Remarks on a recent paper titled: “On the split common fixed point problem for strict pseudocontractive and asymptotically nonexpansive mappings in Banach spaces”

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Abstract
In a recently published theorem on the split common fixed point problem for strict pseudocontractive and asymptotically nonexpansive mappings, Tang et al. (J. Inequal. Appl. 2015:305, 2015) studied a uniformly convex and 2-uniformly smooth real Banach space with the Opial property and best smoothness constant $\kappa$ satisfying the condition $0 < \kappa < \frac{1}{\sqrt{2}}$, as a real Banach space more general than Hilbert spaces. A well-known example of a uniformly convex and 2-uniformly smooth real Banach space with the Opial property is $E = l^p$, $2 \leq p < \infty$. It is shown in this paper that, if $\kappa$ is the best smoothness constant of $E$ and satisfies the condition $0 < \kappa \leq \frac{1}{\sqrt{2}}$, then $E$ is necessarily $l^2$, a real Hilbert space. Furthermore, some important remarks concerning the proof of this theorem are presented.

MSC: 47H09; 47H05; 47J25; 47J05

Keywords: Fixed point; Accretive; Uniformly smooth

1 Introduction
Let $H_1$ and $H_2$ be two real Hilbert spaces, $C$ and $Q$ be nonempty closed and convex subsets of $H_1$ and $H_2$, respectively. Let $A : H_1 \to H_2$ be a bounded linear mapping. The split feasibility problem (SFP) is the following:

\[
\text{find } x^* \in C \text{ such that } Ax^* \in Q.
\]

Let $T : C \to C$ and $S : Q \to Q$ be two mappings with $F(T) := \{x \in C : Tx = x\} \neq \emptyset$, $F(S) := \{x \in Q : Sx = x\} \neq \emptyset$. Then the split common fixed point problem (SCFPP) for mappings $T$ and $S$ is to find a point

\[
q^* \in F(T) \text{ such that } Aq^* \in F(S).
\]
We shall denote the set of solutions of the SCFPP for mappings $T$ and $S$ by $\Omega$, that is,

$$\Omega := \{x^* \in F(T) \text{ such that } Ax^* \in F(S)\}.$$ 

The SFP was first introduced by Censor and Elfving [3] in finite dimensional real Hilbert spaces for modeling inverse problems which arise from phase retrievals and medical image reconstruction [2]. The split common fixed point problem in Hilbert spaces was introduced by Moudafi [9] in 2010. It is now well known that the SFP and SCFPP have applications in very important real life problems. Consequently, these problems have attracted the interest of many researchers.

These problems and their generalizations have been studied in real Hilbert spaces and iterative methods for approximating their solutions, assuming existence, in this setting abound in the literature (see, for example, Tang et al. [12], and the references therein).

Studies of SFP and SCFPP in real Banach spaces more general than Hilbert spaces are very scanty in the literature. Part of the reason for this may be that most of the tools used in real Hilbert spaces for studying them are confined to real Hilbert spaces. Consequently, new tools have to be developed for solving these problems in real Banach spaces more general than Hilbert spaces. Some of the earliest successes in this direction are the results of Takahashi [10] and Takahashi and Yao [11]. In 2015, Tang et al. [12] studied the SFP and SCFPP in real Banach spaces more general than Hilbert spaces and by using hybrid methods and Halpern-type methods, they proved strong and weak convergence of the sequence generated by their algorithm to solutions of these problems. For other results obtained in studying the SFP, the SCFPP and some of their generalizations, in real Banach spaces more general than Hilbert spaces, the reader may see any of the following recent papers [5, 6], and the references therein.

In 2014, Cui and Wang [7] studied the SCFPP of $\tau$-quasi-strictly pseudocontractive mappings in the setting of Hilbert space. Motivated and inspired by this study and the research going on in the domain of split feasibility problems and SCFPP, Tang et al. [12] studied the SCFPP for a $\tau$-quasi-strictly pseudocontractive mapping and an asymptotically nonexpansive mapping in the setting of two real Banach space which they assumed are more general than Hilbert spaces. Their setting is the following:

1. $E_1$ is a uniformly convex and 2-uniformly smooth real Banach space which has the Opial property and the best smoothness constant $\kappa$ satisfying $0 < \kappa < \frac{1}{\sqrt{2}}$.
2. $E_2$ is a real Banach space.
3. $A : E_1 \to E_2$ is a bounded linear mapping and $A^*$ is the adjoint of $A$.
4. $S : E_1 \to E_1$ is an $l_n$-asymptotically nonexpansive mapping with $\{l_n\} \subset [1, \infty)$, and $l_n \to 1$, as $n \to \infty$. $T : E_2 \to E_2$ is a $\tau$-quasi-strict pseudocontractive mapping with $F(S) \neq \emptyset$ and $F(T) \neq \emptyset$, and $T$ is demiclosed at zero.

They proved the following theorem.

**Theorem 1.1** (Theorem 3.1 of [12]) Let $E_1, E_2, A, S, T$ and $\{l_n\}$ be the same as above. For each $x_1 \in E_1$, let $\{x_n\}$ be the sequence generated by

$$x_n = x_n + \gamma J^{-1}_1 A^* J_2 (T - I) Ax_n,$$

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n S_n x_n, \quad \forall n \geq 1,$$
where \(\{\alpha_n\}\) is a sequence in \((0, 1)\) with \(\liminf_{n \to \infty} \alpha_n(1 - \alpha_n) > 0\), \(\gamma\) is a positive constant satisfying

\[
0 < \gamma < \min\left\{ 1 - 2\kappa^2, 1 - \tau \right\}^{1/2},
\]

\(\{l_n\}\) is a sequence in \([1, \infty)\) with \(L := \sup_{n \geq 1} l_n\) and \(\sum_{n=1}^{\infty} (l_n - 1) < \infty\).

(I) If \(\Gamma := \{p \in F(S) : Ap \in F(T)\} \neq \emptyset\) (the set of solutions of the SCFPP is nonempty), then the sequence \(\{x_n\}\) converges weakly to a point \(x^* \in \Gamma\).

(II) In addition, if \(\Gamma \neq \emptyset\), and \(S\) is semi-compact, then \(\{x_n\}\) converges strongly to a point \(x^* \in \Gamma\).

We first make the following remarks concerning the proof of this theorem in Tang et al. [12].

**Remark 1** Definition 2.3(ii) of the \(\tau\)-strict pseudocontractive mapping for \(T : C \to C, C \subset E\), where \(E\) is a real Banach space, given by inequality (2.2), is an error. This definition is restricted to real Hilbert spaces. The definition for \(\tau\)-strict pseudocontractive mapping in real Banach spaces, \(E\), more general than Hilbert spaces, in the sense of Browder and Pertishyn, is the following: \(T : C \to C, C \subset E\), is said to be \(\tau\)-strict pseudocontractive if, \(\forall x, y \in C\), there exist \(j(x - y) \in J(x - y)\) and \(\tau \in (0, 1)\) such that

\[
\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \tau \left\| (I - T)x - (I - T)y \right\|^2,
\]

where \(J : E \to 2E^*\) is the normalized duality mapping on \(E\). It is easy to show that in a real Hilbert space, inequality (1.1) is equivalent to inequality (2.2) in Tang et al. [12], and it is well known that in real Banach spaces more general than Hilbert spaces, inequality (2.2) in Tang et al. [12] is not equivalent to inequality (1.1). Consequently, the use of inequality (2.2) in the proof of Theorem 3.1 in Tang et al. [12] invalidates the argument of the proof.

**Remark 2** The use of Lemma 2.1 by Tang et al. [12] in the proof of Theorem 1.1 is an error. As correctly stated in Lemma 2.1 (see Tang et al. [12] for the statement of Lemma 2.1) the inequality of the lemma can be used provided that the vectors \(x\) and \(y\) are bounded. They used this inequality in the proof without establishing the boundedness of the vectors \((z_n - p)\) and \((S^n z_n - p)\). This erroneous application of the lemma led to inequality (3.5) which showed that \(\lim_{n \to \infty} \|x_n - p\|\) exists and consequently that \(\{x_n\}\) is bounded, which plays a crucial role in what followed. This use of Lemma 2.1 also invalidates the argument in the proof of Theorem 3.1.

**Remark 3** We begin by recalling the following definition.

**Definition 1.2** Let \(E\) be a normed space with \(\dim(E) \geq 2\). The modulus of smoothness of \(E\) is the function \(\rho_E : [0, \infty) \to [0, \infty)\) defined by

\[
\rho_E(\tau) := \sup_{1 \leq \|x\| = \|y\| = 1} \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x + \tau y\| + \|x - \tau y\|}{2} - 1 : \|x\| = 1; \|y\| = 1 \right\}.
\]
The normed space $E$ is called uniformly smooth if and only if $\lim_{\tau \to 0^+} \frac{\rho_E(\tau)}{\tau} = 0$. For $\alpha > 1$, a normed space $E$ is said to be $\alpha$-uniformly smooth if there exists a constant $c > 0$ such that

$$\rho_E(\tau) \leq c\tau^\alpha, \quad \tau > 0.$$  

It is well known that $L_p$, $l_p$ and the Sobolev spaces $W^m_p(\Omega)$, $1 < p < \infty$, are all $p$-uniformly smooth and that the following estimates hold:

$$\rho_{L_p}(\tau) = \rho_{l_p}(\tau) = \rho_{W^m_p(\Omega)}(\tau) = \begin{cases} (1 + \tau^p)^\frac{1}{p} - 1 & \text{if } 1 < p < 2; \\ \frac{p-1}{2} \tau^2 + o(\tau^2) & \text{if } p \geq 2; \end{cases}$$  

(1.2)

where $\tau \geq 0$ (see, e.g., Lindenstrauss and Tzafriri, [8]; see also Chidume [4], page 44). From (1.2), it is clear that, if $1 < p < 2$, then $E = L_p$, $l_p$ or $W^m_p(\Omega)$ is not 2-uniformly smooth.

Consider now $E = L_p$, $l_p$ or $W^m_p(\Omega)$ for $p \in [2, \infty)$. From (1.2), these spaces are 2-uniformly smooth. Furthermore, for these spaces, the following inequality holds:

$$\|x + y\|^2 \leq \|x\|^2 + 2\|y, j(x)\| + (p - 1)\|y\|^2, \quad \forall x, y \in E,$$

(1.3)

and $(p - 1)$ is the best smoothness constant (see, e.g., Bynun [1], Xu [13]; see also Chidume [4], page 44). Comparing inequality (1.3) with the following inequality in Tang et al. [12]:

$$\|x + y\|^2 \leq \|x\|^2 + 2\|y, j(x)\| + 2\|\kappa y\|^2, \quad \forall x, y \in E,$$

we find that $2\kappa^2 = p - 1$ so that $\kappa = \sqrt{\frac{p-1}{2}}$. The condition that $0 < \kappa \leq \frac{1}{\sqrt{2}}$ now implies that $p \leq 2$. But if $p < 2$, then, from Eq. (1.2), $E$ is not 2-uniformly smooth. So, the only possibility is that $p = 2$, i.e., $E = l_2$, a real Hilbert space.

2 Conclusion

The space $E_1$ described in the paper of Tang et al. [12] as a real Banach space more general than real Hilbert space is, in fact, necessarily a real Hilbert space. Furthermore, there are serious errors in the proof of Theorem 3.1 in Tang et al. [12]. Consequently, the following important problem which Tang et al. [12] tried to solve is still open.

3 Open problem

It is of interest to define a space $E_1$ that is a real Banach space more general than real Hilbert spaces, $E_2$ as defined in the paper of Tang et al. [12] and an iterative algorithm for solving split common fixed point problem involving a quasi-strict pseudocontractive mapping and an asymptotically nonexpansive mapping such that the sequence generated by the algorithm converges strongly to a solution of the problem.

Acknowledgements

The author acknowledges the African Development Bank (AfDB) and the Pan African Material Institute (PAMI), AUST for their financial support.

Funding

This work is supported from the African Development Bank (AfDB) research grant funds to AUST.

Availability of data and materials

Not applicable.

Competing interests

The author declares that there is no competing interests.
Authors’ contributions
The author read and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 3 August 2020 Accepted: 25 February 2021 Published online: 08 March 2021

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