Semilinearity of Families of Languages*

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Techniques are developed for creating new and general language families of only semilinear languages, and for showing families only contain semilinear languages. It is shown that for language families $L$ that are semilinear full trios, the smallest full AFL containing $L$ that is also closed under intersection with languages in $NCM$ (where $NCM$ is the family of languages accepted by $NFA$s augmented with reversal-bounded counters), is also semilinear. If these closure properties are effective, this also immediately implies decidability of membership, emptiness, and infiniteness for these general families. From the general techniques, new grammar systems are given that are extensions of well-known families of semilinear full trios, whereby it is implied that these extensions must only describe semilinear languages. This also implies positive decidability properties for the new systems. Some characterizations of the new families are also given.

Keywords: semilinearity; closure properties; counter machines; pushdown automata; decidability.

1. Introduction

One-way nondeterministic reversal-bounded multicounter machines ($NCM$) operate like $NFA$s with $\lambda$ transitions, where there are some number of stores that each can contain some non-negative integer. The transition function can detect whether each counter is zero or non-zero, and optionally increment or decrement each counter; however, there is a bound on the number of changes each counter can make between non-decreasing and non-increasing. These machines have been extensively studied in the literature, for example in [19], where it was shown that $NCMs$ only accept
semilinear languages (defined in Section 2). As the semilinear property is effective for NCM (in that, the proof consists of an algorithm for constructing a finite representation of the semilinear sets), this implies that NCMs have decidable membership, emptiness, and infiniteness properties, as emptiness and infiniteness can be decided easily on semilinear sets (and membership follows from emptiness by effective closure under intersection with regular languages). NCM machines have been applied extensively in the literature, for example, to model checking and verification [25, 20, 26, 21], often using the positive decidability properties of the family.

More general machine models have been studied with an unrestricted pushdown automaton augmented by some number of reversal-bounded counters (NPCM, [19]). Despite the unrestricted pushdown, the languages accepted are all semilinear, implying they have the same decidable properties. This family too has been applied to several verification problems [5, 22], including model checking recursive programs with numeric data types [12], synchronization- and reversal-bounded analysis of multithreaded programs [10], for showing decidable properties of models of integer-manipulating programs with recursive parallelism [11], and for decidability of problems on commutativity [23]. In these papers, the positive decidability properties — the result of the semilinearity — plus the use of the main store (the pushdown), plus the counters, played a key role. Hence, (effective) semilinearity is a crucial property for families of languages.

The ability to augment a machine model with reversal-bounded counters and to only accept semilinear languages is not unique to pushdown automata; in [13], it was found that many classes of machines $\mathcal{M}$ accepting semilinear languages could be augmented with reversal-bounded counters, and the resulting family $\mathcal{M}_c$ would also only accept semilinear languages. This includes models such as Turing machines with a one-way read-only input tape and a finite-crossing worktape. However, a precise formulation of which classes of machines this pertains to was not given.

Here, a precise formulation of families of languages that can be “augmented” with counters will be examined in terms of closure properties rather than machine models. This allows for application to families described by machine models, or grammatical models. It is shown that for any full trio (a family closed under homomorphism, inverse homomorphism, and intersection with regular languages) of semilinear languages $\mathcal{L}_0$, then the smallest full AFL $\mathcal{L}$ (a full trio also closed under union, concatenation, and Kleene*) containing $\mathcal{L}_0$ that is closed under intersection with languages in NCM, must only contain semilinear languages. Furthermore, if the closure properties and semilinearity are effective in $\mathcal{L}_0$, this implies a decidable membership, emptiness, and infiniteness problem in $\mathcal{L}$. Hence, this provides a new method for creating general families of languages with positive decidability properties.

Several specific models are created by adding counters. For example, indexed

*a worktape is finite-crossing if there is a bound on the number of times the boundary of all neighboring cells on the worktape are crossed.
grammars are a well-studied general grammatical model like context-free grammars except where nonterminals keep stacks of “indices”. Although this system can generate non-semilinear languages, linear indexed grammars (indexed grammars with at most one nonterminal in the right hand side of every production) generate only semilinear languages [6]. Here, we define linear indexed grammars with counters, akin to linear indexed grammars, where every sentential form contains the usual sentential form, plus $k$ counter values; each production operates as usual and can also optionally increase each counter by some amount; and a terminal word can be generated only if it can be produced with all counter values equal. It is shown that the family of languages generated must be semilinear since it is contained in the smallest full AFL containing the intersection of linear indexed languages and NCM languages. A characterization is also shown: linear indexed grammars with counters generate exactly those languages obtained by intersecting a linear indexed language with an NCM and then applying a homomorphism. Furthermore, it is shown that right linear indexed grammars (where terminals only appear to the left of nonterminals in productions) with counters coincide exactly with the machine model NPCM. Therefore, linear indexed grammars with counters are a natural generalization of NPCM containing only semilinear languages. This model is generalized once again as follows: an indexed grammar is uncontrolled finite-index if, there is a value $k$ such that, for every derivation in the grammar, there are at most $k$ occurrences of nonterminals in every sentential form. It is known that every uncontrolled finite-index indexed grammar generates only semilinear languages [31]. It is shown here that uncontrolled finite-index indexed grammars with counters generate only semilinear languages, which is also a natural generalization of both linear indexed grammars with counters and NPCM. This immediately shows decidability of membership, emptiness, and infiniteness for this family.

Lastly, the closure property theoretic method of adding counters is found to often be more helpful than the machine model method of [13] in terms of determining whether the resulting family is semilinear, as here a machine model $M$ is constructed such that the language family accepted by $M$ is a semilinear full trio, but adding counters to the model to create $M_c$ accepts non-semilinear languages. This implies from our earlier results, that $M_c$ can accept languages that cannot be obtained from any accepted by $M$ by allowing any number of intersections with NCMs combined with any of the full AFL operations.

This paper therefore contains useful new techniques for creating new language families, and for showing existing language families only contain semilinear languages, which can then be used to immediately obtain decidable emptiness, membership, and infiniteness problems. Such families can perhaps also be applied to various areas, such as to verification, similarly to the use of NPCM. A preliminary version of this paper appeared in [18]. This version includes all missing proofs omitted due to space constraints, and the new Proposition 4 which allows for some of the other proposition statements to be more general. Section 5 is also new.
2. Preliminaries

In this section, preliminary background and notation is given.

Let \( \mathbb{N}_0 \) be the set of non-negative integers, and let \( \mathbb{N}_0^k \) be the set of all \( k \)-tuples of non-negative integers. A set \( Q \subseteq \mathbb{N}_0^k \) is linear if there exists vectors \( v_0, v_1, \ldots, v_l \in \mathbb{N}_0^k \) such that \( Q = \{ v_0 + i_1 v_1 + \cdots + i_l v_l \mid i_1, \ldots, i_l \in \mathbb{N}_0 \} \). Here, \( v_0 \) is called the constant, and \( v_1, \ldots, v_l \) are called the periods. A set \( Q \) is called semilinear if it is a finite union of linear sets.

Introductory knowledge of formal language and automata theory is assumed such as nondeterministic finite automata (NDFAs), pushdown automata (NPDA), Turing machines, and closure properties \([10]\). An alphabet \( \Sigma \) is a finite set of symbols, a word \( w \) over \( \Sigma \) is a finite sequence of symbols from \( \Sigma \), and \( \Sigma^* \) is the set of all words over \( \Sigma \) which includes the empty word \( \lambda \). A language \( L \) over \( \Sigma \) is any \( L \subseteq \Sigma^* \). The complement of a language \( L \subseteq \Sigma^* \), denoted by \( \overline{L} \), is \( \Sigma^* - L \).

Given a word \( w \in \Sigma^* \), the length of \( w \) is denoted by \( |w| \). For \( a \in \Sigma \), the number of \( a \)'s in \( w \) is denoted by \( |w|_a \). Given a word \( w \) over an alphabet \( \Sigma = \{ a_1, \ldots, a_k \} \), the Parikh image of \( w \) is \( \psi(w) = (|w|_{a_1}, \ldots, |w|_{a_k}) \), and the Parikh image of a language \( L \) is \( \{ \psi(w) \mid w \in L \} \). The commutative closure of a language \( L \) is the language \( \text{comm}(L) = \{ w \in \Sigma^* \mid \psi(w) = \psi(v), v \in L \} \). Two languages are letter-equivalent if \( \psi(L_1) = \psi(L_2) \).

A language \( L \) is semilinear if \( \psi(L) \) is a semilinear set. Equivalently, a language is semilinear if and only if it is letter-equivalent to some regular language \([13]\). A family of languages is semilinear if all languages in it are semilinear, and it is said to be effectively semilinear if there is an algorithm to construct the constant and periods for each linear set from a representation of each language in the family. For example, it is well-known that all context-free languages are effectively semilinear \([28]\).

We will only define NCM and NPCM informally here, and refer to \([19]\) for a formal definition. A one-way nondeterministic counter machine can be defined equivalently to a one-way nondeterministic pushdown automaton \([10]\) with only a bottom-of-pushdown marker plus one other symbol. Hence, the machine can add to the counter (by pushing), subtract from the counter (by popping), and can detect emptiness and non-emptiness of the pushdown. A \( k \)-counter machine has \( k \) independent counters. A \( k \)-counter machine \( M \) is \( l \)-reversal-bounded, if \( M \) makes at most \( l \) changes between non-decreasing and non-increasing of each counter in every accepting computation. Let NCM be the class of one-way nondeterministic \( l \)-reversal-bounded \( k \)-counter machines, for some \( k, l \) (DCM for deterministic machines). Let NPCM be the class of machines with one unrestricted pushdown plus some number of reversal-bounded counters. By a slight abuse of notation, we also use these names for the family of languages they accept.

Notation from AFL (abstract families of languages) theory is used from \([7]\). A full trio is any family of languages closed under homomorphism, inverse homomorphism, and intersection with regular languages. Furthermore, a full AFL is a full trio closed
under union, concatenation, and Kleene-*. Given a language family \( \mathcal{L} \), the smallest family containing \( \mathcal{L} \) that is closed under arbitrary homomorphism is denoted by \( \hat{\mathcal{H}}(\mathcal{L}) \), the smallest full trio containing \( \mathcal{L} \) is denoted by \( \mathcal{T}(\mathcal{L}) \), and the smallest full AFL containing \( \mathcal{L} \) is denoted by \( \hat{\mathcal{F}}(\mathcal{L}) \). Given families \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \), let \( \mathcal{L}_1 \land \mathcal{L}_2 = \{ L_1 \cap L_2 \mid L_1 \in \mathcal{L}_1, L_2 \in \mathcal{L}_2 \} \). We denote by \( \hat{\mathcal{F}}_{\text{NCM}}(\mathcal{L}) \) the smallest full AFL containing \( \mathcal{L} \) that is closed under intersection with languages from NCM.

3. Full AFLs Containing Counter Languages

This section will start by showing that for every semilinear full trio \( \mathcal{L} \), the smallest full AFL containing \( \mathcal{L} \) that is also closed under intersection with NCM is a semilinear full AFL (Proposition 4). First, an intermediate result is required.

**Proposition 1.** If \( \mathcal{L} \) is a semilinear full trio, then \( \mathcal{T}(\mathcal{L} \land \text{NCM}) = \hat{\mathcal{H}}(\mathcal{L} \land \text{NCM}) \) is a semilinear full trio.

**Proof.** Let \( \mathcal{C} = \mathcal{T}(\mathcal{L} \land \text{NCM}) \), and let \( \hat{L} \in \mathcal{C} \) over alphabet \( \Gamma = \{ d_1, \ldots, d_s \} \). By definition, \( \mathcal{C} \) is a full trio. It will be shown that \( \hat{L} \) is semilinear. Then \( \hat{L} \) can be obtained from a language \( L \) in \( \mathcal{L} \land \text{NCM} \) via a finite sequence of operations involving homomorphisms, inverse homomorphisms, and intersections with regular sets. Theorem 3.2.3 of \([7]\) shows that for all non-empty languages \( L \), \( \mathcal{T}(L) = \{ g_2(g_1^{-1}(L) \cap R_1) \mid R_1 \text{ is regular}, g_1, g_2 \text{ are decreasing homomorphisms} \} \).

A homomorphism \( g \) is decreasing if and only if \( |g(a)| \leq 1 \), for all letters \( a \). Such homomorphisms are called weak codings. Hence, it is enough to consider that \( \hat{L} \) is obtained from \( L \) via the following sequence: an application of an inverse weak coding homomorphism \( g_1 \), followed by an intersection with a regular language \( R_1 \), followed by an application of a weak coding homomorphism \( g_2 \). Thus, \( \hat{L} = g_2(g_1^{-1}(L) \cap R_1) \).

Since \( L \) is in \( \mathcal{L} \land \text{NCM} \), there are \( L_1 \in \mathcal{L} \) and \( L_2 \in \text{NCM} \) such that \( L = L_1 \land L_2 \). Let \( L_2 \) be accepted by a \( k \)-counter reversal-bounded NCM \( M_2 \), where, without loss of generality, all counters are \( 1 \)-reversal-bounded \([19]\), all counters increase at least once, and all counters decrease to zero before accepting.

Let \( \Sigma = \{ a_1, \ldots, a_n \} \) be the alphabet of \( L_1 \cup L_2 \), and so \( L, L_1, L_2 \subseteq \Sigma^* \), \( g_1 \) is from \( \hat{\Sigma}^* \) to \( \Sigma^* \) for some alphabet \( \hat{\Sigma} \), and so \( g_1^{-1}(L) \subseteq \hat{\Sigma}^* \), \( R_1 \subseteq \hat{\Sigma}^* \), and \( g_2 \) is from \( \hat{\Sigma}^* \) to \( \Gamma^* \). Introduce new symbols \( \Delta = \{ C_1, D_1, \ldots, C_k, D_k \} \) \( (k \) is the number of counters).

Let \( h_\Sigma \) be a homomorphism from \( (\Sigma \cup \Delta)^* \) to \( \Sigma^* \) that fixes each letter of \( \Sigma \) and erases all letters of \( \Delta \), and let \( h_\Delta \) be a homomorphism \( (\hat{\Sigma} \cup \Delta)^* \) to \( \hat{\Sigma}^* \) that fixes each letter of \( \hat{\Sigma} \) and erases all letters of \( \Delta \). There exists \( R_2 \subseteq (\Sigma \cup \Delta)^* \), a regular set, accepted by a nondeterministic finite automaton \( M_2 \) that “encodes” the computation of \( M_2 \) (without doing the counting), as follows:

- \( M_2' \) switches states as in \( M_2 \); \( M_2' \) starts by simulating transitions on each counter being zero;
- every time \( M_2 \) adds to counter \( i \), \( M_2' \) instead reads the input letter \( C_i \); this
is forced to happen at least once for each \( i \), and after it reads the first \( C_i \), it simulates transitions of \( M_2 \) where counter \( i \) is positive;

- every time \( M_2 \) subtracts from counter \( i \), \( M_2 \) reads \( D_i \); \( M_2 \) verifies that at least one \( D_i \) is read, and that no \( C_i \) is read afterwards;
- for each \( i, 1 \leq i \leq k \), at some nondeterministically guessed spot after reading some \( D_i \) symbol, \( M_2 \) guesses that counter \( i \) has hit zero, and it no longer reads any \( D_i \) symbol and simulates only transitions on counter \( i \) being zero;
- \( M_2 \) must end in a final state.

Let \( L' = h^{-1}_\Sigma(L_1) \cap R_2 \) (here, \( h^{-1}_\Sigma(L_1) \) has symbols of \( \Delta \) “shuffled in” to \( L_1 \)). Let \( L'' \) be those words in \( L' \) with the same number of \( C_i \)'s as \( D_i \)'s, for each \( i \). Then it is evident that \( h_\Sigma(L'') = L \) as the NFA \( M_2' \) that accepts \( R_2 \) is behaving like \( M_2 \) but without counting; however the counting is occurring using the intersection in \( L'' \), and then the counter symbols of \( \Delta \) are erased with \( h_\Sigma \). But looking only at \( L' \), it must be that \( L' \in \mathcal{L} \) since \( \mathcal{L} \) is a full trio. Let \( \hat{g}_1 \) be the extension of \( g_1 \) to be a homomorphism from \((\Sigma \cup \Delta)^* \) to \((\Sigma \cup \Delta)^* \), where each letter of \( \Delta \) is mapped to itself, and let \( \hat{g}_2 \) be the extension of \( g_2 \) to be a homomorphism from \((\Sigma \cup \Delta)^* \) to \((\Gamma \cup \Delta)^* \), where each letter of \( \Delta \) is mapped to itself. Let \( R'_1 = h^{-1}_\Sigma(R_1) \); that is, it has symbols of \( \Delta^* \) “shuffled in”. Note that \( R'_1 \) is regular since the regular languages are closed under inverse homomorphism.

Then \( L''' = \hat{g}_2(\hat{g}_1^{-1}(L') \cap R'_1) \), which is also in \( \mathcal{L} \), over \((\Gamma \cup \Delta)^* \). Note that since \( g_1 \) is a weak coding homomorphism, \( \hat{g}_1 \) is as well, and therefore \( \hat{g}_1^{-1} \) simply operates as \( g_1^{-1} \) does while fixing all letters of \( \Delta \). If we then take \( L''' \) and intersect it with all words where the number of \( C_i \)'s is equal to the number of \( D_i \)'s for each \( i \), and then erase all \( C_i \)'s and \( D_i \)'s, we obtain \( \hat{L} \). Let the Parikh image order the letters of this alphabet \( d_1, \ldots, d_s, C_1, D_1, \ldots, C_k, D_k \). This Parikh image of \( L''' \) gives a set \( Q''' \subseteq \mathbb{N}_0^{s+2k} \), which is semilinear since \( \mathcal{L} \) is semilinear. Let \( \hat{Q}' \) be the set obtained from \( Q''' \) by enforcing that the number of \( C_i \)'s is equal to \( D_i \)'s for each \( i \), which is also semilinear since the intersection of two semilinear sets is again semilinear [8]. Then \( \hat{Q} \), the set obtained from \( Q''' \) by projection on the first \( s \) coordinates, is also semilinear and this is the Parikh image of \( \hat{L} \). Hence, \( \hat{L} \) is semilinear.

By definition, \( \hat{H}(\mathcal{L} \cap \text{NCM}) \subseteq \hat{T}(\mathcal{L} \cap \text{NCM}) \). To show \( \hat{T}(\mathcal{L} \cap \text{NCM}) \subseteq \hat{H}(\mathcal{L} \cap \text{NCM}) \), let \( \hat{L} \in \hat{T}(\mathcal{L} \cap \text{NCM}) \). Using the proof that \( \hat{T}(\mathcal{L} \cap \text{NCM}) \) is semilinear above, from \( L''' \in \mathcal{L} \), it is possible to then intersect this language with an NCM that verifies that the number of \( C_i \)'s is equal to the number of \( D_i \)'s, for each \( i \). And then, a homomorphism that erases elements of \( \Delta \) can be applied to obtain \( \hat{L} \).

The next result is relatively straightforward from results in [14], however we have not seen it explicitly stated as we have done. From Corollary 2, Section 3.4 of [7], for any full trio \( \mathcal{L} \), the smallest full AFL containing \( \mathcal{L} \) is the substitution of the regular languages into \( \mathcal{L} \). And from [9], the substitution closure of one semilinear family into another is semilinear. Therefore, we obtain:
Lemma 2. If $L$ is a semilinear full trio, then $\hat{\mathcal{F}}(L)$ is semilinear.

For semilinear full trios $L$, $\hat{\mathcal{F}}(L \land \text{NCM})$ is a semilinear full trio by Proposition 1 and starting with this family and applying Lemma 2, the smallest full AFL containing intersections of languages in $L$ with NCM is semilinear.

Proposition 3. If $L$ is a semilinear full trio, then $\hat{\mathcal{F}}(L \land \text{NCM})$ is semilinear.

It is worth noting that this procedure can be iterated, as therefore $\hat{\mathcal{F}}(\hat{\mathcal{F}}(L \land \text{NCM}) \land \text{NCM})$ must also be a semilinear full AFL, etc. for additional levels. However, it is an interesting open question as to whether there is a strict hierarchy with respect to this iteration.

One could also consider the smallest full AFL containing $L$ that is closed under intersection with NCM. Here, the intersections with NCM can occur arbitrarily many times, even after or in between applying the other full AFL operations.

Proposition 4. If $L$ is a semilinear full trio, then $\hat{\mathcal{F}}_{\text{NCM}}(L)$ is semilinear.

Proof. Let $\hat{L} \in \hat{\mathcal{F}}_{\text{NCM}}(L)$. Then $\hat{L}$ is obtained from some language $L \in L$ via some sequence of the full AFL operations, plus some number, $n$ say, of intersections with NCMs. Hence,

$$\hat{L} \in C = \hat{\mathcal{F}}(\hat{\mathcal{F}}(\cdots \hat{\mathcal{F}}(L \land \text{NCM}) \land \text{NCM}) \land \cdots \land \text{NCM})$$

By iterating Proposition 3 $n$ times, $C$ is semilinear, hence $\hat{L}$ is semilinear.

In contrast, it is shown in [7] that $\hat{\mathcal{H}}(\hat{\mathcal{F}}(\{a^n b^n \mid n > 0\}) \land \hat{\mathcal{F}}(\{a^n b^n \mid n > 0\}))$ is equal to the family of recursively enumerable languages. Therefore, $\hat{\mathcal{H}}(\hat{\mathcal{F}}(\text{NCM}) \land \hat{\mathcal{F}}(\text{NCM}))$ is also equal to the family of recursively enumerable languages (which is not semilinear). But in Proposition 4 only intersections with languages in NCMs are allowed, and not intersections with languages in $\hat{\mathcal{F}}(\text{NCM})$, thereby creating the large difference.

Many acceptor and grammar systems are known to be semilinear full trios, such as finite-index ET0L systems [29], indexed grammars with a bound on the number of variables appearing in every sentential form (called uncontrolled finite-index) [3], multi-push-down machines (which have $k$ pushdowns that can simultaneously be written to, but they can only pop from the first non-empty pushdown) [2], a Turing machine variant with one finite-crossing worktape [13], and pushdown machines that can flip their pushdown up to $k$ times [15].

Corollary 5. Let $L$ be any of the following families:

- languages generated by context-free grammars,
- languages generated by finite-index ET0L,
- languages generated by uncontrolled finite-index indexed languages,
- languages accepted by one-way multi-push-down machine languages,
• languages accepted by one-way read-only input nondeterministic Turing machines with a two-way finite-crossing read/write worktape,
• languages accepted by one-way k-flip pushdown automata.

Then $\hat{\mathcal{F}}_{\text{NCM}}(\mathcal{L})$ is a semilinear full AFL.

A simplified analogue to this result is known for certain types of machines \[13\], although the new result here is defined entirely using closure properties rather than machines. Furthermore, the results in \[13\] do not allow Kleene-* type closure as part of the full AFL properties. For the machine models $\mathcal{T}$ above, it is an easy exercise to show that augmenting them with reversal-bounded counters to produce $\mathcal{T}_c$, the languages accepted by $\mathcal{T}_c$ are a subset of the smallest full AFL closed under intersection with NCM containing languages in $\mathcal{T}$. Hence, these models augmented by counters only accept semilinear languages. Similarly, this type of technique also works for grammar systems, as we will see in Section 6.

In addition, in \[9\], it was shown that if $\mathcal{L}$ is a semilinear family, then the smallest AFL containing the commutative closure of languages in $\mathcal{L}$ is a semilinear AFL. It is known that the commutative closure of every semilinear language is in NCM \[23\], and we know now that if we have a semilinear full trio $\mathcal{L}$, then the smallest full AFL containing $\mathcal{L}$ is also semilinear. So, we obtain an alternate proof that is an immediate corollary since we know that the smallest full AFL containing NCM is a semilinear full AFL.

For any semilinear full trio $\mathcal{L}$ where the semilinearity and the intersection with regular language properties are effective, the membership and emptiness problems in $\mathcal{L}$ are decidable. Indeed, to decide emptiness, it suffices to check if the semilinear set is empty. And to decide if a word $w$ is in $L$, one constructs the language $L \cap \{w\}$, then emptiness is decided.

**Corollary 6.** For any semilinear full trio $\mathcal{L}$ where the semilinearity and intersection with regular language properties are effective, then the membership, emptiness, and infiniteness problems are decidable for languages in $\hat{\mathcal{F}}_{\text{NCM}}(\mathcal{L})$. In these cases, $\hat{\mathcal{F}}_{\text{NCM}}(\mathcal{L})$ are a proper subset of the recursive languages.

As membership is decidable, the family must only contain recursive languages, and the inclusion must be strict as the recursive languages are not closed under homomorphism.

As another consequence, we provide an interesting decomposition theorem of semilinear languages into linear parts. Consider any semilinear language $L$, where its Parikh image is a finite union of linear sets $A_1, \ldots, A_k$, and the constant and periods for each linear set can be constructed. Then we can effectively create languages in perhaps another semilinear full trio separately accepting those words in $L_i = \{w \in L \mid \psi(w) \in A_i\}$, for each $1 \leq i \leq k$.

**Proposition 7.** Let $\mathcal{L}$ be a semilinear full trio, where semilinearity is effective. Given $L \in \mathcal{L}$, we can determine representations of disjoint simple sets (ie. disjoint...
linear sets where the periods form a basis) $A_1, \ldots, A_k$ such that the Parikh image of $L$ is $A = A_1 \cup \cdots \cup A_k$, and $L_i = \{w \in L \mid \psi(w) \in A_i\} \in \hat{F}_{NCM}(\mathcal{L})$, for $1 \leq i \leq k$.

**Proof.** Since semilinearity is effective, we can construct a representation of linear sets $A_1, \ldots, A_k$. Moreover, it is known that given any set of constants and periods generating a semilinear set $Q$, it is possible to effectively construct another set of constants and periods that forms a disjoint finite union of simple sets also generating $Q$ [4, 30]. Therefore, we can assume $A_1, \ldots, A_k$ are of this form. An NCM $M_i$ can be created to accept $\psi^{-1}(A_i)$, for each $i$, $1 \leq i \leq k$, as follows: if $L \subseteq \{a_1, \ldots, a_n\}^*$, then $M_i$ has $n$ counters. If $(x_1, \ldots, x_n)$ is the constant of $A_i$, then $M_i$ adds $x_j$ to counter $j$ for each $j$. Then, for each period, $(y_1, \ldots, y_n)$, $M_i$ nondeterministically guesses some number $c$ and adds $cy_j$ to counter $j$ for each $j$. At this point, the counters can contain any value from $A_i$. From here, for every $a_j$ read as input, $M_i$ subtracts one from counter $j$, and accepts at the end of the input if all counters are empty. Hence, $L_i = L \cap L(M_i) \in \hat{F}_{NCM}(\mathcal{L})$, for each $i$, $1 \leq i \leq k$. \hfill \qed

Therefore, by moving to a more general full trio (contained in the recursive languages), it is possible to decompose a language into separate (disjoint) languages such that each has one of the linear sets as its Parikh image.

### 4. Application to General Multi-Store Machine Models

In [7], a generalized type of multitape automata was studied, called multitape abstract families of automata (multitape AFAs). We will not define the notation used there, but in Theorem 4.6.1 (and Exercise 4.6.3), it is shown that if we have two types of automata $M_1$ and $M_2$ (defined using the AFA formalism), accepting language families $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively, then the languages accepted by automata combining together the stores of $M_1$ and $M_2$, accepts exactly the family $\hat{H}(\mathcal{L}_1 \land \mathcal{L}_2)$. This is shown for machines accepting full AFLs in Theorem 4.6.1 of [7], and for union-closed full trios mentioned in Exercise 4.6.3. We will show that this is tightly coupled with this precise definition of AFAs, as we will define two simple types of automata where each on their own accept a semilinear family, but combining the two stores together to form one multitape model accepts non-semilinear languages.

Given a family of one-way acceptors $\mathcal{M}$, let $\mathcal{M}_c$ be those acceptors augmented by reversal-bounded counters. A checking stack automaton (NCSA) $M$ is a one-way NFA with a store tape, called a stack. At each move, $M$ pushes a string (possibly $\lambda$) on the stack, but $M$ cannot pop. And, $M$ can enter and read from the inside of the stack in a two-way read-only fashion. But once the machine enters the stack, it can no longer change the contents. The checking stack automaton is said to be restricted (or no-read using the terminology of [24]), if it does not read from the inside of the stack until the end of the input. We denote by $\hat{R}_{NC\mathcal{S}A}$ the family of machines, as well as the family of languages described by the machines above. A preliminary investigation of $\hat{R}_{NC\mathcal{S}A_c}$ was done in [24].

Here, we will show the following:
(1) RNCSA is a full trio of semilinear languages equal to the regular languages,
(2) $\mathcal{F}(\text{RNCSA} \land \text{NCM})$ and $\mathcal{F}_{\text{NCM}}(\text{RNCSA})$ are semilinear full AFLs,
(3) every language in $\text{RNCSA} \land \text{NCM}$ is accepted by some machine in $\text{RNCSA}_c$,
(4) there are non-semilinear languages accepted by machines in $\text{RNCSA}_c$.

Therefore, $\text{RNCSA}_c$ contains some languages not in the smallest full AFL containing $\text{RNCSA}$ closed under intersection with NCM, and the multitape automata and results from [7] and [13] do not apply to this type of automata.

**Proposition 8.** RNCSA accepts exactly the regular languages, which is a full trio of semilinear languages.

**Proof.** It is clear that all regular languages are in RNCSA. For the other direction, take an RNCSA machine $M$, and assume without loss of generality that the input alphabet $\Sigma$ and the stack alphabet $\Gamma$ are disjoint. Construct a two-way NFA (2NFA) $M'$ over $(\Sigma \cup \Gamma)^*$ whose input is divided into segments $u_1v_1 \cdots u_nv_n$, where $u_i \in (\Sigma \cup \{\lambda\})$ and $v_i \in \Gamma^*$. $M'$ simulates $M$ by first verifying that $M$, when reading $u_i$, writes $v_i$ on the stack. When $M'$ simulates the two-way read-only phase of $M$ (which only occurs in $M$ after reaching the end of the input), it does so by using the two-way NFA and skipping over the segments of $\Sigma$. Since this language accepted by the 2NFA $M'$ is regular, the language obtained by erasing all letters of $\Gamma$ via homomorphism is also regular, which is exactly $L(M)$.

From Proposition 8, the following is true:

**Corollary 9.** $\mathcal{F}(\text{RNCSA} \land \text{NCM}) = \mathcal{F}_{\text{NCM}}(\text{RNCSA}) = \mathcal{F}(\text{NCM})$ is a semilinear full AFL.

Since RNCSA is equal to the family of regular languages, and NCM is closed under intersection with regular languages, the following is true:

**Proposition 10.** $\mathcal{F}_{\text{NCM}}(\text{RNCSA}) = \text{NCM} \subseteq \text{RNCSA}_c$. Furthermore, the latter family contains non-semilinear languages.

**Proof.** Containment is immediate since $\text{RNCSA}_c$ has reversal-bounded counters. The non-semilinear $L = \{a^ib^j \mid i, j \geq 1, j \text{ is divisible by } i\}$ can be accepted by an RDCSA$_c$. $M$ with one counter that makes only one reversal. $M$, on input $x$ checks that $x = a^ib^j$ for some $i, j \geq 1$, copies $a^i$ onto the stack, and increments the counter to $j$. Then $M$ makes multiple left-to-right and right-to-left sweeps on $a^i$ with the stack while in parallel decrementing the counter to check that $j$ is divisible by $i$.

It is concluded that $\text{RNCSA}_c$ contains some languages not in $\mathcal{F}_{\text{NCM}}(\text{RNCSA}) = \text{NCM}$, since NCM is semilinear. Then it is clear that combining together the stores of RNCSA and NCM accepts significantly more than $\mathcal{H}(\text{RNCSA} \land \text{NCM})$ as is the case for multitape AFA [7]. The reason for the discrepancy between this result and Ginsburg’s result is that the definition of multitape AFA allows for reading the
input while performing instructions (like operating in two-way read-only mode in the stack). In contrast, RNCSA does not allow this behavior. And if this behavior is added into the definition, the full capability of checking stack automata is achieved which accepts non-semilinear languages.

A similar analysis can be done using the method developed in [13] for augmenting the machine models with counters. Let $\mathcal{M}$ be a family of one-way acceptors with some type of store structure $X$. For example, if the storage $X$ is a pushdown stack, then $\mathcal{M}$ is the family of nondeterministic pushdown automata (NPDA). In [13], the following was shown for many families $\mathcal{M}$:

(*) If $\mathcal{M}$ is a semilinear family (i.e, the languages accepted by the machines in $\mathcal{M}$ have a semilinear Parikh image), then $\mathcal{M}_c$ is also a semilinear family.

It was not clear in [13] whether the result above is true for all types of one-way acceptors, in general or for which types (*) holds. However, the family RNCSA (equal to the regular languages) is semilinear (Proposition 8), but RDCSA is not semilinear (Proposition 10).

5. Properties of semilinear language families

This section investigates certain properties of semilinear language families.

**Definition 11.** Given a language family $\mathcal{L}$, define the following families:

$\mathcal{T} = \{ T | L \in \mathcal{L} \}$,

$\mathcal{L}_D = \{ L_1 - L_2 | L_1, L_2 \in \mathcal{L} \}$,

$\mathcal{L}_U = \{ L_1 \cup L_2 | L_1, L_2 \in \mathcal{L} \}$,

$\mathcal{L}_\cap = \{ L_1 \cap L_2 | L_1, L_2 \in \mathcal{L} \}$,

$\mathcal{L}_L = \{ L_1 L_2 | L_1, L_2 \in \mathcal{L} \}$,

$\mathcal{L}^* = \{ L^* | L \in \mathcal{L} \}$,

$\mathcal{L}_{RQ} = \{ L_1 L_2^{-1} | L_1, L_2 \in \mathcal{L} \}$, (right quotient),

$\mathcal{L}_{LQ} = \{ L_1^{-1} L_2 | L_1, L_2 \in \mathcal{L} \}$, (left quotient),

$\mathcal{H}(\mathcal{L}) = \{ h(L) | L \in \mathcal{L}, h \text{ a homomorphism} \}$,

$\mathcal{H}^{-1}(\mathcal{L}) = \{ h^{-1}(L) | L \in \mathcal{L}, h \text{ a homomorphism} \}$.

If $\mathcal{L}$ is semilinear, an interesting question is whether the defined families above must also be semilinear. In [9], it is shown that the substitution of one semilinear family into another is again semilinear. This immediately implies that if $\mathcal{L}$ is a semilinear family, then all of $\mathcal{L}^*, \mathcal{L}_U, \mathcal{L}_L$, and $\mathcal{H}(\mathcal{L})$ are also semilinear. For the remaining properties, we have not seen proofs in the literature, and therefore include short proofs here.

**Proposition 12.** If $\mathcal{L}$ is semilinear, then all of the following need not be semilinear: $\mathcal{T}, \mathcal{L}_D, \mathcal{L}_{RQ}, \mathcal{L}_{LQ}, \mathcal{L}_\cap, \mathcal{H}^{-1}(\mathcal{L})$. 

Proof. First, it will be shown for $L$. Let $L = \{a^1 \#a^2 \# \cdots \#a^k \# \mid k \geq 1 \}$ where $a$ is a letter. Then the complement of $L$, $\overline{L}$ can easily be accepted by an NCM with one 1-reversal counter which, when given an input $w$, nondeterministically selects (1) or (2) below:

(1) accepts, if $w$ is not in a valid format, i.e., not of the form $(a^+ \#)^+$. (M does not need the counter.)

(2) accepts $w$ if it is of the form $a^{i_1} \# \cdots \# a^{i_k}$ but $i_r + 1 \neq i_{r+1}$ for some $r$. (M uses a 1-reversal counter.)

Since all NCM languages are semilinear, $\overline{L}$ is semilinear, but $L$ is not semilinear (if it were semilinear, then projecting onto $a$ would be semilinear, but all unary semilinear languages are regular [14] and this language is not regular by the pumping lemma). This also implies non-closure for $L_D$.

Next, for right quotient, it is known that there is a non-recursively enumerable unary language $L \subseteq a^*$ (that is not semilinear) [27]. Let $L' = cLd \cup da^*c$. Then $L'$ is semilinear since it has the same Parikh image as the regular language $da^*c$. But the right quotient of $L'$ with $d$ is $cL$, which is not semilinear.

Similarly, the left quotient of $L'$ with $c$ is $Ld$, which is not semilinear. The result for intersection is also similar.

For inverse homomorphism, take a homomorphism $h$ that maps $b$ to $ca$, $e$ to $a$, and $f$ to $d$, and $g$ to $ac$. Then $h^{-1}(L') = bL''f \cup fa^*g$ where $L'' = La^{-1}$. The language $h^{-1}(L')$ is clearly not semilinear.

In contrast, it can be seen that for inverse homomorphisms where the homomorphisms are weak codings (that is, $|h(a)| \leq 1$ for all $a \in \Sigma$), then the resulting family is semilinear, as inverse homomorphisms act just as substitutions (as mentioned, the substitution closure of a semilinear family is semilinear) with the additional arbitrary insertion of characters erased by $h$ (which can be added in by placing another period in each linear set of the semilinear set with all 0’s except for a 1 for the position of the character erased by the homomorphism).

These closure properties will motivate the next notion that can help define “well-behaved” semilinear languages.

Definition 13. A semilinear language $L$ is well-behaved if $\widehat{T}(L)$ is semilinear; that is, it is well-behaved if closing it under all full trio operations only give semilinear languages.

Some basic facts are in order.

Proposition 14. The following are true:

- if $L \in \mathcal{L}$, a semilinear full trio, then $L$ is well-behaved,
- not all semilinear languages are well-behaved.

Proof. The first property is immediate since $\widehat{T}(L) \subseteq \mathcal{L}$. 
Consider a non-recursively enumerable unary language $L \subseteq a^*$. Therefore, this language is not semilinear (all unary semilinear languages are regular). Consider $L' = bLc \cup ca^*b$. Then $L'$ is semilinear since it has the same Parikh image as the regular language $ca^*b$. But $L' \cap ba^*c = bLc$, which is not semilinear. Thus, the closure of $L'$ by intersection with regular languages gives non-semilinear languages.

Consider the following language family:

$$\mathcal{L}_{WB} = \{ L \mid L \text{ is well-behaved} \}.$$  

**Proposition 15.** $\mathcal{L}_{WB}$ is the largest semilinear full trio. That is, all semilinear full trios are contained in $\mathcal{L}_{WB}$.

**Proof.** First, it is a semilinear full trio since all languages in it are semilinear, and the closure of each under the full trio properties are in it.

Furthermore, it is the largest since any language not in it must either not be semilinear, or closing it under the full trio operations produces languages that are not semilinear.

This is similar to the known result that there is a largest semilinear AFL [9]. This is an interesting language family, as properties that hold for this single language family hold for all semilinear full trios.

For example, consider the following. A bounded language $L \subseteq w_1^* \cdots w_k^*$ is called bounded Ginsburg semilinear (often just called bounded semilinear in the literature) if the set $\{(i_1, \ldots, i_k) \mid w_1^{i_1} \cdots w_k^{i_k} \in L\}$ is a semilinear set. The following is true from [17]:

**Proposition 16.** All bounded languages in $\mathcal{L}_{WB}$ are bounded Ginsburg semilinear languages.

Next, it follows from Theorem 3.2.3 of [7], that for all non-empty languages $L$, $T(L) = \{ h_2(h_1^{-1}(L) \cap R) \mid R \text{ regular}, h_1, h_2 \text{ are decreasing homomorphisms} \}$. The homomorphisms are both weak codings. Also, semilinear languages are closed under homomorphisms. Hence, the following is true:

**Proposition 17.** $L$ is well-behaved if and only if the family

$$\{ h^{-1}(L) \cap R \mid R \text{ regular, } h \text{ a weak coding homomorphism} \}$$

are all semilinear.

It is evident that $h^{-1}(L)$ must be semilinear since it was previously noted that the family obtained from any semilinear family via inverse weak coding homomorphisms must be semilinear. Hence, if one examines the family of semilinear languages $\mathcal{L} = \{ h^{-1}(L) \mid h \text{ a weak coding homomorphism} \}$, then $L$ is well-behaved if and only if $\mathcal{L} \wedge \mathcal{L}(\text{NFA})$.  

6. Applications to Indexed Grammars with Counters

In this section, we describe some new types of grammars obtained from existing grammars generating a semilinear language family $\mathcal{L}$, by adding counters. The languages generated by these new grammars are then shown to be contained in $\bar{F}(\mathcal{L} \land NCM)$, and by an application of Proposition 3, are all semilinear with positive decidability properties.

We need the definition of an indexed grammar introduced in [11] by following the notation of [10], Section 14.3.

**Definition 18.** An indexed grammar is a 5-tuple $G = (V, \Sigma, I, P, S)$, where $V, \Sigma, I$ are finite pairwise disjoint sets: the set of nonterminals, terminals, and indices, respectively, $S$ is the start nonterminal, and $P$ is a finite set of productions, each of the form either

1) $A \rightarrow \nu$, 2) $A \rightarrow Bf$, or 3) $Af \rightarrow \nu$,

where $A, B \in V$, $f \in I$ and $\nu \in (V \cup \Sigma)^*$.

Let $\nu$ be an arbitrary sentential form of $G$, which is of the form

$$\nu = u_1A_1\alpha_1u_2A_2\alpha_2 \cdots u_kA_k\alpha_ku_{k+1},$$

where $A_i \in V, \alpha_i \in I^*, u_i \in \Sigma^*, 1 \leq i \leq k, u_{k+1} \in \Sigma^*$. For a sentential form $\nu' \in (VI^* \cup \Sigma)^*$, we write $\nu \Rightarrow_G \nu'$ if one of the following three conditions holds:

1) There exists a production in $P$ of the form $(1) A \rightarrow w_1C_1 \cdots w_tC_tw_{t+1}$, $C_j \in V, w_j \in \Sigma^*$, and there exists $i$ with $1 \leq i \leq k$, $A_i = A$ and

$$\nu' = u_1A_1\alpha_1 \cdots u_i(w_1C_1\alpha_i \cdots w_tC_t\alpha_tw_{t+1})u_{i+1}A_{i+1}\alpha_{i+1} \cdots u_kA_k\alpha_ku_{k+1}.$$ 

2) There exists a production in $P$ of the form $(2) A \rightarrow Bf$ and there exists $i$, $1 \leq i \leq k$, $A_i = A$ and

$$\nu' = u_1A_1\alpha_1 \cdots u_i(Bf\alpha_i)u_{i+1}A_{i+1}\alpha_{i+1} \cdots u_kA_k\alpha_ku_{k+1}.$$ 

3) There exists a production in $P$ of the form $(3) Af \rightarrow w_1C_1 \cdots w_tC_tw_{t+1}$, $C_j \in V, w_j \in \Sigma^*$, and an $i$, $1 \leq i \leq k$, $A_i = A$, $\alpha_i = fa'^i\alpha'^i \in I^*$, with

$$\nu' = u_1A_1\alpha_1 \cdots u_i(w_1C_1\alpha'_i \cdots w_tC_t\alpha'^i_{t+1})u_{i+1}A_{i+1}\alpha_{i+1} \cdots u_kA_k\alpha_ku_{k+1}.$$ 

Then, $\Rightarrow_G'$ denotes the reflexive and transitive closure of $\Rightarrow_G$. The language $L(G)$ generated by $G$ is the set $L(G) = \{ u \in \Sigma^* | S \Rightarrow_G'^* u \}$.

This type of grammar can be generalized to include monotonic counters as follows:

**Definition 19.** An indexed grammar with $k$ counters is defined as in indexed grammars, except where rules (1), (2), (3) above are modified so that a rule $\alpha \rightarrow \beta$ now becomes:

$$\alpha \rightarrow (\beta, c_1, \ldots, c_k),$$

(1)

where $c_i \geq 0$, $1 \leq i \leq k$. Sentential forms are of the form $(\nu, n_1, \ldots, n_k)$, and $\Rightarrow_G$ operates as do indexed grammars on $\nu$, and for a production in Equation 4, adds
Given an indexed grammar with counters, the underlying grammar is the indexed grammar obtained by removing the counter components from productions.

Although indexed grammars generate non-semilinear languages, restrictions will be studied that only generate semilinear languages.

An indexed grammar $G$ is linear [3] if the right side of every production of $G$ has at most one variable. Furthermore, $G$ is right linear if it is linear, and terminals can only appear to the left of a nonterminal in productions. Let $L$-IND be the family of languages generated by linear indexed grammars, and let $RL$-IND be the family of languages generated by right linear indexed grammars.

Similarly, indexed grammars with counters can be restricted to be linear. An indexed grammar with $k$-counters is said to be linear indexed (resp. right linear) with $k$ counters, if the underlying grammar is linear (resp. right linear). Let $L$-IND$_k$ (resp. $RL$-IND$_k$) be the family of languages generated by linear (resp. right linear) indexed grammars with counters.

**Example 20.** Consider the language $L = \{v\$w \mid v, w \in \{a, b, c\}^*, |v|_a = |v|_b = |v|_c, |w|_a = |w|_b = |w|_c \}$ which can be generated by a linear indexed grammar with $k$ counters $G = (V, \Sigma, I, P, S)$ where $P$ contains

$$
S \rightarrow (S, 1, 1, 0, 0, 0, 0) \mid (S, 0, 0, 0, 1, 1, 1) \mid (T, 0, 0, 0, 0, 0, 0)
$$

$$
T \rightarrow (aT, 1, 0, 0, 0, 0, 0) \mid (bT, 0, 1, 0, 0, 0, 0) \mid (cT, 0, 0, 0, 1, 0, 0) \mid ($$\$R, 0, 0, 0, 0, 0, 0)
$$

$$
R \rightarrow (aR, 0, 0, 1, 0, 0, 0) \mid (bR, 0, 0, 0, 0, 1, 0) \mid (cR, 0, 0, 0, 0, 0, 1) \mid (\lambda, 0, 0, 0, 0, 0, 0).
$$

This language cannot be generated by a linear indexed grammar [3].

Next, a characterization of languages generated by these grammars will be given with a sequence of results used towards the proof of Proposition 20.

In the following, $\Sigma$ is a terminal alphabet, $C = \{c_1, \ldots, c_k\}$ (for some $k \geq 1$) is an alphabet distinct from $\Sigma$, and $h_c$ is a homomorphism on $\Sigma \cup C$ defined by $h_c(a) = a$ for each $a$ in $\Sigma$, and $h_c(c_i) = \lambda$ for each $c_i$ in $C$.

**Lemma 21.** If $L$ is in $NCM$ (resp., $NPCM$), there is regular language (resp., $NPDA$) $R$ over the alphabet $\Sigma \cup C$ such that $L = h_c(\{w \mid w \in R, |w|_{c_1} = \cdots = |w|_{c_k}\})$.

**Proof.** Let $L \subseteq \Sigma^*$ be accepted by an NCM (resp., NPCM) with $n$ 1-reversal counters. Let $C = \{b_1, c_1, \ldots, b_n, c_n\}$ be an alphabet distinct from $\Sigma$. (Thus $k = 2n$.) It follows from the constructions in [19], that there is a regular language $R$ (resp., NPDA) over alphabet $\Sigma \cup C$ such that $L = h_c(\{w \mid w \in R, |w|_{b_1} = |w|_{c_1}, \ldots, |w|_{b_n} = |w|_{c_n}\})$. Now let $R' = (b_1c_1)^* \cdots (b_n c_n)^* R$. Clearly, $R'$ is also regular (resp., NPDA), and $L = h_c(\{w \mid w \in R', |w|_{b_1} = |w|_{c_1} = \cdots = |w|_{b_n} = |w|_{c_n}\})$. $\square$
Lemma 22. If $L_1 \subseteq \Sigma^*$ is in L-IND, and $L_2 \subseteq \Sigma^*$ is in NCM, then $L_1 \cap L_2 \in L\text{-IND}_c$.

Proof. By Lemma 21, since $L_2$ is in NCM, there is regular set $R$ over alphabet $\Sigma \cup C$ such that $L_2 = h_c(\{w \mid w \in R, |w|_{c_1} = \cdots = |w|_{c_k}\})$. Also, $L_1' = h_c^{-1}(L_1)$ is also a linear indexed language since the family is a full trio [6], and $L_3 = L_1' \cap R$ is also a linear indexed language. Let $G_3$ be a linear indexed grammar generating $L_3$.

We can now construct from $G_3$ a linear indexed grammar with counters generating $L = L_1 \cap L_2$, such that, if $\alpha \to \beta$ is a production in $G_3$, then $\alpha \to (h_c(\beta), |\beta|_{c_1}, \ldots, |\beta|_{c_k})$ is a production in $G_4$. Then $L(G_4) = L_1 \cap L_2$.

Since languages generated by linear indexed grammars with counters are clearly closed under homomorphism, the following is true:

Corollary 23. Let $h$ be a homomorphism, $L_1 \in L\text{-IND}$, and $L_2 \in NCM$. Then $h(L_1 \cap L_2) \in L\text{-IND}_c$.

Lemma 24. If $L \in L\text{-IND}_c$, then $L = h(L_1 \cap L_2)$ for some homomorphism $h$, $L_1 \in L\text{-IND}$, and $L_2 \in NCM$.

Proof. Let $L$ be generated by $G$. Construct a linear indexed grammar (without counters) $G_1$ as follows:

If $\alpha \to (\beta, d_1, \ldots, d_k)$ is a rule in $G$, then $\alpha \to \beta'$ is a rule in $G_1$, where $\beta' = c_1^{d_1} \cdots c_k^{d_k} \beta$, (i.e., we append to the left of $\beta$ a terminal string representing the increments in the counters).

Let $L_1$ be the language generated by $G_1$. Let $L_2 = \{w \mid w \in (\Sigma \cup C)^*, |w|_{c_1} = \cdots = |w|_{c_k}\}$. Clearly $L_2$ is an NCM language, and $L = h_c(L_1 \cap L_2)$.

Proposition 25. $L \in L\text{-IND}_c$ if and only if there is a homomorphism $h$, $L_1 \in L\text{-IND}$, and $L_2 \in NCM$ such that $L = h(L_1 \cap L_2)$.

Proof. This follows immediately from Corollary 23 and Lemma 24.

Implied from the above result and Proposition 25 and since L-IND is an effectively semilinear trio [6], is that L-IND$_c \subseteq \tilde{F}(\text{L-IND} \land \text{NCM})$, and therefore L-IND$_c$ is effectively semilinear.

Corollary 26. The languages generated by linear indexed grammar with counters are effectively semilinear, with decidable emptiness, membership, and infiniteness problems.

Next, a machine model characterization of right linear indexed grammars with counters will be provided. Recall that an NPCM is a pushdown automaton augmented by reversal-bounded counters. The proof uses the fact that every context-free language can be generated by a right-linear indexed grammar [6].
Proposition 27. RL-IND$_c$ = NPCM.

Proof. First, it will be show that NPCM $\subseteq$ RL-IND$_c$. Let $L \in$ NPCM. Then, by Lemma 21 there is an NPDA $L_1$ over $\Sigma \cup C$ such that $L = h_c(\{ w \mid w \in L_1, |w|_{c_1} = \cdots = |w|_{c_k} \})$. It is known that every context-free language can be generated by a right-linear indexed grammar [6], and hence there is a right-linear indexed grammar $G_1$ generating $L_1$. Construct from $G_1$, a right-linear indexed grammar $G$ with counters generating $L$, such that, if $\alpha \to \beta$ is a production in $G_1$, then $\alpha \to (h_c(\beta), |\beta|_{c_1}, \ldots, |\beta|_{c_k})$ is a production in $G$. Then $L(G) = L$.

Next, the converse will be shown. Let $G$ be a right-linear indexed grammar with counters. We first construct a right-linear grammar (without counters) $G_1$ generating a language $L_1$ as in the proof of Lemma 21. Then $L(G_1)$ is a context-free language, and can be accepted by an NPDA $M_1$. We then construct an NPCM $M$ which, when given an input $w$, simulates $M_1$ and checks that $|w|_{c_1} = \cdots = |w|_{c_k}$ using 1-reversal counters. Finally, we construct from $M$ another NPCM $M'$ which erases the $c_i$'s. Clearly, $L(M') = L$.

We conjecture that the family of languages generated by right-linear indexed grammars with counters (the family of NPCM languages) is properly contained in the family of languages generated by linear indexed grammars with counters. Candidate witness languages are $L = \{ w$|$w \in \{ a, b, c \}^*, |w|_a + |w|_b = |w|_c \}$ and $L' = \{ w$|$w \in \{ a, b \}^* \}$. It is known that $L'$ is generated by a linear indexed grammar [6], and hence $L$ can be generated by such a grammar with two counters. But, both $L'$ and $L$ seem unlikely to be accepted by any NPCM. Therefore, indexed grammars with counters form quite a general semilinear family as it seems likely to be more general than NPCM.

Next, another subfamily of indexed languages is studied that are even more expressive than linear indexed grammars but still only generate semilinear languages.

An indexed grammar $G = (V, \Sigma, I, P, S)$ is said to be uncontrolled index-$r$ if, every sentential form in every successful derivation has at most $r$ nonterminals. $G$ is uncontrolled finite-index if $G$ is uncontrolled index-$r$, for some $r$. Let U-IND be the languages generated by uncontrolled finite-index indexed grammars.

Uncontrolled finite-index indexed grammars have also been studied under the name of breadth-bounded indexed grammars in [31, 33], where it was shown that the languages generated by these grammars are a semilinear full trio.

This concept can then be carried over to indexed grammars with counters.

Definition 28. An indexed grammar with $k$-counters is uncontrolled index-$r$ (resp. uncontrolled finite-index) if the underlying grammar is uncontrolled index-$r$ (resp. uncontrolled finite-index). Let U-IND$_c$ be the languages generated by uncontrolled finite-index indexed grammar with $k$-counters, for some $k$.

One can easily verify that Proposition 27 also applies to uncontrolled finite-index indexed grammars with counters. Hence, we have:
Proposition 29. \( L \in U-\text{IND}_c \) if and only if there is a homomorphism \( h \), \( L_1 \in U-\text{IND} \), \( L_2 \in \text{NCM} \) such that \( L = h(L_1 \cap L_2) \).

Implied from the above proposition and Proposition 30 also is that these new languages are all semilinear.

Corollary 30. \( U-\text{IND}_c \) is effectively semilinear, with decidable emptiness, membership, and infiniteness problems.

Hence, \( RL-\text{IND}_c \subseteq L-\text{IND}_c \subseteq U-\text{IND}_c \). We conjecture that both containments are strict; the first was discussed previously, and the second is likely true since \( L-\text{IND} \not\subseteq U-\text{IND} \). Hence, \( U-\text{IND}_c \) forms quite a general semilinear family, containing NPCM with positive decidability properties.

7. Conclusions and Future Directions

It has been previously shown that certain types of machine models accepting only semilinear languages can be augmented by reversal-bounded counters to create a more general machine model, while maintaining semilinearity and positive decision properties. However, this approach did not clearly define what types of models would work with this augmentation, and it did not work with other mechanisms for describing languages. Here, a closure property theoretic method is developed, and it is shown that, for every semilinear full trio \( \mathcal{L} \), the smallest full AFL containing \( \mathcal{L} \) also closed under intersection with reversal-bounded multicycle languages (NCM) is semilinear. Furthermore, the semilinearity is effective in the resulting family if it is effective (with other properties) in \( \mathcal{L} \).

This can be applied in numerous ways. For example, it is shown that if certain subclasses of indexed grammars (linear indexed, or uncontrolled finite-index) are augmented by counters with additional components of the grammars that function like counters, then the resulting families are more general, yet they remain semilinear and have decidable emptiness, membership, and infiniteness problems. There are also other applications, such as to analyzing definitions of abstract automata with multitape stores.

Several open problems remain. It is open whether the application of Proposition 3 creates a strict hierarchy. With respect to indexed grammars with counters, it is open as to whether right-linear grammars are strictly weaker than linear grammars, and whether those are weaker than uncontrolled finite-index grammars.

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