Harmonic Map Formulation of
Colliding Electrovac Plane Waves*

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The formulation of the Einstein field equations admitting two Killing vectors in terms of harmonic mappings of Riemannian manifolds is a subject in which Charlie Misner has played a pioneering role. We shall consider the hyperbolic case of the Einstein-Maxwell equations admitting two hypersurface orthogonal Killing vectors which physically describes the interaction of two electrovac plane waves. Following Penrose’s discussion of the Cauchy problem we shall present the initial data appropriate to this collision problem. We shall also present three different ways in which the Einstein-Maxwell equations for colliding plane wave spacetimes can be recognized as a harmonic map. The goal is to cast the Einstein-Maxwell equations into a form adopted to the initial data for colliding impulsive gravitational and electromagnetic shock waves in such a way that a simple harmonic map will directly yield the metric and the Maxwell potential 1-form of physical interest.

* for Charles W. Misner on his 60th birthday
1 Introduction

Charlie Misner was the first to recognize that the subject of harmonic mappings of Riemannian manifolds finds an important application in general relativity. In a pioneering paper with Richard Matzner [1] he found that stationary, axially symmetric Einstein field equations can be formulated as a harmonic map. Eells and Sampson’s theory of harmonic mappings of Riemannian manifolds [2] provides a geometrical framework for thinking of a set of pde’s, in the same spirit as “mini-superspace” that Charlie was to introduce [3] for ode Einstein equations a little later. The subsequent development of the subject of space-times admitting two Killing vectors, that eventually led to its recognition as a completely integrable system [4] - [7] has employed another formulation of the stationary, axi-symmetric field equations due to Ernst [8] which is equivalent to that of Matzner and Misner.

Charlie’s later work on harmonic maps [9] encompasses a scope much broader than this specific problem and its power and elegance is bound to make a major impact on theoretical physics.

I was privileged to be in contact with Charlie’s ideas at that time and worked on the two Killing vector problem [10], [11]. It was the hyperbolic version of gravitational fields admitting two Killing vectors that attracted my attention. This is the problem of colliding impulsive plane gravitational waves for which Khan and Penrose had presented a famous solution [12]. My work finally led to the exact solution for colliding impulsive plane gravitational waves with non-collinear polarizations [13] which is physically the most general solution of this type. It turned out that the Matzner-Misner formalism was the one more readily amenable to the hyperbolic problem, even though it was originally intended for the elliptic case, whereas the Ernst formulation fitted the elliptic problem, ie the exterior field of rotating stars, best. The relationship between these two formalisms is given by a Neugebauer-Kramer involution [14].

A few years after the solution [13] appeared, there was a remarkable avalanche of papers on colliding plane gravitational waves. There were important papers examining the singularity structure of spacetimes resulting from the collision of gravitational waves [18] - [20]. However, there was also a mass of new exact solutions which are all essentially devoid of any physical interest because their authors had chosen not to solve the Cauchy problem with the initial data appropriate to generic plane waves, but rather they
started with a “solution” and derived (!) the initial data. This type of derived initial data for the collision problem describes some very peculiar plane waves indeed. An inordinately large number of such references can be found in [15].

Nevertheless, physically interesting colliding wave problems are still open and waiting for an exact solution! Remarkably enough, the interaction of plane impulsive gravitational and electromagnetic shock waves is in this category. We have the Khan-Penrose and Bell-Szekeres [16] solutions describing the interaction of either two impulsive gravitational, or two electromagnetic shock waves alone and also the solution of Griffiths [17] for the interaction of an impulsive gravitational wave with an electromagnetic shock wave. But the generic case where we must consider the collision of both type of waves is missing even in the case of collinear polarization. The important open problem here is the construction of an exact solution of the Einstein-Maxwell equations that reduces to all, the Khan-Penrose, Bell-Szekeres and Griffiths solutions. There are various unsatisfactory treatments of this problem in the literature [21], [22]. I shall give its harmonic map formulation.

\section{Initial Data}

The problem of colliding plane gravitational waves was proposed and in essence solved by Penrose [23] in 1965 even though most people writing on this subject do not seem to be familiar with it. We shall use Penrose’s formulation of the Cauchy problem [24], [25] to discuss the interaction of two plane waves, every one of which will consist of a superposition of an impulsive gravitational wave and an electromagnetic shock wave. The interaction will be determined by an integration of the Einstein-Maxwell equations with initial data defined on a pair of intersecting null characteristics. The initial values of the fields will be those appropriate to a plane wave which is given by the Brinkman metric [26], [27]

\[ d s^2 = 2 du' dv' - dx'^2 - dy'^2 + 2 H(v', x', y') dv'^2 \]  

(1)

and the superposition of an impulsive gravitational wave and an electromagnetic shock wave, with amplitudes proportional to \( a, b \) respectively, is
obtained for
\[ H = \frac{a}{2} (y'^2 - x'^2) \delta(v') - \frac{b^2}{2} (x'^2 + y'^2) \theta(v') \] (2)

where \( \delta \) is the Dirac delta-function and \( \theta \) is the Heaviside unit step-function.

The Brinkman coordinate system employed in eq.(1) is useful because the superposition of waves travelling in the same direction is obtained simply by addition. However, the Brinkman coordinates are not suitable for the collision problem because of the explicit dependence of the metric on \( x', y' \).

For this purpose we must transform to the Rosen form where the metric coefficients will depend on \( v \) alone. This is accomplished by the Khan-Penrose transformation

\[

t' = v, \\
u' = u + \frac{1}{2} x^2 F F_v + \frac{1}{2} y^2 G G_v, \\
x' = x F, \\
y' = y G,
\] (3)

which results in
\[ ds^2 = 2 d u d v - F^2 d x^2 - G^2 d y^2, \] (4)

provided

\[
F_{vv} = \begin{bmatrix} -a \delta(v) - b \theta(v) \end{bmatrix} F, \\
G_{vv} = \begin{bmatrix} -a \delta(v) + b \theta(v) \end{bmatrix} G. \] (5)

These are linear, distribution-valued ordinary differential equations which can be solved using the Laplace transform
\[ \mathcal{F}(s) = \int_0^\infty e^{sv} F(v) dv \] (6)

and from eq.(5) we find
\[ \mathcal{F}(s) = \frac{1}{s^2 + b^2} [(s - a) F(0) + F_v(0)] \] (7)

where of \( F(0), F_v(0) \) are initial values. They are obtained from the continuity of the metric and its first derivatives across \( v = 0 \) which requires \( F(0) = 1, F_v(0) = 0 \). In this case inverting the Laplace transform we get
\[ F = \cos(bv\theta(v)) - \frac{a}{b} \sin(bv\theta(v)) \] (8)
and the result for $G$ is obtained by letting $a \rightarrow -a$ in eq. (8) as indicated by eqs. (5).

In Rosen coordinates the general form of the metric that admits two hypersurface orthogonal Killing vectors is given by

$$ds^2 = 2 e^{-M} du dv - e^{-U} \left( e^V dx^2 + e^{-V} dy^2 \right)$$

(9)

where $U, V, M$ depend on only $u, v$ and comparison with eqs. (4) and (8) shows that for the initial value problem the data is given by

$$e^{-U} = \cos^2(bv\theta(v)) - \frac{a^2}{b^2} \sin^2(bv\theta(v))$$

$$e^{-V} = \frac{b + a \tan (bv\theta(v))}{b - a \tan (bv\theta(v))}$$

$$e^{-M} = 1$$

(10)

The limiting values of this result are familiar. If we have just an impulsive gravitational wave, we must pass to the limit $b \rightarrow 0$ which yields

$$e^{-U} = 1 - a v^2 \theta(v)$$

$$e^{-V} = \frac{1 + av\theta(v)}{1 - av\theta(v)}$$

$$e^{-M} = 1$$

(11)

as in the case of Khan and Penrose. Furthermore, in the limit $a \rightarrow 0$ we have only an electromagnetic shock wave

$$e^{-U} = 1 - \sin^2(bv\theta(v))$$

$$e^{-V} = 1$$

$$e^{-M} = 1$$

(12)

which is the result for the Bell-Szekeres case.

Spacetimes describing colliding plane waves are divided into four regions:

Region I: $u < 0, v < 0$ empty space before the collision
Region II: $u > 0, v < 0$ a plane wave
Region III: $u < 0, v > 0$ another wave travelling in the opposite direction
Region IV: $u > 0, v > 0$ the interaction region

The initial values given above are on $v = 0$, the boundary between Regions III, IV and similar results hold on $u = 0$, the boundary between Regions II,
IV, determining the $u$-dependence. In the latter case the amplitudes of these waves will be given by different constants, say $a \rightarrow p$ and $b \rightarrow q$, cf eq. (28) in sequel. The case of Griffiths’ solution is a mixture where we have eqs. (11) between Regions III, IV and eqs. (12) with $u$ replacing $v$ on the boundary between Regions II, IV.

3 Einstein-Maxwell Equations

The Einstein-Maxwell field equations governing the interaction of two plane waves is well-known [28]. Starting with the metric (9) and the Maxwell potential 1-form $A$

$$A = Ax$$

where $A$ depends only on $u, v$, we find a set of Einstein field equations which can be grouped into two categories. First we have the initial value equations

$$2U_{vv} - U_v^2 + 2M_v U_v - V_v^2 = 2\kappa e^{U-V}A_v^2$$
$$2U_{uu} - U_u^2 + 2M_u U_u - V_u^2 = 2\kappa e^{U-V}A_u^2$$

and their integrability conditions

$$U_{uv} - U_v U_u = 0$$
$$2A_{uv} - V_v A_u - V_u A_v = 0$$
$$2V_{uv} - U_v V_u - U_u V_v + 2\kappa e^{U-V}A_v A_u = 0$$
$$2M_{uv} + U_v U_u - V_v V_u + 2\kappa e^{U-V}A_u A_v = 0$$

where $\kappa$ is Newton’s constant in geometrical units.

In eqs. (15) we have the wave equation for $e^{-U}$ and its solution is immediate from the initial values. The following two equations are the main equations and the last equations is irrelevant as $M$ can be obtained from quadratures once the main equations are solved.

The problem consists of finding a solution to eqs. (13) satisfying the initial data (11). Finally, we shall remark that eqs. (11), or (12) can be regarded as the solution of an initial value problem themselves, namely one between Region I and either a gravitational impulsive wave, or an electromagnetic shock wave across the null plane $v = 0$ in Region III. In this mini-problem the first one of eqs. (14) serves as the field equation.
4 Harmonic Maps

We refer to [2] and [29] for a review and survey of the principal results on harmonic mappings of Riemannian manifolds. Here we shall briefly recall the most basic definitions in order to fix the notation. We shall consider two Riemannian manifolds endowed with metrics

\[ ds^2 = g_{ik} \, dx^i \, dx^k, \quad i = 1, \ldots, n \]
\[ ds'^2 = g'_{\alpha\beta} \, dy^\alpha \, dy^\beta, \quad \alpha = 1, \ldots, n' \]

and a map

\[ f : \mathcal{M} \to \mathcal{M}' \]

between them. This map is called harmonic if it extremizes the energy functional of Eells and Sampson,

\[ \delta I = 0, \]
\[ I(f) = \int g'_{\alpha\beta} \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^k} g^{ik} \sqrt{g} \, d^n x \]

where the Lagrangian consists of the trace with respect to the metric \( g \) of the induced metric \( f^* g \) on \( \mathcal{M} \). When the target space \( \mathcal{M}' \) is 1-dimensional, harmonic maps satisfy Laplace’s equation on the background of \( \mathcal{M} \) and on the other hand if \( \mathcal{M} \) is 1-dimensional, then harmonic maps coincide with the geodesics on \( \mathcal{M}' \). The nonlinear sigma model corresponds to the harmonic map \( f : R^2 \to S^2 \).

There are at least three different ways in which the Einstein-Maxwell equations (15) can be formulated as a harmonic map.

1. We can take a 4-dimensional target space \( \mathcal{M}' \) with the metric

\[ ds'^2 = e^{-U} \left( 2 \, dM \, dU + dU^2 - dV^2 \right) - 2 \kappa \, e^{-V} \, dA^2 \]

which has sections of constant curvature and the flat metric

\[ ds^2 = 2 \, du \, dv \]

on \( \mathcal{M} \). This is the electrovac analogue of the formulation given in [13]. It is not the most economical approach because \( M \), which can be obtained by quadratures, appears explicitly in the metric (19).

2. We can get rid of \( M \) and consider a 2-dimensional target space with constant curvature at the expense of regarding \( U \) as a given function on
which does not enter into the variational problem as one of the local components of the harmonic map. Thus assuming that $e^{-U}$ satisfies the wave equation, as in eqs. (13), we can take the metric on the target space as

$$ds'{}^2 = e^{-U} \frac{d\mathcal{E} \, d\bar{\mathcal{E}}}{|\text{Re}\mathcal{E}|}$$

where

$$\mathcal{E} = e^{(V-U)/2} + i \sqrt{\frac{\kappa}{2}} A$$

is an Ernst potential type of complex coordinate. The metric (21) is that of a space of constant negative curvature and $\mathcal{E}$ is the complex coordinate for the Poincaré upper half plane. There exists another representation, namely Klein’s unit disk for the space of constant negative curvature. This is obtained by the transformation

$$\mathcal{E} = \frac{\xi + 1}{\xi - 1}$$

and

$$ds'{}^2 = e^{-U} \frac{d\xi \, d\bar{\xi}}{(1 - \xi\bar{\xi})^2}$$

is the resulting form of the metric. Frequently this is the most convenient representation. The original Matzner-Misner [1] as well as the Neu gebauer-Kramer [14] formulations are of this type.

The metric on $\mathcal{M}$ is the same as the one in eq. (20). This approach is by far the most common procedure followed in the literature.

3. It is possible to reformulate the reduced problem by avoiding the ad hoc introduction of $U$ on $\mathcal{M}'$ at the expense of adding a new Killing direction to $\mathcal{M}$ where the magnitude of the Killing vector is $e^{-U}$. In this case we have

$$ds'{}^2 = \frac{d\xi \, d\bar{\xi}}{(1 - \xi\bar{\xi})^2}$$

where the definition of $\xi$ is the same as before and

$$ds^2 = 2 \, du \, dv - e^{-2U} \, d\bar{z}^2$$
where we consider the mapping to be independent of $z$ and once again $U$ is specified \textit{a priori}. It appears that this possibility has not been discussed before in the literature.

The first formulation where it is not necessary to introduce information from outside the variational principle is attractive. However, the close resemblance of the latter formulations to non-linear $\sigma$-models is an advantage.

## 5 Solutions

The metric (20) on $M$ is not in a form most suitable for the construction of harmonic maps. We need to rewrite it using new coordinates in such a way that some information about the initial data (10) is already incorporated into the system with the result that we can look for simple harmonic maps automatically satisfying the initial data. For this purpose we start with the wave equation for $e^{-U}$ and following Szekeres [28] write its solution as

$$e^{-U} = f(u) + g(v)$$

(27)

where from eq.(8) we know that

$$f = \frac{1}{2} - \left(1 + \frac{a^2}{b^2}\right) \sin^2(bv\theta(v)),$$

$$g = \frac{1}{2} - \left(1 + \frac{p^2}{q^2}\right) \sin^2(qv\theta(v)).$$

(28)

We can now consider a trivial coordinate transformation

$$u \rightarrow f(u) \quad v \rightarrow g(v)$$

which amounts to a replacement of $u, v$ by $f, g$ in the Einstein-Maxwell equations (15). We shall further introduce the following definitions which are formally the same as those given by Khan and Penrose

$$p = \sqrt{\frac{1}{2} - f} \quad q = \sqrt{\frac{1}{2} - g}$$

$$r = \sqrt{\frac{1}{2} + f} \quad w = \sqrt{\frac{1}{2} + g}$$

(29)

in terms of which it will be convenient to introduce new coordinates on $M$. 9
In the vacuum case with non-collinear polarization we had found that

$$\sigma = pw - qr \quad \tau = pw + qr$$

(30)

were useful new coordinates because of two reasons. First of all, $\sigma$ and $\tau$ can be recognized as prolate spheroidal coordinates [30] and we have the Kerr-Tomimatsu-Sato solutions of the main equations. Furthermore, the simplest and the most familiar solution of this type

$$\xi = \cos(\alpha)\tau + i\sin(\alpha)\sigma$$

(31)

satisfies the initial data.

This situation changes for the electrovac case. It turns out that a useful definition of new coordinates is

$$\sigma = \frac{p - q}{r + w} \quad \tau = \frac{p + q}{r + w}$$

(32)

in terms of which the metric on $M$ is given by

$$ds^2 = \frac{d\tau^2}{(1 + \tau^2)^2} - \frac{d\sigma^2}{(1 + \sigma^2)^2}$$

(33)

and from eqs.(27) and (28) we have

$$e^{-U} = \frac{(1 - \sigma^2)(1 - \tau^2)}{(1 + \sigma^2)(1 + \tau^2)} = 1 - p^2 - q^2.$$  

(34)

Then the main equations become

$$-\frac{(1 + \sigma^2)}{(1 - \sigma^2)} \frac{\partial}{\partial \sigma} \left( (1 - \sigma^2) \frac{\partial \xi}{\partial \sigma} \right) + \frac{(1 + \tau^2)}{(1 - \tau^2)} \frac{\partial}{\partial \tau} \left( (1 - \tau^2) \frac{\partial \xi}{\partial \tau} \right)$$

$$= \frac{2\xi}{\xi^2 - 1} \left[ -(1 + \sigma^2)^2 \left( \frac{\partial \xi}{\partial \sigma} \right)^2 + (1 + \tau^2)^2 \left( \frac{\partial \xi}{\partial \tau} \right)^2 \right]$$

(35)

which is similar to the prolate spheroidal case but differs from it in some important respects. Its advantage lies in the fact that

$$\xi = \epsilon \sigma, \quad \epsilon^2 = \pm 1$$

(36)
is a solution of eq.(35) that leads to the Bell-Szekeres solution. Furthermore it can be readily verified that $\xi = \epsilon \tau$ is also a solution as in eq.(31) which is again the Bell-Szekeres solution. In terms of these coordinates the Bell-Szekeres solution is given by

$$ds^2 = \frac{d\tau^2}{(1 + \tau^2)^2} - \frac{d\sigma^2}{(1 + \sigma^2)^2} - \left(\frac{1 - \tau^2}{1 + \tau^2}\right)^2 dx^2 - \left(\frac{1 - \sigma^2}{1 + \sigma^2}\right)^2 dy^2$$

(37)

which may help to clarify the meaning of $\sigma$ and $\tau$.

So the remaining problem is to find a one-parameter complex solution of eq.(35) that reduces to eq.(36). Such a solution will be of physical interest as it will automatically satisfy the proper initial data.

6 Conclusion

The only proper conclusion of a paper such as this, namely the exact solution of eqs.(35) satisfying the initial data in eqs.(10) is missing. In this paper I have given a list of the essential properties that we must require from a physically acceptable solution describing the interaction of plane impulsive gravitational and electromagnetic shock waves and presented some preparatory material towards such a solution. In my case this solution has been missing since 1978 and that is why I felt it inappropriate to publish the work reported here earlier. However, I now feel that the abundance of so many irrelevant “solutions” of this problem in the literature has made the presentation of the real problem imperative.

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