Consistent Linearized Gravity in Brane Backgrounds

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Abstract

A globally consistent treatment of linearized gravity in the Randall-Sundrum background with matter on the brane is formulated. Using a novel gauge, in which the transverse components of the metric are non-vanishing, the brane is kept straight. We analyze the gauge symmetries and identify the physical degrees of freedom of gravity. Our results underline the necessity for non-gravitational confinement of matter to the brane.

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1 Introduction and Summary

There has been much interest recently in the studies of non-compact spaces containing 3-branes as domain walls as an alternative to compactification for the treatment of the hierarchy problem \[1, 2, 3, 4, 5\]. These scenarios, in fact, revive older ideas described in \[6, 7, 8\]. In a simple, non-compact scenario, which was proposed by Randall and Sundrum (RS) \[9\], the 3-brane is flat due to fine tuning of either the brane tension or the cosmological constant. Gravity on the brane resembles the usual 4-dimensional Einstein gravity for long distances, owing to a graviton bound state, whereas the Kaluza Klein modes nearly decouple from the gravity on the brane \[10, 11\].

One of the most interesting questions in this and similar scenarios is the coupling of gravity to matter on the brane, because it represents the very mechanism of generating gravitational forces. In fact, several papers have studied this question, and the existence of long-distance Einstein gravity on the brane has been confirmed \[12–42\]. Many of these calculations have used the RS gauge \[9\], as a result of which the brane appears bent owing to matter located on it \[17, 25, 26, 30, 33\]. Moreover, calculations in global Riemannian normal coordinates are simply not consistent \[28\].

In this paper, we would like to follow up on the issue of localization of gravity on the brane and address the following points: First, we present a globally consistent formulation of linearized gravity with matter located on the brane, which is kept straight, both in the RS and an alternative background. Our motivation for studying the alternative background is that the brane attracts ordinary matter in that case, whereas it is repulsive in the Randall-Sundrum case \[28\]. Consistency of our calculation is made possible by adopting a novel gauge fixing (cf. Secs. 2 and 3). In our gauge, the metric components transversal to the brane are non-vanishing. Fluctuations of this kind have been studied already in \[20\]. Second, we determine the physical degrees of freedom of gravity by analyzing the gauge freedom for a straight brane (Sec. 4). The problem of physical degrees of freedom was studied in \[25, 33, 39\]. Last, we find the particular solution for a matter perturbation on the brane and analyze the effective 4-dimensional gravity (Sec. 5). We would like to mention that our solution is only gauge-equivalent to the bent-brane formulation \[17, 24\], if we are far enough from the brane, although also the results for gravity on the brane seem to be identical. Further comments can be found in Sec. 4.

Although our results are not entirely new, in that they confirm the well-established long-distance Einstein gravity, they represent, to our mind, a concise and elegant global formulation and provide important insights into the dynamics on the brane. In particular, a non-gravitational mechanism of confinement for the RS background turns out to be essential not only because of the instability of the geodesics on the brane \[28, 37\], but also for the Newtonian dynamics on the brane, which would be spoiled by extrinsic contributions. On the other hand, Einstein’s equation does not hold for long distances for the alternative background.

Let us now summarize our results. Our metric has the form

\[
ds^2 = e^{-2k|y|} (\eta_{\mu\nu} + \gamma_{\mu\nu}) dx^\mu dx^\nu + 2 n_\mu dx^\mu dy + (1 + \phi) dy^2,
\]  

(1)

where \(\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)\). In eqn. (1) and henceforth, positive \(k, k > 0\), corresponds to the RS background and negative \(k, k < 0\), to the alternative background.

Using the more general ansatz \(1\) instead of the RS gauge for the metric we are able to keep the brane straight at \(y = 0\) and obtain a globally consistent solution for
the linearized Einstein equations. For a matter perturbation \( t_{\mu\nu} \) located on the brane, our solutions for the first order quantities in eqn. (1) are

\[
n_{\mu} = -\frac{\text{sgn} y}{8k} \gamma_{,\mu}, \quad \phi = -\frac{\text{sgn} y}{4k} \gamma_{,y},
\]

and the traceless transversal part of \( \gamma_{\mu\nu} \) satisfies

\[
\partial_y \left( e^{-2k|y|} \partial_y \tilde{\gamma}_{\mu\nu} \right) - 2k y e^{-2k|y|} \partial_y \tilde{\gamma}_{\mu\nu} + \Box \tilde{\gamma}_{\mu\nu} = -16\pi \delta(y) \left[ t_{\mu\nu} - \frac{1}{3} \left( \eta_{\mu\nu} - \partial_\mu \partial_\nu \right) t \right].
\]

Moreover, the trace \( \gamma \) satisfies the boundary conditions

\[
\gamma|_{y=0} = 32\pi k, \quad \gamma|_{y=\infty} = 0,
\]

but can be altered by gauge transformations for \( y \neq 0 \). The step functions in eqn. (2) correspond to an apparent singularity separating two coordinate patches. We shall demonstrate explicitly in Sec. 4 how to obtain a continuous solution.

## 2 General Method

The starting point for our calculation is the action

\[
S = \int d^4x \int dy \sqrt{-g} (R - 2\Lambda) + \int_{\text{brane}} d^4x \sqrt{-\hat{g}} (\sigma + L_{\text{matter}}),
\]

where

\[
\Lambda = -6k^2 \quad \text{and} \quad \sigma = -12k,
\]

so that the metric

\[
ds^2 = e^{-2k|y|} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2
\]

is a background solutions for \( L_{\text{matter}} = 0 \).

In our approach, we would like to use a global coordinate system, in which the brane is located at \( y = 0 \) even when a perturbation is present on the brane. Thus, it is natural to use the time-slicing formalism \[43\], where we slice with respect to \( y = \text{const.} \) hypersurfaces. At least in the RS case \((k > 0)\), we cannot impose an a priori gauge on the metric perturbations, since we would obtain exponentially growing solutions for large \( y \). Instead, consistency forces us to use a very particular, yet elegant, gauge. This shall become clear in the course of our calculation. Let us now give a short review of the time-slicing formalism \[43\] and briefly outline the specific character of our approach.

In the time slicing formalism, one splits up the metric tensor as

\[
(g_{ab}) = \begin{pmatrix} \hat{g}_{\mu\nu} & n_\mu \\ n_\nu & n_\lambda n^\lambda + n^2 \end{pmatrix}, \quad \left( g^{ab} \right) = \frac{1}{n^2} \begin{pmatrix} n^2 \hat{g}^{\mu\nu} + n^\mu n^\nu & -n^\mu \\ -n^\nu & 1 \end{pmatrix},
\]

where \( \hat{g}_{\mu\nu} \) is the induced metric on the hypersurfaces, \( n^\mu = \hat{g}^{\mu\nu} n_\nu \), and \( a, b = 0, 1, 2, 3, 5 \), \( \mu, \nu = 0, 1, 2, 3 \). Henceforth, we shall denote quantities derived from the induced metric \( \hat{g}_{\mu\nu} \) with a hat. \( n \) and \( n_\mu \) are the lapse function and shift vector, respectively.
We consider the hypersurfaces $y = \text{const.}$, which have the induced metric $\hat{g}_{\mu\nu}$ and the tangent and normal vectors
\[
\partial_\mu x^a = \begin{cases} 
\delta^a_\mu & \text{for } a = 0, 1, 2, 3, \\
0 & \text{for } a = 5,
\end{cases}
\] (8)
and
\[
N_a = (0, 0, 0, 0, -n), \quad \hat{N}^a = \frac{1}{n}(n^\mu, -1),
\] (9)
respectively. Then, the second fundamental form measuring the extrinsic curvature of the hypersurfaces has the form
\[
H_{\mu\nu} = \frac{1}{2n} \left( \partial_\mu \hat{g}_{\nu\nu} - \hat{\nabla}_\mu n_\nu - \hat{\nabla}_\nu n_\mu \right).
\] (10)
Einstein’s equation,
\[
R^{ab} - \frac{1}{2} g^{ab} R = -g^{ab} \Lambda + 8\pi T^{ab},
\] (11)
can now be rewritten in terms of $\hat{g}_{\mu\nu}$, $n_\mu$ and $n^2$. First, using the Gauss-Codazzi equations, the normal and mixed components of eqn. (11) with respect to the hypersurfaces become
\[
\hat{R} + H_\mu^\nu H_\nu^\mu - \hat{H}^2 = 2\Lambda,
\] (12)
\[
\partial_\mu H - \hat{\nabla}_\nu H_\nu^\mu = 0,
\] (13)
where we have used the fact that $N_a T^{ab} = 0$ for the energy momentum tensor derived from the action (4). For the tangential components of eqn. (11) we prefer the form
\[
R_{\mu\nu} = 2 \frac{1}{3} \hat{g}_{\mu\nu} \Lambda + 8\pi \left( T_{\mu\nu} - \frac{1}{3} \hat{g}_{\mu\nu} T \right),
\] (14)
because it saves us from calculating the scalar curvature $R$. In the standard approach, one fixes $n_\mu$ and $n^2$ to some convenient value. Then, eqn. (14) is the equation of motion for $\hat{g}_{\mu\nu}$, whereas eqns. (12) and (13) are constraints. Notice that until now all expressions have been exact.

In a previous paper [28], some of us followed the standard approach choosing globally $n_\mu = n^2 - 1 = 0$ and found that the linearized approximation was inconsistent for the Randall-Sundrum background. Therefore, we would now like to take a different approach. Instead of fixing the lapse function $n$ and the shift vector $n_\mu$ a priori, we leave them present at first and fix them in the course of our calculations by the condition that the linearization be consistent. For the linearization of the Randall-Sundrum and the alternative background, we write the induced metric as
\[
\hat{g}_{\mu\nu} = e^{-2k|y|} (\eta_{\mu\nu} + \gamma_{\mu\nu}),
\] (15)
and we consider $\gamma_{\mu\nu}$, $n_\mu$ and $n^2 - 1$ as small perturbations. Furthermore, we assume that the induced metric perturbations, $\gamma_{\mu\nu}$, are continuous at $y = 0$. The necessary linearized expressions for the connections and curvatures are given in the appendix.
Let us start by linearizing Einstein’s equations. The energy momentum tensor, as found from the action (4), has the form

\[ T_{\mu\nu} = -\frac{3k}{4\pi} \sqrt{\hat{g}} \delta(y) \hat{g}^{\mu\nu} + \delta(y) t^{\mu\nu}(x), \quad T^{5\mu} = T^{55} = 0, \]  

where the first term of \( T_{\mu\nu} \) is the background from the brane, and \( t_{\mu\nu} \) is a small matter perturbation sitting on the brane. The covariant conservation law, \( \nabla_a T^{ab} = 0 \) is satisfied to first order, if and only if \( t_{\mu\nu} \) is conserved in the conventional sense, \( \partial_{\mu} t^{\mu\nu} = 0 \).

The constraints, eqns. (12) and (13), take the linearized forms

\[ e^{2k|y|} (\gamma^{\mu\nu}_{,\mu\nu} - \Box \gamma - 6k \sgn y \partial^\mu n_\mu) = -3k \sgn y \gamma_{,y} - 12k^2 \phi \]  

and

\[ e^{2k|y|} (\Box n_\mu - \partial_\mu \partial^\nu n_\nu) = -3k \sgn y \partial_\mu \phi + \partial_y (\gamma^{\nu}_{,\mu\nu} - \gamma_{,\mu}), \]  

respectively, where we have defined

\[ \phi = n^2 - 1. \]  

Next, let us linearize the tangential equation, eqn. (14), with the energy momentum tensor (16). We have \( \hat{g}/g \approx 1 - \phi \), so that one finds the linearized form

\[ \frac{1}{2} (\gamma^\rho_{,\mu\rho} + \gamma^\rho_{,\nu\rho} - \Box \gamma_{\mu\nu} - \gamma_{,\mu\nu}) + 3k^2 \phi e^{-2k|y|} \eta_{\mu\nu} - \frac{1}{2} \partial_y \left( e^{-2k|y|} \gamma_{\mu\nu,y} \right) \]

\[ - \frac{1}{2} \partial_\mu \partial_\nu \phi + \frac{1}{2} \partial_y (n_{\mu,\nu} + n_{\nu,\mu}) - k \sgn y \left( n_{\mu,\nu} + n_{\nu,\mu} + \eta_{\nu\mu} \partial^\lambda n_\lambda \right) \]

\[ + k \sgn y e^{-2k|y|} \left[ \gamma_{\mu\nu,y} + \frac{1}{2} \eta_{\mu\nu} \gamma_{,y} \right] - \frac{1}{2} k \eta_{\mu\nu} \partial_y \left( e^{-2k|y|} \sgn y \phi \right) \]

\[ = 8\pi \delta(y) \left( t_{\mu\nu} - \frac{1}{3} \eta_{\mu\nu} t \right), \]  

where \( t = \eta^{\mu\nu} t_{\mu\nu} \).

In the Randall-Sundrum case, the exponentially increasing terms in eqns. (17) and (18) are potentially problematic, but problems can be avoided by choosing a suitable gauge. Thus, instead of setting \( n_\mu = \phi = 0 \), we impose the gauge conditions

\[ 4k \phi = -\sgn y \gamma_{,y}, \]  

\[ \partial^\mu \bar{\gamma}_{\mu\nu} = 0, \]  

where

\[ \bar{\gamma}_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{4} \eta_{\mu\nu} \gamma \]

is the traceless part of \( \gamma_{\mu\nu} \). Eqn. (22) says that the traceless part \( \bar{\gamma}_{\mu\nu} \) should also be transversal.
Together, the gauge conditions (21) and (22) imply that the right hand sides of eqns. (17) and (18) are zero, so that these equations reduce to

\[ \gamma^{\mu \nu, \mu \nu} - \Box \gamma = 6k \text{sgn} \ y \partial^\mu n_\mu, \] (23)

\[ \Box n_\mu - \partial_\mu \partial^\nu n_\nu = 0, \] (24)

respectively. Eqns. (23) and (24) are equations for the shift vector \( n_\mu \), and their general solution is

\[ n_\mu = -\frac{\text{sgn} \ y}{8k} \gamma_{, \mu} + A_\mu. \] (25)

Here, the vector \( A_\mu \) satisfies the 4-dimensional equations of a free vector field in Lorentz gauge,

\[ \Box A_\mu = \partial_\mu A_\mu = 0, \] (26)

but it can depend also on \( y \).

After inserting the gauge conditions (21) and (22) as well as the solution (25) for the shift vectors into eqn. (20), we find

\[ \partial_y \left( e^{-2k|y|} \partial_y \tilde{\gamma}_{\mu \nu} \right) - 2k \text{sgn} \ y \ e^{-2k|y|} \partial_y \tilde{\gamma}_{\mu \nu} + \Box \tilde{\gamma}_{\mu \nu} \]

\[ - \partial_y (A_{\mu, \nu} + A_{\nu, \mu}) + 2k \text{sgn} \ y \ (A_{\mu, \nu} + A_{\nu, \mu}) \]

\[ = \delta (y) \left[ -16\pi \left( t_{\mu \nu} - \frac{1}{3} \eta_{\mu \nu} \right) - \frac{1}{2k} \gamma_{, \mu \nu} \right]. \] (27)

Notice that by virtue of eqns. (22) and (26) the left hand side of eqn. (27) is traceless and four-divergence-free. Thus, we should expect the same of the right hand side. The former property translates into

\[ \Box \gamma |_{y=0} = \frac{32\pi k}{3} t, \] (28)

whereas the latter property is expressed as

\[ -16\pi \left( \partial^\mu t_{\mu \nu} - \frac{1}{3} \partial_\nu t \right) - \frac{1}{2k} \partial_\nu \Box \gamma |_{y=0} = 0. \] (29)

Using eqn. (28) we find \( \partial^\mu t_{\mu \nu} = 0 \), which is a good check of consistency, since Einstein’s equation should imply the covariant energy-momentum conservation law.

Looking at the equations presented so far we realize that there is no equation of motion for \( \gamma \), and eqns. (23) and (24) are insufficient to determine \( A_\mu \) and \( \tilde{\gamma}_{\mu \nu} \). Therefore, we suspect that there is residual gauge freedom, the determination of which is the subject of the next section.

### 4 Physical Degrees of Freedom

We would like to discuss the residual gauge freedom left after imposing the gauge conditions (21) and (22) and keeping the brane fixed at \( y = 0 \). We shall find that traceless transversal spin-2 excitations are the only physical degrees of freedom. Moreover, we
shall show how to remove the step functions in the solutions for \( n_\mu \) and \( \phi \) [cf. eqns. (21) and (25)].

To start, consider two coordinate systems with the metrics

\[
ds^2 = e^{-2k|y|} (\eta_{\mu\nu} + \gamma_{\mu\nu}) \, dx^\mu \, dx^\nu + 2n_\mu \, dx^\mu \, dy + (1 + \phi)(dy)^2
\]

\[
e^{-2k|y'|} (\eta_{\mu\nu} + \gamma'_{\mu\nu}) \, dx'^\mu \, dx'^\nu + 2n'_\mu \, dx'^\mu \, dy' + (1 + \phi')(dy')^2,
\]

related by an infinitesimal coordinate transformation,

\[
x'^\mu = x^\mu - \xi^\mu(x,y), \quad y' = y - \xi^5(x,y).
\]

The location of the brane remains unchanged, i.e. we have the restriction \( \xi^5|_{y=0} = 0 \).

This also ensures that no normal components of the energy momentum tensor, \( T^{a5} \), are generated.

Under the coordinate transformation (11), the first order elements transform as

\[
\gamma_{\mu\nu} = \gamma'_{\mu\nu} + 2k \, \text{sgn} \, y \, \xi^5 \eta_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu},
\]

\[
n_\mu = n'_\mu - \xi^5 - e^{-2k y} \xi_{\mu,y},
\]

\[
\phi = \phi' - 2\xi^5.
\]

Here, indices have been lowered using the Minkowski metric. With some calculations one can check that the linearized Einstein equations, eqns. (17), (18) and (20), are invariant under this transformation, provided that \( \xi^5|_{y=0} = 0 \).

Coordinate transformations can be applied separately for \( y < 0 \) and \( y > 0 \). For simplicity, we shall restrict our discussion to \( y > 0 \). For \( y > 0 \), the trace of eqn. (32) yields

\[
\gamma = \gamma' + 8k \xi^5 - 2\partial^\lambda \xi_\lambda,
\]

which can be combined with eqn. (32) to obtain the transformation of \( \tilde{\gamma}_{\mu\nu} \),

\[
\tilde{\gamma}_{\mu\nu} = \tilde{\gamma}'_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \frac{1}{2} \eta_{\mu\nu} \partial^\lambda \xi_\lambda.
\]

Thus, imposing the gauge condition (22) on both metrics yields

\[
\Box \xi_\mu + \frac{1}{2} \partial_\mu \partial_\nu \xi_\nu = 0.
\]

Next, we substitute the gauge condition (22) into eqn. (34) and obtain \( \partial_\mu \partial^\lambda \xi_\lambda = 0 \), i.e.

\[
\partial^\lambda \xi_\lambda = f(x)
\]

is a function of \( x \) only. Moreover, from eqn. (37), we find that \( f \) must satisfy \( \Box f = 0 \).

Finally, substituting the solution for \( n_\mu \), eqn. (25), into eqn. (33), one finds that \( A_\mu \) transforms as

\[
A_\mu = A'_\mu - \frac{1}{4k} \partial_\mu f(x) - e^{-2ky} \xi_{\mu,y}.
\]

As a check of consistency, we observe that eqn. (26) and the boundary condition (28) remain unchanged.
Let us now fix the remaining gauge freedom and identify the physical degrees of freedom. First, we can use
\[ \gamma|_{y=0} = \gamma'|_{y=0} - 2f(x) \]
to obtain a unique solution for the boundary condition (28), which we shall formally write as
\[ \gamma|_{y=0} = \frac{32\pi k}{3\Box}t. \] (40)
Second, we make use of \( \xi^5 \) in order to choose a convenient function for \( \gamma \) in the bulk satisfying the boundary conditions (40). For example, one could pick
\[ \gamma = \frac{32\pi k}{3\Box}t \, e^{-ay^2} \] (41)
with some positive coefficient \( a \). Notice that on the brane \( \gamma \) is determined by the matter content, which cannot be gauged away. Last, we make use of the remaining freedom, \( \xi_\mu \) satisfying \( \partial^\mu \xi_\mu = \Box \xi_\mu = 0 \), and
\[ A_\mu = A'_\mu - e^{-2ky} \xi_{\mu,y} \]
in order to pick a convenient function for \( A_\mu \). Of course, the most convenient value is \( A_\mu = 0 \), which we shall adopt.

After this gauge fixing, we are left with only the physical degrees of freedom \( \tilde{\gamma}_{\mu\nu} \), which describe spin-2 gravity excitations.

We would like to point out the following subtle point regarding the bent-brane formulation used by Garriga and Tanaka [17], Giddings, Katz and Randall [26] and others. If one did not impose the condition \( \xi^5|_{y=0} = 0 \) [cf. eqn. (31)], it would seem from eqns. (32)–(34) that one could transform a metric in our gauge into a metric in Randall-Sundrum gauge using \( \xi^5 = -\gamma'/8k \). The obvious effect would be that the brane appears bent to an observer, and non-zero normal components \( T^5_{\mu\nu} \) are generated. However, we would like to emphasize that eqn. (30) is not valid in this case, because one cannot expand \( e^{-2k|y'|+\xi^5(\eta)+g(x)} \approx e^{-2k|y'|}(1 - 2k \text{sgn} \, y \xi^5) \), which would imply that the brane is located again at \( y' = 0 \) (the brane is where the singularity of the curvature is). Rather, one should write
\[ ds^2 = e^{-2k|y'|+\xi^5}(\eta_{\mu\nu} + \gamma''_{\mu\nu})dx'^\mu dx'^\nu + dy'^2. \]
Thus, eqn. (10) of [17] should contain \( \delta[y - \xi^5(x)] \) and not \( \delta(y) \). In our opinion, this seems to be a drawback of the bent-brane formulation and has not been addressed properly.

As a last point in this section, we would like to demonstrate how to remove the step functions in the solutions for \( n_\nu \) and \( \phi \) [cf. eqn. (2)] and to obtain a continuous solution. First let us note that it is enough to remove the discontinuity of \( n_\nu \), because one can choose \( \gamma \) such that \( \phi|_{y=0} = 0 \) [cf. eqn. (41)], i.e. \( \phi \) is already continuous. We take a gauge transformation of the form
\[ \xi^5 = 0, \quad \xi_\mu = \frac{1}{16k^2} \gamma_\mu e^{2k|y|} \psi(y), \] (42)
where \( \psi(y) \) is a smooth function with a compact support such that \( \psi(0) = 1 \). Then, the gauge-transformed components of the metric will be

\[
\gamma'_{\mu\nu} = \gamma_{\mu\nu} + \frac{1}{8k^2} e^{2k|y|} \psi(y) \gamma_{\mu\nu},
\]

\[
n'_\mu = -\frac{\text{sgn} y}{8k} \gamma_{\mu\nu} + \frac{1}{16k^2} e^{-2k|y|} \partial_y \left[ \gamma_{\mu\nu} e^{2k|y|} \psi(y) \right],
\]

\[
\phi' = -\frac{\text{sgn} y}{4k} \gamma_{\mu\nu}.
\]

One can easily see that \( n'_\mu \) is continuous at \( y = 0 \). Thus, we have found an explicit variable transformation for a vicinity of the brane, which transforms our solution into one that is continuous at \( y = 0 \). We can, therefore, conclude that the step functions in eqn. (3) correspond only to an apparent singularity.

### 5 Particular Solution and Gravity on the Brane

Let us now fix the gauge as described in the last section and solve eqn. (27). Substituting \( A_\mu = 0 \) and eqn. (40), eqn. (27) becomes

\[
\partial_y \left( e^{-2k|y|} \partial_y \tilde{\gamma}_{\mu\nu} \right) - 2k \text{sgn} y e^{-2k|y|} \partial_y \tilde{\gamma}_{\mu\nu} + \Box \tilde{\gamma}_{\mu\nu} = -16\pi \delta(y) \left[ t_{\mu\nu} - \frac{1}{3} \left( \eta_{\mu\nu} - \frac{\partial_{\mu} \partial_{\nu}}{\Box} \right) t \right].
\]

We note that the inverse of the d’Alembertian is unique after the residual gauge fixing.

The particular solutions for the source \( t_{\mu\nu} \) are even in \( y \). Therefore, we shall consider eqn. (46) for \( y > 0 \),

\[
\partial_y \left( e^{-2ky} \partial_y \tilde{\gamma}_{\mu\nu} \right) - 2k e^{-2ky} \partial_y \tilde{\gamma}_{\mu\nu} + \Box \tilde{\gamma}_{\mu\nu} = 0
\]

and impose the Neumann boundary condition

\[
\partial_y \tilde{\gamma}_{\mu\nu} |_{y=+0} = -8\pi \left[ t_{\mu\nu} - \frac{1}{3} \left( \eta_{\mu\nu} - \frac{\partial_{\mu} \partial_{\nu}}{\Box} \right) t \right],
\]

which arises from integrating eqn. (46) over the singularity.

Fourier transforming eqn. (47) with respect to the brane coordinates and changing variables to \( z = e^{2ky} \) leads to the differential equation

\[
\left( z^2 \partial_z^2 - z \partial_z - \frac{p^2}{4k^2} z \right) \tilde{\gamma}_{\mu\nu} = 0,
\]

whose solutions are given in terms of Bessel functions [44].

Let us consider the case \( p^2 > 0 \). The static case \( (p_0 = 0) \) is included here, and the case \( p^2 = 0 \) can be obtained as a limit from \( p^2 > 0 \). The two linearly independent solutions of eqn. (49) are

\[
\tilde{\gamma}_{\mu\nu}(p, y) = C_{\mu\nu}(p) e^{2ky} \begin{cases} K_2 \left( e^{ky} |p|/k \right) \\ I_2 \left( e^{ky} |p|/k \right) \end{cases} \quad (p^2 > 0).
\]
Since blowing-up solutions are inconsistent with the linearization (and clearly unphysical), we are led to choose the solution with the K function for \( k > 0 \) (Randall-Sundrum background) and the solution with the I function for \( k < 0 \) (alternative background). Then, we can determine the coefficients \( c_{\mu \nu}(p) \) from the boundary condition (48). After doing so, the final solution for \( \tilde{\gamma}_{\mu \nu} \) reads

\[
\tilde{\gamma}_{\mu \nu}(p, y) = \frac{8\pi}{|p|} \left[ t_{\mu \nu} - \frac{1}{3} \left( \eta_{\mu \nu} - p_\mu p_\nu \right) t \right] e^{2k|y|} \begin{cases} K_2(e^{k|y|}|p|/k) \frac{K_1(|p|/k)}{I_2(e^{k|y|}|p|/k)} \quad &\text{(RS),} \\ I_2(e^{k|y|}|p|/k) \frac{K_1(|p|/k)}{I_1(|p|/k)} \quad &\text{(alternative).} \end{cases}
\]

(51)

We can use the solution (51) to discuss the effective laws of gravity on the brane. First, we would like to know the gravitational potential due to, but far away from a static point source, which we introduce by

\[
t_{00}(x) = M/M_{Pl}^3 \delta(\vec{x}), \quad t_{00}(p) = 2\pi \delta(p_0) M/M_{Pl}^3,
\]

where \( M_{Pl}^3 \) is the Planck mass in five dimensions.

There are two possibilities for the behaviour of a test particle on the brane. First, one could assume that the particle is free to move in five dimensions, i.e. it will follow a geodesic in five-space, which for small velocities is

\[
\frac{d^2 x^i}{dt^2} \approx -\Gamma^i_{00}
\]

(52)

for \( i = 1, 2, 3 \). Then,

\[
\Gamma^i_{00} = \tilde{\Gamma}^i_{00} + k \text{sgn} y n_i = -\frac{1}{2} \partial_i \tilde{\gamma}_{00} - \frac{1}{8} \partial_i \gamma = -\frac{1}{2} \partial_i \tilde{\gamma}_{00},
\]

(53)

where we have used eqn. (25) and the fact that our particular solution is static. Since we are interested in the long distance behaviour on the brane, we set \( y = 0 \) in eqn. (51) and consider small \( |p| \). Using \( K_2(z) \approx 2/z^2 \), \( K_1(z) \approx 1/z \), \( I_2(z) \approx z^2/8 \) and \( I_1(z) \approx z/2 \), we find

\[
\tilde{\gamma}_{00}(p, 0) \approx \begin{cases} \frac{32\pi k}{3p^2} t_{00} + \cdots &\text{(RS),} \\ \frac{4\pi}{3k} t_{00} + \cdots &\text{(alternative).} \end{cases}
\]

The \( 1/p^2 \) term for the Randall-Sundrum background generates a \( 1/r \) potential term about the static point source, whereas there is no \( 1/r \) term in the alternative case. More specifically, we can deduce from eqns. (52) and (53) that

\[
V_{\text{unconstr}}(r) = \begin{cases} -\frac{4kM}{3M_{Pl}^3 r} + \cdots &\text{(RS),} \\ \text{no 1/r term} \quad &\text{(alternative).} \end{cases}
\]

(54)

The second possibility is that the particle is constrained to move along the brane by some non-gravitational mechanism. This would mean that it follows a geodesic on the brane, i.e.

\[
\frac{d^2 x^i}{dt^2} \approx -\tilde{\Gamma}^i_{00}.
\]

(55)
However,
\[ \Gamma'_{00} = -\frac{1}{2} \partial_0 \gamma_{00} = -\frac{1}{2} \partial_0 \tilde{\gamma}_{00} + \frac{1}{8} \partial_0 \gamma, \quad (56) \]
where \( \gamma \) is given by the Dirichlet boundary condition \([10]\). Hence, the gravitational potential is
\[ V_{\text{constr}}(r) = V_{\text{unconstr}}(r) - \frac{4\pi k}{3 \Box} t_{00} \approx \begin{cases} \frac{-kM}{M_{\text{Pl}}(5)r} + \cdots & \text{(RS)}, \\ -\frac{kM}{3M_{\text{Pl}}(5)r} + \cdots & \text{(alternative)}. \end{cases} \quad (57) \]

Obviously, in the unconstrained case the dynamics on the brane is affected by the non-zero shift vectors. For the RS background, the freedom of a test-particle to move in five dimensions would imply that the trajectories on the brane are unstable, since the brane is repulsive \([28]\). Hence, we are lead to favour the situation, in which the test particle is confined to move along the brane by some non-gravitational mechanism. Then, potential shortcuts via the fifth dimension are not allowed, the dynamics is determined by the intrinsic metric only, and eqn. \([57]\) applies. Moreover, in the alternative background, only confinement to the brane leads to a \(1/r\) potential.

Let us also comment on the solution of eqn. \([49]\) for \(p^2 < 0\), i.e. for the case of tachyonic matter sources on the brane. In that case, the linearly independent solutions of eqn. \([43]\) are
\[ \tilde{\gamma}_{\mu
u}(p, y) = c_{\mu
u}(p) e^{2ky} \begin{cases} N_2 \left( e^{ky} |p|/k \right) \\ J_2 \left( e^{ky} |p|/k \right) \end{cases} \quad (p^2 < 0). \quad (58) \]

In the Randall-Sundrum background, both modes diverge for large \(y\). (Although the Bessel functions both go to zero, the exponential factor in front diverges faster.) Interestingly, these modes are integrable, since the norm integral contains a factor \(e^{-4k|y|}\) in the invariant integration measure cancelling the diverging factor; and they describe the massive Kaluza Klein modes used to construct the five dimensional Green’s function in \([17, 26]\). In fact, we have performed the integral over the Kaluza Klein states for \(p^2 > 0\) in the Green’s function \([17, \text{eqn. (13)}]\) and found perfect agreement with our solution \([51]\). However, classical solutions containing the \(p^2 < 0\) modes will diverge for \(y \rightarrow \infty\). Therefore, in the RS scenario, the existence of tachyonic matter on the brane is inconsistent with linearized gravity in the five-dimensional space-time, as is the existence of free Kaluza Klein gravity excitations. The non-existence of both might be quite desirable in a theory describing the real world.

As a second objective we would like to consider the zero-mode truncation of the solution \([51]\) on the brane. The intrinsic Einstein tensor on the brane can be written as
\[ \hat{R}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \hat{R} = -\frac{1}{2} \Box \tilde{\gamma}_{\mu\nu} - \frac{1}{4} \left( \gamma_{,\mu\nu} - \eta_{\mu\nu} \Box \gamma \right), \quad (59) \]
where we have used the gauge \([22]\). In the Randall-Sundrum case, using only the zero-mode of eqn. \([51]\), we find
\[ \Box \hat{\tilde{\gamma}}_{\mu\nu}^{\text{zero-mode}} = -16\pi k \left[ t_{\mu\nu} - \frac{1}{3} \left( \eta_{\mu\nu} - \frac{\partial \mu \partial \nu}{\Box} \right) t \right]. \quad (60) \]
Furthermore, we can substitute the Dirichlet boundary condition (40) for $\gamma$, so that eqn. (59) becomes

$$\hat{R}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \hat{R} = 8\pi k t_{\mu\nu}. \quad (61)$$

The same equation can be obtained using $\gamma'_{\mu\nu}$, which is related to $\gamma_{\mu\nu}$ by a four-dimensional gauge transformation [cf. eqn. (43)]. Thus, the zero-mode truncation of the solution (51) yields Einstein’s equation on the brane for the Randall-Sundrum background, which by now is a well-established result (cf. e.g. [9, 17, 22]).

It is easy to see that this is not the case for the alternative background, as in that case eqn. (60) would not be valid.

6 Conclusions

In this paper, we have used a novel gauge in order to obtain a solution of the linearized Einstein equations in the Randall-Sundrum and an alternative background, where the brane is kept straight in spite of matter perturbations located on it. Our solution is consistent in each of the two half-spaces $y > 0$ and $y < 0$, and the two patches can be connected by making the gauge transformation (43)–(45).

The explicit solution was summarized already by the equations (1), (2) and (3), and a particular solution of eqn. (3) is found in eqn. (51). Our analysis of the gauge degrees of freedom showed that the traceless transversal part of $\gamma_{\mu\nu}$, $\tilde{\gamma}_{\mu\nu}$, represents all physical degrees of freedom. In particular, we conclude that the unphysical graviscalar mode mentioned in [33] is a gauge mode. This was found independently in [39].

Based on our solution, we studied the effective laws of gravity on the brane and found, in the Randall-Sundrum background, that the zero-mode truncation yields Einstein’s equation for the intrinsic metric on the brane. This implies the validity of the Newtonian limit, if the dynamics is determined by intrinsic quantities on the brane only, and agrees with our derivation of the Newton potential. Moreover, it emphasizes the importance of non-gravitational confinement of matter to the brane in order to prevent the extrinsic geometry from entering the dynamics of matter on the brane. A non-gravitational confinement is also necessary for the geodesics along the brane to be stable [28, 37]. In this article, we did not discuss the corrections to the Newton potential on the brane, as this has been done elsewhere [1, 17, 28, 41].

In this work, we have restricted our discussion to thin branes. It would be interesting to study the analogue for thick branes, which would also provide an appropriate regularization.

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Appendix

Here, we list various linearized expressions necessary for the calculations in the main text. The metric tensor has the form (\ref{eq_metric}), where the induced metric $\hat{g}_{\mu\nu}$ is linearized by eqn. (15), and $\gamma_{\mu\nu}$, $n_{\mu}$ and $n^2 - 1$ are small perturbations. We shall henceforth raise and lower the indices of $\gamma_{\mu\nu}$ and of $\partial_{\mu}$ (and only of these) with the Lorentz metric.

First, the connection coefficients intrinsic to the hypersurfaces are

$$\hat{\Gamma}^{\mu}{}_{\nu\lambda} = \frac{1}{2} (\gamma^{\mu}{}_{\nu,\lambda} + \gamma^{\mu}{}_{\lambda,\nu} - \gamma_{\nu\lambda}{}^{\mu}),$$

and the intrinsic Ricci tensor and curvature scalar are, respectively

$$\hat{R}_{\nu\rho} = \frac{1}{2} (\gamma^{\mu}{}_{\nu,\rho,\mu} + \gamma^{\mu}{}_{\rho,\nu,\mu} - \Box \gamma_{\nu\rho} - \gamma_{\nu,\rho}),$$

$$\hat{R} = e^{2k|y|} (\gamma^{\mu\nu}{}_{,\mu\nu} - \Box \gamma).$$

The linearized expression for the second fundamental form, (cf. eqn. (10)) is

$$H^{\mu}{}_{\nu} = \frac{1}{2n} \left[ -2k \text{sgn} y \delta^{\mu}_{\nu} + \gamma^{\mu}{}_{\nu,y} - e^{2k|y|} \eta^{\mu\lambda} (n_{\nu,\lambda} + n_{\lambda,\nu}) \right].$$

The necessary connection coefficients of the five-space are

$$\Gamma^{\mu}{}_{\nu\lambda} = \hat{\Gamma}^{\mu}{}_{\nu\lambda} - k \text{sgn} y \eta_{\nu\lambda} \eta^{\mu\rho} n_{\rho},$$

$$\Gamma^{y}{}_{\nu\lambda} = \frac{1}{n^2} (k \text{sgn} y \hat{g}_{\nu\lambda}) + \frac{1}{2} \left( n_{\nu,\lambda} + n_{\lambda,\nu} - e^{-2k|y|} \gamma_{\nu,\lambda,y} \right),$$

$$\Gamma^{\mu}{}_{\nu y} = -k \text{sgn} y \delta^{\mu}_{\nu} + \frac{1}{2} \left[ \gamma_{\mu}{}_{\nu,y} + e^{2k|y|} \eta^{\mu\lambda} (n_{\lambda,\nu} - n_{\nu,\lambda}) \right],$$

$$\Gamma^{y}{}_{\nu y} = k \text{sgn} y n_{\nu} + \frac{1}{2} (n^2 - 1)_{,\nu},$$

$$\Gamma^{y}{}_{y y} = \frac{1}{2} (n^2 - 1)_{,y}.$$}

Thus, the linearized Ricci tensor becomes

$$R_{\mu\nu} = \hat{R}_{\mu\nu} - \frac{4k^2}{n^2} \hat{g}_{\mu\nu} + \frac{2k}{n^2} \delta (y) \hat{g}_{\mu\nu} - \frac{1}{2} \left( e^{-2k|y|} \gamma_{\mu\nu,y} \right) - \frac{1}{2} (n^2 - 1)_{,\mu\nu}$$

$$+ \frac{1}{2} (n_{\mu,\nu} + n_{\nu,\mu})_{,y} - k \text{sgn} y \left( n_{\mu,\nu} + n_{\nu,\mu} + \eta_{\mu\nu} n_{\lambda,\lambda} \right)$$

$$+ k \text{sgn} y e^{-2k|y|} \left\{ \gamma_{\mu\nu,y} + \frac{1}{2} \eta_{\mu\nu} (\gamma_{y,y} - (n^2 - 1)_{,y}) \right\}.$$
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