Superconvergent flux recovery of the Rannacher-Turek nonconforming element

Yuwen Li

Abstract This work presents superconvergence estimates of the Rannacher-Turek element for second order elliptic equations on any cubical meshes in $\mathbb{R}^2$ and $\mathbb{R}^3$. In particular, a recovered numerical flux is shown to be superclose to the Raviart–Thomas interpolant of the exact flux. We then design a superconvergent recovery operator based on local weighted averaging. Combining the supercloseness and the recovery operator, we prove that the recovered flux superconverges to the exact flux. As a by-product, we obtain a superconvergent recovery estimate of the Crouzeix–Raviart element method for general elliptic equations.

Keywords superconvergence, rectangular meshes, Rannacher–Turek element, Raviart–Thomas element, Crouzeix–Raviart element

Mathematics Subject Classification (2000) 65N15, 65N30

1 Introduction and preliminaries

Finite element superconvergent recovery is quite popular in practice for their simplicity and ability to develop asymptotically exact a posteriori error estimators. The theory of superconvergent recovery for conforming Lagrange elements is well-established, see, e.g., [30,31,4,5,6,26,29]. Let $u_h$ be the finite element solution approximating the PDE solution $u$. The framework of superconvergent recovery is often divided into two steps. First a supercloseness estimate between $u_h$ and some canonical interpolant $u_I$ is obtained. Then a postprocessed solution $R_h u_h$ is shown to superconverge to $u$ in suitable norm, provided $R_h$ is a bounded operator with super-approximation property.

On the other hand, since the interelement boundary continuity of nonconforming elements is very weak, superconvergence analysis of nonconforming
methods is often more difficult and limited. For the nonconforming Crouzeix–Raviart (CR) [11] element method for the Poisson equation, it can be numerically observed that the canonical interpolant $u_I$ and the finite element solution $u_h$ are not superclose in the energy norm. Hence the aforementioned recovery framework does not work. In [27], Ye developed superconvergence estimates of the CR element using least-squares surface fitting [24, 25]. Guo and Huang [14] presented a polynomial preserving gradient recovery method for the CR element with numerically confirmed superconvergence. Based on an equivalence between the CR method and the lowest order Raviart–Thomas (RT) method for Poisson’s equation (cf. [21, 2]), Hu and Ma [16] proved recovery superconvergence estimate for the CR element using superconvergence of RT elements in [7]. This result is then improved and generalized in e.g., [17, 15, 28]. Readers are also referred to e.g., [10, 9, 20, 19] and references therein for superconvergence of other nonconforming elements.

The nonconforming Rannacher–Turek (NCRT) element [23] is a generalization of the CR element on rectangular meshes. It is noted that there is a superconvergence estimate of the NCRT element at some special points under certain mildly distorted square meshes, see [22]. For the Poisson equation, it has been shown in [18] that several rectangular nonconforming methods do not admit natural supercloseness estimates. In particular, $u_I$ and $u_h$ from the NCRT element are superclose in the energy norm only under square meshes. To overcome this barrier, they enriched the NCRT element by one degree of freedom at the centroid of each element and proved superconvergent gradient recovery estimate of the modified nonconforming element.

In this paper, we shall consider the standard NCRT method (1.2) for solving the general elliptic equation (1.1). First we compute a new numerical flux $\sigma_h$ from the NCRT finite element solution, see Theorem 2.1. We shall show that $\sigma_h$ is superclose to $\Pi_h(\nabla u)$ by comparing it with an auxiliary $H(\text{div})$-conforming flux $\bar{\sigma}_h$ and using well-established superconvergence tools and techniques for RT elements in e.g., [12, 7, 17]. Here $\Pi_h$ is the canonical interpolation of the lowest order rectangular RT element. We then construct a local edge-based weighted averaging operator $A_h$, which makes $\|\nabla u - A_h(\nabla u)\|$ supersmall on any rectangular mesh. Hence $A_h \sigma_h$ superconverges to $\nabla u$ on any rectangular mesh by a triangle-inequality argument. To the best of our knowledge, this is the first superconvergent recovery method for the NCRT element on any rectangular mesh. Our supercloseness estimate directly extends to $\mathbb{R}^3$, see Section 4.

For elliptic equations with variable coefficients and lower order terms, Arbogast and Chen in [1] can reformulate various mixed methods as modified nonconforming methods. However, the general equivalence expression is complicated and it is unclear how far the standard nonconforming finite element solution is from the modified one. On the other hand, superconvergence analysis of $H(\text{div})$-conforming mixed finite elements is well established, see, e.g., [12, 7, 17, 3]. Hence we shall relate nonconforming methods to their mixed counterparts as in [16]. In our superconvergence analysis, it is not necessary to rewrite the NCRT method (1.2) as an equivalent mixed method for the general elliptic
Superconvergence of the Rannacher-Turek element. All we need is the equivalence given by Lemma 2.1 for the Poisson equation. As far as we know, it is the first superconvergence estimate of the CR and NCRT element methods for the general elliptic equation.

In the rest of this section, we introduce preliminary definitions and notations. Let $\Omega = [a, b] \times [c, d] \subset \mathbb{R}^2$ be a rectangle. Consider the second order elliptic equation

$$
-\nabla \cdot (a \nabla u) + b \cdot \nabla u + cu = f \quad \text{in } \Omega, \quad (1.1a)
$$

$$
u = g \quad \text{on } \partial \Omega, \quad (1.1b)
$$

where $a(x) \geq a_0 > 0$ for all $x = (x_1, x_2)^T \in \Omega$, $a, b, c$, and $f$ are smooth functions in $\bar{\Omega}$.

Let $T_h$ be a partition of $\Omega$ by rectangles. Given a rectangle $K \in T_h$, let $\ell_{K,1}$ and $\ell_{K,2}$ denote the width and height of $K$ and $h = \max_{K \in T_h} (\ell_{K,1}, \ell_{K,2})$ the mesh size. We assume that $h < 1$ and $T_h$ is nondegenerate, i.e.

$$
\max_{K \in T_h} \max \left\{ \frac{\ell_{K,1}}{\ell_{K,2}}, \frac{\ell_{K,2}}{\ell_{K,1}} \right\} \leq C_{T_h} < \infty,
$$

where $C_{T_h}$ is a constant independent of $h$. Let $E_h$, $E_h^\circ$, and $E_h^\partial$ denote the set of edges, interior edges, and boundary edges, respectively. The following edge-based patch $\omega_E$ will be frequently used.

1. For $E \in E_h^\circ$, let $\omega_E = K^+ \cup K^-$ where $K^+$ and $K^-$ are the two adjacent rectangles sharing $E$.
2. For $E \in E_h^\partial$, let $\omega_E = K$, where $K$ is the rectangle having $E$ as an edge.

Let

$$V_{g,h} := \{ v_h \in L^2(\Omega) : v_h|_K \in \text{span}\{1, x_1, x_2, x_1^2 - x_2^2\} \text{ for all } K \in T_h, \quad \int_E v_h \text{ is single-valued for all } E \in E_h^\circ, \int_E v_h = \int_E g \text{ for all } E \in E_h^\partial\},$$

where $f_E v := \frac{1}{|E|} \int_E v$ is the mean value of $v$ on $E$. The Rannacher–Turek nonconforming method for (1.1) is to find $u_h \in V_{g,h}$, such that

$$
\langle a \nabla h u_h, \nabla v \rangle + \langle b \cdot \nabla h u_h, v \rangle + \langle cu_h, v \rangle = \langle f, v \rangle, \quad \forall v \in V_{0,h}, \quad (1.2)
$$

where $\langle \cdot, \cdot \rangle$ is the $L^2(\Omega)$-inner product and $\nabla h$ denotes the piecewise gradient with respect to $T_h$. Throughout this paper, we adopt the notation $A \lesssim B$ when $A \leq CB$ for some generic constant $C$ that is independent of $h$. We assume that the standard a priori error estimate for the NCRT method holds:

$$
\|u - u_h\| + h\|\nabla h (u - u_h)\| \lesssim h^2\|u\|_{H^2}, \quad (1.3)
$$

where $\|\cdot\|$ denotes the norm $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{H^2}$ abbreviates $\|\cdot\|_{H^2(\Omega)}$, similar for other Sobolev norms. Readers are referred to [8] for the analogue of (1.3) for the CR method.
The NCRT space $\tilde{V}_h$ using degrees of freedom based on pointwise function evaluation will be used in Section 3. 

$\tilde{V}_h := \{ v_h \in L^2(\Omega) : v_h|_K \in \text{span}\{1, x_1, x_2, x_1^2 - x_2^2\} \text{ for all } K \in T_h, v_h \text{ is continuous at the midpoint of each } E \in \mathcal{E}^0_h\}.$

Let $\mathcal{Q}_{k,l}(K)$ denote the set of polynomials of degree $\leq k$ in $x_1$ and of degree $\leq l$ in $x_2$ on the element $K$. Let 

$$H(\text{div}, \Omega) := \{ \tau \in L_2(\Omega) \times L_2(\Omega) : \nabla \cdot \tau \in L_2(\Omega) \}.$$

The lowest order rectangular Raviart–Thomas finite element space is 

$$\mathcal{RT}_h := \{ \tau_h \in H(\text{div}, \Omega) : \tau_h|_K \in Q_{1,0}(K) \times Q_{0,1}(K) \text{ for all } K \in T_h \}.$$

For convenience we introduce the broken Raviart–Thomas space 

$$\mathcal{RT}^{-1}_h := \{ \tau_h \in L_2(\Omega) \times L_2(\Omega) : \tau_h|_K \in Q_{1,0}(K) \times Q_{0,1}(K), \forall K \in T_h \}.$$

Given $\tau \in H^1(\Omega) \times H^1(\Omega)$, the canonical interpolant $\Pi_h \tau$ is the unique function in $\mathcal{RT}_h$ such that 

$$\int_E (\Pi_h \tau) \cdot n = \int_E \tau \cdot n, \quad \forall E \in \mathcal{E}_h,$$

where $n$ is a unit normal to $E$. Let $P_h$ be the $L_2(\Omega)$-projection onto the space of piecewise constant functions. It is well known that 

$$\nabla \cdot \Pi_h \tau = P_h \nabla \cdot \tau. \quad (1.4)$$

Let $E \in \mathcal{E}^0_h$ and $K^+, K^-$ be the two rectangles sharing $E$. Let $n^+$ and $n^-$ denote the outward unit normal induced by $K^+$ and $K^-$ respectively. In the analysis of nonconforming methods, it is convenient to introduce notations for jumps and averages on $E$: 

$$[\tau] := \tau|_{K^+} \cdot n^+ + \tau|_{K^-} \cdot n^-,$$

$$\{\tau\} := (\tau|_{K^+} + \tau|_{K^-})/2,$$

$$[v] := (v|_{K^+} n^+ + v|_{K^-} n^-)/2,$$

$$\{v\} := (v|_{K^+} + v|_{K^-})/2,$$

where $\tau$ is a vector and $v$ is a scalar. For $E \in \mathcal{E}^0_h$, 

$$[\tau] := \tau \cdot n, \quad \{v\} := v, \quad [v] := 0,$$

where $n$ is the outward unit normal to $\partial \Omega$. It is readily checked that 

$$[\tau v] = [\tau] \{v\} + [v] \cdot \{\tau\}. \quad (1.5)$$

By these notations, a useful fact is that 

$$\tau_h \in \mathcal{RT}_h \text{ if and only if } \tau_h \in \mathcal{RT}^{-1}_h \text{ and } [\tau_h] = 0 \forall E \in \mathcal{E}^0_h. \quad (1.6)$$

The rest of this paper is organized as follows. Section 2 discusses the supercloseness estimate in Theorem 2.1. In Section 3, we propose a postprocessing operator and prove the recovery superconvergence estimate in Theorem 3.2. In Section 4, we extend our superconvergence analysis to the CR element and NCRT element in $\mathbb{R}^3$. Numerical experiments are presented in Section 5.
2 Supercloseness

In this section, we derive a supercloseness estimate for NCRT elements, which is essential to develop superconvergent flux recovery. First we need a lemma in the spirit of Marini (cf. \cite{21}).

**Lemma 2.1** Let $\bar{f}$ be a piecewise constant, $\tau_h|_K \in Q_{1,0}(K) \times Q_{0,1}(K)$ and $\nabla \cdot (\tau_h|_K) = 0$ for all $K \in T_h$. Assume that
\[
\langle \tau_h, \nabla_h v \rangle = \langle \bar{f}, v \rangle \tag{2.1}
\]
for all $v \in V_{0,h}$. Then $\tau_h - \bar{f} r_h \in R T_h$, with
\[
r_h|_K := \left( \frac{\ell^2_{K,2}}{\ell^2_{K,1} + \ell^2_{K,2}} (x_1 - x_{K,1}), \frac{\ell^2_{K,1}}{\ell^2_{K,1} + \ell^2_{K,2}} (x_2 - x_{K,2}) \right)^T,
\]
where $K = [x_{1,i}, x_{1,i+1}] \times [x_{2,j}, x_{2,j+1}]$, $\ell_{K,1} = x_{1,i+1} - x_{1,i}$, $\ell_{K,2} = x_{2,j+1} - x_{2,j}$, and $(x_{K,1}, x_{K,2})^T$ is the centroid of $K$.

**Proof** Consider any vertical edge $E \in \mathcal{E}_h^o$ and the two rectangles
\[
K^- = [x_{1,i}, x_{1,i+1}] \times [x_{2,j}, x_{2,j+1}], \quad K^+ = [x_{1,i+1}, x_{1,i+2}] \times [x_{2,j}, x_{2,j+1}]
\]
sharing it. Let $v \in V_{0,h}$ be the basis function such that
\[
\int_E v_E = 1, \quad \int_{E'} v_E = 0 \text{ for } E' \ni E \neq E.
\]
Note that $\tau_h \cdot (1,0)^T$ is a constant on $E$. It then follows from (2.1) with $v = v_E$, $\nabla_h \cdot \tau_h = 0$ and integration by parts that
\[
\int_E [\tau_h] = \int_{K^+ \cup K^-} \bar{f} v_E. \tag{2.2}
\]
Direct calculation shows that
\[
\int_{K^\pm} v_E = \frac{|K^\pm| \ell^2_{K^\pm,2}}{2(\ell^2_{K^\pm,1} + \ell^2_{K^\pm,2})}.
\]
Combining it with (2.2) yields
\[
[\tau_h - \bar{f} r_h] = 0 \text{ on } E. \tag{2.3}
\]
Similarly, (2.3) also holds for horizontal edges. Combining (2.3) with the fact $(\tau_h - \bar{f} r_h)|_K \in Q_{1,0}(K) \times Q_{0,1}(K)$, we conclude that $\tau_h - \bar{f} r_h \in R T_h$.

**Remark 1** It seems that the NCRT method using degrees of freedom based on pointwise function evaluation does not have a similar equivalence.
To apply Lemma 2.1, we then introduce the auxiliary nonconforming method: Find $\bar{u}_h \in \mathcal{V}_{h,h}$, such that

$$
\langle a \nabla_h u_h, \nabla_h v \rangle = \langle P_h(f - cu - b \cdot \nabla u), v \rangle, \quad \forall v \in \mathcal{V}_{0,h}.
$$

The following lemma shows that $u_h$ and $\bar{u}_h$ are superclose in the $H^1$-norm.

**Lemma 2.2** Let $u_h$ and $\bar{u}_h$ solve (1.2) and (2.4), respectively. Then

$$
\| \nabla_h (u_h - \bar{u}_h) \| \lesssim h^2 \| u \|_{H^2}.
$$

**Proof** Subtracting (2.4) from (1.2) gives

$$
\langle a \nabla_h (u_h - \bar{u}_h), \nabla_h v \rangle = \langle f - cu - b \cdot \nabla u, v - P_h(f - cu - b \cdot \nabla u), v \rangle,
$$

where $v \in \mathcal{V}_{0,h}$. It then follows from (1.3) that

$$
\begin{align*}
\langle a \nabla_h (u_h - \bar{u}_h), \nabla_h v \rangle &= \langle f - cu - b \cdot \nabla u - P_h(f - cu - b \cdot \nabla u), v - P_h v \rangle \\
&+ \langle c(u - u_h), v \rangle + \langle b \cdot \nabla_h (u - u_h), v \rangle \\
&= O(h^2)(\|f\|_{H^1} + \|u\|_{H^2})\|\nabla_h v\| + \langle b \cdot \nabla_h (u - u_h), v \rangle.
\end{align*}
$$

It remains to show that $\langle b \cdot \nabla_h (u - u_h), v \rangle$ is supersmall. By integrating by parts, (1.5), and $\int_E [u - u_h] = 0$, we have

$$
\begin{align*}
\langle b \cdot \nabla_h (u - u_h), v \rangle &= \sum_{K \in T_h} \int_{\partial K} (u - u_h)v b \cdot n - \int_K (u - u_h) \nabla \cdot (bv) \\
&= \sum_{E \in T_h} \int_E \{u - u_h\} [vb - c_E] + [u - u_h] \cdot \{vb - d_E\} \\
&- \int_{\Omega} (u - u_h) \nabla \cdot (bv)
\end{align*}
$$

for any constants $c_E \in \mathbb{R}^2$ and $d_E \in \mathbb{R}^2$. In particular, let $c_E = d_E = b(m_E) / \omega_E$, where $m_E$ is the midpoint of $E$. By the trace inequality

$$
\|w\|_{L^2(\partial K)} \lesssim h^{-\frac{1}{2}} \|w\|_{L^2(K)} + h^\frac{1}{2} \|\nabla w\|_{L^2(K)},
$$

we have

$$
\begin{align*}
\|\{u - u_h\}\|_{L^2(E)} + \|\{u - u_h\}\|_{L^2(E)} &\lesssim h^{-\frac{1}{2}} \|u - u_h\|_{L^2(\omega_E)} + h^\frac{1}{2} \|\nabla_h (u - u_h)\|_{L^2(\omega_E)}
\end{align*}
$$

and

$$
\|\{vb - c_E\}\|_{L^2(E)} + \|\{vb - d_E\}\|_{L^2(E)} \lesssim h^\frac{1}{2} \|\nabla_h (bv)\|_{L^2(\omega_E)}.
$$
It follows from the Cauchy–Schwarz inequality, (2.7), (2.8) and (1.3) that
\[
\|b \cdot \nabla_h (u - u_h)\| \\
\lesssim \sum_{E \in \mathcal{E}_h} \left( \|u - u_h\|_{L^2(E)} \|v - c_E\|_{L^2(E)} + \|u - u_h\| \|\nabla_h \cdot (bv)\| \right)
\]
\[
\leq \sum_{E \in \mathcal{E}_h} \left( \|u - u_h\|_{L^2(E)} + h \|\nabla_h (u - u_h)\|_{L^2(E)} \right) \|\nabla_h (bv)\|_{L^2(E)} + \|u - u_h\| \|\nabla_h \cdot (bv)\|
\]
\[
\lesssim h^2 \|u\|_{H^2} (\|v\| + \|\nabla_h v\|).
\]
Combining (2.9) with (2.5) and using the discrete Poincaré inequality (cf. Theorem 10.6.12. in [8]) \(\|v\| \lesssim \|\nabla_h v\|\), we complete the proof. \(\square\)

Now we are in a position to present supercloseness results. Let
\[
\sigma_h := Q_h (a \nabla_h u_h) - r_h P_h (f - cu_h - b \cdot \nabla_h u_h)
\]
be the recovered flux, where \(r_h\) is defined in Lemma 2.1. \(\sigma_h\) is expected to approximate the exact flux \(\sigma := a \nabla u\) very well.

**Theorem 2.1** Let \(Q_h\) be the \(L^2\)-projection onto \(\nabla_h \mathcal{V}_h\). It holds that
\[
\|P_h \sigma - \sigma_h\| \lesssim h^2 \|u\|_{H^3}.
\]

**Proof** Let \(\tilde{\sigma}_h := Q_h (a \nabla_h \bar{u}_h) - r_h P_h (f - cu_h - b \cdot \nabla u).\) Using the definition of \(\bar{u}_h\), \(\nabla_h \cdot \bar{u}_h = 0\) and Lemma 2.1, we conclude that \(\tilde{\sigma}_h \in \mathcal{R} \mathcal{T}_h \subset H(\text{div}, \Omega)\). Let \(\tau_h = P_h \sigma - \tilde{\sigma}_h\). It follows from (1.4) and \(\nabla_h \cdot r_h = 1\) that
\[
\nabla \cdot \tau_h = P_h \nabla \cdot (a \nabla u) - P_h (f - cu - b \cdot \nabla u) = 0.
\]
Hence \(\tau_h|_K = (c_1 x + c_2, -c_1 x^2 + c_3)^T\) for some \(c_i \in \mathbb{R}\) on an element \(K \in \mathcal{T}_h\).

On the other hand, direct calculation shows that
\[
\int_K \tau_h \cdot \tau_h = \int_K \tau_h \cdot (\tau_h - (c_2 + c_1 x, c_3 - c_1 x^2)^T)
\]
\[
= \frac{c_1}{E_{K,1} + E_{K,2}} \int_K (x_1 - x, x_1 - x^2)^2 - \frac{E_{K,2}^2}{E_{K,1}} (x_1 - x^2)^2 = 0.
\]
Therefore
\[
\|P_h \sigma - \sigma_h\|^2 = \langle P_h \sigma - \sigma_h, \tau_h \rangle + \langle \sigma - \sigma_h, \tau_h \rangle
\]
\[
= \langle P_h \sigma - \sigma, \tau_h \rangle + \langle a \nabla_h (u - \bar{u}_h), \tau_h \rangle
\]
\[
:= I + II.
\]
By Lemma 3.1 with \(k = 0\) in [12] and the Bramble–Hilbert lemma,
\[
|I| \lesssim \|\sigma\|_{H^2} \|\tau_h\|.
\]
For part II, since $\nabla \cdot (\tau_h|_K) = 0$, we have

$$II = \sum_{K \in T_h} \int_K a\nabla(u - \bar{u}_h) \cdot \tau_h$$

$$= \sum_{K \in T_h} \int_K (\nabla(a(u - \bar{u}_h)) - (u - \bar{u}_h)\nabla a) \cdot \tau_h$$

$$= \sum_{K \in T_h} \int_{\partial K} a(u - \bar{u}_h) \tau_h \cdot n - \langle (u - \bar{u}_h)\nabla a, \tau_h \rangle$$

$$:= I_{I_1} + I_{I_2}.$$ (2.12)

$I_{I_2}$ is estimated by Lemma 2.2 and the apriori estimate (1.3):

$$|I_{I_2}| \lesssim h^2 \|u\|_{H^3} \|\tau_h\|.$$ (2.13)

Note that the normal component of $\{\tau_h\}$ is constant on $E$ and $[\tau_h] = 0$ by (1.6). It then follows from $\int_E [\bar{u}_h] = 0$, (1.5), the trace inequality (2.6), an inverse inequality, (1.3), and Lemma 2.2, that

$$I_{I_1} = \sum_{E \in \mathcal{E}_h} \int_E [a(u - \bar{u}_h) \tau_h]$$

$$= \sum_{E \in \mathcal{E}_h} \int_E [a - \int_E a] (u - \bar{u}_h) \cdot \{\tau_h\}$$

$$\lesssim h \sum_{E \in \mathcal{E}_h} \|u - \bar{u}_h\|_{L^2(E)} \|\tau_h\|_{L^2(E)}$$

$$\lesssim h^{\frac{1}{2}} \sum_{E \in \mathcal{E}_h} (h^{-\frac{1}{2}} \|u - \bar{u}_h\|_{L^2(\omega_E)} + h^{\frac{1}{2}} \|\nabla_h (u - \bar{u}_h)\|_{L^2(\omega_E)} \|\tau_h\|_{L^2(\omega_E)}$$

$$\lesssim (\|u - \bar{u}_h\| + h \|\nabla_h (u - \bar{u}_h)\|) \|\tau_h\| \lesssim h^2 \|u\|_{H^3} \|\tau_h\|.$$ (2.14)

Combining (2.10)-(2.14), we obtain

$$\|I_{h \sigma} - \bar{\sigma}_h\| \lesssim h^2 \|u\|_{H^3}.$$ (2.15)

On the other hand, Lemma 2.2 implies

$$\|\sigma_h - \bar{\sigma}_h\| \lesssim h^2 \|u\|_{H^2}.$$ (2.16)

The theorem then follows from (2.15) and (2.16).

Note that $Q_h$ is in fact a element-by-element projection and $Q_h(a \nabla_h u_h) = a \nabla_h u_h$ if $a$ is a piecewise constant.
3 Postprocessing and superconvergence

For the rectangular Raviart–Thomas element, Durán [12] gave a postprocessing operator $K_h^D$ satisfying:

\[
\|K_h^D \tau_h\| \lesssim \|\tau_h\| \quad \text{for all } \tau_h \in \mathcal{RT}_h, \quad (3.1a)
\]

\[
\|\sigma - K_h^D \Pi_h \sigma\| \lesssim h^2 |\sigma|_{H^2}. \quad (3.1b)
\]

Here the input for $K_h^D$ needs to be $H(\text{div})$-conforming. Now assume the recovered flux $\sigma_h \in \mathcal{RT}_h$, e.g., $f$ is piecewise constant, $b = 0$, and $c = 0$. Using (3.1), Theorem 2.1, and the triangle inequality

\[
\|a \nabla u - K_h^D \sigma_h\| \lesssim h^2 \|u\|_{H^3}.
\]

However, $\sigma_h \in \mathcal{RT}_{h}^{-1}$ and $\sigma_h \notin \mathcal{RT}_h$ in general and thus $K_h^D$ cannot be directly applied to $\sigma_h$. In this section, we introduce a simple recovery operator $A_h$ by local weighted averaging.

**Definition 3.1** The operator $A_h : \mathcal{RT}_{h}^{-1} \rightarrow \tilde{V}_h$ is defined as follows.

1. For each $E \in \mathcal{E}_h$, let $m$ be the midpoint of $E$. Let $K^+$ and $K^-$ be the two rectangles sharing $E$ as an edge. Define

\[
(A_h \tau_h)(m) := \frac{|K^-|}{|K^+| + |K^-|} \tau_h|_{K^+}(m) + \frac{|K^+|}{|K^+| + |K^-|} \tau_h|_{K^-}(m).
\]

2. For each $E \in \mathcal{E}_h$, let $m$ denote the midpoint of $E$ and $K$ the element having $E$ as an edge. Let $E'$ be the edge of $K$ opposite to $E$ with midpoint $m'$. Let $K'$ be the other element having $E'$ as an edge and $m''$ the midpoint of the edge of $K'$ opposite to $E'$. Define

\[
(A_h \tau_h)(m) := ((A_h \tau_h)(m') - w'(A_h \tau_h)(m''))/w,
\]

where

\[
w = \frac{|K'|}{|K| + |K'|}, \quad w' = \frac{|K|}{|K| + |K'|}.
\]

Then $A_h \tau_h$ is the unique element in $\tilde{V}_h$ whose midpoint values are specified in the above two steps.

Note that the weight constants in Definition 3.1 are not chosen in a standard way. We show that $A_h$ has a super-approximation property on any non-degenerate rectangular meshes.

**Theorem 3.1** For $\tau_h \in \mathcal{RT}_{h}^{-1}$ and $\tau \in H^2(\Omega)$, it holds that

\[
\|A_h \tau_h\| \lesssim \|\tau_h\|, \quad (3.2a)
\]

\[
\|\tau - A_h \Pi_h \tau\| \lesssim h^2 |\tau|_{H^2}. \quad (3.2b)
\]
Proof Consider $K \in \mathcal{T}_h$ and

$$\omega_K := \bigcup_{E \subset \partial K} \omega_E.$$ 

Using the stability of $A_h$ in the $L_\infty$-norm and the inverse inequality, we prove the stability of $A_h$ in the $L_2$-norm:

$$\|A_h \tau_h\|_{L_2(K)} \lesssim h \|A_h \tau_h\|_{L_\infty(K)} \lesssim h \|\tau_h\|_{L_\infty(\omega_K)} \lesssim \|\tau_h\|_{L_2(\omega_K)}.$$  

(3.2a) then follows from the above estimate and sum of squares.

Let $E \in \mathcal{E}_h^b$ with midpoint $m$ and two adjacent elements $K^+, K^-$ sharing $E$. For $\tau_1 \in Q_{1,1}(\omega_E) \times Q_{1,1}(\omega_E)$, we first want to show $(\tau_1 - A_h \Pi_h \tau_1)(m) = 0$. Since $\Pi_h$ preserves functions in $Q_{1,0}(\omega_E) \times Q_{0,1}(\omega_E)$, it suffices to check when $\tau_1 = (y,0)^T$ or $(0,x)^T$. By linearity we can assume $m = 0$ without loss of generality. If $E$ is a horizontal interior edge, let $K^+ = [-\ell_1/2, \ell_1/2] \times [0, \ell_2^+], K^- = [-\ell_1/2, \ell_1/2] \times [-\ell_2^-, 0]$. Then,

$$\Pi_h \begin{pmatrix} y \\ 0 \end{pmatrix} = \begin{cases} \left( \ell_2^+ / 2, 0 \right)^T & \text{if } y > 0 \\ \left( -\ell_2^- / 2, 0 \right)^T & \text{if } y < 0 \end{cases}, \quad \Pi_h \begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$ 

In each case, $(\tau_1 - A_h \Pi_h \tau_1)(m) = 0$. The same argument works for vertical interior edges.

Let $E \in \mathcal{E}_h^b$ and $K$ the element having $E$ as an edge. Let $E'$ be the edge of $K$ opposite to $E$ and $K'$ be the element sharing the edge $E'$ with $K$. Let $E''$ be the edge of $K'$ opposite to $E'$ and $K''$ be the element sharing $E''$ with $K'$. Let $\omega_E = K \cup K' \cup K''$. By similar argument, we have $(\tau_1 - A_h \Pi_h \tau_1)(m) = 0$ when $\tau_1 \in Q_{1,1}(\omega_E) \times Q_{1,1}(\omega_E)$.

Using the property derived in the above three paragraphs, for $\tau_1 \in Q_{1,1}(\omega_K) \times Q_{1,1}(\omega_K)$, we have

$$\|\tau - A_h \Pi_h \tau\|_{L_2(K)} \lesssim h \|\tau - A_h \Pi_h \tau\|_{L_\infty(K)} \lesssim h \|(\text{id} - A_h \Pi_h)\|_{L_\infty(K)} \lesssim h \|\tau - \tau_1\|_{L_\infty(\omega_K)},$$

where $\text{id}$ is the identity mapping. Then by standard finite element approximation theory (cf. Corollary 4.4.7 in [8]),

$$\inf_{\tau_1 \in Q_{1,1}(\omega_K) \times Q_{1,1}(\omega_K)} \|\tau - \tau_1\|_{L_\infty(\omega_K)} \lesssim h \|\tau\|_{H^2(\omega_K)}$$ \hspace{1cm} (3.3)

and thus

$$\|\tau - A_h \Pi_h \tau\|_{L_2(K)} \lesssim h^2 \|\tau\|_{H^2(\omega_K)}.$$ \hspace{1cm} (3.4)

Then (3.2b) follows from (3.4) and sum of squares.

Combining Theorems 2.1 and 3.1, we obtain the superconvergent flux recovery estimate.

**Theorem 3.2** It holds that

$$\|a \nabla u - A_h \sigma_h\| \lesssim h^2 \|u\|_{H^2}.$$
Superconvergence of the Rannacher-Turek element

Proof Combining Theorems 2.1, 3.1 and the triangle inequality
\[ \|a\nabla u - A_h\sigma_h\| \leq \|a\nabla u - A_h\Pi_h\sigma\| + \|A_h(\Pi_h\sigma - \sigma_h)\| \]
completes the proof. \qed

Consider \( \tilde{\sigma}_h \in RT_{-1}^{-1} \), where
\[ \tilde{\sigma}_h |_{K} := Q_h(a\nabla u_{h}) - r_h(f - b \cdot \nabla u_{h} - cu_{h})(x_K), \]  
\[ \text{with } x_K = (x_{K,1}, x_{K,2})^T \text{ being the centroid of } K. \]
Since \( r_h = O(h) \), we have
\[ \|\tilde{\sigma}_h - \sigma_h\| \lesssim h^2 \|u\|_{H^2}. \]
and thus
\[ \|a\nabla u - A_h\tilde{\sigma}_h\| \lesssim h^2 \|u\|_{H^2}. \]
\( \tilde{\sigma}_h \) is favorable because of lower computational cost.

4 Extensions to triangular elements and higher dimensional space

In this section, we extend superconvergence analysis in Section 3 to triangular CR elements and NCRT elements in \( \mathbb{R}^d \) with \( d \geq 3 \).

4.1 Crouzeix–Raviart elements in \( \mathbb{R}^2 \)

Based on the equivalence between mixed and nonconforming methods for Poisson’s equation, a superconvergent recovery for CR elements applied to Poisson’s equation has been developed in [16]. We generalize this result for elliptic equations with lower order terms and variable coefficients. In this subsection, let \( T_h \) be a triangular mesh on \( \Omega \). The CR finite element space is
\[ V_{\Delta}^{\Delta} g,h := \{ v_h \in L_2(\Omega) : v_h|_K \in \text{span}\{1, x_1, x_2\} \text{ for all } K \in T_h, \]
\[ v_h \text{ is continuous at the midpoint of each } E \in E_h, \]
\[ \int_E v_h = \int_E g \text{ for all } E \in E_h^0 \}. \]
The CR method for (1.1) is to find \( u_h^\Delta \in V_{\Delta, h}^\Delta \), such that
\[ (a\nabla u_h^\Delta, \nabla v) + (b \cdot \nabla u_h^\Delta, v) + (cu_h^\Delta, v) = (f, v), \quad \forall v \in V_{\Delta, h}^\Delta. \]
The lowest order triangular RT finite element space is
\[ RT_{\Delta}^3 := \{ \tau_h \in H(\text{div}, \Omega) : \tau_h|_K \in \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} x_1 \\ 0 \\ x_2 \end{pmatrix} \right\} \text{ for all } K \in T_h \}. \]
It has been shown in [21] that CR and RT finite element spaces are closely related by the following lemma.
Lemma 4.1 Let \( \tilde{f} \) and \( \tau_h \) be piecewise constant functions with respect to \( T_h \). Assume that
\[
\langle \tau_h, \nabla_h v \rangle = \langle \tilde{f}, v \rangle
\]
for all \( v \in V_{\Delta h}^0 \). Then \( \tau_h - \bar{f} r_{\Delta h}^1 \in \mathcal{RT}_h^2 \), with
\[
r_{\Delta h}^1|_K := \frac{1}{2} (x_1 - x_{K,1}, x_2 - x_{K,2})^T,
\]
where \((x_{K,1}, x_{K,2})\) is the centroid of \( K \).

We say \( T_h \) is a uniform parallel mesh if each pair of adjacent triangles in \( T_h \) forms a parallelogram. A supercloseness estimate follows from Lemma 4.1, a supercloseness estimate for triangular RT elements in [17,15], and the same procedure in Section 2. By abuse of notation, \( \Pi_h \) denotes the canonical interpolation onto \( \mathcal{RT}_h^2 \).

Theorem 4.1 Let \( T_h \) be a uniform parallel mesh. Let
\[
\sigma_h^\Delta := \bar{a} \nabla_h \bar{u}_h^\Delta - r_{\Delta h}^1 P_h (f - cu_h^\Delta - b \cdot \nabla u_h^\Delta),
\]
where \( \bar{a}|_K = \tilde{f}_K a \) for \( K \in T_h \). It holds that
\[
\| \Pi_h \sigma - \sigma_h^\Delta \| \lesssim h^2 \| \log h \|^{1/2} \| u \|_{W_2^\infty}.
\]

Proof We use similar notations and proceed as in the proof of Theorem 2.1. Let \( \tau_h = \Pi_h \sigma - \sigma_h^\Delta \), where \( \sigma_h^\Delta = \bar{a} \nabla_h \bar{u}_h^\Delta - r_{\Delta h}^1 P_h (f - cu_h^\Delta - b \cdot \nabla u_h^\Delta) \) and \( \bar{u}_h^\Delta \) is the solution to the auxiliary problem (2.4) with \( V_{\Delta h}^0 \) replacing \( V_{h}^0 \).

It then follows from Lemma 4.1 that \( \tau_h \in \mathcal{RT}_h^2 \) with \( \nabla \cdot \tau_h = 0 \). Hence \( \tau_h = \nabla^\perp w_h \) for some continuous piecewise linear function \( w_h \), where \( \nabla^\perp = (-\partial x_2, \partial x_1)^T \). The bound (2.11) for part I is replaced by
\[
| \langle \sigma - \Pi_h \sigma, \nabla^\perp w_h \rangle | \lesssim h^2 \| \log h \|^{1/2} \| \sigma \|_{W_2^\infty} \| \nabla^\perp w_h \|,
\]
which is proved in [17]. The rest of the proof is the same as Theorem 2.1. \( \Box \)

For the recovery purpose, let
\[
V_h := \{ v_h \in L_2(\Omega) : v_h|_K \in \text{span}\{1, x_1, x_2\} \text{ for all } K \in T_h, \ v_h \text{ is continuous at the midpoint of each } E \in \mathcal{E}_h^0 \}.
\]
Then we consider the postprocessing operator \( K_h \) defined in [7], see also [13].

Definition 4.1 Let \( \tau_h \) be a piecewise constant function.

1. For each \( E \in \mathcal{E}_h^0 \), let \( m \) be the midpoint of \( E \). Let \( K^+ \) and \( K^- \) be the two rectangles sharing \( E \) as an edge. Define
   \[
   (K_h \tau_h)(m) := \frac{1}{2} \tau_h|_{K^+}(m) + \frac{1}{2} \tau_h|_{K^-}(m).
   \]
2. For each $E \in \mathcal{E}_h^0$, let $m$ denote the midpoint of $E$ and $K$ the element having $E$ as an edge. Let $E'$ be another edge of $K$ with midpoint $m'$. Let $K'$ be the other element having $E'$ as an edge and $m''$ the midpoint of the edge of $K'$ that is parallel to $E$. Define

$$(K_h \tau_h)(m) := 2(K_h \tau_h)(m') - (K_h \tau_h)(m'').$$

Then $K_h \tau_h$ is the unique element in $V_h^{\Delta}$ whose midpoint values are specified in the above two steps.

Based on Theorem 4.1, we obtain the superconvergent recovery for the CR element.

**Theorem 4.2** Let $T_h$ be a uniform parallel mesh. Then

$$\|a \nabla u - K_h(\bar{a} \nabla K_h u_h^\Delta)\| \lesssim h^2 |\log h|^{\frac{1}{2}} \|u\|_{W^2_\infty}.$$  

**Proof** The operator $K_h$ is known to satisfy Theorem 3.1 with $K_h$ replacing $A_h$, see [7]. It then follows from Theorem 4.1 and the same argument in the proof of Theorem 3.2 that

$$\|a \nabla u - K_h \sigma_h^\Delta\| \lesssim h^2 |\log h|^{\frac{1}{2}} \|u\|_{W^2_\infty}. \quad (4.1)$$

Let $p = f - cu - b \cdot \nabla u$ and $\tilde{\sigma}_h^\Delta := \bar{a} \nabla h u_h^\Delta - r_h^\Delta P_h p$. It follows from $\|r_h\|_{L^\infty} = O(h)$ and (1.3) that

$$\|\sigma_h^\Delta - \tilde{\sigma}_h^\Delta\| \lesssim h^2 \|u\|_{H^2}. \quad (4.2)$$

Let $m$ be the midpoint of any $E \in \mathcal{E}_h^0$. We have

$$[(K_h(r_h^\Delta P_h p))(m) = [K_h(r_h^\Delta p)](m) + [K_h(r_h^\Delta (P_h p - p))](m)$$

$$= (K_h r_h^\Delta)(m) p(m) + O(h^2) \|u\|_{W^2_\infty} = O(h^2) \|u\|_{W^2_\infty}.$$  

In the last equality, we use $(K_h r_h^\Delta)(m) = 0$. Similar argument works for $E \in \mathcal{E}_h^0$. Hence

$$\|K_h(r_h^\Delta P_h p)\| \lesssim \|K_h(r_h^\Delta P_h p)\|_{L^\infty} \lesssim h^2 \|u\|_{W^2_\infty}. \quad (4.3)$$

Combining (4.1)-(4.3) and the triangle inequality

$$\|a \nabla u - K_h(\bar{a} \nabla K_h u_h^\Delta)\| \leq \|a \nabla u - K_h \sigma_h^\Delta\|$$

$$+ \|K_h(\sigma_h^\Delta - \tilde{\sigma}_h^\Delta)\| + \|K_h(r_h^\Delta P_h p)\|$$

completes the proof \qed

It is noted that $K_h$ also superconverges on mildly structured meshes, see, e.g., [17]. However $K_h$ outputs a nonconforming function which is sometimes undesirable. For a vertex $z$ in $T_h$, let $\omega_z$ be the patch which is the union of triangles surrounding $z$. Define

$$\tilde{K}_h(\bar{a} \nabla h u_h^\Delta)(z) := \sum_{K \subset \omega_z} \frac{|K|}{|\omega_z|} \bar{a} \nabla h u_h^\Delta|_K.$$
We then obtain a nodal averaging procedure $\bar{K}_h$ and a continuous piecewise linear function $\bar{K}(\bar{a}\nabla_h u_h^\Delta)$. Following similar argument in this section, it is straightforward to show
\[
\|a
abla u - \bar{K}_h(\bar{a}\nabla_h u_h^\Delta)\| \lesssim h^2\|u\|_{H^1},
\]
provided $T_h$ is uniformly parallel.

4.2 Rannacher–Turek elements in $\mathbb{R}^d$

Let $\Omega \subset \mathbb{R}^d$ be a hypercube where $d \geq 3$ is an integer. Let $a, b, c, f, g$ in (1.1) be functions in $\mathfrak{x} = (x_1, \ldots, x_d) \in \Omega$. Let $T_h$ be a cubical mesh of $\Omega$. Let $F_h, F_h^\partial$, and $F_h^\partial$ denote the set of faces, interior faces, and boundary faces, respectively. The NCRT element space in $\mathbb{R}^d$ is
\[
\mathcal{V}_{g,h}^{(d)} := \{ v \in L^2(\Omega) : v|_K \in \text{span}\{1, x_1, \ldots, x_d, x_1^2 - x_2^2, \ldots, x_1^2 - x_2^2\}\}
\]
for all $K \in T_h$, $\int_F v$ is single-valued for all $F \in F_h^\partial$,
\[
\int_F v = \int_F g \text{ at the centroid of each } F \in F_h^\partial,
\]
where $\int_F v := \frac{1}{|F|} v$ is the surface mean of $v$ on $F$. The NCRT method for (1.1) in $\mathbb{R}^d$ is to find $u_h^{(d)} \in \mathcal{V}_{g,h}^{(d)}$, such that
\[
\langle a\nabla_h u_h^{(d)}, \nabla_h v \rangle + \langle b \cdot \nabla_h u_h^{(d)}, v \rangle + \langle cu_h^{(d)}, v \rangle = \langle f, v \rangle, \quad \forall v \in \mathcal{V}_{0,h}^{(d)}.
\]
Let $Q_1^{(j)}(K)$ is the space of polynomials on $K$ that are linear in $x_j$ and constant in $x_i$ for $i \neq j$. Let
\[
\mathcal{R}T_h^{(d)} := \{ \tau_h \in H(\text{div}; \Omega) : \tau_h|_K \in \Pi_j=1^d Q_1^{(j)}(K) \text{ for all } K \in T_h \}.
\]
The $H(\text{div})$-space in $\mathbb{R}^d$ is $H(\text{div}; \Omega) = \{ \tau \in H_0^1(\Omega) : \nabla \cdot \tau \in L^2(\Omega) \}$. The next lemma is a direct generalization of Lemma 2.1. The proof follows from direct (but tedious) calculation.

**Lemma 4.2** Let $\bar{f}$ be a piecewise constant, $\tau_h|_K \in \Pi_j=1^d Q_1^{(j)}(K)$ and $\nabla \cdot (\tau_h|_K) = 0$ for all $K \in T_h$. Assume that
\[
\langle \tau_h, \nabla_h v \rangle = \langle f, v \rangle
\]
for all $v \in \mathcal{V}_{0,h}^{(d)}$. Then $\tau_h - \bar{f}r_h^{(d)} \in \mathcal{R}T_h^{(d)}$, with
\[
r_h^{(d)}|_K \cdot e_i := \ell_{K,1} \cdots \ell_{K,i} \cdots \ell_{K,d} (x_i - x_{K,i}) / \sum_{j=1}^d \ell_{K,1} \cdots \ell_{K,j} \cdots \ell_{K,d},
\]
where $\cdot$ means the variable below is suppressed, $e_i$ is the i-th unit vector, $K = \Pi_j=1^d [x_{j,i}, x_{j,i+1}]$, $\ell_{K,j} = x_{j,i+1} - x_{j,i}$, $1 \leq j \leq d$, and $(x_{K,1}, \ldots, x_{K,d})$ is the centroid of $K$. ...
By Lemma 4.2 and following exactly the same procedure in Section 3, we have the supercloseness estimate in $\mathbb{R}^d$.

**Theorem 4.3** Let $Q_h^{(d)}$ be the $L_2$-projection onto $\nabla_h \mathcal{V}_0^{(d)}$ and

$$\sigma_h^{(d)} := Q_h^{(d)}(a \nabla_h u_h^{(d)}) - \tau_h^{(d)} P_h(f - c u_h^{(d)} - b \cdot \nabla_h u_h^{(d)}).$$

It holds that

$$\| \Pi_h^{(d)} (a \nabla u) - \sigma_h^{(d)} \| \lesssim h^2 \| u \|_{H^3}.$$ 

In particular, when $d = 3$,

$$\tau_h^{(3)} |_K = \left( \ell_{K,2}^2 \ell_{K,3}^2 (x_1 - x_{K,1}), \ell_{K,3}^2 \ell_{K,1}^2 (x_2 - x_{K,2}), \ell_{K,1}^2 \ell_{K,2}^2 (x_3 - x_{K,3}) \right)^T.$$

Let $A_h^{(3)}$ be the face-based weighed averaging generalized from $A_h$ in Definition 3.1. By very similar argument, one can show $A_h^{(3)} \Pi_h^{(3)} \sigma$ superconverges to $\sigma$. Hence we obtain the superconvergent flux recovery in $\mathbb{R}^3$.

**Theorem 4.4** For $d = 3$, it holds that

$$\| a \nabla u - A_h^{(3)} \sigma_h^{(3)} \| \lesssim h^2 \| u \|_{H^3}.$$ 

**Proof** The proof is same as Theorems 3.1 and 3.2. We require $d = 3$ since the inequality (3.3) with $h^{2-d}$ replacing $h$ does not hold for $d > 3$.

5 Numerical experiments

| ne  | $||u - u_h||$ | $||a \nabla u - a \nabla_h u_h||$ | $||\Pi_h^{(d)} (a \nabla u) - \sigma_h||$ | $||a \nabla u - A_h^{(3)} \sigma_h||$ |
|-----|----------------|-----------------------------|------------------------------------------|------------------------------------------|
| 6   | 3.455e-02      | 1.157e+00                  | 5.551e-01                                | 1.451e+00                                |
| 24  | 8.394e-03      | 5.723e-01                  | 1.366e-01                                | 4.591e-01                                |
| 96  | 2.112e-03      | 2.890e-01                  | 3.509e-02                                | 6.692e-02                                |
| 384 | 5.350e-04      | 1.457e-01                  | 8.122e-03                                | 1.274e-02                                |
| 1536| 1.352e-04      | 7.316e-02                  | 2.272e-03                                | 2.968e-02                                |
| 6144| 3.410e-05      | 3.671e-02                  | 5.698e-04                                | 7.318e-04                                |
| 24576| 8.582e-06     | 1.841e-02                  | 1.419e-04                                | 1.826e-04                                |

In this section, we test the recovery operators $A_h$ and $A_h^{(3)}$ instead of $\sigma_h$. Instead of $\sigma_h$ analyzed in Sections 3 and 4, we compute the pointwise version flux $\tilde{\sigma}_h$ in (3.5) and similar for $\tilde{\sigma}_h^{(3)}$. In tables, ‘ne’ denotes the number of elements in $\mathcal{T}_h$. The order of convergence is the value $p$, such that the error is $\approx C h^p$ for some constant $C$ independent of $h$. We evaluate $p$ by least squares using Tables 1 and 2.
Problem 1: Consider the equation (1.1) with \( \Omega = [0, 1] \times [0, 1] \),

\[
 u = \exp(2x_1 + x_2)x_1^2(x_1 - 1)^2x_2^2(x_2 - 1)^2,
 a(x) = \exp(x_1), \quad b(x) = x, \quad c(x) = \exp(x_1 + x_2),
\]

and corresponding \( g \) and \( f \). The initial rectangular mesh is

\[
 \mathcal{T}_h = \bigcup_{0 \leq i \leq 2, 0 \leq j \leq 1} [x_{1,i}, x_{1,i+1}] \times [x_{2,j}, x_{2,j+1}],
\]

where \( x_{1,0} = 0, x_{1,1} = 0.4, x_{1,2} = 0.8, x_{1,3} = 1, x_{2,0} = 0, x_{2,1} = 0.7, x_{2,2} = 1 \). We refine the mesh by connecting the midpoints of opposite edges of each rectangle. In the refinement, we randomly perturb the mesh along \( x_1 \)- and \( x_2 \)-directions by 20% of the length of the smallest interval in that direction, respectively. Numerical results are presented in Table 1. The first three rows in Table 1 are not used to evaluate the order since they are outside of the asymptotic regime.

**Table 2** Rate of convergence in \( \mathbb{R}^3 \)

| ne  | \( \| u - u_h^{(3)} \| \) | \( \| a \nabla u - a \nabla_h u_h^{(3)} \| \) | \( \| u_h^{(3)}(a \nabla u) - \tilde{\sigma}_h^{(3)} \| \) | \( \| a \nabla u - A_h^{(3)} \tilde{\sigma}_h^{(3)} \| \) |
|-----|-----------------|-----------------|-----------------|-----------------|
| 8   | 9.341e-01       | 1.280e+01       | 1.863e+01       | 2.238e+01       |
| 64  | 4.158e-01       | 9.418e+00       | 5.547e+00       | 1.516e+01       |
| 512 | 1.200e-01       | 5.032e+00       | 1.902e+00       | 3.448e+00       |
| 4096| 3.010e-02       | 2.525e+00       | 4.967e-01       | 8.599e-01       |
| 32768| 7.661e-03     | 1.269e+00       | 1.285e-01       | 1.709e-01       |
| order | 2.085         | 1.044           | 2.042           | 2.274           |

Problem 2: In the second experiment, consider the equation (1.1) with \( \Omega = [0, 1] \times [0, 1] \times [0, 1] \),

\[
 u(x) = \exp(x_1 + x_2) \sin(3\pi x_1) \sin(2\pi x_2) \sin(\pi x_3),
 a(x) = \exp(x_1 + x_2 + x_3), \quad b(x) = 0, \quad c(x) = 0,
\]

and corresponding \( g \) and \( f \). The initial cubical mesh is

\[
 \mathcal{T}_h = \bigcup_{0 \leq i \leq 1, 0 \leq j \leq 1, 0 \leq k \leq 1} [x_{1,i}, x_{1,i+1}] \times [x_{2,j}, x_{2,j+1}] \times [x_{3,k}, x_{3,k+1}],
\]

where

\[
 (x_{1,0}, x_{1,1}, x_{1,2}) = (0, 0, 1), \quad (x_{2,0}, x_{2,1}, x_{2,2}) = (0, 0, 1), \quad (x_{3,0}, x_{3,1}, x_{3,2}) = (0, 0, 1).
\]

We refine the mesh by connecting the centroid of opposite faces of each element. In the refinement, we randomly perturb the mesh along \( x_1 \)-, \( x_2 \)-, and \( x_3 \)-directions by 20% of the length of the smallest interval in that direction,
respectively. Numerical results are presented in Table 2. For similar reason, the first two rows are not used.

In the two experiments, since the mesh is randomly perturbed, computed errors are not exactly the same (but similar) every time. The numerical results show that our superconvergence estimates Theorems 2.1, 3.2, and 4.3 are asymptotically sharp. We also note that the rate of convergence in the last column of Table 2 is slightly larger than 2 predicted by Theorem 4.4. The expected reason is that the mesh size is not small enough since the computational cost on next several levels is out of the computational power of our machine.

References
1. Arbogast, T., Chen, Z.: On the implementation of mixed methods as nonconforming methods for second-order elliptic problems. Math. Comp. 64(211), 943–972 (1995)
2. Arnold, D.N., Brezzi, F.: Mixed and nonconforming finite element methods: implementation, postprocessing and error estimates. RAIRO Modél. Math. Anal. Numér. 19(1), 7–32 (1985)
3. Bank, R.E., Li, Y.: Superconvergent recovery of Raviart–Thomas mixed finite elements on triangular grids. J. Sci. Comput. (2019). doi:10.1007/s10915-019-01068-0
4. Bank, R.E., Xu, J.: Asymptotically exact a posteriori error estimators. I. grids with superconvergence. SIAM J. Numer. Anal. 41(6), 2294–2312 (2003)
5. Bank, R.E., Xu, J.: Asymptotically exact a posteriori error estimators. II. general unstructured grids. SIAM J. Numer. Anal. 41(6), 2313–2332 (2003)
6. Bank, R.E., Xu, J., Zheng, B.: Superconvergent derivative recovery for lagrange triangular elements of degree p on unstructured grids. SIAM J. Numer. Anal. 45(5), 2032–2046 (2007)
7. Brandts, J.H.: Superconvergence and a posteriori error estimation for triangular mixed finite elements. Numer. Math. 68(3), 311–324 (1994)
8. Brenner, S.C., Scott, L.R.: The mathematical theory of finite element methods, Texts in Applied Mathematics, 15, vol. 35, 3 edn. Springer, New York (2008)
9. Chen, C.M.: Structure theory of superconvergence of finite elements (in Chinese). Hunan Science and Technology Press, Changsha (2002)
10. Chen, H., Li, B.: Superconvergence analysis and error expansion for the Wilson nonconforming finite element. Numer. Math. 69(2), 120–140 (1994)
11. Crouzeix, M., Raviart, P.A.: Conforming and nonconforming finite element methods for solving the stationary Stokes equations. RAIRO Anal. Numér. 7(R-3), 33–75 (1973)
12. Durán, R.: Superconvergence for rectangular mixed finite elements. Numer. Math. 58(3), 287–298 (1990)
13. Durán, R., Muschietti, M.A., Rodríguez, R.: On the asymptotic exactness of error estimators for linear triangular finite elements. Numer. Math. 59(2), 107–127 (1991)
14. Guo, H., Zhang, Z.: Gradient recovery for the Crouzeix-Raviart element. J. Sci. Comput. 64(2), 456–476 (2015)
15. Hu, J., Ma, L., Ma, R.: Optimal superconvergence analysis for the Crouzeix-Raviart and the morley elements (2018). URL https://arxiv.org/abs/1808.09810
16. Hu, J., Ma, R.: Superconvergence of both the Crouzeix-Raviart and morley elements. Numer. Math. 132(3), 491–509 (2016)
17. Li, Y.W.: Global superconvergence of the lowest-order mixed finite element on mildly structured meshes. SIAM J. Numer. Anal. 56(2), 792–815 (2018)
18. Lin, Q., Tobiska, L., Zhou, A.: Superconvergence and extrapolation of non-conforming low order finite elements applied to the poisson equation. IMA J. Numer. Anal. 25(1), 160–181 (2005)
19. Liu, H., Yan, N.: Superconvergence analysis of the nonconforming quadrilateral linear constant scheme for stokes equations. Adv. Comput. Math. 29(4), 375–392 (2008)
20. Mao, S., Shi, Z.C.: High accuracy analysis of two nonconforming plate elements. Numer. Math. 111(3), 407–443 (2009)
21. Marini, L.D.: An inexpensive method for the evaluation of the solution of the lowest order raviart-thomas mixed method. SIAM J. Numer. Anal. 22(3), 499–496 (1985)
22. Ming, P., Shi, Z.C., Xu, Y.: Superconvergence studies of quadrilateral nonconforming rotated Q1 elements. Int. J. Numer. Anal. Model. 3(3), 322–332 (2006)
23. Rannacher, R., Turek, S.: Simple nonconforming quadrilateral Stokes element. Numer. Methods Partial Differential Equations 8(2), 97–111 (1992)
24. Wang, J.: Superconvergence analysis for finite element solutions by the least-squares surface fitting on irregular meshes for smooth problems. J. Math. Study 33(3), 229–243 (2000)
25. Wang, J., Ye, X.: Superconvergence of finite element approximations for the stokes problem by projection methods. SIAM J. Numer. Anal. 39(3), 1001–1013 (2001)
26. Xu, J., Zhang, Z.: Analysis of recovery type a posteriori error estimators for mildly structured grids. Math. Comp. 73(247), 1139–1152 (2004)
27. Ye, X.: Superconvergence of nonconforming finite element method for the Stokes equations. Numer. Methods Partial Differential Equations 18(2), 143–154 (2002)
28. Zhang, Y., Huang, Y., Yi, N.: Superconvergence of the Crouzeix-Raviart element for elliptic equation. Adv. Comput. Math. (2019). doi:10.1007/s10444-019-09714-9
29. Zhang, Z., Naga, A.: A new finite element gradient recovery method: superconvergence property. SIAM J. Sci. Comput. 26(4), 1192–1213 (2005)
30. Zienkiewicz, O.C., Zhu, J.Z.: A simple error estimator and adaptive procedure for practical engineering analysis. Internat. J. Numer. Methods Engrg. 24(2), 337–357 (1987)
31. Zienkiewicz, O.C., Zhu, J.Z.: The superconvergent patch recovery and a posteriori error estimates. I. the recovery technique. Internat. J. Numer. Methods Engrg. 33(7), 1331–1364 (1992)