Intersection theorems for \(\{0, \pm 1\}\)-vectors and 
s-cross-intersecting families

Peter Frankl, Andrey Kupavskii

Abstract

In this paper we pursue two possible directions for the extension of the classical
Erdős-Ko-Rado theorem which states that any family of \(k\)-element, pairwise intersect-
ing subsets of the set \([n] := \{1, \ldots, n\}\) has cardinality at most \((n-1)^{n-k} k\).

In the first part of the paper we study families of \(\{0, \pm 1\}\)-vectors. Denote by \(\mathcal{L}_k\) the family of all vectors \(v\) from \(\{0, \pm 1\}^n\) such that \(\langle v, v \rangle = k\). For any \(k\), most \(l\) and sufficiently large \(n\), as well as for small \(k\) and any \(l, n\), we determine the maximal size of a family \(V \subset \mathcal{L}_k\) satisfying \(\langle v, w \rangle \geq l\) for any \(v, w \in V\).

In the second part of the paper we study cross-intersecting pairs of families. We say that two families \(A, B\) are \(s\)-cross-intersecting, if for any \(A \in A, B \in B\) we have \(|A \cap B| \geq s\). We also say that a family \(A\) of sets is \(t\)-intersecting, if for any \(A_1, A_2 \in A\) we have \(|A_1 \cap A_2| \geq t\). For a pair of nonempty \(s\)-cross-intersecting \(t\)-intersecting families \(A, B\) of \(k\)-sets, we determine the maximal value of \(|A| + |B|\) for sufficiently large \(n\).

1 Introduction

Let \([n] = \{1, \ldots, n\}\) be an \(n\)-element set and for \(0 \leq k \leq n\) let \(\binom{[n]}{k}\) denote the collection of all its \(k\)-element subsets. Further, let \(2^n\) denote the power set of \([n]\). Any subset \(F \subset 2^n\) is called a set family or family for short. We say a “\(k\)-set” instead of a “\(k\)-element set” for shorthand. If \(F \cap F' \neq \emptyset\) for all \(F, F' \in F\), then \(F\) is called intersecting. The following classical result is one of the cornerstones of extremal set theory.

**Theorem** (Erdős-Ko-Rado theorem \[2\]). If \(A \subset \binom{[n]}{k}\) is intersecting and \(n \geq 2k\), then \(|A| \leq \binom{n-1}{k-1}\) holds.

This is one of the first results in extremal set theory and probably the first result about intersecting families.

*Moscow Institute of Physics and Technology, École Polytechnique Fédérale de Lausanne; Email: kupavskii@yandex.ru  Research supported in part by the Swiss National Science Foundation Grants 200021-137574 and 200020-14453 and by the grant N 15-01-03530 of the Russian Foundation for Basic Research.*
Numerous results extended the Erdős-Ko-Rado theorem in different ways. One of the
directions is concerned with nontrivial intersecting families. We say that an intersecting
family is nontrivial if the intersection of all sets from the family is empty. Note that the
size of a family of all k-sets containing a single element matches the bound from the Erdős-
Ko-Rado theorem. Probably, the best known result in this direction is the Hilton-Milner
(Hilton and Milner [13]), which determines the maximum size of a non-trivial intersecting family.

In the same paper Hilton and Milner dealt with pairs of cross-intersecting families. We
say that the families \( A, B \subset 2^{[n]} \) are cross-intersecting if for every \( A \in A \) and \( B \in B \) we have
\( A \cap B \neq \emptyset \). Hilton and Milner proved the following inequality:

**Theorem** (Hilton and Milner [13]). Let \( A, B \subset \binom{[n]}{k} \) be non-empty cross-intersecting families
with \( n \geq 2k \). Then \(|A| + |B| \leq \binom{n}{k} - \binom{n-k}{k} + 1\).

This inequality was generalized by Frankl and Tokushige [11] to the case \( A \subset \binom{[n]}{a} \),
\( B \subset \binom{[n]}{b} \) with \( a \neq b \), as well as to more general constraints on the sizes of \( A, B \). A simple proof of the theorem above may be found in [9].

Another natural direction for the generalizations deals with more general constraints on
the sizes of pairwise intersections. A family \( F \) is called \( t \)-intersecting if for any \( F_1, F_2 \in F \) we have
\(|F_1 \cap F_2| \geq t \). The Complete Intersection Theorem due to Alswede and Khachatrian [1],
continuing previous work done by Frankl [3] and Frankl and Füredi [6], gave an exhaustive
answer to the question of how big a \( t \)-intersecting family of \( k \)-sets could be. We discuss it in
more detail in Section 3.

One more example of a result of such kind is a theorem due to Frankl and Wilson [12].
It states that, if \( n \) is a prime power and \( F \subset \binom{[4n-1]}{2n-1} \) satisfies \(|F_1 \cap F_2| \neq n - 1 \) for any
\( F_1, F_2 \in F \), then the size of \( F \) is at most \( \binom{4n-1}{2n-1} \), which is exponentially smaller than \( \binom{4n-1}{2n-1} \).
This theorem has several applications in geometry.

The third direction is to generalize the EKR theorem to more general classes of objects.
Out of the numerous generalizations of such kind we are going to talk only about the ones
concerned with \( \{0, \pm 1\} \)-vectors, that is, vectors that have coordinates from the set \( \{-1, 0, 1\} \).
In this case a natural replacement for the size of the intersection is the scalar product. A
generalization of [12] to the case of \( \{0, \pm 1\} \)-vectors due to Raigorodskii [17] led to improved
bounds in several geometric questions, such as Borsuk’s problem and the chromatic number
of the space. However, it seems plausible to improve Raigorodskii’s result, once one achieves a
better understanding of “intersecting” families of \( \{0, \pm 1\} \)-vectors. Partly motivated by that,
in our recent work [7, 8] we managed to extend EKR theorem to the case of \( \{0, \pm 1\} \)-vectors
with fixed number of 1’s and −1’s. Let us formulate the results. We need the following
notation.

Let \( s \geq t \geq 1 \) be integers and let \( \mathcal{V}(n, s, t) \) denote the set of all \( \{0, \pm 1\} \)-vectors of length
\( n \) and having exactly \( s \) coordinates equal to +1 and \( t \) coordinates equal to −1. Put
\[
g(n, s, t) := \max\left\{|\mathcal{V}| : \mathcal{V} \subset \mathcal{V}(n, s, t) \text{ and } \langle \mathbf{v}, \mathbf{w} \rangle > -2t \text{ for any } \mathbf{v}, \mathbf{w} \in \mathcal{V}\right\}.
\]
Note that, for \( n \geq 2s \), \(-2t \) is the smallest possible scalar product between two vectors in
\( \mathcal{V}(n, s, t) \): indeed, to achieve it, both vectors have to have their \(-1\)'s in front of the 1's of the other vector, and the remaining \( s - t \) 1's in front of 0's.

**Theorem A** (Frankl, Kupavskii [7]). We have

\[
g(n, s, 1) = s \binom{n-1}{s} \quad \text{for } 2s \leq n \leq s^2, \\
g(n, s, 1) = s \left( \frac{s^2-1}{s} \right) + \left( \frac{s^2+1}{s} \right) + \ldots + \left( \frac{n-1}{s} \right) \quad \text{for } n > s^2.
\]

**Theorem B** (Frankl, Kupavskii [8]). We have

\[
\binom{n}{s+t} \left( \frac{s+t-1}{t-1} \right) \leq g(n, s, t) \leq \binom{n}{s+t} \left( \frac{s+t-1}{t-1} \right) + \binom{n}{2t} \binom{2t}{s-t-1}.
\]

The contribution of this paper is two-fold. First, in several scenarios we determine the maximal size of a family of \( \{0, \pm 1\} \)-vectors of fixed length and with restrictions on the scalar product. The main results are stated in Section 2. Second, we prove results analogous to the Hilton-Milner theorem stated above, but for pairs of families that are \( s' \)-cross-intersecting and \( t' \)-intersecting. For details and precise definitions see Section 3.

### 2 Families of \( \{0, \pm 1\} \)-vectors

Denote by \( \mathcal{L}_k \) the family of all vectors \( \mathbf{v} \) from \( \{0, \pm 1\}^n \) such that \( \langle \mathbf{v}, \mathbf{v} \rangle = k \). Note that \( |\mathcal{L}_k| = 2^k \binom{n}{k} \). This section is mostly devoted to the study of the quantity below.

\[
F(n, k, l) := \max \{ |\mathcal{V}| : \mathcal{V} \subset \mathcal{L}_k, \forall \mathbf{v}, \mathbf{w} \in \mathcal{V} \langle \mathbf{v}, \mathbf{w} \rangle \geq l \}.
\]

Recall the following theorem of Katona:

**Theorem** (Katona, [14]). Let \( n > s > 0 \) be fixed integers. If \( \mathcal{U} \subset 2^{[n]} \) is a family of sets such that for any \( U, V \in \mathcal{U} \) we have \( |U \cup V| \leq s \) then

\[
|\mathcal{U}| \leq f(n, s) := \begin{cases} 
\sum_{i=0}^{s/2} \binom{n}{i} & \text{if } s \text{ is even}, \\
2 \sum_{i=0}^{(s-1)/2} \binom{n-1}{i} & \text{if } s \text{ is odd}.
\end{cases}
\]

Moreover, for \( n \geq s + 2 \) the equality is attained only for the following families. If \( s \) is even, then it is the family \( \mathcal{U}' \) of all sets of size at most \( s/2 \). If \( s \) is odd, then it is one of the families \( \mathcal{U}'_j \) of all sets that intersect \([n] - \{j\}\) in at most \( (s-1)/2 \) elements, where \( 1 \leq j \leq n \).
Given two sets $U, V$, we denote the symmetric difference of these two sets by $U \triangle V$, that is, $U \triangle V := U \setminus V \cup V \setminus U$. A theorem due to Kleitman states that the bound (3) holds for a more general class of families.

**Theorem** (Kleitman, [15]). If for any two sets $U, V$ from a family $U \subset 2^n$ we have $|U \triangle V| \leq s$, then the bound (3) holds for $U$.

Note that there is no uniqueness counterpart in Kleitman’s theorem.

### 2.1 Simple properties of $F(n, k, l)$

First we state and prove some simple observations concerning $F(n, k, l)$.

**Proposition 1.** Fix any $n \geq k \geq 1$. Then $F(n, k, -k + 1) = 2^{k-1}\binom{n}{k} = |L_k|/2$.

**Proof.** We split vectors from $L_k$ in pairs $v, w$ so that $\langle v, w \rangle = -k$. We can take exactly one vector out of each pair in the family. □

For any $v = (v_1, \ldots, v_n)$ and any vector family $V$ define $S(v) = \{i : v_i \neq 0\}$ and $V(S) = \{v \in V : S(v) = S\}$. We also define $N(v) = \{i : v_i = -1\}$.

**Proposition 2.** We have $F(2t, 2t, 2s) = F(2t, 2t, 2s - 1) = f(2t, t - s)$ and $F(2t + 1, 2t + 1, 2s) = F(2t + 1, 2t + 1, 2s + 1) = f(2t + 1, t - s)$.

**Proof.** For any $v, w \in L_{2t}$ we have $\langle v, w \rangle = 2t - 2|N(v) \triangle N(w)|$. Therefore, the statement of the proposition follows from Kleitman’s Theorem. □

We say that a family $V \subset L_k$ of vectors is **homogeneous**, if for every $i \in [n]$ the $i$’th coordinates of vectors from $V$ are all non-negative or all non-positive.

**Proposition 3.** For $n > k$ we have $F(n, k, k - 1) = \max\{k + 1, n - k + 1\}$.

**Proof.** First of all note that $u_iv_i = -1$ forces $\langle u, v \rangle \leq k - 2$. Thus $V$ is homogeneous and the condition translates into $|S(u) \cap S(v)| \geq k - 1$.

Since the only $(k - 1)$-intersecting families of $k$-sets are stars around a $(k - 1)$-element set and subsets of $\binom{[k+1]}{k}$, we are done. □

Given a vector family $V$, for any $i \in [n]$ we denote by $d_i$ the **degree** of $i$, that is, the number of vectors from $V$ that have a nonzero coordinate on position $i$.

**Proposition 4.** For $n > k \geq 2$ the following holds:

(i) $F(3, 2, 0) = 4$.

(ii) $F(n, k, k - 2) = \binom{n}{k}$ if $n = k + 2$ or $k \geq 3$ and $n = k + 1$.

(iii) $F(n, k, k - 2) = \max\{\binom{n-k+2}{2}, k(n - k) + 1\}$ for $n \geq k + 3$. 

4
Proof. The first part is very easy to verify. As for the other parts, assume that \( u_i = -v_i \) for some \( u, v \in \mathcal{V} \). Then \( S(u) = S(v) \) and \( u \) and \( v \) must agree in the remaining coordinate positions. Thus, \( d_i = 2 \) in this case. Since \( F(k, k, k-2) = 2 \), for \( n = k + 1 \) a vertex of degree 2 would force \( |\mathcal{V}| \leq 2 + 2 = 4 \). On the other hand, if we assume that \( \mathcal{V} \) is homogeneous, then \( |\mathcal{V}| \leq (\binom{k+1}{k}) \) for \( k \geq 3 \), as desired. Since \( (\binom{k+2}{k}) - (\binom{k+1}{k}) \geq 2 \) for \( k \geq 2 \), the same argument works for \( n = k + 2, k \geq 2 \) as well.

If \( n > k + 2 \), the same argument as above implies that the family \( \{S(u) : u \in \mathcal{V}\} \) is \((k-2)\)-intersecting. If \( |\mathcal{V}| \) is maximal, then it is homogeneous. The bound then follows from the Complete Intersection Theorem (see Section 3).

2.2 Results

Our main result concerning \( \{0, \pm 1\} \)-vectors is the following theorem, which determines \( F(n, k, l) \) for all \( k, l \) and sufficiently large \( n \) and which shows the connection of \( F(n, k, l) \) with the above stated theorems of Katona and Kleitman, as well as the problem of determining \( g(n, k', l') \).

Theorem 5. For any \( k \) and \( n \geq n_0(k) \) we have

1. \( F(n, k, l) = \binom{n - l}{k - l} \) for \( 0 \leq l \leq k \).
2. \( F(n, k, -l) = f(k, l) \binom{n}{k} \) for even \( 0 \leq l \leq k \),
3. \( F(n, k, -l) = g(n, k - \frac{l+1}{2}, \frac{l+1}{2}) + f(k, l - 1) \binom{n}{k} \) for odd \( 0 \leq l \leq k \).

Note that \( f(k, 0) = 1 \) and so the values for \( l = 0 \) in part 1 and \( l = 0 \) in part 2 coincide. We remark that the statement of part 2 of the theorem for \( l = k \) is obvious and for \( l = k - 1 \) it was already derived in Proposition 1. Using Theorem A we get the exact value of \( F(n, k, -1) \). The value of \( F(n, k, -l) \) for other odd \( l \) is determined up to an additive term of order \( O(n^{k-1}) \) using Theorem B.

Using the same technique, we may extend the result of parts 2 and 3 of Theorem 5 in the following way:

Theorem 6. Let \( \mathcal{V} \subset \mathcal{L}_k \) be the set of vectors such that for any \( v, w \in \mathcal{V} \) we have \( \langle v, w \rangle \neq -l - 1 \) for some \( 0 \leq l < k \). Then we have

\[
\max_{\mathcal{V}} |\mathcal{V}| = f(k, l) \binom{n}{k} + O(n^{k-1}).
\]

For \( k = 3 \) the values not covered by Propositions 1, 3, 4 and Theorem 5 are \( F(n, 3, 0) \) and \( F(n, 3, -1) \) for \( n \) not too large. We determine \( F(n, 3, 0) \) in the following theorem, but first we need some preparation.
Let us use the notation \((a, b, c)\) for a set \(\{a, b, c\}\) if we know that \(a < b < c\). Also for \((a, b, c) \subset [n]\) let \(u(a, b, c) = (u_1, \ldots, u_n)\) with \(u_i = 1\) for \(i \in \{a, b, c\}\) and \(u_i = 0\) otherwise. Further let \(v(a, b, c) = (v_1, \ldots, v_n)\) be the vector with \(v_a = v_b = 1, v_c = -1\) and \(v_i = 0\) otherwise.

Let us show two lower bounds for \(F(n, 3, 0)\). Taking all non-negative vectors gives
\[
F(n, 3, 0) \geq \binom{n}{3}.
\]
The other one is based on the following family:
\[
\mathcal{V}(n) = \{u(a, b, c), v(a, b, c) : |(a, b, c) \cap [3]| \geq 2\}.
\]
Note that \(\langle v, v' \rangle \geq 0\) for all \(v, v' \in \mathcal{V}(n)\) and the first few values of |\(\mathcal{V}(n)\)| are |\(\mathcal{V}(3)\)| = 2, |\(\mathcal{V}(4)\)| = 8, |\(\mathcal{V}(5)\)| = 14, |\(\mathcal{V}(6)\)| = 20.

**Theorem 7.** We have
(1) \(F(n, 3, 0) = |\mathcal{V}(n)|\) for \(n = 3, 4, 5\).
(2) \(F(6, 3, 0) = 21\).
(3) \(F(n, 3, 0) = \binom{n}{3}\) for \(n \geq 7\).

As we will see, the proof of (2) is the most difficult.

### 3 Cross-intersecting families

We say that two families \(A, B \subset 2^{[n]}\) are \(s\)-cross-intersecting, if for any \(A \in A, B \in B\) we have \(|A \cap B| \geq s\). In the proof of Theorem 5 we need to estimate the sum of sizes of two \(t\)-intersecting families that are \(s\)-cross-intersecting. For the proof some relatively crude bounds are sufficient but we believe that this problem is interesting in its own right. As a matter of fact the case of non-uniform families was solved by Sali [18] (cf. [4] for an extension with a simpler proof).

To state our results for the \(k\)-uniform case let us make some definitions.

**Definition 1.** For \(k \geq s > t \geq 1\) and \(k \geq 2s - t\) define the Frankl-family \(\mathcal{A}_i(k, s, t)\) by
\[
\mathcal{A}_i(k, s, t) = \{ A \subset \binom{[k]}{s} : |A \cap [t + 2i]| \geq t + i \}, \quad 0 \leq i \leq s - t.
\]

Note that for \(A, A' \in \mathcal{A}_i(k, s, t)\) one has \(|A \cap A' \cap [t + 2i]| \geq t\), in particular, \(\mathcal{A}_i(k, s, t)\) is \(t\)-intersecting. Also, for \(i < s - t\) every vertex of \([k]\) is covered by some \(A \in \mathcal{A}_i(k, s, t)\).

The following result was conjectured by Frankl [3] and nearly 20 years later proved by Ahlswede and Khachatrian.
Theorem (Complete Intersection Theorem ([1])). Suppose that $A \subset \binom{[k]}{s}$ is $t$-intersecting, $k \geq 2s - t$. Then
\[
|A| \leq \max_{0 \leq i \leq s-t} |A_i(k,s,t)| =: m(k,s,t)
\]
holds. Moreover, unless $k = 2s, t = 1$ or $A$ is isomorphic to $A_i(k,s,t)$, the inequality is strict.

Definition 2. For $k \geq 2s - t$, $0 \leq i < s - t$ define
\[
\mathcal{M}_i(k,s,t) = \{A \subset [k] : |A| \geq s, |A \cap [t+2i]| \geq t+i\} \cup \{A \subset [k] : |A| \geq k-s+t\}.
\]

Note that $\{A \in \binom{[k]}{s} : A \in \mathcal{M}_i(k,s,t)\} = \mathcal{A}_i(k,s,t)$ for $k > 2s - t$ and that $\mathcal{M}_i(k,s,t)$ is $t$-intersecting.

For fixed $k, s, t$ and $0 \leq i < s - t$ let us define the pair $\mathcal{A}_i = \{A \in \binom{[n]}{k} : A \cap [k] \in \mathcal{M}_i(k,s,t)\}, B_i = \{[k]\}$. Then these non-empty $t$-intersecting families are $s$-cross-intersecting.

Note that $\mathcal{A}_{s-t}(k,s,t) = \binom{[2s-t]}{s}$. For $i = s - t$ we define
\[
\mathcal{A}_{s-t} = \{A \in \binom{[n]}{k} : |A \cap [2s-t]| \geq s\}
\]
and
\[
\mathcal{B}_{s-t} = \{B \in \binom{[n]}{k} : [2s-t] \subset B\}.
\]

Again the non-empty $t$-intersecting families $\mathcal{A}_{s-t}$ and $\mathcal{B}_{s-t}$ are $s$-cross-intersecting.

With this terminology we prove

Theorem 8. Let $k > s > t \geq 1$ be integers, $k \geq 2s - t$. Suppose that $A, B \subset \binom{[n]}{k}$ are non-empty $t$-intersecting families which are cross $s$-intersecting. Then for $n \geq n_0(s,t,k)$ we have
\[
\max_{A,B} \{|A| + |B|\} = \max_{0 \leq i \leq s-t} \{|A_i| + |B_i|\}.
\]
Moreover, unless $k = 2s, t = 1$ or $A, B$ are isomorphic to $A_i, B_i$, the equality above transforms into a strict inequality.

We note that the conclusive result for $t = 0$ was obtained by the authors in the paper [10], and therefore we do not consider this case here.

Remark. The assumption $k \geq 2s - t$ is in fact not restrictive, since if $A \in A$ and $B \in B$ then any two sets from $A$ intersect in at least $2s - k$ elements inside $B$ (and analogously for $B$ and $A$). If $2s - k \geq t$, then the $t$-intersecting condition is implied by the $s$-cross-intersecting condition, and it reduces to the case studied in [10].

We also note that the problem makes sense only in case $s > t$. Otherwise, we may just take both $A$ and $B$ to be the same $t$-intersecting family of maximal cardinality. Thus, in case $t \geq s$ the problem reduces to a trivial application of the Complete Intersection Theorem.
4 Shifting for vectors

In the proof of Theorem 5 we use two types of shifting. The first one pushes bigger coordinates to the left. For a given pair of indices $i < j$ in $[n]$ and a vector $v = (v_1, \ldots, v_n) \in \{0, \pm 1\}^n$ we define an $(i, j)$-shift $S_{i,j}(v)$ of $v$ in the following way. If $v_i \geq v_j$, then $S_{i,j}(v) = v$. If $v_i < v_j$, then $S_{i,j}(v) := (v_1, \ldots, v_{i-1}, v_j, v_{i+1}, \ldots, v_{j-1}, v_i, v_{j+1}, \ldots, v_n)$, that is, it is obtained from $v$ by interchanging its $i$-th and $j$-th coordinate.

Next, we define an $(i, j)$-shift $S_{i,j}(W)$ for a family of vectors $W \subset \{0, \pm 1\}^n$.

$$S_{i,j}(W) := \{S_{i,j}(v) : v \in W\} \cup \{v : v, S_{i,j}(v) \in W\}.$$

The second is the up-shift: $S_i(v_1, \ldots, v_n) = (v_1, \ldots, v_n)$ if $v_i = 0$ or $1$ and $S_i(v_1, \ldots, v_n) = (v_1, \ldots, v_{i-1}, 1, v_{i+1}, \ldots, v_n)$ if $v_i = -1$. The shift $S_i(W)$ is defined similarly to $S_{i,j}(W)$:

$$S_i(W) := \{S_i(v) : v \in W\} \cup \{v : v, S_i(v) \in W\}.$$

We call a system $W$ shifted, if $W = S_{i,j}(W)$ for all $i < j \in [n]$ and $W = S_i(W)$ for all $i \in [n]$. Any system of vectors may be made shifted by means of a finite number of $(i, j)$-shifts and $i$-up-shifts. Moreover, it was shown in [7] and [15] (and is easy to verify directly) that $(i, j)$-shifts and up-shifts do not decrease the minimal scalar product in $W$.

5 Proof of Theorems 5 and 6

Part 1 of Theorem 5

First we show that the left hand side is at least the right hand side. Take the family of vectors $V = \{v = (v_1, \ldots, v_n) : v_1 = \ldots = v_l = 1, v_i \in \{0, 1\}\}$. It clearly satisfies the condition $\langle v, w \rangle \geq l$ for any $v, w \in V$ and has cardinality $\binom{n-l}{k-l}$.

We proceed to the upper bound. Take any family $V$ of $\{0, \pm 1\}$-vectors having minimal scalar product at least $l$ and define its subfamilies $V_i(\epsilon) = \{w \in V : w_i = \epsilon\}$ for $\epsilon = \pm 1$.

Claim 9. If for some $i$ both $V_i(-1)$ and $V_i(1)$ are nonempty, then

$$|V_i(1)| + |V_i(-1)| \leq 2^k \binom{k-1}{l+1} \binom{n-l-2}{k-l-2} = O(n^{k-l-2}).$$

We call any such $i \in [n]$ bad.

Proof. Consider the set families $A^+ = \{S \in \binom{[n]-\{i\}}{k-1} : \exists w \in V_i(1) \text{ such that } S(w) = S \cup \{i\}\}$ and $A^- = \{S \in \binom{[n]-\{i\}}{k-1} : \exists w \in V_i(-1) \text{ such that } S(w) = S \cup \{i\}\}$. Both $A^+, A^-$ are nonempty. Each of them is $(l-1)$-intersecting, moreover, they must be $(l+1)$-cross-intersecting, since otherwise we would have two vectors in $V$ that have scalar product at most $l-1$. 


Let $A \in \mathcal{A}^+$, $B \in \mathcal{A}^-$. Then $|A \cap B| \geq l + 1$ for all $B \in \mathcal{A}^-$ implies $|\mathcal{A}^-| \leq \binom{|A|}{l+1} \binom{n-l-2}{k-l-2}$ and similarly $|\mathcal{A}^+| \leq \binom{|B|}{l+1} \binom{n-l-2}{k-l-2}$.

The statement of the claim follows from the trivial inequalities $|\mathcal{V}_i(1)| \leq 2^k - 1|\mathcal{A}^+|$, $|\mathcal{V}_i(-1)| \leq 2^{k-1}|\mathcal{A}^-|$. \hfill $\square$

Consider the set $I \subset [n]$ of all bad coordinate positions and define a subfamily $\mathcal{V}_b$ of all vectors that have non-zero values on the bad positions: $\mathcal{V}_b = \{v \in \mathcal{V} : \exists i \in I : v_i \neq 0\}$. Put $\mathcal{V}_g = \mathcal{V} - \mathcal{V}_b$. Since $\mathcal{V}_g$ is homogeneous on $[n] - I$ and is $l$-intersecting, we have $|\mathcal{V}_g| \leq \binom{n-|I|-l}{k-l}$ for $n \geq k^2$ due to the result of [3]. Therefore,

$$|\mathcal{V}| \leq \binom{n-|I|-l}{k-l} + |I|O(n^{k-l-2}) \leq \binom{n-l}{k-l} + |I|\left(\binom{n-|I|-l}{k-l-1} - O(n^{k-l-2})\right) \leq \binom{n-l}{k-l},$$

for $n$ large enough, with the last inequality strict in case $|I| > 0$.

**Parts 2, 3 of Theorem 5 and Theorem 6**

First we show that there is such a family of vectors for which the inequality is achieved. For each $S \subset \binom{n}{k}$ we take in $\mathcal{V}$ all the vectors from $\mathcal{L}_k(S)$ that have at most $l/2$ $-1$'s in case $l$ is even and the sets that have at most $(l-1)/2$ $-1$'s, together with the family in $V(n, k - (l + 1)/2, (l + 1)/2)$ that realizes the value of $g(n, k - (l + 1)/2, (l + 1)/2)$. It is not difficult to see that $\mathcal{V}$ satisfies the requirements of both Theorems 5 and 6 on the size and on the scalar product.

Now we prove the upper bound. The proof is somewhat similar to the previous case. Take any vector family satisfying the conditions of Theorem 5 or Theorem 6. For any set $X \subset [n]$ and any vector $w$ of length $n$ denote by $w|_X$ the restriction of $w$ to the coordinates $i, i \in X$. Obviously $w|_X$ is a vector of length $|X|$. We look at all vectors $v \in \{\pm 1\}^{l+1}$ and split them into groups of antipodal vectors $v$, $\bar{v}$, that is, vectors that satisfy $\langle v, \bar{v} \rangle = -l-1$. Fix a set of coordinates $I = \{i_1, \ldots, i_{l+1}\}$ and define $\mathcal{V}(I, v) = \{w \in \mathcal{V} : w|_I = v\}$.

**Claim 10.** If both $\mathcal{V}(I, v)$ and $\mathcal{V}(I, \bar{v})$ are nonempty, then

$$|\mathcal{V}(I, v)| + |\mathcal{V}(I, \bar{v})| \leq 2^{k-l-1}(k-l)\binom{n-l-2}{k-l-2} + O(n^{k-l-2}).$$

We call any such $I$ bad. In particular, the claim gives that the total cardinality of all such “bad” subfamilies is $O(n^{k-1})$.

**Proof.** Consider the set families $\mathcal{A} = \{S \in \binom{[n] - I}{k-l-1} : \exists w \in \mathcal{V}(I, v) \text{ such that } S(w) = S \cup I\}$ and $\mathcal{A} = \{S \in \binom{[n] - I}{k-l-1} : \exists w \in \mathcal{V}(I, \bar{v}) \text{ such that } S(w) = S \cup I\}$. Both $\mathcal{A}, \mathcal{A}$ are nonempty.
Moreover, they must be cross-intersecting, since otherwise we would have two vectors in $\mathcal{V}$ that have scalar product exactly $-l - 1$. Therefore, by Hilton-Milner theorem,

$$|A| + |\overline{A}| \leq \left(\frac{n-l-1}{k-l-1}\right) - \left(\frac{n-k}{k-l-1}\right) + 1 \leq (k-l)\left(\frac{n-l-2}{k-l-2}\right).$$

The statement of the claim follows from the trivial inequalities $|\mathcal{V}(I, v)| \leq 2^{k-l-1}|A|$, and $|\mathcal{V}(I, \overline{v})| \leq 2^{k-l-1}|\overline{A}|$. □

Define $\mathcal{V}_b = \{w \in \mathcal{V} : \exists I \subset \binom{[n]}{l+1} : \mathcal{V}(I, v_I) \neq \emptyset, \mathcal{V}(I, \overline{v}_I) \neq \emptyset\}$. Consider the remaining family of “good vectors” $\mathcal{V}_g = \mathcal{V} - \mathcal{V}_b$.

First we finish the proof of Theorem 6. The family $\mathcal{V}_g$ is good in the following sense. If we fix $S \in \binom{[n]}{k}$ and consider the family of sets $A = \{N(w) : w \in \mathcal{V}_g(S)\}$, then $A$ satisfies $A_1 \triangle A_2 \leq l$ for any $A_1, A_2 \in A$. Therefore, $|\mathcal{V}_g(S)| \leq f(k, l)$ by Kleitman’s theorem and

$$|\mathcal{V}| \leq \sum_{s \in \binom{[n]}{k}} f(k, l) + |\mathcal{V}_b| = f(k, l)\left(\frac{n}{k}\right) + O(n^{k-1}).$$

The proof of Theorem 6 is complete.

From now on we suppose that the family $\mathcal{V}$ is shifted. Note that, since shifting may increase scalar products, it is impossible to use it in the case of Theorem 6. It is not difficult to see that the family $\mathcal{V}_g$ is up-shifted, since if we deleted some $w \in \mathcal{V}$, then we would have deleted all $u \in \mathcal{V}$ with $S(u) = S(w), N(u) \supset N(w)$. One can see in a similar way that $\mathcal{V}_g$ is shifted.

**Claim 11.** For every $S \in \binom{[n]}{k}$ the family $\mathcal{V}_g(S)$ satisfies $N(w) \cup N(v) \leq l$ for all $v, w \in \mathcal{V}_g(S)$.

**Proof.** Suppose for contradiction $N(w) \cup N(u) \geq l + 1$. Consider any set $I \subset N(w) \cup N(u)$, $|I| = l + 1$. Since $\mathcal{V}_g$ is up-shifted, there is a vector $u'$ in $\mathcal{V}_g(S)$ such that $\langle w_I, u'_I \rangle = -l - 1$, which means that for $v = w_I$ both $\mathcal{V}_g(I, v)$ and $\mathcal{V}_g(I, \overline{v})$ are nonempty, a contradiction. □

Let us first deal with the case of even $l$.

**Claim 12.** Let $l$ be even. If $I \subset \binom{[n]}{l+1}$ is bad, then for at least $\binom{n-k}{k-l-1}$ sets $S \supset I$, $S \in \binom{[n]}{k}$, we have $|\mathcal{V}_g(S)| \leq f(k, l) - 1$.

**Proof.** Consider the family $A \subset 2^S$ of subsets of $S$, that is defined in the following way: $A = \{N(w) \cap S : w \in \mathcal{V}_g(S)\}$. In view of the uniqueness part of Katona’s theorem, it is sufficient to show that $A$ does not contain one of the sets from the extremal family $U^l$ for at least $\binom{n-k}{k-l-1}$ choices of $S$.

If $I$ is bad then there exists a vector $v$ of length $l + 1$, such that both $\mathcal{V}(I, v), \mathcal{V}(I, \overline{v})$ are nonempty. Assume w.l.o.g. that $|N(v)| \leq l/2$ and take a vector $w \in \mathcal{V}$ such that $w_I = \overline{v}$. Then for any $S$ such that $S \cap S(w) = I$ the set $N(v)$ is missing from $A$ (and, consequently, $|\mathcal{V}_g(S)| \leq f(k, l) - 1$). There are exactly $\binom{n-k}{k-l-1}$ such sets. □

10
Assume now that there are $t$ bad sets $I \subset \binom{[n]}{k}$. Then the number of sets $S \subset \binom{[n]}{k}$ that contain one of the bad sets $I$ is at least $t \binom{n-k}{k-l-1} / \binom{k}{l+1}$. Therefore, by Claims 10 and 12 we have

$$|\mathcal{V}| - f(k, l) \binom{n}{k} \leq -t \binom{n-k}{k-l-1} / \binom{k}{l+1} + \sum_{\text{bad } I} \frac{1}{2} \sum_{v \in \{\pm 1\}^{l+1}} \left(|\mathcal{V}(I, v)| + |\mathcal{V}(I, \bar{v})|ight) \leq$$

$$\leq -t \binom{n-k}{k-l-1} / \binom{k}{l+1} - 2^k(k - l) \binom{n - l - 2}{k - l - 2} < 0,$$

provided $n > 2^k k^2 \binom{k}{l+1}$. We note that taking $n > 4^k k^2$ makes the choice of $n$ from which the proof works independent of $l$.

The case of odd $l$ turns out to be harder. We shall need the following variant of Katona’s theorem.

**Theorem 13.** Assume that $\mathcal{F} \subset 2^{[n]} \setminus \binom{[n]}{m+1}$ and for any $F, G \in \mathcal{F}$ we have $|F \cup G| \leq 2m + 1$. Then $|\mathcal{F}| \leq \sum_{i=0}^{m} \binom{n}{i}$, moreover, for $n > 2m + 2$ the only example attaining the bound is $\bigcup_{i=0}^{m} \binom{[n]}{i}$.

**Proof.** W.l.o.g., we may assume that $\mathcal{F}$ is shifted (we discuss the effect of this assumption on the uniqueness at the end of the proof). The proof is by induction. The statement is clear for $m = 0$, moreover, the extremal family is unique. For $n = 2m + 2$, it is easy to see that $2^{[n]} \setminus \binom{[n]}{m+1}$ splits into pairs of complementary sets, which implies the statement.

Assume that the statement holds for $(n-1, m), (n-1, m-1)$, and let us prove it for $(n, m)$, $n > 2m + 2$. We have $|\mathcal{F}| = |\mathcal{F}(n)| + |\mathcal{F}(\bar{n})|$. By induction, we have $|\mathcal{F}(\bar{n})| \leq \sum_{i=0}^{m-1} \binom{n-1}{i}$. Moreover, by shiftedness it follows that $|\mathcal{F} \cup G| \leq 2m - 1$ for any $F, G \in \mathcal{F}(n)$, and thus $|\mathcal{F}(n)| \leq \sum_{i=0}^{m-1} \binom{n-1}{i}$. Since $n - 1 > 2(m - 1) + 1$, this inequality is sharp unless $\mathcal{F}(n) = \bigcup_{i=0}^{m-1} \binom{[n-1]}{i}$. From here it should be clear that $\mathcal{F}(\bar{n}) \subset \bigcup_{i=0}^{m-1} \binom{[n-1]}{i}$ and thus $\mathcal{F}(\bar{n}) = \bigcup_{i=0}^{m-1} \binom{[n-1]}{i}$ in case of equality.

We remark that if $\mathcal{F}$ was not shifted initially, then it could not shift into $\bigcup_{i=0}^{m} \binom{[n]}{i}$, thus the uniqueness part holds for not shifted families as well.

Let us return to the case of odd $l$. Consider the subfamily $\mathcal{V}' \subset \mathcal{V}$ of all vectors from $\mathcal{V}$ that have exactly $\frac{l+1}{2}$ minus ones, and put $\mathcal{V}'' := \mathcal{V} \setminus \mathcal{V}'$. Arguing as in the case of even $l$, but applying Theorem 13, we get that $|\mathcal{V}''| \leq \sum_{i=0}^{l-1} \binom{l-1}{i} \binom{n}{k}$, and the inequality is sharp if $\mathcal{V}'' \neq \mathcal{U}$, where $\mathcal{U}$ consists of all $\{-1, 0, 1\}$-vectors with $k$ non-zero coordinates and at most $\frac{l+1}{2}$ minus ones.

Note also that any vector with $\frac{l+1}{2}$ minus ones has scalar product at least $-l$ with any vector from $\mathcal{U}$. Thus, it is clear that $\mathcal{V}'$ must avoid scalar product $-l-1$ and this is sufficient. Therefore, $|\mathcal{V}'| \leq g(n, k - \frac{l+1}{2}, \frac{l+1}{2})$ and, moreover, the largest $\mathcal{V}$ satisfying the requirements has size $g(n, k - \frac{l+1}{2}, \frac{l+1}{2}) + \sum_{i=0}^{l-1} \binom{l}{i} \binom{n}{k} = g(n, k - \frac{l+1}{2}, \frac{l+1}{2}) + f(k, l - 1) \binom{n}{k}$. 

11
6 Proof of Theorem 7

Let $W \subset \{0, \pm 1\}^n$ satisfy $\langle w, w \rangle = 3, \langle w, v \rangle \geq 0$ for all $w, v \in W$ and suppose that $W$ is shifted.

Claim 14. If $S(w) = (a, b, c)$ for $w \in W$ then either $w = u(a, b, c)$ or $w = v(a, b, c)$ holds.

Proof. Assume the contrary and let $(x, y, z)$ be the non-zero coordinate values of the vector $w$ (at positions $a, b, c$). Using the up-shift w.l.o.g. $(x, y, z) = (-1, 1, 1)$ or $(1, -1, 1)$. In both cases by shifting $(1, 1, -1)$ is also in $W$. However it has scalar product $-1$ with both, a contradiction. \qed

From the claim $F(3, 3, 0) \leq 2$, and $F(4, 3, 0) \leq \binom{4}{2} \times 2 = 8$ follow.

From now on $n \geq 5$ and we use induction on $n$ to prove the statement of the theorem.

Set $G = \{(a, b) \in \binom{[n-1]}{2} : u(a, b, n) \in W\}$, $H = \{(a, b) \in \binom{n-1}{2} : v(a, b, n) \in W\}$. By shiftedness $H \subset G$ and by $\langle v, w \rangle \geq 0$, $G$ and $H$ are cross-intersecting. We may assume that $H$ is non-empty, since otherwise $|W| \leq \binom{n}{3}$ holds due to shiftedness.

Claim 15. If $v(2, 3, n) \in W$ then $|H| = |G| = 3$.

Proof. By shiftedness $\{(1, 2), (1, 3), (2, 3)\} \subset H \subset G$. The statement follows from the fact that no other 2-element set can intersect those three sets. \qed

Since $W(\overline{n}) \overset{\text{def}}{=} \{w \in W : w_n = 0\}$ satisfies $|W(\overline{n})| \leq F(n - 1, 3, 0)$, $v(2, 3, n) \in W$ and Claim 15 imply

\[
|W| \leq F(n - 1, 3, 0) + 6, \quad \text{which gives}
\]
\[
|W| \leq 8 + 6 = 14 \quad \text{for } n = 5,
\]
\[
|W| \leq 14 + 6 < 21 \quad \text{for } n = 6, \quad \text{and}
\]
\[
|W| \leq \binom{n-1}{3} + 6 < \binom{n}{3} \quad \text{for } n \geq 7.
\]

Consequently, we may suppose that the degree of $n$ in $W$ is at least 7, in particular, $H$ is the star, $H = \{(1, 2), (1, 3), \ldots, (1, p)\}$ for some $p \leq n - 1$.

The following lemma is obvious.

Lemma 16. One of the following holds.

(i) $p = 2$, $|H| = 1$, $|G| \leq (n - 2) + (n - 3) = 2n - 5$
(ii) $p = 3$, $|H| = 2$, $|G| \leq (n - 2) + 1 = n - 1$
(iii) $p \geq 4$, $|H| \leq |G| \leq n - 2$.

Since for $n = 5$ in all cases $|H| + |G| \leq 6$ holds, the proof of part (1) of Theorem 7 is complete. Also, for $n \geq 7$ it follows that $|H| + |G| \leq 2n - 4$. Since $\binom{n}{3} - \binom{n-1}{3} = \binom{n-1}{2} > 2n - 4$ for $n \geq 7$, the induction step works fine in this case too.
The only case that remains is \( n = 6 \). Using shiftedness and \( a = 1 \) for all \((a, b, n)\) with \( v(a, b, n) \in \mathcal{W} \) it follows that \( a = 1 \) holds for all \((a, b, c)\) with \( v(a, b, c) \in \mathcal{W} \).

Define \( \mathcal{B} = \{(b, c) \in [6] - \{1\} : v(1, b, c) \in \mathcal{W}\} \) and \( \mathcal{D} = \{(d, e, f) \in [3,6] : u(a, b, c) \in \mathcal{W}\} \). Note that \((b, c) \in \mathcal{B} \) and \( D \in \mathcal{D} \) cannot satisfy \( D \cap \{b, c\} = \{c\} \), since otherwise \( \langle u(d, e, f), v(1, b, c) \rangle = -1 \). Let us note that one can put into \( \mathcal{W} \) all the vectors \( u(1, b, c) \) with \( 2 \leq b < c \leq 6 \) because all the vectors with a \(-1\) position have 1 in the first coordinate.

Therefore \( |\mathcal{W}| = 10 + |\mathcal{B}| + |\mathcal{D}| \) holds. We have to prove
\[
|\mathcal{B}| + |\mathcal{D}| \leq 11. \tag{6}
\]

Let us first exhibit the system with 21 vectors showing \( F(6, 3, 0) \geq 21 \).

\[
\mathcal{U}_6 = \{u(1, b, c), v(1, b, c) : (b, c) \subset [2, 6] \} \cup \{u(2, 3, 4)\}.
\]

We have \( |\mathcal{U}_6| = 2 \times \binom{5}{2} + 1 = 21 \) and the only non-trivial scalar product to check is that \( \langle v(1, b, c), u(2, 3, 4) \rangle \geq 0 \). It is true automatically if \( c \geq 5 \). For \( c = 3, 4 \) it follows that \( b \in \{2, 3\} \) making the scalar product equal to 0 as desired.

To avoid a tedious case by case analysis it is simplest to give a matching from the 9-element set \( \{D \in [3,6] : D \neq (2, 3, 4)\} \) into the 10-element set \( \{(b, c) : 2 \leq b < c \leq 6\} \) such that whenever \( D \) and \( (b, c) \) are matched, \( D \cap (b, c) = \{c\} \) holds. This will prove (6). An example of the desired matching is exhibited below.

\[
\begin{align*}
(2, 3, 5) &- (4, 5) & (2, 4, 5) &- (3, 5) & (2, 3, 6) &- (4, 6) \\
(2, 4, 6) &- (5, 6) & (2, 5, 6) &- (3, 6) & (3, 4, 5) &- (2, 3) \\
(3, 4, 6) &- (2, 4) & (3, 5, 6) &- (2, 5) & (4, 5, 6) &- (2, 6)
\end{align*}
\]

A careful analysis shows that \( \mathcal{U}_6 \) is the only extremal configuration. The proof is complete.

### 7 Proof of Theorem 8

We may assume that the families \( \mathcal{A}, \mathcal{B} \) are shifted and, thus, both contain \([k]\). We define \( \mathcal{A}(T) = \{A \setminus T : A \in \mathcal{A}, A \supseteq T\} \). Note that we had a similar notation for families of vectors, but now we are working with sets only, so it should not cause any confusion. We call a set \( T \in [n] \) a kernel for a set family \( \mathcal{S} \), if \( \mathcal{S}(T) \) contains \( k + 1 \) pairwise disjoint edges. An inequality proved in [5] states that
\[
|\mathcal{A}(T)| \leq k \binom{n-s-1}{k-s-1}, \tag{7}
\]
if \( T \) is not a kernel for \( \mathcal{A} \), and similarly for \( \mathcal{B}(T) \).

The following claim is an easy application of the pigeon-hole principle.

**Claim 17.** Suppose that \( C_0, \ldots, C_k \) form a sunflower with center \( T \), i.e. \( C_i \cap C_j = T \) for all \( 0 \leq i < j \leq k \). Suppose \( D \) is a \( k \)-element set. Then there exists an \( i \) such that \( D \cap C_i = D \cap T \) holds.
We immediately get the following corollary:

**Corollary 18.** If $|T| = s$ and $|D \cap C_i| \geq s$ for all $0 \leq i \leq k$, then $T \subset D$.

**Lemma 19.** If $T_1$ and $T_2$ are kernels for $A$ and $B$ respectively, then $T_1 = T_2$ holds.

**Proof.** From Corollary 18 it follows that for any $B \in B$ we have $T_1 \subset B$. Applying it for $B_1, B_2 \in B$, $B_1 \cap B_2 = T_2$, we get that $T_1 \subset B_1 \cap B_2 = T_2$, but since $|T_1| = |T_2|$, we have $T_1 = T_2$.

From Corollary 18 and Lemma 19 it follows that if both $A$ and $B$ have kernels, then $|A| + |B| \leq 2^\binom{n-s}{k-s}$, which is smaller then the bound in (3). From now on we may w.l.o.g. assume that $B$ does not have a kernel and that $B$ contains the set $[k]$ as an element. Then any kernel of $A$ must be a subset of $[k]$. Let $J \subset \binom{[k]}{s}$ be the family of kernels of $A$. Define $	ilde{A} = \{A \in A : \exists T \in J, A \supset T\}$.

**Claim 20.** We have $|\tilde{A}| = O(n^{k-s})$. Analogously, we have $|B| = O(n^{k-s})$.

**Proof.** Any set from $\tilde{A}$ must intersect $[k]$ in at least $s$ elements. Thus, $|\tilde{A}| \leq \sum_{T \in \binom{[k]}{s}} |\tilde{A}(T)|$, which by inequality (7) is at most $\binom{[k]}{s} \binom{n-s}{k-s-1} = O(n^{k-s})$. The same proof works for $B$.

Due to Claim 17 and the fact that $A$ is $t$-intersecting, for any set $A \in A$ and any $T \in J$ we have $|A \cap T| \geq t$. Moreover, repeating the proof of Lemma 19 it is easy to see that for any $T_1, T_2 \in J$ we have $|T_1 \cap T_2| \geq t$. Therefore, $|J| \leq m(k,s,t)$ (see the formulation of the Complete Intersection Theorem). Due to Claim 20 we have

$$|A| + |B| \leq |J| \binom{n-k}{k-s} + O(n^{k-s}).$$

Indeed, the only additional thing one has to note is that the number of sets from $A$ that intersect $[k]$ in at least $s + 1$ elements is $O(n^{k-s})$.

On the other hand, it is easy to exhibit an example of a family $A$ that is $t$-intersecting and is $s$-cross-intersecting with $\{[k]\}$, and which has cardinality $m(k,s,t)\binom{n-k}{k-s}$. For that one just has to take a maximum $t$-intersecting family $J'$ of $s$-element sets in $[k]$ and put $A = \{A \in \binom{[n]}{k} : \exists T \in J' : A \cap [k] = T\}$.

Therefore, if $|J| < m(k,s,t)$, then by (8) and the previous construction $|A| + |B|$ cannot be maximal for large $n$. Therefore, from now on we may assume that $J = A_i(k,s,t)$ for some $0 \leq i \leq s - t$, where $|A_i(k,s,t)| = m(k,s,t)$.

We remark that out of $A_j(k,s,t)$ the only family that satisfies $|A_j(k,s,t)| = |A_j(k+1,s,t)|$ is the family $A_{s-t}(k,s,t)$. Indeed, it consists of all the sets that have all their $s$ elements among the first $2s - t$ elements and does not use elements $j$ with $j > 2s - t$. For all the other families, as we have already mentioned, the degree of each $j \in [k]$ is positive.
Suppose first that $0 \leq i < s-t$ and that $|A_{s-t}(k, s, t)| < m(k, s, t)$. Most importantly for us, it means that $m(k - 1, s, t) < m(k, s, t)$ due to the discussion in the previous paragraph. Then it is easy to see that $|B| = 1$. Indeed, if $B_1, B_2 \in B$, then due to Corollary 18 we know that $T \subset B_1 \cap B_2$ for each $T \in J$. Since $|B_1 \cap B_2| \leq k - 1$, we have $|J| \leq m(k - 1, s, t) < m(k, s, t)$.

We showed that the extremal pair of families satisfies $B = \{[k]\}$. To complete the proof in this case and to show that the pair of extremal families is one of $A_i, B_i$ for $0 \leq i < s - t$, we have to verify that $\{A \cap [k] : A \in A\} \subset M_i(k, s, t)$ provided that $J = A_i(k, s, t)$.

Take any set $A$ such that $A \notin M_i(k, s, t)$, that is, $|A \cap [k]| < k - s + t$ and $|A \cap [t + 2i]| < t + i$. The following claim completes the proof in the case $m(k, s, t)$ is attained on $M_i(k, s, t)$ with $i < s - t$.

**Claim 21.** In the notations above, there is $T \in J = A_i(k, s, t)$ such that $|T \cap A| \leq t - 1$.

**Proof.** W.l.o.g., let $A = [1, l] \cup [m + 1, k]$, where $l \leq t + i - 1$ and $l + m < k - s + t$. Set $T = [l', l' + s - 1]$ with $l' \leq i + 1$ and with $l'$ chosen in such a way that $|T \cap A|$ is minimized. It is easy to see that, first, $T \in J$ and, second, $|T \cap A| = \max\{l - i, l + m + s - k\} < t$. 

If the only maximal family is $A_{s-t}(k, s, t)$, then $J = A_{s-t}(k, s, t)$ and $A$ cannot contain a set $A$ that satisfies $|A \cap [2s - t]| < s$. Indeed, then $A$ intersects one of the $T \in J$ in less than $s$ elements, which by Claim 17 means that there exists $A' \in A$ such that $|A' \cap A| < t$, a contradiction. Therefore, the family $A$ is contained in $A_{s-t}$, and we are only left to prove that $B \subset B_{s-t}$. Assume that $B$ contains $B$, such that $B \supseteq [2s - t]$. But then there is $T \in J$ such that $|B \cap T| \leq s - 1$, a contradiction.

The last case that remains to verify is when both $A_{s-t}(k, s, t)$ and $A_{s-t-1}(k, s, t)$ have size $m(k, s, t)$. But either $J = A_{s-t-1}(k, s, t)$ or $J = A_{s-t}(k, s, t)$ and we end up in one of the two scenarios described above.

It is clear from the proof that the families that we constructed are the only possible extremal families (excluding the case $n = 2k$, $s = 1$), which is mostly due to the uniqueness in the formulation of the Complete Intersection Theorem.

**Acknowledgements.** We thank Danila Cherkashin and Sergei Kiselev for pointing out the error in the original version of Theorem 15.

**References**

[1] R. Ahlswede, L. Khachatrian, *The complete intersection theorem for systems of finite sets*, Eur. J. Comb. 18 (1997), N2, 125–136.

[2] P. Erdős, C. Ko, R. Rado, *Intersection theorems for systems of finite sets*, The Quarterly Journal of Mathematics, 12 (1961) N1, 313–320.
[3] P. Frankl, The Erdős-Ko-Rado theorem is true for $n=ckt$, Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Vol. I, 365–375, Colloq. Math. Soc. János Bolyai, 18, North-Holland.

[4] P. Frankl, On cross-intersecting families, Discrete Math. 108 (1992), 291-295.

[5] P. Frankl, The shifting technique in extremal set theory, Surveys in combinatorics, Lond. Math. Soc. Lecture Note Ser. 123 (1987), 81–110, Cambridge University Press, Cambridge.

[6] P. Frankl, Z. Füredi, Beyond the Erdős-Ko-Rado theorem, Journal of Combinatorial Theory, Ser. A 56 (1991) N2, 182–194.

[7] P. Frankl, A. Kupavskii, Erdős-Ko-Rado theorem for \{0, \pm1\}-vectors, J. Comb. Theory Ser. A 155 (2018), 157–179.

[8] P. Frankl, A. Kupavskii, Families of vectors without antipodal pairs, Studia Sci. Math. Hungarica 55 (2018), N2, 231–237.

[9] P. Frankl, A. Kupavskii, A size-sensitive inequality for cross-intersecting families, European Journal of Combinatorics 62 (2017), 263–271.

[10] P. Frankl, A. Kupavskii, Uniform s-cross-intersecting families, Combinatorics, Probability and Computing 26 (2017), N4, 517–524.

[11] P. Frankl, N. Tokushige, Some best possible inequalities concerning cross-intersecting families, Journal of Combinatorial Theory, Ser. A 61 (1992), N1, 87–97.

[12] P. Frankl, R. Wilson, Intersection theorems with geometric consequences, Combinatorica 1 (1981), 357–368.

[13] A.J.W. Hilton, E.C. Milner, Some intersection theorems for systems of finite sets, Quart. J. Math. Oxford 18 (1967), 369–384.

[14] G. Katona, Intersection theorems for systems of finite sets, Acta Math. Acad. Sci. Hung. 15 (1964), 329–337.

[15] D. Kleitman, On a combinatorial conjecture of Erdős, J. Combin. Theory 1 (1966) 209–214.

[16] E.I. Ponomarenko, A.M. Raigorodskii, New upper bounds for the independence numbers with vertices at $\{-1, 0, 1\}^n$ and their applications to the problems on the chromatic numbers of distance graphs, Mat. Zametki, 96 (2014), N1, 138 - 147; English transl. in Math. Notes, 96 (2014), N1, 140 - 148.

[17] A.M. Raigorodskii, Borsuk’s problem and the chromatic numbers of some metric spaces, Russian Math. Surveys, 56 (2001), N1, 103 - 139.

[18] A. Sali, Some intersection theorems, Combinatorica 12 (1992), 351–361.