Joint-Tree Model and the Maximum Genus of Graphs

Guanghua Dong\(^1,2\), Ning Wang\(^3\), Yuanqiu Huang\(^1\) and Yanpei Liu\(^4\)

1. Department of Mathematics, Normal University of Hunan, Changsha, 410081, China
2. Department of Mathematics, Tianjin Polytechnic University, Tianjin, 300387, China
3. Department of Information Science and Technology, Tianjin University of Finance and Economics, Tianjin, 300222, China
4. Department of Mathematics, Beijing Jiaotong University, Beijing 100044, China

E-mail: gh.dong@163.com, ninglw@163.com, hyqq@hunnu.edu.cn, ypliu@bjtu.edu.cn

Abstract: The vertex \( v \) of a graph \( G \) is called a 1-critical-vertex for the maximum genus of the graph, or for simplicity called 1-critical-vertex, if \( G - v \) is a connected graph and \( \gamma_M(G - v) = \gamma_M(G) - 1 \). In this paper, through the joint-tree model, we obtained some types of 1-critical-vertex, and get the upper embeddability of the Spiral \( S_n^m \).

Key Words: Joint-tree, maximum genus, graph embedding, Smarandache \( \mathcal{P} \)-drawing.

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§1. Introduction

In 1971, Nordhaus, Stewart and White [12] introduced the idea of the maximum genus of graphs. Since then many researchers have paid attention to this object and obtained many interesting results, such as the results in [2-8,13,15,17] etc. In this paper, by means of the joint-tree model, which is originated from the early works of Liu ([8]) and is formally established in [10] and [11], we offer a method which is different from others to find the maximum genus of some types of graphs.

Surfaces considered here are compact 2-dimensional manifolds without boundary. An orientable surface \( S \) can be regarded as a polygon with even number of directed edges such that both \( a \) and \( a^{-1} \) occurs once on \( S \) for each \( a \in S \), where the power \( "-1" \) means that the direction of \( a^{-1} \) is opposite to that of \( a \) on the polygon. For convenience, a polygon is represented by a linear sequence of lowercase letters. An elementary result in algebraic topology states that

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each orientable surface is equivalent to one of the following standard forms of surfaces:

\[ O_p = \begin{cases} 
  a_0 a_0^{-1}, & p = 0, \\
  \prod_{i=1}^p a_i b_i a_i^{-1} b_i^{-1}, & p \geq 1
\end{cases} \]

which are the sphere \((p = 0)\), torus \((p = 1)\), and the orientable surfaces of genus \(p (p \geq 2)\).

The genus of a surface \(S\) is denoted by \(g(S)\). Let \(A, B, C, D, \) and \(E\) be possibly empty linear sequence of letters. Suppose \(A = a_1 a_2 \ldots a_r, r \geq 1\), then \(A^{-1} = a_r^{-1} \ldots a_2^{-1} a_1^{-1}\) is called the inverse of \(A\). If \(\{a, b, a^{-1}, b^{-1}\}\) appear in a sequence with the form as \(AaBbCa^{-1}Db^{-1}E\), then they are said to be an interlaced set; otherwise, a parallel set. Let \(\widetilde{S}\) be the set of all surfaces. For a surface \(S \in \widetilde{S}\), we obtain its genus \(g(S)\) by using the following transforms to determine its equivalence to one of the standard forms.

**Transform 1** \(Aaa^{-1} \sim A\), where \(A \in \widetilde{S}\) and \(a \notin A\).

**Transform 2** \(AabBb^{-1}a^{-1} \sim AcBc^{-1}\).

**Transform 3** \((Aa)(a^{-1}B) \sim (AB)\).

**Transform 4** \(AabCa^{-1}Db^{-1}E \sim ADCBEba^{-1}b^{-1}\).

In the above transforms, the parentheses stand for cyclic order. For convenience, the parentheses are always omitted when unnecessary to distinguish cyclic or linear order. For more details concerning surfaces, the reader is referred to [10-11] and [14].

For a graphical property \(\mathcal{P}\), a Smarandache \(\mathcal{P}\)-drawing of a graph \(G\) is such a good drawing of \(G\) on the plane with minimal intersections for its each subgraph \(H \in \mathcal{P}\) and optimal if \(\mathcal{P} = G\) with minimized crossings. Let \(T\) be a spanning tree of a graph \(G = (V, E)\), then \(E = E_T + E_T^*\), where \(E_T\) consists of all the tree edges, and \(E_T^* = \{e_1, e_2, \ldots e_\beta\}\) consists of all the co-tree edges, where \(\beta = \beta(G)\) is the cycle rank of \(G\). Split each co-tree edge \(e_i = (\mu_e, \nu_e) \in E_T^*\) into two semi-edges \((\mu_e, \omega_e), (\nu_e, \omega'_e)\), denoted by \(e_i^{-1}\) (or simply by \(e_i\) if no confusion) and \(e_i^{-1}\) respectively. Let \(\widetilde{T} = (V + V_1, E + E_1)\), where \(V_1 = \{\omega_e, \omega'_e | 1 \leq i \leq \beta\}\), \(E_1 = \{(\mu_e, \omega_e), (\nu_e, \omega'_e) | 1 \leq i \leq \beta\}\). Obviously, \(\widetilde{T}\) is a tree. A rotation at a vertex \(v\), which is denoted by \(\sigma_v\), is a cyclic permutation of edges incident on \(v\). A rotation system \(\sigma = \sigma_G\) for a graph \(G\) is a set \(\{\sigma_v \forall v \in V(G)\}\). The tree \(\widetilde{T}\) with a rotation system of \(G\) is called a joint-tree of \(G\), and is denoted by \(\widetilde{T}_\sigma\). Because it is a tree, it can be embedded in the plane. By reading the lettered semi-edges of \(\widetilde{T}_\sigma\) in a fixed direction (clockwise or anticlockwise), we can...
Lemma 1.0 If there is a joint-tree $\overline{T}_\sigma$ of $G$ such that the genus of its associated surface equals $\lfloor \beta(G)/2 \rfloor$ then $G$ is upper embeddable.

Proof According to the definition of joint-tree, associated surface, and upper embeddable graph, Lemma 1.0 can be easily obtained.

Lemma 1.1 Let $AB$ be a surface. If $x \notin A \cup B$, then $g(AXBx^{-1}) = g(AB)$ or $g(AXBx^{-1}) = g(AB) + 1$.

Proof First discuss the topological standard form of the surface $AB$.

(I) According to the left to right direction, let $\{x_1, y_1, x_1^{-1}, y_1^{-1}\}$ be the first interlaced set appeared in $A$. Performing Transform 4 on $\{x_1, y_1, x_1^{-1}, y_1^{-1}\}$ we will get $A'Bx_1y_1^{-1}x_1^{-1}y_1^{-1} (\sim AB)$. Then perform Transform 4 on the first interlaced set in $A'$. And so on. Eventually we will get $\overline{AB} \prod_{i=1}^r x_i y_i x_i^{-1} y_i^{-1} (\sim AB)$, where there is no interlaced set in $\overline{A}$.

(II) For the surface $\overline{AB} \prod_{i=1}^r x_i y_i x_i^{-1} y_i^{-1}$, from the left of $B$, successively perform Transform 4 on $B$ similar to that on $A$ in (I). Eventually we will get $\overline{AB} \prod_{i=1}^r x_i y_i x_i^{-1} y_i^{-1} \prod_{j=1}^s a_j b_j a_j^{-1} b_j^{-1} (\sim AB)$. Again, we can get $\overline{AB}$ by breaking $A, B$ into interlaced sets. Therefore, $g(AB) = g(AXBx^{-1})$. Hence, we get $g(AXBx^{-1}) = g(AB)$.
$AB$), where there is no interlaced set in $\tilde{B}$.

(III) For the surface $\tilde{AB} \prod_{i=1}^{r} x_{i} y_{i} x_{i}^{-1} y_{i}^{-1} \prod_{j=1}^{s} a_{j} b_{j} a_{j}^{-1} b_{j}^{-1}$, from the left of $\tilde{AB}$, successively perform Transform 4 on $\tilde{AB}$ similar to that on $A$ in (I). At last, we will get $\prod_{i=1}^{p} a_{i} b_{i} a_{i}^{-1} b_{i}^{-1}$, which is the topologically standard form of the surface $AB$.

As for the surface $AxBx^{-1}$, perform Transform 4 on $A$ and $B$ similar to that on $A$ in (I) and $B$ in (II) respectively. Eventually $AxBx^{-1} \prod_{i=1}^{r} x_{i} y_{i} x_{i}^{-1} y_{i}^{-1} \prod_{j=1}^{s} a_{j} b_{j} a_{j}^{-1} b_{j}^{-1}$ (similar to that on $\tilde{AB}$ in (III)) will be obtained. Then perform the same Transform 4 on $AxBx^{-1}$ as that on $\tilde{AB}$ in (III), and at last, one more Transform 4 than that in (III) may be needed because of $x$ and $x^{-1}$ in $AxBx^{-1}$.

Eventually $\prod_{i=1}^{p} a_{i} b_{i} a_{i}^{-1} b_{i}^{-1}$ or $\prod_{i=1}^{p+1} a_{i} b_{i} a_{i}^{-1} b_{i}^{-1}$, which is the topologically standard form of the surface $AxBx^{-1}$, will be obtained.

From the above, Lemma 1.1 is obtained. $\square$

**Lemma 1.2** Among all orientable surfaces represented by the linear sequence consisting of $a_{i}$ and $a_{i}^{-1}$ ($i = 1, \ldots, n$), the surface $a_{1} a_{2} \ldots a_{n} a_{1}^{-1} a_{2}^{-1} \ldots a_{n}^{-1}$ is one whose genus is maximum.

**Proof** According to Transform 4, Lemma 1.2 can be easily obtained. $\square$

**Lemma 1.3** Let $G$ be a graph with minimum degree at least three, and $\tilde{G}$ be the graph obtained from $G$ by a sequence of vertex-splitting, then $\gamma_{M}(\tilde{G}) \leq \gamma_{M}(G)$. Furthermore, if $\tilde{G}$ is upper embeddable then $G$ is upper embeddable as well.

**Proof** Let $v$ be a vertex of degree $n(\geq 4)$ in $G$, and $G'$ be the graph obtained from $G$ by splitting the vertex $v$ into two vertices such that both their degrees are at least three. First of all, we prove that the maximum genus will not increase after one vertex-splitting operation, i.e., $\gamma_{M}(G') \leq \gamma_{M}(G)$.

Let $e_{1}, e_{2}, \ldots, e_{n}$ be the $n$ edges incident to $v$, and $v$ be split into $v_{1}$ and $v_{2}$. Without loss of generality, let $e_{1}, e_{2}, \ldots, e_{r}$ be incident to $v_{1}$, and $e_{r+1}, \ldots, e_{n}$ be incident to $v_{2}$, where $2 \leq r \leq n - 2$. Select such a spanning tree $T$ of $G$ that $e_{1}$ is a tree edge, and $e_{2}, \ldots, e_{n}$ are all co-tree edges. As for graph $G'$, select $T'$ to be a spanning tree such that both $e_{1}$ and $(v_{1}, v_{2})$ are tree edges, and the other edges of $T'$ are the same as the edges in $T$. Obviously, $e_{2}, \ldots, e_{n}$ are co-tree edges of $T'$. Let $T = \{T_{a}, T_{b} = (T - v)_{a}\}$, where $(T - v)_{a}$ is a joint-tree of $G - v$, $T' = \{T'_{a}, T'_{b} = (T' - \{v_{1}, v_{2}\})_{a}\}$, where $(T' - \{v_{1}, v_{2}\})_{a}$ is a joint-tree of $G' - \{v_{1}, v_{2}\}$. It is obvious that $T = T'$. Let $S$ be the set of all the associated surfaces of the joint-trees of $G$, and $S'$ be the set of all the associated surfaces of the joint trees of $G'$. Obviously, $S' \subseteq S$. Furthermore, $|S'| = r! \times (n - r)! \times |T'| < |S| = (n - 1)! \times |T|$. So $S' \subset S$, and we have $\gamma_{M}(G') \leq \gamma_{M}(G)$.

Reiterating this procedure, we can get that $\gamma_{M}(\tilde{G}) \leq \gamma_{M}(G)$. Furthermore, because $\beta(G) = \beta(\tilde{G})$, it can be obtained that if $\tilde{G}$ is upper embeddable then $\left\lfloor \frac{\beta(G)}{2} \right\rfloor = \left\lfloor \frac{\beta(\tilde{G})}{2} \right\rfloor = \gamma_{M}(\tilde{G}) \leq \gamma_{M}(G) \leq \left\lfloor \frac{\beta(G)}{2} \right\rfloor$. So, $\gamma_{M}(G) = \left\lfloor \frac{\beta(G)}{2} \right\rfloor$, and $G$ is upper embeddable. $\square$
§2. Results Related to 1-Critical-Vertex

The neckband $N_{2n}$ is such a graph that $N_{2n} = C_{2n} + R$, where $C_{2n}$ is a 2n-cycle, and $R = \{a_i|a_i = (v_{2i-1}, v_{2i+2}). (i = 1, 2, \ldots, n, \ 2i + 2 \equiv r(\text{mod} 2n), \ 1 \leq r < 2n)\}$. The mobius ladder $M_{2n}$ is such a cubic circulant graph with 2n vertices, formed from a 2n-cycle by adding edges (called "rungs") connecting opposite pairs of vertices in the cycle. For example, Fig. 2.1 and Fig. 2.5 is a graph of $N_{8}$ and $M_{2n}$ respectively. A vertex like the solid vertex in Fig. 2.2, Fig. 2.3, Fig. 2.4, Fig. 2.5, and Fig. 2.6 is called an $\alpha$-vertex, $\beta$-vertex, $\gamma$-vertex, $\delta$-vertex, and $\eta$-vertex respectively, where Fig. 2.6 is a neckband.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figs.png}
\caption{Fig. 2.1. \quad Fig. 2.2 \quad Fig. 2.3 \quad Fig. 2.4 \quad Fig. 2.5 \quad Fig. 2.6}
\end{figure}

**Theorem 2.1** If $v$ is an $\alpha$-vertex of a graph $G$, then $\gamma_M(G - v) = \gamma_M(G)$. If $v$ is a $\beta$-vertex, or a $\gamma$-vertex, or a $\delta$-vertex, or an $\eta$-vertex of a graph $G$, and $G - v$ is a connected graph, then $\gamma_M(G - v) = \gamma_M(G) - 1$, i.e., $\beta$-vertex, $\gamma$-vertex, $\delta$-vertex and $\eta$-vertex are 1-critical-vertex.

**Proof** If $v$ is an $\alpha$-vertex of the graph $G$, then it is easy to get that $\gamma_M(G - v) = \gamma_M(G)$. In the following, we will discuss the other cases.

**Case 1** $v$ is a $\beta$-vertex of $G$.

According to Fig. 2.3, select such a spanning tree $T$ of $G$ such that both $a$ and $b$ are co-tree edges. It is obvious that the associated surface for each joint-tree of $G$ must be one of the following four forms:

(i) $AabBa^{-1}b^{-1} \sim ABaba^{-1}b^{-1}$;
(ii) $AabBb^{-1}a^{-1} \sim AcBc^{-1}$;
(iii) $AbaBa^{-1}b^{-1} \sim AcBc^{-1}$;
(iv) $AbaBb^{-1}a^{-1} \sim ABBab^{-1}a^{-1}$.

On the other hand, for each joint-tree $\tilde{T}^*_v$, which is a joint-tree of $G - v$, its associated surface must be the form as $AB$, where $A$ and $B$ are the same as that in the above four forms.
According to (i)-(iv), Lemma 1.1, and \( g(ABaba^{-1}b^{-1}) = g(AB) + 1 \), we can get that \( \gamma_M(G - v) = \gamma_M(G) - 1 \).

**Case 2** \( v \) is an \( \gamma \)-vertex of \( G \).

As illustrated by Fig. 2.4, both \( v_1 \) and \( v_2 \) are \( \gamma \)-vertex. Without loss of generality, we only prove that \( \gamma_M(G - v_1) = \gamma_M(G) - 1 \). Select such a spanning tree \( T \) of \( G \) such that both \( a \) and \( b \) are co-tree edges. The associated surface for each joint-tree of \( G \) must be one of the following 16 forms:

\[
Aabb^{-1}a^{-1}B, \quad Aabb^{-1}Ba^{-1}, \quad Aaba^{-1}Bb^{-1}, \quad AbBa^{-1}b^{-1}, \\
Abab^{-1}a^{-1}B, \quad Abab^{-1}Ba^{-1}, \quad Aba^{-1}Bb^{-1}, \quad Aba^{-1}b^{-1}, \\
Ab^{-1}a^{-1}Bab, \quad Ab^{-1}Ba^{-1}ab, \quad Aa^{-1}Bb^{-1}ab, \quad ABA^{-1}b^{-1}ab, \\
Ab^{-1}a^{-1}Bba, \quad Ab^{-1}Ba^{-1}ba, \quad Aa^{-1}Bb^{-1}ba, \quad ABA^{-1}b^{-1}ba.
\]

Furthermore, each of these 16 types of surfaces is topologically equivalent to one of such surfaces as \( AB, ABaba^{-1}b^{-1} \), and \( ABaba^{-1}b^{-1} \). On the other hand, for each joint-tree \( \overline{T}_\sigma \), which is a joint-tree of \( G - v_1 \), its associated surface must be the form of \( AB \), where \( A \) and \( B \) are the same as that in the above 16 forms. According to Lemma 1.1 and \( g(ABaba^{-1}b^{-1}) = g(AB) + 1 \), we can get that \( \gamma_M(G - v) = \gamma_M(G) - 1 \).

**Case 3** \( v \) is an \( \delta \)-vertex of \( G \).

In Fig. 2.5, let \( a_i = (v_i, v_{n+i}), i = 1, 2, \ldots, n \). Without loss of generality, we only prove that \( \gamma_M(G - v_1) = \gamma_M(G) - 1 \). Select such a joint-tree \( \overline{T}_\sigma \) of Fig. 2.5, which is illustrated by Fig.3, where the edges of the spanning tree are represented by solid line. It is obvious that the associated surface of \( \overline{T}_\sigma \) is \( mnm^{-1}n^{-1}a_2a_3 \ldots a_na_2^{-1}a_3^{-1} \ldots a_n^{-1} \). On the other hand, \( a_2a_3 \ldots a_na_2^{-1}a_3^{-1} \ldots a_n^{-1} \) is the associated surface of one of the joint-trees of \( G - v_1 \). From Lemma 1.2 and \( g(mnm^{-1}n^{-1}a_2a_3 \ldots a_na_2^{-1}a_3^{-1} \ldots a_n^{-1}) = g(a_2a_3 \ldots a_na_2^{-1}a_3^{-1} \ldots a_n^{-1}) + 1 \), we can get that \( \gamma_M(G - v) = \gamma_M(G) - 1 \).

![Fig. 3.](image)

![Fig. 4.](image)

**Case 4** \( v \) is an \( \eta \)-vertex of \( G \).

As illustrated by Fig. 2.6, every vertex in Fig. 2.6 is a \( \eta \)-vertex. Without loss of generality, we only prove that \( \gamma_M(G - v_{2n}) = \gamma_M(G) - 1 \).

A joint-tree \( \overline{T}_\sigma \) of Fig. 2.6 is depicted by Fig.4. It can be read from Fig.4 that the associated surface of \( \overline{T}_\sigma \) is \( S = a_1a_n(\prod_{i=1}^{n-3}a_{i+1}a_i^{-1})a_{n-2}a_{n-1}^rsrs^{-1}s^{-1} \). Performing a sequence of Transform
From (1) and (3) we have
\[ S = a_1a_n\left(\prod_{i=1}^{n-3} a_{i+1}^{-1} a_{n-2}^{-1} a_i^{-1} r s r^{-1} s^{-1}\right) \]

(Transform 4) \[ \sim \left(\prod_{i=2}^{n-3} a_{i+1}^{-1} a_{n-2}^{-1} a_{i}^{-1} s^{-1} a_1 a_n a_1^{-1} a_n^{-1}\right) \]

(Transform 4) \[ \sim \left(\prod_{i=4}^{n-3} a_{i+1}^{-1} a_{n-2}^{-1} a_{i}^{-1} s^{-1} a_1 a_n a_1^{-1} a_n^{-1} a_3 a_2 a_3^{-1} a_2^{-1}\right) \]

(Transform 4) \[ \sim \begin{cases} r s r^{-1} s^{-1} a_1 a_n a_1^{-1} a_n^{-1} \left(\prod_{i=2}^{n-4} a_{i+1} a_i a_i^{-1} a_i^{-1}\right) & n \equiv 0 \pmod{2}; \\ r s r^{-1} s^{-1} a_1 a_n a_1^{-1} a_n^{-1} \left(\prod_{i=2}^{n-3} a_{i+1} a_i a_i^{-1} a_i^{-1}\right) & n \equiv 1 \pmod{2}. \end{cases} \]

(1)

It is known from (1) that
\[ g(S) = \gamma_M(G) \]

(2)

On the other hand, \( S' = a_1 a_n \left(\prod_{i=1}^{n-3} a_{i+1} a_i^{-1} a_{n-2}^{-1} a_n^{-1}\right) \) is the associated surface of \( \tilde{T}_\sigma^* \), where \( \tilde{T}_\sigma^* \) is a joint-tree of \( G - v_{2n} \). Performing a sequence of Transform 4 on \( S' \), we have
\[ S' = a_1 a_n \left(\prod_{i=1}^{n-3} a_{i+1} a_i^{-1} a_{n-2}^{-1} a_n^{-1}\right) \]

\[ \sim \begin{cases} a_1 a_n a_1^{-1} a_n^{-1} \left(\prod_{i=2}^{n-4} a_{i+1} a_i a_i^{-1} a_i^{-1}\right) & n \equiv 0 \pmod{2}; \\ a_1 a_n a_1^{-1} a_n^{-1} \left(\prod_{i=2}^{n-3} a_{i+1} a_i a_i^{-1} a_i^{-1}\right) & n \equiv 1 \pmod{2}. \end{cases} \]

(3)

It can be inferred from (3) that
\[ g(S') = \gamma_M(G - v_{2n}) \]

(4)

From (1) and (3) we have
\[ g(S) = g(S') + 1. \]

(5)

From (2), (4), and (5) we have \( \gamma_M(G - v_{2n}) = \gamma_M(G) - 1. \)

According to the above, we can get Theorem 2.1. \( \square \)

Let \( G \) be a connected graph with minimum degree at least 3. The following algorithm can be used to get the maximum genus of \( G \).

**Algorithm I:**

1. **Step 1** Input \( i = 0 \), \( G_0 = G \).

2. **Step 2** If there is a 1-critical-vertex \( v \) in \( G_i \), then delete \( v \) from \( G_i \) and go to Step 3.

   Else, go to Step 4.
Step 3 Deleting all the vertices of degree one and merging all the vertices of degree two in \( G_i - v \), we get a new graph \( G_{i+1} \). Let \( i = i + 1 \), then go back to Step 2.

Step 4 Output \( \gamma_M(G) = \gamma_M(G_i) + i \).

Remark Using Algorithm I, the computing of the maximum genus of \( G \) can be reduced to the computing of the maximum genus of \( G_i \), which may be much easier than that of \( G \).

§3. Upper Embeddability of Graphs

An ear of a graph \( G \), which is the same as the definition offered in [16], is a path that is maximal with respect to internal vertices having degree 2 in \( G \) and is contained in a cycle in \( G \). An ear decomposition of \( G \) is a decomposition \( p_0, \ldots, p_k \) such that \( p_0 \) is a cycle and \( p_i \) for \( i \geq 1 \) is an ear of \( p_0 \cup \cdots \cup p_i \). A spiral \( S^n_m \) is the graph which has an ear decomposition \( p_0, \ldots, p_n \) such that \( p_0 \) is the \( m \)-cycle \((v_1v_2 \ldots v_m)\), \( p_i \) for \( 1 \leq i \leq m - 1 \) is the 3-path \( v_{m+2i-2}v_{m+2i-1}v_{m+2i}v_i \) which joining \( v_{m+2i-2} \) and \( v_i \), and \( p_i \) for \( i > m - 1 \) is the 3-path \( v_{m+2i-2}v_{m+2i-1}v_{m+2i}v_{2i-m+1} \) which joining \( v_{m+2i-2} \) and \( v_{2i-m+1} \). If some edges in \( S^n_m \) are replaced by the graph depicted by Fig. 6, then the graph is called an extended-spiral, and is denoted by \( \tilde{S}^n_m \). Obviously, both the vertex \( v_1 \) and \( v_2 \) in Fig. 6 are \( \gamma \)-vertex. For convenience, a graph of \( S^n_5 \) is illustrated by Fig. 5, and Fig. 7 is the graph which is obtained from \( S^n_5 \) by replacing the edge \((v_{13}, v_{14})\) with the graph depicted by Fig. 6.

**Fig. 5.**

**Fig. 6.**

**Fig. 7.**

**Theorem 3.1** The graph \( S^n_5 \) is upper embeddable. Furthermore, \( \gamma_M(S^n_5-v_{2n+3}) = \gamma_M(S^n_5)-1 \), i.e., \( v_{2n+3} \) is a 1-critical-vertex of \( S^n_5 \).

**Proof** According to the definition of \( S^n_5 \), when \( n \leq 4 \), it is not a hard work to get the upper embeddability of \( S^n_5 \). So the following 5 cases will be considered.

Case 1 \( n = 5j \), where \( j \) is an integer no less than 1.

Without loss of generality, a spanning tree \( T \) of \( S^n_5 \) can be chosen as \( T = T_1 \cup T_2 \), where \( T_1 \) is the path \( v_2v_1v_3v_5v_7\left( \prod_{i=1}^{j-1} v_{10i+1}v_{10i+2}v_{10i+3}v_{10i+4}v_{10i+5}v_{10i+6}v_{10i+7}v_{10i+8}v_{10i+9}v_{2n+1}v_{2n}v_{2n-1}v_{2n-2}\right) \left( v_{2n-3}v_{2n-4}v_{2n+5}v_{2n+4}v_{2n+3}\right) \), \( T_2 = (v_{2n+1}, v_{2n+2}) \). Obviously, the \( n + 1 \) co-tree edges of \( S^n_5 \) with respect to \( T \) are \( e_1 = (v_2, v_3), e_2 = (v_2, v_9), e_3 = (v_1, v_7), \prod_{i=1}^{j-1} (e_{5i-1} = (v_{10i-5}, v_{10i-4}), e_{5i} = (v_{10i-6}, v_{10i-3}), e_{5i+1} = (v_{10i+1}, v_{10i+2}), e_{5i+2} = (v_{10i}, v_{10i+9}), e_{5i+3} = (v_{10i+5}, v_{10i+6}), e_{5i+4} = (v_{10i+7}, v_{10i+8}) \).
\((v_{10i-2}, v_{10i+7})\), \(e_{n-1} = (v_{2n-5}, v_{2n-4})\), \(e_n = (v_{2n-6}, v_{2n+3})\), \(e_{n+1} = (v_{2n+2}, v_{2n+3})\). Select such a joint-tree \(T_\sigma\) of \(S_5^n\) which is depicted by Fig. 8. After a sequence of Transform 4, the associated surface \(S\) of \(T_\sigma\) has the form as

\[
S = e_1e_2e_1^{-1}e_3e_4e_5\left\{ \prod_{i=1}^{j-2} e_{5i+1}e_{5i+2}e_{5i+3}e_{5i+4}e_{5i+5}e_{5i+6}e_{5i+7}\right\} e_{n-4}e_{n-3}e_{n-2}e_{n-1}e_{n}^{-1}e_{n+1}^{-1}e_{n+2}^{-1}
\]

\[
\sim \prod_{i=1}^{\frac{n+1}{2}} e_{1i}e_{2i}e_{i1}^{-1}e_{i2}^{-1},
\]

where \(e_{ij}, e_{ij}^{-1} \in \{e_1, \ldots, e_{n+1}, e_1^{-1}, \ldots, e_{n+1}^{-1}\}; i = 1, \ldots, \lfloor \frac{n+1}{2} \rfloor; j = 1, 2\). Obviously, \(g(S) = \lfloor \frac{n+1}{2} \rfloor\). So, when \(n = 5j\), \(S_5^n\) is upper embeddable.

---

**Case 2** \(n = 5j + 1\), where \(j\) is an integer no less than 1.

Without loss of generality, select \(T = T_1 \cup T_2\) to be a spanning tree of \(S_5^n\), where \(T_1\) is the path \(v_3v_2v_1\{ \prod_{i=1}^{j} v_{10i-3}v_{10i-4}v_{10i-5}v_{10i-6}v_{10i+3}v_{10i+4}v_{10i+5}v_{10i+6}v_{10i+7}\} v_{2n+5}v_{2n+4}v_{2n+3}, T_2 = (v_{2n+1}, v_{2n+2})\). It is obviously that the \(n + 1\) co-tree edges of \(S_5^n\) with respect to \(T\) are \(e_1 = (v_1, v_3), e_2 = (v_3, v_4), e_3 = (v_3, v_11), e_4 = (v_2, v_9), j \prod_{i=1}^{k} e_{5i} = (v_{10i-3}, v_{10i-2}), e_{5i+1} = (v_{10i-4}, v_{10i+5}), e_{5i+2} = (v_{10i+3}, v_{10i+4}), e_{5i+3} = (v_{10i+2}, v_{10i+11}), e_{5i+4} = (v_{10i}, v_{10i+9}), e_{n-1} = (v_{2n-5}, v_{2n-4}), e_n = (v_{2n-6}, v_{2n+3}), e_{n+1} = (v_{2n+2}, v_{2n+3})\). Similar to Case 1, select a joint tree \(T_\sigma\) of \(S_5^n\). After a sequence of Transform 4, the associated surface \(S\) of \(T_\sigma\) has the form as

\[
S = e_1e_2e_3e_4e_5e_6e_1^{-1}e_2^{-1}e_7e_8e_3^{-1}e_9\left\{ \prod_{i=1}^{j} e_{5i-1}^{-1}e_{5i}e_{5i+1}e_{5i+2}e_{5i+3}e_{5i+4}e_{5i+5}e_{5i+6}e_{5i+7}\right\} e_{n-4}e_{n-3}e_{n-2}e_{n-1}e_{n}^{-1}e_{n+1}^{-1}e_{n+2}^{-1}
\]

\[
\sim \prod_{i=1}^{\frac{n+1}{2}} e_{1i}e_{2i}e_{i1}^{-1}e_{i2}^{-1},
\]

where \(e_{ij}, e_{ij}^{-1} \in \{e_1, \ldots, e_{n+1}, e_1^{-1}, \ldots, e_{n+1}^{-1}\}; i = 1, \ldots, \lfloor \frac{n+1}{2} \rfloor; j = 1, 2\). Obviously, \(g(S) = \lfloor \frac{n+1}{2} \rfloor\). So, when \(n = 5j + 1\), \(S_5^n\) is upper embeddable.
Case 3  $n = 5j + 2$, where $j$ is an integer no less than 1.

Without loss of generality, select a spanning tree of $S^5_0$ to be $T = T_1 \cup T_2$, where $T_1$ is the path $v_1v_5v_4v_3v_2\{\prod_{i=1}^{j} v_{10i-1}v_{10i-2}v_{10i-3}v_{10i-4}v_{10i+5}v_{10i+4}v_{10i+3}v_{10i+2}\}v_{2n+5}v_{2n+4}v_{2n+3}$, $T_2 = (v_{2n+1}, v_{2n+2})$. It is obviously that the $n + 1$ co-tree edges of $S^5_0$ with respect to $T$ are $e_1 = (v_1, v_2)$, $e_2 = (v_1, v_7)$, $e_3 = (v_5, v_6)$, $e_4 = (v_4, v_{13})$, $e_5 = (v_3, v_{11})$, $i=1^{j-1} (e_{5i+1} = (v_{10i+1}, v_{10i}), e_{5i+2} = (v_{10i-2}, v_{10i+7}), e_{5i+3} = (v_{10i+5}, v_{10i+6}), e_{5i+4} = (v_{10i+4}, v_{10i+13}), e_{5i+5} = (v_{10i+2}, v_{10i+11})$, $e_{n-1} = (v_{2n-5}, v_{2n-4})$, $e_n = (v_{2n-6}, v_{2n+3})$, $e_{n+1} = (v_{2n+2}, v_{2n+3})$. Similar to Case 1, select a joint-tree $\overline{T}_\sigma$ of $S^5_0$. After a sequence of Transform 4, the associated surface $S$ of $\overline{T}_\sigma$ has the form as

$$S = e_1e_2e_3e_4e_5e_1^{-1}e_6e_7e_2^{-1}e_3^{-1}e_8e_9e_4^{-1}e_{10}\{\prod_{i=1}^{j-2} e_{5i+1}^{-1}e_{5i+6}e_{5i+7}e_{5i+2}^{-1}e_{5i+3}^{-1}e_{5i+8}^{-1}e_{5i+9}^{-1}e_{5i+10}^{-1}e_{n-1}^{-1}e_{n-2}^{-1}e_{n-3}^{-1}e_{n-4}^{-1}e_{n-5}^{-1}e_{n-6}^{-1}e_{n-7}^{-1}e_{n-8}^{-1}e_{n-9}^{-1}\}^{[\frac{n+1}{2}]} e_{11}^{-1}e_{i2}^{-1}e_{i1}^{-1}e_{i2}^{-1}$$

where $e_{ij}, e_{ij}^{-1} \in \{e_1, \ldots, e_{n+1}, e_1^{-1}, \ldots, e_{n+1}^{-1}\}; i = 1, \ldots, \lfloor \frac{n+1}{2} \rfloor; j = 1, 2$. Obviously, $g(S) = \lfloor \frac{n+1}{2} \rfloor$. So, when $n = 5j + 2$, $S^5_0$ is upper embeddable.

Case 4  $n = 5j + 3$, where $j$ is an integer no less than 1.

Without loss of generality, a spanning tree $T$ of $S^5_0$ can be chosen as $T = T_1 \cup T_2$, where $T_1$ is the path $v_2v_1v_7v_6v_5v_4v_3\{\prod_{i=1}^{j} v_{10i+1}v_{10i}v_{10i-1}v_{10i-2}v_{10i-3}v_{10i-4}v_{10i+5}v_{10i+4}v_{10i+3}v_{10i+2}\}v_{2n+5}v_{2n+4}v_{2n+3}$, $T_2 = (v_{2n+1}, v_{2n+2})$. It is obviously that the $n + 1$ co-tree edges of $S^5_0$ with respect to $T$ are $e_1 = (v_1, v_5)$, $e_2 = (v_2, v_3)$, $e_3 = (v_2, v_9)$, $i=1^{j} (e_{5i} = (v_{10i-3}, v_{10i-2}), e_{5i+1} = (v_{10i-4}, v_{10i+5}), e_{5i+2} = (v_{10i-6}, v_{10i+3}), e_{5i+2} = (v_{10i+1}, v_{10i+2}), e_{5i+3} = (v_{10i+1}, v_{10i+9}), e_n = (v_{2n+2}, v_{2n+3})$. Similar to Case 1, select a joint-tree $\overline{T}_\sigma$ of $S^5_0$. After a sequence of Transform 4, the associated surface $S$ of $\overline{T}_\sigma$ has the form as

$$S = e_1e_2e_3e_4e_5e_1^{-1}e_6e_7e_2^{-1}\{\prod_{i=1}^{j-1} e_{5i+2}e_{5i+3}e_{5i+4}^{-1}e_{5i+5}^{-1}e_{5i+6}^{-1}e_{5i+7}^{-1}e_{5i+8}^{-1}e_{5i+9}^{-1}e_{5i+10}^{-1}e_{n-1}^{-1}e_{n-2}^{-1}e_{n-3}^{-1}e_{n-4}^{-1}e_{n-5}^{-1}\}^{[\frac{n+1}{2}]} e_{11}^{-1}e_{i2}^{-1}e_{i1}^{-1}e_{i2}^{-1}$$

where $e_{ij}, e_{ij}^{-1} \in \{e_1, \ldots, e_{n+1}, e_1^{-1}, \ldots, e_{n+1}^{-1}\}; i = 1, \ldots, \lfloor \frac{n+1}{2} \rfloor; j = 1, 2$. Obviously, $g(S) = \lfloor \frac{n+1}{2} \rfloor$. So, when $n = 5j + 3$, $S^5_0$ is upper embeddable.

Case 5  $n = 5j + 4$, where $j$ is an integer no less than 1.
Without loss of generality, a spanning tree $T$ of $S^n_5$ can be chosen as $T = T_1 \cup T_2 \cup T_3$, where $T_1$ is the path $v_1v_2\{\prod_{i=1}^{j-1}v_{10i-1}v_{10i-2}v_{10i-3}v_{10i-4}v_{10i-5}v_{10i-6}v_{10i+3}v_{10i+2}v_{10i+1}v_{10i}\}v_{2n+1}-v_{2n}v_{2n-1}v_{2n-2}\prod_{i=0}^{m-5}v_{2n+i}v_{2n+4}v_{2n+3}; T_2 = (v_2, v_3); T_3 = (v_{2n+1}, v_{2n+2})$. It is obviously that the $n+1$ co-tree edges of $S^n_5$ with respect to $T$ are $e_1 = (v_1, v_5), e_2 = (v_1, v_7), e_3 = (v_3, v_4), e_4 = (v_3, v_1), \prod_{i=1}^{j-1}(e_{5i} = (v_{10i-1}, v_{10i}), e_{5i+1} = (v_{10i-2}, v_{10i+7}, e_{5i+2} = (v_{10i-4}, v_{10i+5}), e_{5i+3} = (v_{10i+3}, v_{10i+4}), e_{5i+4} = (v_{10i+2}, v_{10i+11}), e_{5i+1} = (v_{2n+2}, v_{2n+3})$. Similar to Case 1, select a joint-tree $T_\sigma$ of $S^n_5$. After a sequence of Transform 4, the associated surface $S$ of $T_\sigma$ has the form as

$$S = e_2e_1e_3e_4e_5e_6−1e_7e_1−1e_3−1\{\prod_{i=1}^{j-1}e_{5i+3}e_{5i+4}e_{5i+1}−1e_{5i+5}e_{5i+6}−1e_{5i+7}e_{5i+2}−1e_{5i+3}\}e_{n−1}e_{n−5}e_{n−4}e_{n−3}e_{n−2}e_{n−1}e_{n}−1e_{n+1}e_{n+1}−1e_{n}−1 \sim \prod_{i=1}^{n+1}e_{i1}e_{i2}−1e_{i1}−1,$$

where $e_{ij}, e_{ij}^{-1} \in \{e_1, \ldots, e_{n+1}, e_1^{-1}, \ldots, e_{n+1}^{-1}\}; i, 1, \ldots, \lfloor n+1/2 \rfloor; j, 1, 2$. Obviously, $g(S) = \lfloor n+1/2 \rfloor$. So, when $n = 5j + 4$, $S^n_5$ is upper embeddable.

From the Case 1-5, the upper embeddability of $S^n_5$ can be obtained.

Similar to the Case 1-5, for each $n \geq 5$, there exists a joint-tree $T_\sigma$ of $S^n_5 - v_{2n+3}$ such that its associated surface is $S' = S - \{e_{n+1}e_{n}^{-1}e_{n+1}^{-1}\}$. It is obvious that $S'$ is the surface into which the embedding of $S^n_5 - v_{2n+3}$ is the maximum genus embedding. Furthermore, $g(S') = g(S) - 1$, i.e., $\gamma_M(S^n_5 - v_{2n+3}) = \gamma_M(S^n_5) - 1$. So, $v_{2n+3}$ is a 1-critical-vertex of $S^n_5$.

Similar to the proof of Theorem 3.1, we can get the following conclusions.

**Theorem 3.2** The graph $S^n_m$ is upper embeddable. Furthermore, $\gamma_M(S^n_m - v_{m+2n-2}) = \gamma_M(S^n_m) - 1$, i.e., $v_{m+2n-2}$ is a 1-critical-vertex of $S^n_m$.

**Corollary 3.1** Let $G$ be a graph with minimum degree at least three. If $G$, through a sequence of vertex-splitting operations, can be turned into a spiral $S^n_m$, then $G$ is upper embeddable.

**Proof** According to Lemma 1.3, Theorem 3.2, and the upper embeddability of graphs, Corollary 3.1 can be obtained.

In the following, we will offer an algorithm to obtain the maximum genus of the extended-spiral $S^n_m$.

**Algorithm II:**

1. **Step 1** Input $i = 0$ and $j = 0$. Let $G_0$ be the extended-spiral $S^n_m$.
2. **Step 2** If there is a $\gamma$-vertex $v$ in $G_i$, then delete $v$ from $G_i$, and go to Step 3. Else, go to Step 4.
3. **Step 3** Deleting all the vertices of degree one and merging some vertices of degree two in $G_i - v$, we get a new graph $G_{i+1}$. Let $i = i + 1$. If $G_i$ is a spiral $S^n_m$, then go to Step 4. Else,
go back to Step 2.

**Step 4** Let $G_{i+j}$ be the spiral $S_n^m$. Deleting $v_{m+2n-2}$ from $S_n^m$, we will get a new graph $G_{i+j+1}$, (obviously, $G_{i+j+1}$ is either a spiral $S_n^{m-2}$ or a cactus).

**Step 5** If $G_{i+j+1}$ is a cactus, then go to Step 6. Else, Let $n = n - 2$, $j = j + 1$ and go back to Step 4.

**Step 6** Output $\gamma_M(\tilde{S}_n^m) = i + j + 1$.

**Remark** 1. In the graph $G$ depicted by Fig.6, after deleting a $\gamma$-vertex $v_1$ (or $v_2$) from $G$, the vertex $v_3$ (or $v_4$) is still a $\gamma$-vertex of the remaining graph.

2. From Algorithm II we can get that the extended-spiral $\tilde{S}_n^m$ is upper embeddable.

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