The Stability Analysis of Brane Induced Gravity with Quintessence Field on the Brane with a Gaussian Potential

A. Ravanpak and G. F. Fadakar

Department of Physics, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran

(Dated: March 25, 2020)

Abstract

In this manuscript we consider a normal branch of DGP cosmological model with a quintessence scalar field on the brane as the dark energy component. Using dynamical system approach we study the stability properties of the model. We find that $\lambda$, as one of our new dimensionless variables that is defined in terms of the quintessence potential has a crucial role in the history of the universe. We divide our discussion into two parts: a constant $\lambda$, and a varying $\lambda$. In the case of a varying $\lambda$, which is the main part of this work, we consider a Gaussian potential for which $\lambda$ goes to infinity, asymptotically. Here, all the critical points which were obtained in the case of a constant $\lambda$, can be assumed as instantaneous critical points. We discuss the evolution of dynamical variables in such a model and conclude that their asymptotic behaviors follow the trajectories of the moving critical points. Also, we find two different possible fates for the universe. In one of them it experiences an accelerated expansion, then enters a decelerating phase and finally reaches a stable matter dominated solution. In the other scenario, the universe approaches the matter dominated critical point without experiencing any accelerating expansion.

Keywords: DGP, quintessence, dynamical system, stability, Gaussian potential
1. INTRODUCTION

The outcomes of cosmological observations, such as the type Ia supernova (SNe Ia) \[1\], the cosmic microwave background radiation (CMBR) \[2\], the large-scale structure \[3\], and so forth, have disclosed that our universe is undergoing an accelerated expanding phase. Cosmologists describe this surprising phenomenon either with the concept of dark energy (DE) \[4\]-\[17\], or with some extended theories of gravity \[18\]-\[24\].

On the other hand, the concept of extra dimension that arises from string theory has attracted a great amount of attention, specially for explaining the so called hierarchy problem \[25\]-\[34\]. In these theories our four dimensional (4D) universe is considered as a brane embedded in a five dimensional (5D) spacetime dubbed bulk. If we add a 4D scalar curvature into the brane action, on top of the matter Lagrangian in it, we are dealing with a brane induced gravity theory, and DGP braneworld model is its most well-known example in which the bulk is an infinite 5D Minkowski spacetime \[35\]. DGP model includes two distinct branches, the self-accelerating one that yields late-time acceleration geometrically, but suffers from the ghost instability, and the normal branch which is healthy, but cannot explain accelerated expansion of the universe without any DE component.

Furthermore, dynamical system analysis which could be a practical method in examining the long-term behavior of the universe qualitatively, has been widely used in literature \[36\]-\[46\]. This qualitative study is based on stability analysis. In this approach, instead of a particular trajectory, one will find and categorize the type of all the possible trajectories of the universe in an appropriate phase space.

In this manuscript we will consider a normal branch of DGP braneworld cosmology with a quintessence scalar field $\phi$ on the brane, as the DE component. To investigate this model, we will follow the dynamical system approach. After introducing some new dimensionless variables, we will write an autonomous system of ordinary differential equations. Then, we will obtain the critical points of the model and related eigenvalues to study their stability. Although a dynamical investigation of DGP model with a scalar field trapped on the brane has already been studied in \[47\] for a constant scalar field potential and an exponential potential distinctly, but the prior is very simple and special, and the latter does not represent the effect of extra dimension. Here, we will consider a Gaussian potential, and show that with this situation, not only our model can indicate different cosmological epochs such as matter
dominated era and DE dominated era, but also can represent the role of extra dimension. Also, we find a few moving critical points in our model that play an important role in the evolution of the universe.

This article is organized as follows: in Sec.2, we review the basic equations of the model. Sec.3, deals with the new variables and respective autonomous differential equations, critical points and their stability conditions, but for two different situations. In the first part of Sec.3, we discuss a constant $\lambda$ case, and in its second part the case of a varying $\lambda$ will be studied. The asymptotic behavior of the model is also investigated in this section. Finally in Sec.4, we express a summary and discuss the results.

2. THE MODEL

Assuming a homogeneous, isotropic and spatially flat brane in a normal branch of DGP model one can reach to the Friedmann equation on the brane as

$$H^2 + \frac{H}{r_c} = \frac{1}{3M_p^2}(\rho_m + \rho_\phi)$$

(1)

in which $\rho_m$, represents the energy density of the matter content of the universe, and $\rho_\phi$ indicates the quintessence scalar field energy density as the DE component. Also, $H$, $M_p$ and $r_c$, are respectively the Hubble parameter, the Planck mass and the crossover distance where the latter determines transition from 4D to 5D regime and is always positive. In the absence of interaction between the dark sectors of the universe, one can utilize the standard conservation equations as

$$\dot{\rho}_m + 3H\rho_m = 0,$$

(2)

$$\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = 0,$$

(3)

in which the dot denotes derivative with respect to the cosmic time, $t$. In Eq.(3), the energy density of the quintessence scalar field $\rho_\phi$, and its pressure $p_\phi$, are defined as

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi),$$

(4)

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi),$$

(5)

respectively, in which $V(\phi)$ is the quintessence potential. Substituting $\rho_\phi$ and $p_\phi$, in Eq.(3), we obtain the equation of motion of the quintessence scalar field as

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi = 0$$

(6)
Here, the derivative of $V(\phi)$ with respect to the scalar field has been denoted by $V_\phi$.

3. THE PHASE SPACE AND THE STABILITY ANALYSIS

In order to analyze the stability characteristics of the model, we first introduce a set of new dimensionless variables to convert the equations of motion of our model into a self-autonomous dynamical system. The auxiliary variables we have chosen here are as follows

$$
x = \sqrt{\frac{\rho_m}{3M_p^2(H^2 + \frac{H}{r_c})}}, \quad y = \sqrt{\frac{V(\phi)}{3M_p^2(H^2 + \frac{H}{r_c})}}, \quad (7)
$$

$$
z = \frac{\dot{\phi}}{\sqrt{6M_p^2(H^2 + \frac{H}{r_c})}}, \quad l = \sqrt{1 + \frac{1}{Hr_c}}, \quad \lambda = M_pV_\phi/V.
$$

Since an expanding universe and a contracting one are independent submanifolds, we can study them separately [47], [49]. In the following, we will focus on the more popular expanding case. With $H > 0$ for an expanding universe, and $r_c > 0$, we find a constraint as $l \geq 1$. One can check that for $r_c \to \infty$, we have $l = 1$. So the subset $(x, y, z, l = 1)$, corresponds to a 4D Einstein-Hilbert theory limit. With the above phase space variables and using Eq.(4), we can express the Friedmann equation on the brane as the constraint below

$$
x^2 + y^2 + z^2 = 1. \quad (8)
$$

Regarding this constraint and with attention to Eq.(7), the new variables satisfy some other constraints, such that $0 \leq x \leq 1, \ 0 \leq y \leq 1$ and $-1 \leq z \leq 1$, while $l \geq 1$. On the other hand, the Raychaudhury equation and the total equation of state (EoS) parameter of the universe, can be obtained and written in terms of the new variables as below

$$
\frac{\dot{H}}{H^2} = -\frac{3l^2}{l^2 + 1}(1 + z^2 - y^2) \quad (9)
$$

$$
w_{tot} = z^2 - y^2 \quad (10)
$$

To build an autonomous system of ordinary differential equations, we differentiate the phase space variables in Eq.(7). Also, we have reduced the number of degrees of freedom of
the model by one, using the Friedmann constraint:

\[ y' = \sqrt{\frac{3}{2}} y z l \lambda + \frac{3}{2} y (1 + z^2 - y^2), \]  
\[ z' = -3z - \sqrt{\frac{3}{2}} y^2 l \lambda + \frac{3}{2} z (1 + z^2 - y^2), \]  
\[ l' = \frac{3}{2} l \left( \frac{l^2 - 1}{l^2 + 1} \right) (1 + z^2 - y^2), \]  
\[ \lambda' = \sqrt{6} l z \lambda^2 (\Gamma - 1) \] 

In these equations, the prime means derivative with respect to \( \ln a \), and \( \Gamma = V V_{\phi \phi}/V_{\phi}^2 \), in which \( V_{\phi \phi} \), indicates the second derivative of the potential with respect to the scalar field. This 4D autonomous system represents the evolution of the DGP model with a quintessence scalar field, indirectly.

According to linear stability analysis, we first solve the equations \( y' = z' = l' = \lambda' = 0 \), simultaneously to determine the critical points of the system of equations above and respective eigenvalues. Then, we study the behavior of the system near the critical points to describe various kinds of possible trajectories in the phase space. Obviously, an important factor in this level is the form of the quintessence potential, because the new variable \( \lambda \), depends on it. From now on, we separate the work to two different cases: a constant \( \lambda \), and a varying \( \lambda \). But, first let us review the case \( \lambda = constant \), as the authors investigated in [47], since our discussions for a varying \( \lambda \), is strongly dependent on it.

3.1. \( \lambda = constant \)

With attention to Eq.(14), \( \lambda = constant \), could be associated with \( \Gamma = 1 \), as well as the special situation \( \lambda = 0 \). The prior relates to an exponential potential while the latter results in a constant quintessence potential. TABLE II shows the critical points of the model, the related eigenvalues and their existence condition, for \( \lambda = constant \). We have to note that only those critical points that satisfy the constraints on phase space variables, in addition to the Friedmann constraint have been mentioned in this table. It is clear that the critical points \( CP_1, CP_2 \) and \( CP_3 \), exist for all values of \( \lambda \), while the existence of the critical points \( CP_5, CP_6 \) and \( CP_7 \), depends on the value of \( \lambda \), and \( CP_4 \), only exists for \( \lambda = 0 \). There is also a critical line \( CL_1 \), which involves \( CP_4 \), and like it exists only for \( \lambda = 0 \). By critical line we mean infinite number of critical points with \( y = 1, z = 0 \), but with \( l \geq 1 \). Also, it is
obvious that all the critical points in TABLE I are the 4D solutions because of \( l = 1 \), and it is just the critical line \( CL_1 \), that shows the effect of extra dimension. Moreover, one can check that \( CP_5 \) coincides with \( CP_1, CP_2, CP_4, CP_6 \) and \( CP_7 \) in the case \( \lambda = -\sqrt{6}, \lambda = \sqrt{6}, \lambda = 0, \lambda = \sqrt{3} \) and \( \lambda = -\sqrt{3} \), respectively.

**TABLE I: Critical points of the model for \( \lambda = constant \)**

| Critical Points | \((y, z, l)\) | Eigenvalues | Existence |
|-----------------|----------------|-------------|-----------|
| \( CP_1 \)      | \((0, 1, 1)\)  | \((3, 3 + \sqrt{3/2}\lambda, 3)\) | any \( \lambda \) |
| \( CP_2 \)      | \((0, -1, 1)\) | \((3, 3 - \sqrt{3/2}\lambda, 3)\) | any \( \lambda \) |
| \( CP_3 \)      | \((0, 0, 1)\)  | \((3/2, -3/2, 3/2)\) | any \( \lambda \) |
| \( CP_4 \)      | \((1, 0, 1)\)  | \((-3, -3, 0)\) | \( \lambda = 0 \) |
| \( CP_5 \)      | \((\sqrt{1 - \lambda^2}/6, -\sqrt{6\lambda}/6, 1)\) | \((\lambda^2 - 3, \lambda^2/2 - 3, \lambda^2/2)\) | \(-\sqrt{6} \leq \lambda \leq \sqrt{6} \) |
| \( CP_6 \)      | \((\sqrt{6}/2\lambda, -\sqrt{6}/2\lambda, 1)\) | \((-3/4 + 3\sqrt{24 - 7\lambda^2}/4\lambda, -3/4 - 3\sqrt{24 - 7\lambda^2}/4\lambda, 3/2)\) | \( \lambda \geq \sqrt{3} \) |
| \( CP_7 \)      | \((-\sqrt{6}/2\lambda, -\sqrt{6}/2\lambda, 1)\) | \((-3/4 + 3\sqrt{24 - 7\lambda^2}/4\lambda, -3/4 - 3\sqrt{24 - 7\lambda^2}/4\lambda, 3/2)\) | \( \lambda \leq -\sqrt{3} \) |
| \( CL_1 \)      | \((1, 0, [1, \infty])\) | \((-3, -3, 0)\) | \( \lambda = 0 \) |

The stability status of the critical points with attention to respective eigenvalues, their physical descriptions and \( w_{tot} \), have been represented in TABLE I. With attention to Eqs.(7) and (8), critical points \( CP_1 \) and \( CP_2 \), are kinetic dominated solutions. Also, since in these cases \( w_{tot} = 1 \), they behave as stiff matter. Given the value of parameter \( \lambda \), they could be an unstable or a saddle point, if all their eigenvalues are positive or they have different signs. \( CP_3 \) is another critical point of our model that with attention to \( w_{tot} = 0 \), represents a matter dominated universe and is always a saddle point. We call it a pure matter dominated universe, because the Friedmann constraint for \( y = 0 \) and \( z = 0 \), yields \( x = 1 \). \( CP_4 \) demonstrates a quintessence potential dominated solution and with attention to \( w_{tot} = -1 \), we can consider it as a DE dominated solution. But, in this situation we cannot identify the stability status using the common linear perturbation method, because one of their eigenvalues is zero. In such cases, one must adopt other stability approaches. Here, we resort to a numerical analysis. FIG I illustrates some trajectories related to various initial conditions in our phase space. As we see all the trajectories start from the critical points \( CP_1 \) and \( CP_2 \), which are repellor for \( \lambda = 0 \). Also, it is clear that \( CP_4 \), is an attractor. The critical point \( CP_5 \), is generally a saddle point, except for \( \lambda = \pm \sqrt{6} \), and \( \lambda = 0 \), because in these
cases it coincides with the critical points \( CP_{2,1} \) and \( CP_4 \). Substituting this critical point into the Friedmann constraint leads to \( x = 0 \), which in turn implies that \( CP_5 \), corresponds to a scalar field dominated solution, and satisfies the relation \( y^2 + z^2 = 1 \). It means that \( CP_5 \), lies on a unit circle in \( yz \)-plane (See FIG[2]). Although for \( CP_6 \) and \( CP_7 \), we find that \( w_{tot} = 0 \), these critical points do not show a pure matter dominated era, because they do not result \( x = 1 \). We call them scaling solutions. The larger the value of \( |\lambda| \), the closer to a pure matter dominated era. It is worthy to note that for \( \lambda = \sqrt{3} \) and \( \lambda = -\sqrt{3} \), respectively, \( CP_6 \) and \( CP_7 \), behave as the scalar field dominated solution, \( CP_5 \). They are always saddle critical points, though for a given range of \( \lambda \) in which their eigenvalues are not real, they show a spiral behavior. The critical subset \( CL_1 \), that like the critical point \( CP_4 \) only exists for \( \lambda = 0 \), has a zero eigenvalue, as well. Again, regarding to FIG[1] one can conclude that \( CL_1 \), is an attractor line. It is a potential dominated solution which include the effect of extra dimension, additionally. These results are similar to the results of [47].

FIG. 1: The critical points of our dynamical system and a few trajectories for \( \lambda = 0 \). The black dash line represents the critical line \( CL_1 \).
### TABLE II: Stability status of the critical points

| Critical Points | $w_{\text{tot}}$ | Stability in 3D | Description |
|-----------------|------------------|----------------|-------------|
| $CP_1$          | 1                | unstable for $\lambda \geq -\sqrt{6}$ | kinetic dominated |
|                 |                  | saddle for $\lambda < -\sqrt{6}$   |             |
| $CP_2$          | 1                | unstable for $\lambda \leq \sqrt{6}$ | kinetic dominated |
|                 |                  | saddle for $\lambda > \sqrt{6}$   |             |
| $CP_3$          | 0                | saddle           | pure matter dominated |
| $CP_4$          | −1               | stable           | DE dominated |
| $CP_5$          | $\lambda^2/3 - 1$ | saddle          | scalar field dominated |
| $CP_6$          | 0                | saddle for $\sqrt{3} \leq \lambda \leq \sqrt{24/7}$ | scaling solution |
|                 |                  | spiral saddle for $\lambda > \sqrt{24/7}$ |           |
| $CP_7$          | 0                | saddle for $-\sqrt{24/7} \leq \lambda \leq -\sqrt{3}$ | scaling solution |
|                 |                  | spiral saddle for $\lambda < -\sqrt{24/7}$ |           |
| $CL_1$          | −1               | stable           | DE dominated |

only for those critical points that depend on $\lambda$, and this plays a crucial role in the evolution of the universe in our model. As we mentioned earlier, $CP_5$, is unstable for $\lambda = \pm \sqrt{6}$, stable for $\lambda = 0$, and saddle for other values of $\lambda$, but in three dimensions (3D). One can check that in 2D $yz$-plane, $CP_5$, will be a stable critical point for $-\sqrt{3} \leq \lambda \leq \sqrt{3}$, as it is clear from FIG.2. Also, $CP_6$ and $CP_7$, are not saddle critical points in 2D $yz$-plane, but rather stable solutions. For instance, $CP_6$, will be a spiral stable critical point for $\lambda > \sqrt{24/7}$, and a stable critical point for $\sqrt{3} \leq \lambda \leq \sqrt{24/7}$, as it can be seen in FIG.2. One can easily check that $CP_7$, is spiral stable for $\lambda < -\sqrt{24/7}$, and stable for $-\sqrt{24/7} \leq \lambda \leq -\sqrt{3}$.

- **critical points at infinity**

Since the new variable $l$ is unbounded, our discussion is incomplete till now and we have to analyze the stability of the system at infinity, too. We have seen the effect of extra dimension in our model in the critical line $CL_1$ which exist only for $\lambda = 0$, but how about other values of $\lambda$? The answer may be related to the analysis at infinity. To this aim, we try to compact our dynamical system defining a new variable as

$$u = \frac{1}{l}$$ (15)
FIG. 2: 2D representation of phase space for various values of λ. The critical points $CP_1$, $CP_2$, $CP_3$, $CP_5$ and $CP_6$, have been demonstrated with a red, green, yellow, pink and blue circles, respectively. For $\lambda = 0$, $\lambda = \sqrt{3}$ and $\lambda = \sqrt{6}$, the critical point $CP_5$, coincides with $CP_4$, $CP_6$ and $CP_2$, severally.

so that for $l = 1$, and in the limit $l \to \infty$, we have $u = 1$ and $u = 0$, respectively, while $u$ satisfies the constraint $0 \leq u \leq 1$. Then, we obtain a new set of ordinary differential
equations as below

\[ \frac{dy}{d\xi} = \sqrt{\frac{3}{2}} yz\lambda + \frac{3}{2} yu(1 + z^2 - y^2), \tag{16} \]
\[ \frac{dz}{d\xi} = -3z u - \sqrt{\frac{3}{2}} y^2 \lambda + \frac{3}{2} z u (1 + z^2 - y^2), \tag{17} \]
\[ \frac{du}{d\xi} = -\frac{3}{2} u^2 \left( \frac{1 - u^2}{1 + u^2} \right) (1 + z^2 - y^2), \tag{18} \]
\[ \frac{d\lambda}{d\xi} = \sqrt{6} z \lambda^2 (\Gamma - 1) \tag{19} \]
in which \( \frac{d}{d\xi} = u \frac{d}{d \ln a} \). When we calculate the critical points of this new system we find one additional critical line for any value of \( \lambda \) as \((CL_2 : u = 0, y = 0, z = z)\), and also one critical plane for only \( \lambda = 0 \) as \((CPN : u = 0, y = y, z = z)\), on top of all the results in TABLE I. Obviously, \( CL_2 \), is part of \( CPN \), for \( \lambda = 0 \). Using the Friedmann constraint we can conclude that they are matter scaling solutions, because both the quintessence scalar field and the matter content have contributions in these solutions. The only difference between them is that for \( CL_2 \), the contribution of the quintessence potential is zero. Also, one can check that for both of them \( 0 \leq w_{\text{tot}} \leq 1 \), and as a result they cannot certainly relate to an accelerated expanding phase. Since at least one of their eigenvalues is zero, we utilize the numerical approach to understand their stability characteristics. FIG 3 illustrates that the critical plane \( CPN \), and also the critical line \( CL_2 \), which is part of \( CPN \) for \( \lambda = 0 \), behave as saddle critical subsets. But the case differs for other values of \( \lambda \). In these situations as one can see in FIG 4, the critical line \( CL_2 \), behaves as an attractor line.

Now, we have completed our analysis. We have understood that in our model the universe always starts from the unstable kinetic dominated critical points \( CP_1 \) and \( CP_2 \). But its fate depends on the value of \( \lambda \), and also the initial conditions. For \( \lambda = 0 \), it even reaches the stable DE dominated critical point \( CP_4 \) if it evolves in 4D, or comes to the stable DE dominated critical line \( CL_1 \) which shows the effect of extra dimension if it evolves in 5D. Also, for other values of \( \lambda \), and in 5D, the universe finally approaches the matter scaling stable critical line \( CL_2 \), which cannot describe the current accelerated expansion. But if the universe evolves in 4D, it even reaches a scalar field dominated stable critical point \( CP_5 \), or comes to a matter scaling stable critical point \( CP_6 \) (or \( CP_7 \), for negative values of \( \lambda \)). Depending upon the value of \( \lambda \), it may show the late time acceleration. This case is important in our following discussions and will be studied in detail.
FIG. 3: The critical points of our dynamical system and a few trajectories for the case $\lambda = 0$, in the new phase space. The black dash line and the gray plane represent the critical line $CL_1$ and the critical plane $CPN$, respectively. $CPN$, come from the analysis at infinity.

3.2. $\lambda = \lambda(\phi)$

If one consider the quintessence potential anything, except the constant or exponential potential, $\lambda$, will be a dynamical quantity. Here, we are interested in studying the behavior of the model in the case of a Gaussian potential. Assuming $\lambda$, changes sufficiently slow such that we can consider it as a constant within any infinitesimal period of time during the evolution of the universe, we can regard all the critical points in TABLE [II] and the ones we obtained at infinity, as the instantaneous critical points of respective dynamical system [49]-[53]. With this assumption, it is clear that $CP_5$, $CP_6$ and $CP_7$, are moving critical points which can indicate where the solution tends to at each instant if it evolves in 4D. Also, it is worthy to note that in the case of a Gaussian potential, $CP_4$, $CL_1$ and $CPN$, correspond to the extremum of the potential where $\lambda = 0$. To understand the evolution of our universe in a varying $\lambda$ situation completely, we need to find the asymptotic behavior of $\lambda$. In other words, it is important to know either $\lambda \to \infty$, or it approaches zero. Various kinds of potential satisfy different asymptotic limits. For potentials of the form $V = V_0 \phi^{-n}$, with $n > 0$, $V_\phi$, approaches zero faster than the potential itself, then $\lambda \to 0$. This is the case has been investigated for instance in [54], but for $n = 1$. Also, a double exponential potential as $V = V_0 \exp(-\alpha e^\phi)$, as an example of the case $\lambda \to \infty$, has been studied in [54], too. Another kind of potential in which $\lambda$, goes to infinity asymptotically, as it has been mentioned in
FIG. 4: The critical points of our dynamical system and a few trajectories for $\lambda = 1, 2, 3$, in the new phase space. The red dashdot line represents the critical lines $CL_2$ which come from the analysis at infinity.

[54], that is the case of interest for us here, is the Gaussian potential, $V(\phi) = V_0 \exp(-\alpha \phi^2)$, in which $V_0$ and $\alpha$, are positive constants. For such a potential, the quintessence scalar field can roll down plus (minus) infinity with $\dot{\phi} > 0$ ($\dot{\phi} < 0$). Also, one can check that in this situation $\lambda = -2\alpha M_p \phi$, that results $\lambda \to -\infty$ ($\lambda \to \infty$), at the limit $\phi \to \infty$ ($\phi \to -\infty$). Furthermore, for a Gaussian potential one can calculate $\Gamma = 1 - 1/(2\alpha \phi^2)$, and therefore, $\Gamma - 1$, is always negative. Thus, with attention to Eq.(14), the sign of $\lambda'$, depends on the sign of $z$, which is proportional to $\dot{\phi}$. Thus we see that for both $z \geq 0$, we have $|\lambda| \to \infty$.

In the following we will only discuss the positive values of $\lambda$, because of the symmetry.

- **Asymptotic behavior $\lambda \to \infty$**

Regarding the shape of the Gaussian potential and since $\lambda$ is increasing one can assume
that it starts from the top of the potential, the state in which $\lambda = 0$. For the case $\lambda = 0$, the universe desires to achieve even the stable DE dominated critical point $CP_4$, or the critical line $CL_1$, with $w_{tot} = -1$, but since $\lambda$, has dynamics, it does not have enough time to reach them. If the universe evolves in 5D, the trajectories end up in the critical line $CL_2$. But the case is more complicated in 4D. The destination moves around in $yz$-plane. It starts from $CP_4$, on the critical line $CL_1$, and continues as a moving stable scalar field dominated solution $CP_5$. At the same time $w_{tot}$, is increasing. As far as $w_{tot}$, is smaller than $-2/3$ the universe experiences an accelerating phase $[55]$. Along with increasing of $\lambda$, $w_{tot}$, grows as well. For $\lambda > 1$, $w_{tot}$, will be greater than $-2/3$, that shows the universe is in a decelerated expanding phase. $CP_5$, keeps moving till $\lambda = \sqrt{3}$. At this stage we will encounter with two moving critical points $CP_5$, and $CP_6$, that coincide with one another and both behave as a single stable point in 2D phase plane with $w_{tot} = 0$, which is still a scalar field dominated solution. Since then, they will separate and move distinctly. In one hand, $CP_5$, moves around in $yz$-plane as a saddle point so that the contribution of the quintessence kinetic term increases while of its potential term decreases, and at the same time $w_{tot}$, grows. Finally, when $\lambda = \sqrt{6}$, and $w_{tot} = 1$, $CP_5$, coincides with $CP_2$, which is a kinetic dominated solution and behaves as stiff matter. On the other hand, along with changing of $\lambda$, $CP_6$, also moves after the separation from $CP_5$, but as a stable scaling solution with $w_{tot} = 0$, in which the contribution of the matter content is increasing and of the scalar field is diluting. At $\lambda = \sqrt{24/7}$, $CP_6$, turns to a spiral saddle in 3D, or in fact a spiral stable critical point in $yz$-plane. As $\lambda$ increases, $CP_6$, becomes slowly close to $CP_3$, while it is still a spiral stable scaling solution. Finally, in the limit $\lambda \to \infty$, it approaches $CP_3$, which is a pure matter dominated solution with $w_{tot} = 0$, while it behaves as a spiral attractor.

This fact that how our universe has evolved in the past and how it will do so in the future, depends on how fast our system reaches a neighborhood of one of these moving stable critical points. As we mentioned in introduction, a lot of cosmological observations have unveiled that the universe is experiencing a very rapidly accelerated expansion today. Therefore, we can conclude that the evolution of the universe was fast enough, so that it has reached a neighborhood of $CP_5$, when $\lambda$ has not yet reached 1 and $w_{tot}$, is still smaller than $-2/3$, to guarantee this acceleration. Thus, we find that the universe will undergo a phase transition from acceleration to deceleration in the future in our model during the evolution of $\lambda$. Otherwise, it will never experience an accelerating expansion.
FIG. 5 and FIG. 6 illustrate the evolution of various cosmological parameters of our model for a specific choice of initial conditions. FIG. 5 demonstrates the evolution of two of our dynamical system variables $y$ and $z$, with respect to $\ln a$, in addition of the behavior of two moving critical points $CP_5$ and $CP_6$. What is important is that both $y$ and $z$, arrive in $CP_5$, so quickly that $\lambda$, has not yet equalled one (See FIG. 6), and as a consequence the universe experiences a phase of accelerated expansion. Then, they both follow the curve of $CP_5$, for a given time. As soon as $CP_6$ appears, they start to recede the curve of $CP_5$, and turn to the one of $CP_6$. Finally, they both catch the curve of $CP_6$ that is approaching $CP_3$.

FIG. 5: The evolutionary curves of dynamical variables $y$ and $z$, for the initial conditions $y = 0.5$, $z = -0.5$, $\lambda = 0$ and $l = 1$.

FIG. 6 left, shows how $\lambda$ changes with $\ln a$. We see that it always increases though the rate of increasing is not uniform and varies from one place to another. FIG. 6 right, illustrates the evolution of our model parameters versus $\ln a$. In the beginning, the contribution of the quintessence potential ($\Omega_V = V/3M_p^2H^2$), is increasing while the contribution of its kinetic term ($\Omega_k = \dot{\phi}^2/6M_p^2H^2$), and also the one of the matter content ($\Omega_m = \rho_m/3M_p^2H^2$), are decreasing. Therefore, the universe enters an accelerated expanding phase very quickly as it is obvious from the curve of decelerating parameter $q$. But after a period of time, $\Omega_V$ and $\Omega_k$, exchanges their role in the evolutionary scenario. During this process, the universe undergoes another phase transition from acceleration to deceleration. As it is clear in FIG. 6 right, $q$, crosses zero line and takes positive values. And the fate of the universe in our model, as we discussed earlier, is a matter dominated era. One can see From FIG. 6 right that $\Omega_m$, is the dominant component of our model at late times.
FIG. 6: Left: the evolutionary curve of $\lambda$. Right: the evolutionary curves of model parameters $\Omega_m$, $\Omega_V$, $\Omega_k$ and the deceleration parameter $q$. The initial conditions have been used are $y = 0.5$, $z = -0.5$, $\lambda = 0$ and $l = 1$.

4. SUMMARY AND REMARKS

In this article we have studied the evolution of a normal DGP cosmological model in the presence of a quintessence scalar field DE component on the brane with a Gaussian potential, in the context of dynamical system analysis. We have derived an autonomous system of ordinary differential equations in terms of some new dimensionless dynamical variables, and obtained the critical points of the model, even the ones at infinity. We have represented that for a Gaussian potential, the parameter $\lambda = M_p V_\phi/V$, has dynamics and approaches infinity, asymptotically. So, assuming a slowly varying $\lambda$, one can consider all the critical points, the critical lines and the critical plane for the case of a constant $\lambda$, as the instantaneous solutions, so that among them $CP_4$, $CL_1$ and $CP_N$, can relate to the top of the Gaussian potential and $CP_5$, $CP_6$ and $CP_7$, are moving critical points.

We have indicated that if our universe evolves in 5D, it ends up in the attractor line $CL_2$, a matter scaling solution in which the potential has no share, with $0 \leq w_{tot} \leq 1$, that clearly does not show an accelerating era. If we consider the 4D evolution on the $yz$-plane, we find that our universe evolves such that it continuously pursues a stable critical point. We have discussed that depending on how fast our universe evolves, it can experience an accelerated expanding phase or not, but in both the cases the fate of our universe is a matter dominated epoch without acceleration. We have illustrated that if the variation of $\lambda$, is slow enough, the
universe first follows the trajectory of \( CP_5 \), and then turns toward to the one of \( CP_6 \). If our universe comes to \( CP_5 \), before the state \( \lambda = 1 \), it will experience the acceleration, but then certainly undergo a phase transition to a decelerating expansion era. This is probably the case in the model under consideration regarding recent observational data and the present acceleration of the universe.

[1] A. G. Riess et al., Astron. J. 116, 1009 (1998).
[2] D. N. Spergel et al., Astrophys. J. Suppl. Ser. 148, 175 (2003).
[3] M. Tegmark et al., Phys. Rev. D 69, 103501 (2004).
[4] V. Sahni and A. A. Starobinsky, Int. J. Mod. Phys. D 9, 373 (2000).
[5] R. R. Caldwell, R. Dave and R. J. Steinhardt, Phys. Rev. Lett. 80, 1582 (1998).
[6] C. Armendariz-Picon, V. Mukhanov and P. J. Steinhardt, Phys. Rev. D 63, 103510 (2001).
[7] T. Padmanabhan, Phys. Rev. D 66, 021301 (2002).
[8] R. R. Caldwell, Phys. Lett. B 545, 23 (2002).
[9] B. Feng, X. L. Wang and X. M. Zhang, Phys. Lett. B 607, 35 (2005).
[10] A. Kamenshchik, U. Moschella and V. Pasquier, Phys. Lett. B 511, 265 (2001).
[11] H. Farajollahi, A. Ravanpak and G. F. Fadakar, Astrophys. Space Sci. 336, 461 (2011).
[12] H. Farajollahi, A. Ravanpak and G. F. Fadakar, Phys. Lett. B 711, 225 (2012).
[13] M. C. Bento, O. Bertolami and A. A. Sen, Phys. Rev. D 66, 043507 (2002).
[14] A. G. Cohen, D. B. Kaplan and A. E. Nelson, Phys. Rev. Lett. 82, 4971 (1999).
[15] H. Wei and R. G. Cai, Phys. Lett. B 663, 1 (2008).
[16] H. Wei and R. G. Cai, Phys. Lett. B 660, 113 (2008).
[17] C. Gao, F. Wu, X. Chen and Y. G. Shen, Phys. Rev. D 79, 043511 (2009).
[18] S. Capozziello and M. Francaviglia, Gen. Relativ. Gravit. 40, 357 (2007).
[19] S. Nojiri and S. D. Odintsov, Phys. Rep. 505, 59 (2011).
[20] S. Capozziello, V. F. Cardone, H. Farajollahi and A. Ravanpak, Phys. Rev. D 84, 043527 (2011).
[21] H. Farajollahi, A. Ravanpak and P. Wu, Astrophys. Space Sci. 338, 195 (2012).
[22] S. Capozziello, O. Luongo, R. Pincak and A. Ravanpak, Gen. Relativ. Gravit. 50, 53 (2018).
[23] P. Wu and H. Yu, Eur. Phys. J. C 71, 1 (2011).
[24] G. R. Bengochea, Phys. Lett. B 695, 405 (2011).
[25] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B 429, 263 (1998).
[26] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 4690 (1999).
[27] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 3370 (1999).
[28] H. Farajollahi and A. Ravanpak, Phys. Rev. D 84, 084017 (2011).
[29] H. Farajollahi and A. Ravanpak, Can. J. Phys. 89, 863 (2011).
[30] H. Farajollahi, A. Ravanpak and G. F. Fadakar, Astrophys. Space Sci. 348, 253 (2013).
[31] H. Farajollahi and A. Ravanpak, Astrophys. Space Sci. 349, 961 (2014).
[32] A. Ravanpak, H. Farajollahi and G. F. Fadakar, Astrophys. Space Sci. 361, 43 (2016).
[33] M. Bouhmadi-Lopez, Nucl. Phys. B 797, 78 (2008).
[34] E. N. Saridakis, Nucl. Phys. B 830, 374 (2010).
[35] D. Dvali, G. Gabadadze and M. Porrati, Phys. Lett. B 485, 208 (2000).
[36] J. Wainwright and G. F. R. Ellis, Dynamical Systems in Cosmology, Cambridge University Press, London 1997.
[37] A. A. Coley, Dynamical Systems and Cosmology, vol. 291 of Astrophysics and Space Science Library, Springer Netherlands, Dordrecht 2003.
[38] H. Zonunmawia, W. Khyllep, J. Dutta and L. Järv, Phys. Rev. D 98, 083532 (2018).
[39] S. K. Biswas and S. Chakraborty, Gen. Relativ. Gravit. 47, 22 (2015).
[40] K. Zhang, P. Wu and H. Yu, Phys. Lett. B 690, 229 (2010).
[41] H. Farajollahi, A. Salehi, F. Tayebi and A. Ravanpak, J. Cosmol. Astropart. Phys. 05, 017 (2011).
[42] I. Quiros, R. Garcia-Salcedo, T. Matos and C. Moreno, Phys. Lett. B 670, 259 (2009).
[43] H. Zhang and Z. H. Zhu, Phys. Rev. D 75, 023510 (2007).
[44] K. Nozari, F. Rajabi and K. Asadi, Class. Quantum Grav. 29, 175002 (2012).
[45] A. Ravanpak and G. F. Fadakar, Mod. Phys. Lett. A 34, 1950105 (2019).
[46] L. P. Chimento, R. Lazkoz, R. Maartens and I. Quiros, J. Cosmol. Astropart. Phys. 0609, 004 (2006).
[47] I. Quiros, R. Garcia-Salcedo, T. Matos and C. Moreno, Phys. Lett. B 485, 208 (2000).
[48] C. Deffayet, Phys. Lett. B 502, 199 (2001).
[49] A. Ravanpak and G. F. Fadakar, Class. Quantum Grav., 36, 235003 (2019).
[50] E. J. Copeland, M. R. Garousi, M. Sami and S. Tsujikawa, Phys. Rev. D 71, 043003 (2005).
[51] E. J. Copeland, M. Sami and S. Tsujikawa, Int. J. Mod. Phys. D 15, 1753 (2006).
[52] B. J. Barros and N. J. Nunes, Phys. Rev. D 93, 043512 (2016).
[53] S. C. C. Ng, N. J. Nunes and F. Rosati, Phys. Rev. D 64, 083510 (2001).
[54] A. de la Macorra and G. Piccinelli, Phys. Rev. D 61, 123503 (2000).
[55] B. Gumjudpai, Gen. Rel. Grav., 36, 747 (2004).