GIT FOR POLARIZED CURVES

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Abstract. We study the GIT quotients of the Hilbert and Chow schemes of curves, as their degree \(d\) decreases with respect to their genus \(g\). We show that the previous results of L. Caporaso hold true up to \(d > 4(2g - 2)\) and we observe that this is sharp. In the range \(2(2g - 2) < d < \frac{7}{3}(2g - 2)\), we get a complete new description of the GIT quotient. As a corollary of our results, we get a new compactification of the universal Jacobian over the moduli space of pseudo-stable curves.

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1. Introduction

1.1. Motivation and previous works. One of the first successful applications of Geometric Invariant Theory (GIT for short), and perhaps also one of the major motivations for its development by Mumford and his co-authors (see [MFK94]), was the construction of the moduli space $M_g$ of smooth curves of genus $g \geq 2$ together with its compactification $\overline{M}_g$ via stable curves, carried out by Mumford ([Mum77]) and Gieseker ([Gie82]). Indeed the moduli space of stable curves was constructed as a GIT quotient of a locally closed subset of a suitable Hilbert scheme (as in [Gie82]) or Chow scheme (as in [Mum77]) parametrizing $n$-canonically embedded curves, for $n$ sufficiently large. More precisely, Mumford in [Mum77] works under the assumption that $n \geq 5$ and Gieseker in [Gie82] requires the more restrictive assumption that $n \geq 10$. However, it was later realized that Gieseker’s approach can also be extended to the case $n \geq 5$ (see [HM98, Chap. 4, Sec. C] or [Mor10, Sec. 3]).

Recently, there has been a lot of interest in extending the above GIT analysis to smaller values of $n$, especially in connection with the so called Hassett-Keel program whose ultimate goal is to find the minimal model of $M_g$ via the successive constructions of modular birational models of $\overline{M}_g$ (see [FS11] and [AH] for nice overviews).

The first work in this direction is due to Schubert, who described in [Sch91] the GIT quotient of the locus of 3-canonically embedded curves (of genus $g \geq 3$) in the Chow scheme as the coarse moduli space $\overline{M}^\text{ps}_g$ of pseudo-stable curves (or $p$-stable curves for short). These are reduced, connected, projective curves with finite automorphism group, whose only singularities are nodes and ordinary cusps, and which have no elliptic tails. Since the GIT quotient analyzed by Schubert is geometric (i.e. there are no strictly semistable objects), it is easy to see that one gets exactly the same description working with 3-canonically embedded curves inside the Hilbert scheme (see [HH] Prop. 3.13). Later, Hassett-Hyeon have constructed in [HH09] a modular map $T: \overline{M}_g \to \overline{M}^\text{ps}_g$ which on geometric points sends a stable curve onto the $p$-stable curve obtained by contracting all its elliptic tails to cusps. Moreover the authors of loc. cit. identified the map $T$ with the first contraction in the Hassett-Keel program for $\overline{M}_g$.

The case of 4-canonical curves was worked out by Hyeon-Morrison in [HM10]. The GIT quotients for both the Hilbert and Chow scheme turn out to be again isomorphic to $\overline{M}^\text{ps}_g$, although the Chow quotient is not anymore geometric and a more refined analysis is required.

Finally, the case of 2-canonical curves was studied by Hassett-Hyeon in [HH], where the authors described the Hilbert GIT quotient $\overline{M}^h_g$ and the Chow GIT quotient $\overline{M}^c_g$ (they are now different), as moduli spaces of $h$-semistable (resp. $c$-semistable) curves (see loc. cit. for the precise description). Moreover, they constructed a small contraction $\Psi: \overline{M}^\text{ps}_g \to \overline{M}^c_g$ and identified the natural map $\Psi^+: \overline{M}^h_g \to \overline{M}^c_g$ as the flip of $\Psi$. These maps are then interpreted as further steps in the Hassett-Keel program for $\overline{M}_g$. For some partial results on the GIT quotient for the Hilbert scheme of 1-canonically...
embedded curves, we refer the reader to the work of Alper, Fedorchuck and Smyth (see [AFS]).

From the point of view of constructing new projective birational models of $\overline{M}_g$, it
is of course natural to restrict the GIT analysis to the locally closed subset inside the
Hilbert or Chow scheme parametrizing $n$-canonical embedded curves. However, the
problem of describing the whole GIT quotient seems very natural and interesting too.
The first result in this direction is the pioneering work of Caporaso [Cap94], where the
author describes the GIT quotient of the Hilbert scheme of connected curves of genus
$g \geq 3$ and degree $d \geq 10(2g-2)$ in $\mathbb{P}^{d-g}$. The GIT quotient obtained by Caporaso in
loc. cit. is indeed a modular compactification of the universal Jacobian $J_{d,g}$, which is
the moduli scheme parametrizing pairs $(C,L)$ where $C$ is a smooth curve of genus $g$
and $L$ is a line bundle on $C$ of degree $d$. Note that recently Li and Wang in [LW] have
given a different proof of the Caporaso’s result for $d >> 0$.

Our work is motivated by the following

**Problem:** Describe the GIT quotient for the Hilbert and Chow scheme of curves of
genus $g$ and degree $d$ in $\mathbb{P}^{d-g}$, as $d$ decreases with respect to $g$.

1.2. Results. In order to describe our results, we need to introduce some notations.
Fix an integer $g \geq 2$. For any natural number $d$, denote by $\text{Hilb}_d$ the Hilbert scheme
of curves of degree $d$ and arithmetic genus $g$ in $\mathbb{P}^{d-g} := \mathbb{P}(V)$; denote by $\text{Chow}_d$
the Chow scheme of 1-cycles of degree $d$ in $\mathbb{P}^{d-g}$ and by

$$\text{Ch} : \text{Hilb}_d \to \text{Chow}_d$$

the map sending a one dimensional subscheme $[X \subset \mathbb{P}^{d-g}] \in \text{Hilb}_d$ to its 1-cycle. The
linear algebraic group $\text{SL}_{d-g+1}$ acts naturally on $\text{Hilb}_d$ and $\text{Chow}_d$ in such a way that
$\text{Ch}$ is an equivariant map; moreover, these actions are naturally linearized (see Section
4.1 for details), so it makes sense to talk about GIT (semi-,poly-)stability of a point
of $\text{Hilb}_d$ and $\text{Chow}_d$.

We aim at giving a complete characterization of the GIT (semi-,poly-)stable points
$[X \subset \mathbb{P}^{d-g}] \in \text{Hilb}_d$ or of its image $\text{Ch}([X \subset \mathbb{P}^{d-g}]) \in \text{Chow}_d$. Our characterization of
GIT (semi-, poly-) stability will require some conditions on the singularities of $X$ and
some conditions on the multidegree of the line bundle $O_X(1)$. Let us introduce the
relevant definitions.

A curve $X$ is said to be **quasi-stable** if it is obtained from a stable curve $Y$ by
“blowing up” some of its nodes, i.e. by taking the partial normalization of $Y$ at some
its nodes and inserting a $\mathbb{P}^1$ connecting the two branches of each node. A curve $X$
is said to be **quasi-p-stable** if it is obtained from a p-stable curve $Y$ by “blowing up”
some of its nodes (as before) and “blowing up” some of its cusps, i.e. by taking the
partial normalization of $Y$ at some of its cusps and inserting a $\mathbb{P}^1$ tangent to the branch

---

1 In particular, when working with $\text{Hilb}_d$, we will always consider the $m$-linearization for $m >> 0$; see Section 4.1 for details.
point of each cusp. Given a quasi-stable or a quasi-p-stable curve \( X \), we call the \( \mathbb{P}^1 \)'s inserted by blowing up nodes or cusps of \( Y \) the *exceptional components*, and we denote by \( X_{	ext{exc}} \subset X \) the union of all of them.

A line bundle \( L \) of degree \( d \) on a quasi-stable or quasi-p-stable curve \( X \) of genus \( g \) is said to be *balanced* if for each subcurve \( Z \subset X \) the following inequality (called the basic inequality) is satisfied

\[
(*) \quad \left| \deg Z \cdot L - \frac{d \cdot \deg Z(\omega_X)}{2g-2} \right| \leq \frac{|Z \cap Z^c|}{2},
\]

where \( |Z \cap Z^c| \) denotes the length of the 0-dimensional subscheme of \( X \) obtained as the scheme-theoretic intersection of \( Z \) with the complementary subcurve \( Z^c := X \setminus Z \).

A balanced line bundle \( L \) on \( X \) is said to be *properly balanced* if the degree of \( L \) on each exceptional component of \( X \) is 1. Moreover, a properly balanced line bundle \( L \) is said to be *strictly balanced* (resp. *stably balanced*) if the basic inequality (*) is strict except possibly for the subcurves \( Z \) such that \( Z \cap Z^c \subset X_{\text{exc}} \) (resp. such that \( Z \) or \( Z^c \) is entirely contained in \( X_{\text{exc}} \)).

Our first main result extends the description of GIT semi-stable (resp. polystable, resp. stable) points \( [X \subset \mathbb{P}^{d-g}] \in \text{Hilb}_d \) given by Caporaso in \cite{Cap94} to the case \( d > 4(2g-2) \) and also to the Chow scheme.

**Theorem A.** Consider a point \( [X \subset \mathbb{P}^{d-g}] \in \text{Hilb}_d \) with \( d > 4(2g-2) \) and \( g \geq 2 \); assume moreover that \( X \) is connected. Then the following conditions are equivalent:

(i) \( [X \subset \mathbb{P}^{d-g}] \) is GIT semi-stable (resp. polystable, resp. stable);
(ii) \( \text{Ch}([X \subset \mathbb{P}^{d-g}]) \) is GIT semi-stable (resp. polystable, resp. stable);
(iii) \( X \) is quasi-stable and \( \mathcal{O}_X(1) \) is balanced (resp. strictly balanced, resp. stably balanced).

In each of the above cases, \( X \subset \mathbb{P}^{d-g} \) is non-degenerate and linearly normal, and \( \mathcal{O}_X(1) \) is non-special.

The above Theorem A follows by combining Theorem 11.1(1), Corollary 11.2(1) and Corollary 11.3(1). Note also that the condition \( d > 4(2g-2) \) is the sharpest condition under which the above Theorem holds true: if \( d = 4(2g-2) \), then there are GIT stable points \( [X \subset \mathbb{P}^{d-g}] \in \text{Hilb}_d \) with \( X \) having cuspidal singularities (see Remark 4.9).

We then investigate what happens if \( d \leq 4(2g-2) \) and we get a complete answer in the case \( 2(2g-2) < d < \frac{7}{2}(2g-2) \) and \( g \geq 3 \).

**Theorem B.** Consider a point \( [X \subset \mathbb{P}^{d-g}] \in \text{Hilb}_d \) with \( 2(2g-2) < d < \frac{7}{2}(2g-2) \) and \( g \geq 3 \); assume moreover that \( X \) is connected. Then the following conditions are equivalent:

(i) \( [X \subset \mathbb{P}^{d-g}] \) is GIT semi-stable (resp. polystable, resp. stable);
(ii) \( \text{Ch}([X \subset \mathbb{P}^{d-g}]) \) is GIT semi-stable (resp. polystable, resp. stable);
(iii) \( X \) is quasi-p-stable and \( \mathcal{O}_X(1) \) is balanced (resp. strictly balanced, resp. stably balanced).
In each of the above cases, \( X \subset \mathbb{P}^{d-g} \) is non-degenerate and linearly normal, and \( O_X(1) \) is non-special.

Moreover, the Hilbert or Chow GIT quotient is geometric (i.e. all the Hilbert or Chow semistable points are stable) if and only if \( \gcd(2g - 2, d - g + 1) = 1 \).

The above Theorem \( \text{B} \) follows by combining Theorem \( \text{11.1(2)} \), Corollary \( \text{11.2(2)} \), Corollary \( \text{11.3(2)} \) and Proposition \( \text{12.5} \).

We note that the conditions on the degree \( d \) and the genus \( g \) in the above Theorem \( \text{B} \) are sharp. Indeed, if \( d = 2(2g - 2) \) then there are GIT stable points \( [X \subset \mathbb{P}^{d-g}] \in \text{Hilb}_d \) with \( X \) having arbitrary tacnodal singularities (see Remark \( \text{5.2} \)). On the other hand, if \( d = \frac{7}{2}(2g - 2) \) (resp. \( d > \frac{7}{2}(2g - 2) \)) then it follows from \( \text{[Gie82, Prop. 1.0.6, Case 2]} \) that a point \( [X \subset \mathbb{P}^{d-g}] \in \text{Hilb}_d \) with \( X \) having a tacnode with a line in the sense of \( \text{1.3} \) (for example a quasi-p-stable curve \( X \) obtained from a p-stable curve \( Y \) by blowing up a cusp) is such that \( \text{Ch}([X \subset \mathbb{P}^{d-g}]) \) is not GIT stable (resp. GIT semistable) \( \text{3} \). Finally, if \( g = 3 \) then Heyon-Lee proved in \( \text{[HL07]} \) that a 3-canonical irreducible p-stable curve \( X \subset \mathbb{P}^4 \) of genus 2 with one cusp is not GIT polystable (while it is GIT semistable), which shows that the description of GIT stable and GIT polystable points given in Theorem \( \text{B} \) is false in this case. Probably the description of GIT semistable points given in Theorem \( \text{B} \) is still true for \( g = 2 \); however for simplicity we restrict in this paper to the case \( g \geq 3 \) whenever dealing with quasi-p-stable curves.

As an application of Theorem \( \text{B} \) we get a new compactification of the universal Jacobian \( J_{d,g} \) over the moduli space of p-stable curves of genus \( g \). To this aim, consider the category fibered in groupoids \( \mathcal{J}_{d,g}^{ps} \) over the category of schemes, whose fiber over a scheme \( S \) is the groupoid of families of quasi-p-stable curves over \( S \) endowed with a line bundle whose restriction to the geometric fibers is properly balanced. In Section \( \text{12} \) we will prove the following

**Theorem C.** Let \( g \geq 3 \) and \( d \in \mathbb{Z} \).

1. \( \mathcal{J}_{d,g}^{ps} \) is a smooth, irreducible and universally closed Artin stack of finite type over \( k \) and of dimension \( 4g - 4 \).
2. \( \mathcal{J}_{d,g}^{ps} \) admits an adequate moduli space \( \mathcal{J}_{d,g}^{ps} \), which is a normal irreducible projective variety of dimension \( 4g - 3 \) containing \( J_{d,g} \) as an open subvariety.
   Moreover, if \( \text{char}(k) = 0 \), then \( \mathcal{J}_{d,g}^{ps} \) has rational singularities, hence it is Cohen-Macauly.
3. There exists a commutative diagram

\[
\begin{array}{ccc}
\mathcal{J}_{d,g}^{ps} & \longrightarrow & \mathcal{J}_{d,g}^{ps} \\
\Psi^{ps} \downarrow & & \Phi^{ps} \\
\mathcal{M}_g^{ps} & \longrightarrow & \mathcal{M}_g^{ps}
\end{array}
\]
where $\Psi^{ps}$ is universally closed and surjective and $\Phi^{ps}$ is projective, surjective and has equidimensional fibers of dimension $g$.

(4) If $\text{char}(k) = 0$ or $\text{char}(k) = p > 0$ is bigger than the order of the automorphism group of any $p$-stable curve of genus $g$, then for any $X \in \overline{M}_g^{ps}$, the fiber $(\Phi^{ps})^{-1}(X)$ is isomorphic to $\overline{\text{Jac}}_d(X)/\text{Aut}(X)$, where $\overline{\text{Jac}}_d(X)$ is the Simpson's compactified Jacobian of $X$ parametrizing $S$-equivalence classes of rank-1, torsion-free sheaves on $X$ that are slope-semistable with respect to $\omega_X$.

(5) If $2(2g - 2) < d < \frac{7}{2}(2g - 2)$ then $\overline{\mathcal{T}}^{ps}_{d,g} \cong [H_d/GL(r+1)]$ and $\overline{\mathcal{T}}^{ps}_{d,g} \cong H_d/GL(r+1)$, where $H_d \subset \text{Hilb}_d$ is the open subset consisting of points $[X \subset \mathbb{P}^{d-g}] \in \text{Hilb}_d$ such that $X$ is connected and $\text{Ch}(X \subset \mathbb{P}^{d-g})$ is GIT-semistable (or equivalently, $[X \subset \mathbb{P}^{d-g}]$ is GIT-semistable).

The above Theorem follows by combining Theorems and Propositions. A couple of comments on the above theorem are in order. First of all, the hypothesis on the characteristic of the base field $k$ in part (4) is needed in order to guarantee that the automorphism groups of the geometric points of $\overline{\mathcal{T}}^{ps}_{d,g}$ are linearly reductive. For more details, we refer the reader to the proof of Theorem 12.12 and the discussion following it. Secondly, the stack $\overline{\mathcal{T}}^{ps}_{d,g}$ is never a Deligne-Mumford stack nor a proper stack and the map $\Psi$ is never proper nor representable. The reason is that the automorphism group of each geometric point of $\overline{\mathcal{T}}^{ps}_{d,g}$ contains the multiplicative group $\mathbb{G}_m$ acting as scalar multiplication on the line bundle. It is then natural to take the rigidification $\overline{\mathcal{T}}^{ps}_{d,g}/\mathbb{G}_m$ and to ask if the above properties hold true for this new stack $\overline{\mathcal{T}}^{ps}_{d,g}/\mathbb{G}_m$ and for the new map $\overline{\mathcal{T}}^{ps}_{d,g}/\mathbb{G}_m \to \overline{M}_g^{ps}$. In Proposition 12.5, we prove that this is indeed the case if and only if the numerical condition $\gcd(d+1-g,2g-2) = 1$ is satisfied.

1.3. Open problems. This work leaves unsolved some natural problems for further investigation, that we discuss briefly here.

The first problem is of course the following

**Problem A.** Describe the GIT (semi-,poly-)stable points of $\text{Hilb}_d$ and $\text{Chow}_d$ in the case $\frac{7}{2}(2g - 2) \leq d \leq 4(2g - 2)$.

Indeed, as observe before, Theorem 5.1 is false in the range $\frac{7}{2}(2g - 2) \leq d \leq 4(2g - 2)$. In Theorem 5.1 we give some necessary conditions for a point $[X \subset \mathbb{P}^{d-g}] \in \text{Hilb}_d$ (or for its image $\text{Ch}([X \subset \mathbb{P}^{d-g}]) \in \text{Chow}_d$) to be GIT semistable. Further progresses have been made by Fabio Felici in [Fel].

By analogy with the contraction map $T : \overline{M}_g^{ps} \to \overline{M}_g$ constructed by Hassett-Hyeon in [HH09] (see also Fact 2.2 in for more details), the following problem seems very natural.
Problem B. Construct a map $\tilde{T}: \mathcal{J}_{d,g} \to \mathcal{J}^{ps}_{d,g}$ fitting into the following commutative diagram

$$
\begin{array}{ccc}
\mathcal{J}_{d,g} & \xrightarrow{\tilde{T}} & \mathcal{J}^{ps}_{d,g} \\
\downarrow{\Phi} & & \downarrow{\Phi^{ps}} \\
\mathcal{M}_g & \xrightarrow{T} & \mathcal{M}^{ps}_g,
\end{array}
$$

where $\mathcal{J}_{d,g}$ is the Caporaso’s compactification of the universal Picard variety.

More generally, one would like to set up a Hassett-Keel program for $\mathcal{J}_{d,g}$ and give an interpretation of the above map $\tilde{T}$ as the first step in this program.

Finally, as we discuss at the beginning of Section 9, if $d > 2(2g - 2)$ then the locus of connected curves in the Hilbert or Chow GIT semistable locus is a connected component (which is moreover irreducible if either $d < \frac{7}{2}(2g - 2)$ or $d > 4(2g - 2)$, by Proposition 10.9). However, we do not know the answer to the following

Problem C. Are there connected components inside the Hilbert or Chow GIT semistable locus made entirely of non-connected curves?

1.4. Outline of the paper. We now give a brief outline of the paper. In Section 2 we discuss the singular curves that will appear throughout the paper. In Section 3 we collect combinatorial results on balanced multidegrees and on the degree class group that will play a crucial role in several proofs. In Section 4 we set up our GIT problem for Hilb$_d$ and Chow$_d$. Moreover, we recall some well known techniques in GIT (e.g. the Hilbert-Mumford’s criterion for GIT (semi)stability and the basin of attraction) as well as some classical results in GIT of curves (e.g. the potential stability theorem and the stability for smooth curves of high degree). In Section 5 we prove the potential pseudo-stability Theorem 5.1 which gives necessary conditions for GIT semistability in the case $2(2g - 2) < d$. In Section 6 we prove that GIT semistable curves do not have elliptic tails if $2(2g - 2) < d < \frac{7}{2}(2g - 2)$ (see Theorem 6.1). In Section 7 we determine the connected component of the identity of the stabilizer subgroups of the pairs $(X, O_X(1))$ belonging to the GIT semistable locus (see Corollary 7.3). In Section 8 we investigate some properties of GIT semistable pairs $(X, O_X(1))$ with $O_X(1)$ stably balanced (see Theorem 8.1) or strictly balanced (see Corollary 8.6). In Section 9 we construct a map from the GIT quotient of Hilb$_d$ or Chow$_d$ for $2(2g - 2) < d$ towards the moduli space of p-stable curves (see Theorem 9.1). In Section 10 we introduce a stratification of the GIT semistable locus and then we study the closure of the strata (see Proposition 10.5) and we prove a completeness result for these strata (see Proposition 10.6). In Section 11 we characterize GIT (semi, poly)-stable points in Hilb$_d$ and Chow$_d$ if either $4(2g - 2) < d$ or $2(2g - 2) < d < \frac{7}{2}(2g - 2)$ and $g \geq 3$, thus proving Theorems A and B. In Section 12 we define and study a new compactification of the universal Jacobian over the moduli space of p-stable curves; in particular we prove Theorem C. The
Appendix 13 contains some positivity results for balanced line bundles on Gorenstein curves which are used throughout the paper and that we find interesting in their own.

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Conventions.

1.1. \( k \) will denote an algebraically closed field (of arbitrary characteristic). All schemes are \( k \)-schemes, and all morphisms are implicitly assumed to respect the \( k \)-structure.

1.2. A curve is a complete, reduced and separated scheme (over \( k \)) of pure dimension 1 (not necessarily connected). The genus \( g(X) \) of a curve \( X \) is \( g(X) := h^1(X, \mathcal{O}_X) \).

1.3. A subcurve \( Z \) of a curve \( X \) is a closed \( k \)-scheme \( Z \subseteq X \) that is reduced and of pure dimension 1. We say that a subcurve \( Z \subseteq X \) is proper if \( Z \neq \emptyset, X \).

Given two subcurves \( Z \) and \( W \) of \( X \) without common irreducible components, we denote by \( Z \cap W \) the 0-dimensional subscheme of \( X \) that is obtained as the scheme-theoretic intersection of \( Z \) and \( W \) and we denote by \( |Z \cap W| \) its length.

Given a subcurve \( Z \subseteq X \), we denote by \( Z^c := X \setminus Z \) the complementary subcurve of \( Z \) and we set \( k_Z = k_{Z^c} := |Z \cap Z^c| \).

1.4. An elliptic tail of a curve \( X \) is a connected subcurve \( Z \) of genus one meeting the rest of the curve in one point; i.e. a connected subcurve \( Z \subseteq X \) such that \( g(Z) = 1 \) and \( k_Z = |Z \cap Z^c| = 1 \).

1.5. Let \( X \) be a curve. A point \( p \) of \( X \) is said to be

- a node if the completion \( \widehat{\mathcal{O}}_{X,p} \) of the local ring \( \mathcal{O}_{X,p} \) of \( X \) at \( p \) is isomorphic to \( k[[x,y]]/(y^2 - x^2) \);
- a cusp if \( \widehat{\mathcal{O}}_{X,p} \cong k[[x,y]]/(y^2 - x^3) \);
- a tacnode if \( \widehat{\mathcal{O}}_{X,p} \cong k[[x,y]]/(y^2 - x^4) \).

A tacnode with a line of a curve \( X \) is a tacnode \( p \) of \( X \) at which two irreducible components \( D_1 \) and \( D_2 \) of \( X \) meet with a simple tangency and in such a way that \( D_1 \cong \mathbb{P}^1 \) and \( k_{D_1} = 2 \) (or equivalently \( p \) is the set-theoretical intersection of \( D_1 \) and \( D_1^c \)).

1.6. A curve \( X \) is called Gorenstein if its dualizing sheaf \( \omega_X \) is a line bundle.

1.7. A family of curves is a proper, flat morphism \( X \to T \) whose geometric fibers are curves. Given a class \( C \) of curves, a family of curves of \( C \) is a family of curves \( X \to T \) whose geometric fibers belong to the class \( C \). For example: if \( C \) is the class
of nodal curves of genus $g$, a family of nodal curves of genus $g$ is a family of curves whose geometric fibers are nodal curves of genus $g$.

1.8. Consider a contravariant functor

$$\mathcal{F} : \text{SCH} \to \text{SET}$$

from the category SCH of schemes to the category SET of sets.

We say that a scheme $X$ represents $\mathcal{F}$, or that $\mathcal{F}$ is represented by $X$, if $\mathcal{F}$ is isomorphic to the functor of points $\text{Hom}(-, X)$ of $X$, i.e. the functor that associates to a scheme $T$ the set of morphisms $\text{Hom}(T, X)$.

We say that a scheme $X$ co-represents $\mathcal{F}$ if there exists a natural transformation of functors $\Phi : \mathcal{F} \to \text{Hom}(-, X)$ that is universal with respect to natural transformations from $\mathcal{F}$ to the functor of points of schemes, i.e. for any natural transformation $\Psi : \mathcal{F} \to \text{Hom}(-, Y)$ where $Y$ is a scheme there exists a unique morphism $f : X \to Y$ such that $\Psi = f_* \circ \Phi$ where $f_* : \text{Hom}(-, X) \to \text{Hom}(-, Y)$ is the natural transformation of functors induced by composing with $f$.

Given two contravariant functors $\mathcal{F}, \mathcal{G} : \text{SCH} \to \text{SET}$, we say that a natural transformation $\mathcal{F} \to \mathcal{G}$ is a local isomorphism if

(i) for every $T \in \text{SCH}$ and $y \in \mathcal{G}(T)$ there exists an étale cover $\{T_i \to T\}$ of $T$ and elements $\{x_i \in \mathcal{F}(T_i)\}$ with $x_i$ mapping to the restriction $y|_{T_i} \in \mathcal{G}(T_i)$.

(ii) for every $T \in \text{SCH}$ and $x, x' \in \mathcal{F}(T)$ mapping to the same element of $\mathcal{G}(T)$ there exists an étale cover $\{T_i \to T\}$ such that the restrictions of $x$ and $x'$ to $T_i$ coincide in $\mathcal{F}(T_i)$.

Equivalently, a local isomorphism is a natural transformation $\mathcal{F} \to \mathcal{G}$ that induces an isomorphism of the étale sheaves associated to $\mathcal{F}$ and $\mathcal{G}$. Using the well-known fact that a representable functor is a sheaf for the étale topology, it follows that if $\mathcal{F} \to \mathcal{G}$ is a local isomorphism then a scheme $X$ co-represents $\mathcal{F}$ if and only if it co-represents $\mathcal{G}$.

2. Singular curves

The aim of this section is to collect the definitions and basic properties of some special curves that will play a key role in the sequel.

2.1. Stable and p-stable curves. We begin by recalling the definition of Deligne-Mumford’s stable curves ([DM69]) and Schubert’s pseudostable curves ([Sch91]) of genus $g \geq 2$.

**Definition 2.1.** A connected curve $X$ of arithmetic genus $g \geq 2$ is

(i) stable if

(a) $X$ has only nodes as singularities;

(b) the canonical sheaf $\omega_X$ is ample.

(ii) $p$-stable (or pseudo-stable) if
(a) $X$ has only nodes and (ordinary) cusps as singularities;
(b) $X$ does not have elliptic tails;
(c) the canonical sheaf $\omega_X$ is ample.

Note that, in both cases, $\omega_X$ is ample if and only if each connected subcurve $Z$ of $X$ of genus zero is such that $k_Z = |Z \cap Z^c| \geq 3$.

Stable curves and p-stable curves have projective coarse moduli schemes, which are related as follows.

**Fact 2.2** ([DM69], [Sch91], [HH09]).

(i) There exists a projective irreducible variety $\overline{M}_g$ which is the coarse moduli space for stable curves of genus $g$. If $g \geq 3$ then there exists a projective irreducible variety $\overline{M}^{ps}_g$ which is the coarse moduli space for p-stable curves of genus $g$.

(ii) If $g \geq 3$ then there exists a natural map
$$T : \overline{M}_g \to \overline{M}^{ps}_g$$
which sends $X \in \overline{M}_g$ to the p-stable curve $T(X) \in \overline{M}^{ps}_g$ obtained by contracting each elliptic tail to an ordinary cusp. In particular, $T$ is an isomorphism outside the divisor $\Delta_1 \subset \overline{M}_g$ of curves having an elliptic tail.

If $g = 2$ then the functor of p-stable curves of genus $g$ is not separated (or equivalently the stack of p-stable curves is not a Deligne-Mumford stack) and therefore it does not admit a coarse moduli space. Nevertheless, Hyeon and Lee have constructed in [HL07] a projective variety $\overline{M}^{ps}_2$ which co-represents the functor of p-stable curves of genus 2 (in the sense of Convention [LS] and they have defined a map $T : \overline{M}_2 \to \overline{M}^{ps}_2$ which contracts the divisor $\Delta_1$ to the unique p-stable rational curve with two cusps. In this paper, however, we will often assume that $g \geq 3$ whenever we will deal with p-stable curves, in order to avoid these technical issues.

2.2. **wp-stable curves and p-stable reduction.** A common generalization of stable and p-stable curves is provided by the wp-stable (=weakly-pseudo-stable) curves, which were first considered in [HM10] Pag. 8.  

**Definition 2.3.** A connected curve $X$ of genus $g \geq 2$ is said to be wp-stable if

(i) $X$ has only nodes and cusps as singularities;
(ii) the canonical sheaf $\omega_X$ is ample.

As before, the condition that $\omega_X$ is ample is equivalent to the fact that each connected subcurve $Z$ of $X$ of genus zero is such that $k_Z = |Z \cap Z^c| \geq 3$.

**Remark 2.4.** Note that stable curves and p-stable curves are wp-stable. More precisely:

(i) stable curves are exactly those wp-stable curves without cusps.
(ii) p-stable curves are exactly those wp-stable curves without elliptic tails.
Given a wp-stable curve $Y$ it is possible to obtain a p-stable curve, called its p-stable reduction and denoted by $ps(Y)$, by contracting the elliptic tails of $Y$ to cusps. The p-stable reduction works even for families.

**Proposition 2.5.** Let $v : Y \to S$ be a family of wp-stable curves of genus $g \geq 3$. There exists a commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\psi} & ps(Y) \\
v \downarrow & & \downarrow ps(v) \\
\downarrow & & \downarrow ps(v) \\
S & & S
\end{array}
$$

where $ps(v) : ps(Y) \to S$ is a family of p-stable curves of genus $g$. The family $ps(v) : ps(Y) \to S$ is called the p-stable reduction of $v : Y \to S$.

For every geometric point $s \in S$, the morphism $\psi_s : Y_s \to ps(Y)_s$ contracts the elliptic tails of $Y_s$ to cusps of $ps(Y)_s$. Moreover, the formation of $ps(v)$ commutes with base change.

**Proof.** If $v : Y \to S$ is a family of stable curves, this is exactly the content of [HH09, Sec. 3]. In what follows we will show how to generalize Hassett-Hyeon’s argumentation in order to work out in our case.

First of all, if $S = k$, then the statement follows from Proposition 3.1 in [HH09], which asserts that given a stable curve $C$ of genus $g \geq 3$, there is a replacement morphism $\xi_C : C \to T(C)$, where $T(C)$ is a pseudo-stable curve of genus $g$, which is an isomorphism away from the loci of elliptic tails and that replaces elliptic tails with cusps. The argumentation is local on the nodes connecting each genus-one subcurve meeting the rest of the curve in a single node. Since in a wp-stable curve all elliptic tails are connected to the rest of the curve via a single node, the same argumentation works also in our case with no further modifications.

The whole question is now how to make it work over an arbitrary base $S$. Hassett and Hyeon’s approach is then to consider the moduli stack of stable curves $\overline{M}_g$ and a faithfully flat atlas $V \to \overline{M}_g$. The general case follows then by base-change from $V \to \overline{M}_g$ to $S$.

In our case, we consider the stack $\overline{M}_g^{wp}$ whose sections over a scheme $S$ consist of families of wp-stable curves of genus $g$ over $S$.

**CLAIM:** $\overline{M}_g^{wp}$ is a smooth, irreducible algebraic stack.

Indeed, $\overline{M}_g^{wp}$ is algebraic since it is an open substack of the stack of all genus $g$ curves, which is well known to be algebraic (see e.g. [Hal]). By [Ser06, Prop. 2.4.8], an obstruction space for the deformation functor $\text{Def}_X$ of a wp-stable curve $X$ is the vector space $\text{Ext}^2(\Omega^1_X, \mathcal{O}_X)$ which is zero according to [DM69] Lemma 1.3] since $X$ is a reduced curve with locally complete intersection singularities. This implies

\[\text{Unlike } \overline{M}_g, \text{ the stack } \overline{M}_g^{wp} \text{ is non separated (although it is universally closed). This however does not interfere with the proof that follows.}\]
that £Def\textsubscript{X} is formally smooth, hence that $\overline{M}^{wp}_g$ is smooth at $X$. Moreover, from [Ser06, Thm. 2.4.1] and [Ser06, Cor. 3.1.13], it follows that a reduced curve with locally complete intersection singularities can always be smoothened; therefore the open substack $\mathcal{M}_g \subset \overline{M}^{wp}_g$ of smooth curves is dense. Since $\mathcal{M}_g$ is irreducible (by [DM69]), we deduce that $\overline{M}^{wp}_g$ is irreducible as well.

Let now $\rho_\pi : U \to \overline{M}^{wp}_g$ be a faithfully flat atlas of $\overline{M}^{wp}_g$ and let $\pi : Z \to U$ be the associated (universal) family of wp-stable curves. The idea is now to consider an invertible sheaf $L$ on $Z$, which will be a twisted version of the relative dualizing sheaf of $\pi$ such that $L$ is very ample away from the locus of elliptic tails, and instead has relative degree 0 over all elliptic tails. Then use $L$ to define an $S$-morphism from $Z$ to a family of p-stable curves which coincides with the previous one over all geometric fibers of $\pi$.

To be precise, denote by $\delta_1 \subset \overline{M}^{wp}_{g,1}$ the boundary divisor of elliptic tails on the universal stack $\overline{M}^{wp}_{g,1}$ over $\overline{M}^{wp}_g$. An argument similar to the proof of the above Claim shows that $\overline{M}^{wp}_{g,1}$ is smooth; hence $\delta_1$ is a Cartier divisor. Let $\mu_\pi : Z \to \overline{M}^{wp}_{g,1}$ be the classifying morphism corresponding to the family $\pi : Z \to U$ and set $L := \omega_\pi(\mu_\pi^*\delta_1)$. The whole point is now to prove that $\pi_*(L^n)$ is locally free and that $L^n$ is relatively globally generated for $n > 0$ and that the associated morphism factors through

$$Z \xrightarrow{\xi_Z} T(Z) \hookrightarrow \mathbb{P}(\pi_*(L^n))$$

where $T(Z)$ is a family of p-stable curves and $\xi_Z$ coincides with the replacement morphism $\xi_C$ for all geometric fibers $C$ of $\pi$. By browsing carefully through Hassett-Hyeon’s argumentation, we easily conclude that everything holds also in our case.

\[ \square \]

2.3. Quasi-wp-stable curves and wp-stable reduction. The most general class of singular curves that we will meet throughout this work is the one given in the following:

\textbf{Definition 2.6.}

(i) A connected curve $X$ is said to be \textit{pre-wp-stable} if the only singularities of $X$ are nodes, cusps or tacnodes with a line.

(ii) A connected curve $X$ is said to be \textit{pre-p-stable} if it is pre-wp-stable and it does not have elliptic tails.

(iii) A connected curve $X$ is said to be \textit{pre-stable} if the only singularities of $X$ are nodes.

Note that wp-stable (resp. p-stable, resp. stable) curves are pre-wp-stable (resp. pre-p-stable, resp. pre-stable) curves. Moreover, if $p \in X$ is a tacnode with a line lying in $D_1 \cong \mathbb{P}^1$ and $D_2$ as in 1.5 then $(\omega_X)|_{D_1} = \mathcal{O}_{D_1}$, hence $\omega_X$ is not ample. From this, we get easily that

\textbf{Remark 2.7.} $X$ is wp-stable (resp. p-stable, resp. stable) if and only if $X$ is pre-wp-stable (resp. pre-p-stable, resp. pre-stable) and $\omega_X$ is ample.
The pre-wp-stable curves that we will meet in this paper, even when non wp-stable, will satisfy a very strong condition on connected subcurves where the restriction of the canonical line bundle is not ample, i.e., on connected subcurves of genus zero that meet the complementary subcurve in less than three points. This justifies the following

**Definition 2.8.** A pre-wp-stable curve $X$ is said to be

(i) **quasi-wp-stable** if every connected subcurve $E \subset X$ such that $g_E = 0$ and $k_E \leq 2$ satisfies $E \cong \mathbb{P}^1$ and $k_E = 2$ (and therefore it meets the complementary subcurve $E^c$ either in two distinct nodal points of $X$ or in one tacnode of $X$).

(ii) **quasi-p-stable** if it is quasi-wp-stable and pre-p-stable.

(iii) **quasi-stable** if it is quasi-wp-stable and pre-stable.

The subcurves $E$ such $E \cong \mathbb{P}^1$ and $k_E = 2$ are called exceptional and the subcurve of $X$ given by the union of the exceptional subcurves is denoted by $X_{\text{exc}}$. The complementary subcurve $X_{\text{exc}}^c = X \setminus X_{\text{exc}}$ is called the non-exceptional subcurve and is denoted by $\tilde{X}$.

Equivalently, a quasi-wp-stable curve is a pre-wp-stable $X$ such that $\omega_X$ is nef (i.e. it has non-negative degree on every subcurve of $X$) and such that all the connected subcurves $E \subseteq X$ such that $\deg_E \omega_X = 0$ (which are called exceptional components) are irreducible. Note that the term quasi-stable curve was introduced in [Cap94, Sec. 3.3].

We summarize the different types of curves that we have so far introduced into the following table.

| SINGULARITIES | $\omega_X$ NEF + IRREDUCIBLE EXCEPTIONAL COMPONENTS | $\omega_X$ AMPLE |
|---------------|--------------------------------------------------|------------------|
| pre-wp-stable = nodes, cusps, tacnodes with a line | quasi-wp-stable | wp-stable |
| pre-p-stable = pre-wp-stable without elliptic tails | quasi-p-stable | p-stable |
| pre-stable = nodes | quasi-stable | stable |

**Table 1.** Singular curves

Given a quasi-wp-stable curve $Y$, it is possible to contract all the exceptional subcurves in order to obtain a wp-stable curve, which is called the wp-stable reduction of $Y$ and is denoted by $\text{wps}(Y)$. This construction indeed works for families.
**Proposition 2.9.** Let $S$ be a scheme and $u : \mathcal{X} \to S$ a family of quasi-wp-stable curves. Then there exists a commutative diagram

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\phi} & \text{wps}(\mathcal{X}) \\
\downarrow u & & \downarrow \text{wps}(u) \\
S & & S
\end{array}
$$

where $\text{wps}(u) : \text{wps}(\mathcal{X}) \to S$ is a family of wp-stable curves, called the wp-stable reduction of $u$.

For every geometric point $s \in S$, the morphism $\phi_s : \mathcal{X}_s \to \text{wps}(\mathcal{X})_s$ contracts the exceptional subcurves $E$ of $\mathcal{X}_s$ in such a way that

1. If $E \cap E^c$ consists of two distinct nodal points of $X$, then $E$ is contracted to a node;
2. If $E \cap E^c$ consists of one tacnode of $X$, then $E$ is contracted to a cusp.

The formation of $\text{wps}(u)$ commutes with base change. Furthermore, if $u$ is a family of quasi-$p$-stable (resp. quasi-stable) curves then $\text{wps}(u)$ is a family of $p$-stable (resp. stable) curves.

**Proof.** We will follow the same ideas as in the proof of [Knu83, Prop. 2.1] and of [Mel11, Prop. 6.6]. Consider the relative dualizing sheaf $\omega_u := \omega_{\mathcal{X}/S}$ of the family $u : \mathcal{X} \to S$. It is a line bundle because the geometric fibers of $u$ are Gorenstein curves by our assumption. From Corollary 13.7 in the Appendix we get that for all $i \geq 2$, the restriction of $\omega_u^i$ to a geometric fiber $\mathcal{X}_s$ is non-special, globally generated and, if $i \geq 3$, normally generated. Then, we can apply [Knu83, Cor. 1.5] to get the following properties for $\omega_u$:

(a) $R^1 u_* (\omega_u^i) = (0)$ for all $i \geq 2$;
(b) $u_* (\omega_u^i)$ is $S$-flat for all $i \geq 2$;
(c) for any morphism $T \to S$ there are canonical isomorphisms

$$u_*(\omega_u^i) \otimes_{\mathcal{O}_S} \mathcal{O}_T \to (u \times 1)_*(\omega_u^i \otimes_{\mathcal{O}_S} \mathcal{O}_T)$$

for any $i \geq 2$;
(d) the canonical map $u^* u_* (\omega_u^i) \to \omega_u^i$ is surjective for all $i \geq 3$;
(e) the natural maps $(u_* \omega_u^3)^i \otimes u_* \omega_u^3 \to (u_* \omega_u^3)^{i+1}$ are surjective for $i \geq 1$.

Define now

$$\mathcal{S}_i := u_* (\omega_u^i), \text{ for all } i \geq 0.$$  

By (a) and (b) above, we know that $\mathcal{S}_i$ is locally free and flat on $S$ for $i \geq 2$. Consider

$$\mathbb{P}(\mathcal{S}_3) \to S.$$  

Since, by (d) above, the natural map

$$u^* u_* (\omega_u^3) \to \omega_u^3$$  

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is surjective, we get that there is a natural $S$-morphism

$$
\begin{array}{c}
\mathcal{X} \\
\downarrow q \\
\mathcal{Y}
\end{array}
\quad
\begin{array}{c}
P(S_3) \\
\downarrow u
\end{array}
\quad
\begin{array}{c}
S
\end{array}
$$

Denote by $\mathcal{Y}$ the image of $\mathcal{X}$ via $q$ and by $\phi$ the (surjective) $S$-morphism from $\mathcal{X}$ to $\mathcal{Y}$. By (e) above, we get that

$$
\mathcal{Y} = \text{Proj}(\bigoplus_{i \geq 0} S_i).
$$

So, $\phi : \mathcal{Y} \to S$ is flat since the $S_i$’s are flat for $i \geq 2$. To conclude that $\mathcal{Y} \to S$ is a family of wp-stable curves note that the restriction of $\omega^3_u$ to the geometric fibers of $u$ has positive degree in all irreducible components except the exceptional ones, where it has degree 0. Indeed, it is easy to see (see for example [Cat82, Rmk. 1.20]) that, on each geometric fiber $\mathcal{X}_s$, $\phi$ contracts an exceptional component $E \subseteq \mathcal{X}_s$ to a node if $E$ meets the complementary curve in two distinct nodal points and to a cusp if $E$ meets the complementary subcurve in one tacnode. Moreover, $\Phi$ is an isomorphism outside the non exceptional locus. We conclude that $\mathcal{Y} \to S$ is a family of wp-stable curves, so we set $\text{wps}(\mathcal{X}) := \mathcal{Y}$ and $\text{wps}(u) := \mathcal{Y} \to S$.

Property (c) above implies that forming wps is compatible with base-change.

The last assertion is clear from the above geometric description of the contraction $\phi_s : \mathcal{X}_s \to \text{wps}(\mathcal{X})_s$ on each geometric point of $u$.

□

**Remark 2.10.** If $u : \mathcal{X} \to S$ is a family of quasi-stable curves then the wp-stable reduction $\text{wps}(u) : \text{wps}(\mathcal{X}) \to S$ coincides with the usual stable reduction $s(u)$ of $u$ (see e.g. [Knu83]).

The wp-stable reduction allows us to give a more explicit description of the quasi-wp-stable curves.

**Corollary 2.11.** A curve $X$ is quasi-wp-stable (resp. quasi-p-stable, resp. quasi-stable) if and only if can be obtained from a wp-stable (resp. p-stable, resp. stable) curve $Y$ via an iteration of the following construction:

(i) Normalize $Y$ at a node $p$ and insert a $\mathbb{P}^1$ meeting the rest of the curve in the two branches of the node.

(ii) Normalize $Y$ at a cusp and insert a $\mathbb{P}^1$ tangent to the rest of the curve at the branch of the cusp.

In this case, $Y = \text{wps}(X)$. In particular, given a wp-stable (resp. p-stable, resp. stable) curve $Y$ there exists only a finite number of quasi-wp-stable (resp. quasi-p-stable, resp. quasi-stable) curves $X$ such that $\text{wps}(X) = Y$, which we call quasi-wp-stable (resp. quasi-p-stable, resp. quasi-stable) models of $Y$. 
Note that the above operation (i) cannot occur for quasi-stable curves. With a slight abuse of terminology, we call the above operation (i) (resp. (ii)) the blow-up of a node (resp. of a cusp).

**Proof.** We will prove the Corollary only for quasi-wp-stable curves. The remaining cases are dealt with in the same way.

Let $X$ be a quasi-wp-stable curve and set $Y := \text{wps}(X)$. By Proposition 2.9, the wp-stabilization $\phi : X \to Y = \text{wps}(X)$ contracts each exceptional component $E$ of $X$ to a node or a cusp according to whether $E \cap E^c$ consists of two distinct points or one point with multiplicity two. Therefore $X$ is obtained from $Y$ by a sequence of the two operations (i) and (ii).

Conversely, if $X$ is obtained from a wp-stable curve $Y$ by a sequence of operations (i) and (ii), then clearly $X$ is quasi-wp-stable and $Y = \text{wps}(X)$.

The last assertion is now clear. □

We end this section with an extension of the p-stable reduction of Proposition 2.5 to families of quasi-wp-stable curves.

**Definition 2.12.** Let $S$ be a scheme and $u : \mathcal{X} \to S$ be a family of quasi-wp-stable curves of genus $g \geq 3$. Then there exists a commutative diagram

\[
\begin{array}{ccc}
\phi : \mathcal{X} & \xrightarrow{\phi} & \text{wps}(\mathcal{X}) \\
& \searrow^{u} & \text{wps}(u) := \text{wps}(\text{wps}(u)) \\
& & S \\
\psi : \text{ps} \circ \text{wps}(\mathcal{X}) & \xrightarrow{\psi} & \text{ps}(\text{wps}(\mathcal{X})) := \text{ps}(\mathcal{X})
\end{array}
\]

where the family wps(u) is the wp-stable reduction of the family $u$ (see Proposition 2.9) and the family ps(wps(u)) is the p-stable reduction of the family wps(u) (see Proposition 2.5).

We set $\text{ps}(u) := \text{ps}(\text{wps}(u))$ and we call it the $p$-stable reduction of $u$.

3. **Combinatorial results**

The aim of this section is to collect all the combinatorial results that will be used in the sequel.

3.1. **Balanced multidegree and the degree class group.** Let us first recall some combinatorial definitions and results from [Cap94, Sec. 4]. Although the results in loc. cit. are stated for nodal curves, a close inspection of the proofs reveals that the same results are true more in general for Gorenstein curves.

So, we fix a connected Gorenstein curve $X$ of genus $g \geq 2$ and we denote by $C_1, \ldots, C_\gamma$ the irreducible components of $X$. A multidegree on $X$ is an ordered $\gamma$-tuple of integers

\[
d = (d_{C_1}, \ldots, d_{C_\gamma}) \in \mathbb{Z}^\gamma.
\]
We denote by $|d| = \sum_{i=1}^{\gamma} d_{C_i}$ the total degree of $d$. Given a subcurve $Z \subseteq X$, we set $d_Z := \sum_{C_i \subseteq Z} d_{C_i}$. The term multidegree comes from the fact that every line bundle $L$ on $X$ has a multidegree $\deg L$ given by $\deg L := (\deg_{C_1} L, \ldots, \deg_{C_\gamma} L)$ whose total degree $|\deg L|$ is the degree $\deg L$ of $L$.

We now introduce an inequality condition on the multidegree of a line bundle which will play a key role in the sequel.

**Definition 3.1.** Let $d$ be a multidegree of total degree $|d| = d$. We say that $d$ is balanced if it satisfies the inequality (called basic inequality)

$$\left|d_Z - \frac{d}{2g - 2}\deg_Z \omega_X \right| \leq \frac{k_Z}{2},$$

for every subcurve $Z \subseteq X$.

We denote by $\tilde{B}^d_X$ the set of all balanced multidegrees on $X$ of total degree $d$.

Following [Cap94, Sec. 4.1], we now define an equivalence relation on the set of multidegrees on $X$. For every irreducible component $C_i$ of $X$, consider the multidegree $C_i = ((C_i)_1, \ldots, (C_i)_\gamma)$ of total degree 0 defined by

$$(C_i)_j = \begin{cases} |C_i \cap C_j| & \text{if } i \neq j, \\ -\sum_{k \neq i} |C_i \cap C_k| & \text{if } i = j. \end{cases}$$

More generally, for any subcurve $Z \subseteq X$, we set $Z := \sum_{C_i \subseteq Z} C_i$.

Denote by $\Lambda_X \subseteq \mathbb{Z}^\gamma$ the subgroup of $\mathbb{Z}^\gamma$ generated by the multidegrees $C_i$ for $i = 1, \ldots, \gamma$. It is easy to see that $\sum_i C_i = 0$ and this is the only relation among the multidegrees $C_i$. Therefore, $\Lambda_X$ is a free abelian group of rank $\gamma - 1$.

**Definition 3.2.** Two multidegrees $d$ and $d'$ are said to be equivalent, and we write $d \equiv d'$, if $d - d' \in \Lambda_X$. In particular, if $d \equiv d'$ then $|d| = |d'|$.

For every $d \in \mathbb{Z}$, we denote by $\Delta^d_X$ the set of equivalence classes of multidegrees of total degree $d = |d|$. Clearly $\Delta^0_X$ is a finite group under component-wise addition of multidegrees (called the degree class group of $X$) and each $\Delta^d_X$ is a torsor under $\Delta^0_X$.

The following two facts will be used in the sequel. The first result says that every element in $\Delta^d_X$ has a balanced representative. The second result investigates the relationship between balanced multidegrees that have the same class in $\Delta^d_X$.

**Fact 3.3** (Caporaso). For every multidegree $d$ on $X$ of total degree $d = |d|$, there exists $d' \in \tilde{B}^d_X$ such that $d \equiv d'$.

For a proof see [Cap94, Prop. 4.1]. Note that in loc. cit. the above result is only stated for a nodal curve $X$ and $d = 0$. However, a closer inspection of the proof shows that it extends verbatim to our case. See also [MV, Prop. 2.8] for a refinement of the above result.
Fact 3.4 (Caporaso). Let \( d, d' \in \overline{B}^d_X \). Then \( d \equiv d' \) if and only if there exist subcurves \( Z_1 \subseteq \ldots \subseteq Z_m \) of \( X \) such that

\[
\begin{cases}
  d_{Z_i} = \frac{d}{2g-2} \deg_{Z_i} \omega_X + \frac{k_{Z_i}}{2} & \text{for } 1 \leq i \leq m, \\
  d' = d + \sum_{i=1}^{m} Z_i.
\end{cases}
\]

Moreover, the subcurves \( Z_i \) can be chosen in such a way that \( Z_i^c \cap Z_j = \emptyset \) for \( i > j \).

For a proof see [Cap94, p. 620 and p. 625]. In loc. cit., the result is stated for DM-semistable curves but the proof extends verbatim to our case.

3.2. Stably and strictly balanced multidegrees on quasi-wp-stable curves.

We now specialize to the case where \( X \) is a quasi-wp-stable curve of genus \( g \geq 2 \) (see Definition 2.8).

Given a balanced multidegree \( d \) on \( X \), the basic inequality (3.1) gives that \( d_E = -1, 0, 1 \) for every exceptional subcurve \( E \subset X \). The multidegrees such that \( d_E = 1 \) on each exceptional subcurve \( E \subset X \) will play a special role in the sequel; hence they deserve a special name.

Definition 3.5. We say that a multidegree \( d \) on \( X \) is properly balanced if \( d \) is balanced and \( d_E = 1 \) for every exceptional component \( E \) of \( X \).

We denote by \( B^d_X \) the set of all properly balanced multidegrees on \( X \) of total degree \( d \).

The aim of this subsection is to investigate the behavior of properly balanced multidegrees on a quasi-wp-stable curve \( X \), which attain the equality in the basic inequality (3.1) relative to some subcurve \( Z \subset X \). Let us denote the two extremes of the basic inequality relative to \( Z \) by

\[
\begin{cases}
  m_Z := \frac{d}{2g-2} \deg_Z \omega_X - \frac{k_Z}{2}, \\
  M_Z := \frac{d}{2g-2} \deg_Z \omega_X + \frac{k_Z}{2},
\end{cases}
\]

Note that \( m_Z = M_Z^c \) and \( M_Z = m_Z^c \). The definitions below will be important in what follows.

Definition 3.6. A properly balanced multidegree \( d \) on \( X \) is said to be

(i) strictly balanced if any proper subcurve \( Z \subset X \) such that \( d_Z = M_Z \) satisfies \( Z \cap Z^c \subset X_{\text{exc}} \).

(ii) stably balanced if any proper subcurve \( Z \subset X \) such that \( d_Z = M_Z \) satisfies \( Z \subset X_{\text{exc}} \).

\[\text{Footnote: Actually, the reader can easily check that all the results of this subsection are valid more in general if } X \text{ is a G-quasistable curve of genus } g \geq 2 \text{ in the sense of Definition 13.1.}\]
In the case where $X$ is a quasi-stable curve, the above Definition 3.6(i) coincides with the definition of extremal in [Cap94, Sec. 5.2], while the Definition 3.6(ii) coincides with the definition of G-stable in [Cap94, Sec. 6.2]. Here we adopt the terminology of [BFV11, Def. 2.4].

**Definition 3.7.** We will say that a line bundle $L$ on $X$ is balanced if and only if its multidegree $\deg L$ is balanced. Similarly for properly balanced, strictly balanced, stably balanced.

**Remark 3.8.** In order to check that a multidegree $d$ on $X$ is balanced (resp. strictly balanced, resp. stably balanced), it is enough to check the conditions of Definitions 3.1 and 3.6 only for the subcurves $Z \subset X$ such that $Z$ and $Z^c$ are connected. This follows easily from the following facts. If $Z$ is a subcurve of $X$ and we denote by $\{Z_1, \ldots, Z_c\}$ the connected components of $Z$, then

(i) The upper (resp. lower) inequality in (3.1) is achieved for $Z$ if and only if the upper (resp. lower) inequality in (3.1) is achieved for every $Z_i$. This follows from the (easily checked) additivity relations

\[
\begin{align*}
\deg_Z L &= \sum_i \deg_{Z_i} L, \\
\deg_Z \omega_X &= \sum_i \deg_{Z_i} \omega_X, \\
k_Z &= \sum_i k_{Z_i}.
\end{align*}
\]

(ii) $Z \cap Z^c \subseteq X_{\text{exc}}$ if and only if $Z_i \cap Z_i^c \subseteq X_{\text{exc}}$ for every $i$. Similarly, $Z \subseteq X_{\text{exc}}$ if and only if $Z_i \subseteq X_{\text{exc}}$ for every $Z_i$.

(iii) If $Z^c$ is connected, then $Z_i^c = \bigcup_{j \neq i} Z_j \cup Z^c$ is connected for every $Z_i$.

The next result explains the relationship between stably balanced and strictly balanced line bundles.

**Lemma 3.9.** A multidegree $d$ on a quasi-wp-stable curve $X$ of genus $g \geq 2$ is stably balanced if and only if $d$ is strictly balanced and $\tilde{X} = X \setminus X_{\text{exc}}$ is connected.

**Proof.** The proof of [BFV11, Lemma 2.7] extends verbatim from quasi-stable curves to quasi-wp-stable curves. □

The importance of strictly balanced multidegrees is that they are unique in their equivalence class in $\Delta^d_X$, at least among the properly balanced multidegrees.

**Lemma 3.10.** Let $d, d' \in B^d_X$ be two properly balanced multidegrees of total degree $d$ on a quasi-wp-stable curve $X$ of genus $g \geq 2$. If $d \equiv d'$ and $d$ is strictly balanced, then $d = d'$.  

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Proof. According to Fact 3.4, there exist subcurves $Z_1 \subseteq \ldots \subseteq Z_m$ of $X$ such that

\[(3.3)\quad d' = d + \sum_{i=1}^{m} Z_i,\]

\[(3.4)\quad d_{Z_i} = \frac{d}{2g-2} \deg_{Z_i} \omega_X + \frac{k_{Z_i}}{2} \text{ for } 1 \leq i \leq m,\]

\[(3.5)\quad Z_i^c \cap Z_j = \emptyset \text{ for } i > j.\]

Assume, by contradiction, that $d \not\equiv d'$; hence, using (3.3), we can assume that $Z := Z_1$ is a proper subcurve of $X$. From (3.4) and the fact that $d$ is strictly balanced, we deduce that $Z \cap Z^c \subset X_{\text{exc}}$. Therefore, there exists an exceptional subcurve $E \subseteq X_{\text{exc}}$ such that one of the following four possibilities occurs:

- Case (I): $E \subseteq Z$ and $|E \cap Z^c| = 1$,
- Case (II): $E \subseteq Z$ and $|E \cap Z^c| = 2$,
- Case (III): $E \subseteq Z^c$ and $|E \cap Z| = 1$,
- Case (IV): $E \subseteq Z^c$ and $|E \cap Z| = 2$.

Note that in Cases (II) or (IV), we have that the intersection of $E$ with $Z$ or $Z^c$ consists either of two distinct points or of one point of multiplicity two.

Claim: Cases (III) and (IV) cannot occur.

By contradiction, assume first that case (III) occurs. Consider the subcurve $Z \cup E$ of $X$. We have clearly that

\[
\begin{align*}
    d_{Z \cup E} &= d_Z + 1, \\
    \deg_{Z \cup E} \omega_X &= \deg_Z \omega_X, \\
    k_{Z \cup E} &= k_Z.
\end{align*}
\]

Therefore, using (3.4), we have that

\[
d_{Z \cup E} = d_Z + 1 = \frac{d}{2g-2} \deg_Z \omega_X + \frac{k_Z}{2} + 1 = \frac{d}{2g-2} \deg_{Z \cup E} \omega_X + \frac{k_{Z \cup E}}{2} + 1,
\]

which contradicts the basic inequality (3.1) for $d$ with respect to the subcurve $Z \cup E \subseteq X$.

Assume now that case (IV) occurs. For the subcurve $Z \cup E \subseteq X$, we have that

\[
\begin{align*}
    d_{Z \cup E} &= d_Z + 1, \\
    \deg_{Z \cup E} \omega_X &= \deg_Z \omega_X, \\
    k_{Z \cup E} &= k_Z - 2.
\end{align*}
\]

Therefore, using (3.4), it follows that

\[
d_{Z \cup E} = d_Z + 1 = \frac{d}{2g-2} \deg_Z \omega_X + \frac{k_Z}{2} + 1 = \frac{d}{2g-2} \deg_{Z \cup E} \omega_X + \frac{k_{Z \cup E}}{2} + 2,
\]

which contradicts the basic inequality (3.1) for $d$ with respect to the subcurve $Z \cup E \subseteq X$. The claim is now proved.
Therefore, only cases (I) or (II) can occur. Note that

\[(3.6) \quad Z_E = -|E \cap Z^c| = \begin{cases} -1 & \text{if case (I) occurs,} \\ -2 & \text{if case (II) occurs.} \end{cases} \]

Note also that, in any case, we must have that \(E \subseteq Z = Z_1\). Using (3.5), we get that \(E \cap Z_i^c = \emptyset\) for any \(i > 1\), which implies that

\[(3.7) \quad Z_{i,E} = 0 \quad \text{for any } i > 1. \]

We now evaluate (3.3) at the subcurve \(E\): using that \(d_E = 1\) because \(d\) is strictly balanced and equations (3.6) and (3.7), we conclude that

\[d'_E = \begin{cases} 0 & \text{if case (I) occurs,} \\ -1 & \text{if case (II) occurs.} \end{cases} \]

In both cases, this contradicts the assumption that \(d'_E\) is properly balanced, q.e.d. \(\Box\)

We conclude this subsection with the following Lemma, that will be used several times in what follows.

**Lemma 3.11.** Let \(X, Y\) and \(Z\) be quasi-wp-stable curves of genus \(g \geq 2\). Let \(\sigma : Z \to X\) and \(\sigma' : Z \to Y\) be two surjective maps given by blowing down some of the exceptional components of \(Z\). Let \(d\) (resp. \(d'\)) be a properly balanced multidegree on \(X\) (resp. on \(Y\)). Denote by \(\tilde{d}\) the pull-back of \(d\) on \(Z\) via \(\sigma\), i.e., the multidegree on \(Z\) given on a subcurve \(W \subseteq Z\) by

\[\tilde{d}_W = \begin{cases} d_{\sigma(W)} & \text{if } \sigma(W) \text{ is a subcurve of } X, \\ 0 & \text{if } W \text{ is contracted by } \sigma \text{ to a point.} \end{cases} \]

In a similar way, we define the pull-back \(\tilde{d}'\) of \(d'\) on \(Z\) via \(\sigma'\). The following is true:

(i) \(\tilde{d}\) and \(\tilde{d}'\) are balanced multidegrees.

(ii) If \(d\) is strictly balanced and \(\tilde{d} \equiv \tilde{d}'\) then there exists a map \(\tau : X \to Y\) such that the following diagram commutes

\[
\begin{array}{ccc}
Z \\
\sigma \downarrow \quad \quad \sigma' \downarrow \\
X & \xleftarrow{\tau} & Y
\end{array}
\]

**Proof.** Part (i): let us prove that \(\tilde{d}\) is balanced; the proof for \(\tilde{d}'\) being analogous. Consider a connected subcurve \(W \subseteq Z\) and let us show that \(\tilde{d}\) satisfies the basic inequality (3.1) with respect to the subcurve \(W \subseteq Z\). If \(W\) is contracted by \(\sigma\) to a point, then \(W\) must be an exceptional component of \(Z\). In this case, we have that \(d_Z = 0\), \(k_W = 2\) and \(\deg_W(\omega_Z) = 0\) so that (3.1) is satisfied. If \(\sigma(W)\) is a subcurve of \(X\), then \(\tilde{d}_W = d_{\sigma(W)}\) and, since \(\sigma\) contracts only exceptional components of \(Z\), it is easy to see that \(\deg_W(\omega_Z) = \deg_{\sigma(W)}(\omega_X)\) and that \(|W \cap W^c| = |\sigma(W) \cap \sigma(W)^c|\) as...
it is easily seen from the fact that . Therefore, in this case, the basic inequality for \( \tilde{d} \) with respect to \( W \) follows from the basic inequality for \( d \) with respect to \( \sigma(W) \).

Part (ii): start by noticing that if every exceptional component \( E \subset Z \) which is contracted by \( \sigma \) is also contracted by \( \sigma' \) then \( \sigma' \) factors through \( \sigma \), so the map \( \tau \) exists. Let us now prove that in order for the map \( \tau \) to exist, it is also necessary that every exceptional component \( E \subset Z \) which is contracted by \( \sigma \) is also contracted by \( \sigma' \). By contradiction, assume that \( \tau \) exists and that there exists an exceptional subcurve \( E \subset Z \) which is contracted by \( \sigma \) but not by \( \sigma' \). Then we have that

\[
\begin{cases}
\tilde{d}_E = 0, \\
\tilde{d}'_E = d'_{\sigma(E)} = 1,
\end{cases}
\]

where in the last equation we have used that \( \sigma(E') \) is an exceptional component of \( Y \) and that \( d' \) is properly balanced.

Since \( \tilde{d} \) is equivalent to \( \tilde{d}' \) by assumption, Fact 3.3 implies that we can find subcurves \( W_1 \subset \ldots \subset W_m \subseteq Z \) such that

\[
\tilde{d} = \tilde{d}' + \sum_{i=1}^{m} W_i,
\]

\[
\tilde{d}'_{W_i} = \frac{d}{2g - 2} \deg_{W_i} \omega_Z + \frac{k_{W_i}}{2} \text{ for } 1 \leq i \leq m,
\]

\[
W_i^c \cap W_j = \emptyset \text{ for } i > j.
\]

From (3.8) and (3.9), we get that

\[
\sum_{i=1}^{m} W_{iE} = -1.
\]

Denote by \( C_1 \) and \( C_2 \) the irreducible components of \( Y \) that intersect \( E \), with the convention that \( C_1 = C_2 \) if there is only one such irreducible component of \( Y \) that meets \( E \) in two distinct points or in one point with multiplicity 2. It follows directly from the definition of \( W \) (see Section 3) that for any subcurve \( W \subset Z \) with complementary subcurve \( W^c \) we have that

\[
W_{E} = \begin{cases}
2 \text{ if } E \subseteq W^c \text{ and } C_1 \cup C_2 \subseteq W, \\
1 \text{ if } E \subseteq W^c \text{ and exactly one among } C_1 \text{ and } C_2 \text{ is a subcurve of } W, \\
0 \text{ if } E \cup C_1 \cup C_2 \subseteq W^c \text{ or } E \cup C_1 \cup C_2 \subseteq W, \\
-1 \text{ if } E \subseteq W \text{ and exactly one among } C_1 \text{ and } C_2 \text{ is a subcurve of } W, \\
-2 \text{ if } E \subseteq W \text{ and } C_1 \cup C_2 \subseteq W^c.
\end{cases}
\]

Using this formula, together with (3.12) and (3.11), it is easy to see that \( C_1 \) must be different from \( C_2 \) and that, up to exchanging \( C_1 \) with \( C_2 \), there exists an integer
1 \leq q \leq m \text{ such that }
\begin{align*}
\begin{cases}
E \cup C_1 \cup C_2 \subseteq W_i^c & \text{if } i < q, \\
E \cup C_1 \subset W_q \text{ and } C_2 \subseteq W_q^c, \\
E \cup C_1 \cup C_2 \subseteq W_i & \text{if } i > q.
\end{cases}
\end{align*}

(3.13)

Let us now compute $\tilde{d}_{W_q}$. From (3.11), we get that
$$W_i \cap W_q = \begin{cases}
-k_{W_q} & \text{if } i = q, \\
0 & \text{if } i \neq q.
\end{cases}$$

Combining this with (3.9) and (3.10), we get that
$$\tilde{d}_{W_q} = d_{W_q} - k_{W_q} = \frac{d}{2g-2} \deg_{W_q} \omega_Z - \frac{k_{W_q}}{2}. 
\tag{3.14}
$$

Consider now the subcurve $\sigma(W_q)$ of $X$. By (3.14), we have that
$$d_{\sigma(W_q)} = \tilde{d}_{W_q} = \frac{d}{2g-2} \deg_{W_q} \omega_Z - \frac{k_{W_q}}{2} = \frac{d}{2g-2} \deg_{\sigma(W_q)} \omega_X - \frac{k_{\sigma(W_q)}}{2},
$$

and by (3.13) we have that
$$\sigma(W_q) \cap \sigma(W_q)^c \not\subseteq X_{\text{exc}}.$$

This contradicts the fact that $\tilde{d}$ is strictly balanced, q.e.d.

\begin{flushright}
$\Box$
\end{flushright}

4. Preliminaries on GIT

4.1. Hilbert and Chow schemes of curves. Fix, throughout this paper, two integers $d$ and $g \geq 2$ and write $d := v(2g-2) = 2v(g-1)$ for some (uniquely determined) rational number $v$. Set $r + 1 := d - g + 1 = (2v - 1)(g-1)$.

Let $\text{Hilb}_d$ the Hilbert scheme parametrizing subschemes of $\mathbb{P}^r = \mathbb{P}(V)$ having Hilbert polynomial $P(m) := md + 1 - g$, i.e., subschemes of $\mathbb{P}^r$ of dimension 1, degree $d$ and arithmetic genus $g$. An element $[X \subset \mathbb{P}^r]$ of $\text{Hilb}_d$ is thus a 1-dimensional scheme $X$ of arithmetic genus $g$ together with an embedding $X \overset{i}{\hookrightarrow} \mathbb{P}^r$ of degree $d$. We let $\mathcal{O}_X(1) := i^*\mathcal{O}_{\mathbb{P}^r}(1) \in \text{Pic}^d(X)$.

It is well-known (see [MS, Lemma 2.1]) that for any $m \geq M := \binom{d}{2} + 1 - g$ and any $[X \subset \mathbb{P}^r] \in \text{Hilb}_d$ it holds that:
- $\mathcal{O}_X(m)$ has no higher cohomology;
- The natural map
$$\text{Sym}^m V^\vee \to \Gamma(\mathcal{O}_X(m)) = H^0(X, \mathcal{O}_X(m))$$
is surjective.
Under these hypothesis, the \( m \)-th Hilbert point of \([X \subset \mathbb{P}^r]\) in \( \text{Hilb}_d \) is defined to be

\[
[X \subset \mathbb{P}^r]_m := \left[ \text{Sym}^m V^\vee \to \Gamma(O_X(m)) \right] \in \text{Gr}(P(m), \text{Sym}^m V^\vee) \hookrightarrow \mathbb{P} \left( \bigwedge^{P(m)} \text{Sym}^m V^\vee \right),
\]

where \( \text{Gr}(P(m), \text{Sym}^m V^\vee) \) is the Grassmannian variety parametrizing \( P(m) \)-dimensional quotients of \( \text{Sym}^m V^\vee \), which lies naturally in \( \mathbb{P} \left( \bigwedge^{P(m)} \text{Sym}^m V^\vee \right) \) via the Plücker embedding.

For any \( m \geq M \), we get a closed SL(V)-equivariant embedding (see [Mum66, Lect. 15]):

\[
j_m : \text{Hilb}_d \hookrightarrow \text{Gr}(P(m), \text{Sym}^m V^\vee) \hookrightarrow \mathbb{P} \left( \bigwedge^{P(m)} \text{Sym}^m V^\vee \right) := \mathbb{P}^{\left( \bigwedge^{P(m)} \text{Sym}^m V^\vee \right)}.
\]

Therefore, for any \( m \geq M \), we get an ample SL(V)-linearized line bundle \( \Lambda_m := j_m^* O_{\mathbb{P}^r}(1) \) and we denote by

\[
\text{Hilb}^{s,m}_d \subseteq \text{Hilb}^{ss,m}_d \subseteq \text{Hilb}_d
\]

the locus of points that are, respectively, stable and semistable with respect to \( \Lambda_m \). If \([X \subset \mathbb{P}^r] \in \text{Hilb}^{s,m}_d \) (resp. \([X \subset \mathbb{P}^r] \in \text{Hilb}^{ss,m}_d \)), we say that \([X \subset \mathbb{P}^r] \) is \( m \)-Hilbert stable (resp. semistable).

The ample cone of \( \text{Hilb} \) admits a finite decomposition into locally-closed cells, such that the stable and the semistable loci are constant for linearizations taken from a given cell [DH98, Theorem 0.2.3(i)]. In particular, \( \text{Hilb}^{s,m} \) and \( \text{Hilb}^{ss,m} \) are constant for \( m \gg 0 \). We set

\[
\left\{ \begin{array}{ll}
\text{Hilb}^s_d := \text{Hilb}^{s,m}_d & \text{for } m \gg 0, \\
\text{Hilb}^{ss}_d := \text{Hilb}^{ss,m}_d & \text{for } m \gg 0.
\end{array} \right.
\]

If \([X \subset \mathbb{P}^r] \in \text{Hilb}^s_d \) (resp. \([X \subset \mathbb{P}^r] \in \text{Hilb}^{ss}_d \)), we say that \([X \subset \mathbb{P}^r] \) is Hilbert stable (resp. semistable). If \([X \subset \mathbb{P}^r] \in \text{Hilb}^{ss}_d \) is such that the SL(V)-orbit of \([X \subset \mathbb{P}^r] \) is closed inside \( \text{Hilb}^{ss}_d \) then we say that \([X \subset \mathbb{P}^r] \) is Hilbert polystable.

Let \( \text{Chow}_d \hookrightarrow \mathbb{P}(\otimes^2 \text{Sym}^d V^\vee) := \mathbb{P}^r \) the Chow scheme parametrizing 1-cycles of \( \mathbb{P}^r \) of degree \( d \) together with its natural SL(V)-equivariant embedding \( j \) into the projective space \( \mathbb{P}(\otimes^2 \text{Sym}^d V^\vee) \) (see [Mum66, Lect. 16]). Therefore, we have an ample SL(V)-linearized line bundle \( \Lambda := j^* O_{\mathbb{P}^r}(1) \) and we denote by

\[
\text{Chow}^s_d \subseteq \text{Chow}^{ss}_d \subseteq \text{Chow}_d
\]

the locus of points of \( \text{Chow}_d \) that are, respectively, stable and semistable with respect to \( \Lambda \).

There is an SL(V)-equivariant cycle map (see [MK94, §5.4]):

\[
\text{Ch} : \text{Hilb}_d \to \text{Chow}_d \quad [X \subset \mathbb{P}^r] \mapsto \text{Ch}([X \subset \mathbb{P}^r]).
\]

We say that \([X \subset \mathbb{P}^r] \in \text{Hilb}_d \) is Chow stable (resp. semistable) if \( \text{Ch}([X \subset \mathbb{P}^r]) \in \text{Chow}^s_d \) (resp. \( \text{Chow}^{ss}_d \)). We say that \([X \subset \mathbb{P}^r] \in \text{Hilb}_d \) is Chow polystable if \( \text{Ch}([X \subset \mathbb{P}^r]) \in \text{Chow}^{ss}_d \) and its SL(V)-orbit is closed inside \( \text{Chow}^{ss}_d \). Clearly this is equivalent to
asking that \([X \subset \mathbb{P}^r] \in \text{Ch}^{-1}(\text{Chow}^g_d)\) and that the SL(V)-orbit of \([X \subset \mathbb{P}^r]\) is closed inside \(\text{Ch}^{-1}(\text{Chow}^g_d)\).

The relation between asymptotically Hilbert (semi)stability and Chow (semi)stability is given by the following (see [HH, Prop. 3.13])

**Fact 4.1.** There are inclusions

\[
\text{Ch}^{-1}(\text{Chow}^g_d) \subseteq \text{Hilb}^g_d \subseteq \text{Hilb}^{g*}_d \subseteq \text{Ch}^{-1}(\text{Chow}^g_d).
\]

In particular, there is a natural morphism of GIT-quotients

\[
\text{Hilb}^g_d/\text{SL}(V) \rightarrow \text{Ch}^{-1}(\text{Chow}^g_d)/\text{SL}(V).
\]

Note also that in general there is no obvious relationship between Hilbert and Chow polystability.

### 4.2. Hilbert-Mumford’s criterion for \(m\)-Hilbert and Chow (semi)stability.

Let us now review the Hilbert-Mumford’s numerical criterion for the \(m\)-Hilbert (semi)stability and Chow (semi)stability of a point \([X \subset \mathbb{P}^r] \in \text{Hilb}_d\), following [Gie82, Sec. 0.B] and [Mum77, Sec. 2] (see also [HM98, Chap. 4.B]). Although the criterion in its original form involves one parameters subgroups (in short 1ps) of SL(V), it is technically convenient to work with 1ps of GL(V) (see [Gie82, pp. 9-10] for an explanation on how to pass from 1ps of SL(V) to 1ps of GL(V), and conversely).

Let \(\rho : \mathbb{G}_m \rightarrow \text{GL}(V)\) be a 1ps and let \(x_0, \ldots, x_r\) be coordinates of \(V\) that diagonalize the action of \(\rho\), so that for \(i = 0, \ldots, r\) we have

\[
\rho(t) \cdot x_i = t^{w_i} x_i \quad \text{for} \quad i = 0, \ldots, r;
\]

where \(w_i \in \mathbb{Z}\) and \(w_0 \geq \ldots \geq w_r = 0\). The total weight of \(\rho\) is by definition

\[
w(\rho) := \sum_{i=0}^{r} w_i.
\]

Given a monomial \(B = x_0^{\beta_0} \ldots x_r^{\beta_r}\), we define the weight of \(B\) with respect to \(\rho\) to be

\[
w_{\rho}(B) = \sum_{i=0}^{r} \beta_i w_i.
\]

For any \(m \geq M\) as in Section 4.1 and any 1ps \(\rho\) of \(\text{GL}(V)\), we introduce the following function

\[
W_{X,\rho}(m) := \min \left\{ \sum_{i=1}^{P(m)} w_{\rho}(B_i) \right\},
\]

where the minimum runs over all the collections of \(P(m)\) monomials \(\{B_1, \ldots, B_{P(m)}\} \in \text{Sym}^m V^\vee\) which restrict to a basis of \(H^0(X, \mathcal{O}_X(m))\). It is easy to check that \(W_{X,\rho}(m)\) coincide with the filtered Hilbert function of [HH, Def. 3.15]. In the sequel, we will often write \(W_{\rho}(m)\) instead of \(W_{X,\rho}(m)\) when there is no danger of confusion.

The Hilbert-Mumford’s numerical criterion for \(m\)-th Hilbert (semi)stability translates into the following (see [Gie82, p. 10] and also [HM98, Prop. 4.23]).
Fact 4.2 (Numerical criterion for $m$-Hilbert (semi)stability). Let $m \geq M$ as before. A point $[X \subset \mathbb{P}^r] \in \text{Hilb}_d$ is $m$-Hilbert stable (resp. semistable) if and only if for every 1ps $\rho : \mathbb{G}_m \to \text{GL}(V)$ of total weight $w(\rho)$ we have that

$$\mu([X \subset \mathbb{P}^r],\rho) := \frac{w(\rho)}{r+1} mP(m) - W_{X,\rho}(m) > 0$$

(resp. $\geq$).

Indeed, the function $\mu([X \subset \mathbb{P}^r],\rho)$ introduced above coincides with the Hilbert-Mumford index of $[X \subset \mathbb{P}^r] \in \mathbb{P}\left(\bigwedge^P(m) \text{Sym}^m V^\vee\right)$ relative to the 1ps $\rho$ (see [MFK94 2.1]).

The function $W_{X,\rho}(m)$ also allows to state the numerical criterion for Chow (semi)stability. According to [Mum77, Prop. 2.11] (see also [HHi Prop. 3.16]), the function $W_{X,\rho}(m)$ is an integer valued polynomial of degree 2 for $m \gg 0$. We define $e_{X,\rho}$ (or $e_\rho$ when there is no danger of confusion) to be the normalized leading coefficient of $W_{X,\rho}(m)$, i.e.,

$$W_{X,\rho}(m) - e_{X,\rho} m^2 < C m,$$

for $m \gg 0$ and for some constant $C$.

The Hilbert-Mumford’s numerical criterion for Chow (semi)stability translates into the following (see [Mum77, Thm. 2.9]).

Fact 4.3 (Numerical criterion for Chow (semi)stability). A point $[X \subset \mathbb{P}^r] \in \text{Hilb}_d$ is Chow stable (resp. semistable) if and only if for every 1ps $\rho : \mathbb{G}_m \to \text{GL}(V)$ of total weight $w(\rho)$ we have that

$$e_{X,\rho} < 2d \cdot \frac{w(\rho)}{r+1}$$

(resp. $\leq$).

Remark 4.4. Observe that $2d \cdot \frac{w(\rho)}{r+1}$ is the normalized leading coefficient of the polynomial $\frac{w(\rho)}{r+1} mP(m) = \frac{w(\rho)}{r+1} m(dm + 1 - g)$. Therefore, combining Fact 4.3 and Fact 4.2 for $m \gg 0$, one gets a proof of Fact 4.1.

4.3. Basins of attraction. Basins of attraction represent a useful tool in the study of the orbits which are identified in a GIT quotient. We review the basic definitions, following the presentation in [HHi Sec. 4].

Definition 4.5. Let $[X_0 \subset \mathbb{P}^r] \in \text{Hilb}_d$ and $\rho : \mathbb{G}_m \to \text{GL}_{r+1}$ a 1ps of $\text{GL}_{r+1}$ that stabilizes $[X_0 \subset \mathbb{P}^r]$. The $\rho$-basin of attraction of $[X_0 \subset \mathbb{P}^r]$ is the subset

$$A_\rho([X_0 \subset \mathbb{P}^r]) := \{[X \subset \mathbb{P}^r] \in \text{Hilb}_d : \lim_{t \to 0} \rho(t) \cdot [X \subset \mathbb{P}^r] = [X_0 \subset \mathbb{P}^r]\}.$$

Clearly, if $[X \subset \mathbb{P}^r] \in A_\rho([X_0 \subset \mathbb{P}^r])$ then $[X_0 \subset \mathbb{P}^r]$ belongs to the closure of the $\text{SL}_{r+1}$ orbit of $[X \subset \mathbb{P}^r]$. Therefore, if $[X_0 \subset \mathbb{P}^r]$ is Hilbert semistable (resp. Chow semistable) then every $[X \subset \mathbb{P}^r] \in A_\rho([X_0 \subset \mathbb{P}^r])$ is Hilbert semistable (resp. Chow...
semistable) and is identified with $[X_0 \subset \mathbb{P}^r]$ in the GIT quotient $\text{Hilb}^a_d/\text{SL}_{r+1}$ (resp. $\text{Ch}^{-1}(\text{Chow}^a_d)/\text{SL}_{r+1}$).

The following well-known properties of the basins of attraction (see e.g. [HH, p. 24-25]) will be used in the sequel.

**Fact 4.6.** Same notations as in Definition 4.5 and let $m \geq M$ as in Section 4.1.

(i) If $\mu([X_0 \subset \mathbb{P}^r], \rho) < 0$ (resp. $e_{X_0, \rho} > 2d \cdot \frac{w(\rho)}{r+1}$) then every $[X \subset \mathbb{P}^r] \in A_\rho([X_0 \subset \mathbb{P}^r])$ is not $m$-Hilbert semistable (resp. not Chow semistable).

(ii) If $\mu([X_0 \subset \mathbb{P}^r], \rho) = 0$ (resp. $e_{X_0, \rho} = 2d \cdot \frac{w(\rho)}{r+1}$) then $[X_0 \subset \mathbb{P}^r]$ is $m$-Hilbert semistable (resp. Chow semistable) if and only if every $[X \subset \mathbb{P}^r] \in A_\rho([X_0 \subset \mathbb{P}^r])$ is $m$-Hilbert semistable (resp. Chow semistable).

4.4. **Stability of smooth curves and Potential stability.** Here we recall two basic results due to Mumford and Gieseker: the stability of smooth curves of high degree and the (so-called) potential stability theorem.

**Fact 4.7.** If $[X \subset \mathbb{P}^r] \in \text{Hilb}_d$ is connected and smooth and $d \geq 2g + 1$ then $[X \subset \mathbb{P}^r]$ is Chow stable.

For a proof, see [Mum77, Thm. 4.15]. In [Gie82] Thm. 1.0.0], a weaker form of the above Fact is proved: if $[X \subset \mathbb{P}^r] \in \text{Hilb}_d$ is connected and smooth and $d \geq 10(2g - 2)$ then $[X \subset \mathbb{P}^r]$ is Hilbert stable. See also [HM98, Chap. 4.B] and [Mor10, Sec. 2.4] for an overview of the proof.

**Fact 4.8 (Potential stability).** If $d > 4(2g - 2)$ and $[X \subset \mathbb{P}^r] \in \text{Ch}^{-1}(\text{Chow}^a_d) \subset \text{Hilb}_d$ with $X$ connected then:

(i) $X$ is pre-stable, i.e. it is reduced and has at most nodes as singularities;
(ii) $X \subset \mathbb{P}^r$ is non-degenerate, linearly normal (i.e., $X$ is embedded by the complete linear system $|O_X(1)|$) and $O_X(1)$ is non-special (i.e., $H^1(X, O_X(1)) = 0$);
(iii) The line bundle $O_X(1)$ on $X$ is balanced (see Definition 3.7).

For a proof, see [Mum77, Prop. 4.5]. In [Gie82] Thm. 1.0.1, Prop. 1.0.11], the same conclusions are shown to hold under the stronger hypothesis that $[X \subset \mathbb{P}^r] \in \text{Hilb}^a_d$ and $d \geq 10(2g - 2)$. See also [HM98, Chap. 4.C] and [Mor10, Sec. 3.2] for an overview of the proof.

**Remark 4.9.** The hypothesis that $d > 4(2g - 2)$ in Fact 4.8 is sharp: in [HM10] it is proved that all the 4-canonical p-stable curves (which in particular can have cusps) belong to $\text{Hilb}^8_{4(2g-2)}$.

5. **Potential pseudo-stability theorem**

The aim of this section is to generalize the Potential stability theorem (see Fact 4.8) for lower values of $d$. The main result is the following theorem, which we call potential pseudo-stability Theorem for its relations with the pseudo-stable curves (see Definition 2.1(ii)).
Theorem 5.1. (Potential pseudo-stability theorem) If \( d > 2(g-2) \) and \([X \subset \mathbb{P}^r] \in \text{Ch}^{-1}(\text{Chow}_d^{ss}) \subset \text{Hilb}_d\) with \( X \) connected, then

(i) \( X \) pre-wp-stable, i.e. it is reduced and its singularities are at most nodes, cusps and tacnodes with a line.

(ii) \( X \subset \mathbb{P}^r \) is non-degenerate, linearly normal (i.e., \( X \) is embedded by the complete linear system \( |O_X(1)| \)) and \( O_X(1) \) is non-special (i.e., \( H^1(X, O_X(1)) = 0 \));

(iii) The line bundle \( O_X(1) \) on \( X \) is balanced (see Definition 3.3).

Proof. The proof is an adaptation of the results in [Mum77], [Gie82], [Sch91] and [HH, Sec. 7]. Let us indicate the different steps of the proof. Suppose that \([X \subset \mathbb{P}^r] \in \text{Ch}^{-1}(\text{Chow}_d^{ss}) \subset \text{Hilb}_d\) with \( X \) connected.

• \( X \) is generically reduced: same proof of [Sch91, Lemma 2.4] which works under the assumption that \( d > 2(g-2) \) or [HH, Lemma 7.4] which works under the assumption that \( d > \frac{3}{2}(2g-2) \).

• \( X \) does not have triple points: same proof of [Mum77, Prop. 3.1, p. 69] (see also [Sch91, Lemma 2.1]) or [Gie82, Prop. 1.0.4], both of which are easily seen, by direct inspection, to work under the assumption that \( d > \frac{3}{2}(2g-2) \).

• \( X \) does not have non-ordinary cusps: same proof of [Sch91, Lemma 2.3] which works for \( d > 2(g-2) \) or [HH, Lemma 7.2] which works for \( d > \frac{25}{14}(2g-2) \).

• \( X \) does not have higher order tacnodes or tacnodes in which one of the two branches does not belong to a line: same proof of [Sch91, Lemma 2.2], which works for \( d > 2(g-2) \).

• \( X \) is reduced and [5.1][HH] and [5.1][III] hold: Mumford’s argument in the proof of [Mum77, Prop. 5.5] goes through except for the proof that if \( C_1 \) is an irreducible component of \( X_{\text{red}} \) then \( H_1(C_1, O_{C_1}(1)) = 0 \) (compare also with [Sch91 Lemma 2.5] and [III, Prop. 7.6]). So let us see how to modify the argument of Mumford to get the same conclusion also in our case. Suppose, by contradiction, that \( C_1 \) is an irreducible component of \( X_{\text{red}} \) such that \( H^1(C_1, O_{C_1}(1)) \neq 0 \). By the Clifford’s theorem for reduced curves with nodes, cusps and tacnodes of [III, Thm. 7.7] (generalizing the Clifford’s theorem of Gieseker-Morrison for nodal curves in [Gie82, Thm. 0.2.3]), we get that

\[
(5.1) \quad h^0(C_1, O_{C_1}(1)) \leq \frac{\deg C_1 O(1)}{2} + 1.
\]

By the inequality (5.7) of [Mum77, Pag. 96], whose proof works without any assumption on \( d \), we get that

\[
(5.2) \quad k_{C_1} + 2\deg C_1 (O(1)) \leq 2 \frac{2v}{2v - 1} h^0(C_1, O_{C_1}(1)),
\]

where \( d = v(2g-2) \). Combining the above inequalities (5.1) and (5.2) and using our assumption that \( v > 2 \), we get that

\[
2\deg C_1 (O(1)) \leq k_{C_1} + 2\deg C_1 (O(1)) < \frac{4}{3} h^0(C_1, O_{C_1}(1)) \leq \frac{4}{3} \left( \frac{\deg C_1 O(1)}{2} + 1 \right),
\]

\[
24
\]
which gives $\deg C_1 \mathcal{O}(1) < 4$. Substituting in (5.1), we get that $h^0(C_1, \mathcal{O}_{C_1}(1)) < 3$. Since $\mathcal{O}_{C_1}(1)$ is very ample, we must have that $C_1 \cong \mathbb{P}^1$ and $\mathcal{O}_{C_1}(1) = \mathcal{O}_{\mathbb{P}^1}(1)$. However, $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) = 0$ and we get a contradiction.

\[\square\]

**Remark 5.2.** The hypothesis that $d > 2(2g - 2)$ in the above Theorem (5.1) is sharp: in [IHI] Thm. 2.14 it is proved that all the 2-canonical h-stable curves in the sense of [IHI] Def. 2.5, Def. 2.6] (which in particular can have arbitrary tacnodes and not only tacnodes with a line) belong to $\text{Hilb}^2_{2(2g-2)}$.

### 5.1. Balanced line bundles and quasi-wp-stable curves.

The aim of this subsection is to study the following

**Question 5.3.** Given a pre-wp-stable curve $X$, what kind of restrictions does the existence of an ample balanced line bundle $L$ impose on $X$?

The following result gives an answer to the above question.

**Proposition 5.4.** Let $X$ be a pre-wp-stable curve of genus $g \geq 2$. If there exists an ample balanced line bundle $L$ on $X$ of degree $d \geq g - 1$ then $X$ is quasi-wp-stable and $L$ is properly balanced.

**Proof.** Let $Z$ be a connected rational subcurve of $X$ (equivalently $Z$ is a chain of $\mathbb{P}^1$’s) such that $k_Z \leq 2$. Clearly $k_Z \geq 1$ since $X$ is connected and $Z \neq X$ because $g \geq 2$.

If $k_Z = 1$ then $\deg_Z(\omega_X) = -1$ and the basic inequality (3.1) together with the hypothesis that $d \geq g - 1$ gives that

$$\deg_Z(L) \leq \frac{d}{2g-2} \deg_Z(\omega_X) + \frac{k_Z}{2} = -\frac{d}{2g-2} + \frac{1}{2} \leq 0.$$ 

This contradicts the fact that $L$ is ample.

If $k_Z = 2$ then $\deg_Z(\omega_X) = 0$ and the basic inequality (3.1) gives that

$$\deg_Z(L) \leq \frac{d}{2g-2} \deg_Z(\omega_X) + \frac{k_Z}{2} = 1.$$ 

Since $L$ is ample, it has positive degree on each irreducible component of $Z$; therefore $Z$ must be irreducible which implies that $Z \cong \mathbb{P}^1$ and $\deg_Z L = 1$. \[\square\]

Combining the previous Proposition 5.4 with the potential stability Theorem (see Fact 4.8) and the potential pseudo-stability Theorem 5.1, we get the following

**Corollary 5.5.**

(i) If $d > 2(2g - 2)$ and $[X \subset \mathbb{P}^r] \in \text{Ch}^{-1}(\text{Chow}_d^{ss}) \subset \text{Hilb}_d$ with $X$ connected then $X$ is a quasi-wp-stable curve and $\mathcal{O}_X(1)$ is properly balanced.

(ii) If $d > 4(2g - 2)$ and $[X \subset \mathbb{P}^r] \in \text{Ch}^{-1}(\text{Chow}_d^{ss}) \subset \text{Hilb}_d$ with $X$ connected then $X$ is a quasi-stable curve and $\mathcal{O}_X(1)$ is properly balanced.

Note that, by Proposition 13.3(ii) of the Appendix, we have the following Remark, which can be seen as a partial converse to Proposition 5.4.
Remark 5.6. A balanced line bundle of degree \( d > \frac{3}{2}(2g - 2) \) on a quasi-wp-stable curve \( X \) is properly balanced if and only if it is ample. Therefore, for \( d > \frac{3}{2}(2g - 2) \), the set \( B_X^d \) is the set of all the multidegrees of ample balanced line bundles on \( X \).

6. Elliptic tails

The aim of this section is to prove the following

**Theorem 6.1.** If \( 2(2g - 2) < d < \frac{7}{2}(2g - 2) \) and \( [X \subset \mathbb{P}^r] \in \text{Ch}^{-1}(\text{Chow}^{ss}_d) \subset \text{Hilb}_d \) with \( X \) connected, then \( X \) does not have elliptic tails.

**Proof.** Assume that \( 2(2g - 2) < d < \frac{7}{2}(2g - 2) \) and let \( [X \subset \mathbb{P}^r] \in \text{Ch}^{-1}(\text{Chow}^{ss}_d) \subset \text{Hilb}_d \) with \( X \) connected. We known that \( X \) is quasi-wp-stable by Corollary 5.5 and \( L := \mathcal{O}_X(1) \) is very ample, non special and balanced of degree \( d \) by the potential pseudo-stability Theorem 5.1.

Suppose that \( X \) has an elliptic tail, i.e. we can write \( X = Y \cup F \) where \( F \subset X \) is a connected subcurve of arithmetic genus 1, \( Y \subset X \) is a connected subcurve of arithmetic genus \( g - 1 \) and \( F \cap Y = \{p\} \) where \( p \) is a nodal point of \( X \) which is smooth in \( F \) and \( Y \). We want to show, by contradiction, that \( [X \subset \mathbb{P}^r] \not\in \text{Ch}^{-1}(\text{Chow}^{ss}_d) \). Note that since \( \text{Ch}^{-1}(\text{Chow}^{ss}_d) \) is open in \( \text{Hilb}_d \), we can assume that \( F \) is a generic connected curve of arithmetic genus one, and in particular that it is a smooth elliptic curve.

Let \( \nu := \deg L_{|F} \). Since \( L \) (and hence \( L_{|F} \)) is very ample by construction, we must have \( \nu \geq 3 \). On the other hand, by applying the basic inequality (3.1) to the subcurve \( F \subset X \) we get

\[
|\nu - \frac{d}{2g - 2}| \leq \frac{1}{2},
\]

which together with our assumptions on \( d \) gives that \( \nu \leq 3 \). We conclude that \( \nu = 3 \).

Since \( F \) is an elliptic curve, we can write

\[
L_{|F} = \mathcal{O}_F(2p + q)
\]

for some (uniquely determined) \( q \in F \). By our generic assumption on \( F \), we can assume that \( q \neq p \).

Consider now the linear spans \( V_F := \langle F \rangle \) of \( F \) and \( V_Y := \langle Y \rangle \) of \( Y \) on \( \mathbb{P}^r = \mathbb{P}(V) \). It follows from Riemann-Roch theorem, using that \( L \) (hence \( L_{|Y} \) and \( L_{|F} \)) is non special, that \( V_F \) has dimension 2 and \( V_Y \) has dimension \( d - 3 - (g - 1) = r - 2 \). Therefore, we can choose coordinates \( \{x_1, \ldots, x_{r+1}\} \) of \( V \) such that \( V_F = \{x_1 = \ldots = x_{r+1} = 0\} \), \( V_Y = \{x_1 = x_2 = 0\} \) and \( p \) is the point where all the \( x_i \)'s vanish except \( x_3 \). For \( 1 \leq i \leq 3 \), we will confound \( x_i \) with the section of \( H^0(F, L_{|F}) \) it determines and we will denote by \( \text{ord}_p(x_i) \) the order of vanishing of \( x_i \) at \( p \). By Riemann-Roch theorem applied to the line bundles \( L_{|F}(-ip) \) for \( i = 0, \ldots, 3 \) and using (6.1) with \( q \neq p \), we may choose the first three coordinates \( \{x_1, \ldots, x_3\} \) of \( V \) in such a way that

\[
\text{ord}_p(x_i) = 3 - i \quad \text{for} \quad 1 \leq i \leq 3.
\]
Consider the one parameter subgroup $\rho : \mathbb{G}_m \to \text{GL}(V)$ which, in the above coordinates, has the diagonal form $\rho(t) \cdot x_i = t^{w_i} x_i$ for $i = 1, \ldots, r+1$, with weights $w_i$ such that

\[
\begin{align*}
  w_1 &= w_\rho(x_1) = 1, \\
  w_2 &= w_\rho(x_2) = 2, \\
  w_j &= w_\rho(x_j) = 3 \quad \text{for } j \geq 3.
\end{align*}
\]

(6.3)

The total weight of $\rho$ is equal to

\[
  w(\rho) = 1 + 2 + 3(r + 1 - 2) = 3r.
\]

(6.4)

We want now to compute the polynomial $W_{X, \rho}(m)$ defined in (4.1). To that aim, consider the filtration of $H^0(X, L^m)$ (for $m > 0$):

\[
  F^0 \subseteq F^1 \subseteq \ldots \subseteq F^{3m-1} \subseteq F^{3m} = H^0(X, L^m),
\]

where $F^r$ is the subspace of $H^0(X, L^m)$ generated by the image of the monomials $B$ of weight $w_\rho(B) \leq r$ via the surjection $\mu_m : k[x_1, \ldots, x_{r+1}]^m = H^0(\mathbb{P}^r, O_{\mathbb{P}^r}(m)) \to H^0(X, L^m)$. By the definition (4.1) of $W_{X, \rho}(m)$, it follows that

\[
  W_{X, \rho}(m) = \sum_{r=1}^{3m} r \left[ \dim(F^r) - \dim(F^{r-1}) \right] = 3m \dim(H^0(X, L^m)) - \sum_{r=0}^{3m-1} \dim(F^r) = \]

\[
  = 3m(dm + 1 - g) - \sum_{r=0}^{3m-1} \dim(F^r).
\]

(6.5)

We need to compute the dimension of $F^r$ for $r < 3m$. Note that if a monomial $B$ of degree $m$ in the above basis of $V$ has total weight $w_\rho(B) < 3m$ then it must contain at least one factor $x_i$ with $i \leq 3$ by (6.3). Hence such a $B$ vanishes identically on $Y$ by our choice of the coordinates on $V$. Moreover, if such $B$ contains also one factor $x_j$ with $j \geq 4$, then $B$ vanishes identically also on $F$, hence on the entire curve $X$; or in other words $\mu_m(B) = 0$. This discussion shows that $F^r$ for $r < m\nu$ is mapped isomorphically via the restriction map $H^0(X, L^m) \to H^0(F, L^m_{|F})$ onto the subspace $W^r \subseteq H^0(F, L^m_{|F})$ generated by the image of the monomials in $\{x_1, x_2, x_3\}$ of degree $m$ and weight at most $r$ via the multiplication map $\tau_m : k[x_1, x_2, x_3]^m = \text{Sym}^m H^0(F, L_{|F}) \to H^0(F, L^m_{|F})$. In particular

\[
  \dim F^r = \dim W^r \quad \text{for } r < 3m.
\]

(6.6)

Similarly to (4.1), define now the following function

\[
  W^F_{\rho}(m) := \min \left\{ \sum_{i=1}^{3m} w_\rho(B_i) \right\},
\]

(6.7)
where the minimum runs over all the collections of $3m$ monomials $\{B_1, \ldots, B_{3m}\} \in k[x_1, x_2, x_3]_m$ which restrict to a basis of $H^0(F, L^m_{|F})$. As in (6.5) above, we have that (6.8)

$$W^F_{\rho}(m) = \sum_{r=1}^{3m} r \left[ \dim(W^r) - \dim(W^{r-1}) \right] = 3m \dim(H^0(F, L^m_{|F})) - \sum_{r=0}^{3m-1} \dim(W^r) = 9m^2 - \sum_{r=0}^{3m-1} \dim(W^r).$$

Combining (6.5), (6.6), (6.8), we get that (6.9)

$$W_{X,\rho}(m) = W^F_{\rho}(m) + 3(d - 3)m^2 + 3(1 - g)m.$$ 

In particular, the normalized leading coefficient $e_{\rho}$ of $W_{\rho}(m)$ is given by (6.10)

$$e_{X,\rho} = e^F_{\rho} + 6(d - 3),$$

where $e^F_{\rho}$ is the normalized leading coefficient of the degree 2 polynomial $W^F_{\rho}(m)$.

In order to compute the polynomial $W^F_{\rho}(m)$, and in particular its normalized leading coefficient $e^F_{\rho}$, consider the embedding of $F$ as a cubic curve in $\mathbb{P}^2 = \mathbb{P}(H^0(F, L_{|F})^*)$ given by the complete linear system $|L_{|F}|$. Let $f \in k[x_1, x_2, x_3]_3$ be a homogenous polynomial of degree 3 defining $F$. The conditions (6.2) on the order at $p$ of the coordinates $\{x_1, x_2, x_3\}$ translate directly into conditions on the polynomial $f$. More specifically, the point $p$ has coordinates $(0,0,1)$ and $p \in F$ if and only if the coefficient of $x_3^2$ in $f$ is equal to zero. The condition that $\ord_{\rho} x_1 \geq 2$ says that the tangent space of $F$ at $p = (0,0,1)$ is given by $\{x_1 = 0\}$ which translates into the fact that the coefficient of $x_3^2 x_1$ in $f$ is zero while the coefficient of $x_3^2 x_1$ is not zero. Finally we have that $\ord_{\rho} x_1 = 2$ (i.e. $p$ is not a flex point of $F$) if and only if the coefficient of $x_3^2 x_1$ in $f$ is zero. Summing up, every polynomial $f$ such that the coordinates $\{x_1, x_2, x_3\}$ satisfy (6.2) is of the form (6.11)

$$f = a_{300} x_1^3 + a_{210} x_1^2 x_2 + a_{201} x_1^2 x_3 + a_{120} x_1 x_2^2 + a_{102} x_1 x_3^2 + a_{111} x_1 x_2 x_3 + a_{030} x_3^3 + a_{021} x_2 x_3^2,$$

where $a_{102} \neq 0$ and $a_{021} \neq 0$.

Because of the choice (6.3) of the one parameter subgroup $\rho$, it is easy to see that the monomial $x_3^2 x_1$ has the maximal $\rho$-weight among all the monomials appearing in the above equation (6.11) of $f$. Moreover, the same monomial appears with non-zero coefficient in $f$. Therefore a collection of $3m$ monomials that compute the polynomial $W^F_{\rho}(m)$, according to the formula (6.7), is represented by those monomials that are not divisible by $x_3^2 x_1$, i.e.

$$\left\{ x_1^{m-k}, x_3^k \right\}_{0 \leq k \leq m}, \left\{ x_2 x_1^{m-1-h}, x_3^h \right\}_{0 \leq h \leq m-1}, \left\{ x_2 x_1^{m-2-j}, x_2 x_3^j \right\}_{0 \leq j \leq m-2}.$$

Applying formula (6.7), we get

$$W^F_{\rho}(m) = \sum_{k=0}^{m} [w_1 (m-k)+kw_3] + \sum_{h=0}^{m-1} [w_2 + (m-1-h)w_1 + hw_3] + \sum_{j=0}^{m-2} [(j+2)w_2 + (m-2-j)w_1].$$
from which it is easy to compute the normalized leading coefficient

\[ e^*_F = 2w_3 + w_2 + 3w_1 = 11. \]

Combining with (6.10), we get

\[ e_{X, \rho} = 11 + 6(d - 3) = 6d - 7. \]

Let us now look at the right hand side of the numerical criterion for Chow (semi)stability (see Fact 4.3). Using that \( v := \frac{d}{2g - 2} < \frac{7}{2} \) by our assumptions on the degree \( d \), we get that

\[ \frac{d}{r + 1} = \frac{d}{d - g + 1} = \frac{2v}{2v - 1} > \frac{7}{6}. \]

From this and (6.14), we compute

\[ 2d \frac{w(\rho)}{r + 1} = 2d \frac{3r}{r + 1} = 6d - \frac{6d}{r + 1} < 6d - 7. \]

From (6.13) and (6.14), we deduce that the chosen 1ps \( \rho \) satisfies

\[ e_{X, \rho} > 2d \frac{w(\rho)}{r + 1}. \]

In other words, \( \rho \) violates the numerical criterion for Chow semistability of \( [X \subset \mathbb{P}^r] \) (see Fact 4.3); hence \( [X \subset \mathbb{P}^r] \notin \text{Ch}^{-1}(\text{Chow}^{ss}_d) \) which is the desired contradiction.

Combining the previous Theorem 6.1 with Corollary 5.5(i) and Definition 2.8(ii), we get the following

**Corollary 6.2.** If \( 2(2g - 2) < d < \frac{7}{2}(2g - 2) \) and \( [X \subset \mathbb{P}^r] \in \text{Ch}^{-1}(\text{Chow}^{ss}_d) \subset \text{Hilb}_d \) with \( X \) connected, then \( X \) is a quasi-p-stable curve.

7. Stabilizer subgroups

Let \( [X \subset \mathbb{P}^r] \) be a Chow semistable point of \( \text{Hilb}_d \) with \( X \) connected and \( d > 2(2g - 2) \). Note that \( X \) is a quasi-wp-stable curve by Corollary 5.5(ii), \( L := \mathcal{O}_X(1) \) is balanced and \( X \) is non-degenerate and linearly normal in \( \mathbb{P}^r \) by the potential pseudo-stability Theorem 5.1. Moreover:

- If \( 4(2g - 2) < d \) then \( X \) is quasi-stable by Corollary 5.5(ii);
- If \( 2(2g - 2) < d < \frac{7}{2}(2g - 2) \) then \( X \) is quasi-p-stable by Corollary 6.2.

The aim of this section is to describe the stabilizer subgroup of an element \( [X \subset \mathbb{P}^r] \in \text{Hilb}_d \) as above. We denote by \( \text{Stab}_{\text{GL}_{r+1}}([X \subset \mathbb{P}^r]) \) the stabilizer subgroup of \( [X \subset \mathbb{P}^r] \) in \( \text{GL}_{r+1} \), i.e. the subgroup of \( \text{GL}_{r+1} \) fixing \( [X \subset \mathbb{P}^r] \). Similarly, \( \text{Stab}_{\text{PGL}_{r+1}}([X \subset \mathbb{P}^r]) \) is the stabilizer subgroup of \( [X \subset \mathbb{P}^r] \) in \( \text{PGL}_{r+1} \). Clearly \( \text{Stab}_{\text{PGL}_{r+1}}([X \subset \mathbb{P}^r]) = \text{Stab}_{\text{GL}_{r+1}}([X \subset \mathbb{P}^r])/G_m \), where \( G_m \) denotes the diagonal subgroup of \( \text{GL}_{r+1} \) which clearly belongs to \( \text{Stab}_{\text{GL}_{r+1}}([X \subset \mathbb{P}^r]) \).

It turns out that the stabilizer subgroup of \( [X \subset \mathbb{P}^r] \in \text{Hilb}_d \) is related to the automorphism group of the pair \( (X, \mathcal{O}_X(1)) \), which is defined as follows.
Given a variety $X$ and a line bundle $L$ on $X$, an automorphism of $(X, L)$ is given by a pair $(\sigma, \psi)$ such that $\sigma \in \text{Aut}(X)$ and $\psi$ is an isomorphism between the line bundles $L$ and $\sigma^*L$. The group of automorphisms of $(X, L)$ is naturally an algebraic group denoted by $\text{Aut}(X, L)$. We get a natural forgetful homomorphism

\[ F : \text{Aut}(X, L) \to \text{Aut}(X) \]

\[ (\sigma, \psi) \mapsto \sigma \]

whose kernel is the multiplicative group $\mathbb{G}_m$, acting as fiberwise multiplication on $L$, and whose image is the subgroup of $\text{Aut}(X)$ consisting of automorphisms $\sigma$ such that $\sigma^*(L) \cong L$. The quotient $\text{Aut}(X, L)/\mathbb{G}_m$ is denoted by $\text{Aut}(X, L)$ and is called the reduced automorphism group of $(X, L)$.

The relation between the stabilizer subgroup of an embedded variety $X \subset \mathbb{P}^r$ and the automorphism group of the pair $(X, \mathcal{O}_X(1))$ is provided by the following well-known result.

**Lemma 7.1.** Given a projective embedded variety $X \subset \mathbb{P}^r$ which is non-degenerate and linearly normal, there are isomorphisms of algebraic groups

\[
\left\{
\begin{array}{l}
\text{Aut}(X, \mathcal{O}_X(1)) \cong \text{Stab}_{\text{GL}_{r+1}}([X \subset \mathbb{P}^r]), \\
\text{Aut}(X, \mathcal{O}_X(1)) \cong \text{Stab}_{\text{PGL}_{r+1}}([X \subset \mathbb{P}^r]).
\end{array}
\right.
\]

**Proof.** This result is certainly well-known to the experts. However, since we could not find a suitable reference, we sketch a proof for the reader’s convenience.

Observe first that the natural restriction map $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) \to H^0(X, \mathcal{O}_X(1))$ is an isomorphism because by assumption the embedding $X \subset \mathbb{P}^r$ is non-degenerate and linearly normal. Therefore we identify the above two vector spaces and we denote them by $V$. Note that $\mathbb{P}^r = \mathbb{P}(V^\vee)$ and that the standard coordinates on $\mathbb{P}^r$ induce a basis of $V$, which we call the standard basis of $V$.

Let us now define a homomorphism

\[ \eta : \text{Aut}(X, \mathcal{O}_X(1)) \to \text{Stab}_{\text{GL}_{r+1}}([X \subset \mathbb{P}^r]) \subseteq \text{GL}_{r+1} = \text{GL}(V^\vee). \]

Given $(\sigma, \psi) \in \text{Aut}(X, \mathcal{O}_X(1))$, where $\sigma \in \text{Aut}(X)$ and $\psi$ is an isomorphism between $\mathcal{O}_X(1)$ and $\sigma^*\mathcal{O}_X(1)$, we define $\eta((\sigma, \psi)) \in \text{GL}(V^\vee)$ as the composition

\[ \eta((\sigma, \psi)) : V^\vee = H^0(X, \mathcal{O}_X(1))^\vee \xrightarrow{\psi^{-1}} H^0(X, \sigma^*\mathcal{O}_X(1))^\vee \xrightarrow{\sigma^*} H^0(X, \mathcal{O}_X(1))^\vee = V^\vee, \]

where $\psi^{-1}$ is the dual of the isomorphism induced by $\psi^{-1}$ and $\sigma^*$ is the dual of the isomorphism induced by $\sigma^*$. Let us denote by $\phi_{|\mathcal{O}_X(1)}$ (resp. $\phi_{|\sigma^*\mathcal{O}_X(1)}$) the embedding of $X$ in $\mathbb{P}^r$ given by the complete linear series $|\mathcal{O}_X(1)|$ (resp. by $|\sigma^*\mathcal{O}_X(1)|$) with respect to the basis of $H^0(X, \mathcal{O}_X(1))$ (resp. $H^0(X, \sigma^*\mathcal{O}_X(1))$) induced by the standard basis.
of $V$ via the above isomorphisms. By construction, the following diagram commutes:

\[
\begin{array}{ccc}
X^c & \xrightarrow{\phi|_{\mathcal{O}_X(1)}} & \mathbb{P}(H^0(X, \mathcal{O}_X(1))^\vee) \\
\downarrow{\sigma} & & \downarrow{\psi^{-1}} \\
X^c & \xrightarrow{\phi|_{\sigma^*\mathcal{O}_X(1)}} & \mathbb{P}(H^0(X, \sigma^*\mathcal{O}_X(1))^\vee) \\
\end{array}
\]

Thus we get that $\eta((\sigma, \psi))$ belongs to $\text{Stab}_{\text{GL}_{r+1}}([X \subset \mathbb{P}^r]) \subseteq \text{GL}(V^\vee)$ and $\eta$ is well-defined.

Conversely, we define a homomorphism

\[
\tau : \text{Stab}_{\text{GL}_{r+1}}([X \subset \mathbb{P}^r]) \to \text{Aut}(X, L)
\]
as follows. An element $g \in \text{Stab}_{\text{GL}_{r+1}}([X \subset \mathbb{P}^r]) \subseteq \text{GL}_{r+1} = \text{GL}(V^\vee)$ will send $X$ isomorphically onto itself, and thus induces an automorphism $\sigma \in \text{Aut}(X)$. Consider now the isomorphism

\[
\tilde{\psi} : V = H^0(X, \mathcal{O}_X(1)) \xrightarrow{g^{-1}} V = H^0(X, \mathcal{O}_X(1)) \xrightarrow{\sigma^*} H^0(X, \sigma^*\mathcal{O}_X(1)),
\]

where $g^{-1}$ is the dual of $g^{-1}$ and $\sigma^*$ is the isomorphism induced by $\sigma$. The isomorphism $\tilde{\psi}$ induces an isomorphism $\psi$ between $\mathcal{O}_X(1)$ and $\sigma^*\mathcal{O}_X(1)$ making the following diagram commutative

\[
\begin{array}{ccc}
H^0(X, \mathcal{O}_X(1)) \otimes \mathcal{O}_X & \xrightarrow{\tilde{\psi}} & \mathcal{O}_X(1) \\
\downarrow{\psi} & & \downarrow{\psi} \\
H^0(X, \sigma^*\mathcal{O}_X(1)) \otimes \mathcal{O}_X & \xrightarrow{\sigma^*} & \sigma^*\mathcal{O}_X(1).
\end{array}
\]

We define $\tau(g) := (\sigma, \psi) \in \text{Aut}(X, \mathcal{O}_X(1))$.

We leave to the reader the task of checking that the homomorphisms $\eta$ and $\tau$ are induced by morphisms of algebraic groups and that they are one the inverse of the other.

The map $\eta$ sends the subgroup $\mathbb{G}_m \subseteq \text{Aut}(X, \mathcal{O}_X(1))$ of scalar multiplications on $\mathcal{O}_X(1)$ into the diagonal subgroup $\mathbb{G}_m \subset \text{GL}_{r+1}$ and therefore it induces an isomorphism $\text{Aut}(X, \mathcal{O}_X(1)) \cong \text{Stab}_{\text{PGL}_{r+1}}([X \subset \mathbb{P}^r])$. \hfill $\Box$

In Theorem 7.2 below, we describe the connected component $\text{Aut}(X, L)^0$ of $\text{Aut}(X, L)$ containing the identity for the pairs we will be interested in. Recall, from Definition 2.8 that for a quasi-wp-stable curve $X$ we denote by $X_{\text{exc}} \subset X$ the subcurve of $X$ consisting of the union of the exceptional components $E$ of $X$, i.e., the subcurves $E \subset X$ such that $E \cong \mathbb{P}^1$ and $k_E = 2$. We denote by $\tilde{X} := X_{\text{exc}}^c$ the complementary subcurve of $X_{\text{exc}}$ and by $\gamma(\tilde{X})$ the number of connected components of $\tilde{X}$.
Theorem 7.2. Let $X$ be either a quasi-stable curve of genus $g \geq 2$ or a quasi-p-stable curve of genus $g \geq 3$ and let $L$ be a properly balanced line bundle on $X$. Then the connected component $\text{Aut}(X, L)^0$ of $\text{Aut}(X, L)$ containing the identity is isomorphic to $\mathbb{G}_m^\gamma(X)$.

Proof. Consider the wp-stable reduction $X \to \text{wps}(X)$ of $X$ (see Proposition 2.9). Note that since $\text{wps}(X) = \text{Proj}(\oplus_{i \geq 0} H^0(X, \omega_X^i))$, an automorphism of $X$ naturally induces an automorphism of $\text{wps}(X)$, so by composing the homomorphism $F$ (see (7.1)) with the homomorphism $\text{Aut}(X) \to \text{Aut}(\text{wps}(X))$ induced by the wp-stable reduction, we get a homomorphism

$$G : \text{Aut}(X, L) \to \text{Aut}(\text{wps}(X)).$$

Note that if $X$ is quasi-stable of genus $g \geq 2$ then $\text{wps}(X)$ is stable of genus $g \geq 2$ and that if $X$ is quasi-p-stable of genus $g \geq 3$ then $\text{wps}(X)$ is p-stable of genus $g \geq 3$. In any case, we get that $\text{Aut}(\text{wps}(X))$ is a finite group, which is well-known for stable curves and proved in [Sch91, Proof of Lemma 5.3] for p-stable curves.

Therefore the result follows from the claim below.

CLAIM: $\text{Ker}(G) = \mathbb{G}_m^\gamma(X)$.

Recall from Proposition 2.9 that the wp-stable reduction $X \to \text{wps}(X)$ is the contraction of every exceptional component $E \cong \mathbb{P}^1$ of $X$ to a node or a cusp according to whether $E \cap E^c$ consists of two nodes or one tacnode. We can factor the wp-stable reduction of $X$ as

$$X \to Y \to \text{wps}(X),$$

where $c : X \to Y$ is obtained by contracting all the exceptional components $E$ of $X$ such that $E \cap E^c$ consists of two nodes and $Y \to \text{wps}(X)$ is obtained by contracting all the exceptional components $E$ of $Y$ such that $E \cap E^c$ consists of a tacnode. Now, since an automorphism of $X$ must send exceptional components of $X$ meeting the rest of $X$ in two distinct points to exceptional components of the same type, we can factor the map $G$ of (7.5) as

$$G : \text{Aut}(X, L) \xrightarrow{G_1} \text{Aut}(Y) \xrightarrow{G_2} \text{Aut}(\text{wps}(X)).$$

This gives an exact sequence

$$(7.6) \quad 0 \to \text{Ker}(G_1) \to \text{Ker}(G) \xrightarrow{G_1|_{\text{Ker}(G)}} \text{Ker}(G_2).$$

The same proof of [BFV11, Lemma 2.13] applied to the contraction map $X \to Y$ gives that

$$(7.7) \quad \text{Ker}(G_1) = \mathbb{G}_m^\gamma(\bar{X}).$$

Using (7.6) and (7.7), Claim 1 follows if we prove that

$$(7.8) \quad \text{Im}(G_1) \cap \text{Ker}(G_2) = \{id\}.$$
In order to prove (7.8), we need first to describe explicitly Ker\((G_2)\). Recall that, by construction, all the exceptional components \(E \cong \mathbb{P}^1\) of \(Y\) are such that \(E \cap E^c\) consists of a tacnode of \(Y\) and all of them are contracted to a cusp of \(\text{wps}(X)\) by the map \(Y \to \text{wps}(X)\). Therefore Ker\((G_2)\) consists of all the automorphisms \(\gamma \in \text{Aut}(Y)\) such that \(\gamma\) restricts to the identity on \(\overline{Y \setminus \bigcup E}\) where the union runs over all the exceptional subcurves \(E\) of \(Y\). Consider one such exceptional component \(E \subset Y\) and let \(\{p\} = E \cap E^c\). Since \(p\) is a tacnode of \(Y\), there is an isomorphism (see [a], Sec. 6.2)

\[
i : T_p E \xrightarrow{\cong} T_p E^c,
\]

where \(T_p E\) is the tangent space of \(E\) at \(p\) and similarly for \(T_p E^c\). Any \(\gamma \in \text{Aut}(Y)\) preserves the isomorphism \(i\). If moreover \(\gamma \in \text{Ker}(G_2) \subset \text{Aut}(Y)\), then \(\gamma\) acts trivially on the irreducible component of \(E^c\) containing \(p\), hence it acts trivially also on \(T_p E^c\).

Therefore the restriction of \(\gamma \in \text{Ker}(G_2)\) to \(E\) will be an element \(\phi \in \text{Aut}(E)\) that fixes \(p\) and induces the identity on \(T_p E\). If we fix an identification \((E, p) \cong (\mathbb{P}^1, 0)\), the set of all such elements forms a subgroup of \(\text{Aut}(E)\) which is isomorphic to the additive subgroup \(G_\alpha\) of \(\text{Aut}(\mathbb{P}^1) = \text{PGL}_2\) given by all the transformations \(\phi_\lambda\) (for \(\lambda \in \mathbb{C}\)) of the form

\[
\phi_\lambda(z) = \frac{z}{\lambda z + 1}.
\]

Conversely, every such \(\phi\) extends to an automorphism of \(\text{Aut}(Y)\) that is the identity on \(E^c\) and therefore lies on \(\text{Ker}(G_2)\). From this discussion, we deduce that

\[
\text{Ker}(G_2) = \prod_E G_\alpha,
\]

where the product runs over all the exceptional components \(E\) of \(Y\).

We can now prove (7.8). Take an element \((\sigma, \psi) \in \text{Aut}(X, L)\) such that \(G_1(\sigma, \psi) \in \text{Ker}(G_2)\). Consider an exceptional component \(E\) of \(Y\); let \(\{p\} = E \cap E^c\) and let \(C\) be the irreducible component of \(E^c\) containing \(p\). By (7.10) and the discussion preceding it, we get that \(G_1(\sigma, \psi)|_E = \phi_\lambda\) for some \(\lambda \in \mathbb{C}\) (as in (7.9)) and \(G_1(\sigma, \psi)|_C = \text{id}_C\). By construction, the map \(c : X \to Y\) is an isomorphism in a neighborhood of \(E \subset Y\).

Therefore, abusing notation, we identify \(E\) with its inverse image via \(c\), similarly for \(p\), and we call \(C'\) the irreducible component of \(X\) such that \(\{p\} = E \cap C'\). From the above properties of \(G_1(\sigma, \psi)\), we deduce that \(\sigma|_E = \phi_\lambda\) and \(\sigma|_C = \text{id}_C\). Consider now \(\hat{X} \cong E \coprod E^c\) the partial normalization of \(X\) at \(p\) and let \(\nu : \hat{X} \to X\) be the natural map. We have an exact sequence

\[
0 \to G_\alpha \to \text{Pic}(X) \xrightarrow{\nu^*} \text{Pic}(\hat{X}) = \text{Pic}(E) \times \text{Pic}(E^c) \to 0.
\]

By looking at the gluing data defining line bundles on \(X\), it is easy to check that the above automorphism \(\sigma \in \text{Aut}(X)\) acts as the identity on \(\text{Pic}(\hat{X})\) and that it acts on \(G_\alpha\) by sending \(\mu\) into \(\mu + \lambda\). Since there exists an isomorphism \(\psi\) between \(\sigma^*(L)\) and \(L\) by assumption, we must have that \(\lambda = 0\), or in other words that \(\sigma|_E = \phi_0 = \text{id}_E\).
Since this is true for all the exceptional components $E$ of $Y$, from (7.10) we get that $G_1(\sigma, \psi) = \text{id}$ and (7.8) is now proved.

By combining Corollary 5.5(ii), Corollary 6.2, Lemma 7.1 and Theorem 7.2, we get the following Corollary 7.3. Let $[X \subset \mathbb{P}^r] \in \text{Ch}^{-1}(\text{Chow}_{d}) \subset \text{Hilb}_{d}$ with $X$ connected and assume that either $d > 4(2g - 2)$ or $2(2g - 2) < d < \frac{7}{2}(2g - 2)$ and $g \geq 3$. Then the connected component $\text{Stab}_{\text{GL}_{r+1}}([X \subset \mathbb{P}^r])^0$ of $\text{Stab}_{\text{GL}_{r+1}}([X \subset \mathbb{P}^r])$ containing the identity is isomorphic to $G_{\gamma}(\tilde{X})$.

8. BEHAVIOUR AT THE EXTREMES OF THE BASIC INEQUALITY

Recall from Corollary 5.5(i) that if $[X \subset \mathbb{P}^r] \in \text{Hilb}_{d}$ is Chow semi-stable with $X$ connected and $d > 2(2g - 2)$, then $X$ is quasi-wp-stable and $O_X(1)$ is properly balanced.

The aim of this section is to investigate the properties of the Chow semi-stable points $[X \subset \mathbb{P}^r] \in \text{Hilb}_{d}$ such that $O_X(1)$ is stably balanced or strictly balanced (see Definition 3.7).

Our first result is the following Theorem 8.1. If $d > 2(2g - 2)$ and $[X \subset \mathbb{P}^r] \in \text{Hilb}_{d}^s \subseteq \text{Hilb}_{d}$ with $X$ connected, then $O_X(1)$ is stably balanced.

Proof. The proof uses some ideas from [Gie82, Prop. 1.0.7] and [Cap94, Lemma 3.1].

Let $[X \subset \mathbb{P}^r] \in \text{Hilb}_{d}^s \subseteq \text{Hilb}_{d}$ with $X$ connected and assume that $d > 2(2g - 2)$. By the potential pseudo-stability Theorem 5.1 and Corollary 5.5(ii), we get that $X$ is a quasi-wp-stable curve and $L := O_X(1)$ is properly balanced and non-special.

By contradiction, suppose that $O_X(1)$ is not stably balanced. Then, by Definition 3.6 and Remark 3.8, we can find a connected subcurve $Y$ with connected complementary subcurve $Y^c$ such that

\[
\begin{cases}
Y^c \not\subset X_{\text{exc}} \text{ or equivalently } g_{Y^c} = 0 \Rightarrow k_{Y^c} = k_Y \geq 3, \\
\deg_{Y^c}L = M_Y = \frac{d}{2g - 2}\deg_{Y^c}\omega_X + \frac{k_{Y^c}}{2} = \frac{d}{2g - 2}(2g_{Y^c} - 2 + k_{Y^c}) + \frac{k_{Y^c}}{2}, \\
\deg_{Y}L = m_Y = \frac{d}{2g - 2}\deg_{Y}\omega_X - \frac{k_Y}{2} = \frac{d}{2g - 2}(2g_Y - 2 + k_Y) - \frac{k_Y}{2}.
\end{cases}
\]

In order to produce the desired contradiction, we will use the numerical criterion for Hilbert stability (see Fact 4.2). Let $V := H^0(\mathbb{P}^r, O_{\mathbb{P}^r}(1)) = H^0(X, O_X(1))$ and consider the vector subspace

\[ U := \text{Ker} \left\{ H^0(\mathbb{P}^r, O_{\mathbb{P}^r}(1)) \rightarrow H^0(Y, L|_Y) \right\} \subseteq V. \]
Set $N + 1 := \dim U$. Choose a basis $\{x_0, \ldots, x_N, \ldots, x_r\}$ of $V$ relative to the filtration $U \subseteq V$, i.e., $x_i \in U$ if and only if $0 \leq i \leq N$. Define a 1ps $\rho$ of $GL_{r+1}$ by

$$
\rho(t) \cdot x_i = \begin{cases} 
    x_i & \text{if } 0 \leq i \leq N, \\
    tx_i & \text{if } N + 1 \leq i \leq r.
\end{cases}
$$

We will estimate the two polynomials appearing in Fact \[4.2\] for the 1ps $\rho$.

First of all, the total weight $w(\rho)$ of $\rho$ satisfies $w(\rho) = r - N = \dim V - \dim U \leq h^0(Y, L|_Y)$. Since $L$ is non special and $H^0(X, L) \to H^0(Y, L|_Y)$ because $X$ is a curve, we get that $h^0(Y, L|_Y) = \deg_Y L + 1 - g_Y$. Therefore we conclude that

$$
(8.2) \quad \frac{w(\rho)}{r + 1} mP(m) \leq \frac{h^0(Y, L|_Y)}{r + 1} m(d + 1 - g) = \frac{\deg_Y L + 1 - g_Y}{d + 1 - g}[dm^2 + (1 - g)m].
$$

In order to compute the polynomial $W_{\rho}(m)$ for $m \gg 0$, consider the filtration of $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m))$:

$$
0 \subseteq U^m = U^{m - 1}V \subseteq \ldots \subseteq U^{m - i}V^i \subseteq \ldots \subseteq V^m = H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m)),
$$

where $U^{m - i}V^i$ is the subspace of $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m))$ generated by the monomials containing at least $(m - i)$-terms among the variables $\{x_0, \ldots, x_N\}$. Note that for a monomial $B$ of degree $m$, it holds that

$$
(8.3) \quad B \in U^{m - i}V^i \setminus U^{m - i + 1}V^{i - 1} \iff w(\rho)(B) = i.
$$

Via the surjective restriction map $\mu_m : H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m)) \to H^0(X, L^m)$, the above filtration on $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m))$ induces a filtration

$$
0 \subseteq F^0 \subseteq F^1 \subseteq \ldots \subseteq F^i \subseteq \ldots \subseteq F^m = H^0(X, L^m),
$$

where $F^i := \mu_m(U^{m - i}V^i)$. Using (8.3), we get that

$$
(8.4) \quad W_{\rho}(m) = \sum_{i=1}^{m} \left[ \dim(F^i) - \dim(F^{i-1}) \right] = m \dim(F^m) - \sum_{i=1}^{m-1} \dim(F^i) =
$$

$$
= m(dm + 1 - g) - \sum_{i=0}^{m-1} \dim(F^i).
$$

It remains to estimate $\dim F^i$ for $0 \leq i \leq m - 1$. To that aim, consider the partial normalization $\tau : \hat{X} \to X$ of $X$ at the nodes laying on $Y \cap Y^c$. Observe that $\hat{X}$ is the disjoint union of $Y$ and $Y^c$. We denote by $\hat{D}$ the inverse image of $Y \cap Y^c$ via $\tau$. Since $Y \cap Y^c$ consists of $k_Y$ nodes of $X$, $\hat{D}$ is the disjoint union of $D_Y$ and $D_{Y^c}$, where $D_Y$ consists of $k_Y$ smooth points on $Y$ and $D_{Y^c}$ consists of $k_Y$ smooth points on $Y^c$. Consider now the injective pull-back morphism

$$
\tau^* : H^0(X, L^m) \hookrightarrow H^0(\hat{X}, \tau^*L^m) = H^0(Y, L|_Y^m) \oplus H^0(Y^c, L|_{Y^c}^m),
$$

which clearly coincides with the restriction maps to $Y$ and $Y^c$.

Note that if $B$ is a monomial belonging to $U^{m - i}V^i \subseteq H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m))$ for some $i \leq m - 1$, then $B$ contains at least $m - i \geq 1$ variables among the $x_j$’s such that
$x_j \in U$; hence the order of vanishing of $B$ along the subcurve $Y$ is at least equal to $m - i$. This implies that any $s \in F^i \subseteq H^0(X, L^m)$ with $i \leq m - 1$ vanishes identically on $Y$ and vanishes on the points of $D_{Y^c}$ with order at least $(m - i)$. We deduce that

$$
\tau^*(F^i) \subseteq H^0(Y^c, L^m_{|Y^c}((i-m)D_{Y^c})) \quad \text{for } 0 \leq i \leq m - 1.
$$

**CLAIM:** $H^1(Y^c, L^m_{|Y^c}((i-m)D_{Y^c})) = 0$ for $0 \leq i \leq m - 1$ and $m \gg 0$.

Let us prove the claim. Clearly if the claim is true for $i = 0$ then it is true for every $i > 0$; so we can assume that $i = 0$. According to Fact [13.4](#) of the Appendix, it is enough to prove that for any connected subcurve $Z \subseteq Y^c$, we have that

$$
\deg(Z(L^m_{|Z}(-mD_Z))) > 2g_Z - 2 \quad \text{for } m \gg 0,
$$

where $D_Z := D_{Y^c} \cap Z$. Indeed (8.6) is equivalent to

$$
\deg_Z L \geq |D_Z| \quad \text{with strict inequality if } g_Z \geq 1.
$$

Observe that, since each point of $D_Z$ is the intersection of $Z$ with $Y = X \setminus Y^c$ and $Z \cap X = Z \neq \emptyset$ unless $Z = Y^c$ because $Y^c$ is connected, it holds:

$$
|D_Z| \leq k_Z \quad \text{with equality if and only if } Z = Y^c,
$$

where $k_Z$ is, as usual, the length of the schematic intersection of $Z$ with the complementary subcurve $X \setminus Z$ in $X$. In order to prove (8.7), we consider different cases.

If $g_Z \geq 1$ then using the basic inequality (3.1) for $L$ relative to the subcurve $Z$ and the assumption $d > 2(2g - 2)$, we compute

$$
\deg_Z L \geq \frac{d}{2g - 2} \deg\omega_X - \frac{k_Z}{2} > 2(2g - 2 + k_Z) - \frac{k_Z}{2} \geq \frac{3k_Z}{2} \geq \frac{3|D_Z|}{2} \geq |D_Z|,
$$

which shows that (8.7) holds in this case.

If $g_Z = 0$ and $Z = Y^c$ then, using that $\deg_{Y^c} L = M_{Y^c}$ and $k_{Y^c} \geq 3$ by (8.1), we get

$$
\deg_{Y^c} L = M_{Y^c} = \frac{d}{2g - 2} (2g_{Y^c} - 2 + k_{Y^c}) + \frac{k_{Y^c}}{2} > 2(k_{Y^c} - 2) + \frac{k_{Y^c}}{2} > k_{Y^c} = |D_{Y^c}|,
$$

which shows that (8.7) holds also in this case.

It remains to consider the case $g_Z = 0$ and $Z \subsetneq Y^c$. If $k_Z \leq 2$ then, since $X$ is quasi-wp-stable and $Z$ is connected, we must have that $Z$ is an exceptional subcurve of $X$, i.e., $Z \cong \mathbb{P}^1$ and $k = 2$. By Proposition 5.1 it follows that $\deg_Z L = 1$. Since $|D_Z| \leq 1$ by (8.8), we deduce that (8.7) is satisfied also in this case. Finally, assume that $k_Z \geq 3$. Consider the subcurve $W := Z^c \cap Y^c \subset Y^c$. It is easy to check that

$$
k_{Y^c} - k_W = |Z \cap Y| - |W \cap Z| = |D_Z| - (k_Z - |D_Z|) = 2|D_Z| - k_Z.
$$

Using the basic inequality of $L$ with respect to $W$ together with (8.1), (8.9) and $k_Z \geq 3$, we get

$$
\deg_Z L = \deg_{Y^c} L - \deg_W L \geq \frac{d}{2g - 2} \deg_{Y^c} \omega_X + \frac{k_{Y^c}}{2} - \frac{d}{2g - 2} \deg_W \omega_X - \frac{k_W}{2} =
$$

$$
= \frac{d}{2g - 2} \deg_{Z} \omega_X + |D_Z| - \frac{k_Z}{2} > 2(k_Z - 2) + |D_Z| - \frac{k_Z}{2} > |D_Z|.
$$

40
The claim is now proved.

Using the claim above, we get from (8.5) that
\[
\text{dim } F^i = \text{dim } \tau^* (F^i) \leq m \text{deg}_Y \cdot L + (i - m) k_Y + 1 - g_Y \quad \text{for } 0 \leq i \leq m - 1 \text{ and } m \gg 0.
\]

Combining (8.10) and (8.4), we get that
\[
W_{\rho}(m) \geq m (dm + 1 - g) - \sum_{i=0}^{m-1} \left[ m \text{deg}_Y \cdot L + (i - m) k_Y + 1 - g_Y \right] = m (dm + 1 - g) - m \left[ m \text{deg}_Y \cdot L - mk_Y + 1 - g_Y \right] - k_Y \frac{m(m-1)}{2} =
\]
\[
= m^2 \left[ \text{deg}_Y \cdot L + \frac{k_Y}{2} \right] + m \left[ 1 - g_Y - \frac{k_Y}{2} \right].
\]

(8.11)

where in the last equality we have used \( d = \text{deg} L = \text{deg}_Y \cdot L + \text{deg}_Y \cdot \cdot L \) and \( g = g_Y + g_Y + k_Y - 1 \).

Using that \( \text{deg}_Y \cdot L = m_Y \) by (8.1), we easily check that
\[
\text{deg}_Y \cdot L + \frac{k_Y}{2} = \frac{d \text{deg}_Y \cdot L + 1 - g_Y}{d + 1 - g}
\]

and
\[
1 - g_Y - \frac{k_Y}{2} = (1 - g) \frac{d \text{deg}_Y \cdot L + 1 - g_Y}{d + 1 - g}.
\]

(8.12)

(8.13)

By combining (8.2), (8.11), (8.12), (8.13), we get for \( m \gg 0 \):
\[
W_{\rho}(m) \geq m^2 \left[ \text{deg}_Y \cdot L + \frac{k_Y}{2} \right] + m \left[ 1 - g_Y - \frac{k_Y}{2} \right] =
\]
\[
= \frac{\text{deg}_Y \cdot L + 1 - g_Y}{d + 1 - g} \left[ dm^2 + (1 - g)m \right] \geq \frac{w(\rho)}{r + 1} mP(m),
\]

which contradicts the numerical criterion for Hilbert stability (see Fact 4.2), q.e.d.

\[
\Box
\]

8.1. **Closure of orbits.** Given a point \( [X \subset \mathbb{P}^r] \in \text{Hilb}_d \), denote by \( \text{Orb}_G([X \subset \mathbb{P}^r]) \) the orbit of \( [X \subset \mathbb{P}^r] \) under the action of \( G = \text{SL}(V) = \text{SL}_{r+1} \). Clearly \( \text{Orb}_G([X \subset \mathbb{P}^r]) \) depends only on \( X \) and on the line bundle \( L := \mathcal{O}_X(1) \) and not on the chosen embedding \( X \subset \mathbb{P}^r \).

The aim of this subsection is to investigate the following

**Question 8.2.** Given two points \( [X \subset \mathbb{P}^r], [X' \subset \mathbb{P}^r] \in \text{Ch}^{-1}(\text{Chow}_d) \) with \( X \) and \( X' \) connected, when does it hold that
\[
[X' \subset \mathbb{P}^r] \in \overline{\text{Orb}_G([X \subset \mathbb{P}^r])}?
\]

We start by introducing an order relation on the set of pairs \( (X, L) \) where \( X \) is a quasi-wp-stable curve and \( L \) is a properly balanced line bundle on \( X \) of degree \( d \).
**Definition 8.3.** Let \((X', L')\) and \((X, L)\) be two pairs consisting of a quasi-wp-stable curve together with a properly balanced line bundle of degree \(d\) on it.

(i) We say that \((X', L')\) is an **elementary isotrivial specialization** of \((X, L)\), and we write \((X, L) \xrightarrow{\text{el}} (X', L')\), if there exists a proper connected subcurve \(Z \subset X'\) with \(\deg Z = m_Z\), \(Z^c\) connected and \(Z \cap Z^c \subseteq X'_{\text{exc}}\) such that \((X, L)\) is obtained from \((X', L')\) by smoothing some nodes of \(Z \cap Z^c\), i.e., there exists a smooth pointed curve \((B, b_0)\) and a flat projective morphism \(X \to B\) together with a line bundle \(L\) on \(X\) such that \((\mathcal{X}, L)_{b_0} \cong (X', L')\) and \((\mathcal{X}, L)_b \cong (X, L)\) for every \(b_0 \neq b \in B\).

(ii) We say that \((X', L')\) is an **isotrivial specialization** of \((X, L)\), and we write \((X, L) \xrightarrow{\sim} (X', L')\) if \((X', L')\) is obtained from \((X, L)\) via a sequence of elementary isotrivial specializations.

There is a close relationship between the existence of isotrivial specializations and strictly balanced line bundles, as explained in the following

**Lemma 8.4.** **Notations as in Definition 8.3**

(i) If \((X, L) \xrightarrow{\sim} (X', L')\) then \(L\) is not strictly balanced.

(ii) If \(L\) is not strictly balanced then there exists an isotrivial specialization \((X, L) \xrightarrow{\sim} (X', L')\) such that \(L'\) is strictly balanced.

**Proof.** Part (i). Clearly, it is enough to consider the case where \((X, L) \xrightarrow{\text{el}} (X', L')\) is an elementary isotrivial specialization as in Definition 8.3(i). For \(Z \subseteq X'\) as in Definition 8.3(i), decompose \(Z^c\) as the union of all the exceptional components \(\{E_i\}_{i=1,\ldots,k_Z}\) of \(X'\) that meet \(Z\) and a subcurve \(W\). By applying Remark 3.8(i) to the subcurve \(E_1 \cup \cdots \cup E_{k_Z}\), where the basic inequality achieves its maximal value, it is easy to see that \(\deg_W L' = m_W\). Let now \(\widetilde{W}\) be the subcurve of \(X\) given by the union of the irreducible components of \(X\) that specialize to an irreducible component of \(W \subset X'\). Since \((X, L)\) is obtained from \((X', L')\) by smoothing some nodes which belong to \(Z \cap \bigcup_i E_i\) and therefore are not in \(W\), we clearly have that \(\widetilde{W} \cong W\), \(k_{\widetilde{W}} = k_W\) and \(L_{\widetilde{W}} \cong L_W\). Hence \(\deg_{\widetilde{W}} L' = m_{\widetilde{W}}\) and, since \(\widetilde{W} \cap \widetilde{W}^c \not\subset X_{\text{exc}}\), we conclude that \(L\) is not strictly balanced.

Part (ii). If \(L\) is not strictly balanced, we can find a subcurve \(Y \subset X\) such that \(\deg_Y L = M_Y\) and \(Y \cap Y^c \subseteq X_{\text{exc}}\). Using that \(\deg_Y L = M_Y\), or equivalently that \(\deg_Y L = m_{Y^c}\), it is easy to check that if \(n \in Y \cap Y^c \cap X_{\text{exc}}\) then there exists a unique exceptional component \(E\) of \(X\) such that \(n \in E \subset Y\).

Let us denote by \(\{n_1, \ldots, n_r\}\) the points belonging to \(Y \cap Y^c \setminus X_{\text{exc}}\). Let \(X'\) be the blow-up of \(X\) at \(\{n_1, \ldots, n_r\}\) and let \(E_Y := E_1 \cup \cdots \cup E_r\) be the new exceptional components of \(X'\). Given a subcurve \(Z \subseteq X\) denote by \(Z'\) the strict transform of \(Z\) via the blow-up morphism and define \(k_{Z'}^Y := |Z' \cap E_Y \cap Y|\).
Define a multidegree $d$ on $X'$ such that $d_{E_i} = 1$, for $i = 1, \ldots, r$ and, given an irreducible component $C$ of $X$,

$$d_{C'} = \deg_C L - k_{C'}^Y.$$ 

From [Cap94, Important Remark 5.1.1] we know that there is a flat and proper family $\mathcal{X} \to B$ over a pointed curve $(B, b_0)$ and a line bundle $\mathcal{L}$ over $\mathcal{X}$ such that $(\mathcal{X}_b, \mathcal{L}|_{\mathcal{X}_b}) \cong (X, L)$ for $b \neq b_0$ and $(\mathcal{X}_{b_0}, \mathcal{L}|_{\mathcal{X}_{b_0}}) \cong (X', L')$ where $X'$ is the blow-up of $X$ at $\{n_1, \ldots, n_r\}$ and $\deg L' = d$.

Let us check that $L'$ is properly balanced. It is clear that the degree of $L'$ on all the exceptional components of $X'$ is equal to one. Let $W \subseteq X'$ and let us check that $L'$ satisfies the basic inequality (8.1). Start by assuming that $W = Z'$ for some $Z \subseteq Y$. Then we have that

$$\deg_Z L' = \deg Z L - k_{Z'}^Y = \deg Y L - \deg Y \setminus Z L - k_{Z'}^Y = Y - \deg Y \setminus Z L - k_{Z'}^Y \geq M_Y - \deg Y \setminus Z L - k_{Z'}^Y.$$

Suppose now that $W = Z_{Yc} Y_c \cup Z_Y Y_c \cup E_W$ where $Z_{Yc} \subseteq Y_c$, $Z_Y \subseteq Y$ and $E_W \subseteq E_Y$. Then, $\deg W L' = \deg Z_{Yc} L' + \deg Z_Y L' + |E_W|$ and, by (8.15), it follows that

$$\deg W L' = \deg Z_{Yc} L' + m_Z + |Z_Y \cap Y| + |E_W| \geq$$

$$\frac{d \omega_W}{2g-2} - \frac{k_{Z_{Yc}}}{2} + |Z_Y \cap Y| + |E_W| = \frac{m_W}{2} + |E_W| - |E_W \cap Y| \geq m_W$$

Analogously we can show that $\deg W L' \leq M_W$, so we conclude that $L'$ is properly balanced.

Now, if $L'$ is strictly balanced we are done. If not, we repeat the same procedure and after a finite number of steps we will find the desired pair $(X'', L'')$ with $L''$ strictly balanced.

We can now give a partial answer to Question 8.2.

**Theorem 8.5.** Let $[X \subset \mathbb{P}^r], [X' \subset \mathbb{P}^r] \in \mathcal{H}_{\mathbb{P}^r}$ and assume that $X$ and $X'$ are quasi-wp-stable curves and $\mathcal{O}_X(1)$ and $\mathcal{O}_{X'}(1)$ are properly balanced and non-special. Suppose that $(X, \mathcal{O}_X(1)) \sim (X', \mathcal{O}_{X'}(1))$. Then

(i) $[X' \subset \mathbb{P}^r] \in \mathcal{O}_{\mathbb{P}^r}([X \subset \mathbb{P}^r])$.

(ii) $[X \subset \mathbb{P}^r] \in \text{Ch}^{-1}(\text{Chow}_{ss}^d)$ (resp. $\text{Hilb}_{ss}^d$) if and only if $[X' \subset \mathbb{P}^r] \in \text{Ch}^{-1}(\text{Chow}_{ss}^d)$ (resp. $\text{Hilb}_{ss}^d$).
Proof. It is enough, in view of Fact 4.6 to find a 1ps \( \rho : \mathbb{G}_m \to \text{GL}_{r+1} \) that stabilizes \([X' \subset \mathbb{P}^r]\) and such that \( \mu([X' \subset \mathbb{P}^r], \rho) \leq 0 \) for \( m \gg 0 \) and \([X' \subset \mathbb{P}^r] \in A_\rho([X' \subset \mathbb{P}^r]) \).

We can clearly assume that \((X, \mathcal{O}_X(1)) \overset{\rho}{\cong} (X', \mathcal{O}_{X'}(1))\). Using the notations of Definition 8.3(i), this means that there exists a connected subcurve \( Z \subset X' \) with \( Z^c \) connected and \( Z \cap Z^c \subset X'_{\text{exc}} \) and \( \deg Z' = m_Z \) such that \((X, \mathcal{O}_X(1))\) is obtained from \((X', \mathcal{O}_{X'}(1))\) by smoothing some of the nodes of \( Z \cap Z^c \). Moreover, we can decompose the connected complementary subcurve \( Z^c \) as

\[
Z^c = \bigcup_{1 \leq i \leq k_Z} E_i \cup W_i
\]

where the \( E_i \)'s are the exceptional subcurves of \( X' \) that meet the subcurve \( Z \) and \( W := Z^c \setminus \bigcup E_i \) is clearly connected as well. Since \( \deg E_i' = 1 \), it follows from Remark 3.8 applied to the subcurve \( E_1 \cup \cdots \cup E_{k_Z} \) that \( \deg W' = m_W \).

The 1ps \( \rho \) of \( \text{GL}_{r+1} \) we are looking for is similar to the 1ps considered in the proof of Theorem 8.1. More precisely, consider the restriction map

\[
\text{res}: H^0(X', \mathcal{O}_{X'}(1)) \longrightarrow H^0(Z, \mathcal{O}_Z(1)) \oplus H^0(W, \mathcal{O}_W(1)).
\]

The map \( \text{res} \) is injective since the complementary subcurve of \( Z \cup W \) is made of the exceptional components \( E_i \cong \mathbb{P}^1 \), each of which meets both \( Z \) and \( W \) in one point. Moreover, since \( \mathcal{O}_{X'}(1) \) is non-special by assumption, which implies that also \( \mathcal{O}_Z(1) \) and \( \mathcal{O}_W(1) \) are non-special, we have that

\[
\dim H^0(Z, \mathcal{O}_Z(1)) + \dim H^0(W, \mathcal{O}_W(1)) = \deg Z \mathcal{O}_{X'}(1) - g_Z + 1 + \deg W \mathcal{O}_{X'}(1) - g_W + 1 = m_Z - g_Z + 1 + m_W - g_W + 1 = d - g + 1 = \dim H^0(X', \mathcal{O}_{X'}(1)),
\]

where we have used that \( m_Z + m_W = d = k_Z \) and \( g = g_W + g_Z + k_Z - 1 \). This implies that \( \text{res} \) is an isomorphism. Define now the 1ps \( \rho: \mathbb{G}_m \to \text{GL}_{r+1} \) in such a way that

\[
\begin{cases}
\rho(t)|_{H^0(W, \mathcal{O}_W(1))} = t \cdot \text{Id}, \\
\rho(t)|_{H^0(Z, \mathcal{O}_Z(1))} = \text{Id}.
\end{cases}
\]

Let us check that the above 1ps \( \rho \) satisfies all the desired properties.

**CLAIM 1:** \( \mu([X' \subset \mathbb{P}^r], \rho) \leq 0 \) for \( m \gg 0 \).

This is proved exactly as in Theorem 8.1 (see (8.14)) and the equation for \( \mu([X \subset \mathbb{P}^r], \rho) \) given in Fact (4.2).

**CLAIM 2:** \( \rho \) stabilizes \([X' \subset \mathbb{P}^r] \in \text{Hilb}_d \).

Using Lemma 7.1, it is enough to check that

\[
\text{Im}\rho \subseteq \text{Aut}(X', \mathcal{O}_{X'}(1)) \cong \text{Stab}_{\text{GL}_{r+1}}([X' \subset \mathbb{P}^r]) \subseteq \text{GL}_{r+1}.
\]

Since the non exceptional subcurve \( \tilde{X}' \subset X' \) is contained in \( Z \sqcup W \), it follows from the proof of Theorem 7.2 that \( \text{Aut}(X', \mathcal{O}_{X'}(1)) \) contains a subgroup \( H \) isomorphic to \( \mathbb{G}_m^2 \) and such that \( (\lambda, \mu) \in H \cong \mathbb{G}_m^2 \) acts via multiplication by \( \lambda \) on \( H^0(W, \mathcal{O}_W(1)) \) and by \( \mu \) on \( H^0(Z, \mathcal{O}_Z(1)) \). By construction, it follows that \( \text{Im}\rho \subseteq H \) and we are done.

**CLAIM 3:** \([X \subset \mathbb{P}^r] \in A_\rho([X' \subset \mathbb{P}^r])\).
Recall that, by assumption, \((X, \mathcal{O}_X(1))\) is obtained from \((X', \mathcal{O}_{X'}(1))\) by smoothing some of the nodes of \(Z \cap Z' = \bigcup_i (Z \cap E_i)\). Denote by \(n_i\) the node given by the intersection of \(Z\) with \(E_i\) and by \(\text{Def}_{(X', n_i)}\) the functor of infinitesimal deformations of the complete local ring \(\hat{\mathcal{O}}_{X', n_i}\) (see [Ser06, Sec. 2.4]). According to [Ser06, Cor. 3.1.2, Exa. 3.1.4(a)], if we write \(\hat{\mathcal{O}}_{X', n_i} = k[[u_i, v_i]]/(u_iv_i)\), then \(\text{Def}_{(X', n_i)}\) has a semiuniversal ring equal to \(k[[a_i]]\) with universal family given by \(k[[u_i, v_i, a_i]]/(u_iv_i - a_i)\).

Consider now the local Hilbert functor \(H^r_{X'}\) parametrizing infinitesimal deformations of \(X'\) in \(\mathbb{P}^r\) (see [Ser06, Sec. 3.2.1]). Clearly, \(H^r_{X'}\) is pro-represented by the complete local ring of \(\text{Hilb}_d\) at \([X \subset \mathbb{P}^r]\). Since \(X'\) is a curve with locally complete intersection singularities and \(\mathcal{O}_{X'}(1)\) is non-special, from [Kol96, 1.6.10] we get that the natural morphism of functors

\[(8.16)\]

\[H^r_{X'} \rightarrow \text{Def}_{X'}\]

is formally smooth, where \(\text{Def}_{X'}\) is the functor of infinitesimal deformations of \(X'\). It follows easily from [Ser06, Thm. 2.4.1], that also the natural morphism of functors

\[(8.17)\]

\[\text{Def}_{X'} \rightarrow \prod_i \text{Def}_{(X', n_i)}\]

is formally smooth. Moreover, since \(\rho\) stabilizes \([X' \subset \mathbb{P}^r]\) by Claim 2, the above morphisms \((8.16)\) and \((8.17)\) are equivariant under the natural action of \(\rho\) on each functor. Therefore, in order to prove that \([X \subset \mathbb{P}^r] \in A_\rho([X' \subset \mathbb{P}^r])\), it is enough to prove that \(\rho\) acts on each \(k[[a_i]]\) with positive weight (compare also with the proof of [HM10, Lemma 4] and of [HH1 Cor. 7.9]).

Fix a node \(n_i = E_i \cap Z\) for some \(1 \leq i \leq k_Z\). We can choose coordinates \(\{x_1, \ldots, x_{r+1}\}\) of \(V = H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) = H^0(X', \mathcal{O}_{X'}(1))\) in such a way that \(x_i\) is the unique coordinate which does not vanish at \(n_i\), the exceptional component \(E_i\) is given by the linear span \(\langle x_i, x_{i+1} \rangle\) and the tangent \(T_{Z, n_i}\) of \(Z\) at \(n_i\) is given by the linear span \(\langle x_{i-1}, x_i \rangle\). Then the completion of the local ring \(\mathcal{O}_{X', n_i}\) is equal to \(k[[u_i, v_i]]/(u_iv_i)\) where \(u_i = x_{i-1}/x_i\) and \(v_i = x_{i+1}/x_i\). Since \(T_{Z, n_i}\) is contained in the linear span \(\langle Z \rangle\) of \(Z\) and \(\rho(t)|_{(H^0(W, \mathcal{O}_W(1)))} = \text{Id}\) by construction, we have that \(\rho(t) \cdot x_i = x_i\) and \(\rho(t) \cdot x_{i-1} = x_{i-1}\); hence \(\rho(t) \cdot u_i = u_i\). On the other hand, the point \(q_i\) defined by \(x_k = 0\) for every \(k \neq i + 1\) is clearly the node given by the intersection of \(E_i\) with \(W\). Since \(\rho(t)|_{H^0(W, \mathcal{O}_W(1))} = t \cdot \text{Id}\) by construction, we have that \(\rho(t) \cdot x_{i+1} = tx_{i+1}\); hence \(\rho(t) \cdot v_i = tv_i\). Since the equation of the universal family over \(k[[a_i]]\) is given by \(u_iv_i - a_i = 0\) and \(\rho\) acts on this universal family, we deduce that \(\rho(t) \cdot a_i = ta_i\), which concludes our proof.

\[\Box\]

From the above theorem, we deduce now the following

**Corollary 8.6.** Let \([X \subset \mathbb{P}^r] \in \text{Hilb}_d\) with \(X\) connected and \(d > 2(2g - 2)\). If \([X \subset \mathbb{P}^r]\) is Chow polystable or Hilbert polystable then \(\mathcal{O}_X(1)\) is strictly balanced.
Proof. Let us prove the statement for the Chow polystability; the Hilbert polystability being analogous.

Let \([X \subset \mathbb{P}^r] \in \text{Hilb}_d\) for \(d > 2(2g - 2)\) with \(X\) connected and assume that \([X \subset \mathbb{P}^r]\) is Chow-polystable. Recall that \(X\) is quasi-wp-stable by Corollary 5.5(i) and \(\mathcal{O}_X(1)\) is properly balanced by Theorem 5.1 and Proposition 5.4. By Lemma 8.4, we can find a pair \((X', L')\) consisting of a quasi-wp-stable curve \(X'\) and a strictly balanced line bundle \(L'\) on \(X'\) such that \((X, \mathcal{O}_X(1)) \sim (X', L')\). Note that \(L'\) is ample by Remark 5.6; moreover \(X'\) does not have elliptic tails if \(d < 5/2(2g - 2)\) because \(X\) satisfies the same property by Theorem 6.1 and in an isotrivial specialization no new elliptic tails can be created (see Definition 8.3). Therefore, we can apply Theorem 13.5 which allows to conclude that \(L'\) is non-special and very ample; we get a point \([X' | L' | \to \mathbb{P}^r] \in \text{Hilb}_d\). The above Theorem 8.5 gives that \([X' \subset \mathbb{P}^r] \in \text{Orb}_G([X \subset \mathbb{P}^r])\) and \([X' \subset \mathbb{P}^r] \in \text{Ch}^{-1}(\text{Chow}^{ss}_d)\). Since \([X \subset \mathbb{P}^r]\) is Chow polystable, we must have that \([X' \subset \mathbb{P}^r] \in \text{Orb}_G([X \subset \mathbb{P}^r])\); hence \(X' = X\) and \(\mathcal{O}_X(1) = \mathcal{O}_{X'}(1) = L'\) is strictly balanced.

\(\Box\)

9. The map towards the moduli space of p-stable curves

Consider the subscheme of \(\text{Ch}^{-1}(\text{Chow}^{ss}_d) \subset \text{Hilb}_d\) defined by

\[
H_d := \{X \in \text{Ch}^{-1}(\text{Chow}^{ss}_d) \subset \text{Hilb}_d : X\text{ is connected}\}.
\]

Note that if \(d > 2(2g - 2)\) then the condition of being connected is both closed and open in \(\text{Ch}^{-1}(\text{Chow}^{ss}_d) \subset \text{Hilb}_d\): it is closed because of its natural interpretation as a topological condition; it is open because the connected curves belonging to \(\text{Ch}^{-1}(\text{Chow}^{ss}_d) \subset \text{Hilb}_d\) are reduced curves by the potential pseudo-stability Theorem 6.1 and therefore \(X \in \text{Ch}^{-1}(\text{Chow}^{ss}_d)\) is connected if and only if \(h^0(X, \mathcal{O}_X) = 1\), which is an open condition by upper-semicontinuity. Therefore, \(H_d\) is both open and closed in \(\text{Ch}^{-1}(\text{Chow}^{ss}_d)\); or, in other words, it is a disjoint union of connected components of \(\text{Ch}^{-1}(\text{Chow}^{ss}_d)\). Indeed, we will prove later (see Proposition 10.10) that \(H_d\) is irreducible if \(d > 2(2g - 2)\).

Since \(H_d \subset \text{Ch}^{-1}(\text{Chow}^{ss}_d)\) is clearly an \(\text{SL}(V)\)-invariant subscheme, GIT tells us that there exists a projective scheme

\[
\overline{Q}_{d,g} := H_d/\text{SL}(V)
\]

which is a good categorical quotient of \(H_d\) by \(\text{SL}(V)\) (see e.g. [Dol03, Sec. 6.1]).

Theorem 9.1. Assume that \(d > 2(2g - 2)\) and \(g \geq 3\). Then:

(i) There exists a surjective natural map \(\Phi_{\text{ps}} : \overline{Q}_{d,g} \to \overline{M}^\text{ps}_g\).

(ii) If \(d > 4(2g - 2)\) then the above map \(\Phi_{\text{ps}}\) factors as

\[
\Phi_{\text{ps}} : \overline{Q}_{d,g} \xrightarrow{\Phi_c} \overline{M}_g \xrightarrow{T} \overline{M}^\text{ps}_g,
\]

where \(T\) is the map of Fact 2.2.
(iii) We have that

$$(\Phi^{ps})^{-1}(M^0_g) \cong J^0_{d,g},$$

where $M^0_g$ is the open subset of $M_g$ parametrizing curves without non-trivial automorphisms and $J^0_{d,g}$ is the degree $d$ universal Jacobian over $M^0_g$. In particular, $(\Phi^{ps})^{-1}(C) \cong \text{Pic}^d(C)$ for every geometric point $C \in M^0_g \subset M^{ps}_g$.

If $d > 4(2g - 2)$ then the same conclusions hold for the morphism $\Phi^s$.

(iv) $H_d$ is non-singular of pure dimension $r(r + 2) + 4g - 3$.

(v) $\overline{Q}_{d,g}$ is reduced and normal of dimension $4g - 3$. Moreover, if $\text{char}(k) = 0$, then $\overline{Q}_{d,g}$ has rational singularities, hence it is Cohen-Macaulay.

Indeed, the proof below will show that the morphism $\Phi^s$ exists also for $g = 2$ (and $d > 4(2g - 2)$).

Proof. The proof is an adaptation of the ideas from [Cap94, Sec. 2].

Part (i): consider the restriction to $H_d$ of the universal family over Hilb$_d$ and denote it by

$$\begin{align*}
C_d &\hookrightarrow H_d \times \mathbb{P}^r \\
u_d &\downarrow \\
H_d &
\end{align*}$$

The morphism $u_d$ is flat, proper and its geometric fibers are quasi-wp-stable curves by Corollary 6.2(ii). Consider the p-stable reduction of $u_d$ (see Definition 2.12):

$$\begin{align*}
\xymatrix{C_d &\ar[r] &\text{ps}(C_d) \\
u_d &\ar[u] &\text{ps}(u_d) \ar[l] &\ar[u] \\
H_d &}
\end{align*}$$

The morphism ps($u_d$) is flat, proper and its geometric fibers are p-stable curves of genus $g$. Therefore, by the modular properties of $M^{ps}_g$, the family ps($u_d$) induces a modular map $\phi^{ps} : H_d \to \overline{M}^{ps}_g$. Since the group SL($V$) = SL$_{r+1}$ acts on the family $C_d$ by only changing the embedding of the fibers of $u_d$ into $\mathbb{P}^r$, the map $\phi^{ps}$ is SL$_{r+1}$-invariant and therefore it factors via a map $\Phi^{ps} : \overline{Q}_{d,g} \to \overline{M}^{ps}_g$.

Let us show that $\Phi^{ps}$ is surjective. Let $C$ be any connected smooth curve over $k$ of genus $g \geq 2$ and $L$ be any line bundle on $C$ of degree $d > 2(2g - 2)$. Note that $d = \deg L \geq 2g + 1$ since $g \geq 2$. Hence $L$ is very ample and non-special and therefore it embeds $C$ in $\mathbb{P}^r = \mathbb{P}^{d-g}$. By Fact 4.7 the corresponding point $[C \xrightarrow{[L]} \mathbb{P}^r] \in \text{Hilb}_d$ belongs to $H_d$ and clearly it is mapped to $C \in M_g \subset M^{ps}_g$ by $\Phi^{ps}$. We conclude that the image of $\Phi^{ps}$ contains the open dense subset $M_g \subset M^{ps}_g$. Moreover, $\Phi^{ps}$ is projective since $\overline{Q}_{d,g}$ is projective. Therefore, being projective and dominant, $\Phi^{ps}$ has to be surjective. This finishes the proof of part (i).

Consider now Part (ii). If $d > 4(2g - 2)$, then the potential stability Theorem (see Fact 4.8) says that the geometric fibers of the morphism $u_d$ are quasi-stable curves.
From Definition 2.12 and Proposition 2.9 it follows that the p-stable reduction $\text{ps}(u_d)$ of $u_d$ factors through the wp-stable reduction $\text{wps}(u_d)$ of $u_d$ and that the latter one is a family of stable curves. This implies that the map $\Phi_{\text{ps}}^d : \overline{Q}_{d,g} \rightarrow \overline{M}_{g}^{\text{ps}}$ factors via a map $\Phi^* : \overline{Q}_{d,g} \rightarrow \overline{M}_g$ followed by the contraction map $T : \overline{M}_g \rightarrow \overline{M}_{g}^{\text{ps}}$.

Part (iii): the proof of [Cap94, Thm. 2.1(2)] extends verbatim to our case.

Part (iv): the fact that $H_g$ is non-singular of pure dimension $r(r+2) + 4g-3$ is proved exactly as in [Cap94, Lemma 2.2], whose proof uses only the fact that if $X \in H_d$ then $X$ is reduced, a local complete intersection and embedded by a non-special linear system; these conditions are satisfied by the potential pseudo-stability Theorem 5.1. See also [HH, Cor. 6.3] for another proof.

Part (v): $Q_{d,g}$ is reduced and normal since $H_d$ is (see e.g. [Dol03, Prop. 3.1]). The dimension of $\overline{Q}_{d,g}$ is $4g-3$ since $H_d$ has dimension $r(r+2) + 4g-3$, $\text{SL}_{r+1}$ has dimension $r(r+2)$ and the action of $\text{SL}_{r+1}$ has generically finite stabilizers. If $\text{char}(k) = 0$ then $\overline{Q}_{d,g}$ has rational singularities by [Bou87], using that $H_d$ is smooth. This implies that $\overline{Q}_{d,g}$ is Cohen-Macauly since, in characteristic zero, a variety having rational singularities is Cohen-Macauly (see [KoM98, Lemma 5.12]). Alternatively, the fact that $\overline{Q}_{d,g}$ is Cohen-Macauly follows from [HR74], using the fact that $H_d$ is smooth.

Further properties of $\overline{Q}_{d,g}$ and of the morphisms $\Phi_{\text{ps}}^d$ and $\Phi^*$ will be proved later on in Proposition 10.9.

10. A Stratification of the Semistable Locus

Inspired by [Cap94, Sec. 5], we introduce in this section an $\text{SL}_{r+1}$-invariant stratification of $H_d$ (see (9.1)) and we establish some properties of it.

Recall that every $X \in H_d$ is quasi-wp-stable and that $\mathcal{O}_X(1)$ is properly balanced by Corollary 5.5(ii). Recall also that $B^d_X$ denotes the set of multidegrees of properly balanced line bundles on $X$ of total degree $d$ (see Definition 3.5).

Following [Cap94, Sec. 5.1], consider, for any quasi-wp-stable curve $X$ of genus $g$ and any $d \in B^d_X$, the (locally closed) stratum

\[(10.1) \quad M^d_X := \{ [X \subset \mathbb{P}^r] \in H_d : \deg \mathcal{O}_X(1) = d \} \subset H_d \subseteq \text{Hilb}_d.\]

Each stratum $M^d_X$ is $\text{SL}_{r+1}$-invariant since $\text{SL}_{r+1}$ acts on $H_d$ by changing the embedding of $X$ inside $\mathbb{P}^r$ and thus it preserves $X$ and the multidegree $d$. Note that $M^d_X$ may be empty for certain pairs $(X, d)$ as above.

10.1. Specializations of strata. The aim of this subsection is to describe all pairs $(X', d')$ with $X'$ quasi-wp-stable of genus $g$ and $d' \in B^d_X$, such that $M^d_{X'} \subseteq M^d_X$.

Generalizing the refinement relation of [Cap94, Sec. 5.2], we define an order relation on the sets of pairs $(X, d)$ where $X$ is a quasi-wp-stable curve of genus $g$ and $d \in B^d_X$.

**Definition 10.1.** Let $X'$ and $X''$ be two quasi-wp-stable curves of genus $g$. 
(i) We say that $X'' \preceq X'$ if they have the same wp-stable reduction $X = \text{wps}(X') = \text{wps}(X'')$ and there exists a surjective morphism $\sigma : X'' \to X'$ commuting with their wp-stable reduction morphisms

$$X'' \xrightarrow{\phi''} \xrightarrow{\sigma} X' \xleftarrow{\phi'} X$$

$\text{wps}(X'') = X = \text{wps}(X')$

(ii) Assume moreover that $d'' \in B^d_X$, and that $d'' \in B^d_{X'}$. We say that $(X'', d'') \preceq (X', d')$ if $X'' \preceq X'$ and there exists a surjective morphism $\sigma : X'' \to X'$ as before such that for every subcurve $Y' \subseteq X'$ there exists a subcurve $Y'' \subseteq X''$ with $Y' = \sigma(Y'')$ and $d''_Y = d'_Y$.

The order relation $\preceq$ can be described in terms of elementary operations as follows.

**Lemma 10.2.** With the same notations as in the above Definition 10.1, we have that $(X'', d'') \preceq (X', d')$ if and only if $X''$ is obtained from $X'$ via a sequence of blow-ups of nodes and cusps of $X'$ and the the multidegree $d''$ is obtained from $d'$ at each step according to the rules depicted in Figures 1 and 2.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{Blow-up of a node: external and internal cases.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{Blow-up of a cusp.}
\end{figure}

**Proof.** Using the explicit description of the wp-stable reduction of Proposition 2.9 it is easy to see that $X'' \preceq X'$ if and only if $X''$ is obtained from $X'$ via a sequence of blow-ups of nodes and cusps. According to Definition 10.1, it is now clear that at each blow-up $d''$ must be obtained from $d'$ according to the rules depicted in Figures 1 and 2.

\qed

From the above description it is easy to see that there is a relation between the isotrivial specialization introduced in Definition 8.3 and the order relation $\preceq$. More precisely, we have the following
Remark 10.3. Let \((X', L')\) and \((X'', L'')\) be two pairs consisting of a quasi-wp-stable curve of genus \(g\) and a properly balanced line bundle of degree \(d\). If \((X', L') \leadsto (X'', L'')\) then \((X'', \deg L'') \preceq (X', \deg L')\).

The following elementary property of the order relation \(\preceq\) will be used in what follows.

Lemma 10.4. Notations as in Definition [10.1]. If \(X'' \preceq X'\) and \(d'' \in B^d_X\), then there exists \(d' \in B^d_X\), such that \((X'', d'') \preceq (X', d')\).

Proof. By Lemma [10.2] above it is enough to assume that \(X''\) is obtained from \(X'\) by blowing-up a node (which can be internal or external) or a cusp. Start by assuming that \(X''\) is obtained from \(X'\) by blowing up an external node \(N\), as in the picture on the left of Figure [1].

Denote by \(\{C'_1, \ldots, C'_\gamma\}\) the irreducible components of \(X'\), by \(\{C''_1, \ldots, C''_\gamma\}\) their proper transforms in \(X''\) and by \(E\) the exceptional component that is contracted to the node \(N\) by the map \(\sigma: X'' \to X'\). Assume that \(C'_1\) and \(C'_2\) are the two irreducible components of \(X'\) that contain the node \(N\). Define a multidegree \(d'\) on \(X'\) in the following way:

\[
d'_{C'_i} := \begin{cases} 
d''_{C''_i} & \text{for } i \neq 1, \\
d''_{C''_1} + 1 & \text{for } i = 1. 
\end{cases}
\]

It is clear that \(|d'| = d\), so we must check that \(d'\) satisfies the basic inequality \((3.1)\). Given a subcurve \(Z'\) of \(X'\), we denote by \(Z''\) the subcurve of \(X''\) that is the proper transform of \(Z'\) under the blow-up map \(X'' \to X'\). Define \(W_{Z''}\) to be the subcurve of \(X''\) such that \(W_{Z''} = Z''\) if \(C''_1 \subseteq Z'\) and \(W_{Z''} = Z'' \cup E\) if \(C''_1 \not\subseteq Z'\).

Then it is easy to see that

\[
\begin{align*}
d'_{Z''} &= d''_{W_{Z''}}, \\
g_{Z''} &= g_{W_{Z''}}, \\
k_{Z''} &= k_{W_{Z''}}.
\end{align*}
\]

Hence the basic inequality \((3.1)\) for \(d'\) relative to the subcurve \(Z''\) is the same as the basic inequality for \(d''\) relative to the subcurve \(W_{Z''}\). We conclude that if \(d'' \in B^d_{X''}\), then \(d' \in B^d_{X'}\).

The cases where \(X''\) is obtained from \(X'\) by blowing up an internal node or a cusp are similar (and easier) and are therefore left to the reader. \(\Box\)

We will now prove that the above order relation \(\preceq\) determines the inclusion relations among the closures of the strata \(M^d_X \subset H_d\) of \((10.1)\). The result that follows is a generalization of \([\text{Cap94}], \text{Prop. 5.1}\).

Proposition 10.5. Assume that \(d > 2(2g-2)\) and moreover that \(g \geq 3\) if \(d < 4(2g-2)\). Let \(X'\) and \(X''\) be two quasi-wp-stable curves of genus \(g\) and let \(d' \in B^d_X\) and \(d'' \in B^d_{X''}\). Assume that \(M^d_{X''} \neq \emptyset\). Then

\[
M^d_{X''} \subseteq M^d_X \iff (X'', d'') \preceq (X', d').
\]
**Proof.** $\iff$ From Lemma [10.2] above, it is enough to assume that $X''$ is obtained from $X'$ by blowing up a node or a cusp. Assume that $X''$ is obtained from $X'$ by blowing up a node, say $N$. Let $B$ be a smooth curve and consider the trivial family $X' \times B$ over $B$. By blowing up the surface $X \times B$ on the node $N$ belonging to the fiber over a point $b_0 \in B$, we get a family $u : X \to B$ whose geometric fiber $X_b$ over a point $b \in B$ is such that $X_b \cong X'$ for all $b \neq b_0$ and $X_{b_0} \cong X''$ as in the figure below (where we have depicted an external node, but the case of an internal node is completely similar).

Consider the relative Picard scheme $\pi : \text{Pic}_{X/B} \to B$ of the family $u : X \to B$, which exists by a well-known result of Mumford (see [BLR90, Sec. 8.2, Thm. 2]). Since $H^2(X_b, \mathcal{O}_{X_b}) = 0$ for any $b \in B$ because $X_b$ is a curve, we get that $\pi : \text{Pic}_{X/B} \to B$ is smooth by [BLR90, Sec. 8.4, Prop. 2].

Let now $[X'' \subset \mathbb{P} = \mathbb{P}(V)] \in M_{X''}^{d''}$ and set $L'' = \mathcal{O}_{X''}(1) \in \text{Pic}^{d''}(X'')$. Note that the embedding $X'' \subset \mathbb{P}$ defines an isomorphism $\phi : H^0(X'', L'') \cong V$.

We can view $L''$ as a geometric point of $(\text{Pic}_{X/B})_{b_0} \cong \text{Pic}(X'')$. Since the morphism $\pi : \text{Pic}_{X/B} \to B$ is smooth, up to shrinking $B$ (i.e., replacing it with an étale open neighborhood of $b_0$), we can find a section $\sigma$ of $\pi$ such that $\sigma(b_0) = L''$. Moreover, by definition of the order relation $\preceq$ (see Figure 1), it is clear that we can choose the section $\sigma$ in such a way that $\sigma(b)$ is a line bundle of multidegree $d'$ on $X_b \cong X'$ for every $b \neq b_0$.

Up to shrinking $B$ again, we can assume that the section $\sigma$ corresponds to a line bundle $\mathcal{L}$ over $X$ such that $\mathcal{L}_{|X_{b_0}} \cong L''$ and $\mathcal{L}_{|X_b}$ has multidegree $d'$ for $b \neq b_0$. Since $L''$ is very ample and non-special and these conditions are open, up to shrinking $B$ once more, we can assume that $\mathcal{L}$ is relatively very ample and we can fix an isomorphism $\Phi : u_*\mathcal{L} \cong \mathcal{O}_B \otimes V$ of sheaves on $B$ such that $\Phi_{|b_0} = \phi$. Via the isomorphism $\Phi$, the relatively very ample line bundle $\mathcal{L}$ defines an embedding

$$
\begin{array}{c}
\mathcal{X} \\
\downarrow i
\end{array} \cong \begin{array}{c}
\mathbb{P}(\mathcal{O}_B \otimes V) = \mathbb{P}^r_B
\end{array} \quad \begin{array}{c}
\downarrow u
\end{array} \quad \begin{array}{c}
B
\end{array}
$$

whose restriction over $b_0 \in B$ is the embedding $X'' \subset \mathbb{P}$. The family $u : X \to B$ together with the embedding $i$ defines a morphism $f : B \to H_d$ such that $f(b_0) = [X'' \subset \mathbb{P}] \in M_{X''}^{d''}$ and $f(b) \in M_X^{d'}$ for every $b \neq b_0$, so we conclude that $M_{X''}^{d''} \subseteq M_X^{d'}$.

In the case when $X''$ is obtained from $X'$ by blowing up a cusp we proceed in the same way as in the previous case: we consider a family $u : X \to B$ such that $X_{b_0} \cong X''$ and $X_b \cong X'$ for $b \neq b_0$ as in the figure below.

---

[10.2]: Reference to the relevant lemma.
[BLR90]: Reference to the book by Mumford.
[Pic]: Reference to the Picard scheme.
[Pic_{X/B}]: Reference to the Picard scheme over $B$.
[H^0]: Reference to the zeroth cohomology group.
[\mathcal{O}_{X_b}]: Reference to the structure sheaf of $X_b$.
[\mathcal{O}_B]: Reference to the structure sheaf of $B$.
[\mathbb{P}]: Reference to the projective space.
[H_d]: Reference to the Hilbert scheme.
[M_X]: Reference to the Hilbert scheme of curves.
[M_{X''}]: Reference to the Hilbert scheme of curves for $X''$.
[M_X^{d'}]: Reference to the Hilbert scheme of curves of multidegree $d'$ over $X$.
[M_{X''}^{d''}]: Reference to the Hilbert scheme of curves of multidegree $d''$ over $X''$. 

and we apply the same argument as before.

\[ \implies \text{Consider the map (see Theorem 9.1):} \]

\[ \phi^{ps} : H_d \to \overline{Q_{d,g}} := H_d/SL_{r+1} \xrightarrow{\phi^{ps}} \overline{M^g_{\mathbb{P}^r}}. \]

Clearly \( M^{d'}_{X,\nu} \) is contained in the fiber \((\phi^{ps})^{-1}(X)\) where \( X := ps(X') \in \overline{M^g_{\mathbb{P}^r}} \). Therefore \( M^{d''}_{X,\nu} \subseteq M^{d'}_{X,\nu} \) and \( H_d \subseteq (\phi^{ps})^{-1}(X) \), which implies that \( ps(X'') = X = ps(X') \).

By assumption, we can find a smooth curve \( B \) and a morphism \( f : B \to H_d \) such that \( f(b_0) \in M^{d''}_{X,\nu} \) for some \( b_0 \in B \) and \( f(b) \in M^{d'}_{X,\nu} \) for every \( b_0 \neq b \in B \). By pulling back the universal family above \( H_d \) along the morphism \( f \) we get a family

\[ X \xrightarrow{u} B \times \mathbb{P}^r \]

such that \( X_{b_0} = X'' \) and \( X_b = X' \) for every \( b \neq b_0 \). The ps-stable reduction \( ps(u) \) of \( u \) is an isotrivial family of p-stable curves by what observed before. This implies that \( X \) is obtained from an isotrivial family \( \overline{X} \) with fibers isomorphic to \( X' \) by blowing up some nodes and cusps of the central fiber \( \overline{X}_{b_0} = X' \). We get a surjective morphism \( \sigma : X'' = X_{b_0} \to \overline{X}_{b_0} = X' \) which clearly commutes with the p-stable reduction morphisms; in other words \( X'' \preceq X' \).

Consider now the line bundles \( L_{b_0} := O_{X}(1)|_{X_{b_0}} \in \text{Pic}^{d''}(X'') \) and \( L_b := O_{X}(1)|_{X_b} \in \text{Pic}^{d'}(X') \) for any \( b_0 \neq b \in B \). Let \( Y' \subseteq X' \) be a subcurve of \( X' \). Consider the subcurve \( Y'' \subseteq X'' = X_{b_0} \) given by the union of all the irreducible components \( C_i \) of \( X'' \) for which there exists a section \( s \) of \( u : X \to B \) such that \( s(b_0) \in C_i \) and \( s(b) \in Y' \subseteq X' = X_b \) for every \( b \neq b_0 \). By construction, we get that \( \sigma(Y'') = Y' \) and \( d''_{Y''} = \deg_{Y''}L_{b_0} = \deg_{Y''}L_{b} = d'_{Y'} \). We have proved that \((X'',d'') \preceq (X',d')\).

\[ \square \]

10.2. **A completeness result.** Each stratum \( M^d_X \) of (10.1) admits a morphism

\[ M^d_X \to \text{Pic}^{d}(X) \]

\[ [X \subset \mathbb{P}^r] \mapsto (X, O_X(1)) \]

whose fibers are exactly the \( SL_{r+1} \)-orbits on \( M^d_X \). The aim of this subsection is to prove the following completeness result, which generalizes [Cap94 Prop. 5.2].

**Proposition 10.6.** Let \( X \) be a quasi-up-stable curve and \( d \in B^d_X \). Assume that \( d > 2(2g - 2) \) and that \( g \geq 3 \) if \( d < 4(2g - 2) \). Then either \( M^d_X = \emptyset \) or the natural map \( M^d_X \to \text{Pic}^{d}(X) \) is surjective.
We can assume that if \( d < \frac{2}{5}(2g - 2) \) then \( X \) does not contain elliptic tails; in this case every \( L \in \text{Pic}^d(X) \) is non-special and very ample.

Indeed, according to Theorem [13.5](Kl03), \( L \in \text{Pic}^d(X) \) is non-special since \( X \) is quasi-wp-stable, hence \( G \)-semistable, and \( \deg L = d > 2(2g - 2) > 2g - 2 \) (recall that \( g \geq 2 \)). Now, if \( d < \frac{2}{5}(2g - 2) \) and \( X \) contains some elliptic tail \( F \), then from the basic inequality it follows easily that \( d_F = 2 \). But no line bundle of degree 2 on a curve of genus 1 is very ample, hence no line bundle of multidegree \( d \) on \( X \) can be very ample. Otherwise, since any \( L \in \text{Pic}^d(X) \) is ample by Remark [5.6], it follows from Theorem [13.5](Kl03) that \( L \) is very ample, q.e.d.

**Reduction 2:** We can assume that \( d \) is strictly balanced.

Indeed, suppose the proposition is true for all strictly balanced line bundles on quasi-wp-stable curves and let us show that it is true for our multidegree \( d \). By Lemma 8.4(ii) there exists an isotrivial specialization \( (X, L) \rightsquigarrow (X', L') \) such that \( d' := \deg L' \) is a strictly balanced multidegree on \( X' \). Moreover, from the proof of the cited Lemma, it follows easily that the curve \( X' \) and the multidegree \( d' \) depend only on \( X \) and \( d \) and not on \( L \in \text{Pic}^d(X) \). Note that, since \( X' \) is obtained from \( X \) by blowing up some nodes of \( X \), then \( X \) has some elliptic tails if and only if \( X' \) has some elliptic tails. Therefore, according to Reduction 1, \( L \) and \( L' \) are non-special and very ample. Up to the choice of a basis of \( L \) and of \( L' \), we get two points of \( \text{Hilb}_d \), namely \( [X \xrightarrow{L} \mathbb{P}^r] \) and \( [X' \xrightarrow{L'} \mathbb{P}^r] \). These two points are indeed well-defined only up to the action of the group \( \text{SL}_{r+1} \). Note that \( L \in \text{Im}(M^d_X \xrightarrow{\beta} \text{Pic}^d(X)) \) if and only if \( [X \xrightarrow{L} \mathbb{P}^r] \in \text{Ch}^{-1}(\text{Chow}_{ss}^d) \), and similarly \( L' \in \text{Im}(M^d_X \xrightarrow{\beta^'} \text{Pic}^d(X')) \) if and only if \( [X' \xrightarrow{L'} \mathbb{P}^r] \in \text{Ch}^{-1}(\text{Chow}_{ss}^d) \). Therefore, Theorem 8.5(ii) gives that \( L \in \text{Im}(p) \) if and only if \( L' \in \text{Im}(p') \). In other words, we have defined a set-theoretic map

\[ \Upsilon : \text{Pic}^d(X) \to \text{Pic}^d(X') \]

\[ L \mapsto L' \]

such that \( \Upsilon^{-1}(\text{Im}(p')) = \text{Im}(p) \). The proposition for \( d' \) is equivalent to the fact that either \( \text{Im}(p') = \emptyset \) or \( \text{Im}(p') = \text{Pic}^{d'}(X') \). Using the above map \( \Upsilon \), it is easy to see that the above properties hold also for \( d \), hence we can assume that \( d \) is strictly balanced, q.e.d.

We now prove the proposition for a pair \((X,d)\) satisfying the properties of Reduction 1 and Reduction 2. Assume that \( M^d_X \neq \emptyset \), for otherwise there is nothing to prove. Let us first prove the following

**Claim:** The image of the morphism \( p : M^d_X \to \text{Pic}^d(X) \) is open and dense.

Consider a Poincaré line bundle \( \mathcal{P} \) on \( X \times \text{Pic}^d(X) \), i.e., a line bundle \( \mathcal{P} \) such that \( \mathcal{P}_{|X \times \{L\}} \cong L \) for every \( L \in \text{Pic}^d(X) \) (see [Kl05, Ex. 4.3]). By Reduction 1, it follows that \( \mathcal{P} \) is relatively very ample with respect to the projection \( \pi_2 : X \times \text{Pic}^d(X) \to \)}
Pic\(\mathcal{M}(X)\) and that \((\pi_2)_*(\mathcal{P})\) is locally free of rank equal to \(r + 1 = d - g\). We can therefore find a Zariski open cover \(\{U_i\}_{i \in I}\) of \(\text{Pic}\(\mathcal{M}(X)\)\) such that \((\pi_2)_*(\mathcal{P})|_{U_i} \cong \mathcal{O}_{U_i}^{r+1}\) and the line bundle \(\mathcal{P}\) induces an embedding

\[
\begin{array}{ccc}
X \times U_i & \xrightarrow{\eta_i} & \mathbb{P}(\mathcal{O}_{U_i}^{r+1}) = \mathbb{P}^{r+1}_{U_i} \\
\downarrow \pi_2 & & \downarrow \\
U_i & & 
\end{array}
\]

The above embedding corresponds to a map \(f_i : U_i \rightarrow \text{Hilb}_d\) and it is clear that

\[
\text{Im}(p : M^d_X \rightarrow \text{Pic}\(\mathcal{M}(X)\)) = \bigcup_i f_i^{-1}(H_d).
\]

Since \(H_d\) is open inside \(\text{Chow}^{-1}(\text{Chow}_{d}^{ss})\) (by the discussion at the beginning of Section 9) and \(\text{Ch}^{-1}(\text{Chow}_{d}^{ss})\) is open in \(\text{Hilb}_d\) (because any GIT-semistability condition is open), it follows that \(f_i^{-1}(H_d)\) is open inside \(U_i\); hence \(\text{Im}(p) \subset M^d_X\) is open as well. Moreover, since \(\text{Pic}\(\mathcal{M}(X)\)\) is irreducible and \(M^d_X \neq \emptyset\), we get that \(\text{Im}(p)\) is also dense, q.e.d.

In order to finish the proof, it remains to show that \(\text{Im}(p) \subset \text{Pic}\(\mathcal{M}(X)\)\) is closed. Since \(\text{Im}(p)\) is open by the Claim, it is enough to prove that \(\text{Im}(p)\) is closed under specializations (see [Har77], Ex. II.3.18(c)), i.e., if \(B \subset \text{Pic}\(\mathcal{M}(X)\)\) is a smooth curve such that \(B \setminus \{b_0\} \subset \text{Im}(p)\) then \(b_0 \in \text{Im}(p)\). The same construction as in the proof of the Claim gives, up to shrinking \(B\) around \(b_0\), a map \(f : B \rightarrow \text{Hilb}_d\) such that \(f(B \setminus \{b_0\}) \subset H_d \subset \text{Chow}^{-1}(\text{Chow}_{d}^{ss})\). We denote by \(\mathcal{L}\) the relatively ample line bundle on \(\mathcal{X} := X \times B \rightarrow B\) which gives the embedding into \(\mathbb{P}^B\).

We can now apply a fundamental result in GIT, called *semistable replacement property* (see e.g. [Hil91] Thm. 4.5), which implies that, up to replacing \(B\) with a finite cover ramified over \(b_0\), we can find two maps \(g : B \rightarrow H_d\) and \(h : B \setminus \{b_0\} \rightarrow \text{SL}_{r+1}\) such that

\[
(10.3) \quad f(b) = h(b) \cdot g(b) \text{ for every } b_0 \neq b \in B,
\]

\[
(10.4) \quad g(b_0) \text{ is Chow polystable.}
\]

We denote by \(\mathcal{Y} \rightarrow B\) the pull-back of the universal family over \(H_d\) via the map \(g\) and by \(\mathcal{M}\) the line bundle on \(\mathcal{Y}\) which is the pull-back of the universal line bundle via \(g\). Property \((10.3)\) implies that \(X \cong \mathcal{Y}_{b_0}\) and \(\deg \mathcal{M}_{|\mathcal{Y}_b} = d\) for every \(b_0 \neq b \in B\). Moreover, if we set \(Y := \mathcal{Y}_{b_0}\), \(M := \mathcal{M}_{|\mathcal{Y}_0}\) and \(d' := \deg M\), then Proposition \((10.5)\) implies that \((Y, d') \preceq (X, d)\). Therefore, there exists a map \(\Sigma : \mathcal{Y} \rightarrow \mathcal{X}\) over \(B\) whose restriction over \(b_0 \in B\) is the contraction map \(\sigma : Y \rightarrow X\) of Definition \((10.1)\). Observe also that \((10.4)\) together with Corollary \((8.6)\) imply that \(M\) is strictly balanced.

Consider the line bundle \(\widetilde{\mathcal{L}} := \Sigma^*({\mathcal{L}})\) on \(\mathcal{Y}\) and set \(\widetilde{L} := \widetilde{\mathcal{L}}_{|\mathcal{Y}_{b_0}} = \sigma^*(L)\) and \(\tilde{d} = \deg(\widetilde{L})\). Property \((10.3)\) implies that, up to shrinking \(B\) around \(b_0\), \(\widetilde{\mathcal{L}}\) and \(\mathcal{M}\) are
isomorphic away from the central fiber \( Y_{b_0} = Y \); hence, by Lemma 10.7, we can find a Cartier divisor \( T \) on \( Y \) supported on the central fiber \( Y_{b_0} = Y \) such that
\[
\tilde{L} = \mathcal{M} \otimes \mathcal{O}_Y(T).
\]
This implies that the multidegrees \( d' \) and \( \underline{d} \) on \( Y \) are equivalent in the sense of Definition 3.2. Since \( \underline{d} \) is strictly balanced by Reduction 1, we can now apply Lemma 3.11 (with \( Z = Y \) and \( \sigma' = \text{id} \)) in order to conclude that \( X = Y \) or, equivalently, \( \mathcal{X} = \mathcal{Y} \). Since we have already observed that \( (Y, d') \leq (X, \underline{d}) \), we must have that \( d = d' \). Combining this with (10.5), we get that \( L := \mathcal{L}_{X_{b_0}} = \mathcal{M}_{X_{b_0}} = M \). The line bundle \( L \) corresponds to the point \( b_0 \) under the embedding \( B \subseteq \text{Pic}^d(X) \); we deduce that \( L \in \text{Im}(p) \) since \( L = M \) and \( M \) is the image, via the map \( p \), of the point \( g(b_0) \) which belongs to \( M^d_X \) in virtue of (10.4).

\[\square\]

The following well-known Lemma (see e.g. the proof of [Ray70, Prop. 6.1.3]) was used in the above proof of Proposition 10.6.

**Lemma 10.7.** Let \( B \) be a smooth curve and let \( f : \mathcal{X} \to B \) be a flat and proper morphism. Fix a point \( b_0 \in B \) and set \( B^* = B \setminus \{b_0\} \). Let \( \mathcal{L} \) and \( \mathcal{M} \) be two line bundles on \( \mathcal{X} \) such that \( \mathcal{L}_{|f^{-1}(B^*)} = \mathcal{M}_{|f^{-1}(B^*)} \). Then
\[
\mathcal{L} = \mathcal{M} \otimes \mathcal{O}_X(D),
\]
where \( D \) is a Cartier divisor on \( X \) supported on \( f^{-1}(b_0) \).

The following result is an immediate consequence of Proposition 10.6.

**Corollary 10.8.** Let \( \mathbb{X} \rightarrow \mathbb{P}^r \) with \( d > 2(2g - 2) \) and \( g \geq 3 \). Assume that \( \mathbb{X} \) is quasi-wp-stable and that \( \deg i^* \mathcal{O}_{\mathbb{P}^r}(1) = \deg (i')^* \mathcal{O}_{\mathbb{P}^r}(1) \). Then \( \mathbb{X} \rightarrow \mathbb{P}^r \) belongs to \( \text{Ch}^{-1}(\text{Chow}_{d}^{ss}) \) (resp. \( \text{Hilb}_{d}^{ss} \)) if and only if \( \mathbb{X} \rightarrow \mathbb{P}^r \) belongs to \( \text{Ch}^{-1}(\text{Chow}_{d}^{ss}) \) (resp. \( \text{Hilb}_{d}^{ss} \)).

**Proof.** Let us first prove the statement for the Chow semistability. Assume that \( \mathbb{X} \rightarrow \mathbb{P}^r \) with \( d > 2(2g - 2) \) and \( g \geq 3 \). This is equivalent to say that \( [X \rightarrow \mathbb{P}^r] \in M_X^d \) where \( d := \deg i^* \mathcal{O}_{\mathbb{P}^r}(1) = \deg (i')^* \mathcal{O}_{\mathbb{P}^r}(1) \). In particular, \( M_X^d \neq \emptyset \); hence, from Proposition 10.6 we deduce that there exists \( [X \rightarrow \mathbb{P}^r] \in M_X^d \) such that \( j^* \mathcal{O}_{\mathbb{P}^r}(1) = (i')^* \mathcal{O}_{\mathbb{P}^r}(1) \). However, this implies that \( [X \rightarrow \mathbb{P}^r] \) is in the orbit of \( [X \rightarrow \mathbb{P}^r] \). Since each stratum \( M_X^d \) is \( \text{SL}_{r+1} \)-invariant, we get that \( [X \rightarrow \mathbb{P}^r] \in M_X^d \), q.e.d.

The proof for the Hilbert semistability is similar: we can define a stratification of \( \tilde{H}_d := \{[X \subset \mathbb{P}^r] \in \text{Hilb}_{d}^{ss} : X \text{ is connected} \} \leq H_d \) whose strata are given by
\[
\tilde{M}_X^d = \{[X \subset \mathbb{P}^r] \in \tilde{H}_d : \deg \mathcal{O}_X(1) = d \} \leq M_X^d.
\]
It is clear that Propositions 10.5 and 10.6 remain valid if we substitute \( M_X^d \) with \( \tilde{M}_X^d \). Therefore, the above proof for the Chow semistability extends verbatim to the Hilbert semistability.

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10.3. Further properties of $\overline{Q}_{d,g}$. From the above results, we can deduce the irreducibility of the GIT quotient $\overline{Q}_{d,g} := H_d/SL_{r+1}$ and further properties of the maps $\Phi^s : \overline{Q}_{d,g} \to \overline{M}_g$ for $d > 4(2g-2)$ and of $\Phi^{ps} : \overline{Q}_{d,g} \to \overline{M}^{ps}_g$ for $2(2g-2) < d < \frac{7}{2}(2g-2)$ and $g \geq 3$.

**Proposition 10.9.**

(i) Assume that $d > 4(2g-2)$. The morphism $\Phi^s : \overline{Q}_{d,g} \to \overline{M}_g$ has equi-dimensional fibers of dimension $g$ and, if $\text{char}(k) = 0$, $\Phi^s$ is flat over the smooth locus of $\overline{M}_g$.

(ii) Assume that $2(2g-2) < d < \frac{7}{2}(2g-2)$ and $g \geq 3$. The morphism $\Phi^{ps} : \overline{Q}_{d,g} \to \overline{M}^{ps}_g$ has equi-dimensional fibers of dimension $g$ and, if $\text{char}(k) = 0$, $\Phi^{ps}$ is flat over the smooth locus of $\overline{M}^{ps}_g$.

In both cases, we get that $\overline{Q}_{d,g}$ (hence $H_d$) is irreducible.

**Proof.** The proof is a generalization of [Cap94, Cor. 5.1, Lemma 6.2, Thm. 6.1(2)].

Assume first that $d > 4(2g-2)$. Consider the map (see Theorem 9.1)

$$\phi^s : H_d \to \overline{Q}_{d,g} \xrightarrow{\Phi^s} \overline{M}_g.$$  

From Corollary 5.5(ii) it follows that the fiber of $\phi^s$ over a stable curve $X \in \overline{M}_g$ is equal to

$$(\phi^s)^{-1}(X) = \bigcup_{s(X') = X} \overline{M}^d_{X'},$$

where the union runs over the quasi-stable curves $X'$ whose stable reduction $s(X')$ is equal to $X$ and $d' \in B^d_{X'}$. Using Lemma 10.3 for every pair $(X', d')$ appearing in the above decomposition there exists $d \in B^d_X$ such that $(X', d') \preceq (X, d)$. This implies that

$$(\phi^s)^{-1}(X) = \bigcup_{d \in B^d_X} \overline{M}^d_X \cap H_d.$$  

We deduce that the fiber $(\Phi^s)^{-1}(X)$ contains an open dense subset isomorphic to

$$\left( \bigcup_{d \in B^d_X} M^d_X \right) / SL_{r+1} = \bigcup_{d \in B^d_X} M^d_X / SL_{r+1}.$$  

Since the above map (10.2) is $SL_{r+1}$-equivariant, the natural map $M^d_X \subset H_d \to \overline{Q}_{d,g}$ factors through it. Therefore, $\dim M^d_X / SL_{r+1} \leq \dim \text{Pic}(X) = g$; hence all the irreducible components of $(\Phi^s)^{-1}(X)$ have dimension at most $g$. On the other hand, since the general fiber of $\Phi^s$ has dimension $g$ by Theorem 9.1(iii), all the irreducible components of $(\Phi^s)^{-1}(X)$ must have dimension at least $g$ by the upper semicontinuity of the dimension of the fibers. We conclude that $(\Phi^s)^{-1}(X)$ is of pure dimension $g$.

Let us now prove the irreducibility of $\overline{Q}_{d,g}$. We will use the following elementary Fact, whose proof is left to the reader.

□

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Fact A: Let \( f: X \to Y \) be a surjective morphism between two varieties. Assume that \( Y \) is irreducible. If \( f \) has equi-dimensional fibers of the same dimension and the generic fiber is irreducible, then \( X \) is irreducible.

We apply the above Fact A to the morphism \( \Phi^s \) and we use the fact that \( M_g \) is an irreducible variety (see Fact 2.2(i)), that \( \Phi^s \) has equidimensional fibers of dimension \( g \) by what we have just proved and that the generic fiber of \( \Phi^s \) is irreducible by Theorem 9.1(iii). The irreducibility of \( \mathcal{Q}_{d,g} \) follows.

In order to prove the flatness of \( \Phi^s \) over the smooth locus \((M_g)^{sm}\) of \( M_g \), we will use the following well-known flatness’s criterion.

Fact B (see [Mat89, Cor. of Thm 23.1, p. 179]): Let \( f: X \to Y \) be a dominant morphism between irreducible varieties. If \( X \) is Cohen-Macauly, \( Y \) is smooth and \( f \) has equi-dimensional fibers of the same dimension, then \( f \) is flat.

We apply the above Fact B to the restriction morphism \( \Phi^s : (\mathcal{Q}_{d,g})^{-1}((M_g)^{sm}) \to (M_g)^{sm} \) and we use that \( \Phi^s \) has equidimensional fibers of dimension \( g \) as we proved above and that \( \mathcal{Q}_{d,g} \) is Cohen-Macauly if \( \text{char}(k) = 0 \) by Theorem 9.1(iv).

Assume now that \( 2(2g - 2) < d < \frac{7}{2}(2g - 2) \). The proof is entirely analogous to the previous proof noticing that, by Corollary 6.2, the fiber of the morphism

\[
\phi^p_s : H_d \to \mathcal{Q}_{d,g} \xrightarrow{\Phi^p_s} \mathcal{M}_g^{ps}
\]

over a p-stable curve \( X \in \mathcal{M}_g^{ps} \) is given by

\[
(\phi^p_s)^{-1}(X) = \bigcup_{ps(X') = X, d' \in B^d_{X'}} M_{X'}^{d'}
\]

where the union is over the possible quasi-p-stable curves \( X' \) whose p-stable reduction \( ps(X') = wps(X') \) is equal to \( X \) and \( d' \in B^d_{X'} \). We leave the details to the reader.

\[\square\]

11. Semistable, polystable and stable points

The aim of this section is to describe the points of \( \text{Hilb}_d \) that are Hilbert or Chow semistable, polystable and stable. Let us begin with the semistable points.

Theorem 11.1. Consider a point \([X \subset \mathbb{P}^r] \in \text{Hilb}_d\) and assume that \( X \) is connected.

(1) If \( d > 4(2g - 2) \) then the following conditions are equivalent:

(i) \([X \subset \mathbb{P}^r]\) is Hilbert semistable;

(ii) \([X \subset \mathbb{P}^r]\) is Chow semistable;

(iii) \( X \) is quasi-stable, non-degenerate and linearly normal in \( \mathbb{P}^r \) and \( \mathcal{O}_X(1) \) is properly balanced and non-special;

(iv) \( X \) is quasi-stable and \( \mathcal{O}_X(1) \) is properly balanced;

(v) \( X \) is quasi-stable and \( \mathcal{O}_X(1) \) is balanced.

(2) If \( 2(2g - 2) < d < \frac{7}{2}(2g - 2) \) and \( g \geq 3 \) then the following conditions are equivalent:
(i) \([X \subset \mathbb{P}^r] \) is Hilbert semistable;
(ii) \([X \subset \mathbb{P}^r] \) is Chow semistable;
(iii) \(X \) is quasi-p-stable, non-degenerate and linearly normal in \(\mathbb{P}^r\) and \(\mathcal{O}_X(1)\)
	is properly balanced and non-special;
(iv) \(X \) is quasi-p-stable and \(\mathcal{O}_X(1)\) is properly balanced;
(v) \(X \) is quasi-p-stable and \(\mathcal{O}_X(1)\) is balanced.

**Proof.** Let us first prove part (1).

1(i) \(\Rightarrow\) (1ii) follows from Fact 4.1.

1(ii) \(\Rightarrow\) (1iii) follows from the potential stability theorem (see Fact 4.8) and Corollary 5.5(ii).

1(iii) \(\Rightarrow\) (1iv) is clear.

1(iv) \(\Leftrightarrow\) (1v) follows from Remark 5.6, using that \(\mathcal{O}_X(1)\) is ample.

1(iv) \(\Rightarrow\) (1i) First of all, we make the following Reduction:

We can assume that \(\mathcal{O}_X(1)\) is strictly balanced. Indeed, by Lemma 8.4(ii), there exists an isotrivial specialization \((X, \mathcal{O}_X(1)) \rightarrow (X', L')\) such that \(X'\) is quasi-stable and \(L'\) is a strictly balanced line bundle on \(X'\) of total degree \(d\). According to Theorem 13.5 and using that \(d > 4(2g - 2)\), we conclude that \(L'\) is very ample and non-special. Therefore, by choosing a basis of \(H^0(X', L')\), we get a point \([X' \subset \mathbb{P}^r] \in \text{Hilb}_d\). According to Theorem 8.5, \([X \subset \mathbb{P}^r] \in \text{Hilb}_d^{ss}\) if and only if \([X' \subset \mathbb{P}^r] \in \text{Hilb}_d^{ss}\). Therefore, up to replacing \(X\) with \(X'\), we can assume that \(\mathcal{O}_X(1)\) is strictly balanced, q.e.d.

Now, since \(X\) is quasi-stable, we can find a smooth curve \(B \xrightarrow{f} \text{Hilb}_d\) and a point \(b_0 \in B\) such that, if we denote by \(\mathbb{P}^r \times B \xleftarrow{\pi'} \mathcal{X} \xrightarrow{\pi} B\) the pull-back via \(f\) of the universal family over \(\text{Hilb}_d\) and we set \(\mathcal{L} := i^*(\mathcal{O}_{\mathbb{P}^r(1)} \boxtimes \mathcal{O}_B)\), then \([X \leftarrow \mathbb{P}^r \times B |_{b_0} = [X \subset \mathbb{P}^r]\) and \(\mathcal{X}|_{\pi^{-1}(b)}\) is a connected smooth curve for every \(b \in B \setminus \{b_0\}\). Note that, by construction, \(\pi\) is a family of quasi-stable curves of genus \(g\). As in the proof of Proposition 10.6 we can now apply the semistable replacement property, which implies that, up to replacing \(B\) with a finite cover ramified over \(b_0\), we can find two maps \(g : B \rightarrow \text{Hilb}_d\) and \(h : B \setminus \{b_0\} \rightarrow \text{SL}_{q+1}\) such that

\[
(11.1) \quad f(b) = h(b) \cdot g(b) \quad \text{for every } b_0 \neq b \in B,
\]

\[
(11.2) \quad g(b_0) \text{ is Hilbert polystable}.
\]

We denote by \(\mathbb{P}^r \times B \xleftarrow{\pi'} \mathcal{Y} \xrightarrow{\pi} B\) the pull-back via \(g\) of the universal family over \(\text{Hilb}_d\) and we set \(\mathcal{M} := (i')^*(\mathcal{O}_{\mathbb{P}^r(1)} \boxtimes \mathcal{O}_B)\). Property (11.1) implies that, up to shrinking again \(B\) around \(b_0\), we have that

\[
(11.3) \quad (\mathcal{X}, \mathcal{L})|_{\pi^{-1}(B \setminus \{b_0\})} \cong (\mathcal{Y}, \mathcal{M})|_{((\pi')^{-1}(B \setminus \{b_0\}))}.
\]

Note that this fact together with (11.2) and the potential stability Theorem (Fact 4.8) implies that \(\pi'\) is also a family of quasi-stable curves of genus \(g\).
Consider now the stable reductions \( s(\pi) : s(\mathcal{X}) \to B \) of \( \pi : \mathcal{X} \to B \) and \( s(\pi') : s(\mathcal{Y}) \to B \) of \( \pi' : \mathcal{Y} \to B \) (see Remark 2.10). From (11.3), it follows that \( s(\pi) \) and \( s(\pi') \) are two families of stable curves which are isomorphic away from the fibers over \( b_0 \). Since the moduli space \( \overline{M}_g \) of stable curves is separated, we conclude that

\[
(11.4) \quad \begin{array}{c}
\mathcal{X} \\
\downarrow s(\pi) \\
\downarrow s(\pi') \\
B
\end{array} \xrightarrow{\cong} \begin{array}{c}
\mathcal{Y} \\
\downarrow s(\pi') \\
\downarrow s(\pi) \\
B
\end{array}
\]

Therefore \( \pi \) and \( \pi' \) are two families of quasi-stable curves with the same stable reduction (from now on, we identify \( s(\mathcal{X}) \xrightarrow{s(\pi)} B \) and \( s(\mathcal{Y}) \xrightarrow{s(\pi')} B \) via the above isomorphism). If we blow-up all the nodes of the fiber over \( b_0 \) of the stable reduction \( s(\pi) = s(\pi') \), we get a new family of quasi-stable curves \( \pi : Z \to B \) with the same stable reduction as that of \( \pi \) and of \( \pi' \), which moreover dominates \( \pi \) and \( \pi' \), i.e., such that there exists a commutative diagram

\[
(11.5) \quad \begin{array}{c}
Z \\
\Sigma \downarrow \downarrow \Sigma' \\
\mathcal{X} & \pi & \mathcal{Y} \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
\pi' & B
\end{array}
\]

where the maps \( \Sigma \) and \( \Sigma' \) induce an isomorphism of the corresponding stable reductions. Equivalently, the maps \( \Sigma \) and \( \Sigma' \) are obtained by blowing down some of the exceptional components of the fiber of \( Z \) over \( b_0 \). If we set \( \tilde{\mathcal{L}} := \Sigma^*(\mathcal{L}) \) and \( \tilde{\mathcal{M}} := (\Sigma')^*(\mathcal{M}) \), then (11.3) gives that

\[
\tilde{\mathcal{L}}_{\pi^{-1}(B\setminus b_0)} \cong \tilde{\mathcal{M}}_{\pi^{-1}(B\setminus b_0)}. \]

Lemma 10.7 now gives that there exists a Cartier divisor \( D \) on \( Z \) supported on \( \pi^{-1}(b_0) \) such that

\[
(11.6) \quad \tilde{\mathcal{L}} = \tilde{\mathcal{M}} \otimes \mathcal{O}_Z(D).
\]

We now set \( (X, L) := (\mathcal{X}, \mathcal{L})_{b_0} \) and \( d := \deg L \), \( (Y, M) := (\mathcal{Y}, \mathcal{M})_{b_0} \) and \( d' := \deg M \), \( Z := Z_{b_0}, \tilde{L} := \mathcal{L}_{b_0} \) and \( \tilde{d} := \deg \tilde{L}, \tilde{M} := \mathcal{M}_{b_0} \) and \( \tilde{d}' := \deg \tilde{M} \). Equation (11.6) gives that \( \tilde{d} \) and \( \tilde{d}' \) are equivalent on \( Z \). Moreover, \( d \) is strictly balanced by the above Reduction and \( d' \) is strictly balanced by the assumption (11.2) together with Corollary 8.6. Therefore, we can apply Lemma 8.11 twice to conclude that \( \mathcal{X} = \mathcal{Y} \). Now, the relation (11.3) together with the Lemma 10.7 imply that there exists a Cartier divisor \( D' \) on \( \mathcal{X} = \mathcal{Y} \) supported on \( \pi^{-1}(b_0) \) such that

\[
(11.7) \quad L = M \otimes \mathcal{O}_X(D').
\]
In particular, we get that $d$ is equivalent to $d'$. Since $d$ and $d'$ are strictly balanced, Lemma 3.10 implies that $d = d'$. Since $[Y \hookrightarrow \mathbb{P}^r \times B]_{b_0} = [Y \hookrightarrow \mathbb{P}^r] \in \text{Hilb}_{d}^{ss}$ by assumption (11.12), Corollary 10.8 gives that $[X \subset \mathbb{P}^r] \in \text{Hilb}_{d}^{ss}$, q.e.d.

The proof of part (2) is similar: it is enough to replace quasi-stable curves by quasi-p-stable curves (using Corollary 6.2), to replace the stable reduction by the p-stable reduction and using the fact that the moduli space $\overline{M}_g^{ps}$ of p-stable curves of genus $g$ is separated. □

From the above Theorem 11.1, we can deduce a description of the Hilbert and Chow polystable and stable points of $\text{Hilb}_d$.

**Corollary 11.2.** Consider a point $[X \subset \mathbb{P}^r] \in \text{Hilb}_d$ and assume that $X$ is connected.

1. If $d > 4(2g - 2)$ then the following conditions are equivalent:
   (i) $[X \subset \mathbb{P}^r]$ is Hilbert polystable;
   (ii) $[X \subset \mathbb{P}^r]$ is Chow polystable;
   (iii) $X$ is quasi-stable, non-degenerate and linearly normal in $\mathbb{P}^r$ and $\mathcal{O}_X(1)$ is strictly balanced and non-special;
   (iv) $X$ is quasi-stable and $\mathcal{O}_X(1)$ is strictly balanced.

2. If $2(2g - 2) < d < \frac{7}{2}(2g - 2)$ and $g \geq 3$ then the following conditions are equivalent:
   (i) $[X \subset \mathbb{P}^r]$ is Hilbert polystable;
   (ii) $[X \subset \mathbb{P}^r]$ is Chow polystable;
   (iii) $X$ is quasi-p-stable, non-degenerate and linearly normal in $\mathbb{P}^r$ and $\mathcal{O}_X(1)$ is strictly balanced and non-special;
   (iv) $X$ is quasi-p-stable and $\mathcal{O}_X(1)$ is strictly balanced.

**Proof.** Let us prove part (1).

(1i) $\iff$ (1ii): from Theorem 11.1(1) we get that the Hilbert semistable locus inside $\text{Hilb}_d$ is equal to the Chow semistable locus. Since a point of $\text{Hilb}_d$ is Hilbert (resp. Chow) polystable if and only if it is Hilbert (resp. Chow) semistable and its orbit is closed inside the Hilbert (resp. Chow) semistable locus, we conclude that also the locus of Hilbert polystable points is equal to the locus of Chow polystable points.

(1ii) $\Rightarrow$ (1iii) follows from the potential stability theorem (see Fact 4.8), Corollary 6.5(1) and Corollary 8.6.

(1iii) $\Rightarrow$ (1iv) is obvious.

(1iv) $\Rightarrow$ (1i): from Theorem 11.1(1), we get that $[X \subset \mathbb{P}^r] \in \text{Hilb}_d^{ss}$. We have to prove that the $\text{SL}_{r+1}$-orbit of $[X \subset \mathbb{P}^r]$ is closed inside $\text{Hilb}_d^{ss}$. For this reason it is enough to prove that if $B \subset \text{Hilb}_d^{ss}$ is a smooth curve and $b_0$ is a point of $B$ such that all the points of $B \setminus b_0$ are in the same orbit of $[X \subset \mathbb{P}^r]$ then also $b_0$ is in the orbit of $[X \subset \mathbb{P}^r]$. Since $\text{SL}_{r+1}$ acts by changing the embedding of a point $[Y \subset \mathbb{P}^r] \in \text{Hilb}_d$ via a projective change of coordinates, it is enough to prove the following.
Claim: If \( \mathbb{P}^r_B := \mathbb{P}^r \times B \overset{i}{\hookrightarrow} \mathcal{Y} \overset{\pi}{\rightarrow} B \) is a polarized family in \( \text{Hilb}^*_d \) such that for any \( b \in B \setminus b_0 \) we have that \( (\mathcal{Y}, \mathcal{M}) := i^*(\mathcal{O}_{\mathbb{P}^r_B}(1)) \cong (X, \mathcal{O}_X(1)) \), then \( (\mathcal{Y}, \mathcal{M})_{b_0} \cong (X, \mathcal{O}_X(1)) \).

With the same argument as in the proof of Proposition \[10.5\] we get that \( \mathcal{Y} \to B \) is obtained from the constant family \( X := X \times B \overset{\pi}{\rightarrow} B \) by blowing up some of the nodes of the central fiber. In other words there exists a morphism of families of quasi-stable curves

\[
\begin{array}{c}
\mathcal{Y} \\
\Sigma \\
\pi
\end{array} \xymatrix{ & \mathcal{X} := X \times B \ar[dl]_{\pi'} \ar[dr]^{\pi} & }
\]

which is an isomorphism outside \( b_0 \) and which is the contraction of some of the exceptional components of \( \mathcal{Y}_{b_0} \). Consider the constant line bundle \( \mathcal{L} := \mathcal{O}_X(1) \boxtimes \mathcal{O}_B \) on \( \mathcal{X} \).

In view of our assumptions on \( (\mathcal{Y}, \mathcal{M}) \), we deduce that, up to shrinking \( B \) around \( b_0 \), we have an isomorphism \( \mathcal{M}_{\pi^{-1}(b \setminus b_0)} = \Sigma^*(\mathcal{L})_{\pi^{-1}(b \setminus b_0)} \). Lemma \[10.7\] implies then that there exists a Cartier divisor on \( \mathcal{Y} \) supported on the central fiber such that

\[
(11.8) \quad \mathcal{M} = \Sigma^*(\mathcal{L}) \otimes \mathcal{O}_Y(D).
\]

In particular, the multidegrees of the line bundles \( \mathcal{M}_{b_0} \) and of \((\Sigma)^*(\mathcal{L})_{b_0} = (\Sigma_{b_0})^*(\mathcal{O}_X(1)) \) on \( Y := \mathcal{Y}_{b_0} \) are equivalent. Lemma \[8.11\] implies now that \( \Sigma_{b_0} : Y \to X \) is an isomorphism, which indeed is equivalent to the fact that \( \Sigma \) induces an isomorphism between \( \mathcal{Y} \) and the constant family \( \mathcal{X} = X \times B \). Equation \[11.8\] implies now that the multidegree of the line bundles \( \mathcal{M}_{b_0} \) and \( \mathcal{O}_X(1) \) on \( X \) are equivalent. Since \( \deg \mathcal{O}_X(1) \) is strictly balanced by assumption and \( \deg \mathcal{M}_{b_0} \) is properly balanced by the Potential stability Theorem (see Fact \[4.8\] together with Remark \[5.6\]) we can apply Lemma \[8.11\] in order to conclude that \( \deg \mathcal{O}_X(1) = \deg \mathcal{M}_{b_0} \). From what we have proved so far, we deduce that the polarized family \( \mathbb{P}^r_B := \mathbb{P}^r \times B \overset{i}{\hookrightarrow} \mathcal{Y} \overset{\pi}{\rightarrow} B \) we started with is induced by a map \( f : B \to M^d_X \), where \( d := \deg \mathcal{O}_X(1) \). Moreover the original assumption that \( (\mathcal{Y}, \mathcal{M}) := i^*(\mathcal{O}_{\mathbb{P}^r_B}(1)) \cong (X, \mathcal{O}_X(1)) \) for any \( b \in B \setminus b_0 \) translates into the fact that

\[
f(B \setminus b_0) \subseteq p^{-1}(\mathcal{O}_X(1)) ,
\]

where \( p : M^d_X \to \text{Pic}^d(X) \) is the map of \((10.2)\). Since \( p \) is a morphism between algebraic varieties, its fibers are closed and therefore we get that \( f(b_0) \in p^{-1}(\mathcal{O}_X(1)) \), which is equivalent to \( (\mathcal{Y}, \mathcal{M})_{b_0} \cong (X, \mathcal{O}_X(1)) \), q.e.d.

The proof of part \((2)\) is similar: it is enough to replace quasi stable curves by quasi-p-stable curves (which is possible by Corollary \[6.2\] and use Theorem \[11.12\] and the potential pseudo-stability Theorem \[5.1\].

\[\square\]

**Corollary 11.3.** Consider a point \( [X \subset \mathbb{P}^r] \in \text{Hilb}_d \) and assume that \( X \) is connected.

\((1)\) If \( d > 4(2g - 2) \) then the following conditions are equivalent:

\( (i) \) \( [X \subset \mathbb{P}^r] \) is Hilbert stable;
(ii) \([X \subset \mathbb{P}^r] \) is Chow stable;
(iii) \(X\) is quasi-stable, non-degenerate and linearly normal in \(\mathbb{P}^r\) and \(\mathcal{O}_X(1)\) is stably balanced and non-special;
(iv) \(X\) is quasi-stable and \(\mathcal{O}_X(1)\) is stably balanced.

(2) If \(2(2g-2) < d < \frac{7}{2}(2g-2)\) and \(g \geq 3\) then the following conditions are equivalent:

(i) \([X \subset \mathbb{P}^r] \) is Hilbert stable;
(ii) \([X \subset \mathbb{P}^r] \) is Chow stable;
(iii) \(X\) is quasi-p-stable, non-degenerate and linearly normal in \(\mathbb{P}^r\) and \(\mathcal{O}_X(1)\) is stably balanced and non-special;
(iv) \(X\) is quasi-p-stable and \(\mathcal{O}_X(1)\) is stably balanced.

Proof. Let us prove part (1).

\((1i) \Rightarrow (1ii)\) follows from Fact 4.1.

\((1ii) \Rightarrow (1iii)\) follows from the potential stability theorem (see Fact 4.8) and Theorem 8.1.

\((1iii) \Rightarrow (1iv)\) is obvious.

\((1iv) \Rightarrow (1ii)\): from Corollary 11.2(1), we get that \([X \subset \mathbb{P}^r]\) is Chow polystable. Lemma 3.9 gives that \(\tilde{X} := X \setminus \text{exc} \) is connected; hence, from Corollary 7.3, we deduce that \(\text{Stab}_{\text{PGL}_{r+1}}([X \subset \mathbb{P}^r])\) is a finite group. This implies that \([X \subset \mathbb{P}^r] \in \text{Ch}^{-1}(\text{Chow}_d^* \mathcal{O}_X)\) since a point of \(\text{Hilb}_d\) is Hilbert (resp. Chow) stable if and only if it is Hilbert (resp. Chow) polystable and it has finite stabilizers with respect to the action of \(\text{PGL}_{r+1}\).

The proof of part (2) is similar, using the potential pseudo-stability Theorem 5.1 and Corollary 11.2(2).

\(\square\)

12. A NEW COMPACTIFICATION OF THE UNIVERSAL JACOBIAN OVER THE MODULI SPACE OF PSEUDO-STABLE CURVES

Fix integers \(d\) and \(g \geq 2\). Consider the stack \(\mathcal{J}_{d,g}\), called the universal Jacobian stack of genus \(g\) and degree \(d\), whose section over a scheme \(S\) is the groupoid of families of smooth curves of genus \(g\) over \(S\) together with a line bundle of relative degree \(d\). We denote by \(J_{d,g}\) its coarse moduli space, and we call it the universal Jacobian variety (or simply the universal Jacobian) of degree \(d\) and genus \(g\).

From the work of Caporaso (\[Cap94\]), it is possible to obtain a modular compactification of the universal Jacobian stack and of the universal Jacobian variety. Denote by \(\overline{\mathcal{J}}_{d,g}\) the category fibered in groupoids over the category of schemes whose section over

\[\text{In } \overline{\mathcal{J}}_{d,g}, this variety is called the universal Picard variety and it is denoted by \(P_{d,g}\). We prefer to use the name universal Jacobian, and therefore the symbol \(J_{d,g}\), because the word Jacobian variety is used only for curves while the word Picard variety is used also for varieties of higher dimensions and therefore it is more ambiguous. Accordingly, we will denote the Caporaso's compactified universal Jacobian by \(\overline{\mathcal{J}}_{d,g}\) instead of \(P_{d,g}\) as in \[Cap94\] (see Fact 12.1).

In \[Cap94\], this variety is called the universal Picard variety and it is denoted by \(P_{d,g}\). We prefer to use the name universal Jacobian, and therefore the symbol \(J_{d,g}\), because the word Jacobian variety is used only for curves while the word Picard variety is used also for varieties of higher dimensions and therefore it is more ambiguous. Accordingly, we will denote the Caporaso's compactified universal Jacobian by \(\overline{\mathcal{J}}_{d,g}\) instead of \(P_{d,g}\) as in \[Cap94\] (see Fact 12.1).
a scheme $S$ is the groupoid of families of quasi-stable curves over $S$ of genus $g$ endowed with a line bundle whose restriction to each geometric fiber is a properly balanced line bundle of degree $d$. We summarize all the known properties of $\overline{J}_{d,g}$ into the following

**Fact 12.1.** Let $g \geq 2$ and $d \in \mathbb{Z}$.

1. $\overline{J}_{d,g}$ is a smooth, irreducible, universally closed Artin stack of finite type over $k$, having dimension $4g - 4$ and containing $J_{d,g}$ as an open substack.
2. $\overline{J}_{d,g}$ admits an adequate moduli space $\overline{J}_{d,g}$ (in the sense of [Alp2]), which is a normal irreducible projective variety of dimension $4g - 3$ containing $J_{d,g}$ as an open subvariety.
3. There exists a commutative diagram

$$
\begin{array}{ccc}
\overline{J}_{d,g} & \longrightarrow & \overline{J}_{d,g} \\
\Psi \downarrow & & \Phi^* \downarrow \\
\overline{M}_g & \longrightarrow & \overline{M}_g
\end{array}
$$

where $\Psi^*$ is universally closed and surjective and $\Phi^*$ is projective, surjective with equidimensional fibers of dimension $g$.
4. If $\text{char}(k) = 0$, then for any $X \in \overline{M}_g$ we have that

$$(\Phi^*)^{-1}(X) \cong \overline{\text{Jac}_d}(X)/\text{Aut}(X),$$

where $\overline{\text{Jac}_d}(X)$ is the Simpson’s compactified Jacobian of $X$ parametrizing $S$-equivalence classes of rank-1, torsion-free sheaves on $X$ that are slope-semistable with respect to $\omega_X$.
5. If $4(2g - 2) < d$ then we have that

$$
\begin{cases}
\overline{J}_{d,g} \cong [H_d/GL(r + 1)], \\
\overline{J}_{d,g} \cong H_d/GL(r + 1) = Q_{d,g},
\end{cases}
$$

where $H_d \subset \text{Hilb}_d$ is the open subset consisting of points $[X \subset \mathbb{P}^r] \in \text{Hilb}_d$ such that $X$ is connected and $[X \subset \mathbb{P}^r]$ is Hilbert semistable (or equivalently, Chow semistable).

Parts (1), (2), (3) follow by combining the work of Caporaso ([Cap94], [Cap05]) and of Melo ([Mel09]). Part (5) follows as well from the previous quoted papers if $d \geq 10(2g - 2)$ and working with Hilbert semistability. The extension to $d > 4(2g - 2)$ and to the Chow semistability follows straightforwardly from our Theorem 11.1(1).

Part (4) was observed by Alexeev in [Ale04, Sec. 1.8] (see also [CMKV, Sec. 2.9] for a related discussion and in particular for a discussion about the need for the assumption $\text{char}(k) = 0$).

We call $\overline{J}_{d,g}$ (resp. $\overline{J}_{d,g}$) the Caporaso’s compactified universal Jacobian (resp. Caporaso’s compactified universal Jacobian) of genus $g$ and degree $d$. 

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The aim of this subsection is to define and study a new compactification of \( J_{d,g} \) (resp. of \( J_{d,g}^{ps} \)) over the stack \( \overline{M}_g^{ps} \) (resp. the variety \( \overline{M}_g^{ps} \)) of p-stable curves of genus \( g \geq 3 \).

12.1. The moduli stack of properly balanced line bundles over quasi-p-stable curves. Let \( J_{ps}^{d,g} \) be the category whose sections over a \( k \)-scheme \( S \) are pairs \((f : X \to S, L)\) where \( f \) is a family of quasi p-stable curves of genus \( g \geq 3 \) and \( L \) is a line bundle on \( X \) of relative degree \( d \) that is properly balanced on the geometric fibers of \( f \). Arrows between such pairs are given by cartesian diagrams

\[
\begin{array}{ccc}
X & \xrightarrow{h} & X' \\
f \downarrow & & \downarrow f' \\
S & \xrightarrow{} & S'
\end{array}
\]

and an isomorphism \( L \cong h^*L' \). Note that \( J_{ps}^{d,g} \) is a category fibered in groupoids over the category of \( k \)-schemes.

The aim of this subsection is to prove that \( J_{ps}^{d,g} \) is an algebraic stack and to study its properties. Let us first show that \( J_{ps}^{d,g} \) is periodic in \( d \) with period \( 2g - 2 \).

**Lemma 12.2.** For any integer \( n \), there is a natural isomorphism \( J_{ps}^{d,g} \cong J_{d+n(2g-2),g}^{ps} \) of categories fibered in groupoids.

**Proof.** The result follows immediately by noticing that a line bundle \( L \) on a quasi-p-stable curve \( X \) is properly balanced if and only if \( L \otimes \omega_X^n \) is properly balanced, for any integer \( n \). The required isomorphism will then consist of associating to any section \((f : X \to S, L) \in J_{ps}^{d,g}(S)\) the section \((f : X \to S, L \otimes \omega_X^n) \in J_{ps}^{d+n(2g-2),g}(S)\), where by \( \omega_f \) we denote the relative dualizing sheaf of the morphism \( f \). \( \square \)

We will now show that if \( 2(2g - 2) < d < \frac{7}{2}(2g - 2) \) then \( J_{ps}^{d,g} \) is isomorphic to the quotient stack \([H_d/GL_{r+1}]\), where \( GL_{r+1} \) acts on \( H_d \) via its projection onto \( PGL_{r+1} \). Recall that, given a scheme \( S \), \([H_d/GL_{r+1}](S)\) consists of \( GL_{r+1} \)-principal bundles \( \phi : E \to S \) with a \( GL_{r+1} \)-equivariant morphism \( \psi : E \to H_d \). Morphisms are given by pullback diagrams which are compatible with the morphism to \( H_d \).

**Theorem 12.3.** If \( 2(2g - 2) < d < \frac{7}{2}(2g - 2) \) and \( g \geq 3 \) then \( J_{ps}^{d,g} \) is isomorphic to the quotient stack \([H_d/GL_{r+1}]\).

**Proof.** To shorten the notations, we set \( G := GL_{r+1} \). We must show that, for every \( k \)-scheme \( S \), the groupoids \( J_{ps}^{d,g}(S) \) and \([H_d/G](S)\) are equivalent. Our proof goes along the lines of the proof of \( [Mel09] \) Thm. 3.1, so we will explain here the main steps and refer to loc. cit. for further details.

Given \((f : X \to S, L) \in J_{ps}^{d,g}(S)\), we must produce a principal \( G \)-bundle \( E \) on \( S \) and a \( G \)-equivariant morphism \( \psi : E \to H_d \). Notice that since \( d > 2(2g - 2) \), Theorem 13.5(iii) implies that \( H^1(X, L_{X_0}) = 0 \) for any geometric fiber \( X_0 \) of \( f \), so \( f_*^a(L) \) is locally free of rank \( r + 1 = d - g + 1 \). We can then consider its frame bundle \( E \), which is a
principal $GL_{r+1}$-bundle: call it $E$. To find the $G$-equivariant morphism to $H_d$, consider the family $\mathcal{X}_E := \mathcal{X} \times_S E$ of quasi-$p$-stable curves together with the pullback of $\mathcal{L}$ to $\mathcal{X}_E$, call it $\mathcal{L}_E$, whose restriction to the geometric fibers is properly balanced.

By definition of frame bundle, $f_{E*}(\mathcal{L}_E)$ is isomorphic to $\mathbb{A}^r_k \times_k E$. Moreover, the line bundle $\mathcal{L}_E$ is relatively ample by Remark [5.6] hence it is relatively very ample by Theorem [13.5(3)]. Therefore, $\mathcal{L}_E$ gives an embedding over $E$ of $\mathcal{X}_E$ in $\mathbb{P}^r \times E$. By the universal property of the Hilbert scheme $\text{Hilb}_d$, this family determines a map $\psi: E \to \text{Hilb}_d$ whose image is contained in $H_d$ by Theorem [11.1(2)]. It follows immediately from the construction that $\psi$ is a $G$-equivariant map.

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{f} & \mathcal{X}_E := \mathcal{X} \times_S E \\
S & \xrightarrow{\phi} & E & \xrightarrow{\psi} & \text{Hilb}_d \\
\end{array}
\]

Let us check that isomorphisms in $\mathcal{F}_{d,g}^{s}(S)$ lead canonically to isomorphisms in $[H_d/G](S)$. Consider an isomorphism between two pairs $(f : \mathcal{X} \to S, \mathcal{L})$ and $(f' : \mathcal{X}' \to S, \mathcal{L}')$, i.e., an isomorphism $h : \mathcal{X} \to \mathcal{X}'$ over $S$ and an isomorphism of line bundles $\mathcal{L} \cong h^* \mathcal{L}'$. Since $f' h = f$, we get a unique isomorphism between the vector bundles $f_*(\mathcal{L})$ and $f'_*(\mathcal{L}')$. As taking the frame bundle gives an equivalence between the category of vector bundles of rank $r+1$ over $S$ and the category of principal $GL_{r+1}$-bundles over $S$, the isomorphism $f_*(\mathcal{L}) \cong f'_*(\mathcal{L'})$ leads to a unique isomorphism between their frame bundles, call them $E$ and $E'$ respectively. It is clear that this isomorphism is compatible with the $G$-equivariant morphisms $\psi: E \to H_d$ and $\psi': E' \to H_d$.

Conversely, given a section $(\phi : E \to S, \psi : E \to H_d)$ of $[H_d/G]$ over a $k$-scheme $S$, let us construct a family of quasi-$p$-stable curves of genus $g$ over $S$ and a line bundle whose restriction to the geometric fibers is properly balanced of degree $d$.

Let $\mathcal{C}_d$ be the restriction to $H_d$ of the universal family on $\text{Hilb}_d$. By Theorem [11.1(2)], the pullback of $\mathcal{C}_d$ by $\psi$ gives a family $\mathcal{C}_E$ on $E$ of quasi-$p$-stable curves of genus $g$ and a line bundle $\mathcal{L}_E$ on $\mathcal{C}_E$ whose restriction to the geometric fibers is properly balanced. As $\psi$ is $G$-invariant and $\phi$ is a $G$-bundle, the family $\mathcal{C}_E$ descends to a family $\mathcal{C}_S$ over $S$, where $\mathcal{C}_S = \mathcal{C}_E / G$. In fact, since $\mathcal{C}_E$ is flat over $E$ and $E$ is faithfully flat over $S$, $\mathcal{C}_S$ is flat over $S$ too.

Now, since $G = GL_{r+1}$, the action of $G$ on $\mathcal{C}_d$ is naturally linearized. Therefore, the action of $G$ on $E$ can also be linearized to an action on $\mathcal{L}_E$, yielding descent data for $\mathcal{L}_E$. Since $\mathcal{L}_E$ is relatively very ample and $\phi$ is a principal $G$-bundle, a standard descent argument shows that $\mathcal{L}_E$ descends to a relatively very ample line bundle on $\mathcal{C}_S$, call it $\mathcal{L}_S$, whose restriction to the geometric fibers of $\mathcal{C}_S \to S$ is properly balanced by construction.

It is straightforward to check that an isomorphism on $[H_d/G](S)$ leads to an unique isomorphism in $\mathcal{F}_{d,g}^{s}(S)$.

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We leave to the reader the task of checking that the two functors between the groupoids \([H_d/G](S)\) and \(\mathcal{J}_{d,g}^{ps}(S)\) that we have constructed are one the inverse of the other, which concludes the proof.

From Theorem 12.3 and Lemma 12.2 we deduce the following consequences for \(\mathcal{J}_{d,g}^{ps}\).

**Proposition 12.4.** \(\mathcal{J}_{d,g}^{ps}\) is a smooth and irreducible universally closed Artin stack of finite type over \(k\) and of dimension \(4g - 4\), endowed with a universally closed morphism \(\Psi^{ps}\) onto the moduli stack of \(p\)-stable curves \(\mathcal{M}^{ps}_g\).

**Proof.** Using Lemma 12.2, we can assume that \(2(2g - 2) < d < \frac{7}{2}(2g - 2)\) and hence that \(\mathcal{J}_{d,g}^{ps} \cong [H_d/GL_{r+1}]\) by Theorem 12.3.

The fact that \(\mathcal{J}_{d,g}^{ps}\) is a universally closed Artin stack of finite type over \(k\) follows from Theorem 12.3 and general facts of GIT. \(\mathcal{J}_{d,g}^{ps}\) is smooth and irreducible since \(H_d\) is smooth by Theorem 9.1(iv) and irreducible by Proposition 10.9. Using again Theorem 9.1(iv), we can compute the dimension of \(\mathcal{J}_{d,g}^{ps}\) as follows:

\[
\dim \mathcal{J}_{d,g}^{ps} = \dim H_d - \dim GL_{r+1} = r(r + 2) + 4g - 3 - (r + 1)^2 = 4g - 4.
\]

Now, given \((f : X \to S, \mathcal{L}) \in \mathcal{J}_{d,g}^{ps}(S)\), we get an element of \(\mathcal{M}^{ps}_g(S)\) by forgetting \(\mathcal{L}\) and by considering the \(p\)-stable reduction \(ps(f) : ps(X) \to S\) of \(f\) (see Definition 2.12). This defines a morphism of stacks \(\Psi^{ps} : \mathcal{J}_{d,g}^{ps} \to \mathcal{M}^{ps}_g\), which is universally closed since \(\mathcal{J}_{d,g}^{ps}\) is so.

Notice that \(\mathbb{G}_m\) acts on \(\mathcal{J}_{d,g}^{ps}\) by scalar multiplication on the line bundles and leaving the curves fixed. So, \(\mathbb{G}_m\) is contained in the stabilizers of any section of \(\mathcal{J}_{d,g}^{ps}\). This implies that \(\mathcal{J}_{d,g}^{ps}\) are never DM (= Deligne-Mumford) stacks. However, we can quotient out \(\mathcal{J}_{d,g}^{ps}\) by the action of \(\mathbb{G}_m\) using the rigidification procedure defined by Abramovich, Corti and Vistoli in [ACV01]: denote the rigidified stack by \(\mathcal{J}_{d,g}^{ps} / \mathbb{G}_m\). Then from Theorem 12.3 it follows that \(\mathcal{J}_{d,g}^{ps} / \mathbb{G}_m\) is isomorphic to the quotient stack \([H_d/PGL_{r+1}]\) if \(2(2g - 2) < d < \frac{7}{2}(2g - 2)\). Note that, using Proposition 12.4, we get

\[
\dim \mathcal{J}_{d,g}^{ps} / \mathbb{G}_m = \dim \mathcal{J}_{d,g}^{ps} + 1 = 4g - 3.
\]

From the modular description of \(\mathcal{J}_{d,g}^{ps}\) it is straightforward to check that the stack \(\mathcal{J}_{d,g}^{ps} / \mathbb{G}_m\) is the stackification of the prestack whose sections over a scheme \(S\) are given by pairs \((f : X \to S, \mathcal{L})\) as in \(\mathcal{J}_{d,g}^{ps}\) and whose arrows between two such pairs are given by cartesian diagrams

\[
\begin{array}{ccc}
X & \xrightarrow{h} & X' \\
f \downarrow & \cong & \downarrow f' \\
S & \longrightarrow & S'
\end{array}
\]

and an isomorphism \(\mathcal{L} \cong h^*\mathcal{L}' \otimes f^*M\), for some \(M \in \text{Pic}(S)\). We refer to [Mel09, Sec. 4] for more details.

We can now determine when the stack \(\mathcal{J}_{d,g}^{ps} / \mathbb{G}_m\) is a DM-stack.
Proposition 12.5. Assume that \( g \geq 3 \). The following conditions are equivalent:

(i) \( \gcd(d + 1 - g, 2g - 2) = 1 \);
(ii) For any \( d' \equiv d \mod 2g - 2 \) with \( 2(2g - 2) < d' < \frac{7}{2}(2g - 2) \), the GIT quotient \( H_{d'}/\text{PGL}_{r+1} \) is geometric, i.e., there are no strictly semistable points;
(iii) The stack \( \overline{\mathcal{J}}_{d,g}/\mathbb{G}_m \) is a DM-stack;
(iv) The stack \( \overline{\mathcal{J}}_{d,g}/\mathbb{G}_m \) is proper;
(v) The natural morphism \( \overline{\mathcal{J}}_{d,g}/\mathbb{G}_m \to \overline{\mathcal{M}}_g^{\text{ps}} \) is representable.

Proof. \( (i) \Leftrightarrow (ii) \): the GIT quotient \( H_{d'}/\text{PGL}_{r+1} \) is geometric if and only if every polystable point is also GIT-stable. From Corollaries 11.2 (2) and 11.3 (2), this happens if and only if, given a quasi-p-stable curve \( X \) of genus \( g \) and a line bundle \( L \) on \( X \) of degree \( d' \), \( L \) is stably balanced whenever it is strictly balanced. Recalling Definition 3.6, it is easy to see that this occurs if and only if, given a quasi-p-stable curve \( X \) of genus \( g \), any proper connected subcurve \( Y \subset X \) such that

\[
m_Y = \frac{d'}{2g - 2}\deg_Y \omega_X - \frac{k_Y}{2} \in \mathbb{Z},
\]

is either an exceptional subcurve or the complementary subcurve of an exceptional subcurve. Now the combinatorial proof of \cite[Lemma 6.3]{Cap94} shows that this happens precisely when \( \gcd(d' + 1 - g, 2g - 2) = 1 \). We conclude since \( \gcd(d + 1 - g, 2g - 2) = \gcd(d' + 1 - g, 2g - 2) \) for any \( d \equiv d' \mod 2g - 2 \).

For the remainder of the proof, using Lemma 12.2, we can and will assume that \( 2(2g - 2) < d < \frac{7}{2}(2g - 2) \).

Let us now show that the conditions (iii), (iv) and (v) are equivalent. From Theorem 7.2 and its proof, we get that for any \( [X \subset \mathbb{P}^r] \in H_d \), if we set \( L := O_X(1) \), then we have an exact sequence

\[
0 \to \mathcal{G}_m(\overline{\mathcal{X}})^{-1} \to \overline{\text{Aut}(X,L)} \cong \text{Stab}_{\text{PGL}_{r+1}}([X \subset \mathbb{P}^r]) \to \text{Aut}(\text{ps}(X)),
\]

where \( \gamma(\overline{X}) \) denotes, as usual, the connected components of the non-exceptional subcurve \( \overline{X} \) of \( X \). Note that \( \overline{\text{Aut}(X,L)} \) is the automorphism group of \((X,L) \in (\mathcal{J}_{d,g}^{\text{ps}} \sslash \mathbb{G}_m)(k)\) by the definition of the \( \mathbb{G}_m \)-rigidification.

We claim that each of the conditions (iii), (iv) and (v) is equivalent to the condition (*)

\[
\gamma(\overline{X}) = 1 \text{ for any } [X \subset \mathbb{P}^r] \in H_d \text{ or, equivalently, for any } (X,L) \in (\mathcal{J}_{d,g}^{\text{ps}} \sslash \mathbb{G}_m)(k).
\]

Indeed:

- Condition (iii) is equivalent to (*) by Lemma 3.9.
- Condition (iv) implies (*) because the geometric points of a DM-stack have a finite automorphism group scheme. Conversely, if (*) holds then \( \overline{\text{Aut}(X,L)} \subset \text{Aut}(\text{ps}(X)) \), which is a finite and reduced group scheme since \( \overline{\mathcal{M}}_g^{\text{ps}} \) is a DM-stack if \( g \geq 3 \). Therefore, also \( \overline{\text{Aut}(X,L)} \) is a finite and reduced group scheme, which implies that \( \mathcal{J}_{d,g}^{\text{ps}} \sslash \mathbb{G}_m \) is a DM-stack.
• Condition (vi) is equivalent to the injectivity of the map \( \text{Aut}(X, L) \to \text{Aut}(\text{ps}(X)) \)
for any \((X, L) \in (\mathcal{J}^{\text{ps}}_{d,g}) \sslash G_m(k)\). This is equivalent to condition (*) by the
exact sequence (12.1).

\( \text{By} \Rightarrow \text{iv} \): this follows from the well-known fact that the quotient stack associated
to a geometric projective GIT quotient is a proper stack.

\( \text{iv} \Rightarrow \text{By} \): the automorphism group schemes of the geometric points of a proper
stack are complete group schemes. From (12.1), this is only possible if \( \gamma(\tilde{X}) = 1 \)
for any \((X, L) \in (\mathcal{J}^{\text{ps}}_{d,g}) \sslash G_m(k)\), or equivalently if condition (*) is satisfied. This implies
that (iii) holds by what proved above.

**Remark 12.6.** Notice that even if the existence of strictly semistable points in \( H_d \)
prevents \( \mathcal{J}^{\text{ps}}_{d,g} \sslash G_m \) to be separated, the fact that it can be realized as a GIT quotient
implies that its non-separatedness is, in a sense, quite mild. Indeed, according to the
recent work of Alper, Smyth and van der Wick in [ASvdW], we have that both the stack
\( \mathcal{J}^{\text{ps}}_{d,g} \sslash G_m \) and the morphism \( \mathcal{J}^{\text{ps}}_{d,g} \sslash G_m \to \mathcal{M}^g_{\text{ps}} \) are weakly separated, which roughly
means that sections of \( \mathcal{J}^{\text{ps}}_{d,g} \sslash G_m \) over a punctured disc have unique completions that
are closed in \( \mathcal{J}^{\text{ps}}_{d,g} \sslash G_m \) (see [ASvdW] Definition 2.1) for a precise statement). Since
both \( \mathcal{J}^{\text{ps}}_{d,g} \sslash G_m \) and \( \mathcal{J}^{\text{ps}}_{d,g} \sslash G_m \to \mathcal{M}^g_{\text{ps}} \) are also universally closed, then according to
loc. cit. we get that they are weakly proper.

### 12.2. Existence of the moduli space \( \mathcal{J}^{\text{ps}}_{d,g} \) for the moduli stack \( \mathcal{J}^{\text{ps}}_{d,g} \).

Since from Theorem 12.3 above we have that, for \( 2(2g - 2) < d < \frac{7}{2}(2g - 2) \), the stack \( \mathcal{J}^{\text{ps}}_{d,g} \) is
isomorphic to the quotient stack \( [H_d/GL_{r+1}] \), it follows that there is a natural morphism

\[
\mathcal{J}^{\text{ps}}_{d,g} \to \overline{Q}_{d,g} := H_d/GL_{r+1} \text{ for any } 2(2g - 2) < d < \frac{7}{2}(2g - 2).
\]

From the work of Alper (see [Alp] and [Alp2]), we deduce that the morphism (12.2)
realizes \( \overline{Q}_{d,g} \) as the *adequate* moduli space of \( \mathcal{J}^{\text{ps}}_{d,g} \) and even as its *good* moduli space
if the characteristic of our base field \( k \) is equal to zero or bigger than the order of the
automorphism group of every p-stable curve of genus \( g \) (because in this case, all the
stabilizers are linearly reductive subgroups of \( GL_{r+1} \), as it follows from Lemma 7.1
and the proof of Theorem 7.2). We do not recall here the definition of an adequate
or a good moduli space (we refer to [Alp] and [Alp2] for details). We limit ourselves
to point out some consequences of the fact that (12.2) is an adequate moduli space,
namely:

• The morphism (12.2) is surjective and universally closed (see [Alp2] Thm. 5.3.1);
• The morphism (12.2) is universal for maps from \( \mathcal{J}^{\text{ps}}_{d,g} \) to locally separated algebraic spaces (see [Alp2] Thm. 7.2.1);
• For any algebraically closed field \( k' \) containing \( k \), the morphism (12.2) induces
a bijection

\[
\mathcal{J}^{\text{ps}}_{d,g}(k')/\simeq \to \overline{Q}_{d,g}(k')
\]
where we say that two points $x_1, x_2 \in \mathcal{J}_{d,g}(k')$ are equivalent, and we write $x_1 \sim x_2$, if $\{x_1\} \cap \{x_2\} \neq \emptyset$ in $\mathcal{J}_{d,g} \times_k k'$ (see [Alp2, Thm. 5.3.1]).

Moreover, if the GIT-quotient is geometric, which occurs if and only if $\gcd(d - g + 1, 2g - 2) = 1$ by Proposition [12.5], then it follows from the work of Keel-Mori (see [KeM97]) that actually $Q_{d,g}$ is the coarse moduli space for $\mathcal{J}_{d,g}$, which means that the morphism (12.2) is universal for morphisms of $\mathcal{J}_{d,g}$ into algebraic spaces and moreover that (12.2) induces bijections

$$\mathcal{J}_{d,g}(k') \cong Q_{d,g}(k')$$

for any algebraically close field $k'$ containing $k$.

It follows from the above universal properties of the morphism (12.2) that if $2(2g - 2) < d, d' < \frac{7}{2}(2g - 1)$ are such that $\mathcal{J}_{d,g} \cong \mathcal{J}_{d',g}$ then $Q_{d,g} \cong Q_{d',g}$. In particular, using this fact and the periodicity of $\mathcal{J}_{d,g}$ in $d$ (see Lemma [12.2]), we can now give the following

**Definition 12.7.** For any $d \in \mathbb{Z}$ and any $g \geq 3$, we set $\mathcal{J}_{d,g} := Q_{d,g} = H_d/GL_{r+1}$ for any $d' \equiv d \mod 2g - 2$ such that $2(2g - 2) < d < \frac{7}{2}(2g - 2)$.

Note that for any $d \in \mathbb{Z}$, we have a natural morphism

(12.3)

$$\mathcal{J}_{d,g} \rightarrow \mathcal{J}_{d,g}$$

which is an adequate moduli space in general and a coarse moduli space if (and only if) $\gcd(d - g + 1, 2g - 2) = 1$.

We collect some of the properties of $\mathcal{J}_{d,g}$ in the following proposition.

**Proposition 12.8.** Let $g \geq 3$ and $d \in \mathbb{Z}$. Then:

(i) $\mathcal{J}_{d,g}$ is a normal irreducible projective variety of dimension $4g - 3$. Moreover, if $\text{char}(k) = 0$, then $\mathcal{J}_{d,g}$ has rational singularities, hence it is Cohen-Macauly.

(ii) There exists a surjective map $\Phi_{ps} : \mathcal{J}_{d,g} \rightarrow \mathcal{M}_g$ whose geometric fibers are equi-dimensional of dimension $g$. Moreover $(\Phi_{ps})^{-1}(C) \cong \text{Pic}^d(C)$ for every geometric point $C \in \mathcal{M}_g^0 \subset \mathcal{M}_g$ and, if $\text{char}(k) = 0$, the restriction $\Phi_{ps} : (\Phi_{ps})^{-1}(\mathcal{M}_g^0) \rightarrow (\mathcal{M}_g^0)^0$ is flat.

(iii) The $k$-points of $\mathcal{J}_{d,g}$ are in natural bijection with isomorphism classes of pairs $(X, L)$ where $X$ is a quasi-$p$-stable curve of genus $g$ and $L$ is a strictly balanced line bundle of degree $d$ on $X$.

**Proof.** Clearly, the above properties are preserved by the isomorphisms $\mathcal{J}_{d,g} \cong \mathcal{J}_{d+n(2g-2),d}$ induced by Lemma [12.2]. Therefore, we can assume that $2(2g - 2) < d < \frac{7}{2}(2g - 2)$ so that $\mathcal{J}_{d,g} = Q_{d,g} = H_d/GL_{r+1}$.

Parts (i) and (ii) follow by combining Theorem [9.1] and Proposition [10.9].

Part (iii) follows from Corollary [11.2(2)] together with the fact that in a GIT quotient the geometric points naturally correspond to polystable points. 

\[\square\]
12.3. Fibers of $\Phi^{ps} : \overline{\mathcal{M}}_{g,d}^{ps} \to \overline{\mathcal{M}}_{g}^{ps}$. The aim of this subsection is to give a description of the fiber of $\Phi^{ps}$ over a $p$-stable curve $X \in \overline{\mathcal{M}}_{g}^{ps}$ in terms of the Simpson’s compactified Jacobian of $X$, $\overline{\text{Jac}}_{d}(X)$, that we will now describe.

Let $X$ be a $p$-stable curve of genus $g$. Let $I$ be a coherent sheaf on $X$. We say that $I$ is torsion-free if $\text{supp}(I) = X$ and $I$ does not have non-zero subsheaves whose support has dimension zero. Clearly, a torsion-free sheaf $I$ can be not free only at the nodes and cusps of $X$. We say that $I$ is of rank-1 if $I$ is invertible on a dense open subset of $X$. Each line bundle on $X$ is torsion-free of rank-1.

For each subcurve $Y$ of $X$, let $I_{Y}$ be the restriction $I|_{Y}$ of $I$ to $Y$ modulo torsion. If $I$ is a torsion-free (resp. rank-1) sheaf on $X$, so is $I_{Y}$ on $Y$. We let $\deg_{Y}(I)$ denote the degree of $I_{Y}$, that is, $\deg_{Y}(I) := \chi(I_{Y}) - \chi(\mathcal{O}_{Y})$.

Definition 12.9. Let $X$ be a $p$-stable curve of genus $g \geq 2$ and $I$ a rank-1 torsion-free sheaf of degree $d$ on $X$. We say that $I$ is $\omega_{X}$-semistable if for every proper subcurve $Y$ of $X$, we have that

$$\deg_{Y}(I) \geq d\frac{\deg_{Y}(\omega_{X})}{2g - 2} - \frac{k_{Y}}{2}$$

where $k_{Y}$ denotes, as usual, the length of the subscheme $Y \cap X \setminus Y$ of $X$.

Consider the covariant functor

$$\mathcal{J}_{d,X} : \text{SCH} \to \text{SET}$$

which associates to a scheme $T$ the set of $T$-flat coherent sheaves on $X \times T$ which are rank-1 torsion-free sheaves and $\omega_{X}$-semistable on the geometric fibers $X \times \{t\}$ of the second projection morphism $X \times T \to T$.

From the work on Simpson in [Sim94], we have the following result concerning the co-representability of the moduli functor $\mathcal{J}_{d,X}$ (in the sense of Convention [1.8]).

Fact 12.10 (Simpson). For any integer $d$, there is a projective variety $\overline{\text{Jac}}_{d}(X)$ which co-represents the functor $\mathcal{J}_{d,X}$. The geometric points of $\overline{\text{Jac}}_{d}(X)$ parametrize $S$-equivalence classes of rank-1 torsion-free sheaves on $X$ which are $\omega_{X}$-semistable.

For the definition of $S$-equivalence of sheaves, we refer the interested reader to [Sim94].

In the next Lemma we will describe torsion-free, rank-1 sheaves on $X$ via certain line bundles on quasi-$p$-stable models of $X$.

Lemma 12.11. Let $I$ be a rank-1 torsion-free sheaf on a $p$-stable curve $X$. Then there exists a quasi $p$-stable curve $X'$ and a line bundle $L$ on $X'$ such that

(i) $\text{ps}(X') = X$;
(ii) $\deg_{E}L = 1$ for all exceptional subcurves $E$ of $X'$;
(iii) $I = \pi_{*}(L)$ where $\pi : X' \to \text{ps}(X') = X$ is the natural morphism.

Moreover, $X'$ and $L|_{X'}$ are unique.
(ii) Furthermore, $I$ is $\omega_X$-semistable if and only if $L$ is properly balanced.

Proof. We start by proving (i). Denote by Sing($I$) the set of singular points of $X$ where $I$ is not locally-free. Then the partial normalization of $X$ at Sing($I$), $\nu : Y \rightarrow X$, and the blow-up (in the sense of Corollary 2.11) of $X$ at Sing($I$), $\pi : X' \rightarrow X$, fit into the following commutative diagram

\[
\begin{array}{ccc}
Y' & \xrightarrow{i} & X' \\
\downarrow{\nu} & & \downarrow{\pi} \\
X & & \\
\end{array}
\]

where $i$ represents the natural inclusion morphism.

**CLAIM 1:** There is a unique line bundle $M$ on $Y$ such that $\nu_* (M) = I$.

This is certainly well-known (see [Kas] and the references therein), so we only give a sketch of the proof. Consider the sheaf $\mathcal{E}nd(I)$ of endomorphisms of $I$. Scalar multiplication induces a natural inclusion $\mathcal{O}_X \hookrightarrow \mathcal{E}nd(I)$ and this inclusion makes $\mathcal{E}nd(I)$ into a sheaf of finite commutative $\mathcal{O}_X$-algebras. Moreover, there exists a unique rank-1 torsion-free sheaf $J$ on $\text{Spec}(\mathcal{E}nd(I))$ with the property that $f^*(J) = I$, where $f : \text{Spec}(\mathcal{E}nd(I)) \rightarrow X$ is the natural map (see [Kas, Lemma 3.7]). The claim now follows from the following two facts

(*) $\mathcal{E}nd(I) = \nu_* (\mathcal{O}_Y)$ and $J$ is a line bundle.

Indeed, if (*) is true then $Y = \text{Spec}(\mathcal{E}nd(I))$ and we can take $M = J$. Property (*) is a local property, i.e. it is enough to prove that for any $p \in X$ with $\nu^{-1}(p) = \{ q_1, \ldots, q_r \} \subset Y$, we have that

(**) \[
\begin{cases}
\text{End}(I_p) \cong \bigoplus_{i} \mathcal{O}_{Y, q_i} \text{ as } \mathcal{O}_{X, p} \text{-modules}, \\
I_p \text{ is a free module over } \text{End}(I_p).
\end{cases}
\]

If $p \notin \text{Sing}(I)$ then (**) is clear: $\nu$ is an isomorphism above $p$ and $I_p = \mathcal{O}_{X, p}$ is a free module over $\text{End}(I_p) = \mathcal{O}_{X, p}$. If $p \in \text{Sing}(I)$ (hence $p$ is a node or a cusp of $X$), then it is well-known (see e.g. [Kas, Prop. 5.7]) that $I_p$ is isomorphic to the maximal ideal $m_p$ of $\mathcal{O}_{X, p}$, $\text{End}(m_p)$ is isomorphic to the normalization $\mathcal{O}_{\tilde{X}, p}$ of $\mathcal{O}_{X, p}$ and $m_p$ is a free module over $\mathcal{O}_{\tilde{X}, p}$. Property (**) is proved also in this case, q.e.d.

Let now $E := E_1 \cup \cdots \cup E_n$ be the union of the exceptional subcurves of $X'$. Then we can find a line bundle $L$ on $X'$ such that $L|_Y = M$ and $\deg_{E_i} L = 1$, $i = 1, \ldots, n$.

The proof of part (i) is now implied by the following

**CLAIM 2:** The natural restriction morphism

\[\text{res} : \pi_* L \rightarrow \nu_*(L|_Y) = \nu_*(M)\]

is an isomorphism of sheaves on $X$.

We must show that for every open subset $U \subseteq X$, the restriction map

\[\text{res} : L(\pi^{-1}(U)) \rightarrow L|_Y(\nu^{-1}(U)) = L|_Y(\nu^{-1}(U))\]
is an isomorphism of $\mathcal{O}_X(U)$-modules. Suppose for simplicity that $U$ contains a unique point $p \in \text{Sing}(I)$ and let $E_0$ be its pre-image under $\pi$. Then every section $s \in L(\pi^{-1}(U))$ can be seen as a couple $(\text{res}(s), s|_{E_0})$ plus a compatibility condition. In the case when $p$ is a node, this condition just says that the value of $s|_{E_0}$ in $\pi^{-1}(p)$ must coincide with the values of $\text{res}(s)$ on those points. In the case when $p$ is a cusp, the condition says that the value of $s|_{E_0}$ on the pre-image $\nu^{-1}(p)$ of the cusp must coincide with the value of $\text{res}(s)$ on that point and the same for their derivatives at that point. We conclude using the fact that a section $s \in H^0(P^1, \mathcal{O}_{P^1}(1))$ is determined either by its value at two distinct points of $P^1$ or by its value at one point together with its derivative at that point. The general case, where $U$ contains several points of $\text{Sing}(I)$, is dealt with similarly.

Now to prove (ii) we start by recalling that if $I$ is a torsion-free rank-1 sheaf on a $p$-stable curve $X$ and if $Z = Z_1 \cup \cdots \cup Z_r$ is a subcurve of $X$ with $Z_i$ irreducible, then

$$\deg_Z(I) = \sum_{i=1}^r \deg_{Z_i}(I) + i_Z$$

where $i_Z$ is the number of points in $\text{Sing}(I)$ lying in two different irreducible components of $Z$ (see for instance [MV, Lemma 4.4]). Assume first that $I$ is $\omega_X$-semistable. Let $Z'$ be a subcurve of $X'$ and denote by $Z$ the image of $Z'$ under $\pi$. Then by (i) and (12.7) we have that

$$\deg_{Z'} L = \deg_{\pi(Z)} I + e_{Z'}$$

where $e_{Z'}$ is the number of exceptional subcurves in $Z'$ meeting the rest of $Z'$ in less than 2 points. Since by hypothesis $I$ is $\omega_X$-semistable we get that

$$\deg_{Z'} L \geq d \frac{\omega_{Z'}}{2g-2} - \frac{k_{Z'}}{2} + e_{Z'}.$$  

Let $e^0_{Z'}$ (resp. $e^1_{Z'}$) be the number of exceptional subcurves in $Z'$ meeting the rest of $Z'$ in exactly 0 (resp. 1) points. Then $k_{Z'} = k_Z + 2e^0_{Z'}$, so from (12.8) we get that

$$\deg_{Z'} L \geq d \frac{\omega_{Z'}}{2g-2} - \frac{k_{Z'}}{2} + e^1_{Z'},$$

which shows that $L$ is balanced on $X'$. Since, by construction, the degree of $L$ on each exceptional subcurve of $X'$ is equal to 1, we get that $L$ is properly balanced.

Now, suppose that $L$ is properly balanced and let us see that $I$ is $\omega_X$-semistable. Let $Z$ be a proper subcurve of $X$. Then by the above discussion it is clear that since $L$ is properly balanced there is a subcurve $Z' \subseteq X'$ such that $\pi(Z') = Z$ and $\deg_{Z'} L = \deg_Z I$. It immediately follows that $I$ is $\omega_X$-semistable. \qed

The following theorem yields a modular description of the fibers of the map $\Phi^{ps}: \mathcal{M}_{d,g}^{ps} \to \mathcal{M}_{g}^{ps}$. In the proof, we will use the terminology recalled in Convention 1.8.

**Theorem 12.12.** Let $g \geq 3$ and $d \in \mathbb{Z}$. Assume that $\text{char}(k) = 0$ or that $\text{char}(k) = p > 0$ is bigger than the order of the automorphism group of every $p$-stable curve of
genus $g$. Then the fiber $(\Phi^\text{ps})^{-1}(X)$ of the morphism $\Phi^\text{ps} : \mathcal{J}^\text{ps}_{d,g} \to \mathcal{M}^\text{ps}_g$ over $X \in \mathcal{M}^\text{ps}_g$ is isomorphic to $\text{Jac}_d(X)/\text{Aut}(X)$.

Proof. Consider the contravariant functor $\mathcal{F}_{d,X} : \text{SCH} \to \text{SET}$ which associates to a scheme $S$ the set of isomorphism classes of pairs given by a family of quasi-$p$-stable curves $f : \mathcal{Y} \to S$ such that the $p$-stable reduction of all geometric fibers of $f$ is isomorphic to $X$ together with a line bundle $\mathcal{L}$ over $\mathcal{Y}$ such that the restriction of $\mathcal{L}$ to the geometric fibers of $f$ is properly balanced of degree $d$.

CLAIM 1: $(\Phi^\text{ps})^{-1}(X)$ co-represents the functor $\mathcal{F}_{d,X}$ if $\text{char}(k) = 0$ or if $\text{char}(k) = p > 0$ is bigger than the order of the automorphism group of every $p$-stable curve of genus $g$.

Using Lemma 12.2 we can assume that $2(2g-2) < d < \frac{7}{2}(2g-2)$, in which case $\mathcal{J}^\text{ps}_{d,g}$ is isomorphic to the GIT quotient $\mathcal{Q}_{d,g} = H_d/\text{GL}_{d+1}$ by Definition 12.7. Under our assumptions on the characteristic of $k$, the stabilizers of the action of $\text{GL}_{d+1}$ on $H_d$ are linearly reductive as it follows from Lemma 7.1 and the proof of Theorem 7.2. This implies that the GIT quotient $H_d/\text{GL}_{d+1}$ is a universal categorical quotient (in the sense of [MPK94, Chap. 0, Def. 0.7]) and the result now follows as in [CMKV, Fact 2.8], q.e.d.

Consider now the contravariant functor $\mathcal{J}_{d,X} : \text{SCH} \to \text{SET}$ that associates to a scheme $S$ the set of isomorphism classes of pairs given by a family of $p$-stable curves $f : \mathcal{X} \to S$ with all geometric fibers isomorphic to $X$ together with an $S$-flat coherent sheaf $\mathcal{I}$ on $\mathcal{X}$ such that the restriction of $\mathcal{I}$ to any geometric fiber $\mathcal{X}_s$ of $f$ is rank-1, torsion-free and $\omega_{\mathcal{X}_s}$-semistable.

CLAIM 2: $\text{Jac}_d(X)/\text{Aut}(X)$ co-represents the functor $\mathcal{J}_{d,X}$.

Let us first define a natural transformation $\Phi : \mathcal{J}_{d,X} \to \text{Hom}(-, \text{Jac}_d(X)/\text{Aut}(X))$. Consider a section $(f : \mathcal{X} \to S, \mathcal{I}) \in \mathcal{J}_{d,X}(S)$ and let $\{U_i \to S\}$ be an étale cover of $S$ that trivializes the family $f$, i.e. such that there is an isomorphism $\alpha_i : \mathcal{X}|_{U_i} \cong \times U_i$ over $U_i$. Therefore we get an element $(X \times U_i \to U_i, (\alpha_i)_*(\mathcal{I})) \in \mathcal{J}_{d,X}(U_i)$, which, since $\text{Jac}_d(X)$ co-represents $\mathcal{J}_{d,X}$ by Fact 12.10, determines a morphism $\psi_i : U_i \to \text{Jac}_d(X)$. Consider now two open subsets $U_i$ and $U_j$ of the above étale cover and set $U_{ij} = U_i \times_S U_j$. The restrictions of the sheaves $(\alpha_i)_*(\mathcal{I})$ and $(\alpha_j)_*(\mathcal{I})$ to $X \times U_{ij}$ differ by an automorphism of $X$. This is equivalent to say that the restrictions of the morphism $\psi_i$ and $\psi_j$ to $U_{ij}$ differ by an automorphism of $X$. Therefore, the compositions $\phi_i : U_i \to \text{Jac}_d(X) \to \text{Jac}_d(X)/\text{Aut}(X)$ agree on the pairwise intersections $U_{ij}$ and hence glue together to give a map $\phi : S \to \text{Jac}_d(X)/\text{Aut}(X)$ such that the restriction of $\phi$ to $U_i$ coincide with $\psi_i$. By defining $\Phi(f : \mathcal{X} \to S, \mathcal{I}) := \phi$, we get the required natural transformation of functors $\Phi$. The fact that $\Phi$ is universal with respect to natural transformations from $\mathcal{J}_{d,X}$ to functors of points of schemes follows now easily from Fact 12.10 we leave the details to the reader.

CLAIM 3: There is a local isomorphism $\mathcal{F}_{d,X} \to \mathcal{J}_{d,X}$. 


Let now \( (f : Y \to S, \mathcal{L}) \in \mathcal{T}_{d,X}(S) \) and consider the p-stable reduction of \( f \) (see Definition 2.12).

\[
\begin{array}{ccc}
Y & \xrightarrow{\pi} & \text{ps}(Y) \\
\downarrow f & & \downarrow \text{ps}(f) \\
S & & S
\end{array}
\]

Together with the sheaf \( \pi_*(\mathcal{L}) \) on \( \text{ps}(Y) \) whose restriction to the geometric fibers is torsion-free and rank-1. By sending \( (f : Y \to S, \mathcal{L}) \) into \( (\text{ps}(f) : \text{ps}(Y) \to S, \pi_*(\mathcal{L})) \), we get a natural transformation of functors \( \mathcal{T}_{d,X} \to \tilde{J}_{d,X} \). Lemma 12.11 implies that this natural transformation of functors is a local isomorphism, q.e.d.

The proof of the Theorem now follows by combining the above three Claims and using the fact (recalled in Convention 1.8) that two locally isomorphic functors are co-represented by the same scheme.

\[\square\]

It would be interesting to know if the above Theorem 12.12 is true in any characteristic. This would follow if one could prove that \( J^{ps}_{d,g} \) is a good moduli scheme for the stack \( \tilde{J}^{ps}_{d,g} \) in the sense of Alper (\[Alp\]).

### 13. Appendix: Positivity properties of balanced line bundles

The aim of this Appendix is to investigate positivity properties of balanced line bundles of sufficiently high degree on reduced Gorenstein curves. The results obtained here are applied in this paper only for quasi-wp-stable curves. However, we decided to present these results in the Gorenstein case for two reasons: firstly, we think that these results are interesting in their own (in particular we will generalize several results of \[Cap10\] and \[Mel11, Sec. 5\] in the case of nodal curves); secondly, our proof extends without any modifications to the Gorenstein case.

So, throughout this Appendix, we let \( X \) be a connected reduced Gorenstein curve of genus \( g \geq 2 \) and \( L \) be a balanced line bundle on \( X \) of degree \( d \), i.e., a line bundle \( L \) of degree \( d \) satisfying the basic inequality

\[
\left| \deg_Z L - \frac{d}{2g-2} \deg_Z \omega_X \right| \leq \frac{k_Z}{2},
\]

for any (connected) subcurve \( Z \subseteq X \), where \( k_Z \) is as usual the length of the scheme-theoretic intersection of \( Z \) with the complementary subcurve \( Z^c := X \setminus Z \) and \( \omega_X \) is the dualizing invertible (since \( X \) is Gorenstein) sheaf.

The following definitions are natural generalizations to the Gorenstein case of the familiar concepts for nodal curves.

**Definition 13.1.** Let \( X \) be a connected reduced Gorenstein curve of genus \( g \geq 2 \). We say that
(i) $X$ is $G$-semistable\footnote{The letter G stands for Gorenstein to suggest that these notions are the natural generalizations of the usual notions from nodal to Gorenstein curves.} if $\omega_X$ is nef, i.e. $\deg_Z \omega_X \geq 0$ for any (connected) subcurve $Z$. The connected subcurves $Z$ such that $\deg_Z \omega_X = 0$ are called exceptional subcurves.

(ii) $X$ is $G$-quasistable if $X$ is $G$-semistable and every exceptional subcurve $Z$ is isomorphic to $\mathbb{P}^1$.

(iii) $X$ is $G$-stable if $\omega_X$ is ample, i.e. $\deg_Z \omega_X > 0$ for any (connected) subcurve $Z$.

Note that $G$-semistable (resp. $G$-stable) curves are called semi-canonically positive (resp. canonically positive) in \cite[Def. 0.1]{Cat82}. The terminology $G$-stable was introduced in \cite[Def. 2.2]{CCE08}. We refer to \cite[Sec. 1]{Cat82} for more details on $G$-stable and $G$-semistable curves.

Observe also that quasi-wp-stable, quasi-p-stable and quasi-stable curves are $G$-quasistable; similarly wp-stable, p-stable and stable curves are $G$-stable.

**Remark 13.2.** Given a subcurve $i : Z \subseteq X$ with complementary subcurve $Z^c$, consider the exact sequence

$$0 \to \omega_X \otimes I_{Z^c} \to \omega_X \to (\omega_X)|_{Z^c} \to 0,$$

where $I_{Z^c}$ is the ideal sheaf of $Z^c$ in $X$. By the definition of the dualizing sheaf $\omega_Z$ of $Z$, it is easy to check that $i_*(\omega_Z) = \omega_X \otimes I_{Z^c}$ which, by restricting to $Z$, gives

$$\omega_Z = (\omega_X \otimes I_{Z^c})|_Z = (\omega_X)|_Z \otimes I_{Z \cap Z^c}/Z,$$

where $I_{Z \cap Z^c}/Z$ is the ideal sheaf of the scheme theoretic intersection $Z \cap Z^c$ seen as a subscheme of $Z$. By taking degrees, we get the adjunction formula

$$\deg_Z \omega_X = 2g_Z - 2 + k_Z.$$

Using the above adjunction formula and recalling that $g_Z \geq 0$ if $Z$ is connected, it is easy to see that:

(i) $X$ is $G$-semistable if and only if for any connected subcurve $Z$ such that $g_Z = 0$ we have that $k_Z \geq 2$.

(ii) $X$ is $G$-stable if and only if for any connected subcurve $Z$ such that $g_Z = 0$ we have that $k_Z \geq 3$.

Our first result says when a balanced line bundle of sufficiently high degree is nef or ample.

**Proposition 13.3.** Let $X$ be a connected reduced Gorenstein curve of genus $g \geq 2$ and let $L$ be a balanced line bundle on $X$ of degree $d$. The following is true:
\( (i) \) If \( d > \frac{1}{2} (2g - 2) = g - 1 \) then \( L \) is nef if and only if \( X \) is \( G \)-semistable and for every exceptional subcurve \( Z \) it holds that \( \deg_Z L = 0 \) or 1.

\( (ii) \) If \( d > \frac{3}{2} (2g - 2) = 3(g - 1) \) then \( L \) is ample if and only if \( X \) is \( G \)-quasistable and for every exceptional subcurve \( Z \) it holds that \( \deg_Z L = 1 \).

**Proof.** Let us first prove part \((i)\). Let \( Z \subseteq X \) be a connected subcurve of \( X \). If \( Z = X \) then \( \deg_Z L = \deg L = d > (g - 1) > 0 \) by assumption. So we can assume that \( Z \subsetneq X \).

Notice that, since \( X \) is connected, this implies that \( k_Z \geq 1 \).

If \( \deg_Z \omega_X = 2g_Z - 2 + k_Z > 0 \) then, using the basic inequality (13.1) and the assumption \( d > \frac{1}{2} (2g - 2) \), we get

\[
\deg_Z L \geq d \cdot \frac{2g_Z - 2 + k_Z}{2g - 2} - \frac{k_Z}{2} > \frac{2g_Z - 2 + k_Z}{2} - \frac{k_Z}{2} \geq \begin{cases} 
0 & \text{if } g_Z \geq 1, \\
-1 & \text{if } g_Z = 0,
\end{cases}
\]

hence \( \deg_Z L \geq 0 \). If \( g_Z = 0 \) and \( k_Z = 1 \) then, using the basic inequality and the assumption on \( d \), we get that

\[
\deg_Z L \leq \frac{d}{2g - 2} (-1) + \frac{1}{2} < 0.
\]

Therefore if \( L \) is nef then \( X \) must be \( G \)-semistable. Finally, if \( Z \) is any exceptional subcurve of \( X \), then the basic inequality gives

(13.3) \quad \abs{\deg_Z L} \leq 1,

from which we deduce that if \( L \) is nef then \( \deg_Z L = 0 \) or 1. Conversely, it is also clear that if \( X \) is \( G \)-semistable and \( \deg_Z L = 0 \) or 1 for every exceptional subcurve \( Z \) of \( X \) then \( L \) is nef.

Let us now prove part \((ii)\). Let \( Z \subseteq X \) be a connected subcurve of \( X \). If \( Z = X \) then \( \deg_Z L = \deg L = d > 3(g - 1) > 0 \) by assumption. So we can assume that \( Z \subsetneq X \).

Notice that, since \( X \) is connected, this implies that \( k_Z \geq 1 \).

If \( \deg_Z \omega_X = 2g_Z - 2 + k_Z > 0 \) then, using the basic inequality (13.1) and the inequality \( d > \frac{3}{2} (2g - 2) \), we get

\[
\deg_Z L \geq d \cdot \frac{2g_Z - 2 + k_Z}{2g - 2} - \frac{k_Z}{2} > \frac{3(2g_Z - 2 + k_Z)}{2} - \frac{k_Z}{2} \geq \begin{cases} 
\frac{k_Z}{2} & \text{if } g_Z \geq 1, \\
\frac{2k_Z - 6}{2} & \geq 0 & \text{if } g_Z = 0 \text{ and } k_Z \geq 3,
\end{cases}
\]

hence \( \deg_Z L > 0 \). From part \((i)\) and equation (13.3), we get that if \( L \) is ample then \( X \) is \( G \)-semistable and for every exceptional subcurve \( Z \) we have that \( \deg_Z L = 1 \). Note that every exceptional subcurve \( Z \) of \( X \) is a chain of \( \mathbb{P}^1 \). Assume that this chain has length \( l \geq 2 \) and denote by \( W_i \) (for \( i = 1, \ldots, l \)) the irreducible components of \( Z \). Then each of the \( W_i \)'s is an exceptional subcurve of \( X \). Therefore, the same inequality as before gives that if \( L \) is ample then \( \deg_{W_i} L = 1 \). This is a contradiction since \( 1 = \deg_Z L = \sum_i \deg_{W_i} L = l > 1 \). Hence \( Z \cong \mathbb{P}^1 \) and \( X \) is \( G \)-quasistable. Conversely, it is clear that if \( X \) is \( G \)-semistable and \( \deg_Z L = 1 \) for every exceptional subcurve \( Z \) of \( X \) then \( L \) is ample.

\( \square \)
We next investigate when a balanced line bundle on a reduced Gorenstein curve is non-special, globally generated, very ample or normally generated. To this aim, we will use the following criteria, due to Catanese-Franciosi [CF96], Catanese-Franciosi-Hulek-Reid [CFHR99] and Franciosi-Tenni [FT] (see also [Fra04] and [Fra07]) which generalize the classical criteria for smooth curves.

**Fact 13.4.** ([CF96], [CFHR99], [FT]) Let $L$ be a line bundle on a reduced Gorenstein curve $X$. Then the following holds:

(i) If $\deg_Z L > 2g_Z - 2$ for all (connected) subcurves $Z$ of $X$, then $L$ is non-special, i.e., $H^1(X, L) = 0$.

(ii) If $\deg_Z L > 2g_Z - 1$ for all (connected) subcurves $Z$ of $X$, then $L$ is globally generated;

(iii) If $\deg_Z L > 2g_Z$ for all (connected) subcurves $Z$ of $X$, then $L$ is very ample.

(iv) If $\deg_Z L > 2g_Z$ for all (connected) subcurves $Z$ of $X$, then $L$ is normally generated, i.e. the multiplication maps

$$\rho_k : H^0(X, L)^\otimes k \to H^0(X, L^k)$$

are surjective for every $k \geq 2$.

Recall that if $Z$ is a subcurve that is a disjoint union of two subcurves $Z_1$ and $Z_2$ then $g_Z = g_{Z_1} + g_{Z_2} - 1$. From this, it is easily checked that if the numerical assumptions of (i), (ii), (iii) and (iv) are satisfied for all connected subcurves $Z$ then they are satisfied for all subcurves $Z$. With this in mind, part (i) follows from [CF96, Lemma 2.1]. Note that in loc. cit. this result is only stated for a curve $C$ embedded in a smooth surface; however, a closer inspection of the proof reveals that the same result is true for any Gorenstein curve $C$. Parts (ii) and (iii) follow from [CFHR99, Thm. 1.1]. Part (iv) follows from [FT, Thm. 4.2], which generalizes the previous results of Franciosi (see [Fra04, Thm. B] and [Fra07, Thm. 1]) for reduced curves with locally planar singularities.

Using the above criteria, we can now investigate when balanced line bundles are non-special, globally generated, very ample or normally generated.

**Theorem 13.5.** Let $L$ be a balanced line bundle of degree $d$ on a connected reduced Gorenstein curve $X$ of genus $g \geq 2$. Then the following properties hold:

(i) If $X$ is $G$-semistable and $d > 2g - 2$ then $L$ is non-special.

(ii) Assume that $L$ is nef. If $d > \frac{5}{2}(2g - 2) = 3(g - 1)$ then $L$ is globally generated.

(iii) Assume that $L$ is ample. Then:

(a) If $d > \frac{5}{2}(2g - 2) = 5(g - 1)$ then $L$ is very ample and normally generated.

(b) If $d > \max\{\frac{3}{2}(2g - 2) = 3(g - 1), 2g\}$ and $X$ does not have elliptic tails (i.e., connected subcurves $Z$ such that $g_Z = 1$ and $k_Z = 1$) then $L$ is very ample and normally generated.

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Proof. In order to prove part (i), we apply Fact 13.3(iii). Let $Z \subseteq X$ be a connected subcurve. If $Z = X$ then $\deg_z L = d > 2g - 2$ by assumption. Assume now that $Z \subsetneq X$ (hence that $k_Z \geq 1$). Since $X$ is G-semistable, we have that $\deg_z(\omega_X) = 2g - 2 + k_Z \geq 0$. If $\deg_z(\omega_X) > 0$ then the basic inequality (13.1) together with the hypothesis on $d$ gives that

$$\deg_z L \geq \frac{d}{2g - 2}(2g - 2 + k_Z) - \frac{k_Z}{2} > 2g - 2 + \frac{k_Z}{2} > 2g - 2.$$  

If $\deg_z(\omega_X) = 0$ (which happens if and only if $Z$ is exceptional, i.e., $g_Z = 0$ and $k_Z = 2$) then the basic inequality gives that

$$\deg_z L \geq \frac{d}{2g - 2}(2g - 2 + k_Z) - \frac{k_Z}{2} = -1 > -2 = 2g - 2.$$  

In order to prove part (ii), we apply Fact 13.4(ii). Let $Z \subseteq X$ be a connected subcurve. If $Z = X$ then we have that $\deg_z L = d > 3(g - 1) \geq 2g - 1$ by the assumption on $d$. Assume now that $Z \subsetneq X$ (hence that $k_Z \geq 1$). If $g_Z = 0$ then $\deg_z L > -1 = 2g - 1$ since $L$ is nef. Therefore, we can assume that $g_Z \geq 1$. By applying the basic inequality (13.1) and using our assumption on $d$, we get that

$$\deg_z L \geq \frac{d}{2g - 2}(2g - 2 + k_Z) - \frac{k_Z}{2} > \frac{3}{2}(2g - 2 + k_Z) - \frac{k_Z}{2} = 3(g - 1) + k_Z \geq 2g - 1.$$  

In order to prove parts (iiiA) and (iiiB), we apply Facts 13.3(iii) and 13.3(iv). Let $Z \subseteq X$ be a connected subcurve. If $Z = X$ then, in each of the cases (iiiA) and (iiiB), we have that $\deg_z L = d > 2g$ by the assumption on $d$ (note that $5(g - 1) > 2g$ since $g \geq 2$). Assume now that $Z \subsetneq X$ (hence that $k_Z \geq 1$). If $g_Z = 0$ then $\deg_z L > 0 = 2g$ since $L$ is ample. Therefore, we can assume that $g_Z \geq 1$.

In the first case (iiiA), by applying the basic inequality (13.1) and the numerical assumption on $d$, we get that

$$\deg_z L \geq \frac{d}{2g - 2}(2g - 2 + k_Z) - \frac{k_Z}{2} > \frac{5}{2}(2g - 2 + k_Z) - \frac{k_Z}{2} = 5(g - 1) + 2k_Z \geq 2g.$$  

In the second case (iiiB), from the basic inequality (13.1) and the numerical assumption on $d$, we get that

$$\deg_z L \geq \frac{d}{2g - 2}(2g - 2 + k_Z) - \frac{k_Z}{2} > \frac{3}{2}(2g - 2 + k_Z) - \frac{k_Z}{2} = 3(g - 1) + k_Z \geq 2g,$$

where in the last inequality we used that $g_Z, k_Z \geq 1$ and $(g_Z, k_Z) \neq (1, 1)$ because $X$ does not contain elliptic tails.

\[ \square \]

Remark 13.6. Theorem 13.5(ii) recovers [Cap10, Thm. 2.3(i)] in the case of nodal curves. Theorem 13.5(ii) combined with Proposition 13.3(iii) recovers and improves [Cap10, Thm. 2.3(iii)] in the case of nodal curves. Theorem 13.5(iii) improves [Mel11, Cor. 5.11] in the case of nodal curves. See also [Bal09], where the author gives some criteria for the global generation and very ampleness of balanced line bundles on quasi-stable curves.
The previous results can be applied to study the positivity properties of powers of the canonical line bundle on a reduced Gorenstein curve, which is clearly a balanced line bundle.

**Corollary 13.7.** Let $X$ be a connected reduced Gorenstein curve of genus $g \geq 2$. Then the following holds:

(i) If $X$ is $G$-semistable then $\omega_X^i$ is non-special and globally generated for all $i \geq 2$;
(ii) If $X$ is $G$-stable then $\omega_X^i$ is very ample for all $i \geq 3$;
(iii) If $X$ is $G$-quasistable then $\omega_X^i$ is normally generated for all $i \geq 3$.

**Proof.** Part (i) follows from Theorem 13.5(i) and Theorem 13.5(ii).

Part (ii) follows from Theorem 13.5(iiia).

Let us now prove part (iii). If $X$ is $G$-stable, then this follows from Theorem 13.5(iiia). In the general case, since $\omega_X^i$ is globally generated by part (i), it defines a morphism

$$q : X \to \mathbb{P} := \mathbb{P}(H^0(X, \omega_X^i)^\vee),$$

whose image we denote by $Y := q(X)$. Since $X$ is $G$-quasistable, the degree of $\omega_X^i$ on a connected subcurve $Z$ of $X$ is zero if and only if $Z = E$ is an exceptional subcurve, i.e., if $E \cong \mathbb{P}^1$ and $k_E = 2$. The map $q$ will contract such an exceptional subcurve $E$ to a node if $E$ meets the complementary subcurve $E^c$ in two distinct points and to a cusp if $E$ meets $E^c$ in one point with multiplicity two. Moreover, using Fact 13.4(iii), it is easy to check that $\omega_X$ is very ample on $X \setminus \cup E$, where the union runs over all exceptional subcurves $E$ of $X$. We deduce that $Y$ is $G$-stable. By what proved above, $\omega_X^i$ is normally generated. Clearly $q^* \omega_Y^i = \omega_X^i$ and moreover, since $q$ has connected fibers, we have that $q_* \mathcal{O}_X = \mathcal{O}_Y$. This implies that $H^0(X, (\omega_X^i)^k) = H^0(Y, (\omega_Y^i)^k)$ from which we deduce that $\omega_X^i$ is normally generated. \qed

**Remark 13.8.** Part (i) of the above Corollary 13.7 recovers [Cat82, Thm. A and p. 68], while part (ii) recovers [Cat82, Thm B]. Part (iii) was proved for nodal curves in [Mel11, Cor. 5.9].

A closer inspection of the proof reveals that parts (ii) and (iii) continue to hold for $\omega_X^2$ if, moreover, $g \geq 3$ and $X$ does not have elliptic tails (see also [Cat82, Thm. C] and [Fra04, Thm. C]).

Let us end this Appendix by mentioning that it is possible to generalize the above results in order to prove that a balanced line bundle of sufficiently high degree is $k$-very ample in the sense of Beltrametti-Francia-Sommese ([BFS89]). Recall first the definition of $k$-very ampleness.

**Definition 13.9.** Let $L$ be a line bundle on $X$ and let $k \geq 0$ be an integer. We say that $L$ is $k$-very ample if for any 0-dimensional subscheme $S \subset X$ of length at most $k + 1$ we have that the natural restriction map

$$H^0(X, L) \to H^0(S, L|_S)$$
is surjective. In particular 0-very ample is equivalent to being globally generated and 1-very ample is equivalent to being very ample.

The proof of the following Theorem is very similar to the proof of the Theorem [13.5 above, using again [CFHR99] Thm. 1.1], and therefore we omit it.

**Theorem 13.10.** Let \( k \geq 2 \) and assume that \( X \) is \( G \)-stable. Then:

(i) If \( d > \frac{2k+3}{2}(2g-2) = (2k + 3)(g - 1) \) then \( L \) is \( k \)-very ample.

(ii) If \( d > \frac{2k+1}{2}(2g-2) = (2k + 1)(g - 1) \) and \( X \) does not have elliptic tails then \( L \) is \( k \)-very ample.

**References**

[ACV01] D. Abramovich, A. Corti, A. Vistoli: *Twisted bundles and admissible covers.* Comm. Algebra 31 (2003), no. 8, 3547–3618.

[Ale04] V. Alexeev: *Compactified Jacobians and Torelli map.* Publ. RIMS, Kyoto Univ. 40 (2004), 1241–1265.

[Alp] J. Alper: *Good moduli spaces for Artin stacks.* Preprint available at arXiv:0804.2242v2.

[Alp2] J. Alper: *Adequate moduli spaces and geometrically reductive group schemes.* Preprint available at arXiv:1005.2398.

[AFS] J. Alper, M. Fedorchuk, D. Smyth: *Finite Hilbert stability of (bi)canonical curves.* Preprint available at arXiv:1109.4986.

[AH] J. Alper, D. Hyeon: *GIT construction of log canonical models of \( \overline{M}_{g} \).* Preprint available at arXiv:1109:2173.

[ASvdW] J. Alper, D. I. Smyth, F. van der Wyck: *Weakly proper moduli stacks of curves.* Preprint available at arXiv:1012.0538.

[Bal09] E. Ballico: *Very ampleness of balanced line bundles on stable curves.* Riv. Mat. Univ. Parma (8) 2 (2009), 81–90.

[BFS89] M. Beltrametti, P. Francia, A. J. Sommese: *On Reider’s method and higher order embeddings.* Duke Math. J. 58 (1989), 425–439.

[BFV11] G. Bini, C. Fontanari, F. Viviani: *On the birational geometry of the universal Picard variety.* International Mathematics Research Notices 2011, article ID rnq188, doi:10.1093/imrn/rnq188.

[BLR90] S. Bosch, W. Lütkebohmert, M. Raynaud: *Néron models.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Vol. 21. Springer-Verlag, Berlin, 1990.

[Bou87] J.F. Boutot: *Singularités rationnelles et quotients par les groupes réductifs.* Invent. Math. 88 (1987), 65–68.

[Cap94] L. Caporaso: *A compactification of the universal Picard variety over the moduli space of stable curves.* J. Amer. Math. Soc. 7 (1994), 589–660.

[Cap05] L. Caporaso: *Néron models and compactified Picard schemes over the moduli stack of stable curves.* Amer. J. Math. 130 (2008), no. 1, 1–47.

[Cap10] L. Caporaso: *Linear series on semistable curves.* International Mathematics Research Notices (2010), doi: 10.1093/imrn/rnq188.

[CCE08] L. Caporaso; J. Coelho; E. Esteves: *Abel maps of Gorenstein curves.* Rend. Circ. Mat. Palermo (2) 57 (2008), no. 1, 33–59.

[CMKV] S. Casalaina-Martin, J. Kass, F. Viviani: *The Local Structure of Compactified Jacobians: Deformation Theory.* Preprint available at arXiv:1107.4166.
[Cat82] F. Catanese: Pluricanonical-Gorenstein-curves. Enumerative geometry and classical algebraic geometry (Nice, 1981), pp. 51–95, Progr. Math. Vol. 24, Birkhäuser Boston, Boston, MA, 1982.

[CF96] F. Catanese, M. Franciosi: Divisors of small genus on algebraic surfaces and projective embeddings. Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993), 109–140, Israel Math. Conf. Proc., 9, Bar-Ilan Univ., Ramat Gan, 1996.

[CFHR99] F. Catanese; M. Franciosi; K. Hulek; M. Reid: Embeddings of curves and surfaces. Nagoya Math. J. 154 (1999), 185–220.

[Del03] I. Dolgachev: Lectures on invariant theory. London Mathematical Society Lecture Note Series Vol. 296. Cambridge University Press, Cambridge, 2003.

[DH98] I. V. Dolgachev, Y. Hu: Variation of geometric invariant theory quotients. Inst. Hautes Études Sci. Publ. Math. 87 (1998), 5–56. With an Appendix by Nicolas Ressayre.

[DM69] P. Deligne, D. Mumford: The irreducibility of the space of curves of given genus. Inst. Hautes Études Sci. Publ. Math. 36 (1969), 75–109.

[Fel] F. Felici: GIT for curves of low degree. In progress.

[FS11] M. Fedorchuk, D. I. Smyth: Alternate compactifications of moduli space of curves. To appear in Handbook of Moduli (G. Farkas and I. Morrison, editors), available at arXiv:1012.0329.

[Fra04] M. Franciosi: Adjacent divisors on algebraic curves (with an Appendix of F. Catanese). Adv. Math. 186 (2004), 317–333.

[Fra07] M. Franciosi: Arithmetically Cohen-Macaulay algebraic curves. Int. J. Pure Appl. Math. 34 (2007), 69–86.

[FT] M. Franciosi, E. Tenni: The canonical ring of a 3-connected curve. Preprint available at arXiv:1107.5535.

[Gie82] D. Gieseker: Lectures on moduli of curves. Tata Institute of Fundamental Research Lectures on Mathematics and Physics, Volume 69. Tata Institute of Fundamental Research, Bombay, 1982.

[Hal] J. Hall: Moduli of Singular Curves. Preprint available at arXiv:1011.6007v1.

[HM98] J. Harris, I. Morrison: Moduli of curves. Graduate text in mathematics 187. Springer-Verlag, New York-Heidelberg, 1998.

[Har77] R. Hartshorne: Algebraic geometry. Graduate Texts in Mathematics 52. Springer-Verlag, New York-Heidelberg, 1977.

[HH09] B. Hassett, D. Hyeon: Log canonical models for the moduli space of curves: first divisorial contraction. Trans. Amer. Math. Soc. 361 (2009), 4471–4489.

[HH] B. Hassett, D. Hyeon: Log canonical models for the moduli space of curves: the first flip. Preprint available at arXiv:0806.3444.

[HL07] D. Hyeon, Y. Lee: Stability of tri-canonical curves of genus two. Math. Ann. 337 (2007), 479–488.

[HR74] M. Hochster, J. L. Roberts: Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay. Adv. Math. 13 (1974), 115–175.

[HM10] D. Hyeon, I. Morrison: Stability of Tails and 4-Canonical Models. Math. Res. Lett. 17 (2010), no. 4, 721–729.

[Kas] J. Kass: Good completions of Néron models. Harvard University, 2009.

[KeM97] S. Keel, S. Mori: Quotients by groupoids, Ann. of Math. (2) 145 (1997), no. 1, 193–213.

[Kle05] S. L. Kleiman: The Picard scheme. Fundamental algebraic geometry, 235–321, Math. Surveys Monogr. Vol. 123, Amer. Math. Soc., Providence, RI, 2005.

[Knu83] F. F. Knudsen: The projectivity of the moduli space of stable curves. II. The stacks M_{g,n}. Math. Scand. 52 (1983), no. 2, 161–199.

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[KoM98] J. Kollár, S. Mori: Birational geometry of algebraic varieties. With the collaboration of C. H. Clemens and A. Corti. Translated from the 1998 Japanese original. Cambridge Tracts in Mathematics, Vol. 134. Cambridge University Press, Cambridge, 1998.

[Li] J. Li, X. Wang: Hilbert-Mumford criterion for nodal curves. Preprint available at arXiv:1108.1727v1.

[Mat89] H. Matsumura: Commutative ring theory. Translated from the Japanese by M. Reid. Second edition. Cambridge Studies in Advanced Mathematics, Vol. 8. Cambridge University Press, Cambridge, 1989.

[Mel09] M. Melo: Compactified Picard stacks over $\overline{M}_g$. Math. Zeit. 263 (2009), No. 4, 939–957.

[Mel11] M. Melo: Compactified Picard stacks over the moduli stack of stable curves with marked points. Adv. Math 226 (2011), 727–763.

[MV] M. Melo, F. Viviani: Fine compactified Jacobians. To appear in Math. Nach. (preprint available at arXiv:1009.3205v3).

[Mor10] I. Morrison: GIT Constructions of Moduli Spaces of Stable Curves and Maps. Ji, Lizhen (ed.) et al., Geometry of Riemann surfaces and their moduli spaces. Somerville, MA: International Press. Surveys in Differential Geometry 14, 315–369 (2010).

[MS] I. Morrison, D. Swinarski: Gröbner techniques for low degree Hilbert stability. Preprint available at arXiv:0910.2047.

[Kol96] J. Kollár: Rational curves on algebraic varieties. Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 32. Springer-Verlag, Berlin, 1996.

[MFK94] D. Mumford, J. Fogarty, F. Kirwan: Geometric invariant theory. Ergebnisse der Mathematik und ihrer Grenzgebiete (2), Vol. 34. Springer-Verlag, Berlin, third edition, 1994.

[Mum66] D. Mumford: Lectures on curves on an algebraic surface. Annals of Mathematics Studies, No. 59. Princeton University Press, Princeton, N.J., 1966.

[Mum77] D. Mumford: Stability of projective varieties. Enseignement Math. (2) 23 (1977), 39–110.

[Ray70] M. Raynaud: Spécialisation du foncteur de Picard. Inst. Hautes Études Sci. Publ. Math. No. 38 (1970), 27–76.

[Sch91] D. Schubert: A new compactification of the moduli space of curves. Compositio Math. 78 (1991), 297–313.

[Ser06] E. Sernesi: Deformations of algebraic schemes. Grundlehren der mathematischen Wissenschaften 334, Springer, New York, 2006.

[Sim94] C. T. Simpson. Moduli of representations of the fundamental group of a smooth projective variety. Inst. Hautes Études Sci. Publ. Math., No. 80 (1994), 5–79.

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