A STABLE RANGE DESCRIPTION OF THE SPACE OF LINK MAPS

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Abstract. Version: October 19, 2009 We study the space Link(\(P, Q; N\)) of link maps, maps from \(P \coprod Q\) to \(N\) such that the images of \(P\) and \(Q\) are disjoint. We give a range of dimensions, interpreted as the connectivity of a certain map, in which the cobordism class of the “linking manifold” is enough to distinguish the homotopy class of one link map from another.

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1. Introduction

Let \(N\) be a smooth manifold and let \(P\) and \(Q\) be smooth compact manifolds. A (smooth) link map of \(P\) and \(Q\) in \(N\) is a pair \((f : P \to N, g : Q \to N)\) of smooth maps such that \(f(P)\) is disjoint from \(g(Q)\). The set of link maps, denoted by Link\((P, Q; N)\), is an open subspace of Map\((P, N) \times Map(Q, N) = Map(P \coprod Q, N)\).

For brevity here we will write \(\mathcal{M}\) for Map\((P, N) \times Map(Q, N)\) and denote the complement of the set of link maps in \(\mathcal{M}\) by \(\mathcal{B}\). We prove that a certain “linking number” map

\[
\ell : hofiber(f_1, g_1)(\mathcal{M} - \mathcal{B} \to \mathcal{M}) \to \Omega Q^{T(N-(TP \oplus TQ))} \underset{\text{holim}}{\text{holim}}(P f_1 \rightarrow N g_1 Q)
\]

is \((2(n - p - q) - 3)\)-connected, where \(p\), \(q\), and \(n\) are the dimensions of the manifolds. The map was defined by the second author in [5], although the version we reference below is a homotopy-theoretic one give in [3]. Its domain is the homotopy fiber of the inclusion \(\mathcal{M} - \mathcal{B} \to \mathcal{M}\) with respect to any point \((f_1, g_1) \in \mathcal{M} - \mathcal{B}\). Its codomain is the infinite loopspace associated to the Thom spectrum of a virtual vector bundle. Both of these

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Theorem 1.1. The map

\[ \Lambda : \Sigma \text{hofiber}(\mathcal{M} - \mathcal{B} \rightarrow \mathcal{M}) \rightarrow Q^\top_{+}N^{-(\top P \oplus TQ)} \text{holim}(P \rightarrow N \leftarrow Q) \]

adjoint to \( \ell \) is \( 2(n - p - q) - 1 \)-connected.

The fact that \( \ell \) is \( 2(n - p - q) - 3 \)-connected then follows immediately by the Freudenthal theorem, since the domain of \( \ell \) is \( (n - p - q - 1) \)-connected. Note that the connectivity claimed for \( \Lambda \) is negative if \( p + q \geq n \), so it is no loss to assume \( p + q < n \).

1.1. Conventions. A space \( X \) is \( k \)-connected if for every \( j \) with \(-1 \leq j \leq k\) every map \( S^j \rightarrow X \) can be extended to a map \( D^{j+1} \rightarrow X \). (In other words, \((-1)\)-connected means nonempty and if \( k \geq 0 \) then \( k \)-connected means that there is exactly one path-component and that the homotopy groups vanish through dimension \( k \).) A map is \( k \)-connected if each of its homotopy fibers is \((k-1)\)-connected. A (weak) equivalence is an \( \infty \)-connected map.

We write \( QX = \Omega^{\infty}\Sigma^{\infty}X \) if \( X \) is a based space. If \( X \) is unbased, then \( X_+ \) means \( X \) with a disjoint basepoint added and \( Q_+X \) means \( Q(X_+) \). For a vector bundle \( \xi \) over a space \( X \), the unit disk bundle and the unit sphere bundle are \( D(X; \xi) \) and \( S(X; \xi) \). The Thom space \( X^\xi \) is the quotient \( D(X; \xi)/S(X; \xi) \), or equivalently the homotopy cofiber of the projection \( S(X; \xi) \rightarrow X \). If \( \xi \) and \( \eta \) are two vector bundles on \( X \), then by choosing a vector bundle monomorphism \( \eta \rightarrow \epsilon^i \) to a trivial bundle we can define \( Q^{\xi - \eta}_{+}X = \Omega^{i}QX^{\xi \oplus \epsilon^i/\eta} \). This is essentially independent of the choice of \( i \geq 0 \) and vector bundle monomorphism, in the sense that for large \( i \) the weak homotopy type of this space is independent of those choices.

2. Sketch of the proof of Theorem 1.1

To prove Theorem 1.1 we will use the diagram \( 1 \) below and obtain the connectivity of \( \Lambda \) from the connectivities of all the other maps. For this we must introduce another closed set \( \mathcal{V} \subset \mathcal{B} \). Recall that a point \((f, g) \in \mathcal{M}\) belongs to \( \mathcal{B} \) if the statement \( f(x) = z = g(y) \) holds for some pair \((x, y) \in P \times Q\) and some point \( z \in N \). The closed set \( \mathcal{B} \) has codimension \( n - p - q \) in \( \mathcal{M} \) in some sense. Inside this space of “bad” maps is a set \( \mathcal{V} \) of “very bad” maps, having codimension in \( \mathcal{M} \) is \( 2(n - p - q) \). A point \((f, g) \in \mathcal{V}\) if either the statement \( f(x) = z = g(y) \) holds for more than one choice of \((x, y, z) \) or else it holds for one such choice in such a way that the associated map of tangent spaces \( T_xP \oplus T_yQ \rightarrow T_zN \) is not injective. The set \( \mathcal{B} - \mathcal{V} \) may be regarded as a submanifold of \( \mathcal{M} \). It has maps to \( P \),
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Q, and N given by x, y, and z. Pulling back tangent bundles via these maps, we obtain vector bundles on B − V, which we will denote simply by TP, TQ, and TN. There is also a monomorphism TP ⊕ TQ → TN, and its cokernel TN/(TP ⊕ TQ) may be thought of as the normal bundle of B − V in M.

The next result immediately implies Theorem 1.1.

**Theorem 2.1.** In the diagram below, the maps F and H are equivalences, the maps G, C, and D are \(2(n − p − q) - 1\)-connected, and the map E is \(3(n − p − q) - 2\)-connected.

\[
\begin{align*}
\Sigma \text{hofiber}(M − B → M) & \xrightarrow{\Lambda} Q_+^{TN−(TP⊕TQ)} \xrightarrow{\text{holim}(P → N ← Q)} \\
\text{hofiber}(B − V → M − V) & \xrightarrow{G} Q_+^{TN−(TP⊕TQ)} \xrightarrow{\text{hofiber}(B − V → M)} \\
\Sigma \text{hofiber}(M − B → M − V) & \xrightarrow{F} Q_+^{TN−(TP⊕TQ)} \xrightarrow{\text{hofiber}(B − V → M)} \\
\text{hofiber}(B − V → M − V)^{TN/(TP⊕TQ)} & \xrightarrow{E} \text{hofiber}(B − V → M)^{TN/(TP⊕TQ)}
\end{align*}
\]

We now briefly define the maps in the diagram and explain about their connectivities. Steps that are sketchy here will be filled in in the following sections. Let \(c = n − p − q\).

The equivalence F is essentially an instance of the following general fact. If \(Y\) is a smooth submanifold of \(X\) and also a closed subset, then the suspension of the homotopy fiber of the inclusion \(X − Y → X\) is equivalent to the Thom space, over the homotopy fiber of \(Y → X\), of the normal bundle of \(Y\) in \(X\). This general fact will be proved, and adapted to the present function-space setting, in Section 4.

G is an inclusion map. Since \(V\) has codimension \(2c\) in \(M\), the inclusion \(M − V → M\) is \((2c − 1)\)-connected. (This will be worked out in detail in Section 3.) Therefore the map of homotopy fibers is \((2c − 2)\)-connected and the map G of suspensions is \((2c − 1)\)-connected.

E is a map of Thom spaces. For a \(k\)-connected map \(Z → W\) of spaces and a vector bundle \(ξ\) on \(W\) with fiber dimension \(d\), the associated map \(Z^ξ → W^ξ\) is \((k + d)\)-connected. In our case \(d = c\) and \(k = 2c − 2\); the inclusion of hofiber\((B − V → M − V)\) into hofiber\((B − V → M)\) is \((2c − 2)\)-connected, again because the inclusion of \(M − V\) into \(M\) is \((2c − 1)\)-connected.

C is the canonical map \(Z → QZ\), where the space \(Z\) is \((c − 1)\)-connected, being the Thom space of a vector bundle of rank \(c\). By the Freudenthal Theorem, the map is \((2c − 1)\)-connected.

The equivalence H is simply a matter of rewriting the Thom spectrum of a virtual vector bundle \(ξ − η\) as the suspension spectrum of the Thom space of \(ξ/η\) when \(η\) is a subbundle of \(ξ\).
The map $D$ arises from a $(c - 1)$-connected map from $\text{hofiber}(B - V \to M)$ to $\text{holim}(P \to N \leftarrow Q)$. To explain further, we introduce the space $\tilde{B}$ of all $((f, g), x, y, z) \in M \times P \times Q \times N$ such that $f(x) = z = g(y)$. Projection to $M$ gives a map from $\tilde{B}$ onto $B$. Let $\tilde{V} \subset \tilde{B}$ be the preimage of $V$. The projection $\tilde{B} - V \to B - V$ is an isomorphism. The inclusion $\tilde{B} - \tilde{V} \to \tilde{B}$ is $(c - 1)$-connected for reasons of codimension (again, the details are in section 3), and therefore the induced map $\text{hofiber}(B - V \to M) \to \text{hofiber}(\tilde{B} - M) \simeq \text{holim}(P \to N \leftarrow Q)$ is also $(c - 1)$-connected. There are vector bundles $TP, TQ$, and $TN$ on $\tilde{B}$ pulling back to their namesakes on $B - V$. (The monomorphism $d f \oplus dg : TP \oplus TQ \to TN$ is not available on the $\text{holim}(P \to N \leftarrow Q)$ side, which we why switched from Thom spaces to Thom spectra).

We end this section with a brief account of the commutativity of diagram (I). First we need to define the map $\Lambda$. As mentioned in Section 1, $\Lambda$ is adjoint to a map $\ell : \text{hofiber}_{(f_1, g_1)}(M - B \to M)$, which is a composite defined as follows. Let $(f_1, g_1) \in \text{hofiber}_{(f_1, g_1)}(M - B \to M)$. The map $M \to \text{Map}(P \times Q, N \times N)$ given by $(f, g) \mapsto f \times g$ induces a map $\text{hofiber}_{(f_1, g_1)}(M - B \to M) \to \text{hofiber}_{(f_1, g_1)}(\text{Map}(P \times Q, N \times N) \to \text{Map}(P \times Q, N \times N))$. We can identify the latter homotopy fiber as a space of sections as follows.

Let $E = \text{holim}(P \times Q, f_1 \times g_1 \circ N \times N \leftarrow N \times N - \Delta_N)$. The projection map $E \to P \times Q$ is a fibration with fiber over $(p, q)$ the space $\Phi_2(N) = \text{hofiber}_{(f_1, g_1)}(N \times N - \Delta N \to N \times N)$. Let $\Gamma(P \times Q, E)$ its space of sections. This space of sections has a preferred basepoint given by $(f_1, g_1)$. It is equivalent to $\text{hofiber}_{f_1 \times g_1}(\text{Map}(P \times Q, N \times N - \Delta_N) \to \text{Map}(P \times Q, N \times N))$ by inspection. Let $Qf_1Sf_1E \to P \times Q$ be the fibration whose fibers are $Q \Phi_2(N)$, where $S$ stands for the unreduced suspension. The canonical map $\Gamma(P \times Q, E) \to \Omega \Gamma(P \times Q, Qf_1Sf_1E)$ is easily shown to be $(2n - p - q - 1)$-connected, and there is an equivalence $\Omega \Gamma(P \times Q, Qf_1Sf_1E) \simeq \Omega Q^{\text{TN} - (TP \oplus TQ)}\text{holim}(P \times Q, f_1 \times g_1 \circ N \times N \leftarrow \Delta_N)$ which is the identity on the loop coordinate. Moreover, there is a homeomorphism $\text{holim}(P \times Q, f_1 \times g_1 \circ N \times N \leftarrow \Delta_N) \cong \text{holim}(P \xrightarrow{f_1} N \xrightarrow{g_1} Q)$. The composite map $\text{hofiber}_{(f_1, g_1)}(M - B \to M) \to \Omega Q^{\text{TN} - (TP \oplus TQ)}\text{holim}(P \xrightarrow{f_1} N \xrightarrow{g_1} Q)$ is the map $\ell$, and $\Lambda$ is its adjoint. See Section 9).

Now let $(f_1, g_1, v) \in \text{hofiber}(B - V \to M - V)^{TN/TP \oplus TQ}$. Here $v$ is a vector of length $0 \leq |v| \leq 1$, and $(f_1, g_1, v)$ is identified to a point when $|v| = 1$. After applying the maps $E, C, H$, and $D$ in diagram (I), it is clear that $(f_1, g_1, v)$ is sent to $((x, \beta, y), v) \in Q^{\text{TN} - (TP \oplus TQ)}\text{holim}(P \to N \leftarrow Q)$, where $(x_0, y_0) \in P \times Q$ is the unique pair such that $f_0(x_0) = g_0(y_0)$ and $\beta : I \to N$ is the path defined by $\beta(s) = f_{1 - 2s}(x_0)$ for $0 \leq s \leq 1/2$ and $\beta(s) = g_{2s - 1}(y_0)$ for $1/2 \leq s \leq 1$.

Now we must apply $F, G, \Lambda$ to $(f_1, g_1, v)$. A careful examination of the material in Section 4 reveals that $F$ sends $(f_1, g_1, v)$ to the point $s \land (\tilde{f}_1, \tilde{g}_1)$, where $s = 1 - |v|$ and $(\tilde{f}_1, \tilde{g}_1) \in \text{hofiber}(M - B \to M)$ is defined as follows. For $s \leq t \leq 1$, $(\tilde{f}_1, \tilde{g}_1) = (f_{(t - s)/(1 - s)}, g_{(t - s)/(1 - s)})$. For $0 \leq t \leq s$, $(\tilde{f}_1, \tilde{g}_1)$ has the following properties: $(\tilde{f}_1, \tilde{g}_1) \in M - B$ for $t < s$, $(\tilde{f}_1, \tilde{g}_1) = (f_0, g_0)$ has a unique pair $(x_0, y_0) \in P \times Q$ such that $f_0(x_0) = g_0(y_0) = z_0 \in N$ and such that $f_0(x_0) - g_0(y_0) \in T_{z_0}N$, when projected to
that it corresponds to a homotopy of pairs. If we can show that the condition

\[
\text{guarantee that there are maps}
\]

Consider the adjoint map \( \Phi : \mathcal{F} \to (\mathcal{M}, \mathcal{M} - \mathcal{V}) \) for the first, it suffices if for every smooth manifold \( W \) of dimension \( j < 2n - 2p - 2q \), for every map of pairs \( \phi : (W, \partial W) \to (\mathcal{M}, \mathcal{M} - \mathcal{V}) \), there is a homotopy of pairs to a map that is disjoint from \( \mathcal{V} \).

The proof outlined above uses that the pair \((\mathcal{M}, \mathcal{M} - \mathcal{V})\) is \((2n - 2p - 2q - 1)\)-connected and that the pair \((\mathcal{B}, \mathcal{B} - \mathcal{V})\) is \((n - p - q - 1)\)-connected. We now justify these statements more carefully.

For the first, it suffices if for every smooth manifold \( W \) of dimension \( j < 2n - 2p - 2q \), for every map of pairs \( \phi : (W, \partial W) \to (\mathcal{M}, \mathcal{M} - \mathcal{V}) \), there is a homotopy of pairs to a map that is disjoint from \( \mathcal{V} \).

Consider the adjoint map \( \Phi : W \times (P \coprod Q) \to N \). By a preliminary homotopy we can assume that \( \Phi \) is smooth, and we can make the homotopy small enough in the \( C^0 \) sense so that it corresponds to a homotopy of pairs. If we can show that the condition \( \Phi^{-1}(\mathcal{V}) = \emptyset \) holds for a dense set of all such smooth maps \( \Phi \), then another small homotopy will complete the job. For the density statement we will use the multijet transversality theorem of Mather [11, Proposition 3.3] (which appears in [1] as Theorem 4.13).

Recall the setup: Two smooth maps \( \Phi, \Psi : X \to Y \) have the same \( m \)-jet at \( x \in X \) if \( \Phi(x) = \Psi(x) \) and \( \Phi \) and \( \Psi \) have the same derivatives through order \( m \). Let \( X^{(r)} \subset X^r \) be the space of configurations of \( r \) distinct points in \( X \). The maps \( \Phi \) and \( \Psi \) have the same \( m \)-multijet at \( (x_1, \ldots, x_r) \in X^{(r)} \) if for every \( i \in \{1, \ldots r\} \) they have the same \( m \)-jet at \( x_i \). The manifold \( J_m^{(r)}(X, Y) \) of multijets has a point for each \( r \)-tuple \( (x_1, \ldots, x_r) \) and each equivalence class of maps as above. A smooth map \( \Phi : X \to Y \) determines a smooth map \( j_m^{(r)}(\Phi) : X^{(r)} \to J_m^{(r)}(X, Y) \). The multijet transversality theorem asserts that, for every submanifold \( Z \) of \( J_m^{(r)}(X, Y) \), the set of all \( \Phi \) such that \( j_m^{(r)}(\Phi) \) is transverse to \( Z \) is a countable intersection of dense open sets in the function space \( \text{Map}(X, Y) \). It follows that such a set, or even the intersection of countably many such sets, is dense.

We now introduce various submanifolds \( Z \) of \( J_m^{(r)}(W \times (P \coprod Q), N) \), for various values of \( r \) and \( m \). The point is that the condition \( \phi^{-1}(\mathcal{V}) = \emptyset \) will hold if and only if for each of these the set \( j_m^{(r)}(\Phi) \) is disjoint from \( Z \). The codimension of \( Z \) will always be big enough so that in order for \( j_m^{(r)}(\Phi) \) to be transverse to \( Z \) it must be disjoint. Therefore the theorem will guarantee that there are maps \( W \to \mathcal{M} - \mathcal{V} \) arbitrarily close to a given map \( \Phi : W \to \mathcal{M} \).

Let \( k \) be the dimension of \( W \). We consider the various ways in which \( \phi \) could hit \( \mathcal{V} \).

1. There might exist distinct \( x_1 \) and \( x_2 \) in \( P \) and distinct \( y_1 \) and \( y_2 \) in \( Q \) such that for some \( w \in W \) we have \( \Phi(w, x_1) = \Phi(w, y_1) \) and \( \Phi(w, x_2) = \Phi(w, y_2) \). Then the point

\[
((w, x_1), (w, x_2), (w, y_1), (w, y_2)) \in (W \times (P \coprod Q))^{(4)}
\]

maps into a certain submanifold of \( J_0^{(4)}(W \times (P \coprod Q), N) \) whose codimension is \( 3k + 2n \). (That is \( 3j \) to make four points of \( W \) equal to each other and \( 2n \) for two coincidences in \( N \).) This codimension is greater than the dimension \( 4k + 2p + 2q \) of
(the relevant open and closed part of) \((W \times (P \coprod Q))^{(4)}\), so that transverse means disjoint.

(2) There might exist distinct \(x_1\) and \(x_2\) in \(P\) and \(y\) in \(Q\) such that \(\Phi(w, x_1) = \Phi(w, y) = \Phi(w, x_2)\). This leads to a submanifold of \(J_0^{(3)}(W \times (P \coprod Q), N)\) whose codimension is \(2k + 2n\), greater than the dimension \(3k + 2p + q\) of (part of) \((W \times (P \coprod Q))^{(3)}\).

(3) There might exist \(x\) in \(P\) and distinct \(y_1\) and \(y_2\) in \(Q\) such that \(\Phi(w, x) = \Phi(w, y_1) = \Phi(w, y_2)\). The relevant manifold has codimension \(2k + 2n\) in \(J_0^{(3)}(W \times (P \coprod Q), N)\), greater than \(3k + p + 2q\).

(4) There might exist \(x\) in \(P\) and \(y\) in \(Q\) such that \(\Phi(w, x) = z = \Phi(w, y)\) and such that the linear map \(T_xP \oplus T_yQ \to T_zN\) given by differentiation of \(\phi(w)\) at \(x\) and \(y\) has rank less than \(p + q\). For each fixed rank \(r < p + q\) this leads to a submanifold of \(J_1^{(2)}(W \times (P \coprod Q), N)\) whose codimension \(k + (n - r)(p + q - r)\) is greater than \(2k + p + q\).

This completes the proof that the pair \((\mathcal{M}, \mathcal{M} - \mathcal{V})\) is \((2n - 2p - 2q - 1)\)-connected.

To prove that the pair \((\mathcal{B}, \mathcal{B} - \mathcal{V})\) is \((n - p - q - 1)\)-connected, essentially the same kind of standard dimension-counting will succeed, but a simple reference as before to the multijet transversality theorem will not suffice because \(\mathcal{B}\) is not simply the space of maps from one manifold to another.

First observe that both the projection \(\mathcal{B} \to P \times Q \times N\) and its restriction \(\mathcal{B} - \mathcal{V} \to P \times Q \times N\) are fibrations. It therefore suffices if, for a point \((x_0, y_0, z_0) \in P \times Q \times N\), the pair \((\mathcal{B}_0, \mathcal{B}_0 - \mathcal{V}_0)\) of fibers is \((n - p - q - 1)\)-connected. Here \(\mathcal{B}_0 \subset \mathcal{M}\) is the set of all \(\phi\) such that \(\phi(x_0) = z_0 = \phi(y_0)\), and \(\mathcal{V}_0 \subset \mathcal{B}_0\) is the set of all \(\phi\) such that in addition at least one of the following is true:

1. \(\phi(x) = \phi(y)\) for some \(x \in P - x_0\) and some \(y \in Q - y_0\)
2. \(\phi(x) = z_0\) for some \(x \in P - x_0\)
3. \(\phi(y) = z_0\) for some \(y \in Q - y_0\)
4. The linear map \(T_{x_0}P \oplus T_{y_0}Q \to T_{z_0}N\) has rank less than \(p + q\).

To deal first with (4), note that \(\mathcal{B}_0\) is fibered over the space \(\mathcal{L}\) of all linear maps \(T_{x_0}P \oplus T_{y_0}Q \to T_{z_0}N\). Let \(\mathcal{L}^{\text{max}} \subset \mathcal{L}\) be the open set of maps of rank \(p + q\) and let \(\mathcal{B}_0^{\text{max}} \subset \mathcal{B}_0\) be its preimage. The pair \((\mathcal{B}_0, \mathcal{B}_0^{\text{max}})\) is \((n - p - q - 2)\)-connected (one better than needed), because the pair \((\mathcal{L}, \mathcal{L}^{\text{max}})\) is \((n - p - q - 2)\)-connected, because the closed set \(\mathcal{L} - \mathcal{L}^{\text{max}}\) is the union of finitely many submanifolds having codimension at least \(n - p - q - 1\).

It remains to show that the pair \((\mathcal{B}_0^{\text{max}}, \mathcal{B} - \mathcal{V})\) is \((n - p - q - 1)\)-connected. Both \(\mathcal{B}_0^{\text{max}}\) and \(\mathcal{B} - \mathcal{V}\) fiber over \(\mathcal{L}^{\text{max}}\), so we can replace the two spaces by their fibers, say \(\mathcal{B}_L\) and \(\mathcal{B}_L - \mathcal{V}_L\) over a given \(L \in \mathcal{L}\). Now given a map \(\phi: W \to \mathcal{B}_L\), we want to perturb it slightly so as to eliminate behaviors (1), (2) and (3). None of these can occur for \(x\) near \(x_0\) or \(y\) near \(y_0\) anyway, given the choice of \(L\), so we look for perturbations that are fixed near \(x_0\) and \(y_0\). In other words, we look for a small compactly supported change in the map \(\Phi: W \times ((P - x_0) \coprod (Q - y_0)) \to N\). This goes as before: case (1) leads to a submanifold of \(J_0^{(2)}(W \times ((P - x_0) \coprod (Q - y_0)), N)\) with
codimension \(j + n\), greater than \(2j + p + q\); case (2) leads to a submanifold of \(J_0^{(1)}(W \times ((P - x_0) \sqcup (Q - y_0)), N)\) with codimension \(n\), greater than \(j + p\); and case (3) leads to a submanifold of \(J_0^{(1)}(W \times ((P - x_0) \sqcup (Q - y_0)), N)\) with codimension \(n\), greater than \(j + q\).

4. Normal bundles and homotopy cofibers

Suppose that \(X\) is a smooth manifold, and that the closed subset \(Y \subset X\) is a smooth submanifold with normal bundle \(\nu\).

Of course, the Thom space \(Y^\nu\) is equivalent to the homotopy cofiber of the inclusion map \(X - Y \rightarrow X\). This follows from the fact that there is a homotopy pushout square

\[
\begin{array}{ccc}
S(Y; \nu) & \rightarrow & D(Y; \nu) \\
\downarrow & & \downarrow \\
X - Y & \rightarrow & X.
\end{array}
\]

The homotopy fibers over \(X\) of the four spaces above form another homotopy pushout square

\[
\begin{array}{ccc}
\text{hofiber}(S(Y; \nu) \rightarrow X) & \rightarrow & \text{hofiber}(D(Y; \nu) \rightarrow X) \\
\downarrow & & \downarrow \\
\text{hofiber}(X - Y \rightarrow X) & \rightarrow & \text{hofiber}(X \rightarrow X) \simeq *.
\end{array}
\]

Comparing homotopy cofibers of the rows in this square, we obtain an equivalence

\[
\text{hofiber}(Y \rightarrow X)^\nu \rightarrow \Sigma \text{hofiber}(X - Y \rightarrow X)
\]

Here we have written \(\nu\) for the pullback of \(\nu\) to \(\text{hofiber}(Y \rightarrow X)\).

We need statements like those above in which the manifolds \(X\) and \(Y\) are replaced by the function spaces \(M - V\) and \(B - V\) and the role of the normal bundle is played by the vector bundle \(TN/(TP \oplus TQ)\) on \(B - V\). The only little difficulty is that the square (2) depended on having a tubular neighborhood. We will write down a substitute for (2) that avoids this dependence.

Let \(P(Y, X)\) be the space of all smooth paths \(\gamma : [0, 1] \rightarrow X\) such that \(\gamma^{-1}(Y) = 0\) and \(\gamma'(0)\) is not tangent to \(Y\). We have the homotopy-commutative square

\[
\begin{array}{ccc}
P(Y, X) & \rightarrow & Y \\
\downarrow & & \downarrow \\
X - Y & \rightarrow & X
\end{array}
\]

in which the top and left maps are evaluation at 0 and at 1 respectively.

There are equivalences

\[
\text{hofiber}(P(Y, X) \rightarrow Y) \rightarrow \text{hofiber}(S(Y; \nu) \rightarrow Y) = Y^\nu
\]
and

\[(5) \quad \text{hocofiber}(P(Y, X) \to Y) \to \text{hocofiber}(X - Y \to X).\]

The logic is as follows:

For \((4)\) we use the map \(P(Y, X) \to S(Y; \nu)\) that sends \(\gamma\) to the projection of \(\gamma'(0)\) in the direction perpendicular to \(Y\), normalized to have unit length. It is a map over \(Y\) between two spaces fibered over \(Y\), and it is an equivalence because for each point in \(Y\) the map of fibers is an equivalence.

For \((5)\) we need to see that the homotopy-commutative square \((3)\) is a homotopy pushout, in the sense that the associated map from the homotopy colimit of

\[X - Y \leftarrow P(Y, X) \to Y\]

to \(X\) is an equivalence. After choosing a tubular neighborhood of \(Y\) in \(X\), one can map \(S(Y; \nu)\) to \(P(Y, X)\) by using radial paths perpendicular to \(Y\). This map is an equivalence because it is a one-sided inverse to an equivalence. It follows that in showing that the square is a homotopy pushout we may consider instead the square

\[
\begin{array}{ccc}
S(Y; \nu) & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X - Y & \longrightarrow & X
\end{array}
\]

But this comes down to considering the same strictly commutative square \((2)\) that we began with.

Note that although a tubular neighborhood was used in proving \((5)\) to be an equivalence, the definitions of \((4)\) and \((5)\) did not use it. This is the point of introducing \(P(Y, X)\).

Now for the function spaces: Again we will obtain equivalences

\[
\text{hocofiber}(P(B - V, M - V) \to M - V) \to (B - V)'
\]

(where \(\nu\) now means the bundle \(TN/(TP \oplus TQ)\) on \(B - V\) and

\[
\text{hocofiber}(P(B - V, M - V) \to M - V) \to \text{hocofiber}(M - B \to M - V)
\]

We define the space \(P(B - V, M - V)\). A point in it is a map \(\gamma : [0, 1] \to M\) meeting the following conditions. Write \(\gamma(t) = (f_t, g_t)\). The conditions are:

1. \(\gamma\) is smooth in the sense that the adjoint maps \((t, x) \mapsto f_t(x)\) and \((t, x) \mapsto g_t(x)\) from \([0, 1] \times P\) and \([0, 1] \times Q\) to \(N\) are smooth.
2. For every \(t > 0\), \(\gamma_t\) is in \(M - B\), that is, \(f_t(P) \cap g_t(Q) = \emptyset\).
3. \(\gamma_0 \in B - V\), that is, \((3a)\) there is exactly one point \((x_0, z_0, y_0) \in P \times N \times Q\) such that \(f_0(x) = z_0 = g_0(y)\) and \((3b)\) \(d f_0 \oplus d g_0 : T_{x_0}P \oplus T_{y_0}Q \to T_{z_0}N\) is injective.
4. \(\gamma'(0)\) is not tangent to \(B - V\), that is, the vector \(f_0'(x_0) - g_0'(y_0) \in T_{z_0}(N)\) does not belong to the subspace \((d_{x_0} f)(T_{x_0}P) \oplus (d_{y_0} g)(T_{y_0}Q)\). Here \(f'\) and \(g'\) are derivatives with respect to \(t\).
Consider the homotopy-commutative square

\[
\begin{array}{ccc}
P(B-V, M-V) & \rightarrow & B-V \\
\downarrow & & \downarrow \\
M-B & \rightarrow & M-V,
\end{array}
\]

where the upper map and the left map take \( \gamma = (f, g) \) to \((f_0, g_0)\) and \((f_1, g_1)\) respectively.

We argue much as in the finite-dimensional case.

First, there is an equivalence \( P(B-V, M-V) \rightarrow S(B-V; \nu) \) that respects the projection to \( B-V \), namely the map that takes \( \gamma = (f, g) \) to the unit vector in \( T_{x_0}N/(T_{x_0}P \oplus T_{y_0}Q) \) determined by the element \( f'_0(x_0) - g'_0(y_0) \) of \( T_{x_0}P \oplus T_{y_0}Q \). It is an equivalence because it is a map between spaces fibered over \( B-V \) and it induces equivalences fiber by fiber.

Second, the square is a homotopy pushout. For this step, instead of trying to come up with a tubular neighborhood we reduce to the finite-dimensional case.

To show that the map from the homotopy colimit of

\[
M-B \leftarrow P(B-V, M-V) \rightarrow B-V
\]

to \( M-V \) is surjective on homotopy groups, let \( X = S^k \) and take any map \( \phi : X \rightarrow M-V \), with adjoint \( \Phi = (F,G) \), \( F : X \times P \rightarrow N \), \( G : X \times Q \rightarrow N \). Deforming by a homotopy that stays within \( M-V \), make \( \Phi \) “transverse to \( B-V \)” in the sense that \( F \) and \( G \) together give a map \( X \times P \times Q \rightarrow N \times N \) which is transverse to the diagonal. The preimage of the diagonal in \( X \times P \times Q \) is a submanifold, and it is embedded in \( X \) by the projection. Call its image \( Y \). The normal bundle of \( Y \) in \( X \) is the pullback of \( T_N/(TP \oplus TQ) \) by \( \phi \).

Now inverting the equivalence

\[
\text{hocolim}(X - Y \leftarrow P(Y, X) \rightarrow Y) \rightarrow X
\]

and composing with the obvious map

\[
\text{hocolim}(X - Y \leftarrow P(Y, X) \rightarrow Y) \rightarrow \text{hocolim}(M - B \leftarrow P(B-V, M-V) \rightarrow M-V)
\]

we get

\[
X \rightarrow \text{hocolim}(M - B \leftarrow P(B-V, M-V) \rightarrow M-V)
\]

a lifting (up to homotopy) of \( \phi \). Essentially the same argument serves to lift a homotopy and prove the injectivity.

Taking homotopy fibers over \( M-V \) all around, we obtain the needed equivalence \( F \).

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