HIGHER STEENROD SQUARES FOR KHOVANOV HOMOLOGY

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Abstract. We describe stable cup-i products on the cochain complex with $F_2$ coefficients of any augmented semi-simplicial object in the Burnside category. An example of such an object is the Khovanov functor of Lawson, Lipshitz and Sarkar. Thus we obtain explicit formulas for cohomology operations on the Khovanov homology of any link.

1. Introduction

In 2014 [LS14a], Lipshitz and Sarkar, using framed flow categories, defined a new invariant of knots and links valued in spectra that refined Khovanov homology: they associated to each link a cellular spectrum $\mathcal{X}$ whose cellular cochain complex was the Khovanov complex, and so its cohomology was the Khovanov homology of the link. As a consequence, Khovanov homology became endowed with stable operations, such as Steenrod squares when cohomology is taken with coefficients in the field $F_2$ with two elements.

Shortly after, Lipshitz and Sarkar [LS14b] were able to give a combinatorial formula for the second Steenrod square on the Khovanov homology of any link $L$

$$\text{Sq}^2 : \text{Kh}^*(L; F_2) \longrightarrow \text{Kh}^{*+2}(L; F_2),$$

in terms of the Khovanov complex and an extra datum called ladybug matching. They also showed (see also [See12]) that $\text{Sq}^2$ distinguishes some pairs of knots that are not distinguished by Khovanov homology.

Three years later, together with Lawson [LLS15], they gave two new constructions of Khovanov spectra that simplified the original construction of Lipshitz and Sarkar. In their second construction, they associated to each link diagram $D$ a strictly unital lax 2-functor $F_D$ from a cube poset to the Burnside 2-category, and associated a realisation spectrum $|F_D|$ to each such 2-functor. The spectrum $|F_D|$ is homotopy equivalent to the spectrum $\mathcal{X}$ first constructed by Lipshitz and Sarkar. This construction was revisited in [LS17] and in [LLS17], where they asked the following question:

Are there nice formulations of the action of the Steenrod algebra on $\text{Kh}^*(L; F_2)$, purely in terms of the Khovanov functor to the Burnside category?

The author would like to thank the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the programme Homotopy Harnessing Higher Structures and the Biblioteca Miguel González García at A Coruña, where work on this paper was undertaken. The author was supported by EPSRC grant number EP/R014604/1, by project MTM2016-76453-C2 (AEI/FEDER, UE), and by the Spanish Ministry of Economy and Competitiveness through the María de Maeztu Programme for Units of Excellence in R&D (MDM-2014-0445).
Symmetric multiplications. In order to make the question concrete, we introduce the following nice formulation of Steenrod squares on a cochain complex: A symmetric multiplication on a cochain complex \((C^*, d)\) of \(\mathbb{F}_2\)-modules is a family of operations

\[\sim_i : C^n \otimes C^i \to C^{n+i}, \quad i \in \mathbb{Z},\]

satisfying that

\[(1.1) \quad \alpha \sim_i \beta = 0 \quad \text{for } i < 0,\]
\[(1.2) \quad d(\alpha \sim_i \beta) = d\alpha \sim_i \beta + \alpha \sim_i d\beta + \alpha \sim_{i-1} \beta + \beta \sim_{i-1} \alpha \quad \text{for all } i.\]

Such structure endows the cohomology groups of the cochain complex with Steenrod squares, which are operations

\[\text{sq}^i : H^n(C^*) \to H^{n+i}(C^*), \quad \text{sq}^i([\alpha]) = [\alpha \sim_{-i} \alpha],\]

defined for \(i \geq 0\). As a consequence of (1.1), \(\text{sq}^i([\alpha]) = 0\) if \(i > n\) and the 0th operation \(\sim_0\) gives a well-defined graded multiplication on the cohomology of \((C^*, d)\).

The prominent example of these structures appears in the normalised cochain complex \(N^*(Y_\bullet; \mathbb{F}_2)\) of a simplicial set \(Y_\bullet\), which becomes endowed with a symmetric multiplication using the cup-\(i\) product formulas of Steenrod \cite{St47}.

The normalisation process in the construction of \(N^*(X_\bullet; \mathbb{F}_2)\) is done in two steps: first, kill the image of the degeneracies of the simplicial set \(Y_\bullet\) thus obtaining a semi-simplicial set \(X_\bullet\) (a simplicial set without degeneracies), and then take the dual of the chain complex \(C_\bullet(X_\bullet; \mathbb{F}_2)\) of alternating sums of face maps on the semi-simplicial set \(X_\bullet\). The cup-\(i\) products are defined out of the semi-simplicial structure and involve only face maps, so the cochain complex of any semi-simplicial set is also enhanced with cup-\(i\) products.

As an example, here are formulas for \(\sim_0\) and \(\sim_{-2}\). If \(\alpha \in C^p(X_\bullet; \mathbb{F}_2)\) and \(\beta \in C^q(X_\bullet; \mathbb{F}_2)\), then \(\alpha \sim_0 \beta \in C^{p+q}(X_\bullet; \mathbb{F}_2)\) is the Alexander–Whitney product of \(\alpha\) and \(\beta\), whose value on a chain \(\sigma \in C_{p+q}(X_\bullet; \mathbb{F}_2)\) is

\[(\alpha \sim_0 \beta)(\sigma) = \alpha(\partial_{p+1} \cdots \partial_{p+1} \sigma) \cdot \beta(\partial_0 \cdots \partial_0 \sigma).\]

If \(\alpha \in C^n(X_\bullet; \mathbb{F}_2)\) is a cocycle, then the first Steenrod square \(\text{sq}^1([\alpha])\) of \([\alpha]\) can be computed as \([\alpha \sim_{-2} \alpha]\), which is defined as

\[(\alpha \sim_{-2} \alpha)(\sigma) = \sum_{j<k \text{ even}} \alpha(\partial_j \sigma) \cdot \alpha(\partial_k \sigma) + \sum_{j>k \text{ odd}} \alpha(\partial_j \sigma) \cdot \alpha(\partial_k \sigma).\]

On the other hand, Steenrod squares on a topological space \(X\) can be defined as natural transformations \(\text{Sq}^i : H^*(X; \mathbb{F}_2) \to H^{*+i}(X; \mathbb{F}_2)\) that satisfy certain axioms. The fact that the Steenrod squares \(\text{sq}^i\) for a simplicial set \(X_\bullet\) that arise from the cup-i products coincide with the axiomatic Steenrod squares \(\text{Sq}^i\) for the topological space \([X_\bullet]\) is not immediate, and uses the singular chain functor of Eilenberg \cite{El44} to compare the cohomology operations in both settings.

Observe also that the simplicial structure is crucial to define the symmetric multiplication. In contrast, the cellular cochain complex of a CW-complex does not have in general a symmetric multiplication.
Stable symmetric multiplications. Condition (1.1) above implies that \( \sim_0 \) is a well-defined product on cohomology, but is not necessary for the definition of the Steenrod squares. A stable symmetric multiplication on a cochain complex \( C^* \) of \( \mathbb{F}_2 \)-modules is a family of operations

\[ \sim_i : C^p \otimes C^q \longrightarrow C^{p+q-i}, \quad i \in \mathbb{Z}, \]

satisfying (1.2). Such a structure gives again operations \( \operatorname{Sq}^i : H^n(C^*) \longrightarrow H^{n+i}(C^*) \), \( \operatorname{Sq}^i([\alpha]) = [\alpha \sim_{n-i} \alpha] \) defined for \( i \geq 0 \).

In Section 2.1 we explain how to associate to the Khovanov functor \( F_D \) of Lawson, Lipshitz and Sarkar an augmented semi-simplicial object in the Burnside category \( X_* \) whose cochain complex \( C^*(X; \mathbb{F}_2) \) is an iterated suspension of the Khovanov complex. These objects are defined in Section 2 and are generalisations of augmented semi-simplicial sets, where face maps \( \partial_i : X_n \longrightarrow X_{n-1} \) are replaced by zig-zags \( X_n \leftarrow Q^n \rightarrow X_{n-1} \) plus some higher categorical data. When each map \( X_n \leftarrow Q^n \) is a bijection one recovers the concept of augmented semi-simplicial set, and if additionally the augmentation is trivial, one recovers the concept of semi-simplicial set. An order on a augmented semi-simplicial object in the Burnside category is a choice of order on each \( Q^n \) and \( X_n \) whose cochain complex has a natural stable symmetric multiplication, i.e., there are explicit operations

\[ \sim_i : C^p(X_*; \mathbb{F}_2) \otimes C^q(X_*; \mathbb{F}_2) \longrightarrow C^{p+q-i}(X_*; \mathbb{F}_2), \quad i \in \mathbb{Z} \]

satisfying (1.2), and, for every free order-preserving map \( f : X_* \longrightarrow Y_* \),

\[ f^*(\alpha \sim_i \beta) = f^*(\alpha) \sim_i f^*(\beta). \]

If \( X_* \) is a semi-simplicial set, then these operations are the Steenrod cup-i products.

The explicit formulas for the \( \sim_i \) products are given in Section 3 and the naturality is proven in Proposition 6.4. In Section 5 we prove that the Steenrod squares induced by the symmetric multiplication are invariant under suspension, satisfy a Cartan formula and the first square induced by the symmetric multiplication are invariant under suspension, satisfy the Bockstein homomorphism. In Theorem 6.6 we improve their naturality properties obtaining the following result.

Theorem B. The Steenrod operations \( \operatorname{Sq}^i : H^*(X_*; \mathbb{F}_2) \rightarrow H^{*+i}(X_*; \mathbb{F}_2) \) associated to the stable symmetric multiplication are natural with respect to maps of augmented semi-simplicial objects in the Burnside category. In particular, they do not depend on the chosen order on \( X_* \).

In Corollary 7.10 we obtain the following consequence for Khovanov homology.

Corollary C. There is an explicit stable symmetric multiplication on the Khovanov complex of any oriented link diagram \( D \) with \( n_- \) negative crossings

\[ \sim_i : C^{p-n_-+1}(D; \mathbb{F}_2) \otimes C^{q-n_-+1}(D; \mathbb{F}_2) \longrightarrow C^{p+q-i-n_-+1}(D; \mathbb{F}_2) \quad i \in \mathbb{Z}. \]

Therefore Khovanov homology becomes endowed with the Steenrod squares

\[ \operatorname{Sq}^i : Kh^*(D; \mathbb{F}_2) \longrightarrow Kh^{*+i}(D; \mathbb{F}_2), \quad \operatorname{Sq}^i([\alpha]) = [\alpha \sim_{n+n_-+i} \alpha] \]

associated to this stable symmetric multiplication, which are invariant under Reidemeister moves and reordering of the crossings.
The techniques of this paper do not allow us to prove that the Steenrod squares of Corollary C coincide with the Steenrod squares

\[ \text{Sq}^i : H^n(\mathcal{F}_D; \mathbb{F}_2) \to H^{n+i}(\mathcal{F}_D; \mathbb{F}_2), \quad i \geq 0 \]

of the realisation spectrum of \( \mathcal{F}_D \). Such comparison will be developed in the companion paper [CG], where we will construct a “singular chain functor” from the category of spectra to the category of augmented semi-simplicial objects in the Burnside category. The results in the present paper are purely combinatorial, and have to be compared with the constructions of Steenrod in [Ste47] for simplicial complexes, whereas the results of the companion paper [CG] are mainly homotopy-theoretic and have to be compared with the constructions of Eilenberg [Eil44] for topological spaces. In particular, spectra will be essentially absent from this paper, and will only be barely mentioned in some examples in Section 8.

Outline of the paper. In Section 2 we first explain how to translate the framework of [LLS15], which is expressed in terms of cubes in the Burnside category, to our framework in terms of augmented semi-simplicial objects in the Burnside category. Then we introduce several definitions and constructions that will be used through the paper. In Section 3 we present formulas for cup-i products, and we prove in Section 4 that they endow the cohomology of any augmented semi-simplicial object in the Burnside category with a stable symmetric multiplication. In Section 5 we define Steenrod squares and we prove that they are stable under suspension, that they satisfy a Cartan formula and that the first square is the Bockstein homomorphism. The proofs of naturality are deferred to Section 6. In Section 7 we apply the previous results to the Khovanov functor of Lawson, Lipshitz and Sarkar and we prove Corollary C. The paper finishes with several examples in Section 8. The reader only interested on explicit formulas for operations on Khovanov homology will find them in Section 3 after having got used to the terminology introduced in Section 2 and may afterward safely skip Sections 4, 5 and 6 and proceed directly to Section 7 and the examples in Section 8.

Acknowledgments. The author is especially grateful to Aníbal Medina-Mardones for the inspiration received while reading his paper [MM18]. He is also grateful to Javier Gutiérrez, Carles Casacuberta, Joana Cirici and Marithania Silvero from the Topology group at Barcelona, and thanks Tyler Lawson, Clemens Berger, and Oscar Randal-Williams for their feedback during his stay at the Isaac Newton Institute for Mathematical Sciences.

2. Khovanov functors and semi-simplicial objects in the Burnside category

This section begins with a quick explanation of how to translate the context in [LLS15, LLS17, LS17] (cubes in the Burnside category) to our context (augmented semi-simplicial objects in the Burnside category). Sections 2.2 to 2.9 are devoted to present and prove the concepts and claims used in this explanation, while Sections 2.10 and 2.11 will set up the notation for augmented semi-simplicial objects in the Burnside category. In this exposition, \( R \) denotes a commutative ring with unit.
2.1. Khovanov spectra. In [Kho00, BN02], Khovanov associated to each link diagram $D$ with $c$ ordered crossings and $n_-$ negative crossings, a contravariant functor $F_D$ from the cube category $2^c$ to the category of $R$-modules. Every such functor has a totalisation $\text{Tot} F_D \in \text{Ch}(R)$, which is a bounded chain complex of $R$-modules. Khovanov proved that if two link diagrams $D$ and $D'$ are obtained from each other using some Reidemeister move, then the chain complexes $\Sigma^{-n_-} \text{Tot} F_D$ and $\Sigma^{-n_-'} \text{Tot} F_{D'}$ are quasi-isomorphic. The Khovanov homology of a link $L$ is then defined as the cohomology of $\Sigma^{-n_-} \text{Tot} F_D$, for any diagram $D$ representing $L$, and it is a link invariant up to isomorphism.

The construction of the Khovanov spectrum of Lawson, Lipshitz and Sarkar [LLS15, LLS17] first associates to a link diagram $D$ with $c$ ordered crossings a lax strictly unital contravariant functor $F_D: 2^c \to B$ from the cube category $2^c$ to the Burnside 2-category; and to a Reidemeister move from $D$ to a diagram $D'$ a “stable equivalence” between $F_D$ and $F_{D'}$. They showed that postcomposing with a certain functor $A_R: B \to R\text{-Mod}$, one obtains back the Khovanov construction: $F_D = A_R \circ F_D$. Moreover, they constructed a realisation functor $|\cdot|$ that converts every contravariant strictly unital lax functor $F: 2^c \to B$ into a spectrum $|F|$ and every strictly unital lax natural transformation into a map of spectra, with the property that

$$H_*(\text{Tot} A_R \circ F_D) \cong H_*(|F_D|; R).$$

Finally, they proved that the stable equivalences associated to the Reidemeister moves are converted into weak equivalences of spectra, thus concluding that the stable homotopy type of $\Sigma^{-n_-} |F_D|$ is a link invariant.

An augmented semi-simplicial object in a 2-category $C$ is a strictly unital lax functor from the category of possibly empty finite ordinals $\Delta_{\text{inj}}^*$ to $C$. There is a cofinal functor $2^c \to \Delta_{\text{inj}}^*$, and, if $C$ has finite coproducts, taking left Kan extension along it defines a functor

$$\Lambda: C^{(2^c)^{op}} \to C^{\Delta_{\text{inj}}^{op}}$$

between categories of strictly unital lax functors. When $C$ is the category of $R$-modules, we let

$$M: R\text{-Mod}^{\Delta_{\text{inj}}^{op}} \longrightarrow \text{Ch}(R)$$

be the functor that takes a semi-simplicial $R$-module to its Moore chain complex of alternating sums of face maps. Then, the upper part of the following diagram commutes:

$$\begin{array}{ccc}
B(2^c)^{op} & \xrightarrow{A_R} & (R\text{-Mod})(2^c)^{op} \\
\downarrow \Lambda & & \downarrow \Lambda \\
B^{\Delta_{\text{inj}}^{op}} & \xrightarrow{A_R} & (R\text{-Mod})^{\Delta_{\text{inj}}^{op}} \\
\downarrow * & & \downarrow M \\
\text{Spectra} & \xrightarrow{H_*(-; R)} & \text{Graded } R\text{-Mod} \\
\end{array}$$

In this paper we are only interested in the upper part of the diagram. We give the following comment without proof regarding the bottom part: The functor $*$ is constructed in [Bar17] and the composition of $\Lambda$ and that functor is homotopy
equivalent in an $\infty$-categorical sense to the desuspension of the realisation construction $|\cdot|$ of Lawson, Lipshitz and Sarkar. Additionally, the bottom square commutes.

Let $X_\bullet \in \mathcal{B}^{op}_{\text{pair}}$, and let $C^*(X_\bullet; \mathbb{F}_2)$ be the dual of the Moore chain complex $M(A_R \circ X_\bullet)$. We will construct natural operations

$$\sim_i : C^p(X_\bullet; \mathbb{F}_2) \otimes C^q(X_\bullet; \mathbb{F}_2) \to C^{p+q-i}(X_\bullet; \mathbb{F}_2)$$

analogous to the classical cup-$i$ products for semi-simplicial sets. If $\mathcal{F}_D \in \mathcal{B}^{op}_{(2^c)}$ is the functor of Lipshitz, Lawson and Sarkar, then, since the above diagram commutes, we will have operations $\sim_i$ defined on

$$M \circ A_R \circ \Lambda(\mathcal{F}_D)) = \Sigma^{-1} \text{Tot} \circ A_R(\mathcal{F}_D) = \Sigma^{-1} \text{Tot}(\mathcal{F}_D),$$

the $(n-1)$st suspension of the Khovanov complex. Thus, we will obtain, for any link $L$, cohomology operations

$$\eta^1 : Kh^*(L; \mathbb{F}_2) \to Kh^{*+i}(L; \mathbb{F}_2), \quad i \geq 0.$$

### 2.2. Sequences.

Let $P_q(n)$ be the set of all increasing sequences $U = (u_1, \ldots, u_q)$ such that $0 \leq u_i \leq n$ for each $i = 1, \ldots, q$, and let $P(n) = \bigcup_q P_q(n)$. Suppose that $V \in P_p(n)$ and define

$$\psi_V : \{U \in P_q(n) \mid U \cap V = \emptyset\} \to P_q(n-p)$$

$$(u_1, \ldots, u_q) \mapsto (w_1, \ldots, w_q), \quad w_j = u_j - |\{v \in V \mid v < u_j\}|,$$

$$\gamma_V : P_q(n-p) \to P_q(n)$$

$$(u_1, \ldots, u_q) \mapsto (w_1, \ldots, w_q), \quad w_j = u_j + |\{v \in V \mid v \leq u_j\}|,$$

$$\eta_V : \{U \in P_q(n) \mid V \subseteq U\} \to P_{q-p}(n-p)$$

$$U \mapsto \psi_V(U \setminus V),$$

$$\xi_V : P_q(n) \to P_{q+p}(n+p)$$

$$U \mapsto \gamma_V(U) \cup V.$$

We have that

$$\psi_V(\gamma_V(U)) = U, \quad \gamma_V(\psi_V(U)) = U,$$

$$\eta_V(\xi_V(U)) = U, \quad \xi_V(\eta_V(U)) = U.$$

### 2.3. The augmented semi-simplicial category.

Let $\Delta_{\text{pair}}$ be the category of non-empty finite ordinals and order-preserving injections between them, and let $\Delta_{\text{pair}}$ be the category of finite ordinals and order-preserving maps between them. We write $[n] = \{0, \ldots, n\}$ for the ordinal with $n+1$ elements and note that $[-1]$ stands for the empty ordinal. For each sequence $U \in P_q(n)$, define the $U$th generalised face map $\partial^n_U : [n-q] \to [n]$ as the unique order-preserving injective map that misses $U$. Every morphism in $\Delta_{\text{pair}}$ is a generalised face map, and, if $U \in P(n)$ and $U = V \cup W$, with $V \in P_q(n)$ then

$$\partial^{n-q}_{\psi_V(W)} \circ \partial^n_U = \partial^n_U.$$

If $V \in P_q(n)$ and $W \in P_p(n-q)$, we can rewrite this condition as:

$$\partial^{n-q}_{W} \circ \partial^n_V = \partial^n_{\xi_V(W)}.$$
When $U = \{u\}$ has a single element, $\partial_U$ is called a face map and the equation (2.2) becomes the usual simplicial identities of the face maps. In that case, we write $\partial_u$ instead of $\partial_{\{u\}}$. Every generalised face map is a composition of face maps.

2.4. **The cube poset.** Let $0 < 1$ be the poset with two elements 0, 1 and one morphism from 0 to 1. Let $2^n$ be the $n$th power of this poset, whose elements are tuples $v = (v_1, \ldots, v_n)$ with $v_i \in \{0, 1\}$ and $v \leq v'$ if and only if $v_i \leq v'_i$ for all $i = 1 \ldots n$. The elements of the cube are graded by the Manhattan norm $|v| = \sum_i v_i$. Alternatively, $2^n$ is the poset of subsets of $\{1, \ldots, n\}$ ordered by inclusion. We will denote a subset $A$ of $\{1, \ldots, n\}$ as the ordered sequence $(a_1, \ldots, a_m)$ on the elements of $A$. The grading of a subset $A \subseteq \{1, \ldots, n\}$ is its cardinality $|A|$. The relation between both perspectives is given by identifying a tuple $(v_1, \ldots, v_n)$ with the subset $A = \{i \mid v_i = 1\} \subseteq \{1, \ldots, n\}$. In this paper we will use the second description of $2^n$.

2.5. **2-categories.** In this section we quickly remind the definitions of 2-category, of strictly unital lax functor and of strictly unital lax natural transformation [Bor94 §7.5]. As the 2-morphisms in the 2-categories used in this paper are invertible, it is customary to replace the adjective “lax” by “pseudo-natural”. Moreover, it will be convenient to work with op-lax functors and op-lax natural transformations, i.e., our structural 2-morphisms in a pseudo-functor or a pseudo-natural transformation are the inverses of the usual 2-morphisms. In Sections 2.10 and 2.11 we give a more careful definition of the particular functors and natural transformations that are used in the paper.

Recall that a 2-category $\mathcal{C}$ is a category enriched in the category of small categories, i.e., it consists on the data of

1. a collection of objects,
2. for each pair of objects $X, Y$, a category of morphisms $\text{hom}(X, Y)$,
3. for each object $X$, an object $\text{Id}_X$ of $\text{hom}(X, X)$ and
4. for each triple of objects $X, Y, Z$, a composition functor

$$\text{hom}(Y, Z) \times \text{hom}(X, Y) \rightarrow \text{hom}(X, Z)$$

satisfying that the composition is associative and unital. The objects of $\text{hom}(X, Y)$ are called 1-morphisms and the morphisms in $\text{hom}(X, Y)$ are called 2-morphisms.

Recall that a strictly unital pseudo-functor $F$ from a category $\mathcal{D}$ to a 2-category $\mathcal{C}$ with invertible 2-morphisms consists of the data:

1. for each object $a$ of $\mathcal{D}$, an object $F(a)$ in $\mathcal{C}$,
2. for each morphism $f$ of $\mathcal{D}$, a morphism $F(f)$ in $\mathcal{C}$,
3. for each decomposition $f = g_2 \circ g_1$, a 2-morphism $F(g_1, g_2)$ from $F(f)$ to $F(g_2) \circ F(g_1)$

satisfying that

4. for each object $a$ of $\mathcal{D}$, $F(\text{Id}_a) = \text{Id}_{F(a)}$,
5. for each morphism $f : a \rightarrow b$ of $\mathcal{D}$, both $\mu_{\text{Id}_a, f}$ and $\mu_{f, \text{Id}_b}$ are the identity 2-morphisms.
6. for each decomposition $f = g_3 \circ g_2 \circ g_1$, we have

$$(\text{Id} \times F(g_2, g_3)) \circ F(g_1, g_3 \circ g_2) = (F(g_1, g_2) \times \text{Id}) \circ F(g_2 \circ g_1, g_3)$$

Recall that a strictly unital pseudo-natural transformation between two 2-functors $F, G : \mathcal{D} \rightarrow \mathcal{C}$ consists of the data:
(1) for each object \(a\) of \(D\), a morphism \(\alpha_a : F(a) \to G(a)\) and
(2) for each morphism \(f : a \to b\) of \(D\), a 2-morphism \(\alpha_f\) from \(\alpha_b \circ F(f)\) to \(G(f) \circ \alpha_a\), satisfying that
(3) for each object \(a\) of \(D\), \(\alpha_{Id_a}\) is the identity 2-morphism on \(\alpha_a\):
(4) for each decomposition \(f = g_2 \circ g_1\), we have:
\[
G(g_1, g_2) \circ \alpha_f = \alpha_{g_2} \circ \alpha_{g_1} \circ F(g_1, g_2)
\]
Let \(C^{op}\) be the category whose objects are contravariant strictly unital pseudo-functors from \(D\) to \(C\) and whose morphisms are strictly unital pseudo-natural transformations between them.

2.6. The Burnside 2-category. Given two sets \(X, Y\), a locally finite span from \(X\) to \(Y\) is a pair of functions
\[
\begin{array}{ccc}
X & \xleftarrow{\text{source}} & Q & \xrightarrow{\text{target}} & Y.
\end{array}
\]
such that \(\text{source}^{-1}(x)\) is a finite set for every \(x \in X\). A locally finite span is free if the source map is an injection. A fibrewise bijection between two locally finite spans \(X \xleftarrow{\text{source}} Q \xrightarrow{\text{target}} Y\) and \(X \xleftarrow{\text{source'}} Q' \xrightarrow{\text{target'}} Y\) is a bijection \(\tau : Q \to Q'\) such that \(\text{source}' \circ \tau = \text{source}\) and \(\text{target}' \circ \tau = \text{target}\):

\[
\begin{array}{ccc}
X & \xleftarrow{\text{source}} & Q & \xrightarrow{\text{target}} & Y \quad & X & \xleftarrow{\text{source}} & Q' & \xrightarrow{\text{target'}} & Y.
\end{array}
\]

The composition of two fibrewise bijections of locally finite spans is the composition of bijections:

\[
\begin{array}{ccc}
X & \xleftarrow{\text{source}} & Q_1 & \xrightarrow{\tau_1} & Q_2 & \xleftarrow{\text{source}} & Y \\
& \downarrow{\tau_2} & & & & \downarrow{\tau_3} & \\
X & \xleftarrow{\text{source}} & Q_3 & \xrightarrow{\text{target}} & Y,
\end{array}
\]

and the identity morphism of a locally finite span is the identity bijection. This defines a category of locally finite spans from a set \(X\) to a set \(Y\).

Definition 2.1. We denote by \(B\) the Burnside 2-category for the trivial group, whose objects are sets, and the category of morphisms from a set \(X\) to a set \(Y\) is the category of locally finite spans from \(X\) to \(Y\). Composition of locally finite spans is given by taking their fibre product:

\[
\begin{array}{ccc}
X_1 & \xleftarrow{\text{source}} & Q_1 \times_X Q_2 & \xrightarrow{\text{target}} & Q_2 \xrightarrow{\text{target}} X_3,
\end{array}
\]
composition of fibrewise bijections is their fibre product

\[
\begin{array}{ccc}
Q_1 & \xleftarrow{\tau_1} & Q_2 \\
X_1 & \xrightarrow{\tau_2} & X_3 \\
Q_3 & \xrightarrow{\tau_1 \times \tau_2} & Q_4 & \xrightarrow{\tau_1 \times \tau_2} & Y.
\end{array}
\]

and the identity on a set \(X\) is the span \(X \xleftarrow{\text{id}} X \xrightarrow{\text{id}} X\).

**Warning.** In [LLS15] and [LLS17] the objects of the Burnside category are required to be finite sets. This restriction is unnecessary for the results in this paper.

The Burnside 2-category has a coproduct

\[ \Pi : \mathcal{B} \times \mathcal{B} \to \mathcal{B} \]

that sends a pair of finite sets to their disjoint union and a pair of spans \(X \xleftarrow{Q} Y\) and \(X' \xleftarrow{Q'} Y'\) to the span

\[ X \Pi X' \to Q \Pi Q' \to Y \Pi Y'. \]

The categorical product in \(\mathcal{B}\) coincides with the coproduct and there is a tensor product

\[ \times : \mathcal{B} \times \mathcal{B} \to \mathcal{B} \]

that sends a pair of finite sets to their product and a pair of spans \(X \xleftarrow{Q} Y\) and \(X' \xleftarrow{Q'} Y'\) to the span

\[ X \times X' \to Q \times Q' \to Y \times Y'. \]

The category of pointed sets \(\text{Set}^\ast\) includes into the Burnside 2-category \(\mathcal{B}\) by sending a pointed set \((X, x_0)\) to \(X \setminus \{x_0\}\) and a morphism \(f : (X, x_0) \to (Y, y_0)\) to the span

\[ X \setminus \{x_0\} \xleftarrow{X \setminus f^{-1}(y_0)} \xrightarrow{f} Y \setminus \{y_0\}. \]

This inclusion induces an equivalence of categories between \(\text{Set}^\ast\) and the subcategory \(\mathcal{B}_{\text{free}}\) of \(\mathcal{B}\) whose objects are sets and whose 1-morphisms are free spans. Observe that there is at most one 2-morphism between any two spans.

The category of sets \(\text{Set}\) includes into the category \(\text{Set}^\ast\) by sending a set to its disjoint union with some fixed basepoint. The composition of these two inclusions induces an equivalence of categories between \(\text{Set}\) and the subcategory of \(\mathcal{B}\) whose objects are sets and whose 1-morphisms are spans for which the source map is a bijection.

2.7. **From the Burnside category to the category of \(R\)-modules.** Define the \(R\)-linearization functor

\[ A_R : \mathcal{B} \to \text{R-Mod} \]

that takes a set to the free \(R\)-module on it, and a span \(X \xleftarrow{\text{source}} Q \xrightarrow{\text{target}} Y\) to the homomorphism \(R(X) \to R(Y)\) whose value on \(x \in X\) is

\[ \sum_{y \in Y} | \text{source}^{-1}(x) \cap \text{target}^{-1}(y) | \cdot y. \]

Any two spans connected by a 2-morphism are sent to the same homomorphism and the sum is well-defined because \(\text{source}^{-1}(x)\) is finite, so \(A_R\) is well-defined.
Moreover, this functor is symmetric monoidal with respect to the tensor products $\Pi, \times$ in $\mathcal{B}$ and the tensor products $\otimes, \otimes$ in $\text{R-Mod}$, so:

$$A_R(X \Pi Y) \cong A_R(X) \oplus A_R(Y), \quad A_R(X \times Y) \cong A_R(X) \otimes A_R(Y).$$

Let $f, g: X \to Y$ be the spans $X \leftarrow Q \to Y$ and $X \leftarrow Q' \to Y$. We denote by $f + g$ the span $X \leftarrow Q \Pi Q' \to Y$. This operation commutes with $R$-linearization too:

$$A_R(f + g) = A_R(f) + A_R(g).$$

**Definition 2.2.** Two spans $X \leftarrow Q \to Y$ and $X \leftarrow Q' \to Y$ are *equivalent* if there is a $2$-morphism between them. Alternatively, they are equivalent if they have the same $\mathbb{Z}$-linearization. They are $\mathbb{F}_2$-equivalent if they have the same $\mathbb{F}_2$-linearization.

**Notation.** We use the symbol “$\equiv$” to denote the relation of being equivalent, and the symbol “$\sim$” for the relation of being $\mathbb{F}_2$-equivalent.

### 2.8. The functor $\Lambda$.

Let $\mathcal{C}$ be a $2$-category with invertible $2$-morphisms. An $n$-cube in $\mathcal{C}$ is a strictly unital pseudo-functor $F: \Delta^{op}_{n} \to \mathcal{C}$, a *semi-simplicial object in $\mathcal{C}$* is a strictly unital pseudo-functor $X: \Delta_{\text{inj}}^{op} \to \mathcal{C}$ and an *augmented semi-simplicial object in $\mathcal{C}$* is a strictly unital pseudo-functor $X: \Delta_{\text{inj}}^{op} \to \mathcal{C}$. As customary, if $X$ is an augmented semi-simplicial object in a category $\mathcal{C}$, we will write $X_{\bullet}$ for $X$, $X_n$ for $X(n)$ and $\partial^i_n$ for $X(\partial^i_n)$.

A *map between two $n$-cubes in $\mathcal{C}$* is a strictly unital pseudo-natural transformation between them. A *map between two augmented semi-simplicial objects in the Burnside category in $\mathcal{C}$* is a strictly unital pseudo-natural transformation between them.

If $A = (a_1, \ldots, a_m)$, let $\varphi_A: A \to \{0, \ldots, m-1\}$ be the function $\varphi_A(a_i) = i - 1$. There is a functor $\lambda: \Delta^{op}_{n} \to \Delta_{\text{inj}}^{op}$ that sends a vertex $A \subset \{1, \ldots, n\}$ of $\Delta^{op}_{n}$ to the ordinal $|A| - 1$ and a morphism $B \subset A$ to the morphism $\partial^{|A| - 1}_{|B|}$. If $\mathcal{C}$ has finite coproducts, left Kan extension along this functor, defines a functor

$$\Lambda: \mathcal{C}(\Delta^{op}_{n}) \to \mathcal{C}(\Delta_{\text{inj}}^{op}),$$

given explicitly on an object $F: (\Delta^{op}_{n}) \to \mathcal{C}$ as

$$\Lambda(F)(k) = \prod_{|A| = k+1} F(A) \quad \quad \quad k \geq -1,$$

$$\Lambda(F)(\partial^k_{U}) = \prod_{|A| = k+1} F(A \backslash \varphi_A^{-1}(U) \subset A) \quad \quad \quad U \in P_q(n),$$

$$\Lambda(F)(\partial^k_{V}, \partial^k_{W}) = \prod_{|A| = k+1} F(B \subset A, C \subset B) \quad \quad \quad V \in P_q(n), W \in P(n - q),$$

where $B = A \backslash \varphi_A(V)$ and $C = B \backslash \varphi_B(W)$. A similar formula holds for maps between them. Setting $\mathcal{C}$ to be either the Burnside 2-category or the category of $R$-modules $\text{R-Mod}$, seen as a $2$-category with only identity $2$-morphisms, we obtain the upper left square in $(2.7)$:

$$\begin{array}{ccc}
\mathcal{B}(\Delta^{op}_{n}) & \xrightarrow{A_{\mathcal{R}}^{op}} & (\text{R-Mod})^{(\Delta^{op}_{n})}
\end{array}$$

which commutes because $A_{\mathcal{R}}: \mathcal{B} \to \text{R-Mod}$ preserves finite coproducts.
2.9. Realisations and totalisations. Define the Moore functor

\[ M : \text{R-Mod}^{\Delta^m_{op}} \longrightarrow \text{Ch}(R) \]

as the functor that sends an augmented semi-simplicial R-module \( X \) to the following chain complex \( C_*(X) \):

\[ C_n(X) = X_n, \quad d_n = \sum_{i=0}^{n} (-1)^i \partial^n_i, \]

and sends a map \( f : X \to Y \) to the homomorphism of chain complexes which in degree \( n \) is

\[ f_n : C_n(X) \to C_n(Y). \]

Define the totalisation functor \( \text{Tot} : \text{R-Mod}^{2^n_{op}} \longrightarrow \text{Ch}(R) \) as the functor that sends a cube of R-modules \( F \) to the following chain complex \( C_*(F) \):

\[ C_m(F) = \bigoplus_{|A|=m} F(A), \quad d_m = \sum_{|A|=m} \sum_{i=1}^{n} (-1)^i F(A \setminus \{a_i\} \subset A) \]

and sends a map \( f : F \to G \) to the homomorphism of chain complexes which in degree \( m \) is

\[ \bigoplus_{|A|=m} f_A : \bigoplus_{|A|=m} F(A) \longrightarrow \bigoplus_{|A|=m} G(A). \]

When \( X = \Lambda(F) \), the Moore chain complex and the totalisation agree up to suspension: \( C_*(\Lambda(F)) = \Sigma^{-1} C_*(F) \), so the upper right square in (2.1) commutes.

**Definition 2.3.** If \( R \) is a commutative ring, the \( R \)-realisation functor is the composition

\[ | \cdot |_R : \mathcal{B}^{\Delta^m_{op}} \xrightarrow{\Lambda_R^{op}} \text{R-Mod}^{\Delta^m_{op}} \xrightarrow{M} \text{Ch}(R). \]

Explicitly, the \( R \)-realisation \( |X|_R \) of an augmented semi-simplicial object in the Burnside category \( X \) is the chain complex

\[ C_n(X; R) = \Lambda(R(X_n)), \quad d_n = \sum_{i=0}^{n} (-1)^i \Lambda(R(\partial^n_i)). \]

The \( R \)-totalisation functor is the composition

\[ \text{Tot}_R : \mathcal{B}^{2^n_{op}} \xrightarrow{\Lambda_{R^{op}}^{op}} \text{R-Mod}^{2^n_{op}} \xrightarrow{\text{Tot}} \text{Ch}(R). \]

Explicitly, the \( R \)-totalisation \( \text{Tot}_R(F) \) of a cube in the Burnside category \( F : 2^n \to \mathcal{B} \) is the chain complex

\[ C_m(F; R) = \bigoplus_{|A|=m} R(F(A)), \quad d_m = \sum_{|A|=m} \sum_{i=1}^{n} (-1)^i \Lambda_{\mathcal{B}}(F(A \setminus \{a_i\} \subset A)). \]

**Definition 2.4.** A map \( f : X \to Y \) of augmented semi-simplicial objects in the Burnside category is an equivalence if its \( \mathbb{Z} \)-realisation \( C_*(f; \mathbb{Z}) : C_*(X; \mathbb{Z}) \to C_*(Y; \mathbb{Z}) \) is a chain homotopy equivalence. Two augmented semi-simplicial objects in the Burnside category \( X, Y \) are equivalent if there is a zig-zag of equivalences between them, in which case we write \( X \simeq Y \).
Definition 2.5. A map \( f : F \to G \) of cubes in the Burnside category, is an equivalence if the induced map \( C_*(f) : C_*(F; \mathbb{Z}) \to C_*(G; \mathbb{Z}) \) is a chain homotopy equivalence. Two cubes \( F, G : 2^n \to B \) are equivalent if there is a zig-zag of equivalences between them, in which case we write \( F \simeq G \).

The inclusions \( \text{Set} \subset \text{Set}_* \subset B \) of Section 2.6 induce inclusions of categories

\[
\text{Set}^\Delta_{\text{op}*} \to \text{Set}_*^\Delta_{\text{op}} \to B^\Delta_{\text{op}*}
\]

which commute with the realisation functors.

2.10. Augmented semi-simplicial objects in the Burnside category. An augmented semi-simplicial object in the Burnside category \( X : \Delta^\text{op}_{\text{inj}*} \to B \) will be denoted \( X_* \). Additionally, the set \( X(n) \) is denoted \( X_n \), the locally finite span \( X(\partial^n_p) \) is denoted \( \partial^n_p \) and, if \( \partial^n_p = \partial^n_W \circ \partial^n_p \), the fibrewise bijection \( X(\partial^n_{V_1}, \partial^n_{V_2}) \) is denoted \( \mu_{V_1,V_2}^{n} \). With this notation, an augmented semi-simplicial object in the Burnside category consists of the following data:

1. For each \( n \geq -1 \) a set \( X_n \),
2. For each \( n \geq -1 \) and each \( U \in P_q(n) \), a locally finite span \( X_n \leftarrow Q^n_U \to X_{n-q} \) that we denote by \( \partial^n_U \).
3. For each \( n \geq -1 \) and each \( U \in P_q(n) \), and each partition \( U = V_1 \cup V_2 \) with \( V_1 \in P_{q_1}(n) \), a fibrewise bijection
   \[
   \mu_{V_1,V_2}^{n} : Q^n_{V_1} \to Q^n_{V_1} \times Q^n_{V_2} \times Q^n_{V_1}(V_2),
   \]
   i.e., a fibrewise bijection over \( X_n \) and \( X_{n-q_1-q_2} \)
4. For each \( n \geq -1 \), \( \partial^n_U \) is the identity span on \( X_n \),
5. For each \( n \geq -1 \), and each \( V \in P_q(n) \), \( \mu_{0,V} \) and \( \mu_{V,0} \) are the identity bijections
6. For each \( n \geq -1 \), and each \( U \in P_q(n) \) and each partition \( U = V_1 \cup V_2 \cup V_3 \) with \( V_i \in P_{q_i}(n) \), the following square commutes:

\[
\begin{array}{ccc}
\partial^n_U & \xrightarrow{\mu_{V_1, V_2}^{n}} & \partial^n_{W_1} \circ \partial^n_{V_1} \\
\mu_{V_1, V_3}^{n} & \circ \partial^n_{W_2} & \circ \partial^n_{V_3}
\end{array}
\]

\[
\begin{array}{ccc}
\partial^n_{W_1} & \circ \partial^n_{V_2} \times \text{Id} & \mu_{V_1, V_3}^{n} \circ \partial^n_{W_2} \circ \partial^n_{V_3}
\end{array}
\]

where

\[
\begin{align*}
n_j &= n - \sum_{i=1}^{j} q_i, \\
W_2 &= \psi_{V_1}(V_2), \\
W_3 &= \psi_{V_1 \cup V_2}(V_3) \\
V_{12} &= V_1 \cup V_2, \\
V_{23} &= V_2 \cup V_3, \\
W_{23} &= \psi_{V_1}(V_2 \cup V_3)
\end{align*}
\]

Notation. We will systematically refer to the set \( Q^n_U \) with the name \( \partial^n_U \) of the whole span. In particular, “\( x \in \partial^n_U \)” will be used instead of “\( x \in Q^n_U \)”. We will denote the diagonal bijection in (6) as

\[
\mu_{V_1, V_2, V_3}^{n} : \partial^n_U \to \partial^n_{W_3} \circ \partial^n_{W_2} \circ \partial^n_{V_1}.
\]
If \( V_1 \subseteq U \) are disjoint, we write \( V_2 = U \setminus V_1 \) and define
\[
\lambda^{n_1}_{V_1} : \partial^0_U \xrightarrow{\mu^{n_1}_{V_1, V_2}} \partial^{n_1}_{W_2} \circ \partial^{n_1}_{V_1} \xrightarrow{\text{proj}} \partial^{n_1}_{V_1}.
\]
If \( V_1, V_2 \subseteq U \) are disjoint, we write \( V_3 = U \setminus (V_1 \cup V_2) \) and define
\[
\lambda^{n_1}_{V_1, V_2} : \partial^0_U \xrightarrow{\mu^{n_1}_{V_1, V_2, V_3}} \partial^{n_1}_{W_2} \circ \partial^{n_1}_{W_1} \xrightarrow{\text{proj}} \partial^{n_1}_{W_2} = \partial^{n_1}_{V_1} \circ (\Sigma V_2).
\]
We will omit the superscript \( n \) in \( \partial^0_U \) or \( \mu^{n_1}_{V_1, V_2} \) or \( \lambda^{n_1}_{V_1, V_2} \) whenever it agrees with the the variable \( n \) in the context.

**Definition 2.6.** An ordered augmented semi-simplicial object in the Burnside category is an augmented semi-simplicial object in the Burnside category together with an ordering of each span \( \partial^0_U \) (i.e., of the set \( Q^0_U \)).

**Definition 2.7.** The suspension of an (ordered) augmented semi-simplicial object in the Burnside category \( X_\bullet \) is the (ordered) augmented semi-simplicial object in the Burnside category
\[
(\Sigma X_\bullet)_{-1} = \emptyset, \quad (\Sigma X_\bullet)_n = X_{n-1} \quad \text{if} \quad n > -1,
\]
with face maps \( \partial_U^0 \) given by
\[
\partial_U^0 = \begin{cases} 
\partial^{n-1}_{\psi_0(U)} & \text{if} \ 0 \not\in U \\
\emptyset & \text{if} \ 0 \in U.
\end{cases}
\]
Here \( \emptyset \) is the empty span \( (\Sigma X_\bullet)_n \leftarrow \emptyset \rightarrow (\Sigma X_\bullet)_{n-\{U\}} \). If \( x \in X_{n-1} \), we write \( \Sigma x \) for the corresponding element in \( (\Sigma X_\bullet)_n \). Note that \( C_*(\Sigma X_\bullet; R) = \Sigma C_*(X_\bullet; R) \).

2.11. Maps between augmented semi-simplicial objects in the Burnside category. Let \( X_\bullet \) and \( Y_\bullet \) be augmented semi-simplicial objects in the Burnside category, and write \( \partial^0_U \) and \( \mu^{n_1}_{V_1, V_2} \) for the structure maps in \( X_\bullet \) and \( \partial^0_U \) and \( \mu^{n_1}_{V_1, V_2} \) for the structure maps in \( Y_\bullet \).

A map \( f \) from \( X_\bullet \) to \( Y_\bullet \) consists on the following data:

1. For every \( n \geq -1 \), a span \( f_n : X_n \leftarrow F_n \rightarrow Y_n \),
2. For every \( n \geq -1 \), every \( q \geq 0 \) and every \( U \in P_q(n) \), a fibrewise bijection \( f^0_U : f_{n-q} \circ \partial^0_U \rightarrow \partial^0_U \circ f_n \). In other words, a 2-morphism

\[
\begin{array}{ccc}
X_n & \xrightarrow{f_n} & Y_n \\
\downarrow^{\partial^0_U} & \iff & \downarrow^{\partial^0_U} \\
X_{n-q} & \xrightarrow{f_{n-q}} & Y_{n-q}
\end{array}
\]

such that

3. for each \( n \geq -1 \), \( f^0_U \) is the identity bijection,
4. for each \( n \geq -1 \), each \( U \in P_q(n) \) and each partition \( U = V_1 \cup V_2 \), we have, using the notation of 2.10, that the following diagram of 2-morphisms
commutes:

\[
\begin{array}{c}
X_n \xrightarrow{f_n} Y_n \\
\downarrow \partial_U^n \quad \downarrow \mu_{V_1, V_2}^n \quad \downarrow \bar{\partial}_U^n \\
X_{n_1} \xrightarrow{f_{n_1}} Y_{n_1} \\
\downarrow \partial_{V_1}^{n_1} \quad \downarrow f_{V_1}^{n_1} \quad \downarrow \partial_{V_1}^{n_1} \\
X_{n_2} \xrightarrow{f_{n_2}} Y_{n_2}
\end{array}
\]

in other words, the following diagram of bijections commutes:

\[
\begin{array}{c}
f_{n_2} \circ \partial_U^n \\
\downarrow \mu_{V_1, V_2}^n \downarrow f_{V_1}^{n_1} \downarrow \bar{\partial}_{V_1}^{n_1} \\
f_{n_2} \circ \partial_{V_2}^{n_1} \circ \partial_{V_1}^{n_1} \downarrow f_{V_1}^{n_1} \downarrow \bar{\partial}_{V_1}^{n_1} \downarrow \bar{f}_{n_1} \circ f_n \\
\downarrow \bar{\partial}_{V_1}^{n_1} \downarrow \bar{\partial}_{W_2}^{n_1} \downarrow \bar{\partial}_{V_1}^{n_1} \downarrow f_n \\
\end{array}
\]

(2.4)

(here we have omitted the subscripts under the \( \times \) symbols to lighten the diagram).

**Remark 2.8.** If \( U = V_1 \cup V_2 \cup V_3 \) and \( n - |V_1| = n_1 \) and \( n - |V_1| - |V_2| = n_2 \), the following diagrams commute

\[
\begin{array}{c}
\partial_U^n \leftarrow f_{n_2} \circ \partial_U^n \\
\downarrow \lambda_{V_1} \downarrow f_{V_1}^n \downarrow \lambda_{V_1} \\
\partial_{V_1}^n \leftarrow f_{n_2} \circ \partial_{V_2}^{n_1} \circ \partial_{V_1}^{n_1} \\
\downarrow \lambda_{V_1} \downarrow \lambda_{V_1} \\
\partial_{V_1}^n \leftarrow \partial_{V_1}^n \circ f_n \\
\downarrow \bar{\partial}_{V_1}^{n_1} \downarrow \bar{\partial}_{V_1}^{n_1} \downarrow f_n \\
\end{array}
\]

(2.5)

\[
\begin{array}{c}
\partial_U^n \leftarrow f_{n_2} \circ \partial_U^n \\
\downarrow \lambda_{V_1, V_2} \downarrow f_{V_2}^{n_1} \circ \partial_{V_1}^{n_1} \\
\partial_{V_1}^n \leftarrow f_{n_2} \circ \partial_{V_2}^{n_1} \circ \partial_{V_1}^{n_1} \\
\downarrow \lambda_{V_1, V_2} \downarrow \lambda_{V_1, V_2} \\
\partial_{V_1}^n \leftarrow \partial_{V_1}^n \circ f_n \\
\downarrow \bar{\partial}_{V_1}^{n_1} \downarrow \bar{\partial}_{V_1}^{n_1} \downarrow f_n \\
\end{array}
\]

(2.6)

3. CUP-\( i \) OPERATIONS

In this section we state the bulk of our main theorem: the cochain complex of any augmented semi-simplicial object in the Burnside category has a stable symmetric multiplication. The proof is given in the next section. Our presentation is parallel
to the presentation of the symmetric multiplication on the cochain complex of a simplicial set given in [MMIS].

3.1. Sequences. Let \( P_q(n) \) be the set of \( q \)-tuples \( U \) of non-decreasing non-negative integers bounded by \( n \) where every number appears at most twice, i.e., an element is a sequence \( (0 \leq u_1 \leq u_2 \leq \ldots \leq u_q \leq n) \) where \( u_{i-1} = u_i = u_{i+1} \) never occurs. The cardinal of a sequence \( U \) will be denoted \( |U| \).

If \( U \in P_q(n) \), let \( \tilde{U} \subset U \) be the subset of non-repeated numbers and let \( \bar{U} \subset U \) be the subset of repeated numbers. For example, if \( U = (1, 2, 2, 3, 4, 4) \), then \( \bar{U} = (2, 4) \) and \( \tilde{U} = (2, 4) \). Define \( P^q(n) = \{ U \in P_q(n) \mid |\tilde{U}| = r \} \).

If \( U \in P_q^0(n) \), define the index of \( u_i \) in \( U \) as \( \text{ind}_U(u_i) = u + i \). If \( U \in P_q^r(n) \), let \( \tilde{U} \in P_q^{0-r}(n) \) be the result of removing one instance of each repeated number. If \( u_i \in \tilde{U} \), define the index of \( u_i \) in \( U \) as \( \text{ind}_U(u_i) = \{ u + i, u + i + 1 \} \). Set \( U^+, U^- \subset U \) as the subsets that contain all elements of positive (negative) index.

3.2. Wedge products of face spans. Let \( X_\bullet \) be an ordered augmented semi-simplicial object in the Burnside category, as defined in Section 2.10.

**Definition 3.1.** Let \( U, V \in P(n) \) and let \( (s, t) \in \partial U \times \partial V \). A pair of disjoint subsets \( W = (W^+, W^-) \) of \( U \cap V \) is \((s, t)\)-good if

\[
\lambda_{W^+}(s) = \lambda_{W^-}(t) \quad \quad \lambda_{W^+-W^-}(s) \neq \lambda_{W^+-W^-}(t).
\]

Furthermore, such pair is \((s, t)\)-positive (or positive, for short) if

\[
n + |U| + |V| \text{ is even and } \lambda_{W^+, W^-}(s) < \lambda_{W^+, W^-}(t) \text{ or } n + |U| + |V| \text{ is odd and } \lambda_{W^+, W^-}(s) > \lambda_{W^+, W^-}(t)
\]

and \((s, t)\)-negative (or negative, for short) if

\[
n + |U| + |V| \text{ is even and } \lambda_{W^+, W^-}(s) > \lambda_{W^+, W^-}(t) \text{ or } n + |U| + |V| \text{ is odd and } \lambda_{W^+, W^-}(s) < \lambda_{W^+, W^-}(t)
\]

Clearly, if \( \lambda_{W^+}(s) = \lambda_{W^-}(t) \) and \( W_1^+ \subset W^+ \), then \( \lambda_{W_1^+}(s) = \lambda_{W^-}(t) \) as well, and if \( \lambda_{W^+, W^-}(s) \neq \lambda_{W^+, W^-}(t) \) and \( W^+ \subset W^+_1 \), then \( \lambda_{W^+, W^-}(s) \neq \lambda_{W^+_1, W^-}(t) \) too. In this line, we have the following straightforward lemma:

**Lemma 3.2.** Let \( U, V \in P(n) \), let \( (s, t) \in \partial U \times \partial V \), let \((W^+, W^-)\) be a \((s, t)\)-good pair and suppose that \( 0 \leq w \leq n \) with \( w \notin W^- \).

1. The pair \((W^+, W^- \cup \{w\})\) is \((s, t)\)-good if and only if \( w \notin W^+ \), and
2. the pair \((W^+ \setminus \{w\}, W^- \cup \{w\})\) is \((s, t)\)-good if and only if \( w \in W^+ \).

Two pairs \((W_0^+, W_0^-)\) and \((W_1^+, W_1^-)\) are consecutive if \( W_1^- \) is obtained from \( W_0^- \) by adding an element and \( W_1^+ \) is the result of removing that element (if it is there) from \( W_0^+ \). We write \((W_0^+, W_0^-) \prec (W_1^+, W_1^-)\) if the two pairs are consecutive. Let \( \mathcal{O}_{s,t}(U \cap V) \) be the collection of maximal chains

\[
(W_1^+, W_1^-) \prec \ldots \prec (W_r^+, W_r^-), \quad r = |U \cap V|
\]

of \((s, t)\)-good pairs. A maximal chain is positive (resp. negative) if all its entries are positive (resp. negative). A maximal chain is almost positive (resp. almost negative) if \((W_i^+, W_i^-)\) is positive (resp. negative) for all \( i \neq r \). For every maximal chain \( W \),
necessarily \((W^n, W^o) = (\emptyset, U \cap V)\). We say that \((s, t)\) is positive (negative) if \((s, t) = (\emptyset, U \cap V)\) is positive (negative). We introduce the following subsets of \(O_{s,t}(U \cap V)\):

\[
\begin{align*}
O_{s,t}(U \cap V)^\pm & \quad \text{the set of positive (negative) maximal chains,} \\
O_{s,t}(U \cap V)^\pm_o & \quad \text{the set of almost positive (negative) maximal chains,} \\
O_{s,t}(U \cap V)^\pm_\emptyset & \quad \text{the set of almost positive (negative) maximal chains such that } W^n_{r-1} = \emptyset, \\
O_{s,t}(U \cap V)^\pm_o & \quad \text{the set of almost positive (negative) maximal chains such that } W^n_{r-1} \neq \emptyset, \\
O_{s,t}(U \cap V)^\pm_o \{x\} & \quad \text{the set of almost positive (negative) maximal chains such that } W^n_{r-1} = \emptyset \text{ and } U \cap V \setminus W^n_{r-1} = \{x\}, \\
O_{s,t}(U \cap V)^\pm_o \{x\} & \quad \text{the set of almost positive (negative) maximal chains such that } W^n_{r-1} \neq \emptyset \text{ and } U \cap V \setminus W^n_{r-1} = \{x\}.
\end{align*}
\]

**Definition 3.3.** Let \(U, V\) be sequences in \(P(n)\) and let \(x \in U \cap V\). If the symbol \(\square\) denotes either \(\odot, \Delta, \vee\) or the absence of a symbol, define the spans

\[
\begin{align*}
&\partial_U \odot \partial_V \quad X_n \times X_n \leftrightarrow \sum_{(s,t) \in \partial U \times \partial V} O_{s,t}(U \cap V)^\pm_\square \rightarrow X_{n-|U|} \times X_{n-|V|} \\
&\partial_U \odot \partial_V \quad X_n \times X_n \leftrightarrow \sum_{(s,t) \in \partial U \times \partial V} O_{s,t}(U \cap V)^\pm_\square \rightarrow X_{n-|U|} \times X_{n-|V|} \\
&\partial_U \odot \partial_V \quad X_n \times X_n \leftrightarrow \sum_{(s,t) \in \partial U \times \partial V} O_{s,t}(U \cap V)^\pm_o \{x\} \rightarrow X_{n-|U|} \times X_{n-|V|} \\
&\partial_U \odot \partial_V \quad X_n \times X_n \leftrightarrow \sum_{(s,t) \in \partial U \times \partial V} O_{s,t}(U \cap V)^\pm_o \{x\} \rightarrow X_{n-|U|} \times X_{n-|V|} \\
&\partial_U \odot \partial_V \quad X_n \times X_n \leftrightarrow \sum_{(s,t) \in \partial U \times \partial V} O_{s,t}(U \cap V)^\pm_o \{x\} \rightarrow X_{n-|U|} \times X_{n-|V|},
\end{align*}
\]

where the value of the source and target maps at \(\alpha \in O_{s,t}(U \cap V)^\pm_\square\) is

\[
\begin{align*}
\text{source}(\alpha) & = (\text{source}(s), \text{source}(t)), & \text{target}(\alpha) & = (\text{target}(s), \text{target}(t)).
\end{align*}
\]

Note that if \(U \cap V = \emptyset\), then there is a single maximal chain of length 0 that is \(0\)th positive and negative, hence the first two spans are equivalent to \(\partial_U \times \partial_V\), and the last two are the empty span.

3.3. **Stable symmetric comultiplications.** Let \((C_*, d)\) be a chain complex of \(\mathbb{F}_2\)-modules, let \(T: C_* \otimes C_* \rightarrow C_* \otimes C_*\) be the twist homomorphism \(T(a \otimes b) = b \otimes a\), and let \(1: C_* \otimes C_* \rightarrow C_* \otimes C_*\) denote the identity map. A **stable symmetric comultiplication** on \((C_*, d)\) is a family of homomorphisms

\[
\nabla_i: C_* \rightarrow C_* \otimes C_*
\]

with \(i \in \mathbb{Z}\), such that \(\nabla_i\) has degree \(i\) and

\[
\nabla_i \circ d = d \circ \nabla_i + (1 + T) \nabla_{i-1}.
\]

The homomorphisms \(\sim_i: C_* \otimes C_* \rightarrow C_*\) dual to \(\nabla_i\) endow the dual cochain complex with a stable symmetric multiplication.

Let \(X_\bullet\) be an augmented semi-simplicial object in the Burnside category, and let \(C_* := C_*(X_\bullet; \mathbb{F}_2)\) be its \(\mathbb{F}_2\)-realisation. Let \(n \geq -1\) and \(q \geq 0\). Define the span
\( \nabla^{(q)} \) from the set \( X_n \times X_n \) to the set \( \bigcup_{i+j=q} X_{n-i} \times X_{n-j} \) as

\[
\nabla^{(q)} = \sum_{U \in \mathcal{P}_q(n)} \partial_U \wedge \partial_U^+. \tag{3.1}
\]

The \( \mathbb{F}_2 \)-realisation of \( \nabla^{(q)} \) is then a homomorphism \( C_n \otimes C_n \to (C_* \otimes C_*)_{2n-q} \).

Define the homomorphism \( \nabla_k : C_* \to C_* \otimes C_* \) of graded modules of degree \( k \)

\[
\nabla_k = A_{\mathbb{F}_2} \left( \sum_n \nabla^{(n-k)} \circ \Delta_n \right),
\]

where \( \Delta_n : X_n \to X_n \times X_n \) is the diagonal map.

**Theorem 3.4.** The chain complex with \( \mathbb{F}_2 \) coefficients of an ordered augmented semi-simplicial object in the Burnside category \( X_* \) has a symmetric comultiplication whose \( i \)-th operation is the homomorphism \( \nabla_i \).

If \( U, V \in \mathcal{P}_q(n) \) and \( \partial_{U \cup V} \) is a free span, then the composition \( (\partial_{U \cup V} \times \partial_{U \cup V}) \circ \Delta \)

is a free span as well. Hence, if \( U \cap V \neq \emptyset \), and \( (s, t) \in \partial_U^+ \times \partial_V^+ \) and \( \text{source}(s) = \text{source}(t) \), then \( \lambda_{U \cup V}(s) = \lambda_{U \cup V}(t) \), so \( \mathcal{O}_{s,t}(U \cap V) \) is empty and therefore the span \( (\partial_U \wedge \partial_V) \circ \Delta \) is empty.

**Corollary 3.5.** Let \( X_* \) be an ordered augmented semi-simplicial object in the Burnside category. If \( U \in \mathcal{P}_q(n) \) with \( \bar{U} \neq \emptyset \) and \( \partial_{U} \) is a free span, then \( \partial_{U^-} \wedge \partial_{U^+} \circ \Delta \)

is the empty span. As a consequence, if \( X_* \) is a semi-simplicial set, one may replace \( \nabla_{(q)} \) by

\[
\nabla^{(q)} = \sum_{U \in \mathcal{P}_q(n)} \partial_U^+ \wedge \partial_U^-\tag{3.1}
\]

which is the formula given in [MM18] to define the symmetric comultiplication on the chain complex of a simplicial set.

**4. Proof**

That \( 1_k \to \nabla_k \) is a homomorphism of differential graded modules is a consequence of the following equation of spans, which we prove true along this section.

\[
\partial^{2n-q} \circ \nabla^{(q)} \circ \Delta_n + \nabla^{(n-1)} \circ \Delta_{n-1} \circ \partial^n \equiv (1 + T)\nabla^{(n+1)} \circ \Delta_n. \tag{4.1}
\]

Here, \( \partial^n \) denotes the span \( \sum_{x=0}^n \partial_x \) from \( X_n \) to \( X_{n-1} \), whose \( \mathbb{F}_2 \)-linearization is the differential \( C_n \to C_{n-1} \). On the other hand, \( \partial^{2n-q} \) is the span from \( \bigcup_{i+j=2n-q} X_i \times X_j \) to \( \bigcup_{i+j=2n-q-1} X_i \times X_j \) given by the union of all spans

\[
X_i \times X_j \leftarrow \sum_{y=0}^i \partial_i^y \times \text{Id}_j + \sum_{y=0}^j \text{Id}_i \times \partial_j^y \rightarrow X_{i-1} \times X_j \cup X_i \times X_{j-1},
\]

whose \( \mathbb{F}_2 \)-linearization is the differential \( (C_* \otimes C_*)_{2n-q} \to (C_* \otimes C_*)_{2n-q-1} \). Here and during the proof we write \( \text{Id}_n \) for the identity span \( \partial^n \).
4.1. **Rewriting the formula in terms of wedge products.** Recall from Section 2.2 that for each \( x \in \{0, \ldots, n\} \) there are functions

\[
\gamma_x : P_q(n) \xrightarrow{\sim} \{ V \in P_q(n+1) \mid x \notin V \} \subset P_q(n+1)
\]

\[
\xi_x : P_q(n) \xrightarrow{\sim} \{ V \in P_{q+1}(n+1) \mid x \in V \} \subset P_{q+1}(n+1)
\]

with \( \xi_x(U) = \gamma_x(U) \cup \{ x \} \). Define a new function

\[
\xi_{xx} : \mathcal{P}_{q-1}(n-1) \hookrightarrow \mathcal{P}_{q+1}(n)
\]

as \( \xi_{xx}(U) = \gamma_x(U) \cup \{ x, x \} \), and a new span

\[
\partial_x \| \partial_x = \{(a, b) \in \partial_x \times \partial_x \mid a = b\}.
\]

**Lemma 4.1.** If \( U \in \mathcal{P}_{q-1}(n-1) \) and \( x \in \{0, \ldots, n\} \), there are equivalences of spans

\[
\Delta_{n-1} \circ \partial_x = (\partial_x \| \partial_x) \circ \Delta_{n}
\]

\[
\partial_{U-}^n \land \partial_{U+}^n \circ \partial_x = \partial_{\xi_{xx}(U)-} \land \partial_{\xi_{xx}(U)+},
\]

**Proof.** The first equation is clear. For the second, let \( \ell = n - |U^-| - 1 \), let \( m = n - |U^+| - 1 \) and let \( V = \xi_{xx}(U) \in \mathcal{P}_{q}^r(n) \), and observe that

\[
V^- = \xi_x(U^-) \quad V^+ = \xi_x(U^+) \quad \check{V} = \check{U} \cup \{ x \}.
\]

By definition, the left hand-side is

\[
(4.2) \quad X_n \times X_n \leftarrow \sum_{(a, a) \in \partial_x \| \partial_x} \{(a, a)\} \times \mathcal{O}_{s.t}(\check{U})^+ \to X_k \times X_m
\]

and the right hand-side is

\[
\sum_{(s', t') \in \partial_{U-} \times \partial_{U+}} \mathcal{O}_{s', t'}(\check{V})^-\check{x} \to X_k \times X_m.
\]

Observe first that \( \mathcal{O}_{s', t'}(\check{V})^-\check{x} \) is non-empty only if \( \lambda_x(s') = \lambda_x(t') \). Moreover, \( \mu_{x,V^{-}\cup U} \times \mu_{x,V^{+}\cup (x)} \) induces a bijection between the set of those \((s',t')\) such that \( \lambda_x(s') = \lambda_x(t') \) and the indexing set of the summation in (4.2), given by sending \((s',t')\) to \(((a,a),(s,t))\), where

\[a = \lambda_x(s') = \lambda_x(t'), \quad s = \lambda_{x,U^{-}\cup x}(s'), \quad t = \lambda_{x,U^{+}\cup x}(t').\]

To finish the proof, we construct a bijection

\[
\varphi_{a,s,t} : \{(a, a)\} \times \mathcal{O}_{s,t}(\check{U})^+ \to \mathcal{O}_{s', t'}(\check{V})^-\check{x}
\]

given by sending a maximal \((s,t)\)-good chain \((W_{1}^n, W_{1}^\gamma) \prec \ldots \prec (W_{r-1}^n, W_{r-1}^\gamma)\) in \( \mathcal{O}_{s,t}(\check{U})^+ \) to the maximal sequence

\[
(\xi_x(W_{1}^n), \gamma_x(W_{1}^\gamma)) \prec \ldots \prec (\xi_x(W_{r-1}^n), \gamma_x(W_{r-1}^\gamma)) \prec (\emptyset, \check{V})
\]

which is \((s',t')\)-good because for each \( i = 1, \ldots, r-1 \)

\[
\lambda_{\xi_x(W_{i}^n)}(s') = (a, \lambda_{x,W_{i}^n}(s)), \quad \lambda_{\xi_x(W_{i}^n)}(s') = \lambda_{x,W_{i}^n}(s)
\]

and the pair \((\emptyset, \check{V})\) is also good because
The left hand-side of the left column should be $\mu_{x,\gamma_x(U)}(\lambda_{\xi_{x}(W_{r-})}(s'))$ up and $\mu_{x,\gamma_x(U)}(\lambda_{\xi_{x}(W_{r+})}(t'))$ down, but we find the complete notation too burdened. Additionally, since all pairs in the original $(s,t)$-good chain were positive and

$$n - 1 + |U^-| + |U^+| = n - 1 + q - 1, \quad n + |V^-| + |V^+| = n + q + 1$$

have different parity, we have that all the new pairs are negative except possibly the last one, therefore the new chain is almost negative. Moreover $(\xi_{x}(W_{r}^{-}), \gamma_x(W_{r}^{0})) = (\xi_{x}(\emptyset), \gamma_x(\emptyset))$, so the new chain is in $O_{s',t'}(\bar{V}_{\emptyset})^{-,x}$. The maximal chains $(A_{i}^{-}, A_{i}^{0}) \prec \ldots \prec (A_{r}^{-}, A_{r}^{0})$ in $O_{s',t'}(\bar{V}_{\emptyset})^{-,x}$ are precisely the almost negative $(s,t)$-good maximal chains that satisfy that $x \in A_i^{-}$ and $x \notin A_i^{0}$ for all $i < r$. Hence, for every maximal chain $(A_{i}^{-}, A_{i}^{0}) \prec \ldots \prec (A_{r}^{-}, A_{r}^{0})$ in $O_{s',t'}(\bar{V}_{\emptyset})^{-,x}$ and every $0 < i < r$, the inverse function $\gamma_{x}^{-1}$ is well-defined on $A_{i}^{-}$ and the inverse function $\xi_{x}^{-1}$ is well-defined on $A_{i}^{0}$. Hence we can define an inverse of $\phi_{n,s,t}$ by sending a sequence $(A_{1}^{-}, A_{1}^{0}) \prec \ldots \prec (A_{r}^{-}, A_{r}^{0})$ to the sequence $(\emptyset, \emptyset) \prec (\xi_{x}^{-1}(A_{1}^{-}), \gamma_{x}^{-1}(A_{1}^{0})) \prec \ldots \prec (\xi_{x}^{-1}(A_{r}^{-1}), \gamma_{x}^{-1}(A_{r}^{0}-1)).$

\begin{proposition}
The span $\nabla^{n-1}_{n-1} \circ \Delta_{n-1} \circ \partial^{n}$ is equivalent to the following spans

\begin{align*}
\sum_{x=0}^{n} \nabla^{n-1}_{n-1} \circ \Delta_{n-1} \circ \partial_x &= \sum_{x=0}^{n} \left( \sum_{U \in \mathcal{P}_{q-1}(n-1)} \partial_{U}^{-} \land \partial_{U}^{+} \right) \circ \Delta_{n-1} \circ \partial_x \\
&= \sum_{U \in \mathcal{P}_{q-1}(n-1)} \sum_{x=0}^{n} \partial_{U}^{-} \land \partial_{U}^{+} \circ \partial_x \| \partial_x \circ \Delta_n \\
&= \sum_{U \in \mathcal{P}_{q-1}(n-1)} \sum_{x=0}^{n} \partial_{\xi_{x}(U)}^{-} \bar{\lambda}_{\emptyset} \partial_{\xi_{x}(U)}^{+} \circ \Delta_n \\
&= \sum_{U \in \mathcal{P}_{q+1}(n+1)} \sum_{x \in U} \partial_{U}^{-} \bar{\lambda}_{\emptyset} \partial_{U}^{+} \circ \Delta_n
\end{align*}

\end{proposition}

\begin{proof}
The first equality is the definition, the second and third follow from Lemma 4.1. the fourth comes from sending the summand indexed by $U$ and $x$ to the summand indexed by $\xi_{x}(U)$ and $x$, and the fifth holds by definition.
\end{proof}

\begin{lemma}
Let $U \in \mathcal{P}_{q}(n)$ and let $\ell = |U^-|$ and $m = |U^+|$, and let $y \in \{0, \ldots, n-\ell\}$ or $y \in \{0, \ldots, n-m\}$ and let $x = \gamma_{U^-}(y)$ in the first case and $x = \gamma_{U^+}(y)$ in the second case. Then the following spans are equivalent

\begin{align*}
(\partial_{y}^{n-\ell} \times \text{Id}_{n-m}) \circ (\partial_{U}^{-} \land \partial_{U}^{+}) &= \begin{cases}
\partial_{\xi_{x}(\emptyset)}^{-} \bar{\lambda} \partial_{U}^{+} & \text{if } x \notin U^+ \\
\partial_{\xi_{x}(\emptyset)}^{-} \bar{\lambda}_{\emptyset} \partial_{U}^{+} & \text{if } x \in U^+
\end{cases} \\
(\text{Id}_{n-\ell} \times \partial_{y}^{n-m}) \circ (\partial_{U}^{-} \land \partial_{U}^{+}) &= \begin{cases}
\partial_{U}^{-} \bar{\lambda} \partial_{\xi_{x}(\emptyset)}^{+} & \text{if } x \notin U^- \\
\partial_{U}^{-} \bar{\lambda}_{\emptyset} \partial_{\xi_{x}(\emptyset)}^{+} & \text{if } x \in U^-
\end{cases}
\end{align*}

\end{lemma}
Proof. We give the proof of the first equation, the other being completely analogous. By definition, the left hand-side is

\[ X_n \times X_n \leftarrow \sum_{(s,t) \in \partial U^- \times \partial U^+} O_{s,t}(U^- \cap U^+)^+ \times \{a\} \rightarrow X_{\ell-1} \times X_m \]

and the right hand-side is, depending on whether \( x \in U^+ \) or not,

\[ X_n \times X_n \leftarrow \sum_{(s',t') \in \partial U_-(y) \times \partial U_+} O_{s',t'}(\xi_{U^-}(y) \cap U^+)^\sim \rightarrow X_{\ell-1} \times X_m \]

The bijection

\[ \mu_{U^- x}: \partial_{\xi_U^- (y)} \ni \theta \mapsto \partial_{\theta} \cap \partial_U^- \]

induces a bijection between the indexing set of the summation in (4.3) an the indexing set of the summation in (4.4) (or in (4.5)) given by sending \((s',t')\) to \(((s,t),a)\), where

\[ s = \lambda_{U^-}(s'), \quad t = t' \quad a = \lambda_{U^- x}(s'). \]

Therefore it remains to build, in each case, bijections

\[ O_{s,t}(\bar{U})^+ \rightarrow O_{s',t'}(\xi_{U^-}(y) \cap U^+)^\sim, \]

\[ O_{s,t}(\bar{U})^+ \rightarrow O_{s',t'}(\xi_{U^-}(y) \cap U^+)^\sim^x. \]

In the first case, as \( x \notin U^+ \), we have that

\[ \xi_{U^-}(y) \cap U^+ = (\{x\} \cup U^-) \cap U^+ = U^- \cap U^+ = \bar{U}. \]

Define (4.6) by sending a maximal chain to itself. This sends \((s,t)\)-good chains to \((s',t')\)-good chains because for any pair \((W^u, W^s)\) of disjoint subsets of \(U^- \cap U^+\), we have that

\[ \lambda_{W^u}(s) = \lambda_{W^s}(s'), \quad \lambda_{W^u \cap W^s}(s) = \lambda_{W^u \cap W^s}(s') \]

\[ \lambda_{W^u}(t) = \lambda_{W^s}(t'), \quad \lambda_{W^u \cap W^s}(t) = \lambda_{W^u \cap W^s}(t'), \]

and it sends positive pairs to negative pairs and vice versa because

\[ n + |U^+| + |U^-| = n + q + r, \quad n + |\xi_{U^-}(x)| + |U^+| = n + q + r + 1 \]

have different parity.

In the second case, as \( x \in U^+ \), we have that

\[ \xi_{U^-}(y) \cap U^+ = (\{x\} \cup U^-) \cap U^+ = \{x\} \cup (U^- \cap U^+) = \{x\} \cup \bar{U}. \]

Define (4.7) by sending a maximal chain \((W^u_1, W^s_1) \prec \ldots \prec (W^u_r, W^s_r)\) in \(O_{s,t}(\bar{U})^+\) to the maximal chain

\[ (W^u_1, W^s_1) \prec \ldots \prec (W^u_r, W^s_r) \prec (\emptyset, \{x\} \cup \bar{U}). \]

in \(O_{s',t'}(\{x\} \cup \bar{U})\). For \( i = 1, \ldots , r \), we have that

\[ \lambda_{W^u_1}(s) = \lambda_{W^s_1}(s'), \quad \lambda_{W^u_1 \cap W^s_1}(s) = \lambda_{W^u_1 \cap W^s_1}(s') \]

\[ \lambda_{W^u_1}(t) = \lambda_{W^s_1}(t'), \quad \lambda_{W^u_1 \cap W^s_1}(t) = \lambda_{W^u_1 \cap W^s_1}(t'). \]
Therefore, as \((W_i^n, W_i^\circ)\) is \((s, t)\)-good for \(i = 1, \ldots, r\), it is also \((s', t')\)-good for \(i = 1, \ldots, r\). If \(i = r + 1\), we have that \(\lambda_{W_i^\circ}(s') \neq \lambda_{W_i^\circ}(t')\) because of Lemma 3.2 \([1]\), taking \(W^v = W_r^\circ\) and \(w = x\) so that \(W_{r+1}^v = W^v \cup \{w\}\). Therefore, the maximal chain \((4.8)\) is \((s', t')\)-good.

Since for each \(i = 1 \ldots r\), \((W_i^n, W_i^\circ)\) is positive for \((s, t)\), we have that \((W_i^n, W_i^\circ)\) is negative for \((s', t')\) because of the right hand-side of \((4.9)\) and because

\[
n + |U^+| + |U^-| = n + q + r, \quad n + |\xi_{U^+}(x)| + |U^+| = n + q + r + 1
\]

have different parity. Finally, by construction \(\{x\} \cup \bar{U} \setminus W_r^\circ = \{x\}\). Hence \((4.7)\) is well-defined. Its inverse sends a maximal chain \((A_{11}^n, A_1^\circ) \prec \ldots \prec (A_{r+1}^n, A_{r+1}^\circ)\) to the maximal chain \((A_1^1, A_1^\circ) \prec \ldots \prec (A_r^n, A_r^\circ)\).

If \(U \in P_q(n)\) and \(x \in U\), let us write \(U(x) \in P_q(n)\) for the result of removing one instance of \(x\) from \(U\). For example, if \(U = (1, 2, 2, 3)\), then \(U(2) = (1, 2, 3)\) and \(U(3) = (1, 2, 2)\).

**Proposition 4.4.** The span \(\partial^{2n-q} \circ \nabla^{(n)}\) is equivalent to the following spans:

\[
\sum_{U \in P_q(n)} \left( \sum_{y=0}^{n-|U^-|} (\partial_y^{n-|U^-|} \times \text{Id}_{n-|U^+|}) \circ (\partial_U^- \land \partial_U^+) \right)
\]

\[
= \sum_{U \in P_q(n)} \left( \sum_{x \notin U^-} \partial_{\{x\} \cup U^-} \bar{\lambda} \partial_U^+ + \sum_{x \notin U^+} \partial_U^- \bar{\lambda} \partial_{\{x\} \cup U^+} \right)
\]

\[
= \sum_{U \in P_q(n)} \left( \sum_{x \notin U} \partial_{\{x\} \cup U^-} \bar{\lambda} \partial_U^+ + \partial_{U^+} \bar{\lambda} \partial_{\{x\} \cup U^+} \right)
\]

\[
+ \sum_{x \in U^+ \setminus U^-} \partial_{\{x\} \cup U^-} \bar{\lambda} \bar{\nu} \partial_U^+
\]

\[
+ \sum_{x \in U^- \setminus U^+} \partial_U^- \bar{\lambda} \bar{\nu} \partial_{\{x\} \cup U^+}
\]

\[(4.11)\]

\[
= \sum_{U \in P_q+1(n)} \left( \sum_{x \in U} \partial_{\{x\} \cup U^-} \bar{\lambda} \partial_{U(x)^+} + \partial_{U(x)^-} \bar{\lambda} \partial_{\{x\} \cup U(x)^+} \right)
\]

\[
+ \sum_{x \in U \setminus \text{ind}_{U}(x) \text{ even}} \partial_{\{x\} \cup U^-} \bar{\lambda} \bar{\nu} \partial_{U(x)^+}
\]

\[
+ \sum_{x \in U \setminus \text{ind}_{U}(x) \text{ odd}} \partial_{U(x)^+} \bar{\lambda} \bar{\nu} \partial_{\{x\} \cup U(x)^+}
\]

\[(4.12)\]

\[(4.13)\]

**Proof.** The first term is the definition of \(\partial^{2n-q} \circ \nabla^{(n)}\). The first equality holds by Lemma 4.3 by sending the summand indexed by \(U\) and \(y\) to the summand indexed by \(U\) and either \(x = \gamma_{U^-}(y)\) or \(x = \gamma_{U^+}(y)\), depending on the case. The second
equality is a rearrangement of the terms, and the third equality is given by sending the summand indexed by \( U \) and \( x \) to the summand indexed by \( U \cup \{x\} \).

We immediately have:

**Lemma 4.5.** The summands (4.12) and (4.13) add up to:

\[
\sum_{U \in \mathcal{P}_{q+1}(n)} \partial U_- \wedge \partial U_+.
\]

**Lemma 4.6.** The summand (4.11) is \( \mathbb{F}_2 \)-equivalent to:

\[
\sum_{U \in \mathcal{P}_{q+1}(n)} \partial U_- \wedge \partial U_+ + \partial U_+ \wedge \partial U_+.
\]

**Proof.** The summand (4.11) is empty unless it is indexed by a sequence \( U \in \mathcal{P}_{q+1}(n) \) such that \( \hat{U} \neq \emptyset \). Let \( U \) be such a sequence and let \( x \in \hat{U} \). In order to lighten the notation, write \( U[x]^\pm = \{x\} \cup U(x)^\pm \) and define

\[
\hat{U}^\pm_{<x} = \{u \in \hat{U}^\pm \mid u < x\} \quad \hat{U}^\pm_{>x} = \{u \in \hat{U}^\pm \mid u > x\},
\]

and note that

\[
U(x)^+ = \hat{U}^+_x \cup \hat{U}^-_{>x} \cup \hat{U}, \quad U(x)^- = \hat{U}^-_x \cup \hat{U}^+_<x \cup \hat{U},
\]

(4.14)

Assume first that \( x \) is either the first or the last number in \( \hat{U} \) (which need not to be different). Then

(4.15) \( \partial U_- \wedge \partial U_+ = \begin{cases} \partial U[x]^+ \wedge \partial U(x)^+ & \text{if } x \text{ is odd and the last number in } \hat{U} \\ \partial U[x]^+ \wedge \partial U(x)^+ & \text{if } x \text{ is even and the last number in } \hat{U} \end{cases} \)

(4.16) \( \partial U_+ \wedge \partial U_- = \begin{cases} \partial U[x]^+ \wedge \partial U(x)^+ & \text{if } x \text{ is even and the first number in } \hat{U} \\ \partial U[x]^+ \wedge \partial U(x)^+ & \text{if } x \text{ is odd and the first number in } \hat{U} \end{cases} \)

In every other case not treated in (4.15) or (4.16), write \( l(x) \) for the element in \( \hat{U} \) that precedes \( x \) and \( r(x) \) for the element in \( \hat{U} \) that succeeds \( x \). From (4.14) it follows that

\[
\begin{align*}
U[l(x)]^- &= U[x]^-, & U[l(x)]^+ &= U(x)^+ & \text{if } l(x) \in U^- \text{ and } x \in U^+ \\
U(l(x))^- &= U[x]^-, & U(l(x))^+ &= U(x)^+ & \text{if } l(x) \in U^+ \text{ and } x \in U^+ \\
U[l(x)]^- &= U(x)^-, & U[l(x)]^+ &= U[x]^+ & \text{if } l(x) \in U^- \text{ and } x \in U^- \\
U(l(x))^- &= U(x)^-, & U[l(x)]^+ &= U[x]^+ & \text{if } l(x) \in U^+ \text{ and } x \in U^- 
\end{align*}
\]

and therefore, all the terms of (4.11) indexed by \( U \) that do not appear in (4.15) or (4.16) are paired as follows:

\[
\partial U[x]^+ \wedge \partial U(x)^+ = \begin{cases} \partial U[l(x)]^- \wedge \partial U[l(x)]^+ & \text{if } l(x) \in U^- \text{ and } x \in U^+ \\ \partial U[r(x)]^- \wedge \partial U[r(x)]^+ & \text{if } x \in U^- \text{ and } r(x) \in U^+ \\ \partial U[l(x)]^- \wedge \partial U[l(x)]^+ & \text{if } l(x) \in U^+ \text{ and } x \in U^+ \\ \partial U[r(x)]^- \wedge \partial U[r(x)]^+ & \text{if } x \in U^- \text{ and } r(x) \in U^- 
\end{cases}
\]
that we want to prove translates into those summands labeled by $U$.

Using the simplifications of Lemmas 4.5 and 4.6, together with the identity (4.11) is $F$-equivalent to the sum of (4.15) and (4.16). As a consequence, (4.17) becomes $F$-equivalent to the sum of (4.15) and (4.16).

4.2. Summing everything up. For each $U \in \mathcal{P}_{q+1}(n)$, let $S(U)$ be the sum of those summands indexed by $U$ in Propositions 4.2 and 4.4. Then the equality (4.1) that we want to prove translates into

$$\sum_{U \in \mathcal{P}_{q+1}(n)} S(U) \equiv \sum_{U \in \mathcal{P}_{q+1}(n)} \partial_{U^-} \land \partial_{U^+} + \partial_{U^+} \land \partial_{U^-}. \tag{4.17}$$

Using the simplifications of Lemmas 4.5 and 4.6 together with the identity

$$\partial_{U^-} \land \partial_{U^+} + \partial_{U^+} \land \partial_{U^-} = \partial_{U^-} \land \partial_{U^+},$$

the summand indexed by $U$ in the left hand-side of (4.17) becomes

$$S(U) \equiv \partial_{U^-} \land \partial_{U^+} + \partial_{U^+} \land \partial_{U^-} \quad \text{if} \quad \hat{U} = \emptyset$$

$$S(U) = \partial_{U^-} \land \partial_{U^+} + \partial_{U^+} \land \partial_{U^-} \quad \text{if} \quad \hat{U} = \emptyset$$

$$S(U) \equiv \partial_{U^-} \land \partial_{U^+} + \partial_{U^+} \land \partial_{U^-} \land \partial_{U^-} \quad \text{if} \quad \hat{U} \neq \emptyset \text{ and } \hat{U} \neq \emptyset.$$  

If $\hat{U} = \emptyset$, then $\partial_{U^-} \land \partial_{U^+} = \partial_{U^-} \land \partial_{U^+} = \partial_{U^-} \land \partial_{U^+}$ so (4.17) holds. Otherwise, (4.17) becomes

$$\partial_{U^-} \land \partial_{U^+} \equiv \partial_{U^-} \land \partial_{U^+} + \partial_{U^+} \land \partial_{U^-} \quad \text{if} \quad \hat{U} = \emptyset$$

$$\partial_{U^-} \land \partial_{U^+} + \partial_{U^-} \land \partial_{U^+} = \partial_{U^-} \land \partial_{U^+} \land \partial_{U^-} \quad \text{if} \quad \hat{U} \neq \emptyset.$$  

We may now cancel both appearances of $\partial_{U^+} \land \partial_{U^-}$ in the second line, whereas in the first line, $U^- = U = U^+$, so $\partial_{U^-} \land \partial_{U^+} = \partial_{U^-} \land \partial_{U^+}$. Therefore we are left with the same equation in both cases:

$$\partial_{U^-} \land \partial_{U^+} + \partial_{U^-} \land \partial_{U^+} \land \partial_{U^-} \equiv \emptyset.$$  

We will prove that this equation holds in Proposition 4.12, finishing the proof of the theorem.

4.3. Colored graphs. Let $\Gamma$ be a graph with vertices $V(\Gamma)$ and edges $E(\Gamma)$. Let $\Delta^{k-1}$ be the standard simplex of dimension $k-1$, whose set of faces we identify with $2^k := \text{Map}(\{1, \ldots, k\}, \{0, 1\})$. Let $I(k)$ be the $k$-dimensional cube whose vertices are the elements of $2^k$, and let $dI(k)$ be the $(k-1)$-dimensional polyhedron that results from removing the maximal face of $I(k)$.

**Definition 4.7.** A $k$-coloring on $\Gamma$ is a pair of functions (which we refer to as “labels”) $\nu: V(\Gamma) \to 2^k$ and $\epsilon: E(\Gamma) \to \{1, \ldots, k\}$ such that

1. each vertex has $k$ incident edges, all labeled differently;
2. if $v_1$ and $v_2$ are the two ends of an edge $e$ and $\nu(e) = i$, then $\nu(v_1)[j] = v(v_2)[j]$ for all $j \neq i$. 

We define $\nu(\Gamma)$ to be the product of these functions. 

We will now turn our attention to the proof of Theorem 4.12.
An ordered simplicial complex is a simplicial complex together with an ordering of the vertices of each simplex. Note that this is not the usual terminology.

Recall that the barycentric subdivision of a polyhedron \( P \) is the ordered simplicial complex \( \text{sd} P \) whose vertices are the non-empty faces of \( P \) and a sequence of faces \((\sigma_0, \ldots, \sigma_k)\) forms an ordered simplex of \( \text{sd} P \) if \( \sigma_{i-1} \) is a face of \( \sigma_i \) for all \( i = 1 \ldots k \).

If \((\Gamma, v, \epsilon)\) is a \( k \)-colored graph, define an ordered simplicial complex \( S\Gamma \) whose set of vertices is the union of \( V(\Gamma) \) and the set \( \partial I(k) > 0 \) of faces of \( \partial I(k) \) of positive dimension. A sequence of vertices \((v_1, \ldots, v_j)\) forms an ordered simplex if and only if \( v_1 \in V(\Gamma), \{v_2, \ldots, v_j\} \) is a collection of faces of positive dimension of \( \partial I(k) \) and \((v(v_1), v_2, \ldots, v_j)\) is a simplex in \( \text{sd} \partial I(k) \).

The following lemma is immediate from the definition of a \( k \)-colored graph.

**Lemma 4.8.** Let \((\Gamma, v, \epsilon)\) be a \( k \)-colored graph. Then

1. The 1-skeleton of \( S\Gamma \) is \( \text{sd} \Gamma \).
2. The star of each vertex of \( S\Gamma \) is simplicially isomorphic to the star of any vertex of \( \partial I(k) \), in particular \( S\Gamma \) is a simplicial manifold of dimension \( k - 1 \).
3. There is a map of ordered simplicial complexes \( \varphi: S\Gamma \to \text{sd} I(k) \) given by sending a simplex \((v_1, v_2, \ldots, v_j)\) to the simplex \((\varphi(v_1), v_2, \ldots, v_j)\).
4. If \( v \in V(\partial I(k)) \), then \( \varphi^{-1}(v) \subset V(\Gamma) \).
5. For each vertex \( v \) of \( \Gamma \), the restriction \( \varphi|_{\text{star}_{\Gamma}(v)}: \text{star}_{\Gamma}(v) \to \text{star}_{\partial I(k)}(\varphi(v)) \) is an isomorphism of simplicial complexes. In particular, the map \( \varphi \) is regular at the vertices of \( \Gamma \).

Now, observe that the value of \( \varphi \) at a vertex is the same as the value of \( v \) at that vertex, and that, by [2], \( S\Gamma \) is a simplicial manifold of dimension \( (k - 1) \) and that, by [4] and [5], the map \( \varphi \) is PL-regular at every vertex of \( \Gamma \), so the inverse images of any two vertices are cobordant 0-dimensional PL-manifolds (cf. [Mil97] p. 24) for a smooth version of this statement), hence their cardinalities have the same parity:

**Corollary 4.9.** Let \( \Gamma \) be a \( k \)-colored graph. Then, for each pair of vertices \( v_1, v_2 \in \partial I(k) \), we have that \( |v^{-1}(v_1)| \equiv |v^{-1}(v_2)| \mod 2 \).

Let \( \Gamma \) and \( \Theta \) be \( k \)-colored graphs. A graph map \( f: \Gamma \to \Theta \) is color-preserving if for every edge \( e \) of \( f \), we have that \( \epsilon(e) = \epsilon(f(e)) \). Such a map induces a simplicial map \( S\Gamma \to S\Theta \), and the same argument as before yields the following generalisation of the previous Corollary.

**Corollary 4.10.** Let \( \Gamma \) and \( \Theta \) be \( k \)-colored graphs and let \( f: \Gamma \to \Theta \) be a color preserving graph map. If \( \Theta \) is connected, then for any two vertices \( v_1, v_2 \) of \( \Theta \),

\[ |f^{-1}(v_1)| \equiv |f^{-1}(v_2)| \mod 2 \]

4.4. Application. Take \( k = r - 1 \), and, for each \((s, t) \in \partial_I^- \times \partial_I^+ \) such that \( s \neq t \) construct a \( k \)-colored graph \( \Gamma(s, t) \) whose vertices are the maximal chains of \((s, t)\)-good pairs and there is an edge between two different maximal chains \((W_{j_0}, W_{j_0}^o)\) and \((V_{j_0}^0, V_{j_0}^r)\) if there is a \( i \in 1 \ldots r - 1 \) such that \((W_{j_0}^i, W_{j_0}^i) = (V_{j_0}^i, V_{j_0}^i) \) for all \( j \neq i \). We label such an edge with the number \( i \) and each vertex
with the positiveness function of the chain:

\[ v((W^i_j, W^o_j))_{j=1}^r[i] = \begin{cases} 
0 & \text{if the pair } (W^i_j, W^o_j) \text{ is positive}, \\
1 & \text{if the pair } (W^i_j, W^o_j) \text{ is negative},
\end{cases} \quad i = 1, \ldots, r - 1. \]

**Lemma 4.11.** $\Gamma(s,t)$ is a $k$-colored graph.

**Proof.** First, two chains connected with an edge labeled by $i$ may only change positivity in their $i$th pairs, so Condition (2) holds. For Condition (1), we need to prove that every chain $\{(W^i_j, W^o_j)\}_{j=1}^r$ has a unique edge labeled by $i$. Assume first that this edge exists and let the other end be the chain $\{(V^i_j, V^o_j)\}_{j=1}^r$. Then, we have that

\[ V_j = W_j \quad \text{for all } j \neq i. \tag{4.18} \]

Write
\[
w_{i+1} = W^o_{i+1} \setminus W^o_i \quad w_i = W^o_i \setminus W^o_{i-1} \\
v_{i+1} = V^o_{i+1} \setminus V^o_i \quad v_i = V^o_i \setminus V^o_{i-1}.
\]

Then, because of (4.18) and because we assume that both chains are different, we have necessarily that $v_i = w_{i+1}$ and $w_i = v_{i+1}$. Therefore, we have that

\[ V^o_i = V^o_{i-1} \cup \{v_i\}, \quad V^o_i = \begin{cases} 
V^o_{i-1} & \text{if } v_i \notin V_{i-1} \\
V^o_{i-1} \setminus \{v_i\} & \text{if } v_i \in V_{i-1}.
\end{cases} \]

Therefore, if it exists, then it is uniquely given by this formula. To see that it exists, we have to see that all $V_j$’s are $(s,t)$-good. That is clear for all $j \neq i$ because $W_j$ is $(s,t)$-good, and also for $j = i$, because of Lemma 3.2.

**Proposition 4.12.** Let $U \in P^r(n)$ be a sequence with $\hat{U} \neq U$. Then

\[ \partial U^- \sim_\hat{o} \partial U^+ + \partial U^- \sim \partial U^+ + \partial U^- \land \partial U^+ \equiv \emptyset. \]

**Proof.** The left hand-side is, by definition,

\[ \sum_{(s,t) \in \partial U^- \times \partial U^+} O_{(s,t)}(\hat{U})^- + O_{(s,t)}(\hat{U})^+ + O_{(s,t)}(\hat{U})^+ \]

and using the $k$-coloring of Lemma 4.11,

\[ O_{(s,t)}(\hat{U})^- = v^{-1}(1,1,\ldots,1,1) \]
\[ O_{(s,t)}(\hat{U})^+ = \begin{cases} 
v^{-1}(0,0,\ldots,0,0) & \text{if } (s,t) \text{ is positive}, \\
\emptyset & \text{if } (s,t) \text{ is negative},
\end{cases} \]
\[ O_{(s,t)}(\hat{U})^- = \begin{cases} 
v^{-1}(1,1,\ldots,1,1) & \text{if } (s,t) \text{ is negative}, \\
\emptyset & \text{if } (s,t) \text{ is positive},
\end{cases} \]

Hence the sum

\[ O_{(s,t)}(\hat{U})^- + O_{(s,t)}(\hat{U})^- + O_{(s,t)}(\hat{U})^+ \]

is the union of the inverse image under $v$ of two vertices of $\partial I(r-1)$. Therefore, by Corollary 4.9, we have that the cardinality of both preimages have the same parity, hence (4.20) is even, and so (4.19) is $\mathbb{F}_2$-equivalent to the empty span. \qed
5. Steenrod squares

If $\alpha$ is a cocycle in a cochain complex $C^*$ of $\mathbb{F}_2$-modules with a symmetric multiplication $\sim_i: C^* \otimes C^* \to C^*$, then it follows from (1.2) that $\alpha \sim_i \alpha$ is also a cocycle and that if $\alpha$ and $\beta$ are cohomologous, then $\alpha \sim_i \alpha$ is cohomologous to $\beta \sim_i \beta$. Therefore, we obtain for each $i \geq 0$ a well-defined operation

$$\text{sq}^i: H^n(C^*) \to H^{n+i}(C^*), \quad [\alpha] \mapsto [\alpha \sim_{n-i} \alpha]$$

which is called the $i$-th Steenrod square of $C^*$.

**Proposition 5.1.** If $\alpha, \beta \in C^*(X_\bullet; \mathbb{F}_2)$ and $\Sigma(\alpha), \Sigma(\beta) \in C^*(\Sigma X_\bullet; \mathbb{F}_2)$ denote their suspensions, then $\Sigma(\alpha \sim_i \beta) = \Sigma(\beta) \sim_{i+1} \Sigma(\alpha)$. As a consequence, if $[\alpha] \in H^*(X_\bullet; \mathbb{F}_2)$, then $\Sigma \text{sq}^i([\alpha]) = \text{sq}^i \Sigma([\alpha])$.

**Proof.** Let $\nabla_i$ denote the symmetric comultiplication on $C^*(X_\bullet; \mathbb{F}_2)$ and let $\tilde{\nabla}_i$ denote the symmetric comultiplication on $C^*(\Sigma X_\bullet; \mathbb{F}_2) = \Sigma C^*(X_\bullet; \mathbb{F}_2)$. Then

$$(\Sigma \otimes \Sigma) \circ \nabla(n) = T \cdot \tilde{\nabla}^{(n+1)} \circ \Sigma$$

because, writing $\partial_U$ for the $U$th generalised face map of $\Sigma X_\bullet$,

$$\sum_{U \in P_q(n+1)} \partial_U^- \otimes \partial_U^+ = \sum_{U \in P_q(n+1)} \partial_{\psi_0^{-1}(U)^-} \otimes \partial_{\psi_0^{-1}(U)^+} = \sum_{U \in P_q(n)} \partial_{U^+} \otimes \partial_{U^-},$$

where the first equality holds because the span $\partial_U$ is non-empty only if $U$ is in the image of $\psi_0$, and the second equality holds because if $U = \{u_1, \ldots, u_q\}$, then $\psi_0(U) = \{u_1 + 1, \ldots, u_q + 1\}$, therefore the index of every entry changes by one. Additionally, the sum $n + |U^+| + |U^-|$ used to define the positivity of a $(s,t)$-good pair in Definition 3.1 also changes by one.

**Proposition 5.2.** The first Steenrod square is the Bockstein homomorphism.

**Proof.** The Bockstein homomorphism

$$\beta: H^n(X_\bullet; \mathbb{F}_2) \to H^{n+1}(X_\bullet; \mathbb{F}_2)$$

is defined on a cocycle $\alpha$ by first considering $\alpha$ as a cocycle of $C^n(X_\bullet; \mathbb{Z})$, then observing that the coboundary of $\alpha$ is even, because it is a cocycle, and then taking the reduction mod 2 of $\delta(\alpha)/2$. Additionally, writing

$$\delta^{\text{even}} = \sum_{0 \leq i \leq n, i \text{ even}} \delta^i(\alpha), \quad \delta^{\text{odd}} = \sum_{0 \leq i \leq n, i \text{ odd}} \delta^i$$

we have that

$$\delta(\alpha)/2 = \frac{1}{2} \sum_{i=0}^n (-1)^i \delta^i(\alpha) = \frac{1}{2} \left( \delta^{\text{even}}(\alpha) - \delta^{\text{odd}}(\alpha) \right)$$

where the equality $*$ holds because of the following: As $\alpha$ is a cocycle with $\mathbb{F}_2$ coefficients, $\delta(\alpha)$ is even, and therefore both $\delta^{\text{even}}(\alpha)$ and $\delta^{\text{odd}}(\alpha)$ have the same parity. Now, let $e_0$ and $e_1$ be the first two digits in the binary expansion of $\delta^{\text{even}}(\alpha)$ and let $o_0$ and $o_1$ be the first two digits in the binary expansion of $\delta^{\text{odd}}(\alpha)$. As $\delta(\alpha)$ is even, $o_0 - o_1 = 0$, so the parity of $(\delta^{\text{even}}(\alpha) - \delta^{\text{odd}}(\alpha))/2$ is $e_1 + o_1$. Finally, by Lucas’ Theorem, we have that $e_1 = \left( \frac{\delta^{\text{even}}(\alpha)}{2} \right)$ and $o_1 = \left( \frac{\delta^{\text{odd}}(\alpha)}{2} \right)$. 

On the other hand, the first Steenrod square of \( \alpha \) is

\[
\text{sq}^1(\alpha) = \alpha \sim_{n-2} \alpha.
\]

Write \( \nabla_{n-2}[1,1] \) for the component of \( \nabla_{n-2} \) that lands in \( X_{n-1} \times X_{n-1} \), and observe that if \( U = (u_1, u_2) \in \mathcal{P}_2(n) \), then \( u_1 \) and \( u_2 \) have the same parity and if they are even, then \( u_1 \in U^- \) and \( u_2 \in U^+ \), whereas if they are odd, \( u_1 \in U^+ \) and \( u_2 \in U^- \).

Then we have:

\[
\nabla_{n-2}[1,1] = \sum_{U \in \mathcal{P}_2(n+1)} \partial_{U^-} \land \partial_{U^+}
\]

\[
= \sum_{U \in \mathcal{P}_2(n+1)} \partial_{U^-} \land \partial_{U^+} + \sum_{U \in \mathcal{P}_1(n+1)} \partial_{U^-} \land \partial_{U}^+
\]

\[
= \sum_{0 \leq u, v \leq n} \partial_u \times \partial_v + \sum_{0 \leq u < v \leq n, u, v \text{ even}} \partial_v \times \partial_u + \sum_{0 \leq u \leq n} \left( \partial_u \right)_2
\]

\[
= \left( \partial_{\text{even}} \right)_2 + \left( \partial_{\text{odd}} \right)_2
\]

where, as before, we write

\[
\partial_{\text{even}} := \sum_{0 \leq i \leq n, i \text{ even}} \partial_i \quad \partial_{\text{odd}} := \sum_{0 \leq i \leq n, i \text{ odd}} \partial_i.
\]

In the remaining of this section we prove a Cartan formula for these Steenrod squares. The natural product operation on augmented semi-simplicial objects in the Burnside category is the join, that we introduce now. If \( m \leq n \) and \( U \in \mathcal{P}(n) \) or \( U \in \mathcal{P}(n) \), define

\[
U_{\leq m} = \{ u \in U \mid u \leq m \} \quad U_{> m} = \{ u \mid u + m + 1 \in U \}.
\]

If \( n = n_1 + n_2 + 1 \), then there is a bijection \( \mathcal{P}(n) \to \mathcal{P}(n_1) \times \mathcal{P}(n_2) \) given by sending a sequence \( U \) to the pair of sequences \( (U_{\leq n_1}, U_{> n_1}) \). If \( U \in U_{\leq n_1} \), then its index in \( U_{\leq n_1} \) coincides with its index in \( U \), whereas if \( u \in U_{> n_1} \), then its index in \( U_{> n_1} \), and its index in \( U \) coincide if \( n_1 \) and \( |U_{\leq n_1}| \) have different parity, and are opposite if \( n_1 \) and \( |U_{\leq n_1}| \) have the same parity, therefore, writing \( U_{\leq n_1}^{\pm} \) for \( (U_{\leq n_1})^{\pm} \) and \( U_{> n_1}^{\pm} \) for \( (U_{> n_1})^{\pm} \), we have

\[
U_{\leq n_1}^{-} = (U^{-})_{\leq n_1} \quad U_{> n_1}^{-} = \begin{cases} (U^{-})_{> n_1} & \text{if } n_1 + |U_{\leq n_1}| \text{ is odd} \\ (U^{+})_{> n_1} & \text{if } n_1 + |U_{\leq n_1}| \text{ is even} \end{cases}
\]

\[(5.2) \]

\[
U_{\leq n_1}^{+} = (U^{+})_{\leq n_1} \quad U_{> n_1}^{+} = \begin{cases} (U^{+})_{> n_1} & \text{if } n_1 + |U_{\leq n_1}| \text{ is odd} \\ (U^{-})_{> n_1} & \text{if } n_1 + |U_{\leq n_1}| \text{ is even} \end{cases}
\]

**Definition 5.3.** The join product of two augmented semi-simplicial objects in the Burnside category \( X_\bullet, Y_\bullet \) is the augmented semi-simplicial object in the Burnside
category \((X \ast Y)_*\) given by
\[
(X \ast Y)_n = \bigoplus_{n_1 + n_2 = n - 1} X_{n_1} \times Y_{n_2}
\]
\[
\partial^n_u = \bigoplus_{n_1 + n_2 = n - 1} \partial^n_{U_{\leq n_1}} \times \partial^n_{U_{> n_1}},
\]
\[
\mu^n_{V,W} = \bigoplus_{n_1 + n_2 = n - 1} \mu^n_{V_{\leq n_1},W_{\leq n_1}} \times \mu^n_{V_{> n_1},W_{> n_1}}.
\]

There is a canonical isomorphism \(C^*((X \ast Y)_*; R) \cong C^*(X_*; R) \otimes \Sigma C^*(Y_*; R)\), therefore, when \(R = \mathbb{F}_2\), the Künmeth formula gives an isomorphism \(H^*((X \ast Y)_*; \mathbb{F}_2) \cong \Sigma (H^*(X_*; \mathbb{F}_2) \otimes H^*(Y_*; \mathbb{F}_2))\).

**Remark 5.4.** The product of two cubes in the Burnside category \(F: 2^{n_1} \to B, G: 2^{n_2} \to B\) (\cite{LLS17} Definition 5.4), \cite{LLS15} Def. 4.20) is another cube \(F \times G: 2^{n_1+n_2} \to B\) in the Burnside category. If \(|\cdot|: \mathcal{B}_{fin}(2^{n_1+n_2}) \to \textbf{Sp}\) denotes the realisation functor from the finite Burnside category (see Warning in page 9) to the category of spectra of \(\text{\cite{LLS13}},\) then \cite{LLS15} Prop. 4.23
\[|F \times G| \simeq |F| \wedge |G|\]

It is easy to check that \(\Lambda(F \times G) = \Lambda(F) \ast \Lambda(G)\), and one could prove that, writing again \(|\cdot|: \mathcal{B}^{n_1+n_2} \to \textbf{Sp}\) for the functor \(*\) in Diagram 2.1
\[(X \ast Y)_*|_S \simeq \Sigma |X_*|_S \wedge |Y_*|_S,\]

though this computation is beyond the scope of this paper.

**Definition 5.5.** If \(X_*\) and \(Y_*\) are ordered, then we order \((X \ast Y)_*\) lexicographically, i.e., if \((s_1, s_2)\) and \((t_1, t_2)\) are elements in \(\partial^n_u\), and
\[
(s_1, s_2) \in \partial^n_{U_{\leq n_1'}} \times \partial^n_{U_{> n_1'}}, \quad (t_1, t_2) \in \partial^n_{U_{\leq n_1'}} \times \partial^n_{U_{> n_1'}},
\]
then \((s_1, s_2) < (t_1, t_2)\) if and only if one of the following holds:
\[
n_1 < n_1' \quad \text{or} \quad n_1 = n_1', s_1 < t_1 \quad \text{or} \quad n_1 = n_1', s_1 = t_1, s_2 < t_2.
\]
Elements of \(\partial^n_{U_{\leq n_1}}\) are called **strong** and elements of \(\partial^n_{U_{> n_1}}\) are called **weak**.

**Proposition 5.6.** Let \(n = n_1 + n_2 + 1\) and let \(q \geq 0\). The span \(\nabla^{(n)}|_{X_{n_1} \times Y_{n_2}}\) is \(\mathbb{F}_2\)-equivalent to:
\[
\sum_{q_1+q_2=q} \sum_{r_1 \vdash r_2} \sum_{u_1 \in \mathcal{P}_{q_1}^{(n_1)}(n_1)} \sum_{u_2 \in \mathcal{P}_{q_2}^{(n_2)}(n_2)} \left(\partial^n_{U_{\leq n_1'}} \wedge \partial^n_{U_{> n_1'}}\right) \times T^{q_1+n_1+r_1+1}(\partial^n_{U_{> n_2}} \wedge \partial^n_{U_{> n_2}})
\]

So, in general, it is not true that if \(\alpha, \alpha' \in C^*(X_*; \mathbb{F}_2)\) and \(\beta, \beta' \in C^*(Y_*; \mathbb{F}_2)\), then
\[
\alpha \otimes \beta \sim_i \alpha' \otimes \beta' = \sum_{j=0}^{i} (\alpha \sim_j \alpha') \otimes (\beta \sim_{i-j} \beta'),
\]

so the symmetric comultiplication does not satisfy a Cartan formula. But if we forget the twist \(T^{n_1+q_1+r_1+1}\), Proposition 5.6 becomes
\[
\nabla^{(n)}|_{X_{n_1} \times Y_{n_2}} = \sum_{q_1+q_2=q} \nabla^{(n_1)} \times \nabla^{(n_2)}
\]
and since the twist is irrelevant if $\alpha = \alpha'$ and $\beta = \beta'$, we do have a Cartan formula for the squaring operations:

$$\alpha \otimes \beta \sim_{i} \alpha \otimes \beta = \sum_{j=0}^{i} (\alpha \otimes j \alpha) \otimes (\beta \otimes i-j \beta).$$

**Corollary 5.7.** If $\alpha \otimes \beta$ is a cohomology class in $H^{*}(\mathbb{X} \ast \mathbb{Y}, \mathbb{F}_2)$ via the isomorphism (5.3), then

$$sq^{i}(\alpha \otimes \beta) = \sum_{j=0}^{i} sq^{j}(\alpha) \otimes sq^{i-j}(\beta).$$

5.1. **Proof of Proposition 5.6.** There is a bijection between the indexing sets of both summations given by sending $U \in \mathcal{P}_{q}(n)$ to

$$U_{1} = U_{\leq n_{1}}, \quad U_{2} = U_{> n_{1}}, \quad q_{1} = |U_{1}|, \quad q_{2} = |U_{2}|.$$  

Therefore the proposition will follow if for every $U \in \mathcal{P}_{q}(n)$ with $|\hat{U}_{\leq n_{1}}| = r_{1},$

$$(5.6) \quad \partial_{U}^{n} - \partial_{U}^{n} \equiv \left(\partial_{U_{\leq n_{1}}}^{n} \wedge \partial_{U_{> n_{1}}}^{n}\right) \times T_{n_{1}+n_{1}+r_{1}+1} \left(\partial_{U_{> n_{1}}}^{n} \wedge \partial_{U_{> n_{1}}}^{n}\right),$$  

which, developing each term, is the same as

$$\sum_{(s,t) \in \partial_{U_{-}}^{n} \times \partial_{U_{+}}^{n}} \mathcal{O}_{s,t}(\hat{U})^{\pm} \equiv \sum_{(s_{1}, t_{1}) \in \partial_{U_{\leq n_{1}}}^{n} \times \partial_{U_{\leq n_{1}}}^{n}} \mathcal{O}_{s_{1}, t_{1}}(\hat{U}_{\leq n_{1}})^{\pm} \times \mathcal{O}_{s_{2}, t_{2}}(\hat{U}_{> n_{1}})^{\pm},$$

where the sign $\pm$ is positive if $q_{1} + n_{1} + r_{1}$ is odd and negative otherwise. Therefore, it will be enough to prove that for each $s, t \in \partial_{U_{-}}^{n} \times \partial_{U_{+}}^{n},$

$$(5.7) \quad |\mathcal{O}_{s,t}(\hat{U})^{\pm}| = |\mathcal{O}_{s_{1}, t_{1}}(\hat{U}_{\leq n_{1}})^{\pm} \times \mathcal{O}_{s_{2}, t_{2}}(\hat{U}_{> n_{1}})^{\pm}| \mod 2.$$  

From now on we assume that $q_{1} + n_{1}$ is odd, so that (5.2) gives

$$\hat{U}_{\leq n_{1}} = (U^{-})_{\leq n_{1}}, \quad \hat{U}_{> n_{1}} = (U^{-})_{> n_{1}},$$  

$$\hat{U}_{\leq n_{1}} = (U^{+})_{\leq n_{1}}, \quad \hat{U}_{> n_{1}} = (U^{+})_{> n_{1}}.$$  

The proof of the other case is formally the same.

**Definition 5.8.** Let $\mathbb{X}_{\bullet}$ be an ordered augmented semi-simplicial object in the Burnside category, let $U \in \mathcal{P}_{q}(n),$ and let $(s,t) \in \partial_{U_{-}}^{n} \times \partial_{U_{+}}^{n}$. Say that a pair $W = (W^{n}, W^{o})$ of disjoint subsets of $\hat{U}$ is $(s,t)$-parallel if

$$\lambda_{W,s}^{n}(s) = \lambda_{W,s}^{o}(t) \quad \lambda_{W,s}^{n}(s) = \lambda_{W,s}^{o}(t).$$

Note that being parallel is downwards closed: if $W \prec W'$ and $W'$ is $(s,t)$-parallel, then $W$ is $(s,t)$-parallel too. Note also that if $\lambda_{W,s}^{n}(s) = \lambda_{W,s}^{o}(t)$ and $W^{o}$ is empty, then $W$ is $(s,t)$-parallel.

Let $(s,t) \in \partial_{U_{-}}^{n} \times \partial_{U_{+}}^{n}$ with $s = (s_{1}, s_{2})$ and $t = (t_{1}, t_{2})$. An $(s,t)$-good pair $W = (W^{n}, W^{o})$ decomposes as

$$W^{n} = (W^{n})_{\leq n_{1}} \cup (W^{n})_{> n_{1}}, \quad W^{o} = (W^{o})_{\leq n_{1}} \cup (W^{o})_{> n_{1}},$$

and we refer to the pairs

$$W_{\leq n_{1}} := ((W^{n})_{\leq n_{1}}, (W^{o})_{\leq n_{1}}) \quad W_{> n_{1}} := ((W^{n})_{> n_{1}}, (W^{o})_{> n_{1}})$$
Lemma 5.10. The cardinality of the subset \( O \) defines a free involution on \( \mathbb{F} \). This defines a free involution on \( \mathbb{F} \).

Proof. If \( \mathbb{W} \in \mathcal{O}_{s,t}(\tilde{U})^+ \), let \( i \leq r \) be the smallest index such that \( \mathbb{W}_i \) is semi-parallel. This index cannot be 1 because in that case \( \mathbb{W}_1 \) would not be \((s,t)\)-positive. On the other hand, if \( \mathbb{W}_i \) is strong (alternatively, weak), then \( i \) is also the smallest index such that \( \mathbb{W}_i \) is strong (alternatively, weak), because the property of being parallel is downwards closed. Observe now that the following are equivalent,

1. \( w_i \in \mathbb{W}_j^o \) for all \( j < i \),
2. \( w_i \in \mathbb{W}_j^x \) for some \( j < i \),

and define another maximal chain \( \mathbb{W} \in \mathcal{O}_{s,t}(\tilde{U})^+ \) as follows:

\[
\begin{align*}
\mathbb{W}_j^o &= \mathbb{W}_j^o & \text{for all } j \\
\mathbb{W}_j^x &= \mathbb{W}_j^x & \text{for all } j \geq i \\
\mathbb{W}_j^w &= \begin{cases} 
\mathbb{W}_j^x \setminus \{w_i\} & \text{if } w_i \in \mathbb{W}_j^x \\
\mathbb{W}_j^x \cup \{w_i\} & \text{if } w_i \notin \mathbb{W}_j^x 
\end{cases} & \text{for all } j < i
\end{align*}
\]

This defines a free involution on \( \mathcal{O}_{s,t}(\tilde{U})^+ \), hence this set has even cardinality. \( \square \)

Lemma 5.11. The cardinality of the subset \( \mathcal{O}_{s,t}(\tilde{U})^{\pm} \subseteq \mathcal{O}_{s,t}(\tilde{U})^+ \) has the same parity as the cardinality of \( \mathcal{O}_{s,t}(\tilde{U})^{\pm} \times \mathcal{O}_{s,t}(\tilde{U})^{\pm} \), where the sign \( \pm \) is positive if \( r_1 \) is even and negative if \( r_1 \) is odd.

Proof. Let us construct first an \((r-1)\)-colored graph \( \Theta_{r_1,r_2} \) (see Definition 4.7). Its vertices are pairs of functions \((\varphi, \nu)\), where

\[
\varphi: \{1, \ldots, r\} \rightarrow \{\text{weak, strong}\} \quad \nu: \{1, \ldots, r\} \rightarrow \{0, 1\}
\]

and \(|\varphi^{-1}(\text{strong})| = r_1\) and \(|\varphi^{-1}(\text{weak})| = r_2\). The \(s\)th incident edge to the vertex \((\varphi, \nu)\) connects \((\varphi, \nu)\) with \((\bar{\varphi}, \bar{\nu})\), where
(1) If \( \varphi(i) = \varphi(i+1) \), then
\[
\bar{\varphi}(j) = \varphi(j) \text{ for all } j,
\bar{\nu}(i) \neq \nu(i),
\bar{\nu}(j) = \nu(j) \text{ if } j \neq i.
\]

(2) If \( \varphi(i) \neq \varphi(i+1) \), then
\[
\bar{\varphi}(i) = \varphi(i+1), \quad \bar{\varphi}(i+1) = \varphi(i), \quad \bar{\varphi}(j) = \varphi(j) \text{ if } j \neq i, i+1
\]
\[
\bar{\nu}(i) = \nu(i+1), \quad \bar{\nu}(i+1) = \nu(i), \quad \bar{\nu}(j) = \nu(j) \text{ if } j \neq i, i+1.
\]

Observe that \( \Theta_{1,r_2} \) has 4 connected components: if we write
\[
h^s_{\varphi} = \max \{ i \mid \varphi(i) = \text{strong} \}, \quad h^w_{\varphi} = \max \{ i \mid \varphi(i) = \text{weak} \},
\ell^s_{\varphi} = \min \{ i \mid \varphi(i) = \text{strong} \}, \quad \ell^w_{\varphi} = \min \{ i \mid \varphi(i) = \text{weak} \},
\]
then the vertices of \( (\varphi, \nu) \) and \( (\bar{\varphi}, \bar{\nu}) \) are in the same component if and only if
\[
\nu(h^s_{\varphi}) = \bar{\nu}(h^s_{\bar{\varphi}}), \quad \nu(h^w_{\varphi}) = \bar{\nu}(h^w_{\bar{\varphi}}).
\]
Recall the definition of the graph \( \Gamma(s,t) \) from Section 4.4. The vertices of \( \Gamma(s,t) \) that are in \( O_{s,t}(\bar{U}) \neq \) span a connected component of \( \Gamma(s,t) \) that we call \( \Gamma(s,t) \neq \). There is a color-preserving graph map
\[
f : \Gamma(s,t) \neq \rightarrow \Theta_{1,r_2}
\]
that sends a non-semi-parallel \( (s,t) \)-good maximal chain \( \mathcal{W} \) to the vertex \( (\varphi, \nu) \) with
\[
\varphi(i) = \begin{cases} 
\text{strong} & \text{if } W_i \text{ is strong} \\
\text{weak} & \text{if } W_i \text{ is weak}
\end{cases}
\]
\[
\nu(i) = \begin{cases} 
0 & \text{if } W_i \text{ is semi-positive} \\
1 & \text{if } W_i \text{ is semi-negative}
\end{cases}
\]
Let \( \Theta^+_{1,r_2} \) be the set of those vertices of \( \Theta_{1,r_2} \) such that

(1) \( \nu(j) = 0 \) if \( \varphi(j) = \text{strong} \),
(2) \( \nu(j) = 0 \) for all \( j < \ell^s_{\varphi} \).

Then, using (5.8), we have that
\[
f^{-1}(\Theta^+_{1,r_2}) = O_{s,t}(\bar{U})^+.
\]
If we let
\[
\Theta^+_{1,r_2} = \{ (\varphi, \nu) \in \Theta^+_{1,r_2} \mid \ell^s_{\varphi} = \ell \},
\]
and we let
\[
m^w_{\varphi} = \max \{ \varphi(i) = \text{weak}, i \neq h^w_{\varphi} \}
\]
them \( m^w_{\varphi} > \ell^s_{\varphi} \) if and only if \( \ell^s_{\varphi} < r_2 \). If \( \ell < r_2 \), construct a free involution of \( \Theta^+_{1,r_2} \) by sending a vertex \( (\varphi, \nu) \) to the vertex \( (\bar{\varphi}, \bar{\nu}) \) with
\[
\bar{\varphi}(j) = \varphi(j) \text{ for all } j
\]
\[
\bar{\nu}(j) = \nu(j) \text{ if } j = m^w_{\varphi}
\]
\[
\bar{\nu}(j) \neq \nu(j) \text{ if } j \neq m^w_{\varphi}.
\]
The remaining elements of $\Theta_{r_1,r_2}$ are as follows: $\Theta_{r_1,r_2}^+$ has $2r_1$ elements, classified by the pair $(h^w_r - r_2, \nu(h^w_r)) \in \{1, \ldots, r_1\} \times \{0, 1\}$, and given by
\[
\varphi(i) = \begin{cases} 
\text{weak} & \text{if } i < r_2 \text{ or } i = h^w_r \\
\text{strong} & \text{if } i \geq r_2 \text{ and } i \neq h^w_r 
\end{cases}
\]
\[
\nu(i) = \begin{cases} 
0 & \text{if } i \neq h^w_r \\
0 \text{ or } 1 & \text{if } i = h^w_r 
\end{cases}
\]

On the other hand, $\Theta_{r_1,r_2}^{+,r_2+1}$ has a single element given by
\[
\varphi(i) = \begin{cases} 
\text{weak} & \text{if } i \leq r_2 \\
\text{strong} & \text{if } i > r_2 
\end{cases}
\]
\[
\nu(i) = 0 \text{ for all } i.
\]

Define an involution on $\Theta_{r_1,r_2}^+ \cup \Theta_{r_1,r_2}^{+,r_2+1}$ as follows: if $r_1$ is even, then send an element $(\varphi, \nu)$ of $\Theta_{r_1,r_2}^+$ classified by $(h^w_r - r_2, \nu(h^w_r))$ to the element $(\bar{\varphi}, \bar{\nu})$ of $\Theta_{r_1,r_2}$ with
\[
h^w_r = 2 \cdot \left\lceil \frac{n^r_w}{2} \right\rceil
\]
and send the unique element of $\Theta_{r_1,r_2}^{+,r_2+1}$ to itself. Let us denote this latter element by $a$. If $r_1$ is odd, then send an element $(\varphi, \nu)$ of $\Theta_{r_1,r_2}^{+,r_2+1}$ classified by $(h^w_r - r_2, \nu(h^w_r))$ with $h^w_r > r_2 + 1$ to the element $(\bar{\varphi}, \bar{\nu})$ of $\Theta_{r_1,r_2}$ with
\[
h^w_r = 2 \cdot \left\lceil \frac{n^r_w}{2} \right\rceil
\]
and send the element classified by $(r_2 + 1, 0)$ to the unique element of $\Theta_{r_1,r_2}^{+,r_2+1}$, and send the element classified by $(r_2 + 1, 1)$ to itself. Let us denote this latter element by $b$.

Altogether, we have defined an involution on $\Theta_{r_1,r_2}^+$ with a unique fixed point, and such that $(\varphi, \nu)$ and $(\bar{\varphi}, \bar{\nu})$ lie in the same connected component of $\Theta_{r_1,r_2}$.

Hence, by Corollary 4.10, we have that if $(\varphi, \nu) \in \Theta_{r_1,r_2}^+$, then
\[
|f^{-1}(\varphi, \nu)| \equiv |f^{-1}(\bar{\varphi}, \bar{\nu})| \mod 2,
\]
therefore
\[
|f^{-1}(\Theta_{r_1,r_2}^+)| = \begin{cases} 
|f^{-1}(a)| & \text{mod 2 if } r_1 \text{ is even} \\
|f^{-1}(b)| & \text{mod 2 if } r_1 \text{ is odd} 
\end{cases}
\]
Moreover, the vertex $b$ lies in the same connected component of $\Theta_{r_1,r_2}$ as the vertex $c$ defined as
\[
\varphi(i) = \begin{cases} 
\text{weak} & \text{if } i \leq r_2 \\
\text{strong} & \text{if } i > r_2 
\end{cases}
\]
\[
\nu(i) = 1 \text{ for all } i
\]
and therefore $|f^{-1}(b)| \equiv |f^{-1}(c)| \mod 2$. Finally, there are bijections
\[
f^{-1}(a) \mapsto \mathcal{O}_{s_1,t_1}(\bar{U}_{\leq n_1})^+ \times \mathcal{O}_{s_2,t_2}(\bar{U}_{\geq n_1})^+
\]
\[
f^{-1}(c) \mapsto \mathcal{O}_{s_1,t_1}(\bar{U}_{\leq n_1})^- \times \mathcal{O}_{s_2,t_2}(\bar{U}_{\geq n_1})^-
\]
given by sending a maximal chain $W_1 \prec \ldots \prec W_r$ to the pair of maximal chains
\[
W_{r_2+1} \prec \ldots \prec W_r \quad W_1 \prec \ldots \prec W_{r_2}.
\]
The statement now follows from (5.9), (5.10) and (5.11).
6. Naturality

The cup-i products constructed are not natural under maps of augmented semi-simplicial objects in the Burnside category. This is no surprise, because the realisation functor $| \cdot |_2: B^{\Delta_{op}} \to \text{Ch}(\mathbb{Z})$ is not faithful (compare to the case of semi-simplicial sets, where the functor is faithful). The following is a simple situation in which naturality fails:

**Example 6.1.** Let $X_\bullet$ be given by $X_{-1} = \{a\}, X_0 = \{b\}$ and $X_i = \emptyset$ otherwise, and let $\partial_0^b$ be the span $\{b\} \leftarrow Q \rightarrow \{a\}$ with $Q = \{q_1, q_2\}$ and $q_1 < q_2$. Let $Y_\bullet$ be given by $Y_{-1} = \{c\}, Y_0 = \{d\}$ and $Y_i = \emptyset$ otherwise, and let $\partial_0^d$ be the span $\{d\} \leftarrow P \rightarrow \{c\}$ with $P = \{p_1, p_2, p_3\}$ and $p_1 < p_2 < p_3$. We claim that $\sim_{-2}$ is not natural for any non-trivial map from $X_\bullet$ to $Y_\bullet$.

Any such map $f: X_\bullet \to Y_\bullet$ will consist on spans
\[
\{b\} \leftarrow S \rightarrow \{d\} \quad \{a\} \leftarrow T \rightarrow \{c\}
\]
and some bijection between $S \times P$ and $Q \times T$. If $f$ is non-trivial (i.e., $S$ and $T$ are non-empty), then $S$ has cardinality 2 and $T$ has cardinality 3. To compute $\sim_{-2}$ in $X_\bullet$ and $Y_\bullet$ we start with their duals:

\[
\nabla_{-2}(b) = \mathcal{A}_{\bar{S}_2}(\partial_0^b \land \partial_0^d \circ \Delta)(b) \quad \nabla_{-2}(d) = \mathcal{A}_{\bar{S}_2}(\partial_0^d \land \partial_0^b \circ \Delta)(d)
\]

and as the spans $\partial_0^b \land \partial_0^d$ and $\partial_0^d \land \partial_0^b$ are

\[
\{b\} \leftarrow \{(q_1, q_2)\} \rightarrow \{a\} \quad \{d\} \leftarrow \{(p_1, p_2), (p_1, p_3), (p_2, p_3)\} \rightarrow \{c\}
\]
we have that

\[
\nabla_{-2}(b) = 1 \cdot a = a \quad \nabla_{-2}(d) = 3 \cdot c = c
\]

and therefore, writing $a^*, b^*, c^*, d^*$ for the duals of $a, b, c, d$, we have

\[
a^* \sim_{-2} a^* = b^* \quad c^* \sim_{-2} c^* = d^*
\]

but $f^*(a^* \sim_{-2} c^*) = 2d^* = 0$ whereas $f^*(c^*) \sim_{-2} f^*(c^*) = a^* \sim_{-2} a^* = b^*$.

6.1. Naturality of the cup-i products. Let $X_\bullet, Y_\bullet$ be a pair of augmented semi-simplicial objects in the Burnside category and let $f: X_\bullet \to Y_\bullet$ be a map as in Section 2.11, so the face maps in $X_\bullet$ will be denoted $\partial_U^0$ and the face maps in $Y_\bullet$ will be denoted $\bar{\partial}_U^0$.

**Definition 6.2.** The map $f$ is free if for each $n \geq -1$, $f_n: X_n \to Y_n$ is a function of pointed sets (which is the same as a free span, that is a span of the form $X_n \leftarrow F_n \to Y_n$).

Let us now endow both $X_\bullet$ and $Y_\bullet$ with an order, and endow the spans $f_{n-q} \circ \partial_U^0$ and $\bar{\partial}_U^0 \circ f_n$ with the partial order induced by the projection maps

\[
f_{n-q} \circ \partial_U^0 \longrightarrow \partial_U^0 \quad \bar{\partial}_U^0 \circ f_n \longrightarrow \bar{\partial}_U^0.
\]

**Definition 6.3.** A free map from $X_\bullet$ to $Y_\bullet$ is order-preserving if for each $n \geq -1$ and for each $U \in P_q(n)$, the 2-morphism (which is a fibrewise bijection) $f_U^0: f_{n-q} \circ \partial_U^0 \to \bar{\partial}_U^0 \circ f_n$ is the (unique) order preserving fibrewise bijection.

**Proposition 6.4.** The cup-i products are natural with respect to order-preserving free maps.
**Proof.** Let \( f : X_{\bullet} \to Y_{\bullet} \) be such order-preserving free map and let \( \alpha, \beta \) be a \( p \)-cochain and a \( q \)-cochain in \( Y_{\bullet} \), and let \( \sigma \) be a \((p + q - i)\) simplex in \( X_{\bullet} \). Then

\[
(f^*(\alpha \leadsto \beta))(\sigma) = (\alpha \otimes \beta)(\nabla_i(f_*(\sigma)))
\]

\[
= (\alpha \otimes \beta) ((f_* \otimes f_*)(\nabla_i(\sigma)))
\]

\[
= (f^* \alpha \otimes f^* \beta)(\nabla_i(\sigma))
\]

\[
= (f^* \alpha \leadsto_i f^* \beta)(\sigma),
\]

where every equality is formal except for \(*\), which we prove true now: We claim that if \( f \) is free and order-preserving, there is an equivalence of spans

\[
\nabla(\sigma) \circ \Delta \circ f_n = (f \times f) \circ \nabla(\sigma) \circ \Delta.
\]

Note first that if \( f \) is a free map, then

\[
\Delta \circ f_n = (f_n \times f_n) \circ \Delta,
\]

so we have to prove,

\[
\nabla(\sigma) \circ (f_n \times f_n) \circ \Delta = (f \times f) \circ \nabla(\sigma) \circ \Delta
\]

that is,

\[
\sum_{U \in \mathcal{P}_n(n)} \partial_{U^-} \land \partial_{U^+} \circ (f_n \times f_n) \circ \Delta = \sum_{U \in \mathcal{P}_n(n)} (f \times f) \circ (\partial_{U^-} \land \partial_{U^+}) \circ \Delta.
\]

Now, if \( X \xleftarrow{Q} \xrightarrow{\text{source}} Y \) is a locally finite span \( Q \) and \( x \in X \) and \( y \in Y \), define the restriction set \( Q_{x,y} \) as \( \text{source}^{-1}(x) \cap \text{target}^{-1}(y) \). In order to check whether two spans \( Q, Q' \) from \( X \) to \( Y \) are equivalent, it is enough to check that for each \( x \in X \) and each \( y \in Y \), the restriction sets \( Q_{x,y} \) and \( Q'_{x,y} \) are in bijection. Hence, it is enough to check that for each \( U \in \mathcal{P}_n(n) \) with \( n - |U^-| = m \) and \( n - |U^+| = \ell \), each \( x \in X_n \) and each \((y, y') \in Y_m \times Y_{\ell}\), the following sets are in bijection

(6.1) \[
\partial_{U^-} \land \partial_{U^+} \circ (f_n \times f_n)|_{x,y,y'} \quad (f_m \times f_\ell) \circ (\partial_{U^-} \land \partial_{U^+})|_{x,y,y'}.
\]

If \( f \) is a free map, \( U \in \mathcal{P}_n(n) \), \( x \in X_n \) and \( y \in Y_{n-q} \), either the projections

\[
\partial^n_{U} \circ f_n|_{x,y} \quad \partial^n_{U}|_{x,y}
\]

\[
f_{n-q} \circ \partial^n_{U}|_{x,y} \quad \prod_{f(z)=y} \partial^n_{U}|_{x,z}
\]

that forget the component of \( f \) are bijections, or the domain of each projection is empty. In the second situation both sides of (6.2) are empty, so we assume the first situation, in which case there are bijections

(6.3) \[
\varphi : \partial^n_{U}|_{f(x),y} \cong \partial^n_{U} \circ f_n|_{x,y} \cong f_{n-q} \circ \partial^n_{U}|_{x,y} \cong \prod_{f(z)=y} \partial^n_{U}|_{x,z}.
\]

Under the isomorphisms (6.2), each side of (6.1) becomes isomorphic to

\[
\prod_{(s,t) \in \partial^n_{U^-} \times \partial^n_{U^+}} \mathcal{O}_{s,t}(U) \quad \prod_{(s,t) \in \partial^n_{U^-} \times \partial^n_{U^+}} \mathcal{O}_{s,t}(U)
\]

\[
\text{source}(s) = \text{source}(t) = f(x) \quad \text{target}(s) = \text{source}(t) = y
\]

\[
\text{source}(s) = f_m(\text{target}(s)) = y \quad \text{source}(t) = f_z(\text{target}(t)) = y
\]

Now, the bijection \( \varphi \) induces an isomorphism between these two sets because

- a pair \((W^n, W^o)\) is \((s, t)\)-good if and only if it is \((\varphi(s), \varphi(t))\)-good.
A pair \((W^o, W^o)\) is \((s, t)\)-positive if and only if it is \((\varphi(s), \varphi(t))\)-positive.

These two assertions follow from diagrams [2.5] and [2.6], using that \(f\) is order-

\[\square\text{ preserving and noting that, after taking restriction sets, the upper and bottom rows}\]

\[\text{in those diagrams are the bijections [6.3].}\]

\textbf{Corollary 6.5.} If \(f \colon X_\bullet \to Y_\bullet\) is an order-preserving free map of ordered aug-

mented semi-simplicial objects in the Burnside category, then \(f^*\text{sq}^i = \text{sq}^i f^*\).

\[\text{\textbf{6.2. A mapping cylinder construction.}}\]

Recall that if \(f \colon C_\bullet \to C'_\bullet\) is a homo-

morphism of chain complexes of \(R\)-modules, the mapping cylinder of \(f\) is the chain

\[\text{complex } M(f)_\bullet \text{ with } M(f)_n = C_{n-1} \oplus C_n \oplus C'_n \text{ and differential } d'' \text{ given by}\]

\[d''(x, y, z) = (-d(x), d(y) + x, d'(c) - f(x)).\]

This chain complex comes with maps

\[C_\bullet \xrightarrow{i_f} M(f)_\bullet \xrightarrow{g_f} C'_\bullet\]

given by including \(C_\bullet\) as the second summand and by sending an element \((x, y, z)\) to \(f(y) + z\). Moreover, the last map has a homotopy inverse \(h_f \colon C'_\bullet \to M(f)_\bullet\) given by including \(C'_\bullet\) as the third summand, and therefore \(g_f\) is a chain homotopy equivalence. Additionally, \(f = h_f \circ i_f\).

This construction has the following counterpart \(M(\Sigma^2 f)_\bullet\) for the double suspen-

sion of a map \(f \colon X_\bullet \to Y_\bullet\) of augmented semi-simplicial objects in the Burnside category (again, we use the notation of Section 2.11): \(M(\Sigma^2 f)_\bullet\) is the augmented semi-simplicial object in the Burnside category whose set of \(n\)-simplices is

\[M(\Sigma^2 f)_n = X_{n-3} \cup X_{n-2} \cup Y_{n-2}.\]

Write \(\psi_{01}\) and \(\psi_{012}\) for \(\psi_{\{0,1\}}\) and \(\psi_{\{0,1,2\}}\). If \(U \in \text{P}_q(n)\), the generalised face map \(\bar{\partial}^n_0\) of \(M(\Sigma^2 f)_\bullet\) is the sum of the following three spans (note that the second and third spans are precisely the generalised face maps of \(\Sigma^2 X_\bullet\) and \(\Sigma^2 Y_\bullet\)):

\[
\bar{\partial}^n_0|_{X_{n-3}} = \begin{cases} 
\partial^n_{\psi_{012}}(U) & \text{if } \{0, 1, 2\} \cap U = \emptyset \\
\emptyset & \text{if } 2 \in U \\
\partial^n_{\psi_{012}}(U \cup \{0\}) & \text{if } 0 \in U \text{ and } \{1, 2\} \cap U = \emptyset \\
\partial^n_{\psi_{012}}(U \cup \{1\}) \circ f_{n-3} & \text{if } 1 \in U \text{ and } \{0, 2\} \cap U = \emptyset \\
\emptyset & \text{if } \{0, 1\} \subset U 
\end{cases}
\]

\[
\bar{\partial}^n_0|_{X_{n-2}} = \begin{cases} 
\partial^{n-2}_{\psi_{01}}(U) & \text{if } \{0, 1\} \cap U = \emptyset \\
\emptyset & \text{if } 0 \in U \text{ or } 1 \in U 
\end{cases}
\]

\[
\bar{\partial}^n_0|_{Y_{n-2}} = \begin{cases} 
\partial^{n-2}_{\psi_{01}}(U) & \text{if } \{0, 1\} \cap U = \emptyset \\
\emptyset & \text{if } 0 \in U \text{ or } 1 \in U 
\end{cases}
\]

Let \(U = V_1 \cup V_2\) with \(U \in \text{P}_q(n)\) and \(V_1 \in \text{P}_p(n)\), and write as usual \(W_2 = \psi_{V_1}(V_2)\) and observe that

\[\psi_{\psi_A(B)}(\psi_A(C)) = \psi_A(\psi_B(C)),\]

\[(6.4)\]
and if \( i \in \{0, 1\} \) and \( U \cap \{0, 1\} = \{i\} \), then, depending on whether \( i \in V_1 \) or \( i \in V_2 \), we have:

\[
\psi_{01}(\psi_{V_1}(V_2)) = \psi_{012}(\psi_{V_1 \setminus \{i\}}(V_2))
\]

\[
\psi_{012}(\psi_{V_1}(V_2) \setminus \{i\}) = \psi_{012}(\psi_{V_2}(V_2 \setminus \{i\})).
\]

Define the structural 2-morphisms

\[
\bar{\mu}_{V_1, V_2} : \bar{\partial}_U \to \bar{\partial}_W \circ \bar{\partial}_{V_1}
\]

of \( M(\Sigma^2f) \) as follows:

1. if \( 2 \in U \) and \( \{0, 1\} \cap U \neq \emptyset \), then at least two of the three spans are empty, therefore both sides of (6.7) are empty.
2. if \( 2 \in U \) and \( \{0, 1\} \cap U = \emptyset \), then
   \[
   \bar{\mu}_U^n = \mu_{\psi_{01}(U)}^{n-2} \cup \bar{\mu}_{\psi_{01}(U)}^{n-2}
   \]
3. if \( 2 \notin U \) and \( \{0, 1\} \cap U = \emptyset \) then
   \[
   \bar{\mu}_U^n = \mu_{\psi_{12}(U)}^{n-3} \cup \mu_{\psi_{01}(U)}^{n-2} \cup \bar{\mu}_{\psi_{01}(U)}^{n-2}
   \]
4. if \( 2 \notin U \) and \( 1 \notin U \) and \( 0 \in V_1 \), then
   \[
   \bar{\partial}_U^n = \partial_{\psi_{12}(U \setminus \{0\})}^{n-2}
   \]
   \[
   \bar{\partial}_{V_1}^{n-2} = \partial_{\psi_{12}(V_1 \setminus \{0\})}^{n-3}
   \]
   \[
   \bar{\partial}_W^{n-p} = \partial_{\psi_{12}(W_2)}^{n-p-2} \cup \partial_{\psi_{01}(W_2)}^{n-p-2} \cup \bar{\partial}_{\psi_{01}(W_2)}^{n-p-2}
   \]

and since the target of the span \( \bar{\partial}_{V_1}^{n-2} \) is contained in \( X_{n-p-2} \) (the second factor of \( M(\Sigma^2f)_{n-p} \)), we have that

\[
\bar{\partial}_W^{n-p} \circ \bar{\partial}_{V_1}^{n-2} = \partial_{\psi_{012}(V_1 \setminus \{0\})}^{n-p-2} \circ \partial_{\psi_{012}(V_1 \setminus \{0\})}^{n-3}
\]

and setting \( U' = \psi_{012}(U \setminus \{0\}) \), \( V'_1 = \psi_{012}(V_1 \setminus \{0\}) \), \( V'_2 = \psi_{012}(V_2) \), define (6.7) as follows, using the identities (6.4) and (6.5):

\[
\mu_{V_1', V_2'} : \partial_{U'} \to \partial_{W_2'} \circ \partial_{V_1'}
\]

5. if \( 2 \notin U \), \( 1 \notin U \) and \( 0 \in V_2 \), then
   \[
   \bar{\partial}_U^n = \partial_{\psi_{012}(U \setminus \{0\})}^{n-3}
   \]
   \[
   \bar{\partial}_{V_1}^{n-3} = \partial_{\psi_{12}(V_1 \setminus \{0\})}^{n-3} \cup \partial_{\psi_{01}(V_1)}^{n-2} \cup \bar{\partial}_{\psi_{01}(V_1)}^{n-2}
   \]
   \[
   \bar{\partial}_W^{n-p} = \partial_{\psi_{12}(W_2 \setminus \{0\})}^{n-3} \cup \partial_{\psi_{01}(W_2 \setminus \{0\})}^{n-2}
   \]

and since the source of the span \( \bar{\partial}_W^{n-p} \) is contained in \( X_{n-p-3} \) (the first factor of \( M(\Sigma^2f)_{n-p} \)), we have that

\[
\bar{\partial}_W^{n-p} \circ \bar{\partial}_{V_1}^{n-3} = \partial_{\psi_{012}(W_2 \setminus \{0\})}^{n-p-3} \circ \partial_{\psi_{012}(V_1)}^{n-3}
\]

and setting \( U' = \psi_{012}(U \setminus \{0\}) \), \( V'_1 = \psi_{012}(V_1) \), \( V'_2 = \psi_{012}(V_2 \setminus \{0\}) \), define (6.7) as follows, using the identities (6.4) and (6.6):

\[
\mu_{V_1', V_2'} : \partial_{U'} \to \partial_{W_2'} \circ \partial_{V_1'}
\]
(6) if $2 \notin U$, $0 \notin U$ and $1 \in V_1$, then
\[
\tilde{\partial}^n_U = \partial_{\psi_{12}(U\setminus\{1\})}^{n-3} \circ f_{n-3} \\
\tilde{\partial}^n_{V_1} = \partial_{\psi_{01}(V_1\setminus\{1\})}^{n-3} \circ f_{n-3} \\
\tilde{\partial}^{n-p}_{W_2} = \partial_{\psi_{01}(W_2)}^{n-p-3} \cup \partial_{\psi_{01}(W_2)}^{n-p-2} \cup \tilde{\partial}_{\psi_{01}(W_2)}^{n-p-2}
\]
and since the target of the span $\tilde{\partial}^n_{V_1}$ is contained in $Y_{n-p-2}$ (the third factor of $M(\Sigma^2 f)_{n-p}$), we have that
\[
\tilde{\partial}^{n-p}_{W_2} \circ \tilde{\partial}^n_{V_1} = \partial_{\psi_{01}(W_2)}^{n-p-3} \circ \partial_{\psi_{12}(V_1\setminus\{1\})}^{n-3} \circ f_{n-3}
\]
and setting $U' = \psi_{12}(U \setminus \{1\})$, $V'_1 = \psi_{01}(V_1 \setminus \{1\})$, $V'_2 = \psi_{01}(V_2)$, define \([6.7]\) as follows, using the identities \([6.4]\) and \([6.5]\):
\[
\tilde{\rho}^{n-3}_{V_1, V_2} \circ \text{Id}: \tilde{\partial}^{n-3}_{U'} \circ f_{n-3} \rightarrow \tilde{\partial}^{n-p-2}_{W_2} \circ \tilde{\partial}^{n-3}_{V_1'} \circ f_{n-3}
\]
(7) if $2 \notin U$, $0 \notin U$ and $1 \in V_2$, then
\[
\tilde{\partial}^n_U = \partial_{\psi_{12}(U\setminus\{1\})}^{n-3} \circ f_{n-3} \\
\tilde{\partial}^n_{V_1} = \partial_{\psi_{01}(V_1\setminus\{1\})}^{n-3} \cup \partial_{\psi_{01}(V_1)}^{n-2} \cup \tilde{\partial}_{\psi_{01}(V_1)}^{n-2} \\
\tilde{\partial}^{n-p}_{W_2} = \partial_{\psi_{01}(W_2\setminus\{1\})}^{n-p-3} \circ f_{n-p-3}
\]
and since the source of the span $\tilde{\partial}^{n-p}_{W_2}$ is contained in $X_{n-p-3}$ (the first factor of $M(\Sigma^2 f)_{n-p}$), we have that
\[
\tilde{\partial}^{n-p}_{W_2} \circ \tilde{\partial}^n_{V_1} = \partial_{\psi_{01}(W_2\setminus\{1\})}^{n-p-3} \circ f_{n-p-3} \circ \partial_{\psi_{12}(V_1)}^{n-3}
\]
and setting $U' = \psi_{12}(U \setminus \{1\})$, $V'_1 = \psi_{01}(V_1)$, $V'_2 = \psi_{01}(V_2 \setminus \{1\})$, define \([6.7]\) as follows, using the identities \([6.4]\) and \([6.6]\):
\[
\tilde{\partial}^{n-3}_{U'} \circ f_{n-3} \xrightarrow{\mu_{V_1, V_2}} \tilde{\partial}^{n-p-3}_{W_2} \circ \tilde{\partial}^{n-3}_{V_1'} \circ f_{n-3} \xleftarrow{f_{V_1}} \tilde{\partial}^{n-p-3}_{W_2} \circ \tilde{\partial}^{n-3}_{V_1'}
\]
(8) If $2 \notin U$ and either $\{0, 1\} \subset V_1$ or $\{0, 1\} \subset V_2$, then at least two of the three spans in \([6.7]\) are the empty span, and therefore both sides are empty.
(9) If $0 \in V_1$ and $1 \in V_2$
\[
\tilde{\partial}^n_U = \emptyset \quad \tilde{\partial}^n_{V_1} = \partial_{\psi_{01}(V_1\setminus\{0\})}^{n-3} \quad \tilde{\partial}^{n-p}_{W_2} = \partial_{\psi_{01}(W_2\setminus\{1\})}^{n-p-3} \circ f_{n-p-3}
\]
and since the source of the span $\tilde{\partial}^{n-p}_{W_2}$ is contained in $X_{n-p-3}$ while the target of the span $\tilde{\partial}^n_{V_1}$ is contained in $X_{n-p-2}$,
\[
\tilde{\partial}^{n-p}_{W_2} \circ \tilde{\partial}_{V_1} = \emptyset
\]
so \([6.7]\) is the unique bijection between the empty spans.
(10) If $2 \notin U$, $1 \in V_1$ and $0 \in V_2$
\[
\tilde{\partial}^n_U = \emptyset \quad \tilde{\partial}^n_{V_1} = \partial_{\psi_{12}(V_1\setminus\{1\})}^{n-3} \circ f_{n-3} \quad \tilde{\partial}^{n-p}_{W_2} = \partial_{\psi_{12}(W_2\setminus\{0\})}^{n-p-3} \circ f_{n-p-3}
\]
and since the source of the span $\tilde{\partial}^{n-p}_{W_2}$ is contained in $X_{n-p-3}$ while the target of the span $\tilde{\partial}_{V_1}^{n-p}$ is contained in $Y_{n-p-2}$,
\[
\tilde{\partial}^{n-p}_{W_2} \circ \tilde{\partial}_{V_1}^{n-p} = \emptyset
\]
so \([6.7]\) is the unique bijection between the empty spans.
The verification of Conditions (4), (5) and (6) in the definition of augmented semi-simplicial object in the Burnside category are not made explicit, because in each case they will be compositions of 2-morphisms $\mu_{iV_i, iV_j}$ and $f_{ij}$, and the diagrams will commute because $f$ satisfies (2.4) and $X_\bullet$ and $Y_\bullet$ satisfy the aforementioned Condition (6).

The augmented semi-simplicial objects in the Burnside category $\Sigma^2 X_\bullet$ and $\Sigma^2 Y_\bullet$ are included in $M(\Sigma^2 f)_\bullet$ as the second and third factors. Let us denote by

$$\Sigma^2 X_\bullet \xrightarrow{i_{\Sigma^2 f}} M(\Sigma^2 f)_\bullet \xrightarrow{h_{\Sigma^2 f}} Y_\bullet$$

these inclusions. Then, by construction, the $R$-realisation of $M(\Sigma^2 f)_\bullet$ will be the mapping cylinder of the $R$-realisation of $f$:

$$|M(\Sigma^2 f)_\bullet|_R = M(\Sigma^2 f|_R)_\bullet$$

in particular $h_{\Sigma^2 f}$ is an equivalence.

6.3. Naturality of Steenrod squares.

**Theorem 6.6.** If $f : X_\bullet \to Y_\bullet$ is a map of augmented semi-simplicial objects in the Burnside category, and both $X_\bullet$ and $Y_\bullet$ are ordered, then $f^* sq^i = sq^i f^*$.

**Proof.** Let $| \cdot |$ denote the $F_2$-realisation $| \cdot |_{F_2} : B^{\Delta^{|X|}}_{\bullet} \to \text{Ch}(F_2)$. The diagram

$$\begin{array}{ccc}
\Sigma^2 X_\bullet & \xrightarrow{i_{\Sigma^2 f}} & M(\Sigma^2 f)_\bullet \\
& \searrow^{h_{\Sigma^2 f}} & \downarrow \\
\Sigma^2 f & \xrightarrow{\Sigma^2 f} & \Sigma^2 Y_\bullet
\end{array}$$

does not commute, but after taking $F_2$-realisations, the diagram of chain complexes

$$\begin{array}{ccc}
|\Sigma^2 X_\bullet| & \xrightarrow{|i_{\Sigma^2 f}|} & |M(\Sigma^2 f)_\bullet| \\
& \searrow^{|h_{\Sigma^2 f}|} & \downarrow \\
|\Sigma^2 f| & \xrightarrow{|\Sigma^2 f|} & |\Sigma^2 Y_\bullet|
\end{array}$$

does commute up to homotopy because, by construction, this is the mapping cylinder diagram of $|\Sigma^2 f|$. Moreover, there is a chain map $g : |M(\Sigma^2 f)_\bullet| \to |\Sigma^2 Y_\bullet|$ that is a homotopy inverse of $|h_{\Sigma^2 f}|$, and therefore $|\Sigma^2 f| \simeq g \circ |i_{\Sigma^2 f}|$. Let $M(\Sigma^2 f)_\bullet$ be endowed with any order extending the order of its subobjects $\Sigma^2 X_\bullet$ and $\Sigma^2 Y_\bullet$. Then, the inclusion $h_{\Sigma^2 f}$ is free and order-preserving, so $h_{\Sigma^2 f}$ preserves Steenrod squares, and therefore $g$ preserves Steenrod squares: for every cohomology class $\alpha \in H^*(\Sigma^2 Y_\bullet, F_2)$,

$$\begin{align*}
\text{sq}^i(h^* f^* \alpha) &= |h_{\Sigma^2 f}| \text{sq}^i(g^* \alpha) \\
\Rightarrow g^* \text{sq}^i(h_{\Sigma^2 f}^* g^* \alpha) &= g^* \text{sq}^i(g^* \alpha) \\
\Rightarrow g^* \text{sq}^i(\alpha) &= \text{sq}^i(g^* \alpha).
\end{align*}$$

On the other hand, since $i_{\Sigma^2 f}$ is free and order-preserving we have that $|i_{\Sigma^2 f}|^* \circ \text{sq}^i = \text{sq}^i \circ |i_{\Sigma^2 f}|^*$, and since $|\Sigma^2 f|$ is homotopy equivalent to $g \circ |i_{\Sigma^2 f}|$, we deduce that $|\Sigma^2 f|^* \circ \text{sq}^i = \text{sq}^i \circ |\Sigma^2 f|^*$. As by Proposition 5.1 Steenrod squares commute with suspension, we have $|f|^* \circ \text{sq}^i = \text{sq}^i \circ |f|^*$.

$\square$
As a consequence, if \( X_\bullet \) is an augmented semi-simplicial object in the Burnside category, the Steenrod squares \( sq^i : H^n(X_\bullet ; \mathbb{F}_2) \to H^{n+i}(X_\bullet ; \mathbb{F}_2) \) are independent of the order on \( X_\bullet \) used to define them, hence are well-defined natural operations on the cohomology of any augmented semi-simplicial object in the Burnside category.

7. Cubes and Khovanov homology

Using the construction of Lawson, Lipshitz and Sarkar and the functor \( \Lambda \), the cup-i products of Theorem 3.4 may be defined on the Khovanov cochain complex associated to an oriented link diagram \( D \) with ordered crossings. In this section we prove that the Steenrod squares \( sq^i \) do not depend on the order of the crossings and are invariant under Reidemeister moves.

7.1. Stable equivalences of stable functors. We start by reviewing the concept of stable functor [LLS17, Section 5], and proving that stable equivalences of stable functors induce, after applying \( \Lambda \), zig-zags of equivalences between iterated suspensions of augmented semi-simplicial objects in the Burnside category.

**Definition 7.1.** A face inclusion of degree \( r \) is a functor \( \iota : 2^n \to 2^{n'} \) that is injective on objects and for every \( A \subset \{1, \ldots, n\} \), \( |\iota(A)| = |A| + r \). A face inclusion is sequential if, writing \( \bar{a} \) for the unique element of the singleton \( \iota(\{a\}) \setminus \iota(\emptyset) \), we have that \( \bar{a} < \bar{b} \) if \( a < b \), and \( b < \bar{T} \) if \( b \in \iota(\emptyset) \).

**Definition 7.2.** If \( \iota : 2^n \to 2^{n'} \) is a face inclusion, and \( F : 2^n \to B \), then there is a unique functor \( \iota_* F \) such that \( F = (\iota_* F) \circ \iota \) and \( \iota_* F(A) = \emptyset \) if \( A \in 2^{n'} \setminus 2^n \). This assignment is natural and defines a functor \( \iota_* : B(2^n)^{op} \to B(2^{n'})^{op} \).

**Lemma 7.3.** Let \( \iota : 2^n \to 2^{n'} \) be a face inclusion of degree \( r \) and let \( F,G : 2^n \to B \) be cubes in the Burnside category.

1. There is a permutation \( \omega \) of \( \{1, \ldots, n\} \) such that the composition \( \omega \circ \iota \) is sequential.
2. If \( \omega \) is a permutation of \( \{1, \ldots, n\} \), then \( (\iota_* F) \circ \omega = (\omega \circ \iota)_* (F) \).
3. If \( \iota \) is sequential, then \( \Sigma^r \Lambda(F) = \Lambda(\iota_* F) \).
4. If \( f : F \to G \) is an equivalence of cubes in the Burnside category, then \( \Lambda(f) : \Lambda(F) \to \Lambda(G) \) is an equivalence of augmented semi-simplicial objects in the Burnside category.

**Proof.** The first two assertions are clear. The third follows because

\[
\Lambda(\iota_* F)_n = \Lambda(F)_{n-r}
\]

and, if \( \partial_U \) denotes the generalised face map on the left and \( \partial_U \) the generalised face map on the right, then

\[
\partial_U^n = \begin{cases}
\partial^r_{\psi(0, \ldots, r-1)}(U) & \text{if } \{0, \ldots, r-1\} \cap U = \emptyset \\
\emptyset & \text{otherwise}
\end{cases}
\]

\[
\partial^r_{\psi(0, \ldots, r-1)}(V, \psi(0, \ldots, r-1)(W)) = \begin{cases}
\partial^r_{\psi(0, \ldots, r-1)}(V) \cup \psi(0, \ldots, r-1)(W) & \text{if } \{0, \ldots, r-1\} \cap (V \cup W) = \emptyset \\
\emptyset & \text{otherwise}
\end{cases}
\]

The fourth is a consequence of the commutativity of the right square in [2.1], explained in Section 2.9. \( \square \)
The proof of the following proposition is given in Section \[7.2\] and is based on [Ste17, Section 7].

**Proposition 7.4.** If $\omega$ is a permutation of \{1, \ldots, n\}, there are sequential face inclusions $\Phi, \Psi^\omega: 2^n \to 2^{2n}$ of degree 0 such that for every $F: 2^n \to B$ there is a $G: 2^{2n} \to B$ and a zig-zag of equivalences of cubes in the Burnside category

$$\Phi_* F \to G \leftarrow \Psi^\omega_*(F \circ \omega).$$

**Corollary 7.5.** If $\iota: 2^n \to 2^{n'}$ is a face inclusion of degree $r$ and $F: 2^n \to B$ is a cube in the Burnside category, then there is a zig-zag of equivalences of augmented semi-simplicial objects in the Burnside category

$$\Lambda(\iota_* F) \to \Lambda(G) \leftarrow \Sigma' \Lambda(F).$$

**Proof.** By Lemma \[7.3\] \[1\], there is a permutation $\omega$ of \{1, \ldots, n\} such that $\omega \circ \iota$ is a sequential face inclusion, and by Proposition \[7.4\] there is a pair of sequential inclusions $\Phi_*, \Psi^\omega_*: 2^n \to 2^{2n}$ and a $G: 2^{2n} \to \hat{B}$ and a zig-zag of equivalences between cubes in the Burnside category

$$\Phi_*(\iota_* F_i) \to G \leftarrow \Psi^\omega_*((\iota_* F) \circ \omega).$$

Therefore, by Lemma \[7.3\] \[2\] we have equivalences of augmented semi-simplicial objects in the Burnside category

$$\Lambda(\iota_* F_i) = \Lambda(\Phi_*(\iota_* F_i)) \simeq \Lambda(G) \simeq \Lambda(\Psi^\omega_*((\iota_* F) \circ \omega)) = \\
= \Lambda((\iota_* F) \circ \omega) = \Lambda((\omega \circ \iota)_* F) = \Sigma' \Lambda(F).$$

**Definition 7.6.** A stable functor is a triple $(F, n, r)$ where $F: 2^n \to B$ for some $n \geq 0$ and $r \in \mathbb{Z}$. A stable map from a stable functor $(F, n, r)$ to a stable functor $(F', n', r')$ is a pair $(f, \iota)$, where $\iota: 2^n \to 2^{n'}$ is a face inclusion of degree $r - r'$ and $f: \iota_* F \to F'$ is a map of cubes. The composition $(f', \iota') \circ (f, \iota)$ is $(f' \circ (\iota'_* f), \iota' \circ \iota)$. A stable map $(f, \iota): (F, n, r) \to (F', n', r')$ is a stable equivalence if $f: \iota_* F \to F'$ is an equivalence of cubes in the Burnside category. Two cubes in the Burnside category are stably equivalent if there is a zig-zag of stable equivalences between them.

From Corollary \[7.5\] and Lemma \[7.3\] \[1\], we deduce

**Corollary 7.7.** If $(F, r)$ and $(F', r')$ are stable functors that are stably equivalent, then there exists some $k \geq 0$ such that $\Sigma^{r+k} \Lambda(F)$ and $\Sigma^{r'+k} \Lambda(F')$ are equivalent.

**7.2. Proof of Proposition 7.4** Let $F: 2^n \to B$ be a cube in the Burnside category, with morphisms $F(B \subset A): F(A) \to F(B)$ for each $B \subset A \subset \{1, \ldots, n\}$ and higher morphisms $F(B \subset A, C \subset B): F(C \subset A) \to F(C \subset B) \circ F(B \subset A)$. Let $\omega$ be a permutation of \{1, \ldots, n\}, which induces a maximal face inclusion $\omega: 2^n \to 2^n$ and let $\omega' = F \circ \omega$. Let

$$\Phi: \{1, \ldots, n\} \to \{1, \ldots, 2n\} \quad \Phi(i) = i$$

$$\Psi: \{1, \ldots, n\} \to \{1, \ldots, 2n\} \quad \Psi(i) = \omega^{-1}(i) + n$$

$$\Gamma: \{1, \ldots, 2n\} \to \{1, \ldots, n\} \quad \Gamma(i) = \begin{cases} i & \text{if } i \leq n \\ \omega(i - n) & \text{if } i > n \end{cases}$$

and observe that $\Gamma \circ \Phi = \text{Id}$ and $\Gamma \circ \Psi = \text{Id}$ and that both $\Phi$ and $\Psi^\omega := \Psi \circ \omega$ induce sequential face inclusions of degree 0 and that $\Gamma$ induces a functor from $2^{2n}$
to $2^n$. If $A,B \subset \{1,\ldots,2n\}$, write $A < B$ (resp. $A \leq B$) if for every $a \in A$ and every $b \in B$, $a < b$ (resp. $a \leq b$). A vertex $A \subset \{0,\ldots,2n\}$ of $2^n$ is well-ordered if $\Phi^{-1}(A) \leq \Psi^{-1}(A)$ and it is very well-ordered if $\Phi^{-1}(A) < \Psi^{-1}(A)$. Note that these properties are hereditary: if $B \subset A$ and $A$ is (very) well-ordered, then $B$ is (very) well-ordered too. Define a new cube $G: 2^n \to B$ as

$$G(A) = \begin{cases} F \circ \Gamma(A) & \text{if } A \text{ is well-ordered} \\ \emptyset & \text{otherwise.} \end{cases}$$

with

$$G(B \subset A) = F \circ \Gamma(B \subset A) \quad G(B \subset A, C \subset B) = F \circ \Gamma(B \subset A, C \subset B)$$

whenever $A$ is well-ordered, and the empty span and the unique 2-morphism between empty spans otherwise. Since $F = G \circ \Phi$ and $F^\omega = G \circ \Psi^\omega$, we have that the sequential face inclusions $\Phi$ and $\Psi^\omega$ induce maps of cubes

$$f: \Phi_* F \to G \quad f^\omega: (\Psi^\omega)_* F \to G.$$ 

The induced maps $f_*: C_*(F; R) \to C_*(G; R)$ and $f^\omega_*: C_*(F^\omega; R) \to C_*(G; R)$ have left inverses

$$g: C_*(G; R) \to C_*(F; R) \quad g^\omega: C_*(G; R) \to C_*(F^\omega; R)$$

whose value on a generator $\sigma \in G(A)$ is

$$g(\sigma) = \begin{cases} \sigma \in F(\Gamma(A)) & \text{if } A \text{ is very well-ordered} \\ 0 & \text{otherwise.} \end{cases}$$

$$g^\omega(\sigma) = \begin{cases} \sigma \in F^\omega(\omega^{-1} \circ \Gamma(A)) & \text{if } A \text{ is very well-ordered} \\ 0 & \text{otherwise.} \end{cases}$$

We now construct a chain homotopy $D$ from the identity on $C_*(G; \mathbb{F}_2)$ to $f_* \circ g_*$ and a chain homotopy $D^\omega$ from the identity to $f^\omega_* \circ g^\omega_*$. As a consequence, both $f$ and $f^\omega$ induce isomorphisms on homology, so $f$ and $f^\omega$ are equivalences of cubes.

If $A = (a_1,\ldots,a_m) \subset \{1,\ldots,2n\}$ is a vertex of $2^n$ and $j \in \{1,\ldots,m\}$, define $A_{<j} = (a_1,\ldots,a_j)$ and $A_{\geq j} = (a_j,\ldots,a_m)$. If $A$ is very well-ordered, then the value of $G$ at $A$ and at $\Theta_j(A) = \Phi \circ \Gamma(A_{<j}) \cup \Psi \circ \Gamma(A_{\geq j})$ coincides, hence we may define a homomorphism

$$D_j: C_*(G; R) \to C_{*+1}(G; R)$$

whose value on a generator $\sigma \in G(A)$ is $\sigma \in G(\Theta_j(A))$ if $A$ is very well-ordered and 0 otherwise. Finally, define the chain homotopies

$$D: C_*(G; R) \to C_{*+1}(G; R) \quad D^\omega: C_*(G; R) \to C_{*+1}(G; R)$$

whose value on a generator $\sigma \in G(A)$ with $A = \{a_1,\ldots,a_m\}$ is

$$D(\sigma) = \sum_{j=1}^\ell (-1)^j D_j(\sigma) \quad D^\omega(\sigma) = \sum_{j=\ell+1}^m (-1)^{j+1} D_j(\sigma),$$

where $\ell$ is the number such that $a_\ell \leq n$ and $a_{\ell+1} > n$. We have

$$\partial \circ D + D \circ \partial = f \circ g - \text{Id} \quad \partial \circ D^\omega + D^\omega \circ \partial = f^\omega \circ g^\omega - \text{Id}.$$
The first equation holds because the summands in both sides are paired as follows

\[(−1)^{k+j}∂_k ∘ D_j = (−1)^{j+k}D_{j−1} ∘ ∂_k \quad \text{if } k < j, j + 1\]

\[(−1)^{k+j}∂_k ∘ D_j = (−1)^{j+k}D_j ∘ ∂_{k−1} \quad \text{if } k > j, j + 1\]

\[(−1)^{2j}∂_j ∘ D_j = (−1)^{2j−1}∂_j ∘ D_{j−1}\]

\[(−1)^{2}∂_1 ∘ D_1 = f ∘ g\]

\[(−1)^{2}∂_{t+1} ∘ D_t = −Id,\]

and the second equation is obtained similarly.

7.3. **Khovanov homology.** Let \( D \) be an oriented knot diagram with \( c \) crossings and \( n_− \) negative crossings whose crossings have been ordered. Let \( Λ : 2^c → B \) be the Khovanov functor of Lawson, Lipshitz and Sarkar [LLS15, LLS17], from which one obtains the stable functor \( (F_D, −n_−) \). If we write \( C_∗(D; F_2) \) for the dual of the Khovanov cochain complex of \( D \) with \( F_2 \) coefficients, then

\[C_∗(D; F_2) = Σ^{−n_−}C_∗(F_D; F_2).\]

Applying the functor \( Λ \) of Section 2.8 to \( F \), we obtain an augmented semi-simplicial object in the Burnside category \( X_∗ \) and, the following chain complexes are equal (see Section 2.9)

\[C_∗(X_∗; F_2) = Σ^{−1}C_∗(F_D; F_2).\]

Now, after choosing and order on \( X_∗ \), the cochain complex \( C_∗(X_∗; F_2) \) becomes endowed with a symmetric multiplication via the cup-\( i \)-products of Theorem 3.4 and therefore so does the Khovanov cochain complex of \( D \):

\[\sim_i : C^{p−n−i+1}(D; F_2) ⊗ C^{q−n−i+1}(D; F_2) → C^{p+q−i−n−i+1}(D; F_2) \quad i ∈ \mathbb{Z}.\]

As a consequence, the Khovanov homology of \( D \) is enhanced with the Steenrod squares associated to this symmetric multiplication

\[sq^i : Kh^{n−n−i+1}(D; F_2) → Kh^{n+i−n−i+1}(D; F_2), \quad sq^i([α]) = [α \sim_{n−i} α], \quad i ≥ 0.\]

**Proposition 7.8.** [LLS17, p. 14] If \( D \) and \( D′ \) are two oriented link diagrams with ordered crossings and \( n_− \) and \( n′_− \) negative crossings, and \( D \) and \( D′ \) are related by a Reidemeister move, then \( (F_D, −n_−) \) and \( (F_{D′}, −n′_−) \) are stably equivalent.

Note also that if \( D \) is a link diagram with \( c \) ordered crossings and \( n_− \) negative crossings, and \( D′ \) is the same diagram but whose crossings have been ordered differently, then there is a permutation \( ω \) of \( \{1, \ldots, c\} \) such that \( F_{D′} = ω ∘ F_D \), hence \( (F_D, −n_−) \) and \( (F_{D′}, −n_−) \) are stably equivalent. Therefore, by Corollary 7.7.

**Corollary 7.9.** If \( D \) and \( D′ \) are two oriented link diagrams with ordered crossings and \( n_− \) and \( n′_− \) negative crossings, and \( D \) and \( D′ \) are related by a Reidemeister move or by a reordering of the crossings, then there is some \( k ≥ 0 \) such that \( Σ^{k−n−} ∘ Λ(F_D) \) and \( Σ^{k−n′−} ∘ Λ(F_{D′}) \) are equivalent.

**Corollary 7.10.** If \( D \) is a link diagram with ordered crossings and \( n_− \) negative crossings, then the Steenrod squares \( sq^i \) applied to the augmented semi-simplicial object in the Burnside category \( Λ(F_D) \) give operations

\[sq^i : Kh^{n−n−i+1}(D; F_2) → Kh^{n+i−n−i+1}(D; F_2), \quad sq^i([α]) = [α \sim_{n−i} α]\]

that are independent of the chosen diagram and the ordering of the crossings.
8. Examples

Write $|\cdot|_S$ for the functor labeled $*$ in diagram (2.1). We will use this functor together with the commutativity of the bottom square of diagram (2.1) to be able to discuss the expected values of the Steenrod squares in spectra (though the computations are independent of these discussions).

**Example 8.1.** Consider the ordered span $f$ given by $\{x\} \leftarrow [a,b] \rightarrow \{y\}$, and construct the ordered augmented semi-simplicial object in the Burnside category $X_\bullet$ by declaring that

$$
X_{-1} = \{y\} \quad X_0 = \{x\} \quad X_i = \emptyset \quad \text{for all } i > 0
$$

and that

$$
\partial_0^0 = f : X_0 \longrightarrow X_{-1}.
$$

Then, we have that $C_\ast(X_\ast; \mathbb{Z})$ is the chain complex

$$
\cdots \longrightarrow 0 \longrightarrow \mathbb{Z}\langle x \rangle \overset{-2}{\longrightarrow} \mathbb{Z}\langle y \rangle \longrightarrow 0
$$

where $x$ is in degree 0 and $y$ is in degree $-1$. The homology of this complex is concentrated in degree $-1$ and $H_{-1}(X_\ast; \mathbb{Z}) = \mathbb{Z}_2$, therefore $|X_\ast|_S$ is a Moore spectrum $M(\mathbb{Z}_2, -1)$ for the group $\mathbb{Z}_2$ in degree $-1$, and, by the uniqueness of Moore spectra, we have determined the homotopy type of $|X_\ast|_S$. A model of $M(\mathbb{Z}_2, -1)$ is, for example, $\Sigma^{-2} \Sigma^\infty \mathbb{R}P^2$.

The complex of cochains with $\mathbb{F}_2$ coefficients is

$$
0 \longrightarrow \mathbb{F}_2\langle y^* \rangle \overset{-2}{\longrightarrow} \mathbb{F}_2\langle x^* \rangle \longrightarrow 0 \longrightarrow \cdots
$$

By definition,

$$
sq^1(y^*) = y^* \rightsquigarrow_{-2} y^*
$$

and

$$
\rightsquigarrow_{-2} : \mathbb{F}_2\langle y^* \rangle \otimes \mathbb{F}_2\langle y^* \rangle \longrightarrow \mathbb{F}_2\langle x^* x \rangle
$$

is the dual of

$$
\nabla : \mathbb{F}_2\langle x \rangle \rightarrow \mathbb{F}_2\langle y \rangle \otimes \mathbb{F}_2\langle y \rangle.
$$

Now, $\mathcal{P}_2(0)$ has a single element $(0,0)$, therefore we have

$$
\nabla_{-2} = A_{\mathbb{F}_2}(\partial_0 \land \partial_0 \circ \Delta)
$$

and, since $a < b$ and $n + |U^+| + |U^-| = 0 + 1 + 1$ is even, we have that $\mathcal{O}_{a,b}(\bar{U})^+$ has a single element $(\emptyset, \{0\})$ and that $\mathcal{O}_{a,b}(\bar{U})^-$ is empty, therefore $\partial_0 \land \partial_0$ is equivalent to the span

$$
\{x\} \times \{x\} \leftarrow \{(a,b)\} \rightarrow \{y\} \times \{y\},
$$

whose $\mathbb{F}_2$-realisation is the homomorphism $(x,x) \mapsto (y,y)$. Therefore

$$
\nabla_{-1}(x) = A_{\mathbb{F}_2}(\partial_0 \land \partial_0 \circ \Delta)(x) = A_{\mathbb{F}_2}(\partial_0 \land \partial_0)(x,x) = (y,y)
$$

so $y^* \rightsquigarrow_{-2} y^* = x^*$ and $\text{sq}^1(y^*) = x^*$, as it should be.
Example 8.2. Let us take now another copy \( X' \) of the object \( X \) studied in Example 8.1 and take the join product \( X \ast X' \), which is (Definition 5.3):

\[
X_1 = \{xx'\}, \quad X_0 = \{yx', xy\}, \quad X_{-1} = \{yy'\}
\]

with generalised face maps

\[
\begin{align*}
\partial_1^0 & : xx' \rightarrow \{a, b\} \leftarrow yx' \\
\partial_1^1 & : xx' \rightarrow \{a', b'\} \leftarrow xy'
\end{align*}
\]

\[
\begin{align*}
\partial_0 & : yx' \rightarrow a'b' \\
\partial_{01} & : xx' \rightarrow \{aa', ab', ba', bb'\}
\end{align*}
\]

and we endow each span with the order that results from reading the entries left to right or up to down. This is summarised in the following diagram:

The 2-morphisms are:

\[
\begin{align*}
\partial_1^0 \times X_0 \partial_0 & \xrightarrow{\mu_{b,1}} \partial_{01} \partial_1^1 \times X_0 \partial_0 \\
(a, a') & \leftarrow aa' \rightarrow (a', a) \\
(a, b') & \leftarrow ab' \rightarrow (b', a) \\
(b, a') & \leftarrow ba' \rightarrow (a', b) \\
(b, b') & \leftarrow bb' \rightarrow (b', b)
\end{align*}
\]

By (5.4), \(|X_\ast X'|| \lesssim \Sigma^{-3} \Sigma^{\infty} \mathbb{R}P^2 \wedge \mathbb{R}P^2\), so we expect that \( sq^2((yy')^*) = (xx')^* \). By definition, we have that (5.1)

\[
\text{sq}^2((yy')^*) = (yy')^* \quad \text{and that the operation } \quad \text{is dual to the operation } \nabla_{-3}.
\]

Since \( P_4(1) \) has only one element \( U = (0, 0, 1, 1) \), writing \( \partial_{01} \) for \( \partial_{1(0,1)} \)

\[
\begin{align*}
\nabla_{-3} = \mathbb{A}_{\mathbb{F}_2} (\partial_{01} \wedge \partial_{01} \circ \Delta).
\end{align*}
\]

now, to compute this wedge product,

\[
\partial_{01} \wedge \partial_{01} = \sum_{(s,t) \in \partial_{01} \times \partial_{01}} \mathcal{O}_{s,t}(\mathring{U})^+
\]

we need to compute

\[
\mathcal{O}_{s,t}(\mathring{U})^+ = \{\text{positive maximal chains that are } (s, t)-\text{good}\}
\]
All maximal chains have length 3, so they take the form \((W^i_1, W^0_1)\). Every element \((s, t)\) with \(s \neq t\) has two \((s, t)\)-good maximal chains: one in which either \(W^m_1 = \{0\}\) or \(W^l_1 = \emptyset\) and \(W^0_1 = \{0\}\); and another in which either \(W^l_1 = \{1\}\) or \(W^m_1 = \emptyset\) and \(W^0_1 = \{1\}\). We call the first maximal chain “left” and the second “right”. By definition, since \(n + |U^-| + |U^+| = 1 + 2 + 2 = 5\), we have that \((W^m, W^0)\) is a positive \((s, t)\)-good pair if
\[
\lambda_{W^-}(s) = \lambda_{W^-}(t) \quad \lambda_{W^-,W^-}(s) > \lambda_{W^-,W^-}(t)
\]
Now, \(\partial_{01} \times \partial_{01}\) has 42 elements. The elements in the diagonal have no maximal chains, so \(O_{s,t}(\tilde{U})^+ = \emptyset\) for them. For the remaining 12, we have that for half of them (those \((s, t)\) such that \(s < t\)) the pair \((\emptyset, \{0, 1\})\) is negative, so \(O_{s,t}(\tilde{U})^+ = \emptyset\) for them too. In the following table we give the positivity of the middle pair \((W^m_1, W^0_1)\) of the left and right maximal chains of the remaining 6 elements. As each maximal chain is determined by its first pair \((W^m_1, W^0_1)\), we give only this datum.

| \((s, t)\) | max. chain (left) | max. chain (right) | positivity |
|----------|-----------------|-----------------|----------|
| \((bb', ba')\) | \((\emptyset, \{1\})\) | \((\emptyset, \{0\})\) | ++ |
| \((bb', ab')\) | \((\emptyset, \{0\})\) | \((\{1\}, \{0\})\) | ++ |
| \((bb', aa')\) | \((\emptyset, \{0\})\) | \((\emptyset, \{1\})\) | ++ |
| \((ba', ab')\) | \((\emptyset, \{0\})\) | \((\emptyset, \{1\})\) | ++ |
| \((ba', aa')\) | \((\emptyset, \{0\})\) | \((\{1\}, \{0\})\) | ++ |
| \((ab', aa')\) | \((\emptyset, \{0\})\) | \((\emptyset, \{1\})\) | ++ |

For example, for \((s, t) = (bb', ab')\), we have that
\[
\lambda_0(bb') = b > a = \lambda_0(ab') \quad \Rightarrow \quad (\emptyset, \{0\}) \text{ is good}
\]
\[
\lambda_1(bb') = b' = b = \lambda_1(ab'), \lambda_{1,0}(bb') = b > a = \lambda_{1,0}(ab') \quad \Rightarrow \quad (\{1\}, \{0\}) \text{ is good and positive}
\]
and both are positive, and if \((s, t) = (ba', ab')\), we have that
\[
\lambda_0(ba') = b > a = \lambda_0(ab') \quad \Rightarrow \quad (\emptyset, \{0\}) \text{ is good and positive}
\]
\[
\lambda_1(ba') = a' < b' = \lambda_1(ab'), \quad \Rightarrow \quad (\emptyset, \{1\}) \text{ is good and negative.}
\]
As a consequence, there are 11 positive maximal chains, so \(\partial_{01} \cap \partial_{01}\) has odd cardinality, so \(A_{\mathbb{F}_2}(\partial_{01} \wedge \partial_{01} \circ \Delta)\) is the homomorphism that sends \(xx'\) to \(yy' \otimes yy'\).

Therefore,
\[
\nabla_{-3}(xx') = yy' \quad (yy')^* \prec_{-3} (yy')^* = (xx')^* \quad sq^2((yy')^*) = (xx')^*.
\]

**Example 8.3.** Iterating the construction of the previous example, we obtain models of \(\Sigma^{-k-1}\Sigma^\infty \mathbb{R}P^2 \wedge \mathbb{R}P^2 \wedge \ldots \mathbb{R}P^2\), for which the operation \(sq^k \) applied to the element in degree \(-1\) is non-trivial. This operation comes from a cup-\(i\) product of degree \(-2k\), so these iterations give examples of non-trivial cup-\(i\) operations of all negative degrees.

**Example 8.4.** Let us consider now an ordered augmented semi-simplicial object in the Burnside category \(X_\bullet\) with
\[
X_{-1} = \{a\} \quad X_0 = \{b_1, \ldots, b_k\} \quad X_1 = \{c\} \quad X_i = \emptyset \quad i > 1
\]
Endow \(\partial_0^1 \circ \partial_0^0\) and \(\partial_1^1 \circ \partial_1^0\) with the lexicographic order. Let \(N\) be the cardinal of \(\partial_1^1\{0,1\}\). Using the ordering, we can identify \(\mu_{0,1}^1\) and \(\mu_{1,0}^1\) as elements of the
symmetric group on $N$ letters, and we let $\sigma(\mu_{0,1}^1)$ and $\sigma(\mu_{1,0}^1)$ be the sign of these permutations. We claim that

$$(8.1) \quad \nabla - \delta = (\sigma(\mu_{0,1}^1) + \sigma(\mu_{1,0}^1)) \cdot a \otimes a.$$ 

so

$$(8.2) \quad \text{sq}^2(a^*) = (\sigma(\mu_{0,1}^1) + \sigma(\mu_{1,0}^1)) \cdot c^*.$$ 

To prove (8.1), note first that, as in Example 8.2, $\mathcal{P}(1)$ has only one sequence $U = (0,0,1,1)$, so

$$\nabla - \delta = A_2 (\nabla \circ \Delta_1) = A_2 (\partial^1_{1,0} \wedge \partial^1_{0,1} \circ \Delta_1).$$

To compute $\partial^1_{1,0} \wedge \partial^1_{0,1}$, note that, for each $(s,t) \in \partial^1_{1,0} \times \partial^1_{0,1}$ the set $O_{s,t}(U)$ has exactly two elements (cf. Example 8.2): the left and the right. The left element is positive if $s > t$ and $\mu_{0,1}^1(s) > \mu_{1,0}^1(t)$ in the lexicographic order. The right element is positive if $s > t$ and $\mu_{1,0}^1(s) > \mu_{0,1}^1(t)$.

Assume first that $\mu_{0,1}^1$ and $\mu_{1,0}^1$ are the identity permutations. Then, for each $(s,t)$, either $s > t$, in which case both the left and right chains are positive, or $s < t$, in which case none of them is positive. Therefore, $O_{s,t}(U)^+$ has even cardinality and $\partial^1_{1,0} \wedge \partial^1_{0,1} \equiv \emptyset$, so (8.3) holds in this case.

Assume now that (8.1) holds for some permutations $\mu_{0,1}^1$ and $\mu_{1,0}^1$, and let $\bar{\mu}_{0,1}^1$ be the result of changing $\mu_{0,1}^1$ by a transposition of two consecutive elements $s,t \in \partial^1_{0,1}$ with $s < t$, i.e.,

$$\bar{\mu}_{0,1}^1(s) = \mu_{0,1}^1(t), \quad \bar{\mu}_{0,1}^1(t) = \mu_{0,1}^1(s), \quad \bar{\mu}_{0,1}^1(r) = \mu_{0,1}^1(r) \quad r \neq s,t.$$ 

Then, for any $(r,r')$ different from $(s,t)$ or $(t,s)$, the positivity of the maximal $(r,r')$-good chains remains the same. Additionally, no $(t,s)$-good maximal chain in $O_{r,s}(U)^+$ was positive for, and none of them becomes positive, because $t > s$. On the other hand, the right $(s,t)$-good maximal chain is as positive as before whereas the left $(s,t)$-good maximal chain changes its positivity. Therefore the parity of $\partial^1_{1,0} \wedge \partial^1_{0,1}$ changes by one, so the cup-i product in the new semi-simplicial object changed by one, as does the sign of the permutation $\bar{\mu}_{0,1}^1$. A symmetric argument shows that the same holds for $\mu_{1,0}^1$, and since the symmetric group is generated by transpositions of consecutive elements, we conclude that (8.4) holds always.

This simple formula in terms of the sign of the permutations raises the following question:

**Is it possible to give an interpretation of the products $\partial_U \wedge \partial_V$ in terms of the homology of the permutation groups $\Sigma_k$ with $\mathbb{F}_2$ coefficients?**

**Example 8.5.** We know compute a second Steenrod square in the Khovanov homology of the disjoint union of two right-handed trefoils $T \amalg T$ (cf. [LLS15, p. 60]). In this example and the next one we assume that the reader is familiar with Khovanov homology and with the Khovanov functor of Lawson, Lipshitz and Sarkar [LLS15, LLS17]. We will use the following knot diagram $D$ of $T \amalg T$:
After choosing an ordering of the crossings, its 1-resolution is

Let $C^{\ast \ast}(D; \mathbb{F}_2)$ be the Khovanov complex with $\mathbb{F}_2$ coefficients associated to this diagram, which is concentrated in degrees 0, 1, 2, 3, 4, 5 and 6 because the number of negative crossings $n_-$ of this diagram is 0. Figure 1 shows the cube of resolutions in degrees 4, 5 and 6, and below are the ranks of the subcomplex $C^{14 \ast \ast}(D; \mathbb{F}_2)$ generated by the generators of quantum grading 14 and its homology:

|  | 6 | 5 | 4 | 3 |
|---|---|---|---|---|
| $C^{14 \ast \ast}(D; \mathbb{F}_2)$ | $\mathbb{F}_2^{15}$ | $\mathbb{F}_2^{30}$ | $\mathbb{F}_2^{15}$ | $\mathbb{F}_2^2$ |
| $Kh^{14 \ast \ast}(D; \mathbb{F}_2)$ | $\mathbb{F}_2$ | $\mathbb{F}_2^2$ | $\mathbb{F}_2$ | 0 |

Consider now the Khovanov functor $\mathcal{F}^{14}: \mathcal{B}^6 \to \mathcal{B}$ for $D$ in quantum grading 14, and let $X_\ast = \Lambda(\mathcal{F}^{14})$ be the associated augmented semi-simplicial object in the Burnside category (see Section 2.8). Let $C_\ast$ be the $\mathbb{F}_2$-realisation of $X_\ast$, which by construction is the one-fold desuspension of the chain complex $\text{Tot}_{\mathbb{F}_2}(\mathcal{F}^{14})$, whose $n_-$-desuspension is the Khovanov complex. Since $n_-$ = 0, we have that the dual complex of $C_\ast$ is:

$$C^\ast \cong \Sigma^{-1}C^{14 \ast \ast}(D; \mathbb{F}_2).$$

The generators of $C^{14 \ast \ast}(D; \mathbb{F}_2)$ are as follows:

- a generator in homological degree 6 (semi-simplicial degree 5) enhances two circles with $x_-$ and four circles with $x_+$.
- a generator in homological degree 5 (semi-simplicial degree 4) enhances one circle with $x_-$ and four circles with $x_+$.
- a generator in homological degree 4 (semi-simplicial degree 3) enhances each of the four circles with $x_+$.

Hence, there is a unique generator $z_u$ in quantum grading 14 in each vertex $u$ with semi-simplicial grading 3, which enhances every circle with $x_+$. Let us compute the second Steenrod square of the following cocycle

$$\alpha = z_{011110}^* + z_{011011}^* + z_{110011}^* + z_{110110}^*.$$

As $\alpha$ has semi-simplicial degree 3, by definition

$$\text{sq}^2([\alpha]) = [\alpha \smile_{-2} \alpha].$$
Figure 1. Cube of resolutions of $D$ in homological degrees 4, 5 and 6 (column $h$), which under the semi-simplicial convention are degrees 3, 4 and 5 (column $n$). Each arrow $a < b$ is labeled with the face map $\Lambda(a < b)$. The cube is split in four rows to avoid clut-tering. The edges are oriented downwards instead of upwards, as in [LLS15] and [LLS17]. We name the six circles in $D$ as $a, b, c, d, e, f$ as shown in the bottom left part of the figure.
Now, the \(\sim_1\) product is dual to \(\nabla_1\), and we want to compute it on the generators of \(C_\ast\) of semi-simplicial degree 5 (which is where the second Steenrod square of \(\alpha\) lives), and again, by definition,

\[
\nabla_1|_{\Delta_5} = A_{\mathbb{F}_2} \left( \nabla_1(\beta) \circ \Delta_5 \right).
\]

and

\[
\nabla_1(\beta) = \sum_{U \in P_4(5)} \partial_{U^-}^5 \land \partial_{U^+}^5.
\]

We only need to look at those \(U\)'s such that \(|U^-| = |U^+| = 2\) because we are feeding both sides of \(\sim_1\) with the cochain \(\alpha\) of semi-simplicial degree 3. Moreover, every face map in the cube is a merging, so it defines a span that is of the form \(A \leftarrow A \rightarrow B\), i.e., a function of sets. Therefore, by Corollary 3.5, every summand indexed by a \(U \in P_4(5)\) with \(U \neq \emptyset\) is trivial. There are exactly three sequences in \(P_4(5)\) for which \(|U^-| = |U^+| = 2\) and \(U = \emptyset\), which are:

\[
\begin{array}{ccc}
\text{U} & \text{U}^- & \text{U}^+ \\
(0, 1, 3, 4) & (0, 1) & (3, 4) \\
(0, 2, 3, 5) & (0, 5) & (2, 3) \\
(1, 2, 4, 5) & (4, 5) & (1, 2) \\
\end{array}
\]

Write \(\partial_{i,j}^5\) for \(\partial_{\{i,j\}}^5\). First, note that the span \(\partial_{01}^5\) is non-trivial on a generator \(z\) if and only if \(z\) enhances two of the circles \(a, b, c\) with \(x_{\sim}\). Similarly, the span \(\partial_{23}^5\) is non-trivial on a generator \(z\) if and only if \(z\) enhances two of the circles \(d, e, f\) with \(x_{\sim}\). Since these two conditions are mutually excluding, we have that \((\partial_{01}^5 \land \partial_{23}^5) \circ \Delta_5\) is the empty span. A similar argument shows that \(\partial_{02}^5 \land \partial_{13}^5 \circ \Delta_5\) is trivial as well. The remaining span is \(\partial_{03}^5 \land \partial_{12}^5 \circ \Delta_5\). The span \(\partial_{05}^5\) is non-trivial on a generator \(z\) if and only if \(z\) enhances one of the circles \(a, c\) and one of the circles \(e, f\) with \(x_{\sim}\). The span \(\partial_{23}^5\) is non-trivial on a generator \(z\) if and only if \(z\) enhances two of the circles \(b, c\) and one of the circles \(d, f\) with \(x_{\sim}\). As a consequence, the span \(\partial_{05}^5 \land \partial_{23}^5 \circ \Delta_5\) is non-trivial only on the generator \(x_{c,f}\) that enhances the circles \(c, f\) with an \(x_{\sim}\) and every other circle with an \(x_{+}\). The target of the span is precisely the singleton \(\{(z_u, z_v)\}\), with \(u = 011110\) and \(v = 110011\). Therefore we have that for each generator \(x\) of \(C_5\),

\[
\nabla_1(x) = \begin{cases} 
0 & \text{if } x \neq x_{c,f} \\
z_u \otimes z_v & \text{if } x = x_{c,f}.
\end{cases}
\]

Since \(\alpha(z_u) = \alpha(z_v) = 1\), we have that

\[
(\alpha \sim_1 \alpha)(x) = \begin{cases} 
0 & \text{if } x \neq x_{c,f} \\
1 & \text{if } x = x_{c,f}
\end{cases}
\]

and therefore letting \(\beta\) be the dual of \(x_{c,f}\), we have

\[
\text{sq}^2(\alpha) = [\beta].
\]

Since \([\beta]\) is a non-zero generator of \(Kh^{14,6}(D; \mathbb{F}_2)\), we additionally deduce that \(\text{sq}^2\) is non-trivial on \([\alpha]\).

**Example 8.6.** In the previous example, every map involved was a map of sets. That simplified drastically the computations. In this example we introduce several
splittings, which will give rise to spans that are not maps of sets. Consider the following diagram $D$ of the unlink:

\[ \begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\end{array} \]

whose 1-resolution is

\[ \begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\end{array} \]

We take the following order on the chords of the 1-resolution:

\[ \begin{array}{c}
0 \\
1 \\
2 \\
3 \\
\end{array} \]

Let $C^{*,*}(D; F_2)$ be the Khovanov complex with $F_2$ coefficients associated to this diagram, which is concentrated in degrees $-2, -1, 0, 1$ and 2 because the number of negative crossings $n_-$ of this diagram is 2. Let $C^{1,*}(D; F_2)$ be the subcomplex generated by the generators of quantum grading 1, which is concentrated in homological degrees $-1, 0, 1, 2$ where it attains the following ranks:

\[
C^{1,2}(D; F_2) \cong F_2 \\
C^{1,1}(D; F_2) \cong F_2^3 \\
C^{1,0}(D; F_2) \cong F_2^4 \\
C^{1,-1}(D; F_2) \cong F_2^4.
\]

The homology of this subcomplex is the Khovanov homology of the unlink in quantum grading 1, which takes the following values:

\[
Kh^{1,i}(D; F_2) \cong \begin{cases} F_2 & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}
\]

Consider now the Khovanov functor $F^1: 2^4 \to B$ for $D$ in quantum grading 1, and let $X_* = \Lambda(F^1)$ be the associated augmented semi-simplicial object in the Burnside category (see Section 2.8). Let $C_*$ be the $F_2$-realisation of $X_*$, which by construction is the one-fold desuspension of the chain complex $\text{Tot}_{F_2}(F^1)$, whose $n_-$-desuspension is the Khovanov complex. Since $n_- = 2$, we have that the dual complex of $C_*$ is:

\[
C^* \cong \Sigma C^{1,*}(D; F_2).
\]

Figure 2 shows the cube of resolutions of $D$. We order the circles at each vertex of the cube as follows: if the circles are not nested, order them left-to-right, if the circles are nested, order them outside-to-inside. This rule is well-defined except at the vertices 1001 and 1100, where we choose any order, as it will be irrelevant in the computation.

The generators of $C^{1,*}(D; F_2)$ are as follows:
Figure 2. Cube of resolutions of $D$. We label each edge $u < v$ with the face map $\Lambda(u < v)$ and with a number $k$ that indicates which coordinate of the cube is changed along each edge. Note that we write the cube following the convention of [LLS15] and [LLS17], so the face maps go downwards instead of upwards.
there is a single generator \( x \) in homological degree 2 (semi-simplicial degree 3) that enhances the only circle with \( x_- \).

- Each vertex \( u \) in homological degree 1 (semi-simplicial degree 2) contributes with two generators, each of them enhances one circle with \( x_- \) and the other circle with \( x_+ \). For the \( j \)th vertex (read left to right, \( j = 1 \ldots 5 \)), let \( y_{j,1} \) be the generator that enhances the first circle with \( x_- \) and let \( y_{j,2} \) be the other generator.

- Read left to right, the first, fourth and sixth vertices in homological degree 0 (semi-simplicial degree 1) contribute with three generators, each of them enhances one circle with \( x_- \) and the remaining two circles with \( x_+ \). For the \( j \)th vertex (\( j = 1, 4, 6 \)), let \( z_{j,1}, z_{j,2} \) and \( z_{j,3} \) be the generators that enhance the first, second and third circles with an \( x_- \), respectively.

- Read left to right, the second, third and fifth vertices in homological degree 0 (semi-simplicial degree 1) contribute with one generator, that enhances the only circle with \( x_+ \). For the \( j \)th vertex (\( j = 2, 3, 5 \)), let \( z_j \) denote that generator.

- Each vertex in homological degree \(-1\) (semi-simplicial degree 0) contributes with a single generator, that labels every circle with \( x_+ \).

Let \( \alpha = z_{1,1}^* + z_{1,3}^* + z_{4}^* + z_{5}^* \in C^1 \), which is a cocycle because

\[
\delta(z_{1,1}^*) = y_{1,1}^* + y_{2,1}^* \\
\delta(z_{1,3}^*) = y_{1,2}^* + y_{2,2}^* \\
\delta(z_4^*) = y_{1,1}^* + y_{1,2}^* + y_{4,1}^* + y_{4,2}^* \\
\delta(z_5^*) = y_{2,1}^* + y_{2,2}^* + y_{4,1}^* + y_{4,2}^*.
\]

Let us compute the second Steenrod square of \( [\alpha] \), using the formula

\[
\text{sq}^2(\alpha) = [\alpha \cap -1 \alpha].
\]

The \( \cap -1 \) product is dual to \( \nabla_{-1} \), and we want to compute it on the generators of \( C_* \) of semi-simplicial degree 3 (which is where the second Steenrod square of \( \alpha \) lives), so we have:

\[
\nabla_1|_{C_3} = \mathcal{A}_2 \left( \nabla^{(3)} \circ \Delta_3 \right).
\]

and, by definition,

\[
\nabla^{(3)} = \sum_{U \in \mathcal{P}_3(3)} \partial_U^{\hat{3}} \wedge \partial_U^{\hat{3}}.
\]

The target of this span is the union of \( X_{-1} \times X_3, X_0 \times X_2, X_1 \times X_1, X_2 \times X_0 \) and \( X_3 \times X_{-1} \). We are interested only on its restriction to \( X_1 \times X_1 \), so we only need to consider the summands indexed by the following \( U \in \mathcal{P}_3(3) \):

\[
\begin{align*}
(0, 0, 1, 1) & \quad (0, 0, 2, 2) & \quad (0, 0, 3, 3) & \quad (1, 1, 2, 2) & \quad (1, 1, 3, 3) \\
(2, 2, 3, 3) & \quad (0, 0, 1, 3) & \quad (0, 1, 1, 3) & \quad (0, 2, 2, 3) & \quad (0, 2, 3, 3)
\end{align*}
\]
which are characterised by the property that \(|U^-| = |U^+| = 2\):

\[
\begin{align*}
(0, 0, 1, 1)^- &= (0, 1) & (2, 2, 3, 3)^- &= (2, 3) \\
(0, 0, 1, 1)^+ &= (0, 1) & (2, 2, 3, 3)^+ &= (2, 3) \\
(0, 0, 2, 2)^- &= (0, 2) & (0, 0, 1, 3)^- &= (0, 1) \\
(0, 0, 2, 2)^+ &= (0, 2) & (0, 0, 1, 3)^+ &= (0, 3) \\
(0, 0, 3, 3)^- &= (0, 3) & (0, 1, 1, 3)^- &= (0, 1) \\
(0, 0, 3, 3)^+ &= (0, 3) & (0, 1, 1, 3)^+ &= (1, 3) \\
(1, 1, 2, 2)^- &= (1, 2) & (0, 2, 2, 3)^- &= (0, 2) \\
(1, 1, 2, 2)^+ &= (1, 2) & (0, 2, 2, 3)^+ &= (2, 3) \\
(1, 1, 3, 3)^- &= (1, 3) & (0, 2, 3, 3)^- &= (0, 3) \\
(1, 1, 3, 3)^+ &= (1, 3) & (0, 2, 3, 3)^+ &= (2, 3)
\end{align*}
\]

Since the class \(a\) is supported at the first, third and fifth vertices, the span \(\partial^3_{U^-} \cap \partial^3_{U^+}\) is irrelevant unless both \(U^-\) and \(U^+\) belong to \{(0, 1), (0, 3), (1, 3)\}. Therefore the only relevant \(U^-\)’s are

\[
(0, 0, 1, 1) \quad (0, 0, 3, 3) \quad (1, 1, 3, 3) \quad (0, 0, 1, 3) \quad (0, 1, 1, 3)
\]

and \(\partial_{U^-}\) and \(\partial_{U^+}\) are one of the following three spans, where we write \(\partial^k_{i,j}\) for \(\partial^k_{\{i,j\}}\):

\[
\begin{align*}
\partial^3_{01} &\quad \xymatrix{ & z_{1,1} \ar[r] & z_{1,1} \\
& z_{1,2} \ar[ru] & z_{1,2} \\
z_{1,3} & z_{1,3} \ar[lu] & z_{1,3}} \\
\partial^3_{03} &\quad \xymatrix{ & a \ar[r] & z_{3} \\
x & z_{1,2} \ar[ru] & z_{1,2} \\
a & z_{1,3} \ar[lu] & z_{1,3}} \\
\partial^3_{13} &\quad \xymatrix{ & c \ar[r] & z_{5} \\
x & z_{1,2} \ar[ru] & z_{1,2} \\
d & z_{1,3} \ar[lu] & z_{1,3}}
\end{align*}
\]

We order the elements of these spans up-to-bottom, i.e.: \(z_{1,1} < z_{1,2} < z_{1,3}\) and \(a < b\) and \(c < d\). In order to understand the 2-morphisms involved, we first make explicit the following spans. We also pick an ordering of each span, which is indicated on the right (recall that, at the end, the Steenrod squares will not depend on the chosen order). The orders indicated may be extended arbitrarily to the whole spans \(\partial^3_0, \partial^3_1\) and \(\partial^3_3\). Since the computations do not depend on this extended order, we do not indicate it, in order to keep the example short.

\[
\begin{align*}
\partial^3_0 &\quad \xymatrix{ & y_{1,1} \ar[r] & y_{1,1} \\
x & y_{1,2} \ar[ru] & y_{1,2} \\
y_{1,2} & y_{1,2} \ar[lu] & y_{1,2} \\
y_{1,1} \ar[lu] & y_{1,1} & y_{1,1} \ar[lu]} \\
\partial^3_1 &\quad \xymatrix{ & y_{2,1} \ar[r] & y_{2,1} \\
x & y_{2,2} \ar[ru] & y_{2,2} \\
y_{2,2} & y_{2,2} \ar[lu] & y_{2,2} \\
y_{2,1} \ar[lu] & y_{2,1} & y_{2,1} \ar[lu]} \\
\partial^3_3 &\quad \xymatrix{ & y_{4,1} \ar[r] & y_{4,1} \\
x & y_{4,2} \ar[ru] & y_{4,2} \\
y_{4,2} & y_{4,2} \ar[lu] & y_{4,2} \\
y_{4,1} \ar[lu] & y_{4,1} & y_{4,1} \ar[lu]}
\end{align*}
\]
and the bijections relevant for the spans $\partial_{011}^3$, $\partial_{03}^3$, and $\partial_{13}^3$ are (we omit the superscript 3 on $\mu_{i,j}^3$):

\[
\begin{align*}
\mu_{0,1} : & \partial_{01}^3 \to \partial_0^2 \circ \partial_0^3 \\
& z_{1,1} \mapsto (y_{1,1}, z_{1,1}) \\
& z_{1,2} \mapsto (y_{1,2}, z_{1,2}) \\
& z_{1,3} \mapsto (y_{1,3}, z_{1,3}) \\
\mu_{1,0} : & \partial_{01}^3 \to \partial_0^2 \circ \partial_1^3 \\
& z_{1,1} \mapsto (y_{2,1}, z_{1,1}) \\
& z_{1,2} \mapsto (y_{2,1}, z_{1,2}) \\
& z_{1,3} \mapsto (y_{2,2}, z_{1,3}) \\
\mu_{0,3} : & \partial_{03}^3 \to \partial_0^2 \circ \partial_0^3 \\
& a \mapsto (y_{1,1}, y_{1,1}) \\
& b \mapsto (y_{1,2}, y_{1,2}) \\
\mu_{3,0} : & \partial_{03}^3 \to \partial_0^2 \circ \partial_3^3 \\
& a \mapsto (y_{4,2}, y_{4,2}) \\
& b \mapsto (y_{4,1}, y_{4,1}) \\
\mu_{1,3} : & \partial_{13}^3 \to \partial_1^2 \circ \partial_1^3 \\
& c \mapsto (y_{2,1}, y_{2,1}) \\
& d \mapsto (y_{2,2}, y_{2,2}) \\
\mu_{3,1} : & \partial_{13}^3 \to \partial_1^2 \circ \partial_3^3 \\
& c \mapsto (y_{4,2}, y_{4,2}) \\
& d \mapsto (y_{4,1}, y_{4,1}).
\end{align*}
\]

To determine $\mu_{0,3}, \mu_{3,0}, \mu_{1,3}$ and $\mu_{3,1}$, we have used the ladybug matching. Now, to compute $\partial_{U^-} \wedge \partial_{U^+}$, we need to understand $O_{s,t}(U^+)$ for each $(s, t) \in \partial_{U^-} \times \partial_{U^+}$. As in this case $n + |U^-| + |U^+| = 3 + 2 + 2 = 7$, we have that an $(s, t)$-good pair $(W^v, W^o)$ is positive if

\[
\lambda_{W^o}(s) = \lambda_{W^o}(t) \quad \lambda_{W^o,W^v}(s) > \lambda_{W^o,W^v}(t)
\]

If $\bar{U} = (u_1, u_2)$ has two elements, then each $(s, t)$ has exactly two maximal $(s, t)$-good chains, which we call “left” and “right”: the left maximal chain is either $(\emptyset, \{u_1\}) \prec (\emptyset, \{u_1, u_2\})$ or $(\{u_1\}, \{u_2\}) \prec (\emptyset, \{u_1, u_2\})$, whereas the right maximal
chain is either \((\emptyset, \{u_2\}) \prec (\emptyset, \{u_1, u_2\})\) or \((\{u_2\}, \{u_1\}) \prec (\emptyset, \{u_1, u_2\})\). We will refer to any of these maximal chains by its first pair, as the second pair is always \((\emptyset, \{u_1, u_2\})\). We will write \((W^n, W^\circ)\) instead of \((W^n_1, W^\circ_1)\) if \(U = \{u\}\) has a single element, then \((\emptyset, \{u\})\) is the only maximal chain. Here are the spans \(\partial^3_{U^-} \land \partial^3_{U^+}\) for the five relevant cases.

- If \(U = (0, 0, 1, 1)\). Then \(\partial^3_{U^-} = \partial^3_{U^+} = \partial^3_{01} = \{z_{1,1}, z_{1,2}, z_{1,3}\}\). Now, if \((s, t)\) is any of the pairs \((z_{1,1}, z_{1,3}), (z_{1,1}, z_{2,1}), (z_{1,2}, z_{1,3})\), then \(\mathcal{O}_{s,t}(U)^+ = \emptyset\), because as \(z_{1,1} < z_{1,2}\), \(z_{1,1} < z_{1,3}\), and \(z_{1,2} < z_{1,3}\) we have that the pair \((\emptyset, \{u_1, u_2\})\), present in all maximal chains, is not positive. On the other hand, if \((s, t)\) is any of the pairs \((z_{1,2}, z_{1,1}), (z_{1,3}, z_{1,1}), (z_{1,3}, z_{1,2})\), we have the following table indicating the positiveness of each maximal chain. The column “span” specifies the target of \(\lambda_{W^n, W^\circ}\) (its source is always \(\partial^3_{01}\)). To compute the values \(\lambda_{W^n, W^\circ}(s)\) and \(\lambda_{W^n, W^\circ}(t)\), one uses the bijections \(\mu_{01}\) and \(\mu_{10}\) above.

| \((s, t)\) | maximal chain | \(\lambda_{W^n, W^\circ}(s)\) | \(\lambda_{W^n, W^\circ}(t)\) | span | positiveness |
|-----------|---------------|----------------|----------------|------|-------------|
| \((z_{1,2}, z_{1,1})\) | \(\emptyset, \{0\}\) | \(y_{1,2}\) | \(y_{1,1}\) | \(\partial^3_{0}\) | \(-\) |
| \((z_{1,2}, z_{1,1})\) | \(\{1\}, \{0\}\) | \(z_{1,2}\) | \(z_{1,1}\) | \(\partial^3_{1011}\) | \(\pm\) |
| \((z_{1,3}, z_{1,1})\) | \(\emptyset, \{0\}\) | \(y_{1,2}\) | \(y_{1,1}\) | \(\partial^3_{0}\) | \(-\) |
| \((z_{1,3}, z_{1,1})\) | \(\emptyset, \{1\}\) | \(y_{2,2}\) | \(y_{2,1}\) | \(\partial^3_{i}\) | \(+\) |
| \((z_{1,3}, z_{1,2})\) | \(\{0\}, \{1\}\) | \(z_{1,3}\) | \(z_{1,2}\) | \(\partial^3_{1011}\) | \(+\) |
| \((z_{1,3}, z_{1,2})\) | \(\emptyset, \{1\}\) | \(y_{2,2}\) | \(y_{2,1}\) | \(\partial^3_{i}\) | \(+\) |

For example, the first row in the table computes the left maximal chain of the pair \(s = z_{1,2}, t = z_{1,1}\). The first step is to find \(W^n_1\), which will be empty if \(\lambda_0(s) \neq \lambda_0(t)\) and equal to \(\{0\}\) if \(\lambda_0(s) = \lambda_0(t)\). The projection \(\lambda_0\) is defined as the composition

\[
\lambda_0: \partial^3_{01} \xrightarrow{\mu_{01}} \partial^3_{0} \circ \partial^3_{0} \longrightarrow \partial^3_{0}
\]

that sends \(s\) and \(t\) first to \((y_{1,2}, z_{1,2})\) and \((y_{1,1}, z_{1,1})\), and then to \(y_{1,2}\) and \(y_{1,1}\). Since \(y_{1,2} \neq y_{1,1}\), we deduce that \(W^n_1 = \emptyset\), hence \((W^n_1, W^\circ_1) = (\emptyset, \{0\})\).

The second step is to compute the image of \(s\) and \(t\) under the projection

\[
\lambda_{0,\{0\}}: \partial^3_{01} \longrightarrow \partial^3_{0} \circ \partial^3_{0} \circ \partial^3_{0} \longrightarrow \partial^3_{0}.
\]

As \(\partial^3_{0}\) is the identity morphism, the first bijection \(\mu_{01, \{0\}, \{1\}}\) may be replaced by \(\mu_{01}\), and therefore the value of \(\lambda_{0,\{0\}}\) on \(s\) and \(t\) is

\[
s \mapsto (y_{1,2}, z_{1,2}) \mapsto y_{1,2} \quad t \mapsto (y_{1,1}, z_{1,1}) \mapsto y_{1,1}.
\]

For the fourth step one checks that \(y_{1,2} < y_{1,1}\) in the span \(\partial^3_{0}\), and therefore the pair is negative.

As another example, the second row in the table computes the right maximal chain of the pair \(s = z_{1,2}, t = z_{1,1}\). The first step is to find \(W^n_1\), which will be empty if \(\lambda_1(s) \neq \lambda_1(t)\) and equal to \(\{1\}\) if \(\lambda_1(s) = \lambda_1(t)\). The projection \(\lambda_1\) is defined as the composition

\[
\lambda_1: \partial^3_{01} \xrightarrow{\mu_{01}} \partial^3_{0} \circ \partial^3_{0} \longrightarrow \partial^3_{i}
\]

that sends \(s\) and \(t\) first to \((y_{2,1}, z_{1,2})\) and \((y_{2,1}, z_{1,1})\), and then to \(y_{2,1}\) and \(y_{2,1}\). Since \(y_{2,1} = y_{2,1}\), we deduce that \(W^n_1 = \{1\}\), hence \((W^n_1, W^\circ_1) =\)
\((\{1\}, \{0\})\). The second step is to compute the image of \(s\) and \(t\) under the projection
\[
\lambda_{\{1\},\{0\}}: \partial^3_0 \rightarrow \partial^2_0 \circ \partial^1_0 \rightarrow \partial^2_0.
\]
As \(\partial^2_0\) is the identity morphism, the first bijection \(\mu_{\{1\},\{0\}},\emptyset\) may be replaced by \(\mu_{\{1\},\emptyset}\). Additionally, the image of the bijection is contained in the subspace \(\partial^2_0|_{1011}\) of \(\partial^2_0\), and the value of \(\lambda_{\{1\},\{0\}}\) on \(s\) and \(t\) is
\[
s \mapsto (y_{2,1}, z_{1,2}) \mapsto z_{1,2} \in \partial^2_0|_{1011}
\]  
\[
t \mapsto (y_{2,1}, z_{1,1}) \mapsto z_{1,1} \in \partial^2_0|_{1011}.
\]

For the fourth step one checks that \(z_{1,2} > z_{1,1}\) in the span \(\partial^2_0|_{1011}\), and therefore the pair is positive.

Going back to the table, we deduce that for the pairs \((z_{1,2}, z_{1,1})\) and \((z_{1,3}, z_{1,1})\), the set \(O_{s,t}(\vec{U})^+\) has a single element, whereas for the pair \((z_{1,3}, z_{1,2})\) the set \(O_{s,t}(\vec{U})^+\) has two elements. Therefore \(\partial^2_0 \wedge \partial^3_0\) is isomorphic to the following span:

\[
\begin{array}{c}
(x, x) \\
\downarrow \\
(z_{1,2}, z_{1,1}) \\
\downarrow \\
(z_{1,3}, z_{1,1}) \\
\downarrow \\
(z_{1,3}, z_{1,2}) \\
\end{array}
\]

- \(U = (0, 0, 3, 3)\). Then \(\partial^3_{U^-} = \partial^3_{U^+} = \partial^3_{03} = \{a, b\}\). Now, since \(b > a\), then \(O_{s,t}(\vec{U})^+ = \emptyset\) for \((s, t) = (a, b)\). On the other hand, if \((s, t) = (b, a)\) then

\[
\begin{array}{|c|c|c|c|c|}
\hline
(s, t) & \text{maximal chain} & \lambda_{W^-,W^+}(s) & \lambda_{W^-,W^+}(t) & \text{span} & \text{positiveness} \\
\hline
(b, a) & (\emptyset, \{0\}) & y_{1,2} & y_{1,1} & \partial^3_{0} & - \\
\hline
(b, a) & (\emptyset, \{3\}) & y_{4,1} & y_{4,2} & \partial^3_{3} & + \\
\hline
\end{array}
\]

Therefore, \(O_{s,a}(\vec{U})^+\) has a single element, hence \(\partial^2_{03} \wedge \partial^3_{03}\) is isomorphic to the following span:

\[
(x, x) \mapsto (z_{3,3}) \mapsto (z_{3,3})
\]

- \(U = (1, 1, 3, 3)\). Then \(\partial^3_{U^-} = \partial^3_{U^+} = \partial^3_{13} = \{c, d\}\). Now, since \(d > c\), then \(O_{s,t}(\vec{U})^+ = \emptyset\) for \((s, t) = (c, d)\). On the other hand, if \((s, t) = (d, c)\) then

\[
\begin{array}{|c|c|c|c|c|}
\hline
(s, t) & \text{maximal chain} & \lambda_{W^-,W^+}(s) & \lambda_{W^-,W^+}(t) & \text{span} & \text{positiveness} \\
\hline
(d, c) & (\emptyset, \{1\}) & y_{2,2} & y_{2,1} & \partial^3_{1} & + \\
\hline
(d, c) & (\emptyset, \{3\}) & y_{4,1} & y_{4,2} & \partial^3_{3} & + \\
\hline
\end{array}
\]

Therefore, \(O_{d,c}(\vec{U})^+\) has two elements, so \(\partial^3_{13} \wedge \partial^3_{13}\) is isomorphic to the following span:

\[
(x, x) \mapsto (d, c) \mapsto (z_{5,5})
\]

- \(U = (0, 0, 1, 3)\). Then \(\partial^3_{U^-} = \partial^3_{01} = \{z_{1,1}, z_{1,2}, z_{1,3}\}\) and \(\partial^3_{U^+} = \partial^3_{03} = \{a, b\}\) and we have that for each \((s, t) \in \partial^3_{U^-} \times \partial^3_{U^+}\), the positiveness of the only
maximal chain is as follows:

| (s, t) | maximal chain | λ_{W^+, W^+}(s) | λ_{W^+, W^+}(t) | span | positiveness     |
|-------|---------------|-----------------|-----------------|------|-----------------|
| (z_{1,1}, a) | (0, {0}) | y_{1,1} | y_{1,1} | ∂_0^3 | not (s, t)-good |
| (z_{1,2}, a) | (0, {0}) | y_{1,2} | y_{1,1} | ∂_0^3 | -               |
| (z_{1,3}, a) | (0, {0}) | y_{1,2} | y_{1,1} | ∂_0^3 | -               |
| (z_{1,1}, b) | (0, {0}) | y_{1,1} | y_{1,2} | ∂_0^3 | +               |
| (z_{1,2}, b) | (0, {0}) | y_{1,2} | y_{1,2} | ∂_0^3 | not (s, t)-good |
| (z_{1,3}, b) | (0, {0}) | y_{1,2} | y_{1,2} | ∂_0^3 | not (s, t)-good |

Therefore, O_{z_{1,1}, b}(\tilde{U})^+ has a single element, and every other O_{s,t}(\tilde{U})^+ is empty, so ∂_0^3 \wedge ∂_0^3 is isomorphic to the following span:

$$ (x, x) \longleftrightarrow (z_{1,1}, b) \longrightarrow (z_{1,1}, z_3) $$

- U = (0, 1, 1, 3). Then ∂_{U^-}^3 = ∂_{01}^3 = \{z_{1,1}, z_{1,2}, z_{1,3}\} and ∂_{U^+}^3 = ∂_{13}^3 = \{c, d\}

and we have that for each (s, t) ∈ ∂_{U^-}^3 × ∂_{U^+}^3, the positiveness of the only maximal chain is as follows:

| (s, t) | maximal chain | λ_{W^+, W^+}(s) | λ_{W^+, W^+}(t) | span | positiveness     |
|-------|---------------|-----------------|-----------------|------|-----------------|
| (z_{1,1}, c) | (0, {1}) | y_{2,1} | y_{2,1} | ∂_1^3 | not (s, t)-good |
| (z_{1,2}, c) | (0, {1}) | y_{2,1} | y_{2,1} | ∂_1^3 | not (s, t)-good |
| (z_{1,3}, c) | (0, {1}) | y_{2,1} | y_{2,1} | ∂_1^3 | +               |
| (z_{1,1}, d) | (0, {1}) | y_{2,1} | y_{2,2} | ∂_1^3 | -               |
| (z_{1,2}, d) | (0, {1}) | y_{2,1} | y_{2,2} | ∂_1^3 | -               |
| (z_{1,3}, d) | (0, {1}) | y_{2,2} | y_{2,2} | ∂_1^3 | not (s, t)-good |

Therefore, O_{z_{1,3}, c}(\tilde{U})^+ has a single element, and every other O_{s,t}(\tilde{U})^+ is empty, so ∂_0^3 \wedge ∂_0^3 is isomorphic to the following span:

$$ (x, x) \longleftrightarrow (z_{1,3}, c) \longrightarrow (z_{1,3}, z_5) $$

As a consequence, we have that

| U | \mathcal{A}_{F_2}(∂_{U^-}^3 \wedge ∂_{U^+}^3 \circ \Delta_3)(x) |
|---|------------------------------------------------------------|
| (0, 0, 1, 1) | z_{1,2} \otimes z_{1,1} + z_{1,3} \otimes z_{1,1} + 2 \cdot z_{1,3} \otimes z_{1,2} |
| (0, 0, 3, 3) | 1 \cdot z_3 \otimes z_3 |
| (1, 1, 3, 3) | 2 \cdot z_5 \otimes z_5 |
| (0, 0, 1, 3) | z_{1,1} \otimes z_3 |
| (0, 1, 1, 3) | z_{1,3} \otimes z_5 |
Dualising:
\[
\alpha \sim -1 \alpha = \nabla_{-1}^\ast ( (z_{1,1}^\ast + z_{1,3}^\ast + z_{3,1}^\ast + z_5^\ast) \otimes (z_{1,1}^\ast + z_{1,3}^\ast + z_3^\ast + z_5^\ast))
\]
\[
= \nabla_{-1}^\ast (z_{1,1}^\ast \otimes z_{1,1}^\ast + z_{1,1}^\ast \otimes z_{1,3}^\ast + z_{1,3}^\ast \otimes z_{1,1}^\ast + z_{1,3}^\ast \otimes z_3^\ast + z_{1,3}^\ast \otimes z_5^\ast + z_3^\ast \otimes z_{1,3}^\ast + z_3^\ast \otimes z_3^\ast + z_3^\ast \otimes z_5^\ast + z_3^\ast \otimes z_5^\ast + z_5^\ast \otimes z_{1,3}^\ast + z_5^\ast \otimes z_3^\ast + z_5^\ast \otimes z_3^\ast + z_5^\ast \otimes z_5^\ast)
\]
\[
= (0 + 0 + 1 + 0
+ 1 + 0 + 0 + 1
+ 0 + 0 + 1
+ 0 + 0 + 0 + 0) \cdot x^4
= 0
\]
So we conclude that \(sq^2([\alpha]) = [0]\).

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