Abstract

The low-temperature representation for the quark condensate in a weak magnetic field $H$ is known up to two-loop order. Remarkably, at one-loop order, the published series for the quark condensate in the chiral limit and $H \ll T^2$ are inconsistent. Using an alternative representation for the kinematical Bose functions, we derive the series to arbitrary order in $H/T^2$, and also determine which of the published results is correct.

1 Motivation

The low-energy behavior of quantum chromodynamics in the presence of a magnetic field has been explored by many authors in great detail. The partition function has been evaluated up to two-loop order within chiral perturbation theory, and low-temperature series for the quark condensate have been presented. A comprehensive list of references can be found in the nice review by Andersen et al. [1].

Of particular interest is the low-temperature expansion of the quark condensate in the chiral limit in a weak magnetic field $H$. The weak magnetic field limit is implemented by $|qH| \ll T^2$ where $|q|$ stands for the electric charge of the pion. The relevant quantity is

$$\frac{\langle \bar{q}q \rangle}{\langle 0|\bar{q}q|0 \rangle},$$

where $\langle 0|\bar{q}q|0 \rangle$ is the quark condensate at zero temperature and zero magnetic field. Two series are available in the literature for the above quantity in the chiral limit: up
to one-loop order, the author of Refs. [2–5] obtains
\[
\frac{\langle \bar{q}q \rangle}{\langle 0 | \bar{q}q | 0 \rangle} = 1 + \frac{C |qH|}{16\pi^2 F^2} - \frac{T^2}{8F^2} - \frac{7 \sqrt{|qH|T}}{48\pi F^2} - \frac{|qH|}{16\pi^2 F^2} \log \frac{|qH| T^2}{T^2}, \quad |qH| \ll T^2,
\]
(1.2)
while the author of Refs. [6, 7] ends up with
\[
\frac{\langle \bar{q}q \rangle}{\langle 0 | \bar{q}q | 0 \rangle} = 1 + \frac{|qH| \log 2}{16\pi^2 F^2} - \frac{T^2}{8F^2} + \frac{5 \sqrt{|qH|T}}{48\pi F^2} + \ldots, \quad |qH| \ll T^2.
\]
(1.3)
Both series contain the leading term at zero temperature,
\[
\frac{|qH| \log 2}{16\pi^2 F^2},
\]
(1.4)
which is linear in the magnetic field and positive, and has been derived in the pioneering paper by Shushpanov and Smilga [8]. As far as finite-temperature corrections are concerned, we first have a term that does not involve the magnetic field,
\[-\frac{T^2}{8F^2},\]
(1.5)
derived a long time ago in the original article by Gasser and Leutwyler [9]. However, in nonzero magnetic field, the two series disagree with respect to the leading contribution at finite temperature: the coefficients of the $\sqrt{HT}$-term are different both in magnitude and sign. Finally, according to Refs. [2–5], logarithmic terms of the form $H \log(H/T^2)$ also emerge.

In order to make the low-temperature expansion in the weak magnetic field limit ($|qH| \ll T^2$) more transparent, we factorize out temperature and use the relevant expansion parameter $\epsilon < 1$,
\[
\epsilon = \frac{|qH|}{T^2}.
\]
(1.6)
The two published series can then be cast into the general form
\[
\frac{\langle \bar{q}q \rangle}{\langle 0 | \bar{q}q | 0 \rangle} = 1 + \frac{|qH| \log 2}{16\pi^2 F^2} + \left\{ q_1 \sqrt{\epsilon} + q_2 \epsilon \log \epsilon + q_3 \epsilon + q_4 \epsilon^2 + q_5 \epsilon^3 + O(\epsilon^4) \right\} T^2 - \frac{1}{8F^2} T^2 + O(T^4).
\]
(1.7)
Let us consider the quantity
\[
Q(\epsilon) = \frac{1}{\sqrt{\epsilon} T^2} \left( \frac{\langle \bar{q}q \rangle}{\langle 0 | \bar{q}q | 0 \rangle} - 1 - \frac{|qH| \log 2}{16\pi^2 F^2} + \frac{1}{8F^2} T^2 - O(T^4) \right).
\]
(1.8)
\footnote{Note that the constant $C$ in Eq. (1.2) involves further terms: $-2\gamma_E + 2 \log 4\pi + \frac{1}{3}$. As we comment at the end of Section 2, these terms do not contribute at zero temperature.}
In the limit \( \epsilon \to 0 \), we have
\[
\lim_{\epsilon \to 0} Q(\epsilon) = q_1.
\] (1.9)

Irrespective of whether or not a logarithmic contribution is present, \( Q(\epsilon) \) should converge to the leading coefficient \( q_1 \). The authors of Refs. [2–5] and Refs. [6, 7] end up with different values for \( q_1 \). The motivation for the present study is to decide which of the two published results is correct, and to go to higher orders in the weak magnetic field expansion. Our calculation is based on chiral perturbation theory, much like Refs. [2–7], but relies on an alternative representation for the kinematical Bose functions that appear at one-loop order – our approach then allows for a systematic and very transparent expansion in the limit \( |qH| \ll T^2 \).

As it turns out, our leading coefficient \( q_1 \) is yet different from the two published results, and higher-order terms in our series disagree with the Agasian series [2–5]. We have checked that our series perfectly coincides with the exact result that we have evaluated numerically. We stress that the criticism is not directed towards the one-loop evaluation of the partition function – rather, our intention is to point out that, in the low-temperature expansion of the quark condensate at one-loop order, errors exist concerning the weak magnetic field expansion \( |qH| \ll T^2 \) in the chiral limit. More important, the correct series is derived in the present study for the first time.

### 2 Quark Condensate in Weak Magnetic Fields

The essentials of chiral perturbation theory have been outlined in many excellent reviews where the interested reader is referred to (see, e.g., Refs. [10, 11]). Here we merely provide a brief sketch of the method and the one-loop evaluation.

The QCD Lagrangian for two flavors reads
\[
\mathcal{L}_{QCD} = -\frac{1}{2g^2} \text{tr} G^{\mu \nu} G_{\mu \nu} + \bar{q} i \gamma^\mu D_\mu q - \bar{q} m q, \quad (q = u, d).
\] (2.1)

In the present study we focus on the isospin limit \( m_u = m_d \). The quark condensate,
\[
\langle 0 | \bar{q} q | 0 \rangle,
\] (2.2)

is the order parameter associated with the spontaneously broken chiral symmetry \( SU(2) \times SU(2) \to SU(2) \). The corresponding Goldstone bosons are the three pions.

In the effective field theory, the pion fields \( \pi^i (i = 1, 2, 3) \) are contained in the SU(2) matrix \( U = \exp(i\tau^i \pi^i / F) \), where \( \tau^i \) are the Pauli matrices and \( F \) is the (tree-level) pion decay constant. The leading term in the effective Lagrangian is of momentum order \( p^2 \) and reads
\[
\mathcal{L}_{eff}^2 = \frac{1}{4} F^2 \text{Tr} \left[ (D_\mu U)^\dagger (D_\mu U) - M^2 (U + U^\dagger) \right],
\] (2.3)
Figure 1: QCD partition function diagrams contributing up to one-loop order in the low-temperature expansion. The filled circle refers to $\mathcal{L}_{eff}^2$, while the number 4 in the box corresponds to $\mathcal{L}_{eff}^4$.

where $M$ is the (tree-level) pion mass. It should be pointed out that the magnetic field $H$ is taken into account by the covariant derivative

$$D_\mu U = \partial_\mu U + i [Q, U] A^{EM}_\mu .$$

(2.4)

Where $Q$ stands for the charge matrix of the quarks, $Q = \text{diag}(2/3, -1/3)e$, and the gauge field $A^{EM}_\mu = (0, 0, -Hx, 0)$ incorporates the constant magnetic field in Landau gauge $\{1\}$. The next-to-leading piece in the effective Lagrangian $-\mathcal{L}_{eff}^4$ is of momentum order $p^4$, and involves various next-to-leading order effective constants $l_i$ and $h_i$ that require renormalization (see Appendix A). The explicit form of $\mathcal{L}_{eff}^4$ can be found, e.g., in Refs. $\{11, 12\}$.

Chiral perturbation theory refers to low temperatures, small quark masses and weak magnetic fields. We first consider the free energy density from where the quark condensate can be derived. The corresponding Feynman diagrams, up to one-loop order $p^4$, are depicted in Fig. 1. Their evaluation leads to the following low-temperature representation for the free energy density:\footnote{The present evaluation parallels the evaluation of the partition function in zero magnetic field described in much detail in Refs. $\{13, 14\}$.}

$$z = z_0(M, 0, H) - \frac{2}{4\pi^2} g_0(M, T, 0) - \tilde{g}_0(M, T, H) + O(p^6) .$$

(2.5)

The quantity $z_0$ is the free energy density at zero temperature, while the two other contributions are finite-temperature corrections: the first one refers to zero magnetic field, the second one incorporates the magnetic field.

As we show in Appendix A.3, the quark condensate can then be obtained from the free energy density. Up to one-loop order we get

$$\langle \bar{q}q \rangle / \langle 0 | \bar{q}q | 0 \rangle = 1 - \frac{|qH|}{16\pi^2 F^2} \int_0^\infty dt t^{-1} \left( \frac{1}{\sinh(t)} - \frac{1}{t} \right) - \frac{3g_1(0, T, 0)}{2F^2} - \frac{\tilde{g}_1(0, T, H)}{F^2} + O(p^4) ,$$

(2.6)
where $\langle 0|\bar{q}q|0 \rangle$ is the quark condensate at zero temperature and zero magnetic field. The first line in Eq. (2.6) refers to zero temperature, while the kinematical functions $g_1(0,T,0)$ and $\tilde{g}_1(0,T,H)$ describe the behavior of the system at finite temperature. We now analyze in detail the structure of the above terms.

The basic object in the evaluation of the partition function – or, equivalently, free energy density – is the thermal propagator $G(x)$ for the pions in the background of a magnetic field. It can be constructed from the zero-temperature propagator $\Delta(x)$ in Euclidean space by

$$G(x) = \sum_{n=-\infty}^{\infty} \Delta(\vec{x}, x_4 + n\beta), \quad \beta = \frac{1}{T}. \quad (2.7)$$

The propagator $\Delta^0(x)$ referring to the neutral pion is not affected by the magnetic field and takes the simple form

$$\Delta^0(x) = \frac{(2\pi)^d}{4\pi} \int \frac{d^d p}{e^{ipx} (M^2 + p^2)^{-1}} = \int_0^\infty d\rho (4\pi\rho)^{-d/2} e^{-\rho M^2 - x^2/4\rho}. \quad (2.8)$$

As for the two charged pions, it is convenient to start with the representation for the zero-temperature propagator in Minkowski space given in Refs. [1, 15],

$$\Delta^\pm = \exp[is \Phi(x_\perp)] \int \frac{d^d p}{(2\pi)^d} e^{-ipx} \Delta^\pm(p_\parallel, p_\perp),$$

$$\Delta^\pm(p_\parallel, p_\perp) = i \int_0^\infty \frac{ds}{\cos(|qH|s)} \exp \left( is(p_\parallel^2 - M^2) - ip_\perp^2 \tan(|qH|s) \right), \quad (2.9)$$

where

$$\Phi(x_\perp) = \frac{|qH|}{2} x_1^2 x_2^2 \quad (2.10)$$

is the so-called Schwinger phase, and the other quantities are

$$p_\parallel^2 = p_0^2 - p_3^2, \quad p_\perp^2 = p_1^2 + p_2^2, \quad s = \text{sign}(qH). \quad (2.11)$$

The point is that the summation over the Landau levels – associated with the magnetic field – has already been performed in $\Delta^\pm$. In the thermal propagator there is then only one sum left: the one induced by finite temperature. This simplifies the calculation considerably. After integration over the momenta, and going from Minkowski to Euclidean space, we obtain

$$\Delta^\pm(x) = \frac{|qH|}{(4\pi)^2} e^{-s_\perp |qH|x_1 x_2/2} \int_0^\infty ds \frac{e^{-sM^2}}{s \sinh(|qH|s)} \exp \left( -\frac{x_1^2 + x_2^2}{4s} - \frac{|qH|(x_1^2 + x_2^2)}{4 \tanh(|qH|s)} \right), \quad (2.12)$$

from where the thermal propagator for the charged pions can be constructed via Eq. (2.7).
Up to one-loop order, the thermal propagator $G(x)$ only has to be evaluated at the origin $x=0$, where it can be decomposed into the $T=0$ contribution and a second piece that refers to finite temperature,

$$G(0) = \Delta(0) + g_1(M, T, H).$$

The latter belongs to the class of kinematical Bose functions $g_r(M, T, H)$ defined by

$$g_r(M, T, H) = \frac{T^{d-2r-2}}{(4\pi)^{r+1}} |qH| \int_0^\infty dt \frac{t^{r-\frac{d}{2}}}{\sinh(|qH|t/4\pi T^2)} \exp\left(-\frac{M^2}{4\pi T^2} t \right) \left[ S\left(\frac{1}{t}\right) - 1 \right].$$

Here $S(z)$ is the Jacobi theta function

$$S(z) = \sum_{n=-\infty}^{\infty} \exp(-\pi n^2 z).$$

The kinematical Bose functions $g_r(M, T, H)$ describe the thermodynamic properties of the pions in presence of magnetic fields. For the effective theory to be consistent, the quantities $T, M$ and $H$ must be small with respect to the underlying QCD scale $\Lambda \approx 1$ GeV. In this study, we are particularly interested in the chiral limit $M \to 0$ and the weak magnetic field limit $|qH| \ll T^2$.

We proceed with the evaluation of the functions $g_r(M, T, H)$. Since the Taylor expansion of the inverse hyperbolic sine starts with

$$\frac{1}{\sinh(t)} = \frac{1}{t} + \mathcal{O}(t),$$

we perform the following subtraction in the integrand

$$g_r(M, T, H) = \frac{T^{d-2r-2}}{(4\pi)^{r+1}} |qH| \int_0^\infty dt t^{r-\frac{d}{2}} \left( \frac{1}{\sinh(|qH|t/4\pi T^2)} - \frac{4\pi T^2}{|qH|t} \right) \exp\left(-\frac{M^2}{4\pi T^2} t \right) \left[ S\left(\frac{1}{t}\right) - 1 \right]$$

$$+ \frac{T^{d-2r}}{(4\pi)^r} \int_0^\infty dt t^{r-\frac{d}{2}-1} \exp\left(-\frac{M^2}{4\pi T^2} t \right) \left[ S\left(\frac{1}{t}\right) - 1 \right].$$

The second term describes pions in zero magnetic field, and has been evaluated before in Ref. [16],

$$g_r(M, T, 0) = 2 \int_0^\infty \frac{d\rho}{(4\pi \rho)^{\frac{d}{2}}} \rho^{r-1} \exp(-\rho M^2) \sum_{n=1}^{\infty} \exp(-n^2/4\rho T^2).$$

Footnote: Note that we just subtract and re-add a term.
We thus consider the first term that depends on the magnetic field,

$$
\tilde{g}_r(M, T, H) = \frac{|qH|}{(4\pi)^2} \int_0^\infty dt t^{-\frac{1}{2}} \left( \frac{1}{\sinh(t)} - \frac{1}{t} \right) \exp \left( - \frac{M^2}{|qH|} t \right) \times \left[ S\left( \frac{|qH|}{4\pi T^2} \right) - 1 \right].
$$

(2.19)

Since the analysis of $\tilde{g}_r(M, T, H)$ is rather technical, we relegate it to an appendix. In the same appendix A, we also discuss the structure of the $T=0$ contribution in the free energy density $z_0(M, 0, H)$. Here we just provide the final representation for the quark condensate in the chiral limit and $|qH| \ll T^2$. The latter limit is implemented by expanding the various quantities in Eq. (2.6) in the parameter $\epsilon$,

$$
\epsilon = \frac{|qH|}{T^2}.
$$

(2.20)

Up to one-loop order, the low-temperature expansion of the quark condensate in the chiral limit and $|qH| \ll T^2$ then takes the form

$$
\frac{\langle \bar{q}q \rangle}{\langle 0|\bar{q}q|0 \rangle} = 1 + \frac{|qH| \log 2}{16\pi^2 F^2} + \left\{ \frac{|I_{\frac{1}{2}}|}{8\pi^{3/2} F^2} \sqrt{\epsilon} - \frac{\log 2}{16\pi^2 F^2} \epsilon - \frac{a_1}{F^2} \epsilon^2 - \frac{a_2}{F^2} \epsilon^4 + \mathcal{O}(\epsilon^6) \right\} T^2,
$$

$$
- \frac{1}{8F^2} T^2 + \mathcal{O}(T^4),
$$

(2.21)

where

$$
I_{\frac{1}{2}} = \int_0^\infty dt t^{-1/2} \left( \frac{1}{\sinh(t)} - \frac{1}{t} \right) \approx -1.516256,
$$

$$
a_1 = -\frac{\zeta(3)}{384\pi^4}, \quad a_2 = \frac{7\zeta(7)}{98304\pi^8}.
$$

(2.22)

The analytical representation for the coefficients $a_p$ can be found in the appendix, along with the numerical values for the first few coefficients $a_1, \ldots, a_5$ in Table 5. The series at finite temperature in nonzero magnetic field is thus dominated by the square-root term $\propto \sqrt{\epsilon}$, followed by a term linear in $\epsilon$. The remaining corrections involve even powers of $\epsilon$.

The temperature-independent contribution in the quark condensate that involves the magnetic field,

$$
\frac{|qH| \log 2}{16\pi^2 F^2},
$$

(2.23)

is the Shushpanov-Smilga term derived a long time ago [8], and later confirmed in Refs. [2–7], among others. However, comparing our leading temperature-dependent contribution,

$$
\frac{\sqrt{|qH| T}}{8\pi^{3/2} F^2} |I_{\frac{1}{2}}|,
$$

(2.24)
with the respective leading terms in the two published series, Eq. (1.2),
\[ -\frac{7\sqrt{|qH| T}}{48\pi F^2}, \]
and Eq. (1.3),
\[ \frac{5\sqrt{|qH| T}}{48\pi F^2}, \]
we observe disagreement with either result. Still, it is interesting to note that the leading term obtained by Andersen, Eq. (2.26), numerically almost coincides with ours,
\[ \frac{5}{48\pi} \approx 0.0331573, \quad \frac{1}{8\pi^{3/2}} |I_\frac{1}{2}| \approx 0.0340375, \]
in particular, it is also positive. As far as higher-order contributions are concerned, we cannot confirm the emergence of logarithmic terms of the form \( H \log(H/T^2) \) as suggested in Refs. [2–5].

To underline the correctness of our series, we perform some simple numerical tests. First of all, we establish the connection between our kinematical functions and those in the literature. The representation for the kinematical functions used by the authors of Refs. [2–7] is the same as the one used in Ref. [17] where numerical data is available. The relevant Bose function for the quark condensate in the chiral limit reads
\[ R(0, T, H) = \frac{\epsilon T^2}{2\pi} \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} \frac{dz}{2\pi} \frac{1}{\sqrt{z^2 + (2k + 1)^2}} \exp \left[ \frac{1}{\sqrt{z^2 + (2k + 1)^2}} \right] - 1. \]

Using Table I of Ref. [17], we have verified that the connection between the kinematical function \( R(0, T, H) \) and our representation \( \tilde{g}_1(0, T, H) \),
\[ \tilde{g}_1(0, T, H) = \frac{\epsilon T^2}{16\pi^2} \int_0^\infty dt \frac{1}{t} \left( \frac{1}{\sinh(\epsilon t/4\pi)} - \frac{4\pi}{\epsilon t} \right) \left[ S\left( \frac{1}{t} \right) - 1 \right], \]
is given by\(^4\)
\[ \frac{R(0, T, H)}{T^2} - \frac{1}{12} = \frac{\tilde{g}_1(0, T, H)}{T^2}. \]

The point is that the function \( R(0, T, H) \) contains a temperature-dependent contribution that does not involve the magnetic field: this term has to be subtracted in order to compare with our representation \( \tilde{g}_1(0, T, H) \) that describes the purely \( H \neq 0 \)-part by definition, Eq. (2.17). Having established equivalence between previous analyses and ours through Eq. (2.30), any discrepancies in the weak magnetic field limit \( |qH| \ll T^2 \) can be traced back to the expansion of the kinematical functions in the parameter \( \epsilon = |qH|/T^2 \).

\(^4\)Note that the ratio \( |qH|/T^2 = \epsilon \), both in \( R(0, T, H) \) and \( \tilde{g}_1(0, T, H) \), is arbitrary – we are not necessarily referring to the weak magnetic field limit \( |qH| \ll T^2 \).
The first numerical test consists in comparing our series with the exact result. More precisely, we consider successive approximations in the brace
\[
\left\{ \frac{|I_1|}{8\pi^{3/2}F^2} \sqrt{\epsilon} - \frac{\log 2}{16\pi^2F^2} \epsilon - \frac{a_1}{F^2} \epsilon^2 + \mathcal{O}(\epsilon^4) \right\}
\]
(2.31)
of the expansion (2.21), and compare them with the exact result given by the Bose function
\[
-\frac{\tilde{g}_1(0, T, H)}{T^2}.
\]
(2.32)

In Table 1 we provide numerical data from the series (2.21) by including terms up to order $\mathcal{O}(\sqrt{\epsilon})$, $\mathcal{O}(\epsilon)$, and $\mathcal{O}(\epsilon^2)$, respectively. We notice a clear hierarchy: the $\sqrt{\epsilon}$-term yields a very good leading approximation, while subsequent terms are heavily suppressed – our series hence converges very fast.

| $\epsilon$ | $-\tilde{g}_1/T^2$ | $\mathcal{O}(\sqrt{\epsilon})$ | $\mathcal{O}(\epsilon)$ | $\mathcal{O}(\epsilon^2)$ |
|---|---|---|---|---|
| 0.1 | 0.0103249857050 | 0.01076360492 | 0.01032466434 | 0.0103249857058 |
| 0.05 | 0.00739162808212 | 0.007611018032 | 0.007391547742 | 0.00739162808217 |
| 0.01 | 0.00335985989497 | 0.003403750739 | 0.003359856681 | 0.00335985989497 |
| 0.005 | 0.00238486900367 | 0.002384868200 | 0.00238486900367 |
| 0.001 | 0.00107197111872 | 0.00107197111872 | 0.00107197111872 |
| 0.0005 | 0.000758907108297 | 0.000758907108297 |
| 0.0001 | 0.000339936133675 | 0.000339936133675 |

Table 1: Leading terms in our series (2.21) for the finite-temperature quark condensate in the limit $|qH| \ll T^2$. The $\sqrt{\epsilon}$-term provides a very good approximation for the exact result, and the series converges rapidly.

In a second test we compare our series with the one-loop results in the literature. While Andersen in Refs. [6, 7] provides the leading term in the weak magnetic field lim...
expansion (which numerically is very close to our result), Agasian in Refs. [2–5] furthermore derives higher-order corrections for the limit \(|qH| \ll T^2\). In Table 2 we list the numerical values obtained from the Agasian series, Eq.(54) of Ref. [2].

\[
\left\{ \begin{array}{l}
- \frac{7}{48\pi F^2} \sqrt{\epsilon} - \frac{1}{16\pi^2 F^2} \epsilon \log \epsilon - \frac{2(\gamma_E - \log 4\pi - \frac{1}{6})}{16\pi^2 F^2} \epsilon \\
\end{array} \right.,
\]

(2.33)

Since the series includes terms up to \(O(\epsilon)\), we also go up to linear order in our series (2.21). Inspecting Table 2, one notices that the Agasian series does not correctly describe the quark condensate in the weak magnetic field limit.

A final remark concerns the structure of the low-temperature series. At zero temperature, our series (2.21) reduces to the Shushpanov-Smilga term as it should. On the other hand, as one approaches zero temperature, the Agasian series Eq. (1.2) formally reduces to

\[
\frac{C |qH|}{16\pi^2 F^2}, \quad C = \log 2 - 2\gamma_E + 2 \log 4\pi + \frac{1}{3},
\]

(2.34)

which contradicts the original Shushpanov-Smilga result [8]. Moreover, the series – as it stands in Refs. [2–5] – also diverges as zero temperature is approached, because of the logarithmic contribution.

3 Conclusions

Expansions for the quark condensate at low temperatures, small pion masses, and weak magnetic fields have been presented up to two-loop order in the literature. Still, since discrepancies between two published results concern the one-loop level, the present analysis is perfectly justified.

We emphasize that our approach is based on an alternative representation for the kinematical Bose functions – different from the representations used in Refs. [2–7]. Remarkably, we find that the leading term at finite temperature in the expansion of the quark condensate in a weak magnetic field (\(|qH| \ll T^2\)), and in the chiral limit, does not coincide with either of the two published terms. As far as higher-order corrections are concerned, our approach allows for a systematic derivation of these contributions, that illuminates the structure of the series.

The low-temperature series is dominated by a square-root term \(\sqrt{|qH|}/T^2\) that is positive, much like the (zero-temperature) Shushpanov-Smilga term. The next term is linear in \(|qH|/T^2\) and negative, while subsequent corrections involve even powers of \(|qH|/T^2\). Higher-order terms are heavily suppressed such that our series converges rapidly.

\[ \text{See footnote 5.} \]
Invoing the exact one-loop expression for the quark condensate – valid for arbitrary ratio $|qH|/T^2$ – we have numerically verified that our expansion correctly describes the quark condensate in weak magnetic fields. We have also observed that the series published in Refs. [2–5] fails to approximate the exact result.

Acknowledgments

The author thanks J. O. Andersen for correspondence and R. A. Sáenz for helpful comments.

A Explicit Calculations

In this appendix we first discuss the free energy density at zero temperature. We then consider the kinematical Bose functions $\tilde{g}_r(M, T, H)$ in the chiral limit and analyze their behavior in weak magnetic fields ($|qH| \ll T^2$). Collecting results, we provide the representation for the quark condensate in the chiral limit and $|qH| \ll T^2$. Finally, we show how to extract the leading terms in the expansion of the quark condensate in a straightforward way.

A.1 Zero Temperature

The free energy density at zero temperature, up to one-loop order, amounts to

$$z_0(M, 0, H) = -F^2 M^2 - (l_3 + h_1) M^4 + 4 k_2 |qH|^2 + \frac{1}{2} M^4 \lambda + I_1 + I_2 + O(p^6),$$

where

$$I_1 = -|qH|^\frac{d}{2} \int_0^\infty dt t^{-\frac{d}{2}-1} \exp\left( - \frac{M^2}{|qH|} t \right),$$

$$I_2 = -|qH|^\frac{d}{2} \int_0^\infty dt t^{-\frac{d}{2}} \left( \frac{1}{\sinh(t)} - \frac{1}{t} \right) \exp\left( - \frac{M^2}{|qH|} t \right).$$

The integral $I_1$ can be written as

$$I_1 = M^4 \lambda - \frac{M^4}{64\pi^2},$$

where

$$\lambda = \frac{1}{2} (4\pi)^{-\frac{d}{2}} \Gamma(1 - \frac{1}{2}d) M^{d-4}$$

and

$$= \frac{M^{d-4}}{16\pi^2} \left[ \frac{1}{d-4} - \frac{1}{2} \ln 4\pi + \Gamma'(1) + 1 \right] + O(d-4).$$

(A.3)
contains a pole at \(d=4\). It should be stressed that factors of \(|qH|\) cancel: the integral \(I_1\) does not depend on the magnetic field. The UV-divergence in \(I_1\) – along with the UV-divergence in the term \(\frac{1}{2} M^4 \lambda\) of Eq. (A.1) – can be absorbed into the next-to-leading order effective constants \(l_3\) and \(h_1\) in the standard manner, i.e., in chiral perturbation theory where no magnetic field is present (for details see, e.g., Ref. [12]).

The integral \(I_2\), that does depend on the magnetic field, also diverges in the limit \(d \to 4\). The singularity is proportional to \(|qH|^2\) and can be absorbed into the next-to-leading order effective constant \(h_2\). Explicitly, we subtract the next Taylor term in the expansion of the inverse hyperbolic sine in \(I_2\), such that the integral

\[
- \frac{|qH|^2}{(4\pi)^2} \int_0^\infty dt t^{-\frac{3}{2}} \left( \frac{1}{\sinh(t)} - \frac{1}{t} + \frac{t}{6} \right) \exp\left( - \frac{M^2}{|qH|} t \right)
\]

(A.4)

becomes finite if one approaches the physical dimension \(d \to 4\). The remainder,

\[
\hat{I}_2 = \frac{|qH|^2}{6(4\pi)^2} \int_0^\infty dt t^{-\frac{3}{2}+1} \exp\left( - \frac{M^2}{|qH|} t \right),
\]

(A.5)

can be expressed in terms of \(\lambda\) as

\[
\hat{I}_2 = -\frac{|qH|^2}{3} \lambda - \frac{|qH|^2}{96\pi^2}.
\]

(A.6)

Gathering results, the renormalized free energy density at zero temperature takes the form

\[
z_0(M, 0, H) = -F^2 M^2 + \frac{M^4}{64\pi^2} (\overline{l}_3 - 4\overline{l}_1 - 1) + \frac{|qH|^2}{96\pi^2} (\overline{h}_2 - 1)
\]

\[
- \frac{|qH|^2}{16\pi^2} \int_0^\infty dt t^{-2} \left( \frac{1}{\sinh(t)} - \frac{1}{t} + \frac{t}{6} \right) \exp\left( - \frac{M^2}{|qH|} t \right) + O(p^6).
\]

(A.7)

Up to the factors \(\gamma_3/32\pi^2, \delta_1/32\pi^2,\) and \(\delta_2/32\pi^2,\) the constants \(\overline{l}_3, \overline{l}_1,\) and \(\overline{h}_2\) are the running coupling constants at the fixed renormalization scale \(\mu = M_\pi\), where \(M_\pi \approx 139.6\,\text{MeV}\) is the physical pion mass (for details see, e.g., Ref. [16]).

### A.2 Finite Temperature

We now turn to finite temperature where the kinematical Bose functions

\[
g_r(M, T, H) = \frac{|qH|^{\frac{d-r}{2}}}{(4\pi)^{\frac{d}{2}}} \int_0^\infty dt t^{-\frac{d-r}{2}} \left( \frac{1}{\sinh(t)} - \frac{1}{t} \right) \exp\left( - \frac{M^2}{|qH|} t \right)
\]

\[
\times \left[ \frac{\left( |qH| \right)}{4\pi T^2 t} - 1 \right] - \frac{|qH|^2}{16\pi^2} \int_0^\infty dt t^{-2} \left( \frac{1}{\sinh(t)} - \frac{1}{t} + \frac{t}{6} \right) \exp\left( - \frac{M^2}{|qH|} t \right) + O(p^6).
\]

(A.8)
become relevant. Our analysis proceeds along the lines of Ref. [18]. From the very 
start, we refer to the chiral limit that we implement by $M \to 0$, while keeping $T$ and $|qH|$ fixed. Changing integration variables, and defining $\epsilon = |qH|/T^2$, we first write

$$\tilde{g}_r(0, T, H) = \frac{\epsilon}{(4\pi)^{d-1}} T^{d-2r} \int_0^\infty dt \, t^{-\frac{d}{2} + r} \left( \frac{1}{\sinh(\epsilon t/4\pi)} - \frac{4\pi}{\epsilon t} \right) \left[ S \left( \frac{1}{t} \right) - 1 \right].$$  (A.9)

The integral is split into two pieces, namely $0 \leq t \leq 1$ and $1 \leq t < \infty$. In the second interval we use the Jacobi identity

$$S(t) = \frac{1}{\sqrt{t}} S(1/t),$$  (A.10)

and change the integration variable $t \to 1/t$. This then leads to

$$\tilde{g}_r(0, T, H) = \frac{\epsilon}{(4\pi)^{d-1}} T^{d-2r} \int_0^1 dt \, t^{-\frac{d}{2} + r} \left( \frac{1}{\sinh(\epsilon t/4\pi)} - \frac{4\pi}{\epsilon t} \right) \left[ S \left( \frac{1}{t} \right) - 1 \right] + \frac{\epsilon}{(4\pi)^{d-1}} T^{d-2r} \left\{ I_A + I_B + I_C \right\},$$  (A.11)

with

$$I_A = \int_0^1 dt \, t^{\frac{d}{2} - r - \frac{d}{2}} \left( \frac{1}{\sinh(\epsilon t/4\pi)} - \frac{4\pi t}{\epsilon} \right) \left[ S \left( \frac{1}{t} \right) - 1 \right],$$

$$I_B = \int_0^1 dt \, t^{\frac{d}{2} - r - \frac{d}{2}} \left( \frac{1}{\sinh(\epsilon t/4\pi)} - \frac{4\pi t}{\epsilon} \right),$$

$$I_C = -\int_0^1 dt \, t^{\frac{d}{2} - r - 2} \left( \frac{1}{\sinh(\epsilon t/4\pi)} - \frac{4\pi t}{\epsilon} \right).$$  (A.12)

The integral $I_B$ we decompose as

$$I_B = \int_0^\infty dt \, t^{\frac{d}{2} - r - \frac{d}{2}} \left( \frac{1}{\sinh(\epsilon t/4\pi)} - \frac{4\pi t}{\epsilon} \right) - \int_1^\infty dt \, t^{\frac{d}{2} - r - \frac{d}{2}} \left( \frac{1}{\sinh(\epsilon t/4\pi)} - \frac{4\pi t}{\epsilon} \right).$$  (A.13)

After a few trivial manipulations, we end up with

$$I_B = I_{B1} + I_{B2},$$

$$I_{B1} = \frac{\epsilon^{\frac{d}{2} - r - \frac{d}{2}}}{(4\pi)^{\frac{d}{2} - r - \frac{d}{2}}} \int_0^\infty dt \, t^{-\frac{d}{2} + r + \frac{1}{2}} \left( \frac{1}{\sinh(t)} - \frac{1}{t} \right),$$

$$I_{B2} = -\int_0^1 dt \, t^{-\frac{d}{2} + r + \frac{1}{2}} \left( \frac{1}{\sinh(\epsilon t/4\pi)} - \frac{4\pi t}{\epsilon} \right).$$  (A.14)

The integral $I_C$ is processed in an analogous way, with the result

$$I_C = I_{C1} + I_{C2},$$

$$I_{C1} = -\frac{\epsilon^{\frac{d}{2} - r - 1}}{(4\pi)^{\frac{d}{2} - r - 1}} \int_0^\infty dt \, t^{-\frac{d}{2} + r} \left( \frac{1}{\sinh(t)} - \frac{1}{t} \right),$$

$$I_{C2} = \int_0^1 dt \, t^{-\frac{d}{2} + r} \left( \frac{1}{\sinh(\epsilon t/4\pi)} - \frac{4\pi t}{\epsilon} \right).$$  (A.15)
A.3 Representation for the Quark Condensate

We now focus on the quark condensate in the chiral limit,

$$\langle \bar{q}q \rangle = \langle 0 | \bar{q}q | 0 \rangle \left[ 1 - \frac{1}{F^2} \frac{\partial}{\partial M^2} \left( z - z_0(M, 0, 0) \right) \right]_{M^2 = 0}, \quad (A.16)$$

where

$$z = z_0(M, 0, H) - \frac{2}{7} g_0(M, T, 0) - \tilde{g}_0(M, T, H) + \mathcal{O}(p^6) \quad (A.17)$$

is the (total) free energy density and

$$z_0(M, 0, 0) = -F^2 M^2 + \frac{M^4}{64 \pi^2} (7_3 - 4 \tilde{h}_1 - 1) \quad (A.18)$$

is the $T=0$ contribution that is independent of the magnetic field. Accordingly, the one-loop representation for the quark condensate in the chiral limit reads

$$\frac{\langle \bar{q}q \rangle}{\langle 0 | \bar{q}q | 0 \rangle} = 1 - \frac{|qH|}{16 \pi^2 F^2} \int_0^\infty dt t^{-1} \left( \frac{1}{\sinh(t)} - 1 \right) - \frac{3 g_1(0, T, 0)}{2F^2} - \frac{\tilde{g}_1(0, T, H)}{F^2} + \mathcal{O}(p^4). \quad (A.19)$$

The first line refers to zero temperature, the second line refers to finite temperature.

Our final task is to explore the weak magnetic field limit $|qH| \ll T^2$ that we obtain by expanding in the parameter $\epsilon$

$$\epsilon = \frac{|qH|}{T^2}. \quad (A.20)$$

A common factor in the various integrands considered in subsection A.2 is

$$\frac{1}{\sinh(\epsilon t/4\pi)} - \frac{4\pi}{\epsilon t}, \quad (A.21)$$

that we expand into

$$\frac{1}{\sinh(\epsilon t/4\pi)} - \frac{4\pi}{\epsilon t} = c_1 t \epsilon + c_2 t^3 \epsilon^3 + c_3 t^5 \epsilon^5 + \mathcal{O}(\epsilon^7). \quad (A.22)$$

The first few Taylor coefficients read

$$c_1 = -\frac{1}{24\pi} \approx -1.33 \times 10^{-2},$$

$$c_2 = \frac{7}{23 040 \pi^3} \approx 9.80 \times 10^{-6},$$

$$c_3 = -\frac{31}{15 482 880 \pi^5} \approx -6.54 \times 10^{-9},$$

$$c_4 = \frac{127}{9 909 043 200 \pi^7} \approx 4.24 \times 10^{-12},$$

$$c_5 = -\frac{73}{896 909 967 360 \pi^9} \approx -2.73 \times 10^{-15}. \quad (A.23)$$
Introducing the quantities $\tilde{\alpha}_p$, $\hat{\alpha}_p$, and $\beta_p$ by

$$
\tilde{\alpha}_p = \int_0^1 dt \, c_p \, t^{2p-2} \left[ S \left( \frac{1}{t} \right) - 1 \right],
$$

$$
\hat{\alpha}_p = \int_0^1 dt \, c_p \, t^{-2p-\frac{1}{2}} \left[ S \left( \frac{1}{t} \right) - 1 \right],
$$

$$
\beta_p = \int_0^1 dt \, c_p \, (t^{-1} - t^{-\frac{1}{2}}) t^{2p-1},
$$

(A.24)

and defining the coefficients $a_p$ as

$$
a_p = \frac{\tilde{\alpha}_p + \hat{\alpha}_p + \beta_p}{16\pi^2},
$$

(A.25)

the low-temperature representation for the quark condensate in the chiral limit and $|qH| \ll T^2$ then takes the form

$$
\frac{\langle \bar{q}q \rangle}{\langle 0 | \bar{q}q | 0 \rangle} = 1 + \frac{|qH| \log 2}{16\pi^2 F^2} + \left\{ \frac{|I_{\frac{1}{2}}|}{8\pi^{3/2} F^2} \sqrt{\epsilon} - \frac{\log 2}{16\pi^2 F^2} \epsilon - \frac{a_1}{F^2} \epsilon^2 + \frac{a_2}{F^2} \epsilon^4 + \mathcal{O}(\epsilon^6) \right\} T^2,

$$

$$
- \frac{1}{8F^2} T^2 + \mathcal{O}(T^4),
$$

(A.26)

The integral $I_{\frac{1}{2}}$ amounts to

$$
I_{\frac{1}{2}} = \int_0^\infty dt \, t^{-1/2} \left( \frac{1}{\sinh(t)} - \frac{1}{t} \right) \approx -1.516256,
$$

(A.27)

while numerical values for the first few coefficients $a_p$ in the above expansion are provided in Table 3.

| $p$ | $a_p$ |
|-----|-------|
| 1   | $-3.21361844712 \times 10^{-5}$ |
| 2   | $7.56726355863 \times 10^{-9}$ |
| 3   | $-8.00051395855 \times 10^{-12}$ |
| 4   | $1.87869037118 \times 10^{-14}$ |
| 5   | $-7.80774216239 \times 10^{-17}$ |

Table 3: Numerical values for the coefficients $a_p$ defined by Eq. (A.25).

Processing integrals in the same manner as described in the previous subsection, and using the identity

$$
\frac{2}{\pi^2} \Gamma \left( \frac{z}{2} \right) \zeta(z) = \int_0^\infty dt \, t^{\frac{z}{2} - 1} \left[ S(t) - 1 \right],
$$

(A.28)

15
we can express the coefficients $a_p$ in terms of the Riemann $\zeta$-function as

$$a_p = \frac{c_p}{8\pi^{2p+\frac{3}{2}}} \Gamma(2p - \frac{1}{2}) \zeta(4p - 1).$$  \hspace{1cm} (A.29)

The final representation for the quark condensate in the chiral limit and $|qH| \ll T^2$ thus reads

$$\langle \bar{q}q \rangle \langle 0 | \bar{q}q | 0 \rangle = 1 - \frac{1}{8F^2} T^2 + \frac{|qH| \log 2}{16\pi^2 F^2}$$

$$+ \left\{ \frac{|I_1|}{8\pi^{3/2} F^2} \sqrt{\epsilon} - \frac{\log 2}{16\pi^2 F^2} \epsilon + \frac{\zeta(3)}{384\pi^4 F^2} \epsilon^2 - \frac{7\zeta(7)}{98304\pi^8 F^2} \epsilon^4 + O(\epsilon^6) \right\} T^2$$

$$+ O(T^4).$$  \hspace{1cm} (A.30)

### A.4 Straightforward Derivation of the Leading Terms

In order to readily derive the leading terms in the quark condensate in the chiral limit and $|qH| \ll T^2$, we consider the relevant Bose function

$$\tilde{g}_1(0, T, H) = \frac{|qH|}{16\pi^2} \int_0^{\infty} dt \frac{1}{t} \left( \frac{1}{\sinh(t)} - \frac{1}{t} \right) \left( S\left( \frac{|qH|}{4\pi T^2 t} \right) - 1 \right),$$

that we write as

$$\tilde{g}_1(0, T, H) = -\frac{|qH|}{16\pi^2} \int_0^{\infty} dt \frac{1}{t} \left( \frac{1}{\sinh(t)} - \frac{1}{t} \right)$$

$$+ \frac{\sqrt{|qH|T}}{8\pi^{3/2}} \int_0^{\infty} dt \frac{1}{t^{1/2}} \left( \frac{1}{\sinh(t)} - \frac{1}{t} \right) S\left( \frac{4\pi T^2}{|qH| t} \right).$$  \hspace{1cm} (A.32)

Note that the Jacobi theta function

$$S(z) = \sum_{n=-\infty}^{\infty} \exp(-\pi n^2 z)$$

satisfies the identity

$$S(z) = \frac{1}{\sqrt{z}} S\left( \frac{1}{z} \right).$$  \hspace{1cm} (A.33)

The integral in the first line, Eq. (A.32), is known analytically,

$$\int_0^{\infty} dt \frac{1}{t^{1/2}} \left( \frac{1}{\sinh(t)} - \frac{1}{t} \right) = - \log 2,$$  \hspace{1cm} (A.35)

and gives rise to the correction linear in $\epsilon$ in Eq. (A.26). Regarding the second line, Eq. (A.32), in the limit $|qH| \ll T^2$, all contributions in the Jacobi theta function – except $n=0$ – are exponentially suppressed: the corresponding integral hence reduces to $I_1 \frac{T}{|qH|}$. We immediately obtain the leading temperature-dependent term in the weak magnetic field expansion $|qH| \ll T^2$ in the chiral limit,

$$\frac{\sqrt{|qH|T}}{8\pi^{3/2} F^2} |I_1|.$$  \hspace{1cm} (A.36)
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