Phasor field waves: A statistical treatment for the case of a partially coherent optical carrier

SYED AZER REZA,1,* SEBASTIAN BAUER,2 AND ANDREAS VELTEN2,3,*

1Department of Physics and Optical Engineering, Rose-Hulman Institute of Technology, Terre Haute, IN 47803, USA
2Department of Biostatistics and Medical Informatics, University of Wisconsin – Madison, Madison, WI 53706, USA
3Department of Electrical and Computer Engineering, University of Wisconsin – Madison, Madison, WI 53706, USA
*reza@rose-hulman.edu
*velten@wisc.edu
https://biostat.wisc.edu/~compoptics/

Abstract: This paper presents a statistical treatment of phasor fields (P-fields) – a wave-like quantity denoting the slow temporal variations in time-averaged irradiance (which was recently introduced to model and describe non-line-of-sight (NLoS) imaging as well as imaging through diffuse or scattering apertures) – and quantifies the magnitude of a spurious signal which emerges due to a partial spatial coherence of the underlying optical carrier. This spurious signal is not described by the Huygens-like P-field imaging integral which assumes optical incoherence as a necessary condition to describe P-field imaging completely (as was shown by Reza et al. [1–3]). In this paper, we estimate the relationship between the expected magnitude of this spurious signal and the degree of partial roughness within the P-field imaging system. The treatment allows us to determine the accuracy of the estimate provided by the P-field integral for varying degrees of partial coherence and allows to define a P-field signal-to-noise ratio as a figure-of-merit for the case of a partially coherent optical carrier. The study of partial coherence also enables to better relate aperture roughness to P-field noise.

© 2020 Optical Society of America

1. Introduction

Non-line-of-sight (NLoS) around-the-corner imaging was first experimentally demonstrated by Velten et al. [4]. NLoS imaging is based on diffuse reflections off a surface which is usually called the relay wall. An exemplary NLoS setup is shown in Fig. 1. In the configuration discussed here, the relay wall is scanned by a picosecond laser, and the travel time of the photons returning to the relay wall at one specific spot is recorded using an ultrafast detector such as a Single-Photon Avalanche Diode (SPAD). Over the years, various techniques including filtered backprojection, speckle correlations etc. [5–11] have been proposed for NLoS reconstructions. While each of these various approaches entail respective benefits and drawbacks, all of them treat NLoS imaging as a separate class of imaging to conventional line-of-sight (LoS) imaging which is described by the solution to the wave equation; namely the Huygens’ integral. An NLoS reconstruction method based on virtual, so-called phasor field (P-field) waves was introduced in [12], where the required sinusoidal light intensity modulation was performed in post-processing. This enables the description of NLoS reconstruction as a (virtual) wave propagation problem where the relay wall can be treated as the aperture plane of a virtual camera.

A more thorough theoretical analysis of phasor fields as well as practical experiments with real modulated light sources have been presented as well [1, 3]. In [1], the propagation of P-fields was shown to be described by the P-field integral, an integral which is analogous to
the Huygens’ integral with spherical $\mathcal{P}$-field contributions replacing the spherical Electric field (E-field) contributions. This $\mathcal{P}$-field integral only holds exactly true if the underlying optical carrier can be considered as spatially incoherent [1]. In this case the $\mathcal{P}$-field contributions add linearly as is the case for E-field contributions with the Huygens’ integral. The effect of temporal coherence of the optical carrier on $\mathcal{P}$-field propagation and interference has been briefly discussed by Teichman [13]. Furthering the theory of $\mathcal{P}$-field imaging, Dove et al. [14] recently proposed $\mathcal{P}$-field occlusion-aided NLoS imaging.

All the aforementioned contributions share the assumption that the relay wall is rough on an optical scale, meaning that the optical phase is shifted by the reflection such that it is uniformly distributed over the range $[0, 2\pi]$ when reduced to modulo-$2\pi$. In this case, the reflected light is incoherent and addition of the sinusoidally modulated intensities reflected off different relay wall points is sufficient to describe the resulting wave front in front of the relay wall. However, there are cases where the relay wall is not optically rough, meaning that the reflected light cannot be considered incoherent any more. For this reason, in this paper, we discuss the effect of partial spatial coherence of the underlying optical carrier (E-field) on $\mathcal{P}$-field imaging. This allows us to calculate the magnitude of an additive $\mathcal{P}$-field noise that is introduced by partial optical coherence and enables us to calculate an expected $\mathcal{P}$-field signal-to-noise ratio (SNR) for varying degrees of partial spatial coherence. This statistical treatment can also be useful to describe the relation between a certain degree of partial-roughness of the $\mathcal{P}$-field aperture and the expected $\mathcal{P}$-field SNR. This allows for calculating the limits on minimum aperture roughness that satisfy any desired $\mathcal{P}$-field SNR tolerance value. We want to remark that our work could be extended in the future by incorporating the second-order statistics (auto-correlation) calculations developed for speckle effects [15, 16]. These allow for example for the calculation of the 2D spatial intensity auto-correlation after the reflection based on the 2D auto-correlation of the surface profile.
2. \(P\)-field imaging for incoherent and partially coherent E-field

2.1. Introduction to \(P\)-fields

The notion of the \(P\)-fields was introduced in [1, 2]. The goal of this is to describe an NLoS imaging system analogously to a LoS imaging system - which enables us to describe and evaluate NLoS system performance and limitations from the well-known treatment of LoS systems.

It is well-known that the Huygens’ integral is a solution to the scalar wave equation and describes the E-field at a location \((x,y)\) in a detection plane \(\Sigma(z = Z)\) as a sum of E-field spherical wavelet contributions from all locations \((x',y')\) of a specular or non-rough aperture \(\mathcal{A}\) defined by the plane \(z = 0\). In the context of imaging, the Huygens’ integral

\[
E(x, y, z = Z) = \frac{j}{\lambda_E} \int_{\mathcal{A}} E(x', y', 0) e^{jk|\mathbf{r}|} \chi_E d\mathbf{x}' d\mathbf{y}'
\]  

(1)

describes E-field propagation and interference resulting in image formation in \(\Sigma\) as is shown in Fig. 2. In (1), \(|\mathbf{r}| = \sqrt{(x-x')^2 + (y-y')^2 + Z^2}\) is the absolute distance between any unique pair of locations \((x',y',0)\) \(\in\mathcal{A}\) and \((x,y,Z)\) \(\in \Sigma\), \(z = Z\) is the separation distance between \(\mathcal{A}\) and \(\Sigma\), \(k\) is the E-field wave number expressed in terms of the E-field (optical) wavelength \(\lambda_E\) as \(k = 2\pi/\lambda_E\) and \(\chi_E\) is the E-field obliquity factor.

For the case of a diffuse aperture plane \(\mathcal{A}\), the notion of \(P\)-field propagation was described in [1] where a \(P\)-field signal is described as the baseband envelope of the amplitude-modulated optical irradiance of the carrier. It has been shown in [1] that this is equivalent to effectively amplitude modulating the optical carrier E-field to yield a modulated scalar time-harmonic real-valued field \(\text{Re}[E_0(x', y', t)e^{j\Omega t}]\) where \(E_0(x', y', t) = E_0(x', y')\cos(\Omega t/2)\) is the baseband modulating signal of angular frequency \(\Omega/2\) and an amplitude \(E_0\) scaled by the impedance \(\zeta\) of the medium to effectively have units of \(\sqrt{W/m^2}\) in the \(z = 0\) plane.

The quantity phasor field is defined as the slowly-varying signed envelope of the time-averaged optical irradiance of frequency \(\Omega\) – which depicts the signal that is used to directly amplitude-modulate the optical carrier. At a diffuse aperture (or origin) plane at \(z = 0\) the \(P\)-field is described as

\[
\text{Re}(P_{0,\Omega}(x', y')e^{j\Omega t}) = \Delta I_0(x', y', t) = I_0(x', y', t) - \langle I_0(x', y', t)\rangle,
\]  

(2)

where

\[
I_0(x', y', t) = \frac{1}{T} \int_{t-T/2}^{t+T/2} |E_0(x', y', \tau)|^2 d\tau = \frac{1}{2} |E_0(x', y', t)|^2 [1 + \cos(\Omega t)]
\]  

(3)

and

\[
\langle I_0(x', y', t)\rangle = \lim_{T\to\infty} \frac{1}{T} \int_{t-T/2}^{t+T/2} |E_0(x', y', t)|^2 dt = \frac{1}{2} |E_0(x', y', t)|^2
\]  

(4)

are the short time and the long time averages of the optical irradiance contribution from \((x', y', 0)\) \(\in \mathcal{A}\) under the condition that \(\Omega \ll \omega\) and that the integration time \(T\) of the detector is much longer than the time period of the optical carriers’ E-field oscillation and much smaller than the time period of the \(P\)-field signal, i.e.,

\[
\frac{2\pi}{\omega} \ll T \ll \frac{2\pi}{\Omega}.
\]  

(5)

The resulting \(P\)-field contribution \(\text{Re}(P_{0,\Omega}(x', y')e^{j\Omega t}) = \Delta I_0(x', y', t)\) from \((x', y')\) (the same quantity as stated in (2)) is hence a real signed time-harmonic function of frequency \(\Omega\) and
amplitude \( P_{0,\Omega}(x', y') = P_0|E_0(x', y')|^2 / 4 \zeta \) with \( E_0(x', y') \) representing the E-field amplitude contribution from \((x', y')\) and \( \zeta \) the impedance of the propagation medium. Hence, \( \Delta I_0(x', y', t) \) is effectively expressed as

\[
\Delta I_0(x', y', t) = P_{0,\Omega}(x', y') \cos(\Omega t). \tag{6}
\]

In (6), the operation of subtracting the long-time average from the short-time average is equivalent to physically detecting with an AC-coupled detector which removes the DC offset from the received slowly-varying optical carrier envelope. It was also shown in [1] that the expected value of the total \( \mathcal{P} \)-field – \( \mathcal{P}_{\text{Sum}}(x, y, z = Z, t) \) (denoted as \( I_{\text{Tot.-F}}(x, y, Z) \) in [1]) – received at a location \((x, y)\) within the detection plane \( \Sigma \) located at \( z = Z \) for a spatially incoherent optical carrier is expressed as the sum of all \( \mathcal{P} \)-field spherical contributions \( \mathcal{P}(r) \) from \( \mathcal{A} \) in a Huygens’-like integral as

\[
\mathcal{P}_{\text{Sum}}(x, y, Z, t) \propto \frac{1}{|r(x, y, Z)|_{\text{AV}}} \int_{\mathcal{A}} P_{0,\Omega}(x', y') \frac{e^{i\beta|r|}}{|r|} \chi_{P} d x' d y'. \tag{7}
\]

In (7), \( \text{Re}[\mathcal{P}(r)e^{i\Omega t}] = \Delta I_0(x', y', t) / |r| \) is one time-harmonic real \( \mathcal{P} \)-field contribution from \((x', y')\) scaled down by \(|r|\) expressed in the complex phasor notation. Also

\[
\text{Re} \left( \frac{1}{|r(x, y, Z)|_{\text{AV}}} \int_{\mathcal{A}} P_{0,\Omega}(x', y') \frac{e^{i\beta|r|}}{|r|} \chi_{P} d x' d y' e^{i\Omega t} \right) = \Delta I_Z(x, y, t) = I_Z(x, y, t) - \langle I_Z(x, y, t) \rangle,
\]

is again the difference between the short time averaged and the long time statistically averaged optical irradiance quantities \( I_Z(x', y, t) \) and \( \langle I_Z(x', y, t) \rangle \) at \((x, y, z = Z)\), \(|r| = \sqrt{Z^2 + (x - x')^2 + (y - y')^2}\) is the distance between any locations \((x', y', z = 0) \in \mathcal{A}\) and \((x, y, Z) \in \Sigma\), \(|r(x, y, Z)|_{\text{AV}} = \sqrt{Z^2 + (x - x')^2 + (y - y')^2}\) is the average distance between a location \((x, y, Z) \in \Sigma\) from an average location \((x', y, z = 0) \in \mathcal{A}\), \( \chi_{P}(x', x, y', y) \) is the \( \mathcal{P} \)-field obliquity factor, and \( \beta \) is the \( \mathcal{P} \)-field wave number expressed in terms of the \( \mathcal{P} \)-field wavelength \( \lambda_P \) as \( \beta = 2\pi / \lambda_P \). This integral in (8) is referred to as the \( \mathcal{P} \)-field integral which clearly depicts that spherical \( \mathcal{P} \)-field contributions from a diffuse surface \( \mathcal{A} \) interfere analogously to spherical E-field contributions from a specular surface as was originally described by the Huygens’ integral presented in (1).

### 2.2. Effect of partial coherence of E-fields on the \( \mathcal{P} \)-field sum

For the case of partial optical coherence, we will show during the course of our mathematical treatment that the total \( \mathcal{P} \)-field \( [P_{0,\Omega}(x, y, Z, t)]_{\text{Tot}} \) at any location \((x, y, Z) \in \Sigma\) is described as a \( \mathcal{P} \)-field integral sum \( \mathcal{P}_{\text{Sum}}(x, y, Z, t) \) described in [1] with an additive spurious sum \( \mathcal{P}_{\text{Sp}}(x, y, Z, t) \) from the cross-interference of all \( \mathcal{P} \)-field contributions from \( \mathcal{A} \). This cross-interference due to the partial spatial coherence of the optical carrier is the underlying basis of E-field and \( \mathcal{P} \)-field speckle and is a result of aperture roughness and uncertainties in phases accumulated in propagation from \( \mathcal{A} \) to \( \Sigma \). In other words, we will demonstrate that the total \( \mathcal{P} \)-field contribution \([P_{0,\Omega}(x, y, Z, t)]_{\text{Tot}} \) is equal to the Huygens-like \( \mathcal{P} \)-field sum added with a spurious \( \mathcal{P} \)-field noise signal \( \mathcal{P}_{\text{Sp}}(x, y, Z, t) \), i.e.,

\[
[P_{0,\Omega}(x, y, Z, t)]_{\text{Tot}} = \mathcal{P}_{\text{Sum}}(x, y, Z, t) + \mathcal{P}_{\text{Sp}}(x, y, Z, t) \tag{9}
\]
and relate the magnitude of this spurious $P$-field noise to the degree of roughness in the aperture plane $A$. In the case of a diffuse surface/aperture $A$ in transmission or reflection, the optical carrier as well as its slowly varying envelope experience a phase shift which is spatially-dependent on the roughness profile of $A$, i.e., the roughness in each location within $A$ determines the magnitudes of the phase shifts $\delta \phi_k$ and $\delta \phi_\beta$ imparted to the optical carrier and its slowly-varying envelope respectively. Let $\omega$ denote the frequency of the optical carrier and $\Omega$ the frequency of the slowly-varying envelope of amplitude-modulated carrier irradiance (a $P$-field contribution).

In the case $\omega \gg \Omega$, the random phase $\delta \phi_k = \phi_R$ of the E-field accumulated due to diffuse reflections or transmission is much larger than the random phase $\delta \phi_\beta$ of the $P$-field because of the difference of a few orders of magnitude between the optical and $P$-field wavelengths; $\phi_R \gg \delta \phi_\beta$. We will show in subsequent sections that in the case of a partially spatially coherent optical carrier $[P_{0,\Omega}(x, y, Z, t)]_{Tot}$ is described as

$$\left| [P_{0,\Omega}(x, y, z = Z, t)]_{Tot} \right| = \left| \sum_{n=1}^{N} \frac{P_{0n,\Omega}}{|r_n|} \cos \left[ \Omega t + 2 \phi_{\beta n} \right] \, dA \right| + \left. \sigma \left[ \langle I_{Cross} (x, y, Z, t) \rangle \right] \, \right|_{Spurious \ P-field \ Signal \ Noise} \left| P_{Sp(x,y,Z,t)} \right|$$

where

$$\sigma \left[ \langle I_{Cross} (x, y, Z, t) \rangle \right] = \sqrt{\frac{\theta^2 + 8 \theta^4 + 8 \cos (2\theta) - (2 + \theta^2) \cos (4\theta) - 6 \theta^2 + \cos (4\theta) - 1}{16\theta^4} + \sin^4 (\theta)} \cdot \left[ \sum_{m=1}^{N} \sum_{n=1 \neq m}^{N} \sqrt{\frac{P_{0m,\Omega} P_{0n,\Omega}}{|r_m||r_n|}} \cos (\Omega t) \, dA \right]$$

is the standard deviation of the zero-mean cross interference from all pairs of the $m^{th}$ and $n^{th}$ ($m \neq n$) $P$-field (and E-field) contributions from $(x_m, y_m, 0)$ and $(x_n, y_n, 0) \in A$ respectively out of a total of $N$ contributions and $[-\theta, \theta]$ is the range of random phase introduced exclusively by the roughness at each location of $A$. In (11), it is assumed that these contributions stem from infinitesimally small relay wall (aperture in the context of NLoS imaging) patches and are independent from one another. The phases $\phi_{km}$, $\phi_{kn}$ and $\phi_{\beta m}$, $\phi_{\beta n}$ denote the E-field and $P$-field phases accumulated by these respective contributions with corresponding magnitudes of $P_{0m,\Omega}$ and $P_{0n,\Omega}$. \[\]
and \( \mathcal{P}_{0n} \). The variables \(|r_n|\) and \(|r_s|\) are the distances between \((x_m, y_m, 0), (x_n, y_n, 0) \in \mathcal{A}\) and \((x, y, z = Z) \in \Sigma\).

In Section 3, the spurious \( \mathcal{P} \)-field signal \( \mathcal{P}_{Sp} (x, y, Z, t) \) will be derived as a function of the allowable range \([-\theta, \theta]\) of values that a random variable \( \phi_R \) denotes the phase difference (imparted solely by aperture roughness) between any pair of \( \mathcal{P} \)-field contributions originating from two different locations in \( \mathcal{A} \). Hence, we derive a direct relationship in (11) between the magnitude of the spurious \( \mathcal{P} \)-field noise \( \mathcal{P}_{Sp} (x, y, Z, t) \) and the degree of randomness in \( \mathcal{A} \).

Moreover – analogous to conventional signal theory – we calculate a \( \mathcal{P} \)-field signal-to-noise ratio (SNR) as a figure-of-merit of the \( \mathcal{P} \)-field NLoS imaging system as

\[
\mathcal{P}_{SNR} (x, y, Z) = \frac{|\mathcal{P}_{Sum} (x, y, Z, t)|}{|\mathcal{P}_{Sp} (x, y, Z, t)|} \tag{12}
\]

and show that the \( \mathcal{P} \)-field SNR saturates beyond a certain degree of surface roughness and the performance of a \( \mathcal{P} \)-field imaging system remains consistent beyond this threshold aperture roughness as indicated by a saturation in the \( \mathcal{P} \)-field SNR.

3. Deriving the relationship between partial spatial E-field coherence and the \( \mathcal{P} \)-field noise

3.1. Interference of unmodulated E-fields

We begin by reminding the reader of how the interference of \( N \) discrete unmodulated E-field contributions from \( \mathcal{A} \) when observed at \( \Sigma \) can be described. The irradiance \( I(x, y, Z) \) at a generic location \((x, y, z = Z) \in \Sigma \) is the square of the sum of all E-field contributions from \( \mathcal{A} \) and it is stated as

\[
I(x, y, Z) = \frac{1}{\xi} \left| \sum_{n=1}^{N} \frac{E_{0n}}{|r_n|} \right|^2. \tag{13}
\]

In (13), \(|r_n| = \sqrt{(x - x'_n)^2 + (y - y'_n)^2 + Z^2}\) is the distance between any location \((x'_n, y'_n, 0) \in \mathcal{A}\) and \((x, y, z) \) and the E-field contribution from \((x'_n, y'_n, 0) \) has magnitude \( |E_{0n}| \) and phase \( \phi_{kn} = k |r_n| \). Hence the \( n\textsuperscript{th} \) E-field contribution – whose polarization is expressed by a unit vector \( \hat{e} \) – can be stated as \( E_{0n} = |E_{0n}| \cos(\omega t + \phi_{kn}) \hat{e} \). From (13), the irradiance at \((x, y, Z) \) can be expressed as

\[
I(x, y, Z) = \frac{1}{4\xi} \left| \sum_{n=1}^{N} \frac{|E_{01}|}{|r_1|} e^{j(\omega t + \phi_{11})} + \frac{|E_{02}|}{|r_2|} e^{j(\omega t + \phi_{12})} + \ldots + \frac{|E_{0N}|}{|r_N|} e^{j(\omega t + \phi_{1N})} + c.c. \right|^2, \tag{14}
\]

where ‘c.c.’ denotes the complex conjugate sum of each of the preceding E-field terms. A time-averaged irradiance \( \langle I(x, y, Z) \rangle \) is calculated by integrating \( I(x, y, Z) \) over the detector integration time \( T \)

\[
\langle I(x, y, Z) \rangle = \frac{I_1}{|r_1|^2} + \frac{I_2}{|r_2|^2} + \frac{I_3}{|r_3|^2} + \ldots + \frac{I_N}{|r_N|^2} + 2\sqrt{T_1 \sqrt{T_2}} \cos \Delta \phi_{1,2} + 2\sqrt{T_1 \sqrt{T_3}} \cos \Delta \phi_{1,3} + \ldots + 2\sqrt{T_1 \sqrt{T_N}} \cos \Delta \phi_{1,N} + 2\sqrt{T_2 \sqrt{T_N}} \cos \Delta \phi_{2,N} + \ldots + 2\sqrt{T_{N-1} \sqrt{T_N}} \cos \Delta \phi_{N-1,N}, \tag{15}
\]

as the average of all high frequency sinusoidal terms with a frequency of \( 2\omega \) is approximately zero if \( T \gg 2\pi/\omega \). In (15), each irradiance term \( I_n \) represents the time-averaged irradiance contribution from \((x'_n, y'_n, 0) \in \mathcal{A}\) given by \( I_n = |E_{0n}|^2/2\xi \). \( \langle I(x, y, Z) \rangle \) can be simply expressed
When optical irradiance is directly amplitude-modulated using a low frequency signal within the detector plane when the underlying E-field carrier contributions from AC-coupled value of irradiance (the $\Omega$ term for $\Omega$ instead of $\Omega_0$)

\[ \langle I(x, y, Z) \rangle = \sum_{n=1}^{N} \frac{I_n}{|r_n|^2} + 2 \sum_{p=1}^{N} \sum_{q=1}^{N} \sqrt{T_p} \sqrt{T_q} \cos \Delta \phi_{p,q} \cdot \tag{16} \]

In Eqns. (15) and (16), $\Delta \phi_{p,q} = \phi_{k_p} - \phi_{k_q}$ denotes the optical (E-field) phase difference between the $p^{th}$ and $q^{th}$ E-field contributions from $\mathcal{A}$ while $I_p$ and $I_q$ denote the time-averaged irradiance contributions from the $p^{th}$ and $q^{th}$ locations in $\mathcal{A}$ respectively. Note that the expression in (16) has been separated into two terms; the first term is the sum of individual irradiance contributions (i.e., Term 1 for $p = q$) from each location $(x', y', 0) \in \mathcal{A}$ and a second term which is a sum of $\mathcal{P}$-field cross-multiplication terms between any two non-identical locations $(x_p, y_p, 0)$ and $(x_q, y_q, 0) \in \mathcal{A}$.

In this paper, we will derive a similar general expression for the interference of $N$ $\mathcal{P}$-field contributions for the case of a partially coherent optical carrier. This will allow us to establish a relationship between the partial coherence of the E-field and the magnitude of the additive $\mathcal{P}$-field cross-interference between $\mathcal{P}$-field contributions from different locations in $\mathcal{A}$. To re-emphasize, this sum of $\mathcal{P}$-field cross-interference is not explained by the $\mathcal{P}$-field integral because of speckle-averaging which results in the expected value of the cross-interference term to 0 for fully incoherent summation.

### 3.2. Interference of multiple $\mathcal{P}$-field contributions

When optical irradiance is directly amplitude-modulated using a low frequency signal

\[ P(t) = P_0 [1 + \cos\Omega t], \tag{17} \]

the time-averaged irradiance at a location $(x, y, Z) \in \Sigma$ is simply calculated by multiplying the Poynting vector $S(x, y, Z, t)$ at $(x, y, Z)$ by the $\mathcal{P}$-field modulating signal contribution from a location $(x', y', 0) \in \mathcal{A}$ and calculating a time-average over the integration time of the detector. In order to calculate the time-averaged irradiance at $(x, y, Z)$ as a result of multiple contributions from various locations within $\mathcal{A}$, all E-field contributions from $\mathcal{A}$ have to be added, the result is squared and integrated over the integration time window $T$ of the detector. Moreover, it was shown in [1], it was shown that when all the E-field contributions from $\mathcal{A}$ are incoherent, the resulting time-averaged AC-coupled value of irradiance (the $\mathcal{P}$-field sum) at any location $(x, y, Z) \in \Sigma$ is simply the linear sum of individual irradiance $\mathcal{P}$-field contributions from all locations in $\mathcal{A}$. Hence the subsequent $\mathcal{P}$-field sum is expressed as a Huygens-like $\mathcal{P}$-field integral.

Here, we determine the time-averaged irradiance and the resulting $\mathcal{P}$-field sum at any location within the detector plane when the underlying E-field carrier contributions from $\mathcal{A}$ cannot be considered completely incoherent. If the phase accumulated due to propagation between $\mathcal{A}$ and $\Sigma$ by an $n^{th}$ $\mathcal{P}$-field contribution of amplitude $P_{0n}$ can be denoted by $\phi_{\mathcal{P},n} = \beta|r_n|$ and the relatively smaller random phase shift due to aperture roughness or any other reason denoted by $\delta \phi_p(x', y', 0)$, then as was shown in [1], the $n^{th}$ $\mathcal{P}$-field contribution at $(x, y, Z)$ is simply stated as

\[ P_n(r_n, t) = P_{0n} \cos [\Omega t + \beta|r_n| + \delta \phi_p(x', y', 0)]. \tag{18} \]

It was also shown in [1], that irradiance contribution in $\Sigma$ that is amplitude-modulated with $\mathcal{P}(t)$ is equivalent to modulating the E-field of the underlying optical carrier by

\[ Q_n(r_n, t) = \sqrt{P_{0n} \cos [\Omega' t + \beta|r_n|/2 + \delta \phi_p(x', y', 0)/2]}, \tag{19} \]

where $\Omega' = \Omega/2$. Knowing this fundamental $\mathcal{P}$-field framework developed earlier, we first derive an expression for the $\mathcal{P}$-field sum (i.e., total AC coupled time-averaged optical irradiance) at any
where \( k |r_n| \) is the phase accumulated by the \( n^{th} \) E-field contribution in propagating a distance \( |r_n| \) between locations \((x', y', 0) \in \mathcal{A} \) and \((x, y, Z) \in \Sigma \), \( \delta \phi_k(x', y', 0) \) is the random phase added to this \( n^{th} \) E-field contribution and \( E_{0n} = |E_{0n}| \hat{e} \) represents a vector of magnitude \( |E_{0n}| \) in the direction of the unit vector \( \hat{e} \) that denotes the polarization of the E-field. If the total phases accumulated by the \( n^{th} \) E-field and \( Q_n(r_n, t) \) contributions are stated as \( \phi_{kn} = k|r_n| + \delta \phi_k(x', y', 0) \) and \( \phi_{bn} = \beta |r_n|/2 + \delta \phi_{b,n}(x', y', 0)/2 \) respectively, then the time-averaged irradiance \( \langle I(x, y, Z, t) \rangle \) measured at a location \((x, y, Z) \in \Sigma \) due to just two modulated E-field contributions is simply expressed as

\[
\langle I(x, y, Z, t) \rangle = \frac{1}{\xi T} \int_{-T/2}^{+T/2} \frac{P_0 |E_{01}|^2}{|r_1|^2} \cos^2 (\Omega' \tau + \phi_{b1}) \cos^2 (\omega \tau + \phi_{k1}) + \frac{P_0 |E_{02}|^2}{|r_2|^2} \cos^2 (\Omega' \tau + \phi_{b2}) \cos^2 (\omega \tau + \phi_{k2}) + \frac{2 |E_{01}| |E_{02}| P_0}{|r_1| |r_2|} \cos (\Omega' \tau + \phi_{b1}) \cos (\Omega' \tau + \phi_{b2}) \cos (\omega \tau + \phi_{k1}) \cos (\omega \tau + \phi_{k2}) \right) d\tau.
\] (21)

Note that \( \langle I(x, y, Z, t) \rangle \) is still a function of time \( t \) as the modulation components of \( \langle I(x, y, Z, t) \rangle \) of frequency \( \Omega' \) do not average out for the chosen detector integration time window of length \( T \). Assuming identical polarization of each of the E-field contributions, (21) can be further expressed as

\[
\langle I(x, y, Z, t) \rangle = \frac{1}{\xi T} \int_{-T/2}^{+T/2} \left[ \frac{P_0 |E_{01}|^2}{|r_1|^2} \cos^2 (\Omega' \tau + \phi_{b1}) \cos^2 (\omega \tau + \phi_{k1}) + \frac{P_0 |E_{02}|^2}{|r_2|^2} \cos^2 (\Omega' \tau + \phi_{b2}) \cos^2 (\omega \tau + \phi_{k2}) + \frac{2 |E_{01}| |E_{02}| P_0}{|r_1| |r_2|} \cos (\Omega' \tau + \phi_{b1}) \cos (\Omega' \tau + \phi_{b2}) \cos (\omega \tau + \phi_{k1}) \cos (\omega \tau + \phi_{k2}) \right] d\tau.
\] (22)

Using the product-of-cosines identity

\[
\cos (\alpha_1) \cos (\alpha_2) = \frac{1}{2} \left[ \cos (\alpha_1 + \alpha_2) + \cos (\alpha_1 - \alpha_2) \right],
\] (23)

and the cosine double angle identity

\[
\cos^2 (\alpha) = \frac{1 + \cos (2\alpha)}{2},
\] (24)
we can express (22) as

\[
\langle I(x, y, Z, t) \rangle = \frac{1}{T} \int_{T/2}^{T/2 + \tau} \left( \frac{P_0|E_{01}|^2}{4|p_1|^2} \left[ 1 + \cos \left( \Omega \tau + 2\phi_{B1} \right) \right] \left[ 1 + \cos \left( 2\omega \tau + 2\phi_{k1} \right) \right] + 
  \frac{P_0|E_{02}|^2}{4|p_2|^2} \left[ 1 + \cos \left( \Omega \tau + 2\phi_{B2} \right) \right] \left[ 1 + \cos \left( 2\omega \tau + 2\phi_{k2} \right) \right] + 
  \right) d\tau.
\]

After performing time-integration over an integration window \( T \) that satisfies the condition in (5), i.e.,

\[
\frac{2\pi}{\omega} \ll T \ll \frac{2\pi}{\Omega},
\]
we eliminate all high frequency terms oscillating at \( 2\omega \) as their mean over several cycles can be approximated to zero. Moreover, considering the fact that \( T \ll \frac{2\pi}{\Omega} \), the values of the low frequency sinusoidal terms with frequency \( \Omega \) almost remain approximately constant during each of the detector integration time interval \( T \) and these sinusoidal terms are retrieved almost perfectly over successive time integrations. Hence, for the condition in (26), we can express (25) as

\[
\langle I(x, y, Z, t) \rangle = \frac{P_0|E_{01}|^2}{4\zeta|p_1|^2} + \frac{P_0|E_{02}|^2}{4\zeta|p_2|^2} \left[ \Omega \right] + \frac{P_0|E_{01}|^2}{4\zeta|p_1|^2} \left[ \Omega \right] + \frac{P_0|E_{02}|^2}{4\zeta|p_2|^2} \left[ \Omega \right]
\]

When the detector is AC-coupled, the constant Terms 1, 2 and 6 from (27) are eliminated. The operation of short-time integration to eliminate high frequency terms in (25) in conjunction with the AC-coupling operation in (27) is equivalent to subtracting the long-time average from the short-time average as was stated in (8). If the \( P \)-field amplitude \( P_{0n,\Omega} \) of the \( n \)th contribution is defined as \( P_{0n,\Omega} = P_0|E_{0n}|^2/4\zeta \), we obtain

\[
\Delta I_Z (x, y, t) = \frac{P_{01,\Omega}}{|p_1|^2} \cos \left[ \Omega \tau + 2\phi_{B1} \right] + \frac{P_{02,\Omega}}{|p_2|^2} \cos \left[ \Omega \tau + 2\phi_{B2} \right] + 
  \right)
\]

In the general case with \( N \) \( P \)-field contributions from \( A \) contributing to \( \Delta I_Z (x, y, t) \), (28) can be
expanded and expressed as the following Riemann sum:

\[
\Delta I_Z(x, y, t) = \sum_{n=1}^{N} P_{0n, \Omega} \cos \left[ \Omega t + 2\phi_{kn} + \phi_{m} + \phi_{n} \right] dA + \\
+ 2 \sum_{m=1}^{M} \sum_{n=1}^{N} \frac{\sqrt{P_{0m, \Omega} P_{0n, \Omega}}}{|r_m||r_n|} \cos \left[ \phi_{km} - \phi_{kn} \right] \cos \left[ \Omega t + \phi_{m} + \phi_{n} \right] dA, \quad (29)
\]

where \(dA = dx'dy'\) represents an infinitesimally small area in \(A\) over which \(\phi_{k}\) and \(\phi_{r}\) remain almost constant. Each such small area \(dA\) is responsible for one of many \(P\)-field contributions that sum a detector plane location \((x, y, Z)\). In (29), the first summation term represents the \(P\)-field summation for an incoherent optical carrier as was presented in [1]. The next term represents the summation of all \(P\)-field cross-interference terms previously not considered in [1].

### 3.3. Estimate of the \(P\)-field speckled-based noise as a function of aperture roughness

Now that we have split the total \(P\)-field at any location \((x, y, z = Z)\) into two distinct sums - i.e. the desired \(P\)-field sum and an additive spurious cross-interference \(P\)-field sum - we can proceed to focus on the sum of cross-interference contributions \(\langle I_{\text{cross}}(x, y, Z, t) \rangle\) which is the basis of the additive \(P\)-field speckled-based noise. The sum of the product of the mixed \(P\)-field interference terms \(\langle I_{\text{cross}}(x, y, Z, t) \rangle\) from (29)

\[
\langle I_{\text{cross}}(x, y, Z, t) \rangle = 2 \sum_{m=1}^{M} \sum_{n=1}^{N} \frac{\sqrt{P_{0m, \Omega} P_{0n, \Omega}}}{|r_m||r_n|} \cos \left[ \phi_{km} - \phi_{kn} \right] \cos \left[ \Omega t + \phi_{m} + \phi_{n} \right] dA \quad (30)
\]

is an additive term to Term 1 in (29) (which represents the \(P\)-field integral in continuous form). Our objective is to determine the expected value and the variance of \(I_{\text{cross}}\) depending on the magnitude of roughness in \(A\). Let us first look into the expression in (30) and determine the effect of a random phase added by each location \((x', y', 0) \in A\).

We know from our definitions that

\[
\cos \left[ \phi_{km} - \phi_{kn} \right] =\cos \left[ (k|r_m| - k|r_n|) + \delta \phi_{k,m}(x'_m, y'_m, 0) - \delta \phi_{k,n}(x'_m, y'_m, 0) \right],
\]

\[
\cos \left[ \Omega t + \phi_{m} + \phi_{n} \right] =\cos \left[ \Omega t + \beta |r_m|/2 + \beta |r_n|/2 + \delta \phi_{\beta,m}/2 + \delta \phi_{\beta,n}/2 \right].
\]

If additional phase shifts in the \(P\)-field and the E-field contributions can be simply attributed to aperture roughness and assuming that this roughness in \(A\) is much smaller than the \(P\)-field wavelength \(\lambda_p\) (see [1]), we can assume that \(\delta \phi_{\beta,m} + \delta \phi_{\beta,n} \approx 0\). On the other hand the phase term \((\beta |r_m| + \beta |r_n|)\) is deterministic because its assumed that the degree of roughness is much smaller than the \(P\)-field carrier modulation wavelength. The term \((k|r_m| - k|r_n|)\) denoting difference in phase accumulated due to propagation exclusively by E-field carrier contributions from \(A\) is also considered to be a random variable owing to minute path length differences in the order of the E-field wavelength compared to an ideal-world scenario where these would be perfectly accounted for. The uncertainty in \((k|r_m| - k|r_n|)\) gives rise to optical speckle in most E-field spatial distributions arising from any interference and propagation of E-fields.

Lastly, but most importantly, the phase term \(\phi_{\beta}(x'_{(m,n)}, y'_{(m,n)}, 0)\) is most affected by minor random path changes (exclusively due to roughness in \(A\)) as these path changes result in phase
changes to the E-field contributions which are much larger than one E-field phase cycle. If random phase shifts \( \delta \phi_{km}(x_m', y_m', 0) \) and \( \delta \phi_{kn}(x_n', y_n', 0) \) to the \( m^{th} \) and \( n^{th} \) E-field contributions are each considered to be uniformly-distributed random variables that exist in the range \([-\theta, \theta]\) – where \( \theta \) depends on the amount of roughness in the diffuse surface \( \mathcal{A} \) that determines the degree of partial coherence of the carrier – then \( \phi_R(x_{(m,n)}, y_{(m,n)}, 0) \) is a triangularly-distributed random variable in the range \([-2\theta, 2\theta]\) (Recall that the probability distribution of a difference of two uniformly-distributed random variables is the convolution of these two distributions). For simplicity, we denote this triangularly-distributed random variable as \( \phi_R \). We can express from (31)

\[
\cos (\phi_{km} - \phi_{kn}) \cos (\Omega t + \phi_{bm} + \phi_{bn}) = \\
\cos (\Omega t + \beta |r_m|/2 + \beta |r_n|/2) \left\{ \cos (k |r_m| - k |r_n|) \cos (\phi_R) - \sin (k |r_m| - k |r_n|) \sin (\phi_R) \right\}.
\]

(32)

This results in

\[
\cos (\phi_{km} - \phi_{kn}) \cos (\Omega t + \phi_{bm} + \phi_{bn}) = \\
\frac{1}{2} \left\{ \cos ((k + \beta/2)|r_m| - [k - \beta/2]|r_n| + \Omega t) + \cos ((k - \beta/2)|r_m| - [k + \beta/2]|r_n| - \Omega t) \right\} \cos (\phi_R) \\
- \frac{1}{2} \left\{ \sin ((k + \beta/2)|r_m| - [k - \beta/2]|r_n| + \Omega t) + \sin ((k - \beta/2)|r_m| - [k + \beta/2]|r_n| - \Omega t) \right\} \sin (\phi_R).
\]

(33)

For the condition \( \beta/2 \ll k \), we assume that \( k + \beta/2 \approx k \) and \( k - \beta/2 \approx k \) and given that \textit{cosine} and \textit{sine} are even and odd functions respectively, (33) can be expressed as

\[
\cos (\phi_{km} - \phi_{kn}) \cos (\Omega t + \phi_{bm} + \phi_{bn}) = \\
\cos (\Omega t) \cos (k |r_m| - k |r_n|) \cos (\phi_R) - \cos (\Omega t) \sin (k |r_m| - k |r_n|) \sin (\phi_R).
\]

(34)

It is now that we are in a position to calculate the expected value of \( \langle I_{\text{Cross}}(x, y, Z) \rangle \) as a function of the range of allowable values \([-2\theta, 2\theta]\) which the continuous random variable \( \phi_R \) can assume. From (34), we can express \( \langle I_{\text{Cross}}(x, y, Z, t) \rangle \) as

\[
\langle I_{\text{Cross}}(x, y, Z, t) \rangle = 2 \sum_{n=1}^{N} \sum_{m=1, m \neq n}^{N} \sqrt{\frac{P_{m,n}}{|r_m||r_n|}} \times \\
\cos (\Omega t) \left\{ \cos (k |r_m| - k |r_n|) \cos (\phi_R) - \sin (k |r_m| - k |r_n|) \sin (\phi_R) \right\}.
\]

(35)

Consider \( \gamma_1 \) and \( \gamma_2 \) to be two random variables that represent the quantities

\[
\gamma_1 = \cos (k |r_m| - k |r_n|) \quad \forall \ m \neq n,
\]

(36)

and

\[
\gamma_2 = \sin (k |r_m| - k |r_n|) \quad \forall \ m \neq n.
\]

(37)

Moreover, let \( \eta_1 \) and \( \eta_2 \) represent two random variables which describe the quantities

\[
\eta_1 = \cos (\phi_R) \quad \eta_2 = \sin (\phi_R).
\]

(38)
For clarity, a distinction between $\eta_1$, $\eta_1$ and $\gamma_1, \gamma_2$ has to be made. The difference in carrier propagation phases (for reasons other than aperture roughness) from the $m^{th}$ and $n^{th}$ contributions from $\mathcal{A}$ is denoted by random variables $\gamma_1$ and $\gamma_2$ which are zero-mean variables with a finite variance for most practical imaging purposes. Assuming that uncertainties (or randomness) in propagation phases accumulated by contributions from $\mathcal{A}$ are uniformly distributed in the modulo $2\pi$ range $[-\alpha, \alpha]$, the resulting random variables $\gamma_1$ and $\gamma_2$ denoting phase differences accumulated due to propagation path difference between pairs of contributions from $\mathcal{A}$ to a location $x, y, z = Z \in \Sigma$ are triangularly distributed in the modulo $2\pi$ range $[-2\alpha, 2\alpha]$ because variations in each of the constituent propagation phases $k|r_m|$ and $k|r_n|$ are uniformly distributed over the modulo-$2\pi$ range $[-\alpha, \alpha]$. These random differences in the propagation phases describe the well-known optical speckle. $\gamma_1$ and $\gamma_2$ do not account for additional phase changes induced due to surface roughness at $\mathcal{A}$.

On the other hand, recalling from (31), the random variable $\phi_R$ (and its derived random variables $\eta_1$ and $\eta_2$) denotes the difference in the phase induced solely by the aperture $\mathcal{A}$ for any given pair of $\mathcal{P}$-field contributions. Assuming that each phase change induced due to aperture roughness is uniformly distributed within the range $[-\theta, \theta]$, the random phase difference $\phi_R$ is triangularly distributed from $[-2\theta, 2\theta]$.

As the random variables $\eta_1$ and $\gamma_1$ are mutually independent (as well as $\eta_2$ and $\gamma_2$), the expected value of $\langle I_{\text{Cross}}(x, y, Z, t) \rangle$ over the entire aperture plane can now be expressed as

$$
\mathbb{E}[\langle I_{\text{Cross}}(x, y, Z, t) \rangle] = 2 \mathbb{E} \left[ \sum_{n=1}^{N} \sum_{m \neq n}^{N} \left[ \gamma_1 \eta_1 \frac{\mathcal{P}_{\text{un,} \Omega} \mathcal{P}_{\text{un,} \Omega}}{|r_m||r_n|} \cos(\Omega r) - \gamma_2 \eta_2 \frac{\mathcal{P}_{\text{un,} \Omega} \mathcal{P}_{\text{un,} \Omega}}{|r_m||r_n|} \cos(\Omega r) \right] \right] \, dA.
$$

$$
= 2 \mathbb{E}[\langle I_{\text{Cross}}(x, y, Z, t) \rangle] = 2 \mathbb{E}[\langle I_{\text{Cross}}(x, y, Z, t) \rangle] - \mathbb{E}[\langle I_{\text{Cross}}(x, y, Z, t) \rangle]
$$

Equation (39)

Knowing from Appendix A that $\mathbb{E}[\eta_2] = 0$, $\mathbb{E}[\langle I_{\text{Cross}}(x, y, Z, t) \rangle]$ can be expressed as

$$
\mathbb{E}[\langle I_{\text{Cross}}(x, y, Z, t) \rangle] = 2 \mathbb{E} \left[ \sum_{n=1}^{N} \sum_{m \neq n}^{N} \frac{\mathcal{P}_{\text{un,} \Omega} \mathcal{P}_{\text{un,} \Omega}}{|r_m||r_n|} \cos(\Omega r) \mathbb{E}[\gamma_1] \mathbb{E}[\eta_1] \, dA. \right.
$$

Equation (40)

Furthermore, as is commonly the case in optical wave propagation $\alpha = 2\pi$ in the modulo-$2\pi$ sense. This is because the propagation path differences between different optical components are multiple times the optical wavelength. As $\mathbb{E}[\gamma_1] = \text{sinc}^2(\alpha)$ from (64), this results in $\mathbb{E}[\gamma_1] = 0$ for $\alpha = 2\pi$ for a triangularly-distributed $\gamma_1$. Therefore the expected value of the cross-interference sum $\mathbb{E}[\langle I_{\text{Cross}}(x, y, Z, t) \rangle] = 0$. This is a well-known result of first order optical speckle and as $\mathcal{P}$ fields are slowly-varying envelopes of the underlying optical carrier, this resulting cross-interference expected value is zero regardless of whether optical carrier is modulated or not.

As is the case for studying optical speckle, the quantity of interest for describing $\mathcal{P}$-field speckle is the variance (and the resulting standard deviation) of $\langle I_{\text{Cross}}(x, y, Z, t) \rangle$. From (35), we obtain

$$
\sigma[\langle I_{\text{Cross}}(x, y, Z, t) \rangle] = 2\sigma \left[ \sum_{n=1}^{N} \sum_{m \neq n}^{N} \left[ \gamma_1 \eta_1 \frac{\mathcal{P}_{\text{un,} \Omega} \mathcal{P}_{\text{un,} \Omega}}{|r_m||r_n|} \cos(\Omega r) - \gamma_2 \eta_2 \frac{\mathcal{P}_{\text{un,} \Omega} \mathcal{P}_{\text{un,} \Omega}}{|r_m||r_n|} \cos(\Omega r) \right] \right] \, dA.
$$

Equation (42)
\[ \sigma \left[ \langle I_{\text{cross}}(x, y, Z, t) \rangle \right] = 2\sigma[G_1 \eta_1 - G_2 \eta_2] \sum_{n=1}^{N} \sum_{m=1}^{N} \frac{\sqrt{P_{\Omega, n}^\Omega P_{\Omega, n}^\Omega}}{|r_m||r_n|} \cos \left( \Omega r \right) dA. \quad (43) \]

If we define random variables \( \mu_1 = G_1 \eta_1 \) and \( \mu_2 = G_2 \eta_2 \), which may necessarily not be considered independent for the moment. We can define

\[ \sigma_{\mu_1}^2 = \left( \sigma_{G_1}^2 + \mathbb{E}^2[\eta_1] \right) \cdot \left( \sigma_{G_1}^2 + \mathbb{E}^2[\eta_1] \right) - \mathbb{E}^2[\eta_1] \mathbb{E}^2[\eta_1], \]

and

\[ \sigma_{\mu_2}^2 = \left( \sigma_{G_2}^2 + \mathbb{E}^2[\eta_2] \right) \cdot \left( \sigma_{G_2}^2 + \mathbb{E}^2[\eta_2] \right) - \mathbb{E}^2[\eta_2] \mathbb{E}^2[\eta_2]. \]

For \( \mathbb{E}[\eta_1] = E[\eta_2] = 0 \) for \( \alpha = 2\pi \) in the modulo-2\( \pi \) sense and \( \mathbb{E}[\eta_2] = 0 \), we can express (44) and (45) as

\[ \sigma_{\mu_1}^2 = \left( \sigma_{G_1}^2 + \mathbb{E}^2[\eta_1] \right), \]

and

\[ \sigma_{\mu_2}^2 = \left( \sigma_{G_2}^2 + \mathbb{E}^2[\eta_2] \right). \]

Moreover, we can express

\[ \sigma^2[\eta_1 \eta_1 - \eta_2 \eta_2] = \sigma^2[\mu_1 - \mu_2] = \sigma_{\mu_1}^2 + \sigma_{\mu_2}^2 - 2\text{Cov}[\mu_1, \mu_2], \]

where 'Cov' denotes the covariance of \( \mu_1 \) and \( \mu_2 \). For \( \text{Cov}[\mu_1, \mu_2] = \mathbb{E}[\mu_1 \mu_2] - \mathbb{E}[\mu_1] \mathbb{E}[\mu_2] \)

\[ \sigma^2[\eta_1 \eta_1 - \eta_2 \eta_2] = \sigma_{\mu_1}^2 + \sigma_{\mu_2}^2 - 2(\mathbb{E}[\mu_1 \mu_2] - \mathbb{E}[\mu_1] \mathbb{E}[\mu_2]). \]

The term \( \mathbb{E}[\mu_1 \mu_2] \) can be expressed as

\[ \mathbb{E}[\mu_1 \mu_2] = \mathbb{E}[\cos(k|r_m| - k|r_n|) \sin(k|r_m| - k|r_n|) \cos(\phi_R) \sin(\phi_R)] = \\
\mathbb{E}\left[ \frac{1}{4} \sin(2[k|r_m| - k|r_n|]) \sin(2\phi_R) \right] = 0. \quad (50) \]

Also

\[ \mathbb{E}[\mu_1] = \mathbb{E}[\eta_1] \mathbb{E}[\eta_1] = 0, \]

\[ \mathbb{E}[\mu_2] = \mathbb{E}[\eta_2] \mathbb{E}[\eta_2] = 0. \]

This results in

\[ \sigma^2[\eta_1 \eta_1 - \eta_2 \eta_2] = \sigma_{\mu_1}^2 + \sigma_{\mu_2}^2. \]

Substituting \( \sigma_{\mu_1}^2 \) and \( \sigma_{\mu_2}^2 \) from (46) and (47) into (53) yields

\[ \sigma^2[\eta_1 \eta_1 - \eta_2 \eta_2] = \left( \sigma_{G_1}^2 + \mathbb{E}^2[\eta_1] \right) \cdot \left( \sigma_{G_1}^2 + \mathbb{E}^2[\eta_1] \right) \cdot \left( \sigma_{G_2}^2 + \mathbb{E}^2[\eta_2] \right). \]

This results in

\[ \sigma \left[ \langle I_{\text{cross}}(x, y, Z, t) \rangle \right] = 2 \sqrt{\left( \sigma_{G_1}^2 \right) \cdot \left( \sigma_{G_1}^2 + \mathbb{E}^2[\eta_1] \right) \cdot \left( \sigma_{G_2}^2 \right)} \sum_{n=1}^{N} \sum_{m=1}^{N} \frac{\sqrt{P_{\Omega, n}^\Omega P_{\Omega, n}^\Omega}}{|r_m||r_n|} \cos \left( \Omega r \right) dA. \quad (55) \]

Now we only need to substitute for all the quantities in (55) in terms of \( \theta \) (taking \( \alpha = 2\pi \)) to obtain an expression for the the spurious \( P \)-field noise sum which is additive to the \( P \)-field sum provided by the \( P \)-field integral in [1]. Substituting all the terms in (55) from Appendix A, we
obtain a rather complicated expression for the \( P \)-field speckle noise term \( \sigma \left( \langle I_{\text{Cross}}(x, y, Z, t) \rangle \right) \). Substituting for \( \alpha = 2\pi \) in (65) and (66), we obtain \( \sigma_{\gamma_2}^2 = 0.5 \) and \( \sigma_{\gamma_1}^2 = 0.5 \). Also using (64), (65), and (66), we substitute for \( \sigma_{\gamma_2}^2, \mathbb{E}^2 \left[ \eta_1 \right] \), and \( \sigma_{\gamma_1}^2 \) in (55) to obtain \( \sigma \left( \langle I_{\text{Cross}}(x, y, Z, t) \rangle \right) \) as a function of aperture roughness \( \theta \) as

\[
\sigma \left( \langle I_{\text{Cross}}(x, y, Z, t) \rangle \right) = \sqrt{2} \left( \sigma_{\gamma_1}^2 + \mathbb{E}^2 \left[ \eta_1 \right] \right) + \left( \sigma_{\gamma_2}^2 \right) \sum_{n=1}^{N} \sum_{m=1, m \neq n}^{N} \sqrt{P_{\text{om}, \Omega} P_{\text{om}, \Omega}} \frac{\cos \left( \Omega t \right) \cos \left( \frac{\pi}{r_m} \right)}{|r_m||r_n|} \mathrm{d}A
\]

\[
= \sqrt{2} \left( \frac{\theta^2 + 8\theta^4 + 8 \cos (2\theta) - (2 + \theta^2) \cos (4\theta) - 6}{16\theta^4} + \frac{8\theta^2 + \cos (4\theta) - 1}{16\theta^2} + \frac{\sin^4 (\theta)}{8} \right).
\]

This terms represents \( P \)-field speckle in light of the carrier speckle. We note that

\[
\sigma^2 \left[ \gamma_1 \eta_1 - \gamma_2 \eta_2 \right] \approx \frac{1}{2} \left[ \left( \sigma_{\gamma_1}^2 + \mathbb{E}^2 \left[ \eta_1 \right] \right) + \left( \sigma_{\gamma_2}^2 \right) \right],
\]

for \( \alpha = 2\pi \). Moreover, as \( \theta \) even slightly large, \( \mathbb{E}^2 \left[ \eta_1 \right] = \sin^4 (\theta) \to 0 \) leaving

\[
\sigma^2 \left[ \gamma_1 \eta_1 - \gamma_2 \eta_2 \right] \approx \frac{1}{2} \left[ \left( \sigma_{\gamma_1}^2 \right) + \left( \sigma_{\gamma_2}^2 \right) \right],
\]

which indicates that the \( P \)-field speckle directly depends on carrier speckle only which is due to the sum of variances in phases accumulated as a result of propagation and aperture roughness. In other words, \( P \)-field speckle is no worse than carrier speckle in most practical scenarios with \( \alpha = 2\pi \).

The general expression for the magnitude of the total \( P \)-field sum \( |P_{0, \Omega}(x, y, z = Z, t) |_{\text{Tot}} \) (where \( |P_{0, \Omega}(x, y, z = Z, t) |_{\text{Tot}} = |I(x, y, Z, t) | \)) is given by

\[
|P_{0, \Omega}(x, y, z = Z, t) |_{\text{Tot}} = \sum_{n=1}^{N} \frac{P_{\text{om}, \Omega}}{|r_m|^2} \cos \left( \Omega t + 2 \beta_{\eta_n} \right) \mathrm{d}A + \sigma \left( \langle I_{\text{Cross}}(x, y, Z, t) \rangle \right) \mathrm{Spurious \ P-field\ Signal\ Noise} |P_{\text{Sp}}(x, y, Z, t) |_{\text{Sat}}
\]

(59)

The total \( P \)-field in (59) comprises of what has already been described as the Huygens-like \( P \)-field in [1] and the additional additive \( P \)-field interference sum of contributions from different locations within \( A \). This we denote as the spurious term \( P_{\text{Sp}}(x, y, Z, t) \) and for the case of an incoherent optical carrier (i.e. \( \theta = \pi \)), this term reaches a saturation value \( |P_{\text{Sp}}(x, y, Z, t) |_{\text{Sat}} \) resulting in a quasi-steady \( P \)-field SNR which we discuss next. A complete spatial randomization of the carrier phase due to roughness in \( A \) is explained when \( \theta = \pi \) and consequently \( |\phi_R| \leq \pi \). Under this condition

\[
|P_{0, \Omega}(x, y, z = Z, t) |_{\text{Tot}} = \iint_A \frac{P_{\text{om}}(x', y', 0) \cos \left( \Omega t + \beta \right)}{|r|^2} \mathrm{d}x' \mathrm{d}y' + |P_{\text{Sp}}(x, y, Z, t) |_{\text{Sat}},
\]

\[
= \iint_A \frac{P_{\text{om}}(x', y', 0) e^{i\beta |r|}}{|r|^2} \mathrm{d}x' \mathrm{d}y' + |P_{\text{Sp}}(x, y, Z, t) |_{\text{Sat}},
\]

(60)
and the $\mathcal{P}$-field integral completely describes $\mathcal{P}$-field imaging. Another observation is that $\mathcal{P}_{Sp}(x, y, Z, t)$ has a frequency of $\Omega$ which is the same as the $\mathcal{P}$-field modulating signal frequency as well as the $\mathcal{P}$-field sum $\mathcal{P}_{Sum}(x, y, Z, t)$ that the $\mathcal{P}$-field integral yields. Therefore, the spurious $\mathcal{P}$-field noise exists within the same frequency bandwidth as the actual $\mathcal{P}$-field modulating signal and cannot be simply filtered out and remains as additive noise to the desired $\mathcal{P}_{Sum}(x, y, Z, t)$ quantity.

The model presented here can also be extended to incorporate the spatial structure of the interference patterns to come up with NLoS reconstruction algorithms that can work with partially specular surfaces. The speckle analysis discussed in [15, 16] can provide a starting point for this.

### 3.4. $\mathcal{P}$-field signal-to-noise ratio

We are now also in a position to mathematically express the ratio between the magnitudes of desired $\mathcal{P}$-field sum from the $\mathcal{P}$-field integral and the spurious $\mathcal{P}$-field signal as a function of $\theta$. We are tempted to call this ratio the $\mathcal{P}$-field SNR (signal-to-noise ratio) represented by $\mathcal{P}_{SNR}(x, y, Z)$ and expressed as

$$\mathcal{P}_{SNR}(x, y, Z) = \frac{\mathcal{P}_{Sum}(x, y, Z, t)}{|\mathcal{P}_{Sp}(x, y, Z, t)|}$$

(61)

It is pertinent to plot $1/|\mathcal{P}_{Sp}(x, y, Z, t)|$ versus $\theta$ to describe the change in the $\mathcal{P}$-field SNR with an increasing aperture roughness. We do so in Fig. 3 where we make two key observations. Firstly, the SNR for low aperture roughness is remarkably low as is expected. At these low aperture roughness levels, the $\mathcal{P}$-field integral $|\mathcal{P}_{Sum}(x, y, Z, t)|$ fails and does not model NLoS imaging analogous to how Huygens integral describes conventional LoS imaging. Spatial incoherence is a fundamental underlying condition which is required for the $\mathcal{P}$-field integral to accurately describe NLoS imaging.

Secondly, we observe that the $\mathcal{P}$-field SNR reaches a steady state value when the aperture roughness allows induces a random phase shift of up to $\phi_{R} = \pi/2$. It remains steady and shows minimal change for increasing aperture roughness thereafter.
3.5. Useful insights from estimating a relationship between \( P \)-field noise and aperture roughness

If we attribute the partial coherence of the E-field solely to the roughness in the aperture plane \( \mathcal{A} \), we are able to establish a relationship between this roughness and the magnitude of the \( P \)-field noise \( P_{\text{sp}}(x, y, Z, t) \). For this purpose, we model an aperture as a collection of infinitesimally small statistically independent regions each imparting a random phase to the carrier E-field. For the case of a single random aperture plane, we can simply calculate the \( P \)-field noise from (56) by knowing the maximum roughness variation in \( \mathcal{A} \) and setting the limits of the random variable \( \phi_R \) as \([−\theta, \theta]\) accordingly.

For a random aperture which is modeled as a collection of \( N \) mutually-independent small regions, we can also determine the error in the estimate of the \( P \)-field noise from (56). Eqn.(56) can also be used to determine the maximum roughness of the wall to achieve the saturation level of the \( P \)-field signal-to-noise ratio.

Due to the saturation of the \( P \)-field SNR, most NLoS experiments conducted so far use diffuse relay walls (which exhibit surface roughness resulting in a phase shift by multiple periods). Reconstruction methods have been developed for this highly-diffuse aperture scenario. A saturation of the \( P \)-field SNR for such highly diffuse surfaces, as we show here, explains the high quality \( P \)-field reconstruction such as the ones demonstrated in [12]. While these methods yield very good results in this case, there are also more specularly reflective surfaces in the real world. Hopefully, this work provides the necessary theoretical insight that allows for the future development of reconstruction methods that can handle more specular surfaces (but not mirrors – otherwise, NLoS imaging would be pointless).

Optical NLoS imaging around corners generally involves relay surfaces which are typically rough enough - owing to the small wavelength scales at optical frequencies - to obtain a stable \( P \)-field SNR. Outside of the realm of optical carrier-based NLoS imaging, our analysis can prove to be a valuable tool for future imaging around corner applications under consideration involving low carrier frequencies such as applications involving ultrasound-based NLoS imaging around corners where the \( P \)-field SNR performance can be evaluated for semi-rough relay wall apertures at the ultrasound signal wavelength range and help design aperture roughness profiles to obtain a desired minimal \( P \)-field imaging SNR.

4. Conclusion

This paper expands on the topic of phasor field propagation and interference for NLoS imaging when, unlike previous treatment in [1] where the E-field was considered as spatially incoherent, the E-field is considered as partially coherent. We determine the magnitude of an increasing \( P \)-field additive noise with an increase in E-field coherence. This deviation from a complete \( P \)-field integral-based solution to NLoS imaging is very important to study the effects of aperture roughness on NLoS imaging scene reconstruction as well as quantifying the degree of spatial coherence and its effects on occlusion-aided imaging. The statistical treatment presented in this paper is critical in establishing a unified \( P \)-field behavior under various different levels of optical coherence for NLoS imaging and other types of LoS imaging techniques such as \( P \)-field imaging through fog. Via this statistical treatment of \( P \)-fields, we also validate optical incoherence as a necessary condition for the \( P \)-field integral to completely describe NLoS imaging as was described in [1]. The magnitude of the \( P \)-field noise that is introduced by virtue of deviating from this condition also allows us to determine a \( P \)-field signal-to-noise ratio.
Appendix A: Expected value and standard deviation of \( \cos(\phi_R) \) and \( \sin(\phi_R) \) for a triangularly distributed random variable \( \phi_R \)

In this section, we summarize and enlist (omitting proofs) the fundamental expressions for the expected values of \( \cos(\phi_R) \) and \( \sin(\phi_R) \) and the variance of \( \cos \theta \), where \( \phi_R \) is treated as a triangularly distributed random variable in the modulo \( 2\pi \) range \([−2\theta, 2\theta]\) for the possible values of \( \theta \). We require the analytical forms of these expressions for determining the expected value of the sum of all \( \mathcal{P} \)-field cross-multiplication terms in (29). As a reminder, the probability density function of \( \phi_R \) is considered triangularly distributed because \( \phi_R \) denotes the difference between two uniformly-distributed random variables (that each denote phase difference added through random propagation path lengths which are considered uniformly distributed respectively) in the range modulo \( 2\pi \) \([−\theta, \theta]\). The triangularly distributed random phase \( \phi_R \) has the probability density function

\[
P_D(\phi_R) = \begin{cases} 
\left( \frac{1}{2\theta} \right)^2 (\phi_R + 2\theta) & -2\theta \leq \phi_R \leq 0, \\
\left( \frac{1}{2\theta} \right)^2 (2\theta - \phi_R) & 0 \leq \phi_R \leq 2\theta, \\
0 & \text{otherwise.}
\end{cases}
\]  

\[\text{(62)}\]

**Expected value of \( \cos(\phi_R) \)**

When \( g(U) \) denotes the function of the random variable and \( P_D(U) \) its probability density function, the Fundamental theorem of expectation [17]

\[
\mathbb{E}\{g(U)\} = \int_{-\infty}^{\infty} g(U)P_D(U) \, dU,
\]  

\[\text{(63)}\]

can be used to determine the expected values of \( \cos(\phi_R) \) and \( \sin(\phi_R) \). If \( \eta_1 = \cos(\phi_R) \), then the expected value \( \mathbb{E}(\eta_1) = \mathbb{E}(\cos(\phi_R)) \) is given by

\[
\mathbb{E}(\eta_1) = \int_{-2\theta}^{0} \cos(\phi_R) \left( \frac{1}{2\theta} \right)^2 (\phi_R + 2\theta) \, d\phi_R + \int_{0}^{2\theta} \cos(\phi_R) \left( \frac{1}{2\theta} \right)^2 (2\theta - \phi_R) \, d\phi_R
\]  

\[= \text{sinc}^2(\theta).\]  

**Expected value of \( \sin(\phi_R) \)**

Since \( \sin \) is an odd function, it is quite evident that \( \mathbb{E}[\sin(\phi_R)] = 0 \) which can also be derived from the fundamental theorem of expectation.

**Standard deviation of \( \cos(\phi_R) \)**

Similarly, we can determine the standard deviation of \( \eta_1 = \cos(\phi_R) \). The full proof is omitted, but the result for triangularly distributed \( \phi_R \) is given by

\[
\sigma_{\eta_1} = \sqrt{\frac{\theta^2 + 8\theta^4 + 8\cos(2\theta) - (2 + \theta^2) \cos(4\theta) - 6}{16\theta^4}}.
\]  

\[\text{(65)}\]

**Standard deviation of \( \sin(\phi_R) \)**

We can also determine the standard deviation of \( \eta_1 = \sin(\phi_R) \). We state it here without proof that

\[
\sigma_{\eta_2} = \sqrt{\frac{8\theta^2 + \cos(4\theta) - 1}{16\theta^2}}.
\]  

\[\text{(66)}\]
References

1. S. A. Reza, M. La Manna, S. Bauer, and A. Velten, “Phasor field waves: A Huygens-like light transport model for non-line-of-sight imaging applications,” Opt. Express 27, 29380–29400 (2019).
2. S. A. Reza, M. La Manna, and A. Velten, “Imaging with phasor fields for non-line-of-sight applications,” in Imaging and Applied Optics 2018 (3D, AO, AIO, COSI, DH, IS, LACSEA, LS&C, MATH, pcAOP), (Optical Society of America, 2018), p. CM2E.7.
3. S. A. Reza, M. La Manna, S. Bauer, and A. Velten, “Phasor field waves: Experimental demonstrations of wave-like properties,” Opt. Express 27, 32587–32608 (2019).
4. A. Velten, T. Willwacher, O. Gupta, A. Veeraraghavan, M. G. Bawendi, and R. Raskar, “Recovering three-dimensional shape around a corner using ultrafast time-of-flight imaging,” Nature Communications 3, 1–8 (2012).
5. O. Katz, P. Heidmann, M. Fink, and S. Gigan, “Non-invasive single-shot imaging through scattering layers and around corners via speckle correlations,” Nat. Photonics 8, 784 (2014).
6. M. La Manna, F. Kine, E. Breitbach, J. Jackson, T. Sultan, and A. Velten, “Error backprojection algorithms for non-line-of-sight imaging,” IEEE Transactions on Pattern Analysis Mach. Intell. 41, 1615–1626 (2018).
7. O. Gupta, T. Willwacher, A. Velten, A. Veeraraghavan, and R. Raskar, “Reconstruction of hidden 3D shapes using diffuse reflections.” Optics Express 20, 19096–19108 (2012).
8. M. O’Toole, D. B. Lindell, and G. Wetzstein, “Confocal non-line-of-sight imaging based on the light-cone transform,” Nature 555, 338–341 (2018).
9. D. B. Lindell, G. Wetzstein, and M. O’Toole, “Wave-based non-line-of-sight imaging using fast fk migration,” ACM Transactions on Graph. (TOG) 38, 1–13 (2019).
10. F. Heide, L. Xiao, W. Heidrich, and M. B. Hullin, “Diffuse mirrors: 3D reconstruction from diffuse indirect illumination using inexpensive time-of-flight sensors,” in Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition, (2014), pp. 3222–3229.
11. M. Batarseh, S. Sukhov, Z. Shen, H. Gemar, R. Rezvani, and A. Dogariu, “Passive sensing around the corner using spatial coherence,” Nat. Commun. 9, 1–6 (2018).
12. X. Liu, I. Guillén, M. La Manna, J. H. Nam, S. A. Reza, T. H. Le, A. Jarabo, D. Gutierrez, and A. Velten, “Non-line-of-sight imaging using phasor-field virtual wave optics,” Nature 572, 620–623 (2019).
13. J. A. Teichman, “Phasor field waves: a mathematical treatment,” Opt. Express 27, 27500–27506 (2019).
14. J. Dove and J. H. Shapiro, “Paraxial theory of phasor-field imaging,” Opt. Express 27, 18016–18037 (2019).
15. J. W. Goodman, Statistical Properties of Laser Speckle Patterns (Springer, Berlin, Heidelberg, 1975), pp. 9–75.
16. J. W. Goodman, Speckle phenomena in optics: theory and applications (SPIE Press, 2020).
17. J. K. Blitzstein and J. Hwang, Introduction to probability (Chapman and Hall/CRC, 2014).