Weakly nonlinear observables in dark energy cosmologies

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What are the fundamental limitations of reconstructing the properties of dark energy, given cosmological observations in the weakly nonlinear regime in a range of redshifts, to be as precise as required? The aim of this paper is to address this question by constructing model-independent observables, whilst completely ignoring practical problems of real-world observations. Non-Gaussianities already present in the initial conditions are not directly accessible from observations, because of a perfect degeneracy with the non-Gaussianities arising from the nonlinear matter evolution in generalized dark energy models. By imposing a specific set of evolution equations that should cover a range of dark energy cosmologies, we however find a constraint equation for the linear structure growth rate $f_1$ expressed in terms of model-independent observables. Entire classes of dark energy models which do not satisfy this constraint equation could be ruled out, and for models satisfying it we could reconstruct e.g. the nonlocal bias parameters $b_1$ and $b_2$.

I. INTRODUCTION

Gravity is a nonlinear phenomenon that is also responsible for today’s observed large-scale structure. On very large scales, gravitational interactions should be close to linear, even if there are significant non-Gaussian features in the initial conditions for structure formation. The linearity of gravitational interactions on large scales is mainly due to a suppressed interaction rate of energy fluctuations close to the causality horizon. On smaller scales well inside the horizon, gravitational interactions grow exponentially and, since dark matter seems to behave as being pressureless, we observe nonlinear amplifications of over- and underdensities, accompanied by increasing tidal interactions.

Cosmological structures such as filaments, clusters and voids emerge on scales that connect the linear and nonlinear regime. Galaxies are tracers of the underlying matter distribution, and the explicit bias relation is unknown. Generally, galaxy bias could depend on the scale and on nonlocal physical processes, such as galaxy formation and hydrodynamical interactions, whose specific mechanisms are not yet comprehensively understood. Simplified bias models such as the local model can be very accurate, especially on large scales, but need to be revised when investigating cosmological models beyond ΛCDM. The reason for the necessary revision is that departures from ΛCDM usually imply scale-dependent matter growth, which also renders the bias to be scale-dependent and nonlocal (see [1] for a review).

Dark energy (DE) could affect all of the above. So far the simple ΛCDM model has been remarkably successful at explaining a host of astrophysical observations on a wealth of scales, but more sophisticated DE models are not ruled out and should be further investigated [2–4]. Much effort has been made to understand DE and possible modifications at the level of background, linear and nonlinear perturbation observables, often with the premise to fix a particular DE model and investigate the resulting phenomenological consequences (e.g. [5–12]). In the literature there are also many approaches to investigate DE modifications in a model-independent way [13–24], but they are usually restricted to the linear regime. One of the tasks of the present study is to extend the model-independent approach by allowing weak nonlinearities in the analysis.

By also allowing weak nonlinearities in our model-independent analysis, we of course not only provide access to a wealth of additional cosmological information, but also introduce many more unknowns that should be taken into account. Such unknowns could arise from e.g. the bias model or the nonlinear matter evolution within the DE model. Furthermore, also non-Gaussian modifications could be present already in the initial conditions of structure formation. These modifications are usually dubbed as primordial non-Gaussianity (PNG), and in the present paper we refrain to use any simplified parametrization of PNG. Rather we show, amongst other things, that nonlinearities in the matter evolution and PNG are indistinguishable, because of a perfect degeneracy in DE models. However, by going beyond linear order we derive new observables that constrain the combined effect of gravity and PNG and, furthermore, find a novel constraint equation that also gives insight into the linear regime of structure formation — much more in-

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In Fourier space is w.r.t. time. If not otherwise stated, the functional dependence on two wavevectors \( k \) we make use of the integral shorthand notation of DE cosmologies.

We adopt metric signature \((-+++)\), cosmic time is \( t \) and its corresponding partial derivative is the overdot, \( \dot{a} \), and \( \dot{z} \) the redshift. The subscript 0 denotes present time. If not otherwise stated, the functional dependence in Fourier space is w.r.t. \( k = |k|\). The shorthands \( k_{12} \) and \( k_{123} \) stand for \( k_1 + k_2 \) and \( k_1 + k_2 + k_3 \), respectively, and we make use of the integral shorthand notation \( \int d^3k_{12} = \int d^3k_1 \int d^3k_2 \). For a given function \( \mathcal{F}(k_i, k_j) \) that depends on two wavevectors \( k_i \) and \( k_j \), where \( i, j \in \{1, 2, 3\} \) and \( i \neq j \), the shorthand \( \mathcal{F}^{eq} \) denotes equal dependence for which \( k_1 = k_2 = k_3 \equiv k \). We apply a similar shorthand \( \mathcal{F}^{eq}_{ij} \equiv \mathcal{F}^{eq}(k_i, k_j) \) for triangle dependences in the squeezed limit, where \( k_1 = k_2 \approx k \) and \( k_3 = \Delta k \), and \( \Delta k/k \to 0 \).

II. ASSUMPTIONS

(a) The background geometry of the Universe is well described by a Friedmann–Lemaître–Robertson–Walker metric; its evolution is parametrized by the cosmic scale factor \( a(t) \). The Hubble parameter \( H = \dot{a}/a \) is governed by the Friedmann equation

\[
H^2 - H_0^2 \Omega_0 a^{-2} = \frac{1}{3} (\rho_m + \rho_x)
\]

(setting \( 8\pi G = 1 \)), where \( H_0 \) and \( \Omega_0 \) are respectively the present day values of the Hubble parameter and curvature, \( \rho_m \sim a^{-3} \) is the background density of matter, and \( \rho_x \) is the combined background density of an unspecified modification of gravity. Background observations can generally measure \( H(z) \) up to a multiplicative constant (see e.g., [21]), and thus we assume in the following that the dimensionless Hubble function \( \mathcal{E}(z) \equiv H(z)/H_0 \) is an observable. Combining measurements of the luminosity or angular-diameter distance with \( H(z) \), we can furthermore determine \( \Omega_0 \) [25]. By contrast, it is impossible to measure \( \Omega_m \) without invoking an explicit parametrization for \( \rho_x \), as the problem is perfectly degenerated [25].

(b) We are interested in the cosmological evolution within the weakly nonlinear regime where perturbation theory should be valid. We only take the leading nonlinearities into account and consequently ignore any loop contributions. This implies that we only need to go up to second order in the fluid variables, i.e.,

\[
\delta_m = \delta_{m1} + \delta_{m2}, \quad \theta_m = \theta_{m1} + \theta_{m2},
\]

where \( \delta_m \equiv (\rho_m - \bar{\rho}_m)/\rho_m \) is the matter density contrast and \( \theta_m \) the divergence of the matter peculiar velocity. In the following section we provide some evolution equations for these fluid variables, although we stress that explicit evolution equations are only required for Sec.VIII when we derive a novel constraint equation.

We furthermore restrict our analysis to the sub-horizon regime to minimize contamination of secondary GR corrections (e.g., arising on the matter level [27], through radiation [28], or through light-cone effects [29]). Relativistic corrections in the initial conditions could generate nonzero intrinsic bispectra, which we do allow in our analysis; see (d).

(c) We apply the so-called plane-parallel limit when projecting fluid variables from real-space coordinates \( x \) to redshift-space coordinates \( s \) [30],

\[
s = x + \nabla^{-2} \nabla \theta,
\]

where the inverse Laplacian is w.r.t. the real-space coordinates. For the bias relation between matter and galaxy, we allow the bias function to be scale- and time-dependent. This means that the galaxy density can be written as \( \delta_g = \delta_{g1} + \delta_{g2} \), which is in Fourier space [31]

\[
\delta_{g1}(z; k) = b_1 \delta_{m1},
\]

\[
\delta_{g2}(z; k) = b_1 \delta_{m2} + \frac{1}{2} \int \frac{d^3k'}{(2\pi)^3} b_2(k', k - k') \times \delta_{m1}(k') \delta_{m1}(k - k'),
\]

where, again, the unknown bias functions \( b_1 \) and \( b_2 \) are generally scale- and time-dependent. At the same time we assume that there is no bias between the matter and galaxy velocity. This means that, on the scales we consider, both baryons and dark matter respond in the same way to gravity.

(d) Although there is currently no sign of any significant nonzero primordial non-Gaussianity (PNG), we allow in the present analysis for the most general deviation of Gaussian initial conditions, that is we do not invoke any parametrization of PNG. Rather we assume that there exists a possible nonzero intrinsic matter bispectrum,

\[
(\delta_{m1} \delta_{m1} \delta_{m1})_{\ell} \sim B_{m111}(z; k_1, k_2, k_3),
\]

which is related in some arbitrary way to the initial curvature perturbation on superhorizon scales. For example, \( B_{m111} \) could originate from PNG of the local type in which case one would expect large contributions to the bispectrum in the squeezed limit. We neglect PNG contributions to the initial trispectrum and higher-order correlators, as they usually involve loop correlations; see assumption (b).
III. EQUATIONS IN REAL & REDSHIFT SPACE

Although not required for the bulk part of this paper (except Sec. VII), let us assume the following explicit evolution equations for matter,

\[ \delta_m' + \nabla \cdot (1 + \delta_m) v_m = 0, \quad (6a) \]

\[ v_m' + (v_m \cdot \nabla) v_m = - \left( 2 + \frac{H'}{H} \right) v_m - \nabla \Psi. \quad (6b) \]

Here \( \delta_m \equiv (\rho_m - \bar{\rho}_m)/\bar{\rho}_m \) is the matter density contrast and \( v_m \) the peculiar velocity of matter, \( H \) the Hubble parameter, a prime denotes a partial derivative w.r.t. the time variable \( \ln a \), and \( a \) is the cosmic scale factor itself determined by the Friedmann equation [1]. We make use of the modified Poisson equation,

\[ \nabla^2 \Psi(x) = \frac{3}{2} \Omega_m \int d^3 y Y(x-y) \delta_m(y), \quad (6c) \]

where \( Y \) is a scale- and time-dependent clustering function. The function \( Y \) is by definition equal to unity in a Universe that is prescribed by a cosmological constant (\( \Lambda \)) and a CDM component; by contrast, for a realistic \( \Lambda \)CDM Universe, where generally not just CDM but also other fluid components are present (e.g., massive neutrinos), \( Y \) differs (mildly) from unity reflecting the fact that CDM couples to other fluid components gravitationally. In addition, \( Y \neq 1 \) can be established by a wealth of modified gravity scenarios (see [2] and references therein). In the following, we will make no model-dependent assumptions how \( Y \) might look alike and thus leave it as a free function.

To solve Eqs. (6a)-(6c), we assume that the fluid motion is irrotational and thus, the velocity can be fully described by its divergence, \( \theta = \nabla \cdot v \). Perturbing the density and velocity according to [2], we obtain to first order in Fourier space

\[ \delta_m'' + \left( 2 + \frac{H'}{H} \right) \delta_m' - \frac{3}{2} \Omega_m Y \delta_m = 0. \quad (7) \]

The growing mode solution for the density can be formally written as \( \delta_m(z; k) = D(z; k) \delta_0(z_0; k) \), where \( D \) is the linear growth function which is normalized to unity today, and \( \delta_0 \) is the present matter density. We note that \( D \) is not only time-dependent but in general also scale-dependent. Using the solution for the density, one immediately gets for the first-order velocity \( \theta_m = -\delta_m' = -f_1 \delta_m \), where the linear structure growth rate \( f_1 \) is defined by \( f_1 \equiv D'/D \).

Second-order solutions can be formally written as

\[ \delta_{m2}(z; k) = \int \frac{d^3 k_{12}}{(2\pi)^3} \delta_D^{(3)}(k - k_{12}) F_2(z; k_1, k_2) \times \delta_m(z; k_1) \delta_m(z; k_2), \quad (8a) \]

\[ \theta_{m2}(z; k) = \int \frac{d^3 k_{12}}{(2\pi)^3} \delta_D^{(3)}(k - k_{12}) G_2(z; k_1, k_2) \times \delta_m(z; k_1) \delta_m(z; k_2), \quad (8b) \]

where \( \delta_D^{(3)} \) is the Dirac-delta distribution, \( F_2 \) and \( G_2 \) are perturbation kernels with symmetric \( k \)-dependence in their arguments, and the matter density and velocity only depend on the magnitude of the wavevector \( k \equiv |k| \), due to statistical isotropy. For an Einstein-de Sitter (EdS) universe the above kernels become time-independent, and read in our sign convention

\[ F_{2,\text{EdS}} = \frac{5}{7} + \frac{k_1 \cdot k_2}{2k_1 k_2} \left[ k_1 \cdot k_3 + k_2 \cdot k_3 \right] + \frac{2}{7} \left( \frac{k_1 \cdot k_2}{k_1 k_2} \right)^2, \quad (9) \]

\[ G_{2,\text{EdS}} = -\frac{3}{7} - \frac{k_1 \cdot k_2}{2k_1 k_2} \left[ k_1 \cdot k_3 + k_2 \cdot k_3 \right] - \frac{4}{7} \left( \frac{k_1 \cdot k_2}{k_1 k_2} \right)^2. \quad (10) \]

For a standard \( \Lambda \)CDM model or in modified gravity, however, these kernels generally do depend on time. Note that to construct the observables we do not require explicit solutions for \( F_2 \) and \( G_2 \); we only assume that solutions for \( \delta_m \) and \( \theta_m \) can be written in terms of a power series in the linear density.

Up to this point we have dealt with matter perturbations in real space, but what we observe are galaxies, measured in redshift space. We deal with the galaxy description as outlined in Sec. IIII, see in particular Eqs. (1a)-(4b), where we employ a nonlocal bias description between the matter density \( \delta_m \) and the galaxy density \( \delta_g \), whereas \( v_m = v_{\text{rel}} \equiv v \). The next step is to incorporate the effects of redshift-space distortions, resulting from the fact that the observed comoving positions of galaxies \( s \) are modified by their peculiar motion according to \( s = x + v(z) \hat{z} \) in the plane-parallel limit, where \( \hat{z} \) is the unit vector along the line of sight and \( v \) is the projection of the peculiar velocity along the \( z \)-axis. This leads to the following relation between the galaxy density in redshift space, \( \delta_g^{(r)}(z; k) \), and the one in real space, \( \delta_g(z; x) \)

\[ \delta_g^{(r)}(z; k) = \int d^3 x e^{-ik \cdot x} [1 + \delta_g(z; x)] e^{-ik \cdot v(z)}. \quad (11) \]

Taylor expanding the fluid variables and the exponential in the last expression, we obtain

\[ \delta_g^{(r)}(z; k) = S_1(z; k_1) \delta_m(z; k_1), \quad (12) \]

\[ \delta_g^{(r)}(z; k) = \int \frac{d^3 k_{12}}{(2\pi)^3} \delta_D^{(3)}(k - k_{12}) S_2(z; k_1, k_2) \times \delta_m(z; k_1) \delta_m(z; k_2), \quad (13) \]

with the kernels

\[ S_1 = b_1 + f_1 \mu_1^2, \quad (14) \]

\[ S_2 = -\mu_2^2 G_2 + b_1(k_{12}) F_2 + \frac{1}{2} \mu_1 k_{12} \left[ \frac{\mu_1}{k_1} b_1(k_2) f_1(k_1) + \frac{\mu_2}{k_2} b_1(k_1) f_1(k_2) \right] \]

\[ + \left( \mu_1 k_{12} \right)^2 \left[ \frac{\mu_1}{k_2} b_1(k_1) f_1(k_2) + \frac{1}{2} b_2(k_1, k_2) \right], \quad (15) \]

where \( \mu = k \cdot \hat{z}/k \) is the cosine of the angle formed by the direction of the observation \( \hat{z} \) and the wavevector \( k \) and
\[ \mu_i = k_i \cdot \hat{z}/k_i. \] In the case when \( f_1 \) is scale-independent (e.g., in \( \Lambda \)CDM), these kernels are formally equivalent with the ones in Ref. [21].

IV. POWER SPECTRUM IN REDSHIFT SPACE

To understand our methodology in the following sections, it is instructive to first investigate the linear observables that can be constructed from the galaxy power spectrum [21]. The galaxy power spectrum in redshift space is defined as

\[ \langle \delta^2 (k_1) \delta (k_2) \rangle_c = (2\pi)^3 \delta^2_D (k_1) P^g (z; \mu_1, k_1), \] (16)

where we note that \( P^g \) depends not only on the magnitude but also on the cosine of the wavevector w.r.t. the direction of the observation, since it acquires an angular dependence due to the redshift-space distortions. The matter power spectrum \( P_m (k) \), by contrast, depends only on the modulus \( k \) due to the assumption of statistical isotropy.

As for the perturbations of field variables, we can formulate the power spectrum in terms of a power series within perturbation theory, e.g., for the matter power spectrum we have, to the leading order, that \( P_m = P_{m11} \sim \delta^2_{m11} \). In the linear regime and for scales much smaller than the survey characteristic size, one can write for the galaxy power spectrum

\[ P^g (z; k, \mu) = (b_1 + f_1 \mu^2)^2 P_{m11}, \] (17)

where we remind the reader that the functions \( f_1 \) and \( b_1 \) depend generally on space and time. This expression can be written in terms of a polynomial in \( \mu \):

\[ P^g (z; k, \mu) = P_{m11} (z; k) \sum_i P_i \mu^i, \] (18)

with the only nonvanishing coefficients

\[ P_0 = b_1^2, \quad P_2 = 2b_1 f_1, \quad P_4 = f_1^2. \] (19)

Observations can be made in principle at all values of \( \mu \). This means that one can measure individually each term in the \( \mu \) expansion. Taking ratios of the various terms in Eq. (18) one gets rid of \( P_{m11} \) (and the unknown normalization \( \sigma_8 \)), whose shape depends in general on initial conditions. One obtains, for example, the quantity

\[ P_1 = f_1/b_1 \] (20)

from \((2P_4)/P_2\). The same procedure, extended to the bispectrum, is at the core of the method presented below. In addition to galaxy spectra, we will take into account also shear lensing spectra and cross-correlation spectra of lensing and galaxy clustering, in order to identify which quantities can be measured directly from observations without assumptions on the shape of the (bi-)spectra.

Further linear observables are reviewed in Sec. VII A.

V. BISPECTRUM IN REDSHIFT SPACE

The galaxy bispectrum in redshift space is defined as

\[ \langle \delta^2 (k_1) \delta (k_2) \delta (k_3) \rangle_c = (2\pi)^3 \delta^3_D (k_1, k_2, k_3) B_g (k_1, k_2, k_3). \] (21)

The bispectrum is nonzero only when nonlinearities are present. This is especially the case in the weakly nonlinear regime of structure formation, which expands roughly from several Mpc scales up to the cosmological horizon. As mentioned in Sec. III, we allow in the present analysis of nonlinearities arising from the initial condition (PNG) and of the nonlinear evolution of matter.

For the galaxy bispectrum, we get to the leading order

\[ B_g = 2S_2 (k_1, k_2) S_1 (k_1) S_1 (k_2) P_{m11} (k_1) P_{m11} (k_2) \]
\[ \quad + \text{two perms} + S_1 (k_1) S_1 (k_2) S_1 (k_3) B_{m111}, \] (22)

where

\[ \langle \delta_{m1} (k_1) \delta_{m1} (k_2) \delta_{m1} (k_3) \rangle_c = (2\pi)^3 \delta^3_D (k_1, k_2, k_3) B_{m111}. \] (23)

is the said non-Gaussian component arising from primordial/unknown physics.

The galaxy bispectrum in redshift space is a function of five variables. The shape of the triangle is defined by three variables: the length of two sides, i.e., the magnitude of two wavevectors, \( k_1 \) and \( k_2 \), and the angle between them, \( \cos \theta_{12} = k_1 \cdot k_2 / (|k_1| |k_2|) \). The two remaining variables characterize the orientation of the triangle with respect to the line of sight: we take them to be the polar angle of \( k_1 \), \( \omega = \arccos \mu_1 \), and the azimuthal angle \( \phi \) around \( k_1 \). All the angles between the wavevectors and the line of sight can be written in terms of \( \mu_1 \) and \( \phi \) [22],

\[ \mu_1 = \frac{k_1 \cdot \hat{z}}{k_1}, \quad \mu_2 = \mu_1 \cos \theta_{12} - \sqrt{1 - \mu_1^2} \sin \theta_{12} \cos \phi, \]
\[ \mu_3 = -\frac{k_1}{k_3} \mu_1 - \frac{k_2}{k_3} \mu_2. \] (24)

We now determine the explicit expressions for the galaxy bispectrum for two fixed triangle configurations, namely for the equilateral and the squeezed type.

A. The equilateral bispectrum

In the equilateral configuration all the wavevectors have the same magnitude which we take to be \( k_1 = k_2 = k_3 = k \), from which it follows that \( k_i \cdot k_j / (k_i k_j) = -1/2 \), for \( i \neq j \). Furthermore, the relation (21) between the three \( \mu_i \)’s simplifies to

\[ \mu_2 = -\frac{\mu_1}{2} - \sqrt{3 - 3\mu_1^2} \cos \phi/2, \quad \mu_3 = -\mu_1 - \mu_2, \] (25)

which we use to replace all \( \mu_2 \)’s and \( \mu_3 \)’s in the general expression for the bispectrum [22] in terms of \( \mu_1 \). We
are thus left with a bispectrum that depends only on two angles, namely on $\mu_1$ and on the azimuthal angle $\phi$. We integrate out the azimuthal angle because of statistical isotropy around the redshift axis. Thus, one finally arrives at the equilateral bispectrum which is given in terms of a polynomial in $\mu_1$, 

$$B^\text{eq}_g = P^2_{m11} \sum_i B^\text{eq}_i \mu_1^i,$$  

(26)

with nonvanishing coefficients $B^\text{eq}_0, B^\text{eq}_2, B^\text{eq}_4, B^\text{eq}_6$ and $B^\text{eq}_8$. In the main text we only need the last two coefficients 

$$B^\text{eq}_6 = -\frac{177}{1024} f_1^2 \left( f_1^2 + 16 G^\text{eq}_2 - \frac{8}{3} Q^\text{eq}_{m111} f_1 \right),$$  

(27)

$$B^\text{eq}_8 = -\frac{87}{1024} f_1^4,$$  

(28)

where $G^\text{eq}_2$ is the second-order velocity kernel in the equilateral configuration, and we have defined the reduced intrinsic bispectrum 

$$G^\text{eq}_{m11} \equiv B^\text{eq}_{m111} / P^2_{m11}.$$

The complete list of bispectrum coefficients is given in Appendix A.

B. The squeezed bispectrum

The squeezed bispectrum is a specific limit that correlates density perturbations on essentially two different scales to each other. In that limit, two density perturbations which are usually taken to be well inside the horizon, are correlated with another perturbation close to the horizon (or beyond). The corresponding triangle configuration in that limit is such that one wavevector, $\Delta k$, is much smaller than the other two. We choose $k_1 = k_2 = k$, and $k_3 = \Delta k$. We leave $\Delta k$ as a free parameter but note that the squeezed approximation becomes more accurate when $\Delta k / k \to 0$. In the present paper we assume that the correlation length $k$ is in the weakly nonlinear regime, where second-order perturbation theory is a good approximation of the underlying physics, whereas $\Delta k$ is on sufficiently large scales where perturbations should mostly follow the overall Hubble flow and are otherwise well described by linear perturbation theory. For the squeezed bispectrum, we thus assume the existence of an intermediate regime where we can use the linear observables as linear operators on functions which depend on the squeezed bispectrum triangle side $\Delta k$ (see the following).

From the $\mu_j$ relations $[24]$, we get $\mu_2 \simeq -\mu_1$ for all values of the azimuthal angle $\phi$, and the latter drops out. Thus, we can write the squeezed bispectrum as a polynomial of two cosines, 

$$B^\text{eq}_g = \sum_{i,j} B^\text{eq}_{ij} \mu_1^i \mu_2^j,$$  

(30)

with the only nonvanishing coefficients 

$$B^\text{eq}_{0\Delta k} = a_1 b_1 \Delta k + b^\text{sq}_{2,12} b_1^2 P^2_{m11} + B^\text{sq}_{m111} b_1^3 \Delta k,$$  

(31)

$$B^\text{eq}_{1\Delta k} = a_1 b_1 \Delta k + B^\text{eq}_{m111} b_1 \Delta k,$$  

(32)

$$B^\text{eq}_{2\Delta k} = a_1 f_1 b_1 \Delta k + a_2 b_1 \Delta k + 2b^\text{sq}_{2,12} b_1 f_1^2 P^2_{m11} + 2B^\text{sq}_{m111} f_1 b_1 \Delta k,$$  

(33)

$$B^\text{eq}_{2\Delta k} = a_1 f_1 b_1 \Delta k + B^\text{eq}_{m111} f_1 b_1 \Delta k,$$  

(34)

$$B^\text{eq}_{2\Delta k} = a_2 f_1 b_1 \Delta k + b^\text{sq}_{2,12} f_1^2 P^2_{m11} + B^\text{eq}_{m111} f_1^2 b_1 \Delta k,$$  

(35)

$$B^\text{eq}_{2\Delta k} = a_2 f_1 b_1 \Delta k + B^\text{eq}_{m111} f_1^2 b_1 \Delta k,$$  

(36)

where we have introduced the shorthand notation $b^\text{eq}_{2,12} = b^\text{eq}_2(k_1, k_2)$, and $P_{m11, \Delta k} \equiv P_{m11}(\Delta k)$ etc., and defined 

$$a_1 = \left( b^\text{sq}_{2,13} + b^\text{sq}_{2,23} + 4b_1 F^\text{eff}_2 P_{m11} P_{m11, \Delta k},$$  

(37)

$$a_2 = \left( 2b_1 \Delta k f_1 - G^\text{sq}_{2,12} \right) P_{m11} P_{m11, \Delta k}.$$  

(38)

The bar indicates the ratio $\bar{B}^\text{eq}_{0\Delta k} = P_{1, \Delta k}^{-1} B^\text{eq}_{0\Delta k}$ etc., where $P_{1, \Delta k} = f_1 \Delta k / b_1 \Delta k$ is the linear observable in the $\Delta k$ mode, which is assumed to be in the quasi-linear regime. As promised above, $P_{1, \Delta k}$ is thus to be understood as an operator acting on given functions. By contrast, we do not make use of the operator $P_{1} \equiv P_{1}(k)$ as the $k$-mode is assumed to be well inside the nonlinear regime. We have defined $2F^\text{eff}_2 \equiv F^\text{eq}_{2,13} + F^\text{eq}_{2,23}$ and $2G^\text{sq}_{2,12} \equiv G^\text{sq}_{2,13} + G^\text{sq}_{2,23}$ which are free of infrared divergences even in the vicinity of $\Delta k \to 0$. We note that in deriving the above expressions, we have assumed that $F_{2,12} = F_2(k, -k) = 0$, a relation which is trivial to see in an EdS universe but generally holds also in DE models, as we shall prove in Appendix C.

The galaxy bispectrum coefficients contain mostly too cluttered information about unknowns, and this is why we investigate in the following more sources of potential observables. Nevertheless, some of the above coefficients will become essential when determining our observables.

VI. LENSING AND LENSING-GALAXY CROSS-SPECTRA

Weak lensing, together with cross-correlations, provides another important tool in our analysis to gain further knowledge of nonlinear structure formation. To discuss weak lensing we make use of the scalar line element $ds^2 = -(1 + 2\Psi) dt^2 + a^2 (1 + 2\Phi) d\vec{x}^2$ up to second order. We neglect vector and tensor modes as we are usually interested in DE modifications of the scalar type. Secondary vector and tensor modes, even present in standard ΛCDM cosmologies (see e.g. [33]), are ignored as well, as their impact should be vanishingly small on the scales we consider.

Dark energy models usually modify the source term in the Poisson equation [30], and on top of that, modifications in the gravitational slip are expected as well, the
latter defined by
\[
\eta = -\frac{\Phi}{\Psi},
\]
In ΛCDM we have \( \eta = 1 \) to a very good approximation. Gravitational lensing is unaffected by redshift-space distortions or the (unknown) bias, and is instead only sensitive to the total matter perturbation,
\[
k^2 \Phi_{\text{lens}} = k^2 (\Psi - \Phi) = -\frac{3}{2} \Sigma \Omega_m \delta_m,
\]
where we have defined the modified lensing function \( \Sigma = Y (1 + \eta) \), with \( Y \rightarrow 1 \) in ΛCDM. What we truly observe in a measurement of gravitational lensing is the projection of the 3D power spectrum and the bispectrum on a 2D sphere integrated along the line of sight. The integral involves a window function that depends on the survey specification and the geometry of the background space-time. Assuming a perfect knowledge of the window function one can differentiate the integral relation between the 3D and the 2D spectra and therefore link the un-projected 3D bispectrum to the actual observations.

A. Lensing bispectrum

We define the lensing bispectrum as
\[
\Omega_m^2 \left( \Sigma(k_1) \delta_m(k_1) \Sigma(k_2) \delta_m(k_2) \Sigma(k_3) \delta_m(k_3) \right)_c \equiv (2\pi)^3 \delta^{(3)}(k_{123}) B_{\text{lens}}(k_1, k_2, k_3).
\]
Since the lensing signal is not sensitive to redshift-space distortions, the lensing bispectrum will not be affected by any projection effects. We obtain at the leading order for the equilateral configuration
\[
B^\text{eq}_{\text{lens}} = \Omega_m^2 \left( 6 F^\text{eq}_{1m11} P^2_{m11} + B^\text{eq}_{m111} \right),
\]
and for the squeezed configuration
\[
B^\text{eq}_{\text{lens}} = \Omega_m^2 \Sigma^2 \delta_{\Delta k} \left( 4 F^\text{eq}_{2\text{edf}} P_{m11} P_{m11,\Delta k} + B^\text{eq}_{m111} \right).
\]

B. Lensing-galaxy cross bispectra

We also consider two types of cross-correlations between galaxy and lensing signal, the first is the galaxy-galaxy-lensing bispectrum, defined by
\[
\Omega_m \left( \delta^g(k_1) \delta_m(k_2) \Sigma(k_3) \delta(k_3) \right)_c \equiv (2\pi)^3 \delta^{(3)}(k_{123}) B^{\text{ggl}}(k_1, k_2, k_3),
\]
which is in the equilateral configuration
\[
B^{\text{ggl,eq}} = \Omega_m \Sigma P^2_{m11} \sum_i B_{i}^{\text{ggl,eq}} \mu_i,
\]
with nonvanishing coefficients \( B^{\text{ggl,eq}}_0, B^{\text{ggl,eq}}_2, B^{\text{ggl,eq}}_4 \) and \( B^{\text{ggl,eq}}_6 \). All coefficients are reported in Appendix A in the following we only need the last one, i.e.,
\[
B^{\text{ggl,eq}}_6 = -\frac{59}{128} f_1^3.
\]
We have also derived the squeezed limit of that cross-bispectra, together with the other cross-bispectrum, the lensing-lensing-galaxy bispectrum, defined by
\[
\Omega_m^2 \left( \Sigma(k_1) \delta(k_1) \Sigma(k_2) \delta(k_2) \delta_m(k_3) \right)_c \equiv (2\pi)^3 \delta^{(3)}(k_{123}) B^{\text{llg}}(k_1, k_2, k_3),
\]
and we report all coefficients in Appendix A. We note that in deriving the equilateral coefficients for the cross-correlators \( B^{\text{ggl}} \) and \( B^{\text{llg}} \), we have integrated out the azimuthal angular dependence \( \phi \) as explained around Eq. (45).

A comment on stochastic biasing models (see e.g. [34]) is in order. In that class of phenomenological models, one introduces correlation coefficients, usually dubbed \( r \), that parametrize the disknowledge of the underlying deterministic formation process of biased tracers [31]. Since we do not assume any simplified bias model, our nonlocal bias model incorporates the stochasticity between matter and galaxy fields, and thus, we do not need to introduce these correlation coefficients for our cross-correlators. See Sec. II E in Ref. [31] for a highly related discussion.

VII. OBSERVABLES

The cosine-independent coefficients of the various sorts of bispectra are not directly observable since they are proportional to the model-dependent matter power spectrum and to the unknown normalization of the density fluctuation amplitude. However, taking ratios of these coefficients, these unknowns drop out. Taking the time derivative of a coefficient by subsequent division by a coefficient is another useful operation, since unknowns disappear. Thus, this method provides access to a wealth of cosmological information in a model-independent way.

In the following we briefly summarize the findings of linear observables that can be obtained from the galaxy and lensing power spectrum, then we extend the set of observables into the weakly nonlinear regime, by the use of the above bispectrum coefficients.

A. Linear observables

This section summarizes the findings from the literature [14, 19, 35, 36], and we follow in particular the procedure of Ref. [21]. There it has been shown that a set of model-independent observables can be obtained by taking ratios of the power spectrum coefficients [19]; in...
that spirit, also, time differentiation is a fruitful operation. For example, taking the ratio of \( P_2 \) and \( 2P_4 \), one gets \( b_1/f_1 \), whereas taking the time derivative of \( P_4 \) divided by \( P_4 \) gives \( f_1 + f_1' / f_1 \). Another important linear observable is \( \Omega_m \Sigma / f_1 \), which is obtained by taking the ratio of the lensing power spectrum and \( P_4 \). In summary, the linear observables are \[ P_1 = f_1 / h_1, \quad P_2 = \Omega_m \Sigma / f_1, \quad P_3 = f_1 + f_1' / f_1. \]

Interestingly, we obtain these (and many more) observables also from the bispectrum coefficients, with the important difference, that the bispectrum coefficients should hold on a wider range of scales, simply because they are obtained by using a better perturbation approximation.

**B. Nonlinear observables**

It is straightforward to confirm from our bispectrum coefficients the findings of \( P_1-P_3 \), but now obtained from a wider range of cosmological scales,

\[
B_1 = \frac{B_{40}^\text{eq} - B_{42}^\text{eq}}{B_{00}^\text{eq} - B_{02}^\text{eq}} = \frac{f_1^2}{b_1^1},
\]

\[
B_2 = \frac{87\epsilon^2 \Omega_0 \Sigma B_{20}^\text{eq}}{472(1 + z)^3 B_{20}^\text{eq}} = \frac{\Omega_0 \Sigma}{f_1},
\]

\[
B_3 = \frac{1}{4} \left( B_{30}^\text{eq} P_{20}^\text{eq} \right)^{' eq} = f_1 + f_1'.
\]

Evidently, these observables are constructed from nonlinear quantities but are identical to the linear observables. This equivalence can be used to establish a consistency relation in various ways. For example if the actual measurements from both the linear and nonlinear regime yield inconsistent results, there could be some unresolved systematic in the theory or analysis.

Observe furthermore that the above nonlinear observables are independent of the unknown intrinsic bispectrum contribution \( B_{m111} \) (or \( Q_{m111} \)). In fact, since we do not want to specify the DE model, second-order perturbations arising from the nonlinear matter evolution are indistinguishable from PNG modifications, as it is also evident from the following two observables,

\[
B_4 = -\frac{29B_{000}^\text{eq}}{2048B_{00}^\text{eq} P_{20}^\text{eq} B_{20}^\text{eq}} = f_1^{-1} F_2^\text{eq} + \frac{Q_{m111}^\text{eq}}{6 f_1},
\]

\[
B_5 = \frac{29B_{20}^\text{eq}}{944 B_{40}^\text{eq}} - \frac{1}{16} = f_1^{-2} G_2^\text{eq} - \frac{Q_{m111}^\text{eq}}{6 f_1}.
\]

However, summing up these two observables, we obtain another important observable that is independent of \( Q_{m111} \),

\[
B_6 = B_4 + B_5 = f_1^{-1} F_2^\text{eq} + f_1^{-2} G_2^\text{eq}.
\]

This observable will become crucial in the following section when we establish a nonlinear model-independent constraint.

What knowledge can be gained on the nonlocal bias coefficients? We find model-independent constraints for the following bias ratios,

\[
B_7 = \frac{b_{23}^\text{eq}}{b_1^2}, \quad B_8 = \frac{b_{23}^\text{eq}}{b_1^2},
\]

which we shall derive in Appendix D where we also provide even more observables. What is missing is a similar uncluttered observable involving \( b_{23}^\text{eq} \) or \( b_{23}^\text{eq} \), which, however, we have been unable to find.

**VIII. MODEL-INDEPENDENT CONSTRAINTS**

**A. Linear regime**

Observe that the PDE for the linear matter density, Eq. (7), can be rewritten in terms of a PDE for \( f_1 \),

\[
f_1' + f_1^2 + f_1 \left( 2 + \frac{\epsilon'}{\epsilon} \right) = \frac{3}{2} \Omega_m Y,
\]

where we remind the reader that \( \epsilon = H/H_0 \), and we have

\[
\Omega_m = \Omega_{m0} \left( \frac{1 + z}{\epsilon} \right)^3.
\]

In Ref. [21] it has been shown, using the set of linear observables [48], that the above equation turns into a relation for the anisotropic stress \( \eta \),

\[
\frac{3P_2(1 + z)^3}{2\epsilon^2 (P_3 + 2 + \epsilon'/\epsilon)} - 1 = \eta.
\]

This relation implies a model-independent constraint of \( \eta \) in terms of linear observables, a powerful result that can be used e.g. to rule out entire classes of DE models.

**B. Nonlinear regime**

Here we seek a similar relation as above, now obtained from our novel nonlinear observables. To achieve this, we use the linear result \( \theta_{m1} = -f_1 \delta_{m1} \) and the fully general Ansatz (cf. Eqs. [8], here suppressing the integrals and Dirac-deltas because of notational simplicity)

\[
\delta_{m2} = F_2 \delta_{m1} \delta_{m1} \quad \theta_{m2} = G_2 \delta_{m1} \delta_{m1}
\]

in Eqs. [60]–[61] together with the modified Poisson equation [62]. Taking the divergence of Eq. [60], expanding Eqs. [60]–[61] in perturbation theory and Fourier transforming the resulting expressions, these equations become respectively at second order
\[
\{ F_2' + F_2[f_1(k_1) + f_1(k_2)] \} \delta_{m1} \delta_{m1} = \left\{ \frac{1}{2} \frac{k_1 \cdot k_2}{k_1 k_2} \left[ \frac{f_1(k_1)k_2}{k_1} + \frac{f_1(k_2)k_1}{k_2} \right] - G_2 + \frac{1}{2} f_1(k_1) + \frac{1}{2} f_1(k_2) \right\} \delta_{m1} \delta_{m1}, \quad (60)
\]
\[
\{ G_2' + G_2[f_1(k_1) + f_1(k_2)] \} \delta_{m1} \delta_{m1} = \left\{ - \left( 2 + \frac{E'}{E} \right) G_2 - f_1(k_1) f_1(k_2) \left( \frac{k_1 \cdot k_2}{k_1 k_2} \right)^2 \right. \\
- \left. \frac{1}{2} f_1(k_1) f_1(k_2) \left( \frac{k_1 \cdot k_2}{k_1 k_2} \right) \left[ \frac{k_1}{k_2} + \frac{k_2}{k_1} \right] - \frac{3}{2} \Omega_m Y F_2 \right\} \delta_{m1} \delta_{m1}. \quad (61)
\]

These relations must also hold for specific configurations and without loop integrals (see App. D for a rigorous proof).

For example, in the equilateral case we get the two relations
\[
F_2^{eq} + 2 f_1 F_2^{eq} = \frac{f_1}{2} - G_2^{eq}, \quad (62)
\]
\[
G_2^{eq} + 2 f_1 G_2^{eq} = - \left( 2 + \frac{E'}{E} \right) G_2^{eq} + \frac{f_1^2}{4} - \frac{3}{2} \Omega_m Y F_2^{eq}. \quad (63)
\]

Now, making use of these equations, and the quantity \( f_1^2 B_0 = f_1 F_2^{eq} + G_2^{eq} \) with its time derivative, \( G_2^{eq} = \left( f_1^2 B_0 \right)' - f_1^2 F_2^{eq} - f_1^2 B_0 \) we obtain a model-independent realization of \( f_1 \). For this we first use Eq. (62) to get an expression for \( F_2^{eq} \), and then plug this into the expression for \( G_2^{eq} \) in terms of \( B_0 \). We get
\[
G_2^{eq} = \left( f_1^2 B_0 \right)' - f_1^2 F_2^{eq} - \frac{f_1^2}{2} + f_1^3 B_3 - f_1^2 B_0^{eq}. \quad (64)
\]

Plugging this in (63) we finally get after a little algebra
\[
\frac{3}{4 B_0} - \frac{B_3'}{B_3} - 2 B_3 - \left( 2 + \frac{E'}{E} \right) = f_1, \quad (65)
\]

where we have used Eqs. (31) and (66). This is our main result. We stress that the l.h.s. is obtained from model-independent observables, and thus this equation delivers a model-independent measurement of \( f_1 \).

Having obtained \( f_1 \), we get from the nonlinear observables (40) and (53) the bias parameters \( b_1, b_2^{eq} \) and \( b_2^{eq} \), as well as the quantity \( \Omega_{m0}^2 \Sigma \). If we furthermore use the linear relation (58) that gives \( \eta \), we also get \( \Omega_{m0} Y \), by virtue of \( \Sigma = Y (1 + \eta) \), i.e.,
\[
\Omega_{m0} Y = \frac{2 f_1 B_2 E^2 (P_3 + 2 + E'/ E)}{3 P_2 (1 + z)^3}. \quad (66)
\]

**IX. CONCLUSIONS**

We have shown that, without imposing any DE parametrization, cosmological observations can measure only (1) \( \Omega_0 \) and \( E = H/H_0 \) at the background level; (2) the combinations \( P_1 = f_1/b_1, P_2 = \Omega_{m0} \Sigma / f_1 \) and \( P_3 = f_1 + f_1' / f_1 \) at the linear level; and (3) the novel observables \( B_1 - B_0 \) and \( C_1 - C_4 \) that are applicable in the weakly nonlinear regime. (A concise list of these nonlinear observables is given in Appendix D). The observables \( \mathcal{P}_1 - \mathcal{P}_3 \) with \( B_1 - B_2 \) are formally identical, however with the crucial difference that the former are measured from linear scales, and the latter from nonlinear scales. Thus, different scales are probed but the respective measurements of the observables on these scales must be identical. From this one could perform consistency tests that rule out entire classes of DE models.

Many unknowns remain unknowns, especially \( \Omega_{m0} \), the DE density parameter \( \Omega_\Lambda \), and we are left with a degeneracy between non-Gaussianities in the initial conditions (arising from PNG) and non-Gaussianities from the matter evolution. From our nonlinear observables, however, we can derive a model-independent constraint equation given in Eq. (65). This relation should hold for a wide range of DE models, and, if verified by cosmological observations, can be used to obtain a model-independent measure of \( f_1 \). That in turn, in combination with our observables, enables us to reconstruct the bias parameters \( b_1 \) and \( b_2 \), and the quantities \( \Omega_{m0} Y \) and \( \Omega_{m0} \Sigma \). Lastly, having \( f_1 \) one gets the normalization dependent quantity \( R = D f_1 \sigma_8 \delta_{m0} [21] \), from which one gets \( \sigma_8^{2} P_{m1} \) as well.

All the practical limitations of a real measurement, that we neglect here, are of course the most challenging problem to handle (see e.g., [37, 39]). This paper thus should be understood as a potential starting point for a long journey with many hurdles ahead, however with the final goal to reconstruct or reject entire classes of DE cosmologies.

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Appendix A: All bispectrum coefficients

In the main text we have mentioned only bispectrum coefficients that are relevant in deriving our main results, whilst skipping other coefficients. Here we give a complete list of bispectrum coefficients.

The galaxy bispectrum in the equilateral configuration reads

\[ B^q_{\text{eq}} = P^2_{m11} \sum_i B^q_i \mu^i_1, \]  

(A1)

with the nonvanishing coefficients

\[
B^q_{0} = \frac{27}{128} f^2_1 b^2_1 + \frac{3}{4} f_1 b^3_1 + 6b^3_i F^q_2 + 3b^2_i b^q_2 + \frac{27}{64} f^2_1 b_1 + \\
+ 3f_1 b^3_2 F^q - \frac{3}{2} b^3_1 G^q_2 + \frac{3}{2} f_1 b_2 b^q_2 + 27 \frac{f^2_1 b^q_2}{64} f^2_1 b_1 \\
- \frac{27}{32} f_1 b_1 G^q_2 + Q^q_{m111} \left( b^3_1 + \frac{3}{4} b^2_1 f_1 + \frac{27}{128} f^2_1 \right), \quad (A2)
\]

\[
B^q_{2} = \frac{3}{4} f_1 b^3_1 + \frac{9}{32} f^2_1 b_2 + 3f_1 b^2_1 F^q - \frac{3}{2} b^2_2 G^q_2 + \frac{3}{2} f_1 b_2 b^q_2 \\
+ \frac{9}{32} f^2_1 b_2 F^q - \frac{9}{16} f_1 G^q_2 + \frac{9}{64} f^2_1 b^q_2 - 135 \frac{f^2_1 f^2_1}{1024} \right) \\
- \frac{81}{128} f^2_1 G^q_2 + \frac{3 f_1}{Q^q_{m111}} (32b^2_1 + 6b_1 f_1 + 9f^2_1), \quad (A3)
\]

\[
B^q_{4} = \frac{27}{128} f^2_1 b^2_1 + \frac{351}{1024} f^1_1 + \frac{27}{64} f^2_1 b^q_2 - \frac{27}{32} f_1 G^q_2 \\
+ \frac{27}{64} f^2_1 b^2_1 + \frac{117}{32} f^2_1 G^q_2 + \frac{3 f^2_1}{128} Q^q_{m111} (9b_1 - 26f_1), \quad (A4)
\]

\[
B^q_{6} = -\frac{177}{1024} f^1_1 \left( f^2_1 + 16G^q_2 - \frac{8}{3} Q^q_{m111} f_1 \right), \quad (A5)
\]

\[
B^q_{8} = -\frac{87}{1024} f^1_1, \quad (A6)
\]

where

\[ Q^q_{m111} = B^q_{m111} / P^2_{m111}. \]  

(A7)

For the squeezed galaxy bispectrum we have

\[ B^q_{\text{sq}} = \sum_{ij} B^q_{ij} \mu^i_1 \mu^j_{1,\Delta k}, \]  

(A8)

with the only nonvanishing coefficients

\[
B^q_{00} = a_1 b_1 b_1,1,\Delta k + b^q_{12} b^1_1 P^q_{m11} + B^q_{m111} b^2_1 b_1,1,\Delta k, \quad (A9)
\]

\[
B^q_{0i} = a_1 b_1 b_1,1,\Delta k + B^q_{m111} b^2_1 b_1,1,\Delta k, \quad (A10)
\]

\[
B^q_{20} = a_1 f_1 b_1,1,\Delta k + a_2 b_1 b_1,1,\Delta k + 2b^q_{21} b_1 f_1 P^q_{m11} \\
+ 2B^q_{m111} f_1 b_1,1,\Delta k, \quad (A11)
\]

\[
B^q_{22} = a_1 f_1 b_1,1,\Delta k + a_2 b_1 b_1,1,\Delta k + 2B^q_{m111} f_1 b_1,1,\Delta k, \quad (A12)
\]

\[
B^q_{40} = a_2 f_1 b_1,1,\Delta k + b^q_{12} f_1 P^q_{m11} + B^q_{m111} f^2_1 b_1,1,\Delta k, \quad (A13)
\]

\[
B^q_{42} = a_2 f_1 b_1,1,\Delta k + B^q_{m111} f^2_1 b_1,1,\Delta k, \quad (A14)
\]

where

\[ a_1 = \left( b^q_{1,13} + b^q_{2,23} + 4b_1 F^q_{2,\text{eff}} \right) P_{m11} P_{m11,1,\Delta k}, \]  

(A15)

\[ a_2 = \left( 4f_2 F^q_{2,\text{eff}} - 2[f, 1, \Delta k] / 2 + 2b_1,1,\Delta k f_1 \right) \times P_{m11} P_{m11,1,\Delta k}. \]  

(A16)

For the pure lensing bispectra, we get in the equilateral configuration

\[ B^q_{\text{lens}} = \Omega^3 m^3 \left( 6F^q_{2,\text{eff}} P^2_{m11} + B^q_{m111} \right), \]  

(A17)

and in the squeezed configuration

\[ B^q_{\text{lens}} = \Omega^3 m^2 \Sigma^2 \Delta_k \left( 4F^q_{2,\text{eff}} P_{m111} P_{m11,1,\Delta k} + B^q_{m111} \right), \]  

(A18)

which, evidently, have no angular dependence.

Next is the cross-bispectrum 'galaxy-galaxy-lensing' which is in the equilateral configuration

\[ B^q_{\text{ggl,eq}} = \Omega m \Sigma^2 \Delta_k \sum_i B^q_{i,\text{ggl,eq}} \mu^i_1, \]  

(A19)

with

\[ B^q_{0,\text{ggl,eq}} = \frac{3}{8} f_1 f^2_1 + \frac{3}{8} f_1 b^2_1 + 6b^2_1 F^q + 2b^2_2 + \frac{3}{2} f_1 b_2,1,\Delta k \]  

\[ + \frac{3}{8} b_1 G^q_2 + \frac{1}{8} Q^q_{m111} (8b_2^2 + 3b_1 f_1), \]  

(A20)

\[ B^q_{2,\text{ggl,eq}} = \frac{7}{8} f_1 b^2_1 + \frac{7}{8} f_1 F^q - \frac{27}{128} f^3_1 + \frac{3}{4} f_1 F^q + \frac{9}{16} f_1 b_1 \]  

\[ + \frac{7}{2} f_1 b_1 F^q - \frac{7}{4} b_1 G^q_2 - \frac{3}{2} f_1 G^q_2 \]  

\[ + \frac{1}{8} Q^q_{m111} (7b_1 f_1 + 3f^2_1), \]  

(A21)

\[ B^q_{4,\text{ggl,eq}} = \frac{1}{16} f_1 b_1 + \frac{39}{64} f_1 - \frac{1}{4} f_1 F^q + \frac{1}{2} f_1 G^q_2 \]  

\[ - \frac{1}{8} Q^q_{m111} f^2_1, \]  

(A22)

\[ B^q_{6,\text{ggl,eq}} = -\frac{59}{128} f^3_1, \]  

(A23)

and in the squeezed configuration

\[ B^q_{\text{ggl,eq}} = \Omega m \Sigma^2 \Delta_k \sum_i B^q_{i,\text{ggl,eq}} \mu^i_1, \]  

(A24)

with coefficients

\[ B^q_{0,\text{ggl,eq}} = b_1 a_1 + B^q_{m111} b^2_1, \]  

(A25)

\[ B^q_{2,\text{ggl,eq}} = f_1 a_1 + b_1 a_2 + 2B^q_{m111} b_1 f_1, \]  

(A26)

\[ B^q_{4,\text{ggl,eq}} = f_1 a_2 + B^q_{m111} f^2_1, \]  

(A27)

with \( a_1 \) and \( a_2 \) as above.

The second cross-bispectrum we consider is the lensing-lensing-galaxy bispectrum, defined by

\[ \Omega^2 \left( \Sigma(k_1) \delta(k_1) \Sigma(k_2) \delta(k_2) \delta^2(k_3) \right) c \equiv \]  

\[ (2\pi)^3 \delta_D^{(3)}(k_{123}) B^\text{lgl}(k_1, k_2, k_3), \]  

(A28)
which is in the equilateral configuration

\[ B_{\text{eq}}^{\text{llg}} = \Omega^2_m \Sigma^2 P_{m11}^2 \sum_i B_i^{\text{llg,eq}} \mu_i, \]  

(A29)

with

\[ B_0^{\text{llg,eq}} = b_0^2 + \frac{3}{8} f_1 b_1 + 6 b_1 \mathcal{F}_{2}^{\text{eq}} + \frac{3}{2} f_1 \mathcal{F}_{-2}^{\text{eq}} - \frac{3}{4} G_{2}^{\text{eq}}, \]

\[ B_1^{\text{llg,eq}} = \frac{3}{16} f_1^2 - \frac{1}{8} f_1 b_1 - \frac{1}{2} f_1 \mathcal{F}_{2}^{\text{eq}} + \frac{3}{4} G_{-2}^{\text{eq}} - \frac{1}{8} Q_{m11}^{\text{eq}} f_1, \]

\[ B_2^{\text{llg,eq}} = \frac{5}{16} f_1^2, \]

(A32)

and in the squeezed configuration

\[ B_{\text{eq}}^{\text{llg,eq}} = \Omega^2_m \Sigma^2 \sum_i B_i^{\text{llg,eq}} \mu_i^{\Delta_i}, \]  

(A33)

with

\[ B_0^{\text{llg,eq}} = 4 b_1 \Delta_k \mathcal{F}_{2,\text{eff}} P_{m11} P_{m11,\Delta_k} + b_{2,12}^2 P_{m11}^2 \]

\[ + B_{2m11}^{\text{llg,eq}} b_1, \]  

(A34)

\[ B_2^{\text{llg,eq}} = 4 f_1 \Delta_k \mathcal{F}_{2,\text{eff}} P_{m11} P_{m11,\Delta_k} + B_{m11}^{\text{llg,eq}} f_1, \Delta_k. \]  

(A35)

**Appendix B: More nonlinear observables**

Here we report the full list of nonlinear observables including their derivations,

\[ B_1 = \frac{B_{00}^{\text{gg}} - B_{02}^{\text{gg}}}{B_{00}^{\text{gg}} - B_{02}^{\text{gg}}} = f_1^2, \]  

(B1)

\[ B_2 = -\frac{87 \mathcal{E}^2 \Omega_m^2 \Sigma B_{g00}^{\text{gl}}} {472(1 + z)^3 B_{g00}^{\text{gg}}} = \frac{\Omega_m^2 \Sigma}{f_1}, \]  

(B2)

\[ B_3 = \frac{1}{4} \left( \frac{B_{02}^{\text{gg}} P_{m11}^2}{B_{02}^{\text{gg}}} \right)' = f_1 + f_1', \]  

(B3)

\[ B_4 = -\frac{29 B_{\text{len}}^{\text{gg}}}{2048 B_{g00}^{\text{gg}} P_{m11}^2 B_3} = f_1^{-1} \mathcal{F}_{2}^{\text{eq}} + \frac{Q_{m11}^{\text{eq}}}{6 f_1}, \]  

(B4)

\[ B_5 = \frac{29 B_{\text{len}}^{\text{gg}}}{944 B_{g00}^{\text{gg}}} - \frac{1}{16} = f_1^{-2} \mathcal{F}_{2}^{\text{eq}} - \frac{Q_{m11}^{\text{eq}}}{6 f_1}, \]  

(B5)

\[ B_6 = B_1 + B_5 = f_1^{-1} \mathcal{F}_{2}^{\text{eq}} + f_1^{-2} \mathcal{F}_{2}^{\text{eq}}, \]  

(B6)

\[ B_7 = \frac{3 B_{11}^{1/2} C_1 / 16 - 3/8 - 6 b_1}{3/8 + 2 B_{11}^{1/2}}, \]  

(B7)

\[ B_8 = B_1 C_2 (1 + C_3) = \frac{b_{2,12}^{\text{eq}}}{b_1^{\text{eq}}}, \]  

(B8)

\[ B_9 = \frac{B_{\text{len}}^{\text{eq}} B_{1,\Delta_k}^{1/2} C_4 - 1}{B_1 C_2^{\text{eq}} P_{m11} - B_0^{\text{llg,eq}}} = b_1 \frac{4 \mathcal{F}_{2,\text{eff}} + Q_{m11}^{\text{eq}}}{b_{2,12}^{\text{eq}} + b_{2,23}^{\text{eq}}}, \]  

(B9)

\[ B_{10} = \frac{B_{00}^{\text{gg}} - 2 B_{12}^{\text{gg}} B_{11}^{1/2} - G B_{11}^{1/2}} {b_{2,12}^{\text{eq}} + b_{2,23}^{\text{eq}}} = \frac{4 f_1 \mathcal{F}_{2,\text{eff}} - 2 f_1 b_1 \Delta_k + 4 G_{2}^{\text{eq}}}{f_1}, \]  

(B10)

and

\[ C_1 = -\frac{5 B_{2}^{\text{llg,eq}} + 3 B_{2}^{\text{llg,eq}}}{B_{2}^{\text{llg,eq}}} + 2 B_{1/2}^{\text{llg,eq}} = -8 f_1^{-1} \mathcal{F}_{2}^{\text{eq}} + 4 f_1^{-2} \mathcal{F}_{2}^{\text{eq}} - 2 f_1^{-1} Q_{m11}^{\text{eq}}, \]  

(B11)

\[ C_2 = -\frac{177 \mathcal{E}_{40}^{\text{gg}} - B_{00}^{\text{gg}}}{1024 B_{00}^{\text{gg}} P_{m11}^2} = \frac{f_1}{f_1} + 16 G_{2}^{\text{eq}} + 8 f_1 Q_{m11}^{\text{eq}} / 3, \]  

(B12)

\[ C_3 = \frac{87 B_{00}^{\text{gg}}}{177 B_{00}^{\text{gg}}} - 1 = 16 f_1^{-2} \mathcal{F}_{2}^{\text{eq}} - \frac{8}{3} f_1^{-1} Q_{m11}^{\text{eq}}, \]  

(B13)

\[ C_4 = \frac{(1 + z)^3}{2} B_2. \]  

(B14)

In deriving the above we have defined a quantity that is depend on the normalization of density fluctuations and thus generally not an observable,

\[ G = B_{00}^{\text{gg}} - C_4^{1/2} B_{11}^{1/2} \Omega_m^2 \Sigma^2 B_0^{\text{llg,eq}} = (b_{2,13}^{\text{gg}} + b_{2,23}^{\text{gg}}) b_1 b_1 \Delta_k \mathcal{P} \mathcal{P} \Delta_k. \]  

(B15)

**Appendix C: Evolution equations in squeezed limit**

In Sec. VIII we have derived a relation from the fluid equations by applying the equilateral limit to the wave-dependence of the kernels. Here we repeat the analysis for the squeezed case, also to prove that \( F_{2,12}^{\text{eq}} = F_{2} (k, -k) = 0 \).

Using Eqs. (60) and (61) as a starting point and taking the squeezed limit, we obtain respectively

\[ \left\{ F_{2,12}^{\text{eq}} + 2 f_1 F_{2,12}^{\text{eq}} \right\} \delta_{m1}^2 = -C_{2,12}^{\text{eq}} \delta_{m1}^2, \]  

(C1)

\[ \left\{ C_{2,12}^{\text{eq}} + 2 f_1 C_{2,12}^{\text{eq}} \right\} \delta_{m1}^2 = \left\{ -2 + H' \right\} C_{2,12}^{\text{eq}} \delta_{m1}^2 - 3 \Omega_m Y F_{2,12}^{\text{eq}} \delta_{m1}^2. \]  

(C2)

Defining \( \delta_{m1}^2 = F_{2,12}^{\text{eq}} \delta_{m1}^2 + \theta_{m1}^2 = G_{2,12}^{\text{eq}} \delta_{m1}^2 \), we can combine these equations into the following PDE,

\[ \delta_{m1}^{2 \prime} + \left( 2 + H' \right) \delta_{m1}^{2} \delta_{m1}^{2} - 3 \Omega_m Y \delta_{m1}^{2} = 0. \]  

(C3)
This PDE coincides exactly with the one obtained for the linear matter density, Eq. (7), thus its solution will grow with the same linear amplitude $D$. But since $\delta_2$ is of second order with the fastest growing mode potentially of the order of $D^2$, we conclude that the above PDE for $\delta_2^{\text{eq}}$ excites nothing more but decaying modes, and thus we can set $\delta_2^{\text{eq}} = 0$, from which follows that $F_{2,12}^{\text{eq}} = 0$.

Appendix D: Evolution equations for the matter bispectrum

In Sec. VIIIB we have determined evolution equations that lead subsequently to the constraint equation (65). For its derivation we have argued that we can drop two loop integrals and a Dirac-delta. Here we provide a more rigorous derivation that obviously leads to the identical final result (65).

Actually, to understand our methodology, it is sufficient to focus on the l.h.s. term of Eq. (60) for which we again restore the double integrals and the Dirac-delta. In that term we interchange some dependences according to $k \to k_3$, $k_1 \to k_4$, $k_2 \to k_5$, and write equivalently

$$\int \frac{d^3k_{15}}{(2\pi)^3} \delta_D^{(3)}(k_3 - k_{15}) \left\{ F_2'(k_4, k_5) + F_2(k_4, k_5) \left[ f_1(k_4) + f_1(k_5) \right] \right\} \delta_{m1}(k_4) \delta_{m1}(k_5).$$

(D1)

Multiplying this by $\delta_{m1}(k_1) \delta_{m1}(k_2)$ and taking the correlator of the resulting expression, we have

$$\int \frac{d^3k_{15}}{(2\pi)^3} \delta_D^{(3)}(k_3 - k_{15}) \left\{ F_2'(k_4, k_5) + F_2(k_4, k_5) \left[ f_1(k_4) + f_1(k_5) \right] \right\} \langle \delta_{m1}(k_1) \delta_{m1}(k_2) \delta_{m1}(k_4) \delta_{m1}(k_5) \rangle_c
= 2(2\pi)^3 \delta_D^{(3)}(k_{123}) \left\{ F_2'(k_1, k_2) + F_2(k_1, k_2) \left[ f_1(k_1) + f_1(k_2) \right] \right\} P_{m1}(k_1) P_{m1}(k_2),$$

(D2)

where we have used Wick’s theorem [40] and discarded a zero-mode term $\sim \delta_D^{(3)}(k_3)$. The r.h.s. term in Eq. (D2) is only nonzero if the closure condition, dictated by $\delta_D^{(3)}(k_{123})$, is satisfied. This is indeed the case for the bispectrum where the three wavevectors form a closed triangle in Fourier space. By contrast, the omitted zero-mode term that is proportional to $\delta_D^{(3)}(k_3)$ dictates $k_3 = 0$ and no triangle closure condition.

The same technique applies to the r.h.s. of Eq. (60) [and, of course, to the whole Eq. (61) as well]; dropping the Dirac-delta, some constant factors and the two power spectra, we then obtain for the equilateral triangle configuration Eq. (62) [and Eq. (63), respectively], which concludes the proof.

Finally we note that the above technique delivers evolution equations not for the fluid variables but for the bispectrum. In general, this technique of course applies not only to the bispectrum but to any polyspectrum.
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