Abstract

We consider the problem of fairly dividing a two dimensional heterogeneous good among multiple players. Applications include division of land as well as ad space in print and electronic media. Classical cake cutting protocols primarily consider a one-dimensional resource, or allocate each player multiple infinitesimally small “pieces”. In practice, however, the two dimensional shape of the allotted piece is of crucial importance in many applications (e.g. squares or bounded aspect-ratio rectangles are most useful for building houses, as well as advertisements). We thus introduce and study the problem of fair two-dimensional division wherein the allotted plots must be of some restricted two-dimensional geometric shape(s). Adding this geometric constraint re-opens most questions and challenges related to cake-cutting. Indeed, even the elementary proportionality fairness criteria can no longer be guaranteed in all cases. In this paper we thus examine the level of proportionality that can be guaranteed, providing both impossibility results (for proportionality that cannot be guaranteed), and algorithmic constructions (for proportionality that can be guaranteed). We focus primarily on the case when the cake is a rectilinear polygon and the allotted plots must be squares or bounded aspect-ratio rectangles.

September 17, 2014

Erel Segal-Halevi∗ Avinatan Hassidim† Yonatan Aumann‡

September 17, 2014

∗erelsgl@gmail.com
†avinatanh@gmail.com
‡yaumann@gmail.com
§Computer Science Department, Bar Ilan University, Israel
# 1 Introduction

Fair division of land has been an important issue since the dawn of human history. One of the classic fair division protocols, “I cut - you choose”, is already described in the Bible (Genesis 13) as a method for dividing land between two people. The modern study of fair division, commonly termed cake cutting, began in the 1940’s. The first challenge was conceptual - how should “fairness” be defined when the cake is heterogeneous and different people may assign different values to subsets of the cake? Steinhaus [1948] introduced the elementary and most basic fairness requirement, now termed proportionality: each of the $n$ players should get a share which he values as worth at least $1/n$ of the value of the entire cake. Steinhaus [1948] also presented an elegant algorithm for proportionally dividing a cake among an arbitrary number of players. Since then, many other desirable properties of cake partitions have been studied, including: envy-freeness [e.g. Brams and Taylor, 1995, Su, 1999], equitability [e.g. Jones, 2002], social welfare maximization [e.g. Kohler et al., 2011, Bei et al., 2012, Caragiannis et al., 2012], computational efficiency [e.g. Even and Paz, 1984, Edmonds et al., 2008, Balkanski et al., 2014] and truthfulness [e.g. Mossel and Tamuz, 2010, Chen et al., 2013, Cole et al., 2013]. See the books by Brams and Taylor [1995, 2008], Robertson and Webb [1998] and a recent survey by Procaccia [2013] for more information.

Interestingly, over decades of research on the cake-cutting problem, little to no attention has been given to the case of dividing a two-dimensional good. Indeed, most all work on cake cutting explicitly assumes that the cake and the allotted pieces are one-dimensional intervals. This is usually justified by the claim that higher dimensional settings can always be projected onto one dimension, and hence, fairness in one-dimension implies fairness in higher dimensions. However, projecting back from the one dimension, the resulting two-dimensional plots are thin rectangular slivers, of little use in most practical applications; it is hard to build a house on a $10 \times 1,000$ meter plot even though its area is a full hectare, and a thin 1-inch wide advertisement space would ill-serve most advertisers, regardless of its height. Thus, in most applications, the geometric shape of the allotted piece is of prime importance. Hence, we seek divisions in which the allotted plots must be of some restricted family of “usable” two-dimensional shapes, e.g. squares or rectangles of bounded aspect-ratio.

Adding this geometric constraint re-opens most questions and challenges related to cake-cutting. Indeed, even the elementary proportionality fairness criteria can no longer be guaranteed, as illustrated by the following example. Consider a square cake and two players, both with a uniform valuation function over the entire cake. It is easy to see that if each player must get a square plot then the most both players can get is $1/4$ of the cake (if one gets more - the other must get less). Thus, even in this simple case, proportionality cannot be attained.

In this work we initiate the study of two-dimensional fair division with geometric constraints, focusing on proportionality. Since full proportionality cannot be guaranteed, we ask what level of proportionality can be guaranteed? That is, since a $1/n$ fraction is not attainable, what (other) fraction of the value of entire cake can we guarantee that each player get? Note that a-priori it is not even clear that a $1/O(n)$ fraction can be guaranteed. For this question, we provide both impossibility results (for proportionality levels that cannot be attained), and algorithmic constructions (for the proportionality that can be guaranteed).

 Naturally, the attainable level of proportionality may depend on both the desired shape of the allotted plots and the shape of the entire cake. In this work we focus on case where the allotted plots must be square or rectangles of some bounded aspect-ratio (fat rectangles) and the original cake is square, rectangular, a rectilinear polygon, a quarter plane, a half plane or the entire unbounded plane. For each, we provide upper and lower bounds on the guaranteeable proportionality level. All the bounds are tight up to constant factors, with a multiplicative gap.

---

1Several authors studied a circular cake [Thomson, 2007, Brams et al., 2008, Barbanel et al., 2009], but it is still a one-dimensional circle and the pieces are contiguous one-dimensional arcs.
of at most 2.

1.1 Our results

We prove a multitude of upper and lower bounds, for different original cake shapes and different
prescribed plot shapes. Our main results can be summarized in the following two theorems.

**Theorem 1.** With a square original cake that must be divided among \( n \geq 2 \) players, each of
whom must get a square plot:

(a) There exists a polynomial time algorithm that allots each player a plot worth at least
\( \frac{1}{4n-4} \)
of the value of the entire cake (according to the player’s own valuation).

(b) The largest proportion that can be guaranteed to all players (simultaneously) is
\( \frac{1}{2n} \), regardless of run-time considerations.

Thus, the algorithm of (a) is tight up to a factor of 2. We note that the algorithm of (a)
does not require full knowledge of the entire valuation functions, but rather only makes use of
a polynomial number of queries to the players, where the queries are analogous to the standard
ones used in the cake-cutting literature [e.g. Even and Paz 1984, Robertson and Webb 1998].

We also note that the algorithm of (a) uses a recursive halving technique that can also be
used with cakes and plots of other shapes, such as fat triangles, trapezoids or multi-dimensional
cubes. The upper bound of (b) holds even when we allow the pieces to be fat rectangles - rect-
angles with a bounded aspect ratio. It, too, can be extended to multi-dimensional cubes.

**Theorem 2.** For an original cake in the shape of the unbounded plane, to be divided among
\( n \geq 4 \) players, each of whom must get a square piece, there exists a polynomial time algorithm
that allots each player a plot worth at least \( \frac{1}{2n-4} \) of the value of the entire cake (according to the
player’s own valuation).

The trivial upper bounds is \( \frac{1}{n} \), so this algorithm too is tight up to a factor of 2.

The algorithm for dividing an unbounded plane is based on an algorithm for dividing a
quarter-plane, which is in turn based on an algorithm for dividing a rectilinear polygon with two
sides unbounded. This latter algorithm is interesting in its own right. It guarantees each player
a value of at least \( \frac{1}{2n-2+\text{CoverNum}} \) where \( \text{CoverNum} \) is the smallest number of squares required
to cover the polygon. This guarantee is tight.

**Techniques.** Introducing the geometric constraints not only re-opens all the known cake-
cutting results, but also deems most of its techniques unusable. Thus, we must develop new
proof and algorithmic techniques, aimed at this new problem. For the upper bounds, we develop
a new generic proof technique, which employs a recursive structure of “water pools in the desert”
for the construction of hard instances. For the algorithmic constructions, we develop two new
algorithmic techniques. The one, employed for the bounded cake shapes, uses recursive guillotine
cuts - axis-parallel cuts running from one end of a piece to another. The cuts are determined
by queries which resemble the mark and eval queries in the classic Robertson-Webb model
[Robertson and Webb 1998]. The other technique, employed for the unbounded cake shapes,
uses a process by which players are allowed to draw minimal squares in specified locations within
the cake and a certain square is selected and “carved” from the cake using non-guillotine cuts.

1.2 Related research

Both algorithmic techniques make heavy use of notions from computational geometry, such as
minimum polygon covering and maximum independent set. They are also related to problems
such as optimal square packing. Relevant references are given in Section 4.

We found only few cake-cutting papers which explicitly consider a two-dimensional cake. Two of them discuss the problem of dividing a disputed territory between several bordering countries, with the constraint that each country should get a piece adjacent to its current border. [Hill 1983] proved that such a partition exists and [Beck 1987] complemented this proof with a division procedure. [Iyer and Huhns 2009] describe a procedure that asks each of the n players to draw n disjoint rectangles on the map of the two-dimensional cake. These rectangles are supposed to represent the “desired areas” of the player. The procedure tries to give each player one of his n desired areas. However, it does not succeed unless each rectangle proposed by an individual intersects at most one other rectangle drawn by any other player. If even a single rectangle of Alice intersects two rectangles of George (for example), then the procedure fails and no player gets any piece.

1.3 Paper structure

We proceed by formally describing our model in section 2. Then we present our negative results (impossibility proofs) in section 3 and our positive results (division algorithms) in section 4. In both sections, we present both generic lemmas that are relevant to every cake and every family of usable shapes, and instantiations of these lemmas for cakes that are rectilinear polygons and pieces that are squares or fat rectangles.

2 The Model

There is a cake \( C \) which is a measurable subset of the 2-dimensional plane. The cake can be infinite. \( C \) has to be divided among a group of players \( N = \{1, ..., n\} \), giving every player \( i \in N \) a measurable piece \( P_i \subseteq C \) such that the pieces are pairwise disjoint. Some parts of the cake may remain unallocated (i.e. free disposal is allowed).

There is a family \( S \) of geometric shapes which are considered the “usable shapes”. Every allocated piece must be usable, i.e. \( \forall i: P_i \in S \).

Every player \( i \in N \) has a subjective value measure \( V_i \), assigning to every measurable subset of \( C \) a non-negative finite number. In the context of land division, the value measure represents the subjective quality of the land and may depend upon factors such as the amount of fertile soil it contains, the probability of finding oil in it, etc. We require \( V_i(C) \) to be finite even when \( C \) is infinite. The value measures \( V_i \) are additive, i.e. the value of a whole is equal to the sum of the values of its parts. We also assume that the measures are absolutely continuous with respect to area (or just continuous for short), i.e., the value of a piece with an area of 0 is 0; hence there is no difference in value between open and closed shapes with the same interior. Continuity implies that there are no singular points with positive value - a property usually termed non-atomicity. [Hill and Morrison, 2010]

The fairness of a given allocation \( \{P_1, ..., P_n\} \) is evaluated in terms of its proportionality level, defined as the proportion of the cake value enjoyed by the least fortunate player:

\[
\text{Prop}(\{P_1, ..., P_n\}, \{V_1, ..., V_n\}) = \min_{i \in N} \frac{V_i(P_i)}{V_i(C)}
\]

An allocation with proportionality \( \frac{1}{n} \) is usually called a proportional allocation. It is well-known that without geometric constraints a proportional allocation exists for every combination of continuous value measures. However, this is not necessarily true when the plots are required to have a certain geometric shape. Hence, our research question is: given a cake \( C \) and a family \( S \), what is the largest proportionality level that can be attained for every combination of continuous value measures? We call this number the proportionality level of \( C \) relative to \( S \):
Proposition (C, n, S) = inf sup \( Prop(\{P_1, ..., P_n\}, \{V_1, ..., V_n\}) \)

where the infimum is taken over all \( n \)-tuples of continuous value measures \( \{V_1, ..., V_n\} \) and the supremum is taken over all allocations \( \{P_1, ..., P_n\} \) of \( S \)-shaped pieces of \( C \) to \( n \) players with these value measures.

The classic algorithms for proportional cake-cutting [e.g. Steinhaus, 1948] prove that \( Prop(\text{Any shape}, n, \text{all shapes}) = \frac{1}{n} \). One-dimensional algorithms with contiguous pieces [e.g. Su, 1999] prove that \( Prop(\text{Interval}, n, \text{intervals}) = \frac{1}{n} \) and when translated to two dimensions they yield: \( Prop(\text{Rectangle}, n, \text{rectangles}) = \frac{1}{n} \). However, none of these algorithms considers constraints that are two-dimensional in nature such as square pieces. Such two-dimensional constraints are the focus of our research.

Our challenge in the rest of this paper will be to calculate \( Prop(C, n, S) \) for various cakes and families of shapes. The calculation is done in two steps: (a) Proving an upper bound \( Prop(C, n, S) \leq f \). This is done by showing a single combination of \( n \) value measures on \( C \), for which no allocation of shapes from \( S \) can have proportionality above \( f \). (b) Proving a lower bound \( Prop(C, n, S) \geq f \). This is done by describing an algorithm which finds, for every combination of \( n \) value measures on \( C \), an allocation of shapes from \( S \) with proportionality at least \( f \). These two steps are handled in sections 3 and 4 respectively.

Most of our results consider square pieces. While a square is a convenient and usable shape, a rectangular land-plot can also be useful if it is not too thin. The following definition, adapted from the computational geometry literature [e.g. Agarwal et al., 1995, Katz, 1997], quantifies this intuition:

**Definition 2.1.** For \( R \geq 1 \), a \( d \)-dimensional object is called \( R \)-fat if it contains a \( d \)-cube with side-length \( x \) and is contained in a \( d \)-cube with parallel faces and side-length \( R \cdot x \), for some \( x > 0 \). In particular, an \( R \)-fat rectangle is a rectangle having length/width ratio at most \( R \).

So a square is 1-fat and also 2-fat; a 10-by-20 rectangle is 2-fat but not 1-fat; a right-angled isosceles triangle is 2-fat; a circle is \( \sqrt{2} \)-fat; etc.

### 3 Impossibility Results

#### 3.1 One player

In this section we prove upper bounds relevant to Theorems 1 and 2. We start our discussion with some simple examples that provide intuition on proving impossibility results.

First, consider Robinson Crusoe, a single player on a lonely island. Without geometric constraints he can of course have the entire island so his proportion is 1, but with geometric constraints he might have to settle for less. For example, if the island is 10-by-20 and the usable shapes are squares, then for some value measures (e.g. the uniform measure) only half

2Note that this is different than the egalitarian value, which is the largest proportionality level for specific value measures, i.e. only the supremum without the infimum. Finding the egalitarian value for specific value measures is a difficult problem even without geometric constraints [see e.g. Dall’Aglio and Di Luca, 2012, Aumann et al, 2013].
the value can be used: $\text{Prop}(10 \times 20 \text{ rectangle}, n = 1, \text{Squares}) \leq \frac{1}{2}$. If the island is 10-by-15, a similar upper bound can be proved by considering the island in Figure 1/Left. This is a desert with two equal-sized water pools (the small blue squares). The water pools are far apart so no square in $C$ intersects more than one pool. If Robinson’s value measure is proportional to the amount of water then he cannot have more than half the total value; hence $\text{Prop}(10 \times 15 \text{ rectangle}, n = 1, \text{Squares}) \leq \frac{1}{2}$. Note that the pools in the figure are squares, but we would get the same result if the pools were circles or any other shape, as long as it is sufficiently small.

This construction can be generalized for arbitrary rectangular islands, proving that $\text{Prop}(L \times 1 \text{ rectangle}, n = 1, \text{Squares}) \leq \frac{1}{\lceil L \rceil}$. A generalization to arbitrary shapes is given by the following definition and lemma:

**Definition 3.1.** Let $C$ be a shape and $S$ a family of shapes. (a) An $S$-independent set in $C$ is a set of squares in $C$ no two of which are covered by a single shape from $S$ contained in $C$. (b) The $S$-independence number of $C$, $\text{IndepNum}(C, S)$, is the maximum cardinality of an $S$-independent set in $C$.

The independence number is a geometric property of the shape of $C$ and $S$. For example, for $L \geq 1$, $\text{IndepNum}(L \times 1 \text{ rectangle}, \text{Squares}) = \lceil L \rceil$ because $\lceil L \rceil$ sufficiently small squares can be arranged in equal intervals along a long side of the rectangle such that no two of them can be covered by one square. Some more examples are illustrated in Figure 2/Left. This geometric property is related to proportionality by the following lemma, which is easily proved using the water-pools scheme we used in previous example:

**Lemma 3.1.** For every shape $C$ and family $S$: $\text{Prop}(C, n = 1, S) \leq \frac{1}{\text{IndepNum}(C, S)}$

Hence, for example, in an L-shaped island, Robinson Crusoe may have to settle for a utility of only $\frac{1}{3}$ (if he wants squares) or $\frac{1}{2}$ (if he wants rectangles).

### 3.2 Many players

Now consider what happens when Friday joins the island. Now, even if the island is square, a proportional division with square plots may be impossible. In the introduction we already discussed an example which proves that $\text{Prop}(\text{Square}, n = 2, \text{Squares}) \leq \frac{1}{4}$. When the island is 10-by-20, a similar bound can be proved by considering a desert with pools as in Figure 1/Middle. There are 4 equal-sized pools. At most one square intersects two or more pools: a square intersecting the pool at the far east cannot intersect any other pool, and a square intersecting two or more pools at the west does not leave room for any other square intersecting two pools. Hence, at least one of the partners will have at most a single pool and a proportion of $\frac{1}{4}$, so $\text{Prop}(10 \times 20 \text{ rectangle}, n = 2, \text{Squares}) \leq \frac{1}{4}$.

In this paper we consider not only finite cakes but also infinite ones, such as a quarter-plane or even an unbounded plane. When dividing an infinite cake we allow the pieces to be infinite too. For example, when the usable shapes are squares we treat a quarter-plane as a square (with infinite side-length). Even with infinite pieces, a proportional division is not always guaranteed. Consider for example the quarter-plane in Figure 1/Middle, having 3 pools. Assume the horizontal and vertical distance between the pools is 1 and the diameter of each pool is $\epsilon$. Then every square intersecting at least two pools must have side-length at least $1 - 2\epsilon$. It is clear that there is room for at most one such square. Hence $\text{Prop}(\text{Quarter plane}, n = 2, \text{Squares}) \leq \frac{1}{4}$. Note that the bound is valid both for axis-parallel squares and for rotated squares, although the geometric calculations required for proving the latter case are more laborious.

In order to prove impossibility results for an arbitrary number of players, we use the following scheme, which is a generalization of the pools scheme used in the previous examples. Given a cake $C$ and a family of usable shapes $S$, we find an arrangement of $m$ pools contained in $C$. By
“pool” we mean a small circle or square or any other shape having positive area. If at most \( n - 1 \) disjoint shapes from \( S \) intersect two or more pools, then \( Prop(C, n, S) \leq \frac{1}{m} \), because at least one of \( n \) players will have to live with at most a single pool. Note that this scheme requires only a single piecewise-uniform value measure. Hence all impossibility results proved by this scheme are valid even for \( n \) players having an identical valuation function which is piecewise-uniform.

As an example of using this scheme, consider Figure 1/Right, having 5 pools where at most 2 disjoint squares intersect two pools. This shows that \( Prop(Quarter\ plane, n = 3, Squares) \leq \frac{1}{4} \). This construction can be generalized recursively and we get the following upper bound, which is proved in Appendix A:

**Claim 3.1.** For every \( n \geq 1 \), \( Prop(Quarter\ plane, n, Squares) \leq \frac{1}{2n-1} \)

This upper bound is also trivially true when the cake is a (bounded) square. But for a square we can add a pool at the north-eastern corner and get a slightly tighter bound:

**Claim 3.2.** For every \( n \geq 2 \), \( Prop(Square, n, Squares) \leq \frac{1}{2n} \)

Lemma 3.1 shows the dependency of proportionality on the geometry of the cake. Claims 3.1 and 3.2 show the dependency of proportionality on the number of players. The next claim combines the two factors. It is about a rectilinear polygon - a polygon in which all angles are multiples of 90°. Proof details are given in Appendix A. Figure 2/Right illustrates the construction for \( k = 3 \) and \( n = 3 \), having 7 pools where at most 2 disjoint squares intersect two pools.

**Claim 3.3.** Let \( C \) be a rectilinear polygon and let \( k = \text{IndepNum}(C, Squares) \). Then for every \( n \geq 1 \): \( Prop(C, n, Squares) \leq \frac{1}{2n-2+k} \)

So far we focused on square pieces. Intuitively we could think that allowing fat rectangles (see Definition 2.1) instead of just squares should considerably increase the attainable proportionality level. Surprisingly, this is not the case:

**Claim 3.4.** \( \forall n \geq 1, R \geq 1, Prop(Quarter\ plane, n, R\ fat\ rectangles) \leq \frac{1}{2n-1} \)

The proof is essentially the same as Claim 3.1 - we just have to use sufficiently small pools. This upper bound obviously also holds for bounded squares, though the slightly tighter bound of Claim 3.2 holds only for \( R < 2 \).

Since an \( \infty \)-fat rectangle is just an arbitrary rectangle, we have from classic cake-cutting: \( Prop(Square, n, \infty\ fat\ rectangles) = \frac{1}{n} \). Contrasting this with Claim 3.3 we see a surprising discontinuity of the \( Prop \) function at \( R = \infty \). Thus, if the players agree to use any rectangular plot, they can receive their proportional share of \( \frac{1}{n} \), but if they insist on using \( R \)-fat rectangles, even when \( R \) is very large, they might have to settle for about half of this share.

Claim 3.3 generalizes to \( R \)-fat rectangles only when the rectilinear polygon \( C \) is simply-connected (hole-free). The proof relies on a geometric lemma which guarantees the existence of an independent set with a corner square [Gill et al., 2013]:
Claim 3.5. Let \( C \) be a hole-free rectilinear polygon and let \( k = \text{IndepNum}(C, R \text{ fat rectangles}) \).

Then for every \( n \geq 1 \) and \( R < \infty \): \( \text{Prop}(C, n, R \text{ fat rectangles}) \leq \frac{1}{2n-2+2^k} \).

In particular: \( \text{Prop}(L \times 1 \text{ rectangle}, n, R \text{ fat rectangles}) \leq \frac{1}{2n-2+1} \).

For the case \( R = \infty \): \( \text{Prop}(C, n, \text{ rectangles}) \leq \frac{1}{n-1+2^k} \).

Regarding a half-plane, the following claim can be proved very similarly to Claim \ref{claim:halfplane}.

Claim 3.6. \( \forall n \geq 1, R \geq 1 \), \( \text{Prop(Half plane, n, R \text{ fat rectangles})} \leq \frac{1}{2n}|\bar{V}(C)|-1 \).

We mention without proof the following generalization of Claims \ref{claim:halfplane} and \ref{claim:squares} to multi-dimensional cubes. Let \( C \) be a \( d \)-cube with \( \delta \leq d \) unbounded sides, all in different dimensions (i.e. at most one side is unbounded in each dimension). Then:

Claim 3.7. \( \forall n \geq 2 \): \( \text{Prop}(d \text{ cube with } \delta \text{ unbounded sides, } d \text{ cubes}) \leq \frac{1}{d(n-2)+(2^\delta-\delta)} \).

4 Algorithms

4.1 One player

We begin the algorithmic part of this paper in Robinson Crusoe’s lonely island. Robinson can achieve a valuable land-plot using the following definition and lemma, which are duals of Definition \ref{def:covering} and Lemma \ref{lem:covering}.

Definition 4.1. Let \( C \) be a shape and \( S \) a family of shapes. (a) An \( S \)-cover of \( C \) is a set of shapes in \( S \) whose union equals \( C \). (b) The \( S \)-cover number of \( C \), \( \text{CoverNum}(C, S) \), is the minimum cardinality of an \( S \)-cover of \( C \).

Lemma 4.1. \((\text{Covering Lemma})\) For every shape \( C \) and family \( S \): \( \text{Prop}(C, n = 1, S) \geq \frac{1}{\text{CoverNum}(C, S)} \).

Proof. Let \( k = \text{CoverNum}(C, S) \) and \( \{C_1, \ldots, C_k\} \) be an \( S \)-cover of \( C \). By definition \( \bigcup_{i=1}^k C_i = C \). By additivity \( \sum_{i=1}^k V(C_i) \geq V(C) \). Hence for at least one \( i \): \( V(C_i) \geq \frac{V(C)}{k} \). By selecting this \( C_i \), which is a member of \( S \), Robinson can enjoy a proportion of at least \( \frac{1}{k} \).

In general, the computational problem of finding a minimum covering is NP-hard.\footnote{The Wikipedia page https://en.wikipedia.org/wiki/Polygon_covering provides an up-to-date survey.} There are two important cases in which it is known to be polynomial: (a) When \( C \) is a hole-free rectilinear polygon and \( S \) the family of squares \cite{BarYehuda1990}; (b) When \( C \) is an orthogonally-convex rectilinear polygon and \( S \) the family of rectangles \cite{Franzblau1984}.

Comparing the Covering Lemma to Lemma \ref{lem:covering} raises the question of when the two bounds for a single player coincide, i.e. when does \( \text{CoverNum}(C, S) = \text{IndepNum}(C, S) \)? Two results from computational geometry provide two important cases in which they are equal:\footnote{We are grateful to Joseph O’Rourke for kindly providing these references. Note that in the references, an \( S \)-independent set when \( S \) is the family of squares is termed “anti-square” and similarly an \( S \)-independent set when \( S \) is the family of rectangles is termed “anti-rectangle”.

\text{Prop}(C, n = 1, S) = \frac{1}{\text{IndepNum}(C, S)} = \frac{1}{\text{CoverNum}(C, S)} \). Note that the latter two papers also provide concrete examples of rectilinear polygons for which \( \text{CoverNum}(C, S) > \text{IndepNum}(C, S) \). We do not know the exact value of \( \text{Prop}(C, n = 1, S) \) in these cases. Indeed, the problem of cake-cutting with geometric constraints provides interesting open questions even when there is only a single player!
4.2 Many players

Following the common practice in cake-cutting since Steinhaus [1948], our protocols for $n > 1$ players provide worst-case guarantees, i.e., each player has a strategy that guarantees a piece having a certain value, regardless of what the other players do. When describing such a strategy, we refer to it as the safe strategy of the player.

Consider first a case of two players on a square island. In section 3 we saw an upper bound of $\frac{1}{4}$ on the proportionality in this case. The problem of how to attain this proportionality is a nice puzzle which can be solved by a very simple protocol: ask each player to draw a square corner, give the smallest square to its owner and the rest of the cake to the other player.

Having described the protocol, we now describe a safe strategy: draw the smallest corner square having a value of $\frac{1}{4}$. To find this square, partition the cake to a $2 \times 2$ grid. By additivity at least one of the quarters must have value at least $\frac{1}{4}$. Shrink it towards its corner until the shrunk square has value exactly $\frac{1}{4}$ (see Figure 3). Now, either you receive this square, or the other player receives a smaller square and you receive the remaining L-shape which has value at least $\frac{3}{4}$. But the cover number of the L-shape is 3, hence by the Covering Lemma you can get from it a utility of at least $\frac{1}{4}$. The protocol and the safe strategy together prove that: \[ \text{Prop}(\text{Square}, n = 2, \text{Squares}) = \frac{1}{4}. \]

Our algorithms use two types of queries, which are analogous to the “mark” and “eval” queries used by classic cake-cutting algorithms [e.g. Robertson and Webb, 1998, Woeginger and Sgall, 2007].

In a mark query, each player should mark a piece he bids to receive from the cake. The protocol rules restrict the marks such that each bid must either contain or be contained in each other bid. Now the protocol selects a smallest bid (which is contained in all other bids), gives it to its bidder and divides the remainder of the cake recursively among the remaining $n - 1$ players.

As an example, if $C$ is a rectangle and $S$ the family of rectangles, a possible mark query is: “bid a rectangle whose western side coincides with the western side of $C$”. A safe strategy for a player using this strategy is to bid a rectangle $r$ with $V(r) = \frac{V(C)}{n}$. This is always possible thanks to the continuity of $V(C)$. A player using this strategy is guaranteed a rectangle with value at least $\frac{V(C)}{n}$, thus re-proving a result already known from 1-dimensional cake-cutting:

\[ \text{Prop}(\text{Rectangle}, \text{Rectangles}) = \frac{1}{n}. \]

In an eval query, the protocol partitions $C$ to $m \geq 2$ disjoint sub-cakes $C_j$, having $\bigcup_{j=1}^{m} C_j = C$. Each player $i$ should evaluate every sub-cake $C_j$ by providing a non-negative integer $P_i^j$. $P_i^j$ stands for “partners” and represents the number of players with which $i$ is willing to share $C_j$ (including player $i$ himself). The protocol rules obligate each player to make sure the sum of his partner numbers is at least $n$, i.e.: $\forall i: \Sigma_{j=1}^{m} P_i^j \geq n$. The protocol then partitions the players to $m$ disjoint groups $G_j$ in a way that respects the players’ preferences: $\forall i, j: i \in G_j \Rightarrow P_i^j \geq |G_j|$. I.e., every player in group $j$ has indicated a willingness to share the sub-cake $C_j$ with the number of partners in the group. This partition can be carried out by Algorithm 1 described in Appendix B. Finally the protocol recursively divides every sub-cake $C_j$ to the players in group $G_j$.

As an example, if $C$ is an archipelago made of disjoint $m$ rectangular islands and $S$ is the family of rectangles, a possible eval query is: “evaluate each of the $m$ islands”. A safe strategy for
a player with value measure $V$ is to let $P_j = \left\lfloor \frac{V(C_j)}{V(C)} \right\rfloor \cdot (n + m - 1)$. By the properties of the floor function, $P_j \geq \frac{V(C_j)}{V(C)} \cdot (n + m - 1)$ and the sum satisfies: $\sum_{j=1}^{m} P_j \geq (n + m - 1) \sum_{j=1}^{m} \frac{V(C_j)}{V(C)} - m = (n + m - 1) \cdot 1 - m = n - 1$. But the sum is an integer so it must be at least $n$, in compliance with the protocol rules. Again by the properties of the floor function, $P_j \leq \frac{V(C_j)}{V(C)} \cdot (n + m - 1)$, so a player using this strategy is guaranteed to share a rectangle valued $V(C_j)$ with at most $\frac{V(C_j)}{V(C)} \cdot (n + m - 1)$ partners and receive a rectangle valued $\frac{V(C)}{(n + m - 1)}$. This proves the following new result: $\text{Prop}(m \text{ disjoint rectangles}, \text{Rectangles}) \geq \frac{1}{n + m - 1}$.

The algorithms we describe in the following subsections use sophisticated combinations of mark and eval queries to handle more complicated scenarios.

### 4.3 Guillotine algorithms

In this subsection we prove the lower bounds of Theorem 1. To give some intuition, we start with a looser bound which is simpler to prove.

**Claim 4.1.** For every $n \geq 2$, $\text{Prop}(\text{Square}, n, \text{Squares}) \geq \frac{1}{6n-8}$.

**Proof.** (Sketch) A square cake $C$ can be divided using the following sequence of two queries:

1. **Eval query**: $C$ is partitioned to 4 sub-squares in a $2 \times 2$ grid. Each player provides 4 integers indicating the number of partners with which he is willing to share each of the 4 quarters; the numbers must sum up to at least $n$. The players are partitioned to 4 groups using Algorithm 1. If not all players are in the same group then we can just divide each sub-square recursively to its players. If all $n$ players are in the same group, corresponding to a single quarter $C_j$, we proceed to the next query:

2. **Mark query**: each player should bid an L-shape using an L-shaped mark inside the quarter $C_j$. For example, if $C_j$ is the south-western quarter of $C$ then each player should draw a square cornered at the south-western corner of $C$ and bid the L-shape outside that square. The protocol picks a smallest L-shape $C_L$ and allows its bidder to select a single square contained in $C_L$. The protocol then uses a guillotine cut to separate this square from the $C \setminus C_L$ (note that $C \setminus C_L$ is a corner square contained in the quarter $C_j$). The winning bidder receives his selected square and $C \setminus C_L$ is divided recursively among the remaining $n-1$ players. A flow-chart of the protocol is illustrated in Figure 4. Appendix D explains the safe strategy and proves that this strategy guarantees a value of at least $\frac{V(C)}{6n-8}$. \hfill $\square$

We call the coefficient of $n$ in the denominator of $\text{Prop}$, the *proportionality coefficient*. The proportionality coefficient in the impossibility results of section 3 is 2 while the proportionality coefficient in the algorithm of Claim 4.1 is 6; this is a large proportionality gap which we want to close. One way to narrow the proportionality gap is to allow the pieces to be fat rectangles (Definition 2.1). The improved bound is valid not only for square cakes but also for cakes that are 2-fat rectangles. The algorithm uses a sequence of four queries: an eval query, a mark query, another eval query and another eval query. The details are given in Appendix E and illustrated by a flow chart in Figure 6. We get:

**Claim 4.2.** For every $n \geq 2$, $\text{Prop}(2 \text{ fat rectangle}, n, 2 \text{ fat rectangles}) \geq \frac{1}{4n-5}$.

With some more effort, the proportionality coefficient can be reduced to 4 even for square pieces. To this end we use a pair of algorithms that call each other recursively: an algorithm for dividing a 2-fat rectangle with all sides bounded and an algorithm for dividing a rectangle with one of its longer sides unbounded (the meaning of an unbounded side is that some of the pieces may flow over the border). The algorithm details are given in Appendix F and illustrated in Figures 7 and 8. We get a pair of results:

5It is easy to show that this bound is tight by using our water pools scheme from Section 6.
Figure 4: Left: Staircase with \( k = 4 \) corners (numbered 1,...,4), and its 4 covering squares. 

Middle: Corner squares (solid) with a diagonal (dashed) representing \( a_t \) (the taxicab distance from the origin). The blue square at corner 2 is the winning square as its taxicab distance is minimal. 

Right: Cutting the blue square and its shadow. The remainder has \( k' = 3 \) corners.

Claim 4.3. \( \forall n \geq 2 \): 
(a) \( \text{Prop}(2 \text{ fat rectangle with all sides bounded, } n, \text{ squares}) \geq \frac{1}{4n-4} \) 
(b) \( \text{Prop}(\text{Rectangle with a long side unbounded, } n, \text{ squares}) \geq \frac{1}{4n-5} \)

Theorem 1 is proved by combining Claims 3.2 and 3.3 (a).

We briefly mention without proof two generalizations of Theorem 1. First, if \( C \) is a rectangle with aspect ratio \( L \) and the pieces are fat rectangles then:

Claim 4.4. \( \text{Prop}(L \times 1 \text{ rectangle, } n, \text{ R fat rectangles}) \geq \frac{1}{4n-6+\lceil \max(2L, R) \rceil} \)

Second, let \( C \) be a \( d \)-dimensional cube with \( \delta \leq d \) unbounded sides, all in different dimensions (i.e. at most one side is unbounded in each dimension). Then:

Claim 4.5. \( \forall n \geq 2 \): \( \text{Prop}(d \text{ cube with } \delta \text{ unbounded sides, } d \text{ cubes}) \geq \frac{1}{(2d^2+1+2)(n-2)+2d^2+\delta} \)

Currently this is the best we know to do with guillotine cuts. In the following section we introduce a new algorithmic technique with which we can both narrow the proportionality gap and handle more diverse cakes.

4.4 Staircase algorithms

In this section we prove Theorem 2. We start by an intuitive presentation of the ideas behind the algorithm which we use to fairly divide the plane. It is easier to think about the algorithm as if it’s trying to divide the north-eastern quarter-plane, i.e. the cake is like a square with two sides unbounded. Recall from Claim 3.1 that the upper bound for this case is \( \frac{1}{2n-1} \). We want to begin by giving someone a square starting at the south western corner, which is worth \( \frac{1}{2n-1} \) to her. Ideally, we would like to give that square to a player who would take the smallest such square, and recurse. However, when we try to do this we run into trouble, as the cake is no longer square - instead it is an L-shape. Worse, if (say) we carve a second square next to the first which is larger than the first square, the cake no longer resembles an L, and becomes a more complicated shape.

To be able to run recursively, the algorithm accepts as input a rectilinear polygonal domain with two unbounded sides, which for brevity we call “staircase” because of its shape (see Figure 4/Left). A staircase can be characterized by the number of inner corners which we denote by \( k \). Clearly \( \text{CoverNum}(C, Squares) = \text{IndepNum}(C, Squares) = k \), hence by Claim 3.3 the upper bound on proportionality is \( \frac{1}{2n-2+k} \). Fortunately, this upper bound can be attained by our staircase algorithm, whose details are given in Appendix G:

\footnote{Algorithms for partially-unbounded squares has been recently studied by computational geometers with relation to the task of online square packing. For example, [Pekete and Hoffmann, 2013] studies online square packing with zero or two unbounded sides while [Pekete et al, 2014] studies online square packing with one unbounded side.}
Claim 4.6. If $C$ is a staircase-shaped polygonal domain having $k$ corners, then $\text{Prop}(C, n, \text{Squares}) = \frac{1}{2n^2 + k}$. In particular, $\text{Prop}(\text{Quarter plane}, n, \text{Squares}) = \frac{1}{2n - 1}$.

The algorithm uses only mark queries: it lets each player draw squares in one or more corners of the staircase and selects the “best” square. Surprisingly, it turns out that the best square to choose is not the smallest one but the one with the smallest taxicab distance from its center to the origin (see Figure 4/Middle). After a square is allocated, the “shadow” of the square should also be discarded in order to keep the cake in a staircase shape (see Figure 4/Right). It is possible to prove that, if the square with the smallest taxicab distance is removed together with its shadow, then the value of the remaining cake to the remaining $n - 1$ agents is sufficiently large such that they can continue the division recursively. See Appendix [G] for more details.

To divide a half-plane, we can partition it to two quarter-planes:

Claim 4.7. $\text{Prop}(\text{Half plane}, n, \text{Squares}) \geq \frac{1}{2n - 2}$

Proof. Start with a single mark query. Assume the cake is the half-plane $y \geq 0$ and there are $n$ players who value it as $2n - 2$. Ask each player to draw a vertical line such that the value on the west of the line is exactly 1. Cut the cake at the most western line and give the entire quarter-plane to the west of the line to the player who drew that line. The remaining $n - 1$ players value the remaining cake, which has two unbounded sides, as at least $2n - 3 = 2(n - 1) - 1$. Divide it among them using the staircase algorithm of Claim 4.6.

Similarly we can divide an unbounded plane by partitioning it to two half-planes. This gives our Theorem 2: $\text{Prop}(\text{Plane}, n, \text{Squares}) \geq \frac{1}{2n - 4}$

Comparing our lower bounds to the upper bounds of Section 3 reveals several gaps waiting to be closed. First, we currently don’t know how to generalize the staircase technique to an arbitrary rectilinear polygon or to a square with all sides bounded, so we don’t know if the proportionality coefficient for is 2 (as in the impossibility result) or 4 (as in the algorithm of subsection 4.3). Second, for an unbounded plane we currently only have the trivial upper bound $\frac{1}{n}$, so we don’t know if the proportionality coefficient is 1 or 2 (as in Theorem 2). Additionally, we don’t know how to generalize the staircase technique to three or more dimensions, so we have a gap with the upper bounds of Section 3.

An open question which we find most intriguing is: does there always exist a proportional division when the cake is unbounded and the pieces are convex (not necessarily rectangular) fat shapes? We conjecture that the answer is negative and that the proportionality coefficient when the pieces are $d$-dimensional fat shapes is $d$.

5 Conclusion and Future Work

We defined a problem of fair cake-cutting with geometric constraints, inspired by the task of fairly dividing multi-dimensional land resources. We presented several proof techniques and algorithmic methods for achieving a partially-proportional division in various scenarios. Our main proportionality results are summarized in the appendices in Table [1]. Some of our division algorithms were implemented. An on-line demo is available in which the user can generate random value measures specified by interest points and verify that the resulting division indeed satisfies the partial-proportionality guarantees.

The new cake-cutting problem introduced in this paper has a large potential for future research. First, still in the realm of proportionality, it is possible to extend the problem definition by allowing each player to have a different geometric constraint (a different family $S$ of usable shapes). Additionally, instead of defining partial proportionality relative to the total value of the entire cake, it is possible to define it relative to the total usable value that a single player

\footnote{http://162.243.116.217/geometry/fair-division.html}
can derive when given the entire cake. This could potentially lead to different algorithms and techniques.

In addition to proportionality, every problem that has been studied with relation to the classic cake-cutting problem can be studied again with the additional geometric constraints. This includes, in particular: envy-freeness, social welfare maximization, computational efficiency and truthfulness.

6 Acknowledgments

Erel Segal-Halevi author is grateful to the Doctoral Fellowships of Excellence Program and to the ISF grant 1083/13 for their financial support and Prof. Shmuel Nitzan from the Bar Ilan Economics department for helpful comments and discussions.

A.H. is supported by ISF grant 1241/12 and by BSF grant 2012344.

Many members of the math.stackexchange.com and mathoverflow.com communities kindly helped in coping with specific geometry-related issues arising in this paper. In particular, we are grateful to: Christian Blatter, Ilya Bogdanov, Boris Bukh, Anthony Carapetis, Christopher Culter, David Eppstein, Peter Franek, Nick Gill, John Gowers, Dafin Guzman, Marcus Hum, Robert Israel, Barry Johnson, V. Kurchatkin, Raymond Manzoni, Ross Millikan, Mariusz Nowak, Boris Novikov, Joseph O’Rourke, Emanuele Paolini, Steven Taschuk, Marc van Leeuwen, Hagen von Eitzen, Martin von Gagern, Jared Warner, Ittay Weiss and Phoemue X.

A Proofs of impossibility results

Claim 3.1: For every \( n \geq 1 \), Prop(Quarter plane, n, Squares) \( \leq \frac{1}{2n-1} \).

Proof. Assume the cake is the north-eastern quadrant \([0, \infty) \times [0, \infty]\). It is sufficient to prove that for every \( n \) there is an arrangement of \( 2n - 1 \) square pools in that quadrant such that at most \( n-1 \) disjoint squares intersect two or more pools. The proof is by induction on \( n \). The case \( n = 1 \) is trivial and the case \( n = 2 \) was already proved (see Figure 1/Middle). Assume there is such an arrangement with \( 2(n-1) - 1 \) square pools. Shrink the entire arrangement towards the origin until it is contained in the square \([0, \epsilon] \times [0, \epsilon]\), where \( \epsilon > 0 \) is a sufficiently small constant. Add two new square pools with side-length \( \epsilon \) cornered at \((1,0)\) and \((0,1)\). We now have an arrangement of \( 2n - 1 \) pools. By the induction assumption, there are at most \( n-2 \) disjoint squares intersecting two pools in the shrunk square. Each square intersecting one of the new pools and another pool (either new or old) must have side-length at least \( 1 - 2\epsilon \); at most one such square fits into \( C \). The total number of squares is thus at most \( n-1 \). Figure 1/Right illustrates the construction for \( n = 3 \), having 5 pools where at most 2 disjoint squares intersect two pools.

Claim 3.3: Let \( C \) be a rectilinear polygon and let \( k = \text{IndepNum}(C, \text{Squares}) \). Then for every \( n \geq 1 \): Prop(\( C \), \( n \), \( \text{Squares} \)) \( \leq \frac{1}{2n-2+k} \).

Proof. Let \( I \) be an independent set of \( k \) square pools in \( C \). Let \( c \) be a certain convex corner of \( C \) (a corner with internal angle 90°). Let \( s \) be a maximal square having \( c \) as one of its corners. By definition of independent set, \( s \) covers at most one square pool from \( I \). Remove that pool from \( I \). Now there is a small square subset of \( s \) adjacent to the corner \( c \), that is disjoint from any square containing a pool from \( I \). Put inside that area a set \( J \) of \( 2n - 1 \) pools constructed as in Claim 3.1. No square in \( C \) intersects two pools from \( I \), or a pool from \( I \) and a pool from \( J \). At most \( n-1 \) squares in \( C \) intersect two pools from \( J \). Hence we have now \((2n-1) + (k-1) = 2n-2+k\) pools with at most \( n-1 \) squares intersecting two or more of them. Figure 2/Right illustrates the construction for \( k = 3 \) and \( n = 3 \), having 7 pools where at most 2 disjoint squares intersect two pools.
Algorithm 1 Partitioning Players to Rooms. A generalization of [Culter 2013].

INPUT: A group of players $N = \{1, ..., n\}$ and a set of parts $M = \{1, ..., m\}$.

for every $i \in N$, a set of $m$ non-negative integers $P_{ij}$ (with $j \in M$) such that $\sum_{j \in M} P_{ij} \geq n$.

OUTPUT: A partitioning of $N$ to $m$ disjoint groups $G_j$ (some possibly empty) such that for every $i \in N, j \in M$: $i \in G_j \implies P_{ij} \geq |G_j|$.

ALGORITHM:
- Initialize $G_m := \emptyset$.
- Order the players in decreasing order of $P_{im}$.
- While $P_{im} > |G_m|$: let $G_m := G_m \cup \{i\}$.
- Recursively call the algorithm with the players in $N \setminus G_m$ and the parts in $M \setminus \{m\}$.

B Partitioning Players to Rooms

Suppose there are $n$ players and $m$ rooms. Each player $i \in \{1, ..., n\}$ provides $m$ numbers $P_{ij}$ summing up to at least $n$: $\forall i : \Sigma_{j=1}^{m} P_{ij} \geq n$. $P$ stands for “partners” and represents the number of players with which the player $i$ is willing to share room number $j$. Algorithm 1, which is a generalization of an algorithm by [Culter 2013], can be used to divide the players into rooms in a way that respects their preferences, i.e. for every player $i$ in room $j$, the total number of players in room $j$ is at most $P_{ij}$.

Proof. The algorithm fills the rooms $G_j$ one after the other recursively. The pre-condition at the beginning of each recursive iteration is the requirement $\forall i : \Sigma_{j=1}^{m} P_{ij} \geq n$. When the while-loop terminates, all players in $G_m$ have $P_{im} \geq |G_m|$ and all players not in $G_m$ have $P_{im} \leq |G_m|$. Because of the pre-condition, for every player $i$ not in $G_m$: $\Sigma_{j=1}^{m-1} P_{ij} \geq n - |G_m|$. Hence, we can call our algorithm recursively with the $n - |G_m|$ remaining players and fill the remaining $m - 1$ rooms.

C Introduction to Flow-Charts and Tables

This appendix aims to make the division algorithms of Section 4 more understandable by providing intuitive flow-charts describing the division process. In each flow-chart:

- Dashed lines are lines drawn by the protocol during eval queries;
- Dotted lines are lines drawn by the players during mark queries;
- Solid lines are cuts made by the protocol after processing the answers to queries;
- The numbers $k_1, k_2$ etc. are the numbers of players in the groups $G_1, G_2$ etc.

Not all cases are shown; some cases are omitted because they are symmetrically similar to the illustrated cases. The red text hints at how the constants in the Prop formula were calculated. In addition to the flow-charts, we describe the algorithms in tables in which:

- The left column describes the protocol steps. A square □ next to a protocol step means that the protocol ends at that step.
- The right column describes the corresponding safe strategy for a single player Alice/George. The right column also contains the proof that the safe strategy indeed fulfills the proportionality guarantee.
**D  Dividing a Square to Squares** (see Claim 4.1 and Figure 5)

INPUT: A square cake $C$, assumed to be $[0,1] \times [0,1]$. $n \geq 1$ with $\forall i: V_i(C) \geq \max(1,6n-8)$.

OUTPUT: $n$ disjoint squares contained in $C$ such that $\forall i: V_i(P_i) \geq 1$.

| Protocol | Safe strategy for George and proof that it works |
|----------|-----------------------------------------------|
| If $n = 1$: give the entire cake to the single player. | $V(C) \geq 1$ and $C$ is a square. |
| **Eval query**: Partition the square cake $C$ to four quarters, $\{C_j\}_{j=1}^4$, in a grid of $2 \times 2$. Ask each player to evaluate each quarter by inputting the integers $\{P_{ij}\}_{j=1}^4, n_{ij}$. Partition the players to 4 groups $G_j$ using Algorithm 1. (a) If $V(C_j) > V(C) - 3$ then let $P_j = n$; (b) If $V(C_j) < 1$ then let $P_j = 0$; (c) Otherwise let $P_j = \lfloor \frac{V(C_j) + 8}{6} \rfloor$. To prove that the strategy complies with the protocol rules, let $k$ be the number of sub-cakes with $V(C_j) < 1$. Obviously $k \leq 3$. If $k = 3$ then the 4th quarter has value more than $V(C) - 3$ so its $P_j$ is $n$ and the rules are kept. If $k = 2$ then the total value of the remaining quarters is more than $V(C) - 2 = 6n - 10$ and $\sum_{j=1}^m P_j > \frac{\sum_{j=1}^m (V(C_j) + 8)}{6} - 2 = \frac{6n - 10 + 2 \cdot 8}{6} - 2 = n - 1$; hence the sum must be $n$ in accordance with the protocol rules. The cases $k = 1, 0$ are similar (in fact, the constants 6 and 8 were calculated based on the case $k = 2$, which happens to be the worst case here). |
| If the number of players in all groups is less than $n$, then cut the cake to 4 quarters using two guillotine cuts and divide each quarter recursively to the players in its group. | By (b), if George is alone in a group then $V(C_j) \geq 1$; by (c), if George is with at most $P_j < n$ players then $V(C_j) \geq 6P_j - 8$. |
| Otherwise, all $n$ players are in a single group; w.l.o.g, assume it is the group corresponding to the south-western square $[0,\frac{1}{2}] \times [0,\frac{1}{2}]$. Proceed with a **Mark query**: Ask each player $i$ to bid an L-shape by drawing a corner square $s_i$ at the south-western corner. Bid an L-shape with value exactly 3. Note George can get here only by saying $P_j = n$, which by (a-c) above implies that $V(C \setminus [0,\frac{1}{2}] \times [0,\frac{1}{2}]) \leq 3$. |
| Pick a smallest L-shape $C_L$ and allow its bidder to select a single square. | Select a square with value 1. Note that George can get here only if $V(C_L) = 3$. $C_L$ is an L-shape coverable by 3 squares so by the Covering Lemma George can get a square with value 1. |
| Divide $C \setminus C_L$ recursively among the remaining $n - 1$ players. | George can get here only if $V(C \setminus C_L) \geq V(C) - 3 = 6n - 11 > 6(n - 1) - 8$. |
Figure 5: Dividing a square to squares. See details near Claim 4.1.
INPUT: A square and $n$ players that value it as $V_n \geq 6n - 8$.
OUTPUT: Each player can get a square with value $\geq 1$. 
E Dividing a 2-fat rectangle to 2-fat rectangles (see Figure 6)

INPUT: A 2-fat rectangle $C$, assumed to be $[0, L] \times [0, 1]$ with $L \in [1, 2]$. $n \geq 1$ with $\forall i: V_i(C) \geq \max(1, 4n - 5)$.
OUTPUT: $n$ disjoint 2-fat rectangles contained in $C$ such that $\forall i: V_i(P_i) \geq 1$.

| Protocol | Safe strategy for George and proof that it works |
|----------|-----------------------------------------------|
| If $n = 1$: give the entire cake to the single player. | $V(C) \geq 1$ and $C$ is a 2-fat rectangle. |
| **Eval query:** Partition $C$ along its longer side to a western half ($x \leq \frac{1}{2}$) and an eastern half. Ask each player to evaluate each half by inputting two integers summing up to $n$. Partition the players to two groups using Algorithm 4 | (a) If $V(C_j) > V(C) - 1$ then let $P_j = n$; (b) If $V(C_j) < 1$ then let $P_j = 0$; (c) Otherwise let $P_j = \frac{V(C_j) + 1}{2}$. Note $\frac{V(C_j) + 1}{2} < P_j$ so the sum $P_1 + P_2$ is larger than $\frac{V(C_1^2 + 2}{4} > n - 1$, hence the sum is at least $n$. |
| If the number of players in all groups is less than $n$ then make a vertical cut at $x = \frac{1}{2}$. The two halves are 2-fat rectangles. Use the 4 walls algorithm recursively to divide each half to the players in its group. | By (b), if George is alone in a group then $V(C_j) \geq 1$; by (c), if George is with at most $P_j < n$ players then $V(C_j) \geq 4P_j - 5$. |
| Otherwise, all $n$ players are in a single group; w.l.o.g. assume it is the group corresponding to the western half $[0, \frac{1}{2}] \times [0, 1]$. Proceed with a **mark query:** Ask each player $i$ to bid a rectangle $[x_i, L] \times [0, 1]$. Let $x^* = \max x_i$. | Select $x$ such that $V([x, L] \times [0, 1]) = 1$. Note that George can get here only by saying $P_i = n$, which by (a-c) above implies that $V([0, \frac{1}{2}] \times [0, 1]) > V(C) - 1$ so $V([\frac{1}{2}, L] \times [0, 1]) < 1$. |
| If $x^* \geq \frac{1}{2}$ then make a vertical cut at $x^*$ and give the eastern part to its bidder. | If this player is George, then by his selection of $x$ he receives a 2-fat rectangle with value 1. |
| The western part is a 2-fat rectangle; recursively divide it among the remaining $n - 1$ players. | If George is one of these players, then by his selection of $x$ the value of the western part is at least $4n - 6 > 4(n - 1) - 5$. |
| Otherwise ($\forall i: x_i < \frac{1}{2}$), proceed with an **Eval query:** partition the “far west” of $C$ ($x \leq \frac{1}{2}$) to two squares: $[0, \frac{1}{2}] \times [0, \frac{1}{2}]$ and $[0, \frac{1}{2}] \times [\frac{1}{2}, 1]$. Input two integers from each player and partition the players to two groups using Algorithm 4 | (d) If $V(C_j) > V(C) - 2$ then let $P_j = n$; (e) If $V(C_j) < 1$ then let $P_j = 0$; (f) Otherwise let $P_j = \frac{V(C_j) + 1}{2}$. Note George can get here only by saying $x < \frac{1}{2}$, which implies $V(C_1) + V(C_2) = V([0, \frac{1}{2}] \times [0, 1]) > V(C) - 1$. Hence if one square has $V < 1$ then the other square has $V > V(C) - 2$. Also $\frac{V(C_1) + 1}{2} < P_j$ so $P_1 + P_2 > \frac{V(C_1^2 + 2}{4} = n - 1$ and the sum is at least $n$. |
| If the number of players in all groups is less than $n$ then make a horizontal cut at $y = \frac{1}{2}$. Recursively divide each square to the players in its group. | By (e), if George is alone in a group then $V(C_j) \geq 1$; by (f), if George is with at most $P_j < n$ players then $V(C_j) \geq 4P_j - 5$. |
| Otherwise, all $n$ players are in a single group; w.l.o.g. assume it is the group corresponding to the south-western square $[0, \frac{1}{2}] \times [0, \frac{1}{2}]$. Proceed with a **mark query:** Ask each player $i$ to bid an L-shape by drawing a corner square $s_i$ at the south-western corner. | Bid an L-shape with value exactly 1. Note George can get here only by saying $P_i = n$, which by (d-e) above implies that $V(C \setminus [0, \frac{1}{2}] \times [0, \frac{1}{2}]) \leq 2$. |
| Select the smallest L-shape $C_L$. Let its bidder select either its east or its north. Separate the selected part from $C \setminus C_L$ using a single guillotine cut and give it its bidder. | Select the east if its value is at least 1; select the north otherwise. Note George can get here only if $V(C_L) = 2$, so either the east or the north must have value at least 1. Both are 2-fat rectangles. |
| Recursively divide $C \setminus C_L$ among the remaining $n - 1$ players. | George can get here only if $V(C \setminus C_L) \geq V(C) - 2 = 4n - 7 > 4(n - 1) - 5$. |
Figure 6: Dividing a 2-fat rectangle to 2-fat rectangles (Claim 4.2).

INPUT: A 2-fat rectangle and n players that value it as $V_n \geq 4n - 5$.

OUTPUT: Each player can get a 2-fat rectangle with value $\geq 1$. 
Quarter-proportional division with guillotine cuts

We prove Claim 4.3 by describing a pair of algorithms calling each other recursively. All algorithms get as input:

- A cake \( C \) which is assumed to be the rectangle \([0, L] \times [0, 1]\);
- \( n \) players with different continuous value measures \( \{V_i\}_{i=1}^n \).

All algorithms return as output \( n \) square plots \( \{P_i\}_{i=1}^n \), such that \( \forall i : V_i(P_i) = 1 \).

The algorithms differ in the number of “walls” (bounded sides): in the 4 walls algorithm, all output squares should be contained in \( C \), while in the 3 walls algorithm, some output squares may flow over the eastern border \( x = L \), although there is no value in the range \( x > L \).
F.1 4 Walls Algorithm (see Figure 7)

PRE-CONDITIONS: C is a 2-fat rectangle (1 ≤ L ≤ 2). ∀i : \( V_i(C) \geq \max(2, 4n - 4) \). n ≥ 1.

POST-CONDITION: All n output squares are contained in \( C = [0, L] \times [0, 1] \).

| Protocol | Safe strategy for Alice and proof that it works |
|----------|-----------------------------------------------|
| If \( n = 1 \): give the entire cake to the single player. | \( V(C) \geq 2 \) and \( C \) is 2-fat. |
| **Eval query**: Partition \( C \) along its longer side to a western half \( (x \leq \frac{L}{2}) \) and an eastern half. Ask each player to evaluate each half by inputting two integers summing up to \( n \). Partition the players to two groups using Algorithm 1. | (a) If \( V(C_j) > V(C) - 2 \) then let \( P_j = n \); (b) If \( V(C_j) < 2 \) then let \( P_j = 0 \); (c) Otherwise let \( P_j = \left\lfloor \frac{V(C_j) + 1}{4} \right\rfloor \). Note \( \frac{V(C_j)}{4} < P_j \) so the sum \( P_1 + P_2 \) is larger than \( \frac{V(C)}{4} = n - 1 \), hence the sum is at least \( n \). |
| If the number of players in all groups is less than \( n \) then make a vertical cut at \( x = \frac{L}{2} \). The two halves are 2-fat rectangles. Use the 4-walls algorithm recursively to divide each half to the players in its group. | By (b), if Alice is alone in a group then \( V(C_j) \geq 2 \); by (c), if Alice is with at most \( P_j < n \) players then \( V(C_j) \geq 4P_j - 4 \). |
| Otherwise, all \( n \) players are in a single group; w.l.o.g. assume it is the group corresponding to the western half \( [0, \frac{L}{2}] \times [0, 1] \). Proceed with a **mark query**: Ask each player \( x \) to bid a rectangle \( [x, L] \times [0, 1] \). Let \( x^* = \max x_i \). | Select \( x \) such that \( V([x, L] \times [0, 1]) = 2 \). Note that Alice can get here only by saying \( P_j = n \), which by (a-c) above implies that \( V([0, \frac{L}{2}] \times [0, 1]) > V(C) - 2 \) so \( V((\frac{L}{2}, L] \times [0, 1]) < 2 \). |
| If \( x^* \geq \frac{L}{2} \) then make a vertical cut at \( x^* \) and give the eastern rectangle to its bidder. | If this player is Alice, then by her selection of \( x \) she receives a 2-fat rectangle with value 2. |
| The western part is a 2-fat rectangle; use the 4-walls algorithm to divide it among the remaining \( n - 1 \) players. | If Alice is one of these players, then by her selection of \( x \) the value of the western part is at least \( 4n - 6 > 4(n - 1) - 4 \). |
| Otherwise (\( \forall i : x_i < \frac{L}{2} \)), proceed with an **Eval query**: partition the “far west” of \( C (x \leq \frac{L}{2}) \) to two squares: \( [0, \frac{L}{2}] \times [0, \frac{L}{2}] \) and \( [0, \frac{L}{2}] \times [\frac{L}{2}, 1] \). Input two integers from each player and partition the players to two groups using Algorithm 1. | (d) If \( V(C_j) > V(C) - 3 \) then let \( P_j = n \); (e) If \( V(C_j) < 1 \) then let \( P_j = 0 \); (f) Otherwise let \( P_j = \left\lfloor \frac{V(C_j) + 1}{4} \right\rfloor \). Note Alice can get here only by saying \( x < \frac{L}{2} \), which implies \( V(C_1) + V(C_2) = V([0, \frac{L}{2}] \times [0, 1]) > V(C) - 2 \). Hence if one square has \( V < 1 \) then the other square has \( V > V(C) - 3 \). Also \( \frac{V(C_j) + 1}{4} < P_j \) so \( P_1 + P_2 > \frac{V(C) + 2 - 2}{4} = n - 1 \) and the sum is at least \( n \). |
| If the number of players in all groups is less than \( n \) then make a horizontal cut at \( y = \frac{L}{2} \). Use the 3-walls algorithm to divide each square to the players in its group. The output squares may flow over the eastern border of the squares \( (x = \frac{L}{2}) \), but this is harmless as there are no other squares there. | By (e), if Alice is alone in a group then \( V(C_j) \geq 1 \); by (f), if Alice is with at most \( P_j < n \) players then \( V(C_j) \geq 4P_j - 5 \). |
| Otherwise, all \( n \) players are in a single group; w.l.o.g. assume it is the group corresponding to the south-western square \( [0, \frac{L}{2}] \times [0, \frac{L}{2}] \). Proceed with a **mark query**: Ask each player \( x \) to bid an L-shape by drawing a corner square \( s \), at the south-western corner. | Bid an L-shape with value exactly 3. Note Alice can get here only by saying \( P_j = n \), which by (d-f) above implies that \( V(C \setminus [0, \frac{L}{2}] \times [0, \frac{L}{2}]) \leq 3 \). |
| Select a smallest L-shape, \( C_L \). Let its bidder select either its east or its north. Separate the selected part from \( C \setminus C_L \) using a single guillotine cut and give it to its bidder. | Select the east, which is a 2-fat rectangle, if its value is at least 2; select the north, which is a square, otherwise. Note Alice can get here only if \( V(C_L) = 3 \), so if the north is selected its value is at least 1. |
| Now either the north or the east of \( C_L \) are unused; use the 3-walls algorithm in the appropriate direction to divide \( C \setminus C_L \) among the remaining \( n - 1 \) players. | Alice can get here only if \( V(C \setminus C_L) \geq V(C) - 3 = 4n - 7 > 4(n - 1) - 5 \). |
Figure 7: 4 Walls Algorithm.

INPUT: A 2-fat rectangle with all 4 sides bounded; n players that value it as $V^D_n \geq 4n - 4$.
OUTPUT: Each player can get a square with value $\geq 1$.
This algorithm uses the algorithm illustrated in Figure 8 to divide a square unbounded in one side.
### F.2 3 Walls Algorithm (see Figure 8)

**PRE-CONDITIONS:** $C$ is an arbitrary rectangle with $L \leq 1$. For every player $i$, $V_i(C) \geq \max(1, 4n − 5)$. $n \geq 1$.

**POST-CONDITION:** All $n$ output squares are contained in $[0, L + 1] \times [0, 1]$, i.e.: they are within three walls of $C$ but may flow over the eastern wall $x = L$.

| Protocol | Safe strategy for Alice and proof that it works |
|----------|-----------------------------------------------|
| If $n = 1$: give to the single player the square $[0, 1] \times [0, 1]$. | $V(C) \geq 1$ and $C$ is contained in the allocated square. |
| **mark query:** Ask each player $i$ to bid a square $[x_i, x_i + 1] \times [0, 1]$ where $x_i \in [0, L]$. Let $x^* = \max x_i$. | Bid a square with value 1 by selecting $x$ such that $V([x, L] \times [0, 1]) = 1$. Note that $V([x, L] \times [0, 1]) = V([x, x + 1] \times [0, 1])$. |
| If $x^* \geq \frac{1}{2}$ then make a vertical cut at $x^*$. Give the eastern square $[x^*, x^* + 1] \times [0, 1]$ to its bidder. | If this player is Alice, then by her selection of $x$ she receives a square with value 1. |
| The western part is a 2-fat rectangle; use the 4-walls algorithm to divide it among the remaining $n − 1$ players. | If Alice is one of these players, then by her selection of $x$ the value of the western part is at least $4n − 6 > 4(n − 1) − 4$. |
| Otherwise ($\forall i : x_i < \frac{1}{2}$), proceed with an **Eval query:** partition the “far west” of $C$ ($x \leq \frac{1}{2}$) to two squares: $[0, \frac{1}{2}] \times [0, \frac{1}{2}]$ and $[0, \frac{1}{2}] \times [\frac{1}{2}, 1]$. Input two integers from each player and partition the players to two groups using Algorithm 1. | (g) If $V(C_j) > V(C) − 2$ then let $P_j = n$; (h) If $V(C_j) < 1$ then let $P_j = 0$; (i) Otherwise let $P_j = \lfloor \frac{V(C_j) + 1}{4} \rfloor$. Note Alice can get here only by saying $x < \frac{1}{2}$, which implies $V(C_1) + V(C_2) = V([0, \frac{1}{2}] \times [0, 1]) > V(C) − 1$. Hence if one square has $V < 1$ then the other square has $V > V(C) − 2$. Also $\frac{V(C_j) + 1}{4} < P_j$ so $P_1 + P_2 > \frac{V(C) + 2}{4} = n − 1$ and the sum is at least $n$. |
| If the number of players in all groups is less than $n$ then make a horizontal cut at $y = \frac{1}{2}$. Use the 3-walls algorithm to divide each square to the players in its group. The resulting squares are contained in $[0, 1] \times [0, \frac{1}{2}]$ and $[0, 1] \times [\frac{1}{2}, 1]$, which are both contained in $[0, L + 1] \times [0, 1]$. | By (h), if Alice is alone in a group then $V(C_j) \geq 1$; by (i), if Alice is with at most $P_j < n$ players then $V(C_j) \geq 4P_j − 5$. |
| Otherwise, all $n$ players are in a single group; w.l.o.g., assume it is the group corresponding to the south-western square $[0, \frac{1}{2}] \times [0, \frac{1}{2}]$. Proceed with a **mark query:** Ask each player $i$ to bid an L-shape by drawing a corner square $s_i$ at the south-western corner. | Bid an L-shape with value exactly $V(C) − 2$. Note Alice can get here only by saying $P_j = n$, which by (g-i) above implies that $V(C \setminus [0, \frac{1}{2}] \times [0, \frac{1}{2}]) \leq 2$. |
| Select a smallest L-shape, $C_L$. Let its bidder select either its east or its north. Separate the selected part from $C \setminus C_L$ using a single guillotine cut and give it to its bidder. | Select the east, in which a square can be allocated, if its value is at least 1; select the north, which is a square, otherwise. Note Alice can get here only if $V(C_L) = 2$, so if the north is selected its value is at least 1. Both the east and the north are squares. |
| Now either the north or the east of $C_L$ are unused; use the 3-walls algorithm in the appropriate direction to divide $C \setminus C_L$ among the remaining $n − 1$ players. The output squares are contained either in $[0, \frac{1}{2}] \times [0, 1]$ or in $[0, 1] \times [0, \frac{1}{2}]$, which are both contained in $[0, L + 1] \times [0, 1]$. | Alice can get here only if $V(C \setminus C_L) \geq V(C) − 2 = 4n − 7 > 4(n − 1) − 5$. |
Figure 8: 3 Walls Algorithm.
INPUT: A rectangle with one of the long sides unbounded; n players that value it as $V^C_n \geq 4n - 5$.
OUTPUT: Each player can get a square with value $\geq 1$.
This algorithm uses the algorithm illustrated in Figure 7 to divide a square bounded in all sides.
G Staircase algorithm

Claim 4.9 If C is a rectilinear polygonal domain with two unbounded sides having k corners, then \( \text{Prop}(C, n, \text{Squares}) = \frac{1}{2n-2+k} \).

Proof. The following protocol can be used:

(1) Each player \( i \) draws a square \( s_i^j \) in every corner \( j \in \{1, \ldots, k\} \). Let \((x^i, y^i)\) be the coordinates of corner \( j \) and \( l_i^j \) the side-length of square \( s_i^j \).

(2) For every square \( s_i^j \), calculate \( t_i^j := x^i + y^i + l_i^j \). The winning square \( s^* \) is a square \( s_i^j \) for which \( t_i^j \) is smallest (breaking ties arbitrarily). The winning square is allocated to its owner. See Fig. 4 Middle-Right.

(3) Discard the shadow of the winning square by cutting from its north-western corner towards the western boundary of \( C \) and from its south-eastern corner towards the southern boundary of \( C \). Thus the entire box \([0, x^* + l^*] \times [0, y^* + l^*]\) is removed from the cake. See Fig. 4 Right.

(4) The remainder is a staircase with possibly a different number of corners. Divide it recursively to the remaining \( n - 1 \) players.

For the safe strategy, consider a certain player George who values the entire cake as \( V(C) \geq 2n - 2 + k \). The safe strategy for George is simply to draw all corner squares with value exactly 1 (possibly including squares with infinite side-length). Because \( V(C) \geq k \), there is at least one corner such that George values the quarter-plane at that corner as at least 1. Hence he can draw at least one square. We now prove that the safe strategy guarantees George a value of at least 1.

If the selected square in step (2) belongs to George then we are done. Otherwise, another square is selected and an entire box is removed from the cake. This removes from the cake a certain number \( m \geq 1 \) of corners - all corners having \((x \leq x^* + l^*) \land (y \leq y^* + l^*)\). Additionally, two new corners are added - one with \((x \leq x^*) \land (y = y^* + l^*)\) and one with \((y \leq y^*) \land (x = x^* + l^*)\); see Figure 4 Right. The new number of corners is thus \( k + 2 - m \).

George loses the value contained in the removed box. We now prove that this lost value is at most \( m \). Partition the removed box to \( m \) disjoint rectangular components, starting from the winning square \( s^* \) and going to the north-west and to the south-east, such that each component is located in a different corner. The components to the north-west of \( s^* \) are of the form: \([x, x^*] \times [y, y^* + l^*]\) and the components to its south-east are: \([x, x^* + l^*] \times [y, y^*]\). The winning square \( s^* \) itself is a single component; for example, the box corresponding to the blue square in Fig. 4 Middle has three components, in corners 1 2 and 3. We now prove that, for each of the \( m \) removed corners \((x, y)\), if George has drawn a square in that corner then the component of the removed box in \((x, y)\) is contained in George’s square. We consider first a component to the north-west of \( s^* \). This is a box: \([x, x^*] \times [y, y^* + l^*]\). George’s square in that corner, say \([x, x + l] \times [y, y + l]\), was not selected, hence necessarily \( x + y + l \geq x^* + y^* + l^* \). But here \( x < x^* \) and \( y < y^* + l^* \), hence necessarily \( y^* + l^* < y + l \) and \( x^* < x + l \), hence the component is contained in George’s square. A very similar calculation holds for a component to the south-east of \( s^* \). For example, in Fig. 4 Middle-Right, the winning square is the blue square. Its component in corner 3 is contained in the cyan square and its component in corner 1 is contained in the green square. Since the value of George’s square in each corner is 1, the value of the component is at most 1, hence the total value of the removed box is at most \( m \).

This implies that the new value of the cake is at least \( 2n - 2 + k - m = 2(n - 1) - 2 + (k + 2 - m) \), which is just the value that George needs in order to recursively divide the remaining cake with \( n - 1 \) players.

\[^8\] \( t_i^j \) can be interpreted as the taxicab distance between the origin to the center of \( s \), or equivalently to the south-eastern corner of \( s \) or the north-western corner of \( s \).

\[^9\] A similar property incidentally holds also for the cyan square, but not for the green and red squares.
| Cake ($C$) | Usable shapes ($S$) | Impossibility | Algorithm |
|-----------|---------------------|---------------|-----------|
| Square, all sides bounded | Squares | $\frac{1}{2n}$ | $\frac{1}{2n}$ (guillotine) * |
| Square, all sides bounded | 2-fat rectangles | $\frac{1}{2n}$ | $\frac{1}{3n}$ (guillotine) * |
| Rect. polygon, $k = \text{IndepNum}(C, S)$ | $R$-fat rectangles | $\frac{2n-2+k}{2n}$ | ? * |
| Quarter-plane (2 unbounded sides) | Squares | $\frac{1}{2n}$ | $\frac{1}{2n}$ (staircase) * |
| Rect. polygon, 2 unbounded sides, $k$ corners | Squares | $\frac{2n-2+k}{2n}$ | $\frac{1}{2n}$ (staircase) * |
| Half-plane (3 unbounded sides) | Squares | $\frac{1}{2n-1}$ | $\frac{1}{2n}$ (staircase) * |
| Plane (4 unbounded sides) | Squares | $\frac{1}{2n-1}$ | $\frac{2n-1}{4n-4}$ (staircase) * |
| $L \times 1$ rectangle ($L > 1$) | $R$-fat rectangles | $\frac{2n-2+\left\lceil \frac{L}{R} \right\rceil}{4n-6+\left\lceil \frac{2n-2}{2R} \right\rceil}$ | $\frac{2n-2+\left\lceil \frac{L}{R} \right\rceil}{4n-6+\left\lceil \frac{2n-2}{2R} \right\rceil}$ (staircase) * |
| $d$-cube with $\delta$ unbounded sides | $d$-cubes | $d(n-2)+(2d^2-\delta)$ | $(2d^2-\delta)$ (staircase) * |

Table 1: Bounds on the partial-proportionality $\text{Prop}(C, n, S)$ in several geometric scenarios. Impossibility results (upper bounds) are explained in Section 3. Algorithms (lower bounds) are explained in Section 4. Stars in the rightmost column denote scenarios in which we have open questions.

References

Pankaj K. Agarwal, Matthew J. Katz, and Micha Sharir. Computing depth orders for fat objects and related problems. *Computational Geometry*, 5(4):187–206, November 1995.

Michael O. Albertson and Claire J. O’Keefe. Covering Regions with Squares. *SIAM Journal on Algebraic Discrete Methods*, 2(3):240–243, September 1981.

Yonatan Aumann, Yair Dombb, and Avinatan Hassidim. Computing Socially-Efficient Cake Divisions. In AAMAS, May 2013.

Eric Balkanski, Simina Brânzei, David Kurokawa, and Ariel D. Procaccia. Simultaneous Cake Cutting. In AAAI, 2014.

Reuven Bar-Yehuda and Eyal Ben-Hanoch. A linear-time algorithm for covering simple polygons with similar rectangles. *International Journal of Computational Geometry & Applications*, 6.01:79–102, 1996.

Julius B. Barbanel, Steven J. Brams, and Walter Stromquist. Cutting a pie is not a piece of cake. *American Mathematical Monthly*, 116(6):496–514, 2009.

Anatole Beck. Constructing a Fair Border. *The American Mathematical Monthly*, 94(2):157+, February 1987.

Xiaohui Bei, Ning Chen, Xia Hua, Biaoshuai Tao, and Endong Yang. Optimal Proportional Cake Cutting with Connected Pieces. In AAAI, 2012.

Steven J. Brams and Alan D. Taylor. *Fair Division: From Cake-Cutting to Dispute Resolution*. Cambridge University Press, February 1996.

Steven J. Brams, Michael Jones, and Christian Klamler. Proportional pie-cutting. *International Journal of Game Theory*, 36(3):353–367, March 2008.

Ioannis Caragiannis, Christos Kaklamanis, Panagiotis Kanellopoulos, and Maria Kyropoulou. The Efficiency of Fair Division. *Theory of Computing Systems*, 50(4):589–610, September 2012.

Seth Chaiken, Daniel J. Kleitman, Michael Saks, and James Shearer. Covering Regions by Rectangles. *SIAM Journal on Algebraic Discrete Methods*, 2(4):394–410, December 1981.
Yiling Chen, John K. Lai, David C. Parkes, and Ariel D. Procaccia. Truth, justice, and cake cutting. *Games and Economic Behavior*, 77(1):284–297, January 2013.

Yuga J. Cohler, John K. Lai, David C. Parkes, and Ariel D. Procaccia. Optimal Envy-Free Cake Cutting. In *AAAI (25th AAAI Conference on Artificial Intelligence)*, 2011.

Richard Cole, Vasilis Gkatzelis, and Gagan Goel. Mechanism Design for Fair Division, February 2013.

Chris Culter. Dividing people to rooms. Mathematics Stack Exchange, August 2013. URL:http://math.stackexchange.com/q/461675 (version: 2013-08-07).

Marco Dall’Aglio and Camilla Di Luca. Finding maxmin allocations in cooperative and competitive fair division, August 2012.

Jeff Edmonds, Kirk Pruhs, and Jaisingh Solanki. Confidently Cutting a Cake into Approximately Fair Pieces. 2008.

Shimon Even and Azaria Paz. A note on cake cutting. *Discrete Applied Mathematics*, 7(3):285–296, March 1984.

Sándor P. Fekete and Hella-Franziska Hoffmann. Online Square-into-Square Packing. In Prasad Raghavendra, Sofya Raskhodnikova, Klaus Jansen, and Joséd Rolim, editors, *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques*, volume 8096 of *Lecture Notes in Computer Science*, pages 126–141. Springer Berlin Heidelberg, 2013.

Sándor P. Fekete, Tom Kamphans, and Nils Schweer. Online Square Packing with Gravity. 68 (4):1019–1044, 2014.

Deborah S. Franzblau and Daniel J. Kleitman. An Algorithm for Constructing Regions with Rectangles: Independence and Minimum Generating Sets for Collections of Intervals. In *Proceedings of the Sixteenth Annual ACM Symposium on Theory of Computing, STOC ’84*, pages 167–174, New York, NY, USA, 1984. ACM.

Nick Gill, Marcus Hum, and Dömötör Pálvölgyi. Is there always a maximum anti-rectangle with a corner square? MathOverflow, November 2013. URL:http://mathoverflow.net/q/149020 (version: 2013-11-17).

Theodore P. Hill. Determining a Fair Border. *The American Mathematical Monthly*, 90(7):438+, August 1983.

Theodore P. Hill and Kent E. Morrison. Cutting Cakes Carefully. *The College Mathematics Journal*, pages 281–288, September 2010.

Karthik Iyer and Michael N. Huhns. A Procedure for the Allocation of Two-Dimensional Resources in a Multiagent System. *International Journal of Cooperative Information Systems*, 18, 2009.

Michael A. Jones. Equitable, Envy-Free, and Efficient Cake Cutting for Two People and Its Application to Divisible Goods. *Mathematics Magazine*, 75(4):275+, October 2002.

Matthew J. Katz. 3-D vertical ray shooting and 2-D point enclosure, range searching, and arc shooting amidst convex fat objects. *Computational Geometry*, 8(6):299–316, November 1997.

Elchanan Mossel and Omer Tamuz. Truthful Fair Division. In Spyros Kontogiannis, Elias Koutsoupias, and Paul G. Spirakis, editors, *SAGT*, volume 6386, pages 288–299, Berlin, Heidelberg, July 2010. Springer Berlin Heidelberg.
Ariel D. Procaccia. Cake Cutting: Not Just Child’s Play. *Communications of the ACM (CACM)*, 2013.

Jack M. Robertson and William A. Webb. *Cake-Cutting Algorithms: Be Fair if You Can*. A K Peters/CRC Press, first edition, July 1998.

Hugo Steinhaus. The problem of fair division. *Econometrica*, 16(1), January 1948.

Francis E. Su. Rental Harmony: Sperner’s Lemma in Fair Division. *The American Mathematical Monthly*, 106(10):930+, December 1999.

William Thomson. Children Crying at Birthday Parties. Why? *Economic Theory*, 31(3):501–521, June 2007.

Gerhard J. Woeginger and Jiří Sgall. On the complexity of cake cutting. *Discrete Optimization*, 4(2):213–220, June 2007.