Long Ladders do not have the edge-Erdős-Pósa property

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Abstract
We prove that ladders with 3 rungs and a minor of it (the house graph) have the edge-Erdős-Pósa property, while ladders with 14 rungs or more have not. Additionally, we prove that the latter bound is optimal in the sense that the only known counterexample graph does not permit a better result.

1 Introduction
For a graph $H$, can we find many (vertex-)disjoint subgraphs that contain $H$ as a minor in a graph $G$? If not, can we find a bounded (vertex) set $S$ in $G$, called the hitting set, such that $G - S$ does not contain $H$ as a minor? If at least one of those statements holds for every $k \in \mathbb{N}$ and every graph $G$, then we say that $H$ has the Erdős-Pósa property. When we replace vertices with edges and search for edge-disjoint graphs and an edge hitting set, we arrive at the edge-Erdős-Pósa property.

The vertex version is well researched. Robertson and Seymour [12] have shown as early as 1986 that $H$ has the (vertex-)Erdős-Pósa property if and only if $H$ is a planar graph. The edge-Erdős-Pósa property, however, behaves quite differently. It is still true that non-planar graphs do not have the edge-Erdős-Pósa property, as stated by Raymond and Thilikos in 2017 [11], but the edge version fails even for some very simple planar graphs: Bruhn, Heinlein and Joos [3] have shown in 2018 that subcubic trees with large enough pathwidth and also ladders (see Figure 1) with 71 rungs or more do not have the edge-Erdős-Pósa property.

The key ingredient for these proofs is a gadget that Bruhn et al. call condensed wall (see Figure 2). As of now, all planar graphs that have the edge-Erdős-Pósa property are minors of the condensed wall and all that do not have

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it are not minors of it. Therefore, the condensed wall might play a key role in trying to characterize the graphs that have the edge-Erdős-Pósa property. This leads to a conjecture by Bruhn et al.

**Conjecture 1.** Let $H$ be a planar graph such that there is an integer $r$ such that the condensed wall of size $r$ contains $H$ as a minor. Then $H$ has the edge-Erdős-Pósa property.

Graphs that are known to have the edge-Erdős-Pósa property are cycles [7], long cycles [4] [14], $K_4$ [1] and $Θ_r$ [11]. All of those are minors of the condensed wall.

![Figure 2: A condensed wall of size 5.](image)

In this paper, we will strengthen the result of Bruhn et al.:

**Theorem 2.** Ladders with 14 rungs or more do not have the edge-Erdős-Pósa property.

The theorem is optimal in the sense that every counterexample based on a condensed wall (and recall, we do not know others) will not work for ladders with fewer than 14 rungs: Indeed such ladders are minors of the condensed wall, see Proposition 27. We also prove a positive result:

**Theorem 3.** The ladder with 3 rungs has the edge-Erdős-Pósa property.

We also show that the house graph has the edge-Erdős-Pósa. One other result that we obtain and might be of interest, is that $A$-$m$-trees, trees that contain $m$ vertices of a specified set $A$, have the edge-Erdős-Pósa. These are a generalization of $A$-paths.

Do ladders with 4 to 13 rungs have the edge-Erdős-Pósa property? We do not know, but this question can be seen as a test for the condensed wall gadget. If it is proven that ladders with 13 rungs still do not possess the edge-Erdős-Pósa property, then clearly there is a counterexample graph not based on a condensed wall. However, if it is shown that ladders with 13 rungs have the edge-Erdős-Pósa property, then this is yet another strong indication that the condensed wall plays a key role for this property.

For an overview and a comparison of the ordinary Erdős-Pósa property and its edge counterpart, we recommend the appendix of [2] and [10]. A large list of Erdős-Pósa results can be found on Raymond’s webpage [9].
2 Definitions

In this section, we will state the basic definitions needed for this paper. Throughout, we will use standard notation as introduced in Diestel’s textbook [6].

A graph $H$ has the edge-Erdős-Pósa property if there exists a function $f : \mathbb{N} \to \mathbb{R}$ such that for every graph $G$ and every integer $k$, there are $k$ edge-disjoint graphs in $G$ each containing a minor isomorphic to $H$ or there is an edge set $X$ of $G$ of size at most $f(k)$ meeting all subgraphs in $G$ containing a minor isomorphic to $H$. Note that for a graph $H$ with a maximum degree of 3, a graph $G$ contains $H$ as a minor if and only if it contains a subdivision of $H$ [3].

Next, we define an elementary ladder $L$ as a simple graph consisting of a vertex set $V(L) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and edge set $E(L) = \{u_iu_{i+1} | i \in [n-1]\} \cup \{v_iv_{i+1} | i \in [n-1]\} \cup \{u_iv_i | i \in [n]\}$ for some $n \in \mathbb{N}, n \geq 1$. The $n$ paths (of length one) $u_iv_i, i \in [n]$ are called rungs of $L$, where the two paths (of length $n$) $u_1u_2u_3 \ldots u_n$ and $v_1v_2v_3 \ldots v_n$ are called stringers of $L$. The size or length of a ladder is given by its number of rungs, i.e. the size of $L$ is $n$. A ladder is a subdivision of an elementary ladder. We adapt the notion of rungs, stringers and size for ladders from their counterparts in elementary ladders. A subladder $L'$ a subgraph $L' \subseteq L$, where both $L'$ and $L$ are ladders and every rung $R$ of $L'$ is also an entire rung of $L$.

The main gadget used for proving that long ladders do not have the edge-Erdős-Pósa property is a wall-like structure called condensed wall introduced by Bruhn et. al. [3]. A condensed wall $W$ of size $r \in \mathbb{N}$ is the graph consisting of the following:

- For every $j \in [r]$, let $P_j = w_1^j, \ldots, w_{2r}^j$ be a path of length $2r - 1$ and for $j \in \{0\} \cup [r]$, let $z_j$ be a vertex. Moreover, let $a, b$ be two further vertices.
- For every $i, j \in [r]$, add the edges $z_{j-1}^iw_{2i-1}^j, z_ju_{2i}^j, z_{i-1}z_i, au_1^i$ and $bu_{2r}^j$.

We define $c = z_0$ and $d = z_r$ and refer to

$$W_j = W([w_1^j, \ldots, w_{2r}^j, z_{j-1}, z_j])$$

as the $j$-th layer of $W$. Note that the layers of $W$ are precisely the blocks of $W - \{a, b\}$. We will refer to the vertices $a, b$ quite often, and whenever we write $a$ or $b$ in this article, we refer to those vertices in a condensed wall. The vertices connecting the layers of $W$ are $z_i, i \in \{0\} \cup [r]$, and we will call those bottleneck vertices. This includes the vertices $c$ and $d$. The edges $z_{i-1}z_i, i \in [r]$ are called jump-edges.

For vertices $a, b, c, d$, an $(a-b, c-d)$-linkage is the vertex-disjoint union of an $a$-$b$-path with a $c$-$d$-path.

3 Toolbox

To show that a ladder $L$ does not have the edge-Erdős-Pósa property, we explicitly construct a counterexample graph $G^*$ for every given length $l$ of the ladder and size $r-1$ of the hitting set. So let $r \geq 2$ and $l \geq 14$ be some integers. Then we construct $G^*$ in the following way:
• Let \( l_A \geq 7 \) and \( l_C \geq 4 \) s. t. \( l_A + l_C + 3 = l \)

• Let \( L_A \) be an elementary ladder of length \( l_A \) and let \( L_C \) be an elementary ladder of length \( l_C \)

• Let \( G \) be the union of \( L_A \) and \( L_C \), where every edge is replaced by \( r \) internally disjoint paths of length 2.

• Add a condensed wall \( W \) of size \( 2r \) to \( G^* \)

• Connect the stringers at one end of \( L_A \) and \( L_C \) to \( W \) with \( r \) internally disjoint paths of length 2 each as in Figure 3.

A depiction of \( G^* \) can be found in Figure 3. Let \( A \) be the component of \( G^* - W \) that contains a ladder with \( l_A \) rungs, and let \( C \) be the other component of \( G^* - W \).

![Figure 3: Construction in Theorem 2 for a ladder with 14 rungs. Note that all edges drawn represent \( r \) internally disjoint paths of length 2.](image)

To show that a ladder \( L \) does not have the edge-Erdős-Pósa property, we first show that we can always find a ladder of length \( l \) in our counterexample graph \( G^* \), even with \( r - 1 \) edges being deleted. This implies there can be no edge hitting set regardless of its size, as \( r \) was arbitrary. Afterwards, we will show that there can never be two edge-disjoint ladders of length \( l \) in \( G^* \), implying that the edge-Erdős-Pósa property fails even for \( k = 2 \). The first part will be relatively easy.

### 3.1 Finding one ladder

For our embedding in \( W \), we will introduce a graph called \( X\text{-wing} \). An \( X\text{-wing} \) is a subgraph \( G \) of a condensed wall that consists of ladder \( L_1 \) with 3 rungs.
Furthermore, let there be two disjoint paths connecting the first rung of $L_1$ to $a$ and $b$, while there are two more disjoint paths connecting the last rung of $L_1$ to $c$ and $d$. For an illustration, see Figure 4.

![Figure 4: An (a-b, c-d)-linkage in an X-wing](image)

**Lemma 4.** In every condensed wall $W$ of size $2r$, there exists an X-wing, even when $r - 1$ edges of $W$ are deleted.

**Proof.** After the deletion of $r - 1$ edges, there are still two complete adjacent layers $W_{i-1}, W_i$ of $W$ whose edges to $a$ and $b$ are still present. There, we can embed a graph $X$ as in Figure 5. In $W$, there were $2r$ edge-disjoint paths connecting $z_{i-2}$ to $c$ and another $2r$ edge-disjoint paths connecting $z_i$ to $d$ without using $a$ or $b$. So with $r - 1$ edges being deleted, there is still at least one (in fact $r + 1$) of each kind left. Together with $X$, they form an X-wing. □

![Figure 5: An X-wing (thick edges) in a condensed wall of size 2. Edges belonging to rungs are labeled.](image)

The following lemma yields one part of the proof of Theorem 2.

**Lemma 5.** For every integer $l \geq 14$, there is no function $f : \mathbb{N} \to \mathbb{R}$ such that for every graph $G$ and every integer $k$, there is an edge-hitting set $B$ of size at most $f(k)$ such that $G - B$ does not contain a ladder with $l$ rungs as a minor.

**Proof.** Let $r - 1 \leq f(2)$ be the size of $B$. Recall $G^*$ as defined in the beginning of Section 3. In $G^*$ we can clearly find a ladder $L_1$ with $l_A$ rungs in $A$ and another ladder $L_2$ with $l_C$ rungs in $C$. Lemma 4 yields an X-wing $X$ in $W$. As there are $r$ disjoint connections between each stringer of $L_1, L_2$ and $W$, we can connect $L_1, L_2$ and $X$ to form a ladder with $l_A + 3 + l_C = l$ rungs. □
### 3.2 Excluding two ladders

The tactic for the proof of Theorem 2 will be to show that every ladder $U$ in $G^*$ must contain an X-wing and thus an $(a-b, c-d)$-linkage in $W$, which can be only present once (see Lemma 6). However, to prove that $U$ contains an X-wing, we must exclude every other possibility of an embedding of $U$ in $G^*$. Therefore, we will need several technical lemmas to get an upper bound for how large a ladder in $W$ can be without containing an X-wing.

**Lemma 6** (Bruhn et al. [3]). A condensed wall $W$ (of any size) does not contain two edge-disjoint $(a-b, c-d)$-linkages.

When considering how large a ladder in a condensed wall can be, the two vertices $a$ and $b$ will be of great importance. Without them, we cannot form large ladders, as the next lemma will show:

**Lemma 7.** Let $W$ be a condensed wall. In $W - \{a, b\}$, the following holds:

(i) Every ladder $L$ in $W - \{a, b\}$ has at most 5 rungs. (Bruhn et al. [3] )

![Figure 6: A ladder with 5 rungs and a path](image)

(ii) There is no embedding of a graph as in Figure 6 in $W - \{a, b\}$ such that the branch vertex corresponding to $v_1$ or $v_2$ is a bottleneck vertex.

(iii) There is no embedding of a graph as in Figure 7 or in Figure 8 in $W - \{a, b\}$ such that the branch vertices corresponding to one of $v_1$ and $v_2$ and one of $v_3$ and $v_4$ are bottleneck vertices.

![Figure 7: A ladder with 4 rungs and two paths](image)

**Proof.** (i) Every cycle in $W - \{a, b\}$ must contain either $z_i$ or $z_{i-1}$ (or both). When considering the smallest cycles in a ladder (those that contain two adjacent rungs), every vertex can only be part of two of them. Therefore, there can be at most four of those small cycles, which implies $L$ has size at most 5. (The same result has also been proven in Lemma 9 in Bruhn et al. [3].)
(ii) + (iii) Let $W_i$ be the layer of $W$ that contains $L$. $W_i$ is connected to $W - W_i$ by only four vertices: $a, b, z_{i-1}$ and $z_i$. Therefore, each ladder stringer that is continued outside of $W_i$ without using $a$ or $b$ must use one of $z_{i-1}$ and $z_i$.

Now when a ladder stringer of $L$ leaves $W_i$ (more precisely, when there is a path connected to the end of the ladder stringer that leaves $W_i$) through a vertex $z$, $z$ must necessarily lie on the end of $L$ (or not on $L$ at all). When $z_{i-1}$ or $z_i$ lie at the end of $L$, this implies that $L$ is actually shorter than 5, as every cycle in a layer $W_i$ of $W$ must contain either $z_i$ or $z_{i-1}$ (or both).

Therefore, $L$ has length at most 4 if $z_i$ or $z_{i-1}$ lies on one of its ends. If both $z_i$ and $z_{i-1}$ lie on (different) ends of the ladder, $L$ is reduced to have a length of at most 3. Finally, $L$ has at most 2 rungs if both $z_i$ and $z_{i-1}$ were on the same end.

The following lemma will already yield a useful (and tight, see Proposition 27) upper bound on the size of a ladder $L$ in a condensed wall $W$.

**Lemma 8.** In every condensed wall $W$, every ladder $L$ contains at most 13 rungs.

**Proof.** Suppose there were a ladder $L$ in $W$ with $l \geq 14$ rungs. Then $L - \{a, b\}$ contains at most three (disjoint) inclusion-maximal subladders. By construction, each of them contains neither $a$ nor $b$. Additionally, as $a$ and $b$ were part of at most 2 rungs of $L$, those subladders must together contain at least $l - 2 \geq 12$ rungs.

First, suppose there were at most two such subladders $L_1, L_2$. Then by Lemma 7 (i), each may only contain up to 5 rungs, which sums up to $5 + 5 = 10 < 12$ rungs, a contradiction.

So we can assume that there are exactly three (disjoint inclusion-maximal) subladders $L_1, L_2$ and $L_3$ of $L - \{a, b\}$. Then two of them (say $L_1$ and $L_3$) must contain the ends of $L$, while one of them (say $L_2$) lies in between. Via the stringers of $L$, $L_2$ has four disjoint paths connecting it to $L_1$ and $L_3$. By Lemma 7 (iii), $L_2$ can therefore only contain up to 3 rungs. (Note that only two paths may contain $a$ or $b$.)

Similarly, $L_1$ and $L_3$ have each two (disjoint) paths connecting them to $L_2$. One of them must contain $a$ or $b$ for each of $L_1$ and $L_3$, which implies that the other cannot contain $a$ or $b$. By Lemma 7 (ii), we conclude that $L_1$ and $L_3$ may therefore contain at most 4 rungs. This sums up to $4 + 4 = 11 < 12$ rungs for the three of them, a contradiction.

In the following, we will prove a series of lemmas that deal with ladders and connecting paths in a couple of special cases. They will be used for the proof of Theorem 2.
Lemma 9. Let $L_1$ be a ladder in a condensed wall $W$. Let $R$ be a rung at the end of $L_1$, with endvertices $v_1, v_2$. Let $P_1$ be a $v_1$-a-path and let $P_2$ be a $v_2$-b-path (both in $W$). Furthermore, let $P_1, P_2$ and $L_1$ be internally disjoint. Finally, let $L$ be the union of $L_1, P_1$ and $P_2$. (See Figure 9)

![Figure 9: A ladder with 6 rungs and paths to $a$ and $b$](image)

Then $L_1$ contains at most 6 rungs.

Proof. Let $L_2$ be the maximum subladder of $L_1 - R$. Then $L_2$ cannot contain $a$ and $b$. By Lemma 7 (i), $L_2$ can therefore contain at most 5 rungs. Together with $R$, this implies at most 6 rungs for $L_1$.

Lemma 10. Let $L_1$ be a ladder in a condensed wall $W$. Let $R$ be a rung at the end of $L_1$, with endvertices $v_1, v_2$. Let $P_1$ be a $v_1$-a-path and let $P_2$ be a $v_2$-z-path (both in $W$), where $z$ is a bottleneck vertex. Furthermore, let $P_1, P_2$ and $L_1$ be internally disjoint. Finally, let $L$ be the union of $L_1, P_1$ and $P_2$ (See Figure 10). Furthermore, let $L$ not contain $b$.

![Figure 10: A ladder with 5 rungs and paths to $a$ and a bottleneck vertex $z$](image)

Then $L_1$ contains at most 5 rungs.

Proof. Let $L_2$ be the maximum subladder of $L_1 - R$. Then $L_2$ cannot contain $a$ and $b$. If $L_2$ contains at least 2 rungs, it must be entirely contained in a single layer $W_i$ of $W$ by Lemma 7 (i). Furthermore, the stringers connecting $L_2$ to $R$ and the paths $P_1$ and $P_2$ form two disjoint paths continuing the stringers of $L_2$ to vertices outside of $W_i$. By Lemma 7 (ii), $L_2$ can therefore contain at most 4 rungs. Together with $R$, this implies at most 5 rungs for $L_1$.

Lemma 11. Let $L_1, L_2$ be disjoint ladders in a condensed wall $W$. Let $R_1$ be a rung at the end of $L_1$, with endvertices $v_1, v_2$. Similarly, let $R_2$ be a rung at the end of $L_2$, with endvertices $v_3, v_4$. Let $P_1$ be a $v_1$-a-path and let $P_2$ be a $v_4$-z-path (both in $W$), where $z$ is some vertex in a layer that is not occupied by
$L_1$. Additionally, let $P_a$ be a $v_2$-$v_3$-path in $W$. Furthermore, let $P_1, P_2, P_3, L_1$ and $L_2$ be internally disjoint. Finally, let $L$ be the union of $L_1, L_2, P_1, P_2$ and $P_3$. (See Figure 11)

![Figure 11: Two ladders with paths to $a$ and a bottleneck vertex $z$](image)

Then the sum of rungs in $L_1$ and $L_2$ is at most 12 rungs.

Proof. Let $L_1'$ be the maximum subladder of $L_1 - R_1$. Then both $L_2$ and $L_1'$ cannot contain $a$.

Now suppose neither of them would contain $b$, either. Then by Lemma 7 (i), each of $L_2$ and $L_1'$ can contain at most 5 rungs. Together with $R_1$, this means up to 11 rungs for $L_1$ and $L_2$.

Now suppose $b$ would be contained in $L_2$. If $b$ is contained in $R_2$, we define $L_2'$ as the (unique) maximal subladder of $L_2 - R_2$. Again, we apply Lemma 7 (i) to see that $L_1'$ and $L_2'$ can have at most 5 rungs each. This implies at most 6 rungs each for $L_1$ and $L_2$, which sums up to at most 12 rungs in total.

If $b$ is contained in $L_2 - R_2$, there is a (maximal) proper subladder $L_2'$ of $L_2 - b$ that contains $R_2$ and has four disjoint paths continuing its stringers to vertices outside of its layer: $P_2, P_3$ and the part of the stringers of $L_2$. However, as $a$ is contained in neither of them (as $a$ is on $P_1$), at least three of them contain neither $a$ nor $b$. This is a contradiction.

Finally, we can suppose $b$ is contained in $L_1'$. Then $L_2$ contains at most 2 rungs due to Lemma 7 (iii).

Now $L_1' - b$ contains at most two disjoint maximal subladders. If it would contain only one such maximal subladder $L_3$, then $L_3$ can contain at most 5 rungs by Lemma 7 (i) as it contains neither $a$ nor $b$. This implies up to 7 rungs for $L_1$, which sums up to at most $7 + 2 = 9$ rungs for $L_1$ and $L_2$.

So let us suppose $L_1' - b$ contains exactly two disjoint maximal subladders $L_3$ and $L_4$, where $L_3$ is the subladder adjacent to $R_1$. Then $L_4$ can contain at most 4 rungs by Lemma 7 (ii), and $L_3$ can contain at most 3 rungs by Lemma 7 (iii).

Together with at most 1 rung that can be incident with $b$ and $R_1$, this implies up to $4 + 1 + 3 + 1 = 9$ rungs for $L_1$. With 2 rungs in $L_2$, we get a sum of rungs of at most 11.

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Lemma 12. Let $L_1$ be a ladder in a condensed wall $W$. Let $R$ be a rung at the end of $L_1$, with endvertices $v_1, v_2$. Let $P_1$ be a $v_1$-$c$-path and let $P_2$ be a $v_2$-$d$-path (both in $W$). Furthermore, let $P_1, P_2$ and $L_1$ be internally disjoint. Finally, let $L$ be the union of $L_1, P_1$ and $P_2$. (See Figure 12)

Let $Q = (P_1 - v_1) + (P_2 - v_2)$.

(i) If both $L_1$ and $Q$ contain neither $a$ nor $b$, $L_1$ has at most 2 rungs.

(ii) If $L_1$ contains neither $a$ nor $b$, and there is exactly one of $a$ and $b$ on $Q$, $L_1$ has at most 4 rungs.

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(iii) If $L_1$ contains neither $a$ nor $b$, and there are both $a$ and $b$ on $Q$, $L_1$ has at most 5 rungs.

(iv) If $L_1$ contains exactly one of $a$ and $b$, but $Q$ contains neither of them, $L_1$ has at most 5 rungs.

(v) If $L_1$ contains exactly one of $a$ and $b$, and there is exactly one of $a$ and $b$ on $Q$, too, $L_1$ may contain up to 8 rungs.

(vi) If $L_1$ contains both $a$ and $b$ (which implies $Q$ contains neither of them), $L_1$ has at most 9 rungs.

Proof. (i) This follows from Lemma 7 (iii).

(ii) As $L_1$ contains neither $a$ nor $b$, if $L_1$ has more than one rung, it must be entirely contained in a single layer of $W$. Additionally, as $Q$ is allowed to contain only one of $a$ and $b$, either $P_1$ or $P_2$ must not contain $a$ and $b$. This implies $L_1$ has a stringer continued to $c$ or $d$ without using $a$ and $b$. By Lemma 7 (ii), $L_1$ can thus have at most 4 rungs.

(iii) As in (ii), $L_1$ must be contained in a single layer of $W$. By Lemma 7 (i), $L_1$ can therefore have at most 5 rungs.

Before we continue with the proof of (iv), we make some observations that will prove useful for the next three cases. Let $P$ be the path in $L$ formed by $P_1, P_2$ and $R$.

Claim 1: If $P$ uses neither $a$ nor $b$, $P$ must use all the vertices $z_i, i \in \{0\} \cup [r]$, where $z_0 = c$ and $z_r = d$.

By definition, $P$ uses $z_0 = c$ and $z_r = d$. Furthermore, $a, b$ and $z_i$ disconnect $c$ from $d$ for every $i \in \{1, \ldots, n - 1\}$.

Claim 2: If $P$ contains neither $a$ nor $b$, and $L_1$ contains at most one of them, then $L_1$ has at most 2 rungs.

Without using a bottleneck vertex, all cycles in $W$ must contain both $a$ and $b$. As $L$ contains at most one of $a$ and $b$, if $L_1$ has more than one rung, all rungs of $L_1$ must be on a cycle with a bottleneck vertex. As $R$ is the only rung of $L_1$ that contains bottleneck vertices, all cycles of $L_1$ must contain $R$. As $R$ is required to be on the end of $L_1$, $L_1$ cannot have more than 2 rungs, no matter whether we use $a$ or $b$ or not.

Claim 3: If $P$ contains neither $a$ nor $b$, but $L$ contains both, then $L$ has at most 4 rungs.

Now there can be additional cycles that use no bottleneck vertex, but those must then contain both $a$ and $b$. When considering the smallest cycles on $L_1$
(those that use two adjacent rungs), there can be only two cycles that contain both $a$ and $b$. Together with the single cycle that may be incident with a bottleneck vertex that we also got in the upper case, we are left with at most 3 smallest cycles in $L_1$. This means we get an upper bound of 4 rungs for $L_1$.

Now, we can continue with the last three proofs:

(iv) If $P$ uses neither $a$ nor $b$ as in (i), $L_1$ has at most 2 rungs as seen in Claim 2. Therefore, we must only consider the case where $P$ uses $v \in \{a, b\}$. As we know that $v$ is not a part of $Q$, we can conclude that $v$ lies on $r$. Then we can apply Lemma 10 to see that $L_1$ contains at most 5 rungs.

(v) Again, there is $v \in \{a, b\}$ contained in $L_1$. If $v$ were contained in $R$, $L_1 - R$ would contain neither $a$ nor $b$. Therefore, $L_1 - R$ could only contain a subladder with at most 5 rungs due to Lemma 7 (i). This would imply at most $5 + 1 = 6$ rungs for $L_1$.

Now suppose $v$ is not in $R$. Then it is not a part of $P_1$ and $P_2$, either. As in (ii), since $Q$ is allowed to contain only one of $a$ and $b$, either $P_1$ or $P_2$ must not contain $a$ and $b$. Let $L_2, L_3$ be the two maximal subladders of $L_1 - v$, where $L_3$ is the one that is incident with $P_1$ and $P_2$. The stringers of $L_1$ form two disjoint paths connecting $L_2$ and $L_3$, where only one of them can contain $v$. If $L_2$ or $L_3$ contains more than one rung, it is entirely contained in a single layer of $W$. We can therefore apply Lemma 11 (ii) and (iii) to see that $L_2$ can only contain up to 4 rungs, while there are at most 3 rungs in $L_3$. Together with at most one rung that is incident with $v$, we get a total of $4 + 1 + 3 = 8$ rungs as an upper bound for $L_1$.

(vi) If $P$ uses neither $a$ nor $b$, $L_1$ has at most 4 rungs as seen at the beginning of the proof. Therefore, we can conclude $P$ uses at least one of $a$ and $b$. Thus, $L_1 - R$ contains exactly one maximal subladder $L_2$ with only rung less that $L_1$, but only one of $a$ or $b$ can be contained in $L_2$. As in the proof of (v), we can argue that $L_2$ can only contain at most 8 rungs as it has two stringers continued via disjoint paths to $c$ and $d$, but only one of the paths may contain $a$ or $b$. This implies there are at most $8 + 1 = 9$ rungs for $L_1$.

Lemma 12 can also be used when there are two ladders, as in the following lemma:

**Lemma 13.** Let $L_1, L_2$ be two disjoint ladders in a condensed wall $W$. Let $R_1$ be a rung at the end of $L_1$ with endvertices $v_1, v_2$. Similarly, let $R_2$ be a rung at the end of $L_2$ with endvertices $v_3, v_4$.

Now, let $P_1$ be a $v_1$-$c$-path in $W$ and let $P_2$ be a $v_4$-$d$-path in $W$. Additionally, let $P_3$ be a $v_2$-$v_3$-path in $W$. Finally, let $P_1, P_2, P_3, L_1$ and $L_2$ be internally disjoint. (See Figure 13)

![Figure 13: Two ladders with paths to $c$ and $d$](image)

Then the sum of rungs in $L_1$ and $L_2$ is at most 13.
Proof. When the sum of rungs in $L_1$ and $L_2$ is more than 13, we can conclude that at least one of them (say $L_1$, as the situation is symmetric) must contain at least 7 rungs. Then $L_1 - R_1$ must contain a subladder $L'_1$ with at least 6 rungs. Due to Lemma 12 (i), this is only possible if $L'_1$ contains $a$ or $b$. Say $L'_1$ contains $a$, as the situation is again symmetric. Now $P_3, R_2$ and $P_2$ form together a path $P$ that continues a ladder stringer of $L_1$ to $d$. Additionally, $L_1$ has its other stringer continued at the same end via $P_1$ to $c$. Let $Q_1 = (P_1 - v_1) \cup (P - v_2)$.

Now, we observe that $L_2$ is in a similar situation. $P_3, R_1$ and $P_1$ form a path $P'$ that continues a stringer of $L_2$ to $c$. Additionally, $L_2$ has its other stringer continued at the same end via $P_2$ to $d$. Let $Q_2 = (P_2 - v_4) \cup (P' - v_3)$. As $a$ is part of $L_1 - R_1$, we note that $Q_2, L_2$ and $Q_1$ cannot contain $a$. Next, we take a look at where $b$ might be situated. We divide three cases:

First, $b$ might be part of $L_1$. Then $L_1$ contains both $a$ and $b$. By Lemma 12 (vi), $L_1$ can contain at most 9 rungs. In this case, $b$ cannot be part of $L_2$ anymore. However, it could be a part of $R_1 \subset Q_2$. In this case, $L_2$ can contain up to 4 rungs by Lemma 12 (ii). This sums up to $9 + 4 = 13$ rungs in total, which was what we wanted. Alternatively, $b$ could not be part of $R_1$. Then $b$ is not a part of $Q_2$, which implies that $L_2$ can only contain two rungs by Lemma 12 (i). This implies up to $9 + 2 = 11$ rungs in total.

Second, $b$ might be part of $L_2$. Then $Q_2$ contains neither $a$ nor $b$. Using Lemma 12 (iv), $L_2$ can have at most 5 rungs. Similar to the first case, $b$ might lie on $R_2 \subset Q_1$, which implies that $L_1$ can contain at most 8 rungs due to Lemma 12 (v). This means we get at most $8 + 5 = 13$ rungs in total. Alternatively, $b$ might not lie on $R_2$, implying it is not on $Q_1$, either. Then we use Lemma 12 (iv) to see that $L_1$ could only contain 5 rungs, which implies up to $5 + 5 = 10$ rungs in total.

Third and finally, $b$ might be a part of neither $L_1$ nor $L_2$. If it is on $P_1, P_2$ or $P_3$, it is part of both $Q_1$ and $Q_2$. We can therefore conclude that $L_1$ contains at most 8 rungs by Lemma 12 (v), while $L_2$ contains at most 4 rungs by Lemma 12 (ii). This sums up to $8 + 4 = 12$ rungs in total. Alternatively, $b$ could not be part of $P_1, P_2$ or $P_3$. Then, it is neither in $L_1, L_2, Q_1$ nor in $Q_2$. Therefore, we use Lemma 12 (iv) to see that $L_1$ can contain only 5 rungs. Similarly, Lemma 12 (i) implies that $L_2$ can only contain 2 rungs. In total, this means at most $5 + 2 = 7$ rungs in total.

As we have seen that an upper bound of 13 rungs in total hold true for all three cases, we have proven the lemma.

Lemma 14. Let $L$ be a ladder where the pairs $\{a, b\}$ and $\{c, d\}$ are situated on the stringers of $L$ between different rungs, but both vertices of each pair lie between the same two rungs and on the same stringer of $L$.

Except for the part of the stringers that is between the two vertices of each pair, all of $L$ is situated in a condensed wall $W$. An example for the part of $L$ in $W$ is depicted in Figure 14.

Then $L$ has at most 11 rungs.

Proof. Let the maximal subladders of $L$ be named as in Figure 14. We begin by observing that $L_1$ and $L_2$ can only contain at most 5 rungs each by Lemma 10. Additionally, we get an upper bound of 3 rungs for $L_3$ with Lemma 7 (iii). Therefore, if one of $L_1$ or $L_2$ had less than 3 rungs, or if $L_3$ had only one single rung, we get an upper bound of 11 rungs in total, which was what we wanted.
We can therefore assume that each of $L_1$ and $L_2$ contains at least 4 rungs, while $L_3$ must contain at least two rungs. This implies that $L_3$ is situated in a single layer of $W$ as it contains neither $a$ nor $b$. Let $R_2$ be the rung of $L_2$ which is closest to $b$. We conclude $L_2 - R_2$ has a proper maximal subladder $L'_2$ with at least three rungs that contains neither $a$ nor $b$. It must therefore be also entirely contained in a single layer of $W$. The same holds true for a maximal subladder $L'_1$ of $L_1 - R_1$, where $R_1$ is the rung of $L_1$ closest to $a$. But then $L'_2$ has four disjoint paths to vertices outside of its own layer, with only one of them containing $a$ or $b$. This is a contradiction.

4 Proof of Theorem 2

Theorem 2. Ladders with 14 rungs or more do not have the edge-Erdős-Pósa property.

Proof. As a counterexample, we use the graph $G^*$ as introduced in the beginning of Section 3. In Lemma 5, we have seen that there exists no upper bound on the size of the hitting set.

The only thing that is left to prove is that every ladder in $G^*$ must contain an X-wing. So let $U$ be a ladder with $l$ rungs in $G^*$. First of all, we observe that $U$ cannot be entirely contained in $W$ due to Lemma 5. Even more clearly, it cannot be entirely contained in $G^* - W$ as there is only room for $l_A$ or $l_C$ rungs there.

We call $U_{in}$ the maximal subgraph of $U$ in $W$, while we call the one in $G^* - (W - \{a, b, c, d\})$ $U_{out}$. $U_{in}$ and $U_{out}$ are edge-disjoint, $U_{in} + U_{out} = U$ and both are non-empty as seen above.

Claim 1: $U \cap C \neq \emptyset$

Suppose $U$ would be disjoint from $C$. Then $U$ is entirely contained in $G^* - C$. We have seen before that in this case, it cannot be disjoint from $A$ or $W$. As $U$ is two-connected, it must therefore use both $a$ and $b$ to connect $U_{in}$ and $U_{out}$.

Now we distinguish three cases:

First, $a$ and $b$ could be on the same ladder stringer. Then either $U_{in}$ or $U_{out}$ must contain two subladders which together contain all $l$ rungs and are connected at one stringer via a path. Clearly, they cannot be entirely contained in $A$. If they were in $W$ instead, we can use Lemma 10 to conclude that each of them can only contain 5 rungs, which sums up to $10 < l$ rungs in total, a contradiction.

Second, one of $U_{in}$ and $U_{out}$ might contain only (part of) a single rung, while the other contains a ladder with $l - 1$ rungs. So many rungs cannot be
contained in $A$. If they were contained in $W$ instead, we observe that this ladder can be split into (at most) two (inclusion-)maximal subladders where both have paths connecting their stringers to $a$ and $b$. By Lemma 9, each of those subladders can only contain up to 6 rungs. Therefore, $U_{in}$ can only contain up to $12 \leq l - 2 < l - 1$ rungs, a contradiction.

Finally, $a$ and $b$ could split $U$ into two subladders, where one is contained in $U_{in}$, and the other in $U_{out}$. Together, they must contain all $l$ rungs. In $U_{out}$, there can be only $l_A$ rungs. In $U_{in}$, we use again Lemma 9 to see there can be only 6 rungs. Together, this yields an upper bound of $l_A + 6 \leq l - 1 < l$ rungs for $U$, a contradiction.

As we arrived at a contradiction in all cases, we know that $U$ cannot be disjoint from $C$. Next, we want to see that the same holds true for $A$.

**Claim 2:** $U \cap A \neq \emptyset$

Suppose $U$ would be disjoint from $A$. Then $U$ is entirely contained in $G^* - A$. Again, we can conclude that $U$ must use both $c$ and $d$ to connect $U_{in}$ in $W$ and $U_{out}$ in $C$. As before, we divide the same three cases:

First, $c$ and $d$ could be on the same ladder stringer. Then $U_{in}$ or $U_{out}$ must contain two subladders $L_1, L_2$ which together contain all $l$ rungs and are connected at one stringer via a path. Again, it cannot be entirely contained in $C$. We conclude it is in $W$ instead. We apply Lemma 13 to see that there can be only 13 rungs in $W$ in this case, a contradiction.

Second, one of $U_{in}$ and $U_{out}$ might contain only a single rung, while the other contains a ladder with $l - 1$ rungs. Again, so many rungs cannot be contained in $C$. If they were contained in $W$ instead, we observe that this ladder can be split into (at most) two (inclusion-)maximal subladders $L_1, L_2$, where both have paths connecting their stringers to $c$ and $d$.

This time, we need to distinguish where $a$ and $b$ lie. As the situation is symmetric, suppose that $L_1$ contains at least as many vertices of $a$ and $b$ as $L_2$ does.

If $L_1$ contains both $a$ and $b$, we conclude that they cannot lie in $L_2$ or its paths to $c$ and $d$. Therefore, we can apply Lemma 12 (i) to see that $L_2$ can only contain 2 rungs, while $L_1$ has at most 9 by Lemma 12 (vi) in this case. This sums up to at most 11 rungs in total.

If $L_1$ contains exactly one of $a$ and $b$ (say $a$), then $b$ might lie on $L_2$ or the paths to $c$ and $d$. If $b$ is in $L_2$, we get at most 5 rungs for each of $L_1$ and $L_2$ by Lemma 12 (iv), which sums up to 10 rungs in total. If $b$ is not in $L_2$, it may lie on the paths to $c$ and $d$ for both of $L_1$ and $L_2$. Even so, we get at most 8 rungs for $L_1$ by Lemma 12 (v) and at most 4 rungs for $L_2$ by Lemma 12 (ii), which sums up to 12 rungs in total. In every case, we got an upper bound of $12 < l - 1$ rungs or less for $U_{in}$, a contradiction.

Finally, $c$ and $d$ could split $U$ into two subladders, where one is contained in $U_{in}$ and the other in $U_{out}$. Again, they must together contain all $l$ rungs. In $U_{out}$, there can be only $l_C$ rungs. In $U_{in}$, we use Lemma 12 (vi) to see there can be only 9 rungs. Together, this yields an upper bound of $l_C + 9 \leq l - 1$ rungs for $U$, a contradiction. We conclude:

$U$ has edges in each of $A$, $C$ and $W$.

As $A$ and $C$ are different (and therefore disconnected) components of $G^* - W$, 14
we can conclude that $U_{out}$ consists of at least two components, where at least one is in each of $A$ and $C$.

**Claim 3:** At least one of $A$ and $C$ must contain at least one edge of a rung of $U$.

Suppose $A$ and $C$ do not contain a single edge of any rung of $U$. Our last conclusion shows that there must be some edges of $U$ in both $A$ and $C$, so those must be part of a ladder stringer. As there are only two vertices each ($a, b$ and $c, d$) separating the parts of $U_{out}$ from $U_{in}$, these parts of a ladder stringer of $U$ must each be between two adjacent rungs.

Now there are two cases:

First, the parts of the ladder stringers in $U_{out}$ could lie between the very same rungs of $U$.

Now they could lie on the same ladder stringer, meaning that $U_{in}$ contains two (inclusion-) maximal subladders with $l$ rungs in total, connected via a path at one stringer. Furthermore, each subladder has one more stringer continued via a path. One of them contains $a$ or $b$, the other contains $c$ or $d$. By Lemma 11, the subladders can only have $12 \leq l - 2 < l$ rungs in total, a contradiction.

Otherwise, $a, b$ and $c, d$ must lie on different ladder stringers. Then $U_{in}$ is disconnected into two components, each containing an (inclusion-) maximal subladder that has one ladder stringer continued to one of $a$ and $b$ and the other to $c$ or $d$. The sum of rungs of both subladders is again $l$. By Lemma 10, each of those subladders can contain at most 5 rungs. This results in an upper bound of $10 \leq l - 4$ rungs for $U$, a contradiction.

Second, the ladder stringer parts in $U_{out}$ could lie between different rungs, resulting in $U_{in}$ being still connected. Moreover, $U_{in}$ will then contain exactly three (inclusion-) maximal subladders that together contain all $l$ rungs of $U$. By Lemma 14, we get at most $11 \leq l - 3$ rungs for this collection, a contradiction.

**Claim 4:** $A$ must contain at least one edge of a rung of $U$.

Suppose $A$ only contains part of a ladder stringer. As we have proven Claim 3, we know that $C$ must then contain some edge of a rung of $U$. Now there are two cases:

First, $C$ might only contain part of a single rung of $U$. Then all other $l - 1$ rungs must be contained in $U_{in}$. $U_{in}$ must then contain exactly two subladders which are connected at one stringer via a path. We apply Lemma 10 to see that each subladder contains at most 5 rungs, which means $U_{in}$ can only contain $10 \leq l - 4 < l - 1$ rungs, a contradiction.

Second, $C$ might contain part of several rungs of $U$. This is only possible if $c$ and $d$ are situated on the ladder stringers of $U$ and $C$ contains a proper subladder of $U$. Now $U_{out}$ can contain up to $l_C$ rungs. Then $U_{in}$ must again contain exactly two (inclusion-) maximal subladders $L_1, L_2$ which are connected via a path at one ladder stringer. This time, however, one of them (say $L_1$) also has both of its ladder stringers continued to $c$ and $d$ at the other end of $L_1$. Moreover, each of them also has one stringer continued via a path to $a$ or $b$.

Let $L'_1$ be the (unique) maximal subladders of $L_1 - \{a, b\}$. If $L'_1$ contains more than one rung, it is situated in a single layer of $W$. But $L'_1$ contains four paths continuing its stringers, where only one of them may contain $a$ or $b$. This is a contradiction. Therefore, $L'_1$ may only contain 1 rung, which implies at most 2 rungs for $L_1$. By Lemma 10, $L_2$ has at most 5 rungs. This sums up to at
most $2 + 5 = 7$ rungs for $U_{in}$. Together with $l_C$ rungs in $U_{out}$, we get at most $7 + l_C \leq l - 3$ rungs for $U$, a contradiction.

**Claim 5:** Both $A$ and $C$ must each contain at least one edge of a rung of $U$.

We have seen that the claim is true for $A$. Therefore, assume it would not hold for $C$. Similar to Claim 4, there are two cases:

First, $A$ might only contain part of a single rung of $U$. Then all other $l - 1$ rungs must be contained in $U_{in}$. $U_{in}$ must then contain exactly two (inclusion-) maximal subladders $L_1, L_2$ which are connected at one stringer via a path. As $a$ and $b$ are incident with the same rung $R$ of $U$ and $R$ is not a rung of $L_1$ or $L_2$, we can conclude that $a$ and $b$ are either both in $L_1$ (or both in $L_2$), but not on the rung that is incident with the paths to $c$ and $d$, or $a$ and $b$ are both not on $L_1$ or $L_2$ at all.

If $a$ and $b$ are both in $L_1$ or $L_2$, then each of $L_1$ and $L_2$ has two disjoint paths continuing their stringers to $c$ and $d$. Furthermore, none of those paths contains $a$ or $b$. (One path goes through the other ladder, and we use that $a$ and $b$ are not on the first rung.) This implies that one of $L_1$ and $L_2$ contains at most 9 rungs by Lemma 12 (vi), while the other can contain only 2 rungs by Lemma 12 (i). This sums up to at most 11 rungs for $U_{in}$, a contradiction.

If $a$ and $b$ were not in $L_1$ or $L_2$, each of the ladders can contain at most 5 rungs by Lemma 7 (i). This sums up to at most $10 < l - 1$ rungs for $U$, which is again a contradiction.

Second, $A$ might contain part of several rungs of $U$, resulting in $A$ containing a proper subladder of $U$. This subladder can contain up to $l_A$ rungs. $U_{in}$ now contains exactly two (inclusion-) maximal subladders $L_1, L_2$ which are connected at one stringer via a path. Furthermore, one of the subladders (say $L_1$) has stringers continued to $a$ and $b$.

We use the argument from the first case again to see that each of $L_1, L_2$ has a path connecting it to $c$ and another to $d$, both starting at its ladder stringers at the same end of the subladder. Furthermore, $L_2$ cannot contain $a$ or $b$. In $L_1$, $a$ and $b$ are used to continue its stringers at the same end, so $L_1 - a - b$ contains a subladder $L'_1$ which has at most one rung less than $L_1$ and contains neither $a$ nor $b$. We can therefore apply Lemma 12 (i) to see that $L'_1$ and $L_2$ can only contain up to 2 rungs each. This means $L_1$ can only contain up to 3 rungs, which implies an upper bound of $3 + 2 = 5$ rungs for $U_{in}$. Together with $l_A$ rungs in $U_{out}$, we arrive at an upper bound of $5 + l_A \leq l - 2$ rungs for $U$, a contradiction.

For our next claim, we pick among all possible rungs $R_A$ that are (at least partly) contained in $A$ and among all possible rungs $R_C$ that (are at least partly) contained in $C$ those two that are closest to each other (measured in the number of rungs between them).

Let $L_W$ be the subladder of $U$ that contains exactly all rungs between $R_A$ and $R_C$. Let $L_{R_A}$ be the (inclusion-) maximal subladder of $U - L_W$ that contains $R_A$, and let $L_{R_C}$ be the (inclusion-) maximal subladder of $U - L_W$ that contains $R_C$. Then all rungs of $U$ are contained in exactly one of $L_{R_A}, L_W$ and $L_{R_C}$.

**Claim 6:** $L_W$ contains at least 3 rungs.

First, we will see how many rungs there can be situated in $L_{R_A}$ and $L_{R_C}$. If $A$ contains (part of) more than one rung, it must contain all rungs completely.
Furthermore, all rungs of $L_{R_A}$ must then be in $A$, resulting in $l_A$ rungs. Alternatively, $A$ might contain part of only one rung. Then the rest of $L_{R_A}$ must be contained in $W$. There, it contains exactly one (inclusion-) maximal subladder $L_1$ that must contain all but one rung of $L_{R_A}$. Furthermore, $L_1$ has both stringers at one end continued by one path each to $a$ and $b$. By Lemma 9 there can be only 6 rungs of $L_{R_A}$ in $W$, resulting in up to $6 + 1 = 7 \leq l_A$ rungs for $L_{R_A}$.

The same argumentation also holds for $C$: $C$ might contain all rungs of $L_{R_C}$, resulting in $l_C$ rungs. The only alternative would be that $C$ only contains part of a single rung of $L_{R_C}$. But then all other rungs of $L_{R_C}$ must be part of a single subladder $L_2$. At one end of $L_2$, both stringers are continued via one path each to $c$ and $d$, respectively. Furthermore, $L_2$ is not allowed to contain $a$ or $b$ as those vertices are somewhere on the border of $L_{R_A}$ and $L_W$. We can therefore conclude that $L_2$ can only contain up to 2 rungs by Lemma 12(i). This results in an upper bound of $2 + 1 = 3 < l_C$ rungs for $L_{R_C}$.

Now, we can conclude that $L_{R_A}$ can only contain up to $l_A$ rungs, while $L_{R_C}$ can contain at most $l_C$ rungs. This means there must be at least 3 rungs left for $L_W$, proving Claim 6.

**Claim 7:** $W$ contains an X-wing.

Let $v_1, v_2$ be the endvertices of the part of $R_A$ that is contained in $A$. Similarly, let $v_3, v_4$ be the endvertices of the part of $R_C$ that is contained in $C$. In $L_{R_A}$, $v_1$ has two (internally) disjoint paths to the ladder stringers $S_1, S_2$ of $U$. One of them crosses $v_2$, so we pick the other path $Q_1$. Say $Q_1$ connects $v_1$ to $S_1$. Similarly, there is a path $Q_2$ on $L_{R_A}$ that does not cross $v_1$ and connects $v_2$ to $S_2$.

$S_1$ is connected to all rungs of $U$, so in particular, we can find a path $T_1$ on $S_1$ that connects $Q_1$ with the first rung of $L_W$ (the one that is closest to $L_{R_A}$). Together the paths $Q_1$ and $T_1$ form a path $P'_1$ that connects $v_1$ to a stringer of $L_W$. Thereby, $P'_1$ needs to enter $W$ via $a$ or $b$. Therefore, $P'_1$ contains $a$ or $b$. This means that $P'_1$ contains a subpath $P_1$ in $W$ connecting $a$ or $b$ to the end of a stringer of $L_W$. Similarly, we can find a path $P_2$ in $W$ connecting the other vertex of $a$ and $b$ to the other end of a stringer of $L_W$ (situated at the same end of $L_W$).

On the other side of $L_W$, we apply the same approach to $v_3$ and $v_4$ to find a path $P_3$ in $W$ connecting $c$ to the end of a stringer of $L_W$, and another path $P_4$ connecting $d$ to the last stringer end of $L_W$. Note that by construction, $P_1, P_2, P_3, P_4$ and $L_W$ are internally disjoint and lie in $W$. Together, they form an X-wing.

### 5 Small Ladder

From now on, ladder will always mean a subdivision of a ladder with three rungs. In this chapter we will prove that the ladder has the edge-Erdős-Pósa property. The proof goes as follows: We start by constructing a tree with some special property in a graph that contains a vertex that intersects all ladders. If that tree is large, then we find many edge-disjoint ladders and if it is small, then our graph has a very simple structure (after removing some edges). In the latter case we can quite easily find $k$ edge-disjoint ladders or a bounded set of
edges that intersects all ladders. In the end we can generalize this to not only work for graphs that contain a vertex that intersects all ladders but all graphs. Almost the same proof can be used to prove that the house graph (Figure 15a) has the edge-Erdős-Pósa property as well.

(a) The house graph.  
(b) The diamond graph.  
(c) A short $\Theta_6$

By Mader’s theorem, non-trivial paths that have their endvertices in a vertex set $A$, called $A$-paths, have the edge-Erdős-Pósa property. A generalization of $A$-paths that we need are $A$-$m$-trees. Let $A$ be a vertex set in a graph $G$, an $A$-$m$-tree is a tree that contains $m$ vertices of $A$. We will show that these have the edge-Erdős-Pósa property using the very close notion of $A$-Steiner-trees. Again let $A$ be a vertex set in a graph $G$, an $A$-Steiner-tree is a tree that contains all vertices of $A$. Kriesell [8] conjectured that if no edge-set of size at most $2k - 1$ separates $A$ then there are $k$ edge-disjoint $A$-Steiner-trees in $G$. This conjecture is almost proven: DeVos, McDonald and Pivotto showed that this is true if every separator of $A$ has size at least $5k + 4$.

**Theorem 15** (Devos et al. [5]). $A$-Steiner-trees have the edge-Erdős-Pósa property with hitting set bound $5k + 3$.

We also need a few other tree results. The first one is a very basic lemma and, as far as we know, has not been written down anywhere else. A segment in a tree is a path between two vertices that are not of degree 2 and such that all its interior vertices have degree 2. Basically these are shortest paths between branch vertices and/or leaves of the tree.

**Lemma 16.** The number of segments in a tree with at least two vertices is at most 2 times the number of leaves.

**Proof.** It suffices to show that for every tree with no vertex of degree 2, the number of its edges is bounded by 2 times the number of its leaves. Then for any other tree, contract all vertices of degree 2 and their incident edges to a single edge to obtain the statement. Let $\ell$ be the number of leaves and $r$ the number of non-leaves in a tree $T$ with no vertices of degree 2. Through a very simple induction we obtain that $r < \ell$. Then $|E(T)| = |V(T)| - 1 = r + \ell - 1 < 2\ell$. 

There is a lemma in [10] about splitting a tree into large but not too large subtrees. A mark is a function $f : V(G) \to \{0, 1\}$, a vertex $v$ is marked if $f(v) = 1$.

**Lemma 17** (Raymond et al. [10]). Let $T$ be a tree, $m \in \mathbb{N}$ and let $f : V(T) \to \{0, 1\}$ be a mark in $T$. Let there be at least $2m$ marked vertices in $T$. There exist edge-disjoint subtrees $T_1, T_2$ of $T$ such that $T_1 \cup T_2 = T$ and $T_1$ contains $r$ marked vertices where $m \leq r \leq 2m$. 

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We need a generalization of this lemma, which can be proven very easily by induction.

**Lemma 18.** Let \( T \) be a tree, \( k, m \in \mathbb{N} \) and let \( f : V(T) \to \{0, 1\} \) be a mark in \( T \). Let there be at least \( 2mk \) marked vertices. There exist edge-disjoint subtrees \( T_1, \ldots, T_k \) of \( T \) such that \( \bigcup_{p=1}^{k} T_p = T \) and each \( T_p \) contains at least \( m \) marked vertices \( a^1_i, \ldots, a^m_i \) such that \( a^i_i \neq a^j_j \) for all \( i, j \) and \( p \neq q \).

Essentially what the lemma means is that each tree contains \( m \) unique marked vertices. A marked vertex may be contained in multiple trees but it counts as a marked vertex only for one of those. Now we can prove that \( A-M \)-trees have the edge-Erdős-Pósa property.

**Theorem 19.** Let \( A \) be a set of vertices in a graph \( G \). Then there are either \( k \) edge-disjoint \( A-m \)-trees or a set of at most \( 2m^2k^2 \) edges that intersect all \( A-m \)-trees.

**Proof.** Let \( k \in \mathbb{N} \) and let \( A \) be a vertex set in some graph \( G \). We do induction on the size of \( A \) to show that either there are \( k \) edge-disjoint \( A-m \)-trees or a hitting set of size \( (m - 1)|A| \) in \( G \). We may assume that \( G \) is connected, otherwise apply this to each component separately. If the size of \( A \) is smaller than \( m \), then there is no \( A-m \)-tree in all of \( G \) and, hence, there is a hitting set of size zero; so the induction start is true.

Now let \( |A| \geq m \) and let \( a \) be a vertex in \( A \). For each vertex \( v \in A \setminus \{a\} \) add \( k \) new vertices that only \( v \) is adjacent to and let \( A_v \) be the set of these \( k \) vertices. Assume there are \( (m - 1)k \) edge-disjoint paths from \( A \) to the vertices in \( \bigcup_{v \in V \setminus \{a\}} A_v \). As the size of each \( A_v \) is \( k \) there are at most \( k \) paths that end in a set \( A_v \) and, therefore, there are at least \( m - 1 \) different sets such that a path ends there. Order the sets \( A_v \) such that \( A_v, \ldots, A_{v_{m-1}} \) are the sets where the most paths end. At least one path ends in each of these. Combining these paths yields a graph that contains \( a, v_1, \ldots, v_{m-1} \) (maybe even more vertices of \( A \)) and, therefore, also a tree that contains \( m \) vertices of \( A \). Remove the paths used to construct that tree and observe that in each set \( A_v \) at most \( k - 1 \) of the remaining paths can end. Inductively we can then find \( k \) edge-disjoint \( A-m \)-trees; they are edge-disjoint since the paths we used to construct them are.

So assume we do not find \( (m - 1)k \) edge-disjoint paths between \( a \) and \( \bigcup_{v \in V \setminus \{a\}} A_v \). By Menger’s theorem we find a set \( X \) of at most \( (m - 1)k - 1 \) edges that intersects all such paths; remove them. If \( a \) is still connected to another vertex \( v \in A \), then all the edges from \( v \) to \( A_v \) have to be in \( X \). These are \( k \) edges and since \( |X| \leq (m - 1)k - 1 \) edges, this can happen at most \( m - 2 \) times. Therefore, the component that contains \( a \), after removing \( X \), contains fewer than \( m \) vertices of \( A \) and, thus, can be deleted from the graph. Hence, we get a graph with fewer vertices in \( A \) and, using induction, it follows that there are either \( k \) edge-disjoint \( A-m \)-trees or a set of at most \( (m - 1)k(|A| - 1) + (m - 1)k = (m - 1)k|A| \) edges that intersect all \( A-m \)-trees. This finishes the induction.

The last thing we need to do is to bound the size of \( A \). Clearly, if \( A \) was allowed to be arbitrarily large, then also the hitting set would be arbitrarily large. Let the size of \( A \) be at least \( 2mk \) and mark all vertices of \( A \) (define a mark that assigns 1 to the vertices of \( A \)). By Lemma 25 one can assume that \( G \) is connected and hence there is a spanning tree in \( G \), which contains all vertices of \( A \). Use Lemma 15 to decompose that spanning tree into \( k \) edge-disjoint subtrees such that each of them contains \( m \) marked vertices. As the
marked vertices belong to $A$, we are done. Therefore, we may assume that the size of $A$ is bounded by $2mk$. Then the size of the hitting set can be bounded by $2m^2k^2$.

A short $\Theta_r$ is a simple graph that is a subdivision of $\Theta_r$ where each edge has been subdivided at most once (see Figure 15c). We call the two branch vertices of the $\Theta_r$ endvertices and the other vertices interior vertices. For technical reasons we also call a diamond a short $\Theta_2$, where the endvertices are the vertices of degree 2. The order of a $\Theta_r$ is the number of vertices that are not endvertices, that is $|V(\Theta_r)| - 2$.

A circular ordering of short $\Theta_r$’s is a sequence of short $\Theta_r$’s, say $\Theta^1, \ldots, \Theta^n$ with endvertices $v_1, v_2$ where we identify $v_2^i$ and $v_2^{i+1}$ and also $v_1^n$ and $v_1^1$, see Figure 16. Circular orderings of $\Theta_r$’s are exactly the 2-connected graphs that do not contain a ladder, see Lemma 21.

![Figure 16: A circular ordering of $\Theta_3$’s.](image)

Whenever we say that there is a ladder in a graph during a proof, we will only tell you which the two vertices of degree 3 are. One only has to check that there three disjoint paths between these two vertices, two of which have at least length 3. The following remark helps finding ladders.

**Remark 20.** A cycle $C$ of length at least 5 together with a $C$-path of length at least 3 is a ladder.

**Lemma 21.** Let $G$ be a 2-connected graph that does not contain a ladder. Then $G$ is a circular ordering of short $\Theta_r$’s or has fewer than six vertices. Additionally, if the length of a longest cycle in $G$ is at most 4 then $G$ is also a short $\Theta_r$.

**Proof.** Let $C$ be a longest cycle in $G$.

**Claim 1:** Every vertex outside of $C$ has all its neighbours on $C$.

Assume otherwise. Let $w_1, w_2$ be adjacent vertices outside of $C$. As $G$ is 2-connected there are disjoint paths $P_i$ from $w_i$ to $c_i \in C$ for $i = 1, 2$. The path $P = P_1 \cup w_1w_2 \cup P_2$ has length at least 3. If the length of $C$ is at most 4, then there is a path of length at most 2 between $c_1$ and $c_2$ on $C$. Replacing this path by $P$, yields a longer cycle, which is a contradiction. If the length of $C$ is at least 5 then adding $P$ to $C$ gives us a ladder as mentioned in Remark 20. This contradicts the assumption.
Claim 2: The degree of every vertex outside of $C$ is exactly 2. And it is adjacent to two vertices $v, w \in C$ with distance 2 on $C$.

Let $u$ be a vertex outside of $C$. As $G$ is 2-connected, the degree of $u$ has to be at least 2 and by Claim 1 all of its neighbours have to be on $C$. If $u$ is adjacent to two vertices with distance 1 in $C$, that is they are adjacent in $C$, we can increase the length of $C$ by additionally visiting $u$. If the distance of two of its neighbours is at least 3 in $C$, then $C$ together with $u$ yields a ladder. Here the two vertices of degree 3 are those two neighbours of $u$. In each case we get a contradiction.

So whenever we pick two neighbours of $u$ we know that their distance is exactly 2 in $C$. Assume that $u$ has at least three neighbours on $C$. Take any path that contains three neighbours of $u$ and let $v$ be the neighbour that is in between the other two neighbours on that path. This is a ladder where the vertices of degree 3 are $u$ and $v$, again a contradiction. Therefore, the claim is true.

Claim 3: All chords of $C$ are between vertices at distance exactly 2 in $C$.

An edge between two vertices of distance at least 3 in $C$ together with $C$ is a ladder-expansion and, hence, not present. So every chord of $C$ can only be between vertices of distance exactly 2.

Claim 4: If $v_1, v_2, v_3, v_4$ are vertices in $C$ and occur in this order in $C$ (after picking any orientation on $C$), then there are no vertices $z_1, z_2$ outside of $C$ such that $z_1$ is adjacent to $v_1$ and $v_3$ and $z_2$ is adjacent to $v_2$ and $v_4$.

This would give us a ladder-expansion with $v_2$ and $v_3$ as the vertices of degree 3, thus, this does not occur.

Now we are ready to prove the statement of the lemma. First, let the length of $C$ be at least 5. Remove all chords of $C$ for now. We will start with some vertex of the cycle and show that we can go around the cycle and construct a circular ordering of $\Theta_v$'s.

If there is no vertex of degree 3 in $G$, then $G$ is a cycle and therefore a circular ordering of $\Theta_v$'s of order zero. So let $v_1$ be a vertex of degree at least 3. By Claim 2, $v_1$ has to be contained in $C$ and let $v_1v_2\ldots v_n$ be an ordering of the vertices of $C$. The neighbourhood of $v_1$ can be partitioned into vertices that are adjacent to only $v_1$ and $v_3$ and into vertices that are adjacent to only $v_1$ and $v_{n-1}$. Without loss of generality assume that $v_1$ and $v_3$ have at least one common neighbour outside $C$. Now $v_1, v_3$ and their common neighbours are a short $\Theta_v$. We immediately get that all common neighbours, except for $v_2$, have no further neighbours. Suppose that $v_2$ has a further neighbour; note that we removed all chords from $C$. So $v_2$ has to have a neighbour outside of $C$ and that neighbour has to be adjacent to either $v_4$ or $v_n$. But this is exactly the construct in Claim 4 and that was impossible. So $v_2$ does not have any further neighbours.

Continue with $v_3$. If $v_3$ has any neighbour that is not already in the short $\Theta_v$ between $v_1$ and $v_2$ then that neighbour is a common neighbour of $v_3$ and $v_5$ and, by the same argument as before, we get that there is a short $\Theta_v$ between $v_3$ and $v_5$ and all the interior vertices have no further neighbours. If $v_3$ has no other neighbours then we get a short $\Theta_v$ between $v_3$ and $v_4$. Continue with $v_5$ or $v_4$ respectively until we get back to $v_1$. Note that as $v_2$ is only adjacent to $v_1$ and
there cannot be a short \( \Theta_r \) such that \( v_1 \) is in its interior. Therefore, we get a circular ordering of \( \Theta_r \)'s. Note that by this construction there is no short \( \Theta_1 \) of order 1; this is a path of length 2 and we would have split it into two single edges which are both short \( \Theta_1 \).

Now we need to deal with the chords of \( C \). First let the length of \( C \) be 3. Then \( G \) is a triangle as every vertex outside would allow us to increase the length of that cycle, which means \( G \) contains fewer than six vertices. So let its length be 4 now and let \( v_1, \ldots, v_4 \) be an ordering of the vertices of \( C \). If there is no further vertex, then \( G \) contains only four vertices, which are fewer than six. By Claim 2, all vertices outside are adjacent to either \( v_1, v_3 \) or \( v_2, v_4 \). Note that if there is a vertex of each of those types, then we get a contradiction to Claim 4. So assume that all vertices outside of \( C \) are adjacent to \( v_1, v_3 \). The chord \( e_2e_4 \) cannot exist as one could find a cycle of length 5 then. It follows that this is a short \( \Theta_r \) between \( v_1 \) and \( v_3 \).

Hence, the length of \( C \) is at least 5. Let \( e \) be a chord of \( C \). By Claim 3 the distance of the endvertices of \( e \) is exactly 2; we may assume that \( e = e_2e_4 \). First, let \( e_2 \) and \( e_4 \) be endvertices of some short \( \Theta_r \). In that case \( v_3 \) is adjacent only to \( e_2 \) and \( e_4 \). After adding the edge \( e_2e_4 \), there is a short \( \Theta_r \) between \( v_2 \) and \( v_4 \) with interior vertex \( v_3 \) and the graph still is a circular ordering of short \( \Theta_r \)'s.

Now let one endvertex of \( e \), say \( e_2 \), be a vertex in the interior of a short \( \Theta_r \) of order at least 2. By construction, there is a short \( \Theta_r \) of order at least 2 between \( v_1 \) and \( v_3 \). There is a ladder with \( v_1 \) and \( v_2 \) as vertices of degree 3, which is a contradiction, so this case will never happen.

Next let only one of the endvertices of \( e \), again \( e_2 \), be in the interior of a short \( \Theta_r \) of order 1. Since this short \( \Theta_r \) cannot be a short \( \Theta_1 \), the vertices \( v_1 \) and \( v_3 \) are adjacent. Moreover, we may assume that \( v_3 \) is the endvertex of some short \( \Theta_r \); otherwise we can go to the last case. Hence there is a short \( \Theta_r \) between \( v_3 \) and \( e_4 \) and, by Claim 2, it is just an edge. What we get is a diamond between \( v_1 \) and \( v_4 \), which is a short \( \Theta_r \), and the graph is still a circular ordering of short \( \Theta_r \)'s.

Lastly, let both endvertices of \( e \) be vertices in the interior of a short \( \Theta_r \) of order 1. Again \( v_1 \) and \( v_3 \) are adjacent as well as \( v_3 \) and \( v_5 \). If the length of \( C \) is at least 6, then we get a ladder where the vertices of degree 3 are \( v_1 \) and \( v_3 \) (\( v_3 \) and \( v_5 \) is also a possible choice). So suppose the length of \( C \) is exactly 5. The graph induced by the vertices in \( C \) contains a 4-wheel (a cycle of length 4 together with a vertex that is adjacent to all vertices on that cycle). There is a Hamiltonian path (path that contains all vertices of a graph) between any two vertices in the 4-wheel. If there was a vertex outside of \( C \), then that vertex would need to have two neighbours on \( C \), but together with the Hamiltonian path between these two neighbours, we get a cycle of length 6. This is a contradiction. Therefore, there are no vertices outside of \( C \) and the graph contains fewer than 6 vertices.

Using the same approach it is not that hard to see that the following characterization is also true.

**Lemma 22.** Let \( G \) be a 2-connected graph that does not contain a house-expansion, then \( G \) is a cycle or a short \( \Theta_r \).
There are some helpful techniques when dealing with the edge-Erdős-Pósa property. Essentially, we just have to look at a small subset of all graphs and see if the edge-Erdős-Pósa property holds for that subset, then we immediately get that it is also true for all graphs (although with a worse hitting set bound usually). We say a class $\mathcal{F}$ has the edge-Erdős-Pósa property in a family of graphs $\mathcal{G}$ if there is a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for every $k \in \mathbb{N}$ every graph $G \in \mathcal{G}$ either contains $k$ edge-disjoint members of $\mathcal{F}$ or a hitting set for those of size at most $f(k)$. We will only give the proof ideas as these techniques have been used before (see for example [1]).

**Lemma 23.** Let $\mathcal{G}_H^m$ be the family of graphs that do not contain a member of a class of graphs $\mathcal{F}$ with at most $m$ edges. If $\mathcal{F}$ has the edge-Erdős-Pósa property in $\mathcal{G}_H^m$ then it already has the edge Erdős-Pósa property in the family of all graphs. If $g(k)$ is a bound on the hitting set size in $\mathcal{G}_H^m$ and satisfies $g(k) \geq g(k-1) + m$ for all $k \geq 2$ then $g(k)$ is also a bound on the hitting set size for the family of all graphs.

**Proof.** We know that the Erdős-Pósa property holds in the class $\mathcal{G}_H^m$, so take a graph that contains an expansion with at most $m$ edges. Remove that expansion and do induction on $k$. $$

**Lemma 24** (1-vertex-hitting-set). Let $\mathcal{G}_x^*$ be the family of graphs that contain a vertex that intersects all members of a class of graphs $\mathcal{F}$ and let $\mathcal{F}$ have the vertex-Erdős-Pósa property. If $\mathcal{F}$ has the edge-Erdős-Pósa property in $\mathcal{G}_x^*$, then it has the edge-Erdős-Pósa property in the family of all graphs. Furthermore, if $f(k)$ is a bound on the vertex hitting set size and $g(k)$ is a bound on the edge hitting set size in $\mathcal{G}_x^*$, then $f(k) \cdot g(k)$ is a bound on the edge hitting set size for the family of all graphs.

**Proof.** Let $k$ be a positive integer and let $G$ be any graph. Let $X$ be a minimal vertex hitting set in $G$. Show by induction on the size of $X$ that there are either $k$ edge-disjoint subgraphs of $G$ that belong to $\mathcal{F}$ or a set of at most $|X|g(k)$ edges that intersect all subgraphs of $G$ that belong to $\mathcal{F}$. That $\mathcal{F}$ has the edge-Erdős-Pósa property in $\mathcal{G}_x^*$ is the induction start.

Since the class $\mathcal{F}$ has the vertex-Erdős-Pósa property, there are either $k$ disjoint subgraphs of $G$ that belong to $\mathcal{F}$, which are also edge-disjoint, or a set of at most $f(k)$ vertices that intersect all subgraphs of $G$ that belong to $\mathcal{F}$. In that case we find an edge hitting set of size at most $f(k)g(k)$.

**Lemma 25.** Let $\mathcal{G}_i$ be the family of $i$-connected graphs and let $\mathcal{F}$ be a class of graphs such that each member is $i$-connected, $i \in \{1, 2\}$. If $\mathcal{F}$ has the edge-Erdős-Pósa property in $\mathcal{G}_i$, then it has the edge-Erdős-Pósa property in the family of all graphs. If $g(k)$ is a bound on the hitting set order in $\mathcal{G}_i$ and it is a polynomial (lowest degree $\geq 2$) with only positive coefficients then it is also a bound on the hitting set order for the family of all graphs.

**Proof.** Let $\mathcal{F}$ have the edge-Erdős-Pósa property in $\mathcal{G}_i$ with hitting set bound $g$. Let $G$ be a graph with multiple components. Since all members of $\mathcal{F}$ are connected, we can deal with the components separately. First remove all components that do not contain a member of $\mathcal{F}$, as they are not needed. Let $\ell_j$ be the maximum number of edge-disjoint members of $\mathcal{F}$ in the $j$-th component of $G$. If the sum of the $\ell_j$ is at least $k$, then we find $k$ edge-disjoint members of $\mathcal{F}$. So
Claim 2: Let \( T \) be a tree and let \( B_T \) be the set of 1-preleaves in \( T \). From here on, we write \( A_T \) and \( B_T \). First we want to bound the size of \( A_T \).

\[ \text{Claim 1: The size of } A_T \text{ is bounded by } 6k, \text{ that is } |A_T| \leq 6k. \]

Assume the contrary. We mark all vertices of \( A_T \); there are at least \( 2 \cdot 3k \) marked vertices. Use Lemma 24 to decompose \( T \) into \( k \) edge-disjoint trees \( T_1, \ldots, T_k \) such that each \( T_i \) contains at least three marked vertices \( a^1_i, a^2_i, a^3_i \) and \( a^i_n \neq a^i_m \) for \( i \neq j \).

Now take any tree \( T_i \) and its three marked vertices \( a^1_i, a^2_i, a^3_i \). Take the unique path \( P \) from \( a^1_i \) to \( a^2_i \) and the unique path from \( a^1_i \) to \( P \) in \( T_i \) and let the intersection be \( r \). By construction, the paths from \( r \) to \( a^1_i \), to \( a^2_i \) and to \( a^3_i \) are disjoint (except for \( r \)). At least two of those paths have length at least 1, say the paths from \( r \) to \( a^1_i \) and \( a^3_i \). Contract these paths until there is only one edge left and then contract the path from \( r \) to \( a^2_i \) until it is just a vertex, that means \( r = a^2_i \) after contraction. As each marked vertex is a preleaf we can add one leaf of \( a^1_i, a^2_i \) and \( a^3_i \) and also the vertex \( v \), yielding the graph in Figure 17 which is a ladder.

As the marked vertices we use are different for each tree \( T_i \) and no path between the \( a^j_i \) uses any leaves we get \( k \) edge-disjoint ladders, which means we are done.

\[ \text{Claim 2: If } P \text{ is a path in } G \text{ - } v \text{ that contains } 5n \text{ neighbours of } v \text{ then there are } n \text{ ladders in } P \cup v. \]

Let \( p_1, \ldots, p_n \) be the first five vertices on \( P \) (starting in an arbitrary endvertex of \( P \)) that are adjacent to \( v \). Let \( P_i \) be the subpath of \( P \) between \( p_i \) and \( p_{i+1} \).
\( i = 1, \ldots, 4 \). Contract all \( P_i \) until they are just an edge and delete the edges \( v p_2 \) and \( v p_4 \); this is a ladder (see Figure 18). We can do this again for the next five neighbours of \( v \) and the two ladders we find are edge-disjoint because they only intersect in \( v \). Since we have \( 5n \) neighbours of \( v \) on \( P \) we get \( n \) edge-disjoint ladders.

**Claim 3:** The number of neighbours of \( v \) that are not leaves of \( T \) is bounded by 80k.

Let \( T' \) be the subtree of \( T \) that we obtain by removing all leaves of \( T \). We immediately get that all leaves of \( T' \) are preleaves of \( T \). Hence, the number of leaves of \( T' \) is bounded by 6k and, by Lemma 16, the tree \( T' \) contains at most 12k segments. If there are any edges from \( v \) to endvertices of segments, remove them; these are at most 24k edges. For each segment of \( T' \) count the number of full sets of five neighbours of \( v \) and let \( \ell \) be the sum of these numbers over all segments. If \( \ell \geq k \) then we find \( k \) edge-disjoint ladders by applying Claim 2 to each segment. So there are at most \( k - 1 \) full sets of five neighbours of \( v \) on all segments; remove the edge from \( v \) to these \( 5(k - 1) \) (or fewer) neighbours. Afterwards there are fewer than five neighbours of \( v \) on each segment and as there are at most 12k segments we have at most 48k remaining edges. All in all, fewer than 80k edges from \( v \) to vertices of \( T' \). Since \( T' \) contains all vertices that are no leaves of \( T \), we have proven the claim.

Delete all edges from \( v \) to these vertices. All neighbours of \( v \) are now leaves of \( T \). Observe that \( G \) without these edges is still 2-connected. The next thing we want to prove is that leaves of \( T \) belonging to preleaves of order at least 4 collectively do not have many neighbours.

**Claim 4:** There are at most 6k leaves of \( T \) belonging to preleaves of order at least 4 that are adjacent to vertices other than \( v \) and their preleaf.
First we show that:

If a leaf $\ell$ of a preleaf $w$ of order at least 4 has a neighbour $x \not\in \{v, w\}$, then there is a path from $x$ to the tree that does not use $\ell$. All such paths end in a leaf of a 1-preleaf and do not use $v$. \hfill (1)

To start the proof of this statement we want to find a path from $x$ to the tree that does not use $\ell$. If $x \in V(T)$, we are done. If $x \not\in V(T)$, we know that there are two disjoint paths from $x$ to $T$ and only one of them can use $\ell$. Additionally none of them uses $v$ as all neighbours of $v$ are contained in $T$ and $x \neq v$.

Let $P$ be a path from $x$ to a vertex $t \neq \ell$ of the tree that does not contain $\ell$. Add $P$ and the edge $x\ell$ to $T$ and remove the edge $w\ell$. It is easy to see that this gives us another tree $T'$ that contains all neighbours of $v$ and such that all its leaves are neighbours of $v$.

First, if $x$ is a preleaf of $T$ of order at least 2, then there is a ladder in $G$ with at most 7 edges, see Figure 19. By the assumptions in the beginning of the proof this is a contradiction. So let $x$ be a 1-preleaf of $T$. In $T'$ the vertex $x$ is a preleaf of order at least 2 and $w$ stays a preleaf in $T'$ as its order in $T$ was at least 4 and it lost only one leaf. Thus, the amount of preleaves of $T'$ is the same as in $T$ but there is one fewer 1-preleaf, which is a contradiction to the choice of $T$.

Now assume that $x$ is not a preleaf of $T$. After exchanging the edge $w\ell$ for the path $P$ and the edge $x\ell$, the vertex $x$ becomes a preleaf of $T'$. If $t$ is not a leaf, then every preleaf of $T$ is still a preleaf in $T'$ and, therefore, $T'$ has more preleaves than $T$, again a contradiction to the choice of $T$.

If $t$ is a leaf of some preleaf $w_0 \neq w$ that has at least two leaf neighbours, then $w_0$ loses one leaf neighbour but since it had at least two it is still a preleaf in $T'$, which is a contradiction. Lastly if $t$ is a leaf of $w$ then $w$ actually loses two leaves but as its order is at least 4 it still has at least one leaf left. Like before, this is a contradiction which proves (1).

The next thing we want to do is to bound the number of paths that go into a leaf of a 1-preleaf. All the paths we look at are constructed as above, that is they start in a leaf of a preleaf of order at least 4 and end in a leaf of a 1-preleaf and do not use intersect other vertices from the tree.

There are no two distinct leaves $\ell_1, \ell_2$ of preleaves of order at least 4 such that there is a leaf $\ell^*$ of a 1-preleaf $w_0$ and paths $P_i$ from $\ell_i$ to $\ell^*$ that are internally disjoint from $T$. \hfill (2)
Assume the contrary. Follow $P_i$ from $\ell_1$ until it intersects $P_2$ for the first time, let $x$ be this intersection. Add the subpath of $P_i$ from $\ell_1$ to $x$ and $P_2$ to $T$ and remove the edges from $\ell_1$ and $\ell_2$ to their preleaves, let $T'$ be this tree. Observe that $w_0$ is not a preleaf anymore and the preleaves (or preleaf) of $\ell_1$ and $\ell_2$ are still preleaves of order at least 2. If $x$ is adjacent to both $\ell_1$ and $\ell_2$ then $x$ is a preleaf of order at least 2 in $T'$. Hence, $T'$ would be a better choice for $T$ as it has the same amount of preleaves but fewer 1-preleaves.

So let $x$ not be adjacent to both $\ell_1$ and $\ell_2$. Let $p_i$ be the neighbour of $\ell_i$ on $P_i$ for $i = 1, 2$. Then it follows that $p_1 \neq p_2$ and both of those vertices become preleaves. Therefore, the amount of preleaves in $T'$ is larger than in $T$. This is a contradiction and, thus, proves (2).

By Claim 1 we have at most $6k$ 1-preleaves and thus at most $6k$ leaves of 1-preleaves. Combining this with (1) and (2), we get that there are at most $6k$ leaves of preleaves of order at least 4 that are adjacent to vertices other than $v$ and their preleaf, thus proving Claim 4.

Delete the edges from $v$ to the leaves that are adjacent to vertices other than $v$ and their preleaf and also delete all edges from leaves of preleaves of order at most 3 to $v$. Together these are at most $24k$ edges. Iteratively remove all leaves of $T$ that are not adjacent to $v$ (we get a minimal subtree that contains all remaining neighbours of $v$). All remaining leaves are adjacent only to their preleaf and $v$ and since the set of leaves coincides with the set of neighbours of $v$, the same can be said about the neighbours of $v$.

Let $w_1, \ldots, w_n$ be the remaining preleaves. Split $v$ into vertices $v_1, \ldots, v_n$ such that $v_i$ is adjacent to all common neighbours of $w_i$ and $v$. Let $A_v = \{v_1, \ldots, v_n\}$. Let $T_0$ be an $A_v$-3-tree that contains $v_1, v_2$ and $v_3$, identify these three vertices again. Using that all neighbours of $v$ are adjacent only to $v$ and their preleaf and the fact that we therefore have to use the three different preleaves $w_1, w_2, w_3$ for $T_0$, it follows that this gives us a ladder in the same way as has been done in Figure 17. Note that there is a natural bijection between the edge sets before and after splitting $v$. Use Theorem 19 to either find $k$ edge-disjoint $A_v$-3-trees or a set of at most $18k^2$ edges that intersect all such trees. In the first case we immediately get $k$ edge-disjoint ladders. In the second case remove all these edges; there are no $A_v$-3-trees anymore. Identify all vertices in $A_v$ again. Let $X$ be the set of edges that have been removed from $G$ and let $G' = G - X$.

**Claim 5:** In every block of $G'$, there are either $\ell$ edge-disjoint ladders or a set of at most $3\ell + 1$ edges that intersect all ladders for all $\ell \in \mathbb{N}$.

Assuming this claim is true, we want to show that this proves the theorem. As every ladder is contained wholly in a block of $G'$ and all blocks are edge-disjoint, we can look at each block separately. First, remove each block that does not contain a ladder. As each ladder contains $v$, we remove all blocks that do not contain $v$. For each of the remaining blocks $B_1, \ldots, B_n$ count the maximum number of edge-disjoint ladders $\ell_1, \ldots, \ell_n \geq 1$. If the sum of the $\ell_i$ exceeds $k - 1$ then we find $k$ edge-disjoint ladders. Hence we may assume that the sum is at most $k - 1$. Using Claim 5, we can find a set of at most $3(\ell_i + 1) + 1$ edges that intersects all ladders in $B_i$. So altogether there is a set of at most $\sum_{i=1}^n (3\ell_i + 4) \leq 3(k - 1) + 4n \leq 7(k - 1)$ that intersects all ladders (note that we use $n \leq k - 1$ here). Up to this point, we had already removed $18k^2 + 104k$ edges from the graph and, thus, there is a set of at most $18k^2 + 11k$ edges that
intersects all ladders in $G$.

Now we still need to incorporate Lemma 23, Lemma 24, and Lemma 25. Since the hitting set bound satisfies $f(k) \geq f(k - 1) + 7$ and since it is a polynomial where all coefficients are positive, the Lemmas 23 and 25 do not change the hitting set size. To apply Lemma 24 we need a bound on the vertex hitting set for ladders. It has been shown in [13] that the optimal bound for this is in $Θ(k \log k)$. Hence, the bound we obtain for the edge hitting set is $O(k^3 \log k)$ and we are done.

So the last thing we need to do, is to prove the claim. Any block of $G'$ that does not contain a ladder, contains a set of zero edges that intersects all ladders. Thus, the claim is true in this case.

Let $B$ be a block of $G'$ that contains a ladder and, therefore, also contains $v$. We want to find the number of preleaves in $B$ that are adjacent to neighbours of $v$. If there is no such preleaf in $B$, then $B$ contains only $v$ and clearly there is no ladder. If there is exactly one preleaf in $B$, then $B$ is a short $Θ_r$ and again there is no ladder. Note that we used that neighbours of $v$ are adjacent only to $v$ and their preleaf. On the other hand, there are at most two preleaves in $B$ that are adjacent to neighbours of $v$, otherwise take a tree in $B - v$ that contains three of those preleaves and connect each preleaf to $v$ through one of its leaves. This yields an $A_r$-3-tree that is not hit by $X$ if we split $v$, which is a contradiction.

Hence, we can assume that $B$ contains two preleaves $w_1, w_2$ that are still adjacent to neighbours of $v$. Observe that between $v$ and $w_1$ and between $v$ and $w_2$ there is a short $Θ_r$. Since all neighbours of $v$ are only adjacent to their preleaf and $v$ and by 2-connectedness of $B$, it follows that after removing $v$ and its neighbours the remaining graph is still connected. We call all the blocks of this graph and also the short $Θ_r$ between $v$ and $w_1$ and $v$ and $w_2$, parts of $B$. All parts can be ordered cyclically as seen in Figure 20 as otherwise $B$ would not be 2-connected. Therefore, there are exactly two vertices in each part that are adjacent to vertices outside the part, we call them passing points. All parts by themselves are 2-connected and as no part contains a ladder, either because it does not contain $v$ or because it is a short $Θ_r$ it follows, by Lemma 21 that each part is either a short $Θ_r$, a circular ordering of those, or contains fewer than six vertices.

![Figure 20: This shows the parts and passing points.](image-url)
Suppose there are two passing points belonging to the same part such that there are fewer than $3\ell + 2$ edge-disjoint paths between them in that part. By Menger’s theorem, there is a set of at most $3\ell + 1$ edges that intersects all these paths. After removing these edges every block inside of $B$ that contains $v$ is a short $\Theta_r$. Hence, there is no ladder in $B$ anymore and the claim is true. So we may assume that there are at least $3\ell + 2$ edge-disjoint paths between any two passing points belonging to the same part. Thus, there are at least six vertices in each part and each passing point has degree at least 3. So each part is either a short $\Theta_r$ or a circular ordering of those. Furthermore, each passing point is an endvertex of some $\Theta_r$.

Suppose there are $\ell$ edge-disjoint cycles $C_1, \ldots, C_\ell$ of length at least 5 such that each of them is contained wholly in some part. The parts where these cycles are found cannot be short $\Theta_r$’s, so these parts have to be circular orderings of short $\Theta_r$’s. Moreover, each cycle of length at least 5 in a part passes through all endvertices of the $\Theta_r$’s in that part and, thus, also the passing points of that part. Because of the structure of a circular ordering of $\Theta_r$’s, removing any cycle in it can reduce the number of edge-disjoint paths between the passing points by at most 2. As there are at least $3\ell$ edge-disjoint paths in a part there are still at least $\ell$ edge-disjoint paths that are also edge-disjoint from $C_1, \ldots, C_\ell$ in each part. Taking one such path from each part and combining them, we get a cycle that is edge-disjoint from $C_1, \ldots, C_\ell$ and passes through all passing points. Hence, there are $\ell$ edge-disjoint cycles $D_1, \ldots, D_\ell$ that pass through all passing points and are edge-disjoint from $C_1, \ldots, C_\ell$.

As all parts containing $v$ are short $\Theta_r$’s, it follows that the distance from each $C_i$ to $v$ is at least 2. In each cycle $D_i$, there is a $C_r$-path that goes through $v$ and, therefore, has length at least 4. By Remark 20, the graph $C_i \cup D_i$ contains a ladder. As $C_1, \ldots, C_\ell, D_1, \ldots, D_\ell$ are pairwise edge-disjoint, we obtain $\ell$ edge-disjoint ladders.

So assume that there are fewer than $\ell$ edge-disjoint cycles of length at least 5 in the parts of $B$. As observed before such cycles have to go through all endvertices of the $\Theta_r$’s in a circular ordering. Note Therefore, if the maximum number of edge-disjoint cycles of length at least 5 in a part is $2\ell + 2$. Remove $n + 1$ edges from this short $\Theta_r$. Do this for all parts that contain a cycle of length at least 5. The number of edges that were removed is at most $2\ell - 2$. Then the only remaining block of $B$ that contains $v$ is a circular ordering of short $\Theta_r$’s. This graph does not contain a ladder.

The same proof (with a small modification) also yields the following result:

**Theorem 26.** The house graph has the edge Erdős-Pósa property with hitting set bound $O(k^2 \log k)$.

**Proof.** Until after the proof of Claim 4, everything stays the same (note that any ladder is a subdivision of the house). Then instead of looking for $A_r$-3-trees, look for $A_r$-paths. Any such path yields a house-expansion. If there are $2k$ edge-disjoint paths, these can be made into $k$ edge-disjoint house-expansions. Otherwise there is a set of $4k$ edges that intersect all these paths. All remaining blocks that contain $v$ are short $\Theta_r$ and, therefore, do not contain a house-expansion, by Lemma 22. □
6 Discussion

Is Theorem 2 optimal? We will see in Proposition 27 that it is impossible to improve upon our bound of 14 rungs when using a condensed wall.

**Proposition 27.** For every \( n \in \mathbb{N} \), every condensed wall \( W \) of size \( r \geq 5 \cdot n \) contains \( n \) edge-disjoint subdivisions of ladders \( L_1, L_2, \ldots, L_n \) which all have exactly 13 rungs.

**Proof.** We will split our wall into \( n \) chunks \( C_i, i \in [n] \) each containing exactly 5 layers of \( W \) plus \( a \) and \( b \). If \( r > 5 \cdot n \), we can discard any additional layers by not placing any part of a ladder in them. Let \( C_i \) be the subgraph of \( W \) induced by \( W_{5i-4}, W_{5i-3}, \ldots, W_{5i}, a \) and \( b \) for all \( i \in \mathbb{N} \). The only vertices shared by all chunks \( C_i \) are \( a \) and \( b \), otherwise the chunks only overlap at a bottleneck vertex. In particular, all chunks are edge-disjoint.

Each ladder \( L_i, i \in [n] \) will be placed in its own chunk \( C_i \) as in Figure 21.

![Figure 21: A ladder (thick edges) of size 13 in a condensed wall of size 5. Edges belonging to rungs are labeled.](image)

As the upper definition yields a ladder \( L_i \) with 13 rungs for every \( i \in [n] \) and those ladders do not share a single edge, we have proven that \( W \) contains \( n \) edge-disjoint ladders.

In Conjecture 1, we have posed the question whether the condensed wall plays a key role for characterizing the edge-Erdős-Pósa property. We have completed an alternative proof for Theorem 2 which does not use a standard condensed wall anymore, but omits all jump-edges \( z_{i-1}z_i \). This is a hint that the condensed wall might not be as crucial as it seemed to be.

Curiously, our proof is the only known case for which the edge-Erdős-Pósa property does not fail for \( k = 2 \), but instead for \( k = 3 \).

With respect to ladders, we can also take a look at the other side of the problem: Can we prove that ladders with more than three rungs have the edge-Erdős-Pósa property? We do not know, but at least the proof in this paper will not carry over to a larger ladder. This is because the way we construct the tree in the proof does not help for larger ladders. Additionally, it looks like it is much...
harder to characterize the graphs without longer ladders. However, it might be possible to generalize the approach to other subdivisions of $\Theta_3$.

We also want to quickly discuss the hitting set size for ladders with three rungs. Let $\mathcal{F}$ be the set of ladders with three rungs. The hitting set bound for these in $\mathcal{G}_k^*$ is at least $k - 1$ (take for example $k - 1$ ladders and choose one vertex from each ladder and identify them with each other) and as we have mentioned in the proof of Theorem 3 the optimal vertex hitting set bound for the vertex-Erdős-Pósa property of ladders is $\Theta(k \log k)$ [13]. So if we are using Lemma 24 the hitting set bound will be in $\Omega(k^2 \log k)$. We could quite easily obtain this bound if we could obtain a linear bound in $k$ for Theorem 19, that is, for $A$-$m$-trees, as then the hitting set bound in $\mathcal{G}_k^*$ would also be linear instead of quadratic. And actually we do believe that this is true. On the other hand the bound for the ladders has to be in $\Omega(k \log k)$ as in the vertex version [10], so there is not that much room to improve.

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