Quasi-SMC based on MPC for a constrained continuous-time nonlinear system with external disturbances

Huan Meng, Jinhui Zhang

Abstract—In this article, a quasi-sliding mode control (QSMC) based on MPC is proposed for the constrained continuous-time nonlinear system with external disturbances. The MPC problem is formulated relating to the design of QSMC, to generate the control input, which can imitate the control process of QSMC and guarantee the satisfaction of state and input constraints. Meanwhile, the cost function of MPC problem is reconstructed, in which the QSMC based on MPC can show better convergence rate by tuning the weight parameters. Finally, a simulation case is provided to demonstrate the effectiveness of the proposed approach.

Index Terms—Quasi-sliding mode control (QSMC), model predictive control (MPC), constrained continuous-time nonlinear system.

I. INTRODUCTION

Sliding mode control (SMC) [1] has been widely employed in industrial applications such as power systems [2], unmanned vehicle [3] and spacecraft [4], for its robustness and simplicity. The control process of SMC consists of reaching mode and sliding mode. In the reaching mode, the state system is driven onto the a manifold called “sliding surface”, and then the state slides to the origin along the surface in the sliding mode. The main advantages of SMC lies in the simplicity of its feedback control law design after choosing the sliding variable and its robustness on the sliding mode surface. However, the drawback of SMC is incapable of dealing with the constraints on both input and state, which may cause control performance deterioration and even system instability. Contrast to SMC, model predictive control (MPC), has attracted more attention in recent years since it can be able to manage both input and state constraints.

MPC, known as the receding horizon control (RHC), is to obtain the optimal control sequence by solving online a finite horizon open-loop optimization problem subject to system dynamics and constraints. Then the first part of the sequence is applied to system to update the control input. When the system is nominal, MPC is an effective control technique. Nevertheless, the theoretical properties and system stability may not be guaranteed when existing external disturbances. To cope with the problem, several robust MPC strategy has been investigated, like min-max MPC [5]–[7], tube-MPC [8]–[10] and inherent robust MPC [11]–[13]. However it has to be point out that those robust MPC approaches still suffer from low robustness problem. It is worth noting that the advantages and disadvantages of MPC and SMC are complementary. Therefore, several works about combining SMC and MPC have emerged.

The works of merging SMC and MPC can be divided into two types. One type is to compose a separate SMC and MPC law into a single control channel [14]–[18]. Here, the inner-outer loop structure is adopted, where the SMC restrains the disturbances in the inner loop and MPC provides the control law for the system in the outer loop. However, by this approach, the control domain has to be tightened, which will effect the control performance. Meanwhile, the SMC controller is only to enhance the robustness of MPC, and the advantage of MPC structure has not been exploited. The other type is to merge SMC and MPC into a single control law. In [19], [20], considering an unconstrained linear system, MPC controller generates the control law based on SMC reaching law to improve the performance of SMC. The linear system with constraints is further considered in [21], [22], which guarantee the convergence of state to the sliding mode band. In [23], considering the unconstrained nonlinear system, the linear approximation is first used and then applies the same approach in [19]. However, the system error can be caused by the linear approximation. In [24], the sliding variable is used to define the cost function of MPC problem. However, the control can only drive the state into the boundary layer of the sliding surface instead of the origin. In [25], the MPC is used to ‘imitate’ the control law of SMC for a constrained discrete-time nonlinear system. It is proved that the MPC law has the same properties of SMC law, and the constraints are satisfied.

Inspired by the work [25], we exert the idea into the constrained continuous-time nonlinear system. For the ideal SMC, it has to be pointed out that the discontinuous function $\text{sgn}(\cdot)$ has to be applied to achieve finite-time convergence of the sliding variable to zero, which can lead to chattering problem. Meanwhile, the discontinuous function can also lead mixed integer nonlinear programming (MINLP) problem for merging SMC and MPC, which is intractable to deal with. Therefore, the quasi-sliding mode control is introduced [26]–[28], in which the discontinuous function is replaced by the smooth function and the sliding variable converges to the vicinity of the sliding surface. Furthermore, there is only control input variable in the cost function of MPC problem in [25], which means that the convergence rate of state is unadjustable. Therefore, the cost function has be reconstructed to achieve better convergence rate. The contributions of this paper can be summarized as follows:
(1) For the constrained nonlinear continuous-time system with external disturbances, the QSMC based on MPC is proposed. By referring to the design of QSMC, the MPC problem is formulated to generate the control input, which imitates the control process of QSMC and guarantees the satisfaction of both state and input constraints.

(2) The cost function in the MPC problem is reconstructed. The state is introduced in the cost function and by tuning the weight parameters of the cost function, the system can show better convergence rate. The recursive feasibility of MPC problem and input-to-state stability of closed-loop system is strictly proven.

The notation adopted in this paper are as follows. \( \mathbb{R} \) and \( \mathbb{R}^n \) are the real space and its n-dimensional Euclidean space respectively. For \( r_1 \in \mathbb{R} \), \( \mathbb{R} \geq r_1 \), denotes the set \( \{ r \in \mathbb{R} | r \geq r_1 \} \). \( \mathbb{N} \) denotes the collection of all natural number. For a vector \( x \), \( \| x \| \overset{\Delta}{=} \sqrt{x^T x} \) denotes the Euclidean norm and \( \| x \|_Q \overset{\Delta}{=} \sqrt{x^T Q x} \) with \( Q > 0 \) denotes the \( Q \)-weighted norm. For a matrix \( M \in \mathbb{R}^{n \times n} \), \( \lambda(M) \) and \( \lambda(M) \) are its maximum and minimum eigenvalues. A function \( \partial(\cdot) : R_{\geq 0} \rightarrow R_{\geq 0} \) is a \( K \)-function if it is continuous, strictly increasing and \( \partial(0) = 0 \). A function \( \varsigma(\cdot) : R_{\geq 0} \rightarrow R_{\geq 0} \) is a K-function if it is a K-function and \( \lim_{s \rightarrow +\infty} \varsigma(s) \rightarrow +\infty \). A function \( \beta(\cdot, \cdot) : R_{\geq 0} \times R_{\geq 0} \rightarrow R_{\geq 0} \) is a K-L function if for any fixed \( t \geq 0 \), \( \beta(s, t) \) is a K-function and for any fixed \( s \geq 0 \), \( \beta(s, \cdot) \) is decreasing and \( \lim_{t \rightarrow +\infty} \beta(s, t) \rightarrow 0 \). The Pontryagin set difference of two sets \( A \in \mathbb{R}^n \), \( B \in \mathbb{R}^n \) is defined as \( A \sim B \overset{\Delta}{=} \{ z \in \mathbb{R}^n | z + b \in A, \forall b \in B \} \). The function \( \text{sgn}(s) \) is defined as \( \text{sgn}(s) \overset{\Delta}{=} \{ 1, s > 0 \}
\{ -1, s \leq 0 \} \cdot x(\tau | t) \) indicates the value of variable \( x \) at time \( \tau \) predicted from time \( t \). Moreover, we mark feasible variables satisfying all constraints as \( \overset{\circ}{\cdot} \) and mark optimal variables attained by solving optimization problem as \( \overset{\circ}{\cdot} \). To simplify notation, time dependence is omitted when not relevant.

II. PROBLEM FORMULATION

A. System description

Consider the following constrained continuous-time nonlinear system

\[ \dot{x} = f(x, u) + w, \]  

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \) (\( n \geq m \)) and \( w \in \mathbb{R}^n \) are the state, control input and external disturbances respectively, and \( f(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \) is a smooth function. Both the state and control input are constrained by \( x \in \mathcal{X}, u \in \mathcal{U} \), where \( \mathcal{X} \) and \( \mathcal{U} \) are compact and convex sets containing the origin. The disturbances is assumed to be bounded by \( \| w \| \leq \mu \).

Assumption 1. The function \( f(x, u) \) is Lipschitz for its first argument \( x \in \mathcal{X} \), and the Lipschitz constant is \( \nu \), i.e.

\[ \| f(x_1, u) - f(x_2, u) \| \leq \nu \| x_1 - x_2 \| \]

In order to effectively control the system, the SMC strategy is applied in this paper. Since the SMC is incapable of dealing with the constraints, the following unconstrained SMC strategy is first considered.

B. Unconstrained SMC

The sliding variable \( s \) is designed as

\[ s = g(x) \]  

where \( g(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is a smooth function. The sliding surface \( S \) associated with (2) is given by

\[ S \overset{\Delta}{=} \{ x | s = g(x) = 0 \} . \]  

Definition 1. An ideal sliding mode is said to take place on (3) if there exists a finite \( t_r \in \mathbb{R}^+ \) such that \( s = 0 \) for all \( t \geq t_r \).

Definition 2. The control function, which needs to be applied to system (1) after reaching the sliding surface \( s = 0 \), to ensure the system trajectory stays on the surface thereafter, is called the equivalent control law.

Based on the design principle of SMC in [29], the equivalent control law \( u_{eq}(x) \) can be obtained by solving \( g(f(x, u_{eq}(x)) + w) = 0 \). Then the closed-loop dynamics of (1) under \( u_{eq} \) is written as

\[ \dot{x} = f(x, u_{eq}(x)) + w \]

Remark 1. In case of the nonlinear system \( \dot{x} = \tilde{f}(x) + b(x)u + w \), the sliding variable \( s \) is designed as \( s = Cx \), where \( \tilde{f}(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n \), \( b(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m} \), \( C \in \mathbb{R}^{m \times n} \). Assuming that \( Cb(x) \) is invertible, then the equivalent control law can be obtained as

\[ u_{eq}(x) = -\frac{1}{Cb(x)}C \left( \tilde{f}(x) + \bar{\mu} \text{sgn}(Cx) \right) \]

with \( \bar{\mu} \geq \mu \).

It is noted that to realize the ideal sliding mode in presence of nonzero disturbances, the discontinuous function \( \text{sgn}(\cdot) \) must be applied, which may lead to chattering problem. In order to eliminate the chattering, the concept of quasi-sliding mode is defined in the following.

Definition 3. A quasi-sliding mode is said to take place on (3) if there exist a finite \( t_r \in \mathbb{R}^+ \) and a K-function \( \partial_1(\cdot) \) such that \( \| s \| \leq \partial_1(\mu) \) for all \( t \geq t_r \).

Remark 2. It is clear that in the quasi-sliding mode, the state is restrained in a boundary layer of \( S \) and the band with is depend on the value of \( w \). When \( \| w \| = 0 \), the quasi-sliding mode turns into the ideal sliding mode.

In the QSMC, the state can only converge to the boundary layer of \( S \). Therefore, the concept of input-to-stable stability [30] is introduced.

Definition 4. The system (1) with the initial state \( x(t_0) \) is said to be (globally) input-to-state stable (ISS), if there exist a K-L function \( \beta(\cdot, \cdot) \) and a K-function \( \partial_2(\cdot) \) such that, it holds that

\[ \| x(t) \| \leq \beta(x(t_0), t) + \partial_2(\mu) \]

for each \( t \geq t_0 \).
**Definition 5.** A smooth function $V(\cdot)$ is called an ISS Lyapunov function: $\mathbb{R}^n \to \mathbb{R}_{\geq 0}$ for the system (1) if there exist $K_{\infty}$-function $\varsigma_1(\cdot), \varsigma_2(\cdot), \varsigma_3(\cdot)$ and $K$-function $\partial_3(\cdot)$ such that

$$
\varsigma_1(||x(t)||) \leq V(x(t)) \leq \varsigma_2(||x(t)||),
$$
$$
\dot{V}(x(t)) \leq -\varsigma_3(||x(t)||) + \partial_3(\mu).
$$

Based on the Definition 4 and 5, the following assumption is presented.

**Assumption 2.** There exist the continuous equivalent control law $u_{eq}(x)$, under which the closed-loop dynamics (4) is ISS.

**Remark 3.** To obtain a continuous equivalent control law, one obvious solution is to approximate the discontinuous function in the ideal SMC by the continuous one. For instance, the equivalent control law in (5) can be replaced by

$$
u(x) = -(Cb(x))^{-1}C \left( f(x) + \mu \tanh(Cx) \right)
$$

and it is easy to prove the closed-loop dynamic (4) is ISS.

Using the QSMC strategy, the control process is divided into two stages: reaching mode and sliding mode. In the reaching mode, the system state is first driven into the boundary layer of sliding surface. Once the boundary layer is reached, the control stage turns into the sliding mode and the state slides to origin along this boundary layer. Thus, the total control law $u_s$ can be designed as

$$
u_s = \nu_{eq}(x) + \nu_{ro},
$$

where $\nu_{ro}$ is the robust control law responsible for driving the system state into the boundary layer of sliding surface. Then the closed-loop dynamics of (1) under $u_s$ is written as

$$\dot{x} = f(x, u_s) + w.
$$

However, it has to be noted that both state and input constraints may be violated by applying the control law $u_s$ generated directly from QSMC. Therefore, the QSMC based on MPC strategy is investigated in the next part.

**C. QSMC based on MPC**

Define the sampling instant as $\{t_k\}, k \in \mathbb{N}$ and the sampling period $\delta = t_{k+1} - t_k$, the optimization problem is solved at each sampling instant. Then the optimization problem at $t_k$ with prediction horizon $T$ is formulated as follows:

**Problem 1**

$$
\min_{\bar{u}_{ro}(\cdot)} J(x(t_k), \bar{u}_{ro}(\cdot)),
$$

subject to

$$
\bar{x}(t_k|t_k) = x(t_k),
$$
$$
\bar{x}(\tau|t_k) = f(\bar{x}(\tau|t_k), \bar{u}_s(\tau|t_k)),
$$
$$
\bar{u}_s(\tau|t_k) = u_{eq}(\bar{x}(\tau|t_k)) + \bar{u}_{ro}(\tau|t_k),
$$
$$
\bar{x}(\tau|t_k) \in \bar{X}(\tau - t_k),
$$
$$
\bar{u}_s(\tau|t_k) \in \bar{U},
$$
$$
\bar{x}(t_k + T|t_k) \in \Omega_x,
$$

with $\tau \in [t_k, t_k + T]$, $\bar{X}(\tau - t_k) = \bar{X} \sim \bar{Y}(\tau - t_k)$ is the tightened state constraint sets with $\bar{Y}(\tau) = \{x \in \mathbb{R}^n, \|x\| \leq \frac{1}{\delta}(e^{\nu \tau} - 1)\}$, and $\Omega_x = \{\bar{x} \in \mathbb{R}^n, \|\bar{x}\|_p \leq \varepsilon\}$ is the terminal region. The cost function $J(x(t_k), \bar{u}_{ro}(\cdot))$ in Problem 1 is defined as

$$
J(x(t_k), \bar{u}_{ro}(\cdot)) = \int_{t_k}^{t_k+T} L(\bar{x}(\tau|t_k), \bar{u}_{ro}(\tau|t_k)) d\tau + V_f(\bar{x}(t_k + T|t_k)) d\tau,
$$

where $L(\bar{x}(\tau|t_k), \bar{u}_{ro}(\tau|t_k)) = \|\bar{x}(\tau|t_k)\|_Q^2 + \|\bar{u}_{ro}(\tau|t_k)\|_R^2$ is stage cost with the weight matrices $Q > 0, R > 0$ and $V_f(\bar{x}(t_k + T|t_k)) = \|\bar{x}(t_k + T|t_k)\|_P^2$ is terminal cost with the weight matrix $P > 0$.

By solving Problem 1, the optimal input sequence is obtained for $\tau \in [t_k, t_k + T]$ as

$$
\bar{u}_{ro}^*(\tau|t_k) = \arg\min_{\bar{u}_{ro}(\cdot)} J(x(t_k), \bar{u}_{ro}(\cdot)),
$$

and the corresponding QSMC control law is

$$
\bar{u}_s^*(\tau|t_k) = u_{eq}(\bar{x}^*(\tau|t_k)) + \bar{u}_{ro}^*(\tau|t_k), \tau \in [t_k, t_{k+1}].
$$

**Assumption 3.** For the nominal system (11), there exist an invariant set $\Omega_\alpha = \{x \in \mathbb{R}^n, \|x\|_p \leq \alpha\}$ with $\alpha \geq \varepsilon$ and the equivalent control law $\nu_{eq}(\bar{x})$ such that for all $\bar{x} \in \Omega_\alpha$, by implementing the control law $u_{eq}(\bar{x})$, it holds that $u_{eq}(\bar{x}) \in \bar{U}$ and

$$
\dot{V}_f(\bar{x}) + L(\bar{x}, 0) \leq 0.
$$

Based on the discussions above, the procedure of QSMC based on MPC is described in Algorithm 1.

**Algorithm 1** QSMC based on MPC

1. Set $k = 0$ and initialize the nominal system state by the actual state $x(t_0)$.
2. While $x(t) \notin \Omega_x$
3. Solve Problem 1.
4. Apply $u_s(t_k) = \bar{u}_s^*(t_k|t_k)$ to the real system for $t \in [t_k, t_{k+1})$.
5. Update $k = k + 1$.
6. End while
7. Apply the control law $u = u_{eq}(x)$.

**Remark 4.** The optimization variable $\bar{u}_{ro}$ in Problem 1 has two purposes. The one is to be the robust control law to drive the sliding variable into the boundary layer of sliding surface. The other is to serve as a regulator, which guarantees the state and control input satisfy the constraints.

**Remark 5.** It is noted that the cost function in [31] is designed as

$$
J = \min_{c_t} c_t^T \Psi c_t,
$$

where $c_t = \bar{u}_{ro}$ and $\Psi$ is $R$ in this paper. However, the convergence rate is unadjustable because there is only control input in the cost function. Different from (19), the cost function
consists of the stage cost and terminal cost are added, where the terminal cost is to penalize the undesired terminal state, and thus realizes quasi-infinite horizon prediction (referring to [32]). In the stage cost, Q and R are tuning parameters, where Q larger than R leads to faster convergence rate, while R larger than Q leads to less control energy consumption.

III. THEORETICAL ANALYSIS

In this section, the recursive feasibility of the Problem 1 and the stability of closed-loop system is developed.

A. Recursive feasibility analysis

Theorem 1. For system (1), the Problem 1 is recursive feasible for any \( t_k \) if the Problem 1 is feasible at the initial time \( t_0 \) and the following conditions are satisfied: (1) \( \lambda \left( \sqrt{P} \right) \frac{\lambda(Q)}{2\lambda(P)} \delta \geq \ln \varepsilon \), (2) \( Q \) is constructed as follows: where \( \lambda \) and \( \mu \) are tuning parameters, \( Q \) leads to faster convergence rate, while \( \mu \) larger than \( \lambda \) leads to less control energy consumption.

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Part 3: To show the state $\tilde{x}(\tau|t_{k+1})$ under the control law (22) satisfies the state constraint (13).

For $\tau \in [t_{k+1}, t_k + T]$, due to
$$\|\tilde{x}(\tau|t_{k+1}) - \bar{x}^*(\tau|t_k)\| \leq \frac{\mu}{\sqrt{V}} (e^{(\nu_0 - 1)e^{(\tau - t_{k+1})}} - 1),$$
we have $\tilde{x}(\tau|t_{k+1}) \in \bar{x}(\tau - t_{k+1})$ by referring to Lemma 3.

Consider $\tau \in [t_k + T, t_{k+1} + T]$. According to Assumption 3, $\Omega_\alpha$ is a invariant set under the control law $u_{eq}(\tilde{x}(\tau|t_{k+1}))$. Since $|\tilde{x}(t_k + T|t_{k+1})| \leq \alpha$ and $\Omega_\alpha \subseteq \bar{x} \sim \bar{y}(T)$, we have $\tilde{x}(t_k + T|t_{k+1}) \subseteq \Omega_\alpha \subseteq \bar{x} \sim \bar{y}(T) \subseteq \bar{x}(\tau - t_{k+1})$.

The proof is completed. □

Remark 6. It is noted that the three conditions (1) $\bar{x}(\sqrt{P}) \geq (e^{(\nu_0 - 1)} e^{(T - \delta)} - 1) \geq v(\alpha - \varepsilon)$, (2) $\bar{x}(\sqrt{P}) \geq \frac{\mu}{\sqrt{V}} \geq \ln \frac{\mu}{\sqrt{V}}$, (3) $\Omega_\alpha \subseteq \bar{x} \sim \bar{y}(T)$ in Theorem 1 are to confine the range of $\alpha$, $\varepsilon$ and $\mu$. The region $\Omega_\alpha$ and $\Omega_\alpha$ can be ensured by applying the condition (1) and (2), and the upper bound of external disturbances $\mu$ can be ensured by applying the condition (1) and (3).

B. Stability analysis

To show the stability of closed-loop system, the following theorem is presented to illustrate the real state of the system converges to the region $\Omega_\varepsilon$.

Theorem 2. For system (1) under the QSMC based on MPC approach, if the conditions in Theorem 1 and the condition $\xi_1 + \xi_2 + \xi_3 < 0$ are satisfied, where
$$\xi_1 = \frac{\lambda(Q)}{\lambda(P)} \frac{1}{2} \mu (e^{(\nu_0 - 1)} e^{(T - \delta)} - 1)$$
$$\xi_2 = \frac{\lambda(P)}{\lambda(Q)} \frac{1}{2} \mu (e^{(\nu_0 - 1)} e^{(T - \delta)} - 1)$$
$$\xi_3 = -\frac{\lambda(Q)}{\lambda(P)} \mu (\alpha - \varepsilon)^2$$
and $\gamma_x = \sup_{\tilde{x} \in \bar{x}, u_r \in U} \{\|\tilde{x}\| = \tilde{f}(\bar{x}, u_r)\}$, then the state system $\tilde{x}$ converges asymptotically to the region $\Omega_\alpha$ under the control law (17).

Proof. Suppose that the state $x$ is outside the region $\Omega_\varepsilon$, i.e. $x \notin \Omega_\varepsilon$, then we define $\Delta J \equiv J(\tilde{x}(\tau|t_{k+1}), u_r(\tilde{x}(\tau|t_{k+1})) - J(\bar{x}^*(\tau|t_k), u_r(\bar{x}^*(\tau|t_k)))$. By expanding the term $\Delta J$, we have
$$\Delta J = \int_{t_{k+1}}^{t_{k+1} + T} L(\tilde{x}(\tau|t_{k+1}), u_r(\tilde{x}(\tau|t_{k+1})))d\tau$$
$$- \int_{t_k}^{t_{k+1} + T} L(\bar{x}^*(\tau|t_k), u_r(\bar{x}^*(\tau|t_k)))d\tau$$
$$+ V_f(\bar{x}(t_{k+1} + T|t_{k+1})) - V_f(\bar{x}^*(t_k + T|t_{k}))$$
$$= \Delta_1 + \Delta_2 + \Delta_3,$$
where
$$\Delta_1 = \int_{t_{k+1}}^{t_{k+1} + T} \|\tilde{x}(\tau|t_{k+1})\|^2 P - \|\bar{x}(\tau|t_k)\|^2 Q d\tau,$$
$$\Delta_2 = \int_{t_{k+1}}^{t_{k+1} + T} L(\tilde{x}(\tau|t_{k+1}), u_r(\tilde{x}(\tau|t_{k+1})))d\tau$$
$$+ V_f(\bar{x}(t_{k+1} + T|t_{k+1})) - V_f(\bar{x}^*(t_k + T|t_{k}))$$
$$\Delta_3 = - \int_{t_k}^{t_{k+1} + T} L(\bar{x}^*(\tau|t_k), u_r(\bar{x}^*(\tau|t_k)))d\tau.$$
For $\Delta_1$, it holds that
$$\Delta_1 \leq \frac{\lambda(Q)}{\lambda(P)} \int_{t_{k+1}}^{t_{k+1} + T} \|\tilde{x}(\tau|t_{k+1}) - \bar{x}(\tau|t_k)\|^2 \|\tilde{x}(\tau|t_k)\| d\tau$$
$$\leq \frac{\lambda(Q)}{\lambda(P)} \int_{t_{k+1}}^{t_{k+1} + T} \|\tilde{x}(\tau|t_{k+1}) - \bar{x}(\tau|t_k)\| d\tau$$
$$\leq \frac{\lambda(Q)}{\lambda(P)} \int_{t_{k+1}}^{t_{k+1} + T} \|\tilde{x}(\tau|t_{k+1}) - \bar{x}(\tau|t_k)\| + 2 \|\bar{x}(\tau|t_k)\| d\tau$$
$$\leq \frac{\lambda(Q)}{\lambda(P)} \int_{t_{k+1}}^{t_{k+1} + T} \frac{\mu}{\sqrt{V}} (e^{(\nu_0 - 1)} e^{(T - \delta)} - 1) d\tau$$
$$\leq \frac{\lambda(Q)}{\lambda(P)} \frac{\mu}{\sqrt{V}} (e^{(\nu_0 - 1)} e^{(T - \delta)} - 1).$$

Considering $\Delta_2$, we have
$$\Delta_2 = \int_{t_{k+1}}^{t_{k+1} + T} L(\tilde{x}(\tau|t_{k+1}), u_r(\tilde{x}(\tau|t_{k+1})))d\tau + V_f(\tilde{x}(t_{k+1} + T|t_{k+1})) - V_f(\tilde{x}(t_k + T|t_{k}))$$
$$\leq V_f(\tilde{x}(t_k + T|t_{k})) - V_f(\tilde{x}(t_k + T|t_{k}))$$
$$\leq \frac{\lambda(P)}{\lambda(Q)} \int_{t_{k+1}}^{t_{k+1} + T} \|\tilde{x}(\tau|t_{k+1}) - \bar{x}(\tau|t_{k})\| d\tau$$
$$\leq \frac{\lambda(P)}{\lambda(Q)} \frac{\mu}{\sqrt{V}} (e^{(\nu_0 - 1)} e^{(T - \delta)} - 1).$$

For $\Delta_3$, we have
$$\Delta_3 \leq - \frac{\lambda(Q)}{\lambda(P)} \int_{t_k}^{t_{k+1} + T} \|\bar{x}^*(\tau|t_k)\|^2 d\tau$$
Since the real state $\|x(\tau|t_k)\|$ satisfies $\|x(\tau|t_k)\| > \varepsilon$ and $\|x(\tau|t_k) - \bar{x}^*(\tau|t_k)\| \leq (\sqrt{P})^{-1} (e^{(\nu_0 - 1)} e^{(T - \delta)}) - 1$, for $\tau \in [t_{k+1}, t_{k+1}], one has
$$\|\bar{x}^*(\tau|t_k)\|^2 \geq \lambda(P) \|x(\tau|t_k)\|^2$$
$$\geq \lambda(P) (\|x(\tau|t_k)\| - \|x(\tau|t_k) - \bar{x}^*(\tau|t_k)\|)^2$$
$$> \lambda(P) (\varepsilon - (\sqrt{P})^{-1} (e^{(\nu_0 - 1)} e^{(T - \delta)} - 1))^2.$$

Then, $\Delta_3$ is bounded as
$$\Delta_3 \leq - \frac{\lambda(Q)}{\lambda(P)} \int_{t_k}^{t_{k+1} + T} \|\bar{x}^*(\tau|t_k)\|^2 d\tau$$
Therefore, $\Delta J = \Delta_1 + \Delta_2 + \Delta_3 < 0$ can be deduced. By following the Theorem 2 of [1], the state $x$ will converge asymptotically to the region $\Omega_\varepsilon$.

The proof is completed.

After the state $x$ enters $\Omega_\varepsilon$, i.e. $x \in \Omega_\varepsilon$, the control law is switched to $u_{eq}(x)$. Then the closed-loop system can be represented as
$$\dot{x} = f(x, u_{eq}(x)) + w.$$
Therefore, the closed-loop dynamics of (1) ISS.
Consider a nonlinear cart-damper-spring system shown in Fig. 1 with the following dynamics
\[
\begin{align*}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= -\frac{h_d}{M_c} x_1(t) x_2(t) - \frac{1}{M_c} x_2(t) + \frac{\omega(t)}{M_c} + \frac{\alpha}{M_c},
\end{align*}
\]
(26)
where \(x_1(t)\) and \(x_2(t)\) denote the displacement and the velocity of the cart satisfying \(|x_1| \leq 2.77, |x_2| \leq 3.5, M_c = 1.25\text{kg}\) represents the mass of the cart, \(\gamma = 0.90\text{N/m}\) is the stiffness factor of spring, and \(h_d = 0.42\text{N.s/m}\) is the damper factor.

The control input \(u(t)\) is constrained by \(|u(t)| \leq 8\), and the upper bound of external disturbances \(|\omega(t)| \leq \mu = 0.1\text{N}\). The Lipschitz constant \(\nu\) of the system (26) is computed as \(\nu = 1.15\) and the initial state of system is set as \(\{2, 3\}\).

The sliding variable is chosen as \(s = c_1 x_1 + c_2 x_2\) with \(c_1 = c_2 = 0.18\). For unconstrained SMC, by referring to [29], the equivalent control law is designed as
\[
\begin{align*}
\dot{u}_{eq}(x) &= -M_c F_s - \frac{c_1}{c_2} M_c x_2 - \mu \tanh(s/\varphi),
\end{align*}
\]
(27)
where \(F_s = -\frac{\gamma}{M_c} e^{-x_1(t)} x_1(t) - \frac{h_d}{M_c} x_2(t), \mu = 0.5, \varphi = 0.05\). Therefore, the boundary layer of sliding surface can be calculated as \(|s| \leq \frac{\mu}{\varphi} \leq 0.01\). Based on the equivalent control law (27), the weight matrices of cost function in Problem 1 are set as \(Q = \text{diag}\{0.35, 0.35\}, R = 0.1\) and \(P = \text{diag}\{0.85, 0.85\}\), respectively. The sampling interval \(\delta\) and prediction horizon \(T\) are set as \(\delta = 0.1s\) and \(T = 2s\). For the invariant set \(\Omega_s\) and the terminal region \(\Omega_c\), the parameters are computed as \(\alpha = 0.0608, \varepsilon = 0.0420\).

The simulation example is conducted by following is conducted by following Algorithm 1, and the nonlinear optimization solver ‘IPOPT’ is used. Meanwhile, the unconstrained SMC law designed as
\[
\begin{align*}
u_s &= u_{eq}(x) - \frac{M_c}{c_2} (\eta_1 |s|^{\gamma_1} \text{sign}(s) + \eta_2 |s|^{\gamma_2} \text{sign}(s)) \quad \text{for } u_{eq}
\end{align*}
\]
(28)
is also executed as comparison, where the \(u_{eq}\) part is called the fixed-time reaching law by referring to [35] with \(\eta_1 = 0.65, \eta_2 = 0.22, \gamma_1 = 1.5, \gamma_2 = 0.5\).

The control input, the displacement state and velocity state are displayed in Fig. 2, 3 and 4 respectively. It can be seen that with the control input satisfying the constraint, the displacement of closed-loop under the unconstrained SMC law

![Fig. 2. Comparison of control signals](image)

![Fig. 3. Comparison of displacements of the closed-loop system](image)

![Fig. 4. Comparison of velocities of the closed-loop system](image)
unconstrained SMC by selecting the tuning weight parameter that satisfies under the QSMC based on MPC. Furthermore, the trajectory of sliding variables is illustrated in Fig. 5. It shows that the sliding variable first converges into the boundary layer of sliding surface, and then slides to the origin within the boundary layer.

V. CONCLUSIONS

This paper presented a quasi-sliding mode control (QSMC) based on MPC for the continuous-time constrained nonlinear system with external disturbances. The MPC problem has been formulated by referring to the design of QSMC, to generate the control input, which can imitate the control process of QSMC and guarantee the satisfaction of state and input constraints. Meanwhile, the cost function of MPC problem has reconstructed, in which the QSMC based on MPC can show better convergence rate by tuning the weight parameters. The simulation case has verified that the effectiveness of the proposed approach. The effectiveness of the proposed approach is demonstrated by using the example of a uav.

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