ACM BUNDLES ON GENERAL HYPERSURFACES IN $\mathbb{P}^5$ OF LOW DEGREE

L. CHIANTINI, C. MADONNA

ABSTRACT. In this paper we show that on a general hypersurface of degree $r = 3, 4, 5, 6$ in $\mathbb{P}^5$ a rank 2 vector bundle $E$ splits if and only if $h^1(E(n)) = h^2(E(n)) = 0$ for all $n \in \mathbb{Z}$. Similar results for $r = 1, 2$ were obtained in [15], [16] and [1].

1. INTRODUCTION

The construction of rank 2 bundles on smooth varieties $X$ of dimension $n > 3$ is strictly related with the structure of subvarieties of codimension 2. When $X$ is a projective space, then there are few examples of these subvarieties which are smooth. The famous Hartshorne’s conjecture suggests that all smooth subvarieties of codimension 2 in $\mathbb{P}^7$ are complete intersection. Rephrased in the language of vector bundles, this means that all rank 2 bundles on $\mathbb{P}^7$ decompose in a sum of two line bundles.

Also in $\mathbb{P}^5, \mathbb{P}^6$, we do not have examples of indecomposable rank 2 bundles. In $\mathbb{P}^4$, only the Horrocks-Mumford’s indecomposable bundle is known. This bundle has some non–zero cohomology group, since it is well known that a rank 2 bundle $E$ on $\mathbb{P}^r$ ($r \geq 3$) splits if and only if it is “arithmetically Cohen–Macaulay” (ACM for short), i.e. $h^i(E(n)) = 0$ for all $n, i = 1, \ldots, r - 1$.

ACM property does not imply a decomposition when we replace the projective space with other smooth threefolds. There are examples of indecomposable ACM bundles of rank 2 on a general hypersurface of degree $r = 2, 3, 4, 5$ in $\mathbb{P}^4$. On the other hand we proved in [6] that all ACM rank 2 bundles on a general sextic in $\mathbb{P}^4$ splits.

In this paper we examine the similar problem for general hypersurfaces $X$ in $\mathbb{P}^5$, in some sense the easiest examples of smooth 4-folds different from $\mathbb{P}^4$.

It is well known that a general quadric hypersurface $X$ in $\mathbb{P}^5$ contains families of planes. Since any plane $S$ has a canonical class which is a twist of the restriction of the canonical class of the quadric (in other words: a plane is ”subcanonical” in $X$), then $S$ corresponds via the Serre’s construction to a rank 2 bundle $E$ on $X$ which is indecomposable (for $S$ is not complete intersection of $X$ and some other hypersurface) and ACM (for $S$ is arithmetically Cohen–Macaulay).

On the other hand, since any indecomposable ACM rank 2 bundle on a general sextic hypersurface in $\mathbb{P}^5$ would restrict to an indecomposable rank 2 ACM bundle on a general hyperplane section of $X$, which is a general sextic hypersurface of $\mathbb{P}^4$, then by the main result of [6] we know that such bundles cannot exist (see proposition 3.6 below).

1991 Mathematics Subject Classification. 14J60.
Thus we are led to consider general hypersurfaces $X_r \subset \mathbb{P}^5$ of low degree $r$ and study ACM rank 2 bundles on $X_r$. Our main result shows that none of such vector bundles lives on $X_r$ for $2 < r < 7$.

**Theorem 1.1.** Let $E$ be a rank 2 vector bundle on a general hypersurface $X_r \subset \mathbb{P}^5$ of degree $r = 3, 4, 5, 6$. Then $E$ splits if and only if

$$h^i(E(n)) = 0 \quad \forall n \in \mathbb{Z} \quad i = 1, 2.$$  

Notice that one finds indecomposable ACM rank 2 bundles on general hypersurfaces of degree 3, 4, 5 in $\mathbb{P}^4$. So we prove in fact that they do not lift from a general hyperplane section of $X$ to $X$ itself.

The proof is achieved using the tools of [6], since we have a classification of possible ACM indecomposable rank 2 bundles on a general hypersurface of low degree in $\mathbb{P}^4$. It has been obtained by Arrondo and Costa in degree 3 (see [2]), by the second author in degree 4 (see [13]) and by both authors in degree 5 ([5]). This implies a numerical characterization of the possible Chern classes of indecomposable ACM bundles of rank 2 on $X$ (see also [12]), and we conclude with a case by case analysis.

In the language of codimension 2 subvarieties, we get the following characterization of complete intersections, which is the analogue of the classical Gherardelli’s criterion for curves in $\mathbb{P}^3$:

**Corollary 1.2.** Let $S$ be a surface contained in a general hypersurface $X_r \subset \mathbb{P}^5$ of degree $r = 3, 4, 5, 6$. Then $S$ is complete intersection in $X_r$ if and only if it is subcanonical (i.e. its canonical class $\omega_S$ is $\mathcal{O}_S(e)$, for some $e \in \mathbb{Z}$) and $h^i(I_{S/X_r}(n)) = 0$ for all $n \in \mathbb{Z}$ and $i = 1, 2$, where $I_{S/X_r}$ is the ideal sheaf defining $S$ in $X_r$.

Let us finish with some remarks.

The non–existence of indecomposable ACM rank 2 bundles on hypersurfaces of degree $r \geq 7$ in $\mathbb{P}^4$ has not been settled yet simply because the technicalities introduced in [6] become odd as the degree $r$ grows. Indeed first of all the number of Chern classes which are not excluded using the main result of [12] grows as a linear function of $r$. Furthermore, as $r$ grows, for any value of $c_1$ one has to exclude an increasing number of second Chern classes. This is easy when $c_2$ is big, but becomes hard for low $c_2$ (compare the proof of case 5.11 in [6]), as we have to exclude the existence of some curves on $X$, which could be reducible or even non–reduced. We did not find a general argument for this step: only a careful ad hoc examination led us to conclude the case of sextic threefolds in $\mathbb{P}^4$.

On the other hand, there are strong evidences that ACM rank 2 indecomposable bundles cannot exist on general hypersurfaces of degree 7 or more. We were unable to prove this statement in $\mathbb{P}^4$. Could it be easier to find a direct proof for hypersurfaces in $\mathbb{P}^5$?

In any event, the main theorem implies easily:

**Corollary 1.3.** On a general hypersurface $X$ of degree $3, 4, 5, 6$ in $\mathbb{P}^n$, $n \geq 5$, a rank 2 vector bundle splits if and only if it is arithmetically Cohen–Macaulay.

Finally observe that Evans and Griffith proved in [8] that a rank 2 bundle $E$ on $\mathbb{P}^4$ splits if and only $h^1(E(n)) = 0$ for all $n$. This condition is considerably weaker than ACM. We wonder if a similar result could work on a general hypersurface of low degree in $\mathbb{P}^5$. 


2. Notations and generalities

We work over the field of complex numbers $\mathbb{C}$. Let $X_r \subset \mathbb{P}^5$ be a smooth 4-dimensional hypersurface of degree $r \geq 1$. The letter $H$ will denote the class of a hyperplane section of $X_r$. We have $\text{Pic}(X_r) \cong \mathbb{Z}[H]$, and $H^4 = r$. Recall that the canonical class of $X_r$ is $\omega_{X_r} = (r - 6)H$. Given a vector bundle $E$ on $X_r$ we introduce the notion of stability.

\begin{equation}
(2.1) \quad b(E) = b = \max\{n \mid h^0(E(-n)) \neq 0\}.
\end{equation}

**Definition 2.1.** We say that the vector bundle $E$ is normalized if $b(E) = 0$.

Notice that changing $E$ with $E(-b)$, we may always assume that $E$ is normalized. From now on we will assume this.

We denote by $c_1 = c_1(E)$ the first Chern class of $E$ identified with an integer via the isomorphism $\text{Pic}(X_r) \cong \mathbb{Z}[H]$. When $E$ has rank 2, the number

\begin{equation}
(2.2) \quad 2b - c_1 = 2b(E) - c_1(E)
\end{equation}

is invariant by twisting and measures the “level of stability of $E$”. Indeed $E$ is stable (semistable) if an only if $0 > 2b - c_1$ ($0 \geq 2b - c_1$).

We say that $E$ is “arithmetically Cohen–Macaulay (ACM)” when for all $n \in \mathbb{Z}$ we have $h^1(E(n)) = h^2(E(n)) = 0$. Clearly this implies, by duality, $h^3(E(n)) = 0$ for all $n \in \mathbb{Z}$.

Take a global section $s$ of $E$ whose zero-locus $S$ has codimension 2. We have the following exact sequence (see e.g. [17]):

\begin{equation}
(2.3) \quad 0 \rightarrow \mathcal{O}_{X_r} \rightarrow E \rightarrow I_{S/X_r}(c_1(E)) \rightarrow 0
\end{equation}

which relates the cohomology of $E$ with the geometric properties of $S \subset X_r$ encoded by the cohomology groups of the ideal sheaf $I_{S/X_r}$ of $S$.

In particular $S$ is subcanonical, i.e. $K_S \cong \mathcal{O}_S(c_1(E) + r - 6)$, moreover $c_2(E) = \text{deg} S$. Also we have a formula for the sectional genus $g$ of the surface $S$:

\begin{equation}
(2.4) \quad 2g - 2 = c_2 + K_S \cdot H \cdot S = c_2 + (c_1 + r - 6)H \cdot H \cdot S = c_2(c_1 + r - 5)
\end{equation}

Conversely, starting with a locally complete intersection and subcanonical surface $S$ contained in $X_r$ one can reconstruct a rank 2 vector bundle having a global section vanishing exactly on $S$. In these cases we will say that the vector bundle “$E$ is associated with $S$”.

We notice that when $E$ is normalized, then every global section of $E$ has zero-locus of codimension 2.

If $Y_r$ is a general hyperplane section of $X_r$ and $E$ is a rank two vector bundle on $X_r$, we denote by $E'$ the restriction of $E$ to $Y_r$. We know that $\text{Pic}(Y_r) \cong \mathbb{Z}[h]$, where $h$ is the hyperplane class of $Y_r$. Under the isomorphism $\text{Pic}(X_r) \cong \text{Pic}(Y_r)$ we have $c_1(E) = c_1(E')$ and $c_2(E) = c_2(E')$.

We recall here the main results of [12] and [6], which we are going to use several times in the sequel:

**Theorem 2.2.** (see [12]) Let $Y_r$ be a smooth hypersurface of degree $r$ in $\mathbb{P}^4$. If $E$ is an ACM and normalized rank 2 vector bundle on $Y_r$, then $E$ splits unless $r > c_1 > 2 - r$. 


Theorem 2.3. (see [6]) Let $Y$ be a general hypersurface of degree 6 in $\mathbb{P}^4$. Then a rank 2 vector bundle $E$ on $Y$ splits in a sum of line bundles if and only if $E$ is ACM.

3. SOME PRELIMINARY GENERAL RESULTS

Remark 3.1. Consider the exact sequence which links $E$ with its restriction $E'$ to a general hyperplane section $Y_r$ of $X_r$:

$$0 \to \mathcal{E}(-1) \to \mathcal{E} \to \mathcal{E}' \to 0$$

Then $b(E') \geq b(E)$ and equality holds when $h^1(E(-b(E) - 2)) = 0$, which is true when $E$ is ACM.

Notice that, by the sequence, if $E$ is ACM on $X_r$ then $E'$ is also ACM on $Y_r$. It is clear that $E'$ splits when $E$ splits. Conversely assume that $E$ is ACM and $E'$ splits. Take a global section $s' \in H^0(E'(a))$ with empty zero-locus. The surjection $H^0(E(a)) \to H^0(E'(a)) \to 0$ derived from sequence (3.1) shows that $s'$ lifts to a global section $s \in H^0(E(a))$, whose zero-locus must be empty, since otherwise it had at most codimension 2, a contradiction for it does not intersect a hyperplane.

It follows that we may apply the main result of [12], getting:

**Proposition 3.2.** If $E$ is an ACM and normalized rank 2 vector bundle on $X_r$, then $E$ splits unless $r > 1 > 2 - r$.

Some well known facts about the non-existence of surfaces of low degree on general hypersurfaces of $\mathbb{P}^5$ together with a numerical analysis leads us to the following refinement of the previous result:

**Proposition 3.3.** Let $E$ be a normalized ACM rank 2 vector bundle on a general hypersurface $X_r \subset \mathbb{P}^5$ of degree $r \geq 3$. Then $E$ splits unless $3 - r < c_1 < r$.

**Proof.** We need to exclude the case $c_1 = 3 - r$.

Consider a global section $s$ and its zero-locus $S$. The exact sequence (2.3) here reads

$$0 \to \mathcal{O}_{X_r} \to \mathcal{E} \to \mathcal{I}_{S/X_r}(3 - r) \to 0$$

and implies $h^0(E(r - 3)) = h^0(O_{X_r}(r - 3))$. By Serre duality $h^4(E(r - 3)) = h^0(E(r - 6)) = h^0(O_{X_r}(r - 6))$. Moreover $h^1(E(r - 3)) = h^1(O_{X_r}(r - 6))$. By Riemann-Roch one is thus able to compute the second Chern class of $E(r - 3)$, hence also the second Chern class $c_2$ of $E$. It turns out $c_2 = 1$. So $S$ is a plane. Since a general $X_r$ of degree $r \geq 3$ contains no planes (see e.g. [7]), then $X_r$ has no indecomposable and normalized rank 2 ACM bundles with $c_1 = -r + 3$. □

Next we use the link between ACM bundles with $c_1 = r - 1$ and pfaffian hypersurfaces.

**Definition 3.4.** A hypersurface $X_r \subset \mathbb{P}^5$ is pfaffian if its equation is pfaffian of a skew-symmetric matrix of linear forms in $\mathbb{P}^5$.

The results proved by Beauville in [9] exclude the existence of ACM rank 2 bundles with $c_1 = r - 1$ on a general hypersurface $X_r$, $r \geq 3$.

**Proposition 3.5.** When $r \geq 3$ then a general hypersurface $X_r \subset \mathbb{P}^5$ has no normalized indecomposable rank 2 ACM bundles $E$ with $c_1(E) = r - 1$ and $c_2 = r(r - 1)(2r - 1)/6$. 

\textbf{Proof.} It follows soon by the following two facts proved in \cite{3}. \(X_r\) is pfaffian if and only if there exists an indecomposable ACM rank 2 vector bundle on \(X_r\) with Chern classes as in the statement. Moreover the general hypersurface of degree \(r \geq 3\) in \(\mathbb{P}^5\) is not pfaffian. \(\blacksquare\)

Let us now turn to hypersurfaces of low degree. We want to exclude the existence of indecomposable ACM rank 2 bundles on general hypersurfaces. This follows easily on sextic hypersurfaces, using the main result of \cite{6}.

\textbf{Proposition 3.6.} On a general sextic hypersurface \(X \subset \mathbb{P}^5\) all ACM rank 2 bundles \(\mathcal{E}\) split.

\textbf{Proof.} A general hyperplane section \(Y\) of \(X\) is a general sextic hypersurface of \(\mathbb{P}^4\). By remark \ref{3.3} we know that an indecomposable ACM rank 2 bundle on \(X\) restricts to an indecomposable ACM rank 2 bundle on \(Y\). In \cite{6} we excluded the existence of such bundles. \(\blacksquare\)

For hypersurfaces of degree \(r < 6\) we cannot use the same procedure, since there exist indecomposable ACM rank 2 bundles on general cubics, quartics and quintics of \(\mathbb{P}^4\).

We use instead an examination of the family of surfaces associated to ACM rank 2 bundles. Let us set some more pieces of notation.

Call \(H(d, g)\) the Hilbert scheme of arithmetically Cohen–Macaulay (ACM) surfaces in \(\mathbb{P}^5\) of degree \(d = c_2\) and sectional genus \(g\) such that \(2g - 2 = c_2(c_1 + r - 5)\). This is a smooth quasi-projective subvariety of the Hilbert scheme.

Let \(P(r)\) be the scheme which parametrizes hypersurfaces of degree \(r\) in \(\mathbb{P}^5\). In the product \(H(d, g) \times P(r)\) one has the incidence variety

\begin{equation}
I(r, d, g) = \{(S, X) : X \text{ is smooth and } S \subset X \in H(d, g) \times P(r)\}
\end{equation}

with the two obvious projections \(p(r) : I(r, d, g) \rightarrow H(d, g)\) and \(q(r) : I(r, d, g) \rightarrow P(r)\). The fibers of \(q(r)\) are projective spaces of fixed dimension (by Riemann–Roch).

We will show that \(I(r, d, g)\) does not dominate \(P(r)\) for all choices of \(d, g\) corresponding to surfaces associated with an indecomposable ACM rank 2 bundle on a general \(X_r\). This is achieved in the next sections by computing the dimension of \(I(r, d, g)\) and observing that it is smaller than \(\dim(P(r))\).

Let us see, for instance, what happens for quadric surfaces.

\textbf{Remark 3.7.} Any quadric surface \(S\) contained in a general hypersurface \(X_r, r \geq 3\), is reduced since \(X_r\) contains no planes. Hence it is a surface in \(\mathbb{P}^3\), that is \(S\) is a complete intersection of type \((1, 1, 2)\) in \(\mathbb{P}^5\).

Thus we may compute the normal bundle \(N_S\) of \(S\):

\[h^0(N_S) = h^0(O_S(2) \oplus O_S(1) \oplus O_S(1)) = 9 + 4 + 4 = 17\]

hence \(\dim(H(2, 0)) \leq 17\).

\textbf{Proposition 3.8.} On a general hypersurface \(X_r \subset \mathbb{P}^5\) of degree \(r \geq 3\) there are no indecomposable normalized ACM rank 2 bundles \(\mathcal{E}\) with \(c_1(\mathcal{E}) = 4 - r\).

\textbf{Proof.} First we show that any such bundle \(\mathcal{E}\) is associated with a complete intersection quadric surface.

Consider a global section \(s \in H^0(\mathcal{E})\) and its zero-locus \(S\). The exact sequence \((2.3)\) here reads

\[0 \rightarrow O_{X_r} \rightarrow \mathcal{E} \rightarrow \mathcal{I}_{S/X_r}(4 - r) \rightarrow 0\]
and implies \( h^0(\mathcal{E}(r - 4)) = h^0(\mathcal{O}_X(r - 4)). \) \( h^1(\mathcal{E}(r - 4)) \) and \( h^2(\mathcal{E}(r - 4)) \) vanish by assumptions. By duality \( h^4(\mathcal{E}(r - 4)) = h^0(\mathcal{E}(r - 6)) = h^0(\mathcal{O}_X(r - 6)). \) Hence just as in proposition 4.1 we use Riemann-Roch to prove that \( c_3(\mathcal{E}) = 2. \) So \( S \) has degree 2. Since a general \( X_r \) contains no planes, \( S \) is reduced and the claim is proved.

Let \( \mathcal{I}_S \) indicate the ideal sheaf of \( S \) in \( \mathbb{P}^5. \) One computes from the resolution of \( \mathcal{I}_S: \)

\[
h^0(\mathcal{I}_S(r)) = 2h^0(\mathcal{O}_{\mathbb{P}^5}(r - 1)) - 2h^0(\mathcal{O}_{\mathbb{P}^5}(r - 3)) + h^0(\mathcal{O}_{\mathbb{P}^5}(r - 4))
\]

and thus one easily sees that:

\[
\dim(I(r, 2, 0)) \leq h^0(\mathcal{I}_S(r)) - 1 + h^0(N_S) < \dim(\mathbb{P}(r))
\]

for \( r > 2, \) which means that the map \( q(r) \) above is not dominant. The conclusion follows. \( \Box \)

With the results above we dispose of the case of cubic hypersurfaces:

**Proposition 3.9.** On a general hypersurface \( X := X_3 \subset \mathbb{P}^5 \) of degree 3 there are no indecomposable ACM rank 2 bundles.

**Proof.** By proposition 3.8 we know that any normalized indecomposable ACM rank 2 bundle on a smooth cubic hypersurface satisfies \( 3 > c_1(\mathcal{E}) > 0. \) So only the cases \( c_1(\mathcal{E}) = 1 \) and \( c_1(\mathcal{E}) = 2 \) are left. But on a general cubic hypersurface the case \( c_1(\mathcal{E}) = 2 \) is excluded by proposition 3.5 while the case \( c_1(\mathcal{E}) = 1 \) is excluded by proposition 3.8. \( \Box \)

4. **Quartic Hypersurfaces**

In this section we fix \( r = 4. \) Our goal is to exclude the existence of indecomposable ACM rank 2 bundles \( \mathcal{E} \) on a general quartic fourfold \( X := X_4. \) We also assume that \( \mathcal{E} \) is normalized.

Arguing as in proposition 3.8 we know that for such a bundle \( \mathcal{E} \) the only possibilities for the first Chern classes are \( c_1(\mathcal{E}) = 2 \) and \( c_1(\mathcal{E}) = 1. \)

We dispose of these cases using a computation for the normal bundle of the zero–locus of a general global section of \( \mathcal{E}. \)

**Remark 4.1.** If \( \mathcal{E} \) is an ACM rank 2 bundle on a smooth hypersurface \( X_r, \) then the zero–locus \( S \) of a global section of \( \mathcal{E} \) has codimension at most 3 in \( \mathbb{P}^5. \) If it has codimension 3, then it is an “arithmetically Gorenstein” subscheme of codimension 3 in the projective space \( \mathbb{P}^5. \) Thus its ideal sheaf \( \mathcal{I}_S \) in \( \mathbb{P}^5 \) has a self dual free resolution of type

\[
0 \to \mathcal{O}(-e - 6) \to \bigoplus_{j=1}^r \mathcal{O}(-m_j) \to \bigoplus_{i=1}^r \mathcal{O}(-n_i) \to \mathcal{I}_S \to 0
\]

where \( e \) is the number such that the canonical class of \( S \) is \( e \) times the hyperplane section and \( e + 6 - m_i = n_i \) for all \( i. \)

Using the previous resolution one can compute the cohomology of the normal bundle \( N_S \) of \( S \) in \( \mathbb{P}^5. \) Indeed by 3 and Theorem 2.6 of 11 we have the following:

**Proposition 4.2.** (Kleppe - Miró-Roig) With the notation of the previous remark, order the integers \( n_i \) and \( m_j \) so that:

\[
n_1 \leq n_2 \leq \ldots \leq n_r \quad \text{and} \quad m_1 \geq m_2 \geq \ldots \geq m_r.
\]
Then:

$$h^0 N_S = \sum_{i=1}^{r} h^0 O_S(n_i) + \sum_{1 \leq i < j \leq r} \left( -n_i + m_j + \frac{5}{5} \right) +$$

$$- \sum_{1 \leq i < j \leq r} \left( n_i - m_j + \frac{5}{5} \right) - \sum_{i=1}^{r} \left( n_i + \frac{5}{5} \right).$$

(4.2)

**Remark 4.3.** If $S$ is an ACM subscheme of $\mathbb{P}^5$ and $C$ is a general hyperplane section of $S$, then a minimal resolution of the ideal sheaf of $C$ in $\mathbb{P}^4$ lifts to a minimal resolution of the ideal sheaf of $S$ in $\mathbb{P}^5$.

Let us go back to general quartic hypersurfaces $X$.

A general hyperplane section $Y$ of $X$ is a general quartic threefold in $\mathbb{P}^4$ and the restriction $\mathcal{E}'$ of $\mathcal{E}$ to $Y$ is an indecomposable ACM bundle of rank 2. These bundles are classified in [13], where the possibilities for the second Chern classes of $\mathcal{E}'$, hence also of $\mathcal{E}$, are listed. These possibilities are:

$$(c_1, c_2) \in \{(-1,1), (0,2), (1,3), (1,4), (1,5), (2,8), (3,14)\}.$$

The cases $(c_1, c_2) = (-1,1), (0,2), (3,14)$ cannot occur on a general quartic fourfold, by propositions 3.3, 3.8 and 3.10.

We explore the remaining possibilities case by case.

**Case 4.1.** $c_1(\mathcal{E}) = 1$, $c_2(\mathcal{E}) = 3$.

By [13] $\mathcal{E}'$ is associated with a plane cubic curve, hence $\mathcal{E}$ is associated with a cubic surface $S \subset \mathbb{P}^3$. It turns out $h^0(N_S) = h^0(O_S(3) \oplus O_S^2(1)) = 27$ while the ideal sheaf $I_S$ has $h^0(I_S(4)) = 95$. Thus in this case dim$(I(4,3,1)) \leq 121$. Since dim$(\mathbb{P}(4)) = 125$, the projection $q(4) : I(4,3,1) \to \mathbb{P}(4)$ cannot be dominant and this case is excluded on a general quartic hypersurface $X$.

**Case 4.2.** $c_1(\mathcal{E}) = 1$, $c_2(\mathcal{E}) = 4$.

By [13] $\mathcal{E}'$ is associated with a quartic curve, complete intersection of 2 quadrics in $\mathbb{P}^3$. Hence $\mathcal{E}$ is associated with a complete intersection of two quadrics $S \subset \mathbb{P}^4$. It turns out $h^0(N_S) = h^0(O_S^2(2) \oplus O_S(1)) = 31$ while $h^0(I_S(4)) = 85$, so that dim$(I(4,4,1)) \leq 115$ and the projection $q(4) : I(4,4,1) \to \mathbb{P}(4)$ cannot be dominant.

**Case 4.3.** $c_1(\mathcal{E}) = 1$, $c_2(\mathcal{E}) = 5$.

By [13] $\mathcal{E}'$ is associated to an elliptic quintic curve, whose ideal sheaf $J$ in $\mathbb{P}^4$ has resolution:

$$0 \to O_{\mathbb{P}^4}(-5) \to O_{\mathbb{P}^4}^5(-3) \to O_{\mathbb{P}^4}^3(-2) \to J \to 0$$

(4.3)

from which we have the resolution for the ideal sheaf $I_S$ of a quintic surface $S$ associated with $\mathcal{E}$. Now we use proposition 4.2 to compute $h^0(N_S) = 35$ while from the resolution we get $h^0(I_S(4)) = 75$ so that dim$(I(4,5,1)) \leq 109$ and again $q(4)$ does not dominate $\mathbb{P}(4)$.

**Case 4.4.** $c_1(\mathcal{E}) = 2$, $c_2(\mathcal{E}) = 8$.

By [13] $\mathcal{E}'$ is associated to a curve of degree 8 in $\mathbb{P}^4$, whose ideal sheaf $J$ has resolution:

$$0 \to O_{\mathbb{P}^4}(-6) \to O_{\mathbb{P}^4}^3(-4) \oplus O_{\mathbb{P}^4}^5(-3) \to O_{\mathbb{P}^4}^3(-3) \oplus O_{\mathbb{P}^4}^5(-2) \to J \to 0$$

(4.4)

from which we have the resolution for the ideal sheaf $I_S$ of a surface $S$ of degree 8 associated with $\mathcal{E}$. Notice that we do not know the number of minimal generators
of degree 3 for the ideal sheaf of $S$ (if any). Nevertheless we may use proposition 4.2 to compute $h^0(\mathcal{N}_S)$. Indeed in the computation it turns out that the contribution of cubic generators disappears and one gets $h^0(\mathcal{N}_S) = 54$. Also one sees that $h^0(\mathcal{J}_S(4)) = 60$ so that $\dim(I(4,8,5)) \leq 113$ and again $q(4)$ does not dominate $\mathbb{P}(4)$.

No other cases may occur, by [13]. Hence we conclude:

**Proposition 4.4.** On a general hypersurface $X \subset \mathbb{P}^5$ of degree 4 there are no indecomposable ACM rank 2 bundles.

5. Quintic hypersurfaces

In this section we exclude the existence of indecomposable ACM rank 2 bundles $\mathcal{E}$ on a general quintic fourfold $X$. As usual we assume that $\mathcal{E}$ is normalized.

In this case, we are left with several cases for the first Chern class, namely $c_1(\mathcal{E}) = 0,1,2,3$.

Again a general hyperplane section $Y$ of $X$ is a general quintic threefold in $\mathbb{P}^4$ and the restriction $\mathcal{E}'$ of $\mathcal{E}$ to $Y$ is an indecomposable ACM bundle of rank 2. These bundles are classified in [5]. In particular for the Chern classes we have the following possibilities:

| $c_1$ | $c_2$ |
|-------|-------|
| 0     | 3, 4, 5 |
| 1     | 4, 6, 8 |
| 2     | 11, 12, 13, 14 |
| 3     | 20 |

We explore again the situation case by case.

**Case 5.1.** $c_1(\mathcal{E}) = 0$, $c_2(\mathcal{E}) = 3$.
By [5] $\mathcal{E}'$ is associated with a plane cubic curve, hence $\mathcal{E}$ is associated to a cubic surface $S \subset \mathbb{P}^3$. Then as above one computes $h^0(\mathcal{N}_S) = 27$ while the ideal sheaf $\mathcal{J}_S$ has $h^0(\mathcal{J}_S(5)) = 206$. Thus in this case $\dim(I(5,3,1)) \leq 232$. Since $\dim(\mathbb{P}(5)) = 251$, the projection $q(5) : I(5,3,1) \to \mathbb{P}(5)$ cannot be dominant and this case is excluded on a general quintic hypersurface.

**Case 5.2.** $c_1(\mathcal{E}) = 0$, $c_2(\mathcal{E}) = 4$.
By [5] $\mathcal{E}'$ is associated with a quartic curve, complete intersection of 2 quadrics in $\mathbb{P}^3$. Hence $\mathcal{E}$ is associated to a complete intersection of two quadrics $S \subset \mathbb{P}^4$. It turns out $h^0(\mathcal{N}_S) = 31$ while $h^0(\mathcal{J}_S(5)) = 191$, so that $\dim(I(5,4,1)) \leq 221$ and $q(5)$ is not dominant.

**Case 5.3.** $c_1(\mathcal{E}) = 0$, $c_2(\mathcal{E}) = 5$.
By [5] $\mathcal{E}'$ is associated with a quintic elliptic curve and as above one gets a resolution (5.1) $0 \to \mathcal{O}_{\mathbb{P}^5}(-5) \to \mathcal{O}_{\mathbb{P}^5}(5) \to \mathcal{O}_{\mathbb{P}^5}(2) \to \mathcal{I} \to 0$ for the ideal sheaf of a surface associated with $\mathcal{E}$. Then $h^0(\mathcal{N}_S) = 35$ while $h^0(\mathcal{J}_S(5)) = 176$, so that $\dim(I(5,5,1)) \leq 210$ and $q(5)$ is not dominant.

**Case 5.4.** $c_1(\mathcal{E}) = 1$, $c_2(\mathcal{E}) = 4$.
By [5] $\mathcal{E}'$ is associated with a plane quartic curve, hence $\mathcal{E}$ is associated with a quartic surface $S \subset \mathbb{P}^3$. It turns out $h^0(\mathcal{N}_S) = h^0(\mathcal{O}_S(4) \oplus \mathcal{O}_S^2(1)) = 42$ while the
ideal sheaf $J_S$ has $h^0(J_S(5)) = 200$. Thus in this case $\dim(I(5, 4, 3)) \leq 241$ and the projection $q(5) : I(5, 4, 3) \to \mathbb{P}(5)$ cannot be dominant.

**Case 5.5.** $c_1(\mathcal{E}) = 1$, $c_2(\mathcal{E}) = 6$.

By [5] equation 4.2 one is able to compute $h^0(J_S(5)) = 175$ so that $\dim(I(5, 6, 4)) \leq 222$ and the projection $q(5)$ cannot be dominant.

**Case 5.6.** $c_1(\mathcal{E}) = 1$, $c_2(\mathcal{E}) = 8$.

By [5] $\mathcal{E}'$ is associated to a curve of degree 8 in $\mathbb{P}^4$, whose ideal sheaf $\mathcal{J}$ has resolution:

\begin{equation}
0 \to \mathcal{O}_{\mathbb{P}^4}(-6) \to \mathcal{O}_{\mathbb{P}^4}^3(-4) \oplus \mathcal{O}_{\mathbb{P}^4}^3(-3) \to \mathcal{O}_{\mathbb{P}^4}^3(-2) \to \mathcal{J} \to 0.
\end{equation}

As above one gets $h^0(N_S) = 54$ while $h^0(J_S(5)) = 150$ so that $\dim(I(5, 8, 5)) \leq 203$ and $q(5)$ is not dominant.

Consider now the case $c_1(\mathcal{E}) = 2$ and $c_2(\mathcal{E}) = 11, 12, 13, 14$. Let $S$ be a surface associated with $\mathcal{E}$ and call $C$ a general hyperplane section of $S$, which is thus associated with $\mathcal{E}'$. One computes:

\[ h^0(J_S(5)) = 245 - 10 \deg(S) = 245 - 10c_2(\mathcal{E}) \]

so that we only need to prove that:

\begin{equation}
(5.3) \quad h^0(N_S) < 10c_2(\mathcal{E}) + 7.
\end{equation}

We use the results of [5] §4 and [5] case 5.7 to compute a minimal resolution for the ideal sheaf of $C$ in $\mathbb{P}^4$, hence also a resolution of $J_S$, which leads to the computation of $h^0(N_S)$, via proposition 1.2.

**Case 5.7.** $c_1(\mathcal{E}) = 2$, $c_2(\mathcal{E}) = 11$.

By [5] §4 the resolution of the ideal sheaf $J_S$ is:

\begin{equation}
0 \to \mathcal{O}_{\mathbb{P}^4}(-7) \to \mathcal{O}_{\mathbb{P}^4}(0) \oplus \mathcal{O}_{\mathbb{P}^4}(1) \oplus \mathcal{O}_{\mathbb{P}^4}(2) \to \mathcal{J} \to 0.
\end{equation}

Comparing the first Chen classes in the exact sequence, one finds $c = b - 2$. Using equation 4.2 one is able to compute $h^0(N_S)$. It turns out that $b$ and $c$ cancel and one finds $h^0(N_S) = 83 < 117$ so that $\dim(I(5, 11, 12)) \leq 214$ and $q(5)$ is not dominant.

**Case 5.8.** $c_1(\mathcal{E}) = 2$, $c_2(\mathcal{E}) = 12$.

By [5] §4 the resolution of the ideal sheaf $J_S$ is:

\begin{equation}
0 \to \mathcal{O}_{\mathbb{P}^4}(-7) \to \mathcal{O}_{\mathbb{P}^4}(0) \oplus \mathcal{O}_{\mathbb{P}^4}(1) \oplus \mathcal{O}_{\mathbb{P}^4}(2) \to \mathcal{J} \to 0.
\end{equation}

where $b = c - 1$ and $b = 0, 1$, according with the existence of a cubic syzygy between the two quadrics. In both cases, using equation 4.2 one computes $h^0(N_S) = 81 < 127$ so that $\dim(I(5, 12, 13)) \leq 205$ and $q(5)$ is not dominant.

**Case 5.9.** $c_1(\mathcal{E}) = 2$, $c_2(\mathcal{E}) = 13$.

In this case we have only one quadric containing $S$ and the resolution of $J_S$ is given by:

\begin{equation}
0 \to \mathcal{O}_{\mathbb{P}^4}(-7) \to \mathcal{O}_{\mathbb{P}^4}(0) \oplus \mathcal{O}_{\mathbb{P}^4}(1) \oplus \mathcal{O}_{\mathbb{P}^4}(2) \to \mathcal{J} \to 0.
\end{equation}

where $b = c - 1$ and $b = 0, 1$, according with the existence of a cubic syzygy between the two quadrics. In both cases, using equation 4.2 one computes $h^0(N_S) = 81 < 127$ so that $\dim(I(5, 12, 13)) \leq 205$ and $q(5)$ is not dominant.
So one computes $h^0(N_S) = 79 < 137$. It follows that $q(5)$ is not dominant.

**Case 5.10.** $c_1(\mathcal{E}) = 2$, $c_2(\mathcal{E}) = 14$.

By [5] §4 the resolution of the ideal sheaf $\mathcal{I}_S$ is:

$$0 \to \mathcal{O}_{P^5}(-7) \to \oplus \mathcal{O}_{P^5}(-4) \to \oplus \mathcal{O}_{P^5}(-3) \to \mathcal{I}_S \to 0$$

and one computes $h^0(N_S) = 77 < 147$ so that $q(5)$ is not dominant.

Finally for $c_1 = 3$ we have:

**Case 5.11.** $c_1(\mathcal{E}) = 3$, $c_2(\mathcal{E}) = 20$.

By [5] we know the resolution of the ideal sheaf of a curve associated with $\mathcal{E}'$, so that the ideal sheaf of a surface associated with $\mathcal{E}$ is:

$$0 \to \mathcal{O}_{P^5}(-8) \to \mathcal{O}_{P^5}(-5) \to \mathcal{O}_{P^5}(-3) \to \mathcal{I}_S \to 0.$$ 

One computes $h^0(N_S) = 110$ and $h^0(\mathcal{I}_S(5)) = 80$ so that dim($I(5,20,31)$) $\leq 189$ and $q(5)$ is not dominant.

Hence we may conclude

**Proposition 5.1.** On a general hypersurface $X \subset \mathbb{P}^5$ of degree 5 there are no indecomposable ACM rank 2 bundles.

The main theorem follows.

**Remark 5.2.** By [4] there exists a non discrete family (up to twist) of isomorphism classes of indecomposable ACM vector bundles on any smooth projective hypersurface of degree $r \geq 3$ in the 5–dimensional complex projective space $\mathbb{P}^5$. On a general $X_r$ the rank of the bundles constructed in [4] is 16 (cfr. [14]).

The problem of determining the minimum rank $BGS(X_r)$ for ACM bundles on $X_r$ moving in a non–trivial family (the $BGS$ invariant) is still open.

We prove in this paper that $BGS(X_r) > 2$ for general hypersurfaces in $\mathbb{P}^5$ of degree $r \leq 6$.

**References**

[1] E. Arrondo, *On congruences of lines in the projective space*, Mém. Soc. Math. France (N.S.) No. 50 (1992), 96 pp.

[2] E. Arrondo and L. Costa, *Vector bundles on Fano 3-folds without intermediate cohomology*, Comm. Algebra 28 (2000), no. 8, 3899–3911.

[3] A. Beauville, *Determinantal hypersurfaces. Dedicated to William Fulton on the occasion of his 60th birthday*, Michigan Math. J. 48 (2000), 39–64.

[4] R.O. Buchweitz, G.M. Greuel, and F.O. Schreyer, *Cohen-Macaulay modules on hypersurface singularities II*, Invent. Math. 88 (1987), 165–182.

[5] L. Chiantini and C. Madonna, *ACM bundles on a general quintic threefold. Dedicated to Silvio Greco on the occasion of his 60th birthday (Catania, 2001)*, Matematiche (Catania) 55 (2000), no. 2, 239–258.

[6] L. Chiantini and C. Madonna, *A splitting criterion for rank 2 bundles on a general sextic threefold*, Internat. J. Math. 15 (2004), no. 4, 341–359.

[7] O. Debarre, *Sur la variété des espaces linéaires contenus dans une intersection complète*, Math. Ann. 312 (1998), no. 3, 549–574.

[8] E. G. Evans and P. Griffith, *The syzygy problem*, Ann. of Math. 214 (1981), 323–333.

[9] D. Buchsbaum and D. Eisenbud, *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3*, Amer. J. Math. 99 (1977) no.3 447–485.

[10] R. Hartshorne, *Stable vector bundles of rank 2 on $\mathbb{P}^3$*, Math. Ann. 238 (1978), 229–280.

[11] J.O. Kleppe and R.M. Miró-Roig, *The dimension of the Hilbert scheme of Gorenstein codimension 3 subschemes*, J. Pure Appl. Alg. 127 (1998), 73–82.
[12] C.Madonna, A splitting criterion for rank 2 vector bundles on hypersurfaces in $\mathbb{P}^4$, Rend. Sem. Mat. Univ. Politec. Torino 56 (1998), no. 2, 43–54.

[13] C.Madonna, Rank-two vector bundles on general quartic hypersurfaces in $\mathbb{P}^4$, Rev. Mat. Complut. 13 (2000), no. 2, 287–301.

[14] C.Madonna, Rank 4 ACM bundles on a smooth quintic hypersurface in $\mathbb{P}^4$, preprint.

[15] C.Okonek, M.Schneider and H.Spindler, Vector bundles on complex projective spaces, Progress in Mathematics 3, 1980.

[16] G.Ottaviani, Some extension of Horrocks criterion to vector bundles on Grassmannians and quadrics, Ann. Mat. Pura Appl. (4) 155 (1989), 317–341.

[17] M.Roggero and P.Valabrega, The speciality lemma, rank 2 bundles and Gherardelli-type theorems for surfaces in $\mathbb{P}^4$, Compositio Math. 139 (2003), no. 1, 101–111.

L. Chiantini, Dipartimento di Scienze Matematiche e Informatiche, Università di Siena, Pian dei Mantellini 44, 53100 SIENA (Italy)
E-mail address: chiantini@unisi.it

C.K. Madonna, Dipartimento di Matematica, Università degli Studi di Roma "La Sapienza", P.le A. Moro 1, 00100 ROMA (Italy)
E-mail address: madonna@mat.uniroma1.it