FUNDAMENTAL SOLUTIONS TO $\Box_b$ ON CERTAIN QUADRICS

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Abstract. The purpose of this article is to expand the number of examples for which the complex Green operator, that is, the fundamental solution to the Kohn Laplacian, can be computed. We use the Lie group structure of quadric submanifolds of $\mathbb{C}^n \times \mathbb{C}^m$ and the group Fourier transform to reduce the $\Box_b$ equation to ones that can be solved using modified Hermite functions. We use Mehler’s formula and investigate 1) quadric hypersurfaces, where the eigenvalues of the Levi form are not identical (including possibly zero eigenvalues), and 2) the canonical quadrics in $\mathbb{C}^4$ of codimension two.

1. Introduction

The Heisenberg group in $\mathbb{C}^n$ is the primary example for which the fundamental solution to the Kohn Laplacian $\Box_b$ can be explicitly written down. The purpose of this article is to expand this library of examples to include: 1) quadric hypersurfaces, where the eigenvalues of the Levi form are not identical (including possibly zero eigenvalues), and 2) the canonical quadrics in $\mathbb{C}^4$ of codimension two. It should be noted that $\Box_b$ does not transform well under biholomorphic changes of coordinates. In particular, the fundamental solution to $\Box_b$ in the case of a strictly pseudoconvex quadric hypersurface where the eigenvalues are different cannot be obtained simply by rescaling the variables in the Heisenberg case.

The techniques used in this paper involve representation theory to reduce $\Box_b$ to a modified Hermite equation, where a spectral expansion by Hermite functions provides a quick answer to the solvability of $\Box_b$. This approach has been used by many authors (see for example, Peloso and Ricci [PR03] and the references therein). Our work then goes one step further to compute a (nearly) closed form expression for the fundamental solution to $\Box_b$ for the example quadrics, mentioned above, by using this spectral expansion together with a modification of Mehler’s formula. The formulas obtained clearly exhibit the solution operator for $\Box_b$ in terms of the eigenvalues of the Levi form and should make it easier to obtain more precise estimates in terms of these eigenvalues.

In [FS74b], Folland and Stein give a formula for the fundamental solution of a family of second order operators $\mathcal{L}_\alpha$ that include the fundamental solution to the Kohn Laplacian at every form level (for which a solution exists). Since Folland and Stein simply present their solution and demonstrate that it works, it is not adaptable to many circumstances. On the other hand, in [PR03] Peloso and Ricci give a systematic treatment of $\Box_b$ on quadrics using representation theory and Hermite expansions. Our motivation is to use the Peloso/Ricci approach but generate formulas in the spirit of [FS74b]. We realized from our earlier work on
the Heisenberg group and quadric submanifolds \[BR09\, BR11\] that we could adopt Mehler’s formula, one of the workhorse equations with Hermite functions and the $\Box_b$-heat equation, to the $\Box_b$ equation.

Previously to our work, authors attempted to find formulas for the fundamental solution to $\Box_b$ via the $\Box_b$-heat equation or Hamilton-Jacobi equations. In particular, since $e^{-s\Box_b}$ sol\es the $\Box_b$-heat equation, if $P$ is the projection onto $\text{Ker}(\Box_b)$, then

$$
\int_0^\infty e^{-s\Box_b}(I - P)\, ds
$$

is the fundamental solution to $\Box_b$. The difficulty with this approach is that the techniques to solve for $e^{-s\Box_b}$ do so up to a partial Fourier transform in the totally real tangent direction. Calin et. al. are able to recover the solution for the Heisenberg group and find some formulas for a more general class of quadrics \[CC106\]. The authors are able to use this technique to prove estimates on the $\Box_b$-heat kernel on certain decoupled polynomial models in $\mathbb{C}^n$ \[Rai06\, Rai07\, Rai\, BR11\]. Beals et. al. use an approach through Hamilton-Jacobi equations to generate integral formulas for fundamental solutions for certain subelliptic equations \[BGG96\]. Earlier work by Beals and Greiner in this direction used pseudodifferential operators \[BG88\].

2. Statement of Results

In this section, we summarize the type of results we can obtain from our techniques. A precise definition of $\Box_b$ and further background is given in the following sections.

2.1. Hypersurface Results. We consider the case of a quadric hypersurface in $\mathbb{C}^{n+1}$ of the form

$$M = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}; \text{Im} w = \sum_{j=1}^n \sigma_j|z_j|^2\}, \quad \sigma_j \in \mathbb{R}, \ 1 \leq j \leq n.$$ 

We shall identify a point $(z, w) \in M$ with the point $(z, t = \text{Re} w)$ lying in the tangent space. The case where all the $\sigma_j$ are equal and nonzero is the Heisenberg group, and the fundamental solution to $\Box_b$ on $(0, q)$ forms for $1 \leq q \leq n - 1$ is well known in this case (\[FS74a\, FS74b\, Ste93\]). To simplify the notation of our result, we will (mostly) restrict ourselves to $n = 2$.

**Theorem 1.** Suppose $n = 2$, and $M$ is the hypersurface presented as above with $\sigma_1, \sigma_2 > 0$. Then, on the space of differential $(0, 1)$ forms, spanned by $d\bar{z}_1, \Box_b$ is solvable via a convolution with a kernel $N(z, t)$ where

$$
N(z, t) = \frac{1}{\pi^3} \int_0^1 \frac{\sigma_1 \sigma_2}{(it + s_1(r)\sigma_1|z_1|^2 + s_2(r)\sigma_2|z_2|^2)^2 (1 - r^{\sigma_1})(1 - r^{\sigma_2})} r^{\sigma_1 - 1} \, dr \\
+ \frac{1}{\pi^3} \int_0^1 \frac{\sigma_1 \sigma_2}{(-it + s_1(r)\sigma_1|z_1|^2 + s_2(r)\sigma_2|z_2|^2)^2 (1 - r^{\sigma_1})(1 - r^{\sigma_2})} r^{\sigma_2 - 1} \, dr
$$

where

$$s_j(r) = \frac{1 + r^{\sigma_j}}{1 - r^{\sigma_j}} \text{ for } j = 1, 2.$$ 

In the case of forms spanned by $d\bar{z}_2$, the factor of $r^{\sigma_1 - 1}$ in the numerator is replaced by $r^{\sigma_2 - 1}$, and vice versa.
Theorem 3. The solvability is possible, our next theorem provides the explicit fundamental solution.

for \((0, q)\) of \(M\) for one exceptional direction where one of the eigenvalues is zero. Again, for \((0, q)\) the change of variables

\[
s = \frac{1 + r}{1 - r}, \quad ds = \frac{2dr}{(1 - r)^2}
\]

allows one to compute this integral explicitly to obtain

\[
N(z, t) = \frac{1}{\pi^3(|z|^2 + it)(|z|^2 - it)} = \frac{1}{\pi^3(|z|^4 + |t|^2)},
\]

the well-known fundamental solution for the Heisenberg group in \(\mathbb{C}^3\) on \((0, 1)\)-forms.

Although we have specialized to the case where \(n = 2\) and \(\sigma > 0\), it is clear from the presentation below that the same techniques can handle \(n \geq 2\) and \(\sigma_j \in \mathbb{R}\). If one or more of the \(\sigma_j\) vanish, then the fundamental solution to \(\Box_b\) takes a somewhat different form. To simplify the notation, we consider the case \(n = 3\) with one vanishing eigenvalue.

Theorem 2. Suppose \(n = 3\), and \(M\) is the hypersurface presented as above with \(\sigma_1 = \sigma_2 = 1\) and \(\sigma_3 = 0\). Then, on the space of differential \((0, 1)\) forms, spanned by \(a\bar{z}_1\) and \(a\bar{z}_2\), \(\Box_b\) is solvable via a convolution with a kernel \(N(z, t)\) where

\[
N(z, t) = \frac{8}{\pi^3} \int_0^1 \frac{dr}{|\ln r|(1 - r)^2} \frac{1}{|z'|^2 (\frac{1 + r}{1 - r}) + |z_3|^2 \frac{2}{|\ln r|} + it^3}
\]

2.2. Higher Codimension Examples. Here we focus on the case of a quadric of codimension two in \(\mathbb{C}^4\), given by

\[
M = \{(z, w) \in \mathbb{C}^2 \times \mathbb{C}^2; \quad \text{Im } w = \phi(z, z)\}
\]

where \(\phi: \mathbb{C}^2 \times \mathbb{C}^2 \mapsto \mathbb{C}^2\) is a sesquilinear form (i.e. \(\phi(z, z') = \overline{\phi(z', z)}\)). If the image of the Levi form (i.e. \(\phi\)) is not contained in a one-dimensional cone, then \(M\) is biholomorphic to one of the following three canonical examples (see [Bog91]):

- \(M_1\) where \(\phi(z, z) = (|z_1|^2, |z_2|^2)\)
- \(M_2\) where \(\phi(z, z) = (2 \Re(z_1\overline{z_2}), |z_1|^2 - |z_2|^2)\)
- \(M_3\) where \(\phi(z, z) = (2|z_1|^2, 2 \Re(z_1\overline{z_2}))\)

The first case, \(M_1\), is just the Cartesian product of two Heisenberg groups in \(\mathbb{C}^2\) where solvability in any dimension is not possible. In the second case, the Levi form of \(M_2\) has one positive and one negative eigenvalue in each totally real direction. Therefore \(\Box_b\) is solvable for \((0, q)\)-forms when \(q = 0, 2\), but not when \(q = 1\) (see [PR03]). In the third case, the Levi form of \(M_3\) has one positive and one negative eigenvalue for each totally real direction, except for one exceptional direction where one of the eigenvalues is zero. Again, \(\Box_b\) is solvable for \((0, q)\)-forms when \(q = 0, 2\), but not when \(q = 1\) (again see [PR03]). In the cases where solvability is possible, our next theorem provides the explicit fundamental solution.

Theorem 3. For \(M_2\), the fundamental solution kernel to the Kohn Laplacian \(\Box_b\) for \(q = 0, 2\) is given by

\[
N(z, t) = \frac{1}{4\pi^3} \frac{1}{(|z|^4 + |t|^2)^{3/2}}.
\]
For $M_3$, the fundamental solution kernel to the Kohn Laplacian $\Box_b$ for $q = 0$ is given by
\[
N(z, t) = \frac{1}{\pi^2} \int_0^1 \int_0^{2\pi} \frac{\sigma_1(\theta)\sigma_2(\theta)}{(1 - r^{\sigma_1})(1 - r^{\sigma_2})} \frac{r^{\sigma_1-1}}{2 \, d\theta dr}
\]
\[
\times \frac{2 \, d\theta dr}{(-i(t_1 \cos \theta + t_2 \sin \theta) + E_1(\theta, r)|z_1|^2 + E_2(\theta, r)|z_2|^2)^3}
\]
where
\[
\sigma_1 = \sigma_1(\theta) = 1 + \cos \theta, \quad \sigma_2 = \sigma_2(\theta) = 1 - \cos \theta, \quad E_j(\theta, r) = \frac{\sigma_j(1 + r^{\sigma_j})}{1 - r^{\sigma_j}}, \quad j = 1, 2.
\]
If $q = 2$, then the expression for $N$ is the same except that the factor of $r^{\sigma_1-1}$ is replaced by $r^{\sigma_2-1}$.

The integrands in Theorems 1 and 3 are integrable in $r$ and $\theta$, respectively when $z_1$ and $z_2$ are not zero, and in each case, $N$ can be shown to be locally integrable. More detailed estimates of these formulas in terms of the volumes of nonisotropic balls of the control metric (see [NSW85, NS06] will be given in a future paper. The outline of this paper is as follows. Section 3 contains precise definitions of quadrics and their group structure. Section 4 contains a brief description of unitary representation theory, which is one of the key tools which allows us to transform $\Box_b$ to a Hermite operator. The basic facts about the spectral decomposition of Hermite operators are given in Section 5. In section 6, we explicitly evaluate the spectral decompositions in the cases mentioned in the above theorems.

3. Quadric Submanifolds and $\Box_b$

3.1. Quadric submanifolds. Let $M$ be the the quadric submanifold in $\mathbb{C}^n \times \mathbb{C}^m$ defined by
\[
M = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m; \ Im \ w = \phi(z, z)\}
\]
where $\phi : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}^m$ is a sesquilinear form (i.e. $\phi(z, z') = \overline{\phi(z', z)}$). For emphasis, we sometimes write $M_\phi$ to denote the dependence of $M$ on the quadratic function $\phi$. Note that $M_{-\phi}$ is biholomorphic to $M_\phi$ by the change of variables $(z, w) \mapsto (z, -w)$.

3.2. Lie Group Structure. By projecting $M \subset \mathbb{C}^n \times \mathbb{C}^m$ onto $G = \mathbb{C}^n \times \mathbb{R}^m$, the Lie group structure of $M$ is isomorphic to the following group structure on $G$:
\[
gg' = (z, t)(z', t') = (z + z', t + t' + 2 \, \text{Im} \, \phi(z, z')).
\]
Note that $(0, 0)$ is the identity in this group structure and that the inverse of $(z, t)$ is $(-z, -t)$.

The right invariant vector fields are given as follows: let $g \in G$; if $X$ is a vector field, then we denote its value at $g$ by $X(g)$ as an element of the tangent space of $M$ at $g$. Define $R_g : G \rightarrow G$ by $R_g(g') = gg'$; a vector field $X$ is right invariant if and only if $X(g) = (R_g)_*\{X(0)\}$, where $(R_g)_*$ denotes the differential or push forward operator at $g$ as a map from the tangent space at the origin to the tangent space at $g$. Let $v$ be a vector in $\mathbb{C}^n \approx \mathbb{R}^{2n}$ which can be identified with the tangent space of $M$ at the origin. Let $\partial_v$ be the real vector field given by the directional derivative in the direction of $v$. Then the right invariant vector field at an arbitrary $g = (z, w) \in M$ corresponding to $v$ is given by
\[
X_v(g) = \partial_v + 2 \, \text{Im} \, \phi(v, z) \cdot D_t = \partial_v - 2 \, \text{Im} \, \phi(z, v) \cdot D_t
\]
(see Section 1 in Peloso/Ricci [PR03]). Let $Jv$ be the vector in $\mathbb{R}^{2n}$ which corresponds to $iv$ in $\mathbb{C}^n$ (where $i = \sqrt{-1}$). The CR structure on $G$ is then spanned by vectors of the form:

$$Z_v(g) = (1/2)(X_v - iX_{Jv}) = (1/2)(\partial_v - i\partial_{Jv}) - i\phi(z,v) \cdot D_t$$

and

$$\overline{Z}_v(g) = (1/2)(X_v + iX_{Jv}) = (1/2)(\partial_v + i\partial_{Jv}) + i\phi(z,v) \cdot D_t.$$ 

An easy computation gives:

$$[Z_v, Z_{v'}] = 0, \quad [\overline{Z}_v, \overline{Z}_{v'}] = 0, \quad [Z_v, \overline{Z}_{v'}] = 2i\phi(v,v') \cdot D_t.$$ 

The last expression on the right is the Levi form of $M$ as a map from the complex tangent space ($v \in \mathbb{R}^{2n} = \mathbb{C}^n$) to the totally real directions (spanned by $D_t$).

For $\lambda \in \mathbb{C}^m$, let

$$\phi^\lambda(z, z') = \phi(z, z') \cdot \lambda$$

where $\cdot$ is the ordinary dot product (without conjugation). If $\lambda \in \mathbb{R}^m$, then $\phi^\lambda(z, z')$ is a sesquilinear scalar-valued form with an associated Hermitian matrix. Let $v_1^\lambda, \ldots, v_n^\lambda$ be an orthonormal basis for $\mathbb{C}^n$ which diagonalizes this matrix and we write

$$\phi^\lambda(v_j^\lambda, v_k^\lambda) = \delta_{jk} \mu_j^\lambda$$

where $\mu_j^\lambda, 1 \leq j \leq n$ are its eigenvalues.

3.3. Special Coordinates. For $\lambda \in \mathbb{R}^m$, define the function $\nu(\lambda)$ by

$$\nu(\lambda) = \text{rank}(\phi^\lambda).$$

The function $\nu(\lambda)$ satisfies $0 \leq \nu(\lambda) \leq n$. We assume the eigenvalues are ordered so that $\mu_1^\lambda, \ldots, \mu_{\nu(\lambda)}^\lambda \neq 0$ and $\mu_{\nu(\lambda)+1}^\lambda, \ldots, \mu_n^\lambda = 0$. We identify $x$ with $(x_1^\lambda, \ldots, x_n^\lambda)$ and $y$ with $(y_1^\lambda, \ldots, y_n^\lambda)$. We also write $z = \sum_{j=1}^n (x_j^\lambda + iy_j^\lambda)v_j^\lambda$ for $z = x + iy \in \mathbb{C}^n$. Additionally, we let $z' = (z_1, \ldots, z_{\nu(\lambda)})$, $z'' = (z_{\nu(\lambda)+1}, \ldots, z_n)$ and similarly for $x$ and $y$. Although for many canonical examples, the choice of coordinates will vary smoothly in $\lambda$, this is not the case in general.

3.4. $\square_b$ Calculations. Let $v_1, \ldots, v_n$ be any orthonormal basis for $\mathbb{C}^n$ (later, this choice will be a special coordinate basis mentioned just before Section 3.3). Let $X_j = X_{v_j}, Y_j = X_{Jv_j}$, and let $Z_j = (1/2)(X_j - iY_j), \overline{Z}_j = (1/2)(X_j + iY_j)$ be the right invariant vector fields defined above (which are also the left invariant vector fields for the group structure with $\phi$ replaced by $-\phi$). Also let $dz_j$ and $d\overline{z}_j$ be the dual basis. A $(0,q)$-form can be expressed as $\sum_{K \in I_q} \phi_K \, d\overline{z}^K$ where $I_q = \{K = (k_1, \ldots, k_q) : 1 \leq k_1 < \cdots < k_q \leq n\}$. We recall the following proposition [PR03, Proposition 2.1]:

**Proposition 4.**

$$\square_b(\sum_{K \in I_q} \phi_K \, d\overline{z}^K) = \sum_{K, L \in I_q} \square_{LK} \phi_K \, d\overline{z}^L$$

(1)

where

$$\square_{LK} = -\delta_{LK} L + M_{LK}$$

(2)

and $L$ is the sub-Laplacian on $G$:

$$L = (1/2) \sum_{k=1}^n \overline{Z}_k Z_k + Z_k \overline{Z}_k$$
and

\[
M_{LK} = \begin{cases} 
\frac{1}{2} \left( \sum_{k \in K} [Z_k, \overline{Z}_k] - \sum_{k \notin K} [Z_k, \overline{Z}_k] \right) & \text{if } K = L \\
\epsilon(K, L)[Z_k, \overline{Z}_l] & \text{if } |K \cap L| = q - 1 \\
0 & \text{otherwise.}
\end{cases}
\]

Here, \(\epsilon(K, L)\) is \((-1)^d\) where \(d\) is the number of elements in \(K \cap L\) between the unique element \(k \in K - L\) and the unique element \(l \in L - K\).

The above proposition is stated and proved in [PR03] for left-invariant vector fields. If right invariant vector fields are used, then the above proposition provides a formula for \(\Box_b\) associated to \(M - \phi\).

For later, we record the diagonal part of \(\Box_b\), i.e., \(\Box_{LL}\). Using (2) with \(L = K\) and the above formulas for \(Z_k\), we obtain

\[
\Box_{LL} = -\frac{1}{4} \sum_{k=1}^{n} \left( X_k^2 + Y_k^2 \right) + i \left( \sum_{k \in L} \phi(v_k, v_k) \cdot D_t - \sum_{k \notin L} \phi(v_k, v_k) \cdot D_t \right). \tag{3}
\]

For the case of \(\Box_b\) on the Heisenberg group, \(\phi(z, z) = |z|^2\) and \(Z_k = D_{z_k} - i\overline{z}_k D_t\). In this case, \(\Box_b\) is a diagonal operator (since \([Z_k, \overline{Z}_l] = 0\) when \(k \neq l\)) and the above formula for \(\Box_{LL}\) gives the coefficient of \(\Box_b\) acting on forms of the type \(\phi_L(z) dz^L\).

We will also need the adjoint of \(\Box_{LK}\), which is defined as

\[
\int_{(z, t) \in G} \Box_{LK} \{ f(z, t) \} g(z, t) \, dx \, dy \, dt = \int_{(z, t) \in G} f(z, t) \Box_{LK}^{\text{adj}} \{ g(z, t) \} \, dx \, dy \, dt
\]

(note: this is the “integration by parts” adjoint, not the \(L^2\) adjoint, since there is no conjugation). An easy computation shows

\[
\Box_{LL}^{\text{adj}} = -\frac{1}{4} \sum_{k=1}^{n} \left( X_k^2 + Y_k^2 \right) - i \left( \sum_{k \in L} \phi(v_k, v_k) \cdot D_t - \sum_{k \notin L} \phi(v_k, v_k) \cdot D_t \right). \tag{4}
\]

Note the minus sign instead of the plus sign in front of the imaginary term at the end. For later use, note that

\[
\Box_{LL} = \overline{\Box}_{LL}^{\text{adj}} \quad \text{and} \quad \Box_{LL} \{ f(-z, -t) \} = (\Box_{LL}^{\text{adj}} f)(-z, -t) \]

for any smooth function \(f\).

4. Representation Theory.

4.1. Unitary Representations.

**Definition 5.** For a Lie group, \(G\), a unitary representation is a homomorphism \(\pi\) from \(G\) to the space of unitary operators on \(L^2(\mathbb{R}^{n'})\) for some \(n' \geq 0\).

For our quadric Lie group \(G\), we will fix \(\lambda \in \mathbb{R}^m\) as above and take \(n' = \nu(\lambda)\) (the number of nonzero eigenvalues, \(\mu_j^\lambda\) as in Section 3.3). Suppose \(z^\lambda = z = x + iy \in \mathbb{C}^n\) is the
special coordinate system mentioned in Section 3.3. Let \( t, \lambda \in \mathbb{R}^m \) and \( \eta \in \mathbb{C}^{n-\nu(\lambda)} \). For \( g = (x, y, t) \in G \), define \( \pi_{\lambda, \eta}(x, y, t) : L^2(\mathbb{R}^{\nu(\lambda)}) \rightarrow L^2(\mathbb{R}^{\nu(\lambda)}) \) by

\[
\pi_{\lambda, \eta}(x, y, t)(h)(\xi) = e^{i(\lambda \cdot t + 2 \text{Re}(e^{\eta \cdot \eta}))} e^{-2i \sum_{j=1}^{\nu(\lambda)} \mu_j^2 \xi_j^2 (\xi_j + x_j)} h(\xi + 2x')
\]

for \( h \in L^2(\mathbb{R}^{\nu(\lambda)}) \) (so \( \xi \in \mathbb{R}^{\nu(\lambda)} \)). Note that if \( \eta = \zeta + i\zeta \), then \( \text{Re}(z'' \cdot \bar{\eta}) = x'' \cdot \zeta + y'' \cdot \zeta \). It is a straightforward computation that \( \pi \) is a representation for \( G \). For further background information on representation theory (in particular for the Heisenberg group), see [Tay86].

If \( X \) is a differential operator on \( G \) comprised of right invariant vector fields, then we can compute how \( X \) “transforms” via \( \pi_{\lambda, \eta} \) to an operator on \( L^2(\mathbb{R}^{\nu(\lambda)}) \) denoted by \( d\pi_{\lambda, \eta}X \), which means that

\[
X_g \{ \pi_{\lambda, \eta}(g) \} = d\pi_{\lambda, \eta}X \circ \pi_{\lambda, \eta}(g).
\]

In words, the \( X_g \) on the left side differentiates \( \pi_{\lambda, \eta}(g) \) with respect to the variable \( g \) whereas on the right side, \( d\pi_{\lambda, \eta}X \) is an operator on \( L^2(\mathbb{R}^{\nu(\lambda)}) \). The next proposition identifies \( d\pi_{\lambda, \eta}(X) \) for our basis of right invariant vector fields of \( G \).

**Proposition 6.** For the right invariant vector fields, \( X_j, Y_j, D_{t_k} \) defined in the last section, the following identities hold as operators on \( L^2(\mathbb{R}^{\nu(\lambda)}) \):

\[
X_j \{ \pi_{\lambda, \eta}(g) \} = d\pi_{\lambda, \eta}X_j = \begin{cases} 2D_{\xi_j} \circ \pi_{\lambda, \eta}(g) & 1 \leq j \leq \nu(\lambda) \\ 2i\zeta_j \circ \pi_{\lambda, \eta}(g) & \nu(\lambda) + 1 \leq j \leq n \end{cases}
\]

\[
Y_j \{ \pi_{\lambda, \eta}(g) \} = d\pi_{\lambda, \eta}Y_j = \begin{cases} -2i\mu_j^2 \xi_j \circ \pi_{\lambda, \eta}(g) & 1 \leq j \leq \nu(\lambda) \\ 2i\zeta_j \circ \pi_{\lambda, \eta}(g) & \nu(\lambda) + 1 \leq j \leq n \end{cases}
\]

\[
D_{t_k} \{ \pi_{\lambda, \eta}(g) \} = d\pi_{\lambda, \eta}D_{t_k} = i\lambda_k \circ \pi_{\lambda, \eta}(g) \quad 1 \leq k \leq m.
\]

**Remark.** If we had used left invariant vector fields instead of right invariant vector fields, then the order of the operators on the right would have been reversed (i.e. the \( D_{\xi_j} \) would appear on the right of \( \pi_{\lambda, \eta}(g) \), etc.). See, for example [PR03, equation (8)]. We prefer the above ordering of the operators on the right and therefore have chosen to use right invariant vector fields.

Equation (9) is immediate. Equations (7) and (8) are easily shown to hold at the origin, \( g = 0 \), since \( X_j(0) = D_{x_j} \) and \( Y_j(0) = D_{y_j} \), then use right invariance to conclude these equations hold at all \( g \in G \).

Now we compute the “transform” of \( \Box_{LK} \) and its adjoint, via \( d\pi_{\lambda, \eta} \), using (3) and (4). Note that the coordinates \( (z_1^\lambda, \ldots, z_n^\lambda) \) were chosen to diagonalize the form \( \phi(z, \bar{z}) \cdot \lambda \) with eigenvalues \( \mu_j^\lambda \). This observation, together with formulas (7), (8), and (9) easily establish the following proposition

**Proposition 7.**

\[
d\pi_{\lambda, \eta} \Box_{LK} = \begin{cases} -\Delta_\xi + |\eta|^2 + \sum_{j=1}^{\nu(\lambda)} (\mu_j^\lambda)^2 \xi_j^2 - (\sum_{j \in L} \mu_j^\lambda + \sum_{j \in \bar{L}} \mu_j^\lambda) & \text{if } K = L \\ 0 & \text{if } K \neq L \end{cases}
\]

\[
d\pi_{\lambda, \eta} \Box_{adj}^{LK} = \begin{cases} -\Delta_\xi + |\eta|^2 + \sum_{j=1}^{\nu(\lambda)} (\mu_j^\lambda)^2 \xi_j^2 + (\sum_{j \in L} \mu_j^\lambda + \sum_{j \in \bar{L}} \mu_j^\lambda) & \text{if } K = L \\ 0 & \text{if } K \neq L \end{cases}
\]
Note that $\Box_{LK}$ and its adjoint transform to operators that only differ by a sign change in the zeroth order terms involving the $\mu_j^\lambda$.

4.2. **Group Fourier Transform.** For $(z, t) \in G$ and fixed $\lambda \in \mathbb{R}^m$, we express $(z, t) = (x, y, t) = (x', y', x''', y''', t) = (x', y', z', t)$ as in Section 3.3. The coordinate $z'''$ may be thought of as in $\mathbb{C}^{n-\nu(\lambda)}$ or $\mathbb{R}^{2(\nu-\nu(\lambda))}$.

For an integrable function $f : G \mapsto \mathbb{C}$, we define the group Fourier transform of $f$ as the operator $\pi^{\lambda,\eta}(f) : L^2(\mathbb{R}^{\nu(\lambda)}) \mapsto L^2(\mathbb{R}^{\nu(\lambda)})$ where for $h \in L^2(\mathbb{R}^{\nu(\lambda)})$,

$$\pi^{\lambda,\eta}(f)\{h\}(\xi) = \int_{(z = x + iy, t) \in G} f(z, t)\pi^{\lambda,\eta}(z, t)(h)\, dx\, dy\, dt$$

$$= \int_{(z = x + iy, t) \in G} f(z, t)e^{i\lambda t + 2\Re(z'' \cdot \eta)}e^{-2i\sum_{j=1}^{\nu(\lambda)} \mu_j^\lambda \xi_j + x_j^\lambda}h(\xi + 2x')\, dx\, dy\, dt.$$  

As before, $x_j, y_j$ are the coordinates for $x, y \in \mathbb{R}^n$ relative to the basis $v_1^\lambda, \ldots, v_n^\lambda$.

The following proposition (see [BR11, equation (12)]) relates the group transform to the usual Fourier transform and easily follows from the definition of $\pi^{\lambda,\eta}$. For us, the usual Fourier transform on $\mathbb{R}^d$ and the notation that we use to denote it is

$$\hat{f}(\xi) = f(\hat{\xi}) = \mathcal{F}f(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx.$$  

**Proposition 8.** For $\xi \in \mathbb{R}^{\nu(\lambda)}$ and $h \in L^2(\mathbb{R}^{\nu(\lambda)})$,

$$\pi^{\lambda,\eta}(f)\{h\}(\xi) = (2\pi)^{(2n+m-\nu(\lambda))/2} \int_{x' \in \mathbb{R}^{\nu(\lambda)}} \mathcal{F}_{x',y',t}\{f(x, y, t)e^{-2i\sum_{j=1}^{\nu(\lambda)} \mu_j^\lambda \xi_j + x_j^\lambda}\}(x', 2\mu^\lambda \circ \xi, -2\eta, -\lambda)h(\xi + 2x')\, dx'.$$  

(12)

where $\mathcal{F}_{x',y',t}$ indicates the Fourier transform in the $(x'', y, t)$ variables (but not $x'$) and where $\mu^\lambda \circ \xi = (\mu_1^\lambda \xi_1, \ldots, \mu_{\nu(\lambda)}^\lambda \xi_{\nu(\lambda)})$.

As with the classical Fourier transform, the above process can be reversed to identify $f \in L^2(G)$ from its group Fourier transform. To see this, assume that $\pi^{\lambda,\eta}(f)$ is known as an operator on $L^2(\mathbb{R}^{\nu(\lambda)})$ and we wish to identify $f$. For each $a \in \mathbb{R}^{\nu(\lambda)}$, define $h_a(\xi) = (2\pi)^{-n-m/2}e^{-i\xi \cdot a}$. Set

$$u^{\lambda,\eta}(a, \xi) = \pi^{\lambda,\eta}(f)(h_a)(\xi).$$

The above definition needs justification since $h_a \notin L^2(\mathbb{R}^\nu)$. However, we can multiply $h_a$ by an increasing sequence of cut-off functions (approaching 1 everywhere) and then take the limit as our definition of the left side. This technique is carried out carefully in [BR11, equation (16)]. Using (12), we obtain

$$u^{\lambda,\eta}(a, \xi) = \pi^{\lambda,\eta}(h_a)(\xi) = \mathcal{F}\left\{f(x, y, t)e^{-2i\sum_{j=1}^{\nu(\lambda)} \mu_j^\lambda \xi_j + x_j^\lambda}\right\}(2a, 2\mu^\lambda \circ \xi, -2\eta, -\lambda)e^{-i\xi \cdot a}$$  

(13)

where $\mathcal{F}$ is the Fourier transform in all variables. The motivation for the choice of $h = h_a$ is that it offers the “missing” exponential needed to relate the full Fourier transform $f$ with $u^{\lambda,\eta}$. By inverting the Fourier transform appearing in (13), we obtain the following proposition.
Proposition 9. Let \( \tilde{u}^{\lambda,\eta}(a,b) = u^{-\lambda,-\frac{1}{2}\eta}(a/2,b/(2\mu^{-\lambda})) \) where \( b/(2\mu^{-\lambda}) \) is the vector quantity whose \( j \)th component is \( b_j/(2\mu_j^{-\lambda}) \). Then
\[
f(x', y', \widehat{\eta}, \widehat{\lambda}) = e^{-2i\sum_{j=1}^n \nu_j x_j y_j} F_{a,b}^{-1} \left( e^{-\frac{i}{2} \sum_{j=1}^n a_j b_j / \mu_j} \tilde{u}^{\lambda,\eta}(a,b) \right) (x', y')
\] (14)
where \( f(x', y', \widehat{\eta}, \widehat{\lambda}) \) is the Fourier transform of \( f \) in the variables \( \eta \) and \( \lambda \) (but not \( x' \) and \( y' \)).

We can further recover \( f \) using the inverse Fourier transform in the \( \eta \), \( \lambda \) variables provided that the orthonormal basis \( v_1^\lambda, \ldots, v_n^\lambda \), which diagonalizes \( \phi^\lambda \), depends continuously on \( \lambda \) (as it does in the examples to follow).

4.3. The Transformed \( \square \). We start with a general proposition which describes how right invariant operators transform via the group transform.

Proposition 10. Suppose \( X \) is a differential operator of order at least one which is comprised of right invariant vector fields. Let \( X^{\text{adj}} \) denote the “integration by parts” adjoint of \( X \): i.e., for \( f_1 \in \text{Dom}(X) \cap L^2(G) \) and \( f_2 \in \text{Dom}(X^{\text{adj}}) \cap L^2(G) \),
\[
\int_{g \in G} Xf_1(g)f_2(g) \, dg = \int_{g \in G} f_1(g)X^{\text{adj}}f_2(g) \, dg
\]
where \( dg = dx \, dy \, dt \). Then
\[
d\pi^\lambda,\eta(X^{\text{adj}}) \circ \pi^\lambda,\eta(f) = \pi^\lambda,\eta(Xf).
\]

Proof. If \( h \in L^2(\mathbb{R}^{\nu(\lambda)}) \), then
\[
\pi^\lambda,\eta(Xf)(h) = \int_{g \in G} Xf(g)(\pi^\lambda,\eta(g))(h) \, dg
\]
\[
= \int_{g \in G} f(g)X^{\text{adj}}g(\pi^\lambda,\eta(g))(h) \, dg
\]
\[
= \int_{g \in G} f(g)d\pi^\lambda,\eta(X^{\text{adj}}) \circ (\pi^\lambda,\eta(g))(h) \, dg
\]
\[
= d\pi^\lambda,\eta(X^{\text{adj}})\left\{ \int_{g \in G} f(g)\pi^\lambda,\eta(g)(h) \, dg \right\}
\]
which establishes the proposition.

We shall apply this proposition to \( \square_{LL} \) using (11) (note that the off-diagonal terms transform to zero). We are interested in finding the the operator \( N_L \) which satisfies the equation \( \square_{LL} \circ N_L = I - P_L \), where \( P_L : L^2(G) \to L^2(G) \) is the orthogonal projection onto the \( \text{Ker}(\square_{LL}) \). For now, we will fix the index \( L \). The operators \( N_L \) and \( P_L \) are given by group convolution with functions, denoted by \( N(z,t) \) and \( P(z,t) \), respectively. So we are interested in solving \( \square_{LL}N(z,t) = \delta(z,t) - P(z,t) \) where \( \delta \) is the Dirac delta function supported at the origin.

Corollary 11. \( \square_{LL}N(z,t) = \delta(z,t) - P(z,t) \) if and only if
\[
d\pi^\lambda,\eta(\square_{LL}^{\text{adj}})(\pi^\lambda,\eta N) = I - P^\lambda,\eta
\] (15)
as operators on $L^2(R^d(\lambda))$, where $P_{\lambda,\eta}$ is the $L^2$ projection onto the kernel of $d\pi_{\lambda,\eta}(\square_{LL}^{\text{adj}})$.

**Proof.** We apply Proposition [10] to $\square_{LL}$. Let $h \in L^2(R^d(\lambda))$. The existence of $N$ solving $\square_{LL}N(z,t) = \delta(z,t) - P(z,t)$ implies:

$$d\pi_{\lambda,\eta}(\square_{LL}^{\text{adj}})\pi_{\lambda,\eta}(h) = \int_{(z,t) \in G} \delta(z,t) - P(z,t)\pi_{\lambda,\eta}(z,t)h(\xi) \, dz \, dt$$

$$= h(\xi) - \int_{(z,t) \in G} P(z,t)\pi_{\lambda,\eta}(z,t)h(\xi) \, dz \, dt$$

(since $\pi_{\lambda,\eta}(0,0)$ is the identity operator). We further claim that the integral on the right is the same as the $L^2$ projection of $h$ onto the Ker $\left(d\pi_{\lambda,\eta}(\square_{LL}^{\text{adj}})\right)$. This is established by proving the following:

1. $d\pi_{\lambda,\eta}(\square_{LL}^{\text{adj}})\left\{ \int_{(z,t) \in G} P(z,t)\pi_{\lambda,\eta}(z,t)h(\xi) \, dz \, dt \right\} = 0$ for all $h \in L^2(G)$
2. if $h \in L^2(G)$ is orthogonal to Ker $\left(d\pi_{\lambda,\eta}(\square_{LL}^{\text{adj}})\right)$, then

$$\int_{(z,t) \in G} P(z,t)\pi_{\lambda,\eta}(z,t)h(\xi) \, dz \, dt = 0.$$

The first fact is an easy consequence of Proposition [10]. To establish the second fact, it is enough to show

$$\left\langle \int_{(z,t) \in G} P(z,t)\pi_{\lambda,\eta}(z,t)h(\xi) \, dz \, dt , \, g(\xi) \right\rangle_{\xi} = 0$$

for all $g$ belonging to the Ker $\left(d\pi_{\lambda,\eta}(\square_{LL}^{\text{adj}})\right)$. We have

$$\left\langle \int_{(z,t) \in G} P(z,t)\pi_{\lambda,\eta}(z,t)h(\xi) \, dz \, dt , \, g(\xi) \right\rangle_{\xi} = \left\langle h(\xi), \int_{(z,t) \in G} \overline{P(z,t)}\pi_{\lambda,\eta}^*(z,t)g(\xi) \, dz \, dt \right\rangle_{\xi}.$$

Since $h$ is orthogonal to the Ker $\left(d\pi_{\lambda,\eta}(\square_{LL}^{\text{adj}})\right)$, it suffices to show

$$d\pi_{\lambda,\eta}(\square_{LL}^{\text{adj}})\left\{ \int_{(z,t) \in G} \overline{P(z,t)}\pi_{\lambda,\eta}^*(z,t)g(\xi) \, dz \, dt \right\} = 0.$$

Now $\pi_{\lambda,\eta}^*(z,t) = \pi_{\lambda,\eta}(-z,-t)$, so after a change of variables, the left side becomes

$$d\pi_{\lambda,\eta}(\square_{LL}^{\text{adj}})\left\{ \int_{(z,t) \in G} \overline{P(-z,-t)}\pi_{\lambda,\eta}(z,t)g(\xi) \, dz dt \right\}$$

$$= \int_{(z,t) \in G} \overline{P(-z,-t)}(\square_{LL}^{\text{adj}}\pi_{\lambda,\eta}(z,t))g(\xi) \, dz dt$$

$$= \int_{(z,t) \in G} \overline{P(z,t)}(\square_{LL}^{\text{adj}}\pi_{\lambda,\eta})(-z,-t)g(\xi) \, dz dt.$$
A chain rule calculation shows that \( (\Box_{LL}^\text{adj} f)(-z,-t) = \Box_{LL} \{ f(-z,-t) \} \) (recall (5)). Therefore, we can integrate by parts to show that the right side equals
\[
\int_{(z,t) \in G} \Box_{LL}^\text{adj} \{ P(z,t) \} \pi_{\lambda,\eta}(-z,-t) g(\xi) \, dz \, dt.
\]
Since \( \Box_{LL}^\text{adj} = \Box_{LL} \), the above term is zero, as desired. This proves the second fact and establishes (15). The converse can be established similarly.

5. THE HERMITE OPERATOR

Our starting point is equation Corollary (11) which allows us to transfer the analysis of \( \Box_{LL} \) to the following operator equation
\[
d\pi_{\lambda,\eta}(\Box_{LL}^\text{adj}) \pi_{\lambda,\eta} N = I - P_{\lambda,\eta},
\]
where according to (11),
\[
d\pi_{\lambda,\eta}(\Box_{LL}^\text{adj}) = -\Delta_\xi + |\eta|^2 + \sum_{j=1}^{\nu(\lambda)} (\mu_j^\lambda)^2 \xi_j^2 + (\sum_{j \in L} \mu_j^\lambda - \sum_{j \notin L} \mu_j^\lambda).
\]
The operator on the right is a modified Hermite operator which has a well known spectral decomposition which we now describe. On the real line, and for \( \ell \) a nonnegative integer, define the \( \ell \)-th Hermite function
\[
\psi_\ell(x) = \frac{(-1)^\ell}{2^{\ell/2} \pi^{1/4} (\ell)!^{1/2}} \frac{d^\ell}{dx^\ell} \{ e^{-x^2} \} e^{x^2}/2, \quad x \in \mathbb{R}.
\]
Each \( \psi_\ell \) has unit \( L^2 \)-norm on the real line and satisfies the equation
\[
-\psi_\ell''(x) + x^2 \psi_\ell(x) = (2\ell + 1) \psi_\ell(x),
\]
(see Thangavelu’s book [Tha93]). The Hermite operator, \( -D_{xx} + x^2 \) is a nonnegative, self-adjoint operator and the \( \psi_\ell, \ \ell = 0, 1, \ldots \) forms a complete orthonormal basis of eigenfunctions with eigenvalues \( 2\ell + 1 \). In several variables, we let \( \ell = (\ell_1, \ldots, \ell_{\nu(\lambda)}) \) where each \( \ell_i \) is a non-negative integer. For each \( \lambda \in R^m \setminus \{0\} \), define
\[
\psi_{\ell_i}^\lambda(\xi_i) = \psi_{\ell_i}(|\mu_j^\lambda|^{1/2} \xi_j)|\mu_j^\lambda|^{1/4}; \quad \Psi_\ell^\lambda(\xi) = \prod_{j=1}^{\nu(\lambda)} \psi_{\ell_j}^\lambda(\xi_j).
\]
Each \( \psi_{\ell_i}^\lambda(\cdot) \) has unit \( L^2 \)-norm on \( \mathbb{R} \) and hence \( \Psi_\ell^\lambda \) has unit \( L^2 \)-norm on \( \mathbb{R}^{\nu(\lambda)} \). An easy calculation shows that
\[
(-D_{\xi_j\xi_j} + (\mu_j^\lambda)^2) \{ \psi_{\ell_i}^\lambda(\xi_j) \} = (2\ell_j + 1) \psi_{\ell_j}^\lambda(\xi_j)|\mu_j^\lambda|.
\]
Therefore
\[
d\pi_{\lambda,\eta}(\Box_{LL}^\text{adj}) \{ \Psi_\ell^\lambda(\xi) \} = \Lambda_\ell^\lambda \Psi_\ell^\lambda(\xi)
\]
where
\[
\Lambda_\ell^\lambda = \sum_{j=1}^{\nu(\lambda)} (2\ell_j + 1)|\mu_j^\lambda| + \left( \sum_{j \in L} \mu_j^\lambda - \sum_{j \notin L} \mu_j^\lambda \right) + |\eta|^2.
\]
The collection of functions \( \Psi_\ell^\lambda(\xi) \) form a complete set of orthonormal eigenfunctions for \( d\pi_{\lambda,\eta}(\Box_{LL}^\text{adj}) \) with eigenvalues \( \Lambda_\ell^\lambda,\eta \). Note that \( \Lambda_{\ell,\eta}^\lambda \geq 0 \) for all \( \ell \) and \( \Lambda_{\ell,\eta}^\lambda > 0 \) for all nonzero
multiindices \( \ell \). When \( \ell = 0 \), \( \Lambda_0^{\lambda,\eta} \) can equal zero only if \( \eta = 0 \), all the \( \mu_k^\lambda < 0 \) for \( k \in L \), and all \( \mu_k^\lambda > 0 \) for \( k \notin L \). This cannot occur if either the number of negative eigenvalues is not equal to \( q = \text{length of } (L) \) or if the number of positive eigenvalues is not equal to \( n - q \). This condition is the hypothesis of the first part of the following solvability theorem in Peloso/Ricci [PR03].

**Theorem 12.** (1) (Theorem 1 in [PR03]) Suppose \( |L| = q \). If the number of negative eigenvalues of \( \phi^\lambda \) is not equal to \( q \) or if the number of positive eigenvalues of \( \phi^\lambda \) is not equal to \( n - q \), then the \( \text{Ker} \left( d\pi_{\lambda,\eta}(\Box_{LL}^{\text{adj}}) \right) = \{0\} \) its inverse is given by the operator

\[
\mathcal{J}^{\lambda,\eta} = \sum_{\ell} \frac{1}{\lambda^{\lambda,\eta}_{\ell}} P^\lambda_{\ell}
\]

where \( P^\lambda_{\ell} \) is the orthogonal projection onto the space spanned by \( \Psi^\lambda_{\ell}(\xi) \).

(2) (Theorem 5.2 in [PR03]) Suppose \( \nu(\lambda) = n \) (so there is no \( \eta \)); if \( \mu_j^\lambda < 0 \) for \( j \in L \) and \( \mu_j^\lambda > 0 \) for \( j \notin L \), then \( \text{Ker} \left( d\pi_{\lambda,\eta}(\Box_{LL}^{\text{adj}}) \right) \) is the space spanned by

\[
\Psi^\lambda_{0}(\xi) = \prod_{j=1}^{n} |\mu_j^\lambda|^{1/4} e^{-\mu_j^\lambda |\xi_j|^2/2}.
\]

In the first case of this theorem, we will use Proposition 9 to identify \( N(z,t) \) (or at least its \( t \)-Fourier transform) by applying the the operator \( \mathcal{J}^{\lambda,\eta} \) to the function \( h_a(\xi) = (2\pi)^{-n-m/2} e^{-i\xi a} \), as in the discussion leading up to Proposition 9. The first step is to compute

\[
u(\lambda) \sum_{\ell \in \mathbb{Z}^+_L} \frac{1}{\lambda^{\lambda,\eta}_{\ell}} \prod_{j=1}^{n} |\mu_j^\lambda|^{1/4} P^\lambda_{\ell} \{e^{-i\xi_j a_j}\}
\]

where \( \mathbb{Z}^+_n \) is the set of \( n \)-tuples of nonnegative integers and where \( P^\lambda_{\ell} \) is the orthogonal projection onto the space of \( L^2 \) functions in the variable \( \xi_j \) spanned by \( \psi^\lambda_{\ell_j}(\xi_j) \). Each projection term on the right is

\[
P^\lambda_{\ell_j}(e^{-i\xi_j a_j}) = \left( \int_{\mathbb{R}} e^{-i\xi_j a_j} \psi^\lambda_{\ell_j}(|\mu_j^\lambda|^{1/2} \xi_j) \right) |\mu_j^\lambda|^{1/4} \psi^\lambda_{\ell_j}(|\mu_j^\lambda|^{1/2} \xi_j)
\]

where the last equality uses a standard fact that a Hermite function equals its Fourier transform up to a factor of \( (-i)^{\ell_j} \) (see [Tha93]). Substituting this expression on the right into the above expression for \( u^{\lambda,\eta}(a,\xi) \), we obtain

\[
u(\lambda) \sum_{\ell \in \mathbb{Z}^+_L} \frac{1}{\lambda^{\lambda,\eta}_{\ell}} \prod_{j=1}^{n} \psi^\lambda_{\ell_j}(a_j/|\mu_j^\lambda|^{1/2}) \psi^\lambda_{\ell_j}(|\mu_j^\lambda|^{1/2} \xi_j)
\]

In view of (14), to compute \( N(x',y',\tilde{\eta},\tilde{\lambda}) \), we need to determine

\[
u(\lambda) \sum_{\ell \in \mathbb{Z}^+_L} \frac{1}{\lambda^{\lambda,\eta}_{\ell}} \prod_{j=1}^{n} \psi^\lambda_{\ell_j}(a_j/|\mu_j^\lambda|^{1/2}) \psi^\lambda_{\ell_j}(|\mu_j^\lambda|^{1/2} \xi_j).
\]

(17)
where $b/(2\mu^{-\lambda})$ is the vector quantity whose $j$th component is $b_j/(2\mu_j^{-\lambda})$. From the previous equality, we have

$$\tilde{u}^{\lambda,\eta}(a, b) = (2\pi)^{-\frac{1}{2}(2n+m-\nu(\lambda))} \sum_{\ell \in \mathbb{Z}_+^n} (-i)^{\ell} \prod_{j=1}^{\nu(\lambda)} \psi_{\ell_j} \left( a_j / 2 |\mu_j^\lambda|^{1/2} \right) \psi_{\ell_j} \left( b_j |\mu_j^\lambda|^{1/2} / 2 \mu_j^\lambda \right). \quad (18)$$

Using (14), the formula for the partial transform of $N$ is given in the following proposition.

**Proposition 13.** The partial $(z'', t)$-Fourier transform of the fundamental solution to $\Box_L$ is given by

$$N(x', y', \widehat{\eta}, \widehat{\lambda}) = e^{-2i \sum_{j=1}^{\nu(\lambda)} a_j y_j} \mathcal{F}_{a,b}^{-1} \left( e^{-\frac{4}{\pi} \sum_{j=1}^{\nu(\lambda)} a_j b_j / \mu_j^\lambda} \tilde{u}^{\lambda,\eta}(a, b) \right) (x', y'). \quad (19)$$

where $\tilde{u}^{\lambda,\eta}(a, b)$ is given in (18).

In subsequent sections, this formula will be explicitly computed in the examples mentioned in the introduction.

In a similar fashion, the Szegö operator, $P$, representing the projection onto the zero eigenspace in the second case of Theorem 12 can be computed. We obtain the following result.

**Proposition 14.** Suppose $\nu(\lambda) = n$ (so there is no $\eta$); if $\mu_j^\lambda < 0$ for $j \in L$ and $\mu_j^\lambda > 0$ for $j \not\in L$, then the operator representing the projection onto the zero-eigenspace of $\Box_L$ is given by a (group) convolution with the kernel $P(z, t)$ whose $t$-Fourier transform is given by

$$P(x, y, \widehat{\lambda}) = (2\pi)^{-(n+m/2)} \prod_{j=1}^{n} |\mu_j^\lambda| \left| e^{-|\mu_j^\lambda|(x_j^2+y_j^2)} \right|.$$ 

This can be easily inverted using the inverse Fourier transform in $\lambda$ to recover the classical formulas for the Szegö kernel.

### 6. Calculation of $N$

**6.1. Reductions for the General Case.** Our goal is now to try to unravel the formula (19) for $N(x, y, \widehat{\eta}, \widehat{\lambda})$ under the hypothesis in the first part of Theorem 12. The discussion in this section will apply to the general case. Subsequent sections will address the specific cases given in Theorems 4 and 3.

We rewrite (19) as

$$N(x', y', \widehat{\eta}, \widehat{\lambda}) = (2\pi)^{-\frac{1}{2}(2n+m-\nu(\lambda))} e^{-2i \sum_{j=1}^{n} a_j x_j y_j} \sum_{\ell \in \mathbb{Z}_+^n} \frac{(-i)^{\ell}}{\prod_{j=1}^{\nu(\lambda)} \mathcal{F}_{a_j}^{-1} \left( \psi_{\ell_j} (a_j / 2 |\mu_j^\lambda|^{1/2}) \mathcal{F}_{b_j}^{-1} \left( e^{(-i/4) a_j b_j / \mu_j^\lambda} \psi_{\ell_j} (b_j |\mu_j^\lambda|^{1/2} / -2 \mu_j^\lambda) \right) (y_j) \right) \left( x_j \right).$$

Here, we are using the following notation for the one-variable Fourier transform:

$$\mathcal{F}_b^{-1}(g)(y) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} g(b) e^{iyb} \, db$$


and similarly for $F^{-1}_a(g)(x)$. We compute the above partial inverse Fourier transform expression in $a_j$:

$$
F^{-1}_a \left\{ \psi_{\ell_j} \left( \frac{a}{2|\mu|^{1/2}} \right) \frac{1}{2} \left( e^{-i \frac{\mu b}{|\mu|^{1/2}}} \right) \left( \psi_{\ell_j} \left( \frac{b|\mu|^{1/2}}{2\mu} \right) \right) (y) \right\} (x)
$$

$$
= 4|\mu| F^{-1}_a \left\{ \psi_{\ell_j} (a) \psi_{\ell_j} \left( a - 2 \frac{\mu}{|\mu|^{1/2}} y \right) \right\} (2|\mu|^{1/2} x) 
$$

$$
= 4|\mu| (-1)^{\ell_j} F^{-1} \left\{ F \psi_{\ell_j} (a) F \left( e^{i \frac{\mu b}{|\mu|^{1/2}} y a} \psi_{\ell_j} (a) \right) \right\} (2|\mu|^{1/2} x) 
$$

$$
= \frac{4|\mu|}{\sqrt{2\pi}} (-1)^{\ell_j} \int_{\mathbb{R}} \psi_{\ell_j} (a) e^{i \frac{\mu b}{|\mu|^{1/2}} y (2|\mu|^{1/2} x - a)} \psi_{\ell_j} (2|\mu|^{1/2} x - a) \, da 
$$

$$
= \frac{4|\mu|}{\sqrt{2\pi}} (-1)^{\ell_j} e^{i 4 \mu x y} \int_{\mathbb{R}} \psi_{\ell_j} (a) e^{-2i a y \mu / |\mu|^{1/2}} \psi_{\ell_j} (2|\mu|^{1/2} x - a) \, da.
$$

Thus, we have the following lemma.

**Lemma 15.** In the first case of Theorem 12 where the eigenvalues, $\Lambda_{\ell}^{\lambda, n}$, are nonzero for $\lambda \neq 0$,

$$
N(z, \tilde{\eta}, \lambda) = \left(2\pi\right)^{-n - \frac{m}{2}} 4^{\nu(\lambda)} e^{2i \sum_{j=1}^{n} \mu_j^2 x_j y_j}
$$

$$
\sum_{\ell \in \mathbb{Z}_F, \lambda \neq 0} (-1)^{\ell_j} \prod_{j=1}^{n} \left| \psi_{\ell_j} (a_j) \psi_{\ell_j} (2|\mu_j|^{1/2} x_j - a_j) e^{-2i a y \mu / |\mu|^{1/2}} \right| da_j
$$

where

$$
\Lambda_{\ell}^{-\lambda} = \sum_{j=1}^{n} (2\ell_j + 1)|\mu_j^2| - \left( \sum_{k \in L} \mu_k^2 - \sum_{k \notin L} \mu_k^2 \right) + \left| \eta \right|^2 / 4.
$$

6.2. **The Case when the Eigenvalues have the same Absolute Value.** In this section, we assume $\nu(\lambda) = n$ (so there is no $\eta$ variables) and that all the $\mu_j^2$ have the same absolute value, for each $\lambda \in \mathbb{R}^m$. In this case, we can rescale and assume

$$
|\mu_j^2| = |\lambda|, \text{ for all } 1 \leq j \leq n
$$

and write

$$
\mu_j^\lambda = \sigma_j^\lambda |\lambda| \text{ where } \sigma_j^\lambda = \pm 1.
$$

This assumption, though restrictive, will allow us to recover $N$ for the Heisenberg group, as well as for the case of $M_2$ with real codimension two, given in Theorem 3. We will further assume there are no zero eigenvalues as in the first case of Theorem 12. In this case, we have

$$
\Lambda_{\ell}^{-\lambda} = 2(\sum_{j=1}^{n} \ell_j + J)|\lambda|
$$

where $J = J(\lambda)$ is a positive integer. In the case of the Heisenberg group with $\phi(z, z) = -|z|^2$ and $L$ is an index of length $q$, then $J = q$ if $\lambda < 0$ and $J = n - q$ if $\lambda > 0$. If $1 \leq q \leq n - 1$, then $J$ is a positive integer.
From Lemma 15, \( N \) can be written as

\[
N(z, \lambda) = (2\pi)^{-n - \frac{m}{2}} 4^n |\lambda|^{n-1} e^{2i(\sigma x - y) |\lambda|} \sum_{k=0}^{\infty} \frac{(-1)^k}{2(k + J)} \sum_{|\ell| = k} \int_{a \in \mathbb{R}^n} \Psi_\ell(a) \Psi_\ell(2|\lambda|^{1/2}x - a) e^{-2i(\sigma y - a)|\lambda|^{1/2}} \]

where

\[
\Psi_\ell(t) = \psi_{\ell_1}(t_1) \cdots \psi_{\ell_n}(t_n) \quad \text{and} \quad \sigma x = (\sigma_1 x_1, \ldots, \sigma_n x_n).
\]

Now we use Mehler’s formula (see [Tha93]), for each fixed \( x, y \in \mathbb{R}^n \)

\[
\sum_{k=0}^{\infty} \frac{(-1)^k}{2(k + J)} \sum_{|\ell| = k} \Psi_\ell(x) \Psi_\ell(y) = \phi(-r), \quad |r| < 1
\]

where

\[
\phi(r) = \frac{1}{\pi^{n/2}(1 - r^2)^{n/2}} e^{-(\frac{1}{1+r})^2(x+y)^2/4 - (\frac{1}{1+r})^2(x-y)^2/4}.
\]

The key idea is to now multiply Mehler’s formula by \( r^{J-1} \) and integrate \( r \) over the interval \( 0 \leq r < 1 \):

\[
\sum_{k=0}^{\infty} \frac{(-1)^k}{2(k + J)} \sum_{|\ell| = k} \Psi_\ell(x) \Psi_\ell(y) = \int_0^1 r^{J-1} \phi(-r) \, dr.
\]

Note that there is no problem with convergence of the integral on the right since \( J \) is a positive integer. Applying this formula to (20) with \( x \) replaced by \( a \) and \( y \) replaced by \( 2|\lambda|^{1/2}x - a \) gives

\[
N(z, \lambda) = \frac{(2\pi)^{-n - \frac{m}{2}} 4^n |\lambda|^{n-1}}{2\pi^{n/2}} e^{2i(\sigma x - y) |\lambda|} \int_{a \in \mathbb{R}^n} \int_0^1 \frac{r^{J-1}}{(1 - r^2)^{n/2}} e^{-2i(\sigma y - a)|\lambda|^{1/2}} e^{-(\frac{1}{1+r})|\lambda|^{1/2}(x-y)^2} \, da \, dr.
\]

Now integrate out \( a \) (completing the square in the exponential, etc.) to obtain

\[
N(z, \lambda) = \frac{(2\pi)^{-n - \frac{m}{2}} 4^n |\lambda|^{n-1}}{2} \int_0^1 \frac{r^{J-1}}{(1 - r^2)^{n/2}} \left( 1 + r \right)^{n/2} \left( \frac{1 + r}{1 - r} \right)^{n/2} e^{-(\frac{1}{1+r})|\lambda|^2} \, dr.
\]

If \( z \neq 0 \), then the above integral converges. Now set \( s = (1 + r)/(1 - r) \) and then translate \( s \) by one unit and we obtain the following lemma.

**Lemma 16.** In the case where \( J > 0 \), as above

\[
N(z, \lambda) = (2\pi)^{-n - \frac{m}{2}} 2^n |\lambda|^{n-1} \int_0^s s^{J-1} (s + 2)^{n-J-1} e^{-(s+1)|\lambda|^2} \, ds.
\]

This integral can be computed using integration by parts. \( N \) can then be computed using the inverse Fourier transform in the \( \lambda \) variable, provided the coordinates \( z^\lambda \) from Section 3.3 vary continuously in \( \lambda \). In the Heisenberg group (where \( \phi(z, z) = -|z|^2 \)), the classical formulas for \( N \) (see [Ste93]) can be determined by computing the inverse Fourier transform in \( \lambda \) and separating out the integral over \( \lambda > 0 \), where \( J = q \) and \( \lambda < 0 \), where \( J = n - q \).
6.3. Proof of Theorem 3 for $M_2$. For $\lambda \in \mathbb{R}^2$, let $\phi^\lambda : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{R}$ be defined by $\phi^\lambda(z, z') = \phi(z, z') \cdot \lambda$ where $\phi$ is the defining function for $M_2$ given in Theorem 3. It is easy to compute that the Hermitian form associated with $\phi^\lambda$ has two eigenvalues: $\mu_1^\lambda = |\lambda|$ and $\mu_2^\lambda = -|\lambda|$. For $\lambda = |\lambda|(\cos \theta, \sin \theta) \in \mathbb{R}^2$, the associated eigenvectors are $v_1^\lambda = \frac{1}{\sqrt{2}} \left( \frac{\cos \theta}{\sqrt{1 - \sin^2 \theta}}, \sqrt{1 - \sin^2 \theta} \right)$ and $v_2^\lambda = \frac{1}{\sqrt{2}} \left( -\sqrt{1 - \sin^2 \theta}, \frac{\cos \theta}{\sqrt{1 - \sin^2 \theta}} \right)$, respectively. These eigenvectors vary smoothly with $\theta$.

Since the eigenvalues have the same absolute value, we can compute $N$ using the techniques in the previous section. Solvability of $\Box_b$ is expected for $(0, q)$-forms where $q = 0, 2$, but not for $q = 1$. For both $q = 0, 2$, $\Lambda^{-\lambda} = 2(\ell_1 + \ell_2 + 1)$ and so we can use Lemma 16 with $J = 1$ to compute

$$N(z, \lambda) = \frac{1}{2\pi^3} |\lambda| \int_0^\infty e^{-\lambda|z|^2} ds = \frac{2e^{-|\lambda||z|^2}}{|z|^2}.$$  

Using the inverse Fourier transform in $t \in \mathbb{R}^2$, we obtain

$$N(z, t) = \frac{1}{4\pi^4|z|^2} \int_{\lambda \in \mathbb{R}^2} e^{-|\lambda||z|^2 + i\lambda t} d\lambda.$$  

Polar coordinates can be used to reduce the above integral to

$$N(z, t) = \frac{1}{4\pi^4|z|^2} \int_0^{2\pi} \frac{d\theta}{(|z|^2 - i(t_1 \cos \theta + t_2 \sin \theta))^2}.$$  

The above integrand is periodic in $\theta$. A shift in $\theta$ (specifically, $\theta \mapsto \theta - A$ where $\cos A = t_1/|t|$ and $\sin A = t_2/|t|$) can be used to reduce the denominator of the integrand to $(|z|^2 - i(|t| \cos \theta))^2$. From here, a residue calculation (or Maple) gives

$$N(z, t) = \frac{1}{2\pi^3} \frac{1}{(|z|^4 + |t|^2)^{3/2}},$$  

as stated in Theorem 3. This concludes the proof of the first part of Theorem 3. The second part (for $M_3$) will be given after the next section where we introduce techniques for handling eigenvalues which are not equal in absolute value.

6.4. Proof of Theorem 4. Here, $n = 2$ and the multiindex $L = (1, 0)$. The eigenvalues are $\mu_1^\lambda = \sigma_1 \lambda, \mu_2^\lambda = \sigma_2 \lambda$ with $\sigma_1, \sigma_2 > 0$. We first consider the case when $\lambda > 0$ and we obtain

$$\Lambda_{\ell_1, \ell_2}^{-\lambda} = 2(\ell_1|\mu_1| + (\ell_2 + 1)|\mu_2|) = 2\lambda(\ell_1 \sigma_1 + (\ell_2 + 1) \sigma_2)$$

(in this case, there is some cancellation in $\mu_1$-terms in the formula for $\Lambda_{\ell_1}^{-\lambda}$ but not in $\mu_2$). From Lemma 15, the operator $N$ becomes

$$N(z, \lambda) = \frac{8}{(2\pi)^{3/2}} e^{2i(\sigma x \cdot y)\lambda} \sum_{\ell_1, \ell_2} \frac{(-1)^{\ell_1 + \ell_2} \sigma_1 \sigma_2}{(\ell_1 \sigma_1 + (\ell_2 + 1) \sigma_2)} \int_{a \in \mathbb{R}^2} E(a, x, y, \lambda) da$$

(21)

where

$$E(a, x, y, \lambda) = \prod_{j=1}^2 \psi_{\ell_j}(a_j) \psi_{\ell_j}(2\sigma_j^{1/2} \lambda^{1/2} x_j - a_j) e^{-2i\sigma_j^{1/2} y_j a_j \lambda^{1/2}}.$$  

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This time, we use Mehler’s formula with fractional powers of \( r \):
\[
\sum_{\ell_1, \ell_2} (-r^{\sigma_1})^{\ell_1} (-r^{\sigma_2})^{\ell_2} \psi_{\ell_1}(X_1) \psi_{\ell_1}(Y_1) \psi_{\ell_2}(X_2) \psi_{\ell_2}(Y_2)
\]
\[
= \frac{1}{\pi} \prod_{j=1}^{2} \frac{1}{(1 - r^{2\sigma_j})^{1/2}} e^{\left(\frac{1 - \sigma_j^2}{1 + \sigma_j^2}\right)(X_j + Y_j)^2/4 - \left(\frac{1 - \sigma_j^2}{1 + \sigma_j^2}\right)(X_j - Y_j)^2/4}
\]

where \( X_j, Y_j \in \mathbb{R} \). Multiplying this expression by \( r^{\sigma_2 - 1} \), integrating from \( r = 0 \) to \( r = 1 \) and setting \( X_j = a_j \) and \( Y_j = 2\lambda^{1/2} \sigma_j^{1/2} x_j - a_j, j = 1, 2 \), gives
\[
\sum_{\ell_1, \ell_2} (-1)^{\ell_1 + \ell_2} \prod_{j=1}^{2} \psi_{\ell_j}(a_j) \psi_{\ell_j}(2\lambda^{1/2} \sigma_j^{1/2} x_j - a_j) e^{-2i(\sigma_j^{1/2} y_j a_j)\lambda^{1/2}}
\]
\[
= \frac{1}{\pi} \int_{0}^{1} \frac{r^{\sigma_2 - 1}}{(1 - r^{2\sigma_1})^{1/2}(1 - r^{2\sigma_2})^{1/2}} \prod_{j=1}^{2} e^{-\lambda \sigma_j \left| \frac{j^2}{1 + \sigma_j} \right| |z|^2} dr.
\]

We then integrate out \( a \in \mathbb{R}^2 \) (completing the squares in the exponent, etc.) and simplify to obtain
\[
N(z, \lambda) = \frac{8}{(2\pi)^{5/2}} \lambda \sigma_1 \sigma_2 \int_{0}^{1} \frac{r^{\sigma_2 - 1}}{(1 - r^{\sigma_1})(1 - r^{\sigma_2})} \prod_{j=1}^{2} e^{-\lambda \sigma_j \left| \frac{j^2}{1 + \sigma_j} \right| |z|^2} dr. \quad (22)
\]

When \( \lambda \) reverses sign, then so do the \( \mu_j^{\lambda} \). As a result, the \( \mu_j^{2} \) cancels instead of the \( \mu_j^{\lambda} \) in the expression for \( \Lambda_{\ell}^{-\lambda} \). The above computation goes through unchanged except that \( r^{\sigma_2 - 1} \) replaces \( r^{\sigma_2 - 1} \) and \( |\lambda| \) replaces \( \lambda \). We then obtain
\[
N(z, \lambda) = \frac{8}{(2\pi)^{5/2}} |\lambda| \sigma_1 \sigma_2 \int_{0}^{1} \frac{r^{\sigma_2 - 1}}{(1 - r^{\sigma_1})(1 - r^{\sigma_2})} \prod_{j=1}^{2} e^{-|\lambda| \sigma_j \left| \frac{j^2}{1 + \sigma_j} \right| |z|^2} dr.
\]

We can evaluate the inverse Fourier transform in \( \lambda \) by computing two integrals: one where \( \lambda > 0 \) and the other where \( \lambda < 0 \). After a simple integration by parts we obtain:
\[
N(z, t) = \frac{1}{\pi^3} \int_{0}^{1} \frac{\sigma_1 \sigma_2}{(it + s_1(r)\sigma_1 |z_1|^2 + s_2(r)\sigma_2 |z_2|^2)^2} \left(\frac{1 - r^{\sigma_1}}{1 - r^{\sigma_2}}\right) r^{\sigma_1 - 1} dr
\]
\[
+ \frac{1}{\pi^3} \int_{0}^{1} \frac{\sigma_1 \sigma_2}{(it + s_1(r)\sigma_1 |z_1|^2 + s_2(r)\sigma_2 |z_2|^2)^2} \left(\frac{1 - r^{\sigma_1}}{1 - r^{\sigma_2}}\right) r^{\sigma_2 - 1} dr
\]

where
\[
s_j(r) = \frac{1 + r^{\sigma_j}}{1 - r^{\sigma_j}} \text{ for } j = 1, 2.
\]

This completes the proof of Theorem \( \square \)

**Remark.** The same process can be used to evaluate \( N(z, t) \) in cases where \( n > 2 \) or where the eigenvalues of the Levi form (the \( \sigma_j \)) have both positive and negative terms.
6.5. **Proof of Theorem 2.** The quadric hypersurface of interest in Theorem 2 is

\[ M = \{(z_1, z_2, z_3, w) \in \mathbb{C}^4; \ \text{Re} \ w = |z_1|^2 + |z_2|^2\}. \]

Note that this quadric has a Levi form with diagonal entries 1, 1, 0 corresponding to the directions \(z_1, z_2, z_3\), respectively. The \((0,1)\) forms under consideration are spanned by \(dz_1\) and \(dz_2\) (but not \(dz_3\)). Our starting point is the formula for \(N(x', y', \hat{\eta}, \hat{\lambda})\) given in Lemma 15 with \(n = 3, m = 1, \nu(\lambda) = 2\). In addition, \(z' = x' + iy' = (z_1, z_2); \eta\) is the Fourier transform variable associated to the zero-eigendirection variable \(z_3\); \(\lambda\) is the Fourier transform variable associated with \(t = \text{Re} \ w; \mu_1^\lambda = \mu_2^\lambda = \lambda\). We therefore have

\[ \Lambda_{r-\lambda,-\eta/2} = 2|\lambda|(|\ell| + 1) + \frac{|\eta|^2}{4} \]

(note that the term \(\sum_{j \in L} \mu_j^\lambda - \sum_{j \not\in L} \mu_j^\lambda\) is zero since \(\mu_j^\lambda = \lambda, j = 1, 2\) and one of these indices belongs to \(L\) and one does not).

By Lemma 15,

\[ N(z', \hat{\eta}, \hat{\lambda}) = \frac{16|\lambda|^2}{(2\pi)^7/2} e^{2i \sum_{j=1}^{\infty} \lambda x_j y_j} \sum_{k=0}^{\infty} \frac{(-1)^k}{2|\lambda|(k+1) + |\eta|^2/4} \sum_{|\ell|=k} \prod_{j=1}^{2} \int_\mathbb{R} \psi_{\ell_j}(a_j) \psi_{\ell_j}(2|\lambda|^{1/2} x_j - a_j) e^{-2i\lambda y_j a_j/|\lambda|^{1/2}} da_j. \]

Now the idea is to write

\[ \frac{(-1)^k}{2|\lambda|(k+1) + |\eta|^2/4} = \frac{1}{2|\lambda|} \int_0^1 (-1)^k r^{k + |\eta|^2/8|\lambda|^2} dr. \]

The computations in the previous section (using Mehler’s formula and integrating out \(a\)) can then be repeated with minor modifications to show the following formula for \(N(z', \hat{\eta}, \hat{\lambda})\) which is analogous to (22):

\[ N(z', \hat{\eta}, \hat{\lambda}) = \frac{8|\lambda|}{(2\pi)^7/2} \int_0^1 e^{-|\ln r| |\eta|^2/8|\lambda|^2} \frac{1}{(1 - r)^2} e^{-|\lambda| z'^2(1 + \frac{1}{1 - r})} dr. \]

Note that the above expression is a Gaussian in \(\eta\) and so its inverse Fourier transform in \(\eta\) is easily calculated:

\[ N(z', z_3, \hat{\lambda}) = \frac{32|\lambda|^2}{(2\pi)^7/2} \int_0^1 \frac{1}{|\ln r|(1 - r)^2} e^{-|\lambda| (|z'|^2(1 + \frac{1}{1 - r}) + |z_3|^2 \frac{2}{|\ln r|})} dr. \]

The inverse Fourier transform in \(\lambda\) is now easily calculated:

\[ N(z, t) = \frac{8}{\pi^4} \int_0^1 \frac{dr}{|\ln r|(1 - r)^2} \text{Re} \left\{ \frac{1}{\left[|z'|^2 \left(\frac{1}{1 - r}\right) + |z_3|^2 \frac{2}{|\ln r|} + it\right]^3} \right\} \]

which establishes Theorem 2.
6.6. Proof of Theorem 3 for $M_3$. For $\lambda \in \mathbb{R}^2$, let $\phi^\lambda : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{R}$ be defined by $\phi^\lambda(z, z') = \phi(z, z') \cdot \lambda$ where $\phi$ is the defining function for $M_3$ given in Theorem 3. We write $\lambda \in \mathbb{R}^2$ in polar coordinates $\lambda = |\lambda|(\cos \theta, \sin \theta)$. It is easy to compute that the Hermitian form $\phi^\lambda$ has two eigenvalues: $\mu_1^\lambda = |\lambda|(\cos \theta + 1)$ and $\mu_2^\lambda = |\lambda|(\cos \theta - 1)$. The associated eigenvectors are $v_1^\lambda = \frac{1}{\sqrt{2}} \left( \frac{\sin \theta}{\sqrt{1 - \cos \theta}}, \frac{\sqrt{1 - \cos \theta}}{\sqrt{1 - \cos \theta}} \right)$, and $v_2^\lambda = \frac{1}{\sqrt{2}} \left( \frac{\sqrt{1 - \cos \theta}}{\sqrt{1 - \cos \theta}}, \frac{\sin \theta}{\sqrt{1 - \cos \theta}} \right)$, which vary smoothly in $\theta$. Since the eigenvalues have opposite sign for all $\theta$ except $\theta = 0$, we expect solvability of $\overline{\partial}_b$ on $(0, q)$-forms for $q = 0, 2$. We first assume $q = 0$. We see that

$$\Lambda^{-\lambda}_{\ell_1, \ell_2} = (2\ell_1 + 1)|\mu_1^\lambda| + (2\ell_2 + 1)|\mu_2^\lambda| + \mu_1^\lambda + \mu_2^\lambda = 2|\lambda|((\ell_1 + 1)\sigma_1 + \ell_2\sigma_2)$$

where $\sigma_1 = (1 + \cos \theta)$ and $\sigma_2 = (1 - \cos \theta)$. Repeating the same arguments from the last section, we obtain

$$N(z, \lambda) = \frac{\sigma_1 \sigma_2 |\lambda|}{\pi^3} \int_0^1 \frac{r^{\sigma_1 - 1}}{(1 - r^{\sigma_1})(1 - r^{\sigma_2})} \prod_{j=1}^2 e^{-|\lambda| r^{\sigma_j} \frac{\sigma_j - |\lambda|}{1 - r^{\sigma_j}} |z_j|^2} dr.$$

We now take the inverse Fourier transform of this expression in $\lambda \in \mathbb{R}^2$ using polar coordinates. Integrating $|\lambda|$ from 0 to infinity is a straight forward integration by parts. However, the $\theta$ and $r$ integrals cannot be evaluated in closed form. The result is

$$N(z, t) = \frac{1}{\pi^2} \int_0^1 \int_0^{2\pi} \sigma_1(\theta) \sigma_2(\theta) \frac{r^{\sigma_1 - 1}}{(1 - r^{\sigma_1})(1 - r^{\sigma_2})} \times \frac{2 d\theta dr}{(-i(t_1 \cos \theta + t_2 \sin \theta) + E_1(\theta, r)|z_1|^2 + E_2(\theta, r)|z_2|^2)^3}$$

where

$$\sigma_1 = \sigma_1(\theta) = 1 + \cos \theta, \quad \sigma_2 = \sigma_2(\theta) = 1 - \cos \theta, \quad E_j(\theta, r) = \frac{\sigma_j(1 + r^{\sigma_j})}{1 - r^{\sigma_j}}, \quad j = 1, 2$$

as stated in Theorem 3. In the case where $q = 2$, $\Lambda^{-\lambda}_{\ell_1, \ell_2}$ becomes $2|\lambda|((\ell_1 + 1)\sigma_1 + (\ell_2 + 1)\sigma_2)$. This change results in a factor of $r^{\sigma_1 - 1}$ in the numerator of $N$ instead of the factor of $r^{\sigma_1 - 1}$. This completes the proof of Theorem 3.

References

[BG88] R. Beals and P. Greiner. Calculus on Heisenberg manifolds, volume 119 of Annals of Mathematics Studies. Princeton University Press, 1988.

[BGG96] Richard Beals, Bernard Gaveau, and Peter Greiner. The Green function of model step two hypoelliptic operators and the analysis of certain tangential Cauchy Riemann complexes. Adm. Math., 121(2):288–345, 1996.

[Bog91] Albert Boggess. CR Manifolds and the Tangential Cauchy-Riemann Complex. Studies in Advanced Mathematics. CRC Press, Boca Raton, Florida, 1991.

[BR09] A. Boggess and A. Raich, A simplified calculation for the fundamental solution to the heat equation on the Heisenberg group. Proc. Amer. Math. Soc., 137(3):937–944, 2009.

[BR11] A. Boggess and A. Raich, The $\overline{\partial}_b$-heat equation on quadric manifolds. J. Geom. Anal., 21:256–275, 2011.

[CCT06] O. Calin, D.-C. Chang, and J. Tie. Fundamental solutions for Hermite and subelliptic operators. J. Anal. Math., 100:223–248, 2006.

[FS74a] G.B. Folland and E. Stein. Parametrics and estimates for the $\overline{\partial}_b$ complex on strongly pseudoconvex boundaries. Bull. Amer. Math. Soc., 80:253–258, 1974.
[FS74b] G.B. Folland and E.M. Stein. Estimates for the $\partial_b$-complex and analysis on the Heisenberg group. Comm. Pure and Appl. Math., 27:429–522, 1974.

[NS06] A. Nagel and E.M. Stein. The $\partial_b$-complex on decoupled domains in $\mathbb{C}^n$, $n \geq 3$. Ann. of Math., 164:649–713, 2006.

[NSW85] A. Nagel, E.M. Stein, and S. Wainger. Balls and metrics defined by vector fields I: Basic properties. Acta Math., 155:103–147, 1985.

[PR03] M. Peloso and F. Ricci. Analysis of the Kohn Laplacian on quadratic CR manifolds. J. Funct. Anal., 2003(2):321–355, 2003.

[Rai] A. Raich. Heat equations and the weighted $\partial$-problem. to appear, Commun. Pure Appl. Anal. arXiv:0704.2768.

[Rai06] A. Raich. Heat equations in $\mathbb{R} \times \mathbb{C}$. J. Funct. Anal., 240(1):1–35, 2006.

[Rai07] A. Raich. Pointwise estimates of relative fundamental solutions for heat equations in $\mathbb{R} \times \mathbb{C}$. Math. Z., 256:193–220, 2007.

[Ste93] Elias M. Stein. Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton Mathematical Series; 43. Princeton University Press, Princeton, New Jersey, 1993.

[Tay86] Michael E. Taylor. Noncommutative Harmonic Analysis, volume 22 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, Rhode Island, 1986.

[Tha93] Sundaram Thangavelu. Lectures on Hermite and Laguerre Expansions, volume 42 of Mathematical Notes. Princeton University Press, Princeton, New Jersey, 1993.

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