Experimental realization of non-Abelian non-adiabatic geometric gates

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The geometric aspects of quantum mechanics are emphasized most prominently by the concept of geometric phases, which are acquired whenever a quantum system evolves along a path in Hilbert space, that is, the space of quantum states of the system. The geometric phase is determined only by the shape of this path1–3 and is, in its simplest form, a real number. However, if the system has degenerate energy levels, then matrix-valued geometric state transformations, known as non-Abelian holonomies—the effect of which depends on the order of two consecutive paths—can be obtained4. They are important, for example, for the creation of synthetic gauge fields in cold atomic gases5 or the description of non-Abelian anyon statistics6,7. Moreover, there are proposals8 to exploit non-Abelian holonomic gates for the purposes of noise- resilient quantum computation. In contrast to Abelian geometric operations8, non-Abelian ones have been observed only in nuclear quadrupole resonance experiments with a large number of spins, and without full characterization of the geometric process and its non-commutative nature11,12. Here we realize non-Abelian non-adiabatic holonomic quantum operations13,14 on a single, superconducting, artificial three-level atom15 by applying a well-controlled, two-tone microwave drive. Using quantum process tomography, we determine fidelities of the resulting non-commuting gates that exceed 95 per cent. We show that two different quantum gates, originating from two distinct paths in Hilbert space, yield non-equivalent transformations when applied in different orders. This provides evidence for the non-Abelian character of the implemented holonomic quantum operations. In combination with a non-trivial two-quantum-bit gate, our method suggests a way to universal holonomic quantum computing.

A cyclic evolution of a non-degenerate quantum system is in general accompanied by a phase change of its state. The acquired Abelian phase can be divided into two parts: the dynamical phase, which is proportional to the evolution time and the energy of the system, and the geometric phase, which depends only on the path of the system in Hilbert space. If the system is guided along two different loops in a row, the overall accumulated geometric phase is independent of their sequential arrangement, because the Abelian phases associated with the loops are additive. The situation changes drastically if the energy spectrum of the system contains degenerate subspaces. In this case, the system can undergo a path-dependent unitary transformation by acquiring a non-Abelian holonomy. Because the holonomic transformations do not commute, the order of the two successive loops makes a difference.

The ability to realize non-commuting quantum operations by choosing different paths can be employed for holonomic quantum computation8, which has attracted particular attention8 because of the resilience of geometric phases to certain fluctuations during the evolution of the system19–21. In this scheme, quantum bits are encoded in a doubly degenerate eigenspace of the system Hamiltonian, \(\hat{h}(\lambda)\). The parameters encoded in the vector \(\lambda\) are varied to induce a cyclic evolution of the system. In the original proposal on holonomic quantum computation8, the parameters \(\lambda\) are changed adiabatically in time to guarantee the persistence of the degeneracy. Adiabatic holonomic gates have been proposed for trapped ions8, superconducting quantum bits12,22 (qubits) and semiconductor quantum dots12. However, they are difficult to realize experimentally because of the long evolution time needed to fulfill the adiabatic condition. Instead, a scheme based on non-adiabatic, non-Abelian holonomies13 has been proposed14. Such holonomies, like non-Abelian geometric gates15, combine universality and speed and can thus be implemented conveniently in experiments.

The main idea is to generate a non-adiabatic and cyclic state evolution in a three-level system that results in a purely geometric operation on the degenerate subspace spanned by the computational basis states, \(|0\rangle\) and \(|1\rangle\). The third state, \(|e\rangle\), acts as an auxiliary state and remains unpopulated after the gate operation. This is achieved by driving the system using two resonant microwave pulses (Fig. 1a) with identical time-dependent envelopes, \(\Omega(t)\), but different amplitudes, \(a\) and \(b\), satisfying \(|a|^2 + |b|^2 = 1\) (Fig. 1b). The Hamiltonian of the system in the interaction picture is

\[
h(t) = \frac{h\Omega(t)}{2} (a|e\rangle \langle 0| + b|e\rangle \langle 1|) + \text{H.c.}
\]

where \(h\) denotes Planck’s constant divided by 2\(\pi\), \(\Omega\) stands for Hermitian conjugate and we have used the rotating-wave approximation. This Hamiltonian causes the initial basis vectors to evolve to the states \(|\psi(t)\rangle = \exp\left(-i/h \int_0^t \Omega(t')\,dt'\right) |i\rangle\) (if, \(j = 0, 1\)). Unlike in adiabatic schemes, the \(|\psi(t)\rangle\) are not instantaneous eigenstates of \(h(t)\). By keeping \(a/b\), the complex amplitude ratio of the pulses, constant, no transitions between states are induced and the evolution satisfies the parallel-transport condition, \(\langle\psi(t')|h(t)|\psi(t)\rangle = 0\). As a consequence, the evolution is purely geometric, with vanishing dynamic contributions (Supplementary Information). If the pulse length, \(\tau\), is chosen such that \(\int_0^\tau \Omega(t)\,dt = 2\pi\), the degenerate subspace undergoes a cyclic evolution, and the matrix representation of the final operator that acts on the basis states \(|0\rangle\) and \(|1\rangle\) is

\[
U = \begin{pmatrix}
\cos(\theta) & e^{i\phi} \sin(\theta) \\
-e^{-i\phi} \sin(\theta) & -\cos(\theta)
\end{pmatrix}
\]

where we have parameterized the drive amplitudes via the relation \(e^{i\phi}\tan(\theta/2) = a/b\). A geometric interpretation of the dynamics of the system is visualized in Fig. 1c, in which different values of \(\theta\) and \(\phi\) correspond to different paths \(C\) in Hilbert space.

In our experiments, we realize the holonomic gates using a three-level superconducting artificial atom of the transmon type, embedded in a three-dimensional cavity23 (Fig. 2a). The cavity is made of aluminium and has inner dimensions of 32 mm \(\times 15.5\) mm \(\times 5\) mm. The frequency of the fundamental mode is \(\nu_{0\text{c}}/2\pi = 8.999\) GHz as measured by transmission spectroscopy using the circuit shown in Fig. 2b. The quality factor of the resonator is \(Q = 21,000\). The transmon is made of two 500 \(\mu\)m \(\times 250\) \(\mu\)m aluminium electrodes separated by...
A phase shift corresponding to a phase-flip gate, denoted by the Pauli operator $\sigma_z$, is caused by the deformation of the loop pulses on the $|0\rangle$ state is prepared by sequentially applying pulses on the $|0\rangle$ state. The transport condition fixes the choice of basis states along the loop subspaces remain unchanged under a unitary basis transformation; equivalent subspaces are represented by a fibre associated to each point in $G$. The difference between the initial ($t = 0$) and final ($t = \pi$) points lying on a single fibre corresponds to the holonomy matrix $\Gamma \in U(2)$, which is fully determined by the loop $C$. The experimentally obtained results are in good agreement with theory. For $\theta = 0$, a single drive on the $|\psi\rangle \leftrightarrow |\gamma\rangle$ transition causes a phase shift corresponding to $\gamma = 1$ and the operation $U(0, 0) = \sigma_z$. For $\theta = \pi/4$, the transformation $H = (\sigma_x - \sigma_y) / \sqrt{2}$ is generated (Fig. 3b), which is equivalent to the Hadamard gate. For $\theta = \pi/2$, the pulses are of equal amplitude and the transformation is a NOT gate, $\sigma_x$ (Fig. 3c). Because of dephasing and relaxation of both excited states as well as the finite fidelities of the microwave pulses, the state $|\psi\rangle$ is slightly populated after the gate operation. This population ‘leakage’ is quantified by computing the trace $\text{Tr}(\tilde{\chi}) = 0.96$ of the reduced process matrix $\tilde{\chi}$ (Fig. 3a, black dots).
The experimentally obtained fidelities of the geometric transformations are $F_H = 95.4 \pm 0.6\%$ and $F_{\text{NOT}} = 97.5 \pm 0.9\%$. The numerical solution of a master equation including dissipative processes results in a fidelity of $F = 97.6\%$ for both processes, in good agreement with the experimental values. From this, we conclude that decoherence and decay processes along with dynamical contributions (Supplementary Information) are the main limiting factors for gate performance.

To show explicitly that different loops in Hilbert space result in non-commuting gates, we sequentially apply the geometric transformations $H$ and $\text{NOT}$ in different orders. The non-Abelian character of the operation yields either the operation $\text{NOT} \cdot H = -(i\sigma_3 + 1)/\sqrt{2}$ (Fig. 4a) or $H \cdot \text{NOT} = (i\sigma_3 - 1)/\sqrt{2}$ (Fig. 4b), where $I$ is the identity matrix. We visualize the operations on the Bloch sphere in Fig. 4c, d. The operation $\text{NOT} \cdot H$ corresponds to a $\pi$-rotation about the $H$ axis (the line bisecting the $x$ and $-x$ axes), followed by a $\pi$-rotation about the $x$ axis. This is equivalent to a rotation about the $y$ axis by $\pi/2$. The operation $H \cdot \text{NOT}$ corresponds to a rotation in the opposite direction.

In general, by concatenation of two geometric operations, rotations about arbitrary axes corresponding to a representation of the complete SU(2) group can be realized. Applying the scheme presented here—or, alternatively, cavity-induced geometric phase shifts—to two coupled three-level systems—will complete the universal set of geometric quantum gates and allow for the execution of all-geometric quantum algorithms, which are potentially resilient to noise when short pulses are used. Moreover, holonomic gates demonstrated for superconducting quantum devices could also be applied to other three-level systems with similar energy level structure.

METHODS SUMMARY

To characterize the gates, we perform full process tomography on the three-level system and reconstruct the process matrix, $\rho_{\text{exp}}$, using a maximum-likelihood procedure. In the presence of dissipative processes, the final density matrix, $\rho_f$, is given by the quantum dynamical map $\rho_f = \sum_{\alpha} z_{\alpha} F_{\alpha}(\rho) E_{\alpha}$ acting on the initial state $\rho_0$. The full set of nine orthogonal basis operators $\{E_i\} \in \text{SU(3)}$ is chosen as $\{E_i\} = \{I, i\sigma_1, -i\sigma_1, i\sigma_2, -i\sigma_2, i\sigma_3, -i\sigma_3, -I\}$, where $\sigma_i$ are Pauli operators acting on the levels $i$ and $j$, $\langle i| = |i\rangle \langle i|$ and $I = \langle\rho\rangle |\langle\rho\rangle\rangle$. The process is fully determined by its action on a complete set of nine input states, $|0\rangle$, $|1\rangle$, $|i\rangle$, $|0\rangle + |1\rangle/\sqrt{2}$, $|0\rangle + i|1\rangle/\sqrt{2}$, $|0\rangle + i|1\rangle/\sqrt{2}$, $|0\rangle + |1\rangle/\sqrt{2}$, $|0\rangle + |1\rangle/\sqrt{2}$, $|0\rangle + |1\rangle/\sqrt{2}$. These states are prepared by sequentially applying the identity, the $\pi$-rotation $R_H(\pi)$ and the $\pi/2$-rotations $R_H(\pi/2)$ and $R_H(\pi/2)$ to the $|0\rangle \leftrightarrow |e\rangle$ and $|0\rangle \leftrightarrow |s\rangle$ transitions. After applying the geometric operation to the input states, we perform full state tomography on the respective output states.

The length of a typical sequence is 206 ns (five 40-ns pulses with 2-ns separation). We calibrate the π-pulses on the $|0\rangle \leftrightarrow |e\rangle$ and $|0\rangle \leftrightarrow |s\rangle$ transitions by measuring Rabi oscillations between the corresponding states. From the recorded $\chi^2 = 81$ measurements, we reconstruct the process matrix, $\rho_{\text{exp}}$. The process is compared to the ideal one in equation (1) by calculating its fidelity as $F = Tr(\rho_{\text{exp}} \cdot \rho_0)$. To show explicitly that different loops in Hilbert space result in non-commuting gates, we sequentially apply the geometric transformations $H$ and $\text{NOT}$ in different orders. The non-Abelian character of the operation yields either the operation $\text{NOT} \cdot H = -(i\sigma_3 + 1)/\sqrt{2}$ (Fig. 4a) or $H \cdot \text{NOT} = (i\sigma_3 - 1)/\sqrt{2}$ (Fig. 4b), where $I$ is the identity matrix. We visualize the operations on the Bloch sphere in Fig. 4c, d. The operation $\text{NOT} \cdot H$ corresponds to a $\pi$-rotation about the $H$ axis (the line bisectiong the $x$ and $-x$ axes), followed by a $\pi$-rotation about the $x$ axis. This is equivalent to a rotation about the $y$ axis by $\pi/2$. The operation $H \cdot \text{NOT}$ corresponds to a rotation in the opposite direction.

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Figure 4 | Non-commutativity of holonomic gates. a, b, Process matrices for $\text{NOT} \cdot H$ (a) and $H \cdot \text{NOT}$ (b) gates with fidelities of 95.9%. Because of the non-Abelian character of the geometric operations, the resulting processes are different. c, d, This can be visualized on the Bloch sphere by two rotations around the $x$ and $H$ axes (a represented by $c$, b represented by $d$). The initial state of the system (1) is rotated around the red axis first (red dotted line) towards the intermediate state (2), and is then rotated around the green axis (green dashed line) to the final state (3). The effect of the combined rotations is shown by the thick black line.

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Supplementary Information is available in the online version of the paper.

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