Initial boundary value problem for the focusing nonlinear Schrödinger equation with Robin boundary condition: half-line approach

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We consider the initial boundary value problem for the focusing nonlinear Schrödinger equation in the quarter plane $x > 0, t > 0$ in the case of decaying initial data (for $t = 0$, as $x \to +\infty$) and the Robin boundary condition at $x = 0$. We revisit the approach based on the simultaneous spectral analysis of the Lax pair equations and show that the method can be implemented without any a priori assumptions on the long-time behaviour of the boundary values.

1. Introduction

The inverse scattering transform (IST) method for studying initial value (IV) problems for certain nonlinear evolution equations—integrable nonlinear equations possessing a Lax pair representation—is known as a powerful tool for obtaining rigorous results concerning the most subtle issues about the behaviour of solutions of these problems, including detailed long-time behaviour. The most efficient implementation of the—method turns out to be based on the Riemann–Hilbert (RH) problem method, which is essentially the reformulation of the scattering problem for one of the Lax pair equations—the $x$-equation—in terms of an analytic (matrix) factorization problem of the RH type.

The studies on the adaptation of the IST method to initial boundary value (IBV) problems yield particular classes of boundary conditions, under which the IBV problem remains completely integrable, i.e. solving it...
reduces to solving a series of linear problems. At the beginning of 1990s, it was realized [1–5] that this is the case when the boundary conditions allow an appropriate continuation, based on the Bäcklund transformation, of the given initial data to the whole axis, which reduces the study of the IBV problem to the study of the associated IV problem.

In a recent paper, Deift & Park [6] thoroughly revised this Bäcklund transformation approach and adjusted it to the modern, RH framework. This allowed them to apply the nonlinear steepest descent method [7] and analyse in great detail several important asymptotic and stability questions [8] in the theory of the focusing nonlinear Schrödinger equation (NLS).

A general approach to IBV problems for integrable nonlinear equations was originated in Fokas [9], and it has been actively developed since then [10–13]. In the simplest form of this approach, which is usually referred to as the unified method, or the Fokas method, one treats the IBV problems with general boundary conditions by using the spectral analysis of both linear equations constituting the Lax pair. The relevant RH problem is posed now on the union (cross) of real and imaginary axis, rather than on the real axis only as in the case of the IV problems. A key factor that affects the efficiency of the method is that the construction of the underlying RH problem requires the knowledge of the spectral functions that are determined, in general, by an excessive number of boundary values. These boundary values cannot be prescribed arbitrarily for a well-posed IBV problem and thus the problem of compatibility of the boundary and IVs arises. It turns out that this compatibility can be expressed, in a rather simple, explicit way, in terms of the associated spectral functions [13], which in turn allows obtaining, particularly, the detailed asymptotic pictures. But translating this description of compatibility into the physical space (of boundary and IVs) requires solving, in general, nonlinear problems [14], which makes the whole problem non-integrable.

However, for particular boundary conditions, an additional symmetry in the spectral space allows bypassing the nonlinear step of resolving the compatibility issue and thus making the IBV problem as integrable as the associated IV problem [10,13,15,16].

In Fokas [15] (see also [13]), the boundary conditions (called linearizable) leading to integrable IBV for the NLS equation have been specified by applying a symmetry analysis to the associated Lax pair equations. Particularly, the Dirichlet, Neumann and Robin boundary conditions were selected in this way, and the associated RH problem formalism was presented, under assumption that the boundary values being considered as a ‘potential’ in the time-type equation from the Lax pair decay as $t \to \infty$ and, moreover, generate the associated spectral functions with appropriate analytic properties. Furthermore, in Fokas & Kamvissis [17], it was shown that for the integrable boundary conditions and, specifically, for the Robin boundary condition, the original master RH problem posed on the cross can be unfolded to the problem posed on the real line only. This, in principle, allows one to reproduce the earlier results of Tarasov [5] within the general scheme of Fokas [9].

The important issue that has not been fully addressed in Fokas [15], Fokas et al. [13], and Fokas & Kamvissis [17] is the following. As it has been already mentioned, the original master RH problem is formulated on the cross under a priori assumption that the boundary data rapidly decay as $t \to \infty$ (actually, this assumption is needed only if one wants to study the large-$t$ asymptotics). On the other hand, using simple explicit soliton-like solutions of the NLS equation, one can easily see that this assumption is not necessarily true for solutions satisfying a linearizable boundary condition. Indeed, the two-parameter solutions

$$u(x,t) = \frac{\alpha}{\cosh(\alpha x + \phi_0)} e^{i\alpha^2 t}$$

with $\alpha \in \mathbb{R}$ and $\phi_0 \in \mathbb{R}$, which are obviously not decaying, for any $x$, as $t \to \infty$, satisfy the Robin condition $u_x(0,t) + q \cdot u(0,t) = 0$ with $q = \alpha \cdot \tanh \phi_0$. Moreover, in the case $q > 0$, the large-$t$ behaviour of the solution is oscillatory for generic initial data (see the discussion in §4). Hence, the necessity to produce an independent proof that the RH problem obtained in Fokas et al. [13] and Fokas & Kamvissis [17] for linearizable boundary conditions indeed yields the solution of the
IBV problem in question. In this paper, using some ideas that go back to the late 1980s studies on algebro-geometric solutions of integrable partial differential equations (PDEs), we show how this independent proof can be achieved and hence complete the program originated in Fokas [15].

We revisit the IBV for the NLS equation with linearizable (Robin) boundary condition

\[
iu_t + u_{xx} + 2|u|^2u = 0, \quad x > 0, t > 0,\]

\[
\begin{cases}
u(x, 0) = u_0(x), & x \geq 0 \\u_s(0, t) + qu(0, t) = 0, & t \geq 0,
\end{cases}
\]  

(1.1)

where (a) \(u_0(x)\) decays to 0 as \(x \to +\infty\) and (b) \(q\) is a real constant. Solving (1.1) by the RH method consists of three steps:

(i) Provide a family of the RH problems parametrized by \(x\) and \(t\) such that the solution \(u(x, t)\) of (1.1) is expressed in terms of the solutions of these problems.

(ii) Prove that \(u\) satisfies the initial condition \(u(x, 0) = u_0(x)\).

(iii) Prove that \(u\) satisfies the boundary condition \(u_s(0, t) + qu(0, t) = 0\).

Now let us comment on this procedure. Concerning (i), the construction of the RH problem has to involve only the spectral functions associated with the initial condition \(u_0(x)\). In the general framework of the simultaneous spectral analysis of the Lax pair equations, the RH problem is naturally formulated (see Fokas et al. [13]) on the contour consisting of two lines: \(k \in \mathbb{R}\) and \(k \in i\mathbb{R}\); this reflects the fact that the spectrum of the \(t\)-equation from the Lax pair with coefficients that are finitely supported or decaying at infinity consists of these lines. But for particular boundary conditions, the contour can be deformed (two rays of the imaginary axis can be folded down to the positive real axis) to the real axis only [17]. Then, it is the inherited symmetry property of the jump matrix for the deformed problem that allows verifying directly that the boundary condition holds. As for the verification of the initial condition, it is based on the fact that for \(x = 0\), the original RH problem can be deformed in the opposite way (the rays of the imaginary axis are folded down to the negative real axis), thus reducing the problem to that associated with the \(x\)-equation of the Lax pair with the potential \(u_0(x)\) [13].

In this way, we re-derive the results of Tarasov [5], Deift & Park [6] and hence show that the approach to linearizable IBV problems stemming from the general methodology of simultaneous spectral analysis of the Lax pair equations [13,15] can be implemented without any \(a\ priori\) assumptions about the long-time behaviour of the boundary values.

**Remark 1.1.** For problems involving odd spatial dimensions, such as Kortweg–de Vries, the method of extension to the line fails, whereas the unified method still works.

**Remark 1.2.** For even linear PDEs with non-homogeneous Robin boundary conditions, the unified method is more effective, because the method of extensions yield a solution that is non-uniformly convergent at the boundary (here this problem does not arise owing to the homogeneity of the boundary conditions).

2. The Riemann–Hilbert formalism for initial boundary value problems

First, let us recall the RH formalism for IBV problems on the half-line \(x \geq 0\) for the NLS equation [13].

The focusing NLS equation

\[
iu_t + u_{xx} + 2|u|^2u = 0
\]

is the compatibility condition of two linear equations (Lax pair) for a \(2 \times 2\)-valued function \(\Psi\) [18]:

\[
\Psi_x + ik\sigma_3\Psi = U\Psi
\]

(2.2)
with
\[ U = \begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix} \] (2.3)
and
\[ \Psi_1 + 2\imath k^2 \sigma_3 \Psi = V \Psi \] (2.4)
with
\[ V = \begin{pmatrix} \imath |u|^2 & 2k\mu + iu_x \\ -2k\mu + iu_x & -\imath |u|^2 \end{pmatrix}. \] (2.5)

Assuming that \( u(x, t) \) satisfies (2.1) for \( x > 0 \) and \( 0 < t < T \) with some \( T < \infty \), define the solutions \( \Psi_j(x, t, k) \), \( j = 1, 2, 3 \) of (2.2)–(2.5) as follows: \( \Psi_j(x, t, k) = \Phi_j(x, t, k) e^{(-\imath kx - 2\imath k^2 t)\sigma_3} \), where \( \Phi_j \) solve the integral equations
\[ \Phi_1(x, t, k) = I - e^{-\imath kx\sigma_3} \int_t^T e^{-2\imath k^2 (t-r)\sigma_3} V(0, r, k) \Phi_1(0, r, k) e^{2\imath k^2 (t-r)\sigma_3} e^{\imath kx\sigma_3} \mathrm{d}r + \int_0^x e^{-\imath k(x-y)\sigma_3} U(y, t) \Phi_1(y, t) e^{\imath k(x-y)\sigma_3} \mathrm{d}y, \] (2.6a)
\[ \Phi_2(x, t, k) = I + e^{-\imath kx\sigma_3} \int_0^t e^{-2\imath k^2 (t-r)\sigma_3} V(0, r, k) \Phi_2(0, r, k) e^{2\imath k^2 (t-r)\sigma_3} e^{\imath kx\sigma_3} \mathrm{d}r + \int_0^x e^{-\imath k(x-y)\sigma_3} U(y, t) \Phi_2(y, t) e^{\imath k(x-y)\sigma_3} \mathrm{d}y, \] (2.6b)
and
\[ \Phi_3(x, t, k) = I - \int_x^\infty e^{-\imath k(x-y)\sigma_3} U(y, t) \Phi_3(y, t, k) e^{\imath k(x-y)\sigma_3} \mathrm{d}y \] (2.6c)
(here \( I \) is the \( 2 \times 2 \) identity matrix).

Define the scattering matrices \( s(k) \) and \( S(k), k \in \mathbb{R}, \) as the matrices relating the eigenfunctions \( \Phi_j(x, t, k) \) for all \( x \) and \( t \):
\[ \Psi_3(x, t, k) = \Psi_2(x, t, k) s(k), \quad \Psi_1(x, t, k) = \Psi_2(x, t, k) S(k). \] (2.7)
The symmetry
\[ \Psi_{11}(x, t, k) = \Psi_{22}(x, t, k), \quad \Psi_{12}(x, t, k) = -\Psi_{21}(x, t, k) \] (2.8)
(here and below, a two-figure subscript denote the corresponding matrix entry) implies that
\[ s(k) = \begin{pmatrix} \bar{a}(k) & b(k) \\ -\bar{b}(k) & a(k) \end{pmatrix}, \quad S(k) = \begin{pmatrix} \bar{A}(k) & B(k) \\ -\bar{B}(k) & A(k) \end{pmatrix}. \] (2.9)

From (2.6) and (2.7), it follows that the spectral functions \( a(k) \) and \( b(k) \) can be analytically continued into the upper half-plane \( k \in \mathbb{C}^+ \) as bounded functions, with \( a(k) \to 1 \) and \( b(k) \to 0 \) as \( k \to \infty \). Moreover, they are determined by \( u(x, 0), x \geq 0 \) only, via
\[ s(k) = (\Psi_2)^{-1}(x, 0, k) \Psi_3(x, 0, k). \]

Similarly, the spectral functions \( A = A(k; T) \) and \( B = B(k; T) \) are entire functions bounded in the domains \( I \) and \( III \), where \( I = \{ k : \text{Im} \, k > 0, \text{Re} \, k > 0 \} \) and \( III = \{ k : \text{Im} \, k < 0, \text{Re} \, k < 0 \} \), with \( A(k; T) \to 1 \) and \( B(k; T) \to 0 \) as \( k \to \infty \). Moreover, they are determined by \( u(0, t) \) and \( u_x(0, t) \) for \( 0 \leq t \leq T \) only, via
\[ S(k; T) = (\Psi_2)^{-1}(0, t, k) \Psi_1(0, t, k). \]

The compatibility of the set of functions \( \{ u(x, 0), u(0, t), u_x(0, t) \} \) as traces of a solution \( u(x, t) \) of the NLS equation can be expressed in terms of the associated spectral functions as follows:
\[ A(k; T)b(k) - a(k)B(k; T) = c(k; T) e^{4i k^2 T^2}, \quad \text{Im} \, k \geq 0, \] (2.10)
with some analytic \( c(k; T) = O(1/k) \) as \( k \to \infty \) (actually, \( c(k; T) \) can be expressed in terms of \( \Phi_3 \) as follows: \( c(k; T) = -\int_0^\infty e^{2i y} u(y, T) \Phi_3(22)(y, T, k) \mathrm{d}y \) cf. (2.34) in Fokas et al. [13]). In the general scheme [15] of analysis of IBV problems, (2.10) is called the global relation.
Define
\[ d(k) = a(k)A(k) + b(k)B(k), \quad k \in II = \{ k : \text{Im } k > 0, \text{Re } k < 0 \}. \] (2.11)

Finally, assuming that \( d(k) \) has at most a finite number of simple zeros in \( \Pi \) and \( a(k) \) has at most a finite number of simple zeros in \( \mathbb{C}^+ \), define a piecewise meromorphic function (the superscripts denote the column of the respective matrix)
\[
M(x, t, k) = \begin{cases} 
\left( \frac{\Phi_2^{(1)}}{a} \Phi_2^{(1)} \right), & \text{Im } k > 0, \text{Im } k^2 > 0 \\
\left( \frac{\Phi_1^{(1)}}{d} \Phi_3^{(2)} \right), & \text{Im } k > 0, \text{Im } k^2 < 0 \\
\left( \frac{\Phi_1^{(2)}}{d} \Phi_3^{(2)} \right), & \text{Im } k < 0, \text{Im } k^2 > 0 \\
\left( \frac{\Phi_2^{(1)}}{a} \Phi_3^{(1)} \right), & \text{Im } k < 0, \text{Im } k^2 < 0.
\end{cases} \] (2.12)

Then, the scattering relations (2.7) imply that the limiting values of \( M \) on the cross \( \text{Im } k^2 = 0 \) satisfy the jump relations
\[
M_+(x, t, k) = M_-(x, t, k) e^{-i(kx + 2kt^2)\sigma_3} j_0(k) e^{i(kx + 2k^2t)\sigma_3}, \] (2.13)

where
\[
j_0(k) = \begin{cases} 
\left( \begin{array}{c} 1 + |r(k)|^2 \\
r(k) \\
\Gamma(k) \\
0 \end{array} \right), & k > 0, \\
\left( \begin{array}{c} 1 \\
0 \end{array} \right), & k \in i\mathbb{R}_+, \\
\left( \begin{array}{c} 1 \\
0 \end{array} \right), & k \in i\mathbb{R}_-, \\
\left( \begin{array}{c} 1 + |r(k) + \Gamma(k)|^2 \\
r(k) + \Gamma(k) \\
\Gamma(k) \\
0 \end{array} \right), & k < 0,
\end{cases} \] (2.14)

where \( r(k) = \tilde{b}(k)/a(k) \),
\[
\Gamma(k) = -\frac{\tilde{b}(k)}{a(k)d(k)}. \] (2.15)

The orientation of the contour is chosen as from \(-\infty\) to \(+\infty\) along \( \mathbb{R} \) and away from 0 along \( i\mathbb{R} \).

Complemented with the normalization condition \( M = I + O(1/k) \) as \( k \to \infty \) and the respective residue conditions at the zeros of \( a(k) \) and \( d(k) \) (see Fokas et al. [13] for details), the jump relation (2.13) can be viewed as the RH problem: given \( \{a(k), b(k), A(k), B(k)\} \), find \( M(x, t, k) \) for all \( x \geq 0 \) and \( t \geq 0 \). Then, the solution of the NLS equation, \( u(x, t) \), is given in terms of \( M(x, t, k) \) by
\[
u(x, t) = 2i \lim_{k \to \infty} k M_{12}(x, t, k). \] (2.16)

Moreover, \( u(x, 0) \) generates \( \{a(k), b(k)\} \) and \( \{u(0, t), u_x(0, t)\} \) generates \( \{A(k), B(k)\} \) as the corresponding spectral functions provided the latter verify the global relation (2.10). Therefore, the RH problem approach gives the solution of the overdetermined IBV problem
\[
iu_t + u_{xx} + 2|u|^2 u = 0, \quad x > 0, t > 0, \\
u(x, 0) = u_0(x), \quad x \geq 0, \\
u(0, t) = v_0(t), \quad 0 \leq t \leq T \\
u_x(0, t) = v_1(t), \quad 0 \leq t \leq T \] (2.17)

and provided that the spectral functions \( \{a(k), b(k), A(k), B(k)\} \) constructed from \( \{u_0(x), v_0(t), v_1(t)\} \) satisfy the global relation (2.10).
3. The Riemann–Hilbert formalism for Robin boundary condition

Even in a conditional context presented in §2, the RH method allows obtaining useful information about the solution, e.g. that concerning the large-time behaviour, see Fokas et al. [13]. However, there are cases when one can overcome the conditional nature of the solution and solve a well-posed IBV problem. The key factor for making this possible is the existence of an additional symmetry in the spectral problem for the t-equation of the Lax pair [13,15].

This holds in the case of Robin boundary condition. Indeed, if \( u + qu_x = 0 \) with some \( q \in \mathbb{R} \), then the matrix \( \tilde{V} := V - 2ik^2\sigma_3 \) of the t-equation \( \psi_t = \tilde{V}\psi \) satisfies the symmetry relation Fokas et al. [13]

\[
\tilde{V}(x, t, -k) = N(k)\tilde{V}(x, t, k)N^{-1}(k), \quad (3.1)
\]

where \( N(k) = \text{diag}(N_1(k), N_2(k)) \) with \( N_1(k) = 2k + iq \) and \( N_2(k) = -2k + iq \). In turn, (3.1) implies the symmetry for \( S: S(-k; T) = N(k)S(k; t)N^{-1}(k) \), which reads in terms of \( A \) and \( B \) as

\[
A(-k; T) = A(k; T); \quad B(-k; T) = -\frac{2k + iq}{2k - iq}B(k; T).
\]

Now we note that although \( \Gamma(k) \) is defined, for general boundary values, only for \( k \in I \), the function \( \tilde{\Gamma}(k) \), in the generic case of absence of the zeros of the denominator in (3.4), is analytic (and bounded) for all \( k \in \mathbb{C}^+ \) (and continuous up to the boundary). On the other hand, the exponentials in \( \frac{1}{\tilde{\Gamma}(k)e^{2ikx + 4ik^2t}} \) and \( \frac{1}{\tilde{\Gamma}(k)e^{-2ikx - 4ik^2t}} \) are bounded in I and IV, respectively. Thus, we can deform the RH problem with jump (3.3) on the cross to that on the real axis by defining

\[
\tilde{M}(x, t, k) = \begin{cases} 
\tilde{M}(x, t, k), & k \in \mathbb{R}, \\
\tilde{M}(x, t, k)\left( \frac{1}{\tilde{\Gamma}(k)e^{2ikx + 4ik^2t}}, 0 \right), & k \in I, \\
\tilde{M}(x, t, k)\left( 1, -\tilde{\Gamma}(k)e^{-2ikx - 4ik^2t} \right), & k \in IV.
\end{cases}
\]

The resulting jump conditions take the form

\[
\tilde{M}_+(x, t, k) = \tilde{M}_-(x, t, k)e^{-i(kx + 2k^2t)\sigma_3}\tilde{j}_0(k)e^{i(kx + 2k^2t)\sigma_3}, \quad k \in \mathbb{R}, \quad (3.6)
\]

where

\[
\tilde{j}_0(k) = \begin{pmatrix} 1 + |r_e(k)|^2 & r_e(k) \\
-\overline{r_e}(k) & 1 \end{pmatrix}
\]

and

\[
\tilde{\Gamma}(k) = \frac{b(-k)}{a(k)}\frac{2k + iq}{(2k - iq)a(-k) - (2k + iq)b(-k)}.
\]

\[
\tilde{\Gamma}(k) = e^{2ikx + 4ik^2t}.
\]

\[
\tilde{\Gamma}(k) = e^{-2ikx - 4ik^2t}.
\]
with
\[ r_c(k) = r(k) + \tilde{\Gamma}(k) = \frac{(2k - i\eta)b(k)a(-k) + (2k + i\eta)b(-k)a(k)}{(2k - i\eta)a(k)a(-k) - (2k + i\eta)b(-k)b(k)}. \] (3.8)

If the denominator in (3.8) (more precisely, the function \( a_c(k) \) defined in (3.14) below) has zeros in \( \mathbb{C}^+ \), then the formulation of the RH problem, normalized by \( \hat{M} \rightarrow I \) as \( k \rightarrow \infty \), is to be complemented by the residue conditions at these points (see (6.24c,d) in Fokas et al. [13]; here, because \( a_c(-\bar{k}) = a_c(k) \), each condition at \( \lambda_j \) is to be complemented by a corresponding condition at \( -\bar{\lambda}_j \); also notice that now there is no objection for some \( \lambda_j \) to sit on the imaginary axis). In this case, we make the genericity assumption, as in Fokas et al. [13] and Deift & Park [6], that these zeros are simple and finite in number.

**Remark 3.1.** As has already been mentioned in §1, the important observation that under the symmetry relations (3.2) the RH problem on the cross can be deformed to the RH problem on the real axis was first made in Fokas & Kamvissis [17].

**Remark 3.2.** We emphasize that the analytical continuation \( \tilde{\Gamma}(k) \) of the function \( \Gamma(k) \) is not obliged to satisfy equation (2.15) on the positive real axis where, in view of the global relation, it would have led to the erroneous conclusion that \( r_c(k) \) must vanish for all positive \( k \).

Notice that the RH problem (3.6)–(3.8) coincides with the RH problems obtained in Tarasov [5] and Deift & Park [6] via the Bäcklund technique mentioned in the introduction (cf. (3.8) and (3.14), (3.15) below with (4.51) and (4.52) in Deift & Park [6], where the notations \( A \) and \( B \) correspond to our \( a \) and \( b \). Our derivation is different: it is based on the general IBV methodology. On the one hand, this implies that the derivation inherits the restrictions imposed on the data according to the general scheme. Indeed, to be able to push \( T \rightarrow \infty \) in (2.10) and hence replace \( (B/A)(k; T) \) by \( (b/a)(k) \) for \( k \in I \), a fast \( t \)-decay of the boundary data is to be assumed in order that the limiting spectral functions \( A(k) \) and \( B(k) \) (corresponding to \( T = \infty \)) be well defined. But this decay is not necessarily true for the Robin boundary conditions (indeed, it is generally not true; see §4).

On the other hand, the resulting RH problem (3.6)–(3.8) involves only the spectral functions \( a(k) \) and \( b(k) \) associated with the IVs. This suggests to consider this RH problem in a wider setting, without relying on the spectral problem associated with the boundary values, as it takes place in the general scheme for analysing IBV problems [13]. Namely, we will prove directly that the solution of (3.6)–(3.8) leads to the function \( u(x, t) \) that satisfies (i) the NLS equation, (ii) the given initial conditions, and (iii) (most challenging) the Robin boundary condition.

In implementing this program, the first part is easy. The RH problems (3.3) and (3.6) both give the solution of the NLS equation in the domain \( x > 0 \), \( t > 0 \) via (2.16); this is a standard fact based on ideas of the dressing method [19].

In order to verify the initial condition \( u(x, 0) = u_0(x) \), one observes that for \( t = 0 \), the exponentials in \( \left( \frac{1}{\tilde{\Gamma}(k)} e^{2i\eta x} 1 \right) \) and \( \left( \frac{1}{\tilde{\Gamma}(k)} e^{-2i\eta x} 1 \right) \) are also bounded in II and III, respectively. Thus, we can deform the RH problem with jump (3.3) on the cross to that on the real axis by defining

\[
\tilde{M}(x, t, k) = \begin{cases} 
\hat{M}(x, t, k), & k \in I \cup IV, \\
\hat{M}(x, t, k) \left( \begin{array}{cc} 1 & 0 \\ -\tilde{\Gamma} e^{2i\eta x} & 1 \end{array} \right), & k \in II, \\
\hat{M}(x, t, k) \left( \begin{array}{cc} 1 & \tilde{\Gamma} e^{-2i\eta x} \\ 0 & 1 \end{array} \right), & k \in III, 
\end{cases}
\] (3.9)

which results in the jump condition

\[
\hat{M}_+(x, 0, k) = \hat{M}_-(x, 0, k) e^{ik\sigma_3} \tilde{f}_0(k) e^{ik\sigma_3}, \quad k \in \mathbb{R},
\] (3.10)

where

\[
\tilde{f}_0(k) = \begin{pmatrix} 1 + |r(k)|^2 & \tilde{r}(k) \\ r(k) & 1 \end{pmatrix}.
\] (3.11)
But the resulting RH problem (with residue conditions modified appropriately, see Fokas et al. [13]) coincides with that for the spectral mapping \( \{ u_0(x) \} \mapsto \{ a(k), b(k) \} \), which yields \( u(x, 0) = u_0(x) \) owing to the uniqueness of the solution of the RH problem.

As we noticed above, the most challenging part of the program is to show that \( u(x, t) \) satisfies the Robin boundary condition. We are going to show that it is possible by using a symmetry of \( r_e(k) \) followed from its construction (3.8). This symmetry is as follows:

\[
    r_e(-k) = r_e(k) \frac{\alpha(k)}{\alpha(-k)},
\]

where

\[
    \alpha(k) = (2k - iq)a(k)a(-k) - (2k + iq)b(k)b(-k).
\]

It is convenient to normalize \( \alpha(k) \), which is analytic in \( \mathbb{C}^+ \), in such a way that it approaches 1 as \( k \to \infty \) and that it has neither a zero nor a pole at \( k = i|q|/2 \). Depending on the sign of \( q \) and the behaviour of \( a(k) \) and \( b(k) \) at \( k = i|q|/2 \), different normalizing factors are needed. Indeed, if one introduces \( a_e(k) \) and \( \beta \) by

\[
    a_e(k) = \begin{cases}
        \frac{\alpha(k)}{2k - iq} = a(k)a(-k) - \frac{2b + iq}{2k - iq}b(-k), & \text{if } q < 0, a \left( -\frac{iq}{2} \right) \neq 0 \\
        \text{or } q > 0, b \left( \frac{iq}{2} \right) = 0
    \end{cases}
\]

and respectively

\[
    \beta = \begin{cases}
        \frac{q}{2}, & \text{if } q < 0, a \left( -\frac{iq}{2} \right) \neq 0 \text{ or } q > 0, b \left( \frac{iq}{2} \right) = 0 \\
        -\frac{q}{2}, & \text{if } q > 0, b \left( \frac{iq}{2} \right) \neq 0 \text{ or } q < 0, a \left( -\frac{iq}{2} \right) = 0
    \end{cases}
\]

then the requirements above are satisfied for \( a_e(k) \) while the symmetry condition takes the form

\[
    r_e(-k) = r_e(k) \frac{a_e(k) - i\beta}{a_e(k) + i\beta}.
\]

**Remark 3.3.** The choice of the sign for \( \beta \) in (3.15) can also be characterized in terms of the number of zeros of \( a_e(k) \) in \( \mathbb{C}^+ \), see Deift & Park [6]. Indeed, it follows from (3.14) (recall \( |a|^2 + |b|^2 = 1 \)) that \( a_e(0) = 1 \) in the first case, which corresponds to \( \beta = q/2 \), and \( a_e(0) = -1 \) in the second case, which corresponds to \( \beta = -q/2 \); thus \( \beta = a_e(0)(q/2) \) in the both cases. On the other hand, \( a_e(0) = (-1)^n \), where \( n \) is the number of zeros of \( a_e(k) \). This follows from the formula

\[
    a_e(k) = \prod_{i=1}^{n} \frac{k - k_i}{k - k_i} \exp \left\{ -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log(1 + |r_e(s)|^2)}{s - k} \, ds \right\}, \quad k \in \mathbb{C}^+,
\]

where \( k_i, i = 1, \ldots, n \) are the zeros of \( a_e(k) \) (notice that \( 1 + |r_e(k)|^2 = |a_e(k)|^{-2} \), \( k \in \mathbb{R} \)), and the symmetries

\[
    a_e(-k) = a_e(k), \quad k \in \mathbb{C}^+; \quad |r_e(-k)| = |r_e(k)|, \quad k \in \mathbb{R},
\]

cf. Deift & Park [6]. Therefore, \( \beta = (-1)^n q/2 \).

The symmetry (3.16) yields a certain \( k \to -\bar{k} \) symmetry of the solution \( \hat{M}(x, t, k) \) of the RH problem (3.6). The relevant symmetry relation has been established in Deift & Park [6] in the case \( t = 0 \) and \( x \in \mathbb{R} \). We shall need a version of that relation for the ‘complimentary’ case, i.e. \( x = 0 \), and \( t > 0 \). We shall perform the derivation in this case following practically the same arguments as in Deift & Park [6] (cf. the proof of proposition 4.28 of Deift & Park [6]).
Denote $\hat{J}(x, t, k)$ the jump matrix of problem (3.6), i.e.

$$\hat{J}(x, t, k) = e^{-i(kx + 2kt^2)t_0(k)} e^{i(kx + 2kt^2)t_0(k)}.$$

From (3.16), it follows that

$$\hat{J}(0, t, -k) = \begin{pmatrix} \frac{1}{|a_c(k)|^2} & \frac{r_c(k) a_c(k)}{a_c(k)} \frac{k - i \beta}{k + i \beta} e^{i4\kappa t} \\ \frac{r_c(k) a_c(k)}{a_c(k)} \frac{k + i \beta}{k - i \beta} e^{-i4\kappa t} & 1 \end{pmatrix}$$

(3.17)

where

$$C(k) = \begin{cases} \begin{pmatrix} a_c(k) & 0 \\ 0 & \frac{1}{a_c(k)} \end{pmatrix} \begin{pmatrix} k - i \beta & 0 \\ 0 & k + i \beta \end{pmatrix} \sigma_1, & \text{Im } k > 0, \\ \begin{pmatrix} 1 & 0 \\ \frac{1}{a_c(k)} & 0 \end{pmatrix} \begin{pmatrix} k - i \beta & 0 \\ 0 & k + i \beta \end{pmatrix} \sigma_1, & \text{Im } k < 0, \end{cases}$$

(3.19)

with $\sigma_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$. This implies that the function

$$\tilde{M}(t, k) := \mathcal{E}(t, k) \tilde{M}(0, t, -k) C(k)$$

(3.20)

satisfies the same jump condition as $\hat{M}(0, t, k)$ does:

$$\tilde{M}_+(t, k) = \tilde{M}_-(t, k) \hat{J}(0, t, k), \quad k \in \mathbb{R}.$$

The factor $\mathcal{E}(t, k)$ can be chosen so that $\tilde{M}(t, k) \to I$ as $k \to \infty$ and that $\tilde{M}(t, k)$ has no singularities at $k = \pm i \beta$. Indeed, introducing

$$\mathcal{E}(t, k) = \sigma_1 P(t) \begin{pmatrix} 1 & 0 \\ \frac{1}{k + i \beta} & 1 \end{pmatrix} P^{-1}(t)$$

(3.21)

with $P(t)$ to be defined, one has

$$\tilde{M}(t, k) = \sigma_1 P(t) \begin{pmatrix} (P^{-1} \tilde{M}(0, t, -k))_{11} & (P^{-1} \tilde{M}(0, t, -k))_{12} \\ (P^{-1} \tilde{M}(0, t, -k))_{21} & (P^{-1} \tilde{M}(0, t, -k))_{22} \end{pmatrix} D(k) \sigma_1,$$

where

$$D(k) \equiv \text{diag}(d_1(k), d_2(k)) = \begin{cases} \begin{pmatrix} a_c(k) & 0 \\ 0 & \frac{1}{a_c(k)} \end{pmatrix}, & \text{Im } k > 0, \\ \begin{pmatrix} 1 & 0 \\ \frac{1}{a_c(k)} & 0 \end{pmatrix}, & \text{Im } k < 0. \end{cases}$$

(3.22)

Therefore, there are no singularities at $k = \pm i \beta$ provided

$$(P^{-1} \tilde{M}(0, t, i \beta))_{12} = 0 \quad \text{and} \quad (P^{-1} \tilde{M}(0, t, -i \beta))_{21} = 0.$$  

This suggests to determine $\hat{P}^{-1}(t)$ as follows:

$$\hat{P}^{-1}(t) = \begin{pmatrix} \hat{M}_{22}(0, t, i \beta) & -\hat{M}_{12}(0, t, i \beta) \\ -\hat{M}_{21}(0, t, -i \beta) & \hat{M}_{11}(0, t, -i \beta) \end{pmatrix},$$
which gives

$$\hat{P}(t) = \frac{1}{\Delta(t)} \begin{pmatrix} \hat{M}_{11}(0, t, -i\beta) & \hat{M}_{12}(0, t, i\beta) \\ \hat{M}_{21}(0, t, -i\beta) & \hat{M}_{22}(0, t, i\beta) \end{pmatrix}$$

(3.23)

with

$$\Delta(t) = \hat{M}_{11}(0, t, -i\beta)\hat{M}_{22}(0, t, i\beta) - \hat{M}_{12}(0, t, i\beta)\hat{M}_{21}(0, t, -i\beta)$$

$$= |\hat{M}_{11}(0, t, -i\beta)|^2 + |\hat{M}_{21}(0, t, -i\beta)|^2 > 0.$$

If the setting of the RH problem includes the poles, then, similar to the case considered in Deift & Park [6], one can verify that \(\tilde{M}(t,k)\) defined by (3.20)–(3.23) satisfies the same residue conditions as \(\tilde{M}(0,t,k)\) and hence the uniqueness of the solution of the RH problem gives

$$\tilde{M}(t,k) = \hat{M}(0,t,k),$$

which reads in terms of \(\hat{M}(0,t,k)\) only as

$$\overline{\hat{M}(0,t,-k)} = \sigma_1 \hat{P}(t) \begin{pmatrix} k - i\beta & 0 \\ 0 & k + i\beta \end{pmatrix} \hat{P}^{-1}(t)\hat{M}(0,t,k) \begin{pmatrix} k - i\beta & 0 \\ 0 & k + i\beta \end{pmatrix} D(k)\sigma_1. \quad (3.24)$$

**Remark 3.4.** The above arguments allow actually to prove the general symmetry formula that is valid for all \(x\) and \(t\),

$$\overline{\tilde{M}(-x,t,-k)} = \sigma_1 \hat{P}(t) \begin{pmatrix} k - i\beta & 0 \\ 0 & k + i\beta \end{pmatrix} \hat{P}^{-1}(t)\tilde{M}(x,t,k) \begin{pmatrix} k - i\beta & 0 \\ 0 & k + i\beta \end{pmatrix} D(k)\sigma_1.$$

In the case \(t = 0\) and \(x \in \mathbb{R}\), this is (up to the notations) the formula proved in Deift & Park [6].

We shall now show how the symmetry relation (3.24) alone can be used to establish the Robin boundary condition for \(u(x,t)\). To this end, we first evaluate, using (3.24), the entries \(\tilde{M}_{11}(0, t, -i\beta)\) and \(\tilde{M}_{21}(0, t, -i\beta)\). We have:

$$\overline{\tilde{M}_{11}(0, t, -i\beta)} = \lim_{k \to -i\beta} \left( -\sigma_1 \tilde{E}(t,-k)\tilde{M}(0,t,k) \begin{pmatrix} 0 & k + i\beta \\ k - i\beta & 0 \end{pmatrix} D(k) \right)_{22}$$

$$= \tilde{P}_{22}(t)\hat{P}^{-1}(t)\tilde{M}(0,t,-i\beta))_{22}d_2(-i\beta) = \frac{\hat{M}_{22}(0,t,i\beta)d_2(-i\beta)}{\Delta(t)}$$

$$\overline{\tilde{M}_{21}(0, t, -i\beta)} = \lim_{k \to -i\beta} \left( -\sigma_1 \tilde{E}(t,-k)\tilde{M}(0,t,k) \begin{pmatrix} 0 & k + i\beta \\ k - i\beta & 0 \end{pmatrix} D(k) \right)_{12}$$

$$= \tilde{P}_{12}(t)\hat{P}^{-1}(t)\tilde{M}(0,t,-i\beta))_{22}d_2(-i\beta) = \frac{\hat{M}_{12}(0,t,i\beta)d_2(-i\beta)}{\Delta(t)}$$

(3.25)

where we have used the basic symmetry (2.8). Similarly, we have

$$\overline{\tilde{M}_{21}(0, t, -i\beta)} = \lim_{k \to -i\beta} \left( -\sigma_1 \tilde{E}(t,-k)\tilde{M}(0,t,k) \begin{pmatrix} 0 & k + i\beta \\ k - i\beta & 0 \end{pmatrix} D(k) \right)_{12}$$

$$= \tilde{P}_{12}(t)\hat{P}^{-1}(t)\tilde{M}(0,t,-i\beta))_{22}d_2(-i\beta) = \frac{\hat{M}_{12}(0,t,i\beta)d_2(-i\beta)}{\Delta(t)}$$

(3.26)

Comparing (3.25) and (3.26) gives

$$\overline{\tilde{M}_{11}(0, t, -i\beta)(1-\theta)} = 0 \quad \text{and} \quad \overline{\tilde{M}_{21}(0, t, -i\beta)(1+\theta)} = 0,$$

(3.27)

where

$$\theta = \frac{d_2(-i\beta)}{\Delta},$$

(3.28)

which implies that either \(\tilde{M}_{11}(0, t, -i\beta) = 0\) or \(\tilde{M}_{21}(0, t, -i\beta) = 0\) for all \(t \geq 0\).
The next (and the last) step is to explore an idea that was first suggested by Bobenko in the late 1980s, and was used then in several works dealing with the algebro-geometric solutions of integrable equations [20].

Recall that $\Psi(t,k):=(\hat{M}_{11}(0,t,k)e^{-2ik^2t},\hat{M}_{21}(0,t,k)e^{-2ik^2t})^T$ satisfies the differential equation (2.4) with $u=u(0,t)$ and $u_x=u_x(0,t)$, i.e.

$$\frac{d\Psi_1}{dt}+2ik^2\Psi_1=i|u(0,t)|^2\Psi_1+(2ku(0,t)+iu_x(0,t))\Psi_2$$

and

$$\frac{d\Psi_2}{dt}-2ik^2\Psi_2=-i|u(0,t)|^2\Psi_2+(-2\tilde{u}(0,t)+i\tilde{u}_x(0,t))\Psi_1.$$ (3.29)

From (3.29), it follows that if $\Psi(t,-i\beta)=0$ for all $t \geq 0$, then $-2i\beta u(0,t)+iu_x(0,t) \equiv 0$, or $u_x(0,t)-2\beta u(0,t) \equiv 0$; if $\Psi(t,-i\beta)=0$, then $2i\beta \tilde{u}(0,t)+i\tilde{u}_x(0,t) \equiv 0$, or $u_x(0,t)+2\beta u(0,t) \equiv 0$. Observe that, according to (3.15), $\beta$ can be either $q/2$ or $-\beta$.

A closer look at (3.27) and (3.28) reveals that one can specify precisely whether (i) $\hat{M}_{11}(0,t,-i\beta)=0$ occurs, depending on the sign of $q$ and the properties of $a(|q|/2)$ and $b(|q|/2)$. Indeed, because $\Delta = |\hat{M}_{11}(-i\beta)|^2 + |\hat{M}_{21}(-i\beta)|^2 > 0$, the choice between (i) and (ii) is determined by the sign of $d_2(-i\beta)$. According to (3.14) and (3.15), one can distinguish four cases.

1. If $q > 0$ and $b(|q|/2) = 0$, then $\beta = q/2 > 0$ and thus $d_2(-i\beta) = a(-i|q|/2)$. In turn, from (3.14) it follows that in this case, $a(-i|q|/2) > 0$ and thus $1 + d_2(-i\beta) = \Delta > 0$, which implies (see (3.27)) that $\hat{M}_{21}(0,t,-i\beta) = 0$.

2. If $q < 0$ and $a(-i|q|/2) \neq 0$, then $\beta = q/2 < 0$ and thus $d_2(-i\beta) = a(-i|q|/2)^{-1} = |a(-i|q|/2)|^{-2} > 0$. Hence, in this case one also has $1 - d_2(-i\beta)/\Delta > 0$ and thus $\hat{M}_{21}(0,t,-i\beta) = 0$.

3. If $q < 0$ and $a(-i|q|/2) = 0$, then $\beta = -q/2 > 0$ and thus $d_2(-i\beta) = b(-i|q|/2) = -|b(-i|q|/2)|^2 < 0$. Hence, in this case $1 - d_2(-i\beta)/\Delta > 0$, which implies that $\hat{M}_{11}(0,t,-i\beta) = 0$.

4. If $q > 0$ and $b(|q|/2) \neq 0$, then $\beta = -q/2 < 0$ and thus $d_2(-i\beta) = a(-i|q|/2)^{-1} = |b(|q|/2)|^{-2} < 0$. Hence, in this case one also has $1 - d_2(-i\beta)/\Delta > 0$, which implies that $\hat{M}_{11}(0,t,-i\beta) = 0$.

Summarizing, we see that $\hat{M}_{21}(0,t,-i\beta) = 0$ corresponds to $\beta = q/2$ while $\hat{M}_{11}(0,t,-i\beta) = 0$ corresponds to $\beta = -q/2$, which is indeed consistent with the fact that (3.29) implies $u_x(0,t)+qu(0,t)=0$ for all $t$.

4. Concluding remarks

1. In the general RH approach to IBV problems for integrable nonlinear equations [10–13], an important step is the verification that the solution of the underlying nonlinear equation obtained from the solution of the associated RH problem indeed satisfies the prescribed boundary conditions. In the general case, this can be performed by mapping the master RH problem, in which the space parameter is taken to correspond to the boundary, to the RH problem associated with the t-equation of the Lax pair with a ‘potential’ constructed from the prescribed boundary values, and showing that they are equivalent, in the sense that they produce the same ‘potentials’. But this means that the latter RH problem must be well-defined, which in particular requires, in the case of semi-infinite time interval $0 < t < \infty$, a precise description of the large-time behaviour of the boundary values. On the other hand, such description is, generally, not available in full from the boundary conditions of a well-posed IBV problem, which forces one to make certain a priori assumptions about this behaviour.

On the other hand, for linearizable boundary conditions, as we have shown on the example of the Robin boundary condition for the NLS equation, it is possible to verify directly that these boundary conditions hold, by using additional symmetry properties of an appropriately deformed
master RH problem thus avoiding restricting a priori assumptions. The importance of this fact can be illustrated by the following simple observation: in the case $q > 0$, if $b(iq/2) \neq 0$ (generic case!), then, as it follows from (3.14), $a_\varepsilon(i\xi) = |b(iq/2)|^2 \neq 0$. On the other hand, $a_\varepsilon(i\xi) \to 1$ as $\xi \to +\infty$. Noticing that $a_\varepsilon(i\xi) \in \mathbb{R}$ for all $\xi > 0$, we conclude that $a_\varepsilon(k)$ must have at least one zero for $k \in i\mathbb{R}_+$. Consequently, as it follows from the asymptotic analysis similar to the one done in Deift & Park [6], stationary solitons generated by these zeros dominate the large-$t$ asymptotics in the direction along the $t$-axis, which prevents from assuming the decaying behaviour of $u(0,t)$ and $u_x(0,t)$ needed for defining the associated spectral functions $A(k)$ and $B(k)$.

2. For the initial data that do not produce zeros of $a_\varepsilon(i\xi)$ and thus stationary solitons, the boundary value of the solution of the Robin problem decays as $t^{-1/2}$, as it again follows from the asymptotic analysis similar to Deift & Park [6]. This is formally not enough in order to proceed with the approach of Fokas [15] and Fokas et al. [13]. However, one can guarantee the needed rate of decay of the boundary value assuming that a certain combination of the spectral functions $a(k)$ and $b(k)$, corresponding to the initial data $u_0(x)$ in the IST formalism, vanishes at $k = 0$ to a high order.

3. In our analysis of the IBV problem formulated for the domain $x \geq 0$, $t \geq 0$, we do not go ‘beyond the domain’, working with $x$ and $t$, as parameters in the RH problem, that stay in the domain prescribed by the problem. This is in contrast with the approach based on an appropriate continuation to $x < 0$ with the help of the Bäcklund transformation allowing to control the necessary conditions at $x = 0$ for all $t$ [1,4–6]. Actually, folding the contour of the RH problem down to the real axis establishes the relationship between these approaches. Indeed, the functions $r_x(k)$ and $a_\varepsilon(k)$ are exactly the reflection coefficient and the inverse transmission coefficient for the RHP problem on the whole axis associated with the IV problem on the whole line with the Bäcklund-continued initial data (see Deift & Park [6]).

4. An alternative way to see that $u(x,t)$ constructed from the solution of the RH problem satisfies the boundary condition is based on using the further terms in the expansion of $\hat{M}(x,t,k)$ as $k \to \infty$ giving not only $u(x,t)$ (as in (2.16)) but also $u_x(x,t)$ [13]:

$$u_x(x,t) = \lim_{k \to \infty} \left[ 4(k^2\hat{M})_{12}(x,t,k) + 2iu(x,t)(k\hat{M})_{22}(x,t,k) \right]$$

Indeed, let us substitute the expansion (for $x = 0$)

$$\hat{M}(0,t,k) = I + \frac{m^1(t)}{k} + \frac{m^2(t)}{k^2} + \cdots, \quad k \to \infty$$

into the symmetry relation (3.24) taking into account that, in view of (3.14) and (3.15),

$$a_\varepsilon(k) = \begin{cases} 1 + O\left(\frac{1}{k^2}\right), & \text{if } \beta = \frac{q}{2}, \\ 1 - \frac{iq}{k} + O\left(\frac{1}{k^2}\right), & \text{if } \beta = -\frac{q}{2}. \end{cases}$$

Equating the terms of order $k^{-1}$, one gets

$$\begin{pmatrix} -2m^1_{11} & 0 \\ 0 & -2m^1_{22} \end{pmatrix} = \beta \sigma_1 \tilde{P} \sigma_3 \tilde{P}^{-1} + \frac{iq}{2} \sigma_3.$$ (4.2)

On the other hand, from (3.23) we have

$$\tilde{P} \sigma_3 \tilde{P}^{-1} = \frac{1}{\Delta} \left( \begin{array}{cc} \hat{M}_{11}(-i\beta) & \hat{M}_{12}(i\beta) \\ \hat{M}_{21}(-i\beta) & \hat{M}_{22}(i\beta) \end{array} \right) \sigma_3 \left( \begin{array}{cc} \hat{M}_{22}(i\beta) & -\hat{M}_{12}(i\beta) \\ -\hat{M}_{21}(i\beta) & \hat{M}_{11}(-i\beta) \end{array} \right),$$

where $\delta = \hat{M}_{11}(-i\beta)\hat{M}_{22}(i\beta) + \hat{M}_{21}(-i\beta)\hat{M}_{12}(i\beta)$; here the fact that either $\hat{M}_{11}(-i\beta) = 0$ or $\hat{M}_{21}(-i\beta) = 0$ implies the diagonal structure of the resulting matrix.
Further, in the case $\dot{M}_{11}(-i\beta) = 0$ (recall that in this case $\beta = -q/2$), one has

$$\delta = \dot{M}_{21}(-i\beta)\dot{M}_{12}(i\beta) = -|\dot{M}_{21}(-i\beta)|^2, \quad \Delta = |\dot{M}_{21}(-i\beta)|^2$$

and thus $\ddot{P}\sigma_3P^{-1} = -\sigma_3$. In the case $\dot{M}_{21}(-i\beta) = 0$ ($\beta = q/2$) one has

$$\delta = \dot{M}_{11}(-i\beta)\dot{M}_{22}(i\beta) = |\dot{M}_{11}(-i\beta)|^2, \quad \Delta = |\dot{M}_{11}(-i\beta)|^2$$

and thus $\ddot{P}\sigma_3P^{-1} = \sigma_3$. In both cases, from (4.2) we have

$$m_{11}^2(t) = m_{22}^2(t) = 0. \quad (4.3)$$

Now, equating the terms of order $k^{-2}$, we have for the off-diagonal part:

$$\left(\begin{array}{cc} 0 & -2m_{21}^2 \\ -2m_{12}^2 & 0 \end{array}\right) = \left(\begin{array}{cc} 0 & m_{12}^2(-2m_{11}^2 + iq) \\ m_{12}^2(-2m_{22}^2 - iq) & 0 \end{array}\right),$$

which, in view of (4.3), reads

$$m_{12}^2(t) = m_{12}^1(t)\left(-\frac{iq}{2}\right). \quad (4.4)$$

Finally, (4.1) and (2.16), in view of (4.3) and (4.4), yield

$$u_x(0,t) = 4m_{12}^2(t) = -2iqm_{12}^1(t) = -qu(0,t).$$

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