New correlation relations in classical and quantum systems with different numbers of subsystems

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Abstract. We present a review of the general approach to the problem of correlations in classical statistics and quantum statistics of systems with different numbers of subsystems and demonstrate the information-entropic relations for systems without subsystems recently obtained for Shannon entropies. We present the example of a single-qudit state corresponding to the $N$-level atom, consider explicitly the qutrit state, and show that qutrit can be interpreted as a set of several qubits. For each of these qubits, there exist corresponding von Neumann entropies, and constraints for these entropies determine the hidden correlations between the qubits in spite of the fact that the qutrit does not contain any subsystem. These constraints are expressed in terms of nonnegativity of the mutual information introduced, which usually exists only for the states of systems with subsystems. The value of information parameterizes the hidden correlations of artificial qubits in the system. We discuss examples of some qudits.

Keywords: Moshinsky shutter, probability distributions, qubit, quantum correlations, entropic inequalities, subadditivity condition.

Ad Memoriam of Marcos Moshinsky
Before considering the main topics of this paper, we recall that ten years ago, on April 1, 2009 Professor Marcos Moshinsky passed away, and we are very sad about this.

Professor Moshinsky made outstanding contributions in quantum theory and its applications in different areas of physics. Theoretical physics and nuclear physics were important subjects of his investigations. In view of his activity, theoretical physics in Mexico has been intensely developed, and till now his former students and colleagues are successfully continuing the scientific researches started due to the influence of Professor Moshinsky.

Marcos Moshinsky Borodiansky was born in Kiev, Soviet Union on April 20, 1921. So his mother’s language was Russian, and he used to speak Russian along with Spanish, English, and so on. Marcos had relatives in Moscow and liked to visit them; they called him as Russian Jewish used to do “Marik.”

1 Based on the talk presented by M. A. Man’ko at the 18th International Symposium Symmetries in Sciences (Bregenz, Vorarlberg, Austria, August 4-9, 2019) [itp.uni-frankfurt.de/symmetries-in-science/].
Marik, school boy (on the left) and Marcos’ family with Marcos, young guy in the middle (on the right).

Three types of Mexican Graduation degrees named by Marcos Moshinsky.

The First Latina Escuela took place from July 9 until August 31, 1956. Now it is called the Latin American School of Physics “Marcos Moshinsky” (ELAF, for its acronym in Spanish).
ELAF was created by three of the most significant physicists in Latin America: Juan Jose Giambiagi from Argentina, Jose Leite Lopes from Brazil and Marcos Moshinsky from Mexico. After its inauguration in 1959 in Mexico City, ELAF moved to Rio de Janeiro in 1960, next to Buenos Aires in 1961, and again to Mexico City in 1962. Since then, this school has taken place every third year in Mexico, becoming one of the most traditional schools of physics in the region. ELAF has counted with the participation of prominent scientists from all over the world presenting in courses and seminars the most recent developments in modern physics, inspiring hundreds of students and young scientists from all over Latin America. In 2017, the UNAM Latin American School of Physics “Marcos Moshinsky” was just devoted to quantum correlations and even named *Quantum Correlations*.

Marcos Moshinsky essentially influenced the creation and development of fundamental researches on quantum mechanics, nuclear physics, and group theory not only in Mexico but at the international level as well. His work [1] performed when he was PhD student and called *Moshinsky shutter* was developed, in turn, with his PhD student when Vladimir Man’ko visited UNAM. They published modern consideration of his shutter problem using the Wigner function and tomographic-probability description of this quantum problem; also called *Diffraction in Time* [2].

Being in Princeton, Marcos was very close to Wigner’s activity during the time when *Moshinsky shutter* was introduced and studied. Also there were theoretical and experimental studies of Moshinsky shutter [3, 4]. Later on, when quantum tomography was discussed in the literature, Marcos suggested to consider, for his shutter problem, the connection with Wigner function and quantum tomography in order to clarify the relation of two approaches – Wigner function and tomographic-probability distributions – on the example of his diffraction in time for quantum systems.

The theory of linear canonical transforms was developed under supervision of Marcos Moshinsky [5–7]. The canonical transforms are closely related to the time-dependent (linear and quadratic in the position and momentum) integrals of motion for parametric oscillator systems; see, e.g., [8–11]. There are different methods of obtaining the solution of the Schrödinger evolution equation for the parametric oscillator found in [12] as well as to find integrals of motion for it. For the classical parametric oscillator, the integral of motion, quadratic in the position and momentum, with time-dependent coefficients was found by Ermakov [13]. The quadratic in the position and momentum operator – an analog of the Ermakov invariant for the parametric oscillator was found in [14]. The approach to obtain the solution of the Schrödinger equation for the parametric oscillators and to construct coherent states for a charge moving in the magnetic field was applied in [15,16].
The Lewis–Riesenfeld invariant application is related to employing the nonlinear Riccati equation. The approach to solving the Schrödinger equation for systems with time-dependent Hamiltonians like the parametric oscillator, using the nonlinear Riccati equation, was developed in [9]. The time-dependent integrals of motion for the parametric oscillator, which are linear in the position and momentum operators, and the connection of these integrals of motion with the Lewis–Riesenfeld invariant were found in [17, 18]; see also [19]. The linear in the position and momentum integrals of motion used in [20] for parametric systems are closely connected with obtaining the system of equations for the Green function of the Schrödinger evolution equation studied in [21]. The applications of the integrals of motion both linear and based on solving the nonlinear Riccati equation were studied in [9, 22–24].

Such systems have many applications, say, the motion of charged particle in varying magnetic and electric fields is one of the important examples of such systems. The canonical transforms and integrals of motion play an important role in obtaining explicit solutions of the Schrödinger equation and provide the formulas for transition probabilities between the energy levels [8]. For varying magnetic fields, transition probabilities obtained by such methods describe the excitation of Landau levels.
Some former PhD students of the Lebedev are professors of Mexican Universities, thanks also to Thomas Seligman, one of the young colleagues of Marcos. Namely Thomas Seligman invited Andrei Klimov, the student of Vladimir Man’ko to the University of Guadalajara.

Pier Achille Mello Picco, Professor Emeritus of UNAM got the Moshinsky medal in 2005 (on the left). Later on Andrei Klimov got the Moshinsky medal in 2014 (on the right).

Being the Member of the Standing Committee of the International Colloquium *Group Theoretical Methods in Physics*, Prof. Moshinsky participated in this Colloquium at Moscow in 1992 along with International seminars on the *Group Theoretical Methods in Physics*. Proceedings of these meetings were published; see, for example, [27] and till now they are interesting for many scientists, since at this time in Moscow we had very famous specialists in the group theory and group representation theory, such as members of Gelfand’s seminar, where the group theory was discussed [28], as well as Prof. Yuri Manin, who developed the algebraic approach to physical problems [29], and his colleagues from the Steklov Mathematical Institute; they used to participate in the International conferences organized in the USSR.

Both Eugene Wigner and Marcos Moshinsky were members of the Standing Committee of the Colloquia Group Theoretical Methods in Physics, which is very closely related to our today Bregenz Symposium *Symmetries in Science*.

M. A. Man’ko and V. I. Man’ko visited E. Wigner in his house in Princeton, New Jersey in 1990.
Celebration of Marcos’ 85 years.
Ad Memoriam of Roy Glauber and George Sudarshan

Professor Roy J. Glauber and Professor E. C. George Sudarshan, founders of quantum optics passed away in 2018. We are extremely sad about this.

In 1963, the notion of coherent state of electromagnetic-field oscillations as well as the terminology “coherent state” of an arbitrary oscillator have been introduced. Roy Glauber and George Sudarshan simultaneously published the papers [32, 33], where the properties of coherent states were discussed. These publications are cited in the majority of the papers, where quantum optics and quantum information technologies are discussed.

The general linear positive map of the density matrix to the other density matrices for finite-dimensional systems was presented by Sudarshan et al. in [34] in the form, which later on was generalized in [35] for arbitrary systems. Also we want to point out that the new evolution equation for open quantum systems, which generalizes the Schrödinger equation for the wave function and the von Neumann equation for the density matrix considered in the case of unitary evolution to the case of nonunitary evolution, was obtained in [36, 37] and called the Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) equation.

The above-mentioned results of Glauber and Sudarshan provided the theoretical basis in studies of the evolution of open quantum systems, theory of quantum channels and applications of these approaches in future quantum technologies. These famous contributions play an important role in the foundations of quantum optics and quantum information and developing future quantum technologies. I am happy to let you know that the famous results concerning the study of entanglement phenomenon, quantum tomography and superposition principle in terms of density operators were published by George Sudarshan with coauthors in the Journal of Russian Laser Research [38, 39]. Also Roy Glauber published his papers in our journals [40, 41].

There exist different representations of quantum states in terms of quasidistributions, such as Wigner functions [42] and Husimi–Kano functions [43, 44]. Also the probability representation of quantum states was suggested in [45]. The theory of light coherence and many applications of coherent states in different areas of physics [46–51] and the formalism of the phase space and probability representations of quantum states are reviewed in recently published Special issue of Open System and Information Dynamics [52] dedicated to the memory of George Sudarshan and the focus issue of Physica Scripta dedicated to the memory of Roy Glauber, which is still under preparation [53].

2 Ad Memoriam of Roy Glauber and George Sudarshan was also presented in the talks of M. A. Man’ko at the 16th International Conference on Squeezed States and Uncertainty Relations {ICSSUR} (Universidad Complutense de Madrid, Spain, June 17–21, 2019) [http://eventos.ucm.es/30364/detail/international-conference-on-squeezed-states-and-uncertainty-relations-2019.html] and at the 26th Central European Workshop on Quantum Optics (CEWQO) (Paderborn University, Germany, June 3–7, 2019) [https://cewqo2019.uni-paderborn.de/] and is published in Quantum Reports (2019) [30] and the Journal of Russian Laser Research (2019) [31], respectively.
Our Community and Some Conferences on the Foundations of Quantum Physics

In fact, all the development in theoretical physics discussed by Roy Glauber, Marcos Moshinsky, and George Sudarshan is connected with foundations of quantum physics, including quantum mechanics, quantum field theory, and their applications in nuclear physics and physics of elementary particles. The substantial role in the foundations of quantum physics is played by the theory of symmetries and mathematical and physical applications of group theory and group representation theory in the basic ingredients of the quantum field theory, solid-state theory, gravitation and cosmology. The group theory development and applications in the fundamental areas of physics was discussed and still is discussing in the series of International Colloqua named “Group Theoretical Methods in Physics” where during several decades the applications of the group representation theory like the $SU(3)$ and $SU(6)$ group representation theories, which created the notion of quarks, were discussed, as well as quantum groups related to $q$-oscillators [54,55] and $f$-oscillators [56] were investigated.

The dynamical groups [57], which provide the possibility to study the spectra of quantum systems like the hydrogen atom spectrum, in view of the $O(4,2)$ symmetry group, were considered in [58–60]: they play the important role in considering all chemical elements in the Universe [61–64]. Such group symmetry aspects are present in the programs of the series of Symposia “Symmetries in Science” working during decades in Bregenz, Austria and having been organized by Bruno Gruber together with Michel Ramek and continued by Dieter Schuch.

The foundation of quantum mechanics and the role of symmetry groups as well as algebraic methods in quantum theory, including actual last years quantum information and quantum technology problems are present in the programs of the Bregenz Symposia. Members of the community associated with International Colloqua “Group Theoretical Methods in Physics” and Symposia “Symmetries in Science” are also discussing the foundations of quantum theory in series of International Conferences on Squeezed States and Uncertainty Relations (ICSSUR) [65] initially organized by Young Shi Kim and Vladimir I. Man’ko as Joint Workshop of the University of Maryland and the Lebedev Physical Institute, Moscow and very fast transformed into famous international conference series; the last one took place in Madrid in 2019 with Luis Sanchez Soto, the Chairman and talks published in the new journal *Quantum Reports*, MDPI, Basel [https://www.mdpi.com/journal/quantumrep/special_issues/quantumrep_ICSSUR2019].

Also it is worth mentioning some analogous very successful conferences, not yet organized as series but also dedicated to the foundations of quantum theory such as *Quantum Nonstationary Systems 2007*, Blaubeuren, Germany organized by Manfred Kleber and Tobias Kramer and *Quantum Nonstationary Systems 2009*, Brasilia DF, Brazil organized by Viktor Dodonov, Vladimir Man’ko and Salomon Mizrahi.

The attention to foundations of quantum theory is related to the wish of better understanding the physical meaning of basic ingredients of quantum mechanics such as the notion of the wave function and the notion of observable. In spite of the fact that almost one century passes after Schrödinger introduced the concept of the wave function, its intuitive understanding meets difficulties since this notion of a particle state contradicts the classical understanding of the particle state as given by the particle’s position and velocity values or, in the case of fluctuations, by the probability density of the position and momentum. This problem is difficult not only for students starting to study quantum mechanics in Universities but also for professors giving lectures for these students. The discussion of problems of symmetries of system states, general for classical and quantum worlds, helps to better understand the quantum–classical relations in the foundations of quantum physics.

Below we present a Kaleidoscope of pictures from some of the meetings described above.
G. Sudarshan (1), Y. S. Kim (2), F. Haake (3) and M. Nieto (4) among the participants of ICSSUR 1992 in Moscow.

P. Tombesi, M. A. Man’ko, R. J. Glauber, Y. S. Kim and D. Han at ICSSUR 1992 in Moscow due to a nice tradition of the Academy of Sciences of the USSR to invite the guests to the Bolshoi Theatre for the Swan Lake ballet.

Y. S. Kim, M. A. Man’ko, Mrs. Kim and V. I. Man’ko at ICSSUR 1991 in College Park, Maryland.

V. I. Man’ko, M. A. Man’ko and Y. S. Kim, at ICSSUR 1992 in Moscow.

V. A. Isakov, M. A. Man’ko, O. V. Man’ko and A. S. Chirkin at ICSSUR 2001 in Boston, Massachusetts.

R. J. Glauber and M. A. Man’ko at ICSSUR 1992 in Moscow.
W. Schleich (1), T. Kramer (2), M. Kleber (3), P. Kramer (4),
M. Man’ko (5), V. Man’ko (6), V. Dodonov (7) and M. Scully (8)
among the participants of Quantum Nonstationary Systems 2007 in Blaubeuren, Germany.

E.S. Fradkin, V.I. Man’ko, F. Gürsey,
M.A. Markov, F. Iachello and A. Bohm
at Group Theory Colloquium 1990 in Moscow.

O. Man’ko (1), S. Biedenharn (3), Mrs. Kim (3), M. Man’ko (4) and E. Moshinsky (5) at ICSSUR 1992 in Moscow.
M. Moshinsky and M. Man’ko in Jardin du Luxembourg, Paris at Group Theory Colloquium 2002.

M. Man’ko and G. Marmo at Universidad Carlos III de Madrid in 2008.

V. Dodonov, M. Man’ko and V. Man’ko at ICSSUR 2011 in Foz do Iguaçu, Brazil.

V. Man’ko (1), M. Man’ko (2), R. Kerner (3) and A. Solomon (4) at Quantum Theory and Symmetries 2011 in Prague.

T. Seligman, V. Man’ko, M. Man’ko and G. Marmo at Problems of Mathematical and Quantum Physics (75 + 75 years of Margarita and Vladimir Man’ko), Cuernavaca, Morelos, Mexico, 2015.

Young S. Kim, Margarita Man’ko and Vladimir Man’ko at ICGTMP-1992 in Salamanca.
Social Program of Symmetries in Science Symposium (Gasthof Hotel Lamm, Bregenz, Austria, August 2013).

“Symmetries of Science” Symposium at Kloster Mehrerau, Bregenz.
Margarita Man’ko, Vladimir Man’ko and Tobias Kramer

Olga and Vladimir Man’ko at ICSSUR 2003 in Puebla.

 Mexico 2016.
G. Sudarshan with his wife Bhamathi, Y. S. Kim and M. Man'ko at Quantumlike Models 1994, Erice, Sicily, Italy.

Maurice Kibler and Vladimir Man’ko with their wives in Alps in 1994 due to a nice tradition of Bruno’s brother.

J. Janszky, J. Klauder and M. A. Man’ko at ICSSUR 2003 in Puebla, Mexico.

A. Vourdas and A. Solomon at ICSSUR 2011 in Foz do Iguacu, Brazil.

ICSSUR 2011 in Foz do Iguacu, Brazil.
1. Introduction to hidden correlations in noncomposite systems

Our aim is to present a review and the development of our approach for the description of the states of noncomposite systems, both classical and quantum, and study the correlations in these systems called “hidden correlations.” The hidden correlations were discussed at the previous symposium “Symmetries in Science” in 2017 [66]. Now we present together with a review of the approach some new aspects of hidden correlation problems like Bayes’ formula for systems without subsystems.

The correlations in composite systems like bipartite systems are related to the influence of the behavior of the first subsystem’s degrees of freedom on the behavior of the second subsystem’s degrees of freedom. We show that an analogous dependence exists in the case of systems without subsystems. Such a phenomenon is not obvious: it was discussed in [67]. For example, “the hidden correlations” in the case of four-level atom or spin-3/2 particle can exhibit an analogous dependence in the case of systems without subsystems.

The violations of Bell inequality in entangled states of these bipartite quantum systems has an analog in the state of spin-3/2 particle [68]. Below we show how such an analogy of classical and quantum correlations existing in the states of composite systems appears also in the states of noncomposite systems. The theory of hidden quantum correlations is closely connected with the foundations of quantum mechanics and quantum optics.

Statement 1. Convex sum of two probability distributions is the probability distribution. Let us consider two sets of numbers, i.e., 0 ≤ P_1, P_2, ..., P_N ≤ 1 and 0 ≤ Q_1, Q_2, ..., Q_N ≤ 1 satisfying the equality \( \sum_{j=1}^{N} P_j = \sum_{j=1}^{N} Q_j = 1 \). These numbers are interpreted as probability distributions of one random variable \( j \) which takes all values 1 ≤ j ≤ N.

Then for two given numbers 0 ≤ \( \lambda_1, \lambda_2 \leq 1 \), where \( \lambda_1 + \lambda_2 = 1 \), interpreted as the probability distribution of two dichotomic random variables \( \alpha = 1, 2 \), one has the set of N numbers \( R_1, R_2, ..., R_N \) such that \( R_j = \lambda_1 P_j + \lambda_2 Q_j; \) being nonnegative 0 ≤ \( R_j \leq 1 \), they satisfy the equality \( \sum_{j=1}^{N} R_j = 1 \).

The set of numbers \( R_j \) can be interpreted as the probability distribution of N random variables \( j \). By construction, this probability distribution is the convex sum determined by dichotomic probabilities \( (\lambda_1, \lambda_2) \) of two probability distributions \( P_j \) and \( Q_j \). An analogous statement is valid for two probability distributions 0 ≤ \( P_j \leq 1 \) and 0 ≤ \( Q_j \leq 1 \), where \( j = 1, 2, ..., N; k = 1, 2, ..., M \), are two random variables.

The convex sum of the probability distributions \( R_{jk} = \lambda_1 P_{jk} + \lambda_2 Q_{jk} \) can be interpreted as the probability distribution of two random variables. Repeating this procedure, we can extend this statement to the case of the convex sum of \( n \) probability distributions \( P_{j_1j_2...j_s}^{(\alpha)}; \alpha = 1, 2, ..., n \) of \( s \) random variables \( j_1 = 1, 2, ..., n_1; j_2 = 1, 2, ..., n_2; ...; j_s = 1, 2, ..., n_s \) constructed by means of the probabilities 0 ≤ \( \lambda_\alpha \leq 1 \), using the expression \( R_{j_1j_2...j_s} = \sum_{\alpha=1}^{N} \lambda_\alpha P_{j_1j_2...j_s}^{(\alpha)} \), where \( \sum_{\alpha=1}^{N} \lambda_\alpha = 1 \). This statement is also valid if random variables take continuous values; in this case, we replace the sums over discrete variables by corresponding integrals.

One can formulate the quantum analog of Statement 1.

Statement 2. The convex sum of two density matrices is the density matrix. Given the density matrix \( \rho_1 \) of a quantum state, e.g., qudit or N-level atom. Given the density matrix \( \rho_2 \) of another state of the qudit. This means that one has two Hermitian matrices \( \rho_1^j = \rho_1 \) and \( \rho_2^j = \rho_2 \), with unit trace \( \text{Tr} \rho_1 = \text{Tr} \rho_2 = 1 \) and nonnegative eigenvalues, i.e., the corresponding density operators \( \hat{\rho}_1 \) and \( \hat{\rho}_2 \) acting in the Hilbert space of the qudit state vectors have the property \( \langle \psi | \hat{\rho}_1 | \psi \rangle \geq 0 \) and \( \langle \varphi | \hat{\rho}_2 | \varphi \rangle \geq 0 \), where \( | \psi \rangle \) and \( | \varphi \rangle \) are arbitrary vectors.

Let us construct the matrix \( \rho = \lambda_1 \rho_1 + \lambda_2 \rho_2 \), where 0 ≤ \( \lambda_1, \lambda_2 \leq 1 \) and \( \lambda_1 + \lambda_2 = 1 \). One can check that \( \rho^j = \rho \) and \( \text{Tr} \rho = \lambda_1 \text{Tr} \rho_1 + \lambda_2 \text{Tr} \rho_2 = 1 \). Also for an arbitrary state vector \( | \psi \rangle \), the
corresponding density operators $\hat{\rho}$, $\hat{\rho}_1$, and $\hat{\rho}_2$ satisfy the inequality

$$\langle \psi | \hat{\rho} | \psi \rangle = \lambda_1 \langle \psi | \hat{\rho}_1 | \psi \rangle + \lambda_2 \langle \psi | \hat{\rho}_2 | \psi \rangle \geq 0;$$

this means that the matrix $\rho$ can be interpreted as the density matrix of the qubit state. Obviously, an analogous proof can be repeated for infinite-dimensional systems.

2. Qutrit-state density matrices as convex sums of artificial qubit-state density matrices

In view of Statements 1 and 2, we present generic qutrit-density matrices in the form of convex sums of specific qutrit-state density matrices with only four nonzero matrix elements. The qutrit-state density matrix $\rho$ reads $\rho = \left[ \begin{array}{ccc} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{array} \right]$, where in the generic case all matrix elements $\rho_{jk}$ are not equal to zero. We consider three specific density $3\times3$-matrices $\rho_1$, $\rho_2$, and $\rho_3$ such that the matrix elements of matrix $\rho_1$ denoted as $\rho_{11}^{(1)} = \rho_{12}^{(1)} = \rho_{13}^{(1)} = 0$. Also in the Hermitian matrices $\rho_2$ and $\rho_3$, we have zero matrix elements $\rho_{12}^{(2)} = \rho_{22}^{(2)} = \rho_{23}^{(2)} = 0$ and $\rho_{13}^{(3)} = \rho_{23}^{(3)} = \rho_{33}^{(3)} = 0$, respectively. The qutrit-state density matrices $\rho_1$, $\rho_2$, and $\rho_3$ can be presented in the probability representation of the qubit-state density matrices as follows:

$$\rho_1 = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & p_3^{(1)} & p_3^{(1)*} \\ 0 & p_3^{(1)*} & 1 - p_3^{(1)} \end{array} \right), \quad \rho_2 = \left( \begin{array}{ccc} p_3^{(2)} & 0 & p_3^{(2)*} \\ 0 & 0 & 0 \\ p_3^{(2)*} & 0 & 1 - p_3^{(2)} \end{array} \right), \quad \rho_3 = \left( \begin{array}{ccc} p_3^{(3)} & p_3^{(3)*} & 0 \\ p_3^{(3)*} & 1 - p_3^{(3)} & 0 \\ 0 & 0 & 0 \end{array} \right).$$

In view of Statement 2, we present generic qutrit-density matrices in the form of convex sum of matrices $\rho_1$, $\rho_2$, and $\rho_3$, namely, $\rho = \lambda_1 \rho_1 + \lambda_2 \rho_2 + \lambda_3 \rho_3$, where $0 \leq \lambda_1, \lambda_2, \lambda_3 \leq 1$, i.e.,

$$\rho = \left( \begin{array}{ccc} \lambda_2 p_3^{(2)} + \lambda_3 p_3^{(3)} & \lambda_3 p_3^{(3)*} & \lambda_2 p_3^{(2)*} \\ \lambda_3 p_3^{(3)} & \lambda_1 p_3^{(1)} + \lambda_3 (1 - p_3^{(3)}) & \lambda_1 p_3^{(1)*} \\ \lambda_2 p_3^{(2)} & \lambda_1 p_3^{(1)} & \lambda_1 (1 - p_3^{(1)}) + \lambda_2 (1 - p_3^{(2)}) \end{array} \right).$$

In this equation, complex numbers $p_j^{(j)} = p_1^{(j)} - 1/2 + i (p_2^{(j)} - 1/2)$; $j = 1, 2, 3$ describe six probability distributions of dichotomic random variables $(p_k^{(j)}$, $1 - p_k^{(j)})$; $j = 1, 2, 3$ and $k = 1, 2$.

According to Statement 1, all matrix elements of matrix $\rho$, which can be considered as convex sums of the probability distributions, can be interpreted as probabilities. In fact, the probability distribution determined by three nonnegative numbers $\lambda_1$, $\lambda_2$, and $\lambda_3$, where $\sum_{j=1}^3 \lambda_j = 1$, can be interpreted as the set of probability distributions $(\lambda_j, 1 - \lambda_j)$; $j = 1, 2, 3$.

Thus, in view of Statement 1 and Statement 2, we proved that matrix elements of any qutrit density matrix depending on eight real parameters can be expressed in terms of probabilities of dichotomic random variables; the sum of real nonnegative diagonal matrix elements of the density matrix is equal to unity.

3. Properties of nonnegative numbers on an example of four ones

The notion of “hidden correlations” [66,69–73] can be considered, using some relations between the positive numbers, namely, let us have four nonnegative numbers $\frac{1}{15}, \frac{3}{15}, \frac{7}{15}$ and $\frac{4}{15}$ satisfying the “normalization condition”

$$\frac{1}{15} + \frac{3}{15} + \frac{7}{15} + \frac{4}{15} = 1.$$
One can get other numbers
\[ \alpha = \frac{1}{15} + \frac{3}{15} = \frac{4}{15}, \quad \beta = \frac{7}{15} + \frac{4}{15} = \frac{11}{15} \]
and
\[ \gamma = \frac{1}{15} + \frac{7}{15} = \frac{8}{15}, \quad \delta = \frac{3}{15} + \frac{4}{15} = \frac{7}{15}, \]
which are also normalized, i.e.,
\[ \alpha + \beta = 1 \quad \text{and} \quad \gamma + \delta = 1. \]

From these initial numbers one can obtain pairs with the normalization conditions
\[ P = \frac{1}{15} \left[ \frac{1}{15} + \frac{7}{15} \right]^{-1} = \frac{1}{8} \quad \text{and} \quad Q = \frac{7}{15} \left[ \frac{1}{15} + \frac{7}{15} \right]^{-1} = \frac{7}{8}. \]

The positive numbers satisfy the condition \( P + Q = 1. \)

There exist other possibilities to construct pairs of numbers from four initial ones.

Now we clarify how this procedure is related to the probabilistic properties of a system with two subsystems or with two random variables. We see that for a given probability distribution of one random variable (four numbers) it is possible to create other probability distributions of dichotomic random variables, i.e., \((\alpha, \beta), (\gamma, \delta), \) and \((P, Q). \)

4. Four nonnegative numbers considered as an example of the probability distribution for two classical coins

Let us have two coins I and II in such a game as coin flipping, coin tossing, or heads (UP, \(\oplus\)) or tails (DOWN, \(\ominus\)), which is the practice of throwing a coin in the air and checking which side is showing when it lands, in order to choose between two alternatives (UP, \(\oplus\)) or (DOWN, \(\ominus\)) associated with the probabilities \(P_k\) or \((1 - P_k); k = 1, 2.\)

There are four possibilities
\[ \oplus\oplus \quad \oplus\ominus \quad \ominus\oplus \quad \ominus\ominus. \]

We denote the probabilities of these results as
\[ P(1, 1) = \frac{1}{15}, \quad P(1, 2) = \frac{3}{15}, \quad P(2, 1) = \frac{7}{15}, \quad P(2, 2) = \frac{4}{15}. \]

The probabilities to have the head (UP, \(\oplus\)) or the tail (DOWN, \(\ominus\)) read for the first coin
\[ P(1) = P(1, 1) + P(1, 2) = \frac{4}{15}, \quad P(2) = P(2, 1) + P(2, 2) = \frac{11}{15}, \]
for the second coin
\[ \Pi(1) = P(1, 1) + P(2, 1) = \frac{8}{15}, \quad \Pi(2) = P(1, 2) + P(2, 2) = \frac{7}{15}. \]

After introducing an analogous notation for the numbers \(P(j, k); j, k = 1, 2,\) we see that the addition of the initial numbers obtains the meaning of probabilities characterizing the situation with two random variables – positions of the first and second coins considered as a composite system containing these coins as subsystems.
The second procedure providing two pairs of nonnegative numbers like \( P = \frac{1}{8} \) and \( Q = \frac{7}{8} \) has also a probabilistic interpretation. In view of the notation introduced, we have

\[
P = \frac{1}{8} = \frac{P(1,1)}{P(1,1) + P(2,1)}, \quad Q = \frac{7}{8} = \frac{P(2,1)}{P(1,1) + P(2,1)}.
\]

We denote the first and second numbers as \( P = P(1 | 1) = \frac{1}{8} \) and \( Q = P(2 | 1) = \frac{7}{8} \).

The introduced notation for numbers \( P \) and \( Q \) provides the possibility to interpret them as conditional probabilities.

5. Bayes’ formula for systems with one random variable

Now we discuss the example of an analog of Bayes’ formula for systems without subsystems. In view of the relation obtained, we see that the number \( P = \frac{P(1,1)}{P(1,1) + P(2,1)} \equiv \frac{P(1)}{P(1) + P(3)} \) is the conditional probability for the first coin to have the position “UP” provided the second coin has position “UP” while the number \( Q = \frac{P(2,1)}{P(1,1) + P(2,1)} \equiv \frac{P(3)}{P(1) + P(3)} \) is the conditional probability that the first coin has the position “DOWN” provided the second coin has the position “UP.”

The Bayes’ formula for two coins (composite system with two subsystems) has the form

\[
P(j | k) = \frac{P(j,k)}{\sum_{j=1}^{2} P(j,k)}; \quad j,k = 1,2 \quad \text{or} \quad P(j,k) = P(j | k) \left( \sum_{j=1}^{2} P(j,k) \right).
\]

It has the following meaning: The joint probability distribution of two random variables \( P(j,k) \) is equal to the product of the conditional probability distribution of the first coin to be in the \( j \)th position (provided that the position of the second coin is the \( k \)th one) times the probability of the second coin to be in the \( k \)th position.

We can see that due to the map \( 1 \leftrightarrow 1, 1 \leftrightarrow 2, 3 \leftrightarrow 2, 1 \leftrightarrow 4, 2 \leftrightarrow 3 \), the Bayes’ formula written in terms of the probability distributions of one random variable \( P(1), P(2), P(3), \) and \( P(4) \) has the form of equalities

\[
P(1) = P(1,1) = \frac{P(1) \left[ P(1) + P(3) \right]}{P(1) + P(3)}, \quad P(3) = P(2,1) = \frac{P(3) \left[ P(1) + P(3) \right]}{P(1) + P(3)}.
\]

For generic probability distributions \( P(j,k) \) and \( P(j | k) \), we have the formula which is also written in terms of the probability distribution \( P(n) \) of one random variable rewritten in the form of Bayes’ formula for two random variables.

6. Four-level-atom states as two artificial two-level-atom states and spin-3/2 state as a state of two artificial spin-1/2 systems

First of all, in this section, we demonstrate hidden correlations in quantum states of the four-level atom or the spin-3/2 system. The four-level-atom state is determined by the \( 4 \times 4 \) density matrix \( \rho_{jk} : j,k = 1,2,3,4 \). The idea of hidden correlations means that we can interpret this matrix \( \rho \) as the density matrix of two artificial qubit states; for this, we rewrite its matrix elements in the form \( \rho_{jk,j'k'} : j,k,j',k' = 1,2 \).

This matrix can be considered as the density matrix of a composite system of two spin-1/2 particles (qubits). The density matrix of the first artificial qubit is given by the partial tracing
over indices of the second qubit, i.e.,
\[
\rho(1) = \left( \begin{array}{cc}
\rho_{11,11} + \rho_{12,12} & \rho_{11,21} + \rho_{12,22} \\
\rho_{21,11} + \rho_{22,12} & \rho_{21,21} + \rho_{22,22}
\end{array} \right) \quad \text{or} \quad \rho(1) = \left( \begin{array}{cc}
\rho_{11} + \rho_{22} & \rho_{13} + \rho_{24} \\
\rho_{31} + \rho_{42} & \rho_{33} + \rho_{44}
\end{array} \right),
\]

using the initial notation.

Analogously, the density matrix of the second artificial qubit is given by the partial tracing of the initial matrix over indices of the first qubit, i.e.,
\[
\rho(2) = \left( \begin{array}{cc}
\rho_{11,11} + \rho_{21,21} & \rho_{11,12} + \rho_{21,22} \\
\rho_{12,11} + \rho_{22,21} & \rho_{12,12} + \rho_{22,22}
\end{array} \right) \quad \text{or} \quad \rho(2) = \left( \begin{array}{cc}
\rho_{11} + \rho_{33} & \rho_{12} + \rho_{34} \\
\rho_{21} + \rho_{43} & \rho_{22} + \rho_{44}
\end{array} \right).
\]

By construction, matrices \( \rho(1) \) and \( \rho(2) \) are Hermitian, \( \rho(1) = \rho^\dagger(1) \), \( \rho(2) = \rho^\dagger(2) \), \( \text{Tr} \rho_1 = \text{Tr} \rho_2 = 1 \), and eigenvalues of these matrices are nonnegative.

Correlations of two qubits in the case, where the \( 4 \times 4 \) density matrix \( \rho_{jk,j'k'} \) is the density matrix of composite system, are characterized by the value of mutual quantum information
\[
I_q = -\text{Tr}(\rho(1) \ln \rho(1)) - \text{Tr}(\rho(2) \ln \rho(2)) + \text{Tr}(\rho \ln \rho) \geq 0.
\]

In the case of no correlations, \( I_q = 0 \).

On the other hand, even if the density matrix \( \rho \) is the density matrix of noncomposite system (like a four-level atom or spin-3/2 system), all numerical relations we employed for the case of two-qubit system are valid.

For example, if the density matrix of the four-level atom or the spin-3/2 system \( \rho_{mm'} \) reads (here, \( m, m' = \pm 1/2, \pm 3/2 \) are spin projections of the spin-3/2 particle onto z axes)
\[
\rho = \frac{1}{2} \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix},
\]
it cannot be presented in a separable form, i.e.,
\[
\rho \neq \sum_k p_k \rho_1^{(k)} \otimes \rho_2^{(k)}, \quad 1 \geq p_k \geq 0, \quad \sum_k p_k = 1,
\]

where \( \rho_1^{(k)} \) and \( \rho_2^{(k)} \) are \( 2 \times 2 \) density matrices due to the Peres–Horodecki criterion. This matrix formally corresponds to entangled state (Bell state) of two artificial qubits.

On the other hand, it is a pure superposition state of the spin-3/2 particle
\[
| \psi \rangle = \frac{1}{\sqrt{2}} (| 3/2, 3/2 \rangle + | 3/2, -3/2 \rangle).
\]

Thus, this state is entangled state of two “artificial qubits.”

7. Hidden correlations in spin-7/2 or eight-level-atom state

If the number \( N = nm\ell \), the partial tracing can be done first using the above described procedure, where we introduce the number \( \tilde{m} = m\ell \) and consider the formulated tool for \( N = nm\tilde{m} \). After this, we can apply the elaborated tool for the \( \tilde{m} \times \tilde{n} \) density matrix; this means that we can describe “hidden correlations” for three artificial qubits associated with \( N \times N \) density matrix if \( N = nm\ell \).
The example of such situation is the $8\times8$ density matrix $\rho$ of spin-$7/2$ (or eight-level atom) system; it can be considered as the density matrix of three artificial spin-$1/2$ (or three two-level atom) systems. We denote the same density matrix $\rho$ written for three qubits (three two-level atoms) as \( \rho_{j_1j_2j_3,k_1k_2k_3} \), where indices $j_1, j_2, j_3 = 1, 2$ and $k_1, k_2, k_3 = 1, 2$; it has the form

\[
\rho = \begin{pmatrix}
\rho_{111,111} & \rho_{111,121} & \rho_{111,122} & \rho_{111,211} & \rho_{111,212} & \rho_{111,221} & \rho_{111,222} \\
\rho_{121,111} & \rho_{121,121} & \rho_{121,122} & \rho_{121,211} & \rho_{121,212} & \rho_{121,221} & \rho_{121,222} \\
\rho_{211,111} & \rho_{211,121} & \rho_{211,122} & \rho_{211,211} & \rho_{211,212} & \rho_{211,221} & \rho_{211,222} \\
\rho_{221,111} & \rho_{221,121} & \rho_{221,122} & \rho_{221,211} & \rho_{221,212} & \rho_{221,221} & \rho_{221,222} \\
\rho_{222,111} & \rho_{222,121} & \rho_{222,122} & \rho_{222,211} & \rho_{222,212} & \rho_{222,221} & \rho_{222,222}
\end{pmatrix}.
\]

The matrix of such a form corresponds to the map of indices

\[
1 \leftrightarrow 111, \ 2 \leftrightarrow 112, \ 3 \leftrightarrow 121, \ 4 \leftrightarrow 122, \ 5 \leftrightarrow 211, \ 6 \leftrightarrow 212, \ 7 \leftrightarrow 221, \ 8 \leftrightarrow 222,
\]

which provides the numerical matrix $\rho$ to be written either as $\rho_{\alpha\beta}$; $\alpha, \beta = 1, 2, \ldots, 8$ or as $\rho_{j_1j_2j_3,k_1k_2k_3}$, where $j_1, j_2, j_3 = 1, 2, k_1, k_2, k_3 = 1, 2$.

It is known that, for such density matrix, the strong subadditivity condition is valid

\[-\text{Tr} \ln \rho - \text{Tr} \rho(2) \leq -\text{Tr} \rho(1,2) \ln \rho(1,2) - \text{Tr} \rho(2,3) \ln \rho(2,3).\]

If the matrix $\rho$ corresponds to the composite-system states of three qubits, in this inequality, the $2\times2$ matrix $\rho(2)$ is the density matrix of the second qubit, the $4\times4$ matrix $\rho(1,2)$ is the density matrix of the first and second qubits, and the $4\times4$ matrix $\rho(2,3)$ is the density matrix of the second and third qubits.

If the matrix $\rho$ describes one qudit (spin-$7/2$ or eight-level atom system), the same numerical matrices in this inequality are the $2\times2$ matrix $\rho(2)$ and the $4\times4$ matrices $\rho(1,2)$ and $\rho(2,3)$ describing the density matrices of artificial qubits.

For the qudit state of a noncomposite system, the inequality presented is a new relation which was not considered in the literature.

Now we are in the position to present explicit forms of matrices $\rho$, $\rho(1,2)$, $\rho(2,3)$, and $\rho(2)$. For this, we denote the $8\times8$ matrix $\rho$ using the notation $\rho_{\alpha\beta}$; $\alpha, \beta = 1, 2, \ldots, 8$ for the $8\times8$ matrix of the eight-level atom state, i.e.,

\[
\rho = \begin{pmatrix}
\rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} & \rho_{15} & \rho_{16} & \rho_{17} & \rho_{18} \\
\rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} & \rho_{25} & \rho_{26} & \rho_{27} & \rho_{28} \\
\rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} & \rho_{35} & \rho_{36} & \rho_{37} & \rho_{38} \\
\rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} & \rho_{45} & \rho_{46} & \rho_{47} & \rho_{48} \\
\rho_{51} & \rho_{52} & \rho_{53} & \rho_{54} & \rho_{55} & \rho_{56} & \rho_{57} & \rho_{58} \\
\rho_{61} & \rho_{62} & \rho_{63} & \rho_{64} & \rho_{65} & \rho_{66} & \rho_{67} & \rho_{68} \\
\rho_{71} & \rho_{72} & \rho_{73} & \rho_{74} & \rho_{75} & \rho_{76} & \rho_{77} & \rho_{78} \\
\rho_{81} & \rho_{82} & \rho_{83} & \rho_{84} & \rho_{85} & \rho_{86} & \rho_{87} & \rho_{88}
\end{pmatrix}.
\]

It is convenient to rewrite this matrix in the block form $\rho = \begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix}$. 

\[\text{doi:10.1088/1742-6596/1612/1/012011}\]
Correspondingly, following the same partial tracing rule, we obtain the 2×2 density matrix of the second and third qubits
\[ \rho(2,3) = \left( \begin{array}{cc} \rho_{11} + \rho_{22} + \rho_{55} + \rho_{66} & \rho_{13} + \rho_{57} + \rho_{24} + \rho_{68} \\ \rho_{31} + \rho_{57} + \rho_{24} + \rho_{68} & \rho_{33} + \rho_{57} + \rho_{24} + \rho_{68} \end{array} \right) \]

Finally we arrive at \( \rho(\tilde{2}) = \left( \begin{array}{cc} \rho_{11} + \rho_{12} + \rho_{13} + \rho_{14} + \rho_{15} + \rho_{16} & \rho_{13} + \rho_{14} + \rho_{57} + \rho_{58} \\ \rho_{21} + \rho_{22} + \rho_{55} + \rho_{56} & \rho_{23} + \rho_{24} + \rho_{56} + \rho_{57} + \rho_{58} \end{array} \right) \).

Using the same procedure, we can easily obtain the 4×4 density matrix \( \rho(1,2) \) of the first and second qubits
\[ \rho(1,2) = \sum_{j_k=1}^{2} \rho_{j_1j_2j_3,k_1k_2} = \sum_{j_k=1}^{2} \rho_{j_1j_2j_3,k_1k_2} \]

To obtain this matrix without any efforts, we can use a mnemonic rule which means the following.

Let us rewrite the matrix \( \rho \) in the other block form, \( \rho = \left( \begin{array}{cc} \Gamma_1 & \Gamma_2 \\ \Gamma_3 & \Gamma_4 \end{array} \right) \), where

2×2 matrices \( \Gamma_s; \ s = 1, 2, \ldots, 16 \) read

\[
\begin{align*}
\Gamma_1 &= \left( \begin{array}{cc} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{array} \right), & \Gamma_2 &= \left( \begin{array}{cc} \rho_{13} & \rho_{14} \\ \rho_{23} & \rho_{24} \end{array} \right), & \Gamma_3 &= \left( \begin{array}{cc} \rho_{15} & \rho_{16} \\ \rho_{25} & \rho_{26} \end{array} \right), & \Gamma_4 &= \left( \begin{array}{cc} \rho_{17} & \rho_{18} \\ \rho_{27} & \rho_{28} \end{array} \right), \\
\Gamma_5 &= \left( \begin{array}{cc} \rho_{31} & \rho_{32} \\ \rho_{41} & \rho_{42} \end{array} \right), & \Gamma_6 &= \left( \begin{array}{cc} \rho_{33} & \rho_{34} \\ \rho_{43} & \rho_{44} \end{array} \right), & \Gamma_7 &= \left( \begin{array}{cc} \rho_{35} & \rho_{36} \\ \rho_{45} & \rho_{46} \end{array} \right), & \Gamma_8 &= \left( \begin{array}{cc} \rho_{37} & \rho_{38} \\ \rho_{47} & \rho_{48} \end{array} \right), \\
\Gamma_9 &= \left( \begin{array}{cc} \rho_{51} & \rho_{52} \\ \rho_{61} & \rho_{62} \end{array} \right), & \Gamma_{10} &= \left( \begin{array}{cc} \rho_{53} & \rho_{54} \\ \rho_{63} & \rho_{64} \end{array} \right), & \Gamma_{11} &= \left( \begin{array}{cc} \rho_{55} & \rho_{56} \\ \rho_{65} & \rho_{66} \end{array} \right), & \Gamma_{12} &= \left( \begin{array}{cc} \rho_{57} & \rho_{58} \\ \rho_{67} & \rho_{68} \end{array} \right), \\
\Gamma_{13} &= \left( \begin{array}{cc} \rho_{71} & \rho_{72} \\ \rho_{81} & \rho_{82} \end{array} \right), & \Gamma_{14} &= \left( \begin{array}{cc} \rho_{73} & \rho_{74} \\ \rho_{83} & \rho_{84} \end{array} \right), & \Gamma_{15} &= \left( \begin{array}{cc} \rho_{75} & \rho_{76} \\ \rho_{85} & \rho_{86} \end{array} \right), & \Gamma_{16} &= \left( \begin{array}{cc} \rho_{77} & \rho_{78} \\ \rho_{87} & \rho_{88} \end{array} \right). 
\end{align*}
\]
provides it in the initial notation, namely,

\[
\rho_{(1, 2)} = \begin{pmatrix}
\rho_{11} + \rho_{22} & \rho_{13} + \rho_{24} & \rho_{15} + \rho_{26} & \rho_{17} + \rho_{28} \\
\rho_{31} + \rho_{42} & \rho_{33} + \rho_{44} & \rho_{35} + \rho_{46} & \rho_{37} + \rho_{48} \\
\rho_{51} + \rho_{62} & \rho_{53} + \rho_{64} & \rho_{55} + \rho_{66} & \rho_{57} + \rho_{68} \\
\rho_{71} + \rho_{85} = 2 & \rho_{73} + \rho_{84} & \rho_{75} + \rho_{86} & \rho_{77} + \rho_{88}
\end{pmatrix} = \begin{pmatrix}
 r_1 \\
 r_2 \\
 r_3 \\
 r_4
\end{pmatrix}.
\]

In view of the rules above formulated, we can rewrite the matrix \(\rho(2)\) through the matrix \(\rho_{(1, 2)}\) as the sum of two 2×2 blocks \(r_1 + r_4\) of the matrix \(\rho_{(1, 2)}\) and arrive at the same result as we obtained above in the case employing the matrix \(\rho(2, 3)\)

\[
\rho(2) = \begin{pmatrix}
\rho_{11} + \rho_{22} & \rho_{13} + \rho_{24} \\
\rho_{31} + \rho_{42} & \rho_{33} + \rho_{44}
\end{pmatrix} + \begin{pmatrix}
\rho_{55} + \rho_{66} & \rho_{57} + \rho_{68} \\
\rho_{75} + \rho_{86} & \rho_{77} + \rho_{88}
\end{pmatrix}.
\]

Thus, for an arbitrary eight-level atom (that is the system without subsystems), we introduced the notion of hidden correlations of artificial three two-level atom system characterized by the entropic inequality

\[-\text{Tr} \rho \ln \rho - \text{Tr} \rho(2) \ln \rho(2) \leq -\text{Tr} [\rho(1, 2) \ln \rho(1, 2)] - \text{Tr} [\rho(2, 3) \ln \rho(2, 3)].\]

8. Conclusions

We demonstrated that quantum systems without subsystems (one qudit) have the density matrices of their states with properties analogous to the properties of the density matrices of composite system states. This means that correlations in composite system states formally have analogs also for the density matrices of noncomposite system states. Some aspects of system statistics, including entropic relations, used in quantum information theory were discussed in [74–80] and in just published papers [81–86].

For these noncomposite system states, we discussed the Bayes’ formula and considered the notion of mutual information characterizing “hidden correlations” and discussed information-entropic inequalities available for one qudit states associated with artificial qudits determined by the initial density matrix.

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References

[1] Moshinsky M 1952 Phys. Rev. 88 625
[2] Man’ko V I, Moshinsky M and Sharma A 1999 1999 Phys. Rev. A 59 1809
[3] Moshinsky M 1976 Am. J. Phys. 44 1037
[4] Moshinsky M and Schuch D 2001 J. Phys. A: Math. Gen. 34 4217
[5] Moshinsky M and Queene C 1971 J. Math. Phys. 12 1772
[6] Moshinsky M 1973 SIAM J. Appl. Math. 25 193
[7] Moshinsky M, Seligman T H and Wolf K B 1972 J. Math. Phys. 13 901
[8] Dodonov V V and Man’ko V I 1989 Invariants and the Evolution of Nonstationary Quantum Systems. Proceedings of the Lebedev Physical Institute Nova Science, New York, Vol. 183
[9] Schuch D 2018 Quantum Theory from a Nonlinear Perspective. Riccati Equations in Fundamental Physics Springer, Switzerland
[10] Malkin I A, Man’ko V I and Trifonov D A 1973 J. Math. Phys. 14 576
[11] Malkin I A, Man’ko V I and Trifonov D A 1973 Phys. Rev. D 2 1371
[12] Husimi K 1953 Progr. Theoret. Phys. (Kyoto) 9 381
[13] Ermakov V 1980 “Second-order differential equation. Conditions of complete integrability” Kiev University Izvestia, Series III, 9 1 (in Russian) [English translation by Harin A O 2008 Appl. Anal. Discrete Math., 2, 123]
[14] Lewis H R and Riesenfeld W B 1969 J. Math. Phys. 10 1458
[15] Malkin I A and Man’ko V I 1968 Zh. Eksper. Teor. Fiz. 55 1014 [English translation: 1969 Sov. Phys. JETP 28 527]
[16] Dodonov V V 2017 “Coherent states in a magnetic field and their generalisation” in: J-P Antoine, F Bagarello and J-P Gazeau (Eds.) Coherent States and Their Applications: A Contemporary Panorama. Proceedings of CIRM Workshop Springer Proceedings in Physics, Book 205
[17] Malkin I A and Man’ko V I 1969 Phys. Lett. A 30 414
[18] Malkin I A and Man’ko V I 1970 Zh. Eksper. Teor. Fiz. 58 721
[19] Guerrero J and F. F. L´ opez-Ruiz F F 2015 Phys. Scr. 90 074046
[20] Akhundova E A, Dodonov V V and Man’ko V I 1982 Physica A 115 215
[21] Dodonov V V, Malkin I A and Man’ko V I 1975 Int. J. Theor. Phys. 14 37
[22] Rosas-Ortiz O 2019 “Coherent and squeezed states: Introductory review of basic notions, Properties and generalizations” in: Kuru S, Negro J and Nieto I L M (Eds.) Integrability, Supersymmetry and Coherent States CRM Series in Mathematical Physics, Springer, p. 18
[23] Castaños O, Schuch D and Rosas-Ortiz O 2013 J. Phys. A: Math. Theor. 46 075304
[24] Castaños O, Schuch D and Rosas-Ortiz O 2013 J. Phys.: Conf. Ser. 442 012058
[25] Moshinsky M 1969 Harmonic Oscillator in Modern Physics: From Atoms to Quarks 1st ed. Gordon and Breach, New York
[26] Moshinsky M and Smirnov Yu F 1996 The Harmonic Oscillator in Modern Physics, Contemporary Concepts in Physics, Harwood Academic Publishers, Amsterdam, The Netherlands, Vol. 9
[27] Markov M A, Man’ko V I and Shabad A E (Eds.) 1983 Group-Theoretical Methods in Physics. Proceedings of the Second Zvenigorod Seminar, USSR, 24–26 November 1983 Harwood Academic Publishers, Chur, Switzerland, Vols. 1, 2, 3
[28] Gel’fand I M, Graev M I and Vershik A M 1983 “The commutative model of the principal representation of the current group \( SL(2, \mathbb{R}) \) with respect to a subgroup” in: Markov M A, Man’ko V I and Shabad A E (Eds.) 1983 Group-Theoretical Methods in Physics. Proceedings of the Second Zvenigorod Seminar, USSR, 24–26 November 1983 Harwood Academic Publishers, Chur, Switzerland, Vol. 1, pp. 3–27
[29] Manin Yu A 1983 “Supersymmetry and supergravity in the superspace of null geodesics” in: Markov M A, Man’ko V I and Shabad A E (Eds.) 1983 Group-Theoretical Methods in Physics. Proceedings of the Second Zvenigorod Seminar, USSR, 24–26 November 1983 Harwood Academic Publishers, Chur, Switzerland, Vol. 3, pp. 461–468
[30] Doskoch I Ya and Man’ko M A 2019 Quantum Reports 1(2), 130
[31] Doskoch I Ya and Man’ko M A 2019 J. Russ. Laser Res. 40 1
[32] Glauber R J 1963 Phys. Rev. Lett. 10 84
[33] Sudarshan E C G 1963 Phys. Rev. Lett. 10 277
[34] Sudarshan E C G, Mathews P M and Rau Jayaseetha 1961 Phys. Rev. 121 920
[35] Kraus K 1988 States, Effects and Operations Springer, Berlin
[36] Gorini V, Kossakowski A and Sudarshan E C G 1976 J. Math. Phys. 17 821
[37] Lindblad G 1976 Commun. Math. Phys. 48 119 (1976)
[38] Man’ko V I, Marmo G, Sudarshan E C G and Zaccaria F 1999 J. Russ. Laser Res. 20 421
[39] Sudarshan E C G 2003 J. Russ. Laser Res. 24 195
[40] Glauber R and Man’ko V I 1984 Sov. Phys. JETP 60 450
[41] Glauber R and Man’ko V I 1986 “Damping and fluctuations in systems of connected quantum oscillators” in: Basov N G (Ser. Ed.), Group Theory, Gravitation and Elementary Particle Physics. Proceedings of the P. N. Lebedev Physical Institute Nauka, Moscow, Vol. 167 [English translation: 1987 Nova Science, Commack, New York, Vol. 167]
[42] Wigner E 1932 Phys. Rev. 40 749
[43] Husimi K 1940 Proc. Phys. Math. Soc. Jpn. 22 264
[44] Kano Y 1965 J. Math. Phys. 6 1913
[45] Mancini S, Man’ko V I and Tombesi P 1996 Phys. Lett. A 213, 1
[46] Marmo G and Pascazio S 2019 Open Sys. Inf. Dyn. 26 1950011
[47] Modi K 2019 Open Sys. Inf. Dyn. 26 1950013
[48] Man’ko M A and Man’ko V I 2019 Open Sys. Inf. Dyn. 26 1950016
[49] Grimaldo R, Man’ko V I, Man’ko M A and Messina A 2020 Phys. Scr. 95 024004
[50] Monir H B and Iqbal S 2020 J. Russ. Laser Res. 41 1
[51] Chernega V N and Man’ko O V 2020 J. Russ. Laser Res. 41 11
[52] Editorial Note 2019 Open Sys. Inf. Dyn. 26 1902001
[53] Scully M, Man’ko V I, Castaños O and Man’ko M A (Eds.) 2020 Phys. Scr. Focus Issue in memory of Roy Glauber [https://iopscience.iop.org/journal/1402-4896/page/Focus-Issue-in-memory-of-Roy-Glauber]
[54] Biedenharn L C 1989 J. Phys. A: Math. Theor. 22 L873
[55] Macfarlane A J 1989 J. Phys. A: Math. Theor. 22 4581
[56] Man’ko V I, Marmo G, Sudarshan E C G and Zaccaria F 1997 Phys. Scr. 55 528
[57] Barut A O 1964 Phys. Rev. 135 B839
[58] Malkin I A and Man’ko V I 1966 Sov. Phys. JETP Lett. 2 146
[59] Barut A O and Kleinert H 1967 Phys. Rev. 156 1546
[60] Barut A, Neeman Y and Bohm A 1988 Dynamical Groups and Spectrum Generating Algebras World Scientific, Singapore, Vols. 1 & 2
[61] Barut A O 1972 “Group structure of the periodic system” in: Wybourne B G (Ed.) Proceedings of the Rutherford Centennial Symposium “The Structure of Matter, Christchurch, New Zealand, July 7–9, 1971 University of Christchurch Press, Christchurch, New Zealand, p. 126
[62] Kondelchenko B G and Rumer Yu B 1979 Uspekhi Fiz. Nauk, 129, 339 [English translation: 1979 Sov. Phys. Usp. 22 837]
[63] Kibler M 1989 J. Mol. Struct. (Theochem.) 187 83
[64] Rumer Yu B and Fet A I 2004 Theor. Math. Phys. 103 1221
[65] Dodonov V V, Kim Y S, Man’ko M A, Man’ko V I and Vourdas A 2007 J. Russ. Laser Res. 28 404
[66] Man’ko M A 2018 J. Phys.: Conf. Ser. 1071 conference 1, 012015
[67] Man’ko M A and Man’ko V I 2018 Entropy 20(9) 692
[68] Man’ko M A and Man’ko V I 2014 Int. J. Quantum Inf. 12 156006
[69] Man’ko M A and Man’ko V I 2013 J. Russ. Laser Res. 34 203
[70] Man’ko M A and Man’ko V I 2014 Phys. Scr. T160 014030.
[71] Man’ko M A 2013 Phys. Scr. T153 014045
[72] Man’ko M A and Man’ko V I 2015 Entropy 17 2876
[73] Man’ko M A and Man’ko V I 2013 J. Phys.: Conf. Ser. 442 012008
[74] Man’ko M A, Man’ko V I, Marmo G, Simoni A and Ventriglia F 2014 J. Russ. Laser Res. 35 79
[75] Man’ko M A and Man’ko V I 2015 Entropy 17 2876
[76] Man’ko M A, Man’ko V I and Marmo G 2015 Nuovo Cimento C 38 iss. 5, paper 167
[77] Man’ko M A and Man’ko V I 2017 J. Russ. Laser Res. 38 211
[78] Man’ko M A and Man’ko V I 2018 J. Russ. Laser Res. 39 1
[79] Man’ko M A and Man’ko V I 2018 Phys. Scr. 93 084002
[80] López-Saldivar J A, Castaños O, Nahmad-Achar E, López-Peña R, Man’ko M A and Man’ko V I 2018 Entropy 20(9) 630
[81] López-Saldivar J A, Castaños O, Man’ko M A and Man’ko V I 2019 Quantum Inf. Process. 18 210
[82] López-Saldivar J A, Castaños O, Man’ko M A and Man’ko V I 2019 Entropy 21 736
[83] Man’ko M A and Man’ko V I 2020 Int. J. Quantum Inf. 18 1941021
[84] Andreev V A, Man’ko M A and Man’ko V I 2020 Phys. Lett. A 384 126349
[85] López-Saldivar J A, Man’ko M A and Man’ko V I 2020 Entropy 22(5) 586
[86] Adam P, Andreev V A, Man’ko M A, Man’ko V I and Mechler M 2020 Symmetry 12, in press