On the Connectivity of Cobordisms and Half-Projective TQFT’s

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Abstract: We consider a generalization of the axioms of a TQFT, so called half-projective TQFT’s, where we inserted an anomaly, $x^\mu_0$, in the composition law. Here $\mu_0$ is a coboundary (in a group cohomological sense) on the cobordism categories with non-negative, integer values. The element $x$ of the ring over which the TQFT is defined does not have to be invertible. In particular, it may be zero.

This modification makes it possible to extend quantum-invariants, which vanish on $S^1 \times S^2$, to non-trivial TQFT’s. Note, that a TQFT in the ordinary sense of Atiyah with this property has to be trivial all together.

We organize our discussions such that the notion of a half-projective TQFT is extracted as the only possible generalization under a few very natural assumptions.

Based on separate work with Lyubashenko on connected TQFT’s, we construct a large class of half-projective TQFT’s with $x = 0$. Their invariants all vanish on $S^1 \times S^2$, and they coincide with the Hennings invariant for non-semisimple Hopf algebras and, more generally, with the Lyubashenko invariant for non-semisimple categories.

We also develop a few topological tools that allow us to determine the cocycle $\mu_0$ and find numbers, $\varrho(M)$, such that the linear map associated to a cobordism, $M$, is of the form $x^{\varrho(M)} f_M$. They are concerned with connectivity properties of cobordisms, as for example maximal non-separating surfaces. We introduce in particular the notions of “interior” homotopy and homology groups, and of coordinate graphs, which are functions on cobordisms with values in the morphisms of a graph category.

For applications we will prove that half-projective TQFT’s with $x = 0$ vanish on cobordisms with infinite interior homology, and we argue that the order of divergence of the TQFT on a cobordism, $M$, in the “classical limit” can be estimated by the rank of its maximal free interior group, which coincides with $\varrho(M)$.

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1) Introduction

Although physical motivations were at its origin, the notion of topological quantum field theories (TQFT’s) has become a part of algebraic topology, since it was axiomatically defined by Atiyah [A]. In the same way as for example homology or homotopy, it is given as a functor from a topological category into an algebraic category. More precisely, it is a functor of the following symmetric tensor categories:
\[ V : \text{Cob}_{d+1} \rightarrow \mathbb{R} \text{-mod}, \]  
(1.1)

where \( \text{Cob}_{d+1} \) has as objects d-manifolds and as morphisms \( d + 1 \)-dimensional cobordisms between them, and \( \mathbb{R} \text{-mod} \) is the usual category of (free) \( \mathbb{R} \)-modules and \( \mathbb{R} \)-linear maps. Their tensor products are the disjoint union \( \sqcup \) and \( \otimes_{\mathbb{R}} \), respectively. (Quite often we only require \( \mathbb{R} \) to be a ring).

We may think of \( V \) as a representation of the algebra of cell-attachments to the boundaries of \( d+1 \)-manifolds. But unlike, e.g., homology it detects algebraically much more involved relations between the cells than just their intersection numbers. In this paper we shall be exclusively concerned with the case \( d = 2 \), where this algebra corresponds to quantum groups. We will make extensive reference to known TQFT’s in 2+1 dimensions, and use results of three-dimensional, geometric topology.

The original purpose of this paper is to resolve a paradox that occurs in several different examples of “quantum-invariants” of 3-manifolds and concrete quantum field theories. It is about a degeneracy that at first sight appears to prevent us from constructing a TQFT in the rigorous, axiomatic sense. The problem is resolved by a seemingly minor modification of the axioms of a TQFT, yielding what we shall call half-projective TQFT’s. The generalization naturally leads us to several questions about the connectivity of cobordisms, for which we will develop several tools that should have applications also to other topological problems.
Specifically, the phenomena that we are interested in is that sometimes the invariant of a “quantum-theory” vanishes on the product of sphere and circle, i.e.,

\[ x := \mathcal{V}(S^1 \times S^2) = 0 \quad (1.2) \]

It is an elementary implication of the axioms of a TQFT, observed in [Wi] but also [A], that for a surface, \( \Sigma \), the invariant of the circle product is the dimension of the associated vector space, i.e.,

\[ \mathcal{V}(S^1 \times \Sigma) = \dim(\mathcal{V}(\Sigma)) \quad (1.3) \]

Hence (1.2) entails also triviality of the vector space, \( \mathcal{V}(S^2) = 0 \). A dramatic consequences of this for a TQFT is that \( \mathcal{V} \equiv 0 \) on all surfaces and cobordisms. The reason is easily seen, if we remove from a general cobordism \( M : \Sigma_s \to \Sigma_t \) a ball so that \( M^* := M - D^3 : \Sigma_s \to \Sigma_t \sqcup S^2 \). Expressing the regluing of \( D^3 \) as a composition of cobordisms, we obtain

\[ M : \Sigma_s \overset{M^*}{\longrightarrow} \Sigma_t \sqcup S^2 \overset{D^3}{\longrightarrow} \Sigma_t \quad (1.4) \]

An application of \( \mathcal{V} \) to this yields the assertion, since the middle vector space is mapped to zero. Clearly, this means that a (non-trivial) invariant can be extended to a TQFT only if it does not vanish on \( S^1 \times S^2 \).

However, the examples, in which the degeneracy of (1.2) is encountered, do appear to have a lot of the structural properties of a TQFT, and are very closely related to situations, where non-trivial TQFT’s actually exist.

An example of a more algebraic nature is the Hennings invariant of 3-manifolds, see [H], for a finite dimensional, quasi-triangular Hopf algebra, \( \mathcal{A} \). Its construction is analogous to that of the invariant of Reshetikhin-Turaev in [RT], also [T], except that special elements of \( \mathcal{A} \) are used directly instead of the representation theory of \( \mathcal{A} \). The two invariants can be put on the same footing [Ke3] via the Lyubashenko invariant [L2], which is defined for abelian braided tensor categories. The invariant of [RT] can be extended to a TQFT, which is usually identified with the Chern Simons quantum field theory. Nevertheless, it is easy to see that the (non-trivial) Hennings invariant vanishes on \( S^1 \times S^2 \) if (and only if) \( \mathcal{A} \) is not semisimple.

Subsequent studies in [L1], [L2], [KL], and [Ke3] showed that the latter invariants can still be associated to representations of mapping class groups and, more generally, TQFT’s for cobordisms of connected surfaces. This reveals that the vanishing paradox in (1.2) has to have its origin in basic connectivity properties of cobordisms.

The situation is analogous for the Kuperberg invariant [Ku] for a Hopf algebra \( \mathcal{B} \), which is believed to be closely related to the Hennings invariant. Again, if \( \mathcal{B} \) is not semisimple, it vanishes on \( S^1 \times S^2 \), although it is constructed in a similar way as the Turaev-Viro invariant [TV], for which we always have TQFT’s.

A second example in the concrete physical framework of quantum field theory is given by conformal field theories and their corresponding Chern Simons theories, whose gauge groups are supergroups. The case of \( U(1,1) \)-WZW-models has been worked out in [RS]. In this model we have the same vanishing for the ratios of partition functions \( \mathcal{V}(S^1 \times S^2) := \mathcal{Z}(S^1 \times S^2)/\mathcal{Z}(S^3) = 0 \). In the attempt to construct the operators of a TQFT (in some regularization with parameter \( \varepsilon \to 0 \)) one is faced in [RS] with serious singularities of matrix elements in \( \varepsilon \) that can only be removed at the price of having degenerate inner products for the physical state spaces \( \mathcal{V}(\Sigma) \).
We show in this article that there is essentially only one way to modify the axioms of a TQFT, such that we preserve the tensor product rule for \( V \), and ensure that \( V \) respects gluings of cobordisms over connected surfaces. The answer is provided by the notion of a half-projective TQFT. By this we mean a map between the category of cobordisms to the category of \( \mathbb{R} \)-modules, which is a functor as in (1.1), except that the composition law is of the form

\[
V(M \circ N) = x^{\mu_0(M,N)} V(M) V(N)
\]  
(1.5)

Here \( \mu_0 \) is a “coboundary” on \( \text{Cob}_3 \) in the sense of group cohomology, when we view categories as generalizations of groupoids. \( \mu_0(M,N) \) can be computed from basic connectivity data of \( M \) and \( N \), and it has values only in the non-negative integers, \( \mu_0 \in \mathbb{Z}^{0,+} \).

If the number \( x \in \mathbb{R} \) is invertible (e.g., \( x \neq 0 \) and \( k = \mathbb{R} \) is a field) then the anomaly can of course be removed by rescaling \( V \). Also, if \( M \) and \( N \) are composed over only one connected component, we find that \( \mu_0(M,N) = 0 \) so that \( V \) behaves like an honest functor. This is consistent with the connected TQFT-functors in [KL]. The identity (1.3), however, is now modified. Repeating the original derivation with (1.5) we find

\[
V(S^1 \times \Sigma) = x \dim(V(\Sigma))
\]  
(1.6)

Hence the above examples are not in contradiction with extensions to half-projective TQFT’s, if we set \( x = 0 \).

One of the main results of this paper is the construction of a large class of non-trivial, half-projective TQFT’s with \( x = 0 \). Our starting point here are the connected TQFT’s from [KL], but as in the \( U(1,1) \)-WZW-models we have to deal with degenerate pairings of the spaces \( V(\Sigma) \). The TQFT’s we find extend, in particular, the Hennings invariant for an arbitrary non-semisimple, finite-dimensional, modular Hopf algebra.

In the quantum-algebraic framework, the element \( x \) (and, especially, the fact whether it is trivial or not) is intimately related to semisimplicity and cointegrals of the respective categories or Hopf algebras. In a concrete quantum field theory \( x \) may be seen as a parameter for the renormalization of the product of field operators.

It is interesting to observe that the two non-semisimple examples from above share a few more common features beyond (1.2). In both cases we find that the representations of the mapping class group \( \text{SL}(2, \mathbb{Z}) \) of the torus contains algebraic summands and tensor factors (the semisimple ones only produce finite representations), and that the invariants of lens spaces and Seifert-manifolds are proportional to the order of the first homology group, see [RS] and [Ke3], and references therein. We will investigate these properties in the general, axiomatic setting in separate work.

1.1) Survey of Contents :

In Chapter 2 we provide the definitions of the cobordism category \( \text{Cob}_3(\ast) \) (Section 2.1), the special cocycle \( \mu_0 \) (Section 2.2), the notion of half-projective TQFT’s (Section 2.3), and various versions of extended (half-projective) TQFT’s (Section 2.4). In Section 2.2 we also give an explicit formula (see Lemma [2]) and an algorithm as in (2.11) for the computation of \( \mu_0 \). Section 2.3 includes a discussion of the basic implications of the axioms of a half-projective TQFT. In particular, it is shown in Corollary [4] that, generically, the only properly half-projective, indecomposable TQFT’s are those with \( x = 0 \), even for general rings, \( \mathbb{R} \).
The purpose of Chapter 3 is to develop the topological means that allow us to treat the connectivity properties of cobordisms relevant to the formalism of half-projective TQFTs. In this we are mainly motivated by the result in Lemma 6 which asserts that $\mathbf{V}(M) = x^e(M) f_M$ for some “regularized” $R$-linear map $f_M$, where $g(M)$ is given by the maximal number of non-separating surfaces in $M$. Note, that in general $x^e(M)$ may generate non-trivial ideals in the respective space of linear maps, seen as an $R$-module. In order to be able to compute the number $g(M)$, we show in Theorem of Section 3.5 that it is identical with the maximal rank of a free interior group. The notion of interior fundamental groups, where we divide by the subgroup coming from the surfaces, is introduced in Section 3.2, and a basic gluing-property under compositions of cobordisms is described in Lemma 8.

In Section 3.4 we define coordinate graphs of manifolds with boundary, which is a rather useful tool to the end of encoding the connectivity properties of a cobordism in a combinatorial way. Coordinate graphs are given by (Morse) functions on cobordisms with values in graphs that belong to the graph category from Section 3.3. A result of particular interest is Lemma 13, which ensures that to a decomposition of coordinate graphs we can always find a corresponding connected decomposition of the cobordisms.

An interesting application of the results in Chapter 3 is Corollary 5, which asserts that if $\mathbf{V}$ is a general, half-projective TQFT with $x = 0$, then $\mathbf{V}(M) = 0$ for any cobordism, for which $\beta^m_1(M) \neq 0$, i.e., with infinite, interior homology. For the special case of the $U_q(s\ell_2)$-Hennings invariant ($q$ a root of unity), evaluated on closed manifolds, $M : \emptyset \to \emptyset$, this vanishing phenomenon was also observed by a direct calculation in [O].

In Chapter 4 we show how non-trivial, half-projective TQFT’s can be constructed from connected ones as, e.g., those in [KL]. We start in Section 4.1 with the discussion of an algebra of special cobordisms between a surface and the connected sum of its components. Using the existence of decompositions as in Lemma 19, these cobordisms allow us to express any cobordism by one that cobords only connected surfaces.

In Section 4.2 we start with the list of Axioms V1-V5 for a generalized TQFT, $\mathbf{V}$, which essentially state that $\mathbf{V}$ respects tensor products as well as compositions over connected surfaces. We show that $\mathbf{V}$ necessarily has to be a half-projective TQFT. Moreover, we exhibit a list of eight properties that have to hold for a connected TQFT, if it should extend to a disconnected one satisfying V1-V5. The latter is true if it is, e.g., a specialization from a half-projective one. In Theorem 6 we show that these properties are in fact also sufficient, in order to guarantee the existence of such a half-projective TQFT.

The purpose of Section 4.3 is to show that all but one of these properties are automatically fulfilled, if the connected TQFT descends from an extended TQFT. In particular, we will use that $\mathbf{V}$ is a factorization of a functor $\mathbf{V}_1 : \text{Cob}_3(1) \to \mathcal{C}$, where the objects of $\text{Cob}_3(1)$ are surfaces with one hole, and $\mathcal{C}$ is an abelian, braided tensor category. For the connected TQFT’s the vector spaces $\mathbf{V}(\Sigma)$ are thus identified with invariances of special objects in $\mathcal{C}$. When we pass to the disconnected case we have to divide by the null spaces of the pairing with the respective coinvariance, as it is stated in the summary in Lemma 22. In the derivation we actually first construct in (4.68) the linear morphism spaces, in which the $\tau$-move of closed surfaces is realized, and then identify these in Lemma 20 as linear maps between the quotiented vector spaces.

In Section 4.4 we tie the last remaining property, regarding the projective factor, to the existence of a natural transformation of the identity functor of $\mathcal{C}$, whose image for each object is a multiple of the unit, see (4.81). In Lemma 23 we show that non-triviality of the value of such a transformation on the unit object itself (which will be the same as $x$) is a necessary and sufficient condition for the semisimplicity of $\mathcal{C}$. In the remainder of this section we establish the existence of such a transformation.
by identification with the cointegral of the coend \( F = \int X^\vee \boxtimes X \), assuming that this is contained in \( C \).

Finally, in Section 4.5 we combine the results of the previous sections and of [KL] in Theorem 9, in which we establish the existence of a large class of truly half-projective TQFT’s. We also use the last section to speculate on the possibility of constructing generalized TQFT’s, where we consider besides the tensor products also derived functors, like \( \text{Tor} \), whose contributions may allow us to salvage some of the TQFT data that is lost in the division by the null spaces. As a further possible application of the formalism of half-projective TQFT’s we give a brief discussion of classical limits, for which \( x \to \infty \). We check for circle products the quality of the estimate \( \| \mathcal{V}(M) \| \geq \text{const.} \, x^{\delta(M)} \), which is suggested by the Lemma 3 and the normalizations used in the canonical construction of the invariants. The estimates turn out to holds in all of the considered cases, and they are roughly half of the true order of divergence.

The proofs for the basic, technical lemmas on coordinate graphs are delivered in Appendix A.1. In Appendix A.2 we compute formulas for the coboundaries \( \mu_1 \) and \( \mu_\partial := \mu_1 - \mu_0 \), which are generalizations in homology of \( \mu_0 \). We find \( \mu_\partial \in \mathbb{Z}^{a,+} \). The corresponding anomaly in homotopy \( \mu_\pi \) (see Section 3.2) counts the number of additional, non-separating surfaces in a product of cobordisms that do not stem from the composites. Thus we pick up an additional factor, \( x^{\mu_\pi} \), besides the one from the usual anomaly from (1.3). The tangle presentations of cobordisms from [Ke2], which we refer to in Sections 4.1 and 4.4, are summarized in Appendix A.3.

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2) Cobordism Categories, and Half-Projective TQFT’s

In this chapter we shall define and discuss generalizations of the TQFT-axioms of Atiyah. To this end we first introduce in Section 2.1 the cobordism category \( \text{Cob}_3(*) \), whose objects are compact Riemann surfaces with boundaries, and whose morphisms are homeomorphism classes of cobording 3-manifolds. Decompositions into connected components are expressed in Section 2.2 in terms of the symmetric tensor structure of \( \text{Cob}_3(*) \). Here we also introduce a coboundary, \( \mu_0 := -\delta_{\beta_0}^{\text{int}} \), on \( \text{Cob}_3(*) \), where the coboundary operator \( \delta \) is again to be understood in the generalized, group cohomological sense, see (2.14). The value of \( \mu_0(M,N) \) is always a non-negative integer, and it is \( K - 1 \), if \( M \) and \( N \) are two connected cobordisms that are glued together along \( K \) connected surfaces.

This allows us in Section 2.3 to define the notion of half-projective TQFT’s, \( \mathcal{V} : \text{Cob}_3(0) \to \mathbb{R} - \text{mod} \), on the category of closed surfaces, \( \text{Cob}_3(0) \), by inserting an anomaly of the form \( x^{\mu_0} \) into the composition law for \( \mathcal{V} \), where \( x \) does not have to be invertible. We show that - except for specified, exceptional situations - a half-projective TQFT-functor, \( \mathcal{V} \), is the sum of functors, \( \mathcal{V}^j \), where each \( \mathcal{V}^j \) maps into the free modules of a summand, \( \mathbb{R}_j \subset \mathbb{R} \), and the component of \( x \) in \( \mathbb{R}_j \) is either zero or invertible.

Finally, in Section 2.4 we also discuss the various formalisms of extended TQFT’s, and how the notion of half-projective TQFT’s can be extended to \( \text{Cob}_3(*) \).
2.1) Categories of Cobordisms, the Structure of $\text{Cob}_3(*)$

The cobordism categories, which we wish to consider here, are slightly more general than the ones defined, e.g., in [Ke1] or [KL]. The objects are as usual given by a set of inequivalent, compact, oriented Riemann surfaces, $\Sigma$. Here we are only interested in $\Sigma$ as a topological manifold. Moreover, we assume that $\Sigma$ is equipped with an ordering of its components.

We also fix parametrizations of the boundary components, $\partial \Sigma \cong \bigsqcup^n S^1$. A morphism, $M : \Sigma_s \rightarrow \Sigma_t$, is now defined between any two such surfaces with $n_s$ and $n_t$ holes, respectively, if the total number, $n_s + n_t$, of boundary components of $\partial M$ is even.

We may organize the the set of holes into pairs, such that only holes of different surfaces are matched. For any such choice we glue in cylinders connecting the boundary components of a pair so that we obtain a closed surface as follows:

$$\Sigma^{cl} = -\Sigma_s \sqcup_{\bigsqcup^n S^1} \bigsqcup{n_s+n_t\over 2} S^1 \times [0,1] \sqcup_{\bigsqcup^n S^1} \Sigma_t .$$

(2.7)

A cobordism consists now of an oriented, compact 3-manifold, $M$, and an orientation preserving homeomorphism, $\psi^0 : \Sigma^{cl} \rightarrow \partial M$. Let us denote also the resulting inclusion of the source and target surfaces into the cobordism:

$$\psi : -\Sigma_s \sqcup \Sigma_t \hookrightarrow M .$$

(2.8)

These maps will be sometimes called charts or parametrizations.

We shall denote the set of homeomorphic classes of such cobordisms with fixed numbers, $n_s$ and $n_t$, of source and target surface components by $\text{Cob}_3(n_s,n_t)$.

Below we depict a typical situation of how the boundary $\Sigma^{cl} \cong \partial M$ is built up, where $M : \Sigma_s \rightarrow \Sigma_t$ is a cobordism in $\text{Cob}_3(3,5)$.

$$\Sigma_s : \quad \Sigma_{1,s} \quad \Sigma_{2,s} \quad \Sigma_t : \quad \Sigma_{1,t} \quad \Sigma_{2,t}$$

(2.9)

In this example the source and target surface both have two components, i.e.,

$$\Sigma_s(3) = \Sigma_{1,s}^{(1)} \sqcup \Sigma_{2,s}^{(2)} \quad \text{and} \quad \Sigma_t(5) = \Sigma_{1,t}^{(3)} \sqcup \Sigma_{2,t}^{(2)} ,$$

where the superscript at a surface displays the number of its holes. A surface component is indicated in diagram (2.9) by a fat, horizontal line with an interruption for each hole. We also introduced a labelling of holes so that the hole with label $\alpha_j$ is connected to the hole with label $\beta_j$ by the $j$-th cylindrical piece from (2.7), with $j = 1, \ldots, 4$. Each of these pieces are depicted in (2.9) by a pair of
thinner lines. For simplicity we have omitted in our example the possibility of crossings, although the cylinders may be arbitrarily braided and knotted inside the cobordism.

Note that in this definition we are allowed to have a cylinder connect two holes from two different target (or source) surfaces, as for example the fourth cylinder between \( \Sigma_{1,t} \) and \( \Sigma_{2,t} \) in (2.9). For these we shall also specify a direction so that there is a distinguished start- (or end-) hole. In the above example we thus have to decide whether \( \alpha_t \) or \( \beta_4 \) is the start hole.

The union of all \( \text{Cob}_3(n, m) \) shall constitute \( \text{Cob}_3(\ast) \) as a set. The composition in \( \text{Cob}_3(\ast) \) is given by gluing the two 3-manifolds along the intermediate surface, requiring that end-holes are glued to start-holes.

In the process we may encounter a situation, where several cylindrical parts combine to form a closed (connected) surface in \( \partial M \), which can only be a torus, \( T^2 \). By construction this torus has a distinguished long meridian, with a given direction, and a distinguished short meridian, which carries also a direction due to the induced orientation. If we have similar data fixed on the respective surface, \( T^2 \), that we chose as an object of \( \text{Cob}_3(\ast) \), then there is (up to isotopy) a unique homeomorphism between the two tori. Hence we can add \( T^2 \) to either the start- or target-surface of the composite cobordism in a well defined manner. See also Section 2.4, where these tori are interpreted as “horizontal objects” of a 2-morphism.

On the morphism set we can also define a filling map

\[
\phi_0 : \text{Cob}_3(n_s, n_t) \to \text{Cob}_3(0),
\]
which behaves nicely under compositions. It is given by gluing a full tube \( D^2 \times [0, 1] \) into the cylindrical parts, such that the holes of \( -\Sigma_s \sqcup \Sigma_t \) are closed with discs \( D^2 \times \{0, 1\} \). The cobordism \( \phi_0(M) \) is thus between the same surfaces without punctures.

In the definitions of [Ke2] and [KL] we only considered the case \( n_s = n_t \), and the cylinders had to connect a source hole to a target hole. There \( \phi \) is a true functor. Also, we considered a central extension of \( \text{Cob}_3(\ast) \) by the cobordism-group \( \Omega_4 \). This was naturally constructed by considering first also four-folds bounding the cobordism, and then retaining their signature as an additional structure to the 3-cobordism, see [Ke2] for details. We shall tacitly assume this extension here, too.

### 2.2) Elementary Compositions, and the Cocycle \( \mu_0 \):

In this section we shall introduce the coboundary \( \mu_0 \) on the cobordism category \( \text{Cob}_3(\ast) \), which enters the definition of a half-projective TQFT. We start with the basics of the symmetric tensor structure of \( \text{Cob}_3(\ast) \). We make the following straightforward observations:

A category of cobordisms between closed surfaces admits a natural tensor product given by the disjoint union. This tensor product is extends to the morphisms by disjoint unions in a functorial way, and it is obviously strictly associative.

Recall that for \( \text{Cob}_3(0) \) we also assumed the connected components to be ordered. Thus if \( \Sigma_j \) and \( \Sigma'_j \) are connected surfaces the tensor product of their ordered union \((\Sigma_1 \sqcup \ldots \sqcup \Sigma_n) \otimes (\Sigma'_1 \sqcup \ldots \sqcup \Sigma'_n)\) shall be the ordered union \( \Sigma_1 \sqcup \ldots \sqcup \Sigma_a \sqcup \Sigma'_1 \sqcup \ldots \sqcup \Sigma'_b \).

In particular, this means that \( \Sigma \otimes \Sigma' \) and \( \Sigma' \otimes \Sigma \) are different objects. However, we can find a natural isomorphism, \( \gamma : \Sigma \otimes \Sigma' \xrightarrow{\cong} \Sigma' \otimes \Sigma \), in between them. In \( \text{Cob}_3(0) \) this means that \( \gamma \) is a cobordism, which has a two-sided inverse. As a three-manifold with boundary it is given by \((\Sigma \times [0, 1]) \sqcup (\Sigma' \times [0, 1])\) and the boundary identifications \( \Sigma \times \{0\} \sqcup \Sigma' \times \{0\} \xrightarrow{\cong} \Sigma \sqcup \Sigma' \) and \( \Sigma \times \{1\} \sqcup \Sigma' \times \{1\} \xrightarrow{\cong} \Sigma' \sqcup \Sigma \). are the obvious canonical maps. Also, it is clear that \( \gamma \circ \gamma = \mathbb{1} \).

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The system of morphisms $\gamma = \gamma(\Sigma, \Sigma')$ has in fact all the properties of a commutativity contraind of a symmetric tensor category. Specifically, it is natural with respect to both arguments, and the triangle equality is readily verified.

Let us summarize in the following lemma the basic terminology that is natural from a categorial point of view and which provides a convenient language to organize reorderings of surface components and decompositions of cobordisms.

In the statements for $\text{Cob}_3(*)$ we actually have to apply the adequate generalizations that take care of the non-trivial vertical 1-arrows, like the extra tori discussed in the previous section or renumberings of holes, see Section 2.4.

**Lemma 1**

1. $\text{Cob}_3(*)$ and $\text{Cob}_3(0)$ are strict, symmetric tensor categories.

2. For any set, $\{\Sigma_j : j = 1, \ldots, K\}$, of surfaces and any permutation, $\pi \in S_K$, we have morphisms $\pi^* : \Sigma_1 \otimes \cdots \otimes \Sigma_K \to \Sigma_{\pi^{-1}(1)} \otimes \cdots \otimes \Sigma_{\pi^{-1}(K)}$, with $\pi \in S_K$, which are natural in every argument, and which represent $S_K$.

3. For every morphism there is a unique, maximal number, $b = \beta_0(M)$, such that there is a decomposition in the form $M = \pi_1^* \circ (M_1 \otimes \cdots \otimes M_b) \circ \pi_2^*$. The cobordisms $M_j$ are then all connected.

The second remark in the lemma follows from the observation that cobordisms of the form $\sigma^* = I \otimes \gamma \otimes I$, where $\gamma$ is the commutativity constraint on the union of two consecutive, connected components, automatically fulfill the relations of the usual generators of the symmetric group. Hence for a permutation $\pi \in S_K$ we can define the cobordism $\pi^*$ as the respective composition of the $\sigma^*$’s.

The last remark is simply expressing in categorial terms the fact that, up to reordering of the boundary components, every cobordism can be given as the union of its connected components.

In later chapters we will return to the notation $\sqcup$ instead of $\otimes$.

Adding a sufficient number of cylinders to the right and left of each $M_j$ in the formula in Part 3 of Lemma 1, we obtain commuting morphisms $\tilde{M}_j = I_X \otimes M_j \otimes I_Y$, such that the tensor product $M$ can be rewritten as the composite:

$$M = \pi_1^* \circ (\tilde{M}_1 \circ \cdots \circ \tilde{M}_b) \circ \pi_2^*.$$  \hspace{1cm} (2.11)

With this it follows that the compositions of two morphism can be obtained as an iteration of two types of elementary composites. The first are of the form $M \circ \pi^*$, where $M$ is connected. The second is given by products as follows:

$$M = (I \otimes M_2) \circ (M_1 \otimes I),$$  \hspace{1cm} (2.12)

where $M_1$ and $M_2$ are both connected, and are glued over $K \geq 1$ boundary components. After having made a gluing along one connected component, the identifications of the remaining $(K - 1)$ boundary components are among boundary components of the same connected manifold. We will relate them to $(K - 1)$ uncancellable one-handle attachments in either a direct decomposition in three-dimensions, or to a four-manifold bounding $M$. 

9
We wish to assign the excess number of connected boundary components over which we glue two
cobordisms as a penalty in the form of a cocycle of $\text{Cob}_3(\ast)$. To this end, let us introduce the interior
Betti-numbers
\[
\beta_j^{\text{int}}(M) = \beta_j(\phi_0(M)) - \frac{1}{2} \beta_j(\partial \phi_0(M)) ,
\] (2.13)
where $\beta_j = \text{dim}(H_j(X))$ are the usual Betti-numbers.

In this section we are interested only in the case $j = 0$, where we can omit the filling functor $\phi_0$
from (2.10) in the formula. Some computations for $j = 1$ are given in Appendix A.2. It is easily seen
that any number $\beta_0^{\text{int}}(M) \in \frac{1}{2}\mathbb{Z}$ is realized. They define a coboundary on $\text{Cob}_3(\ast)$ with coefficients
a-priori in the half integers by
\[
\mu_0(M_2, M_1) := -\delta \beta_0^{\text{int}}(M_1, M_2) = \beta_0^{\text{int}}(M_2 \circ M_1) - \beta_0^{\text{int}}(M_1) - \beta_0^{\text{int}}(M_2) .
\] (2.14)

It is readily seen that (2.14) actually defines an integer cocycle, which is, in fact, also an integer
coboundary of, e.g., $\tilde{\beta}_0^{\text{int}}(M) = \beta_0(M) - \beta_0(\Sigma_t)$. The property that makes $\mu_0$
still an interesting quantity is that it is non-negative on all pairs of cobordisms.

Indeed, for an elementary composition over $K \geq 1$ connected boundary components as in (2.12),
we find that this integer is the desired excess number of boundary-component over which we glue:
\[
\mu_0([\Sigma_t] \otimes M_2, M_1 \otimes [\Sigma_t]) = K - 1 .
\] (2.15)

Also, it is easy to see that $\mu_0(\pi^*, M) = 0$ for a permutation, and $\mu_0([\Sigma_t] \otimes M_2, [\Sigma_s] \otimes [\Sigma_t]) = 0$, if $Y$
is the source of $M_2$ and $X$ the target of $M_1$, so that
\[
\mu_0(M_1 \otimes M_2, N) = \mu_0([\Sigma_t] \otimes M_2, [\Sigma_s] \otimes [\Sigma_t] \circ N) + \mu_0(M_1 \otimes [\Sigma_t], N) .
\] (2.16)

This formula allows us to compute $\mu_0$ recursively from a presentation as in Lemma 4, and also implies
$\mu_0 \geq 0$ for general compositions.

For a more systematic computation of the cocycle, let us introduce the spaces
\[
W_1 := \ker(H_0(\psi_1^s)) , \quad W_2 := \ker(H_0(\psi_2^s)) .
\] (2.17)

Here the maps $\psi_j^{s/t}$ are the restrictions of the charts in (2.8) to source and target surfaces,
\[
\psi^t : \Sigma_t \hookrightarrow M , \quad \psi^s : -\Sigma_s \hookrightarrow M ,
\] (2.18)
for a cobordism $M : \Sigma_s \rightarrow \Sigma_t$. We now have the following:

**Lemma 2** For two cobordisms, $M_1 : \Sigma_{s,1} \rightarrow \Sigma$ and $M_2 : \Sigma \rightarrow \Sigma_{t,2}$, and spaces $W_j$ as above, the
cocycle from (2.14) is given as follows:
\[
\mu_0(M_2, M_1) = \text{dim} \left( W_1 \cap W_2 \right) .
\] (2.19)

**Proof:** The exact sequence $0 \rightarrow W_1 \cap W_2 \rightarrow H_0(\Sigma) \rightarrow H_0(M_2) \oplus H_0(M_1) \rightarrow H_0(M_2 \circ M_1) \rightarrow 0$,
implies the dimension-formula $\beta_0(M_2 \circ M_1) - \beta_0(M_2) - \beta_0(M_1) = \text{dim} \left( W_1 \cap W_2 \right) - \beta_0(\Sigma)$. Counting
also boundary components, this yields the asserted formula for $\mu_0$. \hfill $\Box$

A basic property of this cocycle is that it vanishes on invertible cobordisms, i.e.,
\[
\mu_0(M, G) = \mu_0(G, M) = 0 \quad \text{if} \quad G \in \pi_0(\text{Diff}(\Sigma)^+).\] (2.20)
(Here we identify $G$ with a cobordisms by picking a representing automorphism of the surface, $\psi_G : \Sigma \to \Sigma$. The associated cobordisms is then as a 3-manifold given by $\Sigma \times [0,1]$, and the boundary identification are $id : \Sigma \times \{0\} \to \Sigma$ and $\psi_G : \Sigma \times \{1\} \to \Sigma$. It is a basic fact that this actually establishes an isomorphism between the mapping class group of $\Sigma$ and the group of invertible cobordisms from $\Sigma$ to itself).

Another property of $\mu_0$ is found in Section 3.2 and Appendix A.2: If $\mu_0(M_2, M_1) > 0$, then the composite $M_2 \circ M_1$ will contain paths or 1-cocycles that give rise to additional, infinite generators of the fundamental group or the first homology group, respectively, besides those of $M_2$ and $M_1$.

2.3) Half-Projective TQFT’s, and Generalizations :

As we remarked in the introduction it is not possible to construct non-trivial TQFT’s in the classical sense, which vanish on $S^1 \times S^2$. The purpose of this section is to define the modification that makes allows such a construction and discuss a few basic implications.

**Definition 1** Suppose $R$ is a commutative ring with unit, and $R_{mod}$ the symmetric tensor category of free $R$-modules. Further, let $x$ be an element in $R$, and $\mu$ a 2-cocycle on $\text{Cob}_3(0)$, which takes values only in $\mathbb{Z}^+$. We call $V : \text{Cob}_3(0) \to R_{mod}$ a half-projective TQFT (with respect to $x$ and $\mu$), if it full fills all the requirements for a functor of symmetric tensor categories, except for the preservation of compositions. Instead of this we shall assume the relaxed condition:

$$V(M_2 \circ M_1) = x^{\mu(M_2,M_1)} V(M_2) V(M_1).$$

(2.21)

As usual a projective TQFT is one for which $x \in R$ is invertible and the non-negativity of $\mu$ is dropped. An example of the latter is the well known signature-extension of the 2+1-dimensional cobordisms, which we avoided here by passing to a central extension of $\text{Cob}_3(\ast)$ by $\Omega_4$. In this case $\mu$ is the Wall-cocycle. The number $x \in \mathbb{C}^\times$ for TQFT’s that are associated to Chern-Simons theory with Lie-algebra $g$ and level $\ell$ may be obtained from the representation theory of Kac-Moody algebras, see [KW], and is a phase depending on $\ell$, the dimension, and the dual Coxeter number of $g$. For quantum-group constructions starting from a double $D(B)$ it is the pairing of square roots of the modulus and comodulus of $B$, see [Ke1].

In general, if $\mu$ is a coboundary over $\mathbb{Z}$, we may rescale a projective TQFT-functor, and obtain a functor in the ordinary sense. However, for a half-projective TQFT and a trivial cocycle we can only replace $x$ by $xy^{-1}$ for invertible $y \in R$.

This is the situation, which we are interested in here, as the non-semisimple invariants will be associated to half-projective TQFT’s with respect to the connectivity cocycle $\mu_0$ defined in the previous section. It is completely separated from the signature-extension. E.g., it does not lead to any extensions of the mapping class group. In particular, we have the property expressed in (2.21), and that $\mu_0$ is invariant under the natural $\Omega_4$ action.

Moreover, in the usual constructions of invariants, $R$ is assumed to be a field so that the only non-trivial, half-projective TQFT occurs when $x = 0$, (with the usual convention $0^0 = 1$). Let us, however, continue to assume more general $R$ in the following discussion, in order to give more insight into the underlying structures and show directions of possible, further generalizations.

Let us recall a general implication about the dimension of the $R$-modules associated to a surface that has already been observed by Witten [Wi] in the context of ordinary TQFT’s.
Lemma 3 Suppose $\mathbf{V}$ is a half-projective TQFT w.r.t. $x \in R$ and $\mu_0$. If $V_\Sigma = \mathbf{V}(\Sigma)$ is $R$-module associated to a surface $\Sigma$, then

$$\mathbf{V}(S^1 \times \Sigma) = x \dim(V_\Sigma).$$

Proof: For every connected surface $\Sigma$ let us fix an orientation reversing involution

$$\hat{\chi}_\Sigma \in \pi_0\left(\text{Diff}(\Sigma, \partial \Sigma)\right)_2.$$  

If we consider (disjoint unions of) the $\hat{\chi}_\Sigma$’s as boundary charts of the cylinder $\Sigma \times [0,1]$, we obtain morphisms

$$\chi^\dagger_\Sigma: \emptyset \to \Sigma \otimes \Sigma \quad \text{and} \quad \chi_\Sigma: \Sigma \otimes \Sigma \to \emptyset,$$  

(2.22)

that are inverses of each other, and hence define a rigidity structure on $\text{Cob}_3(*)$. With $\chi^2_\Sigma = 1$ they are also symmetric, in the sense that $\chi_\Sigma \circ \gamma = \chi_\Sigma$. Since $\mathbf{V}$ preserves also the symmetric tensor structure $\emptyset$ has to be associated to $R$ and $\gamma$ is mapped to the transposition of tensor factors. Thus, if we apply $\mathbf{V}$ to the $\chi^{(\dagger)}_\Sigma$ we obtain maps

$$\theta^\dagger_\Sigma: R \to V_\Sigma \otimes_R V_\Sigma \quad \text{and} \quad \theta_\Sigma: V_\Sigma \otimes_R V_\Sigma \to R,$$  

(2.23)

that are symmetric with respect to the ordinary transposition, and are inverses of each other. Since we assumed the $V_\Sigma$ to be free $R$-modules, we find by straightforward algebra that $\theta_\Sigma \theta^\dagger_\Sigma$ is the dimension of $V_\Sigma$. We also know that $S^1 \times \Sigma = \chi_\Sigma \chi^\dagger_\Sigma$. The anomaly of this product is $\mu_0 = 1$ so that we find the asserted formula. $\square$

From the above proof we see that, in fact, we do not have to assume that the $V_\Sigma$ are free-modules. The existence of the morphisms in (2.23) implies an isomorphism $V \cong \text{Hom}_R(V,R)$, which applied to $\theta^\dagger_\Sigma$ gives an element $\sum_{\nu} e_{\nu} \otimes l_{\nu} \in V \otimes_R \text{Hom}_R(V,R)$ that inverts the canonical pairing.

Suppose $X$ and $Z$ are $R$-modules and $f: V_\Sigma \to X$ and $p: Z \to X$ are $R$-morphisms, where $p$ is onto. For $x_{\nu} := f(e_{\nu})$ and $p(z_{\nu}) = x_{\nu}$, we can define a map $h: V_\Sigma \to Z$ by $h(v) = \sum z_{\nu} l_{\nu}(v)$ so that $f = p \circ h$. Hence $V_\Sigma$ is projective and therefore are a direct summand of a free $R$-module.

Note also, that if $R = R^0 \oplus R^1$ then we have a direct sum decomposition $V_\Sigma = V_\Sigma^0 \oplus V_\Sigma^1$ using the idempotents that are given by the units in the $R^j$. Moreover, we have $V \otimes_R W = V^1 \otimes_{R^1} W^1 \oplus V^0 \otimes_{R^0} W^0$, etc. In summary, we find the following:

Lemma 4 Suppose $\mathbf{V}$ is a TQFT into possibly non-free $R$-modules, and $R = \bigoplus_j R^j$, where $R^j$ are indecomposable.

Then $\mathbf{V} = \bigoplus_j \mathbf{V}^j$, where each $\mathbf{V}^j$ is a functor into the category of free $R^j$-modules.

A modification of the prerequisites that would be consistent with different values of $\mathbf{V}(S^1 \times \Sigma)$ and hence $x$, is to allow the symmetry-structure of $R$-mod to be different from that induced by $\mathbf{V}$. Thus the $\theta_{\Sigma}$ are now symmetric only up to isomorphism, i.e., we have $\theta_\Sigma T = \theta_{\Sigma}(\mathbb{1} \otimes P_\Sigma)$, where $T$ is the ordinary transposition and $P_\Sigma \in \text{Aut}_R(V_\Sigma)$. Instead of the dimension of $V_\Sigma$ we then obtain the trace over $P_\Sigma$, which may even be zero.

The induced symmetry structure yields in place of the transposition the morphisms, $\hat{\gamma}_\Sigma \in \text{End}(V_\Sigma \otimes V_\Sigma)$, which are the images of the $\gamma$ as in Lemma 3 with $\Sigma = \Sigma_j$. It is not hard to see that for $R = \mathbb{C}$ the structure is equivalent to the canonical one if and only if $tr(\hat{\gamma}_\Sigma) = \dim(V_\Sigma)$.

It is also apparent from Lemma 3 that we should not write the anomaly-term to the other side of the equation in Definition 4. For $h^{an} = x^{-1}$ this would imply that $h$ divides the dimensions of the
vector spaces for every genus. Under the usual assumption (see also V3 of Section 4.2) that the vector space of the sphere is $\mathbb{R}$, this would imply that $h$ is invertible.

The next lemma only uses the composition rule and applies also to the more general settings alluded to above:

**Lemma 5** Suppose $\mathbf{V}$ is a half-projective TQFT w.r.t. $x \in \mathbb{R}$ and $\mu_0$. For a connected surface $\Sigma$ of genus $g$ we then have:

$$\mathbf{V}(S^1 \times \Sigma) \in x^{\max(g,1)} \mathbb{R}.$$  

**Proof:** Consider the two-dimensional four-holed sphere as a 1+1-dimensional cobordism $H_4 : S^1 \sqcup S^1 \to S^1 \sqcup S^1$, and, further, let $\phi^\dagger : \emptyset \to S^1 \sqcup S^1$ and $\phi : S^1 \sqcup S^1 \to \emptyset$ be given by two-holed spheres. For $g \geq 1$ we have $\Sigma = \phi \circ (H_4)^{g-1} \circ \phi^\dagger$, and thus

$$S^1 \times \Sigma = (S^1 \times \phi) \circ (S^1 \times H_4)^{g-1} \circ (S^1 \times \phi^\dagger).$$

Here, every one of the $g$ compositions is over two tori in the boundaries of two connected cobordisms. Their anomalies are thus always $\mu_0 = 1$ and the assertion follows from the definition of a half-projective TQFT. \hfill $\square$

If we set $d_g = \text{dim}(V_{\Sigma_g})$, where $\Sigma$ has genus $g \geq 1$, the combination of Lemmas 3 and 5 yields that $d_gx \in x^g \mathbb{R}$. Suppose for some $g \geq 2$, we have already $x \in x^g \mathbb{R}$. Then there is $y \in \mathbb{R}$ with $x(1 - xy) = 0$. In particular, $e = xy$ is an idempotent in $\mathbb{R}$, which can be used to write $\mathbb{R}$ as a direct sum $e\mathbb{R} \oplus (1 - e)\mathbb{R}$. Now $x$ lies in the first summand, and $y$ is an inverse of this sub-ring with identity $e$. In summary, we have the following strong restriction on the element $x$ and the dimensions of the vector-spaces.

**Corollary 1** Suppose $\mathbf{V}$, $x$, and $d_g$ are as above. Then at least one of the following two has to be true:

1. The dimensions $d_g$ are zero-divisors in $\mathbb{R}/x^g\mathbb{R}$ for every $g \geq 2$, or
2. The ring is a direct sum $\mathbb{R} = R_1 \oplus R_0$, where the component of $x$ in $R_0$ is zero, and the component in $R_1$ is invertible (in $R_1$).

As in Lemma 4 the second possibility implies for the TQFT-functor that $\mathbf{V} = \mathbf{V}_1 \oplus \mathbf{V}_0$, and we have that $\mathbf{V}_1$ can be rescaled to an ordinary TQFT. The only non-trivial half-projective TQFT we can therefore get if the dimension condition fails to hold (and if we stay strictly within the framework of Definition 1) is one with $x = 0$.

### 2.4 TQFT’s for Cobordisms with Corners:

There are several ways of defining extended TQFT’s, which represent categories of cobordisms with corners, like $\text{Cob}_3(*)$. Most of them are consistent, although not always precisely equivalent. In this section we shall give a brief survey over the structures that are of interest to us.

To begin with the Kazhdan-Reshetikhin ladder is defined on a series of categories, $\text{Cob}_3(n) \subset \text{Cob}_3(n,n)$, for which all of the cylinders in (2.24) start at a hole in the source surface, and end at a hole in the target surface. The extended TQFT is then defined, for a given abelian category $\mathcal{C}$, as a series of functors

$$\mathbf{V}_n : \text{Cob}_3(n) \to \mathcal{C} \circ \ldots \circ \mathcal{C} \quad \text{n times}, \quad (2.24)$$

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where $\odot$ is Deligne’s tensor product of categories, see [D]. In particular, to a surface, $\Sigma$, with $\partial \Sigma = \sqcup^n S^1$, this associates an object $X_\Sigma \in C^{\odot n}$. We also require compatibility with the topological tensor product, i.e., $X_{\Sigma_1 \sqcup \Sigma_2} = X_{\Sigma_1} \odot X_{\Sigma_2}$. Moreover, cobordisms are mapped to morphisms in the respective category, $C^{\odot n}$.

Quite often it is more convenient to consider fiber functors that depend on a coloration. By this we mean an assignment of objects, $\{X_j : j = 1, \ldots, n\}$, to the special, cylindrical pieces in the boundary of a cobordism, $M : \Sigma_s \to \Sigma_t$, in $\text{Cob}_3(n)$. Instead of $f = \mathcal{V}_n(M) : X_{\Sigma_s} \to X_{\Sigma_t}$, we then consider the following linear spaces and maps:

$$
\mathcal{V}^{(\Sigma_1, \ldots, \Sigma_n)}_n : \text{Hom}_{C^{\odot n}}(X_1 \odot \ldots \odot X_n, X_{\Sigma_s}) \xrightarrow{\text{Hom}(\odot_j X_j, f)} \text{Hom}_{C^{\odot n}}(X_{\pi(1)} \odot \ldots \odot X_{\pi(n)}, X_{\Sigma_t}),
$$

(2.25)

where $\pi \in S_n$ is the permutation of the holes, defined by the cylinders in $\partial M$, with respect to standard orderings of the holes in $\Sigma_s$ and $\Sigma_t$.

Notice, that the maps in (2.25) fulfill an obvious naturality condition, w.r.t. any given object, $X_j$, which appears both in the source and the target linear space. Conversely, suppose any functor, $\mathcal{C} \to \mathbb{R} - \text{mod}$, of abelian categories (i.e., not necessarily tensor) is representable. Then any such system of maps with the naturality property stems from a functor like the one in (2.24). Examples of categories with representable fiber functors are all those, for which the coend $\mathcal{I} := \int X^\circ \odot X \in C^{\odot 2}$ exists. See [M], [L1], and [Ke3] for definitions.

If we distinguish between in- and out-holes among the boundary components of a surface, we can view the objects of $\text{Cob}_3(n)$ as 1+1-dimensional cobordisms themselves. Thus it is quite natural to define $\text{Cob}_{1+1+1} := \bigcup_{n_{\text{in}}, n_{\text{out}}} \text{Cob}_3(n_{\text{in}} + n_{\text{out}})$ as a 2-category, where the objects are one-manifolds, the 1-morphisms are 1+1-cobordisms, and the 2-morphisms are 2+1-cobordisms between them. An extended TQFT is now a 2-functor of 2-categories:

$$
\mathcal{V} : \text{Cob}_{1+1+1} \longrightarrow \text{AbCat}
$$

(2.26)

Here, $\text{AbCat}$ is the 2-tensor-category of abelian categories. I.e., the object associated to a one-fold, $S$, is as usual an abelian category, $\mathcal{C}^{(S)} = \mathcal{C}^{\odot \beta_0(S)}$, but to a surface we associate a functor between the category of the in-holes and the category of the out-holes. To a cobordism between two surfaces $\mathcal{V}$ then assigns a natural transformation between the respective functors. This picture may be extracted from the previous one, if we construct from an object $X_\Sigma \in C^{(\text{in})} \odot C^{(\text{out})}$ the functor

$$
\mathcal{F}_\Sigma : C^{(\text{in})} \longrightarrow C^{(\text{out})} : X \mapsto \text{Hom}_{C^{(\text{in})}}(X_\Sigma, X),
$$

and from a morphism $f : X_{\Sigma_1} \to X_{\Sigma_2}$ a transformation $\mathcal{F}_{\Sigma_2} \circ \mathcal{F}_{\Sigma_1}$ in the obvious way.

Furthermore, the 2-categorial description imposes more constraints on $\mathcal{V}$ than the Kazhdan-Reshetikhin picture, as $\mathcal{V}$ has to be compatible not only with compositions of 3-dimensional cobordisms but also with those of the 2-cobordisms.

The observant reader might have noticed that we have suppressed here the permutation that appears in (2.23). Indeed, for a precise description we need to consider a slightly more complicated structure for $\text{Cob}_{1+1+1}$ than that of a 2-category. Specifically, in the more general formalism the objects of two cobordant 1-morphisms are not simply the same but are connected by an arrow from a special category. Hence the 2-arrow-diagram of a 2-morphism is not simply given by a 2-gon, but
by a square as below:

\[
\begin{array}{ccc}
S_s^{(in)} & \xrightarrow{\Sigma_s} & S_s^{(out)} \\
\downarrow^\alpha & & \downarrow^\beta \\
S_t^{(in)} & \xrightarrow{\Sigma_s} & S_t^{(out)}
\end{array}
\]

(2.27)

The new vertical arrows are associated to the special cylindrical pieces of \(\partial M\), and are defined by the permutation they induce on the numbering of the holes, i.e., we have \(\alpha \in S_{n_{in}}\) and \(\beta \in S_{n_{out}}\).

Horizontal compositions are only allowed if the adjacent vertical morphisms are identical. For vertical (ordinary) compositions of 2-morphisms the special, vertical arrows are also multiplied.

The TQFT functor \(\mathbf{V}\) shall now assign to a vertical permutation in \(S_n\) the obvious functor on \(C^{\otimes n}\) that implements the respective permutation of tensor factors. A cobordism \(M\) is then mapped by \(\mathbf{V}\) to a natural transformation between the two composites of functors that start at the category of the upper, left corner of the square in (2.27) and end in the lower, right corner.

Recall, that in Section 2.1 we actually defined a more general class of cobordisms, for which the cylindrical pieces are allowed to run from a hole in a component of, e.g., the source surface, \(\Sigma_s\), to a hole in another component of \(\Sigma_s\).

A natural way to incorporate this possibility in our description is to enlarge the category, from which we may take the vertical arrows, from the symmetric groupoid to the category of singular tangles (i.e., strands for which an overcrossing can be changed to an undercrossing). A closed component of such a tangle, which in this category can be isolated as a circle, corresponds to an interior torus that can be added in a unique way as a closed component to either \(\Sigma_s\) or \(\Sigma_t\) as explained in Section 2.1.

In order to define \(\mathbf{V}\) on a singular tangle it suffices to give the action on a maximum or minimum:

\[
\begin{align*}
\mathbf{V}(\cup) : \ & C \otimes C \longrightarrow R - \text{mod} \\
\mathbf{V}(\cap) : \ & R - \text{mod} \longrightarrow C \otimes C
\end{align*}
\]

(2.28)

where \(\mathcal{I}\) is the coend as above. Here \(\boxtimes\) is a (braided) tensor product in \(C\). Note that (2.28) also implies \(\mathbf{V}(\Box) : R \mapsto \text{Inv}(\mathcal{F}),\) where \(F = \int X^Y \boxtimes X\).

As an alternative to this direct functorial description, we may consider also here the assignments of linear maps depending on a given coloration, as in (2.25). The difference is now that a cylindrical piece, which starts and end in the source surface results in a dependence of the source vector space on \(X_j^Y \otimes X_j\) (instead of only \(X_j\)) and no dependence of the target linear space. The first then full fills a di-naturality condition (instead of a naturality condition), which also plays an important role in liftings to the coend \(\mathcal{I}\).

The notion of a half-projective, extended TQFT is most conveniently defined for the version of functors into \(R - \text{mod}\) that depend on colorations. The generalization from Definition is then literally the same. It is also not hard to generalize the construction and discussion of half-projective TQFT for closed surfaces to the case of punctured surfaces, since all that needs to be checked in this picture is the preservation of naturality. We shall thus content ourselves in this article with a construction of half-projective functors \(\mathbf{V}_0 : \text{Cob}_3(0) \rightarrow R - \text{mod}\). Only in Section 4.3 will we return to the direct description of an extended TQFT in terms of functors as in (2.24).
3) Non-Separating Surfaces, Interior Fundamental Groups, and Coordinate-Graphs

In Lemma 5 we used a decomposition of a manifold to show that it is mapped by \( V \) to a multiple of \( x^\varrho \) for some \( \varrho \in \mathbb{Z}^{+0} \). In this chapter we wish to generalize this result, and identify for any type of cobordism, \( M \), orders in \( x \) for \( V(M) \), that are maximal for this decomposition argument. (It will turn out that the one in Lemma 5 is indeed maximal).

From the discussion in Section 2.3 it seems that \( x=0 \) is the only case we should be worried about, i.e., the only relevant question would be whether \( \varrho = 0 \) or not, but not the precise order for \( \varrho > 0 \).

Nevertheless, we shall stay within the more general framework, not only because of the possible modifications we outlined in Section 2.3, but also because of anticipated applications to “classical limits”, which we will sketch in Section 4.5.

In Section 3.1 we relate the orders, \( \varrho \), to the maximal number of non-separating surfaces in a cobordism. The subsequent sections are devoted to computing these numbers from the fundamental groups of \( M \) and \( \partial M \). Specifically, we will find in Section 3.5 that \( \varrho \) is the maximal rank of a free group, \( F \), for which there is an exact sequence of the form, \( \pi_1(\partial M) \rightarrow \pi_1(M) \rightarrow F \rightarrow 1 \). For a special case see Lemma 6.6 in [He]. As an application we find that a half-projective TQFT, \( V \), with \( x=0 \), vanishes on cobordisms, \( M \), with nontrivial “interior Betti-number”, i.e., \( V(M) = 0 \) if \( \beta_{1\text{int}}(M) \neq 0 \).

An important tool in this discussion are the coordinate graphs, which reduce the relevant connectivity information of cobordisms in \( \text{Cob}_3(*) \) to that of morphisms in a graph-category \( \Gamma \). A useful result, proven in Appendix A.1, is that decomposition along such graphs can also be realized as decompositions of the corresponding cobordisms.

3.1) \( r \)-Diagrams of Non-Separating Surfaces :

Let us begin with a definition the systems of non-separating surfaces we are interested in:

**Definition 2** For a manifold \( M \) with boundary an \( r \)-diagram is an embedding of \( r \) Riemann surfaces \( \Sigma_j \hookrightarrow M \) with \( j = 1, \ldots, r \), such that

1. the surfaces are disjoint from each other, i.e., \( \Sigma_i \cap \Sigma_j = \emptyset \) for \( i \neq j \),
2. they lie in the interior of \( M \), i.e., \( \partial M \cap \Sigma_j = \emptyset \),
3. the surfaces are closed so that the embeddings are proper,
4. every \( \Sigma_j \) is two-sided, and
5. their union is non-separating, i.e., \( M - \bigcup_{j=1}^r \Sigma_j \) is connected.

Let us reformulate the existence of an \( r \)-diagram for a cobordism, \( M : \Sigma_s \rightarrow \Sigma_t \), in \( \text{Cob}_3(0) \) in a more categorial language. If we remove thin, two-sided collars, \( \chi_j := \Sigma_j \times [-\varepsilon, \varepsilon] \), from \( M \), we obtain a manifold \( M^* \), which has \( 2r \) additional boundary components, \( \Sigma_j^\pm = \Sigma \times \{ \pm \varepsilon \} \). Hence, as a morphism in \( \text{Cob}_3(0) \), it can be written as follows:

\[
M^* : \Sigma_2 \longrightarrow \Sigma_t \otimes \Sigma_1^+ \otimes \Sigma_1^- \otimes \ldots \otimes \Sigma_r^+ \otimes \Sigma_r^-.
\]

In terms of the rigidity cobordisms from (2.22) the original manifold is given by

\[
M = \left( \mathbb{I}_{\Sigma_t} \otimes \chi_{\Sigma_1} \otimes \ldots \otimes \chi_{\Sigma_r} \right) \circ M^*. \tag{3.29}
\]
It is easy to see that the total anomaly of this product is $\mu_0 = r$.

Let us also introduce the quantity

$$\varrho(M) := \max\{r : M \text{ admits an } r\text{-diagram}\} \quad (3.30)$$

Note at this point that an $r$-diagram of some $M$ with $r < \varrho(M)$ can usually not be completed to a maximal diagram. An example is $S^1 \times \Sigma_g$. Here we can find a $g$-diagram from the lower curves of a Heegaard diagram on $\Sigma_g$, which contains one torus from every composition in the proof of Lemma 3.9. We will, however, see in the next section that after removing $\{1\} \times \Sigma_g$ we have for the complement $\varrho([0, 1] \times \Sigma_g) = 0$.

From (3.29) we find immediately for the following:

**Lemma 6** Suppose $\mathcal{V}$ is a half-projective TQFT w.r.t. $x \in \mathbb{R}$ and $\mu_0$.

Then we have for a cobordism $M : \Sigma_s \to \Sigma_t$ from $\text{Cob}_3(0)$,

$$\mathcal{V}(M) \in x^{\varrho(M)} \text{Hom}_R(\mathcal{V}_{\Sigma_s}, \mathcal{V}_{\Sigma_t}) \quad (3.31)$$

Here, $V_{\Sigma} = \mathcal{V}(\Sigma)$, and we mean $\bigcap_r x^r \text{Hom}(\ldots)$ if $\varrho(M) = \infty$.

The generalization of Lemma 3.31 to extended TQFT’s as in Section 2.4 also holds true, if we have $M \in \text{Cob}_3(\ast)$ and replace $\text{Hom}_R$ by $\text{Hom}_{C \otimes \mathbb{N}}$ on the objects assigned by $\mathcal{V}$ to the punctured source and target surfaces. For a generalization of the previous arguments to surfaces with punctures let us make the following observations:

The definition of an $r$-diagram of a cobordism in $\text{Cob}_3(\ast)$ is simply an $r$-diagram of $\phi_0(M)$, where $\phi_0$ is as in (2.11). In order to obtain the generalization of the composition in (3.29) we first have to make all the surfaces transversal to the external strands, $\tau = D^2 \times I$, that connects holes of the surfaces $\Sigma_s \sqcup \Sigma_t$ to each other. A given strand, $\tau$, is then divided by its intersections with the $\Sigma_j$ into several components. If one such piece, $\delta$, connects a surface $\Sigma$ to itself from the same side, we can surger $\Sigma$ along a slightly thickened $\delta$, and obtain a two-sided surface $\Sigma' = (\Sigma - D^2 \times S^0) \cup (S^1 \times I)$, which is also disjoint from the other surfaces and, together with them, is non-separating. We can thus assume that strands never connect a surfaces to itself from the same side.

By transversality we may also assume that an external strand, $\tau$, meets a collar, $\chi_j = \Sigma_j \times [-\varepsilon, \varepsilon]$, in a vertical, cylindrical piece, $D^2_r \times [-\varepsilon, \varepsilon]$, where $D^2_r \subset \Sigma_j$. Hence, if we remove all external strands, we have presented $M$ as the composite of the anti-morphisms $\chi_j' = \Sigma_j' \times [-\varepsilon, \varepsilon]$, where $\Sigma_j'$ is obtained from $\Sigma_j$ by removing the discs $D^2_r$, and an admissible, connected cobordisms $M^\ast$ in $\text{Cob}_3(\ast)$, as in (3.29).

### 3.2) Interior Fundamental Groups, and an A-Priori Estimate on $\varrho(M)$:

The maximal number $\varrho(M)$ of non-separating surfaces in a cobordism can be obtained from the fundamental groups of the cobordism $M$ and its boundary $\partial M$. It will be given by the maximal rank of a free group onto which the *interior* fundamental group factorizes. In this section we give the definition of $\pi^{ir}_1(M)$ and a first implication for $\varrho(M)$.

If $M : \Sigma_s \to \Sigma_t$ is a cobordism in $\text{Cob}_3(\ast)$ and $\Sigma_\nu$, with $\nu = 1, \ldots, K$ are the connected components of $-\Sigma_s \sqcup \Sigma_t$ we define the *interior* fundamental group, $\pi^{ir}_1(M)$ as the fundamental group of $M$ with the cones of the boundary components glued to it:

$$\pi^{ir}_1(M) = \pi_1(M \cup C\Sigma_1 \cup \ldots \cup C\Sigma_K) \quad (3.32)$$

The following shows that it is enough to consider cobordisms in $\text{Cob}_3(0)$.
Lemma 7 The inclusion $M \hookrightarrow \phi_0(M)$ induces an isomorphism

$$\pi_1^\text{int}(M) \cong \pi_1^\text{int}(\phi_0(M)).$$

Proof: Suppose a cobordism $M'$ is obtained by filling a tube in $M$ that connects a hole in the component $\Sigma_i$ to one in the component $\Sigma_j$. Since for $M \in \text{Cob}_3(\ast)$ we have $i \neq j$, the cones $C\Sigma_i$ and $C\Sigma_j$ are disjoint. Thus $M'$ with cone-attachments is obtained from $M$ with boundary cones by gluing in a ball along a sphere, which does not affect fundamental groups.

For the practical computation of $\pi_1^\text{int}(M)$, assume that we have marked points $p_0 \in M$ and $p_\nu \in \Sigma_\nu$. Let us call a spider, $\langle \gamma \rangle$, of $M$ a collection of paths $\gamma_\nu$ inside $M$, with $\nu = 1, \ldots, K$, that start at $p_0$ and end at $p_\nu$. Thus $X = \langle \gamma \rangle \cup C\Sigma_1 \cup \ldots \cup C\Sigma_K$ is a contractible space such that $M \cup X$ is the union of $M$ with its boundary cones as in (3.32), and $X \cap M \simeq \Sigma_1 \vee \ldots \vee \Sigma_K$. The interior group $\pi_1^\text{int}(M)$ is then given by Seifert-van Kampen (see, e.g., Theorem 7.40 in [Ro]) as the pushout of the respective fundamental groups, i.e., it is universal among the solutions, $\zeta$ and $G$, of the following diagram:

$$\begin{array}{ccc}
\text{free} \prod_{\nu=1}^K \pi_1(\Sigma_\nu, p_\nu) & \longrightarrow & 0 \\
I_* \downarrow & & \downarrow \\
\pi_1(M, p_0) & \xrightarrow{\zeta} & G
\end{array} \quad (3.33)$$

The image of $I_*$ in $\pi_1(M)$ generally depends on the choice of the spider, but their generators lie in the same conjugacy classes. Hence the smallest normal subgroup $N[\text{im}(I_*)] \subset \pi_1(M)$ that contains the image of $I_*$ does not depend on the choice of a spider. This yields the formula:

$$\pi_1^\text{int}(M) := \pi_1(M)/N[\text{im}(I_*)]. \quad (3.34)$$

We also introduce the notion of a free interior group, $F$, of $M$. By this we mean a solution to the diagram (3.33), where $F = G$ is a free group (non-abelian for rank $> 1$), and $\zeta$ an epimorphism. Clearly, by universality it may also be defined by the existence of an epimorphism

$$\tilde{\zeta} : \pi_1^\text{int}(M) \twoheadrightarrow F. \quad (3.35)$$

Let us denote by $F(k)$ the free group in $k$ generators. The role of the anomaly of Section 2.2 for internal fundamental groups can be described as follows:

Lemma 8 Suppose $M$ and $N$ are cobordisms with anomaly $\mu_0 = \mu_0(M, N)$. Then there is an epimorphism $\xi$, such that the following diagram commutes:

$$\begin{array}{ccc}
\pi_1^\text{int}(M \circ N) & \xrightarrow{\xi} & \pi_1^\text{int}(M) * \pi_1^\text{int}(N) * F(\mu_0) \\
\uparrow & & \uparrow \\
\pi_1(M \circ N) & \leftarrow & \pi_1(M) * \pi_1(N) * F(\mu_0)
\end{array} \quad (3.36)$$
Proof: It is enough to consider only connected cobordisms $M$ and $N$ that are connected over $\mu_0 + 1$ surfaces. Moreover, the assertion is the same, if we glue in the cones for the remaining boundary components of $M$ and $N$, i.e., we may assume that $N : \emptyset \to \Sigma_0 \sqcup \ldots \sqcup \Sigma_{\mu_0}$, and $M$ is a cobordism in reverse direction.

Fix a point $x_0 \in \Sigma_0$ and choose spiders $\langle \gamma^M \rangle$ and $\langle \gamma^N \rangle$ of $M$ and $N$ respectively that originate in $x_0$. Hence a leg, $\gamma_j^M$, is a path in $M$ that connects $x_0 \in \Sigma_0$ to a point $x_j \in \Sigma_j$.

For $\tilde{N} = N \cup \langle \gamma^M \rangle$ we have a natural isomorphism

$$\pi_1(N,x_0) \ast F(\mu_0) \cong \pi_1(\tilde{N},x_0),$$

in which the $j$-th free generator $a_j$ of $F(\mu_0)$ is mapped to $(\gamma_j^N)^{-1}\gamma_j^M$.

Since $M \circ N = M \cup \tilde{N}$ and $M \cap \tilde{N} = \langle \gamma^M \rangle \cup \Sigma_0 \cup \ldots \cup \Sigma_{\mu_0}$, the group $\pi_1(M \circ N) = \pi_1^{\text{int}}(M \circ N)$ can be computed as the push-out of the following diagram:

$$\begin{array}{ccc}
\pi_1(M) & \longrightarrow & \pi_1(N) \ast F(\mu_0) \\
I^*_1 & \uparrow & \pi_1^{\text{int}}(M) \\
\prod_{1}^{\mu_0} \pi_1(\Sigma_j,x_j) & \overset{\tilde{I}^*_N}{\longrightarrow} & \pi_1^{\text{int}}(N) \ast F(\mu_0)
\end{array}$$

(3.37)

where $\tilde{I}^*_N(g) = a_j^{-1}I^*_N(g)a_j$ if $g \in \pi_1(\Sigma_j,x_j)$, and the $I^*_M$ are defined as in (3.33).

Clearly, $\pi_1^{\text{int}}(M) \ast \pi_1^{\text{int}}(N) \ast F(\mu_0)$ is a solution of (3.37), since $\text{im}[\tilde{I}^*_N]$ is also in the kernel for the map onto $\pi_1^{\text{int}}(N) \ast F(\mu_0)$. Hence a surjection $\xi$ exists.

The remainder of the diagram, expressing that $\xi$ is defined naturally, can be completes in the obvious way. In the lower horizontal morphism the free generator $a_j \in F(\mu_0)$ is mapped to the closed path $(\gamma_j^N)^{-1}\gamma_j^M$ in the composite $M \circ N$. \hfill \Box

In analogy to $\varrho(M)$ from (3.30) let us define the maximal rank of a free interior group:

$$\varphi(M) := \max\{\mu : M \text{ has } F(\mu) \text{ as free interior group}\}.$$  

(3.38)

The following are easily found from Lemma 8 and (3.29):

Corollary 2

1. $\varphi(M \circ N) \geq \varphi(M) + \varphi(N) + \mu_0(M,N)$.

2. $\varphi(M) \geq \varrho(M)$.

For $\varphi(M)$ and $\varphi(N)$ the first part implies the existence of an other trivial cocycle with values in the non-negative integers, given by:

$$\mu_\pi := -\delta \varphi - \mu_0$$  

(3.39)

Its computation can be quite intricate, and shall not be attempted here. Instead we shall give the analogous computation for homology in Appendix A.2.

The second part of the corollary implies for one that

$$\varrho(M) \leq \varphi(M) < \infty,$$
since the fundamental group of a compact manifold is finitely generated. This also renders the convention made at the end of Lemma 5 superfluous. Another consequence is that with \( \pi_1(\Sigma) \rightarrow \pi_1(M) \) being onto for invertible cobordisms, we have

\[
\varphi(G) = \rho(G) = 0 \quad \text{if} \quad G \in \pi_0\left(\text{Diff}(\Sigma)^+\right),
\]

which was used in the counter-example in Section 3.1. In the remaining sections of this chapter we shall see that in fact \( \varphi \equiv \rho \).

3.3) The Graph-Category \( \Gamma \):

In this section we shall define a category of graphs. It will be used to encode the basic connectivity properties of \( \text{Cob}_3(0) \).

To begin with let us fix a label set, \( S_l \), that is in one-to-one correspondence with the Riemann surfaces, used as objects for \( \text{Cob}_3(0) \). The objects of the category \( \Gamma \) are then given by strings of (possibly repeated) labels, \([a_1, \ldots, a_K]\), \( a_j \in S_l \).

The morphisms, \( \gamma : [a_1, \ldots, a_K] \rightarrow [b_1, \ldots, b_L] \), are given by one-dimensional cell-complexes, \( \gamma \), taken up to homotopy type, for which \( \partial \gamma \) contains \( K + L \) special points, that are labeled by \( a_1, \ldots, b_L \).

The compositions is, as for cobordisms, given by gluings along the respective boundary components, i.e., end-points with the same labels.

A representing cell-complex can be visualized by a graph, with \( K + L \) distinguished vertices of edge-degree one. A generic example of a representing graph is depicted below. In this form the composition of two graphs is defined by placing them on top of each other.

\[
(3.41)
\]

In analogy to Lemma 9 we also have a natural symmetric tensor structure on \( \Gamma \):

Lemma 9

1. \( \Gamma \) is a strict, symmetric tensor category.

2. For any \( \pi \in S_K \), there is a morphism, \( \pi^* : [a_1, \ldots, a_K] \rightarrow [a_{\pi^{-1}(1)}, \ldots, a_{\pi^{-1}(K)}] \). They are natural in \( \Gamma \) and give rise to a representation of \( S_K \). Any invertible morphisms of \( \Gamma \) is of this form.

3. For every morphism there is a unique maximal number, \( b \), such that there is a decomposition in the form \( \gamma = \pi^+_1 \circ \left( \gamma_1 \otimes \ldots \otimes \gamma_b \right) \circ \pi^+_2 \). A representing graph of a component, \( \gamma_j \), is connected.

The tensor product of \( \Gamma \) is given by the juxtapositions of both labels and graphs. The permutations are given by joining a labels \( a_j \) and \( a_{\pi^{-1}(j)} \) on top and bottom by straight lines. Using the triangle identity, this also defines the commutativity constraint \( \gamma \) on strings of arbitrary length, as the obvious crossing of two sets of parallel strands.

A graph, \( \gamma \), that is invertible cannot connect two different source (or target) labels to each other, and cannot contain internal loops. It follows for \( \Gamma \), that up to homotopy the permutations are the only
possibilities. A similar statement hold true also for $Cob_b(\ast)$, if we include the action of the mapping class groups on the surfaces.

A decomposition as in the last part of Lemma 1 is given for the $k = 5$ component graph in (3.41), by the permutations $\pi_1 = (3, 4)$ and $\pi_2 = (2, 3)$, and the connected morphisms $\gamma_1 : [a_1, a_3] \to [b_1]$, $\gamma_2 : [a_2] \to [b_2, b_4]$, $\gamma_3 : [a_4] \to [b_3]$, $\gamma_4 : [a_5, a_6] \to [0]$, and $\gamma_5 : [0] \to [b_5]$.

As opposed to $Cob_b(\ast)$ it is easy to give a list of the homotopy-inequivalent, connected graphs, thus giving a complete description of the category.

The class of a connected morphism, $\gamma$, is clearly determined by the number of source labels, $K$, the number of target labels, $L$, and the first Betti number $\beta_1(\gamma) = \dim(H_1(\gamma))$. Canonical representatives can be found by shrinking all of the internal edges, until we have at most one internal vertex. The results are given in the next diagram.

$$K + L \geq 3 \quad \text{or} \quad \beta_1(\gamma) \neq 0 \quad K + L = 2, \beta_1(\gamma) = 0 \quad K + L = 1, \beta_1(\gamma) = 0$$

Examples for the first type of graphs, which contain an internal vertex, are $\gamma_1$, $\gamma_2$, and $\gamma_3$ with $\beta_1 = 2$, $\beta_2 = 0$, and $\beta_3 = 1$, respectively. The graph $\gamma_3$ is of the second type with only one edge, and $\gamma_5$ is represented by simply one external vertex without edges. Although we will not always need the graphs to be one of the representatives, we shall always assume below that we have no internal vertices of valency one (as, e.g., $\gamma_2$). Hence we have $\partial \gamma = \partial \gamma_s \cup \partial \gamma_t$, with $|\partial \gamma_s| = K$ and $|\partial \gamma_t| = L$.

As for the cobordisms in (2.14) we also have an anomaly for the Betti-numbers of graphs:

$$\beta_1(\gamma_2 \circ \gamma_1) = \beta_1(\gamma_1) + \beta_1(\gamma_2) + \mu_0(\gamma_2, \gamma_1) \quad .$$

(3.43)

Here, $\mu_0$ is defined exactly as in Lemma 27, where the $W_j$ are now given with respect to the inclusions $\partial \gamma_{s,t} \hookrightarrow \gamma$. There is no $\mu_0$-contribution, and $\beta^1_{\text{int}} = \beta_1$ for graphs, since $H_1(\partial \gamma) = 0$. For the composite of two connected graphs over $k \geq 1$ end-points, we obtain as in (2.14) that $\mu_0 = k - 1$.

It will be convenient to introduce a natural partial order on the morphisms of $\Gamma$. For two graphs, $\gamma_1$ and $\gamma_2$, we say that

$$\gamma_1 \prec \gamma_2 \quad ,$$

(3.44)

iff the $\gamma_j$ belong to the same morphism set, and there is an embedding $\gamma_1 \hookrightarrow \gamma_2$ of some representatives, such that the corresponding maps $H_0(\gamma_1) \to H_0(\gamma_2)$ and $H_1(\gamma_1) \to H_1(\gamma_2)$ are isomorphisms and a monomorphism, respectively. It is clear that $\gamma_2$ is obtained by adding internal edges to a given component of $\gamma_1$. I.e., for representatives as in (3.42), $\gamma_2$ differs from $\gamma_1$ only by adding loops to the internal vertices. An inequality as in (3.44) obviously also implies $\beta_1(\gamma_1) \prec \beta_1(\gamma_2)$, $\gamma_1 \circ \gamma \prec \gamma_2 \circ \gamma$, as well as $\gamma_1 \otimes \gamma \prec \gamma_2 \otimes \gamma$.

3.4) Coordinate Graphs of Cobordisms:

The relation between the categories $Cob_b(\ast)$ and $\Gamma$ cannot be given precisely by a functor because the composition anomaly, $\mu_0 + \mu_\ast$, of $Cob_b(\ast)$ is greater than the anomaly $\mu_0$ of $\Gamma$. We shall, however, attempt to relate the morphisms of the two categories in a way that we can conclude from the
decomposition of a graph also the decomposition of a cobordism. We begin with the definition of the notion of a (faithful) coordinate graph, which will be our principal tool in the description of the connectivity of cobordisms.

**Definition 3** A coordinate-graph of a cobordism $M : \Sigma_s \to \Sigma_t$, is a graph $\gamma \in \Gamma$, together with a continuous function

$$f : (M, \Sigma_s, \Sigma_t) \longrightarrow (\gamma, \partial\gamma_s, \partial\gamma_s),$$

such that we have a one-to-one correspondence between boundary components, (i.e., the induced maps $H_0(\Sigma_s/t) \cong H_0(\partial\gamma_s/t)$ are isomorphisms), and the interior of $M$ is also mapped to the interior of $\gamma$.

We say that $f : M \to \gamma$ is a faithful coordinate graph, if there is an embedding:

$$J : (\gamma, \partial\gamma) \hookrightarrow (M, \partial M),$$

which maps again connected components to each other, and for which the composite $f \circ J$ is homotopic to the identity (with fixed endpoints).

An immediate consequence of the correspondence of boundaries is that we can always write the morphisms of a coordinate graph as unions $M = M' \sqcup N$ and $\gamma = \gamma' \sqcup \kappa$, where $N$ and $\kappa$ have no boundaries, and the coordinate map $f$ induces an isomorphism

$$H_0(M') \cong H_0(\gamma') .$$

For this reason we shall often consider only the case of connected cobordisms with connected coordinate graphs.

Next, let us state some obvious facts about the composition and collapse of coordinate graphs:

**Lemma 10**

1. If $M_1$ and $M_2$ have (faithful) coordinate graphs $\gamma_1$ and $\gamma_2$, respectively, then $\gamma_2 \circ \gamma_1$ is a (faithful) coordinate graph of $M_2 \circ M_1$.

2. Suppose $\gamma = \gamma_2 \circ \gamma_1$, where the $\gamma_j$’s are coordinate graphs of the $M_j$’s as above, is a maximal (faithful) coordinate graph of $M = M_2 \circ M_1$. Then $\gamma_j$ has to be a maximal (faithful) coordinate graph of $M_j$, for both $j = 1, 2$.

3. If $\gamma$ is a coordinate graph of $M$ and $\xi : \gamma \to \gamma'$ a continuous map that preserves endpoints, then $\gamma'$ is also a coordinate graph of $M$.

    If $\gamma$ is in addition faithful and there is an inclusion $\gamma \hookrightarrow \gamma$, whose composition with $\xi$ is homotopic to the identity on $\gamma'$, then $\gamma'$ is also faithful.

Note that the converse of Part 2 is not true. A typical application of the observation in part 3.) is given when $\gamma' \preceq \gamma$ is a subgraph, missing one edge of $\gamma$, and the map $\gamma \to \gamma'$ given by collapsing the additional edge into another path in $\gamma'$, as for example in the following picture:

$$\begin{array}{c}
\gamma \quad \xrightarrow{\xi} \quad \gamma' \\
\gamma' \quad \xrightarrow{\xi} \quad \gamma \\
\end{array}$$

(3.46)

If $\gamma$ is as in (3.42) and $\gamma' \preceq \gamma$ is of the same form with $\beta_1(\gamma) - k$ inner loops, then $\xi$ may be defined by collapsing the $k$ outer loops of $\gamma$ to the interior vertex.

Next, we assure the existence of (maximal and minimal) coordinate graphs.
Lemma 11

1. Every cobordism, $M$, admits a (faithful) coordinate graph, $\gamma^M$, which is minimal among all (faithful) coordinate graphs.

2. Every cobordism has a maximal, faithful coordinate graph.

Proof: For a connected cobordism, $M$, we can choose $\gamma^M$, to be the spider with $\beta_1(\gamma^M) = 0$ as in Section 3.2. To define the coordinate map choose a map $g : \Sigma = \Sigma_s \cup \Sigma_t \to \partial \gamma$, which maps different components to different points. Let $p : M \to v$ be the constant map to a point $v$. We set

$$f : M \cong M \cup_{\Sigma \times \{0\}} (\Sigma \times [0, 1]) \xrightarrow{p \cup (g \times id_{[0,1]})} v \cup_{\sim} (\partial \gamma^M \times [0, 1]) \cong \gamma^M,$$

where we identified $\partial \gamma^M \times \{0\} \sim v$.

In the proof of the second part it is clear that a faithful coordinate graph with $\beta_1(\gamma) = k$, implies a surjection $H_1(M) \to \mathbb{Z}^k$. For a compact $M$ we know, e.g., from a Heegaard decomposition, that $H_1(M)$ is finitely generated so that $N$ must be bounded. \hfill $\square$

In Definition 3 we assumed that for a faithful coordinate graph, $h = f \circ J$ is only homotopic to the identity. The next lemma asserts that we may assume that in this case $h$ is also equal to the identity.

Lemma 12 Suppose $f : M \to \gamma$ is a generic faithful coordinate graph with embedding $J : \gamma \looparrowright M$. Then there exists $f^\delta : M \to \gamma$, such that

$$f^\delta \circ J = id,$$

and $f^\delta$ coincides with $f$ outside a neighborhood of $J(\gamma)$.

The proof, although fairly standard, is rather technical and is thus deferred to Appendix A.1.

An application lies in the proof of the following lemma, asserting that if a coordinate graph is a composite, so is the associated cobordism.

Lemma 13 Let $f : M \to \gamma$ be a generic coordinate graph of a connected cobordism, $M$, and $\gamma = \gamma_2 \circ \gamma_1$ a decomposition in $\Gamma$.

1. There is a graph, $\hat{\gamma} \succ \gamma$, with a collapse map, $c : \hat{\gamma} \to \gamma$, such that $\hat{\gamma} = \gamma_2 \circ \hat{\gamma}_1$, and $c(\hat{\gamma}_j) = \gamma_j$. Moreover, $\hat{\gamma}$ has the property that there exists a coordinate map $\hat{f} : M \to \hat{\gamma}$ - with $f = c \circ \hat{f}$ - such that the $M_j := \hat{f}^{-1}(\hat{\gamma}_j)$ are cobordisms with graphs $\hat{\gamma}_j$ and $M = M_2 \circ M_1$.

2. If a composed coordinate graph, $\gamma = \gamma_2 \circ \gamma_1$, is faithful and $\gamma_2$ and $\gamma_1$ are connected, then there exists a coordinate graph, $\hat{\gamma} \succ \gamma$, and a map, $\hat{f}$, as above (except that $f$ may be different from $c \circ \hat{f}$), such that the $\hat{\gamma}_j$ are also faithful, the $M_j$ are connected, and the embedding of $\hat{\gamma}_j$ extends that of $\gamma_j$.

Proof: We shall prove here only the first part of the lemma. Due to its technical nature the proof of the second statement is again deferred to Appendix A.1. It relies heavily on Lemma 12.

By genericity we may assume that $f$ is differentiable, and that $P = \gamma_2 \cap \gamma_1$ consists only of regular values of $f$. For a point, $p \in P$, we can find a neighborhood $(I_p, p) \cong ([-\varepsilon, \varepsilon], 0)$, such that

$$f^{-1}(I_p) = \Sigma_1^p \times [-\varepsilon, \varepsilon] \cup \ldots \cup \Sigma_m^p \times [-\varepsilon, \varepsilon],$$
where each $\Sigma^p_j$ is a connected Riemann surface, and $f$ acts on this space as the projection on the interval $[-\varepsilon, \varepsilon]$.

In order to obtain a coordinate-graph, for each $p \in P$ we insert into $\gamma$ $n_p - 1$ additional edges, $e^p_j \cong [-\varepsilon, \varepsilon]$, identifying their boundary points, $\{\pm \varepsilon\}$, with those in $\partial I_p$. For the resulting graph $\hat{\gamma}$ we also have a coordinate map $\hat{f} : M \to \hat{\gamma}$, which maps $\Sigma^p_j \times [-\varepsilon, \varepsilon] \to e^p_j$ for $j = 1, \ldots, n_p - 1$, through a projection onto $[-\varepsilon, \varepsilon]$, and which coincides with $f$ outside these regions. It is clear that $\hat{f}$ is continuous, and if we define a collapse $c : \hat{\gamma} \to \gamma$ by mapping $e^p_j$ to $I_p$ with fixed endpoints, then $c \circ \hat{f} = f$.

Now, $M_j = f^{-1}(\gamma_j)$, for $j = 1, 2$, are cobordisms with $M = M_2 \circ M_1$, where we glue over the union of all $\Sigma^p_j$. For the subgraphs $\hat{\gamma}_j = \hat{f}(M_j) \subset \hat{\gamma}$ we have of course $\hat{\gamma} = \hat{\gamma}_2 \circ \hat{\gamma}_1$, and a one-to-one correspondence between the set of boundary points of, e.g., $\hat{\gamma}_1$ and the set of $\Sigma^p_j$'s, since $(\partial \hat{\gamma}_1)_t$ contains in addition to the points of $(\partial \gamma)_t$ the interior points $0 \in e^p_j$. The endpoints of $\hat{\gamma}_1$ hence correspond to the boundary components of $M_1$ so that $\hat{f} : M_1 \to \hat{\gamma}_1$ is a coordinate graph.

The following are useful applications of the decomposition along a graph:

**Corollary 3**

1. Every cobordism is given by the composite of cobordisms, whose maximal faithful coordinate graphs are spiders (with at most three end-points).

2. If $M$ has a faithful coordinate graph with $\beta_1(\gamma) = r$, then $M$ admits an $r$-diagram (see Section 3.1).

**Proof:** Pick a maximal (see Lemma [11]) faithful coordinate-graph $\gamma$ of $M$, and write it as a composite of two trees over $\beta_1(\gamma) + 1$ points as in the diagram to the right.

Since $\gamma$ is already maximal we only have to vary the coordinate map in order find the corresponding decomposition for the cobordisms according to Part 2) of Lemma [13]. By Lemma [11] the sub-graphs are also maximal.

The inner vertex of each tree can of course be resolved, such that we obtain a homotopic tree with vertices that have valencies of at most three. Repeated application of Part 2) of Lemma [13] yields then the decomposition into elementary pieces.

If we use the decomposition over $r + 1$ points as above and reconnect the two cobordisms over only one surface we obtain a connected manifold. Hence the remaining surfaces form an $r$-diagram in $M$.

□

**3.5) Existence of Coordinate Graphs from Interior Groups**

The existence of faithful coordinate graphs and - hence $r$-diagrams - can be derived from projections onto the fundamental groups of graphs. The first observation regarding this connection follows immediately by picking transversal, closed paths in $M$, that represent preimages of the generators of $\pi_1(\gamma)$:
Lemma 14 Suppose $f : M \to \gamma$ is a coordinate graph of connected $M$, and
\[ \pi_1(f) : \pi_1(M) \xrightarrow{\cong} \pi_1(\gamma) \]
is onto. Then $\gamma$ is also a faithful coordinate graph.

Note, that since $f$ is constant on the boundaries, $\pi_1(f)$, also factors through $\pi_1(M) \to \pi_1^{\text{int}}(M)$. Hence, with $\pi_1(\gamma) = F(\beta_1(\gamma))$ we have that $\pi_1(\gamma)$ is in fact a free interior group in the sense of (3.35) of Section 3.2. Next, we show that, conversely, a free interior group also implies the existence of a coordinate graph.

Lemma 15 Suppose that for connected $M$
\[ \zeta : \pi_1^{\text{int}}(M) \xrightarrow{\cong} F(k) \cong \pi_1(\gamma) \]
is a free interior group, with $k = \beta_1(\gamma)$.

Then there exists a faithful coordinate graph $f : M \to \gamma$, such that $\zeta = \pi_1(f)$.

Proof: Assume that $\gamma$ is as in diagram (3.42), and let $\gamma^k \subset \gamma$ be the bouquet of circles without the exterior edges so that $\pi_1(\gamma) = \pi_1(\gamma^k)$. Since $\pi_j(\gamma^k) = 0$ for $j \geq 2$ it follows, e.g., from Theorem 6.39.ii) in [Sz], that there is a continuous map $f^k : M \to \gamma^k$, which induces the map $\pi_1(M) \to \pi_1^{\text{int}}(M) \xrightarrow{\zeta} \gamma^k$. By construction we have $\pi(f^k) = 0$ for the restriction $f^k : \partial M \to \gamma^k$, which in turn implies that $f^k$ is homotopic to the constant map $\partial M \to \{v\}$, where $v \in \gamma^k$ is the interior vertex of the bouquet. Let $F^k_\partial : \partial M \times [0,1] \to \gamma^k$ be a corresponding homotopy, with $F^k_\partial(x,0) = f^k(x)$ and $F^k_\partial(x,1) = v$. Given $\psi : \partial M \times [0,1] \cup_{\partial M \times \{0\}} M \cong M$, whose restriction to $M$ is isotopic to the identity, we can define a function:
\[ f^c = (F^k_\partial \cup f^k) \circ \psi^{-1} : (M, \partial M) \to (\gamma^k, v). \]
As in the proof of Part 1) of Lemma 14 we can define from this (using again $\psi$) a coordinate map $f : M \to \gamma$, with $\pi_1(f) = \pi_1(f^c)$. Together with Lemma 14 this implies the assertion. \hfill $\square$

Let us summarize in the next theorem the results of Lemma 8, Corollary 3, and Lemma 15:

Theorem 4 Suppose $M$ is a connected cobordism. Then the following are equivalent:

1. $M$ admits an $r$-diagram.
2. $M$ admits a free interior group of rank $r$.
3. $M$ has a faithful coordinate graph, $\gamma$, with $\beta_1(\gamma) = r$.

In particular Theorem 4 implies the converse inequality of Part 2) of Corollary 2. We find
\[ \varrho(M) = \varphi(M), \]
i.e., the maximal number of non-separating surfaces is given by the rank of the maximal free interior group of a cobordism. It is thus possible to compute the order in $x$ of a linear map associated to a cobordisms, $M$, by a half-projective TQFT as it appears in Lemma 6 using only the fundamental group of $M$ and $\partial M$.

If $x = 0$ it suffices to consider only homology, since we only have to know whether there is a non-trivial interior group or not. More precisely, with $H_1^{\text{int}}$ and $\beta^{\text{int}}$ as defined in Appendix A.2, we have:
Lemma 16 For a connected cobordism \( M \in \text{Cob}_3(0) \),

\[ \beta = 0 \quad \text{if and only if} \quad g(M) = 0. \]

Proof: By naturality of the Hurewicz map the first square of the following diagram commutes.

\[ \begin{array}{ccc}
\pi_1(\partial M) & \longrightarrow & \pi_1(M) \\
\downarrow & & \downarrow \\
H_1(\partial M, \mathbb{Z}) & \longrightarrow & H_1(M, \mathbb{Z}) \\
\alpha & & \end{array} \]

Since the lower sequence is exact, and the Hurewicz maps are surjective, we can infer the existence of a surjection \( \alpha \) of the interior groups, such that all of (3.48) commutes. From \( H_1^\text{int}(M, \mathbb{Q}) \neq 0 \) we know that there is an epimorphism \( H_1^\text{int}(M, \mathbb{Z}) \longrightarrow \mathbb{Z} \), which, composed with \( \alpha \), gives rise to a free interior group of rank one.

Conversely, if there is an epimorphism \( \pi_1(M) \longrightarrow \mathbb{Z} \), the corresponding map \( \pi_1(M) \longrightarrow \mathbb{Z} \) has the commutator sub-group in its kernel and thus factors into homology. Since the kernel also contains the image of \( \pi_1(\partial M) \), it follows from diagram (3.48) that the epimorphism on homology factors through \( H_1^\text{int} \) so that \( \beta \geq 1. \)

We immediately find from this and Lemma 6 the following.

Corollary 5 Suppose \( \mathcal{V} \) is a half-projective TQFT w.r.t. \( x = 0 \) and \( \mu_0 \), and let \( M \in \text{Cob}_3(0) \) is a cobordism with \( \beta = 0 \). Then

\[ \mathcal{V}(M) = 0. \]

In the special case of the “Hennings-invariant” for \( U_q(sl_2) \), with \( q \) at a root of unity, this vanishing property was observed by Ohtsuki, see [O]. The result there, however, is found more-or-less from a direct computation of the invariant.

From the discussion in Chapter 2 we have seen that \( x = 0 \) and invertible \( x \) are probably the only possibilities so that - from an algebraic point of view - we only have to worry whether there are non-trivial interior groups as, e.g., in Lemma 4, but not about the exact order, as suggested in Lemma 6. Still, as we shall discuss in Section 4.5, the precise order can be of interest, if we consider “classical limits” of TQFT’s. In this case we have \( x \rightarrow \infty \), and \( g(M) \) may yield an estimate on the order in \( x \), by which \( \|\mathcal{V}(M)\| \) diverges.

4) Construction of Half-Projective TQFT’s

The aim of this chapter is to show how half-projective TQFT’s can be constructed from connected ones, as for example those found in [KL]. We shall organize our discussions in a deductive way, including additional assumptions only where needed. Specifically, we shall begin in Section 4.2 by introducing a set of Axioms, V1-V5, on a map \( \mathcal{V} : \text{Cob}_3(0) \rightarrow \mathbb{R} - \text{mod} \), extract from this a list of properties, P1-P8, that ensure the existence of such a map, and conclude that \( \mathcal{V} \) necessarily has to be a half-projective TQFT. In Section 4.3 we show that the existence of extended structures implies all but one of the properties automatically. The missing Property P7 on the projectivity \( x \) is discussed in Section 4.4 as a consequence of the closely related concepts of cointegrals, semisimplicity, and the invariant on \( S^1 \times S^2 \). In the discussions of these sections we attempt to give a clear picture of how the given assumptions influence the existence and uniqueness of the properties we derive for \( \mathcal{V} \). The last section
then summarizes the possible diversions from our axioms that might lead to more general definitions of $\mathbf{V}$, in particular the tensor product rule. We also discuss a possible application of the formalism of half-projective TQFT’s to the study of “classical limits”, where the exact orders $\varrho(M) = \varphi(M)$ become relevant.

4.1) Surface-Connecting Cobordisms :

For two closed, connected Riemann surfaces $\Sigma_j$, with $j = 1, 2$, we can think of their connected sum $\Sigma_1 \# \Sigma_2$ as being the result of a 1-surgery on $\Sigma_1 \sqcup \Sigma_2$, i.e., we cut away a disc from each surface and reglue the cylinder $S^1 \times I$ along the boundaries. The corresponding morphism $\Pi : \Sigma_1 \sqcup \Sigma_2 \to \Sigma_1 \# \Sigma_2$ is constructed by attaching a one-handle to the cylinder over $\Sigma_1 \sqcup \Sigma_2$, i.e.,

$$\Pi = \Sigma_1 \times [0, 1] \sqcup_{D^2 \times \{1\}} D^2 \times I \sqcup_{D^2 \times \{1\}} \Sigma_2 \times [0, 1],$$

or, equivalently, a boundary-connected sum of the $\Sigma_j \times [0, 1]$. More generally, we obtain for every surface $\Sigma$ with $K$ ordered connected components $\Sigma_j$ a morphism

$$\Pi_{\Sigma} : \Sigma = \Sigma_1 \sqcup \ldots \sqcup \Sigma_K \longrightarrow \Sigma^\# := \Sigma_1 \# \ldots \# \Sigma_K.$$

The obvious associativity condition for these morphisms is readily verified. Changing orientations we obtain a cobordism in the opposite direction:

$$\Pi^\dagger_{\Sigma} = -\Pi_{\Sigma} : \Sigma^\# = \Sigma_1 \# \ldots \# \Sigma_K \longrightarrow \Sigma = \Sigma_1 \sqcup \ldots \sqcup \Sigma_K.$$

In the next lemma we evaluate the composites of $\Pi_{\Sigma}$ and $\Pi^\dagger_{\Sigma}$:

**Lemma 17** Suppose $\Sigma$ is a closed Riemann surface with $K$ components, $\Sigma_j$, and the cobordisms $\Pi_{\Sigma}$ and $\Pi^\dagger_{\Sigma}$ are as above. Then we have

1. $\Pi^\dagger_{\Sigma} \circ \Pi_{\Sigma} = \mathbb{I}_{\Sigma_{1}} \# \ldots \# \mathbb{I}_{\Sigma_{K}},$

   i.e., the (interior) connected sum of the cylinders $\Sigma_j \times [0, 1]$.

2. For

   $$\Lambda_{\Sigma} := \Pi_{\Sigma} \circ \Pi^\dagger_{\Sigma} : \Sigma^\# \longrightarrow \Sigma^\#.$$

   we have

   $$\Lambda_{\Sigma} \circ \Pi_{\Sigma} = \Pi_{\Sigma} \# (S^1 \times S^2)^\# \ldots \# (S^1 \times S^2)^{K-1 \text{ times}}$$

   and the analogous equation for $\Pi^\dagger_{\Sigma} \circ \Lambda_{\Sigma}$. ($\#$ is the interior connected sum of 3-folds).

**Proof:** For the proof of the first part it is enough to consider only the case $N = 2$. If we think of $\Pi = \Sigma_1 \times [0, 1] \sqcup_{D^2} \Sigma_2 \times [0, 1]$ as a boundary-connected sum of two parts, the composite $\Pi^\dagger \circ \Pi$ is glued together from four pieces. The cylinders $\Sigma_1 \times [0, 1] \subset \Pi$ and $\Sigma_1 \times [1, 2] \subset \Pi^\dagger$ are glued together along $\Sigma_1 \times \{1\}$ except at the disc $D^2 \subset \Sigma_1$. Their union is thus homeomorphic to $(\Sigma_1 \times [0, 1]) - D^3$, where the union $D^2 \cup_{S^1} D^2 \cong S^2_1$ of the discs, that are not identified, bounds the removed ball $D^3$. In the same way the cylinders over $\Sigma_2$ are glued to give $(\Sigma_2 \times [0, 1]) - D^3$ with an additional boundary component, $S^2_2$. In order to find the total composite, we have to glue the discs together as in the definition of $\Pi$, which amounts to gluing $S^2_1$ onto $S^2_2$. This shows the first part of the lemma.
For the second part we find

\[ \Lambda_{\Sigma} \circ \Pi_{\Sigma} = \Pi_{\Sigma} \circ (I_{\Sigma_1} - D^3) \cup (S^2 \times I) \cup (I_{\Sigma_2} - D^3) \cup \ldots \cup (S^2 \times I) \cup (I_{\Sigma_K} - D^3) , \]

where we rewrote the connected sum as a (3-dimensional) index-1-surgery on the union of the cylinders. Clearly, we can view these in the composite also as index-1-surgeries on \( \Pi_{\Sigma} \cong \Pi_{\Sigma} \circ (I_{\Sigma_1} \cup \ldots \cup I_{\Sigma_K}) \). Since \( \Pi_{\Sigma} \) is connected, the surgery-points can be moved together without changing the homeomorphism type of \( \Lambda_{\Sigma} \circ \Pi_{\Sigma} \). The assertion follows now from the fact that an index-1-surgery in a contractible neighborhood is the same as connected summing with \( S^1 \times S^2 \). The proof for \( \Pi_{\Sigma} \circ \Lambda_{\Sigma} \) is analogous. \( \square \)

Besides the formulas in Lemma 17 and associativity we shall consider another type of relations among the connecting morphisms. The basic example is given next:

**Lemma 18** Suppose \( \Sigma_1, \Sigma_2, \) and \( \Sigma_3 \) are closed, connected surfaces. Then

\[ \left( \Pi_{(\Sigma_1 \sqcup \Sigma_2)} \right) \circ \left( I_{\Sigma_1} \cup \Pi_{(\Sigma_2 \sqcup \Sigma_3)} \right) = \Pi_{(\Sigma_1 \# \Sigma_2 \# \Sigma_3)} \circ \Pi_{(\Sigma_1 \cup (\Sigma_2 \# \Sigma_3))} . \quad (4.49) \]

**Proof:** Denote by \( \Sigma_1^{1/3} = \Sigma_1 - D^2 \), and \( \Sigma_2^{1/3} = \Sigma_2 - (D^2 \sqcup D^2) \), with \( \partial \Sigma_2^{1/3} = S_L^1 \sqcup S_R^1 \), the corresponding surfaces with holes, such that, e.g., \( \Sigma_1 \# \Sigma_2 = \Sigma_1^{1/3} \sqcup S_L^1 \Sigma_2^{1/3} \sqcup S_R^1 \), \( D^2 \). The morphism \( \Pi_{\Sigma_1 \sqcup \Sigma_2} \), for example, can then be seen as \( (\Sigma_1^{1/3} \sqcup S_L^1 \Sigma_2^{1/3} \sqcup S_R^1 \Sigma_3^{1/3}) \times [1, 2] \) with a 2-cell, \( C_{12} \cong D^2 \times I \), attached along the connecting \( S_L^1 \) of the source surface. Similarly, \( \Pi_{\Sigma_2 \# \Sigma_3} \) is \( (\Sigma_2^{1/3} \sqcup S_L^1 \Sigma_2^{1/3} \# \Sigma_3^{1/3}) \times [0, 1] \), with a 2-cell, \( C_{23} \), glued along the \( S_L^1 \) of the target surface. We construct the composite by first gluing the pieces \( (\Sigma_1^{1/3} \sqcup S_L^1 \Sigma_2^{1/3}) \times I \) and \( (\Sigma_2^{1/3} \# S_R^1 \Sigma_3^{1/3}) \times I \) together along the respective \( \Sigma_2^{1/3} \) boundary pieces. The result is homeomorphic to the cylinder over \( \Sigma_1 \# \Sigma_2 \# \Sigma_3 \). Attaching the remaining pieces, the 2-cell \( C_{12} \) combines with \( D^2 \times [0, 1] \subset \Pi_{\Sigma_1 \sqcup \Sigma_2} \) to one thickened disc, as does \( C_{23} \) with the \( D^2 \times I \)-piece of \( \Pi_{\Sigma_2 \# \Sigma_3} \). The composite is thus \( \Sigma_1 \# \Sigma_2 \# \Sigma_3 \times I \) with a 2-cell attached along the \( S_L^1 \) to the target surface and another 2-cell glued along \( S_L^1 \) to the source surface. If we split the middle cylinder of the cobordisms into two, the result is precisely the composite on the right of (4.49). \( \square \)

The connecting morphisms allow us to express a general, connected cobordism by a cobordism between connected surfaces. For this purpose let us introduce the notations

\[ \text{Cob}_3^{\text{conn}}(*) \subset \text{Cob}_3(*) \quad \text{and} \quad \text{Cob}_3^{\text{conn}}(0) \subset \text{Cob}_3(0) \]

for the subcategories, whose objects are connected surfaces. We have the following general presentation:

**Lemma 19** For any connected cobordism \( M : \Sigma_s \to \Sigma_t \) in \( \text{Cob}_3(0) \) there exists a morphism, \( \hat{M} : \Sigma_s^{\#} \to \Sigma_t^{\#} \), in \( \text{Cob}_3^{\text{conn}}(0) \), such that

\[ M = \Pi_{\Sigma_t} \circ \hat{M} \circ \Pi_{\Sigma_s} . \]

**Proof:** The proof is immediate from tangle presentations of cobordisms as in Appendix A.3 or [Ke2]. A direct proof is given by choosing a Morse function \( f : M \to [0, 1] \) with \( \Sigma_s = f^{-1}(0) \) and \( \Sigma_t = f^{-1}(1) \). It follows from the general theory of stratified function spaces that \( f \) can be deformed such that it does not have any index-zero-singularities, and the index-one-singularities have values below all other critical values. Also the order of the critical values of index-one can be freely permuted so that we can assume that the one with values in an interval, \([0, \delta]\), are all fusing singularities. This
means the one-handle attachment given by passing through such a singularities is between different components of the surface. As $M$ is connected there will be exactly $\beta_0(\Sigma_s) - 1$ such handles. The handles of $X = f^{-1}([0, \delta])$ can be freely slid using isotopies of maps on the upper surfaces $f^{-1}(\delta)$ of $X$. Hence we can find a boundary chart, for which $X$ is equivalent to $\Pi_{\Sigma_s}$. The arguments for splitting off $\Pi^\dagger_{\Sigma_s}$ are analogous. 

A useful application of Lemma 19 is the presentation of the symmetric group action on the $\Pi_{\Sigma}$. As in Lemma 1 a permutation of the connected components of $\Sigma$ can be given in terms of a cobordism $\pi^*$. It is not hard to see that they can be induced from a corresponding braid group action on the connected sum, $\Sigma^\#$. More precisely, for $K = \beta_0(\Sigma)$ we have a homomorphism

$$\rho : B_K \longrightarrow \pi_0(Diff(\Sigma^\#)^+)$$

such that $\rho(b) \circ \Pi_{\Sigma} = \Pi_{\Sigma}(\overline{b})^*$,

(4.50)

where $b \mapsto \overline{b}$ is given by the natural map $B_K \longrightarrow S_K$.

It is quite helpful to consider choices of the connecting morphisms in the framework of tangle presentations, see Appendix A.3. For simplicity we consider only closed surfaces, denoting by $\Sigma_g$ a connected standard surface of genus $g$.

Both $\Sigma = \bigcup_{g_1} \ldots \bigcup_{g_K}$ and $\Sigma^\# = \Sigma_{g_1+\ldots+g_K}$ are represented in a tangle category by $g_1 + \ldots + g_K$ pairs of end-points. However, in the first case they are organized in $K$ groups (i.e., we have a $\tau$-move for each group), and for $\Sigma^\#$ all end-points belong to the same group. Indicating groups by braces and admitting through-strands the connecting morphisms can be presented as follows:

$$\Pi^\dagger_{\Sigma} = \begin{array}{c}
2(g_1 + \ldots + g_K) \\
\vdots \\
2g_1 \\
2g_K
\end{array}$$

$$\Pi_{\Sigma} = \begin{array}{c}
2g_1 \\
\vdots \\
2g_K
\end{array}$$

(4.51)

We can reproduce the first assertion of Lemma 17 using the fact that tangles are composed in the naive way, if the intermediate object consists of only one group. The resulting tangle diagram of $\Pi^\dagger_{\Sigma} \circ \Pi_{\Sigma}$ consists thus again of $2(g_1 + \ldots + g_K)$ vertical strands. However, now both the top and bottom end-points are divided into $K$ groups. From the three-dimensional interpretation of the tangles, the $K - 1$ dividing lines of the diagram can be seen as $K - 1$ dividing spheres of the corresponding cobordism. Since the composites with connected objects are easily identified as the punctured identity-cobordisms, we find the connected sum $\bigoplus_{\Sigma_1} \# \ldots \# \bigoplus_{\Sigma_K}$. 

If we consider the opposite composite, we glue over $K$ components, and thus, by the rules given in [Ke2], we have to insert $K - 1$ zero-framed annuli. The first will surround the first $2g_1$ strands, the second the next $2g_2$ strands, etc. Only the last group of $2g_K$ strands is not surrounded by an annulus, as depicted in the following diagram:

$$\Lambda_{\Sigma} = \begin{array}{c}
2(g_1 + \ldots + g_K) \\
\vdots \\
2g_1 \\
2g_K
\end{array}$$

(4.52)
Instead of the last group we could have also chosen any other group as the one without an annulus.

The second assertion of Lemma 17 follows easily from (4.52), since the composition of a connecting morphism from either side has simply the effect of creating \( K \) groups, and hence introduces the \( \tau \)-move at a particular group. It is clear that the \( \tau \)-move at the \( j \)-th group with \( 2g_j \) strands allows us to push the annulus through the strands, and thus separate it from the rest of the diagram. The resulting formula for \( \Lambda_{\Sigma} \circ \Pi_{\Sigma} \) follows now from the fact that an isolated, unframed unknot represents a connected sum with \( S^1 \times S^2 \).

Moreover, the relation (4.49) in Lemma 18 is readily verified in the framework of tangles. We find that in both cases the composition yields vertical strands going from two groups with \( g_1 \) and \( g_2 + g_3 \) pairs to two groups with \( g_1 + g_2 \) and \( g_3 \) pairs of strands.

A cobordism \( \hat{M} \) as in Lemma 19 is also easily found from a given tangle, representing \( M \), by simply interpreting top and bottom strands as one group. The ambiguity in choosing \( \hat{M} \) is expressed by the additional \( \tau \)-moves that arise, when we compose with the connecting morphisms.

Finally, let us illustrate the composites in (4.51) for \( K = 2 \) in the following tangle diagram:

\[
\rho(\sigma_1) \circ \Pi_\Sigma = \begin{array}{c}
\begin{array}{c}
2g_1 \\
\vdots \\
2g_2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
2(g_1 + g_2)
\end{array}
\end{array}
\]  

(4.53)

The cobordism \( \rho(\sigma_1) \) of the generator \( \sigma_1 \in B_2 \) is presented by the crossing of \( 2g_1 \) parallel strands with the the remaining set of \( 2g_2 \) parallel strands. As \( \rho(\sigma_1) \subset Cob_3^{\text{gen}}(0) \), all \( 2(g_1 + g_2) \) belong to the same group. However, if we multiply a connecting morphism, they belong to different groups at one end of the composite. The additional \( \tau \)-move allows us to change the collective overcrossing to a collective undercrossing. Thus \( \rho(\sigma_1)^2 \circ \Pi_\Sigma = \Pi_\Sigma \), and we recover the action of the symmetric group.

### 4.2) Basic Constraints on Generalized TQFT’s:

The purpose of this section is to list a number of natural assumptions on a map from \( Cob_3(0) \) to the category of \( R \) modules, such that they necessarily imply the notion of a half projective TQFT as the only possible generalization over an ordinary TQFT. We shall also give a list of conditions in terms of some elementary morphisms and relations that allow us to construct a unique half-projective TQFT from a given connected one.

Let us begin with the axioms we shall require of an assignment

\[
\mathcal{V} : Cob_3(0) \rightarrow R - (\text{free} \text{mod}),
\]

which maps objects to objects and morphism-sets to the corresponding morphism-sets.

V1) \( \mathcal{V}(M)\mathcal{V}(N) = \mathcal{V}(M \circ N) \), if \( M \) and \( N \) are connected, and if they are composed over a connected surface.

V2) \( \mathcal{V} \) respects the symmetric tensor structure.

V3) \( \dim(\mathcal{V}(S^2)) = 1 \).
V4) $\mathcal{V}(\mathbb{I}_{\Sigma}) = \mathbb{I}_{\mathcal{V}(\Sigma)}$.

For the remainder of the chapter we shall define for a given $\mathcal{V}$:

$$x \equiv x_{\mathcal{V}} := \mathcal{V}(S^1 \times S^2). \quad (4.54)$$

Note that in a (generalized) TQFT with $\mathcal{V}_1$ and $\mathcal{V}_2$, $U := \mathcal{V}(S^2)$ is a commutative algebra over $\mathbb{R}$, which acts on $\mathcal{V}$. I.e., if $U = U_0 \oplus U_1$, then $\mathcal{V}$ also decomposes into a sum in a similar way in Lemma [4]. Hence V3 is, e.g., the same as assuming that $U$ is semisimple and $\mathcal{V}$ is indecomposable. Furthermore, together with V1 Axiom V3 implies

$$\mathcal{V}(M \# N) = \mathcal{V}(M \sqcup N). \quad (4.55)$$

It is, by V2, clearly enough to require V4 only for connected surfaces. Also, it is easy to see [A] that $\mathcal{V}(\mathbb{I}_{\Sigma})$ is in any case a projector on $\mathcal{V}(\Sigma)$, and that by a reduction to $\mathcal{V}^{\text{red}}(\Sigma) = \text{im}\left(\mathcal{V}(\mathbb{I}_{\Sigma})\right)$ we can define a consistent, effective TQFT $\mathcal{V}^{\text{red}}$ which obeys V4. Hence the last axiom simply assumes that we have already carried out this reduction.

Axiom V1 also implies that we have an honest functor

$$\mathcal{V}^{\text{conn}} : \text{Cob}^{\text{conn}}(0) \to \mathbb{R} - \text{mod}, \quad (4.56)$$

where $\text{Cob}^{\text{conn}}(0)$ is the category of cobordisms between connected surfaces as in described in the previous section. Using the decomposition in Lemma [4] and the fact that by V1 the composition in (19) of the lemma should be respected by $\mathcal{V}$, it follows that $\mathcal{V}(M)$ for general $M \in \text{Cob}_{3}(0)$ is given by $\mathcal{V}^{\text{conn}}(M)$ and elementary maps, to be associated to the surface connecting cobordisms:

$$i_{\Sigma} := \mathcal{V}(\mathbb{I}_{\Sigma}) : V_{\Sigma_1} \otimes \ldots \otimes V_{\Sigma_K} \to V_{\Sigma^\#}, \quad (4.57)$$

with $\Sigma = \Sigma_1 \sqcup \ldots \sqcup \Sigma_K$ and $\Sigma^\# = \Sigma_1^\# \ldots \# \Sigma_K$.

Next, let us derive from V1-V3 a set of constraints on these maps, from which we infer the existence of a generalized TQFT for disconnected surfaces. To begin with, observe that the composite in Part 1 of Lemma [17] is over a connected surface so that we have $p_{\Sigma} i_{\Sigma} = \mathcal{V}(\mathbb{I}_{\Sigma_1^\#} \ldots \# \mathbb{I}_{\Sigma_K})$. By (1.35) the connected sums can be replaced by disjoint unions, which yields the identity. Hence, by V4, the $i_{\Sigma}$ and $p_{\Sigma}$ must be injections and projections that identify $V_{\Sigma} = \bigotimes_j V_{\Sigma_j}$ as a complemented subspace in $V_{\Sigma^\#}$. They also satisfy associativity constraints as in

$$i_{\Sigma_A \sqcup \Sigma_B} = i_{\Sigma_A^\# \sqcup \Sigma_B^\#} \left( i_{\Sigma_A^\#} \otimes i_{\Sigma_B^\#} \right), \quad (4.58)$$

where $\Sigma_A$ and $\Sigma_B$ are possibly disconnected surfaces, and $\Sigma_A^\#$ and $\Sigma_B^\#$ are as usual the connected sums of their components.

Given a connected TQFT, $\mathcal{V}^{\text{conn}}$, we define

$$L_{\Sigma} := \mathcal{V}^{\text{conn}}(\Lambda_{\Sigma}) \in \text{End}_{\mathbb{R}}(V_{\Sigma^\#}). \quad (4.59)$$

If $\mathcal{V}$ were an ordinary TQFT, the definition of $\Lambda_{\Sigma}$ in Lemma [17] would imply that this map is identical with the following projector:

$$P_{\Sigma} := i_{\Sigma} p_{\Sigma} \in \text{End}_{\mathbb{R}}(V_{\Sigma^\#}). \quad (4.60)$$
However, the composition for \( \Lambda_\Sigma \) is over disconnected \( \Sigma \) so that we cannot apply V1. Still, the findings in the second part of Lemma \([7]\) allow us to derive from Axioms V1 and \((4.53)\) the relations \( L_\Sigma i_\Sigma = x^{-1}i_\Sigma \) and \( p_\Sigma L_\Sigma = x^{-1}p_\Sigma \), where \( K = \beta_0(\Sigma) \). This means that \( L_\Sigma x^{K-1}P_\Sigma + L_\Sigma (1 - P_\Sigma) \), where \([L_\Sigma, P_\Sigma] = 0\), and \( L_\Sigma^2 = x^{-1}L_\Sigma \). So, in principle we could have a composition anomaly of \( i_\Sigma \) and \( p_\Sigma \) in the form of an operator \( L_\Sigma \) acting on the complement of the space \( V_\Sigma = \bigotimes V_\Sigma_j \), through which the injection and projection normally map. At this point we cannot find a reasonable way to incorporate such an anomaly into our formalism, and it does not appear in the known construction. We shall thus add its absence to the list of axioms:

\[ V5) \quad L_\Sigma^2 = 0 \]

In the case, where each \( \mathcal{V}(\Sigma) \) is generated by all \( \mathcal{V}(M) \), with \( M : \emptyset \to \Sigma \), Axiom V5 can also be inferred from the condition that for any collection of \( M_j : \emptyset \to \Sigma \),

\[ \sum_j c_j \mathcal{V}(M_j) = 0 \quad \text{implies} \quad \sum_j c_j \mathcal{V}^{\text{conn}}(\Pi_\Sigma \circ M_j) = 0 \]

Since the equations in Lemma \([18]\) and in \((4.50)\) involve only compositions over connected surfaces, we shall impose by V1 the corresponding formulas for the maps \( i_\Sigma \) and \( p_\Sigma \) as additional conditions. Moreover, suppose that we have cobordisms, \( M_j : \Sigma_j \to \Sigma_j' \), for connected surfaces \( \Sigma_j \) and \( \Sigma_j' \), and \( M : \Sigma' \to \Sigma' \) for their connected sums, such that

\[ \Pi_\Sigma \circ (M_1 \sqcup \ldots \sqcup M_K) = M \circ \Pi_\Sigma . \quad (4.61) \]

Then V1 and V2 imply that this relation is respected by \( \mathcal{V} \).

Let us summarize the conditions on \( \mathcal{V} \) that we have derived so far from the Axioms V1-V5, with \( x \) as in \((4.54)\):

P1) There is a connected TQFT, \( \mathcal{V}^{\text{conn}} \), as in \((4.56)\).

P2) There are injections and projections, \( V_\Sigma \xleftarrow{i_\Sigma} V_{\Sigma}^{\#} \) and \( V_{\Sigma}^{\#} \xrightarrow{p_\Sigma} V_\Sigma \), with \( p_\Sigma i_\Sigma = 1 \).

P3) \( \tau \)-Invariance: \( \mathcal{V}(M) = p_\Sigma \mathcal{V}^{\text{conn}}(\hat{M}) i_{\Sigma} \) only depends on \( M = \Pi_\Sigma \circ \hat{M} \circ \Pi_\Sigma \).

P4) Associativity: See \((4.58)\) and analogously for \( p_\Sigma \).

P5) Symmetry: \( \mathcal{V}^{\text{conn}}(\rho(b)) i_\Sigma = i_\Sigma (\tilde{b})^* \), (see \((4.59)\)), where \( S_K \) acts on \( V_\Sigma = \bigotimes V_\Sigma_j \) by canonical permutation. (same for \( p_\Sigma \)).

P6) Naturality: \( i_\Sigma \left( \mathcal{V}^{\text{conn}}(M_1) \otimes \ldots \otimes \mathcal{V}^{\text{conn}}(M_K) \right) = \mathcal{V}^{\text{conn}}(M) i_\Sigma \), where \( M \) and \( M_j \) are as in \((4.61)\). (same for \( p_\Sigma \)).

P7) Projectivity: \( \mathcal{V}^{\text{conn}}(\Lambda_\Sigma) = x^{\beta_0(\Sigma)} - 1 \cdot \mathcal{I}_\Sigma \). (See \((4.60)\)).

P8) Commutation: \( \left( i_{\Sigma(\Sigma_j \cup \Sigma_3)} \otimes 1 \right) \left( \mathcal{I} \otimes p_\Sigma(:\Sigma_j \cup \Sigma_3) \right) = p_\Sigma (:\Sigma_j \cup \Sigma_3) \), with each \( \Sigma_j \) connected, etc.

This list of properties is, in fact, also sufficient for the existence of a generalized TQFT. Furthermore, this generalization comes out to be precisely the one defined and discussed in Section 2.3, and, conversely, implies the Axioms V1-V5.
Theorem 6
Suppose there is a functor, $\mathcal{V}^{\text{conn}}$, and maps $i_\Sigma$ and $p_\Sigma$, such that properties P1-P8 are full filled. Then there exists a unique, half-projective TQFT, w.r.t. $x = \mathcal{V}^{\text{conn}}(S^1 \times S^2)$ and $\mu_0$,

$$\mathcal{V} : \text{Cob}_3(0) \rightarrow \mathbb{R} - \text{mod},$$

such that $i_\Sigma = \mathcal{V} (\Pi_\Sigma)$, $p_\Sigma = \mathcal{V} (\Pi_\Sigma^1)$, and $\mathcal{V}$ specializes to $\mathcal{V}^{\text{conn}}$ on $\text{Cob}_{3\text{conn}}^3(0)$.

Proof: The assignment of a map $\mathcal{V}(M)$ to a connected cobordism, $M$, is uniquely determined and well define by Axiom V1 and Property P2. Using V2 and P5 its extension to a disconnected cobordisms, $M$, is found from the decompositions in Lemma [L]. From the discussion in Section 2.2 it follows that compatibility of $\mathcal{V}$ with the symmetric tensor structure allows us to consider only elementary compositions as in (2.12). Assume that the cobordisms in this formula are

$$M_1 : \Sigma_A \rightarrow \Sigma_B \sqcup \Sigma_C \quad \text{and} \quad M_2 : \Sigma_C \sqcup \Sigma_D \rightarrow \Sigma_E$$

so that $(\mathbb{I} \otimes M_2)(M_1 \otimes \mathbb{I}) : \Sigma_A \sqcup \Sigma_D \rightarrow \Sigma_B \sqcup \Sigma_E$. The connectivity cocycle is given by $\mu_0 = \beta_0(\Sigma_C) - 1$. We find

$$x^{\mu_0} \mathcal{V}(M_2) \mathcal{V}(M_1) = x^{\beta_0(\Sigma_C) - 1}(\mathbb{I} \otimes \{ p_{SE} \mathcal{V}^{\text{conn}}(M_2) i_{\Sigma_C \sqcup \Sigma_D} \}) \left( \{ p_{SE} \mathcal{V}^{\text{conn}}(M_1) i_{\Sigma_A} \} \otimes \mathbb{I} \right)$$

by (P4)

$$= x^{\beta_0(\Sigma_C) - 1}(p_{SE} \mathcal{V}^{\text{conn}}(M_2) i_{\Sigma_C \sqcup \Sigma_D}) \times \left( \{ p_{SB} \otimes p_{SE} \} \mathcal{V}^{\text{conn}}(M_1) \right) \otimes \mathbb{I}$$

by (P7)

$$= (p_{SB} \otimes p_{SE}) \left( (\mathbb{I} \otimes \{ \mathcal{V}^{\text{conn}}(M_2) i_{\Sigma_C \sqcup \Sigma_D} \}) \left( \{ \mathcal{V}^{\text{conn}}(M_1) \} \otimes \mathbb{I} \right) \right)$$

by (P8)

$$= (p_{SB} \otimes p_{SE}) \left( (\mathbb{I} \otimes \mathcal{V}^{\text{conn}}(M_2) i_{\Sigma_C \sqcup \Sigma_D}) \left( \mathcal{V}^{\text{conn}}(M_1) \right) \otimes \mathbb{I} \right) \otimes \mathbb{I}$$

by (P6)

$$= (p_{SB} \otimes p_{SE}) \left( (\mathbb{I} \otimes \mathcal{V}^{\text{conn}}(M_2) i_{\Sigma_C \sqcup \Sigma_D}) \left( \mathcal{V}^{\text{conn}}(M_1) \right) \right)$$

by (P4)

$$= p_{SB \sqcup SD} \mathcal{V}^{\text{conn}}(X_2 \circ X_1) i_{\Sigma_A \sqcup \Sigma_D}$$

by (P3)

$$= \mathcal{V}(M)$$

with $M = \Pi_{SB \sqcup SE} \circ X_2 \circ X_1 \circ \Pi_{SA \sqcup SD}$. In order to apply Naturality P6 in this calculation we had to choose cobordisms, $\Lambda^\Sigma_{\Sigma_C} : \Sigma_C^# \# \Sigma_D^# \rightarrow \Sigma_C^# \# \Sigma_D$, $X_1 : \Sigma_A^# \# \Sigma_B^# \rightarrow \Sigma_B^# \# \Sigma_C^# \# \Sigma_D^#$, and $X_2 : \Sigma_B^# \# \Sigma_C^# \# \Sigma_D^# \rightarrow \Sigma_B^# \# \Sigma_E^#$, such that they full fill the following equations

$$\Pi_{\Sigma_C \sqcup \Sigma_D} \circ (\Lambda_{\Sigma_C} \sqcup \mathbb{I}_{\Sigma_D}) = \Lambda_{\Sigma_C} \circ \Pi_{\Sigma_B \sqcup \Sigma_D}^\#$$

(N1)

$$\Pi_{\Sigma_B^# \# \Sigma_C^# \# \Sigma_D^#} \circ (\tilde{M}_1 \sqcup \mathbb{I}_{\Sigma_D}) = X_1 \circ \Pi_{\Sigma_A^# \# \Sigma_D}^\#$$

(N2)

$$\Pi_{\Sigma_D}^\# \circ (\tilde{M}_2 \circ \Lambda_{\Sigma_C}) \circ \Pi_{\Sigma_B^# \# \Sigma_C^# \# \Sigma_D^#}^\# = \Pi_{\Sigma_B^# \# \Sigma_E^#} \circ X_2$$

(N3)

In order to complete the proof that $\mathcal{V}$ is a half-projective functor, we still need to show that the above cobordisms $M$, with $\tilde{M} = X_2 \circ X_1$, is in fact $\tilde{M}_2 \circ \tilde{M}_1$. This is accomplished by basically the same
calculation, only now for cobordisms and in reverse order:

\[
\begin{align*}
M_{\text{assoc.}} &= \Pi_{\Sigma_B \sqcup \Sigma_E}^+ \circ X_2 \circ X_1 \circ \Pi_{\Sigma_A \sqcup \Sigma_D}^+ \\
N2 \& N3 &= \left( \Pi_{\Sigma_B}^+ \sqcup \Pi_{\Sigma_E}^+ \circ \left( \Pi_{\Sigma_B}^+ \circ \Sigma_C^# \circ \Pi_{\Sigma_D}^+ \circ \left( \Pi_{\Sigma_B}^+ \sqcup \Sigma_C^# \circ \Pi_{\Sigma_D}^+ \right) \right) \right) \\
\text{Lemma } &\text{3} = \left( \Pi_{\Sigma_B}^+ \sqcup \Pi_{\Sigma_E}^+ \circ \left( \Pi_{\Sigma_B}^+ \circ \Sigma_C^# \circ \Pi_{\Sigma_D}^+ \circ \left( \Pi_{\Sigma_B}^+ \sqcup \Sigma_C^# \circ \Pi_{\Sigma_D}^+ \right) \right) \right) \\
N1 &= \left( \Pi_{\Sigma_B}^+ \sqcup \left( \Pi_{\Sigma_E}^+ \circ \Sigma_C^# \circ \Pi_{\Sigma_D}^+ \circ \left( \Pi_{\Sigma_B}^+ \sqcup \Sigma_C^# \circ \Pi_{\Sigma_D}^+ \right) \right) \right) \\
\text{Lemma } &\text{5} = \left( \Pi_{\Sigma_B}^+ \sqcup \Pi_{\Sigma_C}^+ \circ \left( \Pi_{\Sigma_B}^+ \circ \Sigma_C^# \circ \Pi_{\Sigma_D}^+ \circ \left( \Pi_{\Sigma_B}^+ \sqcup \Sigma_C^# \circ \Pi_{\Sigma_D}^+ \right) \right) \right) \\
\text{assoc.} &= \left( \Pi_{\Sigma_B}^+ \sqcup \Pi_{\Sigma_C}^+ \circ \left( \Pi_{\Sigma_B}^+ \circ \Sigma_C^# \circ \Pi_{\Sigma_D}^+ \circ \left( \Pi_{\Sigma_B}^+ \sqcup \Sigma_C^# \circ \Pi_{\Sigma_D}^+ \right) \right) \right) \\
&= \left( \Pi_{\Sigma_B}^+ \circ \Pi_{\Sigma_C}^+ \circ \Pi_{\Sigma_D}^+ \circ \Pi_{\Sigma_A}^+ \right) \\
&= \left( \Pi_{\Sigma_B}^+ \circ \Pi_{\Sigma_C}^+ \circ \Pi_{\Sigma_D}^+ \circ \Pi_{\Sigma_A}^+ \right) \\
&= \left( \Pi_{\Sigma_B}^+ \circ \Pi_{\Sigma_C}^+ \circ \Pi_{\Sigma_D}^+ \circ \Pi_{\Sigma_A}^+ \right)
\end{align*}
\]

This completes the proof of the theorem. \(\Box\)

In the remaining sections we use Theorem 5 to construct half-projective TQFT’s from known ones. We will verify the necessary and sufficient properties P1-P8 for a very general class of examples.

4.3) The Example of Extended TQFT’s:

In this section we show that if the connected TQFT from (1.53) originates from an extended structure, most of the properties entering Theorem 6 are already full filled. Thus for the remainder of this and the next section we shall require that there be a series of functors \(\mathcal{V}_n^{\text{conn}}\) as in (2.24), with \(\text{Cob}_3(n)\) replaced by \(\text{Cob}_3^{\text{conn}}(n)\).

Connected, extended TQFT’s exist for a quite general class of abelian, categories:

**Theorem 7 ([KL])** Suppose \(\mathcal{C}\) is an abelian, rigid, balanced, modular, braided tensor category over a field \(\mathbb{R} = k\), for which the coend \(F = \int X^\text{br} X\) exists.

Then there is a series of functors, \(\mathcal{V}_n^{\text{conn}}\), on the categories \(\text{Cob}_3^{\text{conn}}(n)\), as in (2.24), which respects both types of tensor products and 2-categorical compositions, and for which

\[
\mathcal{V}_0^{\text{conn}}(\Sigma_g) = \text{Inv}(F \otimes \ldots \otimes F) \quad \text{g times},
\]

where \(\otimes\) is the (braided) tensor product of \(\mathcal{C}\), and \(\Sigma_g\) is the closed surface of genus \(g\). (We denote \(\text{Inv}(X) = \text{Hom}_\mathcal{C}(1, X)\).)

As explained in [Ke3] this specializes to the Reshetikhin-Turaev [RT] invariant (often identified with the Chern-Simons quantum field theory) if \(\mathcal{C}\) is semisimple, and to the Hennings-invariant [H] if \(\mathcal{C} = \mathcal{A} - \text{mod}\). The coend is in the first case \(F = \bigoplus_j j^\text{br} j\), where \(j\) runs over a representative set of simple objects, and in the second case \(F = \mathcal{A}^*\), equipped with the coadjoint action \(ad^*\).

The part of this generalization that will be relevant for this section is that the connected functor needed in Theorem 5 descends from the functor \(\mathcal{V}_1 : \text{Cob}_3^{\text{conn}}(1) \to \mathcal{C}\), and that the latter
is a functor of braided tensor categories. Including also the intermediate tangle presentations this is made precise in the following commutative diagram.

\[
\begin{array}{c}
\mathcal{C}^\text{conn}_1 : C\text{ob}_{\text{conn}}^3(1) \xrightarrow{\phi_0} T\text{gl}^\infty(1)^{\text{conn}} \xrightarrow{\tau\text{-move}} \mathcal{C} \\
\mathcal{C}^\text{conn}_0 : C\text{ob}_{\text{conn}}^3(0) \xrightarrow{\phi_0} T\text{gl}(0)^{\text{conn}} \xrightarrow{\tau\text{-move}} \mathcal{R} - \text{mod}
\end{array}
\]

(4.63)

All of the vertical arrows are surjections. The functor \(\phi_0\) is the filling functor that was described in (2.10) of Section 2.2. It assigns to a once punctured surface, \(\Sigma^*\), a corresponding closed surface, \(\Sigma\), by gluing in a disc, and to a cobordisms, \(M^*\), with corners a morphism, \(M \in C\text{ob}_{\text{conn}}^3(0)\), by filling in a tube, \(D^2 \times I\). The \(\tau\)-move is described in Appendix A.3. As in [KL] we denote by \(T\text{gl}^\infty(1)^{\text{conn}}\) the tangle category, which has the same generators as \(T\text{gl}(0)^{\text{conn}}\), but which is not subject to the \(\tau\)-move. It represents isomorphically \(C\text{ob}_{\text{conn}}^3(1)\), which generalizes the presentation of the mapping class groups of punctured surfaces, \(\Sigma^*\), from [MP]. The \(\tau\)-move accounts for isotopies over the puncture so that \(T\text{gl}(0)^{\text{conn}}\) actually represents \(C\text{ob}_{\text{conn}}^3(0)\). The functor \(\mathcal{C}^\text{conn}_1\) is finally constructed by assigning to a tangle a system of morphisms with naturality properties, and lifting those to a morphism between \(M\) by gluing in a disc, and to a cobordisms, \(F\) in (2.10) of Section 2.2. It assigns to a once punctured surface, \(\Sigma\) itself is described by the morphism \(\mathcal{C}^\text{conn}_1\) is made precise in the following commutative diagram.

Thus we can construct \(\mathcal{C}^\text{conn}_0\) for \(\Sigma \) or \(M\) by first choosing punctured representatives \(\Sigma^*\) or \(M^*\), apply to these \(\mathcal{C}^\text{conn}_0\), and then map the result into \(\mathcal{R} - \text{mod}\) by the \(\text{Inv}\)-functor so that, e.g., \(\mathcal{C}^\text{conn}_0(\Sigma) = \text{Inv}(\Sigma)\). The assignment

\[
X_{\Sigma_g} = F \boxtimes \ldots \boxtimes F
\]

then explains the formula for the vector spaces in Theorem 7.

The benefit of this description lies in the fact that the horizontal arrows in the top row of (4.63) are functors of braided tensor categories. The tensor product of \(C\text{ob}_{\text{conn}}^3(1)\) is given by gluing the boundaries of two surfaces to a three-holed sphere, \(\Sigma_{0,3} = S^2 - (D^2 \cup D^2 \cup D^2)\), and accordingly the two cylindrical boundary components of two cobordisms with corners to two respective pieces in \(\Sigma_{0,3} \times I\). We shall make identifications with standard surfaces that are compatible with those for the ordered connected sums in Section 4.1, using isomorphisms

\[
\Sigma^* \boxtimes \Sigma^* \cong (\Sigma_1 \# \Sigma_2)^*
\]

(4.65)

The category \(T\text{gl}^\infty(1)^{\text{conn}}\) also admits a natural tensor product, given by the (opposite) juxtaposition of two tangles. The presentation functor can be chosen, such that the tensor structure is strictly respected. Moreover, the construction of \(\mathcal{C}^\text{conn}_1\) is such that it is also a strict tensor functor into \(\mathcal{C}\), i.e.,

\[
X_{\Sigma_1 \# \Sigma_2} = X_{\Sigma_1} \boxtimes X_{\Sigma_2}
\]

(4.66)
A connected cobordism between surfaces with several components, \( M : \Sigma_1 \sqcup \ldots \sqcup \Sigma_K \to \Sigma^{\dagger}_1 \sqcup \ldots \sqcup \Sigma^\dagger_L \) may be similarly first described by a morphism

\[
\hat{M}^\bullet : (\Sigma_1 \# \ldots \# \Sigma_K)^\bullet \mapsto (\Sigma_1 \# \ldots \# \Sigma_L)^\bullet,
\]

which is presented by a tangle \( \mathcal{T}(M) \in \mathcal{T}gl^\infty(1)^{\text{conn}} \). The original cobordism \( M \) is then presented by the image of \( \mathcal{T}(M) \) in the tangle category \( \mathcal{T}gl(0)^{\text{conn}} \), where we have introduced a \( K + L \) additional \( \tau \)-moves, one for every group of strands representing a boundary component.

In order to guarantee invariance under the \( \tau \)-moves at the source ends of the tangle, we may proceed analogously and restrict \( \mathcal{V}^{\text{conn}}_1(M^\bullet) \) to the tensor product of the invariances of the objects associated to the individual groups. If we start by carrying out the reduction to the cobordism \( \hat{M} \) of closed, connected surfaces as in (4.63) we find a first candidate for the inclusion from Theorem 3. Specifically, we have that

\[
\mathcal{V}_0^{\text{conn}}(\hat{M})_{\Sigma}^0 \quad \text{only depends on} \quad \hat{M} \circ \Pi_{\Sigma},
\]

where we use the canonical injection:

\[
i_{\Sigma}^0 : \text{Inv}(X_{\Sigma_1}) \otimes \ldots \otimes \text{Inv}(X_{\Sigma_K}) \hookrightarrow \text{Inv}(X_{\Sigma_1} \boxtimes \ldots \boxtimes X_{\Sigma_K}). \quad (4.67)
\]

The difficulty that remains, is to find a projection in reverse direction in the case that the target surface is also disconnected, i.e., \( L > 1 \). In general, if the vector spaces are given by the invariances as above, a canonical map with these desired properties does not exist. Still, we can define canonical matrix elements. More precisely, for every choice of invariances, \( f_j \in \text{Inv}(X_{\Sigma_j}) \), and coinvariances, \( g_j \in \text{Cov}(X_{\Sigma_j}) \), (denoting \( \text{Cov}(Y) \equiv \text{Hom}_{\mathcal{C}}(Y, 1) \)) we have that also

\[(g_1 \boxtimes \ldots \boxtimes g_L)\mathcal{V}^{\text{conn}}_1(\hat{M}^\bullet)(f_1 \boxtimes \ldots \boxtimes f_K) \quad .\]

only depends on \( M = \Pi_{\Sigma}^\dagger \circ \phi_o(\hat{M}^\bullet) \circ \Pi_{\Sigma} \). This circumstance naturally leads us to first construct \( \mathcal{V}_0 \) on the morphisms spaces, and then reconstruct the vector spaces. Generally, let us define for a set of objects \( A_j, B_j \in \text{obj}(\mathcal{C}) \) the null space:

\[
H^0 := \{ h : A_1 \boxtimes \ldots \boxtimes A_K \to B_1 \boxtimes \ldots \boxtimes B_L : (g_1 \boxtimes \ldots \boxtimes g_L)h(f_1 \boxtimes \ldots \boxtimes f_K) = 0 \quad \text{for all} \quad f_j \in \text{Inv}(A_j), \ g_j \in \text{Cov}(B_j) \} .
\]

From this we define the space of matrices:

\[
H(A_1, \ldots, A_K|B_1, \ldots, B_L) := \text{Hom}_{\mathcal{C}}(A_1 \boxtimes \ldots \boxtimes A_K, B_1 \boxtimes \ldots \boxtimes B_L)/H^0.
\]

For a morphism, \( I \), between the tensor products of the \( A_j \)'s and \( B_j \)'s, let us also denote its image in the above space (i.e., its class modulo \( H_0 \)) by \([I]\). A natural definition of the TQFT-functor for disconnected surfaces on only the morphism spaces is thus

\[
\mathcal{V}_0(M) := [\mathcal{V}^{\text{conn}}_1(\hat{M}^\bullet)] \in H(X_{\Sigma_1}, \ldots, X_{\Sigma_K}|X_{\Sigma_1^{\dagger}}, \ldots, X_{\Sigma_L^{\dagger}}) . \quad (4.68)
\]

Even in the connected case \( H(A|B) \) is usually going to be smaller than \( \text{Hom}_{\mathcal{C}}(A, B) \), if \( \mathcal{C} \) is not semisimple. This is due to the fact that the canonical pairing

\[
\text{Cov}(X) \otimes \text{Inv}(X) \longrightarrow \mathbb{R} \quad (4.69)
\]
is degenerate for most objects $X$. Let us denote the null spaces of this pairing by $\text{Cov}^0(X)$ and $\text{Inv}^0(X)$, respectively. It is easily seen that, e.g., $\text{Inv}^0(X)$ is mapped to $\text{Inv}^0(Y)$ by $\text{Inv}(f)$ for a morphism $f : X \to Y$, and that $[f] = 0$, if all of $\text{Inv}(X)$ is mapped into $\text{Inv}^0(Y)$. Still, we can think of the morphisms in the $H(\square)$-spaces as maps between vector spaces, if we pass to the quotients

\[ \overline{\text{Inv}}(X) := \frac{\text{Inv}(X)}{\text{Inv}^0(X)} \quad \text{and} \quad \overline{\text{Cov}}(X) := \frac{\text{Cov}(X)}{\text{Cov}^0(X)} . \tag{4.70} \]

Assuming that $\mathbb{R}$ acts nicely on these spaces this assertion is made more precise in the following lemma:

**Lemma 20** Suppose that for an abelian tensor category, $\mathcal{C}$, over $\mathbb{R}$ the spaces $\text{Inv}(X)$, $\text{Inv}^0(X)$, $\overline{\text{Inv}}(X)$, $\text{Cov}(X)$, etc., are free $\mathbb{R}$-modules, and that the exact sequence

\[ 0 \to \text{Inv}^0(X) \hookrightarrow \text{Inv}(X) \twoheadrightarrow \overline{\text{Inv}}(X) \to 0 , \tag{4.71} \]

as well as the analogous one for $\text{Cov}(X)$, are split over $\mathbb{R}$.

Then we have

1. Duality: $\overline{\text{Cov}}(X) \cong \text{Hom}_{\mathbb{R}}(\overline{\text{Inv}}(X), \mathbb{R})$ and vice versa.

2. $H(A_1, \ldots, A_K \mid B_1, \ldots, B_L) \cong \text{Hom}_{\mathbb{R}}(\overline{\text{Inv}}(A_1) \otimes \cdots \otimes \overline{\text{Inv}}(A_K), \overline{\text{Inv}}(B_1) \otimes \cdots \otimes \overline{\text{Inv}}(B_L))$.

The proof is standard, and makes use of the fact that the split sequence in (4.71) allows us to choose dual basis in $\overline{\text{Inv}}(X)$ and $\overline{\text{Cov}}(X)$. If $\mathbb{R}$ is a field the prerequisites of Lemma 20 are of course always full filled (assuming finite dimensions). Yet, as usual let us consider a more general situation, in order to indicate the extent to which our constructions are the only possible ones.

The identity in the second part of Lemma 20 show that we have to modify the vector spaces of the connected TQFT from Theorem 4 or, more generally, the diagram in (4.63) by assigning

\[ \mathcal{V}_0(\Sigma) := \overline{\text{Inv}}(X_\Sigma) , \quad \text{if } \Sigma \text{ is connected} , \tag{4.72} \]

and extending this to disconnected surfaces by tensor products. This is then compatible with the assignment of linear maps to a morphisms, $M$, where we first choose a $\hat{M}^* \in \text{Cob}^\text{conn}_3(1)$, compute its image in the $H(\square)$-space as in (4.68), and then apply the above isomorphism into the corresponding space of linear maps. On $\text{Cob}^\text{conn}_3(0)$ the functor $\mathcal{V}_0$ will thus be given by the factored version of $\mathcal{V}^\text{conn}_0$, where we divided out the null spaces as in (4.70).

Using the easily verified property that a canonical injection, as in (4.67), maps, e.g., $\text{Inv}^0(X) \otimes \text{Inv}(Y) \hookrightarrow \text{Inv}^0(X \boxtimes Y)$, we can factorize $i_\Sigma^* \in \text{Cob}_3^\Sigma$ from (4.67) into a map

\[ i_\Sigma^* : \overline{\text{Inv}}(X_{\Sigma_1}) \otimes \cdots \otimes \overline{\text{Inv}}(X_{\Sigma_K}) \hookrightarrow \overline{\text{Inv}}(X_{\Sigma_1 \boxplus \cdots \boxplus} X_{\Sigma_K}) . \tag{4.73} \]

Since $\overline{\text{Inv}}$ and $\overline{\text{Cov}}$ are now dual spaces, we may define canonical projections, $p_\Sigma$, associated to the connecting cobordisms $\Pi_\Sigma^\dagger$. They shall be the adjoints of the corresponding inclusion of coinvariances, i.e.,

\[ p_\Sigma^* : \overline{\text{Cov}}(X_{\Sigma_1}) \otimes \cdots \otimes \overline{\text{Cov}}(X_{\Sigma_K}) \twoheadrightarrow \overline{\text{Cov}}(X_{\Sigma_1 \boxplus \cdots \boxplus} X_{\Sigma_K}) . \tag{4.74} \]

It is also straight forward to see that with the definitions in (4.68), $\mathcal{V}_0(M) = p_\Sigma^* \mathcal{V}_0(\hat{M})i_\Sigma^*$, if $M : \Sigma \to \hat{\Sigma}$, and $\hat{M}$ is a corresponding morphism between the connected sums, such that $M = \Pi_\Sigma^\dagger \circ \hat{M} \circ \Pi_\Sigma$. It is also clear from the construction that $p_\Sigma^*i_\Sigma = \text{id}$ so that we have now the ingredients entering Theorem 3 which full fill properties P1, P2, and P3.
Moreover, associativity P4 follows immediately from the associativity of the canonical inclusions of invariances and coinvariances. Compatibility with symmetry as in P5 can be inferred from the fact that \( \varepsilon(X_{\Sigma_1}, X_{\Sigma_2}) \) acts by naturality as the transposition on \( \text{Inv}(X_{\Sigma_1}) \otimes \text{Inv}(X_{\Sigma_2}) \) and hence also on \( \overline{\text{Inv}}(X_{\Sigma_1}) \otimes \overline{\text{Inv}}(X_{\Sigma_2}) \). Also, property P8 is evident, if we consider it for matrix elements. Specifically, choose \( f \in \text{Inv}(X_1) \), \( g \in \text{Inv}(X_1 \square X_2) \), \( \alpha \in \text{Cov}(X_1 \square X_2) \), and \( \beta \in \text{Cov}(X_3) \), and denote by \( \overline{f}, \overline{g}, \overline{\alpha}, \overline{\beta} \), the images in the quotient spaces. We then have the obvious identities

\[
\langle \overline{\alpha} \otimes \overline{\beta}, \left( i_{(1L,2)} \otimes \mathbb{1} \right) \left( \mathbb{1} \otimes p_{(2L,3)} \right) \overline{f} \otimes \overline{g} \rangle = \langle \alpha \otimes \beta, f \otimes g \rangle = \langle \overline{\alpha} \otimes \overline{\beta}, p_{(1 \square (2L,3))} i_{(1L,2 \square 3)} \overline{f} \otimes \overline{g} \rangle.
\]

Finally, the construction of \( \mathcal{V}_0 \) from the tensor functor, \( \mathcal{V}_1^{\text{conn}} \), as in \( \text{(1.63)} \) allows us to infer naturality P6 from the following relation between cobordisms:

**Lemma 21** Suppose \( M_j^* \), with \( j = 1, 2 \), are morphisms in \( \text{Cob}_3^{\text{conn}}(1) \), \( \Sigma (\tilde{\Sigma}) \) is the union of the closed source (target) surfaces, and the choice of the \( \Pi_{\Sigma} \)'s is as in Section 4.1. Then

\[
\phi_0(M_1^* \otimes M_2^*) \circ \Pi_{\Sigma} = \Pi_{\Sigma} \circ (\phi_0(M_1^*) \sqcup \phi_0(M_2^*)).
\]

**Proof:** The relation is readily verified, given the tangle presentation in \( \text{(1.31)} \), the fact that the tensor product in \( \mathcal{Tgl}^{\infty}(1)^{\text{conn}} \) is given by juxtaposition, and that \( \phi_0 \) only introduces another relation, but may be chosen as identity on representing tangles.

The formula may also be understood directly, by considering both sides of the equation as \( M_1^* \sqcup M_2^* \), to which certain elementary manifolds are attached along the two cylindrical boundary pieces. On the left hand side we glue in \( B = S^2 - (D^2 \sqcup D^2) \times [0, 1] \) to get \( \phi_0(M_1^* \otimes M_2^*) \), and attach a 2-cell, \( C_2 \), to the piece, \( \cong S^2 - (D^2 \sqcup D^2) \), of \( X \) in the source surface, in order to realize the composition with \( \Pi_{\Sigma} \). On the right hand side we first glue in two tubes, \( D^2 \times J \), to get \( \phi_0(M_1^*) \sqcup \phi_0(M_2^*) \) and then describe the composition with \( \Pi_{\Sigma} \) by attaching a 1-cell, \( C_1 \), at the discs, \( D^2 \), in the target surface that belong to the tubes. In both cases the combined glued in piece, \( X \sqcup C_2 \cong (D^2 \times J) \sqcup C_1 \sqcup (D^2 \times J) \cong D^3 \), is a ball, and the cylindrical boundary pieces of \( M_1^* \sqcup M_2^* \) are glued to \( D^3 \) along two annuli that are embedded in its boundary \( \partial D^3 = S^2 \).

Let us summarize the findings of this section in the next lemma:

**Lemma 22** Suppose we have a connected TQFT functor \( \mathcal{V}_0^{\text{conn}} \) that descends from a functor \( \mathcal{V}_1^{\text{conn}} \) of braided tensor categories as in \( \text{(4.63)} \) (e.g., the one proposed in Theorem 3). Then we can construct a map \( \mathcal{V}_0 : \text{Cob}_3(0) \to \mathbb{R} \), which satisfies the properties P1-P6, and P8 from Section 4.2. The vector spaces are the quotient spaces as in \( \text{(4.72)} \).

The only thing left to investigate, in order to complete the construction of a half-extended TQFT, is thus the projectivity property P7. This will be done in the next section, as it relates to more specific properties of the constructions starting from an abelian, braided tensor category \( \mathcal{C} \) over a field \( k \). The triviality of \( x \) will turn out to determine completely the semisimplicity of \( \mathcal{C} \).

Let us conclude this section with an example of how vector spaces change, when we divide out the null spaces as in \( \text{(4.70)} \) for a non-semisimple category. The result in the case of \( \mathcal{C} = U_q(sl_2) - \text{mod} \), with \( q \) a primitive \( 2m + 1 \)-st root of unity, is described, e.g., in [Ke3]. The dimensions of the vector spaces for a torus are given by

\[
\dim(\text{Inv}(F)) = 3m + 1 \quad \text{and} \quad \dim(\overline{\text{Inv}}(F)) = 2m.
\]
It is a quite remarkable fact that at least for prime $2m+1$, the space $\text{Inv}_0^0(F)$ is not only an invariant subspace of the representation of the mapping class group derived from $\mathcal{V}_1^{\text{conn}}$, but that it is also a direct summand. I.e., the sequence in (4.71) is also split as a sequence of $SL(2, \mathbb{Z})$-modules.

Notice also that $\overline{\text{Inv}}(F)$ is naturally identified with an invariance of the semisimple trace subquotient, $\mathcal{C}_{\text{tr}}^r$, which is the starting point for the TQFT extending the Reshetikhin-Turaev invariant. The vector space of the torus of the latter is however only $m$-dimensional since the coend of $\mathcal{C}_{\text{tr}}^r$ is smaller than the image of the coend of $\mathcal{C}$.

4.4) Integrals, Semisimplicity, and $x = \mathcal{V}(S^1 \times S^2)$:

In view of the tangle presentation in (4.52) it is clear that the key to understanding the representation of $\Lambda_\Sigma$ in a TQFT, and hence the projectivity $P7$, is to explain the effect of a trivially framed annulus around a vertical strand as in diagram (4.75) in algebraic terms.

Geometrically, the surgery along the annulus $A$ is equivalent to connecting an $S^1 \times S^2$ to the manifold we surger on. The strand $S$ passing through $A$ then indicates a path that generates $\pi_1(S^1 \times S^2)$.

It also has a natural interpretation in the language of cobordism categories, if we consider the tangle in (4.73) as a morphism in $\mathcal{Tgl}^{\infty}(2)$, i.e., the tangle category with one external strand, $S$, and no $\tau$-move. Here the diagram from (4.73) represents a cobordism, $\lambda^{**} : \Sigma_{0,2} \to \Sigma_{0,2}$, in $\text{Cob}^{\text{conn}}_3(2)$. It is explicitly given by the cylinder $\Sigma_{0,2} \times [0,1]$, inside of which we have performed a surgery along the meridian of one of the cylindrical boundary pieces. It is clear that if we apply a filling functor, which glues a tube to one of these boundary pieces, the result in $\text{Cob}^{\text{conn}}_3(1)$ will be $\phi_1(\lambda^{**}) = (D^2 \times [0,1]) \# (S^1 \times S^2)$. Moreover, we have $\lambda^{**} \circ \lambda^{**} = \lambda^{**} \# (S^1 \times S^2)$. (here $\#$ is always the sum with the interior).

At this point it turns out to be rather instructive to include into our discussion the 2-categorial picture of cobordisms and TQFT’s, that was outlined in Section 2.4. For example, we can think of a surface with a hole as a 1+1-cobordism $\Sigma^* : \emptyset \to S^1$, and of a cobordism $M^* \in \text{Cob}_3(1)$ as a 2-morphism between two such 1-morphisms. Also, $\Sigma_{0,2} : S^1 \to S^1$ may be seen as the identity 1-morphism $I_{S^1}$ on the circle, and $\lambda^{**} : I_{S^1} \Rightarrow I_{S^1}$ is a 2-endomorphism.

A 2-category also implies a composition operation, $\bullet_1$, which is the usual composition on the 1+1-cobordisms, and which is naturally extended to the cobording 2-morphisms. Since $I_{S^1}$ can be composed with any $\Sigma^*$ we can form the $\bullet_1$-composite of any $M^* \in \text{Cob}_3(1)$ with $\lambda^{**}$. It is clear that the result can be obtained from $M^*$ also by doing a surgery along a meridian that is pushed off the special cylindrical piece of $\partial M^*$. From this and the rules of the tangle presentation $\text{Cob}^{\text{conn}}_3(1) \Rightarrow \mathcal{Tgl}^{\infty}(1)^{\text{conn}}$ it follows easily that if $T(M^*)$ presents $M^*$ the tangle for the composite is given by placing a trivially framed annulus around the entire tangle $T(M^*)$.

If we introduce also $\lambda^{*}_2 := \lambda^{**} \bullet_1 I^*_\Sigma$, we easily verify the following identities from the 2-categorial distributive law:

$$\lambda^{**} \bullet_1 M^* = \lambda^{*}_2 \circ M^* = M^* \circ \lambda^{*}_2 .$$  \hspace{1cm} (4.76)

It is thus both natural and useful to think of $\lambda^{**}$ as a natural transformation on $\text{Cob}^{\text{conn}}_3(1)$. In particular, the associated morphism $\lambda^{*}_{2g}$, for the connected surface $\Sigma_g$ of genus $g$, is presented in $\mathcal{Tgl}^{\infty}(1)$ by $2g$ vertical strands with a trivially framed annulus around them. Comparing this to
(4.52) and using the braided tensor structures of \( \text{Cob}_3^{\text{con}}(1) \) and \( \text{Tgl}^{\infty}(1) \), we find
\[
\Lambda^\bullet_\Sigma = \lambda^\bullet_{\Sigma_1} \boxtimes \cdots \boxtimes \lambda^\bullet_{\Sigma_{k-1}} \boxtimes \mathbb{I}_{\Sigma_K},
\] (4.77)
where the \( \Sigma_j \)'s are the connected components of \( \Sigma \).

From the definition of an extended TQFT as in (2.26) in Section 2.4 - as well as what we expect from the property expressed in (4.76) - it follows that \( \lambda^\bullet \) is represented by a natural transformation of the identity functor of \( \mathcal{C} \) to itself. In particular we have the following:
\[
\lambda(X_\Sigma) := \mathcal{Y}_1^{\text{con}}(\lambda^\bullet_\Sigma) \quad (4.78)
\]
(Recall that a natural transformation \( \lambda \in \text{Nat}(\text{id}_\mathcal{C}, \text{id}_\mathcal{C}) \) consists of an endomorphisms \( \lambda(X) \in \text{End}_\mathcal{C}(X) \) for every object, such that \( f \lambda(X) = \lambda(Y) f \), if \( f : X \to Y \).)

Let us determine a few constraints on the transformation \( X \mapsto \lambda(X) \). To begin with, note that the braid morphisms, \( \varepsilon(\Sigma_1^\bullet, \Sigma_2^\bullet) : \Sigma_1^\bullet \boxtimes \Sigma_2^\bullet \to \Sigma_2^\bullet \boxtimes \Sigma_1^\bullet \) of \( \text{Cob}_3^{\text{con}}(1) \), can be presented in \( \text{Tgl}^{\infty}(1)^{\text{con}} \) by the diagram in (4.53), except that we combine the top ends into one group. Moreover, \( \lambda^\bullet_{\Sigma_1^\bullet} \boxtimes \mathbb{I}_{\Sigma_2^\bullet} \) is given by diagram (4.52), with \( K = 2 \). In the composite of these two tangles we can slide the \( 2g_2 \) lower strands one by one over the annulus and hence turn the overcrossing into an undercrossing. I.e., in \( \text{Cob}_3^{\text{con}}(1) \) we have the following identity:
\[
\varepsilon(\Sigma_1^\bullet, \Sigma_2^\bullet)(\lambda^\bullet_{\Sigma_1^\bullet} \boxtimes \mathbb{I}_{\Sigma_2^\bullet}) = \varepsilon(\Sigma_2^\bullet, \Sigma_1^\bullet)^{-1}(\lambda^\bullet_{\Sigma_2^\bullet} \boxtimes \mathbb{I}_{\Sigma_1^\bullet}). \quad (4.79)
\]

The corresponding conditions on the transformation of \( \mathcal{C} \) is given by
\[
\mu(X,Y)(\lambda(X) \boxtimes \mathbb{I}_Y) = \lambda(X) \boxtimes \mathbb{I}_Y \quad \text{and} \quad \mu(X,Y)(\mathbb{I}_X \boxtimes \lambda(Y)) = \mathbb{I}_X \boxtimes \lambda(Y) \quad (4.80)
\]
where \( \mu(X,Y) \) is the square of \( \varepsilon \), as in (1.64). To be precise, we should impose this relation only for the special objects, \( X_\Sigma \). However, they will be sufficiently big so that this implies the general statement by naturality.

Other topological considerations lead us to impose conjugation invariance \( \lambda(X)^t = \lambda(X^\nu) \). These two properties also correspond to generating, elementary moves in a “Bridged Link Calculus”, which replace the 2-handle slices of the conventional Kirby calculus, see [Ke2].

Equation (1.64) also shows that (1.80) can be derived from a stronger condition, namely that each \( \lambda(X) \) has a decomposition into a monomorphism and an epimorphism, going through a multiple of the unit object:
\[
\lambda(X) : X \xrightarrow{p^\lambda_X} 1 \oplus \cdots \oplus 1 \xleftarrow{i^\lambda_X} X.
\] (4.81)
(see, e.g., [M] for existence and uniqueness of monic-epic-decompositions). Instead of the injection and projection in (1.81) we may also consider the injections \( i^\nu_X \in \text{Inv}(X) \) and projections \( p^\nu_X \in \text{Cov}(X) \), with \( \nu = 1, \ldots, n_X \), for the individual summands \( \cong 1 \) so that \( \lambda(X) = \sum_{\nu=1}^{n_X} i^\nu_X p^\nu_X \). Its action on invariance is determined by naturality, i.e., we have \( \lambda(X)f = \lambda(1)f \) for \( f : 1 \to X \), (using \( \lambda(1) \in k \cong \text{End}_\mathcal{C}(1) \)). We thus find the following linear dependence between the \( i^\nu_X \) and \( f \):
\[
\sum_{\nu=1}^{n_X} \langle p^\nu_X, f \rangle i^\nu_X = \lambda(1) f \quad \text{for all} \ f \in \text{Inv}(X). \quad (4.82)
\]

This formula allows us to establish a relation between the number on the right side and semisimplicity of the category:
Lemma 23 Suppose \( C \) is an abelian, rigid, balanced tensor category over a field \( k = \mathbb{R} \), with finite dimensional morphism sets. Assume further that there is a transformation \( \lambda \in \text{Nat}(id_C, id_C) \), with \( \lambda \neq 0 \), such that (4.81) holds for all objects \( X \). Then

\[ C \text{ is semisimple, if and only if } \lambda(1) \neq 0. \]

**Proof:** In a balanced category we can construct traces, \( tr_X : \text{End}_C(X) \to k \), that are generally cyclic, and respect the tensor product. As in [Ke4], we may then define a category to be semisimple, iff all pairings of the form

\[ \text{Hom}_C(Y, X) \otimes \text{Hom}_C(X, Y) \xrightarrow{\text{composition}} \text{End}_C(X) \xrightarrow{tr_X} k, \]

are non-degenerate. Rigidity allows us to reduce this to non-degeneracy of the pairings of invariance and coinvariance, as in (4.68), for all objects. Now, if \( \lambda(1) = 0 \), (4.82) implies that \( p^\nu_X f = 0 \) for all \( \nu \), \( X \), and \( f \in \text{Inv}(X) \), and hence a degeneracy of the pairing, if \( \lambda(X) \neq 0 \). Since we assumed \( \lambda \neq 0 \), this proves one implication.

If \( \lambda(1) \neq 0 \) and \( R = k \) is a field, it follows from (1.82) that \( \{i^\nu_X\} \) and \( \{\frac{1}{\lambda(1)} p^\nu_X\} \) are dual basis so that we have non-degeneracy. \( \square \)

For a semisimple category it not hard to see from the proof that such a transformations always exists, and that it is (up to a total scaling) uniquely given by the projection \( P_X \in \text{End}(X) \) onto the maximal, trivial sub-object. More precisely, we have

\[ \lambda(X) = \lambda(1) P_X \quad \text{.} \tag{4.83} \]

Next, we shall evaluate \( \mathcal{V}_0(\Lambda_\Sigma) \) for a surface \( \Sigma \) with \( K \) connected components, \( \Sigma_j \). (We shall often use of the abbreviation \( X_j \equiv X_{\Sigma_j} \).

Since \( \mathcal{V}_1^{\text{conn}} \) is a functor of tensor categories, we can find from this an expression for \( \Lambda_\Sigma^* \) using (4.77):

\[ \mathcal{V}_1^{\text{conn}}(\Lambda_\Sigma^*) = \lambda(X_{\Sigma_1}) \otimes \ldots \otimes \lambda(X_{\Sigma_{K-1}}) \otimes \mathbb{I}_{X_{\Sigma_K}} \quad \text{.} \tag{4.84} \]

If we use the factorization (4.81), we can write the action of this morphism on an element, \( f \in \text{Inv}(X_1 \otimes \ldots \otimes X_K) \), as follows:

\[ \mathcal{V}_1^{\text{conn}}(\Lambda_\Sigma^*) f = \sum_{\nu_1} i^\nu_{X_1} \otimes \ldots \otimes i^{\nu_{K-1}}_{X_{K-1}} \otimes \xi^{(\nu_1, \ldots, \nu_{K-1})}, \tag{4.85} \]

where \( \xi^{(\nu_1, \ldots, \nu_{K-1})} = (p^\nu_{X_1} \otimes \ldots \otimes p^{\nu_{K-1}}_{X_{K-1}} \otimes \mathbb{I}_{X_K}) f \in \text{Inv}(X_K) \)

Here we used \( X_k \cong 1 \otimes \ldots \otimes 1 \otimes X_K \). In the semisimple case we find from (4.83) that \( \mathcal{V}_1^{\text{conn}}(\Lambda_\Sigma) f = (\mathbb{I} \otimes \ldots \otimes P_{X_k}) \mathcal{V}_1^{\text{conn}}(\Lambda_\Sigma^*) f \) so that with (4.83) we have

\[ \mathcal{V}_1^{\text{conn}}(\Lambda_\Sigma^*) f = \lambda(1)^{K-1} \left( P_{X_1} \otimes \ldots \otimes P_{X_k} \right) f \quad \text{.} \]

Semisimplicity also implies \( \overline{\text{Inv}} = \text{Inv} \) and \( \overline{\text{Cov}} = \text{Cov} \) so that we can use the bases consisting of the \( i^\nu_X \) and \( p^\nu_X \), in order to express the injections and projections in (4.73) and (4.74). It follows immediately that the projection \( P_\Sigma \) in (4.64) is precisely given by the above tensor product of projections, restricted to invariance. Hence,

\[ \mathcal{V}_0(\Lambda_\Sigma) = \lambda(1)^{K-1} P_\Sigma \quad \text{.} \tag{4.86} \]
In the case $\lambda(1) = 0$ it follows from (1.82) that the vectors $i_X^\nu$ and $p_X^\nu$ all lie in the null spaces $Inv^0(X)$ and $Cov^0(X)$, respectively. If $K > 1$, this and (4.85) imply that

$$\mathcal{V}_1^{conn}(t) \in Inv^0(X_1) \otimes \ldots \otimes Inv^0(X_{K-1}) \otimes Inv(X_K) \subset Inv^0(X_1 \boxtimes \ldots \boxtimes X_K) .$$

It follows that $\mathcal{V}_0(t) = 0$, i.e., (4.86) also holds in the non-semisimple case. (For the case $K = 1$ we have $\Lambda = I$, and thus $\mathcal{V}_0(t) = I_{X}$ for either case.)

In order to prove that the property P7 follows from the assignment of a natural transformation with a decomposition as in (4.81), we still have to make the identification

$$\lambda(1) = x \equiv \mathcal{V}_0(S^1 \times S^2) . \tag{4.87}$$

To this end we shall make the assumption that the unit object in $C$ is is irreducible, and that $\mathcal{V}_1^{conn}$ also preserves unit objects, i.e., $X_{S^2} = 1$. Equation (4.87) follows now from $\lambda_{S^2} = (I_{S^2}) \# (S^1 \times S^2)$.

For semisimple categories existence and uniqueness of transformations as in (4.87) is obvious. Still, we wish to understand this assertion also in the general case, and make sure that the construction, e.g., in Theorem 7 actually assigns such a transformation to $\lambda^{**}$. For this purpose it will be both useful and instructive to attribute another algebraic interpretation to $\lambda^{**}$.

This has its origins in the case $C = A - mod$, where $A$ is a finite dimensional Hopf algebra. Here a natural transformation, $\lambda$, of the identity functor is uniquely identified with a central element, $\lambda \in Z(A)$, of the Hopf algebra. As the trivial representation of $A$ is given by its counit $\epsilon$, the relation (4.81) translates to

$$y\lambda = \lambda y = \epsilon(y)\lambda \quad \text{for all} \quad y \in A , \quad \text{and} \quad \lambda(1) = \epsilon(\lambda) \quad . \tag{4.88}$$

Elements $\lambda$ satisfying this relation, which we will call (two-sided) cointegrals, are well known in the theory of Hopf algebras. (In a more common convention $\lambda$ is actually the cointegral of $A^*$, which is the algebra used in the categorical description). Existence and uniqueness of integrals and cointegrals has been proven for finite dimensional Hopf algebras in [Sw]. In [LS] it is also shown that $A$ is semisimple, if and only if $\epsilon(\lambda) \neq 0$. Thus Lemma 23 is just the categorical generalization of this result.

The action of the associated natural transformation $X \mapsto \lambda(X)$ of the identity on $A - mod$ to itself, is given by the canonical application of $\lambda$ to the $A$-module, $V_X$. We have $n_X = \dim(\text{im}(\lambda(X)))$.

For example, if $A = U_q(sl_2)'$, with $q$ a root of unity, it follows from its representation theory, see [Rp], that $n_X$ is given by the number of times $P_0$ occurs as a direct summand in $V_X$, where $P_0$ is the indecomposable, projective representation that contains an invariant vector. In particular, $n_X = 0$ if $V_X$ is fully reducible.

The notion of cointegrals can be generalized to the type of tensor categories, $C$, considered in Theorem 4 for which the coend $F = \int X^{-1} \otimes X$ exists.

In [L1] it is shown that $F$ has the structure of a categorial, braided Hopf algebra in $C$. Moreover, there is a natural Hopf algebra pairing

$$\omega : F \boxtimes F \rightarrow 1 ,$$

for which the left and right null space $\ker(\omega)^* \rightarrow F$ coincide. Let us call $\omega$ balanced, if $\ker(\omega)$ is preserved by the balancing $v \in Nat(I_C)$, where the action of $v$ is given by the lifting of $I_C v(X) \in End(X^{-1} \boxtimes X)$. This property has been introduced in [L1] as axiom (M2). We shall also say that $C$ is strictly modular if $\ker(\omega) = 0$. 

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The algebra of natural transformations of the identity of \( C \) is canonically isomorphic to the elements in \( \text{Cov}(F) \). Hence a cointegral of \( F \) is defined to be a morphism,

\[
\begin{array}{c}
F \xrightarrow{\Delta} F \boxtimes F \\
\lambda : F \to 1, \text{ such that } \lambda \downarrow \text{ commutes,} \\
1 \xrightarrow{e} F
\end{array}
\]

where \( \Delta \) is the coproduct and \( e \) is the unit of \( F \). (The previously discussed case of an ordinary Hopf algebra is implied by the identity \( \text{Cov}(F) = \text{Cov}(A^*, ad^*) \cong Z(A) \).)

We also have the dual notion of a categorial integral for \( F \), which is given by an invariance,

\[
\begin{array}{c}
F \xrightarrow{\mu \boxtimes 1} F \boxtimes F \\
\mu : 1 \to F, \text{ such that } 1^* \downarrow \text{ commutes,} \\
1 \xrightarrow{\mu} F
\end{array}
\]

with \( m \) being the multiplication, and \( 1^* \) the counit of \( F \).

The existence and uniqueness, proven for ordinary Hopf algebras in [Sw], is generalized to the categorial versions in the following theorem:

**Theorem 8 ([L1])** Suppose \( C \) is a category as above, which contains the coend \( F \).

Then there exist (up to scalings) a unique integral, \( \mu : I_\mu \to F \), and a unique cointegral, \( \lambda : F \to I_\lambda \), where \( I_\mu \) and \( I_\lambda \) are invertible.

If the pairing, \( \omega \), is balanced, then \( I_\mu \cong I_\lambda \cong 1 \), and \( \lambda \) and \( \mu \) are as in (4.89) and (4.90), respectively.

In the construction in [L2] and [KL] the transformation that is a-priori associated to the diagram in (4.75) is given as a coinvariance of \( F \) by the composite

\[
\lambda^+ : F \xrightarrow{I_F \boxtimes \mu} F \boxtimes F \xrightarrow{\omega} 1 .
\]

The following lemma compels us to add non-degeneracy of \( \omega \) to our list of requirements on \( C \).

**Lemma 24 ([Ke3])**

Suppose \( C \) is as above, and \( \omega \) is balanced.

Then \( \lambda = \lambda^+ \), if and only if \( C \) is strictly modular.

This now establishes the fact that the construction used in Theorem 8 does in fact associate the cointegral of \( F \) to the diagram in (4.75).

Still, we need to show that the image of \( \lambda(X) \) is actually of the form \( 1 \oplus \ldots \oplus 1 \), in order to complete our construction. For categories over fields, \( k \), with \( \text{char}(k) = 0 \), this is implied by a result of P. Deligne, which asserts that an object, on which the coaction of \( F \) is trivial, necessarily has to be the direct sum of units. It is found [D2] as a corollary to the existence of tensor products of abelian categories. The same result has been proven independently by V. Lyubashenko [L3] by the use of squared coalgebras:
Lemma 25 ([L3],[D2])

The cointegral of a braided tensor category over a field \( k \), with \( \text{char}(k) = 0 \), and with coend \( F \) factors for each object as in \((4.81)\).

It thus follows that the construction in Theorem 7 also implies Property P7.

4.5) Main Result, and Hints to Further Generalizations and Applications:

In this concluding section we shall summarize the various possible deviation from our axioms, that we pointed out throughout this chapter, in order to find more general types of non-semisimple TQFT’s for disconnected surfaces than half-projective TQFT’s. We also discuss possible applications of the half-projective formulation to “classical limits”, where the \( x \) has the role of a renormalization parameter with \( x \to \infty \). To begin with, let us state the main result of this chapter, which follows from results of the preceding three sections, and which completes the construction in [KL]:

**Theorem 9** Suppose that \( \mathcal{C} \) is a abelian, rigid, balanced, strictly modular, braided tensor category, which is defined over a field \( k \) with \( \text{char}(k) = 0 \), and which contains the coend \( F \).

Then there exists a half-projective TQFT-functor, w.r.t. \( \mu_0 \) and \( x \in k \),

\[
\mathcal{V}_0 : \text{Cob}_3(0) \longrightarrow \text{Vect}(k) \quad \text{with} \quad \mathcal{V}_0(\Sigma_g) = \text{Inv}(F \bigcirc \ldots \bigcirc F) \text{,}
\]

such that

\[
x \neq 0 \quad \text{if and only if} \quad \mathcal{C} \text{ is semisimple .}
\]

Moreover, for \( x = 0 \) the functor \( \mathcal{V}_0 \) restricted to \( \text{Cob}_3^{\text{conn}}(0) \) is the null space quotient of \( \mathcal{V}_0^{\text{conn}} \), and for \( x \neq 0 \) it is the TQFT extending the Reshetikhin-Turaev invariant.

The arguments given in Section 4.2 that lead up to P2, and the ones given in Section 4.3, yielding Lemma 20, essentially necessitate the division of the original invariances by the null spaces \( \text{Inv}^0(\Sigma_g) \), in order to achieve compatibility of the TQFT with the tensor structures in \( \text{Cob}_3(0) \) and \( \mathbb{R} - \text{mod} \), as required in the axioms V1-V5. Nevertheless, by Theorem 7 these spaces do carry quite interesting representations of mapping class groups, or other features of cobordisms in \( \text{Cob}_3^{\text{conn}}(0) \).

In Section 2.3 we alluded to the possibility of circumventing the alternative of Corollary 1 by admitting non-canonical symmetry structures on \( \mathbb{R} - \text{mod} \) so that \( x \in \mathbb{R} \) may be, e.g., nilpotent. This would yield a richer filtration of \( \text{Inv}(X) \) with respect to the action of \( \lambda(X) \) on \( X \).

A more promising approach is to relax the Axioms V1-V5. Recall for example that V5 has already been put in by hand for mostly technical reasons. In order to explain what we might anticipate as a generalization of the tensor product rule, let us show that, in some cases, we can think of the \( \text{Inv} \)-spaces as cohomologies. In the non-semisimple case (\( \mathbb{R} = k \) a field) we have that \( \lambda \) acts like a differential, i.e., \( \lambda(X)^2 = 0 \).

Note also that \( \text{im}(\lambda(X)) \subset \text{Inv}^0(X) \) for all \( X \), and that \( \text{Inv}(X) \subset \ker(\lambda(X)) \).

For \( \mathcal{C} = U_q(\mathfrak{sl}_2) - \text{mod} \), and if \( X \) is the sum of only projective and irreducible representations, it follows from the representation theory of \( U_q(\mathfrak{sl}_2) \), see [Rp], that the first inclusion for the image is in fact an isomorphism. Moreover, if we restrict the action of \( \lambda(X) \) to the trivial weight space of \( X \), then the second inclusion for the kernel is an isomorphism for \( X = 1 \) and for \( X = P_0 \) (i.e., the unique indecomposable, projective representation that contains an invariant vector).
It has been worked out in [Os] that the adjoint representation does in fact contain only projective and irreducible summands. Thus we conclude for the vector space of the torus

\[ \mathcal{V}(S^1 \times S^1) \equiv \overline{\text{Inv}(F)} = H^*(F_{00}, \lambda(F)), \]

where the coend \( F \) is as usual given by the functions on \( U_q(s\ell_2) \) with coadjoint representation, and \( F_{00} \) is the intersection of the summand with trivial Casimir value and the trivial weight space.

This formula for the vector space of a torus suggests to consider besides the tensor product also a derived functor, analogous to \( \text{Tor} \), in order to retain some information of the \( \text{Inv}^0 \)-spaces.

The formalism of half-projective TQFT’s may also have applications in the case, where \( \mathcal{C} \) is semisimple so that \( x \in k \) is invertible. For a fixed, finite \( x \) we recover an ordinary TQFT by rescaling the canonically constructed TQFT functor by \( \mathcal{V}(M) \rightarrow x^{-\beta(M)} \mathcal{V}(M) \). Yet, for “classical limits”, in which the renormalization parameter \( x \) will tend to infinity, the anomaly may give us some estimates on the divergence of the canonically defined \( \mathcal{V}(M) \). Unlike the non-semisimple case not only triviality but the exact value of \( g(M) \) should be of interest.

In the construction from [KL] the integral and cointegrals admit a canonical normalization:

\[ \mu = \mathcal{D}^{-1} \sum_{j \in J} d(j) \tilde{\coev}_j \quad \text{and} \quad \lambda = \mathcal{D} \text{ev}_1, \]

where \( \text{ev} \) and \( \tilde{\coev} \) are the evaluation and flipped coevaluation, \( d(j) \) are the quantum-dimensions for simple objects \( j \in J \), and

\[ x = \mathcal{D} := \pm \sqrt{\sum_{j \in J} d(j)^2}. \]

The notation for \( \mathcal{D} \) is the same as in [T]. The normalization is determined by \( \lambda \cdot \mu = 1 \), and relation (4.91), which are imposed by invariance under the local, interior moves of cancellation (or \( \bigcirc \bigcirc \)-move) and modification, see [Ke3].

For the Chern-Simons quantum field theory with a simple, connected, and simply connected gauge group, \( G \), the set \( J \) is identified with the (highest) weights in an elementary, truncated alcove of the weight space of \( G \), whose size depends on the level of the theory (see, e.g., [KW]). In the classical limit the level goes to infinity so that eventually every dominant weight will belong to \( J \). Hence \( |J| \rightarrow \infty \), and we have

\[ x = \mathcal{V}(S^1 \times S^2) \rightarrow \infty. \]

Evidently, this limit of TQFT’s is ill defined on most cobordisms.

Still, there are cases, in which the limit exists in a given sense. For example, if \( G = SU(2) \), and \( J = \{1, \ldots, N\} \), where the labels are given by the dimension of the irreducible \( SU(2) \)-representation, we can consider the (projective) representation of \( SL(2, \mathbb{Z}) = \pi_0(\text{Diff}(S^1 \times S^1)^+) \) on \( \mathcal{V}(S^1 \times S^1) = \mathbb{C}^N \), induced by the TQFT. If we identify the labels with points in \( \mathbb{R}^+ \), via \( x_j = \frac{1}{\sqrt{N}} \), then we can define the limits of the generators of \( SL(2, \mathbb{Z}) \) as unitary operators. In particular, \( S \) is identified with the Fourier transform on \( \mathcal{V}_\infty(S^1 \times S^1) = L^2(\mathbb{R}^+, dx) \), and the limit of \( T \) is given by the multiplication operator of \( e^{ix^2} \) on the same space, where

\[
S := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad T := \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.
\]
For a connected cobordism, $M$, with maximal, free, interior group of rank $\varphi(M)$, we find analogous to Lemma 1 from Theorem 4 and Lemma 13 that

$$V_0(M) = \tau^{\varphi(M)} \cdot V_0(M_2) \cdot V_0(M_1) \quad (4.92)$$

Here, $M_2$ and $M_1$ are connected cobordisms, for which we have $\varphi(M_j) = 0$. Note that the latter also holds true for the invertible cobordisms, for which the classical limit existed.

For general $M$ we may ask by which order $|V_0(M)|$ diverges in the limit with $J, \chi \to \infty$. Assuming that the composition of $V_0(M_1)$ and $V_0(M_2)$ does not degenerate we expect to find from (4.92) that $\varphi(M)$ is at least a lower bound on this order.

It is still quite crude, as we can see in the example of $S^1 \times \Sigma$, where $\varphi(M)$ is roughly half of the true order. More precisely, if we use the Verlinde formula, $\text{dim}(\mathcal{V}_{\Sigma_{g}}) = D^{2g-2} \sum_{j} d(j)^{2-2g}$, (see, e.g., [T]), and that for $SU(2)$ we have $x = D \sim N^{\frac{2}{3}}$, we compute from Lemma 3:

$$V_0(S^1 \times \Sigma_g) \sim x \text{ if } g = 0, \quad \sim x^{\frac{2}{3}} \text{ if } g = 1, \quad \text{and} \quad \sim x^{2g-1} \text{ if } g > 1.$$  

In contrast to that we have that for $g > 0$ the number $\varphi(S^1 \times \Sigma_g)$ is given precisely by the maximal number of non-separating curves on $\Sigma_g$, which is exactly $g$. Also, $\varphi(S^1 \times S^2) = 1$ so that $\varphi(S^1 \times \Sigma_g) = \text{max}(g, 1)$ and the decompositions in Lemma 5 are in fact maximal.

**Appendix**

**A.1) Proofs of Section 3.4 :**

**A.1.1 Proof of Lemma 12 :** Generically we may assume that $f : M \to \gamma$ is a differentiable function, which is transversal to the embedded graph $J(\gamma)$. This implies that there are no critical points in a vicinity of the graph, and that the edges of $J(\gamma)$ are tangential to the level sets of $f$ only at a finite number of interior points. Let us add the latter as vertices of valency two to $\gamma$.

By genericity we may also assume that for the vertices $v_j \in \gamma$, the images $h(v_j) \in \gamma$ (where $h = f \circ J$) are distinct points in the interiors of the edges. We can choose disjoint open intervals $I_j^{*}$ and $\overline{I}_j \subset I_j^{*}$, such that $v_j \in I_j$, and $I_j^{*}$ lies inside of an edge of $\gamma$. Hence we can find a function $\psi : \gamma \to \gamma$, which is the identity outside of all of the $I_j^{*}$, which is strictly monotonous outside of $I_j$, and which collapses $I_j$ to the vertex $v_j$, i.e., $\psi(I_j) = \{v_j\}$. The function $f^* := \psi \circ f$ has the property that it collapses an entire neighborhood of the vertex of the embedded graph $J(\gamma)$ to a point. Since we can choose $f^*$ arbitrarily close to $f$ by making the intervals $I_j^{*}$ smaller and smaller, we shall assume this property, by genericity, already for $f$.

Let us thus consider the compact regions $B_j^{\alpha} = f^{-1}(h(v_j))$ in $M$, for which the $J(v_j) \in B_j^{\alpha}$ are interior points. At points outside of the $B_j^{\alpha}$ and in a vicinity of the edges $f$ is regular, and the level surfaces are transversal to the edges. Hence we may assume that for each edge $e(j, k) \subset \gamma$, joining the vertex $v_j$ to the vertex $v_k$, there is an embedding, $f_{e(j, k)} : D^2 \times [0, 1] \hookrightarrow M$, such that $t \mapsto f_{e(j, k)}(0, t)$ parametrizes $J(e(j, k)) \subset M$, and $f$ is constant on the disc-fibers, i.e., $f(f_{e(j, k)}(p, t)) = f(f_{e(j, k)}(0, t))$. Moreover, we may choose the parametrization, such that $f_{e(j, k)}(B_j^{\alpha}) = D^2 \times [0, \varepsilon]$, and $D^2$ is a disc of radius $\varepsilon$. 

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Next, we choose vicinities of the vertices, given by embeddings, \( \delta_j : (D^3, 0) \hookrightarrow (B^q_j, v_j) \), such that the discs \( \rho_{e(j,k)}(D^2 \times \{\varepsilon\}) \) are (disjointly) contained in \( \delta_j(S^2) \) for every edge joining \( v_j \), and \( \rho_{e(j,k)}(D^2 \times [0, \varepsilon]) \subseteq \text{im}(\delta_j) \).

The union of the images of the \( \delta_j \) and the \( \rho_{e(j,k)} \) forms a neighborhood \( U(\gamma) \) of \( J(\gamma) \) in \( M \). It may be given as a disjoint union of regions

\[
R_{e(j,k)} := \{ \rho_{e(j,k)}(p, t) : |p| < t < 1 - |p|, |p| \leq \varepsilon \},
\]

and their complement, \( \bigcup J_j \). Here, the \( J_j \) are the images of the \( \delta_j \) with the cones \( R^\gamma_{e(j,k)} := \{ \rho_{e(j,k)}(p, t) : |p| < t < \varepsilon, |p| \leq \varepsilon \} \) removed, and are therefore homeomorphic to a cone over a sphere, \( S_j \cong S^2 - \bigcup_j D^2 \), which has a hole for every edge, \( e \), at \( v_j \). We may choose a parametrization, \( \hat{\delta}_j : (S_j \times [0, \varepsilon], S_j \times \{0\}) \rightarrow (CS_j, *) \cong (Y_j, *) \subseteq M \), such that the second parameter is equal to \( t = |p| \) at the common boundary with \( R_{e(j,k)} \).

We define a function \( \overline{\kappa_a} : \gamma \rightarrow \gamma \), which maps an edge, parametrized by \( [0, 1] \rightarrow \gamma : t \mapsto s(t) \), to itself, such that

\[
\overline{\kappa_a}(s([0, a])) = s(0), \quad \text{and} \quad \overline{\kappa_a}(s([1 - a, 1])) = s(1).
\]

Also define the projection

\[
\Pi_{e(j,k)} : R_{e(j,k)} \rightarrow \gamma : \rho_{e(j,k)}(p, t) \mapsto J^{-1}(\rho_{e(j,k)}(0, t)),
\]

and from this the composite

\[
\kappa(\rho_{e(j,k)}(p, t)) := \overline{\kappa_p} \circ \Pi_{e(j,k)}(\rho_{e(j,k)}(p, t)) = J^{-1}\left(\rho_{e(j,k)}\left(0, \frac{t - |p|}{1 - 2|p|}\right)\right).
\]

Moreover, if we define \( \kappa(\text{im}(\hat{\delta}_j)) = v_j \), we obtain a continuous map

\[
\kappa : U(\gamma) \rightarrow M.
\]

Now, \( h : \gamma \rightarrow \gamma \) collapses, in the same way as \( \overline{\kappa_\varepsilon} \), the \( \varepsilon \)-neighborhoods of a vertex onto a point. Hence we can write

\[
h = h^s \circ \overline{\kappa_\varepsilon},
\]

for some \( h^s : \gamma \rightarrow \gamma \), which is again homotopic to one.

Let \( H : [0, \varepsilon] \times \gamma \rightarrow \gamma \) be such a homotopy, i.e., \( H_\varepsilon = h^s \) and \( H_0 = \text{id} \). We then define \( f^s : M \rightarrow \gamma \) by

\[
\begin{align*}
f^s(x) &= f(x), & \text{for } x \in M - U(\gamma) \\
f^s(\rho_{e(j,k)}(p, t)) &= H_p(\kappa(\rho_{e(j,k)}(p, t))) & \text{on } R_{e(j,k)} \\
f^s(\hat{\delta}_j(s, r)) &= H_r(v_j) & \text{on } Y_j.
\end{align*}
\]

For the \( |p| = \varepsilon \)-piece of \( R_{e(j,k)} \) this gives

\[
H_\varepsilon\left(\kappa(\rho_{e(j,k)}(p, t))\right) = h^s \circ \overline{\kappa_\varepsilon} \circ \Pi_{e(j,k)}(\rho_{e(j,k)}(p, t)) = h\left(J^{-1}(\rho_{e(j,k)}(p, t))\right)
= f(\rho_{e(j,k)}(0, t)) = f(\rho_{e(j,k)}(p, t)),
\]

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since \( f \) is constant along the disc-fibers. Also, \( H_{|p|}(\kappa(\rho_{e(j,k)}(p,t))) = H_r(v_j) \), if \( |p| = t = r \), and \( H_\varepsilon(v_j) = h^s(v_j) = h(v_j) = f(Y_j) \), so that \( f^s \) is continuous. Finally, on \( J(\gamma) \) we have

\[
 f^s(\rho_{e(j,k)}(0,t)) = H_0(\kappa(\rho_{e(j,k)}(0,t))) = J^{-1}(\rho_{e(j,k)}(0,t))
\]

so that \( f^s \circ J = id \). \( \square \)

A.1.2 Proof of Part 2) of Lemma 13:

By Lemma 12 we may assume that \( f \circ J \) is the identity. With \( M_n^\circ = f^{-1}(\gamma_n) \), for \( n = 1, 2 \), we thus obtain, as in the proof of the first part, cobordisms with \( M = M_2^\circ \circ M_1^\circ \), and \( \gamma_n \) embeds into \( M_n^\circ \), such that \( \gamma_n \hookrightarrow M_n^\circ \rightarrow \gamma_n \) is the identity. As a first step let us alter \( f \), such that the \( f^{-1}(\gamma_n) \) are connected:

Denote by \( M^1 \) the component of \( M_1^\circ \), which contains \( \gamma_1 \), and by \( B_1^\nu \) the other components. Each of these is a cobordism \( B_1^\nu : \emptyset \rightarrow \Sigma \), where \( \Sigma \neq \emptyset \), and \( \Sigma \subset f^{-1}(P) \), i.e., it is a union of the connected components \( \Sigma^p_j \) from the proof of the first part. For given \( B_1^\nu \), denote by \( Q_\nu \) the set of labels \((p,j)\) that occur in this union. Choose for every \( \nu \) a coordinate graph \( g^\nu : B_1^\nu \rightarrow \gamma_B^\nu \) (e.g., as in Part 1 of Lemma [11]), and extend this to

\[
 \overline{g}^\nu : B_1^\nu = B_1^\nu \cup \coprod_{(p,j) \in Q_\nu} \Sigma^p_j \times [0,\varepsilon] \rightarrow \gamma_B^\nu.
\]

Here we have added the collars of the boundary components, \( \Sigma^p_j \), of \( B_1^\nu \) in \( M_2 \), and elongated the ends of \( \gamma_B^\nu \) by intervals \([0,\varepsilon]\) to obtain \( \overline{\gamma}^\nu \). Each \( \Sigma^p_j \times \{\varepsilon\} \), with \((p,j) \in Q_\nu \), is mapped to an interior point, \( p^\# \), in \( \gamma_2 \). Since \( \gamma_2 \) is connected, there are maps \( h^\nu : \overline{\gamma}_B^\nu \rightarrow int(\gamma_2) \), such that \((p,j) \) is assigned to the respective \( p^\# \). We can thus define a continuous function \( \tilde{f} : M \rightarrow \gamma \), which is given by \( h^\nu \circ \overline{g}^\nu \) on each \( \overline{B}_1^\nu \), and by \( f \) on the rest of \( M \). We now have that \( \tilde{M}_1 := \tilde{f}^{-1}(\gamma_1) \) is connected. In the same way we can define from this coordinate map \( \hat{f} : M \rightarrow \gamma \), for which also \( \hat{f}^{-1}(\gamma_2) \) is connected, and \( \tilde{f}^{-1}(\gamma_1) \) is the composite of \( \tilde{M}_1 \) and the extra components of \( \hat{f}^{-1}(\gamma_2) - \) hence also connected.

We can thus assume that the \( M_j^\nu = f^{-1}(\gamma_j) \) are connected, yet \( f^{-1}(p) \), with \( p \in P = \gamma_1 \cap \gamma_2 \), may still contain several components. As before we have to add an additional edge to \( \gamma \) for each components disjoint from the embedded \( \gamma \). For such a component \( \Sigma \) choose a path \( t \mapsto q(t) \) in \( M_1^* \), which has starting point \( q(0) \in \Sigma \), and which ends in a point in \( J(\gamma_1) \). For the corresponding path \( r(t) := f(q(t)) \) in \( \gamma_1 \), we define the compact subsets \( M_t := f^{-1}(r([0,t])) \), with \( M_t \subset M_s \subset f^{-1}(\gamma_1) \) for \( t < s \). We set

\[
 T := \inf\{t : J(\gamma_1) \text{ and } \Sigma \text{ are connected in } M_t\}.
\]

Since \( M_T = \cap_{t<T} M_t \) is the intersection of compact sets, in which \( J(\gamma_1) \) and \( \Sigma \) are connected, they are also connected in \( M_T \).

Let \( M_T^\Sigma \) be the component of \( \Sigma \) in \( M_t \) for \( t < T \), and \( M_T^\circ \) its complement in \( M_t \). On \( M_T := \bigcup_{t<T} M_t \), we define a function \( f^T \) as follows. It shall be constant on the union \( \overline{M_T}^\Sigma \) of the \( M_T^\Sigma \), with value \( r(T) = f(q(T)) \), and it is \( f \) on \( \overline{M_T}^\circ \). Continuity of \( r \) and \( r(T) \not\in r([0,T]) \) imply that \( \overline{M_T}^\Sigma \) and \( M_T^\circ \) are disjoint open sets in the induced topology of \( M_T \) so that \( f^T \) is continuous. It is also clear that \( f^T|\overline{M_T}^\Sigma \) and \( f^T|M_T^\circ \) have continuous extensions to their closures in \( M_T \), assigning the value \( r(T) \) to the additional boundary points in both cases. In particular, they can be glued to a continuous function on \( M_T \), and, setting \( f^T = f \) on the closure of the complement \( \overline{M_T}^\circ \), to a continuous function on \( M_1^* \).
Define $\gamma'_1$ by attaching an interval, $[0, \varepsilon]$, to $\gamma_1$, such that $\{0\}$ is an endpoint of $\gamma'_1$ and $\{\varepsilon\}$ is identified with $r(T) = \gamma_1$. For a collar, $\Sigma \times [0, \varepsilon] \hookrightarrow f^{-1}(\gamma_1)$, we can then define a modified, continuous function, $f^{\text{??}} : M_1^* \rightarrow \gamma_1'$, by setting $f^{\text{??}} = f^?$ on the complement of the collar, and on $\Sigma \times [0, \varepsilon]$ we define $f^{\text{??}}$ as the projection onto the additional edge, $[0, \varepsilon]$, of $\gamma'_1$. The path $q : [0, T] \rightarrow M_1^*$ is thus connected to $J(r(T))$ in $M_1^*$ by a path $q' : [T, T'] \rightarrow M_1^* - M_1^T$. Clearly, $r'(t) := f^{\text{??}}(q'(t))$ is a closed path in $\gamma_1 \subset \gamma'_1$. Suppose $r''$ is the inverse path (parametrized by $[T', T'']$), in $\gamma_1$, and set $q'' = J \circ r'' : [T', T''] \rightarrow M_1^*$. The path-composite $\hat{q} := q'' * q' * q : [0, T''] \rightarrow M_1^*$ starts at $\Sigma$, and ends in $r(t)$, and, moreover, $f^{\text{??}} \circ \hat{q}$ is homotopic to $[0, \varepsilon] \hookrightarrow \gamma'_1$. Thus if we rescale the parametrization of $\hat{q}$ from $[0, T'']$ to $[0, \varepsilon]$, we have an embedding $J' : \gamma \hookrightarrow M_1^*$, extending $J$, such that $f^{\text{??}} \circ J'$ is homotopic to the identity.

We apply this process to every additional surface, until we arrive at a faithful coordinate graph, $\gamma$, of $M^*$.

\[\square\]

### A.2) The Spaces \(H_{\text{int}}^1(M, G)\), the Numbers \(\beta^\text{int}_j(M)\), and Further Anomalies:

In (2.13) of Section 2.2 we introduced the interior Betti number $\beta^\text{int}_j(M)$ of a cobordism, $M$. The coboundary $\mu_0 = -\delta \beta^\text{int}_0$ turned out to be non-negative and entered the definition of half-projective TQFT’s. In this appendix we shall be interested in the properties of the coboundary $\mu_1 = -\delta \beta^\text{int}_1$, for which we find $\mu_0 := \mu_1 - \mu_0 \geq 0$. If we consider homotopy instead of homology the analog for $\beta^\text{int}_1$ is $\varphi(M)$, which has been introduced in Section 3.2. There we also defined in (3.39) the coboundary $\mu_\pi$, which is in analogy to $\mu_\beta$. Since the explicit computation of $\mu_\pi$ is quite hard, we shall give here instead a derivation of a formula for $\mu_\beta$.

We shall first identify the numbers $\beta^\text{int}_1(M) \in \mathbb{Z}^{+0}$ with the dimensions of the spaces

\[H^\text{int}_1(M, G) := \text{ker}(H_1(\psi, G))\]  

(93)

where we usually consider coefficients $G = \mathbb{Z}$ or $\mathbb{Q}$, and $\psi$ is the inclusion of the punctured boundary pieces, $-\Sigma_\delta \sqcup \Sigma_\tau$, into $M$. In the case, where we have no punctures, i.e., $M = \phi_0(M)$, we know that $H_1(\psi)$ is half-rank, and the dimension is in fact given by $\beta^\text{int}_1$. Hence

\[H^\text{int}_1(M, \mathbb{Z}) = \mathbb{Z}^{\beta^\text{int}_1(M)} \oplus \text{Torsion}\]  

(94)

which, (by naturality of universal coefficients) implies

\[H^\text{int}_1(M, \mathbb{Q}) = H^\text{int}_1(M, \mathbb{Z}) \otimes \mathbb{Q} = \mathbb{Q}^{\beta^\text{int}_1(M)}\]  

(95)

For punctured surfaces these formulae are still correct, since we have in analogy to Lemma 6.1 the following:

**Lemma 26** Suppose $M \in \text{Cob}_3(*)$ is a cobordism, $\phi_0(M) \in \text{Cob}_3(0)$ is as in (2.10), and $i_\phi : M \hookrightarrow \phi_0(M)$ is the respective inclusion.

1. The inclusion induces an isomorphism (for both types of coefficients):

\[H^\text{int}_1(i_\phi) : H^\text{int}_1(M) \isom H^\text{int}_1(\phi_0(M))\]  

2. The first interior Betti-numbers are computed directly from

\[\beta^\text{int}_1(M) = \beta_1(M) - \frac{1}{2} \beta_1(-\Sigma_\delta \sqcup \Sigma_\tau) + \text{dim}(\text{ker}(i_M))\]

where $i_M : H_1(\sqcup^N S^1 \times I) \rightarrow H_1(M)$ is induced by the inclusion of the cylindrical boundary pieces into the cobordism.
Proof: It is enough to prove the identities for gluing a tube $D^2 \times I$ into only one cylinder $S^1 \times I$ of a cobordism $M$. If $\Sigma = -\Sigma_s \sqcup \Sigma_t$, the corresponding surface $\Sigma'$ of the filled cobordism is obtained by gluing discs into the holes of $\Sigma$. Since, the tube and the discs are contractible, the inclusions of $\phi_p$ following sub-spaces of $H_\mu$ from these we can find compositions. An increase in two manifolds over several boundary components as in (2.15). An increase in $\beta_\ast^\dagger$ indicates 1-cycles

\[
\begin{array}{cccccc}
0 \to & H_1(S^1 \sqcup S^1) & \overset{i \Sigma}{\longrightarrow} & H_1(\Sigma) & \overset{p \Sigma}{\longrightarrow} & H_1(\Sigma') \to 0 \\
p_i \downarrow & \psi_1 \downarrow & & \psi_1 \downarrow & & \\
H_1(S^1 \times I) & \overset{i_M}{\longrightarrow} & H_1(M) & \overset{p_M}{\longrightarrow} & H_1(M') \to 0
\end{array}
\]  

(96)

In the top row $p \Sigma$ is onto because the discs were glued to different components of $\Sigma$ (see definition of $Cob_3(\ast)$), and $i \Sigma$ is always into. As $p_1$ is onto, the two sequences factor into the isomorphism in Part 2 of the lemma.

The analog of (96) also holds for the union of all cylindrical pieces $\sqcup^N S^1 \times I$ and $M'$ replaced by $\phi_0(M)$. The formula for $\beta_1^\ast$ in Part 2 is now only a matter of counting dimensions. \qed

For a cobordism $M : \Sigma_s \to \Sigma_t$ there are natural projections $p^l : H_1(-\Sigma_s \sqcup \Sigma_t) \to H_1(\Sigma_t)$, and $p^s$ analogously. For a composite of two cobordisms, $M_2 \circ M_1$, we can define with $\Sigma = \Sigma_{t,1} = \Sigma_{s,2}$ the following sub-spaces of $H_1(\Sigma)$:

\[
V_1 := p_1^l \left( \ker(H_1(\psi_1)) \right), \quad V_2 := p_2^s \left( \ker(H_1(\psi_2)) \right).
\]  

(97)

From these we can find $\mu_1$ as follows:

Lemma 27 For two cobordisms, $M_1 : \Sigma_{s,1} \to \Sigma$ and $M_2 : \Sigma \to \Sigma_{t,2}$, and spaces $V_j$ as above, the cocycles from (2.14) are given by:

\[
\mu_1(M_2, M_1) - \mu_0(M_2, M_1) = \text{codim}(V_1 + V_2).
\]  

(98)

Proof: In order to compute $\mu_1$ let us also introduce the spaces $^t H_1^\ast(M) := \text{coker}(H_1(\psi^t))$, and, correspondingly, $^s H_1^\ast$. We have the following exact sequence:

\[
0 \to V_1 \to H_1(\Sigma) \to ^s H_1^\ast(M_1) \to H_1^\ast(M_1) \to 0
\]  

(99)

The fact that $V_1$ appears here as the kernel is seen, when we divide $0 \to \ker(\psi_1) \to H_1(\Sigma_{s,1} \sqcup \Sigma) \to H_1(M_1)$ by the sub-sequence through $H_1(\Sigma_{s,1})$.

If we divide the Mayer-Vietoris sequence for the gluing of $M_2$ and $M_1$, by the homologies of the outer surface $\Sigma_{s,1} \sqcup \Sigma_{t,2}$, we obtain an exact sequence:

\[
0 \to V_1 \cap V_2 \to H_1(\Sigma) \to ^s H_1^\ast(M_1) \oplus ^t H_1^\ast(M_2) \to H_1^\ast(M_2 \circ M_1) \to G^{\mu_0} \to 0.
\]  

(100)

Combining this with (99) and its analogue for $V_2$, we find the formula for $\mu_1$ by counting dimensions. \qed

Lemma 27 implies that not only $\mu_0$, and $\mu_1$, but also $\mu_\partial := \mu_1 - \mu_0$ are non-negative integers. This means that the interior Betti-numbers $\beta_j^\ast$, and also $\beta_\partial^\ast := \beta_1^\ast - \beta_0^\ast$ can only increase under compositions. An increase in $\beta_0^\ast(M)$ indicates new 1-cycles in $H_1^\ast(M)$ that are created by connecting two manifolds over several boundary components as in (2.13). An increase in $\beta_\partial^\ast$ indicates 1-cycles
from the intermediate surfaces $\Sigma$ that are not killed in the gluing process. The simplest example for
this are two copies of a full torus that are glued together along the same meridians to give $S^1 \times S^2$.
We easily see that in this case $\mu_0 = 1$.

A.3) Summary of Tangle Presentations:

We shall summarize here the tangle presentations of the cobordism category $Cob_3(0)$ of cobordisms
between parametrized, connected surfaces, with central extension by $\Omega_4$, as in [Ke2]. There are several
versions, and we shall choose here the one that involves ordinary, framed tangles, as opposed to bridged
tangles. The bridged versions are more practical for the construction of invariants and connected
TQFT's, as in [KL], but are less common, and involve another type of surgeries.

The presentation is given by an isomorphism functor:

$$\mathcal{T} : Cob_3(0) \cong \mathcal{T} gl(0).$$

On the level of objects we assign to a surface $\Sigma = \Sigma_1 \sqcup ... \sqcup \Sigma_K$, with $K$ ordered, connected
components, a configuration of intervals, called groups, and points on the real line as follows. To
every component $\Sigma_j$ corresponds to a group $G_j \subset \mathbb{R}$, and to $\Sigma$ we associate the disjoint, ordered
union $G_1 \sqcup ... \sqcup G_K \subset \mathbb{R}$. Furthermore, in each interval $G_j$ we pick $2g_j$ points, which we denote in
increasing order $\{a_{j,1}, a'_{j,1}, a_{j,2}, a'_{j,2}, ..., a'_{j,g_j}, a_{j,g_j}\} \subset G_j$.

A connected morphism in $Cob_3(0)$,

$$M : \Sigma_s = \Sigma_{g_1} \sqcup ... \sqcup \Sigma_{g_K} \longrightarrow \Sigma_t = \Sigma'_{g_1} \sqcup ... \sqcup \Sigma_{g_L},$$

is then represented by a generically projected tangle in a strip $\mathbb{R} \times I$, where we have marked the
groups and points of $\Sigma_s$ on the upper boundary, $\mathbb{R}_s$, of the strip, and the data of $\Sigma_t$ on the lower real
line, $\mathbb{R}_t$. The tangle diagram has thus a general form as below, where we indicated groups by braces.

\[\text{R}_s \quad \text{Tangle} \quad \text{R}_t\]

The tangle consists of possibly linked and knotted strands that end in the $2(g_1^s + ... + g_K^s)$ points
at $\text{R}_s$, and the $2(g_1^t + ... + g_L^t)$ points at $\text{R}_t$. To each strand we assign in addition a framing of its
normal bundle, which is the same as considering ribbons. In diagrams we tacitly assume the framing
to be in the plane of projection (blackboard-framing). A component of a tangle in $\mathcal{T} gl(0)$ has to be
one of the following four types:

1. **Source-ribbon**: A strand, that connects a pair in $\text{R}_s$, i.e., it starts at a point $a_{j,\nu} \in G_j^s$ and ends
   in $a'_{j,\nu} \in G_j^s$.

2. **Target-ribbon**: A strand, that connects a pair in $\text{R}_s$, i.e., it starts at a point $a_{j,\nu} \in G_j^s$ and ends
   in $a'_{j,\nu} \in G_j^s$. 

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3. **Closed ribbon**: A strand \((\cong S^1 \times [-\varepsilon, \varepsilon])\) that is disjoint from the boundaries of the diagram.

4. **Through-ribbon-pair**: A pair of strands, connecting a pair in \(R_s\) to a pair in \(R_t\). I.e., one strand connects \(a_{j,\nu} \in G_j^s\) to \(a_{k,\mu} \in G_k^t\) and the other \(a'_{j,\nu} \in G_j^s\) to \(a'_{k,\mu} \in G_k^t\). (Flipped version also admissible)

Furthermore, we require that in all cases the ribbon, as a surface with boundary, has to be oriented, and the orientation shall be compatible with a fixed one at \(R_s\) and \(R_t\), if it ends in the boundary of the strip.

An element in \(T_{gl}(0)\) is then an equivalence class of tangles, where we consider two tangles equivalent, if one can be transformed into the other by the application of a sequence of the following five *moves*:

1. Isotopies with fixed endpoints.
2. The \(O_2\)-move or 2-handle slide of any type of ribbon over a closed ribbon, see [Ki].
3. The \(O_0\)-move, in which we add or remove an isolated Hopf link, for which one component has 0-framing, and the other either 1- or 0-framing.
4. The \(\tau\)-move (at a group \(G_{s/t}^j\)), which allows us to push any type of ribbon through the 2g\(_{s/t}^j\) parallel strands emerging at the group \(G_{s/t}^j\) very close to \(R_{s/t}\), as in the following diagram:

   ![Diagram](image)

   \(\tau\)-move

5. The \(\sigma\)-move (at a pair \(\{a_{j,\nu}, a'_{j,\nu}\} \subset G_{j}^{s/t}\)), in which we replace (say for \(R_s\)) the two parallel strands at a pair by the chain of an upward arc, a 0-framed annulus, and a downward arc as in the following diagram:

   ![Diagram](image)

   \(\sigma\)-move

In general the \(\tau\)-move has to be assumed for all groups both at \(R_s\) and \(R_t\). Yet, if we consider cobordisms in \(Col_{g}^{(s)}(0)\), where we have only one source and one target group, it is easy to see
that one \(\tau\)-move (say at the source) also implies the other. We denote the respective sub-category of tangles with only one group at top and bottom by \(T_{gl}(0)^{\text{conn}}\).

It has been shown in [KL] that the cobordisms \(Cob_3^{\text{conn}}(n)\) are represented by the tangle category \(T_{gl}(n)^{\text{conn}}\). The tangles in there contain in addition to the top-, bottom-, closed, and through-Ribbons, so called exterior ribbons. Each of these starts at a fixed point at \(R_s\) and ends at another fixed point in \(R_t\). They can also be isotoped, and slide over closed ribbons. However, there is no analog of a \(\sigma\)-move, and the groups, which are associated to one-to-one to the components of \(\partial \phi_0(M)\), and onto which we impose the \(\tau\)-move, include the external strands that represent the punctures in the respective surface.

In the special case of \(T_{gl}(1)^{\text{conn}}\) the external ribbon can be isolated, and the class of a tangles is determined by an ordinary tangle without external strand, see [KL]. We therefore introduce the category, \(T_{gl\infty}(1)^{\text{conn}}\), which has the same tangles as \(T_{gl}(0)^{\text{conn}}\) as generators, but where we omitted the \(\tau\)-move, since we cannot isotop through the extra strand. Although their presentations are different we have \(T_{gl}(1)^{\text{conn}} \cong T_{gl\infty}(1)^{\text{conn}}\), which gives us the presentation of \(Cob_3^{\text{conn}}(1)\) we used in Section 4.3, as well as the relation in diagram (4.63). For mapping class groups this coincides with the presentation in [MP].

The composition law in \(T_{gl}(0)\), which makes \(T\) into a functor, has mostly been described already in the discussion given in Section 4.1. If we have two cobordisms,

\[
M : \Sigma_A \rightarrow \Sigma_B \sqcup \Sigma_C \quad \text{and} \quad N : \Sigma_C \sqcup \Sigma_D \rightarrow \Sigma_E,
\]

where \(M\) and \(N\) are connected, but the surfaces \(\Sigma_A, \Sigma_B\) etc., may be disconnected, then the elementary composition \((\mathbb{I} \otimes N)(M \otimes \mathbb{I}) : \Sigma_A \sqcup \Sigma_D \rightarrow \Sigma_B \sqcup \Sigma_E\) is represented by the following tangle:

Here \(T(M)\) and \(T(N)\) are tangles representing \(M\) and \(N\), and the tangle \(\Lambda_{\Sigma_C}\) is as in (4.52) of Section 4.1. The braces in this diagram enclose the union of the groups in the indicated surface, i.e., we have for example \(\beta_0(\Sigma_A) + \beta_0(\Sigma_D)\) groups at \(R_s\), with \(\beta_1(\Sigma_A) + \beta_1(\Sigma_D)\) emerging strands.

The action of the symmetric group is as described in (4.53). The presentation of a disconnected cobordism is given simply by the presentations of its connected components, if those are in a compatible order.
The interior groups can also be computed from the diagrams, in an analogous way as homotopy and homology groups are computed for link presentations of closed manifolds. Using the explicit form of the functor $\mathcal{T}$, see [Ke2], it is not hard to see that, e.g., an $a$-cycle in $H_1(\Sigma_s)$ corresponds to a small meridian around the corresponding source-ribbon, and the $b$-cycle of the same handle corresponds to a path that is pushed off the same source-ribbon.

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