Alice-Bob Peakon Systems

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Abstract

In this letter, we study the Alice-Bob peakon system generated from an integrable peakon system through using the strategy of the so-called Alice-Bob non-local KdV approach [13]. Nonlocal integrable peakon equations are obtained and shown to have peakon solutions.
I. INTRODUCTION

Recently, nonlocal integrable systems have attracted much attention since the pioneer work about nonlocal nonlinear Schrödinger equation [1]. Lou and Huang applied the nonlocal approach to the KdV equation and proposed the so-called Alice-Bob Physics to generate coherent solutions of nonlocal KdV systems [13]. Before discussing AB peakon systems, let us first recall the basic procedure for regular AB integrable systems.

In the study of classical physics theory, most feasible models are locally established around a single space-time point, say, \( \{ x, t \} \). To investigate some related physical phenomena in two or more places, one has to consider some kinds of new models. In papers [12, 13], the authors proposed Alice-Bob (AB) models to study two-place physical problems. Alice-Bob physics makes sense if the physics is related to two entangled events occurred in two places \( \{ x, t \} \) and \( \{ x', t' \} \). The event at \( \{ x, t \} \) is called Alice event (AE) (denoted by \( A(x, t) \)) and the event at \( \{ x', t' \} \) is called Bob event (BE) (denoted by \( B(x', t') \)). The events AE and BE are called correlated/entangled if AE happens, BE can be determined immediately by the correlation condition

\[
B(x', t') = f(A) = A^f = \hat{f} A,
\]

where \( \hat{f} \) stands for a suitable operator, which may be different at different events. In general, \( \{ x', t' \} \) is not a neighbour to \( \{ x, t \} \). Thus, the intrinsic two-place models, the Alice-Bob systems (ABSs), are nonlocal. Some special types of two-place nonlocal models have been proposed. For example, the following nonlocal nonlinear Schrödinger (NLS) equation,

\[
iA_t + A_{xx} \pm A^2B = 0, \quad B = \hat{f} A = \hat{P} \hat{C} A = A^*(-x, t),
\]

was firstly proposed by Ablowitz and Musslimani[1]. This type of NLS systems has strong relationships to the significant PT symmetric Schrödinger equations[14]. The operators \( \hat{P} \) and \( \hat{C} \) are the usual parity and charge conjugation. Afterwards, other nonlocal systems, including the coupled nonlocal NLS systems [18], the nonlocal modified KdV systems [3, 11], the discrete nonlocal NLS system [2], and the nonlocal Davey-Stewartson systems [6, 7, 9], were investigated as well.
In this letter, we shall utilize the Alice-Bob approach to study integrable peakon equations. For our convenience, the peakon system based on the Alice-Bob (AB) approach is called the Alice-Bob peakon (ABP) system. We will take the integrable peakon systems studied in [19] as the ABP examples.

II. AN INTEGRABLE ALICE-BOB PEAKON SYSTEM WITH LAX PAIR AND INFINITELY MANY CONSERVATION LAWS

Let us consider the following model:

\[ m_t = (mH)_x + mH - \frac{1}{2} m(A - A_x)(A + A_x), \quad \bar{A} = A(-x + x_0, -t + t_0), \quad (2) \]
\[ m = A - A_{xx}, \quad (3) \]

where \( H \) is an arbitrary shifted parity (\( \hat{P}_s \)) and delayed time reversal (\( \hat{T}_d \)) invariant functional

\[ \hat{P}_s \hat{T}_d H = \bar{H} = H, \quad (4) \]

and the definitions of \( \hat{P}_s \) and \( \hat{T}_d \) read \( \hat{P}_s x = -x + x_0, \quad \hat{T}_d t = -t + t_0 \). Actually, equation (2) is generated through the first equation of the two-component system (7) proposed in the paper [19] via \( u = A \) and \( v = \bar{A} \).

Through a lengthy computation, one may have a Lax pair for equation (2)

\[ \phi_x = U \phi, \quad U = \begin{pmatrix} -1 & \lambda m \\ -\lambda \bar{m} & 1 \end{pmatrix}, \quad \bar{m} = \bar{A} - \bar{A}_{xx}, \quad (5) \]
\[ \phi_t = V \phi, \quad V = \frac{1}{2} \begin{pmatrix} \frac{1}{2}(A_x - A)(A + A_x) - \lambda^{-2} & \lambda^{-1}(A - A_x) + \lambda mH \\ -\lambda^{-1}(\bar{A} + \bar{A}_x) - \lambda \bar{m}H & \lambda^{-2} + \frac{1}{2}(A - A_x)(\bar{A} + \bar{A}_x) \end{pmatrix}. \quad (6) \]

Following the typical procedure starting from Lax pair, we may obtain the following con-
ervation laws for equation (2):

$$\rho_{jt} = J_{jx}, \quad j = 0, 1, 2, \ldots, \infty,$$

(7)

$$\rho_j = m\omega_j, \quad j = 0, 1, 2, \ldots,$$

(8)

$$J_j = (A - A_x)\omega_{j-2} + H\rho_j, \quad j = 2, 3, \ldots,$$

(9)

$$J_0 = H\rho_0, \quad J_1 = H\rho_1 - \frac{1}{2}(A - A_x)(\bar{A} + \bar{A}_x),$$

(10)

$$\omega_{j+1} = \frac{1}{m\omega_0} \left( \omega_j - \omega_{jx} - \frac{m}{2} \sum_{k=1}^{j} \omega_k \omega_{j+1-k} \right), \quad j = 1, 2, \ldots,$$

(11)

$$\omega_0 = \sqrt{-\bar{m}m^{-1}}, \quad \omega_1 = \frac{m\bar{m}_x - \bar{m}m_x - 2m\bar{m}}{2m^2\bar{m}}.$$  

(12)

It is noticed that the conserved densities $\rho_j = m\omega_j, \quad j = 0, 1, \ldots$, are not explicitly dependent on $H$.

III. PEAKON SOLUTIONS FOR SPECIAL ABP SYSTEMS

Let us now solve concrete peakon solutions from equation (2) for some special functions $H$ listed in the following examples.

**Example 1.** Taking $H = 0$ sends equation (2) to the following system

$$\begin{cases}
m_t = -\frac{1}{2}m(A - A_x)(\bar{A} + \bar{A}_x), \\
m = A - A_{xx}, \quad \bar{A} \equiv A(-x + x_0, -t + t_0).
\end{cases}$$

This system has only one-peakon solutions:

$$A = c \exp \left[ -\frac{1}{3}c^2 \left( t - \frac{t_0}{2} \right) - \left| x - \frac{x_0}{2} \right| \right],$$

(13)

which are non-traveling solitary waves with a fast decayed standing peak and shown in Fig. 1.

No multi-peakon solution is found for this special example though it is integrable with Lax pair and infinitely many conservation laws.

**Example 2.** Selecting $H = \frac{1}{2}(A\bar{A} - A_x\bar{A}_x)$ leads equation (2) to

$$\begin{cases}
m_t = \frac{1}{2}[m(A\bar{A} - A_x\bar{A}_x)]_x - \frac{1}{2}m(A - A_x)(\bar{A} + \bar{A}_x), \\
m = A - A_{xx}, \quad \bar{A} \equiv A(-x + x_0, -t + t_0).
\end{cases}$$

(14)
FIG. 1: Figure 1. The fast decayed non-traveling peakon solution (13) for the first special ABP system (13) with the peakon parameters $x_0 = t_0 = 0$ and $c = 1$.

This equation possesses the following one peakon solutions

$$A = c \exp \left[ - \left| (x - \frac{x_0}{2}) + \frac{1}{3} c^2 \left( t - \frac{t_0}{2} \right) \right| \right]. \quad (15)$$

as well as $N$-peakon dynamical system

$$A = \sum_{j=1}^{N} p_j \exp \left( - \left| x - \frac{x_0}{2} - q_j \right| \right), \quad (16)$$
\[ p_{jt} = \frac{1}{2}p_j \sum_{i,k=1}^{N} p_i \bar{p}_k [sgn(q_j - q_k) - sgn(q_j - q_i)] e^{-|q_j - q_k| - |q_j - q_i|}, \] (17)

\[ q_{jt} = \frac{1}{6}p_j \bar{p}_j - \frac{1}{2} \sum_{i,k=1}^{N} p_i \bar{p}_k [1 - sgn(q_j - q_k) sgn(q_j - q_i)] e^{-|q_j - q_k| - |q_j - q_i|}. \] (18)

Though the integrability problem of the above \( N \)-peakon system is open, we do get the following explicit 2-peakon solutions

\[ p_{1t} = \frac{1}{2}p_1 (p_1 \bar{p}_2 - p_2 \bar{p}_1) sgn(q_1 - q_2) e^{-|q_1 - q_2|}, \] (19)

\[ p_{2t} = \frac{1}{2}p_2 (p_2 \bar{p}_1 - p_1 \bar{p}_1) sgn(q_2 - q_1) e^{-|q_2 - q_1|}, \] (20)

\[ q_{1t} = \frac{1}{3}p_1 \bar{p}_1 - \frac{1}{2} (p_1 \bar{p}_2 + p_2 \bar{p}_1) e^{-|q_1 - q_2|}, \] (21)

\[ q_{2t} = -\frac{1}{3}p_2 \bar{p}_2 - \frac{1}{2} (p_1 \bar{p}_2 + p_2 \bar{p}_1) e^{-|q_2 - q_1|}. \] (22)

Therefore,

\[ q_1 = \frac{3p_1 p_2 sgn(2t - t_0)}{|p_1^2 - p_2^2|} \left[ e^{-\frac{1}{4}|p_1^2 - p_2^2|(2t - t_0)| - 1} - \frac{p_1^2}{6} (2t - t_0), \right] \]

\[ q_2 = \frac{3p_1 p_2 sgn(2t - t_0)}{|p_1^2 - p_2^2|} \left[ e^{-\frac{1}{4}|p_1^2 - p_2^2|(2t - t_0)| - 1} - \frac{p_2^2}{6} (2t - t_0), \right] \] (23)

where \( p_1 \) and \( p_2 \) are two arbitrary constants.

Fig.2 exhibits the single steady traveling peakon solution expressed by (15).

Fig.3 shows the interactional behaviour for two-peakons given by (16) with \( N = 2 \) and (23).

**Example 3.** Choosing \( H = \frac{1}{2}(A \bar{A}_x - A_x \bar{A}) \) casts equation (2) to the following system

\[
\begin{cases}
    m_t = \frac{1}{2} [m(A \bar{A}_x - A_x \bar{A})]_x - \frac{1}{2} m [A \bar{A} - A_x \bar{A}_x], \\
    m = A - A_{xx}, \bar{A} \equiv A(-x + x_0, -t + t_0),
\end{cases} \] (24)

which has fast decayed one-peakon solutions as follows

\[ A = c \exp \left[ -\frac{1}{3} c^2 \left( t - t_0 \right) - \left| x - \frac{x_0}{2} \right| \right]. \] (25)

This single peakon possesses the same form as the one in Example 1.
FIG. 2: Figure 2. The single steady traveling peakon solution (15) for the second special ABP system (14) with the peakon parameters $x_0 = t_0 = 0$ and $c = 1$.

Similar to the example one we have not yet found two-peakon solutions though the model is also integrable.

**Example 4.** Let $H = \frac{1}{2}(A - A_x)(\bar{A} + \bar{A}_x)$. Then we obtain the following system

$$
\begin{align*}
    m_t &= \frac{1}{2} m (A - A_x)(\bar{A} + \bar{A}_x)_x, \\
    m &= A - A_{xx}, \quad \bar{A} \equiv A(-x + x_0, -t + t_0),
\end{align*}
$$

which admits the following one-peakon solutions

$$
A = c \exp \left[ - \left| \left( x - \frac{x_0}{2} \right) + \frac{1}{3} c^2 \left( t - \frac{t_0}{2} \right) \right| \right].
$$
FIG. 3: Figure 3. The two peakon interactional solution (16) for the second special ABP system (14). The corresponding parameters are chosen as $N = 2$, $x_0 = t_0 = 0$, $p_1 = 1$ and $p_2 = 2$.

Furthermore, we can obtain $N$-peakon dynamical system

$$A = \sum_{j=1}^{N} p_j \exp\left(-\left|x - \frac{x_0}{2} - q_j\right|\right)$$  \hspace{1cm} (28)

$$p_{jt} = 0,$$  \hspace{1cm} (29)

$$q_{jt} = \frac{1}{6} p_j p_j - \frac{1}{2} \sum_{i,k=1}^{N} p_i p_k \left[\text{sgn}(q_j - q_k) - 1\right] \text{sgn}(q_j - q_i) + 1 e^{-|q_j - q_k| - |q_j - q_i|}.$$  \hspace{1cm} (30)
In particular, two peakon solutions take on the following form

\[
q_1 = \frac{1}{6} p_1^2 (2t - t_0) + \frac{3p_1 p_2 \mathrm{sgn}(2t - t_0)}{|p_1^2 - p_2^2|} \left( e^{\frac{1}{6} |(p_1^2 - p_2^2)(2t - t_0)|} - 1 \right),
\]

\[
q_2 = \frac{1}{6} p_2^2 (2t - t_0) + \frac{3p_1 p_2 \mathrm{sgn}(2t - t_0)}{|p_1^2 - p_2^2|} \left( e^{\frac{1}{6} |(p_1^2 - p_2^2)(2t - t_0)|} - 1 \right)
\]

where \(p_1\) and \(p_2\) are two arbitrary constants. In this case, both the single peakon and two peakon forms are the same as those in Example 2.

IV. SUMMARY AND DISCUSSION

In this paper, we give a quite general integrable Alice-Bob peakon system (2) with an arbitrary \(\hat{P}_s \hat{T}_d\) invariant functional. The existence of multi-peakon solutions are discussed for some special cases through selecting different invariant functionals \(\hat{P}_s \hat{T}_d\). The Lax pair of the ABP system (2) and infinitely many conservation laws are found. Starting from the Lax pair and the conservation laws, one may readily set up other integrable properties such as the infinitely many symmetries, bi-Hamiltonian structures, recursion operator, and inverse scattering transformation. But, we do not discuss those topics here in our paper, instead, we focus on peakon solutions for the ABP system. It already reveals from Examples 1 and 3 that for the ABP systems if the single peakon is non-travelling (standing) and fast decayed, then there may be no two-peakon interactional solutions. If the single peakon is non-standing (travelling) and not decayed like Examples 2 and 4, then there may exist multi-peakon solutions. Another amazing fact is that for different ABP systems like Examples 2 and 4, their multi-peakon solutions might be completely same.

There are some intrinsic difference between usual peakon systems and ABP systems. The main difference is that usual peakon systems are local while the ABP systems are nonlocal. The nonlocal property is introduced because the model includes two far-away correlated events, namely, AE and BE. Due to the intrusion of two far-away events in one model, the invariant property in both space and time translations is broken. This kind of symmetry-breaking property probably destroys the existence of multiple decayed standing peakons.

The study of regular peakon systems is one of the hot topics in mathematical physics and
many different peakon systems were found in the literature. For every peakon system, there may exist different versions of integrable ABP systems. Here in the following we just list some of them we developed, but leave the details for our near future investigations.

**Alice-Bob Camassa-Holm (ABCH).** Based on the well-known Camassa-Holm (CH) equation [4], one of the integrable ABCH systems can be written as

\[(1 - \partial_x^2)A_t = (A + B)(3A - A_{xx})_x - 2(A + B)_xA_{xx} + H\]

where \(B = \hat{P}_s\hat{T}_dA\) and \(H\) is an arbitrary \(\hat{P}_s\hat{T}_d\) invariant functional.

**Alice-Bob DP (ABDP).** Similar to the ABCH, adopting the well-known Degasperis-Procesi (DP) equation [5] yields the integrable ABDP system in the following form

\[(1 - \partial_x^2)A_t = (A + B)(4A - A_{xx})_x - 3(A + B)_xA_{xx} + H\]

with an arbitrary \(\hat{P}_s\hat{T}_d\) invariant functional \(H\) and \(B = \hat{P}_s\hat{T}_dA\).

**Alice-Bob b-family (ABbf).** Similar to the ABCH and ABDP and considering the \(b\)-family equation [10], one may generate the ABbf system in the following form

\[(1 - \partial_x^2)A_t = (A + B)((b + 1)A - A_{xx})_x - b(A + B)_xA_{xx} + H\]

with an arbitrary \(\hat{P}_s\hat{T}_d\) invariant functional \(H\) and \(B = \hat{P}_s\hat{T}_dA\).

**Alice-Bob Novikov (ABN).** For the Novikov equation [15], there exists the following ABN system

\[(1 - \partial_x^2)A_t = (A + B)^2(4A - A_{xx})_x + \frac{3}{2}(A + B)_xA_{xx} + H\]

with \(\hat{P}_s\hat{T}_d\) invariant functional \(H\) and \(B = \hat{P}_s\hat{T}_dA\).

**Alice-Bob FORQ (ABFORQ).** For the FORQ system [8, 16, 17], we have an integrable ABFORQ extension as follows

\[(1 - \partial_x^2)A_t = \{[(A + B)^2 - (A + B)_x^2](A_{xx} - A)\}_x + H\]

where \(H\) is \(\hat{P}_s\hat{T}_d\) invariant and \(B = \hat{P}_s\hat{T}_dA\).
Acknowledgements

The author are in debt to the helpful discussions with Professors X. B. Hu and Q. P. Liu. The work was sponsored by the Global Change Research Program of China (No.2015CB953904), the National Natural Science Foundations of China (Nos. 11435005), Shanghai Knowledge Service Platform for Trustworthy Internet of Things (No. ZF1213) and K. C. Wong Magna Fund at Ningbo University. The author (Qiao) thanks the UTRGV President’t Endowed Professorship and the UTRGV College of Science seed grant for their partial supports.

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