On a Classification of Irreducible Almost Commutative Geometries

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Abstract

We classify all irreducible, almost commutative geometries whose spectral action is dynamically non-degenerate. Heavy use is made of Krajewski’s diagrammatic language. The motivation for our definition of dynamical non-degeneracy stems from particle physics where the fermion masses are non-degenerate.

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1 Introduction

Within noncommutative geometry pioneered by Connes [45,3], the almost commutative ones play an interesting role. They are defined by spectral triples \((\mathcal{A}, \mathcal{H}, D)\) where the algebra \(\mathcal{A}\) has the form \(\mathcal{A} = C^\infty(M) \otimes \mathcal{A}\) with \(\mathcal{A}\) a direct sum of matrix algebras and \(M\) a (compact Euclidean) spacetime. For instance in the standard model of particle physics, \(\mathcal{A} = \mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C})\). It is important to classify the almost commutative triples because of their applications to physics. Let us sketch some of them.

Einstein’s derivation of general relativity from Riemannian geometry goes in two steps. The first step sets up the kinematics: the equivalence principle uses general coordinate transformations and starts from the flat metric of special relativity to guess curved metrics. The second step constructs a dynamics for the set of all metrics by imposing covariance under general coordinate transformations.

Connes generalizes Einstein’s derivation to noncommutative geometry [6,3]. In this new setting the metric is encoded in a Dirac operator and a coherent definition of the equivalence principle becomes available: the fluctuations of the Dirac operator by algebra automorphisms properly lifted to the Hilbert space of spinors. Indeed in Riemannian geometry, the algebra is the commutative algebra of functions on spacetime, the automorphisms are precisely the general coordinate transformations and fluctuating the flat Dirac operator leads to Dirac operators with curvature and torsion. The second step is the spectral action which in the commutative i.e. Riemannian case reproduces the Einstein-Hilbert action plus a positive cosmological constant and a curvature squared term. The Euclidean spectral action is positive definite and its ground states can be interpreted as a regularization of the initial singularity.

A noncommutative space or geometry is defined by a ‘spectral triple’ consisting of an algebra \(\mathcal{A}\), a Hilbert space \(\mathcal{H}\) and a Dirac operator \(D\). One important property of noncommutative geometry is that it contains discrete spaces, commutative or not. They have finite dimensional algebras and Hilbert spaces. An almost commutative geometry is a tensor product of the infinite dimensional commutative algebra consisting of spacetime functions with a finite dimensional noncommutative algebra, the ‘internal space’. The internal Dirac operator is simply an initial fermionic mass matrix and the internal algebra automorphisms of the tensor product are gauge transformations. The internal fluctuations produce gauge bosons and Higgs scalars and the spectral action of an almost commutative geometry produces – besides the gravitational action – the complete Yang-Mills-Higgs action. The Higgs scalar is therefore the internal metric and its dynamics is the Higgs potential. The initial fermionic mass matrix in general is not a solution of the internal dynamics, the ‘internal Einstein equation’. These solutions are the minima of the Higgs potential which induces the spontaneous symmetry breaking. They yield the true fermionic mass matrix and we have to compute it. Let us remark that in the prior approach of noncommutative Yang-Mills theories without gravity [7], the Dirac operator was not a dynamical variable. Consequently there was no nuance between initial and true fermion masses.

Although only a small subset of all Yang-Mills-Higgs models can be described as an almost commutative geometry this subset is still infinite and difficult to assess. We
propose to reduce it using two constraints. The first is inspired by ‘grand unified theories', in particular \( SO(10) \): the gauge group of the standard model of electro-magnetic, weak and strong forces is embedded into a simple group and the representation of one generation of quarks and leptons is embedded into an irreducible representation. Therefore our first constraint is: take the internal algebra simple and its spectral triple irreducible. As we will see the resulting fermion masses in the ground state are degenerate in flagrant contradiction to experiment. We therefore analyze internal algebras with two and three simple summands and their irreducible spectral triples. Again, in most cases the fermion masses come out degenerate with a few exceptions. Our aim is to list these exceptions. In other words, our second constraint is to impose a non-degenerate fermionic mass spectrum in the ground state, namely to restrict the analysis to dynamically non-degenerate spectral triples. The mathematical definitions of these constraints are given in sections 2 and 4.

2 Irreducibility

A spectral triple is given by \((\mathcal{A}, \mathcal{H}, D)\) such that the real \(*\)-algebra \(\mathcal{A}\) acts on the complex Hilbert space \(\mathcal{H}\), the Dirac operator \(D\) on \(\mathcal{H}\) is selfadjoint and a priori unbounded. These three items satisfy certain constraints of geometrical significance \[4,5,6\]. The commutative examples come from the triple \((\mathcal{A} = C^\infty(M), \mathcal{H} = L^2(M, S), D = i\gamma^\mu \partial_\mu)\) associated to a compact Riemannian spin manifold \(M\). As the resolvent of \(D\) is compact, \((\mathcal{A}, \mathcal{H}, D = 0)\) is never a spectral triple for infinite dimensional \(\mathcal{H}\). However, this degenerate situation can occur in finite cases, but is excluded from our definition of irreducible spectral triples.

**Definition 2.1.**

i) A spectral triple \((\mathcal{A}, \mathcal{H}, D)\) is **degenerate** if the kernel of \(D\) contains a non-trivial subspace of the complex Hilbert space \(\mathcal{H}\) invariant under the representation \(\rho\) on \(\mathcal{H}\) of the real algebra \(\mathcal{A}\).

ii) A non-degenerate spectral triple \((\mathcal{A}, \mathcal{H}, D)\) is **reducible** if there is a proper subspace \(\mathcal{H}_0 \subset \mathcal{H}\) invariant under the algebra \(\rho(\mathcal{A})\) such that \((\mathcal{A}, \mathcal{H}_0, D|_{\mathcal{H}_0})\) is a non-degenerate spectral triple. If the triple is real, \(S^0\)-real and even, we require the subspace \(\mathcal{H}_0\) to be also invariant under the real structure \(J\), the \(S^0\)-real structure \(\epsilon\) and under the chirality \(\chi\) such that the triple \((\mathcal{A}, \mathcal{H}_0, D|_{\mathcal{H}_0})\) is again real, \(S^0\)-real and even.

**Remark 2.2.**

i) \((\mathcal{A} = C^\infty(M), \mathcal{H} = L^2(M, S), D = i\gamma^\mu \partial_\mu)\) is never degenerate.

ii) If \((\mathcal{A}_i, \mathcal{H}_i, D_i)\) are two spectral triples then \((\mathcal{A}_1 \otimes \mathcal{A}_2, \mathcal{H}_1 \otimes \mathcal{H}_2, D_1 \otimes 1)\) is a spectral triple whose kernel of \(D_1 \otimes 1\) is infinite dimensional when \(\mathcal{H}_2\) is of infinite dimension.

iii) A finite dimensional commutative triple is a collection of points. It is non-degenerate if all points have finite distances. The converse is wrong, \(\mathcal{A} = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}\) with the triple given by diagram 15 is a counterexample.

iv) A reducible triple is not necessarily decomposable into a direct sum. For example,

\[
\mathcal{A} = \mathbb{C} \oplus \mathbb{C} \ni (a, b), \quad \rho(a, b) = \begin{pmatrix}
a & 0 \\
0 & \bar{a}
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & b & 0 & 0 \\
0 & 0 & (\bar{a} & 0) & 0 \\
0 & 0 & 0 & \bar{a}
\end{pmatrix},
\]

(2.1)
\[ \mathcal{D} = \begin{pmatrix} 0 & \mathcal{M} & 0 & 0 \\ \mathcal{M}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{M}^* \\ 0 & 0 & \mathcal{M} & 0 \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \]  

(2.2)

Here and throughout, \( \bar{\cdot} \) and \( * \) mean complex conjugation and adjoint.

v) Our definition of irreducibility differs from the one in our favorite book [8]. Indeed every spectral triple coming from a Riemannian spin manifold is irreducible in our case, even if the manifold is not connected, that is when the commutative spectral triple is a direct sum.

vi) In our definition of reducibility, the Dirac operator is not supposed to leave the subspace \( \mathcal{H}_0 \) invariant. Our definition is adapted to the use we will make of spectral triples: the fluctuations under algebra automorphisms properly lifted to the Hilbert space promote the Dirac operator to a dynamical variable and we are interested in its dynamics, the spectral action [3].

Since we are mainly interested in finite or 0-dimensional triples, we only recall the definition for this case and also restricting ourselves to the real and \( S^0 \)-real triples [3, 4]:

**Definition 2.3.** A real, \( S^0 \)-real, finite spectral triple is given by \((\mathcal{A}, \mathcal{H}, \mathcal{D}, J, \epsilon, \chi)\) with a finite dimensional real algebra \( \mathcal{A} \), a faithful representation \( \rho \) of \( \mathcal{A} \) on a finite dimensional complex Hilbert space \( \mathcal{H} \). Four additional operators are defined on \( \mathcal{H} \): the Dirac operator \( \mathcal{D} \) is selfadjoint, the real structure \( J \) is antiunitary, and the \( S^0 \)-real structure \( \epsilon \) and the chirality \( \chi \) are both unitary involutions. These operators satisfy:

- \( J^2 = 1, \quad [J, \mathcal{D}] = [J, \chi] = [\epsilon, \chi] = [\epsilon, \mathcal{D}] = 0, \quad \epsilon J = -J \epsilon, \quad \mathcal{D} \chi = -\chi \mathcal{D}, \)
- \( [\chi, \rho(a)] = [\epsilon, \rho(a)] = [\rho(a), J \rho(b) J^{-1}] = [[\mathcal{D}, \rho(a)], J \rho(b) J^{-1}] = 0, \forall a, b \in \mathcal{A}. \)
- The chirality can be written as a finite sum \( \chi = \sum_i \rho(a_i) J \rho(b_i) J^{-1} \). This condition is called orientability.
- The intersection form \( \cap_{ij} := \text{tr}(\chi \rho(p_i) J \rho(p_j) J^{-1}) \) is non-degenerate, \( \det \cap \neq 0 \). The \( p_i \) are minimal rank projections in \( \mathcal{A} \). This condition is called Poincaré duality.

With the help of the projectors \( (1 \pm \chi)/2 \) and \( (1 \pm \epsilon)/2 \), the Hilbert space is decomposed as

\[ \mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R \oplus \mathcal{H}_L^c \oplus \mathcal{H}_R^c. \]  

(2.3)

The two first components correspond in physics to particles, \( \epsilon = 1 \), the last two correspond to antiparticles, \( \epsilon = -1 \). We use the convention where left-handed spinors have negative and right-handed spinors have positive chirality.

If we denote by \( \rho_L \) the restriction of \( \rho \) to \( \mathcal{H}_L \), ... and by \( \rho_R^c \) the restriction to \( \mathcal{H}_R^c \), we will always write the representation in the following form

\[ \rho := \begin{pmatrix} \rho_L & 0 & 0 & 0 \\ 0 & \rho_R & 0 & 0 \\ 0 & 0 & \rho_L^c & 0 \\ 0 & 0 & 0 & \rho_R^c \end{pmatrix}. \]  

(2.4)
With respect to the decomposition \([2.3]\) of \(\mathcal{H}\), \(\mathcal{D}\) has the form

\[
\mathcal{D} = \begin{pmatrix}
0 & \mathcal{M} & 0 & 0 \\
\mathcal{M}^* & 0 & 0 & 0 \\
0 & 0 & 0 & \overline{\mathcal{M}} \\
0 & 0 & \overline{\mathcal{M}}^* & 0
\end{pmatrix}.
\]

Note that the \(S^0\)-reality of internal spaces is equivalent to the absence of Majorana-Weyl spinors. These are possible in 0-dimensional spaces, but impossible in 4- and \((1+3)\)-dimensional spaces \(M\).

3 Krajewski diagrams

Krajewski and Paschke & Sitarz have classified all finite, thus 0-dimensional, real spectral triples \([14,10]\). Let us summarize this classification for the \(S^0\)-real case using Krajewski’s diagrammatic language.

3.1 Conventions and multiplicity matrices

- The algebra: it is a finite sum of \(N\) simple algebras, \(\mathcal{A} = \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{K}_i)\) and \(\mathbb{K}_i = \mathbb{R}, \mathbb{C}, \mathbb{H}\) where \(\mathbb{H}\) denotes the quaternions.
- The representation: let us start with the easy case, \(\mathbb{K} = \mathbb{R}, \mathbb{H}\) in all components of the algebra. The algebras \(M_n(\mathbb{R})\) and \(M_n(\mathbb{H})\) only have one irreducible representation, the fundamental one on \(\mathbb{C}^{(n)}\), where \((n) = n\) for \(\mathbb{K} = \mathbb{R}\) and \((n) = 2n\) for \(\mathbb{K} = \mathbb{H}\). Therefore \(\rho\) is of the form

\[
\rho(\bigoplus_{i=1}^{N} a_i) := (\bigoplus_{i,j=1}^{N} a_i \otimes 1_{m_{ij}} \otimes 1_{(n_j)}) \oplus (\bigoplus_{i,j=1}^{N} 1_{(n_i)} \otimes 1_{m_{ji}} \otimes \overline{a_j}).
\]  

(3.1)

The multiplicities \(m_{ij}\) are non-negative integers and we denote by \(1_n\) the \(n \times n\) identity matrix and set by convention \(1_0 := 0\). At the same time the real structure \(J\) permutes the two main summands and complex conjugates them, while the \(S^0\)-real structure and the chirality read

\[
\epsilon = (\bigoplus_{i,j=1}^{N} 1_{(n_i)} \otimes 1_{m_{ij}} \otimes 1_{(n_j)}) \oplus (\bigoplus_{i,j=1}^{N} (-1)^{1_{m_{ij}}} \otimes 1_{(n_j)}),
\]

(3.2)

\[
\chi = (\bigoplus_{i,j=1}^{N} 1_{(n_i)} \otimes \chi_{ij} 1_{m_{ji}} \otimes 1_{(n_j)}) \oplus (\bigoplus_{i,j=1}^{N} \chi_{ji} 1_{m_{ji}} \otimes 1_{(n_j)}),
\]

(3.3)

where \(\chi_{ij} = \pm 1\) according to our previous convention on left-(right-)handed spinors.

We define the multiplicity matrix \(\mu \in M_N(\mathbb{Z})\) such that \(\mu_{ij} := \chi_{ij} m_{ij}\). There are \(N\) minimal projectors in \(\mathcal{A}\), each of the form \(p_i = 0 \oplus \cdots \oplus 0 \oplus \text{diag}(1_{(1)}, 0, \ldots, 0) \oplus 0 \oplus \cdots \oplus 0\). With respect to the basis \(p_i/(1)\), the matrix of the intersection form is \(\mu + \mu^T\).

If the algebra has summands with \(\mathbb{K} = \mathbb{C}\), things get more complicated. Indeed \(M_n(\mathbb{C})\) has two non-equivalent irreducible representations, the fundamental one and its complex conjugate, so we change \([3.1]\) into

\[
\rho(\bigoplus_{i=1}^{N} a_i) := (\bigoplus_{i,j=1;\alpha_i, \alpha_j} a_{i\alpha} \otimes 1_{m_{j\alpha\iota\alpha}} \otimes 1_{(n_j)}) \oplus (\bigoplus_{i,j=1}^{N} 1_{(n_i)} \otimes 1_{m_{j\alpha\iota\alpha}} \otimes \overline{a_{j\alpha}}).
\]  

(3.4)
where $\alpha_i = 1$ when $a_i \in M_{n_i}(\mathbb{K})$, $\mathbb{K} = \mathbb{R}, \mathbb{H}$ and $\alpha_i = 1, 2$ when $a_i \in M_{n_i}(\mathbb{C})$, and $a_{i1} := a_i, a_{i2} := \bar{a}_i$.

Therefore the multiplicity matrix is an integer valued square matrix of size equal to the number of summands with $\mathbb{K} = \mathbb{R}$ and $\mathbb{H}$ plus two times the number of summands with $\mathbb{K} = \mathbb{C}$ and decomposes into $N^2$ submatrices of size $1 \times 1, 2 \times 2, 1 \times 2$ and $2 \times 1$. For example $\mathcal{A} = M_{n}(\mathbb{C}) \oplus M_{m}(\mathbb{C}) \oplus M_{q}(\mathbb{R}) \ni (a, b, c)$ has a $5 \times 5$ multiplicity matrix. Let us label its rows and columns with algebra elements:

$$
\mu = \begin{pmatrix}
\mu_{aa} & \mu_{ab} & \mu_{ac} \\
\mu_{ba} & \mu_{bb} & \mu_{bc} \\
\mu_{ca} & \mu_{cb} & \mu_{cc}
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c
\end{pmatrix}.
$$

If both entries $\mu_{ij}$ and $\mu_{ji}$ of the multiplicity matrix are non-zero, then they must have the same sign.

The nonvanishing entries within each submatrix $1 \times 2$ or $2 \times 1$, like $\mu_{ca}$ or $\mu_{ac}$, must have the same sign, while the signs of the nonvanishing entries in each $2 \times 2$ submatrix, e.g. $\mu_{aa}$ or $\mu_{ab}$ must be checker board like: $\left( \begin{array}{cc}
+ & - \\
- & +
\end{array} \right)$ or $\left( \begin{array}{cc}
- & + \\
+ & -
\end{array} \right)$.

The contracted multiplicity matrix $\hat{\mu}$ is the $N \times N$ matrix constructed from $\mu$ by replacing each of the previous submatrices in $\mu$ by the sum of the entries of the submatrix.

- Poincaré duality: The last condition to be satisfied by the multiplicity matrix reflects the Poincaré duality. With respect to the basis $p_i/(1)$ introduced above, $(1) = 1$ for $\mathbb{K} = \mathbb{R}$ and $\mathbb{C}$, $(1) = 2$ for $\mathbb{K} = \mathbb{H}$, the matrix of the intersection form is $\hat{\mu} + \hat{\mu}^T$. Therefore we must have $\det(\hat{\mu} + \hat{\mu}^T) \neq 0$.

- The Dirac operator: The components of the (internal) Dirac operator are represented by horizontal or vertical lines connecting two nonvanishing entries of opposite signs in the multiplicity matrix $\mu$ and we will orient them from plus to minus. Each arrow represents a nonvanishing, complex submatrix in the Dirac operator: For instance $\mu_{ij}$ can be linked to $\mu_{ik}$ or $\mu_{kj}$ by

\[\mu_{ij} \quad \mu_{ik} \quad \mu_{kj}\]

and these arrows represent respectively submatrices of $\mathcal{M}$ in $\mathcal{D}$ of type $M \otimes 1_{(n_i)}$ with $M$ a complex $(n_j) \times (n_k)$ matrix and $1_{(n_j)} \otimes M$ with $M$ a complex $(n_i) \times (n_k)$ matrix.

The requirement of non-degeneracy of a spectral triple means that every nonvanishing entry in the multiplicity matrix $\mu$ is touched by at least one arrow.

- Convention for the diagrams: We will see that (for sums of up to three simple algebras) irreducibility implies that most entries of $\mu$ have an absolute value less than or equal to two. So we will use a simple arrow to connect plus one to minus one and double arrows to connect plus one to minus two or plus two to minus one (Figure 1.)

\[\begin{array}{c}
-1 \\
+1 \\
-2 \\
+1 \\
-1 \\
+2
\end{array}\]

Fig. 1
Our arrows always point from plus, that is right chirality, to minus, that is left chirality. For a given algebra, every spectral triple is encoded in its multiplicity matrix which itself is encoded in its Krajewski diagram, a field of arrows. In our conventions, for particles, $\epsilon = 1$, the column label of the multiplicity matrix indicates the representation, the row label indicates the multiplicity. For antiparticles, the row label of the multiplicity matrix indicates the representation, the column label indicates the multiplicity. Every arrow comes with three algebras: Two algebras that localize its end points, let us call them right and left algebras and a third algebra that localizes the arrow, let us call it colour algebra. For example for the arrow

\[
\mu_{ij} \quad \mu_{ik}
\]

the left algebra is $A_j$, the right algebra is $A_k$ and the colour algebra is $A_i$.

The circles in the diagrams only intend to guide the eye. A black disk on a double arrow indicates that the coefficient of the multiplicity matrix is plus or minus one at this location, “the two arrows are joined at this location”. For example the the following arrows

\[
\mu_{ij} \quad \mu_{ik} \quad \mu_{ij} \quad \mu_{ik}
\]

\[
\mu_{ij} \quad \mu_{ik}
\]

\[
\mu_{ij} \quad \mu_{ik}
\]

\[
\mu_{ij} \quad \mu_{ik}
\]

represent respectively submatrices of $\mathcal{M}$ of type

\[
(M_1 \quad M_2) \otimes 1_{(n_i)} \quad \text{and} \quad (M_1 \quad M_2) \otimes 1_{(n_i)}
\]

with $M_1, M_2$ of size $(n_j) \times (n_k)$ or in the third case, a matrix of type $(M_1 \otimes 1_{(n_i)} \quad 1_{(n_j)} \otimes M_2)$ where $M_1$ and $M_2$ are of size $(n_j) \times (n_k)$ and $(n_i) \times (n_{\ell})$.

According to these rules, we can omit the number $\pm 1, \pm 2$ under the arrows like in Figure 2, since they are now redundant.

Let us give a few examples of the explicit form of the spectral triple associated to a given Krajewski’s diagram:

Take the algebra $\mathcal{A} = \mathbb{H} \oplus M_3(\mathbb{C}) \ni (a, b)$ with the first diagram of Figure 2.

\[
\begin{array}{cccc}
 a & b & \bar{b} & a & b & \bar{b} & a & b & \bar{b} \\
 a & \bullet & \circ & b & \bullet & \circ & a & \bullet & \circ \\
 b & \circ & \circ & \circ & b & \circ & \circ & \circ & b \\
 \bar{b} & \circ & \circ & \circ & \bar{b} & \circ & \circ & \circ & \bar{b} \\
\end{array}
\]

Fig. 2

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Then the multiplicity matrix and its contraction are
\[
\mu = \begin{pmatrix}
-1 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \hat{\mu} = \begin{pmatrix}
-1 & 1 \\
0 & 0
\end{pmatrix}.
\]

Using (2.3), its representation is, up to unitary equivalence
\[
\rho_L(a, b) = a \otimes 1_2, \quad \rho_R(a, b) = b \otimes 1_2, \quad \rho_L^c(a, b) = 1_2 \otimes a, \quad \rho_R^c(a, b) = 1_3 \otimes a.
\]

The Hilbert space is
\[
\mathcal{H} = \mathbb{C}^4 \oplus \mathbb{C}^6 \oplus \mathbb{C}^4 \oplus \mathbb{C}^6.
\]

In its Dirac operator (2.5), \( M = M \otimes 1_2 \), where \( M \) is a nonvanishing complex \( 2 \times 3 \) matrix.

Real structure, \( S^0 \)-real structure and chirality are given by (cc stands for complex conjugation)
\[
J = \begin{pmatrix} 0 & 1_{10} \\
1_{10} & 0 \end{pmatrix} \circ \text{cc}, \quad \epsilon = \begin{pmatrix} 1_{10} & 0 \\
0 & -1_{10} \end{pmatrix}, \quad \chi = \begin{pmatrix}
-1_4 & 0 & 0 & 0 \\
0 & 1_6 & 0 & 0 \\
0 & 0 & -1_4 & 0 \\
0 & 0 & 0 & 1_6
\end{pmatrix}.
\]

The first tensor factor in \( a \otimes 1_2 \) concerns particles, the second concerns antiparticles denoted by \( .^c \). The antiparticle representation is read from the transposed multiplicity matrix.

The second diagram of Figure 2 yields
\[
\mu = \begin{pmatrix}
-1 & 2 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \hat{\mu} = \begin{pmatrix}
-1 & 2 \\
0 & 0
\end{pmatrix},
\]

and its spectral triple reads:
\[
\rho_L(a, b) = a \otimes 1_2, \quad \rho_R(a, b) = \begin{pmatrix} b & 0 \\
0 & b \end{pmatrix} \otimes 1_2, \quad \rho_L^c(a, b) = 1_2 \otimes a, \quad \rho_R^c(a, b) = 1_3 \otimes a,
\]

\( \mathcal{M} = (M_1 \ M_2) \otimes 1_2 \), \( M_1 \) and \( M_2 \) of size \( 2 \times 3 \),
\[
J = \begin{pmatrix} 0 & 1_{16} \\
1_{16} & 0 \end{pmatrix} \circ \text{cc}, \quad \epsilon = \begin{pmatrix} 1_{16} & 0 \\
0 & -1_{16} \end{pmatrix}, \quad \chi = \begin{pmatrix}
-1_4 & 0 & 0 & 0 \\
0 & 1_{12} & 0 & 0 \\
0 & 0 & -1_4 & 0 \\
0 & 0 & 0 & 1_{12}
\end{pmatrix}.
\]

In Krajewski’s notations, equation (3.1), we would have written \( \rho_R(a, b) = b \otimes 1_2 \otimes 1_2 \), the middle factor showing the entry of the multiplicity matrix \( \mu \).

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Finally, still for the same algebra, let us consider the third diagram of Figure 2. It gives

$$\mu = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{\mu} = \begin{pmatrix} -1 & 1 \end{pmatrix},$$

and

$$\rho_L(a, b) = a \otimes 1_2, \quad \rho_R(a, b) = \begin{pmatrix} b \otimes 1_2 & 0 \\ 0 & a \otimes 1_3 \end{pmatrix},$$

$$\rho'_L(a, b) = 1_2 \otimes a, \quad \rho'_R(a, b) = \begin{pmatrix} 1_3 \otimes a & 0 \\ 0 & 1_2 \otimes b \end{pmatrix},$$

$$\mathcal{M} = (M_1 \otimes 1_2 \ 1_2 \otimes M_2), \quad M_1 \text{ and } M_2 \text{ of size } 2 \times 3.$$  

In the three above examples all arrows have left algebra $\mathbb{H}$, right algebra $M_3(\mathbb{C})$ and colour algebra $\mathbb{H}$. The numerous examples below should allow the reader to get familiar with the translation between diagrams and triples.

### 3.2 Multiplicity matrix and irreducibility

Our work is based on the following lemma indicating that a classification of irreducible spectral triples is possible.

**Lemma 3.1.** i) The direct sum of multiplicity matrices is again a multiplicity matrix describing the direct sum of spectral triples.  
ii) For a given algebra there is only a finite list of multiplicity matrices describing irreducible triples.

**Proof.** i) is obvious.

ii) Given an algebra $\mathcal{A}$, denote by $\mathcal{S}$ the set of multiplicity matrices $\mu \in M_n(\mathbb{Z})$ ($n$ is determined by $\mathcal{A}$) associated to irreducible spectral triples. Define a partial order in $M_n(\mathbb{Z})$ by $\mu \geq \nu$ when $\mu_{ij}$ and $\nu_{ij}$ have the same sign and $|\mu_{ij}| \geq |\nu_{ij}|$ for all $i, j = 1, \ldots, n$. The interest of this order is that for two different multiplicity matrices $\mu, \nu$ such that $\mu \geq \nu$ and $\nu \in \mathcal{S}$, $\mu$ corresponds to a reducible triple.

To prove that $\text{card}(\mathcal{S}) < \infty$, we first identify $M_n(\mathbb{Z})$ with $\mathbb{Z}^{n^2}$. We denote by $\{e_i\}_{i \in \{1, \ldots, n^2\}}$ be the canonical basis of $\mathbb{Z}^{n^2}$ and $\mathcal{S}^+ := \mathcal{S} \cap \mathbb{N}^{n^2}$. We now prove that $\text{card}(\mathcal{S}^+) < \infty$, which is sufficient.

Assume $\text{card}(\mathcal{S}^+) = \infty$. Then there exists at least one direction (say along $e_1$) such that $\sup\{(|\mu|_1)|\mu \in \mathcal{S}^+\} = \infty$. Suppose now that in each hyperplane defined by $me_1, m \in \mathbb{N}$, all points $\mu \in \mathcal{S}^+$ remain uniformly bounded, $\mu_i < B, i = 2, \ldots, n^2$. This means that there exists an infinite family of points in an hypertube parallel to $e_1$. Necessarily, there exists an infinite subfamily $\{\mu_k\}_{k \in \mathbb{N}} \subset \mathcal{S}^+$ of points in the hypertube which are aligned: $(\mu_k)_i = (\mu_{k+1})_i, \forall i \neq 1$ and $(\mu_k)_1 < (\mu_{k+1})_1$. But this is impossible since $\mu_k \leq \mu_{k+1}$ cannot happen. As a consequence, there exists a second direction (say along $e_2$ after
renumbering) where the intersection of successive hyperplanes along \( e_1 \) and \( e_2 \) contain an infinite family of points of \( S^+ \). By induction, the same reasoning in each direction \( e_i \) implies that there exits an infinite family \( \{ \mu_k \}_k \) of points in \( S^+ \) with increasing vectors: \( (\mu_k)_i < (\mu_{k+1})_i, \forall i \). Again this yields the contradiction \( \mu_k \leq \mu_{k+1} \) and \( \mu_{k+1} \notin S^+ \). \( \square \)

4 Fluctuations and dynamical non-degeneracy

The aim of this work is two-fold. First, we work out all irreducible real, \( S^0 \)-real diagrams for algebras with one, two and three simple summands. Second, we give all associated spectral triples that are ‘dynamically non-degenerate’. By this we mean the following.

The spectrum of the (internal) Dirac operator \( D \) is always degenerate: all nonvanishing eigenvalues come in pairs of opposite sign due to the chirality that anticommutes with \( D \). All eigenvalues appear twice due to the real structure that commutes with \( D \). There is a further degeneracy, two-fold in the example above, \( M = M \otimes I_2 \), that comes from the first order axiom. Let us call it colour degeneracy. It is absent if and only if the colour algebras of all arrows are commutative. Of course, these three degeneracies survive the fluctuations of the Dirac operator and the minimization of the Higgs potential. By dynamical non-degenerate we mean (see precise definition below) that no minimum of the Higgs potential has degeneracies other than the above three. The first two degeneracies survive quantum fluctuations as well. We also want the colour degeneracies to be protected from quantum fluctuations. A natural protection is unbroken gauge invariance, a requirement that we will include into the definition of dynamical non-degeneracy.

Except for complex conjugation in \( M_n(\mathbb{C}) \) and permutations of identical summands in the algebra \( A = A_1 \oplus A_2 \oplus \ldots \oplus A_N \), every algebra automorphism \( \sigma \) is inner, \( \sigma(a) = uau^{-1} \) for a unitary \( u \in U(A) \). Therefore the connected component of the automorphism group is \( Aut(A)^e = U(A)/(U(A) \cap Center(A)) \). Its lift to the Hilbert space \( L(\sigma) = \rho(u)J\rho(u)J^{-1} \) is multi-valued.

The fluctuation \( f^D \) of the Dirac operator \( D \) is given by a finite collection \( f \) of real numbers \( r_j \) and algebra automorphisms \( \sigma_j \in Aut(A)^e \) such that

\[
f^D := \sum_j r_j L(\sigma_j) D L(\sigma_j)^{-1}, \quad r_j \in \mathbb{R}, \quad \sigma_j \in Aut(A)^e.
\]

The fluctuated Dirac operator \( f^D \) is often denoted by \( \varphi \), the ‘Higgs scalar’, in the physics literature. We consider only fluctuations with real coefficients since \( f^D \) must remain selfadjoint.

To avoid the multi-valuedness in the fluctuations, we allow the entire unitary group viewed as a (maximal) central extension of the automorphism group. We will come back to minimal central extensions in another work.

An almost commutative geometry is the tensor product of a finite noncommutative triple with an infinite, commutative spectral triple. By Connes’ reconstruction theorem \( \square \) we know that the latter comes from a Riemannian spin manifold, which we will take to
be any 4-dimensional, compact, flat manifold like the flat 4-torus. The spectral action of this almost commutative spectral triple reduced to the finite part is a functional on the vector space of all fluctuated, finite Dirac operators:

\[ V(\mathcal{D}) = \lambda \text{tr}[(\mathcal{D})^4] - \mu^2 \frac{1}{2} \text{tr}[(\mathcal{D})^2], \]

where \( \lambda \) and \( \mu \) are positive constants \([3, 11]\). The spectral action is invariant under lifted automorphisms and even under the unitary group \( U(\mathcal{A}) \ni u \),

\[ V([\rho(u)J\rho(u)J^{-1}]\mathcal{D}[\rho(u)J\rho(u)J^{-1}]) = V(\mathcal{D}), \]

and it is bounded from below. Our task is to find the minima \( \mathcal{D} \) of this action, their spectra and their little groups

\[ G_\ell := \left\{ u \in U(\mathcal{A}), \; [\rho(u)J\rho(u)J^{-1}]\mathcal{D}[\rho(u)J\rho(u)J^{-1}] = \mathcal{D} \right\}. \]

**Definition 4.1.** The irreducible spectral triple \((\mathcal{A}, \mathcal{H}, \mathcal{D})\) is **dynamically non-degenerate** if all minima \( \mathcal{D} \) of the action \( V(\mathcal{D}) \) define a non-degenerate spectral triple \((\mathcal{A}, \mathcal{H}, \mathcal{D})\) and if the spectra of all minima have no degeneracies other than the three kinematical degeneracies: left-right, particle-antiparticle and colour. Of course in the massless case there is no left-right degeneracy. We also suppose that the colour degeneracies are protected by the little group. By this we mean that all eigenvectors of \( \mathcal{D} \) corresponding to the same eigenvalue are in a common orbit of the little group (and scalar multiplication and charge conjugation).

In physicists’ language this last requirement means noncommutative colour groups are unbroken. It ensures that the corresponding mass degeneracies are protected from quantum corrections.

## 5 Statement of the result

The main result of this work is the following

**Theorem 5.1.** The sum of simple algebras, \( \mathcal{A} = \bigoplus_{i=1}^{N} \mathcal{A}_i \) with \( N = 1, 2, 3 \) admits a finite, real, \( S^0 \)-real, irreducible and dynamically non-degenerate spectral triple if and only if it is in this list, up to a reordering of the summands:

| \( N = 1 \) | \( N = 2 \) | \( N = 3 \) |
|------------|------------|------------|
| void       | 1 \( \oplus \) 1 | 1 \( \oplus \) 1 \( \oplus \) C |
| 2 \( \oplus \) 1 | 1 \( \oplus \) 1 \( \oplus \) 1 | 1 \( \oplus \) 1 \( \oplus \) 1 |
| 2 \( \oplus \) 1 | 2 \( \oplus \) 1 \( \oplus \) C | 2 \( \oplus \) 1 \( \oplus \) 1 |

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Here 1 is a short hand for \( \mathbb{R} \) or \( \mathbb{C} \) and 2 for \( M_2(\mathbb{R}) \), \( M_2(\mathbb{C}) \) or \( \mathbb{H} \). The ‘colour’ algebra \( \mathcal{C} \) is any simple algebra and has two important constraints:

i) Its representations on corresponding left- and right-handed subspaces of \( \mathcal{H} \) are identical (up to possibly different multiplicities).

ii) The Dirac operator \( D \) is invariant under \( U(\mathcal{C}) \),

\[
\rho(1, 1, w) D \rho(1, 1, w)^{-1} = D, \quad \text{for all } w \in U(\mathcal{C}).
\]

This implies that the unitaries of \( \mathcal{C} \) do not participate in the fluctuations and are therefore unbroken, i.e. elements of the little group.

Let us emphasize that although the 4-dimensional ‘spacetime’ manifold \( M \) used to define the almost commutative geometry does not show up in this result, it is an important ingredient of the spectral action and its asymptotic behavior. In particular the dimension of \( M \) is linked to the order of the polynomial \( V \). Therefore our classification indeed concerns 4-dimensional, almost commutative geometries.

We give in section 9.1 an example of a reducible triple which is dynamically non-degenerate and whose algebra is not in the above list.

6 One simple algebra

From the classification \([14,10]\), we know that \( \mathcal{A} = M_n(\mathbb{C}) \) are the only simple algebras to admit real spectral triples. Up to permutation of \( a \) and \( \bar{a} \) (complex conjugation in \( \mathcal{A} \)), up to permutation of particles and antiparticles (reflection of the diagram with respect to the main diagonal) and up to permutation of left- and right-handed particles (changing the direction of all arrows), all real \( S^0 \)-real and irreducible triples have Krajewski diagrams indicated in Figure 3:

Indeed, a Krajewski diagram must contain at least one arrow which can be put into the position \( \mu = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \) by use of the three above permutations. However, alone this arrow does not fulfill Poincaré duality, \( \hat{\mu} + \hat{\mu}^T = 0 \). There are four ways to add a second arrow. But Poincaré duality can only be satisfied if the two arrows are joined in one point. Adding a third arrow makes the diagram corresponding to a reducible spectral triple.

The first diagram yields:

\[
\rho_L(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \otimes 1_n, \rho_R(a) = \bar{a} \otimes 1_n, \rho_L^*(a) = 1_n \otimes \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \rho_R^*(a) = 1_n \otimes a, \quad (6.1)
\]
\[ \mathcal{M} := M \otimes 1_n := \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \otimes 1_n, \quad M_1, M_2 \in M_n(\mathbb{C}). \]

Real structure, \(S^0\)-real structure and chirality are:

\[ J = \begin{pmatrix} 0 & 1_{3n^2} \\ 1_{3n^2} & 0 \end{pmatrix} \circ \text{cc}, \quad \epsilon = \begin{pmatrix} 1_{3n^2} & 0 \\ 0 & -1_{3n^2} \end{pmatrix}, \quad \chi = \begin{pmatrix} -1_{2n^2} & 0 & 0 & 0 \\ 0 & +1_{n^2} & 0 & 0 \\ 0 & 0 & -1_{2n^2} & 0 \\ 0 & 0 & 0 & +1_{n^2} \end{pmatrix}. \]

Let us write the fluctuations of \(D\) as

\[ fD := \begin{pmatrix} 0 & fM \\ fM^* & 0 \\ 0 & 0 & 0 & fM \end{pmatrix}. \quad (6.2) \]

Here

\[ fM = fM \otimes 1_n = \begin{pmatrix} fM_1 \\ fM_2 \end{pmatrix} \otimes 1_n. \]

Then

\[ fM_1 = \sum_j r_j u_j M_1 u_j^T, \quad fM_2 = \sum_j r_j u_j M_2 u_j^T, \quad \text{so} \quad fM = \sum_j r_j \begin{pmatrix} u_j & 0 \\ 0 & u_j \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} u_j^T. \]

and

\[ V(fD) = \lambda \text{tr}[fD^4] - \frac{\mu^2}{2} \text{tr}[fD^2] = 4n (\lambda \text{tr}[fM^* fM]) - \frac{\mu^2}{2} \text{tr}[fM^* fM]). \]

The matrix \(fM\) is of size \(2n \times n\). Therefore the fluctuation \(fD\) has at least \(n\) vanishing eigenvalues each still coming with its \(n\)-fold colour degeneracy and all triples with \(n \geq 2\) are dynamically degenerate since \(fM fM^* \in M_{2n}(\mathbb{C})\) has at least \(n\) zero eigenvalues.

For \(n = 1\), all minima of the action \(V\) are of the form \(|fM_1|^2 + |fM_2|^2 = \frac{\mu^2}{4\lambda}\). But the corresponding fluctuated Dirac operator has a nontrivial invariant subspace in its kernel and the triple is degenerate. To make the subspace explicit apply a unitary change of basis to set \(fM_2\) to zero. In every minimum the little group is \(Z_2\).

At this point, an overkill is instructive. We will show that the case \(n \geq 2\) also features dynamical degeneracy in the non-zero eigenvalues.

In general, the set of all possible fluctuations \(fD\), i.e. the image under the fluctuations \(6.3\), is difficult to describe. However, the action \(V\) only depends on the positive \(n \times n\) matrix \(C := fM^* fM\) and is a sum of \(n^2\) polynomials of fourth order in the matrix elements of \(C\):

\[ V(C) = 4n (\sum_{i=1}^{n} (\lambda C_{ii}^2 - \frac{\mu^2}{2} C_{ii}) + \sum_{i \neq j} \lambda |C_{ij}|^2). \quad (6.3) \]
If $C = \frac{u^2}{4\lambda} 1_n$ is in this image then it is the unique minimum in terms of the variable $C$.

To compute the minima of the action, we now distinguish cases:

1. At least one diagonal element of one of the two matrices is non-zero:

   After a suitable renumbering of the basis of the Hilbert space $\mathcal{H}$, we may assume $(M_1)_{11} \neq 0$. With a first fluctuation,

   $$r_1 = \frac{1}{2}, r_2 = \frac{1}{2}, \quad u_1 = 1_n, \quad u_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1_{n-1} \end{pmatrix},$$

   we obtain for $\hat{f}M$ a block diagonal matrix with $2 \times 2$ blocks. By means of the fluctuation

   $$r_1 = \frac{1}{2}, r_2 = \frac{1}{2}, \quad u_1 = 1_n, \quad u_2 = \begin{pmatrix} 1 & 0 \\ 0 & i_{1_{n-1}} \end{pmatrix},$$

   we isolate the $(1,1)$ elements of $M_1$ and $M_2$ and with $r_1 = r_2 = \ldots = r_n = 1, u_1 = 1_n,$

   $$u_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1_{n-2} \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-3} \end{pmatrix}, \quad u_4 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_{n-4} \end{pmatrix}, \ldots$$

   we distribute them over the entire diagonal obtaining

   $$\hat{f}M = \begin{pmatrix} (M_1)_{11} 1_n \\ (M_2)_{11} 1_n \end{pmatrix}.$$  

Then $C$ is a non-vanishing multiple of the identity and a suitable multiple of the above $\hat{f}M$ is a minimum. The spectrum of this minimum $\hat{f}D$ has an additional dynamical degeneracy and up to the sign, $\hat{f}D$ has one single eigenvalue. The little group is $G_\ell = O(n)$. Note also that $\hat{f}M_1$ and $\hat{f}M_2$ are proportional, $\hat{f}M_1 = \alpha \hat{f}M_2, \alpha \in \mathbb{C},$ so there is a unitary change of basis in the Hilbert space $\mathcal{H}$ such that $\hat{f}M_2 = 0$ in the new basis and the spectral triple $(M_n(\mathbb{C}), \mathcal{H}, \hat{f}D)$ is degenerate.

2. All diagonal elements of $M_1$ and $M_2$ vanish but $M_1$ and $M_2$ are not both skewsymmetric:

   After a suitable renumbering, we may assume $(M_1)_{12} = \beta - \gamma, (M_1)_{21} = \beta + \gamma$, $(M_1)_{11} = (M_1)_{22} = 0$, with $\beta \neq 0$. As in case 1 we can isolate this block. Fluctuating with

   $$r_1 = 1, \quad u_1 = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1_{n-2} \end{pmatrix},$$

we obtain $(\hat{f}M_1)_{11} = -\beta$ and conclude as in case 1.

3. $M_1$ and $M_2$ are skewsymmetric and linearly dependent:

   Thus $M$ can be written as $M = M_1 \otimes \begin{pmatrix} 1 \\ \alpha \end{pmatrix}, \alpha \in \mathbb{C}$ and by a unitary change of basis, we may assume $\alpha = 0$ that is $M_2 = 0$ without changing the representation (6.1). Then
the spectral triple is degenerate. Let us nevertheless finish this case. By another unitary change of basis (or fluctuation), cf. appendix [A.1], we put

$$M_1 = \begin{pmatrix} 0 & -\lambda_1 & & \\ -\lambda_1 & 0 & & \\ & \ddots & \ddots & \\ & & & 0 \end{pmatrix},$$

where the zero in the lower right corner concerns the case $n$ odd. Let us suppose that $\lambda_1$ is not zero. By isolating the upper left corner and by distributing it over the entire diagonal we obtain a $fM_{1ii} = \lambda_1$ for all $i$. If $n$ is even, we then have the minimum $\hat{C} = \frac{\mu}{\sqrt{2\lambda}} \mathbf{1}_n$ for $V(C)$. Its spectrum is again completely degenerate with little group $G_\ell = USp(\frac{n}{2})$. If $n$ is odd the spectrum of $fD$ contains 4 vanishing eigenvalues, all others being of same absolute value and $G_\ell = USp(\frac{n-1}{2}) \times U(1)$.

4: $M_1$ and $M_2$ are skew-symmetric and linearly independent:

Then $fM_1$ and $fM_2$ vary independently over all skew-symmetric matrices (cf. Lemma A.2). For $n = 3$ the minimization can be done by direct calculation. All minima are gauge equivalent to

$$fM = \frac{\mu}{\sqrt{6\lambda}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \hat{C} := fM^* fM = \frac{\mu^2}{6\lambda} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$  

Although the spectrum has a two-fold dynamical degeneracy the little group is only $G_\ell = U(1) \ni \text{diag}(e^{i\vartheta}, e^{-i\vartheta}, e^{-i\vartheta})$. For general $n$, we were unable to compute a minimum explicitly, but we will show that its spectrum is dynamically degenerate. To alleviate notations let us rescale variables, $fD \rightarrow \frac{\mu}{\sqrt{2\lambda}} fD$. Then the spectral action reads for $C_i := fM_i^* fM_i$

$$V(fM_1, fM_2) = \frac{\mu n}{\lambda} \left( \text{tr}[C_1^2] - \text{tr}[C_1] + \text{tr}[C_2^2] - \text{tr}[C_2] + 2 \text{tr}[C_1 C_2] \right).$$

All minima of this fourth order polynomial have vanishing partial derivatives with respect to $fM_1^*$ and $fM_2^*$. These equations read

$$fM_1 = 2 fM_1 fM_1^* fM_1 + fM_1 fM_2^* fM_2 + fM_2 fM_2^* fM_1, \quad (6.4)$$

$$fM_2 = 2 fM_2 fM_2^* fM_2 + fM_2 fM_1^* fM_1 + fM_1 fM_1^* fM_2. $$

Let us put $X := \hat{C}_1$, $Y := \hat{C}_2$ and $Z := \hat{M}_1^* \hat{M}_2$. We multiply equation (6.4) from the left by $fM_1$ and likewise for (6.5) to get

$$X = 2X^2 + XY + ZZ^*, \quad (6.5)$$

$$Y = 2Y^2 + XY + Z^* Z.$$
Subtracting (6.5) from its Hermitian conjugate, we have that $X$ and $Y$ commute and subtracting (6.6) from (6.5) we get

$$2(X - Y)(X + Y - \frac{1}{2}1_n) = [Z^*, Z].$$

Multiplying (6.5) from the left with $\hat{f}M_1^*$ and multiplying the Hermitian conjugate of (6.4) from the right with $\hat{f}M_2^*$, we have

$$Z = 2ZY + ZX + XZ,$$

Taking the difference, we find that $Z$ commutes with $X + Y$.

We are to show that the spectrum of $\hat{C} = \hat{f}M_1^* \hat{f}M = X + Y$ is degenerate: Let us suppose that it is non-degenerate. Take an orthonormal basis of eigenvectors of $\hat{C}$. Then it is also an eigenbasis of $Z$ implying that $[Z, Z^*] = 0$ and by (6.6) the eigenvalues $x_j$ and $y_j$ of $X$ and $Y$ corresponding to the $j$-th basis vector satisfy at least one of the equations $x_j = y_j$, $x_j + y_j = \frac{1}{2}$. But as $\hat{f}M_1$ and $\hat{f}M_2$ are skewsymmetric, each eigenvalue of $X$ and $Y$ is doubly degenerate with the exception of one vanishing eigenvalue if $n$ is odd. This contradicts the non-degeneracy of $X + Y$.

The triple of the second diagram of figure 3 differs from the first only with respect to representation and Dirac operator

$$\rho_L(a) = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \otimes 1_n, \quad \rho_R(a) = \bar{a} \otimes 1_n, \quad \rho_L^*(a) = 1_n \otimes \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}, \quad \rho_R^*(a) = 1_n \otimes a,$$

$$\mathcal{M} = \begin{pmatrix} M_1 \otimes 1_n \\ 1_n \otimes M_2 \end{pmatrix}, \quad M_1, M_2 \in M_n(\mathbb{C}).$$

Again, non-degeneracy of the zero eigenvalue requires $n = 1$ and all minima, $|\hat{f}M_1|^2 + |\hat{f}M_2|^2 = \frac{n^2}{4}$, have little group $\mathbb{Z}_2$. But now all minima of the action $V$ are not gauge equivalent. Indeed when $\hat{f}M_1 = \mu(4\lambda)^{-1/2}$, $\hat{f}M_2 = 0$ and $\text{Ker}(\mathcal{D}) = \text{Span}\{\mathcal{H}_L, \mathcal{H}_c \}$. Thus the eigenvector in the image of $\frac{1}{\sqrt{2}}(1 - \epsilon)$ associated to the zero eigenvalue of the fluctuated Dirac operator, which is in $\mathcal{H}_L$, transforms under $\rho(u)\mathcal{D}\rho(u)^{-1}$ as multiplication by $u^{-2}$, while it transforms as multiplication by $u^2$ when $\hat{f}M_2 = \mu(4\lambda)^{-1/2}$. According to our definition the triple is dynamically degenerate because in both cases, the eigenvectors define a one-dimensional complex subspace invariant under $\mathcal{A}$ and contained in the kernel of the fluctuated Dirac operator $\mathcal{D}$.

The last two diagrams of Figure 3 are treated as the first two and yield the same conclusions.

### 7 Two simple algebras

Again our starting point is the list, Figure 4, of all irreducible Krajewski diagrams up to the three mentioned types of permutations and up to permutations of the two algebras and disregarding any direct sum of two diagrams from Figure 3.
Let $k$ and $\ell$ be the size of the matrices of $A_1 = M_n(\mathbb{K}) \ni a$ and $A_2 = M_m(\mathbb{K}) \ni b$. E.g. $k = n$ for $A_1 = M_n(\mathbb{R})$ or $M_n(\mathbb{C})$ and $k = 2n$ for $A_1 = M_n(\mathbb{H})$.

The first diagram of figure 4 yields:

$$\rho_L(a,b) = a \otimes 1_k, \quad \rho_R(a,b) = b \otimes 1_k, \quad \rho^{\circ}_L(a,b) = 1_k \otimes a, \quad \rho^{\circ}_R(a,b) = 1_\ell \otimes a,$$

$$\mathcal{M} = M \otimes 1_k, \quad M \in M_{k \times \ell}(\mathbb{C}).$$

$M$ is non-zero and we may assume $M_{11} \neq 0$. Except for the $k$-fold colour degeneracy, we accept at most one zero eigenvalue of $M^* M$. Therefore we must have $k = \ell$ or $k = \ell \pm 1$.

We assume $\ell \leq k$. Let in (6.2)

$$f_M = fM \otimes 1_k, \quad fM = \sum_j r_j u_j M v_j^{-1}, \quad u_j \in U(A_1), \quad v_j \in U(A_2).$$

If $k \geq 2$, $\ell > 2$ or $k > 2$, $\ell \geq 2$, we may isolate the upper $2 \times 2$ block by fluctuations: With the first fluctuation,

$$r_1 = \frac{1}{2}, \quad r_2 = \frac{1}{2}, \quad u_1 = 1_k, \quad v_1 = 1_\ell, \quad u_2 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_{k-2} \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_{\ell-2} \end{pmatrix},$$

we obtain for $M$ a block diagonal type matrix. By means of the fluctuation

$$r_1 = \frac{1}{2}, \quad r_2 = \frac{1}{2}, \quad u_1 = 1_k, \quad v_1 = 1_\ell, \quad u_2 = 1_k, \quad v_2 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_{\ell-2} \end{pmatrix}.$$
we isolate the upper block. If $k = \ell = 2$ this step is empty.

Now we may distinguish cases.

1: $\mathcal{A}_1 = M_n(\mathbb{R})$ or $M_n(\mathbb{C})$, $\mathcal{A}_2 = M_m(\mathbb{R})$ or $M_m(\mathbb{C})$: like above, we isolate $M_{11}$ and as in case 1 for one algebra we distribute $M_{11}$ over the entire diagonal, obtaining thus $fM^*fM$ proportional to the identity. The spectrum of the fluctuated Dirac operator $fD$ minimizing the action $V$ has an $\ell$-fold dynamical degeneracy.

2: $\mathcal{A}_1 = M_n(\mathbb{H})$, $\mathcal{A}_2 = M_m(\mathbb{H})$, define the fluctuation

$$r_1 = \frac{1}{2}, r_2 = \frac{1}{4}, r_3 = \frac{1}{4}, \quad u_1 = 1_k, \quad v_1 = 1_\ell, \quad u_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 1_{k-2} \end{pmatrix}, \quad v_2 = \begin{pmatrix} \pm \left( \begin{array}{ccc} 0 & 1 \\ -1 & 0 \\ 0 & 1_{\ell-2} \end{array} \right) \\ \begin{array}{c} 0 \\ i \\ 0 \end{array} \\ 0 & 1_{\ell-2} \end{pmatrix}, \quad v_1 = \begin{pmatrix} \pm \left( \begin{array}{ccc} 0 & 1 \\ i & 0 \\ 0 & 1_{k-2} \end{array} \right) \\ \begin{array}{c} 0 \\ i \\ 0 \end{array} \\ 0 & 1_{k-2} \end{pmatrix}.$$ 

the upper block is proportional to $1_2$. The plus signs in $\pm$ are used if $M_{11} = M_{22}$. We distribute the block over the diagonal and get an $\ell$-fold dynamical degeneracy (recall that $\ell \leq k$).

3: $\mathcal{A}_1 = M_n(\mathbb{R})$ or $M_n(\mathbb{C})$, $\mathcal{A}_2 = M_m(\mathbb{H})$. If $M_{12} = 0$ we fluctuate with

$$r_1 = r_2 = \frac{1}{2}, r_3 = \frac{1}{2}, \quad u_1 = 1_k, \quad v_1 = 1_\ell, \quad u_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1_{k-2} \end{pmatrix}, \quad v_2 = 1_\ell,$$

$$u_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1_{k-2} \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1_{\ell-2} \end{pmatrix},$$

and obtain

$$fM = \begin{pmatrix} M_{11} & 0 \\ 0 & -M_{11} \\ 0 & 0 \end{pmatrix}.$$ 

If $M_{12} \neq 0$ we fluctuate with

$$r_1 = r_2 = \frac{1}{2}, r_3 = \frac{1}{2}, \quad u_1 = 1_k, \quad v_1 = 1_\ell, \quad u_2 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_{k-2} \end{pmatrix}, \quad v_2 = 1_\ell,$$

$$u_3 = 1_k, \quad v_3 = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \\ 0 & -1_{\ell-2} \end{pmatrix},$$
with $\theta := \frac{1}{2}(\text{Arg}(M_{11}) - \text{Arg}(M_{12}))$ and obtain

$$f_M = \begin{pmatrix} e^{i\theta}M_{11} & e^{-i\theta}M_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$ 

In both cases, we distribute the $2 \times 2$ block over the diagonal and achieve $f_M^* f_M$ proportional to the identity.

In all three cases, $\ell$ must be one to avoid dynamical degeneracy.

The second diagram of Figure 4 is treated in the same fashion. The last two diagrams of Figure 4 have no ‘letter changing arrow’, an arrow connecting an $a$ to a $b$. They are treated as the triples with one simple algebra: Counting neutrinos, that is requiring at most one zero eigenvalue (up to a possible colour degeneracy) yields $\mathcal{A} = \mathbb{C} \oplus \mathbb{C}$ and degeneracy.

Finally using the permutations we get the list of all irreducible, dynamically non-degenerate triples with two algebras:

- There are the commutative triples, that is the two-point spaces, $\mathcal{A} = \mathbb{C} \oplus \mathbb{C} \ni (a, b)$:

  $$\rho(a, b) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & \bar{a} & 0 \\ 0 & 0 & 0 & \bar{a} \end{pmatrix}, \text{ and in } \mathcal{D}, \mathcal{M} \in \mathbb{C}. \quad (7.1)$$

  There is a second one with the same algebra:

  $$\rho(a, b) = \begin{pmatrix} \bar{a} & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & \bar{a} & 0 \\ 0 & 0 & 0 & \bar{a} \end{pmatrix}. \quad (7.2)$$

  And there are the real versions, $\mathcal{A} = \mathbb{C} \oplus \mathbb{R}$, $\mathbb{R} \oplus \mathbb{C}$ and $\mathbb{R} \oplus \mathbb{R}$. 

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- The noncommutative triples have $A = M_2(\mathbb{C}) \oplus \mathbb{C} \ni (a, b)$ with four irreducible triples:

$$\rho(a, b) = \begin{pmatrix} a \otimes 1_2 & 0 & 0 & 0 \\ 0 & b_{12} & 0 & 0 \\ 0 & 0 & 1_2 \otimes a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} 0 \\ m \end{pmatrix} \otimes 1_2, \quad m \in \mathbb{C}, \quad (7.3)$$

$$\rho(a, b) = \begin{pmatrix} \bar{a} \otimes 1_2 & 0 & 0 & 0 \\ 0 & b_{12} & 0 & 0 \\ 0 & 0 & 1_2 \otimes \bar{a} & 0 \\ 0 & 0 & 0 & \bar{a} \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} 0 \\ m \end{pmatrix} \otimes 1_2, \quad (7.4)$$

$$\rho(a, b) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b_{12} & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} 0 \\ m \end{pmatrix}, \quad (7.5)$$

$$\rho(a, b) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & \bar{b} & 0 & 0 \\ 0 & 0 & b_{12} & 0 \\ 0 & 0 & 0 & \bar{b} \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} 0 \\ m \end{pmatrix}. \quad (7.6)$$

In all four cases, all minima $\hat{f}\mathcal{D}$ are gauge equivalent to the Dirac operator $\mathcal{D}$ with the absolute value of $m$ fixed in terms of $\lambda$ and $\mu$ and the little groups are

$$G_\ell = U(1) \times U(1) \ni \left( e^{i\alpha} \begin{pmatrix} 0 \\ e^{i\beta} \end{pmatrix}, e^{i\beta} \right).$$

In the two triples (7.3) and (7.4), the unitaries of the colour algebra, $M_2(\mathbb{C})$, are spontaneously broken, they do not leave any minimum invariant, $U(2) \not\subset G_\ell$. According to our definition these two triples are dynamically degenerate.

- Then there are diverse real versions: replace the complex $2 \times 2$ matrices $M_2(\mathbb{C})$ by quaternions $\mathbb{H}$ or by real matrices $M_2(\mathbb{R})$ and/or replace $\mathbb{C}$ by $\mathbb{R}$. For the real forms, of course, we have no complex conjugations in the representations. We summarize the little groups (arrows mean group homomorphisms):

$$\begin{align*}
\mathbb{C} \oplus \mathbb{C} & \ni U(1) \times U(1) \rightarrow U(1), \\
\mathbb{C} \oplus \mathbb{R} & \ni U(1) \times \mathbb{Z}_2 \rightarrow U(1), \\
\mathbb{R} \oplus \mathbb{R} & \ni \mathbb{Z}_2 \times \mathbb{Z}_2 \rightarrow \mathbb{Z}_2, \\
M_2(\mathbb{C}) \oplus \mathbb{C} & \ni U(2) \times U(1) \rightarrow U(1) \times U(1), \\
M_2(\mathbb{C}) \oplus \mathbb{R} & \ni U(2) \times \mathbb{Z}_2 \rightarrow U(1) \times \mathbb{Z}_2, \\
\mathbb{H} \oplus \mathbb{C} & \ni SU(2) \times U(1) \rightarrow U(1), \\
\mathbb{H} \oplus \mathbb{R} & \ni SU(2) \times \mathbb{Z}_2 \rightarrow \mathbb{Z}_2, \\
M_2(\mathbb{R}) \oplus \mathbb{C} & \ni O(2) \times U(1) \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2, \\
M_2(\mathbb{R}) \oplus \mathbb{R} & \ni O(2) \times \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2. \quad (7.7)
\end{align*}$$
The triples (7.3) or its commutative version (7.1), (7.4) or its commutative version (7.2), (7.5), and (7.6) are represented by the four diagrams of figure 5.

Fig. 5

8 Three simple algebras

8.1 Proof for $N = 3$

So far, we found that all irreducible, dynamically non-degenerate triples were associated to diagrams with letter changing arrows only, i.e. arrows connecting two different algebras. These arrows are stable under contraction of the multiplicity matrix. Therefore we start by constructing all irreducible, contracted diagrams, Figure 6. In other words we neglect the complex conjugate representations.

This list becomes exhaustive upon permutations of the three algebras $\mathcal{A}_1 = M_n(\mathbb{K}) \ni a$, $\mathcal{A}_2 = M_m(\mathbb{K}) \ni b$, $\mathcal{A}_3 = M_q(\mathbb{K}) \ni c$, upon permuting left and right, i.e. changing the directions of all two or three arrows simultaneously, and upon permutations of particles and antiparticles independently in every connected component of the diagram.

Let $k$, $\ell$, $p$ be the sizes of the matrices $a$, $b$, $c$.

Diagram 1 yields:

$$\rho_L(a, b, c) = \begin{pmatrix} a \otimes 1_k & 0 \\ 0 & b \otimes 1_\ell \end{pmatrix}, \quad \rho_R(a, b, c) = \begin{pmatrix} b \otimes 1_k & 0 \\ 0 & c \otimes 1_\ell \end{pmatrix}.$$
\[ \rho_L^c(a, b, c) = \begin{pmatrix} 1_k \otimes a & 0 \\ 0 & 1_\ell \otimes b \end{pmatrix}, \quad \rho_R^c(a, b, c) = \begin{pmatrix} 1_\ell \otimes a & 0 \\ 0 & 1_p \otimes b \end{pmatrix}, \]

and

\[ \mathcal{M} = \begin{pmatrix} M_1 \otimes 1_k & 0 \\ 0 & M_2 \otimes 1_\ell \end{pmatrix}, \quad M_1 \in M_{k \times \ell}(\mathbb{C}), \quad M_2 \in M_{\ell \times p}(\mathbb{C}). \]

The fluctuations,

\[ f_{M_1} = \sum_j r_j u_j M_1 v_j^{-1}, \quad u_j \in U(A_1), \quad v_j \in U(A_2), \]
\[ f_{M_2} = \sum_j r_j v_j M_2 w_j^{-1}, \quad w_j \in U(A_3), \]

produce two decoupled fields \( f_{M_1} \) and \( f_{M_2} \) as can be seen by applying the fluctuation:

\[ r_1 = \frac{1}{2}, \quad u_1 = 1_k, \quad v_1 = 1_\ell, \quad w_1 = 1_p, \quad r_2 = \frac{1}{2}, \quad u_2 = 1_k, \quad v_2 = 1_\ell, \quad w_2 = -1_p. \]

Since the arrows \( M_1 \) and \( M_2 \) are disconnected, the action is a sum of an action in \( f_{M_1} \) and of an action in \( f_{M_2} \). Proceeding as in the preceding section we find that a minimum \( \hat{f}_{M_1} \) has min\{\( k, \ell \)\} eigenvalues \( \mu(4\lambda)^{-\frac{1}{2}} \) and \( |k - \ell| \) eigenvalues zero and \( \hat{f}_{M_2} \) has min\{\( \ell, p \)\} eigenvalues \( \mu(4\lambda)^{-\frac{1}{2}} \) and \( |\ell - p| \) eigenvalues zero. All triples associated to the first diagram are therefore dynamically degenerate.

For the same reason, we can discard diagrams 2, 3, 4, 6, because they also have two disconnected horizontal arrows not vertically aligned.

**Diagram 5** yields:

\[ \rho_L(a, b, c) = \begin{pmatrix} a \otimes 1_k & 0 \\ 0 & b \otimes 1_p \end{pmatrix}, \quad \rho_R(a, b, c) = b \otimes 1_k, \]
\[ \rho_L^c(a, b, c) = \begin{pmatrix} 1_k \otimes a & 0 \\ 0 & 1_\ell \otimes c \end{pmatrix}, \quad \rho_R^c(a, b, c) = 1_\ell \otimes a, \]

\[ \mathcal{M} = \begin{pmatrix} M_1 \otimes 1_k \\ 1_\ell \otimes M_2 \end{pmatrix}, \quad M_1 \in M_{k \times \ell}(\mathbb{C}), \quad M_2 \in M_{p \times k}(\mathbb{C}). \]

Again the fluctuations,

\[ f_{M_1} = \sum_j r_j u_j M_1 v_j^{-1}, \quad u_j \in U(A_1), \quad v_j \in U(A_2), \]
\[ f_{M_2} = \sum_j r_j v_j M_2 w_j^{-1}, \quad w_j \in U(A_3), \]

produce two decoupled fields \( f_{M_1} \) and \( f_{M_2} \) but now the arrows are connected and consequently, the action does not decouple,

\[ V(C_1, C_2) = 4k [\lambda \text{tr}(C_1^2) - \frac{1}{2} \mu^2 \text{tr}(C_1)] + 4\ell [\lambda \text{tr}(C_2^2) - \frac{1}{4} \mu^2 \text{tr}(C_2)] + 8\lambda \text{tr}(C_1) \text{tr}(C_2), \]

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where \( C_i := f M_i^* f M_i \). Let \( x_1, x_2, ..., x_\ell \) be the eigenvalues of \( C_1 \) and \( y_1, y_2, ..., y_k \) be the eigenvalues of \( C_2 \). The action only depends on these variables and in its minimum all \( x_i \) are equal or vanish and all \( y_i \) are equal or vanish. The spectrum of the minimal Dirac operator \( iD \) contains at most three non-vanishing numbers: \( \sqrt{x}, \sqrt{y}, \sqrt{x + y} \) implying that \( k \) and \( \ell \) are less than or equal to two. The fermionic mass matrix \( \mathcal{M} \) is of size \((k^2 + \ell p) \times (k \ell)\). To get at most one zero eigenvalue we must require \(|k^2 + \ell p - k \ell| \leq 1\) implying \( k = p = 1 \). For \( \ell = 1 \) the minimum is at \(|f M_1|^2 + |f M_2|^2 = \frac{\mu^2}{4x}\), for \( \ell = 2 \) the minimum is at \( f M_1 = 0, |f M_2|^2 = \frac{\mu^2}{4x} \). In both cases, \( f M_1 = 0 \), the triple is degenerate in the sense that the Dirac operator has an invariant subspace in the kernel.

Diagram 7 falls in the same way.

Diagram 8 yields the representations

\[
\rho_L(a, b, c) = \begin{pmatrix} a \otimes 1_k & 0 & 0 \\ 0 & c \otimes 1_k & 0 \\ 0 & 0 & b \otimes 1_p \end{pmatrix}, \quad \rho_R(a, b, c) = \begin{pmatrix} b \otimes 1_k & 0 & 0 \\ 0 & c \otimes 1_k & 0 \\ 0 & 0 & 1_p \otimes 1_c \end{pmatrix},
\]

\[
\rho_L^*(a, b, c) = \begin{pmatrix} 1_k \otimes a & 0 & 0 \\ 0 & 1_p \otimes a & 0 \\ 0 & 0 & 1_l \otimes c \end{pmatrix}, \quad \rho_R^*(a, b, c) = \begin{pmatrix} 1_l \otimes a & 0 \\ 0 & 1_p \otimes c \end{pmatrix}.
\]

The possible complex conjugations in the representation will not be important in this diagram. The mass matrix is

\[
\mathcal{M} = \begin{pmatrix} M_1 \otimes 1_k & 0 \\ M_2 \otimes 1_k & 0 \\ 0 & M_3^* \otimes 1_p \end{pmatrix}, \quad M_1, M_2, M_3 \in M_{k \times \ell}(\mathbb{C}), \quad M_1, M_2, M_3 \in M_{p \times \ell}(\mathbb{C}).
\]

The fluctuations are

\[
f M_1 = f \sum_j r_j u_j M_1 v_j^{-1}, \quad u_j \in U(A_1), \quad v_j \in U(A_2),
\]

\[
f M_2 = f \sum_j r_j w_j M_2 v_j^{-1}, \quad w_j \in U(A_3),
\]

\[
f M_3 = f \sum_j r_j w_j M_3 v_j^{-1},
\]

and the action is, with \( C_i := f M_i^* f M_i \)

\[
V(C_1, C_2, C_3) = 4k [\lambda \text{tr}(C_1 + C_2)^2 - \frac{1}{2} \mu^2 \text{tr}(C_1 + C_2)] + 4p [\lambda \text{tr}(C_3)^2 - \frac{1}{2} \mu^2 \text{tr}(C_3)].
\]

Requiring at most one zero eigenvalue (up to a possible colour degeneracy) implies \( k = 1, \ell = p + 1 \) or \( k = 1, \ell = p \). In both cases \( f M_1 \) and \( f M_2 \) vary independently. The colour group consists of the \( u \)s and \( w \)s. As they are spontaneously broken we must have \( k = p = 1 \), leaving \( \ell \) and \( \ell = 2 \). In the commutative case with no complex conjugations in the
representation, \( \hat{f}M_3 = \beta \hat{f}M_2 \) for a complex constant \( \beta \neq 0 \). If \( |\beta| \geq 1 \) the minimum is given by
\[
|\hat{f}M_1|^2 = \frac{(1 - |\beta|^{-2})\mu^2}{4\lambda}, \quad |\hat{f}M_2|^2 = \frac{|\beta|^{-2}\mu^2}{4\lambda}, \quad \text{and} \quad |\hat{f}M_3|^2 = \frac{\mu^2}{4\lambda}.
\]
Its mass spectrum \( \{0, \frac{\mu}{\sqrt{4\lambda}}, \frac{\mu}{\sqrt{4\lambda}}\} \) is dynamically degenerate. If \( |\beta| \leq 1 \) the minimum is given by
\[
|\hat{f}M_1|^2 = 0, \quad |\hat{f}M_2|^2 = \frac{1 + |\beta|^2}{1 + |\beta|^4} \frac{\mu^2}{4\lambda}, \quad \text{and} \quad |\hat{f}M_3|^2 = \frac{|\beta|^2(1 + |\beta|^2)}{1 + |\beta|^4} \frac{\mu^2}{4\lambda}.
\]
The triple is degenerate because of the invariant subspace in the kernel of the Dirac operator. With additional complex conjugations in the representation, \( \hat{f}M_3 \) may decouple from \( \hat{f}M_2 \) and the triple becomes dynamically degenerate as in diagram 1.

In the noncommutative case \( k = 1, \ell = 2, p = 1, M_1 \) and \( M_2 \) must be linearly independent. To get a nondegenerate triple for any choice of \( \mathcal{A}_1 \) and \( \mathcal{A}_3 = \mathbb{R} \) or \( \mathbb{C} \) and \( \mathcal{A}_2 = M_2(\mathbb{R}), M_2(\mathbb{C}) \) or \( \mathbb{H} \) it is sufficient to take the example: \( M_1 = (m_1, 0), M_2 = (0, m_2), M_3 = \beta M_2 \). The minimum is given by \( \hat{f}M_1 \hat{f}M_2^* = 0 \) and
\[
|\hat{f}M_1|^2 = \frac{\mu^2}{4\lambda}, \quad |\hat{f}M_2|^2 = \frac{1 + |\beta|^2}{1 + |\beta|^4} \frac{\mu^2}{4\lambda}, \quad |\hat{f}M_3|^2 = \frac{|\beta|^2(1 + |\beta|^2)}{1 + |\beta|^4} \frac{\mu^2}{4\lambda},
\]
dynamically non-degenerate for \( \beta \neq 0 \) and \( |\beta| \neq 1 \).

Let us note a new phenomenon: the three eigenvalues of a minimum \( \hat{f}D \) are tied together by a mass relation. Its origin is clear, when we add more and more irreducible components to the Hilbert space the number of possible fluctuations does not change, the number of components in the Dirac operator increases. This phenomenon does not only occur for the particular choice of the mass matrices \( M_1, M_2, M_3 \) above, but is generic for diagram 8. The little groups are same as the last six in (7.7).

**Diagram 9** yields the representations
\[
\rho_L(a, b, c) = \begin{pmatrix} a \otimes 1_k & 0 & 0 \\ 0 & a \otimes 1_p & 0 \\ 0 & 0 & b \otimes 1_p \end{pmatrix}, \quad \rho_R(a, b, c) = \begin{pmatrix} b \otimes 1_k & 0 & 0 \\ 0 & a \otimes 1_\ell & 0 \\ 0 & 0 & c \otimes 1_p \end{pmatrix}.
\]
\[
\rho^c_L(a, b, c) = \begin{pmatrix} 1_k \otimes a & 0 & 0 \\ 0 & 1_k \otimes c & 0 \\ 0 & 0 & 1_\ell \otimes c \end{pmatrix}, \quad \rho^c_R(a, b, c) = \begin{pmatrix} 1_\ell \otimes a & 0 & 0 \\ 0 & 1_k \otimes b & 0 \\ 0 & 0 & 1_p \otimes c \end{pmatrix},
\]
with possible complex conjugations. The mass matrix is
\[
\mathcal{M} = \begin{pmatrix} M_1 \otimes 1_k & 0 & 0 \\ 0 & 1_k \otimes M_2^* & 0 \\ 0 & 0 & M_3 \otimes 1_p \end{pmatrix}, \quad M_1 \in M_{k \times \ell}(\mathbb{C}), M_2, M_3 \in M_{\ell \times p}(\mathbb{C}).
\]
The fluctuations are
\[
\hat{f}M_1 = \sum_j r_j u_j M_1 v_j^{-1}, \quad u_j \in U(\mathcal{A}_1), \quad v_j \in U(\mathcal{A}_2),
\]

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\[ fM_2 = \sum_j r_j v_j M_2 w_j^{-1}, \quad w_j \in U(A_3), \]
\[ fM_3 = \sum_j r_j v_j M_3 w_j^{-1}, \]

and the action \( V(C_1, C_2, C_3) \) is equal to
\[
4k \left[ \lambda \text{tr}(C_1)^2 - \frac{1}{2} \mu^2 \text{tr}(C_1) \right] + 4k \left[ \lambda \text{tr}(C_2)^2 - \frac{1}{2} \mu^2 \text{tr}(C_2) \right] + 4p \left[ \lambda \text{tr}(C_3)^2 - \frac{1}{2} \mu^2 \text{tr}(C_3) \right].
\]

Counting neutrinos and imposing broken colour to be commutative leaves only one case, \( k = \ell = p = 1 \). We choose \( M_3 = \beta M_2 \) and get the minima at
\[
|\hat{f}M_1|^2 = \frac{\mu^2}{4\lambda}, \quad |\hat{f}M_2|^2 = \frac{1 + |\beta|^2}{1 + |\beta|^4} \frac{\mu^2}{4\lambda}, \quad |\hat{f}M_3|^2 = |\beta|^2 \frac{1 + |\beta|^2}{1 + |\beta|^4} \frac{\mu^2}{4\lambda},
\]

so it is dynamically non-degenerate for \( \beta \neq 0 \) and \( |\beta| \neq 1 \).

**Diagram 10** yields the representations
\[
\rho_L(a, b, c) = \begin{pmatrix} a \otimes 1_k & 0 & 0 \\ 0 & a \otimes 1_k & 0 \\ 0 & 0 & b \otimes 1_p \end{pmatrix}, \quad \rho_R(a, b, c) = \begin{pmatrix} b \otimes 1_k & 0 & 0 \\ 0 & c \otimes 1_p \end{pmatrix},
\]
\[
\rho_L^c(a, b, c) = \begin{pmatrix} 1_k \otimes a & 0 & 0 \\ 0 & 1_k \otimes a & 0 \\ 0 & 0 & 1_\ell \otimes c \end{pmatrix}, \quad \rho_R^c(a, b, c) = \begin{pmatrix} 1_\ell \otimes a & 0 & 0 \\ 0 & 1_p \otimes c \end{pmatrix},
\]

with possible complex conjugations. The mass matrix is
\[
\mathcal{M} = \begin{pmatrix} M_1 \otimes 1_k & 0 \\ M_2 \otimes 1_k & 0 \\ 0 & M_3 \otimes 1_p \end{pmatrix}, \quad M_1, M_2 \in M_{k \times \ell}(\mathbb{C}), \quad M_3 \in M_{\ell \times p}(\mathbb{C}).
\]

The fluctuations are
\[
fM_1 = \sum_j r_j u_j M_1 v_j^{-1}, \quad u_j \in U(A_1), \quad v_j \in U(A_2),
\]
\[
fM_2 = \sum_j r_j u_j M_2 v_j^{-1},
\]
\[
fM_3 = \sum_j r_j v_j M_3 w_j^{-1}, \quad w_j \in U(A_3),
\]

and the action is
\[
V(C_1, C_2, C_3) = 4k \left[ \lambda \text{tr}(C_1 + C_2)^2 - \frac{1}{2} \mu^2 \text{tr}(C_1 + C_2) \right] + 4p \left[ \lambda \text{tr}(C_3)^2 - \frac{1}{2} \mu^2 \text{tr}(C_3) \right].
\]

Neutrino counting implies \( k = 1, \ell = 2, p = 1 \) or \( k = \ell = p = 1 \). In both cases \( fM_3 \) varies independently of \( fM_1 \) and \( fM_2 \).
The commutative case has always \(|\hat{\beta}M_1|^2 + |\hat{\beta}M_2|^2 = |\hat{\beta}M_3|^2 = \mu^2(4\lambda)^{-1}\) and is dynamically degenerate.

In the noncommutative case \(k = 1, \ell = 2, p = 1, M_1\) and \(M_2\) must be linearly independent. For \(A_1 \oplus A_2 = \mathbb{R} \oplus M_2(\mathbb{C}), \mathbb{C} \oplus M_2(\mathbb{R})\) and \(\mathbb{C} \oplus M_2(\mathbb{C})\), lemma A.1 decouples \(\hat{\beta}M_1\) and \(\hat{\beta}M_2\) and the triples are dynamically degenerate, \(\hat{\beta}M_1 = (\mu(4\lambda)^{-1/2}, 0), \hat{\beta}M_2 = (0, \mu(4\lambda)^{-1/2}), \hat{\beta}M_3 = (0, \mu(4\lambda)^{-1/2})\). For \(A_1 \oplus A_2 = \mathbb{R} \oplus \mathbb{H}\) and \(\mathbb{C} \oplus \mathbb{H}\) we choose

\[
\rho_L(a, b, c) = \begin{pmatrix}
a \otimes 1_k & 0 & 0 \\
0 & a \otimes 1_k & 0 \\
0 & 0 & b \otimes 1_p
\end{pmatrix}, \quad M_1 = (m_1, 0), \quad M_2 = (0, \alpha m_1).
\]

Its minimum has the non-degenerate spectrum \(\{1^{+1/|\alpha|^2}, \frac{|\alpha|^2(1+|\alpha|^2)}{1+|\alpha|^2}, 1, 0\}\) in units of \(\frac{\mu^2}{4\lambda}\) with one mass relation. Finally for \(A_1 \oplus A_2 = \mathbb{R} \oplus M_2(\mathbb{R})\) we choose \(M_1 = (i, 1)\) and \(M_2 = (1, 0)\) to obtain the non-degenerate spectrum \(\{2^{+\sqrt{2}}, 1, 0\}\) in units of \(\frac{\mu^2}{4\lambda}\).

**Diagram 11** yields the representations

\[
\rho_L(a, b, c) = \begin{pmatrix}
a \otimes 1_k & 0 & 0 \\
0 & a \otimes 1_k & 0 \\
0 & 0 & b \otimes 1_p
\end{pmatrix}, \quad \rho_R(a, b, c) = \begin{pmatrix}
b \otimes 1_k & 0 & 0 \\
0 & b \otimes 1_k & 0 \\
0 & 0 & c \otimes 1_p
\end{pmatrix},
\]

\[
\rho_L^{\ell}(a, b, c) = \begin{pmatrix}1_k \otimes a & 0 & 0 \\
0 & 1_\ell \otimes c & 0 \\
0 & 0 & 1_p \otimes c
\end{pmatrix}, \quad \rho_R^{\ell}(a, b, c) = \begin{pmatrix}1_\ell \otimes a & 0 & 0 \\
0 & 1_\ell \otimes a & 0 \\
0 & 0 & 1_p \otimes c
\end{pmatrix},
\]

with possible complex conjugations. The mass matrix is

\[
\mathcal{M} = \begin{pmatrix}
M_1 \otimes 1_k & M_2 \otimes 1_k & 0 \\
0 & 0 & M_3 \otimes 1_p
\end{pmatrix}, \quad M_1, M_2 \in M_{k \times \ell}(\mathbb{C}), \quad M_3 \in M_{\ell \times p}(\mathbb{C}).
\]

The fluctuations are

\[
\hat{\beta}M_1 = \sum_j r_j u_j M_1 v_j^{-1}, \quad u_j \in U(A_1), \quad v_j \in U(A_2),
\]

\[
\hat{\beta}M_2 = \sum_j r_j u_j M_2 v_j^{-1},
\]

\[
\hat{\beta}M_3 = \sum_j r_j v_j M_3 w_j^{-1}, \quad w_j \in U(A_3),
\]

and the action is

\[
V(C_1, C_2, C_3) = 4k [\lambda \text{tr}(C_1 + C_2)^2 - \frac{1}{2} \mu^2 \text{tr}(C_1 + C_2)] + 4p [\lambda \text{tr}(C_3)^2 - \frac{1}{2} \mu^2 \text{tr}(C_3)].
\]

Neutrino counting and imposing broken colour to be commutative leaves only one possibility: \(k = \ell = p = 1\), which is treated as the case \(k = \ell = p = 1\) of diagram 10 with the same conclusion, degeneracy.
Diagram 12 yields the representations

\[ \rho_L(a, b, c) = \left( \begin{array}{ccc} a \otimes 1_k & 0 & 0 \\ 0 & b \otimes 1_p & 0 \\ 0 & 0 & a \otimes 1_p \end{array} \right), \quad \rho_R(a, b, c) = \left( \begin{array}{ccc} b \otimes 1_k & 0 & 0 \\ 0 & a \otimes 1_\ell & 0 \\ 0 & 0 & c \otimes 1_p \end{array} \right), \]

\[ \rho_L^c(a, b, c) = \left( \begin{array}{ccc} 1_k \otimes a & 0 & 0 \\ 0 & 1_\ell \otimes c & 0 \\ 0 & 0 & 1_\ell \otimes c \end{array} \right), \quad \rho_R^c(a, b, c) = \left( \begin{array}{ccc} 1_\ell \otimes a & 0 & 0 \\ 0 & 1_k \otimes b & 0 \\ 0 & 0 & 1_k \otimes b \end{array} \right), \]

with possible complex conjugations. The mass matrix is

\[ \mathcal{M} = \left( \begin{array}{ccc} M_1 \otimes 1_k & 0 & 0 \\ 0 & M_2 \otimes 1_k & 0 \\ 0 & 0 & M_3 \otimes 1_p \end{array} \right), \quad M_1, M_2 \in M_{k \times \ell}(\mathbb{C}), \quad M_3 \in M_{\ell \times p}(\mathbb{C}). \]

The fluctuations are

\[ \begin{align*}
\delta M_1 &= \sum_j r_j u_j M_1 v_j^{-1}, \quad u_j \in U(A_1), \quad v_j \in U(A_2), \\
\delta M_2 &= \sum_j r_j u_j M_2 v_j^{-1}, \\
\delta M_3 &= \sum_j r_j v_j M_3 w_j^{-1}, \quad w_j \in U(A_3).
\end{align*} \]

The action \( V(C_1, C_2, C_3) \) equals

\[ 4k [\lambda \text{tr}(C_1)^2 - \frac{1}{2} \mu^2 \text{tr}(C_1)] + 4k [\lambda \text{tr}(C_2)^2 - \frac{1}{2} \mu^2 \text{tr}(C_2)] + 8\lambda \text{tr}(C_1) \text{tr}(C_2) + 4p [\lambda \text{tr}(C_3)^2 - \frac{1}{2} \mu^2 \text{tr}(C_3)]. \]

Since \( \delta M_3 \) decouples, \( \ell \) and \( p \) can at most differ by one. The neutrino count and imposing broken colour to be commutative implies \( k = \ell = p = 1 \). Then \( M_2 = \alpha M_1 \) and we must distinguish two cases: \( \delta M_2 = \alpha \delta M_1 \) or \( \delta M_1 \) and \( \delta M_2 \) independent. Both possibilities have a dynamically degenerate minimum: \( |\delta M_1|^2 + |\delta M_2|^2 = |\delta M_3|^2 = \frac{\mu^2}{4\lambda} \).

Diagram 13 has a ladder form, i.e. it consists of horizontal arrows, vertically aligned. Its representations are

\[ \rho_L(a, b, c) = \left( \begin{array}{ccc} a \otimes 1_k & 0 & 0 \\ 0 & a \otimes 1_k & 0 \\ 0 & 0 & a \otimes 1_p \end{array} \right), \quad \rho_R(a, b, c) = \left( \begin{array}{ccc} b \otimes 1_k & 0 & 0 \\ 0 & b \otimes 1_\ell & 0 \\ 0 & 0 & b \otimes 1_p \end{array} \right), \]

\[ \rho_L^c(a, b, c) = \left( \begin{array}{ccc} 1_k \otimes a & 0 & 0 \\ 0 & 1_k \otimes a & 0 \\ 0 & 0 & 1_k \otimes c \end{array} \right), \quad \rho_R^c(a, b, c) = \left( \begin{array}{ccc} 1_\ell \otimes a & 0 & 0 \\ 0 & 1_\ell \otimes c & 0 \\ 0 & 0 & 1_\ell \otimes c \end{array} \right), \]

with possible complex conjugations here and there. The mass matrix is

\[ \mathcal{M} = \left( \begin{array}{ccc} M_1 \otimes 1_k & 0 & 0 \\ M_2 \otimes 1_k & 0 & 0 \\ 0 & M_3 \otimes 1_p \end{array} \right), \quad M_1, M_2, M_3 \in M_{k \times \ell}(\mathbb{C}). \]
The fluctuations are
\[ fM_1 = \sum_j r_j u_j M_1 v_j^{-1}, \quad u_j \in U(A_1), \quad v_j \in U(A_2), \]
\[ fM_2 = \sum_j r_j u_j M_2 v_j^{-1}, \]
\[ fM_3 = \sum_j r_j u_j M_3 v_j^{-1}, \]
and the action is
\[ V(C_1, C_2, C_3) = 4k [\lambda \text{tr}(C_1 + C_2)^2 - \frac{1}{2} \mu^2 \text{tr}(C_1 + C_2)] + 4p [\lambda \text{tr}(C_3)^2 - \frac{1}{2} \mu^2 \text{tr}(C_3)]. \]

The neutrino count implies \( k = 1, \ell = 2 \) or \( k = \ell = 1 \).

The case \( k = \ell = 1 \) has the following possibilities:

1: \( \mathbb{R} \oplus \mathbb{R} \oplus \mathcal{C} \), where \( \mathcal{C} \) is any simple ‘colour’ algebra. All possible triples are degenerate in the sense that the Dirac operator has an invariant subspace in its kernel.

2: \( \mathbb{R} \oplus \mathcal{C} \oplus \mathcal{C} \), all possible triples are degenerate.

3: \( \mathbb{C} \oplus \mathbb{R} \oplus \mathcal{C} \ni (a, b, c) \). The non-degenerate triples have:

\[ \rho_L(a, b, c) = \begin{pmatrix} a & 0 & 0 \\ 0 & \bar{a} & 0 \\ 0 & 0 & \bar{a} \otimes 1_p \end{pmatrix}, \quad \rho_R(a, b, c) = \begin{pmatrix} b & 0 \\ 0 & b \otimes 1_p \end{pmatrix}. \]

The fluctuations respect the mass ratios: \( |M_1| : |M_2| : |M_3| = |fM_1| : |fM_2| : |fM_3| \). If \( M_1 \) and \( M_2 \) are different from zero, the kernel of the Dirac operator, \( \mathbb{C}(-M_2, \bar{M}_1, 0, 0, 0; -M_2, M_1, 0, 0, 0)^T \), is not invariant under \( a \in \mathbb{C} \). If \( |M_3|^2 \neq |M_1|^2 + |M_2|^2 \), the triple is dynamically non-degenerate.

4: \( \mathbb{C} \oplus \mathbb{C} \oplus \mathcal{C} \ni (a, b, c) \). The only not obviously degenerate triples have the representations:

\[ \rho_L(a, b, c) = \begin{pmatrix} a & 0 & 0 \\ 0 & \bar{a} & 0 \\ 0 & 0 & \bar{a} \otimes 1_p \end{pmatrix}, \quad \rho_R(a, b, c) = \begin{pmatrix} b & 0 \\ 0 & b \otimes 1_p \end{pmatrix}. \]

and

\[ \rho_L(a, b, c) = \begin{pmatrix} a & 0 & 0 \\ 0 & \bar{a} & 0 \\ 0 & 0 & \bar{a} \otimes 1_p \end{pmatrix}, \quad \rho_R(a, b, c) = \begin{pmatrix} b & 0 \\ 0 & \bar{b} \otimes 1_p \end{pmatrix}. \]

The fluctuations do not respect all mass ratios, in fact \( fM_1 \) and \( fM_2 \) are independent variables and only the ratio \( M_2/M_3 = fM_2/fM_3 =: 1/\beta \) is invariant under fluctuations via (8.1). If \( |\beta| \geq 1 \) the minima are given by

\[ |fM_1|^2 = \frac{(1 - |\beta|^2)\mu^2}{4\lambda}, \quad |fM_2|^2 = \frac{\mu^2}{|\beta|^2 4\lambda}, \quad \text{and} \quad |fM_3|^2 = \frac{\mu^2}{4\lambda}. \]
Its mass spectrum \( \{0, \frac{\mu}{2\sqrt{\lambda}}, \frac{\mu}{2\sqrt{\lambda}} \} \) p times is dynamically degenerate. If \( |\beta| \leq 1 \) the minima are given by

\[
|\hat{f}M| = 0, \quad |\hat{f}M_2| = \frac{1 + p|\beta|^2}{1 + p|\beta|^4} \mu^2, \quad \text{and} \quad |\hat{f}M_3| = \frac{|\beta|^2(1 + p|\beta|^2)}{1 + p|\beta|^4} \frac{\mu^2}{4\lambda}.
\]

The triple is degenerate since there is an invariant subspace in the kernel of the Dirac operator. For the representation \(8.2\) the computations are identical to the case above, \(8.1\), after permuting \( M_1 \) and \( M_2 \) and after replacing \( M_3 \) by \( \overline{M_3} \).

In the case \( k = 1, \ell = 2 \), the three submatrices \( M_1, M_2, M_3 \) are linearly dependent over \( \mathbb{C} \). If \( M_1 \) is proportional to \( M_2 \) then both \( M_3 \) and \( (M_1, M_2) \) have a zero eigenvalue. Therefore \( M_1 \) and \( M_2 \) are linearly independent over \( \mathbb{C} \) and \( M_3 =: \alpha M_1 + \beta M_2 \) with complex coefficients \( \alpha \) and \( \beta \).

5: \( \mathbb{C} \oplus M_2(\mathbb{C}) \oplus \mathbb{C} \ni (a,b,c) \). By lemma A.1, \( \hat{f}M_1 \) and \( \hat{f}M_2 \) vary independently in \( \mathbb{C}^2 \). For the representations with

\[
\rho_L(a,b,c) = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \otimes 1_p \end{pmatrix}, \quad \rho_R(a,b,c) = \begin{pmatrix} b & 0 \\ 0 & b \otimes 1_p \end{pmatrix}, \quad (8.3)
\]

we get \( \hat{f}M_3 =: \alpha \hat{f}M_1 + \beta \hat{f}M_2 \). For instance for \( \alpha = 0 \) we get \( \hat{f}M_1 \hat{f}M_2^* = 0 \) as for diagram 8 and a similar mass relation,

\[
|\hat{f}M_1|^2 = \frac{\mu^2}{4\lambda}, \quad |\hat{f}M_2|^2 = \frac{1 + p|\beta|^2}{1 + p|\beta|^4} \frac{\mu^2}{4\lambda}, \quad |\hat{f}M_3|^2 = |\beta|^2 \frac{1 + p|\beta|^2}{1 + p|\beta|^4} \frac{\mu^2}{4\lambda}. \quad (8.4)
\]

For general \( \alpha \), we get the same relations with \( |\beta|^2 \) replaced by \( |\alpha|^2 + |\beta|^2 \). For the representations with

\[
\rho_L(a,b,c) = \begin{pmatrix} a & 0 & 0 \\ 0 & \tilde{a} & 0 \\ 0 & 0 & \tilde{a} \otimes 1_p \end{pmatrix}, \quad \rho_R(a,b,c) = \begin{pmatrix} b & 0 \\ 0 & b \otimes 1_p \end{pmatrix},
\]

and \( \alpha \neq 0 \), all three doublets, \( \hat{f}M_1, \hat{f}M_2 \) and \( \hat{f}M_3 \) vary independently. The minima, \( |\hat{f}M_1|^2 = |\hat{f}M_2|^2 = |\hat{f}M_3|^2 = \frac{\mu^2}{4\lambda} \) produce a dynamical degeneracy in this case. The other case, \( \alpha = 0 \) is dynamically non-degenerate with the same mass relation as above, equations \(8.4\). All other representations of this algebra are treated the same way and they either have a mass relation or are dynamically degenerate, which means a particularly simple mass relation.

6: \( \mathbb{R} \oplus M_2(\mathbb{C}) \oplus \mathbb{C} \). This case is identical to case 5 with representation \(8.3\).

7: \( \mathbb{C} \oplus \mathbb{H} \oplus \mathbb{C} \ni (a,b,c) \). For example, all triples with

\[
\rho_L(a,b,c) = \begin{pmatrix} a & 0 & 0 \\ 0 & \tilde{a} & 0 \\ 0 & 0 & \tilde{a} \otimes 1_p \end{pmatrix}, \quad \rho_R(a,b,c) = \begin{pmatrix} b & 0 \\ 0 & b \otimes 1_p \end{pmatrix},
\]

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$M_1 = (m_1, 0), M_2 = (0, m_2)$ and $M_3 = (0, m_3)$ have no mass relation. Indeed like in case 3, the mass ratios are stable under fluctuations: $|M_1| : |M_2| : |M_3| = |f M_1| : |f M_2| : |f M_3|$. Other triples behave like in case 5.

8: $\mathbb{R} \oplus \mathbb{H} \oplus C$ has the same examples without mass relations as in case 7.

9: $C \oplus M_2(\mathbb{R}) \oplus C$. For representations (8.3) and $\alpha = 0$, we get minima with mass relation (8.4).

10: $\mathbb{R} \oplus M_2(\mathbb{R}) \oplus C$. Here we take a representation (8.3) with $M_1 = (m_1, 0)$, $M_2 = (0, m_2)$ and $M_3 = (0, \beta m_2)$ and get again the mass relation (8.4).

Up to different multiplicities, we have the same conclusion for the diagrams 18, 17, 22. After permuting $A_1$ and $A_2$ we also have the same results for the other four ladders, diagrams 14, 19, 16, 21.

**Diagram 15** yields the representations

$$
\rho_L(a, b, c) = \left(\begin{array}{ccc}
1_k \otimes a & 0 & 0 \\
0 & 1_p \otimes b & 0 \\
0 & 0 & 1_k \otimes 1_p
\end{array}\right), \quad \rho_R(a, b, c) = \left(\begin{array}{ccc}
b \otimes 1_k & 0 & 0 \\
0 & a \otimes 1_\ell & 0 \\
0 & 0 & b \otimes 1_p
\end{array}\right),
$$

with possible complex conjugations. The mass matrix is

$$
\mathcal{M} = \left(\begin{array}{ccc}
M_1 \otimes 1_k & 1_k \otimes M_2 & 0 \\
0 & 0 & M_3 \otimes 1_p
\end{array}\right), \quad M_1, M_2, M_3 \in M_{k \times \ell}(\mathbb{C}).
$$

The fluctuations are

$$
f M_1 = \sum_j r_j u_j M_1 v_j^{-1}, \quad u_j \in U(A_1), \quad v_j \in U(A_2),
$$

$$
f M_2 = \sum_j r_j u_j M_2 v_j^{-1},
$$

$$
f M_3 = \sum_j r_j u_j M_3 v_j^{-1}.
$$

Neutrino counting and imposing broken colour to be commutative implies $k = \ell = 1$. This case with $A_1$ and $A_2 = \mathbb{R}$ or $\mathbb{C}$ is treated as in diagram 13, 3 yielding a non-degenerate triple without mass relation. After replacing $M_3^*$ by $M_3$, **diagram 20** has identical computations.

**Diagram 23** yields the representations

$$
\rho_L(a, b, c) = \left(\begin{array}{ccc}
a \otimes 1_k & 0 & 0 \\
0 & b \otimes 1_\ell & 0 \\
0 & 0 & b \otimes 1_p
\end{array}\right), \quad \rho_R(a, b, c) = \left(\begin{array}{ccc}
b \otimes 1_k & 0 & 0 \\
0 & a \otimes 1_\ell & 0 \\
0 & 0 & a \otimes 1_p
\end{array}\right),
$$

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\[ \rho_L(a, b, c) = \begin{pmatrix} 1_k \otimes a & 0 & 0 \\ 0 & 1_\ell \otimes b & 0 \\ 0 & 0 & 1_\ell \otimes c \end{pmatrix}, \quad \rho_R(a, b, c) = \begin{pmatrix} 1_p \otimes a & 0 \\ 0 & 1_k \otimes c \end{pmatrix}, \]

with possible complex conjugations. The mass matrix is

\[ \mathcal{M} = \begin{pmatrix} M_1 \otimes 1_k & 0 & 0 \\ 1_\ell \otimes M_2^* & 0 \\ 0 & 0 & M_3^* \otimes 1_p \end{pmatrix}, \quad M_1, M_2, M_3 \in M_{k \times \ell}(\mathbb{C}). \]

The fluctuations are

\[ f_{M_1} = \sum_j r_j u_j M_1 v_j^{-1}, \quad u_j \in U(A_1), \quad v_j \in U(A_2), \]
\[ f_{M_2} = \sum_j r_j u_j M_2 v_j^{-1}, \]
\[ f_{M_3} = \sum_j r_j u_j M_3 v_j^{-1}. \]

Neutrino counting implies \( k = \ell = 1 \). This model with \( A_1 \) and \( A_2 = \mathbb{R} \) or \( \mathbb{C} \) is treated as in diagram 13, 3 yielding a non-degenerate triple without mass relation. After replacing \( M_3^* \) by \( M_3 \), diagram 24 has identical computations.

We must now extend our analysis to include the possibility of complex conjugate representations. As in the case of two algebras, one shows that any diagram containing a connected component consisting of only letter unchanging arrows leads to degenerate spectra. Therefore within the class of irreducible and dynamically non-degenerate triples the leitmotiv of a Krajewski diagram is still carried by its letter changing arrows. For three algebras there are two additional diagrams, diagrams 25, 26, figure 7, that involve only letter changing arrows. Their contracted diagrams resemble diagrams 4 and 9 of figure 6 except for the change of chirality in one arrow and the computation of their triples is similar.

Note that without the presence of conjugate representations, this change violates the condition that nonvanishing entries of the multiplicity matrix and its transposed must have same signs. Figure 8 lists the contractions of all irreducible diagrams whose letter unchanging arrows are connected to at least one letter changing arrow. The blow up of the new symbols is given in Figure 9.

All triples attached to the eleven diagrams of Figure 8 are degenerate or dynamically degenerate:

**Diagram 27** with the first blow up yields

\[ \rho_L(a, b, c) = \begin{pmatrix} c \otimes 1_k & 0 \\ 0 & b \otimes 1_p \end{pmatrix}, \quad \rho_R(a, b, c) = \begin{pmatrix} b \otimes 1_k & 0 & 0 \\ 0 & \bar{c} \otimes 1_k & 0 \\ 0 & 0 & a \otimes 1_p \end{pmatrix}, \]
\[ \mathcal{M} = \begin{pmatrix} M_1 \otimes 1_k & M_2 \otimes 1_k & 0 \\ 0 & 0 & M_3 \otimes 1_p \end{pmatrix}, \]

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$M_1 \in M_{p \times \ell}(\mathbb{C})$, $M_2 \in M_{p \times p}(\mathbb{C})$, $M_3 \in M_{\ell \times k}(\mathbb{C})$. Counting neutrinos leads to $k = \ell = 1$. To get a handle on $p$ we repeat the overkill from section 4. In the worst case $M_2$ and consequently $fM_2$ is skewsymmetric and $p$ is odd. By fluctuations we can obtain

$$fM_2 = \begin{pmatrix}
(0 & -1 \\
1 & 0 \\
0 & -1 \\
1 & 0 \\
& & & & \ddots \\
& & & & & & 0
\end{pmatrix} =: \begin{pmatrix}
A & 0 \\
0 & 0
\end{pmatrix}.$$  

Now $fM_1$ fluctuates independently and we may obtain for its transpose $fM_1^T = (0, ..., 0, 1)$. We get $(fM_1, fM_2)(fM_1, fM_2)^* = 1_p$ and the minimum of the action is dynamically degenerate if $p \geq 1$. The commutative case $k = \ell = p = 1$ is obviously degenerate. For the second blow up the computations are similar with $fM_2$ now proportional to $1_p$ from the start.

**Diagram 28** is treated as diagram 27.

**Diagram 29** has $k = \ell = p = 1$ by neutrino count and admits only degenerate triples.

**Diagrams 30, 31, 32, 34, 35, 36** have $k = \ell = 1$ by neutrino count. As in the commutative case of diagram 8, all their triples are degenerate.
Diagram 33 with the first of the six possible blow ups yields:

\[ \rho_L(a, b, c) = \begin{pmatrix} a \otimes 1_k & 0 & 0 \\ 0 & a \otimes 1_p & 0 \\ 0 & 0 & \bar{b} \otimes 1_p \end{pmatrix}, \]

\[ \rho_R(a, b, c) = \begin{pmatrix} b \otimes 1_k & 0 & 0 & 0 \\ 0 & b \otimes 1_p & 0 & 0 \\ 0 & 0 & b \otimes 1_p & 0 \\ 0 & 0 & 0 & b \otimes 1_p \end{pmatrix}, \]

\[ \rho_L^c(a, b, c) = \begin{pmatrix} 1_k \otimes a & 0 & 0 \\ 0 & 1_k \otimes c & 0 \\ 0 & 0 & 1_\ell \otimes c \end{pmatrix}, \]

\[ \rho_R^c(a, b, c) = \begin{pmatrix} 1_\ell \otimes a & 0 & 0 & 0 \\ 0 & 1_\ell \otimes c & 0 & 0 \\ 0 & 0 & 1_\ell \otimes c & 0 \\ 0 & 0 & 0 & 1_\ell \otimes c \end{pmatrix}, \]

\[ \mathcal{M} = \begin{pmatrix} M_1 \otimes 1_k & 0 & 0 & 0 \\ 0 & M_2 \otimes 1_p & 0 & 0 \\ 0 & M_3 \otimes 1_p & M_4 \otimes 1_p & M_5 \otimes 1_p \end{pmatrix}. \]

\( M_1, M_2 \in M_{k \times \ell}(\mathbb{C}), \ M_3, M_4, M_5 \in M_{\ell \times \ell}(\mathbb{C}). \) Counting neutrinos and imposing broken colour to be commutative gives \( k = \ell = 1. \) This case is degenerate: the kernel of the Dirac operator contains the invariant subspace with elements \( (0, 0, 0, 0, M_5v, -M_4v; 0, 0, 0, 0, M_5w, -M_4w)^T, \) \( v, w \in \mathbb{C}^p. \)
With the second blow up diagram 33 yields

\[ \rho_L(a, b, c) = \begin{pmatrix} a \otimes 1_k & 0 & 0 \\ 0 & a \otimes 1_p & 0 \\ 0 & 0 & \bar{b} \otimes 1_p \end{pmatrix}, \]

\[ \rho_R(a, b, c) = \begin{pmatrix} b \otimes 1_k & 0 & 0 & 0 \\ 0 & b \otimes 1_p & 0 & 0 \\ 0 & 0 & b \otimes 1_p & 0 \\ 0 & 0 & 0 & \bar{b} \otimes 1_p \end{pmatrix}, \]

\[ \rho_L^c(a, b, c) = \begin{pmatrix} 1_k \otimes a & 0 & 0 \\ 0 & 1_k \otimes c & 0 \\ 0 & 0 & 1_\ell \otimes c \end{pmatrix}, \]

\[ \rho_R^c(a, b, c) = \begin{pmatrix} 1_\ell \otimes a & 0 & 0 & 0 \\ 0 & 1_\ell \otimes c & 0 & 0 \\ 0 & 0 & 1_\ell \otimes c & 0 \\ 0 & 0 & 0 & 1_\ell \otimes \bar{c} \end{pmatrix}, \]

\[ M = \begin{pmatrix} M_1 \otimes 1_k & 0 & 0 & 0 \\ 0 & M_2 \otimes 1_p & 0 & 0 \\ 0 & M_3 \otimes 1_p & M_4 \otimes 1_p & 1_k \otimes M_5 \end{pmatrix}. \]

\[ M_1, M_2 \in M_{k \times \ell}(\mathbb{C}), \quad M_3, M_4 \in M_{\ell \times \ell}(\mathbb{C}), \quad M_5 \in M_{p \times p}(\mathbb{C}). \] Counting neutrinos and imposing broken colour to be commutative gives \( k = \ell = p = 1. \) In the notations of Corollary 11.4, the action reads

\[ V(a, b, c) = 4\lambda \left| |M_1ab|^4 + |M_2ab|^4 + 2|M_2ab|^2|M_3b|^2 + (|M_3b|^2 + |M_4b|^2 + |M_5c|^2)^2 \right| - 2\mu^2 \left[ |M_1ab|^2 + |M_2ab|^2 + (|M_3b|^2 + |M_4b|^2 + |M_5c|^2)^2 \right]. \]

Its minimum is degenerate, \( \hat{b} = 0. \)

The other blow ups as well as diagram 37 lead to the same conclusion: degeneracy. This completes the proof of the theorem. \( \square \)

We summarize the possible algebras with \( N = 3 \) and the corresponding Krajewski diagrams of all their irreducible, dynamically non-degenerate triplets in a table:
Note that relaxing the hypothesis of unbroken noncommutative colour does not add any algebra to the list with $N = 1$ and 2. It adds only few algebras to the list with $N = 3$ coming from diagrams 8, 9 and 11. We were unable to treat some of their triples, in particular quaternionic ones.

### 8.2 The standard model of electro-weak and strong forces

Let us close this section by remarking that diagram 17 of Figure 6 with flipped chirality and with algebra $\mathcal{A} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \ni (b, a, c)$, with representation

$$\rho_L(a, b) = \left( \begin{array}{cc} a \otimes 1_3 & 0 \\ 0 & a \end{array} \right), \quad \rho_R(a, b) = \left( \begin{array}{ccc} b_{13} & 0 & 0 \\ 0 & b_{13} & 0 \\ 0 & 0 & \bar{b} \end{array} \right),$$

$$\rho_c^L(a, b) = \left( \begin{array}{cc} 1_2 \otimes c & 0 \\ 0 & \bar{b}_{12} \end{array} \right), \quad \rho_c^R(a, b) = \left( \begin{array}{ccc} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & \bar{b} \end{array} \right),$$

and with mass matrix

$$\mathcal{M} = \left( \begin{array}{ccc} (m_u \otimes 1_3) & 0 & (0 \otimes 1_3) \\ 0 & (m_d \otimes 1_3) \end{array} \right) \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

produces the standard model of electro-magnetic, weak and strong forces with one generation of quarks and leptons, $u, d, \nu$ and $e$. The neutrino is a massless Weyl spinor. The
intersection form written with respect to the basis of projectors

\[ p_1 = (0, 1_2, 0), \quad p_2 = (1, 0, 0), \quad p_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

is

\[ \cap = -2 \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 0 \end{pmatrix}, \]

and non-degenerate. The colour group \( U(3) \) is unbroken and its representations on corresponding left- and right-handed fermions are identical. In physicists’ language this means that gluons are massless and couple vectorially. Further details on the standard model as an almost commutative geometry can be found in [13, 2].

9 Beyond irreducible triples

For the standard model, allowing reducible triples has two important physical consequences:

i) Suppose we want to render the neutrino massive. Majorana masses are incompatible with the axiom that the Dirac operator anticommutes with the chirality. Therefore we must increase the Hilbert space by adding a right-handed neutrino. Then the triple becomes reducible, but worse Poincaré duality breaks down: the intersection form becomes

\[ \cap = -2 \begin{pmatrix} 0 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & 0 \end{pmatrix}, \]

degenerate.

ii) We may add more generations of quarks and leptons. Then the Cabibbo-Kobayashi-Maskawa matrix makes its appearance. Now we may add right-handed neutrinos in some but not in all generations and give Dirac masses to the corresponding neutrinos without violating Poincaré duality.

So far we have no clue to why the standard model comes with three colours, with three generations of quarks and with three generations of leptons. Note however that anomaly cancellations [1] imply further constraints that are satisfied with three colours and with a number of quark generations equal to the number of lepton generations.

9.1 A reducible triple with non-degenerate spectrum

The criterion of dynamical degeneracy loses its meaning in presence of reducible triples as illustrated by the following example: \( \mathcal{A} = M_3(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus \mathbb{C} \ni (a, b, c), \)

\[ \rho(a, b, c) = \begin{pmatrix} a \otimes 1_2 & 0 & 0 & 0 \\ 0 & b \otimes 1_3 & 0 & 0 \\ 0 & 0 & \bar{c}1_3 \otimes 1_2 & 0 \\ 0 & 0 & 0 & \bar{c}1_2 \otimes 1_3 \end{pmatrix}, \]
The fluctuations of the Dirac operator generate an 18-dimensional complex vector space in which the spectral action \( V(f D) \) has to be minimized. We used a steepest descend method of Mathematica for this task and found an absolute minimum at 
\[
\hat{f} D = D \text{ with } m = \sqrt{\frac{\mu}{\lambda}} 
\]
and 
\[
V(\hat{f} D) = -\frac{431}{340} \mu^4. 
\]
The spectrum of this minimum \( \hat{f} D \) is non-degenerate: in units of \( \sqrt{\frac{\mu}{\lambda}} \) we have \{1, \frac{4}{5}, \frac{5}{17}, \frac{20}{17}, \frac{9}{10} \pm \frac{\sqrt{17}}{10} \}. All six values of course appear twice with a positive and twice with a negative sign in the spectrum of \( \hat{f} D \). The little group of this minimum is \( G_\ell = U(1) \times U(1) \subset U(3) \times U(2) \times U(1) \) with generic element \((e^{i\alpha}1_3,e^{i\beta}1_2,e^{i\gamma})\). The spectrum of the minimum appears completely rigid, i.e. there are mass relations.

## 10 Conclusion

Suppose we want to apply conventional, perturbative quantum field theory to the Yang-Mills-Higgs models coming from almost commutative geometries. Then after renormalization, fermion masses are functions of energy and the colour degeneracy is compatible with this energy dependence only if all noncommutative colour groups are unbroken. Furthermore the renormalization of fermion masses is incompatible with mass relations, in particular with the completely rigid reducible spectral triple of section 9. In irreducible spectral triples, mass relations other than degeneracies only appear starting with \( N = 3 \). All triples without such mass relations come from ladder diagrams and have algebras \( 1 \oplus 1 \oplus C \) or \( H \oplus 1 \oplus C \).

Let us suppose that also for \( N \geq 4 \) the irreducible triples without mass relations have contracted multiplicity matrices of ladder type:

\[
\hat{\mu} = \begin{pmatrix}
\alpha & \beta & 0 & 0 \\
\gamma & \delta & 0 & 0 \\
\rho & \sigma & 0 & 0 \\
\theta & \xi & 0 & 0 \\
\end{pmatrix}, \quad N = 4.
\]

Then \( \det(\hat{\mu} + \hat{\mu}^T) = (\rho\xi - \sigma\theta)^2 \) and irreducibility implies \( \alpha = \beta = \gamma = \delta = 0 \). For \( N \geq 5 \), all contracted multiplicity matrices \( \hat{\mu} \) of ladder type have \( \det(\hat{\mu} + \hat{\mu}^T) = 0 \) leading us to the

### Conjecture 10.1

The sum of \( N \) simple algebras, \( \mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus ... \oplus \mathcal{A}_N \) admits a finite, real, \( S^0 \)-real, irreducible and dynamically non-degenerate spectral triple free of mass relations if and only if it is in the list, up to a reordering of the summands:
Here $\mathcal{C}$, $\mathcal{C}_1$ and $\mathcal{C}_2$ are three arbitrary simple algebras. The colour algebras $\mathcal{C}$ for $N = 3$ and $\mathcal{C}_1 \oplus \mathcal{C}_2$ for $N = 4$ have two constraints:

i) Their representations on corresponding left- and right-handed subspaces of $\mathcal{H}$ are identical (up to possibly different multiplicities).

ii) The Dirac operator $\mathcal{D}$ is invariant under $U(\mathcal{C})$ or $U(\mathcal{C}_1 \oplus \mathcal{C}_2)$, 

$$\rho(1,1,w) \mathcal{D} \rho(1,1,w)^{-1} = \mathcal{D}, \quad \text{for all } w \in U(\mathcal{C}) \text{ or } U(\mathcal{C}_1 \oplus \mathcal{C}_2).$$

This implies that the unitaries of $\mathcal{C}$ or $\mathcal{C}_1 \oplus \mathcal{C}_2$ do not participate in the fluctuations and are therefore unbroken, i.e. elements of the little group.

We must admit that our brute force proof by exhaustion is not suitable for $N = 4$ and it seems already a formidable task to write down the list of all contracted irreducible diagrams.

Besides renormalizability, there are two other important items on the physicist’s shopping list, which will further constrain the model building kit:

i) The electric charge of a massless particle must be zero.

ii) The representation of the little group on the Hilbert space of fermions must be complex.

Recall that a unitary representation is called real if it is equal to its complex conjugate and pseudo-real if it is unitarily equivalent to its complex conjugate. Otherwise the representation is complex. For example the fundamental representation of $SU(2)$ is pseudo-real. An irreducible, unitary representation of $U(1)$ is complex if and only if its charge is non-zero.

Before we can examine these two criteria in the irreducible context, we must compute the minimal central extensions [12][15] that allow the lift of algebra automorphisms to the Hilbert space of fermions to have at most a finite number of values. This calculation is under way.
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11 Appendix

Since this paper deals with matrices, let us briefly recall two standard results, one on the singular value decomposition of rectangular matrices and the second on the standard form of skewsymmetric matrices.

We write $O(n) := U(M_n(\mathbb{R}))$, $U(n) := U(M_n(\mathbb{C}))$ and $USp(n) := U(M_n(\mathbb{H}))$.

Lemma 11.1. i) Let $M \in M_{n \times m}(\mathbb{C})$. Then there exist $U \in U(n), V \in U(m)$ such that $M = UDV$ where $D \in M_{n \times m}(\mathbb{R})$ satisfies $D_{ij} = 0$ for $i \neq j$ and $D_{11} \geq D_{22} \geq D_{kk} > D_{k+1,k+1} = \cdots = D_{qq} = 0$ where $k = \text{rank}(M)$ and $q = \text{min}(n,m)$. The $D_{ii}^2$ are the eigenvalues of $M^*M$, the columns of $U$ (resp. $V$) are the eigenvectors of $MM^*$ (resp. $M^*M$) arranged in the same order as the eigenvalues $D_{ii}^2$. In particular, when $M \in M_{n \times m}(\mathbb{R})$, we may assume $U \in O(n), V \in O(m)$ (\textit{[M]} 7.3.5).

ii) Let $M \in M_n(\mathbb{C})$ be a skewsymmetric matrix. Then, there exists $U \in U(n)$ such that

$$UMU^T = \left( \bigoplus_{i=1}^{p} m_i x \right) \oplus 0 \cdots \oplus 0$$

where $x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $m_i \in \mathbb{C}^*$ \hspace{1cm} (11.1)

and the numbers of zeros equals $n - 2p$ (\textit{[M]} 4.4, Problem 26).

For our purpose, $M$ is the complex fermionic mass matrix. Then, in i), the diagonal elements $D_{jj}$ are the Dirac masses and the unitaries $U$ and $V$ are related to the Cabibbo-Kobayashi-Maskawa matrix.

Definition 11.2. Let $M \in M_{n \times m}(\mathbb{C})$ and

$$f \in \mathcal{F}_{\mathbb{K},\mathbb{K}'} := \{(r_j, u_j, v_j)_{j \in J} \in \mathbb{R} \times U(M_n(\mathbb{K})) \times U(M_m(\mathbb{K}')) \mid J \text{ finite}\}$$

where $\mathbb{K}, \mathbb{K}'$ are $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. The fluctuation of $M$ is defined by

$$fM := \sum_j r_j u_j M v_j.$$ 

In the case that $\mathbb{C}$ and $\mathbb{H}$ are involved, we assume of course that $\mathbb{C} \subset \mathbb{H}$.

Note that for a given $M$,

$$\{fM \mid f \in \mathcal{F}_{\mathbb{R},\mathbb{C}}\} = \{fM \mid f \in \mathcal{F}_{\mathbb{C},\mathbb{R}}\} = \{fM \mid f \in \mathcal{F}_{\mathbb{C},\mathbb{C}}\}.$$

Lemma 11.3. Let $\text{Span}_{\mathbb{R}}(E)$ be the real vector space spanned by the set $E$. Then, \hspace{1cm}

i) $\text{Span}_{\mathbb{R}}(O(n)) = M_n(\mathbb{R})$.

ii) $\text{Span}_{\mathbb{R}}(U(n)) = M_n(\mathbb{C})$.

iii) $\text{Span}_{\mathbb{R}}(USp(n)) = M_n(\mathbb{H})$.
Proof. i) It is sufficient to prove that any \( a = \pm a^T \in M_n(\mathbb{R}) \) is in \( \text{Span}_{\mathbb{R}}(O(n)) \).

When \( a = a^T \), there exists \( v \in O(n) \) such that \( a = vdvd^T \) where \( d \) is a real diagonal matrix. Since

\[
d = \sum_{i=1}^{n} \frac{d_i}{2} (2p_i - 1_n) + \frac{d_i}{2} 1_n,
\]

where \( p_i \) is the projection on the \( i \)-th vector basis, so \( d \) is in \( \text{Span}_{\mathbb{R}}(O(n)) \) and so is \( a \).

When \( a = -a^T \), there exist \( v \in O(n) \) and a family \( r_k \in \mathbb{R}, k \leq \frac{n}{2} \) such that

\[
a = v \text{diag}(0, \ldots, 0, r_1b, \ldots, r_kb) v^T
\]

where \( b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). Thus for \( r = \sum_i r_i \),

\[
v^T av = b_1 \text{diag}(1, \ldots, 1, b, 1, \ldots, 1) + r_2 \text{diag}(1, \ldots, 1, 1, b, 1, \ldots, 1) + \cdots +
+r_k \text{diag}(1, \ldots, 1, 1, b, 1, \ldots, 1) - \text{diag}(r, \ldots, r, (r-r_1)1_2, \ldots, (r-r_k)1_2)
\]

is a real linear combination of matrices in \( O(n) \) by i). So \( a \in \text{Span}_{\mathbb{R}}(O(n)) \).

ii) This follows by i) since \( O(n) \) and \( iO(n) \) are included in \( U(n) \).

iii) Let \( 1, e_1, e_2, e_3 \) be the canonical basis of \( \mathbb{H} \) such that \( e_i e_j = \delta_{ij}1 - \epsilon_{ijk}e_k \) and \( 1e_i = e_i 1 = e_i \). Since \( M_n(\mathbb{H}) \) is an \( \mathbb{H} \)-vector space, \( M_n(\mathbb{H}) = 1M_n(\mathbb{R}) + e_1M_n(\mathbb{R}) + e_2M_n(\mathbb{R}) + e_3M_n(\mathbb{R}) \) and the result follows from \( e_iO(n) \subset USp(n) \). \( \square \)

**Corollary 11.4.** \( \{fM \mid f \in F_{K,K'} \} = \{ \sum_i a_i M b_i \mid a_i \in M_n(\mathbb{K}), b_i \in M_m(\mathbb{K'}) \} \).

**Remark 11.5.** If \( \mathbb{K} = \mathbb{K'} \), then for any \( 0 \neq M \in M_{n \times m}(\mathbb{K}) \),

\[ \{fM \mid f \in F_{K,K} \} = M_{n \times m}(\mathbb{K}) \].

Nevertheless, we have a priori

\[ \{fM \mid f \in F_{K,K'} \} \not\subset \{ aMb \mid a \in M_n(\mathbb{K}), b \in M_m(\mathbb{K'}) \} \],

while the converse inclusion is true by the previous corollary. Actually, for \( n = m = 2 \) and \( \mathbb{K} = \mathbb{K'} = \mathbb{C} \), if \( M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \), \( \text{Rank}(aMb) \leq 1 \) for any \( a \) and \( b \) in \( M_2(\mathbb{C}) \), but for the fluctuation

\[
f = \{ r_1 = r_2 = 1, u_1 = u_1 = 1, u_2 = v_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \}, \text{Rank}(fM) = \text{Rank}(1_2) = 2.
\]

**Lemma 11.6.** Given a family of \( k \) \( \mathbb{R} \)-linearly independent matrices \( M_i \in M_{n \times m}(\mathbb{R}) \), \( i = 1, \ldots, k \), there exists a fluctuation \( f \in F_{R,R} \) such that \( fM_i = 0 \) for all \( i \neq 1 \) and \( fM_1 \neq 0 \).

**Proof.** Let \( \{c_i\}_{i \in \{1, \ldots, p\}} \) be the canonical basis of column vectors in \( \mathbb{R}^p \). (We use abusively the same notation for different \( p \)-s.) Remark first that the fluctuation defined for given \( r_1, r_2 \in \mathbb{R}, i, k \in \{1, \ldots, n\} \) and \( j, l \in \{1, \ldots, m\} \) by

\[
M := r_1 M + r_2 c_i c_k^T M c_j c_l^T
\]

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\[ r_1 M + r_2 M_{kl} c_i c_j^T \]
satisfies \( (fM)_{pq} = r_1 M_{pq} \) for all \( p \neq i \) and \( q \neq j \) and \( (fM)_{ij} = r_1 M_{ij} + r_2 M_{kl} \).

For any \( M \in M_{n \times m}(\mathbb{R}) \) let

\[
W(M) := \begin{pmatrix}
M_{11} \\
\vdots \\
M_{1m} \\
M_{21} \\
\vdots \\
M_{nm}
\end{pmatrix} \in \mathbb{R}^{nm}.
\]

Given a family of matrices \( M_1, \ldots, M_k \in M_{n \times m}(\mathbb{R}) \), let \( N \) be the matrix in \( M_{nm \times k}(\mathbb{R}) \) defined by the columns \( W(M_i) \):

\[
N := (W(M_1) \ | \ W(M_2) \ | \ldots \ | \ W(M_k)).
\]

Thus, if a fluctuation \( f \in \mathcal{F}_{\mathbb{R}, \mathbb{R}} \) is defined simultaneously on all \( M_i \)'s, \( N \) is transformed in \( f(N) := (W(fM_1) \ | \ W(fM_2) \ | \ldots \ | \ W(fM_k)) \). By the previous remark, adding a multiple of a line of \( N \) to a multiple of another one correspond precisely to a fluctuation. Using Gauß’ method, if the \( M_i \)'s are linearly independent (thus \( k \leq nm \)), so are the \( W(M_i) \)'s and there exists a fluctuation \( f \) such that

\[
f(N) = \begin{pmatrix}
1_{k \times k} \\
0_{(nm-k) \times k}
\end{pmatrix}
\]
since the rank of \( N \) is \( k \). This means that a second fluctuation given by \( gM := c_1 c_1^T \ M \ c_1 c_1^T \)
will give \( g f M_1 = f M_1 \neq 0 \) while \( g f M_i = 0 \) for all \( i \neq 1 \).

Remark 11.7. This lemma is false when the matrices have complex entries: Let \( M_1 = \begin{pmatrix} i \\ 1 \end{pmatrix} \) and \( M_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \). Then \( M_1 \) and \( M_2 \) are \( \mathbb{R} \)- and \( \mathbb{C} \)-linearly independent. Nevertheless, \( f M_1 = 0 \) always yields \( f M_2 = \text{Im}(f M_1) = 0 \). There only remains the following

Lemma 11.8. Let \( M_i \in M_{n \times m}(\mathbb{C}) \), \( i = 1, \ldots, k \) be \( k \) matrices such that their real and imaginary parts are \( 2k \) \( \mathbb{R} \)-linearly independent matrices. Then there exists a fluctuation \( f \in \mathcal{F}_{\mathbb{R}, \mathbb{R}} \) such that \( f M_1 \neq 0 \) and \( f M_i = 0 \) for all \( i \neq 1 \).

Proof. According to the previous lemma, there exists a fluctuation \( f \in \mathcal{F}_{\mathbb{R}, \mathbb{R}} \) such that \( \text{Re}(f M_1) \neq 0 \) while \( \text{Im}(f M_1) = \text{Re}(f M_i) = \text{Im}(f M_i) = 0 \) for all \( i \neq 1 \), yielding the conclusion since real and imaginary extractions commute with fluctuations.

Within fluctuations, there are the symmetric ones in the following sense:
**Definition 11.9.** Let $\mathcal{F}_{C,C}^T := \{(r_j, u_j)_{j \in J} \in \mathbb{R} \times U(n) \mid J \text{ finite}\}$ and define fluctuations $f^T \in \mathcal{F}_{C,C}^T$ on $n \times n$ square matrices by
\[ f^T M := \sum_j r_j u_j M u_j^T. \]

**Lemma 11.10.** Let $M_1, M_2$ be two skewsymmetric matrices in $M_n(\mathbb{C})$. Then
i) If the constraints $f^T M_1 = 0$ for $f \in \mathcal{F}_{C,C}^T$ always implies $f^T M_2 = 0$, then $M_2$ is $\mathbb{C}$-colinear to $M_1$.

ii) If $M_2$ is not colinear to $M_1$, then there exists a fluctuation $f^T \in \mathcal{F}_{C,C}^T$ such that $f^T M_1 = 0$ and $f^T M_2 \neq 0$.

**Proof.** Note that ii) is a consequence of i).
To prove i), we may assume that $M_1$ has the form as in (11.11).
Define $s := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $t := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are two unitaries satisfying $wxw^T = -x$ for $w = s, t$.
Define $u = v + 1 \oplus \cdots \oplus 1 \in U(n)$ with $v = \bigoplus_{i=1}^p v_i \in U(2p)$ where $v_i \in \{s, t\}$. Then $M_1 + u M_1 u^T = 0$, so if $M_2$ has the form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, with $A \in M_{2p}(\mathbb{C}), B \in M_{2p,n-2p}(\mathbb{C}), C \in M_{n-2p,2p}(\mathbb{C}), D \in M_{n-2p,n-2p}(\mathbb{C})$, then $0 = M_2 + u M_2 u^T$. We deduce $0 = A + v A v^T = B + v B = C + C v^T = D + D$. Thus choosing $v_s = \bigoplus_{i=1}^p s$ and $v_t = \bigoplus_{i=1}^p t$, we have $(1_{2p} + v_s)B + (1_{2p} + v_t)B = 0$, so $B = 0$ since $2 1_{2p} + v_s + v_t$ is invertible. Similarly, $\mathbb{C} = D = 0$.

If $A_{kl}$ is the partition of $A$ in $2 \times 2$ matrices, the constraint $A + v A v^T = 0$ implies $0 = A_{kl} + u_k A_{kl} u_{k}^T$. When $k \neq l$, $A_{kl}$ is necessarily zero since we may choose independently $u_k$ and $u_l$ in $\{s, t\}$. When $k = l$, $A_{kk} = \begin{pmatrix} \alpha_k & \gamma_k \\ -\gamma_k & \beta_k \end{pmatrix}$ where the $\alpha, \beta, \gamma$’s are complex numbers, since $A$ is skewsymmetric. If $u_k = t$, $0 = A_{kk} + u_k A_{kk} u_{k}^T = 2 \begin{pmatrix} \alpha_k & 0 \\ 0 & \beta_k \end{pmatrix}$ and $A_{kk} = \gamma_k x$. Thus $M_2 = \bigoplus_{k=1}^p \gamma_k x \oplus 0 \oplus \cdots \oplus 0$ and it remains to prove that $\gamma_k = c m_k$ for some constant $c$.

Define
\[
\begin{align*}
  u_k & := s \oplus 1_2 \oplus \cdots \oplus 1_2 \oplus 1 \oplus \cdots \oplus 1, \\
  v_k & := 1_2 \oplus \cdots \oplus s \oplus \cdots \oplus 1_2 \oplus 1 \oplus \cdots \oplus 1, \\
  w_k & := \begin{pmatrix} 0 & 0 & \cdots & 0 & 1_2 \\ 0 & 1_2 & & & 0 \\ & & \ddots & & \vdots \\ 0 & & & 1_2 & 0 \\ 1_2 & 0 & \cdots & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1_2 & \cdots & \cdots & 1_2 \end{pmatrix} \oplus \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}
\end{align*}
\]
three unitaries where the perturbation in $v_k$ is put at the k-th entry. Then
\[
0 = (2m_1)^{-1} (M_1 - u M_1 u^T) - (2m_k)^{-1} w_k (M_1 - v_k M_1 v_{k}^T) w_{k}^T
\]
and the same relation for $M_2$ yields $0 = 2(\gamma_1 m_1^{-1} - \gamma_k m_k^{-1})$, so $\gamma_k = \gamma_1 m_1^{-1} m_k$ and $M_2 = (\gamma_1 m_1^{-1}) M_1$. \(\square\)
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\[
\begin{array}{cccc}
  a & b & c & a & b & c & a & b & c \\
  a & o & o & a & o & o & a & o & o \\
  b & o & o & b & o & o & b & o & o \\
  c & o & o & c & o & o & c & o & o \\
  \text{diag. 1} & \text{diag. 2} & \text{diag. 3} & \text{diag. 4} \\
  a & b & c & a & b & c & a & b & c \\
  a & o & o & a & o & o & a & o & o \\
  b & o & o & b & o & o & b & o & o \\
  c & o & c & c & o & c & c & o & c \\
  \text{diag. 5} & \text{diag. 6} & \text{diag. 7} & \text{diag. 8} \\
  a & b & c & a & b & c & a & b & c \\
  a & o & o & a & o & o & a & o & o \\
  b & o & o & b & o & o & b & o & o \\
  c & o & c & c & o & c & c & o & c \\
  \text{diag. 9} & \text{diag. 10} & \text{diag. 11} & \text{diag. 12} \\
\end{array}
\]
Fig. 6.2

Fig. 7
\[ a \ b \ c \]
\[ a \circ \circ \]
\[ b \circ \circ \circ \]
\[ c \circ \circ \circ \]

diag. 27

\[ a \ b \ c \ a \ b \ c \]
\[ a \circ \circ \circ \ a \circ \circ \circ \ a \circ \circ \circ \]
\[ b \circ \circ \circ \ b \circ \circ \circ \ b \circ \circ \circ \]
\[ c \circ \circ \circ \ c \circ \circ \circ \ c \circ \circ \circ \]

diag. 28         diag. 29

\[ a \ b \ c \ a \ b \ c \ a \ b \ c \]
\[ a \circ \circ \circ \ a \circ \circ \circ \ a \circ \circ \circ \]
\[ b \circ \circ \circ \ b \circ \circ \circ \ b \circ \circ \circ \]
\[ c \circ \circ \circ \ c \circ \circ \circ \ c \circ \circ \circ \]

diag. 30         diag. 31         diag. 32         diag. 33

\[ a \ b \ c \ a \ b \ c \ a \ b \ c \]
\[ a \circ \circ \circ \ a \circ \circ \circ \ a \circ \circ \circ \]
\[ b \circ \circ \circ \ b \circ \circ \circ \ b \circ \circ \circ \]
\[ c \circ \circ \circ \ c \circ \circ \circ \ c \circ \circ \circ \]

diag. 34         diag. 35         diag. 36         diag. 37

Fig. 8
etc.

Fig. 9