Adaptivity Gaps for Stochastic Probing:
Submodular and XOS Functions

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August 3, 2016

Abstract

Suppose we are given a submodular function $f$ over a set of elements, and we want to maximize its value subject to certain constraints. Good approximation algorithms are known for such problems under both monotone and non-monotone submodular functions. We consider these problems in a stochastic setting, where elements are not all active and we can only get value from active elements. Each element $e$ is active independently with some known probability $p_e$, but we don’t know the element’s status a priori. We find it out only when we probe the element $e$—probing reveals whether it’s active or not, whereafter we can use this information to decide which other elements to probe. Eventually, if we have a probed set $S$ and a subset active$(S)$ of active elements in $S$, we can pick any $T \subseteq$ active$(S)$ and get value $f(T)$. Moreover, the sequence of elements we probe must satisfy a given prefix-closed constraint—e.g., these may be given by a matroid, or an orienteering constraint, or deadline, or precedence constraint, or an arbitrary downward-closed constraint—if we can probe some sequence of elements we can probe any prefix of it. What is a good strategy to probe elements to maximize the expected value?

In this paper we study the gap between adaptive and non-adaptive strategies for $f$ being a submodular or a fractionally subadditive (XOS) function. If this gap is small, we can focus on finding good non-adaptive strategies instead, which are easier to find as well as to represent. We show that the adaptivity gap is a constant for monotone and non-monotone submodular functions, and logarithmic for XOS functions of small width. These bounds are nearly tight. Our techniques show new ways of arguing about the optimal adaptive decision tree for stochastic problems.

1 Introduction

Consider the problem of maximizing a submodular function $f$ over a set of elements, subject to some given constraints. This has been a very useful abstraction for many problems, both theoretical (e.g., the classical k-coverage problem [WS11]), or practical (e.g., the influence maximization problem [KKT15], or many problems in machine learning [Kra13]). We now know how to perform constrained submodular maximization, both when the function is monotone [FNW78, CCPV11], and also when the function may be non-monotone but non-negative [FMV11, LMNS09, FNS11]. In this paper we investigate how well we can solve this problem if the instance is not deterministically known up-front, but there is uncertainty in the input.

Consider the following setting. We have a submodular function over a ground set of elements $U$. But the elements are not all active, and we can get value only for active elements. The bad news is that a priori we don’t know the elements’ status—whether it is active or not. The good news is that each element $e$ is active independently with some known probability $p_e$. We find out an element $e$’s status only by probing it. Once we know its status, we can use this information to decide which other elements to probe next, and in what order.

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We have some constraints on which subsets we are allowed to probe. Eventually, we stop with some probed set $S$ and a known subset $\text{active}(S)$ of the active elements in $S$. At that time we can pick any $T \subseteq \text{active}(S)$ and get value $f(T)$. What is a good strategy to probe elements to maximize expected value?

Since this sounds quite abstract, here is an example. In the setting of influence maximization, the ground set is a set of email addresses (or Facebook accounts), and for a set $S$ of email addresses $f(S)$ is the fraction of the network that can be influenced by seeding the set $S$. But not all email addresses are still active. For each email address $e$, we know the probability $p_e$ that it is active. (Based, e.g., on when the last time we know it was used, or some other machine learning technique.) Now due to time constraints, or our anti-spam policies, or the fact that we are risk-averse and do not want to make introductory offers to too many people—we can only probe some $K$ of these addresses, and make offers to the active ones in these $K$, to maximize our expected influence. Observe that it makes sense to be adaptive—if $t$.theorist@cs.cmu.edu happens to be active we may not want to probe $t$.theorist@cmu.edu, since we may believe they are the same person.

Another example is robot path-planning. We have a robot that can travel at most distance $D$ each day and is trying the maximize value by picking items. The elements are locations, and an element is active if the location has an item to be picked up. (Location $e$ has an item with probability $p_e$ independent of all others.) Having probed $S$, if $T$ is the subset of active locations, the value $f(T)$ is some submodular function of the set of elements—e.g., the number of distinct items.

There are other examples: e.g., Bayesian mechanism design problems (see [GN13] for details) and stochastic set cover problems that arise in database applications [LPRY08, DHK14].

The question that is of primary interest to us is the following: Even though our model allows for adaptive queries, what is the benefit of this adaptivity? Note that there is price to adaptivity: the optimal adaptive strategy may be an exponentially-large decision tree that is difficult to store, and also may be computationally intractable to find. Moreover, in some cases the adaptive strategy would require us to be sequential (probe one email address, then probe the next, and so on), whereas a non-adaptive strategy may be just a set of $K$ addresses that we can probe in parallel. So we want to bound the adaptivity gap: the ratio between the expected value of the best adaptive strategy and that of the best non-adaptive strategy. Secondly, if this adaptivity gap is small, we would like to find the best non-adaptive strategy efficiently (in polynomial time). This would give us our ideal result: a non-adaptive strategy that is within a small factor of the best adaptive strategy.

The goal of this work is to get such results for as broad a class of functions, and as broad a class of probing constraints as possible. Recall that we were not allowed to probe all the elements, but only those which satisfied some problem-specific constraints (e.g., probe at most $K$ email addresses, or probe a set of locations that can be reached using a path of length at most $D$.)

1.1 Our Results

In this paper, we allow very general probing constraints: the sequence of elements we probe must satisfy a given prefix-closed constraint—e.g., these may be given by a matroid, or an orienteering constraint, or deadline, or precedence constraint, or an arbitrary downward-closed constraint—if we can probe some sequence of elements we can probe any prefix of it. We cannot hope to get small adaptivity gaps for arbitrary functions (see §B for a monotone $0 – 1$ function where the gap is exponential in $n$ even for cardinality constraint), and hence we have to look at interesting sub-classes of functions.

Submodular Functions. Our first set of results are for the case where the function $f$ is a non-negative submodular function. The first result is for monotone functions.

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1If the function is monotone, clearly we should choose $T = \text{active}(S)$. 

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Theorem 1.1 (Monotone Submodular). For any monotone non-negative submodular function $f$ and any prefix-closed probing constraints, the stochastic probing problem has adaptivity gap at most 3.

The previous results in this vein either severely restricted the function type (e.g., we knew a logarithmic gap for matroid rank functions [GNS16]) or the probing constraints (e.g., Asadpour et al. [AN16] give a gap of $\frac{e}{e-1}$ for matroid probing constraints). We discuss these and other prior works in §1.3.

There is a lower bound of $\frac{e}{e-1}$ on the adaptivity gap for monotone submodular functions with prefix-closed probing constraints (in fact for the rank function of a partition matroid, with the constraint being a simple cardinality constraint). It remains an interesting open problem to close this gap.

We then turn to non-monotone submodular functions, and again give a constant adaptivity gap. While the constant can be improved slightly, we have not tried to optimize it, focusing instead on clarity of exposition.

Theorem 1.2 (Non-Monotone Submodular). For any non-negative submodular function $f$ and any prefix-closed probing constraints, the stochastic probing problem has adaptivity gap at most 40.

Both Theorems 1.1 and 1.2 just consider the adaptivity gap. What about the computational question of finding the best non-adaptive strategy? This is where the complexity of the prefix-closed constraints come in. The problem of finding the best non-adaptive strategy with respect to some prefix-closed probing constraint can be reduced to the problem of maximizing a submodular function with respect to the same constraints.

XOS Functions. We next consider more general classes of functions. We conjecture that the adaptivity gap for all subadditive functions is poly-logarithmic in the size of the ground set. Since we know that any subadditive function can be approximated to within a logarithmic factor by an XOS (a.k.a. max-of-sums, or fractionally subadditive) function [Dob07], and every XOS function is subadditive, it suffices to focus on XOS functions. As a step towards our conjecture, we show a nearly-tight logarithmic adaptivity gap for monotone XOS functions of small "width", which we explain below.

A monotone XOS function $f : 2^{[n]} \rightarrow \mathbb{R}_{\geq 0}$ is one that can be written as the maximum of linear functions: i.e., there are vectors $a_i \in \mathbb{R}^n_{\geq 0}$ for some $i = 1 \ldots W$ such that

$$f(S) := \max_i (a_i^T \chi_S) = \max_i (\sum_{j \in S} a_i(j)).$$

We define the width of (the representation of) an XOS function as $W$, the number of linear functions in this representation. E.g., a width-1 XOS function is just a linear function. In general, even representing submodular functions in this XOS form requires an exponential width [BH11, BDF+12].

Theorem 1.3 (XOS Functions). For any monotone XOS function $f$ of width $W$, and any prefix-closed probing constraints, the adaptivity gap is $O(\log W)$. Moreover, there are instances with $W = \Theta(n)$ where the adaptivity gap is $\Omega\left(\frac{\log W}{\log \log W}\right)$.

In this case, we can also reduce the computation problem to linear maximization over the constraints.

Theorem 1.4. Suppose we are given a width-$W$ monotone XOS function $f$ explicitly in the max-of-sums representation, and an oracle to maximize positive linear functions over some prefix-closed constraint. Then there exists an algorithm that runs in time $\text{poly}(n, W)$ and outputs a non-adaptive strategy that has expected value at least an $\Omega(\log W)$-fraction of the optimal adaptive strategy.

1.2 Our Techniques

Before talking about our techniques, a word about previous approaches to bounding the adaptivity gap. Several works, starting with the work of Dean et al. [DGV08] have used geometric “relaxations” (e.g., a linear program
for linear functions [DGV08], or the multilinear extension for submodular settings [ANS08, ASW14]) to get an estimate of the value achieved by the optimal adaptive strategy. Then one tries to find a non-adaptive strategy whose expected value is not much less than this relaxation. This is particularly successful when the probing constraints are amenable to being captured by linear programs—e.g., matroid or knapsack constraints. Dealing with general constraints (which include orienteering constraints, where no good linear relaxations are known) means we cannot use this approach.

The other approach is to argue about the optimal decision-tree directly. An induction on the tree was used, e.g., by Chen et al. [CIK+09] and Adamczyk [Ada11] to study stochastic matchings. A different approach is to use concentration bounds like Freedman’s inequality to show that for most paths down the tree, the function on the path behaves like the path-mean—this was useful in [GKNR12], and also in our previous work on adaptivity gaps of matroid rank functions [GNS16]. However, this approach seems best suited to linear functions, and loses logarithmic factors due to the need for union bounds.

Given that we prove a general result for any submodular function, how do we show a good non-adaptive strategy? Our approach is to take a random path down the tree (the randomness coming from the element activation probabilities) and to show the expected value of this path, when viewed as a non-adaptive strategy, to be good. To prove this, surprisingly, we use induction. It is surprising because the natural induction down the tree does not seem to work. So we perform a non-standard inductive argument, where we consider the all-no path (which we call the stem), show that a non-adaptive strategy would get value comparable to the decision tree on the stem, and then induct on the subtrees hanging off this stem. The proof for monotone functions, though basic, is subtle—requiring us to change representations and view things “right”. This appears in §3.

For non-monotone submodular functions, the matter is complicated by the fact that we cannot pick elements in an “online” fashion when going down the tree—greedy-like strategies are bad for non-monotone functions. Hence we pick elements only with some probability, and show this gives us a near-optimal solution. The argument is complicated by the fact that having picked some elements $X$, the marginal-value function $f_X(S) := f(X \cup S) - f(X)$ may no longer be non-negative.

Finally, for monotone XOS functions of small width, we use the approach based on Freedman’s concentration inequality to show that a simple algorithm that either picks the set optimizing one of the linear functions, or a single element, is within an $O(\log W)$ factor of the optimum. We then give a lower bound example showing an (almost-)logarithmic factor is necessary, at least for $W = O(n)$.

1.3 Related Work

The adaptivity gap of stochastic packing problems has seen much interest: e.g., for knapsack [DGV08, BGK11, Ma14], packing integer programs [DGV05, CIK+09, BGL+12], budgeted multi-armed bandits [GM07, GMR11, LY13, Ma14] and orienteering [GM09, GMR12, BN14]. All except the orienteering results rely on having relaxations that capture the constraints of the problem via linear constraints.

For stochastic monotone submodular functions where the probing constraints are given by matroids, Asadpour et al. [AN16] bounded the adaptivity gap by $\frac{1}{e-1}$; Hellerstein et al. [HKL15] bound it by $\frac{1}{\tau}$, where $\tau$ is the smallest probability of some set being materialized. (See also [LPRY08, DHK14].)

The work of Chen et al. [CIK+09] (see also [Ada11, BGL+12, BCN+15, AGM15]) sought to maximize the size of a matching subject to $b$-matching constraints; this was motivated by applications to online dating and kidney exchange. More generally, see, e.g. [RSU05, AR12], for pointers to other work on kidney exchange problems. The work of [GN13] abstracted out the general problem of maximizing a function (in their case, the rank function of the intersection of matroids or knapsacks) subject to probing constraints (again, intersection of matroids and knapsacks). This was improved and generalized by Adamczyk, et al. [ASW14] to submodular objectives. All these results use LP relaxations, or non-linear geometric relaxations for the submodular settings.
The previous work of the authors [GNS16] gave results for the case where \( f \) was the rank function of matroids (or their intersections). That work bounded the adaptivity gap by logarithmic factors, and gave better results for special cases like uniform and partition matroids. This work both improves the quantitative bounds (down to small constants), generalizes it to all submodular functions with the hope of getting to all subadditive functions, and arguably also makes the proof simpler.

2 Preliminaries and Notation

We denote the ground set by \( X \), with \( n = |X| \). Each element \( e \in X \) has an associated probability \( p_e \). Given a subset \( S \subseteq X \) and vector \( \mathbf{p} = (p_1, p_2, \ldots, p_n) \), let \( S(\mathbf{p}) \) denote the distribution over subsets of \( S \) obtained by picking each element \( e \in S \) independently with probability \( p_e \). (Specifying a single number \( p \in [0, 1] \) in \( S(p) \) indicates each element is chosen with probability \( p_e = p \).

A function \( f : 2^X \to \mathbb{R} \) is

- monotone if \( f(S) \leq f(T) \) for all \( S \subseteq T \).
- linear if there exist \( a_i \in \mathbb{R} \) for each \( i \in X \) such that \( f(S) = \sum_{i \in S} a_i \).
- submodular if \( f(A \cup B) + f(A \cap B) \leq f(A) + f(B) \) for all \( A, B \subseteq X \). We will normally assume that \( f \) is non-negative and \( f(\emptyset) = 0 \).
- subadditive if \( f(A \cup B) \leq f(A) + f(B) \). A non-negative submodular function is clearly subadditive.
- fractionally subadditive (or XOS) if \( f(T) \leq \sum_{i \in S} \alpha_i f(S_i) \) for all \( \alpha_i \geq 0 \) and \( \chi_T = \sum_i \alpha_i \chi_{S_i} \).

An alternate characterization: a function is XOS if there exist linear functions \( a_1, a_2, \ldots, a_w : 2^X \to \mathbb{R} \) such that \( f(X) = \max_j \{a_j(X)\} \). The width of an XOS function is the smallest number \( W \) such that \( f \) can be written as the maximum over \( W \) linear functions.

All objective functions \( f \) that we deal with are non-negative with \( f(\emptyset) = 0 \).

Given any function \( f : 2^X \to \mathbb{R} \), define \( f^{\max}(S) := \max_{T \subseteq S} f(T) \) to be the maximum value subset contained within \( S \). The function \( f \) is monotone if and only if \( f^{\max} = f \). In general, \( f^{\max} \) may be difficult to compute given access to \( f \). However, Feige et al. [FMV11] show that for submodular functions

\[
\frac{1}{4} f^{\max}(S) \leq \mathbb{E}_{R \sim S(\frac{1}{2})} [f(R)] \leq f^{\max}(S).
\] (1)

Also, for a subset \( S \), define the “contracted” function \( f_S(T) := f(S \cup T) - f(S) \). Note that if \( f \) is non-monotone, then \( f_S \) may be negative-valued even if \( f \) is not.

Adaptive Strategies

An adaptive strategy tree \( T \) is a a binary rooted tree where every internal node \( v \) represents some element \( e \in X \) (denoted by elt(\( v \)) = \( e \)), and has two outgoing arcs—the yes arcs are indicating the node to go to if the element \( e = \text{elt}(v) \) is active (which happens with probability \( p_e \)) when probed, and the no arcs are indicating the node to go to if \( e \) is not active (which happens with the remaining probability \( q_e = 1 - p_e \)). No element can be represented by two different nodes on any root-leaf path. Moreover, any root-leaf path in \( T \) should be feasible according to the constraints. Hence, each leaf \( \ell \) in the tree \( T \) is associated with the root-path \( P_{\ell} \): the elements probed on this path are denoted by elt(\( P_{\ell} \)). Let \( A_{\ell} \) denote the active elements on this path \( P_{\ell} \)—i.e., the elements represented by the nodes on \( P_{\ell} \) for which we took the yes branch.

The tree \( T \) naturally gives us a probability distribution \( \pi_T \) over its leaves: start at the root, at each node \( v \), follow the yes branch with probability \( p_{\text{elt}(v)} \) and the “no” branch otherwise, to end at a leaf.

\[\text{Our definitions of fractionally subadditive/XOS differ slightly from those in the literature, since we allow non-monotonicity in our functions. See §A for a discussion.}\]
Given a submodular function \( f \) and a tree \( T \), the associated adaptive strategy is to probe elements until we reach a leaf \( \ell \), and then to pick the max-value subset of the active elements on this path \( P_\ell \). Let \( \text{adap}(T, f) \) denote the expected value obtained this way; it can be written compactly as

\[
\text{adap}(T, f) := \mathbb{E}_{\ell \leftarrow \pi_T} [f^{\max}(A_\ell)].
\] (2)

**Definition 2.1 (stem of \( T \)).** For any adaptive strategy tree \( T \) the stem represents the all-no path in \( T \) starting at the root, i.e., when all the probed elements turn out inactive.

**Definition 2.2 (deepness of \( T \)).** The deepness of a strategy tree \( T \) is the maximum number of active nodes that \( \text{adap} \) sees along a root-leaf path of \( T \).

Note that this notion of deepness is not the same as that of depth used for trees: it measures the number of yes-\( s \)-arcs on the path from the root to the leaf, rather than just the number of arcs seen on the path. This definition is inspired by the induction we will do in the submodular sections.

We can also define the natural non-adaptive algorithm given the tree \( T \): just pick a leaf \( \ell \leftarrow \pi_T \) from the distribution given by \( T \), probe all elements on that path, and choose the max-value subset of the active elements. We denote the expected value by \( \text{alg}(T, f) \):

\[
\text{alg}(T, f) := \mathbb{E}_{\ell \leftarrow \pi_T} [\mathbb{E}_{R \sim X(\rho)} [f^{\max}(R \cap \text{elt}(P_\ell))]].
\] (3)

### 3 Monotone Non-Negative Submodular Functions

We now prove Theorem 1.1, and bound the adaptivity gap for monotone submodular functions \( f \) over any prefix-closed set of constraints. The idea is a natural one in retrospect: we take an adaptive tree \( T \), and show that the natural non-adaptive strategy (given by choosing a random root-leaf path down the tree, and probing the elements on that path) is within a factor of 3 of the adaptive tree. The proof is non-trivial, though. One strategy is to induct on the two children of the root (which, say, probes element \( e \)), but note that the adaptive and non-adaptive algorithms recurse having seen different sets of active elements.\(^3\) This forced previous results to proceed along different lines, using massive union bounds over the paths in the decision tree, and hence losing logarithmic factors. They were also restricted to matroid rank functions, instead of all submodular functions.

A crucial insight in our proof is to focus on the stem of the tree (the all-no path off the root, see Definition 2.1), and induct on the subtrees hanging off this stem. Again we have issues of adaptive and non-adaptive recursing with different active elements, but we control this by giving the adaptive strategy some elements for free, and contracting some elements in the non-adaptive strategy without collecting value for them. The proof for non-monotone functions in §4 will be even more tricky, and will build on ideas from this monotone case. Formally, the main technical result is the following:

**Theorem 3.1.** For any adaptive strategy tree \( T \), and any monotone non-negative submodular function \( f : 2^X \rightarrow \mathbb{R}_{\geq 0} \) with \( f(\emptyset) = 0 \),

\[
\text{alg}(T, f) \geq \frac{1}{3} \text{adap}(T, f).
\]

Theorem 1.1 follows by the observation that each root-leaf path in \( T \) satisfies the prefix-closed constraints, which gives us a feasible non-adaptive strategy. Some comments on the proof: because the function \( f \) is

\(^3\)Adaptive sees \( e \) as active when it takes the yes branch (with probability \( p_e \)), and nothing as active when taking the no branch. Non-adaptive recurses on the yes branch with the same probability \( p_e \) because it picks a random path down the tree—but it then also probes \( e \). So when it takes the yes branch, it has either seen \( e \) as active (with probability \( p_e \)) or not (with probability \( 1 - p_e \)). Hence the set of active elements on both sides are quite different.
monotone, \( f^{\text{max}} = f \). Plugging this into (2) and (3), we want to show that

\[
E_{\ell \leftarrow \pi_T} [E_{R \sim X(p)}[f(R \cap \text{elt}(P_i))]] \geq \frac{1}{3} E_{\ell \leftarrow \pi_T} [f(A_\ell)].
\] (4)

Since both expressions take expectations over the random path, the proof proceeds by induction on the deepness of the tree. (Recall the definition of deepness in Definition 2.2.) We argue that for the stem starting at the root, \( \text{alg} \) gets a value close to \( \text{adap} \) in expectation (Lemma 3.3). However, to induct on the subtree that the algorithms leave the stem on, the problem is that the two algorithms may have picked up different active elements on the stem, and hence the “contracted” functions may look very different. The idea now is to give \( \text{adap} \) the elements picked by \( \text{alg} \) for “free” and disallow \( \text{alg} \) (just for the analysis) to pick elements picked by \( \text{adap} \) after exiting the stem. Now both the algorithms work after contracting the same set of elements in \( f \), and we are able to proceed with the induction.

### 3.1 Proof of Theorem 3.1

**Proof.** We prove by induction on the deepness of the adaptive strategy tree \( T \). For the base case of deepness 0, \( T \) contains exactly one node, and there are no internal nodes representing element. Hence both \( \text{alg} \) and \( \text{adap} \) get zero value, so the theorem is vacuously true.

To prove the induction step, recall that the stem is the path in \( T \) obtained by starting at the root node and following the no arcs until we reach a leaf. (See Figure 1.) Let \( v_1, v_2, \ldots, v_\ell \) denote the nodes along the stem of \( T \) with \( v_1 \) being the root and \( v_\ell \) being a leaf; let \( e_i = \text{elt}(v_i) \). For \( i \geq 1 \), let \( T_i \) denote the subtree hanging off the yes arc leaving \( v_i \). The probability that a path following the probability distribution \( \pi_T \) enters \( T_i \) is \( p_i \prod_{j < i} q_j \), where \( p_i = 1 - q_i \) denotes the probability that the \( i^{th} \) element is active.

![Adaptive strategy tree](image)

Figure 1: Adaptive strategy tree \( T \). The thick line shows the all-no path. The arrows show the path taken by \( \text{adap} \). In this example \( i = 4 \) and \( S_i = \{e_1, e_2, e_3, e_4\} \).

Let \( S_i = \{e_1, e_2, \ldots, e_i\} \) be the first \( i \) elements probed on the stem, and \( R_i \sim S_i(p) \) be a random subset of \( S_i \) that contains each element \( e \) of \( S_i \) independently w.p. \( p_e \). We can now rewrite \( \text{adap} \) and \( \text{alg} \) in a form more convenient for induction. Here we recall the definition of a marginal with respect to subset \( Y \): \( f_Y(S) := f(Y \cup S) - f(Y) \). Note that the leaf \( v_\ell \) has no associated element; to avoid special cases we define a dummy element \( e_\ell \) with \( f(\{e_\ell\}) = 0 \) and \( f(e_\ell) = f \).

**Claim 3.2.** Let \( I \) be the r.v. denoting the index of the node at which a random walk according to \( \pi_T \) leaves the stem. (If \( I = \ell \) then the walk does not leave the stem, and \( T_\ell \) is a deepness-zero tree.) Then,

\[
\text{adap}(T, f) = E_I \left[ f(e_I) + \text{adap}(T_I, f(e_I)) \right]
\] (5)
Proof. Equation (5) follows from the definition of adap; (6) follows from the monotonicity of \( f \). (We are giving the adaptive strategy elements in \( R \) “for free”.) Equation (7) follows from the definition of \( \text{alg} \), and (8) uses the consequence of submodularity that marginals can only decrease for larger sets.

Observe the expressions in (6) and (8) are ideally suited to induction. Indeed, since the function \( f_{R,j} \) also satisfies the assumptions of Theorem 3.1, and the height of \( T_\ell \) is smaller than that of \( T \), we use induction hypothesis on \( T_\ell \) with monotone non-negative submodular function \( f_{R \cup \ell} \) to get

\[
\mathbb{E}_{I,R \sim S_{I}(p)}[\text{alg}(T_\ell, f_{R,j})] \geq \frac{1}{3} \mathbb{E}_{I,R \sim S_{I}(p)}[\text{adap}(T_\ell, f_{R,j})].
\]

Finally, we use the following Lemma 3.3 to show that

\[
\mathbb{E}_{I,R \sim S_{I}(p)}[f(R)] \geq \frac{1}{3} \mathbb{E}_{I,R \sim S_{I}(p)}[f(R) + f(I)].
\]

Substituting these two into (6) and (8) finishes the induction step.

Lemma 3.3. Let \( I \) be the r.v. denoting the index of the node at which a random walk according to \( \pi_T \) leaves the stem. (If \( I = \ell \) then the walk does not leave the stem.) Then,

\[
\mathbb{E}_{I,R \sim S_{I}(p)}[f(R)] \geq \frac{1}{2} \mathbb{E}_{I}[f(I)].
\]

Proof. For brevity, we use \( \mathbb{E}_{I,R}[:] \) as shorthand for \( \mathbb{E}_{I,R \sim S_{I}(p)}[:] \) in the rest of the proof. We prove this lemma by showing that

\[
\mathbb{E}_{I,R}[f(R)] \geq \mathbb{E}_{I,R}[\max_{e_j \in R} f(e_j)] \geq \frac{1}{2} \mathbb{E}_{I}[f(I)].
\]

The first inequality uses monotonicity. The rest of the proof shows the latter inequality.

For any real \( x \geq 0 \), let \( W_x \) denote the indices of the elements \( e_j \) on the stem with \( f(e_j) \geq x \), and let \( \overline{W}_x \) denote the indices of stem elements not in \( W_x \). Then,

\[
\mathbb{E}_{I}[f(e_I)] = \int_0^\infty \mathbb{P}_I[f(e_I) \geq x] \, dx = \int_0^\infty \mathbb{P}_I[I \in W_x] \, dx = \int_0^\infty \sum_{i \in W_x} (p_i \prod_{j<i} q_j) \, dx,
\]

where the last equality uses that the probability of exiting stem at \( i \) is \( p_i \prod_{j<i} q_j \).

On the other hand, we have

\[
\mathbb{E}_{I,R}[\max_{e_j \in R} f(e_j)] = \int_0^\infty \mathbb{P}_I,\mathbb{P}_{e_j \in R}[\max_{e_j \in R} f(e_j) \geq x] \, dx = \int_0^\infty \mathbb{P}_I,R[R \cap W_x \neq \emptyset] \, dx
\]

\[
= \int_0^\infty \sum_{k \in W_x} \mathbb{P}_I[R[k] \geq k] \cdot \mathbb{P}_I[ek \text{ active}] \cdot \mathbb{P}_I[e_j \text{ inactive for all } j < k \text{ with } j \in W_x] \, dx
\]

\[
= \int_0^\infty \sum_{k \in W_x} (\prod_{j<k} q_j) \cdot p_k \cdot \left( \prod_{j<k \& j \in W_x} q_j \right) \, dx = \int_0^\infty \sum_{k \in W_x} \left( \prod_{j<k} q_j^2 \right) \cdot (\prod_{j<k \& j \in W_x} q_j) \cdot p_k \, dx.
\]
Recall that $R \sim S_I(p)$. Above (11) is because, for $e_k$ to be the first element in $W_x \cap R$ (i) the index $I$ must go past $k$, (ii) $e_k$ must be active, and (iii) all elements before $k$ on the stem with indices in $W_x$ must be inactive (which are all independent events). Equation (12) is by definition of these probabilities. Renaming $k$ to $i,$

$$
\mathbb{E}_{I,R} \left[ \max_{e_j \in R} f(e_j) \right] = \int_0^\infty \sum_{i \in W_x} \left( p_i \left( \prod_{j < i \& j \in W_x} q^2_j \right) \left( \prod_{j < i \& j \not\in W_x} q_j \right) \right) \, dx. \tag{13}
$$

To complete the proof, we compare equations (10) and (13) and want to show that for every $x,$

$$
\sum_{i \in W_x} \left( p_i \left( \prod_{j < i \& j \in W_x} q^2_j \right) \left( \prod_{j < i \& j \not\in W_x} q_j \right) \right) \geq \frac{1}{2} \sum_{i \in W_x} \left( p_i \prod_{j < i} q_j \right). \tag{14}
$$

While the expressions look complicated, things simplify considerably when we condition on the outcomes of elements outside $W_x$. Indeed, observe that the LHS of (14) equals

$$
\mathbb{E}_{W_x} \left[ \sum_{i \in W_x} \left( p_i \left( \prod_{j < i \& j \in W_x} q^2_j \right) \left( \prod_{j < i \& j \not\in W_x} q_j \right) \right) \right], \tag{15}
$$

where $1_{q_j}$ is an independent indicator r.v. taking value 1 w.p. $q_j$, and we take the expectation over coin tosses for elements of stem outside $W_x$. Similarly, the RHS of (14) is

$$
\frac{1}{2} \sum_{i \in W_x} \left( p_i \prod_{j < i} q_j \right) = \mathbb{E}_{W_x} \left[ \frac{1}{2} \sum_{i \in W_x} \left( p_i \left( \prod_{j < i \& j \in W_x} q_j \right) \left( \prod_{j < i \& j \not\in W_x} 1_{q_j} \right) \right) \right]. \tag{16}
$$

Hence, after we condition on the elements outside $W_x$, the remaining expressions can be related using the following claim.

**Claim 3.4.** For any ordered set $A$ of probabilities $\{a_1, a_2, \ldots, a_{|A|}\},$ let $b_j$ denote $1 - a_j$ for $j \in [1, |A|]$. Then,

$$
\sum_i a_i \left( \prod_{j < i} b_j \right)^2 \geq \frac{1}{2} \sum_i a_i \left( \prod_{j < i} b_j \right)
$$

*Proof.*

$$
\sum_i a_i \left( \prod_{j < i} b_j \right)^2 = \sum_i \frac{1 - b_i^2}{1 + b_i} \left( \prod_{j < i} b_j \right)^2 \geq \frac{1}{2} \sum_i (1 - b_i^2) \left( \prod_{j < i} b_j^2 \right)
$$

$$
=^{(*)} \frac{1}{2} \left( 1 - \prod_i b_i^2 \right) = \frac{1}{2} \left( 1 - \prod_i b_i \right) \left( 1 + \prod_i b_i \right)
$$

$$
\geq \frac{1}{2} \left( 1 - \prod_i b_i \right) =^{(*)} \frac{1}{2} \sum_i a_i \left( \prod_{j < i} b_j \right),
$$

where we have repeatedly used $a_j + b_j = 1$ for all $j$. The equalities marked $(*)$ move between two ways of expressing the probability of at least one “heads” when the tails probability is $b_j^2$ and $b_j$ respectively. $\blacksquare$

Applying the claim to the elements in $W_x$, in order of their distance from the root, completes the proof. $\blacksquare$
3.2 Lower Bounds

Our analysis cannot be substantially improved, since Claim 3.4 is tight. Consider the setting with $|A|$ being infinite for now, and $a_i = \varepsilon$ for all $i$. Then the LHS of Claim 3.4 is $\varepsilon \sum_{i} (1 - \varepsilon)^{2(i-1)} = \frac{1}{1 - (1 - \varepsilon)} \approx \frac{1}{\varepsilon} + O(\varepsilon)$, whereas the sum on the right is 1. Making $|A|$ finite but large compared to $\frac{1}{\varepsilon}$ would give similar results.

However, there is still hope that a smaller adaptivity gap can be proved using other techniques. The best lower bound on adaptivity gaps for monotone submodular functions we currently know is $\frac{\varepsilon}{1 - \varepsilon}$. The function is the rank function of a partition matroid, where the universe has $k$ parts (each with $k^2$ elements) for a total of $n = k^3$ elements. Each element has $p_e = \frac{1}{k}$. The probing constraint is a cardinality constraint that at most $k^2$ elements can be probed. In this case the optimal adaptive strategy can get $(1 - o(1))k$ value, whereas any non-adaptive strategy will arbitrarily close to $(1 - \frac{1}{k})k$ in expectation. (See, e.g., [AN16, Section 3.1].)

3.3 Finding Non-Adaptive Polices

A non-adaptive policy is given by a fixed sequence $\sigma = (e_1, e_2, \ldots, e_k)$ of elements to probe (such that $\sigma$ satisfies the given prefix-closed probing constraint. If $A$ is the set of active elements, then the value we get is $\mathbb{E}_{A \sim X(p)}[f^{\max}(A \cap \{e_1, e_2, \ldots, e_k\})] = \mathbb{E}_A[f(A \cap \{e_1, e_2, \ldots, e_k\})]$, the inequality holding for monotone functions. If we define $g(S) := \mathbb{E}_{A \sim X(p)}[f(X \cap A)]$, $g$ is also a monotone submodular function. Hence finding good non-adaptive policies for $f$ is just optimizing the monotone submodular function $g$ over the allowed sequences.

E.g., for the probing constraint being a matroid constraint, we can get a $\frac{\varepsilon}{1 - \varepsilon}$ approximation [CCPV11]; for it being an orienteering constraint we can get an $O(\log n)$-approximation in quasi-polynomial time [CP05].

For non-monotone functions (discussed in the next section), we can approximate the $f^{\max}(S)$ by $\mathbb{E}_{R \sim X(p)}[f(S \cap R)]$, and losing a factor of 4, reduce finding good non-adaptive strategies to (non-monotone) submodular optimization over the probing constraints.

4 Non-Monotone Non-Negative Submodular Functions

We now prove Theorem 1.2. The proof for the monotone case used monotonicity in several places, but perhaps the most important place was to claim that going down the tree, both adap and alg could add all active elements to the set. This “online” feature seemed crucial to the proof. In contrast, when the adaptive strategy adap reaches a leaf in the non-monotone setting, it chooses the best subset within the active elements; a similar choice is done by the non-adaptive algorithm. This is why we have $f^{\max}(A_{\ell})$ in (2) versus $f(A_{\ell})$ in (4).

Fortunately, Feige el al. [FMV11] show that for non-negative non-monotone submodular functions, the simple strategy of picking every active element independently w.p. half gives us a near-optimal possible subset. Losing a factor of four, this result allows us to analyze the performance relative to an adaptive online algorithm adap$_{on}$ which selects (with probability $\frac{1}{2}$) each probed element that happens to be active. The rest of the proof is similar (at a high level) to the monotone case: to relate adap$_{on}$ and alg we bound them using comparable terms (adapt and alg in Definition 4.1) and apply induction. Altogether we will obtain:

$$\text{alg} \geq \frac{1}{5} \cdot \text{adap} \geq \frac{1}{10} \cdot \text{adap}_{on} \geq \frac{1}{10} \cdot \text{adap}.$$  

In the inductive proof, we will work with “contracted” submodular functions $g$ obtained from $f$, which may take negative values but have $g(0) = 0$. In order to deal with such issues, the induction here is more complex than in the monotone case.

We first define the surrogates adap and alg for adap and alg recursively as follows.

**Definition 4.1.** For any strategy tree $T$ and submodular function $g$ with $g(0) = 0$, let
• I be the node at which a random walk according to \( \pi_T \) exits the stem.
• \( R \sim S_I(p) \) where \( S_I \) denotes the elements on the stem until node \( I \).
• \( J = \arg\max\{g(e) \mid e \in R, g(e) > 0\} \) w.p. \( \frac{1}{2} \) and \( J = \perp \) w.p. \( \frac{1}{2} \).

Then we define:

\[
\overline{\text{adap}}(T, g) := \mathbb{E}_{I,J}[g(I) + g(J)] + \overline{\text{adap}}(T_I, g_{I\cup J}) \quad \text{and} \quad \overline{\text{alg}}(T, g) := \mathbb{E}_{I,J}[g(J) + \overline{\text{alg}}(T_I, g_{I\cup J})].
\]

Above we account for the non-monotonicity of the function, via this process of random sampling used in the definition of \( \overline{\text{adap}} \) and \( \overline{\text{alg}} \). One problem with following the proof from \( \S 3 \) is that when we induct on the “contracted” function \( f_S \) for some set \( S \), this function may not be non-negative any more. Instead, our proof considers the entire path down the tree and argues about it at one shot; to make the analysis easier we imagine that the non-adaptive algorithm picks at most one item from the stem, i.e., the one with the highest marginal value.

**Lemma 4.2.** For any strategy tree \( T \), the following hold:

(i) For any non-negative submodular function \( f \), \( \overline{\text{adap}}(T, f) \geq \frac{1}{2}\text{adap}_{on}(T, f) \).

(ii) For any submodular function \( g \), \( \overline{\text{alg}}(T, g) \geq \overline{\text{alg}}(T, g) \).

We make use of the following property of submodular functions.

**Lemma 4.3 ([BFNS14], Lemma 2.2).** For any non-negative submodular function \( h : 2^A \rightarrow \mathbb{R}_{\geq 0} \) (possibly with \( h(\emptyset) \neq 0 \)) let \( S \subseteq A \) be a random subset that contains each element of \( A \) with probability at most \( p \) (and not necessarily independently). Then, \( \mathbb{E}_S[h(S)] \geq (1-p) \cdot f(\emptyset) \).

**Proof of Lemma 4.2.** We condition on a random leaf \( \ell \) drawn according to \( \pi_T \). Let \( I_1, \ldots, I_d \) denote the sequence of nodes that correspond to active elements on the path \( P_\ell \), i.e., \( I_1 \) is the point where \( P_\ell \) exits the stem of \( T \), \( I_2 \) is the point where \( P_\ell \) exits the stem of \( T_{I_1} \) etc. Then, the adaptive online value is exactly \( f(I_1, \ldots, I_d) \).

For any \( k = 1, \ldots, d \) let \( P_\ell[I_{k-1}, I_k] \) denote the elements on path \( P_\ell \) between \( I_{k-1} \) and \( I_k \). Also let \( R \) denote the random subset where each element \( e \) on path \( P_\ell \) is chosen independently w.p. \( p_e \).

For \( k = 1, \ldots, d \), define \( J_k \) as follows:

\[
J_k = \arg\max\{f_{L_{k-1}}(e) \mid e \in R \cap \ell[I_{k-1}, I_k], f_{L_{k-1}}(e) > 0\} \quad \text{w.p.} \quad \frac{1}{2} \quad \text{and} \quad J_k = \perp \quad \text{w.p.} \quad \frac{1}{2},
\]

where \( L_{k-1} := \{I_1, \ldots, I_{k-1}\} \cup \{J_1, \ldots, J_{k-1}\} \). In words, the sets \( L \) contain the exit points from the stems, and for each stem also the element with maximum marginal value (if any) with probability half.

For (i), by Definition 4.1, the value of \( \overline{\text{adap}}(T, f) \) conditioned on path \( P_\ell \) and elements \( J_1, \ldots, J_d \) is

\[
\sum_{k=1}^d f_{L_{k-1}}(I_k) + f_{L_{k-1}}(J_k) \geq \sum_{k=1}^d f_{L_{k-1}}(\{I_k, J_k\}) = f(\{I_1, J_1, \ldots, I_d, J_d\}).
\]

The inequality follows from the following two cases:

• If \( I_k \neq J_k \), then by submodularity of \( f_{L_{k-1}} \),

\[
f_{L_{k-1}}(I_k) + f_{L_{k-1}}(J_k) \geq f_{L_{k-1}}(\{I_k, J_k\}) \quad \text{and} \quad f_{L_{k-1}}(\emptyset) = f_{L_{k-1}}(\{I_k, J_k\}).
\]
• If $I_k = J_k$, then by choice of $J_k$ we have $f_{L_{k-1}}(J_k) > 0$ and
\[ f_{L_{k-1}}(I_k) + f_{L_{k-1}}(J_k) = 2 \cdot f_{L_{k-1}}(J_k) > f_{L_{k-1}}(J_k). \]

Using (17) and taking expectation over the $J$s, $\text{adap}(T, f)$ conditioned on path $P_\ell$ is at least
\[ \mathbb{E}_{J_1, \ldots, J_d} \left[ f\left(\{I_1, J_1, \ldots, I_d, J_d\}\right) \right] \geq \frac{1}{2} \cdot f\left(\{I_1, \ldots, I_d\}\right). \]

Above we used Lemma 4.3 on the non-negative submodular function $h(S) := f(S \cup \{I_1, \ldots, I_d\})$, using the fact that the set $\{J_1, \ldots, J_d\}$ contains each element with probability at most half. Finally, deconditioning over $\ell$ (i.e., over $I_1, \ldots, I_d$) proves part (i).

For part (ii), by Definition 4.1, the value of $\text{alg}(T, g)$ conditioned on path $P_\ell$ and elements $J_1, \ldots, J_d$ is
\[ \sum_{k=1}^{d} g_{L_{k-1}}(J_k) \leq \sum_{k=1}^{d} g_{J_1, \ldots, J_{k-1}}(J_k) = g(\{J_1, \ldots, J_d\}), \]

where the inequality is by submodularity of $g$. Since alg chooses the maximum value subset in $R$ and $\{J_1, \ldots, J_d\} \subseteq R$, taking expectations over $\ell$ and $R$, we prove part (ii).

**Lemma 4.4.** For any strategy tree $T$ and submodular function $g$ with $g(\emptyset) = 0$, $\text{alg}(T, g) \geq \frac{1}{2} \cdot \text{adap}(T, g)$.

**Proof.** We proceed by induction. Recall the notation in Definition 4.1. For each node $i$ on the stem of $T$ define $a_i := \max\{g(i), 0\}$. Note that $g(J) = a_J$ by choice of $J$: if $J \neq \bot$ we have $g(J) > 0$ and if $J = \bot$, $g(J) = g(\emptyset) = 0 = a_J$. We will show that
\[ \mathbb{E}_{I,J}[a_J] \leq 4 \cdot \mathbb{E}_{I,J}[a_J]. \]

Then the definition of $\text{adap}(T, g)$ and $\text{alg}(T, g)$, and induction on $T_I$ and $g_{I \cup J}$, would prove the lemma.

Let $K = \arg\max\{a_e \mid e \in R\}$ be the r.v. denoting the maximum weight active (i.e., in $R$) element on the stem. Then, by definition of $J$, we have $\mathbb{E}_{I,J}[a_J] = \frac{1}{2} \mathbb{E}_{I,K}[a_K]$. Finally we can use Lemma 3.3 from Section 3 to obtain $\mathbb{E}_{I,K}[a_K] \geq \frac{1}{2} \mathbb{E}_{I,J}[a_J]$, which proves (18).

**5 Monotone XOS Functions**

In this section we study adaptivity gaps for monotone non-negative XOS functions. To recall, a function is monotone XOS if there exist linear functions $a_1, a_2, \ldots, a_W : X \rightarrow \mathbb{R}^+$ such that $f(S) = \max_{i=1}^{W}\{\sum_{e \in S} a_i(e)\}$.

To simplify notation we use $a_i(S) := \sum_{e \in S} a_i(e)$ for any $i$ and subset $S \subseteq X$. The width of an XOS function is the smallest number $W$ such that $f$ can be written as the maximum over $W$ linear functions. Let $T^*$ denote the optimal adaptive strategy. By monotonicity $f^{\max} = f$ and (2) gives
\[ \text{adap}(T^*, f) = \mathbb{E}_{I,e \in T^*}[f(A_I)]. \]

The following is our main result in this section.

**Theorem 5.1.** The stochastic probing problem for monotone XOS functions of width $W$ has adaptivity gap $O(\log W)$ for any prefix-closed constraints. Moreover, there are instances with $W = O(n)$ and adaptivity gap $\Omega(\frac{\log W}{\log \log W})$.

In §5.2, we also present an efficient non-adaptive algorithm for XOS functions of width $W$ that makes $O(W + \log n)$ calls to the following linear oracle.

**Definition 5.2 (Oracle $O$).** Given a prefix-closed constraint family $\mathcal{F}$ and linear function $a : X \rightarrow \mathbb{R}^+$, oracle $O(\mathcal{F}, a)$ returns a set $S \in \mathcal{F}$ that maximizes $\sum_{e \in S} a(e)$. 

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5.1 Adaptness Gap Upper Bound

We first state a useful property that is used critically later.

**Assumption 5.3** (Subtree property). For any node \( u \) in the optimal adaptive strategy tree \( T^* \), if we consider the subtree \( T' \) rooted at \( u \) then the expected value of \( T' \) is at most that of \( T^* \):

\[
\text{adap}(T', f) \leq \text{adap}(T^*, f), \text{ when } T' \text{ is a subtree of } T^*.
\]

This is because otherwise a better strategy would be to go directly to \( u \) (probing all the element along the way, so that we satisfy the prefix-closed constraint, but ignore these elements), and then to run strategy \( T' \).

**Proof Idea.** The proof consists of three steps. In the first step we argue that one can assume that every coefficient in every linear function \( a_i \) is smaller than \( O\left(\frac{\text{adap}(T^*, f)}{\log W}\right) \) (else there is a simple non-adaptive strategy that is comparable to the adaptive value obtained from a single active item). The second step shows that by losing a constant factor, one can truncate the tree \( T^* \) to obtain tree \( T \), where the instantiated value at each leaf is at most \( 2 \cdot \text{adap}(T^*, f) \). The combined benefit of these steps is to ensure that root-leaf paths have neither high variance nor too large a value. In the third step, we use Freedman’s concentration inequality (which requires the above properties of \( T \)) to argue that for any linear function \( a_i \), the instantiated value on a random root-leaf path is close to its mean with high probability. Taking union bound over the \( W \) linear functions, we can then show that (again with high probability), no linear function has an instantiation much more than its mean. Hence, for a random root-leaf path, adap gets value (the maximum instantiation over linear functions) that is not much more than the corresponding mean, which is a lower bound on the non-adaptive value.

Below we use \( \text{OPT} = \text{adap}(T^*, f) \) to denote the optimal adaptive value.

**Small and large elements.** Define \( \lambda := 10^3 \log W \). An element \( e \in X \) is called large if \( \max_{i=1}^W a_i(e) \geq h := \frac{\text{OPT}}{\lambda} \); it is called small otherwise. Let \( L \) be the set of large elements, and let \( \text{OPT}_L \) (resp., \( \text{OPT}_S \)) denote the value obtained by tree \( T^* \) from large (resp., small) elements. By subadditivity, we have \( \text{OPT}_L + \text{OPT}_S \geq \text{OPT} \).

Lemma 5.4 shows that that when \( \text{OPT}_L \geq \text{OPT}/2 \), a simple non-adaptive strategy proves that the adaptivity gap is \( O(\log W) \). Then Lemma 5.5 shows that when \( \text{OPT}_S \geq \text{OPT}/2 \), the adaptivity gap is \( O(1) \). Choosing between the two by flipping an unbiased coin gives a non-adaptive strategy that proves the adaptivity gap is \( O(\log W) \). This would prove the first part of Theorem 5.1.

**Lemma 5.4.** Assuming that \( \text{OPT}_L \geq \text{OPT}/2 \), there is a non-adaptive solution of value \( \Omega(1/\log W) \cdot \text{OPT} \). Moreover, there is a solution \( S \) satisfying the probing constraint with \( h \cdot \min\{\sum_{e \in S \cap L} a_e, 1\} \geq \frac{\text{OPT}}{O(\log W)} \).

**Proof.** We restrict the optimal tree \( T^* \) to the large elements. So each node in \( T^* \) either contains a large element, or corresponds to making a random choice (and adds no value). The expected value of this restricted tree is \( \text{OPT}_L \). We now truncate \( T^* \) to obtain tree \( T^* \) as follows. Consider the first active node \( u \) on any root-leaf path, remove the subtree below the \( y \in \Sigma \) (active) arc from \( u \), and assign exactly a value of \( h \) to this instantiation. The subtree property (Assumption 5.3) implies that the expected value in this subtree below \( u \) is at most \( \text{OPT} \). On the other hand, just before the truncation at \( u \), the adaptive strategy gains value of \( h = \frac{\text{OPT}}{\lambda} \) since it observed an active large element at node \( u \). By taking expectations, we obtain that the value of \( T^* \) is at least \( \frac{1}{1+\lambda} \cdot \text{OPT}_L \).

Note that \( T^* \) is a simpler adaptive strategy. In fact \( T^* \) is a feasible solution to the stochastic probing instance with the probing constraint \( \mathcal{F} \) and a different objective \( g(R) = h \cdot \min\{|R \cap L|, 1\} \) which is the rank function of the uniform-matroid of rank 1 (scaled by \( h \)) over all large elements. As any matroid rank function is a monotone submodular function, Theorem 1.1 implies that there is a non-adaptive strategy which probes a feasible sequence of elements \( S \in \mathcal{F} \), having value \( \mathbb{E}_{R \sim S(p)}[g(R)] \geq \frac{1}{4} \cdot \text{adap}(T^*, g) \geq \frac{1}{4} \cdot \frac{\text{OPT}_L}{1+\lambda} \). Note that
for any subset $R \subseteq L$ of large elements $f(R) \geq \max_{e \in R} \{ \max_{i=1}^{W} a_i(e) \} \geq h \cdot \min(|R|, 1) = g(R)$; the first inequality is by monotonicity of $f$ and the second is by definition of large elements. So we have:

$$\mathbb{E}_{R \sim \mathcal{S}(p)}[f(R)] \geq \mathbb{E}_{R \sim \mathcal{S}(p)}[g(R)] = \Omega(1/\log W) \cdot \text{OPT}_f.$$ 

It follows that $S$ is the claimed non-adaptive solution for the original instance with objective $f$.

We now show the second part of the lemma using the above solution $S$. Note that

$$\frac{\text{OPT}}{O(\log W)} = \mathbb{E}_{R \sim \mathcal{S}(p)}[g(R)] = h \cdot \mathbb{E}_{R \sim \mathcal{S}(p)}[1(R \cap L \neq \emptyset)] \leq h \cdot \min \left\{ \sum_{e \in S \cap L} p_e, 1 \right\},$$

as desired. 

In the rest of this section we prove the following, which implies an $O(\log W)$ adaptivity gap.

**Lemma 5.5.** Assuming that $\text{OPT}_s \geq \text{OPT}/2$, there is a non-adaptive solution of value $\Omega(1) \cdot \text{OPT}$. 

**Proof.** We start with the restriction of the optimal tree $T^*$ to the small elements; recall that $\text{OPT}_s$ is the expected value of this restricted tree. The next step is to truncate tree $T^*$ to yet another tree $T$ with further useful properties. For any root-leaf path in $T^*$ drop the subtree below the first node $u$ (including $u$) where $f(A_u) > 2 \cdot \text{OPT}$; here $A_u$ denotes the set of active elements on the path from the root to $u$. The subtree property (Assumption 5.3) implies that the expected value in the subtree below $u$ is at most $\text{OPT}$. On the other hand, before the truncation at $u$, the adaptive value obtained is more that $2 \cdot \text{OPT}$. Hence, the expected value of $T^*$ obtained at or above the truncated nodes is at least $\frac{2}{3} \cdot \text{OPT}_s$. Finally, since all elements are small and thus the expected value from any truncated node itself is at most $h \leq 0.01 \cdot \text{OPT}$, the tree $T$ has at least $(\frac{2}{3} - 0.01)\text{OPT}_s \geq \frac{1}{3}\text{OPT}_s$ value. This implies the next claim:

**Claim 5.6.** Tree $T$ has expected value at least $\frac{1}{2} \cdot \text{OPT}_s \geq \frac{1}{3} \cdot \text{OPT}$ and $\max_{\ell \in T} \max_{i=1}^{W} a_i(P_\ell) \leq 2 \cdot \text{OPT}$. 

Next, we want to claim that each linear function behaves like its expectation (with high probability) on a random path down the tree. For any $i \in [W]$ and root-leaf path $P_\ell$ in $T$, define

$$\mu_i(P_\ell) := \mathbb{E}_{R \sim X(p)}[a_i(R \cap P_\ell)] = \sum_{v \in P_\ell} \left( p_{\text{elt}(v)} \cdot a_i(\text{elt}(v)) \right).$$

**Claim 5.7.** For any $i \in [W],$

$$\Pr_{\ell \sim \tau} \left[ |a_i(A_\ell) - \mu_i(P_\ell)| > 0.1 \cdot \text{OPT} \right] \leq \frac{1}{W^2}. \quad (19)$$

**Proof.** Our main tool in this proof is the following concentration inequality for martingales.

**Theorem 5.8** (Freedman, Theorem 1.6 in [Fre75]). Consider a real-valued martingale sequence $\{X_t\}_{t \geq 0}$ such that $X_0 = 0$, and $\mathbb{E} [X_{t+1} \mid X_t, X_{t-1}, \ldots, X_0] = 0$ for all $t$. Assume that the sequence is uniformly bounded, i.e., $|X_t| \leq M$ almost surely for all $t$. Now define the predictable quadratic variation process of the martingale to be $W_t = \sum_{j=0}^{t} \mathbb{E} [X_j^2 \mid X_{j-1}, X_{j-2}, \ldots, X_0]$ for all $t \geq 1$. Then for all $\ell \geq 0$ and $\sigma^2 > 0$, and any stopping time $\tau$ we have

$$\Pr \left[ \left| \sum_{j=0}^{\tau} X_j \right| \geq \ell \text{ and } W_\tau \leq \sigma^2 \right] \leq 2 \exp \left( -\frac{\ell^2/2}{\sigma^2 + M\ell/3} \right).$$

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Consider a random root-leaf path \( P_\ell = \langle r = v_0, v_1, \ldots, v_\ell = \ell \rangle \) in \( T \), and let \( e_\ell = \text{elt}(v_\ell) \). Now define a sequence of random variables \( X_0, X_1, \ldots, \) where
\[
X_\ell = (1(e_\ell \in A) - p_{e_\ell}) \cdot a_\ell(e_\ell).
\]
Let \( H_\ell \) be a filter denoting the sequence of variables before \( X_\ell \). Observe that \( \mathbb{E}[X_\ell \mid H_\ell] = 0 \), which implies \( \{X_\ell\} \) forms a martingale. Clearly \( |X_\ell| \leq |a_\ell(e_\ell)| \leq h \). Now,
\[
\sum_{j=0}^t \mathbb{E} [X_j \mid H_j] \leq \sum_{j=0}^t (p_{e_\ell}(1 - p_{e_\ell}) + (1 - p_{e_\ell})p_{e_\ell}) \cdot a_\ell(e_\ell)
\]
\[
\leq \frac{1}{2} \sum_{j=0}^t a_\ell(e_\ell) \leq \frac{1}{2} \cdot \max_{i=1}^W a_\ell(P_\ell) \leq \text{OPT},
\]
where the last inequality is by Claim 5.6. We use \( |X_j| \leq h = \frac{\text{OPT}}{\lambda} \) and the above equation to bound the variance,
\[
\sum_{j=0}^t \mathbb{E} [X_j^2 \mid H_j] \leq h \cdot \sum_{j=0}^t \mathbb{E} [X_j \mid H_j] \leq \frac{\text{OPT}^2}{\lambda}.
\]
Applying Theorem 5.8, we get
\[
\Pr \left[ \sum_{j=0}^r X_j > 0.1 \text{OPT} \right] = \Pr[|a_\ell(A_\ell) - \mu_\ell(P_\ell)| > 0.1 \text{OPT}]
\]
\[
\leq 2 \exp \left( -\frac{(0.1 \text{OPT})^2/2}{\text{OPT}^2/(\lambda) + (\text{OPT}/(\lambda)) \cdot (0.1 \text{OPT})/3} \right)
\]
\[
\leq \frac{1}{W^2}.
\]
This completes the proof of Claim 5.7.

Now we can finish the proof of Lemma 5.5. We label every leaf \( \ell \) in \( T \) according to the linear function \( a_\ell \) that achieves the value \( f(A_\ell) \), breaking ties arbitrarily. I.e., for leaf \( \ell \) we define
\[
c_\ell^{\max} := a_\ell, \text{ where } a_\ell(A_\ell) = f(A_\ell).
\]
Also define \( \mu_\ell^{\max} := \mu_i \) for \( i \) as above. Using Claim 5.7 and taking a union bound over all \( i \in [W] \),
\[
\Pr_{\ell \gets \pi_T} \left[ |c_\ell^{\max}(A_\ell) - \mu_\ell^{\max}(P_\ell)| > 0.1 \text{OPT} \right] \leq \frac{1}{W}. \quad (20)
\]
Consider the natural non-adaptive solution which selects \( \ell \leftarrow \pi_T \) and probes all elements in \( P_\ell \). This has expected value at least:
\[
\mathbb{E}_{\ell \gets \pi_T} [\mu_\ell^{\max}(P_\ell)] \geq \mathbb{E}_{\ell \gets \pi_T} [c_\ell^{\max}(A_\ell)] - 0.1 \text{OPT} - \frac{1}{W}(2 \text{OPT}) \geq (0.15 - \frac{2}{W}) \cdot \text{OPT}.
\]
If \( W \geq 20 \) then we obtain the desired non-adaptive strategy. The remaining case of \( W < 20 \) is trivial: the adaptivity gap is 1 for a single linear function, and taking the best non-adaptive solution among the \( W \) possibilities has value at least \( \frac{1}{W} \cdot \text{OPT} \). This completes the proof of Lemma 5.5.

Let us record an observation that will be useful for the non-adaptive algorithm.

**Remark 1.** Observe that the above proof shows that when \( \text{OPT}_n \geq \text{OPT}/2 \), there exists a path \( Q \in T^* \) (i.e. \( Q \) satisfies the probing constraints) and a linear function \( a_j \) with mean value \( \mathbb{E}_{R \sim Q|P}[a_j(R)] = \Omega(\text{OPT}) \).
5.2 Polynomial Time Non-adaptive Algorithm

Consider any instance of the stochastic probing problem with a width-$W$ monotone XOS objective and prefix-closed constraint $\mathcal{F}$. Our non-adaptive algorithm is the following (here $\lambda = 10^3 \log W$ as is in §5.1).

Algorithm 5.1 Non-adaptive Algorithm for XOS functions

1: define $m := \max_{e \in X} \{p_e \cdot \max_{i \in [W]} a_i(e)\}$
2: for $j \in \{0, \ldots, 1 + \log n\}$ do
3: define $b_j$ with $b_j(e) = p_e$ if $\max_{i \in [W]} a_i(e) \geq 2^jm$ and $b_j(e) = 0$ otherwise.
4: $T_j \leftarrow O(\mathcal{F}, b_j)$ and $v(T_j) \leftarrow 2^jm \cdot \min\{b_j(T_j), 1\}$.
5: end for
6: for $i \in \{1, \ldots, W\}$ do
7: define $c_i$ with $c_i(e) = p_e \cdot a_i(e)$
8: $S_i \leftarrow O(\mathcal{F}, c_i)$ and $v(S_i) \leftarrow c_i(S_i)$.
9: end for
10: return set $S \in \{S_1, \ldots, S_W, T_0, T_1, \ldots, T_{1+\log n}\}$ that maximizes $v(S)$.

Case I: $\text{OPT}_I \geq \text{OPT}/2$. Lemma 5.4 shows that in this case it suffices to consider only the set of large elements and to maximize the probability of selecting a single large element. While we do not know $\text{OPT}$, and the large elements are defined in terms of $\text{OPT}$, we do know $m = \max_{e \in X} \{p_e \cdot \max_{i \in [W]} a_i(e)\} \leq \text{OPT} \leq n \cdot m$. In the above algorithm, consider the value of $j \in \{0, \ldots, 1 + \log n\}$ when $2^j \cdot m/\lambda$ is between $h$ and $2h$. Let $L$ denote the set of large elements; note that these correspond to the elements with positive $b_j(e)$ values. By the second part of Lemma 5.4, the solution $T_j$ returned by the oracle will satisfy $v(T_j) \geq \text{OPT}/O(\log W)$.

Now interpreting this solution $T_j$ as a non-adaptive solution, we get an expected value at least:

$$h \cdot E_{R \sim T_j(p)}[1(R \cap L \neq \emptyset)] = h \cdot (1 - \prod_{e \in T_j}(1 - b_j(e))) \geq h \cdot (1 - e^{-b_j(T_j)})$$

$$\geq (1 - 1/e)h \cdot \min\{b_j(T_j), 1\} = (1 - 1/e) \cdot v(T_j) \geq \frac{\text{OPT}}{O(\log W)}$$

Case II: $\text{OPT}_s \geq \text{OPT}/2$. In this case Remark 1 following the proof of Lemma 5.5 shows that there exists a solution $Q$ satisfying the probing constraints $\mathcal{F}$ and a linear function $a_j$ with mean value $c_j(Q) = \mathbb{E}_{R \sim Q(p)}[a_j(R)] = \Omega(\text{OPT})$. Since the above algorithm calls $O(\mathcal{F}, c_i)$ for each $i \in [W]$ and chooses the best one, it will return a set with value $\Omega(\text{OPT})$.

5.3 Adapivity Gap Lower Bound

Consider a $k$-ary tree of depth $k$, whose edges are the ground set. Each edge/element has probability $p_e = \frac{1}{k}$. Here, imagine $k = \Theta(\frac{\log n}{\log \log n})$, so that the total number of edges is $\sum_{i=1}^k k^i = n$. For each of the $k^k$ leaves $l$, consider the path $P_l$ from the root to that leaf. The XOS function is $f(S) := \max_l |P_l \cap S|$. Note that the width $W = \Theta(n)$ in this case.

Suppose the probing constraint is the following prefix-closed constraint: there exists a root-leaf path $P_l$ such that all probed edges have at least one endpoint on this path. This implies that we can probe at most $k^2$ edges.

- For an adaptive strategy, probe the $k$ edges incident to the root. If any one of these happens to be active, start probing the $k$ edges at the next level below that edge. (If none were active, start probing the edges below the left-most child, say.) Each level will have at least one active edge with probability $1 - (1 - \frac{1}{k})^k \geq 1 - 1/k$, so we will get an expected value of $\Omega(k)$.
• Now consider any non-adaptive strategy: it is specified by the path $P_l$ whose vertices hit every edge that is probed. There are $k^2$ such edges, we can probe all of them. But the XOS function can get at most 1 from an edge not on $P_l$, and it will get at most $k \cdot 1/k = 1$ in expectation from the edges on $P_l$.

This shows a gap of $\Omega(k) = \Omega(\frac{\log n}{\log \log n})$ for XOS functions with a prefix-closed (in fact subset-closed) probing constraint.

5.3.1 A Lower Bound for Cardinality Constraints

We can show a near-logarithmic lower bound for XOS functions even for the most simple cardinality constraints. The setup is the same as above, just the constraint is that a subset of at most $k^2$ edges can be probed.

• The adaptive strategy remains the same, with expected value $\Omega(k)$.

• We claim that any non-adaptive strategy gets expected value $O(\log k)$. Such a non-adaptive strategy can fix any set $S$ of $k^2$ edges to probe. For each of these edges, choose an arbitrary root-leaf path passing through it, let $T$ be the edges lying in these $k^2$ many root-leaf paths of length $k$. So $|T| \leq k^3$. Let us even allow the strategy to probe all the edges in $T$—clearly this is an upper bound on the non-adaptive value.

The main claim is that the expected value to be maximized when $T$ consists of $k^2$ many disjoint paths. (The $k$-ary tree does not have these many disjoint paths, but this is just a thought-experiment.) The claim follows from an inductive application of the following simple fact.

Fact 5.9. Given independent non negative random variables $X, X', Y, Z$, where $X'$ and $X$ have the same distribution, the following holds:

$$E_{X,Y,Z}[\max\{X + Y, X + Z\}] \leq E_{X,X',Y,Z}[\max\{X + Y, X' + Z\}]$$

Proof. Follows from the fact that $\{\max\{X + Y, X + Z\} > c\} \subseteq \{\max\{X + Y, X' + Z\} > c\}$. ■

Finally, for any path with $k$ edges, we expect to get value 1 in expectation. The probability that any one path gives value $c \log k$ is $\frac{1}{k^c}$, for suitable constant $c$. So a union bound implies that the maximum value over $k^2$ path is at most $c \log k$ with probability $1/k$. Finally, the XOS function can take on value at most $k$, so the expected value is at most $1 + c \log k$.

This shows an adaptivity gap of $\Omega(\frac{k}{\log k}) = \Omega(\frac{\log n}{(\log \log n)^2})$ even for cardinality constraints.

6 Conclusions

In this paper we saw that submodular functions, both monotone and non-monotone, have a constant adaptivity gap, with respect to all prefix-closed probing constraints. Moreover, for monotone XOS functions of width $W$, the adaptivity gap is $O(\log W)$, and there are nearly-matching lower bounds for all $W = O(n)$.

The most obvious open question is whether for all XOS functions, the adaptivity gap is $O(\log^c n)$ for some constant $c \geq 1$. This would immediately imply an analogous result for all subadditive functions as well. (In §A we show that it suffices to bound the adaptivity gap for monotone XOS and subadditive functions.)

Other questions include: can we get better bounds for special submodular functions of interest? E.g., for matroid rank functions, can we improve the bound of $3$ from Theorem 1.1. We can improve the constants of $40$ for the non-monotone case with more complicated analyses, but getting (near)-tight results will require not losing the factor of $4$ from (1), and may require a new insight. Or can we do better for special prefix-closed constraints. Our emphasis was to give the most general result we could, but it should be possible to do quantitatively better for special cases of interest.
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A Monotonicity of XOS and Subadditive Functions

Our definition of fractionally subadditive/XOS differs from the usual one, since it allows the function to be non-monotone. To show the difference, here is the usual definition:

- A function $f$ is monotone fractionally subadditive if $f(T) \leq \sum_i \alpha_i f(S_i)$ for all $\chi_T \leq \sum_i \alpha_i \chi_{S_i}$ with $\alpha_i \geq 0$. Note the subtle difference: we now place a constraint when the set $T$ is fractionally covered by the sets $S_i$. Note that such a function is always monotone: for $T \subseteq S$ we have $\chi_T \leq \chi_S$ and hence $f(T) \leq f(S)$.

Similarly, we can define a function $f$ to be monotone XOS (a.k.a. max-of-sums) if there exist linear functions $a_1, a_2, \ldots, a_m : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ such that $f(X) = \max_j \{a_j(X)\}$. The difference is that we only allow non-negative coefficients in the linear functions.

The equivalence of these definitions is shown in [Fei09]. It is also known that the class of monotone XOS functions lies between monotone submodular and monotone subadditive functions. These same proofs, with minor alterations, show that XOS functions are the same as fractionally subadditive functions (according to the definitions in §2), and lie between general submodular and general subadditive functions.

Finally, if $f$ satisfies the XOS definition in §2, and $f$ is monotone, it also satisfies the definition above. Indeed, by duplicating sets and dropping some elements, we can take the sets $T, \{S_i\}$ and values $\alpha_i \geq 0$ satisfying $\chi_T \leq \sum_i \alpha_i \chi_{S_i}$, and get sets $S'_i \subseteq S_i$ satisfying $\chi_T = \sum_i \alpha_i \chi_{S'_i}$. By the general XOS definition, we get $f(T) \leq \sum_i \alpha_i f(S'_i)$, which by monotonicity is at most $\sum_i \alpha_i f(S_i)$. Hence, the definitions in §2 and above are consistent.

A.1 Adaptivity Gaps for Non-Monotone Functions

It suffices to prove the adaptivity gap conjecture for monotone XOS or subadditive functions, since for any XOS function $f$, the function $f^{\text{max}}$ is also XOS (shown below). Note that $f^{\text{max}}$ is clearly monotone. So we can just deal with the monotone XOS/subadditive function $f^{\text{max}}$. (We note that such a property is not true for submodular functions, i.e. $f$ being submodular does not imply that $f^{\text{max}}$ is.)

Consider any (possibly non-monotone) XOS function $f$. We will show that $f^{\text{max}}$ is fractionally subadditive, i.e., for any $T \subseteq X$, $\{S_i \subseteq X\}$ and $\{\alpha_i \geq 0\}$ with $\chi_T = \sum_i \alpha_i \chi_{S_i}$, $f^{\text{max}}(T) \leq \sum_i \alpha_i f^{\text{max}}(S_i)$.

Consider any $T, \{S_i\}, \{\alpha_i\}$ as above. Let $U \subseteq T$ be the set achieving the maximum in $f^{\text{max}}(T)$, i.e. $f^{\text{max}}(T) = f(U)$. Now consider the linear combination of the sets $\{S_i \cap U\}$ with multipliers $\{\alpha_i\}$. We have $\chi_U = \sum_i \alpha_i \chi_{S_i \cap U}$. So, by the fractionally subadditive property of $f$,

$$f(U) \leq \sum_i \alpha_i \cdot f(S_i \cap U) \leq \sum_i \alpha_i \cdot f^{\text{max}}(S_i).$$

The last inequality above is by definition of $f^{\text{max}}$ as $S_i \cap U \subseteq S_i$. Thus we have $f^{\text{max}}(T) \leq \sum_i \alpha_i f^{\text{max}}(S_i)$ as desired.
B Large Adaptivity Gap for Arbitrary Functions

Consider a monotone function $f$ on $k = \sqrt{n}$ types of items, with $k$ items of each type (total $k^2 = n$ items). On any set $S$ of items, function $f$ takes value 1 if $S$ contains at least one item of every type, and takes value 0 otherwise. Suppose each item is active independently w.p. $1/2$ and the constraint allows us to probe at most $4k$ items. The optimal non-adaptive strategy here is to probe 4 items of each type. This strategy has an expected value of $(\frac{15}{16})^k$. On the other hand, consider an adaptive strategy that arbitrarily orders the types and probes items of a type until it sees an active copy, and then moves to the next type. Since in expectation this strategy only probes 2 items of a type before moving to the next, with constant probability it will see an active copy of every type within the $4k$ probes. Hence, the adaptivity gap for this example is $\Omega(16/15)^k$. 
