Abstract: We introduce a class of regularisable infinite dimensional principal fibre bundles which includes fibre bundles arising in gauge field theories like Yang-Mills and string theory and which generalise finite dimensional Riemannian principal fibre bundles induced by an isometric action. We show that the orbits of regularisable bundles have well defined, both heat-kernel and zeta function regularised volumes. We introduce two notions of minimality (which extend the finite dimensional one) for these orbits, using both heat-kernel and zeta function regularisation methods and show they coincide. For each of these notions, we give an infinite dimensional version of Hsiang’s theorem which extends the finite dimensional case, interpreting minimal orbits as orbits with extremal (regularised) volume.

0. Introduction

This article is concerned with the notions of regularisability and minimality of orbits for an isometric action of an infinite dimensional Lie group $G$ on an infinite dimensional manifold $\mathcal{P}$. Our study is based on heat-kernel regularisation methods. Notions of regularisability and minimality have already been studied by other authors (see [KT], [MRT]) in a particular context and using zeta function regularisation methods. We shall confront the different approaches to these notions as we go along.

We shall introduce a class of principal fibre bundles called (resp. pre-)regularisable fibre bundles which generalise to the infinite dimensional case finite dimensional Riemannian principal fibre bundles arising from a free isometric action. We show that the fibres of these (resp. pre-)regularisable bundles have a well defined (both heat-kernel and zeta function) regularised (resp. preregularisable) volume which is Gâteaux differentiable. This class of (pre-) regularisable fibre bundles includes some infinite dimensional principal bundles arising from gauge field theories such as Yang-Mills and string theory.

We introduce various notions of minimality, heat-kernel minimality and strong heat-kernel minimality using heat kernel regularisation methods on one hand and zeta function minimality, using zeta function regularisation methods on the other hand, all of which extend the finite dimensional notion and coincide in the finite dimensional case. Whenever the structure group is equipped with a fixed Riemannian metric, we show that (strongly) minimal fibres of a (pre-)regularisable principal fibre bundle coincide with the ones with extremal (pre-)regularised volume among orbits of the same type for the group action,
the regularisation being taken in the heat-kernel sense. This gives an infinite dimensional version of Hsiang’s theorem on (pre-) regularisable principal fibre bundles with structure group equipped with a fixed Riemannian metric, which we extend (adding a term which reflects the variation of the metric on the structure group) to any (pre-)regularisable principal bundle.

Starting from a systematic review of the notions of heat-kernel and zeta-function regularised determinants in section I, in section II we introduce the notions of regularisable principal fibre bundle, heat-kernel (pre) regularisability and heat-kernel (strong) minimality of orbits, relating (strong) minimality with the Gâteaux-differentiability of heat-kernel (pre-)regularised determinants interpreted as volumes of fibres. In section III, we compare these notions to zeta-function regularisability and minimality, relating the latter to Gâteaux-differentiability of zeta-function regularised determinants. We show that the two notions of regularisability and minimality coincide on the class of fibre bundles we consider. The relations we set up between the regularised mean curvature vector and the directional gradients of the regularised determinants yield an infinite dimensional version of Hsiang’s theorem from both the heat-kernel and the zeta function point of view. In Appendix A, we apply these results to the coadjoint action of a loop group thus recovering some results concerning regularisability and minimality of fibres studied in [KT]. In Appendix B, we investigate minimality of the orbits in the case of Yang-Mills action, for which a notion of (zeta function) minimality had been suggested in [MRT] from which our notion differs slightly. We point out the fact that when the underlying manifold is of dimension 4, only if the irreducible connections are Yang-Mills, do the notion of zeta function and heat-kernel minimality coincide. In both examples, the space $\mathcal{P}$, resp. the group $\mathbf{G}$ are modelled on a space of sections of a vector bundle $\mathcal{E}$, resp. $\mathcal{F}$ with finite dimensional fibres on a closed finite dimensional manifold $M$ and $\mathbf{G}$ acts on $\mathcal{P}$ by isometries.

One could show, in a similar way to the Yang-Mills case, using results of [RS] that the bundle $\mathcal{M}_{-1} \to \mathcal{M}_{-1}/\text{Diff}_0$ (described in [FT]) arising in bosonic string theory (where $\mathcal{M}_{-1}$ is the manifold of smooth Riemannian metrics with curvature $-1$ on a compact boundaryless Riemannian surface of genus greater than 1 and $\text{Diff}_0$ is the group of smooth diffeomorphisms of the surface which are homotopic to zero), is also a regularisable fibre bundle so that most results of this paper can be applied to this fibre bundle. However since, unlike the case of Yang-Mills theory, its structure group $\text{Diff}_0$ is not equipped with a fixed Riemannian structure but with a family of Riemannian metrics which is parametrised by $g \in \mathcal{M}_{-1}$, minimality of the fibres is not equivalent to extremality of the volumes of the fibres (see Proposition 2.2), and we chose not to treat this example in detail in this paper.

The geometric notions developed in this paper play a important role when projecting a class of semi-martingales defined on the total manifold onto the orbit space for a certain class of infinite dimensional group actions. The heat-kernel regularisation method yields natural links between the geometric and the stochastic picture, which we investigate in [AP2]. The stochastic picture described in [AP2] leads to a stochastic interpretation of the Faddeev-Popov procedure used in gauge field theory to reduce a formal volume measure on path space to a measure on the orbit space, the formal density of which is a regularised “Faddeev-Popov” determinant.
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I. Heat-kernel and Zeta-function regularized determinants

In this section, we recall some basic facts about heat-kernel and zeta function regularised determinants, comparing the two regularisations. Although the results presented here are well known and frequently used in the physics literature, it seemed necessary to us to give a clear and precise presentation of the heat-kernel and zeta function regularisation procedures for later use.

Let us first introduce some notations. For a function \( f(t) \), defined on an interval of \( \mathbb{R}^+ \) containing \([-1, 1]\), we shall write

\[
|f(t) - \sum_{j=-J}^{-1} a_j t^m| < Ct^K \quad \forall \ 0 < t < 1
\]

(1.0)

In the following, we shall always assume that \( J \geq m \).

Lemma 1.0: Let \((A_\varepsilon), \varepsilon \in [0, 1]\) be a one parameter family of trace-class operators on a separable Hilbert space \( H \) (in particular \( \text{tr}(A_1) \) is finite) such that:

1) \( \varepsilon \to \text{tr}A_\varepsilon \) is differentiable on \([0, 1]\)

2) \( \exists J \in \mathbb{N}, m \in \mathbb{N}^*, (a_j)_{j\in\{-J, \ldots, -1\}}, a_j \in \mathbb{R} \) such that

\[
\frac{d}{d\varepsilon}(\text{tr}A_\varepsilon) \simeq_0 \sum_{j=-J}^{-1} \varepsilon^\frac{m}{j} a_j,
\]

(1.1)

Then the expression \( \text{tr}A_\varepsilon - \sum_{j=-J+m}^{-1} \frac{ma_j - m}{j} \varepsilon^\frac{1}{m} - a_m \log \varepsilon \) converges when \( \varepsilon \to 0 \).

Remark: In the following, we shall not distinguish the two cases and adopt the convention that the sum from \(-J + m\) to \(-1\) is zero whenever \( J = m \).

Proof: To show this, let us set for \( 0 < \varepsilon < 1 \), \( g_\varepsilon = \text{tr}A_\varepsilon - \sum_{j=-J+m}^{-1} \frac{ma_j - m}{j} \varepsilon^\frac{1}{m} - a_m \log \varepsilon \). For \( 0 < \varepsilon < \varepsilon' < 1 \), we have:

\[
|g_\varepsilon - g_{\varepsilon'}| \leq \int_\varepsilon^{\varepsilon'} \left| \frac{d}{dt}(\text{tr}A_t) - \sum_{j=-J}^{-1} t^\frac{m}{j} a_j \right| dt
\]

\[
\leq C(\varepsilon' - \varepsilon) \leq C\varepsilon' \quad \text{by (1.1)}
\]
so that \((g_\varepsilon)\) is a Cauchy sequence and hence converges when \(\varepsilon \to 0\). From the convergence of \(g_\varepsilon\) then follows the convergence of \(\text{tr} A_\varepsilon - \sum_{j=-J+m \neq 0}^{-1} \frac{m_{j-m}}{j} \varepsilon^\frac{j}{m} - a_m \log \varepsilon\) when \(\varepsilon\) goes to zero since the terms indexed by 0, 1, \(\cdots\), \(-1 + m\) converge when \(\varepsilon \to 0\).

**Definition:** Whenever \(\text{tr} A_\varepsilon - \sum_{j=-J+m}^{-1} \frac{m_{j-m}}{j} \varepsilon^\frac{j}{m} - a_m \log \varepsilon\) converges when \(\varepsilon \to 0\). We shall call the limit the *regularized limit trace* of the family \(A = (A_\varepsilon)\) and denote it by \(\text{tr}_{\text{reg}}(A)\) so that

\[
\text{tr}_{\text{reg}}(A) = \lim_{\varepsilon \to 0} \left( \text{tr} A_\varepsilon - \sum_{j=-J+m}^{-1} \frac{m_{j-m}}{j} \varepsilon^\frac{j}{m} - a_m \log \varepsilon \right)
\]

(1.2)

This regularised limit trace depends of course on the whole one parameter family and on the choice of the parameter \(\varepsilon\).

Let \(B = (B_\varepsilon)\) be a one parameter family of strictly positive self adjoint operators such that \(A = (A_\varepsilon)\) with \(A_\varepsilon \equiv \log B_\varepsilon\) is a family of trace class operators. We can define the determinant of \(B_\varepsilon\) as \(\det B_\varepsilon = e^{\text{tr} \log B_\varepsilon}\). If the family \(A = (A_\varepsilon)\) has a regularized limit trace, we shall define the *regularized limit determinant* of the family \(B\) by

\[
\det_{\text{reg}} B \equiv e^{\text{tr}_{\text{reg}}(A)}
\]

(1.3)

We now introduce a family of heat-kernel operators which play a fundamental role in this paper. For this we define for \(\varepsilon > 0\) a function \(h_\varepsilon\) by:

\[
h_\varepsilon : \mathbb{R}^+ \to \mathbb{R} \\
\lambda \mapsto e^{-\frac{\int_{\varepsilon}^{\infty} e^{-t\lambda} dt}{\varepsilon}}
\]

Notice that \(h_\varepsilon\) is \(C^\infty\), non decreasing and \((\log h_\varepsilon)'(\lambda) = \lambda^{-1} e^{-\varepsilon \lambda}\). Writing \(\log h_\varepsilon(\lambda) - \log \varepsilon = -\int_{\varepsilon}^{\infty} \frac{e^{-t}}{t} dt - \int_{\varepsilon}^{\lambda} \frac{e^{-\lambda t}}{t} dt + \int_{\varepsilon}^{1} \frac{1}{t} dt\), we find that

\[
\lim_{\varepsilon \to 0} \frac{h_\varepsilon(\lambda)}{\varepsilon} = \lambda e^{\int_{0}^{1} \frac{1-e^{-t}}{t} dt - \int_{1}^{\infty} \frac{e^{-t}}{t} dt}
\]

(1.3bis)

For a strictly positive self adjoint operator \(B\) on a Hilbert space \(H\), we can define \(h_\varepsilon(B)\) which yields a one parameter family of operators \(B_\varepsilon \equiv h_\varepsilon(B)\).

**Definition:** If \(\log h_\varepsilon(B)\) is trace class, we can define

\[
\det_{\varepsilon}(B) = e^{\text{tr} \log h_\varepsilon(B)}
\]

(1.4)
**Definition:** Let $B$ be a strictly positive self-adjoint operator on a separable Hilbert space. Whenever the one parameter family $B = (h_\varepsilon(B))$ has a regularized limit determinant, we shall call this limit the *heat-kernel regularized determinant* of $B$ and we denote it by $\det_{reg}(B)$.

We have

$$\det_{reg} B = \det_{reg}(B) = e^{\text{tr}_{reg}(A)}$$

with $B = (h_\varepsilon(B))$, $A = (\log h_\varepsilon(B))$.

In the following, we give conditions under which we can define the heat-kernel regularized determinant of an operator $B$. But before that, let us state an easy lemma which will prove to be useful for what follows.

**Lemma 1.1:** Let $B$ be a strictly positive self-adjoint operator on a separable Hilbert space such that $e^{-\varepsilon B}$ is trace class for some $\varepsilon > 0$. Then $A_\varepsilon \equiv \log h_\varepsilon(B)$ is also trace-class.

**Proof:** Since $e^{-\varepsilon B}$ is trace class, it is compact and hence has a purely discrete spectrum $\{\mu_n, n \in \mathbb{N}\}$, $\mu_n > 0$, $\mu_n$ tending to zero. Hence $B = -\varepsilon^{-1} \log e^{-\varepsilon B}$ also has purely discrete spectrum $\lambda_n = -\varepsilon^{-1} \log \mu_n$ which tends to infinity and is bounded from below by a strictly positive constant. Let $\Lambda_{n_0}$ be a non zero eigenvalue and let us index the eigenvalues in increasing order so that $\lambda_n \leq \lambda_{n+1}$. We have:

$$\sum_{n \geq n_0} \int_{\varepsilon}^{\infty} e^{-t\lambda_n} \frac{1}{t} dt \leq \varepsilon^{-1} \sum_{n \geq n_0} \lambda_n^{-1} e^{-\varepsilon \lambda_n}$$

$$\leq \varepsilon^{-1} \lambda_0^{-1} \sum_{n \geq n_0} e^{-\varepsilon \lambda_n}$$

The last expression is finite by assumption so that $\sum_{n \geq n_0} \left| \int_{\varepsilon}^{\infty} e^{-t\lambda_n} \frac{1}{t} dt \right|$ is finite and $A_\varepsilon$ is trace class.

**Lemma 1.2:** Let $B$ be a strictly positive self-adjoint operator on a separable Hilbert space such that $e^{-\varepsilon B}$ is trace class for any $\varepsilon > 0$.

1) $e^{-\varepsilon B}$ is trace class for any $\varepsilon > 0$.

2) There is a family $(b_j)_{j=-J,\ldots,0}$, $b_j \in \mathbb{R}$ and an integer $m > 0$ such that

$$\text{tr} e^{-\varepsilon B} \simeq 0 \sum_{j=-J}^{-1} b_j \varepsilon^\frac{j}{m}.$$ 

Then the operator $B$ has a heat-kernel regularised determinant and we have

$$\det_{reg} B = \lim_{\varepsilon \to 0} \left( \det_{\varepsilon} Be^{-\sum_{j=-J}^{-1} \frac{mb_j}{m} \frac{j}{m} - b_0 \log \varepsilon} \right)$$

$$= e^\left(-\sum_{j=-J, j \neq 0}^{m-1} \frac{mb_j}{m} \int_{\varepsilon}^{\infty} \text{tr} e^{-tB} \frac{1}{t} dt - \int_{0}^{1} \frac{F(t)}{t} dt \right)$$

(1.6)
with
\[ F(t) = \text{tr} e^{-tB} - \sum_{j=-J}^{m-1} b_j t^{\frac{j}{m}}. \] (1.7)

**Remark:** If the Hilbert space \( H \) the operator \( B \) acts on is finite dimensional of dimension \( d \), since \( \lim_{\varepsilon \to 0} \text{tr} e^{-\varepsilon B} = d = b_0 \), (1.6) yields \( \det_{\text{reg}} B = \lim_{\varepsilon \to 0} (\det \varepsilon B \varepsilon^{-d}) \).

**Proof:** By Lemma 1.1, we know that \( A_\varepsilon = -\int_{\varepsilon}^{\infty} \frac{e^{-tB}}{t} dt = \log h_\varepsilon(B) \) is trace-class. Since all the terms involved are positive, we can exchange the integral and sum symbols so that \( \text{tr} A_\varepsilon = -\int_{\varepsilon}^{\infty} \text{tr} e^{-tB} dt \). We now apply lemma 1.0 to show that the family \( A_\varepsilon \) has a regularized limit trace. The map \( t \to \text{tr} A_t \) is differentiable and by assumption 2), for \( t > 0 \), we have
\[ \frac{d}{dt} \text{tr} A_t = \text{tr} \frac{e^{-tB}}{t} = \sum_{j=-J}^{m-1} b_j t^{\frac{j}{m}} + F(t) \]
(1.8)
where \( F \) is as in (1.7) and \( |F(t)| \leq Ct \). Thus, taking \( K = 0 \) in (1.0), we have:
\[ \frac{d}{dt} \text{tr} A_t \equiv \sum_{j=-J}^{m-1} b_j t^{\frac{j}{m}} \]
so that we can apply Lemma 1.0 from which follows (replacing \( J \) by \( J + m \) and \( a_j \) by \( b_{j+m} \)) that the one parameter family \( A = (A_\varepsilon) \) has a finite regularized limit trace \( \text{tr}_{\text{reg}}(A) \equiv \lim_{\varepsilon \to 0} (\text{tr} A_\varepsilon - \sum_{j=-J}^{m-1} \frac{mb_j}{j} \varepsilon^{\frac{1}{m}} - b_0 \log \varepsilon) \). Hence, by (1.5) \( B \) has a heat-kernel regularized determinant:
\[ \det_{\text{reg}} B = e^{\text{tr}_{\text{reg}} A} = \lim_{\varepsilon \to 0} \left( e^{\text{tr} A_\varepsilon} e^{-\sum_{j=-J}^{m-1} \frac{mb_j}{j} \varepsilon^{\frac{1}{m}} - b_0 \log \varepsilon} \right) \]
\[ = \lim_{\varepsilon \to 0} \left( \det \varepsilon B e^{-\sum_{j=-J}^{m-1} \frac{mb_j}{j} \varepsilon^{\frac{1}{m}} - b_0 \log \varepsilon} \right). \]

Since \( \text{tr} A_1 = -\int_{1}^{\infty} \frac{e^{-tB}}{t} dt \), integrating (1.8) between \( \varepsilon \) and 1 yields
\[ \text{tr} A_\varepsilon - \sum_{j=-J}^{m-1} \frac{mb_j}{j} \varepsilon^{\frac{1}{m}} - b_0 \log \varepsilon = -\sum_{j=-J}^{m-1} \frac{mb_j}{j} - \int_{\varepsilon}^{1} \frac{F(t)}{t} dt - \int_{1}^{\infty} \frac{e^{-tB}}{t} dt. \] (1.9)
Since \( m \geq 1 \), we have
\[ \lim_{\varepsilon \to 0} \left( \text{tr} A_\varepsilon - \sum_{j=-J}^{m-1} \frac{mb_j}{j} \varepsilon^{\frac{1}{m}} - b_0 \log \varepsilon \right) = \lim_{\varepsilon \to 0} \left( \text{tr} A_\varepsilon - \sum_{j=-J}^{m-1} \frac{mb_j}{j} \varepsilon^{\frac{1}{m}} - b_0 \log \varepsilon \right) \]
which combined with (1.9) and using (1.4) yields (1.6).
Notice that the integral $\int_0^1 \frac{F(t)}{t} dt$ is absolutely convergent. Indeed, by assumption 2), $|F(t)| \leq Ct$ for some positive constant $C$. 

The following lemma gives a class of operators which fit in the framework described above.

**Lemma 1.3:** Let $B$ be a strictly positive self adjoint elliptic operator of order $m > 0$ on a compact boundaryless manifold. For any $\varepsilon > 0$, $e^{-\varepsilon B}$ is trace class and $B$ has a well defined heat-kernel regularised determinant.

**Proof:** We shall show that the assumptions of Lemma 1.2 are fulfilled.

Condition 1) in Lemma 1.2 follows from the fact that a strictly positive s.a elliptic operator on a compact boundaryless manifold has purely discrete spectrum $(\lambda_n)_{n \in \mathbb{N}}$, $\lambda_n > 0$, $\lambda_n \simeq C n^\alpha$, for some $C > 0$, $\alpha > 0$ (see e.g [G] Lemma 1.6.3). Indeed, from this fact easily follows that $\text{tr} e^{-\varepsilon B} = \sum_n e^{-\varepsilon \lambda_n}$ is finite.

Conditions 2) of Lemma 1.1 follow from the fact that for a s.a elliptic operator $B$ of order $m$ on a compact manifold of dimension $d$ without boundary, $\text{tr} e^{-tB} \simeq 0 \sum_{j=0}^{K-1} a_j t^{\frac{m}{m}}$ for any $K > 0$ (this follows for example from Lemma 1.7.4 in [G]). Applying lemma 1.0, we can therefore define the heat-kernel regularized determinant of $B$.

The above definition extends to a class of positive self-adjoint operators which satisfy requirements 1) and 2) of Lemma 1.2 and have possibly non zero kernel. Requirement 1) of the lemma implies that this kernel is finite dimensional. Let $P_B$ the orthogonal projection onto the kernel of the operator $B$ acting on $H$ and let us set $H^\perp \equiv (I - P_B)H$. Let us assume that $H^\perp$ is invariant under the action of $B$ so that we can consider the restriction $B' \equiv B/H^\perp$. The operator $B'$ satisfies requirements of Lemma 1.2, namely

1) $e^{-\varepsilon B'}$ is trace class for any $\varepsilon > 0$.

2) There is a family $(b'_j)_{j=-J,-J+1, \ldots, 0}$, $b'_j \in \mathbb{R}$ and an integer $m > 0$ such that

$$\text{tr} e^{-\varepsilon B'} \simeq 0 \sum_{j=-J}^{m-1} b'_j \varepsilon^{\frac{d}{m}}.$$

where $b'_j = b_j$ for $j \neq 0$ and $b'_0 = b_0 - \dim \text{ker} B$.

Under assumption 1), we can extend definition (1.4) and define:

$$\det' \varepsilon B \equiv e^{\text{tr log} h_{\varepsilon}(B')}$$

Under assumptions 1) and 2), the operator $B'$ has a heat-kernel regularised determinant

$$\det_{reg} B \equiv \lim_{\varepsilon \to 0} \left( \det_{\varepsilon} B' e^{-\sum_{j=-J}^{m-1} \frac{mb_j}{m} \varepsilon^{\frac{d}{m}} - b_0 \log \varepsilon + (\dim \text{ Ker} B) \log \varepsilon} \right)$$
Let us at this stage compare the heat-kernel regularised determinant with the zeta-function regularised one. We refer the reader to [AJPS], [G] for a precise description of the zeta-function regularisation procedure and only describe the main lines of this procedure here.

Recall that for a strictly positive self-adjoint operator $B$ acting on a separable Hilbert space with purely discrete spectrum given by the eigenvalues $(\lambda_n, n \in \mathbb{N})$ with the property $\lambda_n \geq C n^\alpha, C > 0, \alpha > 0$ for large enough $n$, we can define the zeta function of $B$ by:

$$
\zeta_B(s) \equiv \sum_n \lambda_n^{-s}, \quad s \in \mathbb{C}, \quad \text{Re } s > \frac{1}{\alpha}
$$

Furthermore, $\zeta_B(s)$ admits a meromorphic continuation on the whole plane (see e.g [G] Lemma 1.10.1) which is regular at $s = 0$ and one can define the zeta function regularized determinant of $B$ by

$$
\det_{\text{reg}}(B) = e^{-\zeta_B'(0)} \quad (1.10
$$

**Remark**: From the definition, easily follows that in the finite dimensional case the zeta-function regularised and the ordinary determinants coincide.

The following lemma compares the two regularizations.

**Lemma 1.4**: Let $B$ be a strictly positive self-adjoint densely defined operator on a Hilbert space $H$ such that

1) $B$ has purely discrete spectrum $(\lambda_n)_{n \in \mathbb{N}}$ with $\lambda_n \geq C n^\alpha, C > 0, \alpha > 0$ for large enough $n$,

2) $
\exists J > 0, m > 0, (b_j)_{j=-J,\ldots,-m} \quad \text{such that } \quad \text{tr} e^{-\varepsilon B} \simeq 0 \sum_{j=-J}^{m-1} b_j \varepsilon^{\frac{j}{m}}
$

Then

$$
\det_{\text{reg}} B = e^{-\gamma b_0} (\det_{\text{reg}} B) = e^{-(\gamma b_0 + \sum_{j=J}^{m-1} \frac{mb_j}{j} + \int_1^{\infty} \text{tr} e^{-t B} \frac{dt}{t} + \int_0^1 \frac{F(t)}{t} dt)} \quad (1.10\text{bis})
$$

where $\gamma = \lim_{n \to \infty} (1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n)$ is the Euler constant and $b_0$ is the coefficient arising in the heat-kernel expansion of $B$,

$$
\text{tr} e^{-\varepsilon B} \simeq 0 \sum_{j=-J}^{m-1} b_j \varepsilon^{\frac{j}{m}}
$$

for some $J \in \mathbb{N}, m > 0$.

**Remark**: A proof of this result for the Laplace operator on a compact Riemannian surface without boundary can be found in [AJPS].
Proof: Before starting the proof, let us recall that the function Gamma is defined by 
\[ \Gamma(z) = \int_0^\infty \frac{e^{-t}t^z}{t} dt \] for \( 0 < \text{Re} z \). Moreover \( \Gamma(z)^{-1} \) is an entire function and we have

\[ \Gamma(z)^{-1} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)e^{-\frac{z}{n}} \]

where \( \gamma \) is the Euler constant. From this follows that in a neighborhood of zero, we have the asymptotic expansion \( \Gamma(s)^{-1} = s + \gamma s^2 + O(s^3) \).

Using the Mellin transform of the function

\[ \lambda^{-s} = \Gamma(s)^{-1} \int_0^{+\infty} t^{s-1}e^{-t\lambda} dt \]

we can write:

\[ \Gamma(s)\zeta_B(s) = \int_0^1 t^{s-1} \text{tr}e^{-tB} dt + \int_1^{\infty} t^{s-1} \text{tr}e^{-tB} dt \quad (1.11) \]

Notice that the last expression on the r.h.s converges for \( \text{Res} \leq R, R > 0 \) for, setting \( C_R = \sup_{s \geq 1} t^R - e^{-\frac{1}{z}\lambda_n} \), we have \( \int_1^{\infty} t^{R-1}e^{-\frac{1}{z}\lambda_n} \leq C_R \int_1^{\infty} e^{-\frac{1}{z}\lambda_n} = 2C_R \lambda_n^{-1}e^{-\frac{1}{z}\lambda_n} \)

which is the general term of a convergent series.

As before we set

\[ F(t) \equiv \text{tr}e^{-tB} - \sum_{j=-J}^{m-1} b_j \frac{t^m}{n} \]

Using (1.11) and (1.12), we can write for \( s \in \mathfrak{C} \) with large enough real part, \( \text{Res} > \frac{J}{m} \): 

\[ \zeta_B(s) = \Gamma(s)^{-1} \left( \sum_{j=-J}^{m-1} \frac{b_j}{\frac{J}{m} + s} + \int_1^{\infty} t^{s-1} \text{tr}e^{-tB} dt + \int_0^1 t^{s-1} F(t) dt \right) \]

This equality then extends to an equality of meromorphic functions on \( \text{Res} > 0 \) with poles \( s = \frac{-J}{m} \). Using the asymptotic expansion of the inverse of the Gamma function \( \Gamma(s)^{-1} \) around zero, we have:

\[ \zeta'_B(s) = (1 + 2\gamma s + O(s^2)) \left( \sum_{j=-J}^{m-1} \frac{b_j}{\frac{J}{m} + s} + \int_1^{\infty} t^{s-1} \text{tr}e^{-tB} dt + \int_0^1 t^{s-1} F(t) dt \right) \]

\[ + (s + \gamma s^2 + O(s^3)) \left( -\sum_{j=-J}^{m-1} \frac{b_j}{\left(\frac{J}{m} + s\right)^2} + \int_0^1 t^{s-1} F(t) \ln(t) dt + \int_1^{\infty} \ln(t) t^{s-1} \text{tr}e^{-tB} dt \right) \]

Letting \( s \) tend to zero, \( s > 0 \), since the divergent terms \( \frac{b_0}{s} \) and \( -s \frac{b_0}{s^2} \) arising in each of the terms of this last sum compensate, we get:

\[ \zeta'_B(0) = b_0 \gamma + \left( \sum_{j=-J, j \neq 0}^{m-1} \frac{mb_j}{j} + \int_0^1 \frac{F(t)}{t} dt + \int_1^{\infty} \frac{\text{tr}e^{-tB}}{t} dt \right) \]

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Hence, comparing with the expression of $\det_{\text{reg}} B$ given in (1.6), we find:

$$\log \text{Det}_{\text{reg}} (B) = -\zeta_B'(0) = -b_0\gamma + \log \det_{\text{reg}} (B)$$

and hence the equality of the lemma.

**Remarks:**

1) Notice that the same proof as in the lemma replacing the function $\zeta_B(s)$ by $\zeta_\lambda(s) = \lambda^{-s}$ for $\lambda > 0$ (which boils down to taking a one dimensional space $H$ and $F(t) = e^{-t\lambda} - 1$) yields

$$-\log \lambda = \gamma + \int_0^1 \frac{e^{-t\lambda} - 1}{t} dt + \int_1^\infty \frac{e^{-t\lambda}}{t} dt$$

and when choosing $\lambda = 1$, the integral representation of the Euler constant.

$$\gamma = \int_0^1 \frac{1 - e^{-t}}{t} dt - \int_1^\infty \frac{e^{-t}}{t} dt.$$

2) In the finite dimensional case, $\dim H = d$, since $\lim_{\varepsilon \to 0} \text{tr} e^{-\varepsilon B} = d = b_0$, from the result of lemma 1.4 and the fact that the zeta function regularised determinant coincides with the ordinary one, follows that

$$\det_{\text{reg}} B = e^{d\gamma} \text{Det}_{\text{reg}} B = e^{d\gamma} \det B = \lim_{\varepsilon \to 0} (\det_\varepsilon (B) \varepsilon^{-d})$$

where $\det B$ denotes the ordinary determinant of $B$. This agrees with (1.3 bis) using the integral representation given above of the Euler constant.

3) Let $M$ be a Riemannian manifold of dimension $d$ and $B$ a positive self-adjoint elliptic operator with smooth coefficients acting on sections of a vector bundle $V$ on $M$ with finite dimensional fibres of dimension $k$. We know by [G] Theorem 1.7.6 (a) that $b_0 = 0$ if $n$ is odd. However, in general the coefficient $b_0$ is a complicated expression given in terms of the jets of the symbol of the operator $B$. In the following we shall be concerned with the dependence of $b_0$ on the geometric data given on that manifold.

The notion of $\zeta$ function regularized determinant therefore extends to an operator satisfying assumptions of Lemma 1.2, by setting:

$$\text{Det}_{\text{reg}} B = e^{\gamma b_0} \text{det}_{\text{reg}} B$$

It furthermore extends to positive operators satisfying assumptions of Lemma 1.2 with non zero (finite dimensional) kernel and which leave its orthogonal supplement invariant, for in that case, we can set:

$$\text{Det}'_{\text{reg}} B \equiv e^{\gamma (b_0 \cdot \dim \ker B)} \text{det}'_{\text{reg}} B$$
II Regularisable principal fibre bundles

The aim of this section is to define a class of principal fibre bundles for which we can define a notion of regularised volume of the fibres and for which these regularised volumes have differentiability properties.

Let $P$ be a Hilbert manifold equipped with a (possibly weak) right invariant Riemannian structure. The scalar product induced on $T_p P$ by this Riemannian structure will be denoted by $\langle \cdot , \cdot \rangle_p$. We shall assume this Riemannian structure induces a Riemannian connection denoted by $\nabla$ and an exponential map with the usual properties. In particular, for all $p_0$, $\exp_{p_0}$ yields a diffeomorphism of a neighborhood of 0 in the tangent space $T_{p_0} P$ onto a neighborhood of $p_0$ in the manifold $P$.

Let $G$ be a Hilbert Lie group (in fact a right semi-Hilbert Lie group i.e a Hilbert Lie group in the usual sense up to the fact that only right multiplication is required to be smooth in the sense of $[P]$ is enough here) acting smoothly on $P$ on the right by an isometric action $\Theta : G \times P \rightarrow P$

$$(g, p) \rightarrow p \cdot g$$

$(2.0)$

Let for $p \in P$

$$\tau_p : G \rightarrow T_p P$$

$$u \mapsto \frac{d}{dt}(p \cdot e^{tu})_{t=0}$$

$(2.0bis)$

where $\mathcal{G}$ denotes the Lie algebra of $G$.

We shall assume that the action $\Theta$ is free (so that $\tau_p$ is injective on $G$) and that it induces a smooth manifold structure on the quotient space $P/G$ and a smooth principal fibre bundle structure given by the canonical projection $\pi : P \rightarrow P/G$.

Let us furthermore equip the group $G$ with a smooth family of equivalent (possibly weak) $\text{Ad}_g$ invariant Riemannian metrics indexed by $p \in P$. The scalar product induced on $\mathcal{G}$ by the Riemannian metric on $G$ indexed by $p \in P$ will be denoted by $\langle \cdot , \cdot \rangle_p$. Since the metrics are all equivalent, the closure of $\mathcal{G}$ w.r.t $\langle \cdot , \cdot \rangle_p$ does not depend on $p$ and we shall denote it by $H$.

Since $\mathcal{G}$ is dense in $H$, $\tau_p$ is a densely defined operator on $H$ and we can define its adjoint operator $\tau_p^*$ w.r. to the scalar products $\langle \cdot , \cdot \rangle_p$ and $\langle \cdot , \cdot \rangle_p$. We shall assume that $\tau_p^* \tau_p$ has a self adjoint extension on a dense domain $D(\tau_p^* \tau_p)$ of $H$.

We shall assume that $\tau_p$ is injective.

Warning: Although $\tau_p$ is injective on $\mathcal{G}$, the operator $\tau_p^* \tau_p$ might not be injective on the domain $D(\tau_p^* \tau_p)$ as we shall see in applications (cfr.Appendix A).

Definition: The orbit of a point $p_0$ is volume preregularisable if the following assumptions on the operator $\tau_p^* \tau_p$ are satisfied (We refer the reader to Appendix 0 for the definition of Gâteaux-differentiability and related notions):
1) **Assumption on the spectral properties of** $\tau_{p_0}^* \tau_{p_0}$

The operator $e^{-\varepsilon \tau_{p_0}^* \tau_{p_0}}$ is trace class for any $\varepsilon > 0$ and for any vector $X$ at point $p_0$, there is a neighborhood $I_0$ of $p_0$ on the geodesic $p_\kappa = \exp_{p_0} \kappa X$ such that for all $p \in I_0$, $e^{-\varepsilon \tau_{p_0}^* \tau_{p_0}}$ is trace class.

2) **Regularity assumptions**

We shall assume that the maps $p \mapsto \tau_p$ and $p \mapsto \tau_p^* \tau_p$ are Gâteaux differentiable and that for any $t > 0$, the function $p \mapsto \text{tr} e^{-t \tau_p^* \tau_p}$ is Gâteaux differentiable at point $p_0$. We furthermore assume that the Gâteaux-differentials at point $p_0$ in the direction $X$ of these operators are related as follows:

$$\delta X (\text{tr} e^{-\varepsilon \tau_p^* \tau_p}) = -\varepsilon \text{tr} (\delta X (\tau_p^* \tau_p) e^{-\varepsilon \tau_p^* \tau_p})$$

(2.1)

Moreover, for any vector $X$ at point $p_0$, there are constants $C > 0$, $u > 0$ and a neighborhood $I_0$ of $p_0$ on the geodesic $p_\kappa = \exp_{p_0} \kappa X$ such that for any $p \in I_0$:

$$\text{tr} e^{-t \tau_p^* \tau_p} \leq Ce^{-tu}$$

(2.2)

and

$$M_{I_0}(t) \equiv \sup_{p \in I_0} ||| \delta X(p) (\tau_p^* \tau_p) e^{-t \tau_p^* \tau_p} |||_\infty$$

(2.3)

is finite and a decreasing function in $t$.

Here $||| \cdot |||_\infty$ denotes the operator norm on $G$ induced by $(\cdot, \cdot)_p$. $\bar{X}$ is a local vector field defined in a neighborhood of $p_0$ by $\bar{X}(p_\kappa) = \exp_{p_\kappa} (\kappa X)(X)$.

The orbit $O_{p_0}$ is called **volume-regularisable** if $\dim \text{Ker} \tau_p^* \tau_p$ is constant on some neighborhood of $p_0$ on any geodesic containing $p_0$ and if the following assumption is satisfied:

3) **Assumption on the asymptotic behavior of the heat-kernel traces**

There is an integer $m > 0$ and a family of maps $p \mapsto b_j(p), j \in \{-J, \cdots, m-1\}$ which are Gâteaux differentiable in the direction $X$ at point $p_0$ such that

$$\text{tr} e^{-\varepsilon \tau_p^* \tau_p} \simeq_0 \sum_{j=-J}^{m-1} b_j(p) \varepsilon^{\frac{j}{m}}$$

(2.4)

in a neighborhood $I_0$ of $p_0$ on the geodesic $p = \exp_{p_0} \kappa X$, and

$$\delta X \text{tr} e^{-\varepsilon \tau_p^* \tau_p} \simeq_0 \sum_{j=-J}^{m-1} \delta X b_j(p) \varepsilon^{\frac{j}{m}}.$$  

(2.5)

Furthermore, setting

$$F_p(t) \equiv \text{tr} e^{-t \tau_p^* \tau_p} - \sum_{j=-J}^{m-1} b_j(p) t^{\frac{j}{m}}$$
for any vector $X$ at point $p_0$, there is a constant $K > 0$, and a neighborhood $I_0$ of $p_0$ on the geodesic $\kappa \to p_\kappa = \exp_{p_0} \kappa X$ such that:

$$\sup_{p \in I_0} \|\delta \bar{X}(p) F_p(t)\|_\infty \leq K t.$$  \hfill (2.5bis)

A principal bundle as described above with all its orbits volume-preregularisable (resp. volume-regularisable) will be called preregularisable (resp. regularisable).

**Remark:** Since the Riemannian structure on $\mathcal{P}$ is right invariant and the one on $G$ is $Ad_g$ invariant, the above assumptions do not depend on the point chosen in the orbit for we have $\tau_{p,g} = R_{g_*} \tau_p Ad_g$.

Although most fibre bundles we shall come across are not only preregularisable but also regularisable so that the notion of preregularisability might seem somewhat artificial, in applications (see Appendices A and B), it is often enough to verify the conditions required for preregularisability in order to prove a certain minimality of the orbits, namely strong minimality, a notion which will be defined in the following.

Natural examples of regularisable fibre bundles arise in gauge field theories (Yang-Mills, string theory). In gauge field theories, $\mathcal{P}$ and $G$ are modelled on spaces of sections of vector bundles $\mathcal{E}$ and $\mathcal{F}$ based on a compact finite dimensional manifold $M$ and the operators $\tau_p^* \tau_p$ arise as smooth families of Laplace operators on forms. As elliptic operators on a compact boundaryless manifold, they have purely discrete spectrum which satisfies condition 1) (see [G] Lemma 1.6.3 and (2.4) (see [G] Lemma 1.7.4.b)). By classical results concerning one parameter families of heat-kernel operators, they satisfy (2.1) (see [RS] proposition 6.1) and (2.2) (see proof of Theorem 5.1 in [RS]). Since $\delta X B_p$ is also a partial differential operator, by [G] lemma 1.7.7, $\delta X \text{ tr}_e^{-\varepsilon} B$ satisfies (2.5). Assumptions on the Gâteaux-differentiability and assumptions (2.3), (2.5 bis) are fulfilled in applications. Indeed, the parameter $p$ is a geometric object such as a connection, a metric on $M$ and choosing these objects regular enough (of class $H^k$ for $k$ large enough) ensures that the maps $p \mapsto \tau_p, p \mapsto \tau_p^* \tau_p, p \mapsto \text{ tr}_e^{-\varepsilon} B$ etc.. are regular enough for they involve these geometric quantities and their derivatives, but no derivative of higher order.

**Remark:** In the context of gauge field theories, the underlying Riemannian structure w.r.t. which the traces (arising in (2.2)-(2.5bis) are taken are weak $L^2$ Riemannian structures, the ones that also underly the theory of elliptic operators on compact manifolds. In [AP2], we discuss in how far this weak Riemannian structure could be replaced by a strong Riemannian structure, in order to set up a link between this geometric picture and a stochastic one developped in [AP2].

**Proposition 2.1:** Let $O_{p_0}$ be a volume-preregularisable orbit such that for any geodesic containing $p_0$, there is a neighborhood of $p_0$ on this geodesic on which $\tau_p^* \tau_p$ is injective. Then

1) $\det_\varepsilon (\tau_p^* \tau_p)$ is well defined for any $\varepsilon > 0$ and for $p$ in a neighborhood of $p_0$ on any geodesic of $p_0$.

2) The map

$$p \mapsto \det_\varepsilon (\tau_p^* \tau_p)$$

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is Gâteaux-differentiable at point $p_0$, the operator $\int_{\varepsilon}^{+\infty} \delta_X(\tau^*_p \tau_p) e^{-t \tau^*_p \tau_p} dt$ is trace class for any $p$ in a neighborhood of $p_0$ on any geodesic of $p_0$. For any tangent vector $X$ at point $p_0$, we have:

$$\delta_X \log \det_\varepsilon(\tau^*_p \tau_p) = \int_{\varepsilon}^{\infty} \text{tr} (\delta_X \tau^*_p \tau_p) e^{-t \tau^*_p \tau_p} dt$$

$$= \text{tr} \int_{\varepsilon}^{+\infty} (\delta_X \tau^*_p \tau_p) e^{-t \tau^*_p \tau_p} dt$$

(2.6a)

3) If the orbit $O_{p_0}$ is moreover volume-regularisable, the map $p \mapsto \det_{\text{reg}}(\tau^*_p \tau_p)$ is Gâteaux differentiable in all directions at point $p_0$, and with the notations of (2.4) $\delta_X[ \log \det_\varepsilon(\tau^*_p \tau_p) - \sum_{j=-J}^{-1} m_j b_j \varepsilon^{\frac{m}{m}} - b_0 \log \varepsilon]$ converges when $\varepsilon \to 0$ and we have

$$\lim_{\varepsilon \to 0} \delta_X[ \log \det_\varepsilon(\tau^*_p \tau_p) - \sum_{j=-J}^{-1} m_j b_j \varepsilon^{\frac{m}{m}} - b_0 \log \varepsilon] = \delta_X \log \det_{\text{reg}}(\tau^*_p \tau_p)$$

$$= - \sum_{j=-J, j \neq 0} m_j \delta_X b_j - \int_{1}^{\infty} \delta_X \left( \text{tr} \left( \frac{e^{-t \tau^*_p \tau_p}}{t} \right) \right) dt - \int_{0}^{1} \delta_X F_p(t) \frac{t}{t} dt$$

(2.6b)

**Proof**: We set $B_p = \tau^*_p \tau_p$.

1) By the first assumption for volume-preregularisable orbits, we know that $e^{-\varepsilon B_p}$ is trace class so that by lemma 1.1 so is $A^\varepsilon_p \equiv \log h_\varepsilon(B_p)$. Hence $\det_\varepsilon(B_p) = e^{\text{tr} A^\varepsilon_p}$ is well defined.

2) Let us show the first equality in (2.6 a). The orbit $O_{p_0}$ being preregularisable, assumption 2) for volume-preregularisability yields that for any $p \in I_0$ and any $t > \varepsilon > 0$

$$|\text{tr}(\delta_X(p) B_p e^{-tB_p})| \leq CM_{I_0} \left( \frac{t}{2} \right) e^{-\frac{t}{2} u}.$$ 

Here, we have used the fact that $|\text{tr}(UV)| \leq |||U|||\text{tr}V|$ for any bounded operator $U$ and any trace class operator $V$ applied to $U = \delta_X(p) B_p e^{-\frac{1}{2} B_p p_0}$ and $V = e^{-\frac{1}{2} B_p}$. Hence, by Lebesgue dominated convergence theorem, the map $p \mapsto \int_{\varepsilon}^{\infty} t^{-1} e^{-tB_p} dt$ is Gâteaux-differentiable in the direction $X$ at point $p_0$ and

$$\delta_X \int_{\varepsilon}^{\infty} t^{-1} e^{-tB_p} dt = \int_{\varepsilon}^{\infty} t^{-1} \delta_X t e^{-tB_p} dt$$

$$= - \int_{\varepsilon}^{\infty} \text{tr}(\delta_X B_p) e^{-tB_{p_0}} dt$$

using (2.2). Using the fact that $\log \det_\varepsilon(B_p) = - \int_{\varepsilon}^{+\infty} t^{-1} e^{-tB_p} dt$ then yields the first equality in (2.6 a).
The second equality in (2.6 a) and the fact that we can swap the trace and the integral follow from the estimate:

\[ \| \delta X B_p e^{-tB_{p0}} \|_1 \leq \| \delta X B_p e^{\frac{\varepsilon}{2} B_{p0}} \|_\infty \| e^{-\frac{3}{4}tB_{p0}} \|_1 \leq C \| \delta X B_p e^{\frac{\varepsilon}{2} B_{p0}} \|_\infty \| e^{-tu} \]  

valid for \( t \geq \varepsilon \), using assumption (2.2). We finally obtain by dominated convergence:

\[ \text{tr} \int_{\varepsilon}^{+\infty} \delta X B_p e^{-tB_{p0}} dt = \int_{\varepsilon}^{+\infty} \text{tr} \delta X B_p e^{-tB_{p0}} dt. \]

3) Let us first check that the map \( p \mapsto \det_{\text{reg}} B_p \) is Gâteaux differentiable at point \( p_0 \) in the direction \( X \). By (1.6), we have

\[ \log \det_{\text{reg}} B_p = -\sum_{j=-J, j \neq 0}^{m-1} \frac{b_j(p)}{j} - \int_1^{\infty} \text{tr} \frac{e^{-tB_p}}{t} dt - \int_0^1 \frac{F_p(t)}{t} dt \]

The first term on the r.h.s. is Gâteaux differentiable in the direction \( X \) by the assumption on the maps \( p \mapsto b_j(p) \). The second term on the r.h.s. is Gâteaux differentiable by the result (applied to \( \varepsilon = 1 \) of part 2) of this lemma which tells us that \( p \mapsto \det_{\varepsilon}(B_p) \) is Gâteaux differentiable. The Gâteaux differentiability of the last term follows from the local uniform upper bound (2.5 bis).

We now check (2.6 b). The map \( p \mapsto \log \det_{\varepsilon}(B_p) - \sum_{j=-J, j \neq 0}^{m-1} \frac{mb_j \varepsilon^\frac{1}{m}}{J} - b_0 \log \varepsilon \) is Gâteaux differentiable in the direction \( X \) and we can write

\[ \delta X (\log \det_{\varepsilon}(B_p) - \sum_{j=-J, j \neq 0}^{m-1} \frac{mb_j \varepsilon^\frac{1}{m}}{J} - b_0 \log \varepsilon) \]

\[ = \delta X \left( -\sum_{j=-J, j \neq 0}^{m-1} \frac{mb_j \varepsilon^\frac{1}{m}}{J} - b_0 \log \varepsilon \right) \]

\[ = \delta X \left( -\sum_{j=-J, j \neq 0}^{m-1} \frac{m b_j \varepsilon^\frac{1}{m}}{j} - \int_1^{\infty} \text{tr} e^{-tB_p} dt - \int_0^1 \frac{F_p(t)}{t} dt \right) \]

\[ = - \sum_{j=-J, j \neq 0}^{m-1} \delta X b_j \frac{m}{j} - \int_1^{\infty} \delta X \text{tr} e^{-tB_p} dt - \int_0^1 \delta X \frac{F_p(t)}{t} dt \]

which tends to \( \delta X \log \det_{\text{reg}} B_p \) by (1.6) and dominated convergence. Here we have used the results of point 2) of the proposition applied to \( \varepsilon = 1 \) to write

\[ \delta X \int_1^{\infty} \text{tr} e^{-tB_p} dt = \int_1^{\infty} \delta X \text{tr} e^{-tB_p} dt \]

and (2.5 bis) to write \( \delta X \int_0^1 \frac{F_p(t)}{t} dt = \int_0^1 \delta X \frac{F_p(t)}{t} dt. \)
Remark These results extend to the case when instead of assuming that $\tau^*_p \tau_p$ is injective locally around $p_0$, one considers orbits of an action at points $p_0$ for which the dimension of the kernel of $\tau_p$ is constant on some neighborhood of $p_0$ on each geodesic starting at point $p_0$. For this, one should replace $\det_\epsilon \tau^*_p \tau_p$ and $\det_{\text{reg}} \tau^*_p \tau_p$ by $\det'_\epsilon \tau^*_p \tau_p$ and $\det'_{\text{reg}} \tau^*_p \tau_p$. This extension is useful for applications (see Appendix A.).

A direct generalisation of the notion of volume for volume-preregularisable or regularisable orbits would give infinite quantities. But for volume-preregularisable or regularisable orbits, one can define a notion of preregularised or regularised volume, which justifies a posteriori the term “volume-preregularisable or volume-regularisable orbits” for these orbits. Since $\tau_{p,g} = R_g \tau_p \text{Ad}_g$ and since the metric on $G$ is $\text{Ad}_g$ and that on $P$ right invariant, for any $\epsilon > 0$, we have $\det_\epsilon (\tau^*_p \tau_p) = \det_\epsilon (\tau^*_p \tau_p)$ so that it makes sense to set the following definitions:

**Definition:**
1) Let $O_p$ be a volume-preregularisable orbit, then

$$\text{vol}_\epsilon(O_p) \equiv \sqrt{\det'_\epsilon(\tau^*_p \tau_p)}$$

defines a one parameter family of (heat-kernel) preregularised volumes of $O_p$.

2) Let $O_p$ be a volume-regularisable orbit, then

$$\text{vol}_{\text{reg}}(O_p) = \sqrt{\det'_{\text{reg}}(\tau^*_p \tau_p)}$$

defines the heat-kernel regularised volume of $O_p$.

3) Let $O_p$ be a volume-regularisable orbit, then

$$\text{Vol}_{\text{reg}}(O_p) = \sqrt{\text{Det}'_{\text{reg}}(\tau^*_p \tau_p)}$$

defines the zeta function volume-regularised volume of $O_p$.

From lemma 1.4 follows that

$$\text{Vol}_{\text{reg}}(O_p) = e^{-\frac{1}{2} \gamma b'_0(p)} \text{vol}_{\text{reg}}(O_p)$$

where $\gamma$ is the Euler constant and $b'_0(p) = b_0(p) - \text{dim Ker}(\tau^*_p \tau_p)$ is the coefficient arising from the heat-kernel asymptotic expansion of $\tau^*_p \tau_p$ given by (2.4).

**Remarks:**
1) In finite dimension, $\text{dim}H = d$ and $\tau_p$ is injective, we have by (1.13):

$$\lim_{\epsilon \to 0} (\epsilon^{\frac{d}{2}} \text{vol}_\epsilon(O_p)) = \text{vol}_{\text{reg}}(O_p) = e^{\frac{d}{2} \gamma} \text{Vol}_{\text{reg}}(O_p) = e^{\frac{d}{2} \gamma} \text{vol}(O_p)$$

(2.7)
where $\mu$ is the volume measure and $\text{vol}(O_p)$ is the ordinary volume of the fibre $O_p$.

2) If the coefficients $b_j(p)$ arising in the heat-kernel expansion of $\tau_p^*\tau_p$ are independent of $p$, we have for two regularisable orbits $O_{p_0}$ and $O_{p_1}$:

$$\lim_{\varepsilon \to 0} \frac{\text{vol}_\varepsilon(O_{p_0})}{\text{vol}_\varepsilon(O_{p_1})} = \frac{\text{vol}_{\text{reg}}(O_{p_0})}{\text{vol}_{\text{reg}}(O_{p_1})}$$

**Proposition 2.2:** The heat-kernel (pre)-regularised and zeta function regularised volume of a volume-(pre)regularisable orbits $O_p$ is Gâteaux-differentiable at the point $p$.

**Proof:** It follows from Proposition 2.1. 

Let us now introduce a notion of extremality of orbits which generalises the corresponding finite dimensional notion [H].

**Definition:** A *strongly extremal orbit* is a volume-preregularisable orbit, the heat-kernel preregularised volume of which is extremal, i.e $O_p$ is strongly extremal if $\delta_X \text{vol}_\varepsilon(O_p) = 0$ for any horizontal vector $X$ at point $p$ and any $\varepsilon > 0$.

A heat-kernel (resp. zeta function) *extremal orbit* of a volume-preregularisable bundle is an orbit, the heat-kernel (respectively zeta function) regularised volume of which is extremal, i.e $\delta_X \text{vol}_{\text{reg}}(O_p) = 0$ for any horizontal vector $X$ at point $p$.

Notice that whenever $b_0$ does not depend on $p$, the zeta-function regularised volume of an extremal orbit is also heat-kernel extremal. From (2.7) also follows that this notion generalises the finite dimensional notion of extremality of the volume of the fibre.

### III. Minimal regularizable orbits as orbits with extremal regularized volume

We shall consider a preregularisable principal fibre bundle $\mathcal{P} \to \mathcal{P}/G$. By assumption, the bundle is equipped with a Riemannian connection given by a family of horizontal spaces $H_p, p \in \mathcal{P}$ such that

$$T_p\mathcal{P} = H_p \oplus V_p$$

where $V_p$ is the tangent space to the orbit at point $p$ and the sum is an orthogonal one.

For a horizontal vector $X$ at point $p$, we define the shape operator

$$\mathcal{H}_X : V_p \to V_p$$

$$Y \mapsto -(\nabla_Y X)^v(p)$$

where the subscript $v$ denotes the orthogonal projection onto $V_p$ and $\bar{X}$ is a horizontal field with value $X$ at $p$. Similarly, we define the second fundamental form:

$$S^p : V_p \times V_p \to H_p$$

$$(Y, Y') \mapsto (\nabla_Y \bar{Y})^h(p)$$
where $\bar{Y}$, $\bar{Y}'$ are vertical vector fields such that $\bar{Y}(p) = Y$, $\bar{Y}'(p) = Y'$. These definitions are independent of the choice of the extensions of $X, Y$ and $Y'$.

An easy computation shows that the shape operator and the second fundamental form are related as follows:

$$< \mathcal{H}_X(Y), Y' >_p = < S^p(Y, Y'), X >_p \quad (3.1)$$

Note that this explicitly shows that $\mathcal{H}_X$ only depends on $X$ and not on the extension $\bar{X}$ of $X$. Since $S^p$ is symmetric, so is $\mathcal{H}_X$.

As in the finite dimensional case, one can define the notion of totally geodesic orbit, an orbit $O_p$ being totally geodesic whenever the second fundamental form $S^p$ vanishes.

**Definition:** The orbit $O_p$ of a point $p \in \mathcal{P}$ will be called *heat-kernel preregularisable* or for short *preregularisable* if for any horizontal vector $X$ at $p$, $\forall \varepsilon > 0$,

$$\mathcal{H}_X^\varepsilon \equiv e^{-\frac{1}{2} \varepsilon \tau_p^* \tau_p} \mathcal{H}_X e^{-\frac{1}{2} \varepsilon \tau_p^* \tau_p} \quad (3.2)$$

is trace class. A preregularisable orbit $O_p$ will be called *strongly minimal* if moreover for any $q \in O_p$ and $X$ a horizontal vector at point $q$, $\text{tr} \mathcal{H}_X^\varepsilon = 0 \ \forall \varepsilon > 0$.

**Remarks:**

1) The preregularisability of the orbits (namely $\mathcal{H}_X^\varepsilon$ trace class) is automatically satisfied if the manifold $\mathcal{P}$ is equipped with a strong smooth Riemannian structure, since in that case the second fundamental form is a bounded bilinear form and its weighted trace is well defined (see also [AP2] where this is discussed in further details).

2) Since on a preregularisable bundle, the Riemannian structure on $\mathcal{P}$ is right invariant and the one on $G$ is $Ad_g$ invariant, the notion of (pre) regularizability and (strong) minimality of the orbit does not depend on the orbit chosen on the orbit. Indeed, let $p$ be a point, $g \in G$ and $X$ a horizontal vector at point $p$. Let $\bar{X}$ be a right invariant horizontal vector field coinciding with $X$ at $p$. Since $\tau_{p-g} = R_{g\varepsilon}^* \tau_p Ad_g$ and $\mathcal{H}_{\bar{X}(p-g)} = R_{g\varepsilon} \mathcal{H}_{\bar{X}(p)} R_{g\varepsilon}^{-1}$, the vector field $\bar{X}$ being right invariant, we have $\tau_{p-g}^* \tau_p^* g = R_{g\varepsilon}^* \tau_p^* R_{g\varepsilon}^{-1}$. Hence $\mathcal{H}_{\bar{X}(p-g)}^\varepsilon = R_{g\varepsilon} \mathcal{H}_{\bar{X}(p)}^\varepsilon R_{g\varepsilon}^{-1}$ is trace class w.r. to $< \cdot, \cdot >_{p-g}$ whenever $\mathcal{H}_{\bar{X}(p)}^\varepsilon$ is trace class w.r. to $< \cdot, \cdot >_p$ and $\text{tr} \mathcal{H}_{\bar{X}(p-g)}^\varepsilon = \text{tr} \mathcal{H}_{\bar{X}(p)}^\varepsilon$.

3) Notice that if $\mathcal{H}_X$ is trace class, as in the finite dimensional case, strong minimality implies that $\text{tr} \mathcal{H}_X = 0$ and hence ordinary minimality. The fact that strong minimality implies minimality in the finite dimensional case motivates the choice of the adjective "strong".

4) This preregularised shape operator $\mathcal{H}_X^\varepsilon$ and the second fundamental form are related as follows:

$$< \mathcal{H}_X^\varepsilon(Y), Y' >_p = < S^p(e^{-\frac{1}{2} \varepsilon \tau_p^* \tau_p} Y, e^{-\frac{1}{2} \varepsilon \tau_p^* \tau_p} Y'), X >_p$$

Since $\tau_p^* \tau_p^*$ is an isomorphism of the tangent space to the fibre $T_pO_p$, $\mathcal{H}_X^\varepsilon$ vanishes whenever the second fundamental form vanishes and an orbit is totally geodesic whenever this regularised shape operator vanishes on the orbit for some $\varepsilon > 0$.

**Definition:** A preregularizable orbit $O_p$ will be said to be *regularisable* if furthermore, the one parameter family $\mathcal{H}_X^\varepsilon, \varepsilon \in [0, 1]$ admits a regularized limit-trace $\text{tr}_{\text{reg}} \mathcal{H}_X$. 

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For a preregularizable orbit $O_p$ such that for any $\varepsilon > 0$, $X \rightarrow \text{tr}H^\varepsilon_X$ is a bounded linear form on $T_p\mathcal{P}$ for the norm induced by $\langle \cdot, \cdot \rangle_p$, by Riesz theorem we can define the \textit{preregularised mean curvature vector} $S_\varepsilon$ in the closure $H_p$ of $T_p\mathcal{P}$ for this norm by the relation
\[
\langle S_\varepsilon(p), X \rangle_p = \text{tr}H^\varepsilon_X
\] (3.3)
In the same way, for a regularisable orbit $O_p$ such that $X \rightarrow \text{tr}_\text{reg}H_X$ is a bounded linear form on $T_p\mathcal{P}$, by Riesz theorem, we can define the \textit{regularized mean curvature vector} $S_{\text{reg}}(p)$ in $H_p$ by the relation
\[
\langle S_{\text{reg}}(p), X \rangle_p = \text{tr}_\text{reg}H_X
\] (3.4)
for any horizontal vector $X$ at point $p$. Of course, if the Riemannian structure is strong, both $S_\varepsilon(p)$ and $S_{\text{reg}}(p)$ lie in $T_p\mathcal{P}$.

\textbf{Remark}: In the finite dimensional case, we have $S_0(p) = S_{\text{reg}}(p) = \text{tr}S^p$ where $S^p$ is the second fundamental form. In the infinite dimensional case, the family of preregularised mean principal curvature vectors $S_\varepsilon(p)$ coincides with a family of preregularised traces and the regularised mean principal curvature vector $S_{\text{reg}}(p)$ with the regularised trace of the second fundamental form $S^p$.

\textbf{Definition}: A regularisable orbit $O_p$ will be called \textit{heat-kernel minimal} if $\text{tr}_\text{reg}H_X = 0$ for any horizontal vector at point $X$.

\textbf{Remarks}:
1) Since on a preregularisable bundle, the Riemannian structure on $\mathcal{P}$ is right invariant and the one on $G$ is $Ad_g$ invariant, regularizability and minimality of the orbit $O_p$ does not depend on the point $p$ chosen on the orbit. As before, we have $\text{tr}_\text{reg}H_X(p \cdot g) = \text{tr}_\text{reg}H_X(p)$.
2) Here again, in the finite dimensional case, the one parameter family $H^\varepsilon_X$ admits a regularised limit trace given by the ordinary trace $\text{tr}_\text{reg}H_X = \text{tr}H_X$ and heat-kernel minimality is equivalent to the finite dimensional notion of minimality.

Note that a strongly minimal preregularisable orbit $O_p$ is regularisable and minimal since setting $A_\varepsilon \equiv H^\varepsilon_X$ in (1.1), we have $a_j = 0, \forall j$ and hence $\text{tr}_\text{reg}H_X = 0$.

The regularisation of the mean principal curvature vector for orbits of group actions in the infinite dimensional case has been discussed in the literature before. King and Terng in [KT] introduced a notion of regularisability and minimality for submanifolds of path spaces using zeta-function regularisation methods. They in particular show zeta function regularisability and minimality for the orbits of the coadjoint action of a (based) loop group on a space of loops in the corresponding Lie algebra. We shall show later on that these orbits within this framework are regularisable and strongly minimal (hence minimal).

We now introduce a notion of zeta function regularisability which is a slight variation of the one introduced by Maeda, Rosenberg and Tondeur in [MRT1] (see also [MRT2]) in the case of orbits of the gauge action in Yang-Mills theory. This modification is natural in our context as we shall see later on.
Definition: The orbit \( O_p \) of a point \( p \) is \textit{zeta function regularisable} whenever

\[
\lim_{s \to 1} -\frac{1}{2} \left[ \Gamma(s)^{-1} \int_0^\infty t^{s-1} \sum_{\lambda_n \neq 0} e^{-t\lambda_n^p} \delta_X \lambda_n^p dt + (s-1)^{-1}\delta_X b'_0(p) dt \right]
\]

exists for any horizontal field \( X \) at point \( p \) and the limit shall be denoted by \( Tr_{\text{reg}} \mathcal{H}_X \) so that

\[
Tr_{\text{reg}} \mathcal{H}_X = -\frac{1}{2} \lim_{s \to 1} \left[ \Gamma(s)^{-1} \int_0^\infty t^{s-1} \sum_{\lambda_n \neq 0} e^{-t\lambda_n^p} \delta_X \lambda_n^p dt + (s-1)^{-1}\delta_X b'_0(p) \right]
\]

(3.5)

Since on a regularisable principal fibre bundle, the Riemannian structure on \( \mathcal{P} \) is \( \mathcal{G} \) invariant and that on \( \mathcal{G} \) is \( \text{Ad}\mathcal{G} \) invariant the orbit \( O_p \) is zeta function regularisable whenever the above holds for any \( q \in O_p \).

Definition: A zeta function regularisable orbit \( O_p \) will be called \textit{zeta function minimal} if \( Tr_{\text{reg}} \mathcal{H}_X = 0 \) for any horizontal vector at point \( X \).

Remark: This definition coincides with that of [MRT Proposition 5.9] whenever \( \delta_X b_0(p) \) is zero. This notion of regularisability is of course less restrictive than that of [MRT] and we shall see that on a regularisable fibre bundle, there is no obstruction to zeta function regularisability of the orbits.

Let us introduce some notations. Let \( \mathcal{P} \to \mathcal{P}/\mathcal{G} \) be a preregularisable principal fibre bundle and let \( (T^p_n)_{n \in \mathbb{N}} \) be a set of eigenvectors of \( \tau^*_p \tau_p \) in \( \mathcal{G} \) corresponding to the eigenvalues \( (\lambda^p_n)_{n \in \mathbb{N}} \) counted with multiplicity and in increasing order. Let \( p_0 \) be a fixed point in \( \mathcal{P} \) and let \( T^p_{p_0} \) be the isometry from \( (\mathcal{G}, (\cdot, \cdot)_{p_0}) \) into \( (\mathcal{G}, (\cdot, \cdot)_p) \) which takes the orthonormal set \( (T^p_{p_0})_n \) of eigenvectors of \( \tau^*_p \tau_{p_0} \) to the orthonormal set of eigenvectors \( (T^p_n)_n \) of \( \tau^*_p \tau_p \). Notice that \( T^p_{p_0} = I \).

Lemma 3.1: Let \( \mathcal{P} \to \mathcal{P}/\mathcal{G} \) be a preregularisable principal fibre bundle. Let \( p_0 \in \mathcal{P} \) be a point at which the map \( p \mapsto T^p_{p_0} u \) is Gâteaux-differentiable for any \( u \in \mathcal{G} \). Let \( X \) be a horizontal vector at \( p_0 \). We shall consider eigenvalues \( \lambda^p_n \) that correspond to eigenvectors that do not belong to \( \mathcal{I}^p_{p_0} \text{Ker}\tau^*_p \tau_{p_0} \).

1) The maps \( p \mapsto \lambda^p_n \) are Gâteaux-differentiable in the direction \( X \) at point \( p_0 \),

\[
\delta_X \lambda^p_n = (\delta_X (\tau^*_p \tau_p) T_{p_0}^p, T^p_n)_{p_0}
\]

and

\[
\delta_X \log h_\varepsilon(\lambda^p_n) = \int_{\varepsilon}^{+\infty} (\delta_X (\tau^*_p \tau_p) e^{-t\tau^*_p \tau_{p_0} T^p_{p_0}, T^p_n})_{p_0} dt.
\]

2) Furthermore, we have

\[
-< \mathcal{H}_X \hat{U}^p_n, \hat{U}^p_n >_{p_0} + e^{-\varepsilon \lambda^p_{p_0}} (\delta_X \mathcal{I}^p_{p_0} T^p_{p_0}, T^p_{p_0})_{p_0} = \frac{1}{2} \delta_X \log h_\varepsilon(\lambda^p_n)
\]

(3.6a)
where we have set $\tilde{U}_n^p = \|\tau_p T_n^p\|^{-1}\tau_p T_n^p$.

3) If the Riemannian structure on $G$ is fixed (independent of $p$), then $\delta_X T^p_{p_0}$ is antisymmetric and

$$\frac{1}{2} \int_\varepsilon^\infty (\delta_X (\tau^* \tau_p e^{-t\tau_p} T^p_n, T^p_n)_{p_0}) dt = - \langle H^\varepsilon_{X \tilde{U}_n^p}, \tilde{U}_n^p \rangle_{p_0}$$

$$= \frac{1}{2} \delta_X \log h_\varepsilon (\lambda_n^p) = \frac{1}{2} \lambda_n^p \delta_X \log \lambda_n^p e^{-\varepsilon \lambda_n^p}$$

(3.6b)

**Proof:** As before, we shall set $B_p = \tau^* \tau_p$. Since $p_0$ is fixed, we drop the index $p_0$ in $T^p_{p_0}$ and denote this isometry by $T^p$. Notice that $T^{p_0} = I$. As before, we denote by $(T^p_n)_{n \in N}$ the orthonormal set of eigenvectors of $\tau^* \tau_p$ which correspond to the eigenvalues $(\lambda_n^p)_{n \in N}$ in increasing order and counted with multiplicity. We shall set $\tilde{T}_n^p = \tau_p T_n^p$, $T_n^p = \tau_p T_n^{p_0}$.

1) Using the relations $(T^p, T^p, p) = (\cdot, p, p), T^p (T^{p_0}) = T^p, T^{p*} T^p = I$, we can write $\lambda_n^p = (B_p T^p_n, T^p_n) = (B_p T^{p_0} T^{p_0}_n, T^{p_0} T^{p_0}_n)_{p_0}$ and the map $p \mapsto \lambda_n^p$ is Gâteaux differentiable in all directions at point $p_0$ since $p \mapsto B_p, p \mapsto T^p$ are Gâteaux-differentiable by assumption on the bundle. Furthermore

$$\delta_X (B_p T_n^p, T_n^p) = \delta_X (T^{p*} B_p T^{p_0} T_n^p, T_n^p)_{p_0}$$

$$= \delta_X (\delta_X B_p, T_n^p, T_n^p)_{p_0} + \delta_X (\delta_X T^{p*} B_p T_n^{p_0}, T_n^p)_{p_0}$$

$$\delta_X (\delta_X T^{p*} B_p T_n^{p_0}, T_n^p, T_n^p)_{p_0} = (\delta_X B_p T_n^p, T_n^p)_{p_0} + \lambda_n^p ((T^{p*} \delta_X T^p) (T^{p_0}) T_n^{p_0})_{p_0}$$

Since $T^{p*} T^p = I$, we have $\delta_X T^{p*} T^{p_0} + T^{p*} \delta_X T^p = 0$ so that finally $\lambda_n^p$ is Gâteaux-differentiable and $\delta_X \lambda_n^p = ((\delta_X B_p) T^{p_0} T^{p_0}_n, T_n^{p_0})_{p_0}$.

Using the local uniform estimate (2.3), and with the same notations, we have for $t > \varepsilon$:

$$\|(\delta_X (p) B_p) e^{-t B_{p_0} T_n^{p_0}, T_n^{p_0})_{p_0}\| \leq M_{I_0} (\frac{1}{2} t) e^{-\frac{1}{2} t \lambda_n^p}$$

so that the map $p \mapsto \log h_\varepsilon (\lambda_n^p)$ is Gâteaux-differentiable at point $p_0$ in the direction $X$ and

$$\delta_X \log h_\varepsilon (\lambda_n^p) = -\delta_X \int_\varepsilon^\infty t^{-1} (e^{-t B_{p_0} T_n^{p_0}, T_n^{p_0}}) dt$$

$$= (\int_\varepsilon^\infty \delta_X (B_p) e^{-t B_{p_0} T_n^{p_0}, T_n^{p_0})_{p_0} dt$$

2)

By definition of $h_\varepsilon$ we have:

$$\delta_X \log h_\varepsilon (\lambda_n^p) = (\log h_\varepsilon)' (\lambda_n^p) \delta_X \lambda_n^p$$

$$= (\lambda_n^p)^{-1} e^{-\varepsilon \lambda_n^p} \delta_X \lambda_n^p$$

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But, with the notations of Appendix 0:

\[
\delta_X \lambda_n^p = \delta_X < \tilde{T}_n^p, \tilde{T}_n^p >_p = 2 < \delta_X (\tau_p^p) T_n^p, T_n^p >_{p_0} \\
= 2 < \delta_X \tilde{T}_n^p, \tilde{T}_n^p >_{p_0} + 2 < \tau_p \delta_X \tilde{T}_n^p, \tilde{T}_n^p >_{p_0} \\
= -2 < \nabla \tau_p \tilde{X}, \tilde{T}_n^p >_{p_0} + 2 < \tau_p \delta_X \tilde{T}_n^p, \tilde{T}_n^p >_{p_0} \\
= -2 \lambda_n^p < H_X \tilde{U}_n^p, \tilde{U}_n^p >_{p_0} + 2 \lambda_n^p (\delta_X \tilde{T}_n^p, \tilde{T}_n^p)_{p_0}
\]

where for the third equality, we have used the fact that, \( \tilde{X} \) being right invariant, we have \([\tilde{T}_n^p, \tilde{X}] = 0\).

Hence we have:

\[
\delta_X \log h_\varepsilon (\lambda_n^p) = -2 e^{-\varepsilon \lambda_n^p} < H_X (\lambda_n^p) \tilde{U}_n^p, \tilde{U}_n^p >_{p_0} + \\
+ 2 e^{-\varepsilon \lambda_n^p} (\delta_X \tilde{T}_n^p, \tilde{T}_n^p)_{p_0}
\]

which yields 2).

3) On one hand, since the scalar product on the Lie algebra is fixed, we have \( \delta_X \tilde{T}_n^p \subset (\delta_X \tilde{T}_n^p)^* \) (see Appendix 0). On the other hand, since \( \tilde{T}_n^p = I \), we have \( -\delta_X \tilde{T}_n^p \subset \delta_X \tilde{T}_n^p \) (see Appendix 0) so that the second term in the l.h.s of (3.6a) vanishes.

**Definition:** We shall call an orbit \( O_{p_0} \) of a preregularised bundle an orbit of type \((T)\) whenever the following conditions are satisfied:

1) The map \( p \mapsto \tilde{T}_n^p \) is Gâteaux-differentiable at point \( p_0 \).

2) The operator \( \delta_X \tilde{T}_n^p e^{-\varepsilon \tau_{p_0}} \) is trace class for any \( p_0 \in \mathcal{P} \) and \( \varepsilon > 0 \).

3) For any \( p \in \mathcal{P} \), \( \text{tr} \tilde{T}_n^p e^{-\varepsilon \tau_{p_0}} \) is Gâteaux-differentiable at point \( p_0 \in \mathcal{P} \) and

\[
\delta_X \text{tr}(\tilde{T}_n^p e^{-\varepsilon \tau_{p_0}}) = \text{tr}(\delta_X \tilde{T}_n^p e^{-\varepsilon \tau_{p_0}})
\]

Whenever the Riemannian structure on \( \mathcal{G} \) is independent of \( p \), any orbit satisfying condition 1) is of type \((T)\), for in that case the traces involved in 2) and 3) vanish, \( \delta_X \tilde{T}_n^p \) being an antisymmetric operator.

Let us interpret the trace \( \text{tr}(\delta_X \tilde{T}_n^p e^{-\varepsilon \tau_{p_0}}) \) as a variation of a relative volume. Since the map \( \kappa \rightarrow (\tilde{T}_n^p, \tilde{T}_n^p)_{p_0} \) is differentiable at \( \kappa = 0 \), it is continuous at this point. Hence for a family of points \( p_\kappa = \exp_{p_0}(\kappa X) \) on the geodesic at point \( p_0 \) generated by \( X \), there is a constant \( \eta > 0 \) such that for \( \alpha \) small enough, \( \sup_{\kappa \in [0, \alpha]} \| \tilde{T}_n^p - \tilde{T}_n^p \| \leq \eta \) (where the operator norm is taken w.r.t. \( \| \cdot \|_{p_0} \) so that for \( \kappa \) small enough \( (\tilde{T}_n^p, \tilde{T}_n^p)_{p_0} \lambda_n^{p_0} \geq (1 - 2\eta) \lambda_n^{p_0} \). Thus \( \sum_n \log h_\varepsilon [(\tilde{T}_n^p, \tilde{T}_n^p)_{p_0} \lambda_n^{p_0}] \) converges, since \( h_\varepsilon \) is non decreasing. We can therefore define a notion of *preregularised relative volume* of the orbit \( O_{p_\alpha} \) with respect to \( O_{p_0} \):

\[
\text{vol}_\varepsilon^{p_0}(O_{p_\alpha}) = \prod_{\lambda_n \neq 0} \sqrt{h_\varepsilon(\lambda_n^{p_0} (\tilde{T}_n^p, \tilde{T}_n^p)_{p_0})}
\]
Notice that it coincides with $\text{Vol}_\varepsilon(O_{p_0})$ for $\kappa = 0$. The fact that $|||\delta X T^p_{p_0}|||$ is locally bounded on a geodesic starting at point $p_0$ implies that the map $\kappa \mapsto \log \text{vol}_\varepsilon^p(O_{p_\kappa})$ is differentiable at point $\kappa = 0$ and an easy computation yields:

$$
\delta X \log \text{vol}_\varepsilon^p(O_p) = \text{tr}'[\delta X T^p_{p_0} e^{-\varepsilon \tau^*_{p_0} \tau_{p_0}}]
$$

where $\text{tr}'$ means that we have restricted to the orthogonal of $\text{Ker} \tau_{p_0}^* \tau_p$.

In the next proposition, we investigate the relation between the shape operator and the variation of the volume of the orbit.

**Proposition 3.2**: Let $\mathcal{P} \to \mathcal{P}/G$ be a preregularisable principal fibre bundle. Then

1) any orbit of type $(\mathcal{T})$ is preregularisable.

More precisely, if $O_{p_0}$ is an orbit of type $(\mathcal{T})$, for any horizontal vector $X$ at point $p_0$, the operator $H^\varepsilon_X$ is trace class, the maps $p \mapsto \text{vol}_\varepsilon^p(O_p)$ and $p \mapsto \text{vol}_\varepsilon(O_p)$ are Gâteaux differentiable in the direction $X$ at point $p_0$ and

$$
\text{tr} H^\varepsilon_X - \delta X \log \text{vol}_\varepsilon^p(O_p) = -\delta X \log \text{vol}'_\varepsilon(O_p) = -\frac{1}{2} \int_\varepsilon^{+\infty} \text{tr}'[\delta X (\tau^*_p \tau_p) e^{-t \tau^*_p \tau_{p_0}}] dt
$$

(3.7)

If the maps $X \mapsto \delta X \log \text{vol}_\varepsilon(O_p)$ and $X \mapsto \delta X \log \text{vol}_\varepsilon^p(O_p)$ are bounded linear maps on the closure $H_p$ of $T_p \mathcal{P}$ for the norm induced by $\langle \cdot , \cdot \rangle_p$, then the preregularised mean curvature vector $S^\varepsilon$ is a vector in $H_p$ defined by $\text{tr} H^\varepsilon_X = \langle S^\varepsilon, X \rangle_p$.

2) If the Riemannian structure on $G$ is independent of $p$, the orbit of any point $p_0$ is a preregularisable orbit and

$$
\text{tr} H^\varepsilon_X = -\delta X \log \text{vol}'_\varepsilon(O_p) = -\frac{1}{2} \int_\varepsilon^{+\infty} \text{tr}'[\delta X (\tau^*_p \tau_p) e^{-t \tau^*_p \tau_{p_0}}] dt
$$

(3.7bis)

where $\text{tr}'$ means we have restricted to the orthogonal of the kernel of $\tau^*_p \tau_{p_0}$ and $\text{vol}'_\varepsilon$ means that only consider eigenvalues $\lambda^p_n$ that correspond to eigenvectors that do not belong to $T^p_{p_0} \text{Ker} \tau^*_p \tau_{p_0}$.

**Remarks:**

1) In finite dimensions, for a compact connected Lie group acting via isometries on a Riemannian manifold $\mathcal{P}$ of dimension $d$, we have for any $\varepsilon > 0$ and using the various definitions of the volumes:

$$
\lim_{\varepsilon \to 0} \delta X \log \text{vol}_\varepsilon(O_p) = \delta X \log \text{vol}_{\text{reg}} O_p
$$

$$
= \delta X \log \text{Vol}_{\text{reg}} O_p
$$

$$
= \delta X \log \text{vol} O_p
$$

Hence going to the limit $\varepsilon \to 0$ on either side of (3.7bis) we find:

$$
\text{tr} H_X = -\delta X \log \text{vol} O_p.
$$
If the Gâteaux-differentiability involved is a $C^1$- Gâteaux-differentiability, this yields
\[ \text{tr} S^p = -\text{grad} \log \text{vol} O_p \]

This leads to a well known result, namely (Hsiang’s theorem [H]) that the orbits of $G$ whose volume are extremal among nearby orbits is a minimal submanifold of $M$.

2) Equality (3.7) tells us that whenever the Riemannian structure on $G$ is independent of $p$ (as in the case of Yang-Mills theory), strongly minimal orbits of a preregularisable principal fibre bundle are pre-extremal orbits. This gives a weak (in the sense that we only get a sufficient condition for strong minimality and not for minimality) infinite dimensional version of Hsiang’s [H] theorem.

3) If both the spectrum of $\tau^*_p \tau_p$ and the Riemannian structure on $G$ are independent of $p$, as in the case of Yang-Mills theory in the abelian case (where the spectrum only depends on a fixed Riemannian structure on the manifold $M$), the orbits are strongly minimal (see also [MRT] par.5).

**Proof of Proposition 3.2:** We set $B_p = \tau^*_p \tau_p$. for the sake of simplicity, we assume that $B_p$ is injective on its domain, the general case then easily follows.

1) From the preregularisability of the principal bundle follows (see proposition 2.1) that the map $p \mapsto \det_\varepsilon(B_p)$ is Gâteaux-differentiable in the direction $X$ at point $p_0$ and
\[ \delta_X \log \det_\varepsilon(B_p) = \int_{\varepsilon}^{+\infty} dt \text{tr}(\delta_X B_p e^{-tB_p}) \]

On the other hand, by lemma 3.1
\[ \frac{1}{2} < \int_{\varepsilon}^{+\infty} dt (\delta_X B_p e^{-tB_p}) T_p, T_p >_p - e^{-\varepsilon \lambda_n^p} (\delta_X T^p T_p, T_p)_{p_0} = - < \mathcal{H}^\varepsilon_X, \tilde{U}_p, \tilde{U}_p >_p \]

The fibre bundle being preregularisable, by the results of proposition 2.0, the first term on the left hand side is the general term of an absolutely convergent series. On the other hand, the orbit being of type $(T)$, the series with general term $e^{-\varepsilon \lambda_n^p} (\delta_X T^p T_p, T_p)_{p_0}$ is also absolutely convergent. Hence the right hand side of (*) is absolutely convergent and $\mathcal{H}^\varepsilon_X$ is trace class since $(\tilde{U}_n)_{n \in \mathbb{N}}$ is a complete orthonormal basis of $\text{Im} \tau_p$.

\[ - \int_{\varepsilon}^{+\infty} dt \text{tr}(\delta_X B_p e^{-\varepsilon B_p}) = \text{tr} \mathcal{H}^\varepsilon_X - \delta_X \log \text{Vol}_p^p(B_p) = -\delta_X \log \det_\varepsilon(B_p) \]

which then yields (3.7).

The second part of point 1) of the proposition follows from the definition of the mean curvature vector in the case of a Hilbert manifold.

2) This follows from the above and point 3) of lemma 3.1 and holds for any orbit $O_p$ of a regularisable fibre bundle since it does not involve $\delta_X I_p$.

The following proposition gives an interpretation of $\text{tr}_{reg} H_X$ in terms of the variation of $\text{vol}_{reg}(O_p)$.
Proposition 3.3: The fibres of a regularisable principal fibre bundle with structure group equipped with a fixed (p-independent) Riemannian metric are heat-kernel and zeta-function regularisable.

1) Orbits are heat-kernel minimal whenever they are heat-kernel extremal. More precisely, for any point $p_0 \in P$ and any horizontal vector $X$ at point $p_0$, The one parameter family $H^\varepsilon X$ has a limit trace $\text{tr}_{\text{reg}} H^\varepsilon X$ and

$$\text{tr}_{\text{reg}} H^\varepsilon X = -\delta_X \log \text{vol}_{\text{reg}}(O_p)$$

$$= \frac{1}{2} \left[ \sum_{j=-J, j \neq 0}^{m-1} \delta_X \frac{b_j(p)}{j} + \int_0^1 \frac{\delta_X F_p(t)}{t} dt + \int_1^\infty dt t^{-1} \delta_X \text{tr}_{\text{reg}} \tau_p \right]$$

(3.8)

If the coefficients $b_j(p)$ are extremal at point $p_0$ and if $\mathcal{H}_X$ is trace-class, then $\text{tr} \mathcal{H}_X = -\delta_X \log \text{vol}_{\text{reg}} O_p$.

$$\text{Tr}_{\text{reg}} \mathcal{H}_X = -\delta_X \log \text{vol}_{\text{reg}}(O_p).$$

(3.9)

2) The operator $\mathcal{H}_X$ has a well defined zeta function regularised trace and the orbit is zeta function minimal if and only if it is zeta function extremal. More precisely, we have:

$$\text{Tr}_{\text{reg}} \mathcal{H}_X = -\delta_X \log \text{Vol}_{\text{reg}}(O_p)$$

$$= \text{tr}_{\text{reg}} \mathcal{H}_X + \frac{1}{2} \gamma \delta_X b_0$$

(3.10)

If moreover $\delta_X b_0 = 0$ for any horizontal vector $X$ at point $p_0$, an orbit is heat-kernel minimal whenever it is zeta function minimal.

3) Whenever the map $X \mapsto \text{tr}_{\text{reg}} H_X$ is a bounded linear map on $H_p$ (with the notations of proposition 2.2), the regularised mean curvature vector $S_{\text{reg}}$ is a vector in $H_p$ defined by $< S_{\text{reg}}, X >_p = \text{tr}_{\text{reg}} H_X$.

Remarks:

1) In the case of a compact connected Lie group acting via isometries on a finite dimensional Riemannian manifold $P$ of dimension $d$, the two notions of minimality coincide since $b_0 = d$, $\text{Vol}_{\text{reg}}(O_p) = \text{vol}(O_p)$ and (1.10) yields:

$$\text{tr} S^p = -\text{grad} \log \text{vol}(O_p)$$

where $S^p$ is the second fundamental form. It tells us that the orbits of $G$, the volume of which are extremal among nearby orbits is a minimal submanifold of $P$. This proposition therefore gives an infinite dimensional version of Hsiang’s [H] theorem.

2) A zeta function formulation of Hsiang’s theorem in infinite dimensions was already discussed in [MRT1] in the context of Yang-Mill’s theory. However, there was an obstruction due to the factor $b_0(p)$ in the zeta-function regularisation procedure which does not appear here since it has been taken care of in definition (3.5) (see also [MRT2]). A formula similar to (3.10) (but using zeta function regularisation) can be found in [GP] (see formula (3.17) combined with formula (A.3)).
3) Proposition 3.3 puts zeta function regularisability and heat-kernel regularisability on the same footing, showing that for regularisable principal fibre bundles defined by an isometric group action both notions of regularisability hold. It also shows that the two notions of minimality do not coincide in general, since they differ by a local term \( \text{grad} b_0 \), they coincide whenever \( b_0 \) is independent of \( p \).

**Proof of Proposition 3.3:** As before, we set \( B_p = \tau_p^* \tau_p \). As before, we shall assume for simplicity that \( B_p \) is injective; the proof then easily extends to the case when the dimension of the kernel is locally constant on each geodesic containing \( p_0 \).

1) Since the fibre bundle is regularisable, we know by Proposition 2.1 that the map \( p \mapsto \det_{\text{reg}}(B_p) \) is Gâteaux-differentiable in the direction \( X \). Let us now check that \( H_{\text{reg}}^\varepsilon \) has a regularized limit trace, applying lemma 1.0. For this, we first investigate the differentiability of the map \( \varepsilon \mapsto \text{tr} H_{\text{reg}}^\varepsilon \). By the result of Proposition 3.2, we have

\[
\text{tr} H_{\text{reg}}^\varepsilon = \frac{1}{2} \int_{\varepsilon}^{\infty} dt \delta_X \frac{e^{-tB_p}}{t} = -\frac{1}{2} \delta_X \log \det_{\varepsilon}(B_p)
\]

The differentiability in \( \varepsilon \) easily follows from the shape of the middle expression. Setting as before \( F_p(t) = \text{tr} e^{-tB_p} - \sum_{j=-J}^{m-1} b_j t^{\frac{j}{m}} \), we have furthermore

\[
\frac{\partial}{\partial \varepsilon} \text{tr} H_{\text{reg}}^\varepsilon = -\frac{1}{2} \varepsilon^{-1} \delta_X \text{tr} e^{-\varepsilon B_p}
\]

\[
= -\frac{1}{2} \delta_X \frac{F_p(\varepsilon)}{\varepsilon} - \frac{1}{2} \sum_{j=-J}^{m-1} \delta_X b_j \varepsilon^{\frac{j-m}{m}}.
\]

From the regularisability of the fibre bundle follows that \( |\delta_X F_p(\varepsilon)| \leq K \) for some \( K > 0 \) and \( 0 < \varepsilon < 1 \) (see assumption (2.5 bis)) which in turn implies that

\[
\frac{\partial}{\partial \varepsilon} \text{tr} H_{\text{reg}}^\varepsilon \simeq_0 -\frac{1}{2} \sum_{j=-J}^{m-1} \delta_X b_j \varepsilon^{\frac{j-m}{m}}.
\]

Setting \( A_\varepsilon \equiv H_{\text{reg}}^\varepsilon \) in Lemma 1.0, we can define the regularised limit trace (replacing \( J \) by \( J + m \) and \( a_j \) by \( -\frac{1}{2} \delta_X b_j + m \))

\[
\text{tr}_{\text{reg}} H_X = \lim_{\varepsilon \to 0} (\text{tr} H_{\text{reg}}^\varepsilon + \frac{1}{2} \sum_{j=-J}^{m-1} m \frac{\delta_X b_j}{j} \varepsilon^{\frac{j}{m}} + \frac{1}{2} \delta_X b_0 \log \varepsilon)
\]

\[
= \lim_{\varepsilon \to 0} -\frac{1}{2} \left( \delta_X \log \det_{\varepsilon}(B_p) - \sum_{j=-J}^{m-1} m \delta_X b_j \varepsilon^{\frac{j-m}{m}} - \delta_X b_0 \log \varepsilon \right) \quad \text{by (3.7 bis)}
\]

\[
= \lim_{\varepsilon \to 0} -\frac{1}{2} \delta_X \left( \log \det_{\varepsilon}(B_p) - \sum_{j=-J}^{m-1} m b_j \varepsilon^{\frac{j-m}{m}} - b_0 \log \varepsilon \right)
\]

\[
= -\frac{1}{2} \delta_X \log \det_{\text{reg}}(B_p) \quad \text{by (1.6)}
\]

\[
= \frac{1}{2} \left[ \sum_{j=-J, j \neq 0}^{m-1} \frac{m \delta_X b_j}{j} + \int_0^1 dt \frac{\delta_X F_p(t)}{t} + \int_1^\infty t^{-1} \delta_X \text{tr} e^{-t\tau_p^* \tau_p} dt \right] \quad \text{by (2.6.b)}
\]
When the coefficients \( b_j \) are extremal at \( p_0 \), we have

\[
\text{tr} \mathcal{H}^\varepsilon_X = \text{tr} \mathcal{H}^\varepsilon_X + \frac{1}{2} \sum_{j=-J}^{-1} \frac{m \delta_X b_j}{j} \varepsilon^\frac{1}{m} + \frac{1}{2} \delta_X b_0 \log \varepsilon
\]

so that \( \lim_{\varepsilon \to 0} \text{tr} \mathcal{H}^\varepsilon_X = \text{tr}_{\text{reg}} \mathcal{H}_X \). Since \( \mathcal{H}_X \) is trace class by assumption, going back to the definition of \( \mathcal{H}^\varepsilon_X \), one sees that \( \lim_{\varepsilon \to 0} \text{tr} \mathcal{H}^\varepsilon_X = \text{tr} \mathcal{H}_X \), which yields the second point in 1) of proposition 3.3.

2) It is well known that the expression \( \Gamma(s)^{-1} \int_0^\infty t^{s-1} \sum_n e^{-t \lambda_n} \) is finite for \( \text{Re} \) large enough and that it has a meromorphic continuation to the whole plane. Since \( \Gamma(s) = (s-1) \Gamma(s-1) \), we have for \( s \) with large enough real part:

\[
\Gamma(s)^{-1} \int_0^\infty t^{s-1} \sum_n e^{-t \lambda_n} \delta_X \lambda_n^p dt = (s-1)^{-1} \frac{1}{\Gamma(s-1)} \int_0^\infty t^{s-1} \sum_n e^{-t \lambda_n^p} \delta_X \lambda_n^p dt
\]

\[
= -(s-1)^{-1} \frac{1}{\Gamma(s-1)} \int_0^\infty t^{s-2} \delta_X \text{tr} e^{-tB_p} dt \quad \text{see assumption (2.2) and lemma 3.1}
\]

\[
= -(s-1)^{-1} \frac{1}{\Gamma(s-1)} \left( \sum_{j=-J}^{-1} \int_0^1 t^{\frac{1}{m}+s-2} \delta_X b_j dt + \int_1^\infty t^{s-2} \delta_X \text{tr} e^{-tB_p} dt + \int_1^1 \delta_X F_p(t) t^{s-2} dt \right) \quad \text{by (2.5)}
\]

\[
= -(s-1)^{-1} \frac{1}{\Gamma(s-1)} \left[ \sum_{j=-J}^{-1} \frac{1}{s-m} \delta_X b_j + \int_1^1 t^{s-2} \delta_X F_p(t) dt \right]
\]

setting \( F_p(t) = \text{tr} e^{-tB_p} - \sum_{j=-J}^{-1} b_j(p) \frac{t^j}{m} \). Hence, since \( \Gamma(s)^{-1} = s + \gamma s^2 + O(s^3) \) around \( s = 0 \), going to the limit \( s \to 1 \), we find:

\[
\lim_{s \to 1} \Gamma(s)^{-1} \int_0^\infty t^{s-1} \sum_n e^{-t \lambda_n^p} \delta_X \lambda_n^p dt + (s-1)^{-1} \delta_X b_0(p) = \\
= \lim_{s \to 0} (-1 - \gamma s + O(s^2)) \left[ \sum_{j=-J,j\neq 0} \frac{1}{s-m} \delta_X b_j + \int_1^\infty t^{s-1} \delta_X \text{tr} e^{-tB_p} dt \right. \\
+ \left. \int_0^1 t^{s-1} \delta_X F_p(t) dt - \gamma \delta_X b_0 \right]
\]

\[
= \delta_X \det_{\text{reg}}(B_p) - \gamma \delta_X b_0 \quad \text{by formula (1.6) and (2.6 b)}
\]

\[
= -2 \text{tr}_{\text{reg}} \mathcal{H}_X - \gamma \delta_X b_0
\]

\[
= \delta_X \log \text{Det}_{\text{reg}}(B) \quad \text{by formula (1.10 bis)}
\]

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where $\lim_{s \to 0} \int_1^\infty t^{s-1} \delta_X \text{ tr } e^{-tB_p} dt = \int_1^\infty t^{-1} \delta_X \text{ tr } e^{-tB_p} dt$ holds using estimate (*) arising in the proof of proposition 2.0 and $\lim_{s \to 0} \int_0^1 t^{s-1} \delta_X F_p(t) dt + s^{-1} \delta_X b_0 = \int_0^1 t^{-1} \delta_X F_p(t) dt$ by (2.4 b) and using dominated convergence.

The rest of the assertions of 2) then easily follow.

3) This is a consequence of 2) using the definition of the regularised mean principal curvature vector in a Hilbert manifold.  

\[\blacksquare\]
Appendix 0: Gâteaux-differentiability

We extend here the classical notion of Gâteaux-differentiability on Hilbert spaces to Hilbert manifolds. We refer the reader to [AMR] for example for the case of Banach spaces and [E] for a strong version of this notion on infinite dimensional differentiable manifolds.

Let $P$ denote a Hilbert manifold modelled on a Hilbert space $K$ and equipped with a (possibly weak) Riemannian structure, which induces on $T_pP$ a scalar product denoted by $<\cdot, \cdot>_p$. Denote by $H_p$ the closure of $T_pP$ for the norm induced by $<\cdot, \cdot>_p$. We shall assume that this Riemannian structure induces an exponential map and a connection with the usual properties. In particular this yields a local diffeomorphism from the tangent bundle to the manifold.

Gâteaux-differentiability of a function

A function $p \rightarrow f(p) \in \mathcal{R}$ is said to be Gâteaux-differentiable in the direction $X$ where $X$ is a vector at point $p_0 \in P$ if

$$\delta_X f \equiv \lim_{\kappa \rightarrow 0} \kappa^{-1}(f(p_\kappa) - f(p_0))$$

exists with $p_\kappa = \exp_{p_0} \kappa X$, $\exp$ denoting the geodesic exponential on $P$. A function $p \rightarrow f(p) \in \mathcal{R}$ is said to be Gâteaux-differentiable at a point $p_0$ if it is Gâteaux-differentiable in all directions at that point.

The map $X \rightarrow \delta_X f$ defined on $H_p$ is denoted by $\delta_{p_0} f$. If $f$ is Gâteaux differentiable at point $p_0$ and if the map $\delta_{p_0} f$ is a bounded linear form on $H_{p_0}$, by Riesz theorem, we can identify it with a vector $G_{p_0} f$ in $H_{p_0}$:

$$< G_{p_0} f, X >_{p_0} = \delta_X f \quad \forall X \in T_{p_0} P$$

If the Riemannian structure is strong, $G_{p_0} f$ lies in $T_{p_0} P$.

If $f$ is differentiable at point $p_0$, $f$ is Gâteaux-differentiable at $p_0$ and $\delta_{p_0} f = d_{p_0} f$ is a bounded linear map on $T_p P$. Hence, if the Riemannian structure is strong, $d_{p_0} f$ is identified to a vector $\operatorname{grad}_{p_0} f \in T_{p_0} P$ by $d_{p_0} f(X) = < \operatorname{grad}_{p_0} f, X >_{p_0}$ and we have $\operatorname{grad}_{p_0} f = G_{p_0} f$.

If for any $p_0 \in P$ and for any vector $X$ at $p_0$, the map $\delta_{p_0} f$ is a bounded linear form on the Hilbert space $H_p$ and if for any point $p_0$, the map $p \mapsto \delta_p f \circ \exp_{p_0}^*$ is continuous on exponential charts at $p_0$ for the operator norm, the function $f$ will be called $C^1$-Gâteaux-differentiable. Although Gâteaux-differentiability is weaker than differentiability, $C^1$ Gâteaux-differentiability implies $C^1$ differentiability.

**Lemma 0.1**: A $C^1$-Gâteaux-differentiable function is $C^1$ differentiable.

**Proof**: The proof goes as in the case of a Hilbert space (see [AMR] Corollary 2.4.10) using the exponential map.

Let $V(p_0)$ be a neighborhood of $p_0$ small enough so as to be in the image of the exponential map at point $p_0$ and on which $p \mapsto G_p f$ is uniformly continuous. For $p \in V(p_0)$, there is
a vector \( X \in T_{p_0} \mathcal{P} \) such that \( \exp_{p_0} X = p \) and we shall set \( p_\kappa = \exp_{p_0} \kappa X \). Then

\[
|f(p) - f(p_0) - \delta_{p_0} f X| = \left| \int_0^1 \left( \frac{d}{d\kappa} f(p_\kappa) - \delta_{p_0} X \right) d\kappa \right|
\]

\[
\leq \int_0^1 |\delta_{p_\kappa} f \exp_{p_0*}(\kappa X)(X) - \delta_{p_0} f X| d\kappa
\]

\[
\leq \int_0^1 \|\delta_{p_\kappa} f \exp_{p_0*}(\kappa \cdot)(\cdot) - \delta_{p_0} f\| \|X\| d\kappa
\]

\[
\leq \sup_{\kappa \in [0,1]} \|\delta_{p_\kappa} f \exp_{p_0*}(\kappa \cdot)(\cdot) - \delta_{p_0} f\| \|X\|
\]

and the r.h.s converges to zero as \( X \) goes to 0. This says that \( df(p_0) \) exists and is equal to \( G_{p_0} f \) so that \( f \) is differentiable and \( C^1 \).

**Remark:** In finite dimensions, one can show in a similar way that \( C^1\)-Gâteaux-differentiability implies \( C^1\)-differentiability.

**Gâteaux-differentiability of operators**

For a family of bounded operators \( B_p \) on a given Hilbert space \( H \), \( p \) varying in \( \mathcal{P} \), we shall say that the map \( p \rightarrow B_p \) is Gâteaux differentiable in the direction \( X \) at point \( p_0 \) if \( \kappa^{-1}(B_{p_\kappa} - B_{p_0}) \) converges as \( \kappa \) goes to zero in norm to an operator \( \delta_X B \).

A family of unbounded operators \( B_p \) defined on a common dense domain \( D \) in a given Hilbert space \( H \) to a given Hilbert space \( K \), will be called Gâteaux- differentiable at point \( p_0 \) in the direction \( X \) if for every \( u \) in \( D \), \( \kappa^{-1} B_{p_\kappa} u - B_{p_0} u \) converges to a vector in \( K \) when \( \kappa \rightarrow 0 \), thus defining a densely defined operator \( \delta_X B \) on \( H \). Furthermore, if \( p \mapsto B_p^* \) is Gâteaux-differentiable and if the spaces \( H \) and \( K \) are equipped with fixed scalar products, then \( \delta_X B_p^* \subset (\delta_X B_p)^* \). Indeed, for any \( u \) in the domain of \( \delta_X B_p \), \( v \) in the domain of \( \delta_X B_p^* \), we have:

\[
< \delta_X B_p u, v >_K = \lim_{\kappa \rightarrow 0} < \frac{B_{p_\kappa} - B_{p_0}}{\kappa} u, v >_K
\]

\[
= \lim_{\kappa \rightarrow 0} < u, \frac{B_{p_\kappa}^* - B_{p_0}^*}{\kappa} v >
\]

\[
= < u, \delta_X B_p v >
\]

In particular, the Gâteaux-differential at point \( p_0 \) of a family of rotations \( R_p \) of \( H \) which coincide with identity at point \( p_0 \) is antisymmetric. Indeed, differentiating the relation \( R_p^* R_p = I \) at point \( p_0 \), we obtain \( \delta_X R_p^* = -\delta_X R_p \). Since \( \delta_X R_p^* \subset (\delta_X R_p)^* \), we find that \( -\delta_X R_p \subset (\delta_X R_p)^* \), which says that \( R_p \) is antisymmetric.

**Covariant Gâteaux-differentiability**

Let \( O(\mathcal{P}) \) be the orthonormal bundle and let us associate to any \( C^1 \) curve \( \sigma \) in \( \mathcal{P} \) its horizontal lift \( \tilde{\sigma} \) in \( O(\mathcal{P}) \) given \( \tilde{\sigma}(0) \). Thus, to a curve \( \sigma(\kappa) \), we associate \( \tilde{\sigma}(\kappa) \in \text{Isom}(H, T_{p_\kappa} \mathcal{P}) \) where \( p_\kappa = \sigma(\kappa) \).
For $p_0 \in \mathcal{P}$ and $X \in T_{p_0} \mathcal{P}$, define the curve $\sigma(\kappa) = \exp_{p_0} \kappa X$.

**Definition:** Let $p_0 \in \mathcal{P}$ and $X$ be a vector at point $p_0$. A vector field $V$ on $\mathcal{P}$ will be called Gâteaux-differentiable at point $p_0$ in the direction $X$ if, setting $\sigma(\kappa) = \exp_{p_0} \kappa X = p_\kappa$, if the expression $\kappa^{-1} \ddot{\sigma}(0)(\dot{\sigma}(\kappa)^{-1} V(p_\kappa) - \dot{\sigma}(0)^{-1} V(p_0))$ has a well defined limit in $T_{p_0} \mathcal{P}$ when $\kappa$ goes to zero. This limit is denoted by $\delta_X V$ so that

$$
\delta_X V = \dot{\sigma}(0) \frac{d}{d\kappa_{\kappa=0}} (\dot{\sigma}(\kappa)^{-1} V(p_\kappa)).
$$

Let $p \mapsto \mathcal{T}_p$ be a field of operators acting from a dense space $D \subset H$ of a fixed Hilbert space $H$ into $T_p \mathcal{P}$. The map $\mathcal{T}_p$ will be called Gâteaux-differentiable at point $p_0$ in the direction $X$ if for every vector $u \in D$, the vector field $\mathcal{T}_p u$ is Gâteaux-differentiable at point $p_0$ in the direction $X$. We shall write $\delta_X (\mathcal{T}_p u) = \delta_X \mathcal{T}_p u$. 

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Appendix A. Regularizability and minimality of the orbits for a loop group coadjoint action

In this paragraph, we want to investigate the heat-kernel regularizability and minimality of the orbits for the coadjoint action of the loop group in agreement with the work by King and Terng [KT], who showed they are zeta function regularisable and minimal submanifolds. We shall take the same notations as in [KT].

Let $G$ be a connected, compact Lie group, $\mathfrak{g}$ its Lie algebra, $(\cdot, \cdot)_0$ a fixed $Ad$ invariant inner product on $\mathfrak{g}$. Let

$$\Omega_e([0, 1], G) \equiv \{ g \in H^1([0, 1], G), g(0) = g(1) = e \}$$

With the notations of the previous paragraph, we take $G = \Omega_e([0, 1], G)$, which is a Hilbert Lie group. The exponential map is given by $(\text{Exp}_\xi)(t) = \exp t \xi$. We take $P = H^0([0, 1], \mathfrak{g})$.

$\Omega_e([0, 1], G)$ acts on $H^0([0, 1], \mathfrak{g})$ on the left by

$$(g, \gamma) \mapsto g \gamma g^{-1} - g' g^{-1} = g \ast \gamma.$$  

The action is free, smooth and isometric (see [KT] and references therein). The orbit space is $G$ and the map

$$\pi : H^0([0, 1], \mathfrak{g}) \to G$$

$$\gamma \mapsto g(1) \quad \text{with} \quad g^{-1} g' = \gamma, \quad g(0) = e$$

yields a principal fibre bundle structure of $H^0([0, 1], \mathfrak{g})$ with structure group $\Omega_e([0, 1], G)$, the fibres of which are congruent w.r.t isometries of $H^0([0, 1], \mathfrak{g})$. For $g \in H^1([0, 1], \mathfrak{g})$, $\gamma \in H^1([0, 1], G)$ with $g(0) = e$, we have $\pi(\gamma \ast g) = \pi(\gamma) g(1)^{-1}$. Hence for another $\tilde{\gamma} \in H^1([0, 1], \mathfrak{g})$, choosing $g(1) = \pi(\tilde{\gamma})^{-1} \pi(\gamma)$, we have $\pi(\gamma \ast g) = \pi(\tilde{\gamma})$.

In the following, $\hat{a}$ denotes the constant loop with fixed value $a$.

For fixed $\gamma \in H^0([0, 1], \mathfrak{g})$, the tangent map at unit element $\hat{e} \in \Omega_e([0, 1], G)$ to the map:

$$\theta_\gamma : \Omega_e([0, 1], G) \to H^0([0, 1], \mathfrak{g})$$

$$g \mapsto g \ast \gamma$$

is given by

$$\tau_\gamma : \Omega_0([0, 1], \mathfrak{g}) \to H^0([0, 1], \mathfrak{g})$$

$$u \mapsto [u, \gamma] - u'$$  

(A.1)

where $\Omega_0([0, 1], \mathfrak{g}) \equiv T\hat{e}(\Omega_e([0, 1], G)) = \{ u \in H^1([0, 1], \mathfrak{g}), u(0) = u(1) = 0 \}$.

Notice that $\tau_\gamma$ is the sum of a first order differential operator independent of $\gamma$ and of a bounded operator which depends on $\gamma$. Of course, if $G$ is abelian the $\gamma$-dependent part vanishes.
Let us equip $G$ with a fixed weak right invariant Riemannian metric defined by the inner product
\[ <u, v> = \int_0^1 (u(t), v(t))_0 dt \]
on the space
\[ T_g(\Omega_e([0, 1], G)) = \{ u \cdot g, u \in H^1([0, 1], g), u(0) = u(1) = 0 \} \]
by
\[ <u_1 \cdot g, u_2 \cdot g> = \int_0^1 (u_1(t), u_2(t))_0 dt \]
where $u \cdot g = R_{g^*}u$.

The adjoint $\tau^*_\gamma$ of $\tau_\gamma$ w.r.t. this scalar product is given by $\tau^*_\gamma(\xi) = [\gamma, \xi] + \xi'$ for $\xi \in H^1([0, 1], g)$, $\xi(0) = \xi(1)$ and $\tau^*_\gamma \tau_\gamma$ is self adjoint on
\[ D(\tau^*_\gamma \tau_\gamma) \equiv \{ u \in H^2([0, 1], g), u(0) = u(1), \quad u'(0) = u'(1) \}. \]

Notice that it is not injective on its domain since it contains the constant loops.

**Proposition A.1**: The orbits are heat-kernel preregularizable and strongly minimal in $H^0([0, 1], g)$. The preregularized volumes of the orbits are constant.

**Proof**:

Since every orbit contains a constant loop (see [Se] Proposition 8.2) and since the action is isometric, it is enough to consider the orbits of constant loops. Since the orbits are congruent via isometries (see [KT]), is also enough show the heat-kernel regularisability and strong minimality of the orbit containing the 0-loop, namely the orbit of $\gamma_0 = \hat{0}$.

Since $\text{Ker} \tau^*_0 = \{ \hat{a}, a \in g \}$, the normal space to the zero orbit at point $\hat{0}$ is the space of constant loops. We therefore want to show that for any $\varepsilon > 0$ the operator $\mathcal{H}_\varepsilon$ is trace-class for $\varepsilon > 0$ and $\text{tr} \mathcal{H}_\varepsilon = 0$ for any constant loop $\hat{a}$, $\hat{x}$.

In order to apply proposition 2.2, we first check the preregularisability of the bundle. From the above considerations, follows that it is enough to check assumptions (2.1)-(2.3) for $p_0 = \hat{0}$. Let $x \in g$ and let $t$ be the maximal abelian subalgebra of $g$ containing $x$. The constant loop $\hat{x}$ will play the role of the tangent vector $X$ at point $p_0$ used in the main bulk of the paper. Let $a = sx$, $s \in [0, 1]$, so that $\hat{a}$ lies on the geodesic starting at $\hat{0}$ in direction $\hat{x}$. The complexified Lie algebra $g\mathfrak{f}$ has an orthonormal basis built up from $z_\alpha, \alpha \in \Delta^+$, and $a_k, k = 1, \cdots, r$, $r$ being the dimension of $t\mathfrak{f}$, and $\Delta^+$ the set of positive roots of $G$. They satisfy the following anticommuting relations:

\[ [a, z_\alpha] = -i\alpha(a)z_\alpha. \]

Setting $z_\alpha = x_\alpha + iy_\alpha$, this yields:
\[ [a, x_\alpha] = \alpha(a)y_\alpha, [a, y_\alpha] = -\alpha(a)x_\alpha, [a, a_i] = 0. \quad (A.2) \]
Let \( \Omega_e([0, 1], G)_\mathfrak{A} \) denote the complexification of \( \Omega_e([0, 1], G) \) and \( \overline{\Omega_e([0, 1], G)}_\mathfrak{A} \) its closure w.r. to the hermitian form \( H(\cdot, \cdot) \) induced by the \( L^2 \) scalar product \( \langle \cdot, \cdot \rangle \). An orthonormal basis of \( \Omega_e([0, 1], G)_\mathfrak{A} \) is given by

\[
z_{n,\alpha}(t) = z_{\alpha}e^{2i\pi nt}, \quad a_{k,n}(t) = a_ke^{2i\pi nt}, \quad n \in \mathbb{Z}, \alpha \in \Delta^+, k = 1, \ldots, r.
\]

This yields an orthonormal basis of the (real) \( L^2 \) closure \( \overline{\Omega_e([0, 1], G)} \) of \( \Omega_e([0, 1], G) \):

\[
r_{\alpha,n} = \sqrt{2}\text{Re}(z_{\alpha,n}), \quad s_{\alpha,n} = \sqrt{2}\text{Im}(z_{\alpha,n}),
\]

\[
\rho_{k,n} = \sqrt{2}\text{Re}(a_{k,n}), \quad \sigma_{k,n} = \sqrt{2}\text{Im}(a_{k,n})(n \neq 0)
\]

An easy computation using definition (A.1) and relations (A.2) shows that for \( u = \sum_{\alpha,n} u_{\alpha,n}z_{\alpha,n} + \sum_{k,n} \eta_{k,n}a_{k,n} \in \Omega_e([0, 1], G)_\mathfrak{A} \):

\[
\tau_{\alpha}(u) = \sum_{\alpha,n} (i\alpha(a) - 2i\pi n)u_{\alpha,n}z_{\alpha,n} - \sum_{k,n} (2i\pi n)\eta_{k,n}a_{k,n}
\]

which yields:

\[
\tau_{\alpha}(r_{\alpha,n}) = (2\pi n - \alpha(a))s_{\alpha,n}, \quad \tau_{\alpha}^*\tau_{\alpha}(r_{\alpha,n}) = (2\pi n - \alpha(a))^2(r_{\alpha,n})
\]

\[
\tau_{\alpha}(s_{\alpha,n}) = (-2\pi n + \alpha(a))r_{\alpha,n}, \quad \tau_{\alpha}^*\tau_{\alpha}(s_{\alpha,n}) = (2\pi n - \alpha(a))^2(s_{\alpha,n})
\]

\[
\tau_{\alpha}(\rho_{k,n}) = -2\pi n\sigma_{k,n}, \quad \tau_{\alpha}^*\tau_{\alpha}(\rho_{k,n}) = (2\pi n)^2(\rho_{k,n})
\]

\[
\tau_{\alpha}(\sigma_{k,n}) = 2\pi n\rho_{k,n}, \quad \tau_{\alpha}^*\tau_{\alpha}(\sigma_{k,n}) = (2\pi n)^2(\sigma_{k,n}), (n \neq 0).
\]

This proves that the operator \( \tau_{\alpha}^*\tau_{\alpha} \) has purely discrete spectrum given by

\[
(2\pi n - \alpha(a))^2, \quad (2\pi n + \alpha(a))^2, \quad n \in \mathbb{N}^+ \text{ (each taken with multiplicity 2)},
\]

\[
(\alpha(a))^2, \quad \alpha \in \Delta^+ \text{ (taken with multiplicity 2)},
\]

\[
(2\pi n)^2, \quad n \in \mathbb{N}^+ \text{ (each taken with multiplicity 4r)},
\]

\[
0 \text{ (taken with multiplicity r)}
\]

Clearly the eigenvalues behave asymptotically as \( (2\pi n)^2 \) so that the first assumption for a prergeralisable bundle is satisfied. Each of these eigenvalues is clearly differentiable as a function of \( a \) and we set

\[
\beta_{n}^{\varepsilon,\tilde{\varepsilon}}(t) \equiv \delta_{\varepsilon}(\lambda_{n}^{\tilde{\varepsilon}})e^{-t\lambda_{n}^{\tilde{\varepsilon}}}
\]

For \( \lambda_{n}^{\tilde{\varepsilon}} = (2\pi n - \alpha(a))^2 \) or \( \lambda_{n}^{\varepsilon} = (2\pi n + \alpha(a))^2 \), we have \( \beta_{n}^{\varepsilon,\tilde{\varepsilon}} = 4\pi n\alpha(x)e^{-t(2\pi n)^2} \). For \( \lambda_{n}^{\tilde{\varepsilon}} = (2\pi n)^2 \), \( \beta_{n}^{\varepsilon,\tilde{\varepsilon}} = 0 \). Hence \( \beta_{n}^{\varepsilon,\tilde{\varepsilon}} \) which is independent of \( a \) satisfies \( \sum_{n} |\beta_{n}^{\varepsilon,\tilde{\varepsilon}}| < \infty \) which proves (2.1) and (2.2) then follows easily. The bundle is therefore prergeralisable.
From lemma 3.1 and the fact that $G$ is equipped with a fixed Riemannian structure then follows that the spectrum of $H^\varepsilon_\hat{x}$ is purely discrete and given by

$$(\mu^\varepsilon_n, n \in \mathbb{N}) \equiv \left( \frac{\alpha(x)}{2\pi n} e^{-\varepsilon(2\pi n)^2}, -\frac{\alpha(x)}{2\pi n} e^{-\varepsilon(2\pi n)^2}, \alpha \in \Delta^+, n \in \mathbb{N}^* \right)$$

$0, \ldots, 0$ ($r$ times), $n \in \mathbb{N}^*$

which coincides with the expression obtained by [KT].

Applying proposition 3.2, we have that $H^\varepsilon_\hat{x}$ is trace class, $a \mapsto \det_\varepsilon(\tau_\hat{a}^*\tau_\hat{a})$ is Gâteaux-differentiable in the direction $\hat{x}$ at $\hat{0}$ and

$$\text{tr}H^\varepsilon_\hat{x} = -\frac{1}{2} \delta_\varepsilon \log \text{vol}'_\varepsilon(O_{\hat{a}}) \quad (*)$$

where $O_{\hat{a}}$ is the orbit of $\hat{a}$ and $\text{vol}'_\varepsilon$ is to be understood in the sense of proposition 3.2.

Since the l.h.s of $(*)$ is the general term of an absolutely convergent series, we can reorder the terms and write

$$\delta_\varepsilon h_\varepsilon(\tau_\hat{a}^*\tau_\hat{a}) = 2 \sum_{n \in \mathbb{N}} (\mu^\varepsilon_{n} \hat{x} + \mu^\varepsilon_{-n} \hat{x}) = 0$$

so that

$$\text{tr}H^\varepsilon_\hat{x} = \text{< grad log det}_\varepsilon(\tau_\hat{a}^*\tau_\hat{a}), \hat{x} \hat{x} >_{\hat{0}} = 0$$

Thus the orbits are heat minimal the preregularized volumes of the orbits are constant. ■

**Remark:** This confirms the fact proved by King and Terng [KT] that the orbits are zeta function regularizable and minimal and that the zeta function regularized volume of the fibres are constant.

**Appendix B. Gauge orbits for Yang-Mills theory**

In this appendix, we confront the notions of regularity and minimality based on heat-kernel regularizations developed above with those developed in [MRT1] based on zeta function regularizations in the case of Yang-Mills theory. This appendix does not offer new results, its only purpose being to relate the general framework developed in the bulk of this paper with the concrete example of Yang-Mills theory. This appendix is loosely written and we refer the reader to [MRT1] for a precise presentation of the case of Yang-Mills orbits.

The comparison of these two approaches shows that even in this particular example, they are not equivalent in general. Only in very particular cases does the notion of heat-kernel minimality coincide with that of zeta function minimality.

For the description and results that follow, we refer the reader to [KR], [FU] and [MV] (and many other references mentioned therein) where the gauge action in Yang-Mills theory was investigated in details.

Let $\pi : P \to P/G \simeq M$ be a smooth principal bundle over an $n$-dimensional smooth oriented manifold $M$, where $G$ and $M$ are compact connected, $G$ being a Lie group with Lie algebra $\mathfrak{g}$. We shall specialise $G$ to a matrix group here.
If we allow manifolds with boundary, we recover the framework dual to the one described in section II (in the sense that the action on the left on $g$ valued vector fields there is replaced here by an action on the right on $g$-valued forms), when applying the setting we are about to describe to $M = [0, 1]$ with Dirichlet boundary conditions. However, for the sake of clarity, we take $M$ boundaryless here and have treated the dimension 1 case with boundary separately in section II.

Automorphisms of $P$ are described equivalently

1) as fibre preserving homeomorphisms $\phi : P \to P$ commuting with the $G$-action

2) as continuous sections of the principal fibre bundle $E_G \equiv P \times_G G$ associated to $P$, given by the quotient of $P \times G$ by the right action

$$(P \times G) \times G \to P \times G$$

$$(p, g_1) \cdot g \to (p \cdot g, g^{-1}g_1g)$$

which is free and proper.

3) as continuous functions $\phi : P \to G$ satisfying

$$\phi(p \cdot g) = g^{-1}\phi(p)g$$

A connection on $P$ can be described equivalently as

1) a subbundle $A \subset TP$ such that

a) $\pi_* A_p = T_{\pi(p)}M$ and $A_p \cap T_p F_{\pi(p)} = \{0\}$,

b) $A_{p \cdot g} = (R_g)_* A_p$, for $g \in G, p \in P$.

where $F_x$ is the fibre over $x$.

2) a $g$-valued one form $A$ on $P$ such that

a) $\forall X \in g, \forall p \in P, A(\tau_p X) = X$

b) $\forall g \in G, \forall v \in T_p P, A_{p \cdot g}(R_g)_* v = Ad g^{-1} A_p (v)$.

The action of automorphisms on connections can be described as follows. An automorphism $\phi$ of class $C^2$ acts on a connection of class $C^1$ in the following way:

$$(\phi \cdot A)_{\phi(p)} = T_p \phi \cdot A_p \quad (B.1)$$

Expressing $A$ in terms of $g$-valued one forms and $\phi$ as a $G$ valued function on $P$, we have

$$(\phi \cdot A)_p = Ad \phi (p) \circ A_p - R_{\phi(p)*}^{-1} \circ T_p \phi$$

It is clear from this formula that $\phi \cdot A = A \forall A$ if and only if $\phi$ is constant and $Ad \phi = Id$, i.e $\phi \in Z(G)$, the center of $G$.

Let us, with the notations of section I, describe the group $G$. Let $k > \frac{n}{2} + 1$ be an integer and let $G^k$ denote the automorphisms of $P$ of class $H^k$, i.e the group of $H^k$ sections of the bundle $E_G$, which is a Hilbert Lie group modelled on $H^k(E_g)$, the space of $H^k$ sections of
the adjoint bundle $E_{\mathfrak{g}}$ which is the vector bundle associated to $P$ with fibre $\mathfrak{g}$ defined as the quotient of $P \times \mathfrak{g}$ for the action of the group $G$ given by:

$$(P \times \mathfrak{g}) \times G \to E_{\mathfrak{g}}$$

$$((p, \xi), g) \to (p \cdot g, adg\xi)$$

$H^k(E_{\mathfrak{g}})$ can also be seen as the space of $H^k$ $\mathfrak{g}$ valued equivariant functions on $P$:

$$H^k(E_{\mathfrak{g}}) = \{ \phi \in H^k(P, \mathfrak{g}), \forall g \in G, \forall p \in P, \phi(p \cdot g) = g^{-1}\phi(p)g \}$$

Let now $G^{k+1}$ denote the quotient of $G^k$ by its center. It is also a Hilbert lie group modelled on $H^k(E_{\mathfrak{g}})$. With the notations of section I, we set $G \equiv G^{k+1}$.

Let us now, with the notations of section I, describe the manifold $P$ on which $G$ acts. The space $A^k$ of connections of class $H^k$ is a closed affine subspace of the Hilbert space $H^k(\mathfrak{g} \otimes T^*P)$ of $H^k$. By the Sobolev embedding theorems, $H^k$ connections are of class $C^1$ and $H^{k+1}$ automorphisms of $P$ of class $C^2$. Hence $G^{k+1}$ acts on $A^k$ and we shall call a connection $A \in A^k$ irreducible when $\{ \phi \in G^{k+1}, \phi \cdot A = A \}$ is reduced to the constant map with value 1. Let $\overline{A}^k$ denote the space of irreducible connections in $A^k$. It is a Hilbert manifold modelled on $H^k(\mathfrak{g} \otimes T^*P)$. We shall set $P = \overline{A}^k$.

We shall consider the following action of $G$ on $P$:

$$\overline{G}^{k+1} \times \overline{A}^k \to \overline{A}^k$$

$$(\phi, A) \to \phi \cdot A$$

where $\phi \cdot A$ is given by (B.1).

One can show that for $k > \frac{n}{2} + 3$, the action of $G$ on $P$ is free, smooth and proper and that the canonical projection $P \to P/G$ yields a principal fibre bundle structure on the moduli space $P/G$.

The space $\overline{A}^k$ can be equipped with a $G$-invariant $L^2$ (hence weak) Riemannian structure given by

$$< \alpha, \beta > \equiv \int_P (\alpha(p), \beta(p))d\mu(p)$$

where $\mu$ is a smooth $G$-invariant measure on $P$ (recall that $G$ is compact), $\alpha, \beta \mathfrak{g}$ valued one forms on $P$ and $(\cdot, \cdot)$ the bundle metric in $\mathfrak{g} \otimes T^*P$ induced by an $\text{Ad}G$ invariant scalar product on $\mathfrak{g}$ and a $G$ invariant Riemannian metric on $P$. The group $G$ is equipped with a fixed $L^2$ $G$-invariant structure in a similar way. The Riemannian structure on $\overline{A}^k$ induces a Riemannian connection (see e.g [KR]).

For given $A \in \overline{A}^k$, the tangent map at the constant map with value 1$_G$ to

$$\theta_A : \overline{G}^{k+1} \to \overline{A}^k$$

$$\phi \to \phi \cdot A$$
reads
\[ \tau_A : H^{k+1}(E_g) \to H^k(g \otimes T^*P) \]
\[ \lambda \to \text{ad}\lambda \circ A - T\lambda = -\nabla A \lambda \]

where \( \nabla_A \) is the covariant derivative in the associated bundle \( E_g \) defined by the connection \( A \) on \( P \) or equivalently \( \nabla_A = \nabla + [A, \cdot] \) where \( \nabla \) is the covariant derivative \( g \to g \otimes T^*P \) defined by the Riemannian metric on \( P \). Notice that taking \( M = S^1 \), we recover formula (A.1). Notice that when \( G \) is abelian, \( \tau_A = -\nabla \) so that \( \tau_A \) is independent of the choice of the connection \( A \). It is easy to check that for an irreducible connection \( A \), the map \( \tau_A \) is injective.

For a \( C^\infty \) connection, if \( \tau_A^* \) denotes the adjoint of \( \tau_A \) w.r. to the \( L^2 \) structure on \( P \) and on \( G \), the operator \( \tau_A^* \tau_A \) is an elliptic operator of order 2 with \( C^\infty \) coefficients (see e.g [KR]). It is by now a well known result that \( \tau_A^* \tau_A \) satisfies assumptions (2.1)-(2.5) (see e.g [MRT1]) so that an orbit \( O_{A_0} \) for the action (B.2) is regularisable given the assumption that the dimension of the kernel of \( \tau_A^* \tau_A \) is locally constant around \( A_0 \). This last assumption is discussed in [MRT1]. Hence the following conclusions:

1) The orbit of any smooth irreducible connection is regularisable. It is minimal if and only if it has minimal heat kernel regularised volume among orbits of the same type. If the group \( G \) is abelian, the orbit of any smooth irreducible connection is (strongly) minimal.

The proof of this statement in the general case follows from proposition 3.3. The result in the abelian case follows from proposition 3.2 using the fact that \( \tau_A^* \tau_A \) is independent of \( A \) and hence so are its eigenvalues since they only involve the Riemannian structures on \( P \) and \( g \) but not connections.

2) If the dimension of \( M \) is odd or equals 2, the two notions of minimality (heat-kernel an zeta function minimality) coincide and if the dimension of \( M \) is 4, this holds for smooth irreducible Yang-Mills connections.

To prove this statement, we check that the Gâteaux differential in any horizontal direction \( X \) of the coefficient \( b_0 \) in the heat-kernel expansion vanishes which then yields the result by proposition 3.3.

By [G], we know that it vanishes when the dimension of \( M \) is odd and that when \( \dim M = 2 \), \( b_0(A) = c_1 \int_M s(g) + (\dim AdP)(\text{Vol}M) \) where \( c_1 \) is a constant, \( s(g) \) the scalar curvature of a metric \( g \) on \( M \) so that it is independent of \( A \) and \( \delta_X b_0 = 0 \).

When \( \dim M = 4 \), \( b_0(A) = c_2(g) + c_3 \int_M \text{tr}|F_A|^2 \), where \( F_A \) is the curvature of \( A \), where \( c_2(g) \) only depends on the metric \( g \) and \( c_3 \) is a nonzero constant. Since Yang-Mills connections are exactly the ones which extremise the Yang-Mills functional \( \int_M |F_A|^2 \), we have \( \delta_X b_0 = \delta_X \int_M |F_A|^2 = 0 \). (see [MRT1]).
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