Sampling discretization of integral norms and its application

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Abstract

The paper addresses a problem of sampling discretization of integral norms of elements of finite-dimensional subspaces satisfying some conditions. We prove sampling discretization results under two standard kinds of assumptions – conditions on the entropy numbers and conditions in terms of the Nikol’skii-type inequalities. We prove some upper bounds on the number of sample points sufficient for good discretization and show that these upper bounds are sharp in a certain sense. Then we apply our general conditional results to subspaces with special structure, namely, subspaces with the tensor product structure. We demonstrate that applications of results based on the Nikol’skii-type inequalities provide somewhat better results than applications of results based on the entropy numbers conditions. Finally, we apply discretization results to the problem of sampling recovery.

Keywords and phrases: Sampling discretization, entropy numbers, Nikol’skii inequality, recovery.

MSC classification 2000: Primary 65J05; Secondary 42A05, 65D30, 41A63.

1 Introduction

Let $\Omega$ be a subset of $\mathbb{R}^d$ with the probability measure $\mu$. By $L_p$, $1 \leq p < \infty$, norm we understand

$$
\|f\|_p := \|f\|_{L_p(\Omega, \mu)} := \left(\int_{\Omega} |f|^p d\mu\right)^{1/p}.
$$

*The first named author’s research was partially supported by NSERC of Canada Discovery Grant RGPIN-2020-03909. The second named author’s research was supported by the Russian Federation Government Grant No. 14.W03.31.0031.
By $L_\infty$ norm we understand the uniform norm of continuous functions

$$\|f\|_\infty := \max_{x \in \Omega} |f(x)|$$

and with some abuse of notation we occasionally write $L_\infty(\Omega)$ for the space $C(\Omega)$ of continuous functions on $\Omega$.

By discretization of the $L_p$ norm we understand a replacement of the measure $\mu$ by a discrete measure $\mu_m$ with support on a set $\xi = \{\xi^j\}_{j=1}^m \subset \Omega$ in such a way that the error $|\|f\|_p^p(\Omega, \mu) - \|f\|_p^p(\Omega, \mu_m)|$ is small for functions from a given class. In this paper we focus on discretization of the $L_p$ norms of elements of finite-dimensional subspaces. Namely, we work on the following problem.

**The Marcinkiewicz discretization problem.** Let $\Omega$ be a compact subset of $\mathbb{R}^d$ with the probability measure $\mu$. We say that a linear subspace $X_N$ (index $N$ here, usually, stands for the dimension of $X_N$) of $L_p(\Omega, \mu)$, $1 \leq p < \infty$, admits the Marcinkiewicz-type discretization theorem with parameters $m \in \mathbb{N}$ and $p$ and positive constants $C_1 \leq C_2$ if there exists a set

$$\{\xi^j \in \Omega : j = 1, \ldots, m\}$$

such that for any $f \in X_N$ we have

$$C_1 \|f\|_p^p \leq \frac{1}{m} \sum_{j=1}^m |f(\xi^j)|^p \leq C_2 \|f\|_p^p. \quad (1.1)$$

**The Bernstein discretization problem.** In the case $p = \infty$ we define $L_\infty$ as the space of continuous functions on $\Omega$ and ask for

$$C_1 \|f\|_\infty \leq \max_{1 \leq j \leq m} |f(\xi^j)| \leq \|f\|_\infty. \quad (1.2)$$

We will also use the following brief way to express the above properties: The $\mathcal{M}(m, p)$ (more precisely the $\mathcal{M}(m, p, C_1, C_2)$) theorem holds for a subspace $X_N$, written $X_N \in \mathcal{M}(m, p)$ (more precisely $X_N \in \mathcal{M}(m, p, C_1, C_2)$).

There are known results on the Marcinkiewicz discretization problem proved for subspaces $X_N$ satisfying some conditions. There are two types of conditions used in the literature: (I) Conditions on the entropy numbers
and (II) Conditions in terms of the Nikol’skii-type inequalities. We now describe these conditions in detail.

I. Entropy conditions. We begin with the definition of the entropy numbers. Let $X$ be a Banach space and let $B_X$ denote the unit ball of $X$ with the center at 0. Denote by $B_X(y, r)$ the ball with center $y$ and radius $r > 0$, \( \{ x \in X : \| x - y \| \leq r \} \). For a compact set $A \subset X$ we define the entropy numbers $\varepsilon_k(A, X)$, $k = 0, 1, \ldots$

\[
\varepsilon_k(A, X) := \inf \left\{ \varepsilon > 0 : \exists y^1, \ldots, y^{2^k} \in A, A \subseteq \bigcup_{j=1}^{2^k} B_X(y^j, \varepsilon) \right\}.
\]

In our definition of $\varepsilon_k(A, X)$ we require $y^j \in A$. In a standard definition of $\varepsilon_k(A, X)$ this restriction is not imposed. However, it is well known (see [8], p.208) that these characteristics may differ at most by the factor 2.

Throughout the paper we use the following notation for the unit $L_q$ ball of $X$

\[
X^q_N := \{ f \in X^N : \| f \|_q \leq 1 \}.
\]

Here is a standard entropy assumption in discretization theory: Suppose that a subspace $X_N$ satisfies the condition (\( B \geq 1 \))

\[
\varepsilon_k(X^q_N, L_\infty) \leq B(N/k)^{1/q}, \quad 1 \leq k \leq N. \tag{1.3}
\]

II. Nikol’skii-type inequalities. Let $q \in [1, \infty)$ and $X_N \subset L_\infty(\Omega)$. The inequality

\[
\| f \|_\infty \leq M\| f \|_q, \quad \forall f \in X_N \tag{1.4}
\]

is called the Nikol’skii inequality for the pair (\( q, \infty \)) with the constant $M$. It is convenient to write the constant $M$ in the form $M = BN^{1/q}$. If $X_N$ satisfies (1.4) with $M = BN^{1/q}$, then we say that $X_N$ satisfies Condition NqB (see Section 4 below) and write $X_N \in \mathcal{N}(q, \infty, B)$.

It is well known that in the case $q \in [2, \infty)$ the above entropy condition and Nikol’skii-type inequality are closely related. We now comment on the relation between the Nikol’skii inequality and the entropy condition (1.3). On one hand it is easy to see (see [3]) that the entropy condition (1.3) for $k = 1$ implies the following Nikol’skii inequality

\[
\| f \|_\infty \leq 4BN^{1/q}\| f \|_q \quad \text{for all } f \in X_N.
\]

On the other hand we note that the Nikol’skii-type inequality condition (1.4) with $M = BN^{1/q}$ implies the entropy condition (1.3) with $k = 1$. Thus, the
Nikol’skii-type inequality condition is equivalent to the entropy condition (1.3) for \( k = 1 \). Moreover, Lemma 2.2 (see below) from [11] shows that in the case \( q \in [2, \infty) \) the Nikol’skii inequality (1.4) combined with an extra mild condition implies the entropy condition (1.3) with \( B \) replaced by \( C(\log N)^{1/q}B \).

We refer the reader to the recent survey papers [1] and [5] for a detailed description of known results on the Marcinkiewicz discretization problem under conditions I and II. In this paper we only cite those results which are directly related to our new results. We now give brief comments on results obtained in Sections 2–4.

In Section 2 we discuss the following setting. Assume that parameters \( 1 \leq p, q < \infty \) are given. We would like to solve the Marcinkiewicz discretization problem for the \( L_p \) norm under assumption that either Conditions I or II is satisfied with the parameter \( q \). Typically, the known results address either the case \( p = q \) or the case when the Nikol’skii-type inequality for the pair (2, \( \infty \)) is imposed. In Section 2 (see Corollary 2.1) we show how a simple argument allows us to derive a discretization result in the case \( 1 \leq q \leq p < \infty \) under assumptions I from the corresponding result for \( p = q \). The main result of Section 2 is Theorem 2.2, which improves known results in the case when \( p = q \) and \( 3 < q < \infty \) under Conditions II. Then we use Theorem 2.2 to deduce discretization results in the case \( 2 \leq q \leq p < \infty \) (see Corollary 2.2).

In Section 3 we apply results of Section 2 to subspaces with special structure, namely, subspaces with the tensor product structure. We demonstrate that applications of results based on Conditions II provide somewhat better results than applications of results based on Conditions I. This observation is based on a known result on the Nikol’skii-type inequalities for subspaces with the tensor product structure (see Lemma 3.1 below).

In Section 4 we apply discretization results of Section 2 to the problem of sampling recovery. Recently, it was observed in [12] how discretization results can help to prove general inequalities between optimal sampling recovery and the Kolmogorov widths. Namely, it was proved in [12] that the optimal error of recovery in the \( L_2 \) norm of functions from a class \( F \) can be bounded above by the value of the Kolmogorov width of \( F \) in the uniform norm. In Section 4 we demonstrate how to derive some general inequalities for the optimal sampling recovery in \( L_p \) from the corresponding discretization results of Section 2.
2 A generalization

We start with the following Theorem 2.1, which was proved in [9] in the case of \( q = 1 \) with the help of the chaining technique, and in [2] for the general case.

**Theorem 2.1.** Let \( 1 \leq q < \infty \). Suppose that a subspace \( X_N \) satisfies the condition
\[
\varepsilon_k(X_N^q, L_\infty) \leq B(N/k)^{1/q}, \quad 1 \leq k \leq N, \tag{2.1}
\]
where \( B \geq 1 \). Then for a large enough constant \( C(q) \) there exists a set of
\[
m \leq C(q)B^qN(\log_2(2BN))^2
\]
points \( \xi^j \in \Omega, \ j = 1, \ldots, m \), such that for any \( f \in X_N \) we have
\[
\frac{1}{2} \| f \|_q^q \leq \frac{1}{m} \sum_{j=1}^{m} |f(\xi^j)|^q \leq \frac{3}{2} \| f \|_q^q.
\]

Here is a direct corollary of Theorem 2.1

**Corollary 2.1.** Let \( 1 \leq q \leq p < \infty \). Suppose that the condition (2.1) is satisfied. Then for a large enough constant \( C(p, q) \) there exists a set of
\[
m \leq C(p, q)B^pN^{p/q}(\log_2(2BN))^2
\]
points \( \xi^j \in \Omega, \ j = 1, \ldots, m \), such that for any \( f \in X_N \) we have
\[
\frac{1}{2} \| f \|_p^p \leq \frac{1}{m} \sum_{j=1}^{m} |f(\xi^j)|^p \leq \frac{3}{2} \| f \|_p^p.
\]

**Proof.** Let \( 1 \leq q \leq p < \infty \). Then condition (2.1) implies
\[
\varepsilon_k(X_N^p, L_\infty) \leq \varepsilon_k(X_N^q, L_\infty) \leq BN^{1/q-1/p}(N/k)^{1/p}, \quad 1 \leq k \leq N. \tag{2.2}
\]
Thus, condition (2.1) is satisfied for \( p \) with \( B' = BN^{1/q-1/p} \). Theorem 2.1 gives
\[
m \leq C(p)(B')^pN(\log_2(2B'N))^2 \leq C(p, q)B^pN^{p/q}(\log_2(2BN))^2.
\]

\[\square\]
We now prove an analog of Theorem 2.1 for $q \in (2, \infty)$ under a condition on $X_N$ in terms of the Nikol’skii inequality instead of the entropy condition (2.1) in Theorem 2.1.

**Theorem 2.2.** Let $2 \leq q < \infty$. Suppose that a subspace $X_N$ satisfies the Nikol’skii type inequality

$$\|f\|_{\infty} \leq BN^{1/q}\|f\|_{q}, \quad \forall f \in X_N,$$

(2.3)

where $B \geq 1$. Then for any $\epsilon \in (0, 1)$ there is a large enough constant $C = C(p, q, \epsilon)$ such that there exists a set of

$$m \leq CB^qN(\log_2(2BN))^3$$

(2.4)

points $\xi_j \in \Omega$, $j = 1, \ldots, m$, such that for any $f \in X_N$ we have

$$(1 - \epsilon)\|f\|_{q}^q \leq \frac{1}{m} \sum_{j=1}^{m} |f(\xi_j)|^q \leq (1 + \epsilon)\|f\|_{q}^q,$$

Proof. We derive Theorem 2.2 from the $\epsilon$-version of Theorem 2.1. The following Remark 2.1 is from [2].

**Remark 2.1.** Under the assumptions of Theorem 2.1, we can deduce a slightly stronger result, namely, that for any $\epsilon \in (0, 1)$, there exists a set of $m$ points $\{\xi_j\}_{j=1}^{m} \subset \Omega$ with

$$m \leq C(q, \epsilon)B^qN(\log_2(2BN))^2$$

such that

$$(1 - \epsilon)\|f\|_{q}^q \leq \frac{1}{m} \sum_{j=1}^{m} |f(\xi_j)|^q \leq (1 + \epsilon)\|f\|_{q}^q, \quad \forall f \in X_N,$$

where $C(q, \epsilon)$ is a positive constant depending only on $\epsilon$ and $q$.

At the first step we use the following Lemma 2.1 from [3].

**Lemma 2.1.** Let $1 \leq p < \infty$ be a fixed number. Assume that $X_N$ is an $N$-dimensional subspace of $L_\infty(\Omega)$ satisfying the following condition for some parameter $\beta > 0$ and constant $K \geq 2$:

$$\|f\|_{\infty} \leq (KN)^{\beta/p}\|f\|_{p}, \quad \forall f \in X_N.$$  

(2.5)
Let \( \{x^j\}_{j=1}^{\infty} \) be a sequence of independent random points selected from \( \Omega \) according to \( \mu \). Then there exists a positive constant \( C_\beta \) depending only on \( \beta \) such that for any \( 0 < \epsilon \leq \frac{1}{2} \) and

\[
m \geq C_\beta K^\beta \epsilon^{-2} (\log(2/\epsilon)) N^{\beta + 1} \log N,
\]

the inequality

\[
(1 - \epsilon) \|f\|_p^p \leq \frac{1}{m} \sum_{j=1}^m |f(x^j)|^p \leq (1 + \epsilon) \|f\|_p^p,
\]

holds for all \( f \in X_N \) with probability \( \geq 1 - m^{-N/\log K} \).

By Lemma 2.1 with \( p = q, \beta = 1, K = B^q \), and \( m = S \) we replace the \( L_q(\Omega, \mu) \) space by the \( L_q(\Omega_S, \mu_S) \) with \( \Omega_S = \{x^j\}_{j=1}^S, \mu_S(x^j) = 1/S, j = 1, \ldots, S \) with the relations

\[
(1 - \epsilon) \|f\|_{L_q(\Omega, \mu)}^q \leq \|f\|_{L_q(\Omega_S, \mu_S)}^q \leq (1 + \epsilon) \|f\|_{L_q(\Omega, \mu)}^q,
\]

\[
S \leq C_2 B^q \epsilon^{-2} (\log(2/\epsilon))^2 \log N.
\]

At the second step we apply the following Lemma 2.2 from [11], which is based on the corresponding results from [6], with \( s = S \) to the restriction \( X_N(\Omega_S) \) of the subspace \( X_N \) onto the \( \Omega_S \) in the case of \( L_q(\Omega_S, \mu_S) \).

**Lemma 2.2.** Let \( q \in [2, \infty) \). Assume that for any \( f \in X_N \) we have

\[
\|f\|_\infty \leq BN^{1/q} \|f\|_q
\]

with some constant \( B \geq 1 \). Also, assume that \( X_N \in M(s, \infty, C_1) \) with \( s \geq 1 \). Then for \( k \in [1, N] \) we have

\[
\varepsilon_k(X_N^q, L_\infty) \leq C(q, C_1)(\log s)^{1/q} B(N/k)^{1/q}.
\]

This gives us the bound

\[
\varepsilon_k(X_N^q(\Omega_S), L_\infty) \leq C(q, \epsilon, C_2) B(\log(2BN))^{1/q} (N/k)^{1/q}.
\]

At the third step we apply Remark 2.1 to the \( X_N(\Omega_S) \) and complete the proof.
Here is a direct corollary of Theorem 2.2.

**Corollary 2.2.** Let $2 \leq q \leq p < \infty$. Suppose that condition (2.3) is satisfied with $B \geq 1$. Then for any $\epsilon \in (0, 1)$ there is a large enough constant $C = C(p, q, \epsilon)$ such that there exists a set of

$$m \leq C B^p N^{p/q} (\log_2 (2BN))^3$$

(2.13)

points $\xi_j \in \Omega$, $j = 1, \ldots, m$, such that for any $f \in X_N$ we have

$$(1 - \epsilon) \|f\|_p^p \leq \frac{1}{m} \sum_{j=1}^{m} |f(\xi_j)|^p \leq (1 + \epsilon) \|f\|_p^p.$$

**Proof.** Let $2 \leq q \leq p < \infty$. Then condition (2.3) implies

$$\|f\|_\infty \leq BN^{1/q} \|f\|_q \leq BN^{1/q - 1/p} N^{1/p} \|f\|_p, \quad 1 \leq k \leq N.$$  

(2.14)

Thus, condition (2.3) is satisfied for $p$ with $B' = BN^{1/q - 1/p}$. Theorem 2.2 gives

$$m \leq C(p, \epsilon) (B')^p N (\log_2 ((2B'N)))^3 \leq C(p, q, \epsilon) B^p N^{p/q} (\log_2 (2BN))^3.$$  

(2.15)

**Remark 2.2.** Corollary 2.2 provides bound (2.13) on $m$ that consists of three factors: $N^{p/q}$, $(\log_2 (2BN))^3$, which grow with $N$, and $B^p$, which may grow with $N$. We do not know if the factor $(\log_2 (2BN))^3$ can be dropped in (2.13). However, we know that in the case $q > 2$ neither $N^{p/q}$ can be replaced by $N^{p/q-\delta}$ nor $B^p$ can be replaced by $B^{p-\delta}$ with any $\delta > 0$. In the case $q = 2$ the factor $N^{p/2}$ cannot be replaced by $N^{p/2-\delta}$ with any $\delta > 0$.

**Proof.** We begin with the case $q = 2$. Let $\Lambda_n = \{k_j\}_{j=1}^n$ be a lacunary sequence: $k_1 = 1$, $k_{j+1} = bk_j$, $b > 1$, $j = 1, \ldots, n - 1$. Denote

$$\mathcal{T}(\Lambda_n) := \left\{ f : f(x) = \sum_{k \in \Lambda_n} c_k e^{ikx}, \quad x \in \mathbb{T} \right\}.$$

It is clear that $\mathcal{T}(\Lambda_n) \in \mathcal{N}(2, \infty, 1)$. Indeed, for $f(x) = \sum_{k \in \Lambda_n} c_k e^{ikx}$ we have

$$\|f\|_\infty \leq \sum_{k \in \Lambda_n} |c_k| \leq n^{1/2} \left( \sum_{k \in \Lambda_n} |c_k|^2 \right)^{1/2} = n^{1/2} \|f\|_2.$$  

(2.15)
It is proved in [5] (see D.20. A lower bound) that the condition \( T(\Lambda_n) \in \mathcal{M}(m, p, C_1, C_2) \) with \( p > 2 \) and fixed \( C_1 \) and \( C_2 \) implies \( m \geq C(p, C_1, C_2)n^{p/2} \). This completes the proof of Remark 2.2 in the case \( q = 2 \). We now let \( q > 2 \) and rewrite (2.15) in the form
\[
\| f \|_\infty \leq n^{1/2} \| f \|_2 \leq n^{1/2-1/q} n^{1/q} \| f \|_q.
\]
Therefore, \( T(\Lambda_n) \in \mathcal{N}(q, \infty, n^{1/2-1/q}) \). Then, on one hand we know from the above that \( m \) must grow in the sense of order as \( n^{p/2} \), on the other hand if we replace in (2.13) \( N^{p/q} \) by \( N^{p/q-\delta} \), then we obtain
\[
m \leq C n^{p(1/2-1/q)} n^{p/q-\delta} (\log n)^3 \leq C n^{p/2-\delta} (\log n)^3.
\]
We get a contradiction. In the same way we get a contradiction in the case of replacement \( B^p \) by \( B^{p-\delta} \). The proof is complete.

A comment. Sometimes it is convenient to have the entropy bound (2.1) for all \( k \in \mathbb{N} \), namely, the bound
\[
\varepsilon_k(X^q_N, L_\infty) \leq B_1(N/k)^{1/q}, \quad 1 \leq k < \infty.
\] (2.16)
We now prove that (2.1) implies (2.16) with \( B_1 = 6B \). We begin by pointing out that the assumption (2.1) for \( k = N \) implies the inequality
\[
\varepsilon_k(X^q_N, L_\infty) \leq 6B 2^{-k/N} \quad \text{for} \quad k > N.
\] (2.17)
This follows directly from the facts that for each Banach space \( X \) (see [10, (7.1.6), p. 323]),
\[
\varepsilon_k(A, X) \leq \varepsilon_N(A, X) \varepsilon_{k-N}(B_X, X), \quad k > N,
\]
and for each \( N \)-dimensional space \( X \) (see [10 Corollary 7.2.2, p. 324]),
\[
\varepsilon_m(B_X, X) \leq 3(2^{-m/N}).
\]
Next, we use the inequality \( 2^x \geq 1 + x \), for \( x \geq 1 \). This inequality follows from \( (x \geq 1, a = \ln 2) \)
\[
e^{ax} = 1 + ax + \frac{(ax)^2}{2!} + \cdots \geq 1 + x \left( a + \frac{a^2}{2!} + \cdots \right) = 1 + x(e^a - 1) = 1 + x.
\]
Therefore, condition (2.1) implies condition (2.16) with \( B_1 = 6B \).
3 Discretization in subspaces with tensor product structure

Let $s \in \mathbb{N}$. Suppose that we have $s$ subspaces $X(N_i, i) \subset C(\Omega_i)$ with\[\dim X(N_i, i) = N_i, \quad i = 1, \ldots, s.\]
Denote $N := (N_1, \ldots, N_s)$ and $X_N := \text{span}\{f_1(x^1) \times \cdots \times f_s(x^s) : f_i \in X(N_i, i), \quad i = 1, \ldots, s\}$ a subspace of $C(\Omega_1 \times \cdots \times \Omega_s)$. Consider a product measure $\mu = \mu_1 \times \cdots \times \mu_s$ on $\Omega := \Omega_1 \times \cdots \times \Omega_s$ with $\mu_i$ being a probability measure on $\Omega_i$, $i = 1, \ldots, s$.

First, we prove some discretization results for the $X_N$ under the conditions\[[\varepsilon_k(X^q(N_i, i), L_{\infty}) \leq B_i(N_i/k)^{1/q}, \quad 1 \leq k < \infty, \quad i = 1, \ldots, s, \quad (3.1)\]
where $B_i \geq 1$, $i = 1, \ldots, s$. Note, that as it is explained at the end of Section 2, conditions (3.1) are equivalent to the same conditions with a weaker restrictions on $k$: instead of $1 \leq k < \infty$ we can take $1 \leq k \leq N_i$. We begin with a simple observation.

**Proposition 3.1.** Let $1 \leq q \leq p < \infty$. Suppose that conditions (3.1) are satisfied. Then for a large enough constant $C(p, q, s)$ there exists a set $\xi(m)$ with a tensor product structure of\[m \leq C(p, q, s) \prod_{i=1}^s B_i^p N_i^{p/q} (\log_2(2B_i N_i))^2\]
points $\xi^j \in \Omega$, $j = 1, \ldots, m$, such that for any $f \in X_N$ we have\[\left(\frac{1}{2}\right)^s \|f\|_p^p \leq \frac{1}{m} \sum_{j=1}^m |f(\xi^j)|^p \leq \left(\frac{3}{2}\right)^s \|f\|_p^p. \quad (3.2)\]

**Proof.** By Corollary 2.1 for each subspace $X(N_i, i), i = 1, \ldots, s$, we find a set $\xi(m_i, i) = \{\xi^j(m_i, i)\}_{j=1}^{m_i} \subset \Omega_i$ of $m_i$ points such that\[m_i \leq C(p, q) B_i^p N_i^{p/q} (\log_2(2B_i N_i))^2 \quad (3.3)\]
and for any $f \in X(N_i, i)$ we have\[\frac{1}{2} \|f\|_p^p \leq \frac{1}{m_i} \sum_{j=1}^{m_i} |f(\xi^j(m_i, i))|^p \leq \frac{3}{2} \|f\|_p^p. \quad (3.4)\]
Then applying (3.4) successively with respect to $i = 1, \ldots, s$ we obtain for $f \in X_N$ inequalities (3.2) with $\xi(m) = \xi(m_1, 1) \times \cdots \times \xi(m_s, s)$ and by (3.3)

$$m = \prod_{i=1}^{s} m_i \leq C(p, q)^s \prod_{i=1}^{s} B_i^p N_i^{p/q} (\log_2(2B_i N_i))^2.$$

This completes the proof.

Second, we discuss some results, when instead of conditions (3.1) we impose conditions in terms of Nikol’skii-type inequalities: For any $f \in X(N_i, i)$

$$\|f\|_\infty \leq B_i N_i^{1/q} \|f\|_q, \quad i = 1, \ldots, s. \quad (3.5)$$

In the same way as Proposition 3.1 was derived from the discretization result – Corollary 2.1 – we derive from Corollary 2.2 the following statement.

**Proposition 3.2.** Let $2 \leq q \leq p < \infty$. Suppose that conditions (3.5) are satisfied with $B_i \geq 1$, $i = 1, \ldots, s$. Then for a large enough constant $C = C(p, q, s)$ there exists a set $\xi(m)$ with a tensor product structure of

$$m \leq C \prod_{i=1}^{s} B_i^p N_i^{p/q} (\log_2(2B_i N_i))^3$$

(3.6)

points $\xi^j \in \Omega$, $j = 1, \ldots, m$, such that for any $f \in X_N$ we have

$$\left(\frac{1}{2}\right)^s \|f\|_p^p \leq \frac{1}{m} \sum_{j=1}^{m} |f(\xi^j)|^p \leq \left(\frac{3}{2}\right)^s \|f\|_p^p. \quad (3.7)$$

In a particular case when $N_i = N$ and $B_i = B$ for $i = 1, \ldots, s$, the extra logarithmic factor in (3.6) will be of order $(\log(2BN))^3$. We now show how it could be reduced to $(\log(2BN))^3$. We need the following lemma, which is a particular case of Theorem 3.3.3 from [10, p.107] in the case of periodic functions with $\Omega_i = T$, $i = 1, \ldots, s$.

**Lemma 3.1.** Let $1 \leq q < \infty$. Suppose that conditions (3.5) are satisfied. Then for $f \in X_N$ we have

$$\|f\|_\infty \leq \left(\prod_{i=1}^{s} (B_i N_i^{1/q})\right) \|f\|_q.$$
Proof. Let \( f \in X_\mathbb{N} \). Then for any \( x = (x^1, x^2) \in \Omega_1 \times \Omega_2 \), we have

\[
|f(x^1, x^2)|^q \leq B_1^q N_1 \int_{\Omega_1} |f(y^1, x^2)|^q \, d\mu_1(y^1)
\]

\[
\leq B_1^q N_1 B_2^q N_2 \int_{\Omega_1} \left[ \int_{\Omega_2} |f(y^1, y^2)|^q \, d\mu_2(y^2) \right] \, d\mu_1(y^1),
\]

where we used (3.5) for \( i = 1 \) and the fact that \( f(\cdot, x^2) \in X(N_1, 1) \) in the first step, and (3.5) for \( i = 2 \) and the fact that \( f(y^1, \cdot) \in X(N_2, 2) \) for each fixed \( y^1 \in \Omega_1 \) in the second step. This proves the stated inequality for \( s = 2 \).

The inequality for the general case \( s \geq 2 \) follows by induction.

\[
\square
\]

Using Lemma 3.1, we may apply Corollary 2.2 to the space \( X_\mathbb{N} := X_{\mathbb{N}} \) with \( N = \prod_{i=1}^s N_i \) and \( B = \prod_{i=1}^s B_i \). We then obtain the following version of Proposition 3.2.

**Proposition 3.3.** Let \( 2 \leq q \leq p < \infty \). Suppose that the conditions (3.5) are satisfied with \( B_i \geq 1, \ i = 1, \ldots, s \). Then for a large enough constant \( C = C(p, q) \) there exists a set \( \xi(m) \) of

\[
m \leq C \left( \prod_{i=1}^s B_i^p N_i^{p/q} \right) \left( \log_2 \left( \prod_{i=1}^s (B_i N_i) \right) \right)^3
\]

(3.8)

points \( \xi_j \in \Omega, \ j = 1, \ldots, m \), such that

\[
\frac{1}{2} \|f\|_p^p \leq \frac{1}{m} \sum_{j=1}^m |f(\xi_j)|^p \leq \frac{3}{2} \|f\|_p^p, \ \forall f \in X_{\mathbb{N}}.
\]

(3.9)

Clearly, Proposition 3.3 is a better version of Proposition 3.2. However, there is no guarantee that the set \( \xi(m) \) in Proposition 3.3 has the tensor product structure. It would be interesting to prove an analog of Proposition 3.3 where the set \( \xi(m) \) of points is given by a tensor product of subsets of \( \Omega_i, \ i = 1, \cdots, s \) and the power of the log factor in the estimate (3.8) is independent of \( s \). An affirmative answer to this question would yield a significant reduction of the log factor in the estimate (2.4) of Theorem 2.2 as can be seen from the following simple lemma.
Lemma 3.2. Let $X_N \subset C(\Omega)$ be an $N$-dimensional subspace satisfying
\[
\|f\|_\infty \leq BN^{1/p}\|f\|_{L_p(\Omega, \mu)}, \quad \forall f \in X_N,
\]
for some $1 \leq p < \infty$, and constant $B \geq 1$. Assume that $1 \in X_N$ and there exists a positive constant $\alpha$ such that for each integer $s \geq 2$, there exists a finite subset $\Lambda = \Lambda_1 \times \cdots \times \Lambda_s \subset \Omega^s$ such that $|\Lambda| = |\Lambda_1| \cdots |\Lambda_s| \leq C(p, s, \alpha)(B^p N^s (\log_2 (2BN)))^\alpha$ and
\[
\frac{1}{2}\|F\|^p_{L_p(\Omega^s, \mu^s)} \leq \frac{1}{|\Lambda|} \sum_{\omega \in \Lambda} |F(\omega)|^p \leq \frac{3}{2}\|F\|^p_{L_p(\Omega^s, \mu^s)}, \quad \forall F \in X_N,
\]
where $X_N := \text{span}\{f_1(x^1) \times \cdots \times f_s(x^s) : f_i \in X_N, i = 1, \ldots, s\}$. Then for any $\delta \in (0, 1)$, there exist $m \leq C(p, \delta, \alpha)B^p N (\log_2 (2BN))^\delta$ points $\xi^j \in \Omega$, $j = 1, \ldots, m$, such that for any $f \in X_N$ we have
\[
\frac{1}{2}\|f\|^p_p \leq \frac{1}{m} \sum_{j=1}^m |f(\xi^j)|^p \leq \frac{3}{2}\|f\|^p_p.
\]

Proof. Let $s \geq 2$ be an integer, and let $\Lambda = \Lambda_1 \times \cdots \times \Lambda_s$ be a finite subset of $\Omega^s$ with the stated properties in Lemma 3.2. Without loss of generality, we may assume that $|\Lambda_1| = \min_{1 \leq i \leq s} |\Lambda_i|$. Then
\[
|\Lambda_1| \leq C(p, s, \alpha)^{1/s}(B^p N (\log_2 (2BN)))^{\alpha/s}.
\]
Let $f \in X_N$, and define
\[
F(x^1, \cdots, x^s) := f(x^1), \quad x := (x^1, \cdots, x^s) \in \Omega^s.
\]
Clearly, $\|F\|^p_{L_p(\Omega^s)} = \|f\|^p_{L_p(\Omega)}$ and
\[
\frac{1}{|\Lambda|} \sum_{\omega \in \Lambda} |F(\omega)|^p = \frac{1}{|\Lambda_1|} \sum_{\omega_1 \in \Lambda_1} |f(\omega_1)|^p.
\]
Since $1 \in X_N$, we have $F \in X_N$. It then follows that
\[
\frac{1}{2}\|f\|^p_p \leq \frac{1}{|\Lambda_1|} \sum_{\omega_1 \in \Lambda_1} |f(\omega_1)|^p \leq \frac{3}{2}\|f\|^p_p.
\]
4 Sampling recovery

We first recall the setting of the optimal recovery. For a fixed integer \( m \) and a set of points \( \xi := \{\xi_j\}_{j=1}^m \subset \Omega \), let \( \Phi \) be a linear operator from \( C^m \) into \( L_p(\Omega, \mu) \). For a class \( F \subset L_p(\Omega, \mu) \) (usually, centrally symmetric and compact subset of \( L_p(\Omega, \mu) \)), define

\[
\rho_m(F, L_p) := \inf_{\text{linear } \Phi; \xi \in F} \sup_{f \in F} \| f - \Phi(f(\xi^1), \ldots, f(\xi^m)) \|_p.
\]

The above described recovery procedure is a linear procedure. The following modification of the above recovery procedure is also of interest. We now allow any mapping \( \Phi : C^m \to X_N \subset L_p(\Omega, \mu) \), where \( X_N \) is a linear subspace of dimension \( N \leq m \), and define

\[
\rho^*_m(F, L_p) := \inf_{\Phi; \xi; N, N \leq m \in F} \sup_{f \in F} \| f - \Phi(f(\xi^1), \ldots, f(\xi^m)) \|_p.
\]

In both of the above cases we build an approximant, which comes from a linear subspace of dimension at most \( m \). It is natural to compare the quantities \( \rho_m(F, L_p) \) and \( \rho^*_m(F, L_p) \) with the Kolmogorov widths. Let \( F \subset L_p \) be a centrally symmetric compact. The quantities

\[
d_n(F, L_p) := \inf_{\{u_i\}_{i=1}^n \subset L_p \text{ and } f \in F} \sup_{c_i} \left\| f - \sum_{i=1}^n c_i u_i \right\|_p, \quad n = 1, 2, \ldots,
\]

are called the Kolmogorov widths of \( F \) in \( L_p \). In the definition of the Kolmogorov widths we take the element of best approximation of \( f \in F \) as an approximating element from \( U := \text{span}\{u_i\}_{i=1}^n \). This means that in general (i.e. if \( p \neq 2 \)) this method of approximation is not linear.

We have the following obvious inequalities

\[
d_m(F, L_p) \leq \rho^*_m(F, L_p) \leq \rho_m(F, L_p).
\]

The main result of the paper [12] is the following general inequality.

**Theorem 4.1.** There exist two positive absolute constants \( c \) and \( C \) such that for any compact subset \( \Omega \) of \( \mathbb{R}^d \), any probability measure \( \mu \) on it, and any compact subset \( F \) of \( C(\Omega) \) we have

\[
\rho_{cn}(F, L_2(\Omega, \mu)) \leq Cd_n(F, L_\infty).
\]
We now formulate a conditional result from [12], which was used for the proof of Theorem 4.1. Let $X_N$ be an $N$-dimensional subspace of the space of continuous functions $C(\Omega)$. For a fixed $m$ and a set of $m$ points $\xi := \{\xi^\nu\}_{\nu=1}^m \subset \Omega$, we associate a function $f \in C(\Omega)$ with a vector

$$S(f, \xi) := (f(\xi^1), \ldots, f(\xi^m)) \in \mathbb{C}^m.$$ Define

$$\|S(f, \xi)\|_p := \left(1 \over m \sum_{\nu=1}^m |f(\xi^\nu)|^p \right)^{1/p}, \quad 1 \leq p < \infty,$$

and

$$\|S(f, \xi)\|_\infty := \max_{\nu} |f(\xi^\nu)|.$$

For a positive weight $w := (w_1, \ldots, w_m) \in \mathbb{R}^m_+$, consider the following norm

$$\|S(f, \xi)\|_{p,w} := \left(\sum_{\nu=1}^m w_\nu |f(\xi^\nu)|^p \right)^{1/p}, \quad 1 \leq p < \infty.$$

Define the best approximation of $f \in L^p(\Omega, \mu), 1 \leq p \leq \infty$ by elements of $X_N$ as follows

$$d(f, X_N)_p := \inf_{u \in X_N} \|f - u\|_p.$$

It is well known that there exists an element, which we denote by $P_{X_N,p}(f) \in X_N$, such that

$$\|f - P_{X_N,p}(f)\|_p = d(f, X_N)_p.$$

The operator $P_{X_N,p} : L^p(\Omega, \mu) \to X_N$ is called the Chebyshev projection.

**Theorem 4.2** below was proved in [12] under the following assumptions.

**A1. Discretization.** Let $1 \leq p \leq \infty$. Suppose that $\xi := \{\xi^j\}_{j=1}^m \subset \Omega$ is such that for any $u \in X_N$ in the case $1 \leq p < \infty$ we have

$$C_1 \|u\|_p \leq \|S(u, \xi)\|_{p,w}$$

and in the case $p = \infty$ we have

$$C_1 \|u\|_\infty \leq \|S(u, \xi)\|_\infty$$

with a positive constant $C_1$ which may depend on $d$ and $p$.

**A2. Weights.** Suppose that there is a positive constant $C_2 = C_2(d, p)$ such that $\sum_{\nu=1}^m w_\nu \leq C_2$. 

15
Consider the following well known recovery operator (algorithm)

\[ \ell_p^w(\xi)(f) := \ell_p^w(\xi, X_N)(f) := \arg \min_{u \in X_N} \| S(f - u, \xi) \|_{p,w}. \]

Note that the above algorithm \( \ell_p^w(\xi) \) only uses the function values \( f(\xi^\nu), \nu = 1, \ldots, m \). In the case \( p = 2 \) it is a linear algorithm – orthogonal projection with respect to the norm \( \| \cdot \|_2 \), \( w \). Therefore, in the case \( p = 2 \) approximation error by the algorithm \( \ell_2^w(\xi) \) gives an upper bound for the recovery characteristic \( \varrho_m(\cdot, L_2) \). In the case \( p \neq 2 \) approximation error by the algorithm \( \ell_p^w(\xi) \) gives an upper bound for the recovery characteristic \( \varrho_m^*(\cdot, L_p) \).

**Theorem 4.2.** Under assumptions **A1** and **A2** for any \( f \in \mathcal{C}(\Omega) \) we have for \( 1 \leq p < \infty \)

\[ \| f - \ell_p^w(\xi, X_N)(f) \|_p \leq (2C_1^{-1}C_2^{1/p} + 1)d(f, X_N)_\infty. \]

Under assumption **A1** for any \( f \in \mathcal{C}(\Omega) \) we have

\[ \| f - \ell_\infty(\xi, X_N)(f) \|_\infty \leq (2C_1^{-1} + 1)d(f, X_N)_\infty. \]

Theorem 4.2 is devoted to recovery by weighted least squares algorithms \( \ell_p^w(\xi) \). It requires a discretization theorem with positive weights \( w \). There is such a theorem from [7] for \( p = 2 \) and a general subspace \( X_N \) of \( L_2(\Omega, \mu) \), which we formulate as follows.

**Theorem 4.3.** There exist three absolute positive constants \( C_0, c_0, C'_0 \) such that for every \( N \)-dimensional subspace \( X_N \) of \( L_2(\Omega, \mu) \), there exist \( m \leq C'_0N \) points \( \xi^1, \ldots, \xi^m \in \Omega \) and positive weights \( w_1, \ldots, w_m \) such that

\[ c_0\| f \|_2^2 \leq \sum_{j=1}^m w_j |f(\xi^j)|^2 \leq C_0\| f \|_2^2, \quad \forall f \in X_N. \tag{4.2} \]

For a fixed integer \( m \geq 1 \) and a class \( \mathbf{F} \subset \mathcal{C}(\Omega) \) (usually, a centrally symmetric compact in \( \mathcal{C}(\Omega) \)), we define

\[ \ell_{m}^{w_{ls}}(\mathbf{F}, L_2) := \inf_{\xi,w,X_N,N \leq m \atop f \in \mathbf{F}} \sup_{f \in \mathbf{F}} \| f - \ell_2^w(\xi, X_N)(f) \|_2, \]

where the infimum is taken over all \( N \)-dimensional subspaces \( X_N \subset L_2(\Omega, \mu) \) with \( N \leq m \), all collections \( \xi := \{\xi^j\}_{j=1}^m \subset \Omega \) of \( m \) points in \( \Omega \), and all positive weights \( w = (w_1, \ldots, w_m) \in \mathbb{R}_+^m \).
Note that if we assume in addition that $1 \in X_N$ in Theorem 4.3, then the weights $w_j$ in (4.2) satisfy $\sum_{j=1}^m w_j \leq C_0$. As a result, Theorem 4.3 combined with Theorem 4.2 gives an analog of the following Theorem 4.4 from [12].

**Theorem 4.4.** There exist two positive absolute constants $c$ and $C$ such that for any compact subset $\Omega$ of $\mathbb{R}^d$, any probability measure $\mu$ on it, and any compact subset $F$ of $C(\Omega)$ we have

$$\varrho^{uls}_{cn}(F, L_2(\Omega, \mu)) \leq C d_n(F, L_\infty).$$

We may want the recovery algorithm $\ell_2 w(\xi)$ to be the classical least square algorithm, i.e. $w = w_m := (1/m, \ldots, 1/m)$. For that we need an $L_2$-discretization theorem with equal weights. There is such a theorem from [7] under an extra assumption on the subspace $X_N$:

**Condition E($t$).** We say that an orthonormal system $\{u_i(x)\}_{i=1}^N$ defined on $\Omega$ satisfies Condition E($t$) with a constant $t > 0$ if for all $x \in \Omega$

$$\sum_{i=1}^N |u_i(x)|^2 \leq N t^2.$$

Under Condition E($t$), we have the following discretization theorem, which was proved in [7]:

**Theorem 4.5.** Let $\Omega \subset \mathbb{R}^d$ be a compact set with the probability measure $\mu$. Assume that $\{u_i(x)\}_{i=1}^N$ is a real (or complex) orthonormal system in $L_2(\Omega, \mu)$ satisfying Condition E($t$) for some constant $t > 0$. Then there exists a set $\{\xi^j\}_{j=1}^m \subset \Omega$ of $m \leq C_1 t^2 N$ points such that for any $f = \sum_{i=1}^N c_i u_i$, we have

$$C_2 \|f\|_2 \leq \frac{1}{m} \sum_{j=1}^m |f(\xi^j)|^2 \leq C_3 t^2 \|f\|_2^2,$$

where $C_1, C_2$ and $C_3$ are absolute positive constants.

Recall that $w_m = (\frac{1}{m}, \ldots, \frac{1}{m}) \in \mathbb{R}^m$. For a fixed positive integer $m$ and a class $F \subset C(\Omega)$ (usually, a centrally symmetric compact in $C(\Omega)$), define

$$\varrho^*_{m}(F, L_2) := \inf_{\xi, X_N, N \leq m} \sup_{f \in F} \|f - \ell_2 w_m(\xi, X_N)(f)\|_2.$$
where the infimum is taken over all $N$-dimensional subspaces $X_N \subset L_2(\Omega, \mu)$ with $N \leq m$, and all collections $\xi := \{\xi_j\}_{j=1}^m \subset \Omega$ of $m$ points in $\Omega$. We now define $E(t)$-conditioned Kolmogorov widths by

$$d^{E(t)}_N(F, L_p) := \inf_{\{u_1, \ldots, u_N\} \text{ satisfies Condition } E(t)} \sup_{f \in F} \inf_{c_1, \ldots, c_N} \|f - \sum_{i=1}^{N} c_i u_i\|_p.$$ 

Now combining Theorem 4.5 with Theorem 4.2 we obtain the following result, which was proved in [12].

**Theorem 4.6.** Let $F$ be a compact subset of $C(\Omega)$. There exist two positive constants $c$ and $C$ which may depend on $t$ such that

$$\rho_{\text{ES}}(F, L_2) \leq C d^{E(t)}_n(F, L_\infty).$$

We now present some results on the sampling recovery in $L_p$, $2 < p < \infty$. For a fixed positive integer $m$, and a class $F \subset C(\Omega)$ (usually, a centrally symmetric compact in $C(\Omega)$), we define

$$\rho_{\text{SN}}^p(F, L_p) := \inf_{\xi, X_N, N \leq m} \sup_{f \in F} \|f - \ell p w_m(\xi, X_N)(f)\|_p,$$

where the infimum is taken over all $N$-dimensional subspaces $X_N \subset L_p(\Omega, \mu)$ with $N \leq m$, and all collections $\xi := \{\xi_j\}_{j=1}^m \subset \Omega$ of $m$ points in $\Omega$.

**Condition NpB.** We say that an $N$-dimensional subspace $X_N \subset L_p(\Omega, \mu)$ satisfies Condition NpB (Nikol’skii-type inequality for the pair $(p, \infty)$) if for all $x \in \Omega$ we have for all $f \in X_N$

$$|f(x)| \leq B N^{1/p} \|f\|_p,$$

where $B \geq 1$ is a constant. It is well known that Condition $E(t)$ is equivalent to Condition N2t (see, for instance, [7] for an explanation and [4] for a detailed discussion).

We now define the NpB-conditioned Kolmogorov width by

$$d^{\text{NpB}}_N(F, L_p) := \inf_{X_N \text{ satisfies Condition NpB}} \sup_{f \in F} \inf_{g \in X_N} \|f - g\|_p.$$ 

Theorem 2.2 combined with Theorem 4.2 gives the following analog of Theorem 4.6.
Theorem 4.7. Let $F$ be a compact subset of $C(\Omega)$. For $p \in (2, \infty)$ and every constant $B \geq 1$, there exist two positive constants $c$ and $C$, which may depend on $p$, such that

$$\ell_{m}^{lp}(F, L_p) \leq Cd_n^{NpB}(F, L_\infty)$$

provided $m \geq cB^p n(\log(2Bn))^3$.

Corollary 2.2 combined with Theorem 4.2 gives the following generalization of Theorem 4.7.

Theorem 4.8. Let $F$ be a compact subset of $C(\Omega)$, and let $2 \leq q \leq p < \infty$. Let $B \geq 1$ be a given constant. Then there exist two positive constants $c, C$, which may depend on $q, p$, such that

$$\ell_{m}^{lp}(F, L_p) \leq Cd_n^{NqB}(F, L_\infty)$$

provided $m \geq cB^p n^{p/q}(\log(2Bn))^3$.

Condition EqB. We say that an $N$-dimensional subspace $X_N \subset L_p(\Omega, \mu)$ satisfies Condition EqB (Entropy condition with parameters $q$ and $B$) if it satisfies inequalities (2.1).

Using Corollary 2.1 and Theorem 4.2 we obtain the following version of Theorem 4.8.

Theorem 4.9. Let $F$ be a compact subset of $C(\Omega)$ and let $1 \leq q \leq p < \infty$. There exist two positive constants $c$ and $C$ which may depend on $p$ and $q$ such that

$$\ell_{m}^{lp}(F, L_p) \leq Cd_n^{EqB}(F, L_\infty)$$

provided $m \geq cB^p n^{p/q}(\log(2Bn))^2$.

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