Blackbody radiation in a nonextensive scenario

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Abstract

An exact analysis of the N-dimensional blackbody radiation process in a nonextensive à la Tsallis scenario is performed for values of the nonextensive’s index in the range \(0 < q < 1\). The recently advanced “Optimal Lagrange Multipliers” (OLM) technique has been employed. The results are consistent with those of the extensive, \(q = 1\) case. The generalization of the celebrated laws of Planck, Stefan-Boltzmann, and Wien are investigated.

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Planck’s blackbody radiation studies constitute one of the milestones in the history of Physics. Planck’s law is satisfactorily accounted by recourse of Bose-Einstein statistics. The possible existence of small deviations from this law in the cosmic blackbody radiation has recently been discussed. They would have arisen, at the time of the matter-radiation decoupling as a consequence of the fact that long-range interactions seem to be associated to a nonextensive scenario. The concomitant thermostatistical treatment is by now recognized as a new paradigm for statistical mechanical considerations. It revolves around the concept of Tsallis’ information measure $S_q$, a generalization of Shannon’s one, that depends upon a real index $q$ and becomes identical to Shannon’s measure for the particular value $q = 1$.

The blackbody radiation problem was discussed within the Tsallis’ nonextensive framework in. It was there conjectured that the deviations from Planck’s law detected in the pertinent data could be attributed to a long-range gravitational influence. The study reported in employs the so-called Curado-Tsallis unnormalized expectation values in the limit $q \to 1$. Afterwards, Lenzi advanced an exact treatment of the problem, also within the Curado-Tsallis framework.

Nowadays, it is believed that this framework has been superseded by the so-called normalized one, advanced in, which seems to exhibit important advantages. This normalized treatment, in turn, has been considerably improved by the so-called “Optimal Lagrange Multipliers” (OLM) approach. It is then natural to revisit the problem from such a new context.

In such a vein we shall consider blackbody radiation in equilibrium within an enclosure of volume $V$ with the goal of ascertaining the possible $q$-dependence of the Planck spectrum, Steffan-Boltzmann’s law and Wien’s one. The OLM treatment seems to be indicated in view of the findings of regarding the particular nature of the Lagrange multiplier associated to the temperature.
The paper is organized as follows: In Sect. II we present a brief OLM primer, while in Sect. III we obtain the partition function, internal energy and energy density of our problem. In Sect. IV first-order corrections in \(1 - q\) are discussed and, finally, some conclusions are drawn in Sect. V.

II. MAIN RESULTS OF THE OLM FORMALISM

For a most general quantal treatment, in a basis-independent way, consideration is required of the density operator \(\hat{\rho}\) that maximizes Tsallis’ entropy

\[
\frac{S_q}{k} = 1 - \frac{Tr(\hat{\rho}^q)}{q - 1}, \tag{1}
\]

subject to the \(M\) generalized expectation values \(\langle \hat{O}_j \rangle\), where \(\hat{O}_j\) \((j = 1, \ldots, M)\) denote the relevant observables.

Tsallis’ normalized probability distribution \([13]\), is obtained by following the well known MaxEnt route \([18]\). Instead of effecting the variational treatment of Tsallis-Mendes-Plastino (TMP) \([13]\), we pursue the alternative path developed in \([15]\). One maximizes Tsallis’ generalized entropy \((1)\) \([4,5,19]\) subject to the constraints \([4,15]\)

\[
Tr(\hat{\rho}) = 1 \tag{2}
\]

\[
Tr \left[ \hat{\rho}^q \left( \hat{O}_j - \langle \hat{O}_j \rangle_q \right) \right] = 0, \tag{3}
\]

whose generalized expectation values \([13]\)

\[
\langle \hat{O}_j \rangle_q = \frac{Tr(\hat{\rho}^q \hat{O}_j)}{Tr(\hat{\rho}^q)}, \tag{4}
\]

are (assumedly) a priori known. The resulting density operator reads \([15]\)

\[
\hat{\rho} = Z_q^{-1} \left[ 1 - (1 - q) \sum_j \lambda_j \left( \hat{O}_j - \langle \hat{O}_j \rangle_q \right) \right]^{\frac{1}{1-q}}, \tag{5}
\]

where \(\{\lambda_j\}\) stands for the Optimal Lagrange Multipliers’ set and \(Z_q\) is the partition function

\[
\bar{Z}_q = Tr \left[ 1 - (1 - q) \sum_j \lambda_j \left( \hat{O}_j - \langle \hat{O}_j \rangle_q \right) \right]^{\frac{1}{1-q}}. \tag{6}
\]
It is shown in [13] that

$$\text{Tr}(\hat{\rho}^q) = \bar{Z}_q^{1-q},$$

(7)

and that Tsallis' entropy can be cast as

$$S_q = k \ln_q \bar{Z}_q,$$

(8)

with $\ln_q \bar{Z}_q = (1 - \bar{Z}_q^{1-q})/(q - 1)$. These results coincide with those of TMP [13] in their normalized treatment. If, following [13], we define now

$$\ln_q Z_q = \ln_q \bar{Z}_q - \bar{Z}_q^{1-q} \sum_j \lambda_j \left< \hat{O}_j \right>_q,$$

(9)

we are straightforwardly led to [15]

$$\frac{\partial}{\partial \left< \hat{O}_j \right>_q} \left( \frac{S_q}{k} \right) = \bar{Z}_q^{1-q} \lambda_j$$

(10)

$$\frac{\partial}{\partial \lambda_j} (\ln_q Z_q) = -\bar{Z}_q^{1-q} \left< \hat{O}_j \right>_q.$$

(11)

Equations (10) and (11) are modified Information Theory relations like those used to build up an à la Jaynes [18], Statistical Mechanics. The basic Legendre-structure relations can be recovered in the limit $q \to 1$.

III. BLACKBODY RADIATION

A. The standard situation $q = 1$

In order to analyze the blackbody radiation in the generalized statistical context we first consider the standard situation, $q = 1$. We look for the equilibrium properties of the blackbody electromagnetic radiation. The appropriate thermodynamical variables are the volume $V$ and the temperature $T$ [20,21].

The Hamiltonian of the electromagnetic field, in which there are $n_{k,\epsilon}$ photons of momentum $k$ and polarization $\epsilon$, is given by
\[ \hat{H} = \sum_{k,\epsilon} \hbar \omega \hat{n}_{k,\epsilon}, \]  

(12)

where the frequency is \( \omega = c |k| \) (\( c \) is light’s speed) and \( n_{k,\epsilon} = 0, 1, 2, \ldots \), with no restrictions on \( \{n_{k,\epsilon}\} \). The partition function reads

\[ Z_1 = Tr \left( e^{-\beta \hat{H}} \right). \]  

(13)

For a macroscopic volume we define a density of states in the \( n \)-dimensional space \( g_n(\omega) = A_n \omega^{n-1} \) with

\[ A_n = \frac{2\tau_n V}{(4\pi c^2)^{n/2} \Gamma(n/2)}, \]  

(14)

where \( \tau_n = n - 1 \) is the number of linear-independent polarizations. Eq. (13) can be written as

\[ Z_1 = \exp \left\{ \int_0^\infty d\omega g_n(\omega) \ln \left[ 1 - \exp \left( -\beta \hbar \omega \right) \right] \right\} = e^{\xi_n}, \]  

(15)

where

\[ \xi_n = \frac{I_n A_n}{(\hbar \beta)^n}, \]  

(16)

with

\[ I_n = -\int_0^\infty dx x^{n-1} \ln \left( 1 - e^{-x} \right) = \Gamma(n) \zeta(n + 1), \]  

(17)

\( \zeta \) stands for the Riemann z-function and \( \Gamma \) is the Gamma function.

B. Generalized situation \( q \neq 1 \) (\( 0 < q < 1 \))

1. Generalized Partition Function

The OLM-Tsallis generalized partition function (see Section II) is

\[ \tilde{Z}_q = Tr \left[ 1 - (1 - q) \beta \left( \hat{H} - U_q \right) \right]^{1\over 1-q}, \]  

(18)
where ̃H is the Hamiltonian \(\{12\}\), \(U_q\) is the mean energy \(\{4\}\), and \(\beta\) is the optimal multiplier Lagrange. The photons’ chemical potential is, of course, taken as zero.

With the aim of calculating \(\bar{Z}_q\) we follow the steps of \[12\]. We work again in the large volume limit and regard Eq. \(\{18\}\) as an integral that can be evaluated using the relation \[22\]

\[
e^{-ab b^{1-z}} 2\pi \int_{-\infty}^{\infty} dt \frac{e^{itb}}{(a + it)^2} = \begin{cases} \frac{1}{\Gamma(z)} & \text{for } b > 0 \\ 0 & \text{for } b < 0, \end{cases}
\]

with \(a > 0\), \(\text{Re}(z) > 0\), and \(-\pi/2 < \arg(a + it) < \pi/2\).

If we set \(a = 1\), \(b = 1 - (1 - q)\beta(\hat{H} - U_q)\) (the cut-off condition is naturally fulfilled \[6\]) and \(z = 1/(1 - q) + 1\), the generalized partition function adopts the appearance

\[
\bar{Z}_q(\beta) = \int_{-\infty}^{\infty} dt K_q(t)e^{\tilde{\beta}U_q}Z_1(\tilde{\beta}),
\]

with \(Z_1\) given by Eq. \(\{13\}\),

\[
K_q(t) = \frac{\Gamma[(2 - q)/(1 - q)]\exp(1 + it)}{2\pi(1 + it)^{(1 + q)/(1 - q)}},
\]

and

\[
\tilde{\beta} = (1 + it)(1 - q)\beta.
\]

In order to evaluate the integral in \(20\) we expand the exponential term and obtain

\[
\bar{Z}_q(\beta) = \sum_{m=0}^{\infty} \xi_n^m \frac{\Gamma[(2 - q)/(1 - q)]}{m!} \int_{-\infty}^{\infty} dt \frac{e^{(1 + (1 - q)\beta U_q)(1 + it)}}{(1 + it)^{(2 - q)/(1 - q) + nm}},
\]

where \(\xi_n\) is given by Eq. \(\{16\}\).

Using again \(\{13\}\) with \(b = 1 + (1 - q)\beta U_q\), \(a = 1\), and \(z = (2 - q)/(1 - q) + nm\) we arrive to

\[
\bar{Z}_q(\beta) = \sum_{m=0}^{\infty} \xi_n^m \frac{\Gamma[(2 - q)/(1 - q)]}{m! \Gamma[(2 - q)/(1 - q) + nm]} [1 + (1 - q)\beta U_q]^\frac{1}{1 - q} + nm,
\]

and notice that an additional cut-off’s like condition must be considered \(1 + (1 - q)\beta U_q > 0\). Otherwise, \(\bar{Z}_q(\beta) = 0\).
2. The internal energy

The generalized internal energy can be cast in the fashion

$$U_q = \frac{T_1}{T_2},$$

(25)

where we have introduced the definitions

$$T_1 = Tr \left\{ \left[ 1 - (1 - q)\beta(\hat{H} - U_q) \right]^{\frac{1}{1-q}} \hat{H} \right\},$$

(26)

$$T_2 = Tr \left\{ \left[ 1 - (1 - q)\beta(\hat{H} - U_q) \right]^{\frac{2}{1-q}} \right\}.$$  

(27)

We first analyze $T_2$. The expression $\left[ 1 - (1 - q)\beta(\hat{H} - U_q) \right]^{\frac{2}{1-q}}$ is evaluated by recourse to (19), where we set $a = 1$, $b = 1 - (1 - q)\beta(\hat{H} - U_q)$, and $z = 1/(1-q)$.

$$\left[ 1 - (1 - q)\beta(\hat{H} - U_q) \right]^{\frac{2}{1-q}} = \frac{\Gamma[1/(1-q)]}{2\pi} \int_{-\infty}^{\infty} dt \frac{e^{(1+it)[1-(1-q)\beta(\hat{H} - U_q)]}}{(1 + it)^{1/(1-q)}},$$

(28)

The trace operation yields afterwards

$$T_2 = \frac{\Gamma[1/(1-q)]}{2\pi} \int_{-\infty}^{\infty} dt \frac{e^{(1+it)[1+(1-q)\beta U_q]}}{(1 + it)^{1/(1-q)}} Z_1(\tilde{\beta}),$$

(29)

with $\tilde{\beta}$ defined by Eq. (22). Expanding exponents and employing once again (19) leads finally to

$$T_2 = \sum_{m=0}^{\infty} \frac{\xi_m}{m!} \frac{\Gamma[1/(1-q)]}{\Gamma[1/(1-q) + nm]} \left[ 1 + (1 - q)\beta U_q \right]^{\frac{2}{1-q} + nm}.$$  

(30)

As for $T_2$, we recast $T_1$ as

$$T_1 = \frac{n\Gamma[1/(1-q)]}{2\pi (1 - q)\beta} \sum_{m=0}^{\infty} \frac{\xi_{m+1}}{m!} \int_{-\infty}^{\infty} dt \frac{e^{(1+it)[1+(1-q)\beta U_q]}}{(1 + it)^{1/(1-q) + n(m+1)}} Tr \left[ Z_1(\tilde{\beta})\hat{H} \right].$$

(31)

We proceed as follows: i) take advantage of the fact that

$$Tr \left[ Z_1(\tilde{\beta})\hat{H} \right] = -\frac{\partial Z_1(\tilde{\beta})}{\partial \tilde{\beta}},$$

(32)

ii) use Eq. (13), and iii) expand again the exponential term. We find

$$T_1 = \frac{n\Gamma[1/(1-q)]}{2\pi (1 - q)\beta} \sum_{m=0}^{\infty} \frac{\xi_{m+1}}{m!} \int_{-\infty}^{\infty} dt \frac{e^{(1+it)[1+(1-q)\beta U_q]}}{(1 + it)^{1/(1-q) + n(m+1)}}.$$  

(33)
The integral is evaluated as before, which yields

\[ T_1 = \frac{n}{1 - q} \sum_{m=0}^{\infty} \frac{\xi_n^{m+1}}{m!} \frac{\Gamma[1/(1-q)]}{\Gamma[1/(1-q) + n(m+1) + 1]} \left[ 1 + (1 - q)\beta U_q \right]^{1/n(m+1)} , \]  

(34)

so that the internal energy becomes

\[ U_q = \frac{n \xi_n}{1 - q} \sum_{m=0}^{\infty} \frac{\xi_n^m}{m!} \frac{1}{\Gamma[1/(1-q) + n(m+1) + 1]} \left[ 1 + (1 - q)\beta U_q \right]^{n(m+1)+1} \frac{1}{\sum_{m=0}^{\infty} \frac{\xi_n^m}{m!} \Gamma[1/(1-q) + nm] \left[ 1 + (1 - q)\beta U_q \right]^{nm}} , \]

(35)

although compliance with the Tsallis’cut-off condition \( 1 + (1 - q)\beta U_q > 0 \) is always to be demanded. The above is a non-linear equation to be tackled in numerical fashion. Some pertinent results are displayed in Fig. 1 for different \( q \)-values, they include the standard \( q = 1 \) case that yields the Steffan Boltzmann law

\[ U_1 = \sigma T^4 \]

(36)

with \( \sigma = 5.67 \times 10^{-8}W/m^2K^4 \), the so-called Steffan-Boltzmann constant.

We notice that a Steffan Boltzmann-like law holds, if \( kT \) is small enough, for all \( q \in (0 < q < 1) \), with \( \sigma = \sigma(q) \). For high \( T \)-values and \( q \ll 1 \) the power law behavior seems to persist, but the corresponding power, let us call it \( a_q \), is no longer equal to 4. Fig. 2 is a graph of \( a_q \) versus \( q \) for

\[ U_q \propto T^{a_q} . \]

(37)

Fig. 2 (a) shows \( a_q \) for small \( kT \)-values. The validity of Stefan Boltzmann’s Law for a wide range of \( q \)-values is easily appreciated. Fig. 2 (b) depicts, instead, \( a_q \) for large \( kT \)-values. In this limit, important deviations from the \( a_q = 4 \) value are detected.

An intermediate temperature range is observed in Fig. 1 in which the power law behavior (of \( U_q \) vs. \( kT \)) is violated. The corresponding transition is the more abrupt the larger \( |q - 1| \). As will be seen in the next section, these violations can be attributed to a Tsallis cut-off in the energy densities.
3. Energy densities

The generalized spectral energy distribution of blackbody radiation \( u_q \) is given by

\[
U_q = \int_0^\infty d\omega u_q. 
\]  
(38)

In order to obtain \( u_q \) we work with \( U_q = T_1/T_2 \), with \( T_1 \) and \( T_2 \) given by Eqs. (26) and (27), respectively. The denominator will not suffer any change, but in the numerator we will keep the integral form of \( Z_1 \) as given in Eq. (15). Then, Eq. (32) will lead to

\[
Tr \left( e^{-\tilde{\beta} \hat{H} \hat{H}} \right) = \bar{\hbar} A_n e^{\xi_n} \int_0^\infty d\omega \omega^n \int_0^\infty dt \frac{e^{(1+it)(1-(1-q)\beta(\hbar \omega-U_q))}}{(1 + it)^{1-q}} \frac{e^{\xi_n}}{1 - e^{-\tilde{\beta} \hbar \omega}},
\]  
(39)

allowing us to cast \( T_1 \) in the fashion

\[
T_1 = \frac{\Gamma[1/(1-q)]}{2\pi} \bar{\hbar} A_n \int_0^\infty d\omega \omega^n \sum_{s=0}^{\infty} \sum_{m=0}^{\infty} \frac{\xi_m m!}{\Gamma[1/(1-q) + nm]} \int_0^\infty dt \frac{e^{(1+it)(1-(1-q)\beta(\hbar \omega(s+1)-U_q))}}{(1 + it)^{1-q}} \frac{1}{1 - e^{-\tilde{\beta} \hbar \omega}}.
\]  
(40)

Recourse to the identity

\[
\frac{1}{1 - e^{-\tilde{\beta} \hbar \omega}} = \sum_{s=0}^{\infty} \left( e^{-\tilde{\beta} \hbar \omega} \right)^s,
\]  
(41)

yields now

\[
T_1 = \frac{\Gamma[1/(1-q)]}{2\pi} \bar{\hbar} A_n \int_0^\infty d\omega \omega^n \sum_{s=0}^{\infty} \sum_{m=0}^{\infty} \frac{\xi_m m!}{\Gamma[1/(1-q) + nm]} \int_0^\infty dt \frac{e^{(1+it)(1-(1-q)\beta(\hbar \omega(s+1)-U_q))}}{(1 + it)^{1-q}} \frac{1}{1 - e^{-\tilde{\beta} \hbar \omega}}.
\]  
(42)

If we use Eq. (19) the integral above is easily calculated, and we obtain

\[
T_1 = \frac{\Gamma[1/(1-q)]}{2\pi} \bar{\hbar} A_n \int_0^\infty d\omega \omega^n \sum_{s=0}^{\infty} \sum_{m=0}^{\infty} \frac{\xi_m m!}{\Gamma[1/(1-q) + nm]} \frac{1 - (1-q)\beta(\hbar \omega(s+1) - U_q)}{\Gamma[1/(1-q) + nm]}.
\]  
(43)

According to (38) the ratio \( T_1/T_2 \) implies that

\[
u_q = \frac{\sum_{m=0}^{\infty} B_m S_m}{\sum_{m=0}^{\infty} B_m},
\]  
(44)

where

\[
B_m = \frac{\xi_m}{m!} \frac{[1 + (1-q)\beta U_q]^{nm}}{\Gamma[1/(1-q) + nm]},
\]  
(45)

\[
S_m = \sum_{s=1}^{\infty} \left[ \frac{1 - (1-q)\beta[\hbar \omega(s+1) - U_q]}{1 + (1-q)\beta U_q} \right]^{rac{m}{1-q} + nm}.
\]  
(46)
Figs. 3 and 4 are plots of $u_q$ vs. the frequency $\omega$. Notice that they quite resemble the semi-empirical Planck law for small $|q - 1|$-values. Fig. 5 clearly shows that the frequency shifts of the maxima of the above referred to curves (located at $\omega = \omega^*$) comply with the Wien law

$$\omega^* \propto T,$$

(47)

for such $q$-values. Up to $q = 0.8$ this nice behaviour persists. As $q$ departs from 1 an anomalous behaviour ensues. Local maxima begin to appear in the curves $u_q(\omega)$ (Figs. 3 and 4). As $T$ grows the global maximum is shifted and the secondary ones become more pronounced, until one of them replaces the former one as the maximum maximorum. An abrupt jump in the curve $\omega^*$ vs. $T$ ensues as a consequence (Fig. 5).

The departures from Planck’s law as $|q - 1|$ grows, could in principle be subject to experimental verification, thus $q$ would be fixed.

The above referred to discontinuities in the “Wien-like” behaviour are due to the Tsallis’ cut-off operative in terms of the sums $T_1$ and $T_2$.

IV. FIRST ORDER CORRECTIONS

We have seen in the preceding section that (44) yields, for $q$-values close to unity, results that quite resemble the ones given by Planck’s law. It is then reasonable to look for a perturbative expansion in $1 - q$.

Let us start with $S_m$ (Cf. Eq. (44)) and write it in the form

$$S_m = \sum_{s=0}^{\infty} \exp \left\{ \left( \frac{q}{q - 1} + nm \right) \ln \left[ 1 - (1-q) \frac{\beta h \omega (1+s)}{1 + (1 - q) \beta U_q} \right] \right\}. \quad (48)$$

A first order expansion of the logarithm yields

$$S_m \approx \sum_{s=0}^{\infty} \exp \left\{ -[q + nm(1-q)] \frac{\beta h \omega (1+s)}{1 + (1 - q) \beta U_q} \right\}, \quad (49)$$

a power series in $s$ of guaranteed convergence for $1 - q < 1$. Neglecting the term $nm(1-q)$ on account of the fact that the concomitant series rapidly converges so that terms for which $m$ could compete with $1/(1 - q)$ are indeed negligible, we have
\[ S_m \approx \frac{1}{\exp \left[ \frac{q\beta h \omega}{1 + (1-q)\beta U_q} \right] - 1}, \quad (50) \]

and replacement in Eq. (44) yields

\[ u_q \approx \frac{\hbar A_q \omega^n}{\exp \left[ \frac{q\beta h \omega}{1 + (1-q)\beta U_q} \right] - 1}, \quad (51) \]

a first order correction for a “generalized Planck law” that yields the classical result for \( q \to 1 \). Eq. (51) provides one then with an approximate energy density.

Using now this equation as a starting point we also get first order corrections in \( 1 - q \) to the Wien law. We can locate the maximum with respect to \( \omega \) of Eq. (51) with the auxiliary definition

\[ x = \frac{\omega q\beta h}{1 - (1-q)\beta U_q}, \quad (52) \]

and immediately find

\[ e^{-x} + \frac{x}{n} = 1, \quad (53) \]

whose solution is a constant \( b \) for each fixed value of \( n \). For \( n = 3 \) we find, for instance, \( b = 2.82 \). The maximum of \( u_q(\omega) \) for different \( T \)'s is located at distinct frequencies according to

\[ \omega_i = \frac{b k}{q h} T_i - \frac{1 - q}{q} \frac{U_q}{h}, \quad (54) \]

a first order correction to Wien’s law.

If we integrate Eq. (51) over frequencies we find \( U_q \) as a function of \( T_q \), and a concomitant first order correction to Planck’s law

\[ U_q = \hbar A_q \alpha \left[ \frac{1 + (1-q)\beta U_q}{q\beta h} \right]^{n+1}, \quad (55) \]

that reduces trivially to Planck’s one in the limit \( q \to 1 \). We call \( \alpha \) the integral

\[ \alpha = \int_0^\infty dx \frac{x^n}{e^x - 1} = \Gamma(n+1)\zeta(n+1), \quad (56) \]

with \( x \) given by (52).
It may be convenient to point out that the first order relationships here obtained ignore possible cut-off problems. Anyway, in the $0 < q < 1$ range (with $q \to 1$) that is relevant to the present considerations, such problems should not arise.

**V. CONCLUSIONS**

We have presented here exact results for the mean energy and the energy density of the blackbody problem in a nonextensive environment. A *Stefan Boltzmann-like law* is obtained, for all $0 < q < 1$, although with a $q$-dependent SB constant $\sigma(q)$, for almost the whole temperature range, although for some temperatures a deviation from the power law behaviour is detected.

The energy density curves correspond to Planck’s law for $q$ values close to unity. For other $q$’s a gradual departure from the typical Planck curve is observed, which can be attributed to Tsallis’ cut-off effects. *Wien’s law* is also obtained for $q$ values close to unity.

Finally, we have presented first order perturbative results in the limit $q \to 1$ for the energy density $u_q$ (the parameter is, of course, $1 - q$). These could be of some utility in obtaining first order nonextensive corrections to the Bose-Einstein equation.

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FIG. 1. The internal energy $U_q$ as a function of $T = 1/\beta$ for different values of the non-extensivity parameter $q$.

FIG. 2. Power law exponent $a_q$ as a function of $q$ (see text for details). The graph (a) corresponds to small temperatures, and the (b)-one to high $T$-values.
FIG. 3. Energy density $u_q$ as a function of the frequency $\omega$. Each plot corresponds to a different value for $q$. $kT$ equals unity in all cases.

FIG. 4. Energy density $u_q$ as a function of the frequency $\omega$. Each plot correspond to different values of $q$ parameter. In all of them, $kT = 10^2$ was kept fixed.
FIG. 5. We depict the frequency $\omega^*$ for which the energy density reaches its maximum, as a function of $kT$, for different values of $q$. 