The dispersive approach to QCD is applied to the study of the hadronic vacuum polarization function $\Pi(q^2)$. This approach provides unified integral representations for $\Pi(q^2)$ and related functions, which embody the intrinsically nonperturbative constraints originating in the kinematic restrictions on the respective physical processes. The obtained hadronic vacuum polarization function proves to be in a good agreement with pertinent lattice simulation data. The calculated hadronic contributions to the muon anomalous magnetic moment and to the shift of the electromagnetic fine structure constant conform with recent estimations of these quantities.

PACS numbers: 11.55.Fv, 12.38.Lg, 13.40.Em, 14.60.Ef

I. INTRODUCTION

The theoretical description of a number of the strong interaction processes is inherently based on the hadronic vacuum polarization function $\Pi(q^2)$. In particular, this function plays a crucial role in the studies of the heaviest lepton hadronic decays and of the annihilation of a pair of elementary particles into hadrons, that provides decisive self–consistency tests of quantum chromodynamics (QCD). At the same time, the function $\Pi(q^2)$ enters in the analysis of the hadronic contributions to such quantities of the precise particle physics as the muon anomalous magnetic moment and the running of the electromagnetic fine structure constant, that, in turn, puts strong limits on the effects due to a possible new physics beyond the standard model (SM). Additionally, the theoretical exploration of the aforementioned processes constitutes a natural framework for a thorough investigation of both perturbative and intrinsically nonperturbative aspects of hadron dynamics.

The strong interactions possess the feature of the asymptotic freedom, that makes it possible to apply the perturbation theory to the study of the ultraviolet behavior of the function $\Pi(q^2)$. However, there is still no reliable method of theoretical description of hadron dynamics at low energies, which would have provided one with rigorous unambiguous results. This fact eventually forces one to invoke a variety of nonperturbative approaches in order to examine the strong interactions in the infrared domain. For example, an insight into the low–energy behavior of the hadronic vacuum polarization function can be gained from such methods as lattice simulations [1–4], operator product expansion [5–8], as well as some others (see, e.g., Refs. [9, 10]).
Certain nonperturbative hints on the low–energy hadron dynamics are also embedded within corresponding dispersion relations. The latter render the kinematic restrictions on the relevant physical processes into the mathematical form. As a result, dispersion relations impose stringent constraints on the pertinent quantities [such as \( \Pi(q^2) \) and related functions], that should definitely be accounted for when one comes out of the applicability range of perturbation theory. These constraints have been embodied within dispersive approach to QCD [11, 12], which provides unified integral representations for the functions on hand. It is worthwhile to mention also that the methods based on the dispersion relations are widely employed in theoretical particle physics. Among the recent applications of such methods are, for instance, extension of the applicability range of chiral perturbation theory [13, 14], precise determination of parameters of resonances [15], as well as the assessment of the hadronic light–by–light scattering [16].

The primary objective of this paper is to obtain the hadronic vacuum polarization function within dispersive approach, to compare the obtained result with relevant lattice simulation data, as well as to evaluate the corresponding hadronic contributions to the muon anomalous magnetic moment and to the shift of the electromagnetic fine structure constant.

The layout of the paper is as follows. In Sec. II the dispersive approach to QCD is briefly overviewed. Section III contains the juxtaposition of the hadronic vacuum polarization function obtained in the framework of dispersive approach with pertinent lattice simulation data and elucidates the qualitative distinctions between the approach on hand, its massless limit, and perturbative approach. Section IV presents the evaluation of the aforementioned hadronic contributions to electroweak observables. In the Conclusions (Sect. V) the basic results are summarized. Auxiliary material is given in the Appendix.

II. DISPERSIVE APPROACH TO QUANTUM CHROMODYNAMICS

The hadronic vacuum polarization function \( \Pi(q^2) \) is defined as the scalar part of the hadronic vacuum polarization tensor

\[
\Pi_{\mu\nu}(q^2) = i \int d^4x \ e^{iqx} \langle 0 \mid T\{J_\mu(x) J_\nu(0)\} \mid 0 \rangle = \frac{1}{12\pi^2} (q_\mu q_\nu - g_{\mu\nu} q^2) \Pi(q^2).
\]

The kinematics of the process on hand determines the cut structure of \( \Pi(q^2) \) in the complex \( q^2 \)–plane. Specifically, the function \( \Pi(q^2) \) has the only cut along the positive semiaxis of real \( q^2 \) starting at the hadronic production threshold \( q^2 \geq m^2 \) (discussion of this issue can be found in, e.g., Ref. [17]). Proceeding from this fact and bearing in mind the asymptotic ultraviolet behavior of the hadronic vacuum polarization function one can write down the corresponding dispersion relation [see Eq. (2) below] by making use of the once–subtracted Cauchy integral formula.

For practical purposes, it is convenient to define the Adler function \( D(Q^2) \) [see Eq. (3) below] and the related function \( R(s) \), which is identified with the so–called \( R \)–ratio of electron–positron annihilation into hadrons [see Eq. (4) below]. Eventually, the complete set of relations, which express the functions \( \Pi(q^2) \), \( R(s) \), and \( D(Q^2) \) in terms of each other,
acquires the following form (see papers [18–20] as well as [12] and references therein):

$$\Delta \Pi(q^2, q_0^2) = (q^2 - q_0^2) \int_{m^2}^{\infty} \frac{R(\sigma)}{(\sigma - q^2)(\sigma - q_0^2)} d\sigma$$

$$= -\int_{-q_0^2}^{-q^2} D(\zeta) \frac{d\zeta}{\zeta},$$

$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0^+} \left[ \Pi(s + i\varepsilon) - \Pi(s - i\varepsilon) \right]$$

$$= \frac{1}{2\pi i} \lim_{\varepsilon \to 0^+} \int_{s+i\varepsilon}^{s-i\varepsilon} D(-\zeta) \frac{d\zeta}{\zeta},$$

$$D(Q^2) = -\frac{d \Pi(-Q^2)}{d \ln Q^2}$$

$$= Q^2 \int_{m^2}^{\infty} \frac{R(\sigma)}{(\sigma + Q^2)^2} d\sigma.$$

In these equations \(\Delta \Pi(q^2, q_0^2) = \Pi(q^2) - \Pi(q_0^2)\), whereas \(Q^2 = -q^2 > 0\) and \(s = q^2 > 0\) denote the spacelike and timelike kinematic variables, respectively. The common prefactor \(N_c \sum_{f=1}^{n_f} Q_f^2\) is omitted throughout the paper, where \(N_c = 3\) is the number of colors, \(Q_f\) stands for the electric charge of \(f\)-th quark (in units of the elementary charge \(e\)), and \(n_f\) denotes the number of active flavors. The integration contour in Eq. (5) lies in the region of analyticity of its integrand (see Fig. 1). Note that the derivation of relations (2)–(7) requires only the knowledge of the cut structure of hadronic vacuum polarization function \(\Pi(q^2)\) and its asymptotic ultraviolet behavior. It is worth mentioning also that Eqs. (2) and (7) can be used for extracting the functions \(\Delta \Pi(q^2, q_0^2)\) and \(D(Q^2)\) from the experimental data on \(R(s)\).

As noted in the Introduction, the dispersion relations (2)–(7) embody the kinematic restrictions on the respective physical processes and impose intrinsically nonperturbative constraints on the functions \(\Pi(q^2)\), \(R(s)\), and \(D(Q^2)\), that should certainly be taken into account when one oversteps the limits of applicability of perturbation theory. These constraints have been properly accounted for within dispersive approach to QCD [11, 12], which provides the following integral representations for the functions on hand:

$$\Delta \Pi(q^2, q_0^2) = \Delta \Pi^{(0)}(q^2, q_0^2) + \int_{m^2}^{\infty} \rho(\sigma) \ln \left( \frac{\sigma - q^2 m^2 - q_0^2}{\sigma - q_0^2 m^2 - q^2} \right) \frac{d\sigma}{\sigma},$$

$$R(s) = R^{(0)}(s) + \theta(s - m^2) \int_{s}^{\infty} \rho(\sigma) \frac{d\sigma}{\sigma},$$

$$D(Q^2) = D^{(0)}(Q^2) + \frac{Q^2}{Q^2 + m^2} \int_{m^2}^{\infty} \rho(\sigma) \frac{\sigma - m^2 d\sigma}{\sigma + Q^2 \sigma},$$

1 Including the correct analytic properties in the kinematic variable, that implies the absence of unphysical singularities in Eqs. (8–10), see Refs. [11, 12].

2 Its preliminary formulation was discussed in Refs. [21, 22].
FIG. 1. The integration contour in Eq. (5). The physical cut \( \zeta \geq m^2 \) of the Adler function \( D(-\zeta) \) is shown along the positive semiaxis of real \( \zeta \).

with \( \rho(\sigma) \) being the spectral density

\[
\rho(\sigma) = \frac{1}{2\pi i} \frac{d}{d \ln \sigma} \lim_{\epsilon \to 0^+} \left[ p(\sigma - i\epsilon) - p(\sigma + i\epsilon) \right]
= -\frac{d}{d \ln \sigma} r(\sigma)
= \frac{1}{2\pi i} \lim_{\epsilon \to 0^+} \left[ d(-\sigma - i\epsilon) - d(-\sigma + i\epsilon) \right].
\] (11)

In these equations \( p(q^2), r(s), \) and \( d(Q^2) \) denote the strong corrections to the functions \( \Pi(q^2), R(s), \) and \( D(Q^2) \), respectively, whereas \( \theta(x) \) is the unit step–function \( \theta(x) = 1 \) if \( x \geq 0 \) and \( \theta(x) = 0 \) otherwise. The leading–order terms in Eqs. (8)–(10) have the following form [17, 23]:

\[
\Delta \Pi^{(0)}(q^2, q_0^2) = 2 \frac{\varphi - \tan \varphi}{\tan^3 \varphi} - 2 \frac{\varphi_0 - \tan \varphi_0}{\tan^3 \varphi_0},
\] (12)

\[
R^{(0)}(s) = \theta(s - m^2) \left(1 - \frac{m^2}{s}\right)^{3/2},
\] (13)

\[
D^{(0)}(Q^2) = 1 + \frac{3}{\xi} \left[ \frac{1}{1 + \xi^{-1}} \sinh^{-1}(\xi^{1/2}) \right].
\] (14)

where \( \sin^2 \varphi = q^2/m^2, \sin^2 \varphi_0 = q_0^2/m^2, \) and \( \xi = Q^2/m^2 \), see also Refs. [12, 24, 25]. The integral representations (8)–(10) have been obtained by employing only the relations (2)–(7) and the asymptotic ultraviolet behavior of the hadronic vacuum polarization function. Neither additional approximations nor phenomenological assumptions were involved in the derivation of Eqs. (8)–(10), see papers [11, 12] and references therein for the details. Note also that three integral representations for the functions on hand (8)–(10) satisfy all six relations (2)–(7) by construction, see discussion of this issue in the Appendix.
It is worthwhile to outline that the dispersive approach provides the expression for the Adler function, which agrees with the relevant experimental prediction in the entire energy range \([11, 34, 35]\). At the same time, the representations \((8)–(10)\) conform with the results of Ref. \([36]\) as well as of Ref. \([37]\). Additionally, the dispersive approach has proved to be capable of describing OPAL (update 2012, Ref. \([38]\)) and ALEPH (update 2014, Ref. \([39]\)) experimental data on inclusive \(\tau\) lepton hadronic decay in vector and axial–vector channels in a self–consistent way \([12, 40]\) (see also Refs. \([24, 25]\)).

So far, the method, which would enable one to retrieve the unique unabridged expression for the spectral density \(\rho(\sigma)\) appearing in Eqs. \((8)–(10)\) is still unavailable, see discussion of this issue in, e.g., Refs. \([25, 35, 41, 42]\). Nonetheless, the perturbative part of the spectral density \(\rho(\sigma)\) can be calculated by making use of the perturbative expression for either of the strong corrections \(p(q^2), r(s),\) and \(d(Q^2)\) (see, e.g., Refs. \([43, 44]\)):

\[
\rho_{\text{pert}}(\sigma) = \frac{1}{2\pi i} \frac{d}{d\ln \sigma} \lim_{\varepsilon \to 0^+} \left[ p_{\text{pert}}(\sigma - i\varepsilon) - p_{\text{pert}}(\sigma + i\varepsilon) \right] = -\frac{d}{d\ln \sigma} r_{\text{pert}}(\sigma) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0^+} \left[ d_{\text{pert}}(-\sigma - i\varepsilon) - d_{\text{pert}}(-\sigma + i\varepsilon) \right].
\]

It is worth mentioning here that in the massless limit \((m^2 = 0)\) for the case of perturbative spectral function \((15)\) Eqs. \((9)\) and \((10)\) become identical to those of the analytic perturbation theory (APT) \([45]\) (see also Refs. \([46–57]\)). However, the massless limit loses some of the nonperturbative constraints, which relevant dispersion relations impose on the functions on hand, that turns out to be essential for the studies of hadron dynamics at low energies, see discussion of this issue in Refs. \([11, 12, 25, 35]\).

### III. COMPARISON OF \(\Pi(q^2)\) WITH LATTICE SIMULATION DATA

As mentioned in the Introduction, the lattice QCD simulations constitute an efficient method of investigation of the nonperturbative aspects of strong interactions. Over past time this method has been applied to an extensive study of a broad range of topics\(^4\), including the low–energy behavior of the hadronic vacuum polarization function \(\Pi(q^2)\). It is of a particular interest to juxtapose the function \(\Pi(q^2)\) obtained within dispersive approach \([8]\) with relevant lattice simulation data. To achieve this objective, it is convenient to proceed with the subtracted at zero form of Eq. \((8)\), namely

\[
\bar{\Pi}(Q^2) = \Delta \Pi(0, -Q^2) = \Delta \Pi^{(0)}(0, -Q^2) + \int_{m^2}^{\infty} \rho(\sigma) \ln \left( \frac{1 + Q^2/m^2}{1 + Q^2/\sigma} \right) d\sigma.
\]

In what follows we shall employ the perturbative expression for the spectral function \([15]\). At the one–loop level it assumes a simple form \(\rho_{\text{pert}}^{(1)}(\sigma) = (4/\beta_0)[\ln^2(\sigma/\Lambda^2) + \pi^2]^{-1}\), where \(\beta_0 = 11 - 2n_f/3\) and \(\Lambda\) denotes the QCD scale parameter. The explicit expressions for the spectral function \([15]\) up to the four–loop level can be found in Ref. \([43]\).

\(^3\) The studies of the Adler function within other approaches can be found in, e.g., Refs. \([26, 33]\).

\(^4\) For a recent overview see, e.g., Ref. \([58]\).
FIG. 2. Comparison of the four–loop hadronic vacuum polarization function obtained within dispersive approach (solid curve) with lattice simulation data (circles). The APT prediction of $\Pi(q^2)$ (dashed curve) is denoted by dashed curve, whereas its perturbative approximation is shown by dot–dashed curve. Values of parameters: $\Lambda = 419 \text{ MeV}$, $n_f = 2$.

As one can infer from Fig. 2, the hadronic vacuum polarization function (solid curve) is in a good agreement with lattice simulation data (circles) (the rescaling procedure described in Refs. [59, 60, 61] was applied). The presented result corresponds to the four–loop level, $\overline{\text{MS}}$ subtraction scheme, and $n_f = 2$ active flavors (recently completed calculation of the respective perturbative coefficient [62] was employed). To elucidate the qualitative distinctions between the approaches mentioned above, Fig. 2 also displays the one–loop perturbative approximation of $\Pi(q^2)$ (dot–dashed curve)

$$
\Delta \Pi^{(1)}_{\text{pert}}(-Q^2, -Q_0^2) = \Delta \Pi^{(0)}_{\text{pert}}(-Q^2, -Q_0^2) - \frac{4}{\beta_0} \ln \left[ \frac{a^{(1)}_{\text{pert}}(Q_0^2)}{a^{(1)}_{\text{pert}}(Q^2)} \right]
$$

(17)

and the one–loop Eq. (8) in the massless limit, which, in the considered case, corresponds to APT (dashed curve)

$$
\Delta \Pi^{(1)}_{\text{APT}}(-Q^2, -Q_0^2) = \Delta \Pi^{(0)}_{\text{pert}}(-Q^2, -Q_0^2) - \frac{4}{\beta_0} \ln \left[ \frac{a^{(1)}_{\text{APT}}(Q_0^2)}{a^{(1)}_{\text{APT}}(Q^2)} \right].
$$

(18)

In Eqs. (17) and (18) the leading–order term reads

$$
\Delta \Pi^{(0)}_{\text{pert}}(-Q^2, -Q_0^2) = - \ln \left( \frac{Q^2}{Q_0^2} \right).
$$

(19)

$a(Q^2) = \alpha(Q^2) \beta_0 / (4\pi)$, $a^{(1)}_{\text{pert}}(Q^2)$ denotes the one–loop perturbative running coupling

$$
\alpha^{(1)}_{\text{pert}}(Q^2) = \frac{4\pi}{\beta_0} \frac{1}{\ln z}, \quad z = \frac{Q^2}{\Lambda^2},
$$

(20)

and

$$
\alpha^{(1)}_{\text{n}}(Q^2) = \frac{4\pi}{\beta_0} \frac{z - 1}{z \ln z}.
$$

(21)
It is interesting to note here that the expression (21) was first obtained in Ref. [41] and has been independently rediscovered (proceeding from entirely different reasoning) in Ref. [63] as well as in Ref. [64]. It should be emphasized that the perturbative approximation (17) contains infrared unphysical singularities, whereas Eqs. (16) and (18) do not. However, the APT prediction (18) is undefined at $Q_0^2 \to 0$, that originates in the mathematical fact that in the massless limit the function $\Pi(q^2)$ cannot be subtracted at the beginning of its branch cut.

IV. HADRONIC CONTRIBUTIONS TO ELECTROWEAK OBSERVABLES

A. Muon anomalous magnetic moment

The theoretical description of the muon anomalous magnetic moment $a_\mu = (g_\mu - 2)/2$ is a long–standing challenging issue of the elementary particle physics, which engages the entire pattern of interactions within SM. Both experimental measurements [65, 66] and theoretical evaluations [67, 68] of $a_\mu$ have achieved an impressive accuracy, and the remaining discrepancy of the order of few standard deviations between them may be a signal for a new fundamental physics beyond SM. The uncertainty of the theoretical estimation of $a_\mu$ is mainly dominated by the leading–order hadronic contribution $a_{HLO}^\mu$, which involves the integration of hadronic vacuum polarization function $\Pi(q^2)$ over the range inaccessible within perturbation theory (see, e.g., Ref. [69]):

$$a_{HLO}^\mu = \frac{1}{3} \left( \frac{\alpha}{\pi} \right)^2 \int_0^\infty f(x) \frac{\zeta}{4m_\mu^2} d\zeta = \frac{1}{3} \left( \frac{\alpha}{\pi} \right)^2 \int_0^1 (1 - x) \Pi \left( \frac{m_\mu^2}{1 - x} \right) dx. \quad (22)$$

In this equation

$$f(x) = \frac{1}{x^3} \frac{y^5(x)}{1 - y(x)}$$

and $y(x) = x(\sqrt{1 + x^{-1}} - 1)$ is a monotonously nondecreasing function of its argument, $0 \leq y(x) < 1/2$.

As mentioned above, in the framework of dispersive approach the hadronic vacuum polarization function $\Pi(q^2)$ is free of the unphysical singularities. Thus, the integration in Eq. (22) can be performed in a straightforward way, that eventually results in

$$a_{HLO}^\mu = (696.3 \pm 5.4) \times 10^{-10}, \quad (24)$$

the particle data entering Eq. (22) being taken from Refs. [70, 71]. The obtained value of the leading–order hadronic contribution to $a_\mu$ (24) corresponds to the four–loop level and appears to be in a good agreement with recent estimations of this quantity [72–74].

The complete SM prediction of the muon anomalous magnetic moment comprises the QED contribution $a_{QED}^\mu = (11658471.8951 \pm 0.0080) \times 10^{-10}$ [75], the electroweak contribution

To obviate this difficulty one may express $a_{HLO}^\mu (22)$ in terms of $R(s)$ by making use of Eq. (2) and replace the low–energy behavior of $R(s)$ with relevant experimental data, see reviews [67, 68] and references therein.
The subtracted muon anomalous magnetic moment $\Delta a_\mu = a_\mu - a_0$. Theoretical predictions are denoted by circles, averaged experimental value \[26\] is shown by vertical shaded band, and $a_0 = 11659 \times 10^{-7}$.

$$a_\mu^{\text{EW}} = (15.36 \pm 0.10) \times 10^{-10} \text{[76]},$$

as well as the leading–order $a_\mu^{\text{HLO}} \text{[24]}$, higher–order $a_\mu^{\text{HHO}} = (-9.84 \pm 0.07) \times 10^{-10} \text{[72]}$, and light–by–light $a_\mu^{\text{Hlbl}} = (11.6 \pm 4.0) \times 10^{-10} \text{[77]}$ hadronic contributions, that eventually results in

$$a_\mu = (11659185.3 \pm 6.7) \times 10^{-10}. \quad (25)$$

The difference between this value and the Brookhaven E821 experimental measurement \[66\]

$$a_\mu^{\text{exp}} = (11659208.9 \pm 6.3) \times 10^{-10} \quad (26)$$

is $(23.6 \pm 9.2) \times 10^{-10}$, that corresponds to the discrepancy of 2.6 standard deviations. As one can infer from Fig. 3, the recent theoretical estimations of the muon anomalous magnetic moment $a_\mu$ fairly agree with each other and imply that a few $\sigma$ deviation from the BNL experimental measurement still persists.

### B. Electromagnetic fine structure constant

The electromagnetic running coupling $\alpha_{em}(q^2)$ plays a central role in various issues of precision particle physics. The vacuum polarization effects screen the electric charge and make the electromagnetic coupling $\alpha_{em}$ dependent on the energy scale $q^2$:

$$\alpha_{em}(q^2) = \frac{\alpha}{1 - \Delta \alpha_{\text{lep}}(q^2) - \Delta \alpha_{\text{had}}(q^2)}, \quad (27)$$

with $\alpha = e^2/(4\pi) \simeq 1/137.036$ being the fine structure constant. In Eq. \[27\] the leptonic contribution $\Delta \alpha_{\text{lep}}(q^2)$ can reliably be calculated by making use of perturbation theory \[79\].

---

6 The averaged experimental value \[26\] accounts for the updated ratio of the muon–to–proton magnetic moment, see Ref. \[78\].
FIG. 4. Theoretical predictions of the hadronic contribution to the shift of the electromagnetic fine structure constant at the scale of $Z$ boson mass.

However, similarly to the aforementioned case of the muon anomalous magnetic moment, the hadronic contribution to Eq. (27)

$$\Delta \alpha_{\text{had}}(q^2) = -\frac{\alpha}{3\pi} q^2 \mathcal{P} \int_{m^2 - q^2}^{\infty} \frac{R(s)}{s} \, ds,$$

($\mathcal{P}$ stands for the “Cauchy principal value”) involves the integration over the low–energy range and constitutes the prevalent source of the uncertainty of $\alpha_{\text{em}}(q^2)$, see discussion of this issue in, e.g., papers [72, 80] and references therein.

In the framework of dispersive approach the evaluation of five–flavor hadronic contribution to the shift of electromagnetic fine structure constant at the scale of $Z$ boson mass yields

$$\Delta \alpha_{\text{had}}^{(5)}(M_Z^2) = (274.5 \pm 1.4) \times 10^{-4},$$

the pertinent particle data being taken from Refs. [70, 71]. The result (29) corresponds to the four–loop level and its comparison with recent theoretical predictions of $\Delta \alpha_{\text{had}}^{(5)}(M_Z^2)$ is displayed in Fig. 4. In turn, by making use of the leptonic $\Delta \alpha_{\text{lep}}(M_Z^2) = (314.979 \pm 0.002) \times 10^{-4}$ [79] and top quark $\Delta \alpha_{\text{top}}(M_Z^2) = (-0.70 \pm 0.05) \times 10^{-4}$ [82] terms, one eventually arrives at $\alpha_{\text{em}}^{-1}(M_Z^2) = 128.967 \pm 0.019$, that appears to be in a good agreement with recent estimations of this quantity [72, 74, 81].

V. CONCLUSIONS

The hadronic vacuum polarization function obtained within dispersive approach proves to be in a good agreement with relevant lattice simulation data. The calculated value of the leading–order hadronic contribution to the muon anomalous magnetic moment conforms with its recent assessments and leads to 2.6 $\sigma$ discrepancy between the theoretical prediction

\footnote{The respective contribution of the top quark is added separately, see Ref. 82.}
of $a_\mu$ and its BNL experimental measurement. The evaluated hadronic contribution to the shift of the electromagnetic fine structure constant at the scale of $Z$ boson mass agrees with recent estimations of this quantity.

ACKNOWLEDGMENTS

The author is grateful to R. Kaminski, E. Passemar, M. Passera, J. Portoles, and H. Wittig for the stimulating discussions and useful comments.

Appendix: Correspondence between two sets of relations for $\Pi(q^2)$, $R(s)$, and $D(Q^2)$

As mentioned in Sec. IIII, three integral representations (8)–(10) for the functions on hand satisfy by construction all six relations (2)–(7). It is straightforward to verify explicitly that the set of relations (2)–(7) holds for the leading–order terms (12)–(14) as well as for the most of the strong corrections (8)–(10). In particular, to show that the relations (3) and (6) are valid for the pair of the strong corrections [(8), (10)] one has to apply directly the integration and differentiation, respectively. To demonstrate that the relations (2) and (7) hold between pairs of expressions [(8), (9)] and [(9), (10)] the integration by parts is required. The validity of relation (5) for the pair [(9), (10)] can be shown by employing

$$\lim_{\varepsilon \to 0^+} \frac{1}{x \pm i \varepsilon} = \mp i \pi \delta(x) + \mathcal{P} \frac{1}{x},$$

in the respective integrand. The remaining relation (11) between the pair of the strong corrections [(8), (9)] is somewhat more laborious to demonstrate than the others, and will be addressed in this section.

For the strong corrections $p(q^2)$ and $r(s)$ the relation (11) can be written as

$$r(s) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0^+} \left[ \Delta p(s + i \varepsilon, q_0^2) - \Delta p(s - i \varepsilon, q_0^2) \right],$$

where $\Delta p(q^2, q_0^2) = p(q^2) - p(q_0^2)$. By virtue of Eq. (8)

$$\Delta p(s \pm i \varepsilon, q_0^2) = \int_{m^2}^{\infty} \rho(\sigma) \left[ \ln \left( \frac{s - \sigma \pm i \varepsilon}{s - m^2 \pm i \varepsilon} \right) + \ln \left( \frac{m^2 - q_0^2}{\sigma - q_0^2} \right) \right] \frac{d \sigma}{\sigma}.$$

Then, since

$$\lim_{\varepsilon \to 0^+} \ln(x \pm i \varepsilon) = \ln |x| \pm i \pi \theta(-x),$$

the first term in the square brackets of Eq. (A.3) can eventually be represented as (the limit $\varepsilon \to 0^+$ is assumed hereinafter)

$$\ln \left( \frac{s - \sigma \pm i \varepsilon}{s - m^2 \pm i \varepsilon} \right) = \ln \left| \frac{s - \sigma}{s - m^2} \right| \pm i \pi \theta(s - m^2) \theta(\sigma - s).$$

Thus, Eq. (A.3) acquires the form

$$\Delta p(s \pm i \varepsilon, q_0^2) = \int_{m^2}^{\infty} \rho(\sigma) \ln \left( \frac{s - \sigma}{s - m^2} \right) \left( \frac{m^2 - q_0^2}{\sigma - q_0^2} \right) \frac{d \sigma}{\sigma} \pm i \pi \theta(s - m^2) \int_{s}^{\infty} \rho(\sigma) \frac{d \sigma}{\sigma}.$$
and, therefore, Eq. (A.2) reads

\[ r(s) = \theta(s - m^2) \int_s^\infty \rho(\sigma) \frac{d\sigma}{\sigma}, \]

(A.7)

that coincides with the integral representation \[^9\].

---

[1] T. Blum, Phys. Rev. Lett. 91, 052001 (2003); C. Aubin and T. Blum, Phys. Rev. D 75, 114502 (2007); C. Aubin, T. Blum, M. Golterman, and S. Peris, *ibid.* 86, 054509 (2012); 88, 074505 (2013); M. Golterman, K. Maltman, and S. Peris, *ibid.* 88, 114508 (2013).

[2] M. Gockeler et al. [QCDSF Collaboration], Nucl. Phys. B 688, 135 (2004); Nucl. Phys. B (Proc. Suppl.) 94, 571 (2001); 129, 293 (2004).

[3] E. Shintani et al. [JLQCD and TWQCD Collaborations], Phys. Rev. D 79, 074510 (2009); 82, 074505 (2010); P. Boyle, L. Del Debbio, E. Kerrane, and J. Zanotti, Phys. Rev. D 85, 074504 (2012).

[4] X. Feng, K. Jansen, M. Petschlies, and D. Renner, Nucl. Phys. B (Proc. Suppl.) 225–227, 269 (2012); X. Feng, S. Hashimoto, G. Hotzel, K. Jansen, M. Petschlies, and D.B. Renner, Phys. Rev. D 88, 034505 (2013); PoS (LATTICE 2013), 464 (2013).

[5] M.A. Shifman, A.I. Vainshtein, and V.I. Zakharov, Nucl. Phys. B 147, 385 (1979); 147, 448 (1979).

[6] L.J. Reinders, H. Rubinstein, and S. Yazaki, Phys. Rept. 127, 1 (1985).

[7] S. Narison, World Sci. Lect. Notes Phys. 26, 1 (1989); Nucl. Phys. B (Proc. Suppl.) 164, 225 (2007); 207, 315 (2010).

[8] P. Colangelo and A. Khodjamirian, [arXiv:hep-ph/0010175](http://arxiv.org/abs/hep-ph/0010175).  
[9] A.E. Dorokhov and W. Broniowski, Eur. Phys. J. C 32, 79 (2003); Phys. Rev. D 78, 073011 (2008); A.E. Dorokhov, *ibid.* 70, 094011 (2004); A.E. Dorokhov, A.E. Radzhabov, and A.S. Zhevlakov, Eur. Phys. J. C 72, 2227 (2012); JETP Lett. 100, 133 (2014).

[10] A.E. Dorokhov, Acta Phys. Polon. B 36, 3751 (2005); Nucl. Phys. A 790, 481 (2007); [arXiv:hep-ph/0601114](http://arxiv.org/abs/hep-ph/0601114).

[11] A.V. Nesterenko and J. Papavassiliou, J. Phys. G 32, 1025 (2006).

[12] A.V. Nesterenko, Phys. Rev. D 88, 056009 (2013).

[13] F. Guerrero and A. Pich, Phys. Lett. B 412, 382 (1997); A. Pich and J. Portoles, Phys. Rev. D 63, 093005 (2001); D. Gomez Dumm and P. Roig, Eur. Phys. J. C 73, 2528 (2013).

[14] V. Bernard and E. Passemar, Phys. Lett. B 661, 95 (2008); V. Bernard, M. Oertel, E. Passemar, and J. Stern, Phys. Rev. D 80, 034034 (2009).

[15] R. Garcia–Martin, R. Kaminski, J.R. Pelaez, and J.R. Elvira, Phys. Rev. Lett. 107, 072001 (2011); R. Garcia–Martin, R. Kaminski, J.R. Pelaez, J.R. Elvira, and F.J. Yndurain, Phys. Rev. D 83, 074004 (2011).

[16] G. Colangelo, M. Hoferichter, M. Procura, and P. Stoffer, JHEP 1409, 091 (2014); G. Colangelo, M. Hoferichter, A. Nyffeler, M. Passera, and P. Stoffer, Phys. Lett. B 735, 90 (2014); G. Colangelo, M. Hoferichter, B. Kubis, M. Procura, and P. Stoffer, *ibid.* 378, 6 (2014).

[17] R.P. Feynman, *Photon–hadron interactions*, Reading, MA: Benjamin (1972) 282 p.
[18] S.L. Adler, Phys. Rev. D 10, 3714 (1974).
[19] F.J. Gilman, SLAC–PUB–1650 (1975).
[20] A.V. Radyushkin, preprint JINR E2–82–159 (1982); JINR Rapid Commun. 78, 96 (1996); N.V. Krasnikov and A.A. Pivovarov, Phys. Lett. B 116, 168 (1982).
[21] A.V. Nesterenko and J. Papavassiliou, Phys. Rev. D 71, 016009 (2005).
[22] A.V. Nesterenko and J. Papavassiliou, Int. J. Mod. Phys. A 20, 4622 (2005); Nucl. Phys. B (Proc. Suppl.) 152, 47 (2005); 164, 304 (2007).
[23] A.I. Akhiezer and V.B. Berestetsky, Quantum electrodynamics, Interscience, NY (1965) 868 p.
[24] A.V. Nesterenko, Nucl. Phys. B (Proc. Suppl.) 234, 199 (2013); SLAC eConf C1106064, 23 (2011); arXiv:1110.3415 [hep-ph].
[25] A.V. Nesterenko, PoS (Confinement X), 350 (2013); arXiv:1401.0620 [hep-ph].
[26] C.J. Maxwell, Phys. Lett. B 409, 382 (1997); D.M. Howe and C.J. Maxwell, Phys. Rev. D 70, 014002 (2004); P.M. Brooks and C.J. Maxwell, ibid. 74, 065012 (2006).
[27] S. Eidelman, F. Jegerlehner, A.L. Kataev, and O. Veretin, Phys. Lett. B 454, 369 (1999); A.L. Kataev, arXiv:hep-ph/9906534 Phys. Lett. B 668, 350 (2008); JETP Lett. 94, 789 (2011).
[28] K.A. Milton, I.L. Solovtsov, and O.P. Solovtsova, Phys. Rev. D 64, 016005 (2001); Mod. Phys. Lett. A 21, 1355 (2006).
[29] G. Cvetic and T. Lee, Phys. Rev. D 64, 014030 (2001); G. Cvetic, C. Dib, T. Lee, and I. Schmidt, ibid. 64, 093016 (2001); G. Cvetic, ibid. 89, 036003 (2014); G. Cvetic and C. Valenzuela, J. Phys. G 32, L27 (2006); G. Cvetic, C. Valenzuela, and I. Schmidt, Nucl. Phys. B (Proc. Suppl.) 164, 308 (2007).
[30] M. Beneke and M. Jamin, JHEP 0809, 044 (2008).
[31] I. Caprini and J. Fischer, Eur. Phys. J. C 64, 35 (2009); Phys. Rev. D 84, 054019 (2011).
[32] G. Abbas, B. Ananthanarayan, and I. Caprini, Phys. Rev. D 85, 094018 (2012); G. Abbas, B. Ananthanarayan, I. Caprini, and J. Fischer, ibid. 87, 014008 (2013).
[33] A.V. Nesterenko, SLAC eConf C0706044, 25 (2007).
[34] A.V. Nesterenko, Nucl. Phys. B (Proc. Suppl.) 186, 207 (2009).
[35] M. Baldicchi, A.V. Nesterenko, G.M. Prosperi, D.V. Shirkov, and C. Simolo, Phys. Rev. Lett. 99, 242001 (2007); M. Baldicchi, A.V. Nesterenko, G.M. Prosperi, and C. Simolo, Phys. Rev. D 77, 034013 (2008).
[36] B. Blossier et al., Phys. Rev. D 85, 034503 (2012); 89, 014507 (2014); Nucl. Phys. B (Proc. Suppl.) 234, 217 (2013).
[37] K. Ackerstaff et al. [OPAL Collaboration], Eur. Phys. J. C 7, 571 (1999); D. Boito, et al., Phys. Rev. D 84, 113006 (2011); 85, 093015 (2012); arXiv:1410.3528 [hep-ph]; arXiv:1410.8415 [hep-ph].
[38] S. Schael et al. [ALEPH Collaboration], Phys. Rept. 421, 191 (2005); M. Davier, A. Hocker, and Z. Zhang, Rev. Mod. Phys. 78, 1043 (2006); M. Davier, A. Hocker, B. Malaescu, C. Yuan, and Z. Zhang, Eur. Phys. J. C 74, 2803 (2014).
[39] A.V. Nesterenko, in Proc. of 11th International Conference on Quark Confinement and the Hadron Spectrum, (Saint–Petersburg, Russian Federation, 2014) (to be published); Nucl.
Phys. B (Proc. Suppl.) (to be published); [arXiv:1409.0687] [hep-ph].
[41] A.V. Nesterenko, Phys. Rev. D 62, 094028 (2000); 64, 116009 (2001).
[42] A.V. Nesterenko, Int. J. Mod. Phys. A 18, 5475 (2003); Nucl. Phys. B (Proc. Suppl.) 133, 59 (2004).
[43] A.V. Nesterenko and C. Simolo, Comput. Phys. Commun. 181, 1769 (2010); 182, 2303 (2011).
[44] A.P. Bakulev and V.L. Khandramai, Comput. Phys. Commun. 184, 183 (2013); C. Ayala and G. Cvetic, [arXiv:1408.6868] [hep-ph]; [arXiv:1411.1581] [hep-ph].
[45] D.V. Shirkov and I.L. Solovtsov, Phys. Rev. D 62, 094028 (2000); 64, 116009 (2001).
[46] A.V. Nesterenko, Int. J. Mod. Phys. A 18, 5475 (2003); Nucl. Phys. B (Proc. Suppl.) 133, 59 (2004).
[47] A.V. Nesterenko and C. Simolo, Comput. Phys. Commun. 181, 1769 (2010); 182, 2303 (2011).
[48] A.P. Bakulev and V.L. Khandramai, Comput. Phys. Commun. 184, 183 (2013); C. Ayala and G. Cvetic, [arXiv:1408.6868] [hep-ph]; [arXiv:1411.1581] [hep-ph].
[49] D.V. Shirkov and I.L. Solovtsov, Phys. Rev. D 62, 094028 (2000); 64, 116009 (2001).
[50] A.V. Nesterenko, Int. J. Mod. Phys. A 18, 5475 (2003); Nucl. Phys. B (Proc. Suppl.) 133, 59 (2004).
[51] A.V. Nesterenko and C. Simolo, Comput. Phys. Commun. 181, 1769 (2010); 182, 2303 (2011).
[52] A.P. Bakulev and V.L. Khandramai, Comput. Phys. Commun. 184, 183 (2013); C. Ayala and G. Cvetic, [arXiv:1408.6868] [hep-ph]; [arXiv:1411.1581] [hep-ph].
[53] D.V. Shirkov and I.L. Solovtsov, Phys. Rev. D 62, 094028 (2000); 64, 116009 (2001).
[54] A.V. Nesterenko, Int. J. Mod. Phys. A 18, 5475 (2003); Nucl. Phys. B (Proc. Suppl.) 133, 59 (2004).
[55] A.V. Nesterenko, Int. J. Mod. Phys. A 18, 5475 (2003); Nucl. Phys. B (Proc. Suppl.) 133, 59 (2004).
[56] A.V. Nesterenko, Int. J. Mod. Phys. A 18, 5475 (2003); Nucl. Phys. B (Proc. Suppl.) 133, 59 (2004).
[57] A.V. Nesterenko, Int. J. Mod. Phys. A 18, 5475 (2003); Nucl. Phys. B (Proc. Suppl.) 133, 59 (2004).
[58] A.V. Nesterenko, Int. J. Mod. Phys. A 18, 5475 (2003); Nucl. Phys. B (Proc. Suppl.) 133, 59 (2004).
[59] M. Della Morte, B. Jaeger, A. Juttner, and H. Wittig, AIP Conf. Proc. 1343, 337 (2011); PoS (LATTICE 2011), 161 (2011); PoS (LATTICE 2012), 175 (2012); JHEP 1203, 055 (2012).

[60] A. Francis, B. Jaeger, H.B. Meyer, and H. Wittig, Phys. Rev. D 88, 054502 (2013); PoS (LATTICE 2013), 305 (2013); H. Horch, G. Herdoiza, B. Jaeger, H. Wittig, M. Della Morte, and A. Juttner, ibid. 304 (2013); M. Della Morte, A. Francis, G. Herdoiza, H. Horch, B. Jaeger, A. Juttner, H. Meyer, and H. Wittig, PoS (LATTICE 2014), 162 (2014).

[61] X. Feng, K. Jansen, M. Petschlies, and D.B. Renner, Phys. Rev. Lett. 107, 081802 (2011); D. Bernecker and H.B. Meyer, Eur. Phys. J. A 47, 148 (2011).

[62] P.A. Baikov, K.G. Chetyrkin, and J.H. Kuhn, Phys. Rev. Lett. 101, 012002 (2008); 104, 132004 (2010); P.A. Baikov, K.G. Chetyrkin, J.H. Kuhn, and J. Rittinger, Phys. Lett. B 714, 62 (2012).

[63] F. Schrempp, J. Phys. G 28, 915 (2002); D. Klammer and F. Schrempp, JHEP 0806, 098 (2008).

[64] M. Baldicchi and G.M. Prosperi, AIP Conf. Proc. 756, 152 (2005).

[65] J. Bailey et al. [CERN–Mainz–Daresbury Collaboration], Nucl. Phys. B 150, 1 (1979); F.J.M. Farley and Y.K. Semertzidis, Prog. Part. Nucl. Phys. 52, 1 (2004).

[66] G.W. Bennett et al. [Muon (g − 2) Collaboration], Phys. Rev. Lett. 89, 101804 (2002); 89, 129903(E) (2002); 92, 161802 (2004); Phys. Rev. D 73, 072003 (2006).

[67] M. Knecht, Lect. Notes Phys. 629, 37 (2004); M. Davier and W.J. Marciano, Ann. Rev. Nucl. Part. Sci. 54, 115 (2004).

[68] M. Passera, J. Phys. G 31, R75 (2005); F. Jegerlehner and A. Nyffeler, Phys. Rept. 477, 1 (2009); J.P. Miller, E. de Rafael, B.L. Roberts, and D. Stockinger, Ann. Rev. Nucl. Part. Sci. 62, 237 (2012).

[69] B.E. Lautrup, A. Peterman, and E. de Rafael, Phys. Rept. 3, 193 (1972).

[70] J. Beringer et al. [Particle Data Group Collaboration], Phys. Rev. D 86, 010001 (2012).

[71] P.J. Mohr, B.N. Taylor, and D.B. Newell, Rev. Mod. Phys. 84, 1527 (2012).

[72] K. Hagiwara, R. Liao, A.D. Martin, D. Nomura, and T. Teubner, J. Phys. G 38, 085003 (2011).

[73] F. Jegerlehner and R. Szafron, Eur. Phys. J. C 71, 1632 (2011).

[74] M. Davier, A. Hocker, B. Malaescu, and Z. Zhang, Eur. Phys. J. C 71, 1515 (2011); 72, 1874(E) (2012).

[75] T. Aoyama, M. Hayakawa, T. Kinoshita, and M. Nio, Phys. Rev. Lett. 109, 111808 (2012).

[76] C. Gnendiger, D. Stockinger, and H. Stockinger–Kim, Phys. Rev. D 88, 053005 (2013).

[77] A. Nyffeler, Phys. Rev. D 79, 073012 (2009); Nuovo Cim. C 37, 173 (2014).

[78] B.L. Roberts, Chin. Phys. C 34, 741 (2010).

[79] M. Steinhauser, Phys. Lett. B 429, 158 (1998); C. Sturm, Nucl. Phys. B 874, 698 (2013).

[80] M. Passera, PoS (HEP 2005), 305 (2006).

[81] F. Jegerlehner, Nuovo Cim. C 034S1, 31 (2011).

[82] J.H. Kuhn and M. Steinhauser, Phys. Lett. B 437, 425 (1998).