ON CROSSED PRODUCT OF ALGEBRAS*

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Abstract

The concept of a crossed tensor product of algebras is studied from a few points of views. Some related constructions are considered. Crossed enveloping algebras and their representations are discussed. Applications to the noncommutative geometry and particle systems with generalized statistics are indicated.

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I Introduction

The notion of a crossed product of Hopf algebras is well-known \[1\]. It is also known that there is an algebra analogue for such product called a crossed (braided) product of algebras. It has been used in several constructions in the area of the noncommutative geometry, quantum groups and braided categories. For example this product in braided categories has been studied intensively by Majid \[2, 3, 4, 5\]. It is interesting that there is a more general notion of a crossed product of algebras without the notion of the braided categories. Namely, if \(A\) and \(B\) are two unital and associative algebras over a field \(k\), then such product is formed by a bigger algebra \(W\). This algebra contains algebras \(A\) and \(B\) as subalgebras in such a way that \(W\) is generated as an algebra just by \(A\) and \(B\). This product is called in general a crossed (or a twisted) tensor product and it has been recently studied on an abstract algebraic level by Van Daele and Van Keer \[6\], by Čap, Schichl and Vanžura \[7\]. An application of such product in the area of \(C^*\)-algebras has been considered by Woronowicz \[8\]. A crossed tensor product has also been used by Zakrzewski \[9\] in the study of quantum Lorentz and Poincaré groups. An interesting approach for the study of noncommutative de Rham complexes has been developed by Manin \[10\]. It is based on the notion of the so-called skew product of algebras. Similar concept corresponding to the algebra of differential forms on a full matrix bialgebra has been developed by Sudbery \[11\]. According to his construction such algebra is a skew product of an algebra of functions and an algebra of differential forms with constant coefficients. It is obvious that such skew product provide an example of a crossed tensor product. Related subject has been also considered by Wambst \[12\]. One can see that in general the algebra \(W\) of differential operators acting on an algebra \(A\) can be described as a crossed product of the algebra and the algebra of vector fields corresponding for an arbitrary noncommutative differential calculus \[13, 14, 15, 16\].

In the present paper we are going to study the concept of a crossed tensor product of algebras from a few different points of views: module theory, Hopf algebras, free product of algebras and some constructions related to the noncommutative geometry. Our considerations are motivated by the application to the construction of the so-called crossed enveloping algebras and their representations. Note that Wick algebras considered previously in the study of deformed commutation relations \[17\] are particular examples of crossed enveloping algebras. Such algebras has been also important in the study of systems with generalized quantum statistics \[18, 19, 20\].

The paper is organized as follows. We recall the definition of the crossed product in the Section 2. The corresponding module structures are considered in this section. The relation between this product and the smash product or the semi–simple product of Hopf algebras is given. The connection with the free product of algebras is studied in Section 3. The construction of twisted product for free algebras is described in details in the Section 4. Ideals in the twisted products and corresponding quotient constructions are studied in the Section 5. Consistency conditions for such constructions are described as consequences of axioms for the twisted product. Some examples are given. In the Section 6 representations of the twisted tensor product are considered. Crossed enveloping algebras are described as a twisted tensor product of a pair of conjugated algebras. Representations of crossed enveloping algebras are also considered. As an example the Fock space representation for a system with generalized statistics is given.
II Preliminaries

In this note \( k \) is a field of complex (or real) numbers. All objects considered here are first of all \( k \)-vector spaces. All maps are assumed to be \( k \)-linear maps. The tensor product \( \otimes \) means \( \otimes_k \). In what follows algebra means associative unital \( k \)-algebra and homomorphisms are assumed to be unital. If \( \mathcal{A} \) is an algebra then \( \mathcal{A}^{\text{op}} \) denotes algebra with the opposite multiplication: \( a^{\text{op}} a' = a'a \). For comultiplication \( \Delta \) we shall use a shorthand Sweedler (sigma) notation \( \Delta(a) = \sum_i a_i' \otimes a_i'' \equiv a^{(1)} \otimes a^{(2)} \) (with \( \sum \), omitted). Likewise, throughout the paper we will use the Sweedler type notation for a twisting map (see below) \( \tau : \mathcal{B} \otimes \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{B} \) i.e. we will write \( \tau(b \otimes a) \equiv a^{(1)} \otimes b^{(2)} \); again the summation is assumed here but not written explicitly.

Let \( \mathcal{A} \) and \( \mathcal{B} \) be two unital and associative algebras over \( k \). The multiplication in these algebras is denoted by \( m_{\mathcal{A}} \) and \( m_{\mathcal{B}} \), respectively. Let us recall briefly the concept of a crossed product of algebras [4].

**Definition:** An associative algebra \( \mathcal{W} \) equipped with two injective algebra homomorphisms \( i_{\mathcal{A}} : \mathcal{A} \hookrightarrow \mathcal{W} \) and \( i_{\mathcal{B}} : \mathcal{B} \hookrightarrow \mathcal{W} \) such that the canonical linear map \( \Phi : \mathcal{A} \otimes_k \mathcal{B} \longrightarrow \mathcal{W} \) defined by

\[
\Phi(a \otimes b) := m_{\mathcal{W}} \circ (i_{\mathcal{A}} \otimes i_{\mathcal{B}})(a \otimes b)
\]

is a linear isomorphism is said to be a crossed (twisted) product of \( \mathcal{A} \) and \( \mathcal{B} \).

The above definition means that the crossed product of algebras \( \mathcal{A} \) and \( \mathcal{B} \) is a bigger algebra \( \mathcal{W} \) which contain these two algebras as subalgebras in such a way that \( \mathcal{W} \) is generated by \( \mathcal{A} \) and \( \mathcal{B} \). In particular as a linear space the algebra \( \mathcal{W} \) is isomorphic to \( \mathcal{A} \otimes \mathcal{B} \). The definition is given up to the isomorphism of algebras. As an example, we can consider the standard tensor product of algebras with multiplication given by the formula

\[(a \otimes b)(a' \otimes b') := aa' \otimes bb' \text{ for } a \otimes b, a' \otimes b' \in \mathcal{A} \otimes \mathcal{B}.
\]

For our purposes here, we shall denote by \( \mathcal{A} \otimes_c \mathcal{B} \) an algebra being the standard tensor product of two algebras \( \mathcal{A} \) and \( \mathcal{B} \). One will see in the moment that this example does not exhaust all possible cases.

First, let us study module structures on the above crossed product of algebras. If \( \mathcal{A} \) and \( \mathcal{B} \) are algebras, then an arbitrary linear space \( \mathcal{M} \) is said to be a \((\mathcal{A}, \mathcal{B})\)-bimodule if \( \mathcal{M} \) is left \( \mathcal{A} \)-module and right \( \mathcal{B} \)-module and both structures commute, i.e.

\[
(a.m).b = a.(m.b), \quad a \in \mathcal{A}, b \in \mathcal{B}, m \in \mathcal{M}.
\]

In other words, \((\mathcal{A}, \mathcal{B})\)-bimodules are left modules over \( \mathcal{A} \otimes_c \mathcal{B}^{\text{op}} \). We shall also identify \( \mathcal{A} \)-bimodules with \((\mathcal{A}, \mathcal{A})\)-bimodules. Observe that the tensor product \( \mathcal{A} \otimes \mathcal{B} \) has a canonical left \( \mathcal{A} \)-module and right \( \mathcal{B} \)-module structure defined by

\[
a.(a' \otimes b) := aa' \otimes b, \quad (a \otimes b').b' := a \otimes bb',
\]

respectively. This means that \( \mathcal{A} \otimes \mathcal{B} \) inherits a \((\mathcal{A}, \mathcal{B})\)-bimodule structure in natural way.

The \((\mathcal{B}, \mathcal{A})\)-bimodule structure on \( \mathcal{A} \otimes \mathcal{B} \) is a problem. We use the concept of module cross [10] for the study of this problem. Let \( \tau : \mathcal{B} \otimes \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{B} \) be a linear mapping, then we define the left, right and two-sided universal lift of \( \tau \) as mappings (cf. [10])

\[
\tau^u : \mathcal{A} \otimes \mathcal{B} \otimes \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{B} , \quad \tau^u(a \otimes b \otimes a') := a' \cdot \tau(b \otimes a),
\]

\[
\tau^u : \mathcal{B} \otimes \mathcal{A} \otimes \mathcal{B} \longrightarrow \mathcal{A} \otimes \mathcal{B} , \quad \tau^u(b \otimes a \otimes b) := \tau(b \otimes a).b',
\]

\[
\tau^u : \mathcal{B} \otimes \mathcal{A} \otimes \mathcal{B} \longrightarrow \mathcal{A} \otimes \mathcal{B} , \quad \tau^u(b' \otimes a \otimes b \otimes a') := a' \cdot \tau(b \otimes a).b',
\]

respectively.
Lemma: A mapping \( \tau : B \otimes A \rightarrow A \otimes B \) define on \( A \otimes B \) a structure of: (i) a right \( A \)-module, (ii) left \( B \)-module, (iii) \((A, B)\)-bimodule, if and only if the corresponding universal lift \( "\tau", \tau^u \) or \( "\tau^u" \) is the algebra homomorphism

\[
\begin{align*}
(i) & \quad "\tau \in \text{alg}(A^{op}, \text{End}_k(A \otimes B)), \\
(ii) & \quad \tau^u \in \text{alg}(B, \text{End}_k(A \otimes B)), \\
(iii) & \quad "\tau^u \in \text{alg}(B \otimes_c A^{op}, \text{End}_k(A \otimes B)),
\end{align*}
\]

respectively.

Proof: Let \( M \) be a left \( B \)-module. For each \( b \in B \) define \( L_b \in \text{End}_kM \) by \( L_b(m) := b.m \). Then \( L : B \rightarrow \text{End}_kM \) is an algebra homomorphism. Similarly, right \( A \)-module structures on \( M \) are in one-to-one correspondence with left \( A^{op} \) structures on \( M \). Now (iii) is obvious, since any \((B, A)\)-bimodule structure is in fact, due to commutativity (2), a left \( B \otimes_c A^{op} \) structure. \( \square \)

Definition: A \((A, B)\)-bimodule \( W \) which is at the same time a \((B, A)\)-bimodule, \( A \)-bimodule and \( B \)-bimodule is said to be a crossed \((A, B)\)-bimodule.

Theorem: There is one to one correspondence between crossed \((A, B)\)-bimodule structure on \( A \otimes B \) and linear mappings \( \tau : B \otimes A \rightarrow A \otimes B \) satisfying the following relations

\[
\tau(1 \otimes a) = a \otimes 1 \\
\tau \circ (m_B \otimes id_A) = (id_A \otimes m_B) \circ (\tau \otimes id_B) \circ (id_B \otimes \tau)
\]

and

\[
\tau(b \otimes 1) = 1 \otimes b \\
\tau \circ (id_B \otimes m_A) = (m_A \otimes id_B) \circ (id_A \otimes \tau) \circ (\tau \otimes id_A),
\]

Proof: The left \( A \)-module and a right \( B \)-module acting on \( A \otimes B \) is given by formulae (3). We define a right \( A \)-module and a left \( B \)-module action on \( A \otimes B \) by formulae

\[
(a \otimes b).a' := a'.(\tau(b \otimes a) = "\tau(a \otimes b \otimes a')
\]

\[
b'.(a \otimes b) := \tau(b \otimes a)b' = \tau^u(b' \otimes a \otimes b),
\]

respectively. \( \square \)

Definition: A \( k \)-linear mapping \( \tau : B \otimes A \rightarrow A \otimes B \) satisfying the condition (4) is called a left \( B \)-module cross. Similarly, if the relation (3) is satisfied, then \( \tau \) is called a right \( A \)-module cross. The map \( \tau \) is said to be an algebra cross if it is both a left \( B \)- and right \( A \)-module cross.

It is obvious that the standard twist (switch) \( \tau : B \otimes A \rightarrow A \otimes B \) defined by \( \tau(b \otimes a) := a \otimes b \) satisfies all conditions for the cross. It give rise to the standard tensor product of algebras \( A \otimes_c B \). The second example is a graded algebra version of the previous one. It is provided by the following graded twist

\[
\tau(b \otimes a) := (-1)^{mn}a \otimes b,
\]

where \( a \in A, b \in B \) are homogeneous elements of graded algebras \( A \) and \( B \) of grade \( m \) and \( n \), respectively. Let \( \tau : B \otimes A \rightarrow A \otimes B \) be an algebra cross, then according to the last theorem there exists a structure of a crossed \((A, B)\)-bimodule on \( A \otimes B \). This structure will be denoted by \( A \rtimes_c B \).
Lemma: Let $\mathcal{W}$ be a crossed $(\mathcal{A}, \mathcal{B})$–bimodule. Assume that the algebra $\mathcal{A}$ as $k$-submodule universally generates $\mathcal{W}$ as a left $\mathcal{A}$–module and similarly $\mathcal{B}$ generates $\mathcal{W}$ as a right $\mathcal{B}$–module. Then there exist the unique algebra cross $\tau : \mathcal{B} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{B}$ such that $\mathcal{W} = \mathcal{A} \triangleright \tau \triangleleft \mathcal{B}$.

Proof: We denote by $\Phi : \mathcal{A} \otimes \tau \mathcal{B} \rightarrow \mathcal{W}$ the mapping which is a left $\mathcal{A}$–module and a right $\mathcal{B}$–module isomorphism. We define $\tau(b \otimes a) := \Phi^{-1}(ba)$. □

Theorem: There is one to one correspondence between algebra cross $\tau : \mathcal{B} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{B}$ and crossed product $\mathcal{W}$ of algebras $\mathcal{A}$ and $\mathcal{B}$.

Proof: Let us assume that the algebra $\mathcal{W}$ is universally generated crossed product of algebras $\mathcal{A}$ and $\mathcal{B}$. We define a linear mapping $\tau_{\mathcal{W}} : \mathcal{B} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{B}$ by the following formula

$\tau_{\mathcal{W}}(b \otimes a) := [m_{\mathcal{W}} \circ (i_{\mathcal{A}} \otimes i_{\mathcal{B}})]^{-1}(ba)$. (10)

It can be deduced that the above mapping is an algebra cross. Moreover, the map $m_{\mathcal{W}} \circ (i_{\mathcal{A}} \otimes i_{\mathcal{B}})$ is an algebra isomorphism of $\mathcal{A} \triangleright \tau \triangleleft \mathcal{B}$ onto $\mathcal{W}$. Conversely, let $\tau : \mathcal{B} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{B}$ be an algebra cross, then the tensor product $\mathcal{A} \otimes \mathcal{B}$ of algebras $\mathcal{A}$ and $\mathcal{B}$ equipped with the multiplication $m_\tau : (\mathcal{A} \otimes \mathcal{B}) \otimes (\mathcal{A} \otimes \mathcal{B}) \rightarrow \mathcal{A} \otimes \mathcal{B}$ defined by the formula

$m_\tau := (m_{\mathcal{A}} \otimes m_{\mathcal{B}}) \circ (id_{\mathcal{A}} \otimes \tau \otimes id_{\mathcal{B}})$ (11)

is associative [3, 4]. In this case both relations (7b) and (8b) can be written equivalently by

$\tau \circ (m_{\mathcal{B}} \otimes m_{\mathcal{A}}) = m_{\tau} \circ (\tau \otimes \tau) \circ (id_{\mathcal{B}} \otimes \tau \otimes id_{\mathcal{A}})$. (12)

It is easy to see that $\mathcal{A} \otimes \mathcal{B}$ equipped with the above multiplication is a crossed product of algebras $\mathcal{A}$ and $\mathcal{B}$. The inclusion $i_{\mathcal{A}}$ and $i_{\mathcal{B}}$ are defined as follows

$i_{\mathcal{A}}(a) := a \otimes 1, \quad i_{\mathcal{B}}(b) := 1 \otimes b$. (13)

Observe that the multiplication (11) in the crossed product $\mathcal{A} \triangleright \tau \triangleleft \mathcal{B}$ of algebras $\mathcal{A}$ and $\mathcal{B}$ can be given in the following form

$(a' \otimes b)(a \otimes b') = a'(a_{(1)} \otimes b_{(2)})b'$, (14)

where, as already mentioned, the Sweedler type notation for the cross $\tau$ has been used, i.e.

$\tau(b \otimes a) := a_{(1)} \otimes b_{(2)}$. (15)

In particular, we have the relations

$(a \otimes 1)(1 \otimes b) = a \otimes b, \quad (1 \otimes b)(a \otimes 1) = a_{(1)} \otimes b_{(2)}$. (16)

It is interesting that elements of the algebra $\mathcal{A} \triangleright \tau \triangleleft \mathcal{B}$ can be ordered in such a way that all elements of the algebra $\mathcal{B}$ are to the right and elements of the algebra $\mathcal{A}$ are to the left. Such ordering is said to be Wick ordering.

As we already mentioned before, the crossed product related to the standard twist is known as the tensor product of algebras. If we use the graded twist, then we obtain the
graded tensor product of graded algebras. In the general case, however, an algebra cross
is not so simple. The construction of all possible crossed products for a given pair of
algebras $\mathcal{A}, \mathcal{B}$ is a problem. It is difficult to describe a general method for an arbitrary
pair of algebras. Hence we restrict our attention to some particular classes of algebras.
Let $\mathcal{B}$ be a bialgebra. This means that there is an algebra homomorphism $\Delta : \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B}$
and the counit $\varepsilon$. We have here the following well–known conditions: the coassociativity
\[(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta,\]  
and two relations for the counit
\[(\varepsilon \otimes id) \circ \Delta = 1 \otimes id, \quad (id \otimes \varepsilon) \circ \Delta = id \otimes 1.\]  
An algebra $\mathcal{A}$ is said to be a left $\mathcal{B}$–module algebra if there is an action $\triangleright : \mathcal{B} \rightarrow \mathcal{A}$ such
that
\[b \triangleright (aa') = (b^{(1)} \triangleright a)(b^{(2)} \triangleright a'),\]  
\[1 \triangleright a = a.\]  
We have the following:

**Lemma:** If $\mathcal{A}$ is a left $\mathcal{B}$–module algebra, then there is an algebra cross $\tau : \mathcal{B} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{B}$
defined by the relation
\[\tau(b \otimes a) = (b^{(1)} \triangleright a) \otimes b^{(2)}\]  
for $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

**Proof:** We shall show that two condition (7, 6) for the cross (20) are satisfied. For the
left hand side of the first relation (7) we calculate
\[[\tau \circ (id_{\mathcal{B}} \otimes m_{\mathcal{A}})](b \otimes a \otimes a') = [b^{(1)} \triangleright m(a \otimes a') \otimes b^{(2)}] = m[(b^{(1)} \triangleright a) \otimes (b^{(2)} \triangleright a')] \otimes b^{(3)},\]  
where $(id \otimes \Delta) \circ \Delta(b) = b^{(1)} \otimes b^{(2)} \otimes b^{(3)}$ and the coassociativity condition have been used.
For right hand side we obtain
\[[m_{\mathcal{A}} \otimes id_{\mathcal{B}}] \circ (id_{\mathcal{A}} \otimes \tau) \circ (\tau \otimes id_{\mathcal{A}})(b \otimes a \otimes a') = m[(b^{(1)} \triangleright a) \otimes (b^{(2)} \triangleright a')] \otimes b^{(3)].\]  
The second relation (13) can be calculated in a similar way. \hfill \Box

If $\tau$ is the cross defined by the formula (21), then the multiplication in the corresponding
crossed product $\mathcal{A} \bowtie_{\tau} \mathcal{B}$ is given by
\[(a \otimes b)(a' \otimes b') = a(b^{(1)} \triangleright a') \otimes b^{(2)}b'.\]  
We can see, in this case, that the crossed product $\mathcal{A} \bowtie_{\tau} \mathcal{B}$ is exactly the so–called smash
product $\mathcal{A} \bowtie \mathcal{B}$, see Ref.[22, 1]. If in addition $\mathcal{A}$ and $\mathcal{B}$ are endowed with a Hopf algebra
structure, then the corresponding crossed product is the semi–simple product of Hopf
algebras introduced by Molnar [21].

Let $\mathcal{A}$ and $\mathcal{B}$ be a dual pair of Hopf algebras [25]. This means that we have a bilinear
pairing $<, , > : \mathcal{B} \otimes \mathcal{A} \rightarrow k$ such that
\[< \Delta(b), a \otimes a' > = b, aa' >, \quad < bb', a > = b \otimes b', \Delta(a) >.\]  

Observe that there is a left action of the algebra $B$ on $A$

$$b \triangleright a = < b, a^{(2)} > a^{(1)},$$

(23)

One can prove that the algebra $A$ is a (left) $B$–module algebra and the mapping $\tau : B \otimes A \rightarrow A \otimes B$ defined by

$$\tau(b \otimes a) \equiv a^{(1)} \otimes b^{(2)} := b^{(1)} \triangleright a \otimes b^{(2)} = < b^{(1)}, a^{(2)} > a^{(1)} \otimes b^{(2)}$$

(24)

is a cross. This is interesting point that the corresponding crossed product $A \triangleright \curvearrowright_s B$ contains all information about noncommutative differential operators [24] on $A$. It means that we can forget the Hopf algebra structures in $A$ and $B$ and restrict our attention to the algebra structure only. In this case we obtain the so–called crossed product of algebras [7].

### III Free product of algebras

Let $A$ and $B$ be two unital associative algebras over a field $k$. Then there is an algebra of polynomials containing elements of these two algebras. This algebra is said to be a free product of algebras $A$ and $B$ [23]. Namely, we have here the

**Definition:** An (algebraic) free product of algebras $A$ and $B$ is the algebra $A \ast B$ formed by all formal finite sums of monomials of the form $a_1 \ast b_1 \ast a_2 \ast \ldots \text{ or } b_1 \ast a_1 \ast b_2 \ast \ldots$, where $a_i \in A$, $b_i \in B$, $i = 1, 2, \ldots$ are non-scalar elements.

In other words $A \ast B$ is the algebra generated by two algebras $A$ and $B$ with no relations except for the identification of unit element, i.e. $1_A = 1_B = 1$. One can see that this free product of algebras is commutative and associative

$$A \ast B = B \ast A, \quad (A \ast B) \ast C = A \ast (B \ast C).$$

(25)

Moreover, if $A_1$ is a subalgebra of $A$ and $B_1$ is a subalgebra of $B$, then $A_1 \ast B_1$ is a subalgebra of $A \ast B$. In particular, the algebras $A$ and $B$ are subalgebras of $A \ast B$. It is known that the product $A \ast B$ possesses the following universal property:

**Lemma:** For every pair of algebra maps $u : A \rightarrow C$ and $v : B \rightarrow C$ there exist one and only one algebra map $w$ such that $u = w \circ j_A$ and $v = w \circ j_B$ or in other words the following diagram

$$\begin{array}{ccc}
A & & C \\
\downarrow j_A & \swarrow & \\
A \ast B & w & \\
\uparrow j_B \\
B
\end{array}$$

(26)

commutes. Here, $j_A$ (resp. $j_B$) denotes the natural inclusion of $A$ (resp. $B$) into $A \ast B$. Note that $w$ is onto if and only if $C$ is generated by images $u(A)$ and $v(B)$.

**Proof:** The proof can be immediately seen if one defines

$$w(\ldots b \ast a \ldots) = \ldots v(b)u(a) \ldots$$
Let us consider a simple example of an algebraic free product.

**Example:** Let $U$ and $W$ be two $k$-vector space. Then the tensor algebra over the direct sum $U \oplus W$ is a free product of tensors algebras $TU$ and $TW$, i.e. we have the relation

$$T(U \oplus W) = TU \ast TW.$$ 

Let us consider the free product of maps.

**Definition:** Let $f : A \to B$ and $g : C \to D$ be two algebra maps. Then a mapping $f \ast g : A \ast B \to C \ast D$ defined by

$$f \ast g (\ldots b \ast a \ldots) = \ldots g(b) \ast f(a) \ldots$$

for $a \in A$ and $b \in B$, is called a free product of $f$ and $g$.

We have the following simple lemma.

**Lemma:** The map $f \ast g$ is injective (resp. surjective) if and only if the maps $f$ and $g$ are injective (resp. surjective).

Now we are going to study ideals in the free product of algebras. It is interesting that in a free product $A \ast B$ of algebras $A$ and $B$ may exists an ideal $J$ such that the quotient $(A \ast B)/J$ can be also expressed as a free product of certain algebras.

**Lemma:** Let $I_A$ and $I_B$ be ideals in algebras $A$ and $B$, respectively. Then

$$J(I_A, I_B) := I_A \ast B + A \ast I_B$$

forms an ideal in the free product $A \ast B$ such that

$$A \ast B / J(I_A, I_B) = (A/I_A) \ast (B/I_B).$$

**Proof:** Let $\pi_A : A \to A/I_A$ and $\pi_B : B \to B/I_B$ be canonical projections, i.e. $I_A := ker \pi_A$ and $I_B := ker \pi_B$. It is obvious that the free product $\pi_A \ast \pi_B : A \ast B \to (A/I_A) \ast (B/I_B)$ is surjective and $(A/I_A) \ast (B/I_B) = (A \ast B) / ker (\pi_A \ast \pi_B)$. One can see that $ker (\pi_A \ast \pi_B) = ker (\pi_A) \ast B + A \ast ker (\pi_B) = J(I_A, I_B).$

The ideal $J(I_A, I_B)$ from the above lemma is called a **free ideal** in $A \ast B$ generated by $I_A$ and $I_B$.

If $A$ and $B$ are two $k$-algebras and $\tau : B \otimes A \to A \otimes B$ is a cross, then the crossed product $A \rtimes_\tau B$ of these algebras can be given by their free product modulo certain ideal. More precisely, we have here the following

**Lemma:** For the crossed product $A \rtimes_\tau B$ of algebras $A$ and $B$ we have the formula

$$A \rtimes_\tau B = (A \ast B) / I_\tau,$$

where $I_\tau$ is an ideal generated by the relation

$$I_\tau = \text{gen}\{b \ast a - a_{(1)} \ast b_{(2)}\}$$

for $a \in A, b \in B$ and $\tau(b \otimes a) := a_{(1)} \otimes b_{(2)}$.

**Proof:** We use the universality of the free product. If $C \equiv A \rtimes_\tau B$, then $u \equiv i_A, v \equiv i_B$, and there exist unique morphism $w \equiv i_A \ast i_B$ such that $i_A = w \circ j_A$ and $i_B = w \circ j_B$. Observe that $w$ is a morphism from $A \ast B$ to $A \rtimes_\tau B$, and his kernel is equal to the ideal $I_\tau$. □

□
IV Crossed product of free algebras

Let $\mathcal{A}$ and $\mathcal{B}$ are graded algebras. This means that we have the following decompositions

$$\mathcal{A} = \bigoplus_{k=0}^{\infty} \mathcal{A}^k, \quad \mathcal{B} = \bigoplus_{k=0}^{\infty} \mathcal{B}^k, \quad (32)$$

where $\mathcal{A}^0 \cong \mathcal{B}^0 \cong k$. Let $\tau : \mathcal{B} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{B}$ be an arbitrary algebra cross. Then the algebra cross $\tau$ can be given by the relation

$$\tau = \bigoplus_{k,l=0}^{\infty} \tau_{k,l}, \quad (33)$$

where $\tau_{k,l}$ is the restriction of the algebra cross $\tau$ to the space $\mathcal{B}^k \otimes \mathcal{A}^l$, $(k, l = 1, 2, \ldots)$. In such a way the algebra cross $\tau$ can be reduced by a set of mappings $\{\tau_{i,j} : \mathcal{B}^i \otimes \mathcal{A}^j \rightarrow \mathcal{A} \otimes \mathcal{B}\}$. Observe that we always have

$$\tau_{0,0} \equiv id, \quad \tau_{k,0}(\mathcal{B}^k \otimes 1) := 1 \otimes \mathcal{B}^k, \quad \tau_{0,m}(1 \otimes \mathcal{A}^m) := \mathcal{A}^m \otimes 1. \quad (34)$$

We have here the following problem: Find the conditions under which all $\tau_{k,l}$ for $k, l > 1$ can be constructed starting from $\tau_{1,1}$. The mapping $\tau_{1,1}$ is given as an initial data for such construction. We restrict our attention to certain particular cases. Let us consider the crossed product of free algebras in details. Let $\mathcal{A}$ be a free algebra generated by $x^1, \ldots, x^m$ and let $\mathcal{B}$ be a free algebra generated by $y^1, \ldots, y^n$. We identify these free algebras $\mathcal{A}$, $\mathcal{B}$ with tensor algebras $TE$ and $TF$, respectively, where $E$ is a linear span of generators $x^1, \ldots, x^m$ of $\mathcal{A}$ and $F$ is a linear span of $y^1, \ldots, y^n$. This means that $\mathcal{A}^1 \equiv E$, $\mathcal{A}^k \equiv E^{\otimes k}$, and similarly for $\mathcal{B}$. Note that $E$ and $F$ are said to be generating spaces for algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Let us consider the structure of the crossed product of free algebras $TE$ and $TF$ in more details.

Remark: Let $\tau_{1,1} : F \otimes E \rightarrow E \otimes F$ be a linear mapping, then there is a unique algebra cross $\tau : TF \otimes TE \rightarrow TE \otimes TF$ such that $\tau_{F \otimes E} = \tau_{1,1}$. Indeed, if $\tau_{1,1} : F \otimes E \rightarrow E \otimes F$ is a linear mapping, then we can introduce a set of mappings $\{\tau_{i,j} : F^{\otimes i} \otimes E^{\otimes j} \rightarrow TE \otimes TF\}$ as follows: Obviously $\tau_{0,0} \equiv id$, $\tau_{k,0}(F^{\otimes k} \otimes 1) := 1 \otimes F^{\otimes k}$, and $\tau_{0,m}(1 \otimes E^{\otimes m}) := E^{\otimes m} \otimes 1$. Then the algebra cross $\tau$ can be defined by the relations (6) and (33). For example

$$\tau_{2,1} = (\tau_{1,1} \otimes id) \circ (id \otimes \tau_{1,1}). \quad (35)$$

Similarly

$$\tau_{1,2} = (id \otimes \tau_{1,1}) \circ (\tau_{1,1} \otimes id). \quad (36)$$

We can calculate $\tau_{k,l}$ for arbitrary $k, l$ in a similar way. Let us consider this case in more details.

Definition: Let $\mathcal{A}$ and $\mathcal{B}$ be two graded algebras. An algebra cross $\tau$ is said to be homogeneous if the image of $\tau_{k,l}$ lies in $\mathcal{A}^l \otimes \mathcal{B}^k$ for all $k, l = 1, 2, \ldots$.

It is obvious that he homogeneous cross can be determined uniquely by a set of linear mappings $\tau_{k,l} : B^k \otimes A^l \rightarrow A^l \otimes B^k$ such that

$$\tau_{k,l+m} \circ (id_A \otimes m_A) = (m_A \otimes id_B) \circ (id_A \otimes \tau_{k,m}) \circ (\tau_{k,l} \otimes id_A),$$

$$\tau_{k+l,m} \circ (m_B \otimes id_A) = (id_A \otimes m_B) \circ (\tau_{k,m} \otimes id_B) \circ (id_B \otimes \tau_{l,m}). \quad (37)$$
for arbitrary integers \( k, l, m > 0 \).

Consider two free algebras \( A := TE \) and \( B := TF \) with their natural gradings. Choose a basis \( x^1, \ldots, x^m \) in \( E \) and a basis \( y^1, \ldots, y^n \) in \( F \). Now, the linear operator \( \tau_{1,1} \equiv \hat{\tau} : F \otimes E \rightarrow E \otimes F \) can be expressed by

\[
\hat{\tau}(y^i \otimes x^j) = \hat{\tau}_{kl}^{ij} x^k \otimes y^l,
\]

its matrix elements \( \hat{\tau}_{kl}^{ij} \). Let us calculate all components \( \tau_{k,l} : F^\otimes k \otimes E^\otimes l \rightarrow E^\otimes l \otimes F^\otimes k \) for this cross. Obviously for \( k = 1 \) and arbitrary \( l > 1 \) we obtain the map \( \tau_{1,l} : F \otimes E^\otimes l \rightarrow E^\otimes l \otimes F \), where

\[
\tau_{1,l} := \hat{\tau}_{l}^{(l)} \circ \ldots \circ \hat{\tau}_{1}^{(1)},
\]

and \( \hat{\tau}_{l}^{(i)} : E_{l}^{(i)} \rightarrow E_{l}^{(i+1)} \), \( E_{l}^{(i)} := E \otimes \ldots \otimes E \otimes F \otimes E \otimes \ldots \otimes E \) \((l + 1\text{-factors, } F \text{ on the i-th place, } 1 \leq i \leq l)\) is given by the relation

\[
\hat{\tau}_{l}^{(i)} := \text{id}_E \otimes \ldots \otimes \hat{\tau} \otimes \ldots \otimes \text{id}_E,
\]

where \( \hat{\tau} \) is on the i-th place. One verifies that

\[
\hat{\tau}_{l}^{(i)} \circ \hat{\tau}_{l}^{(j)} = \hat{\tau}_{l}^{(j)} \circ \hat{\tau}_{l}^{(i)}
\]

if \( |i - j| \geq 2 \). For arbitrary \( k \geq 1 \) and \( l \geq 1 \) we obtain the map \( \tau_{k,l} : F^\otimes k \otimes E^\otimes l \rightarrow E^\otimes l \otimes F^\otimes k \), where

\[
\tau_{k,l} := (\tau_{1,l})^{(1)} \circ \ldots \circ (\tau_{1,l})^{(k)},
\]

where \( (\tau_{1,l})^{(i)} \) is defined in similar way like \( \hat{\tau}_{l}^{(i)} \). In this way we obtain the result:

**Lemma:** Let \( TE \) and \( TF \) be free algebras and \( \hat{\tau} \) be a linear operator defined on generators of these algebras by the relation (33), then there is a homogeneous algebra cross \( \tau : TF \otimes TE \rightarrow TE \otimes TF \) which is given by the relations (33) and (40).

**Proof:** We must prove that for the map \( \tau \) defined by relations (33) and (40) the identities (37) hold true. Observe that we have \( m_A(a \otimes a') \equiv a \otimes a' \) for \( a \in E^\otimes l, a' \in E^\otimes m \) and similarly for \( m_B \), i.e. \( m_A \) and \( m_B \) act as identity operators in this case. Therefore, the relations (37) can be rewritten in a simpler form

\[
\tau_{k,l+m} = (\text{id}_{TE} \otimes \tau_{k,m}) \circ (\tau_{k,l} \otimes \text{id}_{TE}),
\]

\[
\tau_{k+l,m} = (\tau_{k,m} \otimes \text{id}_{TF}) \circ (\text{id}_{TF} \otimes \tau_{l,m}).
\]

After substituting the definition (41) of the maps \( \tau_{k,l} \) into (41) and some calculations we obtain our result. \( \square \)

We have here the following

**Theorem:** Let \( \tau : TF \otimes TE \rightarrow TE \otimes TF \) be an arbitrary cross. Then for the corresponding crossed product we have the following relation

\[
TE \rtimes_{\tau} TF = T(E \oplus F)/I_{\tau},
\]

where

\[
I_{\tau} := \text{gen}\{b \otimes a - a_{(1)} \otimes b_{(2)}\}
\]
is an ideal in $T(E \oplus F)$. If the cross $\tau$ is homogeneous, then
\[ I_\tau := \text{gen}\{ y^i \otimes x^j - \hat{\tau}^{ij}_{kl} x^l \otimes y^k \}. \] (44)

**Proof:** According to the last Lemma of the previous Section for the crossed product we have the relation
\[ TE >\triangleleft_{\tau} TF = (TE * TF)/I_\tau = T(E \oplus F)/I_\tau. \]

Now it is obvious that in the study of noncommutative de Rham complexes and noncommutative calculi with partial derivatives there are several examples of algebras which can be described as algebra crossed product \([10, 12, 15, 28, 29, 30]\).

If the operator $\hat{\tau}$ is given by the diagonal matrix $\hat{\tau}^{ij}_{kl} := t^{ij}_{k1} \delta_{i1} \delta_{j1}, t^{ij} \in k \setminus \{0\}$, then we obtain a simple example of cross for free algebras, namely the so called color cross
\[ \tau (y^i \otimes x^j) = t^{ij} x^j \otimes y^i, \] (45)

If we assume that $k \equiv \mathcal{C}$ and $t^{ij} \equiv q, q \in \mathcal{C} \setminus \{0\}$, then we obtain the $q$-cross.

V Ideals in crossed product

Let us assume that $A >\triangleleft_{\tau} B$ and $A' >\triangleleft_{\tau'} B'$ are crossed product of algebras $A, B$ and $A', B'$ with respect to a cross $\tau$ and $\tau'$, respectively. It is natural to define a morphism of such two crossed products of algebras as a map which transform the first crossed product in the second one.

**Definition:** An algebra morphism $h : A >\triangleleft_{\tau} B \rightarrow A' >\triangleleft_{\tau'} B'$ is said to be a crossed product algebra morphism if there exist two algebra morphisms $h_A : A \rightarrow A'$ and $h_B : B \rightarrow B'$ such that $h = h_A \otimes h_B$.

The above definition means that the crossed product algebra morphism $h : A >\triangleleft_{\tau} B \rightarrow A' >\triangleleft_{\tau'} B'$ is described as a pair of algebra homomorphisms $h_A : A \rightarrow A'$ and $h_B : B \rightarrow B'$. Observe that in the opposite case when we have an arbitrary pair of algebra homomorphism, then their tensor product is not a crossed product algebra morphism, however, there is the following lemma:

**Lemma:** Let $h_A : A \rightarrow A'$ and $h_B : B \rightarrow B'$ be two algebra morphism. Then $h = h_A \otimes h_B$ is a crossed product algebra morphism if and only if we have the relation
\[ (h_A \otimes h_B) \circ \tau = \tau' \circ (h_B \otimes h_A), \] (46)

or in other words the following diagram commutes
\[ \begin{array}{ccc}
B \otimes A & \xrightarrow{\tau} & A \otimes B \\
\downarrow h_B \otimes h_A & & \downarrow h_A \otimes h_B \\
B' \otimes A' & \xrightarrow{\tau'} & A' \otimes B' 
\end{array} \] (47)

We introduce the notion of ideals in crossed product of algebras. Let $A >\triangleleft_{\tau} B$ be a crossed product of algebras $A$ and $B$ with respect to a cross $\tau$, then we have the following:
Definition: A two-sided ideal $J$ in $\mathcal{A} >_\tau \mathcal{B}$ is said to be a crossed ideal in $\mathcal{A} >_\tau \mathcal{B}$ if the quotient map $\pi : \mathcal{A} >_\tau \mathcal{B} \longrightarrow (\mathcal{A} >_\tau \mathcal{B})/J$ is a morphism of crossed products of algebras.

The above definition means that the factor algebra $(\mathcal{A} >_\tau \mathcal{B})/J$, where $J$ is a crossed ideal must be a crossed product of certain algebras $\mathcal{A}', \mathcal{B}'$ with respect to a certain new cross $\tau'$. Thus we must have the relation

$$(\mathcal{A} >_\tau \mathcal{B})/J \cong \mathcal{A}' >_{\tau'} \mathcal{B}'. \tag{48}$$

Let us consider this problem in more details. If $\pi : \mathcal{A} >_\tau \mathcal{B} \longrightarrow (\mathcal{A} >_\tau \mathcal{B})/J$ is a surjective morphism of a crossed product of algebras, then there is a pair of surjective algebra homomorphisms $\pi_A : \mathcal{A} \longrightarrow \mathcal{A}'$ and $\pi_B : \mathcal{B} \longrightarrow \mathcal{B}'$. Observe that these mappings are in fact quotient ones. This means that $\mathcal{A}' \equiv \mathcal{A}/I_A$, and $\mathcal{B}' \equiv \mathcal{B}/I_B$ where $I_A$ (the kernel of $\pi_A$) is a two-sided ideal in $\mathcal{A}$ and $I_B$ is a two-sided one in $\mathcal{B}$. One can see that there is a cross $\tau' : \mathcal{B}/I_B \otimes \mathcal{A}/I_A \longrightarrow \mathcal{A}/I_A \otimes \mathcal{B}/I_B$ such that the following diagram is commutative

$$\begin{array}{ccc}
\mathcal{B} \otimes \mathcal{A} & \xrightarrow{\tau} & \mathcal{A} \otimes \mathcal{B} \\
\pi_B \otimes \pi_A \downarrow & & \downarrow \pi_A \otimes \pi_B \\
\mathcal{B}/I_B \otimes \mathcal{A}/I_A & \xrightarrow{\tau'} & \mathcal{A}/I_A \otimes \mathcal{B}/I_B.
\end{array} \tag{49}$$

In this way we obtain the following:

Lemma: If $J$ is a crossed ideal in the crossed product $\mathcal{A} >_\tau \mathcal{B}$, then there is a pair of ideals $(I_A, I_B)$ in algebras $\mathcal{A}$ and $\mathcal{B}$, respectively and the cross $\tau' : \mathcal{B}/I_B \otimes \mathcal{A}/I_A \longrightarrow \mathcal{A}/I_A \otimes \mathcal{B}/I_B$ such that we have the relation (48). \hfill \Box

It is interesting to investigate the opposite statement. For a given ideals $(I_A, I_B)$ in $\mathcal{A}$ and $\mathcal{B}$, respectively find a corresponding ideal in $\mathcal{A} >_\tau \mathcal{B}$. First, we consider a particular case when one of the ideal in the above pair is trivial.

Definition: A two-sided ideal $I_A$ in the algebra $\mathcal{A}$ such that $I_A \otimes \mathcal{B}$ is a crossed ideal in the algebra $\mathcal{W} \equiv \mathcal{A} >_\tau \mathcal{B}$ is said to be a left $\tau$-ideal in $\mathcal{A}$.

Observe that we have the following criterion (cf. Proposition 3.2.4 in [10])

Lemma: An ideal $I_A$ in $\mathcal{A}$ is a left $\tau$-ideal in $\mathcal{A} >_\tau \mathcal{B}$ if and only if

$$\tau(\mathcal{B} \otimes I_A) \subset I_A \otimes \mathcal{B}. \tag{50}$$

Proof: How can be easily seen, the condition (50) is equivalent to the fact that $J := I_A \otimes \mathcal{B}$ is a two-sided ideal in $\mathcal{A} >_\tau \mathcal{B}$. Therefore, one has to prove that (50) implies that $J$ is a crossed ideal as well. Indeed: the vector space quotient $(\mathcal{A} \otimes \mathcal{B})/J$ is isomorphic to $\mathcal{A}/I_A \otimes \mathcal{B}$. Since $J$ is an ideal, the projection map $\pi_A \otimes id_B : \mathcal{A} >_\tau \mathcal{B} \longrightarrow \mathcal{A}/I_A \otimes \mathcal{B}$, where $\pi_A(a) = [a]$ and $[a] \in \mathcal{A}/I_A$ denotes the equivalence class of $a \in \mathcal{A}$, is an algebra map. In particular, $\pi_A \otimes id_B((1 \otimes b)(a \otimes 1)) = ([1] \otimes b) \otimes ([a] \otimes 1) = [a(1)] \otimes b(2)$. This means that $\tau'([a] \otimes b) := [a(1)] \otimes b(2)$ is a new twist converting $\mathcal{A}/I_A \otimes \mathcal{B}$ into a crossed product algebra. Moreover, the following diagram

$$\begin{array}{ccc}
\mathcal{B} \otimes \mathcal{A} & \xrightarrow{\tau} & \mathcal{A} \otimes \mathcal{B} \\
\downarrow id_B \otimes \pi_A & & \downarrow \pi_A \otimes id_B \\
\mathcal{B} \otimes \mathcal{A}/I_A & \xrightarrow{\tau'} & \mathcal{A}/I_A \otimes \mathcal{B}.
\end{array} \tag{51}$$
must commutes. The formula (50) gives the condition for the commutativity of the above diagram.

It follows immediately from the proof of the previous lemma that we have: Lemma: If $I_A$ is a left $\tau$-ideal in the algebra $A$, then there is a new cross $\tau': B \otimes A/I_A \rightarrow A/I_A \otimes B$ such that the quotient algebra $(A \triangleright_{\tau} B)/(I_A \otimes B)$ is isomorphic to the crossed product $A/I_A \triangleright_{\tau'} B$. \hfill $\Box$

This lemma means that for a left $\tau$-ideal $I_A$ in $A$ we have the relation

$$(A \triangleright_{\tau} B)/(I_A \otimes B) \cong A/I_A \triangleright_{\tau'} B.$$ \hfill (52)

This algebra is said to be a left factor of the crossed product $A \triangleright_{\tau} B$. We can define a right $\tau$-ideal $I_B$ in $B$ in a similar way. It is easy to see that for this ideal we have similar results as for the left $\tau$-ideal. In this way we obtain a right factor of the crossed product $A \triangleright_{\tau} B$ as the following quotient

$$(A \triangleright_{\tau} B)/(I_A \otimes B) \cong A \triangleright_{\tau'} B/I_B.$$ \hfill (53)

A two-sided ideal $J_{I_A,I_B} := I_A \otimes B + A \otimes I_B$ in $A \triangleright_{\tau} B$, where $I_A$ is a left $\tau$-ideal in $A$ and $I_B$ is a right $\tau$-ideal in $B$, is said to be a crossed ideal generated by $I_A,I_B$.

Theorem: If $J_{I_A,I_B}$ is a crossed ideal in the algebra $A \triangleright_{\tau} B$ generated by $\tau$-ideals $I_A,I_B$, then there is a new cross $\tau': B/I_B \otimes A/I_A \rightarrow A/I_A \otimes B/I_B$ such that the quotient algebra $(A \triangleright_{\tau} B)/J_{I_A,I_B}$ is isomorphic to the crossed product $A/I_A \triangleright_{\tau'} B/I_B$. \hfill $\Box$

The quotient algebra $(A \triangleright_{\tau} B)/J_{I_A,I_B}$ is said to be a factor of the crossed product $A \triangleright_{\tau} B$ with respect to $\tau$-ideals $(I_A,I_B)$.

Definition: Let $A \triangleright_{\tau} B$ be a crossed product of algebras $A$ and $B$ with respect to a given cross $\tau: B \otimes A \rightarrow A \otimes B$. If there exist a pair of algebras $\tilde{A}$, $\tilde{B}$ and a cross $\tilde{\tau}: \tilde{B} \otimes \tilde{A} \rightarrow \tilde{A} \otimes \tilde{B}$ such that the product $A \triangleright_{\tau} B$ is an image of $\tilde{A} \otimes_{\tilde{\tau}} \tilde{B}$ under certain surjective morphism $h = (h_A,h_B)$ of crossed products, i.e. the following diagram

$$
\begin{array}{ccc}
\tilde{B} \otimes \tilde{A} & \overset{\tilde{\tau}}{\longrightarrow} & \tilde{A} \otimes \tilde{B} \\
\downarrow_{h_B \otimes h_A} & & \downarrow_{h_A \otimes h_B} \\
B \otimes A & \overset{\tau}{\longrightarrow} & A \otimes B
\end{array}
$$

(54)

is commutative, then $\tilde{A} \otimes_{\tilde{\tau}} \tilde{B}$ is said to be a cover crossed product for $A \triangleright_{\tau} B$.

Lemma: Assume that $A$ and $B$ are algebras with presentation $A := TE/I_A$ and $B := TF/I_B$. If $\tilde{\tau}: TF \otimes TE \rightarrow TE \otimes TF$ is a cross, then the corresponding crossed product $TE \triangleright_{\tilde{\tau}} TF$ is cover for a product $A \triangleright_{\tau} B$ with certain cross $\tau: B \otimes A \rightarrow A \otimes B$ if and only if the ideal $I_A$ is a left $\tilde{\tau}$-ideal in $TE$ and $I_B$ is a right $\tilde{\tau}$-ideal in $TF$. \hfill $\Box$

Lemma: Let $A \triangleright_{\tau} B$ be a crossed product of algebras $A$ and $B$ with presentation $A := TE/I_A$ and $B := TF/I_B$, respectively. If the crossed product $TE \triangleright_{\tilde{\tau}} TF$ is a cover for the product $A \triangleright_{\tau} B$, then we have the relation

$$A \triangleright_{\tau} B \equiv T(E \oplus F)/I,$$ \hfill (55)
V IDEALS IN CROSSED PRODUCT

where $I$ is the ideal in the tensor algebra $T(E \oplus F)$ of the form

$$I \equiv I_1 + I_2 + I_\tau,$$

(56)

$I_1 := \langle I_A \rangle_{T(E \oplus F)}$ is an ideal in $T(E \oplus F)$ generated by the $\tau$–ideal $I_A$, similarly $I_2 := \langle I_B \rangle_{T(E \oplus F)}$, and $I_\tau$ is an ideal in $T(E \oplus F)$ defined by the relation

$$I_\tau := \langle \upsilon \otimes u - \tilde{\tau}(\upsilon \otimes u) \rangle_{T(E \oplus F)},$$

(57)

for every $u \in E$ and $\upsilon \in F$.

\[\Box\]

**Lemma:** Assume that $E, F$ are two linear spaces and $R : E \otimes E \rightarrow E \otimes E, S : F \otimes F \rightarrow F \otimes F$ are two linear operators. Let $A$ and $B$ be two quadratic algebras generated by $E$ and $F$. It means that we have the quotients

$$\mathcal{A} := TE/I_R, \quad \mathcal{B} := TF/I_S,$$

(58)

where ideals are given by the quadratic relations

$$I_R = \langle id - R \rangle_{TE}, \quad I_S = \langle id - S \rangle_{TF}.$$

Assume further, that a homogeneous cross $\tilde{\tau} : TF \otimes TE \rightarrow TE \otimes TF$ is induced by a linear operator $C : F \otimes E \rightarrow E \otimes F$. Then there is a cross $\tau : \mathcal{B} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{B}$ and the corresponding crossed product $\mathcal{A} \rtimes_{\tau} \mathcal{B}$ if and only if we have the following relations

$$(id \otimes C) \circ (C \otimes id) \circ (id \otimes id - (id \otimes R)) = (id - (R \otimes id)) \circ (id \otimes C) \circ (C \otimes id),$$

$$(C \otimes id) \circ (id \otimes C) \circ (id - (S \otimes id)) = (id - (id \otimes S)) \circ (C \otimes id) \circ (id \otimes C).$$

(59)

Moreover

$$T(E \oplus F)/I \cong TE/I_R \rtimes_{\tau} TF/I_S,$$

(60)

where $I$ is an ideal of the form

$$I \equiv I_1 + I_2 + I_C,$$

(61)

$I_1 := \langle I_R \rangle_{T(E \oplus F)}, I_2 := \langle I_S \rangle_{T(E \oplus F)}$ and $I_C$ is an ideal given by the relation

$$I_C := \langle \upsilon \otimes u - C(\upsilon \otimes u) \rangle_{T(E \oplus F)},$$

(62)

for every $u \in E$ and $\upsilon \in F$.

\[\text{Proof:}\] One checks that (59) are equivalent to the $\tilde{\tau}$-ideal conditions (50) for $I_R$ and $I_S$ respectively. \[\Box\]

**Lemma:** Let $R, C$ and $S$ be three linear operators like in the previous lemma. If

$$(R \otimes id)(id \otimes C)(C \otimes id) = (id \otimes C)(C \otimes id)(id \otimes R),$$

$$(id \otimes S)(C \otimes id)(id \otimes C) = (C \otimes id)(id \otimes C)(S \otimes id),$$

(63)

then the conditions (57) are satisfied. \[\Box\]

Let us consider an example of an algebra crossed product. It is well–known that the
The notion of ∗-algebras is well-known. An associative algebra $A$ is said to be a ∗-algebra if we have the following relations for the ∗-operation

$$(ab)^* = b^*a^*, \quad a^{**} = a, \quad (aa)^* = \overline{aa}^*,$$

where $a, b, a^*, b^* \in A$, $\alpha \in \mathbb{C}$, $\overline{\alpha}$ is a complex conjugated to $\alpha$. In this section we assume that $k = \mathbb{C}$ is the field of complex numbers. It is obvious that not every algebra is a ∗-algebra. Observe that if $A$ is a ∗-algebra, then the ∗-operation can be described in two equivalent ways: as an involutive anti-isomorphism of $A$ or an involutive isomorphism between $A$ and $\overline{A}^p$. If $A$ is not a ∗-algebra, then it is interesting to describe all possible ∗-algebra extensions of it. We introduce here the concept of conjugated algebras and crossed enveloping algebras for this goal.

**Definition:** If $A$ be an arbitrary associative algebra, then an algebra $B$ is said to be conjugated to $A$ if there is an antilinear anti-isomorphism (in the complex case) $(-)^*: A \rightarrow B$ such that

$$(ab)^* = b^*a^*, \quad (aa)^* = \overline{aa}^*,$$

where $a, b \in A$ and $a^*, b^*$ are their images under the isomorphism $(-)^*$. The inverse isomorphism $B \rightarrow A$ will be denoted by the same symbol, i.e.

$$(a^*)^* = a.$$
If $A$ is an algebra, then the conjugated algebra will be denoted by $A^*$. It follows immediately from the definition that for a given algebra $A$ the conjugate algebra $A^*$ always exists. The algebra $A^*$ as a vector space is isomorphic to the complex conjugate space $\overline{A}$, and as an algebra – to the opposite one $A^{op}$, i.e. $A \equiv \overline{A}^{op}$, (in the real case it coincides with the opposite algebra $A^{op}$).

Consider a crossed product $\mathcal{W}_\tau(A) := A \rtimes_\tau A^*$ of an algebra with its conjugate. We can try to define the natural $*$-operation in $\mathcal{W}_\tau(A)$ by the relation

$$(a \otimes b^*)^* := b \otimes a^*$$

for $a, b \in A$. Then the following holds

**Lemma:** The algebra $\mathcal{W}_\tau(A)$ is a $*$-algebra if and only if

$$(\tau(b^* \otimes a))^* = \tau(a^* \otimes b)$$

for any $a, b \in A$.

**Proof:** One needs the property $[(1 \otimes b^*)(a \otimes 1)]^* = (1 \otimes a^*)(b \otimes 1)$ or

$$b_{(2)} \otimes a^*_{(1)} = b_{(1)} \otimes a^*_{(2)}$$

which is equivalent to the relation (70).

\[\Box\]

An algebra cross $\tau : A^* \otimes A \longrightarrow A \otimes A^*$ satisfying the relation (70) is called a $*$-cross.

**Definition:** If $\tau : A^* \otimes A \longrightarrow A \otimes A^*$ is a $*$ - cross, then the crossed product

$$\mathcal{W}_\tau(A) := A \rtimes_\tau A^*.$$  

is called a crossed enveloping algebra of $A$ with respect to $\tau$.

Note that the switch $\sigma(b^* \otimes a) := a \otimes b^*$ satisfies (70). It implies that crossed enveloping algebra generalizes the concept of enveloping algebras for associative algebra [33].

From now on, we assume that every algebra cross considered below is a $*$-cross. We shall also identify our two "star"-operations and use the symbol "$\star$" for both of them.

Let us consider the crossed enveloping algebra $\mathcal{W}_\tau(A)$, where $A \equiv TE$ is a free algebra and the generating space $E$ is a (finite or infinite dimensional) complex Hilbert space equipped with an orthonormal basis $\{x^i : i = 1, \ldots, N\}$, and $\tau$ is an arbitrary cross. Note that similar algebras have been studied previously by a few authors [33, 34]. Observe that the conjugated algebra $A^*$ can be identified with the tensor algebra $T E^*$, where $E^*$ is the complex conjugation space. The pairing $(\cdot|\cdot) : E^* \otimes E \longrightarrow \mathcal{C}$ and the corresponding scalar product is given by

$$g_E(x^i| x^j) \equiv (x^i|x^j) = \langle x^i|x^j \rangle := \delta^{ij}.$$  

Let $\hat{\tau} : E^* \otimes E \longrightarrow E \otimes E^*$ be a linear and Hermitian operator with matrix elements

$$\hat{\tau}(x^i \otimes x^j) = \Sigma \hat{\tau}_{kl}^{ij} x^k \otimes x^l,$$

then the quotient

$$\mathcal{W}(\hat{\tau}) = T(E \oplus E^*)/I_{\hat{\tau}},$$

where the ideal $I_{\hat{\tau}}$ is given by the relation

$$I_{\hat{\tau}} := gen\{x^i \otimes x^j - \Sigma \hat{\tau}_{kl}^{ij} x^k \otimes x^l - (x^i|x^j)\}$$
is said to be Hermitian Wick algebra [17].

**Theorem.** (Jørgensen, Schmitt and Werner [17]) The Hermitian Wick algebra \( W(\hat{\tau}) \) is isomorphic to the crossed enveloping algebra \( W_\tau(TE) \) of \( TE \) with respect to the (non-homogeneous) cross generated by \( \hat{\tau} + g_E \).

**Proof:** It has been shown in [17] that the Wick ordered monomials form a basis in \( W(\hat{\tau}) \). In our language it means that \( W(\hat{\tau}) \) as a vector space is isomorphic to \( TE \otimes TE^* \). Moreover, \( TE \) and \( TE^* \) are subalgebras in \( W(\hat{\tau}) \). This implies that \( W(\hat{\tau}) \) is a crossed product, i.e. \( W(\hat{\tau}) \cong TE \triangleright_l \tau TE^* \) for a certain cross \( \tau \). Since \( \hat{\tau} \) is Hermitian operator and \( \langle | \rangle \) is Hermitian scalar product then \( I_{\hat{\tau}} \) is \( \star \)-ideal (i.e. \( I_{\hat{\tau}}^* \subset I_{\hat{\tau}} \)). As a consequence, the cross \( \tau \) is \( \star \)-cross. \( \square \)

Let \( A \) be an algebra with the presentation \( A := TE/I_A \), then for the algebra \( A^* := TE^*/I_A^* \). Observe that if \( I_A \) is a left \( \tau \)-ideal then \( I_A^* \) is automatically a right \( \tau \)-ideal (remember that \( \tau \) is a \( \star \)-cross). Then there is a cross \( \tau' : A^* \otimes A \rightarrow A \otimes A^* \) and the corresponding crossed enveloping algebra \( W_{\tau'}(A) = A \triangleright_l \tau A^* \).

Let \( H \) be a \( k \)-vector space. We denote by \( L(H) \) the algebra of linear operators acting on \( H \). Let \( A \) and \( B \) be two arbitrary \( k \)-algebras and \( \tau : B \otimes A \rightarrow A \otimes B \) be a cross.

**Theorem:** Let \( \pi_A \) and \( \pi_B \) be representations of the algebras \( A \) and \( B \) in \( L(H) \), respectively. If the condition

\[
\pi_B(b)\pi_A(a) = \pi_A(a_{(1)})\pi_B(b_{(2)})
\]

(76)

holds for all \( a \in A, b \in B \), then there exist unique representation \( \pi \) of the crossed product \( A \triangleright_l \tau B \) in \( L(H) \) such that \( \pi|_A = \pi_A \) and \( \pi|_B = \pi_B \).

**Proof:** The representation \( \pi : A \triangleright_l \tau B \rightarrow L(H) \) is defined by

\[
\pi(a \otimes b) := \pi_A(a)\pi_B(b).
\]

(77)

\( \square \)

This theorem allows us to introduce the following definition:

**Definition:** The representation \( \pi \) from the above theorem is said to be a crossed product of representations \( \pi_A \) and \( \pi_B \) and it is denoted by \( \pi_A \triangleright_l \tau \pi_B \).

It is not difficult to prove the converse:

**Theorem:** If \( \pi \) is a representation of the crossed product \( A \triangleright_l \tau B \) of algebras \( A \) and \( B \) in \( H \), then there exist representations \( \pi_A \) and \( \pi_B \) of \( A \) and \( B \), respectively, such that

\[
\pi = \pi_A \triangleright_l \tau \pi_B.
\]

(78)

**Proof:** Representations \( \pi_A \) and \( \pi_B \) are defined by the formulae

\[
\pi_A(a) := \pi(a \otimes 1), \quad \pi_B(b) := \pi(1 \otimes b).
\]

(79)

\( \square \)
VI CROSSED ENVELOPINGS AND REPRESENTATIONS

Let us consider representations of crossed enveloping algebras. For a given representation \( \pi : A \to L(H) \) of \( A \) in a Hilbert space \( H \) one can define a conjugate representation \( \pi^+ : \star \rightarrow L(H) \), where \( + \) stands for the Hermitian conjugation in \( L(H) \). Thus we have:

**Theorem:** Let \( \mathcal{W} \equiv A >\!\!<_{\tau} A^* \) be a crossed enveloping algebra. If \( \pi : A \to L(H) \) is a representation in a Hilbert space \( H \) such that
\[
\pi(b)^+ \pi(a) = \pi(a(b_1)) \pi(b_2) \tag{80}
\]
then there is a unique Hermitian or \( \star \)–representation \( \pi_{\mathcal{W}} : \mathcal{W} \to L(H) \) such that
\[
\pi_{\mathcal{W}} = \pi >\!\!<_{\tau} \pi^+ \tag{81}
\]
Conversely, any Hermitian representation of \( \pi_{\mathcal{W}} \) in a Hilbert space has the form \( \pi_{\mathcal{W}}(w^*) = \pi_{\mathcal{W}}(w)^+ \) for \( w \in \mathcal{W} \).

**Proof:** It is a direct consequence of two proceeding Theorems. One also easily verifies the Hermiticity condition:
\[
\pi_{\mathcal{W}}(w^*) = \pi_{\mathcal{W}}(w)^+ \quad \text{for} \quad w \in \mathcal{W}.
\]
\[\blacksquare\]

As an example, we outline the Fock space representation construction of a cross enveloping algebra \( \mathcal{W}_\tau(A) \equiv A >\!\!<_{\tau} A^* \). For this purpose we assume that \( A \) is a pre-Hilbert space with an unitary scalar product \( \langle | \rangle \). Its completion will be denoted by \( H \). In this case we have at our disposal a canonical representation (the quantization) \( \Pi \) acting on the algebra \( A \) by means of the left (or right) multiplications in \( A \). For \( x, f \in A \) it writes
\[
\Pi(x) \equiv a_i, \quad \Pi^+(x^\dagger) \equiv a_i^+, \tag{84}
\]
It is customary to call them creation and annihilation operators. Thus the commutation relations play a role of the compatibility conditions relating \( a_i^+ \), \( \tau \) and \( \langle | \rangle \), since \( a_i \) are Hermitian conjugate to \( a_i^+ \). For non-free algebra \( a_i^+ \)-s have to satisfy a set of generating relations of the algebra \( A \). These give rise to the supplementary commutation relations. For the ground state \( |0\rangle \equiv 1 \in A \) and annihilation operators we usually assume
\[
\langle 0|0 \rangle = 0, \quad a_i |0 \rangle = |0 \rangle. \tag{85}
\]
If the action \( \Pi \) admits non-degenerate, positive definite (pre-) Hermitian scalar product such that the annihilation operators are well defined and the creation and annihilation operators satisfy the commutation relations \( \langle 80 \rangle \) together with \( \langle 85 \rangle \), then we say that we have the well–defined Fock representation for a crossed enveloping algebra. Of course, the canonical commutation relations (CCR) and the canonical anti-commutation relations (CAR) provide the most familiar examples of this type. Some other examples can be found in [17] and references therein. Similarly, systems with generalized statistics can be described as Fock-like representations of crossed enveloping algebras [20].
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