Turaev–Viro TQFT and the Rank versus Genus Conjecture

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Abstract

This paper presents a way to estimate the Heegaard genus of a 3-manifold using the Turaev–Viro state sum TQFT. The Turaev–Viro state sum TQFT is derived from the modular category associated to the quantum group $U_q(sl_2)$, which is unitary for some $q$ by Wenzl. Hence by Turaev and Virelizier the corresponding TQFT is unitary. We modify a proof by Garoufalidis to give a lower bound of the Heegaard genus using a unitary TQFT, and then use the software Regina to provide some known counterexamples to the rank versus genus conjecture.

1 Overview

For a closed orientable 3-manifold $M$, the Heegaard genus $g(M)$ of $M$, is the minimal genus over all Heegaard splittings of $M$. The rank $r(M)$ of $M$ is the rank of the fundamental group $\pi_1(M)$. Hence $r(M) \leq g(M)$.

Conjecture 1. $r(M) = g(M)$?

This conjecture was proposed by Waldhausen [Wal78]. In 1984, Boileau–Zieschang [BZ84] discovered the first counterexamples among Seifert manifolds. Later, for any $n$, graph manifolds of genus $4n$ whose fundamental group is $3n$-generated are constructed by Schultens and Weidmann in [SW07]. In 2009, Namazi and Souto [NS09] showed that rank equals genus if the gluing map of a Heegaard splitting is a high power of a generic pseudo-Anosov map. In 2013, Tao Li showed in [Li13] that there are counterexamples among hyperbolic manifolds, using topological argument.

In this thesis, we apply the Turaev–Viro TQFT to the rank versus genus conjecture. We obtain a lower bound of the Heegaard genus of a 3-manifold, and provide some known counterexamples to the rank versus genus conjecture.

1.1 Background

Inspired by Witten, Reshetikhin and Turaev [RT91] constructed quantum invariants using representations of a quantum group $U_q(sl_2(C))$. This invariant is part of a topological quantum field theory (TQFT). This was generalized to simple Lie algebras of type A, B, C and D in [TW93]. More generally, the TQFT can be constructed using a general modular category, as is explained in [Tur92]
or [Tur10]. There is another approach via Kauffman bracket, see [BHMV92], [BHMV95]. The book [Lus10] is an introduction to quantum groups.

In [TV92], Turaev and Viro constructed a TQFT using quantum 6j-symbols associated to $U_q(sl_2(C))$. More generally, the construction can be performed using a unimodal category, see [Tur10], or using a finite semisimple spherical category of non-zero dimension, see [BW96]. This is called the Turaev–Viro–Barrett–Westbury invariant in the book by Turaev and Virelizier [TV17]. In 2018, Qingtao Chen and Tian Yang [CY18] defined the invariant for a manifold with boundary and proposed volume conjectures for the Reshetikhin–Turaev and the Turaev–Viro invariants.

The process of constructing a modular category from $U_q(sl_2(C))$ using the notion of tilting modules is explained in [BK01]. In [BW96], Barrett and Westbury used the paper by Andersen and Paradowski [HAP95], and showed that the quantum $sl_2$ can be used to produce a state sum TQFT. (This general procedure is needed in the proof of unitarity.) Wenzl [Wen98] showed that, if $g$ is a simple Lie algebra and $q = e^{\pi i/ld}$ where $d = m \in \{1, 2, 3\}$ is the ratio of the square lengths of a long and a short root and $l = x$ is larger than the dual Coxeter number, then the category for $U_qg$ is unitary. (It is shown in [KJ96] that the category is Hermitian.) Note that the ratio is 1 for Lie type $A$. The dual Coxeter number of $A_n$ is $n + 1$, and $sl_2$ corresponds to $A_1$. There is a survey [Row06] on the general case.

It is shown in [TV17], Appendix G, that a unitary fusion category $C$ gives rise to a unitary state sum TQFT.

### 1.2 A Lower Bound of the Heegaard Genus

In [Gar98], Theorem 2.2 gives a lower bound of Heegaard genus using a unitary TQFT. This theorem is not correct in general. Note that Theorem 2.2 uses Lemma 2.1, and the author claims that Lemma 2.1 follows from a paper by Witten [Wit89]. In [Wit89], there is an implicit condition on the TQFT, namely the space associated to $S^2$ must be one-dimensional. This condition is not mentioned in [Gar98].

We modify this proof in [Gar98] to give the bound. Parallel to Lemma 2.1 in [Gar98], our unitary TQFT $Z$ must have the following three additional properties:

1. If $M, N$ are closed 3-manifolds, then $Z(M \# N)Z(S^3) = Z(M)Z(N)$.
2. $Z(S^2 \times S^1) = 1$.
3. $0 < Z(S^3) < 1$.

Assuming the space associated to $S^2$ is one-dimensional, property (1) follows from a cut-and-paste argument (see Lemma 10.2 in [TV17]). This is true at least for the state sum TQFT. Lemma 13.6 in [TV17] states that, over any commutative ring $k$, the $k$-module $|S^2|_C$ is isomorphic to $k$. (Here $C$ is a spherical fusion $k$-category.)

As for property (2), section 13.1.3 in [TV17] shows that $|S^1 \times S^2|_C = 1$.

For property (3), $|S^3|_C = (\dim(C))^{-1}$. If $C$ is spherical, then $\dim(C) = \sum_{i \in I} (\dim(i))^2$, where $I$ is a (finite) representative set of simple objects of $C$. Note that the quantum dimension is not necessarily an integer. Here, we
consider the original Turaev–Viro TQFT [TV92], at \( q = e^{\pi i/r} \), \( r \geq 3 \). Then

\[
|S^3| = -\frac{\left(e^{\pi i/r} - e^{-\pi i/r}\right)^2}{2r} = \frac{2\sin^2\left(\frac{\pi}{r}\right)}{r} \in (0, 1).
\]

Combining the results, we obtain a lower bound of the Heegaard genus. It is worth noting that Corollary 11.7 in [TV13] gives the same bound.

We use the software Regina to search for counterexamples. The confirmed ones have appeared in [BZ84].

Here is the structure of the thesis. Chapter 2 includes necessary definitions and the construction of the category needed, described in [BK01]. In Chapter 3 we construct the state sum TQFT and sketch a proof of unitarity, described in [TV17] and [TV13]. In Chapter 4 we present the estimation and the computer search.

2 Modular Category Associated to \( U_q(sl_2) \)

2.1 Definitions

Our definitions are taken from [TV17], [TV13], [Row06], and [BK01].

A \textit{monoidal category} is a category with a tensor product \( \otimes \) and an identity object \( \mathbb{1} \) satisfying axioms that guarantee that the tensor product is associative (up to specified isomorphism) and that \( \mathbb{1} \otimes X \cong X \otimes \mathbb{1} \cong X \) for any object \( X \).

Let \( \mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1}) \) and \( \mathcal{D} = (\mathcal{D}, \otimes', \mathbb{1}') \) be monoidal categories. A \textit{monoidal functor} from \( \mathcal{C} \) to \( \mathcal{D} \) is a functor \( F : \mathcal{C} \to \mathcal{D} \) endowed with a morphism \( F_0 : \mathbb{1}' \to F(\mathbb{1}) \) in \( \mathcal{D} \) and with a natural transformation

\[
F_2 = \{F_2(X,Y) : F(X) \otimes' F(Y) \to F(X \otimes Y)\}_{X,Y \in \text{Ob} (\mathcal{C})}
\]

satisfying some compatibility conditions. The monoidal functor is \textit{strong} if \( F_0 \) and \( F_2(X,Y) \) are isomorphisms for all \( X,Y \).

Let \( k \) be a commutative ring. A \textit{\( k \)-category} \( \mathcal{C} \) is a category such that all Hom sets are left \( k \)-modules and the composition of morphisms is \( k \)-bilinear. An object \( D \) of \( \mathcal{C} \) is a \textit{direct sum} of objects \( X_1, \ldots, X_n \) if there is a family of morphisms \( p_i : D \to X_i, q_i : X_i \to D \), such that

\[
\text{id}_D = \sum q_i p_i, \quad p_i q_j = \delta_{ij} \text{id}_{X_i}, \quad \forall i, j.
\]

An object \( X \) of \( \mathcal{C} \) is \textit{simple} if \( \text{End}_{\mathcal{C}}(X) \) is isomorphic to \( k \) as a \( k \)-module.

A \textit{monoidal} \( k \)-category is a \( k \)-category which is monoidal and such that monoidal product of morphisms is \( k \)-bilinear. It is \textit{semisimple} if any object is isomorphic to a direct sum of simple ones.

A monoidal category has \textit{left duality} if for any object \( X \) there is an object \( X^\ast \) (the \textit{left dual} of \( X \)) and

\[
b_X = \text{coev}_X : \mathbb{1} \to X \otimes X^\ast, \quad d_X = \text{ev}_X : X^\ast \otimes X \to \mathbb{1}
\]

such that

\[
(id_X \otimes d_X)(b_X \otimes id_X) = id_{X}, \quad (d_X \otimes id_{X^\ast})(id_{X^\ast} \otimes b_X) = id_{X^\ast}.
\]
A left rigid category is a monoidal category with a left duality. A rigid category is a monoidal category which is left rigid and right rigid. Here a right duality is similarly defined using
\[ \tilde{ev}_X : X \otimes X^* \to 1, \quad \tilde{coev}_X : 1 \to X^* \otimes X. \]

For \( f \in \text{Hom}(X, Y) \), let
\[ f^* = (d_Y \otimes \text{id}_{X^*})(\text{id}_{Y^*} \otimes f \otimes \text{id}_{X^*})(\text{id}_{Y^*} \otimes b_X). \]

A braiding in a monoidal category is a collection of natural isomorphisms
\[ c_{X,Y} : X \otimes Y \cong Y \otimes X \]
such that
\[ c_{X,Y} \otimes Z = (\text{id}_Y \otimes c_{X,Z})(c_{X,Y} \otimes \text{id}_Z), \quad c_{X,Y} \otimes Z = (c_{X,Z} \otimes \text{id}_Y)(\text{id}_X \otimes c_{Y,Z}). \]

A braiding is symmetric if for all \( X, Y \), \( c_{Y,X} c_{X,Y} = \text{id} \). In this case the category is symmetric. A braided functor between braided categories \((\mathcal{C}, c)\) and \((\mathcal{C}', c')\) is a monoidal functor \( F : \mathcal{C} \to \mathcal{C}' \) such that for all \( X, Y \in \text{Ob}(\mathcal{C}) \),
\[ F_2(Y, X) c'_{F(Y), F(X)} = F(c_{X,Y}) F_2(X, Y). \]

A symmetric functor is a braided monoidal functor between symmetric categories.

A twist in a braided monoidal category is a natural family of isomorphisms \( \theta_{X} : X \cong X \), such that
\[ \theta_{X \otimes Y} = c_{Y,X} c_{X,Y} (\theta_X \otimes \theta_Y). \]

A ribbon category is a braided monoidal category with a twist and a left duality such that \( \theta_X^* = (\theta_X)^* \).

A pivotal category \( \mathcal{C} \) is a rigid category with distinguished duality such that the induced left and right dual functors coincide as monoidal functors. The left trace of an endomorphism \( f \) of an object \( X \) is
\[ \text{tr}_l(f) = ev_X (\text{id}_{X^*} \otimes f) \tilde{coev}_X \in \text{End}_\mathcal{C}(1). \]

The right trace of \( f \) is
\[ \text{tr}_r(f) = \tilde{ev}_X (f \otimes \text{id}_{X^*}) \text{coev}_X \in \text{End}_\mathcal{C}(1). \]

The left dimension of an object \( X \) is \( \text{dim}_l(X) = \text{tr}_l(\text{id}_X) \). The right dimension is \( \text{dim}_r(X) = \text{tr}_r(\text{id}_X) \).

A spherical category is a pivotal category whose left and right traces coincide. All ribbon categories are spherical.

A pre-fusion \( k \)-category is a monoidal \( k \)-category \( \mathcal{C} \) such that there is a set \( I \) of simple objects of \( \mathcal{C} \) such that \( 1 \in I \), \( \text{Hom}_\mathcal{C}(i, j) = 0 \) for any distinct \( i, j \in I \), and every object of \( \mathcal{C} \) is a direct sum of a finite family of elements of \( I \).

A fusion \( k \)-category is a rigid pre-fusion \( k \)-category which has only a finite number of isomorphism classes of simple objects.
The dimension of a pivotal fusion $k$-category $C$ is defined by
\[
\dim(C) = \sum_{i \in I} \dim(\ell_i) \dim(\ell_i) \in k.
\]

Let $C$ be a ribbon fusion $k$-category. For any $i, j \in I$, set
\[
S_{i,j} = \text{tr}(\alpha_j \alpha_i) \in \text{End}_C(\mathbb{I}) = k.
\]

The category $C$ is said to be modular if this $S$-matrix is invertible over $k$. The modularity of $C$ implies that $\dim(C)$ is invertible in $k$.

A Hermitian fusion category is a spherical fusion category $C$ over $C$ endowed with antilinear homomorphisms
\[
\{f \in \text{Hom}_C(X, Y) \mapsto f^\dagger \in \text{Hom}_C(Y, X)\}_{X,Y \in \text{Ob}(C)}
\]
such that $\tilde{f} = f, g f = f \bar{g}, f \otimes g = f \otimes \bar{g}, e\bar{v}_X = \bar{c}e\bar{v}_X, \bar{c}e\bar{v}_X = \bar{e}v_X$.

A unitary fusion category is a Hermitian fusion category $C$ such that $\text{tr}(f \tilde{f}) > 0$ for any non-zero morphism $f$ in $C$. The dimension of a unitary fusion category is a positive real number.

2.2 Representations of $U_q(\mathfrak{sl}_2)$

We will focus on the case where $q = \mathfrak{sl}_2, q = e^{\pi i/r}, r \geq 3$. We follow [BK01].

Definition 1. A Hopf algebra $A$ over a field $k$ is a bialgebra with an algebra anti-isomorphism $\gamma : A \rightarrow A$, called the antipode, satisfying $\mu(i \delta \gamma) \Delta = \epsilon = \mu(\gamma \otimes i \delta)$.

The category of finite-dimensional representations of $A$ is a rigid monoidal category.

Let $g$ be a finite-dimensional simple Lie algebra over $C$, and $\mathfrak{h}$ be its Cartan subalgebra. Let $\Delta \subset \mathfrak{h}^*$ be the root system, $\Pi = \{\alpha_1, \ldots, \alpha_r\} \subset \Delta$ be the set of simple roots, $h_i = \alpha_i^\vee \in \mathfrak{h}$ be the coroots, and $A = (a_{ij})$ be the Cartan matrix, $a_{ij} = (\alpha_i^\vee, \alpha_j)$.

Let $P \subset \mathfrak{h}^*$ be the weight lattice, $Q \subset \mathfrak{h}^*$ be the root lattice, and $Q^\vee \subset \mathfrak{h}$ be the dual root lattice (coroot lattice). Let $\langle \cdot, \cdot \rangle$ be an invariant bilinear form on $g$ normalized by $\langle (\alpha, \alpha) \rangle = 2$ for short roots $\alpha$. Then $d_i := \langle (\alpha_i, \alpha_i) \rangle / 2 \in \mathbb{Z}$ for all $i = 1, \ldots, r$. Let $C_q$ be the field $C$ $(q^{1/2(1/2)} / q)$ where $q$ is a formal variable.

Definition 2. The quantum group $U_q(\mathfrak{g})$ is the associative algebra over $C_q$ with generators $e_i, f_i (i = 1, \ldots, r), q^h (h \in Q^\vee)$ and relations
\[
q^{h'} q^{h''} = q^{h' + h''}, q^0 = 1, \quad h', h'' \in Q^\vee,
\]
\[
q^h e_i q^{-h} = q^{(h, \alpha_i)} e_i, q^h f_i q^{-h} = q^{-(h, \alpha_i)} f_i, [e_i, f_j] = \delta_{ij} \frac{q^{d_i h_i} - q^{-d_i h_i}}{q^{d_i} - q^{-d_i}},
\]
\[
\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix} e_i^{1-a_{ij}-k} e_j e_i^k = 0, \quad i \neq j,
\]
\[
\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix} f_i^{1-a_{ij}-k} f_j f_i^k = 0, \quad i \neq j.
\]
where 
\[
\begin{bmatrix}
 n \\
 k
\end{bmatrix}_i := \frac{[n]_i!}{[k]_i! [n-k]_i!}, \quad [n]_i! := [1]_i [2]_i \cdots [n]_i, \quad [a]_i := \frac{q^{d_i n} - q^{-d_i n}}{q^{d_i} - q^{-d_i}}.
\]

This is a Hopf algebra with
\[
\begin{align*}
\Delta (q^h) &= q^h \otimes q^h, \\
\Delta (e_i) &= e_i \otimes q^{d_i h_i} + 1 \otimes e_i, \\
\Delta (f_i) &= f_i \otimes 1 + q^{-d_i h_i} \otimes f_i, \\
\epsilon (q^h) &= 1, \quad \epsilon (e_i) = \epsilon (f_i) = 0, \\
\gamma (q^h) &= q^{-h_i}, \quad \gamma (e_i) = -e_i q^{-d_i h_i}, \quad \gamma (f_i) = -q^{-d_i h_i} f_i.
\end{align*}
\]

Let \( A = \mathbb{Z} \left[ q^{\pm 1/\langle P / Q \rangle} \right] \) and let \( U_q (g) \) be the \( A \)-subalgebra of \( U_q (g) \) generated by the elements
\[
e_i^{(n)} = \frac{q^n}{[n]_i!}, \quad f_i^{(n)} = \frac{f^n}{[n]_i!}, \quad q^h, \quad n = 1, 2, \ldots, i = 1, \ldots, r, h \in Q^+.
\]

Fix \( x \in \mathbb{C}^* \) and consider \( C \) as an \( A \)-module via the homomorphism
\[
A \to C, \quad q^a \mapsto e^{a x}, \quad m := \max d_i.
\]

Set
\[
U_q (g) |_{q = e^{\frac{m x}{n}}} := U_q (g) \otimes_{A} C.
\]

We abbreviate \( U_q (g) |_{q = e^{\frac{m x}{n}}} \) to \( U_q (g) \).

The category \( C (g, x) \) of finite-dimensional representations of \( U_q (g) \) over \( C \) possessing a weight decomposition
\[
V = \bigoplus_{\lambda} V^\lambda, \quad q^h |_{V^\lambda} = q^{(h, \lambda)} \text{id}_{V^\lambda},
\]

\[
e_i^{(n)} (V^\lambda) \subset V^{\lambda + n \alpha_i}, \quad f_i^{(n)} (V^\lambda) \subset V^{\lambda - n \alpha_i},
\]
is a ribbon category.

The Weyl modules of \( U_q (\mathfrak{sl}_2) \) are \( V_n = \bigoplus_{i=0}^n \mathbb{C} v_i, \quad n \in \mathbb{Z}_+, \) where \( v_0 \) is the highest weight vector and \( v_i = f^{(i)} v_0 \). The action is
\[
q^h v_i = q^{n-2i} v_i, \quad e v_i = [n-i+1] v_{i-1}, \quad f v_i = [i+1] v_{i+1}.
\]

The module \( V_n \) is irreducible for \( n < x \), and \( \dim_q V_n = [n+1] = 0 \) if and only if \( x \) divides \( n+1 \). Hence for \( 0 \leq n \leq x-2, \) \( V_n \) is irreducible and \( \dim_q V_n \neq 0 \). Correspondingly, let \( C := \{ 0, \frac{1}{x}, \ldots, \frac{x-1}{x} \} \) be the set of weights, where \( g = s_{12}, \quad q = e^{\pi i / x}, \quad x \geq 3 \). From now on we index the Weyl modules by their weights in \( \frac{1}{x} \mathbb{Z} \).

A module \( T \) over \( U_q (\mathfrak{sl}_2) \) is called tilting if both \( T \) and \( T^* \) have composition series with factors Weyl modules. Let \( T \) be the full subcategory of \( C (\mathfrak{sl}_2, x) \) consisting of all tilting modules. If \( \lambda \in C \) then the module \( V_\lambda \) is tilting. It is shown in [HAP95] that the category of tilting modules \( T \) is closed under \( * , \otimes, \oplus \) and direct summands. It is a ribbon category.

A tilting module \( T \) is called negligible if \( \text{tr}_q f = 0 \) for any \( f \in \text{End} T \). A morphism \( f : T_1 \to T_2 \) is called negligible if \( \text{tr}_q (f g) = 0 \) for all \( g : T_2 \to T_1 \).
Let \( C^{\text{int}} := C^{\text{int}}(\mathfrak{sl}_2, \kappa) \) be the category with objects tilting modules and morphisms
\[
\Hom_{C^{\text{int}}}(V, W) = \Hom_{\mathcal{T}}(V, W)/\text{negligible morphisms}.
\]
Then \( C^{\text{int}} \) is a ribbon category. Any object \( V \) in \( C^{\text{int}} \) is isomorphic to \( \bigoplus_{\lambda \in \mathcal{C}} n_{\lambda} V_{\lambda} \).
\( C^{\text{int}} \) is a semisimple abelian category. Indeed, \( C^{\text{int}} \) is a modular tensor category with simple objects \( V_{\lambda}(\lambda \in \mathcal{C}) \).

2.3 Unitarity of the category

For a survey on the general case, see [Row06]. We will focus on the case where \( \mathfrak{g} = \mathfrak{sl}_2, q = e^{\pi i/r}, r \geq 3 \).

The invertibility of \( S \)-matrix is shown in [TW93]. Their work is based on the notion of quasimodular Hopf algebra, which is a weakened version of modular Hopf algebra introduced in the paper [RT91].

In [KJ96], Hermitian structure on the category is defined in Section 4. Note that the notation in [KJ96] is different. In [Wen98], positivity is proved by Wenzl. Xu [Xu98] independently showed some of the cases covered by Wenzl.

**Theorem 2** ([Wen98], [Xu98]). The categories \( C^{\text{int}}(\mathfrak{sl}_2, \kappa) \) are unitary when \( q = e^{\pi i/\kappa} \).

3 Turaev–Viro TQFT

3.1 TQFT

Manifolds considered are assumed to be oriented.

An \( n \)-dimensional cobordism is a quadruple \((M, \Sigma_0, \Sigma_1, h)\), where \( M \) is a compact oriented \( n \)-manifold, \( \Sigma_0, \Sigma_1 \) are closed oriented \((n-1)\)-manifolds, and \( h : (-\Sigma_0) \cup \Sigma_1 \rightarrow \partial M \) is an orientation preserving homeomorphism. Two cobordisms \((M, \Sigma_0, \Sigma_1, h)\) and \((M', \Sigma_0, \Sigma_1, h')\) between \( \Sigma_0 \) and \( \Sigma_1 \) are homeomorphic if there is an orientation preserving homeomorphism \( \alpha : M \rightarrow M' \) such that \( h' = \alpha h \).

Let \( \text{Cob}_3 \) be the category with objects closed oriented 2-manifolds and morphisms cobordisms between them considered up to homeomorphisms. It is a symmetric monoidal category, where the monoidal product is the ordered disjoint union of manifolds and cobordisms. Each closed oriented 3-manifold represents a morphism \( \emptyset \rightarrow \emptyset \).

Let \( \text{Mod}_k \) be the symmetric monoidal category of \( k \)-modules.

**Definition 3.** A 3-dimensional Topological Quantum Field Theory (TQFT) is a symmetric strong monoidal functor
\[
Z : \text{Cob}_3 \rightarrow \text{Mod}_k.
\]

A conjugation in a monoidal category \( \mathcal{C} \) is a family of maps
\[
\{ f \in \Hom_{\mathcal{C}}(X, Y) \rightarrow \bar{f} \in \Hom_{\mathcal{C}}(Y, X) \}_{X, Y \in \text{Ob}(\mathcal{C})}
\]
compatible with the associativity and unitality constraints of \( \mathcal{C} \), such that for all morphisms \( f, g \) in \( \mathcal{C} \) and all \( X \in \text{Ob}(\mathcal{C}) \)
\[
\overline{gf} = \bar{g} \bar{f}, \quad \overline{id_X} = id_X, \quad \bar{f} = f, \quad \bar{f} \otimes g = \bar{f} \otimes \bar{g}.
\]
The category $\text{Cob}_3$ has a conjugation

$$(M, \Sigma_0, \Sigma_1, h : (-\Sigma_0) \sqcup \Sigma_1 \to \partial M)$$

$$\mapsto (-M, \Sigma_1, \Sigma_0, h P : (-\Sigma_1) \sqcup \Sigma_0 \to -\partial M = \partial (-M)),$$

where $P$ is the orientation reversing permutation homeomorphism

$$(-\Sigma_1) \sqcup \Sigma_0 \simeq (-\Sigma_0) \sqcup \Sigma_1.$$

Let $\text{Hilb}$ be the category of finite-dimensional Hilbert spaces over $\mathbb{C}$ and $\mathbb{C}$-linear homomorphisms between them. This is a symmetric ribbon $\mathbb{C}$-category. The conjugation is given by assigning to every morphism its hermitian adjoint.

**Definition 4.** A unitary 3-dimensional TQFT is a symmetric strong monoidal functor

$$Z : \text{Cob}_3 \to \text{Hilb}$$

commuting with the conjugations in $\text{Cob}_3$ and in $\text{Hilb}$.

Note that there is a forgetful functor $\text{Hilb} \to \text{Mod}_C$. Composing such a functor $Z$ with the forgetful functor, we obtain a 3-dimensional TQFT. We call $Z$ a unitary lift of the latter TQFT.

The category $\text{Cob}_3$ has a subcategory isomorphic to the category of closed oriented surfaces and isotopy classes of orientation-preserving homeomorphisms. Any such homeomorphism $f : \Sigma \to \Sigma'$ determines a morphism

$$(C = \Sigma' \times [0, 1], \Sigma, \Sigma', h : (-\Sigma) \sqcup \Sigma' \simeq \partial C)$$

in $\text{Cob}_3$, where $h(x) = (f(x), 0)$ for $x \in \Sigma$ and $h(x') = (x', 1)$ for $x' \in \Sigma'$. Restricting a TQFT $Z : \text{Cob}_3 \to \text{Mod}_k$ to this subcategory, we obtain the action of homeomorphisms, and in particular the action of mapping class groups. We denote the induced map by $f_\# : Z(\Sigma) \to Z(\Sigma')$. It follows that $(f^{-1})_\# = (f_\#)^{-1}$.

If, in addition, $Z$ is unitary, then

**Proposition 3.** $\langle f_\# x, f_\# y \rangle = \langle x, y \rangle$.

**Proof.** The adjoint of $f_\#$ is the same as the inverse of $f_\#$. Indeed there is a homeomorphism from the conjugation of $(C = \Sigma' \times [0, 1], \Sigma, \Sigma', h : (-\Sigma) \sqcup \Sigma' \simeq \partial C)$ to the cobordism associated to $f^{-1}$.

Theorem 10.3.1 in [Tur10] is a similar result.

### 3.2 Construction of Turaev–Viro state sum TQFT

We sketch a construction following [TV17] and [TV13].

Fix a spherical fusion $k$-category $\mathcal{C}$ of invertible dimension. Fix a (finite) representative set $I$ of simple objects of $\mathcal{C}$.
3.2.1 Colored graphs in surfaces

A graph $G$ in an oriented surface $\Sigma$ is a finite graph without isolated vertices embedded in $\Sigma$. A graph $G$ is $\mathcal{C}$-colored, if each edge of $G$ is oriented and endowed with an object of $\mathcal{C}$ called the color of the edge.

For a vertex $v$ of a $\mathcal{C}$-colored graph $G$, the set $E_v$ of half-edges of $G$ incident to $v$ with cyclic order induced by the opposite orientation of $\Sigma$, and the color $c = c_v : E_v \to \text{Ob}(\mathcal{C})$, the orientation $e = e_v : E_v \to \{+,-\}$, together form a cyclic $\mathcal{C}$-set. The sign is $+$ if $e \in E_v$ is oriented towards $v$.

Starting from $e = e_1 \in E_v$, let

$$H_v = \text{Hom}_\mathcal{C} \left( \mathbb{1}, c(e_1)^{e(e_1)} \otimes \cdots \otimes c(e_n)^{e(e_n)} \right),$$

where $X^+ = X$, $X^- = X^*$.

This module can be made into the symmetrized multiplicity module $H(E_v)$, independent of the starting point chosen.

Set $H_v(G) = H(E_v)$ and

$$H(G) = \bigoplus_v H_v(G).$$

Orient the plane $\mathbb{R}^2$ counterclockwise. Let $G$ be a $\mathcal{C}$-colored graph in $\mathbb{R}^2$. Pick any $\alpha_v \in H_v(G)$ and replace $v$ by a box colored with the corresponding element in the Hom set. This transforms $G$ into a planar diagram which determines, by the graphical calculus, an element $f_\mathcal{C}(G) (\otimes_v \alpha_v)$ of $\text{End}_\mathcal{C}(\mathbb{1}) = \mathbb{k}$. See [Tur10], [TV17] or [TV13] for an introduction to the graphical calculus. This procedure defines a vector

$$f_\mathcal{C}(G) \in H(G)^* = \text{Hom}_\mathbb{k}(H(G), \mathbb{k}).$$

As $\mathcal{C}$ is spherical, the invariant $f_\mathcal{C}$ generalizes to graphs in the 2-sphere.

3.2.2 State sums on skeletons of 3-manifolds

A 2-polyhedron is a compact topological space that can be triangulated using a finite number of simplices of dimension $\leq 2$ so that all 0-simplices and 1-simplices are faces of 2-simplices. For a 2-polyhedron $P$, let $\text{Int}(P)$ be the subspace consisting of all points having a neighborhood homeomorphic to $\mathbb{R}^2$.

A stratification of a 2-polyhedron $P$ is a graph $G$ embedded in $P$ so that $P \setminus \text{Int}(P) \subset G$. The vertices and edges of $P^{(1)} := G$ are called respectively the vertices and edges of $P$.

Cutting a stratified 2-polyhedron $P$ along the graph $P^{(1)} \subset P$, we obtain a compact surface $\tilde{P}$ with interior $P \setminus P^{(1)}$. The (connected) components of $\tilde{P}$ are called the regions of $P$. We let $\text{Reg}(P)$ be the (finite) set of all regions of $P$.

A branch of a stratified 2-polyhedron $P$ at a vertex $x$ is a germ at $x$ of an adjacent region. A branch of $P$ at an edge $e$ is a germ at $e$ of an adjacent region. The set of branches of $P$ at $e$ is denoted $P_e$. The number of elements of $P_e$ is called the valence of $e$.

The edges of $P$ of valence 1 and their vertices form a graph called the boundary of $P$ and denoted $\partial P$. We say that $P$ is orientable (resp. oriented) if all regions of $P$ are orientable (resp. oriented).
A skeleton of a closed 3-manifold $M$ is an oriented stratified 2-polyhedron $P \subset M$ such that $\partial P = \emptyset$ and $M \setminus P$ is a disjoint union of open 3-balls.

Any vertex $x$ of a skeleton $P \subset M$ has a closed ball neighborhood $B_x \subset M$ such that $\Gamma_x = P \cap \partial B_x$ is a finite non-empty graph and $P \cap B_x$ is the cone over $\Gamma_x$. The pair $(\partial B_x, \Gamma_x)$ is the link of $x$ in $(M, P)$. If $M$ is oriented, then we endow $\partial B_x$ with orientation induced by that of $M$ restricted to $M \setminus \operatorname{Int} (B_x)$.

Let $M$ be a closed oriented 3-manifold. Fix a skeleton $P$ of $M$. For a map $c : \operatorname{Reg}(P) \to I$ and an oriented edge $e$ of $P$, we define a $k$-module
\[
H_c(e) = H(P_e),
\]
where $P_e$ is the cyclic $C$-set of branches of $P$ at $e$. If $e^{op}$ is the same edge with opposite orientation, then there is a contraction
\[
* : H_c(e)^* \otimes H_c(e^{op})^* \to k.
\]

The link of a vertex $x \in P$ determines a $C$-colored graph $\Gamma_x$ in $\partial B_x \cong S^2$. Hence there is a tensor $\mathbb{F}_C(\Gamma_x) \in H_c(\Gamma_x)^*$. Note that $H_c(\Gamma_x) = \otimes_v H_c(e)$, where $e$ runs over all edges of $P$ incident to $x$ and oriented away from $x$. Hence $\otimes_v \mathbb{F}_C(\Gamma_x) \in \otimes_v H_c(e)^*$, where $e$ runs over all oriented edges of $P$. Set $*P = \otimes_e * : \otimes_v H_c(e)^* \to k$.

Define
\[
|M|_C = \left(\dim(C)\right)^{-|P|} \sum_x \left( \prod_{r \in \operatorname{Reg}(P)} \left( \dim(c(r)) \chi(r) \right) *P \left( \otimes_v \mathbb{F}_C(\Gamma_x) \right) \right) \in k,
\]
where $|P|$ is the number of components of $M \setminus P$, $c$ runs over all maps $\operatorname{Reg}(P) \to I$, and $\chi(r)$ is the Euler characteristic of $r$.

**Theorem 4** ([TV13], [TV17]). The invariant $|M|_C$ is independent of $P$ chosen.

### 3.2.3 The state sum TQFT

Let $M$ be a compact 3-manifold with boundary. Let $G$ be an oriented graph in $\partial M$ such that all vertices of $G$ have valence $\geq 2$. A skeleton of the pair $(M, G)$ is an oriented stratified 2-polyhedron $P \subset M$ satisfying certain requirements including $P \cap \partial M = \partial P = G$.

For any compact oriented 3-manifold $M$ and any $I$-colored graph $G$ in $\partial M$, we define a topological invariant $|M, G| \in k$ as follows. Pick a skeleton $P \subset M$ of the pair $(M, G)$. Pick a map $c : \operatorname{Reg}(P) \to I$ extending the coloring of $G$.

Let $E_0$ be the set of oriented edges of $P$ with both endpoints in $\operatorname{Int}(M)$, and let $E_\partial$ be the set of edges of $P$ with exactly one endpoint in $\partial M$ oriented towards this endpoint. We have
\[
\otimes_{e \in E_0} H_c(e)^* = \otimes_{e} H_v (G^{op}; -\partial M)^* = H (G^{op}; -\partial M)^*,
\]
where $G^{op}$ in $-\Sigma$ is obtained by reversing orientation in all edges of $G$ and in $\Sigma$.

There is a contraction homomorphism
\[
*P : \otimes_{e \in E_0 \cup E_\partial} H_c(e)^* \to \otimes_{e \in E_0} H_c(e)^* = H (G^{op}; -\partial M)^*.
\]
The tensor product $\otimes_x F_C(\Gamma_x)$ over all vertices $x$ of $P$ lying in $\text{Int}(M)$ is a vector in $\otimes_{e \in E_0 \cup E_2} H_e(e)$. Set

$$[M, G] = (\dim(C))^{-|P|} \sum_r \left( \prod_{r \in \text{Reg}(P)} (\dim c(r))^{\chi(r)} \right) * P (\otimes_x F_C(\Gamma_x)),$$

where $|P|$ is the number of components of $M \setminus P$, $c$ runs over all maps $\text{Reg}(P) \to I$ extending the coloring of $G$.

**Theorem 5 ([TV13], [TV17]).** $[M, G] \in H(G^{op}; - \partial M)^*$ does not depend on the choice of $P$.

Let $\Sigma_0, \Sigma_1$ be closed oriented surfaces, and let $f : \Sigma_0 \to \Sigma_1$ be a morphism in $\text{Cob}_3$ represented by a pair $(M, h)$. For $I$-colored graphs $G_0 \subset \Sigma_0$ and $G_1 \subset \Sigma_1$,

$$[M, h (G_0^{op} \cup G_1)] \in H (h (G_0^{op} \cup G_1); \partial M) \simeq \text{Hom}_k (H (G_0), H (G_1)).$$

Set

$$[f, G_0, G_1] = \frac{(\dim(C)^{\Sigma_1 \setminus G_1})}{\dim(G_1)} [M, h (G_0^{op} \cup G_1)] : H (G_0) \to H (G_1),$$

where $|\Sigma_1 \setminus G_1|$ is the number of components of $\Sigma_1 \setminus G_1$ and $\dim(G_1)$ is the product over all edges of $G_1$ of the dimensions of their colors.

A skeleton of a closed surface $\Sigma$ is an oriented graph $G$ embedded in $\Sigma$ such that all vertices of $G$ have valence $\geq 2$ and all components of $\Sigma \setminus G$ are open disks. We let $\text{col}(G)$ be the set of all maps from the set of edges of $G$ to $I$.

For a closed oriented surface $\Sigma$ and a skeleton $G \subset \Sigma$, consider the $k$-module

$$[\Sigma, G] = \bigoplus_{c \in \text{col}(G)} H((G, c); \Sigma).$$

For a morphism $f : \Sigma_0 \to \Sigma_1$ in $\text{Cob}_3$ and skeletons $G_0 \subset \Sigma_0$ and $G_1 \subset \Sigma_1$, consider the $k$-linear homomorphism

$$[f, G_0, G_1] : [\Sigma_0, G_0] \to [\Sigma_1, G_1]$$

whose restriction to every summand $H ((G_0, c_0); \Sigma_0)$ of $[\Sigma_0, G_0]$ is equal to $\sum_{c \in \text{col}(G_1)} [f, (G_0, c_0), (G_1, c)].$

For any skeletons $G, G'$ of a closed oriented surface $\Sigma$, set

$$p (G, G') = [\text{id}_G; G, G'] : [\Sigma, G] \to [\Sigma, G'],$$

$$[\Sigma, G] = \text{Im}(p(G, G)).$$

The $k$-linear homomorphism $p (G, G')$ restricts to an isomorphism $[\Sigma, G] \to [\Sigma, G']$ and the family $\left([\Sigma, G]_G, \{p (G, G')\}_{G,G'}\right)$ is a projective system. The projective limit

$$[\Sigma] = \lim_{\leftarrow} [\Sigma, G]$$

is a projective system.
is a $k$-module depending only on $\Sigma$.

We associate with each morphism $f : \Sigma_0 \to \Sigma_1$ in $\text{Cob}_3$ a homomorphism $|f| : |\Sigma_0| \to |\Sigma_1|$. For each skeleton $G$ of $\Sigma$, we have a $k$-linear cone isomorphism $\tau_G : |\Sigma| \to |\Sigma, G|$. Pick any skeletons $G_0 \subset \Sigma_0$ and $G_1 \subset \Sigma_1$. Set

$$|f| = \tau_{G_1}^{-1} \circ |f, G_0, G_1|^\circ \circ \tau_{G_0} : |\Sigma_0| \to |\Sigma_1|.$$ 

This homomorphism does not depend on the choice of $G_0$ and $G_1$.

**Theorem 6** ([TV13], [TV17]). The functor $| \cdot | = | \cdot |_C$ is a 3-dimensional TQFT.

### 3.3 Unitarity

Let $k$ be an algebraically closed field. Let $C$ be a unitary fusion category (over $C$). Then $\dim(C) \neq 0$.

**Theorem 7** (Theorem G.1 in [TV13]. Corollary 11.6 in [TV13]). The state sum TQFT $| \cdot |_C$ has a unitary lift.

The proof is involved. Recall the construction of the Reshetikhin–Turaev extended TQFT $\tau_B$ associated to a modular category $B$ over $k$ equipped with a distinguished square root of $\dim(B)$, described in detail in [Tur10]. In this case unitarity is proved in [Tur10], Chapter IV, section 11.

The center $Z(C)$ of $C$ has a subcategory the unitary center $Z^u(C)$ of $C$, such that the inclusion is an equivalence of categories. Theorem 17.1 in [TV17] implies that

$$| \cdot |_C \simeq \tau_{Z(C)} \simeq \tau_{Z^u(C)}.$$ 

Since $Z^u(C)$ is an anomaly-free unitary modular category, these TQFTs are unitary.

In this section we introduce the idea of the proof, following [TV13].

### 3.3.1 Extending to graph TQFTs

The goal of this subsection is Theorem 11.1 and Theorem 11.2 in [TV13]:

**Theorem 8** ([TV13]). Let $C$ be a spherical fusion category over an algebraically closed field $k$ such that $\dim(C) \neq 0$. Then $|M|_C = \tau_{Z(C)}(M)$ for any closed oriented 3-manifold $M$. Moreover the TQFTs $| \cdot |_C$ and $\tau_{Z(C)}$ are isomorphic.

Let $C$ be a monoidal category. A **half braiding** is a pair $(A, \sigma)$, where $A \in \text{Ob}(C)$ and $\sigma = \{\sigma_X : A \otimes X \to X \otimes A\}_{X \in \text{Ob}(C)}$ is a natural isomorphism such that $\sigma_{X \otimes Y} = (\text{id}_X \otimes \sigma_Y)(\sigma_X \otimes \text{id}_Y)$ for all $X, Y \in \text{Ob}(C)$.

The **center** of $C$ is the braided category $Z(C)$ whose objects are half braidings. A morphism $(A, \sigma) \to (A', \sigma')$ in $Z(C)$ is a morphism $f : A \to A'$ such that $(\text{id}_X \otimes f)\sigma_X = \sigma'_X(f \otimes \text{id}_X)$ for all $X \in \text{Ob}(C)$. There is a forgetful functor.

The following theorem is Theorem 1.2 and Proposition 5.18 in [Mü03].

**Theorem 9** ([Mü03]). Let $C$ be a spherical fusion $k$-category such that $\dim(C) \neq 0$. Then $Z(C)$ is an anomaly free modular category with $\Delta_+ = \Delta_- = \dim(C)$, $\dim(Z(C)) = \Delta_+ \Delta_- = (\dim(C))^2$. 

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Hence it makes sense to talk about the Reshetikhin–Turaev extended TQFT.

It is necessary to extend the definition of a TQFT. For any category $B$, we define a category $\mathcal{L}_B$ of 3-cobordisms with $B$-colored framed oriented links inside. The links may be empty so that the category $\text{Cob}_3$ is a subcategory of $\mathcal{L}_B$.

By a link TQFT we mean a symmetric monoidal functor $Z : \mathcal{L}_B \to \text{Mod}_k$.

For a category $B$, we define a category $\mathcal{G}_B$ of 3-cobordisms with $B$-colored ribbon graphs inside, see [Tur10]. Here we consider only ribbon graphs disjoint from the bases of cobordisms.

By a graph TQFT we mean a symmetric monoidal functor $\mathcal{G}_B \to \text{Mod}_k$.

The TQFT $\tau_{\mathcal{Z}(\mathcal{C})}$ extends to a graph TQFT still denoted $\tau_{\mathcal{Z}(\mathcal{C})}$, see [Tur10]. This TQFT is non-degenerate, i.e. the vector space $\tau_{\mathcal{Z}(\mathcal{C})}(\Sigma)$ is generated by the vectors $\tau_{\mathcal{Z}(\mathcal{C})}(M, \emptyset, \Sigma)(1)$, where $M$ runs over all compact oriented 3-manifolds with $\mathcal{Z}(\mathcal{C})$-colored ribbon graphs inside and with $\partial M = \Sigma$.

The TQFT $| \cdot |_C$ also can be extended to a graph TQFT. We discuss the extension to a link TQFT, which is similar.

Let $M$ be a compact oriented 3-manifold, and $P$ be a skeleton. A link in $M$ can be presented by possibly intersecting circles in $P$ in general position, called the link diagram. The underlying 4-valent graph may be added to the 1-skeleton of $P$. An enriched link diagram in $P$ is a link diagram with half integers added to record the framing. Two enriched link diagrams in $P$ represent isotopic framed links if and only if these diagrams may be related by a finite sequence of certain moves and ambient isotopies in $P$.

Consider a pair $(M, L)$, where $M$ is a compact oriented 3-manifold (with empty graph $G = \emptyset$ in the boundary) and $L = L_1 \sqcup \cdots \sqcup L_N \subset \text{Int}(M)$ is an oriented framed link whose all components are non-trivial and colored by simple objects $J_1, \ldots, J_N$ of $\mathcal{Z}(\mathcal{C})$. We define $|M, L|_C \in \mathbb{C}$ as follows. Fix a (finite) representative set $I$ of simple objects of $\mathcal{C}$. Pick a special skeleton (see [TV13]) $P$ of $M$ and an oriented enriched link diagram $d$ in $P$ representing $L$. After adding $d$ to $P(1)$, we obtain regions.

Pick distinguished square root $\nu_{J_q} \in \mathbb{C}$ of the twist scalar $v_{J_q}$ of $J_q$. Let $n_q \in \frac{1}{2} \mathbb{Z}$ be the pre-twist of the loop of $d$ representing $L_q$. Set

$$|M, L|_C = (\dim(\mathcal{C}))^{-|P|} \prod_{q=1}^N \nu_{J_q}^{2n_q} \sum_c \prod_{r \in \text{Reg}(d)} (\dim c(r))^{\chi(r)} *_{P} (\otimes_{x} |x|_{c}).$$

This is a topological invariant of the pair $(M, L)$. It extends to the case where links may have trivial components.

It is shown in Theorem 1.2 in [Mü03] that here $\mathcal{Z}(\mathcal{C})$ is semisimple with finitely many simple objects. The invariant extends by linearity to arbitrary colors of the components of $L$.

We extend the TQFT $| \cdot |_C : \text{Cob}_3 \to \text{Mod}_k$ to a link TQFT $\mathcal{L}_{\mathcal{Z}(\mathcal{C})} \to \text{Mod}_k$, and moreover a graph TQFT.

3.3.2 $|M, R|_C = \tau_{\mathcal{Z}(\mathcal{C})}(M, R)$

For a closed (connected) oriented 3-manifold $M$ with $\mathcal{Z}(\mathcal{C})$-colored ribbon graph $R$ without free ends inside, we claim

Lemma 10. $|M, R|_C = \tau_{\mathcal{Z}(\mathcal{C})}(M, R)$. 

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Present $M$ by surgery on $S^3$ along a framed oriented link $L$. Pick a closed regular neighborhood $U$ of $L$, and set $E = S^3 \setminus \text{Int}(U)$. We consider $R \subset E$.

Pick a homeomorphism

$$f_L : (S^1 \times S^1)^{\sqcup n} \to \partial U = -\partial E$$

which carries the $q$-th copy of $S^1 \times \{pt\}$ onto a positively oriented meridian of $L_q$ and carries the $q$-th copy of $\{pt\} \times S^1$ onto a positively oriented longitude of $L_q$ determined by the framing.

Define $V = (- (S^1 \times D^2))^{\sqcup n}$, $W = (D^2 \times S^1)^{\sqcup n}$. Using $f_L$, the gluing of $W$ to $E$ yields $S^3$, while the gluing of $V$ to $E$ yields $M$. The torus vector of a TQFT $Z$ is the vector $w = w(Z) = Z(V)(1_k) \in Z(\partial V) = Z(S^1 \times S^1)$ where $n = 1$.

There are morphisms

$$Z(f_L) = (f_L)_# : Z((S^1 \times S^1)^{\sqcup n}) \to Z(-\partial E),$$

$$Z(E, R) : Z(-\partial E) \to k.$$ 

Define $Z_R$ to be the composition. Hence

$$Z(M, R) = Z_R(w^{\otimes n}).$$

We fix a representative set $J$ of simple objects of $Z(C)$. For $X \in \text{Ob}(Z(C))$, let

$$K_X = \{0\} \times S^1 \subset \text{Int}(D^2 \times S^1)$$

be the $Z(C)$-colored knot with orientation induced by that of $S^1$, constant framing, and color $X$. Set

$$[X] = Z(D^2 \times S^1, K_X)(1_k) \in Z(S^1 \times S^1).$$

Indeed,

**Lemma 11.** The vectors $\{[j]\}_{j \in J}$ form a basis.

One can explicitly expand $w$ in terms of this basis. To calculate $[M, R]_C$, we need $Z_R(w^{\otimes n})$, and it suffices to calculate

$$Z_R([j_1] \otimes_k \cdots \otimes_k [j_n]) = Z(S^3, T_{j_1, \ldots, j_n}) = (\dim(C))^{-1} \langle T_{j_1, \ldots, j_n} \rangle_{Z(C)}$$

$$\in \text{End}_{Z(C)}(1_{Z(C)}) = \text{End}_C(1) = k,$$

where $j_1, \ldots, j_n \in J$, $T_{j_1, \ldots, j_n}$ is the union of $R$ with the link $L$ whose components $L_1, \ldots, L_n$ are colored with $j_1, \ldots, j_n$, and $\langle \cdot \rangle_{Z(C)}$ is the graphical calculus, see Theorem 16.1 in [TV17].

Now Lemma 10 follows from a complete expansion.
3.3.3 Proof of Theorem 8

We claim that there is a monoidal isomorphism of the functors $\tau_Z(C)$ and $\cdot |_C$ from $G_Z(C)$ to $\text{Mod}_k$.

Indeed there is a general criterion establishing isomorphism of two (generalized) TQFTs, see [Tur10], Chapter III, Section 3. If

1. at least one of the TQFTs is non-degenerate,
2. the values of these TQFTs on cobordisms with empty bases are equal, and
3. the vector spaces associated with any object have equal dimensions,

then these TQFTs are isomorphic.

The graph TQFT $\tau_Z(C)$ is non-degenerate. That $|M,K|_C = \tau_Z(C)(M,K)$ for any $Z(C)$-colored ribbon graph $K$ in a closed oriented 3-manifold $M$ is proven. The equality of dimensions is provided by the following lemma. This completes the proof of the theorem.

The following lemma is Lemma 11.3 in [TV13].

**Lemma 12 ([TV13])**. The vector spaces $|\Sigma|_C$ and $\tau_Z(C)(\Sigma)$ associated with any closed connected oriented surface $\Sigma$ have equal dimensions.

Its proof involves more tools like coends and Hopf monads, see [TV13].

3.3.4 Proof of unitarity

A half braiding $(A, \sigma)$ of $C$ is unitary if $\sigma_X = \sigma_X^{-1}$ for all $X \in \text{Ob}(C)$. The unitary center $Z^u(C)$ of $C$ is the full subcategory of $Z(C)$ formed by the unitary half braidings.

The inclusion $Z^u(C) \subset Z(C)$ is a braided equivalence, see Theorem 6.4 in [Mü03]. Therefore $Z^u(C)$ is also modular. The induced conjugation makes $Z^u(C)$ a unitary modular category, and hence the TQFT $\tau_{Z^u(C)}$ is unitary.

Now there are isomorphisms of TQFTs

$$\tau_{Z^u(C)} \simeq \tau_{Z(C)} \simeq | \cdot |_C.$$  

The first one is induced by the braided equivalence $Z^u(C) \simeq Z(C)$. The unitary structure of $\tau_{Z^u(C)}$ is transported to $| \cdot |_C$ via the isomorphism.

By Corollary 11.6, Theorem 11.2, and Theorem 11.5 of [Tur10],

**Corollary 13 ([TV13])**. If $C$ is a unitary fusion category, then

$$||M|_C| \leq (\text{dim}(C))^{g(M)-1}$$

for any closed oriented 3-manifold $M$, where $g(M)$ is the Heegaard genus of $M$.

In the next chapter we give another proof of a similar theorem.

4 Counterexamples to Rank versus Genus Conjecture

4.1 Estimating the Heegaard genus

In this section we estimate the Heegaard genus, using arguments similar to Theorem 2.2 in [Gar98]. This theorem is not correct in general. Note that Theorem 2.2 uses Lemma 2.1, and the author claims that Lemma 2.1 follows
Lemma 15 (Lemma 10.2 in [TV17]). Over any commutative ring \( k \), the \( k \)-module \( |S^2|_C \) is isomorphic to \( k \).

The proof is just a direct calculation using definitions.

Lemma 15 (Lemma 10.2 in [TV17]). Let \( Z : \text{Cob}_{n} \to \text{Mod}_k \) be an \( n \)-dimensional TQFT such that \( Z(S^{n-1}) \simeq k \). Then, for any connected sum \( M = M_0 \# M_1 \), we have

\[
Z(S^n) Z(M, -\partial M_0, \partial M_1) = Z(M_1, \emptyset, \partial M_1) \circ Z(M_0, -\partial M_0, \emptyset).
\]

In particular, if \( \partial M_0 = \partial M_1 = \emptyset \), then \( Z(S^n) Z(M) = Z(M_0) Z(M_1) \).

Sketch of proof. For \( i = 0, 1 \), pick a ball \( B_i \subset \text{Int}(M_i) \) and set \( N_i = M_i \setminus \text{Int}(B_i) \). Let

\[
b_0 : S^{n-1} \to \emptyset, \quad b_1 : \emptyset \to S^{n-1}, \quad n_0 : -\partial M_0 \to S^{n-1}, \quad n_1 : S^{n-1} \to \partial M_1
\]

be the corresponding morphisms.

Pick a \( k \)-linear isomorphism \( z : Z(S^{n-1}) \to Z(\emptyset) \). Then \( Z(b_0) = \lambda_0 z \) and \( Z(b_1) = \lambda_1 z^{-1} \) for some \( \lambda_0, \lambda_1 \in k \). It follows that

\[
Z(S^n) \text{id}_{Z(\emptyset)} = Z(S^n, \emptyset, \emptyset) = Z(b_0 \circ b_1) = Z(b_0) Z(b_1) = \lambda_0 \lambda_1 \text{id}_{Z(\emptyset)}.
\]

Hence

\[
Z(S^n) Z(M, -\partial M_0, \partial M_1) = \lambda_0 \lambda_1 Z(n_1 \circ n_0) = Z \left( n_1 \circ (\lambda_1 z^{-1}) \circ (\lambda_0 z) \circ Z(n_0) \right) = Z \left( n_1 \circ b_1 \circ Z(b_0 \circ n_0) \right) = Z(M_1, \emptyset, \partial M_1) \circ Z(M_0, -\partial M_0, \emptyset).
\]

\( \square \)

A direct calculation in section 13.1.3 in [TV17] shows that

Lemma 16 ([TV17]). \( |S^1 \times S^2|_C = 1 \), and \( |S^3|_C = (\text{dim}(C))^{-1} \).

Here, we consider the original Turaev–Viro TQFT [TV92], at \( q = e^{\pi i/r} \), \( r \geq 3 \).

Then

\[
|S^3| = \frac{(e^{\pi i/r} - e^{-\pi i/r})^2}{2r} = \frac{2\sin^2 \left( \frac{\pi}{r} \right)}{r} \in (0, 1).
\]

Lemma 17. Let \( M, N \) be compact oriented 3-manifolds and \( f : \partial M \to \partial N \) be an orientation-preserving homeomorphism. Let \( Z \) be a unitary TQFT, and write \( f_\# : Z(\partial M) \to Z(\partial N) \) for the isomorphism induced by \( f \). Then

\[
Z(M \cup f - N) = (f_\# Z(M), Z(N))_{Z(\partial N)} \in \mathbb{C}.
\]
Proof. In Cob₃ there are morphisms

\[-N : \partial N \to \emptyset, \quad N : \emptyset \to \partial N.\]

Hence

\[Z(-N) : Z(\partial N) \to Z(\emptyset) = \mathbb{C}, \quad Z(N) : \mathbb{C} \to Z(\partial N).\]

Note that \(N\) and \(-N\) are conjugate. By the definition of unitary TQFT,

\[\langle f^#Z(M), Z(N) \rangle_{Z(\partial N)} = \langle f^#Z(M), Z(N) \rangle_{Z(\partial N)} = \langle Z(-N)f^#Z(M), 1 \rangle_{Z(\partial N)} = Z(M \cup_f -N).\]

In the case where we glue two components \(\Sigma_1, \Sigma_2\) of the boundary of one manifold \(M\), it is equivalent to glue \(\Sigma_1 \times I\) to \(M\), and the lemma applies as well.

**Theorem 18.** Let \(Z\) be the original Turaev–Viro TQFT \([TV92]\), at \(q = e^{\pi i/r}, r \geq 3\). Let \(M\) be a closed oriented 3-manifold. Then

\[|Z(M)| \leq Z(S^3)^{-g(M)+1},\]

where \(g(M)\) is the Heegaard genus of \(M\). Thus

\[g(M) - 1 \geq -\frac{\log |Z(M)|}{\log Z(S^3)}.\]

**Proof.** Let \(M = H \cup_f (-H)\) be a Heegaard splitting of \(M\), where \(H\) is a handlebody of genus \(g\), and \(f : \partial H \to \partial H\) is an orientation-preserving homeomorphism. Let \(u := Z(H) \in Z(\partial H)\). Then

\[|Z(M)| = |\langle u, f^#(u) \rangle| \leq \sqrt{\langle u, u \rangle \langle f^#(u), f^#(u) \rangle} = \langle u, u \rangle = Z(\#_{i=1}^g S^2 \times S^1) = Z(S^2 \times S^1)^g Z(S^3)^{-g+1} = Z(S^3)^{-g+1}.\]

\[\square\]

### 4.2 Searching for counterexamples

First we claim that

**Proposition 19.** There are no counterexamples to the rank versus genus conjecture among manifolds with Heegaard genus 0, 1, 2.
| Symbol | Base orbifold                        |
|--------|-------------------------------------|
| S2     | 2-sphere                            |
| RP2/n2 | Real projective plane               |
| T      | Torus                               |
| KB/n2  | Klein bottle                         |
| D      | Disc with regular boundary          |
| M/n2   | Möbius band with regular boundary   |
| A      | Annulus with two regular boundaries |

Proof. If $\pi_1(M)$ is a finite cyclic group, by elliptisation theorem $M$ is elliptic. The only elliptic 3-manifolds with finite cyclic fundamental group are the lens spaces (and $S^3$).

If $\pi_1(M) = \mathbb{Z}$, $M$ is irreducible and does not contain incompressible torus and hence is geometric. $M$ has $S^2 \times \mathbb{R}$ geometry, and is hence either $S^1 \times S^2$ or $\mathbb{RP}^2 \times S^1$. $\pi_1(S^1 \times S^2) = \mathbb{Z}$, while $\pi_1(\mathbb{RP}^2 \times S^1)$ is not $\mathbb{Z}$. Thus $M = S^1 \times S^2$.

For the theorems used in the proof, see [Mar16]. Manifolds we are looking for have Heegaard genus at least 3.

The software Regina [BBP+21] is used in concrete calculation. Given a triangulation, Regina computes the state sum invariant. The function turanviroApprox() in Regina computes the given Turaev–Viro state sum invariant of a 3-manifold using a fast but inexact floating-point approximation. The function turanviro() in Regina computes the given Turaev–Viro state sum invariant of a 3-manifold using exact arithmetic.

We use the census of all minimal triangulations of all closed prime orientable 3-manifolds that can be built from $\leq 11$ tetrahedra, as tabulated using Regina [Bur11].

We use the original Turaev–Viro TQFT [TV92], at $q = e^{\pi i/r}$, $r = 5$.

4.2.1 Notation

Notation used in naming 3-manifolds is described in Regina.

The name $T \times I / [a, b \quad c, d]$ denotes a torus bundle over the circle. This torus bundle is expressed as the product of the torus and the interval, with the two torus boundaries identified according to the monodromy $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

A Seifert fibred space is denoted by SFS $[B; (p_1, q_1) (p_2, q_2) \cdots (p_k, q_k)]$, where $B$ is the base orbifold and each $(p_i, q_i)$ describes an exceptional fibre.

In the special case where there are no exceptional fibres at all, we simply write the space as $B \times S^1$ (if there are no fibre-reversing twists) or $B \times \mathbb{RP}^1$ (if there are).

The name SFS $[B_1; \ldots] U/m$ SFS $[B_2; \ldots]$, $m = [a, b \quad c, d]$ denotes a graph manifold formed from a pair of Seifert fibred spaces $M_1$, $M_2$. Let $\phi_i$ and $\omega_i$ be curves on the boundary of $M_i$ representing the fibres and base orbifold, $i = 1, 2$. We identify both boundaries so that

$$\begin{pmatrix} \phi_2 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} \phi_1 \\ \omega_1 \end{pmatrix}.$$
3-piece graph manifolds are defined similarly.

### 4.2.2 Data for $r = 5$

Here is a table of manifolds with Turaev–Viro invariant $\geq 7.235$. Their $H_1$ are shown. Here $\star$ means the Heegaard genus is larger than the rank of $H_1$.

| Manifold | $H_1$ | Heegaard genus |
|----------|-------|----------------|
| $SFS[S2: (2,1) (2,1) (2,1) (2,-1)]$ | $Z_2 + Z_4$ | $13.105572809000083$ |
| $SFS[S2: (2,1) (2,1) (2,1)]$ | $Z_2 + Z_8$ | $15.99999999999984$ |
| $T \times S^1$ | $Z$ | $15.99999999999988$ |
| $KB/n2 \times S^1$ | $Z + 2Z_2$ | $15.99999999999998$ |
| $SFS[S2: (2,1) (2,1) (2,1) (2,1)]$ | $Z_2 + Z_{10}$ | $7.99999999999994$ |
| $SFS[S2: (2,1) (2,1) (2,1) (3,-2)]$ | $Z_2 + Z_{14}$ | $7.447213595499956$ |
| $SFS[S2: (2,1) (2,1) (2,1) (3,-1)]$ | $Z_2 + Z_{16}$ | $7.447213595499945$ |
| $SFS[S2: (2,1) (2,1) (2,1) (2,3)]$ | $Z_2 + Z_{20}$ | $7.447213595499940$ |
| $SFS[S2: (2,1) (2,1) (2,1) (2,5)]$ | $Z_2 + Z_{26}$ | $7.447213595499930$ |
| $SFS[S2: (2,1) (2,1) (2,1) (3,2)]$ | $Z_2 + Z_{30}$ | $7.447213595499924$ |
| $SFS[S2: (2,1) (2,1) (2,1) (3,4)]$ | $Z_2 + Z_{34}$ | $7.447213595499917$ |
| $SFS[S2: (2,1) (2,1) (2,1) (5,-9)]$ | $Z_2 + Z_{38}$ | $7.447213595499915$ |
| $SFS[S2: (2,1) (2,1) (2,1) (7,-4)]$ | $Z_2 + Z_{46}$ | $7.447213595499910$ |
| $SFS[S2: (2,1) (2,1) (2,1) (7,-6)]$ | $Z_2 + Z_{48}$ | $7.447213595499906$ |
| $SFS[S2: (2,1) (2,1) (2,1) (7,-12)]$ | $Z_2 + Z_{50}$ | $7.447213595499904$ |
| $SFS[S2: (2,1) (2,1) (2,1) (7,-2)]$ | $Z_2 + Z_{54}$ | $7.447213595499900$ |
| $SFS[S2: (2,1) (2,1) (2,1) (7,-3)]$ | $Z_2 + Z_{58}$ | $7.447213595499898$ |
| $SFS[S2: (2,1) (2,1) (2,1) (7,-4)]$ | $Z_2 + Z_{62}$ | $7.447213595499895$ |
| $SFS[S2: (2,1) (2,1) (2,1) (7,-5)]$ | $Z_2 + Z_{66}$ | $7.447213595499892$ |
| $SFS[S2: (2,1) (2,1) (2,1) (7,-6)]$ | $Z_2 + Z_{68}$ | $7.447213595499889$ |
| $SFS[S2: (2,1) (2,1) (2,1) (7,-7)]$ | $Z_2 + Z_{74}$ | $7.447213595499886$ |
| $SFS[S2: (2,1) (2,1) (2,1) (7,-8)]$ | $Z_2 + Z_{78}$ | $7.447213595499884$ |
| $SFS[S2: (2,1) (2,1) (2,1) (7,-9)]$ | $Z_2 + Z_{82}$ | $7.447213595499881$ |
| $SFS[S2: (2,1) (2,1) (2,1) (7,-10)]$ | $Z_2 + Z_{86}$ | $7.447213595499878$ |
| $SFS[S2: (2,1) (2,1) (2,1) (7,-11)]$ | $Z_2 + Z_{90}$ | $7.447213595499875$ |
| $SFS[S2: (2,1) (2,1) (2,1) (7,-12)]$ | $Z_2 + Z_{94}$ | $7.447213595499872$ |

**Note!**

| Manifold | $H_1$ | Heegaard genus |
|----------|-------|----------------|
| $SFS[S2: (2,1) (2,1) (2,1) (2,1) (2,-3)]$ | $4Z_2$ | $77.78885438199974$ |
| $SFS[S2: (2,1) (2,1) (2,1) (2,1) (2,-1)]$ | $3Z_2 + Z_6$ | $18.89442719099984$ |
| $SFS[RP2/n2: (2,1) (2,1) (2,-1)]$ | $2Z_2 + Z_8$ | $22.894427190999874$ |
2 \( Z_2 + Z_8 \)
SFS [T: (2,3)] : #1: 9.236067977499754
2 \( Z + Z_3 \)
SFS [KB/n2: (2,3)] : #1: 9.236067977499763
\( Z + Z_8 \) ★ False.
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ -1,4 — 0,1 ] : #1: 8.422291236000305
2 \( Z_2 + Z_8 \)
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ 0,1 — 1,0 ] : #1: 8.161300899000896
2 \( Z_2 + Z_4 \)
SFS [S2: (2,1) (2,1) (2,7)] : #1: 15.99999999999968
2 \( Z_2 + Z_20 \)
SFS [S2: (2,1) (2,1) (2,1) (3,7)] : #1: 7.447213595499941
\( Z_2 + Z_{46} \) ★ True by [BZ84].
SFS [S2: (2,1) (2,1) (2,1) (3,8)] : #1: 7.99999999999971
\( Z_2 + Z_{50} \) ★ True by [BZ84].
SFS [S2: (2,1) (2,1) (2,1) (5,1)] : #1: 9.91934955049517
\( Z_2 + Z_{34} \) ★ True by [BZ84].
SFS [S2: (2,1) (2,1) (2,1) (5,4)] : #1: 9.91934955049517
\( Z_2 + Z_{46} \) ★ True by [BZ84].
SFS [S2: (2,1) (2,1) (2,1) (7,2)] : #1: 7.99999999999971
\( Z_2 + Z_{50} \) ★ True by [BZ84].
SFS [S2: (2,1) (2,1) (2,1) (7,3)] : #1: 7.447213595499948
\( Z_2 + Z_{54} \) ★ True by [BZ84].
SFS [S2: (2,1) (2,1) (2,1) (13,-21)] : #1: 7.447213595499955
\( Z_2 + Z_{6} \) ★ True by [BZ84].
SFS [S2: (2,1) (2,1) (2,1) (13,-8)] : #1: 7.447213595499948
\( Z_2 + Z_{46} \) ★ True by [BZ84].
SFS [S2: (2,1) (2,1) (2,1) (2,1)] : #1: 71.99999999999923 Note!
3 \( Z_2 + Z_{10} \)
SFS [S2: (2,1) (2,1) (2,1) (3,-5)] : #1: 37.10557280900008
2 \( Z_2 + Z_4 \)
SFS [S2: (2,1) (2,1) (2,1) (3,-2)] : #1: 37.10557280900008
2 \( Z_2 + Z_{16} \)
SFS [S2: (2,1) (2,1) (2,1) (3,-4)] : #1: 38.89442719099985
2 \( Z_2 + Z_8 \)
SFS [S2: (2,1) (2,1) (2,1) (3,-1)] : #1: 36.0000000000012
2 \( Z_2 + Z_{20} \)
SFS [RP2/n2: (2,1) (2,1) (2,3)] : #1: 16.42229123600026
2 \( Z_2 + Z_8 \)
SFS [RP2/n2: (2,1) (2,1) (3,-2)] : #1: 9.447213595499948
\( Z_4 + Z_{12} \) ★ False.
SFS [RP2/n2: (2,1) (2,1) (3,1)] : #1: 9.447213595499948
\( Z_4 + Z_{12} \) ★ False.
SFS [RP2/n2: (2,1) (2,1) (3,-1)] : #1: 11.447213595499925
\( Z_4 + Z_{12} \) ★ False.
SFS [RP2/n2: (2,1) (2,1) (3,2)] : #1: 8.211145618000165
\( Z_4 + Z_{12} \) ★ False.
SFS [T: (1,5)] : #1: 7.99999999999971
2 Z + Z₅
SFS [T: (3,4)] : #1: 9.527864045000376

2 Z + Z₄
SFS [KB/n2: (1,5)] : #1: 7.999999999999987

Z + Z₄ ★False.
SFS [KB/n2: (3,4)] : #1: 9.52786404500038

Z + Z₂ + Z₆
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ 1,4 — 0,1 ] : #1: 9.105572809000032

2 Z₂ + Z₁₆
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ 4,3 — 1,1 ] : #1: 7.2360679774997605

Z₄ + Z₂₀
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ 5,3 — 0,1 ] : #1: 7.44721359549993

2 Z₄ ★
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ 5,5 — 0,1 ] : #1: 7.788854381999794

2 Z₄ ★
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ -1,3 — -1,1 ] : #1: 7.788854381999803

2 Z₂ + Z₈
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ 5,3 — 1,1 ] : #1: 7.89442719099983

2 Z₂ + Z₈
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ -5,7 — -3,4 ] : #1: 7.236079774997749

Z₄ + Z₂₀
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ -5,8 — -3,5 ] : #1: 10.89442719099983

2 Z₂ + Z₁₈ ★
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ -5,8 — -3,5 ] : #1: 11.44721359549993

2 Z₂ + Z₄
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ -1,3 — 0,1 ] : #1: 7.23607977499761

Z₂ + Z₂₀
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ 1,1 — 0,1 ] : #1: 12.1613008990009

2 Z₂ + Z₁₆
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ 1,1 — 1,0 ] : #1: 15.05572809000077

2 Z₂ + Z₈
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ 0,1 — 1,0 ] : #1: 11.44721359549991

2 Z₂ + Z₆ ★
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ 0,1 — 1,1 ] : #1: 9.44721359549998

Z₂ + Z₄ ★
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ 0,1 — -1,1 ] : #1: 9.44721359549997

Z₂ + Z₃₄ ★
| Expression                                      | Value     |
|------------------------------------------------|-----------|
| \(SFS[D: (2,1) (3,2)] U/m SFS[D: (2,1) (2,1) (2,1)], m = [0,1,1,1] \): #1: 8.080650449500448 | 7.527864045000408 |
| \(Z_2 + Z_{18} \)                              | \(\star\) |
| \(SFS[D: (2,1) (3,2)] U/m SFS[D: (2,1) (2,1) (2,1)], m = [0,1,1,1] \): #1: 7.527864045000408 | 7.527864045000408 |
| \(Z_2 + Z_{10} \)                              | \(\star\) |

**Notes:**
- \(Z_i\) represents a set of integers.
- \(\star\) indicates the expression is true by [BZ84].
SFS \[S2: (2,1) (2,1) (2,1) (2,1) (2,3)\] : #1: 77.788854382
2 Z_2 + Z_3

SFS \[S2: (2,1) (2,1) (2,1) (2,1) (3,2)\] : #1: 38.89442719099981
2 Z_2 + Z_3

SFS \[S2: (2,1) (2,1) (2,1) (3,1) (3,-5)\] : #1: 19.447213595499942
\text{False.}

SFS \[S2: (2,1) (2,1) (2,1) (3,1) (3,-2)\] : #1: 18.552786404500043
\text{False.}

SFS \[S2: (2,1) (2,1) (2,1) (3,1) (3,-4)\] : #1: 18.552786404500001
\text{False.}

SFS \[S2: (2,1) (2,1) (2,1) (3,1) (3,-1)\] : #1: 19.4472135955
\text{False.}

SFS \[S2: (2,1) (2,1) (3,4)\] : #1: 11.447213595499914
\text{False.}

SFS \[T: (2,7)\] : #1: 9.236067977499735
2 Z + Z_7

SFS \[T: (3,8)\] : #1: 18.472135954999473
2 Z + Z_8

SFS \[KB/n2: (2,7)\] : #1: 9.236067977499738
Z + Z_8 \text{False.}

SFS \[KB/n2: (3,7)\] : #1: 9.23606797749975
Z + Z_12 \text{False.}

SFS \[KB/n2: (3,8)\] : #1: 18.472135954999477
Z + Z_2 + Z_6

SFS \[KB/n2: (4,5)\] : #1: 7.99999999999645
Z + Z_{16} \text{False.}

T x I / [ 13,8 — 8,5 ] : #1: 10.472135954999501
Z + 2 Z_4

T x I / [ -13,-8 — -8,-5 ] : #1: 10.472135954999501
Z + 2 Z_2 + Z_10

SFS \[D: (2,1) (2,1)\] U/m SFS \[D: (2,1) (2,1)\], m = [ 1,4 — 0,-1 ] : #1: 11.577708763999615
2 Z_2 + Z_24
9.105572809000037
2 Z_2 + Z_36
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ -3, -4 — 1, 1 ] : #1:
8.422291236000294
2 Z_2 + Z_48
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ 1, 5 — 0, 1 ] : #1: 7.999999999999984
Z_4 + Z_20
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ 4, 5 — 1, 1 ] : #1: 7.447213595499944
Z_4 + Z_28
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ 4, 5 — 3, 4 ] : #1: 7.447213595499939
Z_4 + Z_8
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ 5, 7 — 3, 4 ] : #1: 7.236067977497685
Z_4 + Z_20
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ 3, 8 — 2, 5 ] : #1: 11.57770876399964
2 Z_2 + Z_16
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ 5, 8 — 2, 3 ] : #1: 10.894427190999833
2 Z_2 + Z_32
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ 5, 8 — 3, 5 ] : #1: 14.472135954999498
2 Z_2 + Z_20
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ -7, 12 — -3, 5 ] : #1:
10.894427190999867
2 Z_2 + Z_12
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ -5, 12 — -2, 5 ] : #1:
15.57708763999548
2 Z_2 + Z_16
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ -5, 13 — -2, 5 ] : #1:
7.236067977499755
Z_4 + Z_20
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (3,1)], m = [ -5, 8 — -2, 3 ] : #1:
7.78854381999799
Z_2 + Z_18
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (3,2)], m = [ -5, 8 — -2, 3 ] : #1:
7.236067977499767
Z_2 + Z_30
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ 1, 1 — -1, 0 ] : #1:
17.950155281000672
2 Z_2 + Z_32
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ 1, 1 — 0, -1 ] : #1:
11.478019326001158
2 Z_2 + Z_32
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ 2, 1 — 1, 0 ] : #1:
17.95015528100069
2 Z_2 + Z_28
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ 2, 1 — 1, 1 ] : #1:
15.05572809000081
2 Z_2 + Z_20
SFS [D: (2,1) (2,1)] U/m SFS [D: (2,1) (2,1)], m = [ -1, 3 — 0, 1 ] : #1:
34.04984471899907
2 Z_2 + Z_16
\text{SFS} \ [D: (2,1)] \ U/m \ SFS \ [D: (2,1)], \ m = [2,1] : \ #1: 
12.1613008990009
\text{2} \ Z_2 + Z_4
\text{SFS} \ [D: (2,1)] \ U/m \ SFS \ [D: (2,1)], \ m = [-2,3] : \ #1: 
27.83869910099898
\text{2} \ Z_2 + Z_4
\text{SFS} \ [D: (2,1)] \ U/m \ SFS \ [D: (2,1)], \ m = [-2,3] : \ #1: 
28.52198067398794
\text{2} \ Z_2 + Z_4
\text{SFS} \ [D: (2,1)] \ U/m \ SFS \ [D: (2,1)], \ m = [-1,3] : \ #1: 
36.5219806739852
\text{2} \ Z_2 + Z_4
\text{SFS} \ [D: (2,1)] \ U/m \ SFS \ [D: (2,1)], \ m = [-1,3] : \ #1: 
22.04984718999115
\text{2} \ Z_2 + Z_4
\text{SFS} \ [D: (2,1)] \ U/m \ SFS \ [D: (2,1)], \ m = [-1,3] : \ #1: 
32.94427190999095
Z + 2 \ Z_2
\text{SFS} \ [D: (2,1)] \ U/m \ SFS \ [D: (2,1)], \ m = [1,1] : \ #1: 
7.527864045000402
Z_4 + Z_2 \star
\text{SFS} \ [D: (2,1)] \ U/m \ SFS \ [D: (2,1)], \ m = [1,1] : \ #1: 
7.527864045000403
Z_4 + Z_2 \star
\text{SFS} \ [D: (2,1)] \ U/m \ SFS \ [D: (2,1)], \ m = [1,1] : \ #1: 
8.0806504495000455
\text{2} \ Z_4 \star
\text{SFS} \ [D: (2,1)] \ U/m \ SFS \ [D: (2,1)], \ m = [1,1] : \ #1: 
8.975077640500356
Z_4 + Z_2 \star
\text{SFS} \ [D: (2,1)] \ U/m \ SFS \ [M/n2: (2,1)], \ m = [-2,1] : \ #1: 
8.5527864045000011
Z_4 + Z_2 \star
\text{SFS} \ [D: (2,1)] \ U/m \ SFS \ [M/n2: (2,1)], \ m = [-1,1] : \ #1: 
8.5527864045000011
2 \ Z_8 \star
\text{SFS} \ [D: (2,1)] \ U/m \ SFS \ [M/n2: (2,1)], \ m = [-1,3] : \ #1: 
7.788854381999817
2 \ Z_8 \star
\text{SFS} \ [D: (2,1)] \ U/m \ SFS \ [M/n2: (2,1)], \ m = [1,3] : \ #1: 
11.788854381999764
2 \ Z_8 \star
\text{SFS} \ [D: (2,1)] \ U/m \ SFS \ [M/n2: (2,1)], \ m = [-2,3] : \ #1: 
9.91934955049948
Z_4 + Z_8 \star
\text{SFS} \ [D: (2,1)] \ U/m \ SFS \ [M/n2: (2,1)], \ m = [-2,3] : \ #1: 
10.683281572999679
Z_4 + Z_8 \star
\text{SFS} \ [D: (2,1)] \ U/m \ SFS \ [M/n2: (2,1)], \ m = [-1,3] : \ #1: 
9.024922359499575

25
| Z,4 + Z,12 | SFS [D: (2,1) (2,1)] U/m SFS [M/n2: (3,1)], m = [ 1,1 — 0,-1 ] : #1: 10.633436854000454 |
|------------|----------------------------------------------------------------------------------------------------------------------------------|
| Z,2 + Z,24 | SFS [D: (2,1) (2,1)] U/m SFS [M/n2: (3,1)], m = [ 1,-2 — 0,1 ] : #1: 10.6832815729999697 |
| Z,4 + Z,12 | SFS [D: (2,1) (2,1)] U/m SFS [M/n2: (3,1)], m = [ -1,2 — 1,-1 ] : #1: 8.422912360000298 |
| Z,4 + Z,12 | SFS [D: (2,1) (2,1)] U/m SFS [M/n2: (3,2)], m = [ 0,1 — 1,2 ] : #1: 9.919349550499494 |
| Z,2 + Z,20 | SFS [D: (2,1) (3,1)] U/m SFS [D: (2,1) (2,1) (2,1)], m = [ 0,-1 — 1,0 ] : #1: 11.4472135954999914 |
| Z,2 + Z,54 | SFS [D: (2,1) (3,1)] U/m SFS [D: (2,1) (2,1) (2,1)], m = [ 0,1 — 1,1 ] : #1: 8.211145618000156 |
| Z,2 + Z,26 | SFS [D: (2,1) (3,2)] U/m SFS [D: (2,1) (2,1) (2,1)], m = [ 0,-1 — 1,0 ] : #1: 8.975077640500354 |
| Z,2 + Z,66 | SFS [D: (2,1) (3,2)] U/m SFS [D: (2,1) (2,1) (2,1)], m = [ 1,1 — 0,1 ] : #1: 7.527864045000142 |
| Z,2 + Z,50 | SFS [D: (2,1) (3,2)] U/m SFS [D: (2,1) (2,1) (2,1)], m = [ 1,1 — 1,0 ] : #1: 8.9750776405003354 |
| Z,2 + Z,54 | SFS [D: (3,1) (3,1)] U/m SFS [D: (2,1) (2,1) (2,1)], m = [ -1,1 — 1,0 ] : #1: 8.9750776405003354 |
| Z,6 + Z,18 | SFS [D: (3,2) (3,2)] U/m SFS [D: (2,1) (2,1) (2,1)], m = [ -1,1 — 0,1 ] : #1: 7.527864045000394 |
| Z,2 + Z,30 | SFS [D: (3,2) (3,2)] U/m SFS [D: (2,1) (2,1) (2,1)], m = [ -1,1 — 1,0 ] : #1: 8.080650449500455 |
| Z,2 + Z,6 | SFS [D: (2,1) (2,1)] U/m SFS [A: (4,1)] U/n SFS [D: (2,1) (2,1)], m = [ 0,1 — 1,0 ], n = [ 1,1 — 1,0 ] : #1: 7.7447213595499939 |
| Z,4 + Z,12 | SFS [D: (2,1) (2,1)] U/m SFS [A: (4,3)] U/n SFS [D: (2,1) (2,1)], m = [ 0,1 — 1,0 ], n = [ 1,1 — 1,0 ] : #1: 7.788854381999809 |

\[ 2 Z_6 \]
SFS \[D: (2,1) (2,1)\] U/m SFS \[A: (5,2)\] U/n SFS \[D: (2,1) (2,1)\], \(m = [0,1 — 1,0]\), \(n = [1,1 — 1,0]\) : \#1: 8.21114561800015
\(\mathbb{Z}_4 + \mathbb{Z}_{12}\) ★
SFS \[D: (2,1) (2,1)\] U/m SFS \[A: (5,3)\] U/n SFS \[D: (2,1) (2,1)\], \(m = [0,1 — 1,0]\), \(n = [0,1 — 1,0]\) : \#1: 8.21114561800015
\(\mathbb{Z}_4 + \mathbb{Z}_{28}\) ★
SFS \[D: (2,1) (2,1)\] U/m SFS \[A: (5,3)\] U/n SFS \[D: (2,1) (2,1)\], \(m = [0,-1 — 1,0]\), \(n = [0,1 — 1,0]\) : \#1: 16.42229123600025
\(2 \mathbb{Z}_2 + \mathbb{Z}_8\)
SFS \[D: (2,1) (2,1)\] U/m SFS \[A: (5,3)\] U/n SFS \[D: (2,1) (2,1)\], \(m = [0,-1 — 1,0]\), \(n = [0,1 — 1,0]\) : \#1: 8.21114561800015
\(\mathbb{Z}_4 + \mathbb{Z}_{12}\) ★

4.2.3 Analysis

Here ★ means the Heegaard genus is larger than the rank of \(H_1\). These are potential counterexamples.

**Proposition 20.** (Proposition 1.5 in [BZ84])

\[
G = \langle s_1, \ldots, s_{2q}, f \mid s_1^2, \ldots, s_{2q-1}^2, s_{2q}^2, s_1s_2^2s_3^2 \cdots s_{2q}^2f^\beta, s_1 \cdots s_{2q}f^e \rangle
\]

with \(q \geq 2\), \(\lambda > 0\) and \(\beta\) arbitrary, admits a presentation with \(2q - 2\) generators and \(2q - 2\) defining relators.

Usually the fundamental group considered is a quotient of \(G\). We write True by [BZ84], if the manifold is a counterexample by this proposition.

Observe that among the 92 potential counterexamples, 35 of them are marked as above. By Theorem 3.1 in [BZ84], they are counterexamples. Also, in the list, any manifold that can be judged by Theorem 3.1 are marked.

Note that the Seifert Euler number in [BZ84] differs from the one defined in [Mar16] by a sign.

Theorem 2.1 and Theorem 1.1 in [BZ84] also help, and 17 manifolds are marked with False.

4.2.4 Other remarks

In [Wei03] the writer describes a family of 3-dimensional graph manifolds consisting of two Seifert fibre spaces whose base orbifolds are the M"obius band and a disk and which have one, respectively two, exceptional fibres. The considered spaces have 2-generated fundamental group but are not of Heegaard genus 2. It is not difficult to see that these manifolds are of genus 3. The precise description of those manifolds is given in Theorem 1 there.

Observe that as manifolds, SFS \[D: (2,1) (2,1)\] are the circle bundle over the M"obius strip with total space orientable. (See Proposition 10.3.33 in [Mar16].)

As Seifert fibrations, SFS \[D: (2,k) (2,j)\] for \(k,j\) odd, and they need to be considered. (See Proposition 10.3.11 in [Mar16].) The TQFT method only tells us that among the 92 selected manifolds, genus \(\geq 3\). What we can detect is the case genus \(\geq 3\) and rank = 2.

Accordingly, the author thinks that the following manifold might be another counterexample:
The case \( r = 4 \) has been considered by the author and nothing interesting was found in the database.

There is also a census of smallest known closed hyperbolic 3-manifolds (where “smallest” refers to volume), as tabulated by Hodgson and Weeks \([HW94]\). The cases \( r = 3, 4, 5, 6 \) have been considered by the author and nothing interesting was found in this database.

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**About Regina**

*Regina* is a software for low-dimensional topology, with a focus on triangulations, knots and links, normal surfaces, and angle structures.

Any database used here can be downloaded from:

https://regina-normal.github.io/data.html.

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