MEASURE THEORETIC ASPECTS OF ERROR TERMS

By

Kamalakshya Mahatab

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I, Kamalakshya Mahatab, hereby declare that the investigation presented in this thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree or diploma at this or any other Institution or University.

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List of Publications

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2. Kamalakshya Mahatab, Number of Prime Factors of an Integer. Mathematics Newsletter, Ramanujan Mathematical Society, volume 24 (2013).

Others

1. Kamalakshya Mahatab and Anirban Mukhopadhyay, Measure Theoretic Aspects of Oscillations of Error Terms. arXiv:1512.03144v1 (2015).
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# CONTENTS

Notations xv

Synopsis xvii

I Introduction 1
1 Framework . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 3
2 Applications . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4
2.1 Twisted Divisors . . . . . . . . . . . . . . . . . . . . . . . . 5
2.2 Square Free Divisors . . . . . . . . . . . . . . . . . . . . . . . 6
2.3 Divisors . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 7
2.4 Error Term in the Prime Number Theorem . . . . . . . . . . 8
2.5 Non-isomorphic Abelian Groups . . . . . . . . . . . . . . . . . . 8

II Analytic Continuation Of The Mellin Transform 10
1 Perron’s Formula . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 10
2 Analytic continuation of $A(s)$ . . . . . . . . . . . . . . . . . . . . . 13
2.1 Preparatory Lemmas . . . . . . . . . . . . . . . . . . . . . . . 13
2.2 Proof of Theorem 3 . . . . . . . . . . . . . . . . . . . . . . . . 18
3 Alternative Approches . . . . . . . . . . . . . . . . . . . . . . . . . . 18

III Landau’s Oscillation Theorem 21
1 Landau’s Criterion for Sign Change . . . . . . . . . . . . . . . . . . 21
2 $\Omega_\pm$ Results . . . . . . . . . . . . . . . . . . . . . . . . . . . . 21
3 Measure Theoretic $\Omega_\pm$ Results . . . . . . . . . . . . . . . . . . 27
4 Applications . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 33
4.1 Square Free Divisors . . . . . . . . . . . . . . . . . . . . . . . . 33
4.2 The Prime Number Theorem Error . . . . . . . . . . . . . . . . . 36

IV Influence Of Measure 38
1 Refining Omega Result from Measure . . . . . . . . . . . . . . . . . 38
2 Omega Plus-Minus Result from Measure . . . . . . . . . . . . . . . 40
### The Twisted Divisor Function

| Section | Title                                           | Page |
|---------|------------------------------------------------|------|
| 1       | Applications of $\tau(n, \theta)$              | 49   |
| 1.1     | Clustering of Divisors                          | 50   |
| 1.2     | The Multiplication Table Problem                | 51   |
| 2       | Asymptotic Formula for $\sum_{n \leq x} |\tau(n, \theta)|^2$ | 52   |
| 3       | Oscillations of the Error Term                  | 53   |
| 4       | An Omega Theorem                                | 56   |
| 4.1     | Optimality of the Omega Bound                    | 68   |
| 5       | Influence of Measure on $\Omega_x$ Results      | 70   |
**Notations**

We denote the set of natural numbers by \( \mathbb{N} \), the set of integers by \( \mathbb{Z} \), the set of real numbers by \( \mathbb{R} \), the set of positive real numbers by \( \mathbb{R}^+ \), and the set of complex numbers by \( \mathbb{C} \).

The notation \( i \) stands for \( \sqrt{-1} \), the square root of \(-1\) that belongs to the upper half plane in \( \mathbb{C} \).

We denote the Lebesgue measure on the real line \( \mathbb{R} \) by \( \mu \).

For \( z = \sigma + it \in \mathbb{C} \), we denote \( \sigma \) by \( \Re(z) \) and \( t \) by \( \Im(z) \).

Let \( f(x) \) and \( g(x) \) be a complex valued function on \( \mathbb{R}^+ \). As \( x \to \infty \), we write

- \( f(x) = O(g(x)) \), if \( \lim_{x \to \infty} \frac{|f(x)|}{g(x)} > 0 \);
- \( f(x) = o(g(x)) \), if \( \lim_{x \to \infty} \frac{|f(x)|}{g(x)} = 0 \);
- \( f(x) \ll g(x) \), if \( f(x) = O(g(x)) \);
- \( f(x) \gg g(x) \), if \( g(x) = O(f(x)) \);
- \( f(x) \sim g(x) \), if \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = 1 \);
- \( f(x) \asymp g(x) \), if \( 0 < \lim_{x \to \infty} \frac{|f(x)|}{g(x)} < \infty \).

Let \( f(x) \) be a complex valued function on \( \mathbb{R}^+ \), and let \( g(x) \) be a positive monotonic function on \( \mathbb{R}^+ \). As \( x \to \infty \), we write

- \( f(x) = \Omega(g(x)) \), if \( \limsup_{x \to \infty} \frac{|f(x)|}{g(x)} > 0 \);
- \( f(x) = \Omega_+(g(x)) \), if \( \limsup_{x \to \infty} \frac{f(x)}{g(x)} > 0 \);
- \( f(x) = \Omega_-(g(x)) \), if \( \liminf_{x \to \infty} \frac{f(x)}{g(x)} < 0 \);
- \( f(x) = \Omega_\pm(g(x)) \), if \( f(x) = \Omega_+(g(x)) \) and \( f(x) = \Omega_-(g(x)) \).


SYNOPSIS

This thesis studies fluctuation of error terms that appears in various asymptotic formulas and size of the sets where these fluctuations occur. As a consequence, this approach replaces Landau’s criterion on oscillation of error terms.

**General Theory**

Consider a sequence of real numbers \( \{a_n\}_{n=1}^{\infty} \) having Dirichlet series

\[
D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},
\]

which is convergent in some half-plane. As in Perron summation formula \([37 \text{ II.2.1}]\), we write

\[
\sum_{n \leq x}^* a_n = M(x) + \Delta(x),
\]

where \( M(x) \) is the main term, \( \Delta(x) \) is the error term and \( \sum^* \) is defined as

\[
\sum_{n \leq x}^* a_n = \begin{cases} 
\sum_{n \leq x} a_n & \text{if } x \notin \mathbb{N}, \\
\sum_{n < x} a_n + \frac{1}{2}a_x & \text{if } x \in \mathbb{N}.
\end{cases}
\]

In this thesis, we obtain \( \Omega \) and \( \Omega_\pm \) estimates for \( \Delta(x) \). We shall use the Mellin transform of \( \Delta(x) \) (defined below) to obtain such estimates.

**Definition.** The Mellin transform of \( \Delta(x) \) be \( A(s) \), defined as

\[
A(s) = \int_1^\infty \frac{\Delta(x)}{x^{s+1}} \, dx.
\]

In this direction, under some natural assumptions and for a suitably defined contour \( \mathcal{C} \), we shall show that

\[
A(s) = \int_{\mathcal{C}} \frac{D(\eta)}{\eta(s-\eta)} \, d\eta.
\]
In the above formula, the poles of $D(s)$ that lie left to $C$ are all the poles that contributes to the main term $M(x)$. Landau [26] used the meromorphic continuation of $A(s)$ to obtain $\Omega_\pm$ results for $\Delta(x)$. He proved that if $A(s)$ has a pole at $\sigma_0 + it_0$ for some $t_0 \neq 0$ and has no real pole for $s \geq \sigma_0$, then

$$\Delta(x) = \Omega_\pm(x^{\sigma_0}).$$

We shall show a quantitative version of Landau’s theorem, which also generalizes a theorem of Gautami, Ramaré and Schlage-Puchta [6]. Below we state this theorem in a simplified way. We introduce the following notations to state these theorems.

**Definition.** Let

$$A_+^T(x^{\sigma_0}) := \{ T \leq x \leq 2T : \Delta(x) > \lambda x^{\sigma_0} \},$$

$$A_-^T(x^{\sigma_0}) := \{ T \leq x \leq 2T : \Delta(x) < -\lambda x^{\sigma_0} \},$$

$$A_T(x^{\sigma_0}) := A_+^T(x^{\sigma_0}) \cup A_-^T(x^{\sigma_0}),$$

for some $\lambda, \sigma_0 > 0$.

**Theorem.** Let $\sigma_0 > 0$, and let the following conditions hold:

1. $A(s)$ has no real pole for $\Re(s) \geq \sigma_0$,
2. there is a complex pole $s_0 = \sigma_0 + it_0$, $t_0 \neq 0$, of $A(s)$, and
3. for positive functions $h^\pm(x)$ such that $h^\pm(x) \to \infty$ as $x \to \infty$, we have

$$\int_{A_+^T(x^{\sigma_0})} \frac{\Delta^2(x)}{x^{2\sigma_0+1}} \, dx \ll h^+(T).$$

Then

$$\mu(A_+^T(x^{\sigma_0})) = \Omega\left(\frac{T}{h^+(T)}\right),$$

where $\mu$ denotes the Lebesgue measure.

In the above theorem, Condition 2 is a very strong criterion. In the following theorem, we replace Condition 2 by an $\Omega$-bound of $\mu(A_T(x^{\sigma_0}))$ and obtain an $\Omega_\pm$-result from the given $\Omega$-bound.

**Theorem.** Let $\sigma_0 > 0$, and let the following conditions hold:

1. $A(s)$ has no real pole for $\Re(s) \geq \sigma_0$, and
2. $\mu(A_T(x^{\sigma_0})) = \Omega(T^{1-\delta})$ for $0 < \delta < \sigma_0$. 

\[ \text{xviii} \]
Then
\[ \Delta(x) = \Omega( T^{\sigma_0 - \delta} ) \]
for any \( \delta' \) such that \( 0 < \delta' < \delta \).

The above two theorems are applicable to a wide class of arithmetic functions. Now we mention some results obtained by applying these theorems.

**A Twisted Divisor Function**

Given \( \theta \neq 0 \), define
\[ \tau(n, \theta) = \sum_{d \mid n} d^{\theta}. \]

The Dirichlet series of \( |\tau(n, \theta)|^2 \) can be expressed in terms of Riemann zeta function as
\[
D(s) = \sum_{n=1}^{\infty} \frac{|\tau(n, \theta)|^2}{n^s} = \frac{\zeta^2(s) \zeta(s + i\theta) \zeta(s - i\theta)}{\zeta(2s)} \quad \text{for} \quad \Re(s) > 1.
\]

In [13, Theorem 33], Hall and Tenenbaum proved that
\[
\sum_{1 \leq n \leq x} |\tau(n, \theta)|^2 = \omega_1(\theta)x \log x + \omega_2(\theta)x \cos(\theta \log x) + \omega_3(\theta)x + \Delta(x),
\]
where \( \omega_i(\theta) \)'s are explicit constants depending only on \( \theta \). They also showed that
\[
\Delta(x) = O_x^{\frac{1}{2}}(x^{1/4}).
\]

Here the main term comes from the residues of \( D(s) \) at \( s = 1, 1 \pm i\theta \). All other poles of \( D(s) \) come from zeros of \( \zeta(2s) \). Using a pole on the line \( \Re(s) = 1/4 \), Landau’s method gives
\[
\Delta(x) = \Omega_x^{\frac{1}{2}}(x^{1/4}).
\]

We prove the following bounds for a computable \( \lambda(\theta) > 0 \) and for any \( \epsilon > 0 \):
\[
\mu \left( \{ T \leq x \leq 2T : \Delta(x) > (\lambda(\theta) - \epsilon)x^{1/4} \} \right) = \Omega \left( T^{1/2}(\log T)^{-12} \right),
\]
\[
\mu \left( \{ T \leq x \leq 2T : \Delta(x) < (\lambda(\theta) + \epsilon)x^{1/4} \} \right) = \Omega \left( T^{1/2}(\log T)^{-12} \right).
\]

For a constant \( c > 0 \), define
\[
\alpha(T) = \frac{3}{8} - \frac{c}{(\log T)^{1/8}}.
\]
Applying a method due to Balasubramanian, Ramachandra and Subbarao [5], we prove

$$\Delta(T) = \Omega \left( T^{\alpha(T)} \right).$$

In fact, this method gives \(\Omega\)-estimate for the measure of the sets involved:

$$\mu(\mathcal{A} \cap [T, 2T]) = \Omega \left( T^{2\alpha(T)} \right),$$

where

$$\mathcal{A} = \{ x : |\Delta(x)| \geq M x^{\alpha(x)} \}$$

and \(M > 0\) is a positive constant. We also show that

either \(\Delta(x) = \Omega \left( x^{\alpha(x)+\delta/2} \right)\) or \(\Delta(x) = \Omega_{\pm} \left( x^{3/8-\delta'} \right)\),

for \(0 < \delta < \delta' < 1/8\). For any \(\epsilon > 0\), this result and the conjecture

$$\Delta(x) = O(x^{3/8+\epsilon})$$

proves that

$$\Delta(x) = \Omega_{\pm}(x^{3/8-\epsilon}).$$

**Prime Number Theorem Error**

Let \(a_n\) be the von Mandoldt function \(\Lambda(n)\):

$$\Lambda(n) := \begin{cases} \log p & \text{if } n = p^r, \ r \geq 1, \ p \text{ prime} \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\sum_{n \leq x}^* \Lambda_n = x + \Delta(x).$$

From the Vinogradov’s zero free region for Riemann zeta function, one gets [23, Theorem 12.2]

$$\Delta(x) = O \left( x \exp \left( -c (\log x)^{3/5} (\log \log x)^{-1/5} \right) \right)$$

for some constant \(c > 0\).

Hardy and Littlewood [15] proved that

$$\Delta(x) = \Omega_{\pm} \left( x^{1/2} \log \log \log x \right).$$
But this result does not say about the measure of the sets, where the above $\Omega_\pm$ bounds are attained by $\Delta(x)$. We obtain the following weaker result, but with an $\Omega$-estimates for the measure of the corresponding sets.

Let $\lambda_1 > 0$ denotes a computable constant. For a fixed $\epsilon$, $0 < \epsilon < \lambda_1$, we write

\[
\mathcal{A}_1 := \{ x : \Delta(x) > (\lambda_1 - \epsilon)x^{1/2} \}, \\
\mathcal{A}_2 := \{ x : \Delta(x) < (-\lambda_1 + \epsilon)x^{1/2} \}.
\]

Then

\[
\mu([T, 2T] \cap \mathcal{A}_j) = \Omega(T^{1-\epsilon}) \quad \text{for } j = 1, 2 \quad \text{and for any } \epsilon > 0.
\]

Under Riemann Hypothesis, we have

\[
\mu([T, 2T] \cap A_j) = \Omega\left(\frac{T}{\log T^4}\right) \quad \text{for } j = 1, 2.
\]

We also show the following unconditional $\Omega$-bounds for the second moment of $\Delta$:

\[
\int_{[T, 2T] \cap \mathcal{A}_j} \Delta^2(x) dx = \Omega(T^2) \quad \text{for } j = 1, 2.
\]

**Non-isomorphic Abelian Groups**

Let $a_n$ denote the number of non-isomorphic abelian groups of order $n$. We write

\[
\sum_{n \leq x}^* a_n = \sum_{k=1}^{6} b_k x^{1/k} + \Delta(x).
\]

In the above formula, we define $b_k$ as

\[
b_k := \prod_{j=1, j \neq k}^{\infty} \zeta(j/k).
\]

It is an open problem to show that

\[
\Delta(x) \ll x^{1/6+\delta} \quad \text{for any } \delta > 0. \tag{1}
\]

The best result on upper bound of $\Delta(x)$ is due to O. Robert and P. Sargos \textsuperscript{33}, which gives

\[
\Delta(x) \ll x^{1/4+\epsilon} \quad \text{for any } \epsilon > 0.
\]
Also Balasubramanian and Ramachandra [4] proved that
\[ \Delta(x) = \Omega(x^{1/6}\sqrt{\log x}). \]

Following their method, we prove
\[ \mu \left( \{T \leq x \leq 2T : |\Delta(x)| \geq \lambda_2 x^{1/6}(\log x)^{1/2} \} \right) = \Omega(T^{5/6-\epsilon}), \]
for some \( \lambda_2 > 0 \) and for any \( \epsilon > 0 \). They also obtained
\[ \Delta(x) = \Omega_{\pm}(x^{92/1221}), \]
while it has been conjectured that
\[ \Delta(x) = \Omega_{\pm}(x^{1/6-\delta}), \]
for any \( \delta > 0 \). We shall show that either
\[ \int_T^{2T} \Delta^4(x) dx = \Omega(T^{5/3+\delta}) \text{ or } \Delta(x) = \Omega_{\pm}(x^{1/6-\delta}), \]
for any \( 0 < \delta < 1/42 \). The conjectured upper bound [1] of \( \Delta(x) \) gives
\[ \int_T^{2T} \Delta^4(x) dx \ll T^{5/3+\delta}. \]
This along with our result implies that
\[ \Delta(x) = \Omega_{\pm}(x^{1/6-\delta}) \text{ for any } 0 < \delta < 1/42. \]
[ I ] INTRODUCTION

In 1896, Jacques Hadamard and Charles Jean de la Vallée-Poussin proved that the number of primes up to $x$ is asymptotic to $x / \log x$. This result is well known as the Prime Number Theorem (PNT). Below we state a version of this theorem (PNT*) in terms of the von-Mangoldt function.

**Definition 1.** For $n \in \mathbb{N}$, the von-Mangoldt function $\Lambda(n)$ is defined as
$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^r, \, r \in \mathbb{N} \text{ and } p \text{ prime}, \\ 0 & \text{otherwise}. \end{cases}$$

**Theorem (PNT*).** For a constant $c_1 > 0$, we have
$$\sum_{n \leq x} \Lambda(n) = x + O \left( x \exp \left( -c_1 \left( \log x \right)^{3/5} (\log \log x)^{-1/5} \right) \right),$$
where
$$\sum_{n \leq x} \Lambda(n) = \begin{cases} \sum_{n \leq x} \Lambda(n) & \text{if } x \notin \mathbb{N}, \\ \sum_{n \leq x} \Lambda(n) - \Lambda(x)/2 & \text{otherwise}. \end{cases}$$

For a proof of the above theorem see [23, Theorem 12.2]. Proof of PNT* uses analytic continuation of the function
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$
defined for $\Re(s) > 1$. The function $\zeta(s)$ is called the ‘Riemann zeta function’, named after the famous German mathematician Bernhard Riemann. In 1859, Riemann showed that this has a meromorphic continuation to the whole complex plane. He also showed PNT by assuming that the meromorphic continuation of $\zeta(s)$ does not have zeros for $\Re(s) > \frac{1}{2}$. This conjecture of Riemann is popularly known as the ‘Riemann Hypothesis’ (RH), and is an unsolved problem. Under RH, the upper bound for $\Delta(x)$ in PNT* can be improved as in the following theorem:
Theorem (PNT**). Let $\Delta(x)$ be defined as in PNT*. Further, if we assume RH, then
\[ \Delta(x) = O \left( x^{\frac{3}{4}} \log^2 x \right). \]

Proof. See [40].

In fact, we shall see in Theorem [6] that PNT** is equivalent to RH. At this point, it is natural to ask the following questions:

- Can we obtain a bound for $\Delta(x)$, better than the bound in PNT**?
- Is $\Delta(x)$ an increasing or a decreasing function?
- Can $\Delta(x)$ be both positive and negative depending on $x$?
- How large are positive and negative values of $\Delta(x)$?

We shall make an attempt to answer these question by obtaining $\Omega$ and $\Omega_\pm$ results. The following result was obtained by Hardy and Littlewood [15] in the year 1916:
\[ \Delta(x) = \Omega_\pm \left( x^{\frac{1}{2}} \log \log \log x \right). \] (I.1)

The above $\Omega_\pm$ bound on $\Delta(x)$ gives some answer to our earlier questions. It says that we cannot have an upper bound for $\Delta(x)$ which is smaller than $x^{\frac{1}{2}} \log \log \log x$. It also says that $\Delta(x)$ often takes both positive and negative values with magnitude of order $x^{\frac{1}{2}} \log \log \log x$. This suggests, it is important to obtain $\Omega$ and $\Omega_\pm$ bounds for various other error terms. In this direction, Landau’s theorem [26] (see Theorem [6] below) gives an elegant tool to obtain $\Omega_\pm$ results. Applying this theorem, we have
\[ \Delta(x) = \Omega_\pm \left( x^{\frac{3}{4}} \right). \]

The advantage of Landau’s method as compared to Hardy and Littlewood’s method is in its applicability to a wide class of error terms of various summatory functions. In Landau’s method, the existence of a complex pole with real part $\frac{1}{2}$ serves as a criterion for the existence of above limits. In this thesis, we shall investigate on a quantitative version of Landau’s result by obtaining the Lebesgue measure of the sets where $\Delta(x) > \lambda x^{1/2}$ and $\Delta(x) < -\lambda x^{1/2}$, for some $\lambda > 0$. We shall show that the large Lebesgue measure of the set where $|\Delta(x)| > \lambda x^{1/2}$, for some $\lambda > 0$ replaces the criterion of existence of a complex pole in Landau’s method. This approach has the advantage of getting $\Omega_\pm$ results even when no such complex pole exists. This is evident from some applications which we discuss in this thesis.
1 Framework

In this thesis, we consider a sequence of real numbers \( \{a_n\}_{n=1}^{\infty} \) having Dirichlet series

\[
D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}
\]

that converges in some half-plane. The Perron summation formula (see Theorem \[1\]) uses analytic properties of \( D(s) \) to give

\[
\sum_{n \leq x}^* a_n = \mathcal{M}(x) + \Delta(x),
\]

where \( \mathcal{M}(x) \) is the main term, \( \Delta(x) \) is the error term (which would be specified later) and \( \sum^* \) is defined as

\[
\sum_{n \leq x}^* a_n = \begin{cases} 
\sum_{n \leq x} a_n & \text{if } x \notin \mathbb{N} \\
\sum_{n \leq x} a_n - \frac{1}{2}a_x & \text{if } x \in \mathbb{N}.
\end{cases}
\]

In Chapter \[\text{II}\] we analyze the Mellin transform \( A(s) \) of \( \Delta(x) \), which is defined as:

**Definition 2.** For a complex variable \( s \), the Mellin transform \( A(s) \) of \( \Delta(x) \) is defined by

\[
A(s) = \int_{1}^{\infty} \frac{\Delta(x)}{x^{s+1}} \, dx.
\]

In general, \( A(s) \) is holomorphic in some half plane. We shall discuss a method to obtain a meromorphic continuation of \( A(s) \) from the meromorphic continuation of \( D(s) \). In particular, we shall prove in Theorem \[3\] that under some natural assumptions

\[
A(s) = \int_{C} \frac{D(\eta)}{\eta(s-\eta)} \, d\eta,
\]

where the contour \( \mathcal{C} \) is as in Definition \[3\] and \( s \) lies to the right of \( \mathcal{C} \). Later, this result will complement Theorem \[9\] and Theorem \[11\] in their applications.

In Chapter \[\text{III}\] we revisit Landau’s method and obtain measure theoretic results. Also we generalize a theorem of Kaczorowski and Szydło \[24\], and a theorem of Bhowmik, Ramaré and Schlage-Puchta \[6\] in Theorem \[11\].

Let

\[
\mathcal{A}(\alpha, T) := \{ x : x \in [T, 2T], |\Delta(x)| > x^\alpha \},
\]
and let $\mu$ denotes the Lebesgue measure on $\mathbb{R}$. In Chapter IV we establish a connection between $\mu(A(\alpha,T))$ and fluctuations of $\Delta(x)$. In Proposition 3 we see that

$$\mu(A(\alpha,T)) \ll T^{1-\delta} \implies \Delta(x) = \Omega(x^{\alpha+\delta/2}).$$

However, Theorem 13 gives that

$$\mu(A(\alpha,T)) = \Omega(T^{1-\delta}) \implies \Delta(x) = \Omega_{\pm}(x^{\alpha-\delta}),$$

provided $A(s)$ does not have a real pole for $\Re(s) \geq \alpha - \delta$. In particular, this says that either we can improve on the $\Omega$ result or we can obtain a tight $\Omega_{\pm}$ result for $\Delta(x)$.

In Chapter V we study a twisted divisor function defined as follows:

$$\tau(n,\theta) = \sum_{d|n} d^{i\theta} \text{ for } \theta \neq 0.$$  \hfill (I.2)

This function is used in [13, Chapter 4] to measure the clustering of divisors. We give a brief note on some applications of $\tau(n,\theta)$ in Section V.1. In [13, Theorem 33], Hall and Tenenbaum proved that

$$\sum_{n \leq x} |\tau(n,\theta)|^2 = \omega_1(\theta)x \log x + \omega_2(\theta)x \cos(\theta \log x) + \omega_3(\theta)x + \Delta(x), \quad \text{(I.3)}$$

where $\omega_i(\theta)$s are explicit constants depending only on $\theta$. They also showed that

$$\Delta(x) = O(x^{1/2} \log^6 x). \quad \text{(I.4)}$$

We give a proof of this formula in Theorem 14. Also, we derive $\Omega$ and $\Omega_{\pm}$ bounds for $\Delta(x)$ using techniques from previous chapters. In Theorem 15 we obtain an $\Omega$ bound for the second moment of $\Delta(x)$ by adopting a technique due to Balasubramanian, Ramachandra and Subbarao [5].

The main theorems of this thesis, except Theorem 11, are from [28], which is a joint work of the author with A. Mukhopadhyay.

2 Applications

Now we conclude the introduction by mentioning few applications of the methods given in this thesis.
2.1 Twisted Divisors

Consider the twisted divisor function $\tau(n, \theta)$ defined in the previous section. The Dirichlet series of $|\tau(n, \theta)|^2$ can be expressed in terms of the Riemann zeta function as:

$$D(s) = \sum_{n=1}^{\infty} \frac{|\tau(n, \theta)|^2}{n^s} = \frac{\zeta^2(s)\zeta(s + i\theta)\zeta(s - i\theta)}{\zeta(2s)} \quad \text{for } \Re(s) > 1. \quad (I.5)$$

In Theorem 14, we shall show

$$\sum_{n \leq x} |\tau(n, \theta)|^2 = \omega_1(\theta)x \log x + \omega_2(\theta)x \cos(\theta \log x) + \omega_3(\theta)x + \Delta(x),$$

where $\omega_i(\theta)$s are explicit constants depending only on $\theta$ and

$$\Delta(x) = O_\theta(x^{1/2} \log^6 x).$$

The Dirichlet series $D(s)$ has poles at $s = 1, 1 \pm i\theta$ and at the zeros of $\zeta(2s)$. Using a complex pole on the line $\Re(s) = 1/4$, Landau’s method gives

$$\Delta(x) = \Omega_\pm(x^{1/4}).$$

In order to apply the method of Bhowmik, Ramaré and Schlage-Puchta, we need

$$\int_T^{2T} \Delta^2(x)dx \ll T^{2\sigma_0 + 1 + \epsilon},$$

for any $\epsilon > 0$ and $\sigma_0 = 1/4$; such an estimate is not possible due to Corollary 3. Generalization of this method in Theorem 9 can be applied to get

$$\mu(\mathcal{A}_j \cap [T, 2T]) = \Omega(T^{1/2}(\log T)^{-12}) \quad \text{for } j = 1, 2,$$

and here $\mathcal{A}_j$s' for $\Delta(x)$ are defined as

$$\mathcal{A}_1 = \{x : \Delta(x) > (\lambda(\theta) - \epsilon)x^{1/4}\} \quad \text{and} \quad \mathcal{A}_2 = \{x : \Delta(x) < (\lambda(\theta) + \epsilon)x^{1/4}\},$$

for any $\epsilon > 0$ and $\lambda(\theta) > 0$ as in (V.3). But under Riemann Hypothesis, we show in (V.5) that the above $\Omega$ bounds can be improved to

$$\mu(\mathcal{A}_j) = \Omega(T^{3/4 - \epsilon}) \quad \text{for } j = 1, 2 \quad \text{and for any } \epsilon > 0.$$
Fix a constant $c_2 > 0$ and define 

$$\alpha(T) = \frac{3}{8} - \frac{c_2}{(\log T)^{1/8}}.$$ 

In Corollary 4, we prove that 

$$\Delta(T) = \Omega \left(T^{\alpha(T)}\right).$$

In Proposition 6, we give an $\Omega$ estimate for the measure of the sets involved in the above bound: 

$$\mu(\mathcal{A} \cap [T, 2T]) = \Omega \left(T^{2\alpha(T)}\right),$$

where 

$$\mathcal{A} = \{x : |\Delta(x)| \geq Mx^\alpha\}$$

for a positive constant $M > 0$. In Theorem 17 we show that 

either $\Delta(x) = \Omega \left(x^{\alpha(x)+\delta/2}\right)$ or $\Delta(x) = \Omega_\pm \left(x^{3/8-\delta'}\right)$,

for $0 < \delta < \delta' < 1/8$. We may conjecture that 

$$\Delta(x) = O(x^{3/8+\epsilon}) \text{ for any } \epsilon > 0.$$

Theorem 17 and this conjecture imply that 

$$\Delta(x) = \Omega_\pm \left(x^{3/8-\epsilon}\right) \text{ for any } \epsilon > 0.$$

### 2.2 Square Free Divisors

Let $\Delta(x)$ be the error term in the asymptotic formula for partial sums of the square free divisors: 

$$\Delta(x) = \sum_{n \leq x} \omega(n) - \frac{x \log x}{\zeta(2)} + \left(- \frac{2\zeta'(2)}{\zeta^2(2)} + \frac{2\gamma - 1}{\zeta(2)}\right)x,$$

where $\omega(n)$ denotes the number of distinct primes divisors of $n$. It is known that $\Delta(x) \ll x^{1/2}$ (see [19]). Let $\lambda_1 > 0$ and the sets $\mathcal{A}_j$ for $j = 1, 2$ be defined as in Section 4.1: 

$$\mathcal{A}_1 = \{x : \Delta(x) > (\lambda_1 - \epsilon)x^{1/4}\}, \quad \text{and} \quad \mathcal{A}_2 = \{x : \Delta(x) < (-\lambda_1 + \epsilon)x^{1/4}\}.$$
In (III.14), we show that
\[ \mu (A_j \cap [T, 2T]) = \Omega \left( T^{1/2} \right) \text{ for } j = 1, 2. \]

But under Riemann Hypothesis, we prove the following \( \Omega \) bounds in (III.15):
\[ \mu (A_j \cap [T, 2T]) = \Omega \left( T^{1-\epsilon} \right) \text{, for } j = 1, 2 \text{ and for any } \epsilon > 0. \]

2.3 Divisors

Let \( d(n) \) denotes the number of divisors of \( n \):
\[ d(n) = \sum_{d|n} 1. \]

Dirichlet [17, Theorem 320] showed that
\[ \sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + \Delta(x), \]
where \( \gamma \) is the Euler constant and
\[ \Delta(x) = O(\sqrt{x}). \]

Latest result on \( \Delta(x) \) is due to Huxley [20], which is
\[ \Delta(x) = O(x^{131/416}). \]

On the other hand, Hardy [14] showed that
\[ \Delta(x) = \Omega_+((x \log x)^{1/4} \log \log x), \]
\[ = \Omega_-(x^{1/4}). \]

There are many improvements on Hardy’s result due to K. Corrádi and I. Kátaí [7], J. L. Hafner [12] and K. Sounderarajan [36]. As a consequence of Theorem 13, we shall show in Chapter IV that for all sufficiently large \( T \) and for a constant \( c_3 > 0 \), there exist \( x_1, x_2 \in [T, 2T] \) such that
\[ \Delta(x_1) > c_3 x_1 \text{ and } \Delta(x_2) < -c_3 x_2. \]

In particular, we get
\[ \Delta(x) = \Omega_\pm(x^{1/4}). \]
2.4 Error Term in the Prime Number Theorem

Let $\Delta(x)$ be the error term in the Prime Number Theorem:

$$\Delta(x) = \sum_{n \leq x} \Lambda(n) - x.$$

We know from Landau’s theorem [26] that

$$\Delta(x) = \Omega_\pm (x^{1/2})$$

and from the theorem of Hardy and Littlewood [15] that

$$\Delta(x) = \Omega_\pm \left(x^{1/2} \log \log x\right).$$

We define

$$A_1 = \{ x : \Delta(x) > (\lambda_2 - \epsilon)x^{1/2} \} \quad \text{and} \quad A_2 = \{ x : \Delta(x) < (-\lambda_2 + \epsilon)x^{1/2} \},$$

where $\lambda_2 > 0$ be as in Section 4.2. If we assume Riemann Hypothesis, then the theorem of Kaczorowski and Szydło (see Theorem 8 below) along with PNT** gives

$$\mu(A_j \cap [T, 2T]) = \Omega \left(\frac{T}{\log^4 T}\right) \quad \text{for} \quad j = 1, 2.$$

However, as an application of Corollary [4] we prove the following weaker bound unconditionally:

$$\mu(A_j \cap [T, 2T]) = \Omega \left(T^{1-\epsilon}\right), \quad \text{for} \quad j = 1, 2 \quad \text{and for any} \quad \epsilon > 0.$$

2.5 Non-isomorphic Abelian Groups

Let $a_n$ be the number of non-isomorphic abelian groups of order $n$, and the corresponding Dirichlet series is given by

$$\sum_{n=1}^{\infty} a(n) \frac{1}{n^s} = \prod_{k=1}^{\infty} \zeta(ks) \quad \text{for} \quad \Re(s) > 1.$$
Let $\Delta(x)$ be defined as

$$\Delta(x) = \sum_{n \leq x}^* a_n - \sum_{k=1}^6 \left( \prod_{j \neq k} \zeta(j/k) \right) x^{1/k}.\]$$

It is an open problem to show that

$$\Delta(x) \ll x^{1/6+\epsilon} \text{ for any } \epsilon > 0. \quad (I.6)$$

The best result on upper bound of $\Delta(x)$ is due to O. Robert and P. Sargos [33], which gives

$$\Delta(x) \ll x^{1/4+\epsilon} \text{ for any } \epsilon > 0.$$

Balasubramanian and Ramachandra [4] proved that

$$\int_T^{2T} \Delta^2(x) dx = \Omega(T^{4/3} \log T).$$

Following the proof of Proposition [6] we get

$$\mu \left( \{ T \leq x \leq 2T : |\Delta(x)| \geq \lambda_3 x^{1/6} (\log x)^{1/2} \} \right) = \Omega(T^{5/6-\epsilon}),$$

for some $\lambda_3 > 0$ and for any $\epsilon > 0$. Sankaranarayanan and Srinivas [35] proved that

$$\Delta(x) = \Omega_{\pm} \left( x^{1/10} \exp \left( c \sqrt{\log x} \right) \right)$$

for some constant $c > 0$. It has been conjectured that

$$\Delta(x) = \Omega_{\pm}(x^{1/6-\delta}) \text{ for any } \delta > 0.$$ 

In Proposition [2] we prove that either

$$\int_T^{2T} \Delta^4(x) dx = \Omega(T^{5/3+\delta}) \text{ or } \Delta(x) = \Omega_{\pm}(x^{1/6-\delta}),$$

for any $0 < \delta < 1/42$. The conjectured upper bound (I.6) of $\Delta(x)$ gives

$$\int_T^{2T} \Delta^4(x) dx \ll T^{5/3+\delta}.$$  

This along with Proposition [2] implies that

$$\Delta(x) = \Omega_{\pm}(x^{1/6-\delta}) \text{ for any } 0 < \delta < 1/42.$$  

II Analytic Continuation Of The Mellin Transform

In this chapter, we express the error term \( \Delta(x) \) as a contour integral using the Perron’s formula. This allows us to obtain a meromorphic continuation of \( A(s) \) (see Definition [2]) in terms of the meromorphic continuation of \( D(s) \), which is the main theorem of this chapter (Theorem [3]). This theorem will be used in the next chapter to obtain \( \Omega_\pm \) results for \( \Delta(x) \).

1 Perron’s Formula

Recall that we have a sequence of real numbers \( \{a_n\}_{n=1}^\infty \), with its Dirichlet series \( D(s) \). The Perron summation formula approximates the partial sums of \( a_n \) by expressing it as a contour integral involving \( D(s) \).

**Theorem 1** (Perron’s Formula, Theorem II.2.1 [37]). Let \( D(s) \) be absolutely convergent for \( \Re(s) > \sigma_c \), and let \( \kappa > \max(0, \sigma_c) \). Then for \( x \geq 1 \), we have

\[
\sum_{n \leq x}^* a_n = \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{D(s)x^s}{s} \, ds.
\]

But in practice, we use the following effective version of the Perron’s formula.

**Theorem 2** (Effective Perron’s Formula, Theorem II.2.1 [37]). Let \( \{a_n\}_{n=1}^\infty \), \( D(s) \) and \( \kappa \) be defined as in Theorem [4]. Then for \( T \geq 1 \) and \( x \geq 1 \), we have

\[
\sum_{n \leq x}^* a_n = \int_{\kappa-iT}^{\kappa+iT} \frac{D(s)x^s}{s} \, ds + O\left(x^\kappa \sum_{n=1}^\infty \frac{|a_n|}{n^\kappa(1+T|\log(x/n)|)}\right).
\]

The above formulas are used by shifting the line of integration, and thus by collecting the residues of \( D(s)x^s/s \) at its poles lying to the right of the shifted contour. The residues contribute to the main term \( M(x) \), leaving an expression for \( \Delta(x) \) as a
contour integral. So we write

$$\sum_{n \leq x}^* a_n = M(x) + \Delta(x),$$

where $M(x)$ is the main term and $\Delta(x)$ is the error term. We make the following natural assumptions on $D(s), M(x)$ and $\Delta(x)$.

**Assumptions 1.** Suppose there exist real numbers $\sigma_1$ and $\sigma_2$ satisfying $0 < \sigma_1 < \sigma_2$, such that

(i) $D(s)$ is absolutely convergent for $\Re(s) > \sigma_2$.

(ii) $D(s)$ can be meromorphically continued to the half plane $\Re(s) > \sigma_1$ with only finitely many poles $\rho$ of $D(s)$ satisfying $\sigma_1 < \Re(\rho) \leq \sigma_2$.

We shall denote this set of poles by $P$.

(iii) The main term $M(x)$ is sum of residues of $\frac{D(s)x^s}{s}$ at poles in $P$:

$$M(x) = \sum_{\rho \in P} \text{Res}_{s=\rho} \left( \frac{D(s)x^s}{s} \right).$$

The above assumptions also imply:

**Note 1.** We may also observe:

(i) For any $\epsilon > 0$, we have

$$|a_n|, |M(x)|, |\Delta(x)|, \left| \sum_{n \leq x}^* a_n \right| \ll x^{\sigma_2 + \epsilon}.$$

(ii) The main term $M(x)$ is a polynomial in $x$, and $\log x$:

$$M(x) = \sum_{j \in J} \nu_{1,j} x^{\nu_{2,j}} (\log x)^{\nu_{3,j}},$$

where $\nu_{1,j}$ are complex numbers, $\nu_{2,j}$ are real numbers with $\sigma_1 < \nu_{2,j} \leq \sigma_2$, $\nu_{3,j}$ are positive integers, and $J$ is a finite index set.

To express $\Delta(x)$ in terms of a contour integration, we define the following contour.
Definition 3. Let $\sigma_1, \sigma_2$ be as defined in Assumptions [1]. Choose a positive real number $\sigma_3$ such that $\sigma_3 > \sigma_2$. We define the contour $\mathcal{C}$, as in Figure [II.1], as the union of the following five line segments:

$$\mathcal{C} = L_1 \cup L_2 \cup L_3 \cup L_4 \cup L_5,$$

where

$$L_1 = \{ \sigma_3 + iv : T_0 \leq v < \infty \}, \quad L_2 = \{ u + iT_0 : \sigma_1 \leq u \leq \sigma_3 \},$$
$$L_3 = \{ \sigma_1 + iv : -T_0 \leq v \leq T_0 \}, \quad L_4 = \{ u - iT_0 : \sigma_1 \leq u \leq \sigma_3 \},$$
$$L_5 = \{ \sigma_3 + iv : -\infty < v \leq -T_0 \}.$$

Now, we write $\Delta(x)$ as an integration over $\mathcal{C}$ in the following lemma.

Lemma 1. Under Assumptions [1] the error term $\Delta(x)$ can be expressed as:

$$\Delta(x) = \int_{\mathcal{C}} \frac{D(\eta) x^\eta}{\eta} d\eta.$$

Proof. Follows from Theorem [1].
2 Analytic continuation of $A(s)$

Now, we shall discuss a method to obtain a meromorphic continuation of $A(s)$, which will serve as an important tool to obtain $\Omega_{\pm}$ results for $\Delta(x)$ in the following chapter. Below we present the main theorem of this chapter.

**Theorem 3.** Under Assumptions, we have

$$A(s) = \int_{\mathcal{C}} \frac{D(\eta)}{\eta(s-\eta)} d\eta,$$

when $s$ lies right to $\mathcal{C}$.

2.1 Preparatory Lemmas

We shall need the following preparatory lemmas to prove the above theorem.

From Lemma 1, we have:

$$A(s) = \int_{1}^{\infty} \int \frac{D(\eta)}{\eta} dx \frac{d\eta}{x^{s+1}}. \quad (\text{II.1})$$

To justify Theorem 3, we need to justify the interchange of the integrals of $\eta$ and $x$ in (II.1).

**Definition 4.** Define the following complex valued function $B(s)$:

$$B(s) := \int_{\mathcal{C}} \frac{D(\eta)}{\eta} \int_{1}^{\infty} \frac{dx}{x^{s-\eta+1}} d\eta$$

$$= \int_{\mathcal{C}} \frac{D(\eta)}{(s-\eta)\eta} d\eta \quad \text{for } \Re(s) > \Re(\eta).$$

The integral defining $B(s)$ being absolutely convergent, we have $B(s)$ is well defined and analytic.

**Definition 5.** For a positive integer $N$, define the contour $\mathcal{C}(N)$ as:

$$\mathcal{C}(N) = \{ \eta \in \mathcal{C} : |\Im(\eta)| \leq N \}.$$

**Definition 6.** Integrating the integrals of $\eta$ and $x$, define $B_N(s)$ as:

$$B_N(s) = \int_{\mathcal{C}(N)} \frac{D(\eta)}{\eta} \int_{1}^{\infty} \frac{dx}{x^{s-\eta+1}}$$
\[
= \int_{\varepsilon(N)} D(\eta) \frac{d\eta}{(s-\eta)\eta} \text{ for } \Re(s) > \Re(\eta).
\]

With above definitions we prove:

**Lemma 2.** The functions \(B\) and \(B_N\) satisfy the following identities:

\[
B(s) = \lim_{N \to \infty} B_N(s) \quad \text{(II.2)}
\]

\[
= \lim_{N \to \infty} \int_{\varepsilon(N)}^{1} \int_{\varepsilon(N)}^{\infty} \frac{D(\eta)x^\eta}{\eta} d\eta \frac{dx}{x^{s+1}}. \quad \text{(II.3)}
\]

**Proof.** Assume \(N > T_0\). To show (II.2), note:

\[
\left| B(s) - B_N(s) \right| \leq \left| \int_{\varepsilon(N)}^{\sigma_3+iN} D(\eta) d\eta \right|
\]

\[
\ll \left| \int_{\sigma_3+iN}^{\sigma_3+i\infty} \frac{D(\eta)}{(s-\eta)\eta} d\eta \right| + \left| \int_{\sigma_3-i\infty}^{\sigma_3-iN} \frac{D(\eta)}{(s-\eta)\eta} d\eta \right|
\]

\[
\ll \int_{N}^{\infty} \frac{dv}{v^2} \ll \frac{1}{N}. \quad \text{(substituting } \eta = \sigma_3 + iv)\]

This completes proof of (II.2).

We shall prove (II.3) using a theorem of Fubini and Tonelli [8, Theorem B.3.1, (b)]. To show that the integrals commute, we need to show that one of the iterated integrals in (II.3) converges absolutely. We note:

\[
\int_{\varepsilon(N)}^{\infty} \int_{1}^{\infty} \frac{D(\eta)}{\eta x^{s-\eta+1}} dx \frac{d\eta}{|d\eta|} < \infty
\]

This implies (II.3). \( \Box \)

Let

\[
B'_N(s) := \int_{1}^{\infty} \int_{\varepsilon(N)}^{\infty} \frac{D(\eta)x^\eta}{\eta} d\eta \frac{dx}{x^{s+1}}. \quad \text{(II.4)}
\]

We re-write (II.3) of Lemma 2 as:

\[
\lim_{N \to \infty} B'_N(s) = B(s).
\]

14
Observe that \( A(s) = B(s) \), if
\[
\lim_{N \to \infty} \int_1^\infty \int_{\mathbb{C} - \mathbb{C}(N)} \frac{D(\eta)x^n}{\eta} \frac{dx}{x^{s+1}} = 0;
\]

can be shown by interchanging the integral of \( x \) with the limit. For this, we need the uniform convergence of the integrand, which we do not have. It is easy to see from Theorem \( \mathbb{E} \) that the problem arises when \( x \) is an integer. To handle this problem, we shall divide the integral in two parts, with one part having neighborhoods of integers.

**Definition 7.** For \( \delta = \frac{1}{\sqrt{N}} \) \(( \text{where } N \geq 2 \) \), we construct the following set as a neighborhood of integers:
\[
S(\delta) := [1, 1 + \delta] \cup (\bigcup_{m \geq 2}[m - \delta, m + \delta]).
\]

Write
\[
A(s) - B'_{N}(s) = J_{1,N}(s) + J_{2,N}(s) - J_{3,N}(s), \quad (\text{II}.5)
\]
where
\[
J_{1,N}(s) = \int_{S(\delta)^c} \int_{\mathbb{C} - \mathbb{C}(N)} \frac{D(\eta)x^n}{\eta} \frac{dx}{x^{s+1}},
\]
\[
J_{2,N}(s) = \int_{S(\delta)} \int_{\sigma_3 - i\infty}^{\sigma_3 + i\infty} \frac{D(\eta)x^n}{\eta} \frac{dx}{x^{s+1}},
\]
\[
J_{3,N}(s) = \int_{S(\delta)} \int_{\sigma_3 - i\infty}^{\sigma_3 + iN} \frac{D(\eta)x^n}{\eta} \frac{dx}{x^{s+1}}.
\]

In the next three lemmas, we shall show that each of \( J_{i,N}(s) \to 0 \) as \( N \to \infty \).

**Lemma 3.** For \( \Re(s) = \sigma > \sigma_3 + 1 \), we have the limit
\[
\lim_{N \to \infty} J_{1,N}(s) = 0.
\]

**Proof.** Using Theorem \( \mathbb{E} \) with \( x \in S(\delta)^c \), we have
\[
\left| \int_{\mathbb{C} - \mathbb{C}(N)} \frac{D(\eta)x^n}{\eta} \frac{dx}{x^{s+1}} \right| \ll x^{\sigma_3} \sum_{n=1}^\infty \frac{|a_n|}{n^{\sigma_3}(1 + N|\log(x/n)|)}
\]
\[
\ll \frac{x^{\sigma_3}}{N} \sum_{n=1}^\infty \frac{|a_n|}{n^{\sigma_3}} + \frac{1}{N} \sum_{x/2 \leq n \leq 2x} \frac{x|a_n|}{|x - n|} (\frac{x}{n})^{\sigma_3}
\]

15
\[
\ll \frac{x^{\sigma_3}}{N} + \frac{x^{\sigma_3+1+\epsilon}}{\delta N} \ll \frac{x^{\sigma_3+1+\epsilon}}{\sqrt{N}} \quad (\text{as } \delta = N^{-\frac{1}{2}}).
\]

From the above calculation, we see that
\[
|J_{1,N}| \ll \frac{1}{\sqrt{N}} \int_1^\infty x^{\sigma_3-\sigma+\epsilon} dx \ll \frac{1}{\sqrt{N}}
\]
for \(\sigma = \Re(s) > \sigma_3 + 1 + \epsilon\). This proves our required result.

**Lemma 4.** For \(\Re(s) = \sigma > \sigma_3\),
\[
\lim_{N \to \infty} J_{2,N}(s) = 0.
\]

**Proof.** Recall that
\[
\sum_{n \leq x}^\ast a_n = \begin{cases} 
\sum_{n < x} a_n + a_x/2 & \text{if } x \in \mathbb{N}, \\
\sum_{n \leq x} a_n & \text{if } x \notin \mathbb{N}.
\end{cases}
\]

By Note 1
\[
\sum_{n \leq x}^\ast a_n \ll x^{\sigma_3}.
\]

Using this bound, we calculate an upper bound for \(J_{2,N}\) as follows:
\[
\left| \int_{S(\delta)} \int_{\sigma_3-i\infty}^{\sigma_3+i\infty} \frac{D(\eta)x^{\eta}}{\eta} d\eta \right| \ll \int_{S(\delta)} \left| \sum_{n \leq x}^\ast a_n \right| dx 
\ll \int_{S(\delta)} x^{\sigma_3-\sigma-1} dx \ll \int_1^{1+\delta} x^{\sigma_3-\sigma-1} dx + \sum_{m=2}^\infty \int_{m-\delta}^{-m+\delta} x^{\sigma_3-\sigma-1} dx.
\]

This gives
\[
|J_{2,N}(s)| \ll \delta + \sum_{m \geq 2} \left( \frac{1}{(m-\delta)^{\sigma_3-\sigma}} - \frac{1}{(m+\delta)^{\sigma_3-\sigma}} \right).
\]

Using the mean value theorem, for all \(m \geq 2\) there exists a real number \(\overline{m} \in [m-\delta, m+\delta]\) such that
\[
|J_{2,N}(s)| \ll \delta + \sum_{m \geq 2} \frac{\delta}{\overline{m}^{\sigma_3-\sigma_3+1}} \ll \delta \ll \frac{1}{\sqrt{N}} \quad \text{by choosing } \sigma > \sigma_3.
\]

This implies that \(J_{2,N} \to 0\) as \(N \to \infty\). \(\square\)
Lemma 5. For \( \sigma > \sigma_3 \), we have

\[
\lim_{N \to \infty} J_{3,N}(s) = 0.
\]

Proof. Consider

\[
J_{3,N}(s) = \int_{S(\delta)} \int_{\sigma_3-iN}^{\sigma_3+iN} \frac{D(\eta) x^{\eta}}{\eta} \frac{dx}{x^{s+1}} .
\]

This double integral is absolutely convergent for \( \Re(s) > \sigma_3 \). Using the Theorem of Fubini and Tonelli \( \text{[8, Theorem B.3.1, (b)]} \), we can interchange the integrals:

\[
J_{3,N}(s) = \int_{\sigma_3-iN}^{\sigma_3+iN} \frac{D(\eta)}{\eta} \left\{ \int_{1}^{1+\delta} \frac{x^{\eta}}{x^{s+1}} dx + \sum_{m \geq 2} \int_{m-\delta}^{m+\delta} \frac{x^{\eta}}{x^{s+1}} dx \right\} d\eta .
\]

For any \( \theta_1, \theta_2 \) such that \( 0 < \theta_1 < \theta_2 < \infty \), we have

\[
\int_{\theta_1}^{\theta_2} x^{\eta-s-1} dx = \frac{1}{s-\eta} \left\{ \frac{1}{\theta_1^{s-\eta}} - \frac{1}{\theta_2^{s-\eta}} \right\} = \frac{\theta_2 - \theta_1}{\bar{\theta}^{s-\eta+1}},
\]

for some \( \bar{\theta} \in [\theta_1, \theta_2] \). Applying the above formula to \( J_{3,N}(s) \), we get

\[
J_{3,N}(s) = \int_{\sigma_3-iN}^{\sigma_3+iN} \frac{D(\eta)}{\eta} \sum_{m \geq 1} \frac{2\delta}{m^{s-\eta+1}} d\eta = 2\delta \sum_{m \geq 1} \int_{\sigma_3-iN}^{\sigma_3+iN} \frac{D(\eta)}{m^{s-\eta+1}} d\eta ,
\]

where \( \frac{1}{2} \in [1, 1 + \delta] \) and \( \frac{m}{m} \in [m - \delta, m + \delta] \) for all integers \( m \geq 2 \). In the above calculation, we can interchange the series and the integral as the series is absolutely convergent. So we have

\[
J_{3,N}(s) \ll \delta \sum_{m \geq 1} \int_{-N}^{N} \frac{1}{(1 + |v|) m^{\sigma_3+iN}} dv \quad (\text{substituting } \eta = \sigma_3 + iv)
\]

\[
\ll \delta \log N \sum_{m \geq 1} \frac{1}{m^{\sigma_3+iN}} \ll \frac{\log N}{\sqrt{N}}.
\]

Here we used the fact that for \( \sigma > \sigma_3 \), the series

\[
\sum_{m \geq 1} \frac{1}{m^{s-\eta+1}}
\]
is absolutely convergent. This proves our required result.

2.2 Proof of Theorem 3

Proof. From equation (II.5) and Lemma 3, 4 and 5, we get

\[ A(s) = \lim_{N \to \infty} B'_N(s) \]

for \( \Re(s) > \sigma_3 + 1 \), and where \( B'_N(s) \) is defined by (II.4). From Lemma 2, we have

\[ B(s) = \lim_{N \to \infty} B'_N(s). \]

This gives \( A(s) \) and \( B(s) \) are equal for \( \Re(s) > \sigma_3 + 1 \). By analytic continuation, \( A(s) \) and \( B(s) \) are equal for any \( s \) that lies right to \( C \). □

In this chapter, we shall use the meromorphic continuation of \( A(s) \) derived in Theorem 3 to obtain measure theoretic \( \Omega_\pm \) results for \( \Delta(x) \).

3 Alternative Approches

Theorem 3 gives a way for meromorphic continuation of \( A(s) \) by formulating it as a contour integral. This theorem has its significance in terms of elegance and generality. However, there are alternative and easier ways in many cases. Below we give an example.

Note that

\[ \sum_{n=1}^{\infty} \frac{a_n}{n^s} = -\int_1^{\infty} \left( \sum_{n \leq x} a_n \right) x^{-s} \, dx \quad \text{for } \Re(s) > \sigma_2. \]

This gives

\[ \frac{D(s)}{s} = \int_1^{\infty} \left( \sum_{n \leq x} a_n \right) x^{-s-1} \, dx \quad \text{for } \Re(s) > \sigma_2. \]

So we can express \( A(s) \) as

\[ A(s) = \frac{D(s)}{s} - \int_1^{\infty} M(x)x^{-s-1} \, dx \quad \text{for } \Re(s) > \sigma_2. \quad (II.6) \]

The above formula reduces the problem of meromorphically continuing \( A(s) \) to that of

\[ \int_1^{\infty} M(x)x^{-s-1} \, dx. \]
To demonstrate this method, we consider the case when \( D(\eta) \) has a pole at \( \eta = 1 \) and residue at this pole gives the main term \( M(x) \), i.e. \( \mathcal{P} = \{ 1 \} \). The following meromorphic functions may serve as examples of \( D(\eta) \) in this situation:

\[
\frac{\zeta(s)}{\zeta(2s)}, \frac{\zeta^2(s)}{\zeta(2s)}, \frac{-\zeta'(s)}{\zeta(s)}, \ldots.
\]

For a small positive real number \( r \), we can write \( M(x) \) as

\[
M(x) = \frac{1}{2\pi i} \int_{|\eta-1|=r} \frac{D(\eta)x^\eta}{\eta} \, d\eta.
\]

Thus

\[
\int_1^\infty \frac{M(x)}{x^{s+1}} \, dx = \int_1^\infty \frac{1}{2\pi i} \int_{|\eta-1|=r} \frac{D(\eta)x^\eta}{\eta} \, d\eta \frac{dx}{x^{s+1}}
\]

\[
= \frac{1}{2\pi i} \int_{|\eta-1|=r} \frac{D(\eta)}{\eta} \left( \int_1^\infty \frac{dx}{x^{s-\eta+1}} \right) \, d\eta
\]

(using [8, Theorem B.3.1, (b)]).

\[
= \frac{1}{2\pi i} \int_{|\eta-1|=r} \frac{D(\eta)}{\eta(s-\eta)} \, d\eta.
\]

Let the Laurent series expansion of \( D(\eta) \) at \( \eta = 1 \) be

\[
D(\eta) = \sum_{n \leq N} b_n \frac{(\eta - 1)^n}{\eta} + H(\eta),
\]

where \( H(\eta) \) is holomorphic for \( \Re(\eta) > \sigma_1 \). Plugging in this expression for \( D(\eta) \) in (II.7), we get

\[
\int_1^\infty \frac{M(x)}{x^{s+1}} \, dx = \sum_{n \leq N} b_n \frac{1}{2\pi i} \int_{|\eta-1|=r} \frac{d\eta}{(\eta - 1)^n (s-\eta)^{n+1}}.
\]

Let \( \Re(s) \geq 1 + 2r \), then

\[
\frac{|\eta - 1|}{|s - 1|} \leq \frac{1}{2} \quad \text{for } |\eta - 1| = r.
\]

This gives

\[
\frac{1}{s-\eta} = \sum_{n=0}^\infty \frac{(\eta - 1)^n}{(s - 1)^{n+1}}
\]
is an absolutely convergent series. Using the above expansion of $(s - \eta)^{-1}$ in (II.8), we have

\[
\int_1^\infty \frac{M(x)}{x^{s+1}} \, dx = \sum_{n \leq N} b_n \frac{1}{2\pi i} \int_{|\eta-1|=r} \left\{ \sum_{m=0}^{\infty} \frac{(\eta - 1)^m}{(s - 1)^{m+1}} \right\} \frac{d\eta}{(\eta - 1)^n} \\
= \sum_{n \leq N} \frac{b_n}{(s - 1)^n} \quad (\text{by } [34, \text{Theorem 6.1}]) \\
= \frac{D(s)}{s} - H(s).
\]

Thus we got

\[ A(s) = H(s) \quad \text{for } \Re(s) \geq 1 + 2r. \]

But the right hand side is holomorphic for $\Re(s) > \sigma_1$ hence the formula gives analytic continuation of $A(s)$ in the half plane $Re(s) > \sigma_1$.

Similar calculations can be done when the main term $M(x)$ is more complicated.
In this chapter, we revisit a result due to Landau and obtain $\Omega_{\pm}$ results for $\Delta(x)$ using certain singularities of $D(s)$. Also we shall measure the fluctuations of $\Delta(x)$ in terms of $\Omega$ bounds, which generalizes a result of Kaczorowski and Szydło [24], and a result of Bhowmik, Schlage-Puchta and Ramaré [2].

1 Landau’s Criterion for Sign Change

We begin with a result on real valued functions that do not change sign. This appears in a paper of Landau [26], attributed to Phragmén and stated without a proof. Here we present a proof of this result following [37, II.1.3, Theorem 6].

**Theorem 4** (Phragmén-Landau). Let $f(x)$ be a real valued piecewise continuous function defined for $x \geq 1$, and bounded on every compact intervals. Let $F(s)$ be its Mellin transform:

$$F(s) = \int_1^\infty \frac{f(x)}{x^{s+1}} \, dx,$$

converges absolutely in some complex right half plane. Also assume that $f(x)$ does not change sign for $x \geq x_0$, for some $x_0 \geq 1$. If $F(s)$ diverges for some real $s$, then there exist a real number $\sigma_0$ satisfying the following properties:

1. the integral defining $F(s)$ is divergent for $s < \sigma_0$ and convergent for $s > \sigma_0$,
2. $s = \sigma_0$ is a singularity of $F(s)$,
3. and $F(s)$ is analytic for $\Re(s) > \sigma_0$.

**Proof.** Let $\sigma_0$ be:

$$\sigma_0 = \inf \{ \sigma \in \mathbb{R} : F(\sigma) \text{ converges} \}.$$

We shall show that $\sigma_0$ satisfies the properties given in the theorem.

As $f(x)$ does not change sign for $x \geq x_0$, convergence of $F(\sigma)$ implies the absolute convergence of $F(s)$ for $\Re(s) \geq \sigma$. This proves (1) and (3). To prove (2), we proceed by method of contradiction. Assume that $s = \sigma_0$ is not a singularity of $F(s)$. Then
there exist $\sigma_0' > \sigma_0$ and $r > \sigma_0' - \sigma_0$ such that $F(s)$ has the following Taylor series expansion:

$$F(s) = \sum_{k=0}^{\infty} \frac{1}{k!} F^{(k)}(\sigma_0')(s - \sigma_0')^k,$$

for all $s$ satisfying $|s - \sigma_0'| < r$.

**Claim (1).** For $\sigma_0'$ as above, we have

$$F(s) = \sum_{k=0}^{\infty} \frac{1}{k!} (s - \sigma_0')^k \int_1^{\infty} (-\log x)^k \frac{f(x)}{x^{\sigma_0'+1}} \, dx.$$

**Proof of Claim (1).** By Cauchy’s integral formula, we can write

$$F^{(k)}(\sigma_0') = \frac{k!}{2\pi i} \int_{C} \frac{F(z)}{(z - \sigma_0')^{k+1}} \, dz,$$

where $C$ is a circle with a small enough radius having its center at $\sigma_0'$. So we have

$$F(s) = \sum_{k=0}^{\infty} \frac{(s - \sigma_0')^k}{2\pi i} \int_{C} \frac{1}{(z - \sigma_0')^{k+1}} \int_1^{\infty} \frac{f(x)}{x^{z+1}} \, dx \, dz.$$

Suppose we can exchange the integrals of $x$ and $z$, then

$$F(s) = \sum_{k=0}^{\infty} \frac{(s - \sigma_0')^k}{k!} \int_1^{\infty} \frac{f(x)}{x^{\sigma_0'+1}} \frac{1}{2\pi i} \int_{C} \frac{x^{-z}}{(z - \sigma_0')^{k+1}} \, dz \, dx$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} (s - \sigma_0')^k \int_1^{\infty} (-\log x)^k f(x) \, dx,$$

which proves Claim 1 conditionally. The only thing remains is to show that we can exchange integrals of $x$ and $z$. If we choose $C$ with a small enough radius, then

$$\int_1^{\infty} \frac{f(x)}{x^{c\log c+1}} \, dx$$

is absolutely convergent and so is the double integral

$$\int_{C} \frac{1}{(z - \sigma_0')^{k+1}} \int_1^{\infty} \frac{f(x)}{x^{z+1}} \, dx \, dz.$$

By the theorem of Fubinni and Tonelli [§ Theorem B.3.1, (b)], we can exchange these two iterated integrals. This completes the proof of Claim 1.
Claim (2). For $|s - \sigma'_0| < r$, the integral

$$F(s) = \int_1^\infty \frac{f(x)}{x^{s+1}} dx$$

converges.

Proof of Claim (2). We shall simplify $F(s)$ using Claim 1. We write

$$F(s) = \sum_{k=0}^{\infty} \frac{(\sigma'_0 - s)^k}{k!} \int_1^\infty \frac{(\log x)^k f(x)}{x^{\sigma'_0+1}} dx.$$ 

In the above identity, we can exchange the series and the integral as the series is absolutely convergent. So we have

$$\int_1^\infty \frac{f(x)}{x^{\sigma'_0+1}} \left( \sum_{k=0}^{\infty} \frac{(\sigma'_0 - s)^k}{k!} (\log x)^k \right) dx$$

$$= \int_1^\infty \frac{f(x)}{x^{\sigma'_0+1}} \exp((\sigma'_0 - s) \log x) dx = \int_1^\infty \frac{f(x)}{x^{s+1}} dx.$$ 

This completes the proof of Claim 2.

But Claim 2 implies that we have a real number smaller than $\sigma_0$, say $\sigma''_0$, such that the integral of $F(\sigma''_0)$ converges. This is a contradiction to the definition of $\sigma_0$. So $\sigma_0$ is a singularity of $F(s)$, which proves (2). 

The following theorem appears in [1, Section 2] without a proof and is attributed to Landau. We shall prove this theorem using Theorem 4.

Theorem 5 (Phragmén-Landau-Anderson-Stark). Let $f(x)$ be a real valued piecewise continuous function defined on $[1, \infty)$, bounded on every compact intervals, and does not change sign when $x > x_0$ for some $1 < x_0 < \infty$. Define

$$F(s) := \int_1^\infty \frac{f(x)}{x^{s+1}} dx,$$

and assume that the above integral is absolutely convergent in some half plane. Further, assume that we have an analytic continuation of $F(s)$ in a region containing a part of the real line

$$l(\sigma_0, \infty) := \{ \sigma + i0 : \sigma > \sigma_0 \}.$$

Then the integral representing $F(s)$ is absolutely convergent for $\Re(s) > \sigma_0$, and hence $F(s)$ is an analytic function in this region.
Proof. By Theorem [4] if
\[ \int_1^\infty \frac{f(x)}{x^{\sigma'+1}} \, dx \]
diverges for some \( \sigma' > \sigma_0 \), then there exist a real number \( \sigma'_0 \geq \sigma' > \sigma_0 \) such that \( F \) is not analytic at \( \sigma'_0 \). But this contradicts our assumption that \( F \) is analytic on \( l(\sigma_0, \infty) \). So the integral
\[ \int_1^\infty \frac{f(x)}{x^{\sigma'+1}} \, dx \]
converges \( \forall \sigma' > \sigma_0 \),
and since \( f \) does not change sign for \( x \geq x_0 \), \( F(s) \) converges absolutely for \( \Re(s) > \sigma_0 \). This also gives that \( F(s) \) is analytic for \( \Re(s) > \sigma_0 \).

The above two theorems give some criteria when a function does not change sign. In the next section we will use these results to show the sign changes of \( \Delta(x) \).

2 \( \Omega_\pm \) Results

Consider the Mellin transform \( A(s) \) of \( \Delta(x) \). We need the following assumptions to apply Theorem [5].

Assumptions 2. Suppose there exists a real number \( \sigma_0 \), \( 0 < \sigma_0 < \sigma_1 \), such that \( A(s) \) has the following properties.

(i) There exists \( t_0 \neq 0 \) such that
\[ \lambda := \limsup_{\sigma \to \sigma_0} (\sigma - \sigma_0) |A(\sigma + it_0)| > 0. \]

(ii) At \( \sigma_0 \) we have
\[ l_s := \limsup_{\sigma \to \sigma_0} (\sigma - \sigma_0) A(\sigma) < \infty, \]
\[ l_i := \liminf_{\sigma \to \sigma_0} (\sigma - \sigma_0) A(\sigma) > -\infty. \]

(iii) The limits \( l_i, l_s \) and \( \lambda \) satisfy
\[ l_i + \lambda > 0 \quad \text{and} \quad l_s - \lambda < 0. \]

(iv) We can analytically continue \( A(s) \) in a region containing the real line \( l(\sigma_0, \infty) \).

Remark 1. From Assumptions [2] (i), we see that \( \sigma_0 + it_0 \) is a singularity of \( A(s) \).
We construct the following sets for further use.

**Definition 8.** With \( l_s, l_i \) and \( \lambda \) as in Assumptions 2 and for an \( \epsilon \) such that \( 0 < \epsilon < \min(l_i + \lambda, \lambda - l_s) \), define

\[
A_1 := \{ x : x \in [1, \infty), \Delta(x) > (l_i + \lambda - \epsilon)x^{\sigma_0} \},
\]
and

\[
A_2 := \{ x : x \in [1, \infty), \Delta(x) < (l_s - \lambda + \epsilon)x^{\sigma_0} \}.
\]

Under Assumptions 2 and using methods from [24], we can derive the following measure theoretic theorem.

**Theorem 6.** Let the conditions in Assumptions 2 hold. Then for any real number \( M > 1 \), we have

\[
\mu(A_1 \cap [M, \infty]) > 0,
\]
and

\[
\mu(A_2 \cap [M, \infty]) > 0.
\]

This implies

\[
\Delta(x) = \Omega_\pm(x^{\sigma_0}).
\]

**Proof.** We prove the Theorem only for \( A_1 \) as the other part is similar.

Define

\[
g(x) := \Delta(x) - (l_i + \lambda - \epsilon)x^{\sigma_0},
\]
\[
G(s) := \int_1^\infty \frac{g(x)}{x^{s+1}}dx;
\]
\[
g^+(x) := \max(g(x), 0),
\]
\[
G^+(s) := \int_1^\infty \frac{g^+(x)}{x^{s+1}}dx;
\]
\[
g^-(x) := \max(-g(x), 0),
\]
\[
G^-(s) := \int_1^\infty \frac{g^-(x)}{x^{s+1}}dx.
\]

With the above notations, we have

\[
g(x) = g^+(x) - g^-(x),
\]
and

\[
G(s) = G^+(s) - G^-(s).
\]

Note that

\[
G(s) = A(s) - \int_1^\infty (l_i + \lambda - \epsilon)x^{\sigma_0-s-1}dx
\]
\[
= A(s) + \frac{l_i + \lambda - \epsilon}{\sigma_0 - s}, \quad \text{for } \Re(s) > \sigma_0.
\]

25
So $G(s)$ is analytic wherever $A(s)$ is, except possibly for a pole at $\sigma_0$. This gives
\[
\limsup_{\sigma \searrow \sigma_0} (\sigma - \sigma_0)|G(\sigma + it_0)| = \limsup_{\sigma \searrow \sigma_0} (\sigma - \sigma_0)|A(\sigma + it_0)| = \lambda. \quad (\text{III.1})
\]
We shall use the above limit to prove our theorem. We proceed by method of contradiction. Assume that there exists an $M > 1$ such that
\[
\mu(\mathcal{A}_1 \cap [M, \infty)) = 0.
\]
This implies
\[
G^+(\sigma) = \int_1^\infty \frac{g^+(x)}{x^{s+1}} \, dx = \int_1^M \frac{g^+(x)}{x^{s+1}} \, dx
\]
is bounded for any $s$, and so is an entire function. By Assumptions 2, $A(s)$ and $G(s)$ can be analytically continued on the line $l(\sigma_0, \infty)$. As $G(s)$ and $G^+(s)$ are analytic on $l(\sigma_0, \infty)$, $G^-(s)$ is also analytic on $l(\sigma_0, \infty)$. The integral for $G^-(s)$ is absolutely convergent for $\Re(s) > \sigma_3 + 1$, and $g^-(x)$ is a piecewise continuous function bounded on every compact sets. This suggests that we can apply Theorem 5 to $G^-(s)$, and conclude that
\[
G^-(s) = \int_1^\infty \frac{g^-(x)}{x^{s+1}} \, dx
\]
is absolutely convergent for $\Re(s) > \sigma_0$.

From the above discussion, we summarize that the Mellin transforms of $g, g^+$ and $g^-$ converge absolutely for $\Re(s) > \sigma_0$. As a consequence, we see that $G(\sigma), G^+(\sigma)$ and $G^-(\sigma)$ are finite real numbers for $\sigma > \sigma_0$. For $\sigma > \sigma_0$, we compare $G^+(\sigma)$ and $G^-(\sigma)$ in the following two cases.

Case 1: $G^+(\sigma) < G^-(\sigma)$.

In this case,
\[
(\sigma - \sigma_0)|G(\sigma + it_0)| \leq (\sigma - \sigma_0)|G(\sigma)| = -(\sigma - \sigma_0)G(\sigma) = -(\sigma - \sigma_0)A(\sigma) + l_i + \lambda - \epsilon.
\]
So we have
\[
\limsup_{\sigma \searrow \sigma_0} (\sigma - \sigma_0)|G(\sigma + it_0)| \leq l_i + \lambda - \epsilon - \liminf_{\sigma \searrow \sigma_0} A(\sigma) \leq \lambda - \epsilon.
\]
This contradicts (III.1).

Case 2: $G^+(\sigma) \geq G^-(\sigma)$.
We have,

\[ (\sigma - \sigma_0)|G(\sigma + it_0)| \leq (\sigma - \sigma_0)G^+(\sigma) \]

\[ = O(\sigma - \sigma_0) \quad (G^+(\sigma) \text{ being a bounded integral}). \]

Thus

\[ \lim \sup_{\sigma \searrow \sigma_0} (\sigma - \sigma_0)|G(\sigma + it_0)| = 0. \]

This contradicts (III.1) again.

Thus \( \mu(\mathcal{A}_1 \cap [M, \infty)) > 0 \) for any \( M > 1 \), which completes the proof.

\[ \square \]

3 Measure Theoretic \( \Omega_{\pm} \) Results

Now we know that \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are unbounded. But we do not know how the size of these sets grow. An answer to this question was given by Kaczorowski and Szydło in [24, Theorem 4].

**Theorem 7** (Kaczorowski and Szydło [24]). *Let the conditions in Assumptions 2 hold. Also assume that for a non-decreasing positive continuous function \( h \) satisfying*

\[ h(x) \ll x^\epsilon, \]

*we have*

\[ \int_{T}^{2T} \Delta^2(x)dx \ll T^{2\sigma_0+1}h(T). \]

(III.2)

*Then as \( T \to \infty \),*

\[ \mu(\mathcal{A}_j \cap [1, T]) = \Omega\left(\frac{T}{h(T)}\right) \quad \text{for } j = 1, 2. \]

In [24], Kaczorowski and Szydło applied this theorem to the error term appearing in the asymptotic formula for the fourth power moment of Riemann zeta function. We write this error term as \( E_2(x) \):

\[ \int_{0}^{x} \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 dt = xP(\log x) + E_2(x), \]

where \( P \) is a polynomial of degree 4. Motohashi [31] proved that

\[ E_2(x) \ll x^{2/3+\epsilon}, \]
and further in [32] he showed that

$$E_2(x) = \Omega_{\pm}(\sqrt{x}).$$

Theorem of Kaczorowski and Szydło (Theorem 8) gives that there exist \(\lambda_0, \nu > 0\) such that

$$\mu\{1 \leq x \leq T : E_2(x) > \lambda_0 \sqrt{x}\} = \Omega(T/(\log T)^\nu)$$

and

$$\mu\{1 \leq x \leq T : E_2(x) < -\lambda_0 \sqrt{x}\} = \Omega(T/(\log T)^\nu)$$

as \(T \to \infty\). These results not only prove \(\Omega_{\pm}\)-results, but also give quantitative estimates for the occurrences of such fluctuations. The above theorem of Kaczorowski and Szydło has been generalized by Bhowmik, Ramaré and Schlage-Puchta by localizing the fluctuations of \(\Delta(x)\) to \([T, 2T]\). Proof of this theorem follows from [6, Theorem 2] (also see Theorem 10 below).

**Theorem 8** (Bhowmik, Ramaré and Schlage-Puchta [6]). Let the assumptions in Theorem 7 hold. Then as \(T \to \infty\),

$$\mu(\mathcal{A}_j \cap [T, 2T]) = \Omega\left(\frac{T}{h(T)}\right) \quad \text{for } j = 1, 2.$$

An application of the above theorem to Goldbach’s problem is given in [6]. Let

$$\sum_{n \leq x} G_k(n) = \frac{x^k}{k!} - k \sum_{\rho} \frac{x^{k-1+\rho}}{\rho(1+\rho) \cdots (k-1+\rho)} + \Delta_k(x),$$

where the Goldbach numbers \(G_k(n)\) are defined as

$$G_k(n) = \sum_{n_1 + \cdots + n_k = n} \Lambda(n_1) \cdots \Lambda(n_k),$$

and \(\rho\) runs over nontrivial zeros of the Riemann zeta function \(\zeta(s)\). Bhowmik, Ramaré and Schlage-Puchta proved that under Riemann Hypothesis

$$\mu\{T \leq x \leq 2T : \Delta_k(x) > (c_k + c'_k)x^{k-1}\} = \Omega(T/(\log T)^6)$$

and

$$\mu\{T \leq x \leq 2T : \Delta_k(x) < (c_k - c'_k)x^{k-1}\} = \Omega(T/(\log T)^6) \quad \text{as } T \to \infty,$$

where \(k \geq 2\) and \(c_k, c'_k\) are well defined real number depending on \(k\) with \(c'_k > 0\).

Note that Theorem 7 implies Theorem 8, but both the theorems are applicable to the same set of examples. The main obstacle in applicability of these theorems is the condition (III.2). For example, if \(\Delta(x)\) is the error term in approximating
$\sum_{n \leq x} |\tau(n, \theta)|^2$, we cannot apply Theorem 7 and Theorem 8. However, the following theorem due to the author and A. Mukhopadhyay [28, Theorem 3] overcomes this obstacle by replacing the condition (III.2).

**Theorem 9.** Let the conditions in Assumptions 2 hold. Assume that there is an analytic continuation of $A(s)$ in a region containing the real line $l(\sigma_0, \infty)$. Let $h_1$ and $h_2$ be two positive functions such that

$$\int_{[T,2T] \cap A_j} \frac{\Delta^2(x)}{x^{2\sigma_0+1}} \ll h_j(T) \quad \text{for } j = 1, 2. \quad (\text{III.3})$$

Then as $T \to \infty$,

$$\mu(A_j \cap [T,2T]) = \Omega \left( \frac{T}{h_j(T)} \right) \quad \text{for } j = 1, 2. \quad (\text{III.4})$$

Below we state an integral version of Theorem 8 as in [6].

**Theorem 10** (Bhowmik, Ramaré and Schlage-Puchta [6]). Suppose the conditions in Assumptions 2 hold, and let $h(x)$ be as in Theorem 8. Then as $\delta \to 0^+$,

$$\int_1^\infty \mu(A_j \cap [x,2x])h(4x) \frac{1}{x^{2+\delta}} \, dx = \Omega \left( \frac{1}{\delta} \right), \quad \text{for } j = 1, 2. \quad (\text{III.5})$$

In our next theorem, we generalize Theorem 7, 8, 9 and 10.

**Theorem 11.** Let the conditions in Theorem 9 hold. Then as $\delta \to 0^+$,

$$\int_1^\infty \mu(A_j \cap [x,2x])h_j(x) \frac{1}{x^{2+\delta}} \, dx = \Omega \left( \frac{1}{\delta} \right) \quad \text{for } j = 1, 2. \quad (\text{III.5})$$

*Proof.* We shall prove the theorem for $j = 1$; the proof is similar for $j = 2$. We define $g, g^+, g^-, G, G^+$ and $G^-$, as in Theorem 6. Let

$$m^\#(x) := h_1(x)\mu(A_1 \cap [x,2x])x^{-1}.$$

First, we shall show:

**Claim (1).** As $\delta \to 0$,

$$\sum_{k \geq 0} \frac{m^\#(2^k)}{2^{k\delta}} = \Omega \left( \frac{1}{\delta} \right).$$

Assume that

$$\sum_{k \geq 0} \frac{m^\#(2^k)}{2^{k\delta}} = o \left( \frac{1}{\delta} \right). \quad (\text{III.6})$$

29
From the above assumption, we may obtain an upper bound for $G^+(\sigma)$ as follows:

$$
\int_{A_1} \frac{g^+(x)dx}{x^{\sigma+1}} \leq \sum_{k \geq 0} \int_{A_1 \cap [2^k, 2^{k+1}]} \frac{\Delta(x)dx}{x^{\sigma+1}} \quad \text{(as $\Delta(x) > g(x)$ on $A_1$)}
$$

$$
\leq \sum_{k \geq 0} \left( \int_{A_1 \cap [2^k, 2^{k+1}]} \frac{\Delta^2(x)dx}{x^{2\sigma_0+1}} \right)^{\frac{1}{2}} \left( \frac{\mu(A_1 \cap [2^k, 2^{k+1}])}{2k(2\delta+1)} \right)^{\frac{1}{2}} \quad \text{(where $\sigma - \sigma_0 = \delta > 0$)}
$$

$$
\leq c_3 \sum_{k \geq 0} \left( \frac{h_1(2^k)\mu(A_1 \cap [2^k, 2^{k+1}])}{2k(2\delta+1)} \right)^{\frac{1}{2}} \leq c_3 \sum_{k \geq 0} \left( \frac{m^#(2^k)}{2k\delta} \right)^{\frac{1}{2}}.
$$

From the above inequality, we get

$$
\delta G^+(\sigma) \ll \delta \left( \sum_{k \geq 0} \frac{1}{2k\delta} \right)^{\frac{1}{2}} \left( \sum_{k \geq 0} \frac{m^#(2^k)}{2k(\sigma-\sigma_0)} \right)^{\frac{1}{2}} = o(1) \quad \text{(III.7)}
$$

as $\delta \to 0^+$. Therefore

$$
G^+(s) = \int_1^\infty \frac{g^+(x)dx}{x^{s+1}}
$$

is absolutely convergent for $\Re(s) > \sigma_0$, and so it is analytic in this region. But

$$
G^-(s) = G(s) - G^+(s),
$$

and $G$ is analytic on $l(\sigma_0, \infty)$. So $G^-$ is also analytic on $l(\sigma_0, \infty)$. Using Theorem 5, we get

$$
G^+(s) = \int_1^\infty \frac{g^+(x)dx}{x^{s+1}}
$$

is absolutely convergent for $\Re(s) > \sigma_0$. Absolute convergence of the integrals of $G$ and $G^+$ implies that the Mellin transformation of $g^-(x)$, given by

$$
G^-(s) = \int_1^\infty \frac{g^-(x)dx}{x^{s+1}},
$$

is also absolutely convergent for $\Re(s) > \sigma_0$. As a consequence, we get $G(\sigma), G^+(\sigma)$, and $G^-(\sigma)$ are finite non-negative real numbers for $\sigma > \sigma_0$. As indicated in Case-1 of Theorem 6 we can not have

$$
G^+(\sigma) < G^-(\sigma) \quad \text{when $\sigma > \sigma_0$}.
$$

So we always have

$$
G^+(\sigma) \geq G^-(\sigma).
$$
Using (III.7),
\[
\limsup_{\sigma \searrow \sigma_0} (\sigma - \sigma_0) |G(\sigma + it)| \leq \limsup_{\sigma \searrow \sigma_0} (\sigma - \sigma_0) G(\sigma) = 0.
\]
This is a contradiction to (III.1), and so (III.6) is wrong. This proves our Claim.

Now we are ready to prove the theorem. For \(k \geq 1\), observe that
\[
\int_{k-1}^{k} \frac{m^\#(2^x)}{2^x} dx = \int_{k-1}^{k} \frac{h_1(2^x) \mu(A_1 \cap [2^x, 2^{x+1}]}){2^{x(\delta + 1)}} dx = \int_{k-1}^{k} \int_{2^x}^{2^{x+1}} \frac{h_1(2^x)}{2^{x+\delta}} dA_1(t) dx
\]
where \(A_1(t)\) is the indicator function of \(A_1\)
\[
= \int_{2^k}^{2^{k+1}} \int_{k-1}^{\log_2 t} \frac{h_1(2^x)}{2^{x(1+\delta)}} dxdA_1(t) + \int_{2^k}^{2^{k+1}} \int_{\log_2 t - 1}^{k} \frac{h_1(2^x)}{2^{x(1+\delta)}} dxdA_1(t)
\]
From the above identity, we have
\[
\int_{k-1}^{k} \frac{m^\#(2^x)}{2^x} dx = \int_{2^k}^{2^{k+1}} \int_{k-1}^{\log_2 t} \frac{h_1(2^x)}{2^{x(1+\delta)}} dxdA_1(t)
\]
and
\[
\int_{k}^{k+1} \frac{m^\#(2^x)}{2^x} dx = \int_{2^k}^{2^{k+1}} \int_{\log_2 t}^{k} \frac{h_1(2^x)}{2^{x(1+\delta)}} dxdA_1(t).
\]
So we get
\[
\int_{k-1}^{k+1} \frac{m^\#(2^x)}{2^x} dx \geq \int_{2^k}^{2^{k+1}} \int_{\log_2 t}^{k} \frac{h_1(2^x)}{2^{x(1+\delta)}} dxdA_1(t) + \int_{2^k}^{2^{k+1}} \int_{\log_2 t - 1}^{k} \frac{h_1(2^x)}{2^{x(1+\delta)}} dxdA_1(t)
\]
\[
= \int_{2^k}^{2^{k+1}} \int_{\log_2 t}^{\log_2 \log_2 t} \frac{h_1(2^x)}{2^{x(1+\delta)}} dxdA_1(t).
\]
Now, we may use the fact that \(h_1\) is a monotonically increasing function having polynomial growth, and simplify the above calculation as follows:
\[
\int_{k-1}^{k+1} \frac{m^\#(2^x)}{2^x} dx \geq h_1(2^k) \int_{2^k}^{2^{k+1}} \int_{\log_2 t}^{\log_2 \log_2 t} \frac{dx}{2^{x(1+\delta)}} dA_1(t)
\]
\[
= h_1(2^k) \int_{2^k}^{2^{k+1}} \left(2^{-\log_2 t - 1}(1+\delta) - 2^{-\log_2 t (1+\delta)}\right) dA_1(t)
\]
31
\[ \frac{h_1(2^k)}{\log 2} \int_{2^k}^{2^{k+1}} \frac{2^{1+\delta} - 1}{t^{1+\delta}} \, dA_1(t) \geq \frac{h_1(2^k)}{2(2^k+1)} \mu(A_1 \cap [2^k, 2^{k+1}]) \geq \frac{1}{4} \frac{m^\#(2^k)}{2^k}. \]  

(III.8)

Now using Claim (1), we get

\[ \int_0^\infty \frac{m^\#(2^x)}{2^{2\delta x}} \, dx \gg \sum_{k=1}^\infty \frac{m^\#(2^k)}{2^{k\delta}} = \Omega \left( \frac{1}{\delta} \right). \]

Changing the variable \( x = 2^e \) in the above inequality gives

\[ \frac{1}{\log 2} \int_1^\infty \frac{m^\#(u)}{u^{1+\delta}} \, du = \Omega \left( \frac{1}{\delta} \right), \]

or

\[ \int_1^\infty \frac{\mu(A_j \cap [u, 2u]) h_j(u)}{u^{2+\delta}} \, du = \Omega \left( \frac{1}{\delta} \right). \]

This proves the theorem.

**Corollary 1.** Let the conditions given in Theorem 9 hold. Suppose we have a monotonically increasing positive function \( h \) such that

\[ \Delta(x) = O(h(x)), \]  

(III.9)

then

\[ \mu(A_j \cap [T, 2T]) = \Omega \left( \frac{T^{1+2\sigma_0}}{h^2(T)} \right) \quad \text{for } j = 1, 2. \]  

(III.10)

**Corollary 2.** Similar to Corollary 7, we assume that the conditions in Theorem 9 hold. Then we have

\[ \int_{[T,2T] \cap A_j} \Delta^2(x) \, dx = \Omega(T^{2\sigma_0+1}) \quad \text{for } j = 1, 2. \]  

(III.11)

**Proof.** This Corollary follows from the proof of Theorem 11. We shall prove this Corollary for \( A_1 \), and the proof for \( A_2 \) is similar. Note that in the proof of Theorem 11 we showed that the integral for \( G^+(s) \) is absolutely convergent for \( \Re(s) > \sigma_0 \) by assuming (III.6). Then we got a contradiction which proves Claim (1) of Theorem 11. Now we proceed in a similar manner by assuming (III.11) is false. So we have

\[ \int_{[T,2T] \cap A_1} \Delta^2(x) \, dx = o(T^{2\sigma_0+1}). \]
So for an arbitrarily small constant $\varepsilon$, we have

\[
|G^+(s)| \leq \int_{A_1} \frac{g^+(x)dx}{x^{\sigma+1}} \leq \sum_{k \geq 0} \int_{A_1 \cap [2^k, 2^{k+1}]} \frac{\Delta(x)dx}{x^{\sigma+1}}
\]

\[
\leq \sum_{k \geq 0} \frac{1}{2^{k(\sigma-\sigma_0)}} \left( \int_{A_1 \cap [2^k, 2^{k+1}]} \frac{\Delta^2(x)dx}{x^{2\sigma_0+1}} \right)^{1/2}
\]

\[
\leq c_4(\varepsilon) + \varepsilon \sum_{k \geq k(\varepsilon)} \frac{1}{2^{k(\sigma-\sigma_0)}},
\]

where $c_4(\varepsilon)$ is a positive constant depending on $\varepsilon$. From this we obtain that $G^+(s)$ is absolutely convergent for $\Re(s) > \sigma_0$. Now onwards the proof is same as that of Theorem 11.

\[ \square \]

4 Applications

Now we demonstrate applications of our theorems in the previous section to error terms appearing in two well known asymptotic formulas.

4.1 Square Free Divisors

Let $a_n = 2^{\omega(n)}$, where $\omega(n)$ denotes the number of distinct prime factors of $n$; equivalently, $a_n$ denotes the number of square free divisors of $n$. We write

\[
\sum_{n \leq x} 2^{\omega(n)} = \mathcal{M}(x) + \Delta(x),
\]

where

\[
\mathcal{M}(x) = \frac{x \log x}{\zeta(2)} + \left( -\frac{2\zeta'(2)}{\zeta^2(2)} + \frac{2\gamma - 1}{\zeta(2)} \right) x,
\]

and by a theorem of Hölder [19]

\[
\Delta(x) \ll x^{1/2}.
\]

Under Riemann Hypothesis, Baker [2] has improved the above upper bound to

\[
\Delta(x) \ll x^{4/11}.
\]

33
It is easy to see that the Dirichlet series $D(s)$ has the following form:

$$D(s) = \sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s} = \frac{\zeta^2(s)}{\zeta(2s)}.$$  

Let $A(s)$ be the Mellin transform of $\Delta(x)$ at $s$, and let $s_0$ be the zero of $\zeta(2s)$ with least positive imaginary part:

$$2s_0 = \frac{1}{2} + i14.134\ldots \quad (\text{III.13})$$

We define a contour $\mathcal{C}^{(1)}$ as union of the following five lines:

$$\mathcal{C}^{(1)} := \left( \frac{5}{4} - i\infty, \frac{5}{4} - i2 \right] \cup \left[ \frac{5}{4} - i2, \frac{3}{4} - i2 \right] \cup \left[ \frac{3}{4} - i2, \frac{3}{4} + i2 \right]$$

$$\cup \left[ \frac{3}{4} + i2, \frac{5}{4} + i2 \right] \cup \left[ \frac{5}{4} + i2, \frac{5}{4} + i\infty \right).$$

The contour $\mathcal{C}^{(1)}$ is represented by ‘dashed’ lines in Figure [III.1]. By Theorem 3, we have

$$A(s) = \int_{1}^{\infty} \frac{\Delta(x)}{x^{s+1}} dx = \int_{\mathcal{C}^{(1)}} \frac{D(\eta)}{\eta(s-\eta)} d\eta.$$
Now, we shift the contour \( C^{(1)} \) to form a new contour \( C^{(2)} \), so that
\[
1, \ s_0, \ l \left( \frac{1}{4}, \infty \right)
\]
lie to the right of \( C^{(2)} \). We have represented the contour \( C^{(2)} \) by dotted lines in Figure III.1.

Since \( s_0 \) is a pole of \( D(s) \) and is on the right side of \( C^{(1)} \), we have
\[
A(s) = \int_{C^{(2)}} \frac{D(\eta)}{\eta(s-\eta)} \, d\eta + \text{Res}_{\eta=s_0} \left( \frac{D(\eta)}{\eta(s-\eta)} \right).
\]
From the above formula, we may compute the following limits:
\[
\lambda_1 := \lim_{\sigma \searrow 0} \sigma |A(\sigma + s_0)| = |s_0|^{-1} \left| \text{Res}_{\eta=s_0} D(\eta) \right| > 0
\]
and
\[
\lim_{\sigma \searrow 0} \sigma A(\sigma + 1/4) = 0.
\]

For a fixed \( \epsilon_0 > 0 \),
\[
\mathcal{A}_1 = \{ x : \Delta(x) > (\lambda_1 - \epsilon_0)x^{1/4} \}
\]
and
\[
\mathcal{A}_2 = \{ x : \Delta(x) < (-\lambda_1 + \epsilon_0)x^{1/4} \}.
\]

Using Corollary 1 and (III.12), we get
\[
\mu(\mathcal{A}_j \cap [T, 2T]) = \Omega \left( T^{1/2} \right) \text{ for } j = 1, 2. \tag{III.14}
\]

Under Riemann Hypothesis we may show (also see Proposition 7),
\[
\int_T^{2T} \Delta^2(x) \ll T^{3/2+\epsilon} \text{ for any } \epsilon > 0.
\]
The above upper bound and Theorem 9 give
\[
\mu(\mathcal{A}_j \cap [T, 2T]) = \Omega \left( T^{1-\epsilon} \right), \text{ for } j = 1, 2 \text{ and for any } \epsilon > 0. \tag{III.15}
\]

35
4.2 The Prime Number Theorem Error

Consider the error term in the Prime Number Theorem:

\[ \Delta(x) = \sum_{n \leq x} \Lambda(n) - x. \]

Let

\[ \lambda_2 = |2s_0|^{-1}, \]

where 2s_0 is the first nontrivial zero of \( \zeta(s) \) as in (III.13). We shall apply Corollary 1 to prove the following proposition.

**Proposition 1.** We write

\[ A_1 = \{ x : \Delta(x) > (\lambda_2 - \epsilon_0)x^{1/2} \} \]

and \[ A_2 = \{ x : \Delta(x) < (-\lambda_2 + \epsilon_0)x^{1/2} \} \],

for a fixed \( \epsilon_0 \) such that \( 0 < \epsilon_0 < \lambda_2 \). Then

\[ \mu (A_j \cap [T, 2T]) = \Omega \left( T^{1-\epsilon} \right) \text{ for } j = 1, 2 \text{ and for any } \epsilon > 0. \]

**Proof.** Here we apply Corollary 1 in a similar way as in the previous application, so we shall skip the details.

The Riemann Hypothesis, Theorem 8 and Theorem PNT** give

\[ \mu (A_j \cap [T, 2T]) = \Omega \left( \frac{T}{\log^4 T} \right) \text{ for } j = 1, 2; \]

this implies the proposition. But if the Riemann Hypothesis is false, then there exists a constant \( a \), with \( 1/2 < a \leq 1 \), such that

\[ a = \sup \{ \sigma : \zeta(\sigma + it) = 0 \}. \]

Using Perron summation formula, we may show that

\[ \Delta(x) \ll x^{a+\epsilon}, \]

for any \( \epsilon > 0 \). Also for any arbitrarily small \( \delta \), we have \( a - \delta < \sigma' < a \) such that \( \zeta(\sigma' + it') = 0 \) for some real number \( t' \). If \( \lambda'' := |\sigma' + it'|^{-1} \), then by Corollary 1 we get

\[ \mu \left( \left\{ x \in [T, 2T] : \Delta(x) > (\lambda''/2)x^{\sigma'} \right\} \right) = \Omega \left( T^{1-2\delta-2\epsilon} \right) \]

36
and \( \mu \left( \left\{ x \in [T, 2T] : \Delta(x) < -\left(\lambda''/2\right)x^{\sigma'} \right\} \right) = \Omega \left( T^{1-2\delta-2\epsilon} \right) \).

As \( \delta \) and \( \epsilon \) are arbitrarily small and \( \sigma' > 1/2 \), the above \( \Omega \) bounds imply the proposition.

**Remark 2.** Results similar to Proposition 1 can be obtained for error terms in asymptotic formulas for partial sums of Mobius function and for partial sums of the indicator function of square-free numbers.

**Remark 3.** In Section 4.1 and 4.2, we saw that \( \mu(A_j) \) are large. Now suppose that \( \mu(A_1 \cup A_2) \) is large, then what can we say about the individual sizes of \( A_j \)? We may guess that \( \mu(A_1) \) and \( \mu(A_2) \) are both large and almost equal. But this may be very difficult to prove. In the next chapter, we shall show that if \( \mu(A_1 \cup A_2) \) is large, then both \( A_1 \) and \( A_2 \) are nonempty.
In this chapter, we study the influence of measure of the set where \( \Omega \)-result holds, on its possible improvements. The following proposition is an interesting application of the main theorem (Theorem 13) of this chapter.

Let \( \Delta(x) \) denotes the error term appearing in the asymptotic formula for average order of non-isomorphic abelian groups:

\[
\Delta(x) = \sum_{n \leq x} a_n - \sum_{k=1}^{6} \left( \prod_{j \neq k} \zeta(j/k) \right) x^{1/k},
\]

where \( a_n \) denotes the number of non-isomorphic abelian groups of order \( n \). One would expect that

\[
\Delta(x) = O \left( x^{1/6+\epsilon} \right) \quad \text{for any } \epsilon > 0
\]

(see Section 3.2 for more details), so an analogous \( \Omega_{\pm} \) bound for \( \Delta(x) \) is also expected. The proposition below gives a sufficient condition to obtain such an \( \Omega_{\pm} \) bound.

**Proposition 2.** Let \( \delta \) be such that \( 0 < \delta < 1/42 \), and \( \Delta(x) \) be as in (IV.1). Then either

\[
\int_{T}^{2T} \Delta^4(x)dx = \Omega(T^{5/3+\delta}),
\]

or

\[
\Delta(x) = \Omega_{\pm}(x^{1/6-\delta}).
\]

It may be conjectured that

\[
\int_{T}^{2T} \Delta^4(x)dx = O(T^{5/3+\epsilon})
\]

for any \( \epsilon > 0 \). By the above proposition, this conjecture implies

\[
\Delta(x) = \Omega_{\pm}(x^{1/6-\epsilon}) \quad \text{for any } \epsilon > 0.
\]

We begin by assuming the conditions and notations given in Assumptions 1. Further we have the following notations for this chapter.
Notations. For $i = 0, 1, 2$, let $\alpha_i(T)$ denote positive monotonic functions such that $\alpha_i(T)$ converges to a constant as $T \to \infty$. For example, in some cases $\alpha_i(T)$ could be $1 - 1/\log(T)$, which tend to 1 as $T \to \infty$.

For $i = 0, 1$, let $h_i(T)$ be positive monotonically increasing functions such that $h_i(T) \to \infty$ as $T \to \infty$.

For a real valued and non-negative function $f$, we denote

$$\mathcal{A}(f(x)) := \{ x \geq 1 : |\Delta(x)| > f(x) \}.$$ 

1 Refining Omega Result from Measure

Define an $X$-Set as follows.

Definition 9. An infinite unbounded subset $S$ of non-negative real numbers is called an $X$-Set.

Now we hypothesize a situation when there is a lower bound estimate for the second moment of the error term.

Assumptions 3. Let $S$ be an $X$-Set. Define a real valued positive bounded function $\alpha(T)$ on $S$, such that

$$0 \leq \alpha(T) < M \leq \infty$$

for some constant $M$. For a fixed $T$, we write

$$\mathcal{A}_T := [T/2, T] \cap \mathcal{A}(c_5 x^\alpha(x)) \quad \text{for} \quad c_5 > 0.$$ 

For all $T \in S$ and for constants $c_6, c_7 > 0$, assume the following bounds hold:

(i) \[ \int_{\mathcal{A}_T} \frac{\Delta^2(x)}{x^{2\alpha+1}} \, dx > c_6, \]

(ii) \[ \mu(\mathcal{A}_T) < c_7 h_0(T), \quad \text{and} \]

(iii) the function \[ x^{\alpha+1/2} h_0^{-1/2}(x) \]

is monotonically increasing for $x \in [T/2, T]$.

We note that the first assumption indicates an $\Omega$-estimate. The next two assumptions indicate that the measure of the set on which the $\Omega$ estimate holds is not ‘too big’.
Proposition 3. Suppose there exists an $X$-Set $S$ for $\Delta(x)$, having properties as described in Assumptions 3. Let the constant $c_8$ be given by

$$c_8 := \sqrt{\frac{c_6}{2^{M+1}c_7}}.$$

Then there exists a $T_0$ such that for all $T>T_0$ and $T \in S$, we have

$$|\Delta(x)| > c_8 x^{\alpha+1/2} h_0^{-1/2}(x)$$

for some $x \in [T/2,T]$.

In particular

$$\Delta(x) = \Omega(x^{\alpha+1/2} h_0^{-1/2}(x)).$$

Proof. If the statement of the above proposition is not true, then for all $x \in [T/2,T]$ we have

$$\Delta(x) \leq c_8 x^{\alpha+1/2} h_0^{-1/2}(x).$$

From this, we may derive an upper bound for second moment of $\Delta(x)$:

$$\int_{A_T} \frac{\Delta^2(x)}{x^{2\alpha+1}} \, dx \leq \frac{c_8^2 T^{2\alpha+1} \mu(A_T \cap [T/2,T])}{h_0(T)(T/2)^{2\alpha+1}} \leq c_8^2 2^{2M+1} c_7 \leq c_6.$$

This bound contradicts (i) of Assumptions 3, which proves the proposition. \qed

The above proposition will be used in the next chapter to obtain a result on the error term appearing in the asymptotic formula for $\sum_{n \leq x} |\tau(n, \theta)|^2$.

2 Omega Plus-Minus Result from Measure

In this section, we prove an $\Omega_\pm$ result for $\Delta(x)$ when $\mu(A_T)$ is big. We formalize the conditions in the following assumptions.

Assumptions 4. Suppose the conditions in Assumptions 3 hold. Let $l$ be an integer such that

$$l > \max(\sigma_2, 1),$$

and let $\alpha_1(T)$ be a monotonic function satisfying the inequality

$$0 < \alpha_1(T) \leq \sigma_1.$$

Furthermore:
(i) the Dirichlet series $D(\sigma + it)$ has no pole when $\alpha_1(T) \leq \sigma \leq \sigma_1$;
(ii) if $|t| \leq T^{2l}$ and $\alpha_1(T) \leq \sigma \leq \sigma_1$, we have

$$|D(\sigma + it)| \leq c_9(|t| + 1)^{l-1}$$

for some constant $c_9 > 0$.

**Assumptions 5.** Suppose there exists $\epsilon > 0$ such that the following holds:

if $D(\sigma + it)$ has no pole for $\alpha_1(T) - \epsilon < \sigma \leq \sigma_1$ and $|t| \leq 2T^{2l}$, then there exists a constant $c_{10} > 0$ depending on $\epsilon$ such that

$$|D(\sigma + it)| \leq c_{10}(|t| + 1)^{l-1},$$

when $\alpha_1(T) \leq \sigma \leq \sigma_1$ and $|t| \leq T^{2l}$.

Assumptions 5 says that if $D(s)$ does not have pole in $\alpha_1(T) - \epsilon < \sigma \leq \sigma_1$, then it has polynomial growth in a certain region.

**Lemma 6.** Under the conditions in Assumptions 4 we have

$$\Delta(x) = \int_{\alpha_1-2T^{2l}}^{\alpha_1+2T^{2l}} \frac{D(\eta)x^{\eta}}{\eta} d\eta + O(T^{-1}),$$

for all $x \in [T/2, 5T/2]$.

**Proof.** Follows from Perron summation formula.

**Lemma 7** (Balasubramanian and Ramachandra [4]). Let $T \geq 1$, $\delta_0 > 0$ and $f(x)$ be a real-valued integrable function such that

$$f(x) \geq 0 \quad \text{for } x \in [T - \delta_0 T, \ 2T + \delta_0 T].$$

Then for $\delta > 0$ and for a positive integer $l$ satisfying $\delta l \leq \delta_0$, we have

$$\int_T^{2T} f(x)dx \leq \frac{1}{(\delta T)^l} \int_0^{\delta T} \cdots \int_0^{\delta T} \int_{T-\sum_1^l y_i}^{T+\sum_1^l y_i} f(x)dx \ dy_1 \ldots dy_l.$$

**Proof.** For $0 \leq y_i \leq \delta T$, $i = 1, 2, \ldots, l$

$$\int_T^{2T} f(x)dx \leq \int_{T-\sum_1^l y_i}^{T+\sum_1^l y_i} f(x)dx,$$
as \( f(x) \geq 0 \) in
\[
\left[ T - \sum_{i=1}^{l} y_i, 2T + \sum_{i=1}^{l} y_i \right] \subseteq [T - \delta_0 T, 2T + \delta_0 T].
\]

This gives
\[
\frac{1}{(\delta T)^l} \int_{0}^{\delta T} \cdots \int_{0}^{\delta T} \int_{T - \sum_{i=1}^{l} y_i}^{2T + \sum_{i=1}^{l} y_i} f(x) \, dx \, dy_1 \cdots dy_l \\
\geq \frac{1}{(\delta T)^l} \int_{0}^{\delta T} \cdots \int_{0}^{\delta T} \int_{T}^{2T} f(x) \, dx \, dy_1 \cdots dy_l = \int_{T}^{2T} f(x) \, dx.
\]

The next theorem shows that if \( \Delta(x) \) does not change sign then the set on which \( \Omega \)-estimate holds cannot be ‘too big’.

**Theorem 12.** Suppose the conditions in Assumptions 4 hold. Let \( h_1(T) \) be a monotonically increasing function such that \( h_1(T) \to \infty \). Let \( \alpha_2(T) \) be a bounded positive monotonic function, such that
\[
0 < \alpha_1(T) < \alpha_2(T) \leq \sigma_1, \text{ and} \\
\frac{h_1(T)}{T^{\alpha_1}} \to \infty \text{ as } T \to \infty.
\]

If there exist a constant \( x_0 \) such that \( \Delta(x) \) does not change sign on \( A(h_1(x)) \cap [x_0, \infty) \), then
\[
\mu(A(x^{\alpha_2(x)}) \cap [T, 2T]) \leq 4h_1(5T/2)T^{1-\alpha_2(T)} + O(1 + T^{1-\alpha_2(T)+\alpha_1(T)})
\]
for \( T \geq 2x_0 \).

**Proof.** Trivially we have
\[
\mu(A(x^{\alpha_2}) \cap [T, 2T]) \leq \int_{T}^{2T} |\Delta(x)| \frac{dx}{x^{\alpha_2}}.
\]
Using Lemma 7 on the above inequality, we get
\[
\mu(A(x^{\alpha_2}) \cap [T, 2T]) \leq \frac{1}{(\delta T)^l} \int_{0}^{\delta T} \cdots \int_{0}^{\delta T} \int_{T - \sum_{i=1}^{l} y_i}^{2T + \sum_{i=1}^{l} y_i} \frac{|\Delta(x)|}{x^{\alpha_2}} \, dx \, dy_1 \cdots dy_l,
\]

for \( T \geq 2x_0 \).
where $\delta = \frac{1}{2^2}$.

Let $\chi$ denote the characteristic function of the complement of $A(h_1(x))$:

$$\chi(x) = \begin{cases} 1 & \text{if } x \notin A(h_1(x)), \\ 0 & \text{if } x \in A(h_1(x)). \end{cases}$$

For $T \geq 2x_0$, $\Delta(x)$ does not change sign on

$$\left[ T - \sum_{i=1}^{l} y_i, 2T + \sum_{i=1}^{l} y_i \right] \cap A(h_1(x)),$$

as $0 \leq y_i \leq \delta T$ for all $i = 1, \ldots, l$. So we can write the above inequality as

$$\mu(\mathcal{A}(x^{a_2}) \cap [T, 2T]) \leq \frac{2}{(\delta T)^l} \int_0^{\delta T} \cdots \int_0^{\delta T} \int_{T - \sum_{i=1}^{l} y_i}^{2T + \sum_{i=1}^{l} y_i} \frac{\Delta(x)}{x^{a_2}} \chi(x) dx dy_1 \cdots dy_l$$

$$+ \frac{1}{(\delta T)^l} \int_0^{\delta T} \cdots \int_0^{\delta T} \int_{T - \sum_{i=1}^{l} y_i}^{2T + \sum_{i=1}^{l} y_i} \frac{\Delta(x)}{x^{a_2}} dx dy_1 \cdots dy_l. \quad \text{(IV.2)}$$

Since $x \notin A(h_1(x))$ implies $|\Delta(x)| \leq h_1(x)$, we get

$$\frac{2}{(\delta T)^l} \int_0^{\delta T} \cdots \int_0^{\delta T} \int_{T - \sum_{i=1}^{l} y_i}^{2T + \sum_{i=1}^{l} y_i} \frac{\Delta(x)}{x^{a_2}} \chi(x) dx dy_1 \cdots dy_l$$

$$\leq 4h_1(5T/2)T^{1-a_2}. \quad \text{(IV.3)}$$

We use the integral expression for $\Delta(x)$ as given in Lemma 6 and get

$$\frac{1}{(\delta T)^l} \int_0^{\delta T} \cdots \int_0^{\delta T} \int_{T - \sum_{i=1}^{l} y_i}^{2T + \sum_{i=1}^{l} y_i} \frac{\Delta(x)}{x^{a_2}} dx dy_1 \cdots dy_l$$

$$= \frac{1}{(\delta T)^l} \int_0^{\delta T} \cdots \int_0^{\delta T} \int_{T - \sum_{i=1}^{l} y_i}^{2T + \sum_{i=1}^{l} y_i} \int_{\alpha_1 - iT^{2l}}^{\alpha_1 + iT^{2l}} \frac{D(\eta) x^{\eta - a_2}}{\eta} d\eta dx dy_1 \cdots dy_l + O(1)$$

$$\ll 1 + \frac{1}{(\delta T)^l} \int_{\alpha_1 - iT^{2l}}^{\alpha_1 + iT^{2l}} \frac{D(\eta)}{\eta} \int_0^{\delta T} \cdots \int_0^{\delta T} \int_{T - \sum_{i=1}^{l} y_i}^{2T + \sum_{i=1}^{l} y_i} x^{\eta - a_2} dx dy_1 \cdots dy_l d\eta$$

$$\ll 1 + \frac{1}{(\delta T)^l} \int_{\alpha_1 - iT^{2l}}^{\alpha_1 + iT^{2l}} \frac{D(\eta)(2T + \delta T)^{\eta - a_2 + l + 1}}{\eta \prod_{j=1}^{l+1}(\eta - a_2 + j)} d\eta$$
\[ 1 + \frac{T^{\alpha_1 - \alpha_2 + l + 1}}{(\delta T)^l} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1 + |t|)^{l-1}}{(1 + |t|)^{l+2}} dt \ll 1 + T^{1 - \alpha_2 + \alpha_1}. \]

(IV.4)

The theorem follows from (IV.2), (IV.3) and (IV.4).

**Theorem 13.** Consider \( \alpha_1(T), \alpha_2(T), \sigma_1, h_1(T) \) as in Theorem 12, and \( P \) as in Assumptions 7. Let \( D(s) \) does not have a real pole in \([\alpha_1 - \epsilon_0, \infty) - P\), for some \( \epsilon_0 > 0 \). Suppose there exists an \( X\)-Set \( S \) such that for all \( T \in S \)

\[ \mu(A(x^{\alpha_2}) \cap [T, 2T]) > 5h_1(5T/2)T^{1-\alpha_2}. \]

Then:

(i) under Assumptions 4, we have

\[ \Delta(x) = \Omega_{\pm}(h_1(x)) \]

( In this case \( \Delta(x) \) changes sign in \([T/2, 5T/2] \cap A(h_1(x)) \) for \( T \in S \) and \( T \) is sufficiently large.);

(ii) under Assumptions 3, we have

\[ \Delta(x) = \Omega_{\pm}(x^{\alpha_1 - \epsilon}), \text{ for any } \epsilon > 0. \]

**Proof.** If the conditions in Assumptions 4 hold, then (i) follows from Theorem 12. To prove (ii), choose an \( \epsilon \) such that \( 0 < \epsilon < \epsilon_0 \). Now suppose \( \eta_0 \) is a pole of \( D \) for \( \Re(\eta) \geq \alpha_1(T) - \epsilon \) and \( t \leq 2T^2l \), then by Theorem 6,

\[ \Delta(x) = \Omega_{\pm}(T^{\alpha_1 - \epsilon}). \]

If there are no poles in the above described region of \( \sigma + it \), then we are in the set-up of Assumptions 4 and get

\[ \Delta(x) = \Omega_{\pm}(h_1(x)). \]

We have

\[ T^{\alpha_1(T)} = o(h_1(T)), \]

which gives

\[ \Delta(x) = \Omega_{\pm}(x^{\alpha_1 - \epsilon}). \]

This completes the proof of (ii).
3 Applications

Now we shall see two examples demonstrating applications of Theorem 13.

3.1 Error term of the divisor function

Let \( d(n) \) denote the number of divisors of \( n \):

\[
d(n) = \sum_{d|n} 1.
\]

Dirichlet [17, Theorem 320] showed that

\[
\sum_{n \leq x} \tau(n) = x \log(x) + (2\gamma - 1)x + \Delta(x),
\]

where \( \gamma \) is the Euler constant and

\[
\Delta(x) = O(\sqrt{x}).
\]

Latest result on \( \Delta(x) \) is due to Huxley [20], which is

\[
\Delta(x) = O(x^{13/416}).
\]

On the other hand, Hardy [14] showed that

\[
\Delta(x) = \Omega_+((\log x)^{1/4} \log \log x),
\]

\[
= \Omega_-(x^{1/4}).
\]

There are many improvements of Hardy’s result. Some notable results are due to K. Corrádi and I. Kátai [7], J. L. Hafner [12], and K. Sounderarajan [36]. Below, we shall show that \( \Delta(x) \) is \( \Omega_\pm(x^{1/4}) \) as a consequence of Theorem 13 and results of Ivić and Tsang (see below). Moreover, we shall show that such fluctuations occur in \([T, 2T]\) for every sufficiently large \( T \).

Ivić [21] proved that for a positive constant \( c_{11} \),

\[
\int_T^{2T} \Delta^2(x)dx \sim c_{11}T^{3/2}.
\]
A similar result for fourth moment of $\Delta(x)$ was proved by Tsang [39]:

$$\int_T^{2T} \Delta^4(x)dx \sim c_{12}T^2,$$

for a positive constant $c_{12}$. Let $\mathcal{A}$ denote the following set:

$$\mathcal{A} := \left\{ x : |\Delta(x)| > \frac{c_{11}x^{1/4}}{6} \right\}.$$

For sufficiently large $T$, using the result of Ivić [21], we get

$$\int_{[T,2T] \cap \mathcal{A}} \Delta^2(x)dx \geq \int_{[T,2T] \cap \mathcal{A}} \Delta^4(x)dx \left( \int_{[T,2T] \cap \mathcal{A}} \frac{1}{x^2}dx \right)^{1/2} \left( \int_{[T,2T] \cap \mathcal{A}} \frac{1}{x}dx \right)^{1/2} \geq \frac{c_{11}}{5} - \frac{c_{11}}{6} \geq \frac{c_{11}}{30}.$$

Using Cauchy-Schwarz inequality and the result due to Tsang [39] we get

$$\int_{[T,2T] \cap \mathcal{A}} \Delta^2(x)dx \leq \left( \int_{[T,2T] \cap \mathcal{A}} \frac{\Delta^4(x)}{x^{3/2}}dx \right)^{1/2} \left( \int_{[T,2T] \cap \mathcal{A}} \frac{1}{x^{3/2}}dx \right)^{1/2} \leq \left( \frac{c_{12} \mu([T,2T] \cap \mathcal{A})}{T} \right)^{1/2}.$$

The above lower and upper bounds on second moment of $\Delta$ gives the following lower bound for measure of $\mathcal{A}$:

$$\mu([T,2T] \cap \mathcal{A}) > \frac{c_{11}^2}{901c_{12}}T,$$

for some $T \geq T_0$. Now, Theorem [13] applies with the following choices:

$$\alpha_1(T) = 1/5, \quad \alpha_2(T) = 1/4, \quad h_1(T) = \frac{c_{11}^2}{9000c_{12}} T^{1/4}.$$

Finally using Theorem [13] we get that for all $T \geq T_0$ there exists $x_1, x_2 \in [T,2T]$ such that

$$\Delta(x_1) > h_1(x_1) \quad \text{and} \quad \Delta(x_2) < -h_1(x_2).$$
In particular we get
\[ \Delta(x) = \Omega_{\pm}(x^{1/4}). \]

### 3.2 Average order of Non-Isomorphic abelian Groups

Let \( a_n \) denote the number of non-isomorphic abelian groups of order \( n \). The Dirichlet series \( D(s) \) is given by
\[ D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_{k=1}^{\infty} \zeta(ks), \quad \Re(s) > 1. \]

The meromorphic continuation of \( D(s) \) has poles at \( 1/k \), for all positive integer \( k \geq 1 \). Let the main term \( M(x) \) be
\[ M(x) = \sum_{k=1}^{6} \left( \prod_{j \neq k} \zeta(j/k) \right) x^{1/k}, \]
and the error term \( \Delta(x) \) be
\[ \sum_{n \leq x} a_n - M(x). \]

Balasubramanian and Ramachandra \[4\] proved that
\[ \int_T^{2T} \Delta^2(x) \, dx = \Omega(T^{4/3} \log T), \quad \text{and} \quad \Delta(x) = \Omega_{\pm}(x^{92/1221}). \]

Sankaranarayanan and Srinivas \[35\] improved the \( \Omega_{\pm} \) bound to
\[ \Delta(x) = \Omega_{\pm}\left(x^{1/10} \exp\left(c\sqrt{\log x}\right)\right) \]
for some constant \( c > 0 \). An upper bound for the second moment of \( \Delta(x) \) was first given by Ivić \[22\], and then improved by Heath-Brown \[18\] to
\[ \int_T^{2T} \Delta^2(x) \, dx \ll T^{4/3} (\log T)^{89}. \]

This bound of Heath-Brown is best possible in terms of power of \( T \). But for the fourth moment, the similar statement
\[ \int_T^{2T} \Delta^4(x) \, dx \ll T^{5/3} (\log T)^{C}, \]

47
which is best possible in terms of power of $T$, is an open problem. Another open problem is to show that
\[
\Delta(x) = \Omega_{\pm}(x^{1/6} - \delta) \quad \text{for any} \quad \delta > 0.
\]
For $0 < \delta < 1/42$, we have stated in Proposition 2 that either
\[
\int_{T}^{2T} \Delta^4(x)dx = \Omega(T^{5/3+\delta}) \quad \text{or} \quad \Delta(x) = \Omega_{\pm}(x^{1/6} - \delta).
\]
Below, we present a proof of this proposition.

**Proof of Proposition 2.** If the first statement is false, then we have
\[
\int_{T}^{2T} \Delta^4(x)dx \leq c_{13}T^{5/3+\delta},
\]
for some constant $c_{13}$ depending on $\delta$ and for all $T \geq T_0$. Let $A$ be defined by:
\[
A = \{x : |\Delta(x)| > c_{14}x^{1/6}\}, \quad c_{14} > 0.
\]
By the result of Balasubramanian and Ramachandra [4], we have an $X$-Set $S$, such that
\[
\int_{[T,2T] \cap A} \Delta^2(x)dx \geq c_{15}T^{4/3}(\log T)
\]
for $T \in S$. Using Cauchy-Schwartz inequality, we get
\[
c_{15}T^{4/3}(\log T) \leq \int_{[T,2T] \cap A} \Delta^2(x)dx \leq \left( \int_{T}^{2T} \Delta^4(x)dx \right)^{1/2} (\mu(A \cap [T, 2T]))^{1/2}
\leq c_{13}^{1/2}T^{5/6+\delta/2}(\mu(A \cap [T, 2T]))^{1/2}.
\]
This gives, for a suitable positive constant $c_{16}$,
\[
\mu(A \cap [T, 2T]) \geq c_{16}T^{1-\delta}(\log T)^2.
\]
Now we use Theorem 13 (i), with
\[
\alpha_2 = \frac{1}{6}, \quad \alpha_1 = \frac{13}{84} - \frac{\delta}{2}, \quad \text{and} \quad h_1(T) = T^{1/6-\delta}.
\]
So we get
\[
\Delta(x) = \Omega_{\pm}(x^{1/6} - \delta).
\]
This completes the proof. \qed
[ V ] THE TWISTED DIVISOR FUNCTION

Recall that in Chapter 1, we have defined the twisted divisor function \( \tau(n, \theta) \) as follows:

\[
\tau(n, \theta) = \sum_{d \mid n} d^\theta, \quad \text{for } \theta \in \mathbb{R} - \{0\}, \ n \in \mathbb{N}.
\]

We also have stated the following asymptotic formula:

\[
\sum_{n \leq x} |\tau(n, \theta)|^2 = \omega_1(\theta)x \log x + \omega_2(\theta)x \cos(\theta \log x) + \omega_3(\theta)x + \Delta(x),
\]

where \( \omega_i(\theta) \)s are explicit constants depending only on \( \theta \) and

\[
\Delta(x) = O(x^{1/2} \log^6 x).
\]

In this chapter, we give a proof of this formula (see Section 2, Theorem 14). In Section 3, we use Theorem 9 to obtain some measure theoretic \( \Omega \) results. Further, we obtain an \( \Omega \) bound for the second moment of \( \Delta(x) \) in Section 4 by adopting a technique due to Balasubramanian, Ramachandra and Subbarao [5]. In the final section, we prove that if the \( \Omega \) bound obtained in the previous section can not be improved, then

\[
\Delta(x) = \Omega(x^{3/8 - \epsilon}) \quad \text{for any} \quad \epsilon > 0.
\]

Now we motivate with a brief note on few applications of \( \tau(n, \theta) \).

1 Applications of \( \tau(n, \theta) \)

The function \( \tau(n, \theta) \) can be used to study various properties related to the distribution of divisors of an integer:

\[
\sum_{d \mid n} 1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tau(n, \theta) \frac{e^{-ib\theta} - e^{-ia\theta}}{-i\theta};
\]
here $\sum^*$ means that the corresponding contribution to the sum is $\frac{1}{2}$ if $c^n$ or $c^b$.

Below we present two applications.

1.1 Clustering of Divisors

The following function measures the clustering of divisors of an integer:

$$ W(n, f) := \sum_{d,d'|n} f(\log(d/d')) $$

for some constant $c > 0$ and for a function $f \in L^1(\mathbb{R})$. We assume that $f$ has a Fourier transformation, say $\hat{f}$, and $\hat{f} \in L^1(\mathbb{R})$.

**Proposition 4.** With the above notations:

$$ \sum_{n \leq x} W(n, f) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\theta) \sum_{n \leq x} |\tau(n, \theta)|^2 d\theta. $$

**Proof.** Note that by the Fourier inversion formula, we get

$$ W(n, f) = \sum_{d,d'|n} f(\log(d/d')) = \frac{1}{2\pi} \sum_{d,d'|n} \int_{-\infty}^{\infty} \hat{f}(\theta) \left( \frac{d}{d'} \right)^{i\theta} d\theta $$

$$ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\theta) \left( \sum_{d,d'|n} \left( \frac{d}{d'} \right)^{i\theta} \right) d\theta = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\theta) |\tau(n, \theta)|^2 d\theta. $$

This implies the proposition. \qed

Using Proposition 4 and the formula in (1.3), we may write

$$ \sum_{n \leq x} W(n, f) = \frac{x \log x}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\theta)(\omega_1(\theta)d\theta + \frac{x}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\theta)\left(\omega_2(\theta)\cos(\theta \log x) + \omega_3(\theta)\right) d\theta $$

$$ + \frac{x}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\theta)\Delta(x, \theta)d\theta. $$

(In the above identity, we denoted $\Delta(x)$ by $\Delta(x, \theta)$.)

This gives that the function $\sum_{n \leq x} W(n, f)$ behaves like $x \log x$. Further, if we want to obtain more information on $\sum_{n \leq x} W(n, f)$, we may analyzing other terms in the above formula. But now, we skip the details and refer to [13, Chapter 4].

50
1.2 The Multiplication Table Problem

The multiplication table problem asks for an estimate on the order of the growth of $|\text{Mul}(N)|$ as $N \to \infty$, where

$$\text{Mul}(N) := \{1 \leq m \leq N^2 : m = ab, \ a, b \in \mathbb{Z} \text{ and } 1 \leq a, b \leq N\}.$$ 

The initial attempts in this direction are due to Erdős [9]. He used a result of Hardy and Ramanujan [16] (also see [27]) to show

$$|\text{Mul}(N)| \ll \frac{N^2}{(\log N)^{\nu_0} \sqrt{\log \log N}} \text{ as } N \to \infty,$$

and here

$$\nu_0 = 1 - \frac{1 + \log \log 2}{\log 2}.$$ 

Intuitively, the theorem of Hardy and Ramanujan says that most of the positive integers less than $x$ have around $\log \log x$ prime factors; more precisely

$$\#\{n \leq x : |\omega(n) - \log \log n| < \sqrt{\log \log n}\} \sim x \text{ as } x \to \infty.$$ 

This gives that most of the positive integers less than $N^2$ have around $2 \log \log N$ prime factors, whereas most of the integers in the multiplication table have around $2 \log \log N$ prime factors, which implies $|\text{Mul}(N)| = o(N^2)$. Erdős used a refined version of this argument to obtain the given upper bound for $|\text{Mul}(N)|$.

The best known bound on the asymptotic growth of $|\text{Mul}(n)|$ is due to Ford [10]:

$$|\text{Mul}(N)| \asymp \frac{N^2}{(\log N)^{\nu_0}(\log \log N)^{3/2}}, \quad \text{as } N \to \infty.$$ 

To obtain the expected lower bound for $|\text{Mul}(N)|$, Ford first proved that

$$|\text{Mul}(N)| \gg \frac{N^2}{(\log N)^2} \sum_{n \leq N^{1/8}} \frac{L(n)}{n}, \quad \text{where } L(n) := \mu(\cup_{d|n}[\log(d/2), \log d]).$$ 

We may also observe that

$$\sum_{n \leq N^{1/8}} \frac{L(n)}{n} \geq \frac{\left(\sum_{n \leq N^{1/8}} \frac{d(n)}{n}\right)^2}{6 \sum_{n \leq N^{1/8}} \frac{W(n)}{n}}.$$ 

Rest of the part in Ford’s argument deals with the above sums involving the divisor
function \( d(n) \) and \( W(n) := W\left(n, 1_{\left[\frac{1}{2}, 2\right]}\right) \), where \( 1_{\left[\frac{1}{2}, 2\right]} \) is the indicator function of the interval \( \left[\frac{1}{2}, 2\right] \). We skip the details and refer to [11].

2 Asymptotic Formula for \( \sum_{n \leq x} |\tau(n, \theta)|^2 \)

In this section, we shall prove the following asymptotic formula for \( \sum_{n \leq x} |\tau(n, \theta)|^2 \).

**Theorem 14** (Theorem 33, [13]). Let \( \theta \neq 0 \) be a fixed real number. Then for \( x \geq 1 \), we have

\[
\sum_{n \leq x} |\tau(n, \theta)|^2 = \omega_1(\theta)x \log x + \omega_2(\theta)x \cos(\theta \log x) + \omega_3(\theta)x + O(\theta^{1/2} \log^6 x)
\]

where \( \omega_i(\theta) \)s are explicit constants depending only on \( \theta \).

**Proof.** Recall that the corresponding Dirichlet series \( D(s) \) has the following meromorphic continuation:

\[
D(s) = \sum_{n=1}^{\infty} \frac{|\tau(n, \theta)|^2}{n^s} = \frac{\zeta^2(s)\zeta(s + i\theta)\zeta(s - i\theta)}{\zeta(2s)} , \quad \text{for } s > 1.
\]

For \( x \geq 2 \), we denote \( \kappa = 1 + \frac{1}{\log x} \) and \( T = x + |\theta| + 1 \). By Perron’s formula

\[
\sum_{n \leq x} |\tau(n, \theta)|^2 = \frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} D(s)x^s \frac{ds}{s} + O(x^\epsilon).
\]

After shifting the line of integration to \( \Re(s) = \frac{1}{2} \), we may estimate the contributions from horizontal lines as follows:

\[
T^{-1} \int_{\frac{1}{2}}^{1} |D(\sigma \pm i\epsilon)|x^\sigma d\sigma \ll T^{-1} \int_{\frac{1}{2}}^{1} T^{-1-\sigma+\epsilon} x^\sigma d\sigma \ll x^\epsilon.
\]

To obtain an asymptotic formula for \( \sum_{n \leq x} |\tau(n, \theta)|^2 \), we add up the residues from the poles \( 1, 1 \pm i\theta \) after shifting the line of integration to \( \Re(s) = \frac{1}{2} \):

\[
\sum_{n \leq x} |\tau(n, \theta)|^2 = \mathcal{M}(x) + O\left(x^\epsilon + x^{\frac{1}{2}} \int_{-T}^{T} \left| \zeta^2\left(\frac{1}{2} + it\right)\zeta\left(\frac{1}{2} + i(t + \theta)\right)\zeta\left(\frac{1}{2} + i(t - \theta)\right) \right| dt \right),
\]

where

\[
\mathcal{M}(x) = \omega_1(\theta)x \log x + \omega_2(\theta)x \cos(\theta \log x) + \omega_3(\theta)x.
\]
If we write
\[ \mathcal{J}(a, T) := \int_{-T}^{T} \frac{\zeta^4(\frac{1}{2} + i(a + t))}{\sqrt{t^2 + \frac{1}{4}}} \, dt \quad \text{for } a, T \in \mathbb{R} \text{ and } T \geq 1, \]
then we have [23, Theorem 5.1]
\[ \mathcal{J}(a, T) \ll a \log^5 T. \]  
(V.1)

To express \( \Delta(x) \) in terms of \( \mathcal{J}(a, T) \), observe that
\[
\Delta(x) = \sum_{n \leq x}^* |\tau(n, \theta)|^2 - M(x) \\
\ll x^\epsilon + x^{\frac{3}{2}} \int_{-T}^{T} \left| \frac{\zeta^2(\frac{1}{2} + it)\zeta(\frac{1}{2} + i(t + \theta))\zeta(\frac{1}{2} + i(t - \theta))}{\zeta(1 + it)\zeta(\frac{1}{2} + it)} \right| \, dt \\
\ll x^\epsilon + x^{\frac{3}{2}} \log x \int_{-T}^{T} \frac{\zeta^2(\frac{1}{2} + it)\zeta(\frac{1}{2} + i(t + \theta))\zeta(\frac{1}{2} + i(t - \theta))}{|\frac{1}{2} + it|} \, dt.
\]

From (V.1) and using the Cauchy-Schwartz inequality twice, we get
\[ \Delta(x) \ll x^\epsilon + x^{\frac{3}{2}} \log x \mathcal{J}^{\frac{1}{2}}(0, x) \mathcal{J}^{\frac{1}{2}}(\theta, x) \mathcal{J}^{\frac{1}{2}}(-\theta, x) \ll \theta x^{\frac{1}{2}} \log^6 x, \]
which gives the required result. \( \square \)

In the following sections, we shall obtain various \( \Omega \) and \( \Omega_\pm \) bounds for \( \Delta(x) \).

3 Oscillations of the Error Term

Here we shall apply results in Chapter III to \( \Delta(x) \) and obtain some measure theoretic \( \Omega_\pm \) results. We begin by defining a contour \( \mathcal{C} \) as given in Figure V.1
\[
\mathcal{C} = \left( \frac{5}{4} - i\infty, \frac{5}{4} - i(\theta + 1) \right] \cup \left[ \frac{5}{4} - i(\theta + 1), \frac{3}{4} - i(\theta + 1) \right] \\
\cup \left[ \frac{3}{4} - i(\theta + 1), \frac{3}{4} + i(\theta + 1) \right] \cup \left[ \frac{3}{4} + i(\theta + 1), \frac{5}{4} + i(\theta + 1) \right] \\
\cup \left[ \frac{5}{4} + i(\theta + 1), \frac{5}{4} + i\infty \right).
\]
From Theorem 1 we have

\[ \Delta(x) = \int_{C} \frac{D(\eta)x^{\eta}}{\eta} d\eta. \]

The above identity expresses the Mellin transform \( A(s) \) of \( \Delta(x) \) as a contour integral involving \( D(s) \). Using Theorem 3 we write

\[ A(s) = \int_{1}^{\infty} \frac{\Delta(x)}{x^{s+1}} dx = \int_{C} \frac{D(\eta)}{\eta (s - \eta)} d\eta, \]

when \( s \) lies right to the contour \( C \). Denote the first nontrivial zero of \( \zeta(s) \) with least positive imaginary part by \( 2s_{0} \). An approximate value of this point is

\[ 2s_{0} = \frac{1}{2} + i14.134 \ldots \]

Define the contour \( C(s_{0}) \), as in Figure V.2 such that \( s_{0} \) and any real number \( s \geq 1/4 \) lie in the right side of this contour. A meromorphic continuation of \( A(s) \) to all \( s \) that lies right side of \( C(s_{0}) \) is given by

\[ A(s) = \int_{C(s_{0})} \frac{D(\eta)x^{\eta}}{\eta} d\eta + \frac{\text{Res} D(\eta)}{\eta = s_{0}} \frac{\text{Res} D(\eta)}{s_{0}(s - s_{0})}. \] (V.2)
From (V.2), we calculate the following two limits:

\[
\lambda(\theta) := \lim_{\sigma \to 0} \sigma |A(\sigma + s_0)| = |s_0|^{-1} \left| \text{Res}_{\eta=s_0} D(\eta) \right| > 0 \quad (V.3)
\]

and

\[
\lim_{\sigma \to 0} \sigma A(\sigma + 1/4) = 0.
\]

For a fixed small enough \(\epsilon > 0\), define

\[
\mathcal{A}_1 = \{ x : \Delta(x) > (\lambda(\theta) - \epsilon)x^{1/4} \},
\]

\[
\mathcal{A}_2 = \{ x : \Delta(x) < (-\lambda(\theta) + \epsilon)x^{1/4} \}.
\]

Theorem [14] and Corollary [1] give

\[
\mu(\mathcal{A}_j \cap [T, 2T]) = \Omega \left( T^{1/2} (\log T)^{-12} \right) \quad \text{for} \quad j = 1, 2. \quad (V.4)
\]

Under Riemann Hypothesis, Theorem [9] and Proposition [7] give

\[
\mu(\mathcal{A}_j \cap [T, 2T]) = \Omega \left( T^{3/4-\epsilon} \right) \quad \text{for} \quad j = 1, 2. \quad (V.5)
\]

55
From Corollary 2, we get
\[ \int_{A_j \cap [T,2T]} \Delta^2(x) dx = \Omega \left( T^{3/2} \right) \text{ for } j = 1, 2. \] (V.6)

4 An Omega Theorem

Recall that (see Theorem 14)
\[ \sum_{n \leq x} |\tau(n, \theta)|^2 = M(x) + \Delta(x), \]
where the main term
\[ M(x) = \omega_1(\theta)x \log x + \omega_2(\theta)x \cos(\theta \log x) + \omega_3(\theta)x \]
comes from the poles of \( D(s) \) at \( s = 1, 1 + i\theta \) and \( s = 1 - i\theta \). We may observe from Corollary 2 that if \( D(s) \) has a complex pole at \( s_0 = \sigma_0 + it_0 \), other than \( 1 + i\theta \) and \( 1 - i\theta \), then
\[ \int_T^{2T} \Delta(x) dx = \Omega(x^{2\sigma_0+1}). \]

By Riemann Hypothesis, the only positive value for \( \sigma_0 \) is \( \frac{1}{4} \), which is same as (V.6). In this section, we shall use a technique due to Balasubramanian, Ramachandra and Subbarao [5] to improve this omega bound further. Now we state the main theorem of this section.

**Theorem 15.** For any \( c > 0 \) and for a sufficiently large \( T \) depending on \( c \), we get
\[ \int_T^{\infty} \frac{\Delta(x)^2}{x^{2\alpha+1}} e^{-2x/y} du \gg_c \exp \left( c(\log T)^{7/8} \right), \] (V.7)
where
\[ \alpha = \alpha(T) = \frac{3}{8} - \frac{c}{(\log T)^{1/8}}. \]

In particular, this implies
\[ \Delta(x) = \Omega(x^{3/8} \exp(-c(\log x)^{7/8}), \]
for some suitable \( c > 0 \).

In order to prove the theorem, we need several lemmas, which form the content of this section. We begin with a fixed \( \delta_0 \in (0, 1/16] \) for which we would choose a numerical value at the end of this section.
**Definition 10.** For \( T > 1 \), let \( Z(T) \) be the set of all \( \gamma \) such that

1. \( T \leq \gamma \leq 2T \),
2. either \( \zeta(\beta_1 + i\gamma) = 0 \) for some \( \beta_1 \geq \frac{1}{2} + \delta_0 \)
or \( \zeta(\beta_2 + 2i\gamma) = 0 \) for some \( \beta_2 \geq \frac{1}{2} + \delta_0 \).

Let

\[
I_{\gamma,k} = \{ T \leq t \leq 2T : |t - \gamma| \leq k \log^2 T \} \text{ for } k = 1, 2.
\]

We finally define

\[
J_k(T) = [T, 2T] \setminus \bigcup_{\gamma \in Z(T)} I_{\gamma,k}.
\]

**Lemma 8.** With the above definition, we have for \( k = 1, 2 \)

\[
\mu(J_k(T)) = T + O \left( T^{1-\delta_0/4} \log^3 T \right).
\]

**Proof.** We shall use an estimate on the function \( N(\sigma, T) \), which is defined as

\[
N(\sigma, T) := |\{ \sigma' + it : \sigma' \geq \sigma, 0 < t \leq T, \zeta(\sigma' + it) = 0 \}|.
\]

Selberg [38, Page 237] proved that

\[
N(\sigma, T) \ll T^{1 - \frac{1}{4}(\sigma - \frac{1}{2})} \log T, \text{ for } \sigma > 1/2.
\]

Now the lemma follows from the above upper bound on \( N(\sigma, t) \), and the observation that

\[
| \bigcup_{\gamma \in Z(T)} I_{\gamma,k} | \ll N \left( \frac{1}{2} + \delta_0, T \right) \log^2 T.
\]

\[\square\]

The next lemma closely follows Theorem 14.2 of [38], but does not depend on Riemann Hypothesis.

**Lemma 9.** For \( t \in J_1(T) \) and \( \sigma = 1/2 + \delta \) with \( \delta_0 < \delta < 1/4 - \delta_0/2 \), we have

\[
|\zeta(\sigma + it)|^{\pm 1} \ll \exp \left( \log \log t \left( \frac{\log t}{\delta_0} \right)^{\frac{1-2\delta}{1-2\delta_0}} \right)
\]

and

\[
|\zeta(\sigma + 2it)|^{\pm 1} \ll \exp \left( \log \log t \left( \frac{\log t}{\delta_0} \right)^{\frac{1-2\delta}{1-2\delta_0}} \right).
\]
Proof. We provide a proof of the first statement, and the second statement can be similarly proved.

Let \(1 < \sigma' \leq \log t\). We consider two concentric circles centered at \(\sigma' + it\), with radius \(\sigma' - 1/2 - \delta_0/2\) and \(\sigma' - 1/2 - \delta_0\). Since \(t \in J_1(T)\) and the radius of the circle is \(\ll \log t\), we conclude that

\[
\zeta(z) \neq 0 \text{ for } |z - \sigma' - it| \leq \sigma' - \frac{1}{2} - \frac{\delta_0}{2}
\]

and also \(\zeta(z)\) has polynomial growth in this region. Thus on the larger circle, \(|\log \zeta(z)| \leq c_{17} \log t\), for some constant \(c_{17} > 0\). By Borel-Carathéodory theorem,

\[
|z - \sigma' - it| \leq \sigma' - \frac{1}{2} - \delta_0 \text{ implies } |\log \zeta(z)| \leq \frac{c_{18} \sigma'}{\delta_0} \log t,
\]

for some \(c_{18} > 0\). Let \(1/2 + \delta_0 < \sigma < 1\), and \(\xi > 0\) be such that \(1 + \xi < \sigma'\). We consider three concentric circles centered at \(\sigma' + it\) with radius \(r_1 = \sigma' - 1 - \xi\), \(r_2 = \sigma' - \sigma\) and \(r_3 = \sigma' - 1/2 - \delta_0\), and call them \(C_1, C_2\) and \(C_3\) respectively. Let

\[
M_i = \sup_{z \in C_i} |\log \zeta(z)|.
\]

From the above bound on \(|\log \zeta(z)|\), we get

\[
M_3 \leq \frac{c_{18} \sigma'}{\delta_0} \log t.
\]

Suitably enlarging \(c_{18}\), we see that

\[
M_1 \leq \frac{c_{18}}{\xi}.
\]

Hence we can apply the Hadamard’s three circle theorem to conclude that

\[
M_2 \leq M_1^{1-\nu} M_3^\nu, \text{ for } \nu = \frac{\log(r_2/r_1)}{\log(r_3/r_1)}.
\]

Thus

\[
M_2 \leq \left(\frac{c_{18}}{\xi}\right)^{1-\nu} \left(\frac{c_{18} \sigma' \log t}{\delta_0}\right)^\nu.
\]

It is easy to see that

\[
\nu = 2 - 2\sigma + \frac{4\delta_0(1 - \sigma)}{1 + 2\xi - 2\delta_0} + O(\xi) + O\left(\frac{1}{\sigma'}\right).
\]

58
Now we put
\[ \xi = \frac{1}{\sigma'} = \frac{1}{\log \log t}. \]

Hence
\[ M_2 \leq \frac{c_{18} \log t \log \log t}{\delta_0'} = \frac{c_{19} \log \log t}{\delta_0'} (\log t)^{2-2\sigma + \frac{4\delta_0(1-\sigma)}{1+2\xi-2\delta_0}}, \]
for some \( c_{19} > 0 \). We observe that
\[ 2 - 2\sigma + \frac{4\delta_0(1-\sigma)}{1+2\xi-2\delta_0} < 2 - 2\sigma + \frac{4\delta_0(1-\sigma)}{1-2\delta_0} = 1 - 2\delta. \]
So we get
\[ |\log \zeta(\sigma + it)| \leq c_{19} \log \log t \left( \frac{\log t}{\delta_0} \right)^{\frac{1-2\delta}{1-2\delta_0}}, \]
and hence the lemma.

We put \( y = T^b \), for a constant \( b \geq 8 \). Now suppose that
\[ \int_{T}^{\infty} \frac{|\Delta(u)|^2}{u^{2\alpha+1}} e^{-u/y} du \geq \log^2 T, \]
for sufficiently large \( T \). Then clearly
\[ \Delta(u) = \Omega(u^\alpha). \]

Our next result explores the situation when such an inequality does not hold.

**Proposition 5.** Let \( \delta_0 < \delta < \frac{1}{4} - \frac{\delta_0}{2} \). For \( 1/4 + \delta/2 < \alpha < 1/2 \), suppose that
\[ \int_{T}^{\infty} \frac{|\Delta(u)|^2}{u^{2\alpha+1}} e^{-u/y} du \leq \log^2 T \]
for any sufficiently large \( T \). Then we have
\[ \int_{Re(s)=\alpha} \frac{|D(s)|^2}{|s|^2} ds \ll 1 + \int_{T}^{\infty} \frac{|\Delta(u)|^2}{u^{2\alpha+1}} e^{-2u/y} du. \]

Before embarking on a proof, we need the following technical lemmas.

**Lemma 10.** For \( 0 \leq \Re(z) \leq 1 \) and \( |\Im(z)| \geq \log^2 T \), we have
\[ \int_{T}^{\infty} e^{-u/y} u^{-z} du = \frac{T^{1-z}}{1-z} + O(T^{-b'}) \quad \text{(V.9)} \]
\[ \int_T^\infty e^{-u/y}u^{-z} \log u \, du = \frac{T^{1-z}}{1-z} \log T + O(T^{-b'}), \quad (V.10) \]

where \( b' > 0 \) depends only on \( b \).

**Proof.** By changing variable by \( v = u/y \), we get

\[ \int_T^\infty e^{-u/y}u^{-z} \log u \, du = y^{1-z} \int_{T/y}^\infty e^{-v}v^{-z} \log v \, dv. \]

Integrating the right hand side by parts

\[ \int_{T/y}^\infty e^{-v}v^{-z} \log v \, dv = \frac{e^{-T/y}}{1-z} \left( \frac{T}{y} \right)^{1-z} + \frac{1}{1-z} \int_{T/y}^\infty e^{-v}v^{1-z} \log v \, dv \]

It is easy to see that

\[ \int_{T/y}^\infty e^{-v}v^{1-z} \log v \, dv = \Gamma(2-z) + O \left( \left( \frac{T}{y} \right)^{2-\Re(z)} \right). \]

Hence (V.9) follows using \( e^{-T/y} = 1 + O(T/y) \) and Stirling’s formula along with the assumption that \( |\text{Im}(z)| \geq \log^2 T \).

Proof of (V.10) proceeds in the same line and uses the fact that

\[ \int_{T/y}^\infty e^{-v}v^{1-z} \log v \, dv = \Gamma'(2-z) + O \left( \left( \frac{T}{y} \right)^{2-\Re(z)} \log T \right). \]

Then we apply Stirling’s formula for \( \Gamma'(s) \) instead of \( \Gamma(s) \).

---

**Lemma 11.** Under the assumption (V.8), there exists \( T_0 \) with \( T \leq T_0 \leq 2T \) such that

\[ \frac{\Delta(T_0)e^{-T_0/y}}{T_0^a} \ll \log^2 T, \]

and

\[ \frac{1}{y} \int_{T_0}^\infty \frac{\Delta(u)e^{-u/y}}{u^a} \, du \ll \log T. \]

**Proof.** The assumption (V.8) implies that

\[ \log^2 T \geq \int_T^{2T} \frac{\Delta(u)^2}{u^{2a+1}} e^{-u/y} \, du \]

\[ = \int_T^{2T} \frac{\Delta(u)^2}{u^{2a}} e^{-2u/y} \, du \]

\[ = \int_T^{2T} \frac{\Delta(u)^2}{u^{2a}} e^{-2u/y} \, du \]

60
\[
\geq \min_{T \leq u \leq 2T} \left( \frac{|\Delta(u)|}{u^\alpha} e^{-u/y} \right)^2,
\]
which proves the first assertion. To prove the second assertion, we use the previous assertion and Cauchy-Schwartz inequality along with assumption (V.8) to get
\[
\left( \int_{T_0}^\infty \frac{|\Delta(u)|^2}{u^\alpha} e^{-u/y} du \right)^2 \leq \left( \int_{T_0}^\infty \frac{|\Delta(u)|^2}{u^{2\alpha+1}} e^{-u/y} du \right) \left( \int_{T_0}^\infty u e^{-u/y} du \right) \ll y^2 \log^2 T.
\]
This completes the proof of this lemma.

We now recall a mean value theorem due to Montgomery and Vaughan [30].

**Notation.** For a real number \( \theta \), let \( \|\theta\| := \min_{n \in \mathbb{Z}} |\theta - n| \).

**Theorem 16** (Montgomery and Vaughan [30]). Let \( a_1, \ldots, a_N \) be arbitrary complex numbers, and let \( \lambda_1, \ldots, \lambda_N \) be distinct real numbers such that
\[
\delta = \min_{m,n} \|\lambda_m - \lambda_n\| > 0.
\]
Then
\[
\left| \int_0^T \sum_{n \leq N} a_n \exp(i\lambda_n t) \right|^2 dt = \left( T + O\left( \frac{1}{\delta} \right) \right) \sum_{n \leq N} |a_n|^2.
\]

**Lemma 12.** For \( T \leq T_0 \leq 2T \) and \( \Re(s) = \alpha \), we have
\[
\int_T^{2T} \left| \sum_{n \leq T_0} \frac{|\tau(n, \theta)|^2}{n^s} e^{-n/y} \right|^2 t^{-2} dt \ll 1.
\]

**Proof.** Using theorem 16, we get
\[
\int_T^{2T} \left| \sum_{n \leq T_0} \frac{|\tau(n, \theta)|^2}{n^s} e^{-n/y} \right|^2 t^{-2} dt \leq \frac{1}{T^2} \left( T \sum_{n \leq T_0} |b(n)|^2 + O \left( \sum_{n \leq T_0} n|b(n)|^2 \right) \right),
\]
where \( b(n) = \frac{|\tau(n, \theta)|^2}{n^\alpha} e^{-n/y} \).

Thus
\[
\sum_{n \leq T_0} |b(n)|^2 \leq \sum_{n \leq T_0} \frac{d(n)^4}{n^{2\alpha + \epsilon}} \ll T_0^{1-2\alpha + \epsilon} \quad \text{and} \quad \sum_{n \leq T_0} n|b(n)|^2 \leq \sum_{n \leq T_0} \frac{d(n)^4}{n^{2\alpha-1}} \ll T_0^{2-2\alpha + \epsilon}
\]
for any $\epsilon > 0$, since the divisor function $d(n) \ll n^\epsilon$. As we have $\alpha > 0$, this completes the proof. \hfill \Box

**Lemma 13.** For $\Re(s) = \alpha$ and $T \leq T_0 \leq 2T$, we have

$$\int_{T}^{2T} \left| \sum_{n \geq 0} \int_{0}^{1} \frac{\Delta(n + x + T_0)e^{-(n+x+T_0)/y}}{(n + x + T_0)^{s+1}}dx \right|^2 dt \ll \int_{T}^{\infty} \frac{|\Delta(x)|^2}{x^{2\alpha+1}}e^{-2x/y}dx.$$

**Proof.** Using Cauchy-Schwarz inequality, we get

$$\left| \sum_{n \geq 0} \int_{0}^{1} \frac{\Delta(n + x + T_0)e^{-(n+x+T_0)/y}}{(n + x + T_0)^{s+1}}dx \right|^2 \leq \int_{0}^{1} \left| \sum_{n \geq 0} \frac{\Delta(n + x + T_0)e^{-(n+x+T_0)/y}}{(n + x + T_0)^{s+1}} \right|^2 dx.$$

Hence

$$\int_{T}^{2T} \left| \int_{0}^{1} \sum_{n \geq 0} \frac{\Delta(n + x + T_0)e^{-(n+x+T_0)/y}}{(n + x + T_0)^{s+1}}dx \right|^2 dt \leq \int_{T}^{2T} \left| \int_{0}^{1} \sum_{n \geq 0} \frac{\Delta(n + x + T_0)e^{-(n+x+T_0)/y}}{(n + x + T_0)^{s+1}} \right|^2 dx dt \leq \int_{0}^{1} \left| \int_{T}^{2T} \sum_{n \geq 0} \frac{\Delta(n + x + T_0)e^{-(n+x+T_0)/y}}{(n + x + T_0)^{s+1}} \right|^2 dt dx.$$

From Theorem 16 we can get

$$\int_{T}^{2T} \left| \sum_{n \geq 0} \frac{\Delta(n + x + T_0)e^{-(n+x+T_0)/y}}{(n + x + T_0)^{s+1}} \right|^2 dt \ll T \sum_{n \geq 0} \frac{|\Delta(n + x + T_0)|^2}{(n + x + T_0)^{2\alpha+2}}e^{-2(n+x+T_0)/y} + O\left( \sum_{n \geq 0} \frac{|\Delta(n + x + T_0)|^2}{(n + x + T_0)^{2\alpha+1}} \right)$$

$$\ll \sum_{n \geq 0} \frac{|\Delta(n + x + T_0)|^2}{(n + x + T_0)^{2\alpha+1}}e^{-2(n+x+T_0)/y}.$$
Hence
\[
\int_0^{2T} \left| \sum_{n \geq 0} \int_0^1 \frac{\Delta(n + x + T_0)e^{-(n+x+T_0)/T}}{(n + x + T_0)^{s+1}} \, dx \right|^2 \, dt \\
\ll \int_0^1 \sum_{n \geq 0} \frac{|\Delta(n + x + T_0)|^2}{(n + x + T_0)^{2s+1}} e^{-2(n+x+T_0)/y} \, dx \ll \int_T^\infty \frac{|\Delta(x)|^2}{x^{2s+1}} e^{-2x/y} \, dx,
\]
completing the proof. \hfill \Box

**Proof of Proposition 5.** For \( s = \alpha + it \) with \( 1/4 + \delta/2 < \alpha < 1/2 \) and \( t \in J_2(T) \), we have
\[
\sum_{n=1}^\infty \frac{|\tau(n, \theta)|^2}{n^s} e^{-n/y} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} D(s + w)\Gamma(w)y^w \, dw \\
= \frac{1}{2\pi i} \int_{2-i\log^2 T}^{2+i\log^2 T} + O\left(y^2 \int_{\log^2 T}^\infty |D(s + w)||\Gamma(w)|dw\right).
\]
The above error term is estimated to be \( o(1) \). We move the integral to
\[
\left[\frac{1}{4} + \frac{\delta}{2} - \alpha - i log^2 T, \frac{1}{4} + \frac{\delta}{2} - \alpha + i log^2 T\right].
\]
Let \( \delta' = 1/4 + \delta/2 - \alpha \). In this region \( \Re(2s + 2w) = 1/2 + \delta \). So we can apply Lemma 9 to conclude that \( D(s + w) = O(T^\kappa) \), for some constant \( \kappa > 0 \). Thus the integrals along horizontal lines are \( o(1) \). Since the only pole inside this contour is at \( w = 0 \), we get
\[
\sum_{n=1}^\infty \frac{|\tau(n, \theta)|^2}{n^s} e^{-n/y} = D(s) + \frac{1}{2\pi i} \int_{\delta'-i\log^2 T}^{\delta'+i\log^2 T} D(s + w)\Gamma(w)y^w \, dw + o(1).
\]
Since \( \delta' < 0 \), the remaining integral can be shown to be \( o(1) \) for \( b \geq 8 \). Using \( T_0 \) as in Lemma 11 we now divide the sum into two parts:
\[
D(s) = \sum_{n \leq T_0} \frac{|\tau(n, \theta)|^2}{n^s} e^{-n/y} + \sum_{n > T_0} \frac{|\tau(n, \theta)|^2}{n^s} e^{-n/y} + o(1).
\]
To estimate the second sum, we write
\[
\sum_{n > T_0} \frac{|\tau(n, \theta)|^2}{n^s} e^{-n/y} = \int_{T_0}^\infty \frac{e^{-x/y}}{x^s} d\left(\sum_{n \leq x} |\tau(n, \theta)|^2\right)
\]
63
\[
\int_{T_0}^{\infty} \frac{e^{-x/y}}{x^s} d(M(x) + \Delta(x)) = \int_{T_0}^{\infty} \frac{e^{-x/y}}{x^s} M'(x) dx + \int_{T_0}^{\infty} \frac{e^{-x/y}}{x^s} d(\Delta(x)).
\]

Recall that

\[M(x) = \omega_1(\theta)x \log x + \omega_2(\theta)x \cos(\theta \log x) + \omega_3(\theta)x,\]

thus

\[M'(x) = \omega_1(\theta) \log x + \omega_2(\theta) \cos(\theta \log x) - \theta \omega_2(\theta) \sin(\theta \log x) + \omega_1(\theta) + \omega_3(\theta).\]

Observe that

\[
\int_{T_0}^{\infty} \frac{e^{-x/y}}{x^s} \cos(\theta \log x) dx = \frac{1}{2} \int_{T_0}^{\infty} \frac{e^{-x/y}}{x^{s+i\theta}} dx + \frac{1}{2} \int_{T_0}^{\infty} \frac{e^{-x/y}}{x^{s-i\theta}} dx.
\]

Applying Lemma 10, we conclude that

\[
\int_{T_0}^{\infty} \frac{e^{-x/y}}{x^s} M'(x) dx = o(1).
\]

Integrating the second integral by parts:

\[
\int_{T_0}^{\infty} \frac{e^{-x/y}}{x^s} d(\Delta(x)) = \frac{e^{-T_0/y} \Delta(T_0)}{T_0^s} + \frac{1}{y} \int_{T_0}^{\infty} \frac{e^{-x/y}}{x^s} \Delta(x) dx - s \int_{T_0}^{\infty} \frac{e^{-x/y}}{x^{s+1}} \Delta(x) dx.
\]

Applying Lemma 11, we get

\[
\sum_{n>T_0} \frac{1}{n^s} \frac{|\tau(n, \theta)|^2}{e^{-n/y}} = s \int_{T_0}^{\infty} \frac{\Delta(x) e^{-x/y}}{x^{s+1}} dx + O(\log T)
\]

\[
= s \sum_{n \geq 0} \int_{0}^{1} \frac{\Delta(n + x + T_0) e^{-(n+x+T_0)/y}}{(n + x + T_0)^{s+1}} dx + O(\log T).
\]

Hence we have

\[
D(s) = \sum_{n \leq T_0} \frac{|\tau(n, \theta)|^2}{n^s} e^{-n/y} + s \sum_{n \geq 0} \int_{0}^{1} \frac{\Delta(n + x + T_0) e^{-(n+x+T_0)/y}}{(n + x + T_0)^{s+1}} dx + O(\log T).
\]

64
Squaring both sides, and then integrating on \( J_2(T) \), we get
\[
\int_{J_2(T)} \frac{|D(\alpha + it)|^2}{|\alpha + it|^2} \, dt \ll \int_T^{2T} \left| \sum_{n \leq T_0} \frac{|\tau(n, \theta)|^2}{n^s} e^{-n/y} \right|^2 \, dt
\]
\[
+ \int_T^{2T} \left| \sum_{n \geq N_0} \int_0^1 \frac{\Delta(n + x + T_0) e^{-(n + x + T_0)/y}}{(n + x + T_0)^{s+1}} \, dx \right|^2 \, dt.
\]

The proposition now follows using Lemma 12 and Lemma 13.

We are now ready to prove our main theorem of this section.

**Proof of Theorem 15.** We prove by contradiction. Suppose that (V.7) does not hold. Then, given any \( N_0 > 1 \), there exists \( T > N_0 \) such that
\[
\int_T^\infty \frac{\Delta(x)^2}{x^{2\beta+1}} e^{-2x/y} \, dx \ll \exp \left( c \log T \right)^{7/8},
\]
for all \( c > 0 \). This gives
\[
\int_T^\infty \frac{\Delta(x)^2}{x^{2\beta+1}} e^{-2x/y} \, dx \ll 1,
\]
where
\[
\beta = \frac{3}{8} - \frac{c}{2(\log T)^{1/8}}.
\]

We apply Proposition 5 to get
\[
\int_{J_2(T)} \frac{|D(\beta + it)|^2}{|\beta + it|^2} \, dt \ll 1. \tag{V.11}
\]

Now we compute a lower bound for the last integral over \( J_2(T) \). Write the functional equation for \( \zeta(s) \) as
\[
\zeta(s) = \pi^{1/2-s} \frac{\Gamma((1-s)/2)}{\Gamma(s/2)} \zeta(1-s).
\]

Using the Stirling’s formula for \( \Gamma \) function, we get
\[
|\zeta(s)| = \pi^{1/2-\beta} t^{1/2-\beta} |\zeta(1-s)| \left( 1 + O \left( \frac{1}{T} \right) \right),
\]

65
for $s = \beta + it$. This implies

$$|D(\beta + it)| = t^{2-4\beta} \frac{|\zeta(1-\beta + it)^2 \zeta(1-\beta - it - i\theta) \zeta(1-\beta - it + i\theta)|}{|\zeta(2\beta + i2t)|}.$$ 

Let $\delta_0 = 1/16$, and

$$\beta = \frac{3}{8} - \frac{c}{2(\log T)^{1/8}} = \frac{1}{2} - \delta$$

with

$$\delta = \frac{1}{8} + \frac{c}{2(\log T)^{1/8}}.$$ 

Then using Lemma 9 we get

$$|\zeta(1-\beta + it)| = \left| \zeta \left( \frac{1}{2} + \delta + it \right) \right| \gg \exp \left( \log \log t \left( \frac{\log t}{\delta_0} \right)^{\frac{1-2\delta}{1-2\delta_0}} \right).$$

For $t \in J_2(T)$ we observe that $t \pm \theta \in J_1(T)$, and so the same bounds hold for $\zeta(1-\beta + it + i\theta)$ and $\zeta(1-\beta + it - i\theta)$. Further

$$|\zeta(2\beta + i2t)| = \left| \zeta \left( \frac{1}{2} + \left( \frac{1}{2} - 2\delta \right) + i2t \right) \right| \ll \exp \left( \log \log t \left( \frac{\log t}{\delta_0} \right)^{\frac{4\delta}{1-2\delta_0}} \right).$$

Combining these bounds, we get

$$|D(\beta + it)| \gg t^{2-4\beta} \exp \left( -5 \log \log t \left( \frac{\log t}{\delta_0} \right)^{\frac{1-2\delta}{1-2\delta_0}} \right).$$

Therefore

$$\int_{J_2(T)} |D(\beta + it)|^2 dt \gg \int_{J_2(T)} |D(\beta + it)|^2 dt \gg \int_{J_2(T)} |D(\beta + it)|^2 dt$$

$$\gg T^{4-8\beta} \exp \left( -10 \log \log T \left( \frac{\log T}{\delta_0} \right)^{\frac{1-2\delta}{1-2\delta_0}} \right) \mu(J_2(T))$$

$$\gg T^{5-8\beta} \exp \left( -10 \log \log T \left( \frac{\log T}{\delta_0} \right)^{\frac{1-2\delta}{1-2\delta_0}} \right),$$

where we use Lemma 8 to show that $\mu(J_2(T)) \gg T$. Now putting the values of $\delta$
and $\delta_0$ as chosen above, we get
\[
\int_{j(T)} |D(\beta + it)|^2 \frac{dt}{|\beta + it|^2} \gg \exp\left(3c(\log T)^{7/8}\right),
\]
since $\frac{1-2\delta}{1-2\delta_0} < 7/8$. This contradicts (V.11), and hence the theorem follows. \(\Box\)

The following two corollaries are immediate.

**Corollary 3.** For any $c > 0$ there exists an $X$-Set $S$, such that for sufficiently large $T$ depending on $c$ there exists an
\[
X \in \left[ T, \frac{T^b}{2 \log^2 T} \right] \cap S,
\]
for which we have
\[
\int_X^{2X} \frac{|\Delta(x)|^2}{x^{2\alpha+1}} dx \geq \exp\left( (c - \epsilon)(\log X)^{7/8} \right)
\]
with $\alpha$ as in Theorem 13 and for any $\epsilon > 0$.

**Corollary 4.** For any $c > 0$ there exists an $X$-Set $S$, such that for sufficiently large $T$ depending on $c$ there exists an
\[
x \in \left[ T, \frac{T^b}{2 \log^2 T} \right] \cap S,
\]
for which we have
\[
\Delta(x) \geq x^{3/8} \exp\left( -c(\log x)^{7/8} \right).
\]

We can now prove a "measure version" of the above result.

**Proposition 6.** For any $c > 0$, let
\[
\alpha(x) = \frac{3}{8} - \frac{c}{(\log x)^{1/8}}
\]
and $A = \{x : |\Delta(x)| \gg x^{\alpha(x)}\}$. Then for every sufficiently large $X$ depending on $c$, we have
\[
\mu(A \cap [X, 2X]) = \Omega(X^{2\alpha(X)}).
\]

**Proof.** Suppose that the conclusion does not hold, hence
\[
\mu(A \cap [X, 2X]) \ll X^{2\alpha(X)}.
\]
Thus for every sufficiently large $X$, we get
\[
\int_{A \cap [X,2X]} \frac{|\Delta(x)|^2}{x^{2\alpha+1}} \, dx \ll X^{2\alpha} \frac{M(X)}{X^{2\alpha+1}} = \frac{M(X)}{X},
\]
where $\alpha = \alpha(X) = \alpha_X$ and $M(X) = \sup_{X \leq x \leq 2X} |\Delta(x)|^2$. Using dyadic partition, we can prove
\[
\int_{A \cap [T,y]} \frac{|\Delta(x)|^2}{x^{2\alpha+1}} \, dx \ll \frac{M_0(T)}{T} \log T, \quad \text{where} \quad M_0(T) = \sup_{T \leq x \leq y} |\Delta(x)|^2
\]
and $y = T^b$ for some $b > 0$ and $T$ sufficiently large. This gives
\[
\int_T^\infty \frac{|\Delta(x)|^2}{x^{2\alpha+1}} e^{-2x/y} \, dx \ll \frac{M_0(T)}{T} \log T.
\]
Along with (V.7), this implies
\[
M_0(T) \gg T \exp \left( \frac{c}{2} \left( \log T \right)^{7/8} \right).
\]
Thus
\[
|\Delta(x)| \gg x^{1/2} \exp \left( \frac{c}{4} \left( \log x \right)^{7/8} \right),
\]
for some $x \in [T, y]$. This contradicts the fact that $|\Delta(x)| \ll x^{1/2} \left( \log x \right)^6$. \qed

4.1 Optimality of the Omega Bound

The following proposition shows the optimality of the omega bound in Corollary 3.

Proposition 7. Under Riemann Hypothesis (RH), we have
\[
\int_X^{2X} \Delta^2(x) \, dx \ll X^{7/4+\epsilon},
\]
for any $\epsilon > 0$.

Proof. Theorem 2 (Perron’s formula) gives
\[
\Delta(x) = \frac{1}{2\pi} \int_{-T}^{T} D(3/8 + it)x^{3/8+it} \, dt + O(x^\epsilon),
\]
for any $\epsilon > 0$ and for $T = X^2$ with $x \in [X, 2X]$. Using this expression for $\Delta(x)$, we
write its second moment as
\[
\int_X^{2X} \Delta^2(x)dx = \int_X^{2X} \int_{-T}^{T} \int_{-T}^{T} \frac{D(3/8 + it_1)D(3/8 + it_2)}{(3/8 + it_1)(3/8 + it_2)} x^{3/4 + i(t_1 + t_2)} dx \, dt_1 dt_2 \\
+ O \left( X^{1+\epsilon}(1 + |\Delta(x)|) \right)
\]
\[
\ll X^{7/4} \int_{-T}^{T} \int_{-T}^{T} \frac{D(3/8 + it_1)D(3/8 + it_2)}{(3/8 + it_1)(3/8 + it_2)(7/4 + i(t_1 + t_2))} \big| dt_1 dt_2 + O(X^{3/2+\epsilon}).
\]

In the above calculation, we have used the fact that \( \Delta(x) \ll x^{1/4+\epsilon} \) as in (1.4). Also note that for complex numbers \( a, b \), we have \( |ab| \leq \frac{1}{2}(|a|^2 + |b|^2) \). We use this inequality with
\[
a = \frac{|D(3/8 + it_1)|}{|3/8 + it_1|\sqrt{7/4 + i(t_1 + t_2)}} \quad \text{and} \quad b = \frac{|D(3/8 + it_2)|}{|3/8 + it_2|\sqrt{7/4 + i(t_1 + t_2)}},
\]
to get
\[
\int_X^{2X} \Delta^2(x)dx \ll X^{7/4} \int_{-T}^{T} \int_{-T}^{T} \left| \frac{D(3/8 + it_2)}{(3/8 + it_2)} \right|^2 dt_1 \, dt_2 + O(X^{3/2+\epsilon})
\]
\[
\ll X^{7/4} \log X \int_{-T}^{T} \left| \frac{D(3/8 + it_2)}{(3/8 + it_2)} \right|^2 dt_2 + O(X^{3/2+\epsilon}).
\]

Under RH, \( |D(3/8 + it_2)| \ll |t_2|^{1/2+\epsilon} \). So we have
\[
\int_X^{2X} \Delta^2(x)dx \ll X^{7/4+\epsilon} \quad \text{for any} \quad \epsilon > 0.
\]

Note. The method we have used in Theorem 15 has its origin from the Plancherel’s formula in Fourier analysis. For instance, we may observe from Theorem 1 that under Riemann Hypothesis and other suitable conditions
\[
\frac{\Delta(e^u)}{e^{u\sigma}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{D(\sigma + it)e^{it}}{\sigma + it} \, dt \quad \text{for} \quad \frac{1}{4} < \sigma \leq \frac{1}{2}.
\]
So \( \frac{\Delta(e^u)}{e^{u\sigma}} \) is the Fourier transform of \( \frac{D(\sigma + it)}{\sigma + it} \). By Plancherel’s formula
\[
\int_{-\infty}^{\infty} \left| \frac{\Delta(e^u)}{e^{2u\sigma}} \right| du = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \left| \frac{D(\sigma + it)}{\sigma + it} \right|^2 \, dt.
\]
Now we change the variable $u$ to $\log x$ and use the functional equation for $\zeta(s)$ to get

$$\int_1^\infty \Delta^2(x) \frac{dx}{x^{2\sigma+1}} \lesssim \int_1^\infty \left| \frac{D(\sigma+it)}{\sigma+it} \right|^2 dt \gg \int_1^\infty t^{2-8\sigma-16\epsilon}$$

for any $\epsilon > 0$. We may choose $\sigma = \frac{3}{8} - \epsilon$, then the above integral on the left side is convergent. But if $\Delta(x) \ll x^{3/8-\epsilon}$, then the integral in the right diverges. This gives

$$\Delta(x) = \Omega(x^{\frac{3}{8}-\epsilon}).$$

In [3] and [4], Balasubramanian and Ramachandra used this insight to obtain $\Omega$ bounds for the error terms in asymptotic formulas for partial sums of square-free divisors and counting function for non-isomorphic abelian groups. This method requires the Riemann Hypothesis to be assumed in certain cases. Later Balasubramanian, Ramachandra and Subbarao [5] modified this technique to apply on error term in the asymptotic formula for the counting function of $k$-full numbers without assuming Riemann Hypothesis. This method has been used by several authors including [25] and [35].

5 Influence of Measure on $\Omega_\pm$ Results

In this section, we shall show that for any $\epsilon > 0$,

$$\text{if } \Delta(x) \ll x^{3/8+\epsilon}, \text{ then } \Delta(x) = \Omega_\pm \left(x^{3/8-\epsilon}\right).$$

This improves our earlier result, which says that $\Delta(x)$ is $\Omega_\pm \left(x^{1/4}\right)$. Now, we state the main theorem of this section.

**Theorem 17.** Let $\Delta(x)$ be the error term of the summatory function of the twisted divisor function as in Theorem 14. For $c > 0$, let

$$\alpha(x) = \frac{3}{8} - \frac{c}{(\log x)^{1/8}}.$$

Let $\delta$ and $\delta'$ be such that

$$0 < \delta < \delta' < \frac{1}{8}.$$

Then either

$$\Delta(x) = \Omega \left(x^{\alpha(x)+\frac{\delta}{2}}\right) \text{ or } \Delta(x) = \Omega_\pm \left(x^{\frac{3}{8}-\delta'}\right).$$

To prove the above theorem, we estimate the growth of the Dirichlet series $D(\sigma+it)$
by assuming that it does not have poles in a certain region.

**Lemma 14.** Let \( \delta \) and \( \sigma \) be such that

\[
0 < \delta < \frac{1}{8}, \quad \text{and} \quad \frac{3}{8} - \delta \leq \sigma < \frac{1}{2}.
\]

If \( D(\sigma + it) \) does not have a pole in the above mentioned range of \( \sigma \), then for

\[
\frac{3}{8} - \delta + \frac{\delta}{2(1 + \log \log(3 + |t|))} < \sigma < \frac{1}{2},
\]

we have

\[
D(\sigma + it) \ll_{\delta, \theta} |t|^{2-2\sigma + \epsilon}
\]

for any positive constant \( \epsilon \).

**Proof.** Let \( s = \sigma + it \) with \( 3/8 - \delta \leq \sigma < 1/2 \). Recall that

\[
D(s) = \frac{\zeta^2(s)\zeta(s + i\theta)\zeta(s - i\theta)}{\zeta(2s)}.
\]

Using functional equation, we write

\[
D(s) = \mathcal{X}(s)\frac{\zeta^2(1-s)\zeta(1-s-i\theta)\zeta(1-s+i\theta)}{\zeta(2s)},
\]

where \( \mathcal{X}(s) \) is of order (can be obtained from Stirling’s formula for \( \Gamma \))

\[
\mathcal{X}(\sigma + it) \asymp t^{2-4\sigma}.
\]

Using Stirling’s formula and Phragmén-Lindelöf principle, we get

\[
|\zeta(1-s)| \ll |t|^{\sigma/2} \log t.
\]

So we get

\[
|\zeta^2(1-s)\zeta(1-s-i\theta)\zeta(1-s+i\theta)| \ll t^{2\sigma} (\log t)^4.
\]

Now we shall compute an upper bound for \( |\zeta(2s)|^{-1} \). This can be obtained in a similar way as in Lemma 9. We choose \( t \geq 100 \). Similar computation can be done when \( t \) is negative.

Consider two concentric circles \( C_{1,1} \) and \( C_{1,2} \), centered at \( 2 + it \) with radii

\[
\frac{5}{4} + 2\delta \quad \text{and} \quad \frac{5}{4} + 2\delta - \frac{\delta}{1 + \log \log(|t| + 3)}.
\]
The circle $C_{1,1}$ passes through $3/4 - 2\delta + i2t$ and $C_{1,2}$ passes through $3/4 - 2\delta + \delta(1 + \log \log(|t| + 3))^{-1} + i2t$. By our assumption, $\zeta(z)$ does not have any zero for $|z - 2 - it| \leq 5/4 + 2\delta$. This implies $\log \zeta(z)$ is a holomorphic function in this region. It is easy to see that on the larger circle $C_{1,1}$, we have $\log |\zeta(z)| < \sigma' \log t$ for some positive constant $\sigma'$. We apply Borel-Carathéodory theorem to get an upper bound for $\log \zeta(z)$ on $C_{1,2}$:

$$|\log \zeta(z)| \leq 3\delta^{-1}(1 + \log \log(t + 3)) (\sigma' \log t + |\log \zeta(2 + it)|) \leq 10\delta^{-1}\sigma'(\log \log t) \log t$$

for $z \in C_{1,2}$.

We may also note that if $\Re(z - 3/4 - 2\delta) > \delta(\log \log t)^{-1}$ and $\Im(z) \leq t/2$, then similar arguments give

$$|\log \zeta(z)| < \delta^{-1}\sigma'(\log \log t) \log t,$$

for some positive constant $\sigma'$ that has changed appropriately.

Now we consider three concentric circles $C_{2,1}, C_{2,2}, C_{2,3}$, centered at $\sigma'' + i2t$ and with radii $r_1 = \sigma'' - 1 - \eta$, $r_2 = \sigma'' - 2\sigma$ and $r_3 = \sigma'' - \delta_0$ respectively. Here

$$\delta_0 = \frac{3}{4} - 2\delta + \frac{\delta}{1 + \log \log(t + 3)}.$$

We shall choose $\sigma'' = \eta^{-1} = \log \log t$. Let $M_1, M_2, M_3$ denote the supremums of $|\log \zeta(z)|$ on $C_{2,1}, C_{2,2}, C_{2,3}$ respectively. We have already calculated that

$$M_3 \leq \delta^{-1}\sigma'(\log \log t) \log t.$$

It is easy to show that

$$M_1 \leq \sigma' \log \log t,$$

where $\sigma'$ is again appropriately adjusted. Applying the three circle theorem we conclude

$$M_2 \leq \sigma'(\log \log t)\delta^{-a} \log^a t,$$

where

$$a = \frac{\log(r_2/r_1)}{\log(r_3/r_1)} = \frac{1 - 2\sigma + \eta}{1 - \delta_0 + \eta} + O\left(\frac{1}{\sigma''}\right) = \frac{4(1 - 2\sigma)}{1 + 8\delta} + O_8\left(\frac{1}{\log \log t}\right).$$

This gives

$$|\zeta(2s)|^{-1} \ll \exp\left(c(\log \log t)(\log t)^{\frac{4(1 - 2\sigma)}{1 + 8\delta}}\right),$$

(V.15)
for a suitable constant $c > 0$ depending on $\delta$. The bound in the lemma follows from (V.12), (V.13), (V.14) and (V.15).

Now we complete the proof of Theorem 17.

Proof of Theorem 17. Let $M$ be any large positive constant, and define

$$A := A(Mx^\alpha(x)).$$

Then from Corollary 3, we have

$$\int_{[T,2T] \cap \mathcal{A}} \Delta^2(x) \frac{dx}{x^{2\alpha(T)} + 1} \gg \exp\left(c(\log T)^{7/8}\right).$$

Assuming

$$\mu([T,2T] \cap \mathcal{A}) \leq T^{1-\delta} \quad \text{for} \ T > T_0,$$

(V.16)

Proposition 3 gives

$$\Delta(x) = \Omega(x^{\alpha(x)+\delta/2})$$

as $h_0(T) = T^{1-\delta}$, which is the first part of the theorem. But if (V.16) does not hold, then we have

$$\mu([T,2T] \cap \mathcal{A}) > T^{1-\delta}$$

for $T$ in an $X$-Set. We choose

$$h_1(T) = T^{\frac{3}{8} - \frac{2c}{(\log T)^{1/8}} - \delta}, \quad \alpha_1(T) = \frac{3}{8} - \frac{3c}{(\log T)^{1/8}} - \delta, \quad \alpha_2(T) = \alpha(T).$$

Let $\delta''$ be such that $\delta < \delta'' < \delta'$. If $D(\sigma + it)$ does not have pole for $\sigma > 3/8 - \delta''$ then by Lemma 14 $D(\alpha_1(T) + it)$ has polynomial growth. So Assumptions 5 is valid. Since

$$T^{1-\delta} > 5h_1(5T/2)T^{1-\alpha_2(T)},$$

by case (ii) of Theorem 13 we have

$$\Delta(T) = \Omega_{\pm}\left(T^{\frac{3}{8} - \frac{2c}{(\log T)^{1/8}} - \delta''}\right).$$

The second part of the above theorem follows from the choice $\delta' > \delta''$. 

73
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