Detection of Impulsive Light–Like Signals in General Relativity

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Abstract

The principal purpose of this paper is to study the effect of an impulsive light–like signal on neighbouring test particles. Such a signal can in general be unambiguously decomposed into a light–like shell of null matter and an impulsive gravitational wave. Our results are: (a) If there is anisotropic stress in the light–like shell then test particles initially moving in the signal front are displaced out of this 2-surface after encountering the signal; (b) For a light–like shell with no anisotropic stress accompanying a gravitational wave the effect of the signal on test particles moving in the signal front is to displace them relative to each other with the usual distortion due to the gravitational wave diminished by the presence of the light– like shell. An explicit example for a plane–fronted signal is worked out.

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1 Introduction

The direct detection of gravitational waves will be realised in the near future. The detectors which are presently under construction are designed to observe signals in the frequency range from 10 Hz to 10 kHz. The main source of such signals is considered to be an inspiralling binary neutron star system. However a large variety of other sources of gravitational waves exists [1] and among them are cataclysmic events such as supernovae, which are expected to produce bursts of gravitational radiation and of null matter.

In this paper we consider impulsive gravitational waves and light–like shells of null matter. These phenomena are both impulsive light–like signals and mathematically correspond to a space–time geometry having a Riemann tensor containing a singular part proportional to a Dirac delta function whose support is a light–like hypersurface [2]. This light–like signal can be decomposed unambiguously [3] into a gravitational wave part and a shell or matter part.

To detect such impulsive light–like signals we need to know their effect on the relative motion of neighbouring test particles. This is the main purpose of the present work. We consider a congruence of time–like geodesics in a vacuum space–time crossing a singular null hypersurface, which represents the history of the impulsive gravitational wave and the shell of null matter. We study the relative motion of the test particles as their world lines cross the null hypersurface.

In section 2 we present a summary of the properties of a singular null hypersurface aswell as the decomposition of the signal into an impulsive gravitational wave and a null shell. The details can be found in references [2] and [3]. In section 3 the principle of detection is described and our main results are obtained. These are: (a) If there is anisotropic stress in the light–like shell then test particles initially moving in the signal front are displaced out of this 2-surface after encountering the signal; (b) For a light–like shell with no anisotropic stress accompanying a gravitational wave the effect of the signal on test particles moving in the signal front is to displace them relative to each other with the usual distortion due to the gravitational wave diminished by the presence of the light–like shell. An example of a plane–fronted wave and null shell is worked out in detail in section 4.

The current design of gravity wave detectors is aimed at observing gravitational waves that establish oscillatory motion in the test particles (such as the waves from an inspiralling neutron star system). The light–like signals we are studying displace the test particles but do not establish oscillatory motion. A modification of the detectors would be necessary to observe these signals.
The behavior of geodesics in space–times of impulsive gravitational waves has been studied in [4]–[6]. The present work differs from these because we include a light–like shell in the signal and we use a local coordinate system in which the metric tensor is continuous across the history of the signal in space–time whereas in [4]–[6] a distributional metric is used.

2 Fundamental Assumptions and Equations

The history of a light–like signal in a space–time $M$ is a null hypersurface $N$ with the following properties: In a local coordinate system $\{x^\mu\}$ covering both sides of $N$ let $(g_{\mu\nu})$ be the components of the metric tensor of $M$ and let the equation of $N$ be $u(x^\mu) = 0$. The normal to $N$ has covariant components $n_\mu = \alpha^{-1} u_\mu$, with $\alpha$ any function of $x^\mu$ and the comma denoting partial differentiation with respect to $x^\mu$. Since $N$ is null we have

\[ g^{\mu\nu} n_\mu n_\nu = 0, \]  

(2.1)
on $N$. We define a transversal with contravariant components $N^\mu$ on $N$ as any vector on $N$ which is not tangent to $N$. Thus

\[ N^\mu n_\mu = \eta^{-1} \neq 0, \]  

(2.2)for some function $\eta(x^\mu)$. For $N$ to be a singular null hypersurface we take the first derivatives of the metric tensor to jump across $N$ and we furthermore assume that the jumps in these first derivatives take the form [2]

\[ [g_{\mu\nu,\alpha}] = \eta^\gamma_{\mu\nu} n_\alpha, \]  

(2.3)for some tensor $\gamma_{\mu\nu} = \gamma_{\nu\mu}$ defined on $N$. We see from (2.2) and (2.3) that

\[ N^\alpha [g_{\mu\nu,\alpha}] = \gamma_{\mu\nu}. \]  

(2.4)

As a consequence of (2.3) there is a jump in the Christoffel symbols across $N$ given by

\[ \Gamma^\mu_{\alpha\beta} = \frac{\eta}{2} \left( \gamma^\mu_{\alpha} n_\beta + \gamma^\mu_{\beta} n_\alpha - n^\mu n^\alpha \gamma_{\alpha\beta} \right). \]  

(2.5)

It now follows that in general there is a surface stress–energy tensor concentrated on $N$. This is given by the coefficient of $\alpha \delta(u)$ (where $\delta(u)$ is the Dirac delta function which is singular on $u = 0$) in the expression for the Einstein tensor of $M$. The components $S^{\alpha\beta}$ of this stress–energy tensor are given by [2]

\[ 16\pi\eta^{-1} S^{\alpha\beta} = 2 \gamma^{(\alpha} n^{\beta)} - \gamma^\gamma g^{\alpha\beta} - \gamma n^\alpha n^\beta, \]  

(2.6)
with
\[\gamma^\alpha = \gamma^\alpha_\beta n^\beta, \quad \gamma^\dagger = \gamma^\alpha n_\alpha, \quad \gamma = \gamma_{\alpha\beta} g^{\alpha\beta}.\] (2.7)

If \(\{e_a\}\), with \(a = 1, 2, 3\), are three linearly independent vector fields tangential to \(\mathcal{N}\) then on \(\mathcal{N}\) the induced metric tensor has components
\[g_{ab} = g_{\mu\nu} e^\mu_a e^\nu_b,\] (2.8)
and this is, of course, degenerate. We also have
\[\gamma_{ab} = \gamma_{\mu\nu} e^\mu_a e^\nu_b.\] (2.9)

It can be shown \[2\] that \(\gamma_{ab}\) can also be calculated by a cut and paste technique in which the space–time \(M\) is subdivided into \(M^+(u > 0)\) to the future of \(\mathcal{N}\) and \(M^-(u < 0)\) to the past of \(\mathcal{N}\), described in two different coordinate systems, and then re–attached on \(\mathcal{N}\) with matching conditions which preserve the induced metric tensor. An example of this appears in section 4 below.

We can express the surface stress–energy tensor components in an intrinsic form as \[2\]
\[16\pi \eta^{-1} S^{ab} = (g^a_{\mu} n^b \gamma_{\mu\nu} n^d + g^b_{\mu} n^a \gamma_{\mu\nu} n^d) \gamma_{cd} - \gamma^\dagger g_{\mu
u} n^a g^b_{\nu} g^{cd} \gamma_{cd}.\] (2.10)

Here \(g^a_{\mu}\) is a “pseudo”–inverse of \(g_{ab}\) in (2.8) above, defined (non–uniquely) by
\[g^a_{\mu} g_{\mu\nu} = \delta^a_c - \eta n^a N_c,\] (2.11)
and \(N_a = N_\mu e^\mu_a\), while \(n^a\) are the coefficients in the expansion of the normal on the basis \(\{e_a\}\):
\[n^{\mu} = n^a e^\mu_a.\] (2.12)

We note from (2.8) and (2.12) that since \(\{e_a\}\) are tangential to \(\mathcal{N}\) we have \(g_{ab} n^b = 0\). The components \(S^{\alpha\beta}\) of the surface stress–energy tensor in (2.6) are recovered from the intrinsic components (2.10) using
\[S^{\alpha\beta} = S^{ab} e^a_\alpha e^b_\beta.\] (2.13)

If the surface stress–energy tensor (2.10) is written in the form \[2\]
\[\eta^{-1} S^{ab} = \sigma n^a n^b + P g^a_{\mu} g^b_{\nu} + \Pi^{ab},\] (2.14)
with
\[16\pi \sigma = -g^a_{\mu} \gamma_{\mu\nu} n^b \gamma_{cd},\] (2.15)
\[16\pi P = -\gamma^\dagger,\] (2.16)
\[16\pi \Pi^{ab} = (g^a_{\mu} n^b \gamma_{\mu\nu} n^d + g^b_{\mu} n^a \gamma_{\mu\nu} n^d) \gamma_{cd},\] (2.17)
then we can identify $\sigma, P, \Pi^{ab}$ with the relative matter density, isotropic pressure and anisotropic stress respectively of the light–like shell.

As a consequence of our basic assumption (2.3) the Weyl tensor of $M$ has a term proportional to the Dirac delta function $\delta(u)$. The coefficient of $\delta(u)$ can be written as a sum [3], $W_{\kappa\lambda\mu\nu} + M_{\kappa\lambda\mu\nu}$ where $W_{\kappa\lambda\mu\nu}$ represents the gravitational wave part of the light–like signal and $M_{\kappa\lambda\mu\nu}$ is constructed solely from the surface stress–energy tensor of the signal. We note that the vectors $\{N^\mu, e^a_\mu\}$ constitute a basis for the tangent space to $M$ at each point of $\mathcal{N}$. On this basis we find that the components $W_{\kappa\lambda\mu\nu} e^\kappa_a e^\lambda_b e^\mu_c e^\nu_d$ and $W_{\kappa\lambda\mu\nu} e^\kappa_a e^\lambda_b e^\mu_c N^\nu$ of $W_{\kappa\lambda\mu\nu}$ vanish identically while

$$W_{\kappa\lambda\mu\nu} e^\kappa_a N^\lambda_b e^\mu_c N^\nu = -\frac{1}{2} \eta^{-1} \gamma_{ab}, \quad (2.18)$$

with

$$\gamma_{ab} = \gamma_{ab} - \frac{1}{2} g_{cd}^* \gamma_{cd} g_{ab} - 2 \eta n^d \gamma_{d(a} N_{b)} + \eta^2 \gamma^\dagger N_a N_b - \frac{1}{2} \eta^2 \gamma^\dagger (N_\mu N_\nu) g_{ab}. \quad (2.19)$$

This part of $\gamma_{ab}$ satisfies $\gamma_{ab} n^b = 0 = g_{ab}^* \gamma_{ab}$ and thus (i) it does not contribute to the surface stress–energy tensor (2.10) and (ii) it has two independent components. We find that

$$W_{\kappa\lambda\mu\nu} n^\kappa W_{\kappa\lambda\mu\nu} e^\kappa_a = 0. \quad (2.20)$$

Hence $W_{\kappa\lambda\mu\nu}$ is type N in the Petrov classification, with $n^\mu$ as four-fold degenerate principal null direction. The two independent components of $W_{\kappa\lambda\mu\nu}$ represent the two degrees of freedom of polarisation present in this gravitational wave part of the signal. The components $M_{\kappa\lambda\mu\nu}$ expressed on the basis $\{N^\mu, e^a_\mu\}$ are given in terms of the surface stress–energy tensor in [3]. They are in general Petrov type II and may specialise to type III. The $M_{\kappa\lambda\mu\nu}$ vanish if the surface stress–energy is isotropic (i.e. if $S^{\alpha\beta}$ only has the final term in (2.6) non–zero, or equivalently, if $S^{ab}$ only has the final term in (2.10) non–zero) or if the surface stress–energy vanishes. For future reference we note that we have a decomposition of $\gamma_{ab}$ which, following (2.10) and (2.19), can be written as

$$\gamma_{ab} = \tilde{\gamma}_{ab} + \bar{\gamma}_{ab}, \quad (2.21)$$

with

$$\bar{\gamma}_{ab} = 16 \pi \eta \left\{ g_{ac} S^{cd} N_a N_b + g_{bc} S^{cd} N_d N_a - \frac{1}{2} g_{cd} S^{cd} N_a N_b - \frac{1}{2} g_{ab} S^{cd} N_c N_d \right\}. \quad (2.22)$$
3 Principles of Detection

The principles upon which the detection of a light–like signal is carried out are based on the interaction of neighbouring test particles with the signal described by the geodesic deviation equation. In the space–time \( M \) (which we shall consider to be a vacuum space–time except possibly on the null hypersurface \( N \)) we consider a one–parameter family of integral curves of a vector field with components \( T^\mu \) forming a 2–space \( M_2 \). We take \( T^\mu \) to be a unit time–like vector field, so that

\[
g_{\mu\nu} T^\mu T^\nu = -1 ,
\]

and we take the integral curves of \( T^\mu \) to be time–like geodesics with arc length as parameter along them. Thus

\[
\dot{T}^\mu \equiv T^\mu |_\nu T^\nu = 0 ,
\]

with the stroke denoting covariant differentiation with respect to the Levi–Civita connection associated with the metric tensor \( g_{\mu\nu} \). The dot will denote covariant differentiation in the direction of \( T^\mu \) of any tensor defined along the integral curves of \( T^\mu \). Let \( X^\mu \) be an orthogonal connecting vector joining neighbouring integral curves of \( T^\mu \) and tangent to \( M_2 \). Thus \( g_{\mu\nu} T^\mu X^\nu = 0 \) and

\[
\dot{X}^\mu = T^\mu |_\nu X^\nu .
\]

It follows that \( X^\mu \) satisfies the geodesic deviation equation

\[
\ddot{X}^\mu = -R^\mu_{\lambda\sigma\rho} T^\lambda X^\sigma T^\rho ,
\]

where \( R^\mu_{\lambda\sigma\rho} \) are the components of the Riemann tensor of the space–time \( M \). For consistency with the assumed jumps (2.3) of the partial derivatives of the metric tensor components across \( N \), which lead to a possible Dirac delta function in the Riemann tensor of \( M \) (which is singular on \( N \)) we find that we can assume that the partial derivatives of \( T^\mu \) and \( X^\mu \) jump across \( N \) and that these jumps have the forms

\[
[T^\mu, \lambda] = \eta P^\mu n_\lambda , \quad [X^\mu, \lambda] = \eta W^\mu n_\lambda , \quad (3.5)
\]

for some vectors \( P^\mu, W^\mu \) defined on \( N \) (but not necessarily tangential to \( N \)). Let \( \{ E_a \} \) be three vector fields defined along the time–like geodesics tangent to \( T^\mu \) by parallel transporting \( \{ e_a \} \) along these geodesics. Thus

\[
\dot{E}_a^\mu = 0 ,
\]

(3.6)
and on $\mathcal{N}$ we take $E_a = e_a$. The jump in the partial derivatives of $E^\mu_a$ must take the form

$$[E^\mu_{a,\lambda}] = \eta \, F^\mu_a \, n_{\lambda} \, .$$

(3.7)

for some $F^\mu_a$ defined on $\mathcal{N}$.

The behavior of the orthogonal connecting vector $X^\mu$ as it crosses $\mathcal{N}(u = 0)$ from the past ($u < 0$) to the future ($u > 0$) is important. Let $X^\mu_{(0)}$ denote $X^\mu$ evaluated on $\mathcal{N}$ (i.e. where the time–like geodesic, along which $X^\mu$ moves, intersects $\mathcal{N}$). Let $T^\mu_{(0)}$ denote $T^\mu$ evaluated on $\mathcal{N}$. It is convenient to write

$$X^\mu_{(0)} = X^\alpha_{(0)} T^\alpha_{(0)} + X^a_{(0)} e^\mu_a \, ,$$

(3.8)

for some functions $X^\alpha_{(0)}$, $X^a_{(0)}$ evaluated on $\mathcal{N}$. We obtain the following information on the vectors $P^\mu$, $W^\mu$, $F^\mu_a$ appearing in (3.5) and (3.8): From (3.1) we find that

$$\gamma_{\mu\nu} T^\mu_{(0)} T^\nu_{(0)} + 2 \, P^\mu \, T^\mu_{(0)} = 0 \, .$$

(3.9)

From (3.2) we derive

$$\gamma_{\alpha\beta} T^\alpha_{(0)} T^\beta_{(0)} n^\mu = 2 \, n^\alpha T^\alpha_{(0)} \left\{ P^\mu + \gamma^\mu_{\beta} T^\beta_{(0)} \right\} \, ,$$

(3.10)

and this includes (3.9) as a special case. The orthogonality of $X^\mu$, $T^\mu$ leads to

$$\gamma_{\alpha\beta} X^\alpha_{(0)} T^\beta_{(0)} + T^\alpha_{(0)} W_\alpha + X^\alpha_{(0)} P_\alpha = 0 \, .$$

(3.11)

The propagation equation (3.3) gives

$$W^\mu = X^\alpha_{(0)} P^\mu \, .$$

(3.12)

If (3.12) is substituted in (3.11) then the resulting equation can be obtained from (3.10). The expression for $W^\mu$ obtained using (3.10) and (3.12) can be derived directly from the geodesic deviation equation (3.4). Finally (3.6) yields

$$(n_\alpha T^\alpha_{(0)}) F^\mu_a = - \frac{1}{2} \, \gamma^\mu_{\alpha} e^\alpha_\alpha (n_\beta T^\beta_{(0)}) + \frac{1}{2} \, n^\mu \left( \gamma_{\alpha\beta} e^\alpha_\alpha T^\beta_{(0)} \right) \, .$$

(3.13)

As a consequence of (3.3) we deduce, for small $u$, $X^\mu = -X^\mu + \eta \, \alpha^{-1} u \, \vartheta(u) W^\mu \, ,$

(3.14)

with $-X^\mu$ in general dependent on $u$ and such that when $u = 0$, $-X^\mu = X^\mu_{(0)}$. Here $\vartheta(u)$ is the Heaviside step function which is equal to unity if $u > 0$ and equal to zero if $u < 0$. Similar equations to (3.14) can be obtained for $g_{\mu\nu}$, $T^\mu$, $E^\mu_a$ using (2.3), (3.5) and (3.8). It is convenient to calculate $X^\mu$ on
the basis \( \{ E_\mu^a, T^\mu \} \). Its component in the direction of \( T^\mu \) is, of course, zero and its components in the directions \( \{ E_\mu^a \} \) are
\[
X_a = g_{\mu \nu} X^\mu E_\nu^a. \tag{3.15}
\]
Using (3.8)–(3.14) we calculate (3.15) to read, for small \( u > 0 \),
\[
X_a = \left( \tilde{g}_{ab} + \frac{1}{2} \eta \alpha^{-1} u \gamma_{ab} \right) X_b^{(0)} + u V_{(0)a}. \tag{3.16}
\]
with \( \gamma_{ab} \) given by (2.9), \( -V_{(0)a} = d^-X_a/du \) evaluated at \( u = 0 \) and \( \tilde{g}_{ab} \) given by
\[
\tilde{g}_{ab} = g_{ab} + \left( T_{(0)\mu} e_\mu^a \right) \left( T_{(0)\nu} e_\nu^b \right), \tag{3.17}
\]
with \( g_{ab} \) found in (2.8). Although \( g_{ab} \) is degenerate we note that \( \tilde{g}_{ab} \) is non-degenerate. The final term in (3.16) is present due to the relative motion of the test particles before encountering the signal. It would represent the relative displacement for small \( u > 0 \) if no signal were present.

In parallel with (3.8) we can write
\[
X^\mu = X T^\mu + X_a E_\mu^a. \tag{3.18}
\]
Since \( X^\mu, T^\mu \) are orthogonal we have
\[
X = X_a E_\mu^a T^\mu. \tag{3.19}
\]
Substituting (3.18) with (3.19) into (3.15) we obtain
\[
X_a = \tilde{G}_{ab} X^b, \tag{3.20}
\]
with
\[
\tilde{G}_{ab} = g_{\mu \nu} E_\mu^a E_\nu^b + E_\mu^a T_\mu T^\nu E_\nu^b. \tag{3.21}
\]
However on account of the parallel transport (3.2) and (3.6) of \( T^\mu, E_\mu^a \) we have \( \tilde{G}_{ab} = \tilde{g}_{ab} \) and so we can write
\[
X_a = \tilde{g}_{ab} X^a, \tag{3.22}
\]
and this equation can be inverted.

To see the separate effects of the wave and shell parts of the light-like signal on the relative position of the test particles we use in (3.16) the decomposition of \( \gamma_{ab} \) given by (2.21) and (2.22). It is convenient to specialise the triad \( \{ e_a \} \) by choosing \( e_1^\mu = n^\mu \) and \( e_A, \) with \( A = 2, 3, \) orthogonal to \( T_{(0)}^\mu \). It then follows from (2.12) that \( n^a = \delta_1^a \) and that \( g_{ab} = 0 \) except for \( g_{2A} \) with \( A, B = 2, 3. \) We can specialise \( N^\mu \) by taking \( N^\mu = T_{(0)}^\mu \). Thus \( e_A^\mu N_\mu = 0 \)
and \( N_a = 0 \) except for \( N_1 = \eta^{-1} \). It follows from (2.11) that we can have \( g_{\ast}^{ab} = 0 \) except for \( g_{\ast}^{AB} = g^{AB} \). We shall take the \((2,3)-2\)-surface in \( \mathcal{N} \) to be the signal front and we shall assume that before the signal arrives the test particles are in this \((2,3)-2\)-surface. The metric of this space–like 2–surface has components \( g_{AB} \) and since this is a Riemannian 2–surface we can choose coordinates \( \{ x^A \} \) such that
\[
g_{AB} = \eta^{-1} \left( \eta^{-2}, p^{-2}, p^{-2} \right).
\]

It follows from (2.11) that we can have \( g^*_{ab} = 0 \) except for \( g^*_{AB} = g^{AB} \). We shall take the \((2,3)-2\)-surface in \( \mathcal{N} \) to be the signal front and we shall assume that before the signal arrives the test particles are in this \((2,3)-2\)-surface. The metric of this space–like 2–surface has components \( g_{AB} \) and since this is a Riemannian 2–surface we can choose coordinates \( \{ x^A \} \) such that
\[
g_{AB} = \eta^{-1} \left( \eta^{-2}, p^{-2}, p^{-2} \right).
\]

Since \( \hat{\gamma}_{ab} \) satisfies \( \hat{\gamma}_{ab} n^b = 0 = g^*_{ab} \hat{\gamma}_{ab} \) we have now in our special frame \( \hat{\gamma}_{a1} = 0 \) for \( a = 1, 2, 3 \) and

\[
(\hat{\gamma}_{AB}) = \left( \begin{array}{cc}
\hat{\gamma}_{22} & \hat{\gamma}_{23} \\
\hat{\gamma}_{23} & -\hat{\gamma}_{22}
\end{array} \right)
\]

It follows from (2.22) that, in this special frame, \( \bar{\gamma}_{ab} \) is given by
\[
\bar{\gamma}_{11} = -8\pi \eta^{-1} p^{-2} \left( S^{22} + S^{33} \right),
\]
\[
\bar{\gamma}_{1B} = \bar{\gamma}_{B1} = 16\pi \eta^{-1} S^{B1},
\]
\[
\bar{\gamma}_{AB} = -8\pi \eta^{-1} p^{-2} S^{11} \delta_{AB}.
\]

Using (3.22) we can now write (3.16) as follows:
\[
\eta^{-2} X^1 = X_1 = \eta^{-2} X^1_{(0)} + \frac{1}{2} \eta \alpha^{-1} u \bar{\gamma}_{1B} X^B_{(0)} + u - V_{(0)1},
\]
\[
p^{-2} X^A = X_A = p^{-2} X^A_{(0)} + \frac{1}{2} \eta \alpha^{-1} u \gamma_{AB} X^B_{(0)} + u - V_{(0)A}.
\]

We note that if there is a wave component to the signal it does not contribute to \( X^1 \). If initially in physical space the test particles are moving in the \((2,3)-2\)-surface (the signal front) then \( X^1_{(0)} = 0 = V_{(0)1} \). In this case if \( S^{B1} \neq 0 \) then \( X^1 \neq 0 \). Hence we can say that if there is anisotropic stress in the light–like shell then test particles initially moving in the signal front are displaced out of this 2–surface after encountering the signal.

Suppose that there is no anisotropic stress \( \left( S^{1B} = 0 \right) \) in the light–like shell. Now \( X^1_{(0)} = 0 = V_{(0)1} \) implies \( X^1 = 0 \) and we can we can rewrite (3.27) using (3.25) and the decomposition of \( \gamma_{ab} \) in the form
\[
X^A = \left( 1 - 4\pi \alpha^{-1} u S^{11} \right) \left( \delta_{AB} + \frac{1}{2} \eta \alpha^{-1} u p^2 \hat{\gamma}_{AB} \right) X^B_{(0)},
\]
for small \( u \). The factor \( \delta_{AB} + \frac{1}{2} \eta \alpha^{-1} u p^2 \hat{\gamma}_{AB} \), with \( \hat{\gamma}_{AB} \) given in matrix form above, describes the usual distortion effect of the wave part of the signal on the test particles in the signal front (see [8] and the explicit example given at
the end of section 4 below) while the presence of the light–like shell leads to
an overall diminution factor \(1 - 4\pi \alpha^{-1} u S^{11} < 1\). This latter inequality arises
as follows: Since we have chosen \(u\) to increase as one crosses \(N(u = 0)\) from
\(M^-(u < 0)\) to \(M^+(u > 0)\), we must have \(\alpha^{-1} \eta > 0\) on \(N\). The relative energy
density \(\sigma\), given in \((2.14)\), is positive and so using \((2.14)\) in our special frame
we obtain \(\alpha^{-1} S^{11} = \alpha^{-1} \eta \sigma > 0\). Thus we can conclude that for a light–like
shell with no anisotropic stress accompanying a gravitational wave the effect
of the signal on test particles moving in the signal front is to displace them
relative to each other with the usual distortion due to the gravitational wave
diminished by the presence of the light–like shell.

4 Plane Fronted Signal

A plane fronted signal which incorporates a gravitational wave and a light–
like shell with anisotropic stress can easily be constructed propagating through
flat space–time. In this case \(N\) is the history of a null hyperplane. In coordinates covering both sides of \(N\) the line–element of the space–time \(M\) reads in this case
\[
\mathrm{d}s^2 = \left(\mathrm{d}x + \frac{u_+}{F_v} \mathrm{d}F_x\right)^2 + \left(\mathrm{d}y + \frac{u_+}{F_v} \mathrm{d}F_y\right)^2 - 2 \, \mathrm{d}v \left(\mathrm{d}u - \frac{u_+}{F_v} \mathrm{d}F_v\right).
\]
\[
(4.1)
\]
Here \(u_+ = u \vartheta(u)\) with \(\vartheta(u)\) the Heaviside step function as before. The
equation of \(N\) is \(u = 0\). Also \(F = F(x, y, v)\) with partial derivatives indicated
by subscripts and with \(F_v \neq 0\). For the signal with history \(u = 0\) in the space–
time with line-element \((4.1)\) we find that the surface stress–energy tensor is
given by \((2.14)\) with \(\eta = -1\) and
\[
\sigma = \frac{-1}{8\pi} \frac{F_{xx} + F_{yy}}{F_v}, \quad (4.2)
\]
\[
P = \frac{-1}{8\pi} \frac{F_{vv}}{F_v}, \quad (4.3)
\]
\[
\Pi^{12} = \frac{-1}{8\pi} \frac{F_{xx}}{F_v}, \quad \Pi^{13} = -\frac{1}{8\pi} \frac{F_{yy}}{F_v}, \quad (4.4)
\]
with all other components of \(\Pi^{ab}\) vanishing. The coefficient of the delta
function in the Weyl tensor has a Petrov type N part \((W_{\alpha\beta\mu\nu}\) above) given
in Newman–Penrose notation by
\[
\hat{\Psi}_4 = \frac{1}{2 F_v} \left(F_{xx} - F_{yy} - 2 i F_{xy}\right). \quad (4.5)
\]
There is also a Petrov type II part \((M_{\kappa\lambda\mu\nu} \text{ above})\) given in Newman–Penrose notation by
\[
\Psi_2 = \frac{1}{3} \frac{F_{vv}}{F_v}, \quad \Psi_3 = \frac{1}{\sqrt{2}} \frac{F_{xv} - i F_{yv}}{F_v},
\]
which can be written in terms of \(P\) and \(\Pi^{ab}\). It is interesting to note that using (4.3)–(4.6) the line–element \((\text{I.I})\) can be written in terms of \(\sigma\), \(P\), \(\Pi^{ab}\) and \(\Psi_4\).

We mentioned following (2.9) how the space–time model of a light–like signal could be constructed by a cut and paste technique. This can be seen in the case of \((\text{I.I})\) as follows: The line–element \((\text{I.I})\) can be transformed for \(u > 0\) into the manifestly flat form
\[
ds_+^2 = dx_+^2 + dy_+^2 - 2 du_+ dv_+,
\]
by the transformation
\[
x_+ = x + u \frac{F_x}{F_v}, \quad y_+ = y + u \frac{F_y}{F_v}, \quad v_+ = F + \frac{1}{2} u F_v^{-1} \left( F_{xv}^2 + F_{yv}^2 \right), \quad u_+ = u F_v.
\]
Thus \((\text{I.7})\) is the line–element of \(M^+ (u > 0)\). The line–element \((\text{I.I})\) can be trivially transformed for \(u < 0\) into the manifestly flat form
\[
ds_-^2 = dx_-^2 + dy_-^2 - 2 du_- dv_-,
\]
by the transformation
\[
x_- = x, \quad y_- = y, \quad v_- = v, \quad u_- = u.
\]
Thus \((\text{I.12})\) is the line–element of \(M^- (u < 0)\). We see from (4.9)–(4.11) and (4.14) that on \(u = 0\)
\[
x_+ = x_-, \quad y_+ = y_-, \quad v_+ = F(v_-, x_-, y_-),
\]
and these matching conditions leave the induced line–element of \(N(u = 0)\) invariant:
\[
ds_+^2 + dy_+^2 = ds_-^2 + dy_-^2.
\]
We interpret (4.7)–(4.15) as follows: We have subdivided the space–time $M$ into two halves, $M^+(u > 0)$ and $M^-(u < 0)$, and we have re–attached the halves on $N$ with the mapping (4.14) which preserves the line–element induced on $N$. This is an example of the cut and paste procedure mentioned in section 2.

A simple explicit example of (3.28) is provided by specializing (4.1) with the choice

$$F(x, y, v) = v - \frac{a}{2} (x^2 + y^2) + \frac{b}{2} (x^2 - y^2) + c x y ,$$

(4.16)

with $a, b, c$ constants and $a > 0$. This is a homogeneous signal with vanishing stress and relative energy density

$$\sigma = \frac{a}{4 \pi} ,$$

(4.17)

in the light–like shell and with constant amplitude of each component of the gravitational wave since the entries in

$$(\hat{\gamma}_{AB}) = \begin{pmatrix} 2b & 2c \\ 2c & -2b \end{pmatrix} ,$$

are constants. Assuming zero relative velocity for the test particles before the signal arrives, (3.28) reads

$$X^2 = (1 - a u) (1 + b u) \left( X^2_{(0)} + c u X^3_{(0)} \right) ,$$

(4.18)

$$X^3 = (1 - a u) (1 - b u) \left( c u X^2_{(0)} + X^3_{(0)} \right) .$$

(4.19)

Hence particles at rest on the circle \( \left( X^2_{(0)} \right)^2 + \left( X^3_{(0)} \right)^2 = \text{constant} \), before encountering the light–like signal, undergo a small displacement after encountering the signal which is composed of: (1) a rotation through the small angle $c u$ and (2) a deformation into an ellipse with semi axes of lengths $(1 - a u) (1 + b u)$ and $(1 - a u) (1 - b u)$. If the light–like shell is not part of the signal (i.e. if $a = 0$) then the semi axes of the ellipse are not shortened by the factor $(1 - a u)$. If $a = c = 0$ in (4.16) then (4.1) coincides with the homogeneous plane impulsive wave of Penrose [9].

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