Centrifugally driven relativistic dynamics on curved trajectories

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Motion of test particles along rotating curved trajectories is considered. The problem is studied
both in the laboratory and the rotating frames of reference. It is assumed that the system rotates
with the constant angular velocity $\omega = \text{const}$. The solutions are found and analyzed for the case when
the form of the trajectory is given by an Archimedes spiral. It is found that particles can reach infinity
while they move along these trajectories and the physical interpretation of their behaviour is given.
The analogy of this idealized study with the motion of particles along the curved rotating magnetic
field lines in the pulsar magnetosphere is pointed out. We discuss further physical development (the
conserved total energy case, when $\omega \neq \text{const}$) and astrophysical applications (the acceleration of
particles in active galactic nuclei) of this theory.

I. INTRODUCTION

Rotation and relativity are those two features of motion, which do not easily match with each other. Still in
astrophysics, with its abundance of extremely strong electromagnetic and gravitational fields, there are situations
where motion is both rotational and relativistic. Most prominent examples include swirling astrophysical jets in
active galactic nuclei (AGNs) and quasars, innermost regions of black hole accretion disks, accretion columns in X-ray
pulsars and plasma outflows in radio pulsar magnetospheres. In these kinematically complex astrophysical flows,
rotation is interlaced with the relativistic motion of particles, the coexistence of these two features of the motion leads
to observationally puzzling phenomena with sophisticated and ill-understood physical background. The interest to
these flows is not new, but the upgrade of highly idealized models to more realistic, astrophysically relevant levels is
still related with major theoretical and computational difficulties.

Some important and basic theoretical issues, related with the relativistic rotation, are not uniquely defined and
often evoke controversial interpretations. One of the most notable examples is the “centrifugal force reversal” effect,
originally found in [1], and later [2-4] studied in detail. It was argued that under certain conditions the centrifugal
force attracts towards the rotation axis both for Schwarzschild and Kerr black holes. In Ernst spacetime, which
represents the gravitational field of a mass embedded in a magnetic field, the centrifugal force acting on a particle
in circular orbit was reported [5] to reverse its sign even twice! In the simplest case of the Schwarzschild spacetime,
strictly and essentially speaking, it was found that below the radius of the spatially circular photon orbit an increase
of the angular velocity of a test particle causes more attraction rather than additional centrifugal repulsion. This
effect was interpreted by Abramowicz [4] in terms of the centrifugal force reversal - it was stated that in such cases
the centrifugal force attracts towards the axis of rotation!

This interpretation was criticized by de Felice [6,7] (see also [8]), who argued that the discovered effect could be
attributed to the strength of the gravitational field and be explained in a way which preserves the repulsive character
of the centrifugal force. The spirit of this approach — to save the intuitively appealing nature of the centrifugal force
as of “something which pushes things away” [7] — is theoretically valid and practically convenient. After all, in general
relativity, there is no implicit way to define the centrifugal force: in any case one needs to introduce some sort of
“3+1” spacetime splitting and dub as the “centrifugal force” some Newtonian-like expression, which looks like it [7].
Moreover, de Felice found several interesting examples of the ambiguity of the global concept of ‘outwards and pointed
out at the deep interrelation of this problem with the definition of the centrifugal force in relativity. Abramowicz
studied further the problem of the local and the global meaning of ‘inwards and ‘outwards [9] and showed that the
centrifugal force always repels outwards in the local sense, while it may attract inwards, towards the centre of the
circular motion, in the global sense! The theoretical scheme for the operationally unambiguous definition of the inward
direction was suggested by de Felice and Usseglio-Tomasset [10] and later this approach was used for the geometrical definition of the generalized centrifugal force [11].

Therefore the effect, discovered in [1], is indubitably a genuine relativistic effect, although its interpretation in terms of the “reversal” of the centrifugal force is not implicit and is largely the matter of definition.

Same is true for another rotational effect, disclosed by Machabeli and Rogava [12], on the basis of the relatively simple and idealized special-relativistic gedanken experiment: motion of a bead within a rigidly and uniformly rotating massless linear pipe. It was shown that even if the starting velocity of the bead is nonrelativistic, after an initial phase of usual centrifugal acceleration, while the bead acquires high enough relativistic velocity, it starts to decelerate and after reaching the light cylinder changes the character of its motion from centrifugal to centripetal. It was found that when the initial velocity \( v \geq \sqrt{2}/2 \) the motion of the bead is decelerative all the way from the pivot to the light cylinder.

Certainly no real pipe may stay absolutely rigid, especially nearby the light cylinder. Besides, in order to maintain the uniform rotation of such a device, one needs an infinite amount of energy. Therefore the setup considered in [12] was highly idealized. The constant rotation rate assumption was replaced in [13] with a more realistic one: the total energy of the system “rotator+pipe+bead” was assumed to be constant. It was found that the moving bead acquires energy from the slowing down rotator, but under favorable conditions the bead deceleration still happens.

The results of [12] were interpreted by its authors in terms of the centrifugal force reversal. De Felice disputed this interpretation [14] and argued that, also in this case, like in above-mentioned general-relativistic examples, the generalized definition of the centrifugal force may guarantee the absolutely repulsive character of the force. He pointed out that an inertial observer will never see the bead reaching the light cylinder, because all light signals from the bead are infinitely redshifted. It was also shown that the vanishing of the radial velocity of the bead at the light cylinder can be interpreted in terms of the corresponding vanishing of the bead’s proper time.

Despite the controversy of interpretations it is generally believed that rotational relativistic effects could operate in different astrophysical situations and might, hopefully, lead to detectable observational appearances. Recently, Heyl [15] suggested that the observed QPO frequency shifts in bursters are caused by a geometrical effect of the strong gravity, similar to the Abramowicz-Lasota centrifugal force “reversal”¹. As regards Machabeli-Rogava gedanken experiment, it implies that radially constrained, relativistic and rotationally (centrifugally) driven motion shows inevitable radial deceleration near the light cylinder. Evidently this effect might occur in a number of astrophysical situations, where motion is constrained, rotational and strongly (special) relativistic.

One of the most important class of astrophysical flows, where this effect could show up, is centrifugally driven outflows. In the context of pulsar emission theory they were first considered in the late 1960s by Gold [17,18] (for recent studies see e.g. Ref.19 and 20). For accreting black holes, both of galactic and extragalactic origin, Blandford and Payne [21] first noted that centrifugally driven outflows from accretion disks could be responsible for the launch of jets, if the poloidal field direction is inclined at an angle less than 60° to the radial direction².

Recently Gangadhara and Lesch [23] suggested that centrifugal acceleration, taking place as a consequence of the bead-on-the-wire motion similar to the Machabeli-Rogava [12] gedanken experiment, may account for the acceleration of particles to very high energies by the centrifugal forces while they move along rotating magnetic filed lines of the rotating AGN magnetosphere. They claimed that the highly nonthermal, X-ray and \( \gamma \)-ray emission in AGNs arises via the Comptonization (inverse-Compton scattering) of ultraviolet photons by centrifugally accelerated electrons. The same processes was critically re-examined by Rieger and Mannheim [24] and it was found that the rotational energy gain of charged particles is efficient but substantially limited not only by the Comptonization but also by the effects of the relativistic Coriolis forces. The specific nature of the propagation of electromagnetic radiation in the rotating frame of reference [14,25] is another aspect of this problem, which still needs to be taken into account.

The whole philosophy of the ‘pipe-bead’ (ot ‘rotator-pipe-bead’) gedanken experiments was to mimic the common situation in relativistic and rotating astrophysical flows, where the plasma particles are doomed to move along the field lines of governing magnetic fields. While we consider relatively small length scales, the shape of the field lines can be approximated as being straight. However, on larger length scales the curvature of the field lines turns out to be important for the physics of the plasma streams, which are guided by them. The natural question arises: how the motion of the bead changes when the pipe is curved? In other words, how the dynamics of particles, prescribed to move along the fixed trajectories, change when the shape of their involuntary tracks of motion is not straight!? Obviously, this is not only a mere theoretical curiosity, but the issue which might have a tangible practical importance.

¹However later [16] it was found that the Heyl’s calculations contained the sign error and the real effect could hardly account for the observed frequency shifts in the type I X-ray bursts.
²For a rapidly rotating Kerr black hole the critical angle can be as large as 90° [22].
In astrophysical situations the role of the “pipes” is played by the magnetic field lines, and the latter are always curved. Therefore, it is clear that the study of the motion of test particles along prescribed curved rotating trajectories is a necessary and important step for the ultimate building of a physically meaningful model of centrifugally driven relativistic particle dynamics for rotating magnetospheres of pulsars and AGNs.

It is the purpose of this paper to address the above stated issue. In particular, in the next section, we develop special-relativistic theory of the motion of centrifugally driven particles on fixed nonstraight trajectories. The formalism is developed both for the laboratory frame (LF) and for the frame of reference rotating with the system (rotating frame, or RF). Equations of motion are derived and solved numerically. The detailed study is given only for the case when the angular velocity of the rotation is constant. However, we also outline the formalism for the astrophysically more realistic case of the conservative ‘rotator-pipe-bead’ system with perceptible exchange of energy between the bead and the rotator, leading to the variability of the angular velocity of the whole system. In the final section of the paper we discuss the results, consider the directions and aims of the future study, suggest and discuss those astrophysical situations, where the obtained results could be useful for the clarification of puzzling observational appearances of related astronomical objects.

II. MAIN CONSIDERATION

The ideal two-dimensional system, which we are going to consider, consists of three basic parts: the device of the mass $M$ and the moment of inertia $I$, rotating with the angular velocity $\omega(t)$, hereafter referred as the rotator; the massless but absolutely rigid pipe steadily attached to the rotator; and the small bead of the mass $m$ and the radius equal to the internal cross-section radius of the pipe. The bead is put inside the pipe and can slide along the pipe without a friction. Evidently, instead of the pipe-bead dichotomy, since we are considering the two-dimensional layout, one may think about the ‘wire-on-bead analogy, which is sometimes used [24].

Contrary to the [12], where a straight pipe case was studied, now we let the pipe to be an arbitrarily flat curve, mathematically defined by:

$$\varphi \equiv \varphi(r),$$

(1a)

with $d\varphi/dr \equiv \varphi'(r)$:

$$\Phi \equiv r\varphi'(r).$$

(1b)

The dynamics of the system may be studied basing on two alternative assumptions:

1. It makes the task simpler to suppose that the kinetic energy of the rotator $E_M$ is huge and always $E_M \gg E_m$; i.e., despite the exchange of the energy with the moving bead, $E_M$ stays practically constant. Hence, the angular velocity of the whole “rotator-pipe-bead” system (henceforth referred as the RPB system) stays constant:

$$\omega = \text{const.}$$

(2a)

In this case the first part of the triple RPB system (the rotator) continuously supplies the bead with energy and helps to keep the angular velocity of rotation constant. Therefore, the problem reduces to the study of the pipe-bead double system (the PB system) with the constant rotation rate. In the case of the straight pipe ($\varphi = \varphi_0$) the problem has exact analytic solution, found and analyzed in [12].

2. It is more realistic to assume that the rotator energy $E_M$ is finite, so that the whole RPB system is conservative $E_{\text{tot}} = E_M + E_m = \text{const}$. There is a perceptible energy exchange between the rotator and the bead: both the energy $E_M$ and the angular momentum $L_M$ of the rotator are variable and, consequently, the angular velocity of the rotation can not stay constant:

$$\omega \neq \text{const.}$$

(2b)

The problem with the straight pipe and variable rotation rate $\omega(t)$ has no analytic solution. It was studied numerically in [13].

With either (2a) or (2b) assumptions the pipe is always assumed to be the passive part of the system. In order to mimic a magnetic field line it is assumed to be massless, having no share in the energy and/or momentum balance of the whole system. Still the role of the pipe — as the dynamic link between the rotator and the bead — is significant:
it provides the prescribed “guiding” of the bead motion in the rotating frame of reference and makes the trajectory of the bead known in advance.

In this paper the dynamics of the gedanken system is studied in detail only under the first, easier, assumption of the constant angular velocity. The rout to the solution of the problem under the second assumption is also given, but its full study needs separate consideration and will be published elsewhere.

There are two natural frames of reference, in which the dynamics of this system could be studied. The first, inertial one, is the laboratory frame (LF), where the observer measures the angular velocity of the rotator (and the pipe) to be \( \omega(t) \), while the angular velocity of the bead is equal to:

\[
\Omega(t) = \omega(t) + \varphi'(r)v(t),
\]

and the dynamics of the moving bead is governed by the pipe reaction force acting on it. Note that \( v(t) \equiv dr/dt \) is the radial velocity of the bead relative to the LF.

The second frame, rigidly attached to the rotator and rotating with it (hereafter referred as the rotating frame, or the RF), is non-inertial, but quite convenient for the inspection of the motion of the bead along the curved pipe. This original approach implies embodying of the form of the pipe into the metric of the rotating frame. It was used in [12] for the straight pipe case and proved to be quite efficient for the case when \( \omega(t) \) is assumed to be a constant. On the contrary, as we shall see later, the LF treatment appears to be handier when the second (2b) approximation (\( E_{\text{tot}} = \text{const} \) and \( L_{\text{tot}} = \text{const} \), rather than \( \omega(t) = \text{const} \) ) is chosen. That is why it is important to consider the problem both in the LF and in the RF.

A. Uniformly rotating PB system

First, let us consider the problem in the laboratory frame of reference (LF) and ascertain that it admits full (numerical) solution of the associated initial value problem. Second, let us consider the same problem in the rotating frame of reference (RF). We shall see that when the (2a) assumption of the constancy of the rotation rate is used the latter approach is mathematically easier and provides fuller information about the dynamics of the system.

1. LF treatment

The most straightforward way to approach the problem is to consider it in the laboratory frame of reference, in which the spacetime is Minkowskian:

\[
ds^2 = -dT^2 + dX^2 + dY^2 = -dT^2 + dr^2 + r^2 d\phi^2.
\]

We use geometrical units, in which \( G = c = 1 \). Note that azimuthal angle \( \phi \), as measured in the LF, is related with the azimuthal angle \( \varphi \), measured in the RF, via the obvious expression: \( \phi = \varphi + \omega t \). The pipe reaction force \( \mathbf{F} \) is the dynamic factor constraining the bead to move along the pipe. It is easy to see (from the 4-velocity normalization \( g_{\alpha\beta}U^\alpha U^\beta = -1 \) ) that the Lorentz factor of the moving bead is:

\[
\gamma(t) = \left[ 1 - r^2\Omega^2 - v^2 \right]^{-1/2}.
\]

The angle between the radius-vector of a point of the pipe and the tangent to the same point is given by the relation:

\[
\alpha = \arctan \Phi,
\]

and the components of the reaction force, acting in the radial and azimuthal directions, are

\[
F_r = -|F| \sin \alpha = -\frac{\Phi}{\sqrt{1 + \Phi^2}}|F|, \quad (7a)
\]

\[
F_\phi = |F| \cos \alpha = \frac{1}{\sqrt{1 + \Phi^2}}|F|, \quad (7b)
\]

respectively.
Defining the physical components of the bead relativistic momentum \( [m(t) \equiv m_0 \gamma(t)] \):

\[
P_r \equiv mv, \quad (8a)
\]

\[
P_\phi \equiv mr\Omega, \quad (8b)
\]

we can write the two components of the equation of motion in the following way:

\[
\dot{P}_r - \Omega P_\phi = F_r, \quad (9a)
\]

\[
\dot{P}_\phi + \Omega P_r = F_\phi. \quad (9b)
\]

Combining these equations we can, first, derive the equation:

\[
\dot{P}_r + \Phi \dot{P}_\phi + \Omega (\Phi P_r - P_\phi) = 0. \quad (10a)
\]

It is easy to calculate that:

\[
\dot{\Omega} = \varphi' \dot{v} + \varphi'' v^2, \quad (10b)
\]

\[
\dot{m} = m\gamma^2[(\Omega + r\varphi''v)\Omega v + (v + r^2\varphi'\Omega)\dot{v}]; \quad (10c)
\]

and using these relations together with (8) we can easily derive the explicit equation for the radial acceleration of the bead:

\[
\ddot{r} = \frac{r\omega\Omega - \gamma^2 r v(\varphi' + \omega v)(\Omega + r\varphi''v)}{\gamma^2 \Delta^2}, \quad (11)
\]

where

\[
\Delta \equiv [1 - \omega^2 r^2 + \Phi^2]^{1/2}. \quad (12)
\]

The Eq. (11) being of the form \( \ddot{r} = G(\dot{r}, r) \) admits full numerical solution, as the standard initial value problem, providing the initial position of the bead, \( r_0 \), its initial velocity, \( v_0 \), and the shape of the pipe, \( \varphi(r) \), are specified.

Defining the spatial vector of the 2-velocity \( \mathbf{v} \equiv (v, r\Omega) \), we can calculate the absolute value of the reaction force \( |\mathbf{F}| \) using the equation [26]:

\[
\dot{m} = \mathbf{F} \cdot \mathbf{v}, \quad (13a)
\]

which, in our case, leads to:

\[
\sqrt{\text{grr}} \dot{m} = r\omega |\mathbf{F}|. \quad (13b)
\]

It is also easy to verify that the following quantity:

\[
\Psi \equiv m(t) - \omega r P_\phi = m_0 \gamma (1 - r^2\omega\Omega) = \text{const}(t), \quad (14)
\]

is the constant in time. This allows to find the solutions of the problem as functions of the specific value of this constant. In the next subsection we will see what is the physical meaning of this parameter - it turns out to be proportional to the proper energy of the moving bead in the RF.

One important class of a possible shape of the curved trajectory is Archimedes spiral, given by the formula:

\[
\varphi(r) = ar, \quad a = \text{const}. \quad (15)
\]

In this case, since \( \varphi'' = 0 \), from (11) it is easy to see that \( \ddot{r} \sim \Omega \), while (14) implies that \( |\mathbf{F}| \sim \Omega \) as well. Therefore, in the case of the Archimedes spiral trajectory, we can predict that the asymptotic behavior of the functions \( \Omega(t), v(t), \dot{v}(t), \) and \( |\mathbf{F}(t)| \) will be similar.
2. RF treatment

We see the LF treatment allows to solve the problem and to obtain the complete information about the dynamics of the bead motion along the fixed nonstraight (curved) trajectories. However it is quite instructive and much more convenient to consider the same problem in the frame of reference, rotating with the pipe-bead system (rotating frame - RF). In order to do this we, first, need to switch from (4) to the frame, rotating with the angular velocity \( \omega \).

Employing the transformation of variables:

\[
T = t,
\]

\[
X = r \cos \phi = r \cos (\varphi + \omega t),
\]

\[
Y = r \sin \phi = r \sin (\varphi + \omega t),
\]

we arrive to the metric:

\[
ds^2 = -(1 - \omega^2 r^2)dt^2 + 2\omega r dtd\varphi + r^2 d\varphi^2 + dr^2.
\]

(17)

For the straight pipe (\( \varphi = \varphi_0 \)) case (17) reduces to the metric \( ds^2 = -(1 - \omega^2 r^2)dt^2 + dr^2 \), which was basic metric for the [12] study. Now, for a curved pipe, defined by the equation (1), (17) reduces to the following form:

\[
ds^2 = -(1 - \omega^2 r^2)dt^2 + 2\omega r \Phi dt dr + (1 + \Phi^2)dr^2.
\]

(18)

For the resulting metric tensor

\[
\|g_{\alpha\beta}\| = \begin{pmatrix}
    -(1 - \omega^2 r^2), & \omega r \Phi \\
    \omega r \Phi, & 1 + \Phi^2
\end{pmatrix},
\]

(19)

we can easily find out that

\[
\Delta = \left| -\det(g_{\alpha\beta}) \right|^{1/2} = (1 - \omega^2 r^2 + \Phi^2)^{1/2},
\]

(20)

and, apparently it is the same function \( \Delta \), defined previously by (13).

For this relatively simple, but nondiagonal, two-dimensional spacetime we can develop the “1+1” formalism. Doing so we follow as a blueprint the well-known “3+1 formalism, widely used in the physics of black holes [27-29]. Namely, we introduce definitions of the lapse function:

\[
\alpha \equiv \frac{\Delta}{\sqrt{g_{rr}}} = \sqrt{\frac{1 - \omega^2 r^2 + \Phi^2}{1 + \Phi^2}},
\]

(21)

and the one-dimensional vector \( \vec{\beta} \) with its only component:

\[
\beta^r \equiv \frac{g_{tr}}{g_{rr}} = \frac{\omega r \Phi}{1 + \Phi^2}.
\]

(22)

Within this formalism (18) can be presented in the following way:

\[
ds^2 = -\alpha^2 dt^2 + g_{rr}(dr + \beta^r dt)^2.
\]

(23)

Note that for the metric tensor (19) \( t \) is the cyclic coordinate and, moreover, in the RF the motion of the bead inside the pipe is geodesic - there are no external forces acting on it. Hence the proper energy of the bead, \( E \), must be a conserved quantity. Employing the definition of the four velocity \( U^\alpha \equiv dx^\alpha/d\tau \) we can write:

\[
E \equiv -U_t = -U^t [g_{tt} + g_{tr} v] = \text{const}.
\]

(24)

On the other hand, the basic four-velocity normalization condition \( g_{\alpha\beta} U^\alpha U^\beta = -1 \) requires

\[
U^t = \left[ -g_{tt} - 2g_{tr} v - g_{rr} v^2 \right]^{-1/2},
\]

(25a)
this equation, written explicitly, has the following form:

\[ U^t = \left[ 1 - \omega^2 r^2 - 2\omega r \Phi v - (1 + \Phi^2) v^2 \right]^{-1/2}. \]  

(25b)

Recalling the expression (3) for the angular velocity of the bead \( \Omega(t) \), measured in the LF, and the definition (5) of the Lorentz factor \( \gamma(t) \) in the same frame of reference we can easily see that:

\[ U^t = \left[ 1 - r^2 \Omega^2 - v^2 \right]^{-1/2} = \gamma(t). \]  

(25c)

It is important to note that the conserved proper energy of the bead, \( E \), defined by (24) may be written in terms of the \( \Omega(t) \) and \( \gamma(t) \) functions simply as:

\[ E = \gamma(t) [1 - r^2(t) \omega \Omega(t)] = \text{const.} \]  

(26)

Taking time derivative of this relation and rearranging the terms we will finally arrive to exactly the same Eq. (11) for the radial acceleration of the bead \( \ddot{r} \) as in the LF treatment. Note also that \( \Psi = m_0 E \).

But the convenience of the RF treatment goes much further. From (24) and (25a) we can derive the explicit quadratic equation for the velocity:

\[ (g_{tr}^2 + E^2 g_{rr}) v^2 + 2g_{tt}(gtt + E^2)v + g_{tt} + E^2 = 0, \]  

(27)

with the obvious solution:

\[ v = \dot{r} = \frac{\sqrt{g_{tt} + E^2}}{g_{tr} + E^2 g_{rr}} \left[ -g_{tr} \sqrt{g_{tt} + E^2} \pm E \Delta \right]. \]  

(28)

The “1+1” formalism helps to write equivalents of the same equations in a more elegant form. Namely, if we define the radial velocity

\[ V^r \equiv \frac{1}{\alpha} (v + \beta r), \]  

(29)

and corresponding Lorentz factor:

\[ \tilde{\gamma} \equiv (1 - V^2)^{-1/2}, \]  

(30)

then, instead of (25c), we will simply have:

\[ U^t = \tilde{\gamma}/\alpha, \]  

(31)

while from the (24) we obtain:

\[ E = \tilde{\gamma} [\alpha - (\beta \cdot \vec{V})]. \]  

(32)

Instead of (27) we will have \([V^2 \equiv g_{rr} V^r V^r = V^r, \beta^2 \equiv g_{rr} \beta^r \beta^r = \beta_r \beta^r;](\beta^2 + E^2)V^2 - 2\alpha(\beta \cdot \vec{V}) + (\alpha^2 - E^2) = 0, \]  

(33)

with the solution:

\[ V^r = \frac{1}{\beta^2 + E^2} \left[ \alpha \beta^r \pm E \sqrt{\frac{E^2 + \beta^2 - \alpha^2}{g_{rr}}} \right]. \]  

(34)

Note that the RF Lorentz factor defined by (30) and the LF Lorentz factor, specified by (5) do not equal each other

\[ \tilde{\gamma}(t) \neq \gamma(t), \]  

(35)

which is the manifestation of the obvious fact that Lorentz factor is not an invariant physical quantity. One can see that for the \( \tilde{\gamma}(t) \) the following quadratic equation holds:

\[ (\alpha^2 - \beta^2) \tilde{\gamma}^2 - 2\alpha E \tilde{\gamma} + (E^2 + \beta^2) = 0, \]  

(36)
with the following solution:
\[
\tilde{\gamma}(t) = \frac{1}{\alpha^2 - \beta^2} \left[ \alpha E \pm |\beta| \sqrt{\beta^2 + E^2 - \alpha^2} \right].
\] (37)

Therefore, above developed theory allows us to look for the solution of the initial value problem for the bead moving along an arbitrarily curved pipe. The thorough consideration of different particular cases is beyond the scope of the present paper. Instead, we will give representative solutions for one of the simplest kinds of a spiral – Archimedes spiral – defined by (15).

The scheme for the complete inspection of the problem for any given initial value problem is the following: First, one specifies the initial location of the bead \((r_0)\) and its initial radial velocity \((v_0)\). The values of the \(\varphi'(r_0)\) and \(\Phi(r_0)\) are fixed as soon as we specify the form of the pipe \(\varphi(r)\). The initial values for the \(\Omega(0)\) and \(\gamma(0)\) are given by (3) and (5):
\[
\Omega_0 = \omega + \varphi'(r_0)v_0,
\] (38)
\[
\gamma_0 = [1 - r_0^2\Omega_0^2 - v_0^2]^{-1/2},
\] (39)
while the value of the bead proper energy, according to (26), is given as:
\[
E = \gamma_0[1 - v_0^2\omega\Omega_0].
\] (40)

Working with the Eq.(28), as the first order ordinary differential equation for the radial position \(r(t)\) of the bead at any moment of time, we can subsequently calculate all other physical variables. On the Fig.1 the set of solutions is given for the case of the Archimedes spiral with \(a = -5\), rotating with the angular velocity \(\omega = 2\) for the bead, which initially was situated right over the pivot of the rotator \((r_0 = 0)\) and had initial radial velocity \(v_0 = 0.1\).

These plots tell us that in the limit of large distance from the rotator the value of the radial velocity tends to the asymptotic value:
\[
\lim \inf v(t) = v_\infty \equiv -\omega/a,
\] (41)
which, in this case, is equal to \(v_\infty = 0.4\).

![Graphs](image)

FIG. 1. Graphs for the radial distance \(r(t)\), velocity \(v(t)\), acceleration \(a(t)\), the Lorentz factor \(\gamma(t)\), angular velocity \(\Omega(t)\) and the absolute value of the reaction force \(|F(t)|\) for the rotationally \((\omega = 2)\) driven bead, moving on the Archimedes spiral with \(a = -5, r_0 = 0, v_0 = 0.1\).

The angular velocity of the bead in the LF tends to zero, as well as the absolute value of the pipe reaction force, implying that at infinity the bead asymptotically reaches the limit of the force-free motion. This limit is understandable also analytically, because from (3) and (28) we can see that:
\[
v \rightarrow \frac{\omega}{|a|} + \frac{E\sqrt{a^2 - \omega^2}}{\omega a^2 r^2},
\] (42)
$$\Omega \rightarrow \frac{E \sqrt{a^2 - \omega^2}}{\omega \eta r^2}. \quad (43)$$

From these expressions it is clear that this regime is accessible iff the condition $|a| > \omega$ holds! Otherwise, the particle is not able to reach the infinity.

Since the shape of the function $r(t)$ is almost linear it is instructive to make plots for the functions $v(r)$ for different values of the initial radial velocity $v_0$, but with all other parameters of the initial value problem being the same. On the Fig.2 we plotted these functions for eight different values of the initial radial velocity. We see that when $v_0 = v_\infty$, the movement of the particle is force-free (geodesic) and uniform during the whole course of the motion. Physically it means that for this particular value of the $v_0$ the shape of the pipe follows the geodesic trajectory of the bead, in the RF, for the metric (17) on the rotating 2D disk, so the bead moves freely, without interacting with the walls of the pipe. When $v_0 < v_\infty$, the particle moves with positive acceleration and asymptotically reaches the force-free regime in the infinity. While, when $v_0 > v_\infty$ the character of the motion is decelerative, but the force-free limit is reached, again, when the bead heads to infinity.

One more example of the latter behavior, similar to the case shown on the Fig. 1, but plotted for the initial velocity $v_0 = 0.5 > v_\infty = 0.4$ is given on the Fig.3. Here we see that, unlike the case given on the Fig.1, the acceleration of the bead is negative all the time and it reaches zero “from below”, taking less and less negative values. While the angular velocity of the bead relative to the LF $\Omega(t)$ is also negative from the beginning but its absolute value decreases and reaches the zero as the particle tends to the infinity.

**FIG. 2.** Graphs for the radial velocity $v(r)$, when the initial value of the $v(r)$ is taken to be: 0.01, 0.1, 0.3, 0.4, 0.5 (force-free value), 0.7, 0.9, 0.99. $\omega = 0.1$, $a = -0.2$. 
FIG. 3. Graphs for the radial distance $r(t)$, velocity $v(t)$, acceleration $a(t)$, the Lorentz factor $\gamma(t)$, angular velocity $\Omega(t)$ and the absolute value of the reaction force $|F(t)|$ for the rotationally ($\omega = 2$) driven bead, moving on the Archimedes spiral with $a = -5$, $r_0 = 0$, $v_0 = 0.5$.

B. Conserved energy case

When the bead accelerates, it continuously takes energy from the rotator. So, if one needs to keep the rotation rate constant, one needs to supply the system with energy from outside. This is, certainly, less realistic setup than the assumption that the “rotator-pipe-bead” system is conservative, viz. its total energy $E_{tot}$ is constant. In this case, however, the bead acceleration can not be permanent, because asymptotically it extracts all energy from the rotator and reaches the regime: $\omega(t) \to 0$, $E_M \to 0$ and $E_m \to E_{tot}$. Clearly, in this situation, it is more convenient to study the dynamics in the laboratory frame of reference (LF), in which the rotator and the pipe are rotating rigidly with the time-dependent angular velocity $\omega(t)$. As regards the bead, since the shape of the pipe is curved, its angular velocity relative to the LF is given by (3). Since the pipe is considered to be massless and absolutely rigid, it does not contribute any amount of energy and/or angular momentum to the total energy $E_{tot}$ and angular momentum $L_{tot}$ of the system. The rotator for simplicity is assumed to be a sphere of the radius $R$ and the mass $M$ having the inertia moment

$$ I = \frac{2}{5} MR^2, \quad (44) $$

the energy

$$ E_M = \frac{I}{2} \omega^2(t), \quad (45) $$

and the angular momentum

$$ L_M = I \omega(t). \quad (46) $$

Note that (44-46) are nonrelativistic expressions. If initially, at $t = t_0$, $\omega_0 R \ll 1$, $\omega_0 R/c \ll 1$, $R = 1.2 \times 10^6 \text{cm}$, $\omega_0 = 190.4 \text{Hz}$ and consequently $\omega_0 R/c \simeq 7.6 \times 10^{-3}$. Even for the fastest millisecond pulsars $\omega_0 R/c \leq 0.25$. This justifies the usage of nonrelativistic (44-46) expressions in our analysis.
The remaining part of the threefold system — the bead — is assumed to be of the rest mass $m_0$. Its angular velocity and radial velocity relative to the LF, at any given moment of time, are $\Omega(t)$ and $v(t) = \dot{r}$, respectively. Even when the initial radial velocity of the bead is nonrelativistic ($v_0 \ll 1$), it is still necessary to write relativistic expressions for its energy and angular momentum, because the bead gains energy, accelerates and sooner or later its motion becomes relativistic. Therefore, its energy and angular momentum must be written as:

$$E_m = m(t) = m_0 \gamma(t),$$
$$L_m = m_0 \gamma(t)r^2(t)\Omega(t) = m(t)r^2(t)\Omega(t),$$

where the LF Lorentz factor $\gamma(t)$ is defined by (5).

The system “rotator-pipe-bead” is conservative, there is no energy inflow from outside. In this sense it principally differs from the one considered in the previous section, where either the external energy source was necessary to keep the rotation rate $\omega$ constant or the rotator was assumed to possess an infinite amount of energy. Now, since the system is conservative, its dynamics are governed by the conservation laws of its total energy $E_{tot} = E_M + E_m$ and total angular momentum $L_{tot} = L_M + L_m$:

$$\frac{I}{2} \omega^2(t) + m_0 \gamma(t) = E_{tot},$$
$$I \omega(t) + m_0 \gamma(t)r^2\Omega(t) = L_{tot}.$$

And the solution of the problem reduces to the solution of these equations, linked with (9), for two unknown functions of time $r(t)$ and $\omega(t)$ for an arbitrary initial value problem: initial location of the bead $r_0 = r(0)$ and the initial value of the rotation rate $\omega_0 = \omega(0)$ of the whole system. The detailed study of this problem is beyond the scope of this paper and will be given in a separate publication.

III. CONCLUSION

The purpose of the present paper was to study the dynamics of relativistic rotating particles with prescribed, curved trajectories of motion in the rotating frame of reference. The work is a natural generalization of the gedanken “pipe-bead” experiment considered Machabeli and Rogava [12]. In that paper the authors considered the case of the straight rotating pipe and they found out that when the velocity of the bead, driven by the rotation of the whole device and sliding along/within the pipe, is high enough the character of the motion changes from the accelerated to the decelerated one. In particular, it was found that when the bead starts moving from the pivot ($r = 0$) of the rotating pipe with initial velocity $v_0 > \sqrt{2}/2$, the motion is decelerative from the very beginning.

In this paper we consider the motion of rotationally driven particles along flat trajectories of arbitrarily curved shape. The practical motivation for this approach and its importance are related with the following two facts:

1. The ‘pipe-bead’ (or the ‘bead-on-the-wire’) gedanken experiment considered as a model for the study of dynamics of centrifugally driven relativistic particles in rotating magnetospheres, in various classes of astrophysical objects, like pulsars [17-18,12,20] and AGNs [21-24]. The role of “pipes” is played by the magnetic field lines.

2. The shape of magnetic field lines is always curved. It implies that for the large-scale, global dynamics of charged particles — driven by centrifugal forces and moving along curved field lines of rotating magnetospheres — it is important to know what qualitative changes occur when the form of the field lines is not linear but curved.

In this paper we studied this problem, on the level of the idealized gedanken experiment, both in the laboratory (LF) and in the rotating (RF) frames of reference. For the simple example of the Archimedes spiral we found that the dynamics of such particles may involve both accelerative and decelerative modes of motion.

One important difference from the linear pipe case [12] is that for the case of a curved pipe the motion of the bead is not any more radially bounded: there exist regimes of motion when the bead may reach infinity. This result has simple physical explanation. For the case of the linear pipe, rotating with the constant angular velocity $\omega_0$, the natural limit of the radial motion was given by the light cylinder radius, defined as $R_L \equiv \omega_0^{-1}$. Now, in the case of the curved pipe, even when it rotates with the same constant rate, the bead can slide in the azimuthal direction, following the curvature of the pipe and having a variable angular velocity $\Omega(t)$. It means that now the role of the effective light cylinder is played by $R_L(t) = \Omega(t)^{-1}$, and, hence, all those radial distances become accessible, where
Eventually the pulsar wind. Through its slowing rate, and its radiation losses and energy taken away by centrifugally driven plasma, forming reality of the pulsar environment and could help to relate with each other total energy losses of a pulsar, estimated fluid particles would behave, moving along rotating curved trajectories. This will comprise one more step closer to the theory with the existing empirical (observational) data about the energy deposited by pulsars into their winds and collective plasma effects are essential, so the simple ‘one-particle treatment can give only very approximate picture of the involved physical processes. But taking into account the plasma fluid effects and the role of the radiation on the dynamics of infalling plasma streams, one could try to show how important the rotational (inertial) processes are for the dynamics of the flows infalling on strongly magnetized neutron stars and what is the influence of these processes on the observational appearance of related X-ray sources.

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