Anisotropic Fluid and Bianchi Type III Model in $f(R)$ Gravity

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Abstract

This paper is devoted to study the Bianchi type III model in the presence of anisotropic fluid in $f(R)$ gravity. Exponential and power-law volumetric expansions are used to obtain exact solutions of the field equations. We discuss the physical behavior of the solutions and anisotropy behavior of the fluid, the expansion parameter and the model in future evolution of the universe.

Keywords: $f(R)$ Theory; Bianchi Type III; Anisotropic Fluid.
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1 Introduction

Hubble diagram of type Ia Supernovae (SNela) measured by the Supernovae Cosmology Project [1] and High-z Team, up to the redshift $z \sim 1$ [2], has been the first piece of evidence that current universe is undergoing a phase of accelerated expansion. The other Balloon-born experiments such as BOOMERanG [3] and MAXIMA [4] have detected the anisotropy spectrum of the Cosmic Microwave Background (CMB) radiations representing that universe is spatially flat. These data indicate that present universe is dominated by an un-clustered fluid with large negative pressure called dark energy.

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(DE) which causes expansion. The above picture has been further strengthened by the current measurements of CMB spectrum obtained by Wilkinson Microwave Anisotropy Probe (WMAP) experiment [5], [6] and by extension of the Hubble SNela diagram to redshifts higher than one [7]. There are various attempts to construct the acceptable dark energy models in different directions. For example, traditional cosmological constant, quintessence or phantom models, dark fluid with complicated equation of state, String or M-theory, higher dimensions, brane-world models, etc. In addition to these attempts, the $f(R)$ theory of gravity has also been helpful in describing the evolution of the universe.

The recent enthusiasm in $f(R)$ theory is caused by its success as the gravitational alternative for DE [8]. Although, the cosmological constant, $\Lambda$, is the simplest explanation of the DE and a best fit to some of the astrophysical data [5]. However, the $\Lambda$-Cold Dark Matter ($\Lambda$CDM) fails in explaining why the value of $\Lambda$ is so tiny (120 orders of smaller magnitude) as compared to the vacuum energy predicted by particle physics (the coincidence problem). In order to solve this problem, people replaced the cosmological constant with a scalar field referred to quintessence or phantom models [9]. In order to explain DE of the current universe with the help of the above models, one can use the effective equation of state parameter (EoS) $\omega$. When $w = -1$, the universe passes through $\Lambda$CDM epoch. If $w < -1$ then we live in the phantom-dominated universe and for $w > -1$, the quintessence dark era occurs. It is mentioned here that energy conditions are violated in all these cases. Also, the phantom phase ends at finite-time singularity in the future [10] while quintessence universe may end up at more general singularity. This is also true for the $f(R)$ gravity which leads to DE with the corresponding effective $\omega$ (quintessence or phantom) [8].

The $f(R)$ theory of gravity quite naturally describes the transition from deceleration to acceleration in such an evolving universe [11]. This theory is very useful in high energy physics, for example, to solve hierarchy or gravity-GUTs unification problems [12]. A large body of papers is available in the literature [13]-[15] addressing the well-known issues of gravitational stability [16], Newtonian limit [17], singularity problems [18], solar system test [19], etc, in the context of $f(R)$ gravity. The exact solution of the field equations in metric $f(R)$ theory has been obtained in all the three symmetric versions. Spherically symmetric solutions have been investigated for both vacuum and non-vacuum cases by different authors [20]. Azadi et al. [21] and Momeni [22] have studied vacuum cylindrically symmetric solutions. Sharif
and Shamir [23] have explored plane symmetric solutions both for vacuum and non-vacuum cases.

It is found that some large-angle anomalies appear in CMB radiations which violate the statistical isotropy of the universe [24]. Plane Bianchi models (which are homogeneous but not necessarily isotropic) seem to be the most promising explanation of these anomalies. Jaffe et al. [25] investigated that removing a Bianchi component from the WMAP data can account for several large-angle anomalies leaving the universe to be isotropic. Thus the universe may have achieved a slight anisotropic geometry in cosmological models regardless of the inflation. Further, these models can be classified according to whether anisotropy occurs at an early stage or at later times of the universe. The models for the early stage can be modified in a way to end inflation with a slight anisotropic geometry [26]. For the latter class, the isotropy of the universe, achieved during inflation, can be distorted by modifying DE [27].

The universe acceleration provides information about the major part of the universe which has large negative pressure but without telling anything about the number of cosmic fluids in the universe. This may be explained by considering the accelerating expansion with a single fluid and an equation of state acting like DE. The main benefit of this approach is that a suitable equation of state can be obtained and observational data can be fitted. In General Relativity (GR), people have worked by choosing anisotropic fluid with anisotropic EoS. Akarsu and Kilinc [28] have studied the Bianchi type III model in the presence of single imperfect fluid with dynamical anisotropic EoS parameter. They have concluded that anisotropy of the DE do not always promote anisotropy of the expansion. Sharif and Zubair [29] have investigated the Bianchi type $VI_0$ cosmological models in the presence of electromagnetic field and anisotropic dark energy. They have examined the effects of electromagnetic field on the dynamics of the universe. In a recent work [30], we have obtained solutions of the Bianchi type $VI_0$ universe in $f(R)$ gravity.

Here, our objective is to find exact solutions of the Bianchi type III model to discuss future evolution of the universe in the metric $f(R)$ gravity. We take anisotropic fluid to represent DE. The paper is organized as follows. In section 2, we present some features of the Bianchi type III model and define dynamical quantities describing the evolution of the universe. Section 3 provides the field equations and anisotropic parameter of the expansion. We obtain possible solutions of the field equation and discuss the physical
behavior of these solutions in section 4. In the last section 5, we summarize the results obtained.

2 The Model

The homogeneous and anisotropic Bianchi type III spacetime is described by the line element

\[ ds^2 = dt^2 - A^2(t)dx^2 - e^{-2\alpha x}B^2(t)dy^2 - C^2(t)dz^2, \]  

(2.1)

where the scale factors \( A, B \) and \( C \) are only functions of cosmic time \( t \) and \( \alpha \neq 0 \) is a constant.

The energy-momentum tensor for anisotropic fluid is given as

\[ T^\nu_\mu = \text{diag}[\rho, -p_x, -p_y, -p_z] = \text{diag}[1, -\omega_x, -\omega_y, -\omega_z]\rho, \]

(2.2)

where \( \rho \) is the density of the fluid while \( p_x, p_y \) and \( p_z \) are pressures and \( \omega_x, \omega_y \) and \( \omega_z \) are directional EoS parameters on the \( x, y \) and \( z \) axes respectively. The DE schematically characterized by this parameter. The deviation from isotropy may be obtained by setting

\[ \omega_x = \omega, \quad \omega_y = \omega + \delta \quad \text{and} \quad \omega_z = \omega + \gamma, \]

where \( \omega \) is the deviation-free EoS parameter and \( \delta \) and \( \gamma \) are the deviations from \( \omega \) on \( y \) and \( z \) axes respectively. In this case, the energy-momentum tensor becomes

\[ T^\nu_\mu = \text{diag}[1, -\omega, -(\omega + \delta), -(\omega + \gamma)]\rho. \]  

(2.3)

The skewness parameters \( \delta \) and \( \gamma \) may be constant or functions of cosmic time. The average scale factor and volume of the universe for this model will be

\[ a = (ABC)^{1/3}, \quad V = a^3 = ABC. \]  

(2.4)

The mean Hubble parameter \( H \), deceleration parameter \( q \), expansion scalar \( \Theta \) and the directional Hubble parameters in the \( x, y \) and \( z \) directions turn out to be

\[ H = \frac{\ln \dot{V}}{3} = \frac{1}{3} \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right), \quad q = -\frac{\ddot{a}}{\dot{a}^2}, \]

(2.5)

\[ \Theta = u^a_{;a} = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}, \quad H_x = \frac{\dot{A}}{A}, \quad H_y = \frac{\dot{B}}{B}, \quad H_z = \frac{\dot{C}}{C}. \]  

(2.6)
Here the deceleration parameter \( q \) measures the rate of expansion of the universe. The sign of \( q \) indicates the state of expanding universe. If \( q < 0 \) or \( q > 0 \) respectively, then it represents inflation or deflation of the universe while \( q = 0 \) shows expansion with constant velocity.

To examine whether expansion of the universe is anisotropic or not, we define anisotropic expansion parameter as

\[
\Delta = \frac{1}{3} \sum_{i=1}^{3} \left( \frac{H_i - H}{H} \right)^2,
\]

where \( H_i \) \((i = 1, 2, 3)\) are the directional Hubble parameters in the directions of \( x, y \) and \( z \) axes respectively. If \( \Delta = 0 \), then the universe expands isotropically. Further, any anisotropic model of the universe with diagonal energy-momentum tensor approaches to isotropy if \( \Delta \to 0, V \to +\infty \) and \( \rho > 0 \) as \( t \to +\infty \) \([28, 31]\).

3 The Field Equations in \( f(R) \) Gravity

The Lagrangian in \( f(R) \) gravity is given as \([32]\)

\[
L = \frac{f(R)}{2\kappa} + L_M,
\]

where \( f(R) \) is a function of the Ricci scalar and \( L_M \) describes all kinds of matter including non-relativistic (cold) dark matter. Assuming variation with respect to the metric tensor, \( g_{\mu\nu} \), we have the following fourth order partial differential equations

\[
F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu \nabla_\nu F(R) + g_{\mu\nu} \Box F(R) = \kappa T_{\mu\nu},
\]

where \( F(R) \equiv df(R)/dR \), \( \Box \equiv \nabla^\nu \nabla_\nu \) with \( \nabla_\mu \) representing the covariant derivative and \( \kappa (= \frac{8\pi G}{c^4} = 1) \) is the coupling constant in gravitational units. The trace of Eq.(3.2) yields

\[
F(R)R - 2f(R) + 3\Box F(R) = T.
\]

This equation is helpful to solve the field equations and also to express \( f(R) \) in terms of its derivative, i.e.,

\[
f(R) = \frac{-T + F(R)R + 3\Box F(R)}{2}.
\]
The scalar curvature for Bianchi type III model is given by

\[
R = -2 \left[ \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} - \frac{\alpha^2}{A^2} + \frac{\dot{A} \dot{B}}{AB} + \frac{\dot{A} \dot{C}}{AC} + \frac{\dot{B} \dot{C}}{BC} \right].
\]  

(3.5)

The corresponding field equations become

\[
\left( \frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} \right) F + \frac{1}{2} f(R) - \left( \frac{\dot{A} \dot{B}}{AB} + \frac{\dot{A} \dot{C}}{AC} \right) \dot{F} = -\rho, \tag{3.6}
\]

\[
\left( \frac{\ddot{B}}{B} - \frac{\alpha^2}{A^2} + \frac{\dot{B} \dot{C}}{BC} \right) F + \frac{1}{2} f(R) - \left( \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) \dot{F} = -\omega \rho, \tag{3.7}
\]

\[
\left( \frac{\ddot{C}}{C} + \frac{\dot{A} \dot{C}}{AC} + \frac{\dot{B} \dot{C}}{BC} \right) F + \frac{1}{2} f(R) - \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) \dot{F} = -\left( \omega + \gamma \right) \rho. \tag{3.8}
\]

\alpha \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right) F = 0. \tag{3.10}

The solution of Eq. (3.10) yields

\[
B = c_1 A, \tag{3.11}
\]

where \( c_1 \) is a constant of integration. Using Eq. (3.11) and then subtracting Eq. (3.8) from Eq. (3.9), we obtain \( \delta = 0 \). This indicates that directional EoS parameters \( \omega_x, \omega_y \) along \( x \) and \( y \) axes become equal and hence pressures. Consequently, the field equations turn out to be

\[
\left( \frac{2\ddot{A}}{A} + \frac{\ddot{C}}{C} \right) F + \frac{1}{2} f(R) - \left( \frac{2\dot{A}}{A} + \frac{\dot{C}}{C} \right) \dot{F} = -\rho, \tag{3.12}
\]

\[
\left( \frac{\ddot{A}}{A} - \frac{\alpha^2}{A^2} + \frac{\dot{A} \dot{C}}{AC} \right) F + \frac{1}{2} f(R) - \left( \frac{\dot{A}}{A} + \frac{\dot{C}}{C} \right) \dot{F} = -\omega \rho, \tag{3.13}
\]

\[
\left( \frac{\ddot{C}}{C} + \frac{2\dot{A} \dot{C}}{AC} \right) F + \frac{1}{2} f(R) - \left( \frac{2\dot{A}}{A} \right) \dot{F} = -\left( \omega + \gamma \right) \rho. \tag{3.14}
\]

Subtracting Eq. (3.13) from Eq. (3.14) and integrating the resulting equation, we have

\[
\frac{\dot{A}}{A} - \frac{\dot{C}}{C} = \frac{c}{V F} + \frac{1}{V F} \int \left( \frac{\alpha^2 F}{A^2} + \gamma \rho \right) V dt, \tag{3.15}
\]
where $c$ is a positive constant of integration. In terms of directional Hubble parameters, the above equation can be written as

$$H_x - H_z = \frac{c}{VF} + \frac{1}{VF} \int \left( \frac{\alpha^2 F}{A^2} + \gamma \rho \right) V dt. \quad (3.16)$$

Using Eqs. (3.16) and (2.7), it follows that

$$\Delta = \frac{2}{9H^2} \left[ c + \int \left( \frac{\alpha^2 F}{A^2} + \gamma \rho \right) V dt \right] V^{-2} F^{-2}. \quad (3.17)$$

If we take $\gamma = 0$ and $F(R) = 1$, the anisotropy parameter of expansion reduces to GR for an isotropic fluid

$$\Delta = \frac{2}{9H^2} \left[ c + \int \frac{\alpha^2 F}{A^2} V dt \right] V^{-2} F^{-2}. \quad (3.18)$$

The integral in Eq. (3.17) vanishes for the following value of $\gamma$ [28, 29], i.e.,

$$\gamma = -\frac{\alpha^2 F}{\rho A^2}. \quad (3.19)$$

Consequently, the energy-momentum tensor, anisotropy parameter and Eq. (3.16) will take the form

$$T^\nu_\mu = \text{diag}[1, -\omega, -\omega, -\omega + \frac{\alpha^2 F}{\rho A^2} \rho], \quad (3.20)$$

$$\Delta = \frac{2}{9H^2} \frac{c^2}{V^2} V^{-2} F^{-2}, \quad (3.21)$$

$$H_x - H_z = \frac{c}{VF}. \quad (3.22)$$

It is interesting to mention here that these results will reduce to GR [28] for $F(R) = 1$.

### 4 Solutions of the Field Equations

In this section we would obtain exact solutions of the Bianchi type III model in the presence of anisotropic fluid. Using Eq. (3.19) in Eqs. (3.12) - (3.14)
along with Eq. (3.3), we have

\[
(\frac{2\ddot{A}}{A} + \frac{\dddot{C}}{C})F + \frac{1}{2}f(R) - (\frac{2\dot{A}}{A} + \frac{\dot{C}}{C})\dot{F} = -\rho,
\]

(4.1)

\[
(\frac{\ddot{A}}{A} + \frac{\dot{A}^2}{A^2} + \frac{\dot{A}\dot{C}}{AC})F + \frac{1}{2}f(R) - \ddot{F} - (\frac{\dot{A}}{A} + \frac{\dot{C}}{C})\dot{F} = - (\omega + \gamma)\rho,
\]

(4.2)

\[
(\frac{\ddot{C}}{C} + \frac{2\dot{A}\dot{C}}{AC})F + \frac{1}{2}f(R) - \ddot{F} - (\frac{2\dot{A}}{A})\dot{F} = - (\omega + \gamma)\rho,
\]

(4.3)

\[
-\left(\frac{2\dot{A}}{A} + \frac{\dot{C}}{C} - \frac{\alpha^2}{A^2} + \frac{\dot{A}^2}{A^2} + \frac{2\dot{A}\dot{C}}{AC}\right)F - 2f(R) + 3\dot{F}
+ 3\left(\frac{2\ddot{A}}{A} + \frac{\dddot{C}}{C}\right)\dot{F} = (1 - 3\omega - \gamma)\rho.
\]

(4.4)

This is a system of four differential equations in five unknowns. In order to obtain solution, we make use of two different volumetric expansion laws (in next subsections) to get a system of five equations with five unknowns.

### 4.1 Bianchi Type III Model for Exponential Expansion

We consider the following volumetric exponential expansion law

\[
V = c_2 e^{3kt},
\]

(4.5)

where \(c_2\) and \(k\) are positive constants. Using the value of \(V\) in Eq. (3.22), we have

\[
H_x - H_z = \frac{ce^{-3kt}}{c_2 \dot{F}}.
\]

(4.6)

Solving the system of equations (4.1)-(4.4) and Eq. (4.6), we obtain the scale factors as follows

\[
A = c_3 e^{kt + \frac{\alpha}{3c_2}} \int \frac{e^{-3kt}}{F} dt,
\]

(4.7)

\[
B = c_4 e^{kt - \frac{\alpha}{3c_2}} \int \frac{e^{-3kt}}{F} dt,
\]

(4.8)

\[
C = c_4 e^{kt - \frac{2\alpha}{3c_2}} \int \frac{e^{-3kt}}{F} dt,
\]

(4.9)

where \(c_3\) and \(c_4\) are positive constants of integration. To solve the integral part in the above equations, we assume a relation between \(F\) and \(V\) as \(F V^{m/3}\) \[23\], which gives

\[
F = c_5 V^{m/3} \Rightarrow F = c_5 e^{mkt},
\]

(4.10)
where \( c_5 \) is a positive proportionality constant and \( m \) is a non-zero arbitrary constant. Using Eq.\((4.10)\) in Eqs.\((4.7)-(4.9)\), the scale factors become

\[
A = c_3 e^{kt - \frac{c}{3c_2c_5} \frac{1}{k(3+m)} e^{-(3+m)kt}},
\]

\[
B = c_1 c_3 e^{kt - \frac{c}{3c_2c_5} \frac{1}{k(3+m)} e^{-(3+m)kt}},
\]

\[
C = c_4 e^{kt + \frac{2c}{3c_2c_5} \frac{1}{k(3+m)} e^{-(3+m)kt}}.
\]

This is the first solution of Bianchi type III model with exponential volumetric expansion. When \( m > -3 \), the scale factors admit constant values at early times of the universe, after that start increasing with the increase in cosmic time without showing any type of initial singularity and finally diverges to \( \infty \) for \( t \to \infty \). This shows that at the initial epoch, the universe starts with zero volume and expands exponentially approaching to infinite volume. However, for \( m < -3 \), the scale factors \( A, B \) increase with time while \( C \) approaches to zero as \( t \to \infty \). Also, the model with \( m = -3 \) represents the universe at early stage. Moreover, the expansion scalar for these scale factors exhibits the constant value, i.e., \( \Theta = 3k \) which shows uniform exponential expansion from \( t = 0 \) to \( t = \infty \).

The mean, directional and deceleration parameters turn out to be

\[
H_x = H_y = k + \frac{c}{3c_2c_5} e^{-(3+m)kt}, \quad H_z = k - \frac{2c}{3c_2c_5} e^{-(3+m)kt},
\]

\[
H = k, \quad q = -1.
\]

This shows that the mean Hubble parameter is constant whereas others are dynamical. As time approaches from zero to infinity (for \( m > -3 \)), the directional Hubble parameters start reducing towards the constant value of \( H \) and becomes equal as \( t \to \infty \). For \( m < -3 \), parameters along \( x \) and \( y \) axes will increase from the mean Hubble parameter by a constant factor \( \frac{c}{3c_2c_5} \) while parameter along \( z \)-axis decreases by twice the same factor. Also, the deceleration parameter appears with negative sign which shows accelerating expansion of the universe as we can expect for exponential volumetric expansion.

The anisotropy parameter of the expansion takes the form

\[
\Delta = \frac{2}{9} \left( \frac{c}{k c_2 c_5} \right)^2 e^{-2(m+3)kt}.
\]

Here one can observe that at \( t = 0 \), the anisotropy parameter measures a constant value while it vanishes for \( m > -3 \) at infinite time of the universe.
This indicates that the universe expands isotropically at later times without taking any effect from anisotropy of the fluid. However, for $m < -3$, the anisotropy in the expansion will increase with time.

Inserting the scale factors, Eqs. (4.11)-(4.13), in Eqs. (4.1)-(4.4) and use Eq. (4.10), we obtain the following energy density of anisotropic dark fluid

$$
\rho = \frac{c_5 e^{nk}}{3} \left[ -3m^2 k^2 + 2mk^2 + \frac{2mkc}{3c_2c_5} e^{-(3+m)kt} - \frac{2c^2}{(c_2c_5)^2} e^{-2(3+m)kt} - \frac{2\alpha^2}{c_3^2} e^{-2kt + \frac{2c}{3c_2c_5} e^{-(3+m)kt} + \frac{-6c^2}{c_3^2} e^{-2(3+m)kt}} \right].
$$

We see that the matter density is constant at early stage of the universe ($t = 0$) and shows monotonic behavior in the evolving cosmic times. This behavior of matter density deviates from GR where it becomes constant in exponential expansion. Here, for positive values of $m$, it will positively increase with time and diverges to $\infty$, showing future time singularity. For $-3 < m < 0$, it starts decreasing with cosmic time and eventually approaches to zero as $t \to \infty$. It is mentioned here that isotropic conditions of the model may not be satisfied and hence leading to the conclusion that Bianchi type III model remains anisotropic in the later times whereas in GR it becomes isotropic. The anisotropy of the model is maintained not due to the presence of anisotropic fluid but due to modification in gravity. It is worth mentioning here that the results correspond to GR results for $F(R) = 1$.

The anisotropic EoS parameter obtained from the field equations is given by

$$
\omega = \frac{3m(kc_2c_3c_5)^2 - 2mkcc_2c_5 e^{-(3+m)kt} + 9\alpha^2 c_2c_5 e^{-2kt + \frac{2c}{3c_2c_5} e^{-(3+m)kt}}}{3m(kc_2c_3c_5)^2(-3m + 2) + 2mkcc_2c_5 e^{-(3+m)kt} - 6c^2 c_3^2 e^{-2(3+m)kt} - 6\alpha^2 c_2c_5 e^{-2kt + \frac{2c}{3c_2c_5} e^{-(3+m)kt}}}. \tag{4.17}
$$

When $t \to \infty$ this gives a constant quantity, $\frac{1}{2-3m}$, which can further be characterized into phantom and quintessence regions for $m > 1$ or $-3 < m < 1$ respectively. For $m = 1$, the EoS parameter is $\omega = -1$, corresponds to the vacuum energy which is mathematically equivalent to cosmological constant while $m < -3$ does not yield fruitful results. We see that $\rho$ increases rapidly in the phantom region where $\omega < -1$. It is mentioned here that phantom regime support the observational evidence of a recent supernova data [33]-[35]. However, it decreases in the quintessence region representing relatively slow expansion.
Using Eqs. (4.10), (4.11) and (4.16) in Eq. (3.19), the skewness parameter along $z$-axis becomes

\[
\gamma = \left[ -9\alpha^2 c^2 e^{2c^2} t^{-2kt+\frac{2c}{3\alpha} e^{-\left(3+m\right)kt}} \right] / \\
\left[ 3m(kc_2c_5)^2 (-3m+2) + 2mkcc_2c_5 e^{-(3+m)kt} - 6\alpha^2 c^2 e^{-2(3+m)kt} \\
-6\alpha^2 c^2 e^{-(3+m)kt} \right]. \tag{4.18}
\]

We have already found $\delta = 0$ and we see that $\gamma$ also approaches to zero as $t \to \infty$. Thus the anisotropy of the fluid is completely removed in the future evolution of the universe. The scalar curvature and $f(R)$ function for exponential model becomes

\[
R = -10k^2 - \frac{4}{9c^2 c_5^2} e^{2(3+m)kt} + \frac{4}{3c_2 c_5} e^{-(3+m)kt} + \frac{2\alpha^2}{c_3^2} e^{-2kt+\frac{2c}{3\alpha} e^{-(3+m)kt}} \\
+ \frac{c_2 e^{mk}}{3} \left[ (36m^2 k^2 + 84mk^2 + 9R)c_2^2 c_5^2 - 4mk\lambda c_2 c_5 e^{-(3+m)kt} \\
+6\lambda^2 c_3^2 e^{-2(3+m)kt} + 24\alpha^2 c_2^2 c_5^2 e^{-kt+\frac{2c}{3\alpha} e^{-(3+m)kt}} \right]. \tag{4.19}
\]

This becomes constant in the quintessence region when $t \to \infty$ while for $m < -3$, it diverges giving the same behavior as the energy density. Since the energy density decreases in the quintessence region, thus the universe starts from inflation driven by the anisotropic energy density at the early stage where curvature is very large. The scalar curvature also reduces with the passage of time. After that time, the matter density or radiations become small and the curvature becomes constant.

### 4.2 Bianchi Type III Model for Power-law Expansion

Here we solve the field equations by assuming power-law volumetric expansion given as

\[
V = c_2 t^{3n}, \tag{4.20}
\]

where $c_2$ and $n$ are positive constants. For $n > 1$, Bianchi models exhibit accelerating volumetric expansion. When $n = 1$, the models represent volumetric expansion with constant velocity while for $n < 1$ these show decelerating volumetric expansion. Adopting the same procedure as in the exponential
model, we obtain the scale factors

\begin{align*}
A &= c_3 t^n \frac{e^{\frac{c}{t^2 c_5 (1-3n-mn)}}}{t^{1-3n-mn}}, \quad (4.21) \\
B &= c_1 c_3 t^n \frac{e^{\frac{c}{t^2 c_5 (1-3n-mn)}}}{t^{1-3n-mn}}, \quad (4.22) \\
C &= c_4 t^n \frac{e^{\frac{c}{t^2 c_5 (1-2n-mn)}}}{t^{1-3n-mn}}. \quad (4.23)
\end{align*}

We discuss the behavior of these scale factors in the future evolution by taking very large values of time. It is found that for \( n < \frac{1}{3+m} \), the scale factors \( A \) and \( B \) increase with time while \( C \) approaches to zero provided that \( m \) is always be greater than \(-3\) because \( n \) is assumed to be a positive number. This indicates that the universe is expanding along \( x \) and \( y \) axes with deceleration whereas there is no expansion along \( z \)-axis. Similarly, for \( n > \frac{1}{3+m} \), the behavior of scale factors interchanged, however, the range \(-3 < m \leq -2\) lies in the accelerating phase and \(-2 < m \) comprises the decelerating epoch.

The Hubble parameters, scalar expansion and anisotropy parameter of expansion in this case become

\begin{align*}
H &= \frac{n}{t}, \quad H_x = \frac{n}{t} + \frac{c t^{-(3+m)n}}{3c_2 c_5}, \quad H_z = \frac{n}{t} - \frac{2c}{3c_2 c_5} t^{-(3+m)n}, \quad (4.24) \\
\Theta &= \frac{3n}{t}, \quad \Delta = \frac{2}{9} \left(\frac{c}{nc_2 c_5}\right)^2 t^{2(1-3n-mn)}. \quad (4.25)
\end{align*}

We see that the Hubble parameters and scalar expansion are extremely large at the origin of the universe and start decreasing monotonically with the passage of time and possibly will take zero value in the future (provided that \( m > -3 \)). This shows that, at earlier times of the universe (just after the big bang), the expansion is much faster but slows down for later time of the universe. The anisotropy parameter approaches to zero for \( n > \frac{1}{3+m} \) leaving isotropic expansion in the future while for \( n < \frac{1}{3+m} \), its behavior is switched.

With the same manipulations as in the previous subsection, the unknown
functions in the field equations turn out to be

\[ \rho = \frac{c_5 t^{mn}}{3} \left[ (-3m^2 n^2 + 3mn + 2mn^2 + 6n)t^{-2} + \frac{2mnc}{3c_2c_5} t^{-(3+m)n-1} \right. \\
\left. \quad - \left( \frac{2c^2}{c_2c_5} \right)^2 t^{-2(3+m)n} - \frac{2c^2}{c_3} t^{-2n} e^{-2c \frac{1}{3} \frac{1}{(1-3n-nm)}} \right], \tag{4.26} \]

\[ \omega = \left[ 9\alpha^2 c_2^2 c_5^2 t^{-2n} e^{-2c \frac{1}{3} \frac{1}{(1-3n-nm)}} - 2c_2 c_5 c_3^2 c m n t^{-(3+m)n-1} + 3mn^2 (c_2 c_3 c_5^2)^2 t^{-2} \right] \\
\left/ 3(c_2 c_3 c_5)^2 (-3mn(mn - 1) + 2mn^2 + 6n)t^{-2} + 2mnc c_2 c_5 c_3^2 t^{-(3+m)n-1} \right. \\
\left. - 6c^2 c_3^2 t^{-2(3+m)n} - 6\alpha^2 c_2^2 c_5^2 t^{-2n} e^{-2c \frac{1}{3} \frac{1}{(1-3n-nm)}} \right], \tag{4.27} \]

\[ \gamma = \left[ -9\alpha^2 c_2^2 c_5^2 t^{-2n} e^{-2c \frac{1}{3} \frac{1}{(1-3n-nm)}} / [2(c_2 c_3 c_5)^2 (-3mn(mn - 1) + 2mn^2 + 6n)t^{-2} + 2mnc c_2 c_5 c_3^2 t^{-(3+m)n-1} - 6c^2 c_3^2 t^{-2(3+m)n} \right. \\
\left. - 6\alpha^2 c_2^2 c_5^2 t^{-2n} e^{-2c \frac{1}{3} \frac{1}{(1-3n-nm)}} \right]. \tag{4.28} \]

In the accelerating models, the energy density will increase with time while for the decelerating models it will decrease. The dark energy EoS parameter for the accelerating models gives \( \omega = -1 + \frac{6-3m(mn-1)+3mn}{6-3m(mn-1)+2mn} \) for later times. This indicates that the universe will pass through quintessence epoch with appropriate values of \( m > -3 \). The skewness deviation parameter along z-axis vanish in the future evolution. Thus DE isotropous in the far future.

The scalar curvature is

\[ R = (6n - 10n^2)t^{-2} - \frac{32}{9} \frac{c^2}{c_2^2 c_5^2} t^{-(3+m)n} + \frac{4cn}{3c_2 c_5} t^{-(3+m)n} \]

\[ + \frac{2c^2}{c_3^2} t^{-2n} e^{-2c \frac{1}{3} \frac{1}{(1-3n-nm)}}, \tag{4.29} \]

\[ f(R) = \frac{c_5 t^{mn}}{2} \left[ (4mn(mn - 1) + \frac{25}{3} mn^2 + 6n)t^{-2} + R \right. \\
\left. + 8\alpha^2 \frac{c^2}{c_3^2} t^{-2n} e^{2c \frac{1}{3} \frac{1}{(1-3n-nm)}} + \frac{2\lambda^2}{3c_3} t^{-2(3+m)n} - \frac{2\lambda mn}{9c_2 c_5} t^{-(3+m)n-1} \right]. \tag{4.30} \]

The scalar curvature approaches to zero for the passing times similar to \( \rho \).
5 Summary and Outlook

The aim of this paper is to discuss expansion of the universe due to anisotropic DE in $f(R)$ gravity. For this purpose, we have found exact solutions of Bianchi type III model by assuming exponential and power-law volumetric expansion to cover all types of the expansion histories. These models represent an accelerated expansion of the universe with $V \to \infty$ as $t \to \infty$ and supports the observations of the Type Ia supernova [1, 2] and WMAP data [5, 6]. The whole discussion is made in terms of $F(R)$ which is assumed to have direct proportion with volume of the universe. The results of the paper can be summarized as follows:

- The solution of the field equations for Bianchi type III model yields that the scale factors $A$ and $B$ are equal and the deviation from isotropy along $y$-axis is zero (i.e., $\delta = 0$). This shows that pressure of DE along $x$ and $y$ axes is same.

- The physical behavior of the dynamical quantities depends on the values of $m$. In case of exponential model, the dynamical parameters show different behavior for $m \gtrsim -3$ while in power-law model we can only discuss $m > -3$ for later times to keep $n$ positive. However, here we have further two types of models for $n < \frac{1}{3+m}$ or $n > \frac{1}{3+m}$.

- The scalar expansion is constant for the first solution indicating that the universe expands homogeneously whereas in the second solution it shows that the expansion rate is faster at the beginning but slows down with the passage of time. The anisotropy parameter of expansion at $t = 0$ measures a constant value while it vanishes for $m > -3$ at infinite time of the universe. This shows that the universe expands isotropically at later times. It is interesting to mention here that our solution can be used to understand cosmological phase transitions. It may help to understand the isotropization mechanism which could be responsible for the passage from a possible prior anisotropic phase to the present isotropic era we live in.

- Behavior of EoS parameter and $\rho$ exhibit that the universe will pass through quintessence region and phantom regions for $m > 1$ or $-3 < m < 1$ respectively. It is worth mentioning here that solution in the phantom regime ($\omega < -1$) supports the observational evidence of a
recent supernova data \cite{33}-\cite{35}. Notice that the energy density increases/decreases monotonically with the passage of time depending upon the value of $m$ in $f(R)$ gravity. However, it becomes constant in GR at later times. The skewness parameter along $z$-axis will vanish thus acquiring isotropy in the fluid at infinite times.

- The Bianchi type III model remains anisotropic even in the later times which is different from GR \cite{28} where these become isotropic.
- The scalar curvature becomes constant for $m > -3$ and diverges for $m < -3$ in exponential models while it becomes zero in the power-law models for later times.

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