Classical Renormalization of Codimension-two Brane Couplings

Claudia de Rham

Dept. of Physics & Astronomy, McMaster University, Hamilton ON, Canada
Perimeter Institute for Theoretical Physics, Waterloo, ON, Canada

Abstract. The curvature on codimension-two and higher branes is not regular for arbitrary matter sources. Nevertheless, the low-energy theory for an observer on such a brane should be well-defined and independent to any regularization procedure. This is achieved via appropriate classical renormalization of the brane couplings, and leads to a natural hierarchy between standard model couplings and couplings to gravity.

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General matter sources on codimension-two branes

Codimension-two braneworld models have recently enjoyed an increased interest in their potential resolution of the cosmological constant problem, [1]. However, understanding the cosmology on such objects, requires the introduction of matter sources other than pure tension, which has been shown to lead to metric divergences on the defect, [2]. As a consequence, it is necessary to regularize codimension-two branes before drawing any physical conclusion. A priori this is not a problem since any codimension-two object arises from an underlying theory which naturally provides a regularization scheme. However, we still expect from field theory that the low-energy theory on the brane is independent from such a regularization. Or in other words, at low-energy, one should be able to integrate out the fields responsible for the object regularization, and thus end up with a low-energy physics independent to the high-energy regularization scale. To this end, the same technique as that usually employed in field theory should be followed and the couplings of the theory should be renormalized, [3].

Origin of the Problem

To understand the origin of the problem, let us consider a free massless scalar field $\phi$ (symbolizing the graviton) living in a flat six-dimensional space-time

$$\text{d}s^2 = \eta_{\mu \nu} \text{d}x^\mu \text{d}x^\nu + \text{d}r^2 + r^2 \text{d}\theta^2,$$

with $0 \leq \theta < 2\pi \alpha$, and where $(1 - \alpha)$ is the deficit angle ($\alpha \leq 1$). Its propagator $D(x^a, x'^a)$, satisfying $r \Box_{\text{d}^6} D(x^a, x'^a) = i\delta^{(6)}(x^a - x'^a)$ is finite as long as one of the end point is taken away from $r = 0$, but it contains a logarithmic singularity when trying
to evaluate both points on the tip of the cone. Introducing a momentum cutoff scale \( \Lambda \) in the evaluation of the propagator, one has (see ref. \[3\]),

\[
D_{k}(0,0) = -\int_{0}^{\Lambda} \frac{dq}{2\pi\alpha} \frac{i}{k^{2} + q^{2}} = -\frac{i}{2\pi\alpha} \log \frac{\Lambda}{k},
\]

(2)

where \( k^{\mu} \) is the four-dimensional momentum. In principle, this should not be a problem, unless a physical source is introduced at \( r = 0 \). In that case, gravity diverges on the brane. We can therefore wonder what observers on such a codimension-two brane located at \( r = 0 \) and coupled to gravity will feel.

**Observer on the brane**

To mimic the effect on observers on the brane, we consider a second scalar field \( \chi \), that we call the Standard Model (SM) field, localized on the tip of the cone and coupled to “gravity” (the field \( \phi \)):

\[
S = -\int d^{4}x d\theta dr \left( \frac{1}{2} (\partial_{\mu}\phi)^{2} + \delta^{2}(y) \left[ \frac{1}{2} (\partial_{\mu}\chi)^{2} + \frac{m^{2}}{2} \chi^{2} + \lambda \chi\phi \right] \right).
\]

(3)

In the absence of the coupling \( \lambda \) between the SM field \( \chi \) and the graviton \( \phi \), the SM propagator would simply be \( G_{\chi\chi}^{\text{SM}} = -i(k^{2} + m^{2})^{-1} \). The coupling with gravity will however induce divergences to the SM propagator, leading to the following “dressed” two-point function, \[4\]

\[
G_{\chi\chi}^{\text{dressed}} = -\frac{i}{(k^{2} + m^{2}) - i\lambda^{2}D_{k}(0,0)}.
\]

(4)

This clearly shows how the coupling of gravity to the SM field introduces a singular term \( D_{k}(0,0) \) in the expression of the SM propagator. As it stands, one can consider arbitrarily small energies, the propagator will still depend on the cutoff scale \( \Lambda \), through \( D_{k}(0,0) = -i/2\pi\alpha \log \Lambda/k \). To make sense of the theory at low energies, \( k \gg \Lambda \), this divergence ought therefore to be absorbed in the coupling constants. In particular the divergence disappears if the SM mass \( m \) is renormalized in the following way

\[
m^{2}(\mu) = m^{2}(\Lambda) - \frac{1}{2\pi\alpha} \lambda^{2}(\Lambda) \log \frac{\Lambda}{\mu},
\]

(5)

leading to the following Renormalization Group (RG) flows

\[
\mu \frac{d m^{2}(\mu)}{d\mu} = \frac{\lambda^{2}(\mu)}{2\pi\alpha}, \quad \mu \frac{d \lambda(\mu)}{d\mu} = 0
\]

(6)

where \( \mu \) is the physical scale. This mass renormalization is sufficient to ensure that the propagators for both fields are finite and independent of the cutoff scale. The Green’s function of the matter field on the brane is thus finite despite being coupled with the graviton field which itself has not a well-defined limit on the brane.
Further interactions

To this setup, further brane-bulk interactions can also be considered, which will in turn affect the brane observables. In particular, restricting ourselves to the relevant and marginal operators, the most general brane interactions are then

\[
S = -\int d^6x \left[ \frac{1}{2} (\partial \phi)^2 + \frac{\delta(r)}{2\pi \alpha} L_{\chi \phi} \right],
\]

where

\[
L_{\chi \phi} = \left( \frac{1}{2} (\partial \chi)^2 + \frac{1}{2} m^2 \chi^2 + \beta_3 \chi^3 + \beta_4 \chi^4 \right) + (\lambda \chi \phi + \lambda_2 \chi^2 \phi).
\]

At this point two independent kinds of divergences can be distinguished. Interactions of the form \(\beta_n \chi^n\) with \(n > 2\), induce loop corrections to the standard Green’s functions which diverge in the ultra-violet. These divergences, standard in field theory, can be absorbed by renormalization of the mass \(m\), the coupling \(\beta_4\) as well as the wavefunction. However, such divergences can be treated in a completely independent way to that arising at the tree-level in our codimension-two scenario. Interactions of the form \(\lambda_3 \phi \chi^2\), for instance will typically induce tree-level divergences which can be absorbed by appropriate renormalization of the couplings.

At tree-level, the divergent parts of the brane field three and four-point function are proportional to

\[
\langle \chi^3 \rangle_{\text{div}} \propto (\beta_3 - i\lambda_3 \lambda D_k(0,0))
\]

and

\[
\langle \chi^4 \rangle_{\text{div}} \propto \left( \beta_4 - \frac{i}{2} \lambda_3^2 D_k(0,0) \right),
\]

which again can be absorbed by appropriate renormalization of the couplings \(\beta_3\) and \(\beta_4\):

\[
\mu \frac{d\beta_3(\mu)}{d\mu} = \frac{\lambda(\mu) \lambda_3(\mu)}{2\pi \alpha}, \quad \mu \frac{d\beta_4(\mu)}{d\mu} = \frac{\lambda_3(\mu)^2}{4\pi \alpha},
\]

while the coupling \(\lambda_3\) remains constant \(\partial_\mu \lambda_3 = 0\), similarly as \(\lambda\). Furthermore, these coupling flows is actually sufficient to ensure that every \(n\)-point function (with \(n \geq 2\)) is finite as the codimension-two cutoff scale \(\Lambda\) is sent to infinity. With respect to this divergence, the theory is thus completely renormalizable. We have therefore achieved to derive a low-energy effective field theory on the codimension-two brane, independent to any regularization procedure.

Hierarchy problem

Already at the level of this simple toy-model, one can distinguish between two different kinds of couplings:

- Bulk-Brane couplings \(\lambda_B\), \((\lambda_B = \lambda, \lambda_3, \cdots)\), symbolizing interaction with gravity, that do not flow in this specific example, \(\partial_\mu \lambda = \partial_\mu \lambda_3 = 0\).
• Pure brane couplings $\beta_b$, ($\beta_b = m, \beta_3, \beta_4, \cdots$), symbolizing interactions from the SM, which typically flow as $\mu \partial_\mu \beta \propto \lambda \lambda_B$.

Couplings from the SM thus behave very differently from couplings with bulk fields (i.e. gravity), leading to the important conclusion that interactions with bulk fields are naturally suppressed compared to interactions of brane fields between themselves! Although this suppression is not sufficient to solve the hierarchy problem by itself, it can act constructively to the ADD scenario, [5]. Moreover, in higher-codimension scenarios, the suppression could be much more significant, since the divergence in this case is that of a power law rather than logarithmic. This could potentially lead to a potential hierarchy between SM forces and gravity in braneworld scenarios with codimension greater than two.

**Discussion**

So far, distributional sources on codimension-two branes only made sense as regularized objects, since bulk fields typically diverge logarithmically when evaluated on the brane. We have presented here how classical regularization of brane couplings lead to well-defined observables on the brane, and this independently to any regularization procedure. This provides a well defined low-energy effective theory on the brane which resulting RG flows can offer interesting physical implications. A hierarchy between the SM forces and gravity appears for instance as a natural consequence, and could be amplified in higher-codimension setups.

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