An Accelerated Directional Derivative Method for Smooth Stochastic Convex Optimization

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Abstract

We consider smooth stochastic convex optimization problems in the context of algorithms which are based on directional derivatives of the objective function. This context can be considered as an intermediate one between derivative-free optimization and gradient-based optimization. We assume that at any given point and for any given direction, a stochastic approximation for the directional derivative of the objective function at this point and in this direction is available with some additive noise. The noise is assumed to be of an unknown nature, but bounded in the absolute value. We underline that we consider directional derivatives in any direction, as opposed to coordinate descent methods which use only derivatives in coordinate directions. For this setting, we propose a non-accelerated and an accelerated directional derivative method and provide their complexity bounds. Despite that our algorithms do not use gradient information, our non-accelerated algorithm has a complexity bound which is, up to a factor logarithmic in problem dimension, similar to the complexity bound of gradient-based algorithms. Our accelerated algorithm has a complexity bound which coincides with the complexity bound of the accelerated gradient-based algorithm up to a factor of square root of the problem dimension, whereas for existing directional derivative methods this factor is of the order of problem dimension. We also extend these results to strongly convex problems. Finally, we consider derivative-free optimization as a particular case of directional derivative optimization with noise in the directional derivative and obtain complexity bounds for non-accelerated and accelerated derivative-free methods. Complexity bounds for these algorithms inherit the gain in the dimension dependent factors from our directional derivative methods.

Keywords:
Stochastic programming, Convex programming, Smoothness, Acceleration, Gradient-free optimization

1. Introduction

Zero-order or derivative-free optimization considers problems of minimization of a function using only, possibly noisy, observations of its values. This area of optimization has a long history, starting as early as in 1960 [39], see...
also [11, 42, 13]. Even an older area of optimization, which started in 19th century [11], considers first-order methods which use the information about the gradient of the objective function. In this paper, we choose an intermediate class of problems. Namely, we assume that at any given point and for any given direction, a noisy stochastic approximation for the directional derivative of the objective function at this point in this direction is available. We underline that we consider directional derivatives in any direction, as opposed to coordinate descent methods which rely only on derivatives in coordinate directions. We refer to the class of optimization methods, which use directional derivatives of the objective function, as directional derivative methods. Unlike well developed areas of derivative-free and first-order stochastic optimization methods, the area of directional derivative optimization methods for stochastic optimization problems is not sufficiently covered in the literature. This class of optimization methods can be motivated by at least three situations.

The first one is connected to Automatic Differentiation [43]. Assume that the objective function is given as a computer program, which performs elementary arithmetic operations and elementary functions evaluations. Automatic Differentiation allows to calculate the gradient of this objective function and the additional computational cost is no more than five times larger than the cost of the evaluation of the objective value. The drawback of this approach is that it requires to store in memory the result of all the intermediate operations, which can require large memory amount. On the contrary, calculation of the directional derivative is easier than the calculation of the full gradient and requires the same memory amount as the calculation of the value of the objective [28]. Since a random vector can be a part of the program input or some randomness can be used during the program execution, stochastic optimization problems can also be considered.

Importantly, automatic calculation of the directional derivative does not require the objective function to be smooth. This fact motivates the study of directional derivative methods in connection to Deep Learning. Indeed, learning problem is often stated as a problem of minimization of a loss function. A non-smooth activation function, called rectifier, is frequently used in Deep Learning as a building block for the loss function. Formally speaking, this non-smoothness does not allow to use Automatic Differentiation in the form of backpropagation to calculate the gradient of the objective function. At the same time, directional derivatives can be calculated by properly modified backpropagation.

The second motivating situation is connected to quasi-variational inequalities, which are used in modelling of different phenomena, such as sandpile formation and growth [38], determination of lakes and river networks [6], and superconductivity [5]. It happens that directional derivatives can be calculated for such problems [32] as a solution to some auxiliary problem. Since this subproblem can not always be solved exactly, the noise in the directional derivative naturally arises. If the considered physical phenomenon takes place in some random media, stochastic optimization can be a natural approach to use.

The third motivating situation is connected to derivative-free stochastic optimization. In this situation a gradient approximation, based on the difference of stochastic approximations for the values of the objective in two close points, can be considered as a noisy directional derivative in the direction given by the difference of these two points [19]. In this case, derivative-free stochastic optimization can be considered as a particular case of directional derivative stochastic optimization.

Motivated by potential presence of non-stochastic noise in the problem, we assume that the noise in the directional derivative consists of two parts. Similar to stochastic optimization problems, the first part is of a stochastic nature. On the opposite, the second part is an additive noise of an unknown nature, but bounded in the absolute value. More precisely, we consider the following optimization problem

$$
\min_{x \in \mathbb{R}^n} \left\{ f(x) := \mathbb{E}_\xi[F(x, \xi)] = \int_X F(x, \xi) dP(x) \right\},
$$

where $\xi$ is a random vector with probability distribution $P(\xi)$, $\xi \in X$, and for $P$-almost every $\xi \in X$, the function $F(x, \xi)$ is closed and convex. Moreover, we assume that, for $P$ almost every $\xi$, the function $F(x, \xi)$ has gradient $g(x, \xi)$, which is $L(\xi)$-Lipschitz continuous with respect to the Euclidean norm and $L_2 := \sqrt{\mathbb{E}[L(\xi)^2]} < +\infty$. Under this assumptions, $\mathbb{E}_\xi[g(x, \xi)] = \nabla f(x)$ and $f$ has $L_2$-Lipschitz continuous gradient with respect to the Euclidean norm. Also we assume that

$$
\mathbb{E}_\xi[\|g(x, \xi) - \nabla f(x)\|_2^2] \leq \sigma^2,
$$

where $\| \cdot \|_2$ is the Euclidean norm.
Finally, we assume that an optimization procedure, given a point \( x \in \mathbb{R}^n \), direction \( e \in S_2(1) \) and \( \xi \) independently drawn from \( P \), can obtain a noisy stochastic approximation \( \tilde{f}(x, \xi, e) \) for the directional derivative \( \langle g(x, \xi), e \rangle \):

\[
\tilde{f}(x, \xi, e) = \langle g(x, \xi), e \rangle + \zeta(x, \xi, e) + \eta(x, \xi, e),
\]

\[
\mathbb{E}_\xi(\zeta(x, \xi, e))^2 \leq \Delta_\zeta, \quad \forall x \in \mathbb{R}^n, \forall e \in S_2(1),
\]

\[
\mathbb{E}(|\eta(x, \xi, e)|) \leq \Delta_\eta, \quad \forall x \in \mathbb{R}^n, \forall e \in S_2(1), \text{ a.s. in } \xi,
\]

where \( S_2(1) \) is the Euclidean sphere or radius one with the center at the point zero and the values \( \Delta_\zeta, \Delta_\eta \) are controlled and can be made as small as it is desired. Note that we use the smoothness of \( F(\cdot, \xi) \) to write the directional derivative as \( \langle g(x, \xi), e \rangle \), but we do not assume that the whole stochastic gradient \( g(x, \xi) \) is available.

It is well-known \cite{29, 15, 17} that, if the stochastic approximation \( g(x, \xi) \) for the gradient of \( f \) is available, an accelerated gradient method has complexity bound \( O\left( \max\left\{ \sqrt{n/\epsilon}, \sigma^2/\epsilon^2 \right\} \right) \), where \( \epsilon \) is the target optimization error. The question, to which we give a positive answer in this paper, is as follows.

*Is it possible to solve a smooth stochastic optimization problem with the same \( \epsilon \)-dependence in the complexity and only noisy observations of the directional derivative?*

1.1. Related work

We first consider the related work on directional derivative optimization methods and, then, a closely related class of derivative-free methods with two-point feedback, the latter meaning that an optimization method uses two function value evaluations on each iteration. Since all the considered methods are randomized, we compare oracle complexity bounds in terms of expectation, that is, a number of directional derivatives or function values evaluations which is sufficient to achieve an error \( \epsilon \) in the expected optimization error \( \mathbb{E} f(\hat{x}) - f^* \), where \( \hat{x} \) is the output of an algorithm and \( f^* \) is the optimal value of \( f \).

1.1.1. Directional derivative methods

**Deterministic smooth optimization problems.** In \cite{36}, the authors consider the Euclidean case and propose a non-accelerated and an accelerated directional derivative method for smooth convex problems with complexity bounds \( O(nL_2/\epsilon) \) and \( O(n \sqrt{L_2}/\epsilon) \) respectively. Also they propose a non-accelerated and an accelerated method for problems with \( \mu \)-strongly convex objective and prove complexity bounds \( O(nL_2/\mu \log_2(1/\epsilon)) \) and \( O(n \sqrt{L_2}/\mu \log_2(1/\epsilon)) \) respectively. For a more general case of problems with additional bounded noise in directional derivatives, but also for the Euclidean case, an accelerated directional derivative method was proposed in \cite{19} and a bound \( O(n \sqrt{L_2}/\epsilon) \) was proved.

We also should mention coordinate descent methods. In the seminal paper \cite{35}, a random coordinate descent for smooth convex and \( \mu \)-strongly convex optimization problems were proposed and \( O(L/\epsilon) \) and \( O(L/\mu \log_2(1/\epsilon)) \) complexity bounds were proved, where \( L \) is an effective Lipschitz constant of the gradient varying from \( n \) to some average over coordinates coordinate-wise Lipschitz constant. In the same paper, an accelerated version of random coordinate descent was proposed for convex problems and \( O(n \sqrt{L}/\epsilon) \) complexity bound was proved. Papers \cite{30, 20, 31, 40} generalize accelerated random coordinate descent for different settings, including \( \mu \)-strongly convex problems, and \cite{37, 2, 21} provide a \( O(\sqrt{L}/\epsilon) \) and \( O(\sqrt{L/\mu \log_2(1/\epsilon)}) \) complexity bounds, where \( L \) is an effective Lipschitz constant of the gradient varying from \( n \) to some average over coordinates coordinate-wise Lipschitz constant, and, in the best case, is dimension-independent. An accelerated random coordinate descent with inexact coordinate-wise derivatives was proposed in \cite{15} with \( O(n \sqrt{L}/\epsilon) \) complexity bound and also a unified view on directional derivative methods, coordinate descent and derivative-free methods.

**Stochastic optimization problems.** A directional derivative method for non-smooth stochastic convex optimization problems was introduced in \cite{36} with a complexity bound \( O(n^2/\epsilon^2) \). A random coordinate descent method for non-smooth stochastic convex and \( \mu \)-strongly convex optimization problems were introduced in \cite{14} with complexity bounds \( O(n/\epsilon^2) \) and \( O(n/\mu \epsilon) \) respectively.

1.1.2. Derivative-free methods

**Deterministic smooth optimization problems.** A non-accelerated and an accelerated derivative-free method for this type of problems were proposed in \cite{36} for the Euclidean case with the bounds \( O(nL_2/\epsilon) \) and \( O(n \sqrt{L_2}/\epsilon) \).
respectively. The same paper proposed a non-accelerated and an accelerated method for $\mu$-strongly convex problems with complexity bounds $O(nL_2/\mu \log_2(1/\varepsilon))$ and $O(n \sqrt{L_2/\mu \log_2(1/\varepsilon)})$ respectively. A non-accelerated derivative-free method for deterministic problems with additional bounded noise in function values was proposed in [1] together with $O(nL_2/\varepsilon)$ bound and application to learning parameter of a parametric PageRank model. Deterministic problems with additional bounded noise in function values were also considered in [19], where several accelerated derivative-free methods, including Derivative-Free Block-Coordinate Descent, were proposed and a bound $O(n \sqrt{L_2/\varepsilon})$ was proved, where $L$ depends on the method and, in some sense, characterizes the average over blocks of coordinates Lipschitz constant of the derivative in the block.

Stochastic optimization problems. Most of the authors in this group solve a more general problem of bandit convex optimization and obtain bounds on the so-called regret. It is well known [12] that a bound on the regret can be converted to a bound on the expected optimization error. Non-smooth stochastic optimization problems were considered in [16, 23, 22, 41, 7, 26] and [22, 7] proved a bound $\tilde{O}(n^{2/\varrho}/(\mu_\varepsilon \varepsilon))$ respectively. A non-accelerated derivative-free method with the bounds $O(n \sqrt{L_2/\varepsilon})$ was later extended for $\mu_\varepsilon$-strongly convex w.r.t. to $\varepsilon$-norm problems, the authors of [22] proved a bound $\tilde{O}(n^{2/\varrho}/(\mu_\varepsilon \varepsilon))$.

Intermediate, partially smooth problems with a restrictive assumption of boundedness of $\mathbb{E} \|g(x, \xi)\|^2$, were considered in [16], where it was proved that a proper modification of Mirror Descent algorithm with derivative-free approximation of the gradient gives a bound $O(n^{2/\varrho}/\varepsilon^2)$ for convex problems, improving upon the bound $O(n^{2/\varrho}/\varepsilon^2)$ of [1]. For strongly convex w.r.t 2-norm problems, the authors of [1] obtained a bound $\tilde{O}(n^{2/\varrho}/\varepsilon)$, which was later extended for $\mu_\varepsilon$-strongly convex problems and improved to $\tilde{O}(n^{2/\varrho}/(\mu_\varepsilon \varepsilon))$ in [22].

In the fully smooth case, without the assumption that $\mathbb{E} \|g(x, \xi)\|^2 < +\infty$, papers [25, 24] proposed a derivative-free algorithm for the Euclidean case with the bound

$$\tilde{O}\left(\max\left\{\frac{nL_2\varepsilon}{\varepsilon^2}, \frac{n\sigma^2}{\varepsilon^2}\right\}\right).$$

In [18], the authors proposed a non-accelerated and an accelerated derivative-free method with the bounds

$$\tilde{O}\left(\max\left\{\frac{n^2 L_2 R_p^2 \varepsilon}{\varepsilon^2}, \frac{n^2 \sigma^2 R_p^2 \varepsilon}{\varepsilon^2}\right\}\right), \quad \tilde{O}\left(\max\left\{\frac{n^{\frac{2}{1+\varrho}} L_2 \varepsilon}{\varepsilon^2}, \frac{n^{\frac{2}{1+\varrho}} \sigma^2 R_p^2}{\varepsilon^2}\right\}\right)$$

respectively, where $R_p$ characterizes the distance in $\varepsilon$-norm between the starting point of the algorithm and a solution to [1], $p \in [1, 2]$ and $q \in [2, \infty]$ is the conjugate to $p$, given by the identity $\frac{1}{p} + \frac{1}{q} = 1$.

1.2. Our contributions

As we have seen above, only two results on directional derivative methods for non-smooth stochastic convex optimization are available in the literature, and, to the best of our knowledge, nothing is known about directional derivative methods for smooth stochastic convex optimization, even in the well-developed area of random coordinate descent methods. Our main contribution consists in closing this gap in the theory of directional derivative methods for stochastic optimization and considering even more general setting with additional noise of an unknown nature in the directional derivative.

Our methods are based on two proximal setups [8] characterized by the values $p \in [1, 2]$ and its conjugate $q \in [2, \infty]$, given by the identity $\frac{1}{p} + \frac{1}{q} = 1$. The case $p = 1$ corresponds to the choice of 1-norm in $\mathbb{R}^n$ and corresponding prox-function which is strongly convex with respect to this norm (we provide the details below). The case $p = 2$ corresponds to the choice of the Euclidean 2-norm in $\mathbb{R}^n$ and squared Euclidean norm as the prox-function. As our main contribution, we propose an Accelerated Randomized Directional Derivative (ARDD) algorithm for smooth problems.
stochastic optimization based on noisy observations of directional derivative of the objective. Our method has the complexity bound
\[
\tilde{O}\left(\max\left\{n^{\frac{3}{2}+\frac{1}{q}}, \frac{n^{\frac{3}{2}+\frac{1}{q}}}{\sqrt{\frac{L_{2}R_{p}^{2}}{e}}, \frac{n^{\frac{3}{2}+\frac{1}{q}}}{\sqrt{\frac{\sigma^{2}R_{p}^{2}}{e^2}}}}\right\}\right),
\]
where \(R_{p}\) characterizes the distance in \(p\)-norm between the starting point of the algorithm and a solution to (1). Our algorithm for \(p = 1, q = \infty\) is based on a novel idea of combining gradient step with respect to 2-norm proximal setup and mirror descent step with respect to 1-norm proximal setup. We underline that using different norms and proximal setups for these two steps allows us to gain \(\sqrt{n}\) factor in the case \(p = 1\) in comparison with the standard choice \(p = 2\) and both steps made with respect to 2-norm proximal setup.

As our second contribution, we propose a non-accelerated Randomized Directional Derivative (RDD) algorithm with the complexity bound
\[
\tilde{O}\left(\max\left\{n^{\frac{3}{2}+\frac{1}{q}}, \frac{n^{\frac{3}{2}+\frac{1}{q}}}{\sqrt{\frac{L_{2}R_{p}^{2}}{e}}, \frac{n^{\frac{3}{2}+\frac{1}{q}}}{\sqrt{\frac{\sigma^{2}R_{p}^{2}}{e^2}}}}\right\}\right).
\]
Interestingly, in this case, we obtain a dimension independent complexity bound despite we use only noisy directional derivative observations.

Note that, in the case of (1) having a sparse solution, our bounds for \(p = 1\) allow to gain a factor of \(\sqrt{n}\) in the complexity of the accelerated method and a factor of \(n\) in the complexity of the non-accelerated method in comparison to the Euclidean case \(p = 2\). Indeed, sparsity of a solution \(x^*\) means that \(\|x^*\|_p = O(1) \cdot \|x^*\|_2\) and, if the starting point is zero, we obtain \(R_{p}^1 = \|x^*\|_p^1 = O(1) \cdot \|x^*\|_2^1 = O(1)R_{2}^2\). Hence, the bounds for \(p = 1\) and \(p = 2\) can be compared only based on the corresponding powers of \(n\), the latter being smaller for the case \(p = 1, q = \infty\).

As our third contribution, we extend the above results to the case when the objective function is additionally known to be \(\mu\)-strongly convex w.r.t. \(p\)-norm. For this case, we propose an accelerated and a non-accelerated algorithm which respectively have complexity bounds
\[
\tilde{O}\left(\max\left\{n^{\frac{3}{2}+\frac{1}{q}}, \frac{L_{2}}{\mu_{p}} \log_{2} \frac{\mu_{p}R_{p}^{2}}{e}, \frac{n^{\frac{3}{2}+\frac{1}{q}}}{\sqrt{\frac{\sigma^{2}R_{p}^{2}}{e^2}}}\right\}\right),
\tilde{O}\left(\max\left\{n^{\frac{3}{2}+\frac{1}{q}}, \frac{L_{2}}{\mu_{p}} \log_{2} \frac{\mu_{p}R_{p}^{2}}{e}, \frac{n^{\frac{3}{2}+\frac{1}{q}}}{\sqrt{\frac{\sigma^{2}R_{p}^{2}}{e^2}}}\right\}\right).
\]
As our final contribution, we consider derivative-free smooth stochastic convex optimization with inexact values of the stochastic approximations for the function values as a particular case of optimization using noisy directional derivatives. This allows us to obtain the complexity bounds of [18] as a straightforward corollary of our results in this paper. At the same time we obtain new complexity bounds for the strongly convex case which, to the best of our knowledge, were not known in the literature.

Note that our results for accelerated and non-accelerated methods are somewhat similar to the finite-sum minimization problems of the form
\[
\min_{x \in \mathbb{R}^n} \sum_{i=1}^{m} f_i(x),
\]
where \(f_i\) are convex smooth functions. For such problems accelerated methods have complexity \(\tilde{O}(m + \sqrt{mL/s})\) and non-accelerated methods have complexity \(\tilde{O}(m + L/s)\) (see, e.g. [2] for a nice review on the topic). As we see, acceleration allows to take the square root of the second term but for the price of \(\sqrt{m}\) and the two bounds can not be directly compared without additional assumptions on the value of \(m\).

1.3. Paper organization

The rest of the paper is organized as follows. In Section 2 both for convex and strongly convex problems, we introduce our algorithms, state their convergence rate theorems and corresponding complexity bounds. Section 3 is devoted to proof of the convergence rate theorem for our accelerated method and convex objective functions. Section 4 is devoted to proof of the convergence rate theorem for our non-accelerated method and convex objective functions. Finally, in Section 5 we provide the proofs for the case of strongly convex objective function.
2. Algorithms and main results

In this section, we provide our non-accelerated and accelerated directional derivative methods both for convex and strongly convex problems together with convergence theorems and corresponding complexity bounds. The proofs are rather technical and postponed to next sections.

2.1. Preliminaries

We start by introducing necessary objects and technical results.

**Proximal setup.** Let \( p \in [1, 2] \) and \( \|x\|_p \) be the \( p \)-norm in \( \mathbb{R}^n \) defined as

\[
\|x\|_p = \sum_{i=1}^{n} |x_i|^p, \quad x \in \mathbb{R}^n,
\]

\( \| \cdot \|_p \) be its dual, defined by \( \|g\|_p = \max \{\langle g, x \rangle, \|x\|_p \leq 1 \} \), where \( q \in [2, \infty] \) is the conjugate number to \( p \), given by \( \frac{1}{p} + \frac{1}{q} = 1 \), and, for \( q = \infty \), by definition \( \|x\|_\infty = \max_{i=1, \ldots, n} |x_i| \).

We choose a prox-function \( d(x) \) which is continuous, convex on \( \mathbb{R}^n \) and is 1-strongly convex on \( \mathbb{R}^n \) with respect to \( \| \cdot \|_p \), i.e., for any \( x, y \in \mathbb{R}^n \)

\[
d(y) - d(x) - \langle \nabla d(x), y - x \rangle \geq \frac{1}{4} \|y - x\|^2_p.
\]

Without loss of generality, we assume that \( \min d(x) = 0 \). We define also the corresponding Bregman divergence \( V[z](x) = d(x) - d(z) - \langle \nabla d(z), x - z \rangle, x, z \in \mathbb{R}^n \).

Note that, by the strong convexity of \( d \),

\[
V[z](x) \geq \frac{1}{2} \|x - z\|^2_p, \quad x, z \in \mathbb{R}^n.
\]  

(7)

For the case \( p = 1 \), we choose the following prox-function \([3, 8]\)

\[
d(x) = \frac{en^{(x-1)(2-x)/n} \ln n}{2 \|x\|_2^2}, \quad \kappa = 1 + \frac{1}{\ln n}
\]  

(8)

and, for the case \( p = 2 \), we choose the prox-function to be the squared Euclidean norm

\[
d(x) = \frac{1}{2} \|x\|^2_2.
\]  

(9)

**Main technical lemma.** In our proofs of complexity bounds, we rely on the following lemma which was proved in \([18]\).

**Lemma 1.** Let \( e \in RS_2(1) \), i.e be a random vector uniformly distributed on the surface of the unit Euclidean sphere in \( \mathbb{R}^n \), \( p \in [1, 2] \) and \( q \) be given by \( \frac{1}{p} + \frac{1}{q} = 1 \). Then, for \( n \geq 8 \) and \( \rho_n = \min(q - 1, 16 \ln n - 8)n^{q-1} \),

\[
E_n \|e\|^2_2 \leq \rho_n,
\]  

(10)

\[
E_n \left(\langle s, e \rangle^2 \|e\|^2_2\right) \geq \frac{6\rho_n}{n} \|s\|^2_2, \quad \forall s \in \mathbb{R}^n.
\]  

(11)

**Stochastic approximation of the gradient.** Based on the noisy stochastic observations \([3]\) of the directional derivative, we form the following stochastic approximation of \( \nabla f(x) \)

\[
\tilde{\nabla}^m f(x) = \frac{1}{m} \sum_{i=1}^{m} \tilde{f}(x, \xi_i, e)e,
\]  

(12)

where \( e \in RS_2(1) \), \( \xi_i, i = 1, \ldots, m \) are independent realizations of \( \xi \), \( m \) is the batch size.
2.2. Algorithms and main results for convex problems

Our Accelerated Randomized Directional Derivative (ARDD) method is listed as Algorithm 1. Note that \( y_{k+1} \) is defined by gradient step from \( x_{k+1} \) and \( z_{k+1} \) is defined by mirror descent step from \( z_k \). Thus, our algorithm for \( p = 1 \), \( q = \infty \) is based on a novel idea of combining gradient step with respect to 2-norm proximal setup and mirror descent step with respect to 1-norm proximal setup\(^4\). This combination allows us to gain a factor of the order of \( \sqrt{n} \) for the case \( p = 1 \) in comparison to standard choice \( p = 2 \).

Algorithm 1 Accelerated Randomized Directional Derivative (ARDD) method

Input: \( x_0 \) — starting point; \( N \geq 1 \) — number of iterations; \( m \geq 1 \) — batch size.

Output: point \( y_N \).

1: \( y_0 \leftarrow x_0 \), \( z_0 \leftarrow x_0 \).
2: for \( k = 0, \ldots, N - 1 \) do
3: \( \alpha_{k+1} \leftarrow \frac{4k+2}{48n^2(x_0, L_2^2) + 4} \), \( \tau_k \leftarrow \frac{1}{4k(x_0, L_2^2)} = \frac{2}{x_0^2} \).
4: Generate \( e_{k+1} \in RS_2(1) \) independently from previous iterations and \( \xi_i \), \( i = 1, \ldots, m \) – independent realizations of \( \xi \).
5: Calculate
\[
\tilde{\nabla}^m f(x_{k+1}) = \frac{1}{m} \sum_{i=1}^{m} \tilde{f}(x_{k+1}, \xi_i, e_{k+1}) \epsilon_{k+1}.
\]
6: \( x_{k+1} \leftarrow \tau_k z_k + (1 - \tau_k)y_k \).
7: \( y_{k+1} \leftarrow x_{k+1} - \frac{1}{2\tau_k} \tilde{\nabla}^m f(x_{k+1}) \).
8: \( z_{k+1} \leftarrow \arg\min_{z \in \mathbb{R}^n} \{ \alpha_{k+1} n \tilde{\nabla}^m f(x_{k+1}), z - z_k \} + V[z_k](z) \).
9: end for
10: return \( y_N \).

Theorem 1. Let ARDD method be applied to solve problem (1). Then
\[
\mathbb{E}[f(y_N)] - f(x^*) \leq \frac{384 \Theta_p^2 n^2 p L_2}{N m} + \frac{4L_2}{m} \sigma^2 + \frac{61N}{24L_2} \Delta_\xi + \frac{122N}{4L_2^2} \Delta_\eta + \frac{12}{N} \frac{\sqrt{2n} \Theta_p}{m} \left( \frac{\sqrt{\Delta_\xi}}{2} + 2 \Delta_\eta \right) + \frac{N^2}{12npL_2} \left( \frac{\sqrt{\Delta_\xi}}{2} + 2 \Delta_\eta \right)^2,
\]
where \( \Theta_p = V[z_0](x^*) \) is defined by the chosen proximal setup and \( \mathbb{E}[\cdot] = \mathbb{E}_{\xi_1, \ldots, \xi_N, \xi_{1,1}, \ldots, \xi_{N,1}} \).

Before we proceed to the non-accelerated method, we give the appropriate choice of the ARDD method parameters \( N, m \), and accuracy of the directional derivative evaluation \( \Delta_\xi, \Delta_\eta \). These values are chosen such that the r.h.s. of (13) is smaller than \( \varepsilon \). For simplicity we omit numerical constants and summarize the obtained values of the algorithm parameters in Table 1 below. The last row represents the total number \( Nm \) of oracle calls, that is, the number of directional derivative evaluations, which was advertised in \( (4) \).

Our Randomized Directional Derivative (RDD) method is listed as Algorithm 2.

Theorem 2. Let RDD method be applied to solve problem (1). Then
\[
\mathbb{E}[f(y_N)] - f(x^*) \leq \frac{384p L_2 \Theta_p}{N m} + \frac{2}{L_2} \frac{\sigma^2}{m} + \frac{n}{12L_2} \Delta_\xi + \frac{4n}{3L_2^2} \Delta_\eta + \frac{8}{N} \frac{\sqrt{2n} \Theta_p}{m} \left( \frac{\sqrt{\Delta_\xi}}{2} + 2 \Delta_\eta \right) + \frac{N}{3L_2 p n} \left( \frac{\sqrt{\Delta_\xi}}{2} + 2 \Delta_\eta \right)^2,
\]
where \( \Theta_p = V[z_0](x^*) \) is defined by the chosen proximal setup and \( \mathbb{E}[\cdot] = \mathbb{E}_{\xi_1, \ldots, \xi_N, \xi_{1}, \ldots, \xi_{N,1}} \).

\(^4\)The idea of combining gradient step with dual averaging step to accelerate gradient-based optimization was proposed in \( (13) \). The idea of combining gradient step and mirror descent step for deterministic gradient-based optimization was proposed in \( (3) \). In both these works the same proximal setup was used for both steps.
Let our choice of the prox-function (8), $\Omega_p$ be defined as follows. For $p = 1$ and our choice of the prox-function (8), $\Omega_p = \frac{\nu}{\ln(1+\nu)} \ln n = O(\ln n)$ for our choice of $\kappa = 1 + \frac{\nu}{\ln n}$, see [33, 22]. For $p = 2$ and our choice of the prox-function (8), $\Omega_p = 1$. Our Accelerated Randomized Directional Derivative method for strongly convex problems (ARDDsc) is listed as Algorithm [3].

**Theorem 3.** Let $f$ in problem (1) be $\mu_p$-strongly convex and ARDDsc method be applied to solve this problem. Then

$$
\mathbb{E} f(u_K) - f^* \leq \frac{\mu_p R_p^2}{2} \cdot 2^{-\kappa} + 2\Delta.
$$

### 2.3. Extensions for strongly convex problems

In this subsection, we assume additionally that $f$ is $\mu_p$-strongly convex w.r.t. $p$-norm. Our algorithms and proofs rely on the following fact. Let $x_*$ be some fixed point and $x$ be a random point such that $\mathbb{E}_x[\|x - x_*\|^2_p] \leq R_p^2$, then

$$
\mathbb{E}_x d\left(\frac{x - x_*}{R_p}\right) \leq \frac{\Omega_p}{2}.
$$

where $\mathbb{E}_x$ denotes the expectation with respect to random vector $x$ and $\Omega_p$ is defined as follows. For $p = 1$ and our choice of the prox-function (8), $\Omega_p = \frac{\nu}{\ln(1+\nu)} \ln n = O(\ln n)$ for our choice of $\kappa = 1 + \frac{\nu}{\ln n}$, see [33, 22]. For $p = 2$ and our choice of the prox-function (8), $\Omega_p = 1$. Our Accelerated Randomized Directional Derivative method for strongly convex problems (ARDDsc) is listed as Algorithm [3].
Output: \( \Delta = x - x_0 \)

where \( a = 384 n^2 \rho_n \).

2: for \( k = 0, \ldots, K - 1 \) do

3: Set

\[
N_0 = \left\lfloor \frac{8aL_2\Omega_p}{\mu_p} \right\rfloor, \tag{16}
\]

where \( b = \frac{1}{e} \).

4: Set \( d_k(x) = R_k^2 d \left( \frac{\text{x} - x_k}{R_k} \right) \).

5: Run ARDD with starting point \( u_k \) and prox-function \( d_k(x) \) for \( N_0 \) steps with batch size \( m_k \).

6: Set \( u_{k+1} = y_{N_0}, k = k + 1 \).

7: end for

8: return \( u_K \)

where \( \Delta = \frac{61 N_0}{24 L_2} \Delta_c + \frac{122 N_0}{9 L_2} \Delta_n^2 + \frac{12}{L_2} \sqrt{2a R_1 \Omega_p N_0} \left( \frac{\sqrt{\Delta_c}}{2} + 2 \Delta_n \right) + \frac{N_0^2}{12 a \rho_n L_2} \left( \frac{\sqrt{\Delta_c}}{2} + 2 \Delta_n \right)^2 \).

Moreover, under an appropriate choice of \( \Delta_c \) and \( \Delta_n \) s.t. \( 2 \Delta \leq \varepsilon/2 \), the oracle complexity to achieve \( \varepsilon \)-accuracy of the solution is

\[
O \left( \max \left\{ n^{\frac{1}{p} + \frac{1}{q}} \frac{L_2 \Omega_p}{\mu_p} \log_2 \frac{\mu_p R_0^2}{\varepsilon}, n^{\frac{1}{p} + \frac{1}{q}} \Omega_p \right\} \right). \tag{19}
\]

Before we proceed to the non-accelerated method, we give the appropriate choice of the accuracy of the directional derivative evaluation \( \Delta_c, \Delta_n \) for ARDDsc to achieve an accuracy \( \varepsilon \) of the solution. These values are chosen such that the r.h.s. of (18) is smaller than \( \varepsilon \). For simplicity we omit numerical constants and summarize the obtained values of the algorithm parameters in Table 3 below. The last row represents the total number of oracle calls, that is, the number of directional derivative evaluations, which was stated in (6).

Our Randomized Directional Derivative method for strongly convex problems (RDDsc) is listed as Algorithm 4.
Let $f$ in problem Theorem 4. 

Input: $x_0$ — starting point s.t. $\|x_0 - x_i\|_p^2 \leq R_p^2$; $K \geq 1$ — number of iterations; $\mu_p$ — strong convexity parameter. 

Output: point $u_K$.

1. Set 
   \[ N_0 = \left\lceil \frac{8aL_2\Omega_n}{\mu_p} \right\rceil, \]  
   where $a = 384n\mu_n$.

2. for $k = 0, \ldots, K - 1$ do

3. Set 
   \[ m_k := \max \left\{ 1, \left\lceil \frac{8br^{2-k}}{L_2\mu_pR_p^2} \right\rceil \right\}, \quad R_k^2 := \frac{4\Delta \left( 1 - 2^{-k} \right)}{\mu_p}, \]  
   where $b = 2$.

4. Set $d_k(x) = R_k^2 \left( \frac{r - q}{R_k} \right)$.

5. Run RDD with starting point $u_k$ and prox-function $d_k(x)$ for $N_0$ steps with batch size $m_k$.

6. Set $u_{k+1} = y_{N_0}, k = k + 1$.

7. end for

8. return $u_K$

**Theorem 4.** Let $f$ in problem (1) be $\mu_p$-strongly convex and RDDsc method be applied to solve this problem. Then

\[ \mathbb{E}f(u_K) - f^* \leq \frac{\mu_pR_p^2}{2} \cdot 2^{-K} + 2\Delta. \]  

where $\Delta = \frac{n}{12L_2} \Delta_c + \frac{4n}{L_2} \Delta_q^2 + \frac{8\sqrt{5nR_2\Omega_n}}{\nu_{\max}} \left( \sqrt{\frac{\chi}{2}} + 2\Delta_q \right) + \frac{N_08\sqrt{5nR_2\Omega_n}}{\nu_{\max}} \left( \sqrt{\frac{\chi}{2}} + 2\Delta_q \right)^2$. Moreover, under an appropriate choice of $\Delta_c$ and $\Delta_q$ s.t. $2\Delta_\leq \varepsilon/2$, the oracle complexity to achieve $\varepsilon$-accuracy of the solution is

\[ \tilde{O} \left( \max \left\{ n^2L_2^2 \Omega_p^2 \log \frac{\mu_pR_p^2}{\varepsilon}, n^2\sigma^2 \Omega_p^2 \right\} \right). \]

Before we proceed, we give the appropriate choice of the accuracy of the directional derivative evaluation $\Delta_c$, $\Delta_q$ for RDDsc to achieve an accuracy $\varepsilon$ of the solution. These values are chosen such that the r.h.s. of (21) is smaller than $\varepsilon$. For simplicity we omit numerical constants and summarize the obtained values of the algorithm parameters in Table 4 below. The last row represents the total number of oracle calls, that is, the number of directional derivative evaluations, which was stated in (6).

2.4. Corollaries for derivative-free optimization

In this subsection, following [18], we consider derivative-free smooth stochastic optimization in the two-point feedback situation. We assume that an optimization procedure, given a pair of points $(x, y) \in \mathbb{R}^{2n}$, can obtain a pair
of noisy stochastic realizations \((\tilde{f}(x, \xi), \tilde{f}(y, \xi))\) of the objective function \(f\), where

\[
\tilde{f}(x, \xi) = F(x, \xi) + \Xi(x, \xi), \quad |\Xi(x, \xi)| \leq \Delta, \quad \forall x \in \mathbb{R}^n, \text{ a.s. in } \xi,
\]

and \(\xi\) is independently drawn from \(P\).

Based on these observations of the objective function, we form the following stochastic approximation of \(\nabla f(x)\)

\[
\tilde{\nabla}^nf(x) = \frac{1}{m} \sum_{i=1}^{m} \tilde{f}(x + te, \xi_i) - \tilde{f}(x), e = \left(\sum_{i=1}^{m} (\xi(x, \xi_i, e) + \eta(x, \xi_i, e))\right),
\]

where \(e \in \mathbb{R}_2(1)\), \(\xi_i, i = 1, ..., m\) are independent realizations of \(\xi\), \(m\) is the batch size, \(t\) is some small positive parameter which we call smoothing parameter, \(g^m(x, \xi) := \frac{\nabla f(x)}{m} \sum_{i=1}^{m} g(x, \xi)\), and

\[
\zeta(x, \xi_i, e) = \frac{F(x + te, \xi_i) - F(x)}{t}, \quad (g(x, \xi_i, e) + \eta(x, \xi_i, e)) = \frac{\Xi(x + te, \xi_i)}{t}, \quad i = 1, ..., m.
\]

By Lipschitz smoothness of \(F(\cdot, \xi)\), we have \(|\zeta(x, \xi, e)| \leq \frac{L_2^f}{2}\) for all \(x \in \mathbb{R}^n\) and \(e \in S_2(1)\). Hence, \(E_\xi(\zeta(x, \xi, e))^2 \leq \frac{L_2^f}{2}\) for all \(x \in \mathbb{R}^n\) and \(e \in S_2(1)\). At the same time, from (22), we have that \(|\eta(x, \xi, e)| \leq \frac{\Delta}{m^2}\) for all \(x \in \mathbb{R}^n\), \(e \in S_2(1)\) and a.s. in \(\xi\). Applying Theorem 1 and Theorem 2 with \(\Delta_\xi = \frac{L_2^f}{2}\) and \(\Delta_\eta = \frac{\Delta}{m^2}\), we reproduce respectively the result of Theorem 2 and Theorem 3 in [18]. Applying Theorem 3 and Theorem 4 with \(\Delta_\xi = \frac{L_2^f}{2}\) and \(\Delta_\eta = \frac{\Delta}{m^2}\), we obtain also complexity bounds [6] for derivative-free smooth stochastic strongly convex optimization, which was not yet done in the literature.

### 3. Proof of main result for ARDD method

We divide the proof of Theorem 1 into two large steps. First, to simplify the derivations, we prove this theorem assuming two additional inequalities which connect noisy stochastic approximation of the gradient \((\tilde{\nabla} f)\) with the true gradient and function values. This result is stated as Lemma 2. Then, in Lemma 3 we show that our approximation of the gradient \((\tilde{\nabla} f)\) indeed satisfies these two inequalities.

**Lemma 2.** Let \(\{x_k, y_k, z_k\}\), \(k \geq 0\) be generated by ARDD method. Assume that there exist numbers \(\delta_1 > 0, \delta_2 > 0\) such that, for all \(k \geq 0\)

\[
E \left[\tilde{\nabla}^nf(x_{k+1}, \xi_k) - \tilde{\nabla}^nf(x_{k})\right] \geq \frac{1}{n} E \left[\nabla f(x_{k+1}), z_k - x_k\right] - \delta_1 E \left[||z_k - x_k||\right]
\]

and

\[
E \left[||\tilde{\nabla}^nf(x_{k+1})||^2\right] \leq 96\rho_2 L_2 E \left[||f(x_{k+1}) - f(y_{k+1})||\right] + \delta_2,
\]

where expectation is taken w.r.t. all randomness and \(x^*\) is a solution to (1). Then

\[
E[f(y_N)] - f(x^*) \leq \frac{384\rho_2^2 L_2^2}{N} + \frac{124 \sqrt{8\rho_2^2 L_2^2}}{N^2} \delta_1 + \frac{125 N^2}{3600 \rho_2^2 L_2^2} \delta_2 + \frac{N^2}{3600} \rho_2 L_2^2,
\]

where \(\Theta_p = V[z_0](x^*)\) is defined by the chosen proximal setup and the expectation is taken w.r.t. all randomness.
This result is proved below in subsection 3.1.

**Lemma 3.** Let \( \{x_k, y_k, z_k\}, k \geq 0 \) be generated by ARDD method. Then (24) and (25) hold with
\[
\delta_1 = \frac{\sqrt{\Delta_c}}{2\sqrt{m}} + \frac{2\Delta_r}{\sqrt{n}}
\]
and
\[
\delta_2 = \frac{96\rho_n}{n} \cdot \frac{\sigma^2}{m} + 61\rho_n\Delta_c + 976\rho_n^2\Delta_r.
\]
This result is proved below in subsection 3.2.

**Proof of Theorem 1** Combining Lemma 2 and Lemma 3, we obtain (13).

### 3.1. Proof Lemma 2

The following lemma estimates the progress in step 8 of ARDD method (and in step 5 of RDD method), which is a Mirror Descent step.

**Lemma 4.** Assume that \( z_* = \arg \min_{v \in \mathbb{R}^n} \{ an(\nabla m f(x), v - z) + V[z](v) \} \). Then, for any fixed \( u \in \mathbb{R}^n \),
\[
anE \left[ \nabla \tilde{m} f(x), z - u \right] \leq \frac{\sigma^2}{2} E \left[ \| \nabla \tilde{m} f(x) \|^2 \right] + E \left[ V[z](u) - V[z_*](u) \right],
\]
where expectation is taken w.r.t. all randomness.

**Proof.** For all \( u \in \mathbb{R}^n \), we have
\[
an(\nabla \tilde{m} f(x), z - u) = an(\nabla \tilde{m} f(x), z_* - z) + an(\nabla \tilde{m} f(x), z_* - u)
\]
\[
\overset{\circ}{\leq} an(\nabla m f(x), z - z_*) + \langle -\nabla V[z](z_*), z_* - u \rangle \overset{\circ}{=} an(\nabla \tilde{m} f(x), z - z_*)
\]
\[
+ V[z](u) - V[z_*](u) - V[z_*](u) \leq \left( an(\nabla \tilde{m} f(x), z - z_* - \frac{1}{2}\| z - z_* \|^2) \right)
\]
\[
+ V[z](u) - V[z_*](u) \leq \frac{\sigma^2}{2} \| \nabla \tilde{m} f(x) \|^2 + V[z](u) - V[z_*](u),
\]
where \( \circ \) follows from the definition of \( z_* \), whence \( \langle \nabla V[z](z_*), u - z_* \rangle \geq 0 \) for all \( u \in \mathbb{R}^n \); \( \circ \) follows from the "magic identity" Fact 5.3.3 in [8] for the Bregman divergence; \( \circ \) follows from (7), and \( \circ \) follows from the Fenchel inequality \( \zeta(s, z) - \frac{1}{2}\| z \|^2 \leq \zeta \left( \nabla \tilde{m} f(x), z - z_* \right) \). Taking full expectation we get (29).

Now we prove the following lemma which estimates the one-step iteration progress of the whole algorithm.

**Lemma 5.** Let \( \{x_k, y_k, z_k, \alpha_k, t_k\}, k \geq 0 \) be generated by ARDD method. Then, under assumptions of Lemma 2
\[
48n^2\rho_n L_2 \alpha_k^2 + E[f(y_{k+1})] - (48n^2\rho_n L_2 \alpha_k^2 + \alpha_k)E[f(y_k)]
\]
\[
- E[V[z_k](x_k)] + E[V[z_{k+1}](x_k)] - \alpha_k \delta_1 nE[\| z_k - x_k \|^2] = \frac{\alpha_k^2 n^2}{2} \delta_2 \leq \alpha_k f(x_k),
\]
where expectation is taken w.r.t. all randomness, \( z^* \) is a solution to (1).

**Proof.** Combining (24), (25) and (29), we obtain
\[
\alpha_k E[\nabla f(x_{k+1}), z - x_k] \leq 48n^2\rho_n L_2 (E[f(x_{k+1})] - E[f(y_{k+1})])
\]
\[
+ E[V[z_k](x_k)] - E[V[z_{k+1}](x_k)] + \alpha_k \delta_1 nE[\| z_k - x_k \|^2] + \frac{\alpha_k^2 n^2}{2} \delta_2.
\]
Further,
\[
\alpha_{k+1} \left( \mathbb{E}[f(x_{k+1})] - f(x_k) \right) \leq \alpha_{k+1} \mathbb{E} \left[ \nabla f(x_{k+1}), x_{k+1} - x_k \right] + \alpha_{k+1} \mathbb{E} \left[ \nabla f(x_{k+1}), z_k - x_k \right]
\]
\[
\overset{\circ}{\leq} \left( 1 - \tau_k \right) \mathbb{E} \left[ \nabla f(x_{k+1}), y_k - x_{k+1} \right] + \alpha_{k+1} \mathbb{E} \left[ \nabla f(x_{k+1}), z_k - x_k \right]
\]
\[
\overset{\circ}{\leq} \left( 1 - \tau_k \right) \mathbb{E} \left[ f(y_k) \right] - \mathbb{E} \left[ f(x_{k+1}) \right] + 48a^2 \rho \lambda L \left( \mathbb{E} \left[ f(x_{k+1}) \right] - \mathbb{E} \left[ f(y_k) \right] \right) + \alpha_{k+1} \mathbb{E} \left[ \nabla f(x_{k+1}) \right],
\]
where we denoted
\[
T elescoping (31) for \ \Theta
\]
\[
\text{We define}
\]
\[
+ \alpha_{k+1} \mathbb{E} \left[ f(x_{k+1}) \right] + \mathbb{E} \left[ V_{\xi}(x_i) \right] - \mathbb{E} \left[ V_{\xi}(z_{i+1}) \right] = \alpha_{k+1} \mathbb{E} \left[ \nabla f(x_{k+1}) \right] \]
\[
\overset{\circ}{\leq} \left( 1 - \tau_k \right) \mathbb{E} \left[ f(y_k) \right] - \mathbb{E} \left[ f(x_{k+1}) \right] + 48a^2 \rho \lambda L \left( \mathbb{E} \left[ f(x_{k+1}) \right] - \mathbb{E} \left[ f(y_k) \right] \right) + \alpha_{k+1} \mathbb{E} \left[ \nabla f(x_{k+1}) \right]
\]
\[
\text{Further,}
\]
\[
\text{where we denoted}
\]
\[
\text{We have that} \ \zeta_1 := \delta_l n, \ \zeta_2 := n^2 \delta_2.
\]
\[
\text{We define} \ \Theta := \mathbb{E}[z_0](x^*), \ R_k := \mathbb{E}[x^* - z_k]_2. \ \text{To simplify the notation, we define} \ B_l := \zeta_1 \sum_{k=0}^{l-1} \alpha_{k+1} + \Theta + \frac{\sqrt{\delta_2}}{48a^2 \rho \lambda L}.
\]
\[
\text{we obtain from (33)}
\]
\[
\text{which gives}
\]
\[
\text{Moreover,}
\]
\[
\frac{1}{2} \left( \mathbb{E}[\|z_l - x^*\|_2^2] \right)^2 \leq \frac{1}{2} \mathbb{E}[\|z_l - x^*\|_2^2] \leq \mathbb{E}[V_\xi(z_0)(x^*)] \leq B_l + \zeta_1 \sum_{k=1}^{l-1} \alpha_{k+1} R_k
\]
\[
\text{Note that} \ \alpha_1 = \frac{1}{48a^2 \rho \lambda L} \text{and therefore} \ 48a^2 \rho \lambda L \alpha_1^2 - \alpha_1 = 0.
\]
whence,

\[ R_l \leq \sqrt{2} \cdot \sqrt{B_l + \zeta_l \sum_{k=1}^{l-1} \alpha_{k+1} R_k}. \]  

(38)

Applying Lemma 12 for \( a_0 = \zeta_2 a_1^2 + \Theta + \frac{\sqrt{200}}{96 \pi \rho \|z\|_2} \), \( a_k = \zeta_2 a_1^2 \), \( b = \zeta_1 \) for \( k = 1, \ldots, N - 1 \), we obtain

\[ B_l + \zeta_l \sum_{k=1}^{l-1} \alpha_{k+1} R_k \leq \left( \sqrt{B_l} + \sqrt{2} \zeta_1 \cdot \frac{\rho}{96 \pi \rho \|z\|_2} \right)^2, \quad l = 1, \ldots, N \]  

(39)

Since \( V[\xi](x^*) \geq 0 \), by inequality (35) for \( l = N \) and the definition of \( B_l \), we have

\[ \frac{(N+1)^2}{192 \pi \rho \|z\|_2} (\mathbb{E}[f(Y_N)] - f(x^*)) \leq \left( \sqrt{B_N} + \sqrt{2} \zeta_1 \cdot \frac{N^2}{96 \pi \rho \|z\|_2} \right)^2 \leq 2B_N + 4\zeta_1^2 \cdot \frac{N^4}{96 \pi \rho \|z\|_2}, \]

\[ \leq 2\zeta_2 \sum_{k=0}^{N-1} a_{k+1}^2 + 2\Theta + \frac{\sqrt{200}}{24 \pi \rho \|z\|_2} + 4\zeta_1^2 \cdot \frac{N^4}{96 \pi \rho \|z\|_2}, \]

(40)

where \( \Box \) is due to the fact that \( \forall a, b \in \mathbb{R} \) \( (a + b)^2 \leq 2a^2 + 2b^2 \) and \( \Box \) is because \( \sum_{k=0}^{N-1} a_{k+1}^2 = \frac{1}{96 \pi \rho \|z\|_2} \sum_{k=2}^{N+1} k^2 \leq \frac{\frac{1}{6} (N+1)(N+2)(2N+3)}{96 \pi \rho \|z\|_2} \). Dividing (40) by \( \frac{(N+1)^2}{96 \pi \rho \|z\|_2} \) and substituting \( \zeta_1, \zeta_2 \)

\[ \mathbb{E}[f(Y_N)] - f(x^*) \leq \frac{38 \sqrt{200} \rho \|z\|_2}{(N+1)^2} + \frac{12 \sqrt{200}}{N^2} \zeta_1 + \frac{(N+1)\zeta_2}{24 \pi \rho \|z\|_2} + \frac{N^2 \zeta_2^2}{24 \pi \rho \|z\|_2}, \]

\[ \mathbb{E}[f(Y_N)] - f(x^*) \leq \frac{38 \sqrt{200} \rho \|z\|_2}{(N+1)^2} + \frac{12 \sqrt{200}}{N^2} \zeta_1 + \frac{(N+1)\zeta_2}{24 \pi \rho \|z\|_2} + \frac{N^2 \zeta_2^2}{24 \pi \rho \|z\|_2}. \]

\[ \Box \]

3.2. Proof Lemma 4

We start with the following technical result which connects our noisy approximation (12) of the stochastic gradient with the stochastic gradient itself and also with \( \nabla f \).

Lemma 6. For all \( x, s \in \mathbb{R}^n \), we have

\[ \mathbb{E}_\theta \| \nabla^m f(x) \|_2^2 \leq \frac{12m}{2m} \| g^m(x, \xi_m) \|_2^2 + \frac{m}{2m} \sum_{i=1}^{m} \zeta(x, \xi_i)^2 + 16 \rho \varsigma \Delta^2, \]

(41)

\[ \mathbb{E}_\theta \| \nabla^m f(x) \|_2 \geq \frac{1}{2m} \| g^m(x, \xi_m) \|_2 - \frac{1}{2m} \sum_{i=1}^{m} \zeta(x, \xi_i)^2 - 8 \varsigma^2, \]

(42)

\[ \mathbb{E}_\theta \langle \nabla^m f(x), s \rangle \geq \frac{1}{m} \langle g^m(x, \xi_m), s \rangle - \frac{1}{2m} \sum_{i=1}^{m} [\zeta(x, \xi_i)] - \frac{2\varsigma \varsigma_1 \varsigma_2}{\varsigma_3}, \]

(43)

\[ \mathbb{E}_\theta \| (\nabla f(x), e) - (\nabla^m f(x)) \|_2^2 \leq \frac{1}{2m} \| \nabla f(x) - g^m(x, \xi_m) \|_2^2 + \frac{1}{m} \sum_{i=1}^{m} \zeta(x, \xi_i)^2 + 16 \varsigma^2, \]

(44)

where \( g^m(x, \xi_m) := \frac{1}{m} \sum_{i=1}^{m} g(x, \xi_i), \zeta(x, \xi_i) \) and \( \Delta \) are defined in (3).

Proof. First of all, we rewrite \( \nabla^m f(x) \) as follows

\[ \nabla^m f(x) = \left( g^m(x, \xi_m), e \right) + \frac{1}{m} \sum_{i=1}^{m} \theta(x, \xi_i, e), \]

where

\[ \theta(x, \xi_i, e) = \zeta(x, \xi_i) + \eta(x, \xi_i, e), \quad i = 1, \ldots, m. \]
By (3), we have
\[ |θ(x, ξ_i, e)| ≤ |ζ(x, ξ_i)| + Δ_θ. \] (45)

**Proof of (45).**

\[
\|\mathbb{E}_θ[\nabla^m f(x)]\|_2 ≤ \mathbb{E}_θ\left\| \left( g^m(x, \xi_m^e), e \right) + \frac{1}{m} \sum_{i=1}^m θ(x, ξ_i, e) e \right\|_2^2
\]
\[
≤ 2\mathbb{E}_θ\left\| \left( g^m(x, \xi_m^e), e \right) e \right\|_2^2 + 2\mathbb{E}_θ\left\| \frac{1}{m} \sum_{i=1}^m θ(x, ξ_i, e) e \right\|_2^2
\]
\[
≤ \frac{12m}{\eta} \|g^m(x, \xi_m^e)\|^2_2 + \frac{3m}{\eta} \left( |ζ(x, ξ_i)| + Δ_θ \right)^2 ≤ \frac{12m}{\eta} \|g^m(x, \xi_m^e)\|^2_2 + \frac{m}{\eta} \sum_{i=1}^m |ζ(x, ξ_i)|^2 + 16Δ_θ^2,
\]
where (1) holds since \(|x + y|^2_2 ≤ 2\|x\|^2_2 + 2\|y\|^2_2\), \(∀x, y \in \mathbb{R}^n\); (2) follows from inequalities (10), (11), (45) and the fact that, for any \(a_1, a_2, \ldots, a_m > 0\), it holds that \(\left( \sum_{i=1}^m a_i^2 \right)^2 ≤ m \sum_{i=1}^m a_i^2\).

**Proof of (46).**

\[
\|\mathbb{E}_θ[\nabla^m f(x)]\|_2^2 = \mathbb{E}_θ\left\| \left( g^m(x, \xi_m^e), e \right) + \frac{1}{m} \sum_{i=1}^m θ(x, ξ_i, e) e \right\|_2^2
\]
\[
≥ \frac{1}{m} \mathbb{E}_θ\|g^m(x, \xi_m^e), e\|_2^2 - \frac{1}{m} \sum_{i=1}^m \left( |ζ(x, ξ_i)| + Δ_θ \right)^2 ≥ \frac{1}{m} \mathbb{E}_θ\|g^m(x, \xi_m^e)\|_2^2 - \frac{1}{m} \sum_{i=1}^m |ζ(x, ξ_i)|^2 - 8Δ_θ^2,
\]
where (1) follows from (45) and inequality \(|x + y|^2_2 ≥ \frac{1}{2}\|x\|^2_2 - \|y\|^2_2\), \(∀x, y \in \mathbb{R}^n\); (2) follows from \(e \in S_2(1)\) and Lemma B.10 in [9], stating that, for any \(e \in \mathbb{R}^n\), \(\mathbb{E}(s, e)^2 = \frac{1}{2}\|s\|^2_2\).

**Proof of (47).**

\[
\mathbb{E}_θ[\nabla^m f(x), s] = \mathbb{E}_θ(\langle g^m(x, \xi_m^e), e \rangle, s, e, s) + \mathbb{E}_θ \frac{1}{m} \sum_{i=1}^m θ(x, ξ_i, e)(e, s)
\]
\[
≥ \frac{1}{m} \langle g^m(x, \xi_m^e), s \rangle - \frac{1}{m} \sum_{i=1}^m \left( |ζ(x, ξ_i)| + Δ_θ \right) \mathbb{E}_θ(e, s)
\]
\[
≥ \frac{1}{m} \langle g^m(x, \xi_m^e), s \rangle - \frac{1}{m} \|g^m(x, \xi_m^e)\|^2_2 - \frac{1}{m} \sum_{i=1}^m |ζ(x, ξ_i)|^2 - 2Δ^2\|e\|^2_2,
\]
where (1) follows from \(\mathbb{E}_θ[\|g\|_2^2] = g, ∀g \in \mathbb{R}^n\) and (45); (2) follows from Lemma B.10 in [9], since \(\mathbb{E}(s, e)^2 ≤ \mathbb{E}(s, e)^2\), and the fact that \(|\|x\|_2^2 ≤ \|x\|_p^2\) for \(p ≤ 2\).

**Proof of (48).**

\[
\|\mathbb{E}_θ[\nabla^m f(x) - \nabla^m f(x)]\|_2^2 = \mathbb{E}_θ\left\| \left( \nabla^m f(x), e - \nabla^m f(x) - θ(x, ξ_i, e) e \right) \right\|_2^2
\]
\[
≤ 2\mathbb{E}_θ\left\| \left( \nabla^m f(x) - g^m(x, \xi_m^e) \right) e \right\|_2^2 + 2\mathbb{E}_θ\left\| \frac{1}{m} \sum_{i=1}^m θ(x, ξ_i, e) e \right\|_2^2
\]
\[
≤ \frac{2}{\eta} \|\nabla f(x) - g^m(x, \xi_m^e)\|_2^2 + \frac{m}{\eta} \sum_{i=1}^m |ζ(x, ξ_i)|^2 + 16Δ_θ^2,
\]
where (1) holds since \(|x + y|^2_2 ≤ 2\|x\|^2_2 + 2\|y\|^2_2\), \(∀x, y \in \mathbb{R}^n\); (2) follows from \(e \in S_2(1)\) and Lemma B.10 in [9], and (45).

We continue by proving the following lemma which estimates the progress in step 7 of ARDD, which is a gradient step.

**Lemma 7.** Assume that \(y = x - \frac{1}{2\eta} \nabla^m f(x)\). Then,
\[
\|g^m(x, \xi_m^e)\|^2_2 ≤ 8nL_2^2(f(x) - \mathbb{E}_θ f(y)) + 8\|\nabla f(x) - g^m(x, \xi_m^e)\|^2_2 + \frac{2m}{\eta} \sum_{i=1}^m |ζ(x, ξ_i)|^2 + 80nΔ_θ^2,
\]
(50)

where \(g^m(x, \xi_m^e)\) is defined in Lemma [6], \(ζ(x, ξ_i)\) and \(Δ_θ\) are defined in [3].
From this and Lemma 8.

\[\langle \nabla f(x), y - x \rangle = \langle \nabla f(x), e \rangle y = \langle \nabla f(x), e \rangle (e, y - x) = \langle (\nabla f(x), e \rangle, y - x \rangle.\]

From this and \(L_2\)-smoothness of \(f\) we obtain

\[f(y) \leq f(x) + \langle (\nabla f(x), e \rangle, y - x \rangle + \frac{L_2}{2} ||y - x||^2 \]

\[\leq f(x) + \langle \nabla^m f(x), y - x \rangle + L_2 ||y - x||^2 + \langle (\nabla f(x), e \rangle, e - \nabla^m f(x), y - x \rangle - \frac{L_2}{2} ||y - x||^2 \]

\[\leq f(x) + \langle \nabla^m f(x), y - x \rangle + L_2 ||y - x||^2 + \frac{1}{\sqrt{L_2}} ||(\nabla f(x), e \rangle, e - \nabla^m f(x)||^2,\]

where \(1\) follows form the Fenchel inequality \((s, z) - \frac{1}{2} ||z||^2 \leq \frac{1}{2} ||s||^2\). Using \(y = x - \frac{1}{\sqrt{L_2}} \nabla^m f(x)\), we get

\[\frac{1}{\sqrt{L_2}} ||\nabla^m f(x)||^2 \leq f(x) - f(y) + \frac{1}{\sqrt{L_2}} ||(\nabla f(x), e \rangle, e - \nabla^m f(x)||^2 \]

Taking the expectation in \(e\) and applying \(42, 44\), we obtain

\[\frac{1}{\sqrt{L_2}} \left( \frac{1}{2} ||g^m(x, \xi_m)||_2^2 - \frac{1}{m} \sum_{i=1}^m \zeta(x, \xi, \xi)^2 - 8\Delta \right) \leq \frac{1}{\sqrt{L_2}} \mathbb{E}_e ||\nabla^m f(x)||^2 \]

\[\leq f(x) - \mathbb{E}_e(f(y) + \frac{1}{\sqrt{L_2}} \mathbb{E}_e ||(\nabla f(x), e \rangle, e - \nabla^m f(x)||^2) \]

\[\leq f(x) - \mathbb{E}_e(f(y) + \frac{1}{\sqrt{L_2}} \left( \mathbb{E}[\|\nabla f(x) - g^m(x, \xi_m)\|_2^2] + \frac{1}{m} \sum_{i=1}^m \zeta(x, \xi, \xi)^2 + 16\Delta \right).\]

Rearranging the terms, we obtain the statement of the lemma.

\[\blacksquare\]

**Proof of Lemma 3.** Taking the expectation w.r.t. all randomness of (23) and using inequality

\[\mathbb{E}[|\zeta(x, \xi, \xi)|] \leq \sqrt{\mathbb{E}[|\zeta(x, \xi, \xi)|^2]} \leq \sqrt{\Delta},\]

we obtain inequality (24) with \(\delta_1 = \frac{\Delta}{2\sqrt{m}} + \frac{\Delta}{\sqrt{m}}\). Combining (41) and (50), taking the full expectation and using \(\mathbb{E}[\|\nabla f(x) - g^m(x, \xi_m)\|_2^2] \leq \frac{\rho_m}{m}\), which follows from \(2\), we obtain (25) with \(\delta_2 = \frac{9\rho_m}{n} + \frac{\rho_m}{m} + 61\rho_n \Delta + 976\rho_n \Delta^2\). \(\blacksquare\)

### 4. Proof of main result for RDD method

As in the previous section, we divide the proof of Theorem 2 into large steps. First, to simplify the derivations, we prove this theorem assuming two additional inequalities which connect or noisy stochastic approximation of the gradient (12) with the true gradient and function values. Then we show that our approximation of the gradient (12) indeed satisfies these two inequalities.

**Lemma 8.** Let \(\{x_k, y_k, z_k\}, k \geq 0\) be generated by RDD method. Assume that there exist numbers \(\delta_1 > 0, \delta_2 > 0\) such that, for all \(k \geq 0\)

\[\mathbb{E}\left[\|\nabla^m f(x_k), x_k - x_s\|\right] \geq \frac{1}{n} \mathbb{E}\left[\langle \nabla f(x_k), x_k - x_s\rangle\right] - \delta_1 \mathbb{E}\left[\|x_k - x_s\|\right]\]

\[\mathbb{E}\left[\langle \nabla^m f(x_k)\|_2^2\right] \leq \frac{4\rho_nL_2}{n} \left(\mathbb{E}[f(x_k)] - f(x_s)\right) + \delta_2,\]

where expectation is taken w.r.t. all randomness and \(x_s\) is a solution to (1). Then

\[\mathbb{E}[f(x_N)] - f(x_s) \leq \frac{384\rho_nL_3\Theta_N}{N} + \frac{n}{128L_2} \delta_2 + \frac{8n\sqrt{2\rho_n}}{N} \delta_1 + \frac{nN}{32\rho_n} \delta_1^2,\]

where \(\Theta_p = V[1, 10](x^*)\) is defined by the chosen proximal setup and the expectation is taken w.r.t. all randomness.

\(\footnote{Note that we use \(s = z_k - x\), which does not depend on \(x_1, x_2, \ldots, x_n\) from the \((k + 1)\)-th iterate and it does not depend on \(e_k\). Therefore we can use tower property of mathematical expectation and take firstly conditional expectation w.r.t. \(x_1, \ldots, x_n\) and after that take full expectation.}\)
This result is proved below in subsection 4.1.

**Lemma 9.** Let \( \{x_k, y_k, z_k\} \), \( k \geq 0 \) be generated by RDD method. Then (51) and (52) hold with

\[
\delta_1 = \sqrt{\frac{\alpha^2}{2}} + \frac{2\Delta_\eta}{\sqrt{n}}
\]

and

\[
\delta_2 = \frac{24\rho_n}{n} + \frac{\sigma^2}{m} + \rho_n \Delta c + 16\rho_n \Delta^2_v.
\]

This result is proved below in subsection 4.2.

**Proof of Theorem 2.** Combining Lemma 8 and Lemma 9, we obtain (14).

**4.1. Proof of Lemma 8**

Combining (29), (51) and (52) we get

\[
\alpha E[\nabla f(x_k), x_k - x_\ast] \leq 24\alpha^2 \rho_n L_2 (E[f(x_k)] - f(x_\ast)) + \alpha \delta_1 n E[\|x_k - x_\ast\|_p] + \frac{\alpha^2 \rho_n}{2} \delta_2
\]

whence due to convexity of \( f \) we have

\[
\frac{(\alpha - 24\alpha^2 \rho_n L_2)}{\delta_2} (E[f(x_k)] - f(x_\ast)) \leq \alpha \delta_1 n E[\|x_k - x_\ast\|_p] + \frac{\alpha^2 \rho_n}{2} \delta_2
\]

because \( \alpha = \frac{1}{48\rho_n L_2} \). Summing (56) for \( k = 0, \ldots, l - 1 \), where \( l \leq N \) we get

\[
0 \leq \frac{N\alpha}{4} (E[f(x_l)] - f(x_\ast)) \leq \frac{\alpha^2 \rho_n}{2} \delta_2 + \alpha \delta_1 n \sum_{k=0}^{l-1} E[\|x_k - x_\ast\|_p] + V[x_0](x_\ast) - E[V[x_\ast](x_\ast)],
\]

where \( x_l \overset{\text{def}}{=} \frac{1}{l} \sum_{k=0}^{l-1} x_k \). From the previous inequality we get

\[
\frac{1}{2} (E[\|x_l - x_\ast\|_p])^2 \leq \frac{1}{2} E[\|x_l - x_\ast\|_p^2] \leq E[V[x_\ast](x_\ast)]
\]

\[
\leq \Theta_p + l \cdot \frac{\alpha^2 \rho_n}{2} \delta_2 + \alpha \delta_1 n \sum_{k=0}^{l-1} E[\|x_k - x_\ast\|_p],
\]

whence \( \forall l \leq N \) we obtain

\[
E[\|x_k - x_\ast\|_p] \leq \sqrt{2} \sqrt{\Theta_p + l \cdot \frac{\alpha^2 \rho_n}{2} \delta_2 + \alpha \delta_1 n \sum_{k=0}^{l-1} E[\|x_k - x_\ast\|_p]}.
\]

Denote \( R_k = E[\|x^* - x_k\|_p] \) for \( k = 0, \ldots, N \). Applying Lemma 13 for \( a_0 = \Theta_p + \alpha \delta_1 n E[\|x_0 - x_\ast\|_p] \leq \Theta_p + an \sqrt{2\Theta_p \delta_1} \), \( a_k = \frac{\alpha^2 \rho_n}{2} \delta_2 \), \( b = n\delta_1 \) for \( k = 1, \ldots, N - 1 \) we have for \( l = N \)

\[
\frac{N\alpha}{4} (E[f(x_N)] - f(x_\ast)) \leq \left( \sqrt{\Theta_p + N \cdot \frac{\alpha^2 \rho_n}{2} \delta_2 + an \sqrt{2\Theta_p \delta_1} \sqrt{N} + \sqrt{2n} \delta_1 aN} \right)^2
\]

\[
\leq 2\Theta_p + Na^2 n^2 \delta_2 + 2an \sqrt{2\Theta_p \delta_1} + 4n^2 \delta_1^2 a^2 N^2,
\]

whence

\[
E[f(x_N)] - f(x_\ast) \leq \frac{384\rho_n L_2 a_0}{N} + \frac{an \sqrt{2\Theta_p \delta_1}}{N} \delta_2 + \frac{8n \sqrt{2\Theta_p \delta_1}}{N} \delta_1 + \frac{2nN \delta_1^2}{M \rho_n \delta_1},
\]

because \( \alpha = \frac{1}{48\rho_n L_2} \).
4.2. Proof Lemma

Taking mathematical expectation w.r.t. all randomness from the \(43\) we obtain inequality \(51\) with \(\delta_1 = \frac{\sqrt{N}}{2\sqrt{m}} + \frac{\lambda N}{2\sqrt{m}}\), because \(\mathbb{E}[\ell(x, \xi)] \leq \sqrt{\mathbb{E}[(\ell(x, \xi))^2]} \leq \sqrt{\Delta}.\) Combining \(41\) and
\[
\|g^\eta(x, \bar{\xi}_m)\|_2^2 \leq 2\|\nabla f(x)\|_2^2 + 2\|\nabla f(x) - g^\eta(x, \bar{\xi}_m)\|_2^2 \leq 4L_2 (\mathbb{E}[f(x)] - f(x_*)) + 2\|\nabla f(x) - g^\eta(x, \bar{\xi}_m)\|_2^2,
\]
and taking full mathematical expectation we obtain \(52\) with \(\delta_2 = \frac{2\lambda N}{m} + \frac{\rho_n}{m} \Delta + 16\rho_n \Delta^2_n\).

5. Proofs for strongly convex problems

5.1. Accelerated algorithm

**Lemma 10.** Assume that we start ARDD Algorithm \(7\) from a random point \(x_0\) such that \(\mathbb{E}[\xi_\|x_* - x_0\|_p^2] \leq R_p^2\), use the function \(R_p^2d\left(\frac{x - x_0}{R_p}\right)\) as the prox-function and run ARDD for \(N_0\) iterations. Then
\[
\mathbb{E}[(f(y_N)) - f^*] \leq \frac{aL_2R_p^2\Omega_p}{N_0} + \frac{b\rho_n^2N_0}{mL_2} + \Delta,
\]
where \(a = 384n^2\rho_n, b = \frac{4}{\eta}, \Delta = \frac{61N_0}{2m_\mathcal{L}}\Delta + \frac{122N_0}{2m_\mathcal{L}} \Delta^2 + \frac{12}{m_\mathcal{L}} \left(\frac{\sqrt{N}}{2} + 2\Delta\right) + \frac{N_0}{12m_\mathcal{L}} \left(\frac{\sqrt{N}}{2} + 2\Delta\right)^2\) and the expectation is taken with respect to all the randomness.

**Proof.** Note that \(R_p^2d\left(\frac{x - x_0}{R_p}\right)\) is strongly convex with constant 1 w.r.t \(\|\cdot\|_p\). Since \(0 = \arg\min d(x)\), we have, for the prox-function \(d(x) = R_p^2d\left(\frac{x - x_0}{R_p}\right)\) and corresponding Bregman divergence \(\mathcal{V}[x_0](x)\),
\[
\Theta_p = \mathcal{V}[x_0](x) = \tilde{d}(x) - \tilde{d}(x_0) = \langle \nabla \tilde{d}(x_0), x_* - x_0 \rangle = \tilde{d}(x_0) \leq R_p^2\Omega_p \frac{N_0}{2}.
\]
Applying Theorem \(1\) an taking additional expectation w.r.t to \(x_0\), we finish the proof of the lemma.

**Proof of Theorem 3** We prove by induction that
\[
\mathbb{E}\|x_k - x_*\|_p \leq R_k^2 = R_p^2 - 2^{-k} + \frac{4\Delta}{\mu_p} \left(1 - 2^{-k}\right), \tag{60}
\]
For \(k = 0\), this inequality obviously holds. Let us assume that it holds for some \(k \geq 0\) and prove the induction step. Applying Lemma \(10\) at the step \(k\) of Algorithm \(3\) we obtain that
\[
\mathbb{E}f(u_{k+1}) - f^* = \mathbb{E}f(y_N) - f^* \leq \frac{aL_2R_k^2\Omega_p}{N_0^2} + \frac{b\rho_n^2N_0}{mL_2} + \Delta.
\]
By definition of \(N_0\), we have
\[
\frac{aL_2R_k^2\Omega_p}{N_0^2} = \frac{aL_2R_k^2\Omega_p}{8L_2\Omega_p \mu_p} = \frac{\mu_p R_k^2}{8}.
\]
\(\text{Note that we use } s = x_k - x_0\), which does not depend on \(\xi_1, \xi_2, \ldots, \xi_n\) from the \((k + 1)\)-th iterate and it does not depend on \(e_k\). Therefore we can use tower property of mathematical expectation and take firstly conditional expectation w.r.t. \(\xi_1, \ldots, \xi_n\) and after that take full expectation.
By definition of $m_k$, we have
\[ m_k \geq \frac{8b\sigma^2 N_0}{L_2 \mu_p R_k^2 2^{-k}} \geq \frac{8b\sigma^2 N_0}{L_2 \mu_p} \left( \frac{R_k^2}{2^{-k}} + \frac{4\Delta}{\mu_p} (1 - 2^{-k}) \right) = \frac{8b\sigma^2 N_0}{L_2 \mu_p R_k^2} \]
and
\[ \frac{b\sigma^2 N_0}{m_k L_2} \leq \frac{b\sigma^2 N_0}{L_2 \frac{8b\sigma^2 N_0}{L_2 \mu_p R_k^2}} = \frac{\mu_p R_k^2}{8}. \]
Hence,
\[ \mathbb{E}(u_{k+1}) - f' \leq \frac{\mu_p R_k^2}{4} + \Delta = \frac{\mu_p}{4} \left( \frac{R_k^2}{2^{-k}} + \frac{4\Delta}{\mu_p} (1 - 2^{-k}) \right) + \Delta \geq \frac{\mu_p}{2} \left( \frac{R_k^2}{2^{-k}} + \frac{4\Delta}{\mu_p} (1 - 2^{-k}) \right) = \frac{\mu_p R_k^2}{2}. \]
Since $f$ is strongly convex, we have
\[ \mathbb{E}[\|u_{k+1} - x^*\|^2] \leq \frac{2}{\mu_p} \left( \mathbb{E}(u_{k+1}) - f' \right) \leq \frac{R_k^2}{2}. \]
This finishes the induction step and, as a byproduct, we obtain inequality (18).

It remains to estimate the complexity. To make the right hand side of (18) smaller than $\varepsilon$ it is sufficient to choose $K = \lceil \log_2 \frac{\mu_p R_k^2}{\varepsilon} \rceil$. To estimate the total number of oracle calls, we write
\[
\text{Number of calls} = \sum_{k=0}^{K-1} N_0 m_k = \sum_{k=0}^{K-1} N_0 \left( 1 + \frac{8b\sigma^2 N_0 2^k}{L_2 \mu_p R_k^2} \right) \leq K N_0 + \frac{8b\sigma^2 N_0 2^K}{L_2 \mu_p R_k^2} \leq \frac{8a L_2 \Omega_p}{\mu_p} \log_2 \frac{\mu_p R_k^2}{\varepsilon} + \frac{8b\sigma^2}{L_2 \mu_p} \frac{8a L_2 \Omega_p}{\mu_p} \frac{\mu_p R_k^2}{\varepsilon} \leq \tilde{O} \left( \max \left\{ \log_2 \left( \frac{\mu_p R_k^2}{\varepsilon} \right), \frac{n^2 \|x^* - x_0\|^2}{\mu_p \varepsilon} \right\} \right),
\]
where we used that $a = 384n^2 \rho_n$, $b = \frac{4}{\varepsilon}$ and $\rho_n$ is given in Lemma 11.

5.2. Non-accelerated algorithm

**Lemma 11.** Assume that we start RDD Algorithm 1 from a random point $x_0$ such that $\mathbb{E}[\|x^* - x_0\|^2] \leq R_p^2$, use the function $R_p^2 d \left( \frac{x-x_0}{\|x-x_0\|} \right)$ as the prox-function and run RDD for $N_0$ iterations. Then
\[
\mathbb{E}[f(y_{N_0})] - f^* \leq \frac{a L_2 \Omega_p}{N_0} R_p^2 + \frac{b\sigma^2}{m L_2} + \Delta,
\]
where $a = 192 n \rho_n$, $b = 2$, $\Delta = \frac{n}{2L_2} \Delta_x + \frac{8}{mL_2} \Delta_{\eta} + \frac{8\sqrt{2} R_k^2}{n \rho_n} \left( \frac{\sqrt{n}}{2} + 2\Delta_{\eta} \right) + \frac{N_0}{4m \rho_n} \left( \frac{\sqrt{n}}{2} + 2\Delta_{\eta} \right)^2$ and the expectation is taken with respect to all the randomness.

**Proof.** Note that $R_p^2 d \left( \frac{x-x_0}{\|x-x_0\|} \right)$ is strongly convex with constant 1 w.r.t $\| \cdot \|_p$. Since $0 = \arg \min d(x)$, we have, for the prox-function $\tilde{d}(x) = R_p^2 d \left( \frac{x-x_0}{\|x-x_0\|} \right)$ and corresponding Bregman divergence $\tilde{V}(x_0)(x)$,
\[
\Theta_p = \tilde{V}(x_0)(x) = \tilde{d}(x) - \tilde{d}(x_0) - \langle \nabla \tilde{d}(x_0), x_0 - x_0 \rangle = \tilde{d}(x) - \tilde{d}(x_0) \leq \frac{R_p^2 \Omega_p}{2}.
\]
Applying Theorem 2 and taking additional expectation w.r.t to $x_0$, we finish the proof of the lemma.
Proof of Theorem \[\text{We prove by induction that}
\]
\[
E\|u_k - x^*\|_p^2 \leq R_k^2 = R_p^{2-2^{-k}} + \frac{4\Delta}{\mu_p} \left(1 - 2^{-k}\right).
\]
(61)

For \(k = 0\), this inequality obviously holds. Let us assume that it holds for some \(k \geq 0\) and prove the induction step. Applying Lemma [\ref{lem:induction_step}] at the step \(k\) of Algorithm [\ref{alg:main_algorithm}] we obtain that
\[
E(f(u_{k+1}) - f^* - Ef(y_{k+1}) - f^*) \leq \frac{aL_2^2\Omega_p}{N_0} + \frac{b\sigma^2}{m_kL_2} + \Delta.
\]

By definition of \(N_0\), we have
\[
\frac{aL_2^2\Omega_p}{N_0} \leq \frac{aL_2^2\Omega_p}{8\mu_p^2} = \frac{\mu_p R_k^2}{8}.
\]

By definition of \(m_k\), we have
\[
m_k \geq \frac{8b\sigma^2}{L_2\mu_p R_p^{2-2^{-k}}} \geq \frac{8b\sigma^2}{L_2\mu_p R_p^{2-2^{-k}}} \left(\frac{4\Delta}{\mu_p} \left(1 - 2^{-k}\right)\right) = \frac{8b\sigma^2}{L_2\mu_p R_k^2}.
\]

and
\[
\frac{b\sigma^2}{m_kL_2} \leq \frac{b\sigma^2}{L_2\mu_p R_k^2} = \frac{\mu_p R_k^2}{8}.
\]

Hence,
\[
E(f(u_{k+1}) - f^*) \leq \frac{\mu_p R_k^2}{4} + \Delta = \frac{\mu_p}{4} \left(R_p^{2-2^{-k}} + \frac{4\Delta}{\mu_p} \left(1 - 2^{-k}\right)\right) + \Delta = \frac{\mu_p}{2} \left(R_p^{2-2^{-k}} + \frac{4\Delta}{\mu_p} \left(1 - 2^{-k}\right)\right) = \frac{\mu_p R_k^2}{2}.
\]

Since \(f\) is strongly convex, we have
\[
E\|u_{k+1} - x^*\|_p^2 \leq \frac{2}{\mu_p} (Ef(u_{k+1}) - f^*) \leq R_k^2.
\]

This finishes the induction step and, as a byproduct, we obtain inequality [\ref{ineq:induction_end}] smaller than \(\varepsilon\) it is sufficient to choose \(K = \left\lceil \log \frac{\mu_p R_k^2}{\varepsilon} \right\rceil\). To estimate the total number of oracle calls, we write

Number of calls = \[\sum_{k=0}^{K-1} N_0 m_k \leq \sum_{k=0}^{K-1} N_0 \left(1 + \frac{8b\sigma^2 2^k}{L_2\mu_p R_k^2}\right) \leq KN_0 + \frac{8b\sigma^2 N_0 2^K}{L_2\mu_p R_k^2} \]
\[
\leq \frac{8aL_2\Omega_p}{\mu_p} \log_2 \frac{\mu_p R_k^2}{\varepsilon} + \frac{8b\sigma^2}{L_2\mu_p R_k^2} \frac{8aL_2\Omega_p}{\mu_p} \frac{\mu_p R_k^2}{\varepsilon}
\]
\[
\leq \frac{8aL_2\Omega_p}{\mu_p} \log_2 \frac{\mu_p R_k^2}{\varepsilon} + \frac{64b\sigma^2\Omega_p}{\mu_p\varepsilon} = \tilde{O} \left(\max \left\{ \frac{n\hat{L}_2\Omega_p}{\mu_p \log_2 \frac{\mu_p R_k^2}{\varepsilon}}, \frac{n\hat{\sigma}^2\Omega_p}{\mu_p \varepsilon} \right\} \right),
\]

where we used that \(a = 192\nu_0\), \(b = 2\) and \(\rho_0\) is given in Lemma [\ref{lem:rho0}].

\[\square\]

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Appendix A. Technical Results

Lemma 12. Let \(a_0, \ldots, a_{N-1}, b, R_1, \ldots, R_{N-1}\) be non-negative numbers such that

\[
R_l \leq \sqrt{2} \cdot \left( \sum_{k=0}^{l-1} a_k + b \sum_{k=1}^{l-1} a_{k+1} R_k \right) \quad l = 1, \ldots, N, \tag{A.1}
\]

where \(a_{k+1} = \frac{k+2}{96\rho_n^2 L_2} c(k)\) for all \(k \in \mathbb{N}\). Then for \(l = 1, \ldots, N\)

\[
\sum_{k=0}^{l-1} a_k + b \sum_{k=1}^{l-1} a_{k+1} R_k \leq \left( \sum_{k=0}^{l-1} a_k + \sqrt{2}b \cdot \frac{\rho}{96\rho_n^2 L_2} \right)^2. \tag{A.2}
\]

Proof. For \(l = 1\) it is trivial inequality. Assume that (A.2) holds for some \(l < N\) and prove it for \(l + 1\). From the induction assumption and (A.1) we obtain

\[
R_l \leq \sqrt{2} \left( \sum_{k=0}^{l-1} a_k + \sqrt{2}b \cdot \frac{\rho}{96\rho_n^2 L_2} \right), \tag{A.3}
\]

whence

\[
\sum_{k=0}^{l} a_k + b \sum_{k=1}^{l} a_{k+1} R_k = \sum_{k=0}^{l-1} a_k + b \sum_{k=1}^{l-1} a_{k+1} R_k + a_l + b a_{l+1} R_l \leq \left( \sum_{k=0}^{l-1} a_k + \sqrt{2}b \cdot \frac{\rho}{96\rho_n^2 L_2} \right)^2 + a_l + \sqrt{2}b a_{l+1} R_l
\]

\[
= \sum_{k=0}^{l} a_k + 2 \sum_{k=0}^{l-1} a_k \cdot \sqrt{2}b \cdot \frac{\rho}{96\rho_n^2 L_2} + \left( \frac{\rho}{96\rho_n^2 L_2} + a_{l+1} \right) + 2b^2 \left( \frac{\rho}{96\rho_n^2 L_2} \right)^2 + \left( \frac{\rho}{96\rho_n^2 L_2} \right)^2 \leq \sum_{k=0}^{l} a_k + 2 \sum_{k=0}^{l-1} a_k \cdot \sqrt{2}b \cdot \frac{\rho}{96\rho_n^2 L_2} + 2b^2 \left( \frac{\rho}{96\rho_n^2 L_2} \right)^2 = \left( \sum_{k=0}^{l} a_k + \sqrt{2}b \cdot \frac{\rho}{96\rho_n^2 L_2} \right)^2,
\]

where (i) follows from the induction assumption and (A.3), (ii) is because \(\sum_{k=0}^{l-1} a_k \leq \sum_{k=0}^{l} a_k\) and

\[
\frac{\rho}{96\rho_n^2 L_2} + \frac{a_{l+1}}{96\rho_n^2 L_2} \leq \frac{l^2 + l^2}{96\rho_n^2 L_2} \leq \sum_{k=0}^{l} a_k + \sqrt{2}b \cdot \frac{\rho}{96\rho_n^2 L_2} \leq \frac{(l+1)^2}{96\rho_n^2 L_2}.
\]

\(\Box\)

Lemma 13. Let \(a_0, \ldots, a_{N-1}, b, R_1, \ldots, R_{N-1}\) be non-negative numbers such that

\[
R_l \leq \sqrt{2} \cdot \left( \sum_{k=0}^{l-1} a_k + b a \sum_{k=1}^{l-1} R_k \right) \quad l = 1, \ldots, N. \tag{A.4}
\]

Then for \(l = 1, \ldots, N\)

\[
\sum_{k=0}^{l-1} a_k + b a \sum_{k=1}^{l-1} R_k \leq \left( \sum_{k=0}^{l-1} a_k + \sqrt{2}b a \right)^2. \tag{A.5}
\]
Proof. For $l = 1$ it is trivial inequality. Assume that (A.3) holds for some $l < N$ and prove it for $l + 1$. From the induction assumption and (A.4) we obtain

$$R_l \leq \sqrt{2} \left( \sum_{k=1}^{l-1} a_k + \sqrt{2}b\alpha l \right), \quad (A.6)$$

whence

$$\sum_{k=1}^l a_k + ba \sum_{k=1}^l R_k = \left( \sum_{k=0}^{l-1} a_k + \sqrt{2}b\alpha l \right)^2$$

$$\leq \left( \sum_{k=0}^{l-1} a_k + \sqrt{2}b\alpha l \right)^2 + a\alpha + \sqrt{2}ba \left( \sum_{k=0}^{l-1} a_k + \sqrt{2}b\alpha l \right)$$

$$= \sum_{k=0}^l a_k + 2 \sqrt{\sum_{k=0}^{l-1} a_k \cdot \sqrt{2}b\alpha (l+1) + 2b^2\alpha^2(l+1)^2} = \left( \sum_{k=0}^{l-1} a_k + \sqrt{2}b\alpha (l+1) \right)^2,$$

where $\blacklozenge$ follows from the induction assumption and (A.6), $\blacklozenge$ is because $\sum_{k=0}^{l-1} a_k \leq \sum_{k=0}^l a_k$. \hfill $\blacksquare$

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