Gibbs u-states for the foliated geodesic flow and transverse invariant measures

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Abstract
This paper is devoted to the study of Gibbs u-states for the geodesic flow tangent to a foliation with negatively curved leaves. By definition, they are the invariant measures that have Lebesgue disintegration in the unstable horospheres tangent to the leaves.

On the one hand we give sufficient conditions for the existence of transverse invariant measures. In particular we prove that when this foliated geodesic flow preserves a Gibbs su-state, i.e. a measure with Lebesgue disintegration both in the stable and unstable horospheres, then it has to be obtained by combining a transverse invariant measure and the Liouville measure on the leaves.

On the other hand we study in detail the projections of Gibbs u-states along the unit spheres tangent to the foliation. We show that they have Lebesgue disintegration in the leaves and that the local densities have a very specific form: they possess an integral representation analogue to the Poisson representation of harmonic functions.

1 Introduction
This is the second of a series of three papers in which we study a notion of Gibbs measure for the geodesic flow tangent to the leaves of a closed foliated manifold with negatively curved leaves [Al1, Al2].

The starting point of this work is the following problem, solved in [Al2] in a more general setting: prove that the geodesic flow tangent to the leaves of a foliated \( CP^1 \)-bundle over a closed and negatively curved manifold and without holonomy invariant measure has a unique SRB measure. We were even able to prove that this measure is the unique Gibbs u-state for this foliated geodesic flow i.e. the unique invariant measure with Lebesgue disintegration in unstable horospheres (see theorem B below).

The goal here is twofold. Firstly, we wish to give some new sufficient conditions for a foliation with negatively curved leaves to admit a transverse measure invariant by its holonomy pseudogroup. Secondly, this paper serves as a companion to [Al1]. In the latter, we were led to associate a notion of Gibbs measure for the foliated geodesic flow to the now classical notion of Garnett’s harmonic measure. Here, we associate a new notion of harmonic measure, called \( \phi^n \)-harmonic measure, to the classical notion of Gibbs u-state (see for example [BDV]). More precisely, when \((M,\mathcal{F})\) is a closed foliated manifold with negatively curved leaves, Gibbs u-states on the unit tangent bundle \( T^1\mathcal{F} \) always exist. It occurs that their projections on \( M \) along the unit tangent fibers to the leaves have Lebesgue disintegration in the leaves of \( \mathcal{F} \) and that the local densities have an integral representation very similar to the Poisson representation of harmonic functions.

Existence of invariant measures. One can associate to any foliation \( \mathcal{F} \) a dynamical system on a complete transversal \( \mathcal{T} \) called the holonomy pseudogroup. It is generated by the transition functions and shall be denoted by \( \mathcal{P} \). The existence of invariant measures for such a dynamical system is in general very rare.
Plante was the first to give a sufficient condition for this to happen. Let \((M, \mathcal{F})\) be a compact foliated manifold whose leaves are endowed with a smooth Riemannian metric which varies continuously in the \(C^\infty\)-topology with the transverse parameter (this will be called a leafwise metric in the sequel). He proved that if a leaf has a subexponential growth, then its closure supports a transverse invariant measure. In \([GLW]\), Ghys, Langevin and Walczak developed a notion of geometric entropy for foliations and proved that its vanishing implies the existence of a transverse invariant measure.

When the leaves of \(\mathcal{F}\) are hyperbolic Riemann surfaces there is another condition ensuring the existence of a transverse invariant measure. The unit tangent bundle of the foliation \(T^1\mathcal{F}\) is foliated by the unit tangent bundles of the leaves of \(\mathcal{F}\). We denote by \(\tilde{\mathcal{F}}\) this foliation. It comes with three flows: the foliated geodesic flow \(G_t\) and the foliated stable and unstable horocyclic flows \(H^\pm_t\). The following folklore result may be found for example in \([BG]\) and \([Ma]\).

Any probability measure on \(T^1\mathcal{F}\) which is invariant by the foliated geodesic flow and by both the foliated horocyclic flows is totally invariant: locally it is the product of the Liouville measure by a transverse holonomy invariant measure.

The proof follows from an elegant algebraic argument. Since all the leaves are hyperbolic surfaces they are uniformized by the upper half plane \(H\) endowed with the Poincaré metric. The unit tangent bundle \(T^1H\) can be identified with the Lie group \(PSL_2(\mathbb{R})\). The three flows are then identified with actions of 1-parameter subgroups of \(PSL_2(\mathbb{R})\) by right translations. Moreover they generate all \(PSL_2(\mathbb{R})\). Hence a measure invariant by these three subgroups has to be invariant by the whole action of \(PSL_2(\mathbb{R})\) by right translations. Thus it is a multiple of the bi-invariant Haar measure which is identified with the Liouville measure of \(T^1H\).

The conditional measures in the leaves of any measure on \(T^1\mathcal{F}\) invariant by the joint action of the three foliated flows are proportional to the Liouville measure. Finally such a measure reads locally as the product of the Liouville measure by a transverse holonomy invariant measure. We call it totally invariant.

We are interested in the case where the leaves of \(\mathcal{F}\) are of arbitrary dimension and with negative sectional curvature. In this case there are no horocyclic flows on \(T^1\mathcal{F}\) anymore, but two horospheric subfoliations of \(\tilde{\mathcal{F}}\). We can define a Gibbs su-state as an invariant measure for the foliated geodesic flow \(G_t\) which is both a Gibbs s-state and a Gibbs u-state: it has both Lebesgue disintegration in stable and unstable manifolds. The main result is then:

**Theorem A.** Let \((M, \mathcal{F})\) be a closed foliated manifold endowed with a leafwise metric whose leaves are negatively curved. If the foliated geodesic flow admits a Gibbs su-state, then \(\mathcal{F}\) has a transverse invariant measure.

The algebraic argument above will be replaced by an argument of absolute continuity of horospheric subfoliations, which will allow us to prove that any Gibbs su-state is totally invariant. The principle behind it, which is the cornerstone of \([AL2]\), is that it is impossible to prescribe both the measure classes of the disintegration of a \(G_t\)-invariant measure in the stable and unstable manifolds. Indeed if we could do so this would imply the existence of a transverse invariant measure. The proof of this theorem will be the occasion to work with a weak notion of hyperbolicity introduced by Bonatti, Gómez-Mont and Martínez in \([BGM]\) called foliated hyperbolicity.

**Uniqueness result for foliated bundles.** From this theorem we can deduce uniqueness of SRB measures in the context of projective foliated bundles. Let \(\Pi: M \to B\) be a locally trivial bundle with fiber \(\mathbb{CP}^1\) whose basis \(B\) is a closed and negatively curved manifold. Assume that there is a foliation \(\mathcal{F}\) transverse to the fibers. Lift the metric of the basis to the leaves via the fibration. Then
the differential of $\Pi$ induces a $\mathbb{CP}^1$-bundle $D\Pi: T^1\mathcal{F} \to T^1B$ foliated by the unit tangent bundles of the leaves of $\mathcal{F}$.

As we said earlier Gibbs u-states for $G_t$ always exist. Furthermore, it is possible to prove that ergodic components of Gibbs u-states for $G_t$ are still Gibbs u-states (see theorem 4.2). Since when restricted to a leaf the submersion $\Pi$ is a local isometry, the projection on $T^1B$ of a Gibbs u-state for the foliated geodesic flow is a Gibbs u-state for the geodesic flow of $T^1B$: this is the Liouville measure. In [Al2] we prove the following folklore result.

*When no measure in $\mathbb{CP}^1$ is invariant by the holonomy group, there exist only two invariant measures for the foliated geodesic flow $G_t$, denoted by $\mu^+$ and $\mu^-$, which are ergodic and project down to the Liouville measure. Moreover the measure $\mu^+$ is a Gibbs u-state, and the measure $\mu^-$ is a Gibbs s-state.*

If there is no holonomy invariant measure, theorem A implies that $\mu^-$ can’t be a Gibbs u-state. Or if one prefers, $\mu^+$ is the unique ergodic Gibbs u-state for $G_t$. The above then implies the following uniqueness result (which is a special case of theorem B of [Al2]):

**Theorem B.** Let $(\Pi, M, B, \mathbb{CP}^1, \mathcal{F})$ be a projective foliated bundle over a closed Riemannian manifold $B$ with negative sectional curvature and whose leaves are locally isometric to the basis. Assume moreover that the holonomy group $\rho(\pi_1(B))$ has no invariant probability measure on $\mathbb{CP}^1$. Then the foliated geodesic flow $G_t$ possesses a unique Gibbs u-state on $T^1\mathcal{F}$.

**$\phi^u$-harmonic measures.** As we mentioned earlier, projections of Gibbs u-states for $G_t$ on $M$ along the unit spheres tangent to the foliation have a very specific form. In [BGM], Bonatti, Gómez-Mont and Martínez already proved that they have Lebesgue disintegration in the leaves of $\mathcal{F}$. We wish to describe them in more details. In order to do so, we need to introduce a kernel which plays the role of the Poisson kernel (see [AS] for details). Let $\tilde{L}$ be a leaf of $\mathcal{F}$, and $\tilde{L}$ be its Riemannian universal cover. It is a complete connected and simply connected Riemannian manifold whose sectional curvature is pinched between two negative constants. Therefore it can be compactified by adjonction of a topological sphere $\tilde{L}(\infty)$ which possesses a Hölder structure.

If $\text{Jac}^uG_t$ denotes the Jacobian of the time $t$ of the flow in the unstable direction and if $\beta_\xi$ denotes the Busemann function at $\xi$ (see section 5.1 for the definition), one defines a kernel $k^u$ on $\tilde{L} \times \tilde{L} \times \tilde{L}(\infty)$ by the following formula:

$$k^u(z_1, z_2; \xi) = \lim_{t \to -\infty} \frac{\text{Jac}^uG_{-t-}\beta_\xi(z_1, z_2)(v_\xi, z_2)}{\text{Jac}^uG_{-t}(v_\xi, z_2)},$$

where $v_\xi, z_2$ is the unit vector based at $z_1$ which points to $\xi$ in the past.

A $\phi^u$-harmonic function on $\tilde{L}$ is a function with an integral representation:

$$h(z) = \int_{\tilde{L}(\infty)} k^u(o, z; \xi) d\eta_o(\xi),$$

where $o \in \tilde{L}$ is a base point and $\eta_o$ is a finite Borel measure on $\tilde{L}(\infty)$. When $L$ is hyperbolic, it is possible to prove that the kernel above coincides with the Poisson kernel and that $\phi^u$-harmonic and harmonic functions coincide.

A $\phi^u$-harmonic measure for $\mathcal{F}$ is a probability measure on $M$ which has Lebesgue disintegration in the leaves, and whose local densities in the plaques are $\phi^u$-harmonic. The next theorem provides a canonical one-to-one correspondence between Gibbs u-state and $\phi^u$-harmonic measures. In particular, it provides the existence of $\phi^u$-harmonic measures for foliations with negatively curved leaves. The proof of this theorem is based on a variation of the *unrolling argument* described in [Al1], as well as on the absolute continuity of horospheric subfoliations.
**Theorem C.** Let $(M, F)$ be a closed foliated manifold endowed with a leafwise metric. Assume that all its leaves are negatively curved. Then the projection along the unit tangent fibers of any Gibbs $u$-state is a $\phi^u$-harmonic measure. Moreover for any $\phi^u$-harmonic measure $m$ for $F$, there exists a unique Gibbs $u$-state for $F$, which projects down onto $m$.

We are able to deduce from this theorem some basic properties of $\phi^u$-harmonic measures. Firstly, using the absolute continuity of horospheric subfoliations of $F$ we find a continuous function $h_0 : M \to \mathbb{R}$ which is continuous and $\phi^u$-harmonic in restriction to the leaves (see theorem 5.5). Then we have a notion of totally invariant $\phi^u$-harmonic measure by combining transverse invariant measures with measures with density $h_0$ with respect to the volume in the leaves.

Secondly, by using the theorem above, we prove an ergodic decomposition theorem for $\phi^u$-harmonic measures, where by definition an ergodic measure $\phi^u$-harmonic $m$ satisfies $m(F) = 0$ or $1$ for any Borel set $X \subset M$ which is saturated by $F$: see theorem 5.16.

The name $\phi^u$-harmonic has been chosen because Gibbs $u$-state come from a potential that is usually denoted $\phi^u$ (see [BR] for example and the formula 5.15 for the definition). This choice of terminology is coherent with the notion of $F$-harmonic measures introduced in [AI2] for foliated bundles and general potentials.

**Generalization of a theorem of Matsumoto.** In a nice paper [Mat] Matsumoto considered closed foliated manifolds $(M, F)$ with hyperbolic leaves and their harmonic measures (in the sense of Garnett). He showed that the extension by holonomy of a local harmonic density on a typical leaf $L$ defines, up to multiplication by a constant, a harmonic function on its universal cover $\tilde{L}$ called the characteristic function of $L$ and denoted by $h_L$. Using the Poisson representation of harmonic functions, we see that the function $h_L$ is associated to a measure on $\tilde{L}(\infty)$ denoted by $\eta_L$. Its measure class $[\eta_L]$ only depends on the leaf $L$, and we call it the characteristic class of $L$.

By analysing the properties of Brownian motion tangent to the leaves, he proved the following:

**Theorem 1.1 (Matsumoto).** Let $(M, F)$ be a closed foliated manifold whose leaves are hyperbolic manifolds. Let $m$ be a harmonic measure which is not totally invariant. Then for $m$-almost every leaf $L$, the characteristic class $[\eta_L]$ on $\tilde{L}(\infty)$ is singular with respect to the Lebesgue measure.

Moreover, the characteristic function of $m$-almost every leaf is unbounded.

When the curvature of the leaves of $F$ are variable and $m$ is a $\phi^u$-harmonic measure, we can define the characteristic $\phi^u$-function as well as the characteristic class of almost every leaf $L$ in the same way (see the section 6.2 for the details). The visibility class on $\tilde{L}(\infty)$ is obtained by pushing at infinity the Lebesgue measure on the unit tangent fibers by the geodesic flow. It is well defined by absolute continuity of the center unstable foliation. We can prove the following theorem which gives a sufficient condition on the densities of $\phi^u$-harmonic measures for the existence of transverse invariant measure. This theorem implies Matsumoto’s result (see the proof of corollary 6.3). The proof of this theorem is dynamical and only relies on the absolute continuity of horospheric subfoliations.

**Theorem D.** Let $(M, F)$ be a closed foliated manifold endowed with a leafwise metric such that all leaves are negatively curved. Let $m$ be a non totally invariant $\phi^u$-harmonic measure. Then for $m$-almost every leaf $L$, the characteristic class $[\eta_L]$ on $\tilde{L}(\infty)$ is singular with respect to the visibility class.

**Organization of the paper.** In section 2 we will give the basic properties of transverse measures for foliations as well as of the associated cocycles. In section 3, we introduce the notion of foliated hyperbolicity, we state theorems of existence and absolute continuity of stable and unstable
In the more general case of leafwise hyperbolic flows which preserve the volume inside the leaves. In section 5, we define \( \phi^u \)-harmonic measures, prove theorem C and obtain some basic results about ergodic theory of these measures. In final section 6, we state and prove our generalization of Matsumoto's theorem.

## 2 Transverse measures for foliations

### 2.1 Foliations and holonomy

**Foliations.** A closed manifold \((C^\infty, \text{compact and boundaryless})\) \(M\) of dimension \(n\) possesses a foliation of dimension \(d\) and of class \(C^\infty\) (in short we'll say that \((M, \mathcal{F})\) is a closed foliated manifold) if it is endowed with a finite foliated atlas \(\mathcal{A} = (U_i, \phi_i)_{i \in I}\) consisting of local charts \(\phi_i : U_i \rightarrow P_i \times T_i\) where \(P_i\) and \(T_i\) are cubes of respective dimensions \(d\) and \(n - d\) such that the changes of charts are \(C^\infty\) diffeomorphisms of the form:

\[
\phi_j \circ \phi_i^{-1}(z, x) = (\zeta_{ij}(z, x), \tau_{ij}(x)),
\]

where \((z, x) \in \phi_i(U_i \cap U_j)\). The sets \(\phi_i^{-1}(P_i \times \{x\})\) are called **plaques**. Since the change of charts have this very particular form, we can glue the plaques together, so as to obtain a partition of \(M\) by immersed submanifolds, called the **leaves** of \(\mathcal{F}\). We will denote by \((P_i(x), x)_{x \in T_i}, i \in I\) the plaques of \(\mathcal{F}\) and, abusively, we will always identify the transversals \(T_i\) with their preimages \(\phi_i^{-1}([z_i] \times T_i)\) for some choice \(z_i\). Such a collection of embedded submanifolds transverse to the leaves and whose union meets each leaf is called a **complete system of transversals**.

We say that a closed foliated manifold \((M, \mathcal{F})\) is endowed with a **leafwise metric** if:

- each leaf \(L\) possesses a \(C^\infty\) Riemannian metric denoted by \(g_L\);
- the metric \(g_L\) varies continuously with \(L\) in the \(C^\infty\)-topology.

Since \(M\) is compact, a leafwise metric gives **uniformly bounded geometry** to the leaves of \(\mathcal{F}\): their injectivity radii are bounded uniformly from below, and their sectional curvatures are uniformly pinched.

**Holonomy.** Let \((M, \mathcal{F})\) be a closed foliated manifold. Fix a foliated atlas \(\mathcal{A} = (U_i, \phi_i)_{i \in I}\) of \(M\) and suppose it is a **good atlas** in the sense that:

1. if two plaques intersect each other, their intersection is connected;
2. if two charts \(U_i\) and \(U_j\) intersect each other, then \(U_i \cap U_j\) is included in a foliated chart: a plaque of \(U_i\) intersects at most one plaque of \(U_j\).

We can define a **complete transversal** \(\mathcal{T}\) as the union of all the \(T_i\). The maps \(\tau_{ij}\) generate a **pseudogroup** \(\mathcal{P}\) of local diffeomorphisms of \(\mathcal{T}\), called the **holonomy pseudogroup** of \(\mathcal{F}\). Recall that we chose to identify \(T_i\) and a local transversal \(\phi_i^{-1}([z_i] \times T_i)\), and to consider the points of \(\mathcal{T}\) as points of \(M\). Note that this pseudogroup is symmetric since we have \(\tau_{ij}^{-1} = \tau_{ji}\).

Also, we can define the **holonomy map along a path.** If \(c : [0, 1] \rightarrow M\) is a path tangent to a leaf, and \(T_i\) and \(T_j\) are submanifolds transverse to \(\mathcal{F}\) containing \(c(0)\) and \(c(1)\), there exist two neighbourhoods \(S_j \subset T_j\) and \(S_j \subset T_j\) of \(c(0)\) and \(c(1)\) respectively such that we can send every \(x \in S_j\) onto \(\tau_c(x) \in U_1\) by sliding along the leaves of \(\mathcal{F}\). More precisely, if we consider any chain of charts that cover \(c\), say \(U_{i_1}, ..., U_{i_n}\), then \(\tau_c\) is defined as the composition \(\tau_{i_n \circ i_{n-1} \circ ... \circ i_1}\). The germ at \(c(0)\) of \(\tau_c : S_j \rightarrow S_j\) does not depend on the choice of \(c\) nor does it on the choice of the chain of charts, but only on the homotopy class of \(c\).
2.2 Transverse measures and cocycles

Invariant measures. Let $(M, F)$ be a closed manifold endowed with a good foliated atlas. Then we have a dynamical system given by the action of the holonomy pseudogroup $P$ on the complete transversal $T$. We say that a finite Borel measure $\nu$ on $T$ is invariant by an element $\tau \in P$ if for any Borel set $A$ included in the domain of $\tau$, $\nu(\tau A) = \nu(A)$. A transverse invariant measure is a finite measure on $T$ that is invariant by the action of each element of $P$. In general, the existence of a transverse invariant measure is extremely rare.

In [BM], in order to prove the unique ergodicity of horocycle foliations, Bowen and Marcus introduce an equivalent notion of transverse measure that is maybe more adapted to dynamical systems theory. Let $(T_i)_{i \in I}$ be a complete system of transversals to the foliation. A transverse invariant measure is a family of finite nonnegative measures $(\nu_i)_{i \in I}$ satisfying:

1. $\nu_i(T_i) > 0$ for some $i \in I$;
2. if for $i, j \in I$ there is a holonomy map $h_{T_i \to T_j} : S_i \to S_j$ between two open sets of $T_i$ and $T_j$, then for any Borel set $A_i \subset S_i$ we have $\nu_i(A_i) = \nu_j(h_{T_i \to T_j}(A_i))$.

Totally invariant measures. When the foliation is endowed with a leafwise metric, one can always combine a transverse invariant measure with the volume of the leaves. By doing so we obtain a measure on $M$ called totally invariant. Totally invariant measures are harmonic measures in the sense of Garnett [Gar] (the local densities in the plaques are constant functions: in particular they are harmonic). In that sense, harmonic measures are generalizations of transverse invariant measures.

Quasi-invariant measures. Most foliations don’t possess transverse invariant measures. Thus, we are led to consider invariant measure classes on a complete transversal.

We say that a finite Borel measure $\nu$ on $T$ is quasi-invariant by the holonomy pseudogroup if for any $\tau \in P$ and any Borel set $A$ lying in the domain of $\tau$ such that $\nu(A) = 0$, we still have $\nu(\tau A) = 0$.

We associate to each transverse quasi-invariant measure the so called Radon-Nikodym cocycle, defined for a couple $(\tau, x) \in P \times T$ with belonging to the goal of $\tau$:

$$k(\tau, x) = \frac{d[\tau \ast \nu]}{d\nu}(x).$$

Ghys’ lemma The following lemma, due to Ghys (see [Gh]), will be useful all along this text. He stated it for families of transverse measures induced by harmonic measures. In his proof, that we recall below, he only needs the quasi-invariance of the families of measures.

Lemma 2.1 (Ghys). Let $(M, F)$ be a closed foliated manifold and let $T$ be a complete transversal. Let $\nu$ be a measure on $T$ quasi-invariant by the action of the holonomy pseudogroup $P$. Then there exists a Borel set $\mathcal{X}_0 \subset T$ which is full for $\nu$, and saturated by the action of the pseudogroup $P$, such that for any $x \in \mathcal{X}_0$ and any $\tau \in P$ that fixes $x$, we have:

$$\frac{d[\tau \ast \nu]}{d\nu}(x) = 1.$$

Proof. The Radon-Nikodym cocycle $d[\tau \ast \nu]/d\nu$, $\tau \in P$ allows us to define for any $x$ lying in the support of $\nu$ a morphism $\pi_1(L_x, x) \rightarrow (0, \infty)$: we want to prove that, almost surely, it is trivial.
Let \( \tau \in \mathcal{P} \). Consider the Borel set constituted of points \( x \in \mathcal{T} \) which are fixed by \( \tau \) and satisfy \( d[\tau \ast v]/dv(x) < 1 \). By definition, this set is fixed by \( \tau \), and its measure is contracted by \( \tau \): it has measure zero. By considering \( \tau^{-1} \), we can prove that the set of points \( x \) which are fixed by \( \tau \) and satisfy \( d[\tau \ast v]/dv(x) > 1 \) has also measure zero. It follows that the set of points \( x \) fixed by \( \tau \) and such that \( d[\tau \ast v]/dv(x) \neq 1 \) has zero measure for \( v \).

The pseudogroup \( \mathcal{P} \) is finitely generated: in particular, it is countable. Using the \( \sigma \)-additivity of \( \nu \), we see that the set of points which are fixed by an element of the pseudogroup \( \mathcal{P} \) with Jacobian \( \neq 1 \) is of measure zero for \( \nu \). Denote by \( \mathcal{X}_0 \) the complement of this set. We have to show that it is saturated by the action of the pseudogroup.

The groups of germs of holonomy transformations which fix two points of the same leaf are conjugated. Thus if \( x \in \mathcal{X}_0 \), we see that for any \( y \in L_x \), and \( \tau \in \mathcal{P} \) that fixes \( y \), we also have \( d[\tau \ast v]/dv(y) = 1 \); \( \mathcal{X}_0 \) is in particular saturated by \( \mathcal{P} \).

We can conclude the proof of the lemma.

We have the following useful interpretation of Ghys’ lemma. For \( \nu \)-almost every point \( x \in \mathcal{T} \), and for any \( y \in \mathcal{P}(x) \), we have, if \( \tau_1, \tau_2 \in \mathcal{P} \) contain \( x \) in their domains and satisfy \( \tau_1(x) = \tau_2(x) \):

\[
\frac{d[\tau_1^{-1} \ast v]}{dv}(x) = \frac{d[\tau_2^{-1} \ast v]}{dv}(x).
\]

In other terms, the Radon-Nikodym cocycle is a cocycle on the orbit of almost every point of \( \mathcal{T} \) and doesn’t depend on the path chosen between two points of the same orbit.

**Lebesgue disintegration and cocycles.** Assume now that \((M, \mathcal{F})\) is endowed with a leafwise metric. We say that a probability measure \( m \) on \( M \) has **Lebesgue disintegration** if its conditional measures in the plaques of \( \mathcal{F} \) are equivalent to Lebesgue.

We say that the disintegration of a measure \( m \) in the leaves of \( \mathcal{F} \) is singular with respect to Lebesgue if the conditional measures of \( m \) in the plaques of \( \mathcal{F} \) are singular with respect to Lebesgue.

To such a measure \( m \) can be associated a family of transverse invariant measures as well as a Radon-Nikodym cocycle. Indeed, let \( \mathcal{A} = (U_i, \phi_i)_{i \in I} \) be a good foliated atlas. Consider a chart of the form \( U_i = \bigcup_{x \in T_i} P_i(x) \) satisfying \( m(U_i) > 0 \). In restriction to \( U_i \), \( m \) reads as follows:

\[
m_{|U_i} = (h_i(z, x) \text{Leb}(z)) \nu_i(x).
\]

Then the family of transverse measures \( (\nu_i)_{i \in I} \) is quasi-invariant by the holonomy pseudogroup: indeed, if \( m(U_i \cap U_j) > 0 \), by evaluation on the intersection, we find that for, \( \nu_i \)-almost all \( x \in T_i \):

\[
\frac{d[\tau_i^{-1} \ast v_j]}{dv_i}(x) = \frac{h_i(z, x)}{h_j(z, \tau_{ij}(x))}.
\]

**Extension of local densities.** The proof of the next lemma is a straightforward application of Ghys’ lemma: we gave a proof of it in [Ali]. It states that when a measure \( m \) has Lebesgue disintegration, the local density defined on almost every plaque can be extended to the whole leaf.

**Lemma 2.2.** Let \((M, \mathcal{F})\) be a closed foliated manifold endowed with a leafwise metric. Let \( m \) be a measure for \( \mathcal{F} \). Then there is a full and saturated Borel set \( \mathcal{X} \subset M \) such that for any \( y \in \mathcal{X} \), if \( y \in P_{i_0}(x) \subset U_{i_0} \), for some \( i_0 \in I \), and \( x \in T_{i_0} \), the following formula defines a positive function of \( z \in L_y \):

\[
H_x(z) = \frac{d[\tau_c^{-1} \ast v_i]}{dv_i}(x)h_i(z, \tau_c(x)),
\]

where \( z \in U_i \), and \( c \) is any path joining \( y \) and \( z \).
3 Foliated hyperbolicity

3.1 Definitions

Leafwise hyperbolic flows. The concept of foliated hyperbolicity has been introduced in a recent work by Bonatti, Gómez-Mont and Martínez [BGM]. Their goal is to apply Pesin’s theory for the geodesic flow tangent to the leaves of a transversely conformal foliation with hyperbolic leaves.

Let \((M, \mathcal{F})\) be a closed foliated manifold endowed with a foliated metric. One says that a vector field \(X\) is leafwise hyperbolic if it satisfies the following:

- \(X\) is tangent to the leaves of \(\mathcal{F}\);
- \(X\) is \(C^2\) (one could ask for \(C^1+\eta\) regularity), when restricted to the leaves: denote by \(\Phi_t\) the associated flow;
- \(X\) varies continuously in the \(C^2\)-topology with the transverse parameter;
- there is a continuous splitting of the tangent space of the foliation \(T\mathcal{F} = E^s \oplus E^u \oplus E^c\), such that \(E^c = \mathbb{R}X\), and such that the two bundles \(E^s\) and \(E^u\), respectively called stable and unstable bundles, are uniformly contracted and expanded by the flow, i.e. there are uniform constants \(C_s, C_u > 0\), and \(\chi_u > 0, \chi_s < 0\), such that:

\[
\begin{align*}
||D\Phi_t(v_s)|| &\leq C_s e^{\chi_s t} ||v_s|| & \text{if } v_s \in E^s \text{ and } t > 0 \\
||D\Phi_{-t}(v_u)|| &\leq C_u e^{-\chi_u t} ||v_u|| & \text{if } v_u \in E^u \text{ and } t > 0
\end{align*}
\]

(3.4)

We will respectively denote by \(E^c^u\) and \(E^c^s\) the bundles \(E^u \oplus E^c\) and \(E^s \oplus E^c\). We call them center unstable and center stable bundles. Usually when \(\star = u, s, c, u, s\), and \(t \in \mathbb{R}\), we will denote by \(\text{Jac}^\star \Phi_t\) the restriction \(|\det(D\Phi_t)|_{E^\star}\).

It is possible to prove that the bundles \(E^s\) and \(E^u\) are uniformly Hölder continuous inside the leaves by an adaptation of the proof in the classical case (see for example the appendix of Brin in the book [Ba]). Thus one can prove that for any \(t \in \mathbb{R}\) there exists uniform Hölder constants for logarithms of the Jacobians \(\text{Jac}^\star \Phi_t\).

The notion of foliated hyperbolicity has the flavor of partial hyperbolicity. The transverse direction plays the role of the central direction. Nevertheless, there is a major difference: the dynamics of the flow in the transverse direction has no reason to be dominated by the contraction and the expansion in the stable and unstable bundles.

The foliated geodesic flow. Here we assume that all the leaves of \(\mathcal{F}\) are negatively curved. Since the geometry of the leaves is uniformly bounded we have uniform bounds on the sectional curvature \(-b^2 \leq -a^2 < 0\).

The unit tangent bundle of the foliation \(T^1 \mathcal{F}\) consists of unit vectors tangent to the leaves of \(\mathcal{F}\). It is a closed manifold endowed with a foliation \(\mathcal{F}\) whose leaves are the unit tangent bundles of that of \(\mathcal{F}\). One obtains an atlas \(\mathcal{A}\) of \(\mathcal{F}\) by pulling back a foliated atlas \(\mathcal{A}\) for \(\mathcal{F}\) by the unit tangent bundle. The charts are then trivially subfoliated by spheres: we deduce that the holonomy pseudogroup of \(\mathcal{F}\) and \(\mathcal{F}\) are the same.

In particular, \(\mathcal{F}\) admits a transverse invariant measure if and only if \(\mathcal{F}\) does.

We endow \(T^1 \mathcal{F}\) with the foliated Sasaki metric: this is a leafwise metric. There are two continuous bundles \(E^s\) and \(E^u\) on \(T^1 \mathcal{F}\) such that:
BGM

states that the leaves of will be use to denote a local manifold “with sufficiently small radius”. The following proposition

ball in center stable the F

These manifolds form two subfoliations of hyperbolic flow on M. For any x ∈ s continuous.

These local manifolds vary continuously with the point x in the C3.2 Stable and unstable manifolds

Invariant foliations and local product structure. The usual method of the graph transform (see [HPS]) can be adapted in order to prove that stable and unstable bundles are uniquely integrable (see [BGM]).

Theorem 3.1. Let (M, F) be a foliated manifold endowed with a leafwise metric. Let Φt be a leafwise hyperbolic flow on M. For any x ∈ M, there exists a pair of C2 open discs embedded in the leaf of x which contain x and are denoted by Wsloc(x) and Wu loc(x) such that:

1. Ts Wsloc(x) = Es(x) for * = s, u;
2. for any t > 0, Φt(Wsloc(x)) ⊂ Wsloc(Φt(x)) and Φt(Wu loc(x)) ⊂ Wu loc(Φt(x));
3. there are immersed global manifolds which subfoliate F defined by:

\[ W^s(x) = \bigcup_{t>0} \Phi_{-t}(W^s_{loc}(\Phi_t(x))) \]
\[ W^u(x) = \bigcup_{t>0} \Phi_t(W^u_{loc}(\Phi_{-t}(x))). \]

The sets Wsloc(x) and Wu loc(x) are respectively called the local stable and unstable manifolds of x. These local manifolds vary continuously with the point x in the C2 topology. The invariant global manifolds are characterized by the following dynamical properties:

\[ W^s(x) = \{ y \in L_x; \lim_{t \to -\infty} \text{dist}_F(\Phi_t(x), \Phi_t(y)) = 0 \} \]
\[ W^u(x) = \{ y \in L_x; \lim_{t \to -\infty} \text{dist}_F(\Phi_{-t}(x), \Phi_{-t}(y)) = 0 \}. \]

Note in particular that the global invariant manifolds are independent of the choice of the local manifolds.

Invariant foliations and local product structure. The collections (Ws(x))x∈M and (Wu(x))x∈M form what we call stable and unstable foliations. We denote them by Ws and Wu.

The center stable and center unstable manifolds of a point x will be denoted by Wcs(x) and Wcu(x). They are by definition the saturations of Ws(x) and Wu(x) in the direction of the flow. These manifolds form two subfoliations of F denoted respectively by Wcs and Wcu that we call the center stable and center unstable foliations.

If δ > 0 is small enough, we introduce the notation Wδs,loc(x) for the connected component of the ball in Lx centered at x and with radius δ with Wδs,loc(x) (where * = s, u, c, s, cu). The notation Wδs,loc(x) will be use to denote a local manifold “with sufficiently small radius”. The following proposition states that the leaves of F possess a structure of local product which is uniformly Hölder continuous.
Proposition 3.2. There are two numbers $0 < \delta < \gamma < 2\delta$ such that for any $x$ and $y$ that belong to the same leaf $L$ and are at distance at most $\gamma$, the local center stable manifold of $x$ and unstable manifold of $y$ of size $\delta$ intersect transversally in a unique point denoted by $[x, y]$:

$$W_{\delta}^{cs}(x) \cap W_{\delta}^{u}(y) = [x, y].$$

Furthermore, the coordinate $[., .]$ depends Hölder continuously on $x$ and $y$ with uniform constants.

We will use the following convenient notation. If $x \in M$ and $A_\star \subset W_\delta^\star$, we denote:

$$[A_u, A_{cs}] = \{[x_u, x_{cs}] ; x_u \in A_u, x_{cs} \in A_{cs}\}.$$

Moreover, the rectangle $[W_{\delta}^{u}(x), W_{\delta}^{cs}(x)]$ will be denoted by $\mathcal{R}_x$.

3.3 Absolute continuity and local product structure of the volume in the leaves

Stable and unstable holonomies. When $x, y$ lie in a same leaf, say, of $W_u$, we will use notation for the unstable holonomy between the respective center stable manifolds:

$$\text{hol}_{x \rightarrow y}^u : W_{loc}^{cs}(x) \rightarrow W_{loc}^{cs}(y).$$

Analogous definitions are given for the holonomy maps $\text{hol}_{x \rightarrow y}^\star, \star = s, cu, cs$.

Absolute continuity. Recall that a foliation is said to be absolutely continuous if all the holonomy maps between smooth transversals preserve the Lebesgue class. The usual proof of the absolute continuity of the invariant foliations of a $C^2$ Anosov flow (see the book of Mañé [M]) can be adapted to our foliated context.

Indeed, even if the leaves are not compact, they are immersed in a compact manifold, and the flow varies continuously with the leaf in the $C^2$-topology. We can verify that the distortion controls usually needed in order to treat the classical case (see [M]) are also valid in this foliated context. The precise verification can be found in an appendix of the author’s thesis [Al3]. We state the following theorem in the context of unstable holonomy maps. The case of stable ones follows by reversing the time.

Theorem 3.3. Let $\Phi_t$ be a leafwise hyperbolic flow on a closed foliated manifold $(M, \mathcal{F})$. Then the unstable holonomy maps are absolutely continuous, and their Jacobians satisfy the following.

1. For all $x, y$ lying in the same unstable manifold, one has:

$$\text{Jac}_{x \rightarrow y}^u(x) = \lim_{T \rightarrow \infty} \frac{\text{Jac}_{x_{cs}^\Phi_{-T}}^{cs}(x)}{\text{Jac}_{y_{cs}^\Phi_{-T}}^{cs}(y)}.$$

2. There exists uniform positive constants $C_0, \alpha_0 > 0$ such that for $x, x'$ lying on the same center stable manifold and $y, y'$ lying respectively in $W_u(x), W_u(x')$ such that:

$$\text{dist}_{cs}(x, x'), \text{dist}_u(x, y), \text{dist}_u(x', y') < 1,$$

we have:

$$\left| \log \text{Jac}_{x \rightarrow y}^u(x) - \log \text{Jac}_{x' \rightarrow y'}^u(x') \right| \leq C_0 \text{dist}_{cs}(x, x')^{\alpha_0}.$$
**Local product structure of the volume.** Let $x \in M$. The induced volume on $W^u(x)$ (resp. $W^{cs}(x)$) shall be denoted by $\text{Leb}^u_x$ (resp. $\text{Leb}^{cs}_x$). We can define a finite measure $m_x$ on the rectangle $\mathcal{R}_x$ by requiring that for any Borel sets $A^u \subset W^u(x)$ and $A^{cs} \subset W^{cs}(x)$ the following relation holds:

$$m_x(A^u, A^{cs}) = \text{Leb}^u_x(A^u)\text{Leb}^{cs}_x(A^{cs}).$$

This amounts to the same thing as defining $m_x$ on the rectangle $\mathcal{R}_x$:

- by integration against $\text{Leb}^{cs}_x$ of the measures $\text{hol}^{cs}_{x \to z} \ast \text{Leb}^u_x$,
- by integration against $\text{Leb}^u_x$ of the measures $\text{hol}^u_{x \to y} \ast \text{Leb}^{cs}_x$.

By absolute continuity of the measures, we find the following equivalent formulas, which hold for any Borel subset $A \subset \mathcal{R}_x$:

$$m_x(A) = \int_{W^{cs}_x(x)} \left[ \int_{A \cap W^u(z)} \text{Jac hol}^{cs}_{z \to x}(\zeta) \, d\text{Leb}^u_x(\zeta) \right] \, d\text{Leb}^{cs}_x(z) \quad (3.5)$$

$$= \int_{W^u_x(x)} \left[ \int_{A \cap W^{cs}(z)} \text{Jac hol}^u_{y \to z}(\zeta) \, d\text{Leb}^{cs}_y(\zeta) \right] \, d\text{Leb}^u_y(y). \quad (3.6)$$

The function which associates to $x \in M$ the angle between $E^u(x)$ and $E^{cs}(x)$, that we will denote by $\alpha(x)$, is, like the bundles themselves, Hölder continuous along the leaves.

The goal of this paragraph is to prove the following proposition, which is a direct consequence of the absolute continuity of the invariant foliations:

**Proposition 3.4.** The two measures $\text{Leb}_{L_x}$ and $m_x$ are equivalent inside $\mathcal{R}_x$ with a Radon-Nikodym derivative:

$$\frac{d m_x}{d \text{Leb}_{L_x}}(y) = \frac{1}{\sin \alpha(y)} \text{Jac hol}^u_{y \to x}(y) \text{Jac hol}^{cs}_{y \to x}(y).$$

The proof of this proposition follows from two lemmas.

**Lemma 3.5.** The following quantity exists for any $x$:

$$\frac{d m_x}{d \text{Leb}_{L_x}}(x) = \frac{1}{\sin \alpha(x)}.$$

**Proof.** We denote by $B_{\delta}(x, r)$ the ball inside the leaf $L_x$ with center $x$ and radius $r$. We write:

$$m_x(B_{\delta}(x, r)) = \int_{W^{cs}_x(x)} \left[ \int_{B_{\delta}(x, r) \cap W^u(z)} \text{Jac hol}^{cs}_{z \to x}(\zeta) \, d\text{Leb}^u_x(\zeta) \right] \, d\text{Leb}^{cs}_x(z).$$

Now, when $r$ tends to zero, we know that:

- the Jacobian of the exponential map $\exp_x : T_x L_x \to L_x$ tends to 1: we can choose to work in the Euclidian space $T_x L_x$ with almost no volume distortion;
- by continuity of the Jacobian of the center stable holonomy maps, $\text{Jac hol}^{cs}_{z \to x}$ is uniformly close to 1 when $z \in W^{cs}_x(x)$;
- the Jacobian of $(\exp_x)^{-1} : W^{cs}_x(x) \to E^{cs}(x)$ tends to 1 thus we can assume that the integral above is taken on a Euclidian ball of radius $r$ inside $E^{cs}(x)$;
• by continuity of the unstable foliation, the Jacobians of all maps \((\exp^{\mu})^{-1} : W^u_r(z) \to E^u(z)\)
are all close to 1, and the angle between spaces \(E^u(z)\) and \(E^{cs}(x)\) is close to that between spaces \(E^u(x)\) and \(E^{cs}(x)\), that is \(\alpha(x)\): with little distortion, we may assume that all these slices are parallel.

Hence, consider in \(T_xL_x\), a Euclidian structure obtained by requiring that \(E^u\) and \(E^s\) are orthogonal. The corresponding volume has density \(1/\sin \alpha(x)\) with respect to the usual volume.

By what precedes and by Fubini’s theorem, the previous integral is equivalent when \(r\) tends to zero, to the mass of the ball for this modified volume.

We then have the following limit:
\[
\lim_{r \to 0} \frac{m_x(B_{\mathbb{R}}(x, r))}{\operatorname{Leb}_{L_x}(B_{\mathbb{R}}(x, r))} = \frac{1}{\sin \alpha(x)}.
\]
\[
\quad
\]

Lemma 3.6. Assume that \(x, y\) belong to the same leaf, and that \(\mathcal{R}_x\) and \(\mathcal{R}_y\) intersect. Then \(m_x\) and \(m_y\) are equivalent on \(\mathcal{R}_x \cap \mathcal{R}_y\). More precisely, if \(\xi \in \mathcal{R}_x \cap \mathcal{R}_y\), and if \(\xi^u = [y, \xi] \in W^u_{loc}(y)\) and \(\xi^{cs} = [\zeta, y] \in W^{cs}_{loc}(y)\), then:
\[
\frac{dm_{x}}{dm_{y}}(\zeta) = \frac{\text{Jac} \text{hol}_{\mathcal{R}^u_{y-x}}(\xi^u)}{\text{Jac} \text{hol}_{\mathcal{R}^{cs}_{y-x}}(\xi^{cs})}.\]
\[
\quad
\]

Proof. Suppose that \(x, y\) satisfy the hypothesis of the proposition. Let \(\xi \in \mathcal{R}_x \cap \mathcal{R}_y\), \(\xi^u = [y, \xi]\), \(\xi^{cs} = [\zeta, y]\) and consider two small discs \(D^u \subset W^u_{\delta}(y)\) and \(D^{cs} \subset W^{cs}_{\delta}(y)\) with same (small) radius and centered respectively at \(\xi^u\) and \(\xi^{cs}\).

The rectangle \(D = [D^u, D^{cs}]\) is an open set containing \(\zeta\). We have the following equality:
\[
\frac{m_x(D)}{m_y(D)} = \frac{\operatorname{Leb}_x^u(\text{hol}^u_{y-x}(D^u)) \operatorname{Leb}_x^{cs}(\text{hol}^{cs}_{y-x}(D^{cs}))}{\operatorname{Leb}_y^u(D^u) \operatorname{Leb}_y^{cs}(D^{cs})}.
\]

When the common radius of the discs \(D^u\) and \(D^{cs}\) goes to zero:
\[
\frac{\operatorname{Leb}_x^u(\text{hol}^u_{y-x}(D^u)) \operatorname{Leb}_x^{cs}(\text{hol}^{cs}_{y-x}(D^{cs}))}{\operatorname{Leb}_y^u(D^u) \operatorname{Leb}_y^{cs}(D^{cs})} \to \frac{\text{Jac} \text{hol}_{y-x}^u(\xi^u) \text{Jac} \text{hol}_{y-x}^{cs}(\xi^{cs})}{\text{Jac} \text{hol}_{\mathcal{R}^u_{y-x}}(\xi^u)} \text{Jac} \text{hol}_{\mathcal{R}^{cs}_{y-x}}(\xi^{cs}).
\]

Moreover, the angle between unstable and center stable holonomy maps is uniformly bounded, and the unstable and center stable holonomy maps are uniformly Hölder continuous. Thus as the common radius of the discs \(D^u\) and \(D^{cs}\) tends to zero, the open set \(D\) shrinks nicely between to Riemannian balls such that the quotients of their radii are uniformly bounded. Hence, we can use the Lebesgue density theorem and we get:
\[
\frac{m_x(D)}{m_y(D)} \to \frac{dm_{x}(\zeta)}{dm_{y}(\zeta)},
\]
concluding the proof of the lemma. □

Proof of proposition 3.4. Now, we can conclude the proof of proposition 3.4. If \(y \in \mathcal{R}_x\), then:
\[
\frac{m_x(B_{\mathbb{R}}(y, r))}{\operatorname{Leb}_{L_y}(B_{\mathbb{R}}(y, r))} = \frac{m_y(B_{\mathbb{R}}(y, r))}{\operatorname{Leb}_{L_y}(B_{\mathbb{R}}(y, r))} \frac{m_x(B_{\mathbb{R}}(y, r))}{m_y(B_{\mathbb{R}}(y, r))}.
\]

As \(r\) tends to 0, the first factor converges to \(\sin \alpha(y)^{-1}\) by the first lemma, and the second one converges to \(\text{Jac} \text{hol}_{y-x}^u(y)\text{Jac} \text{hol}_{y-x}^{cs}(y)\) (since here \(\zeta = y\), we have \(\xi^u = \xi^{cs} = y\)). The proof of the proposition is now over. □
4 Gibbs su-states and transverse invariant measures

4.1 Gibbs u-states for leafwise hyperbolic flows

Existence and properties. In [BGM], Bonatti, Gómez-Mont and Martínez define and study Gibbs u-states for leafwise hyperbolic flows and, using the results of Deroin and Kleptsyn [DK], and a previous work of Martínez, Bakhtin-Martínez, [Ma, BMa], they obtain uniqueness results in the case of the foliated geodesic flow of a transversally conformal foliation by hyperbolic Riemann surfaces.

**Definition 4.1.** Let $\Phi_t$ be a leafwise hyperbolic flow of a closed foliated manifold $(M, \mathcal{F})$. A Gibbs u-state for $\Phi_t$ is a probability measure on $M$ which is invariant by the flow, and that has Lebesgue disintegration in the unstable leaves.

Similarly, a Gibbs s-state for $\Phi_t$ is a Gibbs u-state for $\Phi_{-t}$: it has Lebesgue disintegration in the stable leaves.

Finally, a Gibbs su-state for $\Phi_t$ is a probability measure which is both a Gibbs u-state and a Gibbs s-state for $\Phi_t$.

The existence of Gibbs u-states relies on a classical distortion control. Recall that the unstable Jacobian of the flow $\text{Jac}^u \Phi_t$ is uniformly log-Hölder. This allows us to state the following existence theorem, whose proof is a simple adaptation of section 11.2.2 of [BDV].

**Theorem 4.2.** Let $\Phi_t$ be a leafwise hyperbolic flow on a closed foliated manifold $(M, \mathcal{F})$. Then:

1. for every Borel set $D \subset W^u_{\text{loc}}(x)$ with positive Lebesgue measure, any accumulation point of the following family of measures, indexed by $T \in (0, \infty)$, is a Gibbs u-state:

   $$\mu_T = \frac{1}{T} \int_0^T \Phi_t * \left( \frac{\text{Leb}^u_{|D^u}}{\text{Leb}^u(D^u)} \right) dt,$$

   and its local densities, denoted by $\psi^u_y$, are uniformly log-bounded in the unstable plaques and satisfy, for $z_1, z_2 \in W^u_{\text{loc}}(y)$:

   $$\frac{\psi^u_y(z_2)}{\psi^u_y(z_1)} = \lim_{t \to \infty} \frac{\text{Jac}^u \Phi_{-t}(z_2)}{\text{Jac}^u \Phi_{-t}(z_1)}$$ (4.7)

2. all ergodic components of a Gibbs u-states are Gibbs u-states with local densities in the unstable plaques that are uniformly log-bounded and satisfy relation (4.7);

3. each Gibbs u-state for $\Phi_t$ is a measure whose local densities in the unstable plaques are uniformly log-bounded and satisfy the relation (4.7).

**Remark 1.** We can always assume $\psi^u_y(y) = 1$. More precisely, consider a Gibbs u-state for $\Phi_t$, as well as a foliated chart of the form:

$$U = \bigcup_{x \in T} \mathcal{R}_x.$$

We will denote by $\mu_x$ the conditional measure of $\mu$ on $\mathcal{R}_x$. By uniqueness in Rokhlin’s disintegration theorem, and because the local densities are uniformly log-bounded, there exists a unique measure $\nu^c_{cs}(x)$ on $W^c_{\text{loc}}(x)$ which has a uniformly log-bounded derivative with respect to the projection of $\mu_x$ on $W^c_{\text{loc}}(x)$, and such that $\mu_x$ disintegrates as follows in $\mathcal{R}_x$:

$$\mu_x = \left( \psi^u_y \text{Leb}^u_y \right) \nu^c_{cs}(y),$$ (4.8)
with \( \psi_y(y) = 1 \) or if one prefers, for \( \zeta \in W_{\text{loc}}^u(y) \):

\[
\psi_y^u(\zeta) = \lim_{t \to -\infty} \frac{\text{Jac}^u \Phi_{-t}(\zeta)}{\text{Jac}^u \Phi_{-t}(y)}.
\]

Remark 2. A measure \( \mu \) invariant by the flow induces the \( dt \) element on flow lines. Hence, any Gibbs u-state for \( \Phi_t \) is a Gibbs cu-state: it has Lebesgue disintegration in the leaves of \( W^{cu} \). Then there exists densities \( \psi^u_y \), which coincide with \( \psi^u_y \) in the local unstable manifold of \( y \) (in particular, \( \psi^u_y(y) = 1 \)), such that \( \mu_x \) disintegrates as follows in the local center unstable manifolds:

\[
\mu_x = \left( \psi^u_y \text{Leb}^u_y \right) \nu^c_x(y).
\]

These densities are then determined by the following relation: if \( \zeta \in W_{\text{loc}}^{cu}(y) \), and if \( \tau \) is such that \( \Phi_{-\tau}(\zeta) \in W_{\text{loc}}^u(y) \), then:

\[
\psi_y^{cu}(\zeta) = \lim_{t \to -\infty} \frac{\text{Jac}^u \Phi_{-t-\tau}(\zeta)}{\text{Jac}^u \Phi_{-t}(y)}.
\]

4.2 A sufficient condition for existence of transverse invariant measures

The main goal of this paragraph is to prove the following theorem:

**Theorem 4.3.** Let \( \Phi_t \) be a leafwise hyperbolic flow of a closed foliated manifold \((M, \mathcal{F})\), which preserves the Lebesgue measure inside the leaves. Assume that \( \Phi_t \) admits a Gibbs su-state \( \mu \). Then \( \mathcal{F} \) possesses a transverse measure invariant by holonomy.

Consider a good foliated atlas whose charts are of the following form: \( U = \bigcup_{x \in T} \mathcal{R}_x \).

**Identification of the densities.** In what follows, we assume that \( \Phi_t \) is a leafwise hyperbolic flow admitting a Gibbs su-state \( \mu \). Denote by \( (\mu_x)_{x \in T} \) the family of conditional measures of \( \mu \) on rectangles \( \mathcal{R}_x \). For \( x \in T \) there exists a measure \( \nu^c_x \) on \( W_{\text{loc}}^{cu}(x) \), as well as a measure \( \nu^u_x \) on \( W_{\text{loc}}^u(x) \) such that \( \mu_x \) disintegrates in \( \mathcal{R}_x \) as follows:

\[
\mu_x = \left( \psi^c_x \text{Leb}^c_x \right) \nu^c_x(z) = \left( \psi^u_x \text{Leb}^u_x \right) \nu^u_x(y).
\]

The following lemma is an immediate application of the absolute continuity of invariant foliations.

**Lemma 4.4.** The measures \( \nu^c_x \) and \( \nu^u_x \) are respectively equivalent to \( \text{Leb}^c_x \) and \( \text{Leb}^u_x \).

**Proof.** We only prove the fact that \( \nu^u_x \) is equivalent to \( \text{Leb}^u_x \). The other assertion follows from the same argument.

First, by remark 1 above, the projection of \( \mu_x \) on \( W_{\text{loc}}^u(x) \) along the center stable foliation is equivalent to \( \nu^u_x \). It is then enough to prove the lemma for this projection.

Because \( \mu \) is a Gibbs u-state, it has Lebesgue disintegration in unstable leaves. Therefore, by absolute continuity of the holonomy maps \( \text{hol}^c_{z-x} \), \( z \in W_{\text{loc}}^{cu}(x) \), the projections along the center stable foliation on \( W_{\text{loc}}^u(x) \) of the conditional measures of \( \mu_x \) are equivalent to Lebesgue.

We may conclude.
As a consequence there exist two positive functions $f : W^u_{loc}(x) \to (0, \infty)$ and $g : W^c_{cs}(x) \to (0, \infty)$ such that the disintegrations of $\mu_x$ in the local unstable and center stable manifolds read as follows:

$$
\mu_x = \left( \psi^u_y \, d\text{Leb}^u_x \right) g(z) \, d\text{Leb}^{cs}_y(z)
$$

$$
= \left( \psi^c_y \, d\text{Leb}^{cs}_y \right) f(y) \, d\text{Leb}^u_x(y).
$$

**Lemma 4.5.** The measure $\mu_x$ is equivalent to $m_x$ in $\mathcal{R}_x$. More precisely, if $\zeta \in \mathcal{R}_x$, $y = [\zeta, x]$, $z = [x, \zeta]$, and $F$ denotes the Radon-Nikodym derivative $d\mu_x / d\mu_y$, then:

$$
\frac{F(\zeta)}{F(x)} = \frac{\psi^c_{\zeta} \psi^u_x(\zeta) \text{Jac hol}_{\zeta-x}^c(z)}{\text{Jac hol}_{\zeta-x}^u(\zeta)}
$$

(4.9)

$$
= \frac{\psi^c_y \psi^u_x(\zeta) \text{Jac hol}_{y-x}^c(\zeta)}{\text{Jac hol}_{y-x}^u(\zeta)}
$$

(4.10)

**Proof.** Recall that the disintegrations of $m_x$ in the local unstable and in the center stable manifolds are respectively given by the relations (3.5) and (3.6). Thus, we have for all $\zeta \in \mathcal{R}_x$, if $y = [\zeta, x]$ and $z = [x, \zeta]$,

$$
\frac{d\mu_x}{dm_y}(\zeta) = \frac{\psi^u_x(\zeta)}{\text{Jac hol}_{x-y}^u(\zeta)} g(z)
$$

(4.11)

$$
= \frac{\psi^c_y(\zeta)}{\text{Jac hol}_{x-y}^c(\zeta)} f(y).
$$

(4.12)

In particular it comes that:

$$
g(x) = f(x) = \frac{d\mu_x}{dm_y}(x).
$$

By separation of variables it comes that:

$$
\frac{g(z)}{f(y)} = \frac{\psi^c_x(\zeta) \text{Jac hol}_{z-x}^c(\zeta)}{\psi^u_x(\zeta) \text{Jac hol}_{z-y}^u(\zeta)}
$$

and this relation holds for all $\zeta \in \mathcal{R}_x$. If we choose $\zeta \in W^c_{cs}(x)$, then we have $z = 0$ and $y = x$. By replacing in the previous equality we obtain:

$$
\frac{g(z)}{f(x)} = \psi^c_x(\zeta) \text{Jac hol}_{z-x}^c(z).
$$

We obtain the relation (4.9) for the normalized density by injecting the last equality in (4.11).

The other equality can be treated similarly.

**Theorem 4.6.** The measure $\mu_x$ is equivalent to the Lebesgue measure. Moreover, if $G = d\mu_x / d\text{Leb}_1$, then for all $\zeta \in \mathcal{R}_x$ we have, if $y = [\zeta, x]$, $z = [x, \zeta]$, and $\tau_1, \tau_2 \in \mathbb{R}$ satisfy $\Phi_{\tau_1}(y) \in W^s_{loc}(\zeta)$, and $\Phi_{\tau_2}(z) \in W^s_{loc}(x)$:

$$
\frac{G(\zeta)}{G(x)} = \lim_{t \to -\infty} \frac{\text{Jac } \Phi_{-\tau}(y)}{\text{Jac } \Phi_{-\tau}(x)} \lim_{t \to \infty} \frac{\text{Jac } \Phi_{\tau}(\zeta)}{\text{Jac } \Phi_{\tau}(y)}
$$

(4.13)

$$
= \lim_{t \to -\infty} \frac{\text{Jac } \Phi_{-\tau}(z)}{\text{Jac } \Phi_{-\tau}(x)} \lim_{t \to \infty} \frac{\text{Jac } \Phi_{\tau}(\zeta)}{\text{Jac } \Phi_{\tau}(z)}.
$$

(4.14)
**Proof.** Lebesgue-almost surely, we have $d\mu_x/d\text{Leb} = (d\mu_x/dm_x) \times (d\text{m}_y/d\text{Leb})$. Hence, we obtain the density $G$ by multiplying the two densities given by the previous lemma, as well as by the proposition 3.4. The formulas we get are
\[
\frac{G(\zeta)}{G(x)} = \frac{\sin \alpha(x)}{\sin \alpha(\zeta)} \psi^u_x(y) \psi^{cs}_y(\zeta) \text{Jac}^{cs}_{z=x}(\zeta) \text{Jac}^u_{y-x}(y) \]
\[
= \frac{\sin \alpha(x)}{\sin \alpha(\zeta)} \psi^{cs}_x(z) \psi^{cs}_z(\zeta) \text{Jac}^{cs}_{y-x}(\zeta) \text{Jac}^{cs}_{z=x}(z),
\]
where $\zeta \in \mathcal{R}_x$, and $y = [\zeta, x]$, $z = [x, \zeta]$, and $\alpha$ is the angle between the bundles $E^u$ and $E^{cs}$.

But we have precise expressions of local densities of Gibbs states as well as that of the Jacobians of the holonomy maps. For example:
\[
\psi^u_y(y) = \lim_{t \to \infty} \frac{\text{Jac}^u_\Phi_{t-y}(y)}{\text{Jac}^u_\Phi_{t-x}(x)},
\]
and:
\[
\text{Jac}^{cs}_{y-x}(y) = \lim_{t \to \infty} \frac{\text{Jac}^{cs}_{\Phi_{t-y}(y)}}{\text{Jac}^{cs}_{\Phi_{t-x}(x)}}.
\]

By placing ourselves in a basis formed by vectors of $E^u$ and $E^{cs}$ we see that for all $y$ and all $t > 0$ we have:
\[
\text{Jac}_{\Phi_{t-y}(y)} = \frac{\sin \alpha(\Phi_{t-y}(y))}{\sin \alpha(y)} \text{Jac}^u_\Phi_{t-y}(y) \text{Jac}^{cs}_{\Phi_{t-y}(y)}.
\]

But $\alpha$ is uniformly log-Hölder continuous in the leaves of $\mathcal{F}$ because it is uniformly bounded and uniformly Hölder in the leaves. Hence when $x$ and $y$ lie in the same unstable manifold we have that $\lim_{t \to -\infty} \sin \alpha(\Phi_{t-y}(y))/\sin \alpha(\Phi_{t-x}(x)) = 1$. We deduce that for $x$ and $y$ lying on the same local unstable manifold:
\[
\lim_{t \to \infty} \frac{\text{Jac}_{\Phi_{t-y}(y)}}{\text{Jac}_{\Phi_{t-x}(x)}} = \frac{\sin \alpha(x)}{\sin \alpha(y)} \psi^u_x(y) \text{Jac}^u_{y-x}(y).
\]

Similarly we prove that if $y$ lies on the same local center stable manifold as $\zeta$ and if $\tau_1$ satisfies $\Phi_{\tau_1}(y) \in W^{cs}_{loc}(\zeta)$, then the following holds true:
\[
\lim_{t \to \infty} \frac{\text{Jac}_{\Phi_{t+\tau_1}(y)}}{\text{Jac}_{\Phi_{t-y}(y)}} = \frac{\sin \alpha(y)}{\sin \alpha(\zeta)} \psi^{cs}_y(\zeta) \text{Jac}^{cs}_{z=x}(\zeta).
\]

By multiplication of these limits, we get the formula (4.13).

The other formula follows from a symmetric argument. \qed

**End of the proof of theorem 4.3.** Assume now that the leafwise hyperbolic flow preserves the volume inside the leaves, and possesses a Gibbs su-state $\mu$. Then the conditional measures in the plaques of $\mathcal{F}$ have a constant density with respect to the volume: indeed, if $G$ is the density obtained in the previous theorem 4.6, we find that for any $x$ and any $\zeta \in \mathcal{R}_x$, if $y = [\zeta, x]$ satisfies $\Phi_{\tau_1}(y) \in W^{cs}_{loc}(\zeta)$:
\[
\frac{G(\zeta)}{G(x)} = \lim_{t \to \infty} \frac{\text{Jac}_{\Phi_{t-y}(y)}}{\text{Jac}_{\Phi_{t-x}(x)}} \lim_{t \to \infty} \frac{\text{Jac}_{\Phi_{t+\tau_1}(y)}}{\text{Jac}_{\Phi_{t-y}(y)}} = 1.
\]

Now, if we evaluate $\mu$ on the intersection of two foliated charts $U_i$ and $U_j$, which is supposed to be of positive measure, we find the equality of the following transverse measures when restricted to the goal of the holonomy map $\tau_{ij}$:
\[
\text{Leb}(\mathcal{R}_{\tau_{ij}(x)}) \tau_{ij} * v_i(x) = \text{Leb}(\mathcal{R}_x) v_j(x).
\]
Therefore, the family of measures defined on the complete system of transversal \((T_i)_{i \in I}\) by 
\(\text{Leb}(\mathcal{R}_x) v_i(x)\) is invariant by holonomy. The proof of theorem 4.3 is now finished. \(\Box\)

5 Gibbs u-states for the foliated geodesic flow and \(\phi^u\)-harmonic measures

In all what follows, \((M, \mathcal{F})\) denotes a closed foliated manifold endowed with a leafwise metric whose leaves are negatively curved. The unit tangent bundle \(T^1 \mathcal{F}\) is foliated by the unit tangent bundle of the leaves of \(\mathcal{F}\). The foliated geodesic flow shall be denoted by \(G_t\).

5.1 \(\phi^u\)-harmonic measures.

**Potential and harmonic kernel.** The potential associated to Gibbs u-states is usually denoted by \(\phi^u\) (see for example [BR]). It is defined for \(v \in T^1 \mathcal{F}\) by:

\[
\phi^u(v) = -\frac{d}{dt} \bigg|_{t=0} \log \text{Jac}^u G_t(v). \tag{5.15}
\]

The foliated geodesic flow is of class \(C^\infty\) inside the leaves and varies continuously with the transverse parameter in the \(C^\infty\) topology. We deduce that the function \(\phi^u\) is continuous in \(M\) and, like does the unstable bundle, is uniformly Hölder continuous inside the leaves.

We can associate to this potential a kernel which will play the role of the Poisson kernel (see [AS]). Let \(L\) be a leaf of \(\mathcal{F}\), and let \(\tilde{L}\) denote its Riemannian universal cover. We can lift the restricted function \(\phi^u|_{T^1 L}\) so as to obtain a function \(\tilde{\phi}^u : T^1 \tilde{L} \rightarrow \mathbb{R}\). We can define on \(\tilde{L} \times \tilde{L} \times \tilde{L}(\infty)\) the following kernel:

\[
k^u(z_1, z_2; \xi) = \exp \left[ \int_{\xi}^{z_2} \tilde{\phi}^u - \int_{\xi}^{z_1} \tilde{\phi}^u \right],
\]

where the difference of the integrals has the following meaning. Let \(c\) be a geodesic ray asymptotic to \(\xi\). Then the following limit exists and does’nt depend on the choice of the ray \(c\):

\[
\int_{\xi}^{z_2} \tilde{\phi}^u - \int_{\xi}^{z_1} \tilde{\phi}^u = \lim_{t \to \infty} \left( \int_{c(t)}^{z_2} \tilde{\phi}^u - \int_{c(t)}^{z_1} \tilde{\phi}^u \right).
\]

**Lemma 5.1.** For any triple \((z_1, z_2, \xi) \in \tilde{L} \times \tilde{L} \times \tilde{L}(\infty)\) we have the following equality:

\[
k^u(z_1, z_2; \xi) = \lim_{t \to -\infty} \frac{\text{Jac}^u G_{-t} (z_1, z_2)}{\text{Jac}^u G_{-t} (v_{\xi, z_1})} (v_{\xi, z_2}),
\]

where \(v_{\xi, z}\) denotes the unit vector based at \(z_i\) such that \(\lim_{t \to -\infty} G_{-t} (v_{\xi, z_i}) = \xi\).

It is well known that when \(L\) is a hyperbolic manifold, the potential \(\phi^u\) is everywhere equal to the dimension of horospheres, i.e. \(d - 1\) (\(d\) being the dimension of the leaves). It is also well known that in that case, the Poisson kernel on \(\tilde{L}\) coincide with the map which associates to a triple \((z_1, z_2, \xi) \in \tilde{L} \times \tilde{L} \times \tilde{L}(\infty)\) the quantity \(\exp(-d + 1) \beta_\xi(z_1, z_2)\), where \(\beta_\xi\) denotes the Busemann cocycle at \(\xi\):

\[
\beta_\xi(z_1, z_2) = \lim_{t \to -\infty} \text{dist}(c(t), z_2) - \text{dist}(c(t), z_1),
\]

where \(c\) is any geodesic ray tangent to \(\xi\). As a consequence, we have the following proposition:

**Proposition 5.2.** Assume that \(\mathcal{F}\) is a foliation of a closed manifold with hyperbolic leaves. Then for every leaf \(L\), the kernel \(k^u\) coincide with the Poisson kernel.
\( \phi^\mu \)-harmonic functions. With the kernel \( k^\mu \) playing the role of the Poisson kernel, we can define a notion of harmonicity related to the potential \( \phi^\mu \).

**Definition 5.3.** Let \( N \) be a complete connected and simply connected Riemannian manifold whose sectional curvature is pinched between two negative constants. Fix a base point \( o \in N \). A positive function \( h : \mathbb{L} \to \mathbb{R}^+ \) is said to be \( \phi^\mu \)-harmonic if there exists a finite Borel measure on the sphere at infinity \( N(\infty) \), denoted by \( \eta_o \), such that for all \( z \in \mathbb{L} \), we have:

\[
h(z) = \int_{\mathbb{L}(\infty)} k^\mu(o, z; \xi) d\eta_o(\xi).
\]

The function induced by a \( \phi^\mu \)-harmonic function invariant by a discrete subgroup of isometries \( \Gamma \) on the quotient \( N/\Gamma \) will still be called \( \phi^\mu \)-harmonic.

**Remark.** The notion of \( \phi^\mu \)-harmonic function is independent of the choice of the base point \( o \). If \( o' \) is another point of \( N \), if \( \eta_{o'} \) is a finite Borel measure on \( N(\infty) \) and if \( h \) is the corresponding \( \phi^\mu \)-harmonic function, then we can write for any \( z \in \mathbb{L} \),

\[
h(z) = \int_{\mathbb{L}(\infty)} k^\mu(o', z; \xi) d\eta_{o'}(\xi),
\]

where \( \eta_{o'}(\xi) = k^\mu(o, o'; \xi)\eta_o(\xi) \).

**A natural \( \phi^\mu \)-harmonic function on the leaves.** We want to show the existence of a natural positive function on \( M \) which is continuous and whose restriction to any leaf \( L \), is \( \phi^\mu \)-harmonic. This will allow us to define the notion of totally invariant measure, and to show that to any transverse holonomy invariant measure one can associate a natural totally invariant \( \phi^\mu \)-harmonic measure.

Before we state the next proposition we need a notation. Let \( L \) be a leaf of \( \mathcal{F} \) and let \( v \in T^1L \) be based at \( x \in L \). We denote by \( \theta(v) \) the angle between the center unstable manifold \( W^{cu}(v) \) and the unit tangent fiber \( T^2_1L \).

**Proposition 5.4.** Let \( (M, \mathcal{F}) \) be a closed foliated manifold endowed with a leafwise metric. Assume that its leaves are negatively curved.

Let \( T_1, T_2 \) be two transverse sections to \( W^{cu} \) which are included in some unit tangent fibers. Assume that there exists a holonomy map along \( W^{cu} \), \( \text{hol}_{T_1}^{cu} : T_1 \to T_2 \), which is a homeomorphism. Then for all \( w \in T_2 \) and \( w' = \text{hol}_{T_1}^{cu}(w) \in T_1 \):

\[
\frac{d}{d\text{Leb}_{T_2}} \left[ \text{hol}_{T_1}^{cu} \ast \text{Leb}_{T_1} \right](w) = \frac{\sin\theta(w)}{\sin\theta(w')} k^\mu(w, w'; \xi),
\]

where \( \xi = c_\mu(-\infty) \).

**Remark.** Here, the notation is bit abusive. We need to choose lifts to the universal cover of the base points which lie in the same fundamental domain, and evaluate \( k^\mu \) on these points.

**Proof.** Let \( T_1, T_2 \) be two transverse sections to \( W^{cu} \), as stated in the lemma: they are included in unit tangent fibers, and there is a holonomy map along center unstable leaves \( \text{hol}_{T_1}^{cu} : T_1 \to T_2 \), which is a homeomorphism.

By the absolute continuity of the center unstable foliation, we know that this map preserves the Lebesgue class. Better: we know its Jacobian. We have, for all \( w \in T_2 \), and \( w' = \text{hol}_{T_1}^{cu}(w) \):
\[
\frac{d \left[ \text{hol}_{T_1 - T_2}^{cu} \ast \text{Leb}_{T_1} \right]}{d \text{Leb}_{T_2}} (w) = \lim_{t \to -\infty} \frac{\text{Jac}_{T_2}^{G - t - \beta}(w', w)}{\text{Jac}_{T_1}^{G - t}(w)},
\]
where, for \( v \in T_1 \), \( \text{Jac}_{T_1}^{G_t}(v) \) stands for the map \( |\det(DG_t)| \) restricted to the tangent space to \( T_1 \) at \( v \).

By placing ourselves in a basis formed by vectors tangent to unit tangent fibers and to center unstable manifolds, and using the invariance of the Liouville measure by the geodesic flow inside the leaves, we get for all \( w \in T_2 \) and \( t \in \mathbb{R} \),

\[
1 = \frac{\sin \theta_t(G_t(w))}{\sin \theta(w)} \text{Jac}_{T_1}^{G_t}(w) \text{Jac}_{T_2}^{G_u}(w),
\]
where \( \theta_t(G_t(w)) \) denotes the angle at \( G_t(w) \) between the center unstable manifold \( W^{cu}(G_t(w)) \) and the image \( G_t(T_2) \). Note also that the geodesic flow is orthogonal to the unstable manifold and acts by isometries on the orbits. Hence we have for all \( t \in \mathbb{R} \), \( \text{Jac}_{T_2}^{G_t}(w) = \text{Jac}_{T_1}^{G_t}(w) \).

Finally, we notice that when \( t \to -\infty \), the angle \( \theta_{-t}(G_{-t}(w)) \) becomes closer and closer to the angle at \( G_{-t}(w) \) of the stable and center unstable manifolds. In particular, by Hölder continuity of the stable and center unstable bundles, we have for \( w' = \text{hol}_{T_2 - T_1}^{cu}(w) \),

\[
\lim_{t \to -\infty} \frac{\sin \theta_{-t}(G_{-t}(w'))}{\sin \theta(t - \beta_t(w', w)(G_{t-\beta_t(w', w)} w))} = 1.
\]

The following equality then follows:

\[
\lim_{t \to -\infty} \frac{\text{Jac}_{T_2}^{G_{-t-\beta_t(w', w)}}(w)}{\text{Jac}_{T_1}^{G_{-t}}(w')} = \frac{\sin \theta(w)}{\sin \theta(w')} \lim_{t \to -\infty} \frac{\text{Jac}_{T_2}^{G_{-t}}(w')}{\text{Jac}_{T_1}^{G_{-t}}(w')} \frac{\text{Jac}_{T_2}^{G_{-t-\beta_t(w', w)}}(w)}{\text{Jac}_{T_1}^{G_{-t}}(w')},
\]

finishing the proof of the proposition.

Define on the unit tangent fiber \( T_1^1 \mathcal{F} \), \( x \in M \), the measure \( \omega_x \) with a density \( \sin \theta(v) \) with respect to Lebesgue. Consider a couple of transverse sections of the center unstable foliation \( T_1, T_2 \) each included inside a unit tangent fiber, and such that there is a holonomy map \( \text{hol}_{T_1 - T_2}^{cu} : T_1 \to T_2 \) which is a homeomorphism. Let \( w \in T_2 \) be a vector based at \( x \in L \) and let \( w' = \text{hol}_{T_2 - T_1}^{cu}(w) \) be based at \( x' \in L \). Then we have:

\[
\frac{d \left[ \text{hol}_{T_1 - T_2}^{cu} \ast \omega_{x'} \right]}{d \omega_x} (w) = k^u(x', x; \xi),
\]
where \( \xi = c_x(-\infty) \). The following theorem is a direct consequence of proposition 5.4:

**Theorem 5.5.** Let \((M, \mathcal{F})\) be a closed foliated manifold whose leaves are negatively curved. Then the map \( h_0 : M \to \mathbb{R} \) which associates to \( x \in M \) the number:

\[
h_0(x) = \int_{T_1^1 \mathcal{F}} \sin \theta d \text{Leb}_{T_1^1 \mathcal{F}},
\]
is continuous on \( M \) and, when restricted to the leaves of \( \mathcal{F} \), is \( \phi^u \)-harmonic.
In order to prove this theorem we need to introduce the family of finite Borel measures on \( \tilde{L}(\infty) \) induced by \((\omega_z)_{z \in \tilde{L}}\). We can naturally lift this family the universal cover so as to obtain family denoted by \((\omega_{\tilde{z}})_{\tilde{z} \in \tilde{L}}\).

Let \( z \in \tilde{L} \). There is a natural identification \( \pi_z : T^1_z \tilde{L} \to \tilde{L}(\infty) \), which associates to a unit vector \( v \) based at \( z \), the point \( \pi_z(v) = c_v(-\infty) \). Via this identification, \( \tau_{z, z'} = \pi_{z'}^{-1} \circ \pi_z : T^1_z \tilde{L} \to T^1_{z'} \tilde{L} \) is a holonomy map along the center unstable foliation.

Hence, if \( v_z = \pi_z * \omega_z \), we obtain that for any \( z_1, z_2 \in \tilde{L} \) the measures \( v_{z_1} \) and \( v_{z_2} \) lie in the same measure class (the so called visibility class). By proposition 5.4 the associated Radon-Nikodym cocycle is given by:

\[
\frac{d\nu_{z_1}}{d\nu_{\tilde{z}_1}}(\xi) = \frac{d\omega_{z_1}}{d[I_{z_1, z_2} * \omega_{\tilde{z}_1}]}(\pi_{z_1}^{-1}(\xi)) = k^u(z_1, z_2; \xi),
\]

(5.16)

for any \( \xi \in \tilde{L}(\infty) \).

**Proof of theorem 5.5.** First, the unstable and unit tangent bundles of the leaves are continuous. Hence this is also the case of the angle \( \theta \) between them as well as of its sine. Since the metric varies continuously with the transverse parameter, this integral is a continuous function of \( x \in M \).

We have to prove that when restricted to a leaf \( L \), this function is \( \phi^u \)-harmonic. To do so lift \( h_0 \) to the universal cover of \( L \). We obtain a function \( \tilde{h}_0 \) which associates to \( z \in \tilde{L} \) the mass of \( \omega_z \). But by definition, \( \nu_z \) and \( \omega_z \) have equal masses.

Fix a base point \( o \in \tilde{L} \). An application of the formula (5.16) gives that for any \( z \in L \), we have:

\[
\tilde{h}_0(z) = \text{mass}(\nu_z) = \int_{\tilde{L}(\infty)} k^u(o, z; \xi) d\nu_o(\xi).
\]

Hence, the restriction of \( h_0 \) to the leaves is a \( \phi^u \)-harmonic function.

**\( \phi^u \)-harmonic measures.** Now we can define the notion of \( \phi^u \)-harmonic measure for foliations with negatively curved leaves:

**Definition 5.6.** Let \((M, \mathcal{F})\) be a closed foliated manifold endowed with a leafwise metric such that all leaves are negatively curved. A probability measure \( m \) on \( M \) is said to be \( \phi^u \)-harmonic if it has Lebesgue disintegration in the leaves of \( \mathcal{F} \), and if the local densities are \( \phi^u \)-harmonic functions.

The question of existence of these measures will be treated in the next paragraph. Let’s give first some basic properties of such measures.

**Lemma 5.7.** Let \( m_1 \) and \( m_2 \) be two singular \( \phi^u \)-harmonic measures. Then, there exists a Borel set \( \mathcal{X} \subset M \) which is saturated by \( \mathcal{F} \) and such that \( m_1(\mathcal{X}) = 1 \) and \( m_2(\mathcal{X}) = 0 \).

**Proof.** We endow the manifold \( M \) with a good foliated atlas for \( \mathcal{F} \) denoted by \((U_i, \phi_i)_{i \in I}\). A complete system of transversal is denoted by \((T_i)_{i \in I}\).

Let \( m_1 \) and \( m_2 \) be two \( \phi^u \)-harmonic measures. There exists a Borel set \( \mathcal{X}_0 \subset M \) such that \( m_1(\mathcal{X}_0) = 1 \) and \( m_2(\mathcal{X}_0) = 0 \). Since the measures \( m_1 \) and \( m_2 \) have Lebesgue disintegration in the leaves of \( \mathcal{F} \), they induce quasi-invariant families of measures on the transversals \( T_i \) denoted by \((\nu_{i,1})_{i \in I}\) and \((\nu_{i,2})_{i \in I}\).

Up to a set of zero \( m_1 \)-measure, we can suppose that whenever \( m_1(U_i) > 0 \) the projection of \( \mathcal{X}_0 \cap U_i \) on the local transversal \( T_i \), denoted by \( X_i \), is full for \( \nu_{i,1} \) and that these sets are invariant by holonomy.

Let \( \mathcal{X} \) denote the saturated set of \( \mathcal{X}_0 \). Then for all \( i \in I \), \( X_i \) is also the projection on \( T_i \) of \( \mathcal{X} \cap U_i \). Since \( \mathcal{X}_0 \subset \mathcal{X} \) we have \( m_1(\mathcal{X}) = 1 \).

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Now it remains to see that $m_2(\mathcal{X}) = 0$. For all $i$, we have $v_{2,i}(X_i) = m_2(\mathcal{X}_0 \cap U_i) = 0$. But since $X_i$ is also the projection of $\mathcal{X} \cap U_i$ on $T_i$, we have $m_2(\mathcal{X} \cap U_i) = 0$. It comes that $m_2(\mathcal{X}) = 0$. \qed

So far, we did not prove an ergodic decomposition for $\phi^u$-harmonic measure. The next lemma will be useful in the proof of theorem 5.13.

**Lemma 5.8.** Let $\mathcal{A} = (U_i, \phi_i)_{i \in I}$ be a good foliated atlas for $\mathcal{F}$ and $(T_i)_{i \in I}$ an associated complete system of transversals. Then for any $\phi^u$-harmonic $m$, there exists a family of finite measures $m_0, \ldots, m_k$, as well as saturated Borel sets $\mathcal{X}_{i_0}, \ldots, \mathcal{X}_{i_k}$ such that for any $j$,

1. $\mathcal{X}_{i_j} \cap \mathcal{X}_{i_l} = \emptyset$ for $j \neq l$;
2. $m_{i_j}$ is a $\phi^u$-harmonic measure supported by $\mathcal{X}_{i_j}$; in particular, $m_{i_j}$ and $m_{i_l}$ are singular as soon as $j \neq l$;
3. there exists a transversal $T_{i_j}$ such that $\mu_{i_j}$-almost every leaf of $\mathcal{X}_{i_j}$ meets the transversal $T_{i_j}$.

**Proof.** Let $m$ a $\phi^u$-harmonic measure for $\mathcal{F}$. When $m(U_i) > 0$ we denote by $v_i$ the projection of the restriction $m|_{U_i}$ on $T_i$. As in the proof of the previous lemma, we can assume the existence for any $i$ such that $m(U_i) > 0$, of a set $X_i$ full for $v_i$ such that the family $(X_i)_{i \in I}$ is invariant by holonomy.

Let $i_0$ be an index such that $m(U_{i_0}) > 0$. We can define the set $\mathcal{X}_{i_0}$ as the saturated set of $X_{i_0}$: this is a Borel set, on which we can restrict $m$. Call $m_{i_0}$ the resulting measure: we can write $m = m_{i_0} + m'$.

Since $\mathcal{X}_{i_0}$ and $\mathcal{X}'_{i_0}$ are saturated, the measures $m_{i_0}$ and $m' = m|_{\mathcal{X}'_{i_0}}$ are still $\phi^u$-harmonic. Then, either $m' = 0$, and $\mathcal{X}_{i_0}$ is full for $m$, or we can proceed similarly with $m'$.

Finally $m'(T_{i_0}) = 0$ and this process ends after a finite number of steps: we have the desired families of Borel sets and measures. \qed

### 5.2 Bijective correspondence between Gibbs $u$-states and $\phi^u$-harmonic measures

**Induced measures on a complete transversal.** Choose a good foliated atlas $\mathcal{A}$ for $\mathcal{F}$, and denote by $\mathcal{F}'$ a corresponding complete transversal. Pull it back by the unit tangent fibration, and denote by $\mathcal{F}'$ the resulting atlas, whose charts shall be denoted by $(U_i)_{i \in I}$; the plaques are trivially subfoliated by spheres.

When $\mu$ is a Gibbs $u$-state for $G_t$ in $T^1\mathcal{F}$, the induced measure on $\mathcal{F}'$ will be denoted by $\hat{\mu}$. Recall that in restriction to a small transversal $T_i$, it is given by the projection of $\mu|_{U_i}$ along the plaques. We will need the following proposition which is an adaptation of the proposition 4.5. of [Al1] where it is stated for $H$-Gibbs measure. We recall below the main steps of the proof, which is a variation of the Hopf argument.

**Proposition 5.9.** Let $(M, \mathcal{F})$ be a closed foliated manifold endowed with a leafwise metric such all leaves are negatively curved manifolds. Let $\mathcal{A}$ be a good foliated atlas for $\mathcal{F}$, $\mathcal{F}'$ an associated complete transversal, and $\mathcal{F}'$ be the pulled back atlas for $\mathcal{F}$. Then:

1. any Gibbs $u$-state for $G_t$ induces a measure on $\mathcal{F}'$ which is quasi-invariant by the holonomy pseudogroup of $\mathcal{F}'$;
2. two different Gibbs $u$-states for $G_t$ induce different measures on $\mathcal{F}'$.

**Proof.** Let’s summarize below the proof we gave in [Al1] (see this reference for the complete proof), and which can be adapted without modification to our context.
The first part is a rather immediate consequence of the fact that Gibbs u-states have Lebesgue disintegration in the unstable leaves.

The main ingredient of the proof of the second part of the proposition is the absolute continuity of the stable foliation. It is enough to choose two ergodic Gibbs u-states $\mu_1$ and $\mu_2$ and to prove that if the measures they induce on $\mathcal{F}$ (that are denoted by $\tilde{\mu}_1$ and $\tilde{\mu}_2$) aren’t singular, then they are equal.

Assume it is the case. By Birkhoff’s theorem, the basin of attraction of $\mu_1$ is full for $\mu_1$. It induces a Borel set in $\mathcal{F}$ which is positive for $\tilde{\mu}_2$, since this measure is not singular with respect to $\tilde{\mu}_1$.

Hence, since the measures $\mu_1$ and $\mu_2$ have Lebesgue disintegration in the center unstable leaves, we can find a plaque of $\mathcal{A}$ containing two vectors $v$ and $w$ such that $\text{Leb}^{cu}_{\text{loc}}$-almost every point of $W^{cu}_{\text{loc}}(v)$ is in the basin of attraction of $\mu_1$, and that the basin of $\mu_2$ meets $W^{cu}_{\text{loc}}(w)$ in a set of positive $\text{Leb}^{cu}_{w}$-measure.

Let us project this intersection on $W^{cu}_{\text{loc}}(v)$ along the stable foliation. By absolute continuity, we find two points $v'$ and $w'$ lying in the same stable manifold, and such that $v'$ (resp. $w'$) is in the basin of attraction of $\mu_1$ (resp. $\mu_2$). This implies that $\mu_1 = \mu_2$: we can conclude the proof of the proposition.

**Projection of a Gibbs u-state.** It has been noticed by Bonatti, Gómez-Mont and Martínez in [BGM] that the projection of any Gibbs u-state for the foliated geodesic flow $G_t$ along the unit tangent fibers has Lebesgue disintegration in the leaves of $\mathcal{F}$. The next proposition completes this observation by identifying the local densities.

**Proposition 5.10.** Let $(M, \mathcal{F})$ be a closed foliated manifold endowed with a leafwise metric such that the leaves are negatively curved. Let $\mu$ be a Gibbs u-state for the foliated geodesic flow $G_t$. Then the projection of $\mu$ along the unit spheres tangent to the foliation is a $\phi^{cu}$-harmonic measure and induces the same measure on a complete transversal $\mathcal{F}$.

**Proof.** Let $\mu$ a Gibbs u-state for the foliated geodesic flow $G_t$. We know that its conditional measures in the center unstable manifolds are equivalent to Lebesgue. Furthermore we saw that in a chart of the form $U = \bigcup_{x \in S} Q(x)$ for the center unstable foliation, there is a measure $v$ on $S$ such that $\mu_{\mid U} = (\psi_v^{cu} \text{Leb}_v^{cu}) v(v)$, where:

$$\psi_v^{cu}(w) = k^u(v, v; c_v(-\infty))$$

Now choose a foliated atlas $\tilde{\mathcal{A}}$ for $\mathcal{F}$ which trivializes the center-unstable foliation. Let $\tilde{U}$ be a chart of this atlas. The measure $\mu$ has Lebesgue disintegration in the center unstable manifold. Since the center unstable leaves in $T^1L$ are locally isometric to $L$, the projection $m$ of $\mu$ along unit spheres tangent to the foliation still has Lebesgue disintegration in the leaves of $\mathcal{F}$.

Moreover the local densities of conditional measures of $m$ in the plaques are given by integration of the densities $\psi_v^{cu}$ against a measure on the unit tangent sphere $T^1_vL$: this is a $\phi^{cu}$-harmonic function.

Moreover, the measure $\mu$ induces on a transversal $T$ of $\mathcal{F}$ which is given by projection of the measure $v$ along the tangent spheres. This implies that it induces the same measure on $\mathcal{F}$ as its projection $m$.

Existence of Gibbs u-state has been stated in theorem 4.2, this proposition then settles the issue of existence of $\phi^{cu}$-harmonic measures for foliations with negatively curved leaves.

**Corollary 5.11.** Let $(M, \mathcal{F})$ be a closed foliated manifold, endowed with a leafwise metric such that all leaves are negatively curved. The the $\phi^{cu}$-harmonic measures of $M$ form a non empty convex set.
Canonical lift of a $\phi^u$-harmonic measure. We will see that there is a canonical way to lift any $\phi^u$-harmonic measure to $T^1\mathcal{F}$. We will need the following notations:

- $\text{pr} : T^1\mathcal{F} \to M$ denotes the canonical projection along the unit spheres tangent to $\mathcal{F}$;
- $\text{Gibbs}^u$ denotes the set of Gibbs $u$-states for the foliated geodesic flow;
- $\mathcal{H}\ar^{\phi^u}(\mathcal{F})$ denotes the set of $\phi^u$-harmonic measures for $\mathcal{F}$.

**Proposition 5.12.** Let $(M, \mathcal{F})$ be a closed foliated manifold endowed with a leafwise metric such that the leaves are negatively curved. Then the map $\text{pr}_* : \text{Gibbs}^u \to \mathcal{H}\ar^{\phi^u}(\mathcal{F})$ which associates to a Gibbs $u$-state $\mu$ its projection $\text{pr} \ast \mu$, is injective.

**Proof.** This is a straightforward application of propositions 5.9 and 5.10. Indeed two different Gibbs $u$-states induce different measures on a complete transversal, which are the same as the ones induced by their projections on $M$. Hence, these two projections have to induce different measures in a complete transversal: they are in particular different. □

**Theorem 5.13.** Let $(M, \mathcal{F})$ be a closed foliated manifold endowed with a leafwise metric such that the leaves are negatively curved. Then the map $\text{pr}_* : \text{Gibbs}^u \to \mathcal{H}\ar^{\phi^u}(\mathcal{F})$ described in the previous proposition is a bijection.

When $m = \text{pr} \ast \mu$, for $\mu \in \text{Gibbs}^u$, we say that $\mu$ is the canonical lift of $m$ to the bundle $T^1\mathcal{F}$.

**Proof.** The proof follows the same lines as that of proposition 3.6. of [Al1] where we showed how to lift canonically to $T^1\mathcal{F}$ the harmonic measures for $\mathcal{F}$. Let’s give the main steps of the proof:

- We choose a good foliated atlas for $\mathcal{F}$ fine enough so that the plaques trivialize the universal covers of the leaves; by pulling it back by the unit tangent bundle, we obtain an atlas for $\hat{\mathcal{F}}$.
- We choose a $\phi^u$-harmonic measure $m$ such that there is a transversal $T_{i_0}$ which meets $m$-almost all leaves of $\mathcal{F}$; we can restrict ourselves to this case by the lemma 5.8.
- Using Ghys’ lemma 2.1 we extend by holonomy the local $\phi^u$-harmonic densities of $m$-almost every plaque passing through $T_{i_0}$ and we lift these maps to the universal cover.
- Using integral representation of $\phi^u$-harmonic functions, we can “unroll” these densities: there is a canonical way to lift these maps to the unit tangent bundle.
- The conditional measures in center unstable manifolds have densities $k^u(o, z; \xi)$: they are invariant by the geodesic flow and have Lebesgue disintegration in unstable manifolds.
- By projection on $T^1\mathcal{F}$, we define a family of measures on plaques of $\hat{\mathcal{F}}$ which are invariant by the flow and have Lebesgue disintegration in unstable manifolds.
- We normalize these local measures in the plaques by the Radon-Nikodym cocycle induced by $m$, and we integrate them against the transverse measures induced by $m$: we are able to glue these measures together so as to form a probability measure $\mu$ which is a Gibbs $u$-state and projects down onto $m$.
- We get the canonical lift of $m$; this defines a section of projection $\text{pr}_*$: in particular this map is surjective.
- The proposition 5.12 implies that the map $\text{pr}_* : \text{Gibbs}^u \to \mathcal{H}\ar^{\phi^u}(\mathcal{F})$ is injective.
- We may conclude. □
Ergodic decomposition of $\phi^u$-harmonic measures.

**Definition 5.14.** Let $(M,\mathcal{F})$ be a closed manifold endowed with a leafwise metric such that the leaves are negatively curved. A $\phi^u$-harmonic measure $m$ on $M$ is said to be ergodic if any Borel set saturated by $\mathcal{F}$ is full or null for $m$.

**Remark.** Let $\mathcal{T}$ be a complete transversal for $\mathcal{F}$. A $\phi^u$-harmonic measure induces a finite measure $\hat{m}$, on $\mathcal{T}$ which is quasi-invariant by the holonomy pseudogroup. The associated Radon-Nikodym cocycle is defined as a quotient of $\phi^u$-harmonic functions. Saying that $m$ is ergodic amounts to the same thing as saying that any Borel subset of $\mathcal{T}$ which is saturated by the action of the holonomy pseudogroup is full or null for $\hat{m}$.

**Proposition 5.15.** A $\phi^u$-harmonic measure on $M$ is ergodic if and only if its canonical lift to $T^1\mathcal{F}$ is an ergodic Gibbs u-state for $G_t$.

**Proof.** Let $\mathcal{A} = (U_i,\phi_i)_{i\in I}$ be a good foliated atlas for $\mathcal{F}$, that we pull back so as to define a good foliated atlas for $\mathcal{F}$. Let $\mathcal{T}$ be an associated complete transversal. We know that if $m$ is a $\phi^u$-harmonic measure for $\mathcal{F}$ and that if $\mu$ is its canonical lift to $T^1\mathcal{F}$, then the two measures they induce on $\mathcal{T}$, denoted by $\hat{m}$ and $\hat{\mu}$, are the same.

Hence let $\mathcal{X} \subset \mathcal{T}$ be a Borel set which is saturated by the holonomy pseudogroup of $\mathcal{F}$. Then the Borel set of $T^1\mathcal{F}$ defined as the union of the plaques of $\mathcal{F}$ which meet $\mathcal{X}$ is saturated by $\mathcal{F}$, and in particular, by the foliated geodesic flow $G_t$.

Hence, if we assume that the canonical lift of $m$ is ergodic, this Borel set is full or null for $\mu$, and hence $\mathcal{X}$ is full or null for $\hat{\mu} = \hat{m}$. This proves the ergodicity of $m$.

Now assume that a Gibbs u-state $\mu$ is not ergodic: then there are two singular Gibbs u-states, denoted by $\mu_1$ and $\mu_2$, as well as a number $0 < \alpha < 1$ such that $\mu = \alpha \mu_1 + (1 - \alpha)\mu_2$ (recall that ergodic components of Gibbs u-states are still Gibbs u-states).

By projection onto $M$, we obtain $m = \alpha m_1 + (1 - \alpha)m_2$. Since $\mu_1$ and $\mu_2$ are singular, by proposition 5.9 they induce singular measures on the transversal $\mathcal{T}$. Since $m_1$ and $m_2$ induce respectively the same measures on $\mathcal{T}$, they are also singular.

By the lemma 5.7, there exists a saturated Borel set $\mathcal{X} \subset M$ such that $m_1(\mathcal{X}) = 1$, and $m_2(\mathcal{X}) = 0$. Then we find $m(\mathcal{X}) = \alpha \in (0,1)$: $m$ is not ergodic.

We can conclude the proof of the proposition. $\square$

Now, we are ready to prove that all $\phi^u$-harmonic measures may be obtained as a convex combination of $\phi^u$-ergodic measures.

**Theorem 5.16 (Ergodic decomposition).** Let $(M,\mathcal{F})$ be a closed foliated manifold endowed with a foliated metric such that the leaves are negatively curved. The space $\mathcal{H}_{ar}^{\phi^u}(\mathcal{F})$ of $\phi^u$-harmonic measures for $\mathcal{F}$ is a non empty convex set whose extremal points are given by the ergodic measures.

Moreover, there exists a Borel set $\mathcal{X}$ which is full for all $\phi^u$-harmonic measures, as well as a unique family $(m_x)_{x\in\mathcal{X}}$ of probability measures on $M$ such that:

1. for all $x \in \mathcal{X}$, $m_x$ is an ergodic $\phi^u$-harmonic measure;
2. if $x, y \in \mathcal{X}$ belong to a same leaf then $m_x = m_y$;
3. for every $\phi^u$-harmonic measure $m$, we have:

$$m = \int_{\mathcal{X}} m_x \, dm(x).$$
Proof. The fact that extremal points of the convex set $\text{Har}^u$ are the ergodic measures follows from the previous proposition as well as from the fact that it is true for the convex $\text{Gibbs}^u$ of Gibbs $u$-states for $G_t$.

Let $\mathcal{Y} \subset T^1 \mathcal{F}$ be the set of $u$-regular points, i.e. of unit vectors $v$ tangent to $\mathcal{F}$ whose Birkhoff averages $1/T \int_0^T \delta_{G_t(v)} dt$ and $1/T \int_0^T \delta_{G_t(v)} dt$ converge to a same limit $\mu_v$ which, furthermore, is a Gibbs $u$-state.

Any ergodic component of a Gibbs $u$-state is still a Gibbs $u$-state, so $\mathcal{Y}$ is full for any Gibbs $u$-state. Thus, its projection, denoted by $\mathcal{X}$, is full for any $\phi^u$-harmonic measure $m$ (see the theorem 5.13).

We claim that two $u$-regular unit vectors $v_1, v_2$ tangent to a same leaf can be joined by a concatenation of paths inside $\mathcal{W}_s$, and inside $\mathcal{W}_c u$, whose ending points are still $u$-regular. This is a classical argument “à la Hopf”, similar to the one we described in the proof of proposition 5.9, where we stated that two different (hence singular) ergodic Gibbs $u$-states induce two mutually singular measures in a complete transversal. We have to use:

- the fact that in restriction to the center unstable leaves, the set of $u$-regular points is full for the Lebesgue measure;
- the absolute continuity of the stable foliation.

We deduce that when $v_1$ and $v_2$ are $u$-regular and tangent to the same leaf, we have $\mu_{v_1} = \mu_{v_2}$. In particular, for $x \in \mathcal{X}$, the projection $m_x = pr \ast \mu_v$ does not depend on $v \in T^1_x \mathcal{F}$, and if $x$ and $y$ lie on the same leaf then $m_x = m_y$.

All measures $\mu_v$ are ergodic: these are the ergodic components of Gibbs $u$-states. By the previous proposition 5.15, all the measures $m_x$ are ergodic.

In order to conclude, the ergodic decomposition remains to be proven. But to obtain it, we only have to project down onto $M$ the ergodic decomposition of Gibbs $u$-states. □

Totally invariant measures. We said that when $\mathcal{F}$ possesses a transverse measure invariant by holonomy, we can form a harmonic measure by combining it with the volume inside the leaves.

Similarly, we can form a $\phi^u$-harmonic measure by combining it with the measure whose density with respect the volume of the leaves is given by the $\phi^u$-harmonic function $h_0$.

Definition 5.17. Let $(M, \mathcal{F})$ be a closed foliated manifold endowed with a leafwise metric such that all the leaves are negatively curved. A totally invariant $\phi^u$-harmonic is a measure which reads, in restriction to a foliated chart, as the product of a transverse holonomy invariant measure by the measure whose density with respect to Lebesgue in the leaves is given by:

$$h_0(x) = \int_{T^1_x \mathcal{F}} \sin \theta \, d\text{Leb}_{T^1_x \mathcal{F}},$$

where $\theta$ is the angle between center unstable fibers, and unit tangent fibers.

Hence, the argument according which harmonic measures are a good notion which generalize invariant measures can be transposed to the case of $\phi^u$-harmonic measures.
6 More sufficient conditions for the existence of transverse invariant measures

6.1 Sufficient condition on the Gibbs u-states

The theorem 4.3 is stated under the hypothesis of a leafwise hyperbolic flow which preserves the volume inside the leaves. This is the case of the foliated geodesic flow $G_t$, which preserves the Liouville measure inside the leaves. Thus we have the following theorem.

**Theorem 6.1.** Let $(M, F)$ be a closed foliated manifold endowed with a leafwise metric whose leaves are negatively curved. Then if the foliated geodesic flow admits a Gibbs $u$-state, $F$ has a transverse invariant measure.

6.2 Sufficient condition on the $\phi^u$-harmonic densities: a generalization of a result of Matsumoto

**A result of Matsumoto.** In [Mat], Matsumoto considers foliations of compact manifold with hyperbolic leaves. Let $m$ be a harmonic measure for such a foliated manifold $(M, F)$: it has Lebesgue disintegration and its conditional measures in the plaques are harmonic. Then given a typical leaf, it is possible to extend a local harmonic density, thus obtaining what he calls the characteristic function of the leaf. There is an ambiguity because this function depends on the choice of an initial plaque. It is well defined up to a multiplicative constant: whether it is bounded or not is independent of this choice.

A harmonic function of a hyperbolic manifold possesses a Poisson representation: it is determined by a Borel measure $\eta$ on the sphere at infinity. The class of this measure is also independent of the initial choice of a plaque. We call it the characteristic class of the leaf. Matsumoto proves that when $m$ is not totally invariant:

- the characteristic harmonic function of a typical leaf is not bounded;
- the measure $\eta$ associated to the sphere at infinity of a typical leaf is singular with respect to Lebesgue.

A version of this result in the case of transversally conformal foliations by hyperbolic surfaces, may be found in [BGM], and relies on the work of Deroin-Kleptsyn [DK] about the Lyapunov exponent of the foliated Brownian motion.

We intend here to prove an analogous theorem in the case of $\phi^u$-harmonic measures. It will provide a new and dynamical proof of this result where the properties of Brownian motion are not needed: it relies entirely on the absolute continuity of invariant foliations.

**Characteristic functions and classes.** Before we state our generalization, let us first define the objects it involves. Suppose in the sequel that $(M, F)$ is a closed foliated manifold endowed with a leafwise metric such that all the leaves are negatively curved.

Let $m$ be a $\phi^u$-harmonic measure. Thanks to Ghys’ lemma 2.1 we extend the local $\phi^u$ density of a typical leaf. This defines the characteristic function of the leaf: it is not canonical and depends of a choice of plaque.

By definition of a $\phi^u$-harmonic function the lift to the universal cover $h_L$ reads as the integral of the kernel $k^u(o, z; \xi)$ against a finite Borel measure $\eta_L$ defined on $\hat{L}(\infty)$. The measure class $[\eta_L]$ is independent of an initial choice of plaque: this is the characteristic class on the sphere at infinity of a typical leaf.
On the sphere at infinity of a leaf $L$ of $\mathcal{F}$, we may define the visibility class, by pushing by the geodesic flow the Lebesgue measure. Recall that because $L$ is realized as a leaf of a compact manifold, the center unstable foliation is absolutely continuous, and this class of measure is well defined.

**Existence of transverse measure.** The following theorem gives a sufficient condition on the characteristic class on the sphere at infinity for the existence of a transverse invariant measure. Following Matsumoto, we will say that some property is satisfied for $m$-almost every leaf, if there is a Borel set $\mathcal{X}$ saturated by the foliation and which is full for $m$ such that every leaf of $\mathcal{X}$ satisfies this property.

**Theorem 6.2.** Let $(M, \mathcal{F})$ be a closed foliated manifold endowed with a leafwise metric such that all leaves are negatively curved. Let $m$ be a non totally invariant $\phi^u$-harmonic measure. Then for $m$-almost every leaf $L$, the characteristic class $[\eta_L]$ on $\tilde{L}(\infty)$ is singular with respect to the visibility class.

Recall that when the leaves of $\mathcal{F}$ are hyperbolic, harmonic (in the sense of Garnett) and $\phi^u$-harmonic measures coincide. Moreover in that case, the sphere at infinity is a smooth manifold and the visibility class coincides with the Lebesgue class. Hence we obtain a new proof of the following theorem of Matsumoto:

**Corollary 6.3 (Matsumoto).** Let $(M, \mathcal{F})$ be a closed foliated manifold whose leaves are hyperbolic manifolds. Let $m$ be a harmonic measure which is not totally invariant. Then for $m$-almost every leaf $L$, the characteristic class $[\eta_L]$ on $\tilde{L}(\infty)$ is singular with respect to the Lebesgue measure. Moreover, the characteristic function of $m$-almost every leaf is unbounded.

**Proof.** The fact that a typical leaf for a non totally invariant harmonic measure is singular with respect to the Lebesgue measure is an immediate consequence of the theorem 6.2.

The fact that the characteristic function of a typical leaf is unbounded follows from the fact that the characteristic class is singular with respect to Lebesgue, and from the usual Fatou theorem for harmonic functions of hyperbolic manifolds (see [AS] for the negatively curved version). Indeed for Lebesgue-almost every point $\xi \in \tilde{L}(\infty)$, the characteristic function tends to infinity when we converge non-tangentially to $\xi$. \qed

**Link with the Gibbs su-states.** The first step in the proof of the theorem is to reduce the theorem to the case of ergodic $\phi^u$-harmonic measures. To do so, we use our ergodic decomposition theorem 5.16.

Now, let $m$ be an ergodic $\phi^u$-harmonic measure such that there is a Borel set $\mathcal{X}$ saturated by the foliation and with positive measure such that the characteristic class of every leaf of $\mathcal{X}$ is not singular with respect to the visibility class. By ergodicity, this Borel set is full for $m$. The theorem 6.2 is a consequence of the following proposition as well as of the theorem 6.1:

**Proposition 6.4.** Assume that there exists an ergodic $\phi^u$-harmonic measure $m$ such that the characteristic class of $m$-almost every leaf of $\mathcal{F}$ is not singular with respect to the visibility class. Then its canonical lift is a Gibbs su-state.

6.3 **Proof of proposition 6.4**

**Disintegration in stable manifolds.** We will first show that proving proposition 6.4 reduces to proving the following one which is a property of Gibbs u-states:
Proposition 6.5. Let $\mu$ be an ergodic Gibbs $u$-state on $T^1\mathcal{F}$ for the foliated geodesic $G_t$. Then the following alternative holds:

- either $\mu$ is a Gibbs $su$-state;
- or the disintegration of $\mu$ in the stable manifolds is singular with respect to Lebesgue.

This implies proposition 6.4. We will need the two following lemmas which are consequences of the absolute continuity of invariant foliations. Before we state the first one, we need to recall that for any $\phi^u$-harmonic measure $m$, we can project the characteristic class of $m$-almost every leaf $L$ onto the unit tangent fibers to $L$. We will still call this class the characteristic class in the unit tangent fibers. Note that by definition it is invariant by center unstable holonomy maps.

Lemma 6.6. Let $m$ be a $\phi^u$-harmonic measure, and $\mu$ be its canonical lift. Then the disintegration of $\mu$ in the unit tangent fibers of a typical leaf $L$ is absolutely continuous with respect to its characteristic class.

Proof. Let $U$ be a foliated chart for $\mathcal{F}$, and $T$ be a transverse section so that $\hat{U} = \bigcup_{x \in T} T^1 P(x)$ is still a foliated chart for $\hat{\mathcal{F}}$. Then, by definition, the conditional measure of the canonical lift $\mu$ of $m$ in a typical plaque $T^1 P(x)$ is obtained by integration of measures in the center unstable plaques which have a density with respect to Lebesgue (defined in terms of the kernel $k^u$; see for example the proof of proposition 5.10) against a measure at infinity $\eta_L$ which is in the characteristic class of the corresponding leaf.

Using the absolute continuity of the foliation by the unit tangent fibers, we may disintegrate this measure in the fibers with respect to Lebesgue in a local center unstable manifold. The conditional measures are then equivalent to the projection of $\eta_L$ along the geodesics.

By definition these conditional measures all lie in the characteristic class of $L$. □

Lemma 6.7. The disintegration of a Gibbs $u$-state is singular with respect to Lebesgue in the stable manifolds if and only if its disintegration in the unit tangent fibers is so.

Proof. Let $\mu$ be a Gibbs $u$-state for $G_t$. Consider a foliated chart for $\hat{\mathcal{F}}$ whose plaques are small open rectangles $\mathcal{R}_{x_0}$ which possess the local product structure. Consider the conditional measure $\mu_{x_0}$ on this plaque.

The same proof as that of lemma 6.6 shows that the conditional measure on the local stable manifolds of $\mu_{x_0}$ all lie in the projection of harmonic class on stable horospheres along the geodesics. By absolute continuity of the center unstable holonomy maps, they are simultaneously singular with respect with Lebesgue, i.e. if one of them is singular, then they all are.

Hence, the conditional measures of $\mu_{x_0}$ in the local stable manifolds are singular with respect to Lebesgue if and only if the projection of $\mu_{x_0}$ on some local stable leaf along the local center unstable manifolds is singular.

Similarly, $\mu_{x_0}$ has a disintegration singular with respect to Lebesgue in the unit tangent spheres if and only if its projection on a fiber is singular.

But by definition we go from the projection of $\mu_{x_0}$ along center unstable manifolds on a stable plaque to the projection on a unit tangent fiber by a center unstable holonomy map which preserves the Lebesgue class.

In particular, the lemma follows. □
End of the proof of proposition 6.4. Assume that proposition 6.5 is true, and assume that there is an ergodic $\phi^m$-harmonic measure such that the characteristic class of $m$-almost every leaf is not singular with respect to the visibility class.

Then an application of lemma 6.6 gives that when we project down this characteristic class onto unit tangent fibers, the measures we obtain are not singular with respect to Lebesgue.

From lemma 6.7, we deduce that the disintegration of $\mu$ in the stable manifolds is not singular with respect to Lebesgue. But proposition 6.5 then implies that this is only possible if $\mu$ is a Gibbs su-state.

Hence, in order to prove proposition 6.4, and thus theorem 6.2, it is enough to prove proposition 6.5.

Proof of proposition 6.5. It is enough to prove that an ergodic Gibbs u-state whose disintegration in stable manifolds is not singular with respect to Lebesgue is a Gibbs su-state.

Let $\mu$ be such an ergodic Gibbs u-state. There is a Borel set $\mathcal{X}$, full for $\mu$ and such that for any $v \in \mathcal{X}$ and any continuous function $f : T^1 \mathcal{X} \to \mathbb{R}$ we have:

$$\lim_{T \to \infty} \int_0^T f \circ G_t(v) \, dt = \lim_{T \to \infty} \int_0^T f \circ G_{-t}(v) \, dt = \int_{T^1 \mathcal{X}} f \, d\mu.$$

By hypothesis, if we disintegrate $\mu$ in the stable plaques, we can find a point $v_0 \in Y$, as well as a Borel set $D \subset W^s_{loc}(v_0)$ which contains $v_0$ and such that $\text{Leb}^s(D) > 0$ and $D \subset \mathcal{X}$.

Since $D \subset \mathcal{X}$, we obtain that for any $v \in D$ the Birkhoff averages $1/T \int_0^T \delta_{G_{-t}}(v) \, dt$ converges to $\mu$. So by use of dominated convergence, the following family converges to $\mu$: $$\mu_T = \frac{1}{T} \int_0^T \frac{G_{-t} \ast (\text{Leb}^s_{D})}{\text{Leb}^s(D)} \, dt.$$

But by theorem 4.2, we know that every accumulation point of $\mu_T$ is a Gibbs s-state. As a conclusion, $\mu$ is both a Gibbs u-state and a Gibbs s-state: this is a Gibbs su-state.

We have now proven the desired dichotomy: the proof of proposition 6.5, and thus that of the theorem 6.2, is complete.

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References

[Al1] S.Alvarez, Harmonic measures and the foliated geodesic flow for foliations with negatively curved leaves, preprint, [arXiv:1311.3267].

[Al2] S.Alvarez, Gibbs measures for foliated bundles with negatively curved leaves, preprint, [arXiv:1311.3574].

[Al3] S.Alvarez, Mesures de Gibbs et mesures harmoniques pour les feuilletages aux feuilles courbées négativement, Thèse de l’Université de Bourgogne, (2013).

[An] D.Anosov, Geodesic flows on closed Riemannian manifolds with negative curvature, Proc. Steklov Math. Inst. A.M.S Transl., (1969).
M.T. Anderson, R. Schoen, Positive harmonic functions on complete manifolds of negative curvature, *Ann. of Math.*, 121, (1985), 429-461.

Y. Bakhtin, M. Martínez, A characterization of harmonic measures on laminations by hyperbolic Riemann surfaces, *Ann. I. H. Poincaré. Prob. Stat.*, 44, (2008), 1078-1089.

W. Ballmann, *Lectures on Spaces of Nonpositive Curvature*, Birkhäuser Verlag, Basel, 1995. With an appendix by Misha Brin.

C. Bonatti, L. Díaz, M. Viana, *Dynamics Beyond Uniform Hyperbolicity. A global geometric and probabilistic perspective*, Encyclopaedia of Mathematical Sciences, 102. Mathematical Physics, III. Springer Verlag, 2005.

C. Bonatti, X. Gómez-Mont, Sur le comportement statistique des feuilles de certains feuilletages holomorphes, *Monogr. Enseign. Math.*, 38, (2001), 15-41.

C. Bonatti, X. Gómez-Mont, M. Martínez, Foliated hyperbolicity, preprint.

R. Bowen, B. Marcus, Unique ergodicity for horocycle foliations, *Israël J. of Math.*, 26, (1977), 43-67.

R. Bowen, D. Ruelle, The ergodic theory of Axiom A flows, *Invent. Math.*, 29, (1975), 181-202.

C. Camacho, A. Lins Neto: *Geometric Theory of Foliations*, Birkhäuser, Boston Inc., 1985.

A. Candel, Uniformization of surface laminations, *Ann. Scient. Éc. Norm. Sup.*, 26, (1993), 483-516.

M. do Carmo, *Riemannian geometry*, Birkhäuser, Boston Inc., 1976.

B. Deroin, V. Kleptsyn, Random conformal dynamical systems, *Geom. Func. Anal.*, 17, (2007), 1043-1105.

L. Garnett, Foliations, the ergodic theorem and Brownian motion, *J. Funct. Anal.*, 51, (1983), 285-311.

E. Ghys, Topologie des feuilles génériques, *Ann. of Math.*, 141, (1995), 387-422.

E. Ghys, R. Langevin, P. Walczak, Entropie géométrique des feuilletages, *Acta Math.*, 160, (1988), 105-142.

M. Hirsch, C. Pugh, M. Shub, *Invariant manifolds*, in Lecture Notes in Math., 583, Springer Verlag, 1977.

F. Ledrappier, *Structure au bord des variétés à courbure négative*, Sémin. de Th. Spec. et Géom., Grenoble, (1994-1995), 93-118.

R. Mañé, *Ergodic theory and differentiable dynamics*, Springer Verlag, 1987.

M. Martínez, Measures on hyperbolic surface laminations, *Ergod. Th. & Dynam. Sys.*, 26, (2006), 847-867.
[Mat] S.Matsumoto, The dichotomy of harmonic measures of compact hyperbolic laminations, *Tohoku Math. J.*, 64, (2012), 569-592.

[PS] Ya.Pesin, Ya.Sinai, Gibbs measures for partially hyperbolic attractors, *Ergod. Th. & Dynam. Sys.*, (1982), 417-438.

[PI] J.F.Plante, Foliations with measure preserving holonomy, *Ann. of Math.*, 102, (1975), 327-361.

[Ro] V.A.Rokhlin, On the fundamental ideas of measure theory, *Trans. Amer. Math. Soc.*, 10, (1962), 1-52.

[S] Ya.Sinai, Gibbs measures in ergodic theory, *Russ. Math. Surveys*, 166, (1972), 21-69.

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