Simulation of Field Theories in Wavelet Representation

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Abstract

The field is expanded in a wavelet series and the wavelet coefficients are varied in a simulation of the 2D $\phi^4$ field theory. The drastically reduced autocorrelations result in a substantial decrease of computing requirements, compared to those in local Metropolis simulations. The improvement is shown to be the result of an additional freedom in the choice of the allowed range of change at the Metropolis update of wavelet components, namely the range can be optimized independently for all wavelet sizes.

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1 Introduction

In the present work we introduce a new approach to simulations of field theories, based on wavelet expansions. Wavelets form complete sets of localized, orthonormal states. Elements of a set of wavelets differ both in their locations and in their scales. In fact, complete sets contain wavelets at all scales, starting from an elementary scale up to the scale of the system. Wavelets can be labeled by two integers, $n$ and $k$, characterizing their scale and their location. They can be formed from a function $\psi(x)$ as

$$\psi_{k,n}(x) = 2^{-n/2}\psi(2^n x + k).$$ (1)

Wavelets have been applied successfully to linear problems, like signal analysis. In the continuum, wavelets form a complete orthonormal system, allowing the expansion of $L^2$ integrable functions, in a manner analogous to Fourier expansion. The orthogonal functions of the Fourier expansion are not localized, consequently they are not normalizable. Wavelets are localized, albeit on varying scales, from the elementary scale to the scale of the system. They are themselves $L^2$ integrable function. As a rule, they lead to series expansions with convergence considerably faster than that of a Fourier expansion.

The property of wavelets of analysing data at all scales makes them, in principle, attractive for the investigation of lattice problems. One can define wavelet representations, in which fields are expressed by their wavelet expansion coefficients, rather than by their values at given spacetime points. On finite lattices, the range of integers $k$ and $n$ in (1) is finite. It is not obvious that wavelet expansions, designed for investigating linear problems are useful for investigating a highly nonlinear problem such as a lattice field theory. The presence of large scale wavelets introduces nonlocality in the expression for interaction terms. The decrease in autocorrelation time is partially counterbalanced by the increased computational requirements. It is then important to give estimates for the growth of nonlinearity in terms of the lattice size. The question is the balance: whether one gains more by decreasing autocorrelation times than one loses by having to deal with a number of nonlocal terms in the action. This question will be investigated extensively in the remainder of this paper. Wavelets have already been used in lattice problems: as a variational basis in the $XY$ model [4], and as a tool to facilitate gauge fixing [5].

Simulations in wavelet representation are somewhat similar to simulations using collective updating methods, such as Swendsen and Wang’s algorithm [6] or the multigrid method [7]. While the multigrid method is a so called smoothing filter, averaging a large number of points, the wavelet representation is, in a way, the opposite. Wavelets are designed to have their first few moments vanish. Collective updating methods have been designed to decrease the dynamical exponent $z$ appearing in the expression of the autocorrelation time ($\tau$) of physical quantities

$$\tau \simeq c \xi^z,$$ (2)

where $\xi$ is the spacetime correlation length, the inverse of the mass gap. In fact, some of the algorithms are able to reduce $z$ to almost zero. Near second order phase transition points the correlation length is proportional to the linear size of the lattice, $L$. Thus, collective updating algorithms are particularly useful near critical points and for large lattices. In fact, for lattices of small or moderate size or at some distance away from the critical point, most collective updating algorithms are not much superior to local updating methods.
In the next section we will introduce the wavelet formalism and transform the \( \phi^4 \) field theory into wavelet representation. Then we will count the number of nonlocal terms, to get a handle on the increase of computational needs as a function of lattice size. In Section 3, we put the wavelet method to a direct test, we simulate a 2 dimensional \( \phi^4 \) theory, both using the standard Metropolis algorithm and the wavelet method. The last section contains our conclusions.

## 2 The Wavelet Method in Field Theories

Wavelets form a orthonormal system of variables interpolating between coordinate and momentum representations. In fact, wavelet representations (there are many different choices) unify the advantages of both. They are much more local and stable against small perturbations than the momentum representation, and at the same time they can describe long range correlations with a small number of terms, unlike the coordinate representation.

The most widely used wavelet filters were discovered by Daubechies [1]. Daubechies wavelets on discrete sets are defined as orthogonal matrix transforms. For \( p \) of \( 2^n \) spatial components, \( x_1, \ldots, x_{2^n} \) are combined together in two different ways: with coefficients \( c_1 x_1 + c_2 x_2 + \ldots + c_{2^p} x_{2p} \) (smoothed combinations) and with different ordering and signs of the same coefficients \( -c_2 p x_1 + c_{2p-1} x_2 - \ldots + c_1 x_{2p} \) (wavelets). The series is required to have \( p \) vanishing moments, i.e. \( \sum_k (-k)^l c_{2p-k} = 0, \ l = 0, 1, \ldots, p - 1. \) Similarly, one forms such linear combination from subsequent components \( x_i \), shifted by two lattice units an arbitrary number of times. Then requiring the orthogonality of the transformation makes the coefficients essentially unique. The wavelet transform consists of \( n - p \) subsequent orthogonal transformations of the above described type, applied to the smoothed combinations obtained in the previous orthogonal transformation. The coefficients \( c_1, \ldots, c_{2p} \) are called the Daubechies wavelet filter coefficients.

For \( p = 1 \) \( c_0 = c_1 = 1/\sqrt{2}, \) and the wavelets are called Haar-wavelets [2]. The \( r \)th Haar wavelet of length \( 2^k \) on a one dimensional lattice is formed as

\[
\chi_{k,r} = 2^{-k/2} \sum_{i=1}^{2^{k-1}} [\phi(i + r 2^k) - \phi(i + 2^{k-1} + r 2^k)],
\]

where \( \phi(j) \) denotes the field at the \( n \)th site. The range of \( k \) and \( r \) is \( k = 1, 2, \ldots, n, \) and \( r = 0, 2^{n-k} - 1, \) respectively, where \( L = 2^n, \) is the size of the lattice. The orthogonal system is formed from the wavelets \( \chi(k, r), \) and one ”smoothed” combination

\[
\chi = \frac{1}{\sqrt{L}} \sum_{i=1}^{L} \phi(i),
\]

which is the normalized total magnetization.

The structure of the wavelet coefficients for general \( p \) is similar to that of [3]. Thus, the first \( 2^{n-1} \) wavelets are combinations of \( 2p \) neighboring data points shifted by multiples of 2. Then the next \( 2^{n-2} \) wavelets (each shifted from the next by 4 data points) are of length \( l_2 = 1 + (2^2 - 1)(2p - 1), \) etc. The \( 2^{n-k-1} \) \( k \)th type of wavelets are of length \( l_k = 1 + (2^k - 1)(2p - 1). \)
They are shifted from each other by \( 2^k \) points. The last \( p \) combinations are not wavelets, but maximally smoothed combinations of the original data points.

In an application to lattice field theory one has to use wavelets in more than one dimension. In fact, wavelets can easily be defined in any number of dimensions. In each dimension the wavelets are labeled by their scale, \( k_i \), and by their location, \( r \). Thus, a wavelet coefficient \( \chi(k_1, r_1; k_2, r_2; \ldots; k_d, r_d) \) corresponds to a wavelet box of length \( 2^{k_i} \) in the \( i \)th lattice dimension. \( d \) is the number of dimensions. There are \( 2^{n_d-k_1-k_2-\ldots-k_d} \) wavelets labeled by a given scale vector \( \mathbf{k} = \{k_1, k_2, \ldots, k_d\} \). The lattice field has a linear expansion in terms of the coefficients \( \chi(k_1, r_1; k_2, r_2; \ldots; k_d, r_d) \). We will call these coefficients the wavelet representation. It is on an equal footing with the momentum and coordinate representations.

Wavelets are well suited to analyzing linear problems. Consequently a Gaussian model is just as simple in wavelet representation as in coordinate representation. Orthonormality implies that the form of the mass term is unchanged in wavelet representation

\[
S = \frac{m^2}{2} \sum_{j,r} 2|\chi(j, r)|^2
\]

where for the sake of simplicity we have dropped the limits of the summations. The kinetic term is slightly more complicated, and its explicit form is not very enlightening. It has a moderate number of nonlocal terms.

The propagator in wavelet space is quite analogous to the one in coordinate space. Thus, it has the same asymptotic behavior and it also becomes singular at the critical point, \( m^2 = 0 \). Critical phenomena can be investigated in wavelet representation just as well as in coordinate representation. Alternatively, one can easily transform correlation functions back to coordinate representation, using the wavelet transformation.

The question arises, however, how well the wavelet representation is suited to investigate nonlinear problems, like any nontrivial field theory. In the current paper we will investigate a typical example of nonlinear field theories, the \( \phi^4 \) theory. We need to express the interaction term in terms of wavelets. Using the orthogonality of the wavelet transformation an explicit expression for the interaction term of the Lagrangian \( g \sum \phi^4 \) is very complicated. It has many non-local terms, some of them correspond to ‘long-range’ interactions. The success of the wavelet representation depends on the average number of these long range interaction terms per wavelet. If this average number increases slowly with size we have a chance that simulations in wavelet representation will be more efficient than those in coordinate representation.

First, we will calculate the average number of interaction terms per wavelet coefficient on a \( d \)-dimensional lattice. It is sufficient to do the calculation in one dimension. If the number of interaction terms per wavelet is \( N \) in one dimension then it is \( N^d \) in \( d \) dimensions. The interaction term is a quartic combination of the wavelet coefficients. For the sake of simplicity we will do the counting for the Haar-wavelet representation. Then it is easy to see that in every term the coefficient with the lowest value of \( j \) must be squared. Let that coefficient be \( \chi(j, m) \). Then the combinations of other wavelets multiplying \( \chi(j, m)^2 \) can be found as follows: for each \( j \) there is only one wavelet overlapping \( \chi(j, m) \). Then the bilinear multiplier has \( (n - j)(n - j + 1)/2 \) different types of terms. The number of wavelets with
scale $j$ is $L/2^j$. Then the total number of types of non-local interaction terms is

$$LN = \sum_{j=1}^{n-1} \frac{L}{2^j} \frac{(n-j)(n-j+1)}{2}.$$  \hfill (6)

In other words, the average number of non-local interaction terms per wavelet is

$$N \simeq \frac{n^2}{2} = \frac{(\log L)^2}{2(\log 2)^2}. \hfill (7)$$

This number should be compared with 1, the average number of terms per site in the coordinate representation. In $d$ dimension the appropriate factor is

$$N_d = \frac{(\log L)^{2d}}{2^d(\log 2)^{2d}}. \hfill (8)$$

In the next section, before discussing our simulations, we will perform a more precise analysis of computational needs. It will turn out that the number of computations per lattice site in a $d$ dimensional model increases only as $(\log L)^d$ and not as $(\log L)^{2d}$. The application of multiple hits at each wavelet coefficient in the Monte Carlo simulation further decreases the ratio of computational needs.

### 3 Simulations

The difference in the number of cycles needed between local and wavelet simulations is due to the calculation of the change of the action. The largest computational needs arise from the calculation of the interaction term. The action has the form

$$S = \frac{1}{2} \sum_{r} \left[ m^2 \phi_r^2 + \sum_{j} (\phi_r - \phi_{r-\hat{e}_j})^2 + \frac{g}{12} \phi_r^4 \right]$$ \hfill (9)

This expression is very complicated in terms of the wavelet coefficients, even for the simplest, Haar wavelets. For that reason it is not conducive to the calculations to express (9) in terms of $\chi(k_1, r_1, ..., k_d, r_d)$. It is more efficient to store both the coordinate and the wavelet representation coefficients. This doubles storage needs, but accelerates and simplifies the code considerably.

Suppose that during a sweep of the lattice we update the coefficient $\chi = \chi(k_1, r_1, ..., k_d, r_d)$. Then the action has the following dependence on $\chi$:

$$S = \frac{m^2 \chi^2}{2} + \sum_{r} \sum_{j} \left\{ \chi[\phi_r - \phi_{r-\hat{e}_j}] \left[ \frac{\partial \phi_r}{\partial \chi} - \frac{\partial \phi_{r+\hat{e}_j}}{\partial \chi} \right] + \frac{\chi^2}{2} \left[ \frac{\partial \phi_r}{\partial \chi} - \frac{\partial \phi_{r+\hat{e}_j}}{\partial \chi} \right]^2 \right\}$$

$$+ \frac{\chi^4}{4!} \sum_{r} \left( \frac{\partial \phi_r}{\partial \chi} \right)^4 + \frac{\chi^3}{3!} \sum_{r} \left( \frac{\partial \phi_r}{\partial \chi} \right)^3 \phi_r + \frac{\chi^2}{4} \sum_{r} \left( \frac{\partial \phi_r}{\partial \chi} \right)^2 \phi_r^2$$

$$+ \frac{\chi}{6} \sum_{r} \left( \frac{\partial \phi_r}{\partial \chi} \right) \phi_r^3 + \text{terms independent of } \chi,$$ \hfill (10)
The coefficients $\partial \phi_r / \partial \chi$ depend only on the relative position of the lattice points. They are given by products of coefficients $c_i^{(k)}$, for $k = k_1, k_2, ..., k_d$. These coefficients can be easily tabulated with minimal storage requirements, even for general $p$. The coefficient of $\chi^4$, and that of the quadratic part of the kinetic term depend on $k$ only. For Haar wavelets the coefficient of the fourth order term is $2^{(k_1+k_2+...+k_d-dn)/2}$

The action defines the probability distribution of $\chi$. It must be evaluated repeatedly during simulations. The calculation of the interaction part requires 8 floating point operations (5 for Haar wavelets) for every individual coordinate point, $r$. There are $\prod l_i \simeq (2p-1)^d 2^{k_1+...+k_d}$ points inside the wavelet. Now, for a complete sweep one has to update all $2^{nd-k_1-...-k_d}$ coefficients of the same kind, giving the number of floating point operations as $8(2p-1)^d 2^{nd}$. Finally, one has to update the coefficients for all values of $k_1, ..., k_d$, bringing in an additional coefficient of $n^d$, where $n = \log L / \log 2$, where $L$ is the length of an edge of the lattice. Then the average number of floating point operations per lattice site is $N_d = 8(2p-1)^d n^d$. The calculation of the kinetic term requires further $4(2p-1)^d n^d$ operations. For Haar wavelets the calculation of the kinetic term requires only $O(n^{d-1})$ floating point operations.

There is a further requirement in the updating algorithm. After every change of a variable $\chi$ one has to recalculate the affected fields in coordinate representation. It is easy to get a similar estimate for the average number of floating point operations needed for such an updating. It just changes the coefficient of $N_d$ from $8+4d$ to $10+4d$. The ratio to the average number of floating point operations per site in a straightforward simulation in local simulation (coordinate representation) $(6+2d)$ is then

$$\frac{t_w}{t_l} = \frac{10 + 4d}{6 + 2d} \left[ n(2p-1) \right]^d \left[ 1 + O(1/n) \right].$$

where $t_l$ and $t_w$ are the average times a sweep of the lattice takes in local and wavelet simulations, respectively. In particular, for Haar wavelets and two dimensions, $R_d \simeq (n^2/2) \left[ 1 + O(1/n) \right]$. In fact, one can obtain a substantially smaller ratio as follows. Notice that the factor $(\log L)^d$ appears in $[11]$ due to the computation of the coefficients of $\chi$ in action (10). If one performs the Metropolis simulation in such a way that there are several hits at every wavelet $\chi$, then the same coefficients of (10) can be used and the extra calculation becomes not more costly then that of a hit in coordinate representation (local simulation). At the same time, having several hits at the same field component does not lead to any substantial savings in the local updating algorithm. Thus, in the limit of large number of hits the average sweep times for the two types of simulations converge. Of course, after a certain number of hits, the return in decreasing statistical errors diminishes. As usual, one has to find the optimal number of hits, so that the statistical errors were minimized using a fixed amount of computational time. Figure 1. shows the dependence of autocorrelation times on the number of hits. The autocorrelation times have been calculated using two different techniques. One, by direct measurement of the autocorrelation of the total magnetization, and the other by calculating the square of the relative error of the same physical quantity. They provide very similar information. When choosing the number of hits in the simulation, we had to keep in mind that the requirements on computing computing time slowly increase with the number of hits. The product of $\tau_w$ with the average time a sweep, $t_w$, should be
optimized. We chose the optimum value of 5 in all of our simulations. For 5 hits we find the following values for $t_w/t_l$: 1.2, 1.3, 1.54, 1.82, on lattices of sizes $L = 8, 16, 32, 64$, respectively.

The quality of an algorithm is determined by its improvement factor, which is defined as

$$I = \frac{t_w \tau_w}{t_l \tau_l}$$

where $\tau_l$ and $\tau_w$ refer to real time autocorrelation lengths. Having determined (11) in the rest of this paper we will concentrate on the determination of autocorrelation times $\tau_w$ and $\tau_l$.

It is important to emphasize a substantial difference between a simulation using the standard, local (coordinate representation) algorithm and a simulation in wavelet representation. A hidden parameter of all simulations is the size of the window, $\Delta$, in which a new field value is picked during the Metropolis algorithm. In other words, if the value of the field component is $\phi$, then we choose a random new value for $\phi$, $\phi'$, in the interval $\phi - \Delta/2 < \phi' < \phi + \Delta/2$. While in a local simulation there is only one possible parameter $\Delta$, in the wavelet representation there is the possibility, and indeed the necessity, of choosing different values $\Delta_{ij}$, $i, j = 1, \ldots, n+1$, for different sizes of wavelets. The effect of this freedom of choice of the window sizes on autocorrelation times will be discussed later.

The evaluation of the improvement factor $I$ of (12) requires the determination of autocorrelation times. We have performed extensive simulations on lattices of sizes $L = 8, 16, 32, 64$. In wavelet simulation, after a warmup period of $O(10^4)$ sweeps we read out the relevant physical variables in the next $O(10^5)$ sweeps. Using the local updating algorithm we had to have much longer runs, $O(10^6)$ sweeps to be able to determine autocorrelation lengths. We compared the values of various physical quantities, such as $\langle \phi^2 \rangle$, $\langle H \rangle$, $\langle |M| \rangle$, $\langle c_v \rangle$, and $\langle \chi \rangle$, where $M$ is the magnetization, $c_v$ is the specific heat, and $\chi$ is the magnetic susceptibility. All the physical quantities agreed within statistical errors in the two simulations. The simulations were performed at fixed $g = 1$ and varying $m^2$ between the limits of -0.31 and -0.25, a range which includes the critical point.

The purpose of the simulations was to calculate autocorrelation times, obtained from the $t$ dependence of autocorrelation functions

$$\Gamma(t) = \langle X(t')X(t' - t) \rangle_{t'}$$

where $X$ is a physical quantity, $t$ and $t'$ label real time during the simulation, measured in units of sweeps. The average of (13) is over time $t'$. For moderate values of $t$, $\Gamma(t)$ is expected to have an exponential dependence on $t$, $\Gamma(t) \simeq \Gamma(0) e^{-t/\tau}$, where $\tau$ is the autocorrelation time. In fact, the integrated autocorrelation function,

$$\Gamma'(t) = \frac{1}{\Gamma(0)} \sum_{t'=0}^{t} \Gamma(t') \simeq \frac{1}{1 - e^{-1/\tau}}$$

serves as a better measure of the autocorrelation time. The integrated autocorrelation function goes into saturation at large $t$. The saturation value determines the autocorrelation time, $\tau$. 


We calculated autocorrelation times (measured in units of sweeps) for three physical quantities: Total magnetization, $M$, average value of $\phi(x)^2$, and the average value of the action. The dependence of autocorrelation times on the mass parameter of the Lagrangian, near the critical point is shown in Figure 2. Typical errors are about 5-10%. The point of Fig. 2 represent data on lattice size $L = 32$.

It is important to determine the dependence of autocorrelation times on the size of the lattice. Let us define an exponent $\zeta$ in a manner, somewhat analogous to
\begin{equation}
\tau = cL^\zeta.
\end{equation}

At the critical point $\zeta$ coincides with $z$. Away from the critical point $\zeta$ is smaller than $z$ because $L > \xi$. The difference of $z$ and $\zeta$ measures the finite size effect on the correlation length. The exponent $\zeta$, plotted in Fig. 3, in the critical region was calculated using all four lattice sizes we studied. Near the critical point, $m^2 \simeq -0.285$, the exponent is almost 2. In fact, if the smallest lattice, $L = 8$, is omitted from the fit the value exceeds 2. This shows that simulations in wavelet representation do not circumvent critical slowing down. In fact, a plot of exponent $\zeta$ in local simulations is almost identical to Fig. 3. In other words, the possible gain in autocorrelation times in local and wavelet simulations has a large and fairly constant value for all lattice sizes considered. Fig. 4 shows these ratios in the critical region of $m^2$. A multiplier 5 corresponding to the number of hits per wavelet coefficient has been included in $\tau_w$. The ratio has only a moderate variation with lattice size or $m^2$. It is in the range of 10-15. Then the improvement factor, $I$, of (12) at the critical point varies between 7 on the largest lattice to a value over 10 on the smaller lattices. There is a slight tendency to an increase of $I$ when one moves away from critical point toward the disordered phase.

4 Discussion

The general expectation of substantially decreased autocorrelation times in a simulation in wavelet representation was borne out by our simulations. One can demonstrate quite dramatically the effectiveness of our method on the free bosonic theory ($g = 0$). Consider the autocorrelation of the total magnetization. As we mentioned earlier, the magnetization is proportional to one of the wavelet expansion coefficients, the ‘non-wavelet’ component. Let us denote this component by $\chi$. Since the kinetic term is independent of $\chi$, $\chi$ completely decouples. The $\chi$ dependence of the action is
\begin{equation}
S = \frac{1}{2}m^2\chi^2.
\end{equation}

In other words, the action is a Gaussian function of $\chi$. Then the autocorrelation length for magnetization, can only depend on the dimensionless product $m\Delta$, where $\Delta$ is the range of change of $\chi$ in the simulation. The dynamical critical exponent is 2. This can be seen by the following heuristic argument. Suppose $m\Delta << 1$. Then starting from an average value, $\chi \sim 1/m$, after $n$ sweeps the value of $\chi$ wanders away by an average value of $\delta \chi \sim \sqrt{n\Delta}$.
(random walk). Complete decorrelation requires \( \delta \chi \sim \chi \sim 1/m \). Then we obtain the following estimate for the autocorrelation time:

\[
\tau = n \simeq \frac{1}{(m\Delta)^2} = \frac{\lambda^2}{\Delta^2},
\]

where \( \lambda \) is the spatial correlation length on the lattice. (17) shows not only that the critical exponent, \( z = 2 \), but also shows that the numerical value of the autocorrelation length is inversely proportional to \( \Delta^2 \). In other words, if one chooses \( \Delta \) for the non-wavelet coefficient too small one can get very long autocorrelation times. In a wavelet simulation one can control \( \Delta \) independently for different types of coefficients. Indeed, in our simulation we have chosen the window \( \Delta \) for \( \chi \) in such a way that the acceptance was approximately 0.5.

In a local, coordinate representation simulation one has no control over the allowed range of change of the total magnetization. In fact, it is expected to change by a very small amount in every sweep. Since \( \chi = \sum \phi/\sqrt{M} \), thus, \( \delta \chi = \sum \delta \phi/\sqrt{M} \), roughly speaking \( \delta \chi \sim \sqrt{M} \delta \phi/\sqrt{M} = \delta \phi \). We performed some simulations at \( g = 0 \) as well (free bosonic theory), and in those simulations the optimal window \( \Delta \) was independent of the mass, \( m \). Since the autocorrelation can only depend on the product \( m\Delta \), at small \( m \) and constant \( \Delta \) one can use the asymptotic behavior which should be the same as (17). Indeed we obtained the following autocorrelation times for the total magnetization: 865 at \( m^2 = .005 \), 310 at \( m^2 = .01 \), 228 at \( m^2 = .015 \), and 179 at \( m^2 = .02 \). All of these have about 10% error. These autocorrelation times give a good fit to (17). In other words, the effective value of \( \Delta \) for the magnetization was indeed constant, as we assumed from the outset. Notice, however, what happens in a simulation in wavelet representation. The window \( \Delta \) is adjusted for each type of wavelet separately. In particular, one adjusts \( \Delta \) for the magnetization as well. If one requires constant acceptance, then \( \Delta m \) and so the autocorrelation time is kept constant. The critical slowing down, at least for free theories, is completely eliminated.

In the interacting case there is substantial mass renormalization. Still, for moderate values of the \( m\Delta \) one expects a general dependence like (17) on \( \Delta \) where one should use the renormalized mass. Unfortunately, in interacting theories in \( D = 2 \) there is another dimensional quantity, \( g \). The autocorrelation time also depends on the dimensionless quantity \( g/m^2 \). The dependence on this quantity seems to dominate the behavior of exponent \( \zeta \). The constant \( c \) is, however, affected very strongly by the value of \( m\Delta \), leading to a dramatic decrease of autocorrelation times in simulations in wavelet representation.

To check the above ideas, we ran a series of simulations in wavelet representation on an \( L = 32 \) lattice, at the same value of the physical parameters, set at the approximate location of the critical point, \( g = 1 \) and \( \mu^2 = -.285 \), but varying the window size for the wavelet component, that is proportional to the total magnetization. Fig. 5 shows autocorrelation times for magnetization as a function of the window size, \( \Delta \), for the appropriate wavelet component, \( \chi = \sum \phi/\sqrt{M} \), where \( M \) is the total number of lattice sites. The curve has, for these values of \( m^2 \) and \( g \), a broad minimum around an optimal window size.

The optimal value of window size for the change of \( \phi \) in local simulations is about \( \Delta = 3 \). Due to the orthogonality of the transformation from local variables, \( \phi \), to wavelet components, \( \chi \), a local simulation is in some sense equivalent to a wavelet simulation in which every window \( \Delta \) is set equal to the value of \( \Delta \) in the local simulation. For this reason we ran a simulation on a \( 32 \times 32 \) lattice at \( m^2 = -.285 \), setting all windows equal to \( \Delta = 3.0 \).
We obtained an autocorrelation time of $\tau_w = 226$, comparable to $\tau_l \simeq 436$. At the same point, if we optimize all the windows, such that all acceptances are between 0.4 and 0.5, then $\tau_w = 39$. This result supports the idea that a major part of the decrease in autocorrelation time is not the result of relaxing the system at different scales simultaneously. It is rather the consequence of the extra freedom of setting the windows for every scale independently.

In the present paper we have used Haar wavelets. It would be of some interest to investigate simulations with other examples of the Daubechies wavelet series. Because the wavelet expansion coefficients would be all different, the calculation of the action would take a longer time. It is still possible that the improved convergence properties of the wavelet expansion would make the comparison with the local simulation even more favorable.

Finally, we would like to point out that wavelet representations can be used for theories with more complicated order parameters as well. In such simulations the ratio of sweeptimes should be roughly the same as has been obtained in the current paper. As an example, for compact Abelian gauge theories wavelets should be formed from the components of the vector potential. The calculation of the change of the action requires the calculation of trigonometric function in both representations. The estimate we have given for computational requirements in Sec. 2. would still stand. Unfortunately, since the action is nonpolynomial, the ratio of simulation times may not be substantially improved by repeated hits at individual wavelet coefficients.

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FIGURE CAPTIONS

Fig. 1.

$\tau_w$ for magnetization as a function of the number of hits per wavelet coefficient on a $32 \times 32$ lattice. The diamonds represent a direct measurement, while the triangles represent $\tau_{\text{eff}}$ calculated from relative errors.

Fig. 2.

$\tau_w$ as a function of $m^2$ for the total magnetization (diamonds), for $\langle \phi^2 \rangle$ (triangles), and for the total energy (stars).

Fig. 3.

The exponent $\zeta$ as a function of $m^2$.

Fig. 4.

The ratio of autocorrelation times $\tau_l/\tau_w$ as a function of $m^2$. Figures 4a, 4b, 4c, and 4d correspond to lattices of size $L = 8, 16, 32$, and 64, respectively.

Fig. 5.

Autocorrelation time $\tau_w$ as a function of $\Delta$ at $m^2 = -0.285$ on a lattice of size $L = 32$. 