CONVEXITY OF THE SMALLEST PRINCIPAL CURVATURE OF THE
CONVEX LEVEL SETS OF SOME QUASI-LINEAR ELLIPTIC
EQUATIONS WITH RESPECT TO THE HEIGHT

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Abstract. For the \( p \)-harmonic function with strictly convex level sets, we find a test function which comes from the combination of the norm of gradient of the \( p \)-harmonic function and the smallest principal curvature of the level sets of \( p \)-harmonic function. We prove that this curvature function is convex with respect to the height of the \( p \)-harmonic function. This test function is an affine function of the height when the \( p \)-harmonic function is the \( p \)-Green function on the ball. For the minimal graph, we obtain a similar results.

1. Introduction

In this paper, for \( p \)-harmonic function and minimal graph with strictly convex level sets, we shall explore the relation of its smallest principal curvature of the level sets and the height of the function.

The convexity of the level sets of the solutions of elliptic partial differential equations has been studied for a long time. For instance, Ahlfors \cite{Ahlfors} contains the well-known result that level curves of Green function on simply connected convex domain in the plane are the convex Jordan curves. In 1956, Shiffman \cite{Shiffman} studied the minimal annulus in \( \mathbb{R}^3 \) whose boundary consists of two closed convex curves in parallel planes \( P_1, P_2 \). He proved that the intersection of the surface with any parallel plane \( P \), between \( P_1 \) and \( P_2 \), is a convex Jordan curve. In 1957, Gabriel \cite{Gabriel} proved that the level sets of the Green function on a 3-dimensional bounded convex domain are strictly convex. In 1977, Lewis \cite{Lewis} extended Gabriel’s result to \( p \)-harmonic functions in higher dimensions. Caffarelli-Spruck \cite{Caffarelli-Spruck} generalized the Lewis \cite{Lewis} results to a class of semilinear elliptic partial differential equations. Motivated by the result of Caffarelli-Friedman \cite{Caffarelli-Friedman}, Korevaar \cite{Korevaar} gave a new proof on the results of Gabriel and Lewis by applying the deformation process and the constant rank theorem of the second fundamental form of the convex level sets of \( p \)-harmonic function. A survey of this subject is given by Kawohl \cite{Kawohl}. For more recent related extensions, please see the papers by Bianchini-Longinetti-Salani \cite{Bianchini-Longinetti-Salani} and Bian-Guan-Ma-Xu \cite{Bian-Guan-Ma-Xu}.

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Now we turn to the curvature estimates of the level sets of the solutions of elliptic partial differential equations. For 2-dimensional harmonic function and minimal surface with convex level curves, Ortel-Schneider [22], Longinetti [14] and [15] proved that the curvature of the level curves attains its minimum on the boundary (see Talenti [25] for related results). Longinetti also studied the precisely relation between the curvature of the convex level curves and the height of 2-dimensional minimal surface in [15]. In 2008, Jost-Ma-Ou [10] and Ma-Ye-Ye [19] proved that the Gaussian curvature and the principal curvature of the convex level sets of 3-dimensional harmonic function attains its minimum on the boundary. Later, Ma-Ou-Zhang [18] got the Gaussian curvature estimates of the convex level sets on higher dimensional harmonic function, and Wang-Zhang [26] got the similar curvature estimates of some quasi-linear elliptic equations under certain structure condition [3]. Both of their test functions involved the Gaussian curvature of the boundary and the norm of the gradient on the boundary. For the principal curvature estimates in higher dimension, in terms of the principal curvature of the boundary and the norm of the gradient on the boundary, Chang-Ma-Yang [6] obtained the lower bound estimates of principal curvature for the strictly convex level sets of higher dimensional harmonic functions and solutions to a class of semilinear elliptic equations under certain structure condition [3]. Recently, in Guan-Xu [9], they got a lower bound for the principal curvature of the level sets of solutions to a class of fully nonlinear elliptic equations in convex rings under the general structure condition [3] via the approach of constant rank theorem. For the harmonic functions in space forms, in Ma-Zhang [21], they proved that the level sets are strictly convex and obtained a lower bound estimates for the Gaussian curvature of the convex level sets in convex rings.

Naturally, we hope to give a further characterization of the curvature of the level sets. In Ma-Zhang [20], they studied the concavity of the Gaussian curvature of the convex level sets of $p$-harmonic functions with respect to the height of the function, and got the following results.

**Theorem 1.1** (See [20]). Let $u$ satisfy

\[
\begin{aligned}
\text{div}(|\nabla u|^{p-2}\nabla u) &= 0 & \text{in} & & \Omega = \Omega_0 \setminus \overline{\Omega}_1, \\
u &= 0 & \text{on} & & \partial \Omega_0, \\
u &= 1 & \text{on} & & \partial \Omega_1,
\end{aligned}
\]

where $\Omega_0$ and $\Omega_1$ are bounded smooth convex domains in $\mathbb{R}^n, n \geq 2$, $1 < p < +\infty$ and $\overline{\Omega}_1 \subset \Omega_0$. Let $K$ be the Gaussian curvature of the level sets. Then the function

\[
f(t) = \min_{x \in F_t}(|\nabla u|^{n+1-2p}K)^{\frac{1}{n-1}}(x)
\]
is a concave function for $t \in (0,1)$. Equivalently, for any point $x \in \Gamma_t$, we have the following estimate

$$\left(|\nabla u|^{n+1-2p} K\right)^{\frac{1}{n-1}}(x) \geq (1-t) \max_{\partial \Omega_0}(|\nabla u|^{n+1-2p} K) + t \max_{\partial \Omega_1}(|\nabla u|^{n+1-2p} K) + t.$$ 

Furthermore, the function $f(t)$ is an affine function of the height $t$ when the $p$-harmonic function is the $p$-Green function on the ball.

As a counterpart, we shall consider the convexity of the smallest principal curvature of the level sets of $p$-harmonic function and minimal graph in this paper.

Now we state our main theorems.

**Theorem 1.2.** Let $u$ satisfy

$$\begin{cases}
\text{div}(|\nabla u|^{p-2} \nabla u) = 0 & \text{in } \Omega = \Omega_0 \setminus \Omega_1, \\
u = 0 & \text{on } \partial \Omega_0, \\
u = 1 & \text{on } \partial \Omega_1,
\end{cases}$$

where $\Omega_0$ and $\Omega_1$ are bounded smooth convex domains in $\mathbb{R}^n, n \geq 2, 1 < p < +\infty$ and $\Omega_1 \subset \Omega_0$. Let $k_1$ be the smallest principal curvature of the level sets. Then we have the following statements.

(i) The function

$$f(t) = \max_{x \in \Gamma_t}(|\nabla u|^k k^{-1})(x)$$

is a convex function for $t \in (0,1)$. Equivalently, for any point $x \in \Gamma_t$, we have the following estimate

$$\left(|\nabla u|^k k^{-1}(x) \leq (1-t) \max_{\partial \Omega_0}(|\nabla u|^k k^{-1}) + t \max_{\partial \Omega_1}(|\nabla u|^k k^{-1}).
$$

(ii) For $n = 3, p = 2$, the function

$$f(t) = \min_{x \in \Gamma_t} \log k_1(x)$$

is a concave function for $t \in (0,1)$. Equivalently, for any point $x \in \Gamma_t$, we have the following estimate

$$\log k_1(x) \geq (1-t) \min_{\partial \Omega_0} \log k_1 + t \min_{\partial \Omega_1} \log k_1.$$

Now we give an example to show our estimate is sharp in some sense.

**Remark 1.3.** Let $u$ be the standard $p$-Green function on the ball $B_R(0) \subset \mathbb{R}^n$, i.e.

$$u(x) = \begin{cases}
\frac{|x|^{p-n}}{p-1} - R^{p-n}, & \text{for } 1 < p < n; \\
-\log |x| + \log R, & \text{for } p = n.
\end{cases}$$
Then
\[ |\nabla u|(x) = \begin{cases} 
\frac{n-p}{p-1}|x|^\frac{1-n}{p-1}, & \text{for } 1 < p < n, \\
\frac{1}{|x|}, & \text{for } p = n,
\end{cases} \]
and the smallest principal curvature of the level set through \( x \) is
\[ k_1(x) = |x|^{-1}. \]
Hence for \( t = u(x) \) and \( 1 < p < n \),
\[ (|\nabla u|k_1^{-1})(x) = \frac{n-p}{p-1}|x|^{\frac{p-n}{p-1}} \]
\[ = \frac{n-p}{p-1}[u(x) + R^{\frac{p-n}{p-1}}] \]
\[ = \frac{n-p}{p-1} + \frac{n-p}{p-1} R^{\frac{p-n}{p-1}}. \]

For \( p = n \), we have
\[ (|\nabla u|k_1^{-1})(x) = 1. \]

From the above calculation, we know \( |\nabla u|k_1^{-1} \) is an affine function on the height of the \( p \)-Green function.

For the minimal graph, we have the following results.

**Theorem 1.4.** Let \( u \) satisfy
\[
\begin{cases} 
\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0 & \text{in } \Omega = \Omega_0 \setminus \overline{\Omega_1}, \\
u = 0 & \text{on } \partial \Omega_0, \\
u = 1 & \text{on } \partial \Omega_1,
\end{cases}
\]
where \( \Omega_0 \) and \( \Omega_1 \) are bounded smooth convex domains in \( \mathbb{R}^n \), \( n \geq 2 \), and \( \overline{\Omega_1} \subset \Omega_0 \). Let \( k_1 \) be the smallest principal curvature of the level sets. Then we have the following statements.

(i) The function
\[ f(t) = \max_{x \in \Gamma_t} (|\nabla u|k_1^{-1})(x) \]
is a convex function for \( t \in (0,1) \). Equivalently, for any point \( x \in \Gamma_t \), we have the following estimate
\[ (|\nabla u|k_1^{-1})(x) \leq (1-t) \max_{\partial \Omega_0} (|\nabla u|k_1^{-1}) + t \max_{\partial \Omega_1} (|\nabla u|k_1^{-1}). \]

(ii) For the 3-dimensional case, the function
\[ f(t) = \min_{x \in \Gamma_t} \log k_1(x) \]
is a concave function for $t \in (0, 1)$. Equivalently, for any point $x \in \Gamma_t$, we have the following estimate

$$\log k_1(x) \geq (1 - t) \min_{\partial \Omega_0} \log k_1 + t \min_{\partial \Omega_1} \log k_1.$$ 

**Remark 1.5.** For the 2-dimensional minimal graph over annulus, Longinetti [15] proved that $f(t) = \min_{x \in \Gamma_t} \log k(x)$ is a concave function with respect to $t$. More precisely, for any point $x \in \Gamma_t, 0 < t < 1$, he got the following inequality

$$\log k(x) \geq (1 - t) \min_{\partial \Omega_0} \log k + t \min_{\partial \Omega_1} \log k,$$

where $k$ is the curvature of the level curves.

The estimates above contain the norm of the gradient $\nabla u$, but it is well known that $|\nabla u|$ attains its maximum and minimum on the boundary [18].

To prove Theorem 1.2, let $(b_{ij})$ be the second fundamental form of the convex level sets, and set

$$\varphi(x, \xi) = -\alpha \log |\nabla u|(x) + \log \left( \sum_{i,j} b_{ij} \xi_i \xi_j \right)$$

on $\bar{\Omega} \times S^{n-1}$.

For suitable choice of $\alpha$ and $\beta$, in Section 3 we will derive the following differential inequality

$$L(e^{t \varphi}) \geq 0 \mod \nabla \theta \varphi \quad \text{in } \Omega,$$

where $L$ is the elliptic operator associated to the $p$-Laplace operator and we have modified the terms involving $\nabla \theta \varphi$ with locally bounded coefficients. Then by applying a maximum principle argument, we can obtain the desired result.

In Section 2, we first give the brief definitions on the support function of the level sets, then obtain another expression of the $p$-Laplace equation with support function and the associated elliptic operator, which appeared in [7] and [16]. We prove Theorem 1.2 in Section 3. In Section 4, we shall deal with the minimal graph, and complete the proof of Theorem 1.4. The main technique in the proof of the theorems consists of rearranging the second and third derivative terms using the equation and the first derivative condition for $\varphi$. The key idea is the Pogorelov’s method in a priori estimates for fully nonlinear elliptic equations.

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2. Support function

First we start by introducing some basic concepts and notations. Let $\Omega_0$ and $\Omega_1$ be two bounded smooth open convex domains in $\mathbb{R}^n$ such that $\overline{\Omega}_1 \subset \Omega_0$ and let $\Omega = \Omega_0 \setminus \overline{\Omega}_1$. Let
$u : \Omega \to \mathbb{R}$ be a smooth function with $|\nabla u| > 0$ in $\Omega$, and its level sets are strictly convex with respect to the normal direction $\nabla u$.

For the sake of simplicity, we will assume that

$$u = 0 \quad \text{on} \quad \partial \Omega_0, \quad u = 1 \quad \text{on} \quad \partial \Omega_1.$$ 

For $0 \leq t \leq 1$, we set

$$\Omega_t = \{x \in \Omega_0 : u \geq t\};$$

note that, throughout this paper, we systematically extend $u = 1$ in $\Omega_1$. Then every $x \in \Omega$ belongs to the boundary of $\Omega_t(x)$.

Under these assumptions, it is then possible to define a function

$$H : \mathbb{R}^n \times [0, 1] \to \mathbb{R}$$

as follows: for each $t \in [0, 1]$, $H(\cdot, t)$ is the support function of the convex body $\Omega_t$, i.e.

$$H(X, t) = H_{\Omega_t}(X) \quad \forall X \in \mathbb{R}^n, \ t \in [0, 1].$$

We call $H$ the support function of $u$.

The rest of this section is devoted to derive the $p$-Laplace equation by means of support function. Before doing this, we should reformulate the first and second derivatives of $u$ in support function $h$ which is the restriction of $H(\cdot, t)$ to the unit sphere $S^{n-1}$, see [7, 17, 23]. But for convenience of the reader, we report the main steps here.

Recall that $h$ is the restriction of $H$ to $S^{n-1} \times [0, 1]$, so $h(\theta, t) = H(Y(\theta), t)$ where $Y \in S^{n-1}$ and $\theta = (\theta_1, \cdots, \theta_{n-1})$ is a local coordinate system on $S^{n-1}$. Since the level sets of $u$ are strictly convex, we can define the map

$$x(X, t) = x_{\Omega_t}(X),$$

which for every $(X, t) \in \mathbb{R}^n \setminus \{0\} \times (0, 1)$ assigns the unique point $x \in \Omega$ on the level set $\{u = t\}$ where the gradient of $u$ is parallel to $X$ (and orientation reversed).

Let

$$T_i = \frac{\partial Y}{\partial \theta_i},$$

so that $\{T_1, \cdots, T_{n-1}\}$ is a tangent frame field on $S^{n-1}$, and let

$$x(\theta, t) = x_{\Omega_t}(Y(\theta));$$

we denote its inverse map by

$$\nu : (x_1, \cdots, x_n) \to (\theta_1, \cdots, \theta_{n-1}, t).$$

For $h(\theta, t) = \langle x(\theta, t), Y(\theta) \rangle$, since $Y$ is orthogonal to $\partial \Omega_t$ at $x(\theta, t)$, differentiating the previous equation we obtain

$$h_i = \langle x, T_i \rangle.$$
In order to simplify the calculation, at any fixed point \( x \), we can also assume that \( \{T_1, \cdots, T_{n-1}, Y\} \) is an orthonormal frame positively oriented. Hence, from the previous two equalities, we have

\[
x = hY + \sum_i h_i T_i,
\]
and

\[
\frac{\partial T_i}{\partial \theta_j} = -\delta_{ij} Y \quad \text{at } x,
\]
where the summation index runs from 1 to \( n - 1 \) if no extra explanation, and \( \delta_{ij} \) is the standard Kronecker symbol. Following [7], we obtain at the considered point \( x \),

\[
\frac{\partial x}{\partial t} = h_t Y + \sum_i h_{ti} T_i;
\]

\[
\frac{\partial x}{\partial \theta_j} = hT_j + \sum_i h_{ij} T_i, \quad j = 1, \cdots, n - 1.
\]

The inverse of the above Jacobian matrix is

\[
\begin{align*}
\frac{\partial \theta_i}{\partial x_\alpha} &= \sum_j b^{ij} [T_j - h^{-1}_t h_{ij} Y]_\alpha, \quad \alpha = 1, \cdots, n, \\
\frac{\partial t}{\partial x_\alpha} &= h^{-1}_t [Y]_\alpha, \quad \alpha = 1, \cdots, n;
\end{align*}
\]

where \([·]_\alpha\) denotes the \( \alpha \)-coordinate of the vector in the bracket and \( b^{ij} \) denotes the inverse tensor of the second fundamental form

\[
(2.2) \quad b^{ij} = \left< \frac{\partial x}{\partial \theta_i}, \frac{\partial Y}{\partial \theta_j} \right> = h \delta_{ij} + h_{ij}
\]

of the level set \( \partial \Omega_t \) at \( x(\theta, t) \). The eigenvalue of the tensor \( b^{ij} \) are the principal curvatures \( \kappa_1, \cdots, \kappa_{n-1} \) of \( \partial \Omega_t \) at \( x(\theta, t) \) (see [23]).

The first equation of (2.1) can be rewritten as

\[
\nabla u = \frac{Y}{h_t},
\]
where the left hand side is computed at \( x(\theta, t) \) while the right hand side is computed at \( (\theta, t) \), it follows that

\[
|\nabla u| = -\frac{1}{h_t}.
\]

By chain rule and (2.1), the second derivatives of \( u \) in terms of \( h \) can be computed as

\[
(2.3) \quad u_{\alpha\beta} = \sum_{i,j} \left[ -h^{-2}_t h_{ti} Y + h^{-1}_t T_i \right]_\alpha b^{ij} [T_j - h^{-1}_t h_{ij} Y]_\beta - h^{-3}_t h_{tt} [Y]_\alpha [Y]_\beta,
\]
for \( \alpha, \beta = 1, \cdots, n \).
Thus the $p$-Laplace equation reads
\begin{equation}
 h_{tt} = \sum_{i,j} \left( \frac{1}{p-1} h_t^2 \delta_{ij} + h_{t_i} h_{t_j} \right) b^{ij},
\end{equation}
and the associated linear elliptic operator is
\begin{equation}
 L = \sum_{i,j,p,q} \left( \frac{1}{p-1} h_t^2 \delta_{pq} + h_{t_p} h_{t_q} \right) b^{ip} b^{jq} \frac{\partial^2}{\partial \theta_i \partial \theta_j} - 2 \sum_{i,j} h_{t_j} b^{ij} \frac{\partial^2}{\partial \theta_i \partial t} + \frac{\partial^2}{\partial t^2}.
\end{equation}
For more details, please see [7] and [16].

Now we state two well known commutation formulas for the covariant derivatives of a smooth function $u \in C^4(S^{n-1})$:
\begin{align}
 u_{ijk} - u_{ikj} &= -u_k \delta_{ij} + u_j \delta_{ik}, \\
 u_{ijkl} - u_{ijlk} &= u_{ik} \delta_{jl} - u_{il} \delta_{jk} + u_{kj} \delta_{il} - u_{lj} \delta_{ik}.
\end{align}
They will be used in the next section.

3. Convexity of the smallest principal curvature of the convex level sets of $p$-harmonic functions

We first state the following lemma which appeared in [15], then we prove Theorem 1.2. For a continuous function $f(t)$ on $[0,1]$ we define its generalized second order derivative at any point $t$ in $(0,1)$ as
\[
 D^2 f(t) = \limsup_{h \to 0} \frac{f(t+h) + f(t-h) - 2f(t)}{h^2}.
\]

**Lemma 3.1** (See [15]). Let $Q \equiv S^{n-1} \times (0,1)$ and $G(\theta,t)$ be a regular function in $Q$ such that
\[
 \mathcal{L}(G(\theta,t)) \geq 0 \quad \text{for} \quad (\theta,t) \in Q,
\]
where $\mathcal{L}$ is an elliptic operator of the form
\[
 \mathcal{L} = \sum_{i,j} a^{ij} \frac{\partial^2}{\partial \theta_i \partial \theta_j} + \sum_i b^i \frac{\partial^2}{\partial \theta_i \partial t} + \frac{\partial^2}{\partial t^2} + \sum_i c^i \frac{\partial}{\partial \theta_i}
\]
with regular coefficients $a^{ij}, b^i, c^i$.

Set
\[
 \phi(t) = \max \{G(\theta,t) | \theta \in S^{n-1} \}.
\]
Then $\phi(t)$ satisfies the following differential inequality
\[
 D^2 \phi(t) \geq 0.
\]
Moreover, $\phi(t)$ is a convex function with respect to $t$. 

Since the level sets of $u$ are strictly convex with respect to the normal direction $\nabla u$, the matrix of the second fundamental form $(b_{ij})$ is positive definite in $\Omega$. By rotating the coordinate system, we may set

$$\varphi = \alpha \log(-h_t) + \log b_{11},$$

where $b_{11}$ is the largest principal radius of the level sets. For $\alpha = -1$ and $\beta = 1$, it follows that

$$e^{\beta \varphi} = |\nabla u| k_1^{-1},$$

where $k_1$ is the smallest principal curvature of the level sets. We will derive the following differential inequality

$$L(e^{\beta \varphi}) \geq 0 \mod |\nabla \theta \varphi| \text{ in } \Omega,$$

where the elliptic operator $L$ is given in (2.5) and we have modified the terms involving $\nabla \theta \varphi$ with locally bounded coefficients. Then by applying the maximum principle argument in Lemma 3.1, we can obtain the desired result.

In order to prove (3.1) at an arbitrary point $x_0 \in \Omega$, we may assume the matrix $(b_{ij}(x_0))$ is diagonal by choosing suitable orthonormal frame. From now on, all the calculation will be done at the fixed point $x_0$. In the following, we shall prove Theorem 1.2 in two steps.

Proof of Theorem 1.2

**Step 1:** we first compute $L(\varphi)$ in (3.18).

Since

$$\varphi = \alpha \log(-h_t) + \log b_{11},$$

taking first derivative of $\varphi$, we get

$$\frac{\partial \varphi}{\partial \theta_j} = \alpha h_t^{-1} h_{tj} b_{11,j},$$

(3.2)

$$\frac{\partial \varphi}{\partial t} = \alpha h_t^{-1} h_{tt} b_{11,t},$$

(3.3)

Taking derivative of equation (3.2) and (3.3) once more, we have

$$\frac{\partial^2 \varphi}{\partial \theta_i \partial \theta_j} = -\alpha h_t^{-2} h_{tii} h_{tj} + \alpha h_t^{-1} h_{tjj} - \sum_{r,s} b_{11} b_{11,l} b_{11,j} b_{11,i} + b_{11} b_{11,ji},$$

$$\frac{\partial^2 \varphi}{\partial \theta_i \partial t} = -\alpha h_t^{-2} h_{tti} h_{ti} + \alpha h_t^{-1} h_{tti} - \sum_{r,s} b_{11} b_{11,l} b_{11,t} b_{11,i} + b_{11} b_{11,ti},$$

$$\frac{\partial^2 \varphi}{\partial t^2} = -\alpha h_t^{-2} h_{tti} h_{tt} + \alpha h_t^{-1} h_{ttt} - \sum_{r,s} b_{11} b_{11,l} b_{11,t} b_{11,t} + b_{11} b_{11,tt},$$
hence
\[ L(\varphi) = -\alpha h_t^{-2} \left[ \sum_{i,j} \left( \frac{1}{p-1} h_t^2 \delta_{ij} + h_{ti} h_{tj} \right) b^{ij} b^{jj} h_{ti} h_{tj} - 2 \sum_i h_t^2 b^{ii} h_{tt} + h_t^2 \right] \]
\[ + \alpha h_t^{-1} \left[ \sum_{i,j} \left( \frac{1}{p-1} h_t^2 \delta_{ij} + h_{ti} h_{tj} \right) b^{ij} b^{jj} h_{tj} - 2 \sum_i h_t^2 b^{ii} h_{tt} + h_{tt} \right] \]
\[ - (b^{11})^2 \left[ \sum_{i,j} \left( \frac{1}{p-1} h_t^2 \delta_{ij} + h_{ti} h_{tj} \right) b^{ii} b^{jj} b_{11, i} b_{11, j} - 2 \sum_i h_t^2 b^{ii} b_{11, i} b_{11, t} + b_t^2 \right] \]
\[ + b^{11} L(b_{11}) \]
\[ = I_1 + I_2 + I_3 + I_4. \]

In the rest of this section, we will deal with the four terms above respectively. By recalling our equation, i.e.
\[ h_{tt} = \sum_{i,j} \left( \frac{1}{p-1} h_t^2 \delta_{ij} + h_{ti} h_{tj} \right) b^{ij}, \]
so at the point \( x_0 \), we have
\[ h_{tt} = \frac{1}{p-1} h_t^2 \sigma_1 + \sum_i h_t^2 b^{ii}, \]
where \( \sigma_1 = \sum_i b^{ii} \) is the mean curvature of the level sets.

For the term \( I_1 \), we have
\[ I_1 = -\alpha h_t^{-2} \left[ \sum_{i,j} \left( \frac{1}{p-1} h_t^2 \delta_{ij} + h_{ti} h_{tj} \right) b^{ii} b^{jj} h_{ti} h_{tj} - 2 \sum_i h_t^2 b^{ii} h_{tt} + h_t^2 \right] \]
\[ = -\alpha h_t^{-2} \left[ \frac{1}{p-1} h_t^2 \sum_i (h_{ti} b^{ii})^2 + \left( \sum_i h_t^2 b^{ii} - h_{tt} \right)^2 \right] \]
\[ = -\frac{\alpha}{p-1} \sum_i (h_{ti} b^{ii})^2 - \frac{\alpha}{(p-1)^2} h_t^2 \sigma_1^2. \]

Now we treat the term \( I_2 \). Differentiating (3.5) with respect to \( t \), we have
\[ h_{ttt} = \frac{2}{p-1} h_t h_{tt} \sigma_1 + 2 \sum_i h_{ti} h_{tt} b^{ii} - \sum_{i,j} \left( \frac{1}{p-1} h_t^2 \delta_{ij} + h_{ti} h_{tj} \right) b^{ii} b^{jj} b_{ij, t}. \]

By inserting (3.8) into \( I_2 \), we can get
\[ I_2 = \alpha h_t^{-1} \left[ \sum_{i,j} \left( \frac{1}{p-1} h_t^2 \delta_{ij} + h_{ti} h_{tj} \right) b^{ii} b^{jj} h_{tj} - 2 \sum_i h_t^2 b^{ii} h_{tt} + h_{tt} \right] \]
\[ = \alpha h_t^{-1} \left[ \sum_{i,j} \left( \frac{1}{p-1} h_t^2 \delta_{ij} + h_{ti} h_{tj} \right) b^{ii} b^{jj} (h_{tj} - b_{ij, t}) + 2 \frac{1}{p-1} h_t h_{tt} \sigma_1 \right]. \]
Recalling the definition of the second fundamental form, i.e. (3.22), together with the equation (3.10), we obtain

\[
I_2 = \alpha h_t^{-1} \left[ \sum_{i,j} \left( \frac{1}{p-1} h_t^2 \delta_{ij} + h_{ti} h_{tj} \right) b_{ij} b^{ij} (-h_t \delta_{ij}) + \frac{2}{p-1} h_t h_t \sigma_1 \right] \tag{3.9}
\]

Combining (3.7) and (3.9),

\[
I_1 + I_2 = -\frac{p\alpha}{p-1} \sum_i (h_{ti} b^{ii})^2 + \frac{\alpha}{(p-1)^2} h_t^2 \sigma_1^2 - \frac{\alpha}{p-1} h_t^2 \sum_i (b^{ii})^2 + \frac{2\alpha}{p-1} \sigma_1 \sum_i h_t^2 b^{ii}. \tag{3.10}
\]

In order to deal with the last two terms, we shall compute \(L(b_{11})\) in advance. By differentiating (3.5) twice with respect to \(\theta_1\), we have

\[
h_{tt1} = \sum_{i,j} \left( \frac{1}{p-1} h_t^2 \delta_{ij} + h_{ti} h_{tj} \right)_{11} b^{ij} + \sum_{i,j,p,q} \left( \frac{1}{p-1} h_t^2 \delta_{ij} + h_{ti} h_{tj} \right) (-b^{ip} b_{pq,1} b^{qj}),
\]

and

\[
h_{tt11} = \sum_{i,j} \left( \frac{1}{p-1} h_t^2 \delta_{ij} + h_{ti} h_{tj} \right)_{11} b^{ij} + 2 \sum_{i,j,p,q} \left( \frac{1}{p-1} h_t^2 \delta_{ij} + h_{ti} h_{tj} \right) (-b^{ip} b_{pq,1} b^{qj})
\]

\[
+ \sum_{i,j,p,q,r,s} \left( \frac{1}{p-1} h_t^2 \delta_{ij} + h_{ti} h_{tj} \right) (2b^{ir} b_{rs,1} b^{sp} b_{pq,1} b^{qj})
\]

\[
+ \sum_{i,j,p,q} \left( \frac{1}{p-1} h_t^2 \delta_{ij} + h_{ti} h_{tj} \right) (-b^{ip} b_{pq,11} b^{qj})
\]

\[
= J_1 + J_2 + J_3 + J_4.
\]

For the term \(J_1\), we have

\[
J_1 = \sum_{i,j} \left( \frac{1}{p-1} h_t^2 \delta_{ij} + h_{ti} h_{tj} \right)_{11} b^{ij}
\]

\[
= \sum_{i,j} \left( \frac{2}{p-1} h_t h_t \delta_{ij} + h_{ti} h_{tj} + h_{ti1} h_{tj} \right)_{1} b^{ij}
\]

\[
= \frac{2}{p-1} h_t^2 \sigma_1 + \frac{2}{p-1} h_t h_t \sigma_1 + 2 \sum_i h_{ti1} h_{ti} b^{ii} + 2 \sum_i h_{ti1} b^{ii}.
\]
Applying (2.6) for the support function \( h \), we can get

\[
b_{ij,1} = b_{11,j},
\]

(3.12)

\[
h_{i11} = h_{1i1} = b_{11,i} - h_{i1} \delta_{i1},
\]

\[
h_{i111} = h_{i111} = b_{i11,i} - h_{i11} \delta_{i1} = b_{11,1i} - h_{11} \delta_{1i},
\]

hence we obtain

\[
J_1 = 2 \sum_i h_{i1} b_{11,i} b_{11,1i} + 2 \sum_i b_{i1} b_{i1,1i,t} + \frac{2}{p-1} h_{i1} \sigma_1 b_{11,1i,t} - 4 h_{i1} b_{11,1i,t} + \frac{2}{p-1} \sum_{i \geq 2} b_{i1}.
\]

(3.13)

For the term \( J_2 \), we have

\[
J_2 = 2 \sum_{i,j} \left( \frac{2}{p-1} h_{i1} h_{i1} \delta_{ij} + h_{i1} h_{i1} + h_{i1} h_{i1} \right) (-b_{i1} b_{i1,1} b_{i1,1})
\]

\[
= -\frac{4}{p-1} h_{i1} h_{i1} \sum_i (b_{i1})^2 b_{i1,1} - 4 \sum_{i,j} h_{i1} h_{i1} b_{i1,1} b_{i1,1} + 4 h_{i1} b_{11,1} \sum_{i,j} h_{i1} b_{i1,1} b_{i1,1}.
\]

(3.14)

Also we have

\[
J_3 = 2 \sum_{i,j,k} \left( \frac{1}{p-1} h_{i1}^2 \delta_{ij} + h_{i1} h_{i1} \right) b_{i1,1} b_{i1,1} b_{i1,1,1} b_{i1,1,1}.
\]

(3.15)

Again by (2.6), we have the following commutation rule

\[
b_{pq,11} = b_{11,pq} + b_{pq} - b_{11} \delta_{pq} + b_{11} \delta_{1p} - b_{11} \delta_{1q}.
\]

For the term \( J_4 \), we have

\[
J_4 = -\sum_{i,j,p,q} \left( \frac{1}{p-1} h_{i1}^2 \delta_{ij} + h_{i1} h_{i1} \right) b_{i1} b_{i1} (b_{11,pq} + b_{pq} - b_{11} \delta_{pq})
\]

\[
= -\sum_{i,j} \left( \frac{1}{p-1} h_{i1}^2 \delta_{ij} + h_{i1} h_{i1} \right) b_{i1} b_{i1} b_{i1,1} b_{i1,1} - h_{i1}
\]

\[
+ \frac{1}{p-1} h_{i1} b_{11} \sum_i (b_{i1})^2 + b_{11} \sum_i (h_{i1} b_{i1})^2.
\]

(3.16)
Notice that \( h_{11tt} = b_{11,t} - h_t \), by putting (3.13)–(3.16) into (3.11), recalling the definition of the operator \( L \), we obtain

\[
L(b_{11}) = 2 \sum_i b_{i1}^2 b_{11,i,t} + \frac{2}{p-1} h_t \sigma_1 b_{11,t} - 4 h_t b_{11}^2 b_{11,t} + \frac{2p-4}{p-1} h_t^2 b_{11}^2 - \frac{2}{p-1} h_t^2 \sum_{i \geq 2} b_{ii}
\]

\[
+ \frac{4-2p}{p-1} h_t^2 b_{11} + \frac{2}{p-1} h_t \sum_{i \geq 2} b_{ii} - \frac{4}{p-1} h_t h_t \sum_i (b_{ii})^2 b_{ii,1}
\]

\[
- 4 \sum_{i,j} h_t b_{i1} b_{j1} b_{i1,1} b_{11,t} + 4 h_t b_{11} \sum_j h_t b_{11,1} b_{11,j}
\]

\[
+ 2 \sum_{i,j} \left( \frac{1}{p-1} h_t^2 \delta_{ij} + h_t h_t \right) b_{i1} b_{j1} b_{kk,1} b_{jk,1}
\]

\[
+ \frac{1}{p-1} h_t^2 b_{11} \sum_i (b_{ii})^2 + b_{11} \sum_i (h_t b_{ii})^2.
\]

Therefore,

\[
I_4 = 2 b_{11} \sum_i b_{i1}^2 b_{11,i,t} + \frac{2}{p-1} h_t \sigma_1 b_{11}^2 b_{11,t} - 4 h_t b_{11}^2 b_{11,t} + \frac{2p-4}{p-1} h_t^2 b_{11}^2
\]

\[
- \frac{2}{p-1} h_t^2 b_{11} \sum_{i \geq 2} b_{ii} + \frac{4-2p}{p-1} h_t^2 (b_{11})^2 + \frac{2}{p-1} h_t^2 b_{11} \sum_{i \geq 2} b_{ii}
\]

\[
- \frac{4}{p-1} h_t h_t b_{11} \sum_i (b_{ii})^2 b_{ii,1} - 4 b_{11} \sum_{i,j} h_t b_{i1} b_{j1} b_{11,j} + 2 b_{11} \sum_{i,j} \left( \frac{1}{p-1} h_t^2 \delta_{ij} + h_t h_t \right) b_{i1} b_{j1} b_{kk,1} b_{jk,1}
\]

\[
+ \frac{1}{p-1} h_t^2 \sum_i (b_{ii})^2 + \sum_i (h_t b_{ii})^2.
\]

By substituting (3.10) and (3.17) in (3.4), we obtain

\[
L(\varphi) = (b_{11})^2 \left[ \sum_{i,j} \left( \frac{1}{p-1} h_t^2 \delta_{ij} + h_t h_t \right) b_{i1} b_{j1} b_{11,j} - 2 \sum_i h_t b_{i1} b_{11,i} + b_{11,t} \right]
\]

\[
+ 2 b_{11} \left[ \sum_{i,j} \left( \frac{1}{p-1} h_t^2 \delta_{ij} + h_t h_t \right) b_{i1} b_{j1} b_{11,i} - 2 \sum_i h_t b_{i1} + b_{11,t} \right]
\]

\[
+ \frac{2}{p-1} h_t \sigma_1 b_{11} b_{11,t} - 4 h_t (b_{11})^2 b_{11,t} - \frac{4}{p-1} h_t h_t b_{11} \sum_i (b_{ii})^2 b_{ii,1}
\]

\[
+ 4 h_t (b_{11})^2 \sum_i h_t b_{i1} + g_1(\alpha) h_t^2 + g_2(\alpha) h_t^2 + \sum_{i \geq 2} g_3(\alpha) h_{ii}^2.
\]
where
\[
q_1(\alpha) = \left(\frac{\alpha}{(p-1)^2} + \frac{2p-3-\alpha}{p-1}\right)(b^{11})^2 + \left(\frac{2\alpha}{(p-1)^2} - \frac{2}{p-1}\right)b^{11} \sum_{i \geq 2} b^{ii}
\]
\[
+ \frac{1-\alpha}{p-1} \sum_{i \geq 2} (b^{ii})^2 + \frac{\alpha}{(p-1)^2} \left(\sum_{i \geq 2} b^{ii}\right)^2,
\]
\[
q_2(\alpha) = \frac{(2-p)\alpha + 3 - p}{p-1}(b^{11})^2 + \frac{2(p+\alpha)}{p-1} b^{11} \sum_{i \geq 2} b^{ii},
\]
\[
q_{3,i}(\alpha) = \frac{2\alpha}{p-1} \sigma_1 b^{ii} + \frac{p-1-\alpha}{p-1} (b^{ii})^2.
\]

**Step 2:** In this step we shall calculate \( L(e^{\beta \phi}) \) and obtain the formula (3.27), thus complete the proof of Theorem 1.2.

Notice that
\[
L(e^{\beta \phi}) = \beta e^{\beta \phi} \{ L(\phi) + \beta \phi^2 \} + \beta^2 e^{\beta \phi} \sum_{i,j} \left(\frac{1}{p-1} h_{ij}\sigma_1 \delta_{ij} + h_{ii} h_{jj} \right) b^{ij} b^{jj} \frac{\partial\phi}{\partial \theta_i} \frac{\partial\phi}{\partial \theta_j}
\]
\[
- 2\beta^2 e^{\beta \phi} \sum_{i,j} h_{ij} b^{ij} \frac{\partial\phi}{\partial \theta_i} \frac{\partial\phi}{\partial \theta_j}.
\]

For \( \beta = 1, \alpha = -1 \), in order to prove
\[
L(e^{\beta \phi}) \geq 0 \mod \nabla \theta \phi \text{ in } \Omega,
\]
it suffice to prove
\[
L(\phi) + \beta \phi^2 \geq 0 \mod \nabla \theta \phi \text{ in } \Omega,
\]
where we have modified the terms involving \( \nabla \theta \phi \) with locally bounded coefficients.

Now we compute \( \beta \varphi_t^2 \). By (3.3) and the equation (3.6), we have
\[
\beta \varphi_t^2 = \beta \alpha^2 h_t^{-2} h_t^{-2} + 2\beta \alpha h_t^{-1} h_{tt} b^{11,b_{11,t}} + \beta (b^{11,b_{11,t}})^2
\]
\[
= \beta \alpha^2 \frac{2\beta \alpha}{(p-1)^2} h_t^{-2} \sigma_1^2 + \frac{2\beta \alpha^2}{p-1} \sigma_1 \sum_i h_{ti}^2 b^{ii} + \beta \alpha^2 h_t^{-2} \left(\sum_i h_{ti}^2 b^{ii}\right)^2
\]
\[
+ \frac{2\beta \alpha}{p-1} h_t \sigma_1 b^{11,b_{11,t}} + 2\beta \alpha h_t^{-1} \left(\sum_i h_{ti}^2 b^{ii}\right) b^{11,b_{11,t}}
\]
\[
+ \beta (b^{11,b_{11,t}})^2.
\]

Jointing (3.18) with (3.21), we regroup the terms in \( L(\varphi) + \beta \varphi_t^2 \) as follows
\[
L(\varphi) + \beta \varphi_t^2 = P_1 + P_2 + P_3 + P_4,
\]
where

\[ P_1 = 2b^{11} \sum_{l \geq 2} b^l \left( \sum_{i,j} h_{ti} h_{tj} b^{ii} b_{11,i} b_{11,j} - 2 \sum_i h_{ti} b^{ii} b_{11,i} b_{11,t} + b_{11,t}^2 \right) \geq 0, \]

\[ P_2 = (1 + \beta)(b^{11} b_{11,t})^2 - 2 \left( \sum_i h_{ti} b^{ii} b_{11,i} - \frac{1}{p-1} h_t \sigma_1 + 2h_t b^{11} - \frac{\beta \alpha}{p-1} h_t \sigma_1 \right) \]

\[ - \beta \alpha h_t^{-1} \sum_i h_{ti}^2 b^{ii} \right) b^{11} b_{11,t} \]

\[ \geq - \frac{1}{1 + \beta} \left( \sum_i h_{ti} b^{ii} b_{11,i} - \frac{1 + \beta \alpha}{p-1} h_t \sigma_1 + 2h_t b^{11} - \beta \alpha h_t^{-1} \sum_i h_{ti}^2 b^{ii} \right)^2, \]

\[ P_3 = (b^{11})^2 \sum_{i,j} \left( \frac{1}{p-1} h_{ti}^2 \delta_{ij} + h_{ti} h_{tj} \right) b^{ii} b^{jj} b_{11,i} b_{11,j} + \frac{2}{p-1} h_t^2 b^{11} \sum_{l \geq 2} b^l \sum_i (b^{ii})^2 b_{11,l}, \]

and

\[ P_4 = q_1(\alpha) h_t^2 + q_2(\alpha) h_t^2 + \sum_{i \geq 2} q_{3,i}(\alpha) h_t^2 + \frac{\beta \alpha^2}{(p-1)^2} h_t^2 \sigma_1^2 + \frac{2 \beta \alpha^2}{p-1} \sigma_1 \sum_i h_{ti}^2 b^{ii} \]

\[ + \beta \alpha^2 h_t^{-2} \left( \sum_i h_{ti}^2 b^{ii} \right)^2. \]

In the following, we will make use of the first order condition, i.e. (3.2), to calculate the terms \( P_2 \) and \( P_3 \).

By (3.2), we have

\[ b^{11} b_{11,j} = \frac{\partial \varphi}{\partial \theta_j} - \alpha h_t^{-1} h_{tj}, \quad \text{for } j = 1, 2, \ldots, n - 1, \]

hence

\[ P_2 \geq - \frac{1}{1 + \beta} \left( \frac{-\alpha(1 + \beta) h_t^{-1} \sum_i h_{ti}^2 b^{ii} - \frac{1 + \beta \alpha}{p-1} h_t \sigma_1 + 2h_t b^{11} + R_2(\nabla \varphi)}{2} \right)^2 + R_2(\nabla \varphi) \]

\[ = - \alpha^2 (1 + \beta) h_t^{-2} \left( \sum_i h_{ti}^2 b^{ii} \right)^2 - \frac{1}{(p-1)^2} \frac{(1 + \beta \alpha)}{1 + \beta} h_t^2 \sigma_1^2 - \frac{4}{1 + \beta} h_t^2 (b^{11})^2 \]

\[ - \frac{2 \alpha (1 + \beta \alpha)}{p-1} \sigma_1 \sum_i h_{ti}^2 b^{ii} + 4 \alpha b^{11} \sum_i h_{ti}^2 b^{ii} + \frac{1}{p-1} \frac{4(1 + \beta \alpha)}{1 + \beta} h_t^2 b^{11} \sigma_1 \]

\[ + R_2(\nabla \varphi), \]
where
\[
R_2(\nabla \varphi) = -\frac{1}{1+\beta} \left( \sum_i h_{ti} b^{ji} \frac{\partial \varphi}{\partial t_i} \right)^2 - \frac{2}{1+\beta} \left( -\alpha(1+\beta) h_t^{-1} \sum_i h_{ti} b^{ji} \right) \\
- \frac{1+\beta\alpha}{p-1} h_t \sigma + 2 h_t b^{11} \left( \sum_i h_{ti} b^{ji} \frac{\partial \varphi}{\partial t_i} \right).
\]

In a similar way, one can check that
\[
P_3 = \frac{\alpha^2}{p-1} \sum_i (h_{ti} b^{ji})^2 + \alpha^2 h_t^{-2} \left( \sum_i h_{ti}^2 b^{ji} \right)^2 + \frac{2\alpha^2}{p-1} b^{11} \sum_i h_{ti}^2 b^{ji} \\
+ \frac{2}{p-1} h_t^2 b^{11} \sum_{i,t \geq 2} b^{ji} b^{ji} + \frac{4\alpha}{p-1} h_t^2 (b^{11})^2 - \frac{4}{p-1} h_t h_t b^{11} \sum_{i \geq 2} (b^{ji})^2 b_{ti,1} \\
- 4\alpha b^{11} \sum_i h_{ti}^2 b^{ji} + R_3(\nabla \theta \varphi),
\]

where
\[
R_3(\nabla \theta \varphi) = \sum_{i,j} \left( \frac{1}{p-1} h_t^2 \delta_{ij} + h_{ti} h_{tj} \right) b^{ji} b^{ji} \left( \frac{\partial \varphi}{\partial t_i} \frac{\partial \varphi}{\partial t_j} - 2\alpha h_t^{-1} h_{ti} \frac{\partial \varphi}{\partial t_i} \right) \\
+ \frac{2}{p-1} h_{ti}^2 b^{11} \sum_{i \geq 2} b^{ji} \left( \frac{\partial \varphi}{\partial t_i} \right)^2 - 2\alpha h_t^{-1} h_{ti} \frac{\partial \varphi}{\partial t_i} \right) \\
- \frac{4}{p-1} h_t h_t b^{11} \frac{\partial \varphi}{\partial t_1} + 4 h_t b^{11} \sum_i h_{ti} b^{ji} \frac{\partial \varphi}{\partial t_i}.
\]

For the fourth term and sixth term in (3.24), we have
\[
\frac{2}{p-1} h_t^2 b^{11} \sum_{i,t \geq 2} b^{ji} b^{ji} b_{ti,1}^2 b_{ti,1} = \frac{4}{p-1} h_t h_t b^{11} \sum_{i \geq 2} (b^{ji})^2 b_{ti,1} \\
\geq \frac{2}{p-1} b^{11} \sum_{i \geq 2} \left( h_t b^{ji} b_{ti,1}^2 h_t \right)^2 - 2 h_t b^{ji} b_{ti,1} h_t b^{11} \\
\geq - \frac{2}{p-1} h_t^2 b^{11} \sum_{i \geq 2} b^{ji}
\]

By (3.24) and (3.25),
\[
P_3 \geq \frac{\alpha^2}{p-1} \sum_i (h_{ti} b^{ji})^2 + \alpha^2 h_t^{-2} \left( \sum_i h_{ti}^2 b^{ji} \right)^2 + \frac{2\alpha^2}{p-1} b^{11} \sum_{i \geq 2} h_{ti}^2 b^{ji} \\
- \frac{2}{p-1} h_t^2 b^{11} \sum_{i \geq 2} b^{ji} + \frac{4\alpha}{p-1} h_t^2 (b^{11})^2 - 4\alpha b^{11} \sum_i h_{ti} b^{ji} + R_3(\nabla \theta \varphi).
\]
Combining (3.22), (3.23) and (3.26), we obtain

\[ L(\varphi) + \beta \varphi_t^2 \geq r_1(\alpha, \beta) h_t^2 + r_2(\alpha, \beta) h_t^2 + \sum_{i \geq 2} r_{3,i}(\alpha, \beta) h_{ti}^2 + R_2(\nabla_\theta \varphi) + R_3(\nabla_\theta \varphi) + q_1(\alpha) h_t^2 + q_2(\alpha) h_{t1}^2 + \sum_{i \geq 2} q_{3,i}(\alpha) h_{ti}^2, \]

where

\[ r_1(\alpha, \beta) = \frac{\beta \alpha^2 - 2\beta \alpha - 1}{(p-1)^2(1+\beta)} + \frac{4\beta \alpha - 4p + 8}{(p-1)(1+\beta)} (b^{11})^2 + \frac{2\beta \alpha^2 - 4\beta \alpha - 2}{(p-1)^2(1+\beta)} + \frac{4(1+\beta \alpha)}{(p-1)(1+\beta)} b^{11} \sum_{i \geq 2} b_{ii} \]

\[ r_2(\alpha, \beta) = \frac{\alpha^2 + 2\alpha}{p-1} (b^{11})^2 - \frac{2(1+\alpha)}{p-1} b^{11} \sum_{i \geq 2} b_{ii}, \]

and

\[ r_{3,i}(\alpha, \beta) = \frac{\alpha^2}{p-1} (b_{ii})^2 + \frac{2\alpha^2}{p-1} b^{11} b_{ii} - \frac{2\alpha}{p-1} \sigma_1 b_{ii}. \]

Then we finally get

\[ L(\varphi) + \beta \varphi_t^2 \geq h_t^2 \left[ \left( \frac{\beta \alpha^2 - \beta \alpha + \alpha - 1}{(p-1)^2(1+\beta)} + \frac{3\beta \alpha - \alpha + (2p - 3)\beta - 2p + 5}{(p-1)(1+\beta)} \right) (b^{11})^2 \right. \]

\[ + \left. \left( \frac{2\beta \alpha^2 - 2\beta \alpha + 2\alpha - 2}{(p-1)^2(1+\beta)} + \frac{4\beta \alpha - 2\beta + 2}{(p-1)(1+\beta)} \right) b^{11} \sum_{i \geq 2} b_{ii} \right] \]

\[ + \frac{\beta \alpha^2 - \beta \alpha + \alpha - 1}{(p-1)^2(1+\beta)} \left( \sum_{i \geq 2} b_{ii} \right)^2 + \frac{1-\alpha}{p-1} \sum_{i \geq 2} \left( b_{ii} \right)^2 \]

\[ \left. + \frac{\alpha^2 + (4-p)\alpha + 3 - p}{p-1} (h_{t1} b^{11})^2 \right] \]

\[ + \sum_{i \geq 2} h_{ti}^2 \left( \frac{2\alpha^2}{p-1} b^{11} b_{ii} + \frac{\alpha^2 - p\alpha + p - 1}{p-1} (b_{ii})^2 \right) \]

\[ + R_2(\nabla_\theta \varphi) + R_3(\nabla_\theta \varphi). \]

For \( \alpha = -1 \) and \( \beta = 1 \), we have

\[ L(\varphi) + \varphi_t^2 \geq \frac{2}{p-1} h_t^2 \sum_{i \geq 2} b_{ii} (b_{ii} - b^{11}) + \frac{2}{p-1} \sum_{i \geq 2} h_{ti}^2 b_{ii} (b^{11} + p b_{ii}) \]

\[ + R_2(\nabla_\theta \varphi) + R_3(\nabla_\theta \varphi) \]

\[ \geq 0 \quad \text{mod } \nabla_\theta \varphi. \]

So we finish the proof of (i).
To proof (ii), we set $\alpha = \beta = 0$ in (3.27). For $n = 3, \ p = 2$, since all the summations have just one term, we have

$$L(\varphi) \geq (h_{t1}b^{11})^2 + (h_{t2}b^{22})^2 + R_2(\nabla_\theta \varphi) + R_3(\nabla_\theta \varphi) \geq 0 \mod \nabla_\theta \varphi.$$ 

Thus we complete the proof of Theorem 1.2. \hfill \Box

4. Convexity of the smallest principal curvature of the convex level sets of minimal graph

By the notations as in Section 2, the minimal surface equation can be written

$$h_{tt} = \sum_{i,j} [(1 + h_t^2)\delta_{ij} + h_{ti}h_{tj}]b^{ij},$$

and the associated linear elliptic operator is

$$\bar{L} = \sum_{i,j,p,q} [(1 + h_t^2)\delta_{pq} + h_{tp}h_{tq}]b^{ip}b^{jq} \frac{\partial^2}{\partial \theta_i \partial \theta_j} - 2\sum_{i,j} h_{ij}b^{ij} \frac{\partial^2}{\partial \theta_i \partial t} + \frac{\partial^2}{\partial t^2}.$$ 

Set

$$\varphi = \alpha \log(-h_t) + \log b_{11},$$

where $b_{11}$ is the largest principal radius of the level sets. We will use the same techniques as in Section 3 to derive the following differential inequality

$$\bar{L}(e^{\beta \varphi}) \geq 0 \mod \nabla_\theta \varphi \text{ in } \Omega,$$

where the elliptic operator $L$ is given in (4.2) and we have modified the term involving $\nabla_\theta \varphi$ with locally bounded coefficients. Then by applying the maximum principle argument in Lemma 3.1, we can obtain the desired result. As in Section 3, all the calculation will be done at the fixed point $x_0$ and we assume the matrix $(b_{ij})$ is diagonal at $x_0$. In the following, we shall prove the theorem in two steps.

**Proof of Theorem 1.4**

**Step 1**: we first compute $\bar{L}(\varphi)$ in (4.16).

Since

$$\varphi = \alpha \log(-h_t) + \log b_{11},$$

taking first derivative of $\varphi$, we get

$$\frac{\partial \varphi}{\partial \theta_j} = \alpha h_t^{-1}h_{tj} + b^{11}b_{11,j},$$

$$\frac{\partial \varphi}{\partial t} = \alpha h_t^{-1}h_{tt} + b^{11}b_{11,t}.$$
Taking derivative of equation (4.4) and (4.5) once more, we have
\[
\frac{\partial^2 \varphi}{\partial \theta_i \partial \theta_j} = -\alpha h_t^{-2} h_{ii} h_{ij} + \alpha h_t^{-1} h_{iji} - \sum_{r,s} b^{1r} b^{r,s} b^{s1} b_{11,i,j} + b^{11} b_{11,i,j},
\]
\[
\frac{\partial^2 \varphi}{\partial \theta_i \partial t} = -\alpha h_t^{-2} h_{iti} + \alpha h_t^{-1} h_{itt} - \sum_{r,s} b^{1r} b^{r,s} b^{s1} b_{11,i,t} + b^{11} b_{11,i,t},
\]
\[
\frac{\partial^2 \varphi}{\partial t^2} = -\alpha h_t^{-2} h_{tt} + \alpha h_t^{-1} h_{ttt} - \sum_{r,s} b^{1r} b^{r,s} b^{s1} b_{11,t,t} + b^{11} b_{11,t,t},
\]

hence
\[
L(\varphi) = -\alpha h_t^{-2} \left( \sum_{i,j} [(1 + h_t^2) \delta_{ij} + h_i h_j] h_{ij} - 2 \sum_i h_{ti}^2 b^{i} i h_{tt} + h_{tt}^2 \right) + \alpha h_t^{-2} \left( \sum_{i,j} [(1 + h_t^2) \delta_{ij} + h_i h_j] h_{ij} - 2 \sum_i h_{ti}^2 h_{i} h_{tt} + h_{ttt} \right) - (h_{11}^2)^2 \left( \sum_{i,j} [(1 + h_t^2) \delta_{ij} + h_i h_j] h_{ij} b_{11,1,i,j} - 2 \sum_i h_{ti}^2 h_{i} h_{11,1,i,t} + b_{11,t}^2 \right) + b^{11} L(b_{11}) = I_1 + I_2 + I_3 + I_4.
\]

In the rest of this section, we will deal with the four terms above respectively. By recalling our equation, i.e.
\[
h_{tt} = \sum_{i,j} [(1 + h_t^2) \delta_{ij} + h_i h_j] b_{ij},
\]
so at the point $x_0$, we have
\[
h_{tt} = (1 + h_t^2) \sigma_1 + \sum_i h_{ti}^2 b^{i},
\]
where $\sigma_1 = \sum_i b^{i}$ is the mean curvature of the level sets.

For the term $I_1$, we have
\[
I_1 = -\alpha h_t^{-2} \left( (1 + h_t^2) \sum_i (h_i b^{i})^2 + \left( \sum_i h_{ti}^2 b^{i} - h_{tt} \right)^2 \right) = -\alpha h_t^{-2} (1 + h_t^2) \sum_i (h_i b^{i})^2 - \alpha h_t^{-2} (1 + h_t^2)^2 \sigma_1^2.
\]

Now we treat the term $I_2$. Differentiating (4.7) with respect to $t$, we have
\[
h_{tt} = 2h_t h_{tt} \sigma_1 + 2 \sum_i h_{tti} h_i b^{i} - \sum_{i,j} [(1 + h_t^2) \delta_{ij} + h_i h_j] b^{i} b^{j} b_{ij,t,t}.
\]
By inserting (4.10) into $I_2$, we can get

$$
\bar{I}_2 = \alpha h_i^{-1} \left( \sum_{i,j} [(1 + h_i^2) \delta_{ij} + h_t h_{ij} b^{ii} b^{jj} (h_{ij} - b_{ij,t}) + 2 h_t h_{ij} \sigma_{1}^2) \right)
$$

(4.11)

$$
= - \alpha (1 + h_i^2) \sum_i (b^{ii})^2 - \alpha \sum_i (h_t b^{ii})^2 + 2 \alpha (1 + h_i^2) \sigma_1^2
$$

$$
+ 2 \alpha \sigma_1 \sum_i h_{ti}^2 b^{ii},
$$

where we used the definition of the second fundamental form, i.e. (2.2). Combining (4.9) and (4.11), we have

$$
\bar{I}_1 + \bar{I}_2 = - \alpha h_i^2 \sum_i (h_t b^{ii})^2 - 2 \alpha \sum_i (h_t b^{ii})^2 + \alpha (1 + h_i^2)^2 (1 + h_i^2) \sigma_1^2
$$

(4.12)

$$
- \alpha (1 + h_i^2) \sum_i (b^{ii})^2 + 2 \alpha \sigma_1 \sum_i h_{ti}^2 b^{ii}.
$$

As in Section 3, we shall compute $\bar{J}(b_{11})$ in advance in order to deal with the last two terms. By differentiating (4.17) twice with respect to $\theta_1$, we have

$$
h_{tt1} = \sum_{i,j} [(1 + h_i^2) \delta_{ij} + h_t h_{ij}] b^{ij} - \sum_{i,j,k,l} [(1 + h_i^2) \delta_{ij} + h_t h_{ij}] b^{ik} b_{kl,1} b^{lj},
$$

and

$$
h_{tt11} = \sum_{i,j} [(1 + h_i^2) \delta_{ij} + h_t h_{ij}] b^{ij} - 2 \sum_{i,j,k,l} [(1 + h_i^2) \delta_{ij} + h_t h_{ij}] b^{ik} b_{kl,1} b^{lj}
$$

$$
+ 2 \sum_{i,j,k,l,s,r} [(1 + h_i^2) \delta_{ij} + h_t h_{ij}] b^{ir} b_{rs,1} b^{sk} b_{kl,1} b^{lj}
$$

(4.13)

$$
- \sum_{i,j,k,l} [(1 + h_i^2) \delta_{ij} + h_t h_{ij}] b^{ik} b_{kl,11} b^{lj}
$$

$$
= \bar{J}_1 + \bar{J}_2 + \bar{J}_3 + \bar{J}_4.
$$
By a similar computation as in (3.13)–(3.16), we have

\[
J_1 = 2 \sum_i h_{ii} b^{ii} b_{11,ii} + 2 \sum_i b^{ii} b^2_{11,t} + 2 h_t \sigma_1 b_{11,t} - 4 h_t b^{11} b_{11,t} - 2 h_t^2 \sum_{i \geq 2} b^{ii} \\
+ 2 h_{tt}^2 \sum_{i \geq 2} b^{ii},
\]

\[
J_2 = -4 h_t h_{tt} \sum_i (b^{ii})^2 b_{ii,1} - 4 \sum_{i,j} h_{ij} b^{ii} b^{jj} b_{ij,1} b_{11,t} + 4 h_t b^{11} \sum_j h_{ij} b^{jj} b_{11,f},
\]

\[
J_3 = 2 \sum_{i,j,k} ((1 + h_t^2) \delta_{ij} + h_t h_{tt}) b^{ii} b^{jj} b^{kk} b_{ik,1} b_{jk,1},
\]

\[
J_4 = -\sum_{i,j} ((1 + h_t^2) \delta_{ij} + h_t h_{tt}) b^{ii} b^{jj} b_{11,ij} - h_{tt} + (1 + h_t^2) b_{11} \sum_i (b^{ii})^2 \\
+ b_{11} \sum_i (h_{tt} b^{ii})^2.
\]

By (4.13) and (4.14), recalling the definition of the operator \( \bar{L} \) and using \( h_{11tt} = b_{11,tt} - h_{tt} \), we obtain

\[
\bar{L}(b_{11}) = 2 \sum_i b^{ii} b^2_{11,t} + 2 h_t \sigma_1 b_{11,t} - 4 h_t b^{11} b_{11,t} - 2 h_t^2 \sum_{i \geq 2} b^{ii} \\
+ 2 h_{tt}^2 \sum_{i \geq 2} b^{ii} - 4 h_t h_{tt} \sum_i (b^{ii})^2 b_{ii,1} - 4 \sum_{i,j} h_{ij} b^{ii} b^{jj} b_{ij,1} b_{11,t} \\
+ 4 h_t b^{11} \sum_j h_{ij} b^{jj} b_{11,j} + 2 \sum_{i,j,k} ((1 + h_t^2) \delta_{ij} + h_t h_{tt}) b^{ii} b^{jj} b^{kk} b_{ik,1} b_{jk,1} \\
+ (1 + h_t^2) b_{11} \sum_i (b^{ii})^2 + b_{11} \sum_i (h_{tt} b^{ii})^2.
\]

Therefore,

\[
\bar{I}_4 = 2 b^{11} \sum_i b^{ii} b^2_{11,t} + 2 h_t \sigma_1 b^{11} b_{11,t} - 4 h_t (b^{11})^2 b_{11,t} - 2 h_t^2 b^{11} \sum_{i \geq 2} b^{ii} \\
+ 2 h_{tt}^2 b^{11} \sum_{i \geq 2} b^{ii} - 4 h_t h_{tt} b^{11} \sum_i (b^{ii})^2 b_{ii,1} - 4 b_t^{11} \sum_{i,j} h_{ij} b^{ii} b^{jj} b_{ij,1} b_{11,t} \\
+ 4 h_t (b^{11})^2 \sum_j h_{ij} b^{jj} b_{11,j} + 2 b^{11} \sum_{i,j,k} ((1 + h_t^2) \delta_{ij} + h_t h_{tt}) b^{ii} b^{jj} b^{kk} b_{ik,1} b_{jk,1} \\
+ (1 + h_t^2) \sum_i (b^{ii})^2 + \sum_i (h_{tt} b^{ii})^2.
\]
Combining (4.16), (4.12) and (4.15), we obtain

\[(4.16)\]
\[
\bar{L}(\varphi) = (b^{11})^2 \left( \sum_{i,j} [(1 + h_i^2)\delta_{ij} + h_i h_{ij}] b^{ii} b_{11,i} b_{11,j} - 2 \sum_i h_i b^{ii} b_{11,i} b_{11,t} + b_{11,t}^2 \right) \\
+ 2b^{11} \sum_{l \geq 2} b^{ii} \left( \sum_{i,j} [(1 + h_i^2)\delta_{ij} + h_i h_{ij}] b^{ii} b_{11,i} b_{11,j} - 2 \sum_i h_i b^{ii} b_{11,i} b_{11,t} + b_{11,t}^2 \right) \\
+ 2h_i \sigma_1 b_{11,i}^2 b_{11,t} - 4h_i (b^{11})^2 b_{11,t} - 4h_i h_{11} b_{11} \sum_i (b^{ii})^2 b_{ii,1} + 4h_i (b^{11})^2 \sum_i h_i b^{ii} b_{11,i} \\
+ \tilde{q}_1(\alpha) h_i^2 + \tilde{q}_2(\alpha) h_{11}^2 + \sum_{i \geq 2} \tilde{q}_{3,i}(\alpha) h_i^2 - \alpha h_i^2 \sum_i h_i b^{ii}^2 - \alpha h_i^2 \sigma_1^2 \\
+ (1 - \alpha) \sum_i (b^{ii})^2,
\]

where

\[
\tilde{q}_1(\alpha) = (b^{11})^2 - 2(1 - \alpha) b^{11} \sum_{i \geq 2} b^{ii} + (1 - \alpha) \sum_{i \geq 2} (b^{ii})^2 + \alpha \left( \sum_{i \geq 2} b^{ii} \right)^2,
\]

\[
\tilde{q}_2(\alpha) = (b^{11})^2 + 2(1 + \alpha) b^{11} \sum_{i \geq 2} b^{ii},
\]

\[
\tilde{q}_{3,i}(\alpha) = 2\alpha \sigma_1 b^{ii} + (1 - 2\alpha) (b^{ii})^2.
\]

**Step 2:** In this step we shall calculate \(\bar{L}(e^{\beta \varphi})\) and obtain the formula \((4.24)\), thus finish the proof of Theorem 1.4

As in Section 3, for \(\beta = 1\) and \(\alpha = -1\), in order to prove

\[
\bar{L}(e^{\beta \varphi}) \geq 0 \mod \nabla_\theta \varphi \quad \text{in} \ \Omega,
\]

it suffice to prove

\[(4.17)\]
\[
\bar{L}(\varphi) + \beta \varphi_t^2 \geq 0 \mod \nabla_\theta \varphi \quad \text{in} \ \Omega,
\]

where we have modified the terms involving \(\nabla_\theta \varphi\) with locally bounded coefficients.

Now we compute \(\beta \varphi_t^2\). By (4.5) and the equation \(4.7\), we have

\[(4.18)\]
\[
\beta \varphi_t^2 = \beta \alpha^2 h_i^2 \sigma_1^2 + 2\beta \alpha^2 \sigma_1 \sum_i h_i^2 b^{ii} + \beta \alpha^2 h_i^{-2} \left( \sum_i h_i^2 b^{ii} \right)^2 + 2\beta \alpha h_i \sigma_1 b_{11} b_{11,i} \\
+ 2\beta \alpha h_i^{-1} \sum_i h_i^2 b^{ii} b_{11,i} + \beta (b^{11} b_{11,t})^2 + \beta \alpha^2 h_i^{-2} \sigma_1^2 \\
+ 2\beta \alpha h_i^{-1} \sigma_1 b_{11} b_{11,t} + 2\beta \alpha^2 h_i^{-2} \sigma_1 \left( h_i^2 \sigma_1 + \sum_i h_i^2 b^{ii} \right).
\]
Jointing (4.16) with (4.18), we regroup the terms in $\bar{L}(\varphi) + \beta \varphi_t^2$ as follows

$$\bar{L}(\varphi) + \beta \varphi_t^2 = \bar{P}_1 + \bar{P}_2 + \bar{P}_3 + \bar{P}_4,$$

where

$$\bar{P}_1 = 2b_{i1} \left( \sum_{i,j} b_{ij}h_{ij}b_{ij} - \sum_i h_{ii}b_{ii} + 2 \sum_i h_{ii}b_{ii} + b_{ii}^2 \right) \geq 0,$$

$$\bar{P}_2 = (1 + \beta)(b_{i1}b_{i1})^2 - 2 \left( \sum_i h_{ii}b_{ii} - h_{i1} - 2h_{ii}b_{ii} + h_{i1}b_{i1} \right),$$

$$\bar{P}_3 = (b_{i1})^2 \sum_{i,j} [(1 + h_{i1}) \delta_{ij} + h_{ii}h_{ij}b_{ij}b_{ij} - 4h_{ii}b_{ii} + \sum_i (b_{ii})^2 b_{ii}, \ldots, n, 1],$$

and

$$\bar{P}_4 = q_1(\alpha)h_{i1}^2 + q_2(\alpha)h_{i1}^2 + \sum_{i \geq 2} q_3, \ldots, n, 1, a^2 h_{i1}^2 + 2\beta a^2 \sigma_1 \sum_i h_{ii} h_{ii},$$

$$+ \beta a^2 h_t^{-2} \left( \sum_i h_{ii}^2 b_{ii} \right)^2 - a h_t^{-2} \sum_i (h_{ii}^2)^2 - a h_t^{-2} \sigma_1^2,$$

$$+ (1 - \alpha) \sum_i (b_{ii})^2 + \beta a^2 h_t^{-2} \sigma_1^2 + 2\beta a^2 \sigma_1^2 + 2\beta a^2 h_t^{-2} \sigma_1 \sum_i h_{ii}^2 b_{ii}. $$

By the first order condition

$$b_{ii} \sigma_1 = \frac{\partial \varphi}{\partial \theta_i} - \alpha h_t^{-1} h_{ii},$$

for $i = 1, 2, \ldots, n - 1,$

hence we have

$$\bar{P}_2 \geq - \frac{1}{1 + \beta} \left( - \alpha (1 + \beta)h_t^{-1} \sum_i h_{ii}^2 b_{ii} - (1 + \beta)h_t^1 + 2h_{i1}b_{i1} - \beta a h_t^{-1} \sigma_1 \right)^2,$$

$$= - \alpha^2(1 + \beta)h_t^{-2} \left( \sum_i h_{ii}^2 b_{ii} \right)^2 - \frac{(1 + \beta)\sigma_1^2}{1 + \beta} h_t^2 \sigma_1^2 - \frac{4}{1 + \beta} h_t^2 (b_{i1})^2,$$

$$- 2\alpha (1 + \beta) \sigma_1 \sum_i h_{ii}^2 b_{ii} + 4\beta b_{i1} \sum_i h_{ii} b_{ii} + \frac{4(1 + \beta)}{1 + \beta} h_t^2 \sigma_1 b_{i1} - \frac{\beta^2 \sigma_1^2}{1 + \beta} h_t^2 \sigma_1^2,$$

$$- 2\beta a^2 h_t^{-2} \sigma_1 \sum_i h_{ii}^2 b_{ii} - \frac{2\beta (1 + \beta) a}{1 + \beta} \sigma_1^2 + \frac{4\beta a}{1 + \beta} b_{i1} \sigma_1 + \bar{R}_2(\nabla \varphi),$$
where
\[
\bar{R}_2(\nabla \varphi) = -\frac{1}{1+\beta} \left( \sum_i h_{ti} b^{ii} \frac{\partial \varphi}{\partial \theta_i} \right)^2 - \frac{2}{1+\beta} \left( -\alpha(1+\beta) h_t^{-1} \sum_i h_{ti}^2 b^{ii} \right) - (1+\beta \alpha) h_t \sigma_1 + 2 h_t b^{11} - \beta \alpha h_t^{-1} \sigma_1 \left( \sum_i h_{ti} b^{ii} \frac{\partial \varphi}{\partial \theta_i} \right).
\]

In a similar way, we can obtain
\[
\bar{P}_3 = \alpha^2 (1 + h_t^{-2}) \sum_i (h_{ti} b^{ii})^2 + \alpha^2 h_t^{-2} \left( \sum_i h_{ti}^2 b^{ii} \right)^2 + 2 \alpha^2 b^{11} (1 + h_t^{-2}) \sum_{i \geq 2} h_{ti}^2 b^{ii} + 2(1 + h_t^2) b^{11} \sum_{i,l \geq 2} b^{ll} (b^{ii})^2 b_{i,l,i} + 4 \alpha h_t^2 (b^{11})^2 - 4 h_t h_t b^{11} \sum_{i \geq 2} (b^{ii})^2 b_{i,i,1} - 4 \alpha b^{11} \sum_i h_{ti} b^{ii} + \bar{R}_3(\nabla \varphi),
\]
(4.20)

where
\[
\bar{R}_3(\nabla \varphi) = \sum_{i,j} \left( (1 + h_t^2) \delta_{ij} + h_{ti} h_{tj} \right) b^{ii} b^{jj} \left( \frac{\partial \varphi}{\partial \theta_i} \frac{\partial \varphi}{\partial \theta_j} - 2 \alpha h_t^{-1} h_{ti} \frac{\partial \varphi}{\partial \theta_i} \right) + 2(1 + h_t^2) b^{11} \sum_{i,l \geq 2} b^{ll} \left( \frac{\partial \varphi}{\partial \theta_l} \right)^2 - 2 \alpha h_t^{-1} h_{ti} \frac{\partial \varphi}{\partial \theta_i} \right) - 4 h_t h_t b^{11} (b^{11})^2 \frac{\partial \varphi}{\partial \theta_1} + 4 h_t b^{11} \sum_i h_{ti} b^{ii} \frac{\partial \varphi}{\partial \theta_i}.
\]

For the fourth term and sixth term in \( \bar{P}_3 \), we have
\[
2(1 + h_t^2) b^{11} \sum_{i,l \geq 2} b^{ll} (b^{ii})^2 b_{i,l,i} - 4 h_t h_t b^{11} \sum_{i \geq 2} (b^{ii})^2 b_{i,i,1} \geq 2 b^{11} \sum_{i \geq 2} b^{ii} \left[ (h_t b^{ii} b_{i,i,1})^2 - 2 h_t b^{ii} b_{i,i,1} \cdot h_t \right] \geq - 2 h_t^2 b^{11} \sum_{i \geq 2} b^{ii}.
\]
(4.21)

By (4.20) and (4.21), we have
\[
\bar{P}_3 = \alpha^2 (1 + h_t^{-2}) \sum_i (h_{ti} b^{ii})^2 + \alpha^2 h_t^{-2} \left( \sum_i h_{ti}^2 b^{ii} \right)^2 + 2 \alpha^2 b^{11} (1 + h_t^{-2}) \sum_{i \geq 2} h_{ti}^2 b^{ii} - 2 h_t^2 b^{11} \sum_{i \geq 2} b^{ii} + 4 \alpha h_t^2 (b^{11})^2 - 4 \alpha b^{11} \sum_i h_{ti}^2 b^{ii} + \bar{R}_3(\nabla \varphi),
\]
(4.22)

Summing all the \( \bar{P}_i \)'s, \( i = 1, 2, 3, 4 \), and rearrange the terms again, we have
\[
\bar{L}(\varphi) + \beta \varphi_t^2 \geq \bar{Q}_1 + \bar{Q}_2 + \bar{R}_2(\nabla \varphi) + \bar{R}_3(\nabla \varphi),
\]
(4.23)
where

\[
\bar{Q}_1 = h_t^2 \left[ \frac{\beta(1 + \alpha)^2}{1 + \beta} (b^{11})^2 + \frac{2\beta\alpha^2 + 2\beta\alpha + 2\alpha - 2\beta}{1 + \beta} b^{11} \sum_{i \geq 2} b^{ii} + (1 - \alpha) \sum_{i \geq 2} (b^{ii})^2 
\right]
\]
\[
+ \frac{(1 + \beta\alpha)(\alpha - 1)}{1 + \beta} \left( \sum_{i \geq 2} b^{ii} \right)^2 + (1 + \alpha)^2 (h_t b^{11})^2 
\]
\[
+ \sum_{i \geq 2} (h_{ti})^2 [2\alpha^2 b^{11} b^{ii} + (1 - \alpha)^2 (b^{ii})^2],
\]

and

\[
\bar{Q}_2 = \alpha(\alpha - 1) h_t^{-2} \sum_i (h_{ti} b^{ii})^2 + 2\alpha^2 h_t^{-2} b^{11} \sum_{i \geq 2} h_{ti}^2 b_{ii} + \frac{\alpha(\beta\alpha - \beta - 1)}{1 + \beta} h_t^{-2} \sigma_1^2 
\]
\[
+ \frac{2\beta\alpha(\alpha - 1)}{1 + \beta} \sigma_1^2 + (1 - \alpha) \sum_i (b^{ii})^2 + \frac{4\beta\alpha}{1 + \beta} \sigma_1 b^{11}.
\]

Set \(\alpha = -1\) and \(\beta = 1\). we have

\[
\bar{L}(\varphi) + \beta \varphi_t^2 \geq 2(1 + h_t^2) \sum_{i \geq 2} b^{ii} (b^{ii} - b^{11}) + 2 \sum_{i \geq 2} h_{ti}^2 b^{ii} (b^{11} + 2b^{ii})
\]
\[
+ 2h_t^{-2} \sum_i (h_{ti} b^{ii})^2 + 2h_t^{-2} b^{11} \sum_{i \geq 2} h_{ti}^2 b^{ii} + \frac{3}{2} h_t^{-2} \sigma_1^2 + 2\sigma_1^2 
\]
\[
+ \bar{R}_2(\nabla_\theta \varphi) + \bar{R}_3(\nabla_\theta \varphi) 
\]
\[
\geq 0 \quad \text{mod} \nabla_\theta \varphi.
\]

Specially, for the 3-dimensional minimal graph, we can also take \(\alpha = \beta = 0\) in (4.23).

Since all the summations have just one term, we get

\[
\bar{L}(\varphi) \geq (h_t b^{11})^2 + (h_t b^{22})^2 + (b^{11})^2 + (b^{22})^2 + \bar{R}_2(\nabla_\theta \varphi) + \bar{R}_3(\nabla_\theta \varphi) 
\]
\[
\geq 0 \quad \text{mod} \nabla_\theta \varphi.
\]

Thus we finished the proof of Theorem 1.4. \(\square\)

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