A COMMON CYCLE APPROACH FOR SOLVING THE ECONOMIC LOT AND INSPECTION SCHEDULING PROBLEM

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Abstract. In this study, we consider an imperfect production system in which the manager not only faces the Economic Lot Scheduling Problem, but also needs to conduct multiple inspections during a production lot of any product. Inspection plays an important role in an imperfect production system since it saves the cost from producing and restoring defective items though it also incurs extra inspection cost at the same time. In this study, we employ the common cycle approach in which all the products share the same replenishment cycle, and adopt a consensus inspection policy. The focus of this study is to determine the optimal cycle time and an optimal production and inspection schedule that minimize the total cost per unit time. We formulate a mathematical model in which we take into accounts both the production capacity and inspection capacity constraints. Also, we conduct full theoretical analysis and propose an effective search algorithm for solving an optimal solution. Our numerical experiments demonstrate the effectiveness of the proposed search algorithm.

1. Introduction. Since the lot sizing strategies are critical to the efficiency of production and inventory systems problems, they have been studied for more than half of a century in the literature. The Economic Lot Scheduling Problem (ELSP) is one of the most representative problems since it requires both lot sizing and production scheduling decision-making at the same time. A great deal of research
has been devoted to the extensions of the ELSP recently. On the other hand, some production systems may shift from an in-control state into an out-of-control state after an unknown period of time. Therefore, inspection plays an important role as a way of detection to avoid the production system keeps manufacturing defective units. Also, it prevents sending defective units to customers so as to save possible warranty or returning costs. Many researchers have paid attention to the lot sizing problem considering inspection for decades. Most of them assume that only single item is produced when formulating their models. However, to the best of the authors’ knowledge, no ELSP model takes into account inspection in the literature. Therefore, we are motivated to solve the ELSP considering inspection in production system, which is named as the Economic Lot and Inspection Scheduling Problem (ELISP), in this study.

We divide our discussions in this section into three parts. The first part reviews the background and the solution approaches for solving the conventional ELSP. The second part surveys the extensions of the ELSP in the literature. We will present a thorough review on the lot sizing models considering inspection in the third part.

1.1. A brief review of the conventional ELSP. The conventional ELSP is concerned with the scheduling of cyclical production of \( n \geq 2 \) items on a single facility in equal lots over an infinite planning horizon, assuming known and constant demands for each item. The objective of the ELSP is to determine the lot size and the production schedule of each item so as to minimize the total cost incurred per unit time. The costs considered include the setup cost and inventory holding cost. The ELSP has been studied for more than 50 years since it was first introduced by Rogers [23] and it is known as an NP-hard problem [8].

Researchers had proposed several solution methodologies for solving the conventional ELSP. A well-known analytical approach is the Common Cycle (CC) approach, in which all the items must be produced using the same cycle time. The CC approach is a simple method that can guarantee the feasibility of its solution (one may refer to Hanssman [7] for details on the CC approach). To facilitate coordinate production lots of the items, the basic period-based approaches become one of the most popular categories of solution approaches in the literature. The basic period (BP) approach restricts all items must be produced at the first period, but each of the items may or may not be produced at other periods. Also, for any item \( i \), its cycle time is an integer multiplier \( k_i \) of a basic period \( B \). The extended basic period (EBP) approach is similar to the BP approach [34]. But, the former allows the flexibility of scheduling the production of an item not being started at the first basic period.

Interested readers may refer to Elmaghraby [4], Lopez and Kingsmans [16] and Yao [34] for literature review on the solution methodologies for solving the ELSP.

1.2. The extensions of the ELSP. Over the past decades, many researchers had proposed various extensions of the conventional ELSP. We note that many extensions of the ELSP were studied using the CC approach.

Some researchers addressed to the issue of sequence-independent setup times in their ELSP models, e.g., Dobson [3] and Wagner and Davis [30]. Both papers use the CC approach for solving their concerned problems. Some others changed the objective function to pursue for profit maximization. Using the CC approach, Faaland et al. [5] proposed an ELSP model that permits lost sale if it leads to more profits. Salvietti and Smith [26] extended the ELSP to include the pricing of
the items with the objective function of profit maximization. Recently, Soman et al. [27] presented an ELSP model considering the shelf life of the items. Moon et al. [17] proposed a hybrid genetic algorithm (GA) for solving the group technology economic lot scheduling problem (GT-ELSP) based on the CC approach and the time varying lot sizing approach.

Several studies extended the ELSP by considering returns and/or remanufacturing recently, e.g., [28], [29], [36], and [19], etc. All of them employed the CC approach for solving their extended models.

Following our literature review, we learn that the CC approach is the most popular solution approach for the extensions of the ELSP. However, we did not locate any extension of the ELSP that takes into account the inspection of the items in a production run. Therefore, we are motivated to study the Economic Lot Sizing and Inspection Problem in this study.

1.3. Review on the lot Sizing models considering inspection. The production process of a deteriorating production system may transit from the “in-control” (producing defective units) state to the “out-of-control” (producing defective units) state after a period of manufacturing operations. In most of the real-world cases, the production managers monitor and control the state of the production process by periodic inspection or maintenance mechanisms. Obviously, detecting defective units by periodic inspections incurs an inspection cost, and it may grow with the number of inspections for a production run. However, the managers and the operators may tune the production facility and prevent keep producing defecting items as they identify the “out-of-control” state. Therefore, periodic inspections may save the warrantee cost from the returned items or the restoration costs for the detected defective items.

Many researchers were interested in deriving the optimal inspection policy (i.e., an optimal number of inspections in a production run) for the items in a production system over the last two decades.

Porteus [20] and Rosenblatt and Lee [24] considered the manufacturing process is in an in-control state at the beginning of each production run in their decision-making scenario. Both of them formulated the deteriorating manufacturing process as a random process and derived the optimal inspection policy using an EPQ-based model. Later, a series of studies extended their models by considering different scenarios and comparing the difference between the obtained solutions; see [25], [12], [13], [21], [14], [11] and [31], etc.

Researchers proposed different inspection policies in a production run in the literature. Based on Rosenblatt and Lee’s [24] work, several studies tried to find the optimal production run length that assumed all defective units are detected and reworked at some costs after the production run; one may refer to [10], [32], [6], [9]. Djamaludin et al. [2] applied an inspection policy in which the last certain number of units in production lot are sent for burn-in tests and the units are scrapped if they do not pass the test. Yeh and Chen [37] extended Djamaludin et al. [2] and proposed an inspection scheme called “last-K unit inspection” policy for the imperfect production system. Also, many researchers proposed several integrated models considering inspections for imperfect manufacturing systems; for instances, [22], [15], and [1], etc.

From the literature above, we find that most of studies focus on the production lot sizing model with single item. Only few studies concern with the multi-item lot-sizing problems that consider the inspection of the items in an imperfect production
system. Therefore, it motivates us to study the ELSP that considers inspection scheduling for an imperfect production system to fill this research gap.

In this paper, we present a mathematical model for the ELISP under the CC approach by revising a model in Yao and Chen’s [36] paper. Yao and Chen developed a model on determining the optimal replenishment cycle time and the optimal inspection schedule for imperfect production system. They derive an effective search algorithm to obtain the optimal solution and take into accounts the constraint for the production capacity and inspection capacity in the proposed model. In their paper, a fixed portion of sample units be randomly picked for inspection. In this study, we revise the Yao & Chen’s model to a new Economic Lot and Inspection Scheduling Problem (ELISP) model and a consensus inspection policy is performed. The tradeoffs among production setup, inspection setup, inventory, inspection, restoration and defective unit costs are analyzed.

The organization of the rest of this research is as follows. We present a mathematical model for the ELISP under the CC approach in Section 2. Then, Section 3 presents full theoretical analysis on the optimal objective function value curve. Based on our theoretical results, we propose an effective search algorithm in Section 4. The first part of Section 5 demonstrates the implementation of the proposed algorithm via a numerical example. Then, the second part employs randomly generated instances to verify the effectiveness of the proposed search algorithm. Finally, we address our concluding remarks in Section 6.

2. The mathematical model. In this section, we will first define the notations for our model formulation, and then, present the mathematical model.

We define the notation for our model formulation as follows. There are a total of $n$ items in the production system. We denote the demand rate, the production rate, the inspection rate, the production lot size and the inspection lot size of item $i$, as $d_i$, $p_i$, $y_i$, $Q_i$ and $u_i$, respectively. The setup time for the production lot and the inspection lot of item $i$, are denoted as $s_i$ and $z_i$, respectively. We denote $r_i$ and $π_i$ as the restoration cost and the opportunity cost of producing a defective unit of item $i$. The holding cost of item $i$ is $h_i$ per unit, per unit time. The decision variables are $T$, which is a common replenishment cycle time for all the items, and $m_i$, the number of inspection lots in a production run for item $i$, for $i = 1, 2, ..., n$. We also define $m = (m_1, m_2, \cdots, m_n)$ for simplification.

Before presenting the mathematical model, we first introduce the assumptions used for the formulation.

1. The parameters (namely, the demand rates, the production rates, the setup times, the inspection times, the setup costs, the inspection costs and the holding costs, etc.) in the model are known and constant. We assume that $p_i > y_i > d_i$ for all $i$.
2. All items are produced on the same facility, and they use a common replenishment cycle.
3. Only one item can be produced at a time on the facility.
4. The production and inspection setup costs and the production and inspection setup times are independent of their production/inspection sequence and lot sizes. A setup is required for each inspection lot. Also, the inspection setup time must not exceed the production time of inspection sub-lots for each item.
5. The shortage is not allowed.
6. The sizes of inspection lot of an item are equally in a production run. All the inspection lots are sent to a single inspection facility to perform consensus inspection.

7. The production facility will not start the setup and production for the production lots (or sub-lots) of the next item until the inspection facility finished the consensus inspection of the just produced item.

8. At the beginning of each inspection lot, the manufacturing process is always in the in-control state. Once the manufacturing process went out of control, it remained the status for the rest of the sub-lot.

9. Once any defective product is detected during inspection, it will be sent to the Reworking Department for restoration. The Reworking Department is able to fully restore the defective product, and the restored product can be used to meet the demand. The Reworking Department also takes care of the inspection of the restored items. Therefore the restoration will not consume the capacity of the production/inspection facility.

10. The transportation time between the production and the Reworking Departments is extremely short and insignificant.

11. We assume that since all the defects are minor, it takes almost no time to finish restoration. Namely, the restoration time is negligible.

Recall that most of the studies in the literature did not consider the inspection time, and they include no constraint(s) for inspection capacity. (Please refer to Section 1 for details.) Note that we take into accounts the capacity constraints for both inspection and production in the proposed model.

In our decision-making scenario, we consider five categories of cost terms as follows:

1. The setup cost: It incurs a (production) setup cost of $A_i$ for each production run and an (inspection) setup cost of $c_i$ for each inspection sub-lot of item $i$, respectively.

2. The inspection cost: We shall have a total of $c_i m_i$ inspection setup (fixed) cost for item $i$. Since a consensus inspection is performed in this study, it incurs an inspection (variable) cost of $v_i$ for each unit. So, we have a total of $T d_i v_i$ inspection cost for item $i$, and the average inspection cost is $d_i v_i$, which is a constant.

3. The expected restoration cost: We define $\alpha_i$ as the probability with which the manufacturing process shifts into the out-of-control state when producing item $i$. Also, $r_i$ is the restoration cost for the inspection of item $i$. Using the results in [21], we may show that the expected restoration cost is given by $\left(\frac{d_i^2 T^2 \alpha_i r_i}{2 m_i}\right)$.

4. The opportunity cost from producing defective product: It can be shown that for item $i$, the opportunity cost of wasting capacity incurred by producing defective units is given by $\left(\frac{T^2 d_i \alpha_i \pi_i}{2 m_i}\right)$.

5. The inventory holding cost: The average inventory holding cost for item $i$ includes two parts. The first part is for the inventory produced for $m_i$ inspection sub-lots of item $i$, and the average holding cost can be expressed by $Td_i^2 (p_i + y_i) h_i/2m_ip_i y_i$. The other is for the inventory of item $i$ that has finished its inspection (and restoration, if necessary), and the average holding cost is given by $Td_i (m_i p_i y_i - d_i m_i p_i - d_i m_i y_i + d_i y_i) h_i/2m_i p_i y_i$. (One may refer to Fig. 3 in Appendix A for the derivation of the inventory holding costs.)
Following the derivation above, we should have the average total costs given by

\[
\sum_{i=1}^{n} \left\{ \frac{A_i + c_im_i}{T} + \frac{T d_i^2 (p_i + y_i) h_i}{2m_i p_i y_i} + \frac{\alpha_i d_i^2 T}{2m_i} (\pi_i + r_i) \right. \\
+ \left. \frac{d_i T (m_i p_i y_i - d_i m_i p_i - d_i m_i y_i + d_i y_i) h_i}{2m_i p_i y_i} \right\}
\]

Since the term \(d_i v_i\) is a constant and will not play a role in optimization, we will not include in the objective function of the proposed model.

Now, we are ready to formulate a mathematical model for the Economic Lot and Inspection Scheduling Problem (ELISP). It shall assist the decision maker in the determination of the optimal production lot sizing and the optimal production and inspection schedule of each item in the concerned production system.

Minimize

\[
\sum_{i=1}^{n} \left\{ \frac{A_i + c_im_i}{T} + \frac{T d_i^2 (p_i + y_i) h_i}{2m_i p_i y_i} \right. \\
+ \left. \frac{d_i T (m_i p_i y_i - d_i m_i p_i - d_i m_i y_i + d_i y_i) h_i}{2m_i p_i y_i} \right\} + \frac{\alpha_i d_i^2 T}{2m_i} (\pi_i + r_i)
\]

(1)

Subject to

\[
\sum_{i=1}^{n} (s_i + \frac{T d_i}{p_i} + \frac{T d_i}{y_i}) \leq T
\]

(2)

\[
\sum_{i=1}^{n} (z_i + \frac{d_i T}{y_i} + \frac{d_i T (m_i - 1)}{m_i p_i}) \leq T
\]

(3)

\[
z_i \leq \frac{T d_i}{m_i p_i}, \ i = 1, 2, \ldots, n
\]

(4)

\[
m_i \leq T d_i, \ i = 1, 2, \ldots, n
\]

(5)

The objective function in (1) summarizes the cost terms in the total costs per unit time. Besides, we include two capacity constraints for production and inspection in (2) and (3), respectively. The rationale of the constraint (2) is the total occupancy for the scheduled production/inspection activities assigned to the production facility does not exceed the length of the cycle length \(T\). Note that the occupancy of each item \(i\) assigned to the production facility includes the production setup time \((s_i)\) and two types of scheduled production/inspection activities occupy the production facility for all the production lot, namely, the manufacturing of the item (which takes \(T d_i/p_i\)) and the consensus inspection on the item (which takes \(T d_i/y_i\)).

The constraint in (3) makes sure that the total occupancy for the scheduled inspection activities assigned to the inspection facility does not exceed the length of the cycle length \(T\). Note that the occupancy of each item \(i\) assigned to the inspection facility should include: the production setup time \((z_i)\), the duration for \(m_i\) inspection sub-lots (with each sub-lot takes \(T d_i/m_i y_i\)) and the duration for \((m_i-1)\) manufacturing sub-lots (with each sub-lot takes \(T d_i/m_i p_i\)). Inequalities (4) explain the inspection setup time must not exceed the production time of inspection sub-lots for each item. These inequalities follow the fourth assumption. Also, inequalities (5) require that the number of inspection lots must be less than production quantity for each item.

The objective function in (1) summarizes the cost terms in the total costs per unit time. Besides, we include two capacity constraints for production and inspection in (2) and (3), respectively. The rationale of the constraint (2) is the total occupancy for the scheduled production/inspection activities assigned to the production facility does not exceed the length of the cycle length \(T\). Note that the occupancy of each item \(i\) assigned to the production facility includes the production setup time \((s_i)\) and two types of scheduled production/inspection activities occupy the production facility for all the production lot, namely, the manufacturing of the item (which takes \(T d_i/p_i\)) and the consensus inspection on the item (which takes \(T d_i/y_i\)).
3. Theoretical analysis. In this section, we present some theoretical results that provide insights into the characteristics of the optimal value of the objective function in eq.(1).

3.1. Properties of the objective function. We may dichotomize the cost terms in the objective function into two groups: (1) the cost terms that include the decision variable $m_i$ and (2) those cost terms do not. We re-organize the cost terms in the objective function as eq.(6).

\[
\sum_{i=1}^{n} \left\{ \frac{c_i m_i}{T} + \frac{T d_i^2 (p_i + y_i) h_i}{2m_i p_i y_i} + \frac{d_i^2 h_i T}{2m_i p_i} + \frac{\alpha_i d_i^2 T (\pi_i + r_i)}{2m_i} \right\} + \\
\sum_{i=1}^{n} \left\{ \frac{A_i}{T} + \frac{T d_i h_i (p_i y_i - d_i p_i - d_i y_i)}{2 p_i y_i} \right\}
\]

(6)

To facilitate our analysis, we define $TC_i(m_i, T)$ as follows.

\[
TC_i(m_i, T) = \frac{c_i m_i}{T} + \frac{T d_i^2 (p_i + y_i) h_i}{2m_i p_i y_i} + \frac{d_i^2 h_i T}{2m_i p_i} + \frac{\alpha_i d_i^2 T (\pi_i + r_i)}{2m_i}, \ i = 1, 2, \ldots, n
\]

(7)

Therefore, we may express the objective function by

\[
TC(m, T) = \sum_{i=1}^{n} TC_i(m_i, T) + \sum_{i=1}^{n} \left\{ \frac{A_i}{T} + \frac{T d_i h_i (p_i y_i - d_i p_i - d_i y_i)}{2 p_i y_i} \right\}
\]

(8)

3.2. Junction points and the piece-wise convex properties. Using the parameters of item 1 in Table 1, we may graphically display the value of $TC_i(m_i, T)$ using different values of $m_i$ in Fig. 1.

We denote it as $TC_C(T)$, the minimum cost function for item $i$ as a function of $T$ (by taking the best value of $m_i$ for $TC_i(m_i, T)$ for all the values of $T$ on the $T$-axis).

We have

\[
TC_i(T) = \min_{m_i \in \mathbb{N}^+} \{TC_i(m_i, T)\}
\]

(9)

One may easily observe that the curve of $TC_i(T)$ is the lower envelop connecting the “best” $TC_i(m_i, T)$ along the $T$-axis. From Fig. 1, we may observe some ‘junction points’ on the curve of $TC_i(T)$ function. We can define a junction point for the curve of $TC_i(T)$ as a particular value $T$ of where two consecutive $TC_i(m_i, T)$ convex curves concatenate. These junction points determine at ‘what value of $T$’ where one should change the number of inspection for item $i$ from $m_i$ to $m_i + 1$ so as to secure the minimum value for the $TC_i(m_i, T)$ function.

We may easily derive a closed-form for the location of the junction points of item $i$ using the fact that $TC_i(m_i, T) = TC_i(m_i + 1, T)$ as follows.

\[
w_i(m_i) = T = \frac{1}{d_i} \sqrt{\frac{2 c_i m_i (m_i + 1)}{h_i/y_i + 2 h_i/p_i + \alpha_i (\pi_i + r_i)}}, \ i = 1, 2, \ldots, n.
\]

(10)

Lemma 3.1 is an immediate result from (10).
Figure 1. The value of $TC_i(m_i, T)$ using different values of $m_i$.

**Lemma 3.1.** For item $i$, the optimal number of inspections for any $T$ is given by

$$m_i^*(T) = \begin{cases} 1, & T \in \left(0, \frac{2}{d_i} \sqrt{\frac{c_i}{\Phi_i}}\right] \\ x, & T \in \left[\left(\frac{1}{d_i} \sqrt{\frac{2c_i(x-1)}{\Phi_i}} \right), \left(\frac{1}{d_i} \sqrt{\frac{2c_ix(x+1)}{\Phi_i}}\right]\right) \end{cases}$$

where $x \geq 2, \Phi_i = h_i/y_i + 2h_i/p_i + \alpha_i(\pi_i + r_i)$

(11)

Lemma 3.1 helps us in obtaining the item $i$’s optimal number of inspections when searching along the $T$-axis. Theorem 3.2 provides more insights into the characteristic of the curve of $TC_i(T)$.

**Theorem 3.2.** Suppose $m_i^{(L)}$ and $m_i^{(R)}$, respectively, are the optimal numbers of the left-side and right-side convex curves with regard to a junction point of the function $TC_i(T)$. Then $m_i^{(R)} = m_i^{(L)} + 1$.

Proof. Suppose that $T_1$ and $T_2$ are located at the left side and the right side of a certain junction point $w_i(m_i)$ (with $T_1 < w_i(m_i) < T_2$). From Lemma 1, the numbers of inspection for $T_1$ and $T_2$ are $m_i^{(L)} = x$ and $m_i^{(R)} = x + 1$, respectively, for some positive integer $x$. Therefore, it holds that $m_i^{(R)} = m_i^{(L)} + 1$. $\quad \square$

Fig. 2 depicts the curve of $TC_i(T)$. Interestingly, it shows that $TC_i(T)$ is a piece-wise convex function of all items with respect to $T$.

**Lemma 3.3.** $TC_i(T)$ is a piece-wise convex function with respect to $T$. Also, for each $m_i$, one can obtain the local minima for $TC_i(T)$ at $\hat{T}_i(m_i)$ with the minimum cost of $2c_i d_i \sqrt{h_i(p_i + 2y_i)} + \alpha_i p_i y_i(\pi_i + r_i)/\sqrt{2p_i y_i c_i}$ where

$$\hat{T}_i(m_i) = (m_i/d_i) \sqrt{2p_i y_i c_i}/[h_i(p_i + 2y_i) + \alpha_i p_i y_i(\pi_i + r_i)]$$

(12)
Proof. Suppose that $T \in [w_i(\kappa - 1), w_i(\kappa)]$ for a particular value of $m_i = \kappa$. Following Lemma 1, we know that $TC_i(T) = TC_i(\kappa, T)$ for $T \in [w_i(\kappa - 1), w_i(\kappa)]$. From the definition of $TC_i(m_i, T)$ in (7), we have the second derivatives of $TC_i(\kappa, T)$ with respect to $T$ in eq. (13).

$$\frac{\partial^2 TC_i(\kappa, T)}{\partial T^2} = \frac{2\kappa c_i}{T^3}$$

(13)

We assert that $TC_i(\kappa, T)$ is convex function since $\frac{\partial^2 TC_i(\kappa, T)}{\partial T^2} > 0$ with $\kappa, c_i, T > 0$.

Also, by taking $\frac{\partial TC_i(m_i, T)}{\partial T} = 0$, we can obtain the local minima for $TC_i(m_i, T)$ for a given value of $m_i$ at $\bar{T}_i(m_i) = (m_i/d_i) \sqrt{2p_iy_ic_i/h_i(p_i + 2y_i) + \alpha_ip_iy_i(\pi_i + r_i)}$, we have the minimum cost of $2c_i d_i \sqrt{h_i(p_i + 2y_i) + \alpha_ip_iy_i(\pi_i + r_i)}/\sqrt{2p_iy_ic_i}$.

Theorem 3.4 indicates the characteristics of the $TC_i(T)$ function.

**Theorem 3.4.** The $TC_i(T)$ function is piece-wise convex with respect to $T$.

Proof. We first define $$g_i(T) = A_i/T + Td_i h_i (p_i y_i - d_i p_i - d_i y_i)/2p_i y_i$$

(15)
Then, $\text{TC}(T) = \sum_{i=1}^{n} \text{TC}_i(T) + \sum_{i=1}^{n} g_i(T)$.

From eq. (15), we know that since the second derivative of $g_i(T)$ is positive since $\frac{d^2 g_i(T)}{dT^2} = \frac{2A_i}{T^4} > 0$, for $A_i, T > 0$. Therefore, we assert that $g_i(T)$ is a convex with respect to $T$.

Recall that Lemma 3.3 asserts that $\text{TC}_i(T)$ is piece-wise convex with respect to $T$. Following the fact that $\text{TC}(T) = \sum_{i=1}^{n} \text{TC}_i(T) + \sum_{i=1}^{n} g_i(T)$, we conclude that $\text{TC}(T)$ is piece-wise convex with respect to $T$ since it is the sum of $n$ convex functions (namely, $g_i(T)$) and $n$ piece-wise convex functions (namely, $\text{TC}_i(T)$). $\square$

**Theorem 3.5.** All the junction points for each individual item $i$ will be inherited by the $\text{TC}(T)$ curve.

**Proof.** One may easily observe that $\text{TC}(T)$ is a separable function. Without loss of generality, assume that $w$ is a junction point for an item $i$, but not a junction point for the other $(n-1)$ items. Then, there must exist $\varepsilon > 0$ such that the followings hold.

1. The curve for $\sum_{j\neq i} \text{TC}_j(T)$ is convex in the interval of $[w - \varepsilon, w + \varepsilon]$ since each one of $\text{TC}_j(T)$ is convex in $[w - \varepsilon, w + \varepsilon]$ where $j \neq i$.
2. $\text{TC}_i(T)$ is convex in the intervals of $[w - \varepsilon, w]$ and $[w, w + \varepsilon]$.
3. $\sum_{i=1}^{n} g_i(T)$ is convex in the intervals of $[w - \varepsilon, w]$ and $[w, w + \varepsilon]$.

Since $\text{TC}(T) = \sum_{j\neq i} \text{TC}_j(T) + \text{TC}_i(T) + \sum_{i=1}^{n} g_i(T)$, $\text{TC}(T)$ is still convex in the intervals $[w - \varepsilon, w]$ and $[w, w + \varepsilon]$. Therefore, $w$ is a junction point on the curve of $\text{TC}(T)$. $\square$

**4. The proposed search algorithm.** In this section, we present a search scheme, which obtains an optimal solution for our ELISP model. Note that we consider the constraints for the production capacity and the inspection capacity as performing the search algorithm.

Our theoretical results in Section 3 encourage us to solve the problem in (1) to (5) by searching along the $T$-axis. We first need to define the search range by a lower bound and an upper bound on the $T$-axis. Also, if we are able to obtain all of the local minima for each convex curve on $\text{TC}(T)$ in the search range, we surely can obtain an optimal solution by picking the one with the lowest objective value.

In the following subsections, we first derive a lower bound on the search range. Then, we demonstrate how to use the junction points to proceed with the search. In addition, we propose an approach to obtain and revise the upper bound on the search range. Finally, we summarize the proposed search algorithm.

**4.1. Lower bound.** We derive a lower bound $T_{LB}$ of the search range by considering three conditions simultaneously: (1) the minimum cycle time by the common cycle (CC) approach in which it requires that $m_i = 1$, $i = 1, 2, \ldots, n$ (Note that all the items share the same cycle time for inspection in the CC approach.), (2) the minimum cycle time keep production feasible, and (3) the minimum cycle time keep inspection feasible. These three candidate low bounds are explained as follows.

Denote $T_{CC}$ as the optimal production cycle for the CC approach, $T_{PC}$ as the optimal production cycle for keeping production feasible and $T_{IP}$ as the optimal production cycle for keeping inspection feasible. Then one may easily obtain the low bound $T_{LB}$ by maximizing these three optimal production cycles.
$$T_{LB} = \max \{T_{CC}, T_{PC}, T_{IC}\} =$$

$$\max \left\{ \frac{\sum_{i=1}^{n} (A_i + c_i)}{\sum_{i=1}^{n} \left( d_i h_i (d_i + p_i) + \alpha_i d_i^2 p_i (\tau_i + r_i) \right)}, \frac{\sum_{i=1}^{n} s_i}{1 - \sum_{i=1}^{n} \frac{d_i}{y_i}}, \frac{\sum_{i=1}^{n} z_i}{1 - \sum_{i=1}^{n} \frac{d_i}{y_i}} \right\}$$

(16)

**Theorem 4.1.** For the $TCC(T)$ function, there exist no local minimum for $T < T_{LB}$.

**Proof.** Recall that Theorem 3.4 asserts that $TCC(T)$ is piece-wise convex. It implies that the global optimal solution must be one of its local minima. For any given vector $\mathbf{m} = (m_1, m_2, \ldots, m_n)$, we may locate its corresponding local minimum, denoted by $\hat{T}(m_1, m_2, \ldots, m_n)$, on the $T$-axis by taking the first-derivative of the objective function with respect to $T$ and setting it equal to zero. The closed-form for locating $\hat{T}(m_1, m_2, \ldots, m_n)$ can be expressed by

$$\hat{T}(m_1, m_2, \ldots, m_n) = \max \left\{ \frac{\sum_{i=1}^{n} (A_i + c_i m_i)}{\sum_{i=1}^{n} \left( d_i h_i (d_i + p_i) + m_i (p_i y_i - d_i p_i y_i + \alpha_i d_i^2 p_i (\tau_i + r_i)) \right)} \right\}$$

(17)

It is obvious that $\hat{T}(m_1, m_2, \ldots, m_n) \geq T_{CC}$ as $m_i \geq 1$ for all $i$. Therefore, there exists no local minima for $T < T_{CC}$.

Theorem 4.1 implies that we may employ $T_{CC}$ as an initial point for the search algorithm to start the search from $T_{CC}$ toward larger values of $T$ (until it meets an upper bound, denoted by $T_{UB}$). But we have to consider the optimal low bound $T_{LB}$ and it must be feasible for the production capacity constraint $\sum_{i=1}^{n} (s_i + \frac{T_{d_i}}{p_i} + \frac{T_{d_i}}{y_i}) \leq T$ and inspection capacity constraint $\sum_{i=1}^{n} (z_i + \frac{d_i T}{y_i} + \frac{d_i T (m_i - 1)}{m_i p_i}) \leq T$. Therefore, we get the $T_{PC}$ and $T_{IC}$ by $\sum_{i=1}^{n} (s_i + \frac{T_{d_i}}{p_i} + \frac{T_{d_i}}{y_i}) = T$ and $\sum_{i=1}^{n} (z_i + \frac{d_i T}{y_i} + \frac{d_i T (m_i - 1)}{m_i p_i}) = T$.

And the optimal lower bound $T_{LB}$ is calculated by maximizing $T_{CC}$, $T_{PC}$ and $T_{IC}$. There exist no feasible local minimum for $T < T_{LB}$. \qed

### 4.2. Proceeding with the search by junction points

In this section we show how to proceed with the search by utilizing a sequence of (sorted) junction points. By Theorem 3.2 and 3.5, each junction point $w_i(m_i)$ provides the information that one should change the optimal number of inspection of item $i$ from $m_i$ to $m_i + 1$ at $w_i(m_i)$ to obtain the optimal value for the $TCC(T)$ function. Therefore, during the search, we need to keep an $n$-dimensional vector $(w_1(m_1^* + 1), w_2(m_2^* + 1), \ldots, w_n(m_n^* + 1))$ in which each value of $w_i(m_i^* + 1)$ indicates the location of the next junction point of each item $i$ where the item $i$’s optimal number of inspections should be changed. As the algorithm search toward larger values of $T$, one should change the number of inspections for the particular item $i$ with the smallest value of $w_i(m_i^* + 1)$ to currently update the vector of optimal numbers of inspections. Let $T_{e}$ represents the current value of $T$ where the search algorithm reaches. Denote $\delta$ as the item index for the item $i$ with the smallest value of $w_i(m_i^* + 1)$, i.e.,
\[
\delta = \arg \min_i \{ w_i(\delta^*_i + 1) > T_c \} \tag{18}
\]

Denote the vector of the optimal numbers of inspection at \( T_c \) by \( \mathbf{m}(T_c) \), i.e., \( \mathbf{m}(T_c) = (m^*_1(T_c), m^*_2(T_c), \ldots, m^*_n(T_c)) \). Since the optimal number of inspection should not exceed the size of inspection lot for each item, during the search, i.e. \( m_i \leq T d_i, i = 1, 2, \ldots, n \). So, for item \( i \) is reached, then we do not change the inspection number \( m_i \) of item \( i \) in following search process.

If \( m^*_i(T_c) \leq T d_i, i = 1, 2, \ldots, n \), then we have to process with the search from \( T_c \), by Theorem 1, we need to update the vector of optimal numbers of inspection at \( w_j m^*_j + 1 \) by

\[
\mathbf{m}(w_j(m^*_j + 1)) = (\mathbf{m}(T_c) \setminus \{ m^*_j \}) \cup \{ m^*_j + 1 \} \tag{19}
\]

where ‘\( \setminus \)’ denotes set subtraction.

Let \( \{ w_j \} \) be the sequence of points that the algorithm reaches. In addition, by the definition, we have \( w_0 = T_{UB} \). From another point of view, the algorithm searches over the set \( \{ w_j \} \), a sequence of junction points from \( w_0 \), where \( w_{j+1} \geq w_j, j = 0, 1, 2, \ldots \). Note that this array is sorted on the location of the junction points in ascending order except that the initial point \( T_{UB} \) may not be a junction point. Lemma 1 asserts that the vector of optimal numbers of inspection for the \( TC(T) \) function is invariant between \( w_{j+1} \) and \( w_j \). Therefore, \( \mathbf{m}(w_j) \) is the vector of optimal numbers of inspection for the \( TC(T) \) function in the interval \( [w_{j+1}, w_j] \). Denote as \( \hat{T}(\mathbf{m}(w_j)) \) the local minimum for the vector of optimal numbers of inspection \( \mathbf{m}(w_j) \) where \( \hat{T}(\mathbf{m}(w_j)) \) can be obtained by eq. (17).

4.3. Upper bound. Note that the search stops when it surpasses the upper bound of the search range. Therefore, the upper bound serves as the termination condition of the proposed search algorithm. We derive an upper bound for such a purpose in this section.

Suppose that \( \mathbf{m}(w_j) \) is the vector of the optimal inspection number in the interval \([w_j, w_{j+1}]\) (i.e. between two consecutive junction points on \( T \)-axis). Following our presentation in Section 4.2, we obtain a local minimum \( \hat{T}(\mathbf{m}(w_j)) \). We obtain the value of the upper bound, denoted by \( T_{UB} \), at the first local minimum secured in the search (i.e., \( \hat{T}(\mathbf{m}(w_0)) \)). Then, we will try to revise the upper bound at each local minimum (i.e., \( \hat{T}(\mathbf{m}(w_j)) \)) during the search.

Looking ahead from a local minimum \( \hat{T} \), if there exists no solution that can obtain a lower objective function value than \( TC(\mathbf{m}(\hat{T}), \hat{T}) \) for \( T > \beta \), then the value of \( \beta \) may serve as an upper bound of the search. Lemma 4.2 suggests a closed form for the upper bound \( \beta \).

Lemma 4.2. At a local minimum \( \hat{T} \), one may secure an upper bound \( \beta \) on the search range by

\[
\beta = \frac{X + \sqrt{X^2 - 2 \left( \sum_{i=1}^{n} \frac{d_i h_i (p_i y_i - d_i p_i - d_i y_i)}{p_i y_i} \right) \left( \sum_{i=1}^{n} A_i \right)}}{\sum_{i=1}^{n} \left( \frac{d_i h_i (p_i y_i - d_i p_i - d_i y_i)}{p_i y_i} \right)} \tag{20}
\]
where

\[
X = \sum_{i=1}^{n} \frac{A_i}{T} + \sum_{i=1}^{n} \frac{T_d h_i (p_i y_i - d_i p_i - d_i y_i)}{2 p_i y_i} + \sum_{i=1}^{n} \phi_i(m_i^*(\tilde{T}), \tilde{T}) \tag{21}
\]

\[
\phi_i(m_i^*(\tilde{T}), \tilde{T}) = \left\{ \begin{array}{cl}
\frac{\alpha_i}{T} + \frac{T_d^2 [(p_i + y_i) h_i + h_i y_i + \alpha_i p_i y_i (\pi_i + r_i)]}{2 p_i y_i} & - \frac{\Omega_i}{\sqrt{2 c_i y_i c_i}}, m_i^*(\tilde{T}) = 1 \\
\frac{\Omega_i}{\sqrt{2 c_i y_i c_i} - (\frac{m_i^*(\tilde{T})}{m_i^*(\tilde{T} + 1)})} & m_i^*(\tilde{T}) \geq 2
\end{array} \right.
\tag{22}
\]

and

\[
\Omega_i = 2 c_i d_i \sqrt{h_i (p_i + 2 y_i) + \alpha_i p_i y_i (\pi_i + r_i)}
\tag{23}
\]

Proof. Before deriving our upper bound \(\beta\), we have the following observations on the two costs involved as we increase the value of \(T\) from \(\tilde{T}\) to any larger value of \(T\).

1. The value of the holding cost increases by

\[
\sum_{i=1}^{n} \frac{T_d h_i (p_i y_i - d_i + p_i - d_i y_i)}{2 p_i y_i} - \sum_{i=1}^{n} \frac{T_d h_i (p_i y_i - d_i + p_i - d_i y_i)}{2 p_i y_i}
\tag{24}
\]

2. The value of the setup cost decreases by

\[
\sum_{i=1}^{n} A_i / T - \sum_{i=1}^{n} A_i / \tilde{T}
\tag{25}
\]

Next, we should analyze how the value of the function \(TC_i(m_i, T)\) changes as we increase the value of \(T\) from \(\tilde{T}\) to any larger value of \(T\). For each item \(i\), we define the function \(\phi_i(m_i^*(\tilde{T}), \tilde{T})\) such that

\[
\phi_i(m_i^*(\tilde{T}), \tilde{T}) = TC_i(m_i^*(\tilde{T}), \tilde{T}) - TC_i(m_i^*(\tilde{T}), T)
\tag{26}
\]

We note that the function \(\phi_i(m_i^*(\tilde{T}), \tilde{T})\) indicates the maximum magnitude of decrement in \(TC_i(m_i, T)\) from \(\tilde{T}\) to any larger value of \(T\). At the local minimum \(\tilde{T}\), two possible cases exist for \(\phi_i(m_i^*(\tilde{T}), \tilde{T})\) based on \(m_i^*(\tilde{T})\); i.e., the item \(i\)'s optimal number of inspections:

1. The case of \(m_i^*(\tilde{T}) = 1\): From Lemma 3.3, we assert that the minimum value of \(TC_i(m_i, T)\) is \(2 c_i d_i \sqrt{h_i (p_i + 2 y_i) + \alpha_i p_i y_i (\pi_i + r_i)} / \sqrt{2 p_i y_i c_i}\). Then, we easily have \(\phi_i(m_i^*(\tilde{T}), \tilde{T})\) by

\[
\phi_i(m_i^*(\tilde{T}) = 1, \tilde{T}) = \frac{c_i}{T} + \frac{T_d^2 [(p_i + y_i) h_i + h_i y_i + \alpha_i p_i y_i (\pi_i + r_i)]}{2 p_i y_i}
\tag{27}
\]

2. The case of \(m_i^*(\tilde{T}) > 1\): From Fig. 2, we are sure that

\[
TC_i(m_i^*(\tilde{T}), \tilde{T}) \leq \max\{TC_i(m_i^*(\tilde{T}), w_i(m_i^*(\tilde{T}))), TC_i(m_i^*(\tilde{T}), w_i(m_i^*(\tilde{T}) + 1))\}.
\tag{28}
\]
In fact, we can easily show that

$$TC_i(m^*_i(\bar{T}), w_i(m^*_i(\bar{T}))) \geq TC_i(m^*_i(\bar{T}), w_i(m^*_i(\bar{T}) + 1)).$$

(29)

Therefore, we have the value of $\phi_i(m^*_i(\bar{T}), \bar{T})$ given by

$$\phi_i(m^*_i(\bar{T}), \bar{T}) = TC_i(m^*_i(\bar{T}), T) - \frac{2c_i d_i \sqrt{h_i (p_i + 2y_i) + \alpha_i p_i y_i (\pi_i + r_i)}}{\sqrt{2p_i y_i c_i}} \left( m^*_i + 1 \right)$$

$$= \frac{2c_i d_i \sqrt{h_i (p_i + 2y_i) + \alpha_i p_i y_i (\pi_i + r_i)}}{\sqrt{2c_i p_i y_i m^*_i (m^*_i + 1)}} \right)$$

(30)

By summarizing both cases above, we have the expression of $\phi_i(m^*_i(\bar{T}), \bar{T})$ in (22).

The upper bound is derived by asserting that for $T \geq \beta$, the increment in the sum of holding cost, i.e., $\sum_{i=1}^{n} \left( \frac{(T-\bar{T})d_i h_i (p_i y_i - d_i p_i - d_i y_i)}{2p_i y_i} \right)$, must exceed the maximum magnitude of decrement from the setup cost $\sum_{i=1}^{n} A_i / T - \sum_{i=1}^{n} A_i / \bar{T}$ and $\sum_{i=1}^{n} \phi_i(m^*_i(\bar{T}), \bar{T})$, or $\sum_{i=1}^{n} \left( T - \bar{T} \right) \frac{d_i h_i (p_i y_i - d_i p_i - d_i y_i)}{2p_i y_i} \geq \left( \sum_{i=1}^{n} A_i / T - \sum_{i=1}^{n} A_i / \bar{T} \right) + \sum_{i=1}^{n} \phi_i(m^*_i(\bar{T}), \bar{T})$, which gives exactly eq. (20).

After obtaining a new local minimum during the search, we should try to revise the upper bound $T_{UB}$ by $T_{UB} = \min\{T_{UB}, \beta\}$, where $\beta$ is obtained from eq. (20).

4.4. The algorithm. We are now ready to enunciate the proposed search algorithm. Recall that the algorithm searches from $T_{LB}$ toward higher values of $T$ until it meets the last-revised upper bound $T_{UB}$. In the search process, we use a sequence of (sorted) junction points $\{w_j\}$ as the backbone and secure all the local minima of the $TC(T)$ function in the range of $[T_{LB}, T_{UB}]$. Denote as $[m^*, T^*]$ the best-in-hand solution where $T^*$ and $m^*$ are respectively, the value of $T$ and the corresponding set of the optimal numbers of inspections. During the search, we should try to update the best-on-hand solution and the upper bound when obtaining $\tilde{T}(m(w_j))$, i.e., the local minimum for the set of optimal numbers in the interval $(w_{j+1}, w_j]$ (namely, $m(w_j)$) using the following steps.

1. Test if the feasible solution obtains by checking the optimal number of inspection should not exceed the size of inspection lot for each item. If it is feasible, go to step 2; otherwise, fix $m^*_i = u_i$ if $m^*_i(w_j) > u_i$ for item $i$ is found during the search process, then go to step 2.

2. Test the feasibility of the solution $(m(w_j), \tilde{T}(m(w_j)))$ by checking the constraints of production capacity and inspection capacity. If it is feasible, go to step 3; otherwise, replace $\tilde{T}(m(w_j))$ with the minimum value of $T$ that satisfies both constraints on production capacity and inspection capacity to secure a feasible solution (with the minimum cost for $(m(w_j))$, and go to step 3.

3. Check if the feasible solution obtains an objective function value lower than the best-in-hand solution. If it holds, update the best-in-hand solution $(m^*, T^*)$. 

4. Update the upper bound $T_{UB}$ of the search by $\beta$ in eq. (20) to shorten the search range.
We summarize the step-by-step procedure of the proposed search algorithm as follows.

1. The Initialization:
   (a) Utilize $T_{LB}$ in eq. (16) as an initial point and let upper bound $T_{UB} = \infty$.
   (b) Calculate $m^* = m(T_{LB})$ using eq. (11), obtain $\{w_i(m_i(T_{LB}))\}_{i = 1, \ldots, n}$ using eq. (10) and secure $m^*_i(w_i) \leq u_i$.
   (c) Let $\delta = \arg \min \{w_i(m_i) > T_c\}$ where $T_c = T_{LB}$.
   (d) Obtain $\hat{T}(m(T_{LB}))$, set $m^* = m(T_{LB})$ and $T^* = \hat{T}(m(T_{LB}))$. Check the feasibility of $(m^*, T^*)$, update the value of $T^*$ to secure feasibility if necessary, and calculate $TC(m^*, T^*)$.
   (e) Calculate the best-on-hand upper bound by $T_{LB} = \beta$ following eq. (20), obtain all the junction points in $[T_{LB}, T_{UB}]$ using eq. (10) and secure the sorted sequence $\{w_i\}$ by sorting the junction points in $[T_{LB}, T_{UB}]$ in an ascending order; otherwise, let $TC(m^*, T^*) = \infty$.
   (f) Set $j = 1$ and $T_c = w_j$.

2. The search procedure:
   (a) Obtain $m(w_j)$ by $m(w_j) \equiv (m(T_c) \setminus \{m_\delta\}) \cup \{m_\delta + 1\}$ and proceed to $w_{j+1}$ with $\delta = \arg \{w_i(m_i + 1) = w_{j+1}\}$.
   (b) Check the feasibility of $(m(w_j), \hat{T}(m(w_j)))$, update the value of $T$ to secure feasibility if necessary, and calculate $TC(m(w_j), T(c(w_j)))$, and try to revise the best-on-hand upper bound $T_{UB}$.
   (c) If $TC(m(w_j), \hat{T}(m(w_j))) < TC(m^*, T^*)$, update $m^*$ and $T^*$ with $T^* = \hat{T}(m(w_j))$ (or the minimum value of $T$ to secure a feasible solution) and $m^* = m(w_j)$.

3. The termination condition of the search algorithm: If $w_{j+1} > T_{UB}$, output $(m^*, T^*$ and $TC(m^*, T^*)$ and the algorithm terminates; otherwise, set $j = j + 1$ and $T_c = w_j$, then go to Step 2.

5. Numerical experiments. In this section, we first present a numerical example to demonstrate the implementation of the proposed search algorithm. In the second part of this section, we will use random experiments to show the effectiveness of the proposed search algorithm.

5.1. A demonstrative example. The numerical example presented in this subsection demonstrates the implementation of the proposed search algorithm. Table 2 presents a set of parameters used in this numerical example.

Our search algorithm starts with the initial point at $T_{LB} = \max\{T_{CC}, T_{PC}, T_{IC}\} = 4.336$. The set of optimal inspection numbers with respective to $T_{LB}$ is $m(T_{LB}) = (1, 1, 2, 1, 2, 2, 2, 4, 3, 1)$. Also, we obtain the corresponding local minimum at $\hat{T}(m(T_{LB})) = 6.9428$. The next junction point is located at $w_1 = 4.4014$. The proposed search algorithm terminates after it surpasses the 121th junction point from $T_{LB}$. Table 3 summarizes that the second and the last rows represent the first solution and the last solution in the search process. Those five solutions (the forth to the eighth row) in the search process indicate where the proposed search algorithm obtains feasible solutions at the local minima. The second to the eleventh column indicates the optimal number of inspections of items 1 to 10. One may use them to verify Theorem 3.2 that $m_i^{(R)} = m_i^{(L)} + 1$ for each item $i$. 
Table 2. The set of parameters used in the demonstrative example

| Item No. | $A_i$ | $c_i$ | $e_i$ | $h_i$  | $\pi_i$ |
|----------|-------|-------|-------|--------|--------|
|          | thousand dollars |
| 1        | 15    | 120   | 50    | 0.00002 | 0.00001 | 0.0015 |
| 2        | 20    | 50    | 50    | 0.00015 | 0.00007 | 0.0002 |
| 3        | 30    | 35    | 10    | 0.00018 | 0.00005 | 0.0011 |
| 4        | 10    | 45    | 260   | 0.00001 | 0.00004 | 0.0015 |
| 5        | 110   | 70    | 70    | 0.00025 | 0.00116 | 0.0008 |
| 6        | 50    | 60    | 160   | 0.00003 | 0.00116 | 0.0008 |
| 7        | 310   | 120   | 30    | 0.00014 | 0.00063 | 0.0025 |
| 8        | 130   | 24    | 40    | 0.00005 | 0.00246 | 0.0028 |
| 9        | 200   | 33    | 30    | 0.00001 | 0.00038 | 0.0022 |
| 10       | 5     | 250   | 20    | 0.00016 | 0.00002 | 0.0016 |

Table 2. The set of parameters used in the demonstrative example (continued)

| Item No. | $d_i$ | $p_i$ | $s_i$ | $z_i$ | $y_i$ |
|----------|-------|-------|-------|-------|-------|
|          | unit/day | unit/day | day/lot | day/insp. lot | unit/day |
| 1        | 400    | 30000  | 0.0125 | 0.0009 | 50000 |
| 2        | 400    | 12000  | 0.00125 | 0.0009 | 32000 |
| 3        | 800    | 8000   | 0.0025 | 0.0018 | 60000 |
| 4        | 1600   | 3500   | 0.00125 | 0.0023 | 25000 |
| 5        | 80     | 6500   | 0.005  | 0.0022 | 30000 |
| 6        | 80     | 6015   | 0.0025 | 0.0030 | 60000 |
| 7        | 24     | 3000   | 0.001  | 0.0009 | 25000 |
| 8        | 340    | 5000   | 0.005  | 0.0010 | 25000 |
| 9        | 340    | 6500   | 0.0075 | 0.0027 | 16000 |
| 10       | 400    | 16000  | 0.00125 | 0.0014 | 50000 |

Table 3. A summary on the search process for the demonstrative example.

| $[w_j, w_{j+1}]$ | $m$ | $\bar{T}_j$ | $TC(m, \bar{T}_j)$ |
|-----------------|-----|-------------|-------------------|
| (4.336, 4.4014) | (1, 1, 2, 1, 2, 2, 4, 3, 1) | 9.643 | $607.822$ |
| ...             | ... | ...         | ...               |
| (30.045, 30.373) | (2, 8, 14, 8, 15, 11, 10, 30, 18, 4) | 30.649 | $495.550$ |
| (30.373, 30.780) | (3, 8, 14, 8, 15, 11, 10, 30, 18, 4) | 31.135 | $495.521$ |
| (30.780, 31.062) | (3, 8, 14, 8, 15, 11, 10, 31, 18, 4) | 31.232 | $495.515$ |
| (31.062, 31.414) | (3, 8, 14, 8, 16, 11, 10, 31, 18, 4) | 31.514 | $495.545$ |
| (31.414, 32.080) | (3, 8, 14, 8, 16, 11, 10, 32, 18, 4) | 31.610 | $495.545$ |
| ...             | ... | ...         | ...               |
| (35.471, 32.080) | (3, 9, 16, 9, 18, 12, 11, 35, 21, 5) | 35.252 | $495.741$ |

As one may observe that the optimal solution is obtained at the first (feasible) local minimum with $\bar{T} = 31.232$ days. The set of optimal numbers of inspections is $m = (3, 8, 14, 8, 15, 11, 10, 31, 18, 4)$. The value of the optimal objective function is given by $8495.515$. The set of lot sizing for each inspection lot is $u = (1041, 1562, 781, 781, 1666, 2271, 781, 302, 521, 1592)$.

5.2. Random examples. This section will show the effectiveness of the proposed search algorithm using random experiments. Table 4 presents the set of the parameters (and the uniform distributions) to randomly generate experimental instances.
Table 4. The set range of parameters used in the random example.

| Parameters | Distribution | Parameters | Distribution |
|------------|--------------|------------|--------------|
| $A_i$      | UNIF(0.400)  | $d_i$      | UNIF(100,4900) |
| $v_i$      | UNIF(5.400)  | $p_i$      | UNIF(5000,35000) |
| $h_i$      | UNIF(0.00292) | $s_i$      | UNIF(0.06,0.5) |
| $\pi_i$    | UNIF(0.0001,0.003) | $z_i$    | UNIF(0.0002, 0.0008) |
| $\alpha_i$ | UNIF(0.05,0.2) | $y_i$      | UNIF(40000,900000) |
| $r_i$      | UNIF(0.00002,0.00025) | $c_i$     | UNIF(20,250) |

Our experiments were conducted on a personal computer with a Centrino 1.7GHz CPU and 1G memory.

We would like to observe the effectiveness of the proposed search algorithm under different level of utilization rate and different level of the number of items. (Following our decision-making scenario, we define the utilization rate as the ratio of the total production time plus the all inspection sub-lot consuming time for all items to the cycle time in this study.) We randomly generate 50 instances for eight different levels of utilization rate with the range from 0.5 to 0.9 and eight different levels of the number of items from 5 to 40.

In this random experiment, we employ the average total cost $TC(m_{LB},T_{LB})$ as a cost upper bound (denoted as $TC_{UB}$) which the cycle time is used from the $T_{LB}$ in (16). The $T_{LB}$ represents the minimize cycle time for the objective function guarantee to get a feasible solution. We have the first derivative of $TC_i(m_i,T_{LB})$ with respect to $m_i$ which get the $T_{LB}d_i\sqrt{[h_i(p_i + 2y_i) + \alpha_i p_i y_i(\pi_i + r_i)]/2c_i p_i y_i}$, $i = 1,2,\ldots,n$. $m_{LB}^i$ is the optimal number of inspection for item $i$ under $T_{LB}$. As we obtain the values of two variables $T_{LB}$ and $m_{LB}$, then we can calculate the upper bound $TC_{UB}$ as follows.

$$TC_{UB}(m_{LB}^i,T_{LB}) =$$

$$\sum_{i=1}^{n} \left\{ \frac{A_i + c_i m_{LB}^i}{T_{LB}} + \frac{T_{LB}d_i^2(p_i + y_i)h_i}{2m_{LB}^i p_i y_i} + \frac{d_i T_{LB}(m_{LB}^i p_i y_i - d_i m_{LB}^i p_i - d_i m_{LB}^i y_i + d_i y_i)h_i}{2m_{LB}^i p_i y_i} + \frac{\alpha_i d_i^2 m_{LB}^i T_{LB}(\pi_i + r_i)}{2} \right\}$$

(31)

The $TC_{UB}$ is the cost value of an easy and feasible solution can serve as an upper bound on the optimal objective function value. Denote $TC^*$ as the optimal objective function value obtained from the proposed search algorithm for a particular instance. Then, we may calculate the value of $\Delta$ in eq. (32) to measure the deviation of the objective function value of the solution obtained from the proposed search algorithm.

$$\Delta = (TC_{UB} - TC^*)/TC_{UB}\cdot 100\%$$

(32)

We note that the utilization rate may affect the solution quality of the proposed search algorithm since we take into accounts the constraints on production capacity and inspection capacity in the decision-making scenario.

We summarize our results in Table 5 with the “Avg. (%) Δ” and “Avg. Run Time” indicating the average improvement percentage of the average total cost and the average run time of the 50 instances, respectively. One may have the following
observations: the higher the utilization rate, the smaller the average value of $(\%)\Delta$ (for a given level of the number of items). Also, the average value of $(\%)\Delta$ decreases as the number of items increases for any level of utilization rate.

From Table 5, one may observe that the average value of $(\%)\Delta$ is impressively small (very close to upper bound $T\bar{C}_{UB}$) as the utilization rate is high or the number of items is large. Therefore, we recommend the proposed search algorithm for the cases in which the utilization rate is low or the number of items is less.

Table 5 also shows that the average run time of our proposed algorithm grows mild with the problem size (i.e., the number of items). We employ the polynomial-order testing procedure in Chapter 9 Polynomial Regression of Neter, Wasserman and Kutner [7]. The curve-fitting result suggests that the average run time grows quadratically in the problem size. Our numerical experiments demonstrate that the proposed search algorithm is an efficient solution approach for solving the ELISP using the CC approach since it is approximately an $O(n^2)$ algorithm.

6. **Concluding remarks.** In this study, we are interested in an economic lot scheduling problem (ELSP) with inspection. By using CC approach and taking into accounts the production capacity and inspection capacity, we develop a search algorithm to solve the optimal replenishment cycle time and the optimal inspection schedule so as to minimize the total costs per unit time. We would like to indicate the three contributions of this paper as follows.

First, to the best of the authors’ knowledge, this article is the first one that considering inspection work for ELSP. (Please refer to section 1.2 for details.) Our theoretical results in this paper shall establish an important foundation for the other lot sizing problems with considering inspection work in their decision-making scenario. Second, in the literature, this proposed model is among the pioneer works that considers the production lot sizing problem with inspection for multiple items (Please refer to section 1.3 for details.). Third, the computational experiments indicate that our proposed search algorithm and the upper bound can be useful tools to obtain the optimal feasible solution for production managers in their decision scenario.

There exists plenty of room for this study’s extensions. Some recommendations for further research including to extend the work to the extended basic period (EBP) solution approach that always can get better solutions than the CC approach as shown in the literature. An additional line of investigation could be to incorporate stochastic demand and consumption rate that proved to be of value in the case of the classical ELSP. Also, we may apply our search algorithm to solve other extensions like reworking or pricing. The authors are currently working on the above topics.

**Appendix A.** The Inventory level of the produced and the inspected items in the production system are shown in Fig. 3.
Table 5. The experimental results for different level of $\sum \rho _i$. (continued)

| No. of items | $\sum \rho _i$ | [0.50,0.55) | [0.55,0.60) | [0.60,0.65) | [0.65,0.70) |
|--------------|----------------|--------------|--------------|--------------|--------------|
| 5            | Avg. (%) $\Delta$ | 13.40%       | 12.22%       | 9.00%        | 8.07%        |
|              | Avg. Run Time†    | 1.418        | 1.410        | 1.406        | 1.401        |
| 10           | Avg. (%) $\Delta$ | 12.70%       | 10.41%       | 8.30%        | 7.65%        |
|              | Avg. Run Time     | 1.425        | 1.421        | 1.431        | 1.409        |
| 15           | Avg. (%) $\Delta$ | 11.75%       | 8.93%        | 7.68%        | 7.00%        |
|              | Avg. Run Time     | 1.428        | 1.425        | 1.420        | 1.415        |
| 20           | Avg. (%) $\Delta$ | 10.66%       | 8.26%        | 7.37%        | 6.50%        |
|              | Avg. Run Time     | 1.434        | 1.431        | 1.427        | 1.424        |
| 25           | Avg. (%) $\Delta$ | 9.78%        | 7.84%        | 6.47%        | 5.21%        |
|              | Avg. Run Time     | 1.441        | 1.438        | 1.435        | 1.430        |
| 30           | Avg. (%) $\Delta$ | 9.73%        | 7.77%        | 6.62%        | 5.56%        |
|              | Avg. Run Time     | 1.446        | 1.445        | 1.441        | 1.438        |
| 35           | Avg. (%) $\Delta$ | 9.25%        | 7.83%        | 6.14%        | 4.31%        |
|              | Avg. Run Time     | 1.479        | 1.466        | 1.435        | 1.447        |
| 40           | Avg. (%) $\Delta$ | 8.15%        | 7.14%        | 5.55%        | 4.09%        |
|              | Avg. Run Time     | 1.526        | 1.488        | 1.471        | 1.461        |

†Measured in Seconds
Figure 3. The inventory level of the produced and the inspected items in the production system.
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