The evolution of the spectrum of a Frobenius Lie algebra under deformation

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\textbf{ABSTRACT}

The category of Frobenius Lie algebras is stable under deformation, and here we examine explicit infinitesimal deformations of four and six dimensional Frobenius Lie algebras to illustrate that the spectrum of a Frobenius Lie algebra can evolve under deformation.

\textbf{1. Introduction}

A Lie algebra \((\mathfrak{g}, [-, -])\) is Frobenius if there exists a linear functional \(F \in \mathfrak{g}^*\) such that the natural map \(\eta : \mathfrak{g} \to \mathfrak{g}^*\) defined by \(x \mapsto F[x, -]\) is an isomorphism. Such an \(F\) is called a Frobenius functional. The set of Frobenius functionals of a Frobenius Lie algebra \(\mathfrak{g}\) is, in general, quite large; forming an open subset of \(\mathfrak{g}^*\) in the Zariski and Euclidean topologies (see [19] and [9]).

Frobenius Lie algebras were introduced in the 1970s by Ooms who showed, in particular, that the universal enveloping algebra \(U(\mathfrak{g})\) admits a faithful simple representation when \(\mathfrak{g}\) is Frobenius (see [19]). Such algebras also have applications in invariant theory and the geometry of coadjoint orbits in \(\mathfrak{g}^*\) (see [18]). Deformation theorists are interested in Frobenius Lie algebras because each provides a solution to the classical Yang-Baxter equation, which in turn quantizes to a universal deformation formula, i.e., a Drinfel’d twist which deforms any algebra which admits an action of \(\mathfrak{g}\) by derivations (see [15]).

If \(\mathfrak{g}\) is Frobenius and \(F \in \mathfrak{g}^*\) is a Frobenius functional, then the inverse image of \(F\) under the mapping \(\eta\) is called a principal element of \(\mathfrak{g}\) and will be denoted \(\hat{F}\) (see [14]). It is the unique element of \(\mathfrak{g}\) such that

\[ F \circ \text{ad} \hat{F} = F([\hat{F}, -]) = F. \tag{1} \]

In [19], Ooms established that the spectrum of the adjoint of a principal element of a Frobenius Lie algebra is independent of the principal element chosen to compute it (see also [13]). Consequently, we can unambiguously refer to the spectrum of \(\mathfrak{g}\) as the spectrum of the adjoint representation of any principal element \(\hat{F} \in \mathfrak{g}\).
Since the category of Frobenius (index zero) Lie algebras is stable under deformation – the index can only decrease under deformation (see [1]) – we come to the motivating question of this article.

Q: Does the spectrum of a Frobenius Lie algebra evolve under the deformation of the underlying algebra?

To address this question we detail some accessible explicit examples of infinitesimal deformations of Frobenius Lie algebras. For a more exotic deformation setup see [16] where Lie algebras are defined by their Maurer-Cartan equations.

Here, following Czikos’s and Verhoczki’s classification of four and six-dimensional Frobenius Lie algebras ([8], cf. Tables 1 and 2), we investigate the infinitesimal deformation theory of these algebras to find that many of these Frobenius Lie algebras can be deformed. A particularly rich example is provided by a certain one of these four-dimensional algebras, $\Phi'$, for which we provide detailed cohomological and spectral calculations. The calculations are routine but potentially instructive.

The structure of the article is as follows. In Section 2, we recall the well-known infinitesimal deformation theory of Nijenhuis and Richardson (see [11, 12]), wherein the deformation of a Lie algebra is controlled by the graded Chevalley-Eilenberg complex (see [20]). In Section 3, we present the classification of four-dimensional Frobenius Lie algebras by detailing in Table 1 the commutator relations, the dimension of the second and third cohomology groups of the Lie algebra with coefficients in the Lie algebra (the case of interest for deformation theory), the spectrum, and whether or not deformations of the algebra exist. In Section 4, we provide detailed deformation theory calculations associated with $\Phi'$; see Table 2, where we provide the full deformation and spectral values at deformation parameter instance $t$, for all deformations of the four-dimensional Lie algebras here considered. In Section 5, a short Epilogue provides connections between this paper’s examples and topical “spectral” research, along with announcements of new results that will be appearing and will be of interest to researchers in this area. In Appendix A, Table 3 is the analogue of Table 2 for the six-dimensional Frobenius Lie algebras in Czikos’s and Verhoczki’s classification. The calculations are similar to those in the four-dimensional case but more tedious for the parametrized families. We provide the results of our cohomological calculations for one such parametrized family $\Phi_{6,12}(\xi)$ (see Example 1). We find this example interesting because for eight distinct values of $\xi$, both $H^2(\Phi_{6,12}(\xi), \Phi_{6,12}(\xi))$ and $H^3(\Phi_{6,12}(\xi), \Phi_{6,12}(\xi))$ are nonzero. Even so, all infinitesimals are unobstructed.

For the non-parametrized six-dimensional families, we provide the number of inequivalent deformations. As with the four-dimensional Lie algebras, all deformations are linear.

2. Deformation theory

Let $(\mathfrak{g}, [ , ])$ be a Lie algebra over an algebraically closed field $F$, where $\text{char } F \neq 2$. A formal one-parameter deformation of $\mathfrak{g}$ is a power series

$$[g, h]_t = [g, h] + \sum_{k \geq 1} \alpha_k(g, h) t^k,$$

where $\alpha_k \in \text{HOM}_F(\Lambda^2 \mathfrak{g}, \mathfrak{g}) = C^2(\mathfrak{g}, \mathfrak{g})$. The latter refers to the standard Chevalley-Eilenberg cochain complex $(C^*(\mathfrak{g}, \mathfrak{g}), \delta)$ of $\mathfrak{g}$ with coefficients in the adjoint representation of $\mathfrak{g}$. Here, $C^n(\mathfrak{g}, \mathfrak{g})$ consists of forms $F^n : \Lambda^n \mathfrak{g} \to \mathfrak{g}$ satisfying $\delta^2 F^n = 0$, where the coboundary operator $\delta$ is defined by

$$\delta F^n(g_1, \ldots, g_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} [g_i, F^n(g_1, \ldots, \hat{g}_i, \ldots, g_{n+1})]$$

$$+ \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} F^n([g_i, g_j], g_1, \ldots, \hat{g}_i, \ldots, \hat{g}_j, \ldots, g_{n+1}).$$
In this setting, \( Z^n(g, g) = \ker(\delta) \cap C^n(g, g) \), \( B^n(g, g) = \text{Im}(\delta) \cap C^n(g, g) \), and \( H^n(g, g) = Z^n(g, g)/B^n(g, g) \). These comprise, respectively, the \( n \)-cocycles, \( n \)-coboundaries, and \( n \)th cohomology group of \( g \) with coefficients in \( g \). Of course, one requires that the deformation remains in the category of Lie algebras so that the Jacobi identity for \([ \cdot, \cdot ]_t \) is satisfied for all values of \( t \). This is equivalent to the sequence of relations

\[
\delta x_k = -\frac{1}{2} \sum_{i=1}^{k-1} [x_i, x_{k-i}],
\]

where

\[
[\beta, \gamma](g_1, \ldots, g_{p+q-1}) = \sum_{1 \leq i_1 < \ldots < i_{p+q-1}} (-1)^{i_1 + \ldots + i_{p+q-1}} \delta^a \beta(\gamma(g_{i_1}, \ldots, g_{i_q}), g_{i_1} \ldots \hat{g}_{i_1} \ldots g_{i_q} \ldots g_{p+q-1})
\]

\[
+ (-1)^{pq+p+q} \sum_{1 \leq j_1 < \ldots < j_{p+q-1}} (-1)^{i_1 + \ldots + i_{p+q-1}} \gamma(\beta(g_{j_1}, \ldots, g_{j_p}), g_{j_1} \ldots \hat{g}_{j_1} \ldots g_{j_p} \ldots g_{p+q-1}).
\]

In particular, when \( p, q = 2 \) Equation (4) becomes

\[
[\beta, \gamma](g_1, g_2, g_3) = \sum_{1 \leq i_1 < i_2 \leq 3} (-1)^{i_1 + i_2 - 3} \beta(\gamma(g_{i_1}, g_{i_2}), g_{i_1} \ldots \hat{g}_{i_1} \ldots g_{i_2} \ldots g_3)
\]

\[
+ \sum_{1 \leq i_1 < i_2 \leq 3} (-1)^{i_1 + i_2 - 3} \gamma(\beta(g_{i_1}, g_{i_2}), g_{i_1} \ldots \hat{g}_{i_1} \ldots g_{i_2} \ldots g_3).
\]

Two deformations \([g, h]_t\) and \([g, h]_t'\) are called equivalent if there exists a formal one-parameter family \( \{\phi_t\} \) of linear transformations of \( g \),

\[
\phi_t(g) = g + \sum_{k \geq 1} \beta_k(g)t^k,
\]

such that

\[
[g, h]'_t = \phi_t^{-1}[\phi_t(g), \phi_t(h)].
\]

In the deformation (2), the first non-zero \( x_i \) is called the infinitesimal of the deformation and it easy to see that equivalent deformations have cohomologous infinitesimals. So, up to equivalence, the infinitesimal deformations of \( g \) may be regarded as elements of \( H^2(g, g) \) with the obstructions to their propagation to higher-order deformations lying in \( H^3(g, g) \). If each element of \( H^2(g, g) \) is obstructed, then \( g \) is called rigid, and if \( H^2(g, g) = 0 \) then \( g \) is said to be absolutely rigid.

As it happens, all the deformations here considered are linear \((x_i = 0 \text{ for } i > 1)\), so we require only the terms corresponding to \( k = 1 \) and \( 2 \) in (3):

\[
\delta x_1 = 0 \text{ and } \delta x_2 = -\frac{1}{2}[x_1, x_1].
\]

A jump deformation of \( g \) is one such that all specializations, as \( t \) varies over \( \mathbb{F} \), are isomorphic except perhaps, the specialization to \( t = 0 \), which must be isomorphic to \( g \) itself. A linear deformation is trivial if its infinitesimal is cohomologous with zero.

### 3. Classification

The following table contains information about the isomorphism classes of four-dimensional Frobenius Lie algebras over a field \( \mathbb{F} \), where \( \text{char } \mathbb{F} \neq 2 \). Following the notation of [8], there is a single four-dimensional Lie algebra \( \Phi' \), and two families of four-dimensional Lie algebras \( \Phi'' \) and \( \Phi''' \), parametrized by \( \Delta \in \mathbb{F} \) and \( 0 \neq \varepsilon \in \mathbb{F} \), respectively. Note the distinguished \( \Delta \) value of 0, which affects the dimension of the second cohomology group.
Table 1. Four-dimensional Frobenius Lie algebras.

| $\mathfrak{g}$ | Commutator relations | $\dim H^2$ | $\dim H^3$ | Spectrum | Deformation |
|---------------|----------------------|-------------|-------------|----------|-------------|
| $\Phi'$       | $[e_1, e_4] = [e_2, e_3] = -e_1,$ | 3           | 0           | $\{0, t, 1, 1\}$ | Yes         |
| $\Phi'(0)$    | $[e_1, e_4] = [e_2, e_3] = -e_1,$ | 2           | 1           | $\{0, 0, 1, 1\}$ | Yes         |
| $\Phi'(\Delta)$ | $[e_1, e_4] = [e_2, e_3] = -e_1,$ | 1           | 0           | $\{0, 1, 1\}$ | No          |
| $\Phi''(\varepsilon)$ | $[e_1, e_4] = [e_2, e_3] = -e_1,$ | 0           | 0           | $\{0, 0, 1, 1\}$ | No          |

Remark 1. Note that $\Phi''(\varepsilon_1) \cong \Phi''(\varepsilon_2)$ if and only if the quotient $\varepsilon_1/\varepsilon_2$ is the square of an element in $\mathbb{F}$ (always the case when $\mathbb{F}$ is algebraically closed, as we assume here), and all other pairs of Lie algebras in the table are non-isomorphic. Interestingly, $\Phi''(\varepsilon)$ provides an example of a parametrized family of Lie algebras which cannot be represented as a formal deformation.

For $\Phi'$, there are three non-trivial deformations: $\Phi'_{1,t}, \Phi'_{2,t},$ and $\Phi'_{3,t}$, with deformation parameter $t$. The latter two are jump deformations – where the deformed algebras happen to be isomorphic to the initial algebra $\Phi'$. The first deformation will become our prime example. For $\Phi''(\Delta)$, there are two deformations: $\Phi''(\Delta)$ and $\Phi''_{2,0}(0)$. The former exists for all values of $\Delta \in \mathbb{F}$, while the latter is a jump deformation and exists only when $\Delta = 0$.

4. Deformations

Notation: To ease notation, we let $\{e_1, e_2, e_3, e_4\}$ be a basis for the Lie algebra $\mathfrak{g}$ under consideration and use $\Gamma^i$ to represent an $i$-cocycle of $\mathfrak{g}$ with coefficients in the adjoint representation of $\mathfrak{g}$. We will need a battery of coefficients for our calculations. With the exception of sub-scripted $\varepsilon$'s (which we reserve for basis elements), we will use sub-scripted and super-scripted (and sometimes both) lower case roman letters for elements of the ground field.

4.1. Cohomology

In this section, we provide detailed calculations of $H^2(\mathfrak{g}, \mathfrak{g})$ and $H^3(\mathfrak{g}, \mathfrak{g})$ in the particular case where $\mathfrak{g} = \Phi'$ (see Theorems 1 and 2). The calculations for the other algebras in Table 1 are similar.

Theorem 1. A basis for $H^2(\Phi', \Phi')$ is given by $\{[\Gamma^2_{1,1}], [\Gamma^2_{1,2}], [\Gamma^2_{1,3}]\}$, where the $\Gamma$s are defined by

- $\Gamma^2_{1,1}$ is defined by $\Gamma^2_{1,1}(e_2, e_4) = e_2$, $\Gamma^2_{1,1}(e_3, e_4) = -e_3$.
- $\Gamma^2_{1,2}$ is defined by $\Gamma^2_{1,2}(e_3, e_4) = e_2$.
- $\Gamma^2_{1,3}$ is defined by $\Gamma^2_{1,3}(e_2, e_4) = e_3$. 
Proof. The fact that $\Gamma^2$ is a cocycle gives the following conditions on its coefficients:

- $c_{1,2}^1 = c_{2}^2 - 3c_{4}^2$
- $c_{1,2}^3 = -c_{2}^3 + 3c_{4}^3$
- $c_{1,4}^1 = c_{2}^4 - c_{3}^4$

- $c_{1,2}^2 = \frac{1}{2}c_{2}^2$
- $c_{2}^3 = 0$
- $c_{1,4}^2 = \frac{1}{2}c_{3}^3 - \frac{1}{2}c_{2}^4$

- $c_{3,2}^1 = 0$
- $c_{3}^3 = \frac{1}{2}c_{2}^3$
- $c_{4}^4 = c_{3}^4$

- $c_{4}^1 = -2c_{2}^4 - 2c_{3}^4$
- $c_{4}^2 = 2c_{2}^4$
- $c_{4}^3 = 2c_{3}^4$

If $\Gamma^2(e_i, e_j) = \sum_{k=1}^{4} c_{i,j}^k e_k$, $1 \leq i \leq j \leq 4$ is to be a coboundary, there must exist $F^1 \in C^1(\mathfrak{g}, \mathfrak{g})$ defined by $F^1(e_i) = \sum_{k=1}^{4} c_{i,k}^k e_k$, $1 \leq \ell \leq 4$ such that $\delta F^1 = \Gamma^2$. This is equivalent to the following conditions:

1. $c_{1}^1 = c_{2}^2 + c_{3}^2 + c_{4}^2$
2. $c_{2}^3 = -\frac{1}{2}c_{2}^2$
3. $c_{3}^2 = 2c_{4}^2 - 2c_{3}^4$
4. $c_{4}^2 = \frac{1}{2}c_{2}^2 + c_{4}^2$

5. $c_{3}^1 = -\frac{1}{2}c_{2}^3 + c_{4}^3$
6. $c_{4}^1 = c_{3}^4$
7. $c_{4}^2 = 2c_{2}^4$
8. $c_{4}^3 = 2c_{4}^4$

Note that for a fixed $\Gamma^2$, we may choose coefficients of $F^1$ as dictated by conditions 1-8. However, Equations (9)–(11) place independent conditions on $\Gamma^2$, rather than on $F^1$. This means that any $\Gamma^2$ which does not satisfy 9-11 has no $F^1$ such that $\delta F^1 = \Gamma^2$. This gives an upper bound of three on the dimension of $H^2(\mathfrak{g}, \mathfrak{g})$. Choosing $\Gamma^2_{1,1}, \Gamma^2_{1,2},$ and $\Gamma^2_{1,3}$ as in the statement of the theorem yields the result – it is straightforward to verify that these are non-cohomologous cocycles.

For $\Phi'$, the following theorem asserts that the third cohomology group is trivial, so the infinitesimals given in Theorem 1 are unobstructed.

**Theorem 2.** $H^3(\Phi', \Phi') = 0$.

Proof. Let $\Gamma^3(e_i, e_j, e_k) = \sum_{k=1}^{4} c_{i,j,k}^k e_k$, $1 \leq i < j < \ell \leq 4$. By a straightforward computation, we see that, in order for $\Gamma^3$ to indeed be a cocycle, the following conditions on its coefficients must be satisfied:

- $c_{1,2,3}^1 = -c_{2,2,4}^1 - c_{3,3,4}^1 + c_{4,3,4}^1$
- $c_{1,2,3}^2 = -c_{3,1,3}^2$
- $c_{1,2,3}^3 = c_{4,1,3}^3$

With these conditions established for $\Gamma^3 \in Z^3(\mathfrak{g}, \Phi)$, we will show that there exists a 2-cochain $F^2$ for which $\delta F^2 = \Gamma^3$. If $F^2(e_i, e_j) = \sum_{k=1}^{4} c_{i,j}^k e_k$, $1 \leq i < j \leq 4$, and

- $c_{1,2}^1 = 2(c_{2}^4 - c_{4}^2 - c_{1,2,4}^1)$
- $c_{1,2}^3 = -2(c_{2}^4 + c_{4}^4 + c_{1,3,4}^3)$
- $c_{1,4}^1 = c_{2}^4 + c_{3}^4 + c_{4}^2$

- $c_{2}^1 = \frac{1}{2}c_{2}^4 - c_{1,2,4}^2$
- $c_{2,3}^2 = 2c_{2}^4 - c_{3}^4 - 2c_{2,3}^3$

- $c_{3}^2 = 2c_{3}^4 + c_{4}^4 - 2c_{3}^3$

- $c_{4}^2 = -\frac{2}{3}c_{2,4}^4$
- $c_{4}^3 = -\frac{2}{3}c_{4}^3$
- $c_{4}^4 = c_{4}^4$

We conclude that $\delta F^2 = \Gamma^3$. The result follows. \qed
4.2. Deformation and spectrum

The main results of this section are displayed in Table 2 below, where the deformations, infinitesimals, and (deformed) spectra of the Lie algebras \( g \) in Table 1 are detailed.

| \( g \) | Infinitesimals (see Theorem 1) | Commutator relations | Spectrum |
|-------|---------------------------------|----------------------|---------|
| \( \Phi_{1,1}' \) | \( \Gamma_{1,1}^2 \) | \( [e_1, e_4] = [e_2, e_3], \) \( -e_1 \) | \( [e_2, e_4] = -(t - \frac{1}{2})e_2, \) \( e_3, e_4] = -\frac{1}{2}e_3 + te_2 \) | \( 0, 0, 1, 1 \) |

Table 2. Deformations of four-dimensional Frobenius Lie algebras.

Remark 2. Since the category of Frobenius Lie algebras is stable under deformation, each deformed algebra in Table 2 is isomorphic to an algebra given in Table 1. The following list describes these isomorphisms:

- Fixing a value of \( \Delta \in \mathbb{F} \) and replacing the basis \( \{e_1, e_2, e_3, e_4\} \) by \( \{e'_1 = 2te_1, e'_2 = -(t + \frac{1}{2})e_2 + e_3, e'_3 = (t - \frac{1}{2})e_2 + e_3, e'_4 = e_4\} \), we see that \( \Phi''(\Delta) \cong \Phi_{1,1}' \) for \( t = \frac{\sqrt{1 - 4\Delta}}{2} \neq 0 \).
- Replacing the basis \( \{e_1, e_2, e_3, e_4\} \) by \( \{e_1, e'_2 = e_2 - e_4, e'_3 = e_2, e'_4 = e_3\} \), we see that \( \Phi''(\varepsilon) \cong \Phi_{2,1}'(0) \) for \( t = \varepsilon \).
- Replacing the basis \( \{e_1, e_2, e_3, e_4\} \) by \( \{e_1, e_2, e'_3 = e_3 + 2te_2, e_4\} \) we see that \( \Phi' \cong \Phi_{2,1}' \).
- Replacing the basis \( \{e_1, e_2, e_3, e_4\} \) by \( \{e_1, e'_2 = e_2 + 2te_2, e_3, e_4\} \) we see that \( \Phi' \cong \Phi_{3,1}' \). Note that although \( \Phi_{2,1}' \cong \Phi_{3,1}' \) for each value of \( t \), they are inequivalent as deformations since their respective infinitesimals, \( \Gamma_{1,2}^2 \) and \( \Gamma_{1,3}^2 \), are non-cohomologous.

We will only prove the results from Table 2 corresponding to the Lie algebra \( \Phi_{1,1}' \). The proofs for the other algebras are similar.

Theorem 3. \( \Gamma_{1,1}^2 \) is the infinitesimal of a deformation of \( \Phi' \), giving rise to the deformed algebra \( \Phi_{1,1}' \) defined by the relations

- \( [e_1, e_4] = [e_2, e_3] = -e_1 \)
- \( [e_2, e_4] = \left( t - \frac{1}{2} \right) e_2 \)
- \( [e_3, e_4] = -\left( t + \frac{1}{2} \right) e_3 \).
Theorem 4. The spectrum of $\Phi'_{1,t}$ is given by $\{0, \frac{1}{2} - t, \frac{1}{2} + t, 1\}$.

Proof. To determine the spectrum of $\Phi'_{1,t}$, we first choose a Frobenius functional. Let $F = f_1 e_1^* + f_2 e_2^* + f_3 e_3^* + f_4 e_4^* \in \mathfrak{g}^*$ and $B = b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4 \in \Phi'_{1,t} \cap \ker(F([-,-]))$. (6)

The equations in (6) yield the following system of equations:

- $F([e_1,B]) = -b_4 f_1 = 0$
- $F([e_2,B]) = -b_3 f_1 - \frac{1}{2} b_4 f_2 = 0$
- $F([e_3,B]) = b_2 f_1 - \frac{1}{2} b_4 f_3 = 0$
- $F([e_4,B]) = b_1 f_1 + \frac{1}{2} b_2 f_2 + \frac{1}{2} b_3 f_3 = 0$

For $F$ to be a Frobenius functional, it must be the case that $B=0$ so let $f_1 = 1$ and $f_2 = f_3 = f_4 = 0$. It follows that $F = e_1^*$ is a Frobenius functional on $\Phi'_{1,t}$.

Next, we need to determine the principal element $\hat{F} \in \Phi'_{1,t}$ corresponding to $F$. Let

$$\hat{F} = p_1 e_1 + p_2 e_2 + p_3 e_3 + p_4 e_4.$$

Equation (1) then yields the following system of equations:

- $p_4 = F([\hat{F}, e_1]) = F(e_1) = 1$
- $p_3 = F([\hat{F}, e_2]) = F(e_2) = 0$
- $-p_2 = F([\hat{F}, e_3]) = F(e_3) = 0$
- $-p_1 = F([\hat{F}, e_4]) = F(e_4) = 0$.

It follows that $\hat{F} = e_4$.

Finally, to determine the spectrum of $\Phi'_{1,t}$, we calculate the spectrum of $\text{ad}\hat{F} : \Phi'_{1,t} \to \Phi'_{1,t}$. It is straightforward to show that

$$\text{ad}\hat{F} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \frac{1}{2} - t & 0 \\
0 & 0 & 0 & \frac{1}{2} + t
\end{bmatrix},$$

so that the spectrum of $\Phi'_{1,t}$ is
5. Epilogue

Topical investigations regarding the spectrum of a Frobenius Lie algebra have concentrated on seaweed Lie algebras (see [9]), or simply “seaweeds” (elsewhere called biparabolic [17]), and the recently introduced Lie poset algebras (see [3]). In a series of papers by Coll et al (see [4], and [2]), it has been established that the unbroken spectrum property holds for all the classical and exceptional Frobenius seaweeds.

Indeed, the interesting spectral properties of seaweeds were the impetus for the motivating question of this article. However, seaweeds appear to be cohomologically inert so cannot be deformed.1

In contrast, Lie poset algebras, which are necessarily solvable, have a rich deformation theory. However, we have no examples of deformable Frobenius Lie poset algebras. It is also worth noting that the unbroken spectrum property seems to be a property of Frobenius Lie poset algebras as well, although the spectrum is “binary”, consisting of only 0’s and 1’s (see [5, 7, and [6]).

It is interesting to note that the spectrum of \( \Phi'(\Delta) \) is an unbroken sequence of integers if and only if \( \Delta = 0 \) or \(-2\). The proof is straightforward as follows. Recall that the spectrum of \( \Phi'(\Delta) \) is \[ \{0, 1, \frac{1}{2} - t, \frac{1}{2} + t, 1\} \]. Thus, the spectrum of \( \Phi''(\Delta) \) consists of integers if and only if \( 1 - 4\Delta = a^2 \), where \( a \) is an odd integer. When \( a = \pm 1 \), i.e., \( \Delta = 0 \), the spectrum of \( \Phi'(\Delta) \) is \( \{0, 0, 1, 1\} \); and when \( a = \pm 3 \), i.e., \( \Delta = -2 \), the spectrum of \( \Phi'(\Delta) \) is \( \{-1, 0, 1, 2\} \). Since \( \frac{1-a}{2} \) (resp. \( \frac{1+a}{2} \)) strictly decreases (resp. increases) as \( a \) increases, the spectrum can only be unbroken for \( a = \pm 1 \) and \( \pm 3 \), i.e., \( \Delta = 0 \) and \(-2\).

Note that \( \Phi''(\Delta) \), where \( \Delta = 0, -2 \) is not a seaweed, since it deforms. And while \( \Phi''(0) \) and \( \Phi''(-2) \) are both solvable, neither is a Lie poset algebra of classical type since there is exactly one such four-dimensional Frobenius algebra. When the ground field is the complex numbers this algebra is isomorphic to \( \Phi''(\varepsilon) \), for all \( \varepsilon \neq 0 \).

Appendix A: Dimension six

Tables 3 and 4 contain information about the isomorphism classes of non-decomposable six-dimensional Frobenius Lie algebras over \( \mathbb{F} \) which we now take to be an algebraically closed field of characteristic zero. Table 3 details the commutator relations and spectra of each algebra in the classification given in [8]. Table 4 summarizes the cohomology and deformations of the non-parametrized algebras given in Table 3. Example 1 addresses one of the parametrized families given in Table 3.

Remark 3. In Table 3, if the parameters \( \xi, \eta \in \mathbb{F} \) are separated by a colon, the isomorphism class does not change if both parameters are multiplied by a nonzero number. There are curly brackets around the parameters when the isomorphism class does not depend on the order of the parameters. The algebra \( \Phi_{6,11}(\xi,\eta) \) depends only on the set \( \{\xi, 1 - \xi, 1 - \eta\} \). Except for these isomorphisms, the isomorphism classes are pairwise distinct.

The third and fourth columns of the following Table 4 provide the dimensions of the second and third cohomology groups of each non-parametrized algebra described in Table 3. The last column indicates whether or not a deformation of the algebra exists, and parenthetically the number of nonequivalent deformations of the indicated algebra.

1In 2014, Gerstenhaber conjectured that seaweeds are cohomologically trivial. This was verified for type-A seaweeds by Elashvili and Rakviashvili (see [10], 2016).
Table 3. Six-dimensional Frobenius Lie algebras.

| $\alpha$ | Commutator relations | Spectrum |
|---------|---------------------|---------|
| $\Phi_{6.1}$ | $[X_1, Y_1] = Y_3$, $[X_1, Y_3] = Y_5$, $[X_1, Y_4] = 2Y_4$, $[X_2, Y_2] = Y_2$, $[X_2, Y_3] = Y_5$, $[X_2, Y_4] = Y_4$, $[Y_1, Y_2] = Y_3$, $[Y_1, Y_3] = Y_4$ | 0, 0, 1, 1, 1 |
| $\Phi_{6.2} (\zeta \neq \eta)$ | $[X_1, Y_1] = Y_3$, $[X_1, Y_3] = Y_5$, $[X_1, Y_4] = \zeta Y_4$, $[X_2, Y_2] = Y_2$, $[X_2, Y_3] = Y_5$, $[X_2, Y_4] = \eta Y_4$, $[Y_1, Y_2] = Y_3$ | $0, 0, \frac{\zeta - 1}{\zeta + 1}, 1, 1$ |
| $\Phi_{6.3} (\zeta : \eta) \neq (1 : 1)$ | $[X_1, Y_1] = Y_3$, $[X_1, Y_3] = Y_5$, $[X_1, Y_4] = Y_4 + \zeta Y_5$, $[X_2, Y_2] = Y_2$, $[X_2, Y_3] = Y_5$, $[X_2, Y_4] = Y_4 + \eta Y_5$, $[Y_1, Y_2] = Y_3$ | $0, 0, \frac{\zeta - 1}{\zeta + 1}, 0, 0$ |
| $\Phi_{6.4} (\zeta : \eta) \neq (0 : 0)$ | $[X_1, Y_1] = Y_3 + \zeta Y_4$, $[X_1, Y_3] = Y_5$, $[X_1, Y_4] = Y_4$, $[X_2, Y_2] = Y_2$, $[X_2, Y_3] = Y_5$, $[X_2, Y_4] = Y_4$, $[Y_1, Y_2] = Y_3$ | 0, 0, 0, 1, 1, 1 |
| $\Phi_{6.5} (\zeta : \eta), \eta \neq 0$ | $[X_1, Y_1] = \frac{3}{2} Y_1 + \zeta Y_2$, $[X_1, Y_2] = \frac{3}{2} Y_2$, $[X_1, Y_3] = Y_5$, $[X_2, Y_4] = Y_4$, $[Y_1, Y_2] = Y_3$ | $0, 0, \frac{3}{2}, \frac{3}{2}, 1, 1$ |
| $\Phi_{6.6}$ | $[X_1, Y_1] = Y_5$, $[X_1, Y_2] = 2Y_2$, $[X_1, Y_3] = 3Y_3$, $[X_1, Y_4] = 4Y_4$, $[X_2, Y_1] = SY_1$, $[Y_1, Y_2] = Y_2$, $[Y_1, Y_3] = Y_4$, $[Y_1, Y_4] = Y_5$, $[Y_2, Y_3] = Y_5$ | $0, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1$ |
| $\Phi_{6.7} (\zeta)$ | $[X_1, Y_1] = \zeta Y_1$, $[X_1, Y_2] = 2\zeta Y_2$, $[X_1, Y_3] = (1 - 2\zeta) Y_3$, $[X_1, Y_4] = (1 - \zeta) Y_4$, $[X_2, Y_1] = Y_5$, $[Y_1, Y_2] = Y_4$, $[Y_1, Y_3] = Y_5$, $[Y_1, Y_4] = Y_5$, $[Y_2, Y_3] = Y_5$ | $0, \zeta, 2\zeta, 1 - 2\zeta, 1 - \zeta, 1$ |
| $\Phi_{6.8}$ | $[X_1, Y_1] = Y_5$, $[X_1, Y_2] = Y_5 - Y_4$, $[X_1, Y_3] = Y_5$, $[X_1, Y_4] = Y_5$, $[Y_1, Y_2] = Y_4$, $[Y_1, Y_3] = Y_5$, $[Y_1, Y_4] = Y_5$, $[Y_2, Y_3] = Y_5$ | 0, 0, 0, 0, 1, 1 |
| $\Phi_{6.9}$ | $[X_1, Y_1] = \frac{1}{2} Y_1 + Y_2$, $[X_1, Y_2] = \frac{1}{2} Y_2 + Y_4$, $[X_1, Y_3] = \frac{1}{2} Y_3$, $[X_1, Y_4] = \frac{1}{2} Y_4$, $[X_2, Y_1] = Y_5$, $[Y_1, Y_2] = Y_4$, $[Y_1, Y_3] = Y_5$, $[Y_1, Y_4] = Y_5$, $[Y_2, Y_3] = Y_5$ | $0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1$ |
| $\Phi_{6.10}$ | $[X_1, Y_1] = \frac{1}{2} Y_1 + Y_2$, $[X_1, Y_2] = \frac{1}{2} Y_2 + Y_3$, $[X_1, Y_3] = \frac{1}{2} Y_3 + Y_2$, $[X_1, Y_4] = \frac{1}{2} Y_4$, $[X_2, Y_1] = Y_5$, $[Y_1, Y_2] = Y_4$, $[Y_1, Y_3] = Y_5$, $[Y_1, Y_4] = Y_5$, $[Y_2, Y_3] = Y_5$ | $0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1$ |
| $\Phi_{6.11} (\zeta : \eta)$ | $[X_1, Y_1] = \zeta Y_1$, $[X_1, Y_2] = \eta Y_2$, $[X_1, Y_3] = (1 - \eta) Y_3$, $[X_1, Y_4] = (1 - \zeta) Y_4$, $[X_2, Y_1] = Y_5$, $[Y_1, Y_2] = Y_4$, $[Y_1, Y_3] = Y_5$, $[Y_1, Y_4] = Y_5$, $[Y_2, Y_3] = Y_5$ | $0, \zeta, \eta, 1 - \zeta, 1 - \eta, 1$ |
| $\Phi_{6.12} (\zeta) \cong \Phi_{6.13} (1 - \zeta)$ | $[X_1, Y_1] = \zeta Y_1$, $[X_1, Y_2] = \frac{1}{2} Y_2 + Y_3$, $[X_1, Y_3] = Y_4$, $[X_1, Y_4] = (1 - \zeta) Y_4$, $[X_2, Y_1] = Y_5$, $[Y_1, Y_2] = Y_4$, $[Y_1, Y_3] = Y_5$, $[Y_1, Y_4] = Y_5$, $[Y_2, Y_3] = Y_5$ | $0, \frac{1}{2}, \frac{1}{2}, \zeta, 1 - \zeta, 1$ |
| $\Phi_{6.13} (\zeta) \cong \Phi_{6.13} (1 - \zeta)$ | $[X_1, Y_1] = \zeta Y_1 + Y_2$, $[X_1, Y_2] = Y_2$, $[X_1, Y_3] = (1 - \zeta) Y_3 - Y_4$, $[X_1, Y_4] = (1 - \zeta) Y_4$, $[X_2, Y_1] = Y_5$, $[Y_1, Y_2] = Y_4$, $[Y_1, Y_3] = Y_5$, $[Y_1, Y_4] = Y_5$, $[Y_2, Y_3] = Y_5$ | $0, \zeta, \frac{1}{2}, \frac{1}{2}, 1 - \zeta, 1$ |
| $\Phi_{6.14}$ | $[X_1, Y_1] = \frac{1}{2} Y_1 + Y_2$, $[X_1, Y_2] = \frac{1}{2} Y_2 + Y_3$, $[X_1, Y_3] = \frac{1}{2} Y_3 - Y_4$, $[X_1, Y_4] = \frac{1}{2} Y_4$, $[X_2, Y_1] = Y_5$, $[Y_1, Y_2] = Y_4$, $[Y_1, Y_3] = Y_5$, $[Y_1, Y_4] = Y_5$, $[Y_2, Y_3] = Y_5$ | $0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1$ |
| $\Phi_{6.15}$ | $[X_1, Y_1] = \frac{1}{2} Y_1 + Y_4$, $[X_1, Y_2] = \frac{1}{2} Y_2 + Y_3$, $[X_1, Y_3] = \frac{1}{2} Y_3 + Y_2$, $[X_1, Y_4] = \frac{1}{2} Y_4$, $[X_2, Y_1] = Y_5$, $[Y_1, Y_2] = Y_4$, $[Y_1, Y_3] = Y_5$, $[Y_1, Y_4] = Y_5$, $[Y_2, Y_3] = Y_5$ | $0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1$ |
Table 4. Cohomology of non-parametrized six-dimensional Frobenius Lie algebras.

| $\Phi_i$ | Commutator relations | $\dim H^2$ | $\dim H^3$ | Deformation(s) |
|----------|----------------------|------------|------------|----------------|
| $\Phi_{1}$ | $[X_1, Y_1] = Y_1$, $[X_1, Y_3] = Y_3$, $[X_1, Y_4] = 2Y_4$, $[X_2, Y_2] = Y_2$, $[X_2, Y_3] = Y_3$, $[X_2, Y_4] = Y_4$, $[Y_1, Y_2] = Y_3$, $[Y_1, Y_3] = Y_4$ | 0 | 0 | No |
| $\Phi_{6}$ | $[X, Y_1] = Y_1$, $[X, Y_2] = 2Y_2$, $[X, Y_3] = 3Y_3$, $[X, Y_4] = 4Y_4$, $[X, Y_5] = 5Y_5$, $[Y_1, Y_2] = Y_3$, $[Y_1, Y_3] = Y_4$, $[Y_1, Y_4] = Y_5$, $[Y_2, Y_3] = Y_5$ | 0 | 0 | No |
| $\Phi_{8}$ | $[X, Y_1] = Y_2$, $[X, Y_3] = Y_3 - Y_4$, $[X, Y_4] = Y_4$, $[X, Y_5] = Y_5$, $[Y_1, Y_3] = Y_4$, $[Y_1, Y_4] = Y_5$, $[Y_2, Y_3] = Y_5$ | 3 | 2 | Yes(3) |
| $\Phi_{9}$ | $[X, Y_1] = \frac{1}{2}Y_1 + Y_3$, $[X, Y_2] = \frac{1}{2}Y_2 + Y_4$, $[X, Y_3] = \frac{3}{2}Y_3$, $[X, Y_4] = \frac{3}{2}Y_4$, $[X, Y_5] = Y_4$, $[Y_1, Y_3] = Y_4$, $[Y_1, Y_4] = Y_5$, $[Y_2, Y_3] = Y_5$ | 1 | 0 | Yes(1) |
| $\Phi_{10}$ | $[X, Y_1] = \frac{1}{2}Y_1$, $[X, Y_2] = \frac{1}{2}Y_2$, $[X, Y_3] = \frac{1}{2}Y_3 + Y_2$, $[X, Y_4] = \frac{3}{2}Y_4$, $[X, Y_5] = Y_5$, $[Y_1, Y_3] = Y_4$, $[Y_1, Y_4] = Y_5$, $[Y_2, Y_3] = Y_5$ | 1 | 0 | Yes(1) |
| $\Phi_{14}$ | $[X, Y_1] = \frac{1}{2}Y_1 + Y_2$, $[X, Y_2] = \frac{1}{2}Y_2 + Y_3$, $[X, Y_3] = \frac{1}{2}Y_3 - Y_4$, $[X, Y_4] = \frac{1}{2}Y_4$, $[X, Y_5] = Y_4$, $[Y_1, Y_4] = Y_2$, $[Y_2, Y_3] = Y_5$ | 2 | 0 | Yes(2) |
| $\Phi_{15}$ | $[X, Y_1] = \frac{1}{2}Y_1 + Y_4$, $[X, Y_2] = \frac{1}{2}Y_2 + Y_3$, $[X, Y_3] = \frac{3}{2}Y_3$, $[X, Y_4] = \frac{3}{2}Y_4$, $[X, Y_5] = Y_4$, $[Y_1, Y_4] = Y_2$, $[Y_2, Y_3] = Y_5$ | 4 | 0 | Yes(4) |
As with the four-dimensional Lie algebras of Table 2, all deformations associated with Table 4 are linear. The calculations involving the cohomology and deformation theories of the parametrized Lie algebras are nettlesome. We provide information on one such parametrized algebra (see Example 1).

**Example 1.** Consider the family given by $\Phi_{6,12}(\xi)$. Routine calculations yield

$$\dim H^2(\Phi_{6,12}(\xi), \Phi_{6,12}(\xi)) = \begin{cases} 4, & \xi = 0, 1 \\ 3, & \xi = \frac{1}{4} \\ 2, & \text{otherwise,} \end{cases}$$

and

$$\dim H^2(\Phi_{6,12}(\xi), \Phi_{6,12}(\xi)) = \begin{cases} 2, & \xi = -1, 0, 1, 2 \\ 1, & \xi = -\frac{1}{4}, \frac{1}{3}, \frac{5}{4}, \frac{7}{4} \\ 0, & \text{otherwise.} \end{cases}$$

Note that the number of deformations of $\Phi_{6,12}(\xi)$ is equal to $\dim H^2(\Phi_{6,12}(\xi), \Phi_{6,12}(\xi))$ for each value of $\xi$, and each deformation is linear. We illustrate one such deformation as follows.

Let $\Phi_{6,12}(2)$ be one of the two deformations given by $\Phi_{6,12}(\xi)$ when $\xi = 2$. This deformation family is defined by the following collection of commutator relations:

- $[e_1, e_2]_t = (2 + \frac{t}{4}) e_2$
- $[e_1, e_3]_t = (\frac{1}{4} - \frac{t}{2}) e_3 + e_4$
- $[e_1, e_4]_t = (\frac{1}{4} - \frac{t}{2}) e_4$
- $[e_1, e_5]_t = (-1 - t) e_5$
- $[e_1, e_6]_t = (1 - \frac{t}{2}) e_6$
- $[e_2, e_5]_t = e_6$
- $[e_3, e_4]_t = e_6$

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