Systematic analysis of the MNS matrix with diagonal reflection symmetries

Masaki J. S. Yang\textsuperscript{1,2,*}

\textsuperscript{1}Department of Physics, Saitama University, Shimo-okubo, Sakura-ku, Saitama, 338-8570, Japan
\textsuperscript{2}Department of Physics, Graduate School of Engineering Science, Yokohama National University, Yokohama, 240-8501, Japan

In this letter, we systematically analyzed the MNS matrix with diagonal reflection symmetries. If the mass matrix of charged leptons $m_e$ is hierarchical, by neglecting the 13 mixing of $m_e$, the MNS matrix is represented by four parameters and several sign degrees of freedom. By substituting the three observed mixing angles $\theta_{ij}$ as input parameters, the Dirac phase $\delta$ and the Majorana phases $\alpha_2, 3$ become functions of the 12 mixing of charged leptons $s_e$. As a result, we obtain a clear correlations between CP-violating phases $|\sin \delta| \simeq 1.6|\sin \alpha_2| \simeq 2.5|\sin \alpha_3|$.

I. INTRODUCTION

The Dirac phase $\delta$, which represents CP violation in the lepton sector, has been measured recently. Although the phase $\delta$ is not determined experimentally, a generalized CP symmetry (GCP) \cite{1-22} can fix this CP violating phase. One of a notable example is the $\mu - \tau$ reflection symmetry \cite{23-54} that predicts the maximal Dirac phase $\delta = \pm \pi/2$.

On the other hand, diagonal reflection symmetries \cite{55-58} are GCPs that can predict relatively small $\delta$ in a way that can unify quarks and leptons. However, general properties of the symmetries are not yet well understood. Thus, in this letter, we perform a systematic analysis of the DRS.

\textsuperscript{*}Electronic address: yang@krishna.th.phy.saitama-u.ac.jp
II. DIAGONAL REFLECTION SYMMETRIES

In this section, we define the diagonal reflection symmetries (DRS). First, a representation
of the CKM matrix proposed by Fritzsch and Xing is \[59\],

\[
V_{\text{CKM}} = U^\dagger_u U_d = \begin{pmatrix}
 c_u & s_u & 0 \\
 -s_u & c_u & 0 \\
 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
 e^{-i\phi} & 0 & 0 \\
 0 & c_q & s_q \\
 0 & -s_q & c_q
\end{pmatrix}
\begin{pmatrix}
 c_d & -s_d & 0 \\
 s_d & c_d & 0 \\
 0 & 0 & 1
\end{pmatrix},
\]

(1)

where \( s_\theta \equiv \sin \theta, c_\theta \equiv \cos \theta \). By determining the four parameters \( s_{q,u,d} \) and \( \phi \) from the observ-
ables, \( \phi \) is almost equal to \( \pi/2 \). Thus, if we interpret unitary matrices \( U_{u,d} \) diagonalizing the
mass matrix of quarks \( m_{u,d} \) as

\[
U_u = \begin{pmatrix}
 +i & 0 & 0 \\
 0 & c_t & s_t \\
 0 & -s_t & c_t
\end{pmatrix}, \quad U_d = \begin{pmatrix}
 1 & 0 & 0 \\
 0 & c_b & s_b \\
 0 & -s_b & c_b
\end{pmatrix},
\]

(2)

the mass matrices \( m_{u,d} = U_{u,d} m_{u,d}^{\text{diag}} U_{u,d}^\dagger \) reconstructed from \( U_{u,d} \) have diagonal reflection sym-
metries defined as \[55\]

\[
R m_{u,\nu}^* R = m_{u,\nu}, \quad m_{d,e}^* = m_{d,e} \quad R = \text{diag}(-1, 1, 1).
\]

(3)

Under these symmetries, the mass matrix of charged leptons \( m_e \) is real and the that of neutrinos
\( m_\nu \) has the following form;

\[
m_\nu = \begin{pmatrix}
 m_{11} & im_{12} & im_{13} \\
 im_{12} & m_{22} & m_{23} \\
 im_{13} & m_{23} & m_{33}
\end{pmatrix},
\]

(4)

with \( m_{ij} \in \mathbb{R} \). These remnant symmetries are almost renormalization-invariant and are easily
realized by scalar fields with vacuum expectation values \( \langle \theta_u \rangle = iv_u, \langle \theta_d \rangle = v_d \) that couple to
only the first generation \[56\].

Since the singular value decomposition of a real matrix is done by a real orthogonal matrix
$O_f$, the MNS matrix $U$ is

$$U = O_e^T \begin{pmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} O_\nu P,$$

(5)

$$O_\nu P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\nu & s_\nu \\ 0 & -s_\nu & c_\nu \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\phi_1} & 0 & 0 \\ 0 & e^{i\phi_2} & 0 \\ 0 & 0 & e^{i\phi_3} \end{pmatrix}. \quad (6)$$

Here, phases $\phi_i = 0$ or $\pi/2$ in the phase matrix $P$ originate from positive or negative singular mass values after a real diagonalization by $O_\nu$.

Under an approximation that the 13 mixing of $m_e$ is negligible, a combination of the 23 mixings of $O_\nu$ and $O_e$ yields a representation of the MNS matrix as

$$U = \begin{pmatrix} c_e & s_e & 0 \\ -s_e & c_e & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} i & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\phi_1} & 0 & 0 \\ 0 & e^{i\phi_2} & 0 \\ 0 & 0 & e^{i\phi_3} \end{pmatrix}. \quad (7)$$

For a hierarchical $m_e$, the error from this approximation is at most $\sqrt{m_\mu m_e/m_\tau} \simeq 0.004$ and is safely neglected.

Let us consider sign degrees of freedom for these parameters $c_{ij}$ and $s_{ij}$. First, the sign of the phase $i$ can be fixed to positive because it is absorbed to $s_e$. Furthermore, the five signs can be made positive by the redefinition of phases. For example, the sign of $c_{12}$ can be changed by multiplying $\text{diag}(-1,-1,1)$ from the right. For the later convenience, we choose the signs of $c_{12}, c_{13}, c_{23}$ and $s_{13}, s_{23}$ to be positive. The other two signs can also be determined from the signs of $\cos \delta$ and $\sin \delta$ for the Dirac phase $\delta$. From the following calculation, we can choose those of $s_{12}$ and $s_e c_e$. However, it is found that the signs of $c_e$ and $s_e$ cannot be determined independently.
III. SYSTEMATIC ANALYSIS OF $U_{\text{MNS}}$

In this section, we analyze the MNS matrix with DRS. By neglecting the phase matrix $P$ and performing the product of the matrices, $U$ is represented as

$$U = \begin{pmatrix}
-s_e c_{23} s_{12} + c_{12} (-s_e s_{13} s_{23} + i c_e c_{13}) & s_e c_{12} c_{23} + s_{12} (-s_e s_{13} s_{23} + i c_e c_{13}) & s_e c_{13} s_{23} + i c_e s_{13} \\
-c_e c_{23} s_{12} + c_{12} (-c_e s_{13} s_{23} - i s_e c_{13}) & c_e c_{12} c_{23} + s_{12} (-c_e s_{13} s_{23} - i s_e c_{13}) & c_e c_{13} s_{23} - i s_e s_{13} \\
s_{12} s_{23} - c_{12} c_{23} s_{13} & c_{12} s_{23} - c_{23} s_{12} s_{13} & c_{13} c_{23}
\end{pmatrix}.$$  

(8)

On the other hand, the standard PDG parameterization is $[30]$

$$U^{\text{PDG}} = \begin{pmatrix}
c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i \delta} \\
-s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i \delta} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i \delta} & s_{23} c_{13} \\
s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i \delta} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i \delta} & c_{23} c_{13}
\end{pmatrix} \times \text{diag}(1, e^{i \alpha_2/2}, e^{i \alpha_3/2}).$$  

(9)

In this definition, $P$ and the phases $\phi_i$ have a potential to set the Majorana phases $\alpha_i$ to $\alpha_i + \pi$ and have no effect on any other physical quantity.

By comparing the absolute values of the third column of Eq. (8) with the PDG parameterization, $s_{13}$ and $s_{23}$ should satisfy

$$|s_e c_{13} s_{23} + i c_e s_{13}|^2 = (s_{13}^{\text{PDG}})^2,$$

(10)

$$|c_e c_{13} s_{23} - i s_e s_{13}|^2 = (s_{23}^{\text{PDG}} c_{13}^{\text{PDG}})^2,$$

(11)

and we obtain the following solutions;

$$s_{23} = \frac{\sqrt{(s_e s_{13}^{\text{PDG}})^2 - (c_e c_{13}^{\text{PDG}} s_{23}^{\text{PDG}})^2}}{\sqrt{-c_e^4 + (c_e s_{13}^{\text{PDG}})^2 - (s_e c_{13}^{\text{PDG}} s_{23}^{\text{PDG}})^2 + s_c^4}}, \quad s_{13} = \frac{\sqrt{(c_e s_{13}^{\text{PDG}})^2 - (s_e c_{13}^{\text{PDG}} s_{23}^{\text{PDG}})^2}}{\sqrt{c_e^4 - s_c^4}}.$$  

(12)

Similarly, from $|U_{12}|^2 = |U_{12}^{\text{PDG}}|^2$, the condition satisfied by $s_{12}$ is

$$|s_e (c_{12} c_{23} - s_{12} s_{13} s_{23}) + i s_{12} c_e c_{13}|^2 = (s_{12}^{\text{PDG}} c_{13}^{\text{PDG}})^2.$$  

(13)

Solving $s_{12}$ from Eqs. (12) and (13), we obtain four solutions. However, only two solutions are physically inequivalent, and they differ in the sign of $\cos \delta$. Experiments favor the solution of
$s_{12} < 0$, because the sign of $s_{12}$ and $\cos\delta$ are almost same as a result of drawing the plots (Figure 1). Since the condition (13) depends only on $s_e^2$ and $c_e^2$, the solutions of $s_{12}$ is independent of signs of $s_e$ and $c_e$.

As input values, we use the latest global fit without Super-Kamiokande (SK) in the Normal Hierarchy (NH) [61]:

$$\sin^2 \theta_{12}^{PDG} = 0.304, \quad \sin^2 \theta_{23}^{PDG} = 0.573, \quad \sin^2 \theta_{13}^{PDG} = 0.0222. \quad (14)$$

This is due to the reason that the values of Inverted hierarchy (IH) with or without SK are close to these values. Although the inclusion of the SK data makes $s_{23}^{PDG}$ about 0.1 smaller for NH, the qualitative behavior in the following discussion remains the same.

From this, $\sin\delta$ and $\cos\delta$ can be expressed as functions of $s_e$ and some sign degrees of freedom. The parameter $\cos\delta$ is given by

$$\cos\delta = \frac{|U_{12}^{PDG}|^2 - (s_{12}^{PDG} s_{13}^{PDG} s_{23}^{PDG})^2 - (c_{12}^{PDG} c_{23}^{PDG})^2}{-2 s_{12}^{PDG} s_{13}^{PDG} s_{23}^{PDG} c_{12}^{PDG} c_{23}^{PDG}} \quad (15)$$

$$= \frac{|U_{22}|^2 (1 - |U_{13}|^2)^2 - |U_{12}|^2 |U_{23}|^2 - |U_{11}|^2 |U_{33}|^2}{-2 |U_{13}| |U_{12}| |U_{23}| |U_{11}| |U_{33}|}. \quad (16)$$

Also, $\sin\delta$ can be evaluated from the Jarlskog invariant;

$$J = -\text{Im} \left[ U_{\mu 3} U_{\tau 2} U_{\mu 2}^{*} U_{\tau 3}^{*} \right] = \sin\delta s_{12}^{PDG} c_{12}^{PDG} s_{13}^{PDG} (c_{13}^{PDG})^2 s_{23}^{PDG} c_{23}^{PDG} \quad (17)$$

$$= c_{13} c_{23} c_e s_e (c_{12} s_{23} + c_{23} s_{12} s_{13}) (s_{12} s_{23} - c_{12} c_{23} s_{13}). \quad (18)$$

It predicts a proportional relationship between $\sin\delta$ and $s_e$. When $s_{13}$ is small, the invariant is roughly

$$J \simeq c_{12} c_{23} c_e s_{12}^2 s_{23} s_e, \quad \sin\delta \simeq -\frac{c_e s_e s_{23}}{s_{13}^{PDG} (c_{13}^{PDG})^2} \simeq \pm 5 s_e. \quad (19)$$

Then the sign of $s_e c_e$ and $\sin\delta$ are opposite. The minus sign comes from the choice $s_{12} \simeq -s_{12}^{PDG}$.

Figure 1 shows plots of $\cos\delta$ and $\sin\delta$ expressed as functions of $s_e$. In the plot of $\cos\delta$, the red and green lines correspond to $s_{12} > 0$ and $s_{12} < 0$. Since the signs are approximately equal ($\text{sign}(\cos\delta) \simeq \text{sign}(s_{12})$) in the parameter regions, experiments favor $s_{12} < 0$. For $\sin\delta$, the color of the lines depend on the sign of $c_e$. The parameter $|\sin\delta|$ becomes maximal around $|s_e| \simeq 0.2$. In regions where $s_e$ is larger than this, $\sin\delta$ and $s_{13}$ become complex numbers and have no real solution. This is due to the following reasons. In the range $|s_e| \lesssim 0.2$, $s_e$ can be regarded as a
FIG. 1: Plots of $\cos \delta$ and $\sin \delta$ expressed as functions of $s_e$. The bright green and orange region represent the $1\sigma$ regions of NH and IH (without SK).

perturbation. From Eq. (12), $s_{12}$ and $s_{23}$ are approximately equal to those of PDG and $s_{13}$ is constrained as

$$s_{23} \simeq s_{23}^{\text{PDG}}, \quad s_{12} \simeq -s_{12}^{\text{PDG}}, \quad s_{13} \simeq \sqrt{(s_{13}^{\text{PDG}})^2 - (s_{{e}s_{23}}^{\text{PDG}})^2}. \quad (20)$$

Since the maximum value of $s_e$ in this range is realized by $s_{13} = 0$,

$$s_e^{\text{max}} \simeq \frac{s_{13}^{\text{PDG}}}{s_{23}^{\text{PDG}}} \simeq 0.196. \quad (21)$$

There exists other solutions with $s_e \simeq \pm 1$. However, since these solutions imply that the eigenstates of the charged leptons $e$ and $\mu$ are interchanged by diagonalization, it is excluded from a point of view of the natural mass matrix [12].

A. Majorana phases

A similar analysis is performed for the Majorana phases. These phases can be evaluated from the following quantities [63];

$$I_1 = \text{Im} \left[ U_{e2}^2 U_{e1}^* / |U_{e2} U_{e1}|^2 \right] = \sin \alpha_2, \quad (22)$$

$$I_2 = \text{Im} \left[ U_{e3}^2 U_{e1}^* / |U_{e3} U_{e1}|^2 \right] = \sin(\alpha_3 - 2\delta). \quad (23)$$

Expansions of $\sin \alpha_{2,3}$ for small $s_e$ is respectively

$$\sin \alpha_2^0 \simeq -\frac{2s_e c_{23}}{c_e c_{12} s_{12}} \simeq +3s_e, \quad \sin \alpha_3^0 \simeq -\frac{2s_e c_e c_{12} c_{23}}{c_{13} s_{12}} \simeq +2s_e c_e. \quad (24)$$
Since these signs depend on \( \text{sign}(c_e s_e) \), we conclude that the signs of \( c_e \) and \( s_e \) cannot be determined independently. Furthermore, \( \sin \alpha_i \) has sign degrees of freedom due to \( \phi_i = 0 \) or \( \pi/2 \) in Eq. (11):\

\[
\alpha_2 = \alpha_2^0 + 2(\phi_2 - \phi_1), \quad \alpha_3 = \alpha_3^0 + 2(\phi_3 - \phi_1).
\]

Therefore, it is difficult to derive general results on the signs of \( \sin \alpha_i \) and \( \cos \alpha_i \) from the mixing matrix only. Plots of \( \sin \alpha_i \) for \( s_e \) are shown in Figure 2. As a result, the CP-violating observables have the following correlations.

\[
\frac{\sin \alpha_2}{\sin \delta} \simeq \frac{2 s_{13} c_{13} c_{23}}{c_2^2 c_{12} s_{12} s_{23}} \simeq 3 \sqrt{2} s_{13} \simeq 0.6, \tag{26}
\]

\[
\frac{\sin \alpha_3}{\sin \delta} \simeq \frac{2 c_{12} c_{23} c_{13} s_{13}}{s_{12} s_{23}} \simeq 2 \sqrt{2} s_{13} \simeq 0.4. \tag{27}
\]

Since \( U \) (8) is CP-symmetric in the limit where \( s_e \) becomes zero, it is a natural consequence that these CP phases have such correlations.

**IV. SUMMARY**

In this letter, we systematically analyzed the MNS matrix with diagonal reflection symmetries. If the mass matrix of charged leptons \( m_e \) is hierarchical, by neglecting the 13 mixing of \( m_e \), the MNS matrix is represented by four parameters and several sign degrees of freedom. By substituting the three observed mixing angles \( \theta_{ij} \) as input parameters, the Dirac phase \( \delta \) and the Majorana phases \( \alpha_{2,3} \) become functions of the 12 mixing of charged leptons \( s_e \). As a result, we obtain a clear correlations between CP-violating phases \(|\sin \delta| \simeq 1.6|\sin \alpha_2| \simeq 2.5|\sin \alpha_3|\).
Acknowledgment

This study is financially supported by JSPS Grants-in-Aid for Scientific Research No. JP18H01210 and MEXT KAKENHI Grant No. JP18H05543.

[1] G. Ecker, W. Grimus, and W. Konetschny, Nucl. Phys. B 191, 465 (1981).
[2] G. Ecker, W. Grimus, and H. Neufeld, Nucl. Phys. B 247, 70 (1984).
[3] M. Gronau and R. N. Mohapatra, Phys. Lett. B 168, 248 (1986).
[4] G. Ecker, W. Grimus, and H. Neufeld, J. Phys. A 20, L807 (1987).
[5] H. Neufeld, W. Grimus, and G. Ecker, Int. J. Mod. Phys. A 3, 603 (1988).
[6] P. Ferreira, H. E. Haber, and J. P. Silva, Phys. Rev. D 79, 116004 (2009), arXiv:0902.1537.
[7] F. Feruglio, C. Hagedorn, and R. Ziegler, JHEP 07, 027 (2013), arXiv:1211.5560.
[8] M. Holthausen, M. Lindner, and M. A. Schmidt, JHEP 04, 122 (2013), arXiv:1211.6953.
[9] G.-J. Ding, S. F. King, and A. J. Stuart, JHEP 12, 006 (2013), arXiv:1307.4212.
[10] I. Girardi, A. Meroni, S. Petcov, and M. Spinrath, JHEP 02, 050 (2014), arXiv:1312.1966.
[11] N. Nishi, Phys. Rev. D 88, 033010 (2013), arXiv:1306.0877.
[12] G.-J. Ding, S. F. King, C. Luhn, and A. J. Stuart, JHEP 05, 084 (2013), arXiv:1303.6180.
[13] F. Feruglio, C. Hagedorn, and R. Ziegler, Eur. Phys. J. C 74, 2753 (2014), arXiv:1303.7178.
[14] P. Chen, C.-C. Li, and G.-J. Ding, Phys. Rev. D 91, 033003 (2015), arXiv:1412.8352.
[15] G.-J. Ding, S. F. King, and T. Neder, JHEP 12, 007 (2014), arXiv:1409.8005.
[16] G.-J. Ding and Y.-L. Zhou, JHEP 06, 023 (2014), arXiv:1404.0592.
[17] M.-C. Chen, M. Fallbacher, K. Mahanthappa, M. Ratz, and A. Trautner, Nucl. Phys. B 883, 267 (2014), arXiv:1402.0507.
[18] C.-C. Li and G.-J. Ding, JHEP 05, 100 (2015), arXiv:1503.03711.
[19] J. Turner, Phys. Rev. D 92, 116007 (2015), arXiv:1507.06224.
[20] W. Rodejohann and X.-J. Xu, Phys. Rev. D 96, 055039 (2017), arXiv:1705.02027.
[21] J. Penedo, S. Petcov, and A. Titov, JHEP 12, 022 (2017), arXiv:1705.00309.
[22] N. Nath, R. Srivastava, and J. W. Valle, Phys. Rev. D 99, 075005 (2019), arXiv:1811.07040.
[23] P. F. Harrison and W. G. Scott, Phys. Lett. B547, 219 (2002), arXiv:hep-ph/0210197.
[24] W. Grimus and L. Lavoura, Phys. Lett. B579, 113 (2004), arXiv:hep-ph/0305309.
[25] W. Grimus, S. Kaneko, L. Lavoura, H. Sawanaka, and M. Tanimoto, JHEP 01, 110 (2006), arXiv:hep-ph/0510326.
[26] Y. Farzan and A. Yu. Smirnov, JHEP 01, 059 (2007), arXiv:hep-ph/0610337.
[61] M. C. Gonzalez-Garcia, M. Maltoni, and T. Schwetz, Universe 7, 459 (2021), arXiv:2111.03086.

[62] R. D. Peccei and K. Wang, Phys. Rev. D 53, 2712 (1996), arXiv:hep-ph/9509242.

[63] J. F. Nieves and P. B. Pal, Phys. Rev. D 36, 315 (1987).