Two-dimensional gapless spin liquids in frustrated SU(N) quantum magnets

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Abstract. A class of the symmetrically frustrated SU(N) models is constructed for quantum magnets based on the generators of the SU(N) group. The total Hamiltonian lacks SU(N) symmetry. A mean field theory in the quasi-particle representation is developed for spin liquid states. Numerical solutions in two dimensions indicate that the ground states are gapless and the quasi-particles are Dirac particles. The mechanism may be helpful in exploring the spin liquid phases in the spin-1 bilinear-biquadratic model and the spin–orbital model in higher dimensions.

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1. Introduction

Exploring spin liquids has been a recurrent theme in the research of strongly correlated electron systems over the past decades [1]–[8]. The picture for spin liquids is based on the concept of resonating valence bonds (RVB), which says all spins form spin-singlet pairs [9]–[11]. Spin-singlet pairs consist of either nearest-neighbouring spins or long-range separated spins. Quantum coherence of spin-singlets determines the quantum properties of the state. It is realized that a short-range RVB state usually exhibits a finite gap for spin excitations [12], while a long-range RVB state possesses antiferromagnetic long-range order (LRO) [13, 14]. Nowadays many systems are found to exhibit the behaviours of spin liquids. One route to explore spin liquids is to focus on geometrically frustrated spin systems or systems with breaking translational invariance. The former include the Kagome lattice model and the $J_1 - J_2$ model. The latter include the spin Peierls systems and the plaquette RVB systems. In these systems the quantum and/or geometric frustration is the key point to realize spin liquids. Another route is to deal with a series of low-dimensional spin systems with Haldane gap or the spin systems with orbital degeneracy which enhances quantum frustration. Such systems neither break the translational invariance nor exhibit geometrical frustration.

In this paper we propose a class of quantum frustrated SU($N$) models for quantum magnets, and develop a mean field theory for the spin liquid involved in it. The models are constructed by the SU($N$) generators, but has no SU($N$) symmetry because of the competition between the generators in two different representations. For small $N$ it is shown that these models are equivalent to several physical models which have been extensively studied, and fall into the same mathematical structures in the particle representation. To explore the spin liquid, a mean field theory based on the RVB state is developed. In two dimension (2D) it is found that the ground states for $N = 3$ and 4 are gapless spin liquids and the collective excitations are Dirac particles. As examples we apply the theory to spin 1 bilinear-biquadratic system and spin 1/2 systems with double orbital degeneracy.

2. Models

Let us start from a class of Hamiltonians

$$H = J_1 \sum_{\langle ij \rangle, \mu \nu} J^\mu_i(r_i) J^\nu_j(r_j) - J_2 \sum_{\langle ij \rangle, \mu \nu} J^\mu_i(r_i) J^\mu_j(r_j),$$

(1)

which is defined on a lattice. $J^\mu_i(r_i)$ are the $N^2 - 1$ generators of SU($N$) group, and satisfy the algebra

$$[J^\nu_i(r_i), J^\mu_j(r_j)] = \delta_{ij}(\delta^\nu_\beta J^\mu_\beta(r_i) - \delta^\mu_\beta J^\nu_\beta(r_i)).$$

(2)

We choose $J^\mu_i(r_i)$ as the fundamental representation with a single box in a Young tableau. $J_1$ and $J_2$ are two coupling constants.

The first term in (1) possesses the SU($N$) symmetry because the operator, $\sum_{\mu \nu} J^\mu_i(r_i) J^\nu_j(r_j) \equiv P_{ij}$, serves as the permutation operator, which swaps two quantum states at sites $i$ and $j$. This term in various forms has been studied extensively. The case for $N = 2$ is the well-known Heisenberg model, which was solved exactly and exhibits a gapless ground state.
in one dimension (1D) [15]. While in higher dimensions it is well established that the ground state possesses antiferromagnetic LRO on hypercubic lattices. The case for \( N = 3 \) was known as the exchange model for ferromagnetism [16]–[18]. For \( N = 4 \) it was proposed that a spin liquid exists on the triangular lattice [19]–[21], but a Schwinger boson theory gives a state with LRO [22].

The second term in (1) also has the SU(\( N \)) symmetry on a bipartite lattice, which means that \( J_2 \) connects two lattice sites belonging to two different sublattices [23]–[28]. In this case the generators at two sublattice sites are expressed in conjugate representations since \( J_\mu^\nu(r_j) = [J_\nu^\mu(r_j)]^\dagger \), and thus the SU(\( N \)) symmetry survives. In 1D the ground state is gaped for \( N = 4 \). In 2D, the ground states are found to exhibit the Néel-type LRO with broken SU(\( N \)) symmetry for \( N \leqslant 4 \), and possibly to be a spin liquid for \( N > 4 \) [25]–[28].

The mixture of the two terms makes the whole Hamiltonian in (1) deviate from the two SU(\( N \)) symmetries. We propose that this symmetric frustration may also lead to a spin liquid.

To study the phase diagram of the model in (1) we introduce a set of creation and annihilation operators to rewrite the Hamiltonian in the second quantization representation. In the SU(\( N \)) representation each site has \( N \) quantum states |\( \mu \rangle \) so that we may introduce \( N \) pairs of operators \( b^\dagger_i \mu \) and \( b_i \mu \): |\( i, \mu \rangle = b^\dagger_i \mu |0angle \) with the vacuum state |0\rangle at site \( i \). In this way we can construct the operator, \( J_\mu^\nu (r_i) \equiv b^\dagger_i \nu b_i \mu \), with a constraint for single occupancy, \( \sum_\mu b^\dagger_i \mu b_i \mu = 1 \), on each site. This is the so-called hard-core condition even if the particles are bosons. Interestingly, it is found that the generators satisfy the SU(\( N \)) algebra in (2) for both boson or fermion representations.

So the Hamiltonian is reduced to

\[
H = J_1 \sum_{ij} P_{ij} - J_2 \sum_{ij} B^\dagger_{ij} B_{ij} + \sum_i \lambda_i \left( \sum_\mu b^\dagger_i \mu b_i \mu - 1 \right),
\]

(3)

where the bond pairing operator \( B_{ij} = \sum_\mu b_{j\mu} b_{i\mu} \) and the Lagrangian multipliers \( \lambda_i \) are introduced to realize the constraint of single occupancy. The permutation operator can be expressed as \( P_{ij} = \sum_\mu b^\dagger_{ij} b_{i\mu} b_{j\mu} = -\varsigma + \varsigma F^\dagger_{ij} F_{ij} \) with \( F_{ij} = \sum_\mu b^\dagger_{ij} b_{i\mu} \), and \( \varsigma = 1 \) for bosons and \(-1 \) for fermions. Several physical systems are shown to belong to this Hamiltonian. It has been known for a long time that the spin SU(2) operators with \( S^\alpha \) can be used to express the SU(\( N \)) generators. Schrödinger developed an expression for the permutation operator in terms of \((S_i \cdot S_j)^m\) for \( m = 0, 1, \ldots, 2S \) [29] (see appendix A). We present several examples which have the equivalent form in either the fermion or boson representation.

For \( N = 2 \), the model in (1) is equivalent to the spin 1/2 XXZ model,

\[
H = \sum_{ij} [J_{xx} S^x_i S^x_j + J_{xx} S^y_i S^y_j + J_{zz} S^z_i S^z_j].
\]

(4)

By defining \( b^\dagger_1 |0\rangle \) and \( b^\dagger_2 |0\rangle \) as the two eigenstates of the operator \( S^z_i \), the spin operators \( S^\alpha_i \) can be expressed in terms of \( b \) operators,

\[
S^x_i = \frac{1}{2} (b^\dagger_1 b_{i1} - b^\dagger_2 b_{i2}),
\]

(5a)

\[
S^y_i = \frac{1}{2} (b^\dagger_2 b_{i1} + b^\dagger_1 b_{i2}),
\]

(5b)

\[
S^z_i = \frac{1}{2} (b^\dagger_1 b_{i1} - b^\dagger_2 b_{i2}).
\]

(5c)
In this way, one can show that the model has the form of (3) for \( N = 2 \) with \( J_1 = (J_{zz} + J_{xx})/4 \) and \( J_2 = (J_{zz} - J_{xx})/4 \). A model with a similar form was studied by Leone and Zimanyi [30]. When \( J_{zz} = J_{xx} \), it returns to the well-known Heisenberg model.

For \( N = 3 \) it is the spin 1 bilinear–biquadratic model,

\[
H = \sum_{ij} [\cos \phi S_i \cdot S_j + \sin \phi (S_i \cdot S_j)^2],
\]

which is one of the prototype models exhibiting a Haldane gap in the 1D antiferromagnet [31, 32]. The phase diagram for 1D is well established that the energy gap persists in the wide range, \( -\pi/4 < \phi < \pi/4 \) [33]–[35]. The phase diagram for 2D and 3D was studied by means of quantum Monte Carlo [36]. A non-zero quadrupole moment at zero temperature is found in the region, \( -\pi < \phi < 0 \). For spin 1, each site has three states \( |m_i\rangle \) with \( m_i = -1, 0, +1 \) (\( i = 1, 2, 3 \)) according to the eigenvalues of \( S_z^i \). We reorganize the three states and define three operators,

\[
\begin{align*}
b_{1}^{\dagger}|0\rangle & = \frac{i}{\sqrt{2}}(|m_1\rangle + |m_3\rangle), \\
b_{2}^{\dagger}|0\rangle & = |m_2\rangle, \\
b_{3}^{\dagger}|0\rangle & = \frac{1}{\sqrt{2}}(|m_1\rangle - |m_3\rangle).
\end{align*}
\]

In terms of \( b \) operators, the three spin operators can be written as

\[
\begin{align*}
S_{x}^i & = i(b_{2}^{\dagger}b_{1} - b_{1}^{\dagger}b_{2}), \\
S_{y}^i & = i(b_{3}^{\dagger}b_{2} - b_{2}^{\dagger}b_{3}), \\
S_{z}^i & = i(b_{1}^{\dagger}b_{3} - b_{3}^{\dagger}b_{1}).
\end{align*}
\]

On this basis the Hamiltonian (6) is reduced to the form of (3) for \( N = 3 \) with \( J_1 = \cos \phi \) and \( J_2 = \sqrt{2} \cos (\phi + \pi/4) \) [8]. At the points, \( \phi = -3\pi/4, \pi/4, \pm\pi/2 \) the model possesses the SU(3) symmetry [15]. It is solvable at \( \tan \phi = 1/3 \) and the ground state is a valence bond solid [37].

For \( N = 4 \), we can present a model in terms of operators \( S = 3/2 \) just like the cases for \( N = 2 \) and 3. Alternatively, we present the spin–orbital model [38],

\[
H = J \sum_{ij} (S_i \cdot S_j + T_i \cdot T_j) + 4V \sum_{ij} (S_i \cdot S_j)(T_i \cdot T_j),
\]

with spin \( S = 1/2 \) and orbital \( T = 1/2 \), where both of the operators \( S \) and \( T \) satisfy the SU(2) algebra. This model was used extensively to describe the orbital physics in transition metal oxides [39]. At \( J = V \) the model has the SU(4) symmetry. There are four simultaneous eigenstates for spin \( S^z \) and \( T^z \) at each site, \( |S_i^z = \pm \frac{1}{2}, T_i^z = \pm \frac{1}{2}\rangle \). So we introduce four creation operators.
as follows:

\[ b_1^\dagger |0\rangle = \frac{1}{\sqrt{2}} \left( |\frac{1}{2}, \frac{1}{2} \rangle - |\frac{1}{2}, -\frac{1}{2} \rangle \right) \];

\[ b_2^\dagger |0\rangle = \frac{i}{\sqrt{2}} \left( |\frac{1}{2}, -\frac{1}{2} \rangle + |\frac{1}{2}, \frac{1}{2} \rangle \right) \];

\[ b_3^\dagger |0\rangle = \frac{i}{\sqrt{2}} \left( |\frac{1}{2}, \frac{1}{2} \rangle - |\frac{1}{2}, -\frac{1}{2} \rangle \right) \];

\[ b_4^\dagger |0\rangle = \frac{1}{\sqrt{2}} \left( |\frac{1}{2}, \frac{1}{2} \rangle + |\frac{1}{2}, -\frac{1}{2} \rangle \right) \].

By introducing a four-component spinor \( \Psi_1^\dagger = (b_1^\dagger, b_2^\dagger, b_3^\dagger, b_4^\dagger) \), we can express the spin and orbital operators in terms of \( \Psi_1^\dagger \) and \( \Psi \),

\[ S^x = \frac{1}{2} \Psi^\dagger (\sigma_x \otimes \sigma_0) \Psi \],

\[ S^y = \frac{1}{2} \Psi^\dagger (\sigma_y \otimes \sigma_0) \Psi \],

\[ S^z = \frac{1}{2} \Psi^\dagger (-\sigma_z \otimes \sigma_0) \Psi \],

\[ T^x = \frac{1}{2} \Psi^\dagger (-\sigma_y \otimes \sigma_z) \Psi \],

\[ T^y = \frac{1}{2} \Psi^\dagger (-\sigma_x \otimes \sigma_y) \Psi \],

\[ T^z = \frac{1}{2} \Psi^\dagger (\sigma_0 \otimes \sigma_x) \Psi \].

Once again one can show that the spin–orbital model falls into the general form of Hamiltonian as in (3) with \( J_1 = (J + V)/2 \) and \( J_2 = (J - V)/2 \).

So far we have shown that the three physical models can be expressed in a general form in terms of SU(\( N \)) generators as in (1) or in the particle representation as in (2). For larger \( N \), the models are related to some high-spin systems with orbital degeneracy, for example, the spin-\( L \) system with double orbital degeneracy [40].

### 3. Mean field theory

Now we turn to the phase diagram of the model on a \( d \)-dimensional hypercubic lattice. We may choose the operators, \( b \)'s, as either bosons, \( [b_{i\mu}, b^\dagger_{j\nu}] = \delta_{ij}\delta_{\mu\nu} \), or fermions, \( \{b_{i\mu}, b^\dagger_{j\nu}\} = \delta_{ij}\delta_{\mu\nu} \). In principle, the bosons tend to condense to the lowest energy state at low temperatures and form a quantum-ordered state, while the fermions tend to form a Fermi sea and quantum-disordered states. In practice, it is inevitable to introduce adequate approximation schemes to deal with a many-body system in either the Bose or Fermi representation. Since our purpose is to search the regime of the spin liquid phase, we choose the Fermi representation in this paper. Without loss of generality we can assume that the system is defined on a \( d \)-dimensional simple cubic lattice, and take \( J_1, J_2 > 0 \). In this way, the Hamiltonian is semi-negative in the Fermi representation. Now the second term in (1) plays the role of an attractive interaction between pairing valence bonds

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The Hubbard–Stratonovich transformation is performed to decouple the Hamiltonian into a bilinear form. Two types of mean fields are introduced:

\[
\Delta_1(k) = J_1 \sum_\delta \langle F_{i,i+\delta} \rangle e^{ik\cdot \delta} \equiv 2J_1 F \sum_\alpha \cos k_\alpha, \\
\Delta_2(k) = J_2 \sum_\delta \langle B_{i,i+\delta} \rangle e^{ik\cdot \delta} \equiv 2J_2 B \sum_\alpha \sin k_\alpha,
\]

(12a)

(12b)

where \( F = \langle F_{i,i+\delta} \rangle = \langle F_{i,i-\delta} \rangle \) and \( B = i \langle B_{i,i+\delta} \rangle = -i \langle B_{i,i-\delta} \rangle \), and \( \alpha = x, y, \ldots \). The bracket \( \langle \cdots \rangle \) represents the thermodynamic average. The chemical potential \( \lambda_i \) is taken to be site-independent, \( \lambda_i = \lambda \), which can also be regarded as a mean field. In the momentum space, the mean field Hamiltonian is

\[
H = \sum_{k,\mu} \epsilon(k) b^\dagger_{k\mu} b_{k\mu} - \frac{1}{2} \sum_k \Delta_2(k) (b_{k\mu} b_{-k\mu} + b^\dagger_{-k\mu} b^\dagger_{k\mu}) \\
- \lambda N_A + dN_A J_1 F^2 + dN_A J_2 B^2,
\]

(13)

where \( \epsilon(k) = \lambda - \Delta_1(k) \), \( N_A \) is the total number of lattice sites. This is a Bardeen–Cooper–Schrieffer (BCS) type of Hamiltonian for \( N \)-component degenerate fermions. By performing the Bogoliubov transformation,

\[
\gamma_{k\mu} = u_k b_{k\mu} - v_k b^\dagger_{-k\mu}; \quad \gamma^\dagger_{-k\mu} = u_k b^\dagger_{-k\mu} + v_k b_{k\mu}
\]

(14)

with the coherence factors satisfying

\[
u_k^2 = \frac{1}{2} \left[ 1 + \frac{\epsilon(k)}{\omega(k)} \right], \\
u_k^2 = \frac{1}{2} \left[ 1 - \frac{\epsilon(k)}{\omega(k)} \right], \\
2u_k v_k = \frac{\Delta_2(k)}{\omega(k)}.
\]

(15a)

(15b)

(15c)

one can diagonalize the Hamiltonian as

\[
H = \sum_{k,\mu} \omega(k) \gamma^\dagger_{k\mu} \gamma_{k\mu} + E_0,
\]

(16)

where the spectrum and the ground energy are

\[
\omega(k) = \sqrt{\epsilon(k)^2 + \Delta_2^2(k)},
\]

(17)

\[
E_0 = -\frac{N}{2} \sum_k \omega(k) + \frac{N - 2}{2} \lambda N_A + dN_A J_1 F^2 + dN_A J_2 B^2.
\]

(18)

The spectra of \( \omega(k) \) are \( N \)-fold degenerate for the quasiparticles. From the free energy for the fermion gas

\[
\Omega = -\frac{N}{\beta} \sum_k \ln(1 + e^{-\beta \omega(k)}) + E_0,
\]

(19)
we obtain a set of the mean field equations by optimizing the free energy with respect to the mean fields \( F, B, \) and \( \lambda \),

\[
\int \frac{d k}{(2\pi)^{d}} \frac{e(k)}{\omega(k)} \tanh \frac{\beta \omega(k)}{2} = \frac{N - 2}{N}, \tag{20a}
\]

\[
\int \frac{d k}{(2\pi)^{d}} \frac{e(k)(-\sum_{a} \cos k_{a})}{\omega(k)} \tanh \frac{\beta \omega(k)}{2} = \frac{2dF}{N}, \tag{20b}
\]

\[
\int \frac{d k}{(2\pi)^{d}} \frac{\Delta_{2}(k)(\sum_{a} \sin k_{a})}{\omega(k)} \tanh \frac{\beta \omega(k)}{2} = \frac{2dB}{N}. \tag{20c}
\]

Thus the mean field Hamiltonian is solved together with the self-consistent equations for the three types of mean fields.

We have \( N \) degenerate spectra for quasi-fermions. These are not the collective modes, and cannot be measured explicitly. In order to calculate spin susceptibility we define the Matsubara Green’s function in the form of a \( 2 \times 2 \) matrix,

\[
G_{\mu}(k, \tau) = -\left( T_{\tau} \begin{pmatrix} b_{k\mu}(\tau)b_{k\mu}^{\dagger}(0) & b_{k\mu}(\tau)b_{-k\mu}(0) \\ b_{-k\mu}(\tau)b_{k\mu}^{\dagger}(0) & b_{-k\mu}(\tau)b_{-k\mu}(0) \end{pmatrix} \right), \tag{21}
\]

where the bracket \( \langle \cdots \rangle \) means that the thermodynamic average is made, and the factor \( T_{\tau} \) is the imaginary time order operator. The Green’s function of the frequency is obtained as follows:

\[
G_{\mu}(k, i\omega_{n}) = \frac{i\omega_{n}\sigma_{0} + e(k)\sigma_{z} + \Delta_{2}(k)\sigma_{x}}{(i\omega_{n})^{2} - \omega^{2}(k)}, \tag{22}
\]

where \( \omega_{n} = (2n + 1)\pi/\beta \) for all integer \( n \). Due to the degeneracy of the spectra, the Green’s functions are independent of the index \( \mu \). The imaginary time dynamic correlation function for \( J_{\nu}^{\mu} \) is defined as

\[
\chi_{\mu\nu}(q, i\omega_{n}) = \int_{0}^{\beta} d\tau e^{i\omega_{n}\tau} (T_{\tau} J_{\nu}^{\mu}(q, \tau) J_{\nu}^{\mu}(-q, 0)). \tag{23}
\]

The imaginary part of \( \chi_{\mu\nu}(q, \omega) \) is worked out as

\[
\text{Im} \chi_{\mu\nu}(q, \omega) = \pi \int \frac{d k}{(2\pi)^{d}} \delta(\omega - \omega(k) - \omega(k + q))(u_{k}^{2}v_{k+q}^{2} + u_{k}v_{k}u_{k+q}v_{k+q}\delta_{\mu\nu}) \tag{24}
\]

for \( \omega > 0 \) at \( T = 0 \). It is shown that \( \text{Im} \chi_{\mu\nu}(q, \omega) \) becomes non-zero only when \( \omega > \Delta_{\text{gap}} = 2 \min \omega(k) \). Thus if the spectra for quasiparticles have an energy gap the collective excitation for the dynamic correlation function will also have a finite energy gap. The energy gap \( \Delta_{\text{gap}} \) can be evaluated by solving the mean field equations.

4. Two-dimensional gapless spin liquids

It is easy to see that when \( J_{2} = 0 \) or \( B = 0 \) the spectra for fermions are reduced to an ideal fermion gas. In this case, the system is a type of gapless spin liquid as discussed for SU(2) and SU(4)
systems. When $J_2$ increases a non-zero solution for $B$ and $F$ is obtained. We take $J_1 = J \cos \theta$ and $J_2 = J \sin \theta$ with $0 < \theta < \pi/2$. For finite $N$, the mean field equations are solved numerically at $T = 0$. In the one-dimensional case, a finite gap is found in a large regime of the parameter space. In this paper we focus on 2D systems. Numerical solution for non-zero $B$ and $F$ has lower energy than other mean field solutions with $F = 0$ or $B = 0$ when $\theta < 1.075$ for $N = 3$ and $< 0.867$ for $N = 4$. Opposite to the 1D cases non-zero $B$ and $F$ do not produce a gaped phase. Instead they lead to a gapless spin liquid with $\Delta_{gap} = 0$. Notice that along the line $k_x^0 = -k_x^0$, we have $\Delta_2(k) = 0$. And on this line if $\epsilon(k) = 0$ the spectrum becomes $\omega(k) = 0$. So we get the solution,

$$k_x^0 = -k_y^0 = \pm \arccos \frac{\lambda}{4J_1F}. \quad (25)$$

Numerical values of the parameters show that $\lambda/4J_1F \lesssim 1$ always holds for the mean field solution. $|k_x^0|$ is plotted in figure 1. In other words, the Fermi surface actually consists of two points located at $k_x^0 = -k_x^0 = \pm \arccos \lambda/4J_1F$. We expand the dispersion relation near the Fermi point $(k_x^0, k_y^0)$ with

$$\Delta k_x = k_x - k_x^0 = \Delta k \cos \varphi, \quad (26a)$$

$$\Delta k_y = k_y - k_y^0 = \Delta k \sin \varphi, \quad (26b)$$

the spectrum becomes linear with respect to $\Delta k$,

$$\omega(k^0 + \Delta k) \simeq v^0 \Delta k, \quad (27)$$

Figure 1. Above: the value of $k^0_x$ at two points for the fermi surface $(k^0_x, -k^0_x)$ and $(-k^0_x, k^0_y)$ for $N = 3, 4$ in square lattice. Bottom: the velocities of the Dirac quasiparticles along the different directions. For each $N$ there exists a fixed point at which the velocity becomes isotropic. The angle $\theta = \arctan(J_2/J_1)$. 

When $J_2$ increases a non-zero solution for $B$ and $F$ is obtained. We take $J_1 = J \cos \theta$ and $J_2 = J \sin \theta$ with $0 < \theta < \pi/2$. For finite $N$, the mean field equations are solved numerically at $T = 0$. In the one-dimensional case, a finite gap is found in a large regime of the parameter space. In this paper we focus on 2D systems. Numerical solution for non-zero $B$ and $F$ has lower energy than other mean field solutions with $F = 0$ or $B = 0$ when $\theta < 1.075$ for $N = 3$ and $< 0.867$ for $N = 4$. Opposite to the 1D cases non-zero $B$ and $F$ do not produce a gaped phase. Instead they lead to a gapless spin liquid with $\Delta_{gap} = 0$. Notice that along the line $k_x^0 = -k_x^0$, we have $\Delta_2(k) = 0$. And on this line if $\epsilon(k) = 0$ the spectrum becomes $\omega(k) = 0$. So we get the solution,

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the spectrum becomes linear with respect to $\Delta k$,

$$\omega(k^0 + \Delta k) \simeq v^0 \Delta k, \quad (27)$$

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where the anisotropic velocity is

\[ v^0 = \lambda \sqrt{2} \left[ c_1^2 + (c_2^2 - c_1^2) \sin^2(\varphi + \pi/4) \right] \]

with \( c_1^2 = (2J_1 F/\lambda)^2 - 1/4 \) and \( c_2^2 = B^2 \tan^2 \theta/F^2 \). The dispersion relation is linear and the quasiparticles are a type of Dirac particles. Its velocity is in general anisotropic and has its minimum for \( \varphi = \pi/4 \) and its maximum for \(-\pi/4\) for a specific \( \theta \). At \( \theta = 0.951 \) for \( N = 3 \) and 0.786 for \( N = 4 \) the velocity becomes isotropic. Beyond the regime in figure 1 we do not anticipate the present mean field theory to be still valid because it is already known that the long-range order exists in the ground state in 2D when \( J_1 = 0 \) [26, 28].

We have also calculated the static correlation function \( \chi_{\mu\mu} \) and \( \chi_{\mu\nu} (\mu \neq \nu) \) for \( N = 3 \) and 4. We found that the results for the two systems are very similar. So we only present the results for \( N = 4 \) here. In figure 2, we plot the diagonal susceptibility \( \chi_{\mu\mu}(q_x, q_y) \) for different couplings. All values of \( \chi \) are convergent, which indicates the absence of long-range order as expected in a spin liquid state. We observe that \( \chi \) has its maximal value at points \((\pm \pi, \pm \pi)\) for small \( \theta \) (or \( J_1 \)), which shows that the static antiferromagnetic correlation dominates. The maximal values decrease with increasing \( \theta \) (from figure 2(a) to 2(b)), and \( \chi_{\mu\mu} \) becomes almost flat for large \( \theta \). In figure 3, we plot the off-diagonal \( \chi_{\mu\nu} (\mu \neq \nu) \). We found that the maximal value of \( \chi \) changes from the points \((\pm \pi, \pm \pi)\) to \((0,0)\) with \( \theta \) increasing. Thus the antiferromagnetic correlation decreases very quickly, and the ferromagnetic correlation dominates at large \( \theta \). For an intermediate \( \theta \) either the diagonal or the off-diagonal \( \chi \) has a relatively flat structure, which implies that the state is a very well-defined spin liquid.
Though the mean field theory has lots of disadvantages, it is still used extensively to study the many body system with strong correlation after some adequate transformations. So far, there has been no definite conclusion on whether the 2D SU(\(N\)) (\(N > 2\)) antiferromagnetic model with \(J_1 > 0\) and \(J_2 = 0\) possesses an LRO ground state [19]–[22]. In the present mean field theory, it is assumed that the ground state is a spin liquid state. Quantum Monte Carlo simulation may provide helpful information. The phase diagram of the 2D spin 1 bilinear-biquadratic model in (6) was studied numerically for \(-\pi < \phi < 0\) [36]. It is found that the ground state is antiferromagnetic for \(-\pi/2 < \phi < 0\), which corresponds to the regime \(\pi/4 < \theta < \pi/2\) for \(N = 3\) in figure 1. As there is lack of data in the regime \(0 < \theta < \pi/4\) (\(\approx 0.785\)) \(0 < \phi < \pi/4\) the numerical calculation does not exclude the possible existence of a spin liquid in the case of \(J_1, J_2 > 0\) although the antiferromagnetic fluctuations may be very strong.

5. Conclusion

In conclusion, we have proposed a class of frustrated SU(\(N\)) models for quantum magnets based on the generators of SU(\(N\)) group, and showed that several physical models have the same mathematical structure in the fermion or boson representation. The model we constructed breaks SU(\(N\)) symmetry when \(J_1\) and \(J_2\) are non-zero simultaneously, although the two parts in the model do have different SU(\(N\)) symmetries. Usually, geometric quantum frustration on lattices or coupling structures is one of the main driving forces to generate a spin liquid. The mixture of two parts with different SU(\(N\)) symmetries provides us a possibly new way to seek for spin liquid in higher dimensions. In this paper, a mean field theory with paired valence bonds for quasi-particles is developed to explore this kind of spin liquid. A gapless spin liquid exists in a
large-extent regime in 2D. It is believed that the mechanism of the spin liquid is the competition of two interactions with different SU($N$) symmetries. Thus the symmetric frustration is a new possible route to explore the spin liquid in higher dimensions.

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Appendix A. Spin exchange operator

The role of the permutation operator is to swap the two quantum states at two different sites,

$$P_{ij}|i, \alpha; j, \beta\rangle = |i, \beta; j, \alpha\rangle.$$

$P_{ij}^2 = 1$ and its two eigenvalues are ±1. The connection between the permutation operator and spin operator was first established for spin 1/2 by Dirac [41], and then generalized to the case of arbitrary spin $S$ by Schrödinger [29], who showed that

$$P_{ij} = \sum_{n=0}^{2S} A_n (S_i \cdot S_j)^n,$$

where the coefficients $A_n$ typically have the values, for $S = 1/2$, $A_0 = 1/2$, $A_1 = 2$; for $S = 1$, $A_0 = -1$, $A_1 = 1$, $A_2 = 1$; for $S = 3/2$, $A_0 = -67/32$, $A_1 = -9/8$, $A_2 = 11/18$, $A_3 = 2/9$. The general expression for $P_{ij}$ is

$$P_{ij} = [(2S)!]^{-2}[M - (2S - 1)2S][M - (2S - 2)(2S - 1)] \cdots [M - 2]M - [(2S - 1)!]^{-2}$$

$$\{[M - (2S - 2)(2S - 1)] \cdots [M - 2]M + \cdots + (-1)^{2S-1}M + (-1)^{2S}$$

$$= \sum_{n=0}^{2S} (-1)^n[(2S - n)!]^{-2}[M - (2S - 1 - n)(2S - n)] \cdots [M - 2]M,$$

where

$$M = (S_i + S_j)^2.$$

Appendix B. Projection operators for two spins

Using $S_{ij} \equiv S_i \cdot S_j = \frac{1}{2}(S_i + S_j)^2 - S(S + 1)$ and the completeness relation for the projection operators $\sum_{j=0}^{2S} P_{ij}^S(i, j) = 1$, one finds

$$(S_{ij})^m = \sum_{j=0}^{2S} [\frac{1}{2}J(J + 1) - S(S + 1)]^m P_{ij}^S(i, j),$$
where $P^{S}_{j}(i, j)$ is defined as the projection operator for total spin $J$ of two spins $S$. For spin 1/2,

$$P^{S=1/2}_{j=0}(i, j) = \frac{1}{4} - S_{ij}, \quad P^{S=1/2}_{j=1}(i, j) = \frac{3}{4} + S_{ij}.$$ 

For spin 1,

$$P^{S=1}_{j=0}(i, j) = -\frac{1}{4} + \frac{1}{3} S_{ij}; \quad P^{S=1}_{j=1}(i, j) = 1 - \frac{1}{2} S_{ij} - \frac{1}{2} S_{ij}^2; \quad P^{S=1}_{j=2}(i, j) = \frac{1}{3} + \frac{1}{2} S_{ij} + \frac{1}{6} S_{ij}^2.$$ 

For spin 3/2,

$$P^{S=3/2}_{j=0}(i, j) = \frac{33}{128} + \frac{31}{96} S_{ij} - \frac{5}{72} S_{ij}^2 - \frac{1}{18} S_{ij}^3; \quad P^{S=3/2}_{j=1}(i, j) = -\frac{81}{128} - \frac{117}{160} S_{ij} + \frac{9}{40} S_{ij}^2 + \frac{1}{10} S_{ij}^3; \quad P^{S=3/2}_{j=2}(i, j) = \frac{165}{128} + \frac{23}{96} S_{ij} - \frac{17}{72} S_{ij}^2 - \frac{1}{18} S_{ij}^3; \quad P^{S=3/2}_{j=3}(i, j) = \frac{11}{128} + \frac{27}{160} S_{ij} + \frac{29}{360} S_{ij}^2 + \frac{1}{90} S_{ij}^3.$$ 

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