Preferential growth: Solution and application to modeling stock market

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Abstract

We consider a preferential growth model where particles are added one by one to the system consisting of clusters of particles. A new particle can either form a new cluster (with probability $q$) or join an already existing cluster with a probability proportional to the size thereof. We calculate exactly the probability $P_i(k, t)$ that the size of the $i$-th cluster at time $t$ is $k$. We applied our model as a background for a microscopic economic model.

1 Introduction

Preferential growth describes systems consisting of groups of entities where the probability of the attachment of a new entity to one of the groups is an increasing function of the group’s size. Recently these types of models were used to describe networks like the World Wide Web (WWW) \cite{1}, Internet \cite{2}, statistics of scientific citation \cite{3} etc which seems to have the common property of scale invariance. It turned out that this behavior is due to two main facts namely that they are continuously evolving and that the attachment probability is proportional to the group’s size. Many features of the above mentioned systems were analyzed and also analytic calculations have been presented for the most important quantities \cite{4} but they are only valid in the asymptotic time case. Here we present the full-time dependent solutions.

We applied this kind of model to describe herding in an economic system. It was suggested \cite{5} that large price movements on the stock market can be explained by the fact that traders are not individual participants but they prefer to form groups, within a group each trader share the same strategy. Our assumption is that those groups are created according to a preferential growth
model. Preferential growth occurs naturally in economics [6]. The power-law behavior of the resulting group distribution can account for the fat-tail distribution of the return which is one of the stylized fact that characterize the stock market.

In the paper we present the full-time dependent solution of the growth model, and briefly describe the most important results of our economic model.

2 Growth model

The system consists of individual groups of entities. The initial condition is one group with one entity in it. At each time step one new entity is added to the system. With probability $q$ it creates a new group, with $p = 1 - q$ it will belong to one of the existing groups. The probability that it joins the $i$th group is proportional to the $i$th group’s size ($k_i/N$). The system’s dynamic

\[
P_i(k, t) = p \frac{(k - 1)}{t - 1} P_i(k - 1, t - 1) + p \left(1 - \frac{k}{t - 1}\right) P_i(k, t - 1) +
\]

\[
+ (1 - p) P_i(k, t - 1) + (1 - p) \Pi_{i-1}(t - 1) \delta_{k,1}(1 - \delta_{i,1}).
\]  

(1)

The quantity $\Pi_i(t)$ denotes the probability that at time $t$ the system consists of $i$ group: $\Pi_i(t) = \binom{t-1}{i-1} p^{i-1-(i-1)}(1 - p)^{t-1-i}$. Our goal is to determine the analytic full-time dependent form of the following quantities: size distribution, $P_i(k, t)$, and mean, $\langle k_i \rangle(t)$, of the individual groups and the average group size distribution, $P(k, t) = \frac{1}{t} \sum_{i=1}^{t} P_i(k, t)$. Here we do not go into the details of the deduction we only present the results. (For more informations see[7]). For the size distribution of the individual groups one gets:
\[ P_i(k, t) = \sum_{l=1}^{k} (-1)^{l-1} \frac{(k-1)}{l-1} \frac{\Gamma(t-lp)}{\Gamma(t)\Gamma(1-lp)} \times \left[ \sum_{b-i}^{t} \frac{\Gamma(b)\Gamma(1-lp)}{\Gamma(b-lp)} \frac{(b-2)}{(i-2)} p^{b-i} (1-p)^{i-1} \right]. \]  

(2)

From \( P_i(k, t) \) we can easily calculate the analytic form of \( P(k, t) \) and \( \langle k_i \rangle(t) \) by simple substitution in their definitions and we get:

\[ P(k, t) = \sum_{l=1}^{k} (-1)^{l-1} \frac{(k-1)}{l-1} \left[ \frac{1-p}{1+lp} + \frac{p+lp}{1+lp} \frac{\Gamma(t-lp)}{\Gamma(t+1)\Gamma(1-lp)} \right]. \]  

(3)

\[ \langle k_i \rangle(t) = \sum_{l=1}^{t-i+1} (-1)^{l-1} t \frac{(t-i+2)}{l+1} \frac{\Gamma(t-lp)}{\Gamma(t)\Gamma(1-lp)} \times \left[ \sum_{b-i}^{t} \frac{\Gamma(b)\Gamma(1-lp)}{\Gamma(b-lp)} \frac{(b-2)}{(i-2)} p^{b-i} (1-p)^{i-1} \right]. \]  

(4)

In the case when the system is large (\( t \to \infty \)) it is worth to analyze the asymptotic case of the above quantities in order to gain a much simpler form.

i) Individual group size distribution

In Eq. (2) there are two time dependent terms. One is \( \frac{\Gamma(t-lp)}{\Gamma(t)\Gamma(1-lp)} \), the other is the sum in the bracket. For large \( t \) values the second term will be proportional to a hypergeometric function, \( 2F_1(i, i-1; i-lp; p) \), and will be time independent. The first term will converge to \( t^{-lp} \) which is a fast decaying function of \( l \) so for \( t \gg k \) one can assume that only the first term of the sum in Eq. (2) gives non-negligible component:

\[ \lim_{t \to \infty} P_i(k, t) = t^{-p} (1-p)^{i-1} \frac{\Gamma(i)}{\Gamma(i-p)} 2F_1(i, i-1; i-p; p) + O(t^{-2p}) \]  

(5)

For large \( i \) values the above form simplifies further and we finally get:

\[ \lim_{t,i \to \infty} P_i(k, t) = \left( \frac{i}{t} \right)^p \]  

(6)

ii) Mean of individual group size

In the analysis of the mean value we assume that for \( k \ll t \) the distribution can be described by Eq. (5) for larger \( k \) values it has a fast decay so: \( \langle k_i \rangle(t) \approx \)
\[ \sum_{k=1}^{k^*} k \mathcal{P}_i(1, t \to \infty). \] (The value \( k^* \) can be defined e.g. as the inflection point of \( \mathcal{P}_i(k, t) \)). Here again analyzing the large \( i \) case we get:

**iii) Distribution of average group size**

In the determination of the asymptotic case of the average group size distribution, \( \mathbf{P}(k) \), we can apply two ways. One is to simply analyze the limit of Eq. (3) or we can also start directly from the master equation, Eq. (1), by summing it up for \( i = 1 \ldots t \). In both cases we get that in the long time limit the distribution will be time independent and will have a power-law decay:

\[
\mathbf{P}(k) = \frac{\Gamma(k) \Gamma \left( \frac{2 + \frac{1}{p}}{1 + \frac{1}{p}} \right) \Gamma \left( k + 1 + \frac{1}{p} \right) 1 - p}{\Gamma \left( k + 1 + \frac{1}{p} \right) 1 + p} \sim k^{-1 - 1/p}, \quad k \to \infty.
\] (7)

Here we would like to mention that however our growth model is not a network model for particular values of the parameter \( p = 0.5 \) it can be considered as a mean field analogy of the Barabási’s network model. The resulting exponent of the distribution, \((-3)\), agrees with their result [1].

### 3 Economic model

The main properties of stock markets price movements are characterized by the so called stylized facts, the distribution of the logarithmic return fat-tailed, autocorrelation of the return is short range while that of the absolute value of return decays with a power-law. We try to reproduce these properties by a simple model.

Power-law decay of the distribution of the logarithmic return can be explained by the herding behavior on the market [5]. Price changes are assumed to be proportional to the excess demand – the difference between the amount of buying and selling orders. If traders form groups so that within each group everyone shares the same strategy than the logarithmic return can be expressed as:

\[ x \sim \sum_{i=1}^{n} s_i \phi_i, \] where \( \phi_i \) is the strategy (to buy, to sell or not to trade), \( s_i \) is the size of the \( i \)th group. If the number of groups that are active (trading) at a given time step is small the distribution of the logarithmic return, \( P(x) \) will be similar to that of the groups’ size distribution, \( P(s) \). In our model groups are created according to the above mentioned growth model. Due to Eq. (7) \( P(s) \) will have a power-law decay which indicates that for law activity the distribution of logarithmic return is also a power-law. Similar results were
get by [8] but they considered trading groups as two dimensional percolation clusters.

Power-law correlation of the absolute value of price is due to the long range correlation of the trading volume (number of trades in a given time interval) [9], which is in our model the sum of the active groups’ sizes, \( S_a = \sum_{a=1}^{n_a} s_i \). We defined the activity (fraction of groups that are trading at a given time step, \( n_a/n \)) as the function of the deviation of the actual price from some fundamental price, \( p_0 \); see Fig. 2. (We took the fundamental price for constant over time.) We ensure with this definition that the activities of the proceeding time steps are correlated. For a given activity value the corresponding trading volumes can be different but in average larger activity implies larger trading volume which indicates that the sum of the trading groups’ sizes will also be correlated. The form of the function was chosen intuitively, it is based on the hypothesis that small deviation from the fundamental price inspire traders – their activity rises – while for large deviations traders become careful so their activity falls back.

![Fig. 2. Price dependence of the activities.](image)

\[ a^+ \text{ is the fraction of the buying groups, } a^- \text{ is the fraction of selling groups at a given time step. For } \ln[p(t)/p_0] < |b^*| \text{ the activities } a^+ \text{ and } a^- \text{ are equal. If } \ln[p(t)/p_0] > b^* \text{ then } a^+ = 0, a^- = a_0 \text{ while for } \ln[p(t)/p_0] < b^* \Rightarrow a^- = 0, a^+ = a_0. a_0 \text{ is the reciprocal of the number of groups.} \]

The results of our model can be seen in Fig. 3. It is surprising that for the distribution of the logarithmic return we not only got a simple power-law but a distribution with two different exponents (Fig. 3/a) which is similar to what was measured on the real market[10]. This is, however, only a qualitative agreement, the value of the exponents differ from the measured ones [10]. To analyze the autocorrelation of the absolute value of return we study the variance of the sum of these variables (normalized to unit variance):

\[
D^2(\sum_{i=1}^{n} x_i) \equiv D^2(n) = n + 2 \sum_{m=1}^{n-1} (n - m) C(m) \sim n^{2\delta},
\]

where \( C(m) \sim m^{-\kappa} \) is the autocorrelation function. For \( \kappa \geq 1 \) we have \( \delta = 0.5 \),
Fig. 3. a) Distribution of the logarithmic return. It has on the log-log plot two approximately linear regimes. The first has an exponent $\simeq 1.72$ the second $\simeq 3.36$ b) Square root of the variance, $D(n)$. The dashed line corresponds to the uncorrelated, $\delta = 0.5$, time series. The full line is a fit with exponent $\delta = 0.95$.

for variables with long-range correlation ($\kappa < 1$) $0.5 < \delta < 1$, and the correlation exponent, $\kappa$, can be calculated through $\kappa = 2 - 2\delta$; see Fig. 3/b. It is clearly shown that the time series of the absolute value of the logarithmic return is correlated but the exponent is much smaller ($\simeq 0.05$) than the reported one ($\simeq 0.3$) [10].

The advantage of our model is its simplicity and that the results have a qualitative agreement with the stylized facts. However, there are tunable parameters in it and the results – the values of the exponents of the distribution and of the autocorrelation – depend on them. Further study is needed to clarify the role of these parameters.

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