Quantum Coulomb Blockade in Chaotic Systems

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We study the Coulomb blockade in a chaotic cavity connected to a lead by a perfectly transmitting quantum channel. In contrast to the previous theories, we show that the quantum fluctuations of charge, resulting from the perfect transmission, do not destroy the Coulomb blockade completely. The oscillatory dependence of all the observable characteristics on the gate voltage persists, its period is not affected; however, phases of the oscillations are random, reflecting the chaotic electron motion in the cavity. Because of this randomness, the Coulomb blockade shows up not in the averages but in the correlation functions of the fluctuating observables (e.g., capacitance or tunneling conductance).

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The effect of Coulomb blockade in chaotic systems sets up an ideal stage for studying the interplay between the quantum chaos and interaction phenomena in a many-electron system. By tuning the connection between the leads and chaotic cavity, Fig. 1, one can study a rich variety of non-trivial effects. In the weak tunneling limit, discrete charging of the cavity results in a system of sharp conductance peaks, which carry information about the chaotic motion of non-interacting electrons confined inside an almost closed cavity. In the opposite limit of wide channels, charge quantization does not occur, and quantum chaos of free electrons in an open billiard may be studied. In a broad intermediate region, the charge quantization is gradually destroyed, and the chaotic electron motion is affected by fluctuations of charge of the cavity. The modern experimental technique allows one to continuously traverse between these regimes.

The effect of charging is described by means of the Hamiltonian

$$H_C = \frac{E_C}{2} \left( \frac{\hat{Q}}{e} - N \right)^2,$$

where the conventional parameter $N$ is related to the gate voltage $V_g$ and gate capacitance $C_g$ by $N = V_g/eC_g$, and $\hat{Q}$ is the cavity charge. Charging energy $E_C$ for large cavities is much larger than the mean level spacing $\Delta$. If the connection of the cavity with the leads is weak, the charge is well quantized for almost all $N$ except narrow vicinities of the charge degeneracy points (half-integer $N$). The behavior of the differential capacitance of the cavity, $dQ/dV_g$ and of the conductance through the cavity is quite different for the system tuned to the immediate vicinity of charge degeneracy points (Coulomb blockade peaks), or away from these points (Coulomb blockade valleys). The statistics of the peaks can be related to the properties of single electron energies and wave functions, so that the distribution functions for observable quantities can be extracted from the well known random matrices results. The transport in the valleys occurs by virtual transitions of an electron via excited states of the cavity. The statistics of conductance in this case was recently obtained in Ref. [6].

![FIG. 1. Schematic view of a chaotic cavity connected to a lead by an ideal quantum channel. Second lead (triangle) can be attached to the cavity by weak tunnel junction for measurements of two-terminal conductance.](image-url)

All the aforementioned results are obtained neglecting quantum fluctuations of the charge of the cavity. These fluctuations become large with increase of the coupling between the cavity and the lead. Then, the difference between the peaks and valleys becomes less pronounced and eventually instead of the peak structure, one observes only a weak periodic modulation. This situation can be described neither by the properties of the single-electron wave function nor by the lowest order virtual transitions via the excited states. The case of almost perfect transmission of a one-channel point contact connecting a cavity with the lead was analysed recently by Matveev [7]. According to Ref. [7], the Coulomb blockade disappears completely if the transmission coefficient of the point contact is exactly unity and $\Delta = 0$. The strength of the theory is in non-perturbative treatment of the Coulomb interaction; the drawback is in neglecting the chaotic motion of electrons in the cavity.

In this Letter, we account for both the strong quan-
tum charge fluctuations, and the chaotic electron motion within the cavity. As we will see, backscattering of electrons from the walls of the cavity into the channel connecting to it lead results in oscillations of observables with the gate voltage. In the limit of perfect channel transmission, the relative magnitude of the differential capacitance oscillations is $\sqrt{\Delta/E_C}$ and $(\Delta/E_C) \ln^2 (E_C/\Delta)$ for the spinless and for spin 1/2 cases respectively. If the second lead is attached to the cavity (see Fig. 1) by a weak tunnel junction with conductance $G_0 \ll e^2/h$, the two-terminal conductance $G$ can be measured. The average value of the conductance is suppressed by the Coulomb blockade, $(G) \simeq G_0 \Delta/E_C$, fluctuations of the conductance are of the order of its average. This resembles the result for the elastic cotunneling in weak coupling regime [5,6]. However, the distinction between valleys and sharp peaks is lost.

Spinless electrons, qualitative discussion – Here, we use an analogy to the Fermi liquid theory of the Kondo problem due to Nozieres [8]. The dynamics of the system at energies smaller than $E_C$ can be well described by the lead and the cavity effectively separated from each other. Indeed, an electron inside the cavity is not affected by the interaction, unless, in the course of chaotic motion, it encounters the channel entrance. Even if the trajectory reaches the channel entrance (Fig. 1), the electron is scattered back, provided its energy $\epsilon$ (measured from the Fermi level) is much smaller than the charging energy $E_C$. The phase shift $\delta$ of this scattering at $\epsilon = 0$ is fixed by the Friedel sum rule $\delta(0) = \pi N$, where we used the fact that for an ideal transmitting channel the charge of the cavity is $Q = eN$. On the other hand, the variation of the phase of the electron reflection from the channel entrance $\alpha(\epsilon/E_C) = \delta(\epsilon,N) - \pi N$ is of the order of unity in the energy interval $|\epsilon| < E_C$.

Because an electron at low energy $\epsilon$ cannot leave the cavity, its motion in the cavity is quantized. The one-electron energies are periodic functions of $N$, with characteristic amplitude $\Delta$ and random phases (the latter assumption will be further elaborated on). Oscillating with $N$ contribution to the ground state energy $\delta E(N)$ is determined by sum of the eigenvalues in the energy interval $|\epsilon| < E_C$; therefore $\delta E(N)$ is the sum of approximately $E_C/\Delta \gg 1$ random numbers. As the result, the average $\langle \delta E(N) \rangle$ vanishes, and the correlation function of the differential capacitance (in units of $C_g$)

$$K_C(N - M) \equiv \frac{1}{E_C} \partial_M^2 \partial_N^2 \langle \delta E(N) \delta E(M) \rangle \quad (2)$$

takes the form

$$K_C(N_-) = \Lambda(0) \left( \frac{\Delta}{E_C} \right) \cos 2\pi N_- \quad (3)$$

where the coefficient $\Lambda(0)$, see Eq. (1), is the result of the effective action theory outlined later.

In order to explain the functional form of $K_C$ in Eq. (3) and make our argumentation more precise, it is very instructive to evaluate the shift of the energy levels starting from the Gutziwer trace formula [1]. The energy of the ground state is given by $E = -\int_{-\infty}^\infty d\epsilon N(\epsilon)$, where $\epsilon$ is measured from the Fermi level, and the integrated density of states $N(\epsilon) = \sum \delta(\epsilon - \epsilon_i)$ is expressed as a sum over the classical periodic orbits:

$$N(\epsilon) = \text{Re} \sum_j R_j(\epsilon) \exp \left[ \frac{i}{\hbar} S_j(\epsilon) + 2i m_j \delta(\epsilon, N) \right]. \quad (4)$$

Here $R_j$ is the weight associated with $j$-th orbit, $S_j$ is the reduced action for this orbit. The last term in the exponent in Eq. (4) characterizes the reflection from the entrance of the cavity, and $n_j$ is the number of such reflections for $j$-th orbit. We omitted the mean value of $N(\epsilon)$ which is independent of the phase shift $\delta$. In the double sum over the periodic orbits, arising in evaluation of (4), one can retain only diagonal terms [1] because different orbits have different actions; the non-diagonal terms oscillate strongly and vanish upon the averaging. We obtain from Eqs. (4) and (2)

$$K_C(N_-) \simeq \text{Re} \int_{-\infty}^0 \frac{dt_1 dt_2}{E_C} \sum_j |R_j|^2 e^{2\pi i n_j N_-} e^{2\pi i (\epsilon_1 - \epsilon_2) t_j} \times \sum_j e^{2\pi i n_j [\alpha(\epsilon_1/E_C) - \alpha(\epsilon_2/E_C)]}, \quad (5)$$

where we used the expansion $S_j(\epsilon_1) - S_j(\epsilon_2) = (\epsilon_1 - \epsilon_2) t_j$, with $t_j$ being the period of $j$-th orbit.

In a chaotic system the orbits proliferate exponentially with the increasing period and cover all the energy shell. This exponential proliferation is compensated by the damping of the weight $|R_j|^2$ which eventually leads to the classical sum rule [11]

$$\sum_j |R_j|^2 \rightarrow \frac{1}{2\pi^2} \int_0^\infty \frac{dt}{t} \ldots \quad (6)$$

valid for periods $t$ much larger than the period of the shortest orbit $h/E_T$. Energy $E_T$ associated with the time scale at which the classical dynamics becomes ergodic is the counterpart of the Thouless energy for the diffusive system. Typically $E_T \gtrsim E_C$, therefore we adopt approximation $E_T \gg E_C \gg \Delta$. Application of Eq. (1) to Eq. (3) yields

$$K_C(N_-) \simeq \int_0^\infty \frac{dt}{t} F\left( \frac{E_C t}{\hbar} \right) \sum_n n^4 P_n(t) \cos 2\pi n N_- \quad (7)$$

Here $F(\frac{E_C t}{\hbar}) = \int_{-\infty}^0 \frac{dt}{t} e^{\pi |t||2i\alpha(\epsilon/E_C)|^2}$ is a dimensionless function. The number of long orbits is exponentially large and they uniformly cover the available phase space, therefore, we can employ the stastical description of the
degrees of freedom of the cavity and obtain a formally exact expression [12] for the partition function
\[ Z = e^{-\beta H_0} (T_\tau e^{-\hat{S}}), \]  
where the \( \mathcal{N} \)-independent energy \( \Omega_0 \) does not contribute to the differential capacitance and can be neglected, and \( T_\tau \) stands for the imaginary time ordering. The averaging \( \langle \ldots \rangle \) is performed over the effective one-dimensional Hamiltonian
\[ \hat{H}_0(\mathcal{N}) = iv_F \int dx \left\{ \psi_L^\dagger \partial_x \psi_L - \psi_R^\dagger \partial_x \psi_R \right\} \]
\[ + \frac{E_C}{2} \left( \int_{-\infty}^0 dx : \psi_L^\dagger \psi_L + \psi_R^\dagger \psi_R : + \mathcal{N} \right)^2, \]
where \( v_F \) is the Fermi velocity in the channel, and \( \ldots \ldots \) stands for the normal ordering. The effective action \( \hat{S} \) is
\[ \hat{S} = \int_0^\beta d\tau_1 d\tau_2 L (\tau_1 - \tau_2) \tilde{\psi}(\tau_1) \psi(\tau_2), \]
where \( \psi(\tau) = e^{\tau \hat{H}_0} (\psi_L(x = 0) + \psi_R(x = 0)) e^{-\tau \hat{H}_0} \) are the fermionic operators in the Matsubara representation, \( \psi(\tau) = \psi(-\tau) \). The kernel \( L(\tau) \) can be expressed in terms of the exact Green function of the closed cavity:
\[ L(\tau) = \frac{1}{4 m_T} \int dy dy' \phi(y) \phi(y') \partial_{xx}^2 \! \tilde{G}(\tau; \mathbf{r}, \mathbf{r}'), \]
where \( \tilde{G} = \mathcal{G} - \mathcal{G}_\infty \), the exact Matsubara Green function \( \mathcal{G} \) is defined with zero boundary conditions (closed cavity), \( \mathcal{G}_\infty \) is the Green function corresponding to an infinite ballistic cavity, and \( \phi(y) \) is the wave function of the transverse motion in the contact. Physically, kernel [14] accounts for all possible returns of an electron to the contact during its chaotic motion within the cavity.

If conductance of the junction connecting the cavity to the second lead is small, \( G_0 \ll e^2 / \pi h \), its effect can be accounted for in the second order perturbation theory. Derivation similar to that of Eqs. [12]-[14] yields for the two-terminal conductance
\[ G / G_0 = \frac{1}{\pi \nu V} \text{Im} \left[ \lim_{\Omega_0 \to V + i 0} \int_0^\beta d\tau e^{-i \Omega_0 \tau} \sinh \pi T \tau \right], \]
where \( \nu \) is the averaged density of states per unit area, \( \Omega_0 = 2\pi T n \) is the bosonic Matsubara frequency, and function \( \Pi(\tau) \) is given by
\[ \Pi(\tau) = \frac{1}{\langle T_\tau e^{-S} \rangle} \int_0^\beta d\tau_1 d\tau_2 L_e(\tau_1) L^*_e(\tau_2) \]
\[ \times (T_\tau e^{-S} \hat{F}(\tau) \hat{F}(0) \psi(\tau - \tau_1) \psi(\tau_2)). \]
In Eq. [16], we retained only the contribution non-vanishing in the limit \( T \to 0 \). Unitary operator \( \hat{F} \) in Eq. [16] shifts the number of electrons in the cavity by
one \( \hat{F} \hat{H}_0(N) \hat{F} = \hat{H}_0(N + 1) \); the kernel \( L_c(\tau) \) describes the motion of an electron from the tunneling junction \( r_i \) to the entrance of the cavity, and is related to the exact one-electron Green function of the closed cavity:

\[
L_c(\tau) = \frac{1}{2m} \int d\psi(y) \partial_2 G(\tau; r_i, r). \tag{17}
\]

In the case of electrons with spins, we imply summation over the spin indices in Eqs. (2), (3) and (16).

The advantage of the representation (13) and (16) is that the Hamiltonian (12) is exactly solvable by the bosonization technique \([12]\), and the action \( S \) can be treated perturbatively provided that \( \Delta \ll E_C, E_T \). Green function \( G \) is a random quantity, and its statistics can be obtained using the well-known expansion in the diffusion and Cooperon modes \([13]\). In the regime \( E_C \ll E_T \) the results become universal, i.e. independent on the details of chaotic dynamics in the system. Below, we present the results of the calculation; details will be reported elsewhere \([13]\).

**Spinless electrons, results** – For spinless electrons (Zeeman splitting exceeds the Fermi energy of electrons in the channel), it suffices to consider only the first order perturbation theory in \( \delta \) for the calculation of the capacitance which gives Eq. (3). Conductance is found from Eq. (15) neglecting action \( S \) at all. For the average conductance, we find

\[
\langle \langle G \rangle \rangle = G_0 \frac{\Delta}{\gamma^2 E_C} \Lambda(0), \tag{18}
\]

where \( \gamma \approx 1.78 \ldots \) is the Euler constant and function \( \Lambda(x) \) is defined in Eq. (3). The correlation function of the conductance fluctuations is given by

\[
\frac{\langle \delta G_1 \delta G_2 \rangle}{\langle \langle G \rangle \rangle^2} = \left[ \Lambda^2 \left( \frac{B^2_1}{E_C^2} \right) + \Lambda^2 \left( \frac{B^2_2}{E_C^2} \right) \right] \cos^2 \pi N_- / \Lambda^2(0), \tag{19}
\]

where the correlation magnetic field is given by Eq. (5), we introduced the short hand notation, \( G_i = G(N_i, B_i) \), and \( B_{\pm} = B_1 \pm B_2, N_- = N_1 - N_2 \).

**Spin 1/2 case, results** – In the spin 1/2 case, the nonvanishing correlation function of the differential capacitance appears only in the second order of perturbation theory in \( \delta \). In zero magnetic field, we find

\[
K_C^2(N_-) = \frac{2\Delta^2}{3\pi^2 E_C^2} \ln^4 \left( \frac{E_C}{T} \right) \cos 2\pi N_- . \tag{20}
\]

We present here also the correlation function in the unitary limit \( B_+ \gg B_\pm \),

\[
K_C^2 = \frac{16\Delta^2}{3\pi^2 E_C^2} \ln^3 \left| \frac{B_+}{B_-} \right| \ln \left( \frac{E_C}{T} \right) \left( \frac{B_-}{B_+} \right)^{1/2} \cos 2\pi N_- . \tag{21}
\]

valid for the region of fields \( B_+ \sqrt{T/E_C} \ll B_- \ll B_\pm \).

To calculate the average conductance, one can neglect \( \delta \), similarly to the spinless case. This yields

\[
\langle \langle G \rangle \rangle = G_0 \frac{\Delta}{\gamma\gamma E_C} \ln \left( \frac{E_C}{T} \right). \tag{22}
\]

The main contribution to the conductance correlation function is independent on \( N_- \), but depends on the magnetic field; in the unitary limit

\[
\frac{\langle \langle G \rangle \rangle^2}{\langle \langle G \rangle \rangle} = \frac{1}{2} \left[ \ln \left( \frac{B_+}{B_\pm} \right) \ln \left( \frac{E_C}{T} \right) \right]^2. \tag{23}
\]

The characteristic magnitude of oscillating with \( N \) contribution to the correlation function is smaller than Eq. (23) by a parameter \( \Delta / E_C \).

At low temperatures \( T \), the above results diverge, which indicates that the higher order terms in \( \delta \) should be taken into account. Using a variational approach \([12]\), we conclude that \( T \) should be replaced with \( \Delta \ln^2(E_C/\Delta) \) in Eqs. (20)-(23).

In conclusion, Coulomb blockade oscillations persist even if a cavity is connected to a lead by a perfect single-mode channel. This effect is ignored if one approximates the electron spectrum in the cavity by a continuum \([7]\), or neglects the charge quantization \([13]\).

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