Direct scheme for measuring the geometric quantum discord

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Received 22 October 2011, in final form 7 February 2012
Published 1 March 2012
Online at stacks.iop.org/JPhysA/45/115308

Abstract

We propose a scheme to directly measure the exact value of the geometric quantum discord of an arbitrary two-qubit state. We need only to perform the projective measurement in all the anti-symmetric subspace and our scheme is parametrically efficient in contrast to the widely adopted quantum state tomography scheme in the sense of less parameter estimations and projectors. Moreover, the present scheme can be easily realized with the current experimental techniques.

PACS numbers: 03.65.Ta, 03.65.Ud

(Some figures may appear in colour only in the online journal)

1. Introduction

Quantum correlation is believed to be a unique property possessed by a quantum system and plays an essential role in quantum information theory. Traditionally, studies on quantum correlation have mainly focused on entanglement [1], which is considered to be a crucial resource for the vast majority of quantum information processing. However, not all advantages of quantum information tasks are attributable to quantum entanglement. For example, deterministic quantum computation with a one-qubit model is believed to estimate the trace of a unitary matrix more efficiently than any classical information processors, whereas there is little or no entanglement during the whole processing [2]. This implies the possible presence of other types of, beyond quantum entanglement, quantum correlation measured by quantum discord in the separable (unentangled) state [3, 4].

Even though quantum discord has attracted considerable attention in quantum information theory involving quantum computation [5–7], dynamics of quantum discord [8–10], operational interpretations of quantum discord [11–14], etc, only the geometric quantum discord (GQD) defined by the distance between a general quantum state and its closest classical state is analytically calculative in mathematics [15]. However, can we directly measure the GQD in experiment? It does not seem to be an easy task, since quantum discord is a...
complicated nonlinear function of the density matrix of the state. To the best of our knowledge, the only method used to exactly estimate the value of quantum discord in the laboratory is the so-called quantum state tomography (QST) [16–18], where one has to measure a complete set of observables to reconstruct the density matrix, and then evaluate the exact value of quantum discord mathematically. It is obvious that the number of observables will grow exponentially with the increasing dimension. Another more direct method is the witness of quantum discord, but unfortunately it requires some a priori knowledge of the state to be detected [19] and only provides a lower bound of quantum discord [20].

In this paper, we propose a scheme to directly measure the GQD of a completely unknown two-qubit state. The present scheme can be used to measure the exact value of GQD rather than its lower or upper bound. In addition, the GQD is measured by the estimation of three parameters in joint measurements of a small number (not more than six) of copies. Compared with QST, our scheme has the following advantages: (i) it requires fewer projective measurements than QST; (ii) we need only to estimate 3 parameters instead of 15 parameters as in QST; (iii) the GQD can be obtained with only local projective measurements on subsystems with the assistance of classical communications.

The paper is organized as follows. In section 2, we first give a brief review of the GQD, and then demonstrate the scheme for directly measuring the GQD of a two-qubit state in detail. Finally, the conclusion is drawn.

2. Scheme for directly measuring the GQD

2.1. Geometric quantum discord

In [15], Dakić et al introduced the GQD to measure the quantum correlation from the geometric perspective, which has the following form:

$$D(\rho) = \min_{\chi \in \Omega_0} ||\rho - \chi||^2, \quad (1)$$

where $\Omega_0$ denotes the set of classical (zero-discord) states and $||X - Y||^2 = tr(X - Y)^2$ stands for the square norm in the Hilbert–Schmidt space. In particular, it is possible to obtain an explicit expression of the GQD for a general two-qubit state [15]. In the Bloch representation, a two-qubit state $\rho$ can be written as follows:

$$\rho = \frac{1}{4} \left( I \otimes I + \sum_{i=1}^{3} x_i \sigma_i \otimes I + \sum_{i=1}^{3} y_i I \otimes \sigma_i + \sum_{i,j=1}^{3} t_{ij} \sigma_i \otimes \sigma_j \right), \quad (2)$$

where $I$ is the identity matrix, $\sigma_i (i = 1, 2, 3)$ are the Pauli matrices, $x_i = tr(\sigma_i \otimes I) \rho$, $y_i = tr(I \otimes \sigma_i) \rho$ are the components of the local Bloch vectors $\vec{x}$ and $\vec{y}$, respectively, and $t_{ij} = tr(\sigma_i \otimes \sigma_j) \rho$ are the components of the correlation matrix $T$. The classical state in $\Omega_0$ is of the form $\chi = p_1 |\psi_1\rangle_a \langle \psi_1| \otimes \rho_{1b} + p_2 |\psi_2\rangle_a \langle \psi_2| \otimes \rho_{2b}$ with $p_1 + p_2 = 1$, and $\rho_{1b}$ and $\rho_{2b}$ are the marginal density matrices of the subsystem $b$; then the GQD of $\rho$ is given as

$$D(\rho) = \frac{1}{4} (||\vec{x}||^2 + ||T||^2 - \lambda_{\text{max}}), \quad (3)$$

with $\lambda_{\text{max}}$ being the largest eigenvalue of the matrix $K = \vec{x} \vec{x}^t + TT^t$ and $||T||^2 = tr(TT^t)$. The superscript $t$ denotes the transpose of vectors or matrices. The GQD equals the information theoretic quantum discord in identifying the classical state (with no nonclassical correlation). For the states with nonzero quantum correlation, it is shown that the GQD may exhibit different behaviors against the information-theory-based quantum discord [21]. However, one can construct a relation between these two definitions of quantum correlations, that is, by performing a proper unitary operation on $|\psi_1\rangle_a$ and $|\psi_2\rangle_a$, these two orthogonal states will
The GQD presented in equation (3) can be rewritten as follows:

$$D(\rho) = \frac{1}{4} \left( \sum_{i=1}^{3} \lambda_i - \lambda_{\text{max}} \right),$$ \hspace{1cm} (4)$$

where $\lambda_i$s are the eigenvalues of the matrix $K$. In the derivation of equation (4), we have adopted the facts $||\vec{x}||^2 = \text{tr}(\vec{x}\vec{x}^T)$ and $||T||^2 = \text{tr}(TT^T)$.

Recall that the GQD is not a symmetric measure under the permutation of the subsystems; if we choose the zero-discord state in the form of

$$\chi = p_1 \rho_{ab} \otimes |\psi_1\rangle_b \langle \psi_1| + p_2 \rho_{2a} \otimes |\psi_2\rangle_b \langle \psi_2|,$$

then the GQD of $\rho$ under the permutation of the subsystems is given as

$$D(\rho) = \frac{1}{4} \left( \sum_{i=1}^{3} \lambda_i' - \lambda'_{\text{max}} \right),$$ \hspace{1cm} (5)$$

with $\lambda_i'$ being the eigenvalue of the matrix $K' = \vec{y}\vec{y}^T + T^TT$. Without loss of generality, we will mainly focus on the measurement of $D(\rho)$ in the following discussions.

2.2. Directly measuring the GQD

In this subsection, we demonstrate our scheme explicitly. From equation (4), we can see that the acquisition of the eigenvalues of matrix $K$ is sufficient to estimate the GQD of $\rho$; this can be achieved if we can obtain the moments $M_k = \sum_{i=1}^{3} (\lambda_i)^k$, $k = 1, 2, 3$ [22]. The method of estimating the spectrum of a matrix via the moments $M_k$ has been widely used in the schemes of direct measurement of entanglement [23–25]. Although it is known that nonlinear functions of the state parameters cannot be directly obtained by performing measurement on a single copy, we can try to estimate them with several copies. In our scheme, in order to obtain the value of $M_k$, we need to perform several joint measurements on $2^k$ copies of states. For conciseness, we employ $\rho_{mba}$ to denote the $m$th copy and $a, b$ to denote the subsystems of each copy, hereinafter.

Let us start with the attainment of $M_1$; in this procedure, we need two copies for each projective measurement. Because $M_1$ is defined as the summation of the eigenvalues of $K$ which mathematically equals the trace of $K$, we have

$$M_1 = \text{tr}(\vec{x}\vec{x}^T + TT^T) = \text{tr}(\vec{x}\vec{x}^T) + \text{tr}(TT^T)$$

$$= \sum_{i=1}^{3} \left[ \text{tr}(\sigma^i_{a_1} \otimes \mathbb{I}_{b_1}) \rho_{a_1b_1} \right] \cdot \left[ \text{tr}(\sigma^i_{a_2} \otimes \mathbb{I}_{b_2}) \rho_{a_2b_2} \right]$$

$$+ \sum_{i,j=1}^{3} \left[ \text{tr}(\sigma^i_{a_1} \otimes \sigma^j_{a_2}) \rho_{a_1b_1} \cdot \text{tr}(\sigma^j_{a_2} \otimes \sigma^i_{a_1}) \rho_{a_2b_2} \right]$$

$$= \text{tr} \left[ \sum_{i=1}^{3} (\sigma^i_{a_1} \otimes \sigma^i_{a_2} \otimes \mathbb{I}_{b_1} \otimes \mathbb{I}_{b_2}) (\rho_{a_1b_1} \otimes \rho_{a_2b_2}) \right]$$

$$+ \text{tr} \left[ \sum_{i,j=1}^{3} (\sigma^i_{a_1} \otimes \sigma^j_{a_2} \otimes \sigma^j_{a_1} \otimes \sigma^i_{a_2}) (\rho_{a_1b_1} \otimes \rho_{a_2b_2}) \right].$$ \hspace{1cm} (6)$$

In the last equality of equation (6), we have defined

$$U_{a_1a_2} = \sum_{i=1}^{3} \sigma^i_{a_1} \otimes \sigma^i_{a_2} = -4P_{a_1a_2} - \mathbb{I}_{b_1b_2}.$$ \hspace{1cm} (7)
The 11 required projectors are given as follows, see also figure 1.

\[
\begin{align*}
V_{b_a b_b} &= \mathbb{I}_{b_a} \otimes \mathbb{I}_{b_b} + \sum_{i=1}^{3} \sigma^i_{b_a} \otimes \sigma^i_{b_b} = -4P_{b_a b_b}^- + 2\mathbb{I}_{b_a b_b}, \\
\end{align*}
\]

where \( P_{b_a b_b}^- = |\Psi^-\rangle_{s_a s_b} \langle \Psi^-| \) and \(|\Psi^-\rangle_{s_a s_b} = (|0\rangle_{s_a} |1\rangle_{s_b} - |1\rangle_{s_a} |0\rangle_{s_b})/\sqrt{2} \) with \( s = a, b \) and here \( m, n = 1, 2 \). It is obvious that the operators \( U_{a_b a_b} \) and \( V_{b_a b_b} \) are local two-qubit operators that performed on the subsystems \( a \) and \( b \), respectively.

Similarly, we can represent \( M_2 \) and \( M_3 \) in terms of the expectation values of the tensor products of \( U_{a_b a_b} \) and \( V_{b_a b_b} \) with four and six copies of states, respectively. The results are given as follows:

\[
\begin{align*}
M_2 &= \text{tr}(\tilde{\rho}^{\vec{x} \vec{z}'} + TT')^2 \\
&= \text{tr}[ (U_{a_b a_b} \otimes U_{a_b a_b} \otimes V_{b_b b_b} \otimes V_{b_b b_b}) (\rho_{a_b a_b} \otimes \rho_{a_b a_b} \otimes \rho_{a_b a_b} \otimes \rho_{a_b a_b}) ], \tag{9} \\
M_3 &= \text{tr}(\tilde{\rho}^{\vec{x} \vec{z}'} + TT')^3 \\
&= \text{tr}[ (U_{a_b a_b} \otimes U_{a_b a_b} \otimes U_{a_b a_b} \otimes V_{b_b b_b} \otimes V_{b_b b_b}) (\rho_{a_b a_b} \otimes \rho_{a_b a_b} \otimes \rho_{a_b a_b} \otimes \rho_{a_b a_b} \otimes \rho_{a_b a_b} \otimes \rho_{a_b a_b} \otimes \rho_{a_b a_b} \otimes \rho_{a_b a_b}) ], \tag{10}
\end{align*}
\]

By substituting equations (7) and (8) into equations (6), (9) and (10), we can represent each \( M_k \) in terms of the sum of the tensor products of \( P_{s_a s_b}^- \); each term in the summation corresponds to a joint measurement on the multiple copies. It is interesting that the joint measurements are composed of a number of projective measurements which are performed on two qubits in the same subsystem, which means that the value of \( M_3 \) can be obtained by performing a set of local two-qubit projective measurements. Although the new expressions of \( M_k \)'s seem to be complex, even tedious, they can be simplified since the outcomes of the joint measurements are not independent. After simple calculations, we find that only 11 projective measurements are required. The number of the projective measurements is less than that required in QST. The 11 required projectors are given as follows, see also figure 1.

For the two-copy case:

\[
\begin{align*}
P_1 &= P_{a_1 a_2}^- \otimes P_{b_1 b_2}, \\
P_2 &= P_{a_1 a_2}^- \otimes \mathbb{I}_{b_1 b_2}, \\
P_3 &= \mathbb{I}_{a_1 a_2} \otimes P_{b_1 b_2}, \\
\end{align*}
\]

for the four-copy case:

\[
\begin{align*}
P_4 &= P_{a_1 a_2}^- \otimes P_{a_1 a_2}^- \otimes P_{b_1 b_2}^- \otimes P_{b_1 b_2}^-; \\
P_5 &= P_{a_1 a_2}^- \otimes \mathbb{I}_{a_1 a_2} \otimes P_{b_1 b_2}^- \otimes P_{b_1 b_2}^-; \\
P_6 &= \mathbb{I}_{a_1 a_2} \otimes P_{a_1 a_2}^- \otimes P_{b_1 b_2}^- \otimes \mathbb{I}_{b_1 b_2}, \\
P_7 &= \mathbb{I}_{a_1 a_2} \otimes \mathbb{I}_{a_1 a_2} \otimes P_{b_1 b_2}^- \otimes \mathbb{I}_{b_1 b_2}. \\
\end{align*}
\]

for the six-copy case:

\[
\begin{align*}
P_8 &= P_{a_1 a_2}^- \otimes P_{a_1 a_2}^- \otimes P_{a_1 a_2}^- \otimes P_{b_1 b_2}^- \otimes P_{b_1 b_2}^- \otimes P_{b_1 b_2}^-; \\
P_9 &= P_{a_1 a_2}^- \otimes \mathbb{I}_{a_1 a_2} \otimes \mathbb{I}_{a_1 a_2} \otimes P_{b_1 b_2}^- \otimes P_{b_1 b_2}^- \otimes P_{b_1 b_2}^-; \\
P_{10} &= \mathbb{I}_{a_1 a_2} \otimes P_{a_1 a_2}^- \otimes \mathbb{I}_{a_1 a_2} \otimes P_{b_1 b_2}^- \otimes P_{b_1 b_2}^- \otimes \mathbb{I}_{b_1 b_2}, \\
P_{11} &= \mathbb{I}_{a_1 a_2} \otimes \mathbb{I}_{a_1 a_2} \otimes \mathbb{I}_{a_1 a_2} \otimes P_{b_1 b_2}^- \otimes P_{b_1 b_2}^- \otimes \mathbb{I}_{b_1 b_2}. \\
\end{align*}
\]
Figure 1. Schematic diagram of the required projective measurements. The circles in orange (green) boxes denote the qubits of subsystem a (b). In each subplot, a pair of orange and green circles (connected by a black dotted line) at the $m$th row denotes the $m$th copy of the state $\rho_{ambm}$. The blue solid curve connecting two circles denotes the operator $P^-$ and the blue dashed curve denotes the identity operator $I$. For example, the last subplot demonstrates the projective measurements $P_{11} = I_{a1}{\otimes}P_{-a2}{\otimes}P_{-a4}{\otimes}P_{-b1}{\otimes}P_{-b3}{\otimes}I_{b5}{\otimes}I_{b6}$.

Set the outcome of the projective measurement $P_i$ as $c_i$; then the values of $M_k$ are given as follows:

\begin{align*}
M_1 &= 16c_1 - 8c_2 - 4c_3 + 2, \\
M_2 &= 256c_4 + 128c_7 - 128(c_5 + 2c_6) - 16(c_3 + 2c_2) + 16(e^2_3 + 4e^2_2) + 4, \\
M_3 &= 4096c_8 - 16(32e^2_3 + 4e^2_4 - 24e^2_2 - 6c_3^2 + 6c_2^2 - 3c_3) \\
&\quad + 192(8c_5^2 + 16c_2c_6 + 4c_3c_5 - 8c_2c_7 - 4c_3c_7 + c_2c_3) \\
&\quad + 384(c_7 + 8c_11 - c_5 - 2c_6 - 8c_9 - 16c_{10}) + 8. \quad (14)
\end{align*}

Thus, we have obtained the moments $M_k$s with direct projective measurements; the eigenvalues $\lambda_i$s can be sequentially determined from the moments using the techniques of [26], where state spectrum estimation was provided. Actually, it is easy to numerically solve out the eigenvalues of $K$ from the moments with the fact that all the moments are real. Note that we can also
obtain the moments of $K'$ by interchanging $c_2$, $c_5$ and $c_9$ with $c_3$, $c_6$ and $c_{10}$, respectively, in equation (14).

In the analysis above, we find that in principle the GQD can be directly and locally measured in experiments. This will be valuable for the case of the correlated qubits shared by two distant participants. Moreover, different from the previous works on entanglement detection, which requires projective measurements on the multiple-qubit space in certain steps [27], the measurements involved in our scheme are all two-qubit projective measurements, which is more feasible in experiments; for instance, the two-qubit projective measurements have already been used for the detection of lower bounds of entanglement in photonic systems [28, 29]. Moreover, we have also roughly compared our scheme with QST utilizing the criterion mentioned in [30]. In our scheme, the number of the parameters to be measured, $r_p$, is 3, which is less than that in QST, 15. In this sense, we can say that our scheme is parametrically efficient. When the number of copies, $r_c$, is concerned, our scheme is a little more consumptive than QST, that is, in each ‘round’ of our scheme the number of consumed copies is larger than that in QST ($44 > 15$). However, the factor $r = r_pr_c$ here is $(r = 132)$ less than that for the QST scheme ($r = 225$).

Finally, we give a brief discussion on the experimental implementation of our scheme. As an example, we restrict our discussion to the photonic system, which has mature manipulation techniques and wide applications in quantum information processing. The required two-qubit projective measurements can be easily realized with the help of beam splitters and single-photon detectors. There also exist two challenges in the experimental implementation: the preparation of the identical copies of an unknown state and the simultaneity of the performance of joint projective measurements. We adopt the method mentioned in [29] to overcome these two points. The correlated photon pairs sequentially generated by the same photon source (undergo the same decoherence channel) are sent into different optical paths, and by carefully arranging the lengths of these optical paths the photons belonging to the same subsystem but generated at different times will arrive at the measuring device simultaneously. As a consequence, both the requirements of the identity of copies and the coincidence of measurements are satisfied. In addition, since the measurement settings for $P_1$, $P_4$ and $P_8$ can also be used to perform the other eight projective measurements, we need only three measurement settings.

3. Conclusions

In conclusion, we have shown that the exact GQD for an arbitrary unknown two-qubit state instead of a lower bound can be directly and locally measured in experiments, provided that no more than six copies of states to be measured are available. Furthermore, our scheme is more parametrically efficient than the widely adopted QST scheme since we need to measure only three parameters, i.e. three moments of the matrix $K$, to obtain the quantum discord. Moreover, the attainment of quantum discord can be achieved by local projective measurements and classical communications; the number of projective measurements is less than that required in QST. In particular, these measurements are all projected onto the two-qubit space which can be easily implemented in experiments. We also expect that the basic idea of our scheme may provide a new point of view to understand quantum correlation.

Acknowledgment

This work was supported by the National Natural Science Foundation of China, no 11175033.
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