Fixed points of endomorphisms of a free metabelian group

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Abstract

We consider IA-endomorphisms (i.e. Identical in Abelianization) of a free metabelian group of finite rank, and give a matrix characterization of their fixed points which is similar to (yet different from) the well-known characterization of eigenvectors of a linear operator in a vector space. We then use our matrix characterization to elaborate several properties of the fixed point groups of metabelian endomorphisms. In particular, we show that the rank of the fixed point group of an IA-endomorphism of the free metabelian group of rank \( n \geq 2 \) can be either equal to 0, 1, or greater than \( (n - 1) \) (in particular, it can be infinite). We also point out a connection between these properties of metabelian IA-endomorphisms and some properties of the Gassner representation of pure braid groups.

1. Introduction

Let \( F = F_n \) be the free group of a finite rank \( n \geq 2 \) with a set \( X = \{x_i\}, 1 \leq i \leq n \), of free generators. Then, let \( F' = [F, F] \) be the commutator subgroup of the group \( F \). The group \( M = M_n = F/F'' \) is called a free metabelian group.

Let \( \varphi \) be an endomorphism of the group \( F \) given by \( \varphi(x_k) = y_k, 1 \leq k \leq n \). It also induces an endomorphism of the group \( M \) in the natural way. We denote elements of a free group and their natural images in a free metabelian group by the same letters when there is no ambiguity. The same applies to endomorphisms.

Birman [3] and Bachmuth [1] have used matrices over group rings to study endomorphisms of a free and a free metabelian group, respectively.

Birman [3] has given a matrix characterization of automorphisms of a free group among arbitrary endomorphisms (the ‘inverse function theorem’) as follows. Define the matrix \( J_\varphi = (d_j(y_k))_{1 \leq i, j \leq n} \) (the ‘Jacobian matrix’ of \( \varphi \)), where \( d_j \) denotes partial Fox derivation (with respect to \( x_j \)) in the free group ring \( ZF \) (see [6]). Then \( \varphi \) is an automorphism if and only if the matrix \( J_\varphi \) is invertible.

Bachmuth [1] has obtained an inverse function theorem of the same kind on replacing the Jacobian matrix \( J_\varphi \) by its image \( J_\varphi^a \) over the abelianized group ring \( Z(F/F') \).

Recently, matrix methods have been used by a number of authors to produce new interesting results on endomorphisms of free and free metabelian groups. Umirbaev [12] has generalized Birman’s result to partial generating systems (so-called primitive systems) of a free group. In [7] and [10], similar results are obtained for primitive
Proposition 3

The rank of an element to the commutator subgroup is an algorithm for detecting the presence of those non-trivial fixed points that belong to the intersection of the fixed point group \( \text{Fix} \) of the matrix 2. An algorithm for detecting fixed points does exist: when all non-trivial fixed points of \( \varphi \) are inside the commutator subgroup \( M' \) (which is the case, for example, when \( \varphi \) is the conjugation by a non-trivial element from \( M' \)), the fixed point group \( \text{Fix} \varphi \) has infinite rank.

Theorem 1.1. Let \( \varphi \) be an IA-endomorphism of the free metabelian group \( M = M_n \) given by \( \varphi(x_i) = y_i, \ 1 \leq i \leq n \), and let \( J^n_\varphi \) be the corresponding abelianized Jacobian matrix. Then:

(i) \( \det(J^n_\varphi - I) = 0 \), where \( I \) is the \( n \times n \) identity matrix;

(ii) If rank \( (J^n_\varphi - I) \leq n - 2 \), then \( \varphi \) has a non-trivial fixed point inside the commutator subgroup \( M' \);

(iii) If rank \( (J^n_\varphi - I) = n - 1 \), then \( \varphi \) has a non-trivial fixed point inside the commutator subgroup \( M' \) if and only if \([x_1, x_2]u_1 \cdot [x_2, x_3]u_2 \cdot \cdots \cdot [x_{n-1}, x_n]u_{n-1} = [y_1, y_2]v_1 \cdot [y_2, y_3]v_2 \cdot \cdots \cdot [y_{n-1}, y_n]v_{n-1} \) for some elements \( u_k \in Z(M/M') \), not all of them zero.

The situation is most subtle when rank \( (J^n_\varphi - I) = n - 1 \). In this case, anything can happen. For example, if \( \varphi \) is an IA-endomorphism of the group \( M_2 \) given by \( \varphi(x_1) = x_1 s; \ \varphi(x_2) = x_2 s^{-1} \), then \( \varphi \) has no non-trivial fixed points for any \( s \in M'_2 \), \( s \neq 1 \) (see Proposition 3.2). On the other hand, if \( \varphi \) is an inner automorphism induced by an element \( g \notin M'_2 \), then \( \varphi \) has a non-trivial fixed point (outside \( M'_2 \)). In both situations, rank \( (J^n_\varphi - I) = 1 = 2 - 1 \).

Although I was not able to distinguish these possibilities 'by matrix means', an algorithm for detecting fixed points does exist:

Theorem 1.2. There is an algorithm for detecting the presence of non-trivial fixed points of an arbitrary IA-endomorphism of a free metabelian group \( M_n \). Also, there is an algorithm for detecting the presence of those non-trivial fixed points that belong to the commutator subgroup \( M'_n \). In both cases, if an algorithm reveals the presence of non-trivial fixed points, \( \varphi \) has at least one of them can be actually found.

In case of a free group, these questions remain open even when restricted to detecting fixed points of automorphisms.

Although I was not able to distinguish these possibilities 'by matrix means', an algorithm for detecting fixed points does exist:

Theorem 1.3. Let \( \varphi \) be an IA-endomorphism of a free metabelian group \( M \). Then the intersection of the fixed point group \( \text{Fix} \varphi \) with \( M' \) is a normal subgroup of \( M \). In particular, if this intersection is non-trivial, then it is infinitely generated.
Furthermore, if \( \varphi \) has the fixed point group of finite rank, then there is a ‘generation gap’

**Theorem 1-4.** Let \( \varphi \) be an IA-endomorphism of a free metabelian group \( M_n \). If the rank of the fixed point group \( \text{Fix} \, \varphi \) is finite, then it is equal to 0, 1, or is greater than \( (n-1) \).

Then, we have

**Proposition 1-5.** For an arbitrary \( n \geq 3 \), there are non-inner (IA-)automorphisms of the free metabelian group \( M_n \) that have infinitely generated fixed point group.

Note that every IA-automorphism of the group \( M_2 \) is inner [1]. Then, if \( u \in M'_n \), the conjugation by \( u \) is an automorphism of \( M_n \) whose fixed point group coincides with \( M'_n \) and therefore is infinitely generated. The same argument proves the existence of inner automorphisms with infinitely generated fixed point group for any group of the form \( F/[R, R] \) with \( F/R \) infinite.

We should also mention here an example of a finitely presented group [5] whose automorphisms can have infinitely generated fixed point group.

All this makes contrast to the situation in a free group: the fixed point group of any free endomorphism has rank \( n = \text{rank} \, F \) or less. This has been proved in [2] for automorphisms, and in [9] – for arbitrary endomorphisms.

In general, it is an interesting and important question – how far is the similarity between free and free metabelian endomorphisms extended? The desire to ‘lift’ properties of metabelian endomorphisms to those of free endomorphisms leads to several interesting questions of which the following one seems particularly attractive:

**Problem 1.** Suppose \( \varphi \in \text{Aut} \, F \) is a non-inner IA-automorphism with a non-trivial fixed point. Is it true that \( \varphi \) has a fixed point inside \([F, F]!\)

Another interesting question is – how can our Theorem 1-1 be strengthened when restricted to automorphisms; in particular, we ask:

**Problem 2.** Is it true that every IA-automorphism of the group \( M_n \) has a non-trivial fixed point?

Informally speaking, endomorphisms with non-trivial fixed points are rare. On the other hand, endomorphisms which are automorphisms, are also rare (in contrast to the situation in linear algebra). Whether or not automorphisms with non-trivial fixed points are rare, remains a mystery.

Finally, we point out a connection between the properties of metabelian IA-endomorphisms described above, and some properties of the Gassner representation of pure braid groups. For definitions of braids, braid groups, and for a background material, we refer to [4].

Let \( \sigma \in P_n \) be a pure braid on \( n \) strands, and \( \hat{\sigma} \) the corresponding link. Then, let \( \Gamma_{n-1}(\sigma) \) be the image of \( \sigma \) under the reduced Gassner representation of the pure braid group \( P_n \), and \( A_\sigma(t_1, \ldots, t_n) \) the Alexander polynomial of \( \hat{\sigma} \). Then (see [4, theorem 3-11]):

\[
A_\sigma(t_1, \ldots, t_n) = 0 \text{ if and only if } \det (\Gamma_{n-1}(\sigma) - I) = 0.
\]
To explain how this is connected to the subject of the present paper, we mention that the pure braid group $P_n$ is isomorphic to a subgroup of the group of IA-automorphisms of $F_n$, and the Gassner representation maps every IA-automorphism in this subgroup onto its abelianized Jacobian matrix. Then, the matrix $\Gamma_{n-1}(\sigma)$ that corresponds to the reduced Gassner representation, is an $(n-1) \times (n-1)$ matrix which is obtained from that abelianized Jacobian matrix by applying a suitable conjugation, and deleting the last row and the last column of the form $(0, \ldots, 0, 1)$.

Therefore, if we denote the free IA-automorphism corresponding (under the unreduced Gassner representation) to the braid $\sigma$ by the same letter, the above condition takes the form $\text{rank } (J_\sigma^a - I) \leq n - 2$, which points to part (ii) of our Theorem 1.

Thus, we have

Corollary 1.6. The Alexander polynomial of a link $\hat{\sigma}$ is zero if and only if the free automorphism which corresponds to the (pure) braid $\sigma$, has a non-trivial fixed point $g \in F'$ modulo $F''$, i.e. $\sigma(g) = g (\text{mod } F'')$.

2. Preliminaries

Let $ZF$ be the integral group ring of the group $F$ and $\Delta$ its augmentation ideal, that is, the kernel of the natural homomorphism $\epsilon: ZF \to Z$. More generally, when $R < F$ is a normal subgroup of $F$, we denote by $\Delta_R$ the ideal of $ZF$ generated by all elements of the form $(r-1)$, $r \in R$. It is the kernel of the natural homomorphism $\epsilon_R: ZF \to Z(F/R)$.

The ideal $\Delta$ is a free left $ZF$-module with a free basis $\{(x_i - 1)\}$, $1 \leq i \leq n$, and left Fox derivations $d_i$ are projections to the corresponding free cyclic direct summands. Thus any element $u \in \Delta$ can be uniquely written in the form $u = \sum_{i=1}^{n} d_i(u)(x_i - 1)$.

One can extend these derivations linearly to the whole $ZF$ by setting $d_i(1) = 0$.

The next lemma is an immediate consequence of the definitions.

Lemma 2.1. Let $J$ be an arbitrary right ideal of $ZF$, and let $u \in \Delta$. Then $u \in J\Delta$ if and only if $d_i(u) \in J$ for each $i$, $1 \leq i \leq n$.

Proof of the next lemma can be found in [6].

Lemma 2.2. Let $R$ be a normal subgroup of $F$, and let $y \in F$. Then $y - 1 \in \Delta_R \Delta$ if and only if $y \in R'$.

We also need the ‘chain rule’ for Fox derivations (see [6]):

Lemma 2.3. Let $\phi$ be an endomorphism of $F$ (it can be linearly extended to $ZF$) defined by $\phi(x_k) = y_k$, $1 \leq k \leq n$, and let $v = \phi(u)$ for some $u, v \in ZF$. Then:

$$d_j(v) = \sum_{k=1}^{n} \phi(d_k(u))d_j(y_k).$$

Lemma 2.3 implies the following product rule for the Jacobian matrices which looks exactly the same as in the ‘usual’ situation of analytic functions and Leibnitz derivations: if $\phi$ and $\psi$ are two endomorphisms of $F$, then

$$J_{\phi \psi} = \psi(J_{\phi}) \cdot J_{\psi}.$$
If furthermore \( \varphi \) and \( \psi \) are IA-endomorphisms, then considering abelianization of this product rule yields:

\[
J^a_{\varphi(\psi)} = J^a_{\varphi} \cdot J^a_{\psi}.
\]

In particular, there is a (faithful) representation of metabelian IA-automorphisms by matrices from \( GL_n(Z(F/F')) \) cf. [1].

Another version of Lemma 2 is: if \( u, v \in ZF \) and \( v = \varphi(u) \), then

\[
(d_1(v), \ldots, d_n(v)) = (\varphi(d_1(u)), \ldots, \varphi(d_n(u))) \cdot J^a_{\varphi}.
\]

Finally, we recall a well-known action via conjugation of a group ring \( ZF \) over the group ring \( rR \) by matrices from \( R/R' \). This product rule yields:

\[
\text{Lemma 2-4. For } r \in R, \ u \in Z(F/R), \text{ one has } d_i(r^u) = u \cdot d_i(r) \mod \Delta R.
\]

3. Proofs

Proof of Theorem 1.4. (i) Let \( \varphi \) be an IA-endomorphism of the group \( M; \varphi(x_i) = y_i, 1 \leq i \leq n \).

Then, by the definition of Fox derivatives, we have \( y_i - 1 = \sum_{k=1}^n d_k(y_k)(x_k - 1) \) for any \( i, 1 \leq i \leq n \).

Since \( \varphi \) is IA, abelianizing this equality gives \( x_i^a - 1 = \sum_{k=1}^n d_k^a(y_k)(x_k^a - 1) \), or, in the matrix form:

\[
((x_1^a - 1), \ldots, (x_n^a - 1)) \cdot J^a_{\varphi} \cdot ((x_1^a - 1), \ldots, (x_n^a - 1)) = 0,
\]

where \( (\cdot)^t \) means taking the transpose (i.e. we consider a column, not a row). This is equivalent to

\[
(J^a_{\varphi} - I) \cdot ((x_1^a - 1), \ldots, (x_n^a - 1)) = 0,
\]

hence the columns of the matrix \( (J^a_{\varphi} - I) \) are dependent, so \( \det (J^a_{\varphi} - I) = 0 \).

(ii) Let \( \varphi \) take \( x_i \) to \( x_i s_i, \ s_i \in M', 1 \leq i \leq n \). For notational convenience, consider the endomorphism \( \varphi \) being lifted to an endomorphism of \( F \). We shall use the same letter \( \varphi \) for this lifted endomorphism. When \( u \in F \), we denote the abelianization of \( u \) (i.e., the image in the group \( A = F/F' \)) by \( u^a \). This agreement also applies to elements of the group ring \( ZF \).

If \( \det (J^a_{\varphi} - I) = 0 \), then the rows of the matrix \( (J^a_{\varphi} - I) \) are dependent over the group ring \( ZA = Z(F/F') \). Make a new matrix \( J' \) upon multiplying the \( k \)th row of the matrix \( (J^a_{\varphi} - I) \) by \( x_k^a, 1 \leq k \leq n \). It is clear that the rows of \( J' \) are dependent over the group ring \( ZA \), too.

Now, \( J' \) is the abelianized Jacobian matrix of the endomorphism \( \varphi' \) that takes \( x_i \) to \( s_i, 1 \leq i \leq n \). Indeed, \( d_j(x_i s_i) = \delta_{ij} + x_i d_j(s_i), \) so that \( x_i d_j(\varphi'(x_i)) = d_j(\varphi(x_i)) - \delta_{ij} \).

Thus, the abelianized Jacobian matrix \( J^a_{\varphi} \) has dependent rows (over the group ring \( ZA \)). We can write this dependence as follows:

\[
\sum_{i=1}^n u_k \cdot d_i^a(s_k) = 0
\]
in $ZA$ for some $u_k \in ZA$, not all of them 0, and $i$ runs from 1 through $n$.

By Lemmas 2.1, 2.2, 2.4, the system (2) is equivalent to the following relation:

$$\prod_{k=1}^{n} s_{k}^{u_k} = 1$$

in the group $M$.

If rank $J_{\varphi} = m \leq n - 2$, then there are precisely $m$ independent elements (say, $s_1, \ldots, s_m$) among $s_k$ (in other words, these $s_1, \ldots, s_m$ generate a free submodule of the relation module of the group $A$), and for any $j, m + 1 \leq j \leq n$, we have a relation of the form

$$s_{j}^{w_{j}} = \prod_{k=1}^{m} s_{k}^{v_{kj}}$$

for some $w_{j}, v_{kj} \in ZA$.

Now, we are going to find a non-trivial fixed point $g \in M'$ of the endomorphism $\varphi$, in the form

$$g = [x_1, x_2]^{z_1} \cdot [x_2, x_3]^{z_2} \cdot \ldots \cdot [x_{n-1}, x_n]^{z_{n-1}}$$

for $z_k \in ZA$ (to be found).

We need a couple of simple observations:

**Lemma 3.1.** (i) Let $\varphi(g) = g$ for some $g \in M'$. Then $\varphi(g^u) = g^u$ for any $u \in ZA$.

(ii) For any element $h \in M'$, one has $h^w = g$ for some $w \in ZA$ and for $g \in M'$ of the form (4).

**Proof.** (i) Since $\varphi$ is IA, we have $\varphi(u) = u$ in $ZA$. The result follows.

(ii) It is clear that the elements $[x_i, x_j], 1 \leq i < j \leq n$, generate $M'$ as a normal subgroup of $M$. It is well-known that the elements $[x_1, x_2], \ldots, [x_{n-1}, x_n]$ generate a free submodule (of rank $(n - 1)$) of the relation module of the group $A$. On the other hand, there are no free submodule of rank $n$ in this relation module. This means for any pair $i, j$ of indices, one has $[x_i, x_j]^{v_{ij}} = [x_1, x_2]^{u_1} \cdot [x_2, x_3]^{u_2} \cdot \ldots \cdot [x_{n-1}, x_n]^{u_{n-1}}$ for some $v_{ij}, u_k \in ZA$. The result follows.

We continue now with part (ii) of Theorem 1.1. From $\varphi(g) = g$, we get

$$[s_1, x_2]^{z_1} \cdot [x_1, s_2]^{z_2} \cdot \ldots \cdot [s_{n-1}, x_n]^{z_{n-1}} \cdot [x_{n-1}, s_n]^{z_n} = 1,$$

or, equivalently:

$$s_{1}^{(x_1-1)z_1} \cdot s_{2}^{(x_2-1)z_2} \cdot \ldots \cdot s_{n-1}^{(x_{n-1}-1)z_{n-1}} \cdot s_{n}^{(1-x_n)z_n} = 1.$$  

(5)

Conjugate both sides of (5) by $w = \prod_{k=m+1}^{n} w_{j}$, where $w_{j} \in ZA$ come from (3). Then, replace every $s_{j}^{w_{j}}, j \geq m + 1$, in (5), with the corresponding element on the right-hand side of (3). This gives

$$s_{1}^{z'_1} \cdot s_{2}^{z'_2} \cdot \ldots \cdot s_{m}^{z'_m} = 1$$

(6)

for some $z'_k \in ZA$, each of which is a $ZA$-linear combination of $z_i, 1 \leq i \leq n - 1$.

Since we have chosen $s_1, \ldots, s_m$ so that they generate a free submodule of the relation module of the group $A$, the equation (6) is equivalent to a system of equations

$$z'_k = 0, \quad 1 \leq k \leq m.$$  

(7)
This is a system of \( m \) homogeneous \( ZA \)-linear equations in \((n - 1) > m \) unknowns \( z_1, \ldots, z_{n-1} \). It is well known that a system like that has a non-trivial solution over \( ZA \) (since \( ZA \) is a commutative domain). This completes the proof.

(iii) The `if` part is obvious. The `only if` part follows from Lemma 3·1.

The following example shows how subtle a situation might be in the case when rank \((J^a_{\varphi} - I)\) = \( n - 1 \).

Proposition 3·2. In the group \( M_2 \), let \( \varphi \) take \( x_1 \) to \( x_1s \), \( x_2 \) to \( x_2s^{-1} \) for some \( s \in M_2 \), \( s \neq 1 \). Then \( \varphi \) has no non-trivial fixed points.

Proof. First we show that \( \varphi \) has no non-trivial fixed points in \( M_2 \). Suppose \( \varphi(g) = g; \ g \in M_2 \). Since \( g \) has the form \([x_1, x_2]^\varphi\) for some \( z \in ZA \), we have (cf. (5)):

\[
[x_1, x_2]^\varphi = [x_1s, x_2s^{-1}]^z = [x_1, x_2 s^{-1} z (x_2-1) u(z_1)]^z.
\]

(8)

It follows that \( s(x_2-1) z + (x_2-1) z = (x_2+1-2) z = 0 \) in the group ring \( ZA \), which is only possible if \( z = 0 \). Thus, \( g = 1 \).

Now let \( g \in M_2 \) be an arbitrary element. Then \( g \) has the form \([x_1^m x_2^n]^z \) for some \( z \in ZA \). Let us see first what the image of \([x_1^m x_2^n]^z \) looks like.

We have: \( \varphi(x_1^m x_2^n) = x_1^m \cdot s \cdot \ldots \cdot x_1^m \cdot s \cdot x_2^n \cdot s^{-1} \cdot \ldots \cdot x_2^n \cdot s^{-1} = h \). Now we apply to this element \( h \) the following `collecting process`: first we collect all the \( x_1 \) on the left by permuting them with \( s \). This gives: \( h = x_1^m \cdot s \cdot x_1^m \cdot s^{-1} \cdot \ldots \cdot s \cdot x_1^m \cdot s^{-1} \cdot \ldots \cdot x_2^n \cdot s^{-1} \).

Then, in the same manner, we collect all the \( x_2 \) on the right of \( x_1^m \):

\[
h = x_1^m x_2^n \cdot s^{x_1^m x_2^n} \cdot \ldots \cdot s^{x_1^m x_2^n} \cdot s^{x_2^n} \cdot s^{-1} \cdot \ldots \cdot s^{-x_2^n} \cdot s^{-1}.
\]

or, equivalently:

\[
h = x_1^m x_2^n \cdot s^{x_1^m x_2^n + \ldots + x_2^n + x_2^n - x_2^n - \ldots - x_2^n - 1}.
\]

(9)

Now write down the whole image of \( g; \ \varphi(x_1^m x_2^n) = (x_1^m x_2^n)^z \) = \( h \cdot [x_1, x_2^n] \cdot s^{x_1^m x_2^n + \ldots + x_2^n + x_2^n - x_2^n - \ldots - x_2^n - 1} (\text{cf. (8)}). \)

\( \varphi(g) = g \), then combining this with (9) yields:

\[
s^{x_1^m x_2^n + \ldots + x_2^n + x_2^n - x_2^n - \ldots - x_2^n - 1 + (x_2 + x_1 - 2) z} = 1,
\]

or, equivalently:

\[
x_1^m x_2^n + \ldots + x_1 x_2^n + x_2^n - x_2^n - \ldots - x_2^n - 1 + (x_2 + x_1 - 2) z = 0.
\]

(10)

All we have to do now is to show that \( w = x_1^m x_2^n + \ldots + x_1 x_2^n + x_2^n - x_2^n - \ldots - x_2^n - 1 \) is not divisible by \( v = x_2 + x_1 - 2 \) in the ring \( ZA \). This is easy to see upon setting \( x_1 = -1 \); then \( w' = (-1)^n x_2^n - x_2^n - \ldots - x_2^n - 1 \); \( v' = x_2 - 3 \). Since \( x_2 = 3 \) is not a root of the polynomial \( w' \), the result follows. This completes the proof of Proposition 3·2.

Proof of Theorem 1·2. Let \( \varphi \) be an IA-endomorphism of the group \( M = M_2 \).

First of all, we compute the rank of the matrix \( J^a_{\varphi} \). If it is not equal to \((n - 1)\), then we just refer to Theorem 1·1 (i), (ii).

Suppose rank \((J^a_{\varphi} - I)\) = \( n - 1 \). To find out if there is a non-trivial fixed point of \( \varphi \) inside the commutator subgroup \( M' \), we consider a system (7); but this time, it is a system of \((n - 1)\) homogeneous \( ZA \)-linear equations in \((n - 1)\) unknowns \( z_1, \ldots, z_{n-1} \). To find out if it has a non-trivial solution (which happens if and only if \( \varphi \) has a non-trivial fixed point inside \( M' \)), we just compute the corresponding determinant and see if it is equal to 0.
A somewhat more difficult problem is to find out if there is a non-trivial fixed point of $\varphi$ outside $M'$. In this case, we proceed as in the proof of Proposition 3.2, but instead of having just one equation of the form (10), we’ll have a system of $(n-1)$ $ZA$-linear equations (they are no longer homogeneous!) in $(n-1)$ unknowns.

Again, since $ZA$ is a commutative domain, we can resolve this system (by using Kramer’s formula). To apply Kramer’s formula, all we need is to be able to find out for a given pair of polynomials, whether or not one of them is divisible by another. Algorithms like that do exist (in particular, for $ZA$); they are based on what is known as Gröbner reduction process.

In any case, the existing algorithms in (Laurent) polynomial algebras not only tell us whether or not a given system of $ZA$-linear equations has a solution, but if it does, they give a solution (although we may not be able to find all of them). This completes the proof of Theorem 1.2.

**Proof of Proposition 1.3.** First of all, we note that the group $\text{Fix} \varphi \cap M'$ is abelian; therefore, if it were finitely generated, then its every subgroup would be finitely generated, too.

Now suppose $g \in \text{Fix} \varphi \cap M'$, $g \neq 1$. Then, by Lemma 3.1 (ii), $g^m \in \text{Fix} \varphi \cap M'$ for any $u \in ZA$ (i.e. $\text{Fix} \varphi \cap M'$ is a normal subgroup of $M$). We are going to show that the subgroup of $M'$ generated by all $g^k$, $k \in Z$, is not finitely generated.

By way of contradiction, suppose it is generated by $g^{x_1}, \ldots, g^{x_n}$, $m > 0$. But every element from this finitely generated group has a form $g^p$ for some Laurent polynomial $p = p(x_1)$, whose degree does not exceed $m$. Therefore, we don’t have the element $g^{x_1^m}$ in this group, hence a contradiction.

**Proof of Theorem 1.4.** By way of contradiction, suppose $H = \text{Fix} \varphi$ is a non-cyclic subgroup of $M$ generated by $h_1, \ldots, h_r$, $r < n$. Then there is a generator of the group $M$, say $x_1$, such that $x_1^m \notin H \cdot M'$ for any $m \geq 1$.

By Lemma 3.1 (i), the group $S = H \cap M'$ is normal in $M$. If $H$ is non-cyclic, then $S$ is non-trivial since it contains a non-trivial subgroup $H'$ (note that $H \nless M'$ since otherwise, $H$ would be infinitely generated by Proposition 1.3). Let $s \in S$, $s \neq 1$. Then for any $m \geq 1$, we have an equality of the form

$$\prod_j h_{ij}^{c_{ij,m}} = s^{x_1^m}$$

for some integers $c_{ij,m}$. Let $d_k^q(s) \neq 0$; then (by Lemma 2.4) applying the derivation $d_k^q$ to both sides of the last equality gives (in the group ring $ZA$):

$$\sum_q \prod_j h_{ij}^{c_{ij,m}} \cdot c_{ij,m} \cdot d_k^q(h_{ij}) = x_1^m \cdot d_k^q(s)$$

(11)

for some collection of indices $j, q$ (of no particular importance to us).

When $m$ runs through 1 to $\infty$, we are encountering representatives of infinitely many distinct cosets of the group $H \cdot M'/M'$ (as a subgroup of $M/M'$) in the supports of elements on the right-hand side of (11). This is due to the condition $x_1^m \notin H \cdot M'$ for any $m \geq 1$ – see above. (By support of a group ring element $u \in ZG$, $u = \sum c_y \cdot g$, we mean the set $\{g, c_y = 0\}$.)

At the same time, the collection of coset representatives of $H \cdot M'/M'$ in the
supports of elements on the left-hand side of (11) does not depend on \( m \), and is therefore finite. This contradiction completes the proof of Theorem 1-4.

**Proof of Proposition 1-5.** Consider an (IA-)automorphism \( \varphi \) of the group \( M_n, n \geq 3 \), given by \( \varphi(x_i) = x_i[x_2, x_3, x_1]; \varphi(x_i) = x_i, i \geq 2 \). We are going to prove that the group \( \text{Fix} \varphi \) is infinitely generated.

It is clear that \( \text{Fix} \varphi \) contains a subgroup of \( M_n \) generated by \( x_2, \ldots, x_n \). Now we are going to look for (other) fixed points in the form

\[
g = x_1^k \cdot [x_1, x_2]^{u_1} \cdot [x_2, x_3]^{u_2} \cdot \ldots \cdot [x_{n-1}, x_n]^{u_{n-1}} \cdot x_2^{m_1} \cdot \ldots \cdot x_n^{m_{n-1}}.
\]

Starting with \( \varphi(g) = g \) and arguing along the same lines as in the proof of Theorem 1-4 (ii) and Proposition 3-2, we finally arrive at

\[
x_1^k - 1 + (x_1 - 1)(x_2 - 1)\cdot u_1 = 0,
\]

which is only possible if \( k = u_1 = 0 \).

It follows that \( x_1^m \notin \text{Fix} \varphi \cdot M' \) for any \( m \geq 1 \), and applying the argument from the proof of Theorem 1-4 yields the result.

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**REFERENCES**

[1] S. Bachmuth. Automorphisms of free metabelian groups. *Trans. Amer. Math. Soc.* 118 (1965), 93–104.

[2] M. Bestvina and M. Handel. Train tracks and automorphisms of free groups. *Ann. of Math.* 135 (1992), 1–53.

[3] J. S. Birman. An inverse function theorem for free groups. *Proc. Amer. Math. Soc.* 41 (1973), 634–638.

[4] J. S. Birman. Braids, links and mapping class groups. *Ann. Math. Studies* 82 (Princeton Univ. Press, 1974).

[5] D. J. Collins and E. C. Turner. Free product fixed points. *J. London Math. Soc.* (2) 38 (1988), 67–76.

[6] R. H. Fox. Free differential calculus. I. Derivation in the free group ring. *Ann. of Math.* 57 (1953), 547–560.

[7] C. K. Gupta, N. D. Gupta and G. A. Noskov. Some applications of Artamonov-Quillen-Suslin theorems to metabelian inner rank and primitivity. *Canad. J. Math.* 46 (1994), 298–307.

[8] N. Gupta. Free group rings. *Contemporary Math.* 66 (1987).

[9] W. Imrich and E. C. Turner. Endomorphisms of free groups and their fixed points. *Math. Proc. Cambridge Phil. Soc.* 105 (1989), 421–422.

[10] V. A. Roman’kov. Criteria for the primitivity of a system of elements of a free metabelian group. *Ukrain. Math. J.* 43 (1991), 996–1002.

[11] Y. Shul’pin. On monomorphisms of free groups. *Arch. Math.* 64 (1995), 465–470.

[12] U. U. Umirbaev. Primitive elements of free groups. *Russian Math. Surveys* 49 (1994), 184–185.