Abstract

In this paper we will study non-anticommutative perturbative quantum gravity on spacetime with a complex metric. After analysing the BRST symmetry of this non-anticommutative perturbative quantum gravity, we will also analyse the effect of shifting all the fields. We will construct a Lagrangian density which is invariant under the original BRST transformations and the shift transformations in the Batalin-Vilkovisky (BV) formalism. Finally, we will show that the sum of the gauge-fixing term and the ghost term for this shift symmetry invariant Lagrangian density can be elegantly written down in superspace with a single Grassmann parameter.

Key words: BV-Formalism, Non-anticommutative Quantum Gravity
PACS number: 04.60.-m

1 Introduction

Noncommutative field theory arises in string theory due to presence of the $NS$ antisymmetric tensor background [1]-[3]. However, other backgrounds like the $RR$ background can generate non-anticommutative field theory [4]. Quantum gravity on noncommutative spacetime has been thoroughly studied [6]-[9]. In fact, it is hoped that noncommutativity predicts the existence of the cosmological constant which is of the same order as the square of the Hubble’s constant [10]. Perturbative quantum gravity on noncommutative spacetime has been already analysed [11]. It was found that the graviton propagator was the same as that in the commutative case. However, the noncommutative nature of spacetime was experienced at the level of interactions.

The idea of noncommutativity of spacetime has been generalized to non-anticommutativity. In addition to this, quantum field theory has been studied on non-anticommutative spacetime [12]. Non-anticommutativity of spacetime occurs if the metric is complex. Spacetime with complex metric has been studied as an interesting example of nonsymmetric gravity [13]-[15]. Even though nonsymmetric quantum gravity was initially studied in an attempt to unify
electromagnetism and gravity [16, 17], it is now mainly studied due to its relevance to string theory [18–20]. Quantum gravity on this non-anticommutative complex spacetime has also been discussed before [21].

In this paper we will discuss the perturbative quantum gravity on this non-anticommutative complex spacetime. We will first analyse the BRST symmetry of this theory and then study its the invariance under the original BRST and shift transformations in the BV-formalism. Consequently, we will express our results in superspace formalism.

The BRST symmetry for Yang-Mills theories [22–25] and spontaneously broken gauge theories [26] has already been analysed in noncommutative spacetime. The invariance of a theory, under the original BRST transformations and shift transformations, that occurs naturally in background field method can be analysed in the BV-formalism [27–30]. Both the BRST formalism [31, 32] and the BV-formalism can be given geometric meaning by the use of superspace [33–36].

BV-formalism has been used for quantizing $W_3$ gravity [37–39]. It has also been used in quantizing metric-affine gravity in two dimensions [40]. The BRST symmetry for perturbative quantum gravity in four-dimensional flat spacetime have been studied by a number of authors [41–43] and their work has been summarized by N. Nakanishi and I. Ojima [44]. The BRST symmetry in two-dimensional curved spacetime has been studied thoroughly [45–47]. Similarly, the BRST symmetry for topological quantum gravity in curved spacetime [48, 49] and the BRST symmetry for perturbative quantum gravity in both linear and non-linear gauge’s has also been analysed [50]. However, so far no work has been done in analysing the non-anticommutative perturbative quantum gravity in superspace BV-formalism. This is what we aim to do in this paper.

2 Perturbative Quantum Gravity

We shall analyse perturbative quantum gravity with the following hyperbolic complex metric [21],

$g^{(f)}_{bc} = b^{(f)}_{bc} + \omega a^{(f)}_{bc}$,

where $\omega$ is the pure imaginary element of a hyperbolic complex Clifford algebra with $\omega^2 = +1$. Here $\omega$ forms a ring of numbers and not a field which the usual system of complex numbers do. The advantage of using $\omega$ is that the negative energy states coming from the purely imaginary part of the metric will be avoided.

Now we can introduce non-anticommutativity as follows,

$[\hat{x}^a, \hat{x}^b] = 2\hat{x}^a \hat{x}^b + i \omega \tau^{ab}$,

where $\tau^{ab}$ is a symmetric tensor. We use Weyl ordering and express the Fourier transformation of this metric as,

$\hat{g}^{(f)}_{ab}(\hat{x}) = \int d^4k \pi e^{-ik\hat{x}} g^{(f)}_{ab}(k)$.

Now we have a one to one map between a function of $\hat{x}$ to a function of ordinary coordinates $y$ via

$g^{(f)}_{ab}(x) = \int d^4k \pi e^{-ikx} g^{(f)}_{ab}(k)$.
So, the product of ordinary functions is given by
\[ g^{(f)ab}(x) \hat{\otimes} g^{(f)ab}(x) = \exp \frac{\omega}{2} \left( \partial^{ab} \partial_{ac} \partial_{bc} \right) g^{(f)ab}(x), \]
(5)

Now \( R^{(f)ab}_{bcd} \) given as,
\[ R^{(f)ab}_{bcd} = -\partial_d \Gamma^{a}_{bc} + \partial_c \Gamma^{a}_{bd} + \Gamma^{a}_{ce} \hat{\otimes} \Gamma^{e}_{bc} - \Gamma^{a}_{ed} \hat{\otimes} \Gamma^{e}_{bc}, \]
(6)
and we also get \( R_{bc} = R^{d}_{bcd} \). Thus finally \( R^{(f)} \) is given by
\[ R^{(f)} = g^{(f)ab} \hat{\otimes} R^{(f)}_{ab}. \]
(7)

The Lagrangian density for pure gravity with cosmological constant \( \lambda \) can now be written as,
\[ L_c = \sqrt{g^{(f)}} \hat{\otimes} (R^{(f)} - 2\lambda), \]
(8)
where we have adopted units, such that \( 16\pi G = 1 \). In perturbative gravity on flat spacetime one splits the full metric \( g^{(f)}_{ab} \) into \( \eta_{ab} \) which is the metric for the background flat spacetime and \( h_{ab} \) which is a small perturbation around the background spacetime, \( g^{(f)}_{ab} = \eta_{ab} + h_{ab} \).
(9)
Here both \( \eta_{ab} \) and \( h_{ab} \) are complex. The covariant derivatives along with the lowering and raising of indices are compatible with the metric for the background spacetime. The small perturbation \( h_{ab} \) is viewed as the field that is to be quantized.

3 BV-Formalism

All the degrees of freedom in \( h_{ab} \) are not physical as the Lagrangian density for \( h_{ab} \) is invariant under the following gauge transformations,
\[ \delta \Lambda h_{ab} = D^c_{ab} \hat{\otimes} \Lambda^c, \]
\[ = \left[ \delta^c_{a} \partial_{a} + \delta^c_{b} \partial_{b} + g^{ce} \hat{\otimes} (\partial_{c} h_{ab}) + g^{ce} \hat{\otimes} h_{ac} \partial_{b} + \eta^{ce} \hat{\otimes} h_{cb} \partial_{a} \right] \hat{\otimes} \Lambda^c. \]
(10)
In order to remove these unphysical degrees of freedom, we need to fix a gauge by adding a gauge-fixing term along with a ghost term. In the most general covariant gauge the sum of the gauge-fixing term and the ghost term can be expressed as,
\[ L_g = s \Psi, \]
(11)
where
\[ \Psi = \bar{\epsilon} \hat{\otimes} \left( \partial^b h_{ab} - k \partial_a h + \frac{1}{2} b_a \right), \]
(12)
with \( k \neq 1 \). Now the sum of the ghost term, the gauge-fixing term and the original classical Lagrangian density is invariant under the following BRST transformations
\[ s h_{ab} = D^c_{ab} \hat{\otimes} c_c, \]
\[ s c^a = c^a \hat{\otimes} \partial^b \epsilon^b, \]
\[ s \epsilon^a = -b^a, \]
\[ s b^a = 0. \]
(13)
BV-formalism is used to analyse the extended BRST symmetry. This extended BRST symmetry for perturbative
quantum gravity can be obtained by first shifting all the original fields as,
\[ h_{ab} \rightarrow h_{ab} - \tilde{h}_{ab}, \quad c^a \rightarrow c^a - \tilde{c}^a, \]
\[ \bar{c}^a \rightarrow \bar{c}^a - \tilde{c}^a, \quad b^a \rightarrow b^a - \tilde{b}^a, \] (14)
and then requiring the resultant theory to be invariant under both the original BRST transformations and these shift transformations. This can be achieved by letting the original fields transform as,
\[ s h_{ab} = \psi_{ab}, \quad s c^a = \phi^a, \]
\[ s \bar{c}^a = \bar{\phi}^a, \quad s b^a = \rho^a, \] (15)
and the shifted fields transform as
\[ s \tilde{h}_{ab} = \psi_{ab} - D^c_{ab} \delta (c_c - \tilde{c}_c), \quad s \tilde{c}^a = \phi^a + (c_b - \tilde{c}_b) \delta \partial^b (c_a - \tilde{c}_a), \]
\[ s \tilde{\bar{c}}^a = \bar{\phi}^a + (b^a - \tilde{b}^a) \delta \partial^b (c_a - \tilde{c}_a), \quad s \tilde{b}^a = \rho^a. \] (16)
Here \( \psi_{ab}, \phi^a, \bar{\phi}^a \), and \( \rho_a \) are ghosts associated with the shift symmetry and their BRST transformations vanish too,
\[ s \phi^a = s \bar{\phi}^a = s \rho^a = 0. \] (17)
We define antifields with opposite parity corresponding to all the original fields. These antifields have the following BRST transformations,
\[ s h^\ast_{ab} = \bar{h}_{ab}, \quad s c^{\ast a} = B^a, \]
\[ s \bar{c}^{\ast a} = \bar{B}^a, \quad s b^{\ast a} = \bar{b}^a. \] (18)
Here \( \bar{h}_{ab}, B^a, \bar{B}^a \), and \( \bar{b}^a \) are Nakanishi-Lautrup fields and their BRST transformations vanish too,
\[ s \bar{h}_{ab} = s B^a = s \bar{B}^a = s \bar{b}^a = 0. \] (19)
It is useful to define
\[ h^\prime_{ab} = h_{ab} - \tilde{h}_{ab}, \quad c^{\prime a} = c^a - \tilde{c}^a, \]
\[ \bar{c}^{\prime a} = \bar{c}^a - \tilde{c}^a, \quad b^{\prime a} = b^a - \tilde{b}^a. \] (20)
The physical requirement for the sum of the gauge-fixing term and the ghost term is that all the fields associated with shift symmetry vanish. This can be achieved by choosing the following Lagrangian density,
\[ \mathcal{L}_g = -\bar{h}^{\ast}_{ab} \delta h_{ab} - h^{\ast ab} \delta (\psi_{ab} - D^c_{ab} \delta (c_c)) \]
\[ = -\bar{B}^a \delta \tilde{c}_a + \bar{c}^{\ast a} \delta (\rho^a + (c_b - \tilde{c}_b) \delta \partial^b (c_a - \tilde{c}_a) + B^a \delta \bar{b}_a + b^{\ast a} \delta \rho_a. \] (21)
The integrating out the Nakanishi-Lautrup fields in this Lagrangian density will make all the shifted fields vanish.
If we choose a gauge-fixing fermion Ψ, such that it depends only on the original fields and furthermore define \( \mathcal{L}_g = s\Psi \), then we have

\[
\mathcal{L}_g = -\frac{\delta \Psi}{\delta h_{ab}} \partial \psi_{ab} + \frac{\delta \Psi}{\delta c_a} \partial \phi_a + \frac{\delta \Psi}{\delta \bar{c}_a} \partial \bar{\phi}_a - \frac{\delta \Psi}{\delta b_a} \partial \rho_a. \tag{22}
\]

The total Lagrangian density is given by

\[
\mathcal{L} = \mathcal{L}_c(h - \bar{h}) + \tilde{\mathcal{L}}_g + \mathcal{L}_g. \tag{23}
\]

After integrating out the Nakanishi-Lautrup fields, this total Lagrangian density can be written as,

\[
\mathcal{L} = \mathcal{L}_c(h - \bar{h}) + h^*_a \partial D^a_c \epsilon^c + \tilde{c}^a \partial \tilde{c}^a c_a
\]

\[- c^a \partial b_a - \left( h^*_a + \frac{\delta \Psi}{\delta h_{ab}} \right) \partial \psi_{ab} - \left( \bar{c}^a + \frac{\delta \Psi}{\delta c_a} \right) \partial \rho^a
\]

\[- \left( c^a - \frac{\delta \Psi}{\delta \bar{c}_a} \right) \partial \bar{\rho}^a + \left( b^*_a - \frac{\delta \Psi}{\delta b^a} \right) \partial \rho^a. \tag{24}\]

Now integrating out the ghosts associated with the shift symmetry, we get the following expression for the antifields,

\[
h^*_a = -\frac{\delta \Psi}{\delta h_{ab}} b_{ab}, \quad c^a = \frac{\delta \Psi}{\delta c_a},
\]

\[
\tilde{c}^a = -\frac{\delta \Psi}{\delta \bar{c}_a}, \quad b^*_a = \frac{\delta \Psi}{\delta b^a}. \tag{25}\]

These equations along with Eq. (12) fix the exact expressions for the antifields in terms of the original fields.

### 4 Superspace Formulation

In this section we will express the results of the previous section in superspace formalism with one anti-commutating variable. Let \( \theta \) be an anti-commutating variable, then we can define the following superfields,

\[
\omega_{ab} = h_{ab} + \theta \psi_{ab}, \quad \bar{\omega}_{ab} = \bar{h}_{ab} + \theta (\psi_{ab} - D^a_{cb} \bar{c}^b c_c),
\]

\[
\eta_a = c_a + \theta \phi_a, \quad \bar{\eta}_a = \bar{c}_a + \theta (\phi_a + (c_a') \partial \bar{\phi}_a),
\]

\[
\bar{\eta}_a = \bar{c}_a + \theta \bar{\phi}_a, \quad \bar{f}_a = \bar{b}_a + \theta \rho_a,
\]

\[f_a = b_a + \theta \rho_a, \quad \bar{f}_a = \bar{b}_a + \theta \rho_a. \tag{26}\]

and the following anti-superfields,

\[
\tilde{\omega}_{ab} = h_{ab} - \theta \psi_{ab}, \quad \tilde{\eta}_a = c_a - \theta B_a,
\]

\[
\bar{\tilde{\eta}}_a = \bar{c}_a - \theta \bar{B}_a, \quad \bar{\tilde{f}}_a = \bar{b}_a - \theta \bar{\rho}_a. \tag{27}\]

From these two equations, we have

\[
\frac{\partial}{\partial \theta} \tilde{\omega}^a \bar{\omega}_{ab} = -\bar{b}_{ab} \bar{h}^a - h_{ab} \bar{\psi}^b \partial (\psi_{ab} - D^a_{cb} \bar{c}^b c_c),
\]

\[
\frac{\partial}{\partial \theta} \bar{\eta}^a \tilde{\eta}_a = -\bar{b}^a \bar{\tilde{\eta}}_a + \eta^a \partial \eta_a + (c^b_a \partial \rho^b c^a). \tag{28}\]
Now we can express $\tilde{L}_g$ given by Eq. (21) as,

$$\tilde{L}_g = \frac{\partial}{\partial \theta} (\tilde{\omega}^{ab} \hat{\omega}_{ab} + \tilde{\eta}^a \hat{\eta}_a - \tilde{\eta}^a \hat{\eta}_a - \tilde{f}^a \hat{f}_a).$$

(29)

Furthermore, if we define $\Psi$ as,

$$\Phi = \Psi + \theta_s \Psi$$

$$= \Psi + \theta \left( - \frac{\delta \Psi}{\delta h_{ab}} \hat{\psi}_{ab} + \frac{\delta \Psi}{\delta c_a} \hat{\phi}_a + \frac{\delta \Psi}{\delta \bar{c}_a} \hat{\phi}_a - \frac{\delta \Psi}{\delta \rho_a} \right),$$

(30)

then we can express $\mathcal{L}_g$ given by Eq. (22) as,

$$\mathcal{L}_g = \frac{\partial}{\partial \theta} \Phi.$$ 

(31)

Now the complete Lagrangian density in the superspace formalism is given by,

$$\mathcal{L} = \frac{\partial}{\partial \theta} \Phi + \frac{\partial}{\partial \theta} (\tilde{\omega}^{ab} \hat{\omega}_{ab} + \tilde{\eta}^a \hat{\eta}_a - \tilde{\eta}^a \hat{\eta}_a - \tilde{f}^a \hat{f}_a)$$

$$+ \mathcal{L}_c (h_{ab} - \tilde{h}_{ab}).$$

(32)

Upon elimination of the Nakanishi-Lautrup fields and the ghosts associated with shift symmetry, this Lagrangian density is manifestly invariant under the BRST symmetry as well as the shift symmetry.

### 5 Conclusion

In this paper we analysed non-anticommutative perturbative gravity with a complex metric. As this theory contained unphysical degrees of freedom, we added a gauge-fixing term and a ghost term to it. We found that the sum of the original classical Lagrangian density, the gauge-fixing term and the ghost term was invariant under the BRST transformations. As the shifting of fields occurs naturally in the background field method, we analysed the effect of the shift symmetry in the BV-formalism. Finally, we expressed our results in the superspace formalism using a single Grassmann parameter.

It is well known that the sum of the original classical Lagrangian density, the gauge-fixing term and the ghost term for most theories posing BRST symmetry is also invariant under another symmetry called the anti-BRST symmetry [44]. It will be interesting to investigate the anti-BRST version of this theory. Furthermore, the invariance of a gauge theory under the BRST and the anti-BRST transformations along with the shift transformations has already been analysed in the superspace BV-formalism [33]. Thus, after analysing the anti-BRST symmetry for this theory, the invariance of this theory under the original BRST and the original anti-BRST transformations along with shift transformations can be studied in the superspace BV-formalism.

It will also be interesting to generalise the results of this paper to general curved spacetime. The generalisation to arbitrary spacetime might not be so simple as it is still not completely clear how the BRST symmetry works for general curved spacetime. There will also be ambiguities due to the definition of vacuum state. We know it is possible to define a vacuum state called
the Euclidean vacuum in maximally symmetric spacetime [51]. We also know the ghosts in anti-de Sitter spacetime do not contain any infrared divergence. Therefore, the generalisation of this work to anti-de Sitter spacetime can be done easily [52]. However, as the ghosts in de Sitter spacetime contain infrared divergence, this work can not be directly extended to de Sitter spacetime [52]. In order to generalize this work to de Sitter spacetime we will have to modify the BRST transformations accordingly.

References

[1] N. Seiberg and E. Witten, JHEP. 9909, 032 (1999)
[2] C. Hofman and E. Verlinde, JHEP. 12, 10 (1998)
[3] C. S. Chu and P.M. Ho, Nucl. Phys. B550, 151 (1999)
[4] H. Ooguri and C. Vafa, Adv. Theor. Math. Phys. 7, 53 (2003)
[5] N. Seiberg, JHEP. 0306, 010 (2003)
[6] D. V. Vassilevich, Nucl. Phys. B715, 695 (2005)
[7] P. Martinetti, Mod. Phys. Lett. A20, 1315 (2005)
[8] L. Freidel and S. Majid, Class. Quant. Grav. 25, 045006 (2008)
[9] T. Ohl and A. Schenkel, JHEP. 10 052 (2009)
[10] R. Banerjee, S. Gangopadhyay and S. K. Modak, Phys. Lett. B686, 181 (2010)
[11] J. W. Moffat, Phys. Lett. B142, (2000)
[12] J. W. Moffat, Phys. Lett. B506, 193 (2001)
[13] J. W. Moffat, Proc. Camb. Phil. Soc. 52, 623 (1956)
[14] J. W. Moffat, Proc. Camb. Phil. Soc. 53, 473 (1957)
[15] J. W. Moffat, Proc. Camb. Phil. Soc. 53, 489 (1957)
[16] A. Einstein, The Meaning of Relativity, fifth edition, Princeton University Press, (1956)
[17] A. Einstein and E. G. Straus, Ann. Math. 47, 731 (1946)
[18] A. H. Chamseddine, Int. J. Mod. Phys. A16, 759 (2001)
[19] T. Damour, S. Deser and J. McCarthy, Phys. Rev. D47, 1541 (1993)
[20] A. H. Chamseddine, Commun. Math. Phys. 218, 283 (2001)
[21] J. W. Moffat, Phys.Lett. B491, 345 (2000)
[22] R. Amorim, H. Boschi-Filho and N. R. F. Braga, Braz. J. Phys. 35, 645 (2005)
[23] R. Amorim and F. A. Farias, Phys. Rev. D81, 124049 (2010)
[24] M. Soroush, Phys. Rev. D67, 105005 (2003)
[25] O. F. Dayi, Phys. Lett. B481, 408 (2000)
[26] Y. Okumura, Phys. Rev. D54, 4114 (1996)
[27] I. A. Batalin and G. A. Vilkovisky, Phys. Lett. B102, 27 (1981)
[28] I. A. Batalin and G. A. Vilkovisky, Phys. Rev. D28, 2567 (1983)
[29] I. A. Batalin and G. A. Vilkovisky, Phys. Lett. B102, 27 (1981)
[30] M. Faizal and M. Khan, Eur. Phys. J. C71, 1603 (2011)
[31] S. Ferrara, O. Piguet and M. Schweda, Nucl. Phys. B119, 493 (1977)
[32] L. Bonora and M. Tonin, Phys. Lett. B98, 48 (1981)
[33] N. R. F. Braga and A. Das, Nucl. Phys. B442, 655 (1995)
[34] N. R. F. Braga and S. M. de Souza, Phys. Rev. D53, 916 (1996)
[35] E. M. C. Abreu and N. R. F. Braga, Int. J. Mod. Phys. A13, 493 (1998)
[36] E. M. C. Abreu and N. R. F. Braga, Phys. Rev. D54, 4080 (1996)
[37] P. Bouwknegt, J. McCarthy and K. Pilch, Lect. Notes. Phys. M42 (1996)
[38] S. Vandoren and A. Van Proeyen, Nucl. Phys. B411, 257 (1994)
[39] J. Paris, Nucl. Phys. B450, 357 (1995)
[40] F. Gronwald, Phys. Rev. D57, 961 (1998)
[41] N. Nakanishi, Prog. Theor. Phys. 59, 972 (1978)
[42] T. Kugo and I. Ojima, Nucl. Phys. B144, 234 (1978)
[43] K. Nishijima and M. Okawa, Prog. Theor. Phys. 60, 272 (1978)
[44] N. Nakanishi and I. Ojima, Covariant operator formalism of gauge theories and quantum gravity, World Sci. Lect. Notes. Phys. (1990)
[45] Yoshihisa Kitazawa, Rie Kuriki and Katsumi Shigura, Mod. Phys. Lett. A12, 1871 (1997)
[46] E. Benedict, R. Jackiw and H. J. Lee, Phys. Rev. D54, 6213 (1996)
[47] Friedemann Brandt, Walter Troost and Antoine Van Proeyen, Nucl. Phys. B464, 353 (1996)
[48] M. Tahiri, Int. Jou. Theo. Phys. 35, 1572 (1996)
[49] M. Menaa and M. Tahiri, Phys. Rev. D57, 7312 (1998)
[50] Mir Faizal, Found. Phys. 41, 270 (2011)
[51] B. Allen and M. Turyn, Nucl. Phys. B292, 813 (1987)
[52] M. Faizal and A. Higuchi, Phys. Rev. D78, 067502 (2008)