1. Introduction

In this paper we give a new Chern-Weil type construction of characteristic classes for fiber bundles by using homotopy theory of $A_\infty$-algebras. For vector bundles, there are several constructions of characteristic classes. In particular, Chern-Weil theory is a beautiful theory to get characteristic classes from additional objects as connections or metrics. For fiber bundles which are not principal bundles with compact Lie groups, it is difficult to get Chern-Weil-type theory because fiber bundles have diffeomorphism groups as structure groups. Our idea is to replace a family of closed manifolds as fibers to a family of $A_\infty$-morphisms with family of metrics on the fibers. Then, we get Lie algebra valued differential forms on the base manifold and characteristic classes from Maurer-Cartan forms on moduli spaces of $A_\infty$-morphisms.

In order to carry out this program, we need the basics of the homotopy theory on $A_\infty$-algebras, in particular a notion of homotopy between $A_\infty$-maps, which was developed for instance in [2, 3, 6, 8, 9, 10, 11, 17, 18, 20, 22, 24]. Remark that [3] treats filtered $A_\infty$-algebras which are important for an application to Floer homology and mirror symmetry.

A strong tool is the decomposition theorem which states that any $A_\infty$-algebra is $A_\infty$-isomorphic to the direct sum of a minimal $A_\infty$-algebra and a linear contractible $A_\infty$-algebra. The decomposition theorem was first mentioned in [15]. A proof was given in [11, 10]. See [2] for a filtered version. Our construction is based on facts which are obtained from the decomposition theorem. An important fact that any $A_\infty$-quasi-isomorphism has its homotopy inverse was first proved in [3] by a different method. See [7, 9, 22] for related results about homotopy inverse with an other version of homotopy.

Our construction of characteristic classes can be seen as a higher homotopy group version of a construction in [21] which use Chen expansions on fundamental groups. For punctured-surface bundles whose fundamental groups are free, Chen expansions were used to get Morita-Mumford-Miller classes in [12, 13] before [21]. It would be interesting to compare our construction with rational homotopy theory on fiber bundles in a wonderful paper [23].

This paper is organized as follows. In Section 2, we discuss the basics of the homotopy theory on $A_\infty$-algebras and obtain key facts for our construction of characteristic classes from the decomposition theorem. In Section 3, we introduce automorphism groups and coderivation Lie algebras associated to minimal $A_\infty$-algebras. Using these tools, in Section 4 we construct characteristic classes of fiber bundles in the following two cases. In Subsection 4.1 we construct them for fiber bundles which are homologically trivial. This case corresponds to the restriction to the
Torelli groups when the fiber is a surface. In Subsection 4.2, we do the same for a fiber bundle whose the fiber is a formal manifold under an additional technical assumption. In Subsection 4.3, we show that there is a commutative diagram to relate these constructions with the construction using the fundamental group in [21].

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## 2. Homotopy Theory of $A_{\infty}$-algebras

Throughout this section, all vector spaces are those over a fixed base field $K$ of characteristic zero.

### 2.1. $A_{\infty}$-algebras

We begin by recalling the notions of an $A_{\infty}$-algebra and an $A_{\infty}$-morphism. See [14] for an introduction to $A_{\infty}$-algebras.

**Definition 2.1** ($A_{\infty}$-algebra [25, 26]). Let $A$ be a $\mathbb{Z}$-graded vector space and $m = \{m_n : A^\otimes n \rightarrow A\}_{n \geq 1}$ be a family of linear maps with deg $m_n = 2 - n$. The pair $(A, m)$ satisfying the $A_{\infty}$-relations

$$\sum_{k+l=n+1} \sum_{j=0}^{k-1} (-1)^{(l+1)(l+1)} m_k \circ (\text{id}_A^j \otimes m_l \otimes \text{id}_A^{(n-j-l)}) = 0$$

for $n \geq 1$ is called an $A_{\infty}$-algebra. Then $m$ is called an $A_{\infty}$-structure on $H$.

The multilinear map $m_k$ has degree $(2 - k)$ indicates the degree of $m_k(a_1, \ldots, a_k)$ is $|a_1| + \cdots + |a_k| + (2 - k)$. The $A_{\infty}$-relations implies $(m_1)^2 = 0$ for $n = 1$, the Leibniz rule of the differential $m_1$ with respect to the product $m_2$ for $n = 2$, and the associativity of $m_2$ up to homotopy for $n = 3$. These facts further imply that the cohomology $H(A, m_1)$ has the structure of a (non-unital) algebra, where the product is induced from $m_2$.

Note that the product $m_2$ is strictly associative in $A$ if $m_3=0$.

**Definition 2.2.** Let $(A, m)$ be an $A_{\infty}$-algebra.

- If higher products are all zero, i.e. $m_3 = m_4 = \cdots = 0$, $(A, m)$ is called a differential graded algebra (DGA).
- If $m_1 = 0$, $(A, m)$ is called minimal.
- If there exists an element $1 \in A$ which satisfies the equations

$$m_2(x, 1) = m_2(1, x) = x, \quad m_n(x_1, \ldots, 1, \ldots, x_n) = 0 \quad (n \neq 2)$$

for $x, x_1, \ldots, x_n \in A$, the triple $(A, 1, m)$ is called unital $A_{\infty}$-algebra.

**Remark 2.3** (Bar construction of an $A_{\infty}$-algebra). Let $(A, m)$ be an $A_{\infty}$-algebra and $s : A \rightarrow A[1]$ be the suspension map. Defining the suspension of $m_n$ by $\tilde{m}_n := s \circ m_n \circ (s^{-1})^\otimes n$ for all $n \geq 1$, then the degree of $\tilde{m}_n$ is 1 and the $A_{\infty}$-relations are rewritten as the simpler equations

$$\sum_{k+l=n+1} \sum_{j=0}^{k-1} \tilde{m}_k \circ (\text{id}_A[l] \otimes \tilde{m}_l \otimes \text{id}_A[l]) = 0$$
(Getzler-Jones [4]). We denote the (counital) tensor coalgebra of $A[1]$ by

$$T^c(A[1]) := \bigoplus_{n=0}^{\infty} A[1]^{\otimes n},$$

where the coproduct $\Delta : T^c(A[1]) \to T^c(A[1]) \otimes T^c(A[1])$ of $T^c(A[1])$ is defined by

$$\Delta(a_1 \otimes \cdots \otimes a_n) := \sum_{k=0}^{n} (a_1 \otimes \cdots \otimes a_k) \otimes (a_{k+1} \otimes \cdots \otimes a_n)$$

for $a_1, \ldots, a_n \in A[1]$. Then $\bar{m}_n : A[1]^{\otimes n} \to A[1]$ extents the unique coderivation $m_n : T^c(A[1]) \to T^c(A[1])$ by the co-Leibniz rule $\Delta \circ m_n = (m_n \otimes \text{id} + \text{id} \otimes m_n) \circ \Delta$. Setting

$$m = \sum_{n=1}^{\infty} m_n \in \text{Coder}(T^c(A[1])),$$

then $m$ is a degree 1 coderivation, i.e. $m^2 = 0$, from the $A_\infty$-relations of $m$. Thus an $A_\infty$-algebra $(A, m)$ is equivalent to a differential graded coalgebra (DGCA) $(T^c(A[1]), m)$. The DGCA $(T^c(A[1]), m)$ is called the bar construction of $(A, m)$.

**Definition 2.4** ($A_\infty$-morphism). Let $(A, m)$ and $(A', m')$ be $A_\infty$-algebras. A family $f = \{f_n : A^{\otimes n} \to A'\}$ of linear maps of deg $f_n = 1 - n$ satisfying the equations

$$\sum_{\sum i \geq 1, \ k_1 + \cdots + k_i = n} (-1)^{\sum j=1 k_j (l-j) + \sum \nu<\mu k_\nu k_\mu} m'_i \circ (f_{k_1} \otimes \cdots \otimes f_{k_i})$$

$$= \sum_{i+j=k, \ i+l+j=n} (-1)^{j+k+n+(i+1)(l+1)} f_k \circ (\text{id}_A^{\otimes i} \otimes m_l \otimes \text{id}_A^{\otimes j})$$

is called an $A_\infty$-morphism $f : (A, m) \to (A', m')$.

- If $f_1$ is a linear isomorphism, $f$ is called an $A_\infty$-isomorphism.
- If $f_2 = f_3 = \cdots = 0$, $f$ is called a linear $A_\infty$-morphism.
- If $(A, 1, m)$ and $(A', 1', m')$ is unital $A_\infty$-algebras and the equations

$$f_1(1) = 1', \ f_n(x_1, \ldots, 1, \ldots, x_n) = 0 \ (n \neq 1)$$

for $x, x_1, \ldots, x_n \in A$ hold, $f : (A, 1, m) \to (A', 1', m')$ is called a unital $A_\infty$-morphism.

The defining equation for $A_\infty$-morphisms for $n = 1$ implies that $f_1 : A \to A'$ forms a chain map $f_1 : (A, m_1) \to (A', m'_1)$. This together with the defining equation for $n = 2$ implies that $f_1 : A \to A'$ induces a (non-unital) algebra map from $H(A, m_1)$ to $H(A', m'_1)$. We denote it by $H(f_1) : H(A, m_1) \to H(A', m'_1)$.

**Definition 2.5.** An $A_\infty$-morphism $f : (A, m) \to (A', m')$ is called an $A_\infty$-quasi-isomorphism if $f_1 : (A, m_1) \to (A', m'_1)$ induces an isomorphism between the cohomologies of these two complexes.

**Remark 2.6** (Bar construction of an $A_\infty$-morphism). Let $f : (A, m) \to (A', m')$ be an $A_\infty$-morphism. Defining the suspension of $f_n$ by $f'_n := s \circ f_n \circ (s^{-1})^{\otimes n} : A[1]^{\otimes n} \to A'[1]$ for all $n \geq 1$, then the degree of $f_n$ is 0 and the relations for $A_\infty$-morphism are rewritten as the equations

$$\sum_{\sum i \geq 1, \ k_1 + \cdots + k_i = n} \bar{m}'_i \circ (\bar{f}_{k_1} \otimes \cdots \otimes \bar{f}_{k_i}) = \sum_{i+j=k, \ i+l+j=n} \bar{f}_k \circ (\text{id}_A^{\otimes i} \otimes \bar{m}_l \otimes \text{id}_A^{\otimes j}).$$
Constructing the coalgebra map $T^c(A[1]) \to T^c(A'[1])$

$$f = \sum_{n=1}^{\infty} \sum_{k_1+\cdots+k_n=n} \bar{f}_{k_1} \otimes \cdots \otimes \bar{f}_{k_n}$$

from maps $\bar{f}_n$, then $f$ is a DGCA map $(T^c(A[1]), m) \to (T^c(A'[1]), m')$ between bar constructions, i.e., $f \circ m = m' \circ f$ from the condition of $A_\infty$-morphism.

The composition of $A_\infty$-morphisms is defined by the composition of bar constructions of $A_\infty$-morphisms. From the definition, any $A_\infty$-isomorphism has its inverse $A_\infty$-isomorphism uniquely.

On the other hand, it is easy to see that the composition of $A_\infty$-quasi-isomorphisms is an $A_\infty$-quasi-isomorphism. An $A_\infty$-quasi-isomorphism has its inverse $A_\infty$-quasi-isomorphism in a strict sense if and only if it is an $A_\infty$-isomorphism, but always has its homotopy inverse as in Theorem 2.10. These facts imply that $A_\infty$-quasi-isomorphisms define an equivalence relation between $A_\infty$-algebras.

2.2. Decomposition theorem of $A_\infty$-algebras. A pair of minimal $A_\infty$-algebra $(H, m^H)$ and an $A_\infty$-quasi-isomorphism $f : (H, m^H) \to (A, m)$ is called a minimal model of $(A, m)$. If $(H, m^H)$, $(A, m)$ and $f$ is unital, $f : (H, m^H) \to (A, m)$ is called unital. The minimal model theorem [8] states that there always exists a minimal model for any $A_\infty$-algebra.

The following stronger theorem was first mentioned in [15], and is called the decomposition theorem. A proof was given in [11, 10]. See [2] for a filtered version.

**Theorem 2.7.** Any $A_\infty$-algebra $(A, m)$ is $A_\infty$-isomorphic to the direct sum of a minimal $A_\infty$-algebra $M$ and a linear contractible $A_\infty$-algebra $C$. Here, a linear contractible $A_\infty$-algebra $C = (C, m^C)$ is an $A_\infty$-algebra such that $m^C_1 = m^C_2 = \cdots = 0$ and the cohomology $H(C, m^C_1)$ is trivial.

**Proof.** We first choose a Hodge decomposition $(H, \iota, \pi, h)$ of the complex $(A, m_1)$, that is, $H := H(A, m_1)$ is the cohomology, $\iota : H \to A$ and $\pi : A \to H$ are linear maps of degree zero such that $\pi \circ \iota = id_H$, $h : A \to A$ is a linear map of degree minus one and they satisfy

$$m_1 h + hm_1 + P = id_A, \quad h^2 = 0$$

where $P := \iota \circ \pi$. This gives a Hodge decomposition of $(T^c(A[1]), m_1)$, as a complex of vector spaces, such that the cohomology is $T^c(A[1])$. Actually, $\iota$ and $\pi$ extend to the (linear) coalgebra maps $\iota : T^c(H[1]) \to T^c(A[1])$ and $\pi : T^c(A[1]) \to T^c(H[1])$ and one can construct a chain homotopy $h : T^c(A[1]) \to T^c(A[1])$ from $h, P$ and the identity map on $A[1]$. One such chain homotopy $h$ shall be constructed in equation (1).

We put $M := \operatorname{Im} P = \operatorname{Im} \iota$ and $C := \operatorname{Im}(m_1 h + hm_1)$. Let us consider a coalgebra homomorphism $\bar{f}^{(2)} : T^c(A[1]) \to T^c(A[1])$ defined by $\bar{f}^{(2)}_1 = id_{A[1]}$,

$$\bar{f}^{(2)}_2 := \bar{h} \bar{m}_2 - \bar{P} \bar{m}_2 \bar{h},$$

and $\bar{f}^{(2)}_3 = \cdots = 0$. This defines an $A_\infty$-isomorphism $f^{(2)} : (A, m) \to (A, m^{(2)})$, where $m^{(2)} := \bar{f}^{(2)} \circ \bar{m} \circ (\bar{f}^{(2)})^{-1}$. In particular, it turns out that $m^{(2)}_2 = P m_2 (P \otimes P)$. Thus, $m^{(2)}_2$ defines a bilinear map on $M$. Inductively, assume now that $(A, m^{(n)})$ is an $A_\infty$-algebra such that $m^{(n)}_2, \ldots, m^{(n)}_n$ define multilinear maps on $M$. We
set a coalgebra homomorphism \( f^{(n+1)} : T^c(A[1]) \to T^c(A[1]) \) by \( f_1^{(n+1)} = id_{A[1]} \),
\( f_2^{(n+1)} = f_3^{(n+1)} = \cdots = f_n^{(n+1)} = 0 \),
\( f_{n+1}^{(n+1)} := \tilde{h}m_{n+1}^{(n)} - \tilde{P}m_{n+1}^{(n)} h \),
and \( f_{n+2}^{(n+1)} = f_{n+3}^{(n+1)} = \cdots = 0 \). Then, one sees that \( m_k^{(n+1)} = m_k^{(n)} \) for \( k \leq n \) and \( m_{n+1}^{(n+1)} = Pm_{n+1}^{(n)} (P \otimes \cdots \otimes P) \). Thus, the induction is completed. For the details see \([10]\).

The decomposition theorem implies the minimal model theorem as follows. Given an \( A_\infty \)-algebra \((A, m)\) and a Hodge decomposition \((H = M, \iota, \pi, h)\) of \((A, m_1)\), by the decomposition theorem we have an \( A_\infty \)-algebra structure on \( M \oplus C \) and an \( A_\infty \)-isomorphism \( A \simeq M \oplus C \). In this situation, the pair \((\iota, \pi)\) extends to the pair of linear \( A_\infty \)-quasi-isomorphisms

\[
\begin{align*}
M & \xrightarrow{\iota} M \oplus C.
\end{align*}
\]

Thus, the composition of \( \iota \) with the \( A_\infty \)-isomorphism gives a minimal model \( M \to (A, m) \) of \((A, m)\). What is stronger, an \( A_\infty \)-quasi-isomorphism \((A, m) \to M\) is also obtained here.

Given a minimal model \((H, m_H) \to (A, m)\), the composition \((H, m_H) \to (A, m) \to M\) of the \( A_\infty \)-quasi-isomorphisms is an \( A_\infty \)-isomorphism since \( H \simeq M \). Thus, given an \( A_\infty \)-algebra \((A, m)\) , its minimal models are unique (only) up to \( A_\infty \)-isomorphisms. On the other hand, when we choose a Hodge decomposition \((H = M, \iota, \pi, h)\) of \((A, m_1)\), there exists a canonical construction of a minimal model of \((A, m)\) as presented in \([16]\). We employ this fact, too, later to construct characteristic classes of fiber bundles.

2.3. \( A_\infty \)-homotopy. For simplicity, suppose \( K = \mathbb{R} \) in this subsection.

**Definition 2.8.** Let \((C, \Delta), (C', \Delta')\) be coalgebras, and \( f : C \to C' \) be a coalgebra map. A linear map \( D : C \to C' \) satisfying

\[
\Delta' D = (f \otimes D + D \otimes f) \Delta
\]

is a coderivation over \( f \). For example, for a coderivation \( D \) on \( C' \), \( fD \) is a coderivation over \( f \). If \( f \) is a coalgebra isomorphism, all coderivations over \( f \) are obtained in such a way. Similarly for a coderivation \( D \) on \( C \), \( Df \) is a coderivation over \( f \) and the parallel fact holds.

**Definition 2.9 (\( A_\infty \)-homotopy).** Two \( A_\infty \)-morphisms \( f, g : (A, m) \to (A', m') \) are \( A_\infty \)-homotopic if there exist families of \( A_\infty \)-morphisms \( f(t) : (A, m) \to (A', m') \) and coderivations \( \eta(t) : T(A[1]) \to T(A'[1]) \) over \( f(t) \) parametrized piecewise smoothly by \( t \in [0, 1] \) such that \( f(0) = f \), \( f(1) = g \) and

\[
\frac{df}{dt}(t) = m' \circ \eta(t) + \eta(t) \circ m.
\]

Then we denote \( f \sim g \), and \( \{ (f(t), \eta(t)) \}_{t \in [0, 1]} \) is called an \( A_\infty \)-homotopy from \( f \) to \( g \).

The following theorem was first proved in \([3]\). We give a proof of it using the decomposition theorem along \([11]\). See \([7, 9, 22]\) for related results about homotopy inverse with an other version of homotopy.
Theorem 2.10. Let \((A, m)\) and \((A', m')\) be \(A\)-algebras. An \(A\)-morphism \(f : (A, m) \to (A', m')\) is an \(A\)-quasi-isomorphism if and only if \(f\) is an \(A\)-homotopy equivalence, i.e. there exists an \(A\)-morphism \(g : (A', m') \to (A, m)\) such that \(g \circ f \sim id_A\) and \(f \circ g \sim id_{A'}\).

Proof. Given a Hodge decomposition \((H = M, \iota, \pi, h)\) of \((A, m_1)\), by Theorem 2.7 we have an \(A\)-isomorphism \(A \simeq M \oplus C\), and the pair \((\iota, \pi)\) extends to the pair of linear \(A\)-quasi-isomorphisms

\[
\begin{array}{c}
M \xrightarrow{\iota} M \oplus C,
\end{array}
\]

Here we show that the projection \(P = \iota \circ \pi\) also extends to the linear \(A\)-(quasi-isomorphism) \(P : M \oplus C \to M \oplus C\) to \(M\) and it turns out to be \(A\)-homotopic to the identity \(A\)-(iso)morphism \(id_{M \oplus C}\). In fact, setting \(P_t := (1 - t)\bar{P} + t id_{A[1]} : A[1] \to A[1]\), by \(\bar{m}_1 P_t = \bar{P} \bar{m}_1\) we have

\[
\frac{d}{dt}P_t^\otimes = P_t^\otimes (id_{A[1]} - \bar{P}) \otimes P_t^\otimes = P_t^\otimes (\bar{m}_1 \bar{h} + \bar{h} \bar{m}_1) \otimes P_t^\otimes = [m, P_t^\otimes \otimes \bar{h} \otimes P_t^\otimes],
\]

where we express as \(P_t^\otimes\) the coalgebra map corresponding to \(P_t\). Thus, \(id_{M \oplus C}\) and \(P\) is \(A\)-homotopic to each other.

(1) \(\delta := \int_0^1 (P_t^\otimes \otimes \bar{h} \otimes P_t^\otimes) dt : T^c(A[1]) \to T^c(A[1])\)

gives a chain homotopy from \(id_{T^c(A[1])}\) to \(P^\otimes\). Namely, the Hodge decomposition of \((T^c(A[1]), m_1)\) is obtained.

We also choose a Hodge decomposition \((H' = M', \iota', \pi', h')\) of \((A', m'_1)\). Then we have the following diagram of \(A\)-algebras and \(A\)-(quasi-isomorphisms)

\[
\begin{array}{c}
\begin{array}{c}
A \xrightarrow{\sim} M \oplus C \xrightarrow{\pi} M = H(A, m_1) \\
\downarrow f \\
A' \xrightarrow{\sim} M' \oplus C' \xrightarrow{\pi'} M' = H(A', m'_1).
\end{array}
\end{array}
\]

and here we define \(f_H\) so that the diagram commutes. Since any composition of \(A\)-quasi-isomorphisms is an \(A\)-quasi-isomorphism, so is \(f_H\). Furthermore, since \(M\) and \(M'\) are minimal \(A\)-algebras, \(f_H\) is actually an \(A\)-isomorphism. Thus, there exists the inverse \(A\)-isomorphism \((f_H)^{-1}\). Then we define \(g\) by the commutative diagram

\[
\begin{array}{c}
\begin{array}{c}
A \xrightarrow{\sim} M \oplus C \xrightarrow{\pi} M = H(A, m_1) \\
\downarrow g \\
A' \xrightarrow{\sim} M' \oplus C' \xrightarrow{\pi'} M' = H(A', m'_1). \\
\end{array}
\end{array}
\]

Note that, in order to construct this \(g\) we need the decomposition theorem only, not the notion of \(A\)-homotopy.
Now one can show \( g \circ f \sim \text{id}_A \) and \( f \circ g \sim \text{id}_{A'} \) since they correspond to \( P \sim \text{id}_{M \oplus C} \) on \( M \oplus C \) and \( P' \sim \text{id}_{M' \oplus C'} \) on \( M' \oplus C' \), respectively. \( \square \)

From Theorem 2.10, an \( A_\infty \)-quasi-isomorphism has a homotopy inverse.

3. Automorphism groups and coderivation Lie algebras

We consider the case of \( K = \mathbb{R} \) in this section. For simplicity, the cofree coalgebra \( T^c(H[1]) \) generated by a \( \mathbb{Z} \)-graded vector space \( H \) is denoted by \( BH \).

3.1. Lie groups and their Lie rings. Let \( (H, m) \) be an \( A_\infty \)-algebra. We consider graded Lie subalgebras contained in the graded Lie algebra

\[
\text{Coder}^+(BH) := \{ D \in \text{Coder}(BH); D|_{\mathbb{R} \otimes H[1]} = 0 \}.
\]

The degree 0 part \( \text{Coder}^{+,0}(BH) \) of \( \text{Coder}^+(BH) \) is the Lie ring of the Lie group \( \text{IAut}(BH) \) of coalgebra morphisms \( f : BH \to BH \) such that \( f_1 = \text{id}_H \). The exponential map \( \exp : \text{Coder}^{+,0}(BH) \to \text{IAut}(BH) \) is bijective.

Since \( m \) is a minimal \( A_\infty \)-algebra structure, its bar construction \( m \) is in \( \text{Coder}^{+,1}(BH) \).

So the inner derivation \( \text{ad}(m) = [m, -] \) is a degree 1 differential on \( \text{Coder}^+(BH) \).

The group of \( A_\infty \)-isomorphisms \( f : (H, m) \to (H, m) \) such that \( f_1 = \text{id}_H \) is denoted by \( \text{IAut}(H, m) \). Its Lie ring is the Lie algebra of coderivations which is a chain map \( (BH, m) \to (BH, m) \), described by

\[
\text{Coder}^{+,0}_m(BH) := \text{Ker}(\text{ad}(m) : \text{Coder}^{+,0}(BH) \to \text{Coder}^{+,1}(BH)).
\]

The Lie ring of the Lie normal subgroup of \( \text{IAut}(H, m) \)

\[
\text{IAuto}(H, m) := \{ f \in \text{IAut}(H, m); f \text{ is } A_\infty \text{-homotopic to the identity} \}
\]

is \( [m, \text{Coder}^{+, -1}(BH)] \). In fact, for any \( f \in \text{IAuto}(H, m) \), there exist \( f(t) \in \text{IAut}(H, m) \) and a coderivation \( h(t) \) over \( f(t) \) such that

\[
\frac{df(t)}{dt}(t) = [m, h(t)],
\]

\( f(1) = f \) and \( f(0) = \text{id} \). Then we have

\[
\log f = \int_0^1 f(t)^{-1} df(t) = \int_0^1 [m, f(t)^{-1} h(t)] dt = \left[ m, \int_0^1 f(t)^{-1} h(t) dt \right]
\]

and which implies \( f \in \exp([m, \text{Coder}^{+, -1}(BH)]) \). Thus, the Lie ring of the quotient Lie group

\[
\text{QIAut}(H, m) := \text{IAut}(H, m)/\text{IAuto}(H, m)
\]

is the Lie algebra \( Q\text{Coder}^+_m(BH) := H^0(\text{Coder}^+(BH), \text{ad}(m)). \)

4. Construction of characteristic classes of fiber bundles

4.1. Moduli space of \( A_\infty \)-minimal models. Let \( X \) be an \( n \)-dimensional oriented closed manifold. It will be a typical fiber of a fiber bundle. We denote the reduced de Rham cohomology of \( X \) by

\[
H := \tilde{H}_{DR}^\bullet(X) = \bigoplus_{p=1}^\infty H_{DR}^p(X),
\]
which is the cohomology of the reduced de Rham complex of $X$

$$A := \check{A}^\bullet(X) = \left( \bigoplus_{p>0} A^p(X) \right) / (dA^0(X)).$$

For a minimal $A_\infty$-algebra structure $m$ on $H$, the moduli space $Q(X, m)$ of $A_\infty$-quasi-isomorphisms over $m$ is the set of $A_\infty$-homotopy classes of $A_\infty$-quasi-isomorphisms $\tau : (H, m) \to A$ such that $\tau_1$ induces the identity map on the their cohomology $H$.

The Lie group $Q\text{IAut}(H, m)$ acts on $Q(X, m)$ by

$$\tau \cdot f := \tau \circ f$$

for $\tau \in Q(X, m)$, $f \in Q\text{IAut}(H, m)$. This action is free and transitive since an $A_\infty$-quasi-isomorphism has a homotopy inverse. So $Q(X, m)$ has (the inverse limit of) smooth manifold structure which is isomorphic to $Q\text{IAut}(H, m)$.

The set $A_\infty(X)$ of minimal $A_\infty$-structures $m$ on $H$ such that $Q(X, m) \neq \emptyset$ is parametrized by the space $\text{IAut}(H, m) \setminus \text{IAut}(BH)$.

So the moduli space of $A_\infty$-minimal models of the reduced de Rham complex $A$ of $X$

$$Q(X) := \coprod_{m \in A_\infty(X)} Q(X, m)$$

is parametrized by the space $Q(X, m) \times_{\text{IAut}(H, m)} \text{IAut}(BH)$ fixing $m$. It is the space of $A_\infty$-homotopy classes of $A_\infty$-minimal models $\tau : (H, m) \to A$ such that $\tau_1$ induces the identity map on the de Rham cohomology $H$.

The mapping class group of $X$ $\mathcal{M}(X) := \text{Diff}_+(X) / \text{Diff}_0(X) = \pi_0(\text{Diff}_+(X))$ acts on $Q(X)$ as follows:

$$[\varphi] \cdot [\tau, m] := [\varphi \circ \tau \circ |\varphi|^{-1}, |\varphi| \circ m \circ |\varphi|^{-1}]$$

for $[\tau, m] \in Q(X)$ and $[\varphi] \in \mathcal{M}(X)$. Here $|\varphi|$ is the induced map on $H$ from $\varphi$. This action is well-defined since two isotopic diffeomorphisms $\varphi_0, \varphi_1$ of $X$ induce $A_\infty$-homotopic dga maps $A \to A$.

4.2. Construction. Let $E \to B$ be a smooth fiber bundle whose fiber is an oriented closed manifold $X$. For simplicity, we set

$$Q := Q(X), \ A_\infty := A_\infty(X), \ Q(m) := Q(X, m), \ M := \mathcal{M}(X).$$

Choose a smooth fiberwise metric $g$ of $E \to B$. The metric $g_b$ on fiber $E_b$ for $b \in B$ defines a Hodge decomposition on the de Rham complex $A^\bullet(E_b)$. These Hodge decompositions give unital minimal models $(H, m_b) \to A^\bullet(E_b)$ of fibers as in [16]. By restricting $(H^\bullet_{\text{DR}}(E_b), m_b) \to A^\bullet(E_b)$ to the reduced de Rham cohomologies $\check{H}^\bullet_{\text{DR}}(E_b)$, we get (non-unital) minimal models

$$\check{H}^\bullet_{\text{DR}}(E_b) \to \bigoplus_{p>0} A^p(E_b) \to \check{A}^\bullet(E_b)$$
of the reduced de Rham complexes $\hat{A}^\bullet(E_b)$. Here the $A_\infty$-algebra structure on $H^\bullet_{DR}(E_b)$ is the restriction of $m_b$. Then we can obtain the map $B \to S\backslash Q$, where $S$ is the image of the structure group of $E \to B$ in $\mathcal{M}$. Here $S\backslash Q$ plays the role of the usual classifying space of bundles with structure group $S$. Defining the de Rham complex of $S\backslash Q$ by $A^\bullet(S\backslash Q) := A^\bullet(Q)^S$, we have the map $H^\bullet_{DR}(S\backslash Q) \to H^\bullet_{DR}(B)$. Since any two metrics can be connected by a segment, this map is independent of the choice of a metric. Remark that algebraic models of classifying spaces of fibrations in the homotopy category are constructed in [23].

4.2.1. Homologically trivial bundles. We consider the case where the structure group of a fiber bundle acts trivially on the de Rham cohomology group of the fiber. In other words, suppose $S = \mathcal{I} := \text{Ker}(\mathcal{M} \to \text{GL}(H))$. Then we have a map $q : B \to A_\infty$ by giving a smooth fiberwise metric of $E \to B$. Fix $m \in A_\infty$. Since the topological group $\text{IAut}(H, m)$ is contractible, the pullback $q^* \text{IAut}(BH) \to B$ of the principal $\text{IAut}(H, m)$-bundle $\text{IAut}(BH) \to A_\infty$ is trivial. Taking a trivialization of the principal bundle, we get the $\mathcal{I}$-equivariant map

$$s : q^* Q = Q(m) \times_{\text{IAut}(H, m)} q^* \text{IAut}(BH) \simeq Q(m) \times A_\infty \to Q(m).$$

Thus we can obtain the chain map

$$A^\bullet(Q(m))^\mathcal{I} \xrightarrow{s^*} A^\bullet(q^* Q)^\mathcal{I} \to A^\bullet(B).$$

From the action of $\text{QIAut}(H, m)$, the space $Q(m)$ has the Maurer-Cartan form $\eta \in A^1(Q(m); \text{QCoder}_m^+(BH))$. Then we have the chain map

$$\Phi : C^\bullet_{CE}(\text{QCoder}_m^+(BH)) \to A^\bullet(Q(m))^\mathcal{I}.$$

Here $C^\bullet_{CE}(\text{QCoder}_m^+(BH)) := \Lambda^\bullet \text{Hom}(\text{QCoder}_m^+(BH), \mathbb{R})$ is the Chevalley-Eilenberg complex of the Lie algebra $\text{QCoder}_m^+(BH)$ introduced in section 3.1. The differential $d_{CE}$ of the Chevalley-Eilenberg complex is defined by

$$(d_{CE}c)(D_1, \ldots, D_{p+1}) := \sum_{i < j} (-1)^{i+j-1} c([D_i, D_j], D_1, \ldots, \hat{D}_i, \ldots, \hat{D}_j, \ldots, D_{p+1})$$

for $p \geq 0$ and $c \in C^p_{CE}(\text{QCoder}_m^+(BH))$. The chain map $\Phi$ is constructed as follows: for a cochain $c \in C^p_{CE}(\text{QCoder}_m^+(BH))$, we define

$$\Phi(c) := c(\eta^p) = \sum_{i_1 < \cdots < i_p} \eta_{i_1} \wedge \cdots \wedge \eta_{i_p} c(b^{i_1} \wedge \cdots \wedge b^{i_p}),$$

where we set

$$\eta = \sum_i \eta_i b^i$$

using a (topological) basis $\{b^i\}$ of $\text{QCoder}_m^+(BH)$. The $p$-form $\Phi(c)$ is $\mathcal{I}$-invariant since $\mathcal{I}$ acts on $H$ trivially. Then $\Phi$ is a chain map by the flatness of $\eta$

$$d\eta + \frac{1}{2}[\eta, \eta] = \sum_i d\eta_i b_i + \sum_{i < j} \eta_i \wedge \eta_j [b^i, b^j] = 0.$$

So we obtain the following:

**Theorem 4.1.** Let $E \to B$ be a smooth fiber bundle with oriented closed fiber $X$ whose structure group acts trivially on the real cohomology group of $X$. Then the chain map $\Psi : C^\bullet_{CE}(\text{QCoder}_m^+(BH)) \to A^\bullet(B)$ obtained by the construction above...
induces the map \( \Psi\# \) between cohomologies which is independent of the choice of a smooth fiberwise metric.

For each cohomology class \( \alpha \) in \( H^\bullet_{CE}(Q\text{Coder}^+_m(BH)) \), we call the image \( c_\alpha(E) := \Psi\#(\alpha) \) by the \( A_\infty \)-characteristic class of \( E \) with label \( \alpha \).

### 4.2.2. Formal manifold bundles.

We consider the case where \( X \) is a formal manifold, i.e. \( A_\infty = A_\infty(X) \) contains the algebra structure \( m \) of \( H \), and there exists a decomposition of \( S \)-modules

\[
\text{Coder}^{+,0}(BH) = V \oplus \text{Coder}^{+,0}_m(BH),
\]

where \( S \) is the image of \( S \) in \( GL(H) \) and \( V \) is an \( S \)-submodule of \( \text{Coder}^{+,0}(BH) \).

By the same discussion of Lemma 3.5 in [21], we can obtain the following:

**Lemma 4.2.** The \( S \)-equivariant principal \( \IAut(H,m) \)-bundle \( \IAut(BH) \rightarrow A_\infty \) is \( S \)-trivial equivariantly.

Then there exists an \( S \)-equivariant diffeomorphism

\[
Q = Q(m) \times \IAut(H,m) \IAut(BH) \cong Q(m) \times A_\infty.
\]

Since the space \( A_\infty \) is also contractible, the space \( Q \) is homotopic to \( Q(m) \) \( S \)-equivariantly. Then, from the Maurer-Cartan form on \( Q(m) \), we have the chain map

\[
C^\bullet_{CE}(Q\text{Coder}^+_m(BH), S) \rightarrow A^\bullet(Q(m))^S
\]

in the same way as subsection 4.2.1. Here

\[
C^\bullet_{CE}(Q\text{Coder}^+_m(BH), S) := C^\bullet_{CE}(Q\text{Coder}^+_m(BH))^S
\]

is the \( S \)-invariant Chevalley-Eilenberg complex of \( Q\text{Coder}^+_m(BH) \).

**Theorem 4.3.** Let \( E \rightarrow B \) be a smooth fiber bundle with oriented closed formal fiber \( X \). Suppose there exists a decomposition of \( S \)-modules

\[
\text{Coder}^{+,0}(BH) = V \oplus \text{Coder}^{+,0}_m(BH),
\]

where \( m \) is the algebra structure of \( H \) and \( S \) is the image of the structure group in \( GL(H) \). Then the chain map \( C^\bullet_{CE}(Q\text{Coder}^+_m(BH), S) \rightarrow A^\bullet(B) \) obtained by the construction above induces the map between cohomologies which is independent of the choice of a smooth fiberwise metric.

### 4.3. Relation to the construction using the fundamental group.

For any \( [\tau,m] \in Q \), we have the dual of the bar construction of \( \tau \)

\[
(BA)^* \rightarrow (BH)^* = \hat{T}(H^*[−1]),
\]

where \( \hat{T}(H^*[−1]) \) means the completed tensor product generated by \( H^*[−1] \).

Let \( * \) be a basis point of \( X \) and \( A^\bullet(X,*) \) be the dga defined by

\[
A^p(X,*) := \begin{cases} 
\{ f \in A^0(X); f(*) = 0 \} & (p = 0) \\
A^p(X) & (p > 0).
\end{cases}
\]

Since the natural dga map \( A^\bullet(X,*) \rightarrow A \), whose degree 0 part is the zero map and positive degree part is the projection, is a quasi-isomorphism, the map \( BA^\bullet(X,*) \rightarrow BA \) induced by \( A^\bullet(X,*) \rightarrow A \) is also a quasi-isomorphism. According to [1], the
chain map $C_\bullet(\Omega X) \to (BA^\bullet(X, *))^*$ obtained by iterated integrals from the cubical chain complex of the loop space $\Omega X$ induce the isomorphism

$\hat{\mathbb{R}}\pi_1 = \hat{H}_0(\Omega X; \mathbb{R}) \to H_0((BA^\bullet(X, *))^*)$,

where $\pi_1 := \pi_1(X, *)$. So we get the isomorphism

$\hat{\mathbb{R}}\pi_1 \simeq H_0((BA^\bullet(X, *))^*) \simeq H_0((BH)^*) = \hat{T}H_1/I_\delta$, 

where $\delta := m^*$, $H_1 := H_1(X; \mathbb{R})[-1]$ and $I_\delta := \delta(H_2(X; \mathbb{R})[-1])$. Remark that the map associated with an element in $Q$ which comes from a metric $g$ on $X$ is the Chen expansion determined with $g$ following [5]. Then we have the $M$-equivariant map $\theta : Q \to \hat{\Theta}(\pi_1)$. Here the definition of the space $\hat{\Theta}(\pi_1)$ is obtained by replacing “Hopf algebra” with “algebra” from $\Theta(\pi_1)$ in [21].

Fixing $m$, we have the commutative diagram

$T_\tau Q(m) \xrightarrow{\theta_*} T_1 QAut(H, m) \xrightarrow{\eta_1} QCoder_m^+(BH)$

$\uparrow \quad \uparrow \quad \uparrow$

$T_{\theta(\tau)} \hat{\Theta}(\pi_1, I_\delta) \xrightarrow{\theta_*} T_1 IAut(\hat{T}H_1/I_\delta) \xrightarrow{\eta_2} ODer^+(\hat{T}H_1/I_\delta)$.

So we obtain

$\theta_*\eta_1 = \theta^*\eta_2$,

where $\eta_1$ is the Maurer-Cartan form on $Q(m)$ by the action of $IAut(H, m)$ and $\eta_2$ is the one on $\hat{\Theta}(\pi_1)$ by the action of $IAut(\hat{T}H_1/I_\delta)$.

Thus we obtain the following:

**Theorem 4.4.** We have the commutative diagram

$H^\bullet_{CE}(QCoder_m^+(BH)) \xrightarrow{\theta_*} H^\bullet_{DR}(B)$

$\downarrow$

$H^\bullet_{CE}(ODer^+(\hat{T}H_1/I_\delta))$

under the assumption in Theorem 4.1 and

$H^\bullet_{CE}(QCoder_m^+(BH), S) \xrightarrow{\theta_*} H^\bullet_{DR}(B)$

$\downarrow$

$H^\bullet_{CE}(ODer^+(\hat{T}H_1/I_\delta), S)$

under the assumption in Theorem 4.3.

It would be interesting to compare our construction with another approach to diffeomorphism groups from noncommutative geometry in [19].

**References**

[1] K.T. Chen, *Iterated path integrals*, Bull. Amer. Math. Soc. 83 (1977), no. 5, 831-879.

[2] C-H Cho and S. Lee. *Potentials of homotopy cyclic $A_\infty$-algebras*. Homology Homotopy Appl. 14 (2012), no. 1, 203–220.

[3] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono. *Lagrangian intersection Floer theory - anomaly and obstruction*. AMS/IP Studies in Advanced Mathematics, 46. American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2009.
[4] E. Getzler and J. D. S. Jones. $A\infty$-algebras and the cyclic bar complex. Illinois J. Math., 34 (1990), no. 2, 256–283.
[5] V. K. A. M. Gugenheim, L. A. Lambe, and J. D. Stasheff. Algebraic aspects of Chen's twisting cochain. Illinois J. Math. 34 (1990), no. 2, 485–502.
[6] V. K. A. M. Gugenheim and H. J. Munkholm. On the extended functoriality of Tor and Cotor. J. Pure Appl. Algebra, 4 (1974), 9–29.
[7] D. Husemoller, J. C. Moore, and J. Stasheff. Differential homological algebra and homogeneous spaces. J. Pure Appl. Algebra, 5 (1974), no. 2, 113–185.
[8] T. V. Kadeishvili. The algebraic structure in the homology of an $A(\infty)$-algebra. Soobshch. Akad. Nauk Gruzin. SSR 108 (1982), no. 2, 249–252.
[9] T. V. Kadeishvili. The functor $D$ for a category of $A(\infty)$-algebras. Soobshch. Akad. Nauk Gruzin. SSR, 125 (1987), no. 2, 273–276.
[10] H. Kajiura. Noncommutative homotopy algebras associated with open strings. Reviews in Math. Phys., 19 (2007), no. 1, 1–99. math.QA/0606332.
[11] H. Kajiura and Y. Terashima. Homotopy equivalence of $A\infty$-algebras and gauge transformation. preprint, 2003.
[12] N. Kawazumi. Cohomological Aspects of Magnus Expansions. math/0505497.
[13] N. Kawazumi. Harmonic Magnus Expansion on the Universal Family of Riemann Surfaces. math/0603158.
[14] B. Keller. Introduction to A-infinity algebras and modules. Homology Homotopy Appl. 3 (2001), no. 1, 1–35.
[15] M. Kontsevich. Deformation quantization of Poisson manifolds. Lett. Math. Phys., 66 (2003), no. 3, 157–216.
[16] M. Kontsevich and Y. Soibelman. Homological mirror symmetry and torus fibrations. In Symplectic geometry and mirror symmetry (Seoul, 2000), pages 203–263. World Sci. Publishing, River Edge, NJ, 2001. math.SG/0011041.
[17] M. Kontsevich and Y. Soibelman. Notes on $A\infty$-algebras, $A\infty$-categories and noncommutative geometry. In Homological mirror symmetry, Lecture Notes in Phys., 757:153–219, Springer, Berlin, 2009.
[18] K. Lefèvre-Hasgawa. Sur les $A\infty$-catégories. math/0310337.
[19] J. Lott. Diffeomorphisms and noncommutative analytic torsion. Mem. Amer. Math. Soc. 141 (1999), no. 673.
[20] V. Lyubashenko. Category of $A\infty$-categories. Homology Homotopy Appl., 5 (2003), no. 1, 1–48.
[21] T. Matsuyuki and Y. Terashima. Characteristic classes of fiber bundles. math/151106810 (2015), to appear in Algebr. Geom. Topology.
[22] A. Prouté. Algèbres différentielles fortement homotopiquement associatives. Thèse d’Etat, Université Paris VII, 1984.
[23] M. Schlessinger and J. Stasheff. Deformation theory and rational homotopy type. U. of North Carolina preprint, 1979 short version: The Lie algebra structure of tangent cohomology and deformation theory. J. Pure Appl. Alg., 38 (1985), no. 2-3, 313–322.
[24] V. A. Smirnov. Simplicial and operad methods in algebraic topology. Translations of Mathematical Monographs, 198. American Mathematical Society, Providence, RI, 2001. pp235. Translated from the Russian manuscript by G. L. Rybnikov.
[25] J. Stasheff. Homotopy associativity of $H$-spaces, I. Trans. Amer. Math. Soc., 108 (1963), 293–312.
[26] J. Stasheff. Homotopy associativity of $H$-spaces, II. Trans. Amer. Math. Soc., 108 (1963), 313–327.
Faculty of Sciences, Chiba University, 1-33 Yayoi-cho, Inage-ku, Chiba 263-8522, Japan
E-mail address: kajiura@math.s.chiba-u.ac.jp

Department of Mathematics, Tokyo Institute of Technology, 2-12-1 Oh-okayama, Meguro-ku, Tokyo 152-8551, Japan.
E-mail address: matsuyuki.t.aa@m.titech.ac.jp

Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, 2-12-1 Ookayama, Meguro-ku, Tokyo 152-8552, Japan.
E-mail address: tera@is.titech.ac.jp