A solution to the incompressible Navier-Stokes Equation

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Abstract: For the \( N \geq 2 \) dimensional incompressible Navier-Stokes Equation:

\[
\begin{align*}
\frac{\partial v}{\partial t} + v \cdot \nabla v &= - \nabla p + \nu \Delta v \\
\operatorname{div} v &= 0 \\
v|_{t=0} &= v_0
\end{align*}
\]

We have got its solution as \( v = \sum_{n=0}^{\infty} a_n(x) t^n \), where \( a_0(x) = v_0(x) \) and \( a_n(x) = \frac{1}{n} \nu \Delta a_{n-1}(x) - \frac{1}{n} p \left[ \sum_{i=0}^{n-1} (a_i(x) \cdot \nabla a_{n-1-i}(x)) \right] \) are all known functions determined only by \( v_0(x) \). Moreover, the series \( \sum_{n=0}^{\infty} a_n(x) t^n \) is proved to be convergent on \((x, t) \in \mathbb{R}^N \times [0, \infty)\), so that the existence and smoothness problem are solved.

Key Word: The Navier-Stokes Equation; the Euler Equation; the existence and smoothness problem; fluid dynamics; Millennium Problems

1. Introduction

Suppose \( N \geq 2 \) is a integer, then the \( N \) dimensional incompressible Navier-Stokes equation and the Euler equation are as following:

\[
\begin{align*}
\frac{\partial v}{\partial t} + v \cdot \nabla v &= - \nabla p + \nu \Delta v \\
\operatorname{div} v &= 0 \\
v|_{t=0} &= v_0
\end{align*}
\]
Where \( \nu(x, t) = (\nu_1(x, t), \ldots, \nu_N(x, t)) \in \mathbb{R}^N \) is the unknown velocity, \( p(x, t) \in \mathbb{R} \) is the unknown pressure function, they are all defined for position \( x = (x_1, \ldots, x_N) \in \mathbb{R}^N \) and time \( t \geq 0 \). \( \nu_0(x) = (\nu_{1,0}(x), \ldots, \nu_{N,0}(x)) \) is a given, \( C^\infty \) divergence-free vector field on \( \mathbb{R}^N \), it is the initial velocity; \( \Delta = \sum \frac{\partial^2}{\partial x^2} \) is the Laplacian in the space variables; \( \nu > 0 \) is a positive coefficient (the viscosity), and the Euler equations are the above equations with \( \nu = 0 \).

For physically reasonable solutions, it is supposed that:

\[
(p, \nu) \in C^\infty[\mathbb{R}^N \times [0, \infty)]
\]

(1.4)

and \( \nu(x, t) \) doesn’t grow large as \( |x| \to \infty \). Hence, the initial condition \( \nu_0 \) is restricted to satisfy

\[
|\partial_x^\alpha \nu_0(x)| \leq C_{\alpha K} (1 + |x|)^{-K}
\]

(1.5)

on \( \mathbb{R}^N \) for any multi-index \( \alpha \) and \( K > 0 \)

And there is a constant \( C \) such that:

\[
\int_{\mathbb{R}^N} |\nu(x, t)|^2 \, dx < C
\]

(1.6)

If there is a \((p, \nu)\) satisfying (1.1)-(1.6), we say that the Navier-Stokes equation has a smooth solution; otherwise, there are blow-ups in the equation.

Solutions to the Navier–Stokes equations are very important in many practical applications. However, theoretical understanding of the solutions to these equations are incomplete. In particular, solutions of the Navier–Stokes equations often include turbulence, which remains one of the greatest unsolved problems in physics, despite its immense importance in science and engineering.

By now, in the case of \( N = 2 \), the Navier–Stokes problem was solved by the 1960s, there exist smooth and globally defined solutions\(^2\).
However, in the case of $N = 3$, it's very much complicated. Now, there are the following results: 1) If the initial velocity $v_0(x)$ is sufficiently small, then there are smooth and globally defined solutions to the Navier–Stokes equations\cite{1}; 2) for any initial $v_0(x)$, there exists a finite time $T > 0$, such that the Navier–Stokes equation have smooth solutions in $(x, t) \in \mathbb{R}^N \times [0, T)$. It is not known if the solutions exist beyond that "blowup time" $T$\cite{1}.

For general cases, people have neither proved that smooth solutions always exist, nor found any counter-examples. This is called the Navier–Stokes existence and smoothness problem. The Clay Mathematics Institute in May 2000 made this problem one of its seven Millennium Prize problems in mathematics\cite{1}.

This paper will study the existence and smoothness problem in the general $N \geq 2$ dimensional space $\mathbb{R}^N$, and it will be written in six chapters:

(1) Introduction: A brief introduction to the equations and their current states;

(2) Preliminary knowledge: Some elementary results that will be used in following chapters;

(3) Existence of a solution $v$ in some time interval $[0, T]$: To discuss the well-known solution $v$ in a time interval $t \in [0, T]$;

(4) The solution $v$ in $t \in [0, T]$ is a power series of $t$: To prove that the well-known solution $v$ in $t \in [0, T]$ is indeed a power series of $t$, and it's convergent in $t \in [0, T]$;

(5) The solution series is convergent on $(x, t) \in \mathbb{R}^N \times [0, \infty)$: To prove that the series is convergent globally, so that the famous existence and smoothness problem are solved;

(6) Conclusions.
Since the book by A. Bertozzi and A. Majda (Cambridge U. Press, 2002) is fundamental in the study of Navier-Stokes equations, we make use of its notations in our paper. Moreover, the contents of chapter 2 and chapter 3 are mostly from this book.

2. Preliminary knowledge

2.1. Calculus Inequalities for Sobolev Spaces

The Sobolev space $H^m(\mathbb{R}^N), m \in \mathbb{Z}^+ \cup \{0\}$, consists of functions $v \in L^2(\mathbb{R}^N)$ such that $D^\alpha v \in L^2(\mathbb{R}^N), 0 \leq |\alpha| \leq m$, where $D^\alpha$ is the distribution derivative $D^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$.

Let $\|v\|_0 = (\int_{\mathbb{R}^N} |v|^2 \, d\xi)^{1/2}$, then the $H^m$ norm, denoted as $\|\cdot\|_m$, is

$$\|v\|_m = \left( \sum_{0 \leq |\alpha| \leq m} \|D^\alpha v\|_0^2 \right)^{1/2} \tag{2.1}$$

The Sobolev space $H^m$ can generalize to the case $s \in \mathbb{R}$. Consider the functional $\|\cdot\|_s : S(\mathbb{R}^N) \to \mathbb{R}^+ \cup \{0\}$ defined by

$$\|v\|_s = \left( \int_{\mathbb{R}^N} (1 + |\xi|^2)^s |\hat{v}(\xi)|^2 \, d\xi \right)^{1/2} \tag{2.2}$$

acting on the Schwarz space of rapidly decreasing smooth functions $S(\mathbb{R}^N)$. Here $\hat{v}$ denotes the Fourier transform of $v$. The Sobolev space $H^s(\mathbb{R}^N), s \in \mathbb{R}$, is the completion of $S(\mathbb{R}^N)$ with respect to the norm $\|\cdot\|_s$. For $s = m$ the two norms are equivalent.

There are many useful properties of Sobolev spaces, we just list them as following:

**Lemma 2.1.** Sobolev Inequality. The space $H^{s+k}(\mathbb{R}^N), s > N/2, k \in \mathbb{Z}^+ \cup \{0\}$, is continuously embedded in the space $C^k(\mathbb{R}^N)$. That is, there exists $c > 0$ such that

$$|v|_{C^k} \leq c \|v\|_{s+k} \quad \forall v \in H^{s+k}(\mathbb{R}^N) \tag{2.3}$$
Lemma 2.2. Interpolation in Sobolev Spaces. (Adams, 1995). Given \( s > 0 \), there exists a constant \( C_s > 0 \), so that for all \( v \in H^s(\mathbb{R}^N) \) and \( 0 < s' < s \),

\[
\| v \|_{s'} \leq C_s \| v \|_0^{1-s'/s} \| v \|_s^{s'/s}
\] (2.4)

Lemma 2.3. Calculus Inequalities in the Sobolev Spaces.
(i) For all \( m \in \mathbb{Z}^+ \cup \{0\} \), there exists \( c > 0 \) such that, for all \( u, v \in L^\infty \cap H^m(\mathbb{R}^N) \),

\[
\| uv \|_m \leq c \{ |u|_{L^\infty} \| D^m v \|_0 + \| D^m u \|_0 |v|_{L^\infty} \},
\] (2.5)

\[
\sum_{0 \leq |k| \leq m} \| D^k (uv) - uD^k v \|_0 \leq c \left\{ \| \nabla u \|_{L^\infty} \| D^{m-1} v \|_0 + \| D^m u \|_0 |v|_{L^\infty} \right\}
\] (2.6)

(ii) For all \( s > (N/2), H^s(\mathbb{R}^N) \) is a Banach algebra. That is, there exists \( c > 0 \) such that, for all \( u, v \in H^s(\mathbb{R}^N) \),

\[
\| uv \|_s \leq c \| u \|_s \| v \|_s
\] (2.7)

2.2. Properties of Mollifiers

Given any radial function

\[
\rho(|x|) \in C^\infty(\mathbb{R}^N), \quad \rho \geq 0, \quad \int_{\mathbb{R}^N} \rho \, dx = 1
\] (2.8)

define the mollification \( J_\varepsilon v \) of functions \( v \in L^p(\mathbb{R}^N), 1 \leq p \leq \infty \), by

\[
(J_\varepsilon v)(x) = \varepsilon^{-N} \int_{\mathbb{R}^N} \rho \left( \frac{x-y}{\varepsilon} \right) v(y) \, dy, \quad \varepsilon > 0
\] (2.9)

Mollifiers have several well-known properties (see, for example, Taylor, 1991).

Lemma 2.4. Properties of Mollifiers. Let \( J_\varepsilon \) be the mollifier defined in (2.9), then \( J_\varepsilon \) is a \( C^\infty \) function and

(i) for all \( v \in C^0(\mathbb{R}^N) \), \( J_\varepsilon v \rightharpoonup v \) uniformly on any compact set \( \Omega \) in \( \mathbb{R}^N \) and

\[
|J_\varepsilon v|_{L^\infty} \leq |v|_{L^\infty}
\] (2.10)

(ii) Mollifiers commute with distribution derivatives,

\[
D^\alpha J_\varepsilon v = J_\varepsilon D^\alpha v, \quad \forall |\alpha| \leq m, \quad v \in H^m
\] (2.11)
(iii) For all \( u \in L^p(\mathbb{R}^N) \), \( v \in L^q(\mathbb{R}^N) \), \((1/p) + (1/q) = 1\),

\[
\int_{\mathbb{R}^N} (J_\epsilon u) vdx = \int_{\mathbb{R}^N} u (J_\epsilon v)dx \tag{2.12}
\]

(iv) For all \( v \in H^s(\mathbb{R}^N) \), \( J_\epsilon v \) converges to \( v \) in \( H^s \) and the rate of convergence in the \( H^{s-1} \) norm is linear in \( \epsilon \):

\[
\lim_{\epsilon \to 0} \|J_\epsilon v - v\|_s = 0, \tag{2.13}
\]

\[
\|J_\epsilon v - v\|_{s-1} \leq C \epsilon \|v\|_s. \tag{2.14}
\]

(v) For all \( v \in H^m(\mathbb{R}^N) \), \( k \in \mathbb{Z}^+ \cup \{0\} \), and \( \epsilon > 0 \),

\[
\|J_\epsilon v\|_{m+k} \leq \frac{c_{mk}}{\epsilon^k} \|v\|_m, \tag{2.15}
\]

\[
\|J_\epsilon D^k v\|_{L^m} \leq \frac{c_k}{\epsilon^{N/2+k}} \|v\|_0. \tag{2.16}
\]

### 2.3. The Hodge Decomposition in \( H^m \)

**Lemma 2.5** Every vector field \( v \in H^m(\mathbb{R}^N) \), \( m \in \mathbb{Z}^+ \cup \{0\} \), has the unique orthogonal decomposition

\[
v = w + \nabla \varphi
\]

such that the Leray's projection operator \( P v = w \) on the divergence-free functions satisfies

(i) \( P v, \nabla \varphi \in H^m \), \( \int_{\mathbb{R}^N} P v \cdot \nabla \varphi dx = 0 \), \( \text{div} \ P v = 0 \), and

\[
\|P v\|^2_m + \|\nabla \varphi\|^2_m = \|v\|^2_m \tag{2.17}
\]

(ii) \( P \) commutes with the distribution derivatives,

\[
P D^\alpha v = D^\alpha P v, \quad \forall v \in H^m, \quad |\alpha| \leq m \tag{2.18}
\]

(iii) \( P \) commutes with mollifiers \( J_\epsilon \),

\[
P(J_\epsilon v) = J_\epsilon (P v), \quad \forall v \in H^m, \quad \epsilon > 0 \tag{2.19}
\]

(iv) \( P \) is symmetric,

\[
(Pu, v)_m = (u, P v)_m \tag{2.20}
\]

### 2.4. The Picard Theorem
**Theorem 2.1.** Picard Theorem on a Banach Space. Let $O \subseteq B$ be an open subset of a Banach space $B$, and $\|\cdot\|_B$ denotes a norm of $B$. Let $F: O \to B$ be a mapping that satisfies the following conditions:

(i) $F(X)$ maps $O$ to $B$.
(ii) $F$ is locally Lipschitz continuous, i.e., for any $X \in O$ there exists $L > 0$ and an open neighborhood $U_X \subset O$ of $X$ such that

\[ \|F(X) - F(\tilde{X})\|_B \leq L \|X - \tilde{X}\|_B \quad \text{for all} \quad X, \tilde{X} \in U_X \]

Then for any $X_0 \in O$ and $\|X_0\|_B \neq 0$, there exists a time $T$ such that the ODE

\[ \frac{dX}{dt} = F(X), \quad X|_{t=0} = X_0 \in O \]

has a unique (local) solution $X \in C^1[[0, T]; O]$.

**Proof.** Since $U_{X_0} \subset O$ is an open set, hence $\exists 0 < d < 1$, such that $U_{X_0'} = \{X \in U_{X_0'} | \|X - X_0\|_B \leq d\|X_0\|_B \} \subset U_{X_0}$. It can be checked that when $X \in U_{X_0}$, there is $(1 - d)\|X_0\|_B \leq \|X\|_B \leq (1 + d)\|X_0\|_B$. Since $F$ is Lipschitz continuous in $U_{X_0}$, we can define the local norm of $F$ as following:

\[ \|F\|_D \triangleq \sup_{X \in U_{X_0}} \frac{\|F(X)\|_B}{\|X\|_B} \]

Let

\[ T = \frac{d}{\|F\|_D \times (1 + d)} \]

Then, we can build the Picard iteration sequence as following:

\[ X_0(t) = X_0 \]

\[ X_{n+1}(t) = X_0 + \int_0^t F(X_n(s))ds, \quad (n \geq 0) \]

(2.21)

It’s easy to see that if $t \in [0, T]$, then $X_n \in \tilde{U}_{X_0'}$ ($n \geq 0$). Hence, the Lipschitz condition can be used.

Take $M = (1 + d)\|X_0\|_B$. Clearly, we have $\|X_1\|_B \leq M$ and $\|X_1(t) - X_0(t)\|_B \leq M$. 

Since \( X_{n+1}(t) - X_n(t) = \int_0^t (F(X_n(s)) - F(X_{n-1}(s))) ds \), we have

\[
\|X_{n+1}(t) - X_n(t)\|_B \leq L \int_0^t \|X_n(s) - X_{n-1}(s)\|_B ds
\]

By induction, we have

\[
\|X_{n+1}(t) - X_n(t)\|_B \leq \frac{ML^n n^n}{n!} (n \geq 1) \tag{2.22}
\]

Since

\[
X_n = X_1 + \sum_{i=1}^{n-1} [X_{i+1}(t) - X_i(t)]
\]

It can be proved that \( \{X_n\} (n \geq 1) \) is a Cauchy sequence under \( \|\|_B \) so that

\[
X_1 + \sum_{n=1}^{\infty} [X_{n+1}(t) - X_n(t)] = X(t)
\]

is uniformly convergent in \( t \in [0, T] \). That is

\[
X(t) = \lim_{n \to \infty} X_n(t) \tag{2.23}
\]

exist uniformly. By the theory of Banach Space, we can know that \( X(t) \in O \) is a solution to the ODE in \( t \in [0, T] \), and

\[
\|X_n(t) - X(t)\|_B \leq \frac{ML^n n^n}{n!} \tag{2.24}
\]

The Picard Theory plays very important role in this paper, each time it is utilized, the constant \( T \) should be carefully chosen so that the Picard iteration sequence can be rightly built.

3. Existence of a solution \( \nu \) in some time interval \([0,T]\)

It’s well known that there are local-in-time existence solutions for the Navier-Stokes equation and the Euler equation. In this chapter, we will discuss this result.

3.1 A regularization of the equations
The equations in (1.1), (1.2) and (1.3) can be rewritten as following:

\[ \begin{align*}
    v_t + v \cdot \nabla v &= - \nabla p + \nu \Delta v \\
    \text{div } v &= 0 \\
    v|_{t=0} &= v_0
\end{align*} \]  

(3.1)

The Picard theorem is a central tool in dealing with them. However, as the equations contain unbounded operators so that we cannot directly apply the Picard theorem for ODEs in a Banach space. A strategy is to design an approximation of the equations by regularization for which people can easily prove a global existence of solutions, and can also show an analogous energy estimate that is independent of the regularization parameter.

An approximate equation to the Navier-Stokes equation has been created, in which the mollifier \( J_\varepsilon \) defined in (2.9) is used to regularize the equations, so that the new equation can satisfy the conditions of the Picard theorem.

\[ \begin{align*}
    v^\varepsilon_t + J_\varepsilon \left( (J_\varepsilon v^\varepsilon) \cdot \nabla (J_\varepsilon v^\varepsilon) \right) &= - \nabla p^\varepsilon + \nu J_\varepsilon (J_\varepsilon \Delta v^\varepsilon) \\
    \text{div } v^\varepsilon &= 0 \\
    v^\varepsilon|_{t=0} &= v_0
\end{align*} \]  

(3.2)

Equations (3.2) explicitly contain the pressure \( p^\varepsilon \). Following Leray, we can eliminate \( p^\varepsilon \) and the incompressibility condition \( \text{div } v^\varepsilon = 0 \) by projecting these equations onto the space of divergence-free functions:

\[ V^s = \{ v \in H^s(\mathbb{R}^N) : \text{div } v = 0 \}. \]  

(3.3)

Because the Leray projection operator \( P \) commutes with derivatives and mollifiers (Lemma 2.5) and \( P v^\varepsilon = v^\varepsilon \), we have

\[ v^\varepsilon_t + P J_\varepsilon \left( (J_\varepsilon v^\varepsilon) \cdot \nabla (J_\varepsilon v^\varepsilon) \right) = \nu J_\varepsilon^2 \Delta v^\varepsilon \]  

(3.4)

The regularized Euler or Navier-Stokes equation in (3.2) reduces to an ODE in the Banach space \( V^s \):

\[ \frac{dv^\varepsilon}{dt} = F_\varepsilon (v^\varepsilon) \]

\[ v^\varepsilon|_{t=0} = v_0 \]  

(3.5)

where
\[ F_\varepsilon(v^\varepsilon) = \nu \partial^2_{\varepsilon} v - P J_\varepsilon [(J_\varepsilon v^\varepsilon) \cdot \nabla (J_\varepsilon v^\varepsilon)] = F^{(1)}_\varepsilon (v^\varepsilon) - F^{(2)}_\varepsilon (v^\varepsilon). \]  \hfill (3.6)

### 3.2 Global existence of regularized equations

**Theorem 3.1.** Local existence of solutions to the regularized equations.

Consider an initial condition \( v_0 \in V^m, m \in \mathbb{Z}^+ \cup \{0\} \). Then

(i). for any \( \varepsilon > 0 \) there exists the unique solution \( v^\varepsilon \in C^1([0,T_\varepsilon);V^m) \) to the ODE in (3.5), where \( T_\varepsilon = T(\|v_0\|_m, \varepsilon) \);

(ii). on any time interval \([0,T]\) on which this solution belongs to \( C^1([0,T];V^0)\),

\[ \sup_{0 \leq t \leq T} \|v^\varepsilon\|_0 \leq \|v_0\|_0 \]  \hfill (3.7)

**Proof.**

(i). Firstly, we will show that the function \( F_\varepsilon \) in (3.6) maps \( V^m \) into \( V^m \) and is locally Lipschitz continuous.

Note that \( F_\varepsilon: V^m \rightarrow V^m \) because \( \text{div} v^\varepsilon = 0, P \) maps into divergence-free vector fields, and \( J_\varepsilon \) commutes with derivatives.

The definition of Sobolev spaces and estimate (2.15) for mollifiers implies that

\[ \|F^{(1)}_\varepsilon(v^{(1)}) - F^{(1)}_\varepsilon(v^{(2)})\|_m \leq \nu \|\partial^2_{\varepsilon} (v^{(1)} - v^{(2)})\|_m \]
\[ \leq \nu \|\partial^2_{\varepsilon} (v^{(1)} - v^{(2)})\|_{m+2} \]
\[ \leq \frac{\nu}{\varepsilon^2} \|v^{(1)} - v^{(2)}\|_m. \]

Calculus inequality (2.5) and commutation property (2.19) of \( P \) and \( J_\varepsilon \) imply that

\[ \|F^{(2)}_\varepsilon(v^{(1)}) - F^{(2)}_\varepsilon(v^{(2)})\|_m \leq \|PJ_\varepsilon ((J_\varepsilon v^{(1)}) \cdot \nabla [J_\varepsilon (v^{(1)} - v^{(2)})])\|_m + \|PJ_\varepsilon [(J_\varepsilon (v^{(1)} - v^{(2)})) \cdot \nabla (J_\varepsilon v^{(2)})]\|_m \]
\[ \leq c \left\{ \|J_\varepsilon v^{(1)}\|_\infty \|\partial^m J_\varepsilon \nabla J_\varepsilon (v^{(1)} - v^{(2)})\|_0 + \|\partial^m J_\varepsilon v^{(1)}\|_0 \|J_\varepsilon \nabla (v^{(1)} - v^{(2)})\|_0 \right\} \]
\[ + \left| J_\varepsilon (v^{(1)} - v^{(2)}) \right|_{L^\infty} \|\partial^m J_\varepsilon \partial^m \nabla v^{(2)}\|_0 + \|\partial^m J_\varepsilon (v^{(1)} - v^{(2)})\|_0 \|J_\varepsilon \partial^m \nabla v^{(2)}\|_0 \]

Mollifier properties (2.15) and (2.16) then give

\[ \|F^{(2)}_\varepsilon (v^{(1)}) - F^{(2)}_\varepsilon (v^{(2)})\|_m \leq \frac{c}{\varepsilon^{N/2 + 1 + m}} (\|v^{(1)}\|_0 + \|v^{(2)}\|_0) \|v^{(1)} - v^{(2)}\|_m. \]

The final result is
so that \( F_\varepsilon \) is locally Lipschitz continuous on any open set \( O^M = \{ v \in V^m | \| v \|_m < M \} \).

(ii). Secondly, we will prove that there is a \( T_\varepsilon > 0 \), such that there is a solution \( v^\varepsilon \in C^1 ([0, T_\varepsilon); V^m) \) to the ODE in (3.5).

In (3.8), let \( v^{(1)} = v \), \( v^{(2)} = 0 \), we can get

\[
\| F_\varepsilon (v) \|_m \leq \left( \frac{CV}{\varepsilon^2} + \frac{c}{\varepsilon^{N/2+1+m}} \| v \|_0 \right) \| v \|_m \leq \left( \frac{CV}{\varepsilon^2} + \frac{c}{\varepsilon^{N/2+1+m}} \| v \|_m \right) \| v \|_m
\]

Let \( M = 2 \| v_0 \|_m \), and \( O^M = \{ v \in V^m | \| v \|_m \leq M \} \), then \( F_\varepsilon (v) \) is Lipschitz continuous in \( O^M \). Let

\[
T_\varepsilon = (M - \| v_0 \|_m) / \left[ \left( \frac{CV}{\varepsilon^2} + \frac{c}{\varepsilon^{N/2+1+m}} M \right) M \right]
\]

Now, we can build the Picard iteration sequence:

\[
v^\varepsilon_0 = v_0
\]

\[
v^\varepsilon_{n+1} = v_0 + \int_0^t F_\varepsilon (v^\varepsilon_n) ds, (n \geq 0)
\]

It is easy to see that when \( t \leq T_\varepsilon \), we have \( v^\varepsilon_n \in O^M, (n \geq 0) \).

By the Picard theorem, \( v^\varepsilon_n \) converges to a divergence free function

\[
v^\varepsilon = \lim_{n \to \infty} v^\varepsilon_n
\]

It’s the unique solution to (3.2) in \( t \in [0, T_\varepsilon] \). It’s easy to show that \( v^\varepsilon \in C^1 ([0, T_\varepsilon); V^m \cap O^M), m \in \mathbb{Z}^+ \cup \{0\} \).

(iii). Finally, we will prove energy bound (3.7): if \( v^\varepsilon \in C^1 ([0, T); V^0) \), then

\[
\sup_{0 \leq t \leq T} \| v^\varepsilon \|_0 \leq \| v_0 \|_0
\]

Take the \( L^2 \) inner product of (3.5) with \( v^\varepsilon \) to obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |v^\varepsilon|^2 \, dx = \nu \int_{\mathbb{R}^3} v^\varepsilon \mathcal{J}_\varepsilon \Delta v^\varepsilon \, dx - \int_{\mathbb{R}^3} v^\varepsilon \partial_j \mathcal{J}_\varepsilon (\mathcal{J}_\varepsilon v^\varepsilon) \cdot \nabla (\mathcal{J}_\varepsilon v^\varepsilon) \, dx
\]
The properties of mollifiers and the operator $P$ from Lemmas 2.4 and 2.5 imply, after integration by parts, that
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} (v^\epsilon)^2 \, dx = v \int_{\mathbb{R}^3} (J_\epsilon v^\epsilon) \Delta (J_\epsilon v^\epsilon) \, dx + \frac{1}{2} \int_{\mathbb{R}^3} (J_\epsilon v^\epsilon) \cdot \nabla (J_\epsilon v^\epsilon)^2 \, dx
\]
\[
= -v \int_{\mathbb{R}^3} (J_\epsilon \nabla v^\epsilon)^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^3} \text{div}(J_\epsilon v^\epsilon) (J_\epsilon v^\epsilon)^2 \, dx
\]
so that
\[
\frac{d}{dt} \|v^\epsilon\|_0^2 + 2v \|\nabla J_\epsilon v^\epsilon\|_0^2 = 0
\]
Because $v \geq 0$, we obtain energy bound (3.7).
#.

**Theorem 3.2.** Global Existence of Regularized Solutions.

Given an initial condition $v_0 \in V^m, m \in \mathbb{Z}^+ \cup \{0\}$, for any $\epsilon > 0$ there exists for all time a unique solution $v^\epsilon \in C^1([0, \infty); V^m)$ to regularized equation (3.5).

**Proof:** We will extend the local solution $v^\epsilon$ globally. Suppose $T > 0$ is finite, and it's the maximal time that admit a local solution $v^\epsilon$ to the regularized equation (3.5) in $(x, t) \in \mathbb{R}^N \times [0, T]$. Firstly, we can calculate a priori bound on $\|v^\epsilon(\cdot, t)\|_m$.

By using (3.8), and let $v^\epsilon(\cdot, t) = v^{(1)}(x, t), v^{(2)}(x, t) = 0$, there is:
\[
\frac{d}{dt} \|v^\epsilon(\cdot, t)\|_m \leq \left(\frac{c}{\epsilon^2} + \frac{c}{\epsilon^{N/2+1+m}} \|v^\epsilon\|_0\right) \|v^\epsilon\|_m \leq \left(\frac{c}{\epsilon^2} + \frac{c}{\epsilon^{N/2+1+m}} \|v_0\|_0\right) \|v^\epsilon\|_m
\]
Then, by the Grönwall's lemma, there is a priori bound $\|v^\epsilon(\cdot, T)\|_m \leq e^{cT}$.

Now, lets consider the regularized equation with $v^\epsilon(\cdot, T)$ as new initial condition:
\[
\frac{dv^\epsilon}{dt} = F_\epsilon(v^\epsilon)
\]
\[v^\epsilon|_{t=0} = v^\epsilon(\cdot, T)
\]
There will exist a $T'^{\ast} > 0$ and a local solution $\tilde{v}^\epsilon$ in $t \in [0, T']$. With this $T'$ and $\tilde{v}^\epsilon$, $v^\epsilon$ can then extend to $(x, t) \in \mathbb{R}^N \times [0, T + T'^{\ast}]$, this contradicts the assumption.

So that, we have extended $v^\epsilon$ globally to $(x, t) \in \mathbb{R}^N \times [0, \infty)$.
#.
3.3. Existence of Solutions to the Navier-Stokes Equation in a time interval

**Proposition 3.1.** The $H^m$ Energy Estimate. Let $v_0 \in V^m$. Then the unique regularized solution $v^\varepsilon \in C^1([0,\infty); V^m)$ to Equation (3.5) satisfies

$$
\frac{d}{dt} \frac{1}{2} \|v^\varepsilon\|^2_m + \nu \|J_\varepsilon \nabla v^\varepsilon\|^2_m \leq c_m \|\nabla J_\varepsilon v^\varepsilon\|_{L^\infty} \|v^\varepsilon\|^2_m \tag{3.9}
$$

**Proof.** Let $v^\varepsilon$ be a smooth solution to Equation (3.5):

$$
v_t^\varepsilon = \nu J_\varepsilon^2 \Delta v^\varepsilon - PJ_\varepsilon \left( (J_\varepsilon v^\varepsilon) \cdot \nabla (J_\varepsilon v^\varepsilon) \right)
$$

In particular, we used integration by parts by means of the divergence theorem. Following a similar procedure, we take the derivative $D^\alpha, |\alpha| \leq m$ of this equation and then the $L^2$ inner product with $D^\alpha v^\varepsilon$:

$$
(D^\alpha v_t^\varepsilon, D^\alpha v^\varepsilon) = (D^\alpha J_\varepsilon^2 \Delta v^\varepsilon, D^\alpha v^\varepsilon) - \{D^\alpha P J_\varepsilon \left[ (J_\varepsilon v^\varepsilon) \cdot \nabla (J_\varepsilon v^\varepsilon) \right], D^\alpha v^\varepsilon \}
$$

$$
= -\nu \|J_\varepsilon D^\alpha \nabla v^\varepsilon\|_0^2 - \{PJ_\varepsilon [(J_\varepsilon v^\varepsilon) \cdot \nabla (D^\alpha J_\varepsilon v^\varepsilon)], D^\alpha v^\varepsilon \}
$$

$$
- \{D^\alpha P J_\varepsilon \left[ (J_\varepsilon v^\varepsilon) \cdot \nabla (J_\varepsilon v^\varepsilon) \right] - PJ_\varepsilon \left[ (J_\varepsilon v^\varepsilon) \cdot \nabla (D^\alpha J_\varepsilon v^\varepsilon) \right], D^\alpha v^\varepsilon \}
$$

Lemmas 2.4 and 2.5 and the divergence theorem imply that

$$
\{PJ_\varepsilon \left[ (J_\varepsilon v^\varepsilon) \cdot \nabla (D^\alpha J_\varepsilon v^\varepsilon) \right], D^\alpha v^\varepsilon \} = \frac{1}{2} \left[ J_\varepsilon v^\varepsilon, \nabla (J_\varepsilon D^\alpha v^\varepsilon) \right]^2
$$

$$
= -\frac{1}{2} \langle \text{div} J_\varepsilon v^\varepsilon, |J_\varepsilon D^\alpha v^\varepsilon|^2 \rangle = 0.
$$

Summing over $|\alpha| \leq m$, we find that calculus inequality (2.6) implies that

$$
\frac{1}{2} \frac{d}{dt} \|v^\varepsilon\|^2_m + \nu \|J_\varepsilon \nabla v^\varepsilon\|^2_m \leq c_m \|J_\varepsilon \nabla v^\varepsilon\|_{L^\infty} \|v^\varepsilon\|^2_m
$$

so that

$$
\frac{d}{dt} \frac{1}{2} \|v^\varepsilon\|^2_m + \nu \|J_\varepsilon \nabla v^\varepsilon\|^2_m \leq c_m \|J_\varepsilon \nabla v^\varepsilon\|_{L^\infty} \|v^\varepsilon\|^2_m
$$
Theorem 3.3. Local-in-Time Existence of Solutions to the Navier-Stokes equations.

Given an initial condition \( v_0 \in V^m, m \geq \left\lceil \frac{N}{2} \right\rceil + 2 \), then

(i) there exists a time \( T \) with the rough upper bound

\[
T \leq \frac{1}{c_m \| v_0 \|_m},
\]

such that for any viscosity \( 0 \leq \nu < \infty \) there exists the unique solution \( v^\nu \in C([0, T]; L^2(\mathbb{R}^N)) \cap C^1([0, T]; L^1(\mathbb{R}^N)) \) to the Euler or the Navier-Stokes equation. The solution \( v^\nu \) is the limit of a subsequence of approximate solutions, \( v^\varepsilon \), of Equation (3.5) and (3.6).

(ii) The approximate solutions \( v^\varepsilon \) and the limit \( v^\nu \) satisfy the higher-order energy estimates

\[
\sup_{0 \leq t \leq T} \| v^\varepsilon \|_m \leq \frac{\| v_0 \|_m}{1 - c_m T \| v_0 \|_m}, \tag{3.10}
\]

\[
\sup_{0 \leq t \leq T} \| v^\nu \|_m \leq \frac{\| v_0 \|_m}{1 - c_m T \| v_0 \|_m}. \tag{3.12}
\]

Proof. The strategy for the local-existence proof, is to first prove the bounds (3.10) in the high norm, then show that we actually have a contraction in the \( H^0 = L^2 \) norm. We then apply an interpolation inequality to prove convergence as \( \varepsilon \to 0 \).

(1). Firstly, we will show that the family \( (v^\varepsilon) \) of regularized solutions is uniformly bounded in \( H^m \). Energy estimate (3.9) and Sobolev inequality (2.3) imply that the time derivative of \( \| v^\varepsilon \|_m \) can be bounded by a quadratic function of \( \| v^\varepsilon \|_m \) independent of \( \varepsilon \), provided that \( m > N/2 + 1 \):

\[
\frac{d}{dt} \| v^\varepsilon \|_m \leq c_m \| \partial_t \varepsilon \|_m \| v^\varepsilon \|_m \leq c_m \| v^\varepsilon \|_m^2, \tag{3.11}
\]

and hence, for all \( \varepsilon \),

\[
\sup_{0 \leq t \leq T} \| v^\varepsilon \|_m \leq \frac{\| v_0 \|_m}{1 - c_m T \| v_0 \|_m} = \| v_0 \|_m + \frac{\| v_0 \|_m^2 c_m T}{1 - c_m T \| v_0 \|_m}. \tag{3.12}
\]

Thus the family \( (v^\varepsilon) \) is uniformly bounded in \( C([0, T]; H^m), m > N/2 \), provided that

\[
T < \left( c_m \| v_0 \|_m \right)^{-1}.
\]
Furthermore, the family of time derivatives \( \frac{dv^\epsilon}{dt} \) is uniformly bounded in \( H^{m-2} \). Equation (3.5) implies that, for \( m > (N/2) + 2 \),
\[
\frac{dv^\epsilon}{dt} \leq \nu \| J^2 \Delta v^\epsilon \|_{m-2} + \| PJ^\epsilon [(J^\epsilon \nabla v^\epsilon) \cdot \nabla (J^\epsilon v^\epsilon)] \|_{m-2} \\
\leq c \nu \| v^\epsilon \|_m + c \| v^\epsilon \|^2_m
\]
Hence the previous estimates yield that for a given \( 0 \leq \nu < \infty \) the family \( \frac{dv^\epsilon}{dt} \) is uniformly bounded in \( H^{m-2} \).

(2). Secondly, we will show that the solutions \( v^\epsilon \) to regularized equation (3.5) form a contraction in the low norm \( C[[0, T]; L^2(\mathbb{R}^N)] \). To do so, we will prove that the family \( v^\epsilon \) forms a Cauchy sequence in \( C[[0, T]; L^2(\mathbb{R}^N)] \). In particular, there exists a constant \( C \) that depends on only \( \| v_0 \|_m \) and the time \( T \) so that, for all \( \epsilon \) and \( \epsilon' \),
\[
\sup_{0 < t < T} \| v^\epsilon - v^{\epsilon'} \|_0 \leq C \max(\epsilon, \epsilon')
\]
Using (3.5), we have that
\[
\frac{d}{dt} \left( \frac{1}{2} \| v^\epsilon - v^{\epsilon'} \|_0^2 \right) = \nu \left( \| J^\epsilon \Delta v^\epsilon - J^\epsilon \Delta v^{\epsilon'} \|_0^2 \right) - \{ PJ^\epsilon [(J^\epsilon \nabla v^\epsilon) \cdot \nabla (J^\epsilon v^\epsilon)] \}
\]
\[
= T1 + T2
\]
We can estimate the first term, \( T1 \), by means of integration by parts and mollifier property (2.14):
\[
\left( \| J^2 \Delta v^\epsilon - J^2 \Delta v^{\epsilon'} \|_0^2 \right) = \left[ \| J^\epsilon \Delta v^\epsilon - J^\epsilon \Delta v^{\epsilon'} \|_0^2 \right] \leq C \max(\epsilon, \epsilon') \| v^\epsilon \|_4 \| v^\epsilon - v^{\epsilon'} \|_0.
\]
We can estimate the second term, \( T2 \), by also using the same tools and the fact that \( v^\epsilon \) is divergence free:
\[ P \mathcal{J}_\epsilon [ (\mathcal{J}_\epsilon v^\epsilon) \cdot \nabla (\mathcal{J}_\epsilon v^\epsilon)] - P \mathcal{J}_\epsilon [ (\mathcal{J}_\epsilon v^\epsilon) \cdot \nabla (\mathcal{J}_\epsilon v^\epsilon)] = \{ (\mathcal{J}_\epsilon - \mathcal{J}_\epsilon) [ (\mathcal{J}_\epsilon v^\epsilon) \cdot \nabla (\mathcal{J}_\epsilon v^\epsilon)] \cdot (v^\epsilon - v^{\epsilon'}) \}
\]
\[ = \{ (\mathcal{J}_\epsilon - \mathcal{J}_\epsilon) [ (\mathcal{J}_\epsilon v^\epsilon) \cdot \nabla (\mathcal{J}_\epsilon v^\epsilon)] \cdot (v^\epsilon - v^{\epsilon'}) \}
\]
\[ + \{ (\mathcal{J}_\epsilon - \mathcal{J}_\epsilon) (v^\epsilon - v^{\epsilon'}) \cdot \nabla (\mathcal{J}_\epsilon v^\epsilon)] \cdot (v^\epsilon - v^{\epsilon'}) \}
\]
\[ + \{ (\mathcal{J}_\epsilon - \mathcal{J}_\epsilon) [ (\mathcal{J}_\epsilon (v^\epsilon - v^{\epsilon'}) \cdot \nabla (\mathcal{J}_\epsilon v^\epsilon)] \cdot (v^\epsilon - v^{\epsilon'}) \}
\]
\[ + \{ (\mathcal{J}_\epsilon - \mathcal{J}_\epsilon) [ (\mathcal{J}_\epsilon (v^\epsilon - v^{\epsilon'}) \cdot \nabla (\mathcal{J}_\epsilon v^\epsilon)] \cdot (v^\epsilon - v^{\epsilon'}) \}
\]
\[ = R1 + R2 + R3 + R4 + R5 \]

Using calculus inequality (2.5) and Sobolev inequality (2.3), we get
\[ |R1| \leq C \max(\epsilon, \epsilon') \| [v^\epsilon \cdot \nabla (v^\epsilon)] \|_1 \| v^\epsilon - v^{\epsilon'} \|_0 \]
\[ \leq C \max(\epsilon, \epsilon') \left( \| v^\epsilon \|_{L^\infty} + \| \nabla v^\epsilon \|_{L^\infty} \right) \| \nabla (v^\epsilon) \|_1 \| v^\epsilon - v^{\epsilon'} \|_0 \]
\[ \leq C \max(\epsilon, \epsilon') \| v^\epsilon \|_{m}^2 \| v^\epsilon - v^{\epsilon'} \|_0 \]

A similar estimate holds for \( R2 \) and \( R4 \),
\[ |R3| \leq C \| v^\epsilon - v^{\epsilon'} \|_0^2 \| v^\epsilon \|_{m} + C \| v^\epsilon - v^{\epsilon'} \|_0 \max(\epsilon, \epsilon') \| v^\epsilon \|_{m}^2 \]

and, finally, integration by parts and the fact that \( v^{\epsilon'} \) is divergence free shows that
\[ R5 = \{ (\mathcal{J}_\epsilon \cdot v^{\epsilon'} \cdot \nabla (\mathcal{J}_\epsilon \cdot v^{\epsilon'}) \cdot (\mathcal{J}_\epsilon \cdot (v^\epsilon - v^{\epsilon'})) \}
\]
\[ = \frac{1}{2} \int_{\mathbb{R}^3} \mathcal{J}_\epsilon \cdot v^{\epsilon'} \cdot \nabla \left[ (\mathcal{J}_\epsilon \cdot (v^\epsilon - v^{\epsilon'})) \right]^2 \]
\[ dx = 0 \]

Putting this all together gives
\[ \frac{d}{dt} \| v^\epsilon - v^{\epsilon'} \|_0 \leq C(M) \left[ \max(\epsilon, \epsilon') + \| v^\epsilon - v^{\epsilon'} \|_0 \right] \]

where \( M \) is an upper bound, from relation (3.13) for the \( \| v^\epsilon \|_{m} \) on \([0, T]\). Integrating this yields
\[ \sup_{0 < t < T} \| v^\epsilon - v^{\epsilon'} \|_0 \leq e^{C(M)T} \left[ \max(\epsilon, \epsilon') + \| v^\epsilon_0 - v^{\epsilon'}_0 \|_0 \right] - \max(\epsilon, \epsilon') \]
\[ \leq C(M, T) \max(\epsilon, \epsilon') \]

(3.13)

where we establish the final inequality by recalling that \( v^\epsilon_0 = v^{\epsilon'}_0 \).

Thus \( v^\epsilon \) is a Cauchy sequence in \( C([0, T]; L^2(\mathbb{R}^N)) \), so that it converges strongly to a
value \( v^\varepsilon \in C([0, T]; L^2(\mathbb{R}^N)) \).

We have just proved the existence of a \( v \) such that

\[
\sup_{0 \leq t \leq T} \| v^\varepsilon - v \|_0 \leq C\varepsilon \tag{3.14}
\]

(3). Thirdly, we can use the fact that the \( v^\varepsilon \) are uniformly bounded in a high norm so that we have strong convergence in all the intermediate norms.

To do this, we apply interpolation lemma 2.4 to the difference \( v^\varepsilon - v \). Taking \( s = m \) and using relations (3.13) and (3.14) gives

\[
\sup_{0 \leq t \leq T} \| v^\varepsilon - v \|_{m'} \leq C(\| v_0 \|_{m'}, T)\varepsilon^{1-m'/m}
\]

Hence for all \( m' < m \) we have strong convergence in \( C\{[0, T]; H^m(\mathbb{R}^N)\} \). With \( 0 < 7/2 < m' < m \), this implies strong convergence in \( C\{[0, T]; C^2(\mathbb{R}^N)\} \). Also, from the equation

\[
v_t^\varepsilon = \varepsilon \partial^2_x \Delta v^\varepsilon - P\mathcal{J}\varepsilon \left( (\mathcal{J}\varepsilon v^\varepsilon) \cdot \nabla (\mathcal{J}\varepsilon v^\varepsilon) \right)
\]

so that \( v_t^\varepsilon \) converges in \( \{C[0, T], C(\mathbb{R}^N)\} \) to \( v\Delta v - P(v \cdot \nabla v) \). Because \( v^\varepsilon \to v \), the distribution limit of \( v_t^\varepsilon \) must be \( v_t \) so, in particular, \( v \) is a classical solution of the Navier-Stokes equations.

#.

Hence, we have got a constant \( T > 0 \), so that the Navier-Stokes equation has a solution \( v \) in \( t \in [0, T] \).

\[4. \textbf{The solution } v \textbf{ in } t \in [0, T] \textbf{ is a power series of } t \]

In the previous chapter, we have get the solution form solution \( v^\varepsilon \) to the regularized equation (3.5) in a time interval \( t \in [0, T_{\varepsilon}] \) by building a Picard iteration sequence \( v_{\varepsilon}^n \). And based on \( v^\varepsilon \), we have got a solution \( v \) to the Navier-Stoke equation in a time interval \( t \in [0, T] \).
Most studies have utilized this methodology, and are satisfied with the existence of a solution in a time interval, or existence of global solutions under specific conditions. However, through detailed analysis, we can find that each $v^n$ is a polynomial of $t$, and hence $v^\varepsilon$ is a power series of $t$ and is convergent in $t \in [0, T_\varepsilon]$. Using this fact, we can prove that the solution is indeed a power series of $t$. This will provide a good possibility to solve the famous existence and smoothness problem.

**Theorem 4.1.** the solution to the Navier-Stokes equation of Theorem 3.3 is indeed a series $v(x,t) = \sum_{n=0}^\infty a_n(x)t^n$, $(x,t) \in \mathbb{R}^N \times [0, T]$, where $a_0(x) = v_0(x)$, and $a_n(x) = \frac{1}{n}v\Delta a_{n-1}(x) - \frac{1}{n}P\left[\sum_{i=0}^{n-1} \left(a_i(x) \cdot \nabla a_{n-1-i}(x)\right)\right]$ are all known functions determined only by $v_0(x)$.

**Proof:**

Suppose $v^\varepsilon$ is the unique solution to the regularized equation (3.5), and $v$ is the local-in-time solution in theorem 3.3. Let

$$\frac{\|v_0\|_m}{1 - c_n T \|v_0\|_m} = M_0$$

Then From (3.10), we have $\sup_{0 \leq t \leq T} \|v^\varepsilon\|_m \leq M_0$, $\sup_{0 \leq t \leq T} \|v\|_m \leq M_0$.

Let $M = 2M_0$, and $O^M = \{v \in V^m| \|v\|_m \leq M\}$, then $F_\varepsilon(v)$ is Lipschitz continuous in $O^M$. Let

$$T_\varepsilon = (M - M_0) / \left[\left(\frac{c}{\varepsilon^2} + \frac{c}{\varepsilon^{N/2+1+m}}M\right)M\right].$$

We will rebuild $v^n_\varepsilon$ under this condition, it's slightly different from that of chapter 3.

Since $v = \lim_{\varepsilon \to 0} v^\varepsilon$, $v^\varepsilon = \lim_{n \to \infty} v^n_\varepsilon$, and $v^n_\varepsilon$ is a Picard iteration sequence, we will prove the theorem in following steps:
(1). Firstly, to prove that the local solution $v^\epsilon(x, t), t \in [0, T_\epsilon]$ to the regularized equation (3.5) is in series form $v^\epsilon(x, t) = \sum_{n=0}^{\infty} a^\epsilon_n(x) t^n$.

Recalling the proving process of Theorem 3.1, the Picard iteration is $v^\epsilon_0 = v_0$, and
\[ v^\epsilon_{n+1} = v_0 + \int_0^t F^\epsilon(v^\epsilon_n) ds, (n \geq 0) \]

Where $v^\epsilon_n \in \mathcal{O}^M$ and $\|v^\epsilon_n\|_m \leq M, (n \geq 0)$ are uniformly bounded. It can be checked that
\[ v^\epsilon_1 = v_0 + \int_0^t F^\epsilon(v^\epsilon_0) ds = v_0 + t \{v^\epsilon_0 t^2 \Delta v_0 - PJ^\epsilon [(J^\epsilon v_0) \cdot \nabla (J^\epsilon v_0)]\} = a^\epsilon_0(x) + a^\epsilon_1(x) t \]

Generally speaking, if $v^\epsilon_n(x, t) = \sum_{i=0}^{n} a^\epsilon_i(x) t^i$ has been got, then $v^\epsilon_{n+1} = v_0 + \int_0^t F^\epsilon(v^\epsilon_n) ds = v_0 + \int_0^t \{v^\epsilon_0 t^2 \Delta v^\epsilon_n - PJ^\epsilon [(J^\epsilon v^\epsilon_n) \cdot \nabla (J^\epsilon v^\epsilon_n)]\} ds = \sum_{i=0}^{n+1} a^\epsilon_i(x) t^i$.

Where $a^\epsilon_{n+1}(x) = \frac{1}{n+1} v^\epsilon_0 t^2 (\Delta a^\epsilon_n(x)) - \frac{1}{n+1} PJ^\epsilon \left[\sum_{i=0}^{n} (J^\epsilon a^\epsilon_i(x) \cdot \nabla J^\epsilon a^\epsilon_{n-i}(x))\right]$.

Hence,
\[ v^\epsilon = \lim_{n \to \infty} v^\epsilon_n = \sum_{i=0}^{\infty} a^\epsilon_i(x) t^i = \sum_{n=0}^{\infty} a^\epsilon_n(x) t^n, t \in [0, T_\epsilon]. \]

The rate of convergence can be evaluated as following:

Let $C_\epsilon = \left(\frac{c\epsilon^2}{\epsilon^2} + \frac{c}{\epsilon^{n/2+1+m}} M\right)$, and rewrite $v^\epsilon$ as
\[ v^\epsilon = v^\epsilon_0 + \sum_{i=1}^{n} (v^\epsilon_i - v^\epsilon_{i-1}) + \sum_{i=n+1}^{\infty} (v^\epsilon_i - v^\epsilon_{i-1}) = v^\epsilon_n + R^\epsilon_n \]

It can be calculated that $\|v^\epsilon_0\|_m \leq M_0$

\[ \|v^\epsilon_1 - v^\epsilon_0\|_m = \|\int_0^t F^\epsilon(v^\epsilon_0) ds\|_m \leq C_\epsilon M_0 t \leq 2C_\epsilon M_0 t \]
\[ \|v_n^\epsilon - v_{n-1}^\epsilon\|_m \leq \frac{1}{n!} M_0 e^{n+1} t^n \]

Hence, \[ \|v_n^\epsilon\|_m = \|v_0^\epsilon\|_m + \sum_{i=1}^{n} (v_i^\epsilon - v_{i-1}^\epsilon) \leq \|v_0^\epsilon\|_m + \sum_{i=1}^{n} \|v_i^\epsilon - v_{i-1}^\epsilon\|_m \leq M_0 \sum_{i=0}^{n} \frac{1}{i!} C_i^\epsilon 2^i t^i \leq M_0 e^{2C\epsilon}. \]

It can also be calculated that:

\[ \|R_n^\epsilon\|_m \leq \sum_{i=n+1}^{\infty} \|v_i^\epsilon - v_{i-1}^\epsilon\|_m \leq M_0 \frac{e^{2C\epsilon}}{(n+1)!} \sum_{i=n+1}^{\infty} i C_i^\epsilon 2^i t^i \leq M_0 \frac{e^{2C\epsilon}}{(n+1)!} \sum_{i=n+1}^{\infty} C_i^\epsilon 2^i t^i \]

Hence \[ \sup_{t \in [0,T_\epsilon]} \|R_n^\epsilon\|_m \leq \frac{eM_0}{(n+1)!}. \]

(2). Secondly, to extend the series from \( t \in [0,T_\epsilon] \) to \( t \in [0,T] \).

From Theorem 3.2, we know that \( \nu^\epsilon \) can be extended to \( t \in [0,\infty) \). However, this extending is not convenient to evaluate the rate of convergence. Hence, we used a step by step methodology to extend.

Let \( k_\epsilon = \lfloor T/T_\epsilon \rfloor \). If \( k_\epsilon > 1 \), since \( \nu^\epsilon(.,T_\epsilon) \) is already known, we can used \( \nu^\epsilon(.,T_\epsilon) \) as a new initial velocity to resolve \( \nu^\epsilon \) in \( t \in [T_\epsilon,2T_\epsilon] \). Consider the regularized equation

\[ \frac{d\nu^\epsilon}{dt} = F_\epsilon(\nu^\epsilon) \]

\[ \nu^\epsilon|_{t=T_\epsilon} = \nu^\epsilon(.,T_\epsilon) \]

And build another Picard iteration sequence \( \bar{\nu}_0^\epsilon = \nu^\epsilon(.,T_\epsilon) \), \( \bar{\nu}_{n+1}^\epsilon = \nu^\epsilon(.,T_\epsilon) + \int_{T_\epsilon}^{T} F_\epsilon(\bar{\nu}_0^\epsilon) ds, (n \geq 0) \). It can also be checked that \( \|\bar{\nu}_n^\epsilon\|_m \leq M, (n \geq 0) \) are uniformly bounded, and there is a solution \( \bar{\nu}^\epsilon = \lim_{n \to \infty} \bar{\nu}_n^\epsilon = \sum_{i=0}^{\infty} b_i^\epsilon(x)(t-T_\epsilon)^n = \sum_{i=0}^{\infty} c_i^\epsilon(x)t^n, t \in [0,T]. \)
\(T, 2T\). Since \(v^\epsilon\) is equal to \(\bar{v}^\epsilon\) in the vicinity near the point \(t = T^\epsilon\), hence \(c_i^\epsilon(x) = a_i^\epsilon, (i \geq 0)\), so that the series has been extended to \(t \in [T^\epsilon, 2T^\epsilon]\).

Through \(\bar{v}^\epsilon_n\), It can also be calculated that
\[
\sup_{t \in [T^\epsilon, 2T^\epsilon]} \|R^\epsilon_n\|_m \leq \frac{eM_0}{(n+1)!}.
\]

Repeating this process for \(t \in [T^\epsilon, 2T^\epsilon], \ldots, T^\epsilon \in [k_{\epsilon - 1}T^\epsilon, k_{\epsilon}T^\epsilon]\) respectively, the series can then be extended to \(t \in [0,T]\).

\[
v^\epsilon = \sum_{n=0}^\infty a_n^\epsilon(x)t^n, t \in [0,T] \tag{4.1}
\]

\[
v^\epsilon = v_n^\epsilon + R^\epsilon_n \tag{4.2}
\]

\[
\sup_{t \in [0,T]} \|R^\epsilon_n\|_m \leq \frac{eM_0}{(n+1)!} \tag{4.3}
\]

Finally, to compute the solution \(v\) to the Navier-Stokes equation in \(t \in [0,T]\).

Using the sobolev inequality (2.3), we knew that \(|R^\epsilon_n| \leq \frac{ce^M_0}{(n+1)!}\)

In (4.2), let \(\epsilon \to 0\), we have

\[
v = \sum_{i=0}^n a_i(x)t^i + r_n(t), t \in [0,T] \tag{4.4}
\]

Where \(r_n(t) \equiv \lim_{\epsilon \to 0} R^\epsilon_n\), hence \(|r_n(t)| \leq \frac{ceM_0}{(n+1)!}\)

So that, we can take \(n \to \infty\) in (4.4), and got

\[
v = \sum_{i=0}^\infty a_i(x)t^i = \sum_{n=0}^\infty a_n(x)t^n, t \in [0,T] \tag{4.5}
\]

Where \(a_0(x) = v_0(x)\), and \(a_n(x) = \frac{1}{n} \nu \Delta a_{n-1}(x) - \frac{1}{n} P \left[ \sum_{i=0}^{n-1} \left( a_i(x) \cdot \nabla a_{n-1-i}(x) \right) \right]\) are all known functions determined only by \(v_0(x)\).

Thus, the solution \(v\) in \(t \in [0,T]\) is a power series of \(t\).

Since the existence of \(v^\epsilon(x,t)\) in \(t \in [0,T^\epsilon]\) and \(v\) in \(t \in [0,T]\) has been proved in the previous chapter, all computations in proving theorem 4.1 are reasonable.
This result is very good, because we have got an explicit solution to the Navier-Stokes equation so long as the initial condition \( v_0 \) is given.

5. The solution series is convergent on \((x, t) \in \mathbb{R}^N \times [0, \infty)\)

In the study of solutions, a special property of power series is used, it can be described in Lemma 5.1.

**Lemma 5.1.** If \( \sum_{n=0}^{\infty} a_n t^n \) is a series, \( \tau \) is a constant, \( \tau > 0 \) or \( \tau < 0 \), and \( \sum_{n=0}^{\infty} a_n \tau^n \) is convergent, then for all \(|t| < |\tau|\), \( \sum_{n=0}^{\infty} a_n t^n \) is convergent.

**Proof:** From the convergence of \( \sum_{n=0}^{\infty} a_n \tau^n \), we know that \( a_n \tau^n \to 0, (n \to \infty) \), such that \( \exists \overline{M} > 0, |a_n \tau^n| \leq \overline{M}, (n \geq 0) \). Hence, if \(|t| < |\tau|\), then,

\[
|\sum_{n=0}^{\infty} a_n t^n| \leq \sum_{n=0}^{\infty} |a_n t^n| = \sum_{n=0}^{\infty} |a_n \tau^n| \cdot (|t/\tau|)^n \leq \overline{M} \sum_{n=0}^{\infty} (|t/\tau|)^n = \frac{\overline{M}}{1-|t/\tau|}.
\]

#

**Theorem 5.1.** The series solution \( v(x, t) \) in Theorem 4.1 is indeed a globally smooth solution.

\[
v(x, t) = \sum_{n=0}^{\infty} a_n(x) t^n
\]

Where \( a_0(x) = v_0(x) \), and \( a_n(x) = \frac{1}{n} \nu \Delta a_{n-1}(x) - \frac{1}{n} P[\sum_{i=0}^{n-1} \left( a_i(x) \cdot \nabla a_{n-1-i}(x) \right)] \), \( (n \geq 1) \).

**Proof:** From Theorem 4.1, there exists the series solution \( v(x, t) \) in \( t \in [0, T] \)

\[
v(x, t) = \sum_{n=0}^{\infty} a_n(x) t^n
\]

To prove its global existence and smoothness, we need only to prove that

\[
|v(x, t)| < \infty, \quad (x, t) \in \mathbb{R}^N \times [0, \infty)
\]
Assume that there is a \((x_0, t_0)\) such that \(|v(x_0, t_0)| = \infty\). It's obvious that \(t_0 > T\), and \(t_0\) can be chosen to be the minimal, so that \(|v(x_0, t)| = \infty\), \(|v(x_0, t)| < \infty\), \((0 \leq t < t_0)\). Since \(|v(x_0, (t_0 - T/4))| < \infty\), \(|- (t_0 - T/2)| < t_0 - T/4\), hence \(|v(x_0, - (t_0 - T/2))| < \infty\).

Replacing \(v_0(x)\) with \(v(x, T/2)\), the Navier-Stokes equation will become:

\[
\begin{align*}
v_t + v \cdot \nabla v &= - \nabla p + \nu \Delta v \\
\text{div} \, v &= 0 \\
v|_{t=0} &= v(x, T/2)
\end{align*}
\]

From Theorem 4.1, equation (5.1),(5.2) and (5.3) have a series form solution \(\tilde{v}(x, t)\) in \(t \in [0, T_0], T_0 > 0\).

\[
\tilde{v}(x, t) = \sum_{n=0}^{\infty} \tilde{a}_n(x)t^n
\]

Noting that \(|\tilde{v}(x_0, - (t_0 - T/2) - T/2)| = |v(x_0, - (t_0 - T/2))| < \infty\). Since \(|-(t_0 - T/2) - T/2| > t_0 - T/2\). Hence, by Lemma 5.1, \(|\tilde{v}(x_0, t_0 - T/2)| < \infty\).

However, on the other hand, there is \(|\tilde{v}(x_0, t_0 - T/2)| = |v(x_0, t_0)| = \infty\).

Thus, a contradiction has happened.

So, there should be

\(|v(x, t)| < \infty\), \((x, t) \in \mathbb{R}^N \times [0, \infty)\).

#

This means that the Navier-Stokes Equation (1.1),(1.2) and (1.3) has a global smooth solution \(v(x, t)\) in \((x, t) \in \mathbb{R}^N \times [0, \infty)\). Similarly, the Euler Equation also has a global smooth solution \(v(x, t)\) in \((x, t) \in \mathbb{R}^N \times [0, \infty)\).
The technique of replacing $v_0(x)$ with $v(x,T/2)$ can be called “Time Shifting”, it has just been used to show the complex relationship between time and space residing in the Navier-Stokes equation and the Euler equation.

6. Conclusions

We have solved the existence and smoothness of the Navier-Stokes equation and the Euler equation for $(x,t) \in \mathbb{R}^N \times [0,\infty), (N \geq 2)$. Moreover, we have obtained explicit solutions to the Navier-Stokes equation and the Euler equation. By the way, we have strong confidence that our methodology in this paper can also solve the Navier-Stokes Equation and the Euler Equation with an external force $f$.

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