Irrational Non-Abelian Statistics for Non-Hermitian Generalization of Majorana Zero Modes

Xiao-Ming Zhao,1,2 Cui-Xian Guo,1,2 Meng-Lei Yang,2 Heng Wang,2 Wu-Ming Liu,1 and Su-Peng Kou2,3

1Beijing National Laboratory for Condensed Matter Physics, Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China
2Center for Advanced Quantum Studies, Department of Physics, Beijing Normal University, Beijing 100875, China

In condensed matter physics, non-Abelian statistics for Majorana zero modes (or Majorana Fermions) is very important, really exotic, and completely robust. The race for searching Majorana zero modes and verifying the corresponding non-Abelian statistics becomes an important frontier in condensed matter physics. In this letter, we generalize the Majorana zero modes to non-Hermitian (NH) topological systems that show universal but quite different properties from their Hermitian counterparts. Based on the NH Majorana zero modes, the orthogonal and non-local Majorana qubits are well defined. In particular, due to the particle-hole-symmetry breaking, NH Majorana zero modes have irrational non-Abelian statistics with continuously tunable braiding Berry phase from $\pi/8$ to $3\pi/8$. This is quite different from the usual non-Abelian statistics with fixed braiding Berry phase $\pi/4$ and becomes an example of “irrational topological phenomenon”.

The one-dimensional NH Kitaev model is taken as an example to numerically verify the irrational non-Abelian statistics for two NH Majorana zero modes. The numerical results are exactly consistent with the theoretical prediction. With the help of braiding these two zero modes, the $\pi/8$ gate can be reached and thus universal topological quantum computation becomes possible.

Majorana zero modes (MZMs) have recently attracted much attention due to their potential application in topological quantum computations (TQCs) [1,12]. MZMs have been predicted to be induced by vortices in a two-dimensional (2D) spinless $p_x + ip_y$-wave superconductor (SC) [2], or localized at the ends of a one-dimensional (1D) $p$-wave SC [3]. For these two-dimensional superconductors (TSCs) with MZMs, topologically protected degenerate ground states (referred to as Majorana qubits) exist. Based on braiding these MZMs that obey non-Abelian statistics [4,11], a TQC is proposed [6,7]. Unfortunately, because the $\pi/8$ gate cannot be reached by braiding processes, a universal TQC based on MZMs has become unrealistic and still remains a challenge.

Alternatively, in recent years non-Hermitian (NH) physics has become an active research area that has attracted considerable research interest [19,73]. Researchers have investigated some NH effects on topological SCs and MZMs. Previous work has focused mainly on two types of NH terms on TSCs: gain/loss in SCs induced by imaginary chemical potentials [25,30] and imbalanced pairing [31], where the MZMs show similar properties as to those in a Hermitian system. However, many open questions still exist regarding the MZMs in NH TSCs:

1) Can we generalize the MZMs to NH systems that show universal but quite different properties from their Hermitian counterparts?

2) Can the NH effect change the non-Abelian statistics of MZMs?

3) Do NH MZMs provide an alternative approach to universal TQC beyond their Hermitian counterparts?

In this letter, we aim to answer the above questions and develop a theory for the NH generalized MZMs and the corresponding NH generalization for non-Abelian statistics (referred to as irrational non-Abelian statistics).

**Non-Hermitian Majorana zero modes.** In certain TSCs, MZMs always emerge around defects (for example, the quantized vortex in 2D TSCs or the end in 1D TSCs). In general, a single MZM (sometimes referred to as Majorana fermion for Hermitian TSCs) can be described by a real fermionic field $\gamma$, i.e., $\gamma^\dagger = \gamma$. We can label two MZMs by complex fermions as $\gamma_1 = c_1 + c_1^\dagger$, $\gamma_2 = -i(c_2 - c_2^\dagger)$ and use them to represent the basis states of a non-local Majorana qubit:

$$|0\rangle_M \equiv \frac{1}{\sqrt{2}}(|\overline{1}\rangle + |\overline{1}\rangle), \quad |1\rangle_M \equiv \frac{1}{\sqrt{2}}(|\overline{1}\rangle - |\overline{1}\rangle),$$

where $|\overline{m}\rangle = |\overline{m}\rangle_1 \otimes |\overline{m}\rangle_2$ with $m, n = 0, 1, (|\overline{m}\rangle_1, |\overline{1}\rangle_1) = (|\overline{0}\rangle_1, c_1^\dagger |\overline{1}\rangle_1)$ are the eigenstates for complex fermions $c_1^\dagger$, $i = 1, 2$. In addition, $|0\rangle_M$ is a fermion-empty state, $|1\rangle_M = C_M^1 |0\rangle_M$ is a fermion-occupied state where $C_M^1$ is the composite fermionic operator (see Appendix-A), $C_M^1 = (\gamma_1 - i\gamma_2)/2 = (c_1 + c_1^\dagger - c_2 - c_2^\dagger)/2$. The fermion parities for the two states of the Majorana qubit are different: the fermion parity of $|0\rangle_M$ is even and the fermion parity of $|1\rangle_M$ is odd. By introducing the fermionic parity operator $\hat{P}_F = (-1)^\Sigma c_i^\dagger c_i$, we have $\hat{P}_F |0\rangle_M = |0\rangle_M$ and $\hat{P}_F |1\rangle_M = - |1\rangle_M$.

However, can we generalize the MZMs and the corresponding Majorana qubit to NH systems? The answer is yes! For a Hermitian system, a global phase transformation $S$ for fermion operators is defined as

$$(c, c^\dagger) \mapsto S(c, c^\dagger)S^{-1} = (e^{-i\phi}c, e^{i\phi}c^\dagger)$$

*Corresponding author; Electronic address: spkou@bnu.edu.cn*
same as their Hermitian counterpart: \( \gamma \), \( \gamma = \gamma^\dagger \). The PH-symmetry for the "empty" state \( |0\rangle \) and the fermion parity for the NH qubits are the same as their Hermitian counterpart \( \gamma \), \( \gamma = \gamma^\dagger \) and the PH-symmetry for the "occupied" state \( |1\rangle \).

**FIG. 1**: An illustration to show the comparison between the typical Majorana qubits and NH Majorana qubits. We list the MZMs, orthogonality of the qubit states, braiding operator and Berry phase in the row 2-5, respectively.

with a real \( \phi \). Here, we generalize the phase \( \phi \) from a real number to an imaginary number \( \phi = -i\beta \), and the imaginary phase transformation becomes a NH particle-hole (PH) similarity transformation, i.e., \((c, c^\dagger) \to (e^{-\beta}c, e^\beta c^\dagger)\) with real \( \beta \). Therefore, with the help of the NH PH similarity transformation \( S \), we define the NH MZMs as \( \gamma_i^\beta = S\gamma_i S^{-1}, i = 1, 2 \) and we have \( \gamma_1^\beta = e^{-\beta}\gamma_1 + e^{\beta}\gamma_1^\dagger \), \( \gamma_2^\beta = -i(e^{-\beta}\gamma_2 - e^{\beta}\gamma_2^\dagger) \), where the NH strength \( \beta \) is a real number \( (\beta = \beta^* \neq 0) \). In particular, the NH MZMs satisfy \( (\gamma_i^\beta)^\dagger = \gamma_i^\dagger, (\gamma_i^\beta)^\dagger \neq \gamma_i^\dagger \). Therefore, the properties of NH MZMs are characterized by \( \beta \). The NH PH similarity transformation \( S \) breaks intrinsic PH symmetry in a TSC, i.e., the symmetry between \( c \) and \( c^\dagger \) is broken. The corresponding TSCs with NH MZMs are no longer Hermitian and the corresponding operators \( \gamma_1^\beta, \gamma_2^\beta \) are no longer real.

We consider a TSC with two NH MZMs \( \gamma_1^\beta, \gamma_2^\beta \) and the corresponding fermionic operators are defined as \( \hat{C}_M^\beta = (\gamma_1^\beta - i\gamma_2^\beta)/2 \) and \( \hat{C}_M^\gamma = (\gamma_1^\beta + i\gamma_2^\beta)/2 \), with \( \{\hat{C}_M^\beta, \hat{C}_M^\gamma\} = 1 \), \( (\hat{C}_M^\beta)^2 = (\hat{C}_M^\gamma)^2 = 0 \). We therefore introduce a NH Majorana qubit \((|0\rangle_\beta_M, |1\rangle_\beta_M) = (|0\rangle_\beta_M, \hat{C}_M^\beta |0\rangle_\beta_M)\) based on the NH MZMs, which can be derived from a Hermitian case under a global NH PH similarity transformation \( S \):

\[
|0\rangle_\beta_M = S |0\rangle_M, \quad |1\rangle_\beta_M = S |1\rangle_M.
\]

From the definition of the NH MZMs, the energy difference between \( |0\rangle_\beta_M^\beta \) and \( |1\rangle_\beta_M^\beta \) disappears. Thus, there is almost no coupling between \( \gamma_1^\beta \) and \( \gamma_2^\beta \).

For the NH Majorana qubits, according to \( S \hat{P}_F S^{-1} = \hat{P}_F \), the fermion parity is also a good quantum number and the fermion parity for the NH qubits are the same as their Hermitian counterpart \( \hat{P}_F |0\rangle_\beta_M = |0\rangle_\beta_M, \quad \hat{P}_F |1\rangle_\beta_M = -|1\rangle_\beta_M \). In addition, we emphasize that the PH-symmetry for the "empty" state \( |0\rangle_\beta_M \) is broken, but the PH-symmetry for the "occupied" state \( |1\rangle_\beta_M \) is unbro-
ken. Under PH transformation, we have

\[
|0\rangle_\beta_M \to |0\rangle_\beta_M^\beta \neq |0\rangle_\beta_M, |1\rangle_\beta_M \to |1\rangle_\beta_M^\beta = |1\rangle_\beta_M.
\]

The PH-symmetry breaking of the NH Majorana qubit plays an important role in changing the typical non-Abelian statistics to anomalous non-Abelian statistics.

**Irrational non-Abelian statistics.** First, we summarize the quantum properties of MZMs in Hermitian cases. MZMs obey SU(2) level-2 non-Abelian statistics. On the one hand, the fusion rule of MZMs is given by \( \sigma \times \sigma = 1 + \psi, \psi \times \psi = 1, \psi \times \sigma = \sigma \), where \( 1 \) is a vacuum sector, \( \psi \) is the (complex) fermion sector, and \( \sigma \) is the MZM sector. Two \( \sigma \)-particles (MZMs) may either annihilate to the vacuum or fuse into a \( \psi \)-particle; On the other hand, if we exchange two MZMs \( (\gamma_1, \gamma_2) \), the result of the braiding is \( \gamma_1 \to -\gamma_2, \gamma_2 \to \gamma_1 \) and the exchange operator (the braiding operator) \( R_M \) can be described by \( R_M = e^{i\Delta \Phi_{\gamma}} \). We may call \( R_M \) to be Ivanov’s braiding operator [1]. During the braiding process, the Berry phases for \( |0\rangle_M \) and \( |1\rangle_M \) are 0 and \( \pi/2 \), respectively. So for the Majorana qubit \((|0\rangle_M, |1\rangle_M) \), the braiding operator is obtained as \( R_M = \exp[-i\Delta \Phi_{\gamma}] \) which is the Ivanov’s braiding operator. Here, \( \tau_\gamma \) denotes a Pauli matrix on the Majorana qubit \((|0\rangle_M, |1\rangle_M) \). According to the topological feature of SU(2) level-2 non-Abelian statistics, the Berry phase during braiding processes \( \Delta \Phi \) is fixed to be \( \pi/4 \) that cannot be changed.

However, for the NH MZMs \( \gamma_1^\beta, \gamma_2^\beta \), their non-Abelian statistics are different from the Hermitian case and become a new type of non-Abelian statistics, namely, irrational non-Abelian statistics.

On the one hand, there exists a typical fusion rule for the NH MZMs: \( \sigma^\beta \times \sigma^\delta = 1^\beta + \psi^\beta, \psi^\beta \times \psi^\beta = 1^\beta, \psi^\beta \times \sigma^\delta = \sigma^\beta, \sigma^\delta \times \sigma^\beta = \sigma^\beta \), where \( 1^\beta \) is the NH vacuum sector, \( \psi^\beta \) is the NH (complex) fermion sector, and \( \sigma^\beta \) is the NH MZM sector. Two NH \( \sigma^\beta \)-particles may either annihilate to the NH vacuum \( 1^\beta \) or fuse into a NH \( \psi^\beta \)-particle.

On the other hand, anomalous braiding processes exist for the NH MZMs \( \gamma_1^\beta, \gamma_2^\beta \). According to the case with two NH MZMs \( \gamma_1^\beta \) and \( \gamma_2^\beta \), two degenerate quantum states always exist. Consequently, the braiding process for the NH MZMs is also defined by \( \gamma_1^\beta \to -\gamma_2^\beta, \gamma_2^\beta \to \gamma_1^\beta \). Then, a question is: can the braiding operator for NH MZMs \( R_M^\beta \) be derived by performing a similarity transformation on the braiding operator for the Hermitian MZMs \( R_M \)? The answer is no, i.e., \( R_M^\beta \neq S R_M^\beta S^{-1} = e^{-i\pi \tau_\gamma} \).

To show why, let us derive the braiding matrix \( R_M^\beta \) on the Majorana qubit during the braiding processes. The Berry phases for the quantum states of the Majorana qubits \(|0\rangle_M^\beta \) and \(|1\rangle_M^\beta \) from the braiding operation are calculated by the Wilson loop method

\[
|A_n| e^{i\phi_n} = \prod_{n=0}^{N_s} |A_n^\beta M (i(\theta, \varphi_n) i(\theta, \varphi_{n+1})^\beta M |,
\]

where \( i = 0, 1 \), the amplitude \( |A_n| = 1 \) when the evolution step number \( N_s \) is sufficiently large, and \( i(\theta, \varphi_n)^\beta M \) is the \( i\)-th state of the Majorana qubit at the \( n\)-step during the
braiding process which is labeled by the two parameters \( \theta = 2 \arctan(e^{-2\beta}) \) and \( \varphi_n \). In particular, we have

\[
|0(\theta, \varphi_n)\rangle_M^\beta = \frac{1}{\sqrt{N_0}} [e^{i\varphi_n} |\Pi\rangle + e^{-2\beta} |0\rangle],
\]

\[
|1(\theta, \varphi_n)\rangle_M^\beta = \frac{1}{\sqrt{2}} [|\Pi\rangle + e^{-i\varphi_n} |0\rangle].
\]

where \( N_0 = \sqrt{1 + e^{-4\beta}} \) is the self-normalization coefficient of the state \( |0(\theta, \varphi_n)\rangle_M^\beta \).

First, we derive the effects of the braiding operator \( \mathcal{R}_M^\beta \) on the quantum state \( |0\rangle_M^\beta \). We map the states \((|0\rangle_M^\beta, |\Pi\rangle)\) onto a pseudo-spin \((|\uparrow\rangle_0, |\downarrow\rangle_0)\) and use the Bloch sphere to label the quantum states. In the Hermitian case \( \beta = 0 \) (Fig.1), the initial state is \( |0\rangle_M^\beta = |\uparrow\rangle_0 + |\downarrow\rangle_0 \), which is denoted by a spot at the equator on the Bloch sphere \([\theta, \varphi] = [\pi/2, 0]\). During braiding process, \( |0\rangle_M^\beta \) adiabatically deforms into \( (e^{i\varphi_n} |\uparrow\rangle_0 + |\downarrow\rangle_0) \) and finally changes into \( (e^{i\varphi_n/2} |\uparrow\rangle_0 + |\downarrow\rangle_0) \) denoted by another spot \([\theta, \varphi] = [\pi/2, \pi/2]\). So the geometry phase (Berry phase) is \( \Delta \varphi = (1 - \cos \theta)/2 \), where \( \theta = \pi/2 \) and \( \Delta \varphi = \pi/2 \); While, in the NH case \( \beta \neq 0 \) (Fig.1), the initial state becomes \((|\uparrow\rangle_0 + e^{-2\beta} |\downarrow\rangle_0)\), which is denoted by a spot away from the equator of the Bloch sphere, \([\theta, \varphi] = [2 \arctan(e^{-2\beta}), 0]\). During the braiding processes, it adiabatically deforms into \( (e^{i\varphi_n} |\uparrow\rangle_0 + e^{-2\beta} |\downarrow\rangle_0) \), and finally changes into \( (e^{i\varphi_n/2} |\uparrow\rangle_0 + e^{-2\beta} |\downarrow\rangle_0) \) denoted by \([\theta, \varphi] = [2 \arctan(e^{-2\beta}), \pi/2]\). After the braiding processes we obtain the geometry phase as \( \Delta \varphi = (1 - \cos \theta)/2 \) where \( \tan(\theta/2) = e^{-2\beta} \) and \( \Delta \varphi = \pi/2 \).

Second, using a similar operation on \( |1\rangle_M^\beta \), we map the qubit \((|\Pi\rangle, |\uparrow\rangle)\) onto a pseudo-spin \((|\uparrow\rangle_1, |\downarrow\rangle_1)\) as shown in Fig.1(b). The braiding operators for the Majorana qubit \((|0\rangle_M^\beta, |1\rangle_M^\beta)\) are obtained as (see the Appendix-D3)

\[
\mathcal{R}_M^\beta (|0\rangle_M^\beta, |1\rangle_M^\beta) = \left( \begin{array}{cc} e^{i\Delta \phi_0^\beta} & 0 \\ 0 & e^{-i\Delta \phi_1^\beta} \end{array} \right),
\]

where the Berry phases are obtained as \( \Delta \phi_0^\beta = \frac{\pi}{4(1 - e^{-4\beta})} \) and \( \Delta \phi_1^\beta = \pi/4 \). It is obvious that the Berry phase for \( |0\rangle_M^\beta \) is different from \( |0\rangle_M^\beta \). The braiding operator is obtained as \( \mathcal{R}_M^\beta = e^{-i\Delta \Phi} \), which is the NH generalization of the Ivanov’s braiding operator. Here, \( \tau_z \) denotes a Pauli matrix on the Majorana qubit \((|0\rangle_M^\beta, |1\rangle_M^\beta)\). \( \Delta \Phi = \frac{1}{2} (\Delta \phi_0^\beta - \Delta \phi_1^\beta) = \frac{\pi}{8} + \frac{\pi}{8} \left( \frac{1}{4(1 - e^{-4\beta})} \right) \) denotes a Berry phase during braiding processes that can continuously tuned from \( \pi/8 \) to \( 3\pi/8 \). Thus, \( \Delta \Phi \) can be an arbitrary value in the region of \( (\pi/8, 3\pi/8) \), including rational number or irrational number. As a result, we call it irrational non-Abelian statistics. By contrast, the Berry phase from braiding processes for usual non-Abelian statistics is fixed to \( \Delta \Phi = \pi/4 \). Besides, when we fix \( \beta \), for two non-Hermitian Majorana zero modes far away, the braiding rule and the corresponding Berry phase \( \Delta \Phi \) will never change, no matter how you change the braiding path! In this sense, this is a remarkable example of “irrational topological phenomenon” and can be considered as non-Abelian generalization of irrational Abelian statistics for U(1) Abelian anyons according to Wilczek flux-binding picture!

**Example for numerical simulations on verifying the irrational non-Abelian statistics.** A 1D NH Kitaev model \( \mathcal{H}_{NH} \) with imbalanced p-wave pairing is taken as an example to illustrate the anomalous non-Abelian statistics of NH MZMs, and the numerical simulations are performed during the braiding processes. The Hamiltonian is written as

\[
\mathcal{H}_{NHK}(\beta) = -\sum_{j=1}^{N} \left[ t(c_{j}^\dagger c_{j+1} + c_{j+1}^\dagger c_j) + \Delta^z c_{j}^\dagger c_{j+1}^\dagger + \Delta^\pi c_j c_{j+1} + \mu (1 - 2n_j) \right],
\]

where \( c_j (c_j^\dagger) \) annihilates (creates) a fermion on site \( j \), \( \Delta^z, \mu \) and \( N \) denote the hopping amplitude, the strength of p-wave pairing, the chemical potential and the lattice number, respectively. We set \( \Delta^\pi = \Delta_0 e^{\pm \beta^2} \), where \( \beta \in R \) represents the NH strength and \( \Delta_0 > 0 \). When \( \beta \neq 0 \), we have \( \mathcal{H}_{NHK} \neq \mathcal{H}_{NH} \), which can be achieved by the NH similarity transformation from it’s Hermitian counterpart. In this letter, we focus on the case of \( \Delta_0 = 0 \).

The 1D NH SC may have nontrivial topological properties (see Appendix-B for details). For the translation variables ansatz, we transform the fermion Hamiltonian into momentum space, \( \mathcal{H}_{NHK}(k) = \sum_{k} \psi_k^\dagger h(k, \beta) \psi_k \) with

\[
h(k, \beta) = (t \cos k + \mu) \cdot \sigma^z + \Delta_0 \sin k \cdot \sigma^{y, \beta} \quad (9)
\]

by introducing \( \psi_k = (c_k, c_k^\dagger)^T \), where \( \sigma^y, \sigma^{\pm} = S \sigma^y S^{-1} = \cosh(\beta) \sigma^y - i \sinh(\beta) \sigma^x \) is a \( 2 \times 2 \) matrix. With the help of the biorthogonal set, we define right/left eigenstates for the NH systems as \( \mathcal{H}_{NHK} \Psi^R_k = \mu_k \Psi^R_k \), and \( \mathcal{H}_{NHK} \Psi^L_k = E^*_{n_k} \Psi^L_k \), where \( E_{n_k}, E^*_{n_k} \) are the corresponding eigenvalues (with \( n = 0, 1 \) representing the two lowest energy states). To describe the topological structure of \( \mathcal{H}_{NHK} \), we define biorthogonal \( Z_2 \) topological invariant,

\[
\omega = \text{sgn}(\eta_{k=0/\pi} \cdot \eta_{k=\pi/2}) \quad (10)
\]

where \( \eta_{k=0/\pi} = \langle \Psi^L_k | c_{k=0/\pi} c_{k=0/\pi} | \Psi^R_k \rangle \) and \( \eta_{k=0/\pi} (\beta) = \beta \) if \( E_{n_k} = 0 \). Therefore, we have \( \eta_{k=\pi} = \text{sgn}(t + \mu) \), \( \eta_{k=\pi} = \text{sgn}(-t + \mu) \). For the case of \( \omega = 1 \) (\( |t| < |\mu| \)), the SC is trivial; But for \( \omega = -1 \) (\( |t| > |\mu| \)), the SC becomes topological one.

In the topological phase with \( \omega = -1 \), there exist two edge states with (nearly) zero energy, i.e., the NH MZMs at the left end \( \gamma^R_{L} \) and at the right end \( \gamma^R_{L} \). After defining the fermionic operators \( \mathcal{C}_M = (\gamma^R_{L} + i \gamma^R_{L})/2 \), \( \mathcal{C}_M = (\gamma^R_{L} - i \gamma^R_{L})/2 \), we obtain the two ground states with the open boundary condition as \( |\Psi_0(\beta)\rangle = \mathcal{C}_M |F\rangle \), \( |\Psi_1(\beta)\rangle = \mathcal{C}_M^\dagger |\Psi_0(\beta)\rangle \), where \( |F\rangle \) is the NH ground state with occupied single particle states for \( E < 0 \) and empty single particle states for \( E > 0 \). We find
Hamiltonian becomes
\[
\hat{H}_{\text{NHK}}(\beta) = -\frac{1}{4} \sum_j (J \sigma_j^x \sigma_{j+1}^x - 4h \sigma_j^z) \quad (11)
\]
where \(\sigma_j^{x,y,z} = S(\beta) \sigma_j^x S^{-1}(\beta) = \cosh(\beta) \sigma_j^x + i \sinh(\beta) \sigma_j^y\) and \(S(\beta) = \exp(i \sum_j \sigma_j^z)\), \(J = t = \Delta_0, h = \mu\). The non-Hermitian model in Eq. (11) can be simulated using three-level atoms in a variety of setups (see Appendix-E), including trapped ions, cavity QED, and atoms in optical lattices. The dynamics by \(H(\beta)\) can be decomposed as
\[
e^{-iH(\beta)t} = e^{-i\mu \sigma^z t} S(\beta) (\prod_i e^{i \frac{2}{\beta} \sigma_j^z \tau_{j+1}^z \sigma_j^z}) S^{-1}(\beta), \quad (12)
\]
where the nonunitary dynamics \(S(\beta)\) and \(S^{-1}(\beta)\) are from measuring whether a spontaneous decay has occurred \(\text{78-80}\). This process can be measured with a high degree of accuracy \(\text{81-83}\).

Meanwhile, the braiding process for the Majorana qubit \(|0_\beta^M\rangle, |1_\beta^M\rangle\) is mapped onto that for the two degenerate ground states in the spin representation. The irrational Ivanov’s braiding operator \(R_{\beta}^M\) for two MZMs is mapped onto the corresponding operator \(R^\tau(\varphi)\), which rotate the spin \(\varphi\) angle around the -axis in spin representation, i.e., \(R_{\beta}^M \rightarrow R^\tau(\varphi)\). This is the same as the case in the Hermitian system. The Berry phases for the quantum states \(|0_\beta^M\rangle\) and \(|1_\beta^M\rangle\) in the Majorana braiding process are calculated by the Wilson loop method like Eq. (4). The numerical results (dots) are exactly consistent with the theoretical prediction (lines) as shown in Fig. 2(c). In supplementary materials, we showed detailed discussion on numerical simulations and theoretical derivation of the Ivanov’s braiding processes (see Appendix-D2, D3).

**Non-Hermitian assisted topological quantum computation via Non-Hermitian MZMs.** Due to non-locality and orthogonality, the NH MZMs may be utilized as a decoherence-free qubit, which play an important role in the realization of fault-tolerant universal TQC. We propose an alternative approach to universal TQC via NH MZMs – Non-Hermitian assisted TQC. If one can realize \(\hat{H}_{\text{NHK}}^\beta\) with the freely adjustable NH strength \(\beta\), we can adiabatically tune \(\beta\) to construct a universal TQC. For the Hadamard gate, phase gate, and controlled NOT gate, we set the NH strength \(\beta\) to zero. For the \(\pi/8\) gate, we set the NH strength \(\beta\) to certain value and braid NH MZM for \(N\) times. For example, \(N = 4\) for \(\beta = -(\ln(0.6))/4 \approx 0.128\). For this case, during the braiding processes, the \(\pi/8\) gate can be reached \(T = |R_{\beta}^3|^N\). In the end, to perform measurement, the NH strength \(\beta\) returns to zero again. What should be mentioned is that the \(T\) gate from braiding process is based on “irrational topological phenomenon”, which has huge advantages over other non-topological methods, such as the method of “magic state distillation” \(\text{72, 74}\).

In Fig. 2(d), an illustration of two phase gates \(S\) and \(\pi/8\) gate for NH assisted TQC is shown. Here, we take a braiding process with three steps as an example: a phase transformation \(\text{see Appendix-C}\). As a result, the

that in the thermodynamic limit \((N \rightarrow \infty)\) the energy splitting of the two MZMs is zero and \(|\Psi_0(\beta)\rangle\) and \(|\Psi_1(\beta)\rangle\) are orthogonal by calculating the similarity between them, which is defined as \(\chi(\beta) = \langle \Psi_0(\beta) | \Psi_1(\beta) \rangle\). Here, \(|\Psi_0(\beta)\rangle\) satisfies the self-normalization condition \(\langle \Psi_0(\beta) | \Psi_0(\beta) \rangle = 1\). For example, when \(t = \Delta_0\) and \(\mu = 0\) we have \(\chi(\beta) = (\tanh \beta)^N\). It is obvious that \(\chi(\beta) \rightarrow 0\) with \(N \rightarrow \infty\), so the two degenerate ground states are orthogonal. The proof of the orthogonal properties is shown in Fig. 2(a) where the numeric results (the dots) are consistent with the analytic results (the lines).

According to the definition, the fermion parity of \(|\Psi_0(\beta)\rangle\) is even and the fermion parity of \(|\Psi_1(\beta)\rangle\) is odd. Therefore, due to the orthogonality and the parities of the two-fold degenerate ground states, we can use them to construct the two basis states of the NH Majorana qubit (see the Appendix-D1). We introduce the NH Majorana qubit in this system as: \(|0_\beta^M\rangle \equiv |\Psi_0(\beta)\rangle = \tilde{C}_M |F\rangle\), \(|1_\beta^M\rangle \equiv |\Psi_1(\beta)\rangle = \tilde{C}_M |0\rangle_M\).

The non-Abelian statistics of two NH MZMs can be verified in the T-junction Majorana chain systems \(\text{11, 73}\), which contain 4 lattice sites, as shown in Fig. 2(b). Here, the braiding processes of the two NH MZMs are denoted by blue dotted arrows. We perform the numerical simulations to verify the non-Hermitian Ivanov’s braiding operator \(R_{\beta}^M\) for two NH MZMs by mapping the original fermionic model \(\hat{H}_{\text{NHK}}(\beta)\) onto a NH transverse Ising model via the NH Jordan-Wigner transformation \(\text{84}\) (see Appendix-C). As a result, the

FIG. 2: (a) The numerical results (dots) and the analytical results (lines) for the similarity between two degenerate ground states in NH Kitaev model with \(t = \Delta_0, \mu = 0\), and \(\beta = 0.2, 0.8\), and 1.2. These results indicate the orthogonality of two degenerate ground states in thermodynamic limit \((N \rightarrow \infty)\); (b) Schematic diagram for the T-type braiding process to exchange the two NH MZMs. We take a system with 8 Majorana fermions as an example; (c) The Berry phase for the quantum states \(|0_\beta^M\rangle\) and \(|1_\beta^M\rangle\) during the braiding processes. (d) An illustration of NH assisted TQC. In step-2, a \(\pi/8\) gate is realized by tuning the NH strength \(\beta\), dotted lines indicate that multiple braiding operations can be performed.
gate $S$ by exchanging two Hermitian MZMs with $\beta = 0$, a $\pi/8$ gate by exchanging two NH MZMs with $\beta \neq 0$, and a phase gate $S$ by exchanging two MZMs with $\beta = 0$.

Conclusion and discussion: In this letter, we developed a theory for NH generalization of MZMs, i.e., $\gamma^\beta = S\gamma S^{-1}$ where $S$ is the NH PH similarity transformation and $\beta$ is the NH strength. The key point of NH generalization of MZMs is the NH PH similarity transformation. Due to the particle-hole-symmetry breaking, the Berry phase from braiding processes become an arbitrary number in a region, i.e., $\Delta \Phi \in (\pi/8, 3\pi/8)$. The irrational non-Abelian statistics for the NH MZMs indicates that in NH topological systems the theory for usual unitary modular tensor category would be generalized to a theory for certain non-unitary modular tensor category, and the theory for usual topological field theories would be generalized to a theory for certain non-Hermitian topological field theories. In the future, we will study these issues.

In addition, we plan to apply the theory to other TSCs, such as the 2D NH $p_x + ip_y$ TSC and higher-order NH TSCs, and then study the possible physical realization of the NH MZMs in these NH topological systems.

Acknowledgments

This work is supported by NSFC Grant No. 1217040237, 11974053, 61835013, National Key R&D Program of China under grants No. 2016YFA0301500, Strategic Priority Research Program of the Chinese Academy of Sciences under grants Nos. XDB01020300, XDB21030300.

[1] D. A. Ivanov, Phys. Rev. Lett. 86, 268 (2001).
[2] N. Read and D. Green, Phys. Rev. B 61, 10267 (2000).
[3] A. Y. Kitaev, Phys. Usp. 44, 131 (2001).
[4] A. Y. Kitaev, Russ. Math.Surv. 52, 1191 (1997).
[5] S. D. Sarma, C. Nayak, and S. Tewari, Phys. Rev. B 73, 220502 (2006).
[6] B. Lian, X. Q. Sun, A. Vaezi, X. L. Qi, and S. C. Zhang, Proc. Natl. Acad. Sci. U.S.A. 115, 10938 (2018).
[7] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. D. Sarma, Rev. Mod. Phys. 80, 1083 (2008).
[8] L. Fu and C. Kane, Phys. Rev. Lett. 100, 096407 (2008).
[9] A. Stern, Nature, 464, 187 (2010).
[10] J. D. L. Sau, R. M. Tewar, and S. D. Sarma, Phys. Rev. Lett. 104, 040502 (2010).
[11] J. Alicea, Y. Oreg, G. Refael, F. von Oppen, and M. P. A. Fisher, Nat. Phys. 7, 412 (2011).
[12] V. Mourik, K. Zuo, S. M. Frolov, S. R. Plissard, E. A. M. Bakkers, L. P. Kouwenhoven, Science 336, 1003 (2012).
[13] M. T. Deng, C. L. Yu, G. Y. Huang, M. Larsson, P. Caroff, H. Q. Xu, Nano. Lett. 12, 6414 (2012).
[14] L. P. Rokhinson, X. Liu, and J. K. Furdyna, Nat. Phys. 8, 795 (2012).
[15] J. Alicea, Rep. Prog. Phys. 75, 076501 (2012).
[16] H. T. Mebrahtu, I. V. Borzenets, H. Zheng, Y. V. Bonze, A. I. Smirnov, S. Florens, H. U. Baranger G. Finkelstein, Nat. Phys. 9, 732 (2013).
[17] S. Nadji-Perge, et al. Science 346, 602 (2014).
[18] E. J. H. Lee, X. Jiang, M. Houzet, R. Aguado, C. M. Lieber, S. D. Franceschi, Nat. Nano 9, 79 (2014).
[19] M. S. Rudner and L. S. Levitov, Phys. Rev. Lett. 102, 065703 (2009).
[20] K. Esaki, M. Sato, K. Hasebe, and M. Kohmoto, Phys. Rev. B 84, 205128 (2011).
[21] Y. C. Hu and T. L. Hughes, Phys. Rev. B 84, 153101 (2011).
[22] S. D. Liang and G. Y. Huang, Phys. Rev. A 87, 012118 (2013).
[23] B. Zhu, R. Lü, and S. Chen, Phys. Rev. A 89, 062102 (2014).
[24] J. M. Zeuner, M. C. Rechtsman, Y. Plotnik, Y. Lumer, S. Nolte, M. S. Rudner, M. Segev, and A. Szameit, Phys. Rev. Lett. 115, 040402 (2015).
[25] X. Wang, T. Liu, Y. Xiong, et al. Phys. Rev. A 92, 012116 (2015).
[26] C. Yuce, Phys. Rev. A 93, 062130 (2016).
[27] Q. B. Zeng, B. Zhu, S. Chen, L. You, R. Lü, Phys. Rev. A 94, 022119 (2016).
[28] H. Menke, M. M. Hirschmann, Phys. Rev. B 95, 174506 (2017).
[29] K. Kawabata, Y. Ashida, H. Katsura and M. Ueda, Phys. Rev. B 98, 085116 (2018).
[30] S. Lieu, Phys. Rev. B 100, 085110 (2019).
[31] C. Li, X. Z. Zhang, G. Zhang, et al. Phys. Rev. B 97, 115436 (2018).
[32] T. E. Lee, Phys. Rev. Lett. 116, 133903 (2016).
[33] P. S. Jose, J. Cayao, E. Prada, and R. Aguado, Sci. Rep. 6, 21427 (2016).
[34] S. Weimann, M. Kremer, Y. Plotnik, Y. Lumer, S. Nolte, K. G. Makris, M. Segev, M. C. Rechtsman, and A. Szameit, Nat. Mater. 16, 433 (2017).
[35] L. Xiao, X. Zhan, Z. H. Bian, K. K. Wang, X. Zhang, X. P. Wang, J. Li, K. Mochizuki, D. Kim, N. Kawakami, W. Yi, H. Obuse, B. C. Sanders, and P. Xue, Nat. Phys. 13, 1117 (2017).
[36] D. Leykam, K. Y. Bliokh, C. Huang, Y. D. Chong, and F. Nori, Phys. Rev. Lett. 118, 040401 (2017).
[37] H. Shen, B. Zhen, and L. Fu, Phys. Rev. Lett. 120, 146402 (2018).
[38] S. Lieu, Phys. Rev. B 97, 045106 (2018).
[39] Y. Xiong, J. Phys. Commun. 2, 035043 (2018).
[40] Z. Gong, Y. Ashida, K. Kawabata, K. Takasan, S. Higashikawa, and M. Ueda, Phys. Rev. X 8, 031079 (2018).
[41] S. Yao, and Z. Wang, Phys. Rev. Lett. 121, 086803 (2018).
[42] S. Yao, F. Song, and Z. Wang, Phys. Rev. Lett. 121, 136802 (2018).
[43] F. K. Kunst, E. Edvardsson, J. C. Budich, and E. J. Bergholtz, Phys. Rev. Lett. 121, 026808 (2018).
[44] C. Yin, H. Jiang, L. Li, R. Lü, and S. Chen, Phys. Rev. A 97, 052115 (2018).
[45] D. S. Borgnia, A. J. Kruchkov, and R. J. Slager Phys.
[46] K. Kawabata, K. Shiozaki, and M. Ueda, Phys. Rev. B 98, 165148 (2018).
[47] V. M. M. Alvarez, J. E. B. Vargas, M. Berdakin, and L. E. F. F. Torres, Eur. Phys. J. Spec. Top. 227, 1295 (2018).
[48] H. Jiang, C. Yang, and S. Chen, Phys. Rev. A 98, 052116 (2018).
[49] M. A. Bandres, S. Wittek, G. Harari, M. Parto, J. Ren, M. Segev, D. N. Christodoulides, and M. Khajavikhan, Science 359, 4005 (2018).
[50] H. Zhou, C. Peng, Y. Yoon, C. W. Hsu, K. A. Nelson, L. Fu, J. D. Joannopoulos, M. Soljacic, and B. Zhen, Science 359, 1009 (2018).
[51] A. Cerjan, S. Huang, M. Wang, K. P. Chen, Y. Chong, and M. C. Rechtsman, Nat. Photon. 13, 623 (2019).
[52] K. Wang, X. Qiu, L. Xiao, X. Zhan, B. C. Sanders, W. Yi, and P. Xue, Nat. Commun. 10, 2293 (2019).
[53] L. Xiao, T. Deng, K. Wang, G. Zhu, Z. Wang, W. Yi, P. Xue, Nat. Phys. 16, 761 (2020).
[54] T. Helbig, T. Hofmann, S. Imhof, M. Abdelghany, T. Kiessling, L. W. Molenkamp, C. H. Lee, A. Szameit, M. Greiter, and R. Thomale, Nat. Phys. 16, 747 (2020).
[55] A. Ghtak and T. Das, J. Phys.: Condens. Matter 31, 263001 (2019).
[56] J. Avila, F. Peña-Randa, E. Prada, P. San-Jose, and R. Aguado, Commun. Phys. 2, 1 (2019).
[57] L. Jin and Z. Song, Phys. Rev. B 99, 081103 (2019); S. Lin, L. Jin, and Z. Song, Phys. Rev. B 99, 165148 (2019); K. L. Zhang, H. C. Wu, L. Jin, and Z. Song, Phys. Rev. B 100, 045141 (2019).
[58] C. H. Lee and R. Thomale, Phys. Rev. B 99, 201103(R) (2019).
[59] T. Liu, Y. R. Zhang, Q. Ai, Z. Gong, K. Kawabata, M. Ueda, and F. Nori, Phys. Rev. Lett. 122, 076801 (2019).
[60] K. Kawabata, K. Shiozaki, M. Ueda, and M. Sato, Phys. Rev. X 9, 041015 (2019).
[61] H. Zhou and J. Y. Lee, Phys. Rev. B 99, 235112 (2019).
[62] C. H. Liu, H. Jiang, S. Chen, Phys. Rev. B 99, 125103 (2019).
[63] E. Edvardsson, F. K. Kunst, and E. J. Bergholtz, Phys. Rev. B 99, 081302(R) (2019).
[64] L. Herviou, J. H. Bardarson, and N. Regnault, Phys. Rev. A 99, 052118 (2019).
[65] K. Yokomizo and S. Murakami, Phys. Rev. Lett. 123, 066404 (2019).
[66] F. K. Kunst and V. Dwivedi, Phys. Rev. B 99, 245116 (2019).
[67] R. Chen, C. Z. Chen, B. Zhou, and D. H. Xu, Phys. Rev. B 99, 155431 (2019).
[68] T. S. Deng and W. Yi, Phys. Rev. B 100, 035102 (2019).
[69] F. Song, S. Yao, and Z. Wang, Phys. Rev. Lett. 123, 170401 (2019).
[70] X. W. Luo and C. W. Zhang, Phys. Rev. Lett. 123, 073601 (2019).
[71] S. Longhi, Phys. Rev. Research 1, 023013 (2019).
[72] H. Jiang, R. Lü, S. Chen, Eur. Phys. J. B 93, 125 (2020).
[73] Y. Ashida, Z. Gong, and M. Ueda, Adv. Phys. 69, 3 (2020).
[74] X. M. Zhao, J. Yu, J. He, Q. B. Cheng, Y. Liang, S. P. Kou, Mod. Phys. Lett. B. 31, 1750123 (2017).
[75] Based on the Solovay-Kitaev algorithm reval in Ref. [4], to realize universal TQC we just need to construct an arbitrary phase gate $P = \text{diag}\{e^{i\Delta \phi}, e^{-i\Delta \phi}\}$ with phase changing $\Delta \phi \neq 0, \pm \pi/4, \pm \pi/2, \pi$ but not must be fixed to $\pi/8$. As a result, the gate with an arbitrary (irrational) phase changing can be realized by finite $\beta$ but not must be fixed to $\beta \to -\infty$.
[76] S. Bravyi and A. Kitaev, Phys. Rev. A 71, 022316 (2005).
[77] S. D. Sarma, M. Freedman and C. Nayak, npj Quantum Inf 1, 15001 (2015).
[78] T. E. Lee and C. K. Chan, Phys. Rev. X 4, 041001 (2014).
[79] T. E. Lee, F. Reiter, and N. Moiseyev, Phys. Rev. Lett. 113, 250401 (2014).
[80] H. Weimer, M. Muller, I. Lesanovsky, P. Zoller, and H. P. Buchler, Nat. Phys. 6, 382 (2010).
[81] N. Katz, M. Ansman, R. C. Bialczak, E. Lucero, R. McDermott, M. Neeley, M. Steffen, E. M. Weig, A. N. Cleland, J. M. Martinis, and A. N. Korotkov, Science 312, 1498 (2006).
[82] A. H. Myerson, D. J. Szwer, S. C. Webster, D. T. C. Allcock, M. J. Curtis, G. Imregh, J. A. Sherman, D. N. Stacey, A. M. Steane, and D. M. Lucas, Phys. Rev. Lett. 100, 200502 (2008).
[83] J. A. Sherman, M. J. Curtis, D. J. Szwer, D. T. C. Allcock, G. Imregh, D. M. Lucas, and A. M. Steane, Phys. Rev. Lett. 113, 180501 (2013).
[84] J. S. Xu, K. Sun, Y. J. Han, C. F. Li, J. K. Pachos, and G. C. Guo, Nat. Commun. 7, 13194 (2016).