n-1) + O(n)] \},
$$

(8.31)

i.e.

$$
\frac{1}{2} \phi(n - 1, \ell) + 2^{n\ell+1} \left[ \frac{7 \times 17}{24 \times 3^3} 4^{n-1} + O(n) \right]
\leq \psi(n, \ell) \leq \frac{1}{2} \phi(n - 1, \ell) + 2^{n\ell+1} \left[ \frac{7 \times 17}{24 \times 3^3} 4^{n-1} + O(n) \right].
$$

(8.32)

Similarly, using Eqs. (8.4), (8.6) and Proposition VIII.1 we have the upper and lower bounds for $\phi(n, \ell)$ as

$$
\phi(n - 1, \ell) + 2^{(n-1)\ell} \times 2 \left\{ 2R(n - 1) + U(n - 1) + O(n) \right\}
\leq \phi(n, \ell) \leq \phi(n - 1, \ell) + 2^{n\ell} \times 2 \left\{ 2R(n - 1) + U(n - 1) + O(n) \right\},
$$

(8.33)

i.e.

$$
\phi(n - 1, \ell) + 2^{(n-1)\ell+1} \left[ 6 \times \frac{7 \times 17}{24 \times 3^3} \times 4^{n-1} + O(n) \right]
\leq \phi(n, \ell) \leq \phi(n - 1, \ell) + 2^{n\ell+1} \left[ 6 \times \frac{7 \times 17}{24 \times 3^3} \times 4^{n-1} + O(n) \right].
$$

(8.34)

By induction, Eq. (8.34) becomes

$$
\left( \frac{7 \times 17}{24 \times 3^3} \right) \left\{ \frac{(2^{\ell+1})^n}{2^{2+\ell} - 1} + 2^{(n-1)\ell} O(n) \right\} \leq \phi(n, \ell) \leq \left( \frac{7 \times 17}{24 \times 3^3} \right) \left\{ \frac{2^{\ell(2^{\ell+1})^n}}{2^{2+\ell} - 1} + 2^{(n-1)\ell} O(n) \right\}.
$$

(8.35)

Substituting Eq. (8.35) into Eq. (8.32), we have

$$
\left( \frac{7 \times 17}{24 \times 3^3} \right) \times 2^{(\ell+2)n-\ell} \left\{ \frac{2^{1+\ell} + 1}{2^{2+\ell} - 1} + 4^{-(n-1)} O(n) \right\}
\leq \psi(n, \ell) \leq \left( \frac{7 \times 17}{24 \times 3^3} \right) \times 2^{(\ell+2)n} \left\{ \frac{2^{1+\ell} + 1}{2^{2+\ell} - 1} + 4^{-(n-1)} O(n) \right\}
$$

(8.36)

for large $n$, which leads to the following theorem:

**Theorem VIII.1** When $n$ is large, the mean $\ell$ displacement is bounded as

$$
2^{(n-1)\ell} \left\{ \frac{2^{1+\ell} + 1}{2^{2+\ell} - 1} + 4^{-n} O(n) \right\} \leq \xi(n, \ell) \leq 2^{n\ell} \left\{ \frac{2^{1+\ell} + 1}{2^{2+\ell} - 1} + 4^{-n} O(n) \right\}.
$$

(8.37)

Similar to the consideration in [21], we can calculate the value of $\ln \xi(n, \ell)/\ln L(w)$ in the large $n$ limit, where $L(w)$ is defined above Corollary IV.2.

**Corollary VIII.1** In the limit of infinite $n$, the exponent for the mean $\ell$ displacement is given as

$$
\lim_{n \to \infty} \frac{\ln \xi(n, \ell)}{\ln [(3^{n+1} + 1)/2]} = \ell \frac{\ln 2}{\ln 3}.
$$

(8.38)
Acknowledgments

The authors would like to thank Weigen Yan for helpful discussion and Chuan-Hung Chen for use of computer facility. The research of S.C.C. was partially supported by the NSC grant NSC-97-2112-M-006-007-MY3 and NSC-98-2119-M-002-001. The research of L.C.C was partially supported by TJ & MY Foundation and NSC grant. L.C.C. would like to thank PIMS, university of British Columbia for the hospitality.

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