5. Harmonic analysis on algebraic groups over two-dimensional local fields of equal characteristic

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In this section we review the main parts of a recent work [4] on harmonic analysis on algebraic groups over two-dimensional local fields.

5.1. Groups and buildings

Let $K$ ($K = K_2$ whose residue field is $K_1$ whose residue field is $K_0$, see the notation in section 1 of Part I) be a two-dimensional local field of equal characteristic. Thus $K_2$ is isomorphic to the Laurent series field $K_1((t_2))$ over $K_1$. It is convenient to think of elements of $K_2$ as (formal) loops over $K_1$. Even in the case where $\text{char } (K_1) = 0$, it is still convenient to think of elements of $K_1$ as (generalized) loops over $K_0$ so that $K_2$ consists of double loops.

Denote the residue map $\mathcal{O}_{K_2} \to K_1$ by $p_2$ and the residue map $\mathcal{O}_{K_1} \to K_0$ by $p_1$. Then the ring of integers $O_K$ of $K$ as of a two-dimensional local field (see subsection 1.1 of Part I) coincides with $p_2^{-1}(\mathcal{O}_{K_1})$.

Let $G$ be a split simple simply connected algebraic group over $\mathbb{Z}$ (e.g. $G = SL_2$). Let $T \subset B \subset G$ be a fixed maximal torus and Borel subgroup of $G$; put $N = [B, B]$, and let $W$ be the Weyl group of $G$. All of them are viewed as group schemes.

Let $L = \text{Hom}(\mathbb{G}_m, T)$ be the coweight lattice of $G$; the Weyl group acts on $L$.

Recall that $I(K_1) = p_1^{-1}(B(\bar{\mathbb{F}}_q))$ is called an Iwahori subgroup of $G(K_1)$ and $T(\mathcal{O}_{K_1})N(K_1)$ can be seen as the “connected component of unity” in $B(K_1)$. The latter name is explained naturally if we think of elements of $B(K_1)$ as being loops with values in $B$.
Definition. Put
\[
D_0 = p_2^{-1} p_1^{-1}(B(\mathbb{F}_q)) \subset G(O_K),
\]
\[
D_1 = p_2^{-1}(T(O_{K_1}) N(K_1)) \subset G(O_K),
\]
\[
D_2 = T(O_{K_2}) N(K_2) \subset G(K).
\]

Then \( D_2 \) can be seen as the “connected component of unity” of \( B(K) \) when \( K \) is viewed as a two-dimensional local field, \( D_1 \) is a (similarly understood) connected component of an Iwahori subgroup of \( G(K_2) \), and \( D_0 \) is called a double Iwahori subgroup of \( G(K) \).

A choice of a system of local parameters \( t_1, t_2 \) of \( K \) determines the identification of the group \( K^*/O_K^* \) with \( \mathbb{Z} \oplus \mathbb{Z} \) and identification \( L \oplus L \) with \( L \otimes (K^*/O_K^*) \).

We have an embedding of \( L \otimes (K^*/O_K^*) \) into \( T(K) \) which takes \( a \otimes (t_1^i t_2^j) \), \( i, j \in \mathbb{Z} \), to the value on \( t_1^i t_2^j \) of the 1-parameter subgroup in \( T \) corresponding to \( a \).

Define the action of \( \hat{W} \) on \( L \otimes (K^*/O_K^*) \) as the product of the standard action on \( L \) and the trivial action on \( K^*/O_K^* \). The semidirect product
\[
\hat{W} = (L \otimes K^*/O_K^*) \rtimes W
\]
is called the double affine Weyl group of \( G \).

A (set-theoretical) lifting of \( W \) into \( G(O_K) \) determines a lifting of \( \hat{W} \) into \( G(K) \).

Proposition. For every \( i, j \in \{0, 1, 2\} \) there is a disjoint decomposition
\[
G(K) = \bigcup_{w \in \hat{W}} D_i w D_j.
\]
The identification \( D_i \backslash G(K)/D_j \) with \( \hat{W} \) doesn’t depend on the choice of liftings.

Proof. Iterated application of the Bruhat, Bruhat–Tits and Iwasawa decompositions to the local fields \( K_2, K_1 \).

For the Iwahori subgroup \( I(K_2) = p_2^{-1}(B(K_1)) \) of \( G(K_2) \) the homogeneous space \( G(K)/I(K_2) \) is the “affine flag variety” of \( G \), see [5]. It has a canonical structure of an ind-scheme, in fact, it is an inductive limit of projective algebraic varieties over \( K_1 \) (the closures of the affine Schubert cells).

Let \( B(G, K_2/K_1) \) be the Bruhat–Tits building associated to \( G \) and the field \( K_2 \). Then the space \( G(K)/I(K_2) \) is a \( G(K) \)-orbit on the set of flags of type (vertex, maximal cell) in the building. For every vertex \( v \) of \( B(G, K_2/K_1) \) its locally finite Bruhat–Tits building \( \beta_v \) isomorphic to \( B(G, K_1/K_0) \) can be viewed as a “microbuilding” of the double Bruhat–Tits building \( B(G, K_2/K_1/K_0) \) of \( K \) as a two-dimensional local field constructed by Parshin ([7], see also section 3 of Part II). Then the set \( G(K)/D_1 \) is identified naturally with the set of all the horocycles \( \{ w \in \beta_v : d(z, w) = r \} \), \( z \in \partial \beta_v \) of the microbuildings \( \beta_v \) (where the “distance” \( d(z, \cdot) \) is viewed as an element of
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a natural \( L \)-torsor). The fibres of the projection \( G(K)/D_1 \to G(K)/I(K_2) \) are \( L \)-torsors.

5.2. The central extension and the affine Heisenberg–Weyl group

According to the work of Steinberg, Moore and Matsumoto [6] developed by Brylinski and Deligne [1] there is a central extension

\[ 1 \to K_1^* \to \Gamma \to G(K_2) \to 1 \]

associated to the tame symbol \( K_2^* \times K_2^* \to K_1^* \) for the couple \((K_2, K_1)\) (see subsection 6.4.2 of Part I for the general definition of the tame symbol).

**Proposition.** This extension splits over every \( D_i, \ 0 \leq i \leq 2 \).

**Proof.** Use Matsumoto’s explicit construction of the central extension.

Thus, there are identifications of every \( D_i \) with a subgroup of \( \Gamma \). Put

\[ \Delta_i = O_{K_1}^* D_i \subset \Gamma, \quad \Xi = \Gamma/\Delta_1. \]

The minimal integer scalar product \( \Psi \) on \( L \) and the composite of the tame symbol \( K_2^* \times K_2^* \to K_1^* \) and the discrete valuation \( v_{K_1} : K^* \to \mathbb{Z} \) induces a \( W \)-invariant skew-symmetric pairing \( L \otimes K^*/O_K^* \times L \otimes K^*/O_K^* \to \mathbb{Z} \). Let

\[ 1 \to \mathbb{Z} \to \mathcal{L} \to L \otimes K^*/O_K^* \to 1 \]

be the central extension whose commutator pairing corresponds to the latter skew-symmetric pairing. The group \( \mathcal{L} \) is called the **Heisenberg group**.

**Definition.** The semidirect product

\[ \widetilde{W} = \mathcal{L} \rtimes W \]

is called the **double affine Heisenberg–Weyl group** of \( G \).

**Theorem.** The group \( \widetilde{W} \) is isomorphic to \( L_{\text{aff}} \rtimes \mathcal{W} \) where \( L_{\text{aff}} = \mathbb{Z} \oplus L \), \( \mathcal{W} = L \rtimes W \) and

\[ w \circ (a, l') = (a, w(l)), \quad l \circ (a, l') = (a + \Psi(l, l'), l'), \quad w \in W, \quad l, l' \in L, \quad a \in \mathbb{Z}. \]

For every \( i, j \in \{0, 1, 2\} \) there is a disjoint union

\[ \Gamma = \bigcup_{w \in \widetilde{W}} \Delta_i w \Delta_j \]

and the identification \( \Delta_i \backslash \Gamma/\Delta_j \) with \( \widetilde{W} \) is canonical.
5.3. Hecke algebras in the classical setting

Recall that for a locally compact group $\Gamma$ and its compact subgroup $\Delta$ the Hecke algebra $H(\Gamma, \Delta)$ can be defined as the algebra of compactly supported double $\Delta$-invariant continuous functions of $\Gamma$ with the operation given by the convolution with respect to the Haar measure on $\Gamma$. For $C = \Delta \gamma \Delta \in \Delta \backslash \Gamma / \Delta$ the Hecke correspondence $\Sigma_C = \{ (\alpha \Delta, \beta \Delta) : \alpha \beta^{-1} \in C \}$ is a $\Gamma$-orbit of $(\Gamma / \Delta) \times (\Gamma / \Delta)$.

For $x \in \Gamma / \Delta$ put $\Sigma_C(x) = \Sigma_C \cap (\Gamma / \Delta) \times \{ x \}$. Denote the projections of $\Sigma_C$ to the first and second component by $\pi_1$ and $\pi_2$.

Let $\mathcal{F}(\Gamma / \Delta)$ be the space of continuous functions $\Gamma / \Delta \to \mathbb{C}$. The operator $\tau_C : \mathcal{F}(\Gamma / \Delta) \to \mathcal{F}(\Gamma / \Delta), \quad f \mapsto \pi_2 \pi_1^*(f)$ is called the Hecke operator associated to $C$. Explicitly,

$$(\tau_C f)(x) = \int_{y \in \Sigma_C(x)} f(y) d\mu_{C, x},$$

where $\mu_{C, x}$ is the $\text{Stab}(x)$-invariant measure induced by the Haar measure. Elements of the Hecke algebra $\mathcal{H}(\Gamma, \Delta)$ can be viewed as “continuous” linear combinations of the operators $\tau_C$, i.e., integrals of the form $\int \phi(C) \tau_C dC$ where $dC$ is some measure on $\Delta \backslash \Gamma / \Delta$ and $\phi$ is a continuous function with compact support. If the group $\Delta$ is also open (as is usually the case in the $p$-adic situation), then $\Delta \backslash \Gamma / \Delta$ is discrete and $\mathcal{H}(\Gamma, \Delta)$ consists of finite linear combinations of the $\tau_C$.

5.4. The regularized Hecke algebra $\mathcal{H}(\Gamma, \Delta_1)$

Since the two-dimensional local field $K$ and the ring $O_K$ are not locally compact, the approach of the previous subsection would work only after a new appropriate integration theory is available.

The aim of this subsection is to make sense of the Hecke algebra $\mathcal{H}(\Gamma, \Delta_1)$.

Note that the fibres of the projection $\Xi = \Gamma / \Delta_1 \to G(K) / I(K_2)$ are $L_{\text{aff}}$-torsors and $G(K) / I(K_2)$ is the inductive limit of compact (profinite) spaces, so $\Xi$ can be considered as an object of the category $\mathcal{F}_1$ defined in subsection 1.2 of the paper of Kato in this volume.

Using Theorem of 5.2 for $i = j = 1$ we introduce:

**Definition.** For $(w, l) \in \widehat{W} = L_{\text{aff}} \times \widehat{W}$ denote by $\Sigma_{w, l}$ the Hecke correspondence (i.e., the $\Gamma$-orbit of $\Xi \times \Xi$) associated to $(w, l)$. For $\xi \in \Xi$ put

$$\Sigma_{w, l}(\xi) = \{ \xi' : (\xi, \xi') \in \Sigma_{w, l} \}.$$
The stabilizer \( \text{Stab}(\xi) \subseteq \Gamma \) acts transitively on \( \Sigma_{w,l}(\xi) \).

**Proposition.** \( \Sigma_{w,l}(\xi) \) is an affine space over \( K_1 \) of dimension equal to the length of \( w \in \hat{W} \). The space of complex valued Borel measures on \( \Sigma_{w,l}(\xi) \) is 1-dimensional. A choice of a \( \text{Stab}(\xi) \)-invariant measure \( \mu_{w,l,\xi} \) on \( \Sigma_{w,l}(\xi) \) determines a measure \( \mu_{w,l,\xi'} \) on \( \Sigma_{w,l}(\xi') \) for every \( \xi' \).

**Definition.** For a continuous function \( f: \Xi \rightarrow \mathbb{C} \) put

\[
(\tau_{w,l} f)(\xi) = \int_{\eta \in \Sigma_{w,l}(\xi)} f(\eta) d\mu_{w,l,\xi}.
\]

Since the domain of the integration is not compact, the integral may diverge. As a first step, we define the space of functions on which the integral makes sense. Note that \( \Xi \) can be regarded as an \( L_{\text{aff}} \)-torsor over the ind-object \( G(K)/I(K_2) \) in the category pro\((C_0)\), i.e., a compatible system of \( L_{\text{aff}} \)-torsors \( \Xi_\nu \) over the affine Schubert varieties \( Z_\nu \) forming an exhaustion of \( G(K)/I(K_1) \). Each \( \Xi_\nu \) is a locally compact space and \( Z_\nu \) is a compact space. In particular, we can form the space \( \mathcal{F}_0(\Xi_\nu) \) of locally constant complex valued functions on \( \Xi_\nu \) whose support is compact (or, what is the same, proper with respect to the projection to \( Z_\nu \)). Let \( \mathcal{F}(\Xi_\nu) \) be the space of all locally constant complex functions on \( \Xi_\nu \). Then we define \( \mathcal{F}_0(\Xi) = \lim_{\leftarrow} \mathcal{F}_0(\Xi_\nu) \) and \( \mathcal{F}(\Xi) = \lim_{\leftarrow} \mathcal{F}(\Xi_\nu) \). They are pro-objects in the category of vector spaces. In fact, because of the action of \( L_{\text{aff}} \) and its group algebra \( \mathbb{C}[L_{\text{aff}}] \) on \( \Xi \), the spaces \( \mathcal{F}_0(\Xi), \mathcal{F}(\Xi) \) are naturally pro-objects in the category of \( \mathbb{C}[L_{\text{aff}}] \)-modules.

**Proposition.** If \( f = (f_\nu) \in \mathcal{F}_0(\Xi) \) then \( \text{Supp}(f_\nu) \cap \Sigma_{w,l}(\xi) \) is compact for every \( w, l, \xi, \nu \) and the integral above converges. Thus, there is a well defined Hecke operator

\[
\tau_{w,l} : \mathcal{F}_0(\Xi) \rightarrow \mathcal{F}(\Xi)
\]

which is an element of \( \text{Mor}(\text{pro}(\text{Mod}_{\mathbb{C}[L_{\text{aff}}]}) \). In particular, \( \tau_{w,l} \) is the shift by \( l \) and \( \tau_{w,l+l'} = \tau_{w,l} \circ \tau_{l,l'} \).

Thus we get Hecke operators as operators from one (pro-)vector space to another, bigger one. This does not yet allow to compose the \( \tau_{w,l} \). Our next step is to consider certain infinite linear combinations of the \( \tau_{w,l} \).

Let \( T_{\text{aff}}^\vee = \text{Spec}(\mathbb{C}[L_{\text{aff}}]) \) be the “dual affine torus” of \( G \). A function with finite support on \( L_{\text{aff}} \) can be viewed as the collection of coefficients of a polynomial, i.e., of an element of \( \mathbb{C}[L_{\text{aff}}] \) as a regular function on \( T_{\text{aff}}^\vee \). Further, let \( Q \subset L_{\text{aff}} \otimes \mathbb{R} \) be a strictly convex cone with apex 0. A function on \( L_{\text{aff}} \) with support in \( Q \) can be viewed as the collection of coefficients of a formal power series, and such series form a ring containing \( \mathbb{C}[L_{\text{aff}}] \). On the level of functions the ring operation is the convolution. Let \( \mathcal{F}_Q(L_{\text{aff}}) \) be the space of functions whose support is contained in some translation of \( Q \). It is a ring with respect to convolution.
Let $\mathbb{C}(L_{aff})$ be the field of rational functions on $T'_{aff}$. Denote by $F^\text{rat}_{Q}(L_{aff})$ the subspace in $F_{Q}(L_{aff})$ consisting of functions whose corresponding formal power series are expansions of rational functions on $T'_{aff}$.

If $A$ is any $L_{aff}$-torsor (over a point), then $\mathcal{F}_{0}(A)$ is an (invertible) module over $\mathcal{F}_{0}(L_{aff}) = \mathbb{C}[L_{aff}]$ and we can define the spaces $\mathcal{F}_{Q}(A)$ and $\mathcal{F}^\text{rat}_{Q}(A)$ which will be modules over the corresponding rings for $L_{aff}$. We also write $\mathcal{F}^\text{rat}_{0}(A) = \mathcal{F}_{0}(A) \otimes_{\mathbb{C}[L_{aff}]} \mathbb{C}(L_{aff})$.

We then extend the above concepts “fiberwise” to torsors over compact spaces (objects of pro($C_{0}$)) and to torsors over objects of ind(pro($C_{0}$)) such as $\Xi$.

Let $w \in \widehat{W}$. We denote by $Q(w)$ the image under $w$ of the cone of dominant affine coweights in $L_{aff}$.

**Theorem.** The action of the Hecke operator $\tau_{w,l}$ takes $\mathcal{F}_{0}(\Xi)$ into $\mathcal{F}^\text{rat}_{Q(w)}(\Xi)$. These operators extend to operators

$$\tau^\text{rat}_{w,l} : \mathcal{F}^\text{rat}_{0}(\Xi) \to \mathcal{F}^\text{rat}_{0}(\Xi).$$

Note that the action of $\tau^\text{rat}_{w,l}$ involves a kind of regularization procedure, which is hidden in the identification of the $\mathcal{F}^\text{rat}_{Q(w)}(\Xi)$ for different $w$, with subspaces of the same space $\mathcal{F}^\text{rat}_{0}(\Xi)$. In practical terms, this involves summation of a series to a rational function and re-expansion in a different domain.

Let $\mathcal{H}_{\pre}$ be the space of finite linear combinations $\sum_{w,l} a_{w,l} \tau_{w,l}$. This is not yet an algebra, but only a $\mathbb{C}[L_{aff}]$-module. Note that elements of $\mathcal{H}_{\pre}$ can be written as finite linear combinations $\sum_{w \in \widehat{W}} f_{w}(t) \tau_{w}$ where $f_{w}(t) = \sum_{l} a_{w,l} t^{l}$, $t \in T'_{aff}$, is the polynomial in $\mathbb{C}[L_{aff}]$ corresponding to the collection of the $a_{w,l}$. This makes the $\mathbb{C}[L_{aff}]$-module structure clear. Consider the tensor product

$$\mathcal{H}_{\text{rat}} = \mathcal{H}_{\pre} \otimes_{\mathbb{C}[L_{aff}]} \mathbb{C}(L_{aff}).$$

Elements of this space can be considered as finite linear combinations $\sum_{w \in \widehat{W}} f_{w}(t) \tau_{w}$, where $f_{w}(t)$ are now rational functions. By expanding rational functions in power series, we can consider the above elements as certain infinite linear combinations of the $\tau_{w,l}$.

**Theorem.** The space $\mathcal{H}_{\text{rat}}$ has a natural algebra structure and this algebra acts in the space $\mathcal{F}^\text{rat}_{0}(\Xi)$, extending the action of the $\tau_{w,l}$ defined above.

The operators associated to $\mathcal{H}_{\text{rat}}$ can be viewed as certain integro-difference operators, because their action involves integration (as in the definition of the $\tau_{w,l}$) as well as inverses of linear combinations of shifts by elements of $L$ (these combinations act as difference operators).

**Definition.** The regularized Hecke algebra $\mathcal{H}(\Gamma, \Delta_{1})$ is, by definition, the subalgebra in $\mathcal{H}_{\text{rat}}$ consisting of elements whose action in $\mathcal{F}^\text{rat}_{0}(\Xi)$ preserves the subspace $\mathcal{F}_{0}(\Xi)$. 

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5.5. The Hecke algebra and the Cherednik algebra

In [2] I. Cherednik introduced the so-called double affine Hecke algebra $\text{Cher}_q$ associated to the root system of $G$. As shown by V. Ginzburg, E. Vasserot and the author [3], $\text{Cher}_q$ can be thought as consisting of finite linear combinations $\sum_{w \in \hat{W}_{\text{ad}}} f_w(t)[w]$ where $W_{\text{ad}}$ is the affine Weyl group of the adjoint quotient $G_{\text{ad}}$ of $G$ (it contains $\hat{W}$) and $f_w(t)$ are rational functions on $T_{\text{aff}}^{\vee}$ satisfying certain residue conditions. We define the modified Cherednik algebra $\hat{\text{H}}_q$ to be the subalgebra in $\text{Cher}_q$ consisting of linear combinations as above, but going over $\hat{W} \subset \hat{W}_{\text{ad}}$.

**Theorem.** The regularized Hecke algebra $H(\Gamma, \Delta_1)$ is isomorphic to the modified Cherednik algebra $\hat{\text{H}}_q$. In particular, there is a natural action of $\hat{\text{H}}_q$ on $\mathcal{F}_0(\Xi)$ by integro-difference operators.

**Proof.** Use the principal series intertwiners and a version of Mellin transform. The information on the poles of the intertwiners matches exactly the residue conditions introduced in [3].

**Remark.** The only reason we needed to assume that the 2-dimensional local field $K$ has equal characteristic was because we used the fact that the quotient $G(K)/I(K_2)$ has a structure of an inductive limit of projective algebraic varieties over $K_1$. In fact, we really use only a weaker structure: that of an inductive limit of profinite topological spaces (which are, in this case, the sets of $K_1$-points of affine Schubert varieties over $K_1$). This structure is available for any 2-dimensional local field, although there seems to be no reference for it in the literature. Once this foundational matter is established, all the constructions will go through for any 2-dimensional local field.

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