TWO-SIDED WEIGHTED BOUNDS ON FUNDAMENTAL SOLUTION TO FRACTIONAL SCHRÖDINGER OPERATOR

D. KINZEBULATOV AND YU. A. SEMÈNOV

Abstract. We establish sharp two-sided weighted bounds on the fundamental solution to the fractional Schrödinger operator using the method of desingularizing weights.

In [MS0], Milman and Semènov developed an approach to study of the integral kernels of semigroups which are not necessarily ultracontractive by transferring them to appropriately chosen weighted spaces where they become ultracontractive [MS1, MS2]. In the special case of the Schrödinger semigroup generated by $-\Delta - V$, with potential $V(x) = \delta (d-2)^2 |x|^{-2}$, $0 < \delta \leq 1$, $d \geq 3$, having a critical-order singularity at $x = 0$ (which makes invalid the standard two-sided Gaussian bounds on its integral kernel) this method yields sharp two-sided weighted bounds on the integral kernel.

In [KSSz], we employed the method of desingularizing weights to establish sharp two-sided weighted bounds on the fundamental solution to the non-local operator $(-\Delta)^{\frac{\alpha}{2}} + b \cdot \nabla$, $b(x) = \delta (d-\alpha)^{-2} c_{\alpha}^{-2} |x|^{-\alpha} x$, $0 < \delta < 1$, $1 < \alpha < 2$, $d \geq 3$, where $c_{\alpha} := c(\frac{\alpha}{2}, 2, d)$,

$$c(\alpha, p, d) := \frac{\gamma(d p - \alpha)}{\gamma(d)}$$

$$\gamma(\alpha) := \frac{2^{\alpha} \pi^{\frac{d}{2}} \Gamma(\frac{d}{2})}{\Gamma(d - \alpha)}$$

In this paper, we specify our arguments in [KSSz] to the operator $(-\Delta)^{\frac{\alpha}{2}} - V$, $V(x) = \delta c_{\alpha}^{-2} |x|^{-\alpha}$, $0 < \delta < 1$, $0 < \alpha < 2$, and obtain sharp two-sided weighted bounds on its fundamental solution. These bounds are known for $0 < \delta \leq 1$, see [BGJP], where the authors use a different technique. Concerning $(-\Delta)^{\frac{\alpha}{2}} + c |x|^{-\alpha}$, $c > 0$, see [CKSV] and [JW].

1. The method of desingularizing weights relies on two assumptions: the Sobolev embedding property, and a “desingularizing” $(L^1, L^1)$ bound on the weighted semigroup. Let $X$ be a locally compact topological space and $\mu$ a $\sigma$-finite Borel measure on $X$. Let $\Lambda$ be a non-negative selfadjoint operator in the (complex) Hilbert space $L^2 = L^2(X, \mu)$ with the inner product $\langle f, g \rangle = \int_X f \overline{g} d\mu$. We assume that $\Lambda$ possesses the Sobolev embedding property:

There are constants $j > 1$ and $c_S > 0$ such that, for all $f \in D(\Lambda^{\frac{j}{2}})$,

$$c_S \|f\|_{L^2_j} \leq \|\Lambda^{\frac{j}{2}} f\|_{L^2}$$

(M1)
but $e^{-tA} \upharpoonright L^1 \cap L^2$, $t > 0$, cannot be extended by continuity to a bounded map on $L^1$ and the ultracontractive estimate

$$\|e^{-tA}f\|_\infty \leq c_t\|f\|_1, \ f \in L^1 \cap L^\infty, \ t > 0$$

is not valid.

In this case we will be assuming that there exists a family of real valued weights $\varphi = \{\varphi_s\}_{s > 0}$ on $X$ such that, for all $s > 0$,

$$\varphi_s, \ 1/\varphi_s \in L^2_{\text{loc}}(X, \mu) \quad \tag{M2}$$

and there exists a constant $c_1$ independent of $s$ such that, for all $0 < t \leq s$,

$$\|\varphi_se^{-tA}\varphi_s^{-1}f\|_1 \leq c_1\|f\|_1, \ f \in \mathcal{D} := \varphi_sL^\infty_{\text{con}}(X, \mu). \quad \tag{M3}$$

**Theorem A (MS2).** In addition to (M1) – (M3) assume that

$$\inf_{s > 0, x \in X} |\varphi_s(x)| \geq c_0 > 0. \quad \tag{M4}$$

Then $e^{-tA}, t > 0$ are integral operators, and there is a constant $C = C(j, c_s, c_1, c_0)$ such that, for all $t > 0$ and $\mu$ a.e. $x, y \in X$,

$$|e^{-tA}(x, y)| \leq Ct^{-j'}|\varphi_t(x)\varphi_t(y)|, \ j' = j/(j - 1). \quad \tag{NIEw}$$

For the sake of completeness, we recall the proof of Theorem A in Appendix A.

In applications of Theorem A to concrete operators the main difficulty consists in verification of the $(L^1, L^1)$ bound (M3). In MS2, (M3) is proved for the Schrödinger operator by means of the theory of m-sectorial operators and the Stampacchia criterion in $L^2$. However, attempts to apply that argument to $(-\Delta)^{\frac{\alpha}{2}}$, $\alpha < 2$, are quite problematic since $(-\Delta)^{\frac{\alpha}{2}}$ lacks the local properties of $-\Delta$. In [KSSz], we developed a new approach to the proof of (M3) by means of the Lumer-Phillips theorem applied to specially constructed $C_0$ semigroups in $L^1$ which approximate $\varphi_se^{-tA}\varphi_s^{-1}$. Thus, in contrast to MS2, where the $(L^1, L^1)$ bound is proved using the $L^2$ theory, here we stay within the $L^1$ theory. For $\alpha = 2$, the approximation semigroups are constructed by replacing $|x|$ by $|x|_\varepsilon = \sqrt{|x|^2 + \varepsilon}$, $\varepsilon > 0$, both in the potential and in the weights, see below. For $\alpha < 2$, the construction of the approximation semigroups is more subtle, and is a key observation.

2. We now state our result on $(-\Delta)^{\frac{\alpha}{2}} - V$ in detail. According to the Hardy-Rellich inequality $\|(-\Delta)^{\frac{\alpha}{2}}f\|_2^2 \geq c^{-2}(\frac{\alpha}{2}, 2, d)|x|^{-\frac{\alpha}{2}}f\|_2^2$ (see [KPS] Lemma 2.7) the form difference $\Lambda = (-\Delta)^{\frac{\alpha}{2}} - V$ is well defined [Ka] Ch.VI, sect 2.5).

Define $\beta$ by $\delta c_\alpha^{-2} = \frac{\gamma(\beta)}{\alpha(\beta - \alpha)}$, and let $\varphi(x) \equiv \varphi_s(x) = \eta(s^{-\frac{1}{\alpha}}|x|)$, where $\eta \in C^2(\mathbb{R} - \{0\})$ is such that

$$\eta(r) = \begin{cases} \ r^{-d+\beta}, & 0 < r < 1, \\ \ \frac{1}{2}, & r \geq 2. \end{cases}$$

**Theorem 1.** Under constraints $0 < \delta < 1$ and $0 < \alpha < 2$, $e^{-tA}$ is an integral operator for each $t > 0$. The weighted Nash initial estimate

$$e^{-tA}(x, y) \leq ct^{-\frac{d}{2}}\varphi_t(x)\varphi_t(y), \ \ c = c_{d, \delta, \alpha},$$

is valid for all $t > 0$, $x, y \in \mathbb{R}^d - \{0\}$. 
Proof of Theorem 7. We verify the assumptions of Theorem A:

(M1) follows from the Hardy-Rellich inequality and the uniform Sobolev inequality \( \|(-\Delta)^{\frac{1}{2}} f\|_{2} \geq c_{S} \|f\|_{2j}, j = \frac{d}{d-\alpha}. \)

(M2), (M3) are immediate from the definition of \( \varphi_{\varepsilon}. \)

(M3) Our goal is to prove the following \((L^{1}, L^{1})\) bound:

\[
\|\varphi e^{-t\Lambda_{\varepsilon}} \varphi^{-1} h\|_{1} \leq e^{\frac{c_{s}}{\varepsilon}} \|h\|_{1}, \quad h \in L^{1} \cap L^{2}, \quad t > 0. \tag{\(\ast\)}
\]

Proof of \(\ast.\) In \(L^{1}\) define operator \(\Lambda_{\varepsilon} = (-\Delta)^{\frac{1}{2}} - V_{\varepsilon}, V_{\varepsilon}(x) = \delta c_{\alpha}^{-2} |x|^{-\alpha}, \varepsilon > 0, D(\Lambda_{\varepsilon}) = D((-\Delta)^{\frac{1}{2}}), \)

\[
Q = \phi_{n}\Lambda_{\varepsilon}^{\alpha} \phi_{n}^{-1}, \quad D(Q) = \phi_{n}D(\Lambda_{\varepsilon}), \quad F_{\varepsilon,n}^{t} = \phi_{n}e^{-t\Lambda_{\varepsilon}^{\alpha} \phi_{n}^{-1}}, \quad \phi_{n}(x) = e^{-\frac{\Lambda_{\varepsilon}}{n}} \varphi(x), \quad n = 1, 2, \ldots
\]

Here \(\phi_{n}, D(\Lambda_{\varepsilon}) := \{\phi_{n}u \mid u \in D(\Lambda_{\varepsilon})\}. \) We also note that \(e^{-t(-\Delta)^{\frac{1}{2}}}, e^{-t\Lambda_{\varepsilon}} : M \to M \) where \(M = C_{u}\) or \(M = L^{1}; \) and \( \varphi = \varphi^{(1)} + \varphi^{(u)}, \) \( \varphi^{(1)} \in D((-\Delta)^{\frac{1}{2}}), \varphi^{(u)} \in D((-\Delta)^{\frac{1}{2}}_{c}). C_{\mu} \equiv C_{u}(\mathbb{R}^{d}) \) stands for the Banach space of uniformly continuous functions endowed with the supremum norm.

Since \(\phi_{n}, \phi_{n}^{-1} \in L_{\infty}, \) the operators \(Q, F_{\varepsilon,n}^{t}\) are well defined.

1. Clearly, \(F_{\varepsilon,n}^{t}\) is a quasi bounded \(C_{0}\) semigroup in \(L^{1}, \) say \(e^{-tG}. \) Set

\[
M := \phi_{n}(1 + (-\Delta)^{\frac{1}{2}})^{-1}[L^{1} \cap C_{u}] = \phi_{n}(\lambda_{\varepsilon} + \Lambda_{\varepsilon}^{-1})^{-1}[L^{1} \cap C_{u}], \quad 0 < \lambda_{\varepsilon} \in \rho(-\Lambda_{\varepsilon}).
\]

Clearly, \(M\) is a dense subspace of \(L^{1}, M \subset D(Q)\) and \(M \subset D(G). \) Moreover, \(Q \upharpoonright M \subset C. \) Indeed, for \(f = \phi_{n}u \in M,\)

\[
Gf = s - L^{1} - \lim_{t\downarrow 0}(1 - e^{-tG})f = \phi_{n}s - L^{1} - \lim_{t\downarrow 0}(1 - e^{-t\Lambda_{\varepsilon}})u = \phi_{n}\Lambda_{\varepsilon}u = Qf.
\]

Thus \(Q \upharpoonright M\) is closable and \(\tilde{Q} := (Q \upharpoonright M)^{\text{clos}} \subset G.\)

Next, let us show that \(R(\lambda_{\varepsilon} + \tilde{Q})\) is dense in \(L^{1}. \) If \(\langle(\lambda_{\varepsilon} + \tilde{Q})h, v\rangle = 0\) for all \(h \in D(\tilde{Q})\) and some \(v \in L^{\infty}, \|v\|_{\infty} = 1,\) then taking \(h \in M\) we would have \(\langle(\lambda_{\varepsilon} + \tilde{Q})\phi_{n}(\lambda_{\varepsilon} + \Lambda_{\varepsilon}^{-1})^{-1}g, v\rangle = 0, g \in L^{1} \cap C_{u},\) or \(\langle\phi_{n}g, v\rangle = 0.\) Choosing \(g = e^{\frac{2}{\varepsilon}}(\chi_{m}v),\) where \(\chi_{m} \in C_{c}^{\infty}\) with \(\chi_{m}(x) = 1\) when \(x \in B(0, m),\) we would have \(\lim_{n\uparrow \infty}\langle\phi_{n}g, v\rangle = \langle\phi_{n}\chi_{m}, |v|^{2}\rangle = 0,\) and so \(v \equiv 0.\) Thus, \(R(\lambda_{\varepsilon} + \tilde{Q})\) is dense in \(L^{1}.\)

2. The main step:

**Proposition 1.** There is a constant \(\tilde{c} = \tilde{c}(d, \alpha, \delta)\) such that

\[
\lambda + \tilde{Q} \text{ is accretive whenever } \lambda \geq \tilde{c}s^{-1}.
\]

Taking Proposition \(\|\) for granted we immediately establish the bound

\[
\|e^{-tG}\|_{1 \rightarrow 1} \equiv \|\phi_{n}e^{-t\Lambda_{\varepsilon}^{\alpha} \phi_{n}^{-1}}\|_{1 \rightarrow 1} \leq e^{\omega t}, \quad \omega = \tilde{c}s^{-1}. \tag{\(\ast\ast\)}
\]

Indeed, the facts: \(\tilde{Q}\) is closed and \(R(\lambda_{\varepsilon} + \tilde{Q})\) is dense in \(L^{1}\) together with Proposition \(\|\) imply \(R(\lambda_{\varepsilon} + \tilde{Q}) = L^{1}.\) But then, by the Lumer-Phillips Theorem, \(\lambda + \tilde{Q}\) is the (minus) generator of a contraction \(C_{0}\) semigroup, and \(\tilde{Q} = G\) due to \(\tilde{Q} \subset G.\) Incidentally, \(M\) is a core of \(G.\)

In turn, \(\ast\ast\) easily yields

\[
\|\varphi e^{-t\Lambda_{\varepsilon}^{\alpha} \varphi^{-1}} h\|_{1} \leq e^{\omega t} \|h\|_{1}, \quad h \in L^{1} \cap L^{2}. \tag{\(\ast\ast\ast\)}
\]

Indeed, \(\ast\) implies that \(\lim_{n \uparrow \infty} \|\phi_{n}e^{-t\Lambda_{\varepsilon}^{\alpha} v\varepsilon}\|_{1} \leq e^{\omega t} \lim_{n \uparrow \infty} \|\phi_{n}v\|_{1}\) for all \(v \in L^{1} \cap L^{2}.\) But

\[
\lim_{n \uparrow \infty} \|\phi_{n}v\|_{1} = \lim_{n \uparrow \infty} \langle \varphi, e^{-\frac{\Lambda_{\varepsilon}}{n} |v|}\rangle = \langle \varphi, |v|\rangle < \infty,
\]
\[ \lim_{n \to \infty} \| \phi_n e^{-t \Delta^\varepsilon} v \|_1 = \lim_{n \to \infty} \langle \varphi, e^{-\frac{\Delta^\varepsilon}{n}} \rangle_{1} = \langle \varphi, e^{-t \Delta^\varepsilon} v \rangle < \infty. \]

Therefore, taking \( v = \varphi^{-1}h \) we arrive at (**) Finally, it is seen that \( \varphi e^{-t \Delta^\varepsilon} \varphi^{-1} \) preserves positivity, so (●) follows from (**) by noticing that \( e^{-t \Delta^\varepsilon} |g| \uparrow e^{-t \Delta} |g| \) \( L^d \) a.e.

Let us write down a simple consequence of (**):

**Corollary 1.** For all \( t > 0, x \in \mathbb{R}^d - \{0\} \) and all small \( \varepsilon > 0 \), there is a constant \( \hat{c} \), such that

\[ e^{-t \Delta^\varepsilon} \varphi_t \leq \hat{c} \varphi_t \text{ and } \langle e^{-t \Delta^\varepsilon} (x, \cdot) \rangle \leq 2 \hat{c} \varphi_t(x). \]

**Proof of Proposition 11** First we note that, for \( f = \phi_n u \in M \),

\[ \langle Qf, f \rangle = \langle \phi_n \Delta^\varepsilon u, f \rangle = \lim_{t \downarrow 0} \langle \phi_n (1 - e^{-t \Delta^\varepsilon}) u, f \rangle, \]

\[ \Re \langle Qf, f \rangle \geq \lim_{t \downarrow 0} \langle (1 - e^{-t \Delta^\varepsilon}) \rangle_{1} \langle u, \phi_n \rangle \]

\[ = \langle \Lambda^\varepsilon e^{-\frac{\Delta^\varepsilon}{n}} u, \varphi \rangle. \]

Let us emphasize that \( e^{-t \Delta^\varepsilon} \) is a holomorphic semigroup due to the Hille Perturbation Theorem (see e.g. [Kal Ch. IX, sect. 2.2]).

We are going to estimate \( J := \langle \Lambda^\varepsilon e^{-\frac{\Delta^\varepsilon}{n}} u, \varphi \rangle = \langle \varphi, \Lambda^\varepsilon e^{-\frac{\Delta^\varepsilon}{n}} \varphi \rangle \) from below using the equality

\[ (-\Delta)^{\frac{\varepsilon}{2}} \varphi = -I_{2-\alpha} \Delta \varphi, \]

where \( I_v \equiv (-\Delta)^{-\frac{\varepsilon}{2}} \).

Since \( e^{-t \Delta^\varepsilon} \) is a \( C_0 \) semigroup in \( L^1 \) and \( C_u, \) and \( \varphi = \varphi(1) + \varphi(u), \varphi(1) \in D((-\Delta)^{\frac{\varepsilon}{2}}), \varphi(u) \in D((-\Delta)^{\frac{\varepsilon}{2}}), \) \( \Lambda^\varepsilon \varphi \) is well defined and belongs to \( L^1 + C_u = \{ w + v \mid w \in L^1, v \in C_u \} \).

Using the equality \( (-\Delta)^{\frac{\varepsilon}{2}} \varphi_1 = V \varphi_1, \) where \( \varphi_1(x) = |x|^{-d+\beta} \) (see e.g. [KPS]), we have

\[ (-\Delta)^{\frac{\varepsilon}{2}} \varphi_1 = V \varphi_1 - I_{2-\alpha} \Delta (\varphi_1 - \varphi_1) = V \varphi_1 - I_{2-\alpha} 1_{B^c(0,1)} (\varphi_1 - \varphi_1). \]

Routine calculation shows that \( -I_{2-\alpha} (1_{B^c(0,1)} \Delta (\varphi_1 - \varphi_1)) \geq -C_1 \) for a constant \( C_1 \).

Since \( \Lambda^\varepsilon \varphi_1 = (-\Delta)^{\frac{\varepsilon}{2}} \varphi_1 - V \varphi_1 \) and \( V \varphi_1 - V \varphi_1 \geq -V \varepsilon (\varphi_1 - \varphi_1) \geq -\delta \alpha^{-2} \), we obtain by scaling the bound

\[ J = \langle e^{-\frac{\Delta^\varepsilon}{n}} u, \varphi \rangle \geq - (\delta \alpha^{-2} + C_1) s^{-1} \| e^{-\frac{\Delta^\varepsilon}{n}} \|_{1 \to 1} \| f^{-1} \|_1, \]

or due to \( \phi_n \geq \frac{1}{2}, \)

\[ J \geq -2Cs^{-1} \| e^{-\frac{\Delta^\varepsilon}{n}} \|_{1 \to 1} \| f \|_1, \quad C = C_1 + \delta \alpha^{-2}. \]

Noticing that \( \| e^{-\frac{\Delta^\varepsilon}{n}} \|_{1 \to 1} \leq c \varepsilon e^{-\frac{\Delta^\varepsilon}{n}} = 1 + o(n) \) and taking \( \lambda = 3Cs^{-1} \) we arrive at

\[ \Re \langle (\lambda + Q) f, \frac{f}{|f|} \rangle \geq 0 \quad f \in M. \]

Clearly, the latter holds for all \( f \in D(\tilde{Q}) \).

The proof of (●) is completed. We have verified all the assumptions of (M_1)-(M_4) of Theorem A. The latter now yields the assertion of Theorem [11]
Having at hand Theorem I and Corollary I it is a simple matter to obtain the upper and lower bounds of the form
\[ e^{-tA}(x, y) \approx e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x, y)\varphi_t(x)\varphi_t(y). \]
Here \( e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x, y) \approx t^{-\frac{d}{2}} \wedge \frac{t}{|x-y|^d}. \) \((a(z) \approx b(z)\) means that \(c^{-1}b(z) \leq a(z) \leq cb(z)\) for some constant \(c > 1\) and all admissible \(z\).

**Proof of upper bound** \( e^{-tA}(x, y) \leq Ce^{-tA}(x, y)\varphi_t(x)\varphi_t(y) \quad (t > 0, x, y \neq 0).\) (For brevity here and below \((-\Delta)^{\frac{\alpha}{2}} := A\).)

By scaling, it suffices to consider \( t = 1. \) Since \( e^{-A}(x, y) \approx 1 \wedge |x-y|^{-d-\alpha} \quad (x \neq y),\) Theorem I yields, for \(|x|, |y| \leq 2R,
\[ e^{-\Lambda^e}(x, y) \leq CRe^{-A}(x, y)\varphi(x)\varphi(y), \quad (\varphi \equiv \varphi_1) \]
By symmetry, it remains to prove this estimate for \(|x| \leq |y|, |y| > 2R, |z| \leq R\) and \(0 \leq \tau < 1,
\[ e^{-(1-\tau)A}(z, y) \leq e^{-A}(x, y). \]
Thus, by the Duhamel formula
\[ e^{-\Lambda^e} = e^{-A} + \int_0^1 e^{-\tau\Lambda^e}V_{\epsilon}e^{-(1-\tau)A}d\tau, \]

\[ e^{-\Lambda^e}(x, y) \leq e^{-A}(x, y)\left(1 + \int_0^1 e^{-\tau\Lambda^e}V_{\epsilon}(x)d\tau\right) + \int_0^1 \langle e^{-\tau\Lambda^e}(x, z)V_{\epsilon}(z)1_{B^c(0, R)}(z)e^{-(1-\tau)A}(z, y)\rangle_z d\tau \]
\[ \leq e^{-A}(x, y)\left(1 + \int_0^1 e^{-\tau\Lambda^e}V_{\epsilon}(x)d\tau\right) + V(R)\int_0^1 \langle e^{-\tau\Lambda^e}(x, z)e^{-(1-\tau)A}(z, y)\rangle_z d\tau. \]

Now fix \( R \) by \( \delta c_\alpha^{-2} R^{-\alpha} = \frac{1}{2}. \) Then
\[ V(R)\int_0^1 \langle e^{-\tau\Lambda^e}(x, z)e^{-(1-\tau)A}(z, y)\rangle_z d\tau \leq \frac{1}{2} \int_0^1 \langle e^{-\tau\Lambda^e}(x, z)e^{-(1-\tau)\Lambda^e}(z, y)\rangle_z d\tau = \frac{1}{2} e^{-\Lambda^e}(x, y), \]
and so
\[ \frac{1}{2} e^{-\Lambda^e}(x, y) \leq e^{-A}(x, y)\left(1 + \int_0^1 e^{-\tau\Lambda^e}V_{\epsilon}(x)d\tau\right). \]
Next, by the Duhamel formula and Corollary I
\[ 1 + \int_0^1 e^{-\tau\Lambda^e}V_{\epsilon}(x)d\tau = \langle e^{-\Lambda^e}(x, \cdot) \rangle \leq 2e^\epsilon \varphi(x), \]
and hence
\[ e^{-\Lambda^e}(x, y) \leq 4e^\epsilon e^{-A}(x, y)\varphi(x) \leq 8e^\epsilon e^{-A}(x, y)\varphi(x)\varphi(y). \]
Finally, setting \( C = C_R \vee (8e^\epsilon) \) and using \( e^{-\Lambda^e}|f| \uparrow e^{-\Lambda}|f| \) we end the proof of the upper bound.

**Proof of lower bound** \( e^{-tA}(x, y) \geq Ce^{-tA}(x, y)\varphi_t(x)\varphi_t(y) \quad (C > 0, x, y \neq 0).\)

**Proposition 2.** Define \( g = \varphi h, \varphi \equiv \varphi_s, 0 \leq h \in S\)-the L.Schwartz space of test functions. There is a constant \( \hat{\mu} > 0 \) such that, for all \( 0 < t \leq s,\)
\[ e^{-\frac{\hat{\mu}}{2}t}(g) \leq \langle \varphi e^{-tA}\varphi^{-1}g \rangle. \]
Proof of Proposition 2. Set \( g_n = \phi_n h, \phi_n(x) = e^{-\frac{\Lambda^\varepsilon}{\alpha} x} \), \( \varphi \equiv \varphi_\alpha \). Let \( \mu > 0 \) be a constant. Then \( \mu = \frac{\mu_1}{s} \)

\[
\langle g_n \rangle - \langle \phi_n e^{-t(\Lambda^\varepsilon - \mu)} h \rangle = -\mu \int_0^t \langle \varphi, e^{-\tau(\Lambda^\varepsilon - \mu)} e^{-\frac{\Lambda^\varepsilon}{\alpha} x} h \rangle d\tau + \int_0^t \langle \varphi, \Lambda^\varepsilon e^{-\tau(\Lambda^\varepsilon - \mu)} e^{-\frac{\Lambda^\varepsilon}{\alpha} x} h \rangle d\tau.
\]

Note that \( \Lambda^\varepsilon \varphi = \Lambda^\varepsilon \dot{\varphi} + \Lambda^\varepsilon (\varphi - \dot{\varphi}) = 1_{B(0,1)}(V - V_\varepsilon)\varphi + v_\varepsilon \), where \( \dot{\varphi}(x) = (s^{-\frac{1}{T}} |x|)^{-d+\beta} \). Routine calculation shows that \( \|v_\varepsilon\| \lesssim \frac{\mu_1}{s}, \mu_1 \neq \mu_1(\varepsilon) \). Thus

\[
\int_0^t \langle v_\varepsilon, e^{-\tau(\Lambda^\varepsilon - \mu)} e^{-\frac{\Lambda^\varepsilon}{\alpha} x} h \rangle d\tau \leq \frac{\mu_1}{s} \int_0^t \langle e^{-\tau(\Lambda^\varepsilon - \mu)} e^{-\frac{\Lambda^\varepsilon}{\alpha} x} h \rangle d\tau \leq \frac{2\mu_1}{s} \int_0^t \langle \varphi, e^{-\tau(\Lambda^\varepsilon - \mu)} e^{-\frac{\Lambda^\varepsilon}{\alpha} x} h \rangle d\tau.
\]

Taking \( \mu = 2\mu_1 \), we have

\[
\langle g_n \rangle - \langle \phi_n e^{-t(\Lambda^\varepsilon - \mu)} h \rangle \leq \int_0^t \langle 1_{B(0,1)}(V - V_\varepsilon)\varphi, e^{-(\tau+\frac{1}{\varepsilon})\Lambda^\varepsilon} h \rangle e^{\mu \tau} d\tau, \text{ or sending } n \to \infty,
\]

\[
\langle g \rangle - e^{\frac{\mu_1}{\varepsilon}} \langle \varphi e^{-t\Lambda^\varepsilon} h \rangle \leq e^{\frac{\mu_1}{\varepsilon}} \int_0^t \langle 1_{B(0,1)}(V - V_\varepsilon)\varphi, e^{-\tau\Lambda^\varepsilon} h \rangle d\tau.
\]

Set \( W_\varepsilon = 1_{B(0,1)}(V - V_\varepsilon)\varphi^2 \) and \( F^\varepsilon = \varphi e^{-t\Lambda^\varepsilon} \varphi^{-1} \). Note that \( W_\varepsilon \in L^1 \) due to \( 2(d - \beta) + \alpha < d \), and \( \|F^\varepsilon f\|_1 \leq e^{\frac{\mu_1}{\varepsilon}} \|f\|_1, f \in L^1 \) due to Proposition 1. Therefore,

\[
\int_0^t \langle 1_{B(0,1)}(V - V_\varepsilon)\varphi, e^{-\tau\Lambda^\varepsilon} h \rangle d\tau = \int_0^t \langle F^\varepsilon W_\varepsilon, \varphi^{-1} h \rangle \leq 2e^{\frac{\mu_1}{\varepsilon}} \|W_\varepsilon\|_1 \|h\|_\infty \to 0 \text{ as } \varepsilon \downarrow 0.
\]

We also need the following consequence of the upper bound and Proposition 2.

**Corollary 2.** Fix \( t > 0 \). Set \( g := \varphi h, \varphi = \varphi_\alpha, 0 \leq h \in S \) with sprt \( h \in B(0,R_0) \) for some \( R_0 < \infty \). Then there are \( 0 < r_t < R_0 \forall t^\frac{\alpha}{2} < R_t, R_0 \) such that, for all \( r \in [0,r_t] \) and \( R \in [2R_t, R_0, \infty] \),

\[
e^{-\frac{\mu_1}{\varepsilon}} \langle g \rangle \leq \langle 1_{R,R_0} \varphi e^{-t\Lambda^\varepsilon} \varphi^{-1} g \rangle, \quad 1_{R,R_0} := 1_{B(0,R)} - 1_{B(0,r)}, \quad 1_{R,0} := 1_{B(0,R)}.
\]

In particular, \( e^{-\frac{\mu_1}{\varepsilon}} \varphi_t(x) \leq e^{-t\Lambda^\varepsilon} \varphi_1 \) for every \( x \in B(0,R_0) \).

**Proof of Corollary 2.** By the upper bound,

\[
\langle 1_{B(0,r)} \varphi e^{-t\Lambda^\varepsilon} \varphi^{-1} g \rangle \leq C \langle 1_{B(0,r)} \varphi^2, e^{-t\Lambda^\varepsilon} g \rangle
\]

\[
\leq CC_1 e^{-\frac{\mu_1}{\varepsilon}} \|1_{B(0,r)} \varphi^2\|_1 \|g\|_1
\]

\[
= o(r_t) \|g\|_1, \quad o(r_t) \to 0 \text{ as } r_t \downarrow 0;
\]

\[
\langle 1_{B^c(0,R)} \varphi e^{-t\Lambda^\varepsilon} \varphi^{-1} g \rangle \leq C \langle 1_{B^c(0,R)} \varphi^2, e^{-t\Lambda^\varepsilon} g \rangle
\]

\[
\leq C \langle e^{-t\Lambda^\varepsilon} 1_{B^c(0,R)}, g 1_{B(0,R_0)} \rangle, \text{ where } R \geq 2R_t, R_0 \geq 2(R_0 \vee t^\frac{\alpha}{2})
\]

\[
\leq C \sup_{x \in B(0,R_0)} e^{-t\Lambda^\varepsilon} 1_{B^c(0,R)}(x) \|g\|_1
\]

\[
\leq C\tilde{C} \tilde{C} C_d R_t \|g\|_1
\]

\[
= o(R_t, R_0) \|g\|_1, \quad o(R_t, R_0) \to 0 \text{ as } R_t, R_0 \uparrow \infty
\]

due to \( e^{-t\Lambda^\varepsilon}(x,y) \leq \tilde{C}(t|x - y|^{-d-\alpha} \wedge t^{-\frac{\alpha}{2}}) \leq \tilde{C} 2d^\frac{\alpha}{2} |y|^{-d-\frac{\alpha}{2}} \text{ if } |x| \leq R_0 \text{ and } |y| \geq R \).

We are left to apply Proposition 2.\qed
Now we are in position to apply the so-called 3q argument. Set \( q_t(x, \cdot) = e^{-t\Lambda}(x, \cdot)\varphi_t^{-1}(x)\varphi_t^{-1}(\cdot) \).

(a) Let \( x, y \in B^c(0,1) \), \( x \neq y \). Clearly,

\[
q_3(x, y) \geq \varphi_3^{-1}(x)\varphi_3^{-1}(y)e^{-3\Lambda}(x, y) \geq e^{-3\Lambda}(x, y) \geq e^{-3A}(x, y).
\]

(b) Let \( x, y \in B(0,1) \), \( 0 < |x| \leq |y| \). By the reproduction property, since \( e^{-t\Lambda} \) is positivity preserving,

\[
q_3(x, y) \geq \varphi_3^{-1}(x)\varphi_3^{-1}(y)\langle e^{-\Lambda}(x, \cdot)\varphi_1^{-1}(\cdot) e^{-2\Lambda}(\cdot, y) \rangle_{1_{r,R}(\cdot)} \\
= \varphi_3^{-1}(x)\varphi_3^{-1}(y)\langle e^{-\Lambda}(x, \cdot)\varphi_1^{-1}(\cdot) e^{-2\Lambda}(\cdot, y) \rangle_{1_{r,R}(\cdot)} \\
\geq \varphi_3^{-1}(x)\varphi_3^{-1}(y)\langle e^{-\Lambda}(x, \cdot)\varphi_1^{-1}(\cdot) 1_{r,R}(\cdot) \rangle \inf_{r \leq |z| \leq R} \varphi_1^{-1}(z) e^{-2\Lambda}(z, y) \\
\text{(here we are using Corollary 2)} \\
\geq e^{-\hat{\mu}-1} \varphi_3^{-1}(x)\varphi_3^{-1}(y)\inf_{r \leq |z| \leq R} e^{-2\Lambda}(z, y) \\
= C_{r,R}\varphi_3^{-1}(y) \inf_{r \leq |z| \leq R} e^{-2\Lambda}(y, z); \\
e^{-2\Lambda}(y, z) \geq \langle e^{-\Lambda}(y, \cdot)\varphi_1^{-1}(\cdot) e^{-2\Lambda}(\cdot, z) 1_{r,R}(\cdot) \rangle \\
\text{(again we are using Corollary 2)} \\
\geq e^{-\hat{\mu}-1} \varphi_1(y)\varphi_1^{-1}(r)\inf_{r \leq |z| \leq R} e^{-\Lambda}(\cdot, z).
\]

Therefore

\[
q_3(x, y) \geq C'_{r,R} \inf_{r \leq |z| \leq R} e^{-\Lambda}(\cdot, z) \geq C''_{r,R} e^{-3A}(x, y).
\]

(c) Let \( x \in B(0,1) \), \( x \neq 0 \), \( y \in B^c(0,1) \). Then

\[
q_3(x, y) \geq \varphi_3^{-1}(x)\varphi_3^{-1}(y)\langle e^{-\Lambda}(x, \cdot)\varphi_1^{-1}(\cdot) e^{-2\Lambda}(\cdot, y) 1_{r,R}(\cdot) \rangle \\
\geq \varphi_1^{-1}(x)\langle e^{-\Lambda}(x, \cdot)\varphi_1^{-1}(\cdot) e^{-2\Lambda}(\cdot, y) 1_{r,R}(\cdot) \rangle \\
\geq e^{-\hat{\mu}-1} \inf_{r<|z|<R} \varphi_1^{-1}(z)e^{-2\Lambda}(z, y) \geq e^{-\hat{\mu}-1} \varphi_1^{-1}(r) \inf_{r<|z|<R} e^{-2\Lambda}(z, y) \\
\geq C_{r,R} e^{-3A}(x, y).
\]

Finally, by (a),(b),(c), \( q_3(x, y) \geq Ce^{-3A}(x, y) \) or \( e^{-3\Lambda}(x, y) \geq Ce^{-3A}(x, y)\varphi_3(x)\varphi_3(y) \). The scaling argument ends the proof of the lower bound. \( \square \)

**Appendix A. Proof of Theorem A**

Set \( L^2_\varphi = L^2(X, \varphi^2d\mu) \), and define a unitary map \( \Phi : L^2_\varphi \to L^2 \) by \( \Phi f = \varphi f \). Then the operator \( \Lambda_\varphi = \Phi^{-1}\Lambda\Phi \) of domain \( D(\Lambda_\varphi) = \Phi^{-1}\Lambda D(\Lambda) \) is selfadjoint on \( L^2_\varphi \) and \( \|e^{-t\Lambda_\varphi}\|_{2 \to 2,\varphi} = \|e^{-t\Lambda}\|_{2 \to 2} \leq 1 \) for all \( t \geq 0 \). Here and below the subscript \( \varphi \) indicates that the corresponding quantities are related to the measure \( \varphi^2d\mu \).
Let $f = \varphi^{-1} h$, $h \in L^\infty_{com}$, and so $f \in L^2_\varphi \cap L^1_\varphi$ by $(M_2)$. Let $u_t = e^{-tA} f$. Then $\varphi u_t = e^{-tA} \varphi f$ and
\[
\langle \Lambda \varphi u_t, u_t \rangle_\varphi = \| \Lambda^\frac{1}{2} \varphi u_t \|^2_2 \geq c_S \| \varphi u_t \|^2_2 \nonumber \\
\geq c_S \| \varphi u_t \|^2_2 \| \varphi u_t \|_{1, \varphi}^{-\frac{2}{j'}} \nonumber \\
= c_S \langle u_t, u_t \rangle_\varphi \| \varphi^{-1} \varphi e^{-tA} \varphi^{-1} \varphi^2 f \| \nabla \bar{f}, \nonumber 
\]
where $(M_1)$ and H"older’s inequality have been used.

Clearly, $-\frac{1}{2} \frac{d}{dt} \langle u_t, u_t \rangle_\varphi = \langle \Lambda \varphi u_t, u_t \rangle_\varphi$. Setting $w := \langle u_t, u_t \rangle_\varphi$ and using $(M_4)$ we have
\[
\frac{d}{dt} w^{-\frac{1}{j'}} \geq \frac{2}{j'} c_S (c_0^{-1} \| \varphi e^{-tA} \varphi^{-1} \varphi^2 f \|_1)^{-\frac{2}{j'}}. \nonumber 
\]
By our choice of $f$, $\varphi^2 f = \varphi h \in D$. Therefore we can apply $(M_3)$ and obtain
\[
\frac{d}{dt} w^{-\frac{1}{j'}} \geq \frac{2}{j'} c_S (c_1 c_0^{-1} \| f \|_{1, \varphi})^{-\frac{2}{j'}}, \quad t \leq s. \nonumber 
\]
Integrating this inequality over $[0, t]$ gives
\[
\| e^{-tA} f \|_{2, \varphi} \leq c t^{-\frac{j'}{2}} \| f \|_{1, \varphi}, \quad t \leq s. \nonumber 
\]
Since $f \in \varphi^{-1} L^\infty_{com}$ and $\varphi^{-1} L^\infty_{com}$ is a dense subspace of $L^1_\varphi$, the last inequality yields
\[
\| e^{-tA} f \|_{1 \rightarrow 2, \varphi} \leq c t^{-\frac{j'}{2}}, \quad t \leq s \nonumber 
\]
and $(NTE_{\mu})$ follows. $\square$

**References**

[BGJP] K. Bogdan, T. Grzywny, T. Jakubowski and D. Pilarczyk, *Fractional Laplacian with Hardy potential*, Comm. Partial Differential Equations, 44 (2019), p. 20-50.

[CKSV] S. Cho, P. Kim, R. Song, Z. Vondraček, *Factorization and estimates of Dirichlet heat kernels for non-local operators with critical killings*, [arXiv:1809.01782](arXiv:1809.01782) (2018), 43 p.

[JW] T. Jakubowski and J. Wang, *Heat kernel estimates for fractional Schrödinger operators with negative Hardy potential*, [arXiv:1809.02425](arXiv:1809.02425) (2018), 26 p.

[Ka] T. Kato. *Perturbation Theory for Linear Operators*. Springer-Verlag Berlin Heidelberg, 1995.

[KSSz] D. Kinzebulatov, Yu. A. Semënov and K. Szczypkowski, *Heat kernel of fractional Laplacian with Hardy drift via desingularizing weights*, Preprint, [arXiv:1904.07363](arXiv:1904.07363) (2019), 19 p.

[KPS] V. F. Kovalenko, M. A. Perelmuter and Yu. A. Semënov, *Schrödinger operators with $L^1_{1/2}(R^d)$-potentials*, J. Math. Phys., 22 (1981), p. 1033-1044.

[MS0] P. D. Milman and Yu. A. Semënov. *Desingularizing weights and heat kernel bounds*, Preprint (1998).

[MS1] P. D. Milman and Yu. A. Semënov. *Heat kernel bounds and desingularizing weights*, J. Funct. Anal., 202 (2003), p. 1-24.

[MS2] P. D. Milman and Yu. A. Semënov. *Global heat kernel bounds via desingularizing weights*, J. Funct. Anal., 212 (2004), p. 373-398.
Université Laval, Département de mathématiques et de statistique, 1045 av. de la Médecine, Québec, QC, G1V 0A6, Canada
E-mail address: damir.kinzebulatov@mat.ulaval.ca

University of Toronto, Department of Mathematics, 40 St. George Str, Toronto, ON, M5S 2E4, Canada
E-mail address: semenov.yu.a@gmail.com