Gauge Field Fluctuations and First-Order Phase Transition in Color Superconductivity

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We study the gauge field fluctuations in dense quark matter and determine the temperature of the induced first-order phase transition to the color-superconducting phase in weak coupling. We find that the local approximation of the coupling between the gauge potential and the order parameter, employed in the Ginzburg-Landau theory, has to be modified by restoring the full momentum dependence of the polarization function of gluons in the superconducting phase.

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Quantum chromodynamics (QCD) at high baryon density has become an active research area in recent years [1, 2, 3]. The interest has been focussed on exploring the nature of nuclear matter under extreme conditions of temperature and density. The physics of this region of the phase diagram of QCD is relevant to the phenomenology of high-energy nuclear collisions and to the properties of highly compressed nuclear matter inside a compact star.

Despite the fact that a nonperturbative approach is lacking — lattice simulations are notoriously difficult because of the fermion sign problem — a number of interesting results have been obtained based on Nambu–Jona-Lasinio (NJL)–type models which are treated in the mean-field approximation [2]. At these high densities — several times higher than the density of ordinary nuclear matter — a novel, color-superconducting phase of nuclear matter is expected to appear [1, 5].

Since QCD is an asymptotically free theory, reliable perturbative calculations can be performed for asymptotically large quark chemical potentials, \( \mu \gg \Lambda_{\text{QCD}} \). Thus, at asymptotic densities color superconductivity can be quantitatively explored within QCD. At such ultra-high chemical potentials, the interaction between two quarks near their Fermi surface is dominated by one-gluon exchange, which is attractive in the color-antisymmetric channel for both the color-electric and color-magnetic parts. The formulas for the energy gap and the transition temperature have been derived in Refs. [6, 7, 8, 9, 10, 11].

In weak coupling and for three colors and three flavors of quarks, the critical temperature, \( T_c \), that corresponds to the pairing instability of the normal phase is given by

\[
\ln \frac{k_BT_c}{\mu} = -\frac{3\pi^2}{\sqrt{2}g} + \ln \frac{2048\sqrt{2}\pi^3}{9\sqrt{3}g^5} + \gamma - \frac{\pi^2 + 4}{8} + O(g).
\]

The color superconductor in this region is of type I.

The physics of a color superconductor in the vicinity of the transition temperature can be described in terms of the Ginzburg-Landau free energy functional [12, 13], which depends on the expectation values of the order parameter and the gauge potential. As the temperature, \( T \), gets sufficiently close to the critical temperature \( T_c \), the fluctuations of the order parameter and of the gauge potential cannot be ignored. Within the framework of the Ginzburg-Landau approach, we can estimate the free energy density of the fluctuations by the thermal energy \( k_BT \) within a volume \( V \), where \( l \) is the characteristic length of the fluctuation. This volume would be \( \xi^3 \) for the order parameter and \( \lambda^3 \) for the gauge potential, \( \xi \) and \( \lambda \) being the coherence length and the magnetic penetration depth at temperature \( T \), respectively. Since both lengths diverge like \( (T-T_c)^{-1/2} \), the corresponding fluctuation energy density behaves as \( (T-T_c)^{-3/2} \). The condensation energy density, however, behaves as \( (T-T_c)^2 \). Therefore, as \( T \to T_c \), the fluctuation energy density will eventually dominate and the nature of the phase transition will be modified. For a strong type-I superconductor, \( \lambda \ll \xi \), and the fluctuations of the gauge field exceed by far the fluctuations of the order parameter. It is then permissible to retain the fluctuations of the gauge field while neglecting those of the order parameter. As we shall see, a first-order phase transition occurs at \( T_c^* \), where \( T_c^* \) and \( T_c \) are the temperatures of the first- and second-order phase transitions, respectively. A subtle issue arises in the calculation of the transition temperature \( T_c^* \), due to the significance of the relation between the magnetic penetration depth \( \lambda \) and the coherence length of the self-energy of magnetic gluons in the superconducting phase, and the coherence length at zero temperature, \( \xi_0 \sim 1/(k_BT_c) \). The validity of the local coupling between the order parameter and the gauge potential employed in the Ginzburg-Landau approach relies on the inequality

\[
\xi_0 \ll \lambda \ll \xi
\]

being valid at \( T_c^* \). The original calculation of Bailin
write the CJT effective potential as variational parameters, for a homogeneous system we aggregators by malism [17]. Denoting the full gluon and quark propagation modes with momentum higher than \( \xi_0 \) and obtained a weaker first-order transition. Both approaches employ the local-coupling approximation, but their results are not completely consistent with Eq. (2).

In this paper we have restored the full momentum dependence of the magnetic gluon self-energy in our calculation. Due to the forward singularity of one-gluon exchange, a determination of the strength of the first-order phase transition can be achieved within the framework of QCD. We find that in the weak-coupling limit

\[
\frac{T_c^* - T_c}{T_c} = \frac{\pi^2}{12\sqrt{2}} g \simeq 0.58 \, g .
\]

Contrary to the assumption made in previous works, it then follows that, at \( T_c^* \),

\[
\lambda \ll \xi_0 \ll \xi .
\]

The effect of the gauge fluctuations can be incorporated into the free energy of dense quark matter around \( T_c \) by using the Cornwall-Jackiw-Tomboulis (CJT) formalism [17]. Denoting the full gluon and quark propagators by \( D \) and \( S \), respectively, and regarding them as variational parameters, for a homogeneous system we write the CJT effective potential as

\[
\Gamma[D, S] = \frac{k_B T}{2 \Omega} \left\{ \text{Tr} \ln D^{-1} + \text{Tr} (D^{-1}D - 1) - \text{Tr} \ln S^{-1} - \text{Tr} (S^{-1}S - 1) - 2 \Gamma_2[D, S] \right\},
\]

where \( \Omega \) is the 3-volume of the system, \( D^{-1} \) and \( S^{-1} \) are the inverse tree-level propagators for gluons and quarks, respectively, and \( \Gamma_2 \) represents the sum of all 2PI vacuum diagrams built with \( D \) and \( S \). Thermal equilibrium corresponds to

\[
\frac{\delta \Gamma}{\delta D} = 0 \quad , \quad \frac{\delta \Gamma}{\delta S} = 0 .
\]

In the mean-field approximation, \( \Gamma_2 \) contains only the sunset-type diagram of Fig. 1a, \( \Gamma_2[D, S] = -\frac{1}{4} \text{Tr} \left( D \hat{\Gamma} S \hat{\Gamma} S \right) \), where \( \hat{\Gamma} \) is the quark-gluon vertex. The first equation gives rise to \( D^{-1}(K) = D^{-1}(K) + \Pi(K) \), with the self-energy \( \Pi(K) \) given by Fig. 1b. The solid line represents the full quark propagator \( S \); \( K = (\vec{k}, \omega) \) is the Euclidean four-momentum of the gluon and \( \omega \) the discrete Matsubara frequency. At the stationary point, the second term on the right-hand side of Eq. (6) cancels the last term. We proceed by writing

\[
S(K) = S_n(K) + \delta S(K) ,
\]

\[
D^{-1}(K) = D_n^{-1}(K) + \delta \Pi(K) ,
\]

where the subscript \( n \) refers to quantities in the normal phase and \( \delta S(P) \) and \( \delta \Pi(K) \) are functions of the gap parameter \( \Delta \) [18]. The CJT effective potential can be written as a sum of four terms

\[
\Gamma = \Gamma_n + \Gamma_{\text{cond}} + \Gamma_{\text{fluc}} + \Gamma_{\text{fluc}}' ,
\]

where

\[
\Gamma_{\text{cond}} = \frac{k_B T}{2 \Omega} \left[ \text{Tr} (D_n \delta \Pi) - \text{Tr} (S^{-1} \delta S) + \text{Tr} \ln (1 + S^{-1}_n \delta S) \right] ,
\]

\[
\Gamma_{\text{fluc}} = \frac{k_B T}{2 \Omega} \sum_{\vec{k}, \omega=0} \text{tr} \left\{ \ln [1 + D_n(K) \delta \Pi(K)] - D_n(K) \delta \Pi(K) \right\} ,
\]

\[
\Gamma_{\text{fluc}}' = \frac{k_B T}{2 \Omega} \sum_{\vec{k}, \omega \neq 0} \text{tr} \left\{ \ln [1 + D_n(K) \delta \Pi(K)] - D_n(K) \delta \Pi(K) \right\} .
\]

In our formulas \( \text{Tr} \) indicates summation over all indices including momentum and energy while \( \text{tr} \) denotes summation over all indices except momentum and energy.

The term \( \Gamma_{\text{cond}} \) when expanded to the fourth power of \( \lambda/k_BT \) gives rise to the Ginzburg-Landau free energy [12, 13]. The entire quadratic term of \( \Gamma \) is included in \( \Gamma_{\text{cond}} \) and its coefficient vanishes at \( T_c \) of Eq. (1). The term \( \Gamma_{\text{fluc}} \) takes the explicit form

\[
\Gamma_{\text{fluc}} = 8 k_B T \int \frac{d^3 \vec{k}}{(2\pi)^3} \left\{ \ln \left[ 1 + \frac{m_A^2(k)}{k^2} \right] - \frac{m_A^2(k)}{k^2} \right\} ,
\]

where the factor 8 is the number of gluon colors. Furthermore, the small mixing with the ordinary electromagnetic field is ignored here. The momentum-dependent magnetic mass of the gluons is given by \( m_A^2(k) = f \left( \frac{k}{2\pi k_B T} \right)^2 / \lambda^2 \), where

\[
\frac{1}{\lambda^2} = \frac{7\zeta(3)}{24\pi^4} \left( \frac{g\mu\Delta}{k_B T_c} \right)^2 .
\]

The function \( f(y) \) becomes identical to that of an electronic superconductor [19] for \( \Delta \ll k_BT \), i.e.,

\[
f(y) = \frac{6}{\zeta(3)} \sum_{s=0}^{\infty} \int_0^1 dx \frac{1 - x^2}{(s + \frac{1}{2})(s + \frac{1}{2})^2 + y^2 x^2} .
\]

The limiting behavior of this function is \( f(0) = 1 \) (London limit) and \( f(y) \simeq 3\pi^3/[28\zeta(3) y] \) for \( y \gg 1 \) (Pippard limit). The integrand of \( \Gamma_{\text{fluc}} \) in the long-wavelength limit becomes identical to the one used in Ref. [1].

Because of the dynamical screening of the gluon propagator at nonzero Matsubara frequency, the term \( \Gamma_{\text{fluc}}' \) contributes terms of order higher than \( O(g) \). Consequently, it will be neglected in the following. The relevant free energy density, expressed in terms of the gap
energy $\Delta$ of the color-flavor locked (CFL) condensate and the temperature near $T_c$ reads

$$
\Gamma - \Gamma_n = \frac{6\mu^2}{\pi^2} t \Delta^2 + \frac{21\zeta(3)}{4\pi^4} \left(\frac{\mu}{k_BT_c}\right)^2 \Delta^4 + 32\pi(k_BT_c)^4 F \left(\frac{1}{4\pi^2k_B^2T_c^2\lambda^2}\right) \equiv \gamma(t, \Delta),
$$

(13)

where $t \equiv (T - T_c)/T_c$, and

$$
F(z) = \int_0^\infty dx \ x^2 \left\{ \ln \left[ 1 + \frac{z}{x^2} f(x) \right] - \frac{z}{x^2} f(x) \right\}. \quad (14)
$$

The asymptotic behavior of the function $F(z)$ can be inferred from that of $f(z)$,

$$
F(z) \simeq \left\{ \begin{array}{ll}
-\frac{3}{2}\zeta(3/2) & \text{for } z \ll 1, \\
-\frac{3}{2\zeta(3)} z \ln \left( \frac{3\zeta(3)}{2z^2} \right) + \text{const} & \text{for } z \gg 1.
\end{array} \right.
$$

(15)

The $z \ll 1$ behavior of the function, when substituted into Eq. (13), gives rise to the well-known $\Delta^3$ term of Ref. [14]. The transition temperature $T^*$ and the value of the gap $\Delta$ at the first-order phase transition are determined from the nontrivial solution of the pair of equations

$$
\gamma(t^*, \Delta) \equiv 0, \quad \frac{\partial \gamma(t^*, \Delta)}{\partial \Delta} = 0.
$$

(16)

Because of the trivial dependence of $\gamma(t^*, \Delta)$ on $t^*$, we can combine the two equations into one

$$
F \left( \frac{1}{4\pi^2k_B^2T_c^2\lambda^2} \right) = \frac{216\pi^7}{\zeta(3)g^4} \left( \frac{k_BT_c}{\mu} \right)^2, \quad (17)
$$

with $F(z) = -F'(z)/z + F(z)/z^2$. It follows from the explicit form of $F(z)$ that the function $F(z)$ is monotonically decreasing for $z > 0$, i.e.,

$$
F'(z) = -\frac{2}{z^3} \int_0^\infty dx \ x^2 \left[ \ln \left( \frac{1}{1 - w} \right) - w - \frac{1}{2}w^2 \right] \leq 0, \quad (18)
$$

where $w = zf(x)/[x^2 + zf(x)]$. This, together with the asymptotic behavior

$$
F(z) \simeq \left\{ \begin{array}{ll}
\frac{3}{2} & \text{for } z \ll 1, \\
\frac{3}{2\zeta(3)} z^{-1} & \text{for } z \gg 1.
\end{array} \right.
$$

(19)

implied by Eq. (15), demonstrates the existence and the uniqueness of the solution to Eq. (17) and therefore to the set of Eqs. (16). Furthermore, we can show that $t^* > 0$. In the weak-coupling limit, the right-hand side of Eq. (17) becomes small and the solution lies on the side of a large argument of $F(z)$. It follows from Eq. (19) that

$$
\left( \frac{\Delta}{k_BT_c} \right)^2 = \frac{\pi^2}{63\zeta(3)} g^2,
$$

(20)

at the transition, which implies that $\lambda \ll \xi_0$. Substituting Eq. (20) into either one of Eqs. (16), we obtain

$$
t^* \equiv \frac{T^*_c - T_c}{T_c} \simeq \frac{g^2}{12} \left[ \ln \left( \frac{\mu}{k_BT_c} \right)^2 + \text{const} \right], \quad (21)
$$

which, upon substitution of Eq. (11), yields Eq. (3) to leading order in $g$.

The strength of the first-order phase transition measured by Eq. (21) or Eq. (3), though robust and vanishing in the limit $g \to 0$, is much stronger than that estimated in Ref. [16]. The extrapolation of this formula to the accessible baryon density inside a neutron star, say $\mu = 500$ MeV, gives rise to $(T^*_c - T_c)/T_c \simeq 1.8$ for $g = 3.1$, given by the one-loop formula with $\Delta_{QCD} = 200$ MeV. This estimate, though inconsistent with the assumption $t \ll 1$, indicates that the gauge field fluctuations cannot be neglected for the color-superconducting phase transition at moderate baryon density.

It is well known in solid state physics that the local coupling to the electromagnetic gauge potential in the Ginzburg-Landau free energy functional cannot be sustained far away from $T_c$ and the condition for locality could be more stringent than that of the Ginzburg-Landau expansion of the condensation energy, i.e., $\Delta/(k_BT_c) \ll 1$, for a strong type-I superconductor [20]. What we found above demonstrates this subtlety. It is interesting to examine the local approximation employed in Ref. [14] for the electronic superconductor using the formalism developed here. The condensation energy and the fluctuation energy take the form

$$
\gamma(t, \Delta) = \frac{k_B^4}{2\pi^2v_F} t \Delta^2 + \frac{7\zeta(3)k_B^4}{32\pi^4v_Fk_B^2T_c^2} \Delta^4 + \frac{4\pi^4(k_BT_c)^4}{v_F^3} \left( \frac{v_F^2}{4\pi^2k_B^2T_c^2\lambda^2} \right), \quad (22)
$$

with $\lambda^{-2} = \frac{7\zeta(3)v_F^2}{12\pi^2} (k_BT_c/k_BT_c)^2$ with $k_F$ and $v_F$ the Fermi momentum and the Fermi velocity. The zero-temperature coherence length $\xi_0 \sim v_F/(k_BT_c)$. Correspondingly, Eq. (17) is replaced by

$$
F \left( \frac{v_F^2}{4\pi^2k_B^2T_c^2\lambda^2} \right) = \frac{\pi^2\kappa^2}{16\alpha_e v_F}, \quad (23)
$$

where $\alpha_e \simeq \frac{3}{16\pi}$ is the fine structure constant and

$$
\kappa = 3 \sqrt{\frac{2}{\zeta(3)}} \alpha_e \left( \frac{\pi}{\alpha_e} \right) \frac{k_BT_c}{k_F} \quad (24)
$$

is the Ginzburg-Landau parameter. The validity of the local coupling approximation relies on a large value of the right-hand side of Eq. (23), which implies that

$$
\kappa \gg \frac{4\sqrt{\alpha_e v_F}}{\pi}, \quad (25)
$$
For typical metals $v_F \sim \alpha e$ and consequently Eq. (25) implies that $\kappa \gg 0.09$. The local approximation, though marginal for the strongest type-I material like aluminium ($\kappa \approx 0.01 \sim 0.02$), works practically for all laboratory-prepared type-I materials.

In this letter, we have incorporated consistently the fluctuations of the gauge field into the free energy of a homogeneous CFL color superconductor in mean-field approximation. We determined the temperature of the fluctuation-induced first-order phase transition to the color-superconducting phase in weak coupling. We find that the typical momentum of the fluctuations corresponds to the Pippard limit of the magnetic self-energy of gluons and the conventional local coupling approximation of the fluctuation, though applicable for the metallic superconductors, breaks down for color superconductivity.

The phase transition to the CFL phase is accompanied by a spontaneous breaking of the chiral symmetry, which is of first order because of instantons. What we found in this letter is that the fluctuations of the gauge fields induce a much stronger first-order phase transition in weak coupling and that the mechanism is not limited to three flavors.

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