The Inertial Theorem

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We present a new theorem that enables to approximate the evolution of a quantum system, driven by an external field. The theorem, coined as ‘inertial theorem’, is valid for fast driving provided the acceleration rate is small. Two explicit solutions, of an harmonic oscillator and a two-level-system model, are analyzed. These examples demonstrate that the inertial result is superior to the one obtained by the adiabatic approximation. The inertial solution is employed to obtain a Markovian Master equation, extending the description to driven open quantum systems. We explore the consequence of new geometric phases associated with the driving parameters.

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I. INTRODUCTION

Quantum state manipulation is an integral part of contemporary quantum science \cite{1,19}. Manipulation of a quantum state is achieved by engineering the Hamiltonian by means of external driving \cite{20}. The formal difficulty in describing such processes is that typically the Hamiltonian does not commute with itself at different times, $[\hat{H}(t), \hat{H}(t')] \neq 0$. As a result, the solution of the dynamics is confronted with the obstacle of time-ordering \cite{21}.

A remedy for solving such quantum dynamics is the adiabatic approximation \cite{22–26}. The validity of the adiabatic solution requires that the change of the Hamiltonian does not commute with itself at different times, $[\hat{H}(t), \hat{H}(t')] \neq 0$. As a result, the solution of the dynamics is confronted with the obstacle of time-ordering \cite{21}.

A remedy for solving such quantum dynamics is the adiabatic approximation \cite{22–26}. The validity of the adiabatic solution requires that the change of the Hamiltonian does not commute with itself at different times, $[\hat{H}(t), \hat{H}(t')] \neq 0$. As a result, the solution of the dynamics is confronted with the obstacle of time-ordering \cite{21}.

We identify in the inertial solution a dynamical and geometric phase $\phi$. The geometric phase requires an evolution characterized by a closed circuit in a parameter space associated with the driving. These phases are manifested in global character of the quantum dynamics.

II. THE INERTIAL THEOREM

The derivation of the inertial theorem is conducted in a state space of system operators $\{\hat{X}\}$, endowed with an inner product defined by $\langle \hat{X}_i, \hat{X}_j \rangle \equiv \text{tr}(\hat{X}_i^\dagger \hat{X}_j)$. Such a state space is called Liouville space, and is also known as the Hilbert-Schmidt space \cite{34,35}.

In the Liouville representation, the system’s dynamics
are calculated in terms of a chosen basis of orthogonal operators \{\hat{B}\}, spanning the Liouville space. This basis of operators construct a vector \(\vec{v}(t)\). For example, for a two-level system the dynamics is described in the Bloch sphere, where the basis is constructed from the Pauli operators.

Employing the Heisenberg equation of motion, the dynamics of \(\vec{v}\) are given by

\[
\frac{d}{dt} \vec{v}(t) = \left( i \left[ \hat{H}(t), \bullet \right] + \frac{\partial}{\partial t} \right) \vec{v}(t) ,
\]

where the convention \(\hbar = 1\) is used throughout this paper.

Here, we consider a finite time-dependent basis of size \(N\) that also forms a closed Lie algebra, which guarantees that the Heisenberg equations of motion, Eq. (1), can be solved within the basis \(\mathcal{B}\). This applies trivially for any finite Hilbert space, or else, when a closed sub-algebra can be found. For example, the Heisenberg Weyl group which defines the Gaussian states of the quantum harmonic oscillator \(\mathcal{B}\). It is useful to limit the description to the minimal sub-algebra required to solve the system’s dynamics. In the case of compact algebras this greatly simplifies the analysis, while for non-compact algebras, finding a sub-algebra is a prerequisite for constructing the inertial solution.

For a closed Lie algebra, equation (1) obtains the simple form

\[
\frac{d}{dt} \vec{v}(t) = -i \mathcal{M}(t) \vec{v}(t) ,
\]

where \(\mathcal{M}\) is a \(N\) by \(N\) matrix with time-dependent elements and \(\vec{v}\) is a vector of size \(N\). In general, the matrix \(\mathcal{M}\) can be non-Hermitian.

The derivation proceeds in the following strategy: We search for a driving protocol that allows solving Eq. (2) explicitly and then extend the solution for a broad range of protocols employing the inertial approximation. We assume that by choosing a unique driving protocol and the suitable time-dependent operator basis, the time dependency can be factorized out, implying that

\[
\mathcal{M}(t) = \Omega(t) \mathcal{B} \left( \vec{\chi} \right) .
\]

Here, \(\Omega(t)\) is a time-dependent real function, and the matrix \(\mathcal{B}(\vec{\chi})\) is a function of the constant parameters \(\{\chi\}\). These parameters are expressed in a short notation as a vector \(\vec{\chi} = (\chi_1, \chi_2, ..., \chi_k)^T\). We restrict our analysis to the case where the spectrum of \(\mathcal{M}\) is non-degenerate. In the explicit examples presented there is a single parameter \(\chi = \chi\) which is equal to the adiabatic parameter \(\mu\).

Once the decomposition is obtained, the dynamics can be expressed as

\[
\frac{d}{d\theta} \vec{v}(\theta) = -i \mathcal{B} (\vec{\chi}) \vec{v}(\theta) ,
\]

here, \(\theta \equiv \theta(t) = \int_0^t dt' \Omega(t')\) is the scaled time. The solution of Eq. (4) is straightforward, yielding

\[
\vec{v}(\theta) = \sum_k c_k \vec{F}_k (\vec{\chi}) e^{-i\lambda_k \theta} ,
\]

where \(\vec{F}_k\) and \(\lambda_k\) are eigenvectors and eigenvalues of \(\mathcal{B}\) and \(c_k\) are constant coefficients. Each eigenvector \(\vec{F}_k\) corresponds to the eigenoperator \(\vec{F}_k\), obtained by summing over the product of \(\vec{F}_k\) vector elements, \(\vec{f}_k\), and the basis operators \(\vec{v}\), \(\vec{F}_k = \sum_i \vec{f}_k \vec{v}\). The eigenoperators are defined by the left eigenvectors of \(\mathcal{B}\), for compact algebras the matrix \(\mathcal{B}\) is Hermitian and the left and right eigen-vectors are conjugates. Notice that the inertial theorem incorporates the adiabatic theorem, since for slow driving, the matrix \(\mathcal{M}(t)\) can be diagonalized at each instant. We then obtain the eigenoperators associated with \(\mathcal{M}(t)\) and the decomposition in Eq. (5).

The advantage of the factorization of equation (4) is that it allows an explicit solution of the dynamics even for cases when the operator basis is time-dependent. As a result, the solution circumvents the time-ordering operation. However, the solution is limited by the constraint that requires \(\mathcal{B}\) to be a constant matrix, i.e., \(\vec{\chi} = \text{const}\). This condition restricts the relevant driving protocols.

For general protocols when \(\vec{\chi}\) varies with \(\theta\), the solution can be extended for the case of slowly varying \(\vec{\chi}(\theta)\). The entire proof is reported in Appendix A, and follows a similar mathematical construction as the adiabatic theorem \(22, 25, 40, 41\), here, we present the final result and discuss the consequences.

For a state \(\vec{v}\), driven by a protocol with a very slow acceleration, the system’s evolution is given by

\[
\vec{v}(\chi, \theta) = \sum_k c_k e^{-i \int_0^\theta d\theta' \lambda_k} e^{i \phi_k} \vec{F}_k (\chi(\theta)) ,
\]

where the dynamical phase is \(\lambda_k = \lambda_k(\theta)\) with \(\theta_0 = \theta(0)\), \(\theta = \theta(t)\) and the second exponent includes a new geometric phase

\[
\phi_k(\theta) = i \int_{\chi(\theta_0)}^{\chi(\theta)} d\chi \left( \vec{G}_k |\nabla_\chi| \vec{F}_k \right) .
\]

Here, \(\vec{G}_k\) are the bi-orthogonal partners of \(\vec{F}_k\), Cf. Appendix A. The system’s state follows the instantaneous solution determined by the instantaneous \(\chi(\theta)\) and phases associated with the eigenvalues \(\lambda_k\) and eigenoperators \(\vec{F}_k\) of \(\mathcal{B}(\chi(\theta))\). Here, we restrict the analysis to the case where \(\lambda_k\) do not cross, hence, the spectrum of \(\mathcal{B}\) remains non-degenerate throughout the evolution. Substituting the inertial solution, Eq. (6), into Eq. (4) allows to assess the validity of the approximation, in terms of the ‘inertial parameter’ \(\Upsilon\) (Appendix A) reads

\[
\Upsilon = \sum_{n,m} \left| \frac{\langle \vec{G}_k |\nabla_\chi \mathcal{B}| \vec{F}_n \rangle}{(\lambda_n - \lambda_k)^2} \cdot \frac{d\chi}{d\theta} \right| .
\]
This implies that the solution, Eq. (6), remains valid when $\tilde{\chi}$ follows a path in parameter space where the eigenvalues $\lambda_k$ and $\lambda_n$ are sufficiently distant from degeneracy [20].

The inertial theorem is applicable when the dynamics of the system can be cast in the form of Eq. (1). In the following, two such examples are presented.

### A. Parametric driven harmonic oscillator

A demonstration of the inertial theorem is illustrated by the parametric harmonic oscillator. Physically the system can be realized by a particle in a varying harmonic potential [18]. The Hamiltonian reads

$$\hat{H} (t) = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 (t) \hat{q}^2 ,$$

where $\hat{q}$ and $\hat{p}$ are the position and momentum operators, $m$ is the particle mass and $\omega (t)$ is the oscillator frequency.

We consider an initial Gaussian state, such a state is fully defined by the set of time dependent operators: $\hat{L} (t) = \frac{\hat{p}^2}{2m} - \frac{1}{2} m \omega^2 (t) \hat{q}^2$, $\hat{C} (t) = \frac{\omega (t)}{2} (\hat{q} \hat{p} + \hat{p} \hat{q})$, $\hat{K} (t) = \sqrt{\omega(t)} \hat{q}$, $\hat{J} (t) = \frac{\hat{p}}{\omega(t)}$ and the Hamiltonian, $\hat{H} (0) = \hat{K} (0)$ [11, 12]. This set of operators constitutes a basis of the Liouville space and fulfills the requirements that lead to the decomposition of Eq. (4). Writing the dynamics in terms of the vector $\vec{v} = \{ \hat{H}, \hat{L}, \hat{C}, \hat{K}, \hat{J}, I \}^T$ gives

$$\frac{d}{dt} \vec{v} (\theta) = -i \mathcal{B} \vec{v} (\theta)$$

with,

$$\mathcal{B} = i \left[ \begin{array}{cccc} \chi & -\chi & 0 & 0 & 0 & 0 \\ -\chi & \chi & -2 & 0 & 0 & 0 \\ 0 & 2 & \chi & 0 & 0 & 0 \\ 0 & 0 & 0 & \chi & 1 & 0 \\ 0 & 0 & 0 & -1 & -\frac{\chi}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \chi \end{array} \right].$$

Here, $\chi = \mu = \frac{\omega}{\sqrt{\kappa^2 + \omega^2}}$, where $\mu$ is the adiabatic parameter, and $\theta = \int_0^t dt' \omega (t')$. For a constant $\chi$, $\mathcal{B}$ is a constant matrix and the eigenoperators and eigenvalues, $\hat{F}_k$ and $\lambda_k$, are obtained by diagonalization, see Appendix B. The matrix $\mathcal{B}$ has real eigenvalues which possess a non-hermitian degeneracy for $\chi = \mu = 2$ [33]. This limits the solution to avoid the proximity of the degeneracy point. The inertial parameter, Eq. (8), obtains the form $\Upsilon \sim \frac{\mu^4 (t)}{(2\kappa)^3}$, where $\kappa = \sqrt{4 - \chi^2}$, which explicitly becomes

$$\Upsilon = \frac{\mu^2 (\frac{\omega}{\kappa} - 2\mu^2)}{(2\kappa)^2 \left( \frac{\omega}{\kappa^2} \log \left( \frac{\omega (t)}{\kappa} \right) - \mu^2 \left( 2 \log \left( \frac{\omega (t)}{\kappa} \right) + 1 \right) \right)} .$$

When $\Upsilon \ll 1$ the inertial solution, Eq. (8), is a good approximation for the true dynamics.

For the demonstration, we consider a particle of mass $m = 1$ in a varying harmonic potential. The initial condition is the ground state $\rho (0) = |0 \rangle \langle 0 |$, associated with the initial frequency $\omega (0) = 20$.

In order to evaluate the inertial approximation we first obtain an exact numerical solution of the dynamics, defining $\hat{\rho}_N$. The fidelity $\mathcal{F}$ of the inertial and adiabatic solutions are calculated in terms of the Bures distance with respect to $\hat{\rho}_N$, $\mathcal{F} = | \text{tr} (\sqrt{\hat{\rho}_N \hat{\rho} \hat{\rho}_N} \hat{\rho} \hat{\rho}_N) |^{-\frac{1}{2}}$ [44, 46]. The fidelities are compared in Fig. 1.

For the analysis, we use the protocol: $\omega (t) = \omega (0) (\chi (0) t + \frac{1}{2} t^2)$. Such a protocol satisfies

$$\chi (t) = \mu (t) = \chi (0) + a \cdot t .$$

The protocol is designed to give the same initial and final frequency and parameter $a$, while $t_f$ and $\chi (0)$ are adjusted accordingly. Modifying the protocol duration, $t_f$ interpolates between the sudden and adiabatic limits. We neglect the geometric phases in the performed calculation, as they are proportional to $d\chi /dt$, for a non-closed circuit in the parameter space, more details in Sec. IV and Appendix B.1.

Figure 1 shows the fidelity, $\mathcal{F}$, of the final state as a function of the protocol duration, $t_f$. For large $t_f$ both the adiabatic and the inertial parameters are small. While for short protocol duration the adiabatic parameters increases and $\Upsilon$ remains small, therefore, the inertial solution remains valid. This is shown in Figure 2. As described previously, the inertial and adiabatic parameters indicate the quality of the solutions. When $\Upsilon < \mu$ we expect that the inertial approximation leads to improved results. The comparison indicates that the inertial approximation outperforms the adiabatic approximation.

For a general protocol, the relative accuracy of the inertial and adiabatic solutions can be compared using $\Upsilon$, Eq. (12), and the instantaneous $\mu$. The result depends in a non-trivial way on the protocol, determining $\omega (t)$, and its first and second derivative as well as the gap between the eigenvalues of $\mathcal{B}$. The inertial parameter is proportional to $\Upsilon \propto \mu^2 \mu$. This, when $\mu$ is small, the inertial approximation is superior over the adiabatic one, when the adiabatic approximation is valid.

### B. Two-Level-System model

We study a driven two-level system with a Hamiltonian

$$\hat{H} (t) = \omega (t) \hat{S}_z + \epsilon (t) \hat{S}_x ,$$

here, $\hat{S}_i$ is the $i = x, y, z$ spin operator. The time-dependent Rabi frequency is $\Omega (t) = \sqrt{\omega^2 (t) + \epsilon^2 (t)}$.

The dynamics of the system is analyzed employing a time-dependent operator basis $\{ \hat{H}, \hat{L}, \hat{C}, \hat{I} \}$, with $\hat{H} (t) = \omega (t) \hat{S}_z + \epsilon (t) \hat{S}_x$, $\hat{L} (t) = \epsilon (t) \hat{S}_z - \omega (t) \hat{S}_x$, $\hat{C} (t) = \Omega (t) \hat{S}_x$. 


FIG. 1. The fidelity of the final state as a function of the protocol time, \( t_f \), for the harmonic oscillator (A) and two-level-system (B). As the accuracy improves the fidelity reaches unity. (Inset) The quality measure \( -\log (1 - F) \), of the inertial solution, as a function of time. This functional dependence presents the accuracy at a higher resolution. As the fidelity converges to unity the fidelity at small times can be explained by the inertial solution, as a function of time. This functional dependence presents the accuracy at a higher resolution.

The equation of motion for the Liouville vector \( \vec{r}(t) = \{H, L, C\}^T \) is of the form

\[
\frac{1}{\Omega} \frac{d\vec{r}}{dt} = \frac{\Omega}{\Omega^2} \vec{r} - i\mathcal{B}\vec{r},
\]

where

\[
\mathcal{B} = i \begin{bmatrix} 0 & \chi & 0 \\ -\chi & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.
\]

Here, \( \tilde{\chi} = \chi = \tilde{\mu} = \frac{\omega_0 - \omega^2}{\Omega^2} \), where \( \tilde{\mu} \) is the adiabatic parameter of the TLS. To transform Eq. (15) to the factorized form, Eq. (3), we define \( \theta = \int_0^t dt' \Omega \) and a scaled vector

\[
\tilde{u}(\theta) \equiv r(\theta) e^{-\int_0^\theta d\theta' \left( \tilde{\Omega}/\Omega^2 \right)},
\]

for which the dynamics obtains the desired form, \( \frac{d}{d\theta} \tilde{u} = -i\mathcal{B}\tilde{u} \). This procedure is not limited to the TLS and relies on the fact that the identity \( \mathcal{I} \) always commutes with \( \mathcal{B} \).

We consider a protocol of constant \( \varepsilon \) with a linear change in \( \tilde{\chi} \), \( \tilde{\chi}(t) = \mu(t) = \chi(0) + \tilde{a} \cdot t \). This leads to the following protocol \( \omega(t) = \varepsilon \frac{z(t)}{\sqrt{1 - z^2}} \), where \( z = \varepsilon \left( \chi(0) t + \frac{1}{2} t^2 + \frac{\omega(0)}{\sqrt{1 - z^2}} \right) \). Using this protocol the exact, adiabatic and inertial solutions were calculated. Following the same procedure as in Sec. [II.A] Fig. [1B] illustrates the superiority of the inertial solution over the adiabatic result.

III. EXTENDING THE INERTIAL THEOREM TO OPEN SYSTEM DYNAMICS

The inertial solution describes the free propagation of isolated systems. When the system is coupled to an external bath, its dynamics is modified by the system-bath interaction. Our purpose is to obtain a reduced description of the system dynamics, where the influence of the bath is treated implicitly.

The derivation first requires solving the free propagation of isolated systems. Then, in order to obtain the reduced dynamics, we utilize the inertial theorem to expand \( H_I \) in terms of the eingenoperators \( F_k \), Eq. (9). A similar procedure was used in Ref. [28] for the case of zero acceleration, \( d\tilde{\chi}/dt = d\mu/dt = 0 \), so called the Non-Adiabatic Master Equation (NAME). The inertial theorem, described in Sec. [II] directly applies, and allows to extend the validity of the NAME for driving protocol with slow acceleration, \( d\tilde{\chi}/dt = d\mu/dt > 0 \), see Appendix C. The Master equation in the interaction representation

\[
\frac{d}{dt} \tilde{u}(\theta) = -i\mathcal{B}\tilde{u}(\theta),
\]

where

\[
\mathcal{B} = i \begin{bmatrix} 0 & \tilde{\chi} & 0 \\ -\tilde{\chi} & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.
\]
reads
\[
\begin{align*}
\frac{d}{dt} \hat{\rho}_S (t) &= -i \left[ \hat{H}_{LS} (t), \hat{\rho}_S (t) \right] \\
+ \sum_j \gamma_j (\alpha_j (t)) \left( \hat{F}_j \hat{\rho}_S (t) \hat{F}_j^\dagger - \frac{1}{2} \left\{ \hat{F}_j^\dagger \hat{F}_j, \hat{\rho}_S (t) \right\} \right).
\end{align*}
\]
(18)

Here, \( \hat{\rho}_S (t) \) is the system’s density operator in the interaction representation relative to the free evolution, Appendix C and \( \hat{F}_j \equiv \hat{F}_j (0) \). The term \( \hat{H}_{LS} (t) \) is the time dependent Lamb-type shift Hamiltonian in the interaction representation. This Master equation is an explicit time-dependent version of the Markovian GKLS Master equation [32, 33].

Within the derivation of Eq. (18), the inertial theorem eigenoperators, \( \hat{F}_j \), Eq. (6), are identified as the jump operators of the Master equation. These determine the fixed point of the dynamical map and the decay rates [28]. The decay rates \( \gamma (\alpha_j) \) are related to the Fourier transform of the bath correlation functions with the effective frequencies \( \alpha_j (t) \) [17]. These effective frequencies are the derivative of the accumulated phases, associated with the eigenvalues of \( \hat{F}_j \), Eq. (6). In Appendix C the construction of Eq. (18) is demonstrated for a driven system, weakly coupled to a bath.

IV. GEOMETRIC PHASE IN LIOUVILLE SPACE

In 1984 Berry showed that a system transported adiabatically, by varying parameters of the Hamiltonian, around a circuit, acquires an additional geometric phase [18]. Following a similar proof, we show, Appendix B that the operator \( \hat{F}_n (\theta) \) attains a new geometric phase, \( \phi_n \), when the parameters \( \{ \chi \} \) are transformed slowly, relative to \( \theta \), in a closed circuit \( C \) in parameter space. The geometric phase has the form
\[
\phi_n (C) = -\text{Im} \left[ \int_C d\tilde{s} \cdot \mathcal{V}_n (\tilde{\chi}) \right],
\]
(19)

where
\[
\mathcal{V}_n (\tilde{\chi}) = \sum_{m \neq n} \frac{\left( \hat{G}_n |\nabla_{\tilde{\chi}} B | \tilde{F}_m \right) \times \left( \hat{G}_m |\nabla_{\tilde{\chi}} B | \tilde{F}_n \right)}{\left( \lambda_m - \lambda_n \right)^2}
\]
(20)

For a circuit which retraces itself, and thus encloses no area [18], the geometric phase \( \phi_n (C) \) vanishes. The phase \( \phi_1 \) determines the instantaneous effective frequency which appear in the rate \( \gamma (\alpha_j) \) of a driven system weakly coupled to an environment, see Appendix C.

Geometric phases affect the system’s dynamics only when the dimension of the parameter space is \( | \chi | \geq 2 \). In the models studied, Sec. II A, II B the dimension of the parameter space is one, \( \chi = \chi = \mu \), thus, any circuit in parameter space encloses itself and so the geometric phase vanishes. We consider a specific model demonstrating the geometric phase, consisting of two interacting spins. The system is embedded in a \( SU (2) \otimes SU (2) \) algebra [49], and is represented by the Hamiltonian
\[
\hat{H} (t) = \hat{H}^1 (t) \otimes \hat{I}^2 + \hat{I}^1 \otimes \hat{H}^2 (t).
\]
(21)

Here, \( \hat{H}^i (t) = \omega_i (t) \hat{S}^z_i + \varepsilon_i (t) \hat{S}^z_i \) with \( i = 1, 2 \). The driving is of the form, \( \omega_i (t) = \Omega (t) \cos (\alpha_i (t)) \) and \( \varepsilon_i (t) = \Omega (t) \sin (\alpha_i (t)) \), where both spins have identical Rabi frequencies, \( \Omega_1 = \Omega_2 = \Omega (t) \), and \( \alpha_i (t) \) is determined by the parameter \( \chi \), Cf. Appendix C.

The dynamics of such a system can be cast in a factorized form, Eq. (1), which includes two independent constant parameters \( \chi_1 \) and \( \chi_2 \), see Appendix C Eq. (55). The analysis of the dynamics leads to the conclusion that non-local operators, such as \( \hat{L}_1 \otimes \hat{L}_2 \), are affected by the geometric phases, that originate from a trajectory in the parameter space of \( \chi_1 \) and \( \chi_2 \). A further analysis on the effect of the geometric phase will be the subject of future research.

V. DISCUSSION

Description of the system’s dynamics in Liouville space allow to identify a protocol which factorizes the equations of motion Eq. (1). This form enables an explicit solution, characterized by the set of parameters \( \{ \chi \} \). Such a structure is the starting point for the inertial approximation, which solves the equation of motion for a slow acceleration (and de-acceleration) of the parameters \( \{ \chi \} \). The solutions are based on the instantaneous eigenvectors \( \{ \hat{F}_k \} \) and their associated dynamical and geometric phases. This solution is applicable for fast driving, under the condition that the rate of acceleration is small, \( d^2 \chi / dt^2 \ll 1 \). In contemporary quantum studies many processes are governed by time-dependent protocols [11][20], the inertial approach will extend the ability to analyze such processes beyond the adiabatic limit, for closed, Sec. II as well as open quantum systems, Sec. III.

Explicit inertial solutions were obtained for the parametric harmonic oscillator and driven two-level-system, for driving protocols associated with a linear change of \( \chi \), Eq. (13), and \( \chi \). These solutions are compared to the adiabatic solution, showing superior accuracy of the inertial solution. For a general protocol, the adiabatic and inertial parameters can be compared, to indicate which solution achieves higher precision.

We utilized the inertial solution to derive the dynamics of a driven open quantum system coupled weakly to the environment [28]. This supplies an explicit equation of motion, which is a perquisite for optimal control theory in open quantum systems [50][53].

Moreover, the inertial construction defines new geometric phases, \( \{ \phi \} \), these are formulated in terms of a closed circuit in the parameter space of \( \{ \chi \} \). Similarly to
the Berry phase [48], \{φ\} imply that the quantum system records the history of motion in the parameter space. Furthermore, when the circuit C passes in the vicinity of degeneracies in the spectrum \{λ\}, of B, Eq. (14), the geometric phases are dominated by the degeneracies. This is a manifestation of the non-locality of quantum mechanics, i.e., the system is affected by regions in parameter space which have not been visited. A further distinction between φ and the Berry phase is that φ can be witnessed directly in terms of the correlation observables, Eq. (6). This is a property of vectors in Liouville space, unlike vectors in the state Hilbert space, where the phase can be observed only by interference.

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Appendix A: Inertial theorem

The following derivation is in the spirit of the adiabatic theorem as presented by Schiff [40] and the generalization for a non-hermitian Hamiltonian is given by Ibanez [41]. We formulate the derivation in Liouville space, a Hilbert space of operators with a scalar product \( \langle \hat{X}, \hat{Y} \rangle \equiv \text{tr} \{ \hat{X} \hat{Y} \} \). These operators operate on an underlying Hilbert space of functions \( \hat{X} \mid \psi \rangle = |φ\rangle \). The Hilbert space of function is defined by the scalar product \( \langle \psi | \phi \rangle \).

Consider a non-hermitian diagonalizable rank \( N \), parameter dependent Liouville generator \( \mathcal{B} (\chi) \), where \( \chi = \{ χ_1, χ_2, ..., χ_d \}^T \) are real parameters which can be viewed as coordinates of a parameter space. We assume the \( N \) instantaneous eigenvectors of the Liouville generator \( \mathcal{B} \) are non-degenerate (at all times, i.e. there is no level crossing). These are denoted by \( \{ \hat{F}_k (\chi) \} \), \( k = 1, 2, ..., N \), and are associated with the eigenvectors of \( \mathcal{B} \). These eigenvectors satisfy and an eigenvalue equation [28]

\[
\hat{F}^\dagger (t) \hat{F}_j (0) \hat{F} (t) = \beta_j (t) \hat{F}_j (0) .
\]

We introduce a second set of biorthogonal partners \( \{ \hat{\mathcal{G}}_k (\chi) \} \) [54], these sets of eigenvectors satisfy

\[
\mathcal{B} \hat{F}_k = λ_k \hat{F}_k \quad \text{and} \quad \mathcal{B}^\dagger \hat{\mathcal{G}}_k = λ_k^\dagger \hat{\mathcal{G}}_k .
\]

The two sets are biorthogonal, meaning \( \langle \hat{G}_k | \hat{F}_n \rangle = δ_{kn} \), where (\( )\) is the projection in Liouville space.

The quantum state is represented in Liouville space by the vector \( \vec{v} (\chi (t), \theta (t)) \) satisfying the equation of motion

\[
\frac{d}{dθ} \vec{v} (θ) = -i \mathcal{B} (\chi) \vec{v} (θ) .
\]

For a diagonalizable matrix, there exists an invertible matrix \( P \), such that \( P \mathcal{B} P^{-1} \) is diagonal. This allows identifying the eigenvectors of \( \mathcal{B} \) as \( \hat{F}_k = P^{-1} \hat{F}_k \), where the rows of \( P^{-1} \) are the left eigenvectors of \( \mathcal{B} \) [55].

The set of instantaneous eigenvectors constitute a complete basis of the Liouville space, allowing to expand the quantum state in terms of the basis elements

\[
\vec{v} (\chi, θ) = \sum_n c_n (θ) e^{-i \int_{θ(0)}^{θ(t)} dθ' λ_n (\chi(θ'))} e^{i φ_n (θ)} \hat{F}_n (\chi (θ)) .
\]

The first phase is the known dynamical phase and the second exponent includes the new geometric phase

\[
φ_k = i \int_{θ(0)}^{θ(t)} (\hat{G}_k | \mathcal{B} | \hat{F}_k ) d\chi' .
\]

The normalization condition leads to

\[
\left( \mathcal{B} \hat{F}_n \right) = - \left( \hat{G}_k | \mathcal{B} | \hat{F}_n \right) (A6)
\]

for all \( n \) and \( k \) implying that \( \left( \hat{G}_k | \mathcal{B} | \hat{F}_n \right) \) is pure imaginary and \( φ_k \) is therefore real.

Similarly, by deriving the identity \( \left( \hat{G}_n | \mathcal{B} | \hat{F}_k \right) = 0 \) with respect to \( θ \), for \( n ≠ k \), gives

\[
\hat{G}_k | \mathcal{B} | \hat{F}_n = \frac{(\hat{G}_k | \mathcal{B} | \hat{F}_n)}{λ_n - λ_k} . \quad (A7)
\]

We proceed by inserting Eq. (A4) into Eq. (A3), projecting \( \hat{G}_k \) from the left, and utilize the orthogonality condition and the derived identities, to obtain a set of differential equations

\[
\frac{dc_k (θ)}{dθ} = \mathcal{B} c_k \cdot \frac{d\chi'}{dθ} = - \sum_{n ≠ k} c_n (θ) \left( \hat{G}_k | \mathcal{B} | \hat{F}_n \right) \frac{d\chi'}{dθ} e^{-i ξ_{nk}} , \quad (A8)
\]

with \( ξ_{nk} ≡ \int_{θ(0)}^{θ(t)} dθ' [λ_n - λ_k] - (φ_n - φ_k) \).

Assuming the geometric phase is small relative to the dynamical phase, integrating equation (A8) solving iteratively leads to

\[
c_k (θ (t)) ≈ c_k (θ (0)) - \sum_{n ≠ k} \int_{θ(0)}^{θ(t)} dθ' \left( \hat{G}_k | \mathcal{B} | \hat{F}_n \right) \frac{d\chi'}{dθ'} e^{-i ξ_{nk}} \left( λ_n - λ_k \right) . \quad (A9)
\]

The term \( \left( \frac{d\chi'}{dθ'} \right)^{-1} \) diverges in the inertial limit, inducing rapid oscillations in the last term. Assuming the
The phase $\phi_n$ a non-integrable real function, determined by an integral in parameter space $\{\chi\}$.

When the matrix $B$ includes three parameters $\chi = (\chi_1, \chi_2, \chi_3)^T$ the calculation of the geometric phase is simplified by utilizing common vector calculus identities and Stokes’ theorem. Following Berry’s derivation, [48], and identities [A7] and [A9] leads to the final result

$$\phi_n (\theta) = \int_C d\chi \cdot \left(\vec{G}_n | \nabla \chi | \vec{B} \right) .$$

(B1)

The inertial solution, Eq. [6], can be written in terms of $\Lambda_k (t)$, Eq. (C3) as

$$\vec{v} (\chi, \theta) = \sum_k c_k e^{-i\Lambda_k(t)} \vec{F}_k (\chi (\theta)) .$$

(B3)

Two terms contribute to the integrand of $\Lambda_k (t)$, the first, $\lambda_k$, is associated with the dynamical phase, and the second, $-i \left(\vec{G}_n | \nabla \chi | \vec{F}_n \right) \cdot \frac{d\chi}{d\theta}$, with the geometric phase. The second term is proportionate to $\vec{v} (\theta)$, which is negligibly small in the inertial limit. While the first, is independent of the change in $\chi$. Hence, in the inertial limit, when the integration is performed over a non-closed circuit, over a simply connected space, the geometric phase can be neglected.

**Appendix C: Non-Adiabatic Master Equation (NAME)**

We present a derivation of a Master equation for a driven quantum system interacting with the electromagnetic field in a thermal state with a temperature $T$, see Ref. [53] for further analysis.

The composite system is represented by the Hamiltonian

$$\hat{H}_{tot} (t) = \hat{H} (t) + \hat{H}_B + \hat{H}_I$$

(C1)

where $\hat{H}$ is the driven system Hamiltonian, the bath Hamiltonian is composed of all the bath modes of the form $H_B = \sum_{\lambda=1,2} \sum_{k} \omega_k b_{k}^\dagger (\vec{k}) b_{\lambda} (\vec{k})$ and $H_I$ is the system bath interaction term. The interaction term under the dipole approximation can be written as $H_I = \vec{E} \cdot \vec{F}$, where $\vec{E}$ is the system dipole operator and $\vec{F}$ is the electromagnetic field operator.

Such field operator obtains the form, $\vec{E} = i \int \vec{V} \cdot \vec{b} (\vec{k}) (\vec{k}) + b_{k} (\vec{k})$, where $V$ is the volume of the field, $\varepsilon_0$ is the electric constant, $b_{k} (\vec{k})$ and $b_{k}^\dagger (\vec{k})$ are the annihilation and creation operators of a bath mode in the $k$’th direction with a frequency $\omega_k$, $(k \equiv |\vec{k}|)$, and polarization $\lambda$.

Following the microscopic derivation [47, 58, 59], we transform to the interaction picture relative to the free Hamiltonian $\hat{H} (t) + \hat{H}_B$. We assume the conditions are such that the inertial approximation is valid, in this regime the dipole operator in the interaction representation can be decomposed in terms of the time-independent eigenoperators $\{\tilde{F}_n\}$, Eq. (4), [6], as

$$\hat{D} (t) = \sum_n a_n \tilde{F}_n e^{-i\Lambda_n (t)} ,$$

with

$$\Lambda_n (t) \equiv \int_{\theta (0)}^{\theta (t)} d\theta \left[ \lambda_n - i \left(\vec{G}_n | \nabla \chi | \vec{F}_n \right) \cdot \frac{d\chi}{d\theta} \right] .$$

(C3)

Here, $\tilde{F}_n \equiv \tilde{F}_n (0)$, $a_n = tr (\hat{D} (0) \tilde{F}_n^\dagger)$ and a up-script tilde designates operators in the interaction picture. The composite Hamiltonian in the interaction picture is given by

$$\hat{H}_{tot} (t) = \tilde{H}_I (t) = i \sum_{\tilde{k}, \lambda, n} \sqrt{\frac{\hbar \omega_k}{2\varepsilon_0 V}} \vec{e}_k (\vec{k}) a_n \tilde{F}_n e^{-i\Lambda_n (t)}$$

$$\times (b_{k} (\vec{k}) e^{-i\omega_k t} + b_{k}^\dagger (\vec{k}) e^{i\omega_k t}) .$$

(C4)

We proceed by assuming the Born Markov approximation to obtain the quantum Markovian master equation

$$\frac{d}{dt} \hat{\rho} (t) =$$

$$- \frac{1}{\hbar^2} \int_0^\infty ds tr_B \left[ \hat{H}_I (t), \left[ \hat{H}_I (t-s), \hat{\rho} (t) \otimes \hat{\rho}_B \right] \right] .$$

(C5)
where $\rho_B$ is the density operator. Assuming the bath correlation functions decay fast relatively to the external driving we approximate $\theta (t - s)$ as

$$\Lambda_k (t - s) \approx \Lambda_k (t) - \alpha_k (t) s , \quad (C6)$$

where $\alpha_k (t) = \left[ \lambda_k (t) - i \tilde{G}_k (t) \right] |\nabla \vec{F}_k (t) | \delta \rho \over \partial t$. This approximation is justified, as the bath correlation functions decay in a typical time-scale which is much smaller than the time-scale of the change in the system parameters, namely, the function $\Lambda_k (t)$. Thus, the contribution to the integral in Eq. (C5) vanishes when the approximation (C6) is not satisfied, see [28] for further details.

Gathering together equations (C4), (C5) and (C6) leads to

$$\frac{d}{dt} \tilde{\rho}_S (t) = \sum_{i,j} \int_0^\infty \frac{a_i a_j}{12 \pi^2 \hbar^2 \varepsilon_0 c^5} d\omega_k \alpha_3 \left( \frac{1}{\alpha - \omega_k} + \frac{N (\omega_k)}{\alpha + \omega_k} \right) . \quad (C7)$$

The final form of the NAME in the interaction picture

$$\frac{d}{dt} \tilde{\rho}_S (t) = - \frac{i}{\hbar} \left[ \hat{H}_{LS} (t), \tilde{\rho}_S (t) \right] + \sum_j \gamma_j (\alpha_j (t)) \left( \tilde{F}_j \tilde{\rho}_S (t) \tilde{F}_j^\dagger - \frac{1}{2} \{ \tilde{F}_j^\dagger \tilde{F}_j, \tilde{\rho}_S (t) \} \right) . \quad (C11)$$

where $\hat{H}_{LS}$ is the Lamb shift correction term in the interaction representation

$$\hat{H}_{LS} (t) = \sum_j j \omega_3 (t) \tilde{F}_j^\dagger \tilde{F}_j . \quad (C12)$$

Equation (C11) is of the GKLS form guaranteeing a complete positive trace preserving dynamical map [32, 33, 58].

**Appendix D: Eigenoperators and eigenvalues**

a. **Parametric harmonic oscillator**

The matrix $\mathcal{B}$, Eq. (11), can be decomposed to two block matrices, the eigenvectors in the $\{ \hat{H}, \hat{L}, \hat{C} \}$ basis of the upper 3 by 3 matrix are $\hat{F}_1 = - \frac{\mu}{\bar{\nu}} (2, 0, \mu)^T$, $\hat{F}_2 = \frac{1}{\kappa} \left\{ \mu, i \kappa, 2 \right\}^T$ and $\hat{F}_3 = \frac{1}{\bar{\nu} \kappa} \left\{ \mu, -i \kappa, 2 \right\}^T$, corresponding to the eigenvalues $\lambda_1 = 0, \lambda_2 = \kappa$ and $\lambda_3 = -\kappa$. Each eigenvector $\hat{F}_k$ correspond to the eigenoperator $\hat{F}_k$, which is obtained by summing over the product of the coefficients and the basis operators. For $\hat{F}_k = \{ f_1^k, f_2^k, f_3^k \}^T$, $\hat{F}_j = f_1^j \hat{H} + f_2^j \hat{L} + f_3^j \hat{C}$. The eigenvectors which correspond to the eigenoperators of the bottom block 2 by 2 matrix are

$$\hat{F}_+ = \left\{ \frac{1}{2} (\mu - i \kappa), 1 \right\}^T \quad \hat{F}_- = \left\{ \frac{1}{2} (\mu + i \kappa), 1 \right\}^T \quad (D1)$$

corresponding to the eigenvalues $\lambda_+ = \frac{\kappa}{2}$ and $\lambda_- = -\frac{\kappa}{2}$.

b. **Two-Level-System**

The eigenvectors, that correspond to the eigenoperators, and eigenvalues of the propagator, Eq. (16), are $\hat{F}_1 = \frac{\mu}{\bar{\nu} \kappa} (1, 0, \mu)^T$, $\hat{F}_2 = \frac{1}{2 \kappa \bar{\nu}} (\mu, -i \kappa, 1)^T$ and $\hat{F}_3 = \frac{1}{2 \mu} (\mu, i \kappa, 1)^T$, corresponding to the eigenvalues $\lambda_1 = 0, \lambda_2 = \bar{\kappa}$, $\lambda_3 = -\bar{\kappa}$, where $\bar{\kappa} = \sqrt{\mu^2 + 1}$.

**Appendix E: Geometric phase example**

The geometric phases, presented in Sec. IV have a non-zero value only when the dynamics in Liouville space are dependent on at least two parameters $\{ \chi \}$. In the following section we present an example of a system composed of two non-interacting spins, $SU (2) \otimes SU (2)$, for which, the geometric phases directly affect the system’s dynamics. The spin system is represented by a composite Hamiltonian $\hat{H} (t)$, Eq. (21). We obtain the dynamics with the help of the Heisenberg equation, Eq. (1) and write the equations of motion in terms two different vectors in Liouville space. The first is a vector including the local operators

$$\vec{r}_l = \{ \hat{H} \hat{I}^2, \hat{L} \hat{I}^2, \hat{C} \hat{I}^2, \hat{L} \hat{C} \hat{I}^2, \hat{C} \hat{L} \hat{I}^2, \hat{C} \hat{L} \hat{C} \hat{I}^2 \}^T \quad (E1)$$

and a vector with non-local operators

$$\vec{r}_{nl} = \{ \hat{H} \hat{L} \hat{I}^2, \hat{H} \hat{I} \hat{L}^2, \hat{H} \hat{C} \hat{I}^2, \hat{L} \hat{I} \hat{C} \hat{I}^2, \hat{L} \hat{C} \hat{L} \hat{I}^2, \hat{C} \hat{I} \hat{C} \hat{L} \hat{I}^2, \hat{C} \hat{L} \hat{C} \hat{L} \hat{I}^2, \hat{C} \hat{L} \hat{C} \hat{L} \hat{C} \hat{I}^2 \}^T \quad (E2)$$

Introducing the notation $\hat{H} \hat{L} \hat{C} \hat{I}^2 \equiv \hat{H} \hat{L} \hat{C} \hat{I}^2$. The dynamics of the local operators is given by

$$\frac{1}{\Omega} \frac{d}{dt} \vec{r}_l = \frac{\hat{\Omega}}{\Omega^2} \vec{r}_l - i \hat{B}_l \vec{r}_l \quad (E3)$$
with

\[ B_n = i \begin{bmatrix}
0 & \chi_1 & 0 & 0 & 0 & 0 \\
-\chi_1 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \chi_2 & 0 & 0 \\
0 & 0 & 0 & -\chi_2 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 0
\end{bmatrix} \quad (E4) \]

The dynamics of the non-local vector obtains a similar form

\[ \frac{1}{\Omega} \frac{d}{dt} \vec{r}_{nl} = \frac{\hat{\Omega}}{\Omega^2} \hat{I} \vec{r}_{nl} - i \vec{B}_{nl} \vec{r}_{nl}, \quad (E5) \]

where

\[ B_{nl} = i \begin{bmatrix}
0 & \chi_2 & 0 & 0 & 0 & 0 \\
-\chi_2 & 0 & 1 & 0 & \chi_1 & 0 \\
-1 & 0 & 0 & 0 & \chi_1 & 0 \\
0 & -1 & 0 & 0 & \chi_1 & 0 \\
0 & 0 & 0 & \chi_2 & 0 & 1 \\
0 & 0 & -1 & 0 & 0 & \chi_2
\end{bmatrix} \quad (E6) \]

First, consider the local matrix \( B_l \). The matrix can be decomposed to two blocks, one dependent on parameter \( \chi_1 \) and another on \( \chi_2 \). This will lead to two independent sets of eigenoperators each dependent on a different parameter, \( \chi_1 \) or \( \chi_2 \). Therefore, the geometric phases associated with the eigenoperators are independent and are influenced only by a single parameter. In contrast, the non-local matrix \( B_{nl} \) cannot be separated to distinct blocks. Leading to eigenoperators and geometric phases that are dependent on both \( \chi_1 \) and \( \chi_2 \). This allows for non-vanishing geometric phases which may affect the dynamics of the non-local operators. These geometric phases are associated with a closed circuit in the \((\chi_1, \chi_2)\) plane.

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