An Application of Value Distribution Theory to Schrödinger Operators with Absolutely Continuous Spectrum

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Abstract

We use the “Value Distribution” theory developed by Pearson and Breimesser to obtain a sequence of functions in the eigenvalue parameter for some Sturm-Liouville problems which have the property of being “uniformly asymptotically distributed”.

1 Introduction

In this paper we apply a general result of Breimesser and Pearson [1] to obtain a very general asymptotic result for Schrödinger equations with one regular endpoint and one Limit Point endpoint which generates some absolutely continuous spectrum. The software package SLEDGE [2] [4], developed by Pruess and Fulton, has as one of its options the computation of the spectral function for Sturm-Liouville problems having continuous spectrum. For cases where the spectrum is known to be absolutely continuous, more recent research codes described in [3] [6] [7] have proven to be much better than the SLEDGE package. This paper will, in the opinion of the authors, provide a background for development of even further formulas and numerical algorithms for the spectral density function in cases having some a.c. spectrum. The general theory of ‘value distribution’ which is providing the new approach described here has been developed by Pearson and Breimesser [8] [9] [10] [1].

For simplicity we restrict our attention to the problem for \( x \in [a, \infty) , \ a \in \mathbb{R} \), with a Dirichlet boundary condition at \( x = a \):
\[-y'' + q(x)y = \lambda y, \quad a \leq x < \infty, \quad (1.1)\]
\[y(a) = 0. \quad (1.2)\]

We pose the assumptions:

- (i) \(q(x)\) is continuous in \([a, \infty)\),
- (ii) the LP/O-N or LP/O case occurs at \(x = \infty\),
- (iii) there exist one or more intervals where a.c. spectrum is known to exist.

See Fulton and Pruess [3, p. 114] for these general endpoint classifications: here O means equation (1.1) is oscillatory at \(\infty\) for all real \(\lambda\) and O-N means there exists \(\Lambda \in \mathbb{R}\) such that equation (1.1) is oscillatory at \(\infty\) for all \(\lambda \in (\Lambda, \infty)\) and nonoscillatory at \(\infty\) for all \(\lambda \in (-\infty, \Lambda)\). The LP/O-N case includes potentials which satisfy

\[\lim_{x \to \infty} q(x) = L\]

for some finite limit (\(\Lambda = L\) in this case) and periodic potentials \(q(x)\), as well as all quasi-periodic and almost periodic potentials. The LP/O case includes potentials which are LP at \(\infty\) and satisfy

\[\lim_{x \to \infty} q(x) = -\infty\]

In all of the above mentioned cases, it is possible for a.c. spectrum to occur.

We now state a very general theorem on Herglotz functions of Breimesser and Pearson [1, Thm1,p.40] which will be applied to the Titchmarsh-Weyl \(m\)-function associated with the above self-adjoint Sturm-Liouville problem \((1.1)-(1.2)\) in the next section:

**Theorem 1** (Breimesser and Pearson, 2003) Let \(f(z)\) be an arbitrary Herglotz function. Let \(S\) be an arbitrary Borel subset of \(\mathbb{R}\), and let \(\Lambda\) be any set of finite measure. Define the angle \(\theta\) subtended by \(S\) at \(z\) for all \(z \in \mathbb{C}\) (\(\mathbb{C}\) = complex numbers) by

\[\theta(S, z) = \int_S \text{Im} \left( \frac{1}{t-z} \right) dt. \quad (1.3)\]

Letting the boundary value of the Herglotz function on the real \(\lambda\)-axis be

\[f^+(\lambda) := \lim_{\epsilon \to 0} [f(\lambda + i\epsilon)]\]

we define,

\[\omega(\lambda, S, f^+) := \frac{1}{\pi} \theta(S, f^+), \quad (1.5)\]

and

\[\omega(\lambda, S, f(\lambda + i\epsilon)) := \frac{1}{\pi} \theta(S, f(\lambda + i\epsilon)). \quad (1.6)\]

Then we have

\[\left| \int_{\Lambda} \omega(\lambda, S, f^+) d\lambda - \int_{\Lambda} \omega(\lambda, S, f(\lambda + i\epsilon)) d\lambda \right| \leq \frac{1}{\pi} \int_{\Lambda} \theta(\Lambda^c, \lambda + i\epsilon) d\lambda \to 0, \text{ as } \epsilon \to 0. \quad (1.7)\]

where \(\theta(\Lambda^C, \lambda + i\epsilon)\) is the angle subtended at \(\lambda + i\epsilon\) by \(\Lambda^C = \mathbb{R} \setminus \Lambda\).
2 A Uniformly Asymptotically Distributed Sequence of Functions

We define a fundamental system \( \{ u, v \} \) of solutions of the Schrödinger equation (1.1) at \( x = a \) by the initial conditions
\[
\begin{bmatrix}
  u(a, \lambda) & v(a, \lambda) \\
  u'(a, \lambda) & v'(a, \lambda)
\end{bmatrix} = \begin{bmatrix}
  1 & 0 \\
  0 & 1
\end{bmatrix}, \lambda \in \mathbb{C}.
\] (2.1)
The Titchmarsh-Weyl \( m \)-function for the problem (1.1)-(1.2) is defined (for Im \( z \neq 0 \)) by
\[
u(x, z) + m(z)v(x, z) \in L^2(a, \infty).
\] (2.2)

The boundary value of the \( m \)-function on the real \( \lambda \)-axis is then defined by
\[
m^+(\lambda) := \lim_{\epsilon \to 0} [m(\lambda + i\epsilon)] = A(\lambda) + iB(\lambda),
\] (2.3)
where we have introduced the real-valued functions \( A(\lambda), B(\lambda) \) as the real and imaginary parts of \( m^+(\lambda) \).

The standard method of approximating spectral quantities for the half line problem (1.1)-(1.2) is to make use of spectral quantities associated with the truncated regular problem on \( [a, b] \) for \( b \in (a, \infty) \):
\[
y'' - q(x)y = \lambda y, \quad a \leq x \leq b \\
y(a) = 0 \\
y(b) = 0
\] (2.4-2.6)

We let \( m_b(z) \) be the \( m \)-function associated with the Sturm-Liouville problem (2.4)-(2.6), that is, for all \( z \) not eigenvalues,
\[
u(b, z) + m_b(z)v(b, z) = 0,
\] or
\[
m_b(z) := -\frac{u(b, z)}{v(b, z)}.
\] (2.7)
The boundary value of this meromorphic \( m_b \) function on the real \( \lambda \)-axis is then well-defined for all \( \lambda \) not eigenvalues of (2.4)-(2.6) as
\[
m^+_b(\lambda) := \lim_{\epsilon \to 0} [m_b(\lambda + i\epsilon)] = -\frac{u(b, \lambda)}{v(b, \lambda)}
\] (2.8)

Here, \( m^+_b(\lambda) \) is necessarily real-valued.

As in Theorem 1, we let \( S \) be any Borel subset of \( \mathbb{R} \) and \( \Lambda \) be any finite interval in \( \mathbb{R} \). For \( x \in [a, \infty) \) and all \( \lambda \in \mathbb{R} \), not eigenvalues (with \( b = x \)) of (2.4)-(2.6), we put
\[
F(x, \lambda) := -\frac{u(x, \lambda)}{v(x, \lambda)} = m^+_x(\lambda),
\] (2.9)

and define
\[
F^{-1}_x(S) = \{ \lambda \in \mathbb{R} | F(x, \lambda) \in S \}.
\] (2.10)

For any real number \( X \) it follows from (1.3) that the angle subtended by \( S \) at \( X \) is
\[
\theta(X, S) = \begin{cases} 
  \pi & \text{if } X \in S \\
  0 & \text{if } X \notin S.
\end{cases}
\] (2.11)

Hence for \( X = F(x, \lambda) = m^+_x(\lambda) \in \mathbb{R} \) we have
\[
\omega(\lambda, S, m^+_x) := \frac{1}{\pi} \theta(S, F(x, \lambda)) = \chi(F^{-1}_x(S))
\] (2.12)
where $\chi(F_x^{-1}(S))$ is the characteristic function of the set $F_x^{-1}(S)$ on the real $\lambda$-axis. Hence,

$$\int_{\Lambda} \omega(\lambda, S, m_x^+) \, d\lambda = \int_{\Lambda} \chi(F_x^{-1}(S)) \, d\lambda = \mu(\Lambda \cap F_x^{-1}(S)), \tag{2.13}$$

where $\mu(\Lambda \cup F_x^{-1}(S))$ denotes the Lebesgue measure of $\Lambda \cap F_x^{-1}(S)$. We then have the following theorem:

**Theorem 2.** (i) For any Borel set $S$ and any finite interval $\Lambda$ we have

$$\lim_{x \to \infty} \mu(\Lambda \cap F_x^{-1}(S)) = \int_{\Lambda} \omega(\lambda, S, m^+) \, d\lambda \tag{2.14}$$

where $\omega(\lambda, S, m^+)$ is defined as in (1.5) for the boundary value (2.3) of the Titchmarsh-Weyl $m$-function for the half line problem (1.1)-(1.2).

(ii) Let $S_\lambda = (\alpha(\lambda), \beta(\lambda))$, where $\alpha(\lambda)$ and $\beta(\lambda)$ are bounded, Lebesgue measurable functions on the interval $\Lambda$. Then we have

$$\lim_{x \to \infty} \mu(\Lambda \cap F_x^{-1}(S_\lambda)) = \int_{\Lambda} \omega(\lambda, S_\lambda, m^+) \, d\lambda. \tag{2.15}$$

**Proof of (i):** Replacing the left-hand side of (2.14) by the left-hand side of (2.13) we observe that

$$\left| \int_{\Lambda} \mu(\Lambda \cap F_x^{-1}(S)) \, d\lambda - \int_{\Lambda} \omega(\lambda, S, m^+) \, d\lambda \right|$$

$$= \left| \int_{\Lambda} \omega(\lambda, S, m_x^+) \, d\lambda - \int_{\Lambda} \omega(\lambda, S, m^+) \, d\lambda \right|$$

$$\leq \left| \int_{\Lambda} (\omega(\lambda, S, m_x^+) - \omega(\lambda, S, m^+(\lambda + ie)) \, d\lambda \right|$$

$$+ \left| \int_{\Lambda} (\omega(\lambda, S, m^+(\lambda + ie) - \omega(\lambda, S, m^+(\lambda + ie)) \, d\lambda \right|$$

Since $m_x(z)$ and $m(z)$ are Herglotz functions (by virtue of being Titchmarsh-Weyl $m$-functions for the problems (1.1)-(1.2) and (2.4)-(2.6), Theorem 1 applies to give the limit of the first and third terms to be zero as $\epsilon \downarrow 0$. For the second term we have for the integrand (for fixed $\epsilon > 0$), as $x \to \infty$, that:

$$|\omega(\lambda, S, m_x(\lambda + ie)) - \omega(\lambda, S, m(\lambda + ie))|$$

$$= \left| \int_{S} \left[ \text{Im} \left( \frac{1}{t - m_x(\lambda + ie)} \right) - \text{Im} \left( \frac{1}{t - m(\lambda + ie)} \right) \right] \, dt \right|$$

$$= \left| \int_{S} \text{Im} \left( \frac{-m(\lambda + ie) + m_x(\lambda + ie)}{(t - m_x(\lambda + ie))(t - m(\lambda + ie))} \right) \, dt \right|$$

$$\leq r_x(\lambda + ie) \cdot \left( \int_{S} \left| \frac{1}{(t - m(\lambda + ie))} \right| \, dt \right)$$

$$\leq r_x(\lambda + ie) \cdot \left( \int_{S} \left( \frac{1}{(t - m(\lambda + ie))} \right)^2 \, dt + 1 \right)$$

$$\leq K_\epsilon \left( \max_{\lambda \in \Lambda} r_x(\lambda + ie) \right),$$

where $K_\epsilon$ is independent of $\lambda \in \Lambda$, and $r_x(\lambda + ie)$ is the radius of the circle $C_{x,\lambda+ie}$ in the $m$-plane which contracts to the limit point $m(\lambda + ie)$. Hence, for fixed $\epsilon > 0$,

$$\lim_{x \to \infty} \omega(\lambda, S, m_x(\lambda + ie)) = \omega(\lambda, S, m(\lambda + ie))$$
uniformly for $\lambda \in \Lambda$. It follows that the above three terms can be made arbitrarily small by fixing $\epsilon > 0$ sufficiently small and then choosing $x$ sufficiently large. \[ \square \]

Proof of (ii). First we assume that $S_\lambda = (\alpha(\lambda), \beta(\lambda))$, $\alpha(\lambda) < \beta(\lambda)$ for all $\lambda \in \Lambda$, where $\alpha(\lambda)$ and $\beta(\lambda)$ are step functions on $\Lambda$, each taking only finitely many values. Letting $\{\alpha_j, j = 1, \ldots, N\}$, and $\{\beta_k, k = 1, \ldots, M\}$, denote the values which $\alpha(\lambda)$ and $\beta(\lambda)$ may have for $\lambda \in \Lambda$, we define

$$ C_{j,k} = \{ \lambda \in \Lambda \mid \alpha(\lambda) = \alpha_j \text{ and } \beta(\lambda) = \beta_k \}. $$

Then the $C_{j,k}$ are disjoint sets and $\lambda \in C_{j,k} \implies \alpha_j < \beta_k$. Putting $S_{j,k} := (\alpha_j, \beta_k)$ whenever $\alpha_j < \beta_k$, we define for $x \in (a, \infty)$,

$$ F^{-1}_x(S_{j,k}) = \{ \lambda \in \mathbb{R} \mid F(x, \lambda) \in S_{j,k} \}. $$

Then the statement (2.14) from part(i) applies with $S = S_{j,k}$, that is,

$$ \lim_{x \to \infty} \mu(\Lambda \cap F^{-1}_x(S_{j,k})) = \int_{\Lambda} \omega(\lambda, S_{j,k}, m^+) \, d\lambda. $$

Since the $C_{j,k}$ are disjoint sets on the $\lambda$-axis, it follows that

$$ \mu(\Lambda \cap F^{-1}_x(S_{\lambda})) = \sum_{j,k} \mu(\Lambda \cap C_{j,k} \cap F^{-1}_x(S_{j,k})) $$

$$ \to \sum_{j,k} \int_{\Lambda} \omega(\lambda, S_{j,k}, m^+) \, d\lambda $$

$$ = \sum_{j,k} \int_{\Lambda} \int_{\alpha_j}^{\beta_k} \left[ \frac{1}{t+\lambda} \right] \, dt \, d\lambda $$

$$ = \int_{\Lambda} \int_{\alpha(\lambda)}^{\beta(\lambda)} \left[ \frac{1}{t+\lambda} \right] \, dt \, d\lambda $$

$$ = \int_{\Lambda} \omega(\lambda, S_{\lambda}, m^+) \, d\lambda, \text{ as } x \to \infty, $$

which proves the statement (2.14) when $S_{\lambda} = (\alpha(\lambda), \beta(\lambda))$ and $\alpha(\lambda)$ and $\beta(\lambda)$ are simple functions on $\Lambda$. The case where $\alpha(\lambda)$ and $\beta(\lambda)$ are general bounded Lebesgue measurable functions can then be handled by approximating $\alpha(\lambda)$ and $\beta(\lambda)$ above and below by simple functions. \[ \square \]

Similarly to the analysis in Fulton, Pearson and Pruess \[7\] and Al-Naggar and Pearson \[11\], we now introduce the space of quadratic forms of solutions of equation (1.1): For $u \in R$ be the vector space of all quadratic forms of solutions of equation (1.1). If $y_1 = c_1 u + d_1 v$ and $y_2 = c_2 u + d_2 v$, it is readily verified that $y_1^2, y_1 y_2, y_2^2$ also belong to $R$. Hence $\mathcal{R}$ is a 3-dimensional space with $\{u^2, uv, v^2\}$ as a basis. Also associated with $\mathcal{R}$ is the solution space of the first order Appell system (see \[7\] [12]) of equations

$$ \frac{dU}{dx} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} 0 & \lambda - q & 0 \\ -2 & 0 & 2(\lambda - q) \\ 0 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} P \\ Q \\ R \end{bmatrix}, $$

which was used extensively in \[3\] [10] [7] and turns out to be a useful companion system to the Schrödinger equation (1.1) in the study of cases involving some a.c. spectrum. One choice of basis for the solution space of (2.17) is

$$ [U_1, U_2, U_3] = \begin{bmatrix} (u')^2 \\ -2uu' \\ u^2 \end{bmatrix}, $$

$$ \begin{bmatrix} 0 \\ \frac{u'v'}{u} \\ -uv \end{bmatrix}, $$

$$ \begin{bmatrix} (v')^2 \\ -2vv' \\ v^2 \end{bmatrix}. $$

(2.18)
Also on the solution space of (2.17) we can define an indefinite inner product by
\[ \langle U, \tilde{U} \rangle := 2(P\tilde{R} + \tilde{PR}) - QR = \text{const}, \text{ independent of } x \in [a, \infty), \]
(2.19)
and the above solutions \( \{U_1, U_2, U_3\} \) are orthogonal with respect to this inner product. If
\[
U(x, \lambda) = a(\lambda)U_1(x, \lambda) + b(\lambda)U_2(x, \lambda) + c(\lambda)U_3(x, \lambda),
\]
\[
\tilde{U}(x, \lambda) = \tilde{a}(\lambda)U_1(x, \lambda) + \tilde{b}(\lambda)U_2(x, \lambda) + \tilde{c}(\lambda)U_3(x, \lambda),
\]
then this inner product can also be expressed as
\[
\langle U, \tilde{U} \rangle = 2(a\tilde{c} + c\tilde{a}) - \tilde{b}b. \tag{2.20}
\]
A solution \( U \), or equivalently, its third component \( R \) is said to be normalized with respect to this indefinite inner product (compare [7, Equa 4.10] and [11, Equa 7]) if
\[
\langle U, U \rangle := 4PR - Q^2 = 2RR'' - 4(q - \lambda)R^2 = 4ac - b^2 = 4. \tag{2.21}
\]
We call \( 4ac - b^2 \) the “discriminant” of the quadratic form \( R = au^2 + buv + cv^2 \). Also, it is readily shown from (2.17) that the third component of \( R \) of any solution \( U \) of (2.17) must satisfy the third order linear equation
\[
R''' + 4(\lambda - q)R' - 2qR = 0. \tag{2.22}
\]
The solution space of (2.22) is the quadratic form space \( R \) and so there is a one-to-one correspondence between the solution space of (2.22) and (2.17). Indeed, for every element, \( R(x, \lambda) \), of the quadratic form space \( R \) we can generate from the Appell equations (2.17) unique functions \( P(x, \lambda) \) and \( Q(x, \lambda) \) such that \( (P, Q, R)^T \) is a solution of (2.17).

We now cite the following lemma (see Al-Naggar and Pearson [13, Lemma 1 and Theorem 2]) which gives a sufficient condition for existence of a.c. spectrum in a given interval \( I \subset (-\infty, \infty) \):

**DEFINITION.** The Sturm-Liouville equation (1.1) satisfies Condition A for a given real value of \( \lambda \) if and only if there exists a complex-valued solution \( y(x, \lambda) \) of (1.1) for which
\[
\lim_{N \to \infty} \int_0^N y(x, \lambda)^2 dx = 0. \tag{2.23}
\]

**LEMMA (Al-Naggar and Pearson).** Let \( I \subset (-\infty, \infty) \) be an interval on which Condition A holds for the general equation (1.1), and let the fundamental system \( \{u(\cdot, \lambda), v(\cdot, \lambda)\} \) be defined by the initial conditions (2.4) at \( x = a \). Then

(i) There exists a complex valued function \( M(\lambda) \) on \( I \) which is uniquely defined by the properties:

\[ (a) \quad \text{Im}[M(\lambda)] > 0 \quad \text{and} \quad (b) \quad \lim_{N \to \infty} \frac{\int_{x_0}^{x_0} (u(x, \lambda) + M(\lambda)v(x, \lambda))^2 dx}{\int_{x_0}^{x_0} |(u(x, \lambda) + M(\lambda)v(x, \lambda)|^2 dx} = 0. \tag{2.24}
\]

(ii) For \( \lambda \in I \) the function \( M \) in (i) is the boundary value of the Titchmarsh-Weyl \( m \)-function defined by (2.3), that is,
\[
M(\lambda) = \lim_{\varepsilon \downarrow 0} [m(\lambda + i\varepsilon)] = A(\lambda) + iB(\lambda). \tag{2.25}
\]

(iii) \( I \subset \sigma_{ac} \), where \( \sigma_{ac} \) is the absolutely continuous spectrum of the Sturm-Liouville problem (1.1)-(1.2), and on \( I \) the spectral density function is given by
\[
f(\lambda) := \frac{1}{\pi} \lim_{\varepsilon \to 0} (\text{Im}[m(\lambda + i\varepsilon)]) = \frac{\text{Im}[m^+(\lambda)]}{\pi} = \frac{B(\lambda)}{\pi}. \tag{2.26}
\]
In accordance with the above lemma, we can formulate the assumption (iii) of a.c. spectrum as follows:

(iii) Let \( I \subset (-\infty, \infty) \) be an interval on which \( B(\lambda) = \text{Im}[m^+(\lambda)] > 0 \).

Under this assumption, we now introduce a polar coordinate representation of \( u + m^+v = (u+Av) + iBv \), namely

\[
\begin{align*}
  u + Av &= r_0(x, \lambda) \cos(\theta_0(x, \lambda)) \\
  Bv &= r_0(x, \lambda) \sin(\theta_0(x, \lambda))
\end{align*}
\]  

(2.27)

so that \( \theta_0(x, \lambda) \) may be defined for \( \lambda \in I \) and \( x \in [a, \infty) \) by

\[
\cot(\theta_0(x, \lambda)) = \left( \frac{u(x, \lambda) + A(\lambda)v(x, \lambda)}{B(\lambda)v(x, \lambda)} \right).
\]  

(2.28)

Taking the inverse of the cotangent, we have

\[
\theta_0(x, \lambda) = \cot^{-1} \left( \frac{u(x, \lambda) + A(\lambda)v(x, \lambda)}{B(\lambda)v(x, \lambda)} \right)
\]  

(2.29)

from which we deduce that

\[
\theta_0'(x, \lambda) = \frac{B}{(u+Av)^2 + B^2v^2} > 0
\]  

(2.30)

for all \( \lambda \in I \) and all \( x \in [a, \infty) \). This suggests that we look at the quadratic form defined by

\[
R_0(x, \lambda) := \frac{|u + m^+(\lambda)v|^2}{\text{Im} m^+(\lambda)}
\]  

(2.31)

\[
= \frac{|u + (A + iB)v|^2}{B}
\]  

\[
= \frac{(u + Av)^2 + B^2v^2}{B}
\]  

\[
= \frac{1}{B}v^2 + \frac{2A}{B}uv + \frac{A^2 + B^2}{B}v^2.
\]

From (2.30) and (2.31) we see that the monotone increasing function \( \theta_0(x, \lambda), x \to \infty \), is related to \( R_0(x, \lambda) \) by

\[
\theta_0(x, \lambda) = \int_a^x \frac{1}{R_0(t, \lambda)} dt = \int_a^x \frac{B}{(u+Av)^2 + B^2v^2} dt.
\]  

(2.32)

To keep \( \theta_0(x, \lambda) \) lying in \([0, \pi]\) for all \( x \in [a, \infty) \), we restrict it by setting it back to zero whenever \( x \) passes through a zero of \( v(x, \lambda) \), that is, we set

\[
\tilde{\theta}_0(x, \lambda) := \theta_0(x, \lambda) \mod \pi := \theta_0(x, \lambda) - n\pi \text{ for } x \in (x_n, x_{n+1}),
\]  

(2.33)

where \( x_0 = a, x_n = x_n(\lambda) = nth \text{ zero of } v \). We can now prove the following theorem concerning the asymptotic behavior of the sequence of functions of \( \lambda, \{\tilde{\theta}_0(x, \lambda)\} \) as \( x \to \infty \).

**Theorem 3.** Let \( I \) be an interval on which \( B(\lambda) > 0 \). Let \( \Lambda \subset I \) be a bounded interval and assume \( C(\lambda) \) and \( D(\lambda) \) are measurable functions on \( \Lambda \) satisfying for all \( \lambda \in \Lambda \)

\[
0 \leq C(\lambda) < D(\lambda) \leq \pi.
\]

Then the set of functions of \( \lambda, \{\tilde{\theta}_0(x, \cdot) : I \to [0, \pi]|x \in (a, \infty)\} \) has the asymptotic property as \( x \to \infty \)

\[
\lim_{x \to \infty} \mu(\Lambda \cap \{\lambda \mid C(\lambda) < \tilde{\theta}_0(x, \lambda) < D(\lambda)\} = \frac{1}{\pi} \int_{\Lambda} \{D(\lambda) - C(\lambda)\} d\lambda.
\]  

(2.34)
Proof. For \( y = \cot \beta \) with \( \beta \in (0, \pi) \), we define \( \beta(y) = \cot^{-1}(y) \) mapping \((-\infty, \infty)\) onto \((0, \pi)\). Since \( \cot \beta \) is monotonically decreasing from \( \beta = 0 \) to \( \beta = \pi \), it follows that for \( x \in (x_n, x_{n+1}) \),
\[
\cot(C(\lambda)) > \cot(\theta_0(x, \lambda)) = \frac{u + A v}{B v} > \cot(D(\lambda)),
\]
or
\[
B(\lambda) \cot(C(\lambda)) - A(\lambda) > \frac{u(x, \lambda)}{v(x, \lambda)} > B(\lambda) \cot(D(\lambda)) - A(\lambda),
\]
or
\[
A(\lambda) - B(\lambda) \cot(C(\lambda)) < F(x, \lambda) = -\frac{u(x, \lambda)}{v(x, \lambda)} < A(\lambda) - B(\lambda) \cot(D(\lambda)).
\]
Hence,
\[
\mu \left( \Lambda \cap \left\{ \lambda \in \mathbb{R} \mid C(\lambda) < \theta_0(x, \lambda) < D(\lambda) \right\} \right) = \mu(\Lambda \cap \{ \lambda \in \mid F(x, \lambda) \in S_{\lambda} \})
\]
where \( S_{\lambda} := (A(\lambda) - B(\lambda) \cot(C(\lambda)), A(\lambda) - B(\lambda) \cot(D(\lambda))) \), so it follows from Theorem 2(ii) that
\[
\lim_{x \to \infty} \mu \left( \Lambda \cap \left\{ \lambda \in \mathbb{R} \mid C(\lambda) < \theta_0(x, \lambda) < D(\lambda) \right\} \right) = \int_{\Lambda} \omega(\lambda, S_{\lambda}, m^+) d\lambda.
\]
But, making use of the definitions (1.3) and (1.5) we observe that
\[
\omega(\lambda, S_{\lambda}, m^+) = \frac{1}{\pi} \theta(S_{\lambda}, m^+)
\]
\[= \frac{1}{\pi} \int_{S_{\lambda}} \text{Im} \left( \frac{1}{t - (A + iB)} \right) dt
\]
\[= \frac{1}{\pi} \int_{S_{\lambda}} \frac{-B}{(t - A)^2 + B^2} dt,
\]
and making the change of variable \( t = A - B \cot \theta \), we obtain
\[
\omega(\lambda, S_{\lambda}, m^+) = \frac{1}{\pi} \int_{C(\lambda)} \frac{B^2 \csc^2 \theta}{B^2 (\cot^2 \theta + 1)} dt
\]
\[= \frac{1}{\pi} (D(\lambda) - C(\lambda)).
\]
Substitution of this into the integral of \( \omega(\lambda, S_{\lambda}, m^+) \) thus yields the result (2.34).

Theorem 3 gives rise to the concept of a sequence of “uniformly asymptotically distributed functions modulo \( \pi \)”.
N\,amely, when \( g_n(\lambda) \) is a sequence of functions defined on \( I \subset \sigma_c \) and mapping into \([0, \pi]\) (such as \( \theta_0(x, \lambda) \)) which satisfy, for any two Lebesgue measurable functions \( C(\lambda), D(\lambda) \) (with \( 0 \leq C(\lambda) < D(\lambda) \leq \pi \)), and any interval \( \Lambda \subset I \), the property (2.34), that is,
\[
\lim_{x \to \infty} \mu(\Lambda \cap \{ \lambda : C(\lambda) < g_n(\lambda) < D(\lambda) \}) = \frac{1}{\pi} \int_{\Lambda} \{D(\lambda) - C(\lambda)\} d\lambda,
\]
we will say that \( g_n(\lambda) \) is uniformly asymptotically distributed modulo \( \pi \). Further investigations of such functions and how they give rise to formulas for spectral density functions defined as in (2.26) are currently in progress.

3 An example: Bessel’s equation

In this section we give an example for which the spectral density function is known, and show that it can be obtained from the formula in [7, Thm 4.12]. Consider the Bessel equation of order \( \nu \in [0, \infty) \) on \([a, \infty), a > 0\).
\[-y'' + \left( \frac{\nu^2 - 1/4}{x^2} \right) y = \lambda y, \quad a \leq x < \infty \tag{3.1}\]
\[y(a) = 0.\]

Since this problem has a.c. spectrum in \((0, \infty)\) we may take \(I = (0, \infty)\) in Theorem 3. The solutions of this equation defined by the initial conditions \((2.1)\) at \(x = a\) are (compare [14] p. 86),

\[u(x, \lambda) = D_1 \sqrt{x} J_\nu \left( x \sqrt{\lambda} \right) + D_2 \sqrt{x} Y_\nu \left( x \sqrt{\lambda} \right) \tag{3.2}\]
\[v(x, \lambda) = D_3 \sqrt{x} J_\nu \left( x \sqrt{\lambda} \right) + D_4 \sqrt{x} Y_\nu \left( x \sqrt{\lambda} \right), \tag{3.3}\]

where

\[D_1 = -\frac{\pi}{2} a^\frac{\nu}{2} Y_\nu (a \sqrt{\lambda}), \quad D_2 = \frac{\pi}{2} a^\frac{\nu}{2} J_\nu (a \sqrt{\lambda}),\]
\[D_3 = \frac{\pi}{4} a^\frac{\nu}{2} Y_\nu (a \sqrt{\lambda}) + \frac{\pi}{2} a^\frac{\nu}{2} \sqrt{\lambda} Y'_\nu (a \sqrt{\lambda}),\]
\[D_4 = -\frac{\pi}{4} a^\frac{\nu}{2} J_\nu (a \sqrt{\lambda}) - \frac{\pi}{2} a^\frac{\nu}{2} \sqrt{\lambda} J'_\nu (a \sqrt{\lambda}).\]

Using the fundamental system of solutions of the Appell system \([24, 11]\), we can find the unique solution, for \(\lambda \in (0, \infty)\), of the initial value problem at infinity (see [7] Equa (1.6)-(1.7) or [5] Thm 1),

\[
\lim_{x \to \infty} \begin{bmatrix} \sqrt{\lambda} \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{\lambda} \\ 0 \end{bmatrix},
\tag{3.4}
\]

by employing asymptotics for the Bessel functions \(J_\nu\) and \(Y_\nu\), as \(x \to \infty\), and this yields the formulas

\[\tilde{a}(\lambda) = \frac{\pi a}{2} \left( J_\nu^2 \left( a \sqrt{\lambda} \right) + Y_\nu^2 \left( a \sqrt{\lambda} \right) \right),\]
\[\tilde{b}(\lambda) = \frac{\pi}{2} \left[ J_\nu^2 \left( a \sqrt{\lambda} \right) + Y_\nu^2 \left( a \sqrt{\lambda} \right) \right] + \pi a \sqrt{\lambda} \left[ J_\nu \left( a \sqrt{\lambda} \right) J'_\nu \left( a \sqrt{\lambda} \right) + Y_\nu \left( a \sqrt{\lambda} \right) Y'_\nu \left( a \sqrt{\lambda} \right) \right].\]
\[\tilde{c}(\lambda) = \frac{\pi}{8a} \left[ J_\nu^2 \left( a \sqrt{\lambda} \right) + Y_\nu^2 \left( a \sqrt{\lambda} \right) \right] + \frac{\pi a \lambda}{2} \left[ \left( J'_\nu \left( a \sqrt{\lambda} \right) \right)^2 + \left( Y'_\nu \left( a \sqrt{\lambda} \right) \right)^2 \right] + \frac{\pi \sqrt{\lambda}}{2} \left[ J_\nu \left( a \sqrt{\lambda} \right) J'_\nu \left( a \sqrt{\lambda} \right) + Y_\nu \left( a \sqrt{\lambda} \right) Y'_\nu \left( a \sqrt{\lambda} \right) \right].\]

From [7] Thm 4.12, Equa 4.29) it follows that (in agreement with [14] p. 86),

\[f(\lambda) = \frac{1}{\pi \tilde{a}(\lambda)} = \frac{2}{\pi^2 a} \frac{1}{J_\nu^2 \left( a \sqrt{\lambda} \right) + Y_\nu^2 \left( a \sqrt{\lambda} \right)}\tag{3.5}\]

From [7] Equas 4.22 & 4.26) we also have \(A(\lambda) + iB(\lambda) = \frac{\tilde{b}}{2a} + i \frac{\tilde{c}}{2} \). Hence putting these formulas, together with the solutions \(\{u, v\}\) from \([3.2], [3.3]\) into \([2.29]\), we have produced, in the definition \([2.34]\) of \(\theta_0(x, \cdot)\), a concrete example of a uniformly asymptotically distributed sequence of functions modulo \(\pi\).

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