A Calabi-Yau theorem for Vaisman manifolds

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Abstract
A compact complex Hermitian manifold \((M, I, \omega)\) is called Vaisman if \(d\omega = \omega \wedge \theta\) and the 1-form \(\theta\), called the Lee form, is parallel with respect to the Levi-Civita connection. The volume form of \(M\) is invariant with respect to the action of the vector field \(X\) dual to \(\theta\) (called the Lee field) and the vector field \(I(X)\), called the anti-Lee field. The cohomology class of \(\theta\), called the Lee class, plays the same role as the Kähler class in Kähler geometry. We prove that a Vaisman metric is uniquely determined by its volume form and the Lee class, and, conversely, for each Lee class \([\theta]\) and each Lee- and anti-Lee-invariant volume form \(V\), there exists a Vaisman structure with the volume form \(V\) and the Lee class \(c[\theta]\). This is an analogue of the Calabi-Yau theorem claiming that the Kähler form is uniquely determined by its volume and the cohomology class.

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1 Introduction

E. Calabi ([Ca1, Ca2]) has noticed that a Kähler metric is uniquely determined by its Kähler class and its volume form. The Calabi conjecture, proven by S.-T. Yau some 23 years later, claims that on any compact Kähler \(n\)-manifold \(M\) there exists a unique Kähler metric \(\omega\) with a given volume.

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form $V$ and a given Kähler class $[\omega]$ if $\int_M V = \int_M [\omega]^n$. This statement is equivalent to the existence and uniqueness of the solutions of the complex Monge-Ampère equation $(\omega + dd^c \varphi)^n = V$.

From this observation, Calabi obtained that (conditional on the Calabi conjecture) every compact Kähler manifold $M$ with $c_1(M) = 0$ admits a unique Ricci-flat metric in any given Kähler class; this Ricci-flat metric is clearly Einstein. Finding and classifying the Einstein metrics (and, more generally, the “extremal metrics”, defined by Calabi in [Ca3] as a generalization of Einstein and constant scalar curvature Kähler metrics) was one of the central subjects of the complex algebraic geometry since the 1980-ies.

The Calabi-Yau theorem has two facets, equally important. One is related to finding the Ricci-flat metric on manifolds with trivial canonical bundle. The other one is not related to the canonical bundle in any way: it is a result which claims that a Kähler metric is uniquely defined by its volume form and the Kähler class. The existence of the Ricci-flat metrics follows directly from this, more general, result.

We prove a version of the Calabi-Yau theorem for Vaisman manifolds (Theorem 4.1).

Let $M$ be a compact smooth manifold, and $F \subset TM$ a smooth foliation. It is called transversally Kähler if the normal bundle $TM/F$ is equipped with a Hermitian structure (that is, a complex structure and a Hermitian metric) which is locally obtained as the pullback of a Kähler structure on the leaf space. Sasakian manifolds are prime examples of transversally Kähler manifolds (the leaf space of the Reeb foliation on a Sasakian manifolds is Kähler).

A differential form on $M$ is called basic if it vanishes on $F$ and is locally obtained as the pullback of a form on the leaf space. The basic forms are preserved by de Rham differential, and the cohomology of the basic forms is called the basic cohomology.

A foliation is taut if the top basic cohomology is non-zero (this is not the usual definition, but a theorem of Habib and Richardson, see [HR]). For taut foliations, one has also Poincaré duality on the basic cohomology, and the identification between the basic cohomology and the basic harmonic forms if a transversal Riemannian structure is given.

When $F$ is taut and transversally Kähler, the basic cohomology should satisfy all the nice properties of the cohomology of the Kähler manifolds: the $dd^c$-lemma, the Hodge decomposition, the Hodge structure, Lefschetz $\mathfrak{sl}(2)$-action and so on. We proved it in the situation when $F$ is trivialized by a group action which preserves the transversally Kähler structure ([OV3, Theorem 5.4]); a similar result was proven much earlier by A. El Kacimi-
In this situation, A. El Kacimi-Alaoui proves the transversal Calabi-Yau theorem showing that the transversal Kähler structures are uniquely defined by the transversal volume form and the transversal Kähler class. We give the uniqueness part of the proof in Lemma 4.2, and for the existence refer to [El].

We apply these results to Vaisman geometry. Any Vaisman manifold is equipped by a transversally Kähler, holomorphic foliation $\Sigma$ (Theorem 2.7). The transversally Kähler structure of this foliation depends on the Vaisman metric, however, the transversally complex structure depends only on the complex structure of the Vaisman manifold ([Ts1] or [Ts2, Corollary 2.7]). We construct a correspondence between the set of Vaisman metrics on $M$ and the set of transversal Kähler structures.

The transversal Calabi-Yau theorem implies an important result about the Vaisman metrics (Theorem 4.1), showing that the Vaisman metric is defined uniquely, up to a constant multiplier, by the volume and the Lee class $[\theta] \in H^1(M)$. The space of possible Lee classes on $M$ (its “Lee cone”) is described in [Ts1] (see also [OV4]): it is identified with a certain open half-space in $H^1(M, \mathbb{R})$. Then, similarly to the Calabi theorem parametrizing the Kähler forms, the set of all Vaisman structures on $(M, I)$ is parametrized by the cohomological data together with the volumes.

The Vaisman Calabi-Yau theorem is deduced directly from the transversal Calabi-Yau theorem, because the transversally Kähler form of a Vaisman manifold, together with its Lee class, uniquely defines the Vaisman structure (Lemma 4.3). On the other hand, the transversal volume form uniquely defines, and is uniquely defined, by the volume form of a Vaisman manifold Lemma 4.4. This follows from a curious observation, made by K. Tsukada in [Ts2], who proved that the direction of the Lee field of a Vaisman manifold is determined by its complex structure.

## 2 Preliminaries

Let $(M, I, g, \omega)$ be a Hermitian manifold, $\dim_{\mathbb{C}} M \geq 2$. Here $\omega(\cdot, \cdot) = g(I\cdot, \cdot)$.

**Definition 2.1:** The Hermitian manifold $(M, I, g, \omega)$ is **locally conformally Kähler** (LCK) if there exists a closed 1-form $\theta$ such that $d\omega = \theta \wedge \omega$. The 1-form $\theta$ is called the **Lee form** and the $g$-dual vector field $\theta^g$ is called the **Lee field**.
Remark 2.2: One can easily see that this definition is equivalent with the existence of a Kähler cover \( \Gamma \rightarrow (\tilde{M}, \tilde{\omega}) \rightarrow M \) such that the deck group \( \Gamma \) acts by holomorphic homotheties (e.g. \([Va]\)). Therefore, one can define a homothety character \( \chi : \Gamma \rightarrow \mathbb{R}^>0 \) which associates to each deck transform \( \gamma \) the scale factor \( \frac{\gamma^*\tilde{\omega}}{\tilde{\omega}} \). When \( \tilde{M} \) is the universal cover, the homothety character is a representation of \( \pi_1(M) \) and uniquely defines the class \( [\theta] \in H^1(M, \mathbb{R}) \simeq H_1(M) \simeq \frac{\pi_1(M)}{[\pi_1(M), \pi_1(M)]} \) of the Lee form.

In this note, we shall be interested in a particular subclass of LCK manifolds, namely the Vaisman manifolds.

Definition 2.3: The LCK manifold \((M, I, g, \omega)\) is a Vaisman manifold if the Lee form is parallel with respect to the Levi-Civita connection of the metric \( g \).

Example 2.4: Almost all non-Kähler compact complex surfaces are LCK, see e.g. \([VVO]\). Diagonal Hopf surfaces and Kodaira surfaces are Vaisman, but Kato surfaces and Inoue surfaces are not Vaisman. In any dimension, all diagonal Hopf manifolds are Vaisman, see \([OV2]\).

Remark 2.5: \(([Va])\) It is easily seen that on a Vaisman manifold, the Lee and anti-Lee vector fields \( \theta^\sharp \) and \( I\theta^\sharp \) are Killing and holomorphic. Moreover, they commute: \([\theta^\sharp, I\theta^\sharp] = 0\). Therefore, they define a holomorphic 1-dimensional foliation \( \Sigma \). One can also show that \( \Sigma \) is transversally Kähler and its leaves are totally geodesic.

Remark 2.6: Up to a homothety, we can suppose that the length of the Lee form is 1. Then \(([Va, page 242])\), the following identity holds:

\[
\omega = d^c\theta + \theta \wedge \theta^c.
\] (2.1)

Theorem 2.7: Let \( M \) be a compact Vaisman manifold, and \( \Sigma \subset TM \) its canonical foliation. Then \( \Sigma = \ker \omega_0 \), where \( \omega_0 = d^c\theta \).

Proof: \([Va, Theorem 3.1]\) or \([Ve, Proposition 6.4]\). ■

Remark 2.8: \(([Va])\) The Kähler cover of a compact Sasakian manifold is a cone \( S \times \mathbb{R}^>0 \), where \( S \) is a Sasakian manifold, with cone metric \( \tilde{g} = \ldots \).
\[ t^2 g^S + dt \otimes dt, \] where \( g^S \) is the Sasaki metric on \( S \) and \( t \) is the coordinate on \( \mathbb{R}^{>0} \).

In LCK geometry, the analogue of the Kähler cone of a compact Kähler manifold is the “Lee cone”, i.e. the set of classes in \( H^1(M) \) which can be represented by Lee forms of an LCK structure on the fixed complex manifold \((M, I)\) admitting LCK structures. For Vaisman manifolds, the “Lee cone” is known:

**Theorem 2.9:** ([Ts1, OV4]) Let \( M \) be a compact Vaisman manifold. Then:

(i) \[ H^1(M) = H^1_0(M) \oplus \overline{H^1_0(M) + \langle \theta \rangle} \]

(ii) Consider a 1-form \( \mu \in H^1(M)^* \) vanishing on \( H^1_0(M) \oplus \overline{H^1_0(M)} \subset H^1(M) \) and satisfying \( \mu([\theta]) > 0 \), where \( H^1_0(M) \) is the space of closed holomorphic 1-forms. Then a class \( \alpha \in H^1(M, \mathbb{R}) \) is a Lee class for some LCK structure if and only if \( \mu(x) > 0 \).

We shall need the following description of basic cohomology on compact Vaisman manifolds:

**Theorem 2.10:** ([Va], [OV3]) Let \((M, I, g, \theta)\) be a compact Vaisman manifold of complex dimension \( n \), with fundamental form \( \omega \), and canonical foliation \( \Sigma \). Denote by \( \mathcal{H}^i \) the space of all basic \( i \)-forms \( \alpha \in \Lambda^*_\text{kah}(M) \) which satisfy:

- for \( i \leq n \): \( \alpha \) is basic harmonic (i.e. \( \Delta_\text{kah}(\alpha) = 0 \)) and satisfies \( \Lambda_{\omega,0}(\alpha) = 0 \);
- for \( i > n \): \( \alpha = \beta \wedge I\theta \) where \( \beta \) is basic harmonic and satisfies \( L_{\omega,0}(\beta) = 0 \).

Then all elements of \( \mathcal{H}^* \oplus \theta \wedge \mathcal{H}^* \) are harmonic and, moreover, all harmonic forms on \( M \) belong to \( \mathcal{H}^* \oplus \theta \wedge \mathcal{H}^* \).

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### 3 The Lee field on a compact Vaisman manifold

In [Ts2], K. Tsukada has shown that the direction of the Lee field is uniquely determined by the complex structure of the Vaisman manifold. We give another proof of this result here; for other proofs, see [MMP] and [OV1].

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3The proof in [OV1] is valid only for \( b_1(M) = 1 \).
**Proposition 3.1:** Let $M$ be a compact complex manifold of Vaisman type, and $\theta^\sharp$ the Lee field of a Vaisman structure $(\omega, \theta)$. Then $\theta^\sharp$ is determined by the complex structure on $M$ uniquely up to a real multiplier.

**Proof.** **Step 1:** Consider the form $\omega_0 = d\theta^c$. As shown in [Ve], $\omega_0$ is a semi-positive Hermitian form, and its kernel is precisely $\Sigma = \langle \theta^\sharp, I(\theta^\sharp) \rangle$. Let $\omega'_0$ be a form associated in the same way with some other Vaisman structure $(\omega', \theta')$. Then $\eta_0 = \omega_0 + \omega'_0$ is an exact, semi-positive $(1,1)$-form. This form cannot be strictly positive because $\int_M \eta_0^{\dim \mathbb{C} M} = 0$, hence it has a non-trivial kernel, which is contained in $\ker \omega_0 \cap \ker \omega'_0$. However, $\dim \mathbb{C} \ker \omega_0 = \dim \mathbb{C} \ker \omega'_0 = 1$, and therefore these kernel spaces coincide. This implies that the canonical foliation $\Sigma'$ associated with $(\omega', \theta')$ is equal to $\Sigma$.

**Step 2:** Recall that $\theta^\sharp$ is holomorphic and Killing by Remark 2.5. Since $\Sigma$ has a non-degenerate holomorphic section $\theta^\sharp$, it is trivial as a holomorphic line bundle. Then $H^0(M, \Sigma) = \mathbb{C}$, and the space of real holomorphic vector fields tangent to $\Sigma$ has real dimension 2. Since $\theta^\sharp$ is Killing, it acts conformally on the Kähler covering $\tilde{M}$ of $M$. However, a holomorphic conformal vector field on a Kähler manifold multiplies the Kähler form by a constant. This gives a character $\sigma : \mathfrak{s} \rightarrow \mathbb{R}$ on the Lie algebra of all holomorphic vector fields tangent to $\Sigma$. We claim that $\sigma$ is uniquely determined by the cohomology class of $\theta$. Indeed, let $f \in C^\infty \tilde{M}$ be a function such that $df = \theta$, and $X \in \mathfrak{s}$, $X = a\theta^\sharp + bI(\theta^\sharp)$. Then

$$\Lie_X f = \langle \theta, X \rangle = a \langle \theta, \theta^\sharp \rangle = a,$$

because $\nabla \theta = 0$, and $a$ is equal to $\sigma(X)$ since $\sigma(X)\tilde{\omega} = \Lie_X \tilde{\omega} = a\Lie_\theta \tilde{\omega} + b\Lie_{I(\theta)} \tilde{\omega} = a\tilde{\omega}$.

**Step 3:** The anti-Lee field $I(\theta^\sharp)$ is distinguished by $\sigma(I(\theta^\sharp)) = 0$, and the direction of $\theta^\sharp$ is determined by $\sigma(\theta^\sharp) > 0$. Therefore, to show that $\theta^\sharp$ is independent from the choice of the Vaisman structure, it would suffice to prove that $\sigma$ is independent.

Let $\theta_1, \theta_2$ be two Lee classes of the Vaisman metrics $\omega_1, \omega_2$ on $M$. Using the harmonic decomposition for 1-forms ([Ts1]), we obtain that any harmonic 1-form on $(\tilde{M}, \omega_1)$ is proportional to $\theta_1 + \alpha$, where $\alpha$ is $\Sigma$-basic. The set of possible Lee classes for Vaisman structures is a half-space in $H^1(M, \mathbb{R})$ (Theorem 2.9), with the boundary represented by $\Sigma$-basic forms. Therefore,
we can always replace the Lee class \([\theta]\) of a Vaisman manifold by \([\text{const} \theta]\), where \text{const} is positive. Then, for some positive real constant \(A\), the class \([A\theta_1 - \theta_2]\) is equal to \([\alpha]\), where \(\alpha\) is a \(\Sigma\)-basic closed 1-form. Since the form \(\alpha\) is basic, \(\langle X, \alpha \rangle = 0\) for any \(X\) tangent to \(\Sigma\). Therefore, the character \(\sigma_1 : \mathfrak{s} \rightarrow \mathbb{R}\) associated to \([\theta_1]\) is proportional to the character \(\sigma_2 : \mathfrak{s} \rightarrow \mathbb{R}\) associated to \([\theta_2] = [A\theta_1 + \alpha]\).

4 The complex Monge-Ampère equation

We start by introducing a version of a result of \([OV1]\) which proves the uniqueness of the solution of the Monge-Ampère equation on Vaisman manifolds. Using a theorem of A. El Kacimi-Alaoui (\([El]\)), we show that the solution always exists. This result is a complete analogue of the existence and uniqueness of the solutions of the complex Monge-Ampère equations on a compact Kähler manifold proven by S.-T. Yau.

We say that a form \(\eta\) on a Vaisman manifold is Lee-invariant if \(\text{Lie}_\theta \eta = 0\) and anti-Lee invariant if \(\text{Lie}_I \theta \eta = 0\). The following theorem claims that the Vaisman structure is uniquely determined by the cohomological data and the Lee- and anti-Lee-invariant volume form.

**Theorem 4.1:** Let \((M, \omega, \theta)\) be a compact Vaisman manifold, and \(V'\) a Lee- and anti-Lee-invariant volume form on \(M\), satisfying \(\int_M V' = \int_M \omega^n\). Then there exists a unique Vaisman metric \(\omega'\) on \(M\) with the same Lee class and the volume form \((\omega')^n = V'\).

This theorem is proven later in this section.

Recall that a transversal volume form, or basic volume form is a basic form \(V \in \Lambda^k_F(M)\) on a foliated manifold \((M, F)\), \(k = \text{codim} F\), which defines a non-degenerate volume form locally on the leaf spaces of \(F\). The following lemma is a transversal form of Calabi-Yau theorem, essentially due to \([El, \S 3.5.5]\).

**Lemma 4.2:** Let \(M\) be a compact Vaisman \(n\)-manifold, and \(\Sigma\) its canonical foliation. Then for any \(\Sigma\)-basic volume form \(V_0\) which is cohomologous to an \((n - 1)\)-th power of a transversally Kähler form \(\eta_1\), there exists a unique transversally Kähler form \(\eta_2\) in the same basic cohomology class such that \(\eta_2^{n-1} = V_0\).
**Proof:** We start by proving the uniqueness of a transversally Kähler form with a given transversal volume. Using the transversal \(dd^c\)-lemma ([El]), we obtain \(\eta_1 = \eta_2 + dd^cf\), where \(f\) is a \(\Sigma\)-basic function (that is, a function which is constant on the leaves of \(\Sigma\)). Then \(\eta_1^{n-1} - \eta_2^{n-1} = dd^cf \wedge P\), where \(P = \sum_{i=0}^{n-2} \eta_1^i \wedge \eta_2^{n-2-i}\). Consider the operator

\[
f \mapsto D_P(f) := \frac{dd^cf \wedge P}{\eta_1^{n-1}}
\]

taking basic functions to basic functions. Let \(U \subset M\) be a sufficiently small open set, and \(X_U\) the leaf space of \(\Sigma\) on \(U\). Clearly, the map \(D_P : C^\infty(X_U) \to C^\infty(X_U)\) is a second order elliptic operator. By Hopf maximum principle, any non-constant \(f \in \ker D_P\) cannot have a maximum. However, any \(\Sigma\)-basic function on \(M\) has a maximum somewhere, because \(M\) is compact. Therefore, any \(f \in \ker D_P\) is constant. This proves the uniqueness of solutions. The existence of solutions is obtained by repeating Yau’s argument in the transversal setup, as done in [El, §3.5.5 (iv)].

**Lemma 4.3:** Let \(M\) be a compact complex \(n\)-manifold of Vaisman type. Then a Vaisman structure on \(M\) is uniquely determined by its transversal Kähler form \(\omega_0\) (Theorem 2.7) and the Lee class \([\theta] \in H^1(M, \mathbb{R})\).\(^4\)

**Proof:** By (2.1), we have \(\omega = \omega_0 + \theta \wedge \theta^c\). Therefore, it would suffice to show that the Lee form \(\theta\) is uniquely determined by \(\omega_0\) and the Lee class. Let \(\theta\) and \(\theta'\) be two Lee forms of Vaisman manifolds, with the same transversal Kähler form \(\omega_0\). Denote by \(\eta\) the 1-form \(\theta - \theta'\). Since \(\omega_0 = d^c\theta = d\theta'\) this would imply \(d^c\eta = d\eta = 0\). Such a 1-form cannot be exact, because if \(\eta = df\), one has \(dd^cf = 0\); however, pluriharmonic functions are constant on any compact manifold by the maximum principle. Therefore, \(\theta\) cannot be cohomologous to \(\theta'\).

**Lemma 4.4:** Let \((M, \omega, \theta)\) be a Vaisman \(n\)-manifold, and \(\omega_0\) its transversal Kähler form (Theorem 2.7). Then \(i_{I(\theta^c)}i_{\theta^c} \omega^n = n\omega_0^{n-1}\), where \(\theta^\sharp\) is the Lee field.

**Proof:** By (2.1), we have \(\omega = \omega_0 + \theta \wedge \theta^c\). This implies \(\omega^n = \omega_0^{n-1} \wedge \theta \wedge \theta^c\).

\(^4\)The Lee class \([\theta]\) is uniquely determined by the homothety character \(\chi\) (Remark 2.2) and determines it.
The form $\omega_0$ is $\Sigma$-basic, and $|\theta^c| = 1$. Then
\[ i_{I(\theta^c)}i_{\theta^c}\omega^n = i_{I(\theta^c)}i_{\theta^c}(\theta \wedge \theta^c) \wedge \omega_0^{n-1} = n\omega_0^{n-1}. \]

Now we can prove Theorem 4.1.

The transversally Kähler form $\omega_0$ associated with the Vaisman structure is uniquely defined by its transversal volume form $V_0$ and its basic cohomology class $[\omega_0] \in H^2_\Sigma(M)$, assuming that the transversal cohomology classes of $\omega^{n-1}$ and $V$ are equal (Lemma 4.2). However, the class $[\omega_0]$ generates the kernel of the natural map $H^2_\Sigma(M) \to H^2(M)$ (Theorem 2.10). This determines the basic cohomology class of $\omega_0$ up to a constant; the constant is fixed if the transversal volume is fixed. However, the transversal volume form $\omega_0^{n-1}$ is determined uniquely, up to a constant multiplier, by the volume form $V$, as follows from Lemma 4.4. We obtain that the volume form of a Vaisman metric uniquely defines $\omega_0$, in such a way that $\omega_0^{n-1} = \text{const} \cdot i_{I(\theta^c)}i_{\theta^c} V$ and the Lee class of the Vaisman structure associated with $\omega_0$ is proportional to $[\theta]$.

By Proposition 3.1, the Lee field $\theta^c$ of a Vaisman structure is uniquely (up to a constant) determined by the complex structure of $M$. By Lemma 4.4, $V_0 = n^{-1}i_{I(\theta^c)}i_{\theta^c} V$ uniquely (up to a constant) defines the transversal volume form $\omega_0^{n-1}$, hence $\omega_0$ exists and is uniquely defined, up to a constant.\(^5\)

Now, the Vaisman metric $\omega = \omega_0 + \theta \wedge \theta^c$ is uniquely defined by $\omega_0$ and the Lee class (Lemma 4.3), hence the Vaisman metric $\omega$ is uniquely (up to a constant) defined by its volume form $V$ and the Lee class. The constant is also fixed, because $V = \omega^n$.

This proves uniqueness of a Vaisman metric with prescribed volume. To see that a metric with a prescribed volume form $V'$ exists, we write the corresponding transversal volume form $V'_0 := n^{-1}i_{I(\theta^c)}i_{\theta^c} V'$ and solve the transversal Calabi-Yau equation (Lemma 4.2), arriving at a transversal Kähler form $\omega'_0 = d^c\theta + dd^c f = d^c\theta'$ satisfying $(\omega'_0)^{n-1} = V'_0$. Then $\omega' := \omega'_0 + \theta' \wedge (\theta')^c$ is an LCK form which is invariant under the Lee field action; by [KO, Theorem A], any LCK metric $\omega'$ admitting a conformal holomorphic flow, non-isometric on its Kähler covering, is Vaisman.\(\blacksquare\)

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\(^5\)This is where we use the assumption that $V$ is Lee- and anti-Lee-invariant; this is equivalent to $V_0$ being basic.
References

[Ca1] E. Calabi, The Space of Kähler Metrics, Proc. of the Int. Congress of Mathematicians 1954, Volume II, 206-207, E.P. Noordhoff, Groningen, 1956. (Cited on page 1.)

[Ca2] E. Calabi, On Kähler manifolds with vanishing canonical class in Algebraic geometry and topology. A symposium in honor of S. Lefschetz, 78-89. Princeton Univ. Press, Princeton, N.J., 1957 (Cited on page 1.)

[Ca3] E. Calabi, Extremal Kähler metrics, in: Seminar on Differential Geometry, ed. S. T. Yau, Annals of Math. Studies 102, Princeton Univ. Press, Princeton, NJ (1982), 259-290. (Cited on page 2.)

[El] A. El Kacimi-Alaoui, Opérateurs transversalement elliptiques sur un feuilletage riemannien et applications, Compositio Math. 73 (1990), no. 1, 57-106. (Cited on pages 3, 7, and 8.)

[HR] G. Habib, K. Richardson, Modified differentials and basic cohomology for Riemannian foliations, J. Geom. Anal. 23 (2013), no. 3, 1314-1342. (Cited on page 2.)

[KO] Y. Kamishima, L. Ornea, Geometric flow on compact locally conformally Kähler manifolds, Tohoku Math. J., 57 (2) (2005), 201-221. (Cited on page 9.)

[MMP] F. Madani, A. Moroianu, M. Pilca, LCK structures with holomorphic lee vector field on vaisman-type manifolds, Geom. Dedicata 213 (2021), 251-266. arXiv:1905.07300v1. (Cited on page 5.)

[OV1] L. Ornea, M. Verbitsky, Einstein-Weyl structures on complex manifolds and conformal version of Monge-Ampère equation, Bull. Math. Soc. Sci. Math. Roumanie 51 (99) No. 4, (2008), 339-353. (Cited on pages 5 and 7.)

[OV2] L. Ornea, M. Verbitsky, Locally conformally Kähler metrics obtained from pseudocovex shells, Proc. Amer. Math. Soc. 144 (2016), 325-335. (Cited on page 4.)

[OV3] L. Ornea, M. Verbitsky, Supersymmetry and Hodge theory on Sasakian and Vaisman manifolds, arXiv:1910.01621, Manuscripta Math. https://doi.org/10.1007/s00229-021-01358-8 (Cited on pages 2 and 5.)

[OV4] L. Ornea, M. Verbitsky, Lee classes on LCK manifolds with potential, arXiv:2112.03363. (Cited on pages 3 and 5.)

[Ts1] K. Tsukada, Holomorphic forms and holomorphic vector fields on compact generalized Hopf manifolds, Compositio Math. 93 (1994), no. 1, 1-22. (Cited on pages 3, 5, and 6.)

[Ts2] K. Tsukada, Holomorphic maps of compact generalized Hopf manifold s, Geom. Dedicata 68 (1997), 61-71. (Cited on pages 3 and 5.)

[Va] I. Vaisman, Generalized Hopf manifolds, Geom. Dedicata, 13 (1982), 231-255. (Cited on pages 4 and 5.)

[Ve] M. Verbitsky, Theorems on the vanishing of cohomology for locally conformally hyper-Kähler manifolds, Proc. Steklov Inst. Math. no. 3 (246), 54-78 (2004). (Cited on pages 4 and 6.)

[VVO] M. Verbitsky, V. Vuletescu, L. Ornea Classification of non-Kähler surfaces and locally conformally Kähler geometry, Russian Math. Surv. 76 (2021), 201-290. arxiv:1810.05768. (Cited on page 4.)
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