Lower bounds of quantum black-box complexity and degree of approximation polynomials by influence of Boolean variables

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Abstract
We prove that, to compute a Boolean function $f : \{0,1\}^N \rightarrow \{-1,1\}$ with error probability $\epsilon$, any quantum black-box algorithm has to query at least $1 - 2\sqrt{\rho_f} N = 1 - 2\sqrt{\bar{S}_f}$ times, where $\rho_f$ is the average influence of variables in $f$, and $\bar{S}_f$ is the average sensitivity. It’s interesting to contrast this result with the known lower bound of $\Omega(\sqrt{S_f})$, where $S_f$ is the sensitivity of $f$. This lower bound is tight for some functions. We also show for any polynomial $\tilde{f}$ that approximates $f$ with error probability $\epsilon$, $\text{deg}(\tilde{f}) \geq \frac{1}{4}(1 - \frac{3}{\epsilon^2})^2 \rho_f N$. This bound can be better than previous known lower bound of $\Omega(\sqrt{{BS}_f})$ for some functions.

Our technique may be of interest itself: we apply Fourier analysis to functions mapping $\{0,1\}^N$ to unit vectors in a Hilbert space. From this viewpoint, the state of the quantum computer at step $t$ can be written as $\sum_{s \in \{0,1\}^N, |s| \leq t} \hat{\phi}_s(-1)^{s \cdot x}$, which is handy for lower bound analysis.

1 Introduction
To compute a Boolean function $f : \{0,1\}^N \rightarrow \{-1,1\}$ in the black-box model, the only way the computer can access the input is to ask an oracle questions like: what is $x_i$. The complexity measurement is the number of times the computer asks. For example, to compute $f(x_0, x_1, \ldots, x_{N-1}) = x_0 \uplus x_1 \uplus \ldots \uplus x_{N-1}$, any classical computer needs to query $\Omega(N)$ times, while surprisingly, there exists a quantum algorithm that queries only $O(\sqrt{N})$ times.

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The speedup of quantum computers can even be exponential if promise problems are considered [DJ92, Si94]. We study quantum lower bounds for error-bounded computation of total Boolean functions only.

Grover’s algorithm was shown to be optimal by [BBHT96, Za97], and tight bounds for all symmetric Boolean functions, in particular for some familiar ones like *PARITY*, *MAJORITY*, are shown in [BBCMW98] by polynomial method. The latter relates two characterizations of Boolean function complexity, the lowest degree of approximation polynomials and the block sensitivity, to lower bounds of quantum black-box complexity. A Boolean function \( f \) on \( N \) Boolean variables can be uniquely represented as a multi-linear real polynomial that takes \( f(x) \) for \( x \in \{0,1\}^N \). Let \( \tilde{f} \) be a real polynomial that approximates \( f \) and has the least degree. The block sensitivity of \( f \), \( BS_f \), is the maximum (over all possible inputs) number of non-intersecting blocks of variables in an input such that flipping all variables in one block flips the function value.

**Theorem 1.1** [BBCMW98] To compute a Boolean function \( f(x_0,x_1,\ldots,x_{N-1}) \) with error probability \( \epsilon \), any quantum computer needs to query at least \( \frac{1}{2} \deg(\tilde{f}) \) times.

**Theorem 1.2** [BBCMW98] To compute a Boolean function \( f \) on \( N \) variables with error probability \( \frac{1}{3} \), any quantum computer needs to ask at least \( \frac{1}{3} \sqrt{BS_f} \) times.

In the latter theorem, replace \( BS_f \) by the sensibility \( S_f \), we get the best known lower bound by of sensitivity. \( S_f \) is just the maximum (over all possible inputs) number of bits such that flipping each flips the function value.

Just like the lowest approximating degree, block sensitivity, and sensitivity, the influence of variables \( (\Inf_i(f)) \) and average sensitivity \( (\bar{S}_f) \), introduced by [KKL88], are yet other important characterizations of Boolean function complexity. One can show that \( \sum_i \Inf_i(f) = S_f \). Our main result provides the first known quantum black-box complexity lower bound by influence of variables (or equivalently, by average sensitivity):

**Theorem 3** To compute a Boolean function \( f : \{0,1\}^N \rightarrow \{-1,1\} \) with error probability \( \epsilon \), any quantum algorithm needs to query at least \( \frac{1}{2} \cdot \left[ 1 - \frac{1-2\sqrt{\epsilon}}{2} + \frac{1-2\sqrt{\epsilon}}{2} \sum_s \hat{f}_s^2 (1 - 2\lambda_s)^k \right] \cdot N \) times, for any positive odd integer \( k \), and \( \hat{f}_s = E_x [ f(x)(-1)^{x_s} ] \).

Actually we prove a stronger results:

**Theorem 4** To compute Boolean function \( f : \{0,1\}^N \rightarrow \{-1,1\} \) with error probability \( \epsilon \), any quantum algorithm needs to query at least \( \frac{1}{2} \cdot \left[ 1 - \frac{1+2\sqrt{\epsilon}}{2} + \frac{1-2\sqrt{\epsilon}}{2} \sum_s \hat{f}_s^2 (1 - 2\lambda_s)^k \right] \cdot N \) times, for any positive odd integer \( k \), and \( \hat{f}_s = E_x [ f(x)(-1)^{x_s} ] \).
The lowest degree of approximation polynomials for symmetric functions has a tight characterization \[Pa92\]. However, for asymmetric functions, the best general lower bound known is:

**Theorem 1.5** \[NS92\] For any Boolean function \( f \) and its approximation \( \tilde{f} \), \( \deg(\tilde{f}) \geq \sqrt{BS_f/6} \).

Our lower bound by influence can be better than this bound for some functions:

**Theorem 1.6** For any polynomial \( \tilde{f} \) that approximates \( f \) with error probability \( \epsilon \), \( \deg(\tilde{f}) \geq \frac{1}{4}(1 - \frac{3\epsilon}{1 + 3\epsilon})^2 \rho_f N \).

Note that, Theorem 1.6 and Theorem 1.1 together imply a lower bound of quantum black-box complexity that is asymptotically the same as in Theorem 1.3. However, the former gives a better constant.

Now we turn to our proof ideas.

For any oracle \( x \), let \( \phi(x) \) be the state of the quantum computer after \( T \) times of queries. If \( f(x) \neq f(y) \), then, in order to distinguish \( x \) and \( y \), \( \|\phi(x) - \phi(y)\|^2 \) must be large, meaning, lower bounded by \( 2 - 4\sqrt{\epsilon} \). If we pick \( x \) uniformly random from \( \{0, 1\}^N \), and pick a random bit \( x_i \), then flip \( x_i \) to get \( y \), then we make a hard time for the quantum algorithm: on one hand, \( E = E(x,y) \left[ \|\phi(x) - \phi(y)\|^2 \right] \) is large (meaning, lower bounded by \( (2 - 4\sqrt{\epsilon})\rho_f \)), on the other hand, this expected value is upper-bounded by \( \frac{4T}{N} \), which then gives us Theorem 1.3.

Here is how we get the upper bound of \( E \) by \( \frac{4T}{N} \). The key observation of \[BBCMW98\] is that with oracle \( x \), after \( T \) queries, the state of the quantum computer can be expressed as:

\[
\phi(x) = \sum_c p_c(x)|c\rangle,
\]

where \( \{c\} \) is a set of orthonormal basis, and \( p_c \)'s are polynomials over \( x_i \)'s of degree at most \( T \). Since \( p_c \)'s can be represented in Fourier basis \( \{L_s\} \), where \( L_s(x) = (-1)^s \cdot x \), we have:

\[
\phi(x) = \sum_{s \in \{0,1\}^N, |s| \leq T} \hat{\phi}_s (-1)^{s \cdot x},
\]

where \( \hat{\phi}_s \) are constant vectors depending on \( s \) only. The number of queries comes in in the maximum possible weight of \( s \) with \( \hat{\phi}_s \neq 0 \), since \( |s| \) is the degree of polynomial \( (-1)^{x \cdot s} \). This representation of \( \phi \) is implicit in \[FGGS98\], and has a natural interpretation that will be described later. We can now write down \( E \) explicitly in terms of \( \hat{\phi}_s \), and then get the desired upper bound by \( \frac{4T}{N} \).

For the lower bound of \( \deg(\tilde{f}) \) by average sensitivity, we use the same idea. We pick the same random pair \( (x, y) \), and examine the expected value of \( \|\tilde{f}(x) - \tilde{f}(y)\|^2 \) and bound this
Section 2 reviews standard notions and facts about Fourier transform and influence of variables. Section 3 looks at quantum black-box computation from the viewpoint of Fourier transform, and section 4 provides the proofs, which is followed by open problems.

Some miscellaneous notations: for any string $s \in \{0,1\}^N$, $|s|$ equals to the number of 1’s in $s$, and $\lambda_s = |s|/N$, $e_i \in \{0,1\}^N$ is the string with the only 1 in the $i$’th position, for $0 \leq i \leq N - 1$. All probability distributions are uniformly random over the corresponding domain. + usually is bitwise XOR, and $\cdot$ is usually inner product.

2 Fourier transform of Boolean functions and influence of variables

For any $s \in \{0,1\}^N$, let $L_s : \{0,1\}^N \rightarrow \{-1,1\}$ be $L_s(x) = (-1)^{s \cdot x}$ for any $x \in \{0,1\}^N$. $\{L_s\}$ form a basis for all functions mapping $\{0,1\}^N$ to $\mathbb{C}$.

**Lemma 2.1** (Well known) Any $f : \{0,1\}^N \rightarrow \mathbb{C}$ can be uniquely represented as: $f = \sum_s \hat{f}_s L_s$, where $\hat{f}_s = \mathbb{E}_x[f(x)L_s(x)]$, and $\max_{s,f \neq 0} \{|s|\} = \deg(f)$, and $\sum_s \hat{f}_s \hat{f}_s^* = \mathbb{E}_s[f(s)f^*(s)]$.

**Definition 2.2** [KKL88] The influence of variable $x_i$ on Boolean function $f(x_0, x_1, \ldots, x_{N-1})$ is given by: $\text{Inf}_{x_i}(f) = \mathbb{P}_x[f(x) \neq f(x + e_i)]$, the average influence (of variables) is $\rho_f = \mathbb{E}_i[\text{Inf}_{x_i}(f)]$.

**Lemma 2.3** [CG88, KKL88] For any Boolean function $f$ on $N$ variables, if $f = \sum_s \hat{f}_s L_s$, then $\rho_f = \sum_s \hat{f}_s^2 \lambda_s$.

**Definition 2.4** The sensitivity of $f$ on input $x$ is given by $S_f(x) = |\{i : f(x) \neq f(x + e_i)\}|$, and the average sensitivity is $\bar{S}_f = \mathbb{E}_x[S_f(x)]$.

It’s easy to check:

**Lemma 2.5** $\bar{S}_f = \rho_f N$. 

3 Quantum black-box algorithm from a viewpoint of Fourier transform

To compute a Boolean function $f$ on $N$ variables, a quantum computer uses four sets of registers $|i⟩_I |a⟩_A |w⟩_W |r⟩_R$: $I$ (index) has $\log N$ bits, $A$ (answer) and $R$ (result) are of one bit each, and $W$ (working) of a fixed number of bits. The quantum computer is working in the Hilbert space spanned by the base vectors $\{ |i⟩_I |a⟩_A |w⟩_W |r⟩_R \}$. An algorithm $\phi$ that computes $f$ in $T$ queries with error probability at most $\epsilon$ can be represented as a sequence of $2T + 1$ unitary operators on the $H$:

$$\phi : U_0 \rightarrow O \rightarrow U_1 \rightarrow O \rightarrow \ldots \rightarrow O \rightarrow U_T,$$

where $U_i$'s are arbitrary unitary transformations defined by the algorithm, and $O$'s are the query gates:

$$O : |i⟩_I |r⟩_A |w⟩_W |r⟩_R \rightarrow |i⟩_I |r + x_i⟩_A |w⟩_W |r⟩_R.$$

The quantum computer starts from the constant vector $|0⟩_{IAWR}$, apply the sequence of unitary transformations, and then observing the $R$ register yields $f(x)$ with error probability smaller than $\epsilon$.

Essentially, $\phi$ defines a function $\{0, 1\}^N \rightarrow \{ \bar{v} \in H, \| \bar{v} \|^2 = 1 \}$. A key observation in [BBCMW98] is,

**Lemma 3.1** [BBCMW98] On oracle $x$, the final state of the quantum computer can be written as:

$$\phi(x) = \sum_c p_c(x) |c⟩,$$

where $c$ is taking over all possible configurations of all the registers, and $p_c(x)$ is a polynomial mapping $\{0, 1\}^N$ to $C$ with $\deg(p_c) \leq T$.

Therefore, we can think of $\phi$ as a polynomial with coefficients in $H$. Then, any quantum black-box algorithm defines such a polynomial of degree at most $T$. Now we introduce the Fourier transform viewpoint of quantum black-box algorithms.

**Lemma 3.2** Let $\phi$ be defined by a quantum black-box algorithm, then $\phi$ can be represented as

$$\phi = \sum_{s, |s| \leq T} \hat{\phi}_s L_s,$$

where $\hat{\phi}_s = E_x \left[ \phi(x) L_s(x) \right]$, and $\sum_s \langle \hat{\phi}_s | \hat{\phi}_s \rangle = 1$. 

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Here is a natural interpretation of $\hat{\phi}_s$. Let’s shift the basis for $H$ from $\{|c\rangle\}$ to that spanned by eigenvectors of the oracle gates. Denote $F|j\rangle_A$ by $|\zeta_j\rangle_A$, where $F$ is the Hadamard transform on $Z_2^m$. The following lemma is easy to check.

**Lemma 3.3** For a general oracle gate considered in literature, namely, the oracle $x$ is regarded as a function $\{0,1\}^n \to \{-1,1\}$, and the oracle gate behaves in this way:

$$O_x|i\rangle|j\rangle_A = |i\rangle|j + x(i)\rangle_A,$$

Then

$$O_x|i\rangle|\zeta_j\rangle|w\rangle|r\rangle = (-1)^{x\cdot e_j^i}|i\rangle|\zeta_j\rangle|w\rangle|r\rangle,$$

where $e_j^i \in \{0,1\}^{2^n}$ and has $j$ in the $i$th block, and 0 in other blocks (each block’s length is $m$).

Now let’s see what’s going on in a quantum black-box computation, following directions of $\{|i\rangle|\zeta_j\rangle\}$: We start from the constant vector $|\vec{0}\rangle$, make a unitary transformation $U_0$, then we project into the subspace spanned by some $|i_1\rangle|\zeta_{j_1}\rangle$, then we make a query, resulting a sign-flip $(-1)^{x\cdot e_{j_1}^i}$, then we continue our walk in the same manner: make the second unitary transformation $U_1$, then go into subspace of some $|i_2\rangle|\zeta_{j_2}\rangle$, ask the oracle to flip our sign by $(-1)^{x\cdot e_{j_2}^i}$, and so on. One good thing about this walk is that every time we project to a subspace or make a query, for different oracle, the only difference is the sign, and this sign is given by

$$L_x(e_{j_1}^{i_1} + e_{j_2}^{i_2} + \cdots + e_{j_t}^{i_t}),$$

if we’ve gone through the path

$$p = |i_1\rangle|\zeta_{j_1}\rangle \to |i_2\rangle|\zeta_{j_2}\rangle \cdots \to |i_t\rangle|\zeta_{j_t}\rangle.$$

We use $\alpha_p$ to denote the vector reached by path $p$ with oracle 0, and $\pi_p = e_{j_1}^{i_1} + e_{j_2}^{i_2} + \cdots + e_{j_t}^{i_t}$, then it’s easy to check:

**Lemma 3.4** $\hat{\phi}_s = \sum_{p, \pi_p = s} \alpha_p$.

4 Main result

4.1 Lower bound of quantum black-box complexity

We’re given a Boolean function $f : \{0,1\}^N \to \{-1,1\}$, which can be represented as $f = \sum_{s \in \{0,1\}^N} \hat{f}_s L_s$. Let $\phi : \{0,1\}^N \to H$ be defined by a quantum black-box algorithm that uses
queries to compute $f$ with error probability bounded by $\epsilon$, and for any input $x \in \{0,1\}^N$, 
\[ \phi(x) = \sum_{a:|a| \leq T} \phi_a L_a(x). \]

Now we take $x$ uniformly random from $\{0,1\}^N$, and take $i_1, i_2, \ldots, i_k$ uniformly and independently random from $\{0,1,\ldots,N-1\}$, for positive odd integer $k$. Let $\vec{i} = e_{i_1} + e_{i_2} + \cdots + e_{i_k}$. We would like to bound 
\[ E = E_{x,i_1,i_2,\ldots,i_k}[\|\phi(x) - \phi(x + \vec{i})\|^2] \]
from both side – the lower bound is in terms of the average sensitivity, and the upper bound by $T$, and then get the lower bound for $T$.

To bound $E$ from below, first note that when a pair of input $x, y \in \{0,1\}^N$ have different function value, the corresponding vector $\phi(x)$ and $\phi(y)$ are far apart:

**Fact 1** If $f(x) \neq f(y)$, $\|\phi(x) - \phi(y)\|^2 \geq 2 - 4\sqrt{\epsilon}.$

Therefore the fraction such that $f(x) \neq f(x + \vec{i})$ bounds $E$ from below. The fraction is given by the following lemma:

**Lemma 4.1** $Pr_{x,i_1,i_2,\ldots,i_k}[f(x) \neq f(x + \vec{i})] = \frac{1}{2} - \frac{1}{2} \sum_s \hat{f}_s^2 (1 - 2\lambda_s)^k$.

**Proof.**
\[
Pr_{x,i_1,i_2,\ldots,i_k}[f(x) \neq f(x + \vec{i})] = \frac{1}{2} E[1 - f(x)f(x + \vec{i})] \quad (1)
\]
\[
= \frac{1}{2} \frac{1}{2} \sum_{s_1,s_2} \hat{f}_{s_1} \hat{f}_{s_2} E[L_{s_1}(x)L_{s_2}(x + \vec{i})] \quad (2)
\]
\[
= \frac{1}{2} \frac{1}{2} \sum_{s_1,s_2} \hat{f}_{s_1} \hat{f}_{s_2} E[(-1)^{x(s_1+s_2)+\vec{i} \cdot s_2}] \quad (3)
\]
\[
= \frac{1}{2} \frac{1}{2} \sum_s \hat{f}_s^2 (E_{i \in \{0,\ldots,N-1\}}[(-1)^{e_i \cdot s}])^k \quad (4)
\]
\[
= \frac{1}{2} \frac{1}{2} \sum_s \hat{f}_s^2 (1 - 2\lambda_s)^k \quad (5)
\]

When $k = 1$, this probability is exactly the average sensitivity. Putting the fact and the lemma together we have:
Lemma 4.2  $E \geq (2 - 4\sqrt{\epsilon})(\frac{1}{2} - \frac{1}{2} \sum_s \hat{f}_s^2(1 - 2\lambda_s)^k)$

Now let’s bound $E$ from above, and we’ll see how powerful the Fourier representation of $\phi$ is.

Lemma 4.3  $E \leq 2 - 2(1 - \lambda_T)^k$.

Proof.  ($|a| \leq T$.)

\begin{equation}
E = E\left[\left\| \sum_{a,|a| \leq T} L_a(x)[1 - L_a(\vec{i})]\hat{\phi}_a \right\|^2 \right]
\tag{7}
\end{equation}

\begin{equation}
= E\left[\sum_{a,b} L_{a+b}(x)[1 - L_a(\vec{i})][1 - L_b(\vec{i})]\langle \hat{\phi}_a | \hat{\phi}_b \rangle \right]
\tag{8}
\end{equation}

\begin{equation}
= \sum_a E_{i_1, i_2, \ldots, i_k} \left[(1 - L_a(\vec{i}))^2 \right] \| \hat{\phi}_a \|^2
\tag{9}
\end{equation}

\begin{equation}
= \sum_a w_a \| \hat{\phi}_a \|^2
\tag{10}
\end{equation}

Where $w_a = E_{i_1, i_2, \ldots, i_k} [(1 - L_a(\vec{i}))^2]$. Now let’s bound $w_a$ by $\lambda_T$, and we think of $a \in \{-1, 1\}^N$, with the standard interpretation of $-1$ as $1$ in the original $a$, and $1$ as $0$.

\begin{equation}
w_a = 4 \Pr\left[ a_{i_1} \cdot a_{i_2} \cdots a_{i_k} = -1 \right]
\tag{11}
\end{equation}

\begin{equation}
= 4E\left[ \frac{1 - a_{i_1} \cdot a_{i_2} \cdots a_{i_k}}{2} \right]
\tag{12}
\end{equation}

\begin{equation}
= 2 - 2(E_i[a_i])^k
\tag{13}
\end{equation}

\begin{equation}
= 2 - 2(1 - 2\lambda_a)^k
\tag{14}
\end{equation}

\begin{equation}
\leq 2 - 2(1 - 2\lambda_T)^k
\tag{15}
\end{equation}

Note that $\sum_a \| \hat{\phi}_a \|^2 = 1$, therefore,

\begin{equation}
E \leq 2 - 2(1 - 2\lambda_T)^k.
\end{equation}

Now put the above two lemmas together, solve the inequality, then we get Theorem 1.4.  Theorem 1.3 is obtained by Theorem 1.4 and Lemma 2.3.

At the time of writing, we don’t know if we can make use of the theorem for $k > 1$. But for $k = 1$, any lower bound for sum of influence implies lower bound for quantum black-box complexity. Here are two examples:

When $f$ is a random function, $\rho_f = \frac{1}{2}$, therefore we have:
Corollary 4.4  To compute a random function $f$ with error probability $\epsilon$, the expected number of queries of any quantum computer is at least $\frac{1-2\sqrt{\epsilon}}{4}N$.

This matches [Da98] up to a constant factor, though slightly worse than the lower bound of [Am98]. Since the PARITY function has average sensitivity 1, we have

Corollary 4.5  To compute PARITY with error probability $\epsilon$, any quantum computer needs to query at least $\frac{1-2\sqrt{\epsilon}}{2}N$ times.

This is also a known result [BBCMW98], [FGGS98].

4.2  Degree lower bound of approximating polynomials

Definition 4.6  [NS92] Let $f: \{0,1\}^N \rightarrow \{0,1\}$ be a Boolean function, we say $\tilde{f}: \{0,1\}^N \rightarrow \mathbb{R}$ approximates $f$ with error probability $\epsilon$ if for any $x \in \{0,1\}^N$, $|f(x) - \tilde{f}(x)| \leq \epsilon$.

Let $\tilde{f} = \sum_s \hat{\tilde{f}}_s L_s$, and $d = \deg(\tilde{f})$. Define

$$E' = E_{x,i} [ |\tilde{f}(x) - \tilde{f}(x+e_i)|^2 ].$$

Putting the following two lemmas together we get Theorem 1.6.

Lemma 4.7  $E' \geq (1 - 2\epsilon)^2 \rho_f$.

Proof. $E' \geq Pr[f(x) \neq f(x+e_i)](1 - 2\epsilon)^2 = (1 - 2\epsilon)^2 \rho_f$. $\blacksquare$

Lemma 4.8  $E' \leq 4(1 + \epsilon)^2 \frac{d}{N}$.

Proof.

$$E' = E [ \sum_s \hat{\tilde{f}}_s (1 - (-1)^{s \cdot e_i}) L_s(x)]^2$$

$$= \sum_{s_1,s_2} \hat{\tilde{f}}_{s_1} \hat{\tilde{f}}_{s_2} E_x [L_{s_1+s_2}(x)] E_i [1 - (-1)^{s_1 \cdot e_i}(1 - (-1)^{s_2 \cdot e_i})]$$

$$= \sum_s \hat{\tilde{f}}_s^2 E_i [1 - (-1)^{s \cdot e_i}]^2$$

$$= 4 \sum_s \hat{\tilde{f}}_s^2 \lambda_s$$
\[
\begin{align*}
\leq \frac{4d}{N} \sum_s \tilde{f}_s^2 \\
\leq 4(1 + \epsilon)^2 d
\end{align*}
\]

Since most functions have large average influence, our lower bound by influence (Theorem 1.1) is better than the bound by block sensitivity (Theorem 1.5) for most functions. An extreme example is \textit{PARITY}. Here is an example of asymmetric function: Let \( f(x_0, x_1, x_2, x_3) = x_0(x_1 - x_2)^2 + (1 - x_0)(x_2 - x_3)^2 \), and \( f_k \) is obtained by iterate \( f \) \( k \) times: \( f_k(X_0, X_1, X_2, X_3) = f(f_{k-1}(X_0), f_{k-1}(X_1), f_{k-1}(X_2), f_{k-1}(X_3)) \), where \( X_i \) is the \( i \)th block of \( 4^{k-1} \) input variables. Then one can show that \( BS_{f_k} = 3^k \), \( \bar{S}_{f_k} = 2.5^k > \sqrt{BS_{f_k}} \). It is conceivable that our lower bound is beneficial in proving degree lower bounds of asymmetric functions when the bound by block sensitivity is not good. (Although, in our example, \( \bar{S}_{f_k} \) is not a tight bound for \( \tilde{f} \) either: one can show that \( \text{deg}(\tilde{f}) \geq 3^k \).)

5 Open problems

So far the lower bound of quantum black-box complexity by the degree of approximation polynomials implies any other (asymptotic) lower bounds. Is it asymptotically optimal for all Boolean functions?

For polynomials that approximates symmetric Boolean functions, \cite{Pa92} gives a tight characterization on the lowest degree. However, we do not know much about the case of asymmetric functions. Is there any general tight bound for the lowest degree of approximation polynomials?

The relations of block sensitivity and sensitivity and their average cases have been longstanding open problems \cite{NS92, Ru95, Be96, Ve98}. Is there any lower bound of quantum black-box complexity and lower degree of approximation polynomials by average block sensitivity?

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