Abstract. Let $P(sl_2(K))$ be the Poisson enveloping algebra of the Lie algebra $sl_2(K)$ over an algebraically closed field $K$ of characteristic zero. The quotient algebras $P(sl_2(K))/(C_P - \lambda)$, where $C_P$ is the standard Casimir element of $sl_2(K)$ in $P(sl_2(K))$ and $0 \neq \lambda \in K$, are proven to be simple in [32]. Using a result by L. Makar-Limanov [22], we describe generators of the automorphism group of $P(sl_2(K))/(C_P - \lambda)$ and represent this group as an amalgamated product of its subgroups. Moreover, using similar results by J. Dixmier [8] and O. Fleury [11] for the quotient algebras $U(sl_2(K))/(C_U - \lambda)$, where $C_U$ is the standard Casimir element of $sl_2(K)$ in the universal enveloping algebra $U(sl_2(K))$, we prove that the automorphism groups of $P(sl_2(K))/(C_P - \lambda)$ and $U(sl_2(K))/(C_U - \lambda)$ are isomorphic.

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1. Introduction

It is well known that all automorphisms of the polynomial algebra $K[x, y]$ in two variables $x, y$ over a field $K$ are tame [14, 16]. Moreover, the automorphism group $\text{Aut} K[x, y]$ of this algebra admits an amalgamated free product structure [16, 26], i.e.,

$$\text{Aut} K[x, y] = A \ast_C B,$$

where $A$ is the affine automorphism subgroup, $B$ is the triangular automorphism subgroup, and $C = A \cap B$.

Similar results hold for free associative algebras [6, 20], free Poisson algebras (in characteristic zero) [23], and free right-symmetric algebras of rank two [2, 15]. Moreover, the automorphism groups of polynomial algebras, free associative algebras, and free Poisson algebras in two variables are isomorphic.

The automorphism groups of commutative and associative algebras generated by three elements are much more complicated. The well-known Nagata automorphism (see [25])

$$\sigma = (x + 2y(zx - y^2) + z(x - y^2)^2, y + z(x - y^2), z)$$

1Department of Mathematics, L.N. Gumilyov Eurasian National University, Nur-Sultan, Kazakhstan, e-mail: altyngul.82@mail.ru

2Department of Mathematics, Wayne State University, Detroit, MI 48202, USA; Department of Mathematics, Al-Farabi Kazakh National University, Almaty, 050040, Kazakhstan; and Institute of Mathematics and Mathematical Modeling, Almaty, 050010, Kazakhstan, e-mail: umirbaev@wayne.edu

1
of the polynomial algebra $K[x, y, z]$ over a field $K$ of characteristic 0 is proven to be non-tame. Free associative algebras in three variables over a field of characteristic 0 also have non-tame automorphisms. The Nagata automorphism gives an example of a wild automorphism of free Poisson algebras in three variables.

In 1964 P. Cohn proved that all automorphisms of finitely generated free Lie algebras over a field are tame. Later this result was extended to free algebras of Nielsen-Schreier varieties. Recall that a variety of universal algebras is called Nielsen-Schreier, if any subalgebra of a free algebra of this variety is free, i.e., an analog of the classical Nielsen-Schreier theorem is true. The varieties of all non-associative algebras, commutative and anti-commutative algebras, Lie algebras, and Lie superalgebras over a field are Nielsen-Schreier.

The automorphism groups of free non-associative algebras and free commutative algebras of rank two admit an amalgamated free product structure. The groups of automorphisms of free Lie algebras and free anti-commutative algebras of rank three also admit an amalgamated free product structure.

The study of relations between the automorphism groups of Poisson algebras and their deformation-quantizations is motivated by the Belov-Kanel – Kontsevich Conjecture, which asserts that the automorphism group of the Weyl algebra $A_n$ over a field $K$ is isomorphic to the group of automorphisms of the symplectic Poisson algebra $P_n$, i.e.,

$$\text{Aut } A_n \simeq \text{Aut } P_n.$$  

Recall that the Weyl algebra $A_n$ of index $n \geq 1$, is the associative algebra over a field $K$ with generators $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ and defining relations

$$[Y_i, X_j] = \delta_{ij}, \quad [X_i, X_j] = 0, \quad [Y_i, Y_j] = 0,$$

where $\delta_{ij}$ is the Kronecker symbol and $1 \leq i, j \leq n$. The symplectic Poisson algebra $P_n$ of index $n \geq 1$ is the polynomial algebra in the variables $x_1, \ldots, x_n, y_1, \ldots, y_n$ endowed with the Poisson bracket defined by

$$\{y_i, x_j\} = \delta_{ij}, \quad \{x_i, x_j\} = 0, \quad \{y_i, y_j\} = 0,$$

where $1 \leq i, j \leq n$.

The structure of the automorphism group of the Weyl algebra $A_1$ is described in [1, 7, 21]. These results easily imply that the groups of automorphisms of the symplectic Poisson algebra $P_1$ and the Weyl algebra $A_1$ are isomorphic, i.e., the Belov-Kanel – Kontsevich Conjecture is true for $n = 1$. In 2005 Belov-Kanel and Kontsevich [4] proved that the groups of tame automorphisms of the symplectic Poisson algebra $P_n$ and the Weyl algebra $A_n$ are isomorphic.

Let $L$ be an arbitrary Lie algebra over a field $K$ of characteristic zero. Denote by $U(L)$ the universal enveloping algebra of $L$ and by $P(L)$ the Poisson enveloping algebra of $L$ (or the symmetric Poisson algebra of $L$). Recall that the well known symmetrization map [9, p.77]

$$S : P(L) \to U(L)$$

is an isomorphism of $L$-modules. Moreover, $U(L)$ is the most well-known natural deformation-quantization of $P(L)$. 

2
In 1973 J. Dixmier [8] studied the quotients $U_\lambda = U(\text{sl}_2(\mathbb{C}))/ (C_U - \lambda)$ of the universal enveloping algebra $U(\text{sl}_2(\mathbb{C}))$ of the three dimensional simple Lie algebra $\text{sl}_2(\mathbb{C})$ over the field of complex numbers $\mathbb{C}$, where $C_U$ is the standard Casimir element and $0 \neq \lambda \in \mathbb{C}$. The structure of these algebras depend on $\lambda$ and $U_\lambda$ is simple if $\lambda \neq n^2 + 2n$ for any natural $n$. If $\lambda = n^2 + 2n$, then $U_\lambda$ has a unique non-trivial ideal of finite codimension $(n + 1)^2$.

In the same paper Dixmier described generators of the group of automorphisms of the algebra $U_\lambda$ and defined the group of tame automorphisms of $U(\text{sl}_2(\mathbb{C}))$. He also formulated a question on the existence of wild automorphisms of $U(\text{sl}_2(\mathbb{C}))$. In 1976 A. Joseph [13] gave an example of a wild automorphism of $U(\text{sl}_2(\mathbb{C}))$. In 1998 O. Fleury [11] represented the automorphism group of the quotient algebra $U_\lambda$ as an amalgamated product of its subgroups. Using this, she also proved that every finite subgroup of $\text{Aut} U(\text{sl}_2(\mathbb{C}))$ is isomorphic to a subgroup of $\text{Aut} \text{sl}_2(\mathbb{C})$.

Let $P(\text{sl}_2(K))$ be the Poisson enveloping algebra of the Lie algebra $\text{sl}_2(K)$ over an algebraically closed field $K$ of characteristic zero. The quotient algebras $P_\lambda = P(\text{sl}_2(K))/(C_P - \lambda)$, where $C_P$ is the standard Casimir element of $\text{sl}_2(K)$ in $P(\text{sl}_2(K))$ and $0 \neq \lambda \in K$, are proven to be simple in [32]. Notice that the Casimir elements $C_P$ of $P(\text{sl}_2(K))$ and $C_U$ of $U(\text{sl}_2(K))$ correspond to each other under the symmetrization map.

In 1990 L. Makar-Limanov [22] described generators of the automorphism group of the algebraic surface defined by $xy = f(z)$ over an algebraically closed field. Using this result, we describe generators of the automorphism group of the Poisson quotient algebra $P(\text{sl}_2(K))/(C_P - \lambda)$, where $\lambda \in K$, and represent this group as an amalgamated product of its subgroups. Then, using the above described results by J. Dixmier [8] and O. Fleury [11], we prove that the automorphism groups of $P(\text{sl}_2(K))/(C_P - \lambda)$ and $U(\text{sl}_2(K))/(C_U - \lambda)$ are isomorphic.

Unfortunately, the question on the isomorphism of the automorphism groups of $P(\text{sl}_2(K))$ and $U(\text{sl}_2(K))$ remains open.

The paper is organized as follows. In Section 2, we describe generators of the automorphism group of the Poisson quotient algebra $P(\text{sl}_2(K))/(C_P - \lambda)$, where $\lambda \in K$. In Section 3, we prove that the automorphism group of $P(\text{sl}_2(K))/(C_P - \lambda)$ admits an amalgamated free product structure. In Section 4, we show that the automorphism groups of $P(\text{sl}_2(K))/(C_P - \lambda)$ and $U(\text{sl}_2(K))/(C_U - \lambda)$ are isomorphic.

### 2. Generators of the automorphism group of $P_\lambda$

A vector space $P$ over a field $K$ endowed with two bilinear operations $x \cdot y$ (a multiplication) and $\{x, y\}$ (a Poisson bracket) is called a *Poisson algebra* if $P$ is a commutative associative algebra under $x \cdot y$, $P$ is a Lie algebra under $\{x, y\}$, and $P$ satisfies the following identity (the Leibniz identity):

$$\{x, y \cdot z\} = y \cdot \{x, z\} + \{x, y\} \cdot z.$$ 

Let $L$ be an arbitrary Lie algebra with Lie bracket $[, ]$ over a field $K$ and let $e_1, e_2, \ldots$ be a linear basis of $L$. Then there exists a unique bracket $\{,\}$ on the polynomial algebra $K[e_1, e_2, \ldots]$ defined by $\{e_i, e_j\} = [e_i, e_j]$ for all $i, j$ and satisfying the Leibniz identity.
With this bracket

\[ P(L) = \langle K[e_1, e_2, \ldots], \cdot, \{, \} \rangle \]

becomes a Poisson algebra. This Poisson algebra \( P(L) \) is called the Poisson enveloping algebra of \( L \). Note that the bracket \( \{, \} \) of the algebra \( P(L) \) depends on the structure of \( L \) but does not depend on a chosen basis.

Consider the three dimensional simple Lie algebra \( \text{sl}_2(K) \) over an algebraically closed field \( K \) of characteristic zero. Let

\[
E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}
\]

be the standard basis of the Lie algebra \( \text{sl}_2(K) \). We have

\[
[E, F] = H, [H, E] = 2E, [H, F] = -2F.
\]

The center of the Poisson enveloping algebra \( P(\text{sl}_2(K)) \) of the Lie algebra \( \text{sl}_2(K) \) is equal to \( K[C_P] \), where

\[
C_P = 4EF + H^2
\]

is the Casimir element of \( \text{sl}_2(K) \) in \( P(\text{sl}_2(K)) \). This is an easy corollary of the fact that the center of the universal enveloping algebra \( U(\text{sl}_2(K)) \) of \( \text{sl}_2(K) \) is generated by the standard Casimir element

\[
C_U = 4FE + H^2 + 2H = 4EF + H^2 - 2H = 2EF + 2FE + H^2
\]

of the universal enveloping algebra \( U(\text{sl}_2(K)) \) of \( \text{sl}_2(K) \) and the image of \( C_P \) under the symmetrization map is \( C_U \).

For any \( \lambda \in K \) let \( P_\lambda = P(\text{sl}_2(K))/(C_P - \lambda) \) be the quotient algebra of the algebra \( P(\text{sl}_2(K)) \) by the principal ideal \( (C_P - \lambda) \). Denote by \( e, h, f \) the images of \( E, H, F \) in \( P_\lambda \), respectively. Then we have

\[
\{e, f\} = h, \{h, e\} = 2e, \{h, f\} = -2f, 4ef + h^2 = \lambda.
\]

Notice that \( P_\lambda \), as an associative and commutative algebra, is generated by \( e, h, f \) and defined by one relation

\[
ef = -\frac{1}{4} h^2 + \frac{1}{4} \lambda.
\]

Consequently, the set of words of the form

\[
f^m h^n, \ h^n e^r, \ m, n, r \geq 0
\]

is a linear basis of \( P_\lambda \).

Let \( G = \text{Aut}(P_\lambda) \) be the automorphism group of the algebra \( P_\lambda \). Denote by \( \varphi = (f_1, f_2, f_3) \) the automorphism of the algebra \( P_\lambda \) such that \( \varphi(e) = f_1, \varphi(h) = f_2, \varphi(f) = f_3 \). If \( \theta = (f_1, f_2, f_3) \) and \( \varphi = (g_1, g_2, g_3) \), then the product in \( \text{Aut}(P_\lambda) \) is defined by

\[
\theta \circ \varphi = (g_1(f_1, f_2, f_3), g_2(f_1, f_2, f_3), g_3(f_1, f_2, f_3)).
\]

First we describe the linear automorphisms of \( P_\lambda \), i.e., the automorphisms of \( \text{sl}_2(K) \). Let \( A = \text{Aut}(\text{sl}_2(K)) \) be the group of all automorphisms of the Lie algebra \( \text{sl}_2(K) \). Every automorphism of \( \text{sl}_2(K) \) gives a unique automorphism of \( \text{Aut}(P_\lambda) \). Further we identify \( A \) with the corresponding subgroup of \( G \).
It is well known [12, p. 306] that every automorphism of the algebra \( \mathfrak{sl}_2(K) \) is inner, i.e., every automorphism coincides with

\[
\hat{T} : \mathfrak{sl}_2(K) \to \mathfrak{sl}_2(K), X \mapsto T^{-1}XT
\]

for some matrix \( T \in \text{GL}_2(K) \). Consequently, \( A \cong \text{PSL}_2(K) = \text{GL}_2(K)/\{ \alpha I | \alpha \in K^* \} = \text{SL}_2(K)/\{ I, -I \} \), where \( I \) is the identity matrix.

Let \( C \) be the set of all automorphisms \( \hat{T} \), where \( T \) runs over the set of matrices of the form

\[
(4) \quad T = \begin{bmatrix} 1 & 0 \\ \alpha & \beta \end{bmatrix} \in \text{GL}_2(K).
\]

Identifying the matrices \( E, H, F \) with their images \( e, h, f \), we can write

\[
\hat{T} = (\beta e + \alpha h - \alpha^2/\beta f, h - 2\alpha/\beta f, 1/\beta f).
\]

The elements of \( C \) will be called \textit{linear triangular} automorphisms. Denote by \( H \) the subset of \( C \) with \( \alpha = 0 \) and by \( C_1 \) the subset of \( C \) with \( \beta = 1 \). This means that \( H \) consists of all automorphisms of the form

\[
H_\beta = (\beta e, h, 1/\beta f)
\]

and \( C_1 \) consists of all automorphisms of the form

\[
(e + \alpha h - \alpha^2 f, h - 2\alpha f, f).
\]

The automorphisms \( H_\beta \) are called \textit{hyperbolic rotations} [22] and the subgroup \( H \) will be called the group of \textit{hyperbolic rotations}. It is easy to check that

\[
C \cong C_1 \rtimes H.
\]

**Lemma 1.** The system of elements

\[
A_0 = \{ \tau_\alpha = (f, -h + 2\alpha f, e + \alpha h - \alpha^2 f), \text{id} = (e, h, f) \}
\]

is a system of representatives of the left cosets of \( C \) in \( A \).

**Proof.** Let

\[
T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}_2(K).
\]

If \( b = 0 \), then \( \hat{T} \in C \) and \( \text{id} \) is a representative of the left coset \( \hat{T} \circ C = \text{id} \circ C = C \). If \( b \neq 0 \), then we can assume \( b = 1 \) in \( \text{PSL}_2(K) \), i.e.,

\[
T = \begin{bmatrix} a & 1 \\ c & d \end{bmatrix}.
\]

This matrix can be easily represented as \( T = PQ_\alpha \), where

\[
P = \begin{bmatrix} a_1 & 0 \\ c_1 & d_1 \end{bmatrix}, \quad Q_\alpha = \begin{bmatrix} \alpha & 1 \\ 1 & 0 \end{bmatrix}.
\]

Then \( \hat{T} = \hat{Q}_\alpha \circ \hat{P} \in \hat{Q}_\alpha \circ C \). Notice that \( \hat{Q}_\alpha = \tau_\alpha \).

Suppose that \( \hat{Q}_{\alpha_1} \circ C = \hat{Q}_{\alpha_2} \circ C \). This means that

\[
\hat{Q}_{\alpha_1}^{-1} \circ \hat{Q}_{\alpha_2} \in \hat{Q}_{\alpha_1} \circ C.
\]
We have

\[ Q_{\alpha_2}Q_{\alpha_1}^{-1} = \begin{bmatrix} \alpha_2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -\alpha_1 \end{bmatrix} = \begin{bmatrix} 1 & \alpha_2 - \alpha_1 \\ 0 & 1 \end{bmatrix}. \]

Obviously, \( Q_{\alpha_2}Q_{\alpha_1}^{-1} \in C \) if and only if \( \alpha_1 = \alpha_2 \). \( \square \)

Generators of the automorphism group of the quotient algebra \( R = K[x, y, z]/(xy - P(z)) \) of the polynomial algebra \( K[x, y, z] \) over an algebraically closed field \( K \), where \( P(z) \in K[z] \), were given by L. Makar-Limanov [22].

**Theorem 1.** [22, p. 252] The group \( \text{Aut}(R) \) is generated by the following automorphisms:

(a) Hyperbolic rotations \( H(x) = \nu x, H(y) = \nu^{-1} y, H(z) = z, \nu \in K^*; \)

(b) Involution \( V(x) = y, V(y) = x, V(z) = z; \)

(c) Triangular automorphisms \( \Delta(x) = x, \Delta(y) = y + [P(z + xg(x)) - P(z)]x^{-1}, \Delta(z) = z + xg(x), g(x) \in K[x]; \)

(d) If \( P(z) = c(z + a)^d \), then rescalings \( R(x) = x, R(y) = \nu^d y, R(z) = \nu z + (\nu - 1)a, \nu \in K^* \) should be added;

(e) If \( P(z) = (z + a)^nQ((z + a)^n) \) and \( \mu \in K \) such that \( \mu^n = 1 \), then the symmetry \( S(x) = x, S(y) = \mu^a y, S(z) = \mu z + (\mu - 1)a \) should be added;

(f) If \( \text{char} K = p > 0 \) and \( P(z) = Q(z^p - a^{p-1}z) \), then the translation \( T(x) = x, T(y) = y, T(z) = z - a \) should be added.

**Proposition 1.** The group \( G = \text{Aut}(P_\lambda) \) is generated by

\[ \tau(e) = f, \tau(h) = -h, \tau(f) = e \]

and the automorphisms of the form

\[ \Delta_g(e) = e - g(f)h - fg^2(f), \Delta_g(h) = h + 2fg(f), \Delta_g(f) = f, \]

where \( g(x) \in K[x] \).

**Proof.** Notice that \( P_\lambda \) as an associative and commutative algebra is represented as \( R = K[f, e, h]/(fe - P(h)), \) where \( P(h) = -\frac{1}{4}h^2 + \frac{\lambda}{4}. \) Let \( G' = \text{Aut}(R) \) be the group of all automorphisms of the associative and commutative algebra \( R \). Consider the list of generators of the group \( G' \) given by Theorem 1 under the correspondence \( (x, y, z) \to (f, e, h): \)

(a) Hyperbolic rotations \( H_\nu(e) = \nu e, H_\nu(h) = h, H_\nu(f) = \nu^{-1} f, \nu \in K^*. \)

(b) Involution \( V(e) = f, V(h) = h, V(f) = e. \)

(c) Triangular automorphisms

\[ \Delta(e) = e + [P(h + fg(f)) - P(h)]f^{-1}, \Delta(h) = h + fg(f), \Delta(f) = f, \]

where \( g(x) \in K[x] \). Notice, that

\[ [P(h + fg(f)) - P(h)]f^{-1} = -1/2g(f)h - 1/4fg^2(f). \]

Replacing \( 1/2g(f) \) by \( g(f) \), we get automorphisms \( \Delta_g \) of the form (5).
(d) Notice that $P(h)$ can be represented in the form $P(h) = c(h + a)^d$ only if $\lambda = 0$. If $\lambda = 0$, then $P(h) = -\frac{1}{4}h^2$ and rescalings $R_\nu(e) = \nu^2 e, R_\nu(h) = \nu h, R_\nu(f) = f$, where $\nu \in K^*$, are automorphisms of $R$.

(e) Consider representations of $P(h)$ in the form $P(h) = (h + a)^n$. If $n = 1$ Theorem 1(e) gives only the identity automorphism. If $n = 2$, then, obviously, $i = 0, a = 0$, and $\mu = \pm 1$. This gives the automorphism $S(e) = e, S(h) = -h, S(f) = f$.

However, not all automorphisms from this list are automorphisms of the Poisson algebra $P_\lambda$. More exactly, $V, S, R_\nu \notin G$ and $\Delta_g, H_\nu \in G$.

Notice that

$$V^2 = S^2 = \text{id}$$

and that

$$\tau = V \circ S = S \circ V = (f, -h, e)$$

is an automorphism of the Poisson algebra $P_\lambda$.

Denote by $G_1$ the subgroup of $G$ generated by $\tau$ and all automorphisms of the form (5).

We also have

$$H_\nu = (\nu e, h, \nu^{-1} f) = \tau \circ (e - i\sqrt{\nu} h + \nu f, h + 2i\sqrt{\nu} f, f) \circ \tau \circ (e - i\sqrt{\nu} h + \nu f, h + 2i\sqrt{\nu} f, f),$$

where $i = \sqrt{-1}$, i.e., $H_\nu \in G_1$.

It is easy to check that

$$\Delta_{g_1} \circ \Delta_{g_2} = \Delta_{g_1 + g_2}, \quad \Delta_g^{-1} = \Delta_{-g}.$$  

Direct calculations give that

$$S \circ \Delta_g \circ S = (e + g(f)h - fg^2(f), -h + 2fg(f), f) \circ (e, -h, f)$$

and

$$\tau \circ \Delta_{-g} \circ \tau = (f - g(e)h - eg^2(e), -h - 2eg(e), e) \circ (f, -h, e)$$

$$= (e, h + 2eg(e), f - g(e)h - eg^2(e))$$

$$= (f - g(e)h - eg^2(e), h + 2eg(e), e) \circ (f, h, e) = V \circ \Delta_g \circ V.$$ 

Hence

$$\Delta_g \circ S = S \circ \Delta_{-g}, \quad \Delta_g \circ V = V \circ \tau \circ \Delta_{-g} \circ \tau.$$ 

Consider the case $\lambda \neq 0$. In this case $G'$ is generated by $S, V$, and the automorphisms of the form (5). Using (6), (7), and (9), we can represent any element of $G'$ in the form $\omega \circ \varphi$, where $\varphi \in G_1$ and $\omega$ is equal to $S, V, \text{ or } \text{id}$. Notice that $\varphi \in G, V \circ \varphi \notin G$, and $S \circ \varphi \notin G$. It follows that $G = G_1$.


Now consider the case $\lambda = 0$. We have

$$R_{\nu_1} \circ R_{\nu_2} = R_{\nu_1 \nu_2},$$

$$S \circ R_{\nu} = (\nu^2 e, -\nu h, f) = R_{\nu} \circ S,$$

$$V \circ R_{\nu} = (\nu^2 f, \nu h, e) = R_{\nu} \circ V \circ H_{\nu^2},$$

$$\Delta_g \circ R_{\nu} = R_{\nu} \circ \Delta_{\nu g}.$$

Using these relations we can represent any element of $G'$ in the form $R_{\nu} \circ \psi$, where $\psi$ belongs to the subgroup generated by the automorphisms of the form (5), $V$, and $S$. As proven above, $\psi$ can be represented in the form $\psi = \omega \circ \varphi$, where $\varphi \in G_1$ and $\omega$ is equal to $S$, $V$, or $\text{id}$. Notice that $\varphi \in G$, $R_{\nu} \circ V \circ \varphi \notin G$, $R_{\nu} \circ S \circ \varphi \notin G$ for any $\nu$, and $R_{\nu} \circ \varphi \notin G$ for any $\nu \neq 1$. It follows that $G = G_1$. □

Corollary 1. The group of linear automorphisms $A$ is generated by $\tau$ and the linear automorphisms of the form (5).

For any $a \in P_\lambda$ denote by $\text{ad}(a) : P_\lambda \to P_\lambda$ the adjoint operator of $a$, i.e., the inner derivation of $P_\lambda$ such that $\text{ad}(a)(x) = \{a, x\}$ for any $x \in P_\lambda$. If $D$ is a locally nilpotent derivation of $P_\lambda$, then

$$\exp(D) = \sum_{i=0}^{\infty} \frac{1}{i!} D^i$$

is an automorphism of $P_\lambda$ and is called an exponential automorphism.

Lemma 2. (1) For any $g \in K[x]$ the derivation $\text{ad}(g(f))$ is locally nilpotent.

(2) Let $\Delta_g$ be an automorphism of the form (5) and let $\hat{g} = \int_0^x gdx$ be the formal integral of $g$. Then

$$\Delta_g = \exp \text{ad}(\hat{g}(f)).$$

Proof. It is easy to check that $\{f^n, h\} = 2nf^n$, $\{f^n, e\} = -nf^{n-1}h$. These relations easily imply the first statement of the lemma.

Suppose that $g(x) = \sum_{i=0}^{n} \alpha_i x^i$. Then $\hat{g}(x) = \sum_{i=0}^{n} \frac{1}{i+1} \alpha_i x^{i+1}$ and

$$\hat{g}(f) = \sum_{i=0}^{n} \frac{1}{i+1} \alpha_i f^{i+1}.$$

Direct calculation gives that

$$\{\hat{g}(f), h\} = 2 \sum_{i=0}^{n} \alpha_i f^{i+1} = 2fg(f),$$

$$\{\hat{g}(f), e\} = -h \sum_{i=0}^{n} \alpha_i f^i = -h g(f),$$

and

$$\{\hat{g}(f), -h g(f)\} = -2fg^2(f).$$

These relations directly imply the statement of the lemma. □
This lemma clarifies the relations (8). Denote by $J$ the subgroup of all exponential automorphisms of the form (5) of $G = \text{Aut} \, P_\lambda$. Denote by $T = \text{Tr}(P_\lambda)$ the subgroup of $G = \text{Aut} \, P_\lambda$ generated by the subgroup $J$ and by the subgroup of hyperbolic rotations $H$. The subgroup $T$ will be called the subgroup of triangular automorphisms of $P_\lambda$.

**Lemma 3.**

$$T = J \rtimes H.$$  

**Proof.** Direct calculation gives that 

$$H_\nu \circ \Delta_g \circ H_\nu^{-1} = \Delta_q,$$  

where $q = \nu^{-1}g(\nu^{-1}f)$.  

This implies that $T = JH$ and $J$ is a normal subgroup of $T$. Obviously, $J \cap H = \{\text{id}\}$. □

**Corollary 2.** Any triangular automorphism of the algebra $P_\lambda$ can be written uniquely in the form

$$(ae - \alpha g(f)h - \alpha fg^2(f), h + 2fg(f), \alpha^{-1}f), \ g(x) \in K[x], \ \alpha \in K^*.$$  

**Lemma 4.** The system of elements

$$B_0 = \{\Delta_q \mid q(x) \in xK[x]\}$$  

is a system of representatives of the left cosets of $C$ in $T$.

**Proof.** Let $\psi \in T$. By Lemma 3 $\psi = \Delta_g \circ H_\nu$. Let $g(f) = q(f) + \alpha$, where $q(x) \in xK[x]$ and $\alpha \in K$. By (8), we get

$$\psi = \Delta_g \circ H_\nu = \Delta_q \circ \Delta_\alpha \circ H_\nu.$$  

Since $\Delta_\alpha \circ H_\nu \in C$, it follows that $\psi \in \Delta_q \circ C$.

Suppose that $\Delta_{q_1} \circ C = \Delta_{q_2} \circ C$. This means that

$$\Delta_{q_1}^{-1} \circ \Delta_{q_2} \in C.$$  

By (8), we have

$$\Delta_{q_1}^{-1} \circ \Delta_{q_2} = \Delta_{-q_1} \circ \Delta_{q_2} = \Delta_{-q_1 + q_2}.$$  

Obviously, $\Delta_{-q_1 + q_2} \in C$ if and only if $q_1 = q_2$. □

3. **Amalgamated Free Product Structure of $G$**

We introduce a linear order on the set of basis words (3) of the algebra $P_\lambda$. Put $f > h > e$. Let $u$ and $v$ be elements of the form (3). We say that $u < v$ if $\deg(u) < \deg(v)$ or $\deg(u) = \deg(v)$ and $u < v$ with respect to the lexicographic order. Every nonzero element $g \in P_\lambda$ can be uniquely represented in the form

$$g = \alpha_1g_1 + \alpha_2g_2 + ... + \alpha_mg_m,$$  

where $g_i$ are elements the form (3), $0 \neq \alpha_i \in K$ for all $i$ and $g_1 > g_2 > ... > g_m$. Then $g_1$ is called the leading word of the element $g$ and will be denoted by $\bar{g}$.

**Lemma 5.** If $u, v, w$ are elements the form (3) and $u < v$, then $\overline{uvw} < \overline{uw}$. 

9
Proof. The statement of the lemma is obviously true if \( \deg(u) < \deg(v) \). It is also true if \( u, v, w \) are all elements of the form \( f^m h^n \) or all elements of the form \( h^n e^r \). Let \( \deg(u) = \deg(v) \). Consider the following cases:

1) Let \( u = f^{m_1} h^{n_1}, v = f^{m_2} h^{n_2} \) and \( w = h^e \). We have \( m_1 < m_2 \) since \( u < v \). By (2),
\[
\overline{uw} = f^{m_1-r} h^{n_1+n+2r}, \quad \overline{vw} = f^{m_2-r} h^{n_2+n+2r} \quad \text{if} \quad r \leq m_1; \\
\overline{uw} = h^{n_1+n+2m_1} e^{r-m_1}, \quad \overline{vw} = f^{m_2-r} h^{n_2+n+2r} \quad \text{if} \quad m_1 < r < m_2; \\
\overline{uw} = h^{n_1+n+2m_1} e^{r-m_1}, \quad \overline{vw} = h^{n_2+n+2m_2} e^{r-m_2} \quad \text{if} \quad m_2 \leq r.
\]
Consequently, \( \overline{uw} < \overline{vw} \).

2) Let \( u = h^{m_1} e^{r_1}, v = h^{n_2} e^{r_2} \) and \( w = f^m h^n \). We have \( r_1 > r_2 \) since \( u < v \). By (2),
\[
\overline{uw} = h^{n_1+n+2m_1} e^{r_1-m}, \quad \overline{vw} = h^{n_2+n+2m_2} e^{r_2-m} \quad \text{if} \quad r_2 \geq m; \\
\overline{uw} = h^{n_1+n+2m_1} e^{r_1-m}, \quad \overline{vw} = f^{m-r_2} h^{n_2+n+2r_2} \quad \text{if} \quad r_1 > m > r_2; \\
\overline{uw} = f^{m-r_1} h^{n_1+n+2r_1}, \quad \overline{vw} = f^{m-r_2} h^{n_2+n+2r_2} \quad \text{if} \quad m \geq r_1.
\]
Consequently, \( \overline{uw} < \overline{vw} \).

3) Let \( u = f^{m_1} h^{n_1} \) and \( v = h^{n_2} e^{r_2} \). This case contradicts the condition \( u < v \).

4) Let \( u = h^{m_1} e^{r_1}, v = f^{m_2} h^{n_2} \) and \( w = h^e \). We have \( (r_1, m_2) \neq (0, 0) \) since \( u < v \). By (2),
\[
\overline{uw} = h^{n_1+n} e^{r_1+r}, \\
\overline{vw} = f^{m_2-r} h^{n_2+n+2r} \quad \text{if} \quad m_2 \geq r; \\
\overline{uw} = h^{n_2+n+2m_2} e^{r-m_2} \quad \text{if} \quad m_2 < r.
\]
Consequently, \( \overline{uw} < \overline{vw} \).

5) Let \( u = h^{m_1} e^{r_1}, v = f^{m_2} h^{n_2} \) and \( w = f^m h^n \). We have \( (r_1, m_2) \neq (0, 0) \) since \( u < v \). By (2),
\[
\overline{uw} = f^{m-r_1} h^{n_1+n+2r_1} \quad \text{if} \quad m \geq r_1; \\
\overline{uw} = h^{n_2+n+2m_2} e^{r_1-m} \quad \text{if} \quad m < r_1, \\
\overline{uw} = f^{m_2+m} h^{n_2+n}.
\]
Consequently, \( \overline{uw} < \overline{vw} \). \( \square \)

Corollary 3. Let \( 0 \neq g, q \in P_\lambda \). Then \( \overline{gq} = \overline{qg} \) and \( \deg(gq) = \deg(g) + \deg(q) \).

Let \( G \) be an arbitrary group, \( G_0, G_1, G_2 \) be subgroups of the group \( G \) and \( G_0 = G_1 \cap G_2 \). The group \( G \) is called the free product of the subgroups \( G_1 \) and \( G_2 \) with the amalgamated subgroup \( G_0 \) and is denoted by \( G = G_1 *_{G_0} G_2 \) if

(a) \( G \) is generated by the subgroups \( G_1 \) and \( G_2 \);
(b) the defining relations of the group \( G \) consist only of the defining relations of the subgroups \( G_1 \) and \( G_2 \).

If \( S_1 \) is a system of left representatives of \( G_1 \) by \( G_0 \) and \( S_2 \) is a system of left representatives of \( G_2 \) by \( G_0 \) then the group \( G \) is a free product of subgroups \( G_1 \) and \( G_2 \) with the amalgamated subgroup \( G_0 \) (see, for example, [19]) if and only if each \( g \in G \) is uniquely representable in the form
\[
g = g_1 \cdots g_k c,
\]
Lemma 6. Let $A_0$ and $B_0$ be the sets defined in Lemma 4 and Lemma 4, respectively. Then any automorphism $\phi$ of the algebra $P_\lambda$ decomposes into a product of the form

$$
\phi = \alpha_1 \circ \beta_1 \circ \alpha_2 \circ \beta_2 \circ \ldots \circ \alpha_k \circ \beta_k \circ \alpha_{k+1} \circ \lambda,
$$

where $\alpha_i \in A_0$, $\alpha_2, \ldots, \alpha_k \neq id$, $\beta_i \in B_0$, $\beta_1, \ldots, \beta_k \neq id$, $\lambda \in C$.

Proof. By Proposition 4 any automorphism $\phi$ can be represented as

$$
\phi = l_1 \circ t_1 \circ l_2 \circ t_2 \circ \ldots \circ l_n \circ t_n \circ l_{n+1},
$$

where $l_i \in A$, $t_i \in T$, $t_i \notin A$.

We prove by induction on $n$ that $\phi$ can be represented as (10) with $k \leq n$. By Lemma 1 we have $l_1 = \alpha_1 \circ \lambda_1$, where $\alpha_1 \in A_0$, $\lambda_1 \in C$. By (8), $\lambda_1 \circ t_1 \in T$ and $\lambda_1 \circ t_1 \notin A$. Then, by Lemma 4 we have $\lambda_1 \circ t_1 = \beta_1 \circ \lambda'_1$, where $id \neq \beta_1 \in B_0$, $\lambda'_1 \in C$. Hence

$$
l_1 \circ t_1 = \alpha_1 \circ \beta_1 \circ \lambda'_1.
$$

Consequently,

$$
\phi = \alpha_1 \circ \beta_1 \circ (\lambda'_1 \circ l_2) \circ t_2 \circ \ldots \circ l_n \circ t_n \circ l_{n+1}.
$$

By the induction proposition, the product

$$
(\lambda'_1 \circ l_2) \circ t_2 \circ \ldots \circ l_n \circ t_n \circ l_{n+1}
$$

can be written in the form

$$
\alpha_2 \circ \beta_2 \circ \ldots \circ \alpha_k \circ \beta_k \circ \alpha_{k+1} \circ \lambda, \quad k \leq n.
$$

Consequently,

$$
\phi = \alpha_1 \circ \beta_1 \circ \alpha_2 \circ \beta_2 \circ \ldots \circ \alpha_k \circ \beta_k \circ \alpha_{k+1} \circ \lambda.
$$

If $\alpha_2 \neq id$, then the resulting representation has the form (10). Let $\alpha_2 = id$. Since $\beta_1 \circ \beta_2 = \beta'_2 \in B_0$, it follows that

$$
\phi = \alpha_1 \circ \beta_1 \circ \beta_2 \circ \ldots \circ \alpha_k \circ \beta_k \circ \alpha_{k+1} \circ \lambda = \alpha_1 \circ \beta'_2 \circ \ldots \circ \alpha_k \circ \beta_k \circ \alpha_{k+1} \circ \lambda.
$$

Consequently, $\phi$ can be represented as (10) since $k - 1 < n$. □

Lemma 7. Let $\varphi = (f_1, f_2, f_3)$ be an automorphism of the algebra $P_\lambda$ such that

$$
\varphi = (f_1, f_2, f_3) = \beta_1 \circ \alpha_2 \circ \beta_2 \circ \ldots \circ \alpha_k \circ \beta_k,
$$

where $id \neq \alpha_i \in A_0$ and $id \neq \beta_i \in B_0$ for any $i$. If $\beta_i = \Delta_{q_i}$, where $q_i(x) \in xK[x]$ and $\deg(q_i) = n_i > 0$, then

$$
\deg(f_1) = \prod_{i=1}^{k}(2n_i + 1), \quad \deg(f_2) = (n_k + 1) \prod_{i=1}^{k-1}(2n_i + 1), \quad \deg(f_3) = \prod_{i=1}^{k-1}(2n_i + 1).
$$

(here we assume that $\prod_{i=1}^{k-1}(2n_i + 1) = 1$ if $k = 1$).
Proof. Prove the statement of the lemma by induction on $k$. If $k = 1$, then $\varphi = \beta_1$ and $\text{mdeg}(\varphi) = (2n_1 + 1, n_1 + 1, 1)$.

Suppose that the lemma holds for $k - 1$. Put

$$\varphi_1 = \beta_1 \circ \alpha_2 \circ \beta_2 \circ \ldots \circ \alpha_{k-1} \circ \beta_{k-1} = (g_1, g_2, g_3).$$

By the induction hypothesis, we get

$$\text{deg}(g_1) = \prod_{i=1}^{k-1} (2n_i + 1), \quad \text{deg}(g_2) = (n_{k-1} + 1) \prod_{i=1}^{k-2} (2n_i + 1), \quad \text{deg}(g_3) = \prod_{i=1}^{k-2} (2n_i + 1).$$

Then

$$\varphi = (f_1, f_2, f_3) = \beta_1 \circ \alpha_2 \circ \beta_2 \circ \ldots \circ \alpha_k \circ \beta_k = \varphi_1 \circ \alpha_k \circ \beta_k = (g_1, g_2, g_3) \circ \alpha_k \circ \beta_k.$$

We have

$$(u_1, u_2, u_3) = (g_1, g_2, g_3) \circ \alpha_k = (g_3, -g_2 + 2\beta g_3, g_1 + \beta g_2 - \beta^2 g_3).$$

Since $\text{deg}(g_1) > \text{deg}(g_2) > \text{deg}(g_3)$, it follows that

$$\text{deg}(u_1) = \text{deg}(g_3) = \prod_{i=1}^{k-2} (2n_i + 1),$$

$$\text{deg}(u_2) = \text{deg}(g_2) = (n_{k-1} + 1) \prod_{i=1}^{k-2} (2n_i + 1),$$

$$\text{deg}(u_3) = \text{deg}(g_1) = \prod_{i=1}^{k-1} (2n_i + 1).$$

Further,

$$\varphi = (f_1, f_2, f_3) = (u_1, u_2, u_3) \circ \beta_k = (u_1, u_2, u_3) \circ (e - q_k(f)h - f q_k^2(f), h + 2fq_k(f), f) =$$

$$= (u_1 - q_k(u_3)u_2 - u_3 q_k^2(u_3), u_2 + 2u_3 q_k(u_3), u_3).$$

Consequently,

$$\text{deg}(f_1) = \max\{\text{deg}(u_1), \text{deg}(q_k(u_3)u_2), \text{deg}(u_3 q_k^2(u_3))\},$$

$$\text{deg}(f_2) = \max\{\text{deg}(u_2), \text{deg}(u_3 q_k(u_3))\},$$

$$\text{deg}(f_3) = \text{deg}(u_3).$$

Recall that $\text{deg}(q_k) = n_k > 0$ and $\text{deg}(u_3) = \prod_{i=1}^{k-1} (2n_i + 1)$. Then, by Corollary 3

$$\text{deg}(u_3 q_k(u_3)) = \prod_{i=1}^{k-1} (2n_i + 1) + n_k \prod_{i=1}^{k-1} (2n_i + 1) = (n_k + 1) \prod_{i=1}^{k-1} (2n_i + 1),$$

$$\text{deg}(q_k(u_3)u_2) = n_k \prod_{i=1}^{k-1} (2n_i + 1) + (n_{k-1} + 1) \prod_{i=1}^{k-2} (2n_i + 1),$$

$$\text{deg}(u_3 q_k^2(u_3)) = \prod_{i=1}^{k-1} (2n_i + 1) + 2n_k \prod_{i=1}^{k-1} (2n_i + 1) = \prod_{i=1}^{k} (2n_i + 1).$$
Consequently,
\[
\deg(f_1) = \prod_{i=1}^{k}(2n_i + 1), \quad \deg(f_2) = (n_k + 1) \prod_{i=1}^{k-1}(2n_i + 1), \quad \deg(f_3) = \prod_{i=1}^{k-1}(2n_i + 1).
\]

**Lemma 8.** The decomposition \((10)\) of the automorphism \(\phi\) from Lemma \(6\) is unique.

**Proof.** It suffices to show that
\[
\alpha_1 \circ \beta_1 \circ \alpha_2 \circ \beta_2 \circ \ldots \circ \alpha_k \circ \beta_k \circ \alpha_{k+1} \circ \lambda \neq id,
\]
for \(k \geq 1\), \(\alpha_i \in A_0\), \(\alpha_2, \ldots, \alpha_k \neq id\), \(\beta_1 \in B_0\), \(\beta_1, \ldots, \beta_k \neq id\), \(\lambda \in C\).

Suppose that
\[
\alpha_1 \circ \beta_1 \circ \alpha_2 \circ \beta_2 \circ \ldots \circ \alpha_k \circ \beta_k \circ \alpha_{k+1} \circ \lambda = id.
\]
Then
\[
(11) \quad \beta_1 \circ \alpha_2 \circ \beta_2 \circ \ldots \circ \alpha_k \circ \beta_k = \alpha_1^{-1} \circ \lambda^{-1} \circ \alpha_{k+1}^{-1}.
\]
By Lemma \(7\) the automorphism
\[
\varphi = \beta_1 \circ \alpha_2 \circ \beta_2 \circ \ldots \circ \alpha_k \circ \beta_k
\]
has the multidegree
\[
mdeg(\varphi) = \left( \prod_{i=1}^{k}(2n_i + 1), (n_k + 1) \prod_{i=1}^{k-1}(2n_i + 1), \prod_{i=1}^{k-1}(2n_i + 1) \right),
\]
where \(n_i > 0\). The automorphism on the right-hand side of \((11)\) is linear and has the multidegree \((1,1,1)\). This contradicts \((11)\). \(\square\)

**Theorem 2.** The group \(G\) of automorphism of the Poisson algebra \(P_\lambda\) is the free product of subgroups \(A\) and \(T\) with amalgamated subgroup \(C = A \cap T\), i.e.,
\[
G = A \ast_C T.
\]

**Proof.** Recall that \(A_0\) and \(B_0\) are the systems of representatives of the left cosets of \(C\) in \(A\) and \(T\), respectively. By Lemma \(6\) and Lemma \(8\) any automorphism of \(P_\lambda\) can be uniquely represented as \((10)\). According to \((19)\), we have
\[
G = A \ast_C T. \quad \square
\]

**4. ISOMORPHISM OF AUTOMORPHISM GROUPS**

For any \(\lambda \in K\) let \(U_\lambda = U(sl_2(K))/(C_U - \lambda)\) be the quotient algebra of the algebra \(U(sl_2(K))\) by the principal ideal \((C_U - \lambda)\). Denote by \(e, h, f\) the images of \(E, H, F\) in \(U_\lambda\), respectively. Then we have
\[
[f, h] = 2f, \quad [h, e] = 2e, \quad \lambda = 4fe + h^2 + 2h = 4ef + h^2 - 2h = 2ef + 2fe + h^2.
\]

J. Dixmier \(8\) described generators of the group of automorphisms of the algebra \(U_\lambda\).
Theorem 3. [8, p. 563] The group \( \text{Aut}(U_\lambda) \) is generated by the exponential automorphisms \( \Phi_{n,\mu} \) and \( \Psi_{n,\mu} \) \((n > 0, \mu \in K)\), where
\[
\Phi_{n,\mu} = (e - \mu n f^{n-1}h + \mu n(n-1)f^{n-1} - \mu^2n^2f^{2n-1}, h + 2\mu nf^n, f),
\]
\[
\Psi_{n,\mu} = (e, h - 2\mu ne^n, f + \mu ne^{n-1}h + \mu n(n-1)e^{n-1} - \mu^2n^2e^{2n-1}).
\]

Recall that \( A = \text{Aut}(\text{sl}_2(K)) \) is the group of all automorphisms of the Lie algebra \( \text{sl}_2(K) \). In Section 2 we identified \( A \) with a subgroup of automorphisms \( G = \text{Aut}(P_\lambda) \) of the Poisson algebra \( P_\lambda \). Obviously, every automorphism of \( A \) gives a unique automorphism of the algebra \( U_\lambda \). For this reason we identify \( A \) with the corresponding subgroup of the group of automorphisms of \( U_\lambda \). Thus, without loss of generality, we can assume that \( \text{Aut}(P_\lambda) \) and \( \text{Aut}(U_\lambda) \) both contain \( A \). Then the subgroups \( H \) and \( C \) of \( A \) may be considered as subgroups of \( \text{Aut}(P_\lambda) \) and \( \text{Aut}(U_\lambda) \).

Denote by \( J' \) the subgroup of all exponential automorphisms \( \Phi_{n,\mu} \) \((\nu \in K^*, n \in \mathbb{N}^*, \mu \in K)\) of \( \text{Aut} U_\lambda \). Notice that every automorphism \( \Phi_{n,\mu} \) can be written in the form
\[
\delta_g(e) = e - g(f)h - fg^2(f) + fg'(f),
\]
\[
\delta_g(h) = h + 2fg(f),
\]
\[
\delta_g(f) = f,
\]
where \( g(x) \in K[x] \) and \( g'(x) \) is the formal derivative of \( g(x) \).

It is easy to check that
\[
\delta_{g_1} \circ \delta_{g_2} = \delta_{g_1 + g_2}.
\]

Denote by \( T' = \text{Tr}(P_\lambda) \) the subgroup of \( \text{Aut} U_\lambda \) generated by the subgroup \( J' \) and by the subgroup of hyperbolic rotations \( H \). The subgroup \( T' \) will be called the subgroup of \( \text{triangular} \) automorphisms of \( U_\lambda \).

O. Fleury [11] proved that the automorphism group of the algebra \( U_\lambda \) admits an amalgamated free product structure.

Theorem 4. [11] We have
\[
\text{Aut}(U_\lambda) = A \ast_C T',
\]
where \( A = \text{Aut}(\text{sl}_2(K)) \) is the subgroup of all automorphisms of the Lie algebra \( \text{sl}_2(K) \) in \( \text{Aut}(U_\lambda) \) and \( C = A \cap T' \).

Lemma 9. \( T' = J' \rtimes H \).

Proof. Direct calculation gives that
\[
H_\nu \circ \delta_q \circ H_\nu^{-1} = \delta_q, \quad \text{where } q = \nu^{-1}g(\nu^{-1}f).
\]
This gives that \( T' = J'H \) and \( J' \) is a normal subgroup of \( T' \). Obviously, \( J' \cap H = \{\text{id}\} \).

\[\square\]

Corollary 4. Any triangular automorphism of the algebra \( U_\lambda \) can be written uniquely in the form
\[
(\alpha e - \alpha g(f)h - \alpha fg^2(f) + \alpha fg'(f), h + 2fg(f), \alpha^{-1}f), \quad g(x) \in K[x], \quad \alpha \in K^*.
\]
Theorem 5. The automorphism groups of the algebras $P_\lambda$ and $U_\lambda$ are isomorphic, i.e.,

$$\text{Aut}(P_\lambda) \cong \text{Aut}(U_\lambda).$$

Proof. Consider the map $\phi : J' \to J$ given by the rule $\delta_g \mapsto \Delta_g$. By (8) and (13),

$$\phi(\delta_{g_1} \circ \delta_{g_2}) = \phi(\delta_{g_1}) \circ \phi(\delta_{g_2}).$$

Consequently, $J' \cong J$. By Lemma 8 and Lemma 9 we get $T' \cong T$. Then Theorem 2 and Theorem 4 directly imply the statement of the theorem. □

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