A randomly determined unpredictable function

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Abstract

Recently, we have introduced unpredictable oscillations, which are in the basis of Poincaré chaos. For theoretical analysis as well as for applications, it is necessary to provide constructive examples of unpredictable functions. We have already provided such functions utilizing orbits of the logistic map, and in the present paper we suggest another way of construction of the functions by applying the Bernoulli random process. A simulation for a randomly determined unpredictable function is provided.

Keywords: Unpredictable function, Unpredictable sequence, Bernoulli process, Poincaré chaos, Symbolic dynamics

1 Introduction and preliminaries

The theory of oscillations extremely rely on functions, which can be either tabulated or formalized. The ones in the second category are based first of all on the functions which are trigonometric, polynomials, hyperbolic trigonometric and others. All of them have been tabulated in computer memories. Next ones are functions, which can be presented as finite or infinite sums of the former ones. They are evaluated by developing software programs and very helpful in applications. Other are oscillations produced as solutions of differential equations. There exists, even, the large class in the qualitative theory of differential equations, oscillatory differential equations. The solutions are approved as oscillations by special type of criteria for the existence.

In this study, we focus on functions which are shaped through qualitative conditions of definitions. They make the core of the research area in the theory of dynamical systems, issued by A. Poincaré, G. Birkhoff, and others. These are periodic, quasi-periodic, almost periodic oscillations, recurrent and Poisson stable orbits. A special type of Poisson stable orbit called an unpredictable trajectory, which leads to Poincaré chaos in the quasi-minimal set, was introduced in the paper \cite{1}. Moreover, the papers \cite{2,3,4,5} were concerned with unpredictable solutions of various types of quasi-linear differential equations. In the present paper, we introduce a new way for unpredictable functions construction benefiting from the dynamics associated with the discrete distribution \cite{6}. We consider the process with a finite number of possible outcomes to generate

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an unpredictable sequence. The sequence is then used to construct a continuous unpredictable function. Thus, unpredictable oscillations appeared as solutions of linear or quasi-linear differential equations, i.e., as outputs of the systems, provided that there is an unpredictable input. The natural question how it is possible to choose the inputs being unpredictable arises. For this reason in the previous papers \[2, 3, 4, 5\], we introduced unpredictable functions built by applying or bits of the logistic map, which were verified to be unpredictable sequences. One can confirm that in this way we utilize several other discrete equations with dynamics topologically equivalent to symbolic dynamics. This is why, they are in some sense the same as those functions, which have been already determined in our research. For that reason, the task of construction of new unpredictable oscillations is undertaken in the present paper. We utilize the two principal issues for the solution of the problem. The first one is that the set of all orbits of the symbolic dynamics coincides with all possible sequences of the symbols. Moreover, realizations of the Bernoulli random process altogether are the set of sequences. Consequently, constructing an orbit of a random process, we obtain an orbit of the symbolic dynamics and simulate a part of the unpredictable sequence. Thus, we obtain that a single iteration of the Bernoulli shift is the same as a trial for the Bernoulli process.

The next definitions are concerned with unpredictable sequences and functions.

**Definition 1.1** (\[4\]) A bounded sequence \( \{\nu_k\}, k \in \mathbb{Z} \) in \( \mathbb{R}^p \) is called unpredictable if there exist a positive number \( \varepsilon_0 \) and sequences \( \{\zeta_n\}, \{\eta_n\}, n \in \mathbb{N} \), of positive integers both of which diverge to infinity such that
\[
\|\nu_k + \zeta_n - \nu_k\| \to 0 \quad \text{as} \quad n \to \infty \quad \text{for each} \quad k \quad \text{in bounded intervals of integers and} \quad \|\nu_{n+\eta_n} - \nu_{\eta_n}\| \geq \varepsilon_0 \quad \text{for each} \quad n \in \mathbb{N}.
\]

**Definition 1.2** (\[2\]) A uniformly continuous and bounded function \( h : \mathbb{R} \to \mathbb{R}^p \) is unpredictable if there exist positive numbers \( \varepsilon_0, \sigma \) and sequences \( \{t_n\}, \{u_n\} \) both of which diverge to infinity such that
\[
h(t + t_n) \to h(t) \quad \text{as} \quad n \to \infty \quad \text{uniformly on compact subsets of} \quad \mathbb{R} \quad \text{and} \quad \|h(t + t_n) - h(t)\| \geq \varepsilon_0 \quad \text{for each} \quad t \in [u_n - \sigma, u_n + \sigma] \quad \text{and} \quad n \in \mathbb{N}.
\]

Consider the space \( \Sigma_m \) of bi-infinite sequences \( \ldots i_{-2}i_{-1}i_0i_1i_2 \ldots \) on finite number of complex numbers \( a_1, \ldots, a_m \), with the metric
\[
d(I, J) = \sum_{k=-\infty}^{\infty} \frac{|i_k - j_k|}{2^{|k|}}, \quad (1.1)
\]
where \( I = (\ldots i_{-2}i_{-1}i_0i_1i_2 \ldots), J = (\ldots j_{-2}j_{-1}j_0j_1j_2 \ldots) \), and \( |\cdot| \) is the absolute value. Introduce the Bernoulli shift \( \varphi : \Sigma_m \to \Sigma_m \) such that
\[
\varphi((\ldots i_{-2}i_{-1}i_0i_1i_2 \ldots)) = (\ldots i_{-2}i_{-1}i_0i_1i_2i_3 \ldots). \quad (1.2)
\]
The map \( \varphi \) is continuous and the metric space \( \Sigma_m \) is compact \[10\].

Let us, now, build an unpredictable point for the dynamics \( (\Sigma_m, d, \varphi) \). Without loss of generality, we consider a particular case of the space when \( m = 2, a_1 = 0, a_2 = 1 \). We need a collection of finite sequences
\(i_k^r, r \in \mathbb{N}, k = 1, 2, \ldots, 2^r\), consisting of 0’s and 1’s. Let us use the notations \(i_1^1 = (0)\) and \(i_2^1 = (1)\) for the sequences of length 1. For each natural number \(r\), we recursively define \(i_{2k-1}^{r+1} = (i_k^r 0)\) and \(i_{2k}^{r+1} = (i_k^r 1)\), \(k = 1, 2, \ldots, 2^r\), where \(i_{2k-1}^{r+1}\) and \(i_{2k}^{r+1}\) are obtained by respectively inserting 0 and 1 to the end of the sequence \(i_k^r\) of length \(r\). For instance, \(i_1^2 = (i_1^1 0) = (00), i_2^2 = (i_1^1 1) = (01), i_3^2 = (i_1^2 0) = (10), \) and \(i_4^2 = (i_1^2 1) = (11)\) are the sequences of length 2. Now consider the following sequence \(i^* = (\ldots i_1^2 3 2 3 2 3 2 3 2 3 2 3 \ldots)\). In \([1]\) it was proved that \(i^*\) is an unpredictable point of the dynamics.

Because the trajectory which initiates at \(i^*\) is dense in the quasi-minimal set \(\Sigma_m\), the dynamics is Poincaré chaotic according to Theorem 3.1 presented in paper \([1]\). Moreover, there is an uncountable set of unpredictable points in the set. From this discussion it implies that any numerical simulation of a discrete finite distribution is an approximation of an unpredictable sequence. Indeed, the metric peculiarity implies that if one considers the point \(i^*\) in \(\Sigma_m\) as a bi-infinite sequence, then it is easily seen that it is an unpredictable sequence in the sense of Definition \([1,4]\). This is in the base of the construction of an unpredictable function in the next section.

2 Main result

Let us fix a finite string \(i_k, \ldots, i_p, 1 \leq k < p\), on the set of complex numbers \(a_1, \ldots, a_m\). It can be accepted as an arc of a sequence from \(\Sigma_m\). Since of the last section discussion, the string can be approximated with arbitrary precision by a shift of the sequence \(i^*\). This possibility to approximate by shifts of the orbits is the main advantage of the Poincaré chaos against other types of chaos. Taking into account that there are limits for the approximation validity in numerical simulations by computers, we can admit that simulation of the string is simulation of the unpredictable sequence itself. This is why, we accept that finite realizations of the Bernoulli process, which are obtained randomly present the unpredictable sequence, since, at first, they are not periodic even on a sufficiently large interval of discrete time, and, secondly, since of the above explanation the simulation is an approximation of the sequence with arbitrary precision. The arbitrariness guarantees that in applications we can get the simulations as the unpredictable sequence with the attributes listed in the definition. Moreover, we must not be confused with the approximations in the basis of the definition. This is true for all types of functions, which are determined through infinitely long algorithms such as series, for instance.

Fix an unpredictable sequence \(i^*\), which is defined on the two real numbers \(a\) and \(b\). One can find that the unpredictability constant \(\epsilon_0\) can be taken equal to \(|a - b|\). Define the function \(\chi(t) : \mathbb{R} \to \mathbb{R}\) through the equation

\[
\chi(t) = \int_{-\infty}^{t} e^{-(t-s)} \pi(s)ds,
\]

where \(\pi(t) : \mathbb{R} \to \mathbb{R}\) is the piecewise constant function satisfying \(\pi(t) = i_k^r\) for \(t \in [k, k + 1), k \in \mathbb{Z}\). One can confirm that \(\sup_{t \in \mathbb{R}} |\chi(t)| \leq M_x\), where \(M_x = \max\{|a|, |b|\}\).

We will show that the function \(\chi(t)\) defined by \([2,6]\) is unpredictable. Consider a fixed compact interval
and a positive number \( \varepsilon \). We assume without loss of generality that \( \alpha \) and \( \beta \) are integers. Let us fix a positive number \( \xi \) and an integer \( \gamma \) such that the inequalities \( Me^{-2(\alpha-\gamma)} < \varepsilon/4 \) and \( \xi(1-e^{-2(\beta-\gamma)}) < \varepsilon \).

Suppose that \( n \) is a sufficiently large natural number satisfying \( |\pi(t + \zeta_n) - \pi(t)| < \xi \) for every \( t \) in \( [\gamma, \beta] \). Accordingly, we have for \( t \in [\alpha, \beta] \) that

\[
|\chi(t + \zeta_n) - \chi(t)| = |\chi(t) - \chi(t)| \leq \int_{-\infty}^{t_0} e^{-2(t-s)}|\pi(s + \zeta_n) - \pi(s)| \, ds + \int_{t_0}^{\beta} e^{-2(t-s)}|\pi(s + \zeta_n) - \pi(s)| \, ds
\]

\[
\leq \int_{-\infty}^{t_0} e^{-2(t-s)}ds + \int_{t_0}^{\beta} e^{-2(t-s)}|\xi| \, ds
\]

\[
\leq 2Me^{-2(\alpha-\gamma)} + \frac{\xi}{2}[1-e^{-2(\beta-\gamma)}]
\]

\[
< \varepsilon.
\]

Thus, \( |\chi(t + \zeta_n) - \chi(t)| \to 0 \) as \( n \to \infty \) uniformly on the interval \([\alpha, \beta]\).

Let us fix a number \( n \) and consider two alternative cases: (i) \( |\chi(\eta_n + \zeta_n) - \chi(\eta_n)| < \frac{\xi}{8} \) and (ii) \( |\chi(\eta_n + \zeta_n) - \chi(\eta_n)| \geq \frac{\xi}{8} \).

(ii) There exists a positive number \( \kappa < 1 \) such that \( e^{-2\kappa} = \frac{2}{3} \). Using the relation

\[
\chi(t + \zeta_n) - \chi(t) = \chi(\eta_n + \zeta_n) - \chi(\eta_n) + \int_{\eta_n}^{\eta_n + \zeta_n} e^{-2(t-s)}|\pi(s + \zeta_n) - \pi(s)| \, ds
\]

we obtain that

\[
|\chi(t + \zeta_n) - \chi(t)| \geq |\int_{\eta_n}^{\eta_n + \zeta_n} e^{-2(t-s)}|\pi(s + \zeta_n) - \pi(s)| \, ds - |\pi(\eta_n + \zeta_n) - \pi(\eta_n)| \geq
\]

\[
\int_{\eta_n}^{\eta_n + \zeta_n} e^{-2(t-s)} \, ds - \frac{\varepsilon_0}{8} \geq \frac{\varepsilon_0}{2} (1 - e^{-2\kappa}) - \frac{\varepsilon_0}{8} = \frac{\varepsilon_0}{24}
\]

for \( t \in [\eta_n + \kappa, \eta_n + 1] \).

(ii) There exists a positive number \( \kappa < 1 \) such that \( 1 - e^{-2\kappa} = \frac{24}{25} \). From the relation (2.4) we get

\[
|\chi(t + \zeta_n) - \chi(t)| \geq |\chi(t) - \chi(\eta_n)| - |\int_{\eta_n}^{\eta_n + \zeta_n} e^{-2(t-s)}(\pi(s + \zeta_n) - \pi(s)) \, ds| \geq
\]

\[
\frac{\varepsilon_0}{8} - \int_{\eta_n}^{\eta_n + \zeta_n} e^{-2(t-s)} \, ds \geq \frac{\varepsilon_0}{8} - [1 - e^{-2\kappa}] = \frac{\varepsilon_0}{24}
\]

for \( t \in [\eta_n, \eta_n + \kappa] \).

Thus, \( \chi(t) \) is an unpredictable function.

It is easy to see that \( \chi(t) \) is a solution of the differential equation

\[
x' = -x + h(t)
\]

(2.5)
with
\[ \chi(0) = \int_{-\infty}^{0} e^{s}h(s)ds, \] (2.6)
but we do not know the value \( \chi(0) \) precisely, since it cannot be evaluated by the improper integral (2.6). Nevertheless, we utilize that \( \chi(t) \) is an exponentially stable solution of equation (2.5). Therefore, any solution \( \varphi(t) \) of (2.5) approximates \( \chi(t) \). The approximation is better for larger \( t \) such that \( \| \chi(t) - \varphi(t) \| \leq \| \chi(0) - \varphi(0) \| e^{-t}, t \geq 0 \). For that reason we take \( \varphi(0) = 0.5 \) so that \( \| \chi(t) - \varphi(t) \| \leq e^{-50} < 10^{-17} \) for \( t \in [50, 100] \). It is less than Matlab precision between 50 and 100. Hence, the part of the time series of \( \varphi(t) \) for \( 50 \leq t \leq 100 \) can be accepted as the graph of the function \( \chi(t) \).

In Figure 1 we depict the unpredictable function \( \chi(t) \) defined by equation (2.5). For the simulation, we use the function \( \pi(t) = i_k, t \in [\mu(k - 1), \mu k), \mu = 0.1, k \in \mathbb{N} \). The sequence \( i_k \) is generated randomly such that \( i_k = 0, 1 \) for each \( k = 1, 2, \ldots \).

![Figure 1: Time series of the unpredictable function \( \chi(t) \)](image)

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