Boundary One-Point Functions, Scattering Theory and Vacuum Solutions in Integrable Systems

V. A. Fateev and E. Onofri

Abstract

Integrable boundary Toda theories are considered. We use boundary one-point functions and boundary scattering theory to construct the explicit solutions corresponding to classical vacuum configurations. The boundary ground state energies are conjectured.

*Laboratoire de Physique Mathématique, Université Montpellier II, Pl. E.Bataillon, 34095 Montpellier, France.
†On leave of absence from Landau Institute for Theoretical Physics, ul.Kosygina 2, 117940 Moscow, Russia.
‡Dipartimento di Fisica, Università di Parma, and I.N.F.N., Gruppo Collegato di Parma, 43100 Parma, Italy
1 Introduction

There is a large class of 2D quantum field theories (QFTs) which can be considered as perturbed conformal field theories (CFTs). These theories are completely defined if one specifies its CFT data and the relevant operator which plays the role of perturbation. The CFT data contain explicit information about the ultraviolet (UV) asymptotics of the field theory, while the long distance behavior requires careful analysis. If a perturbed CFT contains only massive particles it is equivalent to a relativistic scattering theory and it is completely defined by specifying the S-matrix. Contrary to CFT data, the S-matrix data exhibit information about the long distance properties of the theory in an explicit way while the UV asymptotics have to be derived.

A link between these two kinds of data would provide a good viewpoint for understanding the general structure of 2D QFT. In general this problem does not look tractable. Whereas the CFT data can be specified in a relatively simple way the general S-matrix is a rather complicated object even in two dimensions. However, there exists a rather important class of 2D QFTs (integrable theories) where the scattering theory is factorized and the S-matrix can be described in great detail. These theories admit a rather complete description in the UV and IR regimes.

In this paper we consider the application of the boundary one-point functions and boundary scattering theory to the explicit construction of the classical solutions describing the boundary vacuum configurations. As an example of an integrable QFT we study the simply laced affine Toda theory (ATT), which can be considered as a perturbed CFT (a non-affine Toda theory). This CFT possesses an extended symmetry generated by a $W$-algebra. We consider the boundary conditions that preserve this symmetry. This permits us to apply the “reflection amplitude” approach for the calculation of the boundary vacuum expectation values (VEVs) of exponential fields in ATTs. The reflection amplitudes in a CFT define the linear transformations between different exponential fields corresponding to the same primary field of the full symmetry algebra of theory. They play a crucial role in the calculation of VEVs of the exponential fields in perturbed CFTs as well as for the analysis of UV asymptotics of the observables in these QFTs. The boundary VEVs, in particular, contain the information about the boundary values of the solutions of classical boundary ATTs equations. The information about the long distance behavior of these solutions can be extracted from the boundary $S$-matrix. The explicit solutions which are constructed in this paper provide us the consistency check of CFT and $S$-matrix data in integrable boundary ATTs.

The plan of the paper is as follows: in section we recall some basic facts about Toda theories and one-point functions in ATTs defined on the whole plane. In section we consider integrable boundary ATTs. The boundary one-point functions are used to derive the boundary values of the solutions describing vacuum configurations in ATTs. We calculate the classical boundary ground state energies and give the conjecture for them in the quantum case. In section we describe the boundary scattering theory, which is consistent with this conjecture, and construct the boundary state. The vacuum solutions describe the semiclassical asymptotics of the one-point
functions of the bulk operators in boundary QFTs. This makes it possible to derive the long distance asymptotics of vacuum solutions. These asymptotics determine completely the explicit form of solutions which we construct in section 5. The solutions can be written in terms of \( \tau \)-functions, associated with multisoliton solutions of ATTs. In the last section we consider the opposite (dual) limit of the quantum correlation function of the bulk field with a boundary and check that its boundary value coincides with that given by boundary VEV. At the end of the section we consider the solutions for non-simply laced ATTs which can be obtained by reduction from the solutions for simply laced cases.

2 Affine and Non-Affine Toda Theories

The ATT corresponding to a Lie algebra \( G \) of rank \( r \) is described by the action:

\[
\mathcal{A} = \int d^2x \left[ \frac{1}{8\pi} (\partial_\mu \varphi)^2 + \mu \sum_{i=1}^{r} e^{be_i \cdot \varphi} + \mu e^{be_0 \cdot \varphi} \right],
\]

where \( e_i, i = 1, ..., r \) are the simple roots of \( G \) and \(-e_0\) is a maximal root:

\[
e_0 + \sum_{i=1}^{r} n_i e_i = 0.
\]

The fields \( \varphi \) in Eq. (1) are normalized so that at \( \mu = 0 \)

\[
\langle \varphi_a(x) \varphi_b(y) \rangle = -\delta_{ab} \log |x - y|^2
\]

We will consider later the simply laced Lie algebras \( A, D, E \).

For real \( b \) the spectrum of these ATTs consists of \( r \) particles which can be associated with the nodes of the Dynkin diagram (or fundamental representations) of \( G \). The masses of these particles \( m_i \) \( (i = 1, ..., r) \) are given by:

\[
m_i = m\nu_i, \quad m^2 = \frac{1}{2h} \sum_{i=1}^{r} m_i^2,
\]

where \( h \) is the Coxeter number of \( G \) and \( \nu_i^2 \) are the eigenvalues of the mass matrix:

\[
M_{ab} = \sum_{i=1}^{r} n_i (e_i)^a (e_i)^b + (e_0)^a (e_0)^b.
\]

The exact relation between the parameter \( m \) characterizing the spectrum of physical particles and the parameter \( \mu \) in the action (1) can be obtained by the Bethe ansatz method (see for example [4], [5]). It can be easily derived from the results of Ref. [3] and has the form:

\[
-\pi \mu \gamma(1 + b^2) = \left[ \frac{mk(G) \Gamma \left( \frac{1-x}{h} \right) \Gamma \left( \frac{x}{h} \right) x^{2(1+b^2)}}{2\Gamma \left( \frac{1}{h} \right) h} \right],
\]
where as usual \( \gamma(z) = \Gamma(z)/\Gamma(1 - z) \); \( x = b^2/(1 + b^2) \) and

\[
k(G) = \left( \prod_{i=1}^{r} n_i^{n_i} \right)^{1/2h}
\]

with \( n_i \) defined by the equation (7).

The ATTs can be considered as perturbed CFTs. Without the last term (with root \( e_0 \)) the action (1) describes NATTs, which are conformal. To describe the generator of conformal symmetry we introduce the complex coordinates \( z = x_1 + ix_2 \), \( \bar{z} = x_1 - ix_2 \) and the vector

\[
Q = (b + 1/b)\rho, \quad \rho = \frac{1}{2} \sum_{\alpha > 0} \alpha,
\]

where the sum in the definition of the Weyl vector \( \rho \) runs over all positive roots \( \alpha \) of Lie algebra \( G \).

The holomorphic stress energy tensor

\[
T(z) = -\frac{1}{2}(\partial_z \varphi)^2 + Q \cdot \partial_z^2 \varphi
\]

ensures the local conformal invariance of the NATT.

Besides conformal invariance NATT possesses an additional symmetry generated by two copies of the chiral \( W(G) \)-algebras: \( W(G) \otimes \overline{W}(G) \). The full chiral \( W(G) \)-algebra contains \( r \) holomorphic fields \( W_j(z) \) (\( W_2(z) = T(z) \)) with spins \( j \) which follow the exponents of the Lie algebra \( G \). The explicit representation of these fields in terms of fields \( \partial_z \varphi_a \) can be found in [6]. The exponential fields:

\[
V_a(x) = e^{a \cdot \varphi(x)}
\]

are spinless conformal primary fields with dimensions \( \Delta(a) = Q^2 - (a-Q)^2 \). The fields \( V_a(x) \) are also primary fields with respect to full chiral algebra \( W(G) \). It was shown in Ref. [3] that fields \( V_a(x) \) and \( V_{\hat{s}(a)}(x) \) correspond to the same representation of \( W(G) \)-algebra if their parameters are related by any transformation of the Weyl group \( W \) of the Lie algebra \( G \) acting on the vector \( a \) as follows:

\[
a \rightarrow \hat{s}(a) = Q + \hat{s}(a - Q), \quad \hat{s} \in W.
\]

This means that the fields \( V_{\hat{s}(a)} \) for different \( \hat{s} \in W \) are the reflection image of each other and are related by a linear transformation. The functions that define this transformation are known as “reflection amplitudes”. Reflection amplitudes play an important role in the analysis of perturbed CFTs. In particular they permit to calculate the vacuum expectation values of exponential fields in ATTs:

\[
G(a) = \langle \exp a \cdot \varphi \rangle
\]

These VEVs were calculated in Ref.[7] and they can be represented in the form:

\[
G(a) = \left[ \frac{m \kappa \Gamma \left( \frac{1-a^2}{2h} \right) \Gamma \left( \frac{a}{h} \right) x}{2 \Gamma \left( \frac{a}{h} \right) h} \right]^{-a^2} \exp \left( \int_0^\infty dt \left[ a^2 e^{-2t} - F(a, t) \right] \right)
\]
with

\[ F(a, t) = \sinh((1 + b^2)t) \times \sum_{\alpha > 0} \frac{\sinh(ba, t) \sinh((b(a - 2Q) + h(1 + b^2))t)}{\sinh t \sinh(b^2t) \sinh((1 + b^2)ht)} \]  

(14)

here and below subscript \( \alpha \) denotes the scalar product of the vector with a positive root \( \alpha \), i.e.:

\[ a_\alpha = a \cdot \alpha; \quad (a - 2Q)_\alpha = (a - 2Q) \cdot \alpha. \]  

(15)

This expression satisfies many possible perturbative and non-perturbative tests for one-point function in ATT. For example it can be easily derived from Eq.(3) and from the equation of motion that the bulk vacuum energy \( E(b) \) in ATT is expressed in terms of function \( G(a) \) as:

\[ n_i E(b) = h(1 + b^2)\mu G(be_i). \]  

(16)

The values of the function \( G(a) \) at the special points \( be_i \) can be calculated explicitly and the result coincides with the known expression for the bulk vacuum energy [8]:

\[ E(b) = \frac{m^2 \sin(\pi/h)}{4\sin(\pi x/h) \sin(\pi(1 - x)/h)} \]  

(17)

where \( x = b^2/(1 + b^2) \).

It is easy to see from Eq.(13) that function \( E(b) \) (and the function \( G(a) \) as well) is invariant under duality transformation \( b \rightarrow 1/b \) or \( x \rightarrow 1 - x \). The same is true for the bulk scattering S-matrix in ATT. This remarkable property of duality is very important for the analysis of Toda theories.

In the limit \( b \rightarrow 0 \) the field \( \tilde{\varphi} = b\varphi \) can be described by the classical equations. In particular it can be derived from Eq.(13) that the VEV of this field \( \tilde{\varphi}_0 = \langle b\varphi \rangle \) at that limit can be expressed in terms of the fundamental weights \( \omega_i \) of \( G \) as:

\[ \tilde{\varphi}_0 = b\varphi_0 = \sum_{i=1}^r (\log n_i - 2\log k(G))\omega_i \]  

(18)

and coincides with a classical vacuum of ATT [4]. At this limit we have (see Eq.(3)) that \( \mu = (mk(G))^2/4\pi b^2 + O(1) \) and after rescaling and shifting of the field \( \varphi \) the action (1) can be expressed in terms of the field \( \phi = b\varphi - \tilde{\varphi}_0 \) as:

\[ \mathcal{A}_b = \frac{1}{4\pi b^2} \int d^2x \left[ \frac{1}{2}(\partial_\mu \phi)^2 + m^2 \sum_{i=0}^r n_i e^{\epsilon_i \phi} \right] + O(1). \]  

(19)

The VEVs given by Eq.(13) and scattering theory data were used in Ref. [7] to describe the asymptotics of the cylindrical solutions of classical Toda equations following from the action given by Eq.(19). In the next sections we use a similar strategy to explicitly find the vacuum solutions in integrable boundary ATT.
3 Boundary Toda Theories, One-Point Functions and Classical Vacuum Solutions

In this section we consider simply laced boundary Toda theories defined on the half-plane $H = (x, y; y > 0)$. The integrability conditions for classical simply laced ATT on $H$ were studied in the paper [12]. It was shown there that the action of integrable boundary ATT can be written as:

$$
A_{\text{bound}} = \int_H d^2x \left[ \frac{1}{8\pi} (\partial_\mu \varphi)^2 + \mu \sum_{i=1}^r e^{be_i \varphi} + \mu e^{be_0 \varphi} \right]
$$

$$
+ \mu_B \int dx \sum_{i=0}^r s_i e^{be_i \varphi/2} \tag{20}
$$

where either all the parameters $s_i$ vanish, corresponding to the Neumann boundary conditions:

$$
\partial_y \varphi(x, 0) = 0; \tag{21}
$$

or they are given by $s_i = \pm 1$ and the parameter $\mu_B$ is related to the parameter $\mu$ in the bulk (in the classical case) as:

$$
\mu_B^2 = \mu/\pi b^2. \tag{22}
$$

For the Lie algebra $A_1$ (sinh-Gordon model) the integrability conditions are much less stringent and the parameters $s_0$ and $s_1$ can have arbitrary values [11]. Here we discuss this problem for other Lie algebras where the choice of integrable conditions is rather restrictive. We consider Toda theories with Neumann boundary conditions and in the case when all parameters $s_i = 1$ (with $s_0 = 0$ in the non-affine case). Really, these two quite different classical theories in the quantum case are described by the same theory and are related by a duality transformation ($b \rightarrow 1/b$) [13], [14]. The cases corresponding to different signs of the parameters $s_i$ are more subtle and will be considered elsewhere.

We start from a consideration of the boundary NATTs described by the action (20) without the last term in the bulk action and with parameter $s_0 = 0$ in the boundary term. The boundary ATTs will be considered as perturbed boundary CFTs. On the half-plane with $W$-invariant boundary conditions we have only one $W$-algebra. In this case the currents $W_j(z)$ should be the analytical continuations of the currents $W_j(z)$ into the lower half-plane. In particular, they should coincide at the boundary. These conditions impose very strong restrictions on the form of the boundary terms in the action. It is rather easy to derive from the explicit form of $W$-currents [3] that the Neumann boundary conditions (21) preserve $W$-symmetry. The boundary condition (22) whose quantum modified version has the form [4]

$$
\mu_B^2 = \frac{\mu}{2} \cot \left( \frac{\pi b^2}{2} \right), \tag{23}
$$

5
(with all \( s_i = 1, i = 1, \ldots, r \) and \( s_0 = 0 \)) describes the dual theory and, hence, also preserves \( W \)–symmetry.

In the \( W \)-invariant boundary NATT we have two kinds of exponential fields. The bulk fields \( V_a(x, y) \) and the boundary fields \( B_a(x) \) are defined as:

\[
V_a(x, y) = \exp\{a \cdot \varphi(x, y)\}; \quad B_a(x) = \exp\{\frac{1}{2}a \cdot \varphi(x)\}.
\]

These fields are specified by the same quantum numbers as those for the corresponding fields ([1]) defined on the whole plane. In particular, they possess the similar properties with respect to the action of the Weyl group of \( G \) (see section 2), defined by Eq.(11). Following the standard lines one can introduce and calculate boundary reflection amplitudes which relate the fields \( B_a(x) \) and \( B_{a(a)}(x) \) (see Ref.[9] for details).

Boundary reflection amplitudes can be used for the calculation of vacuum expectation values of the boundary exponential fields in ATTs. One-point functions:

\[
G_B(a) = \langle \exp(\frac{1}{2}a \cdot \varphi) \rangle_B
\]

for \( ADE \) ATTs were calculated in [17] and have the form:

\[
G_B(a) = \left[ \frac{mk\Gamma\left(\frac{1-x}{h}\right)\Gamma\left(\frac{x}{h}\right)x}{2\Gamma\left(\frac{1}{h}\right)h} \right]^{-a^2/2} \exp \left( \int_0^{\infty} \frac{dt}{t} \left[ \frac{a^2}{2} e^{-2t} - F_B(a, t) \right] \right)
\]

with

\[
F_B(a, t) = f(t) \sum_{\alpha > 0} \frac{\sinh(ba_\alpha t) \sinh((b(a - 2Q_\alpha) + h(1 + b^2))t)}{\sinh(2t) \sinh(2b^2t) \sinh((1 + b^2)ht)}
\]

where for the boundary conditions (23) the function \( f(t) \) is:

\[
f(t) = 2e^t \sinh((1 + b^2)t) \cosh(b^2t)
\]

and for the dual theory which corresponds to Neumann boundary conditions (21) we should make the substitution \( f(t) \to f^\vee(t) \):

\[
f^\vee(t) = 2e^{tb^2} \sinh((1 + b^2)t) \cosh t.
\]

It is easy to see from the explicit form of \( G_B(a) \) that in the classical limit (\( b \to 0 \) with \( b\varphi \) is fixed) the boundary VEV \( \tilde{\varphi}_{0,B} \) of the field \( b\varphi \) for the Neumann boundary conditions coincides with the classical vacuum \( \tilde{\varphi}_0 \) ([18]) in the bulk. For the boundary conditions (23) this happens only for Lie algebra \( A_r \), where both these values vanish. For other cases we can derive from Eqs.(26-28) that difference of VEVs of the field \( b\varphi \) at the boundary and on the whole plane has a form:

\[
\Theta = \langle b\varphi \rangle_B - \langle b\varphi \rangle = - \sum_{\alpha > 0} x \int_0^{\infty} dt \frac{x \sinh((h - 2\rho_\alpha)t) \sinh t}{\sinh xt \cosh(1 - x)t \sinh ht}
\]
Vector $\Theta$ can be expressed in terms of elementary functions at the semiclassical limit $x \to 0$, at the self-dual point $x = 1/2$ and in the dual limit $x \to 1$. We discuss the dual limit in the last section. Here we note that at $x \to 0$ vector $\Theta$ has a limit:

$$\vartheta = - \sum_{\alpha > 0} \alpha \int_0^\infty \frac{dt \sinh((h - 2\rho_\alpha)t)}{t \sinh(ht)} \tanh t.$$

(31)

For $D$ and $E$ series, which we consider below the numbers $2l_\alpha = h - 2\rho_\alpha$ are always even. For the following it is convenient to define:

$$E_i = \exp\left(\frac{1}{2} \vartheta \cdot e_i \right) = \prod_{\alpha > 0} (p_{l_\alpha})^{-e_i - \alpha}$$

(32)

where $p_l = 1/p_{-l}$ can be written in terms of trigonometric functions as:

$$p_l^2 = \exp\left(\int_0^\infty \frac{dt \sinh 2lt}{t \sinh ht} \tanh t\right) = \prod_{k=1}^l \frac{\cos(\pi (l+1-2k)/h)}{\cos(\pi (l-2k)/h)}.$$

(33)

In particular:

$$E_0 = \frac{p_{q(D)-1} p_{h/2-1}}{p_{h/2-1}}$$

(34)

here $q(G) = \max_i n_i$; $q(D) = 2, q(E_6) = 3, q(E_7) = 4, q(E_8) = 6$.

The vector $\vartheta$ is simply related to the boundary soliton solution that describes the classical vacuum configuration. The classical problem for this solution $\phi(y), y > 0$ can be formulated in the following way. We are looking for a solution to the classical Toda equations, which decreases at $y \to \infty$ and satisfies at $y = 0$ the boundary conditions that follow from the action (20). After rescaling and shifting (see section 2) the field $\phi = b\varphi - \bar{\varphi}_0$ satisfies the equation:

$$\partial_y^2 \phi + m^2 \sum_{i=0}^r n_i e_i \exp(e_i \cdot \phi); \quad n_0 = 1$$

(35)

and the boundary condition at $y = 0$:

$$\partial_y \phi = m \sum_{i=0}^r \sqrt{n_i} e_i \exp(\frac{1}{2} e_i \cdot \phi).$$

(36)

The vector $\vartheta$ is equal to the boundary value of this solution and hence, it completely fixes the boundary soliton. At $y = 0$ we have

$$\phi(0) = \vartheta; \quad \partial_y \phi(0) = \sum_{i=0}^r \sqrt{n_i} e_i E_i.$$

(37)

This gives us the possibility to study Eq.(35) numerically. The numerical analysis of this equation shows that for these and only for these boundary values a smooth solution decreasing at infinity exist (see appendix C).

It is natural to expect that solution $\phi(y)$ to the Eqs. (35), (36) can be expressed in terms of tau-functions associated with multi-soliton solutions of the classical ATT
equations (see, for example [16], [18] and references there). For the $D$ and $E$ series of algebras (besides the cases $D_4$ and $D_5$ [16]) the explicit form of these solutions was not known. We postpone the discussion of these solutions to section 5. Here we consider the classical boundary ground state energy which can be defined as:

$$
E_{\text{bound}}^{(cl)} = \frac{1}{4\pi b^2} \left[ 2m \sum_{i=0}^{r} \sqrt{n_i} E_i + \int_0^\infty dy \left( \frac{1}{2} (\partial_y \phi)^2 + m^2 \sum_{i=0}^{r} n_i (e^{e_i \phi} - 1) \right) \right].
$$

(38)

We note that numerical values of $\vartheta^2$, defined by Eq. (31) are rather small for all $G$, and the functional (38) can be calculated to a good accuracy using the bilinear approximation:

$$
E_{\text{bound}}^{(cl)} = \frac{m}{4\pi b^2} \sum_{i=0}^{r} \sqrt{n_i} E_i (2 - \log E_i) + O(|\vartheta|^3).
$$

More careful analysis of Eqs. (35), (36) (see section 5) gives us the reasons to write the expression for boundary ground state energy in the following form. Namely, we denote as $\Sigma_m(G)$ the sum of the masses of all particles in the ATT:

$$
\Sigma_m(G) = \sum_{i=1}^{r} m_i = m \sum_{i=1}^{r} \nu_i = m \cdot tr(M^{1/2}).
$$

(39)

This has the following values for simply laced Lie algebras:

$$
\begin{align*}
\Sigma_m(A_n) &= 2m \cot(\pi/2h); \\
\Sigma_m(D_n) &= \frac{2m \cos(\pi/4 - \pi/2h)}{\sin(\pi/2h)}; \\
\Sigma_m(E_6) &= m(6 - 2\sqrt{3})^{1/2} \frac{\cos(\pi/8)}{\sin(\pi/2h)}; \\
\Sigma_m(E_7) &= \frac{2m \sin(2\pi/9)}{\sin(\pi/2h)}; \\
\Sigma_m(E_8) &= 4m(\sqrt{3} \sin(\pi/5) \sin(\pi/30))^{1/2} \frac{\cos(\pi/5)}{\sin(\pi/2h)}.
\end{align*}
$$

(40)

The classical boundary ground state energy (38) can be written in terms of these values as:

$$
E_{\text{bound}}^{(cl)}(G) = \frac{h}{4\pi b^2} \tan(\pi/2h) \Sigma_m(G).
$$

(41)

In the quantum case the boundary ground state energy $E_{\text{bound}}^{(q)}$ will have contributions coming from the boundary term in the Hamiltonian and from the bulk fluctuations around the background solution. The contributions of the first type can be calculated using the explicit expression for vacuum expectation values (26). For small $b$ the first quantum correction of the second type can be expressed through the boundary $S$-matrix at $b = 0$ (see, for example [19], [20]). We will discuss these boundary $S$-matrices below. Here we note that in the strong coupling region $b \gg 1$ our theory is described by the weakly coupled dual ATT with Neumann boundary
conditions (21). In the strong coupling limit the dual theory is a set of $r$ free bosonic theories with masses $m_i$. The boundary ground state energy for the free massive bosonic theory with Neumann boundary conditions and mass $m_i$ can be easily calculated and is equal to $m_i/8$. The first perturbative correction in the weakly coupled dual theory can be also evaluated with the result:

$$E^{(q)}_{\text{bound}}(G) = \frac{\Sigma m(G)}{8} \left( 1 + \frac{\pi}{2hb^2} \cot(\pi/2h) + O(1/b^4) \right). \quad (42)$$

Both asymptotics $b \to 0$ (41) and $b \to \infty$ (42) are in agreement with the following conjecture for boundary ground state energy:

$$E^{(q)}_{\text{bound}}(G) = \frac{\sin(\pi/2h)\Sigma m(G)}{8\sin(\pi x/2h)\cos(\pi (1-x)/2h)} \quad (43)$$

where $x = b^2/(1 + b^2)$.

The nonperturbative check of this conjecture can be made using the boundary Thermodynamic Bethe Ansatz equations [21]. The kernels in these nonlinear integral equations depend on the bulk and boundary $S$-matrices. In particular boundary ground state energy can be expressed in terms of the mass $m_j$ of the particle $j$ multiplied by the ratio of Fourier transforms of logarithms of boundary $S$-matrix $R_j$ and bulk amplitude $S_{jj}$ taken at $\omega = i$. To check our conjecture and to construct the explicit solution of Eq.(35) we need the information about boundary $S$-matrix.

4 Boundary $S$-matrix and Boundary State

The boundary $S$-matrix (reflection coefficient) in the ATT for the particle $j$ corresponding to the fundamental representation $\pi_j(G)$ can be defined as:

$$|j, -\theta\rangle_{\text{out}} = R_j(\theta)|j, \theta\rangle_{\text{in}} \quad (44)$$

where $\theta$ is the rapidity of particle $j$.

The reflection coefficients for the $A_r$ ATT with boundary conditions (23) were conjectured in [22], [23]. They can be written in terms of function:

$$(z) = \frac{\sin\left(\frac{\vartheta}{2}\frac{\varpi}{2h} z\right)}{\sin\left(\frac{\vartheta}{2}\frac{\varpi}{2h} z\right)} \quad (45)$$

in the following way:

$$R_j(\theta) = \prod_{a=1}^{j}(a - 1)(a - h)(-a + x)(-a + h + 1 - x). \quad (46)$$

Unfortunately, we were not able to find in the literature any conjecture for other Lie algebras consistent with duality properties discussed above. So we will give here a conjecture which naturally generalizes the reflection coefficients (46) to other simply laced Lie algebras. To do this we rewrite Eq. (46) in the form:
\[ R_j(\theta) = \exp(-i\delta_j(\theta)) \] (47)

where

\[ \delta_j = \int_0^\infty \frac{dt}{t} \sinh\left(\frac{2h\theta t}{\pi}\right)[\sinh((1-x)t) \sinh((h+x)t)\Delta_j(A_r, t) - 2] \] (48)

and

\[ \Delta_j(A_r, t) = \frac{8 \sinh jt \sinh(h - j)t}{\sinh t \sinh 2ht}. \] (49)

The natural generalization of these equations can be written as:

\[ R_j(\theta) = \Phi_j(\theta) \exp(-i\delta_j(\theta)) \] (50)

where \(\Phi_j(\theta)\) are the CDD factors, satisfying the conditions:

\[ \Phi_j(\theta)\Phi_j(-\theta) = 1; \quad \Phi(\theta)\Phi(\theta + i\pi) = 1 \] (51)

and the function \(\delta_j(\theta)\) is defined by Eq.(48), where we should do the substitution \(\Delta_j(A_r, t) \to \Delta_j(G, t)\) with:

\[ \Delta_j(G, t) = \frac{4}{\cosh ht} \left[ (2\cosh t - 2)\delta_{mn} + e_m \cdot e_n \right]^{-1}_{jj}. \] (52)

The most important part of this conjecture is that for the particles \(j\) corresponding to fundamental representations \(\pi_j(G)\) (or the node \(j\) of Dynkin diagram) with \(n_j\) in Eq.(2) equal to 1, the CDD factors \(\Phi_j(\theta)\) in Eq.(50) are equal to one. This statement fixes completely the boundary S-matrix for Lie algebras \(D_r, E_6, E_7\).

We denote by \(R_f(G, \theta)\) the fundamental reflection coefficients. This means that all other amplitudes \(R_j(\theta)\) can be obtained from the fundamental reflection factors \(R_f(G, \theta)\) by application of the boundary bootstrap fusion procedure \([11, 24]\). It is easy to see that \(R_f(D_r, \theta) = R_r(\theta) = R_{r-1}(\theta)\), where particles \(r\) and \(r-1\) correspond to the spinor representations of \(D_r\); \(R_f(E_6, \theta) = R_1(\theta) = R_7(\theta)\), where particles 1 and 7 form the doublet of lightest particles in the \(E_6\) ATT and \(R_f(E_7, \theta) = R_1(\theta)\), where 1 is the lightest particle in the \(E_7\) ATT. For all these three cases the CDD factors \(\Phi_j(\theta) = 1\) and the reflection coefficients are given by Eqs.(17),(18) with \(j = f\) and

\[ \Delta_f(G, t) = \frac{8 \sinh (ht/2) \sinh((h/2 + q - 1)t)}{\sinh qt \sinh(2ht)} \] (53)

where \(q(G) = \max_i n_i; \quad q(D) = 2, \quad q(E_6) = 3, \quad q(E_7) = 4\).

As an example of the application of the boundary bootstrap equations we give here the reflection coefficients \(R_j\) for the particles \(j = 1, 2, ..., n-2\) in the \(D_r\) ATT, which can be obtained from the amplitude \(R_f(\theta) = R_r(\theta) = R_{r-1}(\theta)\). These functions \(R_j\) have the form \([51]\), where:
\[
\Delta_j(D_r,t) = \frac{16 \sinh jt \cosh((h/2-j)t) \sinh(ht/2)}{\sinh t \sinh 2ht}
\]

and the CDD factor can be written as:

\[
\Phi_j = \exp \left( i \int_0^\infty \frac{dt}{t} \sinh(2\theta t/\pi) \sinh(1-x)t \cosh(xt) \psi_j(D_r,t) \right)
\]

where \( \psi_j(D_r,t) = 8 \sinh(j-1)t \sinh jt/(\sinh 2t \cosh ht) \)

The Lie algebra \( E_8 \) has no fundamental representations with \( n_j = 1 \). The lightest particle in this case is associated with the adjoint representation and \( R_f(E_8,\theta) = R_{ad}(\theta) \). The adjoint representations for Lie algebras \( D,E \) have \( n_{ad} = 2 \) and, hence, CDD factors should appear. The reflection coefficients \( R_{ad}(\theta) \) for Lie algebras \( D \) and \( E \) can be written in the form \((50)\), with:

\[
\Delta_{ad}(G,t) = \frac{8 \cosh((q-1)t) \cosh((h/2-q)t)}{\cosh(ht/2) \cosh ht}
\]

where \( q(G) \) is defined above and the CDD factor \( \Phi_{ad}(\theta) \) is defined by Eq.\((55)\) with function

\[
\psi_{ad}(G,t) = \frac{8 \sinh t}{\cosh ht}
\]

The amplitudes \( R_j(\theta) \) following from the scattering theory described above have no poles corresponding to the bound states of particles \( j \) with boundary. The only poles with positive residues which appear in some of the functions \( R_j(\theta) \) are the poles at \( \theta = i\pi/2 \)

\[
R_j(\theta) = \frac{iD^2_{ad}(b)}{\theta - i\pi/2}
\]

corresponding to the particle boundary coupling with zero binding energy. For example, the lightest particle whose boundary amplitude possess such pole is the particle \( m_{ad} \) corresponding to adjoint representation. It is easy derive from Eqs.\((56,57)\) that corresponding residue has a form:

\[
D^2_{ad}(b) = 2 \cot \left( \frac{\pi x}{2h} \right) \tan \left( \frac{\pi (1-x)}{2h} \right) \tan \left( \frac{\pi}{2h} \right) T_0 T_{q-1} T_{h/2-q}
\]

where \( T_k = \tan \left( \frac{\pi}{4} - \frac{\pi(k+1)}{2h} \right) \tan \left( \frac{\pi}{4} - \frac{\pi(k+1-x)}{2h} \right) \cot \left( \frac{\pi}{4} - \frac{\pi k}{2h} \right) \cot \left( \frac{\pi}{4} - \frac{\pi(k+x)}{2h} \right) \).

The amplitudes \( R_j(\theta) \) and their residues can be used for the construction of the boundary state in integrable QFT. It was shown in Ref. [11] that this state can be written in terms of numbers \( D_j \) and functions \( K_j(\theta) = R_j(i\pi/2 - \theta) \) as:

\[
|B\rangle = \exp \left[ \sum_{j=1}^r D_j(b) A_j(0) + \frac{1}{4\pi} \int d\theta K_j(\theta) A_j(\theta) A_j(-\theta) \right] |0\rangle
\]
where \( A_j(\theta) \) are the operators creating the asymptotic states with rapidity \( \theta \) i.e. \( A_j(\theta)|0\rangle = |j, \theta\rangle_{\text{in}} \).

We will be mostly interested by the linear in the operators \( A_j \) part in the expansion of boundary state \( |B\rangle \)

\[
|B\rangle = [1 + \sum_{j=1}^{r} D_j(b) A_j(0)]|0\rangle + ...
\]

which determines “one particle” contributions to the correlation function:

\[
\mathcal{F}_b(y) = \langle 0|b\varphi(y)|B\rangle - \langle 0|b\varphi|0\rangle
\]

where the variable \( y > 0 \) plays the role of Euclidean time. These “one particle” contributions can be written as:

\[
\mathcal{F}_b(y) = \sum_{j=1}^{r} D_j(b) \langle 0|b\varphi(0)|j,0\rangle_{\text{in}} \exp(-m_j y) + ...
\]

The boundary soliton \( \varphi(y) \) (see section 3) coincides with the semiclassical limit of this correlation function i.e. \( \varphi(y) = \lim_{b \to 0} \mathcal{F}_b(y) \). In the weak coupling limit one particle matrix elements can be easily calculated and have a form:

\[
\langle 0|\varphi(0)|j,0\rangle_{\text{in}} = \sqrt{\pi} \xi_j + O(b^2)
\]

where \( \xi_j \) are the eigenvectors of mass matrix: \( M\xi_j = \nu^2_j \xi_j \), satisfying the conditions:

\[
\xi_i \cdot \xi_j = \delta_{ij}; \quad \rho \cdot \xi_j \geq 0.
\]

The corresponding “one particle” contributions to the boundary solution \( \varphi(y) \) will be:

\[
\varphi(y) = \sum_{j=1}^{r} d_j \xi_j \exp(-m_j y) + ...
\]

where

\[
d_j = \lim_{b \to 0} \sqrt{\pi} b D_j(b).
\]

For example the coefficient \( d_{ad} \) corresponding to the lightest particle \( m_{ad} \) in expansion (64) and defining the main term of the asymptotics at \( y \to \infty \) can be extracted from Eq.(59) and has a form:

\[
d_{ad} = 2\sqrt{h} \tan \left( \frac{\pi}{2h} \right) T_0 T_{q-1} T_{h/2-q}
\]

where \( T_k^2 = \lim_{b \to 0} T_k \) :

\[
T_k = \tan \left( \frac{\pi}{4} - \frac{\pi(k + 1)}{2h} \right) \cot \left( \frac{\pi}{4} - \frac{\pi k}{2h} \right).
\]
The coefficients $d_j$ as well as the vector $\vartheta$ (or numbers $E_j$) fix completely the solution $\phi(y)$ of Eqs. (35, 36). They determine the contribution of the zero modes of the linearized Eq. (33) and make it possible to develop in a standard way the regular expansion at large distances. If our scattering theory is consistent with conformal perturbation theory this expansion should converge to the boundary values $\vartheta$ at $y = 0$

At the end of this section we note that the analysis of the boundary Thermodynamic Bethe Ansatz equations with kernels depending on the reflection coefficients written above gives an exact agreement with Eq. (43) for the boundary ground state energy.

5 Boundary Solutions

It is natural to assume that boundary vacuum solutions can be expressed in terms of $\tau$-functions associated with multisoliton solutions of ATTs equations. This means that field $\phi$ can be written as:

$$\phi(y) = -\sum_{i=0}^{r} e_i \log \tau_i(y); \quad \tau_i(y) \to 0, \quad y \to \infty$$

(68)

where functions $\tau_i(y)$ satisfy the equations:

$$-\tau_i''(y) + (\tau_i')^2 = \left( \prod_{j=0}^{r} \tau_j^{x_{ij}} - \tau_i^2 \right) n_i$$

(69)

here $I_{ij}$ is the incidence matrix of the extended Dynkin diagram of $G$.

The classical boundary state energy can be expressed in terms of numbers $E_i$ and the boundary values of $\tau$-functions as:

$$\mathcal{E}^{(cl)}_{\text{bound}} = \frac{\hbar}{2\pi b^2 n_i} \left( \sqrt{n_i E_i + \tau_i'(0)/\tau_i(0)} \right); \quad i = 0, \ldots, r$$

(70)

Functions $\tau_i$ corresponding to multisoliton solutions to the ATT equations are given by finite order polynomials in the variables

$$Z_j = \exp(-m_j y) = \exp(-mv_j y).$$

(71)

The general ansatz for these functions has a form:

$$\tau_i(y) = \sum_{k_1=0}^{n_i} \ldots \sum_{k_r=0}^{n_i} Y_{k_1 \ldots k_r}^{(i)} (t_1 Z_1)^{k_1} \ldots (t_r Z_r)^{k_r}$$

(72)

where $Y_{0 \ldots 0}^{(j)} = 1$ and all other coefficients can be derived from Eq. (69) if we fix the normalization for the linear in variables $Z_j$ terms in the expansion:

$$\tau_i(y) = 1 + \sum_{j=1}^{r} Y_{0,1 \ldots 0}^{(i)} t_j Z_j + O(Z^2).$$

(73)

These coefficients should satisfy the condition:
\[ - \sum_{i=0}^{r} e_i Y_{0,1,0}^{(i)} = N_j \xi_j. \] (74)

After we fix the factors \( N_j \) all coefficients \( Y^{(i)} \) in Eq. (72) are universal numbers depending only on \( G \). The parameters \( t_j \) in the Eq. (72) are simply related with parameters \( d_j \) defined by the expansion (64). It is easy to see from Eqs. (68,73,74) and Eq. (64) that

\[ t_j = d_j/N_j \] (75)

The parameters \( d_j \) can be extracted from the boundary scattering theory described in the previous section. This gives us the possibility to derive the explicit solution to Eqs. (35,36).

5.1 \( D_r \) boundary solution

The boundary values of this solution can be described by the numbers \( E_k \) (see section 3) which in this case are:

\[ E_0 = E_1 = E_{r-1} = E_r = \frac{2\sqrt{2}}{h \sin(\pi/h)}; \]

\[ E_k = E_{r-k} = \frac{\cos(\pi k/h) \cos(\pi (r-k)/h) \rho_{kk}^2 \rho_{r-k}^2}{\cos(\pi (2k-1)/2h) \cos(\pi (2r-2k-1)/2h)}. \] (76)

These numbers possess all the symmetries of extended Dynkin diagram for \( D_r \). The same is true for the boundary solution \( \phi(y) \). It means that in Eq. (68) \( \tau_0 = \tau_1 \) and \( \tau_k = \tau_{r-k} \). For the Lie algebra \( D_r \) the following relation holds:

\[ - \sum_{i=0}^{r} n_i \cos(j \pi(2i-1)/h) e_i = 2\sqrt{h} \sin(\pi j/h) \xi_j \] (77)

so that it is natural to choose \( N_j(D) = 2\sqrt{h} \sin(\pi j/h) \). It is convenient also to introduce the functions:

\[ \bar{\tau}_0 = \bar{\tau}_1 = \bar{\tau}_{r-1} = \bar{\tau}_r = \tau_0^2; \quad \bar{\tau}_k = \tau_k; \quad k = 2, \ldots r-2. \] (78)

The particles which are invariant with respect to symmetries of Dynkin diagram are particles \( 2l; \ 2l \leq r-2 \) with masses \( m_{2l} = m \sqrt{8} \sin(2\pi l/h) \). The analysis of the boundary \( S \)-matrix (54) gives that only amplitudes \( R_{2l}(\theta) \) have a pole at \( \theta = i\pi/2 \). The parameters \( t_{2l} = d_{2l}/N_{2l}(D) \) can be derived from the Eqs. (54,55) and have a form:

\[ t_{2l} = \frac{\tan(\pi/4 - \pi l/h)}{2 \cos^2(\pi l/h)} \prod_{i=1}^{l} \frac{\tan^2(\pi (2i-1)/2h)}{\tan^2(\pi i/h)} \] (79)

Parameters \( t_{2l} \) and functions \( Z_{2l} = \exp(-ym \sqrt{8} \sin(2\pi l/h)) \) are defined for \( 2l \leq r-2 \). To write the solution in the most short form it is convenient, however, to continue
these values to $l \leq r - 2$. To continue the parameters $t_{2l}$ to $l \leq r - 2$ we can use Eq.\((73)\). In this way we obtain:

$$
t_{h-2l} = -t_{2l}; \quad Z_{h-2l} = Z_{2l}
$$

(80)

We introduce also $\omega = \exp(2\pi i/h)$. Then the boundary solution can be written as:

$$
\phi(y) = -\frac{1}{2} \sum_{i=0}^{r} n_{i} e_{i} \log \tilde{\tau}_{i}(y)
$$

(81)

where:

$$
\tilde{\tau}_{i}(y) = \sum_{\sigma_{r-2}=0}^{1} \prod_{k=1}^{r-2} \omega^{(2i-1)k\sigma_{k}} (t_{2k}Z_{2k})^{\sigma_{k}}
\times \prod_{m<n} \left( \frac{\sin (\pi (m-n)/h)\right)}{\sin (\pi (m+n)/h))^{2\sigma_{m}\sigma_{n}}}
$$

(82)

As an example of application of the last equation we give here the result for $D_{7}$-solution. This solution is characterized by three function $\tilde{\tau}_{0}, \tilde{\tau}_{2}, \tilde{\tau}_{3}$ (78) which are:

$$
\tilde{\tau}_{0,3} = \left(1 \pm \frac{\sqrt{3}}{2} t_{1}Z_{1} + \frac{1}{2} t_{2}Z_{2} \pm \frac{\sqrt{3}}{4} \tan^{2} \left( \frac{\pi}{12} \right) t_{1}Z_{1}t_{2}Z_{2} \right)^{2}
$$

$$
\tilde{\tau}_{2} = 1 - 2t_{2}Z_{2} + \frac{3(t_{1}Z_{1})^{2}}{4} + \frac{(t_{2}Z_{2})^{2}}{4} + \frac{3}{16} \tan^{4} \left( \frac{\pi}{12} \right) (t_{1}Z_{1}t_{2}Z_{2})^{2}
$$

(83)

We note that in the usual basis of roots of $D_{r}$: $e_{0}\phi = -\phi_{1} - \phi_{2}$; $e_{k}\phi = \phi_{k} - \phi_{k+1}$, $k = 1...r-1$; $e_{r}\phi = \phi_{r-1} + \phi_{r}$ the solution can be written as $\phi_{k} = \log \left( \tilde{\tau}_{k-1}/\tilde{\tau}_{k} \right)$. It can be checked that boundary value of $\phi(y)$ defined by Eqs.\((81,82)\) coincides with $\vartheta$, defined by Eq.\((70)\) and the classical boundary ground state energy \((70)\) can be derived using Eq.\((11)\).

### 5.2 $E_{6}$ boundary solution

The boundary values for this solution are:

$$
E_{0} = E_{1} = E_{T} = \left(3 + \sqrt{3}/6 \right)^{1/2}; \quad E_{4} = \frac{\sqrt{3}+1}{6} \left(3+5/\sqrt{3}\right)^{1/2};
$$

$$
E_{2} = E_{3} = E_{\overline{T}} = (E_{4}E_{0})^{-1/2}
$$

(83)

Here and later we shall use the numeration of nodes $j$ of Dynkin diagram corresponding to the particles $m_{j}$ (see Fig.s [1-3]) as follows:

![Dynkin Diagram](Fig.1)
The solution possesses $Z_3$ symmetry of this diagram. It means that: $\tau_0 = \tau_1 = \tau_7$ and $\tau_2 = \tau_3 = \tau_5$. The particles that are invariant with respect this symmetry are $m_2 = m_{ad}$ and $m_4$. The analysis of boundary $S$-matrix shows that only amplitude $R_2 = R_{ad}(\theta)$ possesses the pole at $\theta = i\pi/2$ with the residue \(59\). In this case only one particle contributes to the solution. It is convenient to take $N(E_0) = 2\sqrt{3}h$. Then the parameter $t_2 = d_{ad}/N$ is:

$$t_2 = \tan\left(\frac{\pi}{24}\right) T_0 T_2 T_3/\sqrt{3} = \tan\left(\frac{\pi}{24}\right) \tan\left(\frac{5\pi}{24}\right) \tan\left(\frac{\pi}{12}\right)$$ \hspace{1cm} (84)

The variable $Z_2 = \exp(-m\nu y)$, where $\nu_2^2 = 2(3 - \sqrt{3})$ and the solution has a form:

$$\tau_0 = 1 + (2 + \sqrt{3})t_2 Z_2; \quad \tau_2 = 1 - 2t_2 Z_2 + \left((2 + \sqrt{3})t_2 Z_2\right)^2; \quad \tau_4 = 1 - 3\sqrt{3} t_2 Z_2 - 3\sqrt{3}(2 + \sqrt{3})(t_2 Z_2)^2 + \left((2 + \sqrt{3})t_2 Z_2\right)^3.$$ \hspace{1cm} (85)

It is easy to check that $\phi(0) = \vartheta$ and the classical boundary ground state energy \(70\) coincides with \(41\).

### 5.3 $E_7$ boundary solution

The boundary values \(32\) for this solution are:

$$E_0 = E_1 = \frac{p_3 p_5}{p_8} = \frac{2\sqrt{2}}{9} \cot\left(\frac{\pi}{9}\right); \quad E_2 = E_4 = \frac{p_4 p_6}{p_3 p_7}; \quad E_3 = \frac{p_5 p_6 p_7}{p_1 p_2 p_3 p_4 p_8}; \quad E_5 = E_6 = \frac{p_1 p_2 p_7}{p_4 p_6}; \quad E_7 = \frac{p_3 p_4 p_8}{p_1 p_2 p_5 p_7}.$$ \hspace{1cm} (86)

Numbers $E_k$ possess $Z_2$ symmetry of the extended Dynkin diagram

![Dynkin diagram](Fig.2)

This means that in Eq.\(58\) $\tau_0 = \tau_1$, $\tau_2 = \tau_4$ and $\tau_5 = \tau_6$ and only $Z_2$ even particles $m_2 = m_{ad}$, $m_4$, $m_6$ and $m_7$ can contribute to this solution. The analysis of the boundary $S$-matrix (which can be obtained by boundary bootstrap fusion procedure from the amplitude $R_f(\theta)$ \(53\)) shows that only amplitudes $R_2 = R_{ad}$, $R_4$ and $R_7$ possess the poles at $\theta = i\pi/2$. These amplitudes have a form \(60\) where the phases $\delta_j$ and CDD factors $\Phi_j$ are defined by the functions $\Delta_j(E_7, t)$ and $\psi_j(E_7, t)$ in Eqs.\(48\),\(52\),\(55\). For amplitude $R_2 = R_{ad}$ these functions are given in section 4. For amplitudes $R_4$ and $R_7$ they are:

$$\Delta_4 = \frac{16 \cosh t \cosh 3t \cosh 4t}{\cosh(ht/2) \cosh ht}; \quad \psi_4 = \frac{16 \sinh 5t \cosh 4t}{\cosh ht};$$

$$\Delta_7 = \frac{8 \cosh t \cosh 2t \sinh 6t}{\sinh t \cosh(ht/2) \cosh ht}; \quad \psi_7 = \frac{8 \sinh 3t \sinh 5t \sinh 8t}{\sinh t \sinh 2t \cosh ht}.$$
The coefficients $d_j$ can be extracted from these amplitudes using Eq.(65). It is convenient to take $N(E_7) = 2\sqrt{3}h$. Then the parameters $t_j = d_j/N$ can be written in terms of numbers $T_k$ (71) as:

$$
\sqrt{3}t_2 = \tan \left( \frac{\pi}{2h} \right) T_0T_3T_5; \quad \sqrt{3}t_4 = \tan \left( \frac{\pi}{2h} \right) T_0T_1T_3T_5T_7; \\
\sqrt{3}t_7 = \tan \left( \frac{\pi}{2h} \right) T_0T_1^2T_2T_3^2T_5^2T_7(T_6)^{-1}.
$$

(87)

The corresponding variables $Z_j = \exp(-m\nu_j y)$, where:

$$
\nu_2 = \sqrt{32} \sin \left( \frac{\pi}{9} \right) \sin \left( \frac{2\pi}{9} \right); \quad \nu_4 = \sqrt{32} \sin \left( \frac{\pi}{9} \right) \sin \left( \frac{4\pi}{9} \right); \\
\nu_7 = \sqrt{32} \sin \left( \frac{2\pi}{9} \right) \sin \left( \frac{4\pi}{9} \right).
$$

(88)

We give here the explicit form for function $\tau_0$. All other $\tau$-functions can be derived from it using Eq.(69). For example it holds:

$$
\tau_2 = \tau_0^2 - \tau_0 \tau_0'' + (\tau_0')^2
$$

(89)

with similar relations for higher $\tau$-functions. As an example of application of this equation we give the explicit expression for $\tau_2$ in the form (72) in Appendix A.

For all simply-laced algebras the number $n_0$ is equal to one and function $\tau_0$ in Eq.(72) can be written as:

$$
\tau_0 = \sum_{\sigma_1=0}^{1} \sum_{\sigma_2=0}^{1} \sum_{\sigma_3=0}^{1} Y_{\sigma_1\sigma_2\sigma_3}^{(0)}(t_2Z_2)^{\sigma_1}(t_4Z_4)^{\sigma_2}(t_7Z_7)^{\sigma_3}
$$

(90)

It is convenient to simplify the notations. The variables $\sigma_i$ take the values 0 or 1. We can introduce the variable $1 \leq i \leq 3$ indicating $\sigma_i$ which are equal to one. The corresponding coefficients $Y_{\sigma_1,\ldots,\sigma_3}^{(0)}$ we denote as $y_i, y_{ij}, y_{123}$. For example $y_1 = Y_{100}^{(0)}, y_{13} = Y_{101}^{(0)}$ and so on. With this notation we have:

$$
y_1 = \frac{2 \cos^2(2\pi/9)}{\cos(4\pi/9)}; \quad y_2 = \frac{2 \cos^2(\pi/9)}{\cos(2\pi/9)}; \quad y_3 = \frac{2 \cos^2(4\pi/9)}{\cos(\pi/9)}; \\
y_{12} = \frac{8 \cos^4(2\pi/9)}{\cos(\pi/9)}; \quad y_{13} = \frac{8 \cos^4(\pi/9)}{\cos(2\pi/9)}; \quad y_{13} = (2 \cos(4\pi/9))^7 \\
y_{123} = (2 \cos^2(4\pi/9)/\cos(\pi/9))^4
$$

(91)

It can be checked that $\phi(0) = \vartheta$, where $\vartheta$ is defined by Eq.(86) and the classical boundary ground state energy (70) coincides with that given by Eq.(11).

### 5.4 $E_8$ boundary solution

The boundary values for this solutions follow from Eq.(32), where $E_0$ is given by (14) and

$$
E_1 = \frac{p_3p_6p_{10}}{p_1p_5p_{13}}; \quad E_2 = \frac{p_8p_{10}p_{12}}{p_{14}p_3p_{15}}; \quad E_3 = \frac{p_1p_4p_{11}}{p_3p_6p_{12}}; \\
E_4 = \frac{p_2p_3p_{12}p_{13}}{p_6p_{17}p_{14}}; \quad E_5 = \frac{p_5p_{12}p_{14}}{p_{11}p_2p_{17}}; \quad E_6 = \frac{p_{15}}{p_{24}p_{8}}; \\
E_7 = \frac{p_1p_2p_{17}p_{13}}{p_{4}p_{10}p_{13}}; \quad E_8 = \frac{p_{4}p_{10}p_{11}p_{14}}{p_{5}p_{9}p_{12}p_{13}}.
$$

(92)
The extended Dynkin diagram

![Dynkin diagram](Fig.3)

in this case has no symmetries and all particles can contribute to the solution; however the analysis of the boundary $S$-matrix shows that only amplitudes $R_1 = R_{ad}$, $R_2$, $R_5$ and $R_8$ have the poles at $\theta = i\pi/2$. The amplitude $R_1 = R_{ad}$ is given in section 4. Three other amplitudes with these poles are defined by the functions $\Delta_j(E_8, t)$ and $\psi_j(E_8, t)$, which have a form:

\[
\begin{align*}
\Delta_2 &= \frac{16 \cosh 3t \cosh 5t \cosh 6t}{\cosh(\frac{ht}{2}) \cosh ht}; \\
\psi_2 &= \frac{16 \sinh 7t \cosh 6t}{\cosh ht}; \\
\Delta_5 &= \frac{8 \cosh 4t \cosh 5t \sinh 6t}{\sinh t \cosh(\frac{ht}{2}) \cosh ht}; \\
\psi_5 &= \frac{8 \sinh 3t \cosh 4t}{\cosh ht} \left( \frac{\sinh 11t \sinh 12t}{\sinh t \sinh 4t} - 1 \right); \\
\Delta_8 &= \frac{2 \sinh 6t \sinh 10t}{\sinh^2 t \cosh(\frac{ht}{2}) \cosh ht}; \\
\psi_8 &= \frac{8 \sinh 5t \sinh 9t \sinh 14t}{\sin t \sin 2t \cosh ht}.
\end{align*}
\]

These amplitudes determine the coefficients $d_j$ (see Eq.(65)). It is convenient to take $N(E_8) = 2\sqrt{k}$. Then the parameters $t_j = d_j/N$ can be written in terms of numbers $T_k$ (87) as:

\[
\begin{align*}
t_1 &= \tan \left( \frac{\pi}{2h} \right) T_0 T_5 T_9; \\
t_2 &= \tan \left( \frac{\pi}{2h} \right) T_0 T_3 T_5 T_6 T_5 T_{11}(T_8)^{-1} \\
t_5 &= \tan \left( \frac{\pi}{2h} \tan \left( \frac{3\pi}{2h} \right) \right) T_7 T_{13}(T_4 T_6 T_8 T_{10} T_{12})^{-1} \\
t_8 &= \tan^2 \left( \frac{\pi}{2h} \right) T_1 T_3 T_5 T_7 T_9 T_{11} T_{13}(T_6 T_8 T_{10} T_{12} T_2)^{-1}.
\end{align*}
\]

(93)

The corresponding variables $Z_j = \exp(-m\nu_j y)$, where:

\[
\begin{align*}
\nu_{1,5}^2 &= 3(5 - \sqrt{5})/2 \mp \sqrt{3(5 + \sqrt{5})/2}; \\
\nu_{2,8}^2 &= 3(5 + \sqrt{5})/2 \mp \sqrt{3(25 + 11\sqrt{5})/2}.
\end{align*}
\]

(94)

We give here the explicit form for function $\tau_0$. All other $\tau$-functions can be derived from it using Eq.(69). For example function $\tau_1$ is related with $\tau_0$ by Eq.(89). Function $\tau_0$ can be written in the form:

\[
\tau_0(y) = \sum_{\sigma_1=0}^{1} \cdots \sum_{\sigma_4=0}^{1} Y_{\sigma_1 \cdots \sigma_4}^{(0)} (t_1 Z_1)^{\sigma_1} (t_2 Z_2)^{\sigma_2} (t_5 Z_5)^{\sigma_3} (t_8 Z_8)^{\sigma_4} (95)
\]

The coefficients $Y_{\sigma_1 \cdots \sigma_4}^{(0)}$ for this function are given in Appendix B. It was checked with the accuracy allowed by Mathematica that $\phi(0) = \vartheta$ and the classical boundary state energy (74) coincide with that given by Eq.(11).
6 Concluding Remarks

1. In the previous sections we studied the semiclassical asymptotics of integrable boundary ATTs. In particular, we derived the solution \( \phi(y) \) describing the limit \( b \to 0 \) of one-point function \( F_b(y) \) of the bulk operator with a boundary. Here we shortly consider the opposite (dual) limit \( b \to \infty \) of this function. The boundary value \( \Theta \) of function \( F_b(y) \) is given by Eq. (30), and at this limit is equal

\[
\theta^\vee = -\frac{\pi}{2h} \sum_{\alpha > 0} \alpha \cot \left( \frac{\pi \rho_{\alpha}}{h} \right).
\]  

(96)

For example for the Lie algebra \( D_r \) this equation can be written as:

\[
e_0 \cdot \theta^\vee = e_1 \cdot \theta^\vee = e_{r-1} \cdot \theta^\vee = e_r \cdot \theta^\vee = -\pi/(2h) \cot (2\pi/h); \\
e_k \cdot \theta^\vee = -\pi/(2h)[\cot (2\pi k/h) + \cot (2\pi (r-k)/h)].
\]  

(97)

At the limit \( b \to \infty \) our theory is described by the weakly coupled dual theory, which is boundary ATT with Neumann boundary conditions and function \( \phi^\vee(y) = F_\infty(y) \) can be calculated using the perturbation theory. It can be written in the form:

\[
\phi^\vee(y) = \sum_{j=1}^{r} \xi_j [d_j^\vee e^{-m_j y} - \sum_{i=1}^{r} \int \frac{d\theta}{4\sqrt{h}} \frac{f_{ii}^j \sin(\theta_{ii}) e^{-2m_i y \cosh \theta}}{\cosh^2 \theta - \cos^2(\theta_{ii}/2)}]
\]  

(98)

Here \( f_{ii}^j \) are the fusion constants and \( \theta_{ii} \) are the fusion angles. The constants \( f_{ii}^j \) are different from zero and equal to 1 or \(-1\) only if particle \( j \) is a bound state of two particles \( i \). In this case the fusion angle is defined by relation: \( m_j = 2m_i \cos(\theta_{ii}/2) \). The complete list of constants \( f_{ii}^j \) and angles \( \theta_{ii} \) can be found in Ref. [25]. The constants \( d_j^\vee \) describing one particle contributions are different from zero exactly for the same particles \( j \) that and the constants \( d_j \) calculated in previous section. For these particles they can be expressed in terms of residues of amplitudes \( R_j(\theta) \) (see Eq. (65)) as:

\[
d_j^\vee = \lim_{b \to \infty} \sqrt{\pi b} D_j(b) = \frac{\pi}{\sqrt{h}}
\]  

(99)

As an example of the application of Eq. (98) we give the explicit expression for function \( \phi^\vee(y) \) in \( D_r \) ATTs:

\[
\phi^\vee(y) = \frac{\pi}{\sqrt{h}} \sum_{l=1}^{[r/2]-1} \xi_{2l} \left[ e^{-m_{2l} y} - \int \frac{d\theta}{4\pi} (g_l - g_{r-1-l} + f_l) \right]
\]  

(100)

where

\[
g_l(\theta) = \sin(2\pi l/h) \frac{\exp(-4\sqrt{2} \sin(\pi l/h) my \cosh \theta)}{\cosh^2 \theta - \cos^2(\pi l/h)};
\]

\[
f_l(\theta) = 2 \sin(4\pi l/h) \frac{\exp(-2\sqrt{2} my \cosh \theta)}{\cosh^2 \theta - \sin^2(2\pi l/h)}
\]  

(101)
Figure 4: The functions $\phi_0 = -e_0 \cdot \phi$ and $\phi_0^\vee = -e_0 \cdot \phi^\vee$ for the $D_7$ ATT.

A plot of the functions $\phi_0 = -e_0 \cdot \phi$ and $\phi_0^\vee = -e_0 \cdot \phi^\vee$ for $D_7$ ATT is given in Fig.4. It is straightforward to derive from equations (100,101) that

$$\phi^\vee(0) = \pi \frac{|r/2|^{-1}}{2\sqrt{h}} \sum_{l=1} \xi_{2l} \left(1 - \frac{4l}{h}\right)$$

(102)

If we take into account Eq.(77) we can easily find that boundary value $\phi^\vee(0)$ coincides with vector $\theta^\vee$ defined by Eq.(77). The same result can be obtained for all simply laced ATTs.

2. In this paper we derived the weak and strong coupling limits of function $F_b(y)$. In the intermediate region the long distance behavior of this function can be expressed in terms of boundary amplitudes $R_j$ and form factors [26]. Some of form factors in ATTs are calculated in Ref.[10]. At short distances we can calculate $F_b(0)$ which is equal to vector $\Theta$. Using boundary conditions and equations of motion we can express first two derivatives of this function through the boundary one-point function $G_B(a)$:

$$F_b'(0) = \frac{b^2 \mu_B}{2} \sum_{j=0}^r e_j G_B(be_j); \quad F_b''(0) = b^2 \mu \sum_{j=0}^r e_j G_B(2be_j).$$

(103)

To calculate the next derivatives of this function we need the boundary VEVs of the descendant fields. At present the problem of their calculation is not solved yet. We hope to return to this problem in the next publications.

3. In the main part of this paper we studied boundary simply laced ATTs with dual to Neumann boundary conditions [23]. At the end of this section we consider the
classical limit of non-simply laced ATTs with the same boundary conditions. These theories are dual to the ATTs with dual affine algebras and boundary conditions (21). In the non-simply laced case, the corresponding boundary vacuum solutions \( \phi(y) \) satisfy the same Eqs. (33) with modified boundary conditions at \( y = 0 \) which can be written as:

\[
\partial_y \phi = m \sum_{i=0}^{r} \sqrt{2m_i/c_i^2} e_i \exp(e_i \cdot \phi). \tag{104}
\]

The solutions \( \phi_{nsl} \) of these equations can be obtained from the solutions \( \phi_{sl} \) for simply laced ATTs described in the previous section by the reduction with respect to the symmetries of extended \( DE \) Dynkin diagrams (which are the symmetries of solutions \( \phi_{sl} \)). This reduction can be described in the following way. Affine Dynkin diagrams for non-simply laced algebras can be obtained by folding the nodes of diagrams for simply laced algebras \( G \). Let us the set of nodes \( \{j\} \) of affine Dynkin diagram for \( G \) corresponds after the folding to the node \( j \) of the diagram for non-simply laced algebra \( \mathcal{G} \). We denote as \( e_j \) the root of \( G \) with arbitrary \( J \in \{j\} \) and as \( e_j \) the corresponding root of \( \mathcal{G} \). Then it holds:

\[
e_j \cdot \phi_{nsl}(y) = e_J \cdot \phi_{sl}(y) \tag{105}\]

This relation defines completely the solutions of Eqs. (33,104) for all for non-simply laced algebras \( \mathcal{G} \). The ratio of corresponding boundary ground state energies is equal to the ratio of Coxeter numbers of \( \mathcal{G} \) and \( G \).

Extended Dynkin diagram for the Lie algebra \( B_r \) can be obtained by folding nodes \( r \) and \( r + 1 \) of the diagram for \( D_{r+1} \). In this case \( \mathcal{E}(d)(B_r) = \mathcal{E}(d)(D_{r+1}) \). In the dual limit \( B_r \) ATT with boundary conditions (104) is described by weakly coupled \( B_r^\vee \) (or \( A_{2n-1}^{(2)} \)) ATT with Neumann boundary conditions. The perturbative calculations and analysis of boundary \( S \)-matrix for this theory gives us the reasons to conjecture that in quantum case the boundary ground state energies for \( B_r \) and \( D_{r+1} \) ATTs have a similar form:

\[
\mathcal{E}_{\text{bound}}^{(q)}(B_r) = \frac{m \cos(\pi/4 - \pi/2h)}{4 \sin(\pi x/2h) \cos(\pi(1 - x)/2h)} \tag{106}\]

where \( h \) now depends on \( x \): \( h = 2r - x \).

Folding the nodes \( r + 1 \) with \( r + 2 \) and 0 with 1 of \( D_{r+2} \) diagram we obtain the diagram for \( C_r^\vee \) or ( \( D_{r+1}^{(2)} \)) ATT. The similar consideration of this theory leads us to conjecture that \( \mathcal{E}_{\text{bound}}^{(q)}(C_r^\vee) \) is also described by Eq. (106) where now \( h = 2r + 2(1 - x) \).

Folding the nodes \( k \) and \( 2r - k \) of affine \( D_{2r} \) diagram we obtain the classical solution for \( B_r^\vee \) boundary ATT. In this case \( \mathcal{E}(d)(B_r^\vee) = \mathcal{E}(d)(D_{2r})/2 \). The solution for \( BC_r \) (or \( A_{2n}^{(2)} \)) ATT can be obtained from \( D_{2r+2} \) solution by folding \( D_{2r+2} \) diagram with respect to all symmetries. In this case all \( r \) particles contribute to the solution and \( \mathcal{E}(d)(BC_r) = \mathcal{E}(d)(D_{2r+2})/2 \).

The solution for \( G_2 \) ATT can be obtained from \( D_4 \) solution by folding the nodes 1, 3, 4 of affine \( D_4 \) Dynkin diagram. In this case only the heavy particle \( m_2 \) contributes to the solution and \( \mathcal{E}(d)(G_2) = \mathcal{E}(d)(D_4) \). Contrary to that only the light particle \( m_1 \)
contributes to the solution for \( G_2^\vee \) or \( D_4^{(3)} \) ATT. This solution can be obtained by the reduction with respect threefold symmetry of \( E_6 \) extended Dynkin diagram. In this case \( \mathcal{E}^{(cl)}(G_2^\vee) = \mathcal{E}^{(cl)}(E_6)/3 \). Folding the nodes 1 with \( \mathcal{T} \) and 3 with \( \mathfrak{S} \) of the same diagram we obtain “one particle” solution for \( F_4^\vee \) ATT. The “three particle” solution for \( F_4^\vee \) or \( (E_6^{(2)}) \) ATT can be derived by the reduction with respect to \( \mathbb{Z}_2 \) symmetry of \( E_7 \) extended Dynkin diagram. As a result \( \mathcal{E}^{(cl)}(F_4^\vee) = \mathcal{E}^{(cl)}(E_7)/2 \). The solution for the last ATT corresponding to the Lie algebra \( C_r \) vanishes: \( \phi(y) = 0 \).

In this paper we found explicitly all solutions of Eqs.\((35,104)\). However some of non-simply laced ATTs require more general integrable boundary conditions \([12]\), which form the parametric families. We intend to describe the corresponding boundary one-point functions, scattering theory and boundary solutions in a separate publication.

Acknowledgments

We are grateful to Al. Zamolodchikov V.Riva and B.Ponsot for useful discussions. E.O. warmly thanks A. Neveu, director of L.P.M., University of Montpellier II, and all his colleagues for the kind hospitality at the Laboratory. This work supported by part by the EÚ under contract ERBFMRX CT 960012 and grant INTAS-OPEN-00-00055.

Appendix A

In this appendix we give the explicit form for the function \( \tau_2 \) in \( E_7 \) ATT. This function can be obtained from the function \( \tau_0 \) (see Eqs.\((61,71)\)) using Eq.\((89)\) and is determined by the following coefficients \( Y_{abc}^{(2)}, a, b, c = 0, 1, 2; Y_{000}^{(2)} = 1 \):

\[
\tau_2 = \sum_{a=0}^{2} \sum_{b=0}^{2} \sum_{c=0}^{2} Y_{abc}^{(2)} (t_2 Z_2)^a (t_4 Z_4)^b (t_7 Z_7)^c
\]  

or \((2)\)

where

\[
\begin{align*}
Y_{100}^{(2)} &= y_1^{(2)} = 4 \cos(2\pi/9); & Y_{010}^{(2)} &= y_2^{(2)} = -4 \cos(\pi/9); \\
Y_{001}^{(2)} &= y_3^{(2)} = -4 \cos(4\pi/9); & Y_{110}^{(2)} &= 1/ \cos^2(\pi/9); & Y_{101}^{(2)} &= -1/ \cos^2(2\pi/9); \\
Y_{011}^{(2)} &= -4 \cos^2(4\pi/9)/ \cos^2(\pi/9); & Y_{111}^{(2)} &= 128 \cos^4(4\pi/9)/ \cos^2(\pi/9); \\
Y_{200}^{(2)} &= y_1^2; & Y_{020}^{(2)} &= y_2^2; & Y_{002}^{(2)} &= y_3^2; & Y_{210}^{(2)} &= y_{12} y_2^{(2)} y_1/y_2; \\
Y_{120}^{(2)} &= y_{12} y_1^{(2)} y_2/y_1; & Y_{201}^{(2)} &= y_{13} y_3^{(2)} y_1/y_3; & Y_{102}^{(2)} &= y_{13} y_1^{(2)} y_3/y_1; \\
Y_{021}^{(2)} &= y_{23} y_3^{(2)} y_2/y_3; & Y_{012}^{(2)} &= y_{23} y_2^{(2)} y_3/y_2; & Y_{220}^{(2)} &= y_{12}^2; & Y_{202}^{(2)} &= y_{13}^2; \\
Y_{022}^{(2)} &= y_{23}; & Y_{211}^{(2)} &= y_{12} y_{13} Y_{011}^{(2)}/(y_2 y_3); & Y_{212}^{(2)} &= y_{12} y_2 y_{13} Y_{011}^{(2)}/(y_1 y_3); \\
Y_{112}^{(2)} &= y_{13} y_2 Y_{110}^{(2)}/(y_1 y_2); & Y_{221}^{(2)} &= y_{12} y_{13} y_3^{(2)}/y_3; & Y_{212}^{(2)} &= y_{12} y_{13} y_2^{(2)}/y_2; \\
Y_{122}^{(2)} &= y_{12} y_3 y_3 y_1^{(2)}/y_1; & Y_{222}^{(2)} &= y_{123}^2.
\end{align*}
\]
Appendix B

In this appendix we list the coefficients \( Y_{\sigma_1 \sigma_2 \sigma_3 \sigma_4}^{(0)} \) in the Eq.(93) for \( \tau_0 \) function in \( E_8 \) ATT. It is convenient to simplify the notations. The variables \( \sigma_i \) take the values 0 or 1. We can introduce the variable 1 \( \leq i \leq 4 \) indicating \( \sigma_i \) which are equal to one. The corresponding coefficients \( Y_{\sigma_1 \ldots \sigma_4}^{(0)} \) We denote as \( y_i \), \( y_{ij} \), \( y_{ijk} \) and \( y_{1234} \). For example, \( y_2 = Y_{0100}^{(0)} \); \( y_{13} = Y_{1010}^{(0)} \) and so on. Then \( Y_{0000}^{(0)} = 1 \) and

\[
\begin{align*}
y_1 &= \left( 1 + \sqrt{5} \right)^2 \left( \sqrt{3} + \frac{\sqrt{5}}{3} + \sqrt{4 + \frac{8}{\sqrt{5}}} \right); \\
y_2 &= \left( 1 + \sqrt{5} \right)^2 \left( \frac{7}{\sqrt{3}} - \sqrt{5} + \sqrt{4 + \frac{8}{\sqrt{5}}} \right); \\
y_3 &= \left( 1 + \sqrt{5} \right)^2 \left( \sqrt{3} + \frac{\sqrt{5}}{3} - \sqrt{4 + \frac{8}{\sqrt{5}}} \right); \\
y_4 &= \left( 1 + \sqrt{5} \right)^2 \left( \frac{7}{\sqrt{3}} - \sqrt{5} + \sqrt{4 + \frac{8}{\sqrt{5}}} \right); \\
y_{12} &= \frac{43}{3} - \frac{19}{\sqrt{5}} + \frac{1289}{3} - \frac{9302}{3\sqrt{5}}; \\
y_{13} &= -\frac{44}{3} + \frac{83}{\sqrt{5}} - 4 \sqrt{\frac{298}{3} - \frac{165}{3\sqrt{5}}}; \\
y_{14} &= \frac{71}{3} - \frac{101}{\sqrt{5}} + \frac{7801}{3} - \frac{14342}{3\sqrt{5}}; \\
y_{23} &= \frac{25}{3} - \frac{19}{\sqrt{5}} + 5 \sqrt{\frac{17}{3} - \frac{45}{3\sqrt{5}}}; \\
y_{24} &= \frac{52}{3} + \frac{43}{\sqrt{5}} - 4 \sqrt{\frac{125}{3} - \frac{275}{3\sqrt{5}}}; \\
y_{34} &= \frac{43}{3} + \frac{19}{\sqrt{5}} + \sqrt{\frac{1289}{3} - \frac{9302}{3\sqrt{5}}}; \\
y_{123} &= y_{12} y_{13} y_{23}/(y_{1} y_{2} y_{3}); \\
y_{124} &= y_{12} y_{14} y_{24}/(y_{1} y_{2} y_{4}); \\
y_{134} &= y_{13} y_{14} y_{34}/(y_{1} y_{3} y_{4}); \\
y_{234} &= y_{23} y_{24} y_{34}/(y_{2} y_{3} y_{4}); \\
y_{1234} &= y_{12} y_{13} y_{23} y_{14} y_{24} y_{34}/(y_{1} y_{2} y_{3} y_{4})^2.
\end{align*}
\]

Appendix C

The classical vacuum solutions of Sec.4 have been cross-checked numerically. We used Mathematica’s routine \texttt{NDSolve} which allow good accuracy control through the parameters \texttt{Prec, AccuracyGoal, WorkingPrecision}. The code for the solution of boundary problem (85,86) has been developed for all cases considered in this paper; it is available at \url{http://www.fis.unipr.it/~onofri}.

The inherent instability of the boundary value problem (i.e. the existence of solutions with exponential growth at infinity) makes the direct numerical problem rather ill-conditioned; however this fact is completely under control. The numerical solutions agree with the exact ones until the unstable modes eventually take over by amplifying the unavoidable truncation and algorithmic errors. In Fig.5 we may see that the discrepancy between exact and numerical solution eventually grows with exactly the slope dictated by the square root of the largest eigenvalue of the mass matrix (the pattern is replicated systematically for all the other Toda systems).

In case an exact solution were not available, a possible strategy, is given by “backward-integration”: starting from large values of \( y \) and assuming a general linear combination \( \phi = \sum d_j \xi_j \exp\{-m \nu_j y\} \) one can integrate back to \( y = 0 \) and fit the coefficients \( d_j \) to match the boundary conditions. This technique is not affected by the unstable modes and can give rather accurate values for the asymptotic data.
Figure 5: The exponential growth of the difference between exact and numerical solution: the slope is given by the square root of the highest eigenvalue of the mass matrix.

References

[1] V.Fateev, S.Lukyanov, A.Zamolodchikov and Al.Zamolodchikov, Phys. Letters B406 (1997) 83; Nucl. Phys. B516 (1998) 652.

[2] A.Zamolodchikov and Al.Zamolodchikov, Nucl. Phys. B466 (1996) 577.

[3] C.Ahn , V.Fateev, C.Kim, C.Rim and B.Yang, Nucl. Phys. B565 (2000) 611.

[4] Al.Zamolodchikov, Int. J. Mod. Phys. A10 (1995) 112.

[5] V.Fateev, Phys. Lett. B324 (1994) 45.

[6] V.Fateev and S.Lukyanov, Sov. Sci. Rev. A212 (Physics) (1990) 212.

[7] V.Fateev, Mod. Phys. Lett. A15 (2000) 259.

[8] C.Destri, H.de Vega, Nucl.Phys. B358 (1991) 251.

[9] V.Fateev, “Normalization Factors, Reflection Amplitudes and Integrable Systems” preprint hep-th/0103013

[10] S.Lukyanov, Phys. Lett. B408 (1997) 192.
     V.Brazhnikov, Nucl. Phys. B542 (1999) 694.

[11] S.Ghosdal and A.Zamolodchikov, Int. Jour. Math. Phys. A9 (1994) 3841.
[12] P.Bowcock, E.Corrigan, P.E.Dorey and R.H.Rietdijk, Nucl. Phys. B445 (1995) 469.

[13] E.Corrigan, Int. J. Mod. Phys. A13 (1998) 2709.

[14] G.M.Gandenberger, Nucl. Phys. B542 (1999) 659.

[15] V.Fateev, A.Zamolodchikov and Al.Zamolodchikov, “Boundary Liouville Field Theory 1. Boundary States and Boundary Two-Point Functions” preprint hep-th/0001012.

[16] P.Bowcock, JHEP 05 (1998) 8.

[17] V.Fateev, Mod. Phys. Lett. A16 (2001) 1201.

[18] P.Bowcock and M.Perkins, “Aspects of Classical Backgrounds and Scattering for Affine Toda Theory on a Half Line” preprint hep-th/9909174.

[19] E.Corrigan and G.W. Delius, J. Phys. A32 (1999) 159.

[20] E.Corrigan and A.Taormina, J. Phys. A33 (2000) 8739.

[21] A.Leclair, G.Mussardo, H.Saleur and S.Skorik, Nucl. Phys. B453 (1995) 581.

[22] E.Corrigan, P.E.Dorey, R.H.Rietdijk and R.Sasaki, Phys. Lett. B333 (1994) 83.

[23] G.W.Delius and G.M.Gandenberger, Nucl. Phys. B554 (1999) 325.

[24] A.Fring and R.Köberle, Nucl. Phys. B421 (1994) 159.

[25] H.W.Braden, E.Corrigan and P.E.Dorey, Nucl. Phys. B338 (1990)

[26] P.E.Dorey, M.Pillin, R.Tateo and G.M.T.Watts, Nucl.Phys. B594 (2001) 625.