Ultraviolet and Infrared Finiteness in Two Dimensional Curved Space–Time

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Abstract.

Different models of field theories in two dimensions can be described by the action $Tr \int \varphi F$. In the presence of a curved background, we construct a local supersymmetry–like transformations under which the action is invariant. Furthermore, by analysing the cohomology of the theory we show the absence of anomalies. Also the ultraviolet as well as the infrared finiteness of the theory are proven at all orders of perturbation theory.

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1 Introduction

Two dimensional fields models [1] play an important role in physics as well as in mathematics. For instance, conformal field theories and sigma models lead to the development of string theory. On the other hand, by studying low dimensional theories one can gain experience which could help to tackle more complicated problems already present in the 4–dimensional world.

2D physics has some interesting features, like for instance the action describing the Yang–Mills model with vanishing coupling constant and the action of the Jackiw–Teitelboim model for 2D gravity have the same form, namely $Tr \int \varphi F$. The same action is obtained by compactifying the Chern–Simons model [2] on a circle. This was studied by many authors [3], [4] and [5]. In this work we will generalize the analysis of [5] to be valid in a curved space–time. The main ingredient of our analysis is the use of the Landau gauge where local supersymmetry–like transformations are manifest [6], [7]. Another interesting property of the Landau gauge is the existence of the ghost equation [9] which is very helpful for the analysis. Our strategy is to follow an algebraic way [8] and show the stability of the theory. Furthermore, the calculation of possible counterterms is carried out essentially by using cohomology techniques [10], [11], [12]. The advantage in using the framework of algebraic renormalization is that one does not need to specify a subtraction scheme like for instance the BPHZ or the dimensional regularization schemes. On the other hand, such a scheme should exist, a fact which will allow us to use the above mentioned algebraic renormalization. This fact limited our quantum analysis to be valid only in curved, topologically trivial, and asymptotically flat manifolds.

This work is organized as follows: in section 2 we present the classical analysis of the two dimensional model considered in a curved space–time. We begin by describing the classical theory, and we show that local susy–like transformations do exist. This section is also devoted to the discussion of the infrared regularization of the ghost–antighost propagator and its implications. Then, we generalize the local susy–like transformations of the model and we show that the corresponding Ward identity is linearly broken.

Next, in section 3, we construct the most general counterterm which is, in turn, forbidden by the ghost equation. Therefore no deformations are allowed and the theory is finite.

In section 4, the 2D theory is proved to be anomaly free. This last result allows us to extend the classical analysis to all orders of perturbation theory. Hence the finiteness of the two dimensional model, defined on a curved manifold, is proven at all orders of perturbation theory.
2 The model and its infrared regularization

We devote this section to the classical analysis and the infrared regularization of the 2D theory, considered on a two dimensional curved manifold $\mathcal{M}$, in the Landau gauge. The most important feature of such a field model is that it is topological [17] and possesses the following invariant action, which is metric independent:

$$\Sigma_{\text{inv}} = \frac{1}{2} \int_{\mathcal{M}} d^2 x \varepsilon^{\mu\nu} F_{\mu\nu}^a \phi^a,$$  \hspace{1cm} (1)

where $\varepsilon^{\mu\nu}$ is a second rank antisymmetric tensor density of weight 1, and the field strength $F_{\mu\nu}^a$ is given by:

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c. \hspace{1cm} (2)$$

Here $f^{abc}$ denotes the structure constants of the gauge group, which is supposed to be a compact Lie group and all the fields belong to its adjoint representation. $A_\mu^a$ is the gauge field.

Now, we briefly review some connections of the action (1) with different two dimensional theories.

(I) First, following [4], we consider the action

$$S = \int_{\mathcal{M}} d^2 x \left( -\frac{\lambda^2}{2} \phi^a \phi^a - \frac{1}{2} \varepsilon^{\mu\nu} \phi^a F_{\mu\nu}^a \right), \hspace{1cm} (3)$$

where $\lambda$ plays the role of the coupling constant. After integrating over the field $\phi^a$ in the partition function

$$Z = \int D\phi^a D A_\mu^a e^{-S}, \hspace{1cm} (4)$$

and using the Gaussian integral identity

$$\int_{-\infty}^{+\infty} dx \frac{1}{\sqrt{2\pi}} \exp(-\frac{\lambda^2}{2} x^2 - ixy) = \exp(-\frac{y^2}{2\lambda^2}), \hspace{1cm} (5)$$

one can easily see that the action (3) gives rise to the same partition function as the 2D Yang–Mills action. Moreover, for vanishing coupling constant $\lambda \to 0$ the action (3) reduces to

$$\int_{\mathcal{M}} -\frac{i}{2} \varepsilon^{\mu\nu} \phi^a F_{\mu\nu}^a. \hspace{1cm} (6)$$

\textsuperscript{2}Here are our conventions: we denote the space–time metric by $g_{\mu\nu}$, its inverse by $g^{\mu\nu}$ and its determinant by $g$. The scalar density $\sqrt{g}$ has weight +1, whereas the volume element density $d^2 x$ has weight -1, which means that the invariant volume element $dV = \sqrt{g} d^2 x$ has vanishing weight, then transforms as a scalar under diffeomorphisms.

\textsuperscript{3}For the Levi–Civita density $\varepsilon^{\mu\nu}$ we choose $\varepsilon^{12} = 1$. Furthermore, we have $\varepsilon_{\alpha\beta} = g^{-1} g_{\alpha\mu} g_{\beta\nu} \varepsilon^{\mu\nu}$, where $\varepsilon^{\mu\nu}$ has weight +1 and $\varepsilon_{\mu\nu}$ has weight -1.
Hence, in the limit where \((\lambda \to 0)\) the 2D Yang–Mills action and (3) lead to the same partition function \(Z\).

(II) Second we consider the 3D topological Chern–Simons theory

\[
\Sigma_{C.S} = -\frac{1}{2} \int d^3x \varepsilon^{\alpha \beta \gamma} (A^\alpha_0 \partial_\beta A^\gamma_0 + \frac{1}{3} f^{abc} A^a_\alpha A^b_\beta A^c_\gamma),
\]

where the indices \(\alpha, \beta\) and \(\gamma\) are the 3D space–time indices. Now if we compactify the Chern–Simons model (3) on a circle, we get a two dimensional theory described exactly by an action of the same form as (1) where the field \(\varphi^a\) is nothing but the third component of the 3D gauge field \(A^a_\alpha\) in the compactified direction.

(III) The third possibility we mention is the relation of (1) to 2D gravity. Here we follow Chamseddine and Wyler \([3]\) where the following action was proposed as a model for 2D gravity

\[
\Sigma_G = \frac{1}{2} \int d^2x \varepsilon^{\mu \nu} \varphi^A F^A_{\mu \nu}.
\]

In this case

\[
F^A_{\mu \nu} = \partial_\mu e^A_\nu - \partial_\nu e^A_\mu - \varepsilon^A_{BC} e^B_\mu e^C_\nu,
\]

\(A, B\) and \(C\) take the values (0, 1, 2) and the generators \(\tau^A\) of the group \(SO(1,2)\) give rise to the following algebra

\[
[\tau^A, \tau^B] = -\varepsilon^{ABC} \tau^C.
\]

To explicitly see the connection between a 2D gravity model and the action (8) one let the gauge field \(e^A_\mu\) decompose in the zweibein field \(e^a_\mu\) and the spin connection \(e^2_\mu = \omega_\mu\). After some computations (for details, see \([3]\)) one gets from (8) the action

\[
I_G = \int d^2x \sqrt{h} \varphi (R + \Lambda),
\]

\(h\) is the determinant of the metric \(h^{\mu \nu} = e^a_\mu e^b_\nu \eta_{ab}\) and \(R\) is the corresponding Ricci scalar. \(\Lambda\) stands for the cosmological constant. At this level one recognizes that beginning with (8) one could derive the Jackiw–Teitelboim model for 2D gravity, given by (11).

Above, we have seen some connections between different field models and their relation to the action (1), which seems to be the ‘meeting point’ of different 2D theories. Motivated by these facts, from now on, we will concentrate on the analysis of the action (1), which is invariant under the gauge symmetry

\[
\delta_g A^a_\mu = -(\partial_\mu \theta^a + f^{abc} A^b_\mu \theta^c) \equiv -(D_\mu \theta)^a, \\
\delta_g \varphi^a = f^{abc} \theta^b \varphi^c,
\]

where \(\theta^a\) is the gauge parameter. To fix this gauge freedom we choose a Landau–type gauge such that the gauge fixing part of the action, in which the metric describing the
manifold appears explicitly, writes down as

\[ \Sigma_{gf} = -s \int_{\mathcal{M}} d^2 x \left( \sqrt{g} g^{\mu \nu} \partial_{\mu} c^a A^a_{\nu} \right). \]  

(13)

The gauge fixed action \((\Sigma_{inv} + \Sigma_{gf})\) is invariant under the BRS symmetry:

\[
\begin{align*}
 s A^a_{\mu} &= - (D_{\mu} c)^a, \\
 s \varphi^a &= f^{abc} b^b \varphi^c, \\
 s c^a &= \frac{1}{2} f^{abc} b^b c^c, \\
 s \bar{c}^a &= b^a, \\
 s b^a &= 0, \\
 s^2 &= 0.
\end{align*}
\]  

(14)

c^a denotes the Faddeev-Popov ghost field of ghost number +1, \(\bar{c}^a\) is the antighost field of ghost number -1 and \(b^a\) is the Lagrange multiplier of ghost number 0 enforcing the Landau gauge condition.

As for the other topological gauge field models \([13], [24], [7], [26]\), the metric \(g_{\mu \nu}\) is only present in the gauge fixing part of the action which is nothing else but an exact BRS variation. This fact implies that, here, one can also extend the BRS symmetry \([19]\) by letting the operator \(s\) acting on the background metric as:

\[ s g_{\mu \nu} = \hat{g}_{\mu \nu} \quad s \hat{g}_{\mu \nu} = 0. \]  

(15)

Thus the metric is just a gauge parameter, of which the physical observables are independent. Clearly, \(\hat{g}_{\mu \nu}\) is a symmetric tensor of ghost number one.

In addition to the BRS symmetry, the gauge fixed action \((\Sigma_{inv} + \Sigma_{gf})\) is also invariant under a local supersymmetry–like transformations (called superdiffeomorphisms in \([13], [24], [4]\) and whose parameter we denote by \(\xi^\mu\), which has ghost number +2.

\[
\begin{align*}
 \delta_{(\xi)} A^a_{\mu} &= 0, \\
 \delta_{(\xi)} \varphi^a &= -\varepsilon_{\mu \nu} \xi^\mu \sqrt{\bar{g}} g^{\nu \sigma} \partial_\sigma \bar{c}^a, \\
 \delta_{(\xi)} c^a &= -\xi^\mu A^a_{\mu}, \\
 \delta_{(\xi)} \bar{c}^a &= 0, \\
 \delta_{(\xi)} b^a &= L_\xi \bar{c}^a, \\
 \delta_{(\xi)} \hat{g}_{\mu \nu} &= L_\xi g_{\mu \nu}, \\
 \delta_{(\xi)} g_{\mu \nu} &= 0,
\end{align*}
\]  

(16)

where \(L_\xi\) denotes the Lie derivative. When we anticommute the BRS operator \(s\) with \(\delta_{(\xi)}\) we get the on–shell algebra

\[ \{ s, \delta_{(\xi)} \} = L_\xi + \text{equations of motion}. \]  

(17)
On the other hand, in the context of perturbation theory, the model has an infrared problem. Indeed, in the flat space–time limit the propagator \( \langle c^a \bar{c}^a \rangle \) is logarithmic divergent in the infrared limit. To regularize \( \langle c^a \bar{c}^a \rangle \) one has to introduce a mass \( m \) such that

\[
\langle c^a \bar{c}^a \rangle = \frac{1}{k^2 + m^2}.
\]  

(18)

As remarked by the authors of \([5]\), the physical observables are defined in the limit of vanishing mass. However, as long as \( m \) is not zero, the physical quantities may depend on it. A similar situation was already analyzed in the literature, where it was conjectured \([21]\) and then shown \([20]\) that (in the context of a 2D nonlinear sigma model) local observables are well defined in the vanishing mass limit. For topological field theories, however, we have nonlocal observables. In the spirit of \([21]\), the authors of \([5]\) have extended the conjecture of Elitzur \([21]\) to include nonlocal observables, too.

In this paper we will not worry about such questions but concentrate our effort in proving the perturbative finiteness of the model. Thus, in our case (in the curved space–time) we also introduce a mass term \( m^2 \) which would regularize the propagator \( \langle c^a \bar{c}^a \rangle \). Then of course the action gets modified by adding the integrated local polynomial \( \Sigma_m \)

\[
\Sigma_m = s \int_M d^2x \left( \tau_4^\mu c^a A_\mu^a - \sqrt{g} \tau_2 \bar{c}^a c^a \right)
\]

\[
= \int_M d^2x \left( \tau_3^\mu A_\mu^a + \tau_4^\mu b^a - \frac{1}{2} \sqrt{g} \tau_2 \bar{c}^a c^a + \sqrt{g} (\tau_1 + m^2) \bar{c}^a c^a \right)
\]

such that

\[
s\tau_2 = -(\tau_1 + m^2),
\]

\[
s\tau_1 = 0
\]

\[
s\tau_4^\mu = \tau_3^\mu,
\]

\[
s\tau_3^\mu = 0.
\]

(20)

The external sources \( \tau_1 \) and \( \tau_2 \) are scalars and \( \tau_3^\mu \) and \( \tau_4^\mu \) are contravariant vector densities of weight one. By just looking at the transformations (20) we observe that the BRS operator \( s \) is still nilpotent \( s^2 = 0 \) and then \( \Sigma_m \) is by construction BRS invariant.

Another important remark is that the metric \( g_{\mu\nu} \) does still appear in the action only through the BRS exact term \( (\Sigma_m + \Sigma_g) \). Thus the physical observables still do not depend on \( g_{\mu\nu} \), in turn this fact enables us to maintain the BRS transformation of the

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4 We also want to preserve the algebraic structure (17), which we will promote to the functional level (see below).
metric \( (15) \). In table 1 we give the dimensions, the ghost numbers as well as the weights of the above introduced external sources.

|       | \( \tau_1 \) | \( \tau_2 \) | \( \tau_3^\mu \) | \( \tau_4^\mu \) |
|-------|--------------|--------------|----------------|----------------|
| dim   | 2            | 2            | 1             | 1              |
| \( \Phi \Pi \) | 0         | -1           | 1             | 0              |
| Weight | 0           | 0            | 1             | 1              |

Table 1: Dimensions, ghost numbers and weights of the new external sources.

In order to write down the Slavnov identity we couple the nonlinear BRS transformations in \((14)\) to external sources \((18)\), which leads to the external part of the action \( \Sigma_{\text{ext}} \). Hence, the total action takes the form:

\[
\Sigma = \Sigma_{\text{inv}} + \Sigma_{\text{gf}} + \Sigma_{m} + \Sigma_{\text{ext}}.
\]

where,

\[
\Sigma_{\text{ext}} = \int_{\mathcal{M}} d^2x [\Omega^{\mu}(sA^\mu_a) + L^a(s\varphi^a) + \rho^a(s\varphi^a)]
\]

with \( \Omega^{\mu} \) a contravariant vector density of weight +1, \( L^a \) and \( \rho^a \) are both scalar densities of weight +1. In table 2 we give the dimensions, the Faddeev-Popov charges and the weights of the different fields\(^5\):

|       | \( A^a_\mu \) | \( \varphi^a \) | \( c^a \) | \( \bar{c}^a \) | \( b^a \) | \( \Omega^{a\mu} \) | \( L^a \) | \( \rho^a \) | \( g_{\mu\nu} \) | \( \hat{g}_{\mu\nu} \) | \( \varepsilon^\mu \) | \( \xi^\mu \) |
|-------|---------------|---------------|--------|-------------|--------|--------------|--------|--------|--------------|--------------|-------------|-------------|
| dim   | 1             | 0             | 0      | 0           | 1      | 2            | 2      | 0      | 0            | -1           | -1          |             |
| \( \Phi \Pi \) | 0           | 0             | 1      | -1          | 0      | -1           | -2     | -1     | 0            | 1            | 1           | 2           |
| Weight | 0           | 0             | 0      | 0           | 1      | 1            | 1      | 0      | 0            | 0            | 0           | 0           |

Table 2: Dimensions, ghost numbers and weights of the fields.

Now we are ready to write down the Slavnov identity corresponding to the BRS invariance of the total action \((21)\) at the functional level:

\[
S(\Sigma) = \int_{\mathcal{M}} d^2x \left( \frac{\delta \Sigma}{\delta \varphi^a_a} \frac{\delta \varphi^a}{\delta A^\mu_a} + \frac{\delta \Sigma}{\delta \rho^a} \frac{\delta \rho^a}{\delta \varphi^a} + \frac{\delta \Sigma}{\delta L^a} \frac{\delta L^a}{\delta c^a} + b^a \frac{\delta \Sigma}{\delta c^a} + \hat{g}_{\mu\nu} \frac{\delta \Sigma}{\delta \hat{g}_{\nu\mu}} + \tau^\mu_3 \frac{\delta \Sigma}{\delta \tau^\mu_4} - (\tau_1 + m^2) \frac{\delta \Sigma}{\delta \tau_2} \right) = 0.
\]

From the above Slavnov identity we get the linearized Slavnov operator:

\(^5\) The vector \( \varepsilon^\mu \) is the parameter of the diffeomorphism transformations (see below).
\[
S_\Sigma = \int d^2x \left( \frac{\delta \Sigma}{\delta \Omega^\mu} \frac{\delta}{\delta A^a_\mu} + \frac{\delta \Sigma}{\delta A^a_\mu} \frac{\delta}{\delta \Omega^\mu} + \frac{\delta \Sigma}{\delta \rho^a} \frac{\delta}{\delta \phi^a} + \frac{\delta \Sigma}{\delta \rho^a} \frac{\delta}{\delta \phi^a} + \frac{\delta \Sigma}{\delta \phi^a} \frac{\delta}{\delta \rho^a} + \frac{\delta \Sigma}{\delta \phi^a} \frac{\delta}{\delta \rho^a} + \frac{\delta \Sigma}{\delta \phi^a} \frac{\delta}{\delta \rho^a} + \frac{\delta \Sigma}{\delta \phi^a} \frac{\delta}{\delta \rho^a} \right) + \frac{\delta \Sigma}{\delta \phi^a} \delta L^a + b^a \frac{\delta}{\delta \bar{c}^a} \delta L^a + g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} \tau_3^\mu - (\tau_1 + m^2) \delta \tau_2^2 \right) .
\]

(24)

In the presence of the external sources, the Ward operator corresponding to the local supersymmetry–like transformations writes down as

\[
W^S_\xi = \int d^2x \left( \varepsilon_\mu \xi^\mu \rho^a \frac{\delta}{\delta A^a_\mu} - \varepsilon_\mu \xi^\mu (\Omega^\mu_{\nu} + \sqrt{g} g^{\nu\sigma} \partial_{\nu} \bar{c}^a - \tau_4^\nu \bar{c}^a) \frac{\delta}{\delta \phi^a} - \xi^\mu A^a_\mu + L_\xi \bar{c}^a \frac{\delta}{\delta \phi^a} - L_\xi \bar{c}^a \frac{\delta}{\delta \phi^a} - \chi \bar{c}^a \frac{\delta}{\delta \phi^a} + \chi \bar{c}^a \frac{\delta}{\delta \phi^a} \right) + \left( L_\xi \tau_4^\nu - \xi^\mu \sqrt{g} (\tau_1 + m^2) + \xi^\mu (s \sqrt{g}) \tau_2 \right) \frac{\delta}{\delta \tau_2} \frac{\delta}{\delta \tau_2} + \left( L_\xi \tau_4^\nu - \xi^\mu \sqrt{g} (\tau_1 + m^2) + \xi^\mu (s \sqrt{g}) \tau_2 \right) \frac{\delta}{\delta \tau_2} \frac{\delta}{\delta \tau_2} .
\]

(25)

and when it acts on the total action (21) we get the linear breaking

\[
W_\xi \Sigma = \Delta_\xi ,
\]

(26)

where,

\[
\Delta_\xi = \int d^2x \left( - \Omega^\mu_{\nu} L_\xi A^a_\mu - \rho^a L_\xi \phi^a + L^a L_\xi \phi^a + \varepsilon_\mu \varepsilon_\lambda \rho^a s(\sqrt{g} g^{\mu\nu} \partial_{\nu} \bar{c}^a) - \varepsilon_\mu \varepsilon_\nu \rho^a s(\tau_4^\mu \bar{c}^a) \right) .
\]

(27)

At this level we see that the Ward identity of the local susy–like symmetry is linearly broken. This is not the case for the topological Yang-Mills model [22], the three dimensional BF model [7], and the Chern-Simons model [13] considered in a curved manifold, where one had to do with a hard breaking.

By construction \( \Sigma \) is also invariant under the diffeomorphism transformations

\[
W^D_\varepsilon \Sigma = \int d^2x \sum_f \mathcal{L}_f \frac{\delta \Sigma}{\delta f} = 0 .
\]

(28)

The letter \( f \) stands for all the fields describing the model under investigation, whereas \( \varepsilon_\mu \) is the parameter of the diffeomorphism transformations and \( \mathcal{L}_f \) denotes the corresponding Lie derivative. Furthermore, the action \( \Sigma \) obeys three constraints:

(i) the gauge condition

\[
\frac{\delta \Sigma}{\delta \phi^a} = \partial_\nu (\sqrt{g} g^{\mu\nu} A^a_\mu) + \sqrt{g} \tau_2 \phi^a + \tau_4^\mu A^a_\mu ,
\]

(29)
(ii) the antighost equation
\[
\frac{\delta \Sigma}{\delta c^a} + \partial_\nu (\sqrt{g} g^{\mu \nu} \frac{\delta \Sigma}{\delta \Omega^{a\mu}}) + \tau^a_4 \frac{\delta \Sigma}{\delta \Omega^{2\mu}} - \sqrt{\tau_2} 2 \frac{\delta \Sigma}{\delta L^a} = \partial_\nu \left( s (\sqrt{g} g^{\mu \nu}) A^a_\mu \right) + \sqrt{g} (\tau_1 + m^2) c^a - \tau^a_3 A^a_\mu,
\]

(30)

(iii) and the ghost equation
\[
\int_\mathcal{M} d^2 x \left( \frac{\delta S}{\delta c^a} + f^{abc} \phi \frac{\delta S}{\delta b^c} \right) = \int d^2 x \left( f^{abc} (\Omega^{b\mu} A^c_\mu - L^b c^c + \rho^b \varphi^c) - \sqrt{\tau_1} (1 + m^2) c^a - \sqrt{\tau_2} b^a \right),
\]

(31)

To obtain the ghost equation, one has simply to integrate over the space–time the quantity \( (\frac{\delta \Sigma}{\delta c^a}) \) and then use the gauge condition (29).

We end this section by displaying the algebraic structure of the model. First, consider an arbitrary local functional \( \Gamma \) depending on the fields \( (A^a_\mu, \varphi^a, c^a, \tilde{c}^a, b^a, \Omega^{a\mu}, L^a, \rho^a, g_{\mu\nu}, \tilde{g}_{\mu\nu}, \tau_1, \tau_2, \tau^a_3, \tau^a_4) \), one can derive the following nonlinear algebra
\[
S_\Gamma S(\Gamma) = 0,
\]
\[
S_\Gamma (W^D_\epsilon) + W^D_\epsilon S(\Gamma) = 0,
\]
\[
\{W^D_\epsilon, W^D_{\epsilon'}\} \Gamma = -W^D_{(\epsilon, \epsilon')} \Gamma,
\]
\[
S_\Gamma (W^S_\xi) \Gamma = W^S_\xi \Gamma,
\]
\[
\{W^D_\epsilon, W^S_\xi\} \Gamma = W^S_{(\xi, \epsilon)} \Gamma,
\]
\[
\{W^S_\xi, W^S_{\xi'}\} \Gamma = 0.
\]

(32)

Second, if the functional \( \Sigma \) obeys the Slavnov identity as well as the two Ward identities for diffeomorphisms and local susy–like symmetry, then we get the linear off–shell algebra
\[
\{S_\Sigma, S_\xi\} = 0,
\]
\[
\{S_\Sigma, W^D_\epsilon\} = 0,
\]
\[
\{W^D_\epsilon, W^D_{\epsilon'}\} = -W^D_{(\epsilon, \epsilon')} \xi,
\]
\[
\{S_\Sigma, W^S_\xi\} = W^D_\xi,
\]
\[
\{W^S_\xi, W^D_\epsilon\} = W^S_{(\xi, \epsilon)},
\]
\[
\{W^S_\xi, W^S_{\xi'}\} = 0.
\]

(33)

Here, we have used the following notation
\[
[\epsilon, \epsilon']^\mu = \epsilon^\lambda \partial_\lambda \epsilon'^\mu + \epsilon'^\lambda \partial_\lambda \epsilon^\mu,
\]
\[
[\xi, \epsilon]^\mu = \xi^\lambda \partial_\lambda \epsilon^\mu - \epsilon^\lambda \partial_\lambda \xi^\mu.
\]

(34)

Where \([, ,] \) stands for the graded Lie bracket. Furthermore, for reasons which will be clear in the next section, we have attributed ghost number one to \( \epsilon^\mu \), the vector parameter of the diffeomorphism transformations.
We conclude this section by remarking that the mass $m$, used to regularize the infrared divergent propagator, as in (18) could destroy the algebraic structure (17) of the model at the off-shell level (33). To maintain this structure, in the presence of a curved background, the price to pay was the introduction of four new external sources $\tau_1, \tau_2, \tau_3^\mu$ and $\tau_4^\mu$, which appear in the action through the metric dependent and BRS–exact expression $\Sigma_m$.

3 Cohomology analysis

In this section we will look for all possible quantum corrections for the model. Indeed, the construction of the most general counterterm can be done as follows, first we add a perturbation $\Delta$ to the total action $\Sigma$ such that the perturbed action $\Sigma' = \Sigma + \Delta$ fulfills the Slavnov identity (23), and the two Ward identities (26), (28) as well as the identities (29), (30) and (31). Therefore $\Delta$ must obey the constraints:

\[ \delta \Delta = 0, \]
\[ \frac{\delta \Delta}{\delta b^a} = 0, \]  
\[ \frac{\delta \Delta}{\delta \bar{c}^a} + \partial_\mu (\sqrt{g} g^{\mu \nu} \frac{\delta \Delta}{\delta \Omega_{\nu a}}) + \tau_4^\mu \frac{\delta \Delta}{\delta \Omega_{\mu a}} - \sqrt{g} \tau_2 \frac{\delta \Delta}{\delta L^a} = 0, \]  
\[ \int_\mathcal{M} d^2x \frac{\delta \Delta}{\delta \bar{c}^a} = 0, \]  
\[ S_\Sigma \Delta = 0, \]  
\[ W^D_\varepsilon \Delta = 0, \]  
\[ W^S_\xi \Delta = 0. \]

$\Delta$ is an integrated local polynomial of dimension, weight and ghost number zero. From equation (33) we immediately see that $\Delta$ does not depend on the Lagrange multiplier field $b^a$. On the other hand, from (36) we deduce that the integrated polynomial $\Delta$ can depend on the fields $\Omega^{a\mu}, \bar{c}^a$ and $L^a$ only through the combinations

\[ \tilde{\Omega}^{a\mu} = \Omega^{a\mu} + \sqrt{g} g^{\mu \nu} \partial_\nu \bar{c}^a - \tau_4^\mu \bar{c}^a, \]  
\[ \tilde{L}^a = L^a - \sqrt{g} \tau_2 \bar{c}^a. \]

Concerning the three equations (38) – (40), we put them in a single equation

\[ \delta \Delta = 0 \]  

such that the operator $\delta$ is of the form

\[ \delta = S_\Sigma + W^D_\varepsilon + W^S_\xi + \int_\mathcal{M} d^2x (L_\varepsilon \xi^\mu) \frac{\delta}{\delta \xi^\mu} + \int_\mathcal{M} d^2x \left( \frac{1}{2} L_\varepsilon \bar{c}^\mu - \xi^\mu \right) \frac{\delta}{\delta \bar{c}^\mu}. \]
An easy check is to show its nilpotency,

$$\delta^2 = 0,$$

so that (44) is a cohomology problem possessing two possible solutions. Indeed, (44) possesses solutions of the form $\delta = \delta \hat{\Delta}$. These are called trivial solutions because the nilpotency of $\delta$ immediately implies that any expression of the form $\delta \hat{\Delta}$ is automatically a solution of (44). In what follows we will call cohomology of $\delta$ the space of all solutions of (44) modulo trivial solutions.

The first step in solving the cohomology problem (44) is to introduce a filtering operator $\mathcal{N}$ and assign to each field (including $\varepsilon^\mu$ and $\xi^\mu$) homogeneity degree 1.

$$\mathcal{N} = \int_M d^2 x \sum_f f \frac{\delta}{\delta f} + \int_M d^2 x \left( \varepsilon^\mu \frac{\delta}{\delta \varepsilon^\mu} + \xi^\mu \frac{\delta}{\delta \xi^\mu} \right).$$

(45)

The operator $\mathcal{N}$ induces a decomposition of $\delta$

$$\delta = \delta_0 + \delta_1 + \ldots + \delta_N,$$

(46)

as well as a decomposition of $\Delta$

$$\Delta = \sum_{n \geq 0} \Delta_n,$$

(47)

where the index $n$ refers to the corresponding homogeneity degree. The operator $\delta_0$ in (46) has the property that it does not increase the homogeneity degree when it acts on a field polynomial. On the other hand, due to the nilpotency of $\delta$ we also have

$$\delta_0^2 = \{ \delta_0, \delta_1 \} = 0,$$

(48)

and more generally

$$\sum_{i=0}^k \delta_i \delta_{k-i} = 0; \quad k \leq N.$$

(49)

An obvious identity which follows from (46) and $\delta \Delta = 0$ reads as

$$\delta_0 \Delta = 0,$$

(50)

Due to the nilpotency of $\delta_0$ (48), the above equation (50) defines a new cohomology problem. An interesting result is the following theorem:

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An immediate corollary of the theorem is as follows: if the cohomology of $\delta_0$ is empty (trivial), then that of $\delta$ is also empty.
Theorem:
The cohomology of the operator $\delta$ is isomorphic to a subspace of the cohomology of the operator $\delta_0$.

More concretely, for the 2D model under investigation, we have

$$\delta_0 = \int_M \left( d\alpha^a \frac{\delta}{\delta \alpha^a} + dA^a \frac{\delta}{\delta A^a} + d\varphi^a \frac{\delta}{\delta \varphi^a} + d\hat{\Omega}^a \frac{\delta}{\delta \hat{\Omega}^a} + d\hat{\rho}^a \frac{\delta}{\delta \hat{\rho}^a} \right) +$$

$$+ \int_M d^2x \left( \hat{g}_{\mu\nu} \frac{\delta}{\delta \hat{g}_{\mu\nu}} - \tau_1 \frac{\delta}{\delta \tau_2} + \tau_3 \frac{\delta}{\delta \tau_4} - \xi^\mu \frac{\delta}{\delta \xi^\mu} \right)$$

The first part of the expression of $\delta_0$ is given in terms of forms where,

$$A^a = A^a_{\mu} dx^\mu,$$

$$\hat{\Omega}^a = \varepsilon_{\mu\nu} \hat{\Omega}^{a\mu} dx^\nu,$$

$$\hat{L}^a = \frac{1}{2} \varepsilon_{\mu\nu} \hat{L}^a dx^\mu dx^\nu,$$

$$\hat{\rho}^a = \frac{1}{2} \varepsilon_{\mu\nu} \rho^a dx^\mu dx^\nu,$$

and $d$ is the exterior derivative $d = dx^\mu \partial_\mu$.

One can easily see from (51) that the following couples of fields $(g_{\mu\nu}, \hat{g}_{\mu\nu}), (-\tau_1, \tau_2), (\tau^\mu_3, \tau^\mu_4)$ and $(\varepsilon^\mu, -\xi^\mu)$ appear in $\delta_0$-doublets, and then are not present in the cohomology.

Now, let us solve $\delta_0 \Delta = 0$. The local integrated polynomial $\Delta$ can be written as

$$\Delta = \int_M \omega^0_2$$

where $\omega^p_q$ is a polynomial of ghost number $p$ and form degree $q$. By letting the operator $\delta_0$ acting on (53) and taking into account (50) we get, after using Stock’s theorem

$$\delta_0 \omega^0_2 + d\omega^1_1 = 0$$

Now, by applying once again $\delta_0$ on equation (54) and using the algebraic Poincare lemma and the facts that $\delta^2_0 = 0$ and $\{\delta_0, d\} = 0$ we get the descent equations

$$\delta_0 \omega^0_2 + d\omega^1_1 = 0,$$

$$\delta_0 \omega^1_1 + d\omega^2_0 = 0,$$

$$\delta_0 \omega^2_0 = 0.$$  

It is clear that the solution of the last equation in (55) is

$$\omega^2_0 = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \alpha_{n,m} Tr(\varphi^n c \varphi^m c).$$

7Roughly speaking, the algebraic Poincare lemma states that, in the space of forms depending on the fields and their derivatives, the cohomology of the exterior derivative $d$ is trivial. For the exact formulation and proof of the lemma see [11].

8where $\varphi = \varphi^a T^a$ and $c = c^a T^a$. $Tr$ is the trace defined by $Tr(T^a T^b) = \delta^{ab}$ and $T^a$ are the generators of the gauge group.
where $\alpha_{n,m}$ are constant coefficients. The condition that the solution of (42) must be invariant under the whole operator $\delta$ implies that $\alpha_{n,m} = 0$ unless $m = 0$. In this case we define $\alpha_{n,0} \equiv \alpha_n$. In turn, $\omega_0^2$ takes the form

$$\omega_0^2 = \sum_{n=1}^{\infty} \alpha_n \text{Tr}(\varphi^n c c).$$  \hspace{1cm} (57)

For $n = 0$, of course, one has $\omega_0^2 = \text{Tr}(c^2) = c^a c^a = 0$. The expression (57) leads to the nontrivial counterterm

$$\Delta_c = \sum_{n=1}^{\infty} \alpha_n \text{Tr} \left( \int_{\mathcal{M}} \left( \sum_{j=0}^{n-1} \varphi^{j} \delta \varphi^{j-1} \varphi^{(n-i-1)} + \sum_{j=0}^{n-1} \varphi^{j} \tilde{L} \varphi^{(n-i-1)} + 2 \sum_{j=0}^{n-1} \varphi^{j} \varphi^{(n-i-1)} \{A, c\} + 2 \varphi^n \{\hat{\rho}, c\} + 2 \varphi^n A^2 \right) \right),$$  \hspace{1cm} (58)

which is invariant under the whole operator $\delta$ defined in (43). Furthermore, the trivial solution $\delta \hat{\Delta}$ of (42) is given by

$$\delta \hat{\Delta} = \int_{\mathcal{M}} d^2x \left\{ \sqrt{g} \tau_2 f_1(\varphi) + \frac{1}{\sqrt{g}} g_{\mu \nu} \tau_4 (\sum_{n=1}^{\infty} \beta_{1,n} \text{Tr}(\hat{\varphi}^n) + \sum_{n=1}^{\infty} \beta_{2,n} \text{Tr}(\hat{\varphi}^n) + \sum_{n=1}^{\infty} \beta_{3,n} \text{Tr}(\partial^\mu \hat{\varphi}^n) + \sum_{n=1}^{\infty} \beta_{4,n} \text{Tr}(\hat{\varphi}^{(n-1)} \{A, c\}) + \sum_{n=1}^{\infty} \beta_{5,n} \text{Tr}(\partial^\mu \hat{\varphi}^n) + \sum_{n=1}^{\infty} \beta_{6,n} \text{Tr}(\tilde{L}^a \varphi^n) \right\}.$$

(59)

(\kappa_1, \ldots, \kappa_4) \text{ and } (\beta_{1,n}, \ldots, \beta_{6,n}) \text{ are constant coefficients and the functions } f_i \text{ with } (1 \leq i \leq 5) \text{ are given by}

$$f_i(\varphi) = \sum_{n=1}^{\infty} \alpha_{i,n} \text{Tr} \varphi^n.$$  \hspace{1cm} (60)

On the other hand, the trivial counterterm (59) would depend on the transformation parameters $\xi^\mu$ and $\varepsilon^\mu$ which are present in $\delta$ (43). The requirement that (59) must be independent of this two transformation parameters reduces the trivial counterterm to the
simpler form
\[\delta \Delta = S \Sigma \int_M d^2x \kappa (\rho^a \varphi^a + \tilde{L}^a c^a - \tilde{\Omega}^a \mu A^a),\]
\[\equiv S \Sigma \Delta,\]
with \(\kappa\) a constant. So, the possible deformation of the total action (21) is of the general form
\[\Delta = \Delta_c + S \Sigma \Delta,\]
\[(61)\]
In constructing the counterterm (62) we took into account all the constraints (35)–(40) except the ghost equation (37). In fact, the expression (62) violates (37) unless all the constant coefficients appearing in (62) vanish. Therefore the action (21) admits no deformations and all the quantities present at the classical level remain the same and receive no corrections. Furthermore, if the constraints hold at the quantum level and the symmetries are not broken, then the complete absence of deformations would imply the absence of quantum corrections. In this case the theory is said to be finite.

### 4 Anomaly analysis

The last point to be discussed is the possibility of extending the above analysis to all orders of perturbation theory. This fact is only allowed when anomalies are absent.

The three conditions (35), (36) and (37) can be proven to be renormalizable at all orders of perturbation theory by using the arguments of [8] and [9]. Concerning the Slavnov identity and the two Ward identities for diffeomorphisms and local susy-like symmetry, if there is an anomaly, then for the generating functional of vertex functions \(\Gamma = \Sigma + O(\hbar)\) we must have
\[\delta \Gamma = \mathcal{A}.\]
\[(63)\]
Due to the nilpotency of \(\delta\) we get a new cohomology problem
\[\delta \mathcal{A} = 0,\]
\[(64)\]
where \(\mathcal{A}\) is an integrated local polynomial of form degree 2 and ghost number 1.
\[\mathcal{A} = \int_M \omega_1^1.\]
\[(65)\]
Now, using the same strategy as explained in the previous section, we get the following set of descent equations
\[\delta_0 \omega_1^1 + d \omega_1^1 = 0,\]
\[\delta_0 \omega_1^2 + d \omega_1^2 = 0,\]
\[\delta_0 \omega_1^3 = 0.\]
\[(66)\]
The last equation in (66) is solved by
\[\omega^3_0 = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \alpha_{n,m,r} Tr(\varphi^n c \varphi^m c \varphi^r c).\] (67)

This will yield an expression of \(A\) which is not invariant under the whole operator \(\delta\), i.e.
\[\delta A \neq 0\] unless \(\alpha_{n,m,r} = 0\) for all nonvanishing values of \(n\), \(m\), and \(r\). This particularly means that we are left with the single term \(\omega^3_0 = \alpha f^{abc} c^a c^b c^c\), which leads to
\[\omega^1_2 = \alpha f^{abc} (\hat{\rho}^a c^b c^c + A^a A^b c^c).\] (68)

A quick verification shows that \(\omega^1_2\) is invariant under the whole operator \(\delta\). Hence, the possible anomaly candidate solving (64) takes the form
\[A = \alpha \int_M f^{abc} (\hat{\rho}^a c^b c^c + A^a A^b c^c),\] (69)
where \(\alpha\) stands for a constant coefficient. But this anomaly candidate violates the ghost equation (37), a fact which imposes the restriction \(\alpha = 0\). The final result is as follows: the BRS symmetry, the diffeomorphisms and the local susy–like symmetry are anomaly free, then valid at the quantum level. Therefore, the 2D model is anomaly free and ultraviolet as well as infrared finite at all orders of perturbation theory. However, as already mentioned in the introduction, our proof of the finiteness is only valid in the case of a topologically trivial and asymptotically flat manifolds.

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