Abstract—Reconstruction error bounds in compressed sensing under Gaussian or uniform bounded noise do not translate easily to the case of Poisson noise. Reasons for this include the signal dependent nature of Poisson noise, and also the fact that the negative log likelihood in case of a Poisson distribution (which is directly related to the generalized Kullback-Leibler divergence) is not a metric and does not obey the triangle inequality. In this paper, we develop new performance error bounds for a computationally tractable estimator, for reconstruction from compressed measurements affected by Poisson noise, by replacing the generalized Kullback-Leibler divergence with an information theoretic metric - namely the square root of the Jensen-Shannon divergence, which is related to a symmetrization version of the Poisson log likelihood function. This replacement allows for very simple proofs of the error bounds. Numerical experiments are performed showing the practical use of the technique in signal and image reconstruction from compressed measurements under Poisson noise.

Index Terms—Compressed sensing, Poisson noise, reconstruction error bounds, information theoretic metric, Jensen-Shannon divergence, triangle inequality

I. INTRODUCTION

COMPRESSED sensing is today a very mature field of research in signal processing, with several advances on the theoretical, algorithmic as well as application fronts. The theory essentially considers measurements of the form \( y = \Phi x = \Phi \Psi \theta = A \omega \) where \( y \in \mathbb{R}^N \) is a measurement vector, \( A \in \mathbb{R}^{N \times m} \) is the product of a sensing matrix \( \Phi \) (with much fewer rows than columns, \textit{i.e.,} \( N \ll m \)), \( \Psi \in \mathbb{R}^{m \times m} \) is a signal representation orthonormal basis, and \( \theta \in \mathbb{R}^m \) is a vector that is sparse or compressible. Under suitable conditions on the sensing matrix such as the restricted isometry property (RIP) and sparsity-dependent lower bounds on \( N \), it is proved that \( x \) can be recovered near-accurately given \( y \) and \( \Phi \), even if the measurement \( y \) is corrupted by signal-independent, additive noise \( \eta \) of the form \( y = \Phi x + \eta \) where \( \eta \sim \mathcal{N}(0, \sigma) \) or \( \|\eta\|_2 \leq \epsilon \) (bounded noise). The specific error bound \( \mathbf{P1} \) on \( \theta \) in the case of \( \|\eta\|_2 \leq \epsilon \) is given as:

\[
\|\theta - \theta^*\|_2 \leq C_1 \epsilon + C_2 \frac{\epsilon}{\sqrt{s}} \|\theta - \theta_s\|_1
\]

where \( \theta_s \) is a vector created by setting all entries of \( \theta \) to 0 except for those containing the \( s \) largest absolute values, \( \theta^* \) is the minimum of the following optimization problem denoted as \( \mathbf{P1} \):

\[
\text{(P1): minimize} \|z\|_1 \text{ such that } \|y - A z\|_2 \leq \epsilon, \quad (2)
\]

and \( C_1 \) and \( C_2 \) are constants independent of \( m \) or \( N \) but dependent only on \( \delta_s \), the so-called restricted isometry constant (RIC) of \( A \). These bounds implicitly require that \( N \sim \Omega(s \log m) \).

The noise affecting several different types of imaging systems is, however, known to follow the Poisson distribution. Examples include photon-limited imaging systems deployed in night-time photography [2], astronomy [3], low-dosage CT or X-ray imaging [4], fluorescence microscopy [5] and nuclear medicine [6]. The Poisson noise model is given as follows:

\[
y \sim \text{Poisson}(\Phi x) \quad (3)
\]

where \( x \in \mathbb{R}^m_+ \) is the non-negative signal or image of interest. The likelihood of observing a given measurement vector \( y \) is given as

\[
p(y|\Phi x) = \prod_{i=1}^n \left[ \left( \Phi x \right)_i y_i e^{-(\Phi x)_i} \right] / y_i! \quad (4)
\]

where \( y_i \) and \( \left( \Phi x \right)_i \) are the \( i \)th component of the vectors \( y \) and \( \Phi x \) respectively.

Unfortunately, the beautiful mathematical guarantees for compressive reconstruction from bounded or Gaussian noise are no longer directly applicable to the case where the measurement noise follows a Poisson distribution, which is the case considered in this paper. One important reason for the failure of the mathematical guarantees is a characteristic feature of the Poisson distribution - that the mean and the variance are equal to the underlying intensity, thereby completely deviating from the signal independent or bounded nature of other noise models.

Furthermore, the aforementioned imaging systems essentially act as photon-counting systems. Not only does this require non-negative signals of interest, but it also imposes constraints on the nature of the sensing matrix \( \Phi \):

1) Non-negativity: \( \forall i, j, \Phi_{ij} \geq 0 \)
2) Flux-preservation: The total photon-count of the observed signal \( \Phi x \) can never exceed the photon count of the original signal \( x \), \textit{i.e.,} \( \sum_{i=1}^N (\Phi x)_i \leq \sum_{i=1}^m x_i \).

This in turn imposes the constraint that every column of \( \Phi \) must sum up to a value no more than 1, \textit{i.e.} \( \forall j, \sum_{i=1}^N \Phi_{ij} \leq 1 \).
Now, a randomly generated non-negative and flux-preserving \( \Phi \) matrix does not (in general) obey the RIP. This situation is in contrast to randomly generated Gaussian or Bernoulli (\( \pm 1 \)) random matrices which obey the RIP with high probability [11], and poses several challenges. However following prior work [8], we construct a related matrix \( \tilde{\Phi} \) from \( \Phi \) which obeys the RIP.

A. Main Contributions

The derivation of the theoretical performance bounds in Eqn. 1 based on the optimization problem in Eqn. 2 cannot be used in the Poisson noise model case, as it is well known that the use of the \( \ell_2 \) norm between \( y \) and \( \Phi x \) leads to oversmoothing in the lower intensity regions and undersmoothing in the higher intensity regions. To estimate an unknown parameter set \( x \) given a set of Poisson-corrupted measurements \( y \), one proceeds by the maximum likelihood method. Dropping terms involving only \( y \), this reduces to maximization of the quantity

\[ \sum_{i=1}^{N} y_i \log \frac{y_i}{(\Phi x)_i} - \sum_{i=1}^{N} y_i + \sum_{i=1}^{N} (\Phi x)_i, \]

which is called the generalized Kullback-Leibler divergence [9] between \( y \) and \( \Phi x \) - denoted as \( G(y, \Phi x) \). This divergence measure, however, does not obey the triangle inequality, quite unlike the \( \ell_2 \) norm term in Eqn. 2 which is a metric. This ‘metric-ness’ of the \( \ell_2 \) norm constraint is an important requirement for the error bounds in Eqn. 1 proved in [11]. For instance, the triangle inequality of the \( \ell_2 \) norm is used to prove that \( \| A(\theta - \theta^*) \|_2 \leq 2\epsilon \) where \( \theta^* \) is the minimizer of Problem (P1) in Equation 2. This is done in the following manner:

\[ \| A(\theta - \theta^*) \|_2 \leq \| y - A\theta \|_2 + \| y - A\theta^* \|_2 \leq 2\epsilon. \]  

This upper bound on \( \| A(\theta - \theta^*) \|_2 \) is a crucial step in [11] for deriving the error bounds of the form in Equation 1.

The \( \ell_2 \) norm is however not appropriate for the Poisson noise model for the aforementioned reasons. The first major contribution of this paper is to replace the \( \ell_2 \) norm error term by a term which is more appropriate for the Poisson noise model and which, at the same time, is a metric. The specific error term that we choose here is the square root of the Jensen-Shannon divergence, which is a well-known information theoretic metric [10]. Let \( \theta^* \) be the minimizer of the following optimization problem which we denote as (P2):

\[ \text{(P2): minimize } \| z \|_1 \text{ such that } \sqrt{J(y, Az)} \leq \epsilon, z \geq 0, \]

where \( I \triangleq \sum_{i=1}^{m} x_i \) is the total intensity of the signal of interest. Then we prove that

Relative error(\( \theta, \theta^* \)) \( \leq C_1 \sqrt{\frac{N}{I}} \epsilon + \frac{C_2}{I} \sqrt{\bar{s}} \| \theta - \theta^* \|_1 \)  

where \( C_1 \) and \( C_2 \) are constants that depend only on the RIC of the sensing matrix \( \tilde{\Phi} \) derived from \( \Phi \). This result is proved in Section [11] followed by an extensive discussion. In particular, we explain the reason behind the apparently counter-intuitive \( \sqrt{N} \) term: namely, that a Poisson imaging system distributes the total incident photon flux across the \( N \) measurements, reducing the SNR per measurement and hence affecting the performance. This phenomenon has been earlier observed in [8]. Our performance bounds derived with a completely different method confirm the same phenomenon.

While there exists a body of earlier work on reconstruction error bounds for Poisson corrupted compressive measurements [8], [11], [12], the approach taken in this paper is different, and has the following features:

1) It affords much simpler proofs than existing methods (see Section [IV] for more comparisons).
2) It works with a computationally tractable estimator involving regularization with the \( \ell_1 \) norm of the sparse coefficients representing the signal.
3) It demonstrates successfully (for the first time, to the best of our knowledge) the use of the Jensen-Shannon divergence for Poisson compressed sensing problems, at a theoretical as well as experimental level. Our work exploits several interesting properties of the Jensen-Shannon divergence.

A second contribution of this paper is the demonstration of a simple numerical scheme to actually determine the (sparse) signal of interest by jointly penalizing the signal sparsity and the Jensen-Shannon divergence between \( y \) and \( \Phi x \).

B. Organization of the Paper

The main theoretical result is derived in detail in Section [II]. Numerical simulations are presented in Section [III]. Relation to prior work on Poisson compressed sensing is examined in detail in Section [IV], followed by a discussion in Section [V].

II. MAIN RESULT

A. Construction of Sensing Matrices

We construct a sensing matrix \( \tilde{\Phi} \) ensuring that it corresponds to the forward model of a real optical system, based on the approach in [8]. Therefore it has to satisfy certain properties imposed by constraints of a physically realizable optical system - namely non-negativity and flux preservation. One major difference between Poisson compressed sensing and conventional compressed sensing emerges from the fact that conventional randomly generated sensing matrices which obey RIP do not follow the aforementioned physical constraints (although sensing matrices can be designed to obey the RIP, non-negativity and flux preservation simultaneously as in [13]), and we comment upon this aspect in the remarks following the proof of our key theorem, later on in this section). In the following, we construct a sensing matrix \( \tilde{\Phi} \) which has only zero or (scaled) ones as entries. Let us define \( p \) to be the probability that a matrix entry is 0, then \( 1-p \) is the probability that the matrix entry is a scaled 1. Let \( Z \) be a \( N \times m \) matrix whose entries \( Z_{i,j} \) are i.i.d random variables taking only these two different values, i.e.,

\[ Z_{i,j} = \begin{cases} \sqrt{\frac{1-p}{p}} & \text{with probability } p, \\ \frac{1}{\sqrt{1-p}} & \text{with probability } 1-p. \end{cases} \]
Let us define $\tilde{\Phi} \triangleq \frac{Z}{\sqrt{N}}$. For $p = 1/2$, the matrix $\tilde{\Phi}$ now follows RIP of order $2s$ with a very high probability given as $1 - 2e^{-Nc(1+\delta_z)}$ where $\delta_z$ is its RIC of order $2s$ and function $c(h) \triangleq \frac{h^2}{4} - \frac{h^3}{6}$. In other words, for any $2s$-sparse signal $\rho$, the following holds with high probability
\[
(1 - \delta_z)\|\rho\|_2^2 \leq \|\tilde{\Phi}\rho\|_2^2 \leq (1 + \delta_z)\|\rho\|_2^2.
\]
Given any orthonormal matrix $\Psi$, arguments in [7] show that $\tilde{\Phi}\Psi$ also obeys the RIP of the same order as $\tilde{\Phi}$.

However $\tilde{\Phi}$ will clearly contain negative entries with very high probability, which violates the constraints of a physically realizable system. To deal with this, we can construct the flux-preserving and positivity preserving sensing matrix $\Phi$ from $\tilde{\Phi}$ as follows:
\[
\Phi = \sqrt{\frac{p(1 - p)}{N}}\tilde{\Phi} + \frac{(1 - p)}{N}1_{N \times m}, \quad (9)
\]
which ensures that each entry of $\Phi$ is either 0 or $\frac{1}{N}$. One can easily check that $\Phi$ satisfies both the non-negativity as well as flux-preservation properties.

Note that the construction in Eqn. 9 is not unique, and alternative constructions certainly exist. For example, the following alternative construction for $Z$, and hence $\tilde{\Phi}$ and ultimately $\Phi$, can also be used:
\[
Z_{i,j} = \begin{cases} 
-\sqrt{3} & \text{with probability } 1/6, \\
0 & \text{with probability } 2/3, \\
\sqrt{3} & \text{with probability } 1/6.
\end{cases} \quad (10a, 10b, 10c)
\]
In this construction, we have $\tilde{\Phi} = \frac{Z}{\sqrt{N}}$ and $\Phi = \tilde{\Phi} + \frac{\sqrt{3}}{2\sqrt{3N}}$ which produces $\forall i,j$, $\Phi_{ij} \in \{0, \frac{1}{N}, \frac{2}{N}\}$. In the experiments performed in this paper as well as the theoretical analysis, we have used the model in Eqn. 9 though our analysis extends to other models as well.

B. The Jensen-Shannon Divergence and its Square Root

In this section, we throw light upon the motivation for the choice of the square-root of the Jensen-Shannon divergence in problem (P2) for the task of signal recovery from Poisson-corrupted compressed measurements. For this, we define several quantities and discuss the relationship between them.

The well-known Kullback-Leibler Divergence between vectors $P \in \mathbb{R}_{\geq 0}^{n \times 1}$ and $Q \in \mathbb{R}_{\geq 0}^{n \times 1}$ denoted by $D(P, Q)$ is defined as[11]
\[
D(P, Q) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i}. \quad (11)
\]
The Jensen-Shannon Divergence between $P$ and $Q$ denoted by $J(P, Q)$ is defined as
\[
J(P, Q) = \frac{D(P, M) + D(Q, M)}{2} \quad (12)
\]
where $M \triangleq \frac{1}{2}(P + Q)$.

Consider an underlying noise-free signal $x \in \mathbb{R}_{+}^{m \times 1}$. Consider that a compressive sensing device acquires $N \ll m$ measurements of the original signal $x$ to produce a measurement vector $y \in \mathbb{Z}_{+}^{N \times 1}$. We assume that each entry of the measurement vector is corrupted by independent realizations of Poisson noise, giving us the following equation:
\[
\forall i, 1 \leq i \leq N, y_i \sim \text{Poisson}(\Phi x_i) \quad (13)
\]
where as considered before, $\Phi$ is a non-negative flux-preserving sensing matrix. The main task is to estimate the original signal $x$ from $y$. Assuming statistical independence between the different measurements, a common method is to maximize the following likelihood in order to infer $x$:
\[
\mathcal{L}(y | \Phi x) = \prod_{i=1}^{N} p(y_i | (\Phi x)_i) \quad (14)
\]
\[
= \prod_{i=1}^{N} \frac{(\Phi x)_i^{y_i}}{y_i!} e^{-(\Phi x)_i}. \quad (15)
\]
Now, the negative log-likelihood can be expressed as:
\[
\mathcal{NLL}(y, \Phi x) = \sum_{i=1}^{N} y_i \log \frac{y_i}{(\Phi x)_i} - y_i + (\Phi x)_i + \frac{\log y_i}{2} + \frac{\log 2\pi}{2}. \quad (16)
\]
This expression considers the Stirling’s approximation [14] for $\log y_i$ which is given as
\[
\log y_i! \approx y_i \log y_i - y_i + \frac{\log y_i}{2} + \frac{\log 2\pi}{2}, \quad (16)
\]
which is derived from Stirling’s series given below as follows for some integer $n \geq 1$:
\[
n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2}\right) \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n. \quad (17)
\]
Consider the generalized Kullback-Leibler divergence between $y$ and $\Phi x$, denoted as $G(y, \Phi x)$ and defined as
\[
G(y, \Phi x) = \sum_{i=1}^{N} y_i \log \frac{y_i}{(\Phi x)_i} - y_i + (\Phi x)_i. \quad (18)
\]
The generalized Kullback-Leibler divergence turns out to be the Bregman divergence for the Poisson noise model [15] and is used in maximum likelihood fitting for this noise model as well as in non-negative matrix factorization under the Poisson model [9]. The negative log-likelihood can be expressed in terms of the generalized Kullback-Leibler divergence in the following manner:
\[
\mathcal{NLL}(y, \Phi x) = G(y, \Phi x) + \sum_{i=1}^{N} \frac{\log y_i}{2} + \frac{\log 2\pi}{2} \geq G(y, \Phi x) \quad (19)
\]
The inequality follows under the assumption that $\forall i, y_i \geq 1$ since this is a photon-counting process.

Now, neither $\mathcal{NLL}$ nor $G$ obey the triangle inequality. However, let us consider the symmetrized version of the $\mathcal{NLL}$
given by \( N \mathcal{L}(y, \Phi x) + N \mathcal{L}(\Phi x, y) \). This is given by

\[
\begin{align*}
SN \mathcal{L}(y, \Phi x) &= G(y, \Phi x) + G(\Phi x, y) \\
&\quad + \frac{N}{2} \sum_{i=1}^{N} \left( \log y_i + \log(\Phi x)_i + \log 2\pi \right) \\
&\quad \geq G(y, \Phi x) + G(\Phi x, y) \\
&\quad = D(y, \Phi x) + D(\Phi x, y).
\end{align*}
\]

The inequality above again follows under the assumption that the paranthesized term is non-negative. A sufficient condition for this to be satisfied is that either (1) for each \( i \), we must have \( y_i \geq \frac{1}{4\pi^2(\Phi x)_i} \), or (2) the minimum value for \( y_i \geq d \) where \( d \) is the minimum value in \( \{x_i \} \) as given by \( 1/N \). We denote these conditions as ‘Condition 1’ henceforth. Note that, given the manner in which \( \Phi \) is constructed, we have the guarantee that \( (\Phi x)_i \geq x_{\text{min}} \) with a probability of \( 1 - NP^{\text{un}} \) where \( x_{\text{min}} \) is the minimum value in \( x \). The quantity on the right hand side of the last equality above follows from Equations 11 and 18, and yields a symmetrized form of the Kullback-Leibler divergence which is hereafter denoted as \( D_s(y, \Phi x) \).

We now have the following useful lemma giving an inequality relationship between \( D_s \) and \( J \).

**Lemma 0:** Given non-negative vectors \( u \) and \( v \), we have \( \frac{1}{4} D_s(u, v) \geq J(u, v) \).

**Proof:** Following [16], we have \( \frac{u_i + v_i}{2} \geq \sqrt{u_i v_i} \) by the arithmetic-geometric inequality. Now we have:

\[
\begin{align*}
J(u, v) &= \frac{1}{2} \left( \sum_i u_i \log \frac{u_i}{(u_i + v_i)/2} + v_i \log \frac{v_i}{(u_i + v_i)/2} \right) \\
&\leq \frac{1}{2} \left( \sum_i u_i \log \frac{u_i}{\sqrt{u_i v_i}} + v_i \log \frac{v_i}{\sqrt{u_i v_i}} \right) \\
&\leq \frac{1}{4} \left( \sum_i u_i \log \frac{u_i}{v_i} + v_i \log \frac{v_i}{u_i} \right) \\
&= \frac{1}{4} D_s(u, v).
\end{align*}
\]

(22)

In [16], this proof is presented for probability mass functions, but we observe here that it extends to arbitrary non-negative vectors. \( \Box \)

Combining Equations 21 and 22 we arrive at the following conclusion if ‘Condition 1’ holds true:

\[
\begin{align*}
SN \mathcal{L}(y, \Phi x) &\leq \epsilon \rightarrow J(y, \Phi x) \leq \epsilon/4 \\
&\rightarrow \sqrt{J(y, \Phi x)} \leq \epsilon' \triangleq \sqrt{\epsilon}/2.
\end{align*}
\]

Now let us define the following optimization problem:

**(P3):** minimize \( \|z\| \) such that \( SN \mathcal{L}(y, Az) \leq \epsilon, z \geq 0 \).

Following Eqn. 24, we observe that a solution to (P3) is also a solution to (P2) with slight abuse of notation (i.e., the \( \epsilon \) in (P2) should actually be \( \epsilon' \) defined in Eqn. 24). The performance bounds derived in this paper for reconstruction from Poisson-corrupted measurements deal with the estimate obtained by solving the constrained optimization problem (P2) in Eqn. 7 where we consider an upper bound of \( \epsilon \) on the square root of the Jensen-Shannon divergence. The motivation for this formulation will be evident from the properties of the Jensen-Shannon divergence considered in this section: (1) the metric nature of (including the triangle inequality observed by) its square-root, and (2) its relation with the total variation distance \( V(P, Q) \triangleq \sum_i |p_i - q_i| \). These properties are very useful in deriving the performance bounds in the following sub-section. We now prove these two important properties of the Jensen-Shannon divergence before proceeding to the main theorem.

**Lemma 1** The square root of the Jensen-Shannon Divergence is a metric [10].

**Proof:** The square root of the Jensen-Shannon divergence trivially obeys the properties of symmetry, non-negativity and identity. We would like to point out that the proof of the triangle inequality for the square-root of the Jensen-Shannon divergence given in [10] does not require \( P \) and \( Q \) to be probability distributions. In other words given non-negative vectors \( P, Q, \) and \( R \), we have \( \sqrt{J(P, Q)} \leq \sqrt{J(P, R)} + \sqrt{J(Q, R)} \) even if \( \|P\| \neq 1 \), \( \|Q\| \neq 1 \) and \( \|R\| \neq 1 \). We reproduce a sketch of the proof here.

First, we define the function \( L(p, q) \triangleq p \log \frac{2p}{p + q} + q \log \frac{2q}{p + q} \) where scalars \( p \in \mathbb{R}_+, q \in \mathbb{R}_+ \). Given any scalar \( r \in \mathbb{R}_+ \), it is proved in [10] that \( \sqrt{L(p, q)} \leq \sqrt{L(p, r)} + \sqrt{L(r, q)} \). Now, we can clearly see that \( \sqrt{J(P, Q)} = \sum_i \sqrt{L(p_i, q_i)} \). Starting from this, we have:

\[
\begin{align*}
\sqrt{J(P, Q)} &= \sqrt{\sum_i L(p_i, q_i)} \\
&\leq \sqrt{\sum_i \left( \sqrt{L(p_i, r_i)} + \sqrt{L(q_i, r_i)} \right)^2} \\
&= \sqrt{\sum_i L(p_i, r_i) + \sum_i L(q_i, r_i)} \text{ by Minkowski’s inequality} \\
&= \sqrt{J(P, R) + J(Q, R)} \Box
\end{align*}
\]

**Lemma 2:** Let us define

\[
V(P, Q) \triangleq \sum_{i=1}^{n} |p_i - q_i| \quad \Delta(P, Q) \triangleq \sum_{i=1}^{n} \frac{|p_i - q_i|^2}{p_i + q_i}.
\]

If \( P, Q \succeq 0 \) and \( \|P\| \leq 1, \|Q\| \leq 1 \) then

\[
\frac{1}{2} V(P, Q)^2 \leq \Delta(P, Q) \leq 4 J(P, Q).
\]

**Proof:** The latter inequality can be proved using arguments in [17] (Section III) as these arguments do not require \( P \) and \( Q \).
to be probability distributions in any of the steps. To prove
the first inequality, we prove that $2\Delta(P, Q) - V(P, Q)^2 \geq 0$
as follows. Let us define $z_i \triangleq |p_i - q_i|$ and $w_i \triangleq \frac{1}{2}|p_i + q_i|$. If $||P||_1 \leq 1$, $||Q||_1 \leq 1$ then $\sum_{i=1}^n w_i \leq 1$. Hence $\exists \alpha \geq 0$
such that $\sum_{i=1}^n w_i + \alpha = 1$.

$$2\Delta(P, Q) - V(P, Q)^2 = \frac{1}{\prod_{i=1}^n w_i} \left[ \sum_{i=1}^n z_i^2 \prod_{j \neq i} w_j - \left( \prod_{i=1}^n w_i \sum_{i=1}^n z_i \right)^2 \right]$$

$$= \frac{1}{\gamma} \left[ \sum_{i=1}^n z_i^2 \prod_{j \neq i} w_j - \left( \sum_{i=1}^n z_i \right)^2 \right]$$

where $\gamma \triangleq \prod_{i=1}^n w_i$.

$$= \frac{1}{\gamma} \left[ \sum_{i=1}^n z_i^2 \prod_{j \neq i} w_j - \gamma \left( \sum_{i=1}^n z_i \right)^2 \right]$$

$$= \frac{1}{\gamma} \left[ \sum_{i=1}^n z_i^2 \prod_{j \neq i} w_j - \gamma \left( \sum_{i=1}^n z_i \right)^2 \right] \triangleq \frac{1}{\gamma} \left[ \sum_{i=1}^n z_i^2 \prod_{j \neq i} w_j - \gamma \left( \sum_{i=1}^n z_i \right)^2 \right]$$

Note that the first term in the last step is clearly non-negative as
it is the product of a square-term and a term containing $w_k$ values all of which are non-negative and since $\gamma \geq 0$.
The second term is also non-negative as $\alpha \geq 0$. Thus, the
inequality $\frac{1}{2} V(P, Q)^2 \leq \Delta(P, Q)$ is proved.}$\Box$

C. Key Theorem and its Proof

**Theorem 1.3:** Consider a non-negative signal of interest $x = \Psi \theta$ for orthonormal basis $\Psi$ with sparse vector $\theta$.
Define $A \triangleq \Phi \Psi$ for sensing matrix $\Phi$ defined in Eqn. [9] Suppose $y \sim \text{Poisson}(\Phi \Psi \theta)$, i.e. $y \sim \text{Poisson}(A \theta)$, represents a vector of $N \ll m$ Poisson-corrupted compressive measurements of $x$. Let $\theta^*$ be the solution to the problem (P2) defined earlier. If $\tilde{\Phi}$ constructed from $\Phi$ obeys the RIP of order $2s$ with RIC $\delta_{2s} < \sqrt{2}$, then

$$\frac{\|\theta - \theta^*\|_1}{\|	heta\|_1} \leq C' \sqrt{\frac{N}{I}} + C'' \frac{|\theta|_{1/2}}{I}$$

where $C' \triangleq \frac{4\sqrt{8(1 + \delta_{2s})}}{\sqrt{p(1-p)(1-(\sqrt{2})\delta_{2s})}}$ and $C'' \triangleq \frac{2 - 2\delta_{2s} + 2\sqrt{2\delta_{2s}}}{1 - (1 + \sqrt{2})\delta_{2s}}$, and $\theta_s$ is a vector containing the $s$
largest absolute value elements from $\theta$.

**Proof:** Our proof follows the approach for the proof of the key results in [1] for the case of bounded, signal-independent
noise, but adapted here completely for the case of Poisson noise.

1) First we will prove the following result

$$\|\Phi \Psi (\theta - \theta^*)\|_2 \leq 2\sqrt{8} \epsilon.$$  

We have

$$\|\Phi \Psi \theta - \Phi \Psi \theta^*\|_2 \leq \|\Phi \Psi (\theta - \theta^*)\|_1$$

$$= I \left\| \Phi \Psi (\theta - \theta^*) \right\|_1$$

$$\leq I \sqrt{8J(\Phi \Psi \theta, \Phi \Psi \theta^*)}$$

$$= I \sqrt{8J(\Phi \Psi \theta, y)} + I \sqrt{8J(\Phi \Psi \theta^*, y)}$$

by Lemma 1

$$\leq \frac{I}{\sqrt{I}} \sqrt{8J(\Phi \Psi \theta, y)} + I \sqrt{8J(\Phi \Psi \theta^*, y)}$$

$$\leq 2\sqrt{8} \epsilon.$$  

Note that Lemma 2 can be used in the third step above
because we have imposed that $||\Phi \Psi \theta^*||_1 = ||\Phi \Psi \theta||_1 = I$ and because by the flux-preserving property of $\Phi$, we have $||\Phi \Psi \theta||_1 \leq I$ and $||\Phi \Psi \theta^*||_1 \leq I$.

2) Let us define vector $h \triangleq \theta^* - \theta$ which is the difference between the estimated and true coefficient vectors. Let us denote vector $h_T$ as the vector equal to $h$ only on an index set $T$ and zero at all other indices. Let $T_0$ be the
set of indices containing the $s$ largest entries of $h$ (in terms of absolute value), $T_1$ be the set of indices of the next $s$ largest entries of $h_{T_0}$, and so on. We will now
decompose $h$ as the sum of $h_{T_0}, h_{T_1}, h_{T_2}, \ldots$. We will
denote the complement of an index set $T$ as $T^c$. Our aim will be to prove that both $||h_{(T_0 \cup T_1^c)}||_2$ and $||h_{(T_0 \cup T_1)^{c}}||_2$
are upper bounded by sensible and intuitive quantities.

3) We will first prove the bound on $||h_{(T_0 \cup T_1)^{c}}||_2$, in the
following way:

a) We have

$$||h_{T_1}||_2 \leq \sqrt{\sum_{k \in T_1} h_{T_1,k}^2} \leq s^{1/2} ||h_{T_2}||_\infty,$$

$$s ||h_{T_1}||_\infty \leq \sum_{i} ||h_{T_1,i}|| = ||h_{T_1}||_1.$$ 

Therefore,

$$||h_{T_1}||_2 \leq s^{1/2} ||h_{T_2}||_\infty \leq s^{-1/2} ||h_{T_1}||_1.$$ 

b) Using Step 3(a), we get

$$||h_{(T_0 \cup T_1^c)}||_2 \leq ||h_{T_1}||_2 \leq \sum_{j \geq 2} ||h_{T_j}||_2 \leq \sum_{j \geq 2} ||h_{T_j}||_1 \leq s^{-1/2} \sum_{i \geq 1} ||h_{T_i}||_1 \leq s^{-1/2} ||h_{(T_0)}^c||_1.$$ 

c) Using the reverse triangle inequality and the fact that $\theta^*$ is the solution of (P2), we have

$$||\theta||_1 \geq ||\theta + h||_1$$

$$= \sum_{i \in T_0} |\theta_i + h_i| + \sum_{i \in (T_0)^c} |\theta_i + h_i|$$

$$\geq ||\theta_{T_0}||_1 + ||h_{(T_0)^c}||_1 - ||\theta_{(T_0)^c}||_1.$$
Rearranging the above equation gives us
\[ \| \hat{h}_{(T_0)^c} \|_2 \leq \| \hat{h}_{(T_0)} \|_2 + 2\| \theta - \theta_s \|_2 \]

d) We have
\[ \| \hat{h}_{(T_0^c \cup T_1)} \|_2 \leq s^{-1/2} \| \hat{h}_{(T_0)} \|_2 \]
\[ \leq s^{-1/2}(\| \hat{h}_{(T_0)} \|_2 + 2\| \theta - \theta_s \|_2) \]
Using \( \| \hat{h}_{(T_0)} \|_2 \leq \| h_{T_0 \cup T_1} \|_2 \), we get
\[ \| \hat{h}_{(T_0^c \cup T_1)} \|_2 \leq \| h_{T_0 \cup T_1} \|_2 + 2s^{-1/2}\| \theta - \theta_s \|_2. \]
(31)

4) We will now prove the bound on \( \| h_{(T_0^c \cup T_1)} \|_2 \) in the following way:

a) We have
\[ \Phi = \sqrt{\frac{p(1-p)}{N}} \bar{\Phi} + (1-p)\frac{1}{N}1_{N \times m} \]
\[ \Phi \Psi (\theta - \theta^*) = \sqrt{\frac{p(1-p)}{N}} \bar{\Phi} \Psi (\theta - \theta^*) + (1-p)\frac{1}{N}1_{N \times m} \Psi (\theta - \theta^*) \]
\[ = \sqrt{\frac{p(1-p)}{N}} \bar{\Phi} \Psi (\theta - \theta^*) + (1-p)\frac{1}{N} \| \Psi (\theta) - \| \Psi (\theta^*) \|_1 \|_1 \]
As \( \| \Psi (\theta^*) \|_1 = \| \Psi (\theta) \|_1 = 1 \), we get
\[ \Phi \Psi (\theta - \theta^*) = \sqrt{\frac{p(1-p)}{N}} \bar{\Phi} \Psi (\theta - \theta^*). \] (32)

Let us define \( B \triangleq \bar{\Phi} \Psi \). If \( N \geq O(s \log m) \), then \( \bar{\Phi} \) obeys RIP of order \( 2s \) with overwhelming probability, and so does the product \( B \) since \( \Psi \) is an orthonormal matrix [7].

From Eqn. (32) above we have,
\[ \| B (\theta - \theta^*) \|_2 = \sqrt{\frac{N}{p(1-p)}} \| \Phi \Psi (\theta - \theta^*) \|_2 \]
\[ \leq 2 \sqrt{\frac{8NI}{p(1-p)}} \varepsilon \text{ using Eqn. (28)} \]
\[ \therefore \| Bh \|_2 \leq 2 \sqrt{\frac{8NI}{p(1-p)}} \varepsilon \]
Defining \( C_1 \triangleq 2 \sqrt{\frac{8}{p(1-p)}} \), we have
\[ \| Bh \|_2 \leq C_1 \sqrt{NI} \varepsilon \] (33)

b) The RIP of \( B \) with RIC \( \delta_{2s} \) gives us,
\[ \| Bh_{T_0 \cup T_1} \|_2 \leq \sqrt{1 + \delta_{2s}} \| h_{T_0 \cup T_1} \|_2 \]
Using Eqn. (33) and the Cauchy-Schwartz inequality,
\[ \| Bh_{T_0 \cup T_1} \|_2 \| Bh \|_2 \leq C_1 \sqrt{NI(1 + \delta_{2s})} \| h_{T_0 \cup T_1} \|_2. \] (34)

c) \( h_{T_0} \) and \( h_{T_j} \), \( j \neq 0 \), are vectors with disjoint support. Consider
\[ \| Bh_{T_0} \|_2 \| Bh_{T_j} \|_2 \]
where \( \hat{h}_{T_0} \) and \( \hat{h}_{T_j} \) are unit-normalized vectors. This further yields,
\[ \| Bh_{T_0}, Bh_{T_j} \|_2 \]
\[ = \| h_{T_0} \|_2 \| h_{T_j} \|_2 \left( \frac{B(\hat{h}_{T_0} + \hat{h}_{T_j})}{2} - \frac{B(\hat{h}_{T_0} - \hat{h}_{T_j})}{2} \right) \]
\[ \leq \| h_{T_0} \|_2 \| h_{T_j} \|_2 \]
\[ \frac{(1 + \delta_{2s})(\| \hat{h}_{T_0} \|_2^2 + \| \hat{h}_{T_j} \|_2^2) - (1 - \delta_{2s})(\| \hat{h}_{T_0} \|_2^2 + \| \hat{h}_{T_j} \|_2^2)}{4} \]
\[ \leq \delta_{2s} \| h_{T_0} \|_2 \| h_{T_j} \|_2. \] (35)

Analogously,
\[ \| Bh_{T_1}, Bh_{T_j} \|_2 \leq \delta_{2s} \| h_{T_1} \|_2 \| h_{T_j} \|_2. \] (36)

d) We observe that
\[ Bh_{T_0 \cup T_1} = Bh - \sum_{j \geq 2} Bh_{T_j} \]
\[ \| Bh_{T_0 \cup T_1} \|_2^2 = \langle Bh_{T_0 \cup T_1}, Bh \rangle - \sum_{j \geq 2} \langle Bh_{T_0 \cup T_1}, Bh \rangle. \] (37)

e) Using the RIP of \( B \) and Eqns. (34) (35) (36)
\[ (1 - \delta_{2s}) \| h_{T_0 \cup T_1} \|_2^2 \leq \| Bh_{T_0 \cup T_1} \|_2^2 \]
\[ \leq C_1 \sqrt{NI(1 + \delta_{2s})} \| h_{T_0 \cup T_1} \|_2^2 + \delta_{2s} \| h_{T_0} \|_2 \| h_{T_1} \|_2 \sum_{j \geq 2} \| h_{T_j} \|_2 \]
\[ h_{T_0} \] and \( h_{T_1} \) are vectors with disjoint sets of non-zero indices and hence
\[ \| h_{T_0} \|_2 + \| h_{T_1} \|_2 \leq \sqrt{2} \| h_{T_0 \cup T_1} \|_2 \]

Therefore, we get
\[ (1 - \delta_{2s}) \| h_{T_0 \cup T_1} \|_2^2 \]
\[ \leq \| h_{T_0 \cup T_1} \|_2 \left( C_1 \sqrt{NI(1 + \delta_{2s})} + \sqrt{2}\delta_{2s} \sum_{j \geq 2} \| h_{T_j} \|_2 \right). \] (38)

f) We have
\[ \sum_{j \geq 2} \| h_{T_j} \|_2 \leq s^{-1/2} \| h_{(T_0)^c} \|_1 \]
\[ \leq s^{-1/2} \| h_{(T_0)} \|_1 + 2s^{-1/2} \| \theta - \theta_s \|_1 \]
\[ \leq \| h_{(T_0)} \|_1 \| h_{T_0 \cup T_1} \|_2 \| \theta - \theta_s \|_1 \]
\[ \| h_{T_0 \cup T_1} \|_2 \| \theta - \theta_s \|_1 \]
Combining Eqns. (38) and (39)
\[ \| h_{T_0 \cup T_1} \|_2 \leq C_1 \sqrt{NI(1 + \delta_{2s})} + \frac{2\sqrt{2}\delta_{2s} s^{-1/2}}{1 - (1 + \sqrt{2})\delta_{2s}} \| \theta - \theta_s \|_1 \] (40)
5) Combining the upper bounds on $\|h_{(T_0∪T_1)}\|_2$ and $\|h_{(T_0∪T_1)^c}\|_2$ yields the final result as follows:

$$\|h\|_2 = \|h_{T_0∪T_1} + h_{(T_0∪T_1)^c}\|_2$$

$$\leq \|h_{T_0∪T_1}\|_2 + \|h_{(T_0∪T_1)^c}\|_2$$

$$\leq 2\|h_{T_0∪T_1}\|_2 + 2s^{-1/2}\|\theta - \theta_s\|_1$$

Using Eqn. 40, we get

$$\|h\|_2 \leq 2C_1\sqrt{\frac{NT(1+\delta_N)}{1-(1+\sqrt{2})\delta_N}} + 2\left(1 - \left(1 + \sqrt{2}\right)\delta_N\right)^s s^{-1/2}\|\theta - \theta_s\|_1$$

Let us define $C' \triangleq \frac{4\sqrt{8(1+\delta_N)}}{p(1-p)(1-(1+\sqrt{2})\delta_N)}$ and $C'' \triangleq \left(\frac{2-2\delta_N + 2\sqrt{2}\delta_N}{1-(1+\sqrt{2})\delta_N}\right)$. Finally, we get

$$\|h\|_2 \leq C'\sqrt{NT\epsilon} + C''s^{-1/2}\|\theta - \theta_s\|_1 \quad (41)$$

As a norm is a non-negative quantity, we want $C'$ and $C''$ to be positive, therefore we need $\delta_N < \sqrt{2} - 1$.

Now finally, dividing both sides by $I$ we obtain the following upper bounds on the relative reconstruction error,

$$\frac{\|\theta - \theta^*\|_2}{I} \leq C' \sqrt{\frac{N}{I}} \epsilon + C''s^{-1/2}\frac{\|\theta - \theta_s\|_1}{I}.$$

We make several comments on these bounds in the ensuing paragraphs.

**Remarks:**

1) We have derived upper bounds on the relative reconstruction error, i.e. on $\frac{\|\theta - \theta^*\|_2}{I}$ and not on $\|\theta - \theta^*\|_2$. This not a contrivance, but we have chosen the correct error metric here. It should be noted that as the mean of the Poisson distribution increases, so does its variance, which would cause an increase in the root mean squared error. But this error would be small in comparison to the average signal intensity (the mean of the Poisson distribution). Hence the relative reconstruction error is the correct metric to choose in this context.

2) The usage of $\sqrt{J}$, i.e., the square root of the Jensen-Shannon divergence, plays a critical role in this proof. On one hand $J$ is related to the Poisson likelihood as explained before. On the other hand, $\sqrt{J}$ is a metric and hence obeys the triangle inequality. Furthermore, $J$ also upper-bounds the total variation norm, as shown in Lemma 2. Both these properties are essential for the derivation of Step 1 which is a critical step of the entire proof.

3) The relative reconstruction error $\frac{\|\theta - \theta^*\|_2}{I}$ is upper bounded by two terms - the first term is proportional to the square root of the number of measurements $N$ as well as to the upper bound $\epsilon$ on $\sqrt{J}$, and the second term is determined purely by the compressibility of the signal.

4) Both error terms are inversely proportional to $I$, reflecting the common knowledge that reconstruction under Poisson noise is more challenging if the original signal intensity is lower.

5) It may seem counter-intuitive that the first error term increases with $\sqrt{N}$. However if the original signal intensity remains fixed at $I$, an increase in $N$ simply distributes the photon flux across multiple measurements thereby decreasing the SNR at each measurement and degrading the performance. Similar arguments have been made previously in [8]. This behaviour is a feature of Poisson imaging systems, and is quite different from the Gaussian noise scenario [18] where the error decreases with increase in $N$ owing to no flux-preservation constraints.

6) The first error term is directly proportional to $\epsilon$, which is an upper bound on $\sqrt{J(y, \Phi x)}$, where $y$ is a Poisson corrupted version of $\Phi x$. We have experimentally observed a surprising fact: as $I$ (i.e., the sum total of the values in $x$) increases beyond some small threshold $\tau$, the value of $\sqrt{J(y, \Phi x)}$ remains almost constant on an average. For $I < \tau$, the values of $\sqrt{J(y, \Phi x)}$ are lower than this average value. In experiments across signals of varying sparsity and average intensity, with different random $\Phi$ and across several noisy measurements, we have consistently observed this phenomenon. A sample of these results is shown in Figure 1. This leads us to experimentally conclude that $\epsilon$ does not vary in proportion to $\sqrt{J}$ or $I$.

7) The above bound holds for a signal sparse/compressible in some orthonormal basis $\Psi$. However, for reconstruction bounds for a non-negative signal sparse/compressible in the canonical basis, i.e. $\Psi = I$ and hence $x = \theta$, one can solve the following optimization problem which penalizes the $\ell_q$ ($0 < q < 1$) norm instead of the $\ell_1$ norm:

$$\min_{\theta} \|\theta\|_q \text{ subject to } \sqrt{J(y, A\theta)} \leq \epsilon, \|\theta\|_1 = I, \theta \geq 0$$

Performance guarantees for this case can be developed along the lines of the work in [19]. Other sparsity-promoting terms such as those based on a logarithmic penalty function (which approximates the original $\ell_0$ norm penalty more closely than the $\ell_1$ norm) may also be employed [20, 21].

8) While imposition of the constraint that $\|x\|_1 = I$ with $I$ being known may appear as a strong assumption, it must be noted that in some compressive camera architectures, it is easy to obtain an estimate of $I$ during acquisition. One example is the Rice Single Pixel Camera [22], where $I$ can be obtained by turning on all the micro mirrors, thereby allowing the photo-diode to measure the sum total of all values in the signal. The imposition of this constraint has been considered in earlier works on Poisson compressed sensing such as [8] and [11]. Furthermore, we note that in our experiments in Section III we have obtained excellent reconstructions even
without the imposition of this constraint.

9) Measurement matrices in compressed sensing can be specifically designed to have very low coherence, as opposed to the choice of random matrices. Such approaches have been proposed in [23], [24] and extended to further obey flux preservation in a Poisson setting in [13]. Since the coherence value can be used to put an upper bound on the RIP, one can conclude that such matrices will obey RIP even while obeying non-negativity and flux preservation. In case of such matrices which already obey the RIP, the upper bound on the reconstruction error reduces to the following:

$$
\frac{\|\| \theta - \theta^* \|_2}{I} \leq C' \epsilon \frac{\sqrt{N}}{\sqrt{I}} + C'' s^{-1/2} \| \theta - \theta^* \|_1, \quad (42)
$$

i.e., the first term no more contains $\sqrt{N}$. It must be borne in mind, however, that such matrices are obtained as the output of non-convex optimization problems, and there is no guarantee on how low their coherence, and hence their RIP, will be. Indeed they may not follow the sufficient condition in our proof that $\delta_{2s} < \sqrt{2} - 1$.

III. NUMERICAL EXPERIMENTS

For our numerical experiments, we iteratively solved the following optimization problem:

$$(P4): \min \lambda \| \| \theta \|_1 + J(\| y, \Phi \Psi \theta \|) \text{ w.r.t. } \theta, \quad (43)$$

where $\lambda$ is a regularization parameter. Before describing our actual experimental results, we first prove that solving (P4) is equivalent to solving (P2) for some pair of $(\lambda, \epsilon)$ values, but without the constraint $\| \Psi \theta \|_1 = I$.

Lemma 4: Given $\theta$ which is the minimizer of problem (P4) for some $\lambda > 0$, there exists some value of $\epsilon = \epsilon_0$ for which $\theta$ is the minimizer of problem (P2), but without the constraint $\| \Psi \theta \|_1 = I$.

Proof: Our proof follows [25], proposition 3.2. Define $\epsilon_0 \triangleq J(\| \Psi \theta \|, y)$. Consider vector $\theta'$ such that $J(\| \Psi \theta' \|, y) \leq \epsilon_0$. Now since $\theta$ minimizes (P3), we have $\lambda \| \theta \|_1 + J(\| \Phi \Psi \theta \|, y) \leq \lambda \| \theta' \|_1 + J(\| \Phi \Psi \theta' \|, y)$, yielding $\| \theta \|_1 \leq \| \theta' \|_1$, thereby establishing that $\theta$ is also the minimizer of a version of (P2) without the constraint $\| \Psi \theta \|_1 = I$. □

We solved the optimization problem (P4) using the well-known CVX package [26]. Note that the Jensen-Shannon divergence is a convex function and hence P4 is a convex optimization problem, allowing for the use of the CVX package. Experiments were run on Poisson-corrupted compressed measurements derived from a 1D signal with 100 elements and different levels of sparsity in the canonical (i.e., identity) basis as well as different values of $I$. The sensing matrix followed the architecture discussed in Section II. We plotted a graph of the relative reconstruction error given as $RRMSE(x, x^*) \triangleq \frac{\| x - x^* \|}{\| x \|_2}$ versus $I$ for a fixed number of measurements $N = 50$ in Figure 2. This graph clearly reveals lower and lower reconstruction errors with an increase in $I$ which agrees with the worst cases error bounds we have derived in this paper. Note that the graph shows box-plots for reconstruction errors for a population of 10 different measurements of a sparse signal using different $\Phi$ matrices. Figure 2 also shows a graph with box-plots for $RRMSE(x, x^*)$ versus $N$ for a fixed $I = 10^6$. Here we observe that the relative error does not decrease significantly with increase in $N$ because of poorer signal to noise ratio with an increase of $N$ and keeping $I$ constant. Lastly, we also plotted a graph of average $RRMSE(x, x^*)$ against signals of different sparsity levels for a fixed $I$ and a fixed $N$. Note that in each instance for all these experiments, the value of $\lambda$ was picked clairvoyantly, i.e., we chose the value of $\lambda$ that yielded the best reconstruction results assuming the true signal is known in advance. In practice, this parameter would need to be picked by cross-validation or be a user-choice.

We tested the performance of (P4) on an image reconstruction task from compressed measurements under Poisson
Fig. 2. Box plots of relative reconstruction error (RRMSE) for a 1D signal of 100 elements sparse in the canonical basis. RRMSE versus $I$ (top figure) for a fixed $N = 50$ and fixed sparsity = 5, RRMSE versus $N$ (middle figure) for a fixed $I = 10^8$ and fixed sparsity = 5, RRMSE versus sparsity (bottom figure) for a fixed $I = 10^8$ and a fixed $N = 50$. 
Fig. 3. Sample reconstruction results for Poisson-corrupted compressed measurements of an image using penalized JSD and a 2D-DCT basis. Left to right, top to bottom: original image, reconstructions for $I = 10^4$, $I = 10^5$, $I = 10^6$, $I = 10^7$, $I = 10^8$, $I = 10^9$, $I = 10^{10}$. The respective relative reconstruction errors (RRMSE) are 0.7, 0.1, 0.0622, 0.03, 0.015, 0.012 and 0.011.

noise. Each patch of size $7 \times 7$ from a gray-scale image was vectorized and 25 Poisson-corrupted measurements were generated using the sensing matrix discussed in Section [I]. This model is reminiscent of the architecture of the compressive camera designed in [27] except that we considered overlapping patches here. Each patch was reconstructed from its compressed measurements independently by solving (P4) with sparsity in a 2D-DCT basis. The final image was reconstructed by averaging the reconstructions of overlapping patches. This experiment was repeated for different $I$ values by suitably rescaling the intensities of the original image. We also compared our results with the outputs of the following optimization problems:

$$(P5): \min_{\theta} \|\theta\|_1 + SNLL(y, \Phi \Psi \theta) \text{ w.r.t. } \theta$$

$$(P6): \min_{\theta} \|\theta\|_1 + G(y, \Phi \Psi \theta) \text{ w.r.t. } \theta$$

Problems (P5) and (P6) were implemented in CVX under the same setting as described for (P4) since $SNLL$ and $G$ are convex functions. In addition, we also compared these results to those of the well-known Poisson compressed sensing solver known as SPIRAL-TAP from [28] which essentially solves (P6) but follows a different optimization method. We obtained nearly identical results for (P4), (P5), (P6) and SPIRAL-TAP under all settings. In Figure 3, we show reconstruction results with (P4) under different values of $I$. There is a sharp decrease in relative reconstruction error with increase in $I$. We do not show results with other methods including SPIRAL-TAP because the results had no noticeable difference.

Note that in our experiments, we have not made use of the hard constraint $\|x^*\|_1 = I$ in problem (P2). In practice, we however observed that the estimated $\|x^*\|_1$ was close to the true $I$, especially for higher values of $I \geq 10^6$.

Summarily, these numerical experiments confirm the effi-
cacy of using the Jensen-Shannon divergence or its square-root in Poisson compressed sensing problems. While the theoretical properties of this divergence for Poisson compressed sensing have been established earlier in this paper in Section II, our experiments demonstrate its practical utility as well.

IV. RELATION TO PRIOR WORK

In this section, we put our work in the context of existing work on Poisson compressed sensing. The sub-field of Poisson compressed sensing is a relatively less explored territory. While there exist excellent algorithms for Poisson reconstruction such as [21], [3], [29], [30], there is very little work on theoretical bounds for the same problem. Analytical and theoretical bounds for reconstruction under Poisson noise assuming realistic imaging systems (i.e., taking into account non-negativity and flux preserving property of the sensing matrix) were pioneered in the excellent work in [8] and further expanded upon in [11]. In the following, we explain the differences between the approach in this paper and earlier research.

1) Comparison with [8]: The work in [8] considers upper bounds on the expected risk, i.e. on $E\left|\theta - \theta^\star\right|^2$. The upper bounds have the form $E\left|\theta - \theta^\star\right|^2 \leq \sqrt{N\left(\frac{\log m}{I}\right)^{\frac{1}{\alpha}}} + \sqrt{\frac{\log(m/N)}{N}}$ where $\alpha$ is a factor that expresses the compressibility of the signal. On the other hand, our paper considers worst case bounds. We can however adapt our bounds for the expected error. For this purpose, we have empirically observed that the bound $\epsilon$ on $\sqrt{I}J(y, \Phi x)$ has an expected value of the form $O(\lambda^{0.42})$. Consequently, the upper bounds we derive are comparable, as we do not require any multiplicative or additive terms containing $\log m$. A very important point to note is that the approach in [8] considers a sparsity promoting penalty term which is based on coding theory and which is lower in value for sparse coefficient vectors, i.e., it considers $\ell_0$ sparsity regularizers which are not efficiently implementable. In contrast, our approach considers the $\ell_1$ sparsity regularizer for which efficient and tractable algorithms exist.

2) Comparison with [11]: The work in [11] considers minimax bounds on the expected reconstruction error under exact sparsity having the form $\frac{\sqrt{s\log m}}{\sqrt{I}}$ where $s$ is the signal sparsity. On the other hand, our paper considers worst case bounds for sparse as well as compressible signals. The sparsity promoting penalty term in [11] is dealt with in a coding-theoretic framework similar to the approach in [8]. In addition, the work in [11] considers lower bounds on the reconstruction error as well, which are beyond the scope of this work. Lastly, the bounds in [11] do not correspond to a computationally tractable estimator unlike our work which considers the $\ell_1$ norm regularizer and consequently a computationally tractable estimator.

3) Comparison with [31]: The work in [31] deals with a specific type of sensing matrices called the expander-based matrices, unlike the work in this paper which deals with any randomly generated matrices of the form Eqn. 7. The expander-based matrices obey a specific type of RIP called the $\ell_1$-RIP which approximately preserves the $\ell_1$ norm of sparse vectors. Most importantly, the bounds derived in [31] are for signals that are sparse in the canonical basis, unlike our work which deals with any arbitrary sparsifying orthonormal basis.

4) Comparison with [12]: The work in [12] considers compressed sensing under various different noise models from the exponential family. Although Poisson noise is not explicitly considered, the approach is readily extensible to handle Poisson noise. The first major difference between the approach in our paper and [12] is that the latter does not consider realistic imaging systems obeying non-negativity and flux-preservation unlike our approach. The method in [12] is based on a likelihood term that penalizes the Bregman divergence between $y$ and $\Phi x$, i.e., the Lagrange remainder of a first order Taylor series expansion of $y$ about $\Phi x$. This Lagrange remainder term contains the second derivative of the Legendre conjugate of the log-partition function for the Poisson distribution. The log partition has the form $\phi(x) = x \log x - x$ and its second derivative is $1/x$. The bounds derived in [12] have the following form:

$$\frac{\left|\theta - \theta^\star\right|_1}{I} \leq 2\sqrt{2\epsilon} \max_{\mu \in [\mu^\star, \mu^\star]} \phi''(\hat{\mu}) I \min_{\hat{y} \in [y_{\text{min}}, y_{\text{max}}]} \sqrt{\phi''(\hat{y})}$$

where $y, \mu, \mu^\star$ stand for a value in the vectors $y, \mu \triangleq \Phi x$ and $\mu^\star \triangleq \Phi x^\star$ respectively, and $y_{\text{min}} \triangleq \min \{y, \mu, \mu^\star\}$, $y_{\text{max}} \triangleq \max \{y, \mu, \mu^\star\}$. The second derivative term in the numerator is unbounded in case the noise-free measurement vector $\mu$ contains a small value at any of its indices. The denominator on the other hand produces a value that is proportional to the square root of the maximum value of $\sqrt{J}$ which can be unbounded if the corresponding $\mu$ is large since $y \sim \text{Poisson}(\mu)$. Consequently, the upper bound on the error $\frac{\left|\theta - \theta^\star\right|_1}{I}$ could be extremely large (and hence not optimally tight). The approach in this paper avoids the latter issue as far as the bounds using $J$ are concerned. We do require a lower bound on the values in $\mu$ in order for $J$ to lower bound $S/N2L$, but the bounds derived using $J$ as such, are unaffected unless $I$ is very small.

5) A note on the normal approximation to the Poisson: For larger values of $\lambda$, it is known that Poisson($\lambda$) $\sim N(\lambda, \lambda)$. Therefore from $y \sim \text{Poisson}(\Phi x)$, we can approximate that $\frac{y - \Phi x}{\sqrt{\|\Phi x\|^2}} \sim N(0, 1)$. Thus each element of the measurement vector is approximately distributed as follows: $y_i \sim N(\langle \Phi x_i \rangle, \langle \Phi x_i^2 \rangle)$. Now, assuming a bound of $\epsilon$ such that $\frac{y_i - \langle \Phi x_i \rangle}{\sqrt{\langle \Phi x_i^2 \rangle}} \leq \epsilon$, yields $\|y - \Phi x\|_2 \leq \sqrt{7}\epsilon$ because for each $i$, we have
\((\Phi x)_i \leq \frac{I}{N}\). Therefore a solution to minimizing \(\|\theta\|_1\) such that \(\|y - A\theta\|_2 \leq \epsilon\) yields error bounds of the form: 
\[
\frac{\|\theta - \theta^\dagger\|_2}{I} \leq C_3 \sqrt{\frac{N}{I} \epsilon + C_4 s^{-1/2} \|\theta - \theta^\dagger\|_1}
\]
for constants \(C_3\) and \(C_4\). To the best of our knowledge, the normal approximation to the Poisson has not been considered before in Poisson compressed sensing. It should be noted, however, that this model is an approximation and is not valid for very low values of the mean of a Poisson random variable.

6) A note on variance-stabilizing transforms: Consider a random variable \(y \sim \text{Poisson}(x)\), and define \(z = 2\sqrt{y + \frac{3}{8}}\). Then we have \(z \sim \sqrt{x + \frac{3}{8} + \nu}\), where \(\nu \sim \mathcal{N}(0, 1)\), for \(x \geq 4\). This is called as the Anscombe transformation [32], and it is widely used in Poisson denoising [33]. However using this approach in the context of compressive reconstruction of \(x\) from measurement vector \(y \sim \text{Poisson}(\Phi x)\) leads to a non-linear inversion problem, of the form \(\sqrt{y} \sim \sqrt{\Phi x + \frac{3}{8} + \nu}\) where \(\forall i, \nu_i \sim \mathcal{N}(0, 1)\), for which no known error bounds exist (to the best of our knowledge).

V. CONCLUSION

In this paper, we have presented new upper bounds on the reconstruction error from compressed measurements under Poisson noise, for a computationally tractable estimator using the \(\ell_1\) norm sparsity regularizer. Our bounds are easy to derive and follow the scheme of the technique laid out in [1]. The bounds exploit the unique properties of the square-root of the Jensen-Shannon divergence such as its metric nature, and are applicable to sparse as well as compressible signals in any chosen orthonormal basis. We have presented numerical simulations showing the efficacy of the method in reconstruction from compressed measurements under Poisson noise. We observe that the derived upper bounds decrease with an increase in the original signal flux, i.e. \(I\). However the bounds do not decrease with an increase in the number of measurements \(N\), unlike conventional compressed sensing. This observation, though derived independently and using different techniques, agrees with existing literature on Poisson compressed sensing [8]. The reason for this strange observation is the division of the signal flux across the \(N\) measurements, thereby leading to poorer signal to noise ratio. There exist several avenues for future work, as follows:

1) A major issue is to explore theoretical error bounds in the absence of the knowledge of \(I\), which is an open problem to the best of our knowledge. It should be noted, however, that our experimental simulations show excellent reconstructions under appropriate conditions, all under the absence of knowledge of \(I\) and in fact the estimated \(\ell_1\) norm of the signal was seen to be very close to the true \(I\). Yet, it would be interesting to explore methods for theoretical bounds without employing this constraint actively within the analysis.

2) Another major issue is automatic choice for the regularization parameter \(\lambda\) in Poisson regression or compressed sensing problems. Any given value of \(\lambda\) corresponds to an appropriate \(\epsilon\). One could explore methods based on different model selection criteria [34] or image quality metrics.

3) Derivation of error bounds from variance-stabilizing transforms is also an important open problem, especially because it deals with compressed sensing under measurements that are of the form \(\sqrt{\Phi x + \frac{3}{8}}\) (with unknown \(x\)), and hence are effectively non-linear.

4) Furthermore, it will be useful to derive lower-bounds on the reconstruction error.

ACKNOWLEDGMENT

The authors would like to thank Kalyani Krishnamurthy for useful discussions.

REFERENCES

[1] E. Candès, “The restricted isometry property and its implications for compressed sensing,” *Comptes Rendus Mathematique*, vol. 346, no. 910, pp. 589 – 592, 2008.

[2] F. Alter, Y. Matsushita, and X. Tang, “An intensity similarity measure in low-light conditions,” in *ECCV*, 2006.

[3] J. L. Starck and J. Bobin, “Astronomical data analysis and sparsity: From wavelets to compressed sensing,” *Proceedings of the IEEE*, vol. 98, no. 6, pp. 1021–1030, June 2010.

[4] J. Boone, E. Geraghty, J. Seibert, and S. Wootton-Gorges, “Dose reduction in pediatric CT: A rational approach,” *Radiology*, vol. 228, no. 2, pp. 352–360, 2003.

[5] J. Boulanger, C. Kerrvann, P. Bouthemy, P. Elbau, J.-B. Sibarita, and J. Salamero, “Patch-based nonlocal functional for denoising microscopy image sequences,” *IEEE Trans. Med. Imag.*, vol. 29, no. 2, p. 442454, 2010.

[6] M. Aarsvold and J. Vernick, *Emission Tomography*. Academic Press, 2004.

[7] R. Baraniuk, M. Davenport, R. DeVore, and M. Wakin, “A simple proof of the restricted isometry property for random matrices,” *Constructive Approximation*, vol. 28, no. 3, pp. 253–263, Dec. 2008.

[8] M. Raginsky, R. Willett, Z. Harmany, and R. Marcia, “Compressed sensing performance bounds under poisson noise,” *Signal Processing, IEEE Transactions on*, vol. 58, no. 8, pp. 3900–4002, Aug 2010.

[9] C. Fevotte and A. T. Cemgil, “Nonnegative matrix factorizations as probabilistic inference in composite models,” in *Signal Processing Conference*, 2009 17th European, 2009, pp. 1913–1917.

[10] D. Endres and J. Schindelin, “A new metric for probability distributions,” *IEEE Trans. Inf. Theory*, vol. 49, no. 7, p. 18581860, 2003.

[11] X. Jiang, G. Raskutti, and R. Willett, “Minimax optimal rates for poisson inverse problems with physical constraints,” *IEEE Trans. Information Theory*, vol. 61, no. 8, pp. 4458–4474, 2015.

[12] I. Rish and G. Grabarnik, “Sparse signal recovery with exponential-family noise,” in *Communication, Control, and Computing*, 2009. Allerton 2009. 47th Annual Allerton Conference on, 2009, pp. 60–66.

[13] M. Mordechay and Y. Y. Schechner, “Matrix optimization for poisson compressed sensing,” in *Signal and Information Processing (GlobalSIP), 2014 IEEE Global Conference on*, 2014, pp. 684–688.

[14] “Stirling’s approximation,” [https://en.wikipedia.org/wiki/Stirling%27s_approximation](https://en.wikipedia.org/wiki/Stirling%27s_approximation) online; accessed May 2016.

[15] M. Collins, S. Dasgupta, and R. Schapire, “A generalization of principal component analysis to the exponential family,” in *Advances in Neural Information Processing Systems*, 2001.

[16] M. Lin, “Divergence measures based on the shannon entropy,” *IEEE Transactions on Information Theory*, vol. 37, no. 1, pp. 453–459, 1991.

[17] J. Topsoe, “Some inequalities for information divergence and related measures of discrimination,” *IEEE Transactions on Information Theory*, vol. 29, no. 4, pp. 465–466, 1983.

[18] R. Saab and O. Yilmaz, “Sparse recovery by non-convex optimization instance optimality,” *Applied and Computational Harmonic Analysis*, vol. 29, no. 1, pp. 30 – 48, 2010.
[20] E. Candes, M. Wakin, and S. Boyd, “Enhancing sparsity by reweighted $\ell_1$ minimization,” *Journal of Fourier Analysis and Applications*, vol. 14, no. 5, pp. 877–905, 2008.

[21] D. Lingenfelter, J. Fessler, and Z. He, “Sparsity regularization for image reconstruction with poisson data,” vol. 7246, 2009.

[22] M. Duarte, M. Davenport, D. Takhar, J. Laska, T. Sun, K. Kelly, and R. Baraniuk, “Single pixel imaging via compressive sampling,” *IEEE Signal Processing Magazine*, 2008.

[23] M. Elad, “Optimized projections for compressed-sensing,” *IEEE Trans. on Signal Processing*, vol. 55, no. 12, pp. 5695–5702, 2007.

[24] M. Raginsky, S. Jafarpour, Z. T. Harmany, R. F. Marcia, R. M. Willett, and R. Calderbank, “Performance bounds for expander-based compressed sensing in poisson noise,” *IEEE Transactions on Signal Processing*, vol. 59, no. 9, pp. 4139–4153, Sept 2011.

[25] F. J. Anscombe, “The transformation of poisson, binomial and negative-binomial data,” *Biometrika*, vol. 35, no. 3/4, pp. 246–254, 1948.

[30] S. Sra, D. Kim, and B. Scholkopf, “Non-monotonic poisson likelihood maximization,” Max Planck Institute for Biological Cybernetics, Tech. Rep. 170, 2008.

[34] “Akaike information criterion,” [https://en.wikipedia.org/wiki/Akaike_information_criterion](https://en.wikipedia.org/wiki/Akaike_information_criterion) 2016.