Twisting $\kappa$-deformed phase space\textsuperscript{1}

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Abstract

We briefly discuss the twisting procedure applied to the $\kappa$-deformed space-time. It appears that one can consider only two kinds of such twistings: in space and time directions. For both types of twistings we introduce related phase spaces and consider briefly their properties. We discuss in detail the changes of duality relations under the action of twist. The Jordanian twisted space-time and phase space in $D=2$ are also commented.

1 Introduction

Recently, some suggestions appeared that the classical Lorentz invariance should be treated as an approximate symmetry in ultra-high energy processes (shift of GZK kinematic threshold) and the relativistic space-time symmetries on Planck scale should be modified\textsuperscript{1}. There are also some theoretical predictions coming from the string theory and quantum gravity models that space-time at very short distances of order Planck length should be quantum i.e. noncommutative\textsuperscript{2}. One can modify the standard Lorentz or Poincaré relativistic symmetry in two different ways: obtaining a commutative space-time (as in the standard relativistic theory) or noncommutative space-time with space and time non-commuting variables.

The first type of modified relativistic symmetries follows from the concept of the double special relativistic (DSR) theory introduced by Amelino-Camelia\textsuperscript{3}, where two observer-independent parameters (scales) $c, \lambda_p$ – velocity of light and Planck length play the fundamental role. In this framework, two basic models of such DSR theory are considered: DSR1 theory proposed by Amelino-Camelia and DSR2 theory considered by Magueijo and Smolin\textsuperscript{4}. In both proposals the energy-momentum vector space is extended to the four-momentum algebra considered as enveloping algebra of this linear energy-momentum space. In this algebra the nonlinear transformation of the basis vectors (of energy-momentum space) is realized according to DSR1 or DSR2 theory models. In this way we get four non-linearly transformed generators belonging to the energy-momentum algebra which are assumed to represent physical energy and momentum. Using this transformed four-momentum basis we obtain for instance the standard dispersion relation in a modified form which is, however, quite equivalent (under an inverse nonlinear transformation) to the standard one. One can also consider the action of the relativistic symmetry on physical energy and momentum as a nonlinear realization of the Lorentz group and express the addition laws of energy and momentum by the coproduct (non-linearly transformed trivial coproduct)\textsuperscript{5} and extending in this way the energy-momentum algebra to a bialgebra structure. Space-time related\textsuperscript{1}Supported by KBN grant No. 5 P03B 106 21
to this bialgebra structure is defined as a dual bialgebra where multiplication of the space-time variables follows from the coproduct. The momentum coproduct is symmetric, therefore space-time is commuting. In consistency with conclusions in \[5\], recently it has been observed \[6\] that DSR theories with symmetric coproduct law describe just the standard special relativity framework in nonlinear disguise.

The second type of modified relativistic symmetries bases on the concept of the Hopf algebra (quantum group) \[7\], \[8\] and quantum deformations of the classical Lorentz or Poincaré symmetry \[9\]. In this approach there is a distinguished deformation, the so called \(\kappa\)-deformation of the Poincaré symmetry \[10\], where \(O(3)\)-rotational symmetry is not deformed. The Hopf algebra structure of this deformation will be discussed in detail in Section 2. However, we would like to stress that the four-momentum algebra for DSR1 theory and the \(\kappa\)-deformed theory (in bicrossproduct basis) are the same, and give us the same physical predictions. The main difference between both theories lies in their coalgebra sectors, because in the \(\kappa\)-deformed Poincaré algebra the coproduct of the four-momentum is non-symmetric. This fact has important consequences for a possibility of physical interpretation of the \(\kappa\)-deformed addition law of energy-momentum. On the other hand, this non-symmetricity of coproduct allows us to obtain non-commutativity of space-time in the Hopf algebra framework. The idea of a non-commutativity of space-time appeared in physics since Snyder’s articles \[11\], where the space-time is isomorphic to a quotient of de Sitter \(SO(4,1)\) and Lorentz \(SO(3,1)\) groups.

Further, we shall consider \(\kappa\)-deformed space-time defined as a dual Hopf algebra to the \(\kappa\)-deformed four-momentum algebra considered as a subalgebra of \(\kappa\)-Poincaré Hopf algebra i.e. the space-time is understood in Majid and Ruegg sense \[12\]. Such a space-time has a Hopf algebra structure given by coproduct of trivial form. In this paper we shall consider a twisting operation acting on the \(\kappa\)-deformed space-time algebra. The notion of twisting of Hopf algebra was introduced by Drinfeld \[13\] and then applied to enveloping algebras of simple Lie algebras \[14\] and the applications to physical symmetry algebras one can find in \[15\] and recently in \(\kappa\)-deformations of Weyl and conformal symmetries \[16\].

For a given pair of space-time and four-momentum algebras one can define a phase-space algebra using the notion of cross-product algebra \[9\]. It is well known that although both paired algebras have a Hopf algebra structure, the phase space algebra is no longer a Hopf algebra. The case of \(\kappa\)-deformed phase space is considered in \[17\], \[18\].

In Section 2 we recall the Hopf algebra structure of \(\kappa\)-Poincaré algebra in a very useful bicrossproduct basis \[12\] in which the Lorentz algebra sector has a classical form. We also discuss the \(\kappa\)-deformed space-time underlying the role of duality relations.

In Section 3.1 we show that one can define only two kinds of twisting operations on \(\kappa\)-deformed space-time: twisting in space directions (SD) and in time direction (TD). For both cases we derive their Hopf algebra structure. In Section 3.2 we extend our dual pair to a phase space algebra and we find phase space commutation relations. It appears that the SD-twisted phase space can not be considered as a linear vector space because of nonlinear phase space commutation relations.

Section 4 is devoted to the derivation of duality relations between the twisted space-time algebra and the momentum algebra for simplicity in two dimensional space-time. It is more a computational part of the paper where we find that the space-time twisting effectively changes only duality relations. In this way the basis of twisted space-time algebra becomes non-orthogonal (one can say "twisted") with respect to the momentum basis. We also consider D=2 twisted \(\kappa\)-deformed space-time by nonsymmetric twisting function (Jordanian
2 \( \kappa \)-Poincaré algebra in the bicrosproduct basis and \( \kappa \)-deformed space-time

Let us recall the structure of \( \kappa \)-Poincaré algebra given in the bicrossproduct basis \[12\]

- non-deformed (classical) Lorentz algebra \((g_{\mu\nu} = (1, -1, -1, -1))\)

\[ [M_{\mu\nu}, M_{\rho\tau}] = -i(g_{\mu\rho}M_{\nu\tau} + g_{\nu\tau}M_{\mu\rho} - g_{\mu\tau}M_{\nu\rho} - g_{\nu\rho}M_{\mu\tau}), \]  

- deformed covariance relations \((M_i = \frac{1}{2}\epsilon_{ijk}M_{jk}, \ N_i = M_{i0})\)

\[ [M_i, p_j] = i\epsilon_{ijk}p_k, \quad [M_i, p_0] = 0, \quad [M_i, p_0] = 0, \]  

\[ [N_i, p_j] = i\delta_{ij} \left[ \kappa c \sinh\left(\frac{p_0}{\kappa c}\right)e^{-\frac{p_0}{\kappa c}} + \frac{1}{2\kappa c}(\vec{p})^2 \right] - \frac{i}{\kappa c}p_ip_j, \]  

where \( \kappa \) - massive deformation parameter and \( c \) - light velocity.

One can extend this algebra to a Hopf algebra defining the coalgebra sector by coproduct \[\Delta(X)\]

\[ \Delta(M_i) = M_i \otimes 1 + 1 \otimes M_i, \]

\[ \Delta(N_i) = N_i \otimes 1 + e^{-\frac{p_0}{\kappa c}} \otimes N_i + \frac{1}{\kappa c} \epsilon_{ijk}p_j \otimes M_k, \]

\[ \Delta(p_0) = p_0 \otimes 1 + 1 \otimes p_0, \]

\[ \Delta(p_i) = p_i \otimes 1 + e^{-\frac{p_0}{\kappa c}} \otimes p_i, \]  

with antipode \( S(X) \) and counit \( \epsilon(X) \)

\[ S(M_i) = -M_i, \quad S(N_i) = -e^{\frac{p_0}{\kappa c}}N_i + \frac{1}{\kappa c} \epsilon_{ijk}p_j M_k, \]

\[ S(p_0) = -p_0, \quad S(p_i) = -p_i e^{\frac{p_0}{\kappa c}}, \]  

\[ \epsilon(X) = 0, \quad \epsilon(1) = 1, \]  

for \( X = M_i, N_i, p_\mu \).

Therefore, in the \( \kappa \)-Poincaré Hopf algebra one can distinguish the following three subalgebras

- classical Lorentz algebra \( \mathcal{L} = \{M_{\mu\nu}\} \) given by \[11\],

- non-deformed \( O(3) \)-rotation algebra \( \mathcal{M} = \{M_i, \Delta, S, \epsilon\} \) as a Hopf subalgebra with trivial coproduct \( \Delta(M_i) \),

- abelian four-momentum algebra \( \mathcal{P}_\kappa = \{p_\mu, \Delta, S, \epsilon\} \) as a Hopf subalgebra with non-symmetric coproduct \( \Delta(p_\mu) \) given by the relations

\[ [p_\mu, p_\nu] = 0, \quad \epsilon(p_\mu) = 0, \]

\[ \Delta(p_0) = p_0 \otimes 1 + 1 \otimes p_0, \quad \Delta(p_i) = p_i \otimes 1 + e^{-\frac{p_0}{\kappa c}} \otimes p_i, \]  

\[ S(p_0) = -p_0, \quad S(p_i) = -p_i e^{\frac{p_0}{\kappa c}}. \]  

The Hopf four-momentum algebra \( \mathcal{P}_\kappa \) can be considered as a deformation of universal enveloping algebra of classical translation algebra, generated (in Drinfel’d sense \[3\]) by polynomial functions of linear momentum generators \( p_\mu \) (translations).
To define the $\kappa$-deformed space-time we adopt Majid and Ruegg point of view [12], namely, using this algebra $\mathcal{P}_\kappa$ we define $\kappa$-deformed space-time algebra $\mathcal{X}_\kappa$ in a natural way, as a dual algebra to four-momentum translation algebra $\mathcal{P}_\kappa$. One can assume the standard duality relations between linear bases of both algebras, respecting Hopf algebra structure.

Let $\mathcal{X}_\kappa$ be generated by space-time variables $\{x_0, \vec{x}\}$ dual to four-momentum $p_\mu$ satisfying the following duality relations

\begin{align}
\langle p_\mu, x_\nu \rangle &= i g_{\mu\nu}, \quad g_{\mu\nu} = (1, -1, -1, -1), \\
\langle p_\mu, 1 \rangle &= \langle 1, x_\mu \rangle = 0, \quad \langle 1, 1 \rangle = 1, \\
\langle pq, x \rangle &= \langle p \otimes q, \Delta(x) \rangle, \quad \langle p, xy \rangle = \langle \Delta(p), x \otimes y \rangle,
\end{align}

where $p_\mu, p, q \in \mathcal{P}_\kappa$, $x_\nu, x, y \in \mathcal{X}_\kappa$, or using Sweedler coproduct notation (see for instance [9]) $\Delta(a) = a(1) \otimes a(2)$ relations [12] can be rewritten as

\begin{align}
\langle pq, x \rangle &= \langle p \otimes q, \Delta(x) \rangle = \langle p, x(1) \rangle \langle q, x(2) \rangle, \\
\langle p, xy \rangle &= \langle \Delta(p), x \otimes y \rangle = \langle p(1), x \rangle \langle p(2), y \rangle.
\end{align}

Relations (6-9) do not describe the Hopf algebra structure of space-time algebra uniquely, in particular they describe a wide class of coproducts $\Delta(x_\nu)$. The form of space-time co-product depends on the choice of higher order duality relations and vice versa. If we assume orthogonality of space-time and four-momentum monomial basis in the form ($i, j = 1, 2, 3, \quad k, l, m, n = 0, 1, 2, \ldots$)

\begin{align}
\langle p_i^m p_0^n, x_j^k x_l^l \rangle &= m! n! \delta_{mk} \delta_{nl} \delta_{ij} \langle p_0, x_0 \rangle^n \langle p_i, x_i \rangle^m,
\end{align}

then we get the following form of non-commuting space-time Hopf algebra [12]

\begin{align}
[x_0, x_i] &= \langle e^{-\frac{m}{\kappa c}}, x_0 \rangle x_i = -\frac{i}{\kappa c} x_i, \\
\Delta_0(x_\mu) &= x_\mu \otimes 1 + 1 \otimes x_\mu, \quad S(x_\mu) = -x_\mu, \quad \epsilon(x_\mu) = 0,
\end{align}

with trivial, symmetric coproduct $\Delta_0(x_\mu)$.

In this place we would like to stress the difference between the notion of space-time algebra and the concept of deformed (quantum or $\kappa$-deformed) space-time – the linear span of space-time variables $x_\mu$. For the commuting (classical) space-time both notions are equivalent – commuting space-time algebra is simply the algebra of commuting (polynomial) functions on space-time. Classical space-time can be regarded as a dual to the four-momentum (translation) space because of trivial momentum coproduct. However, in the deformed (quantum) case the momentum coproduct [7] contains the exponential factor which belongs to the momentum algebra. It is the reason why we use the notion of space-time algebra for non-commuting space-time.

Because the space-time variables $x_\mu$ do not commute among themselves, one can also choose a space-time monomial basis with opposite ordering and satisfying the duality relations [17]

\begin{align}
\langle p_i^m p_0^n, x_j^l x_k^k \rangle &= \frac{l! \delta_{km} \delta_{ij}}{(l - n)!} \langle p_i, x_i \rangle^m \langle p_0, x_0 \rangle^n \langle e^{\frac{m p_0}{\kappa c}}, x_0^{l - n} \rangle, \\
\langle p_i^m p_0^n, x_j^l x_k^k \rangle &= 0 \quad \text{for} \quad n > l,
\end{align}

therefore, for this ordering, we obtain non-orthogonal duality relations for $0 \leq n \leq l$. Both relations [12] and [15] one can rewrite in a more convenient form using the exponential
generating function
\[
\langle p_i^m e^{\xi p_0}, x_j^k, x_0^l \rangle = \langle e^{\xi p_0}, x_0^l \rangle \langle p_i^m, x_j^k \rangle,
\]
(16)
\[
\langle p_i^m e^{\xi p_0}, x_0^l, x_j^k \rangle = \langle e^{\xi p_0-m \xi}, x_0^l \rangle \langle p_i^m, x_j^k \rangle = \langle e^{\xi p_0}, (x_0 - i \frac{k}{\kappa c})^l \rangle \langle p_i^m, x_j^k \rangle.
\]
(17)

Comparing these formulae one can easily obtain the general form of the space-time commutation relations
\[
[x_i^k, x_0^l] = x_i^k \left\{ x_0^l - \left( x_0 - i \frac{k}{\kappa c} \right)^l \right\},
\]
(18)
in the right-time-ordered basis (12) (with all powers of the time variable \(x_0\) on the right).

We would like to notice, that the duality relations (12) have the same form as in the case of the classical Poincaré algebra with trivial Hopf structure and its commuting dual space-time algebra. This duality relation can be rewritten in an equivalent form (see [12])
\[
\langle f(p_i, p_0), :\phi(x_j, x_0) : \rangle = \left( f \left( -i \frac{\partial}{\partial x_i}, i \frac{\partial}{\partial x_0} \right) \phi \right) (0, 0),
\]
(19)
for polynomial functions \(f, \phi\) and \(:\phi:\) denotes right-time-ordered polynomial.

Further, we shall consider the coproducts of space-time variables related to \(\Delta_0(x_\mu)\) by twisting procedure, and we shall discuss twisted duality relations.

3 Twisted \(\kappa\)-deformed space-time and phase space

3.1 Twisting procedure

It is well known, that \(\kappa\)-deformed space-time [13] can be extended to a Hopf algebra up to similarity transformation (twisting) in the coalgebra sector. The choice of coproduct in the form [14] is the simplest one, however one can also consider a more general class of twisted coproducts [14], [15] given by the following similarity transformation
\[
\Delta^F(x_\mu) = F \Delta_0(x_\mu) F^{-1},
\]
(20)
where \((a \otimes b)(c \otimes d) = ac \otimes bd\) and an invertible twisting function \(F \in \mathcal{X}_\kappa \otimes \mathcal{X}_\kappa\) satisfies the additional Hopf structure requirements which follow from the properties of twisted coproduct \(\Delta^F(x_\mu)\)

coassociativity
\[
(\Delta^F \otimes 1) \Delta^F(x_\mu) = (1 \otimes \Delta^F) \Delta^F(x_\mu),
\]
(21)
consistency relations
\[
(\epsilon \otimes 1) \circ \Delta^F(x_\mu) = (1 \otimes \epsilon) \circ \Delta^F(x_\mu) = x_\mu,
\]
(22)
\[
(S^F \otimes 1) \circ \Delta^F(x_\mu) = (1 \otimes S^F) \circ \Delta^F(x_\mu) = 0,
\]
(23)
where we denote \((a \otimes b) \circ (c \otimes d) = acbd\). We would like to notice, that twisted space-time is defined by twisted coproduct \(\Delta^F\) and antipode \(S^F\).

The coassociativity condition (21) can be rewritten in a more familiar form as a 2-cocycle condition imposed on the twisting function \(F\) [9]
\[
(1 \otimes F)(1 \otimes \Delta_0) F = (F \otimes 1)(\Delta_0 \otimes 1) F.
\]
(24)
Without loss of generality we assume the following exponential form of the twisting function

\[ F = \exp \left( \sum \phi_n \otimes \phi^n \right), \]  

(25)

where \( \phi_n, \phi^n \in X_\kappa \). Then the coassociativity condition one can express by the formula

\[ e^{1 \otimes \phi_n \otimes \phi^n} e^{\phi_n \otimes \Delta_0 (\phi^n)} = e^{\phi_n \otimes \phi^n \otimes 1} e^{\Delta_0 (\phi_n) \otimes \phi^n}. \]  

(26)

Let us notice that the relations define noncommuting space-time as a four-dimensional Lie algebra and \( \kappa \)-deformed space-time algebra as an enveloping Lie algebra \( X_\kappa \). Therefore, one can use the results of ref. [14] to find twisted space-time as a twisted Lie algebra with the Hopf structure. It is equivalent to put the additional requirements on the twisting function

\[ (\Delta_0 \otimes 1)F = F_{13} F_{23} = F_{23} F_{13}, \quad [F_{12}, F_{23}] = 0, \]  

(27)

where we use the standard notation

\[ F_{12} = F \otimes 1, \quad F_{23} = 1 \otimes F, \quad F_{13} = e^{\phi_n \otimes 1 \otimes \phi^n}. \]  

(28)

In this framework, the elements \( \phi_n, \phi^n \) belong to commutative subalgebra of \( X_\kappa \), with trivial coproduct

\begin{align*}
\Delta_0 (\phi^n) &= \phi^n \otimes 1 + 1 \otimes \phi^n, \quad [\phi_n \otimes \phi^n, \Delta_0 (\phi^m)] = 0, \\
\Delta_0 (\phi_n) &= \phi_n \otimes 1 + 1 \otimes \phi_n, \quad [\phi_n \otimes \phi^n, \Delta_0 (\phi_m)] = 0.
\end{align*}  

(29)

Of course, it is not the most general case of a twisting function \( F \). For instance, if we do not assume commutativity of \( F_{12}, F_{23} \) in (27) (so, we give up a Lie algebra twisting framework) we get

\begin{align*}
\Delta_0 (\phi_n) &= \phi_n \otimes 1 + 1 \otimes \phi_n, \quad [\phi_m, \phi_n] = 0, \\
\Delta^F (\phi^n) &= \phi^n \otimes 1 + 1 \otimes \phi^n, \quad [\phi^m, \phi^n] = 0.
\end{align*}  

(30)

(31)

In this case we obtain more general twisting function \( F \) of noncommuting space-time (so called Jordanian twist, discussed in Section 4).

If we consider the twisting of \( \kappa \)-deformed space time algebra, we deal with symmetric function \( F \) because the dual fourmomentum algebra is commuting one. Therefore, a symmetric twisting function \( F \) satisfying the relations [21] and [27] is given by

\[ F = e^{\phi \otimes \phi} = \exp [(ax_0 + b_i x_i) \otimes (ax_0 + b_j x_j)], \]  

(32)

where the four twisting parameters \( a, b_i \) are in general the complex numbers. For this twisting function \( F \) we immediately obtain the following formulae for twisted coproduct of space-time variables

\begin{align*}
\Delta^F (x_0) &= \Delta_0 (x_0) + \frac{b_i}{a} \left[ x_i \otimes (1 - e^{a \lambda \phi}) + (1 - e^{a \lambda \phi}) \otimes x_i \right], \\
\Delta^F (x_i) &= \Delta_0 (x_i) + x_i \otimes (e^{a \lambda \phi} - 1) + (e^{a \lambda \phi} - 1) \otimes x_i,
\end{align*}  

(33)

(34)

where \( \lambda = -i/\kappa c \) [13].

6
Because a time variable should not depend on the space inversion, therefore it is physically reasonable to assume that the twisting function $F$ is a space-inversion invariant. Using this fact one can construct two twisting functions related to two abelian subalgebras of $\kappa$-deformed space-time. The first one generated by the time variable $x_0$ and the other one by the space variables $x_i$, both with trivial coproducts.

**Twisting of space directions (SD)**

$$F_0(a) = e^{a x_0 \otimes x_0},$$  \hspace{1cm} (35)

**Twisting of time direction (TD)**

$$F(b) = e^{b_{ij} x_i \otimes x_j},$$  \hspace{1cm} (36)

where $a, b_{ij} = b_{b_j} \in \mathbb{C}$ in general case. If we consider the space-time Hopf algebra with involution * satisfying $(a \otimes b)^* = a^* \otimes b^*$ then, the assumption of hermiticity of space-time generators $x_\mu^* = x_\mu$ implies the unitarity of twisting functions i.e. $a, b_{ij} \in i \mathbb{R}$ are pure imaginary complex numbers.

For the twisting function $F_0(a)$, using formulae (20) and (23) we obtain SD-twisted space-time Hopf algebra $\mathcal{X}(\alpha)$ hermitian basis

$$[x_0, x_i] = \frac{-i}{\kappa c} x_i, \hspace{1cm} [x_i, x_j] = 0,$$

$$\Delta F_0(a)(x_0) = \Delta \alpha(x_0) = x_0 \otimes 1 + 1 \otimes x_0,$$

$$\Delta F_0(a)(x_i) = \Delta \alpha(x_i) = x_i \otimes e^{\alpha x_0} + e^{\alpha x_0} \otimes x_i,$$  \hspace{1cm} (37)

$$S F_0(a)(x_i) = S \alpha(x_i) = -x_i e^{\alpha(\frac{1}{\kappa c} - 2 \alpha)},$$

$$S F_0(a)(x_0) = S \alpha(x_0) = -x_0, \hspace{1cm} \epsilon(x_\mu) = 0,$$

where

$$\alpha \equiv a < e^{-\frac{\pi a}{\kappa c}}, x_0 > = \frac{-i a}{\kappa c} \in \mathbb{R},$$  \hspace{1cm} (38)

and similarly, choosing the twisting function as $F(b)$ we get the TD-twisted space-time Hopf-algebra $\mathcal{X}(\beta)$ hermitian basis

$$[x_0, x_i] = \frac{-i}{\kappa c} x_i, \hspace{1cm} [x_i, x_j] = 0,$$

$$\Delta F(b)(x_0) = \Delta \beta(x_0) = x_0 \otimes 1 + 1 \otimes x_0 + \beta_{ij} x_i \otimes x_j,$$

$$\Delta F(b)(x_i) = \Delta \beta(x_i) = x_i \otimes 1 + 1 \otimes x_i,$$  \hspace{1cm} (39)

$$S F(b)(x_i) = S \beta(x_i) = -x_i,$$

$$S F(b)(x_0) = S \beta(x_0) = -x_0 + \beta_{ij} x_i x_j, \hspace{1cm} \epsilon(x_\mu) = 0,$$

where

$$\beta_{ij} = \frac{2i}{\kappa c} b_{ij} \in \mathbb{R}.$$  \hspace{1cm} (40)

It is obvious, that in the limit $\alpha, \beta \to 0$ we obtain a $\kappa$-deformed space-time algebra given by (13 14).

It is well known that a twisting by a symmetric function $F$ is equivalent to nonlinear transformation of space-time, therefore SD-twisted coproduct (37) is given by the function $s(x_\mu)$

$$\Delta \alpha(x_0) = \Delta \alpha(s(x_0)), \hspace{1cm} s(x_0) = x_0,$$

$$\Delta \alpha(x_i) = \Delta \alpha(s(x_i)), \hspace{1cm} s(x_i) = x_i e^{\alpha x_0},$$

$$\Delta \alpha(s(x_i)) = s(x_i) \otimes e^{\alpha x_0} + e^{\alpha x_0} \otimes s(x_i),$$  \hspace{1cm} (41)
and analogously, TD-twisted coproduct (39) can be rewritten using function $t(x_\mu)$

\[
\Delta_\beta(x_0) = \Delta_0(t(x_0)), \quad t(x_0) = x_0 + \frac{1}{2} \beta_{ij}x_ix_j,
\]

\[
\Delta_\beta(x_i) = \Delta_0(t(x_i)), \quad t(x_i) = x_i,
\]

\[
\Delta_0(t(x_0)) = t(x_0) \otimes 1 + 1 \otimes t(x_0) + \beta_{ij}t(x_i) \otimes t(x_j).
\]  

(42)

We see that the twisting of $\kappa$-deformed space-time is equivalent to nonlinear transformation of the space-time variables.

### 3.2 Phase space as cross product algebra

Let us notice, that we have two pairs of dual Hopf algebras $X_\kappa(\alpha) \otimes P_\kappa$ and $X_\kappa(\beta) \otimes P_\kappa$ which in the non-deformed limit $\kappa \to \infty$ turn out to be classical space-time and momentum algebras with multiplication defined by the commutator, therefore we should get the quantum-mechanical phase space with standard Heisenberg commutation relations.

In order to construct such a deformed phase space $\Pi_\kappa$ isomorphic as a vector space to $\Pi_\kappa \sim X_\kappa \otimes P_\kappa$ one has to extend the commutation relations (6) and (37) or (39) by adding a cross commutators between $X_\kappa$ and $P_\kappa$.

It appears that a consistent construction of phase space $\Pi_\kappa$ can be done using the notion of a left (right) cross product (smash product) algebra [9]. For simplicity, we shall consider only left cross product algebra.

One can define a left action (representation) of the momentum algebra $P_\kappa$ on the space-time algebra $X_\kappa$ as a linear map

\[
d : P_\kappa \otimes X_\kappa \to X_\kappa : p \otimes x \to p \triangleright x,
\]  

(43)

such that

\[
(p\tilde{p}) \triangleright x = p \triangleright (\tilde{p} \triangleright x), \quad 1 \triangleright x = x.
\]  

(44)

We choose the following left action

\[
p \triangleright x = x_{(1)} < p, x_{(2)} >,
\]  

(45)

therefore $X_\kappa$ is a left $P_\kappa$-module or even a left $P_\kappa$-module algebra, because $X_\kappa$ and $P_\kappa$ are also Hopf algebras and the left action (45) satisfies

\[
p \triangleright (x\tilde{x}) = (p_{(1)} \triangleright x)(p_{(2)} \triangleright \tilde{x}), \quad p \triangleright 1 = \epsilon(p)1.
\]  

(46)

This implies that we can regard the twisted space-time as the left $\kappa$-deformed momentum $P_\kappa$-module algebra for both choices of twisting functions (35) and (36) with the following left action (45) in the case of SD-twisted space-time (37)

\[
p_0 \triangleright x_0 = i, \quad p_i \triangleright x_0 = 0, \quad p_0 \triangleright x_i = i \alpha x_i, \quad p_i \triangleright x_j = -i \delta_{ij} e^{\alpha x_0},
\]  

(47)

and in the case of TD-twisted space-time (39) we get

\[
p_0 \triangleright x_0 = i, \quad p_k \triangleright x_0 = -i \beta_{ik} x_i, \quad p_0 \triangleright x_i = 0, \quad p_k \triangleright x_i = -i \delta_{ki}.
\]  

(48)

We recall the definition of a left cross product algebra [9].
Let $\mathcal{P}_\kappa$ be a Hopf algebra and $\mathcal{X}_\kappa$ a left $\mathcal{P}_\kappa$-module algebra. A left cross product algebra $\Pi_\kappa = \mathcal{X}_\kappa \rtimes \mathcal{P}_\kappa$ is a vector space $\mathcal{X}_\kappa \otimes \mathcal{P}_\kappa$ with the product (left cross product)

$$(x \otimes p)(\tilde{x} \otimes \tilde{p}) = x(p(1) \triangleright \tilde{x}) \otimes \tilde{p}(2),$$

(49)

and the unit element $1 \otimes 1$, where $x, \tilde{x} \in \mathcal{X}_\kappa$ and $p, \tilde{p} \in \mathcal{P}_\kappa$. It appears that $\Pi_\kappa = \mathcal{X}_\kappa \rtimes \mathcal{P}_\kappa$ is an associative algebra, however it can not be extended to a Hopf algebra ([9]). This fact suggests that one can construct $\kappa$-deformed phase-space $\Pi$ for many particle system in a usual way as a tensor product algebra $\Pi = \Pi_\kappa \otimes \cdots \otimes \Pi_\kappa$. The obvious isomorphism $\mathcal{X}_\kappa \sim \mathcal{X}_\kappa \otimes 1$, $\mathcal{P}_\kappa \sim 1 \otimes \mathcal{P}_\kappa$ allows us to define the commutator for the whole phase space $\Pi_\kappa$

$$[x, p] = x \circ p - p \circ x, \quad x \circ p = x \otimes p, \quad p \circ x = (p(1) \triangleright x) \otimes p(2).$$

(50)

Using this definition and formulae (37) and (47) we obtain the commutation relations for the linear basis of SD-twisted phase space $\Pi_\kappa(\alpha) = \mathcal{X}_\kappa(\alpha) \rtimes \mathcal{P}_\kappa$

$$[x_i, x_j] = -\frac{i}{\kappa} x_i, \quad [x_i, x_j] = 0,$$

$$[x_i, p_0] = -i \alpha x_i, \quad [x_0, p_i] = \frac{i}{\kappa} p_i,$$

$$[x_0, p_0] = -i, \quad [p_\mu, p_\nu] = 0,$$

(51)

$$[x_i, p_j] = i \delta_{ij} e^{\alpha x_0} + \left(1 - e^{-\frac{i}{\kappa} x_0} \right) x_i p_j.$$

Let us notice that because of the exponential term in the last relation, the SD-twisted phase space can be considered only as an algebra.

In the limit $\alpha \to 0$ we obtain the standard $\kappa$-deformed phase space considered in [17], a deformed generalization of Heisenberg algebra. It is interesting to notice that one can also consider the limit $\kappa \to \infty, \alpha = const.$ (i.e. one can assume the linear dependence of the twisting parameter $\alpha$ on the deformation parameter $\kappa$, see (38)) of phase space $\Pi_\kappa(\alpha) \to \Pi_\infty(\alpha)$ given by the non-vanishing commutators

$$[x_0, p_0] = -i, \quad [x_i, p_j] = i \delta_{ij} e^{\alpha x_0}, \quad [x_i, p_0] = -i \alpha x_i,$$

(52)

with commuting space-time and momentum.

Similarly, from formulae (39) and (48) we obtain the commutation relations for the linear basis of TD-twisted phase space $\Pi_\kappa(\beta) = \mathcal{X}_\kappa(\beta) \rtimes \mathcal{P}_\kappa$

$$[x_0, x_i] = -\frac{i}{\kappa} x_i, \quad [x_i, x_j] = 0,$$

$$[x_i, p_0] = 0, \quad [x_0, p_i] = \frac{i}{\kappa} p_i + \beta_{ji} x_j,$$

$$[x_0, p_0] = -i, \quad [p_\mu, p_\nu] = 0,$$

(53)

$$[x_i, p_j] = i \delta_{ij}.$$

We see that commutators are given by linear combinations of $p_i, x_i$ therefore one can regard these formulae as defining phase space (not an algebra) in the classical sense. Also in this case we can consider the limit $\kappa \to \infty, \beta = const.$ (see (40)) of phase space $\Pi_\kappa(\beta) \to \Pi_\infty(\beta)$ given by the non-vanishing commutators

$$[x_0, p_0] = -i, \quad [x_i, p_j] = i \delta_{ij}, \quad [x_0, p_i] = i \beta_{ji} x_j,$$

(54)
with commuting space-time and momentum.

It turns out that both phase spaces \( \Pi_\infty(\alpha) \) and \( \Pi_\infty(\beta) \) can be realized by the standard position and momentum operators \( \hat{x}_\mu, \hat{p}_\nu \) satisfying the Heisenberg commutation relations

\[
[\hat{x}_\mu, \hat{p}_\nu] = -i g_{\mu\nu}
\]

and assuming \( \beta_{ij} = \beta \delta_{ij} \)

\[
x_\mu = \hat{x}_\mu, \quad p_0 = \hat{p}_0, \quad p_i = \hat{p}_i - \beta \hat{p}_0 \hat{x}_i, \quad \text{for} \quad \Pi_\infty(\beta) \tag{56}
\]

We notice that both algebras \( \mathcal{X}_\kappa \) and \( \mathcal{P}_\kappa \) possess the Hopf structure therefore one can also consider a left action of space-time algebra on the momentum algebra \( \triangleright : \mathcal{P}_\kappa \otimes \mathcal{X}_\kappa \to \mathcal{X}_\kappa \) formally a changing the position and momentum generators \( x \leftrightarrow p \). It corresponds in quantum mechanical language to the exchange of momentum for positions representation. In this case one can define a phase space as the cross product algebra \( \mathcal{P}_\kappa \triangleright \mathcal{X}_\kappa \) (see [17]) with slightly different cross commutation relations. However, we do not consider twisting of this kind of phase space.

### 4 Duality for twisted D=2 space-time

Considering the formulae (9) we notice that different choices of coproducts \( \Delta(x) \) or \( \Delta(p) \) provide changes in duality relations \( \langle p_\mu q^n, x \rangle \) and \( \langle p, x_\mu y^\nu \rangle \), respectively. Therefore, we can expect the modified duality relations \( \langle p_\mu p_0^n, x_j \rangle \) between four-momentum \( p_\mu \) in the bicrossproduct basis (4) and SD-twisted or TD-twisted space-time given by relations (37) or (39). We find these twisted duality relations in the case of two dimensional (D=2) space-time applying tensor methods to twisted coproduct. This simplification is convenient because of tedious calculations, however, one can immediately generalize the obtained results to the four-dimensional space-time.

#### 4.1 SD-twisted space-time

In the case of the two dimensional SD-twisted space-time relations (37) take the following form (we assume \( c = 1 \) in order to simplify the notation)

- **space-time**

  \[
  [x_0, x] = \langle f(p_0), x_0 \rangle, \quad \Delta(x_0) = x_0 \otimes 1 + 1 \otimes x_0, \quad \Delta(x) = x \otimes e^{\alpha x_0} + e^{\alpha x_0} \otimes x, \tag{57}
  \]

- **momentum space**

  \[
  [p_0, p] = 0, \quad \Delta(p_0) = p_0 \otimes 1 + 1 \otimes p_0, \quad \Delta(p) = p \otimes 1 + f(p_0) \otimes p \tag{59}
  \]

where we denote \( f(p_0) = \exp(-p_0/\kappa) \),

- **duality relations** are given by (41) for two dimensional case \( (\mu, \nu = 0, 1) \).

One can also describe these duality relations in terms of a nonlinear transformed basis of \( \mathcal{X}_\kappa \)

\[
s(x_0) = x_0, \quad s(x_1) = s(x) = xe^{\alpha x_0}, \tag{61}
\]
and using a trivial coproduct $\Delta_0(x_\mu)$ [42]

$$< p_\mu, s(x_{\nu}) > = ig_{\mu\nu}, \quad g_{\mu\nu} = (1, -1),$$

$$< pq, s > = < p \otimes q, \Delta_0(s) >, \quad < p, ss' > = < \Delta(p), s \otimes s' >,$$

(62)

for $s, s' \in X_\kappa, p, q \in \mathcal{P}_\kappa$.

Taking into account the coproduct relations [58] and [60] we can immediately generalize the relations [58] and obtain

$$< p_\mu^m, 1 > = < 1, x_\mu^m > = \delta_{m0}, \quad m = 0, 1, 2, \ldots.$$  

(63)

In order to find other duality relations we apply the useful relation which expresses the coassociativity of coproduct (see [42])

$$< p_\mu^m, x_\nu^k > = < \Delta^{(k-1)}(p_\mu^m), x_\nu^k > = < \left( \Delta^{(k-1)}(p_\mu) \right)^m, x_\nu^k > =$$

$$= < p_\mu^{\otimes m}, \Delta^{(m-1)}(x_\nu^k) > = < p_\mu^{\otimes m}, \left( \Delta^{(m-1)}(x_\nu) \right)^k >,$$

(64)

where

$$\Delta^{(n)} = (I^{\otimes (n-1)} \otimes \Delta)\Delta^{(n-1)} = (1 \otimes \cdots \otimes 1 \otimes \Delta)\Delta^{(n-1)},$$

(65)

$$x_\nu^{\otimes k} = x_\nu \otimes x_\nu \otimes \cdots \otimes x_\nu, \quad p_\mu^{\otimes m} = p_\mu \otimes p_\mu \otimes \cdots \otimes p_\mu.$$  

(66)

In particular

$$\Delta^{(m-1)}(x) = \sum_{i=1}^{m} x_i^{(m-1)}, \quad x_i^{(m-1)} = (e^{\alpha x_0})^{\otimes (i-1)} \otimes x \otimes (e^{\alpha x_0})^{\otimes (m-i)},$$

(67)

$$\Delta^{(m-1)}(x_0) = \sum_{i=1}^{m} (x_0)_i^{(m-1)}, \quad (x_0)_i^{(m-1)} = I^{\otimes (i-1)} \otimes x_0 \otimes I^{\otimes (m-i)},$$

(68)

$$\Delta^{(k-1)}(p) = \sum_{i=1}^{k} p_i^{(k-1)}, \quad p_i^{(k-1)} = f^{\otimes (i-1)} \otimes p \otimes I^{\otimes (k-i)},$$

(69)

$$\Delta^{(k-1)}(p_0) = \sum_{i=1}^{k} (p_0)_i^{(k-1)}, \quad (p_0)_i^{(k-1)} = I^{\otimes (i-1)} \otimes p_0 \otimes I^{\otimes (k-i)}.$$  

(70)

Using the coproduct formulae [58] for space-time variables, duality relations and relations [60, 68] we obtain

$$< p_0^m, x_0 > = \delta_{n1} < p_0, x_0 >, \quad < p_0^m, x_0 > = 0,$$

(71)

$$< p_0^m, x > = 0, \quad < p_0^m, x > = \delta_{m1} < p, x >.$$  

(72)

For instance

$$< p_0^m, x_0 > = < p_0^{\otimes m}, \Delta^{(m-1)}(x_0) > =$$

$$= \sum_{i=1}^{m} < p \otimes \cdots \otimes p, I^{\otimes (i-1)} \otimes x_0 \otimes I^{\otimes (m-i)} > =$$

$$= m < p, 1 >^{m-1} < p, x_0 > = 0.$$  

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Analogously, using the momentum coproduct (60) and formulae (69)-(70) we derive

\[
\langle p_0, x^l_0 \rangle = \delta_{l1} \langle p_0, x_0 \rangle, \quad \langle p, x^0_0 \rangle = 0, \quad \langle p, x^k \rangle = 0, \quad \langle p, x^k \rangle = \delta_{k1} \langle p, x \rangle.
\]

(73)

(74)

The relations (71)-(72) or (73)-(74) can be generalized to the following form

\[
\langle p^n_0, x^l_0 \rangle = n! \langle p_0, x_0 \rangle^n \delta_{nl}, \quad \langle p^m, x^l_0 \rangle = \delta_{m0} \delta_{0l},
\]

\[
\langle p^m, x^k \rangle = m! \langle p, x >^m \delta_{mk}, \quad \langle p^n_0, x^k \rangle = \delta_{n0} \delta_{k0}.
\]

(75)

(76)

From (70) and the trivial form of coproduct (60) we compute for instance

\[
\langle p^n_0, x^l_0 \rangle = \langle \Delta^{(l-1)}(p^n_0), x^0_0 \rangle =
\]

\[
= \langle \left( \sum_{i=1}^{l} (p_0)^{(i-1)}_i \right) \otimes \cdots \otimes x_0 \rangle =
\]

\[
= n! \langle p^n_0 \otimes \cdots \otimes x_0 \rangle = n! \langle p_0, x_0 \rangle.
\]

(77)

We would like to stress that the duality relations (75)-(76) do not depend on SD-twisting transformation and they have the same form in both, κ-deformed and non-deformed cases.

Let us derive the relation which depends on the twisting parameter. Using (75), (76) and coproduct formula (78) for \( \Delta(x) \) we find

\[
\langle p^n_0 p^m, x \rangle = \langle p^n_0 \otimes p^m, \Delta(x) \rangle = \langle p^n_0 \otimes p^m, x \otimes e^{\alpha x_0} + e^{\alpha x_0} \otimes x \rangle =
\]

\[
= \delta_{m1} \langle p, x \rangle \langle p^n_0, e^{\alpha x_0} \rangle = \delta_{m1} \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} \langle p, x \rangle \langle p^n_0, x^k_0 \rangle =
\]

\[
= \alpha^n \delta_{m1} \langle p, x \rangle \langle p_0, x_0 \rangle^n,
\]

(77)

It is convenient to use instead the power function \( p^n_0 \), its generating function \( g(p_0) = e^{\xi p_0} \).

Then we obtain

\[
\langle p^m e^{\xi p_0}, x \rangle = \delta_{m1} \langle p, x \rangle e^{\alpha \xi \langle p_0, x_0 \rangle},
\]

(78)

or using (69) we find a more general formula

\[
\langle p^m e^{\xi p_0}, x^k \rangle = k! \langle p, x \rangle^k \delta_{mk} e^{\alpha \xi \langle p_0, x_0 \rangle} f(k(k-1)) (\alpha < p_0, x_0 >).
\]

(79)

From (73) we can easy calculate the following duality relation

\[
\langle p^m e^{\xi p_0}, x^l_0 \rangle = \sum_{s=0}^{l} \binom{l}{s} \langle p^m \otimes e^{\xi p_0}, x^{l-s}_0 \otimes x^s_0 \rangle =
\]

\[
= \sum_{s=0}^{l} \binom{l}{s} \langle p^m, x^{l-s}_0 \rangle \langle e^{\xi p_0}, x^s_0 \rangle =
\]

\[
= \delta_{m0} \langle e^{\xi p_0}, x^l_0 \rangle = (\xi < p_0, x_0 >)^l \delta_{m0}.
\]

(80)

and finally we find

\[
\langle p^m e^{\xi p_0}, x^k x^l_0 \rangle = \langle \Delta(p^m e^{\xi p_0}), x^k \otimes x^l_0 \rangle =
\]

\[
= \sum_{s=0}^{m} \binom{m}{s} \langle p^{m-s} f^s(p_0) e^{\xi p_0} \otimes p^s e^{\xi p_0}, x^k \otimes x^l_0 \rangle =
\]

12
In the limit $\alpha \to 0$ of the twisting parameter we obtain the duality relations. Therefore, we see that the twisting map destroys some orthogonal duality relations i.e. roughly speaking twisting changes the orthogonal basis of dual momentum and space-time algebras to a non-orthogonal one.

Because twisted space-time is a non-commuting algebra, one can also consider a polynomial basis with opposite ordering of space and time variables i.e. left-time ordered polynomials. In order to find the duality relations for this case, we would like to stress that space-time commutation relations are related to the momentum coproduct in the bicrossproduct basis and do not depend on the twisting map, therefore we can use these relations to derive duality relations for the opposite ordering. Taking into account and the explicit form of the function $f(p_0)$ we obtain the SD-twisted duality relations

\begin{align}
\langle p^m p_0^n, x^k x^l_0 \rangle &= k! n! \langle p, x \rangle^k \langle p_0, x_0 \rangle^l \delta_{mk} \delta_{nl} (\xi^{\alpha} < p_0, x_0 >) f^{\frac{1}{2} k (k-1)} (\alpha < p_0, x_0 >), \\
\langle p^m e^{\xi p_0}, x^k x^l_0 \rangle &= k! (i)^k \langle p, x \rangle^k \langle p_0, x_0 \rangle^l e^{i k \alpha} \delta_{mk} \delta_{nl} (\xi^{\alpha} < p_0, x_0 >) f^{\frac{1}{2} k (k-1)} (\alpha < p_0, x_0 >).
\end{align}

or equivalently (expanding the generating function in powers of $p_0^n$)

\begin{align}
\langle p^m p_0^n, x^k x^l_0 \rangle &= k! n! \langle p, x \rangle^k \langle p_0, x_0 \rangle^l \delta_{mk} \delta_{nl} (\xi^{\alpha} < p_0, x_0 >) f^{\frac{1}{2} k (k-1)} (\alpha < p_0, x_0 >), \\
\langle p^m e^{\xi p_0}, x^k x^l_0 \rangle &= k! (i)^k \langle p, x \rangle^k \langle p_0, x_0 \rangle^l e^{i k \alpha} \delta_{mk} \delta_{nl} (\xi^{\alpha} < p_0, x_0 >) f^{\frac{1}{2} k (k-1)} (\alpha < p_0, x_0 >).
\end{align}

In the limit $\alpha \to 0$ of the twisting parameter we obtain the duality relations. Therefore, we see that the twisting map destroys some orthogonal duality relations i.e. roughly speaking twisting changes the orthogonal basis of dual momentum and space-time algebras to a non-orthogonal one.

Because twisted space-time is a non-commuting algebra, one can also consider a polynomial basis with opposite ordering of space and time variables i.e. left-time ordered polynomials. In order to find the duality relations for this case, we would like to stress that space-time commutation relations are related to the momentum coproduct in the bicrossproduct basis and do not depend on the twisting map, therefore we can use these relations to derive duality relations for the opposite ordering. Taking into account and the explicit form of the function $f(p_0)$ we obtain the SD-twisted duality relations

\begin{align}
\langle p^m p_0^n, x^k x^l_0 \rangle &= k! n! \langle p, x \rangle^k \langle p_0, x_0 \rangle^l \delta_{mk} \delta_{nl} (\xi^{\alpha} < p_0, x_0 >) f^{\frac{1}{2} k (k-1)} (\alpha < p_0, x_0 >), \\
\langle p^m e^{\xi p_0}, x^k x^l_0 \rangle &= k! (i)^k \langle p, x \rangle^k \langle p_0, x_0 \rangle^l e^{i k \alpha} \delta_{mk} \delta_{nl} (\xi^{\alpha} < p_0, x_0 >) f^{\frac{1}{2} k (k-1)} (\alpha < p_0, x_0 >).
\end{align}

or equivalently

\begin{align}
\langle p^m p_0^n, x^k x^l_0 \rangle &= k! \delta_{km} \delta_{nl} \langle p_0 + i \alpha, x_0 \rangle^l (p_0 + i \alpha)^n, x^l_0 >, \\
\langle p^m e^{\xi p_0}, x^k x^l_0 \rangle &= k! \delta_{km} \delta_{nl} \langle p_0 + i \alpha, x_0 \rangle^l (p_0 + i \alpha)^n, x^l_0 >.
\end{align}

In the limit $\alpha \to 0$ of the twisting parameter we obtain the duality relations. Therefore, we see that the twisting map destroys some orthogonal duality relations i.e. roughly speaking twisting changes the orthogonal basis of dual momentum and space-time algebras to a non-orthogonal one.

Because twisted space-time is a non-commuting algebra, one can also consider a polynomial basis with opposite ordering of space and time variables i.e. left-time ordered polynomials. In order to find the duality relations for this case, we would like to stress that space-time commutation relations are related to the momentum coproduct in the bicrossproduct basis and do not depend on the twisting map, therefore we can use these relations to derive duality relations for the opposite ordering. Taking into account and the explicit form of the function $f(p_0)$ we obtain the SD-twisted duality relations

\begin{align}
\langle p^m p_0^n, x^k x^l_0 \rangle &= k! \delta_{km} \delta_{nl} \langle p_0 + i \alpha, x_0 \rangle^l (p_0 + i \alpha)^n, x^l_0 >, \\
\langle p^m e^{\xi p_0}, x^k x^l_0 \rangle &= k! \delta_{km} \delta_{nl} \langle p_0 + i \alpha, x_0 \rangle^l (p_0 + i \alpha)^n, x^l_0 >.
\end{align}

and a non-vanishing duality relation only for $n \leq l$

\begin{align}
\langle p^m p_0^n, x^k x^l_0 \rangle &= k! \delta_{km} \delta_{nl} \langle p_0 + i \alpha, x_0 \rangle^l (p_0 + i \alpha)^n, x^l_0 >.
\end{align}

The duality relations can also describe in terms of a transformed basis $s(x_\mu)$ as follows

\begin{align}
\langle p^m p_0^n, x^k x^l_0 \rangle &= \langle p^m \otimes p_0^n, \Delta_0 \left(s^k(x) s^l(x_0)\right)\rangle, \\
\langle p^m p_0^n, x^k x^l_0 \rangle &= \langle p^m \otimes p_0^n, \Delta_0 \left(s^l(x_0) s^k(x)\right)\rangle.
\end{align}

### 4.2 TD-twisted space-time

Similarly, in the two-dimensional case the TD-twisted space-time is given by the relations

\begin{align}
[x_0, x] &= \langle f(p_0), x_0 > x, \\
\Delta(x_0) &= x_0 \otimes 1 + 1 \otimes x_0 + \beta x \otimes x, \\
\Delta(x) &= x \otimes 1 + 1 \otimes x,
\end{align}

and additional formulae describing the momentum space and duality relations. Also in this case the relations (see )

\begin{align}
\langle p_0^n, x^l_0 \rangle &= n! \langle p_0, x_0 \rangle^n \delta_{nl}, \\
\langle p^m, x^k \rangle &= m! \langle p, x \rangle^m \delta_{mk},
\end{align}

are valid.
are valid and the remaining ones are changed. It is easy to observe that the form of coproduct \( \Delta(x_0) \) implies a vanishing duality relation for any odd power of the momentum

\[
< p^m, x_0 > = < p^2, x_0 > \delta_{m2} = \beta < p, x >^2 \delta_{m2} , \quad < p^{2k+1}, x_0 > = 0 \iff < \sinh(\xi p), x_0 > = 0 .
\]  

(91)

Let us derive the duality relations for even power of the momentum. Using the generating function we obtain

\[
< e^{\xi p}, x_0 > = < \cosh(\xi p), x_0 > = < \Delta^{(l-1)} \exp(\xi p), x_0^{\otimes l} > = < \exp \left( \xi \Delta^{(l-1)}(p) \right), x_0^{\otimes l} >= < \exp \left( \sum_{i=1}^{l} p_i^{(l-1)} \right), x_0^{\otimes l} > = \prod_{i=1}^{l} < \cosh \left( \xi p_i^{(l-1)} \right), x_0^{\otimes l} > .
\]  

(92)

The non-vanishing duality relations are implied by the square power of the momentum variable (91), therefore we can use the expansion

\[
cosh \left( \xi p_i^{(l-1)} \right) \sim I^{\otimes l} + \frac{1}{2} \xi^2 \left( p_i^{(l-1)} \right)^2 ,
\]  

(93)

and we get

\[
< \cosh(\xi p), x_0 > = \prod_{i=1}^{l} < \left( I^{\otimes l} + \frac{1}{2} \xi^2 \left( p_i^{(l-1)} \right)^2 \right), x_0^{\otimes l} > = \delta_{l0} + \sum_{k=1}^{l} \frac{1}{2^k} \xi^{2k} D_k^l ,
\]  

(94)

where

\[
D_k^l \equiv D^l_k(\kappa, \beta) = \sum_{i_1 \neq i_2 \neq \ldots \neq i_{k-1}}^{l-1} < \left( p_i^{(l-1)} \right)^2 \left( p_i^{(l-1)} \right)^2 \ldots \left( p_i^{(l-1)} \right)^2, x_0^{\otimes l} > ,
\]  

(95)

satisfying

\[
D^l_k(0,0) = D^l_k(\kappa,0) = 0 ,
\]  

(96)

and comparing the appropriate left and right terms in (91) we find nonvanishing relations for \( m \leq l \)

\[
< p^{2m}, x_0 > = \frac{(2m)!}{2^m} D^l_m(\kappa, \beta) .
\]  

(97)

Now, we can derive a more general formula

\[
< e^{\xi p_0} p^m, x_0 > = < e^{\xi p_0} \otimes p^m, \Delta^l(x_0) > = < e^{\xi p_0} \otimes p^m, \Delta^0_0(x_0) > = < e^{\xi p_0} \otimes p^m, (x_0 \otimes 1 + 1 \otimes x_0)^l > = \sum_{s=0}^{l} \binom{l}{s} < e^{\xi p_0} \otimes p^m, x_0^{l-s} \otimes x_0^s > = \sum_{s=0}^{l} \binom{l}{s} < e^{\xi p_0}, x_0^{l-s} > < p^m, x_0^s > ,
\]  

(98)

or expanding we get non-vanishing duality relations for \( n \leq l \)

\[
< p_0^n p^m, x_0 > = \frac{l!}{(l-n)!} < p_0, x_0 >^n < p^m, x_0^{l-n} > .
\]  

(99)
Finally, using this relation andCoprod and coproduct formula we derive the general duality relations for the TD-twisted space-time

\[<p^m p_0^n, x^k x_0^l> = <\Delta^m(p)\Delta^n(p_0), x^k \otimes x_0^l> = \frac{m! n! <p, x>^k <p_0, x_0>^n}{(m-k)! (l-n)!} <p^{m-k}, x_0^{l-n}> , \] (100)

or equivalently we get a nonvanishing duality relations for \(m - k = 2s, s \leq l - n, (s = 0, 1, 2, \ldots)\)

\[<p^m p_0^n, x^k x_0^l> = <\Delta^m(p)\Delta^n(p_0), x^k \otimes x_0^l> = \frac{m! n! <p, x>^k <p_0, x_0>^n (2s)!}{2^s (m-k)! (l-n)!} D_{s-l+n}(\kappa, \beta) . \] (101)

In the limit \(\beta \to 0\) using we obtain the duality relation 12. Therefore we see that TD-twisting appears as an additional term to the conventional duality relations 12. One can also easily find the duality relations for the opposite space-time ordering using formula 13 but they have rather complicated form.

Also in this case one can describe the duality relations 100 in terms of nonlinearly transformed basis 12

\[<p^m p_0^n, x^k x_0^l> = p^m \otimes p_0^n, \Delta_0 \left(\frac{t^k(x) t^l(x_0)}{\delta^l(x_0)}\right) > . \] (102)

### 4.3 Space-time and phase-space; beyond \(\kappa\)-deformed framework

One can give up the assumption of a commutative dual momentum space (required by \(\kappa\)-deformed symmetry) and consider a noncommuting fourmomentum algebra dual to the space-time algebra \(\mathcal{X}_\kappa\) defined by 13-14. In such a case we deal with nonsymmetric space-time coproduct. As an example we shall discuss space-time algebra obtained by the Jordanian twist 10 (see also 20) of two dimensional space-time \(\mathcal{X}_\kappa\). First we notice that the relations 13-14 for the case D=2 describe a Lie algebra \(B(2)\) isomorphic to a Borel subalgebra of \(sl(2)\) therefore, we can apply the Jordanian twisting function (with \(\xi\) as a twisting parameter) in the form 20

\[F_J(\kappa, \xi) = e^{i\kappa x_0 \otimes \sigma_\xi(x)} = e^{i\kappa x_0 \otimes \ln(1+\xi x)} , \] (103)

to the trivial coproduct \(\Delta_0(x_\mu), (\mu = 0, 1, x_1 = x)\)

\[\Delta_J(x_\mu) = F_J(\kappa, \xi) \Delta_0(x_\mu) F_J^{-1}(\kappa, \xi) . \] (104)

This coproduct defines the following two dimensional Jordanian twisted space-time \(\mathcal{X}_\kappa^J(\xi)\)

\[[x_0, x] = -\frac{\bar{\kappa}}{\kappa} x, \quad \epsilon(x_\mu) = 0, \] (105)

\[\Delta_J(x_0) = x_0 \otimes e^{-\sigma_\xi(x)} + 1 \otimes x_0 = \Delta_0(x_0) - \xi x_0 \otimes (1 + \xi x)^{-1} , \] (106)

\[\Delta_J(x) = x \otimes 1 + e^{\sigma_\xi(x)} \otimes x = \Delta_0(x) + \xi x \otimes x , \] (107)

\[S_J(x_0) = -x_0 e^{\sigma_\xi(x)} = -x_0 (1 + \xi x) , \] (108)

\[S_J(x) = -x e^{-\sigma_\xi(x)} = -x (1 + \xi x)^{-1} . \] (109)

From the two dimensional version of duality relations 11-12 one can find a dual momentum algebra \(\mathcal{P}_\kappa^J(\xi)\). Because there is an isomorphism between the Hopf algebras \(\mathcal{X}_\kappa^J(\xi)\) and its
Given by $\sigma_\xi(x) \to p_0/\kappa c$, $x_0 \to p/\kappa c\xi$ therefore we can find the defining relations ($\mu = 0, 1$, $p_1 = p$)

$$ [p_0, p] = i\kappa c\xi \left(1 - e^{-\frac{p_0}{\kappa c}}\right), \quad \epsilon(p_\mu) = 0, \quad (110) $$

$$ \Delta(p_0) = p_0 \otimes 1 + 1 \otimes p_0, \quad S(p_0) = -p_0, \quad (111) $$

$$ \Delta(p) = p \otimes 1 + e^{\frac{p_0}{\kappa c}} \otimes p, \quad S(p) = -e^{\frac{p_0}{\kappa c}} p. \quad (112) $$

However the form of the coproduct is the same as for the $\kappa$-deformed momentum algebra (60), the momentum algebra becomes noncommuting.

This pair of dual algebras we can extend to a Jordanian phase space algebra $\Pi^I_\kappa(\xi)$ by the left cross product construction $\Pi^I_\kappa(\xi) = \mathcal{X}^I_\kappa(\xi) \ltimes \mathcal{P}^I_\kappa(\xi)$ (10) and we find the following commutation relations of the linear basis

$$ [x_0, x] = -\frac{i}{\kappa c} x, \quad [p_0, p] = i\kappa c\xi \left(1 - e^{-\frac{p_0}{\kappa c}}\right), \quad (113) $$

$$ [x, p_0] = 0, \quad [x_0, p] = i \left(\frac{p_0}{\kappa c} - \xi x_0\right), \quad (114) $$

$$ [x_0, p_0] = -i, \quad [x, p] = i (1 + \xi x). \quad (115) $$

In the limit $\xi \to 0$ we obtain the standard $\kappa$-deformed phase space (17).

### 5 Final remarks

It is worthwhile to notice, that one can describe the duality relations between space-time and momentum algebras in terms of linearly transformed momentum basis. This possibility is similar to the description using the functions $s(x_\mu), t(x_\mu)$. We consider the case of SD-twisting in the two-dimensional case. First we notice that the duality relations (85) allow us to define a linear (because of the bilinear form $<,>$) transformation $\Phi_\alpha$ in the momentum algebra $\mathcal{P}_\kappa$ corresponding to the twist operation in the space-time $\mathcal{X}_\kappa(\alpha)$ as follows

$$ < p^m p_0^n, x^k(\alpha)x^l_0(\alpha) > = < \Phi_\alpha(p^m p_0^n), x^k x^l_0 > = < f_{mn}(\alpha), x^k x^l_0 >, \quad (116) $$

where $(x(\alpha), x_0(\alpha))$ are space-time variables (57)-(68) generating the twisted algebra $\mathcal{X}_\kappa(\alpha)$ and $x, x_0 \in \mathcal{X}_\kappa$ (see (13)-(14)) or in explicit form

$$ \Phi_\alpha(p^m p_0^n) = f_{mn}(\alpha) = (p_0 + im\alpha)^n p^m e^{-\frac{im}{\kappa c} m(m-1)}. \quad (117) $$

In particular

$$ \Phi_\alpha(p^m) = f_{m0}(\alpha) = p^m e^{-\frac{m}{\kappa c} m(m-1)}, \quad (118) $$

$$ \Phi_\alpha(p_0^n) = f_{0n}(\alpha) = p_0^n, \quad (119) $$

$$ \Phi_\alpha(1) = 1, \quad (120) $$

due to the action of $\Phi_\alpha$ on the momentum space (generated linearly by $p_0,p$) is trivial because $p_0 = f_{01}(\alpha), p = f_{10}(\alpha)$ and their coproduct is given by (6). The action of $\Phi_\alpha$ on the momentum algebra basis $p^m p_0^n$ allows us to extend this transformation onto an arbitrary polynomial function of momentum $\phi = f_{mn} p^m p_0^n$ in a natural way, by the replacement $p^m p_0^n \to f_{mn}(\alpha)$. Thus, one can consider the dual pair $(\mathcal{X}_\kappa(\alpha), \mathcal{P}_\kappa)$ of the twisted space-time algebra and the $\kappa$-deformed momentum algebra or equivalently the pair of a algebras.
\((X_\kappa, P_\kappa(\alpha))\) of \(\kappa\)-deformed space-time and \(\Phi_\kappa\)-transformed momentum algebra with the same duality relations (110) in both cases. The essential difference in both dual constructions lies in their coalgebra sectors.

Our construction of the SD-twisted phase space algebra \(\Pi_\kappa(\alpha)\) leans on the notion of the cross algebra where the multiplication (49) depends on both coproducts (58) and (60). Therefore, for the second pair of algebras \((X_\kappa, P_\kappa(\alpha))\) we obtain the different phase space algebra although both pairs are equivalent as far as the duality relations are concerned. In derivation of phase space commutation relations we use only the first order duality relations (for instance in the definition of the left action (47)) therefore, in the case of the pair \((X_\kappa, P_\kappa(\alpha))\) we obtain a two-dimensional version of commutation relations (51) for \(\alpha = 0\) i.e. the standard \(\kappa\)-deformed phase space. The same conclusions one can obtain by considering TD-twisting space-time.

Therefore, one can find the linear transformation of the momentum algebra which corresponds the twist operation in the space-time algebra, this construction however does not provide the twisted phase space.

In Section 3 we described two possible twistings (in space and time directions) of the space-time algebra and derived corresponding phase spaces.

The duality relations obtained in Section 4 for two-dimensional space-time and momentum algebras one can immediately extend to the four-dimensional case. They describe explicitly the effect of the twisting operation in the space-time algebra.

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