Cosmic Inflation: Trick or Treat?

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Discovered almost forty years ago, inflation has become the leading paradigm for the early universe. Originally invented to avoid the fine-tuning puzzles of the standard model of cosmology, the so-called hot Big Bang phase, inflation has always been the subject of intense debates. In this article, after a brief review of the theoretical and observational status of inflation, we discuss the criticisms that have been expressed against it and attempt to assess whether inflation can really be viewed as a successful solution to the above mentioned issues.

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I. INTRODUCTION

The theory of cosmic inflation was invented to solve fine-tuning problems [1–7]. Indeed, the pre-inflationary standard model of cosmology, the hot Big Bang model [8, 9], suffers from a number of issues all related to a fragile adjustment of the initial conditions needed to make it work. For instance, it is well-known that, in a cosmological model without inflation, when one looks at the last scattering surface (lss) where the Cosmic Microwave Background (CMB) radiation was emitted, one looks at different causally disconnected patches of the universe. But, despite being causally disconnected, they all share, approximately, the same temperature. Unless one fine tunes artificially the initial conditions, this fact is not understandable.
Soon after its advent, it was also realized that inflation provides a mechanism for structure formation [5–7]. In brief, the unavoidable vacuum quantum fluctuations of the gravitational and inflaton fields are stretched over cosmological distances by the inflationary cosmic expansion and are amplified by gravitational instability to eventually give rise to the large scale structures observed in our universe and to the CMB temperature anisotropy. This simple idea implies a series of remarkable predictions among which is the fact that the cosmological perturbations spend time outside the Hubble radius, implying the disappearance of the decaying mode and the presence of coherent oscillations in the CMB power spectrum, or the fact that the two-point correlation function of the inflationary fluctuations should be close to scale invariance.

In 1992 the CMB anisotropies were discovered by the COsmic Background Explorer (COBE) satellite [10, 11] and this marked the beginning of a very important experimental effort by the international community to measure, with a high accuracy, these anisotropies in order to constrain the physics of the early universe. This culminated recently with the publication of the Planck data which is a cosmic variance limited experiment [12–21]. The results of these 30 years of experimental work is consistent with the predictions of single field slow-roll inflation with a minimal kinetic term. It is worth emphasizing that, in some cases, what has been confirmed are predictions and not postdictions. In particular, the prediction that the scalar spectral index should be close but not equal to one has been shown to be true at more than five sigmas by the Planck experiment since $n_s = 0.9645 \pm 0.0049$ [18].

Despite these important successes and despite the fact that it has become the leading paradigm for the early universe, inflation has always been the subject of doubts and criticisms [22–25]. Soon after its invention, two questions were mainly discussed, the choice of the inflationary parameters (for instance the coupling constant in the potential) needed to match the level of CMB anisotropies, a question related to model building and to the physical nature of the inflaton field, and the question of initial conditions at the beginning of inflation. Another issue, the graceful exit or how to stop inflation, was also a hot topic but, apparently, the theory of reheating (and then preheating) gave a satisfactory answer [26–29]. But the two first questions remain
debated. In addition, in conjunction with the experimental efforts mentioned above, various theoretical developments also took place. In particular, it was realized that single field slow-roll models are not the only way to realize inflation and, gradually, a large zoo of models started to appear on stage [30–34]. Importantly, some of these scenarios make different predictions that single field slow-roll inflation. For instance, the level of Non-Gaussianity (NG), which is negligible for single field slow-roll models, can be significant for a model with a non-minimal kinetic term.

Another major theoretical development is the claim that inflation can be eternal [35–41]. This is based on the fact that, due to quantum fluctuations, the various causally disconnected patches that are produced during inflation can be such that the value of the inflaton field is different from one patch to another. In particular, there can be patches where, due to quantum fluctuations, the field climbs its potential instead of rolling it down as it does classically. And, as a consequence, this means that there are patches where inflation never stops. This idea, coupled to the concept of a string landscape, leads to the multiverse, an idea which is nowadays the subject of hot discussions.

The aim of this article is to review the present status of cosmic inflation and to assess whether it can be considered as successful given the assumptions on which it rests and given what it has achieved. In particular, we discuss whether, driven out by the door, fine-tuning problems do not simply slip in again by the window under a different name. A warning is also in order at this stage. In this manuscript, we will use the word “fine-tuning” in a loose sense and will not attempt to define this concept very rigorously. In fact, this question is related to a more general one, namely what are the measures relevant for inflation and how they can be justified. This is important, for instance, for the flatness problem or for the problem of initial conditions. However, here, we will say very little about it and we refer the reader to Ref. [42] where these issues are discussed in great detail.

The article is organized as follows. In the next section, Sec. II, we briefly present the, pre-inflationary, standard model of cosmology, namely the hot Big Bang model. We first discuss its theoretical foundations in Sec. II A and, then, in Sec. II B, how astrophysical observations can constrain it. In Sec. III, we review the difficulties of
this model, in particular the horizon problem, see Sec. III A and the flatness problem, see Sec. III B. In Sec. IV, we introduce inflation and discuss how it can solve the above mentioned puzzles in Sec. IV A. In Sec. IV B, we study how it can be realized in practice and show that the presence of a scalar field dominating the energy budget of the universe is a likely possibility. In Sec. IV C, we present the theory of inflationary cosmological perturbations of quantum-mechanical origin which is at the heart of the calculation of CMB anisotropy. In Sec. IV D, we briefly review the consequences for inflation of the recently released Planck data. In Sec. V, we discuss whether inflation is a fine-tuned scenario, in particular we address the question of whether the choices of the parameters needed in order to have a satisfactory model of inflation is “natural”. Then, in Sec. VI, we discuss the initial conditions at the beginning of inflation, first in an homogeneous and isotropic situation in Sec. VIA, then in an homogeneous but anisotropic situation in Sec. VIB and, finally, in a general inhomogeneous situation in Sec. VIC. We also consider the question of initial conditions for the quantum perturbations, the so-called trans-Planckian problem of inflation in Sec. VID. In Sec. VII, we discuss various aspects of the multiverse question. In Sec. VII A, we explain stochastic inflation and in Sec. VII B, we show how the backreaction is usually taken into account leading to the concept of an eternal inflating universe. In Sec. VII C, we point out that there are models where inflation is not eternal and in Sec. VII D, we discuss the consequences of the possible existence of a multiverse for inflation itself. Finally, in Sec. VIII, we present our conclusions.

II. THE STANDARD MODEL OF COSMOLOGY

A. Relativistic Cosmology

Inflation is supposed to be a solution to some issues of the standard model of cosmology. In order to understand why this is the case, clearly, it is necessary to start with a presentation of the standard model itself. Only after having understood its main features, will it be possible to appreciate its unsatisfactory aspects.

The shape of the Universe is controlled by gravity which, in General Relativity, is
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Described by a metric tensor $g_{\mu\nu}(x)$. The action of the system is given by

$$S = -\frac{c^4}{16\pi G N} \int d^4x \sqrt{-g} \left( R + 2\Lambda_B \right) + S_{\text{matter}}. \quad (1.1)$$

This so-called Einstein-Hilbert action involves two fundamental constants, $c = 3 \times 10^8$ m/s and the Newton constant $G_N = 6.67 \times 10^{-11}$ m$^3$ kg$^{-1}$ s$^{-2}$, appropriate for a relativistic theory of the gravitational field. Quantum effects, which are controlled by the Planck constant, $\hbar = 1.05 \times 10^{-34}$ m$^2$ kg s$^{-1}$, are not needed to describe the dynamics of background spacetime. But, as we will see, they play a fundamental role at the perturbative level. In the following, we will work in terms of natural units for which $\hbar = c = 1$. In this system of units, everything can be expressed in terms of energy, in particular $m_{\text{Pl}} \equiv \sqrt{\hbar c/G_N} = 2.17 \times 10^{-8}$ kg. We will also use the reduced Planck mass defined by $M_{\text{Pl}} \equiv m_{\text{Pl}}/\sqrt{8\pi} = 2.43 \times 10^{18}$ GeV.

Let us now describe the quantities appearing in the action (1.1). $g$ denotes the metric tensor where we have defined $g = \det(g_{\mu\nu}(x))$ and $R_{\mu\nu}$ is the Ricci tensor, a contraction of the Riemann tensor. Finally, the cosmological constant $\Lambda_B$ is a dimension two. $R_{\mu\nu}$ denotes the Ricci tensor which is a contraction of the Riemann tensor.

The energy conservation amounts to $\nabla_{\alpha} T^\mu_{\alpha} = 0$, where $\nabla_{\alpha}$ denotes the covariant derivative. Let us notice that energy conservation is compatible with the Bianchi identities, $\nabla_{\alpha} g_{\mu\nu} = 0$ and the fact that the metric tensor has a vanishing covariant derivative. We will also use the reduced Planck mass defined by $m_{\text{Pl}} = \sqrt{\hbar c/G_N} = 2.17 \times 10^{-8}$ kg. We will also use the reduced Planck mass defined by $M_{\text{Pl}} = m_{\text{Pl}}/\sqrt{8\pi} = 2.43 \times 10^{18}$ GeV.

Conservation of energy amounts to $\nabla_{\alpha} T^\mu_{\alpha} = 0$, where $\nabla_{\alpha}$ denotes the covariant derivative. Let us notice that energy conservation is compatible with the Bianchi identities, $\nabla_{\alpha} g_{\mu\nu} = 0$ and the fact that the metric tensor has a vanishing covariant derivative. We will also use the reduced Planck mass defined by $m_{\text{Pl}} = \sqrt{\hbar c/G_N} = 2.17 \times 10^{-8}$ kg. We will also use the reduced Planck mass defined by $M_{\text{Pl}} = m_{\text{Pl}}/\sqrt{8\pi} = 2.43 \times 10^{18}$ GeV.
derivative. We see that the Einstein equations are a priori very complicated since they are partial, second order and non linear differential equations for the metric tensor.

However, the cosmological principle states that the Universe is, on large scales, homogeneous and isotropic. Of course, this assumption is not obvious a priori and must be carefully observationally checked. We refer the reader to Ref. [43] where this point is discussed in details. Moreover, it must also be explained, rather than postulated, since it would be rather contrived to assume that the initial state was so peculiar. We will of course come back to this question at length in the following sections since inflation is a scenario where this question can, in principle, be addressed. As a consequence of the cosmological principle, the metric tensor takes the Friedmann-Lemaître-Robertson-Walker (FLRW) form, namely

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) \gamma_{ij}^{(3)} dx^i dx^j, \]  \hspace{1cm} (1.4)

where \( t \) is the cosmic time and \( x^i \) are space-like coordinates. The quantity \( \gamma_{ij}^{(3)} \) is the metric of the three-dimensional spacelike sections which have a constant scalar curvature. From the above equation, we have the relation \( g_{ij} = a^2(t) \gamma_{ij}^{(3)} \). In polar coordinates, the three-dimensional metric can be written as

\[ \gamma_{ij}^{(3)} dx^i dx^j = \left[ \frac{dr^2}{1 - \mathcal{K}r^2} + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right], \]  \hspace{1cm} (1.5)

while in Cartesian coordinates, it reads

\[ \gamma_{ij}^{(3)} = \delta_{ij} \left[ 1 + \frac{\mathcal{K}}{4} (x^2 + y^2 + z^2) \right]^{-2}. \]  \hspace{1cm} (1.6)

The constant \( \mathcal{K} \) describes the curvature of the spacelike sections [since \( (3)R = 6\mathcal{K} \), see below] and, without loss of generality, can be chosen to be \( \mathcal{K} = 0, \pm 1 \). As is apparent from the previous equations, there is only one unknown function left, the scale factor \( a(t) \) and, moreover, this function is a function of time only.

On the other hand, matter is assumed to be a collection of \( N \) perfect fluids and, as a consequence, its stress-energy tensor is given by the following expression

\[ T_{\mu\nu} = \sum_{i=1}^{i=N} T_{\mu\nu}^{(i)} = \sum_{i=1}^{i=N} \left\{ [\rho_i(t) + p_i(t)] u_\mu u_\nu + p_i(t) g_{\mu\nu} \right\}, \]  \hspace{1cm} (1.7)
where \( \rho_i(t) \) and \( p_i(t) \) are respectively the energy density and pressure of the fluid “\( i \)”. The vector \( u_\mu \) is the four velocity and satisfies the relation \( u_\mu u^\mu = -1 \). In terms of cosmic time this means that \( u^\mu = (1, 0) \) and \( u_\mu = (-1, 0) \). In accordance with the cosmological principle, the quantities \( \rho_i(t) \) and \( p_i(t) \) only depend on time. In order to close the system of equations, the relation between energy density and pressure, namely the equation of state \( p_i = w_i \rho_i \), must also be provided.

We are now in a position to explicit Einstein equations. In the case of a FLRW metric, one arrives at

\[
\frac{\dot{a}^2}{a^2} + \frac{K}{a^2} = \frac{1}{3M^2_{\text{Pl}}} \sum_{i=1}^{N} \rho_i + \frac{\Lambda_B}{3},
\]

\( (1.8) \)

\[-\left( 2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K}{a^2} \right) = \frac{1}{M^2_{\text{Pl}}} \sum_{i=1}^{N} p_i - \Lambda_B.\]

\( (1.9) \)

We see that one has obtained ordinary, non-linear, second order differential equation for the scale factor \( a(t) \). The fact that we now deal with ordinary differential equation is of course due to the cosmological principle and to the fact that the only unknown function in the metric, the scale factor, is a function of time only. Combining the two equations of motion obtained above, one gets an equation which gives the acceleration of the scale factor, namely

\[
\frac{\ddot{a}}{a} = -\frac{1}{6M^2_{\text{Pl}}} \sum_{i=1}^{N} (\rho_i + 3p_i) + \frac{1}{3} \Lambda_B.
\]

\( (1.10) \)

This equation is especially interesting because it provides the condition leading to an accelerated expansion, namely

\[
\rho_T + 3p_T < 0,
\]

\( (1.11) \)

where \( \rho_T = \sum_{i=1}^{N} \rho_i \) and \( p_T = \sum_{i=1}^{N} p_i \) denote the total energy density and pressure (assuming a vanishing cosmological constant or including its contribution in an extra fluid, see below). Since the energy density of matter must be positive, we see that the above condition requires a negative pressure, i.e. some exotic form of matter.

Even if the Einstein equations have been considerably simplified by the use of the cosmological principle, they remain difficult to solve analytically. However, it turns
out that, if the curvature term vanishes and if there is only one fluid with a constant equation of state, an exact solution to the Einstein equations is available. Of course, one can always solve these equations numerically, but exact solutions will be interesting when we discuss the puzzles of the hot Big Bang phase in the next sections. For this reason, we briefly present them. Since the equation of state is supposed to be constant, the conservation equation, which can be written as

\[ \dot{\rho} + 3H(1 + w)\rho = 0, \]  

(1.12)

can be integrated exactly and the solution reads

\[ \rho(t) = \rho_t \left( \frac{a_t}{a} \right)^{3(1+w)}, \]  

(1.13)

where \( \rho_t \) and \( a_t \) are the energy density and the scale factor expressed at a fiducial time \( t_t \) that can be chosen arbitrarily. Then, one inserts the above result in the Friedmann equation, namely

\[ \left( \frac{1}{a} \frac{da}{dt} \right)^2 = \frac{\rho_t}{3M_{Pl}^2} \left( \frac{a_t}{a} \right)^{3(1+w)}, \]  

(1.14)

whose solution can also be found and reads

\[ \left( \frac{a}{a_t} \right)^{\frac{3(1+w)}{2}} = \frac{3(1+w)}{2} \frac{\rho_t^{1/2}}{\sqrt{3M_{Pl}}} t + C. \]  

(1.15)

In this expression \( C \) is an integration constant. Requiring that \( a = a_t \) when \( t = t_t \), one finds that \( C = -3(1 + w)\rho_t^{1/2}t_t/(2\sqrt{3M_{Pl}}) + 1 \). Finally noticing that \( H_t = \rho_t^{1/2}/(\sqrt{3M_{Pl}}) \), one arrives at

\[ a(t) = a_t \left[ \frac{3}{2}(1 + w)H_t (t - t_t) + 1 \right]^{\frac{2}{3(1+w)}}. \]  

(1.16)

The corresponding Hubble parameter can be expressed as \( H(t) = H_t/[3(1 + w)H_t (t - t_t)/2 + 1] \). We notice that the scale factor vanishes when \( t = t_{BB} \) with \( t_{BB} = t_t - 2/[3(1 + w)H_t] \). In some sense, “time begins” at \( t_{BB} \) and it would be meaningless to consider times such that \( t < t_{BB} \). This is of course the famous Big Bang point where the classical analysis breaks down. This singularity is of course a serious problem for the hot Big Bang model. However, it is not considered as a problem
for inflation simply because inflation does not aim at addressing it. It could be solved if, prior to inflation, there is a bounce \[44, 45\] or if quantum gravitational effects take over and somehow regularize the singularity as done, for instance, in quantum cosmology \[46\]. We see that the singularity problem can be treated separately and does not involve the inflationary scenario.

For future convenience, it is also interesting to rewrite the scale factor in terms of \( t_{\text{BB}} \) and one obtains

\[
a(t) = a_f \left( \frac{t - t_{\text{BB}}}{t_f} \right)^{\frac{2}{3(1+w)}}
\]

(1.17)

that is to say a power-law function. For radiation, \( w = 1/3 \), the scale factor behaves as \( a(t) \propto t^{1/2} \) and for pressure-less matter, \( w = 0 \), one has \( a(t) \propto t^{2/3} \). We also notice that the previous expressions are ill-defined if \( w = -1 \). This is just because in that case we have an exponential solution, namely

\[
a(t) = a_f \exp \left[ H_f (t - t_f) \right],
\]

known as the de Sitter solution.

Putting aside the particular case \( w = -1 \), let us finally come back to the fact that, for \( t = t_{\text{BB}} \), the scale factor vanishes. This is clearly not an artifact of the coordinate system used, as is confirmed by a calculation of the scalar curvature

\[
R = \frac{4(1 - 3w)}{3(1 + w)^2} \frac{1}{(t - t_{\text{BB}})^2},
\]

(1.18)

which blows up when \( t \to t_{\text{BB}} \). This confirms the fact that \( t = t_{\text{BB}} \) corresponds to a real singularity\(^1\).

Having introduced the theoretical tools needed in order to understand the hot Big Bang model, we now discuss the parameters that describe the model and how their values can be inferred from cosmological data.

\(^1\) Notice also that for radiation \( R \) is identically zero. Of course, this does not mean that there is no singularity in a radiation-dominated epoch. This can be shown by computing another invariant, for instance \( R_{\mu\nu}R^{\mu\nu} \) which reads

\[
R_{\mu\nu}R^{\mu\nu} = R_{00}R^{00} + R_{ij}R^{ij} = 9 \left( \frac{\ddot{a}}{a} \right)^2 + \left( \frac{\ddot{a}}{a} + 2 \frac{\dot{a}^2}{a^2} \right)^2 g_{ij} g^{ij}
\]

\[
= 12 \left( \frac{\ddot{a}}{a} \right)^2 + 12 \frac{\ddot{a}^2}{a^2} + 12 \left( \frac{\dot{a}}{a} \right)^4
\]

\[
= \frac{48(3w^2 + 1)}{27(1 + w)^4} \frac{1}{(t - t_{\text{BB}})^4}.
\]

(1.19)

Clearly, \( R_{\mu\nu}R^{\mu\nu} \) blows up as \( t \to t_{\text{BB}} \) even if \( w = 1/3 \).
B. The Real Universe

In order to describe our Universe, we need to know its energy budget, namely the contribution of the different forms of energy density present in the Universe. Our Universe is made of photons, with energy density $\rho_\gamma$, neutrinos with energy density $\rho_\nu$, baryons with energy density $\rho_b$, cold dark matter with energy density $\rho_c$ and dark energy with energy density $\rho_\Lambda$ (here assumed to be a cosmological constant). Photons and neutrinos have an equation of state $1/3$, baryons and cold dark matter have a vanishing equation of state and, finally, dark energy has an equation of state $-1$. We have therefore three types of fluids, radiation $\rho_r = \rho_\gamma + \rho_\nu$, matter $\rho_m = \rho_b + \rho_{cdm}$ and dark energy $\rho_\Lambda$. Their relative importance must be inferred from observations. In order to describe the results of those observations, it is convenient to introduce new quantities. Let us first define the critical energy density: in order to do so, we rewrite the Friedmann equation, Eq. (1.8), as

$$H^2 + \frac{\mathcal{K}}{a^2} = \frac{1}{3M_{\text{Pl}}^2} \left( \rho_\Lambda + \sum_{i=1}^{N} \rho_i \right),$$

with $\rho_\Lambda = \Lambda B M_{\text{Pl}}^2$ the vacuum energy density. We then define the critical energy density by $\rho_{\text{cri}} \equiv 3H^2M_{\text{Pl}}^2$, which is clearly a time-dependent quantity. Then, the Friedmann equation can be rewritten as

$$1 + \frac{\mathcal{K}}{a^2H^2} = \frac{\rho_T}{\rho_{\text{cri}}},$$

where $\rho_T = \rho_\Lambda + \sum_{i=1}^{N} \rho_i$ is the total energy density [compared to the definition below Eq. (1.11), we have now explicitly included the contribution of the cosmological constant in the total energy density]. This means that, if the spatial curvature vanishes then $\rho_T = \rho_{\text{cri}}$ and if $\mathcal{K} > 0$ (respectively $\mathcal{K} < 0$) then $\rho_T > \rho_{\text{cri}}$ (respectively $\rho_T < \rho_{\text{cri}}$). One can also express the weight of a given form of matter by the quantity $\Omega_i$ defined by

$$\Omega_i \equiv \frac{\rho_i}{\rho_{\text{cri}}},$$

and, as a consequence, the Friedmann equation can be re-written as

$$1 + \frac{\mathcal{K}}{a^2H^2} = \Omega_\Lambda + \sum_{i=1}^{N} \Omega_i.$$
In particular, if the spacelike sections are flat then the sum of all the $\Omega_i$’s should be one. It follows from the previous considerations that the contributions of the different forms of energy density in our Universe are expressed through $\Omega_i^0 = \rho_i^0 / \rho_{\text{cri}}^0$, namely the quantity $\Omega_i$ evaluated at present time. The critical energy density today is $\rho_{\text{cri}}^0 = 3H_0^2M_P^2$ with $H_0 = 100h \text{ km} \cdot \text{s}^{-1} \cdot \text{Mpc}^{-1}$, where $h$ takes into account the uncertainty about $H_0$ (recent measurements indicate that $h \simeq 0.67$ [17]). $H_0$ has clearly the dimension of the inverse of a time (is of dimension one) and the above strange units are used because of the measurement of $H_0$ was historically performed using the Hubble diagram [47–50]. In standard units, one has $H_0 = 3.24h \times 10^{-18} \text{s}^{-1}$ while in natural units $H_0 = 2.12h \times 10^{-42} \text{GeV}$. Therefore, we see that, by high energy standards, the current expansion of the Universe is a low energy phenomenon. Given the value of the reduced Planck mass, this implies that $\rho_{\text{cri}}^0 \simeq 8.0990h^2 \times 10^{-47} \text{GeV}^4$.

Let us now describe the composition of our Universe. Data analysis is complicated as it depends on which data sets is included in the analysis. For the moment, let us say that the Planck 2013 data plus the WMAP data on large scale polarization imply that [12–15]

$$\Omega_K = -0.058^{+0.046}_{-0.026}.$$  

(1.24)

If, in addition, Baryonic Acoustic Oscillations (BAO) data are included [12–15], one obtains $\Omega_K = -0.004 \pm 0.0036$. The conclusion is that everything is consistent with a vanishing spatial curvature. The photon energy density is given by $\pi^2T_0^4/15$ where $T_0$ is the CMB temperature which has been measured to be $T_0 = 2.725\pm 0.00006 \text{ K}$ [51]. This implies that

$$\Omega_\gamma h^2 = 2.47159 \times 10^{-5}.$$  

(1.25)

In the same way, the neutrino energy density is fixed since $\rho_\nu = N_{\text{eff}}(7/8)(4/11)^{4/3}\rho_\gamma \simeq 0.68132\rho_\gamma$ with $N_{\text{eff}} = 3$. This leads to

$$\Omega_\nu h^2 = 1.68394 \times 10^{-5}.$$  

(1.26)

For the baryon and cold dark matter energy densities, Planck 2015 with PlanckTT, TE, EE+lowP has obtained [16–18]

$$\Omega_b h^2 = 0.02225 \pm 0.00016, \quad \Omega_{\text{cdm}}^0 h^2 = 0.1198 \pm 0.0015.$$  

(1.27)
Finally, since the curvature is zero, one must have $\Omega^0_b + \Omega^0_{cdm} + \Omega^0_\gamma + \Omega^0_\nu + \Omega^0_\Lambda = 1.$ from which one deduces that

$$\Omega_\Lambda h^2 = 0.306.$$ (1.28)

The previous considerations describe the current state of our universe. The model is a six parameter model: $\rho_b, \rho_{cdm}, \rho_\Lambda$, the optical depth $\tau$ that controls re-ionization [52] and two parameters that describe the fluctuations, their amplitude $A_S$ and spectral index $n_S$ (we discuss these two parameters in more details in the section on inflationary perturbations). A priori, $\rho_\gamma$ and $\rho_\nu$ are also parameters but they are usually considered as fully determined given the precision of the measurement of the CMB temperature and given the fact that we have only three families of particles. It is impressive that with only six parameters, one can account for all the astrophysical and cosmological data.

From those numbers, using the theoretical description presented in the previous section, one can also infer the past history of the universe. The scaling of the three different types of energy densities are given by $\rho_\gamma \propto 1/a^4$, $\rho_m \propto 1/a^3$ and $\rho_\Lambda$ is a constant. As a consequence, equality between radiation and matter occurs when

$$\left(\rho^0_b + \rho^0_{cdm}\right) \left(\frac{a_0}{a_{eq}}\right)^3 = \left(\rho^0_\gamma + \rho^0_\nu\right) \left(\frac{a_0}{a_{eq}}\right)^4,$$ (1.29)

that is to say

$$1 + z_{eq} = \frac{h^2 \Omega^0_b + h^2 \Omega^0_{cdm}}{h^2 \Omega_\gamma (1 + 0.68132)} \simeq 3417,$$ (1.30)

where $z \equiv a_0/a(t) - 1$ is the redshift. In the same way, equality between pressure-less matter and vacuum energy occurs at

$$1 + z_{\text{vac}} = \left(\frac{h^2 \Omega^0_\Lambda}{h^2 \Omega^0_b + h^2 \Omega^0_{cdm}}\right)^{1/3} \simeq 1.29.$$ (1.31)

We thus have three different eras. In the early Universe, radiation dominates, then matter with vanishing pressure takes over and finally, recently, the expansion of the universe became dominated by vacuum energy. During each of these epochs, it is a good approximation to assume that the equation of state is a constant and, therefore,
the solution of the Einstein equations discussed previously, see Eqs. (1.16) and (1.17), will be very useful.

The model that we have just described, the hot Big Bang model or, in its modern incarnation the ΛCDM model, was the standard model of cosmology before the 80’s (of course, the discovery that \( \Lambda_B \neq 0 \) was in fact made later but, here, we refer to the description of the universe at very high redshifts). It is a very successful model since, with a small number of parameters, it can explain a large number of different observations. Historically, three observational pillars have been the expansion of the universe, the Big Bang Nucleosynthesis (BBN) [53] and the presence of the CMB but, nowadays, the model is supported by a much larger sets of observations. Nevertheless, as we are now going to explain, it possesses some undesirable features. It is not that some predictions of this model are in contradiction with the data; it is rather the fact that the initial conditions that need to be postulated in order for the hot Big Bang model to work appears to be very weird. In the next section, we turn to this question.

III. FINE-TUNING PUZZLES OF THE STANDARD MODEL

A. The Horizon problem

The first puzzle that the hot Big Bang model faces is the horizon problem. As the name indicates, it is has something to do with the causality of initial conditions. A first question is “when” should we fix the initial conditions. A priori, this should be done at the earliest time available in the model, namely just after the Big Bang, say at Planck time where the concept of a background spacetime becomes well-defined. But, in practice, can we “see” what happens just after the Big Bang? The answer is no because, prior to recombination, the Universe was opaque and became transparent only after. Recombination is the process by which free electrons and protons combine to form Hydrogen atoms [54]. Before recombination, light could not propagate freely because the cross-section between photons and free electrons was very large (Compton scattering). However, the cross-section of photons with Hydrogen atoms is much smaller and this is the reason why the universe became transparent after recombination.
Recombination is described by the reaction $p + e^- \rightarrow H + \gamma$ which is itself controlled by the Saha equation [55]

\[
\frac{1 - X_e}{X_e^2} = \frac{2\zeta(3)}{\pi^2} \frac{\eta}{m_e} \left( \frac{2\pi T}{m_e} \right)^{3/2} e^{B_H/T},
\]

(1.32)

where $X_e \equiv n_e/n_B$ with $n_e$ the free electron number density and $n_B$ the baryons one. $m_e = 0.511\text{MeV}$ is the mass of the electron and $B_H = m_p + m_e - m_H \simeq 13.6\text{eV}$, $m_p$ being the proton mass and $m_H$ the Hydrogen atom mass, is the binding energy. Finally $\eta \equiv n_B/n_\gamma$ where $n_\gamma$ is the photons number density. If we require $X_e \simeq 0.1$, namely 90% of the free electrons have formed Hydrogen atoms, then we find $T_{\text{rec}} = 0.3\text{eV}$ which corresponds to $z_{\text{rec}} \simeq 1300$. This is the furthest redshift we can reach or observe by traditional means. We see that this event takes place after equality between radiation and matter, see Eq. (1.30), and during the matter dominated era.

Let us now recall the definition of an horizon in cosmology. For this purpose, let us first rewrite the metric in polar coordinates, see Eq. (1.5). One has, assuming no spatial curvature, namely $K = 0$

\[
ds^2 = -dt^2 + a^2(t) \left[ dr^2 + r^2 \left( d\theta^2 + \sin^2\theta d\varphi^2 \right) \right].
\]

(1.33)

The horizon problem comes from the fact that information propagates with a finite speed given by the speed of light. A photon follows a null geodesic and satisfies $ds^2 = 0$ which implies that its radial comoving coordinate can be written as

\[
r(t) = r_E - \int_{t_E}^{t} \frac{d\tau}{a(\tau)},
\]

(1.34)

where $r_E$ is the comoving radial coordinate of the source and $t_E$ the emission time (there is a minus sign in the above equation because the “distance” between the observer of the photon is decreasing with time as it is heading towards the telescope). Then, at time $t$, the proper distance is defined to be $d_p(t) = a(t)r(t)$. If, without loss of generality, we put the origin of the coordinates on Earth, then, at reception at time $t = t_R$, one has by definition $d_p(t_R) = 0$, which allows us to estimate the comoving radial coordinate at emission, namely $r_E = \int_{t_E}^{t_R} d\tau/a(\tau)$. Clearly, this means that the radial coordinate of the furthest event one can, in principle, observe from Earth is
obtained by taking the emission time to be the Big Bang time, namely \( t_E \to 0 \). This defines the size of the horizon a time \( t_R \)

\[
d_H (t_R) = a (t_R) \int_0^{t_R} \frac{d\tau}{a(\tau)}.
\]  

(1.35)

Clearly, the horizon increases as \( t_R \) increases since there is more time for light to travel and, hence, we have access to more and more remote regions of our Universe.

Then, since we have seen that recombination is the earliest event one can observe in practice, let us calculate the angular size of the horizon at that time. From the metric we know that the apparent size \( D \) of a source is given by \( D^2 = a^2 (t_E) r_E^2 d\theta^2 \), which implies that its angular size is given by \( \delta \theta = D / [a (t_E) r_E] \). As a consequence, the angular size of the horizon at recombination (or on the lss) is given by

\[
\delta \theta = \left[ \int_{t_{	ext{lss}}}^{t_0} \frac{d\tau}{a(\tau)} \right]^{-1} \int_0^{t_{	ext{lss}}} \frac{d\tau}{a(\tau)}.
\]  

(1.36)

We see that one needs to know the behavior of the scale factor \( a(t) \) in order to carry out this calculation. Unfortunately, as was already discussed, an exact, analytic, solution valid at any time is not available for the hot Big Bang model. This is here that a piece-wise approximation, where one has several successive epochs with constant equation of state and a scale factor in each era given by Eq. (1.16), will be useful. In accordance with the description of the hot Big Bang model made before, the first phase (phase I) is a phase dominated by radiation for which the scale factor reads \( a(t) = a_i (2H_i t)^{1/2} \), see Eq. (1.17). The quantities \( a_i \) and \( H_i \) are free parameters. At \( t = 0 \), the scale factor vanishes and the scalar curvature blows up; this corresponds to the Big Bang as already discussed. The scale factor behaves according to the above equation for times such that \( 0 < t < t_i \). At \( t = t_i \), we assume that the behavior of \( a(t) \) changes and, for \( t_i < t < t_{\text{end}} \), we assume it is given by (phase II)

\[
a(t) = a_i \left[ \frac{3}{2} (1 + w) H_i (t - t_i) + 1 \right]^{\frac{2}{3(1+w)}},
\]  

(1.37)

in accordance with Eq. (1.16). Notice that, here, we are using Eq. (1.16) and not Eq. (1.17). Usually, this difference is not important but it is relevant when one considers a piece-wise solution for the scale factor. The “normalization” of time has been chosen
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by using \( a(t) \propto t^{1/2} \) during the initial radiation dominated era and, then, it can no longer be modified hence the use of Eq. (1.16). The scale factor and its derivative (and therefore the Hubble parameter \( H = \dot{a}/a \)) are continuous at the transition. The quantity \( w \) is a free parameter describing the equation of state of matter during phase II. Phase II is not part of the hot Big Bang model and we introduce it just for future convenience. If we do not want to include it in our description of the model, we just have to switch it off by taking \( t_i = t_{\text{end}} \). Then, at \( t = t_{\text{end}} \), phase II is over and the radiation dominated era starts again (or continues). This phase III has a scale factor given by \( a(t) = a_{\text{end}} [2H_{\text{end}}(t - t_{\text{end}}) + 1]^{1/2} \), for times such that \( t_{\text{end}} < t < t_{\text{eq}} \).

The quantity \( a_{\text{end}} \) is the scale factor at \( t = t_{\text{end}} \) where \( a(t) \) and \( H(t) \) are continuous. Again, if one switches off phase II, then there is of course no need to distinguish phase I and phase III. At equality between radiation and matter, at time \( t = t_{\text{eq}} \), the matter dominated era starts (phase IV) and the scale factor can now be expressed as \( a(t) = a_{\text{eq}} [\frac{3}{2}H_{\text{eq}}(t - t_{\text{eq}}) + 1]^{2/3} \). This form is valid for times such that \( t_{\text{eq}} < t < t_{\text{de}} \).

Finally at \( t = t_{\text{de}} \) starts the phase dominated by the cosmological constant (phase V) for which \( a(t) \) is given by \( a(t) = a_{\text{de}} e^{H_0(t-t_{\text{de}})} \). This form is valid until present time so for \( t_{\text{de}} < t < t_0 \). During this phase the Hubble parameter is constant and given by its present value \( H_0 \). We stress again that, if phase II is switched off, then the above simple piece-wise model exactly mimics the behavior of \( a(t) \) for the standard hot Big Bang phase.

One has then to calculate the two integrals appearing at the numerator and denominator of Eq. (1.36). This can easily be done given that the behavior of the piece-wise scale factor described previously is, during each phase, just a power law. The integral at the denominator reads

\[
\int_{t_{\text{lss}}}^{t_0} \frac{d\tau}{a(\tau)} = \int_{t_{\text{lss}}}^{t_{\text{de}}} \frac{d\tau}{a(\tau)} + \int_{t_{\text{de}}}^{t_0} \frac{d\tau}{a(\tau)}
\]

\[
= \frac{2}{a_{\text{eq}} H_{\text{eq}}} \left( \frac{a_0}{a_{\text{eq}}} \right)^{1/2} \left[ \left( \frac{a_{\text{de}}}{a_0} \right)^{1/2} - \left( \frac{a_{\text{lss}}}{a_0} \right)^{1/2} \right] + \frac{1}{a_0 H_0} \left( \frac{a_0}{a_{\text{de}}} - 1 \right). \quad (1.39)
\]
But the chain rule gives that
\[
\frac{2}{a_{\text{eq}} H_{\text{eq}}} = \frac{2}{a_0 H_0} \frac{a_{\text{de}} H_{\text{de}}}{a_{\text{eq}} H_{\text{eq}}} = \frac{2}{a_0 H_0} a_{\text{de}} \left( \frac{a_{\text{de}}}{a_{\text{eq}}} \right)^{-\frac{1}{2}} = \frac{2}{a_0 H_0} a_{\text{de}} \left( \frac{a_{\text{de}}}{a_0} \right)^{-\frac{1}{2}} \left( \frac{a_0}{a_{\text{eq}}} \right)^{-\frac{1}{2}},
\]
where we have used that, for power law scale factors, the Hubble parameter can be expressed as a power law of the scale factor. As a consequence, it follows that the integral can be expressed, as expected, only in terms of scale factor ratios at different times, namely
\[
\int_{t_{\text{eq}}}^{t_0} \frac{d\tau}{a(\tau)} = \frac{2}{a_0 H_0} \left( \frac{a_0}{a_{\text{de}}} \right)^{\frac{3}{2}} \left[ \left( \frac{a_{\text{de}}}{a_0} \right)^{\frac{1}{2}} - \left( \frac{a_{\text{eq}}}{a_0} \right)^{\frac{1}{2}} \right] + \frac{1}{a_0 H_0} \left( \frac{a_0}{a_{\text{de}}} - 1 \right).
\]
(1.40)

The second step consists in calculating the integral appearing at the numerator of Eq. (1.36). Following the same procedure as before, one arrives at
\[
\int_0^{t_{\text{eq}}} \frac{d\tau}{a(\tau)} = \int_0^{t_i} \frac{d\tau}{a(\tau)} + \int_t^{t_{\text{end}}} \frac{d\tau}{a(\tau)} + \int_{t_{\text{eq}}}^{t_{\text{end}}} \frac{d\tau}{a(\tau)} + \int_{t_{\text{lss}}}^{t_{\text{eq}}} \frac{d\tau}{a(\tau)},
\]
(1.42)
and, using the piece-wise solution described before, one obtains the following expression
\[
\int_0^{t_{\text{eq}}} \frac{d\tau}{a(\tau)} = \frac{1}{a_i H_i} + \frac{1}{a_i H_i} \frac{2}{1 + 3w} \left[ \left( \frac{a_{\text{end}}}{a_i} \right)^{\frac{1+3w}{2}} - 1 \right] + \frac{1}{a_{\text{end}} H_{\text{end}}} \left( \frac{a_{\text{eq}}}{a_{\text{end}}} - 1 \right)
+ \frac{2}{a_{\text{eq}} H_{\text{eq}}} \left[ \left( \frac{a_{\text{lss}}}{a_{\text{eq}}} \right)^{\frac{1}{2}} - 1 \right].
\]
(1.43)

Then, using the power law behavior of the scale factor in each phase, it is easy to show that \(1/(a_i H_i) = 1/(a_{\text{end}} H_{\text{end}})(a_i/a_{\text{end}})^{(1+3w)/2}\) and \(1/(a_{\text{end}} H_{\text{end}}) = 1/(a_{\text{eq}} H_{\text{eq}})(a_{\text{eq}}/a_{\text{end}})^{-1}\). As a consequence, the integral at the numerator takes the form
\[
\int_0^{t_{\text{lss}}} \frac{d\tau}{a(\tau)} = \frac{1}{a_{\text{eq}} H_{\text{eq}}} \left[ 1 + \frac{1 - 3w a_{\text{end}}}{1 + 3w a_{\text{eq}}} \right] - \frac{1 - 3w a_{\text{end}}}{1 + 3w a_{\text{eq}}} \left( a_i \right)^{\frac{1+3w}{2}}
+ \frac{2}{a_{\text{eq}} H_{\text{eq}}} \left[ \left( \frac{a_{\text{lss}}}{a_{\text{eq}}} \right)^{\frac{1}{2}} - 1 \right].
\]
(1.44)
Finally, since $1/(a_{eq}H_{eq}) = 1/(a_0H_0)(a_0/a_{de})(a_{eq}/a_{de})^{1/2}$, one can establish the expression of the angular size of the horizon, namely

$$\delta \theta = \left( \frac{a_{eq}}{a_0} \right)^{1/2} \left( \frac{a_0}{a_{de}} \right)^{3/2} \left[ 2 \left( \frac{a_{lss}}{a_{eq}} \right)^{1/2} - 1 + \frac{1 - 3w a_{end}}{1 + 3w a_{eq}} - \frac{1 - 3w a_{end}}{1 + 3w a_{eq}} \left( \frac{a_i}{a_{end}} \right)^{1+3w} \right] \times \left\{ 2 \left( \frac{a_0}{a_{de}} \right)^{3/2} \left[ \left( \frac{a_{de}}{a_0} \right)^{1/2} - \left( \frac{a_{lss}}{a_0} \right)^{1/2} \right] + \frac{a_0}{a_{de}} - 1 \right\}^{-1}$$

(1.45)

As already emphasized, we have introduced the phase dominated by the fluid with equation of state $w$ (i.e. the phase II) for future convenience but in the standard model this phase is absent. So we have to switch it off by assuming $a_i = a_{end}$. It is also a good approximation to take $a_0 \simeq a_{de}$ and $a_{lss} \simeq a_{eq}$. In that case one obtains

$$\delta \theta \simeq \frac{1}{2} \left( 1 + z_{lss} \right)^{-1/2} \simeq 0.0138.$$  

(1.46)

(without the simplifying assumptions $a_0 \simeq a_{de}$ and $a_{lss} \simeq a_{eq}$, one easily checks that $\delta \theta \simeq 0.0153$). This means that we should have about 40000 patches on the celestial sphere with completely different temperatures, meaning, a priori, with temperature fluctuations of order one. This is clearly not the case as revealed by the impressive isotropy of the CMB, see Fig. 1. On the Planck map, one indeed sees that the temperature anisotropy is everywhere of the order $10^{-5}$.

Facing this situation, we have two options: either we say that the initial conditions were the same (meaning were fine-tuned at the $10^{-5}$ level) on super-causal scales or we say that the expansion was, in the early Universe, different from that predicted by the standard model. The first solution corresponds to a fine-tuning (moreover on super-causal scales) while the other one corresponds to inflation. Therefore, in some sense, the concept of fine-tuning is at the heart of inflation: inflation was invented to prevent its appearance.

**B. The Flatness Problem**

We have just discussed the horizon problem. But this problem is not the only one faced by the hot Big Bang model and we now turn to another one, namely the flatness
FIG. 1. Map of the temperature anisotropy measured by the European Space Agency (ESA) Planck satellite. The amplitude of the anisotropy is very small, of the order of $\sim 10^{-5}$, which means that the universe was in fact extremely homogeneous and isotropic on the last scattering surface. Figure taken from Ref. [12].

problem (also discussed in more details in Ref. [42]). Let us now consider Eq. (1.23) again. This equations reads

$$1 + \frac{K}{a^2 H^2} = \Omega_T,$$  

(1.47)

and we know that observations indicate that $|\Omega_T^0 - 1| \lesssim 0.01$. Clearly, this means that we live in a spatially flat Universe to a very good approximation. In the context of the standard model of cosmology, this is problematic. Indeed, using the Friedmann equation, one has in general

$$\Omega_T(t) = \frac{\sum_i \Omega_i^0 \left( \frac{a_0}{a} \right)^{3(1+w_i)}}{\sum_i \Omega_i^0 \left( \frac{a_0}{a} \right)^{3(1+w_i)} - (\Omega_T^0 - 1) \left( \frac{a_0}{a} \right)^2},$$  

(1.48)

In the case of the hot Big Bang model, we have seen that the universe is made of radiation and pressure-less matter. As a consequence, the above expression takes the form

$$\Omega_T(t) = \frac{\Omega_m^0 \left( \frac{a_0}{a} \right)^3 + \Omega_r^0 \left( \frac{a_0}{a} \right)^4}{\Omega_m^0 \left( \frac{a_0}{a} \right)^3 + \Omega_r^0 \left( \frac{a_0}{a} \right)^4 - (\Omega_T^0 - 1) \left( \frac{a_0}{a} \right)^2}.$$  

(1.49)
Then, deep in the radiation era, this equation can be approximately expressed as

$$\Omega_T(t) \simeq 1 + \frac{\Omega^0_{\gamma} - 1}{\Omega^0_{\gamma}} \left(\frac{a}{a_0}\right)^2 + \cdots,$$

(1.50)

which implies that

$$\Omega^0_{\gamma} - 1 \simeq \Omega^0_{\gamma} [\Omega_T(z) - 1] (1 + z)^2 \simeq 2.47 h^{-2} \times 10^{-5} [\Omega_T(z) - 1] (1 + z)^2.$$

(1.51)

This equation clearly shows the problem. We know as an observational fact that $|\Omega^0_{\gamma} - 1| \lesssim 0.01$. As we go backwards in time, the redshift $z$ increases and, in order to satisfy $|\Omega^0_{\gamma} - 1| \lesssim 0.01$, $\Omega_T(z) - 1$ must be less and less. If, for instance, we evaluate $\Omega_T(z) - 1$ at BBN ($z \simeq 10^8$), we obtain $|\Omega^0_{\gamma,\text{BBN}} - 1| \lesssim 10^{-13} \mathcal{O} (< 0.01)$. Obviously, if we increase $z$ (namely consider even earlier times), this fine tuning problem becomes even more severe. Going back all the way down to the Planck scale, one has indeed $|\Omega^0_{\gamma,\text{Pl}} - 1| \lesssim 10^{-57} \mathcal{O} (< 0.01)$. The question is then why was the Universe so flat in the early stages of its evolution?

Another way to see the same question is to notice that Eq. (1.47) implies that the solution $\Omega_T = 1$ is an unstable point. In presence of a single fluid with equation of state $w$ (for simplicity), it can indeed be re-written as

$$\Omega_T(N) = 1 + \frac{K}{a^2_{\text{ini}}} e^{(1+3w)N},$$

(1.52)

where $N \equiv \ln \left(\frac{a}{a_{\text{ini}}}\right)$ is the number of e-folds. We see that, if $1 + 3w > 0$, which is always true in a decelerated Universe, the deviation from $\Omega_T = 1$ exponentially grows. In order to understand why this is physically problematic, let us use an analogy with another unstable system, namely a pencil balancing on its tip. Let us represent the pencil by a rod, whose moment of inertia is given by $I = m\ell^2/3$ where $m$ is the mass of the pencil and $\ell$ is length. The pencil is subject to the force of gravity which acts at its mass center. The equation of motion is given by $\boldsymbol{I} \ddot{\Omega} = \mathbf{r} \wedge \mathbf{F}$ with $\mathbf{r} = (\ell/2 \sin \theta, \ell/2 \cos \theta), \mathbf{F} = (0, -mg)$, $\theta$ being the angle between the pencil and the vertical axis and $g = 9.81 \text{ m/s}^2$ the gravitational acceleration. As a consequence, the

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For simplicity, and without loss of generality, we restrict the motion of the pencil to the two-dimensional plan $(y, z)$. 
equation of motion reads
\[ \ddot{\theta} - \frac{3g}{2\ell} \sin \theta = 0. \] (1.53)

If we assume that, initially, the pencil is vertical \((\dot{\theta}_\text{ini} = 0)\), then the solution for small angles reads
\[ \theta(t) \simeq \frac{\dot{\theta}_\text{ini}}{m} \sinh(\omega t), \] (1.54)
with a fundamental frequency given by \(\omega^2 \equiv \frac{3g}{2(2\ell)\ell} \). We see that, for any non-vanishing initial velocity, the system is strongly unstable and \(\theta(t)\) grows exponentially. Therefore, finding the pencil still balancing on its tip after some time would be surprising and would require an explanation. This argument is, however, sometimes dismissed on the basis that one should first define a measure in order to assess, in a quantitative way, how unlikely is this situation (see Ref. [42] for a full treatment of this issue). For instance, if one believes that the measure in phase space is peaked at \(\dot{\theta}_\text{ini} = 0\), then one might be tempted to say that the pencil will never fall. This leads to argue that, in absence of a well justified measure in the space of initial conditions, one cannot say whether it is surprising or not to find the pencil balancing on its tip.

However, this argument ignores a crucial aspect, which is the presence of unavoidable classical and/or quantum fluctuations. Classical fluctuations, for instance, could be modeled, by a random force the components of which in the \((y,z)\) plan are written as \(\eta = (\eta_1, \eta_2)\) with
\[ \langle \eta_i(t)\eta_j(t') \rangle = \Gamma \delta_{ij} \delta(t - t'), \] (1.55)
where \(\Gamma\) is a parameter describing the amplitude of the correlation function. In presence of this force, the equation of motion (1.53) becomes
\[ \ddot{\theta} - \left( \frac{3g}{2\ell} + \frac{3\eta_2}{2m} \right) \sin \theta = -\frac{3\eta_1}{2m} \cos \theta. \] (1.56)
The point is that, now, even if \(\dot{\theta}_\text{ini} = 0\), and contrary to what happened before, \(\theta(t)\) will always grow. In other words, small initial fluctuations will always cause the pencil to fall. Technically, this can be viewed straightforwardly: if one takes \(\eta_2 = 0\)
(which simplifies the problem with no loss of generality), then using the Green function method, the motion of the pencil with $\theta_{\text{ini}} = \dot{\theta}_{\text{ini}} = 0$ now reads

$$\theta(t) = -\frac{3}{2m\omega} \int_{t_{\text{ini}}}^{t} \sinh \left[ \omega (t - t') \right] \eta_1(t') dt'. \quad (1.57)$$

In order to obtain this solution, we have assumed small angles (namely $\sin \theta \simeq \theta$) and, as a consequence, there will be a value of $t > t_{\text{ini}}$ for which the above solution ceases to apply. But this is just a technical limitation that can easily be fixed if needed. The most important property, which would be shared by the exact solution obtained without the small angle approximation, is that small fluctuations will always push very quickly the system out of the unstable equilibrium. Even if those fluctuations are quantum fluctuations, this is sufficient to insure that a macroscopic pen falls in a couple of seconds [56]. We conclude that, even if one manages to obtain a measure which is peaked over the unstable equilibrium position, finding the pencil balancing on its tip would remain a physical problem that needs an explanation.

In the case of Cosmology, small fluctuations in the early Universe are present and, therefore, based on the previous considerations, observing $\Omega_0^T \simeq 1$ requires an explanation. The flatness problem consists in finding a solution to this problem.

The hot Big Bang model has other puzzles, such as, for instance, the presence of dangerous relics originating from phase transitions taking place in the early universe. Rather than describing all these issues in an exhaustive way, we now turn to a possible solution, namely the theory of cosmic inflation.

IV. INFLATION

A. Solving the Standard Model Puzzles

The main idea of inflation is that the puzzles we have described in the previous sections are an indication that the dynamics of the universe at very high redshifts was different from that implied by the hot Big Bang model. According to this model, at very high energies, the universe was radiation dominated, with a scale factor $a(t) \propto t^{1/2}$. According to inflation, this was not the case. Let us now see how it works in practice.
and let us discuss how inflation can solve the horizon problem. For this purpose, we switch on the phase dominated by the fluid with equation of state $w$ (phase II) and rewrite Eq. (1.45) as

$$
\delta \theta \simeq \frac{1}{2} (1 + z_{\text{ls}})^{-1/2} \left\{ 1 + \frac{1 - 3w a_{\text{end}}}{1 + 3w a_{\text{ls}}} \left[ 1 - e^{-\frac{1}{2} N_T (1 + 3w)} \right] \right\},
$$

where we have introduced the total number of e-folds $N_T = \ln \left( \frac{a_{\text{end}}}{a_i} \right)$ during phase II. The presence of phase II introduces a correction to the standard result (1.46), namely the second factor in the above equation. If we want this correction to play a significant role, then the exponential term must be non-negligible. And this is the case if

$$
1 + 3w < 0,
$$

or, in other words, using Eq. (1.10), if the Universe was accelerating $\ddot{a} > 0$. By definition, a phase of accelerating expansion is called a phase of inflation. But having a phase of acceleration is not sufficient, we also need a phase of acceleration that lasts long enough. Indeed requiring $\delta \theta > 2\pi$ gives $N_T \gtrsim \ln (1 + z_{\text{end}})$ (here, we assume that $w$ is not fine tuned to $\lesssim -1/3$). If we write the energy scale at the end of inflation as $\rho_{\text{end}} \simeq (10^x)^4 \text{GeV}^4$, then the previous condition reduces to $N_T \gtrsim 2.3x + 29$. For the Grand Unified Theory (GUT) scale, namely $x = 15$, this gives $N_T \gtrsim 63$. Therefore, one concludes that the horizon problem is solved if we have a phase of inflation. If this phase of inflation takes place at the GUT scale, then it must last more than $\sim 60$ e-folds. If the energy scale is lower, then we need less e-folds.

Let us now see what would be the consequence for the flatness problem. In agreement with what we have discussed before, this means that we postulate the presence of a new fluid, with an a priori unknown equation of state $w$. This unknown fluid dominates the energy density budget of the Universe if $t_i < t < t_{\text{end}}$, namely during phase II, and is smoothly connected to the standard Big Bang phase which takes place for $t > t_{\text{end}}$. As a consequence, this implies that Eq. (1.51) can only be applied if $z < z_{\text{end}}$ since $t_{\text{end}}$ is the earliest time where the standard evolution is valid. In that case, one has

$$
\Omega_T^0 - 1 \simeq \Omega_T^0 (z_{\text{end}}) - 1 \left( 1 + z_{\text{end}} \right)^2 \simeq 2.47 h^{-2} \times 10^{-5} \left[ \Omega_T (z_{\text{end}}) - 1 \right] \left( 1 + z_{\text{end}} \right)^2.
$$

(1.60)
Now our goal is to calculate $\Omega_T(z_{\text{end}}) - 1$ in terms of $\Omega_T(z_{\text{ini}}) - 1$, namely in terms of the initial conditions at the beginning of inflation. During inflation, one has

$$\Omega_T(t) \simeq \frac{\Omega_X^{\text{ini}} \left( \frac{a_{\text{ini}}}{a} \right)^{3(1+w)}}{\Omega_X^{\text{ini}} - (\Omega_T^{\text{ini}} - 1) \left( \frac{a_{\text{ini}}}{a_{\text{end}}} \right)^{1-3w}},$$

which implies that

$$\Omega_T(z_{\text{end}}) \simeq \frac{\Omega_X^{\text{ini}}}{\Omega_X^{\text{ini}} - (\Omega_T^{\text{ini}} - 1) \left( \frac{a_{\text{ini}}}{a_{\text{end}}} \right)^{1-3w}}.$$

(1.62)

Clearly the only way to solve the flatness problem is if inflation is such that $\Omega_T(z_{\text{end}}) \simeq 1$ and the only way to achieve it is to have $1 + 3w < 0$, that to say the same condition than the one derived to solve the horizon problem, see Eq. (1.59). In that situation, the above equation takes the form

$$\Omega_T(z_{\text{end}}) \simeq 1 - \frac{\Omega_T(z_{\text{ini}}) - 1}{\Omega_X^{\text{ini}}} e^{-\frac{N_T}{1+3w}},$$

(1.63)

and, as a consequence

$$\Omega_T^0 - 1 \simeq 2.47 h^{-2} \times 10^{-5} \frac{\Omega_T(z_{\text{ini}}) - 1}{\Omega_X^{\text{ini}}} e^{-\frac{N_T}{1+3w}} (1 + z_{\text{end}})^2.$$

(1.64)

Requiring $|\Omega_T^0 - 1| \lesssim 0.01$ without postulating that $\Omega_T(z_{\text{ini}}) - 1$ is very small, namely without postulating any fine-tuning of the initial conditions at the beginning of inflation leads to $N_T \gtrsim \ln (1 + z_{\text{end}})$, that is to say, again, the same condition as for the horizon problem. The fact that the conditions for solving the horizon and the flatness problems are the same is very suggestive and is also an argument in favor of inflation.

We conclude that inflation can solve the fine-tuning puzzles of the Big Bang model. In addition, we mentioned before the existence of additional puzzles. One can show that inflation can also fix them. The next question is then which type of matter can produce such a phase.

**B. Realizing a Phase of Inflation**

As explained in detail in the previous sections, a phase of accelerated expansion in the early universe solves the puzzles of the standard model of cosmology. Clearly,
at very high energies, the correct framework to describe matter is field theory and its simplest version, compatible with isotropy and homogeneity, is when a scalar field dominates the energy budget of the Universe. This scalar field is called the “inflaton”. In that case, the energy density and pressure are given by

\[ \rho = \frac{\dot{\phi}^2}{2} + V(\phi), \quad p = \frac{\dot{\phi}^2}{2} - V(\phi). \]  

(1.65)

As a consequence, if the potential energy dominates over the kinetic energy, one obtains a negative pressure and, hence, inflation. This can be achieved when the field moves slowly or, equivalently, when the potential is almost flat.

From a field theory perspective, the micro-physics of inflation should be described by an effective field theory characterized by a cutoff \( \Lambda \). One usually assumes that the gravitational sector is described by General Relativity, which itself is viewed as an effective theory with a cutoff at the Planck scale, then \( \Lambda < M_{\text{Pl}} \). On the other hand, we will see that the CMB anisotropy data suggests that inflation could have taken place at energies as high as the GUT scale and this suggests \( \Lambda > 10^{15} \text{GeV} \). Particle physics has been tested in accelerators only up to scales of \( \sim \text{TeV} \) and this implies that our freedom in building models of inflation will remain very important. A priori, without any further theoretical guidance, the effective action can therefore be written as

\[
S = \int d^4x \sqrt{-g} \left[ M_{\text{Pl}}^2 \Lambda_B + \frac{M_{\text{Pl}}^2}{2} R + aR^2 + bR_{\mu\nu}R^{\mu\nu} + \frac{c}{M_{\text{Pl}}^2} R^3 + \cdots ight.
- \frac{1}{2} \sum_i g^{\mu\nu}\partial_\mu \phi_i \partial_\nu \phi_i - V(\phi_1, \cdots, \phi_n) + \sum_i d_i \frac{O_i}{\Lambda_i^{n_i-4}}
+ S_{\text{int}}(\phi_1, \cdots, \phi_n, A_\mu, \Psi) + \cdots.
\]

(1.66)

In the above equation, the first line represents the effective Lagrangian for gravity (recall that \( \Lambda_B \) is the cosmological constant). In practice, we will mainly work with the Einstein-Hilbert term only. The second line represents the scalar field sector and we have postulated that, a priori, several scalar fields are present. The first two terms represent the canonical Lagrangian while \( O_i \) represents a higher order operator of dimension \( n_i > 4 \), the amplitude of which is determined by the coefficient \( d_i \). Those corrections can modify the potential but also the (standard) kinetic term \[57\]. The last
term encodes the interaction between the inflaton fields and the other fields present in Nature, i.e. gauge fields $A_\mu$ and fermions $\Psi$. Those terms are especially important to describe how inflation ends and is connected to the standard model of cosmology. Finally, the dots stand for the rest of the terms such as kinetic terms of gauge bosons $A_\mu$, of fermions $\Psi$ etc...

Given the complexity of the above Lagrangian, it is clear that it is impossible to single out a model of inflation from theoretical considerations only. However, as we will see, the CMB data have given us precious information. In particular, from the absence of non-adiabatic perturbations and from the fact that the CMB fluctuations are Gaussian, models with a single field, a minimal kinetic term and a smooth potential are favored. This does not mean that more complicated scenarios are ruled out (as a matter of fact they are not) but that, for the moment, they are not needed to describe the data. It is important to emphasize that we are driven to this class of models, which is clearly easier to investigate than the more complicated models mentioned above, not because we want to simplify the analysis but because this is what the CMB data suggest. Then, the Lagrangian \( \mathcal{L} \) can be simplified to

\[
\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) + \mathcal{L}_{\text{int}}(\phi, A_\mu, \Psi). \tag{1.67}
\]

During the accelerated phase, the interaction term is supposed to be sub-dominant and will be neglected. Then, only one arbitrary function remains in the Lagrangian, the potential $V(\phi)$. An example of a potential that supports inflation is given in Fig. 2. From CMB data, one can constrain this function and this will be discussed in the following. As already mentioned, the interaction term plays a crucial role in the process which ends inflation. Indeed, it controls how the inflaton field decays into particles describing ordinary matter. These decay products are then supposed to thermalize and the radiation dominated epoch starts at a temperature which is known as the reheating temperature $T_{\text{rh}}$. This quantity is an important parameter of any inflationary model and we will see that the CMB data can also say something about its value.

Following the above considerations, during inflation itself, the interaction term is neglected and the evolution of the system is controlled by the Friedmann and Klein-
FIG. 2. Example of a potential [the Starobinsky potential (1.100)] that can support inflation. Slow roll inflation occurs along the plateau where the potential is almost flat and the reheating phase takes place when the field oscillates around its minimum, here located at the origin.

Gordon equations, namely

\[ H^2 = \frac{1}{3M_{Pl}^2} \left[ \frac{\dot{\phi}^2}{2} + V(\phi) \right], \] (1.68)

\[ \ddot{\phi} + 3H\dot{\phi} + V_\phi = 0, \] (1.69)

where a subscript \( \phi \) means a derivative with respect to the inflaton field. For an arbitrary potential, this system of equations cannot be solved analytically. This means that we have to use either numerical calculations or a perturbative method. In general, a perturbative method is based on the presence of a small parameter in the problem and on an expansion of the relevant quantities of the theory in terms of this small parameter. In the case of inflation, there exists such a small parameter which physically
expresses the fact that the potential is flat. So it can be chosen as the curvature of the potential or, equivalently, as the kinetic to potential energy ratio or, given that inflation corresponds to an approximately constant Hubble parameter, as the derivative of $H$. Therefore, we introduce the Hubble flow functions $\epsilon_n$ defined by [58, 59]

$$\epsilon_{n+1} \equiv \frac{d \ln |\epsilon_n|}{dN}, \quad n \geq 0,$$

(1.70)

where $\epsilon_0 \equiv H_{\text{ini}}/H$ starts the hierarchy and we remind that $N \equiv \ln(a/a_{\text{ini}})$ is the number of e-folds already introduced before. From the above expression, the first Hubble flow parameter can be written as

$$\epsilon_1 = -\frac{\dot{H}}{H^2} = 1 - \frac{\ddot{a}}{aH^2} = \frac{3\dot{\phi}^2}{2} \frac{1}{\dot{\phi}^2/2 + V(\phi)},$$

(1.71)

and, therefore, inflation ($\ddot{a} > 0$) occurs if $\epsilon_1 < 1$. In terms of the Hubble flow parameters, the Friedmann and Klein-Gordon equations take the form

$$H^2 = \frac{V}{M_{\text{Pl}}^2(3 - \epsilon_1)},$$

(1.72)

$$\left(1 + \frac{\epsilon_2}{6 - 2\epsilon_1}\right) \frac{d\phi}{dN} = -M_{\text{Pl}}^2 \frac{d \ln V}{d\phi}.$$  

(1.73)

It is worth stressing the point that these expressions are exact. The condition $\epsilon_1 < 1$ during $\sim 60$ e-folds is sufficient to solve the fine-tuning problems of the standard model, as discussed above. But, if one wants to describe properly the CMB anisotropy (see the discussion below), one needs $\epsilon_n \ll 1$, which is called the slow-roll regime. In this situation, the first three Hubble flow parameters can be approximated as [60]

$$\epsilon_1 \simeq \frac{M_{\text{Pl}}^2}{2} \left(\frac{V_\phi}{V}\right)^2,$$

(1.74)

$$\epsilon_2 \simeq 2M_{\text{Pl}}^2 \left[\left(\frac{V_\phi}{V}\right)^2 - \frac{V_{\phi\phi}}{V}\right],$$

(1.75)

$$\epsilon_2\epsilon_3 \simeq 2M_{\text{Pl}}^4 \left[\frac{V_{\phi\phi\phi\phi}}{V^2} - \frac{V_{\phi\phi}}{V} \left(\frac{V_\phi}{V}\right)^2 + 2 \left(\frac{V_\phi}{V}\right)^4\right].$$

(1.76)

We see that the first Hubble flow parameter is also a measure of the steepness of the potential and of its first derivative. The second Hubble flow parameter is a measure of the second derivative of the potential and so on. Therefore, if one can observationally
constrain the values of the Hubble flow parameters, we can say something about the shape of the inflationary potential. The slow-roll approximation also allows us to simplify the equations of motion and to analytically integrate the inflaton trajectory. Indeed, in this regime, Eqs. (1.68) and (1.69), which control the evolution of the system, can be approximated by \( H^2 \simeq V/(3M_{\text{Pl}}^2) \) and \( d\phi/dN \simeq -M_{\text{Pl}}^2 d \ln V/d\phi \), from which one obtains

\[
N - N_{\text{ini}} = -\frac{1}{M_{\text{Pl}}^2} \int_{\phi_{\text{ini}}}^{\phi} \frac{V(\chi)}{V_\chi(\chi)} d\chi,
\]

(1.77)

\( \phi_{\text{ini}} \) being the initial value of the inflaton. If the above integral can be performed, one gets \( N = N(\phi) \) and if this last equation can be inverted, one has the trajectory, \( \phi = \phi(N) \).

Let us now describe the end of inflation. As already mentioned, this is the phase during which the inflaton decays into the particles of the standard model. During that phase, the interaction term is obviously crucial. This means that, in principle, in order to have a fair description of that process, one must specify all the interaction terms of \( \phi \) with the other scalars, the gauge bosons and the fermions present in the universe together with the corresponding coupling constants. Then, one must solve the (non linear) equations of motion of all these fields. Clearly, this is a very complicated task. However, in a cosmological context, one can proceed in a simpler way. Indeed, the reheating phase can in fact be described by two numbers, \( \rho_{\text{reh}} \), the energy density at which the radiation dominated era starts (and, therefore, at which the reheating epochs stops) and the mean equation of state \( \bar{w}_{\text{reh}} \). Of course, one should also know at which energy density reheating starts but this is not a new parameter since it is determined by the condition \( \epsilon_1 = 1 \). In the following, we denote this quantity \( \rho_{\text{end}} \). Let us notice that the knowledge of \( \rho_{\text{reh}} \) is equivalent to the knowledge of the reheating temperature since

\[
\rho_{\text{reh}} = g_\ast \frac{\pi^2}{30} T_{\text{reh}}^4,
\]

(1.78)

where \( g_\ast \) encodes the number of relativistic degrees of freedom. On the other hand, the mean equation of state controls the expansion rate of the Universe during reheating. Let \( \rho_T = \sum_i \rho_i \) and \( p_T = \sum_i p_i \) be the total energy density and pressure, where the
sum is over all the species present during reheating. Let us define the “instantaneous” equation of state by \( w_{\text{reh}} \equiv p_T/\rho_T \). Then the mean equation of state parameter, \( \bar{w}_{\text{reh}} \), is given by

\[
\bar{w}_{\text{reh}} \equiv \frac{1}{\Delta N} \int_{N_{\text{end}}}^{N_{\text{reh}}} w_{\text{reh}}(n) dn,
\]

where \( \Delta N \equiv N_{\text{reh}} - N_{\text{end}} \) is the total number of e-folds during reheating. The quantity \( \bar{w}_{\text{reh}} \) allows us to determine the evolution of the total energy density since this quantity obeys

\[
\rho_{\text{reh}} = \rho_{\text{end}} e^{-3(1+\bar{w}_{\text{reh}})\Delta N},
\]

where we recall that \( \rho_{\text{end}} \) can be determined once the model of inflation is known.

In fact, as long as the CMB is concerned, only one parameter can be constrained and this parameter is a combination of \( \rho_{\text{reh}} \) and \( \bar{w}_{\text{reh}} \). It is known as the reheating parameter and is defined by

\[
R_{\text{rad}} \equiv \left( \frac{\rho_{\text{reh}}}{\rho_{\text{end}}} \right)^{(1-3\bar{w}_{\text{reh}})/(12+12\bar{w}_{\text{reh}})}.
\]

The justification for this definition can be found in Refs. [61–65] but a simple argument shows that it makes sense. It is clear that one cannot make the difference between a model of instantaneous reheating where \( \rho_{\text{end}} = \rho_{\text{reh}} \) and a model where reheating proceeds with a mean equation of state of radiation, namely \( \bar{w}_{\text{reh}} = 1/3 \), since in this last case reheating cannot be distinguished from the subsequent radiation dominated era. We see on the above definition that, in both cases, the reheating parameters has the same numerical value, \( R_{\text{rad}} = 1 \), which is consistent.

It may come as a surprise that a very complicated phenomenon such as reheating can be described by only one number. But one should keep in mind that this is the case only if one tries to constrain reheating from the CMB or, to put it differently, the reheating parameter is the only quantity that can be measured if one uses CMB data. Moreover, this is not a new situation. This is indeed very similar to what happens for re-ionization [52] for instance. Clearly, re-ionization is, from a particle physics point of view, a very complicated process. But despite this complexity, as long as one considers CMB data only, it is described by one quantity, the optical depth \( \tau \) [52].
C. Inflationary Cosmological Perturbations

So far, we have described the background spacetime during inflation. We now turn to the perturbations [66–69]. As is well-known, this is a crucial part of the inflationary theory since it gives a convincing explanation for the origin of the large scale structures observed in our Universe. However, in order to deal with this question, one must go beyond homogeneity and isotropy which is a complicated task. But, we know that, in the early Universe, the deviations from the cosmological principle were small as revealed, for instance, by the magnitude of the CMB anisotropy $\delta T/T \sim 10^{-5}$. During inflation, we expect the fluctuations to be even smaller since they grow with time according to the mechanism of gravitational collapse. This means that we can treat the inhomogeneities perturbatively and, in fact, restrict ourselves to linear perturbations. Then, the idea is to write the metric tensor as

$$g_{\mu\nu}(\eta, \mathbf{x}) = g^{\text{FLRW}}_{\mu\nu}(\eta) + \delta g_{\mu\nu}(\eta, \mathbf{x}) + \cdots,$$

where $g^{\text{FLRW}}_{\mu\nu}(\eta)$ represents the metric tensor of the FLRW Universe, see Eq. (1.4), and where $\delta g_{\mu\nu}(\eta, \mathbf{x}) \ll g^{\text{FLRW}}_{\mu\nu}(\eta)$. Here, $\eta$ is the conformal time, related to the cosmic time by $d\eta = adt$. In the same way, the inflaton field is expanded as

$$\phi(\eta, \mathbf{x}) = \phi^{\text{FLRW}}(\eta) + \delta \phi(\eta, \mathbf{x})$$

with $\delta \phi(\eta, \mathbf{x}) \ll \phi^{\text{FLRW}}(\eta)$. In fact, $\delta g_{\mu\nu}(\eta, \mathbf{x})$ can be expressed in terms of three types of perturbations, scalar, vector and tensor. In the context of inflation, only scalar and tensor are important. Scalar perturbations are directly coupled to the perturbed scalar field $\delta \phi(\eta, \mathbf{x})$ while tensor fluctuations represent primordial gravitational waves. The equations of motion of each type of fluctuations are given by the perturbed Einstein equations, namely $\delta G_{\mu\nu} = \delta T_{\mu\nu}/M_{\text{Pl}}^2$. But we also need to specify the initial conditions. A crucial assumption of inflation is that the source of the perturbations are the unavoidable quantum vacuum fluctuations of the gravitational and scalar fields. It is clear that this has drastic implications: it means that the large scale structures in the Universe are nothing but quantum fluctuations made classical and stretched to cosmological scales.

Let us now turn to a quantitative characterization of the cosmological fluctuations. The amplitude of scalar perturbations is described by the curvature perturbations [70, 71] $\zeta(\eta, \mathbf{x}) \equiv \Phi + 2(\mathcal{H}^{-1}\Phi' + \Phi)/(3 + 3w)$, with $w = p/\rho$ the equation of state during inflation and $\Phi$ the Bardeen potential [72] (not to be confused with the scalar
field $\phi$). The Bardeen potential is the quantity that describes scalar perturbations as revealed by writing explicitly the perturbed metric in longitudinal gauge, $ds^2 = a^2(\eta)[-(1 - 2\Phi)d\eta^2 + (1 - 2\Phi)\delta_{ij}dx^i dx^j]$. Since we deal with a linear theory, we can go to Fourier space and follow the time evolution of the Fourier component $\zeta_k(\eta)$. Then, the properties of the fluctuations are described by the power spectrum of scalar perturbations, which is given by

$$P_\zeta(k) = \frac{k^3}{2\pi^2} |\zeta_k|^2. \quad (1.82)$$

The power spectrum depends on the model of inflation that is to say, for the simple class of models discussed here, on the potential $V(\phi)$. Unfortunately, there exists no exact analytic calculation of $P_\zeta(k)$ for an arbitrary $V(\phi)$. Therefore, one must either rely on numerical calculations or on perturbative methods. Here again, the slow-roll approximation can be used and leads to the following result [59]

$$P_\zeta(k) = P_{\zeta0}(k_p) \left[ a_0^{(s)} + a_1^{(s)} \ln \left( \frac{k}{k_p} \right) + \frac{a_2^{(s)}}{2} \ln^2 \left( \frac{k}{k_p} \right) + \cdots \right], \quad (1.83)$$

where $k_p$ is a pivot scale and the overall amplitude can be written as

$$P_{\zeta0} = \frac{H_*^2}{8\pi^2\epsilon_{1*}M_{Pl}^2}. \quad (1.84)$$

In the above expression (and in the subsequent ones), a star means that the corresponding quantity has been evaluated at the time at which the pivot scale crossed out the Hubble radius during inflation, namely $k_p \sim a_*H_*$. The amplitude of the spectrum depends on (the square of) the strength of the gravitational field during inflation which is described by the expansion rate $H_*$. It is also inversely proportional to the first derivative of the potential through the presence of $\epsilon_{1*}$ at the denominator. The main property of $P_{\zeta0}$ is that it is does not depend on the wave number, in other words it is scale independent. This result represents one of the main success of inflation since a scale invariant power spectrum was known for a long time to be in agreement with the observations. But there is even more. We see that the scale invariant piece of the power spectrum receives scale dependent logarithmic corrections the amplitudes of which are controlled by the Hubble flow parameters and are given
by [58, 59, 73–79],
\[ a_0^{(s)} = 1 - 2(C + 1)\epsilon_1 + C\epsilon_2 + \left(2C^2 + 2C + \frac{\pi^2}{2} - 5\right)\epsilon_1^2 + \left(C^2 - C + \frac{7\pi^2}{12} - 7\right)\epsilon_1\epsilon_2 + \left(\frac{1}{2}C^2 + \frac{\pi^2}{8} - 1\right)\epsilon_2^2 + \left(-\frac{1}{2}C^2 + \frac{\pi^2}{24}\right)\epsilon_2\epsilon_3 + \cdots, \] (1.85)
\[ a_1^{(s)} = -2\epsilon_1 + \epsilon_2 + 2(2C + 1)\epsilon_1^2 + (2C - 1)\epsilon_1\epsilon_2 + C\epsilon_2^2 - C\epsilon_2\epsilon_3 + \cdots, \] (1.86)
\[ a_2^{(s)} = 4\epsilon_1^2 + 2\epsilon_1\epsilon_2 + \epsilon_2^2 - \epsilon_2\epsilon_3 + \cdots, \] (1.87)
\[ a_3^{(s)} = O(\epsilon_3^3), \] (1.88)
where \( C \equiv \gamma_E + \ln 2 - 2 \approx -0.7296 \), \( \gamma_E \) being the Euler constant. Since the coefficients \( a_1^{(s)}, a_2^{(s)} \) etc ... are small (being proportional to the Hubble flow parameters), this means that the inflationary power spectrum is not exactly scale-invariant but, in fact, almost scale invariant. This is the main prediction of inflation and it was confirmed recently by the CMB Planck data. We stress that this is a prediction since it was made before it was measured. In terms of spectral index, being defined as the logarithmic derivative of \( \ln P_\zeta(k) \), one has
\[ n_S = 1 - 2\epsilon_1 - \epsilon_2, \] (1.89)
where \( n_S = 1 \) corresponds to exact scale invariance. We see on the above expression that the small deviations from exact scale invariance carry information about the shape of the inflationary potential since \( \epsilon_1 \) and \( \epsilon_2 \) respectively depend on the first and second derivative of \( V(\phi) \). Therefore, an accurate measurement of the power spectrum can provide information about which version of inflation was realized in the early universe.

We have also mentioned that gravitational waves are produced during inflation. The corresponding treatment is very similar to the one we have just described. In particular, the tensor power spectrum \( P_h \) can be written in the same way as Eq. (1.90), namely
\[ P_h(k) = P_{h0}(k_{\text{p}}) \left[ a_0^{(T)} + a_1^{(T)} \ln \left( \frac{k}{k_{\text{p}}} \right) + \frac{a_2^{(T)}}{2} \ln^2 \left( \frac{k}{k_{\text{p}}} \right) + \cdots \right], \] (1.90)
with a scale invariant overall amplitude that can be expressed as
\[ P_{h0} = \frac{2H_0^2}{\pi^2 M_{\text{Pl}}^2}. \] (1.91)
FIG. 3. Multipole moments versus angular scale obtained from the Planck 2015 data. The multipole moments are defined from the following expression of the temperature fluctuation two-point correlation function:
\[ \langle \frac{\delta T}{T}(e_1)\frac{\delta T}{T}(e_2) \rangle = \frac{1}{(4\pi)^2} \sum_{\ell} (2\ell + 1) C_\ell P_\ell (\cos \theta) \]
where \( \theta \) is the angle between the two directions \( e_1 \) and \( e_2 \). The multipole moments \( C_\ell \) represent the power of the signal at a given spatial frequency \( \ell \). Notice that the quantity \( D_\ell \) is defined by
\[ D_\ell = \ell(\ell + 1)C_\ell/(2\pi) \]
The red curve corresponds to the best fit in the parameter space of the \( \Lambda \)CDM model. This result is consistent with the predictions of inflation, for instance because of the presence of the Doppler peaks. Figure taken from Ref. [17].

This time, and contrary to scalar perturbations, the amplitude only depends on the Hubble parameter during inflation. This has a very important implication: if one can measure the amplitude of tensor power spectrum, then one immediately determines the expansion rate during inflation or, in other words, the energy scale of inflation. Unfortunately, the inflationary gravitational waves have not yet been detected. As for
FIG. 4. Multipole moments corresponding to the correlation between temperature and so-called E-mode polarization anisotropies (we refer the reader to Ref. [80] for definitions of polarized CMB quantities) obtained from Planck 2015. The red solid line corresponds to prediction of the $\Lambda$CDM model obtained from the best fit in Fig. 3 (namely with temperature measurements only). The lower panel shows the residual with respect to this best fit. Figure taken from Ref. [17].

scalar perturbations, the tensor power spectrum has small scale dependent logarithmic corrections which can be written as [59]

\begin{align}
    a_0^{(T)} &= 1 - 2(C + 1)\epsilon_{1*} + \left(2C^2 + 2C + \frac{\pi^2}{2} - 5\right)\epsilon_{1*}^2 \\
    &\quad + \left(-C^2 - 2C + \frac{\pi^2}{12} - 2\right)\epsilon_{1*}\epsilon_{2*} + \cdots, \quad (1.92) \\
    a_1^{(T)} &= -2\epsilon_{1*} + 2(2C + 1)\epsilon_{1*}^2 - 2(C + 1)\epsilon_{1*}\epsilon_{2*} + \cdots, \quad (1.93) \\
    a_2^{(T)} &= 4\epsilon_{1*}^2 - 2\epsilon_{1*}\epsilon_{2*} + \cdots, \quad (1.94) \\
    a_3^{(T)} &= O(\epsilon_{n*}^3), \quad (1.95)
\end{align}
corresponding to tensor spectral index given by
\[ n_T = -2\epsilon_1, \tag{1.96} \]
an exact scale invariance corresponding, with these conventions, to \( n_T = 0 \) (and not one as for the scalars). Since, by definition of what inflation is, one has \( \epsilon_1 > 0 \), this means that \( n_T < 0 \), i.e. we say that inflation predicts a red power spectrum (that is to say more power on large scales) for gravitational waves. It is also interesting to measure the relative amplitude of the tensors compared to the scalars and this is done in terms of the parameter \( r \) defined by
\[ r \equiv \frac{P_h}{P_\zeta} = 16\epsilon_1. \tag{1.97} \]
Clearly, since \( \epsilon_1 \ll 1 \), tensor are sub-dominant which is compatible with the fact that they have not yet been detected [16, 81].
D. Constraints on Inflation

After having discussed the main features and predictions of the inflationary scenario, let us now review what the CMB Planck data imply for inflation. The Planck data are represented in Figs. 3, 4 and 5. As already mentioned, the most important discovery made by the Planck satellite is probably the measurement of the scalar spectral index which is found to be [18]

$$n_s = 0.9645 \pm 0.0049.$$  \hspace{1cm} (1.98)

It is a crucial result since this is the first time that a deviation from $n_s = 1$ is measured at a statistical significant level (say, more than $5\sigma$). It is clearly a strong point in favor of inflation. As was discussed previously, inflation also predicts the presence of a background of gravitational waves and, unfortunately, we do not yet have a detection of those primordial gravity waves. This means that we only have an upper bound on the parameter $r$, namely

$$r \lesssim 0.07$$  \hspace{1cm} (1.99)

obtained by combining the Planck data and the BICEP/Keck data [16]. As already mentioned, the Planck data are also compatible with no Non-Gaussianity [15] and no non-adiabatic perturbations [18] which is compatible with the simplest model of inflation.

One can also use the Planck data to constrain the shape of the inflationary potential. The performance of a model can be described by two numbers: the Bayesian evidence [83, 84] which characterizes the ability of the model to fit the data in a simple way and the Bayesian complexity [85] which is related to the number of unconstrained parameters (given a data set). A good model is a model that has a large Bayesian evidence and no unconstrained parameters. In Fig. 6, we have represented the Bayesian evidence and complexity for nearly 200 models of inflation, given the Planck data [34, 61, 64, 81, 82, 86–88]. Based on this analysis, it is found that potentials with a plateau are favored by the data, the prototypical example being the Starobinsky
FIG. 6. Inflationary models in the space \( (N_{\text{uc}}, \ln B_{\text{REF}}^i) \). \( N_{\text{uc}} \) represents the number of unconstrained parameters of a given model while \( B_{\text{REF}}^i \) is the evidence of a given model “i” to evidence of a reference model ratio. Each model is represented by a circle (the radius of which has no meaning) with its acronym, taken from Ref. [82], written inside. The four panels correspond to successive zooms towards the best region (indicated by the dashed rectangles). Figures taken from Ref. [82].

model [2] for which the potential is given by

\[
V(\phi) = M^4 \left( 1 - e^{-\sqrt{2/3}\phi/M_{\text{Pl}}} \right)^2. \tag{1.100}
\]

We recall that this potential is represented in Fig. 2.
Reheating can also be constrained by means of the Planck data \[61–65\], see Fig. 7. We have seen that the only piece of information about the end of inflation that can be extracted from CMB data is the posterior distribution of the reheating parameter. In order to quantify whether the constraint is tight or not, one has then to compare the posterior to the prior. In technical terms, this is given by the Kullback-Leibler divergence, \(D_{\text{KL}}\), between the prior and the posterior. In Fig. 7, we have represented \(D_{\text{KL}}\) as a function of the Bayesian evidence for the nearly 200 models of inflation already studied in Fig. 6. Each model is represented by a circle. The yellow band corresponds to the one-sigma deviation around the mean value, which is given by
\( \langle D_{KL} \rangle = 0.82 \pm 0.13 \). This corresponds to an information of almost one bit, and, therefore, this confirms that reheating is constrained by CMB data. Of course, it is not straightforward to translate these constraints into constraints on the reheating temperature unless one specifies \( \varpi_{\text{reh}} \) explicitly, in which case the reheating parameter and the reheating temperature are in one-to-one correspondence.

V. IS INFLATION FINE-TUNED? CHOOSING THE FREE PARAMETERS OF THE INFLATIONARY POTENTIAL

In this section, we turn to the question of whether inflation, which was invented in order to solve fine-tuning problems, is itself fine tuned. Let us discuss the first aspect of the problem, namely how the parameters of the potential must be chosen and what their numerical values are in order for the model to correctly account for the data. Let us start with a particular model, namely Large Field Model (LFI) for which the potential is given by

\[
V(\phi) = M^4 \left( \frac{\phi}{M_{\text{Pl}}} \right)^p,
\]

where \( M \) and \( p \) are two free parameters. Using Eq. (1.77), one can calculate the slow-roll trajectory and one finds

\[
\phi(N) = \sqrt{\phi_{\text{ini}}^2 - 2M_{\text{Pl}}^2(N - N_{\text{ini}})}.
\]

In order to calculate the spectral index and the scalar-to-tensor ratio, one must calculate the Hubble flow parameters. Using the expressions of \( \epsilon_1 \) and \( \epsilon_2 \) in the slow-roll approximation, one obtains, see Eqs. (1.74) and (1.75),

\[
\epsilon_1 = \frac{p^2M_{\text{Pl}}^2}{2\phi^2}, \quad \epsilon_2 = \frac{2pM_{\text{Pl}}^2}{\phi^2}.
\]

This immediately leads to the vacuum expectation value at which inflation ends since the condition \( \epsilon_1 = 1 \) implies \( \phi_{\text{end}}/M_{\text{Pl}} = p/\sqrt{2} \). Then, we must evaluate the Hubble flow parameters at the time that was previously denoted with a star, namely the time at which the pivot scale crossed out the Hubble radius during inflation. Using the slow-roll trajectory, it is easy to show that \( \phi_*^2/M_{\text{Pl}}^2 = p^2/2 + 2p\Delta N_* \), where \( \Delta N_* = N_{\text{end}} - N_* \),
with $N_{\text{end}}$ the number of e-folds at the end of inflation and $N_*$ the number of e-folds at Hubble radius exit. In terms of $\Delta N_*$, the Hubble flow parameters read

$$
\epsilon_1 = \frac{p}{4(\Delta N_* + p/4)}; \quad \epsilon_2 = \frac{1}{\Delta N_* + p/4}.
$$

(1.104)

As a consequence, one has

$$
n_s - 1 = -\frac{p + 2}{2\Delta N_* + p/2}, \quad r = \frac{4p}{\Delta N_* + p/4}.
$$

(1.105)

The measurements of $n_s$ and the constraints on $r$ can therefore allow us to put constraints on the parameter $p$. But we also see that the spectral index and the tensor-to-scalar ratio do not depend on the other free parameter, namely $M$. This one is in fact fixed by the amplitude of the fluctuations (i.e. the “COBE normalization”), that is to say by the fact that $\delta T/T \sim 10^{-5}$. Using Eq. (1.84) and the slow-roll approximation for the Friedmann equation, one obtains that

$$
\frac{M^4}{M_{Pl}^4} = 12\pi^2p^2\left(\frac{\phi_*}{M_{Pl}}\right)^{-p-2} p_{\zeta_0} = 12\pi^2p^2\left(\frac{p^2}{2} + 2p\Delta N_*\right)^{-p/2-1} p_{\zeta_0}.
$$

(1.106)

The value of $P_{\zeta_0}$ is provided by the Planck 2015 data [16–18]

$$
\ln(10^{10}P_{\zeta_0}) = 3.094 \pm 0.0049,
$$

(1.107)

and one finds that $M/M_{Pl} \simeq 1.3 \times 10^{-3}$ for $p = 2$ and $M/M_{Pl} \simeq 3 \times 10^{-4}$ for $p = 4$. In order to obtain these numbers we have assumed $\Delta N_* = 55$ and a comment is in order at this stage. In principle, one should not assume a value for $\Delta N_*$ since it is determined once the reheating temperature and the mean equation of state parameter during reheating have been chosen [61–65]. It can be quite dangerous to choose a “reasonable” value blindly because, sometimes, it could imply a reheating energy density higher that the energy density at the end of inflation which is clearly meaningless. In fact, the dependence in $\Delta N_*$ of $n_s$ and $r$ is precisely the reason why one can use the CMB to put constraints on the reheating epoch, as explained in the previous sections. Indeed, $\Delta N_*$ cannot take arbitrary values otherwise the corresponding spectral index and tensor to scalar ratio would be incompatible with the data. But since $\Delta N_*$ depends on $T_{\text{reh}}$ and $w_{\text{reh}}$, this means that those quantities cannot take arbitrary values as well or, to put it differently, are constrained by the CMB data. Nevertheless, one can show
that, for large-field inflation, $\Delta N_*$ can vary in a quite small range around the value $\Delta N_* = 55$ and this is the reason why we choose this value. Considering another value would not affect much our numerical estimate and would change nothing to the present discussion.

The estimates of the mass scale $M$ derived above show that inflation in this model takes place around the GUT scale. But let us consider the case $p = 4$ and write the potential as $V(\phi) = \lambda \phi^4$ where $\lambda$ is a dimensionless coupling constant. Clearly, $\lambda = M^4/M_{Pl}^4$ which implies that $\lambda \sim 10^{-13}$. This very small value can be viewed as a fine tuning, at least if one adopts the standard lore that absence of fine tuning means that dimensionless quantities should be “naturally” of order one. Let us now consider the case $p = 2$ and write the corresponding potential as $V(\phi) = m^2 \phi^2/2$ where $m$ is the mass of the inflaton field. In that case one has $m = \sqrt{2}(M/M_{Pl})^2 M_{Pl}$ which leads to $m \sim 2 \times 10^{-6} M_{Pl}$. Is this fine tuning? In absence of a rigorous definition of fine tuning, this is hard to tell. But one can notice that $m/H \sim \sqrt{6}(2 + 4\Delta N_*)^{-1/2} < 1$, which may be viewed as unnatural. Indeed, we expect the mass of the inflaton to be corrected by high-energy physics according to $m^2 \to m^2 + gM^2 \ln(\Lambda/\mu)$, where $\mu$ is the renormalization scale, $M > \Lambda$ the mass of a heavy field, $\Lambda$ the cut-off already discussed in Sec. IV B and $g$ the coupling constant. The presence of these corrections implies $m/H \sim 1$ and keeping $m/H < 1$ may be problematic. This problem is also known as the $\eta$-problem of inflation [89]. But, at least, this illustrates the fact that the fine-tuning of the parameters (if any) can depend on the potential. For this reason, it is worth studying the situation for the Starobinsky potential (1.100) since this is the favored model.

The Starobinsky model can be derived from different assumptions. Historically, it was derived by considering $R^2$ corrections to the Einstein-Hilbert action. However, more recently, it was realized that it can also be viewed as a scenario in which the inflaton field is the Higgs field, this one being non-minimally coupled to gravity. In technical terms, the action of the model reads

$$S = \frac{M_{Pl}^2}{2} \int d^4x \sqrt{-g} \left[ (1 + \xi h^2) R - g^\mu\nu \partial_\mu h \partial_\nu h - 2M_{Pl}^2 \frac{\lambda}{4} \left( h^2 - \frac{v^2}{M_{Pl}^2} \right)^2 \right],$$

where $v$ is the Higgs vacuum expectation value and $\lambda$ the self-interacting coupling
constant. The quantity $\xi$ is a dimensionless constant which describes the non-minimal coupling. If one defines the field $\phi$ by $d[\phi/(\sqrt{2}M_{Pl})]/dh = \sqrt{1 + \xi(1 + 6\xi)h^2}/[\sqrt{2}(1 + \xi h^2)]$ then this field has a standard Lagrangian with a potential which is exactly the potential of Eq. (1.100), the scale $M$ being given by

$$M^4 = \frac{M_{Pl}^4 \lambda}{4\xi^2}. \tag{1.109}$$

Then the COBE normalization, which constrains the value of $M$, leads to

$$\xi \sim 46000\sqrt{\lambda}, \tag{1.110}$$

where $\lambda = m_{H}^2/v^2$ with $v \simeq 175$ GeV and $m_{H} \simeq 125$ GeV. We see that $\xi \gg 1$, which can imply many issues as far as the consistency of the model is concerned.

The overall picture that emerges from this section is that it is difficult to say whether the parameters of the inflationary potential are necessarily fine tuned if one wants to account for the data. It is clear that this question is model dependent. For
some potentials, the fine-tuning seems to be present (at least if one adopts a naive definition of fine-tuning) but for others, and in particular those that fit the data well, it is unclear whether this is the case. The situation of the Starobinsky model is particularly interesting. The coupling between gravity and the Higgs is not small, or is not perturbative, which may lead to technical difficulties but this strong coupling problem is not necessarily associated with a fine-tuning problem. Here, we are just missing an objective definition of what fine tuning is.

In fact, one could argue that such a definition exists and is nothing but the Bayesian evidence considered in Sec. IV D. Technically, the Bayesian evidence is the integral of the likelihood over prior space but its meaning can easily be grasped intuitively. Let us consider a model depending on, say, one free parameter. If, for all values of the parameter in the prior range, one obtains a good fit, then the Bayesian evidence is “good”. This is for instance the case of the model in Fig. 8 (left panel). Different points correspond to different values of the reheating temperature but all points are within the 1σ Planck contour. On the contrary, if one needs to tune the value of the free parameter in order to have a good fit, then the Bayesian evidence will be “bad”. This is the case for the model in Fig. 8 (right panel). In order to have a good compatibility with the data (i.e. points within the 1σ contour), one needs to tune the parameter $A_i$ (which controls the amplitude of the quantum corrections) and the Bayesian evidence is “bad”. In other words the wasted parameter space is penalized. Obviously, the smaller the range of $A_i$ leading to a good fit (compared to the prior), the smaller the evidence. We conclude that the evidence is a good, objective, measure of fine tuning. In this sense, the Starobinsky model is the best model because it is the less fine-tuned one.

VI. INFLATIONARY INITIAL CONDITIONS

A. Homogeneous Initial Conditions

Let us now discuss another type of possible fine tuning, namely the initial conditions (see also Ref. [42] for a detailed discussion of this question). We have seen previously
FIG. 9. Phase space for the Starobinsky model. The red line represents the slow roll trajectory while the other colored lines correspond to exact trajectories (numerically computed) with different initial conditions. It is evident from the plot that the slow-roll trajectory is an attractor in phase space. This question is treated in an exhaustive way in Ref. [42].

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that one of the main motivations for inflation is to avoid the fine tuning of the initial conditions that is needed in order for the standard model to work. If our solution to that issue were also fine tuned then one could wonder whether something has been gained or not. In fact this problem has different facets. If one restricts ourselves to an homogeneous and isotropic solution, then the only question is how we should choose \( \phi_{\text{ini}} \) and \( \dot{\phi}_{\text{ini}} \). The slow-roll trajectory corresponds to \( \dot{\phi}_{\text{ini}} \simeq -\frac{V_\phi(\phi_{\text{ini}})}{3H(\phi_{\text{ini}})} \) and, therefore, there could be the worry that we have to tune the initial velocity to this value. However, this is not the case because the slow-roll trajectory is an attractor...
as can be seen in Fig. 9. It is true that, for some $V(\phi)$, the corresponding basin of attraction is very small. This is for instance the case for Small Field Inflation (SFI) if the size of the hilltop part is sub-Planckian [42]. However, on the contrary, it can be very large for other models, such as Large Field Inflation (LFI) (let us also notice that the existence of an attractor is immune to stochastic effects, see Ref. [90]). The interesting point is that it is also the case for the Starobinsky model and plateau potentials [42], namely the models favored by the data. In this sense, in this restricted framework, there is no fine tuning of the initial condition.

B. Anisotropic Initial Conditions

Obviously, however, the previous analysis is not entirely satisfactory. Indeed, we start from a homogeneous and isotropic situation while inflation is precisely supposed to explain why our Universe is homogeneous and isotropic. The analysis can be improved by considering that, initially, the Universe is not isotropic (but still homogeneous) [91–93]. For this purpose let us consider the following metric (Bianchi I model):

$$ds^2 = -dt^2 + a_i^2(t) \left(dx^i\right)^2,$$

that is to say we now have one scale factor for each space direction. This metric can also be rewritten as

$$ds^2 = -dt^2 + a(t)^2 \gamma_{ij} dx^i dx^j$$

with

$$a(t) \equiv \left[a_1(t) a_2(t) a_3(t)\right]^{1/3},$$

and

$$\gamma_{ij} = \begin{pmatrix} e^{2\beta_1(t)} & 0 & 0 \\ 0 & e^{2\beta_2(t)} & 0 \\ 0 & 0 & e^{2\beta_3(t)} \end{pmatrix},$$

with $\sum_{i=1}^{3} \beta_i = 0$. As usual, one can introduce the conformal time $\eta$ in terms of which the metric can be expressed as

$$ds^2 = a^2(\eta) \left(-d\eta^2 + \gamma_{ij} dx^i dx^j\right).$$

Then, the next step is to introduce the shear $\sigma_{ij}$ which is defined by (as usual a prime denotes a derivative
with respect to conformal time)

\[
\sigma_{ij} = \frac{1}{2} \gamma_{ij}' = \begin{pmatrix}
\beta_1' e^{2\beta_1} & 0 & 0 \\
0 & \beta_2' e^{2\beta_2} & 0 \\
0 & 0 & \beta_3' e^{2\beta_3}
\end{pmatrix}.
\]

Assuming that matter is described by a scalar field, it is then easy to write the Einstein equations. They read

\[
3 \frac{H^2}{a^2} = \frac{\rho}{M_{Pl}^2} + \frac{\sigma^2}{2a^2} = \frac{1}{M_{Pl}^2} \left[ \frac{\dot{\phi}^2}{2a^2} + V(\phi) \right] + \frac{\sigma^2}{2a^2},
\]

\[
-\frac{1}{a^2} (\mathcal{H}^2 + 2\mathcal{H}') = \frac{p}{M_{Pl}^2} + \frac{\sigma^2}{2a^2} = \frac{1}{M_{Pl}^2} \left[ \frac{\dot{\phi}^2}{2a^2} - V(\phi) \right] + \frac{\sigma^2}{2a^2},
\]

\[
\left( \sigma^i_j \right)' + 2\mathcal{H} \sigma^i_j = 0,
\]

where \( \sigma^2 = \sigma_{ij} \sigma^{ij} = \sum_{i=1}^{3} \beta_i'^2 \) and \( \sigma^i_j = \gamma^{ik} \sigma_{kj} \), that is to say

\[
\sigma^i_j = \begin{pmatrix}
\beta_1' & 0 & 0 \\
0 & \beta_2' & 0 \\
0 & 0 & \beta_3'
\end{pmatrix}.
\]

The solution for the shear can easily be found, namely \( \sigma^i_j = S^i_j/a^2 \), where \( S^i_j \) is a constant tensor. This implies that \( \sigma^2 = S^2/a^4 \) where \( S^2 = S^i_j S^j_i \). As a consequence, one sees that the shear is in fact equivalent to a stiff fluid with an equation of state \( w_\sigma = p_\sigma/\rho_\sigma = 1 \) and \( \rho_\sigma = M_{Pl}^2 S^2/(2a^6) \). Therefore, if initially the shear dominates, \( \rho_\sigma \gg \rho_\phi \), then the universe will expand as \( a \propto t^{1/3} \), see Eq. (1.17), and the expansion will not be accelerated. However, since \( \rho_\sigma \propto a^{-6} \) while \( \rho_\phi \) is approximately constant, the scalar field will eventually take over and inflation will start. We conclude that, even if the Universe is not initially isotropic, it will become so in the presence of a scalar field whose energy density is dominated by its potential. In this sense, it is legitimate to start from an isotropic situation as was done previously. This is clearly not a fine tuning, but rather an attractor of the dynamical evolution.

### C. Inhomogeneous Initial Conditions

Despite the fact that taking into account the shear represents an improvement, this still does not allow us to discuss the real issue. For that, we need a framework where
the initial state of the Universe is neither isotropic nor homogeneous. Technically, this is clearly very complicated since we have to solve the Einstein equations in full generality. The only way to study these questions exactly is therefore numerical relativity. However, some schemes of approximation have also been developed and we now discuss them. Of course, the perturbative approach described before, see Sec. IV C, is one way of taking into account the inhomogeneities. However, by definition, these fluctuations must be small while we would like to see whether inflation “homogenizes” the Universe even if it is strongly inhomogeneous initially. Another method is the so-called “effective-density approximation” [94, 95], see also Ref. [42]. The idea is to study an inhomogeneous scalar field on a (isotropic and homogeneous) FLRW background and to add to the Friedmann equation a term which describes the back-reaction of the field gradient on the geometry [94, 95]. In practice, one writes
\[ \phi(t, x) = \phi_0(t) + \Re \left[ \delta \phi(t)e^{ikx/a(t)} \right], \] (1.119)
and assumes that the corresponding Klein-Gordon equation can be split into two equations, namely
\[ \ddot{\phi}_0 + 3H \dot{\phi}_0 + V_\phi(\phi_0) = 0, \] (1.120)
\[ \ddot{\delta \phi} + 3H \dot{\delta \phi} + \frac{k^2}{a^2} \delta \phi = 0. \] (1.121)
The Friedmann equation is then written as
\[ H^2 = \frac{1}{3M_{\text{Pl}}^2} \left[ \frac{1}{2} \dot{\phi}_0^2 + V(\phi_0) + \frac{1}{2} \delta \phi^2 + \frac{1}{2} \frac{k^2}{a^2} \delta \phi^2 \right] - \frac{K}{a^2}. \] (1.122)
The wave number \( k \) should be chosen such that the wavelength of the perturbations is much smaller than the Hubble radius, namely \( 2\pi k/a \ll H^{-1} \). In the opposite limit, the contribution of \( \delta \phi \) should just be added to the background. The energy density of the inhomogeneities \( \rho_{\delta \phi} = \rho_{\delta \phi} + \rho_\gamma \), with \( \rho_{\delta \phi} = \frac{\dot{\delta \phi}^2}{2} \) and \( \rho_\gamma = \frac{k^2 \delta \phi^2}{(2a^2)} \) is supposed to dominate initially (i.e. the Universe is inhomogeneous initially), \( \rho_{\delta \phi} \gg \rho_{\phi_0} \). The question is whether \( \rho_{\delta \phi} \) can decrease (i.e. the Universe becomes homogeneous) such that, at some point, \( \rho_{\phi_0} \) takes over and inflation starts.

Let us now discuss the initial conditions. We take \( \dot{\phi}_0 \) and \( \phi_0 \) such that, in absence of inhomogeneities, slow-roll inflation starts. Initially, the Friedmann equations can be
FIG. 10. Evolution of the scalar field $\delta \phi(t)$ obtained by numerical integration of Eqs. (1.120), (1.121) and (1.122). The potential is chosen to be the Starobinsky one, see Eq. (1.100) with a scale $M = 0.001 M_{\text{Pl}}$ which, roughly speaking, matches the CMB normalization. The initial value of the field $\phi_0$ is $\phi_0 = 4 M_{\text{Pl}}$ and $\dot{\phi}_0 = -V_{\phi}(\phi_0)/[3V(\phi_0)]$ (which is the slow-roll velocity). In absence of inhomogeneities, with these initial conditions, inflation would start and would lead to more than 60 e-folds. The initial value of $\delta \phi$ is taken to be $0.01 M_{\text{Pl}}$ (and is therefore less than the Planck mass as required, see the main text) while the initial velocity of $\delta \phi(t)$ is given by $\dot{\delta \phi}_{\text{ini}} = 0$ (blue line) or $\dot{\delta \phi}_{\text{ini}} = 9 \times 10^{-5} M_{\text{Pl}}^2$ (red line). The scale $k$ is chosen to be $k/a_{\text{ini}} = 10^{-3} M_{\text{Pl}}$ and the curvature is given by $\mathcal{K}/a_{\text{ini}}^2 \simeq 1.36 \times 10^{-12}$. This implies the following Hubble parameter $H_{\text{ini}}/M_{\text{Pl}} \simeq 3.69 \times 10^{-5}$ and $\rho_{\phi,\text{ini}} \simeq 3.33 \times 10^{-13} M_{\text{Pl}}^4$. One easily checks that those initial conditions are such that $H_{\text{ini}}^2 a_{\text{ini}}^2/k^2 \simeq 1.36 \times 10^{-3} < 1$ and $\rho_{\phi,\text{ini}}/\rho_{\delta \phi,\text{ini}} \simeq 2.44 \times 10^{-4}$, namely the inhomogeneities largely dominate initially. Finally, the black line represents $\delta \phi_{\text{ini}}/a = \delta \phi_{\text{ini}} e^{-N}$ for the initial conditions corresponding to the blue line. We see that the envelope of the numerical solution indeed follows Eq. (1.124).
FIG. 11. Evolution of the Hubble parameter and of the various energy densities obtained by numerical integration of Eqs. (1.120), (1.121) and (1.122). The initial conditions are those that lead to the red curve in Fig. 10. Initially the Universe is strongly inhomogeneous since $\rho_{\delta\phi} \gg \rho_{\phi}$. However, $\rho_{\phi}$ (black line) stays approximately constant while $\rho_{\delta\phi} \propto a^{-4}$ (blue line). As a consequence, the expansion is first radiation dominated and then (at $N \simeq 1$ in the above plot), $\rho_{\phi}$ takes over and inflation starts. Therefore, at least in this example, large inhomogeneities initially do not prevent the onset of inflation.

written as

$$\frac{3a^2H^2}{k^2} \simeq \frac{1}{2} \frac{\dot{\delta\phi}^2}{M_{Pl}^2} \frac{a^2}{k^2} + \frac{1}{2} \frac{\delta\phi^2}{M_{Pl}^2},$$

(1.123)

since $\rho_{\delta\phi} \gg \rho_{\phi\phi}$. For simplicity we have taken $K = 0$ but it is straightforward to include the case where curvature is not vanishing. We have already mentioned that
the effective density approximation is valid only if the wavelength of $\delta \phi$ is smaller than the Hubble radius. This means that the left hand side of Eq. (1.123) must be small. This immediately implies that $\delta \phi \ll M_{Pl}$ and $\dot{\delta \phi}^2 / M_{Pl}^2 \ll k^2 / a^2$ initially. Then, the corresponding solution is easily determined: the field $\delta \phi$ oscillates and decays inversely proportional to the scale factor \[42\], namely

$$\delta \phi(t) \simeq \Re \left[ \frac{\delta \phi_{\text{ini}}}{a(t)} e^{ikt/a(t)} \right].$$ \hspace{1cm} (1.124)$$

This immediately implies that $\rho_{\delta \phi}$ behaves as radiation, namely $\rho_{\delta \phi} \propto 1/a^4$. In Fig. 10, Eqs. (1.120), (1.121) and (1.122) have been numerically integrated and the evolution of $\delta \phi(t)$ is displayed and compared to Eq. (1.124). We see that they match very well. In Fig. 11, we have represented the corresponding energy densities. While $\rho_{\phi}$ remains constant, $\rho_{\delta \phi}$ behaves as radiation and, as a consequence, becomes very quickly sub-dominant. As a consequence, after a few e-folds, the Universe becomes homogeneous and inflation starts as can be seen on the evolution of the Hubble parameter (green line) which, initially, decreases and, then, becomes almost constant.

The previous analysis seems to indicate that inflation does indeed homogenize the Universe. However, one should be aware of its limitations. Firstly, and obviously, there is the question of the domain of validity of Eqs. (1.120), (1.121) and (1.122) and whether they can really represent a strongly inhomogeneous situation. Clearly, if $2\pi k/a \simeq H^{-1}$ this is not the case and one has to rely on other techniques. Basically, one has two possibilities: either one obtains exact solutions \[96–98\] but they are very hard to find in the inhomogeneous case or one uses numerical simulations \[94, 95, 99–106\]. These ones are also complicated to study since they involve full numerical relativity.

Historically, the first numerical solutions \[94, 95, 99, 100\] were done under the assumption that spacetime is spherically symmetric. This has the advantage to simplify the equations since they only depend on time and $r$, the radial coordinate. Of course, in that case one still has to numerically solve partial differential equations. The metric considered in Ref. \[95\] reads

$$ds^2 = - \left( N^2 - R^2 \beta^2 \right) dt^2 + 2R^2 \beta d\chi dt + R^2 \left( d\chi^2 + \sin^2 \chi d\Omega^2 \right),$$ \hspace{1cm} (1.125)$$

where $0 \leq \chi \leq \pi$ so that the spacelike sections are closed. The lapse and shift functions $N$ and $\beta$ depend on $t$ and $r$ as well as the “scale factor” $R$. The matter
content assumed in Ref. [95] is a scalar field $\phi$, which is the inflaton, and another scalar field $\psi$ without potential and playing the role of an extra fluid. Some important technical restrictions are also postulated on the initial data. Firstly, it is assumed that the total energy density is constant. Given an initial inhomogeneous distribution for the inflaton $\phi(\chi)$, this is achieved by choosing the initial velocity of $\psi$ to be such that the total energy density is constant. Secondly, the initial momentum is taken to vanish. Based on the previous calculations, see Eqs. (1.120), (1.121) and (1.122), it is argued in Ref. [95] that, at least for large-field models, this does not restrict the significance of the results. Thirdly, the integration is performed for values of the inflaton self-coupling that are larger than the ones needed to CMB normalize the model. Different initial configurations for the inflaton field are considered. In particular, the following Gaussian ansatz

$$\phi_{\text{ini}}(\chi) = \phi_0 + \delta \phi \left[ 1 - \exp \left( -\frac{\sin^2 \chi}{\Delta^2} \right) \right] ,$$

was studied in details. This initial profile depends on three parameters: $\phi_0$, the value of the field at the origin $\chi = 0$, $\delta \phi$ which can be viewed as the value of the field on the other side of the universe, $\phi(\pi/2) = \phi_0 + \delta \phi \left( 1 - e^{-1/\Delta^2} \right)$ and $\Delta$ which represents the width of the Gaussian.

Let us now describe the results obtained for large-field inflation. If $V(\chi = 0)$ and $V(\chi = \pi/2)$, or $\phi_0$ and $\delta \phi$, are such that, in a homogeneous situation, inflation would start, then it also starts in the present case. If, on the contrary, $V(\chi = 0)$ is such that inflation would start in a homogeneous situation but not $V(\chi = \pi/2)$ (therefore, the gradients are important), then Ref. [95] has shown that the outcome crucially depends on the width $\Delta$. More precisely, the numerical simulations show that the crucial parameter is $R\Delta/H^{-1}$ which has to be large enough in order for inflation to start. Moreover, the larger the gradient, the shorter the duration of inflation. For small field models, the sensitivity to the initial conditions is even greater.

Few years later, the analysis was improved in a significant way and, in particular, the assumption of spherical symmetry was relaxed. Indeed, Refs. [104–106] ran simulations of strongly inhomogeneous inflation with a three-dimensional numerical relativity code. These simulations are such that the initial time slice has homogeneous total energy
density which means that \((\nabla \phi)^2/2 < 3M_{Pl}^2H^2\) implying that
\[
\nabla \phi < \sqrt{\frac{3M_{Pl}}{H^{-1}}}. \tag{1.127}
\]
Thus, inhomogeneities that have wavelengths smaller than the Hubble radius must have a small amplitude or, to put it differently, large inhomogeneities must necessarily extend over many Hubble patches. The simulations were carried out for a quartic large field model with an initial configuration given by
\[
\phi_{ini}(t_{ini}, \mathbf{x}) = \phi_0 + \delta \phi \sum_{\ell,m,n=1}^{2} \frac{1}{\ell m n} \sin \left( \frac{2\pi \ell x}{L} + \theta_{x\ell} \right) \sin \left( \frac{2\pi \ell y}{L} + \theta_{ym} \right) \sin \left( \frac{2\pi \ell z}{L} + \theta_{zn} \right), \tag{1.128}
\]
where the \(\theta\)'s are random phases. Two runs have been carried out in Ref. [106], one with \(L = H^{-1}\) and \(\delta \phi = 0.0125m_{Pl}\) and one with \(L = 32H^{-1}\) and \(\delta \phi = 0.4m_{Pl}\). In both cases, one has \(\phi_0 = 5m_{Pl}\) and \(H_0 = 0.1m_{Pl}\). The simulations show that, in the first case, the inhomogeneities oscillate and their amplitude is damped. At the end of the run, the inflaton field is homogeneous. But, in the second case, they do not oscillate (initially there are larger than the Hubble radius) and are not damped.

In conclusion, it seems possible to start inflation with inhomogeneous initial conditions and to homogenize the universe. However, admittedly, the numerical simulations that have been carried out so far all require some technical restrictions. The crucial question that emerges from the simulations is the size of the initial homogeneous patch. There is also a dependence in the model with large field scenarios being the preferred class of scenarios. As a consequence, the Starobinsky model is (again) among the good models. Let us also notice that, even more recently, new simulations have been carried out, see Refs. [107, 108]. These new works bring new insights into an issue that will probably be studied even more in the future.

A last comment is that we have good reasons to believe the quantum effects to play an important role at the beginning of inflation. For this reason, studying the initial conditions at the classical level only is maybe not sufficient and even more elaborated investigations may be needed to settle this question.
D. Initial Conditions for the Perturbations

So far, we have discussed the question of the fine tuning of the initial conditions related to the background. Obviously, there is a similar question for the perturbations. We have seen that they are chosen such that the perturbations are initially placed in the vacuum state. However, if one traces back the scale of astrophysical interest today to the beginning of inflation, one notices that they correspond to physical lengths smaller than the Planck length. Clearly, in this regime, the framework used to derive the predictions of inflation, namely quantum field theory in curved spacetime, is no longer valid. This is the so-called trans-Planckian problem of inflation [109–115]. Notice that, at the same time, one has \( H \ll M_{\text{Pl}} \) and, therefore, the concept of classical background is perfectly well defined. So, a priori, one could argue that the initial conditions for the perturbations are tuned in an artificial way. Then, the next question is what happens if one modifies those initial conditions: does it destroy the inflationary predictions that are so successful? To study the robustness of inflation, one can introduce ad-hoc (since we do not know the theory of quantum gravity which would control the behavior of the perturbations on scales smaller than the Planck length), but reasonable, modifications, then recompute the power spectrum of the fluctuations and see whether we obtain a result which significantly differs from the standard result. Various modifications have been proposed, a modification of the dispersion relations of the perturbations [109, 110], a modification of the commutation relations [116] etc. . . . However, the most general approach consists in parameterizing the initial conditions of the perturbations when they emerge from the quantum foam. Let \( M_C \) be the energy scale at which the regime of quantum field theory in curved spacetime breaks down (possibly the Planck scale or the string scale) [111]. A Fourier mode emerges from the quantum foam when its physical wavelength equals the length scale associated to the scale \( M_C \), namely

\[
\lambda(\eta) = \frac{2\pi}{k} a(\eta) = \ell_C \equiv \frac{2\pi}{M_C}, \tag{1.129}
\]

The initial time satisfying Eq. (1.129) is, contrary to what happens in the usual case, scale-dependent. As a consequence, the corresponding power spectrum at the end of
inflation is modified and it now reads \[111\]

\[
P_\zeta(k) = \frac{H^2}{\pi \epsilon_1 m^2_{Pl}} \left\{ 1 - 2 (C + 1) \epsilon_1 - C \epsilon_2 - (2 \epsilon_1 + \epsilon_2) \ln \frac{k}{k_P} - 2 |x| \frac{H}{M_C} \left[ 1 - 2(C + 1) \epsilon_1 - C \epsilon_2 - (2 \epsilon_1 + \epsilon_2) \ln \frac{k}{k_P} \right] \right\}. \tag{1.130}
\]

This expression should be compared to Eq. (1.90). In this expression, the scale \(k_P\) is the pivot scale and \(a_0\) is the scale factor evaluated at the time where \(k_P/a_0 = M_C\). Finally, the initial quantum state of the perturbations at the new scale-dependent initial time is characterized by a complex number \(x\) that can be written in polar form \(x \equiv |x| e^{i \varphi}\), hence defining \(|x|\) and \(\varphi\). This power spectrum is represented in Fig. 12.

Let us now comment on the power spectrum itself. The most obvious remark is that it is modified by the presence of super-imposed oscillations. These oscillations modify the CMB multipole moments as shown in Fig. 13 and, therefore, have observational consequences \[117–119\]. The amplitude of the oscillations is, roughly speaking, given by \(|x| H/M_C\), while the frequency is proportional to \((H/M_C)^{-1}\). On general grounds, we expect the ratio \(H/M_C\) to be a small number. Indeed, we know from the CMB normalization that \(H \lesssim 10^{-5} M_{Pl}\). The scale \(M_C\) is not known but \(M_C \in [10^{-1} M_{Pl}, 10^{-3} M_{Pl}]\) seems reasonable and this implies that, at most, \(H/M_C \sim 0.01\). Therefore, unless the number \(|x|\) is very large, the amplitude of the oscillations is small and one could argue that inflation is robust against trans-Planckian corrections. In this sense, assuming the vacuum state initially is not a fine tuning. Of course, as already mentioned, \(|x|\) could be large and, in this case, the modification sizable. However, the magnitude of \(|x|\) is limited by the backreaction problem \[114\]. Physically, this is due to the fact that \(|x| \neq 1\) corresponds to an excited state. But the particles present in this quantum state carry energy density and this energy density could prevent inflation to start. Therefore, it has to be smaller than the inflationary energy density \(H^2 M^2_{Pl}\). One can show that this leads to an upper bound
FIG. 12. Trans-Planckian power spectra given by Eq. (1.130). The blue line corresponds to a vanilla model with $|x| = 0$ and $\epsilon_1 = 1/(2\Delta N_*)$, $\epsilon_2 = 1/\Delta N_*$ with $\Delta N_* \simeq 50$ as predicted for the $m^2\phi^2$ inflationary model. The red line corresponds to a model with the same values for the slow-roll parameters and $H/M_C \simeq 0.002$, $|x| \simeq 50$, $\varphi = 3$. Finally, the green line represents a model with $H/M_C \simeq 0.001$, $\varphi = 2$ and the same values for the other parameters.

This upper bound is not sufficient to exclude a possible detection of the oscillations in the data (although for the moment nothing has been seen). And, in this sense, one could argue that inflation is not robust to a change of the initial conditions. However, detecting the oscillations would mean opening a window on physics beyond the quantum gravity scale, clearly a fascinating possibility.
VII. THE MULTIVERSE

A. Stochastic Inflation

The discussion of the previous section about the initial conditions misses a crucial ingredient, namely the fact that the background field is itself a quantum field. So far, the quantum effects have been taken into account but only at the perturbative level. The question is now whether they also play an important role in the evolution of the background. Classically, the inflaton field evolves according to the Klein-Gordon equation and, in the slow-roll regime, the typical variation of $\phi$ is then given by $\Delta \phi_{\text{cl}} \simeq -V_{\phi}/(3H)\Delta t$. On the other hand, the amplitude of the quantum kick received by $\phi$ during one e-fold is, roughly speaking, of the order of the square root of the power spectrum of $\delta \phi$, namely $\Delta \phi_{q} \simeq H/(2\pi)$. If $\Delta \phi_{q} \gg \Delta \phi_{\text{cl}}$, then quantum effects are likely to be dominant. In fact, it is easy to see that

$$\frac{\Delta \phi_{q}}{\Delta \phi_{\text{cl}}} = \sqrt{P_{\zeta_{0}}},$$

(1.132)
where $P_{\zeta 0}$ is the amplitude of scalar perturbations, see Eq. (1.84). This equation just tells us that, when the fluctuations are of order one, quantum effects are relevant even for the background. Notice that if we want to see whether stochastic effects can modify the power spectrum of curvature perturbations, then the criterion is different, see Ref. [120].

If, for instance, we consider the model $V(\phi) = M^4(\phi/M_{Pl})^p$, then the condition $\Delta \phi_q > \Delta \phi_{cl}$ is equivalent to $\phi > \phi_s$ with

$$\frac{\phi_s}{M_{Pl}} = \left[ \frac{\pi p \sqrt{6}}{2} \left( \frac{M_{Pl}}{M} \right)^2 \right]^{2/p}.$$  \hspace{1cm} (1.133)

Then, if one uses the expression of $M$ given in Eq. (1.106), one arrives at

$$\frac{\phi_s}{M_{Pl}} = 2^{-\frac{1}{p+2}} \left( \frac{p^2}{2} + 2p\Delta N_s \right)^{1/2} \left( P_{\zeta 0}^{\text{Planck}} \right)^{-\frac{1}{2+p}},$$  \hspace{1cm} (1.134)

where $P_{\zeta 0}^{\text{Planck}}$ is the amplitude of the spectrum measured by the Planck satellite, see Eq. (1.107). Using this result, namely $\ln \left( 10^{10} P_{\zeta 0} \right) = 3.094 \pm 0.0049$, one obtains $\phi_s/M_{Pl} \simeq 1743$ for the model $p = 2$ (one has taken $\Delta N_s \simeq 50$). It is also interesting to estimate the Hubble parameter for this value of the field and one finds

$$\frac{H_s}{M_{Pl}^2} = 4\pi^2 p^2 \left( \frac{p^2}{2} + 2p\Delta N_s \right)^{-\frac{p}{2} - 1} P_{\zeta 0}^{\text{Planck}} \left( \frac{\phi_s}{M_{Pl}} \right)^{p}.$$  \hspace{1cm} (1.135)

For $p = 2$, this gives $H_s/M_{Pl} \simeq 0.005$, the important point being that we are in a regime where the quantum behavior of the inflaton field must be taken into account but where, at the same time, the concept of a background spacetime is still relevant since $H_s/M_{Pl} \ll 1$.

After these qualitative considerations, let us now try to establish more precisely the equations controlling the evolution of the system in this regime [90, 120–127]. Let us first consider a quantum scalar field in a rigid, de Sitter, background. This means that the backreaction of the quantum scalar field is neglected or, in other words, that it is a test field living in a de Sitter spacetime characterized by $H$. In this spacetime, $H^{-1}$ is a preferred length and can be used to distinguish between short and long wavelengths. Then one writes the scalar field according to [121, 122]

$$\dot{\varphi}(t, \mathbf{x}) = \dot{\varphi}_{IR}(t, \mathbf{x}) + \frac{1}{(2\pi)^{3/2}} \int d\mathbf{k} \Theta (k - \sigma a H) \left[ \mu_k(t) e^{ik \cdot \mathbf{x}} \hat{c}_k + \mu^*_k(t) e^{-ik \cdot \mathbf{x}} \hat{c}^+_k \right],$$  \hspace{1cm} (1.136)
where $\sigma \ll 1$ is a small constant. The quantity $\Theta$ is the Heaviside function, $\mu_k(t)$ is the field mode function and $\hat{c}_k$ and $\hat{c}_k^\dagger$ are the annihilation and creation operators satisfying the standard commutation relations $[\hat{c}_k, \hat{c}_p^\dagger] = \delta(k - p)$. One can then insert this expression into the Klein-Gordon equation to find an equation of motion for the long-wavelength, infrared, part of the field. In fact, it is possible to ignore that the infrared field is a quantum field and see it as a stochastic quantity obeying a Langevin equation given by\cite{121, 122}

$$\frac{d\phi_{IR}(N, x)}{dN} = -\frac{V_{\phi}(\phi_{IR})}{3H^2} + \frac{H}{2\pi} \xi(N, x),$$

(1.137)

where $\xi(N)$ is a white noise sourced by the ultraviolet part of the field with correlation function

$$\langle \xi(N, x)\xi(N', x') \rangle = \delta(N - N')j_0(\sigma aH|x - x'|).$$

(1.138)

Here, $j_0$ is a spherical Bessel function of order zero. By solving the Langevin equation, one can calculate the various correlation functions of the field and show that they coincide with the quantum correlation functions (at least in some limit). This approach, called stochastic inflation, is uncontroversial since it is a fact that the two types of correlation function perfectly match. This is another facet of the general fact that, on super-Hubble scales, the system can be described by a classical stochastic process\cite{90, 128, 129}.

**B. Eternal Inflation**

Then, the next step is to relax the assumption that spacetime is rigid and to take into account the back reaction of the scalar field on the geometry\cite{35–41}. This is at this point that speculations enter the game. Since we study a regime where the inflaton field is viewed as a quantum field, it seems that there are two ways to take into account its backreaction. Either we still view the background as classical, in which case, we need an equation such as $G_{\mu\nu} = \langle \hat{T}_{\mu\nu} \rangle / M_{Pl}^2$, or the background spacetime becomes a quantum object, in which case we need an equation similar to $\hat{G}_{\mu\nu} = \hat{T}_{\mu\nu} / M_{Pl}^2$. In the case of eternal inflation, the second choice is made. Notice that one can even
argue that the first choice is inconsistent, see Sec. VII D. However, since quantum objects are represented by stochastic quantities, we are in fact led to the concept of stochastic geometry (supposed to represent, in this approach, the behavior of a quantum geometry). In this view, the stochastic geometry is sourced by the stochastic scalar field. Then comes the question of which equation controls the behavior of the

FIG. 14. Trajectories (vacuum expectation value of the inflaton field versus number of e-folds) for the inflationary model $V(\phi) = \frac{m^2 \phi^2}{2}$ with $m = 0.1 M_{Pl}$ and different initial conditions, $\phi_{ini} = 10 M_{Pl}$ (red line), $\phi_{ini} = 30 M_{Pl}$ (blue line), $\phi_{ini} = 50 M_{Pl}$ (green line) and $\phi_{ini} = 70 M_{Pl}$ (cyan line). The solid lines represent the stochastic trajectories while the dashed ones correspond to the classical, slow-roll, ones.
Here the common assumption consists in postulating that

\[ H^2 = \frac{1}{3M_{Pl}^2} V(\phi_{IR}), \]  

(1.139)

namely the classical equation promoted to an equation for the stochastic quantities. Here, we really deal with an equation of the type \( \hat{G}_{\mu\nu} = \hat{T}_{\mu\nu}/M_{Pl}^2 \) since \( \phi_{IR} \) and \( H \) are now considered as stochastic quantities. We also notice that, obviously, the above equation is only valid in a cosmological context. Then, the Langevin equation (1.137) becomes

\[ \frac{d\phi_{IR}}{dN} = -\frac{V_{\phi}(\phi_{IR})}{3H^2(\phi_{IR})} + \frac{H(\phi_{IR})}{2\pi} \xi(N). \]  

(1.140)

Clearly this equation is not equivalent to Eq. (1.137) and can even be ambiguous because of the second term which is given by the product of two stochastic quantities. In Fig. 14, we present a numerical integration of this equation for the potential \( V = m^2 \phi^2/2 \) and for different initial conditions. It is easy to see that, in that case, the criterion (1.133) reads \( \phi_s/M_{Pl} \simeq \sqrt{4\pi\sqrt{6}(m/M_{Pl})^{-1}} \). For numerical reasons, in order to clearly illustrate the effect, we choose a value of \( m \) much larger than the one implied by the CMB normalization, namely \( m = 0.1M_{Pl} \). This leads to \( \phi_s \simeq 55M_{Pl} \). Then, we numerically integrate Eq. (1.140) for four different initial conditions, \( \phi_{ini} = 10M_{Pl} \), \( \phi_{ini} = 30M_{Pl} \), \( \phi_{ini} = 50M_{Pl} \) and \( \phi_{ini} = 70M_{Pl} \). Using the trajectory (1.102) and the fact that \( \phi_{end}/M_{Pl} = p/\sqrt{2} \), classically, these four initial conditions respectively correspond to a total of \( \sim 24.5, \sim 224.5, \sim 624.5 \) and \( \sim 1124.5 \) e-folds of inflation. This plot confirms the previous analysis. When \( \phi_{ini} < \phi_s \), we see that the stochastic trajectory (solid line) is very close to the classical one (dashed line). On the contrary, when \( \phi_{ini} \sim \phi_s \) or \( \phi_{ini} > \phi_s \), the stochastic effects dominate, the trajectory becomes “chaotic” and strongly differs from its classical counterpart. In particular, we notice that, due to stochastic effects, the value of the field can increase. This means that the field can in fact climb its potential.

Let us now come back to Eq. (1.137) where we assume that the field is a test field living in a de Sitter spacetime. If \( V(\phi) = m^2 \phi^2/2 \), then this equation can be easily
solved (since it is a linear equation) and the solution reads

$$\phi(N, x) = \phi_{\text{ini}}(N, x)e^{-m^2(N-N_{\text{ini}})/(3H^2)} + \frac{H}{2\pi}e^{-m^2N/(3H^2)} \int_{N_{\text{ini}}}^{N} e^{m^2n/(3H^2)}\xi(n, x)dn.$$  

(1.141)

Using this solution, one can then calculate the two-point correlation function at equal time. One obtains

$$\langle \phi(N, x) \phi(N, x') \rangle = \left[ \phi_{\text{ini}}(N, x)\phi_{\text{ini}}(N, x') - \frac{3H^4}{8\pi^2m^2}j_0 \left( \sigma aH|x - x'| \right) \right] e^{-m^2(N-N_{\text{ini}})} + \frac{3H^4}{8\pi^2m^2}j_0 \left( \sigma aH|x - x'| \right).$$

(1.142)

This expression is made of two pieces. The first one, which depends on the initial conditions, decays away exponentially for $N \gg N_{\text{ini}}$ and quickly becomes sub-dominant. The second piece shows that the ultra large scale structure of the field is made of a collection of nearly homogeneous patches of size $H^{-1}$ (i.e. the Hubble radius) since this is the distance at which the correlation function almost vanishes, thanks to the presence of the Bessel function.

Then, since inflation is an almost de Sitter expansion, what we have just described for a test field should also be true when the back reaction is taken into account, namely for the field the behavior of which is controlled by Eq. (1.140)$^3$. Moreover, each patch is isolated from the others as can be seen by computing the event horizon in de Sitter spacetime. Let us indeed consider a specific observer that we choose, for convenience, to be at the origin. Then, its future horizon (the part of the Universe with which the observer will be able to communicate in the future) is given by

$$d_E = a_0 \int_{t_0}^{\infty} \frac{dt}{a(t)} = a_0 \int_{t_0}^{\infty} dt \frac{1}{a_0} e^{-H(t-t_0)} = \frac{1}{H},$$

(1.145)

namely the size of the patch itself. In other words, each patch is causally disconnected from the others and this forever. These patches are sometimes referred to as “pocket

$^3$ For the potential $V(\phi) = m^2\phi^2/2$, this equation reads

$$\frac{d\phi_{\text{IR}}}{dN} + \frac{2M_{\text{Pl}}^2}{\phi_{\text{IR}}} = \frac{m}{2\pi M_{\text{Pl}} \sqrt{6}} \phi_{\text{IR}} \xi.$$  

(1.143)

It is of the Bernouilli type and, therefore, can be solved explicitly. The solution takes the form

$$\phi_{\text{IR}}^2 = e^{-\frac{m}{\sqrt{6}M_{\text{Pl}}} \int_{N_{\text{ini}}}^{N} \xi(n)dn} \left[ \phi_{\text{ini}}^2 - 4M_{\text{Pl}}^2 \int_{N_{\text{ini}}}^{N} e^{-\frac{m}{\sqrt{6}M_{\text{Pl}}} \int_{N_{\text{ini}}}^{n} \xi(n')dn'}dn \right].$$

(1.144)

However, it is so complicated that it is not very useful. In particular, it seems very difficult to calculate the two-point correlation function of the field from this solution.
universes”. The number of these patches is growing with time. Indeed, in one e-fold, the “size” of the Universe increases by a factor $e^3 \sim 20$ while the “size” of a patch is constant (since the Hubble parameter is constant). As a consequence, each e-fold, one patch gives rise to about twenty new patches, all causally disconnected.

There is also some kind of ergodic argument at play. When, see for instance Fig. 14, we have solved the Langevin equation, each realization of the solution of this equation was supposed to represent a specific configuration of the field over the entire homogeneous and isotropic spacetime. But one can also assume that one realization corresponds to a specific value of the field in a given patch since they are causally disconnected. And, as a consequence, different realizations correspond to different values of the field in different patches. So, in this interpretation, different realizations do not represent an ensemble of different field configurations over an homogeneous and isotropic spacetime but, rather, the spatial distribution of $\phi_{\text{IR}}$ in different patches.

The overall picture that emerges is that of an expanding spacetime where the number of independent patches is increasing, the value of the field in each pocket universe being a stochastic quantity controlled by a Langevin equation. Since we have seen that, due to stochastic effects, the field can climb up its potential, there are patches where inflation will never stop. Obviously, the volume occupied by those patches, compared to the volume occupied by the patches where inflation stops, is growing which means that patches where inflation is taking place occupy more and more regions of spacetime. Globally, inflation will never stop meaning that there are always regions of spacetime undergoing inflation. Of course, there will also be regions of spacetime where inflation stops, those where, by chance, the stochastic fluctuations do not push the field upwards. This structure is referred to as “eternal inflation”. The stochastic effects are said to produce a “multiverse”. Notice that the word “multiverse” is especially awkward in the present context since we do not produce many universes as in the many world interpretation of quantum mechanics for instance but just a specific spatial configuration of our single universe made of causally independent regions, the pocket universes.

Before discussing the reliability and the implications of eternal inflation, we would like to investigate the question of whether it is unavoidable or not.
C. Avoiding Self Replication

Before discussing the robustness of eternal inflation, it is interesting to investigate whether this is an unavoidable consequence of inflation. As recently discussed in Ref. [130], it turns out that this is not the case and, in this section, we closely follow this paper although we also present some new results. We have seen that the quantum-to-classical variation of the field is given by the amplitude of the scalar power spectrum, see Eq. (1.132). If there exists a field value for which this amplitude

\[ P_\zeta(\phi) = \frac{H^2(\phi)}{8\pi M^2_{P}\epsilon_1(\phi)}, \]

is of order one, then this means that the quantum fluctuations of the field are of order one and, if the considerations presented in the previous section are correct, the regime of eternal inflation starts. Usually, this happens in the regime where \( \epsilon_1(\phi) \to 0 \) since
$\epsilon_1(\phi)$ stands at the denominator. But this also implies that, if the shape of the potential is such that there is a field range such that $\epsilon_1 \ll 1$ (in order to have inflation!) but otherwise $\epsilon_1(\phi)$ is large, then there could be no regime where $P_{\zeta_0} > 1$. One example was found by V. Mukhanov in Ref. [130]. The corresponding potential is the following one

$$V(\phi) = M^4 \left(1 - e^{-\phi/M_{Pl}}\right)^2 \left(1 - \frac{\phi}{\phi_m}\right)^{-\alpha},$$

(1.147)

and is represented in Fig. 15. Interestingly enough, it looks like the Starobinsky model corrected by a term $(1 - \phi/\phi_m)^{-\alpha}$. The model depends on three parameters: $M$, $\phi_m$ and $\alpha$. As usual $M$ is fixed by the CMB normalization.
The first two Hubble flow parameters are given by the following expressions

\[
\epsilon_1 = \frac{1}{2} \left[ 2 \frac{e^{-\phi/M_{Pl}}}{1 - e^{-\phi/M_{Pl}}} + \frac{M_{Pl}}{\phi_m} \left( 1 - \frac{\phi}{\phi_m} \right)^{-1} \right]^2,
\]

\[
\epsilon_2 = 4\epsilon_1 + 4e^{-\phi/M_{Pl}} \left( 1 - e^{-\phi/M_{Pl}} \right)^{-1} - 4e^{-2\phi/M_{Pl}} \left( 1 - e^{-\phi/M_{Pl}} \right)^{-2} - 8\alpha \frac{M_{Pl}}{\phi_m} \left( 1 - e^{-\phi/M_{Pl}} \right)^{-1} \left( 1 - \frac{\phi}{\phi_m} \right)^{-1} - 2 \left( \alpha + \alpha^2 \right) \frac{M^2_{Pl}}{\phi^2_m} \left( 1 - \frac{\phi}{\phi_m} \right)^{-2}.
\]

The first Hubble flow parameter is represented in Fig. 16. We see that it has exactly the expected shape. There is a field range where \(\epsilon_1\) is very small and this is the regime during which inflation can take place. But, at large-field values, the corrections play a crucial role and \(\epsilon_1 \to +\infty\) as \(\phi \to \phi_m\). As a consequence, the amplitude of the fluctuations is killed and we never reach the regime of eternal inflation.

Moreover, this model is in perfect agreement with the observations. In Fig. 17, we have compared the predictions of the model for \(\alpha = 4\) and different values of \(\phi_m\) (indicated by the color bar) with the CMB data (the pink contours are the WMAP7 contours while the blue contours are the Planck contours). Evidently, the model is in agreement with the data.

From the previous considerations, as we have already discussed, it should be obvious that the quantum fluctuations are suppressed. In order to check this statement explicitly, we have integrated the Langevin equation with the potential (1.147). The result is represented in Fig. 18 and should be compared to Fig. 14. In both plots, the value of \(M\) has been artificially increased (compared to its CMB value) in order to see the effects more clearly. It is evident that, for the model (1.147), and contrary to what happens for large field models, the quantum fluctuations never play an important role. All the stochastic trajectories always remain close to the classical one.

Therefore, in conclusion, the results presented here clearly indicate that eternal inflation is not mandatory at all and that it is perfectly possible to build a model of inflation which is in perfect agreement with the observations and where self replication never starts. Moreover, from a physical point of view, this scenario seems to make sense. In the slow-roll regime, the potential is flat and this leads to predictions in agreement with CMB data. But, in the UV regime, corrections kick in and modify the potential in such a way that eternal inflation is avoided. The only limitation to the
FIG. 17. Predictions in the \((r, n_s)\) space of the inflationary model with the potential given by Eq. (1.147). The scale \(M\) is CMB normalized, \(\alpha = 4\) and \(\log_{10}(\phi_m/M_{pl}) \in [2, 3]\), its value being indicated by the color bar. Along the same interval, different points represent different reheating temperatures. The pink contours are the 1 and 2\(\sigma\) WMAP7 contours while the blue ones are the 1 and 2\(\sigma\) Planck contours.

The previous argument is that, maybe, the field dependence of the corrections is not such that self-replication is prevented. Indeed, for instance, one has \(P_\zeta \sim V^3/V^2_\phi \sim \phi^{n+2}\) if \(V(\phi) \sim \phi^n\). For \(n > 0\), \(P_\zeta\) always grows with \(\phi\). So if the corrections take the form of monomials, quantum corrections will unavoidably become of order one.

**D. Is the Multiverse a Threat for Inflation?**

In this sub-section, one would like to discuss the implications of the previous considerations for inflation. The main point is that inflation and eternal inflation should not be put on an equal footing. The former provides a phenomenological
FIG. 18. The inflaton field vacuum expectation value versus number of e-folds, for the inflationary model given by Eq. (1.147) with $\alpha = 4$ and $\phi_m = 1000$, calculated by means of the Langevin equation (solid lines) and classically (dashed lines) for different initial conditions, $\phi_{ini} = 900$ (green lines), $\phi_{ini} = 800$ (blue lines) and $\phi_{ini} = 700$ (red lines).

description by means of an effective model of the early universe which seems to be in good agreement with the observations while the latter is, at this stage, only a speculation although definitely an interesting one. The arguments that support this point of view are the following.

Firstly, it is important to make the distinction between stochastic inflation and eternal inflation. Stochastic inflation, which is not a model of inflation, but a technique, appears to be very robust. It is just a fact that the quantum correlation functions in an expanding spacetime can be recovered by focusing on the long wavelength part of the field and by requiring it to obey a Langevin equation. This has been proven beyond any doubt, see for instance Refs. [121, 122]. Stochastic inflation studies test quantum
fields, namely neglects the back reaction of the quantum field on the geometry. In stochastic inflation, the geometry of spacetime is rigid and fixed once and for all.

On the contrary, in the case of eternal inflation, one takes into account the backreaction which means that the geometry (i.e. the gravitational field) must be viewed as a quantum (or stochastic) quantity. Clearly, this is reminiscent of quantum gravity. And, of course, the big question is which theory controls the quantum behavior of the geometry. The theory of eternal inflation just models the coupling between the quantum field and the quantum geometry by equation (1.139), an equation that one could also write as

\[ \hat{H}^2 = \frac{1}{3M_{Pl}^2} V(\phi), \]  

(1.150)

where we have used hats to stress the fact that the geometry should now be viewed as a stochastic quantity and that stochastic quantities are in fact quantum quantities. If this equation happened to be too simplistic, then the previous considerations about eternal inflation could be drastically modified.

Let us now discuss the status of this equation in more detail (here, we follow the treatment of Refs. [131, 132]). Classically, one has \( \dot{H} = -(\rho + p)/(2M_{Pl}^2) \). If \( H \) increases due to quantum jumps, then \( \rho + p < 0 \), which means that one must violate the Null Energy Condition (NEC), namely \( T_{\mu\nu}n^\mu n^\nu < 0 \), where \( n^\mu \) is a null vector. For a scalar field \( T_{\mu\nu}n^\mu n^\nu = (n^\mu \partial_\mu \phi)^2 \geq 0 \) and, classically, the NEC cannot be violated. Quantum mechanically, a natural way to describe the backreaction of quantum matter on the geometry is to write the semi-classical Einstein equations, \( G_{\mu\nu} = \langle \hat{T}_{\mu\nu} \rangle/M_{Pl}^2 \). In this approach, geometry remains classical. Then, let us introduce the NEC operator \( \hat{O} \equiv \hat{T}_{\mu\nu}n^\mu n^\nu = \hat{P}^\dagger \hat{P} \), where \( \hat{P} \equiv n^\mu \partial_\mu \hat{\phi} \). Generically, \( \langle \hat{O} \rangle \) is infinite and must be renormalized. If this is done in a quantum state compatible with the symmetry of de Sitter, then, necessarily, \( \langle \hat{T}_{\mu\nu}^{\text{ren}} \rangle \propto g_{\mu\nu} \) and, therefore, \( \langle \hat{O}_{\text{ren}} \rangle = 0 \) and the NEC cannot be violated. This means that it is necessary to go beyond semi-classical gravity if we want to treat the eternal inflation case and allow for a NEC. Notice that this is what is done in the theory of cosmological perturbations where the equations controlling the evolution of the system are \( \delta \hat{G}_{\mu\nu} = \delta \hat{T}_{\mu\nu}/M_{Pl}^2 \), i.e. quantum operators on both sides. In the linear regime, this has been shown to be consistent and is at the
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origin of the claim that inflation implies an almost scale invariant power spectrum for cosmological perturbations. Of course, eternal inflation corresponds to a situation where the fluctuations are, by definition, not small. A possible way out is to define a smeared NEC operator \[131, 132\],

\[
\hat{O}^{\text{ren}}_W \equiv \int d^4x \sqrt{-g} W(x) \hat{O}^{\text{ren}},
\]

(1.151)

where \(W\) is a window function which has support on a finite part of spacetime. This breaks de Sitter invariance and, as a consequence, one can expect \(\langle \hat{O}^{\text{ren}}_W \rangle \neq 0\). Then, the next step would be to calculate the effects of smeared fluctuations on the metric, a framework which does not yet exist. Despite this, it is usually assumed that this effect will be described by an equation similar to Eq. (1.150). As discussed in Refs. \[131, 132\], the equation (1.150) may describe spacetime before and after the fluctuation happens. But important issues are not addressed as, for instance, the behavior of the metric through the fluctuation or what role the conservation of energy plays in this picture.

As written in Ref. \[133\], “An assumption is that Eq. (28) is sufficient to describe this process” [where “Eq. (28)” refers to Eq. (1.150) and where “this process” refers to the response of quantum geometry to stochastic fluctuations of the field], or “So the heuristic argument, while suggestive, is certainly not sufficient by itself to show that eternal inflation can occur”. We conclude from the above considerations that Eq. (1.150), on which partially rests eternal inflation, is an assumption.

To be completely fair, we should also mention an argument which is in favor of Eq. (1.150). Let us indeed consider the Langevin equation (1.140) again. It can also be used to write a Fokker-Planck equation for \(P(\phi, N)\), the probability density of having the field \(\phi\) at time \(N\). It reads

\[
\frac{\partial}{\partial N} P(\phi, N) = \frac{\partial}{\partial \phi} \left[ \frac{V_\phi}{3H^2} P(\phi, N) \right] + \frac{\partial^2}{\partial \phi^2} \left[ \frac{H^2}{8\pi^2} P(\phi, N) \right].
\]

(1.152)

This equation can also be written as \(\partial P/\partial N = \partial J/\partial \phi\) where \(J\) is a current and a stationary solution \(P_{\text{sta}}(\phi)\) can be obtained by requiring that \(\partial P_{\text{sta}}/\partial N = 0\). Then, the Fokker-Planck equation reduces to a first order differential equation whose solution can be expressed as

\[
P_{\text{sta}}(\phi) \propto \exp \left[ \frac{24\pi^2 M_{\text{pl}}^4}{V(\phi)} \right],
\]

(1.153)
where we have ignored the prefactor which does not play a crucial role in our discussion. Notice that if one considers the Fokker-Planck backward equation, then one obtains the same solution but, crucially, with an overall minus sign in the argument of the exponential, namely

\[ P_{\text{sta}}(\phi) \propto \exp \left[ -\frac{24\pi^2 M_{\text{Pl}}^4}{V(\phi)} \right]. \]  

(1.154)

Both equations (1.153) and (1.154) are relevant for stochastic inflation. Notice that their derivation implicitly assumes Eq. (1.150).

Let us now consider the same situation but from a quantum cosmology point of view [134]. In quantum cosmology, both matter and geometry are supposed to be quantized consistently. The corresponding canonical Hamiltonian can be expressed as

\[ H_c = N \left[ -\frac{\pi_a^2}{48 M_{\text{Pl}}^2 v_K a} + \frac{\pi_\phi^2}{2 v_K a^3} - 12 M_{\text{Pl}}^2 k v_K a + v_K a^3 V(\phi) \right], \]  

(1.155)

where the quantity \( v_K \) represents the volume of the spacelike hypersurfaces and \( N \) is the lapse function. Carrying out Dirac quantization leads to the Wheeler-De Witt equation for the wave-function of the universe, \( \Psi(a, \phi) \), namely

\[ \frac{\partial^2}{\partial a^2} \Psi(a, \phi) + \frac{p}{a} \frac{\partial}{\partial a} \Psi(a, \phi) - 6 M_{\text{Pl}}^2 \frac{\partial^2}{\partial \phi^2} \Psi(a, \phi) \]

\[ - 36 v_K^2 M_{\text{Pl}}^4 a_0^2 \left( \frac{a}{a_0} \right)^2 \left[ K - \left( \frac{a}{a_0} \right)^2 \right] \Psi(a, \phi) = 0. \]  

(1.156)

Here the number \( p \) takes into account the factor ordering ambiguity and \( a_0 \equiv \left[ V(\phi)/(3 M_{\text{Pl}}^2) \right]^{-1/2} \). If one neglects the second derivative with respect to \( \phi \) and chooses \( p = -1 \), then the solution can be found explicitly and reads

\[ \Psi(a, \phi) = \frac{\alpha \text{Ai} \left[ z(a) \right] + \beta \text{Bi} \left[ z(a) \right]}{\alpha \text{Ai} \left[ z(0) \right] + \beta \text{Bi} \left[ z(0) \right]}, \]  

(1.157)

where \( \text{Ai} \) and \( \text{Bi} \) are Airy functions of first and second kinds, respectively, and \( z(0) = z(a = 0) \). The quantity \( z(a) \) is defined by \( z(a) \equiv (3 v_K M_{\text{Pl}}^2 a_0^2)^{2/3} \left( K - a^2/a_0^2 \right) \) and \( \alpha \) and \( \beta \) are complex numbers to be determined by boundary conditions: the tunneling wave function corresponds to \( \alpha = 1 \) and \( \beta = i \) and the no boundary one to \( \alpha = 1 \) and \( \beta = 0 \). In order to make predictions, we need to calculate probabilities but...
the Wheeler-De Witt equation does not lead to positive-definite probabilities. Indeed, the associated current,

\[ j = \frac{i}{2M_{\text{Pl}}^2} a^p \left( \Psi^* \partial_a \Psi - \Psi \partial_a \Psi^* \right), \]

is not positive-definite. However, in the limit \( a \gg \ell_{\text{Pl}} \), the Wentzel-Kramers-Brillouin (WKB) approximation is valid and, in this regime, the probabilities are positive. For the tunneling wave function, this gives

\[ j \simeq \frac{2}{\pi a_0^2 M_{\text{Pl}}^2 |D|^2} \left( 3v_K M_{\text{Pl}}^2 a_0^2 \right)^{2/3} = 6v_K e^{-12v_K M_{\text{Pl}}^4 / V(\phi)}. \]

For the no-boundary wave function, one obtains the same result except that there is no minus in the argument of the exponential. If, in addition, the spacelike section are taken to be spheres, then \( v_K = 2\pi^2 \) and the prediction of quantum cosmology reads

\[ j \propto \exp \left[ \pm \frac{24\pi^2 M_{\text{Pl}}^2}{V(\phi)} \right], \]

which is nothing but Eqs. (1.153) and (1.154). We saw before that the use of an equation \( \dot{H}^2 = V(\dot{\phi}) \) is questionable. The previous argument, however, seems to indicate that this could be reasonable. Indeed, as already mentioned, the stationary distribution of the Fokker-Planck equation was obtained by (implicitly) using this equation. The fact that the Wheeler-De Witt equation, which is an equation where the quantum effects of the geometry are taken into account, leads to results consistent with those obtained from the stochastic formalism retrospectively justifies the use of an equation \( \dot{H}^2 = V(\dot{\phi}) \). Of course, the argument is not completely conclusive since the Wheeler-De Witt equation and the minisuperspace approximation can also be questioned. We conclude that the tools used in order to model backreaction in eternal inflation are, at least for the moment, assumptions. These assumptions may be very reasonable (as seems to be suggested by the above argument) but they remain assumptions.

Let us now discuss a second argument. As is clearly illustrated on the no self-reproduction potential of Sec. VII C, eternal inflation also rests on an extrapolation of the potential \( V(\phi) \) beyond the observable window. By observing the CMB anisotropy, we probe only a limited part of \( V(\phi) \) corresponding to about seven e-folds. Eternal
inflation depends on another region of the potential which is not directly observed. Moreover, this part of the potential is usually relevant at energies higher than the energy scale of inflation (there are exceptions, for instance hybrid inflation, see Ref. [126]) where higher order operators can play a crucial role. For instance, our calculation of eternal inflation in large field models rests on the assumption that, even outside the observational window, the potential is given by $V(\phi) \propto \phi^p$. But nobody knows whether this is true since this is not directly observable. The high-energy corrections could maybe produce terms leading to the Mukhanov’s potential of Sec. VII C, in which case eternal inflation would be irrelevant. Notice that, even if one considers a plateau model, these corrections could play an important role. Indeed, it is true that, a priori, corrections in $V/M_{Pl}^4$ are, by construction, always negligible for plateau models. But the potential itself will generically receive corrections. For instance, if one adds a term $\propto R^3$ to the Starobinsky model, then the effective potential grows with $\phi$. As a consequence, when the field is pushed upwards by the stochastic fluctuations, these corrections will be important.

Thirdly, eternal inflation suffers from a kind of “trans-Planckian problem”. Indeed, as discussed before, one expects the field to be pushed upwards by stochastic fluctuations. Generically, this means that the field will penetrate the region where $V(\phi) \gg M_{Pl}^4$. In this regime, even the notion of a background spacetime is lost. Indeed, in Ref. [38], this problem was already encountered and the potential made steeper by hand in order to prevent the field to penetrate the trans-Planckian region. However, what really happens in this regime remains a matter of debate.

Fourthly, the multiverse is in fact a combination of eternal inflation with the string landscape. A priori, string theory only depends on one parameter, the string tension. All the other parameters of high energy physics, the masses of the particles, the coupling constant etc … should be the vacuum expectation values of some fields appearing in string theory. Since, according to eternal inflation, the fields stochastically fluctuate from patch to patch, it should be the same for the parameters. We are thus led to a picture where what we see as fundamental parameters are in fact stochastic quantities fluctuating from one patch (or one “pocket universe”) to another. This is the famous multiverse. As it turns out, the concept of string landscape is not that
obvious and has been discussed among string theorists [135]. At the moment, the best one could conclude is that the multiverse may pose a question, possibly justifying investigating alternatives to inflation [44]. So the multiverse problem is not only based on an extrapolation, it relies in fact on a combination of extrapolations.

Based on the previous discussion, it seems therefore fair to call the multiverse “problem” of inflation a wild speculation. Even if eternal inflation happens, it is not completely obvious that a multiverse will be present. Indeed, since the question of a stringy landscape remains disputed among string experts, one could imagine a situation where eternal inflation occurs but where there is no stringy landscape. In this case, the inflaton vacuum expectation value would still fluctuate from on patch to another but the fundamental constants would be the same everywhere. This implies that the inflationary predictions would also be the same everywhere (for instance, Doppler peaks in the CMB would be present in each pocket universe), at least in the patches where inflation came to an end. In any case, should we reject single-field slow-roll inflation, a falsifiable, well tested, effective approach to the early universe, in addition in perfect agreement with observations because of the multiverse? To say the least, it would be too hasty. It would be similar to rejecting the standard model of particle physics because (at least for the moment) it cannot be obtained from string theory.

VIII. CONCLUSION

In this article, we have discussed various aspects of inflation. The picture that emerges is that inflation is a very successful model of the early universe. It has all the criterions that a good scientific theory should possess.

First, it is falsifiable. One can indeed quote two possible observations that could potentially rule out inflation. All models of inflation predict the presence of Doppler peaks in the CMB multipole moments. Therefore, if instead of detecting them, we had obtained a bump (as predicted, for instance, if the fluctuations entirely originate from topological defects [136–138]), then inflation would have been ruled out. Another observation that could threaten the basic principles of inflation is the observation that $\Omega_K \neq 0$. It is true that an inflationary model with $\Omega_K \neq 0$ has been constructed
in Ref. [139] but this model is so peculiar that it can be viewed as a curiosity and cannot be considered as representative. Some may argue that it shows the amount of arm-twisting that needs to be done to inflation to make it predict $\Omega_K \neq 0$. In any case, it is the author’s point of view that $\Omega_K \neq 0$ (beyond $10^{-5}$ since, of course, some curvature is present in the perturbed universe) should be considered as a fatal blow for inflation.

Second, inflation has been able to make predictions, most notably the prediction that $n_s$ should be close to one but, and this is the crucial point, excluding one (see, however, the exception [140]). As discussed at length previously, this prediction has been confirmed by the data. It is true that a scale-invariant power spectrum, the so-called Harrisson-Zeldovitch (HZ) power spectrum, was already considered before inflation. But, precisely, the HZ power spectrum has $n_s = 1$ while inflation has $n_s \sim 1$ and, crucially, $n_s - 1 \neq 0$. The prediction $n_s - 1 \neq 0$ was first made by inflation and its observational confirmation is therefore a strong argument in favor of inflation.

Third, the criticisms against inflation do not seem completely compelling (see also Ref. [42] where the initial conditions problem and the measure question are discussed in detail). The initial condition problem does not seem to be very severe, thanks to the presence of an attractor. It is true that the attractor is not present for some models (for instance, small-field inflation with sub-Planckian values) but, precisely, the Planck data have singled out a model (namely the Starobinsky model) where it is present.

The multiverse question is nowadays widely debated and there are claims that its appearance implies that standard inflation makes no prediction and, therefore, is not falsifiable. The argument is that if everything happens, there could be patches in our universe where, for instance, the Doppler peaks are present but there could be others where it is not the case. Or there could be patches where $n_s$ is close to one and others where it is far from one. All that is based on the belief that the multiverse is unavoidable. However, it is, at the moment, unreasonable to put the multiverse and standard inflation on an equal footing. Indeed, at this stage, it is fair to say that the multiverse is a speculation (if it is present at all since we have seen that it can be avoided, see Sec. VII C) and one can argue that it would be awkward to reject a good effective model because of a mere speculation. As already mentioned, this would
be like rejecting the standard model of particle physics because, so far, no one has been able to derive it from string theory. To be completely fair with this analogy and the multiverse criticism, it is true that the potential modifications of the standard model of particle physics suggested by string theory are much less radical that what the multiverse implies for standard inflation.

It is also true that we still do not know the physical nature of the inflaton field even if the latest data raise the intriguing possibility that it could be the Higgs field itself. After all, we are trying to develop a theory the typical energy scale of which could be as high as the GUT scale. So, maybe this problem (if it is indeed one) is not in the inflationary scenario but rather in our lack of understanding of particle physics at $10^{15}\text{GeV}$. In any case, with the recent discovery of the Higgs boson, a common criticism against inflation, namely that no scalar field has ever been seen, has fallen.

Of course, this does not mean that inflation has no drawback and should not be criticized. Admittedly, the question of initial conditions is clearly not completely settled. The question which is left partially unanswered is what happens when one starts from strongly inhomogeneous configurations in the most general situation: impressive numerical simulations of fully inhomogeneous situations have been performed but they do not yet cover all the possibilities. This is technically complicated since this requires numerical relativity. But it is fair to admit that this is a remaining issue which is very important. On the other hand, it is not clear whether this question can be treated classically. Most probably, quantum effects also play an important role in this problem which makes it even more complicated.

Another open issue is the Ultra-Violet (UV) sensitivity of inflation. One example is of course eternal inflation itself. Indeed, we have seen that it can happen or not depending on what we assume about the shape of the potential at high energies, outside the observational window. Another example of UV dependence is the trans-Planckian problem of inflation. If the fluctuations behave in a non standard way when their physical wavelength becomes smaller than the Planck length and if the trans-Planckian physics is non-adiabatic, then the prediction of an almost scale-invariant power spectrum could be modified. Let us nevertheless tone down this conclusion by stressing out that the corresponding modification could be very small. As was
discussed earlier, we have indeed two scales in the problem, the scale $M_C$ at which new physics pops up (typically the Planck scale) and the Hubble parameter during inflation. If the effect scales as the ratio $H/M_C$ to some power, then the correction should be very small. Yet another example of UV dependence is the importance of higher order operators for inflationary model building, see Ref. [141].

Therefore, it is true that inflation has some UV sensitivity. But, after all, this is also the case of the standard model of particle physics where the Higgs mass is not stable against quantum corrections (the hierarchy problem). But no one would reject this model because of this issue. Let us also add that it is inconsistent to claim at the same time that inflation is UV dependent and that the multiverse is unavoidable: if inflation is UV dependent, then one can modify it at high energies to avoid the multiverse and this is exactly what the calculation of Sec. VII C reveals. From a more general perspective concerning the IR/UV connection, it is interesting that inflation seems to provide an example in which the decoupling between physics at different scales, which is at the basis of effective field theory, does not work.

In conclusion, inflation appears to be a robust and reliable scenario for the early universe, not completely free of open issues of course but could it have been different for a theory which is trying to describe the first instants of the universe, at energy scales as high as $10^{15}$GeV? At this stage, admittedly, one cannot yet trust it as we trust, for example, the standard model of particle physics. The situation, however, could change soon if, for instance, we could check the consistency relation, $r = -n_T/8$. This is obviously a difficult task and a first step would clearly be to detect primordial gravitational waves. After all if the pieces of information that we have gathered so far are correct, the next generation of experiments should be able to see them. Indeed their target is $r \sim 10^{-4}$ while, our best model, the Starobinsky model, predicts $r \sim 4 \times 10^{-3}$. Then measuring $n_T$ will be even more difficult but would be very important. The measurement of NG would also be important. The expected level, $f_{NL} \simeq 10^{-2}$, is tiny for our preferred class of models but people are already thinking about experiments that could reach this level.

In brief, inflation continues to be an inspiration for many physicists and continues to fuel new interesting works. So, inflation, trick or treat? Treat, definitively!
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