Basic Gravitational Currents and Killing-Yano Forms

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It has been shown that for each Killing-Yano (KY)-form accepted by an $n$-dimensional (pseudo)Riemannian manifold of arbitrary signature, two basic gravitational currents can be defined. Conservation of the currents are explicitly proved by showing co-exactness of the one and co-closedness of the other. Some general geometrical facts implied by these conservation laws are also elucidated. In particular, the conservation of the one-form currents implies that the scalar curvature of the manifold is a flow invariant for all of its Killing vector fields. It also directly follows that, while all KY-forms and their Hodge duals on a constant curvature manifold are the eigenforms of the Laplace-Beltrami operator, for an Einstein manifold this is certain only for KY 1-forms, $(n - 1)$-forms and their Hodge duals.

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I. INTRODUCTION

Conservation laws are intimately related to the symmetries of the underlying space-time. Killing vector fields which generate local isometries play a prominent role in constructing the usual conserved currents and a conserved charge is associated with each such a current. In an $n$-dimensional space-time, by making use of the asymptotic symmetries, an ADM conserved charge [1] can be written as an integral over $(n - 2)$-spheres at spatial infinity [2]. For example, time translation Killing vector field defines the ADM mass and rotational Killing vector fields define the ADM angular momenta. Antisymmetric generalizations of Killing vector fields to higher order forms are called Killing-Yano (KY) forms. These reflect the hidden symmetries of the metric and can be used in generalizing the conserved currents to $p$-brane space-times that are extended objects with $p$ spatial dimensions [3]. For the most recent and for the earlier seminal references about KY (and conformal KY) forms we refer to three recent PhD dissertations [4, 5, 6].

Abbot-Deser (AD) construction of conserved charges [2] can be extended to KY-forms and in such a case the associated generalized charges are called Y-ADM charges. These are constructed for asymptotically flat and asymptotically anti-de Sitter space-times [3, 4, 5, 6]. In ADM case, the charges can be written in terms of Einstein forms and therefore they can be related to stress-energy forms via Einstein equations. The physical meanings of these charges are obtained from this relation. On the other hand, for the generalized charges, there is no direct relation with Einstein equations and hence their physical interpretations are not yet clear. The ADM charges are extensive quantities since they are obtained from integrals over the $(n - 2)$-spheres at infinity which enclose $(n - 1)$-dimensional regions. However, a Y-ADM charge associated to a KY $p$-form is calculated from the integrals over $(n - p - 2)$-spheres at spatial infinity along the directions transverse to the $p$-brane. Since these spheres constitute the boundaries of $(n - p - 1)$-dimensional regions, the corresponding charges are intensive quantities. This makes it possible to interpret Y-ADM charges as charge densities for $p$-brane space-times.

A conventional way of defining a conserved current and the corresponding conserved charge in an $n$-dimensional space-time can be described as follows. Suppose that a $p$-form $J$ satisfies $d^*J = 0$, where $d$ stands for the exterior derivative and $^*$ denotes the Hodge map. Then, by the Stokes theorem the integral of $^*J$ over the boundary of a $(n - p + 1)$-dimensional region vanishes. This means that $J$ is a conserved current and by integrating $^*J$ over an arbitrary $(n-p)$-dimensional region we can define the corresponding charge by $Q = \int_{\Sigma}^{\delta} J$. $Q$ is a conserved charge in the sense that it is a constant for all regions sharing the same homology class with $\Sigma$. A $p$-form $J$ is co-closed if and only if $^*J = 0$, where $\delta$ represents the co-derivative operator defined below. Noting that the co-closedness of $J$...
is equivalent to the condition $d^* J = 0$, the above remarks can be generalized as follows: any co-closed $p$-form is a conserved current and the corresponding conserved charge is defined as prescribed above.

Evidently, all linear combinations of conserved quantities are trivially conserved and therefore a conservation law must be related to a basic current that can not be constructed from others. Physically meaningful conserved charges must also be constructed from the basic currents. In two recent papers [3, 4], it has been shown that the $p$-form

$$ J = -i_{X_a} i_{X_b} \omega \wedge R^{ab} + 2(-1)^p i_{X_a} \omega \wedge P^a + R \omega, \quad (1) $$

constructed from a KY $p$-form $\omega$ (for definition see equation (5)) and curvature characteristics is a generalized conserved current and can be related to a conserved charge of a $p$-brane space-time. Throughout this study we assume the Einstein summation convention over repeated indices and we use $i_X$ to denote the interior derivative with respect to the vector field $X$. In the expression (1), $R^{ab}$ are the curvature 2-forms and $P^a$ are the Ricci 1-forms, respectively, the Ricci 1-forms and the curvature scalar. The current $J$ is an attempt to generalize the well-known (1-form) current $K_a s^{-1} G^a$ which is constructed from the components of a Killing vector field $K$ and Einstein $(n-1)$-forms $G^a = R_{\mu\nu}^{\alpha\beta} (e^{\mu\nu\alpha\beta})$. Here $s^{-1}$ denotes the inverse of the Hodge map which is defined as $s(-1)^{p(n-p)}$ ($s$ is the sign of the determinant of metric) times the Hodge map when acting on $p$-forms.

In this study we will show that two fundamental currents can be constructed from KY-forms and curvature characteristics of the underlying space-time and prove that $J$ given by (1) is a particular linear combination of gravitational currents proposed here. We furthermore show that one of the currents is co-exact which enables us to discover a number of basic geometric facts some of which seem to be unknown in the literature. These facts can be stated as follows.

(i) The scalar curvature of any (pseudo)Riemannian manifold is a flow invariant for all of its Killing vector fields. (ii) The conserved currents directly provide decompositions for differential forms constructed from “contractions” of KY $p$-forms with the Ricci 1-forms, curvature 2-forms and with the Einstein $(n-1)$-forms (for the values $p = 1, 2$ and $p = n-1$). These contractions are expressible solely in terms of KY-forms themselves and their (co)derivatives. (iii) On Einstein manifolds, duals of all Killing and Yano vector fields are eigenforms of the Laplace-Beltrami operator such that the eigenvalues have multiplicities with well defined lower bounds. Some well known facts related to the spectrum of Laplace-Beltrami operator on constant curvature manifolds also follow directly from the properties of conserved currents.

From here on we assume that the underlying manifold is an $n$-dimensional (pseudo)Riemannian manifold with arbitrary signature. Some of our conventions and notations are fixed above for which, as well as for the remaining ones, we have mainly adopted the conventions of reference [8]. The covariant derivative with respect to a given vector field $X$ will be denoted by $\nabla_X$ in terms of which the exterior derivative and co-derivative can be expressed as

$$ d = e^a \wedge \nabla_{X_a}, \quad \delta = -i_{X^a} \nabla_{X^a}, \quad (3) $$

where the local co-frame $\{e^a\}$ is dual to the tangent frame $\{X_a\}$ such that $i_{X_a} e^a = e^a (X_b) = \delta^a_b$. The action of $i_X$ on an arbitrary $p$-form $\alpha$ is defined, for all vector fields $Y_a$ by $i_{X^a} (Y_b^a) = \delta^a_b$. When acting on an arbitrary $p$-form $\delta$ can be written as $(-1)^{p+1} d\delta$. The rest of the paper is organized as follows. The definitions of two basic (uncomposite) currents and the proofs of co-exactness of the one and co-closedness of the other are given in the next section. Three main operators that play important role in our study are also introduced there and their relevant properties are presented. The basic $p$-form currents for the values $p = 1, n-2, n-1,$ are studied in Section III where some of the general geometric facts, such as the statements (i) and (ii) given above, are explored as well. Conserved currents on some special manifolds, such as Ricci flat, conformally flat and constant curvature manifolds as well as Einstein manifolds, are considered in Section IV. A brief summary of the study and some concluding remarks are given in the last section. Contracted Bianchi identities that play prominent role in the proof of the co-closedness of a basic current are collected in Appendix A and an alternative proof for the statement (i) given above is provided in Appendix B.

II. TWO BASIC CONSERVED CURRENTS

We begin by defining three $p$-forms

$$ j_1 = i_{X_a} i_{X_b} \omega \wedge R^{ab}, \quad j_2 = i_{X_a} \omega \wedge P^a, \quad j_3 = R \omega, \quad (4) $$
for a given KY $p$-form $\omega$. A $p$-form $\omega$ is called a KY $p$-form if and only if
\[ \nabla_X \omega = \frac{1}{p+1} i_X d\omega, \]  is satisfied for all vector fields $X$. Two immediate consequences of this definition are that, all KY-forms are co-closed and satisfy the relation $i_Y \nabla_X \omega + i_X \nabla_Y \omega = 0$, for any two vector fields $X$ and $Y$. The second relation means that all symmetrized covariant derivatives of KY-forms vanish. As is evident from the definition (5), any function is a KY 0-form and a KY $n$-form is a constant multiple of the volume form. As manifestations of the hidden symmetries of the metric of the underlying space-time, co-closedness of KY-forms means that all of them define conserved currents and corresponding conserved charges. Apart from this well-established fact, as we are about to see, their particular combinations with the curvature characteristics, also give rise to important conservation laws.

The main goal of our study is to prove that the $(p$-form) currents
\[ J_1 = -j_1 + (-1)^p j_2, \] \[ J_2 = (-1)^p j_2 + j_3, \] are separately conserved on any (pseudo)Riemannian manifold for all $p$'s. These are the currents that were, and will be, referred to as basic. We shall also prove that $J_1$ is co-exact and elucidate its important geometrical and physical implications. Evidently all linear combinations of $J_1$ and $J_2$ are also conserved and the conservation of (1) is a particular case of this fact since $J = J_1 + J_2$. These enable us to define more conserved charges and of course physically meaningful and fundamental ones must be directly constructed from $J_1$ and $J_2$. Finally, we should note that (6) and (7) can be rewritten more compactly as
\[ J_1 = i_{X_a} (i_{X_b} \omega \wedge R^{ba}), \quad J_2 = (-1)^p i_{X_a} (\omega \wedge P^a). \] These clearly exhibit that for all KY $0$-forms (for all functions) while $J_1$ is identically zero, $J_2$ is equal to $j_3$ and that both currents are identically zero for KY $n$-forms. Moreover, by observing that the current (1) is
\[ J = i_{X_a} i_{X_b} (\omega \wedge R^{ab}), \] we see that for all KY $(n-1)$-forms $J$ is identically zero. But, in our case the currents are non-vanishing but linearly dependent. So, for a general (pseudo)Riemannian manifold we can safely say that, the space of conserved currents that linearly depend on a given KY $p$-form is, for $0 < p < n-1$, at least two dimensional and is spanned by $J_1$ and $J_2$. Of course, in some special cases these may be linearly dependent for some, or even for all values of $p$. For example, as we have shown in Section IV, on the constant curvature manifolds the latter extreme case is observed. More details in this context are given in sections III and IV.

### A. Main Operators

The following three second order differential operators
\[ R(X_a, X_b) = \nabla_{X_a} \nabla_{X_b} - \nabla_{X_b} \nabla_{X_a} - \nabla_{[X_a, X_b]}, \] \[ I(R) = e^{a} \wedge i_{X_a} R(X^b, X_a), \] \[ \nabla^2(X_a, X_b) = \nabla_{X_a} \nabla_{X_b} - \nabla \nabla_{X_a} X_b, \] play prominent roles in this study. The first is the well-known curvature operator of the geometry, the second is known as the curvature endomorphism and the third is the Hessian. All of these operators are degree-preserving when acting on differential forms. The torsion-zero condition
\[ [X, Y] = \nabla_X Y - \nabla_Y X, \] of (pseudo)Riemannian geometry implies that the curvature operator can be written as the anti-symmetric difference
\[ R(X_a, X_b) = \nabla^2(X_a, X_b) - \nabla^2(X_b, X_a), \] of two Hessians. The most important property of the curvature endomorphism is that it can also be written as the difference $I(R) = \nabla^2 - \mathcal{R}$ (the classical Weitzenböck formula) of the trace of Hessian $\nabla^2 = \nabla^2(X_a, X^a)$ and the well-known Laplace-Beltrami operator
\[ \mathcal{R} = -(\delta d + d\delta). \]
An important property of the Hessian that will be used in our analysis is that its double contraction vanishes
\[ i_{X_b} i_{X_a} \nabla^2 (X^b, X^a) = 0. \] (14)
This can be easily proved by using the definition of co-derivative and the general relation
\[ [\delta, i_X]_+ = -i_{X^a} i_{\nabla_{X_a} X}, \] (15)
where \([\cdot, \cdot]_+\) denotes the anti-commutator. This relation easily follows from the definition of \(\delta\) and the relation \([\nabla_X, i_Y] = i_{[X, Y]}\). Then by direct computation we obtain
\[ i_{X_a} i_{X_b} \nabla_{X_b} \nabla_{X_a} = -i_{X_a} \delta \nabla_{X_a} = -\omega_c^a (X_b) i_{X_b} i_{X_a} \nabla_{X_a} = i_{X_a} i_{X_b} \nabla_{X_b} \nabla_{X_a}, \]
which proves (14). Here we have used the Poincare lemma \(\delta^2 = 0\) and \(\omega_c^a\)’s are the connection 1-forms defined by \(\nabla_{X_b} X_a = \omega_c^a (X_b) X_c\). As an aside, from (13) we see that the double contraction of the curvature operator also vanishes.

B. Co-exactness of \(J_1\)

We will show that the co-exactness of \(J_1\) is a direct result of the action of curvature endomorphism on KY-forms and the co-closedness of \(J_2\) also follows from action of these operators on KY-forms and Bianchi identities. The actions of curvature operator and curvature endomorphism on an arbitrary \(p\)-form \(\phi\) are as follows
\[ R(X_a, X_b) \phi = -i_{X_a} R_{ab} \wedge i_{X_b} \phi, \] (16)
\[ I(R) \phi = P_c \wedge i_{X_c} \phi - R_{cb} \wedge i_{X_b} i_{X_c} \phi. \] (17)
The first relation can easily be found in the literature (see [8] equation (8.1.11) and [10]) and the second follows from the first. Indeed, multiplying (16) by \(e^a \wedge\) and using the first Bianchi identity \(R_{ab} \wedge e^b = 0\) we find
\[ e^a \wedge R(X_a, X_b) \phi = -R_{cb} \wedge i_{X_b} \phi, \] (18)
and then by contracting with \(i_{X_b}\) we arrive at (17). A comparison of the right hand side of (17) with (6) shows that \(J_1\) is generated by the action of the curvature endomorphism on the KY \(p\)-form \(\omega\) used in its definition:
\[ J_1 = -I(R) \omega. \] (19)
The last relation implies that the action of other operators on KY-forms may also have important implications. It is convenient to consider the action of Hessian first. By differentiating both sides of the defining relation (5) we obtain
\[ \nabla^2 (X_a, X_b) \omega = \frac{1}{p+1} i_{X_b} \nabla_{X_b} d\omega, \] (20)
\[ R(X_a, X_b) \omega = \frac{1}{p+1} (i_{X_b} \nabla_{X_a} - i_{X_a} \nabla_{X_b}) d\omega. \] (21)
The second relation is obtained from (20) by virtue of (13). Multiplying the both sides of (21) with \(e^a \wedge\), leads us to
\[ e^a \wedge R(X_a, X_b) \omega = -\frac{p}{p+1} \nabla_{X_b} d\omega, \] (22)
where we have used the scaling property \(e^a \wedge i_{X_b} \alpha = k \alpha\) that holds for any \(k\)-form \(\alpha\). By contracting both sides of (22) with \(i_{X_b}\), we obtain
\[ I(R) \omega = \frac{p}{p+1} \delta d\omega, \] (23)
and by comparing with (19) we arrive at
\[ J_1 = -\frac{p}{p+1} \delta d\omega. \] (24)
That is, \(J_1\) is a co-exact \(p\)-form and hence provides a conserved current.
C. Co-Closedness of $\mathcal{J}_2$

In proving the co-closedness of $\mathcal{J}_2$ we shall need the covariant derivatives of $d\omega$ and the contracted Bianchi identities. Some of these identities (that can be derived from the so-called second Bianchi identity) are not easily found in the literature and as they are repeatedly used below, we have collected them in the Appendix A where their derivations are given in some details.

From (18) and (22) we directly read

$$\nabla_{X_b} d\omega = \frac{p + 1}{p} R_{cb} \wedge i_{X^c} \omega,$$

and by taking the covariant derivatives of both sides we obtain

$$\nabla_{X_b} \nabla_{X_a} d\omega = \frac{p + 1}{p} (\nabla_{X_b} R^c_{a} \wedge i_{X^c} \omega + R^c_{a} \wedge \nabla_{X_b} i_{X^c} \omega).$$

We now define

$$T^{bca} = \nabla_{X^b} R^{ca} + \omega^{cl} (X^b) R^a_l + \omega^{al} (X^b) R^c_l,$$

$$Q^{ba} = T^{bca} \wedge i_{X^c} \omega.$$  

$T^{bca}$ is thoroughly investigated in Appendix A. In terms of $Q^{ba}$ we rewrite (26) as

$$\nabla_{X^b} \nabla_{X^a} d\omega = \frac{p + 1}{p} Q^{ba} - \omega^{al} (X^b) \nabla_{X^a} d\omega + \frac{1}{p} R^{ca} \wedge i_{X^c} i_{X^b} d\omega,$$

and by leaving $Q^{ba}$ alone we arrive at

$$Q^{ba} = \frac{1}{p + 1} (p \nabla^2 (X^b, X^a) - R^{ca} \wedge i_{X^c} i_{X^b}) d\omega.$$  

We are now ready to take the co-derivative of $\mathcal{J}_2$ given by (8)

$$\delta \mathcal{J}_2 = \frac{(-1)^p}{i_{X_b} \nabla_{X^b} i_{X^b} (\omega \wedge P^p)}.$$

By making use of the relation $[\nabla_X, i_Y] = i_{\nabla_X Y}$, the definition of the KY-forms and contracted Bianchi identity $i_{X_b} P^a = i_{X_a} P^b$, we can write

$$\delta \mathcal{J}_2 = \frac{(-1)^p}{i_{X_b} i_{X^b} (\omega \wedge S^{ab})},$$

where we have defined $S_{ab} = \nabla_{X_a} P_b + \omega_{bk} (X_a) P^k$. In Appendix A, it is shown that $S^{ab} = S^{ba} + i_{X_a} T^{cab}$ which when substituted into above relation yields

$$\delta \mathcal{J}_2 = \delta \mathcal{J}_2 - \frac{(-1)^p}{i_{X^a} i_{X_b} [\omega \wedge i_{X^a} T^{cab}]}.$$  

Thus, by making use of the cyclic property of $T^{abc}$ given by (A12), we have

$$2 \delta \mathcal{J}_2 = - i_{X_a} i_{X_b} [i_{X^a} (\omega \wedge T^{cab}) - i_{X_b} (\omega \wedge T^{bac})]$$

$$= i_{X_a} i_{X_b} (i_{X^a} (\omega \wedge T^{cab}) - i_{X_b} (\omega \wedge T^{bac}))$$

$$= - i_{X_a} i_{X_b} [i_{X^a} (\omega \wedge (T^{abc} + T^{bca}))],$$

and the anti-symmetry property of $T^{cab}$ with respect to the last two indices (see (A7)), we end up with

$$\delta \mathcal{J}_2 = - i_{X_a} i_{X_b} Q^{ba}.$$  

So, the proof of the co-closedness of $\mathcal{J}_2$ has been reduced to the vanishing of double contraction of $Q^{ba}$. We now observe that, by virtue of (14), the double contraction of the first term at the right hand side of (29) vanishes. For the double contraction of the second term of (29) we obtain

$$i_{X_a} i_{X_b} (R^{ca} \wedge i_{X^c} i_{X^b} d\omega) = i_{X_a} P^c \wedge i_{X^c} i_{X^b} d\omega + i_{X_a} R^{ca} \wedge i_{X^c} i_{X^b} i_{X^b} d\omega.$$

Since $i_{X_a} P^c$ is symmetric, the first term at the right hand side vanishes and as the cyclic sum of $i_{X^c} R^{ca}$ amounts to zero (see (A4)), the second term vanishes as well. These prove $i_{X_a} i_{X_b} Q^{ab} = 0$ and hence $\mathcal{J}_2$ is co-closed.
III. SPECIAL $p$-FORM CURRENTS: GENERAL GEOMETRIC FACTS

In this section we study the basic $p$-form currents for the values $p = 1, n-2, n-1$, and explore some of the general implications that follow from their conservations. Some of these implications which hold for any (pseudo) Riemannian manifold were concisely stated in the introduction by the statements (i) and (ii). It is worth emphasizing that the special cases considered below exhaust all possible cases in four dimensions. Similar investigations for some physically relevant special manifolds are carried out in the next section.

A. One-form Currents

Evidently, for all KY 1-forms $j_1$ is identically zero and therefore, $j_2$ is co-exact. For a given KY 1-form $\omega$ we have $j_2 = K_a P^a$, where

$$K = (i_{X_a} \omega) X^a = K^a X_a,$$

is the Killing vector field dual to $\omega$, and then from (6) and (24) we obtain

$$K_a P^a = \frac{1}{2} \delta d \tilde{K}.$$  \hspace{1cm} (32)

This relation is already known in the literature (see pp.231 in [8]) and the above result may be considered as an alternative derivation. What is more important is that the co-closedness of $J_2$ is equivalent to $\delta j_2 = 0$ and this implies that $j_3$ must also be co-closed, for $J_2$ is co-closed. On the other hand, from the definition $j_3 = \omega R$ we find

$$\delta j_3 = -i_{X_a} \nabla X^a (\omega R) = (\delta \omega) R - i_{X_a} \omega \nabla X^a R$$

and since all KY-forms are co-closed, we arrive at $\delta j_3 = -\nabla K R = 0$. As a result, the covariant derivative of curvature scalar with respect to any Killing vector field $K$ of the underlying manifold must be zero;

$$\nabla K R = 0.$$  \hspace{1cm} (33)

So we have arrived at a general relation of the (pseudo)Riemannian geometry which uncovers an important property of the Killing vector fields that seems to be, as far as we know, unnoticed in the literature. Since the Lie derivative and covariant derivative coincide on 0-forms, this also means that the Lie derivative of the scalar curvature with respect to Killing vector fields is zero for any (finite) dimension and signature. More precisely, the scalar curvature of any (pseudo)Riemannian manifold is a flow-invariant for all of its Killing vector fields. An alternative proof of equation (33) is given in Appendix B.

It is worth mentioning that for 1-form currents above we have

$$J = -K_a (2P^a - Re^a) = K_a^{*^{-1}} G^a,$$

where $K_a^{*^{-1}} G^a$ (see also the equation (38) below) is the current whose conservation is the starting point for the studies referred in the introduction. The above analysis explicitly shows that this current is the sum of the co-exact current $j_2$ given by (32) and the co-closed current $J_2 = -K_a (P^a - Re^a)$. It is perhaps, due to the lack of the relation (33), the compositeness of the current $K_a^{*^{-1}} G^a$ have remained unnoticed. As is shown in Appendix B, (33) directly follows from the contracted Bianchi identities and (32).

B. Currents for duals of Yano vector fields

It is a well established fact that the Hodge map defines a general one-to-one correspondence between the vector space of KY $p$-forms and of the closed conformal KY $(n-p)$-forms [11]. In particular, for each KY $(n-1)$-form $\omega_{(n-1)}$ there exists a uniquely determined closed conformal KY 1-form $\tilde{Y}$ such that

$$\omega_{(n-1)} = ^* \tilde{Y},$$ \hspace{1cm} (34)

where the conformal vector field $Y = Y^a X_a$, known as the Yano vector [12], is the metric dual of $\tilde{Y} = Y_a e^a$. Yano vectors are locally gradient fields whose integral curves are pre-geodesics and they generate special conformal transformations. For (34) we have

$$J_2 = (-1)^{n-1} i_{X_a} (^* \tilde{Y} \wedge P^a) = i_{X_a} (\tilde{Y} \wedge ^* P^a).$$ \hspace{1cm} (35)
Since $i_{X^a} P^a = (P^a \wedge e_a) = 0$ by the Bianchi identity, we deduce from (35) that $Y^*_a P^a$ is co-closed and equivalently, the 1-form $Y_a P^a$ is closed. Therefore, the vector field $Y_a P^a$ is locally a gradient field (in fact, the equation (41) given below proves that $Y_a P^a$ is globally a gradient field).

In order to write $J_1$ in terms of KY $(n-1)$-forms we first express $j_1$ and $j_2$ in terms of KY $(n-1)$-forms $^*Y$ and the Einstein $(n-1)$-forms $G^a$ as follows:

$$
j_1 = i_{X^a} i_{X^b} ^*Y \wedge R_{ab} = -Y_a G^a,
$$

$$
j_2 = i_{X_a} ^*Y \wedge P^a = Y^b e_{ab} \wedge P^a = (-1)^a Y_a (G^a + ^*P^a),
$$

where

$$G^c = R_{ab} \wedge ^*(e^{cab}) = R^e e^c - 2^*P^c. \tag{38}$$

So, by combining (36) and (37) as in (6), $J_1$ is found to be

$$J_1 = -Y_a ^*P^a. \tag{39}$$

By virtue of (24), this proves the co-exactness of $Y_a ^*P^a$:

$$Y_a ^*P^a = \frac{n-1}{n} \delta d^*Y. \tag{40}$$

or, equivalently, the exactness of $Y_a P^a$

$$Y_a P^a = \frac{n-1}{n} \delta \delta Y. \tag{41}$$

Equations (32) and (41) reveal an important general role of Killing and Yano vector fields that may be of significant in geometry as well as in physics. These vector fields are the “integrating vector fields” for Ricci 1-forms, in the sense that if the underlying space-time accepts such vector fields then their contractions with Ricci 1-forms as in (32) and (41) are, respectively, co-exact and exact forms which are also specified in terms of the duals of these vector fields. From (38) we also see that they enable us to make decompositions of the corresponding contractions of the Einstein $(n-1)$-forms which contain exact or co-exact parts. On the other hand, for a KY $(n-2)$-form $\omega$ we can write $\omega = ^*\beta$, where $\beta$ is a closed conformal KY 2-form. In this case we have

$$J = i_{X_a} i_{X_b} (\omega \wedge R^{ba}) = \beta_{ba}^* R^{ba},$$

which exhibits the decomposition of $\beta_{ba}^* R^{ba}$ as a sum of co-exact $J_1$ and co-closed $J_2$.

IV. CONSERVED CURRENTS ON SOME SPECIAL MANIFOLDS

On a Ricci-flat (pseudo)Riemannian manifold for which $P^a = 0$ and hence $R = 0$, $J_2$ is identically zero and $J_1$ is equal to $-j_1$. Therefore, for each KY $p$-form $\omega$ accepted by such a manifold we have

$$(i_{X_a} i_{X_b} \omega) \wedge R^{ab} = \frac{p}{p+1} \delta d\omega.$$  

Let us now consider a conformally flat (pseudo)Riemannian manifold which is characterized by the vanishing of Weyl 2-forms, or equivalently, by

$$R_{ab} = \frac{1}{n-2} [P_a \wedge e_b - P_b \wedge e_a + \frac{R}{n-1} e_a \wedge e_b].$$

This implies that $j_1$ linearly depends on $j_2$ and $j_3$ and after a little computation we get

$$J_1 = (-1)^p \frac{n-2p}{n-2} j_2 - \frac{p(p-1)}{(n-1)(n-2)} j_3.$$  

Thus, when $n$ is even and $p = n/2$, $j_3$ is co-exact. In general, in view of (6) and (7) we can say that for $1 < p < n-1$, $J_1$ and $J_2$ are linearly independent if and only if $j_2$ and $j_3$ are.
A. Constant-curvature space-times

For an $n$-dimensional constant-curvature space-time characterized by $R_{ab} = ce_a \wedge e_b$ with $c$ constant, the relations

$$P_a = c(n-1)e_a, \quad \mathcal{R} = cn(n-1). \quad (42)$$

directly follow from $R_{ab} = ce_a \wedge e_b$. In such a case we have,

$$\mathcal{J}_1 = -c(i_X i_Y \omega) \wedge e_{ab} + (-1)^p c(n-1) i_X \omega \wedge e^a$$

$$= -c p(n-p)\omega, \quad (43)$$

$$\mathcal{J}_2 = (-1)^p c(n-1) i_X \omega \wedge e^a + cn(n-1)\omega$$

$$= c(n-1)(n-p)\omega, \quad (44)$$

where the scaling property has been used several times. Thus, the currents become constant multiples of their defining KY-forms and therefore they are linearly dependent. Moreover, their co-closedness are nothing more than the co-closedness of KY-forms and hence give nothing new. But the co-exactness of $\mathcal{J}_1$ has an important implication.

By comparing (24) and (43) we obtain, for $p \neq 0$

$$\delta d\omega = c(p+1)(n-p)\omega,$$

and by recalling the fact that every KY-form is co-closed, we see that every KY $p$-form is an eigenform of the Laplace-Beltrami operator:

$$d^2 \omega = -c(p+1)(n-p)\omega. \quad (45)$$

This is nontrivial for $p \neq 0$ and $p \neq n$. The eigenvalues depend on $n$, on the degree of the KY-form and on the constant $c$. In particular, the sign of eigenvalues is the opposite of the sign of $c$. Since the connection of a (pseudo)Riemannian geometry is metric compatible, covariant derivatives commute with the Hodge map. This implies that three main operators of section II A and therefore $d^2$ also commute with the Hodge map. Equation (45) shows that the Hodge dual of any KY $p$-form (which need not be a KY $(n-p)$-form, but is certainly a closed conformal KY $(n-p)$-form) is also an $(n-p)$-eigenform of $d^2$ corresponding to the same eigenvalue. If we restrict ourselves to $p$-eigenforms of the Laplace-Beltrami operator we also get another series coming from the Hodge dual of a KY $(n-p)$-form. So, for $\alpha = \star \beta$ where $\beta$ is a KY $(n-p)$-form, we have

$$d^2 \alpha = -c p(n-p+1)\alpha,$$

where $\alpha$ is a closed conformal KY $p$-form. As a result, each eigenvalue corresponding to a KY $p$-form is degenerate with multiplicity (counted in the space of $p$-forms) which is not lower than the number of linearly independent KY $p$-forms. An exceptional case may occur when $n$ is even and $p = n/2$. In such a case, the mentioned lower bound may decrease depending on the existence of dual and anti-self dual KY $p$-forms. By now, it is well-established fact that the number of linearly independent KY $p$-forms is bounded from above, for any dimension and signature, by the binomial number

$$C(n+1,p+1) = \frac{(n+1)!}{(p+1)!(n-p)!}, \quad (46)$$

and these upper bounds are attained on the constant curvature manifolds $\mathbb{R}^n$. In fact, on a constant curvature Riemannian manifold, such as the standard $n$-sphere, the spectrum of $d^2$ on $p$-forms is well known: it consists of two series related to each other by the interchange $p \leftrightarrow (n-p)$ in the eigenvalues and both depend (apart from $n$ and $p$) on a nonnegative integer $k = 0, 1, \ldots$. Eigenforms corresponding to the minimal values of these series (for which $k = 0$) turn out to be KY $p$-forms and the Hodge duals of KY $(n-p)$-forms. What we have found by analyzing one of the currents above correspond to minimal parts of the spectrum for the case of a constant curvature space-time which need not necessarily be Riemannian.

B. Einstein Manifolds

Let us now consider an $n$-dimensional Einstein manifold characterized by $P_a = ke^a$ with constant $k = \mathcal{R}/n$. Since in three dimensions any Einstein space is necessarily of constant curvature here we suppose $n \geq 4$ and $R_{ab} \neq ce_a \wedge e_b$. In three dimensions any Einstein space is necessarily of constant curvature and hence we suppose $n \geq 4$.
to reach nontrivial statements. In this case we have
\[ J_1 = -i_X i_X \omega \wedge R_{ab} - kp \omega, \]
\[ J_2 = k(n - p)\omega. \]
Obviously, the conservation of \( J_2 \) becomes trivial but, the co-exactness of \( J_1 \) implies
\[ j_1 = i_X i_X \omega \wedge R_{ab}, \]
\[ = kp \omega - \frac{p}{p + 1} \delta d\omega. \]

In particular, for any KY 2-form \( \omega = (1/2)\omega_{ab} e^{ab} \) of an Einstein manifold the “contracted” curvature 2-form \( \omega_{ba} R^{ab} \) can be written as
\[ \omega_{ba} R^{ab} = 2k \omega - \frac{2}{3} \delta d\omega. \]  

Note that this is trivial for constant curvature spaces. This relation shows that for each KY 2-form accepted by an Einstein space the contraction of the curvature 2-forms as in equation (47) is decomposed into co-closed and co-exact parts that are determined up to well-defined constants, by these KY 2-forms. On the other hand, the equations (32) and (40) (or (41) in the case of Einstein manifolds) imply that \( \tilde{K} \) and \( \tilde{Y} \) (and hence \( \ast \tilde{K} \) and \( \ast \tilde{Y} \)) are eigenforms of the Laplace-Beltrami operator:
\[ \delta^2 \tilde{K} = -2k \tilde{K}, \quad \delta^2 \tilde{Y} = k \frac{n}{n - 1} \tilde{Y}. \]  

Note that while in the first case the eigenvalues have the opposite sign with \( k \), in the second case they have the same signs. The lower bounds for their degeneracy can be determined from (46) as we did in the previous subsection.

V. SUMMARY AND CONCLUDING REMARKS

In this study we have shown that two linearly independent uncomposite currents for each KY-form can be constructed using the curvature forms of the underlying space-time. The current suggested in the literature (see (1)) is a special linear combination of the currents claimed here. Moreover, while the current (1) identically vanishes for all KY \((n - 1)\)-forms, in our case the currents \( J_1 \) and \( J_2 \) are non-vanishing but become linearly dependent. What is more, the current \( J_1 \) is shown to be co-exact from which several geometric facts follow. Some of these can be summarized as follows.

(i) The scalar curvature \( R \) of any (pseudo)Riemannian manifold is a flow invariant for all of its Killing vector fields; \( \nabla_K R = 0 \).

(ii) The conserved currents directly provide decompositions for index-saturated differential forms constructed by contracting the components of KY \( p \)-forms with the Ricci 1-forms, curvature 2-forms and with the Einstein \((n - 1)\)-forms (for the values \( p = 1, 2 \) and \( p = n - 1 \)). These contractions are solely expressible in terms of KY-forms themselves and their (co)derivatives.

(iii) On an Einstein manifold, duals of all Killing and Yano vector fields are eigenforms of the Laplace-Beltrami operator such that the eigenvalues have multiplicities with well defined lower bounds.

(iv) Generalizations of some well-known facts related to the spectrum of Laplace-Beltrami operator on a constant curvature Riemannian manifold also follow directly from the properties of conserved currents. More precisely, we have shown that in a constant curvature (pseudo)Riemannian manifold all KY-forms and their Hodge duals are eigenforms of the Laplace-Beltrami operator.

The whole attention in this study has inevitably been focused on the basic gravitational currents and their properties. The study of related conserved charges in the framework of this paper remains almost untouched, but deserves to be the subject of a separate study. The basic gravitational currents presented in this paper may shed light on some of the problems related to interpretation of generalized charges.
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**APPENDIX A: CONTRACTED BIANCHI IDENTITIES**

In (pseudo)Riemannian geometry, the identities

\[ R^a_b \wedge e^b = 0 , \]  
\[ DR^{ab} \equiv dR^{ab} + \omega^a_c \wedge R^{cb} + \omega^b_c \wedge R^{ac} = 0 , \]

are known as the first and the second Bianchi identities, respectively. Here \( D \) denotes the covariant exterior derivative.

The following useful identities can easily be verified by taking successive contractions of (A1) [8]:

\[ i_{X_a} i_{X_b} R^{cd} = i_{X_c} i_{X_d} R^{ab} , \quad i_{X_a} P_b = i_{X_b} P_a , \]
\[ i_{X_a} R^{bc} + c.p. = 0 , \quad P_b \wedge e^b = 0 . \]

Here c.p. stands for the cyclic permutations. There are also additional contracted Bianchi identities that can be obtained from (A2). But, these are scattered in the literature and since they are heavily used in our analysis, it will be convenient to give some details of their derivations.

It turns out that the Bianchi identities implied by (A2) can most concisely be expressed in terms of 2-form \( T^{ab}_c \) and its contraction defined by

\[ T^{ab}_c = \nabla_{X_c} R^{ab} + \omega^a_d(X_c) R^{db} + \omega^b_d(X_c) R^{ad} , \]
\[ S^b_c = i_{X_a} T^{ab}_c . \]

By definition, \( T^{ab}_c \) is anti-symmetric in the last two indices

\[ T^{ab}_c = -T^{ba}_c . \]

Since

\[ i_{X_a} T^{ab}_c = \nabla_{X_c} (i_{X_a} R^{ab}) + \omega^b_d(X_c) P^d + \omega^a_d(X_c) i_{X_a} R^{db} - i_{\nabla_{X_c} X_a} R^{ab} , \]

and as the last two terms cancel, we obtain

\[ S^b_c = \nabla_{X_c} P^b + \omega^b_d(X_c) P^d \]
By contracting this relation with \( X_b \) we find
\[
i_X S^{cb} = \nabla_X R. \tag{A9}\]

As an aside, we should note that (A2) can be rewritten as
\[
e^c \wedge T_c^{ab} = DR^{ab} = 0 \quad \text{and} \quad (A8) \implies e^c \wedge S_c^a = DP^a.
\]

We are now ready to take the interior derivative of (A2) and write
\[
T^{cab} = d i_X R^{ab} + \omega^a_d \wedge i_X R^{db} + \omega^b_d \wedge i_X R^{ad} + \omega^{cd} \wedge i_X R^{ab}, \tag{A10}
\]
where we have used (A3). This can be rewritten in a more compact form as
\[
T^{cab} = D i_X \cdot R^{ab}. \tag{A11}
\]

In view of the first relation of (A4) and (A11) easily follows that
\[
T^{cab} + T^{abc} + T^{bca} = 0. \tag{A12}
\]

Two successive contractions of this relation yield
\[
i_X T^{abc} - S^{bc} + S^{cb} = 0, \tag{A13}
i_X S^{cb} - i_X S^{cb} + i_X i_X T^{abc} = 0. \tag{A14}
\]

Noting that the third term of (A14) is equal to the first term, by virtue of (A9) we obtain
\[
i_X S^{cb} = \frac{1}{2} \nabla_X R. \tag{A15}
\]

From (A9) and (A15) we observe an interesting property of \( S^{cb} \): its contraction with respect to second index is twice the contraction with respect to first index.

As a result; (A10) (or, equivalently (A11)), (A12), (A13) and (A15) are additional contracted Bianchi identities resulting from contraction of the second Bianchi identity given by (A2). In particular, the last one will play a prominent role in the next appendix.

**APPENDIX B: AN ALTERNATIVE PROOF OF \( \nabla_K R = 0 \)**

Our alternative proof of (33) proceeds as follows. From (A8) we obtain
\[
i_X S^{cb} = i_X \nabla_X P^b + \omega^b_d (X_c) i_X P^d,
\]
and by using this in (A15), the covariant derivative of \( R \) with respect to an arbitrary Killing vector field \( K \) is found to be
\[
\nabla_K R = 2 K_b i_X S^{cb} = 2 i_X [\nabla_X (K_b P^b) - P^b \nabla_X K_b] + 2 K_b \omega^b_d (X_c) i_X P^d. \tag{B1}
\]
In view of (32) and \( \delta^2 = 0 \), the first term of the second equality vanishes and (B1) reduces to
\[
\nabla_K R = -2 (i_X P^b) i_X K_b \nabla_X \hat{K}. \tag{B2}
\]

In writing this relation we renamed the indices and made use of
\[
i_X \nabla_X \hat{K} = \nabla_X K_b + K_d \omega^d_b (X_c).
\]

By the second identity of (A3) and since the symmetrized covariant derivative of any KY-form is zero we obtain \( \nabla_K R = 0 \) from (B2).