On the linear and weak-field limits of scalar-tensor theories of gravity

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The linear approximation of scalar-tensor theories of gravity is obtained in the physical (Jordan) frame under the 4+0 (covariant) and 3+1 formalisms. Then the weak-field limit is analyzed and the conditions leading to significant deviations of the $1/r^2$ Newton’s law of gravitation are discussed. Finally, the scalarization effects induced by these theories in extended objects are confronted within the weak-field limit.

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I. INTRODUCTION

In the last decade there has been an increasing interest in the so-called scalar-tensor theories of gravity (STT; see Ref. [1] for a review) in view of the possible deviations that Einstein’s general theory of relativity (GR) could show in the framework of several upcoming observations. Among these, there has been a considerable interest in detecting a scalar-wave component (spin-0 waves) in addition to the ordinary gravitational waves (spin-2 waves) predicted by GR (see for instance [2, 3, 4, 5, 6]). Among the potential emitters of such scalar-waves are the compact binary systems and neutron stars. Yet, the binary pulsar has showed no appreciable deviations from GR due to the emission of scalar waves, and then one is only able to put stringent bound in the STT parameters or couplings [7]. Moreover, through the effects of spontaneous scalarization in neutron stars [8, 9, 10], it seems that scalar-waves have also a small chance to be observed in detectors like VIRGO or LIGO within a few hundreds of kiloparsecs [8]. However, there is the hope that resonant mass detectors of spherical shape or interferometers like LISA can resolve the existence of a scalar-gravitational wave [8, 9, 10].

At the large scale (cosmological scales), STT have been proposed as models for dark energy that can replace the cosmological constant [11]. However, detailed analysis show that when all the observational constraints (both cosmological and local) are taken into account, the simpler STT fail or are extremely constrained [11]. At this point one cannot only but recognizes the predictive power of GR, confirmed by experiments and observations that were maybe far from being imagined by Einstein at the time of the creation of his theory. At smaller scales (scales of the order of meters or kilometers) the spurious discovery of a fifth force renewed the idea about the existence of new fundamental fields of meter-range [14]. At this regard, considerable effort has been put in measuring gravitationally such kind of interactions that can mimic a varying gravitational “constant” $G$.

In this report the linear limit of STT is reviewed under the usual 4+0 formalism. The weak-field limit is taken and confronted in view of the past and recent experiments (ground based and satellite mission) intended to test the gravitational interaction between bodies. The analysis is performed in the Jordan frame, and therefore no intermediate unphysical variables or transformations are introduced. A Yukawa potential appears as a new term in addition to the ordinary Newtonian potential due to the non-minimal coupling between the scalar field with gravity, the entire gravitational potential having been identified with the potential of test particles. It is then argued that the Yukawa potential can produce significative deviations on the $1/r^2$ Newton’s gravitation law if the scalar “particles” are massive enough.

An appendix analyzing the linear limit of STT under the 3+1 formulation of GR is to be found at the end.

II. SCALAR-TENSOR THEORIES OF GRAVITY

The general action for a scalar-tensor theory of gravity with a single scalar field is given by

$$S[g_{\mu\nu}, \phi, \psi] = \int \left\{ \frac{1}{16\pi G_0} F(\phi) R - \left( \frac{1}{2} (\nabla \phi)^2 + V(\phi) \right) \right\} \times \sqrt{-g} d^4 x + S[g_{\mu\nu}, \psi] \quad (1)$$

where $\psi$ represents collectively the matter fields (fields other than $\phi$).

The representation of the scalar-tensor theories given by Eq. (1) is called the Jordan frame representation. One
can parametrize the same theories as

\[ S[g_{\mu\nu}, \Phi, \psi] = \frac{1}{16\pi G_0} \int \left\{ \Phi R - \frac{\omega(\Phi)}{\Phi} (\nabla \Phi)^2 + 2\Phi \lambda(\Phi) \right\} \times \sqrt{-g} d^4x + S[g_{\mu\nu}, \psi] \tag{2} \]

where

\[ \Phi := F(\phi) \]  
\[ \omega_{\text{BD}}(\Phi) := \frac{8\pi G_0 \Phi}{(\partial_\phi F)^2} \]  
\[ \lambda(\Phi) := -8\pi V(\phi)/\Phi \] . \tag{5}

For instance, the Jordan-frame representation of the Brans-Dicke theory with \( \omega = \text{const.} \) corresponds to \( F = 2\pi G_0 \phi^2/\omega \) and \( V(\phi) = 0 \). It is also customary to parametrize the scalar-tensor theories in the so-called Einstein frame by introducing non-physical fields as follows,

\[ g^*_{\mu\nu} := F(\phi)g_{\mu\nu} \]  
\[ \phi^* = \int \left[ \frac{3}{4} \frac{1}{F^2(\phi)} (\partial_\phi F)^2 + \frac{4\pi}{F(\phi)} \right]^{1/2} d\phi \]  
\[ F^*(\phi^*) = F(\phi) \] . \tag{8}

so that the action Eq. (6) takes the form

\[ S[g^*_{\mu\nu}, \phi^*, \psi] = \frac{1}{16\pi G_0} \int \left[ R^* - 2(\nabla^* \phi^*)^2 - V^*(\phi^*) \right] \times \sqrt{-g^*} d^4x + S[g^*_{\mu\nu}, F^*(\phi^*), \psi] \] , \tag{9}

where all quantities with \( ^* \) are computed with the non-physical metric \( g^*_{\mu\nu} \) and \( \phi^* \).

We emphasize that although the equations of motions obtained from the Einstein frame are simpler than those from the Jordan frame, in the sense that the field \( \phi^* \) appears to be coupled minimally to the non-physical metric, the matter equations derived from the Bianchi identities \( \nabla^*_\mu G^*_\mu\nu = 0 \) will have sources, i.e., \( \nabla^*_\mu T^*_\mu\nu \neq 0 \), where here \( T^*_\mu\nu = T^\mu\nu/F^2(\phi^*) \) is the non physical energy-momentum tensor of the matter fields \( \psi \). However, in the Jordan frame the matter equations resulting from the Bianchi identities \( \nabla_\mu G^*_\mu\nu = 0 \) turn to satisfy \( \nabla_\mu T^\mu\nu = 0 \), reflecting explicitly the fulfillment of the Einstein’s weak equivalence principle (that is the origin of the name “physical metric”).

In the following the Jordan frame representation of the scalar-tensor theories will only be used. The equations of motion obtained from a variational principle using Eq. (6) are

\[ G^\mu\nu = 8\pi G_0 T^\mu\nu , \tag{10} \]
\[ T^\mu\nu := G_{\text{eff}} \frac{F}{F} (T^\mu\nu + T^\mu\nu_{\text{sf}} + T^\mu\nu_{\text{matt}}) , \tag{11} \]
\[ T^\mu\nu_{\text{sf}} := \left[ \nabla^\nu (\partial_\phi F \nabla^\mu \phi) - g^{\mu\nu} \nabla_\lambda (\partial_\phi F \nabla^\lambda \phi) \right] , \tag{12} \]
\[ G_{\text{eff}} := \frac{G_0}{F} , \tag{14} \]

where \( T_{\text{matt}} \) stands for the trace of \( T^\mu\nu_{\text{matt}} \) and the subscript “matt” refers to the matter fields other that \( \phi \). Now, the Bianchi identities imply

\[ \nabla_\mu T^\mu\nu = 0 . \tag{16} \]

However, the use of the equations of motion leads as mentioned to the energy-conservation equations of matter

\[ \nabla_\mu T^\mu_{\text{matt}} = 0 , \tag{17} \]

which implies in the case of test particles, that bodies are subject to no other long range forces than the gravitational ones (free falling particles). In other words, the scalar field being not directly coupled to matter no direct interaction between the scalar field and matter arises. The scalar field \( \phi \) will only interact with matter gravitationally, i.e., only through the curvature effects.

The final form of the field equations have exactly the same form as in general relativity with an effective energy-momentum tensor. This means that one can formulate the Cauchy problem for scalar-tensor theories exactly in the same manner as in GR (see the Appendix).
III. LINEAR LIMIT OF STT

The linear limit of STT has been analyzed in the past by many authors. Wagoner was one of the first in analyzing the gravitational wave emission in STT and the weak-field approximation (an updated analysis was performed in ). Such an analysis was performed in the Einstein frame. More recently, Pimentel and Obregón performed a similar analysis in the Brans-Dicke representation as in Eq. (3).

As mentioned, the linear limit of STT treated here is performed in the Jordan frame. As usual, we consider first order perturbations of the Minkowski spacetime:

\[ g_{\mu\nu} \approx \eta_{\mu\nu} + \epsilon \gamma_{\mu\nu} \, , \]
\[ T_{\mu\nu} \approx T^0_{\mu\nu} + \varepsilon \tilde{T}_{\mu\nu} \, , \]
\[ \phi \approx \phi_0 + \epsilon \tilde{\phi} \, , \]
\[ F(\phi) \approx F_0 + \epsilon F'_0 \tilde{\phi} \, , \]
\[ \partial_\gamma F(\phi) \approx F'_0 + \epsilon F''_0 \tilde{\phi} \, , \]
\[ \partial^2_{\gamma\gamma} F(\phi) \approx F''_0 + \epsilon F'''_0 \tilde{\phi} \, , \]
\[ V(\phi) \approx V_0 + \epsilon V'_0 \tilde{\phi} \, , \]
\[ \partial_\gamma V(\phi) \approx V'_0 + \epsilon V''_0 \tilde{\phi} \, . \]

where \( \epsilon \ll 1 \) and the knott indicates quantities at zero order. In the 4+0 covariant formulation one can introduce the combination

\[ \tilde{\gamma}_{\mu\nu} := \gamma_{\mu\nu} + \kappa \eta_{\mu\nu} \tilde{\phi} \, , \]
\[ \tilde{\gamma}^{\mu\nu} := \gamma^{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \gamma \, , \]

where \( \gamma = \gamma'^\mu_\mu \) and \( \kappa \) is a gauge constant to be fixed later in order to simplify the equations. The resulting linearized Einstein equations are

\[ \tilde{G}_{\mu\nu} = \partial^\rho \partial_\rho \tilde{\gamma}_{\mu\nu} - \frac{1}{2} \Box_\eta \tilde{\gamma}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \partial^\rho \partial_\rho \tilde{\gamma} \sigma_\alpha \]
\[ = 8\pi G_0 \tilde{T}_{\mu\nu} + \kappa \left( \partial^2_{\mu\nu} \tilde{\phi} - \eta_{\mu\nu} \Box_\eta \tilde{\phi} \right) \, . \]

where \( T^0_{\mu\nu} = 0 \) results from the self-consistency of the perturbations at first order. Here \( \Box_\eta \) is the D’Alambertian operator compatible with the flat metric \( \eta_{\mu\nu} \). Moreover, the flat metric is used to raise and lower indices of first order tensorial quantities.

The Lorentz gauge

\[ \partial^\mu \tilde{\gamma}_{\mu\nu} = 0 \, , \]

generalizes the usual Lorentz gauge of GR, and can be imposed to simplify the equations. Then, from Eq. (28) the resulting wave equation is

\[ \Box_\eta \tilde{\gamma}_{\mu\nu} = -16\pi G_0 \tilde{T}_{\mu\nu} - 2\kappa \left( \partial^2_{\mu\nu} \tilde{\phi} - \eta_{\mu\nu} \Box_\eta \tilde{\phi} \right) \, . \]

The linear approximation of the effective energy-momentum tensor Eq. (1) and the Klein-Gordon Eq. (2) turn to be

\[ \tilde{T}_{\mu\nu} = \frac{\tilde{T}^{\text{matt}}_{\mu\nu}}{F_0} + \frac{F'_0}{8\pi F_0 G_0} \left( \partial^\mu \tilde{\phi} - \eta_{\mu\nu} \Box_\eta \tilde{\phi} \right) \, , \]
\[ \Box_\eta \tilde{\phi} - m_0^2 \tilde{\phi} = 4\pi \alpha \frac{F'_0}{F_0} \tilde{T}^{\text{matt}}_{\mu\nu} \, , \]
\[ m_0^2 := \frac{V''_0}{1 + \frac{3}{16\pi G_0 F_0^2}} \, , \]

where we used the following conditions

\[ V(\phi_0) = 0 = V'_0 = 0 \, , \]

resulting from the consistency at first order of the linearized Einstein and Klein-Gordon equations and assuming that \( T^{\text{matt}}_{\mu\nu} = 0 \). In this way the wave equation Eq. (30) becomes

\[ \Box_\eta \tilde{\gamma}_{\mu\nu} = -16\pi G_0 \tilde{T}^{\text{matt}}_{\mu\nu} - 2 \left( \kappa + \frac{F'_0}{F_0} \right) \left( \partial^\mu \tilde{\phi} - \eta_{\mu\nu} \Box_\eta \tilde{\phi} \right) \, . \]

Note that at this order the application of the ordinary divergence in Eq. (30) and the use of the Lorentz gauge Eq. (29) leads to the energy-conservation of the matter perturbations: \( \partial^\mu T^{\text{matt}}_{\mu\nu} = 0 \)

The choice

\[ \kappa = -\frac{F'_0}{F_0} \, , \]

for the gauge constant simplifies the Eq. (30) considerably. So summarizing, we have the following wave equations for the gravitational and scalar modes:

\[ \Box_\eta \tilde{\gamma}_{\mu\nu} = -16\pi G_0 \tilde{T}^{\text{matt}}_{\mu\nu} \, , \]
\[ \partial^\mu \tilde{\gamma}_{\mu\nu} = 0 \, , \]
\[ \Box_\eta \tilde{\phi} - m_0^2 \tilde{\phi} = 4\pi \alpha \frac{F'_0}{F_0} \tilde{T}^{\text{matt}}_{\mu\nu} \, , \]

with the constants \( m_0^2, \alpha \) and \( \kappa \) given by Eqs. (33), (34) and (37) respectively. The analysis of propagation of gravitational and scalar waves will be not pursued here, this has been done elsewhere (see Refs. [2, 3, 4, 5, 6]).

A. The weak-field approximation

In the weak-field approximation one considers slow varying fields and sources such that the temporal derivatives in the D’Alambertian can be neglected and \( T^{\text{matt}}_{\mu\nu} \approx
\[ \rho \delta^{0} \delta_{\mu} \]. The wave equations read then,
\[ 3 \Delta \tilde{\gamma}_{\mu
u} = \frac{-16\pi G_{0}}{F_{0}} \tilde{T}_{\mu
u} , \quad (41) \]
\[ 3 \Delta \tilde{\phi} - m_{0}^{2} \tilde{\phi} = 4\pi \frac{F_{0}'}{F_{0}} \tilde{T}_{\mu
u} , \quad (42) \]
where \(3 \Delta\) is the three dimensional Euclidean Laplacian. The condition for slow varying sources \(T_{\mu\nu} \approx 0 \approx \tilde{T}_{\mu\nu}\), leads then to the following equations
\[ 3 \Delta \tilde{\gamma}_{00} = -\frac{16\pi G_{0}}{F_{0}} \tilde{\rho} , \quad (43) \]
\[ 3 \Delta \tilde{\phi} - m_{0}^{2} \tilde{\phi} = 4\pi \frac{F_{0}'}{F_{0}} \tilde{\rho} , \quad (44) \]
\[ \tilde{\gamma}_{0i} = \text{const.} = \tilde{\gamma}_{ij} , \quad (45) \]
where \(\tilde{\rho} := \tilde{T}_{\mu\nu} = -\tilde{T}_{\mu\nu}\). The constant solutions for \(\tilde{\gamma}_{\mu\nu}\) arise by demanding a well behavior of the metric at spatial infinity (without lost of generality the constants can be gauged out). Thus, with \(\tilde{\gamma}_{ij} = 0\) and Eqs. (26) and (27), it yields
\[ \tilde{\gamma}_{\mu\nu} = \gamma_{\mu\nu} - \eta_{\mu\nu} \left( \gamma_{00} + 2\kappa \phi \right) , \quad (46) \]
and therefore
\[ \tilde{\gamma}_{00} = 2 \left( \gamma_{00} + \kappa \phi \right) . \quad (47) \]
Since the geodesic equation for slow particles leads to the identification of
\[ \Phi_{N} = -\frac{1}{2} \gamma_{00} , \quad (48) \]
with the Newtonian potential \(\Phi\), it turns
\[ \Phi_{N} = -\frac{1}{4} \tilde{\gamma}_{00} + \frac{\kappa}{2} \tilde{\phi} . \quad (49) \]
The two gravitational degrees of freedom that remain in the weak-field limit are thus \(\gamma_{00}\) and \(\phi\) (\(\gamma_{ij} = 0\), and \(\gamma_{ij}\) is given in terms of \(\gamma_{00}\) and \(\phi\)).

The solutions of Eqs. (43) and (44) are respectively
\[ \tilde{\psi} = -\frac{G_{0}}{F_{0}} \int \frac{\tilde{\rho}(\tilde{x}')}{|\tilde{x} - \tilde{x}'|} d^{3}x' + \text{B.C.} , \quad (50) \]
\[ \tilde{\phi} = \alpha \frac{F_{0}'}{F_{0}} \int \frac{\tilde{\rho}(\tilde{x}') e^{-m_{0}|\tilde{x} - \tilde{x}'|}}{|\tilde{x} - \tilde{x}'|} d^{3}x' + \text{B.C.} , \quad (51) \]
where \(\tilde{\psi} := -\frac{1}{4} \tilde{\gamma}_{00}\). So finally
\[ \Phi_{N} = -\frac{G_{0}}{F_{0}} \int \frac{\tilde{\rho}(\tilde{x}')}{|\tilde{x} - \tilde{x}'|} d^{3}x' - \frac{\alpha}{2} \left( \frac{F_{0}'}{F_{0}} \right)^{2} \int \frac{\tilde{\rho}(\tilde{x}') e^{-m_{0}|\tilde{x} - \tilde{x}'|}}{|\tilde{x} - \tilde{x}'|} d^{3}x' + \text{B.C.} . \quad (52) \]

It is important to mention that the details of the general solution (interior plus exterior solutions, both matched continuously at the surface of the extended object) depend strongly on the boundary conditions.

**B. Regular solutions**

Explicit solutions of the gravitational and scalar fields depend on \(\tilde{\rho}(\tilde{x})\), and as mentioned on the boundary conditions as well. Clearly, a detailed solution of the whole system involves also the equation of hydrostatic equilibrium and the equation of state of matter \([17]\). However, here we do not want to be so explicit and only exhibit the possible deviations that the Newton’s gravitation law can suffer depending on the values of the parameters. For simplicity spherical symmetry is assumed. We seek for solutions for \(\psi\) and \(\phi\) with regularity conditions at \(r = 0\)
\[ \partial_{r} \tilde{\psi} = 0 = \partial_{r} \tilde{\phi} , \quad (53) \]
and asymptotic flatness conditions
\[ \lim_{r \to \infty} \tilde{\psi} \to 0 , \quad (54) \]
\[ \lim_{r \to \infty} \tilde{\phi} \to 0 . \quad (55) \]
The solutions are then given by
\[ \tilde{\psi}(r) = -\frac{G_{0}}{F_{0}} \int_{r}^{\infty} \frac{m(\tilde{r})}{\tilde{r}^{2}} d\tilde{r} , \quad (56) \]
\[ \tilde{\phi}(r) = \int_{r}^{\infty} \frac{q(\tilde{r})}{\tilde{r}^{2}} d\tilde{r} , \quad (57) \]
where
\[ m(r) := 4\pi \int_{0}^{r} \tilde{\rho}(\tilde{r}) \tilde{r}^{2} d\tilde{r} , \quad (58) \]
\[ q(r) := 4\pi \int_{r}^{\infty} \left( \alpha \frac{F_{0}'}{F_{0}} \tilde{\rho}(\tilde{r}) - \frac{m_{0}^{2}}{4\pi} \right) \tilde{r}^{2} d\tilde{r} . \quad (59) \]
One can define global quantities as
\[ M := \lim_{r \to \infty} \left( \frac{r^2}{G_0} \partial_r \hat{\psi} \right) \]
\[ = 4\pi \int_0^{+\infty} \rho(\hat{r}) r^2 d\hat{r} , \quad (60) \]
\[ Q := -\lim_{r \to \infty} \left[ r^2 \partial_r \hat{\phi} \right] \]
\[ = 4\pi \int_0^{+\infty} \left( \frac{F_0}{F_0} \rho(\hat{r}) - \frac{m_0^2}{4\pi} \phi \right) r^2 d\hat{r} . \quad (61) \]

Note that in the minimal-coupling case \((F_0 = 0), Q \equiv 0,\) since the only regular solution of the Helmholtz equation is \(\phi = 0.\)

If the density \(\rho\) has compact support (as it is usually the case for astrophysical bodies), then it vanishes for \(r \geq R,\) where \(R\) is the radius of the body. The solutions for \(\psi\) and \(\phi\) matched continuously at \(r = R\) can be then written as

\[ \hat{\psi}(r) = \begin{cases} \frac{G_0 M}{F_0^r} - \frac{G_0}{F_0} \int_0^r r \frac{m(\hat{r})}{2} d\hat{r} & \text{for } r \leq R , \\ \frac{G_0 M}{F_0^r} + \frac{G_0}{F_0} \int_r^R q(\hat{r}) \frac{1}{2} d\hat{r} & \text{for } r \geq R , \end{cases} \quad (62) \]
\[ \hat{\phi}(r) = \begin{cases} \frac{G_0}{F_0^r} \frac{r(1+m_{0}R)}{(1+\lambda R)} + \frac{F_0}{F_0} \rho(\hat{r}) & \text{for } r \leq R , \\ \frac{G_0}{F_0^r} \frac{1}{1+m_{0}R} \frac{m_{0}^2}{4\pi} \phi & \text{for } r \geq R \end{cases} \quad (63) \]

where
\[ m(r) := 4\pi \int_0^{r \leq R} \rho(\hat{r}) r^2 d\hat{r} , \quad (64) \]
\[ q(r) := 4\pi \int_0^{r \leq R} \left( \alpha \frac{F_0}{F_0} \rho(\hat{r}) - \frac{m_{0}^2}{4\pi} \phi \right) r^2 d\hat{r} , \quad (65) \]
and \(M := m(R), \quad \hat{Q} := q(R).\)

**IV. OBSERVATIONAL CONSTRAINTS**

In the following we consider only the exterior solutions and confronted them with observations. Therefore, the exterior Newtonian potential \(\Phi_N\) is given by

\[ \Phi_N = -\frac{G_0 M}{F_0 r} \left[ 1 + \frac{\alpha \sigma}{2G_0} \frac{(F_0')^2}{F_0} e^{-m_0 r} \right] \quad (66) \]

where \(\sigma\) is a constant that depends on the global properties of the source. Namely,

\[ \sigma = \frac{\hat{Q} e^{m_0 r} F_0}{(1+m_{0}R) M \alpha F_0} = \frac{e^{m_0 r}}{1+m_{0}R} \left( 1 - \frac{m_{0}^2}{M \alpha F_0} \int_0^R \phi(r') r'^2 dr' \right) . \quad (67) \]

It is customary to express the coefficients involving \(F_0'\) in terms of the effective Brans-Dicke parameter Eq. (3)

\[ \omega_{BD}^0 = \frac{8\pi G_0 F_0}{(F_0')^2} . \quad (68) \]

Then,

\[ \alpha = \frac{\omega_{BD}^0}{4\pi (3 + 2\omega_{BD}^0)} , \quad (69) \]
\[ m_0^2 = \frac{2\omega_{BD}^0 V_{0}''}{3 + 2\omega_{BD}^0} , \quad (70) \]

and therefore

\[ \Phi_N = -\frac{G_0 M}{F_0 r} \left[ 1 + \frac{\sigma e^{-m_0 r}}{3 + 2\omega_{BD}^0} \right] . \quad (71) \]

Recently experimental bounds on the strength and range of a Yukawa potential that could arise from “fifth force fields” have been analyzed in two kinds of experiments. The first kind consists in the analysis of gravitational signals induced by variations on the mass of a lake \(\text{[18]}.\) Such experiments probe basically fields with a range \(\lambda\) from meters to some kilometers and strength \(|\beta| \in [10^{-4}, 10^{-2}]\). Here the coefficients corresponds to \(\beta = \sigma e^{-r/\lambda} / (3 + 2\omega_{BD}^0)\) and \(\lambda = 1/m_0\) (cf. the curve \(\beta = \pm \beta(\lambda)\) in Ref. \(\text{[18]}\)). The second kind of experiments probes variations of the Newton’s gravitation law at scales of two Earth’s radii by measuring the gravitational effects on the orbit of the laser-ranged LAGEOS satellite \(\text{[19]}\). Assuming that \(r \ll \lambda,\) so that effects of an “intermediate-range” force are taken only to order \((r/\lambda)^2,\) it turns that \(|\beta| < 10^{-5} - 10^{-8}.\) These bounds are even more restrictive than those from the Earth based experiments quoted above.

At solar-system scales, Viking-like experiments restrict \(\omega_{BD}^0 > \omega_{exp},\) where \(\omega_{exp} \approx 3000\) corresponds to the current lower bound on \(\omega_{BD}^0 \text{[20]}\). This bound results from the parametrized post-Newtonian (PPN) approximation and translates into a bound on \(\beta\) and \(\lambda:\)

\[ \beta \lesssim \frac{\sigma}{(3 + 2\omega_{exp})} e^{-r/\lambda} \sim 1.66 \times 10^{-4} \sigma e^{-r/\lambda} . \quad (72) \]
\[ \lambda \lesssim \sqrt{3 + 2\omega_{exp} V_{0}''} \sim 1/\sqrt{V_{0}''} . \quad (73) \]

If we assume for instance that \(V = m^2 \phi^2\) and \(m_0 \ll eV,\) for example, \(m_0 \approx 10^{-21} eV,\) then it turns that \(\lambda \lesssim 13\) kpc, and moreover from Eq. (17) one can expect that \(\sigma \approx 1.\) Then, the most stringent upper bound \(\sim 1/(3 + 2\omega_{BD}^0),\) imposed on \(\beta\) by the solar system experiments arises for long range scalar fields \(\lambda \sim \text{kpc} (i.e., \text{galactic-scale ranges})\):
or the mass of fermions as the coefficient of $m_{eff}$ which always appear in the Newtonian potential \((1)\) and not the separate quantities. So whether one adopts one or the other definition for $G_{eff}$ or $M_{eff}$, the combination of both provides the same number.

Naively one could then expect that $M_{eff}$ is the total gravitational mass of the system. However, this is not the case, since the total ADM-mass formula in the weak-field limit gives [see Eq. (A64) below]

$$M_{ADM} = \int_{\Sigma_t} \tilde{T}^{00} d^3 x,$$  

(79)

which turns to be

$$M_{ADM} = M \left[1 - \frac{\alpha (F_0^2)^2}{2F_0} \right] = 2M \left( \frac{1 + \omega_{BD}^0}{3 + 2\omega_{BD}^0} \right)$$

$$= M - \frac{F_0^2 Q}{2G_0}.$$  

(80)

This result is not surprising if one takes into account that it is in fact the active mass $M_{eff}$ and not the ADM-mass which is measured by orbiting test particles [23].

It is customary in the literature of gravitational physics to introduce different mass definitions of a body. For instance, tensor, scalar and inertial (or sometimes referred to also as Keplerian) masses $M_T$, $M_S$ and $M_I$ [23].

In the weak-field limit these are given by

$$M_T \approx M,$$  

(81)

$$M_S \approx \frac{M}{3 + 2\omega_{BD}^0} = \frac{F_0^2 Q}{2G_0},$$  

(82)

$$M_I \approx M_{eff}.$$  

(83)

In this limit, the tensor mass corresponds to the total rest mass since $M$ is just the integral of the energy-density of the self-gravitating body which at first order coincides with the integral of the rest-mass density over the proper-volume [23].

The scalar mass [23] is simply proportional to the scalar charge of the body. The ADM-mass then reads in this limit as,

$$M_{ADM} = M_T - M_S = M_I - 2M_S = 2M_T - M_I,$$  

(84)

and the other masses can be expressed in terms of the following combinations:

$$M_T = M_{ADM} + M_S = M_I - M_S,$$  

(85)

$$M_I = M_{ADM} + 2M_S = M_T + M_S,$$  

(86)

$$M_S = M_{ADM} + 2M_S = \frac{1}{2} (M_I - M_{ADM}).$$  

(87)

**Spontaneous scalarization.** The phenomenon of spontaneous scalarization [2, 3, 4, 13, 23] arises in scalar-tensor theories within a dense compact object (e.g. neutron star). This phenomenon corresponds to a “sudden” appearance of a non-trivial configuration of the
scalar configurations: one with a trivial scalar field (in the massless case) and different from zero to find the non-trivial scalarization phenomenon in neutron stars (strong field regime) the phenomenon appeared. In other words, with the asymptotic condition $\phi_0 = 0$, it is impossible to have a trivial scalar field in any limit (weak or strong regime), since a trivial scalar-field $\phi = \text{const}$. would not be a solution of Eq. (15) for $M_{\text{mat}} \neq 0$. Induced scalarization then always ensues. The fact that the induced scalarization is present in the weak-field limit produces a smooth transition to the strong field regime (as opposed to spontaneous scalarization). Namely, the curve $Q$ vs. $M_{\text{mat}}$ (scalar charge vs. total baryon mass) do not show any discontinuity contrary to spontaneous scalarization (cf. Ref. 3).

**V. CONCLUSIONS**

The linear limit of scalar-tensor theories of gravity leads to the prediction of a spin-zero gravitational mode (scalar waves) in addition to the well known spin-2 modes. In the weak-field limit, STT predicts an effective gravitational potential leads essentially to a post-Newtonian force $\sim 1/r^2$ with an effective mass coefficient which is proportional to the rest-mass of the self-gravitating body and which depends explicitly on the effective Brans-Dicke parameter. This latter is highly constrained by the post-Newtonian approximation via Viking-like experiments. Therefore, even if a STT does not produce a deviation of the $1/r^2$ Newton’s law, it can however violate dramatically the post-Newtonian bounds. On the other hand, if the scalar field has “intermediate” ranges (meters to kilometers or even thousands of kilometers), then the Yukawa term leads to a “fifth force” field that produces deviations of the $1/r^2$ Newton’s law and which are severely constrained by Earth based experiments or by satellites, notably, by varying-mass experiments and laser-ranged missions. STT also predicts the phenomenon of spontaneous scalarization in compact objects. For the phenomenon to take place, a STT requires the existence of two possible configurations, one of which corresponds to the trivial $\phi = 0$ solution and another one that is not trivial. Both configurations having the same rest-mass and the null asymptotic solution $\phi = 0$. The phenomenon will not be present in the weak-field limit since it would require that the non-minimal coupling function $F(\phi)$ has a two-valued-derivatives: $F'(\phi)|_{\phi = 0} = 0$ and $F'(\phi)|_{\phi = \phi_0} \neq 0$ in order two obtain the two possible configurations for the perturbation $\phi$. The conclusion is that no-analytic function $F(\phi)$ can fulfill both conditions. As opposed to spontan-
neous scalarization, induced scalarization takes place also in the weak-field limit, since for physical couplings one has \( F'(\phi_0) \neq 0 \) (i.e. physical coupling functions \( F \) will not depend explicitly on \( \phi_0 \)), and then only non-trivial regular solutions are admitted.

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**APPENDIX A: THE 3+1 FORMULATION OF GENERAL RELATIVITY**

Let us consider the 3+1 or Adison-Deser-Misner (ADM) formulation of general relativity in which the spacetime (considered to be globally hyperbolic) is foliated by a family of spacelike hypersurfaces \( \Sigma_t \). We shall not enter into the details of the derivation of the 3+1 equations (see Refs. [16, 26, 27, 28, 29]). The sign convention for the 3+1 splitting of the metric is as follows:

\[
ds^2 = -(N^2 - N^i N_i) dt^2 - 2 N_i dt d\xi^i + h_{ij} dx^i dx^j .
\]

(A1)

The extrinsic curvature of the embeddings \( \Sigma_t \) is given by

\[
K_{ij} = -\nabla_i n_j = -NT^t_{ij}
\]

\[
= -\frac{1}{2N} \left( \frac{\partial h_{ij}}{\partial t} + 3\nabla_j N_i + 3\nabla_i N_j \right) ,
\]

(A2)

where \( 3\nabla_j \) stands for the covariant derivative compatible with the three-metric \( h_{ij} \). This is to be regarded as an evolution equation for \( h_{ij} \). The trace of the extrinsic curvature will be denoted by,

\[
K := K^t_{t} .
\]

(A3)

The orthogonal decomposition of the energy-momentum tensor in components tangent and orthogonal to \( \Sigma_t \) leads to [28]:

\[
T^{\mu\nu} = S^{\mu\nu} + J^{\mu} n^{\nu} + n^{\mu} J^{\nu} + E n^{\mu} n^{\nu} .
\]

(A4)

where \( n^{\mu} \) is the normal to \( \Sigma_t \). The tensor \( S^{\mu\nu} \) is symmetric and often called the tensor of constraints; \( J^{\mu} \) is the momentum density vector and \( E \) is the total energy density measured by the observer orthogonal to \( \Sigma_t \).

As in the 4+0 formalism, \( T^{\mu\nu} \) will be the total energy-momentum tensor of matter which can be composed by the contribution of different types of sources:

\[
T^{\mu\nu} = \sum_i T^{\mu\nu}_i .
\]

(A5)

This means that

\[
E = \sum_i E_i , \quad J^{\mu} = \sum_i J^{\mu}_i , \quad S^{\mu\nu} = \sum_i S^{\mu\nu}_i . \quad (A6)
\]

The projection of Einstein equations \( R_{\mu\nu} = 4\pi G_0 (2T_{\mu\nu} - T^\alpha_{\alpha} g_{\mu\nu}) \) in the directions tangent and orthogonal to \( \Sigma_t \), followed by the use of the Gauss-Codazzi-Mainardi equations leads to the 3+1 form of Einstein equations:

\[
3R + K^2 - K_{ij} K^{ij} = 16\pi G_0 E ,
\]

(A7)

known as the Hamiltonian constraint.

\[
3\nabla_i K^i - 3\nabla_i K = 8\pi G_0 J_i ,
\]

(A8)

known as the momentum constraint equations. Finally, the dynamic Einstein equations read

\[
\partial_t K^i_j + N^l \partial_l K^i_j + K^j_i \partial_j N^l - K^j_l \partial_l N^j
\]

\[
+ 3\nabla^i_3 \nabla_j N - 3R^i j N - NK K^i_j
\]

\[
= 4\pi G_0 N \left[ (S - E) \delta^i_j - 2S^i_j \right] \quad (A9)
\]

where \( S = S^i_j \) is the trace of the tensor of constraints, and all the quantities written with a ‘3’ index refer to those computed with the three-metric \( h_{ij} \). Moreover, under the 3+1 formalism tensor quantities tangent to \( \Sigma_t \) use the three-metric to raise and lower their spatial indices. The equations (A2) and (A3) are the set of the Cauchy-initial-data evolution equations for the gravitational field subject to the constraints Eqs. (A7) and (A8).

We can write an evolution equation for the trace \( K \) by taking the trace in Eq. (A3):

\[
\partial_t K + N^l \partial_l K + 3\Delta N - N (3R + K^2) = 4\pi G_0 N [S - 3E] . \quad (A10)
\]

where 3\( \Delta \) stands for the Laplacian operator compatible with \( h_{ij} \). This can be simplified by using Eq. (A7) to give

\[
\partial_t K + N^l \partial_l K + 3\Delta N - NK_{ij} K^{ij} = 4\pi G_0 N [S + E] . \quad (A11)
\]

For our purposes it will no be necessary to write the 3+1 equations for the matter and the scalar field.

1. Linearized equations

In order to linearize the 3+1 equations we assume first order deviations of the 3+1 metric with respect to the Lorentz metric \( \eta_{\mu\nu} \) as follows

\[
N \approx 1 + \epsilon \bar{N} , \quad (A12)
\]

\[
N_i \approx \epsilon \bar{N}_i , \quad (A13)
\]

\[
h_{ij} \approx \delta_{ij} + \epsilon \bar{h}_{ij} . \quad (A14)
\]
At first order \( h^{ij} \approx \delta_{ij} - \epsilon \hat{h}_{ij} \) in order to satisfy the inverse condition \( h_{ij} h^{ij} = \delta^{ij} \). Therefore, covariant and contravariant components of tensorial quantities having no zero order terms are identical to each other at first order. For instance \( N^i = h^{ik} N_k \approx \epsilon \tilde{N}_i \). In order to compare with the 4+0 (full covariant) linear approximation
\[
g_{\mu\nu} \approx \eta_{\mu\nu} + \epsilon \gamma_{\mu\nu} \quad (A15)
\]
it turns out that
\[
\gamma_{00} = -2 \tilde{N} \quad (A16)
\]
\[
\gamma_{0i} = -\tilde{N}_i \quad (A17)
\]
\[
\gamma_{ij} = \tilde{h}_{ij} \quad (A18)
\]
The 3-Christoffel symbols turn to be
\[
3\Gamma^i_{jk} \approx \epsilon 3\Gamma^i_{jk} \quad (A19)
\]
\[
3\tilde{\Gamma}^i_{ij} := \frac{1}{2} \left( -\partial_i \tilde{h}_{jk} + \partial_k \tilde{h}_{ij} + \partial_j \tilde{h}_{ki} \right) \quad (A20)
\]
Therefore 3-covariant derivatives of 3-tensors having no zero order terms become at first order ordinary derivatives. For instance \( 3\nabla_j N_i \approx \epsilon \partial_j \tilde{N}_i \). Then Eq. (A27) leads to
\[
K_{ij} \approx \epsilon \tilde{K}_{ij} \quad (A21)
\]
\[
\tilde{K}_{ij} := -\frac{1}{2} \left( \partial_i \tilde{h}_{jk} + 2\partial_k \tilde{h}_{ij} \right) \quad (A22)
\]
with
\[
K^i_j \approx \epsilon \tilde{K}^i_j \approx \epsilon \tilde{K}_{ij} \quad (A23)
\]
\[
K := K^i_j = \epsilon \tilde{K} \quad (A24)
\]
\[
\tilde{K} := -\frac{1}{2} \left( \partial_i \tilde{h} + 2\partial_i \tilde{N}_i \right) \quad (A25)
\]
where \( \tilde{h} := \tilde{h}_{ij} \) is the trace of the 3-metric perturbation. Using the definition of the 3-Riemann tensor in terms of the 3-Christoffel symbols it is easy to obtain the linearized approximation for the 3-Ricci tensor and the 3-curvature respectively:
\[
3R_{ij} \approx \epsilon 3\tilde{R}_{ij} \quad (A26)
\]
\[
3\tilde{R}_{ij} := \frac{1}{2} \left( 2\partial^2_{(i,j)} \tilde{h}_{ij} - 3\Delta \tilde{h}_{ij} - \partial^2_{ij} \tilde{h} \right) \quad (A27)
\]
\[
3\tilde{R} \approx \epsilon 3\tilde{R} \quad (A28)
\]
\[
3\tilde{\tilde{R}} := \partial^2_{kl} \tilde{h}_{kl} - 3\Delta \tilde{h} \quad (A29)
\]
where \( 3\Delta := \partial^2_{ij} \) stands for the Euclidean 3-Laplacian.

Concerning the sources (matter fields) we assume
\[
E \approx E_0 + \epsilon \tilde{E} \quad (A30)
\]
\[
J_i \approx J^0_i + \epsilon \tilde{J}_i \quad (A31)
\]
\[
S_{ij} \approx S^0_{ij} + \epsilon \tilde{S}_{ij} \quad (A32)
\]
The Hamiltonian constraint Eq. (A6) when linearized reads
\[
3\tilde{R} = 16\pi G_0 \tilde{E} \quad (A33)
\]
or explicitly
\[
\partial^2_{kli} h_{kl} - 3\Delta \tilde{h} = 16\pi G_0 \tilde{E} \quad (A34)
\]
The linearized version of the momentum constraint Eq. (A8), reads
\[
\partial_i \tilde{R}^i_j - \partial_j \tilde{K} = 8\pi G_0 \tilde{J}_i \quad (A35)
\]
Finally the dynamic Einstein Eqs. (A9) when linearized read
\[
\partial_i \tilde{K}^i_j + \partial^i \partial_j \tilde{N} - 3 \tilde{R}^i_j = 4\pi G_0 \left( \tilde{S} - \tilde{E} \right) \delta^i_j - 2 \tilde{S}^i_j \quad (A36)
\]
The linear limit of evolution Eq. (A11) reads
\[
\partial_t \tilde{K}^i_j + 3\Delta \tilde{N} = 4\pi G_0 \left[ \tilde{S} + \tilde{E} \right] \quad (A37)
\]
The self-consistency of the 3+1 equations up to first order imply that the zero order source fields must vanish identically:
\[
E_0 = 0 = J^0_i = S^0_{ij} \quad (A38)
\]
The above equations (A34), (A33) and (A36) are the 3+1 decomposition of the 4+0 equations (27).

The 3+1 splitting of the perturbations (26) is
\[
\tilde{\gamma}_{00} = \frac{1}{2} \left( \tilde{h} - 2\kappa \tilde{\phi} - 2\tilde{N} \right) \quad (A39)
\]
\[
\tilde{\gamma}_{0i} = -\tilde{N}_i \quad (A40)
\]
\[
\tilde{\gamma}_{ij} = \tilde{h}_{ij} - \frac{1}{2} \delta_{ij} \left( \tilde{h} - 2\kappa \tilde{\phi} + 2\tilde{N} \right) \quad (A41)
\]
\[
\tilde{\gamma} = -\left( \tilde{h} + 2\tilde{N} \right) + 4\kappa \tilde{\phi} \quad (A42)
\]
The 3+1 splitting of the Lorentz gauge Eq. (29)
\[
\partial^i \tilde{\gamma}_{ij} = 0 \quad (A43)
\]
Using Eq. (A25) the zero component of Eq. (A43) reads
\[
\partial_i \tilde{N} + \kappa \partial_i \tilde{\phi} = -\tilde{K} \quad (A44)
\]
From Eqs. (A44) and (A57) one obtains a wave equation for \( \tilde{N} \):
\[
-\partial^2_{ij} \tilde{N} + 3\Delta \tilde{N} = \kappa \partial^2_{ij} \tilde{\phi} + 4\pi G_0 \left[ \tilde{S} + \tilde{E} \right] \quad (A45)
\]
On the other hand, differentiating the spatial components of the Lorentz gauge Eq. (A43) with respect to time and using consecutively Eqs. (A22), (A25) and the momentum constraint Eq. (A33) one obtains a wave equation for $\tilde{N}^i$:

$$-\partial_{tt}^2 \tilde{N}^i + 3 \tilde{\Delta} \tilde{N}^i = 2\kappa \partial_n^2 \tilde{\phi} - 16\pi G_0 \tilde{J}^i .$$ \hspace{1cm} (A46)

Finally, the linearized dynamic Einstein Eqs. (A36) together with Eq. (A22) and the spatial components of the Lorentz gauge Eq. (A43) lead to a wave equation for $\tilde{h}_{ij}$

$$-\partial_{tt}^2 \tilde{h}_{ij} + 3 \tilde{\Delta} \tilde{h}_{ij} = -2\kappa \partial_n^2 \tilde{\phi} + 8\pi G_0 \left( (\tilde{S} - \tilde{E}) \delta_j^i - 2\delta_{ij} \right).$$ \hspace{1cm} (A47)

From Eq. (A47) one obtains

$$-\partial_{tt}^2 \tilde{h} + 3 \tilde{\Delta} \tilde{h} = -2\kappa \partial_n^2 \tilde{\phi} + 8\pi G_0 \left( \tilde{S} - 3 \tilde{E} \right).$$ \hspace{1cm} (A48)

Another wave equation which is not independent from the above is obtained from the linearized Hamiltonian constraint Eq. (A34) and the Lorentz-gauge conditions Eq. (A43):

$$\left( -\partial_{tt}^2 + 3 \tilde{\Delta} \right) \left( \frac{1}{2} \tilde{h} - \frac{3}{2} \tilde{\Delta} \tilde{h} - \tilde{N} \right) = -\kappa \left( \partial_n^2 + 3 \tilde{\Delta} \right) \tilde{\phi} - 16\pi G_0 \tilde{E} .$$ \hspace{1cm} (A49)

which is obtained by combining Eqs. (A45) and (A48). This can be written as

$$\left( -\partial_{tt}^2 + 3 \tilde{\Delta} \right) \left( \frac{1}{2} \tilde{h} - \kappa \tilde{\phi} - \tilde{N} \right) = -2\kappa \partial_n^2 \tilde{\phi} - 16\pi G_0 \tilde{E} .$$ \hspace{1cm} (A50)

The combination of Eqs. (A43), (A47) and (A48) as given by Eq. (A44) provides the wave equation

$$\left( -\partial_{tt}^2 + 3 \tilde{\Delta} \right) \tilde{\gamma}_{ij} = -2\kappa \partial_n^2 \tilde{\phi} + 2\kappa \delta_{ij} \partial_n \tilde{\phi} - 16\pi G_0 \tilde{S}_{ij} .$$ \hspace{1cm} (A51)

Therefore Eqs. (A46), (A50) and (A51) recover the 4+0 wave Eq. (30) in the Lorentz gauge.

Now according to the 3+1 splitting of the energy-momentum tensor, and using Eq. (39) we have

$$\tilde{E} = \tilde{T}_{00} = \frac{\tilde{E}_{\text{matt}}}{F_0} + \frac{F_0'}{8\pi G_0} \tilde{\Delta} \tilde{\phi} .$$ \hspace{1cm} (A52)

$$\tilde{J}_i = -\tilde{T}_{i0} = -\frac{\tilde{T}_{ij}}{F_0} + \frac{F_0'}{8\pi G_0} \partial_{\phi}^2 \tilde{\phi} .$$ \hspace{1cm} (A53)

$$\tilde{S}_{ij} = \tilde{T}_{ij} = \frac{\tilde{T}_{ij}}{F_0} + \frac{F_0'}{8\pi G_0} \left( \partial_{\phi}^2 \tilde{\phi} - \delta_{ij} \partial_n \tilde{\phi} \right) .$$ \hspace{1cm} (A54)

$$\tilde{S} = \tilde{T}_t = \frac{\tilde{T}_{tt}}{F_0} + \frac{F_0'}{8\pi G_0} \left( 3 \partial_{\phi}^2 \tilde{\phi} - 2 \tilde{\Delta} \tilde{\phi} \right) .$$ \hspace{1cm} (A55)

Then Eqs. (A45), (A46), and (A47) read respectively

$$-\partial_{tt}^2 \tilde{N} + 3 \tilde{\Delta} \tilde{N} = 4\pi \frac{G_0}{F_0} \left( \tilde{E}_{\text{matt}} + \tilde{S}_{\text{matt}} \right) + \left( \kappa + \frac{2F_0'}{2F_0} \right) \partial_{\phi}^2 \tilde{\phi} - \frac{F_0'}{2F_0} \tilde{\Delta} \tilde{\phi} .$$ \hspace{1cm} (A56)

$$-\partial_{tt}^2 \tilde{N} + 3 \tilde{\Delta} \tilde{N} = -16\pi \frac{G_0}{F_0} \tilde{j}_i + 2 \left( \kappa + \frac{F_0'}{F_0} \right) \partial_{\phi}^2 \tilde{\phi} .$$ \hspace{1cm} (A57)

$$-\partial_{tt}^2 \tilde{h}_{ij} + 3 \tilde{\Delta} \tilde{h}_{ij} = 8\pi \frac{G_0}{F_0} \left[ (\tilde{S}_{\text{matt}} - \tilde{E}_{\text{matt}}) \delta_{ij} - 2 \tilde{S}_{ij} \right] - 2 \left( \kappa + \frac{F_0'}{F_0} \right) \partial_{\phi}^2 \tilde{\phi} .$$ \hspace{1cm} (A58)

From Eq. (A48) one obtains

$$-\partial_{tt}^2 \tilde{h} + 3 \tilde{\Delta} \tilde{h} = 8\pi \frac{G_0}{F_0} \left( \tilde{S}_{\text{matt}} - 3 \tilde{E}_{\text{matt}} \right) - 3 \tilde{J}_0 \partial_n \tilde{\phi} - 2 \left( \kappa + \frac{F_0'}{F_0} \right) \tilde{\Delta} \tilde{\phi} .$$ \hspace{1cm} (A59)

Moreover, from Eq. (A50) we get

$$\left( -\partial_{tt}^2 + 3 \tilde{\Delta} \right) \left( \frac{1}{2} \tilde{h} - \kappa \tilde{\phi} - \tilde{N} \right) = -16\pi \frac{G_0}{F_0} \tilde{E}_{\text{matt}} - 2 \left( \kappa + \frac{F_0'}{F_0} \right) \tilde{\Delta} \tilde{\phi} .$$ \hspace{1cm} (A60)

Whereas the Eq. (A51) leads to

$$\left( -\partial_{tt}^2 + 3 \tilde{\Delta} \right) \tilde{\gamma}_{ij} = -16\pi \frac{G_0}{F_0} \tilde{S}_{ij} - 2 \left( \kappa + \frac{F_0'}{F_0} \right) \left( \partial_{\phi}^2 \tilde{\phi} - \delta_{ij} \partial_n \tilde{\phi} \right) .$$ \hspace{1cm} (A61)

Therefore Eqs. (A57), (A60), and (A61) are equivalent to Eq. (30). In above equations one can employ the convenient choice $\kappa = -F_0/F_0$ used in the 4+0 formulation to simplify the expressions.

2. The weak-field approximation

As in the 4+0 formulation, we consider slow varying sources and neglect the time-derivatives, and also $\tilde{E}_{\text{matt}} = \tilde{\rho}$, $\tilde{S}_{\text{matt}} \ll \tilde{\rho}$, $\tilde{j}_i \ll \tilde{\rho}$. With such considerations Eq. (A56) reads,

$$3 \tilde{\Delta} \left( \tilde{N} + \frac{F_0'}{2F_0} \tilde{\phi} \right) = 4\pi \frac{G_0}{F_0} \tilde{\rho} .$$ \hspace{1cm} (A62)
As in the 4+0 formulation $\gamma_{ij} = \text{const.} = \tilde{N}^i$ are the regular solutions and the constants can be gauged out. Combining (A62) and (14) we can then obtain the solution given by Eq. (72). Therefore the two non-trivial degrees of freedom are $\tilde{N}$ and $\tilde{\phi}$ ( $\tilde{N}^i = 0$, and, $h_{ij}$ is given in terms of $\tilde{N}$ and $\tilde{\phi}$). The perturbed quantity $\tilde{N}$ has the direct interpretation of the Newtonian potential [cf. Eqs. (48) and (A16)]. In particular, for the massless case $m_0 = 0$, it yields,

$$\Delta \tilde{N} = 4\pi G_0 \tilde{\rho} \left( 1 + \alpha \frac{F_0^2}{2G_0 F_0} \right), \quad (A63)$$

where Eq. (14) was used. In the spherical symmetric case, (73) provides the exterior solution for $\tilde{N}$.

Concerning the mass issue, let us focus on the linearized Hamiltonian constraint Eq. (A34). One can integrate over a coordinate volume and use Gauss theorem to obtain

$$\frac{1}{16\pi} \int \left( \partial h_{kk} - \partial \tilde{h} \right) dS^k = \int d\Sigma \hat{E} d^3x, \quad (A64)$$

where the integrals extend to spatial infinite. The left-hand side corresponds precisely to the weak-field version of the ADM-mass formula [cf. Eq. (11.2.14) of Ref. [16]],

$$M_{\text{ADM}} := \frac{1}{16\pi} \int (\partial h_{kl} - \partial_k h) dS^k, \quad (A65)$$

while the right-hand side of Eq. (A64) is the weak-field approximation of the Komar-mass formula.

In the massless case $m_0 = 0$, from (A52) and (14), it turns

$$\hat{E} = \frac{\tilde{\rho}}{F_0} \left( 1 - \frac{\alpha}{2G_0} \frac{(F_0)^2}{F_0} \right). \quad (A66)$$

Therefore, from Eqs. (A64) and (A66), one recovers Eq. (14) for the ADM-mass in the weak-field limit.

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