Robustly Detecting Changes in Warm Jupiters’ Transit Impact Parameters

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Abstract

Torques from a mutually inclined perturber can change a transiting planet’s impact parameter, resulting in variations in the transit shape and duration. Detection of and upper limits on changes in impact parameter yield valuable constraints on a planetary system’s three-dimensional architecture. Constraints for warm Jupiters are particularly interesting because they allow us to test origins theories that invoke a mutually inclined perturber. Because of warm Jupiters’ high signal-to-noise ratio transits, it is feasible to detect changes in impact parameter. However, here we show that allowing the impact parameter to vary uniformly and independently from transit to transit leads to incorrect inferences about the change, propagating to incorrect inferences about the perturber. We demonstrate that an appropriate prior on the change in impact parameter mitigates this problem. We apply our approach to eight systems from the literature and find evidence for changes in impact parameter for warm Jupiter Kepler-46b. We conclude with our recommendations for light-curve fitting, including when to fit impact parameters versus transit durations.

Unified Astronomy Thesaurus concepts: Transit duration variation method (1707); Transit photometry (1709); Extrasolar gas giants (509); Exoplanets (498)

1. Introduction

When a transiting planet is torqued by a body on a mutually inclined orbit, its transit shape and duration change (Figure 1). These changes give us a rare handle on the three-dimensional architectures of planetary systems, which are essential for testing theories of their dynamical origin. Such constraints are especially meaningful and achievable for a class of planets known as warm Jupiters, giant planets with 10–200 day orbital periods. Popular theories for the origins of warm Jupiters—particularly those on elliptical orbits—predict they will be accompanied by a mutually inclined giant planet at $\sim 1-5$ au (e.g., Dawson & Chiang 2014; Dong et al. 2014; Petrovich & Tremaine 2016; Anderson & Lai 2017). The required orbital properties for these outer planets result in changes in shape and duration that are feasibly detectable with Kepler - light curves for warm Jupiters. For example, a warm Jupiter on a 70 day orbit torqued by a seven Jupiter mass, 60° mutual inclination companion at 2 au would exhibit a 30 minute change in its transit duration over the duration of the prime Kepler Mission. When we inject such transits into KOI-3309, a warm Jupiter host with a typical Kp magnitude of 14.8, we recover the duration of each individual transit with 3–5 minute precision (Figure 2). Several studies of warm Jupiters have considered measurements or upper limits on changes in impact parameter and/or transit duration in studying the system’s three-dimensional architecture (e.g., Nesvorný et al. 2012, 2013, 2014; Dawson et al. 2014; Masuda 2017; Mills & Fabrycky 2017). Changes in transit shape and duration can result from a change in either the impact parameter (the distance of the transit chord from the center of the star; Figure 1) or the transit speed, but we expect the change in impact parameter to dominate. The well-separated perturbers invoked as warm Jupiters’ putative companions cause secular variations in the warm Jupiters’ sky-plane inclinations and eccentricities on timescales of thousands of years or longer. Consider a transiting planet located at 0.5 au from its Sun-like star with a sky-plane inclination of $89.725°$. A mere $0.1°$ $(0.0017$ rad) tweak in the sky-plane inclination changes the impact parameter from 0.52 to 0.70, resulting in hefty 17% change in the transit duration. To get an equivalent change in the duration caused by the transit speed would require a full 180° precession for eccentricity $e = 0.1$, a 35° precession for $e = 0.5$, or an increase in eccentricity from 0.1 to 0.25 (or 0.5 to 0.6). Therefore, using the prior knowledge that the change in impact parameter dominates, we can obtain the most precise constraints on the perturbing companion by allowing for a change in impact parameter while keeping the transit speed constant.

However, here we will show that fitting one transit speed (or, equivalently, planet–star separation or light-curve stellar density) for all transits while allowing each transit to have its own impact parameter leads to flawed inferences about transit parameters. The inferred values can differ from the truth at the tens of sigma level. These incorrect parameters translate into incorrect constraints on the perturbing companion. In Section 2, we demonstrate this problem and explain its origin. In Section 3, we show that an appropriate prior on the change in impact parameter mitigates the problem. Conversely, a uniform prior corresponds to unphysical assumptions about the gravitational dynamics. We also discuss when to fit impact parameters versus transit durations. In Sections 4 and 5, we apply our approach to Kepler and TESS systems from the literature and compare with previous analyses (most of which were not subject to the bias described here). We summarize our findings, including recommendations for light-curve fitting, in Section 6.

2. Origin of Flawed Inferences from Transit Duration Variations

Here we show that when we fit a planet’s transit light curve and assume a uniform prior on the magnitude of the variation in
impact parameter from transit to transit, we make incorrect inferences about transit parameters. These incorrect parameters lead to incorrect inferences about the presence and properties of a perturbing body. In this section, we explain the origin of the flawed inferences from transit duration variations.

2.1. Overview of Light-curve Inference

We deduce the properties of a transiting planet based on the shape, depth, and duration of its transits. Figure 3 displays graphical models of the inference of the light-curve parameters from a photometric time series. The planet-to-star radius ratio, $R_p/R_*$, sets the transit depth and affects the duration of the ingress and egress, the intervals when the planet is entering or leaving the face of the star. Each $i$ of $N$ transits has a central transit time, $t_i$. The average interval between consecutive transits is the orbital period $P$. Transit timing variations (TTVs) are deviations in the interval between transits from $P$. The impact parameter, $b$, is the scale-free distance of the transit chord from the center of the star (Figure 1). An impact parameter $b = 0$ corresponds to a transit across the stellar diameter and $b = 1$ to a transit across the edge of the star. The model in which $b$ is the same from transit to transit is depicted in the top panel (a) of Figure 3. The other light-curve parameter depicted in Figure 3, $\rho_{\text{circ}}$, relates to the transit speed. As we mentioned in Section 1, the transit speed can also be parameterized as the planet–star separation or the light-curve stellar density. Here we use the latter parameter, which we denote as $\rho_{\text{circ}}$. The light-curve stellar density assuming a circular orbit. (If the orbit is elliptical, $\rho_{\text{circ}}$ derived from the light curve will differ from the true stellar density.) A transit model may have additional parameters that describe the stellar limb darkening and dilution by another star in the aperture, which we will consider in later sections. See Winn (2010) for a detailed pedagogical treatment of transit geometry and parameters, including equations relating $\rho_{\text{circ}}$ to the transit duration. We use the Mandel & Agol (2002) transit light-curve model with the Kipping (2013) limb-darkening parameters. We
convert our \( r \) to the Mandel & Agol (2002) normalized planet–star separation \( d/R_\star \) as

\[
\frac{d}{R_\star} = \left[ \frac{\rho_{\text{circ}} (P/P_\odot)}{\rho_\odot} \right]^{1/3} \frac{\text{au}}{R_\odot}
\]

where \( P_\odot \) is the Earth’s orbital period, \( \rho_\odot \) is the mean solar density, and \( R_\odot \) is the Sun’s radius. We employ a uniform prior on the limb-darkening parameters, \( P, b, \) and \( K_P/R_\star \). We use a log-uniform prior on \( \rho_{\text{circ}} \), because it is uninformative, because stellar densities themselves span many orders of magnitude, and because \( \rho_{\text{circ}} \) can differ from \( \rho_\star \) by orders of magnitude if the planet’s orbit is elliptical. Moreover, we find the results are not sensitive to whether we use a uniform or log-uniform prior on \( \rho_{\text{circ}} \). We implement this prior by fitting log \( \rho_{\text{circ}} \) instead of \( \rho_{\text{circ}} \) (but report value for \( \rho_{\text{circ}} \)). Except where otherwise noted, we use the publicly available Kepler simple aperture photometry from the the Mikulski Archive for Space Telescopes (MAST).

2.2. Demonstration of Incorrect Inference

Allowing the impact parameter to vary uniformly and independently from transit to transit (Figure 3, panel (b)) results in incorrect inferences. To demonstrate the problem, we inject transits in the out-of-transit data of Kepler-419 and fit the transits with a modified version of Gazak et al. (2012)’s TAP with the Carter & Winn (2009) wavelet likelihood function. Our parameters are the planet-to-star radius ratio, the light-curve stellar density, two quadratic limb-darkening coefficients, a linear trend for each light curve, and white and red noise parameters for long- and short-cadence data. We employ uniform priors on each linear trend’s slope and intercept and on the white and red noise parameters. See Dawson et al. (2015) for details of our modifications to TAP.

In the first demonstration, we inject 10 transits each with a true impact parameter of \( b = 0.5 \) (Figure 4, top panel; Figure 5, left panel). When we use the model depicted in panel (a) of Figure 3 that assumes the impact parameter is the same in each transit, our recovered values for the impact parameter (red; Figure 4) are consistent with those injected (black circles). The two-dimensional posterior of \((b, \rho_{\text{circ}})\) and marginal posterior \( \rho_{\text{circ}} \) encompass the truth (Figure 5). However, when we use the model depicted in panel (b) of Figure 3, in which the impact parameter can vary from transit to transit, our recovered impact parameters (blue, Figure 4) are inconsistently low. The two-dimensional posterior of \((b, \rho_{\text{circ}})\) and marginal posterior \( \rho_{\text{circ}} \) exclude the truth (Figure 5; i.e., the true, injected values lie outside the 99.9999% credible interval). When we fix \( \rho_{\text{circ}} \) to its true value and fit each \( b \), we recover the injected impact parameters (Figure 4, gray); therefore the problem arises from the covariance of \( \rho_{\text{circ}} \) and \( b \).

In the second demonstration, we inject 10 transits in which the true impact parameter varies linearly from \( b = 0.45 \) to \( b = 0.55 \) (Figure 4, bottom panel; Figure 5, right panel). A model that assumes \( b \) is constant (red) recovers values consistent with the truth to within the uncertainties (but by construction does not capture the change). A model with \( \rho_{\text{circ}} \) fixed to its true value (gray) recovers the injected impact parameters precisely. However, the model that allows the impact parameter to vary from transit to transit (blue) leads to inferred impact parameters that are inconsistently low and,
more importantly, overestimate the change in impact parameter
(Figure 4). The latter would lead to incorrect inferences about
the perturber mass and orbit, including mutual inclination. The
two-dimensional posterior of \( (b, \rho_{\text{circ}}) \) and marginal posterior of
\( \rho_{\text{circ}} \) exclude the truth (Figure 5; i.e., the true, injected values lie
outside the 99.9999% credible interval).

We also inspect the posteriors for variables corresponding to
the unit-free full transit duration (\( T_i \)) and ingress/egress
duration (\( \tau_i \)) of each \( i \) transit. We assume1 the following
relations between \( T_i, \tau_i, b, \) and \( \rho_{\text{circ}} \):

\[
T_i = (1 - b^2)^{1/2} \rho_{\text{circ}}^{-1/3},
\]

\[
\tau_i = R_p \left( 1 - b^2 \right)^{-1/2} \rho_{\text{circ}}^{-1/3} \cdot
\]

(1)

Here \( \rho_{\text{circ}} \) has the units of \( \rho_0 \). We perform inference of \( b \) and
\( \rho_{\text{circ}} \) from a set of \( T_i \) and \( \tau_i \) using \texttt{pystan} (Carpenter et al.
2017; Stan Development Team 2017). We plot the posteriors in
Figures 6 and 7. Because \( T_i \) is well constrained by the data,
different treatments of \( b \) lead to similar inferences. However,
the model that allows the impact parameter to vary from transit
to transit (blue) causes incorrect inferences of \( \tau_i \), which is more
uncertain.

2.3. Simplified Model of Light-curve Inference

To reduce the problem demonstrated in Section 2.2 to its
essentials, we reproduce the problem using a simplified toy
model, depicted graphically in Figure 8. Instead of using the
full light curve and parameter set, we use a data set consisting
of a unit-free full transit duration (\( T_i \)) and ingress/egress
duration (\( \tau_i \)) of each \( i \) transit (Equation (1)). Using the Stan
Bayesian statistical modeling software (Carpenter et al. 2017),
we fit only\(^2\) the parameters \( \rho_{\text{circ}} \) and \( b \). As with our full data set,
we use a uniform prior on \( b \) and a log-uniform prior on \( \rho_{\text{circ}} \)
unless otherwise noted. The inference model with the same \( b \)
for each transit is shown in panel (a) of Figure 8 and with \( b \) that
can vary from transit to transit in panel (b).

In our first demonstration, we set \( b = 0.5 \) and \( \rho_{\text{circ}} = 1 \) for
each transit, compute \( T_i \) and \( \tau_i \), and assign each transit’s \( T_i \) and
\( \tau_i \) an uncertainty of \( \sigma_T = 0.04 T_i \) and \( \sigma_\tau = 0.16 \tau_i \), respectively.
The results, shown in the top panel of Figure 9 and left panel of
Figure 10, are very similar to full light-curve inference in
Figures 4 and 5, demonstrating that our toy model has captured
the fundamental issue. A second demonstration, in which \( b \)
varies linearly from 0.45 to 0.55, is shown in the bottom panel
of Figure 9 and right panel of Figure 10 and also captures the
problem.

2.4. Cause of Incorrect Inference from Transit Duration
Variations

In the single transit case, the mode in \( \rho_{\text{circ}} \) is not at the truth,
but the \( \rho_{\text{circ}} \) posterior includes the truth. The parameters \( b \) and
\( \rho_{\text{circ}} \) are covariant (bottom panels of Figure 11) because they both
affect the transit duration \( T_i \) (Equation (1)). (See Carter et al.
2008 for a detailed exploration of their covariance.) Even
though we can break the degeneracy between \( b \) and \( \rho_{\text{circ}} \) by

\( \frac{R_p}{R_*} \) is also partially degenerate with \( b \) and \( \rho_{\text{circ}} \) because it affects the ingress and egress duration (e.g., Carter et al. 2008). This
degeneracy makes the incorrect inference from the real data set even more
severe than in our simplified model.

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1 In real light curves, these approximate expressions are related to the true
durations by a constant in the limit where \( R_p \ll R_* \ll a \) and
\( |b| < \sqrt{1 - \frac{R_p}{R_*}} \) (Winn 2010).

2 The parameter \( \frac{R_p}{R_*} \) is also partially degenerate with \( b \) and \( \rho_{\text{circ}} \) because it affects the ingress and egress duration (e.g., Carter et al. 2008). This
degeneracy makes the incorrect inference from the real data set even more
severe than in our simplified model.
measuring $\tau$, $\tau$ is less precisely constrained than $T$ because the ingress/egress is shorter and shallower than the full duration. For a given $\rho_{\text{circ}}$, the skewed shape of the $\rho_{\text{circ}}$ versus $b$ covariance corresponds to much more posterior area at low $b$ than at high $b$. Higher values of $\rho_{\text{circ}}$ correspond to larger range of $b$ consistent with the observed duration. Incorrect inferences arise when there are multiple transits, each transit is allowed to have its own impact parameter, and $\rho_{\text{circ}}$ is constant from transit to transit.

The simplified toy model in Section 2.3 elucidates the cause of the incorrect inference. The left panel of Figure 11 shows how the posteriors shift away from the truth as we add more and more transits to our data set. In the top panel, we plot the marginal posterior of $\rho_{\text{circ}}$ and $b$ ($\rho_{\text{circ}} = 1.0$ and constant $b = 0.5$ for each transit) have high probability in our posterior. Adding more transits is equivalent to raising the marginal $\rho_{\text{circ}}$ to the power of the number of transits (yellow dashed line): because of the skewed shape, the mode increases and the posterior shifts away from the truth. The right panel shows the same exercise but with $b$ assumed to be constant from transit to transit. In this case, adding more transits gets us closer to the truth. (Of course, to identify mutually inclined perturbers, we do not want to assume $b$ is constant.)

In the simplified case (Section 2.3) of $N$ transits each with an identical measured $T$ and $\tau$, the marginal posterior of $\rho_{\text{circ}}$ for the case where the impact parameter can change from transit to transit is (Figure 11, left panel):

$$\text{prob}(\rho_{\text{circ}} | T, \tau, N) \propto \left[ \int_0^1 \text{prob}(\rho_{\text{circ}}, b | T, \tau) db \right]^N,$$  

whereas in the case where $b$ is constant from transit to transit (Figure 11 right panel):

$$\text{prob}(\rho_{\text{circ}} | T, \tau, N) \propto \int_0^1 \text{prob}(\rho_{\text{circ}}, b | T, \tau) db.$$

The relationship between the $N$ transit posterior and one transit posterior in each case is overplotted in Figure 11.

The marginal posterior of $b_i$ for the case where the impact parameter can change from transit to transit is:

$$\text{prob}(b_i | T, \tau, N) \propto \int_0^1 \text{prob}(b_i | T, \tau, \rho_{\text{circ}}) \times \left[ \int_0^1 \text{prob}(\rho_{\text{circ}}, b | T, \tau) db \right]^{N-1} d\rho_{\text{circ}},$$  

whereas in the case where $b$ is constant from transit to transit

$$\text{prob}(b | T, \tau, N) \propto \int_0^\infty \text{prob}(\rho_{\text{circ}}, b | T, \tau)^N d\rho_{\text{circ}}.$$  

Note that the proportionality in Equations (2)–(5) do not include the priors on $b$ or $\rho_{\text{circ}}$. The problem arises from how our assumptions interplay with the skewed shape of the $(\rho_{\text{circ}}, b)$ posterior. If we expected $b$ to truly be independent from transit to transit (if the universe randomly drew a $b$ from between 0 and 1 each time the same planet transited), it would indeed be more likely for us to see small variations in transit duration from a relatively wide range of low $b$ than from a relatively narrow range of high $b$. A uniform prior is implicitly assuming a special typical scale for the change, $\Delta b \sim 1$. In reality, favoring this special scale is not

Figure 7. Two-dimensional joint posterior distribution for $T$ and $\tau$ (Equation (1)) for each transit using full (flux vs. time) data set for a constant $b$. Gray dotted lines denote the true values. When the impact parameter is allowed to vary uniformly and independently while $\rho_{\text{circ}}$ is the same for each transit (blue, row 1), the true values are not recovered. When the both $b$ and $\rho_{\text{circ}}$ are assumed to be the same for each transit (red, row 2), by definition the change in duration is not recovered (e.g., red posterior is left of the truth in first column and right of the truth in the second column). A Cauchy prior on the change in impact parameter (Section 3.1; black, row 3) recovers the truth, as does fitting individual parameters to each transit with a joint prior on $\rho_{\text{circ}}$, $b$, and $R_p/R_s$ that preserves a uniform prior on $T$ and $\tau$ (Section 3.2; orange, row 4).
with mean impact parameter magnitude of change in impact parameter. The prior is a Cauchy distribution transit to transit. Bottom same across all transits. Top left that are individual to each interest and gray the observed data. The plate prior on the change in impact parameter. Yellow circles are the parameters of simplification model nonetheless reproduces the problem created by a uniform prior on the change in impact parameter. Yellow circles are the parameters of interest and gray the observed data. The plate (black box) indicates parameters that are individual to each $i$ of $N$ transits. Parameters outside the plate are the same across all transits. Top left (a): impact parameter modeled as constant from transit to transit. Top right (b): impact parameter allowed to vary from transit to transit. Bottom (c): same as middle but with a nonuniform prior on the magnitude of change in impact parameter. The prior is a Cauchy distribution with mean impact parameter $b$ and scale $\gamma$ of the change in impact parameter.

in line with the expected variations in impact parameter: rather, the expected scale of the change$^3$ in impact parameter spans many orders of magnitude and is typically $\ll 1$. In other words, we expect the impact parameters among different transits of the same planet to be correlated.

3. Mitigating the Bias

In the previous section, we demonstrated that incorrect inferences arise when we allow $b$ to vary independently from transit to transit with a uniform prior on its variation scale (while assuming $\rho_{\text{circ}}$ and $R_p / R_*$ do not change detectably). Here we present two approaches for mitigating this bias: using an appropriate prior for the change in impact parameter

$$p(b) = \frac{1}{(\pi \gamma^2 \sqrt{1 + (b - \bar{b})^2 / \gamma^2})}.$$  

Figure 9. Impact parameter vs. time injected (dashed line, circles) and recovered (diamonds with error bars representing the median and 68% credible interval). Same as Figure 4 for a simplified data set ($T$, $\tau$ instead of flux vs. time) depicted in Figure 8, panels (a) and (b). The simplified data set captures the problem: when the impact parameter is allowed to vary (blue), the injected impact parameter is not recovered and the inferred change in impact parameter is too large. Top: constant injected impact parameter; bottom: changing injected impact parameter. When the impact parameter is allowed to change from transit to transit in the model (blue; panel (b) of Figure 8), the injected impact parameter is not recovered and the inferred change (bottom) in impact parameter is too large. When the impact parameter is modeled as constant from transit to transit (red; panel (a) of Figure 8), the recovered values are consistent with those injected but the change in impact parameter is by construction not detectable. When $\rho_{\text{circ}}$ is fixed to its true value (gray), the injected impact parameter is recovered precisely, demonstrating that the problem arises from the covariance of $b$ and $\rho_{\text{circ}}$.

(Section 3.2) and fitting parameters for each individual transit to identify changes in duration (Section 3.2). We discuss when to use which approach and how they can be complementary in Section 3.3.

3.1. An Appropriate Prior for the Change in Impact Parameter

We argued that a uniform prior on $\Delta b$ corresponds to a favored scale for a change in $b$ that we do not truly prefer, is in fact not physically plausible, and does not capture our expectation that impact parameters among different transits of the same planet should be correlated. When we have no prior information about a transiting planet’s perturber (or lack thereof), an uninformative prior on the scale of the change in $b$ is most appropriate. We have found that our results are not sensitive to the functional form of the prior. One such prior that we will show works well is a Cauchy prior, which is similar to a Gaussian prior but with longer tails:

$$\text{prob}(b) = [\pi \gamma (1 + [b - \bar{b}]^2 / \gamma^2)]^{-1}.$$  

The likelihood function includes a product over each of $i$ impact parameters. We use a log-uniform prior for the scale $\gamma$. To capture the expected isotropic distribution of systems throughout the Galaxy, we use a uniform prior on the average impact parameter $\bar{b}$. We depict this model graphically in panel (c) of Figures 3 and 8.

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$^3$ We clarify that a uniform prior for the average impact parameter is appropriate and corresponds to the reasonable assumption that other planetary systems are distributed isotropically in space.
posterior and marginal distribution for $\rho_{\text{circ}}$ when the impact parameter is modeled as constant from transit to transit (red) or allowed to vary (blue). Bottom: two-dimensional posterior distribution for $b$ vs. $\rho_{\text{circ}}$. Dotted lines: true injected values. Left: constant injected parameter; right: changing injected impact parameter. The solid black line (left and right) and solid gray lines (right) are the expected degeneracy between $b$ and $\rho_{\text{circ}}$ from a measurement of the total transit duration $T$ (Equation (1)).

Figure 10 shows that this prior mitigates the problem in the simplified toy model (Figure 3, panel (c)). We obtain impact parameters consistent with those injected, whether our injected $b$ is constant or varying. Figure 13 shows that the two-dimensional $(b, \rho_{\text{circ}})$ posterior and marginal $\rho_{\text{circ}}$ posterior encompass the truth.

Using this more appropriate prior also works well for full light-curve fits (Figure 3, panel (c)). Figures 14 and 15 shows the successful recovery of parameters for transits injected into Kepler-419’s out-of-transit light-curve data. With the appropriate prior on the change in impact parameter, the posterior contains the truth for both constant and changing $b$. In the case of changing $b$, our truth-containing inference satisfies a prerequisite to correctly characterize the perturber causing the TDVs. We infer realistic error bars on $\rho_{\text{circ}}$, necessary for identifying planets on highly elliptical orbits (e.g., Kipping 2010; Dawson & Johnson 2012). With the uniform prior on the change in impact parameter (Figure 15, blue), we might incorrectly conclude from the tight marginal distribution of $\rho_{\text{circ}}$ that the circular injected planet is on a moderately elliptical orbit. Our inferred values of $T$ and $\tau$ are also consistent with the truth (Figure 6).

### 3.2. Fitting Parameters for Each Individual Transit to Identify Changes in Duration

Alternatively, we can fit individual parameters to each transit to obtain robust durations and subsequently fit the collection of transit times and durations with a dynamical model. In this approach, we fit $b$, $\rho_{\text{circ}}$, and $R_p/R_*$, incorporating the following term as a prior to preserve a uniform prior on the transit durations $T$ and $\tau$ (Equation (1)) and transit depth (derived following the Appendix of Burke et al. 2007):

$$\text{prob}(\rho_{\text{circ}}, b, R_p/R_*) \propto (R_p/R_*)^2 \frac{|b|}{1 - b^2} \rho_{\text{circ}}^{-5/3}. \quad (7)$$

We caution that Equation (7) assumes $R_p << R_*$ and $|b| < a$ (Winn 2010). In the case of grazing transits, large planet-to-star radius ratio, and/or very close-in orbits, the equation must be modified.

Preserving a uniform prior on $T$, $\tau$, and depth is desirable because the dynamical model that fits inclination and eccentricity vectors will naturally impose physically realistic priors on $b$ and $\rho_{\text{circ}}$. (Note that the dynamical model will also need to incorporate a prior on $\rho_*$ from a stellar model or simultaneously fit stellar parameters such as the Gaia parallax or effective temperature from the spectrum.) If we also impose priors during the light-curve fit (for example, a uniform prior on $b$ and $\rho_{\text{circ}}$), we are applying the priors twice. However, when $T$ is well constrained by the data—as is typically the case for high signal-to-noise ratio giant planet transits—the prior on $b$ and $\rho_{\text{circ}}$ has a negligible effect on the inferred $T$ for each transit.

An equivalent approach is to fit $T$, $\tau$, and transit depth for each light curve. In practice, we find that the above approach (fitting $b$, $\rho_{\text{circ}}$, and $R_p/R_*$ with Equation (7) as a prior) converges more quickly; in the latter approach, $T$ and $\tau$ can wander off to very large values when $\tau$ is not well constrained. Even with the above approach, we found it necessary to impose limits $-1 < b < 1$, $\rho_{\text{circ}} > 0$, and $R_p/R_* > 0$ to ensure convergence.

We caution that the above approach should not be used to obtain posteriors for $\rho_{\text{circ}}$ and $R_p/R_*$. These posteriors can be obtained concurrently with the dynamical model (if so, we recommend fitting the depths as part of the model) or from the approach described in Section 3.1. They can be obtained less precisely by fitting a model with a joint $\rho_{\text{circ}}$, $b$, and $R_p/R_*$ for all transits (Figure 3, panel (a)) or by fitting a binned, phasefolded light curve with each transit shifted to center the midtransit time (e.g., Masuda 2017; Van Eylen et al. 2019).

These less precise approaches could lead to errors in $b$ and $\rho_{\text{circ}}$ when there are transit duration variations or, in the latter approach, large uncertainties in the TTVs that are not marginalized over (Kipping 2014). Another approach used in the literature is to obtain an averaged posterior distribution by taking the median (Nesvorný et al. 2014) or mean (Nesvorný et al. 2012, 2013) across transits of each posterior sample. We do not recommend using the average planet parameters from this approach, as it tends to bias the derived parameters away from the truth (Appendix, Figure A1).

### 3.3. Comparison of the Two Approaches

The first approach is best when the quantity of interest is the change in impact parameter, when one seeks a robust posterior for $\rho_{\text{circ}}$ in the presence of possible changes in impact parameter, and/or one does not plan to fit a dynamical model. The second approach is better when one seeks durations to use in a dynamical model and/or when it is unclear that changes in duration would be dominated by the change in impact parameter (a resonant system instead of a hierarchical system). As discussed in Section 3.2, the second approach does not directly yield a robust posterior for $\rho_{\text{circ}}$, the average impact parameter, or $R_p/R_*$. 

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**Figure 10.** Same as Figure 5 for a simplified data set ($T$, $\tau$ instead of flux vs. time) depicted in Figure 8, panels (a) and (b). The simplified data set captures the problem: when the impact parameter is allowed to vary (blue), the resulting posteriors are inconsistent with the injected values (dotted lines). Top: marginal posterior distribution for $\rho_{\text{circ}}$ when the impact parameter is modeled as constant from transit to transit (red) or allowed to vary (blue). Bottom: two-dimensional posterior distribution for $b$ vs. $\rho_{\text{circ}}$. Dotted lines: true injected values. Left: constant injected parameter; right: changing injected impact parameter. The solid black line (left and right) and solid gray lines (right) are the expected degeneracy between $b$ and $\rho_{\text{circ}}$ from a measurement of the total transit duration $T$ (Equation (1)).
The two approaches can be complementary and used together. One can use the first approach to obtain robust posteriors for \( r_{\text{circ}}, \) \( \bar{b}, \) and \( R_p/R_\star; \) these quantities, along with the changes in impact parameter, can point to a good starting point for the dynamical model. The dynamical model can then be fully fitted to the set of midtransit times and transit durations from the second approach.

4. Applications: Hierarchical Systems

We have demonstrated that allowing the impact parameter to vary uniformly and independently from transit to transit leads to incorrect inferences (Section 2). Having identified an appropriate prior on the change in impact parameter to mitigate this problem (Section 3), we will now apply this approach to

Figure 11. Allowing each impact parameter to vary uniformly and independently shifts the posterior away from the truth \( (r_{\text{circ}} = 1, \bar{b}, \) and constant \( b = 0.5 \) for each transit) as more transits are added. Left panel: each impact parameter can vary uniformly and independently (Figure 8, panel (a)). In both panels, gray corresponds to the inference from a single transit for the marginal \( r_{\text{circ}} \) posterior (top) and joint \( (r_{\text{circ}}, b) \) posterior (bottom). Black corresponds to 10 transits. In the left panel, the 10 transit posterior is far from the truth. In the right panel, the 10 transit posterior is more accurate and precise than the one transit posterior. (Note: the model in the right panel, by construction, cannot capture a change in impact parameter.)

Figure 12. Impact parameter vs. time injected (dashed line, circles) and recovered (diamonds with error bars representing the median and 68% credible interval) for a simplified data set \( (T, \tau \) instead of flux vs. time) depicted in Figure 8. Top: constant injected impact parameter; bottom: changing injected impact parameter. A Cauchy prior on the change in impact parameter (red; Figure 8, panel (c)) allows us to recover values consistent with those injected, whereas a uniform prior (blue; Figure 8, panel (b)) on the change does not.

Figure 13. Top: marginal posterior distribution for \( r_{\text{circ}} \) when a Cauchy (red) or uniform (blue) prior is imposed on the change in impact parameter using the simplified data set \( (T, \tau \) instead of flux vs. time) depicted in Figure 8. Bottom: two-dimensional posterior distribution for \( b \) vs. \( r_{\text{circ}} \). Dotted lines: true injected values. Left: constant injected impact parameter; right: changing injected impact parameter. The solid black lines (left and right) and solid gray lines (right) are the degeneracy between \( b \) and \( r_{\text{circ}} \) from a measurement of the total transit duration \( T \) (Equation (1)). The Cauchy prior (red) allows the recovery of the injected value in the posterior distribution, whereas the uniform prior (blue) does not.
systems from the literature for which changes in impact parameter or transit durations were considered in characterizing a planetary system. In this section, we will focus on hierarchical systems containing a warm Jupiter and a well-separated, nonresonant perturber that causes secular variations in the warm Jupiter’s orbit. Our approach was motivated by and designed for such systems.

4.1. Kepler-419b, a Highly Elliptical Warm Jupiter Perturbed by a Nontransiting Coplanar Jupiter

Kepler-419b is a warm Jupiter with a 70 day orbital period on a highly elliptical ($e = 0.83 \pm 0.01$) orbit (Dawson et al. 2012; Dawson 2014). A nontransiting giant planet at 2.4 au causes TTVs, which Dawson (2014) used to precisely characterize the three-dimensional architecture of the system. Dawson (2014) found from the TTVs alone that the system is coplanar, and changes in impact parameter did not offer any additional constraints. Dawson (2014) allowed the impact parameter to vary uniformly and independently from transit to transit, which we have demonstrated leads to incorrect inferences (Section 2). Although the changes in impact parameter did not help constrain the dynamical fit, Dawson (2014) argued that changes were detected based on the tighter constraints on $\rho_{\text{circ}}$ when $b$ was allowed to vary from transit to transit. Here we have shown that the tighter constraint on $\rho_{\text{circ}}$ is incorrect (e.g., Figure 15).

We perform new fits on the Kepler-419 data set using the appropriate prior on the change in impact parameter from Section 3. We plot the impact parameter versus time in Figure 16 and the two-dimensional posterior for $(b, \rho_{\text{circ}})$ in Figure 17. Using the Cauchy prior on the change in impact parameter (black) removes the apparent variations in impact parameter inferred from the uniform prior (blue) and also leads to a more uncertain but more realistic inference on $\rho_{\text{circ}}$. The results using the Cauchy prior are similar to the case where we impose $\Delta b = 0$.

The $\rho_{\text{circ}}$ from the light curve can be combined with prior knowledge of the star’s density to infer the planet’s eccentricity. A falsely tight constraint on $\rho_{\text{circ}}$ can in principle translate to incorrect inferences on the eccentricity. In Figure 18, we compare the eccentricity constraints derived from the three treatments of the impact parameter. In this case, we find that the degeneracy between the argument of periapse and eccentricity, as well as the uncertainty in the true stellar density, dominate the uncertainty in $e$. The inferred $e$ is not sensitive to the uncertainty on $\rho_{\text{circ}}$. We obtain similar values of $e = 0.83^{+0.10}_{-0.08}$, $e = 0.85^{+0.08}_{-0.07}$, and $e = 0.83^{+0.09}_{-0.08}$ using $\Delta b = 0$, a uniform $\Delta b$, and a Cauchy prior on $\Delta b$, respectively. (Note that the $e = 0.83 \pm 0.01$ derived by Dawson 2014 is a tighter constraint because it also incorporates
The difference is not very large.

Parameter to be constant from transit to transit (values reported by Dawson and allowing the impact parameter to vary with a uniform prior on the change)

Eccentricity vs. $\omega$ Figure 18. Eccentricity vs. $\omega$ posterior distributions forcing the impact parameter to be constant from transit to transit (red, top left), allowing the impact parameter to vary with a uniform prior on the change (blue, top right), and allowing the impact parameter to vary with a Cauchy prior on the change (black, bottom left). Bottom right: marginal posterior distributions for $\omega$. A uniform prior on the change in impact parameter results in a slightly larger inferred eccentricity, but due to uncertainty in $\omega$ and $\rho_{\text{circ}}$, the difference is not very large.

radial-velocity measurements, which confirm the high eccentricity measured using the “photocentric” effect.)

For Kepler-419b, our new analysis does not qualitatively change the conclusions of Dawson (2014) but leads to more accurate values for parameters and their uncertainties. We report these new parameters in Table 1. Almost all the parameters are consistent with those of Dawson (2014) to within uncertainties but the error bars are larger, particularly (as expected) for $\rho_{\text{circ}}$ and $R_p/R_*$. The only major difference is in the average impact parameter, which is significantly larger than the individual impact parameters reported in Dawson (2014).

This larger impact parameter is also expected from our new approach (e.g., Figure 15).

### 4.2. Kepler-693b, a Moderately Elliptical Warm Jupiter Perturbed by a Nontransiting, Mutually Inclined Brown Dwarf

Kepler-693b is a warm Jupiter that exhibits transit timing and duration variations due to the perturbations of a nontransiting brown dwarf, Kepler-693c, hierarchically separated at several au and with a large mutual inclination (Masuda 2017). The brown dwarf causes secular oscillations in the warm Jupiter, allowing the warm Jupiter’s orbit to periodically get close enough to the star for tidal circularization. Therefore Kepler-693c is exactly the type of companion expected for a warm Jupiter achieving its short period through high eccentricity tidal migration. Masuda (2017)’s analysis of Kepler-693b and Kepler-448b (Section 4.3) was not subject to the bias described in Section 2. They followed the approach described in Section 3.2 of fitting individual parameters to each transit to obtain transit times and durations to fit with a dynamical model.

In Figures 19 and 20, we plot the results of our light-curve fits for Kepler-693. We report our best-fit parameters in Table 2. Our light-curve parameters are consistent with Masuda (2017) to within the uncertainties. Consistent with Masuda (2017)’s TDV detections, we detect a change in impact parameter of Kepler-693b (Figure 21). The change scale is $\gamma = 0.018^{+0.001}_{-0.007}$. If we had allowed the impact parameter to vary uniformly and independently, we would have overestimated the magnitude of the change (Figure 19).

Masuda (2017) derived average values for transit parameters from a fit to a binned, phased-folded light curve (see Section 3.2 for a discussion of this approach). Our constraints on the average $\rho_{\text{circ}}$ and transit impact parameter (Figure 20) are somewhat more precise.

### 4.3. Kepler-448b, an Elliptical Warm Jupiter Perturbed by a Nontransiting Brown Dwarf

Masuda (2017) also detected a nontransiting brown dwarf companion to warm Jupiter Kepler-448b using transit timing variations. Masuda (2017) found that Kepler-448b did not exhibit significant transit duration variations, and that the mutual inclination of Kepler-448c is poorly constrained. Therefore it is uncertain whether secular oscillations allow

### Table 1

| Parameter                      | Value          |
|-------------------------------|----------------|
| Planet-to-star radius ratio, $R_{\text{pl}}/R_*$ | 0.0636 ± 0.0007 |
| Light-curve stellar density, $\rho_{\text{pl}} [\rho_\odot]$ | 7.9 ± 1.1 |
| Average impact parameter, $b$ | 0.37 ± 0.09 |
| Impact parameter change scale, $\gamma$ (10$^{-5}$) | 1.3 ± 1.3 |
| Limb-darkening coefficient, $q_1$ | 0.30 ± 0.07 |
| Limb-darkening coefficient, $q_2$ | 0.30 ± 0.09 |
| Red noise, short-cadence, $\sigma_\text{r} [\text{ppm}]$ | 2400 ± 200 |
| White noise, short-cadence, $\sigma_\text{w} [\text{ppm}]$ | 655 ± 5 |
| Red noise, long-cadence, $\sigma_\text{rl} [\text{ppm}]$ | 400 ± 60 |
| White noise, long-cadence, $\sigma_{\text{wl}} [\text{ppm}]$ | 121 ± 7 |

Note. The uncertainties represent the 68.3% credible interval about the median of the posterior distribution.

This larger impact parameter is also expected from our new approach (e.g., Figure 15).
We fit the light curves and do not detect a significant change in impact parameter (Table 3). Our light-curve parameters are consistent with Masuda (2017) except for a small but significant discrepancy in the radius ratio, which may be due to different approaches for treating correlated noise. We echo Masuda (2017)'s hope that Gaia observations may shed light on the mutual inclination between Kepler-448b and c.

**Figure 21.** Kepler-693b: Left: change in impact parameter from its median value, from a fit using a Cauchy prior on the scale for change in impact parameter. Right: posterior for the impact parameter change scale $\gamma$. We confirm that Kepler-693b exhibits a significant change in impact parameter over the Kepler Mission.

**Table 2**

| Parameter | Value$^*$ |
|-----------|-----------|
| Planet-to-star radius ratio, $R_p/R_*$ | 0.116 $^{+0.004}_{-0.003}$ |
| Light-curve stellar density, $\rho_{\text{lc}} [\rho_\odot]$ | 2.5 $^{+0.4}_{-0.4}$ |
| Average impact parameter, $b$ | 0.57 $^{+0.07}_{-0.09}$ |
| Impact parameter change scale, $\gamma$ | 0.018 $^{+0.011}_{-0.007}$ |
| Limb-darkening coefficient, $q_1$ | 0.5 $^{+0.3}_{-0.2}$ |
| Limb-darkening coefficient, $q_2$ | 0.5 $^{+0.3}_{-0.2}$ |
| Red noise, short-cadence, $\sigma_r$ [ppm] | 7000 $^{+2000}_{-00}$ |
| White noise, short-cadence, $\sigma_w$ [ppm] | 6010 $^{+40}_{-31}$ |
| Red noise, long-cadence, $\sigma_r$ [ppm] | 1300 $^{+500}_{-600}$ |
| White noise, long-cadence, $\sigma_w$ [ppm] | 1190 $^{+30}_{-00}$ |

Note.

$^*$ The uncertainties represent the 68.3% credible interval about the median of the posterior distribution.

**Table 3**

| Parameter | Value$^*$ |
|-----------|-----------|
| Planet-to-star radius ratio, $R_p/R_*$ | 0.08993 $^{+0.00007}_{-0.00008}$ |
| Light-curve stellar density, $\rho_{\text{lc}} [\rho_\odot]$ | 0.282 $^{+0.02}_{-0.02}$ |
| Average impact parameter, $b$ | 0.359 $^{+0.006}_{-0.006}$ |
| Impact parameter change scale, $\gamma$ ($10^{-4}$) | 1.0 $^{+5.6}_{-4.9}$ |
| Limb-darkening coefficient, $q_1$ | 0.221 $^{+0.009}_{-0.008}$ |
| Limb-darkening coefficient, $q_2$ | 0.34 $^{+0.02}_{-0.02}$ |
| Red noise, short-cadence, $\sigma_r$ [ppm] | 3600 $^{+30}_{-00}$ |
| White noise, short-cadence, $\sigma_w$ [ppm] | 247.1 $^{+0.7}_{-0.6}$ |

Note.

$^*$ The uncertainties represent the 68.3% credible interval about the median of the posterior distribution.

### 5. Applications: Near Resonant Systems

Although our approach is designed for hierarchical systems, here we explore its application to systems near orbital resonance. These systems have been more commonly characterized using transit time and duration variations than hierarchical systems. Although sometimes our assumption that the change in transit duration is dominated by a change in impact parameter does not hold, we will show that our approach is nonetheless useful for robustly identifying changes in impact parameter.
5.1. Kepler-46b, a Warm Jupiter Perturbed by a Nontransiting, Nearly Coplanar Warm Saturn: Evidence for TDVs

Kepler-46b, a warm Jupiter, was the first planet to have its nontransiting companion characterized without degeneracy by TTVs (Nesvorný et al. 2012). The warm Jupiter’s nontransiting companion, a warm Saturn, may have small mutual inclination (Saad-Olivera et al. 2017). To assess the TDVs, Nesvorný et al. (2012) fit the data using a model in which each transit had its own $\rho_{\text{inc}}, b$, and $R_p/R_\star$ (Section 3.2). Their analysis was not subject to the bias described in Section 2. They found no significant TDVs.

The lack of TDVs allowed them to rule out one of two solutions that were both consistent with the TTVs. However, the transit durations did not offer a meaningful constraint on that favored solution. Saad-Olivera et al. (2017) further refined the system’s parameters using TTVs alone with a longer baseline of the full Kepler data set and found that favored solution to be a much better fit.

Following the procedure described in Section 4.1, we fit the full data set and find evidence for a change in impact parameter (Figures 22 and 23, Table 4). Allowing the impact parameter to vary uniformly and independently from transit to transit results in a large change in impact parameter. With an appropriate prior, the impact parameter still changes but more modestly yet still significantly. The scale for the change is $\gamma = 0.008^{+0.003}_{-0.004}$. We plot the change in impact parameter and $\gamma$ posterior in Figure 24.

Our results for $R_p/R_\star, \rho_{\text{inc}}$, and $b$ are inconsistent at several sigma with Nesvorný et al. (2012), who find $R_p/R_\star = 0.0887^{+0.0010}_{-0.0012}$, $\rho_{\text{inc}} = 1.09^{+0.18}_{-0.13} \rho_\odot$, and $b = 0.75^{+0.22}_{-0.05}$. The difference in $R_p/R_\star$ may be due to the treatment of dilution from other stars in the aperture. Nesvorný et al. (2012) assumed a dilution factor based on the median of simple aperture photometry versus the median of the presearch data conditioned (PDC) photometry for each quarter, assuming that the latter has been corrected for dilution. Nesvorný et al. (2012) infer a larger radius ratio due to their dilution correction. However, we find that the reported crowding
metric indicates that no dilution correction has been applied to the PDC photometry. The PDC photometry does have a different median, but we find the difference is multiplicative, rather than additive as would be applied to correct for blending. For comparison, we fit light curves from the PDC photometry and find our results do not change significantly.

Our larger $\rho_{\text{arc}}$ and smaller $b$ cannot be accounted for by dilution, which would produce the opposite effect (Kipping & Tinetti 2010). Nor is the difference a result of our different prior on $\rho_{\text{arc}}$ or different methods of combining the posteriors from multiple transits (Section 3.2). The difference could be due to different treatments of correlated noise. We can use $\rho_{\text{arc}}$ as a reality check for our derived values. Nesvorný et al. (2012) note that the TTVs constrain Kepler-46b’s eccentricity to be very small and therefore $\rho_{\text{arc}}$ should match $\rho_*$. We compute an updated value $\rho_*$ by fitting the Dartmouth isochrones (Dotter et al. 2008) to Nesvorný et al. (2012)’s spectroscopic parameters and the Gaia parallax and magnitude (Gaia Collaboration et al. 2016, 2018), following Dawson et al. (2019). We find $R_* = 0.83^{+0.02}_{-0.01} R_{\odot}$, $M_* = 0.89^{+0.02}_{-0.03} M_{\odot}$, and $\rho_* = 1.54^{+0.10}_{-0.16} \rho_{\odot}$, in good agreement with our light-curve stellar density.

Ultimately the small but significant differences in our parameters from those of Nesvorný et al. (2012) do not affect the main conclusion—that the impact parameter is changing modestly—except possibly to raise the concern that the change we detect might be caused by dilution or correlated noise. The fact that our impact parameter is declining steadily over four years rather than oscillating from quarter to quarter gives us some confidence that the change is astrophysical.

Figure 25 shows an example of a dynamical model that provides a good fit ($\chi^2 = 51$ for 66 degrees of freedom) to the midtransit times, average impact parameter, and change in impact parameter. We use the stellar parameters derived above; the other astrocentric model parameters at epoch 55053.2826 BJD are $M_p = 1.0 M_{\text{Jup}}$, $P_p = 33.568$ days, $e_p = 0.022$, $\omega_p = 0$, $i_p = 89^\circ 26$, and mean anomaly $M_p = 89^\circ 51$ and $M_e = 0.36 M_{\text{Jup}}$, $P_e = 57.402$ days, $e_e = 0.037$, $\omega_e = 11^\circ$, $i_e = 90^\circ 23$, and mean anomaly $M_e = 353^\circ 7$ in the transit coordinate system with sky in the X-Y plane and +Z-axis pointing at the observer (e.g., Winn 2010). The mutual inclination $0^\circ 97$ is consistent with Saad-Olivera et al. (2017)’s $0^\circ 43^{+0.09}_{-0.06}$ to within two sigma. The transit duration variations computed from the model are dominated by changes in impact parameter. Future dynamical modeling can more thoroughly explore to what extent the detection of this change in impact parameter allows for better constraints on planet parameters, including the mutual inclination. We recommend that a full exploration of parameter space using the dynamical model fit the durations rather than impact parameters to avoid applying the same prior twice (as discussed in Section 3.2). We also recommend full joint dynamical-photometry modeling4 for this system.

5.2. Kepler-108b and c, a Mutually Inclined Planetary System

The Kepler-108 system contains two transiting warm Saturns on orbits mutually inclined by $I(\circ) = 24^{+11}_{-8}$ (Mills & Fabrycky 2017). Both transiting planets exhibit TTVs. Moreover, planet c exhibits clear TDVs, with the transit duration changing by almost an hour over the course of about three years. Mills & Fabrycky (2017) note that planet b may also have TDVs but the change in duration is smaller and less significant (their Figure 1). Mills & Fabrycky (2017) fit the light curves using a joint dynamical-photometry model: an $N$-body integrator models the orbits of the planets and star, and each light-curve model is generated based on the planet’s instantaneous orbit. This approach naturally generates TDVs in the case of non-coplanar planets. More often, studies first fit the light curves using light-curve model parameters and subsequently fit a dynamical model to these light-curve parameters (e.g., Dawson 2014). The latter two-step approach is faster but the results can be sensitive to the choice of light-curve parameters and their priors (e.g., as we have demonstrated here).

In Figures 26 and 27, we plot the results of our light-curve fits for Kepler-108. Following Mills & Fabrycky (2017), we account for dilution from a background star by including an extra parameter, the dilution factor. We set a uniform prior on the dilution factor. We fit the light curves of both planets simultaneously, with shared values for the stellar limb-darkening parameters, noise parameters, and dilution factor. We report our best-fit parameters in Table 5. Consistent with Mills & Fabrycky (2017), we detect a change in impact parameter of Kepler-108c. The change scale is $g = 0.037^{+0.02}_{-0.01}$. If we had allowed the impact parameter to vary uniformly and independently, we would have overestimated the magnitude of the change (Figure 26). We do not detect a significant change in impact parameter of Kepler-108b.

We note as a caveat that when using the alternative approach of fitting individual parameters to each transit (Section 3.2), if we fit a common dilution factor, we deduce very little dilution.

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4 We avoid the common term “photodynamical” model because the term has sometimes referred to a joint dynamical-photometry model (e.g., Mills & Fabrycky 2017) and sometimes to a two-step (first photometry, then dynamical) model (e.g., Nesvorný et al. 2014).
Our inferred parameters in Table 5 are consistent with those of Mills & Fabrycky (2017)'s mutually inclined fit. Mills & Fabrycky (2017) found an average impact parameter of $b = 0.28^{+0.18}_{-0.14}$ for Kepler-108b and $b = 0.65^{+0.09}_{-0.11}$ for Kepler-108c (S. M. Mills 2020, personal communication, 2017 March 10th). Their average scaled planet–star separation corresponds to $\rho_{\text{arc}} = 0.30^{+0.09}_{-0.07} \rho_\odot$ for Kepler-108b and $\rho_{\text{arc}} = 0.35^{+0.14}_{-0.07} \rho_\odot$ (Sean Mills 2020, personal communication, 2017 March 10th).

Generally our uncertainties are larger. Our larger uncertainties may arise because we include noise parameters, including correlated noise, in our inference. Another possibility is that Mills & Fabrycky (2017) obtain more precise values because the joint dynamical-photometry model naturally imposes constraints on the light-curve parameters (i.e., due to the limited possible variations in transit impact parameter allowed by the physical model).

Using a joint dynamical-photometry model like Mills & Fabrycky (2017) naturally imposes an appropriate prior on the change in impact parameter; the transit speed can vary as well according to the dynamical model. Therefore this approach is not subject to the bias described in Section 2. We recommend the joint dynamical-photometry approach if computationally feasible. However, when it is not computationally feasible due to a large data set, the need to account for correlated noise, or a large sample size of planets, we recommend the approach presented here using the Cauchy prior on the change in impact parameter.

5.3. KOI-319.01, a Transiting Warm Jupiter Perturbed by a Nontransiting Warm Saturn or Warm Neptune

Nesvorný et al. (2014) found that KOI-319.01 exhibits large TTVs caused by a nontransiting warm Saturn or warm Neptune. They detected fluctuating TDVs that are not consistent with the dynamical model, which predicts constant or linearly drifting TDVs. Nesvorný et al. (2014) proposed that their TDV errors may be underestimated or may be caused by an unmodeled effect. To assess the TDVs, Nesvorný et al. (2014) fit the data using a model in which each transit had its own $R_p/R_\text{star}$, $b$, and $K_p/R_\text{star}$ (Section 3.2). Their analysis was not subject to the bias described in Section 2.

Our fit results are shown in Figures 28 and 29 and in Table 6. Our parameters are very similar to and consistent with Nesvorný et al. (2014) except that our uncertainties are several times larger. Our fit without the appropriate prior shows some possible variation, but with an appropriate prior, the change is consistent with zero (Table 6). When we fit each transit individually following Section 3.2, we see a drift in transit duration; because our error bars are larger, the changes are consistent with a linear drift (Figure 30). We conclude that the current data do not contain sufficient evidence to definitively attribute the change in duration to a change in impact parameter. We recommend additional dynamical modeling of the duration variations and full joint dynamical-photometry modeling of this system to tease out if and how the duration changes constrain the orbital parameters.

5.4. Kepler-88b, a Warm Neptune Perturbed by a Nontransiting, Nearly Coplanar Warm Jupiter

Kepler-88b is a warm Neptune perturbed by a nontransiting, nearly coplanar warm Jupiter in a 2:1 orbital resonance (Nesvorný et al. 2013). Kepler-88b is not the type of planet our approach is designed for: rather than being a warm Jupiter with a well-separated companion that causes nodal precession, Kepler-88b is a Neptune with a nearby massive resonant companion that can cause significant changes to the longitude of periapse (and hence $\phi_{\text{arc}}$) on a short timescale. As such it makes an interesting test case for our approach, which assumes that only the impact parameter can change detectably.

Nesvorný et al. (2013) found small but significant TDVs for Kepler-88b, the first TDVs due to planet–planet interactions...
detected to our knowledge. To assess the TDVs, Nesvorný et al. (2013) fit the data using a model in which each transit had its own $\rho_{\text{circ}}$, $b$, and $R_p/R_*$ (Section 3.2). Their analysis was not subject to the bias described in Section 2. The companion is well characterized from the TTVs alone and a dynamical fit to only the TTVs predicts the TDVs too. The TDVs are primarily caused by changes in the transit speed (i.e., $\rho_{\text{circ}}$), rather than the impact parameter. Weiss et al. (2019) recently followed up the system with the radial-velocity method and performed joint dynamical-photometry modeling on the combined data set; they also found significant TDVs.

The results from our fits are shown in Figures 31 and 32 and are tabulated in Table 7. Without an appropriate prior for the change in impact parameter (i.e., blue), we might erroneously conclude that the impact parameter is changing. An appropriate prior (black) allows us to correctly deduce that the impact parameter does not change detectably over the time span of the data set. As Nesvorný et al. (2013) and Weiss et al. (2019) simulate, the impact parameter can change over a much longer timescale such that eventually Kepler-88b no longer transits, but the Kepler data set is not long and/or precise enough to detect a change. We believe that the mutual inclination measurement is primarily coming from the TTVs rather than the TDVs, though the TDVs may be contributing an upper limit. Our parameters in Table 7 are consistent with Nesvorný et al. (2013) and Weiss et al. (2019) to within the uncertainties.

### 5.5. TOI-216b and c, a Pair of Warm Jupiters

TOI-216 hosts a pair of transiting warm, large exoplanets in or near the 2:1 orbital resonance (Dawson et al. 2019; Kipping...
Table 6

| Parameter                              | Value          |
|----------------------------------------|----------------|
| Planet-to-star radius ratio, \( R_p/R_* \) | 0.0471 ± 0.0013 |
| Light-curve stellar density, \( \rho_{\text{circ}} \) | 0.150 ± 0.010   |
| Average impact parameter, \( b \)      | 0.910 ± 0.005   |
| Impact parameter change scale, \( \gamma \) | 5 ± 3          |
| Limb-darkening coefficient, \( q_1 \)    | 0.37 ± 0.06     |
| Limb-darkening coefficient, \( q_2 \)    | 0.46 ± 0.03     |
| Red noise, short-cadence, \( \sigma_r \) | 1940 ± 140      |
| White noise, short-cadence, \( \sigma_w \) | 364 ± 2         |
| Red noise, long-cadence, \( \sigma_l \) | 370 ± 20        |
| White noise, long-cadence, \( \sigma_w \) | 88 ± 2          |

Note.

* The uncertainties represent the 68.3% credible interval about the median of the posterior distribution.

Table 7

| Parameter                              | Value          |
|----------------------------------------|----------------|
| Planet-to-star radius ratio, \( R_p/R_* \) | 0.0353 ± 0.0009 |
| Light-curve stellar density, \( \rho_{\text{circ}} \) | 1.2 ± 0.1      |
| Average impact parameter, \( b \)      | 0.46 ± 0.11    |
| Impact parameter change scale, \( \gamma \) | 7 ± 83         |
| Limb-darkening coefficient, \( q_1 \)    | 0.47 ± 0.07    |
| Limb-darkening coefficient, \( q_2 \)    | 0.30 ± 0.08    |
| Red noise, short-cadence, \( \sigma_r \) | 870 ± 90       |
| White noise, short-cadence, \( \sigma_w \) | 551.5 ± 1.0    |
| Red noise, long-cadence, \( \sigma_l \) | 80 ± 50        |
| White noise, long-cadence, \( \sigma_w \) | 108 ± 3        |

Note.

* The uncertainties represent the 68.3% credible interval about the median of the posterior distribution.
et al. 2019). The inner planet’s grazing transit configuration makes its transit durations particularly sensitive to a small precession of the longitude of ascending node. Moreover, based on the planets’ impact parameters, Dawson et al. (2019) found a minimum mutual inclination of 1.8° ± 0.2° degrees. Neither previous study investigated changes in impact parameter or transit duration variations. We fit the TESS simple aperture photometry from MAST together with the ground-based light curves presented in Dawson et al. (2019). We fit the light curves of both planets simultaneously, with shared values for the stellar limb-darkening parameters and noise parameters. We do not detect a significant change in impact parameter for either planet (Table 8). We recommend continued observations from the ground to monitor the inner planet for changes in impact parameter.

6. Summary

Changes in a transiting planet’s impact parameter can constrain the mutual inclinations of planetary systems, including mutual inclinations between the transiting planets and nontransiting companions. Evidence for changes in impact parameters can be evaluated in existing Kepler and TESS data, future TESS data, and planned PLATO data. We present a demonstration of a problem of incorrect inference of changes in impact parameter from transit light curves (Section 2) and two approaches for mitigating the problem (Section 3).

We apply our results to systems from the literature (Sections 4 and 5), most of which were not subject to the bias described here in their previous studies. We discover evidence for a change in impact parameter for Kepler-46b (Section 5.1). We confirm changes in impact parameter for two planets with detected transit duration variations (TDVs), Kepler-639b (Section 4.2) and Kepler-108b (Section 5.2). We confirm no evidence for a change in impact parameter for Kepler-448b (Section 4.3) and TOI-216 b and c (Section 5.5); for the ambiguous cases of Kepler-419b (Section 4.1), Kepler-108c (Section 5.2), and KOI-319.01 (Section 5.3), which exhibits transit duration variations that cannot be definitively attributed to a change in impact parameter from the data alone; and for Kepler-88b (Section 5.4).

The ideal approach for fitting light curves is to simultaneously use a joint photometry-dynamics model and a regression approach that accounts for correlated noise, but in practice, there is a high computational cost to doing both simultaneously off the bat. We recommend the following approaches to ensure the results are robust to parameter choices and model assumptions without requiring unrealistic computation times:

1. To identify changes in impact parameter and/or to obtain a robust \( \rho_{\text{circ}} \) posterior in the presence of possible changes in impact parameter: fit the light curves with individual transit times; individual impact parameters for each transit; and a Cauchy prior on \( \gamma \), the scale of the change in impact parameter (Section 3.1). Specifically, we recommend fitting midtransit times \( t_i \) and changes in impact parameter \( \Delta b_i = b_i - \bar{b} \) for each of \( i \) transits and a joint \( R_p/R_* \), \( \rho_{\text{circ}} \), average impact parameter \( \bar{b} \), impact parameter change scale \( \gamma \), and limb-darkening and noise parameters among all transits. Include Equation (6) in the prior. Use an approach that accounts for correlated noise and does not require pre-detrending, such as a wavelet likelihood combined with linear trends fit to each light-curve segment or Gaussian process regression. Use the posteriors for the noise parameters to identify if: (a) white noise dominates, (b) only long timescale correlated noise (i.e., a linear trend or polynomial) is important, or (c) short timescale noise is important too and therefore a wavelet or Gaussian process likelihood (or an alternative approach) should be included.

2. If the goal is to obtain transit durations for use in a dynamical model, fit individual \( t_i \), \( R_p/R_* \), \( \rho_{\text{circ}} \), and \( b_i \) (and joint values only for limb-darkening and noise parameters), applying the prior in Equation (7) to preserve a uniform prior on transit durations (Section 3.2). Compute the transit durations \( T_i \) from Equation (1); (modifying in the case of grazing transits). Do not use this approach to obtain posteriors for \( R_p/R_* \), \( \rho_{\text{circ}} \), and \( b_i \) posteriors for these values should be obtained using the first approach or, less precisely, fitting parameters jointly to all light curves or a binned, phase-folded light curve. See Section 3.2 for further discussion.

3. If fitting a dynamical model, use the transit times, average impact parameter \( \bar{b} \), \( \rho_{\text{circ}} \), and changes in impact parameter \( \Delta b_i \) from step 1 to identify a dynamical model as a starting point (e.g., as we perform for Kepler-46b in Section 5.1). Then directly fit transit times \( t_i \) and durations \( T_i \) from Step 2 to explore the parameter space for the dynamical model. As discussed in 3.2, it is important to fit the transit durations instead of changes in impact parameter to avoid applying priors on the impact parameter twice.

4. If computationally feasible, fit a full joint photometry-dynamics model to the light curves and compare to the previous step to check for consistency. Use the results of Step 1 to assess if and how correlated noise should be accounted for. If short timescale correlated noise needs to

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Table 8

| Parameter | Value ±
|-----------|-----------
| TOI-216b  |           |
| Planet-to-star radius ratio, \( R_p/R_* \) | 0.11 ± 0.04 |
| Light-curve stellar density, \( \rho_{\text{circ}} \) | 1.1 ± 0.03 |
| Average impact parameter, \( b \) | 1.01 ± 0.05 |
| Impact parameter change scale, \( \gamma \) | 1 ± 0.14 |

**System**

| Parameter | Value ±
|-----------|-----------
| TOI-216c  |           |
| Planet-to-star radius ratio, \( R_p/R_* \) | 0.1230 ± 0.0007 |
| Light-curve stellar density, \( \rho_{\text{circ}} \) | 1.73 ± 0.04 |
| Average impact parameter, \( b \) | 0.14 ± 0.07 |
| Impact parameter change scale, \( \gamma \) | 3 ± 0.05 |

**Note.**

* The uncertainties represent the 68.3% credible interval about the median of the posterior distribution.
be accounted for yet it is not computationally feasible to do so, compare $R_p/R_*$ from Step 1 to get a sense of how much the uncertainties may be underestimated.

Ultimately the presence or absence of detectable changes in impact parameter can help constrain the origins of warm Jupiters. More consideration is needed on the best way to incorporate grazing transits into population studies: they can be quite sensitive to small changes in impact parameter but often have poorly constrained radii. For example, Dawson et al. (2015) included them in their population weighted by their probability of having a Jupiter-like radius, but such an approach is sensitive to the assumed prior on radius. It is important not to exclude nearly grazing transits, as they are particularly sensitive to small changes in impact parameter. Ultimately, since changes in impact parameter manifest as long timescale drift, PLATO can play an essential role by following up the Kepler field and revisiting other fields over a long observational baseline. TESS warm Jupiters can be followed up from the ground (e.g., Dawson et al. 2019) or by CHEOPs to increase the observational baseline. In combination with ground-based follow-up, we can also investigate whether orbital architectures correlate with stellar metallicity or other properties.

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Software: TAP (Gazak et al. 2012).

Appendix

Figure A1 demonstrates the bias introduced by averaging posterior samples across individual transits.
Figure A1. Averaging the posterior samples from individual transits using the median (left) or mean (right) shifts the posterior away from the truth as more transits are added. Gray corresponds to the inference from a single transit for the marginal $\mu_{\text{circ}}$ posterior (top) and joint $(\mu_{\text{circ}}, b)$ posterior (bottom). Black corresponds to 10 transits. In the left panel, the 10 transit posterior is shifted away from the truth toward larger $b$ and larger $\mu_{\text{circ}}$. In the left panel, the right transit posterior is shifted away from the truth toward smaller $b$ and larger $\mu_{\text{circ}}$.  

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