Stability of pentaquarks with a two- plus three-body chromoelectric interaction

Fl. Stancu*

University of Liège, Institute of Physics B5, Sart Tilman, B-4000 Liège 1, Belgium

(Dated: March 8, 2018)

We study the stability of pentaquarks within a schematic model based on SU(3) color symmetry, by taking into account algebraic arguments leading to a chromoelectric interaction containing two- and three-body parts. It has already been proven that such an interaction can influence the spectrum of ordinary baryons and the stability of tetraquarks and hexaquarks. Here we discuss its role in the stability of pentaquarks against strong decays.

I. INTRODUCTION

First mentioned by Gell-Mann in his original paper on the quark model of baryons and mesons, the existence of exotic hadrons as systems with more than three quarks (antiquarks) has become an important challenge for QCD-inspired quark models. In practice the problem has been initiated by Jaffe who noticed that the color-magnetic hyperfine interaction produced a stable quark hexaon, called the η-prime, and stable quark systems.

Ten years later, still based on the one-gluon exchange model Gignoux, Silvestre-Brac and Richard independently Lipkin found that the strange anticharm pentaquark is also stable against strong decays. To be stable, these particles had to contain a heavy flavor. Generally, one expects an increase in the stability if a multiquark system contains heavy flavors or . The above pentaquarks have negative parity, i.e., the parity of the antiquark. Later on, it was shown that the Goldstone-boson exchange model favors positive parity pentaquarks and stabilizes systems of type without strange quarks. On the experimental side the H1 Collaboration has found evidence for a narrow resonance interpreted as a quark pentaquark, so far unconfirmed by other experiments. An earlier experiment of the E791 Collaboration failed to obtain statistically significant signals for a narrow strange anticharm pentaquark. An early short review on the stability of multiquark systems can be found, for example, in Ref. The physics of exotic hadrons has recently been reviewed in the spirit of quark models in a more advanced picture with an expansion in the Fock space to allow additional quark-antiquark pairs in the wave function of baryons.

The recent observation of the hidden-charm pentaquarks with and by the LHCb collaboration has triggered a new strong wave of interest in pentaquarks. Competing models have already been proposed, see Ref. for a recent review on the phenomenology of pentaquarks.

The stability of a few quark systems, like the stability of a few-charge systems depends on the masses of the fermions which are involved. Most of the results discussing the mass dependence are related to heavy-light tetraquarks of type as a function of the quark mass ratio . To reach stability one needs a rather large . The results are model dependent and are also influenced by the performance of the variational method, the most elaborate being that of Ref. where the stability of was achieved. The study of is more complicated because the lowest threshold is . In constituent quark models the chromomagnetic interaction also plays an important role. But in the limit of very heavy quarks this interaction vanishes and the result depends on a pure chromoelectric interaction. The kinetic energy decreases by increasing the constituent mass, so that one gets more binding.

Here we discuss a mechanism which could influence the stability of pentaquarks against strong decays in quark models due to a pure chromoelectric interaction, which contains a two- plus a three-body confining potential as suggested in Ref. Usually constituent quark models contain a two-body confining interaction only. Based on the algebraic argument that SU(3) is an exact local symmetry of QCD which implies that QCD-inspired Hamiltonian models are invariant under a global SU(3) symmetry, in Ref. it was suggested that the quark model Hamiltonian should be expressed in terms of the cubic invariant. If added to the Hamiltonian, the three-body force has implications on the spectrum of ordinary baryons and on the stability of tetraquarks and six-quark systems (the nucleon-nucleon system). Here we discuss its effect on the stability of pentaquarks containing both light and/or heavy quarks, inasmuch as the confining interaction is flavor independent.

In Refs. it was shown that the three-body confining interaction introduced in the next section can modify the gap between physical color singlet states and the unphysical color octet states which plague the spectrum of a quark model Hamiltonian containing two-body color confinement only. For negative values of the relative
strength $c$ between the three- and two-body confinement interaction the color states are shifted high up in the spectrum. Then they can be neglected in calculations and the baryon can be described as a $q^3$ state. For $c \geq 0$ the color states are located in the observed region of the spectrum. Hence they have to be considered as giving rise to $(qqq)(q\bar{q})$ configurations to be admixed with $q^3$ configurations. It has been suggested that most of the low lying baryon resonances could have substantial $q\bar{q}$ admixtures, see, for example, Ref. [21]. The three-body force presented here can lower the energy of a $q\bar{q}$ state close to that of a $q^3$ state. Then one can expect significant admixtures of $q\bar{q}$ with $q^3$ configurations in the baryon wave functions, as mentioned in Ref. [10].

In Ref. [19] it was shown that the three-body confining force increases the coupling between physical states and hidden-color (octet-octet) states in $q^0$ systems if $c$ is negative and has therefore some influence on the nucleon-nucleon interaction derived in a microscopic approach based on the quark structure of baryons.

In the present approach the arguments are purely algebraic but a three-body confining interaction could possibly mimic the little information we have from QCD lattice results [22–20]. Also, it would be useful to establish some connection between flux-tube models for pentaquarks and the present work.

The paper is organized as follows. In the next section we briefly recall the three-body confining interaction of Refs. [17–19]. In Sec. III we introduce the basis color states specific to the $q\bar{q}$ system and derive a useful unitary transformation between states obtained in two relevant coupling schemes. In Sec. IV we present results regarding the role of the three-body confinement interaction on the stability of pentaquarks and implications on the spectra of ordinary baryons. An estimate of the effect of this interaction is made in the case of a simple model of harmonic oscillator confinement. The last section is devoted to conclusions. Useful group theoretical details are described in Appendices A–D.

II. TWO- AND THREE-BODY CONFINING INTERACTIONS

Let us consider the Hamiltonian

$$H = \sum_i \frac{\vec{p}_i^2}{2m_i} - \frac{\vec{P}^2}{2M} + V_{2b} + V_{3b}$$

(1)

where $\vec{p}_i$ and $m_i$ are the momentum and the mass of the quark (antiquark) $i$, $\vec{P}$ and $M$ are the total momentum and the mass of the $q\bar{q}$ system and $V_{2b} + V_{3b}$ is the confinement interaction. For the 2-body confinement interaction we choose the form [17]

$$V_{2b} = \sum_{i<j} V_{ij} \left( \frac{7}{3} + F_i^a \cdot F_j^a \right)$$

(2)

where

$$F_i^a = \frac{1}{2} \lambda_i^a, \quad (a = 1, \ldots, 8)$$

(3)

is the color charge operator of the quark $i$. In constituent quark models only a two-body color confinement interaction is usually considered. The common form of its color part is $-3/2 F_i^a \cdot F_j^a$, which differs from the expression above by an additional constant. As pointed out in Ref. [17] the constant $7/3$ ensures the stability of $q\bar{q}$ pairs. One can easily pass from one form to the other.

The three-body color confinement interaction proposed in Ref. [17] has the following form

$$V_{3b} = V_{ijk} = V_{ijk} C_{ijk}$$

(4)

where $V_{ijk}$ is the radial part and $C_{ijk}$ is the three-body color operator. For a $q^3$ system or a multiquark $q^3$ subsystem this has the form

$$C_{ijk} = d^{abc} F_i^a F_j^b F_k^c$$

(5)

where $d^{abc}$ are some real constants, symmetric under any permutation of indices [27]. The three-body color operator acting in a $q\bar{q}$ subsystem is defined as

$$C_{ijk} = -d^{abc} F_i^a F_j^b F_k^c$$

(6)

where

$$F_i^a = -\frac{1}{2} \lambda_i^a, \quad (a = 1, \ldots, 8)$$

(7)

is the color charge operator of an antiquark. These operators can be expressed in terms of the quadratic invariant $C^{(2)}$ and the cubic invariant $C^{(3)}$ of SU(3) as [17, 19]

$$C_{ijk} = \frac{1}{6} \left[ C_{ij+k}^{(3)} - \frac{5}{2} C_{ij+k}^{(2)} + \frac{20}{3} \right]$$

(8)

and

$$C_{ijk} = -\frac{1}{6} \left[ C_{ij+k}^{(3)} - \frac{5}{2} C_{ij+k}^{(2)} + \frac{50}{9} \right]$$

(9)

We recall that for a given irrep of SU(3) labelled by $(\lambda\mu)$, the eigenvalues of these invariants are

$$C^{(2)} = \frac{1}{3} (\lambda^2 + \mu^2 + \lambda \mu + 3\lambda + 3\mu)$$

(10)

and

$$C^{(3)} = \frac{1}{18} (\lambda - \mu)(2\lambda + \mu + 3)(\lambda + 2\mu + 3).$$

(11)

(Note that an extra factor of 1/2 is needed in Ref. [27] for the eigenvalue of the cubic invariant.) Then for a $q^3$ system the expectation value of $C^{(2)}$ is $\frac{18}{9}$ for a singlet $(\lambda\mu) = (00)$ and $-\frac{5}{36}$ for an octet $(\lambda\mu) = (11)$. The expectation values of $C^{(3)}$ are necessary in the study of a $q\bar{q}$ system, will be given in the next section.
This is an exploratory study where a schematic form for the radial part of $V_{ijk}$ of (14) is assumed. As in Refs. 17, 19 we introduce a parameter $c$ representing the strength of the three-body relative to the two-body interaction. For a $q^3$ system we assume that

$$
\langle V_{ijk} \rangle = c\left(\langle V_{ij} \rangle + \langle V_{jk} \rangle + \langle V_{ki} \rangle \right).
$$

(12)

This is consistent with a triangular shape for a three-body interaction in baryons. With such a form the contribution of $V_{26}$ and $V_{38}$ add up together and the expectation value of the color part of this sum is

$$
\chi_1 = \frac{5}{3} + \frac{10}{9}c
$$

for a color singlet $q^3$ system [17, 19]. To avoid multiple counting, in a $q^3\bar{q}$ system we take

$$
\langle V_{ijk} \rangle = \frac{c}{3}\left(\langle V_{ij} \rangle + \langle V_{jk} \rangle + \langle V_{ki} \rangle \right),
$$

(14)

because each two-body interaction contributes three times. These assumptions will be used in Sec. IV. Next we discuss the basis states.

### III. THE BASIS STATES

For describing $q^3\bar{q}$ systems one can introduce various coupling schemes, each being convenient for a particular form of the interaction operator. For our discussion suppose that the particles 1, 2, 3 and 4 are quarks and 5 is an antiquark. Then one can first couple three quarks together and then couple this subsystem to a $q\bar{q}$ pair. In this way one introduces the so called asymptotic channels having a physical color singlet - color singlet state $|(123)_{1}(45)_{1}\rangle$ and two unphysical color octet - color octet states $|(123)_{8}(45)_{8}\rangle$. This coupling scheme is useful to calculate matrix elements of the operator (5) and it gives the appropriate basis at finite distances. At very large distances only the singlet-singlet state survives, the octet-octet states being pushed up by the quark-quark interaction. They were first called hidden-color states in the context of the nucleon-nucleon problem described as a six quark system [28].

Asymptotic channels are convenient to be used for all multiquark systems. For example the tetraquarks have one singlet-singlet and one octet-octet asymptotic channel [27, 29, 30]. The hexaquarks have one octet-octet asymptotic channel when orbital excitations of permutation symmetry [42] are included [19].

The pentaquark system has the particularity that it has two octet-octet channels, as shown in Appendix C. Thus the asymptotic channels are

$$
|1\rangle = |(123)_{1}(45)_{1}\rangle
$$

$$
|2\rangle = |(123)_{8}(45)_{8}\rangle
$$

$$
|3\rangle = |(123)_{8}(45)_{8}\rangle.
$$

(15)

where the $\rho$ and $\lambda$ superscripts, traditionally used for baryons, correspond to the two linear independent basis vectors of the mixed irreducible representation [21] of the permutation group $S_3$. These are basis vectors where the $\rho$ index indicates that the pair 12 is in an antisymmetric state and the $\lambda$ index indicates that the pair 12 is in symmetric state.

On the other hand to estimate the contribution of the three-body interaction (6) it is convenient first to couple two quarks, say 1 and 2 to the antiquark $\bar{q}$ and then to the subsystem of the remaining pair of quarks, 3 and 4, to get again total color singlets. One can construct the following normalized independent states

$$
|\langle(12)_{S}^{A}\bar{q}|_{[21]}(34)_{[11]}\rangle
$$

$$
|\langle(12)_{S}^{A}\bar{q}|_{[21]}(34)_{[11]}\rangle
$$

$$
|\langle(12)_{S}^{A}\bar{q}|_{[22]}(34)_{[2]}\rangle.
$$

(16)

The upper index $S(A)$ indicates that the pair 12 is in a symmetric (antisymmetric) state. Then, the coupling of this pair to the antiquark $\bar{q}$ leads to three possible color states. The first two contain a triplet SU(3) $q^3\bar{q}$ state denoted by [21] and the third contains an antisextet SU(3) state denoted by [22]. Their forms are explicitly given in Appendix C. Other coupling schemes giving rise to a complete set of independent $q^3\bar{q}$ color singlets are also possible [31].

Of course, the states between different coupling schemes are related to each other. In the present case, we found that the asymptotic channels (15) are related to the intermediate coupling channels (16) by the following unitary transformation
principal quantum number \( n = 3 \). For \( n = 3 \) case, this means that the particle is in its ground state.


The energy of the particle in the ground state is

\[
E_n = n^2 + 3
\]

so the energy of the particle in the ground state is

\[
E_3 = 3^2 + 3 = 9 + 3 = 12
\]

Therefore, the energy of the particle in the ground state is

\[
E_3 = 12
\]

\[
\text{Energy} = 12
\]

Finally, the energy of the particle in the ground state is

\[
\text{Energy} = 12
\]
eigenvalues of this matrix are
\[ e_{1,2} = 20 + \frac{10}{9} c \pm \frac{10}{\sqrt{6}} c, \quad e_3 = 20 + \frac{10}{9} c. \] (23)

To get the full contribution of the confinement one must multiply each eigenvalue by \( \sum V_{ij} \). The eigenvector associated with \( e_1 \) is dominantly color singlet - color singlet (the state |1\rangle appears with 91 % probability) and the eigenvector associated with \( e_2 \) is dominantly color octet - color octet (the state |2\rangle appears with 91 % probability) irrespective of the value of \( c \). The eigenvector \( e_3 \) is a pure color octet-color octet state as it can be seen from Eq. (22). It is stable against strong decay into a baryon plus a meson.

In Ref. [17] the range proposed for the relative strength \( c \) was
\[ -\frac{3}{2} < c < \frac{2}{5}. \] (24)

The upper limit ensures that the lowest color singlet appears below the lower color octet in a \( q^3 \) system and the lower limit is required by the stability condition of the nucleon, \( (V_{2b} + V_{3b}) > 0 \). In Ref. [15] some arbitrariness was noticed regarding the lower limit. For our discussion it is enough to restrict the interval to
\[ -1.0 \leq c < 0.4. \] (25)

In the stability problem against strong decays of a \( q^4 \overline{q} \) system we are interested in the quantity
\[ \Delta E = E(q^4 \overline{q}) - E(q^3) - E(q \overline{q}) \] (26)
where \( E(q^4 \overline{q}) \) represents the lowest energy of the \( q^4 \overline{q} \) system and \( E(q^3) + E(q \overline{q}) \) is the threshold energy for the decay of \( q^4 \overline{q} \) into a baryon described as a \( q^3 \) system and a meson \( q \overline{q} \). The condition \( \Delta E < 0 \) is interpreted as stability, otherwise the system is unstable against strong decays. Of course the Hamiltonian (11) must also be used in the calculation of \( E(q^3) \). In each case a hyperfine interaction should be added but this is beyond the scope of the present study. As mentioned in Sec. II the contribution of \( V_{2b} + V_{3b} \) to a color singlet \( q^3 \) state is proportional to \( \chi_1 \) given by (13). In order to keep the ground state \( E(q^3) \) unchanged in the presence of the three-body force we have to rescale \( E(q^3) \) by dividing it by \( \chi_1 \) as in Ref. [18]. For consistency one must also divide the eigenvalues (25) by the same quantity.

It is useful to make an estimate for a harmonic oscillator confinement. For five quarks (antiquarks) of equal masses the eigenvalues of (11) are [32]
\[ E_i = (N + 6)\hbar \omega_i \] (27)
where \( N \) is the number of quanta and we take \( N = 0 \). The frequency \( \omega_i \) of a five body system is related to the frequency \( \omega_0 \) of a three body system by [21]
\[ \omega_i = \sqrt{5/6} \omega_0. \] (28)

We normalize the eigenvalues \( E_i \) of (11) such as in the absence of three-body forces \((c=0)\) to obtain this relation. In addition, as we deal with 10 pairs of quarks we have to divide each eigenvalue of \( H \) by 10 to be consistent with Ref. [18]. Then, in terms of \( e_i \) of Eqs. (23) we have
\[ E_i = 6\hbar \omega_i = 5 \sqrt{\frac{e_i}{10\chi_1}} \hbar \omega_0, \] (29)
where \( \chi_1 \) is defined by Eq. (10). For \( \hbar \omega_0 = 395 \text{ MeV} \) [18] the three resulting eigenvalues are displayed in Fig. 1. Without a three-body force \((c=0)\) the states are degenerate. The three-body confining force \((c \neq 0)\) introduces a splitting. A negative \( c \) increases all \( E_i \) with respect to the values taken at \( c = 0 \). For example one has \( E_1(c = -1) - E_1(c = 0) = 1.06 \text{ GeV} \). This means that the pentaquarks become less and less stable with a decreasing negative \( c \). The lowest eigenvalue is \( E_1 \) which is dominantly color singlet \( q^3 \) color singlet \( q \overline{q} \). For \( c > 0 \) the lowest state becomes \( E_2 \) which is dominantly color octet - color octet. One has \( E_2(c = 0) - E_2(c = 0.35) = 267 \text{ MeV} \), which is a substantial lowering.

Thus the three-body force helps now in stabilizing the \( q^4 \overline{q} \) system against strong decays because \( E_2 \) is dominantly a color octet - color octet state thus it has a small amplitude to decay strongly. However the gap between \( E_2 \) and the other states is not so large. For example, \( E_2(c = 0.35) - E_1(c = 0.35) = -138 \text{ MeV} \). This means that through the addition of a hyperfine interaction to the Hamiltonian (11) the proportion of singlet-singlet and octet-octet contributions might change.

The \( c > 0 \) case also brings a new perspective to baryon spectroscopy. If the energy of the \( q^4 \overline{q} \) states are close to those of the \( q^3 \) states one should expect significant admixtures of \( q^4 \overline{q} \) states in the baryon resonance wave functions. The low lying \( q^4 \overline{q} \) states provide an approximation to the open baryon-meson channels that give rise to large strong decay widths of the empirical resonances [21]. Let us give an estimate for energies relevant for the Roper resonance. In constituent quark models this is
usually described as a 2 $\hbar\omega_0$ excitation. In these models the presence of a negative additional constant $V_0$ in the Hamiltonian is also required for a realistic description of the spectrum. Accordingly, if we define the "unperturbed" energy (no hyperfine interaction) of a $q^3$ system as $E(q^3) = 3m + 3V_0 + 5\hbar\omega_0$ with $m = 340$ MeV and $V_0 = -269$ MeV as in Ref. [21] and use, as above, $\hbar\omega_0 = 395$ MeV we obtain $E(q^3) = 2188$ MeV. If we estimate the energy of the lowest $q^2\bar{q}$ in a similar manner we have $E(q^2\bar{q}) = 5m + 5V_0 + E_2$. At $c = 0.35$ one has $E_2 = 1897$ MeV from which one obtains $E(q^2\bar{q}) = 2252$ MeV, very close to $E(q^3)$. Thus the three-body confining interaction with a positive $c$ could provide a mechanism to lower the $q^2\bar{q}$ states, which is an alternative to the mechanism proposed in Ref. [21] based on a schematic flavor and spin dependent interaction where the strength of this interaction is adjusted to allow admixtures.

V. CONCLUSIONS

Here we have shown that the three-body confining interaction [14] - [18] with a negative $c$ destabilizes the pentaquarks while a positive $c$ helps in stabilizing them. A similar conclusion has been drawn in Ref. [19] in connection with tetraquarks. The conclusion also holds for tetraquarks or pentaquarks containing heavy flavors, inasmuch as the confinement interaction is flavor independent.

The unitary transformation [17] derived here relates two complete sets of color singlet bases needed to describe $q^2\bar{q}$ systems. Each contain three independent states. The set [18] may be useful in diquark-triquark type models [32]. Thus in a $q^4\bar{q}$ system there are three asymptotic channels in the color space as compared to two for $q^2\bar{q}$ systems.

As mentioned in the introduction the stability of multiquark systems depends on the constituent masses as well. In particular, for unequal masses the kinetic term deserves a special attention, by studying the role of both its symmetric and asymmetric parts, as it has been done for tetraquarks in Ref. [12]. We are aware that the problem becomes complex if the pentaquark contains more than two flavors, as for example, those proposed in Refs. [14, 15].

Similar to tetraquarks [29], one can construct full wave functions including the position, the spin and the flavor degrees of freedom to calculate the total energy of $q^2\bar{q}$ by adding a hyperfine interaction to see which is the role of octet-octet states in the stability. In this way one may perhaps avoid the fall apart decay mode of the ground state pentaquarks obtained in Ref. [31].

It would certainly be interesting to make contact with the many-quark confining forces from flux-tube models inspired by the strong coupling limit of QCD as proposed in Ref. [33] for tetraquarks and extended to pentaquarks in Ref. [36]. For pentaquarks two distinct contributions should be considered, as depicted in Fig. 10 of Ref. [10]. These are a disconnected and a connected flux-tube diagrams, the first corresponding to the baryon-meson configuration and the second to a Steiner tree diagram [36]. In a color basis, the first should be a color singlet-color singlet state and the second should be related to two distinct color triplet - color antitriplet states. These must be associated to a cluster of four quarks and the antitriplet must be associated to the antiquark. One has to establish the relation of this basis to that of Eq. (15) each involving different intermediate couplings. Thus more algebraic work is needed to make a link between flux-tube confining potentials and the present study. If successful, a linear radial dependence, as supported by lattice studies for the three-body confining potential, could be considered in the future.

Appendix A: The SU(3) antiquark generators

We find it useful to exhibit the action of the antiquark generators, Eq. (7),

$$\mathcal{F}^a = -\frac{1}{2} \lambda^a_i, \quad (a = 1, \ldots, 8)$$

on antiquark states $\bar{q}_i$, needed in Appendix B. These are

$$\mathcal{F}^3 \bar{q}_1 = -\frac{1}{2} \bar{q}_2, \quad \mathcal{F}^3 \bar{q}_2 = -\frac{1}{2} \bar{q}_1, \quad \mathcal{F}^3 \bar{q}_3 = 0$$

$$\mathcal{F}^2 \bar{q}_1 = \frac{i}{2} \bar{q}_2, \quad \mathcal{F}^2 \bar{q}_2 = -\frac{i}{2} \bar{q}_1, \quad \mathcal{F}^2 \bar{q}_3 = 0$$

$$\mathcal{F}^1 \bar{q}_1 = -\frac{1}{2} \bar{q}_2, \quad \mathcal{F}^1 \bar{q}_2 = -\frac{1}{2} \bar{q}_3, \quad \mathcal{F}^1 \bar{q}_3 = 0$$

$$\mathcal{F}^4 \bar{q}_1 = -\frac{1}{2} \bar{q}_3, \quad \mathcal{F}^4 \bar{q}_2 = 0, \quad \mathcal{F}^4 \bar{q}_3 = -\frac{1}{2} \bar{q}_1$$

$$\mathcal{F}^5 \bar{q}_1 = \frac{i}{2} \bar{q}_3, \quad \mathcal{F}^5 \bar{q}_2 = 0, \quad \mathcal{F}^5 \bar{q}_3 = -\frac{i}{2} \bar{q}_1$$

$$\mathcal{F}^6 \bar{q}_1 = 0, \quad \mathcal{F}^6 \bar{q}_2 = -\frac{1}{2} \bar{q}_3, \quad \mathcal{F}^6 \bar{q}_3 = -\frac{i}{2} \bar{q}_2$$

$$\mathcal{F}^7 \bar{q}_1 = 0, \quad \mathcal{F}^7 \bar{q}_2 = -\frac{i}{2} \bar{q}_3, \quad \mathcal{F}^7 \bar{q}_3 = -\frac{i}{2} \bar{q}_2$$

$$\mathcal{F}^8 \bar{q}_1 = -\frac{1}{2\sqrt{3}} \bar{q}_2, \quad \mathcal{F}^8 \bar{q}_2 = -\frac{1}{2\sqrt{3}} \bar{q}_2, \quad \mathcal{F}^8 \bar{q}_3 = \frac{1}{\sqrt{3}} \bar{q}_3$$

In the flavor space one can identify

$$\mathcal{F}_1 = \bar{u}, \quad \mathcal{F}_2 = \bar{d}, \quad \mathcal{F}_3 = \bar{s}$$

As a check, using the above relations and the symmetric constants $d_{abc}$ from Table I, one can obtain the eigenvalue of the cubic invariant operator for antiquarks, in the Lorentz vector model [17]

$$\mathcal{C}_{ijk} = d_{abc} \mathcal{F}_i \mathcal{F}_j \mathcal{F}_k$$

(A11)
by acting on \( \tilde{d} \), the highest weight state of the antitriplet \((\lambda \mu) = (01)\). The result is

\[
C^{(3)} \tilde{d} = -\frac{10}{9} \tilde{d}
\]

which means that the eigenvalue of the cubic invariant has an opposite sign to that of the triplet \(u, d, s\), having \((\lambda, \mu) = (10)\), consistent with Eq. \((11)\).

Appendix B: Matrix elements for two-body operators

Here we prove Eq. \((19)\) in a simplified way. This is a more explicit proof of the relation derived in Ref. \([34]\). We first need to evaluate the color operator matrix element of a subsystem of four quarks

\[
\langle O^{C} \rangle = \frac{4}{3} \sum_{i < j}^{4} F_{i} \cdot F_{j} = \frac{1}{2} (C^{(2)} - 4 C^{(2)})_q
\]

where \(C^{(2)}\) is the Casimir operator eigenvalue \((10)\) for a system of four quarks and \(C^{(2)}_q = 4/3\) is the eigenvalue of the Casimir operator for a quark, light or heavy. The color state of the four-quark subsystem is \([211]_C\) so that one has \(C^{(2)} = 4/3\). Hence \(\langle O^{C} \rangle_{[211]} = -2\). As there are 6 pairs in \([211]\), each pair of quarks will have

\[
\langle F_{i} \cdot F_{j} \rangle = -\frac{1}{3} .
\]

For the antiquark the Casimir operator eigenvalue is \(C^{(2)}_q = 4/3\), where \(\tilde{q}\) can be light or heavy. Then for a pentaquark, in a color-singlet state \([222]\), formed of a subsystem of four quarks and an antiquark, one can define the color operator eigenvalue

\[
\langle \sum_{i=1}^{4} F_{i} \cdot F_{\tilde{q}} \rangle = (C^{(2)}_{[222]} - C^{(2)}_q - C^{(2)}_{\tilde{q}})/2 = -\frac{4}{3} .
\]

where we have used the Casimir operator eigenvalue \(C^{(2)}_{[222]} = 0\) of the whole system in a color singlet, and \(C^{(2)}_q = C^{(2)}_{\tilde{q}} = 4/3\). As each quark shares 1/4 each quark can conclude that in average, for every asymptotic channel \([15]\) we have

\[
\langle F_{i} \cdot F_{j} \rangle = -\frac{1}{3}, \quad i = 1, 2, 3, 4, \quad j = 5,
\]

which means that the color expectation value for any pair, quark-quark or a quark-antiquark is \(-1/3\). There are in all 10 pairs so that for a two-body color confinement with a constant \(V_{ij}\), the color operator gives

\[
\sum_{i < j} \langle F_{i} \cdot F_{j} \rangle = -\frac{10}{3} .
\]

There is another simplified, but slightly more elaborate, way to derive Eq. \((19)\). We introduce the scalar product \(F_{123} \cdot F_{45}\) defined as

\[
F_{123} \cdot F_{45} = (F_{1} + F_{2} + F_{3}) \cdot (F_{4} + F_{5}),
\]

or alternatively

\[
F_{123} \cdot F_{45} = F_{1} \cdot F_{4} + F_{2} \cdot F_{4} + F_{3} \cdot F_{4} + F_{1} \cdot F_{5} + F_{2} \cdot F_{5} + F_{3} \cdot F_{5}.
\]

In terms of the Casimir operators eigenvalues of the whole system \(C^{(2)}_{[222]}\) and of the separate subsystems \(C^{(2)}_{123}\) and \(C^{(2)}_{45}\) one has

\[
F_{123} \cdot F_{45} = \frac{1}{2} (C^{(2)}_{[222]} - C^{(2)}_{123} - C^{(2)}_{45}).
\]

For the asymptotic channel \([1]\) we get

\[
\langle 1 | F_{123} \cdot F_{45} | 1 \rangle = 0.
\]

On the other hand it is reasonable to assume that

\[
\langle F_{1} \cdot F_{4} \rangle = \langle F_{2} \cdot F_{4} \rangle = \langle F_{3} \cdot F_{4} \rangle,
\]

\[
\langle F_{1} \cdot F_{5} \rangle = \langle F_{2} \cdot F_{5} \rangle = \langle F_{3} \cdot F_{5} \rangle,
\]

so that we have

\[
\langle F_{i} \cdot F_{4} \rangle + \langle F_{i} \cdot F_{5} \rangle = 0, \quad i = 1, 2, 3.
\]

The nonvanishing terms give

\[
\langle 1 | \sum_{i < j} F_{i} \cdot F_{j} | 1 \rangle = \langle 1 | F_{1} \cdot F_{2} | 1 \rangle + \langle 1 | F_{1} \cdot F_{3} | 1 \rangle
\]

\[
+ \langle 1 | F_{2} \cdot F_{3} | 1 \rangle + \langle 1 | F_{4} \cdot F_{5} | 1 \rangle .
\]

As we know

\[
\langle 1 | F_{i} \cdot F_{j} | 1 \rangle = -\frac{2}{3}, \quad \langle 1 | F_{4} \cdot F_{5} | 1 \rangle = -\frac{4}{3},
\]

we obtain

\[
\langle 1 | \sum_{i < j} F_{i} \cdot F_{j} | 1 \rangle = -\frac{10}{3} ,
\]

as above. For the hidden-color (octet-octet) states \([2]\) and \([3]\) Eq. \((15)\) gives

\[
\langle (123)_{8}(45)_{8} | F_{123} \cdot F_{45} | (123)_{8}(45)_{8} \rangle = -3,
\]

from where the analog of the relation \((15)\) becomes

\[
\langle F_{i} \cdot F_{4} \rangle + \langle F_{i} \cdot F_{5} \rangle = -1, \quad i = 1, 2, 3.
\]

Then, for a color octet three quark states we take the average between symmetric and antisymmetric states of particles 1 and 2 to get

\[
\langle F_{i} \cdot F_{j} \rangle = \frac{1}{2} \left( \frac{1}{3} - \frac{2}{3} \right) = -\frac{1}{6}, \quad i = 1, 2, 3.
\]
Then it follows that for any of the hidden color states one has
\[
\langle (123)_1 | (4\overline{5})_8 \sum_{i<j} F_i \cdot F_j | (123)_8 | (4\overline{5})_8 \rangle = 3 \langle F_1 \cdot F_2 \rangle \
+ 3(\langle F_1 \cdot F_3 \rangle + \langle F_1 \cdot F_5 \rangle + \langle F_4 \cdot F_5 \rangle) \
- 3(\frac{1}{6} - 3 + \frac{1}{6} = -\frac{10}{3})
\]
which is the same as for the singlet-singlet state.

This proof is convenient for our purpose where the confinement is a constant in the configuration space. But if a radial shape is introduced then one has to proceed as in the appendix of Ref. [31], which gives a consistent result with ours.

Using the explicit form of the color basis states \(|1\rangle, |2\rangle\) and \(|3\rangle\) of Appendix C one can calculate the required off-diagonal matrix elements of \(V_{26}\). For quark-quark pairs they are
\[
\langle 1|F_1 \cdot F_2 |2\rangle = \langle 1|F_1 \cdot F_3 |3\rangle = \langle 1|F_2 \cdot F_3 |2\rangle = 0,
\]
\[
\langle 1|F_1 \cdot F_4 |2\rangle = \langle 1|F_2 \cdot F_4 |2\rangle = -\frac{\sqrt{2}}{6},
\]
and for quark-antiquark pairs one has
\[
\langle 1|F_1 \cdot F_3 |2\rangle = \langle 1|F_2 \cdot F_3 |2\rangle = 0.
\]
This implies that
\[
\sum i<j \langle 1|F_i \cdot F_j |2\rangle + \sum i \langle 1|F_i \cdot F_3 |2\rangle = 0
\]
which proves that the off-diagonal matrix element of \(V_{26}\) between states \(|1\rangle\) and \(|2\rangle\) is zero. Similarly one can prove that the other off-diagonal matrix element of \(V_{26}\) are vanishing. For this purpose one needs to know that
\[
\langle 1|F_1 \cdot F_2 |3\rangle = \langle 1|F_1 \cdot F_3 |3\rangle = \langle 1|F_2 \cdot F_3 |3\rangle = 0,
\]
\[
\langle 1|F_1 \cdot F_4 |3\rangle = \langle 1|F_2 \cdot F_4 |3\rangle = \frac{1}{\sqrt{6}},
\]
\[
\langle 1|F_3 \cdot F_4 |3\rangle = 0
\]
and
\[
\langle 1|F_1 \cdot F_3 |3\rangle = \langle 1|F_2 \cdot F_3 |3\rangle = \frac{1}{\sqrt{6}},
\]
\[
\langle 1|F_3 \cdot F_4 |3\rangle = 0, \quad \langle 1|F_4 \cdot F_3 |3\rangle = 0.
\]

Appendix C: Two useful color singlet bases for pentaquarks

We have used the tensor method [27] to write down explicit expressions for the normalized basis states appearing in the unitary transformation \([177]\) in terms of their quark content. By equalizing the coefficients of identical terms of both sides we have obtained a system of linear equations for the matrix elements of the unitary transformation.

Let us denote by \(q_i\) and \(\bar{q}_i\) \((i = 1, 2, 3)\) the three possible color states of a quark and an antiquark respectively. By using the tensor method [27] the asymptotic channel states can be easily constructed as scalars\([36]\) from the product between a flavor singlet (flavor octet) \(q\bar{q}\) state and a flavor singlet (flavor octet) \(q\bar{q}\) state. The three possible normalized colorless \(q\bar{q}\) states are
\[
|1\rangle = |(123)_1 \rangle = \sqrt{\frac{1}{18}}(q^1 q^2 q^3 \rho^1 \delta^i \bar{q}_i)
\]
\[
|2\rangle = |(123)_2 \rangle = \frac{1}{12} \times 
\]
\[
\{ 3(q^1 q^2 q^3 q^4 - q^1 q^2 q^1 q^3)^3 \bar{q}_1 + 3(q^3 q^2 q^4 - q^2 q^1 q^4)^3 \bar{q}_2
\]
\[
-3(q^1 q^2 q^3 q^4 - q^1 q^2 q^1 q^3)^3 \bar{q}_3 + 3(q^3 q^2 q^4 - q^2 q^1 q^4)^3 \bar{q}_1 \bar{q}_2
\]
\[
-3(q^1 q^2 q^3 q^4 - q^1 q^2 q^1 q^3)^3 \bar{q}_3 - \bar{q}^2 q^4 \bar{q}_1 q_1 - (q^3 q^2 q^4 - q^2 q^1 q^4)^3 \bar{q}_2
\]
\[
-3(q^1 q^2 q^3 q^4 - q^1 q^2 q^1 q^3)^3 \bar{q}_3 - \bar{q}^2 q^4 \bar{q}_1 q_1 - (q^3 q^2 q^4 - q^2 q^1 q^4)^3 \bar{q}_2
\]
\[
+ 2q^3 q^4 q^3 \bar{q}_2 + q^3 q^4 q^3 \bar{q}_2
\]
\[
+ q^3 q^4 q^3 \bar{q}_2 + q^3 q^4 q^3 \bar{q}_2
\]
\]
\[
|3\rangle = |(123)_3 \rangle = \sqrt{\frac{1}{48}} \times
\]
\[
\{ -(q^1 q^2 q^3 q^4 - q^2 q^1 q^3)^3 \bar{q}_1 + (q^1 q^2 q^3 q^4 - q^2 q^1 q^3)^3 \bar{q}_1
\]
\[
+ (q^1 q^2 q^3 q^4 - q^2 q^1 q^3)^3 \bar{q}_1
\]
\[
- (q^2 q^3 q^4 q^1 q^2 - q^2 q^3 q^4 q^1 q^2)^3 \bar{q}_1
\]
\[
+ (q^2 q^3 q^4 q^1 q^2 - q^2 q^3 q^4 q^1 q^2)^3 \bar{q}_1
\]
\[
+ (q^1 q^2 q^3 q^4 - q^1 q^2 q^3 q^4)^3 \bar{q}_1
\]
\[
+ (q^1 q^2 q^3 q^4 - q^1 q^2 q^3 q^4)^3 \bar{q}_1
\]
\[
- (q^1 q^2 q^3 q^4 - q^1 q^2 q^3 q^4)^3 \bar{q}_1
\]
\[
+ (q^1 q^2 q^3 q^4 - q^1 q^2 q^3 q^4)^3 \bar{q}_1
\]
\]
Note that the normal order of particles is understood everywhere. The difference between \(|C2\rangle\) and \(|C3\rangle\) is that in the first the particles 1 and 2 are in an antisymmetric state while in the second they are in a symmetric state. That is why they carry the superscripts \(\rho\) and \(\bar{\rho}\) respectively, like the mixed symmetry \(q\bar{q}\) states used in baryon spectroscopy.

For the basis vectors in the intermediate coupling we use the tensor method first to construct \(qq\bar{q}\) states.

1) The three components of \(|(123)_2 \rangle\) \([211]\) state are obtained from by contraction with \(\delta_i^k\)
\[
T^{ij}_k \delta_i^k
\]
from the tensor $T_{ik}^{ij}$ which is symmetric in the upper indices

$$T_{ik}^{ij} = (q^i q^j + q^j q^i)\bar{q}_k$$ \hspace{1cm} (C5)

The normalized components are

$$\langle (12)^S \bar{q}_{[211]} \rangle^1 = \frac{1}{2\sqrt{2}}[2q^1 q^2 \bar{q}_1 + (q^1 q^2 + q^2 q^1)\bar{q}_2 + (q^1 q^3 + q^3 q^1)\bar{q}_3].$$

$$\langle (12)^S \bar{q}_{[211]} \rangle^2 = \frac{1}{2\sqrt{2}}[2q^2 q^3 \bar{q}_2 + (q^2 q^3 + q^3 q^2)\bar{q}_3 + (q^1 q^2 + q^2 q^1)\bar{q}_1].$$

$$\langle (12)^S \bar{q}_{[211]} \rangle^3 = \frac{1}{2\sqrt{2}}[2q^3 q^1 \bar{q}_3 + (q^3 q^1 + q^1 q^3)\bar{q}_1 + (q^2 q^3 + q^3 q^2)\bar{q}_2].$$ \hspace{1cm} (C6)

2) The three components of the state $[(12)^A \bar{q}_{[211]}|^3$ form the antisymmetric tensor

$$T^i = \epsilon^{ijk} T_j T_k$$ \hspace{1cm} (C7)

In this antisymmetric product the first factor represents a $qq$ subsystem described by

$$T_j = c_{jlm} q^l q^m$$ \hspace{1cm} (C8)

and the second an antiquark seen as a $qq$ pair

$$\bar{q}_1 = \frac{1}{\sqrt{2}}(q^2 q^3 - q^3 q^2), \quad \bar{q}_2 = \frac{1}{\sqrt{2}}(q^3 q^1 - q^1 q^3), \quad \bar{q}_3 = \frac{1}{\sqrt{2}}(q^1 q^2 - q^2 q^1).$$ \hspace{1cm} (C9)

Then the first normalized components is

$$\langle (12)^A \bar{q}_{[211]} \rangle^1 = \frac{1}{2}\{(q^2 q^3 - q^3 q^2)\bar{q}_3 - (q^1 q^2 - q^2 q^1)\bar{q}_2\}. $$ \hspace{1cm} (C10)

The other two are obtained by circular permutations.

To form a $q^i \bar{q}$ singlet state together with particles 3 and 4 in the above two cases one must construct the scalar product $\frac{1}{\sqrt{6}}T^i T_i$, where $T^i$ is either of the form \((\text{C6})\) or \((\text{C10})\) plus circular permutations. The tensor $T_i$ made of the particles 3 and 4 is defined by \((\text{C8})\).

3) The six components of $[(12)^A \bar{q}_{[211]}]$ are defined by the symmetric tensor

$$T_{ij} = \frac{1}{2}(T_i T_j + T_j T_i)$$ \hspace{1cm} (C11)

where the first factor in the right hand side should be replaced by \((\text{C8})\) and the second by \((\text{C9})\). Then, for example the first component becomes

$$\langle (12)^A \bar{q}_{[211]} \rangle^1 = \frac{1}{2}\{(q^3 q^1 - q^1 q^3)\bar{q}_3 + (q^1 q^2 - q^2 q^1)\bar{q}_2\}. $$ \hspace{1cm} (C12)

### Table II: The contributing terms to the matrix elements of operators defined by Eqs. \((\text{A3})\) and \((\text{A6})\) for the specific cases $(3)d^abc F_1^3 F_2^3 F_5^3 [3]$ and $(3)d^abc F_1^3 F_2^3 F_5^3 [3]$. The first column gives the color indices $(abc)$, the second the corresponding constant $d^{abc}$, the third is the multiplicity of each term and the fourth and fifth columns the values of the matrix element for a given $d^{abc}$.

| $(abc)$ | $d^{abc}$ | $m^{abc}$ | $(3)d^abc F_1^3 F_2^3 F_5^3 [3]$ | $(3)d^abc F_1^3 F_2^3 F_5^3 [3]$ |
|--------|-----------|-----------|----------------------------------|----------------------------------|
| 118    | $\sqrt{3}/3$ | 3 | -1/288 | -1/144 |
| 146    | 1/2 | 6 | -1/384 | -1/192 |
| 157    | 1/2 | 6 | -1/384 | -1/192 |
| 228    | $\sqrt{3}/3$ | 3 | -1/288 | -1/144 |
| 247    | -1/2 | 6 | -1/384 | -1/192 |
| 256    | 1/2 | 6 | -1/384 | -1/192 |
| 338    | $\sqrt{3}/3$ | 3 | -1/288 | -1/144 |
| 344    | 1/2 | 3 | -1/384 | -1/192 |
| 355    | 1/2 | 3 | -1/384 | -1/192 |
| 366    | -1/2 | 3 | -1/384 | -1/192 |
| 377    | -1/2 | 3 | -1/384 | -1/192 |
| 448    | $-\sqrt{3}/3$ | 6 | -1/1152 | -1/576 |
| 558    | $-\sqrt{3}/3$ | 6 | -1/1152 | -1/576 |
| 668    | $-\sqrt{3}/3$ | 6 | -1/1152 | -1/576 |
| 778    | $-\sqrt{3}/3$ | 6 | -1/1152 | -1/576 |
| 888    | $-\sqrt{3}/3$ | 6 | -1/288 | -1/144 |

To form a $q^i \bar{q}$ singlet state one needs the symmetric tensor

$$T^{ij} = q^i q^j + q^j q^i$$ \hspace{1cm} (C13)

associated to the particles 3 and 4. Then the $q^i \bar{q}$ singlet state is defined by the scalar product $\frac{1}{\sqrt{6}}T^{ij} T_{ij}$.

In this way we have obtained all states needed to determine the unitary transformation \((\text{17})\).

### Appendix D: Matrix elements of $C_{ijk}$ and $C_{ij\bar{k}}$

One can calculate the matrix elements of $C_{ijk}$ either from Eq. \((\text{D5})\) or from Eq. \((\text{D8})\). The matrix elements of $C_{ij\bar{k}}$ can be calculated either from Eq. \((\text{D9})\) or from Eq. \((\text{D9})\).

The first method is more tedious but it provides a check for the second. It implies to apply successively the generators $F_i^a$ or $T_i^a$ on the states \([1]\), \([2]\) or \([3]\). As an example, in Table \(\text{II}\) we show the contribution of all nonvanishing terms to Eqs. \((\text{5})\) and \((\text{8})\). Their sums gives the expectation value of $C_{124}$ and of $C_{12\bar{4}}$, respectively, for the color wave function \([3]\) defined in Eq. \((\text{12})\). By taking into account the multiplicity $m^{abc}$ of each term one obtains - 5/36 consistent with Eq. \((\text{5})\) and 5/18 consistent with Eq. \((\text{8})\), respectively. The latter value is exhibited in column 2 of Table \(\text{II}\).
Thus, the third column gives the expectation value of the cubic invariant $C_{124}$ for the state $[3]$. It is very easy to calculate expectation value of $C_{123}$ for the asymptotic state $|1\rangle$ using its definition \( [C1] \). The result will coincide with the column $3$ and for symmetry reasons all $C_{ijk}$ are identical. As a matter of fact the result is also identical with the column $3$ and for symmetry reasons all $|\text{state}\rangle$ using its definition (\( C_{1} \)). The result will coincide with the column $3$ and for symmetry reasons all $|\text{state}\rangle$ are octets in the color space.

Acknowledgments

The author acknowledges support from the Fonds de la Recherche Scientifique - FNRS under the Grant No. 4.4501.05.

[1] M. Gell-Mann, A Schematic Model of Baryons and Mesons, Phys. Lett. **8** (1964) 214.
[2] R. L. Jaffe, Perhaps a Stable Dihyperon, Phys. Rev. Lett. **38** (1977) 195 Erratum: [Phys. Rev. Lett. **38** (1977) 617].
[3] R. L. Jaffe, Multi-Quark Hadrons. 1. The Phenomenology of (2 Quark 2 anti-Quark) Mesons, Phys. Rev. D **15** (1977) 267; Multi-Quark Hadrons. 2. Methods, Phys. Rev. D **15** (1977) 281; $Q^{**}2$ anti-$Q^{*2}$ Resonances in the Baryon - anti-Baryon System, Phys. Rev. D **17** (1978) 1444.
[4] C. Gignoux, B. Silvestre-Brac and J. M. Richard, Possibility of Stable Multi - Quark Baryons, Phys. Lett. B **193** (1987) 323.
[5] H. J. Lipkin, New Possibilities for Exotic Hadrons: Anticharmed Strange Baryons, Phys. Lett. B **195** (1987) 484.
[6] F. Stancu, Positive parity pentaquarks in a Goldstone boson exchange model, Phys. Rev. D **58** (1998) 111501.
[7] A. Aktas et al. [H1 Collaboration], Evidence for a narrow anti-charmed baryon state, Phys. Lett. B **588** (2004) 17.
[8] E. M. Aitala et al. [E791 Collaboration], Search for the pentaquark via the $P0$ (anti-$c$ s) $\to K^{*}$0 $K^{-}$ p decay, Phys. Lett. B **448** (1999) 303.
[9] F. Stancu, Stability of multiquark systems, AIP Conf. Proc. **508** (2000) 34. [hep-ph/9910547]
[10] J. M. Richard, Exotic hadrons: review and perspectives, Few Body Syst. **57** (2016) no.12, 1185.
[11] R. Aaij et al. [LHCb Collaboration], Observation of $J/\psi$($p$) Resonances Consistent with Pentaquark States in $\Lambda_b^0$ $\to J/\psi$K$^{-}$ p Decays, Phys. Rev. Lett. **115** (2015) 072001.
[12] T. J. Burns, Phenomenology of $P_{c}(4380)^{+}$, $P_{c}(4450)^{+}$ and related states, Eur. Phys. J. A **51** (2015) no.11, 152.
[13] J. M. Richard, A. Valcarce and J. Vijande, “String dynamics and metastability of all-heavy tetraquarks, Phys. Rev. D **95** (2017), 054019
[14] S. Zouzou, B. Silvestre-Brac, C. Gignoux and J. M. Richard, Four Quark Bound States, Z. Phys. C **30** (1986) 457.
[15] D. M. Brink and F. Stancu, Tetraquarks with heavy flavors, Phys. Rev. D **57** (1998) 6778.
[16] D. Janc and M. Rosina, The $T_{sc} = DD^{*}$ molecular state, Few Body Syst. **35** (2004) 175.
[17] V. Dimitrasinovic, Cubic Casimir operator of $SU_{c}(3)$ and confinement in the nonrelativistic quark model, Phys. Lett. B **499** (2001) 135.
[18] Z. Papp and F. Stancu, Three body confinement forces in a realistic constituent quark model, Nucl. Phys. A **726** (2003) 327.
[19] S. Pepin and F. Stancu, On a three-body confinement force in hadron spectroscopy, Phys. Rev. D **65** (2002) 054032.
[20] V. Dimitrasinovic, Color SU(3) symmetry, confinement, stability, and clustering in the $q^{**}2$ $q^{*2}$ system, Phys. Rev. D **67** (2003) 114007.
[21] C. Helminen and D. O. Riska, Low lying $q$ $q$ $q$ anti-$q$ states in the baryon spectrum, Nucl. Phys. A **699** (2002) 624.
[22] G. S. Bali, QCD forces and heavy quark bound states, Phys. Rept. **343** (2001) 1.
[23] T. T. Takahashi, H. Suganuma, Y. Nemoto and H. Matsufuru, Detailed analysis of the three quark potential in SU(3) lattice QCD, Phys. Rev. D **65** (2002) 114509.
[24] C. Alexandrou, P. De Forcrand and A. Tsapalis, The Static three quark SU(3) and four quark SU(4) potentials, Phys. Rev. D **65** (2002) 054503.
[25] H. Suganuma, T. Iritani, F. Okiharu, T. T. Takahashi and A. Yamamoto, Lattice QCD Study for Confinement in Hadrons, AIP Conf. Proc. **1388** (2011) 195, arXiv:1103.4015 [hep-lat].
[26] F. Bicudo and M. Cardoso, Tetraquarks, Why It Is so Difficult to Model Them, Acta Phys. Polon. Supp. **8** (2015) no.1, 71.
[27] F. Stancu, Group theory in subnuclear physics, Oxford Stud. Nucl. Phys. **19** (1996) 1.
[28] M. Harvey, On the Fractional Parentage Expansions of Color Singlet Six Quark States in a Cluster Model, Nucl. Phys. A **352** (1981) 301 Erratum: [Nucl. Phys. A **481** (1988) 834].
[29] D. M. Brink and F. Stancu, Role of hidden color states in 2 $q$ 2 anti-$q$ systems, Phys. Rev. D **49** (1994) 4665.
[30] F. Stancu, Can $Y(4140)$ be a $c$ anti-$c$ $s$ anti-$s$ tetraquark?, J. Phys. G **37** (2010) 075017.
[31] W. Park, A. Park, S. Cho and S. H. Lee, $P_{c}(4380)$ in a constituent quark model, Phys. Rev. D **95** (2017) 054027.
[32] F. Stancu, S. Pepin and L. Y. Glozman, The Nucleon-nucleon interaction in a chiral constituent quark model, Phys. Rev. C **56** (1997) 2779.
[33] M. Karliner and H. J. Lipkin, A Diquark - triquark model for the K N pentaquark, Phys. Lett. B **575** (2003) 249.
[34] M. Genovese, J. M. Richard, F. Stancu and S. Pepin, Heavy flavor pentaquarks in a chiral constituent quark model, Phys. Lett. B **425** (1998) 171.
[35] J. Vijande, A. Valcarce and J.-M. Richard, Stability of multiquarks in a simple string model, Phys. Rev. D **76** (2007) (2012) 054019.
(2007) 114013.

[36] J. M. Richard, Stability of the pentaquark in a naive string model, Phys. Rev. C 81 (2010) 015205.