Effects of nonlinear gradient index on radiative heat transfer in a one-dimensional medium by the DRESOR method

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Abstract. The DRESOR (Distribution of Ratios of Energy Scattered by the medium Or Reflected by the boundary surface) method is applied for radiative heat transfer in a one-dimensional medium with a nonlinear gradient index and gray boundary surfaces. In this proposed method, the DRESOR values calculated by the Monte Carlo method express quantitatively the impact of scattering on radiative transfer and the radiative intensity with high directional resolution of high precision can be easily obtained. With given media characteristics and boundary conditions, the temperature and radiative flux distributions inside the medium are calculated under the condition of radiative equilibrium. It is shown, in the cases studied, that the DRESOR method has a good accuracy. The temperature distributions have a node with different kinds of sine changed gradient index distributions under the same boundary emissivity. The impact of the gradient index on the radiative heat transfer is considerable, and the same as that of the ratios of its amplitude and average index. Besides, the effects of optical thickness, boundary emissivity and scattering phase function on radiative transfer also should be paid adequate attention.

1. Introduction

As one basic energy and information transfer mode, radiative heat transfer has been greatly developed and widely used in high-tech fields, especially under vacuum and high temperature condition. Normally, the refractive index of the participating medium is always assumed to be constant. This dealing simplifies the calculation, while brings uncertainty of results and unavoidable errors. The fact is that the refractive index of the medium is affected by its components, temperature and some other factors. With the development of scientific research, the gradient index materials have important application background and broad prospects for development, which makes us try to obtain its transmission characteristics and mechanisms, and grasp quick and convenient calculating methods. However, as the rays propagate in curved paths, which are determined by the Fermat principle [¹], the solution of radiative heat transfer in a medium with gradient index is much more difficult than that in a medium with uniform index.

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A new process, Distribution of Ratios of Energy Scattered by the medium Or Reflected by the boundary surface method, known as the DRESOR method, was proposed by Zhou [2-5] to address the radiative transfer problems. However, the refractive index considered is assumed to be uniform in all of the above studies. Just recently, Wang [6] successfully expanded the DRESOR method to a one-dimensional linear gradient index medium with black boundaries. As the development of multilayer films material and media, cyclicly changed gradient index gradually evoked people’s attention and the sines changed gradient index is one of the most normally considered. However, things turn out to be manifestly difficult when considering complicated gradient index distributions and gray boundaries. The present work is therefore aimed at discussing the effects of nonlinear gradient index on radiative heat transfer using the DRESOR method in a one-dimensional medium with sine changed gradient index. The basic principle of the DRESOR method, the calculating of radiative intensity and the relevant quantities will be shown in Section 2. Radiative transfer with two sine changed gradient index is considered in three cases. The proper discrete grid number and direction number combination is determined first. Then the first case is taken as an example to examine the accuracy of the proposed method. Then the following two cases are taken to discuss the impact of the gradient index, the ratios of its amplitude and average index on the temperature distribution. Besides, the effects of optical thickness, boundary emissivity, scattering phase function and scattering albedo on radiative equilibrium are investigated, too. These will be described in detail in Section 3. Finally, some conclusions will be drawn in Section 4.

Nomenclature

| Symbols | Meanings |
|---------|----------|
| $N$ | discrete elements of medium |
| $\tau$ | optical thickness |
| $R_d$ | DRESOR value, $1/m^3$, or $1/m^2$ |
| $\theta$ | polar angle |
| $I$ | radiative intensity, $W/(m^2$ sr$)$ |
| $\hat{s}$ | directional vector, (rad) |
| $q$ | radiative flux, $W/(m^2$) |
| $T$ | temperature, K |
| $n$ | refractive index |
| $L$ | physical distance, m |
| $M$ | discrete numbers of angle |
| $\rho/\varepsilon$ | surface reflectivity/emissivity |
| $\omega$ | scattering albedo |

2. Principles

2.1. Basic principle of the DRESOR method

The integral form of the radiative transfer equation in an absorbing, emitting-scattering medium with gradient index can be expressed as [1, 7, 8]:

$$
\frac{I(r,s)}{n^2} = I(c)(r,s) + \int_{s'} S(r,s')exp(-\int_{s_c}^{s} \beta ds')/\beta ds',
$$

(1)

Where $S(r',s')$ is the source term, which is:

$$
S(r',s') = (1 - \omega)I_b(r') + \frac{\omega}{4\pi} \int_{4\pi} \frac{1}{n_s} I(r',s)\Phi(s',s) d\Omega',
$$

(2)

The corresponding boundary condition of the non-transparent surface with arbitrary characteristics is [7]:

$$
I_w(r,s) = n_w^2 \varepsilon(r_w)I_b(r_w) + \int_{s_t < 0} \rho(r_w,s',s)I(r_w,s')|n \cdot s'| d\Omega'.
$$

(3)
According to the DRESOR method, after rewriting equations (2) and (3) then submitting them back into equation (1) we can get the following expression. In order to avoid duplication, details can be found in the previous work done by Wang et al. [6].

\[ \frac{I(r, s)}{n_{r_0}^2} = \left\{ \int_{n_{r_0}} n_{r_0}^2 R_{i}^r(r, r', s)[\pi c(r_i, I_0(r_i))]dA' + \int_{n_{r_0}} n_{r_0}^2 R_{i}^r(r', r, s)[4\pi \beta(1 - \omega)I_0(r')]dV' + [\pi c(r_i, I_0(r_i))] \right\} \]

\[ \frac{1}{n_{r_0}^2} \exp[-\int_0^s \beta ds^*] \int_0^s \frac{1}{4\pi \beta} \left( 4\pi \beta(1 - \omega)I_0(r') + \int_{n_{r_0}} R_{i}^r(r', r', s)[\pi c(r_i, I_0(r_i))]dA' \right) \exp[-\int_0^s \beta ds^*] \beta ds'. \]

Where \( R_{i}^r(r, r', s) \), \( R_{i}^r(r', r, s) \), \( R_{i}^r(r, r', s) \) and \( R_{i}^r(r', r', s) \) are all DRESOR values as defined in the DRESOR method \([2, 3]\). The DRESOR values within the system are essential for calculating the radiative intensity in the DRESOR method. The DRESOR value \( R_{i}^r(r, r', s) \) denotes the ratio of the energy scattered by the unit volume around the point \( r' \) into a unit solid angle around the direction \( s \) to that emitted from a unit volume around the point \( r' \). Similarly, we could get the definitions of \( R_{i}^r(r, r', s) \), \( R_{i}^r(r', r, s) \) and \( R_{i}^r(r', r', s) \). The detailed calculation method for the DRESOR values can be found in the literature \([9, 10]\). Calculating quantitatively the impact of scattering on radiative transfer, as combined within the DRESOR values, is one of the main features of the DRESOR method.

2.2. Discretised calculation of intensity and relevant quantities

Taken the sine changed gradient index into consideration, the gradient index can be described as:

\[ n(x) = n_1 + (n_2 - n_1) \sin(\pi x / L) \ (0 \leq x \leq L). \] (5)

As shown in figure 1, a one-dimensional, semi-transparent, gray, absorbing, emitting and scattering slab, with thickness \( x_L \) and gray boundaries, is considered. The system is discretized into \( N \) grids and \( M \) directions in each grid. The physical length of each grid is \( \Delta x = x_L / N \). The midpoint of each grid

![Figure 1](image_url)
is the calculation point, while the two boundaries are named as 0 and \(N+1\) grid, respectively. The emissivity of the boundary walls are \(\varepsilon_0\) and \(\varepsilon_{N+1}\), while the temperatures of the boundary walls are kept at \(T_0\) and \(T_{N+1}\), respectively. The absorption and scattering coefficients of the medium, \(\kappa\) and \(\sigma_s\), are kept constant and the extinction coefficient is \(\beta = \kappa + \sigma_s\). After discretization, the refractive index of the medium in an arbitrary grid \(i\), \(n(i)\), is constant, and varies among different grids.

The intensity at grid \(i1\) in direction \(\theta_i\), \(I(i1, \theta_i)\), can be got by using the discretizing equation (4). The integration is taken along curved path lines for different directions \(\theta_i\), as shown in figure 1.

For the path lines 22' and 33' in figure 1, there will be full-reflection at grid \(i3\) and, because of symmetry, another full-reflection will appear at grid \(N-i3+1\). The lines will propagate between these two grids till the energy they bring is totally absorbed and scattered by the medium. So the discretized formula of intensity can be described as follows:

\[
\frac{I(i1, \theta_i)}{n_i^2} = \frac{1}{4\pi\beta} \sum_{i=1}^{N-1} \left\{ 4\pi\beta(1-\omega)I_s(i2) + \frac{1}{n_i^2} \pi\varepsilon n_i^2 I_s(0)R'_s(0,i2,\theta_i) + \frac{1}{n_i^2} \pi\varepsilon n_i^2 I_s(N+1)R'_s(N+1,i2,\theta_i) \right\}
+ \Delta\varepsilon \sum_{i=1}^{N-1} \frac{1}{n_i^2} 4\pi\beta(1-\omega)n_i^2 I_s(i4)R'_s(i4,i2,\theta_i) \right\}\left[ \exp(-\tau_{i2-i4}) - \exp(-\tau_{i4-i2}) \right].
\] (6)

However, as for the other path lines in figure 1, no matter whether there will be full-reflection, the path lines will end at the boundary surfaces. So the corresponding formula of intensity can be shown as:

\[
\frac{I(i1, \theta_i)}{n_i^2} = \frac{1}{4\pi\beta} \left\{ \pi\varepsilon n_i^2 I_s(0)R'_s(0,i2,\theta_i) + \frac{1}{n_i^2} \pi\varepsilon n_i^2 I_s(N+1)R'_s(N+1,i2,\theta_i) \right\}
+ \frac{1}{n_i^2} \pi\varepsilon n_i^2 I_s(i4)R'_s(i4,i2,\theta_i) \right\}\left[ \exp(-\tau_{i2-i4}) - \exp(-\tau_{i4-i2}) \right].
\] (7)

It is worth noting that along the integration path, \(n_w\) refers to the boundary of the system and the end of the integration path, and varies with different integration lines. Varying with \(\theta_i\), the integration lines such as lines 11', 44', 55', 66', 77' and 88', shown in figure 1, demonstrate different paths according to the distribution of refractive index inside the medium, and end at different surfaces. The optical thickness can be calculated along the paths. The integration paths are curved between the adjacent grids \(i\) and \(j\) according to Descartes-Snell’s law. This is the other main features of the DRESOR method, obtaining the radiative intensity with high directional resolution of high precision easily, which plays a vital role in radiative image processing in some industrial applications.

After the intensity of each grid in every direction is got, it is of no difficulty to calculate the radiative flux \(q(i)\) of each grid using the following formula \([11]\):

\[
q(i) = \sum_{\theta_i=0}^{\pi} 2\pi I(i, \theta_i) \sin(\theta_i) \cos(\theta_i) \Delta\theta.
\] (8)

When the radiation attains equilibrium inside the medium, we can get the temperature distribution of the medium \([11]\):

\[T(i) = \left[ \frac{1}{4n_f^2 \sigma} \sum_{\theta_i=0}^{\pi} 2\pi I(i, \theta_i) \sin(\theta_i) \Delta\theta \right]^{0.25}.
\] (9)
3. Results and discussions

3.1. Objective studied, cases and conditions of calculation

Consider the radiative transfer process inside the one-dimensional medium with two boundaries, as shown in figure 1. The boundaries are diffuse gray walls with different, given temperatures. The scattering albedo, the optical thickness and the nonlinear gradient index distribution inside the medium are set before calculation. The temperature and radiative flux distribution inside the medium are to be analyzed quantitatively.

In these cases, the temperatures of boundaries are \( T(0) = 1000K \) and \( T(N+1) = 1500K \). All the calculations are performed using an Intel Pentium E5200 @ 2.5GHz computer.

3.2. Case 1: the purely emitting and absorbing medium

First, the influence on the accuracy of the results and the computing time are examined to determine the proper discrete grid number \( N \) and direction number \( M \) combination. The scattering albedo is \( \omega = 0.0 \) and the gradient index of the medium is \( n(x) = 1.8 - 0.6 \sin(x/L) \). The boundaries are all black and the optical thickness is \( \tau = 1.0 \). As shown in figure 2 and table 1, it is obvious to see that the results of the DRESOR method are stable under all these conditions. When \( N \geq 100 \) and \( M \geq 18 \), the influence of \( N \) and \( M \) is quite weak. While the computing time of \( M = 180 \) doesn’t increase too much compared with that of \( M = 18 \) under the same \( N = 100 \); however, the computing time of \( N = 1000 \) is unacceptable compared with that of \( N = 100 \) with the same \( M = 180 \). So \( N = 100 \) and \( M = 180 \) are used in the calculations below.

**Figure 2.** The results under different values of \( N \) and \( M \) combination.

**Table 1.** The computing time under different values of \( N \) and \( M \) combination.

| \( (N, M) \) | Computing time   | \( (N, M) \) | Computing time   |
|-------------|------------------|-------------|------------------|
| (10, 18)    | 12.56185 s       | (100, 18)   | 700.3893 s       |
| (100, 18)   | 907.04685 s      | (100, 1800) | 3103.81255 s     |
| (1000, 180) | 273275.311 s     |             |                  |
In this case, two different gradient index distributions and four different kinds of boundary emissivity are considered.

![Temperature field under radiative equilibrium in the purely emitting and absorbing medium.](image)

**Figure 3.** Temperature field under radiative equilibrium in the purely emitting and absorbing medium.

\[ \varepsilon_0 = 0.2, \varepsilon_{N+1} = 1.0; \text{curve 1: } n(x) = 1.2 + 0.6\sin(\pi x/L), \text{curve 2: } n(x) = 1.8 - 0.6\sin(\pi x/L); \]

\[ \varepsilon_0 = 0.7, \varepsilon_{N+1} = 0.7; \text{curve 3: } n(x) = 1.2 + 0.6\sin(\pi x/L), \text{curve 4: } n(x) = 1.8 - 0.6\sin(\pi x/L); \]

\[ \varepsilon_0 = 1.0, \varepsilon_{N+1} = 1.0; \text{curve 5: } n(x) = 1.2 + 0.6\sin(\pi x/L), \text{curve 6: } n(x) = 1.8 - 0.6\sin(\pi x/L); \]

\[ \varepsilon_0 = 1.0, \varepsilon_{N+1} = 0.2; \text{curve 7: } n(x) = 1.2 + 0.6\sin(\pi x/L), \text{curve 8: } n(x) = 1.8 - 0.6\sin(\pi x/L). \]

The results of temperature distribution are shown in figure 3. These are compared with those got by Huang [12] using the discrete curved ray-tracing method (DCRT), Tan [13] with combined curved ray-tracing and pseudo-source adding method (CRTP), Liu [14] using the meshless method and Liu [15] through a finite element method (FEM). The results predicted by the DRESOR method show excellent agreement with these methods under different gradient indexes and boundary emissivity. The relative errors are below 0.35%.

Unlike that of linear gradient index distribution [6], the temperature distributions have a node of these sine changed gradient index distributions under the same conditions of boundary emissivity. The reason is that the sine changed gradient index causes the special ray propagation path: complete symmetry within two sides of the center, which makes the node affected just by the characters of two surfaces rather than the gradient index. Just as shown in figure 3, if \( \varepsilon_0 = \varepsilon_{N+1} \), the node will be the midpoint of the medium and its radiative equilibrium temperature stays constant. However, this node has a shifting trend towards the bigger boundary emissivity side if \( \varepsilon_0 \neq \varepsilon_{N+1} \).

The effect of optical thickness on radiative transfer under nonlinear gradient index is shown in figure 4. As we can see, with the increase of optical thickness, the temperature difference between its two sides increases while the temperature difference between a sine changed gradient index and a constant gradient index first increases and then decreases. In order to incarnate the sensitivity of the optical thickness, \( \tau = 1.0 \) is used in the following cases.

### 3.3. Case 2: the absorbing, emitting and isotropic scattering medium

In this case, the boundary emissivity is assumed to be \( \varepsilon_0 = \varepsilon_{N+1} = 0.7 \), the scattering albedo is \( \omega = 0.3 \). The effect of the gradient index distributions is investigated.
As shown in figure 5, A/P denotes the ratios of amplitude and average index of the gradient index. As we can see, doing isolated research on the impact of the amplitude or the average index is incorrect and the results seem chaotic. So the effect caused by the amplitude coupled the average index is investigated. First, once the A/P is decided, such as when A/P equals 0.5 in figure 5, no matter what the amplitude and the average index are, the temperature distributions are completely overlapped with each other. Then, with the decrease of the A/P, the propagate path of the radiation shown in figure 1(a) increases, which makes the impact of the boundaries on its nearby medium greater. In consequence, the temperature difference between the two sides increases and so does the temperature gradient.

**Figure 4.** Temperature field under radiative equilibrium with different kinds of optical thickness.

**Figure 5.** Temperature field under radiative equilibrium in the isotropic scattering medium.

3.4. Case 3: the absorbing, emitting and anisotropic scattering medium
In this case, the gradient index distribution is \( n(x) = 1.8 - 0.6 \sin(x / L) \). Two linear scattering phase functions; \( \Phi = 1 + b \mu' \), \((b = 1, -1)\), are investigated.

![Figure 6. Temperature field in anisotropic scattering medium.](image)

**Figure 6.** Temperature field in anisotropic scattering medium.

![Figure 7. The corresponding radiative flux distribution of figure 6.](image)

**Figure 7.** The corresponding radiative flux distribution of figure 6.

Figures 6 and 7 show the effect of boundary emissivity on the radiative equilibrium. Here, the scattering albedo is \( \omega = 0.7 \). Three combinations of boundary emissivity are calculated. As we can see, with a decrease in the boundary emissivity, the energy emitted from the two sides reduces, which in turn makes the energy absorbed and scattered by the medium reduce. So the radiative equilibrium flux then decreases, as shown in figure 7. In other words, the heat effect on the medium of the two boundaries is decreasing gradually. Of course, this makes the temperature difference of its two sides lower and the temperature curve flatter.

At the same time, it is also worth noting that; even though the effect of anisotropic scattering phase function on temperature distribution, as shown in figure 6, is weaker than that caused by the boundary
emissivity, it cannot be ignored. Compared with forward scattering $b = 1$, the backward scattering $b = -1$ enhances the radiative intensity in the negative direction, which, according to formula. (8), reduces the radiative flux under radiative equilibrium. However, the backward scattering $b = -1$ increases the transmitting length to some degree, which means that the heat effect on the medium of the two boundaries is greater than that of $b = 1$. That is why under the same conditions, the flux of the backward scattering $b = -1$ is less than that of forward scattering $b = 1$, while the temperature difference of the backward scattering $b = -1$ is higher than that of forward scattering $b = 1$.

![Temperature field under radiative equilibrium in anisotropic scattering medium with $\varepsilon_0 = \varepsilon_{N+1} = 0.7$.](image)

**Figure 8.** Temperature field under radiative equilibrium in anisotropic scattering medium with $\varepsilon_0 = \varepsilon_{N+1} = 0.7$.

![The corresponding radiative flux distribution of figure 8.](image)

**Figure 9.** The corresponding radiative flux distribution of figure 8.

Figures 8 and 9 show the effect of scattering albedo $\omega$ on radiative equilibrium. Here, the boundary emissivities are $\varepsilon_i = \varepsilon_{N+1} = 0.7$. Three different scattering albedos; $\omega = 0.7$, $\omega = 0.5$ and $\omega = 0.2$, are calculated. When the scattering albedo decreases, the absorbing capacity of the medium increases...
correspondingly. With forward scattering $b=1$, when the scattering albedo $\omega$ changes from 0.7, 0.5 to 0.2, the radiative intensity decreases, so does the radiative flux. However, the temperature difference between the two sides increases because of the increase in the absorbing capacity of the medium. However, with backward scattering $b=-1$, the decrease in the scattering albedo $\omega$ reduces the radiative intensity in the negative direction, which makes the radiative flux increase. At the same time, the enhancement effect of backward scattering as we discussed above is likely to decline, which makes the temperature difference between the two sides decrease. Besides, when the scattering albedo $\omega$ reduces, so does the effect of anisotropic scattering. As shown in figure 8, the difference between $b=1$ and $b=-1$ becomes smaller and smaller.

4. Conclusions
In this paper, the DRESOR method has been applied to radiative transfer in nonlinear gradient index media, with a DRESOR value introduced. The temperature and radiative flux distributions inside the medium were calculated in three cases. The impacts of the gradient index distributions, the variation of the amplitude and the average index, the optical thickness, the scattering albedo, the boundaries emissivity and the scattering phase function were all investigated. The main conclusions follow below.

- This method could deal well with the radiative transfer problem in a one-dimensional semi-transparent slab with nonlinear gradient index and complex boundary characteristics.
- The temperature distributions have a node with different sine changed gradient index distributions under the same boundary emissivity.
- The impact of the gradient index on the radiative heat transfer is the same as that of the ratios of its amplitude and average index. The boundary emissivity has a significant impact on radiative heat transfer, while the effect of optical thickness, the scattering phase function is much weaker. However, they cannot be ignored and should be paid adequate attention.

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