On Mechanics and Thermodynamics of a stellar galaxy in a two-component virial system and the Fundamental Plane.

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Abstract

The paper confirms the existence of a special configuration (among the infinitive number of a priori possible virial states) which a B stellar (Baryonic) component may assume inside a given D dark halo potential well. This satisfies the d’Alembert Principle of virtual works and its typical dimension works as a scale length (we call tidal radius) induced on the gravitational field of the bright component by the dark one. Its dynamic and thermodynamic properties are here analyzed in connection with the physical reason for the existence of the Fundamental Plane for ellipticals and, in general, for two-component virialized systems. The analysis is performed by using two-component models with two power-law density profiles and two homogeneous cores. The outputs of this kind of models, at the special configuration, are summarized and compared with some observable scaling relations for pressure supported ellipticals. The problem of extending the results to a general class of models with Zhao (1996) profiles, which are more suitable for an elliptical galaxy system, is also taken into account. The virial equilibrium stages of the two-component system have to occur after a previous violent relaxation phase. If the stellar B component is allowed to cool slowly its virial evolution consists of a sequence of contractions with enough time to rearrange the virial equilibrium after any step. The thermodynamic process during the dynamical evolution is so divided into a sequence of transformations which are irreversible but occur between two quasi-equilibrium stages. Then, it is possible to assign: a mean temperature to the whole B component during this quasi-static sequence and the entropy variation between two consecutive virial steps. The analysis allows the conclusion that the induced scale length is a real confinement for the stellar system. This follows from the application of the I° Thermodynamics Principle under the virial equilibrium constraint, by checking how larger configurations turn out to be forbidden, according to the II° Thermodynamics Principle. The presence of this specific border on the space of the baryonic luminous component has to be regarded as the physical reason why a stellar galaxy belongs to the Fundamental Plane (FP) and why astrophysical objects, with a completely different history and formation, but characterized by a tidal radius (as the globular clusters are) lie on the same FP. An other problem addressed is how this special configuration may be reached. This is strictly connected with the problem of the
end state of the collisionless stellar system after a violent relaxation phase. Even if degeneracy towards the initial conditions is present on the FP, the mechanic and thermodynamic properties of the special configuration suggest this state may be the best candidate for the beginning of the $B$ component virial evolution, and also give a possible explanation for why an elliptical is not completely relaxed in respect to its dark halo.

Key words: Celestial Mechanics, Stellar Dynamics; Galaxies: Clusters.

1 Introduction

As Ogorodnikov (1965) has highlighted, in order to find the most probable phase distribution function for a stellar system in a stationary state, the phase volume has to be truncated in both coordinate and velocity space. While in the velocity space the truncation arises spontaneously due to the existence of the velocity of escape, the introduction of a cut-off in the coordinate space appears, on one side, necessary in order to obtain a finite mass $M$ and radius $R$, but, on the other, very problematic.

A similar difficulty also appears on the thermodynamical side, for which an extensive literature exists (from: Lynden-Bell & Wood, 1968; Horowitz & Katz, 1978; White & Narayan, 1987, until, e.g., Bertin & Trenti, 2003, and references therein). By using the standard Boltzmann-Gibbs entropy:

$$S = - \int f \ln f \, d^3x \, d^3v$$  

(1)

defined by the distribution function in the $6 -$ dimensional phase space , $f(\vec{x}, \vec{v})$ (hereafter $DF$), and looking for what maximizes the entropy of the same stellar system, the conclusion is: the $DF$ which plays this role in (1) is that of the isothermal sphere. But, the maximization of $S$, subject to fixed mass $M$ and energy $E$, leads again to a $DF$ that is incompatible with finite $M$ and $E$ (see, e.g., Binney & Tremaine, 1987, Chapter 4; Merritt 1999, Lima Neto et al. 1999, Marquez et al. 2001, and references therein).

It is beyond the scope of this paper to enter into the very complicate problem of looking for the suitable models for the collisionless stellar systems by an analysis in the phase space and a research of the $DF$ which maximizes the (1), or to examine the thermodynamic properties of the family models which are able to explain the features of partially relaxed anisotropic stellar systems (see,

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e.g., Stiavelli & Bertin, 1987, Bertin & Trenti, 2003, and references therein). Nevertheless, our limited contribution to the wide discussion existing in the literature will be to underline as in a stellar component, embedded in a second dark matter subsystem (as realistically thought, e.g., Ciotti, 1999, and references therein), a truncation is spontaneously introduced in coordinate space, due to the presence of a scale length induced from the dark halo, as long as virial equilibrium holds. That is the tidal radius which we discovered has to exist when two-component models are considered with two different power-law density distributions and two inner homogeneous cores (Secco, 2000; Secco, 2001, hereafter LS1), under some constraints on the exponents.

The consequence of the existence of a special configuration characterized by this tidal radius are analyzed here.

Finally, we will gain more insight into the physical meaning of the special configuration considered by introducing the thermodynamic information quantity (Layzer, 1976). Even if some considerations which follow are more general and may also be extended to spirals, we will limit our considerations to the collisionless stellar systems, as the ellipticals are considered.

2 Looking for a special virial configuration

To introduce the problem in a general way, we start by considering the potential well of a given spherical virialized dark matter (hereafter, $DM$) halo of mass $M_D$ and virial radius $a_D$, with a density radial profile as follows:

$$
\rho(r) = \frac{\rho_o}{(r/r_o)^\gamma[1 + (r/r_o)^\alpha]^\delta}; \quad \delta = (\beta - \gamma)/\alpha \tag{2}
$$

where $\rho_o$ and $r_o$ are its characteristic density and its scale radius, respectively. These kinds of profiles have already been introduced by Zhao (1996) and by Kravtsov et al. (1998) in order to generalize the universal profile proposed by Navarro, Frenk & White (hereafter, NFW) (Navarro et al. 1996, Navarro et al. 1997) which is obtained from eq. (2) as soon as $(\alpha = 1; \beta = 3; \gamma = 1; \delta = 2)$. Hereafter, we will name them Zhao profiles.

The question which arises is the following: Does a special virial configuration exist among the infinitive number of a priori possible virial configurations which the luminous (Baryonic) component $(B)$ may assume inside the given dark one $(D)$?
Fig. 1. Two-component model: $D$ is the dark matter ($DM$) halo spheroidal component, $B$ is the bright (Baryonic) inner one. Here the $B$ component is non-homothetic to the $D$ one instead of what occurs in this paper. Fig.1 shows a $B$ spheroid with an axis ratio smaller than that of the $D$ one in order to show what is, in the general case, the dark matter fraction which, according to Newton’s first theorem, exerts dynamical effect on the embedded $B$ subsystem. In this general case the dark matter fraction is inside the surface $\Sigma^*$ which, in the homothetic case, coincides with the $B$ contour (Raffaele, 2003).

2.1 Tensor virial formalism

In order to find the answer we need to use the tensor virial theorem extended to two components: $D + B$ (Brosche et al. 1983; Caimmi et al. 1984; Caimmi & Secco, 1992). In a two-component system in which one ($B$) (Baryonic or Bright; in this context also: stellar) is completely embedded in the other ($D$) (of $DM$) and each of them is penetrated by the other, the following tensor virial stationary equations hold:

$$2(T_u)_{ij} = -(V_u)_{ij}; \quad (u = B, D; \quad i, j = x, y, z)$$

(3)

where $(T_u)_{ij}$ is the kinetic-energy tensor and $(V_u)_{ij}$ is the Clausius’ virial tensor which splits into two terms: the self potential-energy tensor, $(\Omega_u)_{ij}$, and the
tidal potential-energy tensor, \((V_{uv})_{ij}\) \((u, v = B, D)\), due to the gravitational force which the \(v\) subsystem exerts on the \(u\) one. The eqs. (3) yield the following pair of tensor equations:

\[
2(T_B)_{ij} = -\Omega_B)_{ij} - (V_{BD})_{ij} \tag{4}
\]
\[
2(T_D)_{ij} = -\Omega_D)_{ij} - (V_{DB})_{ij} \tag{5}
\]

Therefore, e.g., in the case of the inner \(B\) component, we have:

\[
(V_{BD})_{ij} = \int \rho_B \bar{x}_i \frac{\partial \Phi_D}{\partial x_j} d\bar{x}_B ; \tag{6}
\]

where \(\Phi_D\) is the gravitational potential due to the mass distribution of the \(D\) component.

Fig. 2. The energy trends of the \(B\)-system as a function of size ratio of Baryonic to \(DM\) components, \(x = a_B/a_D\), normalized at the factor \((GM_B^2F)/a_D\). The \(V\)-curves represent the Clausius’ virial energies, the \(E\)-curves the corresponding total potential energies, in the cases of Tab.1 (Raffaele, 2003).
Even if we do not look for the distribution functions which correspond to
the models considered in the next subsection, it may be useful to remember
the link between the tensor virial quantities and the phase space, given
by considering the moment equations of the second order, in the coordinate
space, of the $\Pi^o$ Jeans equations which, in turn, are the second order mo-
ments, in the velocity space, of the Boltzmann equation (Binney & Tremaine,
1987, Chap.4; Chandrasekhar, 1969, Chapt.2). It follows that in a system of
collisionless particles with instantaneous $\vec{x}$ position and $\vec{v}$ velocity, character-
ized by a distribution function $f(\vec{x}, \vec{v})$, with spatial density in phase space,
$\nu(\vec{x}) = \int f(\vec{x}, \vec{v}) \, d\vec{v}$, and mass density $\rho(\vec{x}) = m_* \nu(\vec{x})$ ($m_*$ being the average
stellar mass), the kinetic-energy tensors $T_{ij}$, $T'_{ij}$, $\Pi_{ij}$, are defined as:

\begin{align}
T_{ij} &= \frac{1}{2} \int \rho \overline{v_i v_j} \, d\vec{x} = T'_{ij} + \frac{1}{2} \Pi_{ij} \quad ; \\
T'_{ij} &= \frac{1}{2} \int \rho \overline{v_i v_j} \, d\vec{x} \quad ; \quad \Pi_{ij} = \int \rho \sigma^2_{ij} \, d\vec{x} \quad ;
\end{align}

(7)

(8)

that is by the mean of square velocities ($\overline{v_i v_j}$), by the square of mean velocities
($\overline{\nu_i \nu_j}$) of streaming motions, and by the random square velocity components
($\sigma^2_{ij}$), respectively. The general expression for the mean $\overline{v_i v_j}$ is as usual:

$$\overline{v_i v_j} = \frac{1}{\nu} \int f v_i v_j \, d\vec{x}$$

(9)

In the case of one single component, the Clausius’ virial tensor, $V_{ij}$, matches
the self-potential energy tensor, $\Omega_{ij}$, that is:

$$V_{ij} = \Omega_{ij} = -\frac{1}{2} \int \rho \Phi_{ij} \, d\vec{x}$$

(10)

$\Phi_{ij}$ is the tensor potential (e.g., Chandrasekhar, 1969) defined as:

$$\Phi_{ij} = G \int \rho(\vec{x}') \frac{(x_i - x'_i)(x_j - x'_j)}{|\vec{x} - \vec{x}'|^3} \, d\vec{x}$$

with $G$ the gravitational constant. It should be underlined that, in general, the
presence of one external component causes the non-equality between the total
potential energy tensors of the subsystems and their Clausius’ virial tensors,
as occurs in the case of the single component (see, e.g., LS1). Moreover, we
should remember that Clausius’ tensors are made only by forces and positions
and then the mass fraction of the dark outer component which enters in the
$B$ Clausius’ tensor is only that which exerts dynamic effects on $B$, according
to Newton’s first theorem.
2.2 Two-component models

Then we need to model the two components (Fig.1). The models we consider here are the same as in LS1. That is they are built up of two homothetic similar strata spheroids with two power-laws and two different homogeneous cores in the central regions. It should be noted that they correspond, to a good extent, \(^1\) to deal with two spheroids with smoothed profiles of this kind (spherical case\(^2\)):

\[
\rho_D = \frac{\rho_{oD}}{1 + \left(\frac{r}{r_{oD}}\right)^d}; \quad C_D = \frac{a_D}{r_{oD}} \\
\rho_B = \frac{\rho_{oB}}{1 + \left(\frac{r}{r_{oB}}\right)^b}; \quad C_B = \frac{a_B}{r_{oB}}
\]

(C\(_B\) and C\(_D\) are the two concentrations of the two components. In the LS1 models, \(r_{oB}\) and \(r_{oD}\) were the radii of the two different homogeneous cores, which typically assumed one tenth of the virial radii, \(a_B\) and \(a_D\), respectively. According to these, the concentrations in the smoothed profiles both become equal to ten. The density profiles of kind (11) and (12), have the advantage that they may be considered as a generalization of pseudo-isothermal profiles which, in turn, may be regarded as sub-cases of the more general Zhao profiles when: \(\gamma = 0; \beta = \alpha = b, d\).

But a realistic elliptical model has to be: e.g., a stellar component with a Hernquist (1990) (hereafter, Her) density profile and a dark halo with a cored or non-cored NFW profile (that means, according to eq.(2), respectively: \(\alpha = 1; \beta = 4; \gamma = 1; \delta = 3\) and \(\alpha = 1; \beta = 3; \gamma = 0; \delta = 3\); for the cored NFW, \(\alpha = 1; \beta = 3; \gamma = 1; \delta = 2\), for the non-cored NFW) (as in Marmo, 2003, where two homeoidally striated ellipsoids are considered).

Then, the problem of transferring the outputs obtained with two cored power-law profiles (which also hold, to a good extent, for the models with smoothed profiles (11), (12)) to the more general class of models with Zhao profiles given by eq.(2), is still open, even if some preliminary considerations will be made in the subsect. 4.5.

Our aim is to try to explain some scaling relations for elliptical galaxies, which are essentially relationships among the exponents of the three quantities:

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\(^1\) Generally, the deviations are only of a few per cent and less than the 10% related to the Clausius’ minimum location and to its value, respectively.

\(^2\) Even if the considerations which follow are more general, for the sake of simplicity, we will often limit ourselves to the spherical case without losing the validity of spheroidal case, which may be recovered simply by introducing a form factor, \(F\), depending on the axis ratio (see, eq.(18)).
The effective radius, 

\[ r_e = \text{the effective radius,} \]

\[ I_e = \frac{L}{2\pi r_e^2} = \text{mean effective surface brightness within } r_e \]

\[ \sigma_0 = \text{the central projected velocity dispersion.} \]

The advantage of a simple cored power-law model is that it is able to extract, in a completely analytical way, some of these main correlations, highlighting the interplay of the parameters. Moreover, this preliminary analysis may also underline what are simply details in the model and what, on the contrary, is strictly connected with the physical reason for the existence of a FP for two-component virialized systems. That allows us to open the road for a generalization of the present results.

3 The special virial configuration

As shown in the previous papers, a special configuration for the \( B \) component arises inside these kind of homothetic (see, LS1) models (for the non-homothetic, see, Secco, 2000), which describe its evolutive pattern, obtained by contraction inside a \( D \) one, under the assumption that \( D \) is at fixed size and shape. For the sake of simplicity, we will assume that the outer component is frozen, without considering this constraint to be too essential in order to determine the main features of the dynamic evolution we are dealing with (see, LS1). The main reason for this assumption is indeed that the masses of the two components are not equal, the outer one being about ten times the inner one. As a consequence, tidal influences between the subsystems are not symmetric; the one acting from inner to the outer is actually weaker than the reverse (Caimmi & Secco 1992). Moreover, even if a contraction effect is induced by inner density distribution on the inner regions of outer halo, during the dynamical evolution, as already underlined by Barnes & White (1984), that effect does not cause a dramatic modification of the outer mass distribution, if there is a supernovae-driven outflow, according to N-body simulations (Lia et al. 2000). On the other side, models in which this constraint has been changed with some less stringent additional conditions (Caimmi, 1994), seem to come to the same conclusion.

The special configuration appears because a maximum in the Clausius virial energy trend, as \( B \) contracts inside \( D \), (and then a minimum in the kinetic energy) exists under the following constraints on the exponents:

\[ 0 \leq b < 3 ; \ 0 \leq d < 2 \Rightarrow (b + d) < 5 \quad (13) \]

The total potential energy \( E_{pot} \), on the contrary is always monothonic
Table 1
Physical parameters of the models considered in the figures 2, 5, 6. The common density profile parameters (eqs. (11, 12)) are: $C_B = C_D = 10, m = 8.5$. Moreover $x_t = a_t/a_D$; for the definition of $\nu_{\Omega B}; \nu_{\Omega D}; \nu'_V$, see text.

| cases | b   | d   | $x_t$ | $\nu_{\Omega B}$ | $\nu_{\Omega D}$ | $\nu'_V$ |
|-------|-----|-----|-------|------------------|------------------|---------|
| 1)    | 0.0 | 0.0 | 0.389 | 0.300            | 0.300            | 0.300   |
| 2)    | 0.0 | 0.5 | 0.346 | 0.300            | 0.312            | 0.333   |
| 3)    | 0.5 | 0.5 | 0.361 | 0.312            | 0.312            | 0.313   |
| 4)    | 1.5 | 0.5 | 0.419 | 0.367            | 0.312            | 0.254   |

(Fig. 2). Indeed, by definition:

$$ (E_{pot})_B = \Omega_B + W_{BD} $$

(14)

where $\Omega_B$ is the self-potential energy tensor trace and $W_{BD}$ is the interaction-energy trace of the tensor defined as:

$$ (W_{BD})_{ij} = -\frac{1}{2} \int_B \rho_B(\Phi_D)_{ij} d\vec{x}_B ; $$

(15)

As already underlined, in general, $(W_{BD})_{ij}$ does not match $V_{ij}$ (see, Caimmi & Secco, 1992).

By referring to the LS1-models with profiles of kind (11), (12), the special configuration (which appears under the constraints on the exponents of the density power-laws and by taking a frozen $D$ subsystem, as seen in the same paper), corresponds to the dimension of the $B$ component characterized by the following semimajor axis, called tidal radius:

$$ a_t = \left( \frac{\nu_{\Omega B}}{\nu'_V} \right) \frac{1}{2 - d} \frac{M_B}{M_D} a_D $$

(16)

where $M$ and $a$ are, respectively, the mass and the major semiaxis of the subsystem considered. The coefficient $\nu_{\Omega B}$ enters the integral which gives the
self-potential energy tensor of the $B$ component (see, LS1), so it is a function only of the $b$ exponent; the coefficient $\nu'_V$ is defined in the following way:

$$
\nu'_V \simeq \frac{9}{2} \left[ (\nu_B)_M (\nu_D)_M \right]^{-1} \frac{C_B^{-(b+d)}}{(3-d)[5-(b+d)]}
$$

(17)

where $(\nu_B)_M, (\nu_D)_M$ are defined in LS1. $\nu'_V$, which is a function of $b$ and $d$ and of the two concentrations $C_B = C_D$, enters in the definitions of the tidal tensor trace:

$$
V_{BD} \simeq -\nu'_V GM_B \tilde{M}_D a_B F;
$$

(18)

where $F$ is the form factor (see, Marmo & Secco, 2003, hereafter MS3) and $\tilde{M}_D$ is the fraction of $D$ matter exerting dynamical effect on $B$, according to Newton’s first theorem. To a good extent it is given by:

$$
\tilde{M}_D = M_D \left(\frac{a_B}{a_D}\right)^{3-d}
$$

(19)

The same mass fraction normalized to $M_B$ becomes:

$$
\tilde{m} = \frac{M_D}{M_B} \left(\frac{a_B}{a_D}\right)^{3-d}
$$

(20)

Moreover we define the total mass inside the $B$-structure as:

$$
M_{\text{tot}}^* = M_B + \tilde{M}_D = M_B (1 + \tilde{m})
$$

(21)

These quantities also enter into the definition of the $B$-Clausius’ virial tensor trace as follows:

$$
V_B \simeq \left[ -\nu_{\Omega B} \frac{GM_B^2}{a_B F} - \nu'_V \frac{GM_B \tilde{M}_D F}{a_B} \right]
$$

(22)

or in the form normalized by the factor $\frac{a_B}{GM_B^2 F}$:

$$
\tilde{V}_B \simeq -\frac{\nu_{\Omega B}}{x} - \nu'_V m x^{2-d}; \quad x = \frac{a_B}{a_D}; \quad m = \frac{\tilde{M}_D}{M_B}
$$

(23)

Why this configuration of the $B$ component inside the fixed dark matter potential well is so special, may be understood by considering, in the next section, its mechanical and thermodynamical properties.
4 Special configuration: mechanical and thermodynamical properties

4.1 Mechanical arguments

In order to understand the full meaning of the tidal radius, which is defined by eq.(16), and to be able to consider the thermodynamical processes of the $B$ component, we have to look at the physics related to Clausius’ virial energy, $V_B$. By definition, the Clausius’ virial tensor trace is given by:

\[ V_B = \Omega_B + V_{BD} \]  

\[ \Omega_B = \int \rho_B \sum_{r=1}^{3} x_r \frac{\partial \Phi_B}{\partial x_r} \, d\vec{x}_B = \int \rho_B (\vec{r}_B \cdot \vec{f}_B) \, d\vec{x}_B \]  

\[ V_{BD} = \int \rho_B \sum_{r=1}^{3} x_r \frac{\partial \Phi_D}{\partial x_r} \, d\vec{x}_B = \int \rho_B (\vec{r}_B \cdot \vec{f}_D) \, d\vec{x}_B; \]  

where $\vec{f}_B$ and $\vec{f}_D$ is the force per unit of bright mass due to self gravity and the dark matter gravity at the point $\vec{r}_B$, respectively. By definition, the work done by the self gravity forces, $L_s$, in order to assemble the $B$-elements from the infinity is given by the self- potential energy $\Omega_B$ and the work done by the tidal gravity forces, $L_t$, in order to put the $B$ component together with the $D$ one from infinity through all the tidal distorsions (see, Caimmi & Secco, 2004), is given by the tidal potential energy $V_{BD}$.

Then a small variation $\delta V_B$ for a small displacement $\delta \vec{r}_B$ of all $B$ points, has the following meaning:

\[ \delta V_B = \delta L_s + \delta L_t \]  

4.2 Small departures from virial equilibrium

We will now consider what is the mathematical form of the $V_B$ potential energy variation as soon as the $B$ inner system contracts or expands its initial volume $S_o$ of a small quantity $\Delta S_o$. Following Chandrasekhar’s analysis (Chandrasekhar, 1969, Chapter 2) the following holds: by definition, the Clausius virial is a global, integral parameter of an extrinsic attribute, that means of a quantity which is not intrinsic to the fluid element (like pressure or density) but something, we name $F(\vec{x})$, which it assumes simply by virtue of its location such as the gravitational potential and its first derivative. The variation
of the integral:

\[
\delta \int_{S_o} \rho_B F \, d\vec{x}_B = \int_{S_o + \Delta S_o} \rho_B F \, d\vec{x}_B - \int_{S_o} \rho_B F \, d\vec{x}_B \tag{28}
\]

when the instantaneously occupied volume by the fluid changes from \(S_o\) to \(S_o + \Delta S_o\) by subjecting its boundary to the displacement \(\vec{\xi}(\vec{x}, t) = \vec{x} - \vec{x}_o\), may be transformed into the integral over the unperturbed volume, that is:

\[
\delta \int_{S_o} \rho_B F \, d\vec{x}_B = \int_{S_o} \rho_B \Delta F \, d\vec{x}_B \tag{29}
\]

where \(\Delta F\) is the Lagrangian change in \(F\) consequent to the displacement \(\vec{\xi}\).

The extension of this analysis to two-component systems has been performed (Caimmi & Secco, 2004) with the following result:

\[
\delta \Omega_B = - \int_{S_o} \rho_B \sum_{r=1}^{3} \xi_r \frac{\partial \Phi_B}{\partial x_r} \, d\vec{x}_B \tag{30}
\]

\[
\delta V_{BD} = - \int_{S_o} \rho_B \sum_{k=1}^{3} \sum_{r=1}^{3} \xi_k \frac{\partial}{\partial x_k} (x_r \frac{\partial \Phi_D}{\partial x_r}) \, d\vec{x}_B - \int_{M_o} \rho_D \sum_{r=1}^{3} \xi'_r \frac{\partial \Phi_B}{\partial x_r} \, d\vec{x}_D \\
+ \int_{M_o} \rho_D \sum_{k=1}^{3} \sum_{r=1}^{3} \xi'_{r} \frac{\partial}{\partial x_k} (x_r \frac{\partial \Phi_B}{\partial x_r}) \, d\vec{x}_D \tag{31}
\]

where the unperturbed volume of \(D\)-component is \(M_o\) and \(\vec{\xi}'(\vec{x}, t)\) is the amount of the perturbation in the point domain of the same component.

Under the assumption of a frozen dark component, \(\vec{\xi}'\) vanishes and the main result holds:

\[
\delta V_B = \delta L_s + \delta L_t \simeq - \int_{S_o} \rho_B \sum_{r=1}^{3} \xi_r \frac{\partial \Phi_B}{\partial x_r} \, d\vec{x}_B \\
- \int_{S_o} \rho_B \sum_{k=1}^{3} \sum_{r=1}^{3} \xi_k \frac{\partial}{\partial x_k} (x_r \frac{\partial \Phi_D}{\partial x_r}) \, d\vec{x}_B = (\delta V_B)_{a_D} \tag{32}
\]

By definition of the tidal radius, which is the \(B\) dimension at the maximum of its Clausius virial energy, at frozen \(a_D\), the \((\delta V_B)_{a_D}\) is stationary at \(a_t\) (Fig.2)(see, LS1). Therefore, by moving of a virtual\(^3\) displacement \(\delta a_B\), from

\(^3\) Here virtual means at frozen D.
\[ a_B = a_t, \text{ we have:} \]
\[ \delta L_s + \delta L_t \simeq (\delta V_B(a_t))_{a_D} = 0 \]  \hspace{1cm} (33)

This means that the configuration at \( a_t \) satisfy the d’Alembert Principle of virtual works (see also LS1). The physical reason is the following: if, e.g., \( B \) contracts, less dark matter enters inside the \( S_B \) surface, in the meanwhile the self gravity increases. The opposite occurs if \( B \) expands itself. Therefore, even if both forces are attractive, the works which correspond to them, for a virtual displacement, are of opposite signs (see, LS1). Therefore, the tidal radius configuration is an equilibrium configuration even if not stable because the total potential energy of \( B \) has not a minimum (Fig.2).

The consequence of these mechanical arguments with the property of the tidal configuration to be able to distribute in about the equal parts the self- and the tidal- energies (see, next subsection and LS1), is to yield some outputs which are in good agreement with the corresponding observable scaling relations related to the elliptical galaxies, and in general to the existence of a FP for two-component virialized systems, as we have already highlighted in the past papers (LS1, MS3).

4.3 Scaling relations at the special configuration

Even if the physical explanation for the main features of FP for virialized structures and in particular for the dynamically hot ellipticals (Bender et al., 1992), is still an open question, the two main roads which are present in the literature (see, e.g., Renzini & Ciotti, 1993; Ciotti et al., 1996; Bertin et al., 2002) may be summarized as:

a) The tilt of the FP is due to the stellar population effect.

b) The tilt is due to a non-homologous structural effect. The light profile is not an universal de Vaucouleurs profile but a Sersic profile (1968) which changes with the galaxy luminosity: the Sersic index \( n \) increases as the luminosity increases \((\text{weak homology}, \text{Bertin et al., 2002})\).

Neither a) nor b) seem to give the proper answer to the scaling relations problem; the population effect disappears in the K-band (Maraston, 1999), contrary to what is observed (Pahre et al., 1998). Therefore, a metallicity sequence of an old stellar population may at most fit the trend observed in the B-band (Gerhard et al., 2001). Moreover with the weak homology the observed tightness of the FP appears to be hard to explain (Bertin et al., 2002).

According to theory performed in the papers: Secco (2000), LS1, MS3, we
tried to explain the *tilt* by assuming a strict homology which does not imply a constant ratio $M_B/L$ due to the presence of a dynamical effect caused by the scale length induced on the gravitational baryonic field from the dark matter halo distribution.

We come back to the virial equations (3), in the trace form, for a B-component completely embedded in a dark halo. The FP we obtain (see, LS1):

$$r_e = c_2 c_1^{-1} \sigma_o^A I_e^B G_2 \frac{L}{M_{tot}}; \ A = 2; \ B = -1$$

$$G_2 = \frac{1 + \tilde{m}}{F[\nu \Omega_B + \tilde{m} \nu_V]}$$

seems, at a first sight, not substantially changed in respect to the one we get by using the virial equations for a single component system.

But if we consider the trace of eq.(4), at the special configuration with the condition that the bright component is pressure supported (i.e., peculiar kinetic energy $T_{pec}$ is dominant with respect to the rotational kinetic energy $T_{rot}$) we obtain:

$$\frac{1}{2} M_B < \sigma^2 > \simeq \left( \frac{-\Omega_B - V_{BD}}{2} \right)_{a_B = a_t} ; \ T_{rot} << T_{pec}$$

where $< \sigma^2 >$ is the mean square velocity dispersion of the stars. By adding at $a_t$ the equipartition between the *self- and tidal energy*, the previous equation becomes:

$$a_t \simeq \left( \frac{\frac{1}{2} M_B \sigma_o^2 a_D^{-3-d}}{\nu_V GM_B M_D F} \right)^\frac{1}{2+d}$$

which, instead of eq.(34) yields the following FP:

$$r_e \sim \sigma_o^A \sigma_o^{\frac{2}{2+d}} a_D^{-\frac{3-d}{2+d}} m^{-\frac{1}{2+d}} M_B^{\frac{1}{2+d}}$$

That means:

$$\begin{cases}
\sigma_o^A \equiv \sigma_o^{\frac{2}{2+d}} \\
I_e^B \sim a_D^{-\frac{3-d}{2+d}} m^{-\frac{1}{2+d}} M_B^{-\frac{1}{2+d}}
\end{cases}$$

$$\text{14}$$
4.4 Output vs. observables

I) From the first equality we have:

\[ A = \frac{2}{2 - d} \]  

(40)

II) Moreover, due to the two relationships which connect the FP coefficients, \( A, B \) with the observed tilt (see, e.g., Djorgovski & Santiago, 1993), that is:

\[
\begin{cases}
A = \frac{2(1 - \alpha_t)}{1 + \alpha_t} \\
B = -\frac{1}{1 + \alpha_t}
\end{cases}
\]  

(41)

we immediately obtain the tilt of the FP:

\[ \alpha_t = \frac{1 - d}{3 - d} \]  

(42)

and the other coefficient:

\[ B = -\frac{3 - d}{2(2 - d)} \]  

(43)

only as functions of the dark matter distribution. First of all it should be noted that the quantities \( A, B, \alpha_t \) which define the FP and its tilt are independent of \( m \). That means the galaxies which belong to the FP may have a different fraction of baryonic matter in respect to the dark one, but their dark matter density profile must be the same. To probe the issue of the first identity we choose, e.g.:

\[ d = 0.5 \Rightarrow A = 1.33, B = -0.83 \text{ and } \alpha_t = 0.20 \]

in good agreement with the observations in the \( B \)-band.

In the range: \( d = 0 \div 1 \Rightarrow A = 1 \div 2 ; -B = 0.75 \div 1 ; \alpha_t = 0.33 \div 0 \) in agreement with the data related to all bands (Marmo, 2003, Tab.1.1 and the references therein).

This allows us to expect (see, LS1) that in other families of galaxies with dark matter halos of the same kind of elliptical galaxies, as, e.g., the spirals, a Fundamental Plane also has to exist which has the same \( A, B \) and \( \alpha \) exponents, totally independent of a completely different luminous mass distribution. This
is in good agreement with the universal FP discovered by Burstein et al. (1997).

III) It should be noted that the values of $A$, $B$, $\alpha_t$ which define the FP as a whole, are not directly linked with the past cosmological conditions, but they are only a function of the dark matter distribution $d$. This is in agreement with what Djorgovski already noted (1992) on the basis of Gott & Rees (1975), Gunn (1987), Coles & Lucchin (1995), occurs for the scaling relations in a CDM scenario. Indeed, from a cosmological point of view the FP means the following relationship (Djorgovsky, 1992), which we also recover in LS1:

$$2n_{\text{rec}} + 10 = A(1 - n_{\text{rec}}) - B(12\alpha_t + 4n_{\text{rec}} + 8),$$

$n_{\text{rec}}$ = effective spectral index of perturbations.

IV) But in the projections of the FP on the coordinate planes the dependence on the cosmological spectral index appears via the parameter, $\gamma'$, we introduced in LS1, as:

$$\frac{1}{\gamma'(M)} = \frac{1 + 3\alpha_{\text{rec}}(M)}{3} = \frac{5 + n_{\text{rec}}}{6} \quad (44)$$

where, according to Gott & Rees (1975b) and Coles & Lucchin (1995, Chapt.s 14, 15), $\alpha_{\text{rec}}$ is the local slope of the CDM mass variance, $\sigma^2_M$, at recombination time $t_{\text{rec}}$, given by:

$$\alpha_{\text{rec}} = -\frac{d\ln \sigma_M(t_{\text{rec}})}{d\ln M},$$

If the total energy of the system is conserved during the transition from a maximum expansion phase to the virialization, with its re-distribution among the collisionless ingredients by a violent relaxation mechanism, the following dependences on $m$ and $M_B$, for the three main quantities of FP, hold:

$$r_e \sim m^r M_B^R ; \quad r = \frac{(3 - d) - \gamma'}{\gamma'(3 - d)} ; \quad R = 1/\gamma' \quad (45)$$

$$I_e \sim m^i M_B^I ; \quad I = i = 2\frac{\gamma' - (3 - d)}{\gamma'(3 - d)} \quad (46)$$

$$\sigma_o \sim m^s M_B^S ; \quad s = \frac{1}{2} \frac{(3 - d) - \gamma'}{\gamma'(3 - d)} ; \quad S = \frac{1}{2} \frac{\gamma' - 1}{\gamma'} \quad (47)$$
Therefore, the projected scaling relation, such as the Faber-Jackson relation (hereafter FJ), turns out to be:

\[ L \sim m^{2 + \frac{(3-d) - \gamma'}{(3-d)\gamma' - 1}} \sigma_o^{\frac{4\gamma'}{(\gamma'-1)(3-d)}} \]  

(48)

not only directly related to the dark matter distribution, via the exponent \(d\), but also to the perturbation spectrum, via \(\gamma'\).

For a typical galaxy dark matter halo of \(M_D \simeq 10^{11} M_\odot\), it turns out that \(\gamma' = 2\) (Gunn, 1987). If \(d = 0.5\), we obtain:

\[ L \sim M_B^{0.8}; \quad r_e \sim M_B^{0.5} \Rightarrow I_e = L / 2\pi r_e^2 \sim M_B^{-0.2} \]

It should be noted that \(I_e\) decreases as \(M_B\) increases as soon as:

\[ 3 - d > \gamma'; \]

that is \(d < 1\). On this mass scale, the FJ becomes:

\[ L \sim m^{0.16} \sigma_o^{3.2} \]

If \(d = 1\), then \(\alpha_t = 0\) and as a consequence:

\[ L \sim \sigma_o^4 \]

without dependence on \(m\).

V) Moreover on the Clausius’ minimum we have:

\[ (M_{tot}^*)_t = M_B \left(1 + \frac{\nu\Omega_B}{\nu'/(2-d)}\right) \]

If \(b\) and \(d\) are universal, then the consequence is the proportionality of the two masses and then of the two ratios:

\[ L/(M_{tot}^*)_t \sim L/M_B \]  

(49)

VI) From an observational point of view, we know (Cappellari et al., 2004) that, for a sample of E and S0, either fast rotators or nonrotating giant ellipticals, the following tight correlation holds:

\[ M/L \sim \sigma_o^{0.8} \]  

(50)
From the previous scaling relations, we obtain:

\[ \frac{M_B}{L} \sim \sigma_o^{\alpha t/\gamma(M_D)^{-1}} \quad (51) \]

which on this dark matter scale yields an exponent of \( \sigma_o \) exactly equal to 0.8 without distinguishing between \( L/(M_{\text{tot}}^*) \) and \( L/M_B \), according to eq.(49). That also has to be compared with Jørgensen’s (1999) value: 0.76 ± 0.08. It should to be noted, as in the relationship (51), that the dependence on the other factor \( m \) is completely negligible. Indeed it turns to be: \( m^{0.04} \).

VII) Another issue is: the ratio of the dark matter fraction over total mass inside the bright radius \( a_B \).

At \( a_t \), it becomes:

\[
\left( \frac{\bar{M}_D}{\bar{M}_{\text{tot}}} \right)_{a_t} \sim \frac{1}{1 + \frac{\nu}{m_B}(2 - d)}
\]

That means that it only depends on the luminous and dark density profiles. If they are both universal for the galaxy family considered, this dark matter fraction has to be the same for all members.

It should be noted that this result is independent of the total mass ratio, dark over bright, \( m \).

If the most probable value for \( d \) is around 0.5 and \( b \) ranges from 2 ÷ 3 (Jaffe 1983; Hernquist 1990), the most probable values for \( \left( \frac{\bar{M}_D}{\bar{M}_{\text{tot}}} \right)_{a_t} \) turn out to range from 0.57 ÷ 0.80, which corresponds to \( \log \left( \frac{\bar{M}_D}{\bar{M}_{\text{tot}}} \right)_{a_t} = 0.37 \pm 0.69 \), with \( \log \left( \frac{\bar{M}_D}{\bar{M}_B} \right)_{a_t} = 0.50 \) at \( b = 2.5 \). The agreement with the Jørgensen’s histogram in Fig.5 (Jørgensen 1999), related to early type galaxies in the central part of the Coma cluster when the same IMF is assumed, appears to be very good.

But one of the most important issues of the present theory appears to be related to the physical reason which does cause the tilt.

In order to have the tilt we need to have the maximum of Clausius’ virial energy. This, in turn, requires to have the equipartition between the self- and the tidal- energy of the stellar component. By considering the derivative of \( \bar{V}_B \) of eq.(23) in respect to \( x \) and according to eq.(19) and the constraints (13), we conclude that the dark matter mass has to increase steeper than \( (a_B/a_D) \).

This means, in turn, that \( \rho_D \) has to decrease less than \( 1/r^2 \) at the border of the bright mass, in order that tidal energy may overcome the self-energy from
this border forwards. But if we enter deeper into the fine play of the exponents we are able to deduce a stricter constraint.

Going back to the dependence of $I_e$ on the mass ratio $m$ and $M_B$, the message of the (46) is: $I_e$ depends on the cosmological history of the galaxies, on dark matter distribution and on the baryonic fraction which is inside. But the ratio $L/M_B$, that is the tilt, is totally independent of the cosmic perturbation spectrum and of the mass ratio $m$. It turns out to depend only on the dark matter density profile. Indeed, if we look at

$$L \sim I_e r_e^2 \implies m^{i+2r} M_B^{I+2R}$$

where the exponents satisfy the following relationships:

$$i + 2r = 0$$
$$I + 2R = 2/(3 - d)$$

Therefore, it is clear how the ratio $L/M_B$ loses its direct connection with the cosmology given by $\gamma'$. Moreover in order to have the positive (=observed) tilt, we need:

$$2/(3 - d) < 1 \implies 0 < d < 1$$

This condition is stricter than the necessary condition for the maximum: $d < 2$

For $d = 1 \implies$ the tilt disappears.

For $2 > d > 1 \implies$ the tilt appears but it is negative (the opposite of that observed).

Therefore, the slope of FP tells us a constraint on the density distribution of dark matter halo. To have a positive tilt we need the $DM$ mass has to increase steeper than $(a_B/a_D)^2$ at the border of the bright mass. This means, in turn, that $\rho_D$ has to decrease less than $1/r$. If the contrary occurs ($\implies \rho_D$ decreases faster than $1/r$ but less than $1/r^2$) the tilt changes its sign. That immediately underlines a problem with a NFW density profile concerning the inner part of the halo. The debate is still open. From the theoretical side what appears relevant is the conclusion of a recent paper by Mücket & Hoeft (2003) in which the constraint on the exponent in the central dark halo region has to be: $0 \leq d \leq 0.5$, obtained by using Jeans’ equations. From the observation side we underline a very strict limit for the exponent $\gamma$ ($\gamma \leq 0.8$), completely in disagreement with the majority of simulations ($\gamma \geq 1$), determined by fitting the rotation curves obtained with high resolution tecnique, for a sample of
spiral galaxies in which dark halos density profiles of Zhao kind have been used (Garrido, 2003).

4.5 On the limits of the model

The problem of generalizing the results obtained in the case of a two-component model with two power-law density profiles and with two homogeneous cores is still open. Some considerations may help in this future work.

A) The existence of Newton’s first theorem (see, Fig.1), which allows us to ignore the mass distribution of the dark halo outside the $B$ volume as regards the dynamical effects on the stellar subsystem and then for the trend of Clausius’ virial trace tensor $V_B$, which is the key of the whole theory.

B) The conclusion of the previous subsection that is the presence of the tilt and its sign depends on the gradient of dark matter distribution at the border of bright mass confinement.

Both together seem to converge on the main role which the central region of the dark halos plays in the present theory. That may occur in a satisfactory way only if the scale radius of the dark halo, $r_{oD}$ is not too small in respect to the dimension which contains the most of the stellar mass. In other words, we will expect that the dark halo concentrations have to be not too high.

We show in Fig.3 and Fig.4 (Marmo, 2003) the same as in Fig.2 in the case of more general models which belong to Zhao models. That in order to prove as the appearance of Clausius’ minimum, for suitable values of the concentrations, seems a common feature of the these most general and most realistic triaxial models where the density profiles are given by eq.(2) for both the system components (e.g., NFW+Her, Fig.3; cored NFW+ Her, Fig.4). But an other problem arises, with this kind of models. That is to recover, analytically, the interplay of the exponents which appear in FP-quantities.

4.6 Thermodynamic arguments

We will now consider the $B$ component thermodynamics inside a two-component system, to look for what characterizes its special virial configuration from the thermodynamical point of view. We begin to analyze the double system when it arrives at virialized stages after a phase of violent relaxation (Binney & Tremaine, 1987). From this time onwards, it may be assumed that the $B$ component begins its virial evolution which consists of a sequence of slow contractions with enough time to rearrange the virial equilibrium after any step.
Fig. 3. The trends of the Clausius' virial energies in the normalized form, vs. $x$ defined as in Fig.2, (dashed curves), in the case of two-component systems built up of a inner stellar component with a Hernquist (1990) density profile and an outer dark halo with $NFW$ profile (eq.(2)). The concentrations are: $C_B = 4$ and, from top to bottom: $C_D = 4, 7, 10$. The mass ratio is $m = 12$. The curve corresponding to $C_B = C_D = 4$, exhibits surely a minimum for the absolute value of Clausius' virial energy. For comparison, the self potential energy of the same $B$ component when single, is shown with a continuous track (Marmo, 2003).

of the sequence. In this way the thermodynamic process of contraction may be divided into a sequence of transformations which are irreversible but occur between quasi-equilibrium stages with the typical character of external thermal irreversibility (Zemansky, 1968). Therefore, it is possible to assign a mean temperature $\overline{T_S}$ to the whole component during this quasi-static sequence of its dynamic evolution in the following way.
Fig. 4. The trends of the Clausius' virial energies in the normalized form, vs. \( x \) defined as in Fig.2, (dashed curves), in the case of two-component systems built up of a inner stellar component with a Hernquist (1990) density profile (\( \alpha = 1; \beta = 4; \gamma = 1; \delta = 3 \)) and an outer dark halo with a \textit{cored NFW} profile (eq.(2):\( \alpha = 1; \beta = 3; \gamma = 0; \delta = 3 \)). The concentrations are: \( C_B = 4 \) and, from top to bottom: \( C_D = 4, 7, 10 \). The mass ratio is \( m = 12 \). For comparison, the self potential energy of the same \( B \) component when single, is shown with a continuous track (Marmo, 2003).

By assuming that the \( B \) stellar system has an isotropic velocity distribution, we may define its mean temperature as:

\[
T_S = \frac{m_* \langle \sigma^2 \rangle}{k}
\] (52)

where \( m_* \) is the mean mass of the stars and \( k \) is the Boltzmann constant (see, e.g., Lima Neto et al. 1999; the \( a \)-\textit{Ansatz}, in Bertin & Trenti, 2003).

Moreover, if we limit ourselves to the macroscopic pressure supported elliptical systems, which are relevant in order to define the FP we are dealing with, it follows that \( T_S \) is related to the dominant peculiar kinetic energy, \( \frac{1}{2} M_B < \)
Fig. 5. Trends of the entropy function \( \tilde{F}(x) \) (eq.(66)) for the \( B \)-system, in arbitrary units (see text), as function of \( x \) (see, Fig.2). The cases are of Tab.1. The maxima occur at the tidal radii (Raffaele, 2003).

\[
\sigma^2 \simeq T_B, \quad \text{in the following way:}
\]

\[
\overline{T_S} \simeq \frac{2T_B}{Nk} \quad (53)
\]

\( N \) being the star number of the \( B \) component. \( T_B \) is, in turn, connected with the central projected velocity dispersion \( \sigma_o \) by the usual factor \( k_v \) which links the kinematic galactic structure with \( \sigma_o \) as follows (e.g., LS1):

\[
\sigma_o^2 = k_v < \sigma^2 > \quad (54)
\]

In the \( I^o \) Thermodynamic Principle equation for the \( B \) system what has to appear is the work done against its pressure by both the forces on the system: the self gravity and the gravity which the dark matter distribution exerts on it. The corresponding potential energy variation, for a small contraction, will
be $\Delta V_B$ in such a way as to have:

$$\Delta E_T = \Delta Q - \Delta V_B$$  \hspace{1cm} (55)$$

where $E_T$ is the internal energy of $B$ and $\Delta Q$ is the small amount of heat the structure is able to exchange with the surrounding medium, in which the two-component system lies, e.g., by cooling processes. Indeed, we have to take into account that the quasi-static evolution of the inner component (the $D$ one is considered non-dissipative) needs to involve some dissipative processes, as Prigogine (1988) has well pointed out, on the general grounds.

In a gas dominated structure, dissipation may easily occur, e.g., by gas clouds collisions, but also in a collisionless stellar component such as an elliptical. Indeed, several evolutionary processes may potentially occur (see, e.g., Bertin & Trenti, 2003), some of them with a dissipative character.

If enough time is available in order to restore virial equilibrium, the variation of the virial quantities of the same component during this quasi-static transition, turns out to be:

$$\Delta T_B = -\Delta V_B/2$$  \hspace{1cm} (56)$$

Combining the two equations (55, 56) and considering that the internal energy may now be identified with the total macroscopic kinetic energy of the stars, $T_B$, according to eq.(53), we obtain the two equations:

$$\Delta Q = \Delta V_B/2$$  \hspace{1cm} (57)$$

$$\Delta T_S \sim -\Delta V_B/2$$  \hspace{1cm} (58)$$

The two requests that the system settles in virial equilibrium and obeys the $I^o$ Principle yields the equipartition of the virial energy variation which now becomes the variation of the Clausius’ virial energy: that is one half of it has to be exchanged with rest of the universe, the other half has to contribute to change the temperature of the system. The result is formally the classical one found by Chandrasekhar (1939) and Schwarzschild (1958) for a single gaseous stellar component. The big difference, for the $B$ component inside the $D$ one, is that the potential energy $V_B$ is now a non-monotonic function of $a_B$ (Fig.2).

The variation of the entropy of the $B$ system, $S_B$, during the transformation between two virial states due to a small contraction $\Delta a_B$, which has the typical character of external thermal irreversibility, is evaluated as:

$$\Delta S_B = \frac{1}{T_S} \Delta Q = 2Nk \frac{\Delta (V_B/2)}{-V_B/2}$$  \hspace{1cm} (59)$$
If $\Delta V_B$ is negative due to the energy lost by radiation, the consequence of the radiation flow is an entropy amount $\Delta S_r$ of the thermal radiation bath in the surroundings, which has the mean temperature $T_r$, equal to:

$$\Delta S_r = \frac{1}{2} \left( -\Delta V_B \right) \frac{1}{T_r} \quad (60)$$

$\Delta S_r$ is of opposite sign of $\Delta S_B$ and greater than it, in absolute value, in a way that, according to the $II^o$ Thermodynamics Principle, the whole universe increases its entropy $S_u$ of:

$$\Delta S_u = \frac{1}{2} (-\Delta V_B) \left( \frac{1}{T_r} - \frac{1}{T_S} \right) \quad (61)$$

That holds because the surrounding thermal bath, in which the radiation flow will at the end thermalize, has a mean temperature lower than that of the stellar structure.

Due to the non-monotonic character of the $V_B$ trend, these very important consequences follow:

A) as soon as the Clausius’ virial energy is stationary $S_B$ must also be so. That occurs at $a_t$ because the corresponding configuration satisfies the d’Alembert Principle of virtual works (see, eq.(33)). Therefore, one expects to obtain a maximum or a minimum for the entropy of the $B$ component when the stellar system is on its tidal radius configuration. It should be underlined that this result is independent of the special class of two-component models we are dealing with but it depends only on the meaning of Clausius’ virial energy and on presence or not of the Clausius’ virial maximum during the quasi-static contraction sequence the chosen models are able to describe the $B$ subsystem evolution. Indeed, the eq.(59) derives from the physical meaning of Clausius’ energy and from the first Thermodynamical Principle together with the virial constraint.

B) An other consequence of eq.(59) is: all the configurations of the stellar system which correspond to a dimension greater than the tidal radius $a_t$ are forbidden. Indeed let us consider any configuration on the right side of the $x_t$ in Fig. 2. Starting from this one we take into account the thermodynamical transformation which follows to a small contraction $\Delta a_B < 0$. According to the eqs.(53, 57, 58, 59), the consequences are:

$$\Delta a_B < 0; \quad \Delta T_S < 0; \quad \Delta Q > 0; \quad \Delta S_B > 0 \quad (62)$$

The stellar structure would have to increase its entropy by taking energy from the radiation bath in which it is embedded and which has a lower temperature.
than $T_S$. It is manifest that this thermodynamical process contradicts the second Principle. Starting from the same configuration, we now consider a small expansion. We would obtain:

$$\Delta a_B > 0; \Delta T_S > 0; \Delta Q < 0; \Delta S_B < 0$$  \hspace{1cm} (63)

From the thermodynamical second Principle that would be possible but the contradiction arises from the conservation of the total energy which is equal to the half of the total potential energy, $\frac{1}{2}E_{pt}$, of the two-component system (see, MS3, Fig.5). Indeed a small expansion would request an increase of the total mechanical energy of the whole system.

The main result is that all the virial configurations greater than $a_t$ are forbidden.

C) On the contrary, all the configurations characterized by a semimajor axis $a_B < a_t$ correspond to obtain, for a small contraction, the transformations:

$$\Delta a_B < 0; \Delta T_S > 0; \Delta Q < 0; \Delta S_B < 0$$  \hspace{1cm} (64)

which are allowed by the second Principle. A small expansion would on the contrary yield to:

$$\Delta a_B > 0; \Delta T_S < 0; \Delta Q > 0; \Delta S_B > 0$$  \hspace{1cm} (65)

which are forbidden both from the energy conservation Principle and from the second Thermodynamical Principle. The last pair of relations are those we may obtain by using the eqs.(57, 58, 59) in the case of one single $B$ component. That is clearly because: for $a_B < a_t$ the regime is characterized by the overcoming of self-gravity on the tidal-gravity.

The main conclusion is: the scale length induced on the stellar component by the dark matter halo works as a real border of this subsystem in the same way as the Hoerner’s (1958) tidal radius, induced by the galaxy, acts as a confinement for the stellar of a globular cluster (see, Appendix A). Therefore, it appears reasonable that ellipticals and GCs belong to the same FP and that King’s (1966) models, which need tidal cut-off, could be shared by both kinds of objects (Djorgovski, 1995; Burstein et al. 1997; Secco, 2003).

Moreover, the possible way in which the special tidal radius configuration may be reached during the violent relaxation phase of the system, gains a deep meaning. Indeed, this configuration is the widest one, the baryonic matter may have inside the dark potential well, and corresponds to the minimum of the macroscopic random velocity pressure the stellar system has to gain in order to virialize itself.
5 Entropy trend and thermodynamic information

By integration of eq.(59) (regarding the small variations as infinitesimals) we may obtain the trend of the entropy, normalized to the factor $2Nk$, $\tilde{S}(x)$, along the quasi-static virial sequence of the $B$ component, as follows (see also, MS3):

$$\tilde{S}(x) - \tilde{S}(1) = \ln \frac{V_B(1)}{V_B(x)} = \tilde{F}(x); \quad x = a_B/a_D$$  \hspace{1cm} (66)

where $x = 1$ is obtained when both the two subsystems coincide. The absolute value of $V_B(x)$ exhibits a minimum at $x = x_t$ along the virial sequence, therefore the function $\tilde{F}(x)$ has its maximum at the special configuration of $B$. The corresponding entropy variation, in physical units, along the sequence is simply $2Nk\tilde{F}(x)$.

In the case of similar heterogeneous models considered, for the four cases of Tab.1, we obtain the trends shown in Fig.5.

The last thermodynamical quantity to be considered will be the thermodynamic information, which, according to Layzer (1976), is:

$$I = S_{\text{max}} - S$$  \hspace{1cm} (67)

where $S_{\text{max}}$ means the maximum value the entropy of the system may have as soon as the constraints on it, which fix the actual value of its entropy to $S$, are relaxed. Therefore, in order to increase its information a system has to decrease its entropy more and more in respect to that of the universe (the maximum available and given essentially by the entropy of CBR; see, e.g.: Coles & Lucchin, 1995; Secco 1999).

For a luminous component such as $B$, which is embedded in an other $D$, a contraction decreases the entropy function $\tilde{F}(x)$ only if its dimension is smaller or equal to $a_t$. If we now transform the information given by the eq.(67), into the same units of $\tilde{F}(x)$, we obtain:

$$\tilde{I}_{\text{min}} = \tilde{S}_{\text{max}} - \tilde{S}(1) - \tilde{F}(x_t)$$  \hspace{1cm} (68)

This means that from tidal radius downwards the component $B$ may gain information by contraction. We plot in Fig.6 the trends of the information $I$ in the same units of $\tilde{F}(x)$ by assuming $\tilde{S}_{\text{max}} = \tilde{S}(x_t)$ instead of the real maximum entropy value corresponding to the CBR. The eq.(67) becomes:

$$\tilde{I} = \tilde{S}(x_t) - \tilde{S}(1) - \tilde{F}(x)$$  \hspace{1cm} (69)
The tidal radius configuration appears again as the best candidate for the beginning of the virial stage because it corresponds to the minimum of thermodynamical information along the whole virial sequence. The stellar subsystem has the real possibility to evolve onwards, with the dissipative processes indicated, e.g., by Bertin & Trenti (2003), by becoming more structured than it does at the end of the relaxation phase, that is at the special configuration given by $x_t$. The same arguments may be transferred to the cut-off spirals which, in this frame, had to begin a significant structure evolution from this stage forwards. Even if this extension has to be made wider and deeper in the future, it may become the ground for the interpretation of the cut-off radius observed in many edge-on spiral galaxies (see, e.g., van der Kruit, 1979; Pohlen et al. 2000a, 2000b and Kregel et al., 2002). A preliminary analysis has been done in Guarise et al. (2001) and in Secco & Guarise (2001).
6 Concluding remarks on the special configuration

The consequence of the existence of a special configuration characterized by the tidal radius have been analyzed. We stress particularly that:

a) at this special configuration the two-component models considered, built up of two homothetic similar strata spheroids with two power-laws and two different homogeneous cores, yield outputs which are relevant for some observable scaling relations of pressure supported ellipticals (subsect. 4.3 and 4.4). Moreover they could explain the physical reason for the existence of a Fundamental Plane for two-component virialized systems (e.g., Dressler et al. 1987; Djorgovski & Devis, 1987; Bender et al. 1992; Djorgovski & Santiago, 1993; Burstein et al. 1997; Bertin, Ciotti & Del Principe, 2002; Borriello, Salucci & Danese, 2003, and references therein).

b) The models belong, once their profiles are smoothed, to the class with profiles of general pseudo-isothermal kind, in turn, a sub-case of the more general class with profiles introduced by Zhao (1996). Indeed the density distributions of a realistic two-component model for an elliptical belong to this last class: both the dark matter halo profile, as proposed by Navarro, Frenk & White (Navarro et al. 1996, Navarro et al. 1997) (see section 2) and the Hernquist (1990) profile for the stellar component.

Even if the analytical properties we consider here are those we deduce by using the non-smoothed power-law profiles with two homogeneous cores, they may be transferred, to a good extent, to the general pseudo-isothermal case. The other problem, to recover analytical, similar results also in the more realistic model with, e.g., NFW+Her profiles, is still open and has been addressed in the subsect. 4.5.

c) The thermodynamical relevance of this induced scale length is high in connection with the Fundamental Plane of pressure supported ellipticals. Indeed, from the point of view of the two-component galaxy thermodynamics, deduced by the I° Thermodynamic Principle under the virial equilibrium constraint, this dimension works like a wall of a vessel, because a larger configuration turns out to be forbidden by the II° Principle of the Thermodynamics. This cut-off on the luminous component space provides the gravitational field, which is intrinsically without any scale length, of a specific border, as it appears for other stellar systems such as the globular clusters. As we have already shown (LS1, Secco, 2003; MS3) this truncation, which King (1966) has also introduced ad hoc in his primordial models for ellipticals, seems to be the common feature of all the astrophysical structures, from the globular clusters to the galaxy clusters, belonging to the cosmic metaplane, (Djorgovski, 1995; Burstein et al. 1997). This is in the former induced by the Galaxy potential well, in the
others by the dark matter halos. Its existence might be able, in principle, to explain some of the fundamental scaling relations of these structures as we already have proved for the ellipticals (LS1) and, tentatively, for the GCs (Secco, 2003). How this special configuration may be reached, during the stellar system evolution, is strictly connected with the problem of the end state of the collisionless stellar system after a violent relaxation phase and then to the problem of the constraints under which this phase occurs (Merritt, 1999). What we have already proposed in LS1, is that the end state has to correspond to the minimum of the macroscopic pressure that the stellar system needs in order to virialize. This is given by the maximum of Clausius’ virial energy configuration which, we will prove, also has the property of maximizing the entropy of the stellar component located in a two-component virialized system. It corresponds to the wider configuration which does not break the second Thermodynamic Principle together with energy conservation. This enforces the idea of looking at this special configuration as the best candidate for the end of a violent relaxation process of a stellar system when it occurs inside a dark massive halo component. Indeed, the presence of this widest allowed configuration for the baryonic matter, which has the least requests for sustaining the structure in virial equilibrium, would justify the fact that the ellipticals are not completely relaxed systems in respect to the collisionless dark halo. As White & Narayan (1987) have pointed out, by studying single power-law stellar structures, the ellipticals seem indeed to have stopped their violent relaxation process before its end, unlike the collisionless dark matter structures. Really, the $NFW$ profile is given by eq.(2) with the exponent $\beta = 3$ instead of the Hernquist profile, which is in agreement with the de Vaucouleurs light profile, and requires $\beta = 4$.

Moreover, it should be underlined that, due to the physical meaning of Clausius’ virial energy, the maximization of entropy inside the virial evolution sequence is guaranteed as soon as the model exhibits the minimum of the absolute value of Clausius’ virial, apart from the specific model we are dealing with (see, subsect. 4.6).

7 Conclusions

The Mechanics and Thermodynamics of a stellar virialized system, embedded in a dark matter halo, have been considered. The models (the same used in LS1) are not derived by looking for the $DF$ which maximizes the standard Boltzmann- Gibbs entropy. They consist of two heterogeneous spheroids with two power-law profiles and with two homogeneous cores. The concentrations are both equal 10. Under some restrictions, they may be considered as belonging to a sub-class of the general class of models endowed with Zhao density profiles. The most relevant aspects are:
- Under the usual constraints on the exponents of the density profiles and the assumption of a frozen dark matter halo, a configuration exists, characterized by having a dimension called *tidal radius* \((a_t)\), which enjoys special properties both from the mechanic and the thermodynamic point of view.

The mechanic properties at \(a_t\)-configuration are:

i)-the absolute value of the Clausius’ virial energy reaches its minimum.

ii)-The d’Alembert Principle of *virtual works* is satisfied, therefore it is a configuration of equilibrium. Nevertheless it is not of *stable* equilibrium because it does not minimize the total potential energy \((E_{pot})_B\) of the stellar subsystem (see, Fig.2 and MS3).

This special size translates to \(FP\)-like relation by yielding outputs very relevant for some observable scaling relations connected with pressure supported ellipticals.

To fit the observed FP would require the dark halo density distribution to be close to a generalized isothermal profile with an exponent \(d\) around the value 0.5 at the border of Baryonic mass confinement. A stringent upper limit appears to be \(d = 1\) for which the gradient of dark matter distribution would produce a *tilt* exponent: \(\alpha_t = 0\). That means: the observed *tilt* disappears.

The thermodynamical properties at \(a_t\)-configuration are the following:

A)-The last mechanical property has the consequence, from the thermodynamical point of view, that the work done from the external forces, i.e. the self gravity and the tidal gravity, against the macroscopic pressure of the \(B\) stellar subsystem, is zero at \(a_t\) for any small contraction or expansion starting from it. This also means that the entropy of \(B\) is stationary at \(a_t\), if one considers the first Thermodynamics Principle under the virial equilibrium constraint for the thermodynamical irreversible transformations occurring between two consecutive quasi-equilibrium stages in which the evolutive sequence may be divided. At every quasi-equilibrium configuration it is possible to associate a mean temperature and at two consecutive stages, an entropy variation.

B)-By integration of the entropy variation, it appears that entropy reaches, inside the evolutive virial sequence, its maximum at \(a_t\).

Therefore, the first conclusion is that: as soon as a class of models is able to exhibit a minimum of the virial energy along an evolutive sequence of the two-component structure they describe, the model of this class which corresponds to the Clausius’ minimum (in absolute value) is also the model which
maximizes the entropy inside the virial sequence. This is verified in spite of the class of models we are dealing with and it appears new in respect to the results, for single structures, which appear in the literature. Moreover, this also allows us to avoid a pragmatic approach in which the thermodynamic request is separated from the mechanical one as in Lima Neto et al. (1999). Indeed the configuration at \( a_t \) enjoys both the best properties, mechanic or thermodynamic.

\( \beta \)

- The second relevant aspect is that the mechanic and thermodynamic properties together reveal the confinement capability of the scale length induced on the stellar subsystem from the dark matter halo, That turns out to have the same role of the tidal radius induced by the Galaxy on a globular cluster, as Hoerner (1958) found, and represents a generalization of his result (see, Appendix A). This seems to be the key for understanding the physical reason for the FP structure and why objects with a completely different history of formation and evolution share the same FP features. Indeed, the second Principle of Thermodynamics and the energy conservation transform this tidal radius in a true cut-off, i.e. the stellar system, as a whole, cannot have a configuration wider than that which corresponds to the special one. That may be regarded as connected with the request, coming from many open problems, as referred in the introduction, of a tidal cut-off on the coordinate space for the \( DF \).

\( \gamma \)

- The third relevant aspect is related to the possible way in which this special configuration may be reached. Since the beginning (LS1) we noted that Clausius virial maximum corresponds to the minimum of the macroscopic pressure a subsystem needs for virialize during the relaxation phase and the consequent conversion of radial ordered velocity into a dispersion velocity field (see, e.g., Huss et al. 1999). This now becomes more than an Ansatz. Indeed, it appears that the widest allowed configuration for the stellar system corresponds to this minimum. This may explain how and why this special configuration will be reached by solving also the problem addressed by White & Narayan (1987). A possible physical reason is the existence of this \( a_t \)-configuration for the baryonic component, needed because it is less massive than the dark halo and therefore more affected by the other component tidal effect than the halo does.

On the other hands the problem of knowing exactly the end state of a collisionless stellar system after a violent relaxation phase, would require the knowledge of the initial conditions under which this mechanism occurs (Merritt, 1999). But as Djorgovski (1992) has pointed out and we have shown in
LS1 a degeneracy exists on the FP towards the cosmological initial conditions, then the possible end state of a stellar system has probably to be only indirectly deduced from the features of FP. The capability of Clausius virial maximum configuration to explain some of the main features of the FP, may highlight what is, otherwise, covered by intrinsic FP degeneracy.

\((\delta)\)

The fourth is: taking into account the thermodynamical information during the evolution sequence considered here for the \(B\) stellar system, it arises that the special configuration at \(a_t\) corresponds to the minimum of this information. That means an elliptical will gain small information due to its small dissipation capability and therefore will stay at a low level of structuration. On the contrary, a gas dominated galaxy will become more structured and differentiated by this configuration forwards due to its dissipation engine given, e.g., by cloud collisions.

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**A Appendix**

*On the Hoerner’s tidal radius*

How is the relationship between the *tidal radius* \(a_H\) induced by the Galaxy on the satellites, as the Globular Clusters (hereafter \(GCs\)), which Hoerner (1958) discovered and the special size which maximizes the Clausius’ virial energy of the Baryonic component? The roles of confinement are the same (see, e.g., \((\beta)\)-statements of Conclusions) and the mathematical structures are

\(^4\) The degeneracy disappears for the ratio of the dark over baryonic mass, \(m\), (see, LS1).
very similar. The Hoerner’s result is indeed:

\[ a_H = \left( \frac{1}{2} \frac{M_c}{M_G} \right)^{1/3} a_G \]  

(A.1)

where, \( M_c \) is the GC mass and \( M_G \) is Galaxy mass contained in the \( a_G \) radius. The eq.(A.1) has to be compared with the general result obtained in the concentric case given by eq.(16). \( a_H \) is obtained by balancing of all the accelerations on the Cluster point \( P \) in Fig.1A, when the variation of centrifugal force, due to a rigid Galaxy rotation, at the point \( P \) is taken as negligible in respect to that of the Cluster center \( O \) (see, e.g., Brosche et al.1999). It should be underlined that the eq.(A.1) is derived in the Hoerner’s point mass model. On the contrary, the eq.(16) is deduced by the non monotonic character of the Clausius’ energy due to a mass extension of a component submitted both to its self gravity and embedded in the gravity of an other one.

The balance to consider is not of forces but between self- and tidal- potential energy with the same sign but with different trend. The question is: \textit{in the concentric case only comparison between energy may be possible (the forces are concordant), but in the off-center case both the definitions are possible? That means: in this last case also the Clausius’virial of the Cluster shows a non-monotonic character?}

By analogy with the concentric case, we already guessed (Secco, 2003) that, for some mass distributions of Cluster and Galaxy, the answers were positive. Detailed computations of the self- and the tidal-energy tensors have been already performed in a previous paper (Caimmi & Secco, 2003) where the general second order theory in \( a_c/R_o \) (\( R_o \) is the mean orbit radius of the \( CG \)) for the two off-center component systems is developed. For heuristic sake, we take now into account the simple case in which both the two off-center components, Cluster and Galaxy, are considered as two homogeneous spherical mass distributions (Caimmi & Secco, 2005).

Generally speaking, the global tidal-potential energy trace due to the dynamical effect of the Galaxy-matter distribution on the cluster, \( V_{cG} \), may be split into:

\[ V_{cG} = V'_{cG} + V^o_{cG} \]  

(A.2)

where \( V^o_{cG} \) is the potential energy of a mass point placed at the cluster barycentre, with same mass as the cluster, due to the Galaxy mass fraction \( M_G/(R_o) \) which is located inside the mean cluster distance \( R_o \), and \( V'_{cG} \) represents an additional term which is due to the cluster mass distribution. The same holds for the global kinetic energy, \( T_c \), which, according to Koenig’s theorem, may
be split into two terms:

\[ T_c = T'_c + T^o \]  \hspace{1cm} (A.3)

one term, \( T'_c \), is the intrinsic kinetic energy of the cluster, the other one, \( T^o \), is
the kinetic energy of the cluster barycentre. Because of the time mean motion
of the cluster barycentre occurs in stationary virial equilibrium, the following
relation holds:

\[ 2T^o = -V^o_{\Omega_G} \]  \hspace{1cm} (A.4)

as soon as the two quantities are averaged over the period of cluster orbit
inside the gravitational Galaxy field. Therefore, the virial equilibrium of the
cluster mass distribution gives:

\[ 2T'_c = -V'_c = -\Omega_c - V'_{cG} \]  \hspace{1cm} (A.5)

where \( V'_c \) is the Clausius’ virial energy of the cluster due to its mass distribution
and \( \Omega_c \) is its self-potential energy.

From the second order theory in \( a_c/R_o \), the following result may be deduced:

\[ V'_{cG} = -\frac{1}{3}G \frac{M_cM_G(R_o)}{a_c} \left( \frac{a_c}{R_o} \right)^3 \]  \hspace{1cm} (A.6)

Then the Clausius’ virial related to the mass distribution of the globular cluster
becomes:

\[ V'_c = -\frac{3}{5} \frac{GM^2_c}{a_c} - \frac{1}{3}G \frac{M_cM_G(R_o)}{a_c} \left( \frac{a_c}{R_o} \right)^3 \]  \hspace{1cm} (A.7)

The non-monotonic character of \( V'_c \) is manifest and then it follows that also
in the case of two off-center component system, the Clausius’ virial exhibits a
maximum given by:

\[ a_{Gt} = \left( \frac{9}{5} \right)^{1/3} \left( \frac{1}{2} \frac{M_c}{M_G} \right)^{1/6} a_G \]  \hspace{1cm} (A.8)

which turns out to be only about a factor 1.2 greater than the \( a_H \) radius.
The corresponding thermodynamical arguments in the case of two off-center
component system are still an open question.
Fig. A.1. Two off-center spherical component model. The satellite (e.g., a Globular Cluster) is embedded in the gravity field of an other subsystem (e.g., the Galaxy). Its mass $M_c$ is distributed inside $a_c$ and its barycenter $O$ moves in a mean circular orbit (dashed circle) of radius $R_o$ around the center $O'$ of the other component with mass $M_G$ inside the radius $a_G$. $P$ is the point on which the accelerations hold the balance when $a_c$ becomes the tidal radius of Hoerner.

B Appendix

**Glossary of Symbols (in order of comparison)**

- **Section 1**
  - $S$ entropy
  - $f(\vec{x}, \vec{v})$ distribution function, $DF$

- **Section 2**
  - $\rho(r)$ density radial profile
  - $\alpha, \beta, \gamma, \delta$ exponents of Zhao density profile
  - $B$ Bright (stellar) or Baryonic component
  - $DM$ Dark Matter
  - $D$ Dark Matter component
  - $(T_B)_{ij}$ kinetic-energy tensor of Baryonic component
  - $(V_B)_{ij}$ Clausius'virial tensor of Baryonic component
  - $(T_D)_{ij}$ kinetic-energy tensor of $DM$ component
  - $(V_D)_{ij}$ Clausius'virial tensor of $DM$ component
• \((\Omega_B)_{ij}\) self potential-energy tensor of Baryonic component
• \((V_{BD})_{ij}\) tidal potential-energy tensor of Baryonic component due to the gravitational DM force on it
• \((\Omega_D)_{ij}\) self potential-energy tensor of DM component
• \((V_{DB})_{ij}\) tidal potential-energy tensor of DM component due to the gravitational force of Baryonic component on it
• \(\Sigma^*\) the surface which contains the DM mass fraction which, according to Newton’s I° theorem, exerts dynamical effect on the stellar component
• \(\bar{M}_D\) approximate DM mass fraction inside \(\Sigma^*\)
• \(\Phi_B\) gravitational potential due to the Baryonic mass distribution
• \(\Phi_D\) gravitational potential due to DM mass distribution
• \(\Phi_{ij}\) gravitational tensor potential
• \(G\) Gravitational constant
• \(T_{ij}\) kinetic-energy tensor of ordered velocities
• \(\Pi_{ij}\) two times kinetic-energy tensor of random velocities
• \(\nu(\vec{x})\) spatial density in phase space
• \(b\) exponent of power-law density profile in the cored Baryonic component
• \(d\) exponent of power-law density profile in the cored DM component
• \(a\) major semiaxis of the virialized subsystem considered
• \(M\) mass of the virialized subsystem considered
• \(\rho_o\) characteristic density of the virialized subsystem considered
• \(r_o\) scale radius of the subsystem considered
• \(C\) concentration of the subsystem considered
• \(r_e\) the effective radius
• \(I_e\) mean effective surface brightness within \(r_e\)
• \(\sigma_o\) the central projected velocity dispersion
• Section 3
• \((E_{pot})_B\) total potential energy of the stellar component
• \(W_{BD}\) interaction-energy between the Baryonic and the DM components
• \((W_{BD})_{ij}\) interaction-energy tensor between the Baryonic and the DM components
• \(a_t\) tidal radius of the Baryonic component
• \(\nu_{\Omega B}\) coefficient of mass-distribution in the Baryonic self potential-energy tensor
• \(\nu_{\Omega D}\) coefficient of mass-distribution in the DM self potential-energy tensor
• \(\nu'_{V}\) coefficient which weights the tidal potential-energy in respect to the self potential-energy in the Clausius’ virial of the Baryonic component
• $m$ mass ratio of DM to Baryonic component
• $\tilde{m}$ ratio of DM mass fraction $\tilde{M}_D$, to total stellar mass
• $M_{\text{tot}}^*$ total dynamical mass inside the size of Baryonic component
• $F$ form factor, the same for $B$ and DM components
• $x$ size ratio of Baryonic to DM components: $x = a_B/a_D$
• $\tilde{V}_B$ Clausius’ virial energy normalized to $a_D/(GM_B^2 F)$

Section 4

• $L_s$ work done by the self gravity forces
• $L_t$ work done by the tidal gravity forces
• $F(\vec{x})$ extrinsic attribute in Chandrasekhar’s analysis; e.g., gravitational potential, its first derivative, etc. which are not intrinsic to the fluid element but related to its spatial location
• $S_o$ initial volume of the inner system
• $\Delta S_o$ small variation of the initial volume of the inner system
• $\xi(\vec{x}, t)$ generic small displacement, at the time $t$, of the fluid element from the unperturbed position $\vec{x}_0$, in the inner component
• $\Delta F$ Lagrangian change of $F$ due to the displacement $\xi$
• $M_o$ initial volume of the outer system
• $\xi'(\vec{x}, t)$ generic small displacement, at the time $t$, of the fluid element from the unperturbed position $\vec{x}_o$, in the outer system

• $FP$ Fundamental Plane
• $A$ exponent of $\sigma_o$ in the equation of $FP$
• $B$ exponent of $I_e$ in the equation of $FP$
• $L$ Luminosity
• $c_1$ $L/(I_e r_e^2)$
• $< \sigma^2 >$ mean square velocity dispersion of the stellar system
• $k_e$ $\sigma_o^2/<\sigma^2>$
• $< R >$ the gravitational radius
• $k_R$ $r_e/<R>$
• $c_2$ $(Gk_R k_e)^{-1}$
• $T_{\text{pec}}$ peculiar kinetic energy dominant in the stellar component
• $T_{\text{rot}}$ rotational kinetic energy negligible in the stellar component
• $\alpha_t$ tilt exponent of the $FP$: $M_B/L \sim M_B^{\alpha_t}$
• $n_{\text{rec}}$ effective spectral index of perturbations in $CDM$ scenario
• $\gamma'$ exponent in the scaling relation for virialized haloes: $a_D \sim M_D^{1/\gamma'}; \gamma' = 6/(5 + n_{\text{rec}})$
• $\alpha_{\text{rec}}$ local slope of the $CDM$ mass variance
• $r$ exponent of $m$ in the scaling relation: $r_e \sim m^r M_B^R$
• $R$ exponent of $M_B$ in the scaling relation: $r_e \sim m^r M_B^R$
\[ R = 1/\gamma' \]

- \( i \) exponent of \( m \) in the scaling relation: \( I_e \sim m^i M_B^j \);
  \[ i = 2 \frac{\gamma'-(3-d)}{\gamma'(3-d)} \]

- \( I \) exponent of \( M_B \) in the scaling relation: \( I_e \sim m^i M_B^j \);
  \[ I = 2 \frac{\gamma'-(3-d)}{\gamma'(3-d)} \]

- \( s \) exponent of \( m \) in the scaling relation: \( \sigma_o \sim m^s M_B^j \);
  \[ s = \frac{1}{2} \frac{\gamma'-(3-d)}{\gamma'(3-d)} \]

- \( S \) exponent of \( M_B \) in the scaling relation: \( \sigma_o \sim m^s M_B^j \);
  \[ S = \frac{1}{2} \frac{\gamma'-1}{\gamma'} \]

- \( FJ \) Faber-Jackson relation
- \( (M_{\text{tot}}^*)_t \) total dynamical mass inside the size corresponding to tidal radius of Baryonic component

- \( m_* \) mean star mass in the \( B \) stellar system
- \( k \) the Boltzmann constant
- \( T_S \) mean temperature of \( B \) stellar system: \( \frac{m_*<s^2>}{k} \)
- \( N \) star number of the \( B \) stellar component
- \( E_T \) internal energy of the \( B \) component
- \( \Delta Q \) small amount of heat exchanged between the \( B \) system and the surrounding medium
- \( \Delta S_B \) small entropy variation of \( B \) system during the small quasi-static variation of its size \( \Delta a_B \)
- \( \Delta S_r \) corresponding small entropy variation of thermal radiation bath in the surroundings
- \( T_r \) mean temperature of the surrounding thermal bath
- \( \Delta S_u \) small entropy variation of the whole universe

- **Section 5**
  - \( \tilde{S}(x) \) entropy of the \( B \) system, normalized to the factor \( 2Nk \), at the size ratio \( x \) of Baryonic to \( DM \) components
  - \( \tilde{S}(1) \) entropy of \( B \) system, normalized to the factor \( 2Nk \), when the size of Baryonic component is the same of the \( DM \) one
  - \( \tilde{F}(x) \) variation of normalized entropy = \( \tilde{S}(x) - \tilde{S}(1) = \ln \frac{V_B(1)}{V_B(x)} \)
  - \( I = S_{\text{max}} - S \) thermodynamical information
  - \( \overline{I} \) thermodynamical information normalized to the same arbitrary units of \( S \)

- **Appendix A**
  - \( a_H \) Hoerner’s tidal radius
  - \( GC \) Globular Cluster
  - \( a_c \) \( GC \) radius
  - \( a_G \) Galaxy radius
  - \( M_c \) \( GC \) mass
  - \( M_G \) Galaxy mass
  - \( V_{cG} \) global tidal-energy of the \( GC \) due to the Galaxy
• $V'_{cG}$ tidal-energy of the GC due to its mass distribution inside the Galaxy gravity field
• $V^o_{cG}$ potential energy due to the Galaxy gravity field of the cluster barycenter
• $T_c$ global kinetic energy of the GC
• $T^o_c$ intrinsic kinetic energy of the GC
• $T^o$ kinetic energy of the cluster barycenter
• $\Omega_c$ self-potential energy of the GC
• $V^o_c$ Clausius’ virial energy of the cluster mass distribution
• $R_o$ mean orbit radius of the GC
• $M_G(R_o)$ Galaxy mass fraction inside the sphere of radius $R_o$
• $a_{Gl}$ tidal radius at the maximum of Clausius’ virial in the two off-center spherical and homogeneous component model

References

Barnes, J., & White, S.D.M. 1984, Mont.Not., 211, 753
Bender, R., Burstein, D. & Faber, S.M., 1992, ApJ, 399, 462
Bertin, G., & Trenti, M., 2003, ApJ, 584, 729
Bertin, G., Ciotti, L. & Del Principe, M., 2002, A&AS, 386, 149
Binney, J., & Tremaine, S. 1987, Galactic Dynamics, Princeton University Press, Princeton
Borriello, A., Salucci, P. & Danese, L., 2003, MNRAS, 1109, 1120
Brosche, P., Caimmi, R., Secco, L., 1983, A&AS, 125, 338
Brosche, P., Odenkirchen, M., Geffert, M., 1999, NewA, 4, 133
Burstein, D., Bender, R., Faber, S.M., & Nolthenius, R., 1997, AJ, 114(4),1365
Caimmi R., 1994, Ap.S.S. 219, 49.
Caimmi, R., Secco, L., & Brosche, P. 1984, A&AS, 139, 411
Caimmi R. & Secco L., 1992, ApJ, 395, 119.
Caimmi R. & Secco L., 2003, Astron. Nachr., 324, 491.
Caimmi R. & Secco L., 2004, in progress.
Caimmi R. & Secco L., 2005, in progress.
Cappellari M. and SAUROM team, 2004, in preparation.
Chandrasekhar S., 1939, Stellar Structure, Dover Publications, Inc.
Chandrasekhar S., 1969, Ellipsoidal Figures of Equilibrium, Dover Publications, Inc., New York.
Ciotti, L., 1999, ApJ, 520, 574
Ciotti, L., Lanzoni, B., Renzini, A., 1996, MNRAS, 282, 1
Coles, P., & Lucchin, F., 1995, Cosmology, ed. Wiley.
Djorgovski, S. 1992, in Morphological and Physical Classification of Galaxies, eds. G.Longo et al. (Kluver Academic Publishers, Netherlands), 337-356
Djorgovski S., 1995, ApJ, 438,L29-L32
Djorgovski S. & Davis, M., 1987, ApJ, 313, 59
Djorgovski, S., & Santiago, B.X. 1993, in Workshop on Structure, Dynamics and Chemical Evolution of Early-type Galaxies, eds. Danziger, I.J. et al., ESO, Garching, pg. 59
Dressler, A., Lynden-Bell, D., Burstein, D., Davies, R. L., Faber, S.M., Terlevich, R.J., Wegner, G., 1987, ApJ, 313, 42
Garrido, O., 2003, PhD Thesis, Observatoire de Marseille, Universite de Provence
Gerhard, O., Kronawitter, A., Saglia, R.P., & Bender, R., 2001, AJ, 121, 1936
Gott, J.R., & Rees, M. 1975a, A&AS, 15, 235
Gott, J.R., & Rees, M. 1975b, A&AS, 45, 365
Guarise G., Galletta G., & Secco L., Astr. Gesellschaft, Jenam 2001, Abstract Series, 18, 202.
Gunn, G.E. 1987, in The Galaxy, NATO ASI Ser.C207, (Reidel Publ. Co., Dordrecht, Holland), 413
Hernquist, L., 1990, Apj, 356,359.
Hoerner, V.S., 1958, ZA 44, 221.
Horowitz, G., & Katz, J. B., 1978, ApJ, 222,94
Huss,A., Jain, B., Steinmetz, M., 1999 ApJ, 517,64
Jaffe, W., 1983, MNRAS, 202, 995
Jørgensen, I.1999, MNRAS, 306, 607
King I., R., 1966, AJ, 71(1),64
Kravtsov, A. V., Klypin, A.A., Bullock, J.S. Primack, J.R., 1998, ApJ, 502, 48
Kregel, M., van der Kruit, P.C., & Grijs, R., 2002, MNRAS, (astro-ph/0204154).
Layzer D., 1976, ApJ, 206, 559
Lia, C., Carraro, G. & Salucci, P. 2000, A&A, 360, 76
Lima Neto, G.B., Gerbal, D., & Marquez, I., 1999, MNRAS, 309, 481
Lynden-Bell, D., & Wood, R., 1968, MNRAS, 138, 415
Maraston, C., 1999, in ASP Conf. Ser. 163, Star Formation in Early-type Galaxies, ed. J. Cepa & P. Carral (San Francisco:ASP), 28
Marmo, C., 2003, PhD Thesis, Astronomy Department, University of Padova.
Marmo, C., & Secco, L., 2003, NewA, 8/7, 629
Marquez, I., Lima Neto, G.B., Capelato, H., Durret, F., Lanzoni, B., & Gerbal, D., 2001, A&A, 379, 767
Merritt, D., 1999, PASP, 111, 129
Mücket, J.P. & Hoeft, M., 2003, A&A, 404, 809
Navarro, J.F., Frenk, C.S., White, S.D.M., 1996, Apj,462,563
Navarro, J.F., Frenk, C.S., White, S.D.M., 1997,Apj,490,493
Pahre, M.A., Djorgovski, S.G., & de Carvalho, R.R., 1998, AJ, 116, 1591
Prigogine, I., 1988, La nascita del tempo, Bompiani
Ogorodnikov, K.F., 1965, Dynamics of Stellar Systems, Pergamon Press
Pohlen M., Dettmar R.-J., Lütüticke R., & Schwarzkopf U., 2000a, A&AS 144, 405.
Pohlen M., Dettmar R.-J. & Lütticke R., 2000b, A&A 357, L1-L4.
Raffaele, A., 2003, Master Thesis, University of Padua, Italy.
Renzini, A., Ciotti, L., 1993, ApJ, 416, L49-52.
Schwarzschild M. 1958, Structure and Evolution of the Stars, Dover
Secco L., 1999, ASP Conf. Ser., Observational Cosmology: The Development
of Galaxy Systems, Vol.176, Giuricin, G., Mezzetti, M., Salucci, P. (Eds.),
ASP, San Francisco, p.449.
Secco L., 2000, NewA 5, 403.
Secco L., 2001, NewA 6, 339.
Secco L., 2003, ASP Conf. Ser., New Horizons in Globular Cluster Astronomy,
Piotto, G., Meylan, G., Djorgovski, D., and Riello, M.(Eds), ASP, vol.296,
San Francisco, p.87.
Secco L. & Guarise G., Astr. Gesellschaft, Jenam 2001, Abstract Series, 18,
208.
Sersic, J.L., 1968, Atlas de Galaxias Australes, Observatorio Astronomico, Cor-
doba
Stiavelli, M., & Bertin, G., 1987, MNRAS, 229, 61
van der Kruit P.C., 1979, A&AS 38,15.
White,S., D., M., & Narayan, N., 1987, MNRAS, 229, 103
Zemansky, M.W., 1968, Heat and Thermodynamics, Chap. 8, McGraw-Hill
Book Company.
Zhao, H.S., 1996, MNRAS, 278, 488.