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The Stationary Boltzmann equation for a two component gas in the slab with different molecular masses.

Stéphane Brull *

Abstract

The stationary Boltzmann equation for hard and soft forces in the context of a two component gas is considered in the slab when the molecular masses of the 2 component are different. An $L^1$ existence theorem is proved when one component satisfies a given indata profile and the other component satisfies diffuse reflection at the boundaries. Weak $L^1$ compactness is extracted from the control of the entropy production term.

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1 Introduction and setting of the problem.

This article is devoted to the proof of an existence theorem for the stationary Boltzmann equation in the situation of a two component gas having different molecular masses for the geometry of the slab. The slab being represented by the interval $[-1, 1]$, the Boltzmann equation reads

\[ \xi \frac{\partial}{\partial x} f_A(x, v) = Q_{AA}(f_A, f_A)(x, v) + Q_{AB}(f_A, f_B)(x, v), \]  
\[ \xi \frac{\partial}{\partial x} f_B(x, v) = Q_{BB}(f_B, f_B)(x, v) + Q_{BA}(f_B, f_A)(x, v), \]

\[ x \in [-1, 1], \ v \in \mathbb{R}^3. \]
The non-negative functions represent the distribution functions $f_A$ and $f_B$ of the $A$ and the $B$ component and $\xi$ is the velocity component in the $x$ direction. For for any $\alpha, \beta \in \{A, B\}$, $Q_{\alpha,\beta}$ corresponds to the Boltzmann collision operator between the species $\alpha$ and $\beta$. It is defined by

$$Q_{\alpha,\beta}(v) = \int_{\mathbb{R}^3 \times S^2} B^{\alpha,\beta}(f_{\alpha}(x,v'_s)f_{\beta}(x,v') - f_{\beta}(x,v_s)f_{\alpha}(x,v)) \, d\omega dv_s \quad (1.3)$$

with

$$v^{(\beta\alpha)} = v + \frac{2m^\beta}{m^\alpha + m^\beta}(v_s - v, \omega)\omega, \quad v_s^{(\beta\alpha)} = v_s - \frac{2m^\beta}{m^\alpha + m^\beta}(v_s - v, \omega)\omega. \quad (1.4)$$

In the formula (1.4), $v^{(\beta\alpha)}$ and $v_s^{(\beta\alpha)}$ represent the post-collisional velocities between the species $\alpha$ and $\beta$ and $m^\alpha$ represents the mass of the species $\alpha$. For more precisons on the model we refer to ([14], [2]).

The function $B^{\alpha,\beta}(v - v_s, \omega)$ is the kernel of the collision operator $Q_{\alpha,\beta}$. It is a nonegative function whose form is determined by the molecular interaction between the species $\alpha$ and $\beta$. Because of the action and reaction principle, it has the symmetry property $B^{A,B} = B^{B,A}$. More precisely, we consider in this paper the following type of kernels

$$\frac{1}{4\sqrt{2\pi}} \left( \frac{d^\alpha + d^\beta}{2} \right)^2 |v - v_s|^\beta b(\theta),$$
with
\[ 0 \leq \beta < 2, \quad b \in L^1((0, 2\pi)), \quad b(\theta) \geq c > 0 \quad \text{a.e.} \]
for hard forces and
\[ -3 \leq \beta < 0, \quad b \in L^1((0, 2\pi)), \quad b(\theta) \geq c > 0 \quad \text{a.e.} \]
for soft forces.

As (2) define the collision frequency as the vector \((\nu_A, \nu_B)\), with
\[
\nu_\alpha = \sum_{\beta \in \{A, B\}} \int B^{\alpha, \beta} f_\beta d\omega d\nu_\alpha, \quad \alpha \in \{A, B\}.
\]

On the boundary of the domain, the two components satisfy different physical properties. Indeed, the \(A\) component is supposed to be a condensable gas whereas the \(B\) component is supposed to be a non condensable gas.

Hence the boundary condition for the \(A\) component is the given indata profile
\[
f_A(-1, v) = kM_-(v), \xi > 0, \quad f_A(1, v) = kM_+(v), \xi < 0, \quad (1.5)
\]
for some positive \(k\). The boundary condition for the \(B\) component is of diffuse reflection type
\[
f_B(-1, v) = \left( \int_{\xi' < 0} |\xi'| f_B(-1, v') dv' \right) M_-(v), \quad \xi > 0, \quad (1.6)
\]
\[
f_B(1, v) = \left( \int_{\xi' > 0} \xi' f_B(1, v') dv' \right) M_+(v), \quad \xi < 0.
\]

\(M_+\) and \(M_-\) are given normalized Maxwellsians
\[
M_-(v) = \frac{1}{2\pi T_-} e^{-\frac{\|v\|^2}{2T_-}} \quad \text{and} \quad M_+(v) = \frac{1}{2\pi T_+} e^{-\frac{\|v\|^2}{2T_+}}.
\]

As a theoretical point of view, existence theorem for single component gases has been firstly considered. These papers are of interest because the case of the stationary Boltzmann equation is not covered by the DiPerna Lions theory established for the time dependant non linear Boltzmann equation (\([15]\), \([13]\)). In (\([6]\)), an \(L^1\) existence theorem is shown for hard and soft forces when the distribution function has a given indata profile. In the case of boundary conditions of Maxwell diffuse reflection type, an analogous theorem is also shown in (\([3]\)). In these two papers the solution are constructed...
in such a way that they have a given weighted mass. Let us mention the
case of the stationary Povzner equation in the case of hard and soft forces
which is investigated in (\cite{20}). The situation of a two component gas has
been considered in (\cite{10}, \cite{11}) when the molecular masses of the two gases
are the same but with different boundary conditions. In these papers, the
strategy of the resolution is to use that the sum of the distribution of the
two components satisfies the Boltzmann equation for a one component gas.
Hence the weak $L^1$ compactness is firstly obtained for the sum and trans-
mitted to the two distribution functions. But in the present, case due to the
different molecular masses, the sum of the distribution functions is not the
solution of the solution of the Boltzmann equation for a single component
gas. Therefore the weak $L^1$ compactness has to be extracted directly on
each component. In (\cite{12}) the situation of a binary mixture close to a local
equilibrium is investigated. In that case the solution of the system is con-
structed as a Hilbert expansion and the rest term is rigorously controled. In
\cite{16} a moment method is applied in the situation of small Knudsen number
to derive a fluid system.

As a physical point of view and as a numerical point of view, a problem
of evaporation condensation for a binary mixture far from equilibrium has
been considered in \cite{21}. The binary mixture composed of vapor and non
condensable gas in contact with an infinite plane of condensed vapor. More-
over the non condensable gas is supposed to be closed to the condensed
vapor. For the numerical simulations the authors used a time-dependant
BGK model for a two component gas until a stationary state is reached.
The situation of a small Knudsen number, has also been investigated in (\cite{1},
\cite{4}, \cite{3}, \cite{25}) and two types of behaviour are pointed out. In a first situation
the macroscopic velocity of the two gases tends to zero when the Knudsen
number tends to zero. But the zero order term of the temperature in its
Hilbert expansion is calculated from the first order term of the macroscopic
velocity. This means that the macroscopic velocity disappears at the limit
but keeps an influence on the limit. This is the ghost effect pointed in \cite{24}
for a one component gas and in (\cite{1}, \cite{4}, \cite{3}) for a two component gas. In
a second case the B component becomes negligeable and the macroscopic
velocity of the A component becomes consistent. Moreover the B component
accumulates in a thin layer called Knudsen layer at the boundary where the
A component blows. In the situation of vapor-vapor mixture ghost-effects
have also been shown in (\cite{23}).

In this paper, weak solutions $(f_A, f_B)$ to the stationary problem in the
sense of Definition \cite{11} will be considered.
Definition 1.1. Let $M_A$ and $M_B$ be given nonnegative real numbers. $(f_A, f_B)$ is a weak solution to the stationary Boltzmann problem with the $\beta$-norms $M_A$ and $M_B$, if $f_A$ and $f_B \in L^1_{\text{loc}}((-1, 1) \times \mathbb{R}^3)$, $\nu_A, \nu_B \in L^1_{\text{loc}}((-1, 1) \times \mathbb{R}^3)$, 
\[
\int (1 + |v|)^\beta f_A(x, v) dv dx = M_A, \quad \int (1 + |v|)^\beta f_B(x, v) dv dx = M_B,
\]
and there is a constant $k > 0$ such that for every test function $\varphi \in C^1_{\text{c}}([-1, 1] \times \mathbb{R}^3)$ such that $\varphi$ vanishes in a neighborhood of $\xi = 0$, and on $\{(1, v) ; \xi > 0\} \cup \{(-1, v) ; \xi < 0\}$,
\[
\int_{-1}^1 \int_{\mathbb{R}^3} (\xi f_A \frac{\partial \varphi}{\partial x} + Q_{AA}(f_A, f_A) + Q_{AB}(f_A, f_B)) (x, v) dv dx
\]
\[
= k \int_{\mathbb{R}^3, \xi < 0} \xi M_+(v) \varphi(1, v) dv - k \int_{\mathbb{R}^3, \xi > 0} \xi M_-(v) \varphi(-1, v) dv,
\]
\[
\int_{-1}^1 \int_{\mathbb{R}^3} (\xi f_B \frac{\partial \varphi}{\partial x} + Q_{BB}(f_B, f_B) + Q_{BA}(f_B, f_A)) (x, v) dv dx,
\]
\[
= \int_{\xi' < 0} |\xi| M_+(v) \varphi(1, v) dv \left( \int_{\xi' > 0} \xi' f_B(1, v') dv' \right)
\]
\[
- \int_{\xi' > 0} \xi M_-(v) \varphi(-1, v) dv \left( \int_{\xi' < 0} \xi' f_B(-1, v') dv' \right).
\]
Renormalized solutions will also been considered. We recall their definition. Let $g$ be defined for $x > 0$ by 
\[
g(x) = \ln(1 + x).
\]

Definition 1.2. Let $M_A$ and $M_B$ be given nonnegative real numbers. $(f_A, f_B)$ is a renormalized solution to the stationary Boltzmann problem with the $\beta$-norms $M_A$ and $M_B$, if $f_A$ and $f_B \in L^1_{\text{loc}}((-1, 1) \times \mathbb{R}^3)$, $\nu_A, \nu_B \in L^1_{\text{loc}}((-1, 1) \times \mathbb{R}^3)$, 
\[
\int (1 + |v|)^\beta f_A(x, v) dv dx = M_A, \quad \int (1 + |v|)^\beta f_B(x, v) dv dx = M_B,
\]
and there is a constant $k > 0$ such that for every test function $\varphi \in C^1_{\text{c}}([-1, 1] \times \mathbb{R}^3)$ such that $\varphi$ vanishes in a neighborhood of $\xi = 0$ and on $\{(1, v) ; \xi > 0\} \cup \{(-1, v) ; \xi < 0\}$,
\{(-1,v); \xi < 0\} \cup \{(1,v); \xi > 0\},

\[
\int_{-1}^{1} \int_{\mathbb{R}^3} \left( \xi g(f_A) \frac{\partial \varphi}{\partial x} + \frac{Q_{AA}(f_A, f_A)}{1 + f_A} \varphi + \frac{Q_{AB}(f_A, f_B)}{1 + f_A} \varphi \right)(x, v) dx dv = \int_{\mathbb{R}^3, \xi < 0} \xi g(kM_+(v)) \varphi(1,v) dv - \int_{\mathbb{R}^3, \xi > 0} g(kM_-(v)) \varphi(-1,v) dv,
\]

\[
\int_{-1}^{1} \int_{\mathbb{R}^3} \left( \xi g(f_B) \frac{\partial \varphi}{\partial x} + \frac{Q_{BB}(f_B, f_A + f_B)}{1 + f_B} \varphi + \frac{Q_{BA}(f_B, f_A)}{1 + f_B} \varphi \right)(x, v) dx dv = \int_{\xi < 0} \xi g\left( \int_{\xi' > 0} \xi' f_B(1, v') M_+(v) \right) \varphi(1,v) dv - \int_{\xi > 0} \xi g\left( \int_{\xi' < 0} \xi' f_B(-1, v') M_-(v) \right) \varphi(-1,v) dv.
\]

The main results of this paper are the following theorems

**Theorem 1.1.** Given \( 0 \leq \beta < 2 \), \( M_A > 0 \) and \( M_B > 0 \) there is a weak solution to the stationary problem with \( \beta \)-norms equal to \( M_A \) and \( M_B \).

**Theorem 1.2.** Given \( -3 < \beta < 0 \), \( M_A > 0 \) and \( M_B > 0 \), there is a renormalized solution to the stationary problem with \( \beta \)-norms equal to \( M_A \) and \( M_B \).

The present paper is organized as follows. The second and the third section are devoted to the proof of the theorems 1.1 and 1.2. In section 2, we perform a fix point step on an approched problem as in ([6], [7], [10], [11]). In the last part we perform a passage to the limit in the sequences of approximation.

## 2 Approximations with fixed total masses

Let \( r > 0, m \in \mathbb{N}^*, \mu > 0, \delta > 0, j \in \mathbb{N}^* \).

By arguing as in ([8]), we can construct a function, \( \chi^{r,m} \in C_0^\infty \) with range \([0,1]\) invariant under the collision transformations \( J_{\alpha,\beta} \), for any \( \alpha, \beta \in \{A, B\} \) where

\[
J_{\alpha,\beta}(v, v_*, \omega) = (v^{(\alpha,\beta)}, v_*(^{\alpha,\beta}), -\omega),
\]

and under the exchange of \( v \) and \( v_* \) and such that

\[
\chi^{r,m}(v, v_*, \omega) = 1, \quad \forall (\alpha, \beta) \in \{A, B\} \min(|\xi|, |\xi_*|, |\xi^{(\alpha,\beta)}|, |\xi_*^{(\alpha,\beta)}| \geq r),
\]

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and

$$\chi^{r,m}(v, v_*, \omega) = 0, \ \forall (\alpha, \beta) \in \{A, B\} \ \text{max}(|\xi|, |\xi_*|, \xi^{\alpha,\beta,\prime}, |\xi_*^{\alpha,\beta,\prime}|) \leq r - \frac{1}{m}.$$  

The modified collision kernel $B^{a,\beta}_{m,n,\mu}$ is a positive $C^\infty$ function approximating $\min(B^{a,\beta}, \mu)$, when

$$v^2 + v_*^2 < \frac{\sqrt{n}}{2}, \ \text{and} \ \frac{|v - v_*|}{|v - v_*|} \cdot \omega > \frac{1}{m} \ \text{and} \ \frac{|v - v_*|}{|v - v_*|} \cdot \omega < 1 - \frac{1}{m}$$

and such that $B^{a,\beta}_{m,n,\mu}(v, v_*, \omega) = 0$, if

$$v^2 + v_*^2 > \sqrt{n} \ \text{or} \ \frac{|v - v_*|}{|v - v_*|} \cdot \omega < \frac{1}{2m}, \ \text{or} \ \frac{|v - v_*|}{|v - v_*|} \cdot \omega > 1 - \frac{1}{2m}.$$

The functions $\varphi_l$ are mollifiers in the $x$-variable defined by $\varphi_l(x) := l \varphi(lx)$, where

$$\varphi \in C^\infty_0(\mathbb{R}^3_v), \ \text{support}(\varphi) \subset (-1, 1), \ \varphi \geq 0, \ \int_{-1}^{1} \varphi(x)dx = 1.$$  

For the sake of clarity Theorems 1.1 and 1.2 will be shown for $M_A = M_B = 1$. The passage to general weighted masses is immediate and we refer to [6], [7], [10], [11].

Non negative functions $g_A, g_B \in K$ and $\theta \in [0, 1]$ being given. By arguing as in [13], we can construct $F_A$ and $F_B$ solutions of the following boundary value problem

$$\delta F_A + \xi \frac{\partial}{\partial x} F_A = \int_{\mathbb{R}^3_v \times S^2} \chi^{r,m} B_{m,n,\mu}^{AA} \frac{F_A}{1 + gA} (x, v') \frac{g_A \ast \varphi}{1 + \frac{gA}{2}}(x, v_*) dv_* d\omega$$
$$+ \int_{\mathbb{R}^3_v \times S^2} \chi^{r,m} B_{m,n,\mu}^{AB} (x, v') \frac{F_A}{1 + gA} \frac{gB \ast \varphi}{1 + \frac{gB}{2}}(x, v_*) dv_* d\omega$$
$$- F_A \int_{\mathbb{R}^3_v \times S^2} \chi^{r,m} B_{m,n,\mu}^{AA} \frac{gA \ast \varphi}{1 + \frac{gA}{2}}(x, v_*) dv_* d\omega$$
$$- F_A \int_{\mathbb{R}^3_v \times S^2} \chi^{r,m} B_{m,n,\mu}^{AB} (x, v_*) \frac{gB \ast \varphi}{1 + \frac{gB}{2}}(x, v_*) dv_* d\omega, \ \ (x, v) \in (-1, 1) \times \mathbb{R}^3_v,$$

$$F_A(-1, v) = \lambda M_-(v), \ \xi > 0, \ F_A(1, v) = \lambda M_+(v), \ \xi < 0,$$

and

$$7$$
\[
\delta F_B + \xi \frac{\partial}{\partial x} F_B = \int_{R^3} \chi^{r,m} B^{BB}_{m,n,\mu} \frac{F_B}{1 + \frac{F_B}{j}} (x, v^\prime) \frac{g_B * \varphi}{1 + \frac{g_B * \varphi}{j}} (x, v^\prime) dv^\prime d\omega \\
+ \int_{R^3} \chi^{r,m} B^{AB}_{m,n,\mu} \frac{F_B}{1 + \frac{F_B}{j}} (x, v^\prime) \frac{g_B * \varphi}{1 + \frac{g_B * \varphi}{j}} (x, v^\prime) dv^\prime d\omega \\
- F_B \int_{R^3} \chi^{r,m} B^{BB}_{m,n,\mu} \frac{g_B * \varphi}{1 + \frac{g_B * \varphi}{j}} (x, v^\prime) dv^\prime d\omega, \quad (x, v) \in (-1, 1) \times R^3
\]

\[F_B(-1, v) = \theta \lambda M_-(v), \quad \xi > 0, \quad F_B(1, v) = (1 - \theta) \lambda M_+(v), \quad \xi < 0, \quad (2.2)\]
as the \(L^1\) limit of sequences. It can also been proven that the equations (2.1) and (2.2) each has a unique solution which is strictly positive. Let

\[f_A = \frac{F_A}{\int_{\min} \min(\mu, (1 + |v|)^{\beta}) F_A(x, v) dv},\]

\[f_B = \frac{F_B}{\int_{\min} \min(\mu, (1 + |v|)^{\beta}) F_B(x, v) dv},\]

Hence it follows that the functions \(f_A\) and \(f_B\) are well defined since \(F_A\) and \(F_B\) strictly positive.

Indeed using that \(\int_{-1}^{1} (\alpha + \nu(x, v)) dx \leq 2 + 2\mu\), it holds that

\[F_A(x, v) \geq \lambda M_-(v) e^{-\frac{2 + 2\mu}{\xi}}, \quad \xi > 0, \quad F_A(x, v) \geq \lambda M_+(v) e^{-\frac{2 + 2\mu}{|\xi|}}, \quad \xi < 0.\]

Analogously, we obtain

\[F_B(x, v) \geq (1 - \theta) \lambda M_+(v) e^{-\frac{2 + 2\mu}{|\xi|}}, \quad \xi < 0.\]

By taking \(\lambda\) as

\[\lambda = \min\left(\frac{1}{\int_{\xi > 0} \min(\mu, (1 + |v|)^{\beta}) e^{-\frac{2 + 2\mu}{\xi}} dv}, \frac{1}{\int_{\xi < 0} \min(\mu, (1 + |v|)^{\beta}) e^{-\frac{2 + 2\mu}{|\xi|}} dv}\right),\]

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we get
\[
\int \min(\mu, (1 + |v|)^\beta) F_A(x,v) dx dv \geq 1
\]
and
\[
\int \min(\mu, (1 + |v|)^\beta) F_B(x,v) dx dv \geq 1.
\]
Hence the functions \(f_A\) and \(f_B\) are solutions to
\[
\delta f_A + \xi \frac{\partial}{\partial x} f_A = \int_{\mathbb{R}^3} \chi^{r,m} B^{AA}_{m,n,\mu} \frac{f_A}{1 + \frac{v}{\lambda}^2} (x,v') \frac{g_A * \varphi}{1 + \frac{g_A * \varphi}{\lambda}} (x,v') dv' d\omega
\]
\[
+ \int_{\mathbb{R}^3} \chi^{r,m} B^{AB}_{m,n,\mu} \frac{f_A}{1 + \frac{v}{\lambda}^2} (x,v') \frac{g_B * \varphi}{1 + \frac{g_B * \varphi}{\lambda}} (x,v') dv' d\omega
\]
\[
- f_A \int_{\mathbb{R}^3} \chi^{r,m} B^{AA}_{m,n,\mu} \frac{g_A * \varphi}{1 + \frac{g_A * \varphi}{\lambda}} (x,v) dv d\omega,
\]
\[
(x,v) \in (-1,1) \times \mathbb{R}^3,
\]
\[
f_A(-1,v) = \frac{\lambda}{\int \min(\mu, (1 + |v|)^\beta) F_A(x,v) dx dv} M_-(v), \quad \xi > 0,
\]
\[
f_A(1,v) = \frac{\lambda}{\int \min(\mu, (1 + |v|)^\beta) F_A(x,v) dx dv} M_+(v), \quad \xi < 0,
\]
and
\[
(2.3)
\]
\[
\delta f_B + \xi \frac{\partial}{\partial x} f_B = \int_{\mathbb{R}^3}^{} \chi^r \mathcal{B}_{m,n,\mu}^{BB}(v,v_\ast,\omega) \frac{f_B}{1 + \frac{v}{F_B}}(x,v') \frac{g_B * \varphi}{1 + \frac{g_B * \varphi}{j}}(x,v') dv_\ast d\omega \\
+ \int_{\mathbb{R}^3}^{} \chi^r \mathcal{B}_{m,n,\mu}^{BA}(v,v_\ast,\omega) \frac{f_B}{1 + \frac{v}{F_B}}(x,v') \frac{g_A * \varphi}{1 + \frac{g_A * \varphi}{j}}(x,v') dv_\ast d\omega \\
-f_B(x,v) \int_{\mathbb{R}^3}^{} \chi^r \mathcal{B}_{m,n,\mu}^{BB} \frac{g_B * \varphi}{1 + \frac{g_B * \varphi}{j}}(x,v') dv_\ast d\omega \\
-f_B(x,v) \int_{\mathbb{R}^3}^{} \chi^r \mathcal{B}_{m,n,\mu}^{BA} \frac{g_A * \varphi}{1 + \frac{g_A * \varphi}{j}}(x,v') dv_\ast d\omega
\]

In order to use a fixed-point theorem, consider the closed and convex subset of \(L^1([[-1,1] \times \mathbb{R}^3])\),

\[
K = \{ f \in L^1([[-1,1] \times \mathbb{R}^3]), \int_{[-1,1] \times \mathbb{R}^3}^{} \min(\mu, (1 + |v|^3)^\beta) F_B(x,v) dx dv = 1 \}.
\]

The fixed-point argument will now be used in order to solve (2.3, 2.4) with \(g_A = f_A\) and \(g_B = f_B\).

Define \(T\) on \(K \times K \times [0,1]\) by \(T(g_A,g_B,\theta) = (f_A,f_B,\tilde{\theta})\) with

\[
\tilde{\theta} = \frac{\int_{\xi < 0}^{} |\xi| f_B(-1,v) dv}{\int_{\xi < 0}^{} |\xi| f_B(-1,v) dv + \int_{\xi > 0}^{} |\xi| f_B(1,v) dv}
\]

(2.5)

where \((f_A,f_B)\) is solution to (2.3, 2.4).

By reasoning as in [10], it can be shown that the map \(T\) is continous from \(K \times K \times [0,1]\) into itself. So from the Schauder fixed point theorem there is \((f_A,f_B,\theta)\) such that

\[
f_A = g_A, \quad f_B = g_B, \quad \theta = \frac{\int_{\xi < 0}^{} |\xi| f_B(-1,v) dv}{\int_{\xi < 0}^{} |\xi| f_B(-1,v) dv + \int_{\xi > 0}^{} |\xi| f_B(1,v) dv}
\]
that satisfy

\[
\delta f_A + \xi \frac{\partial}{\partial x} f_A = \int_{R^3 \times S^2} \chi r, m B_{m, n, \mu} \frac{f_A}{1 + \frac{\delta f_A}{\mu}} (x, v') f_A * \varphi_l (x, v') dv_3 d\omega
\]

\[
+ \int_{R^3 \times S^2} \chi r, m B_{m, n, \mu} \frac{f_A}{1 + \frac{\delta f_A}{\mu}} (x, v') f_B * \varphi_l (x, v') dv_3 d\omega
\]

\[-f_A \int_{R^3 \times S^2} \chi r, m B_{m, n, \mu} \frac{f_A * \varphi_l (x, v)}{1 + \frac{\delta f_A}{\mu}} dv_3 d\omega
\]

\[-f_B \int_{R^3 \times S^2} \chi r, m B_{m, n, \mu} \frac{f_B * \varphi_l (x, v)}{1 + \frac{\delta f_B}{\mu}} dv_3 d\omega
\]

\[
\left(2.6\right)
\]

with

\[
k_A = \frac{\lambda}{\int \min(\mu, 1 + |v|^2) F_A(x, v) dx dv}
\]

and

\[
\delta f_B + \xi \frac{\partial}{\partial x} f_B = \int_{R^3 \times S^2} \chi r, m B_{m, n, \mu} \frac{f_B}{1 + \frac{\delta f_B}{\mu}} (x, v') f_B * \varphi_l (x, v') dv_3 d\omega
\]

\[
+ \int_{R^3 \times S^2} \chi r, m B_{m, n, \mu} \frac{f_B}{1 + \frac{\delta f_B}{\mu}} (x, v') f_A * \varphi_l (x, v') dv_3 d\omega
\]

\[-f_B \int_{R^3 \times S^2} \chi r, m B_{m, n, \mu} \frac{f_B * \varphi_l (x, v)}{1 + \frac{\delta f_B}{\mu}} dv_3 d\omega
\]

\[-f_A \int_{R^3 \times S^2} \chi r, m B_{m, n, \mu} \frac{f_A * \varphi_l (x, v)}{1 + \frac{\delta f_A}{\mu}} dv_3 d\omega
\]

\[
\left(2.7\right)
\]

with

\[
\chi' = \frac{\lambda}{\int \min(\mu, 1 + |v|^2) F_B(x, v) dx dv}
\]
3 The slab solution for $-3 < \beta \leq 0$ and $0 \leq \beta < 2$.

This section is devoted to the passage to the limit in (2.6, 2.7). It is performed in two times. In the first one the solutions of the approached problem are written in their exponential form and averaging lemmas are applied. The second passage to the limit corresponds to the passage to the limit in (3.8, 3.9). One crucial point is to get an entropy estimate on $(f^j_A, f^j_B)$ in order to extract compactness. In ([10]), this control is obtained from a bound on the entropy of $f^j_A = f^j_A + f^j_B$ by using that $f^j$ satisfy the Boltzmann equation for a single component gas. But in the present paper, due to the difference of the molecular masses, this property is not satisfied.

Keeping, $l, j, r, m, \mu$ fixed, denote $f^j, \delta, l, r, m, \mu$ by $f^\delta$ each distribution function and study the passage to the limit when $\delta$ tends to 0. Writing the equations (2.6, 2.7) in the exponential form and using the averaging lemmas together with a convolution with a mollifier ([7], [18]) give that $f^\delta_A$ and $F^\delta_A$ are strongly compact in $L^1([-1,1] \times \mathbb{R}^3_v)$. Denote by $f_A$ and $F_A$ the respective limits of $f^\delta_A$ and $F^\delta_A$. The passage to the limit when $\delta$ tends to 0 in the equation (2.6) yields

$$
\frac{\xi}{\partial x} \frac{\partial}{\partial x} f_A = \int_{\mathbb{R}^3_v \times S^2} \chi^{r,m} B^{AA}_{m,n,\mu} \frac{f_A}{1 + \frac{F_A}{\beta}}(x, v') \frac{f_A * \varphi_l}{1 + \frac{F_A * \varphi_l}{\beta}}(x, v') dv'_s d\omega 
$$

$$
+ \int_{\mathbb{R}^3_v \times S^2} \chi^{r,m} B^{AB}_{m,n,\mu} \frac{f_A}{1 + \frac{F_A}{\beta}}(x, v') \frac{f_B * \varphi_l}{1 + \frac{F_A * \varphi_l}{\beta}}(x, v') dv'_s d\omega 
$$

$$
- f_A \int_{\mathbb{R}^3_v \times S^2} \chi^{r,m} B^{AA}_{m,n,\mu} \left( \frac{f_A * \varphi_l}{1 + \frac{F_A * \varphi_l}{\beta}}(x, v_s) d\omega, \right.
$$

$$
- f_A \int_{\mathbb{R}^3_v \times S^2} \chi^{r,m} B^{AB}_{m,n,\mu} \left( \frac{f_B * \varphi_l}{1 + \frac{F_A * \varphi_l}{\beta}}(x, v_s) d\omega, \right.
$$

with

$$
\int \min(\mu, (1 + |v|)^\beta) f_A(x, v) dxdv = 1.
$$

(3.8)
For the same reasons, the limit $\lim_B f_B$ of $f_B^δ$ satisfies

$$
\frac{\xi}{\partial x} f_B = \int_{\mathbb{R}^3 \times S^2} \chi_{r,m}^B B_{m,n,\mu} \frac{f_B}{1 + \frac{\rho_B}{j}}(x, v') \frac{f_B \ast \varphi_j(x, v_\ast)}{1 + \frac{\int_{\Omega_{\ast,j}}}{j}} \, dv_\ast \, d\omega \\
+ \int_{\mathbb{R}^3 \times S^2} \chi_{r,m}^B B_{m,n,\mu} \frac{f_B}{1 + \frac{\rho_B}{j}}(x, v') \frac{f_A \ast \varphi_j(x, v_\ast)}{1 + \frac{\int_{\Omega_{\ast,j}}}{j}} \, dv_\ast \, d\omega \\
- f_B \int_{\mathbb{R}^3 \times S^2} \chi_{r,m}^B B_{m,n,\mu} \frac{f_B \ast \varphi_j(x, v_\ast)}{1 + \frac{\int_{\Omega_{\ast,j}}}{j}} \, dv_\ast \, d\omega \\
- f_B \int_{\mathbb{R}^3 \times S^2} \chi_{r,m}^B A_{m,n,\mu} \frac{f_A \ast \varphi_j(x, v_\ast)}{1 + \frac{\int_{\Omega_{\ast,j}}}{j}} \, dv_\ast \, d\omega,
$$

$(x, v) \in (-1, 1) \times \mathbb{R}^3_v,$

$$
\begin{align*}
& f_B(-1, v) = \sigma(-1)\lambda' M_-(v), \quad \xi > 0, \quad f_B(1, v) = \sigma(1)\lambda' M_+(v), \quad \xi < 0, \\
& \int_{\mathbb{R}^3 \times S^2} \min(\mu, (1 + |v|)^{\beta}) f_B(x, v) \, dx \, dv = 1,
\end{align*}
$$

where

$$
\begin{align*}
& \sigma(-1) = \frac{\int_{\xi < 0} |\xi| f_B(-1, v) \, dv}{\int_{\xi > 0} f_B(1, v) \, dv + \int_{\xi < 0} |\xi| f_B(-1, v) \, dv}, \\
& \sigma^j(1) = \frac{\int_{\xi > 0} f_B(1, v) \, dv}{\int_{\xi > 0} f_B(1, v) \, dv + \int_{\xi < 0} |\xi| f_B(-1, v) \, dv}
\end{align*}
$$

and

$$
\lambda' = \frac{\lambda}{\int \min(\mu, (1 + |v|)^{\beta}) F_B^j(x, v) \, dx \, dv}.
$$

Multiply (3.8) by $\log \left( \frac{\rho_A}{1 + \frac{\rho_A}{j}} \right)$ and (3.9) by $\log \left( \frac{\rho_B}{1 + \frac{\rho_B}{j}} \right)$ and the two equations
leads to according to (3, 2, 13),

\[
\int_{\mathbb{R}^3} \xi \left( f_A^j \log(f_A^j)(1, v) - j(1 + \frac{f_A^j}{J}) \log(1 + \frac{f_A^j}{J})(1, v) \right)
- \int_{\mathbb{R}^3} \xi \left( f_A^j \log(f_A^j)(-1, v) - j(1 + \frac{f_A^j}{J}) \log(1 + \frac{f_A^j}{J})(-1, v) \right)
+ \int_{\mathbb{R}^3} \xi \left( f_B^j \log(f_B^j)(1, v) - j(1 + \frac{f_B^j}{J}) \log(1 + \frac{f_B^j}{J})(1, v) \right)
- \int_{\mathbb{R}^3} \xi \left( f_B^j \log(f_B^j)(-1, v) - j(1 + \frac{f_B^j}{J}) \log(1 + \frac{f_B^j}{J})(-1, v) \right)
\]

\[
= - \frac{1}{4} I_{AA}^j(f_A^j, f_A^j) - \frac{1}{2} I_{AB}^j(f_A^j, f_B^j) - \frac{1}{4} I_{BB}^j(f_B^j, f_B^j)
+ \int \chi^{r,m} \mathcal{B}_{m,n,\mu}^{AA} \frac{f_A^j(f_A^j - F_A^j)}{j(1 + F_A^j)(1 + f_A^j) 1 + \frac{f_A^j}{J}} \log \frac{f_A^j}{1 + \frac{f_A^j}{J}}
+ \int \chi^{r,m} \mathcal{B}_{m,n,\mu}^{AB} \frac{f_A^j(f_A^j - F_A^j)}{j(1 + F_A^j)(1 + f_A^j) 1 + \frac{f_A^j}{J}} \log \frac{f_A^j}{1 + \frac{f_A^j}{J}}
- \int \chi^{r,m} \mathcal{B}_{m,n,\mu}^{BB} \frac{f_B^j(f_B^j - F_B^j)}{j(1 + F_B^j)(1 + f_B^j) 1 + \frac{f_B^j}{J}} \log \frac{f_B^j}{1 + \frac{f_B^j}{J}}
+ \int \chi^{r,m} \mathcal{B}_{m,n,\mu}^{AB} \frac{f_A^j(f_A^j - F_A^j)}{j(1 + F_A^j)(1 + f_A^j) 1 + \frac{f_A^j}{J}} \log \frac{f_A^j}{1 + \frac{f_A^j}{J}}
- \int \chi^{r,m} \mathcal{B}_{m,n,\mu}^{BB} \frac{f_B^j(f_B^j - F_B^j)}{j(1 + F_B^j)(1 + f_B^j) 1 + \frac{f_B^j}{J}} \log \frac{f_B^j}{1 + \frac{f_B^j}{J}}
- \int \chi^{r,m} \mathcal{B}_{m,n,\mu}^{BA} \frac{f_B^j(f_B^j - F_B^j)}{j(1 + F_B^j)(1 + f_B^j) 1 + \frac{f_B^j}{J}} \log \frac{f_B^j}{1 + \frac{f_B^j}{J}}
- \int \chi^{r,m} \mathcal{B}_{m,n,\mu}^{AB} \frac{f_B^j(f_B^j - F_B^j)}{j(1 + F_B^j)(1 + f_B^j) 1 + \frac{f_B^j}{J}} \log \frac{f_B^j}{1 + \frac{f_B^j}{J}}
\]
with

\[ I_{AA}(f^j_A, f^j_A) = \int \chi^{r,m} B_{m,n,\mu} \left( \frac{f^j_A}{1 + \frac{f^j_A}{j}} - \frac{f^j_A}{1 + \frac{f^j_A}{j}} \right) \log \left( \frac{f^j_A}{1 + \frac{f^j_A}{j}} \right) dx dv d\omega, \]

\[ I_{BB}(f^j_B, f^j_B) = \int \chi^{r,m} B_{m,n,\mu} \left( \frac{f^j_B}{1 + \frac{f^j_B}{j}} - \frac{f^j_B}{1 + \frac{f^j_B}{j}} \right) \log \left( \frac{f^j_B}{1 + \frac{f^j_B}{j}} \right) dx dv d\omega, \]

\[ I_{AB}(f^j_A, f^j_B) = \int \chi^{r,m} B_{m,n,\mu} \left( \frac{f^j_A}{1 + \frac{f^j_A}{j}} - \frac{f^j_B}{1 + \frac{f^j_B}{j}} \right) \log \left( \frac{f^j_A}{1 + \frac{f^j_A}{j}} \right) dx dv d\omega. \]

According to [2], we have \( I_{AA}(f^j_A, f^j_A) \geq 0, I_{AB}(f^j_A, f^j_B) \geq 0, I_{BB}(f^j_B, f^j_B) \geq 0 \) and by reasoning as in [8], we can prove that the terms

\[ \int \chi^{r,m} B_{m,n,\mu} \frac{f^j_\alpha}{j(1 + \frac{f^j_\alpha}{j})} \frac{f^j_\beta}{j(1 + \frac{f^j_\beta}{j})} \log \frac{f_\alpha}{1 + \frac{f_\alpha}{j}} \]

\[ \int \chi^{r,m} B_{m,n,\mu} \frac{f^j_\alpha(f^j_\alpha - F^j_\alpha)}{j(1 + F^j_\alpha)(1 + f^j_\alpha)} \frac{f^j_\beta}{j} \log \frac{f_\alpha}{1 + \frac{f_\alpha}{j}} \]
are controlled uniformly in $j$. Therefore

\[
\int_{\mathbb{R}^3} \xi \left( f_A^j \log(f_A^j)(1,v) - j(1 + \frac{f_A^j}{j}) \log(1 + \frac{f_A^j}{j})(1,v) \right) \\
- \int_{\mathbb{R}^3} \xi \left( f_A^j \log(f_A^j)(-1,v) - j(1 + \frac{f_A^j}{j}) \log(1 + \frac{f_A^j}{j})(-1,v) \right) \\
+ \int_{\mathbb{R}^3} \xi \left( f_B^j \log(f_B^j)(1,v) - j(1 + \frac{f_B^j}{j}) \log(1 + \frac{f_B^j}{j})(1,v) \right) \\
- \int_{\mathbb{R}^3} \xi \left( f_B^j \log(f_B^j)(1,v) - j(1 + \frac{f_B^j}{j}) \log(1 + \frac{f_B^j}{j})(1,v) \right) \leq c
\]

So as in ([6], [7]), it follows that $f_A^j$ and $f_B^j$ are weakly compact in $L^1$.

**Remark 1.** Contrarily to ([10], [11]), the weak compactness of $f_A^j$ and $f_B^j$ is directly obtained. In ([10], [11]), the author shows that the sum $f^j = f_A^j + f_B^j$ is weakly compact in $L^1$ by using that $f^j$ satisfies the Boltzmann equation for a single component gas. In the present paper, the 2 components having different molecular masses, $f^j$ is not solution of the Boltzmann equation for a one component gas.

Let $Q_{\alpha,\beta}^{-j}$ and $Q_{\alpha,\beta}^{+j}$ be defined by

\[
Q_{\alpha,\beta}^{-j}(f_A^j, f_B^j) = f_A^j(x, v) \int_{\mathbb{R}^3 \times S^2} \chi^{r,m} B_{m,n,\mu} \frac{f_A^j}{1 + \frac{f_A^j}{j}}(x, v, \omega) dv \, d\omega,
\]

\[
Q_{\alpha,\beta}^{+j}(f_A^j, f_B^j) = \int_{\mathbb{R}^3 \times S^2} \chi^{r,m} B_{m,n,\mu} \frac{f_A^j}{1 + \frac{f_A^j}{j}}(x, v') \frac{f_B^j}{1 + \frac{f_B^j}{j}}(x, v, \omega) dv \, d\omega.
\]

**Remark 2.** The quantity $\frac{1}{4} I_{AA}^j(f_A^j, f_A^j) + \frac{1}{4} I_{AB}^j(f_A^j, f_B^j) + \frac{1}{4} I_{BB}^j(f_B^j, f_B^j)$ is a generalization of the entropy production term used in ([8]).

In order to pass to the limit in ([8], [9]) weak compactness is required on the terms $Q_{\alpha,\beta}^{-j}$ and $Q_{\alpha,\beta}^{+j}$. The inequalities

\[
Q_{\alpha,\beta}^{-j}(f_A^j, f_B^j) \leq \int |v - v_\alpha|^\beta f_B^j dv_\alpha d\omega, \quad \{\alpha, \beta\} \in \{A, B\},
\]
gives that $Q^{-}_{\alpha, \beta}$ is weakly compact in $L^1$. By arguing as in [11], we can show that

$$Q^{j+}_{A, A}(f^j_A, f^j_A) + Q^{j+}_{A, B}(f^j_A, f^j_B) + Q^{j+}_{B, A}(f^j_B, f^j_A) + Q^{j+}_{B, B}(f^j_B, f^j_B)$$

$$\leq K \left( Q^{-}_{A, A}(f^j_A, f^j_A) + Q^{-}_{A, B}(f^j_A, f^j_B) + Q^{-}_{B, A}(f^j_B, f^j_A) + Q^{-}_{B, B}(f^j_B, f^j_B) \right) + \frac{1}{\ln K} \left( I_{AA}(f^j_A, f^j_A) + I_{BB}(f^j_B, f^j_B) + I_{BA}(f^j_B, f^j_A) \right).$$

(3.10)

From the weak compactness of $Q^{-}_{\alpha, \beta}$ for $\{\alpha, \beta\} \in \{A, B\}$ and the boundeness from above of

$$I_{AA}(f^j_A, f^j_A) + I_{BB}(f^j_B, f^j_B) + I_{BA}(f^j_B, f^j_A),$$

the gain terms $Q^{j+}_{\alpha, \beta}$ are weakly compact in $L^1$. Hence by arguing as in [3], [4] we can pass to the limit in the equations (3.8, 3.9) and we get that there is $(f_r^j, \mu^j_A, f_r^j, \mu^j_B)$ solution to

$$\xi \frac{\partial}{\partial x} f^{r, \mu}_A = \int_{\mathbb{R}^3 \times S^2} \chi^r B^A_{\mu}(v - v_s, \omega) f^{r, \mu}_A(x, v') f^{r, \mu}_A(x, v_s) dv_s d\omega$$

$$+ \int_{\mathbb{R}^3 \times S^2} \chi^r B^A_{\mu}(v - v_s, \omega) f^{r, \mu}_B(x, v') f^{r, \mu}_A(x, v_s) dv_s d\omega$$

$$- f^{r, \mu}_A \int_{\mathbb{R}^3 \times S^2} \chi^r B^A_{\mu}(v - v_s, \omega) f^{r, \mu}_A(x, v_s) dv_s d\omega,$$

$$- f^{r, \mu}_A \int_{\mathbb{R}^3 \times S^2} \chi^r B^A_{\mu}(v - v_s, \omega) f^{r, \mu}_B(x, v_s) dv_s d\omega, \quad (x, v) \in (-1, 1) \times \mathbb{R}^3,$$

$$f^{r, \mu}_A(-1, v) = k_AM_-(v), \quad \xi > 0, \quad f^{r, \mu}_A(1, v) = k_AM_+(v), \quad \xi < 0,$$

(3.11)

with

$$\int \min(\mu, (1 + |v|)^3) f^{r, \mu}_A(x, v) dx dv = 1,$$
where \( k_A \) is defined in the equation (2.6) before passing to the limit.

\[
\xi \frac{\partial}{\partial x} f_{r,\mu}^B = \int_{\mathbb{R}_+^3 \times S^2} \mathcal{I}^B_{\mu}(v - v_\ast, \omega) f_{r,\mu}^B(x, v') f_{r,\mu}^B(x, v) dv_\ast d\omega \\
+ \int_{\mathbb{R}_+^3 \times S^2} \mathcal{I}^A_{\mu}(v - v_\ast, \omega) f_{r,\mu}^A(x, v') f_{r,\mu}^B(x, v) dv_\ast d\omega \\
- f_{r,\mu}^B \int_{\mathbb{R}_+^3 \times S^2} \mathcal{I}^{BB}_{\mu}(v - v_\ast, \omega) f_{r,\mu}^B(x, v) dv_\ast d\omega \\
- f_{r,\mu}^B \int_{\mathbb{R}_+^3 \times S^2} \mathcal{I}^{BA}_{\mu}(v - v_\ast, \omega) f_{r,\mu}^A(x, v) dv_\ast d\omega, \\
(x, v) \in (-1, 1) \times \mathbb{R}_v^n,
\]

\[
f_{r,\mu}^B(-1, v) = \sigma(-1) \lambda' M_-(v), \xi > 0, \quad f_{r,\mu}^B(1, v) = \sigma(1) \lambda' M_+(v), \xi < 0,
\]

(3.12)

with

\[
\int \min(\mu, (1 + |v|)^\beta) f_{r,\mu}^B(x, v) dxdv = 1.
\]

Here,

\[
\sigma(-1) = \frac{\int_{\xi < 0} \xi |f_{r,\mu}^B(-1, v)| dv}{\int_{\xi > 0} \xi |f_{r,\mu}^B(1, v)| dv + \int_{\xi < 0} \xi |f_{r,\mu}^B(-1, v)| dv}
\]

and

\[
\sigma(1) = \frac{\int_{\xi > 0} \xi |f_{r,\mu}^B(1, v)| dv}{\int_{\xi > 0} \xi |f_{r,\mu}^B(1, v)| dv + \int_{\xi < 0} \xi |f_{r,\mu}^B(-1, v)| dv}.
\]

By using the mass conservation as in ([10]) we can prove that the boundary conditions of (3.12) writes

\[
f_{r,\mu}^B(-1, v) = M_-(v) \int_{\xi < 0} \xi |f_{r,\mu}^B(-1, v)| dv, \xi > 0,
\]

\[
f_{r,\mu}^B(1, v) = M_+(v) \int_{\xi > 0} \xi |f_{r,\mu}^B(1, v)| dv, \xi < 0.
\]

(3.13)

From here the arguments of ([6], [7], [10]) can be used to pass to the limit in the parameters \((r, \mu)\) and to prove that \((f_A, f_B)\) satisfies (1.1, 1.2) in the weak sense for \(0 \leq \beta < 2\) and in the renormalized sense for \(-3 < \beta \leq 0\).

But for the sake of clarity we explain the passage to the limit in the terms (3.13) i.e we prove the weak convergence in \(L^1(\{v \in \mathbb{R}_v^n, \xi > 0\})\) (
resp $L^1\{\{v \in \mathbb{R}^3, \xi < 0\}\}$ of $f^j_B(1, .)$ (resp. $f^j_B(-1, .)$) to $f_B(1, .)$ (resp. $f_B(-1, .)$). First, it is important to check that the fluxes $\int_{\xi > 0} \xi |f^j_B(1, v)| dv$ and $\int_{\xi < 0} |\xi| f^j_B(-1, v) dv$ are controlled. From (3.12) written in the exponential form, it holds that

\[
f^j_B(x, v) \geq f^j_B(-1, v) e^{- \frac{1}{\xi} \int_{\mathbb{R}^3_{x,v} \times S^2} \chi^r (B^\mu_{BA} f^r A (x+s, v, \xi) + B^\mu_{BB} f^r B (x+s, v, \xi)) dv, d\omega ds},
\]

\[
\xi > \frac{1}{2}, |v| \leq 2,
\]

\[
f^j_B(x, v) \geq f^j_B(1, v) e^{- \frac{1}{\xi} \int_{\mathbb{R}^3_{x,v} \times S^2} \chi^r (B^\mu_{BA} f^r A (x+s, v, \xi) + B^\mu_{BB} f^r B (x+s, v, \xi)) dv, d\omega ds},
\]

\[
\xi < -\frac{1}{2}, |v| \leq 2.
\]

(3.14)

For $v$ satisfying $|v| \leq 2$ with $\xi > \frac{1}{2}$ or $\xi < -\frac{1}{2}$,

\[
\int_{-1}^1 \int_{\mathbb{R}^3_{x,v} \times S^2} \frac{\chi^r}{|\xi|} (B^\mu_{BA} f^r A (z, v) + B^\mu_{BB} f^r B (z, v)) dv, d\omega dz
\]

is uniformly bounded from above. Hence, using the definition of the boundary conditions (1.6) in (3.14),

\[
f^j_B(x, v) \geq c M_-(v) \int_{\xi < 0} |\xi| f^j_B(-1, v) dv, \quad \xi > \frac{1}{2}, |v| \leq 2,
\]

\[
f^j_B(x, v) \geq c M_+(v) \int_{\xi > 0} \xi f^j_B(1, v) dv, \quad \xi < -\frac{1}{2}, |v| \leq 2.
\]

So,

\[
c \int_{\{\xi > \frac{1}{2}, |v| \leq 2\} \cup \{\xi < -\frac{1}{2}, |v| \leq 2\}} f^j_B(x, v) dx dv \geq \int_{\xi > 0} \xi f^j_B(1, v) dv + \int_{\xi < 0} |\xi| f^j_B(-1, v) dv.
\]

For $f^j_B$ being non-negative,

\[
c \int_{-1}^1 \int_{\mathbb{R}^3_{x,v}} \min(\mu, (1 + |v|)^\beta) f^j_B(x, v) dx dv \geq \int_{\xi > 0} \xi f^j_B(1, v) dv + \int_{\xi < 0} |\xi| f^j_B(-1, v) dv.
\]
Since \( \int_{-1}^{1} \int_{\mathbb{R}^2} \min(\mu, (1 + |v|)^3) f_B^j(x, v) dx dv = 1 \), the fluxes \( \int_{\xi>0} \xi f_B^j(1, v) dv \) and \( \int_{\xi<0} \xi f_B^j(-1, v) dv \) are bounded uniformly w.r.t \( j \). Furthermore, the energy fluxes are also controlled. Indeed, from property 1.1, the conservation of the energy for \( (f_A^j, f_B^j) \), gives

\[
\int_{\xi>0} \xi v^2 f_B^j(1, v) dv + \int_{\xi<0} |\xi| v^2 f_B^j(-1, v) dv \\
\leq \int_{\xi>0} \xi v^2 (m_B f_A^j(-1, v) + m_B f_B^j(-1, v)) dv \\
+ \int_{\xi<0} |\xi| v^2 (m_A f_A^j(1, v) + m_B f_B^j(1, v)) dv.
\]

By definition of the boundary conditions (3.11) and (3.12),

\[
\int_{\xi>0} \xi v^2 f_B^j(1, v) dv + \int_{\xi<0} |\xi| v^2 f_B^j(-1, v) dv \\
\leq (k^j + \int_{\xi<0} |\xi| f_B^j(-1, v') dv') \int_{\xi>0} \xi v^2 M_-(v) dv \\
+(k^j + \int_{\xi'>0} \xi' f_B^j(1, v') dv') \int_{\xi<0} |\xi| v^2 M_+(v) dv.
\]  

The right-hand side of (3.13) being bounded, the energy fluxes are also bounded. Finally, the entropy fluxes can also be controlled. Indeed

\[
\xi \frac{\partial}{\partial x} (f_A^j(\log(f_A^j) - 1)) = Q_A^j_A(f_A^j, f_A^j) \log(f_A^j) + Q_A^j_B(f_A^j, f_B^j) \log(f_A^j), \\
\xi \frac{\partial}{\partial x} (f_B^j(\log(f_B^j) - 1)) = Q_B^j_B(f_B^j, f_B^j) \log(f_B^j) + Q_B^j_A(f_B^j, f_A^j) \log(f_B^j).
\]  

Using a Green’s formula and an entropy estimate in the system (3.16), leads to

\[
\int_{\xi>0} \xi f_B^j(1, v) \log f_B^j(1, v) dv + \int_{\xi<0} |\xi| f_B^j(-1, v) \log f_B^j(-1, v) dv \\
\leq (\int_{\xi'>0} \xi' f_B^j(1, v') dv' + k^j) \\
\int_{\xi<0} |\xi| M_+(v) \log(M_+(v)) (\int_{\xi'>0} \xi' f_B^j(1, v') dv' + k^j) dv \\
+ (\int_{\xi<0} |\xi| f_B^j(-1, v') dv' + k^j) \\
\int_{\xi>0} M_-(v) \log(M_-(v)) (\int_{\xi'<0} \xi' f_B^j(-1, v') dv' + k^j) dv.
\]
By the Dunford-Pettis criterion ([13]), $f_j^B(1,.)$ is weakly compact in $L^1(\{v \in \mathbb{R}^3, \xi > 0\})$. Let one of its subsequence still denoted by $f_j^B(1,.)$, converging weakly to some $g_+ \in L^1(\{v \in \mathbb{R}^3, \xi > 0\})$. From now the identification between $g_+$ and $f_B(1,v)$ is analogous to the proofs given in ([10], [11]). This concludes the proof of Theorems 1 and 2.

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