CONGRUENCES FOR COEFFICIENTS OF MODULAR FUNCTIONS

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Abstract. We examine canonical bases for weakly holomorphic modular forms of weight 0 and level \( p = 2, 3, 5, 7, 13 \) with poles only at the cusp at \( \infty \). We show that many of the Fourier coefficients for elements of these canonical bases are divisible by high powers of \( p \), extending results of the first author and Andersen. Additionally, we prove similar congruences for elements of a canonical basis for the space of modular functions of level 4, and give congruences modulo arbitrary primes for coefficients of such modular functions in levels 1, 2, 3, 4, 5, 7, and 13.

1. Introduction and Statement of Results

A holomorphic modular form of level \( N \) and weight \( k \) is a function \( f(z) \) which is holomorphic on the complex upper half-plane, satisfies the modular equation

\[
f \left( \frac{az + b}{cz + d} \right) = (cz + d)^k f(z)
\]

for all \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \), and is holomorphic at the cusps of \( \Gamma_0(N) \). Here, as usual,

\[
\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.
\]

If \( f(z) \) is meromorphic at the cusps of \( \Gamma_0(N) \), then we say \( f \) is a weakly holomorphic modular form; additionally, if \( f \) is weakly holomorphic of weight zero, we say \( f \) is a level \( N \) modular function. We denote by \( M_k(N) \) the space of holomorphic level \( N \) modular forms and by \( M^!_k(N) \) the space of weakly holomorphic modular forms of level \( N \). As a subspace of \( M^!_k(N) \), we define the space \( M^\#_k(N) \) to be the space of all modular forms of weight \( k \) and level \( N \) which are holomorphic except possibly at the cusp at \( \infty \).

Every modular form has a Fourier expansion \( f(z) = \sum_{n=n_0}^{\infty} a(n)q^n \), where \( q = e^{2\pi iz} \); the coefficients \( a(n) \) often encode arithmetic information and have been widely studied. As an example, the classical \( j \)-invariant \( j(z) = q^{-1} + \sum c(n)q^n \) is a modular function for \( SL_2(\mathbb{Z}) \). In 1949, Lehner proved [14], [15] that its Fourier coefficients \( c(n) \) satisfy the congruence

\[
c(2^a3^b5^c7^d) \equiv 0 \pmod{2^{3a+8}3^{2b+3}5^{c+1}7^d},
\]

showing that many of the coefficients \( c(n) \) are divisible by large powers of small primes. Kolberg [12], [13] and Aas [1] refined Lehner’s work to give stronger congruences for the coefficients \( c(n) \) modulo large powers of \( p \) for \( p \in \{2, 3, 5, 7\} \). In [10], Griffin further extended these results by proving such congruences for every function in a canonical basis for \( M^!_0(1) \).

For higher levels, Lehner showed that similar congruences hold for the coefficients of modular functions in \( M^!_0(p) \) with \( p \in \{2, 3, 5, 7\} \) if the functions have integral Fourier coefficients and the the order of the pole at infinity is bounded appropriately. Andersen and the first
author [3] extended Lehner’s theorem to include all elements of a canonical basis for \( M_0(p) \), proving the following congruences, from which Lehner’s results follow as a corollary.

**Theorem (3, Theorem 2).** Let \( p \in \{2, 3, 5, 7\} \), and let \( f_{0, m}(z) \in M_0(p) \) be the unique weakly holomorphic modular form with Fourier expansion

\[
f_{0, m}(z) = q^{-m} + \sum_{n=1}^{\infty} a_{0}^{(p)}(m, n)q^n.
\]

Suppose that \( m = p^\alpha m' \) and \( n = p^\beta n' \) with \( (m', p) = 1 \) and \( (n', p) = 1 \). Then for \( \beta > \alpha \), we have

\[
\begin{align*}
a_{0}^{(2)}(2^\alpha m', 2^\beta n') &\equiv 0 \pmod{2^{3(\beta-\alpha)+8}} & \text{if } p = 2, \\
a_{0}^{(3)}(3^\alpha m', 3^\beta n') &\equiv 0 \pmod{3^{2(\beta-\alpha)+3}} & \text{if } p = 3, \\
a_{0}^{(5)}(5^\alpha m', 5^\beta n') &\equiv 0 \pmod{5^{(\beta-\alpha)+1}} & \text{if } p = 5, \\
a_{0}^{(7)}(7^\alpha m', 7^\beta n') &\equiv 0 \pmod{7^{(\beta-\alpha)}} & \text{if } p = 7.
\end{align*}
\]

Since this theorem gives congruences only for \( \beta > \alpha \), it is a natural question whether similar congruences hold for the other coefficients. A quick glance at \( f_{0, 4}^{(2)}(z) \) shows that

\[
f_{0, 4}^{(2)}(z) = q^{-4} - 196608q + 21491712q^2 - 864288768q^3 + \cdots.
\]

For these first few coefficients, \( \alpha = 2 \) and \( \beta < \alpha \), so the hypotheses of the theorem are not satisfied. Yet

\[
\begin{align*}
196608 &= 2^{16} \cdot 3, \\
21491712 &= 2^{12} \cdot 3^2 \cdot 11 \cdot 53, \\
864288768 &= 2^{18} \cdot 3 \cdot 7 \cdot 157.
\end{align*}
\]

From this and other examples, it appears that when \( \alpha > \beta \), the corresponding coefficients are also divisible by high powers of \( p \). The main result of this paper confirms this observation.

**Theorem 1.** Let \( p \in \{2, 3, 5, 7, 13\} \) and let \( f_{0, m}(z) = q^{-m} + \sum_{n=1}^{\infty} a_{0}^{(p)}(m, n)q^n \) be a weakly holomorphic modular form in \( M_0(p) \). Let \( m = p^\alpha m' \) and \( n = p^\beta n' \) with \( m', n' \) not divisible by \( p \). Then for \( \alpha > \beta \), we have

\[
\begin{align*}
a_{0}^{(2)}(2^\alpha m', 2^\beta n') &\equiv 0 \pmod{2^{4(\alpha-\beta)+8}} & \text{if } p = 2, \\
a_{0}^{(3)}(3^\alpha m', 3^\beta n') &\equiv 0 \pmod{3^{3(\alpha-\beta)+3}} & \text{if } p = 3, \\
a_{0}^{(5)}(5^\alpha m', 5^\beta n') &\equiv 0 \pmod{5^{2(\alpha-\beta)+1}} & \text{if } p = 5, \\
a_{0}^{(7)}(7^\alpha m', 7^\beta n') &\equiv 0 \pmod{7^{2(\alpha-\beta)}} & \text{if } p = 7, \\
a_{0}^{(13)}(13^\alpha m', 13^\beta n') &\equiv 0 \pmod{13^{\alpha-\beta}} & \text{if } p = 13.
\end{align*}
\]

We remark that Theorem 1 includes a congruence for \( p = 13 \), while, as noted in [3], for \( \beta > \alpha \) the analogous result is a trivial congruence modulo \( 13^{0(\beta-\alpha)} \). Additionally, we note that the theorem makes no divisibility predictions when \( \alpha = \beta \).

When such a canonical basis is defined for \( M_0^2(4) \), similar congruences hold, giving the following theorem.
Theorem 2. Let \( f_{0,m}^{(4)}(z) = q^{-m} + \sum_{n=1}^{\infty} a_0^{(4)}(m, n) q^n \) be a weakly holomorphic modular form in \( M_0^4(4) \). Let \( m = 2^\alpha m' \) and \( n = 2^\beta n' \) with \( m', n' \) odd. Then
\[
a_0^{(4)}(2^\alpha m', 2^\beta n') \equiv \begin{cases} 0 \pmod{2^{(\alpha-\beta)+8}} & \text{if } \alpha > \beta, \\ 0 \pmod{2^{(\beta-\alpha)+8}} & \text{if } \beta > \alpha. \end{cases}
\]
This result follows from a natural relationship between the canonical bases for \( M_0^4(4) \) and \( M_0^4(2) \).

From these theorems, it is clear that many of the coefficients of these canonical bases are divisible by high powers of primes which divide the level. It is a natural question whether congruences exist modulo powers of primes not dividing the level. For example, consider the following modular form of level 7:
\[
f_{0,5}^{(7)}(z) = q^{-5} - 50q - 180q^2 + 210q^3 + 860q^4 - 1428q^5 + 8400q^6 - 3675q^7 - \cdots.
\]
It is easy to see that each of the coefficients except that of \( q^5 \) is divisible by 5. We prove the following theorem, which holds for any prime \( p \) not dividing the level.

Theorem 3. Let \( f_{0,m}^{(N)}(z) = q^{-m} + \sum_{n=1}^{\infty} a_0^{(N)}(m, n) \in M_0^N(N) \), where \( N \in \{1, 2, 3, 4, 5, 7, 13\} \).
Let \( p \) be a prime not dividing \( N \), and let \( r \in \mathbb{Z}^+ \). If \( p \nmid n \), we have
\[ p^r | a_0^{(N)}(mp^r, n). \]
This result and its proof are analogous to similar divisibility results for the weights \( k \in \{4, 6, 8, 10, 14\} \) appearing in [7].

The rest of the paper now proceeds as follows: in Section 2, we explicitly construct canonical bases for \( M_0^2(p) \) and present some necessary background; Section 3 proves Theorem 1; Section 4 describes the space \( M_0^2(4) \) and proves Theorem 2; Section 5 contains the proof of Theorem 3.

2. Background and Canonical Bases

For \( p \in \{2, 3, 5, 7, 13\} \), the congruence subgroup \( \Gamma_0(p) \) has genus zero, and the space \( M_0^2(p) \) is generated by powers of a single modular function known as a Hauptmodul. A convenient Hauptmodul for \( \Gamma_0(p) \) is given by
\[
\psi^{(p)}(z) := \left( \frac{\eta(z)}{\eta(pz)} \right)^{24} = q^{-1} + O(1),
\]
where \( \eta(z) \) is the Dedekind eta function. The modular form \( \psi^{(p)}(z) \) is a modular function on \( \Gamma_0(p) \) with a simple pole at \( \infty \) and a simple zero at 0; its Fourier coefficients are integers.

We define a canonical basis \( \{ f_{0,m}^{(p)}(z) \}_{m=0}^{\infty} \) for the space \( M_0^2(p) \) by letting \( f_{0,m}^{(p)}(z) \) be the unique modular form in \( M_0^2(p) \) with Fourier expansion beginning \( q^{-m} + O(q) \). It is straightforward to see that \( f_{0,m}^{(p)}(z) \) can be written as \( F(\psi^{(p)}(z)) \), where \( F(x) \) is a polynomial in \( x \) of degree \( m \) with integer coefficients. We write
\[
f_{0,m}^{(p)}(z) = q^{-m} + \sum_{n=1}^{\infty} a_0(m, n) q^n,
\]
so that \( a_q(m, n) \) is the Fourier coefficient of \( q^n \) in the \( m \)th basis element.

We note that this is an extension of the basis given in [6] for \( M_0(p) \), and that similar bases can be defined for \( M_k^2(p) \) for any even weight \( k \). In all weights, these bases consist of forms \( f^{(p)}_k(z) \) whose first few Fourier coefficients are 1, 0, ..., 0, with the number of zeros as large as possible. Thus, we have \( f^{(p)}_{k,m}(z) = q^{-m} + \sum_{n \geq n_0} a^{(p)}_{k}(m, n) q^n. \)

A similar construction gives a basis \( \{g^{(p)}_{k,m}(z)\} \) for the subspace of \( M_k^2(p) \) consisting of forms which vanish at all cusps except possibly at \( \infty \). These \( g^{(p)}_{k,m}(z) \) have Fourier expansion

\[
g^{(p)}_{k,m}(z) = q^{-m} + \sum_{n \geq n_0} b^{(p)}_{k}(m, n) q^n.
\]

We note that in weight 0, the only difference between these bases is the constant term, so we have

\[
a^{(p)}_{0}(m, n) = b^{(p)}_{0}(m, n) \quad \text{if } n \neq 0.
\]

We additionally recall that for a prime \( p \), the \( U_p \) and \( V_p \) operators (see [4]) are defined as follows: for a modular form \( f(z) = \sum_{n=0}^{\infty} a(n) q^n \in M_k^1(N) \), we have

\[
U_p(f(z)) = \sum_{n=0}^{\infty} a(pn) q^n \in M_k^1(pN),
\]

\[
V_p(f(z)) = \sum_{n=0}^{\infty} a(n) p^n q^n \in M_k^1(pN).
\]

If \( p|N \), then we actually have \( U_p(f(z)) \in M_k^1(N) \), while if \( p^2|N \), then \( U_p(f(z)) \in M_k^1(N/p) \). Additionally, for a form \( f(z) \in M_k^1(N) \) and a prime \( p \nmid N \), the action of the standard Hecke operator \( T_p \) is given by

\[
(2.2) \quad T_p(f(z)) = U_p(f(z)) + p^{k-1}V_p(f(z)) \in M_k^1(N).
\]

We also make use of the Ramanujan theta operator [2], which acts on a modular form \( f(z) \) by the rule

\[
\theta f(z) = q \frac{d}{dq} f(z),
\]

so that

\[
\theta \left( \sum a(n) q^n \right) = \sum na(n) q^n.
\]

The theta operator maps a modular form of level \( k \) to a quasi-modular form of weight \( k + 2 \); it preserves holomorphicity but not modularity.

3. Proof of Theorem [1]

We begin the proof of Theorem [1] with the following Zagier-type duality result for the Fourier coefficients of the basis elements \( f^{(p)}_{k,m}(z) \) and \( g^{(p)}_{k,m}(z) \). This was proven in [9] for levels 2 and 3; it follows from work of El-Guindy [8], which allows for easy extension to levels 5, 7, and 13 as well.

Lemma. Let \( k \) be an even integer and let \( p \in \{2, 3, 5, 7, 13\} \). For all integers \( m \) and \( n \), the equality

\[
a^{(p)}_{k}(m, n) = -b^{(p)}_{2-k}(n, m)
\]
Theorem. If \( f(z) \in M_1^0(N) \), then \( \theta(f) \in M_2^0(N) \).

This follows from Bol's identity and the fact that \( M_1^0(N) \) is a subspace of the space of harmonic weak Maass forms.

Applying this theorem to the basis elements \( f_{0,m}(z) \), we have the following corollary.

Corollary 1. We have \( \theta(f_{0,m}) = -m \cdot f_{2,m} \).

Proof. Applying the \( \theta \)-operator to \( f_{0,m}(z) = q^{-m} + O(q) \) gives a modular form of weight 2 and level \( p \) with Fourier expansion beginning \(-mq^{-m} + O(q)\). As the \( f_{2,n} \) with \( n \geq 0 \) form a basis for \( M_2^0(p) \), this must be \(-mf_{2,m} \).

Looking at the action of the \( \theta \)-operator on the Fourier coefficients of these functions, it follows that \( a_0(m,n) = -m \cdot a_2(m,n) \).

We now prove the main theorem, which we restate here for convenience.

Theorem. Let \( p \in \{2,3,5,7,13\} \) and let \( f_{0,m}(z) = q^{-m} + \sum_{n=1}^{\infty} a_0^{(p)}(m,n)q^n \in M_0^0(\Gamma_0(p)) \) be an element of the basis described previously with \( m = p^\alpha m' \) and \( n = p^\beta n' \) with \( m', n' \) not divisible by \( p \). Then for \( \alpha > \beta \), we have

\[
\begin{align*}
a_0^{(2)}(2^\alpha m', 2^\beta n') &\equiv 0 \pmod{2^{4(\alpha-\beta)+8}} \quad \text{if } p = 2, \\
a_0^{(3)}(3^\alpha m', 3^\beta n') &\equiv 0 \pmod{3^{3(\alpha-\beta)+3}} \quad \text{if } p = 3, \\
a_0^{(5)}(5^\alpha m', 5^\beta n') &\equiv 0 \pmod{5^{2(\alpha-\beta)+1}} \quad \text{if } p = 5, \\
a_0^{(7)}(7^\alpha m', 7^\beta n') &\equiv 0 \pmod{7^{2(\alpha-\beta)}} \quad \text{if } p = 7, \\
a_0^{(13)}(13^\alpha m', 13^\beta n') &\equiv 0 \pmod{13^{\alpha-\beta}} \quad \text{if } p = 13.
\end{align*}
\]

Proof. Let \( m = p^\alpha m' \) and \( n = p^\beta n' \) with \( m', n' \) not divisible by \( p \). Let \( a_0^{(p)}(m,n) = a_0^{(p)}(p^\alpha m', p^\beta n') \) be any coefficient where \( \alpha > \beta \). Looking at the coefficient of \( q^n \) in Corollary 1, we have \( ma_0^{(p)}(m,n) = -ma_2^{(p)}(m,n) \). On the other hand, using Zagier duality and equation (2.1), we have \( a_2^{(p)}(m,n) = -b_0^{(p)}(n,m) = -a_0^{(p)}(n,m) \). Thus, we have \( a_0^{(p)}(m,n) = -\frac{m}{n}a_0^{(p)}(n,m) = \frac{\nu}{p^\alpha}a_0^{(p)}(n,m) \). Note that all coefficients are integers here.

Recall that \( a_0^{(p)}(n,m) \) represents the coefficient of \( q^n \) in the weight zero basis element starting with \( q^{-m} \). Since \( \alpha > \beta \), a higher power of \( p \) divides the exponent than divides the order of the pole, and we can apply Theorem 2 of [3]. For instance, for \( p = 2 \) we find that \( 2^{2(\alpha-\beta)+8} | a_0^{(2)}(2^\alpha m', 2^\beta n') \), and we multiply this by an extra factor of \( 2^{\alpha-\beta} \). Therefore, we have \( 2^{4(\alpha-\beta)+8} | a_0^{(2)}(m,n) \), as desired. The argument for \( p = 3, 5, 7, 13 \) is similar. \( \square \)
To illustrate these results, the first four basis elements for $M_0^2(2)$ and $M_2^2(2)$ are given below.

$$ f_{0,1}^{(2)}(z) = q^{-1} + 276q - 2048q^2 + 11202q^3 - 49152q^4 + \cdots, $$

$$ f_{0,2}^{(2)}(z) = q^{-2} - 4096q + 98580q^2 - 1228800q^3 + 10745856q^4 - \cdots, $$

$$ f_{0,3}^{(2)}(z) = q^{-3} + 33606q - 1843200q^2 + 43434816q^3 - 648216576q^4 + \cdots, $$

$$ f_{0,4}^{(2)}(z) = q^{-4} - 196608q + 21491712q^2 - 864288768q^3 + 20246003988q^4 - \cdots. $$

$$ f_{2,1}^{(2)}(z) = q^{-1} - 276q + 4096q^2 - 33606q^3 + 196608q^4 - \cdots, $$

$$ f_{2,2}^{(2)}(z) = q^{-2} + 2048q - 98580q^2 + 1843200q^3 - 21491712q^4 + \cdots, $$

$$ f_{2,3}^{(2)}(z) = q^{-3} - 11202q + 1228800q^2 - 43434816q^3 + 864288768q^4 - \cdots, $$

$$ f_{2,4}^{(2)}(z) = q^{-4} + 49152q - 10745856q^2 + 648216576q^3 - 20246003988q^4 + \cdots. $$

By comparing rows of coefficients in weight 0 to columns of coefficients in weight 2, the duality is clear; for example, $a_0^{(2)}(1, 2) = b_0^{(2)}(1, 2) = -a_2^{(2)}(2, 1)$. The effect of the theta operator is also clear if the coefficients are factored; for example,

$$ f_{0,3} = q^{-3} + 33606q - 1843200q^2 + 43434816q^3 - 648216576q^4 + \cdots, $$

$$ = q^{-3} + 3 \cdot 11202q - \frac{3}{2} \cdot 1228800q^2 + 1 \cdot 43434816q^3 - \frac{3}{4} \cdot 864288768q^4 + \cdots, $$

$$ f_{2,3} = q^{-3} - 11202q + 1228800q^2 - 43434816q^3 + 864288768q^4 - \cdots. $$

4. Level 4

The group $\Gamma_0(4)$ has genus zero and 3 cusps, which can be taken to be at 0, at $\frac{1}{2}$, and at $\infty$. We construct a similar canonical basis for $M_0^2(4)$ by letting $f_{0,m}^{(4)}(z)$, for all $m \geq 0$, be the unique modular form in this space with Fourier expansion

$$ f_{0,m}^{(4)}(z) = q^{-m} + \sum_{n=1}^{\infty} a_0^{(4)}(m, n) q^n. $$

Similarly, we define a basis for the subspace of forms in $M_2^2(4)$ which vanish both at 0 and at $\frac{1}{2}$ by defining

$$ g_{2,m}^{(4)}(z) = q^{-m} + \sum_{n=0}^{\infty} b_2^{(4)}(m, n) q^n $$

for all $m \geq 1$. Given this notation, the first author and Haddock [11] proved the following results.

**Theorem** ([11], Theorem 2). For all integers $m, n$, we have the duality of coefficients

$$ a_0^{(4)}(m, n) = -b_2^{(4)}(m, n). $$

**Theorem** ([11], Theorem 3). If $n \not\equiv m \pmod{2}$, then $a_0^{(4)}(m, n) = 0$.

We now describe the action of the $U_2$ operator on these basis elements.
Theorem 4. For any nonnegative integer \( m \), we have \( U_2(f^{(4)}_{0,2m}(z)) = f^{(2)}_{0,m}(z) \) and \( U_2(f^{(4)}_{0,2m+1}(z)) = 0 \).

Proof. Let \( f^{(4)}_{0,2m}(z) \in M^2_0(\Gamma_0(4)) \), and note that \( V_2U_2 \) acts as the identity on \( f^{(4)}_{0,2m}(z) \) (see section 3 of \( \ddagger \)). Since \( V_2(f^{(2)}_{0,m}(z)) \) is also a modular form in \( M^2_0(4) \) with principal part \( q^{-2m} \), the difference

\[
V_2 \left( U_2(f^{(4)}_{0,2m}(z)) - f^{(2)}_{0,m}(z) \right)
\]

is a modular form in \( M^2_k(4) \) which vanishes at \( \infty \) and must therefore be zero. The first result follows.

To see that \( U_2(f^{(4)}_{k,2m+1}(z)) = 0 \), note that the order of the pole is odd, so all of the nonzero exponents are in the Fourier expansion are odd. Applying \( U_2 \), all of the terms vanish. \( \square \)

We now prove Theorem \( \ddagger \).

Theorem. Let \( f^{(4)}_{0,m}(z) = q^{-m} + \sum_{n=1}^{\infty} a^{(4)}_0(m,n)q^n \in M^2_0(4) \) be an element of the canonical basis. Suppose that \( m = 2^s m' \) and \( n = 2^t n' \) with \( m' \) and \( n' \) odd. Then for \( \alpha \neq \beta \), we have

\[
a^{(4)}_0(2^{\alpha} m', 2^{\beta} n') \equiv 0 \pmod{2^{4(\alpha-\beta)+8}} \quad \text{if } \alpha > \beta,
\]

\[
a^{(4)}_0(2^{\alpha} m', 2^{\beta} n') \equiv 0 \pmod{2^{3(\beta-\alpha)+8}} \quad \text{if } \beta > \alpha.
\]

Proof. Let \( f^{(4)}_{0,m}(z) \in M^2_0(4) \). By Theorem \( \ddagger \) we know that \( f^{(4)}_{0,2m}(z)|U_2 = f^{(2)}_{0,m}(z) \). Looking at coefficients, we find that

\[
a^{(4)}_0(2^{\alpha} m', 2^{\beta} n') = a^{(2)}_0(2^{\alpha-1} m', 2^{\beta-1} n'),
\]

for any odd \( m', n' \) and any \( \alpha, \beta \geq 1 \). By Theorem \( \ddagger \) we know that for \( \alpha > \beta \), we have the congruence

\[
a^{(2)}_0(2^{\alpha-1} m', 2^{\beta-1} n') \equiv 0 \pmod{2^{4(\alpha-\beta)+8}},
\]

giving the first result. Similarly, when \( \beta > \alpha \), we know by Theorem 2 in \( \ddagger \) that

\[
a^{(2)}_0(2^{\alpha-1} m', 2^{\beta-1} n') \equiv 0 \pmod{2^{3(\beta-\alpha)+8}},
\]

giving the second part. \( \square \)

We note that this theorem applies only when \( m \) is even, since when \( m \) is odd, all of the exponents appearing in \( f^{(4)}_{0,m}(z) \) are also odd.

5. Arbitrary Primes

Theorem \( \ddagger \) will follow from the following result.

Lemma 1. Let \( N \in \{1, 2, 3, 4, 5, 7, 13\} \) and let \( p \) be a prime not dividing \( N \). Let \( f^{(N)}_{0,m}(z) = q^{-m} + \sum_{n=0}^{\infty} a^{(N)}_0(m,n)q^n \in M^2_k(N) \) be a basis element as before. Then for any positive integer \( r \) we have

\[
p^r \left( a_0(m, np^r) - a_0 \left( m, np^{r-1} \right) \right) = a_0(mp^r, n) - a_0 \left( mp^{r-1}, \frac{n}{p} \right).
\]

(5.1)
Proof. We proceed as in Lemma 1 of [7]. Applying the $T_p$ operator to the basis element $f_{0,m}(z)$ and using (2.2), we find that the coefficient of $q^n$ in $T_p(f_{0,m}(z))$ is $a_0^{(N)}(m,p) + p^{-1}a_0^{(N)}(m,n/p)$. Additionally, applying (2.2) to the $q^{-m}$ term allows us to conclude that $T_p(f_{0,m}(z)) = p^{-1}q^{-mp} + q^{-m/p} + O(q)$, where the second term is omitted if $p \nmid m$. A straightforward calculation similar to that in section 3 of [11] shows that the coefficient of $q^n$ in $T_p(f_{0,m}(z))$ is also given by $p^{-1}a_0^{(N)}(mp,n) + a_0^{(N)}(n/p,n)$. Combining these two expressions for the coefficient of $q^n$ in $T_p(f_{0,m}(z))$, we find that

\begin{equation}
(5.2) \quad a_0^{(N)}(m,np) = p^{-1} \left( a_0^{(N)}(mp,n) - a_0^{(N)} \left( m \frac{n}{p} \right) \right) + a_0^{(N)} \left( \frac{m}{p},n \right).
\end{equation}

Note that for $1 \leq i \leq r - 1$, replacing $m$ with $mp^i$ and $n$ with $p^{r-i-1}n$ in (5.2) gives

\begin{equation}
(5.3) \quad p^{-i}(a_0^{(N)}(mp^i,np^{r-i}) - a_0^{(N)}(mp^{i-1},np^{r-i-1})) = p^{-i}(a_0^{(N)}(mp^{i+1},np^{r-i-1}) - a_0^{(N)}(mp^i,np^{r-i-2})).
\end{equation}

We now replace $n$ with $np^{r-1}$ in equation (5.2) to obtain

\begin{equation}
(5.4) \quad a_0^{(N)}(m,np^r) = p^{-1}(a_0^{(N)}(mp,np^{r-1}) - a_0^{(N)}(m,np^{r-2})) + a_0^{(N)} \left( \frac{m}{p},np^{r-1} \right),
\end{equation}

and use (5.3) a total of $(r - 1)$ times to obtain

\begin{equation}
(5.5) \quad a_0^{(N)}(m,np^r) = p^{-r}(a_0^{(N)}(mp^r,n) - a_0^{(N)} \left( mp^{r-1},\frac{n}{p} \right)) + a_0^{(N)} \left( \frac{m}{p},np^{r-1} \right).
\end{equation}

Multiplying by $p^r$ and rearranging proves the lemma. \hfill \Box

We remark that this lemma relies only on the existence of the canonical basis and the fact that the $T_p$ operator preserves the space $M_k^f(N)$.

Theorem 3 now follows from this lemma, noting that if $p \nmid n$, then $a_0^{(N)}(mp^{r-1},n/p) = 0$. 

References

1. Hans-Fredrik Aas, Congruences for the coefficients of the modular invariant $j(\tau)$, Math. Scand. 14 (1964), 185–192. MR 0179137 (31 #3388)
2. Scott Ahlgren, The theta-operator and the divisors of modular forms on genus zero subgroups, Math. Res. Lett. 10 (2003), no. 5-6, 787–798. MR 2024734 (2004m:11059)
3. Nickolas Andersen and Paul Jenkins, Divisibility properties of coefficients of level $p$ modular functions for genus zero primes, Proc. Amer. Math. Soc. 141 (2013), no. 1, 41–53. MR 2988709
4. A. O. L. Atkin and J. Lehner, Hecke operators on $\Gamma_0(m)$, Math. Ann. 185 (1970), 134–160. MR 0268123 (42 #3022)
5. Jan H. Bruinier, Ken Ono, and Robert C. Rhoades, Differential operators for harmonic weak Maass forms and the vanishing of Hecke eigenvalues, Math. Ann. 342 (2008), no. 3, 673–693. MR 2430995 (2009f:11046)
6. Darrin Doud and Paul Jenkins, $p$-adic properties of coefficients of weakly holomorphic modular forms, Int. Math. Res. Not. IMRN (2010), no. 16, 3184–3206. MR 2673723 (2011j:11081)
7. W. Duke and Paul Jenkins, *On the zeros and coefficients of certain weakly holomorphic modular forms*, Pure Appl. Math. Q. 4 (2008), no. 4, Special Issue: In honor of Jean-Pierre Serre. Part 1, 1327–1340. MR 2441704 (2010a:11068)

8. Ahmad El-Guindy, *Fourier expansions with modular form coefficients*, Int. J. Number Theory 5 (2009), no. 8, 1433–1446. MR 2582984 (2011i:11066)

9. Sharon Garthwaite and Paul Jenkins, *Zeros of weakly holomorphic modular forms of levels 2 and 3*, Math. Res. Lett. 20 (2013), no. 4, 657–674.

10. Michael Griffin, *Divisibility properties of coefficients of weight 0 weakly holomorphic modular forms*, Int. J. Number Theory 7 (2011), no. 4, 933–941. MR 2812644 (2012i:11048)

11. Andrew Haddock and Paul Jenkins, *Zeros of weakly holomorphic modular forms of level 4*, Int. J. Number Theory 10 (2014), no. 2, 455–470.

12. O. Kolberg, *The coefficients of j(τ) modulo powers of 3*, Arbok Univ. Bergen Mat.-Natur. Ser. 1962 (1962), no. 16, 7. MR 0158061 (28 #1288)

13. **Further congruence properties of the Fourier coefficients of the modular invariant j(τ)**, Math. Scand. 10 (1962), 173–181. MR 0143735 (26 #1287)

14. Joseph Lehner, *Divisibility properties of the Fourier coefficients of the modular invariant j(τ)*, Amer. J. Math. 71 (1949), 136–148. MR 0027801 (10,357a)

15. **Further congruence properties of the Fourier coefficients of the modular invariant j(τ)**, Amer. J. Math. 71 (1949), 373–386. MR 0027802 (10,357b)