Strongly Multiplicatively P-function and Some New Inequalities

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Abstract In this study, we present a new definition of convexity. This definition is the class of strongly multiplicatively P-functions. Some new Hermite-Hadamard type inequalities are derived for strongly multiplicatively P-functions. Some applications to special means of real numbers are given. Ideas of this paper may stimulate further research.

Keywords: Convex function, multiplicatively P-function, strongly multiplicatively P-function, Hölder and Power-Mean Integral inequalities, Hermite-Hadamard inequality

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1. Introduction

In this section, we firstly give several definitions and some known results.

Definition 1: A function \( f: I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) is said to be convex if the inequality

\[
f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)
\]

is valid for all \( x,y \in I \) and \( t \in [0,1] \). If this inequality reverses, then the function \( f \) is said to be concave on interval \( I \neq \emptyset \).

This definition is well known in the literature. Convexity theory has appeared as a powerful technique to study a wide class of unrelated problems in pure and applied sciences.

One of the most important integral inequalities for convex functions is the Hermite-Hadamard inequality. The classical Hermite–Hadamard inequality provides estimates of the mean value of a continuous convex function \( f: [a,b] \rightarrow \mathbb{R} \). The following double inequality is well known as the Hermite-Hadamard inequality in the literature.

Definition 2: Let \( f: [a,b] \rightarrow \mathbb{R} \) be a convex function, then the inequality

\[
f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}
\]

is known as the Hermite-Hadamard inequality.

Some refinements of the Hermite-Hadamard inequality on convex functions have been extensively investigated by a number of authors [1-6] and the Authors obtained a new refinement of the Hermite-Hadamard inequality for convex functions.

Definition 3: A nonnegative function \( f:I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) is said to be \( P \)-function if the inequality

\[
f(tx + (1-t)y) \leq f(x) + f(y)
\]

holds for all \( x,y \in I \) and \( t \in (0,1) \).

We will denote by \( P(I) \) the set of \( P \)-functions on the interval \( I \). Note that \( P(I) \) contain all nonnegative convex and quasi-convex functions.

In [7], Dragomir et al. proved the following inequality of Hadamard type for class of \( P \)-functions.

Theorem 1: Let \( f \in P(I) \), \( a,b \in I \) with \( a < b \) and \( f \in L[a,b] \). Then

\[
f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a} \int_a^b f(x) \, dx \leq 2\left[ f(a) + f(b) \right].
\]

Definition 4: [8] Let \( I \subseteq \mathbb{R} \) be an interval and \( c \) be a positive number. A function \( f:I \rightarrow \mathbb{R} \) is called strongly convex with modulus \( c \) if

\[
f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(x-y)^2
\]

for all \( x,y \in I \) and \( t \in [0,1] \).

In [9], Kadakal gave the definition of multiplicatively \( P \)-function (or \( log \cdot P \)-function) and related Hermite-Hadamard inequality. It should be noted that the concept of \( log \cdot P \)-convex, which we consider in our study and given below, was first defined by Noor et al in 2013 [10]. Then, the algebraic properties of this definition with the name of multiplicatively \( P \)-function are examined in detail by us.

Definition 5: [9,10] Let \( I \neq \emptyset \) be an interval in \( \mathbb{R} \). The function \( f:I \rightarrow (0,\infty) \) is said to be multiplicatively \( P \)-function (or \( log \cdot P \)-function), if the inequality

\[
f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - c\left(t log(x) + (1-t) log(y)\right)
\]

holds for all \( x,y \in I \) and \( t \in (0,1) \).
\( f(tx + (1 - t)y) \leq f(x)f(y) \)
holds for all \( x, y \in I \) and \( t \in [0,1] \).

Denote by \( MP(I) \) the class of all multiplicatively \( P \)-functions on \( I \). Clearly, \( f:I \to (0,0,\infty) \) is multiplicatively \( P \)-function if and only if \( \log f \) is \( P \)-function. The range of the multiplicatively \( P \)-functions is greater than or equal to 1.

**Theorem 2:** Let the function \( f:I \to (1,\infty) \), be a multiplicatively \( P \)-function and \( a, b \in I \) with \( a < b \). If \( f \in L[a,b] \), then the following inequalities hold:

\begin{align*}
\text{i) } & \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq f\left(\frac{a+b}{2}\right) \\
\text{ii) } & \quad f\left(\frac{a+b}{2}\right) \leq f(a)f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq f\left(\frac{a+b}{2}\right)^2.
\end{align*}

Dragomir and Agarwal in [11] used the following lemma to prove Theorems.

**Lemma 1:** The following equation holds true:

\[
\frac{f(a)+f(b)}{2} - \frac{b-a}{b-a} \int_a^b f(x) dx = \frac{b-a}{2} \int_0^1 \left[f'(ta+(1-t)b)\right] dt.
\]

In [12], U. S. Kırmacı used the following lemma to prove Theorems.

**Lemma 2:** Let \( f:I \to \mathbb{R} \) be a differentiable mapping on \( I' \), \( a, b \in I' \) (\( I' \) is the interior of \( I \)) with \( a < b \). If \( f \in L[a,b] \), then we have the following equation holds true:

\[
\frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) = (b-a) \left[\int_0^1 f'(ta+(1-t)b) dt + \left(\frac{1}{2} - (1-t) f'(ta+(1-t)b)\right) dt\right].
\]

The main purpose of this paper is to establish new estimations and refinements of the Hermite–Hadamard inequality for functions whose derivatives in absolute value are strongly multiplicatively \( P \)-function.

### 2. Strongly Multiplicatively \( P \)-functions and Their Some Properties

In this section, we begin by setting some algebraic properties for strongly multiplicatively \( P \)-functions.

**Definition 6:** Let \( I \neq \emptyset \) be an interval in \( \mathbb{R} \). The function \( f:I \to (0,1,\infty) \) is said to be strongly multiplicatively \( P \)-function with modulus \( c > 0 \), if the inequality

\[
f(tx + (1 - t)y) \leq f(x)f(y) - ct(1-t)(x-y)^2
\]
holds for all \( x, y \in I \) and \( t \in [0,1] \).

We will denote by \( SMP(I) \) the class of all strongly multiplicatively \( P \)-functions on interval \( I \).

**Remark 1:** The range of the strongly multiplicatively \( P \)-functions is greater than or equal to 1.

**Proof:** Using the definition of the strongly multiplicatively \( P \)-function, for \( t = 1 \):

\[
f(x) \leq f(x)f(y) \Rightarrow f(x)[1-f(y)] \leq 0.
\]

Here, \( f(x) \geq 0 \), so we obtain \( f(y) \geq 1 \). Similarly, for \( t = 0 \),

\[
f(y) \leq f(x)f(y) \Rightarrow f(y)[1-f(x)] \leq 0.
\]

Since \( f(y) \geq 0 \), we get \( f(x) \geq 1 \).

### 3. Hermite-Hadamard Type Inequalities for Strongly Multiplicatively \( P \)-functions

The goal of this paper is to develop concept of the strongly multiplicatively \( P \)-functions and to establish some inequalities of Hermite-Hadamard type for these classes of functions.

**Theorem 3:** Let the function \( f:I \to (1,\infty) \), be a multiplicatively \( P \)-function and \( a, b \in I \) with \( a < b \). If \( f \in L[a,b] \), then the following inequalities hold:

\begin{align*}
\text{i) } & \quad f\left(\frac{a+b}{2}\right) + c \left(\frac{a-b}{12}\right)^2 \\
& \quad \leq \frac{1}{b-a} \int_a^b f(x) f(a+b-x) dx \\
& \quad \leq \left[ f\left(\frac{a+b}{2}\right) \right]^2 - c \left(\frac{a-b}{12}\right)^2 \left(\frac{2}{3} - \frac{b-a}{a} \int_a^b f(x) dx - \frac{2}{3} \left(\frac{a-b}{3}\right)^2 \right)
\end{align*}

**Proof:** i) Since the function \( f \) is a strongly multiplicatively \( P \)-function, we write the following inequality:

\[
f\left(\frac{a+b}{2}\right) = f\left(\frac{ta+(1-t)b}{2} + \frac{tb+(1-t)a}{2}\right) \\
\leq f\left(\frac{ta+(1-t)b}{2} + \frac{tb+(1-t)a}{2}\right) - c \left(\frac{2t-1}{2}\right)^2 (a-b)^2.
\]

By integrating this inequality on \([0,1]\) and changing the variable as \( x = ta + (1-t)b \), then

\[
\frac{1}{b-a} \int_a^b f(x) f(a+b-x) dx \\
= \frac{1}{b-a} \int_a^b f(x) f(a+b-x) dx - c \left(\frac{2t-1}{2}\right)^2 (a-b)^2.
\]

Moreover, a simple calculation give us that

\[
\frac{1}{b-a} \int_a^b f(x) f(a+b-x) dx \\
\leq \left[ f\left(\frac{a+b}{2}\right) \right]^2 - c \left(\frac{a-b}{12}\right)^2 \left(\frac{2}{3} - \frac{b-a}{a} \int_a^b f(x) dx - \frac{2}{3} \left(\frac{a-b}{3}\right)^2 \right).
\]
So, we get the desired result.

**ii)** Similarly, as \( f \) is a strongly multiplicatively \( P \)-function, we write the following:

\[
f\left(\frac{a+b}{2}\right) \\
\leq f\left(ta+(1-t)b\right)f\left(tb+(1-t)a\right)-ct(1-t)(a-b)^2 \\
\leq f(a)f(b)f\left((1-t)\right)\left(2(1-t)^2\right)(a-b)^2.
\]

Here, by integrating this inequality on \([0,1]\) and changing the variable as \( t = ta + (1-t)b \), then, we have

\[
f\left(\frac{a+b}{2}\right) \leq f(a)f(b)\frac{b}{b-a}f(x)dx - \frac{c}{12}(a-b)^2.
\]

Since,

\[
\frac{1}{b-a}f(x)dx \leq f(a)f(b),
\]

we obtain

\[
f\left(\frac{a+b}{2}\right) \leq f(a)f(b)\frac{b}{b-a}f(x)dx \\
\leq \left[f(a)f(b)\right]^2 - \frac{c}{12}(a-b)^2.
\]

This completes the proof of the theorem.

**Remark 2:** Above Theorem (i) and (ii) can be written together as follows:

\[
f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}f(x)f(a+b-x)dx \\
\leq f(a)f(b)\frac{b}{b-a}f(x)dx \\
\leq \left[f(a)f(b)\right]^2 - \frac{c}{12}(a-b)^2.
\]

**Proof:** Using Lemma 1, since \( |f'| \) is strongly multiplicatively \( P \)-function, we obtain

\[
\left|f(a)+f(b)\right| - \left|\frac{b}{b-a}\int f(x)dx\right| \\
\leq \left|f(a)+f(b)\right| - \left|\frac{b}{b-a}\int f'(x)dx\right| \\
\leq \left|f'(a)|f'(b)|\right|\left|1-2t\right|dt.
\]

This completes the proof of theorem.

**Theorem 5:** Let \( f: I \rightarrow \mathbb{R} \) be a differentiable function on \( I^* \). Assume \( q \in \mathbb{R}, q > 1 \), is such that the function \( \left|f'\right|^q \) is strongly multiplicatively \( P \)-function. Suppose that \( a, b \in I \) with \( a < b \) and \( f' \in L[a,b] \). Then the following inequality holds:

\[
\left|f(a)+f(b)\right|^q - \frac{1}{p+1}\left|f'(a)|f'(b)|\right|^q - \frac{c(a-b)^2}{6}\frac{1}{q},
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Proof:** Let \( a, b \in I \). By assumption, Hölder’s integral inequality, Lemma 1 and the inequality

\[
\left|f'(a)+(1-t)b\right|^q \\
\leq \left|f'(a)|f'(b)|\right|^q - ct(1-t)(a-b)^2,
\]

we have

\[
f(a)+f(b) - \frac{1}{b-a}\int f(x)dx.
\]
Theorem 6: Let $f: I \to \mathbb{R}$ be a differentiable function on $I$. Assume $q \in \mathbb{R}$, $q \geq 1$, such that the function $|f'|^q$ is strongly multiplicatively $P$-function. Suppose that $a, b \in I$ with $a < b$ and $f' \in L[a, b]$. Then the following inequality holds:

$$\frac{1}{2} \left| \int_{a}^{b} (|f'|^q + |f'|^q) \frac{1}{q} \right| \leq \frac{b-a}{2} \left( \int_{a}^{b} |f'|^q \right)^{\frac{1}{q}} \leq \frac{1}{2} \left| \int_{a}^{b} |f'|^q \right|$$

where

$$\int_{0}^{1} |1-2t^p| dt = \frac{1}{p+1}.$$

This completes the proof of theorem.

A more general inequality using Lemma 1 is as follows. 

Theorem 7: Let $f: I \to \mathbb{R}$ be a differentiable function on $I$. Assume $q \in \mathbb{R}$, $q \geq 1$, such that the function $|f'|^q$ is strongly multiplicatively $P$-function. Suppose that $a, b \in I$ with $a < b$ and $f' \in L[a, b]$. Then the following inequality holds:

$$\frac{1}{2} \left| \int_{a}^{b} (|f'|^q + |f'|^q) \frac{1}{q} \right| \leq \frac{b-a}{2} \left( \int_{a}^{b} |f'|^q \right)^{\frac{1}{q}} \leq \frac{1}{2} \left| \int_{a}^{b} |f'|^q \right|$$

where

$$\int_{0}^{1} |1-2t^p| dt = \frac{1}{p+1}.$$

This completes the proof.

Corollary 1: If we take $q = 1$ in inequality (3.2), we obtain the following inequality:

$$\left| \int_{a}^{b} (|f'|^q + |f'|^q) \right| \leq \frac{b-a}{2} \left( \int_{a}^{b} |f'|^q \right)^{\frac{1}{q}} \leq \frac{1}{2} \left| \int_{a}^{b} |f'|^q \right|$$

This inequality coincides with the inequality (3.1).

Theorem 8: Let $f: I \to \mathbb{R}$ be a differentiable function on $I$ such that the function $|f'|$ is strongly multiplicatively $P$-function. Suppose that $a, b \in I$ with $a < b$ and $f' \in L[a, b]$. Then the following inequality holds:

$$\left| \int_{a}^{b} (|f'|^q + |f'|^q) \right| \leq \frac{b-a}{2} \left( \int_{a}^{b} |f'|^q \right)^{\frac{1}{q}} \leq \frac{1}{2} \left| \int_{a}^{b} |f'|^q \right|$$

Proof: Using Lemma 2, since $|f'|$ is strongly multiplicatively $P$-function, we obtain

$$\left| \int_{a}^{b} (|f'|^q + |f'|^q) \right| \leq \frac{b-a}{2} \left( \int_{a}^{b} |f'|^q \right)^{\frac{1}{q}} \leq \frac{1}{2} \left| \int_{a}^{b} |f'|^q \right|$$

This completes the proof of theorem.

Theorem 8: Let $f: I \to \mathbb{R}$ be a differentiable function on $I$. Assume $q \in \mathbb{R}$, $q \geq 1$, such that the function $|f'|^q$ is strongly multiplicatively $P$-function. Suppose that $a, b \in I$
with $a < b$ and $f' \in L[a, b]$. Then the following inequality holds:

$$
\left| \frac{1}{b-a} \int_a^b f(x) \, dx - f \left( \frac{a+b}{2} \right) \right| 
\leq 2(b-a) \left( \frac{1}{(p+1)2^{p+1}} \right)^\frac{1}{p} \left[ \left( \int_0^1 f'(ta+(1-t)b)^q \, dt \right)^\frac{1}{q} \right],
$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

**Proof:** Since the function $|f'|^q$ is a strongly multiplicatively $P$-function, from Lemma 2 and the Hölder’s inequality, we have

$$
\left| \frac{1}{b-a} \int_a^b f(x) \, dx - f \left( \frac{a+b}{2} \right) \right| 
\leq 2(b-a) \left( \frac{1}{(p+1)2^{p+1}} \right)^\frac{1}{p} \left[ \left( \int_0^1 f'(ta+(1-t)b)^q \, dt \right)^\frac{1}{q} \right],
$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

This completes the proof of theorem.

**Theorem 9:** Let $f : I \to \mathbb{R}$ be a differentiable function on $I$. Assume $q \in \mathbb{R}$, $q > 1$, is such that the function $|f'|^q$ is multiplicatively $P$-function. Suppose that $a, b \in I$ with $a < b$ and $f' \in L[a, b]$. Then the following inequality holds:

$$
\left| \frac{1}{b-a} \int_a^b f(x) \, dx - f \left( \frac{a+b}{2} \right) \right| 
\leq \frac{b-a}{4} \left[ \left( f'(a)^q - c(a-b)^2 \right) \right]^\frac{1}{q},
$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

**Proof:** Since the function $|f'|^q$ is a multiplicatively $P$-function, from Lemma 2 and the power-mean integral inequality, we obtain

$$
\left| \frac{1}{b-a} \int_a^b f(x) \, dx - f \left( \frac{a+b}{2} \right) \right| 
\leq \frac{b-a}{4} \left[ \left( f'(a)^q - c(a-b)^2 \right) \right]^\frac{1}{q},
$$

where $\frac{1}{p} + \frac{1}{q} = 1$. 

\[ \int_0^1 t^{p-1} \, dt = \frac{1}{p}, \quad \int_0^1 (1-t)^{p-1} \, dt = \frac{1}{p} \cdot \frac{1}{2^{p+1}}. \]
\[
\int_0^\infty t dt = \int_0^\frac{1}{2} |t-4|dt = \frac{1}{8}
\]

\[
\int_0^\infty t^2 (1-t) dt = \int_0^\frac{1}{2} t(1-t)^2 dt = \frac{5}{192}
\]

**Corollary 2:** If we take \( q = 1 \) in inequality (3.4), we obtain the following inequality:

\[
\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left( \frac{a+b}{2} \right) \right| \leq \frac{b-a}{4} \left( \left| f'(a) \right| \| f''(b) \| - \frac{c(a-b)^2}{24} \right).
\]

This inequality coincides with the inequality (3.3).

### 4. Conclusion

We derived some new Hermite-Hadamard type inequalities for strongly multiplicatively \( P \)-functions. Similar method can be applied to the different type of convex functions.

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