ACYLINDRICAL HYPERBOLICITY OF CUBICAL SMALL CANCELLATION GROUPS

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ABSTRACT. We provide an analogue of Strebel’s classification of geodesic triangles in classical $C^*(\frac{1}{6})$ groups for groups given by Wise’s cubical presentations satisfying sufficiently strong metric cubical small cancellation conditions. Using our classification, we give conditions guaranteeing that a cubical small cancellation group is acylindrically hyperbolic.

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1. Introduction

A cubical presentation of a group is a high-dimensional generalization of both a “classical” and a “graphical” presentation of a group in terms of generators and relators. Cubical presentations, and the cubical small cancellation theory developed by Wise [Wis21, Section 3], plays a significant role in geometric group theory, following spectacular solutions of the virtual Haken conjecture by Agol and of Baumslag’s conjecture on one-relator groups with torsion by Wise.

A classical presentation of a group $G$ consists of a wedge $X$ of circles and a collection of combinatorial immersions $Y_i \to X$ of circles so that the presentation complex $X^*$ formed from $X$ by coning off the various $Y_i$ satisfies $\pi_1 X^* \cong G$. The 1-skeleton $\text{Cay}(X^*)$ of the universal cover $\tilde{X}^*$ of $X^*$ is a Cayley graph of $G$ with respect to the generating set implicit in the choice of $X$. A graphical presentation is a natural generalization of this: $X$ is allowed to be an arbitrary graph, and each $Y_i \to X$ becomes an immersion of graphs.

In [Wis21], it is observed that allowing even more flexibility in the choice of $X$ leads to more tractable “presentations”. This leads to the notion of a cubical presentation (see Section 3): $X$ is now a connected nonpositively curved cube complex and each $Y_i$ is a connected nonpositively curved cube complex equipped with a local isometry $Y_i \to X$. The presentation complex $X^*$ is defined analogously, and there is a generalized Cayley graph $\text{Cay}(X^*)$ which is the cubical part of the universal cover of $X^*$. The analogy with classical presentations is: the cube complex $X$ is a kind of “high-dimensional generating set”, the CAT(0) cube complex $\widetilde{X}$ is the “high-dimensional

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tree" taking the place of the free group on the generating set in the classical case, and $\text{Cay}(X^*)$ corresponds to a Cayley graph.

One can then impose cubical small cancellation conditions, in which “generalized overlaps” between the various $Y_i$ (i.e. shadows of $Y_i$ on $Y_j$, as propagated through the intervening cubes) are small in the appropriate metric sense. In this setting, there are powerful tools – specifically, the ladder theorem and the cubical Greendlinger lemma/diagram trichotomy (see Section 3) – that allow one to extract information about a group from a small cancellation cubical presentation. The small cancellation conditions of interest in this paper are the cubical $C'(\alpha)$ conditions, for $\alpha > 0$. These say that $|P| < \alpha \|Y_i\|$ for all $P, i$, where $\|Y_i\|$ denotes the length of a shortest essential (i.e. not homotopic to a constant map) closed combinatorial path in $Y_i$ and $|P|$ is the length of the geodesic piece $P$ in $Y_i$. A piece is, roughly, a path in the “generalized overlap” between distinct elevations to $\widetilde{X}$ of the various $Y_i$, or between such elevations and carriers of hyperplanes in $\widetilde{X}$ that do not cross $\widetilde{Y}_i$. We also use the stronger uniform cubical $C''(\alpha)$ condition, which asks that any geodesic piece $P$ in any $Y_i$ is shorter than $\alpha \|Y_j\|$, for any relator $Y_j$. See Definition 3.12.

Many groups that do not admit classical presentations satisfying strong small cancellation conditions nonetheless admit cubical presentations with these properties. For example, if $G$ is the fundamental group of a nonpositively curved cube complex $X$, then $G$ admits a cubical presentation with no relators, and therefore satisfies arbitrarily strong cubical small cancellation conditions; on the other hand, $G$ does not in general satisfy strong classical small cancellation conditions, as can be seen by considering, for instance, right-angled Artin groups. Later in this introduction, we list more examples of cubical small cancellation groups.

**Classifying triangles.** Our first result is geometric. We classify geodesic triangles in $\text{Cay}(X^*)$ in terms of the disc diagrams that they bound in $\widetilde{X^*}$. This is a cubical analogue of Strebel’s classification of geodesic triangles in $C'(\frac{1}{4})$ groups (Theorem 43 of [Str90]), which says that any geodesic triangle bounds a disc diagram of one of a small number of specific combinatorial types:

**Theorem A (Classification of triangles).** There exists $\alpha > 0$ so that the following holds. Let $X$ be a connected nonpositively curved cube complex, let $\mathcal{I}$ be a (possibly infinite) index set and let $\{Y_i \to X\}_{i \in \mathcal{I}}$ be a set of local isometries of connected nonpositively curved cube complexes. Suppose that the cubical presentation $\langle X \mid \{Y_i\}_{i \in \mathcal{I}} \rangle$ satisfies the cubical $C'(\alpha)$ condition. Let $X^*$ be the presentation complex and $x, y, z$ be $0$–cells of the universal cover $\widetilde{X^*}$. Then there exists a geodesic triangle $\Delta$ in $\widetilde{X^*}$, with corners $x, y, z$, so that $\Delta$ is the boundary path of a disc diagram $D \to X^*$ of one of $9$ types; in particular, $D$ is the union of $3$ padded ladders. Moreover, any other geodesic triangle with corners $x, y, z$ is square-homotopic to such a $\Delta$.

The precise statement is Theorem 3.25, which explains what the “9 types” of disc diagram are (the nondegenerate ones are shown in Figure 1); a padded ladder is a disc diagram of the type in Figure 2; see Definition 3.21.

The $\alpha$ required in our proof is $\frac{1}{44}$. Conceptually, our theorem says that any geodesic triangle bounds a disc diagram which is square-homotopic (fixing corners) to a disc diagram which is a "thickened tripod". Theorem A reduces to existing results in when $X$ or $\mathcal{I}$ are restricted.

- If $\mathcal{I} = \emptyset$, then Theorem A says that any three vertices in a CAT(0) cube complex determine a geodesic tripod, which is a consequence of the fact that CAT(0) cube complexes are simply connected cube complexes whose 1–skeleta are median graphs [Che00].
- If $X$ is a wedge of circles and each $Y_i$ is an immersed circle, then $\langle X \mid \{Y_i\}_{i \in \mathcal{I}} \rangle$ is a classical $C'(\frac{1}{44})$ presentation, and the original Strebel classification for classical $C''(\frac{1}{4})$ groups applies, and follows from Theorem A.
- If $X$ is a wedge of circles and each $Y_i$ is a graph with $Y_i \to X$ an immersion of graphs, then we have instances of graphical presentations. In this setting, there is a classification of
triangles that holds under weaker small cancellation conditions than are required in the
cubical setting. Indeed, the classification of triangles is completely combinatorial, and
Strebel’s proof actually applies in the setting of the \((3, 7)\)–diagrams used by Gruber-Sisto
in their proof of acylindrical hyperbolicity for graphical small cancellation groups \([\text{GS14}];\)
this combinatorial observation was made by Gruber \([\text{Gru15}; \text{Remark 3.11}].\) While the
result about \((3, 7)\)–diagrams suffices for graphical small cancellation groups, one cannot
extend it directly to disk diagrams over cubical presentations since the presence
of squares means that such diagrams need not satisfy the \((3, 7)\) condition. Moreover,
Theorem A includes the result on graphical presentations in their full generality (where
\(X\) is an arbitrary graph), under our small cancellation assumption.

One can construct examples of small cancellation cubical presentations covered by Theorem A
but not by the classical or graphical small cancellation conditions, and one cannot deduce
Theorem A from the corresponding purely cubical or graphical results. Some explicit examples
of cubical small cancellation groups to which the theorem applies are discussed below.

Applications of the classification. One can imagine applications of Theorem A to the thorough
investigation of cubical small cancellation groups analogous to applications of Strebel’s
classification of triangles in classical small cancellation theory, e.g. conformal dimension of the
boundary \([\text{Mac12}],\) growth tightness \([\text{Sam02}],\) the rapid decay property \([\text{AD12}],\) etc.

In this paper, we focus on acylindrical hyperbolicity, inspired by Gruber-Sisto’s result for
graphical small cancellation groups \([\text{GS18}].\) A group \(G\) is \(acylindrically\) \(hyperbolic\) if it admits
a nonelementary acylindrical action on a hyperbolic space (acylindricity generalizes \(uniform\)
properness). Acylindrically hyperbolic groups were defined by Osin \([\text{Osi16}],\) and the notion

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**Figure 1.** The enumerated cases from Theorem 3.25 are shown, clockwise from
the top left: cases \(1, 3, 5, 7, 8, 6, 4, 2.\) Any of the spurs may instead
be exposed squares and vice versa. The ladder case is not shown. In each case,
there is a “central” cell — a 2–cell or 0–cell — such that the diagram is formed
from three padded ladders, each containing the central cell.

**Figure 2.** A padded ladder, see Definition 3.21.
of an acylindrical action goes back at least to the work of Bowditch [Bow08]. The notion of acylindrical hyperbolicity unifies several generalizations of relative hyperbolicity [BF02 DGO17 Ham08 Sis18] and provides a class of groups with many strong properties: if $G$ is acylindrically hyperbolic, then $G$ is SQ-universal, contains normal free subgroups, and is $C^*$--simple if and only if it has no finite normal subgroup [DGO17]; $G$ contains Morse elements and thus all asymptotic cones of $G$ contain cut-points [Sis14]; the bounded cohomology of $G$ has infinite dimension in dimensions $2$ [HO13] and $3$ [FPS15]; every commensurating endomorphism of $G$ is an inner automorphism [AMS16], etc.

The class of acylindrically hyperbolic groups is now known to be vast, see e.g. [Bow08 Osi16 DGO17 MO15 Osi15 BF10 GS18 BHS17 PS17]. Our second result adds Wise’s cubical small cancellation groups to this notable list. To state it, we need some preliminary discussion. We refer to Section 3 for the exact definition of a cubical presentation, and to Definition 3.12 for the cubical small cancellation conditions we mention shortly.

**Definition 1.1 (Minimal cubical presentation).** We say that $\langle X \mid \{Y_i\}_{i \in I}\rangle$ is a minimal cubical presentation if for all $i \in I$, the following holds. Let $\hat{Y}_i \to Y_i$ be the universal cover, and let $\hat{Y}_i \to \hat{X}$ be a lift of $\hat{Y}_i \to Y_i \to X$. Then $\text{Stab}_{\pi_1}(\hat{Y}_i)$ is conjugate in $\pi_1(X)$ to $\pi_1(Y_i)$ (rather than to a proper supergroup of $\pi_1(Y_i)$). For classical presentations, being minimal corresponds to the situation where relations are not proper powers.

Let $\langle X \mid \{Y_i\}_{i \in I}\rangle$ be a cubical presentation. Suppose for each $i \in I$ that $\hat{Y}_i \to Y_i$ is a finite connected regular cover. Then the maps $\hat{Y}_i \to Y_i \to X$ give a new cubical presentation $\langle X \mid \{\hat{Y}_i\}_{i \in I}\rangle$. Suppose that $\langle X \mid \{Y_i\}_{i \in I}\rangle$ satisfies the uniform $C''(\alpha)$ condition for some $\alpha \in [0, 1)$. Let $\beta = \frac{\|Y_i\|}{\|\hat{Y}_i\|}$. Then, as can be seen from Definition 3.12, the cubical presentation $\langle X \mid \{\hat{Y}_i\}_{i \in I}\rangle$ satisfies the uniform $C''(\alpha\beta)$ condition.

**Theorem B (Acyllindrical hyperbolicity from cubical small cancellation).** Let $X$ be a compact connected nonpositively curved cube complex and $\{Y_i \to X\}_{i \in I}$ be a collection of local isometries with each $Y_i$ a compact connected nonpositively curved cube complex.

(I) There exists a constant $L_0 = L_0(X)$ so that the following holds. Let $\alpha_0 = \min\{\frac{1}{141}, \frac{1}{L_0}\}$ and let $\alpha \in [0, \alpha_0]$. Let $\langle X \mid \{Y_i\}_{i \in I}\rangle$ be a minimal cubical presentation satisfying the uniform $C''(\alpha)$ condition, and let $G$ be the group given by this cubical presentation. Then one of the following holds:

(a) $G$ is finite or two-ended.

(b) The universal cover $\hat{X}$ of $X$ contains a convex $\pi_1(X)$--invariant subcomplex splitting as a product of unbounded cube complexes, each $Y_i$ is contractible, and $G \cong \pi_1(X)$.

(c) $G$ is acylindrically hyperbolic.

(II) Let $\langle X \mid \{Y_i\}_{i \in I}\rangle$ be a cubical presentation satisfying the uniform $C''(\beta)$ condition for some $\beta < 1$. Then there exists $L_1 = L_1(X, \{Y_i \to X\}_{i \in I})$ such that the following holds. For each $i \in I$, let $\hat{Y}_i \to Y_i$ be a finite connected regular cover such that the cubical presentation $\langle X \mid \{\hat{Y}_i\}_{i \in I}\rangle$ satisfies the uniform $C''(\alpha)$ condition for some $\alpha < \min\{\frac{1}{141}, \frac{1}{\beta}\}$. Let $G$ denote the associated quotient of $\pi_1(X)$. Then $G \cong \pi_1(X)/\langle\langle\{\pi_1(\hat{Y}_i) : i \in I\}\rangle\rangle$ satisfies one of (Ia), (Ib), or (Ic).

In particular, $G$ satisfies one of the given conclusions whenever $\inf_{i \in I} \|\hat{Y}_i\|/\|Y_i\|$ is sufficiently large, in terms of $X$ and the $Y_i \to X$.

The first statement is attractive since the small cancellation condition needed to get acylindrical hyperbolicity depends only on $X$. The second statement is extremely useful in practice: one only needs to find a cubical presentation satisfying the uniform $C''(\beta)$ condition for some $\beta < 1$
(but not necessarily satisfying e.g. the uniform $C^\infty(\frac{1}{144})$ condition), with each $\pi_1 Y_i$ residually finite, and then one can construct infinitely many acylindrically hyperbolic quotients of $\pi_1 X$.

**Remark 1.2** (The constants $L_0$ and $L_1$). We give here a brief preview on the computation of these constants. For simplicity, assume in this remark that $\tilde{X}$ does not contain a proper $\pi_1 X$-invariant convex subcomplex. The general case is handled in the proof of Theorem [B] by replacing $\tilde{X}$ by its Caprace-Sageev essential core. That is, in the general case, the constants $L_0, L_1$ are computed in the same way, except using the essential core of $\tilde{X}$ instead of $\tilde{X}$.

The constant $L_0(X)$ is defined as follows. Let $\mathcal{C} \tilde{X}$ be the contact graph of $\tilde{X}$, which is the intersection graph of the set of hyperplane carriers; this was defined, and shown to be hyperbolic, in $\text{[Hag14]}$. Let $\mathcal{L}_0$ be the set of $\tilde{g} \in \pi_1 X$ that act on $\mathcal{C} \tilde{X}$ as loxodromic isometries. From results in $\text{[CS11]}$ and $\text{[Hag13]}$, $\mathcal{L}_0 \neq \emptyset$ if and only if $\tilde{X}$ does not split as the product of unbounded CAT(0) cube complexes. We can take $L_0(X) = \inf_{\tilde{g} \in \mathcal{L}_0} \inf_{\tilde{a} \in \tilde{X}} d_{\tilde{X}}((\tilde{a}, \tilde{g} \tilde{a}))$. So, $L_0(X) < \infty$ unless $\tilde{X}$ is a product. Moreover, since we are considering combinatorial translation length, $L_0(X) \geq 1$. We need minimality of the presentation only in the proof of Lemma [6.11], where we need to know that an element $\tilde{g} \in \mathcal{L}_0$ of minimal translation length cannot stabilise some $\tilde{Y}_i$. Minimality guarantees that $\tilde{g} \in \text{Stab}_{\pi_1 X}(\tilde{Y}_i)$ is conjugate into $\pi_1 Y_i$; our small cancellation condition, which requires $\|Y_i\| > 7L_0$, rules this out.

In case [I], we examine the elements $\tilde{g} \in \mathcal{L}_0$ that stay loxodromic on the hyperbolic graph obtained from the contact graph by coning off each $\tilde{Y}_i$. Lemma [6.8] says that either $\pi_1 X$ stabilises some $\tilde{Y}_i$ (leading to conclusion [Ia]), or the set $\mathcal{L}_1$ of such $\tilde{g}$ is nonempty. We then take $L_1$ to be the minimal translation length of such $\tilde{g}$ on $\tilde{X}$. Note that $L_1$ depends on $X$ and the initial choices $Y_i \to X$, but the latter dependence is only via the action of $\pi_1 X$ on $\tilde{X}$ and the collection of $\tilde{Y}_i \subseteq \tilde{X}$. So $L_1$ does not depend on the further covers $\tilde{Y}_i \to Y_i$. The constant $L_1$ is used in one spot, in Lemma [6.11] to ensure that $\tilde{g} \in \mathcal{L}_1$ does not stabilise any $\tilde{Y}_i$, and can be chosen with translation length less than $\|\tilde{Y}_i\|/7$.

**Remark 1.3** (The purely cubical case). In the “purely cubical” case, where $I = \emptyset$, Theorem [B] says that the fundamental group of a compact nonpositively curved cube complex $X$ is either elementary or acylindrically hyperbolic, unless the universal cover of $X$ contains a $\pi_1 X$-invariant convex subcomplex decomposing as a product with unbounded factors. This also follows from various results in the literature. In $\text{[BHS17]}$, it is shown that, under natural extra hypotheses, the action of $\pi_1 X$ on the contact graph $\mathcal{C} \tilde{X}$ is acylindrical, and $\mathcal{C} \tilde{X}$ is unbounded in the absence of an invariant product subcomplex. Even without the extra hypotheses, any $g \in \pi_1 X$ acting loxodromically on $\mathcal{C} \tilde{X}$ actually acts as a WPD element in the sense of $\text{[BF02]}$, by $\text{[BHS17]}$ Proposition 5.1. Together with results in $\text{[Hag13]}$ characterizing the loxodromic isometries of $\mathcal{C} \tilde{X}$, and a result of Osin connecting WPD elements to acylindricity $\text{[Osi16]}$, this implies the virtually cyclic/product/acylindrically hyperbolic trichotomy of Theorem [B] in the case where $I = \emptyset$. This trichotomy, in the purely cubical case, also follows from the Caprace-Sageev rank rigidity theorem $\text{[CS11]}$ and general results about groups acting on CAT(0) spaces and containing rank one elements $\text{[Osi16]}$ $\text{[Sis18]}$.

In the case where $I = \emptyset$, our proof of Theorem [B] reduces to a proof that elements that are loxodromic on the contact graph act as WPD elements, along similar lines to the proof in $\text{[BHS17]}$.

**Remark 1.4** (Classical and graphical cases). The comparison with the acylindrical hyperbolicity result of Gruber-Sisto, for graphical small cancellation groups (as formulated in $\text{[Gru15]}$), is interesting; our results about cubical small cancellation groups do not follow from corresponding results about graphical small cancellation presentations, since the latter viewpoint does not fully account for high-dimensional cubes.
On the other hand, restricting Theorems $\mathbb{A}$ and $\mathbb{B}$ to the case where $\dim X = 1$ and each $Y_i$ is a graph, one does not reprove the results of [GS18] or [Str90] in full generality, since the cubical $C'(\frac{1}{144})$ condition is more restrictive than the conditions needed in the classical and graphical cases (which are the classical $C'(\frac{1}{3})$ and the graphical $Gr(7)$ conditions, respectively).

**Remark 1.5** (Acylindrical action on a quasi-tree). Combining Theorem $\mathbb{B}$ with a recent result of Balasubramanya [Bal17] shows that any group covered by Theorem $\mathbb{B}$ either satisfies one of the first two conclusions or acts acylindrically and non-elementarily on a quasi-tree.

**On the proof of Theorem $\mathbb{B}$**. Theorem $\mathbb{B}$ is proved roughly as follows. First, we create a hyperbolic $\pi_1 X^*$–space $H$ by coning off each hyperplane carrier $N(H)$ and each relator $Y_i$ in the generalized Cayley graph $\text{Cay}(X^*)$. This procedure is a common generalization of the constructions used in the purely cubical case and in the graphical case from [GST8].

We apply Theorem $\mathbb{A}$ to show that suitable $g \in \pi_1 X^*$ acting loxodromically on $H$ is a WPD isometry of $H$; Osin’s theorem says that $\pi_1 X^*$ is acylindrically hyperbolic or virtually cyclic. This is done in Section $\mathbb{A}$. Hyperbolicity of $H$ also uses Theorem $\mathbb{A}$ and is checked in Section $\mathbb{5}$.

It remains to find suitable loxodromic isometries of $H$. This is done in Section $\mathbb{6}$. First, we show that if $\tilde{g} \in \pi_1 X$ acts loxodromically on the contact graph $CX$, and axes of $\tilde{g}$ have suitably bounded interaction with elevations of relators $Y_i$, then the image $g \in \pi_1 X^*$ of $\tilde{g}$ is loxodromic on $H$. This is accomplished in Lemma $6.4$, Lemma $6.6$, and Lemma $6.7$.

- In the first lemma, we show that $\tilde{g}$ must remain loxodromic on the graph obtained from the contact graph by coning off the various subgraphs associated to the various $\tilde{Y}_i \subseteq \tilde{X}$, unless $\tilde{g}$ has a power conjugate into some $\pi_1 Y_i$.
- In the second lemma, we show that if $\tilde{g}$ has suitably bounded interaction with elevations of relators, then $g$ has infinite order and $\tilde{g}$ is loxodromic on the coned-off contact graph.
- In the third lemma, we deduce that $g$ must be loxodromic on $H$. The idea is to use disc diagrams in $\tilde{X}^*$ to show that axes of $\tilde{g}$ in the coned-off contact graph project to quasigeodesic axes in $H$ for $g$. Since $\tilde{g}$ was loxodromic upstairs, $g$ must therefore be loxodromic downstairs.

The “suitably bounded interaction” hypothesis is made precise in Definition $6.5$. $\tilde{g}$ must be *asytolic*. Up to this point, we only require the uniform $C''(\frac{1}{144})$ condition.

In Lemma $6.11$, we show that under either of the hypotheses in Theorem $\mathbb{3}$ we can find an asystolic element. The remainder of the proof is essentially an application of results in [CST11] and [Hag13] characterising when $L_0 \neq \emptyset$.

**Remark 1.6** (No proof by cubulation). Cubical small cancellation theory is partly motivated by the fact that groups satisfying strong classical small cancellation conditions act nicely on CAT(0) cube complexes [Wis04, Theorem 1.2]. This generalizes in various ways to cubical presentations: if $\langle X \mid Y_i \rangle_{i \in I}$ satisfies the generalized $B(6)$ condition, one can often cubulate the corresponding group; see, for instance, [Wis21, Corollary 5.45].

It is tempting to try to prove Theorem $\mathbb{B}$ using this approach, together with the above-mentioned results about acylindrical hyperbolicity of groups acting on cube complexes. However, there are various problems with this approach. For example, the generalized $B(6)$ condition requires each $Y_i$ to have a wallspace structure, compatible with the local isometry $Y_i \to X$, generalizing the wallspace structure on a circle in which each wall is a pair of antipodal points. (Compare with the *lacunary walling* condition on graphical presentations from [AOT14].)

No cubical $C'(\alpha)$ small cancellation condition implies the generalized $B(6)$ condition, and indeed there are groups that are covered by Theorem $\mathbb{B}$ but which do not admit an action on a CAT(0) cube complex with no global fixed point. This can already be seen in the 1–dimensional case: Proposition 7.1 of [OW07] yields, for any $\alpha > 0$, a graphical presentation $\langle X \mid Y \rangle$, where $X$ is a graph and $Y \to X$ an immersed graph, satisfying the graphical (hence 1–dimensional cubical)
Remark 1.7 (Relationship with rotating families). One can imagine an alternate approach to Theorem \[\text{B}\] using tools from \[\text{DGO17}\]. The idea would be to consider the action of \(\pi_1X\) on some graph quasi-isometric to the contact graph \(C\bar{X}\), and consider the images of the various \(\bar{Y}_i\) under projection to \(C\bar{X}\). In cases where \(\pi_1X\) acts acylindrically on \(C\bar{X}\), it is not hard to deduce from the small cancellation conditions that \(\pi_1X\) acts acylindrically on the graph \(\hat{H}\) obtained by coning off the \(\bar{Y}_i\)-subgraphs of \(C\bar{X}\), using results in \[\text{DGO17}\].

However, we see the following challenge in a possible proof using the “geometric small cancellation” setup from \[\text{DGO17}\]. Some hyperplane \(H\) crossing \(\bar{Y}_i\) can have unbounded intersection with \(\bar{Y}_i\), because the definition of a wall-piece excludes the case of hyperplanes that actually intersect \(\bar{Y}_i\) (see Definition \[\text{3.7}\]). So, \(\text{Stab}_{\pi_1X}(\bar{Y}_i)\) can contain elements of \(\text{Stab}_{\pi_1X}(H)\), which is a vertex-stabiliser in \(C\bar{X}\). So, \(\text{Stab}_{\pi_1X}(\bar{Y}_i)\) need not be purely loxodromic on \(C\bar{X}\). At the same time, we suspect it might still be possible to use a variant of \(\hat{H}\) where the \(\text{Stab}_{\pi_1X}(\bar{Y}_i)\) subgroups act as a rotating family and allow one to use \[\text{DGO17}\] to conclude that the action of \(\pi_1X^*\) on this variant graph has WPD elements.

An explicit uniform \(C''(\frac{1}{144})\) example where \(\text{Stab}_{\pi_1X}(\bar{Y})\) is not purely loxodromic: let \(X\) be a compact nonpositively curved cube complex so that some unbounded hyperplane \(H\) of \(\bar{X}\) has uniformly bounded coarse intersection with all other hyperplanes (including translates of \(H\)). Let \(\bar{Y} = H\), let \(\bar{Y} = \text{Stab}_{\pi_1X}(H) \backslash N(H)\), and let \(Y \to \bar{Y}\) be a regular cover of sufficiently large girth. Then \(\langle X \mid Y \to X \rangle\) is a cubical \(C''(\frac{1}{144})\) presentation, but \(\pi_1Y\) fixes a point in \(C\bar{X}\). For example, one can take a closed hyperbolic surface \(S\), take a finite filling family of simple closed curves, no two of which are parallel, and take \(\bar{X}\) to be the cube complex dual to the system of walls on \(\bar{S}\) obtained by lifting the curves. Then \(H\) corresponds to an elevation of one of the \(Y\) curves, and \(Y\) corresponds to a very long circle covering that curve.

Examples of cubical small cancellation groups. We list here some examples of cubical small cancellation groups to which Theorem \[\text{A}\] and Theorem \[\text{B}\] apply. We have earlier mentioned classical and graphical small cancellation presentations to which our results apply, as well as the purely cubical case where \(X\) is a nonpositively curved cube complex and \(\mathcal{I} = \emptyset\).

1. Classical/RAAG hybrid: let \(X\) be the Salvetti complex of a right-angled Artin group \(A\), with presentation graph \(\mathcal{G}\) (this means, by definition, that \(A\) has a generator for each vertex of \(\mathcal{G}\), with two generators commuting if and only if the corresponding vertices are adjacent in \(\mathcal{G}\)), and let \(\{g_i\}_{i \in \mathcal{I}}\) be a finite collection of independent elements, none of which is supported on a proper join in \(\mathcal{G}\) (i.e. each \(g_i\) is a rank-one isometry of \(\bar{X}\)). So, each \(\langle g_i \rangle\) has a convex cocompact core \(\bar{Y}_i\) in \(\bar{X}\). Then for each \(i\) there exists \(n_i > 0\) so that, letting \(\bar{Y}_i = \langle g_i^{n_i} \rangle \backslash \bar{Y}_i\), the cubical presentation \(\langle X \mid \{Y_i\}_{i \in \mathcal{I}} \rangle\) is a \(C''(\frac{1}{144})\) presentation. We will see below that \(L_0(X) = 2\) in this case. So, if no \(g_i\) is a proper power, Theorem \[\text{B}\] shows that \(\langle X \mid \{Y_i\}_{i \in \mathcal{I}} \rangle\) presents a quotient of \(A\) which is finite, virtually \(\mathbb{Z}\), or acylindrically hyperbolic. Even if the \(g_i\) are proper powers, then taking \(n_i\) sufficiently large (in terms of the \(g_i\)), we obtain the same conclusion from Theorem \[\text{B}\].

Furthermore, instead of cyclic subgroups, one could use appropriately chosen purely loxodromic subgroups as described in \[\text{KMT17}\], which are necessarily free.

2. More generally, let \(X\) be a compact connected nonpositively curved cube complex. Let \(\{Y_i \to X\}_{i \in \mathcal{I}}\) be a collection of local isometries with each \(Y_i\) a compact connected nonpositively curved cube complex so that the resulting cubical presentation \(\langle X \mid \{Y_i\}_{i \in \mathcal{I}} \rangle\) satisfies the (uniform) cubical \(C''(\alpha)\) condition for some \(\alpha > 0\). Suppose that each \(Y_i\) has a residually finite fundamental group. Thus, for any \(n \in \mathbb{N}\), there is a finite regular
cover $\hat{Y}_i \to Y_i$ with $\|\hat{Y}_i\| \geq n\|Y_i\|$. Thus, the related cubical presentation $\langle X \mid \{\hat{Y}_i\}_{i \in I} \rangle$ satisfies the (uniform) cubical $C^m(\frac{a}{b})$ condition.

If $\langle X \mid \{Y'_i\}_{i \in I} \rangle$ was a minimal presentation, $n$ can be chosen in terms of $X$ only when applying Theorem [B]. If not, then $n$ must be chosen sufficiently large in terms of $X$ and the initial $Y_i$.

(3) Given letters $x, y$ and $m \geq 1$, let $(x, y)^m$ denote the first half of the word $(xy)^m$. Consider the Artin group $A = \langle a_1, a_2, \ldots, a_n \mid (a_i, a_j)^{m_{ij}} = (a_j, a_i)^{m_{ij}} \text{ whenever } i \neq j \rangle$.

(Note that we follow the convention of letting $m_{ij} = \infty$ to indicate that there is no relation between $a_i, a_j$.) Let $\hat{A} = \langle a_1, a_2, \ldots, a_n \mid [a_i, a_j] \text{ whenever } m_{ij} = 2 \rangle$ be the underlying right-angled Artin group, $X$ be its Salvetti complex, and $G$ be its presentation graph (so, a graph with a vertex for each $a_i$ and with an edge from $a_i$ to $a_j$ when $m_{ij} = 2$).

That is, $A$ is a quotient of $\hat{A}$ obtained by adding the relations $(a_i, a_j)^{m_{ij}} = (a_j, a_i)^{m_{ij}}$ whenever $m_{ij} \not\in \{2, \infty\}$. Suppose that $G$ does not decompose as a nontrivial join. Moreover, suppose that for all $i, j$ with $m_{ij} \not\in \{2, \infty\}$, the element $g_{ij} = (a_i, a_j)^{m_{ij}}(a_j, a_i)^{-m_{ij}}$ of $\hat{A}$, is not supported in a join in $G$. Then $g_{ij}$ is a rank-one isometry of $X$.

Hence, there is a convex subcomplex $Y_{ij}$ of $X$ on which $\langle g_{ij} \rangle = \text{Stab}_{\hat{A}}(\hat{Y}_{ij})$ acts cocompactly, and which is just the convex hull of a combinatorial $g_{ij}$-axis. Let $Y_{ij}$ be the quotient of $\hat{Y}_{ij}$ by the $\langle g_{ij} \rangle$-action, so that $\langle X \mid Y_{ij} \text{ whenever } 2 < m_{ij} \leq \infty \rangle$ is a minimal cubical presentation for the Artin group $A$. Clearly, $Y_{ij}$ has systole $2m_{ij}$.

If $P$ is a cone-piece between $\hat{Y}_{ij}$ and $\hat{Y}_{kl}$, then $|P| = 1$. Since $g_{ij}$ is not supported in a join, nontrivial wall-pieces have length 1.

Finally, $L_0(X) = 2$, since $\hat{A}$ contains words of length 2 that represent rank-one elements not stabilising hyperplanes.

Suppose that $A$ satisfies the following: for all $i \neq j$, either $m_{ij} = 2$ or $m_{ij} = \infty$ or $m_{ij} > 72$ and no generator commutes with $a_i$ and $a_j$. Then the above cubical presentation for $A$ is $C^m(\frac{1}{144})$ (and hence $C^m(\frac{1}{776})$). So Theorem [B] implies $A$ is virtually cyclic or acylindrically hyperbolic.

There is a related recipe in Section 20 of [Wis21] for building $C(6)$ cubical presentations of Artin groups (cf. [ASS]) but it is harder to see when these are $C^m(\frac{1}{144})$.

Outline of the paper. In Section 2 we recall background on acylindrical hyperbolicity and WPD elements. Section 3 contains a discussion of cubical presentations, disc diagrams, and the parts of cubical small cancellation theory needed in the proof of the classification of triangles, Theorem [A], which also occurs in this section. The proof uses the theory developed in [Wis21].

In Section 4 we give a list of conditions on a cubical small cancellation group $G$ acting on a hyperbolic space $\mathcal{H}$ sufficient to ensure that $G$ contains an element $g$ acting on $\mathcal{H}$ as a WPD element. Specifically, we use Theorem [A] to show that any $g \in G$ acting loxodromically on a space $\mathcal{H}$ satisfying the given conditions acts as a WPD element. In Section 5 we produce such a space $\mathcal{H}$, formed from a generalized Cayley graph $\text{Cay}(X^*)$ by coning off both the relators and the hyperplane carriers. Theorem [A] is also used here to check that $\mathcal{H}$ is hyperbolic. Finally, in Section 6 we prove Theorem [B].

We assume basic knowledge of CAT(0) and nonpositively curved cube complexes and cubical presentations; we refer the reader to [Wis21] for most of the background. Most of the material that we will need from [Wis21] is restated below, although for some more technical points we will refer the reader to [Wis21] with various citations.

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2. Acylindrical hyperbolicity and WPD elements

The notion of an *acylindrically hyperbolic group* was defined in [Osi16], as follows:

**Definition 2.1** (Acylindrical action, acylindrical hyperbolicity). Let \((X,d)\) be a metric space and let \(G\) act on \(X\) by isometries. Then the action is *acylindrical* if for each \(\epsilon \geq 0\), there exists \(R \geq 0\) and \(N \in \mathbb{N}\) so that for all \(x,y \in X\) for which \(d(x,y) \geq R\), we have

\[
|\{g \in G : d(x,gx) \leq \epsilon, d(y,gy) \leq \epsilon\}| \leq N.
\]

Let \(X\) be Gromov-hyperbolic and let \(G\) act by isometries on \(X\). The action of \(G\) is *elementary* if the limit set of \(G\) in \(\partial X\) has at most two points. If \(G\) acts non-elementarily and acylindrically on a hyperbolic space, then \(G\) is *acylindrically hyperbolic*.

When \(G\) is a cubical small cancellation group, we will construct an explicit action of \(G\) on a hyperbolic space \(\mathcal{H}\), but this will not necessarily be the action that witnesses acylindrical hyperbolicity. Instead, the action will be such that \(G\) contains a WPD isometry of \(\mathcal{H}\).

**Definition 2.2** (WPD element [BF02]). Let \(G\) act by isometries on the space \(X\). Then \(h \in G\) is a *WPD element* if for each \(\epsilon > 0\) and each \(x \in X\), there exists \(M \in \mathbb{N}\) so that

\[
|\{g \in G : d(x,gx) \leq \epsilon, d(h^M x, gh^M x) \leq \epsilon\}| < \infty.
\]

In [Osi16], Osin showed that if \(G\) is not virtually cyclic and acts on a hyperbolic space \(\mathcal{H}\), and some \(g \in G\) acts on \(\mathcal{H}\) as a loxodromic WPD element, then \(G\) is acylindrically hyperbolic. This is instrumental in the proof of Theorem B.

3. Triangles in cubical small cancellation groups

In this section, \(X\) denotes a connected nonpositively curved cube complex with universal cover \(\tilde{X}\). When doing geometry in \(\tilde{X}\), we never use the CAT(0) metric and instead only use the usual graph metric on \(\tilde{X}^{(1)}\) in which each 1–cube has length 1 and a combinatorial path is geodesic if and only if it contains at most one edge intersecting each hyperplane of \(\tilde{X}\).

**Notation 3.1** (Carriers and neighbourhoods). Let \(X\) be a nonpositively curved cube complex and let \(H \to X\) be an immersed hyperplane. The *(abstract)* carrier of \(H\) is \(\mathcal{N}(H) = H \times [-\frac{1}{2}, \frac{1}{2}]\), equipped with the product cubical structure, where \(H\) has a nonpositively curved cubical structure coming from \(X\) and \([-\frac{1}{2}, \frac{1}{2}]\) is a 1–cube. The map \(H \to X\) extends to a cubical map \(\mathcal{N}(H) \to X\), see e.g. [Wis21] Section 2.g]. When \(X\) is CAT(0), this map is an embedding whose image is a convex subcomplex \(\mathcal{N}(H)\), the *carrier* of the hyperplane \(H\). In this case, \(\mathcal{N}(H)\) is just the union of the closed cubes that intersect \(H\).

Given an arbitrary metric space \(M\) and a subspace \(H\), we denote by \(\mathcal{N}_r(H)\) the closed \(r\)–neighbourhood of \(H\) in \(M\). The similarity in notation is justified by the fact that, when \(X\) is a CAT(0) cube complex and \(H\) is a hyperplane, \(\mathcal{N}(H)^{(0)} = \mathcal{N}_{\frac{1}{2}}(H \cap \tilde{X}^{(1)})^{(0)}\).

3.1. Cubical presentations. We fix a (possibly infinite) index set \(I\), and for each \(i \in I\), let \(Y_i \to X\) be a local isometry of connected nonpositively curved cube complexes. Each \(Y_i \to X\) is a \(\pi_1\)-injection [Wis21] Corollary 2.12).

Following [Wis21], the associated *cubical presentation* is \(\langle X | \{Y_i\}_{i \in I}\rangle\) and the corresponding *cubical presentation complex* \(X^*\) is formed as follows. For each \(i \in I\), let \(C(Y_i)\) be the relator on \(Y_i\), i.e. the space formed from \(Y_i \times [0,1]\) by collapsing \(Y_i \times \{1\}\) to a point. This space has an obvious cell-structure so that \(Y_i \simeq Y_i \times \{0\} \hookrightarrow C(Y_i)\) is a combinatorial embedding. For
each \( i \in \mathcal{I} \), we attach \( C(Y_i) \) to \( X \) along \( Y_i \times \{0\} \) using the above local isometry. The resulting complex is \( X^* \). The group of our interest is defined by \( G = \pi_1 X^* \).

We say that \( \langle X \mid \{ Y_i \}_{i \in \mathcal{I}} \rangle \) is a cubical presentation for \( G \). We refer to the complexes \( Y_i \), or to the local isometries \( Y_i \to X \), as relators of such a cubical presentation.

The universal cover \( \tilde{X}^* \) of \( X^* \) is a nonpositively curved cube complex with cones attached. Let \( \text{Cay}(X^*) \) be the part of \( \tilde{X}^* \) consisting only of cubes (i.e. the complement of the open cones). This is the generalized Cayley graph of \( G \) with the given cubical presentation. Note that there are covering maps \( \tilde{X} \to \text{Cay}(X^*) \to X \); the generalized Cayley graph is the nonpositively curved cube complex obtained by taking the cover of \( X \) corresponding to the kernel of \( \pi_1 X \to \pi_1 X^* \).

**Remark 3.2.** (Classical and graphical presentations) If \( X \) is a wedge of circles and each \( Y_i \) is an immersed combinatorial circle, then \( \langle X \mid \{ Y_i \}_{i \in \mathcal{I}} \rangle \) is a group presentation in the usual sense (each \( C(Y_i) \) is a disc) and \( \text{Cay}(X^*) \) is the associated Cayley graph of \( G \). As mentioned in [Wis21, Examples 3.8], if \( X \) is a graph and each \( Y_i \) is an immersed graph, then the above cubical presentation is a graphical presentation in the sense of [RS87, Gro03, Oll06].

**Remark 3.3** (Elevations). The local isometries \( Y_i \to X \) lift to local isometries \( \tilde{Y}_i \to \text{Cay}(X^*) \) (in fact, under the small cancellation conditions we shall soon be assuming, the latter maps are embeddings [Wis21 Section 4]). We use the term elevation to refer to a lift \( \tilde{Y}_i \to \tilde{X} \) of the map \( \tilde{Y}_i \to Y_i \to X \), where \( \tilde{Y}_i \to Y_i \) is the universal covering map. Since \( Y_i \to X \) is a local isometry, it is \( \pi_1 \)-injective and \( \tilde{Y}_i \to \tilde{X} \) is a combinatorial embedding with convex image.

### 3.2. Cubical small cancellation conditions

We now review background about cubical small cancellation theory, following [Wis21]. By a trivial path in a cube complex, we mean a combinatorial path that does not traverse any edges, i.e. one whose image is a single vertex. Throughout the paper, all paths are combinatorial and, if \( P \) is a combinatorial path, then \( |P| \) denotes its length; in particular, \( |P| = 0 \) when \( P \) is trivial.

**Definition 3.4** (Wall-projection). Let \( \tilde{X} \) be a CAT(0) cube complex, and let \( U, V \) be convex subcomplexes. Let \( \text{Proj}(U \to V) \) be the subcomplex of \( V \) defined as follows.

First, two cubes \( c, c' \) of \( \tilde{X} \) are parallel if they intersect exactly the same hyperplanes. In particular, any two 0–cubes are parallel.

A cube \( \bar{u} \) of \( V \) belongs to \( \text{Proj}(U \to V) \) if and only if there is a cube \( u \) of \( U \) such that \( \bar{u} \) and \( u \) are parallel, and \( \bar{u} \) is the nearest cube of \( V \) that is parallel to \( u \) [Wis21, Definition 3.4].

**Remark 3.5** (Gate maps). Throughout, we will often want to closest-point project to convex subcomplexes in a CAT(0) cube complex. The way to do this is to use the gate map. Let \( \tilde{X} \) be a CAT(0) cube complex and let \( C \) be a convex subcomplex. Then there is a map \( g_C : \tilde{X} \to C \) with the following properties:

- \( g_C \) is the closest point projection with respect to the combinatorial metric, i.e. for all \( x \in \tilde{X}^{(0)} \), the vertex \( g_C(x) \) is the unique closest vertex of \( C \) to \( x \).
- \( g_C \) is 1–lipschitz on \( \tilde{X}^{(0)} \), with respect to the combinatorial metric.
- If \( x \in \tilde{X} \) and \( H \) is a hyperplane, then \( H \) separates \( x, g_C(x) \) if and only if \( H \) separates \( x, C \).
- If \( x, y \in \tilde{X} \) and \( H \) is a hyperplane, then \( H \) separates \( g_C(x), g_C(y) \) if and only if \( H \) intersects \( C \) and separates \( x, y \).

More information about gate maps can be found in e.g. [BHS17, Section 2.1].

**Remark 3.6** (Gate maps and wall-projections). Lemma 3.6 in [Wis21] implies that if \( U, V \subseteq \tilde{X} \) are convex subcomplexes, then \( \text{Proj}(U \to V) = g_V(U) \), which is itself convex.

Now we can define pieces. Let \( \langle X \mid \{ Y_i \}_{i \in \mathcal{I}} \rangle \) be a cubical presentation.
**Definition 3.7** (Wall-piece). Let $Y_i \to X$ be a relator. Let $\tilde{Y}_i \to \tilde{X}$ be an elevation. Let $H$ be a hyperplane of $\tilde{X}$ that is disjoint from $\tilde{Y}_i$. Suppose that the subcomplex $\mathcal{P} = \text{Proj}(\mathcal{N}(H) \to \tilde{Y}_i)$ is not a single point. Then $\mathcal{P}$ is an abstract wall-piece in $\tilde{Y}_i$. We have a map $\mathcal{P} \to \tilde{Y}_i \to Y_i \to X$. A path $P \to \mathcal{P}$ is a wall-piece of $H$ in $Y_i$, and gives a path $P \to Y_i$ via the preceding map. If $\mathcal{N}(H) \cap \tilde{Y}_i \neq \emptyset$, then $\mathcal{P} = \mathcal{N}(H) \cap \tilde{Y}_i$, and we say that $\mathcal{P}$ is a contiguous wall-piece.

**Definition 3.8** (Cone-piece). Let $Y_i, Y_j \to X$ be relators. Fix an elevation $\tilde{Y}_i \to \tilde{X}$. An abstract cone-piece of $Y_j$ in $Y_i$ is defined as follows. Let $\tilde{Y}_j \to \tilde{X}$ be an elevation of $Y_j$, and let $\mathcal{P} = \text{Proj}(\tilde{Y}_j \to \tilde{Y}_i)$.

Then:

- If $i \neq j$, the complex $\mathcal{P}$ is an abstract cone-piece of $Y_j$ in $Y_i$.
- If $i = j$ and $\tilde{Y}_i \neq \tilde{Y}_j$ (i.e. they are distinct $\pi_1X$-translates of $\tilde{Y}_i$), then $\mathcal{P}$ is an abstract cone-piece of $Y_j$ in $Y_i$.
- If $i = j$ and $\tilde{Y}_i = \tilde{Y}_j$, then $\mathcal{P}$ is an abstract cone-piece unless the following holds: Let $\mathcal{P} \to Y_i, Y_j$ be obtained by composing $\tilde{Y}_i, \tilde{Y}_j \to Y_i, Y_j$ with the inclusions of $\mathcal{P}$ into $\tilde{Y}_i, \tilde{Y}_j$ respectively. Then there exists an automorphism $Y_i \to Y_j$ such that

$$
\begin{align*}
\mathcal{P} &\to Y_i \\
\downarrow &\searrow \downarrow \\
Y_j &\to X
\end{align*}
$$

commutes.

If $\mathcal{P}$ is an abstract cone-piece in $Y_i$, then we again have a map $\mathcal{P} \to \tilde{Y}_i \to Y_i$. A path $P \to \mathcal{P}$ is a cone-piece in $Y_i$, and gives a path $P \to Y_i$, which we will refer to as a cone-piece in $Y_i$ without reference to the elevations involved. If $\tilde{Y}_i, \tilde{Y}_j$ intersect, then $P \to Y_i$ is a contiguous cone-piece.

**Example 3.9** (Cubical proper powers). The third case in Definition 3.8 generalises to cubical small cancellation setting how relators that are proper powers are treated in classical small cancellation theory. For instance, regard the presentation $(a, b \mid (ab)^2)$ as a cubical presentation by taking $X$ to be a wedge of two oriented circles labelled $a, b$ and $Y$ to be a 4-cycle mapping to $X$ according to the word $abab$. So, in the tree $\tilde{X}$ (the Cayley graph of the free group on $a, b$), we can take $\tilde{Y}$ to be the axis of the element $ab$. The lines $\tilde{Y}$ and $ab\tilde{Y}$ coincide — the second is a translate of the first by $ab$. Since $ab$ descends to an automorphism of $Y \to X$, the line $\tilde{Y} = \text{Proj}(ab\tilde{Y} \to \tilde{Y})$ is not an abstract cone-piece.

**Remark 3.10** (Variant definition of a cone-piece). As explained in Section 3 of [Wis21], cubical small cancellation theory works under a slightly different definition of a cone-piece. Indeed, the definition can be changed in the following way. Given $\tilde{Y}_i$ as in Definition 3.8, we forbid $\text{Proj}(\tilde{Y}_j \to \tilde{Y}_i)$ from being an abstract cone-piece whenever $i = j$ and $\tilde{Y}_i = \tilde{Y}_j$ (i.e. the elevations differ by an element of $\text{Stab}_{\pi_1X}(\tilde{Y}_i)$). However, to use this definition, one needs to insist that each element of $\text{Stab}_{\pi_1X}(\tilde{Y}_i)$ projects to an automorphism of $Y_i \to X$. Convention 3.3 in [Wis21] arranges this by insisting that $\pi_1Y_i$ is normal in $\text{Stab}_{\pi_1X}(\tilde{Y}_i)$. We have opted to avoid this convention, and use the definitions above, following Przytycki-Wise and Jankiewicz [PW18, Jan17].

Under our definition, we still have the following: if $\mathcal{P} = \text{Proj}(\tilde{Y}_j \to \tilde{Y}_i)$, and $\mathcal{P}$ is not an abstract cone-piece in $Y_i$, then $i = j$ and $\tilde{Y}_j = g\tilde{Y}_i$ for some $g \in \text{Stab}_{\pi_1X}(\tilde{Y}_i)$. Indeed, if $\mathcal{P}$ is not an abstract cone-piece in $Y_i$, then by definition, $i = j$ and there is an automorphism $g : Y_i \to Y_i$ such that the diagram in Definition 3.8 commutes. Since the automorphism group of $Y_i \to X$ is

$$\text{Normaliser}_{\text{Stab}_{\pi_1X}(\tilde{Y}_i)}(\pi_1\tilde{Y}_i)/\pi_1\tilde{Y}_i,$$

we can lift $g$ to some $g \in \text{Stab}_{\pi_1X}(\tilde{Y}_i)$. 


However, under our definition, the converse does not hold: if \( g \in \text{Stab}_{\pi_1 X}(\tilde{Y}_i) \) does not normalise \( \pi_1 \tilde{Y}_i \), then let \( \tilde{Y}_j = g\tilde{Y}_i \). Let \( \mathcal{P} = \text{Proj}(\tilde{Y}_j \to \tilde{Y}_i) = \tilde{Y}_i \). Then since \( g: \tilde{Y}_i \to \tilde{Y}_j \) does not descend to an automorphism of \( Y_i \to X \), our definition says that the whole of \( \tilde{Y}_i \) is an abstract cone-piece.

Under the metric small cancellation conditions we’ll be using, this then implies that \( Y_i \) is simply connected, and can be removed from the cubical presentation without changing the group being presented or weakening the small cancellation conditions on what remains.

So when working under any of the small cancellation conditions used in this paper, there would have been no loss of generality in assuming that \( \pi_1 Y_i \) is normal in \( \text{Stab}_{\pi_1 X}(\tilde{Y}_i) \) and using the definition of a cone-piece from \cite{Wis21}. (Relatedly, very often we’ll work with minimal cubical presentations, see Definition 1.1, which have the built-in assumption that \( \text{Stab}_{\pi_1 X}(\tilde{Y}_i) \) is conjugate to \( \pi_1 Y_i \), or with cubical presentations that are obtained from the minimal ones by replacing the \( Y_i \) with finite regular covers.)

**Definition 3.11** (Piece). A path \( P \to Y_i \) which is either a cone-piece or a wall-piece is a piece.

In the case where \( X \) is a wedge of circles and each \( Y_i \) is an immersed circle, all wall-pieces are trivial, and cone-pieces correspond to pieces in the sense of classical small cancellation theory.

**Definition 3.12** (\( C'(\alpha) \) condition, uniform \( C''(\alpha) \) condition). The cubical presentation \( \langle X \mid \{Y_i\}_{i \in I} \rangle \) satisfies the cubical \( C'(\alpha) \) small cancellation condition if the following holds for all \( i \in I \): \( \text{diam}(\mathcal{P}) < \alpha \|Y_i\| \) for all abstract pieces \( \mathcal{P} \) in \( Y_i \), where \( \|Y_i\| \) denotes the infimum of the lengths of essential closed paths in \( Y_i \) (a closed path is essential if it is not homotopic to a constant map). In this case, we say that \( \langle X \mid \{Y_i\}_{i \in I} \rangle \) is a \( C'(\alpha) \) presentation and \( G = \pi_1 X^* \) is a \( C'(\alpha) \) group.

Note that if \( |I| < \infty \), then the \( C'(\alpha) \) condition yields a uniform bound on the length of all pieces, namely \( \alpha \max_{i \in I} \|Y_i\| \).

In Section 3 we will use the stronger uniform \( C''(\alpha) \) condition. The cubical presentation \( \langle X \mid \{Y_i\}_{i \in I} \rangle \) satisfies the uniform \( C''(\alpha) \) small cancellation condition if \( \text{diam}(\mathcal{P}) < \alpha \|Y_i\| \) for all \( i \), whenever \( \mathcal{P} \) is an abstract piece (not necessarily in \( Y_i \)). In this case, we say that \( \langle X \mid \{Y_i\}_{i \in I} \rangle \) is a \( C''(\alpha) \) presentation and \( G = \pi_1 X^* \) is a \( C''(\alpha) \) group. This condition is needed to maintain an upper bound on the sizes of pieces, needed, for example, in the proof of Lemma 6.4.

Another way to phrase the uniform condition is to let \( Y = \bigsqcup_{i \in I} Y_i \), so that the local isometries \( Y_i \to X \) induce a local isometry \( Y \to X \). Then the uniform \( C''(\alpha) \) condition for \( \langle X \mid \{Y_i\}_{i \in I} \rangle \) is equivalent to the (non-uniform) cubical \( C'(\alpha) \) condition for the presentation \( \langle X \mid \{Y_i\}_{i \in I} \rangle \) (except allowing disconnected relators). This should be compared to the small cancellation conditions in \cite[Section 2.2]{GSIS}: in both cases, infinitely many relations can be encoded in a single cube complex (a possibly infinite, disconnected graph in the graphical case, \( Y \) here), and it’s the systole of that complex that is used in the uniform small cancellation condition.

### 3.3. Disc diagrams

The key objects in cubical small cancellation theory are **disc diagrams**.

**Definition 3.13** (Disc diagram, boundary path). A disc diagram is a compact, contractible 2–dimensional cell complex \( D \) equipped with a fixed embedding in \( \mathbb{R}^2 \). We regard \( S^2 \) as \( \mathbb{R}^2 \cup \{\infty\} \), so that \( S^2 \) is obtained from \( D \) by attaching a 2–cell containing \( \infty \). The attaching map of this 2–cell is the boundary path \( \partial_p D \) of \( D \).

Given a cubical presentation \( \langle X \mid \{Y_i\}_{i \in I} \rangle \) and a closed path \( P \to X \) that is nullhomotopic in \( X^* \), van Kampen’s lemma provides a disc diagram \( (D, \partial_p D) \to (X^*, X) \) whose boundary path \( \partial_p D = P \). Closed paths in \( \text{Cay}(X^*) \) bound disc diagrams \( (D, \partial_p D) \to (\bar{X}^*, \text{Cay}(X^*)) \).

The 2–cells of such a diagram are either squares (mapping to 2–cubes of \( X \subseteq X^* \)) or 2–simplices mapping to cones over the various \( Y_i \). Since \( P \) avoids cone-points, the 2–simplices of
$D$ are partitioned into classes: for each vertex of $D$ mapping to a cone-point in $X^*$, the incident 2–simplices are arranged cyclically around the vertex to form a subspace $C$ of $D$ which is equal to the cone on its boundary path (a path in $D$ mapping to $X$). The subspace $C$ is a cone-cell. In practice, we ignore the subdivision of $C$ into 2–simplices and regard $C$ as a 2–cell of $D$.

The complexity of $D$ is the pair $(c, s)$, where $c$ is the number of cone-cells and $s$ is the number of squares. Taking the complexity in lexicographic order, we always consider diagrams $(D, \partial_p D) \to (X^*, X)$ which are minimal in the sense that the complexity of $D$ is lexicographically minimal among all diagrams with boundary path $\partial_p D$. This implies that for each cone-cell $C$, the path $\partial_p C \to D \to Y \to X$ is essential.

In general, the cone-cell $C$ might not be a subdiagram of $D$ (it’s true that $C$ is a subspace of $D$, but $C$ itself might not be a disc diagram, in the situation where $\partial C$ self-intersects). However, under the small cancellation hypotheses used throughout this paper, Remark 3.10 (see also Theorem 4.1) of [Wis21] ensures that in any minimal-complexity disc diagram, the cone-cells are actually homeomorphic to discs and, in particular, are subdiagrams. We will assume this freely throughout.

Remark 3.14 (Dual curves and hexagon moves). Let $D \to X$ be a square diagram, i.e. a disc diagram whose 2–cells are squares. A dual curve in $D$ is a path which is the concatenation of midcubes of squares of $D$ that starts and ends on $\partial_p D$, where a midcube of a square $[-\frac{1}{2}, \frac{1}{2}]^2$ is obtained by restricting exactly one coordinate to 0 and a midcube of a 1–cube is its midpoint. If $X$ is a nonpositively curved cube complex, then each dual curve maps to a hyperplane. If $K$ is a dual curve in $D$, then the union of all closed cubes intersecting $K$ is its carrier (in analogy to the definition for the carrier of a hyperplane).

More generally, if $D \to X^*$ is a disc diagram, then one can define dual curves as above, but any dual curve has its two ends either on $\partial_p D$ or on the boundary path of a cone-cell of $X^*$.

A hexagon move is a homotopy of the diagram $D \to X^*$ that fixes the boundary path and the cone-cells and their boundary paths, while modifying the square part of $D$. Specifically, if $s_1, s_2, s_3$ are squares in $D$ arranged cyclically around a central vertex $v$, forming a hexagonal subdiagram $E$ of $D$, then $X$ must contain a 3–cube $c$ with a corner at the image of $v$ formed by the images of $s_1, s_2, s_3$. The (hexagonal) boundary path of $E$ maps to a combinatorial path in $c$, and we can replace $E$ by a diagram $E'$ formed from the other 3 squares on the boundary of $c$; this yields a new diagram $D' \to X^*$, with the same boundary path as $D$, formed by replacing $E$ by $E'$. This modification is a hexagon move. Hexagon moves are used to reduce area (i.e. the number of squares) of diagrams and, hence, their complexity in various ways; detailed accounts can be found in e.g. [Wis21] [Wis12].

Definition 3.15 (External cone-cell, internal cone-cell, internal path). The cone-cell $C$ of the disc diagram $D$ is external if $\partial_p C = QS$, where $Q$ is a non-trivial subpath of $\partial_p D$ (i.e. containing at least one 1–cell) and $S$ is an internal path in the sense that no 1–cell of $S$ lies on $\partial_p D$. The cone-cell $C$ is internal if $\partial_p C$ and $\partial_p D$ have no common non-trivial subpath. A cone-cell can be internal, external, or neither.

Remark 3.16 (Rectification and angling). Given a disc diagram $(D, \partial_p D) \to (X^*, X)$, one can rectify $D$, to produce a rectified diagram $\bar{D}$, by removing some internal open 1–cells, so that $D$ is subdivided into cone-cells, rectangles which are obtained from square ladders by deleting the internal open 1–cells, and complementary regions called shards. See [Wis21] Section 3.f] for more discussion of rectified diagrams. We will not require further details here.

After rectifying $D$, each corner in each of the resulting 2–cells is assigned an angle according to one of several possible schemes. We follow the split-angling defined in [Wis21] Section 3.h]. Specifically, if $v$ is a vertex of the rectified diagram $\bar{D}$, and $e$ is an edge in the link of $v$ (i.e. a corner of a 2–cell at the vertex $v$), then we assign an angle $\angle(e)$ according to rules discussed in [Wis21] Section 3.h]. Since we will just be using consequences of these angle assignments,
rather than the exact (long) definition, we refer the reader to [Wis21] Section 3.h]. Suffice it to say that:

- the angle \( \angle(e) \) is always \( \pi/2, \pi, 2\pi/3, 3\pi/4, \) or 0;
- the choice of angle is made in a way that guarantees nonpositive curvature at shards, in the sense described momentarily.

**Remark 3.17** (Defects and curvature). We now review some notions of curvature, from [Wis21] Section 3.g], that we will require below. Given a rectified disc diagram \( \tilde{D} \), we assign an angle \( \angle(e) \) – a real number – to each corner \( c \) of each 2–cell (i.e. to each 1–cell of each vertex-link).

In our setting, we always assume that this is done using the split-angling convention.

The defect \( \partial(c) \) at the corner \( c \) is \( \partial(c) = 2\pi - \angle(e) \). The curvature \( \kappa(v) \) at a vertex \( v \) of \( \tilde{D} \) is \( \kappa(v) = 2\pi - \sum \angle(e) - \pi \chi(\text{Lk}(v)) \), where \( \text{Lk}(v) \) is the link of \( v \), the notation \( \chi \) means the Euler characteristic, and the sum is taken over the 1–cells \( c \) of \( \text{Lk}(v) \). The curvature \( \kappa(f) \) at a 2–cell \( f \) of \( \tilde{D} \) is \( \kappa(f) = 2\pi - \sum \partial(c), \) where \( c \) varies over the corners of \( f \).

We will need the following theorem, which follows immediately from the “combinatorial Gauss-Bonnet Theorem” as stated in [MW02, Theorem 4.6]; very similar statements can be found in [Bri48, Ger87, BB96].

**Theorem 3.18** (Gauss-Bonnet for diagrams). Let \( \tilde{D} \rightarrow X^* \) be a rectified disc diagram. Then

\[
\sum_f \kappa(f) + \sum_v \kappa(v) = 2\pi,
\]

where \( f \) varies over the 2–cells of \( \tilde{D} \) and \( v \) varies over the 0–cells of \( \tilde{D} \).

In the case where \( X \) is a wedge of circles and each \( Y_i \) is an immersed circle, i.e. \( X^* \) is an ordinary presentation complex, disc diagrams are ordinary van Kampen diagrams, all rectangles are single edge, and rectifying has no effect on the diagram. In this case, all 2–cells of \( D \) are cone-cells, the split-angling continues to ensure that the curvature at each vertex is nonpositive, and the condition on the curvature of shards is vacuous.

**Definition 3.19** (Generalized corner, spur, shell). A (positively-curved) shell \( C \) in the disc diagram \( D \) is an external cone-cell whose curvature is positive; the boundary path of a shell has the form \( QS \), where the outer path \( Q \) is a subpath of the boundary path of \( D \), and the inner path \( S \) has no open 1–cell on \( \partial_p D \).

A spur in \( D \) is a vertex \( v \) in \( \partial_p D \) so that the incoming and outgoing 1–cells of \( \partial_p D \) map to the same 1–cell of \( X \), i.e. \( v \) is the second vertex in a subpath of \( \partial_p D \) of the form \( ee^{-1} \), where \( e \rightarrow X \) is a 1–cell.

A generalized corner is a path \( ef \) in \( D \), where each of \( e, f \) is an edge, so that the dual curves emanating from \( e, f \) cross inside a square \( s \) of \( D \), as shown in Figure 3, and the subdiagram of \( D \) bounded by the carriers of \( e \) and \( f \) is a square diagram. See Definition 2.5 in [Wis21].

**Remark 3.20** (Pushing generalised corners to the boundary by shuffling). If \( ef \) is a generalized corner of a square \( s \), and \( ef \) lies along \( \partial_p D \), then we can perform a series of hexagon moves (see [Wis21], Section 2.e) to homotop \( D \), fixing its boundary path, so that there is a square with boundary path \( efef' \), i.e. we can move squares to the boundary. In [Wis21], this procedure is called “shuffling”. Later in the paper, we will occasionally invoke shuffling to modify disc diagrams \( D \) so that a generalised corner \( ef \) in \( \partial_p D \) is actually an exposed square, i.e. there is a square \( s \) in \( D \) with boundary path containing \( ef \).

If \( ef \) is a generalized corner of a square \( s \), and \( ef \) lies on the boundary of some cone-cell \( C \) mapping to a relator \( Y \), then we can again shuffle until the square \( s \) has two consecutive edges on \( \partial_p C \). Convexity of \( Y \) allows us to “absorb” the square \( s \) into \( C \), lowering complexity of \( D \).
**Definition 3.21** (Padded ladder, ladder, cut-vertex). A padded ladder is a disc diagram $D \to X^*$ (or $\tilde{X}^*$) with the following structure. First, there is a sequence $C_1, \ldots, C_n$, where each $C_i$ is a cone-cell or vertex of $D$, so that $C_i, C_k$ lie in distinct components of $D - C_j$ whenever $i < j < k$. The diagram $D$ is an alternating union of these vertices and cone-cells with a sequence of subdiagrams $R_0, \ldots, R_n$ called pseudorectangles, so that:

1. The path $\partial_p D$ is a concatenation $P_1P_2^{-1}$, where $P_1, P_2$ start on $R_0$ and end on $R_n$.
2. We have $P_1 = \nu_0\rho_0\alpha_1\rho_1 \cdots \alpha_n\rho_n$ and $P_2 = \vartheta_0\gamma_1\vartheta_1\gamma_2 \cdots \gamma_n\vartheta_n\mu_{n+1}$.
3. We have $\partial_p C_i = \mu_i\alpha_i\nu_{i-1}^{-1}\gamma_{i-1}$.
4. We have $\partial_p R_i = \nu_i\rho_i\mu_{i+1}^{-1}\vartheta_{i-1}$.
5. Each $R_i$ is a square diagram, i.e. contains no cone-cells.
6. For each $i$, any dual curve in $R_i$ emanating from $\rho_i$ ends on $\vartheta_i$ and vice versa. Hence, any dual curve emanating from $\nu_i$ ends on $\mu_{i+1}$ and vice versa.
7. For each $i$, no two dual curves emanating from $\nu_i$ cross.

See Figure 2 for a picture illustrating the notation. We say that $R_i$ is horizontally degenerate if $|\mu_{i+1}| = |\nu_i| = 0$ and vertically degenerate if $|\rho_i| = |\vartheta_i| = 0$. When $C_i$ is a vertex, we call it a cut-vertex of the padded ladder $D$.

If $R_0, R_n$ are vertically degenerate, then $D$ is a ladder. (A padded ladder is a special case of what Jankiewicz calls a generalized ladder in [Jan17]; the definition of ladder here is equivalent to that in [Wis21 Definition 3.42]).

We require the following three crucial facts, due to Wise. These are tailored to our specific situation; the statements in [Wis21] are more general.

**Theorem 3.22** (Ladder theorem). Let $\langle X \mid \{Y_1\}_{i \in \mathbb{Z}} \rangle$ be a cubical $C'\left(\frac{5}{12}\right)$ presentation. Let $D \to X^*$ be a minimal disc diagram such that the corresponding rectified diagram has exactly two positively-curved cells along $\partial_p D$. Then $D$ is a ladder.

**Proof.** This follows by combining [Wis21 Theorem 3.43] with [Wis21 Theorem 3.31]. See also [Wis21 Examples 3.s.(3), Theorem 3.32]. □

From Theorem 3.46 of [Wis21], we also get:

**Theorem 3.23** (Greendlinger’s lemma/diagram trichotomy). Under the hypotheses of Theorem 3.22, if $D \to X^*$ is a minimal disc diagram, then either $D$ consists of a single vertex or cone-cell, or $D$ is a ladder, or $D$ contains at least three features of positive curvature (i.e. shells, spurs, or generalized corners) along $\partial_p D$. 
The next theorem follows directly from Lemma 3.70 of [Wis21]. In fact, it holds under weaker small-cancellation conditions (see [Wis21] Lemma 3.70 or [Jan17]), but we will not require this.

**Theorem 3.24** (Short inner paths). Let $\langle X \mid \lbrace Y_i \rbrace_{i \in I} \rangle$ be a cubical $C'(\frac{1}{144})$ presentation. Let $D \to X^*$ be a disc diagram and let $C$ be a shell in $D$ with boundary path $QS$, with $Q$ a maximal common subpath of $\partial_0C$ and $\partial_0D$, and $S$ an internal path. Suppose that $QS$ is essential in the relator $Y$ to which $C$ maps, and that $S$ is of minimal length among all paths $S' \to Y$ that are square-homotopic rel endpoints in $Y$ to $S$. Finally, suppose that the total curvature contribution from $C$ is positive. Then $\lvert S \lvert < \lvert Q \lvert$.

### 3.4. The classification theorem

Using the above fundamental results, we produce a cubical small cancellation version of Strebel’s classification of triangles in classical small cancellation groups [Str90]. An exposed square in a disc diagram $D$ is a square with two consecutive edges on $\partial_D D$. A tripod is a triangle diagram with no cone-cells or squares. We can now state our classification of triangles in cubical small cancellation groups:

**Theorem 3.25** (Classification of triangles in cubical $C'(\frac{1}{144})$ groups). Let $X$ be a connected nonpositively curved cube complex, let $I$ be a (possibly infinite) index set and let $\lbrace Y_i \to X \rbrace_{i \in I}$ be a set of local isometries of connected complexes.

Let $\langle X \mid \lbrace Y_i \rbrace_{i \in I} \rangle$ be a cubical presentation satisfying the $C'(\frac{1}{144})$ condition. Let $\alpha, \beta, \gamma \to \Cay(X^*)$ be combinatorial geodesics so that $\alpha \beta \gamma$ is a geodesic triangle. Then there exists a disc diagram $(D, \partial_D D) \to (X^*, X)$ with boundary path $\alpha' \beta' \gamma' \to \Cay(X^*) \to X^*$ lying in $X$, so that the following hold. First, $\alpha \to X$ and $\alpha' \to X$ co-bound a bigon $B \to X$ (i.e. they are square-homotopic) and the same is true of $\beta, \beta'$ and $\gamma, \gamma'$. Second, $D$ is of one of the following types.

1. **(3-shell generic:)** $D$ has exactly three external cone-cells, $C_1, C_2, C_3$, respectively containing the points $\alpha' \cap \beta', \beta' \cap \gamma', \gamma' \cap \alpha'$. There is exactly one cone-cell $M$ that intersects $\alpha', \beta'$, and $\gamma'$. Moreover, $D$ is the union of three ladders, $L_1, L_2, L_3$ so that $L_i \cap L_j = M$ for all $i, j$. In particular, every cone-cell except $M$ intersects exactly two of the geodesics $\alpha, \beta, \gamma$.

2. **(3-shell tripod:)** $D$ has exactly three external cone-cells, $C_1, C_2, C_3$, respectively containing the points $\alpha' \cap \beta', \beta' \cap \gamma', \gamma' \cap \alpha'$. Every other cone-cell intersects exactly two of the geodesics $\alpha, \beta', \gamma'$. In this case, $D$ is the union of 3 (possibly padded) ladders $L_1, L_2, L_3$ and a tripod triangle $P_1 P_2 P_3 \to X$ so that $L_i$ intersects the other two ladders in the path $P_i$.

3. **(2-shell generic:)** Same as 3-shell generic, except exactly one of $C_1, C_2, C_3$ is a spur or exposed square instead of a cone-cell.

4. **(2-shell tripod:)** Same as 3-shell tripod, except exactly one of $C_1, C_2, C_3$ is a spur or exposed square instead of a cone-cell.

5. **(1-shell generic:)** Same as 2-shell generic, except exactly two of $C_1, C_2, C_3$ are spurs or exposed squares.

6. **(1-shell tripod:)** Same as 2-shell tripod, except exactly two of $C_1, C_2, C_3$ are spurs or exposed squares.

7. **(No-shell generic:)** Same as 3-shell generic, except $C_1, C_2, C_3$ are all spurs or exposed squares.

8. **(No-shell tripod:)** Same as 3-shell tripod except $C_1, C_2, C_3$ are all spurs or exposed squares. This includes the case where $\alpha' \beta' \gamma'$ is nullhomotopic in $X$, in which case $D$ is a tripod.

9. **(Degenerate triangle:)** $D$ is a single vertex or cone-cell, or $D$ is a ladder.

The diagram $D \to X^*$ is a standard diagram for the triangle $\alpha \beta \gamma$. The eight non-degenerate cases are shown in Figure 7.
Remark 3.26 (Media and small cancellation parameters). The standard diagram depends only on the endpoints of the geodesics $\alpha, \beta, \gamma$. Just as it is usual in $\text{CAT}(0)$ cube complexes to homotop geodesics, fixing their endpoints, in order to minimize the area of diagrams, here we are not married to particular geodesics, just to square-homotopy classes relative to their endpoints. In particular, if $\alpha\beta\gamma$ bounds a disc diagram in $X$, then $D$ is a tripod. When $I = \emptyset$, Theorem 3.25 just says: any three 0–cubes in a $\text{CAT}(0)$ cube complex determine a geodesic tripod.

More generally, as illustrated by Figure 1, Theorem 3.25 should be interpreted as saying that the vertices of the triangle have a “median” which is either a vertex or a cone-cell, and there is a geodesic triangle connecting the given three points, each of whose sides passes within a wall-piece of the “median”. In other words, given 0–cells $a, b, c \in \text{Cay}(X^*)$, the “convex hulls” of the three possible pairs mutually coarsely intersect.

At the other extreme, when $X$ is a wedge of circles and each $Y_i$ is an immersed circle, Theorem 3.25 generalizes a weak version of Strebel’s classification of triangles [Str90, Theorem 43]; specifically, Theorem 3.25 provides the same classification as Strebel’s result, but, because the proof must work in the more general context of cubical presentations, we require stronger metric small cancellation conditions than Strebel needs in the classical setting.

Proof of Theorem 3.25. We consider a disc diagram bounded by the geodesic triangle. The proof is then essentially a meticulous application of Theorem 3.23, Theorem 3.22, and Theorem 3.24, following and followed by appropriately chosen square homotopies. The main points are:

- A curvature computation to eliminate the possibility of internal cone-cells (in the Strebel classification in the case of classical $C'(\frac{1}{6})$ condition, one of the primary features is that the disc diagrams do not have internal cells). We also rule out some other types of cone-cells lying along the boundary of the diagram. This computation, which is a slightly modified version of a computation in the proof of Theorem 3.29 of [Wis21] (same computation, except with different numbers reflecting our strong small-cancellation conditions), is why we need the $C'(\frac{1}{144})$ condition.

- Applications of Theorem 3.24 to rule out positively-curved shells along the boundary of a disc diagram bounded by a geodesic triangle.

- An application of Theorem 3.22 to decompose our diagram as the union of at most three ladders meeting along a central cell.

Choosing $\alpha', \beta', \gamma'$ and constructing $D$: Given a geodesic $P \to \text{Cay}(X^*)$, let $[P]$ be the set of geodesics $Q$ that have the same endpoints as $P$ and the additional property that $PQ^{-1}$ bounds a disc diagram containing no cone-cells, i.e. there is a disc diagram $E \to X$ whose boundary path is $PQ^{-1} \to \text{Cay}(X^*) \to X^*$.

Choose a disc diagram $D \to \tilde{X}^*$ so that $\partial_p D = \alpha'\beta'\gamma'$, where $\alpha' \in [\alpha], \beta' \in [\beta], \gamma' \in [\gamma]$. Choose $D$ so that the complexity is minimal among all disc diagrams with boundary path of the preceding form.

Abusing notation slightly, we now temporarily regard $D$ as the rectified diagram from Remark 3.16. This means that certain square ladders are regarded as single (rectangular) 2–cells, cone-cells are regarded as single 2–cells, and the remaining parts of the diagram are 2–cells (formed by ignoring non-boundary 1–cells in certain square subdiagrams) called shards. Angles are assigned to corners according to the split-angling discussed above.

Applying the Greendlinger lemma: By Theorem 3.23 either $D$ is a single vertex, a single cone-cell, a ladder, or $D$ has at least 3 features of positive curvature – spurs, generalised corners, or shells – along $\partial_p D$. In either of the first three cases, assertion (7) holds. So, assume that $D$ has at least 3 features of positive curvature along the boundary, each of which is a shell, a spur, or a generalized corner.
We may assume that all generalised corners along $\partial_p D$ are actually squares with corners on $\partial_p D$. Indeed, let $s$ be a square in $D$ with a generalised corner on $\partial_p D$, so the dual curves $K_1, K_2$ intersecting $s$ end at consecutive 1–cubes $e_1, e_2$ on $\partial_p D$. By shuffling, we modify $D$ – without changing $\partial_p D$ or increasing complexity – so that $s$ lies along the boundary, i.e. $e_1, e_2$ are consecutive 1–cubes of $s$.

**Square homotopies:** Let $s$ be a square of $D$ so that $\partial_p s$ and $\partial_p D$ have a common subpath $e_1 e_2$. For $i \in \{1, 2\}$, let $e'_i$ be the 1–cube of $s$ opposite $e_i$. If $e_1 e_2$ is a subpath of one of the three constituent geodesics of $\partial_p D$ (say, $\alpha'$), then we can modify $\alpha'$ in its square-homotopy class by replacing $e_1 e_2$ by $e'_2 e'_1$, resulting in a new diagram with the same number of cone-cells and fewer squares. This contradicts our minimality assumption.

Hence we can assume the following: any square $s$ with a corner on the boundary lies at the transition from $\alpha'$ to $\beta'$, or $\beta'$ to $\gamma'$, or $\gamma'$ to $\alpha'$. Also, since $\alpha', \beta', \gamma'$ are geodesic, a spur of the form $ee^{-1}$ cannot occur along any of $\alpha', \beta', \gamma'$ so the only spurs consist of overlaps between $\alpha', \beta'$ or $\beta', \gamma'$ or $\gamma', \alpha'$.

We now rule out cone-cells in $D$ of a few types. First, we rule out positively curved shells with outer path on one of our three geodesics. This puts us in a position where we have at most 3 features of positive curvature (we already removed generalised corners along our geodesics, we will shortly remove shells, and spurs are impossible, so positive curvature can only occur at transitions between successive geodesics). Using this, we rule out internal cone-cells and (nonpositively curved) shells. From this, we deduce that each cone-cell has nonempty connected intersection with each of $\alpha', \beta', \gamma'$. This is summarised in Figure 4.

**Claim 1** (No positive curvature along geodesics (after square homotopy)). Let $C$ be a positively-curved shell in $D$ with outer path $O$ and inner path $I$. Then $O$ cannot be a subpath of $\alpha', \beta'$, or $\gamma'$. Thus, $O = AB$, where $A$ is a nontrivial terminal subpath of $\alpha', \beta'$, or $\gamma'$ and $B$ is a nontrivial initial subpath of $\beta', \gamma'$, or $\alpha'$, respectively.

**Proof of Claim 1** Suppose $O$ is a subpath of $\alpha'$. (The other cases are handled identically.) Theorem 3.24 shows that we can replace $O$ by a geodesic in $Y$ joining the endpoints of $I$ to get a strictly shorter path joining the endpoints of $\alpha'$, a contradiction. ■
Claim[1] and the discussion preceding it, put us in the following situation: $D$ has exactly three features of positive curvature along the boundary, which are subdiagrams $C_1, C_2, C_3$. For each $i, \partial_D C_i = OI$, where $O$ is a subpath of $\partial_D D$ and $I$ is an internal path, and $O$ has at least one 1-cube on each of two distinct subpaths among $\alpha', \beta', \gamma'$ of the boundary.

Let $C$ be a cone-cell of $D$. Recall that $C$ is internal if its boundary path intersects $\partial_D D$ in a set containing no 1–cube. To rule out internal cone-cells, we need an auxiliary claim.

Claim 2 (Internal cone-cell curvature contribution). Let $C$ be an internal cone-cell of $D$. Then the curvature at $C$ is strictly less than $-4\pi$.

Proof of Claim[2] We argue almost exactly as in the proof of [Wis21] Theorem 3.32, except exploiting our stronger small-cancellation assumption. Since $D$ has minimal complexity, it is reduced (in the sense of [Wis21] Definition 3.11), then $\partial_D D$ is not the concatenation of fewer than 145 pieces. Now apply exactly the proof of [Wis21] Theorem 3.31, except that the number 13 in that proof is replaced by 145, yielding a curvature contribution from $C$ of at most $2\pi - \frac{145\pi}{6} < -4\pi$. ■

Claim 3 (No internal cone-cells). The diagram $D$ does not contain an internal cone-cell.

Proof of Claim[3] Suppose that there are $n \geq 0$ internal cone-cells.

Let $v$ be a 0–cube of (the rectified) $D$. Then, by Theorem 3.32 of [Wis21], the curvature contribution from $v$ is:

1. at most 0 if $v$ is internal or not contained in a 2–cell and not a spur;
2. exactly $\pi$ if $v$ is a spur;
3. exactly $\pi/2$ if $v$ is the corner of a square along $\partial_D D$.

Let $f$ be a 2–cell of $D$ (a cone-cell, rectangle, or a shard of the corresponding rectified diagram [Wis21]). Then the curvature contribution is:

1. at most 0 if $f$ is a rectangle or shard;
2. less than $-4\pi$ if $f$ is an internal cone-cell, by Claim 2;
3. at most $2\pi$ if $f$ is a shell.

Hence, our three features of positive curvature contribute a total of at most $6\pi$ of curvature, while the sum of the remaining curvatures is $<-4n\pi$. This contradicts Theorem 3.18 unless $n = 0$. Thus, there are no internal cone-cells. ■

We have to rule out another type of cone-cell. A cone-cell $C$ in $D$ is shortly external if its boundary path has the form $OI$, where $I$ is internal and $O$ is a subpath of $\alpha', \beta', \gamma'$.

Claim 4. $D$ has no shortly-external cone-cell.

Proof of Claim[4] Note that $|O| \leq |I|$ since $\alpha', \beta', \gamma'$ are geodesics. Following the proof of Theorem 3.29 of [Wis21], we write $I$ as a concatenation of $k$ paths, each of which is a concatenation of at most 3 pieces, such that each path contributes an angle defect of at least $\pi/4$. Our small-cancellation condition guarantees that there are more than $144/(2 \cdot 3) = 24$ such paths, so the total curvature is at most $2\pi - k\pi/4 < -4\pi$. Exactly as in the proof of Claim 3, we obtain a contradiction with Theorem 3.18 unless there are no shortly-external cone-cells. ■

At this point, we have completed the curvature computations in the proof, and now regard $D$ as an ordinary (not rectified) diagram.

Analysis of the cone-cells: Let $C$ be a cone-cell in $D$. We would like to show that $C$ has connected intersection with each of $\alpha', \beta', \gamma'$.

Suppose that for some $\delta \in \{\alpha', \beta', \gamma'\}$, there is a subpath $\delta' = PQR$ of $\delta$, where:

- the paths $P, R$ are subpaths of the boundary path of $C$,
- the terminal vertex of $P$ and initial vertex of $R$ subtend a subpath $Q'$ of $\partial_D C$, such that
• the path $QQ'$ bounds a subdiagram $E$ of $D$ between $C$ and $\partial_p D$.
• $Q$ and $\partial_p C$ have no common 1–cell.

If $E$ contains no cone-cell, then $E$ is a square diagram between the relator $Y_i$ to which $C$ maps and the geodesic $Q$, so by local convexity of $Y_i$ in $X$, we have that $E \to X$ factors through $Y_i \to X$. Hence $E$ could have been absorbed into the cone-cell $C$, whence minimality of the complexity of $D$ ensures that $E$ is trivial, i.e. $Q = (Q')^{-1}$. This is a contradiction, so $E$ is not a square diagram.

We can assume $C$ is innermost, in the sense that any cone-cell $C_0$ in $E$ has connected intersection (possibly empty) with $Q$.

Such a cone-cell $C_0$ is not internal in $E$, for then it would be internal in $D$, violating Claim 3. So $C_0$ is shortly-external in $D$, or a positively-curved shell along $Q$, violating Claim 4 or Claim 1.

So, we have proven:

**Claim 5.** For each cone-cell $C$ of $D$, the path $\partial_p C$ has connected intersection with each of $\alpha', \beta', \gamma'$. Moreover, $\partial_p C$ intersects at least two of the paths $\alpha', \beta', \gamma'$.

**The current situation:** Thus far, we have reduced to the following situation:

• $D$ is not a ladder.
• $D$ has precisely 3 features of positive curvature along its boundary path, which are subdiagrams $C_1, C_2, C_3$.
• The subdiagram $C_1$ has boundary path $ABI$, where $A$ is a nontrivial terminal subpath of $\alpha'$, $B$ is a nontrivial initial subpath of $\beta'$, and $I$ is a (possibly trivial) path.
• Either $C_1$ is a single cone-cell (a shell) or $C_1$ is a spur, $|I| = 0$, and $A$ is an edge and $B = A^{-1}$, or $C_1$ is a square, $|A|, |B| \geq 1$, and $|I| \leq 2$. The same description holds for $C_2$ (with $\beta', \gamma'$ replacing $\alpha', \beta'$) and $C_3$ (with $\gamma', \alpha'$ replacing $\alpha', \beta'$).
• Every cone-cell $C$ of $D$ not in $\{C_1, C_2, C_3\}$ has connected intersection with each of $\alpha', \beta', \gamma'$ and intersects at least 2 of these paths. We call $C$ a median-cell if $C$ intersects all three of these paths, and a tail-cell otherwise.

We emphasise that if $C$ is a median cell, then it has nonempty, connected intersection with each of $\alpha', \beta', \gamma'$. Hence, if $C$ is a median-cell, then $C$ separates $D$ into three complementary components, each disjoint from one of the paths $\alpha', \beta', \gamma'$. Hence, $C$ is the unique cone-cell of $D$ intersecting each of $\alpha', \beta', \gamma'$. (We note that there may be other disc diagrams with the same boundary path, containing a different median-cell.)

We now divide into cases. First, if $D$ contains a median-cell, then we are in one of the generic cases, i.e. we will show that one of $\{7, 5, 3, 1\}$ holds, according to how many of $\{C_1, C_2, C_3\}$ are spurs or shells. Otherwise, we will show that one of $\{8, 6, 4, 2\}$ holds.

**The generic cases:** Suppose that $D$ has a (unique) median-cell $M$ and let $i \in \{1, 2, 3\}$. Let $\delta, \delta' \in \{\alpha', \beta', \gamma'\}$ be the parts of the boundary path of $D$ that intersect $C_i$. Let $\partial_p M = AP_i BP_i CP_3$, where $A, B, C$ are respectively subpaths of $\alpha', \beta', \gamma'$ and $P_1, P_2, P_3$ are internal paths. Write $\alpha' = \alpha' A \delta', \beta' = \beta' B \beta', \gamma' = \gamma' C \gamma'$. Consider the subdiagram $L_1$ bounded by $A \delta' B \beta' C \partial_p CP_3$. The ladder theorem, Theorem 3.22 and our above analysis of the possible features of positive curvature in (the rectification of) $D$ shows that $L_1$ is a ladder. The ladders $L_2, L_3$ are constructed analogously.

**The tripod cases:** Suppose there is no median-cell. Then we have a subdiagram $T$ of $D$ with boundary path $AP_1 BP_2 CP_3$, where $A$ is a subpath of $\alpha'$, $B$ a subpath of $\beta'$, $C$ a subpath of $\gamma'$, and $P_1, P_2, P_3$ internal subpaths that lie on innermost cone-cells in $D$ or, if they do not exist, spurs or exposed squares in $\{C_1, C_2, C_3\}$. By construction, $T$ is a possibly degenerate square diagram, and by convexity of relators and minimality, for each path $Q \in \{A, B, C, P_1, P_2, P_3\}$, no two dual curves in $T$ emanating from $Q$ can cross. Moreover, no dual curve travels from $Q$ to $Q$ or to the next named subpath, for otherwise we could reduce complexity. Some possibilities are shown in Figure 5.
Figure 5. Some possibilities for the internal square subdiagram in the tripod cases.

For convenience, we lift $T$ to a diagram $T \to \breve{X}$ (the CAT(0) cube complex $\breve{X}$, not the generalized Cayley graph). Here, an analysis of the dual curves shows that $T$ decomposes as required; the analysis is indicated in Figure 6. First, consider dual curves in $T$ traveling from $A$ to $B$, $B$ to $C$, or $C$ to $A$. Taking the union of all carriers of such dual curves yields rectangles attached to $P_1, P_2, P_3$. Now consider the subdiagram that remains. It is a hexagon bounded by subpaths of $A, B, C$ and parts of carriers of dual curves. Dual curves in the subdiagram must travel from a subpath of $A, B, C$ or to the antipodal dual-curve carrier. Dual curves emanating from the same "syllable" of the boundary path do not cross, and we conclude, as at right in Figure 6, that this subdiagram is a "corner of a subdivided cube". It is now easy to deduce the padded ladder decomposition of $D$, with an application of Theorem 3.22. (Various parts of the picture may be degenerate, as suggested in Figure 5.) □

4. Detecting WPD elements using the classification of triangles

In this section, we adopt the following assumptions and conventions:

1. $(X \mid \{Y_i\}_{i \in I})$ is a cubical presentation satisfying the $C'(\frac{1}{144})$ condition, $X^*$ is the presentation complex, and $\breve{X}^*$ is the universal cover. We assume that $X$ is locally finite, but we do not assume $X$ is uniformly locally finite.

2. Since $\pi_1 X$ is torsion-free, every $g \in \pi_1 X - \{1\}$ has a combinatorial geodesic axis in the first cubical subdivision of $\breve{X}$ (see [Hag07] for the notion of cubical subdivision and a proof of the statement about axes). Since we can replace $X$ and each $Y_i$ by their first cubical subdivisions without changing the group of the cubical presentation or the small-cancellation conditions, we assume in the rest of the paper that each $g \in \pi_1 X - \{1\}$ admits a combinatorial geodesic axis.

3. Denote by $d$ the graph metric on $\text{Cay}(X^*)(1)$, and by $d_{\breve{X}}$ the graph metric on $\breve{X}^{(1)}$. 

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Figure 6. The final square diagram analysis in the tripod cases.
Let there be a $\delta$–hyperbolic graph $\mathcal{H}$ and a coarsely surjective map $\Pi: \text{Cay}(X^*) \to \mathcal{H}$ so that:

(a) We have $d_{\Pi}(\Pi(x), \Pi(y)) \leq d(x, y)$ whenever $x, y \in \text{Cay}(X^*)^{(0)}$.
(b) If $Y_i \subseteq \text{Cay}(X^*)$ is any relator, then $\text{diam}(\Pi(Y_i)) \leq \delta$.
(c) The group $\pi_1 X^*$ acts by isometries on $\mathcal{H}$ in such a way that $\Pi$ is $\pi_1 X^*$–equivariant.
(d) Let $H$ be a hyperplane in $\text{Cay}(X^*)$. Then $\text{diam}(\Pi(N(H))) \leq \delta$.

Under these conditions, we will prove a lemma — Lemma 4.5 — showing that $\pi_1 X^*$ contains a WPD isometry of $\mathcal{H}$ provided it contains a fast loxodromic one (defined below). Later, we choose specific $\mathcal{H}$ and $\Pi$. The proof of Lemma 4.5 will require us to show that certain paths produced by an application of Theorem 3.25 fellow-travel. We need some preliminary lemmas:

**Lemma 4.1 (Ladders are thin between cone-cells).** Let $L \to \tilde{X}^*$ be a minimal complexity padded ladder with boundary path $\alpha\beta^{-1}\gamma$, where $\alpha, \beta: [0, \ell] \to \text{Cay}(X^*)$ are geodesics with $\alpha(\ell) = \beta(\ell)$ and $\gamma$ is a piece. Let $\Delta$ be the maximum length of a subpath of $\alpha$ or $\beta$ that lies on a single cone-cell of $L$. Then there exists $k_0 = \kappa_0(\Delta)$ so that for all $t \leq \ell$, $d(\alpha(t), \beta(t)) \leq k_0$.

**Proof.** Write $\alpha = \alpha_0\eta_1\alpha_1 \cdots \eta_n\alpha_n$ and $\beta = \beta_0\eta'_1 \cdots \eta'_n\beta_n$, where each $\alpha_i, \beta_i$ lies on the top or bottom boundary path of one of the constituent pseudorectangles of $L$ and each $\eta_i, \eta'_i$ lies on the boundary path of a cone-cell or cut-vertex (and hence has length at most $\Delta$). The boundary path of the $i^{th}$ pseudorectangle has the form $\alpha_ip_i\beta_i^{-1}q_i$, as in Figure 7. When $i = 0$, the path $q_0 = \gamma$, which is a piece.

![Figure 7. Ladders are thin relative to cone-cells.](image)

We observe that each $p_i, q_i$ is actually a piece. Indeed, let $R_i$ be the pseudorectangle containing $p_i$, and let $C_i, C_{i+1}$ be the cone-cells adjacent to $R_i$. Let $Y_i, Y_{i+1}$ be the relators in $\tilde{X}^*$ to which $C_i, C_{i+1}$ map. Since it is a square diagram, $R_i$ lifts to a disc diagram $\tilde{R}_i \to \tilde{X}$ and the paths $q_i, p_i$ lie on elevations $\tilde{Y}_i, \tilde{Y}_{i+1}$ of $Y_i, Y_{i+1}$. If $\tilde{Y}_i = \tilde{Y}_{i+1}$, then convexity of $\tilde{Y}_i$ and the fact that $\alpha_i, \beta_i$ are geodesics implies that $R_i$ factors through $\tilde{Y}_i$. So, we could replace $C_i \cup R_i \cup C_{i+1}$ in $L$ by a single cone-cell, reducing complexity. Hence $\tilde{Y}_i, \tilde{Y}_{i+1}$ are different, so $p_i, q_i$ are cone-pieces.

Hence, by the small cancellation conditions, there exists $M = M(\Delta)$ so that $|p_i| \leq M$ for all $i$. Indeed, the boundary path of the cone-cell containing $p_i$ consists of one or two pieces, together with two subpaths of length at most $\Delta$. Letting $\tau$ be the length of this boundary path, the small-cancellation conditions imply that $\tau < 288\Delta/142$, and another application of the small-cancellation conditions gives $|p_i| < \Delta/71$.

Since $\alpha, \beta$ are geodesics, we have $||\eta_i| - |\eta'_i|| \leq 2M$ for all $i$, for otherwise we could construct shortcuts. The lemma now follows easily. \hfill $\Box$

**Remark 4.2 (Rank one elements).** As usual (see e.g. [CST]), $\tilde{g} \in \pi_1 X$ is rank one if it is hyperbolic on $\tilde{X}$ and none of its axes lies in an isometrically embedded Euclidean half-plane. Let $\alpha$ be a combinatorial geodesic axis in $\tilde{X}$ for $\tilde{g}$. Let $\mathcal{W}(\alpha)$ be the set of hyperplanes intersecting
\(\alpha\). Let \(C \alpha\) be the graph with vertex set \(W(\alpha)\), with vertices \(H, V\) adjacent if the corresponding hyperplanes have intersecting carriers.

Let \(\tilde{B}\) be the cubical convex hull of \(\alpha\), which is a CAT(0) cube complex whose hyperplanes are exactly those in \(W(\alpha)\). The graph \(C \alpha\) is exactly the contact graph of \(\tilde{B}\), i.e., the intersection graph of its set of hyperplane carriers. Considering the action of \(\langle \tilde{g} \rangle\) on \(\tilde{B}\), we see that there are finitely many \(\langle \tilde{g} \rangle\) orbits of hyperplanes in \(\tilde{B}\), and each has uniformly bounded coarse intersection with \(\alpha\). Hence, by [Hag13] Theorem 2.4, Proposition 2.5, \(\langle \tilde{g} \rangle\) has unbounded orbits in \(C \alpha\).

Therefore, since \(\langle \tilde{g} \rangle\) acts on \(C \alpha\) with finitely many orbits of vertices (each hyperplane of \(\tilde{B}\) is dual to one of \(\langle \tilde{g} \rangle\)–finitely many 1–cubes in \(\alpha\)), there exists \(N\) such that if \(H, V\) are hyperplanes intersecting \(\alpha\) in 1–cubes lying at distance more than \(N\), then \(H\) and \(V\) cannot cross.

**Lemma 4.3.** Let \(\tilde{g} \in \pi_1 X\) act hyperbolically on \(\tilde{X}\), and suppose that \(\tilde{g}\) is rank one. Then for each \(\tilde{x} \in \tilde{X}^{(0)}\), there exists \(\kappa_1\) so that the following holds: if \(n \geq 0\) and \(P, Q: [0, d] \to \tilde{X}\) are combinatorial geodesics joining \(\tilde{x}, \tilde{g}^n \tilde{x}\), then \(d_{\tilde{X}}(P(t), Q(t)) \leq \kappa_1\) for \(0 \leq t \leq d\).

**Proof.** Let \(\alpha \to \tilde{X}\) be a combinatorial geodesic axis for \(\tilde{g}\) and let \(\tilde{a} \in \alpha\) be a 0–cube. Given \(n \geq 0\), let \(P, Q: [0, d_n] \to \tilde{X}^{(1)}\) (where \(d_n = d_{\tilde{X}}(\tilde{x}, \tilde{g}^n \tilde{x})\)) be combinatorial geodesics joining \(\tilde{x}, \tilde{g}^n \tilde{x}\) and let \(D \to \tilde{X}\) be a minimal-area disc diagram with boundary path \(PQ^{-1}\). Note that \(D \to \tilde{X}\) is actually an isometric embedding on the 1–skeleton. Indeed, every dual curve in \(D\) travels from \(P\) to \(Q\) since \(P, Q\) are geodesics. Hence each dual curve maps to a distinct hyperplane, so that for any vertices \(v, v' \in D\), the number of dual curves of \(D\) separating \(v, v'\) is equal to the number of hyperplanes in \(\tilde{X}\) separating their images.

Fix \(t \in \{0, 1, \ldots, d_n\}\). The above discussion shows that \(d_{\tilde{X}}(P(t), Q(t))\) is bounded by the number of dual curves in \(D\) that travel from \(P([0, t])\) to \(Q([t, d_n])\), plus the number of dual curves from \(Q([0, t]))\) to \(P([t, d_n])\). Each dual curve of the former type crosses each dual curve of the latter type.

Let \(\mathcal{F}\) be the set of dual curves of the former type, and let \(\mathcal{V}\) be the set of dual curves of the latter type. Let \(N_1 = |\mathcal{F}|\) and \(N_2 = |\mathcal{V}|\). Note that since all but at most \(2d_{\tilde{X}}(\tilde{a}, \tilde{x})\) hyperplanes that cross \(P\) cross \(\alpha\), at least \(N_1 + N_2 - 2d_{\tilde{X}}(\tilde{a}, \tilde{x})\) hyperplanes in \(\mathcal{F} \cup \mathcal{V}\) cross \(\alpha\).

Let \(\mathcal{F}' \subseteq \mathcal{F}, \mathcal{V}' \subseteq \mathcal{V}\) be subsets consisting of hyperplanes/dual curves that cross \(\alpha\). Then by Remark 1.2, we have \(\min\{N_1 - 2d_{\tilde{X}}(\tilde{a}, \tilde{x}), N_2 - 2d_{\tilde{X}}(\tilde{a}, \tilde{x})\} \leq N\), where \(N\) depends only on \(\tilde{g}\).

But \(|\mathcal{F}| = |\mathcal{V}|\), since for each dual curve crossing \(Q([0, t])\) and \(P([t, d_n])\), there must be a dual curve crossing \(P([0, t])\) and \(Q([t, d_n])\). Hence \(\mathcal{F} \cup \mathcal{V}\) has bounded cardinality, and we conclude that \(d_{\tilde{X}}(P(t), Q(t))\) is bounded by some \(\kappa_1\) depending only on \(\tilde{g}\) and \(\tilde{x}\).

\(\Box\)

**Definition 4.4** (\(\Delta\)-fast). Let \(g \in \pi_1 X^*\) act on \(\mathcal{H}\) as a loxodromic isometry and let \(\Delta \geq 0\). Then \(g\) is \(\Delta\)-fast if the following holds. Let \(\tilde{A}\) be a combinatorial geodesic axis in \(\tilde{X}\) for some \(\tilde{g} \in \pi_1 X\) mapping to \(g\). Let \(A\) be the image of \(\tilde{A}\) in \(\text{Cay}(X^*)\) and let \(x \in A\) be a 0–cube. Let \(R \geq 0\) and let \(\alpha\) be a geodesic in \(\text{Cay}(X^*)\) from \(x\) to \(g^R x\). Then any subpath of \(\alpha\) lying in a hyperplane carrier or relator has length at most \(\Delta\).

We are now ready for the main lemma:

**Lemma 4.5** (Fast loxodromic implies WPD). Suppose \(g \in \pi_1 X^*\) acts loxodromically on \(\mathcal{H}\) and that \(g\) is \(\Delta\)-fast for some \(\Delta\). Then for all \(\epsilon > 0\), \(\tilde{x} \in \mathcal{H}\), there exists \(R \in \mathbb{N}\) so that

\[|\{h \in G \mid d_{\mathcal{H}}(h \tilde{x}, \tilde{x}) \leq \epsilon, d_{\mathcal{H}}(hg^R \tilde{x}, g^R \tilde{x}) \leq \epsilon\}| < \infty,\]

i.e., \(g\) is a WPD element.

**Proof.** Fix \(\epsilon > 0\) and let \(\tilde{x} \in \mathcal{H}\); since \(\Pi\) is coarsely surjective, we can assume \(\tilde{x} = \Pi(x)\) for some vertex \(x\) of \(\text{Cay}(X^*)\). We will actually show that the claim holds provided \(R\) is chosen sufficiently large (in terms of \(g, x, \epsilon\)).
Let $\tau \geq 1$ be the translation length of $g$ on the graph $\mathcal{H}$.

It suffices to prove the claim for a specific $x$, so we can assume that $x$ lies on the image in $\text{Cay}(X^*)$ of the combinatorial geodesic axis of some lift of $g$ to $\pi_1 X$. Hence, since $g$ is $\Delta$-fast, for any $R$ and any geodesic $\alpha$ from $x$ to $g^R x$, any subpath of $\alpha$ lying in a hyperplane carrier or a relator has length at most $\Delta$.

Fix an integer $R$ satisfying $R \geq 10^9(\epsilon + 2\delta + \Delta + \kappa'_1)/\tau$, where $\kappa'_1$ is a constant depending only on $g$ and $x$ and chosen below.

**Rank one lift:** Let $\tilde{g} \in \pi_1 X$ be any lift of $g$ and let $\tilde{A}$ be a combinatorial geodesic axis for $\tilde{g}$. If $\tilde{g}$ is not rank one, then the image $A$ of $\tilde{A}$ in $\mathcal{H}$ has diameter at most $3\delta$, by property (4d) of the map $\Pi$ together with [Hag13, Proposition 5.1]. This contradicts that $g$ is loxodromic. Hence $\tilde{g}$ is rank one.

**Setup for verifying WPD condition:** Let $y = g^R x$. Fix a combinatorial geodesic $\alpha$ of $X^*$ from $x$ to $y$. Suppose that $h \in \pi_1 X^*$ satisfies $d_H(\Pi(x), \Pi(hx)) < \epsilon$ and $d_H(\Pi(x), \Pi(y)) < \epsilon$.

**The triangle:** Let $\beta$ be a $\text{Cay}(X^*)$-geodesic from $x$ to $hx$, let $\eta$ be a geodesic from $hx$ to $y$, and let $\gamma$ be a geodesic from $hx$ to $y$, so that we have geodesic triangles $\alpha^{-1}\beta\gamma$ and $\eta^{-1}(\alpha\eta)^{-1}\gamma$ with common side $\gamma$, as in Figure 8.

**Figure 8.** The geodesic triangles, showing the orientations of the various paths.

**Applying the classification of triangles:** By Theorem 3.25, we have a disc diagram $D = D_1 \cup \gamma D_2 \to X^*$, shown in Figure 9, with boundary path $\alpha^{-1}\beta(\alpha\eta)\eta$, with the following structure:

- The diagram $D_1$ has boundary path $\alpha^{-1}\beta\gamma$ and $D_2$ has boundary path $\eta^{-1}(\alpha\eta)^{-1}\gamma$.
- The diagrams $D_1$ and $D_2$ are minimal for the given boundary paths.
- For $i \in \{1, 2\}$, the diagram $D_i$ decomposes as $B_i^1 \cup B_i^2 \cup B_i^3 \cup S_i$, where $S_i$ is a standard diagram in the sense of Theorem 3.25 and each $B_i^j$ is a bigon diagram in $X$ (i.e., no cone cells). The boundary path of $S_i$ is a geodesic triangle $A_i B_i C_i$, where $A_i \alpha, B_i \beta^{-1}, C_i \gamma^{-1}$ are the boundary paths of $B_i^1, B_i^2, B_i^3$ respectively, and $A_2(h\alpha)^{-1}, B_2\eta^{-1}$, and $C_2\gamma$ are the boundary paths of $B_2^1, B_2^2, B_2^3$ respectively.
- The diagram $S_i$ contains a constituent padded ladder $L_i$ whose image in $X^*$ projects under $\Pi$ to a set of diameter at least $R - 2(\epsilon + 2\delta)$, along with two ladders projecting to sets of diameter $\leq 10(\epsilon + 2\delta)$. Specifically, the padded ladder $L_1$ is the subdiagram of $D_1$ obtained as follows: either $D_1$ is a ladder, in which case $L_1 = D_1$, or there is a cone-cell or tripod with 3 complementary components, all of whose closures are padded.
Claim 6 (Fellow-traveling of $\alpha, A'_1$). There exist $\kappa_1, s_0 > 0$, depending only on $g, x, R$, such that the following hold:

- Parametrising $h\alpha$ so that $(h\alpha)(t) = h \cdot \alpha(t)$, and parametrising $A_2$ so that it starts at $hx$, we have $d(h\alpha(t), A'_2(t)) \leq \kappa_1$ for $0 \leq t \leq |A'_2|$.
- For $0 \leq t \leq |A'_1|$, we have $d(\alpha(t + s), (A'_1)^{-1}(t)) \leq \kappa_1$ for some $s \leq s_0$. Hence, after enlarging $\kappa_1$ by an amount depending only on $g, x, R$, we have $d(\alpha(t), (A'_1)^{-1}(t)) \leq \kappa_1$. 

The diagram $D$ is formed by gluing the minimal diagrams $D_1, D_2$ along $\gamma$. Minimality of $D_1, D_2$ was used in order to apply Theorem 3.25 to extract the padded ladders and bigons. (It does not matter whether or not the entire diagram $D$ is minimal for its boundary path.)

The above notation is summarized in Figure 9.

**Bounds on cones and pseudorectangles:** Let $A'_1$ be the part of $A_1$ on the boundary path of the ladder $L_1$. (Note that $(A'_1)^{-1}$ starts somewhere on $A_1$, and ends at $y$.)

Then there is a decomposition $(A'_1)^{-1} = \rho_0 \sigma_1 \rho_1 \cdots \sigma_s \rho_s$, where each $\rho_i$ lies on a pseudorectangle and each $\sigma_i$ lies on the boundary path of a cone-cell. Our choice of $\Delta$ ensures that $|\sigma_i|, |\rho_i| \leq \Delta$, with the following exception: we may have $|\rho_i| > \Delta$ if the pseudorectangle carrying $\rho_i$ is horizontally degenerate.

The maximal subpath $A'_2$ of $A_2$ (starting at $hx$) lying on the ladder $L_2$ decomposes as $\vartheta_0 \delta_1 \cdots \vartheta_t \vartheta_t$, where each $\vartheta_i$ lies on a pseudorectangle, each $\delta_i$ lies on a cone-cell, and each $|\vartheta_i|, |\vartheta_i| \leq \Delta$, except that we may have $|\vartheta_i| > \Delta$ if $\vartheta_i$ is carried on a horizontally degenerate pseudorectangle. See Figure 10.

**The paths $C'_1, C'_2$:** For each $i$, let $R_i$ be the pseudorectangle carrying $\rho_i$ and let $\rho'_i$ be the part of the boundary path of $R_i$ parallel to (i.e. crossing the same dual curves as) $\rho_i$. Let $K_i$ be the cone-cell carrying $\sigma_i$ and let $\sigma'_i$ be the part of $\partial_y K_i$ between $\rho'_{i-1}$ and $\rho'_i$, as shown in Figure 10. Let $C'_1 = \rho'_0 \sigma'_1 \rho'_1 \cdots \sigma'_s \rho'_s$ be the part of $C_1$ formed by concatenating these paths. Define $\sigma'_i, \vartheta'_i$, and the resulting subpath $C'_2$ of $C'^{-1}$ analogously.
Proof of Claim 6. Recall that \( \tilde{g} \) is rank one. Now, \( \alpha A_1 \) lifts to a geodesic bigon in \( \tilde{X} \) (because the diagram between them is a square diagram and hence lifts), and Lemma 4.3 shows that \( \alpha \) and \( A_1 \) lie at Hausdorff distance \( \kappa'_1 \) bounded in terms of \( g \) and \( x \).

Choose \( s \) so that \( d(\alpha(s), (A'_1)^{-1}(0)) \leq \kappa'_1 \). For the given \( \alpha \) and choice of diagram \( D \), we have such an \( s \). Since \( D \) is one of finitely many possible diagrams constructed in the given way for the finitely many choices of \( \alpha, \beta, \gamma, \eta \) (\( \text{Cay}(\mathbb{X}^*) \) is locally finite and \( qx^R \) is fixed), there is an upper bound \( s_0 \) on such \( s \) depending only on \( g, x, R \). A computation supplies \( \kappa_1 \) in terms of \( \kappa'_1 \), proving the second assertion.

The first assertion follows similarly, except one does not need \( s \) since \( ha \) and \( A'_2 \) have the same initial point. (The asymmetry is because \( \alpha \) and \( (A'_1)^{-1} \) have the same terminal point; refer to Figure 8.)

(Note that \( s \) is small in the following sense. Since \( \Pi \circ \alpha \) has length at least \( \tau R \) in \( \mathcal{H} \), and \( d_{\mathcal{H}}(\Pi((A'_1)^{-1}(0)), \Pi(x)) \leq 10(\epsilon + 2\delta) + 2\delta \), the path \( \alpha_{[s,|a|]} \) projects to a path in \( \mathcal{H} \) of length at most \( \tau R - (10\epsilon + 22\delta + \kappa'_1) \). Since \( \Pi \) is lipschitz, the same number bounds from below the length of \( \alpha_{[s,|a|]} \). We won’t need this fact, but we will use a similar fact about the paths \( C'_1, C'_2 \) established below.)

Fellow-traveling of \( C'_1, C'_2 \): Next, consider the subdiagram \( E = B^3_1 \cup_{\gamma} B^3_2 \) of \( D \) bounded by \( C_1 C_2 \). Since \( C_1, C_2 \) are geodesics, and \( E \) is a square diagram, every dual curve starting on \( C_1 \) ends on \( C_2 \) and every dual curve starting on \( C_2 \) ends on \( C_1 \).

Let \( K, K' \) be dual curves that emanate from \( C'_1 \) and cross each other. Let \( C''_1 \) be the subpath of \( C'_1 \) between and including the 1–cubes dual to \( K \) and \( K' \). These 1–cubes \( e, f \) respectively lie on common cone-cells or pseudorectangles with points \( a_e, a_f \) on \( A'_1 \). We can choose these so that \( d(e, a_e), d(f, a_f) \leq 2\Delta \).

Now, \( a_e, a_f \) respectively lie at distance at most \( \kappa_1 \) from points on \( b_e, b_f \in \alpha \). Hence \( d(e, b_e) \leq 2\Delta + \kappa_1 \) and \( d(f, b_f) \leq 2\Delta + \kappa_1 \). Thus \( d(b_e, b_f) \geq |C''_1| - 4\Delta - 2\kappa_1 \). On the other hand, \( d_{\mathcal{H}}(\Pi(b_e), \Pi(b_f)) \leq 2\delta + 4\Delta + 2\kappa_1 \), by property 4d of \( \Pi \) and the fact that \( K, K' \) cross. (Indeed,
there is a path in $\mathcal{N}(K) \cup \mathcal{N}(K')$ from $e$ to $f$, so $d_H(\Pi(e), \Pi(f)) \leq 2\Delta$, and $\mathcal{N}(K), \mathcal{N}(K')$ map to hyperplane carriers in $\text{Cay}(X^*)$, whose images in $\mathcal{H}$ have diameter at most $\delta$.

Let $v_\alpha = \max \frac{d(p,q)}{d_H(\Pi(p), \Pi(q))}$, where $p,q$ vary over vertices of $\alpha$ with distinct images in $\mathcal{H}$, and let $v$ be the maximal $v_\alpha$ over the (finitely many) choices of $\alpha$ with the given endpoints (here we are using that $X$, and hence $\text{Cay}(X^*)$, is locally finite). Let $\zeta_\alpha$ be the maximum of $d(p,q)$ as $p,q$ vary over vertices of $\alpha$ with $\Pi(p) = \Pi(q)$, and let $\zeta$ be the maximum of the $\zeta_\alpha$ over all choices of $\alpha$. Note that $\zeta_\alpha$ depend on $g,x$ and $R$, but not $h$ or $\alpha$.

So, $|C_{\alpha}^\prime| \leq (2\delta + 10\Delta + 2k_1)v + \zeta + 10\Delta + 2k_1$. In other words, there exists $N$ depending only on $g,x,R$ such that any two dual curves that emanate from $C_{\alpha}^\prime$ at distance more than $N$ cannot cross.

Parameterise $C_1,C_2$ so that $C_1(0) = C_2^{-1}(0) = hx$. Our choice of $R$ ensures that there exist $t_0, t_0'$, depending only on $R, \epsilon, \Delta, \delta$, $\tau$ so that $C_1(t), C_2^{-1}(t)$ lie on $C_1', C_2'$ respectively when $t_0 \leq t \leq t_0'$.

Let $t = \frac{t_0 - t_0'}{2}$ and let $z = C_1(t)$. Let $z' = C_2^{-1}(t)$. Since $E$ is a square diagram, it lifts to a square diagram in $\tilde{X}$ bounded by geodesics lifting $C_1, C_2^{-1}$. So $d(z,z')$ is bounded by the number of dual curves $K$ in $E$ that cross $C_1$ before $z$ and $C_2^{-1}$ after $z'$, or vice versa. Our choice of $R$ ensures that any such $K$ cannot cross $C_1 - C_1'$ or $C_2 - C_2'$, since property (4d) would then provide a shortcut in $\mathcal{H}$ from $\Pi(hx)$ to $\Pi(gx)$. Let $K$ be a dual curve crossing $C_1$ before $z$ and $C_2'$ after $z$ and let $L$ be a dual curve crossing $C_2'$ before $z'$ and $C_1'$ after $z$. Then the distance along $C_1'$ between $K$ and $L$ is at most $N$, so $K$ and $L$ are $N$–close to $z$. Thus $d(z,z')$ is bounded in terms of $N$, say by some $\kappa_2$.

**Fellow-traveling of** $(A_{\alpha}^\prime)^{-1}, C_1'$ and $A_{\alpha}^\prime, C_2'$: By Lemma 4.1, there exists $\kappa_3$, depending on $\Delta$ and the small-cancellation assumption, so that $d((A_{\alpha}^\prime)^{-1}(t), C_1'(t)) \leq \kappa_3$. The same is true for $A_{\alpha}^\prime, C_2'$.

**Conclusion:** Let $z, z'$ be as above. Then $z$ is uniformly close to $\alpha(t)$ (the distance is bounded by $\kappa_1 + \kappa_3$), and the same is true for $h\alpha(t), z'$. Hence $d(\alpha(t), h\alpha(t)) \leq 2(\kappa_1 + \kappa_3) + \kappa_2$, which does not depend on $h$. Since $\text{Cay}(X^*)$ is locally finite and $\pi_1X^*$ acts freely on $\text{Cay}(X^*)$, the action of $\pi_1X^*$ on $\text{Cay}(X^*)$ is metrically proper and hence there are finitely many such $h$. □

5. The hyperbolic space $\mathcal{H}$ and the projection $\Pi: \text{Cay}(X^*) \to \mathcal{H}$

Let $\langle X | \{Y_i\}_{i \in I} \rangle$ be a cubical $C''(\frac{1}{144})$ presentation and define a space $\mathcal{H}$ as follows. First, let $\mathcal{H}'$ be the 1–skeleton of $\tilde{X}^*$. This consists of the 1–skeleton of $\text{Cay}(X^*)$, together with a combinatorial cone on each lift of each $Y_i$. We form $\mathcal{H}$ from $\mathcal{H}'$ by adding a combinatorial cone on the carrier of each hyperplane.

We also have a projection $\Pi: \text{Cay}(X^*) \to \mathcal{H}$, defined as follows. On the 1–skeleton of $\text{Cay}(X^*)$, we declare $\Pi$ to be the inclusion. If $c$ is a cube of $\text{Cay}(X^*)$ with $\text{dim}(c) \geq 2$, we send $c$ arbitrarily to a point in the image of its 1–skeleton. However, we require this choice to be made $\pi_1X^*$–equivariantly, so that $\Pi$ is $\pi_1X^*$–equivariant. Obviously $\Pi$ is coarsely surjective and Lipschitz on the 1–skeleton of $\text{Cay}(X^*)$. By construction, $\Pi$ sends each cone to a set of diameter $\leq 2$, while hyperplane carriers in $\text{Cay}(X^*)$ are sent to subsets of $\mathcal{H}$ with diameter at most 2. Hence, to see that $\mathcal{H}$ and $\Pi$ satisfy the conditions required in Section 4, we need only to prove that $\mathcal{H}$ is hyperbolic.

**Lemma 5.1** (Square bigons have thin projection). Let $\alpha, \beta \to \text{Cay}(X^*)$ be geodesics with common endpoints, and suppose that $\alpha\beta$ bounds a disc diagram $D \to \tilde{X}^*$ that does not contain any cone-cells. Then $\Pi(\alpha), \Pi(\beta)$ lie at uniformly bounded Hausdorff distance in $\mathcal{H}$.

**Proof.** Let $e$ be a 1–cube of $\alpha$ and let $K$ be the dual curve in $D$ emanating from $e$ and mapping to a hyperplane $H$ of $\text{Cay}(X^*)$. Since $\alpha$ is a geodesic, $K$ terminates at a 1–cube $f$ of $\beta$, whence $d_H(\Pi(e), \Pi(f)) \leq 2$. Hence $\Pi(\alpha) \subseteq \mathcal{N}_2(\Pi(\beta))$ and the proof is complete by symmetry. □
Proposition 5.2. The graph $\mathcal{H}$ is hyperbolic.

Proof. It suffices to prove that the 0–skeleton of $\text{Cay}(X^*)$, with the subspace metric inherited from $\mathcal{H}$, is hyperbolic. First, suppose that $\alpha \beta \gamma$ is a geodesic triangle in $\text{Cay}(X^*)$. Then Lemma 5.1 combines with Theorem 3.25 and the fact that $\Pi$ sends cones to uniformly bounded sets to show that each of $\Pi(\alpha), \Pi(\beta), \Pi(\gamma)$ is contained in the $\delta'$–neighborhood in $\mathcal{H}$ of the union of the other two, for some uniform $\delta'$. The Guessing Geodesics Lemma (see e.g. [Ham07 Proposition 3.5] [Bow14 Proposition 3.1]) now implies that $\mathcal{H}$ is hyperbolic. □

Theorem 5.3. Let $\langle X \mid \{Y_i\}_{i \in I}\rangle$ be a $C'(\frac{1}{144})$ presentation with $X$ locally finite, and let $G = \pi_1X^*$. Then any $g \in G$ acting on the space $\mathcal{H}$ constructed above as a first loxodromic element acts on $\mathcal{H}$ as a WPD element, whence either $G$ is virtually cyclic or acylindrically hyperbolic.

Proof. The assertion that $g$ is a WPD element follows from Lemma 4.5; hyperbolicity of $\mathcal{H}$ comes from Proposition 5.2. Applying [Osi16 Theorem 1.2. ($AH_3 \Rightarrow AH_2$)] completes the proof. □

6. Proof of Theorem B

We now study the question of when $\pi_1X^*$ contains a loxodromic isometry of $\mathcal{H}$, using knowledge of which elements of $\pi_1X$ act loxodromically on the contact graph $\mathcal{C}\hat{X}$ of $\hat{X}$, which is the intersection graph of the set of hyperplane carriers in $\hat{X}$.

Let $p: \hat{X} \to \text{Cay}(X^*)$ be the universal covering map (regarding $\text{Cay}(X^*)$ as the cover of $X$ corresponding to the subgroup $\hat{K} = \langle\{\pi_1Y_i\}_{i \in I}\rangle$ of $\pi_1X$). Let $\hat{\mathcal{H}}$ be the graph obtained from $\hat{X}$ by coning off the 1–skeleton of each hyperplane carrier.

Form a new graph $\hat{\mathcal{H}}$ from $\hat{\mathcal{H}}$ by coning off every subgraph of $\hat{X}(1) \subseteq \hat{\mathcal{H}}$ which is the 1–skeleton of an elevation $\hat{Y}_i \hookrightarrow \hat{X}$ of some $Y_i \to X$. Observe that $p$ induces a quotient map $\hat{\rho}: \hat{\mathcal{H}} \to \hat{\mathcal{H}}$, which restricts to $p$ on $\hat{X}(1)$ and which sends the cone-point $\nu_H$ over the hyperplane carrier $\mathcal{N}(H)$ to the cone-point $\nu_{p(H)}$ over the hyperplane carrier $p(\mathcal{N}(H))$. The map $\hat{\rho}$ also sends the cone over each elevation $\hat{Y}_i$ to the cone over the corresponding lift $Y_i \to \hat{X}^*$ of $Y_i \to X$.

Remark 6.1 (Standing assumptions). In this section, we introduce extra hypotheses on the $C'(\frac{1}{144})$ cubical presentation $\langle X \mid \{Y_i\}_{i \in I}\rangle$. First, we assume that $X$ is compact and that each $Y_i$ is compact, as in Theorem B. Second, we assume that $\langle X \mid \{Y_i\}_{i \in I}\rangle$ satisfies the uniform $C''(\frac{1}{144})$ condition from Definition 3.12 again as in Theorem B. Recall that this implies that each geodesic $P$ in any piece satisfies $|P| < \frac{1}{144} \inf_{i \in I} \|Y_i\|$. In particular, there is a uniform bound on the lengths of such $P$. Later, we will introduce additional conditions.

Specifically, we will replace each $Y_i$ by a finite cover $\tilde{Y}_i \to Y_i \to X$. Note that the collection of elevations $\tilde{Y}_i \to \hat{X}$, and their stabilisers in $\pi_1X$, do not depend on the cover $\tilde{Y}_i$.

6.1. Coned-off spaces. We first relate the various coned-off spaces to the corresponding graphs in a standard way.

Lemma 6.2. Let $\Gamma$ be the graph with a vertex for each hyperplane carrier in $\hat{X}$, and a vertex for each elevation $\tilde{Y}_i \hookrightarrow \hat{X}$ of each $Y_i \to X$, with adjacency corresponding to intersection. Then $\Gamma$ is $\pi_1X$–equivariantly quasi-isometric to $\hat{\mathcal{H}}$, and $\hat{\mathcal{H}}$ is $\pi_1X$–equivariantly quasi-isometric to $\mathcal{C}\hat{X}$.

Proof. Define a map $f: \Gamma^{(0)} \to \hat{\mathcal{H}}$ by sending each vertex (corresponding to a hyperplane carrier or an elevation $\tilde{Y}_i$) to the corresponding cone-point. Since each point of $\hat{X}$ lies in a hyperplane carrier, the map $f$ is quasi-surjective. If $v, w$ are vertices of $\Gamma$, corresponding to subcomplexes $\widehat{C}_v, \widehat{C}_w$, and $v, w$ are adjacent, then $\widehat{C}_v \cap \widehat{C}_w \neq \emptyset$, so $d_{\hat{H}}(f(v), f(w)) \leq 2$. Hence $f$ is coarsely Lipschitz. By sending each cone-point $v$ in $\hat{\mathcal{H}}$ to the vertex of $\Gamma$ corresponding to the subcomplex over which $v$ is the cone, we obtain a coarsely Lipschitz quasi-inverse for $f$, so $f$ is a quasi-isometry. That is $\pi_1X$–equivariant follows immediately from the definition of $f$. This shows
Remark 6.3 (Summary of the graphs). The various graphs are summarised below for reference:

- $\mathcal{C}X$ is the contact graph of $\bar{X}$, i.e. the intersection graph of the hyperplane carriers.
- $\Gamma$ is the intersection graph of the family of subcomplexes of $\bar{X}$ that are either hyperplane carriers or elevations $\bar{Y}_i$ of relators.
- $\mathcal{H}$ is the graph formed from $\bar{X}^{(1)}$ by coning off the 1–skeleta of the various hyperplane carriers.
- $\mathcal{H}$ is the graph formed from $\bar{H}$ by coning off the subgraphs of $\bar{X}^{(1)} \subseteq \bar{H}$ that are 1–skeleta of subcomplexes of the form $\bar{Y}_i$.
- $\mathcal{H}$ is formed from the 1–skeleton of $\bar{X}^*$ by coning off the 1–skeleton of each hyperplane carrier, as in Section 5.

The first and third graphs are $\pi_1 X$–equivariantly quasi-isometric, and the second and fourth graphs are $\pi_1 X$–equivariantly isometric.

The graph $\mathcal{H}$ is the “main object of study”, and was chosen because it generalises the hyperbolic space used in the graphical case [GS14]. It is therefore simplest in many settings to work with $\mathcal{H}$ and $\bar{H}$, because of the map $p$.

But in other settings, we invoke results in the literature about actions on $\mathcal{C}X$. Moreover, for statements like Lemma 6.4 it is a bit simpler to work with $\mathcal{C}X$ and $\Gamma$ rather than $\mathcal{H}$ and $\bar{H}$. This explains why we use all of the different graphs. The reader is encouraged to refer to the above list as needed.

6.2. When do loxodromics on $\bar{H}$ stay loxodromic on coning? Since our ultimate goal is to understand when $\pi_1 X^*$ contains a loxodromic isometry of $\mathcal{H}$, and existing tools (mainly from [CS11, Hag13]) tell us when elements of $\pi_1 X$ act loxodromically on $\mathcal{C}X$, we need to relate these phenomena.

Lemma 6.4 (Loxodromics persist upstairs). Let $\bar{g} \in \pi_1 X$ act loxodromically on $\bar{H}$. Then either $\bar{g}$ acts loxodromically on $\mathcal{H}$ or $\bar{g}$ stabilizes some elevation $\bar{Y}_i \subseteq \bar{X}$ of some $Y_i \to X$.

Proof. By Lemma 6.2 $\bar{H}$ is quasi-isometric to the intersection graph $\Gamma$ of the set of hyperplane carriers and various elevations $\bar{Y}_i$ in $\bar{X}$. Hence it suffices to show that $(\bar{g})$ acts loxodromically on $\Gamma$ provided that $\bar{g}$ doesn’t stabilise any $\bar{Y}_i$.

The axis: Let $\bar{A} \subseteq \bar{X}$ be a combinatorial geodesic axis for $\bar{g}$ (by replacing $\bar{X}$ by its first cubical subdivision, we may assume that such an axis exists [Hag07]). Fix a 0–cube $\bar{a} \in \bar{A}$ and fix $n > 0$. Let $P$ be the subpath of $\bar{A}$ joining $\bar{a}$ to $g^n \bar{a}$.

The corresponding geodesic in $\Gamma$: Let $\bar{Q}$ be a geodesic of $\Gamma$ joining vertices corresponding to subcomplexes containing $\bar{a}$ and $g^n \bar{a}$. Let the vertex-sequence of $\bar{Q}$ be $C_0, \ldots, C_N$, where each $C_i$ is either a hyperplane carrier in $\bar{X}$ or an elevation $\bar{Y}_i \subseteq \bar{X}$. Then we have a combinatorial path $Q = \alpha_0 \cdots \alpha_N$ joining $\bar{a}$ to $g^n \bar{a}$, where each $\alpha_i$ is a geodesic in $C_i$. The closed path $QP^{-1}$ bounds a disc diagram $D \to \bar{X}$.

Claim 7. The geodesic $\bar{Q}$, and the corresponding path $Q$, can be chosen so that $Q$ is a geodesic of $\bar{X}$.

Proof of Claim 7. This argument is entirely about CAT(0) cube complexes (i.e. it takes place in $\bar{X}$). The argument is almost exactly the same as the proof of Proposition 3.1 of [BHS17], with one change: the argument in [BHS17] corresponds to the case $I = \emptyset$, but it only uses convexity of carriers; since the $\bar{Y}_i$ are convex, the same argument works here.
Suppose that \( \tilde{Q} \) and \( Q \) have been chosen as above in such a way that any minimal-area disc diagram \( D \to \tilde{X} \) with boundary path \( QP^{-1} \) has area that is minimal for all such choices of \( \tilde{Q}, Q, \) and \( Q \) has no backtracks. (If there are backtracks, we can always remove them. This does not quite make \( Q \) a geodesic, since it can still have distinct non-consecutive edges dual to the same hyperplane.)

Let \( K \) be a dual curve in \( D \) emanating from \( P \). Then \( K \) ends on \( Q \). Indeed, since \( P \) is a geodesic of \( \tilde{X} \), the hyperplane to which \( K \) maps is dual to at most one 1–cell of \( P \).

Now let \( K \) be a dual curve in \( D \) emanating from \( Q \). Then \( K \) emanates from some \( \alpha_i \). Suppose that \( K \) ends on \( \alpha_i \). Since \( \alpha_i \) is a geodesic, we have \( j \neq i \). If \( |i-j| > 2 \), then since the hyperplane to which \( K \) maps has carrier \( \Gamma \)–adjacent to \( C_i, C_j \), we have contradicted that \( |\tilde{Q}| \) is a \( \Gamma \)–geodesic.

If \( j = i \pm 2 \), then we can replace \( C_{i\pm1} \) with the carrier of the hyperplane to which \( K \) maps (i.e., modify \( \tilde{Q} \) and \( Q \)), providing a new choice of \( \tilde{Q} \) leading to a lower-area choice of \( D \).

Finally, if \( j = i \pm 1 \), then we can apply hexagon moves to show that \( \alpha_i \) has a terminal segment coinciding with an initial segment of \( \alpha_{i+1} \) (say), contradicting that \( Q \) has no backtrack.

We conclude that \( D \) can be chosen so that all dual curves travel from \( Q \) to \( P \). Hence, \( Q \) and \( P \) have the same length, so \( Q \) is a geodesic of \( \tilde{X} \).

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Claim 8. Suppose that \( \tilde{g} \) does not have a positive power that stabilises any elevation \( \tilde{Y}_i \) of any \( Y_i \). Then there exists \( L \geq 1 \), depending only on \( \tilde{g} \), such that the following holds. Let \( \tilde{B} \) be either a hyperplane of \( \tilde{X} \) or an elevation of some \( Y_i \to X \). Then at most \( L \) hyperplanes cross both \( \tilde{B} \) and the axis \( \tilde{A} \).

Proof of Claim. First suppose that \( \tilde{B} \) is a hyperplane. Since \( \tilde{g} \) is loxodromic on \( \tilde{H} \), Lemma 6.2 implies that \( \tilde{g} \) is loxodromic on \( \mathcal{C} \tilde{X} \), and the claim follows.

Suppose that for each \( J \geq 0 \), there exists \( \tilde{B}_J \), an elevation of some \( Y_i \to X \), such that at least \( J \) hyperplanes cross both \( \tilde{B}_J \) and \( \tilde{A} \).

Consider the set of hyperplanes crossing \( \tilde{A} \). We first observe that there exists \( n_0 \geq 1 \) and a hyperplane \( H \) crossing \( \tilde{A} \) such that \( \langle \tilde{g}^{n_0} \rangle \cdot H \) is an infinite collection of disjoint hyperplanes. Indeed, otherwise \( \langle \tilde{g} \rangle \) has a bounded orbit in \( \mathcal{C} \tilde{X} \), contradicting the assumption that \( \tilde{g} \) is loxodromic.

Let \( J' \gg 0 \) be an integer to be determined. Since \( \tilde{g} \) is a rank-one element, Remark 4.2 implies that there exists \( J \) such that at least \( J' \) elements of \( \langle \tilde{g}^{n_0} \rangle \cdot H \) cross both \( \tilde{A} \) and \( \tilde{B}_J \). By translating, we can assume that \( H, \tilde{g}^{n_0}H, \ldots, \tilde{g}^{n_0(J'-1)}H \) cross \( \tilde{B}_J \).

Therefore, \( \tilde{g}^{n_0}H, \ldots, \tilde{g}^{n_0(J'-1)}H \) cross \( \tilde{g}^{n_0} \tilde{B}_J \). Hence, there are \( J' - 1 \) hyperplanes that cross both \( \tilde{B}_J \) and \( \tilde{g} \tilde{B}_J \) and cross the geodesic \( \tilde{A} \). For \( J' \) sufficiently large (in terms of the bound on the diameter of cone-pieces provided by the uniform small-cancellation condition, as in Remark 6.1), this gives a contradiction unless \( \tilde{g}^{n_0} \tilde{B}_J = \tilde{B}_J \). This proves the claim.

Suppose that no positive power of \( \tilde{g} \) stabilises an elevation of a relator.

Letting \( L \) be the constant from Claim 8. Our choice of \( Q \) guarantees that each hyperplane crossing each \( \alpha_i \) crosses both the subcomplex \( C_i \) and the axis \( \tilde{A} \). Hence Claim 8 implies that \( |\alpha_i| \leq L \). Thus, \( |\tilde{Q}| \geq L^{-1} |Q| = L^{-1} |\mathcal{d}_{\tilde{X}}(a, \tilde{g}^n a) \) (and \( L \) is independent of \( n \)). Thus, \( |\tilde{Q}| \geq (10\lambda L)^{-1} |\mathcal{d}_{\tilde{X}}(a, \tilde{g}^n a) \). Since the latter quantity is bounded below by a linear function of \( n \) (because \( \tilde{g} \) acts hyperbolically on \( \tilde{X} \)), so is the former. So, \( \tilde{g} \) is loxodromic on \( \tilde{H} \).
Finally, if for some $n > 0$ we have $\tilde{g}^n \in \text{Stab}_{\pi_1 X}(\tilde{Y}_i)$, then let $\tilde{A}$ be an axis for $\tilde{g}$. So $\tilde{A}$ lies in a regular neighbourhood of both $\tilde{Y}_i$ and $\tilde{g}\tilde{Y}_i$, leading to impossibly large pieces unless $\tilde{g} \in \text{Stab}_{\pi_1 X}(\tilde{Y}_i)$.

\section{Asystolicity and embeddability}

We now consider two properties of lifts of an element of $\pi_1 X^*$ to $\pi_1 X$, along the homomorphism $\pi_1 X \to \pi_1 X^*$. Embeddability of a lift guarantees that its image has infinite order. Asystolicity is stronger and more concrete.

\begin{definition}[Embeddable, asystolic] Fix $g \in \pi_1 X^*$. Any $\tilde{g} \in \pi_1 X$ mapping to $g$ is a lift of $g$. Let $\tilde{A}$ be a combinatorial geodesic axis for $\tilde{g}$, which exists provided $\tilde{g} \neq 1$, since $\pi_1 X$ is torsion-free and isometries of $X$ are combinatorially semisimple \cite{Hag07}. Recall that $p: \tilde{X} \to \text{Cay}(X^*)$ denotes the universal covering map. Let $A$ denote $p(\tilde{A})$.

\begin{enumerate}
\item $\tilde{A}$ is an embeddable axis if $p$ restricts to a cubical isomorphism $\tilde{A} \to A$, i.e. if $A$ is an embeddable combinatorial line in $\text{Cay}(X^*)$.
\item We say that $\tilde{g}$ is an embeddable lift of $g$ if $\tilde{g}$ has at least one embeddable axis.
\end{enumerate}

Now fix $\lambda \in [0,1]$.

\begin{enumerate}[resume]
\item We say that $\tilde{A}$ is a $\lambda$-asystolic axis if for each subpath $P$ of $\tilde{A}$ such that $P \subseteq \tilde{Y}_i$, where $\tilde{Y}_i$ is an elevation of a relator $Y_i$, we have $|P| < \lambda \|Y_i\|$.
\item The lift $\tilde{g}$ is $\lambda$-asystolic if every combinatorial geodesic axis of $\tilde{g}$ is a $\lambda$-asystolic axis.
\end{enumerate}

Note that if $\tilde{g}$ is an embeddable lift, then $g$ has infinite order, and $\tilde{g}^n$ is an embeddable lift of $g^n$ for all $n > 0$. Indeed, if $\tilde{A}$ is an embeddable axis for $\tilde{g}$, it is also an embeddable axis for $\tilde{g}^n$. Note that if $\tilde{g}$ is an $\lambda$-asystolic lift of $g$, then $\tilde{g}^n$ is an $\lambda$-asystolic lift of $g^n$ for all $n > 0$.

\begin{lemma}[Embeddability from $\frac{35}{72}$-asystolicity] Let $\tilde{g} \in \pi_1 X$ and let $g$ be its image in $\pi_1 X^*$. Suppose that $\tilde{A}$ is a $\frac{35}{72}$-asystolic axis for $\tilde{g}$. Then $\tilde{A}$ is an embeddable axis for $\tilde{g}$, so $\tilde{g}$ is an embeddable lift of $g^n$ for all $n \in \mathbb{Z} - \{0\}$, and $g$ has infinite order.

\begin{proof}
It suffices to prove the claim for $n = 1$. Let $\tilde{A}$ be a combinatorial geodesic axis for $\tilde{g}$, and suppose that $\tilde{A}$ is $\frac{35}{72}$-asystolic.

Suppose that $\tilde{A}$ is not an embeddable axis. Then $p: \tilde{A} \to A$ is not injective, so there exist distinct $0$–cubes $\tilde{y}, \tilde{y}' \in \tilde{A}$ such that $p(\tilde{y}) = p(\tilde{y}')$. In other words, letting $\tilde{P}$ be the subpath of $\tilde{A}$ joining $\tilde{y}$ to $\tilde{y}'$, the path $P = p \circ \tilde{P}$ in $\text{Cay}(X^*)$ is a nontrivial closed path. Let $D \to \tilde{X}^*$ be a minimal-complexity disc diagram with boundary path $P$.

\textbf{No spurs:} We claim that $P$ has no spurs. Indeed, if $P$ has a spur $ee^{-1}$, then $\tilde{P}$ contains a subpath $\tilde{e}_1 \tilde{e}_2$, where $\tilde{e}_1, \tilde{e}_2$ are distinct $1$–cubes such that $\tilde{e}_1 \cap \tilde{e}_2$ is a $0$–cube and $p(\tilde{e}_1) = p(\tilde{e}_2)^{-1}$. By choosing $\tilde{y}, \tilde{y}'$ as close as possible, we can assume that the endpoints of $\tilde{e}_1 \tilde{e}_2$ are $\tilde{y}, \tilde{y}'$, and $\tilde{e}_1, \tilde{e}_2$ are lifts of the $1$–cube $e$. Hence $\tilde{y} = h\tilde{y}'$ for some nontrivial $h \in \ker(\pi_1 X \to \pi_1 X^*)$, so $h$ fixes the $0$–cube $\tilde{e}_1 \cap \tilde{e}_2$. This contradicts that $\pi_1 X$ acts on $\tilde{X}$ freely.

\textbf{Applying diagram trichotomy:} Hence Theorem \ref{thm:diagram_trichotomy} implies that $D$ is one of the following:

\begin{itemize}
\item A single vertex. This is impossible since $P$ is nontrivial.
\item A single cone-cell. In this case, $P$ is an essential path in a relator $Y_i$, by minimality of the complexity. Hence, $|\tilde{P}| \geq \|Y_i\|$, contradicting asystolicity.
\item A ladder, or a diagram with at least three features of positive curvature (shells or generalised corners).
\end{itemize}

In either of the latter two cases, there are at least two features of positive curvature, and neither is a spur.

\textbf{No shells:} Suppose that $C$ is a positively curved shell in $D$ with boundary path $OI$, with $O$ the outer path and $I$ the inner path. Let $Y_i$ be the relator to which the path $\partial_0 C$ maps. Let $I'$ be the shortest path in $Y_i$ that is square-homotopic in $Y_i$ (relative to its endpoints) to
I. By short inner paths (Theorem 3.24), $|I'| < |O|$. Now, $O$ lifts to a subpath of $\tilde{P}$ lying in an elevation $\tilde{Y}_i$ of $Y_i$, so by asystolicity, $|O| < \|Y_i\|/2$. So $|I'O| < \|Y_i\|/2$. Hence $I'O$ is inessential in $Y_i$. Since $I, I'$ are square-homotopic in $Y_i$, we have that $IO = \partial_P C$ is also inessential, so we can replace $C$ by a square diagram to reduce the complexity. Thus, the shell $C$ cannot exist.

**Shuffling generalised corners to the boundary:** Hence every feature of positive curvature along $\partial_P D$ is a generalised corner of a square. By shuffling — see Remark 3.20 — we can assume that these are exposed squares, i.e. each generalised corner of a square along $P$ is actually a length–2 subpath of the boundary path of a square.

**Square-homotoping $P$:** Since there are at least two of these squares, at least one, denoted $s$, satisfies the following: $\partial_P s = I\tilde{e}f$, where $ef$ is a subpath of $\partial_P D$ and the vertex in which $e, f$ intersect is not $p(\tilde{y})$. Hence we can perform a square homotopy, removing $s$ from $D$, to obtain a new diagram $D'$ in which $ef$ is replaced by $I$ in the boundary path. Note that $|\partial_P D'| = |P|$, and $p(\tilde{y}) \in \partial_P D'$.

Thus we can replace $\tilde{A}$ by a $\langle \tilde{g} \rangle$–invariant geodesic $\tilde{A}'$ as follows: lift $ef$ to a path $\tilde{e}\tilde{f}$ in $\tilde{P}$, lift $s$ to a square $\tilde{s}$ meeting $\tilde{A}$ in the subpath $\tilde{e}\tilde{f}$, and homotop $\tilde{A}$ across $\tilde{s}$. Do the same at each $\langle \tilde{g} \rangle$–translate of $\tilde{s}$. Let $\tilde{P}'$ be the subpath of $\tilde{A}'$ from $\tilde{y}$ to $\tilde{y}'$.

Note that $\tilde{P}'$ projects to a closed path $P$ in $\tilde{X}^*$ bounding a proper subdiagram of $D$. Repeating finitely many times, we find a subdiagram of $D$ that has at least two features of positive curvature (it has at least three, or is a ladder), none of which is a spur and at most one of which is a generalised corner (whose boundary path contains $y$). So, there is some positively-curved shell in this subdiagram with outer path $O$. By short inner paths (Theorem 3.24), $|O| > \|Y_i\|/2$.

Hence $\tilde{A}$ is square-homotopic in $\tilde{X}$ to a $\langle \tilde{g} \rangle$–invariant combinatorial geodesic $\tilde{B}$ such that $\tilde{B}$ contains a path $\tilde{O}$ such that $\tilde{O}$ lies in some $\tilde{Y}_i$ and satisfies $|\tilde{O}| > \|Y_i\|/2$. Moreover, $\tilde{y}, \tilde{y}' \in \tilde{B}$, and $\tilde{O}$ is a subpath of the subpath $\tilde{Q}$ of $\tilde{B}$ joining $\tilde{y}$ to $\tilde{y}'$.

Hence there exists a subpath $\tilde{Q}_1$ of $\tilde{B}$ with the following properties:

- the path $\tilde{Q}_1$ contains a subpath $\tilde{O}_1$ that lies in $\tilde{Y}_i$ and is maximal with that property;
- we have $|\tilde{O}_1| > \|Y_i\|/2$;
- either $\tilde{O}_1$ is unbounded, or $\tilde{Q}_1$ starts and ends on $\tilde{A}$.

We consider two cases.

**$\tilde{O}_1$ bounded:** First suppose that $\tilde{O}_1$ is bounded and let $\tilde{P}_1$ be the subpath of $\tilde{A}$ subtended by the endpoints of $\tilde{Q}_1$.

Consider the geodesic bigon $\tilde{Q}_1\tilde{P}_1^{-1}$ in $\tilde{X}$. Let $E \to \tilde{X}$ be a minimal-area disc diagram with $\partial_P E = \tilde{Q}_1\tilde{P}_1^{-1}$. Moreover, since $\tilde{Y}_i$ is convex, we make our choice allowing the geodesic $\tilde{O}_1$ to vary, fixing the endpoints; any such geodesic lies in $\tilde{Y}_i$. In particular, if $E$ is chosen to be of minimal area among all disc diagrams with the given boundary path (with $\tilde{O}_1$ allowed to vary as above), then no two dual curves emanating from $\tilde{O}_1$ can cross.

If $\tilde{Q}_1 = \tilde{O}_1$, then $\tilde{P}_1$ lies in $\tilde{Y}_i$, and, since $|\tilde{P}_1| = |\tilde{Q}_1| > \|Y_i\|/2$, this contradicts our hypotheses.

Now write $\tilde{Q}_1 = U\tilde{O}_1 V$, with at least one of $U, V$ a nontrivial path. We now allow $U, V$ to vary, fixing their endpoints, and assume that $E$ had minimal area over all of these choices. Hence no dual curves emanating from $U$ can cross, and the same is true of $V$.

Hence consider 1–cubes $r, s$ immediately preceding and succeeding $\tilde{O}_1$ in $\tilde{Q}_1$. At least one of $r$ or $s$ exists; assume it is $r$. If the hyperplane $H_r$ dual to $r$ crosses $\tilde{Y}_i$, then convexity of $\tilde{Y}_i$ implies $r \subseteq \tilde{Y}_i$, contradicting maximality of $\tilde{O}_1$. (Indeed, considering the gate map to $\tilde{O}_1$ shows that $r$ must project to a 1–cube parallel to $r$, since $H_r$ crosses $\tilde{Y}_i$. On the other hand, the initial point of $r$ is already in $\tilde{Y}_i$, and thus sent to itself. Thus $r$ is equal to its image under the gate map, so $r \subseteq \tilde{Y}_i$.)

Hence, $H_r$ does not cross $\tilde{Y}_i$. 
Let $K_r$ be the dual curve in $E$ dual to $r$. Let $K$ be a dual curve emanating from $\tilde{O}_1$. Let $L$ be a dual curve crossing $K$. Then $L$ cannot cross $\tilde{O}_1$. If $L$ crosses $U$, then either $L = K_r$, or $L$ is separated in $E$ from $\tilde{O}_1$ by $K_r$, so $K$ must cross $K_r$. Hence, if $K$ is a dual curve emanating from $\tilde{O}_1$ and having positive length, then $K$ crosses $K_r$ or (by a symmetric argument) the dual curve $K_s$ emanating from the 1–cube of $V$ following $\tilde{O}_1$.

Now, $H_r$ doesn’t cross $\tilde{Y}_i$, so each dual curve starting at $\tilde{O}_1$ and crossing $K_r$ contributes to the length of a wall-piece in $\tilde{Y}_i$. Hence there is a constant $M$, depending only on the uniform $C''(\frac{1}{144})$ condition, such that $K_r$ can cross at most $M$ dual curves. Similarly, $K_s$ crosses at most $M$ dual curves. So, at most $2M$ of the dual curves $K$ emanating from $\tilde{O}_1$ have positive length.

Hence $\tilde{O}_1, \tilde{P}_1$ have a common subpath of length greater than $||Y_i||/2 - 2M$. Since $M < ||Y_i||/144$, we conclude that $\tilde{P}_1$, and hence $\tilde{A}$, has a subpath that lies in $\tilde{Y}_i$ and has length more than $35||Y_i||/72$, a contradiction.

$\tilde{O}_1$ unbounded: The remaining case is where $\tilde{O}_1$ is unbounded. In other words, $\tilde{Y}_i$ contains a sub-ray of the axis $\tilde{B}$ of $\langle \tilde{g} \rangle$. This implies that some power of $\tilde{g}$ stabilises $\tilde{Y}_i$, so $\tilde{B} \subseteq \tilde{Y}_i$.

Let $\tilde{A}_n$ be the subpath of $\tilde{A}$ between $\tilde{Y}_i$ and $\tilde{g}^n \tilde{y}$, let $U, U'$ be geodesics joining $\tilde{y}, \tilde{g}^n \tilde{y}$ to closest 0–cubes of $\tilde{Y}_i$, and let $V$ be a geodesic of $\tilde{Y}_i$ joining the terminal points of $U, U'$. Let $F \to X$ be a minimal-area disc diagram bounded by the paths $\tilde{A}_n, U, U', V$. Then, by allowing $U, U', V$ to vary, fixing their endpoints, and assuming that $D$ is minimal over all such choices, we have that no two dual curves emanating from $V$ can cross. Now, the number of dual curves intersecting $U, U'$ is bounded independently of $n$, since $\tilde{A}$ lies in a uniform neighbourhood of $\tilde{Y}_i$. Hence, when $n$ is sufficiently large, we see that either some dual curve travelling from $\tilde{A}_n$ to $V$ has length 0, or some hyperplane $H$ crosses $U$ and $U'$.

In the former case, $\tilde{A}$ contains a point of $\tilde{Y}_i$. It follows by convexity of $\tilde{Y}_i$ that $\tilde{A} \subseteq \tilde{Y}_i$, contradicting asystolicity. If the former case does not hold for any $n$, then there is a hyperplane $H$ separating $\tilde{A}$ from $\tilde{Y}_i$. The hyperplane $H$ does not cross $\tilde{Y}_i$, but every hyperplane crossing $\tilde{A}$ crosses $\tilde{Y}_i$ and $H$. Hence $\tilde{Y}_i$ contains arbitrarily large wall-pieces, a contradiction.

Conclusion: We have shown that, if $\tilde{A}$ contains no subpath of any $\tilde{Y}_i$ of length more than $35||Y_i||/72$, then $\tilde{A}$ is an embeddable axis and so $\tilde{g}$ is an embeddable lift of $g$. In particular, $g$ has infinite order. □

6.4. Finding fast loxodromics on $\mathcal{H}$. We are now ready for our main technical lemma, which explains how to identify when an element of $\pi_1 X$ that is loxodromic on the contact graph survives in $\pi_1 X^*$ as an element that is loxodromic on $\mathcal{H}$.

Lemma 6.7 (Asystolic loxodromics persist). Let $g \in \pi_1 X^*$. Suppose that $\tilde{g}$ is a lift of $g$ admitting a $\frac{35}{72}$–asystolic axis. Suppose that $\tilde{g}$ is loxodromic on $\tilde{\mathcal{H}}$. Then $g$ is loxodromic on $\mathcal{H}$.

Now suppose that $\tilde{g}$ is a lift of $g$ with a $\frac{17}{36}$–asystolic axis, and that $\tilde{g}$ is loxodromic on $\tilde{\mathcal{H}}$. Then $g$ acts on $\mathcal{H}$ as a loxodromic WPD element.

Outline of the proof. Since the proof of Lemma 6.7 is rather long, we outline the steps.

- Using asystolicity and Lemma 6.6 $g$ has an embedded axis $A$ in Cay($X^*$) that is the image of the asystolic axis $\tilde{A}$ of $\tilde{g}$.
- Fixing $a \in A$, we consider the subpath $A_n$ of $A$ from $a$ to $g^n a$. We also consider the following path $Q$ from $a$ to $g^n a$: choose a sequence $C_0, \ldots, C_N$ of relators or hyperplane-carriers in $X^*$ so that successive ones intersect, the first and last contain $a, g^n a$, and $N$ is as small as possible. We choose a path $Q \to \tilde{X}^*$ from $a$ to $g^n a$ that is a concatenation of paths in the various $C_i$.
- To prove that $g$ is loxodromic, we need to show that $N$ grows linearly in $n$. This is Claim 17. The idea is that for suitable disc diagrams $D \to \tilde{X}^*$ bounded by $A_n$ and $Q$. 


there cannot be any cone-cells in $D$. Hence $D$ factors through $\text{Cay}(X^*)$ and hence lifts to $\tilde{X}$. The diagram $D \to \tilde{X}$ is bounded by a subpath of an axis $\tilde{A}$ of $\tilde{g}$, and a lift $\tilde{Q}$ of $Q$, which is a concatenation of lifts of the $\alpha_i$. So, there is a sequence of vertices in $\Gamma$ joining the endpoints of $\tilde{A}_n$ and having $n$ vertices. This means that a linear function of $N$ bounds from above the length of $\tilde{A}_n$, as measured in $\tilde{\mathcal{H}}$, which in turn grows linearly in $n$ by Lemma \ref{lemma:3.4}. Hence $N$ must grow linearly in $n$.

- Thus, it suffices to show that $Q$ can be chosen so that $D$ lifts. To do this, we choose $C_0, \ldots, C_N$, the path $Q$, and $D$ so that the complexity of $D$ is minimal over all such choices. We decompose $D$ into three subdiagrams $D', F, D''$ as follows. First, we perform square homotopies in $D$ to replace $A_n$ by a path $P$, of the same length (hence also lifting to a geodesic in $\tilde{X}$). The part between $A_n$ and $P$ is $D'$. In Claims \ref{claim:10}-\ref{claim:11}-\ref{claim:12} we show that the remaining subdiagram has no positively-curved shells. Then we square-homotop $Q$ to a path $Q_1$; the subdiagram between these paths is a square diagram $F$, and the remaining part is $D''$.

- We then invoke the ladder theorem, Theorem \ref{thm:3.22} to show that $D''$ is a ladder, in Claim \ref{claim:15}. The idea is that minimality of $N$ means that any positively-curved shell in $D''$ is the concatenation of at most 5 pieces between the shell and the various relators/hyperplanes $C_i$. This contradicts our small-cancellation assumption, via the short inner paths property.

- Using asystolicity — via the auxiliary Claim \ref{claim:9} — we promote the statement that $D''$ is a ladder to the statement that $D''$ is a square diagram. The idea is that any cone-cell in $D''$ has boundary path consisting of at most 5 pieces on the $Q$ side (as above), 2 pieces on the incident pseudorectangles, and a subpath on the $P$ side that must be short (this uses asystolicity). Since $D'', F, D'$ are all square diagrams, so is $D$, which is what we needed.

- It remains to show that $g$ is fast. Here the argument is similar but easier: we form $D$ as above, except that $Q$ is replaced by a $\text{Cay}(X^*)$–geodesic $S$ from $a$ to $g^n a$. The diagram $D$ is now bounded by $A_n$ and $S$. We again square-homotop $A_n$ and $S$ to geodesics, decomposing $D$ as the union of two square diagrams and a third “central” diagram. Much as before, the central diagram has to be a ladder. This is seen more easily than in Claim \ref{claim:15} because $S$ is a geodesic and Theorem \ref{thm:3.24} takes care of positively-curved shells for free. Once we know this diagram is a ladder, two applications of asystolicity (again using Claim \ref{claim:9}) show that $g$ is fast.

- We conclude using Lemma \ref{lemma:4.5}.

**Proof of Lemma \ref{lemma:6.7}** The proof has several parts. Since $\frac{17}{36}$–asystolicity implies $\frac{35}{72}$–asystolicity, we will assume $\frac{35}{72}$–asystolicity for the purpose of showing that $g$ is loxodromic, and $\frac{35}{72}$–asystolicity only for the purpose of showing that $g$ is fast (recall Definition \ref{def:4.4}).

**Embeddability:** Fix a $\frac{35}{72}$–asystolic axis $\tilde{A}$ for $\tilde{g}$. Lemma \ref{lemma:6.6} implies that $\tilde{A}$ is an embeddable axis, so $\tilde{g}$ is an embeddable lift and hence $g$ has infinite order. Hence, because each relator is compact, no lift of $g$ has a positive power that stabilises an elevation of a relator. Since $\tilde{g}$ is loxodromic on $\tilde{\mathcal{H}}$, Lemma \ref{lemma:6.4} ensures that $\tilde{g}$ (which is necessarily rank one and has no power stabilising a hyperplane) is loxodromic on $\tilde{\mathcal{H}}$.

**Bounded coarse intersections with hyperplanes:** Since $\tilde{g}$ is loxodromic on $\tilde{\mathcal{H}}$, there exists $p_g < \infty$ such that for all hyperplanes $\tilde{H}$, we have $\text{diam}(\tilde{g}_N(\tilde{H}))(\tilde{A})) \leq p_g$. In other words, at most $p_g$ of the hyperplanes crossing $\tilde{A}$ can cross $\tilde{H}$.

**Bounded coarse intersections with elevations of relators:** Let $M$ be the upper bound on diameters of pieces, i.e. $M = \frac{1}{144} \inf_i \|Y_i\|$. We need the following claim:
Claim 9. There exists $q_g < \infty$ such that, for all subcomplexes $\tilde{Y}_i$ that are lifts of relators to $\tilde{X}$,
\[
\text{diam}(g_{\tilde{Y}_i}(A)) < q_g,
\]
and
\[
\text{diam}(g_{\tilde{Y}_i}(A)) < \frac{1}{2}\|Y_i\|.
\]
Moreover, under the additional $\frac{17}{30}$-asystolicity assumption, $\text{diam}(g_{\tilde{Y}_i}(A)) < \frac{27}{40}\|Y_i\|$. 

Proof of Claim 9. Let $\tau_3$ be the combinatorial translation length of $\tilde{g}$. Fix a vertex $\tilde{a} \in \tilde{A}$.

Absolute bound on subpaths of $\tilde{A}$ in relators: First, we will show that there exists $q_g'$ such that $|\tilde{P}| \leq q_g'$ whenever $\tilde{P}$ is a subpath of $\tilde{A}$ that lies in some $\tilde{Y}_i$. Suppose not. Then for all $N \in \mathbb{N}$, there exists $\tilde{Y}^N$ (an elevation of a relator $Y_{i_N}$) and a subpath $\tilde{P}_N$ of $\tilde{A}$ such that $\tilde{P}_N$ lies in $\tilde{Y}^N$ and has length more than $N$. By applying powers of $\tilde{g}$, we can assume that $\tilde{P}_N$ joins $\tilde{a}$ to $\tilde{g}^{-kN} \tilde{a}$, where $k_N \geq N/\tau_3 - 1$.

Hence the subpath $\tilde{Q}_N$ of $\tilde{P}_N$ joining $\tilde{g} \tilde{a}$ to $\tilde{g}^{-kN} \tilde{a}$ lies in $\tilde{Y}^N \cap \tilde{g}^{-1} \tilde{Y}^N$. Thus either $\tilde{Y}^N = \tilde{g} \tilde{Y}^N$, or $\tilde{Q}_N \to \tilde{Y}^N$ is a piece. As mentioned above, $\tilde{g}$ cannot stabilise an elevation of a relator. So $\tilde{Q}_N$ is a piece of length at least $N - 2\tau_3$. For $N > M + 2\tau_3$, this is a contradiction. We conclude that there must exist $q_g'$ with the claimed property.

Relative bound on subpaths of $\tilde{A}$ in relators: Second, just by asystolicity, if $\tilde{P}$ is a subpath of $\tilde{A}$ lying in some $\tilde{Y}_i$, we have $|\tilde{P}| < \frac{35}{40}\|Y_i\|$. (Or $|\tilde{P}| < \frac{17}{30}\|Y_i\|$ under the stronger of the two asystolicity assumptions.)

Absolute bound on projection of $\tilde{A}$ to relators: Fix $\tilde{Y}_i$. Let $R$ be a geodesic in $g_{\tilde{Y}_i}(\tilde{A})$; so, the hyperplanes intersecting $R$ all intersect $\tilde{A}$ and $\tilde{Y}_i$. Initially, we choose $R$ to be the image of some geodesic $R'$ in $\tilde{A}$ joining two 0–cubes $y, y'$. So, $R$ joins $g_{\tilde{Y}_i}(y), g_{\tilde{Y}_i}(y')$.

Next, let $\beta, \beta'$ be geodesics joining $y, g_{\tilde{Y}_i}(y)$ and $y', g_{\tilde{Y}_i}(y')$ respectively. Let $D \to \tilde{X}$ be a minimal-area disc diagram bounded by $\beta, \beta', R, R'$. We allow $\beta, \beta', R$ to vary among geodesics with the given endpoints; varying such a geodesic does not change the hyperplanes it crosses. We make all such choices so that, among them, the resulting $D$ has minimal area. (We emphasise that we do not allow $R'$ to vary, since we need it to be a subpath of $\tilde{A}$.) See Figure 11.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure11.png}
\caption{The diagram in $\tilde{X}$ bounded by $R, R', \beta, \beta'$, along with some allowable dual curves. At least $|R'| - 2M$ of the vertical dual curves do not cross any other dual curve, and thus have length 0. Hence, for each $\tilde{Y}_i$, at most $\|Y_i\|/2$ hyperplanes cross both $\tilde{A}$ and $\tilde{Y}_i$ in $\tilde{X}$.}
\end{figure}

Let $K, K'$ be dual curves in $D$ emanating from $\beta$. Then minimality of area ensures that $K, K'$ do not cross. The same holds with $\beta$ replaced by $\beta'$ or $R$. Next, let $K$ be a dual curve emanating

from \( R \). The hyperplane to which \( K \) maps not separate \( y, g_i^{-1}(y) \), since it crosses \( \tilde{Y}_i \), so \( K \) cannot cross \( \beta \). Similarly, \( K \) cannot cross \( \beta' \). So, every dual curve emanating from \( R \) terminates on \( R' \).

Hence every dual curve \( K \) travels from \( R \) to \( R' \) or from \( R' \) to \( \beta \) or \( \beta' \), or from \( \beta \) to \( \beta' \).

If some dual curve \( K \) crosses \( \beta \) and \( \beta' \), then \( K \) maps to a hyperplane \( H \) that does not cross \( \tilde{Y}_i \) (since it separates \( y, g_i^{-1}(y) \)) and crosses every hyperplane crossing both \( R, R' \). Hence \( R' \) is a wall-piece, and we get \( |R'| \leq M \). So, either we are done, or no such \( K \) exists. Assume the latter.

Now, if \( K \) travels from \( \beta \) to \( R' \), then \( K \) maps to a hyperplane \( H \) not crossing \( \tilde{Y}_i \) (since \( H \) separates \( g_i^{-1}(y) \) from \( y \)), so it yields a wall-piece in \( \tilde{Y}_i \) and hence crosses at most \( M \) of the dual curves traveling from \( R \) to \( R' \). The same holds for dual curves traveling from \( \beta' \) to \( R' \) or \( \beta \) to \( \beta' \). Hence there are at least \( |R'| - 2M \) dual curves that travel from \( R \) to \( R' \) and do not cross any other dual curves. These dual curves thus have length 0, so \( R, R' \) have a common subpath of length at least \( |R'| - 2M \). The preceding discussion shows that \( |R'| - 2M < q'_g \), so \( |R'| < q'_g + 2M \). Taking \( q_g = q'_g + 2M \) therefore suffices.

**Relative bound:** The preceding argument also shows that \( |R'| < \frac{35}{72} ||Y_i|| + 2M \) (under \( \frac{35}{72} \)-asystolicity) or \( |R'| < \frac{144}{17} ||Y_i|| + 2M \) (under \( \frac{144}{17} \)-asystolicity). Hence, under the weaker assumption, we get \( |R'| < ||Y_i||/2 \), since \( 144M < \inf_j ||Y_j|| \). Under the stronger assumption, we get \( |R'| < \frac{35}{72} ||Y_i|| \). This completes the proof of the claim. \( \square \)

Now we resume the proof of the lemma.

**The path** \( A \to \text{Cay}(X^*) \): Let \( \hat{a} \in \tilde{A} \) be a 0–cube. Let \( a = p(\hat{a}) \), so that \( g^n a = p(\hat{g}^n \hat{a}) \) for each \( n > 0 \). Let \( A = p \circ \tilde{A} \). Since \( \tilde{A} \) is embeddable, the path \( A \) is an embedded bi-infinite combinatorial path in \( \text{Cay}(X^*) \).

**Paths in \( \mathcal{H} \):** Fix \( n > 0 \). Choose a sequence \( C_0, \ldots, C_N \) satisfying:

- for each \( j \leq N \), either \( C_j \) is the carrier of a hyperplane in \( \tilde{X}^* \) or \( C_j \to \tilde{X}^* \) is a lift of some \( Y_j \to X \) (abusing notation, we use the same name for \( C_j \) as for its image);
- we have \( a \in C_0 \) and \( g^n a \in C_N \);
- we have \( C_j \cap C_{j+1} \neq \emptyset \) for all \( j \leq N \);
- the number \( N \) is minimal with the above properties.

For each \( j \), let \( \alpha_i \) be a combinatorial path in \( C_j \), chosen so that \( Q = \alpha_0 \alpha_1 \cdots \alpha_N \) is a path from \( a \) to \( g^n a \) with no backtracks.

For proving that \( g \) is loxodromic on \( \mathcal{H} \), our goal will be to prove that \( N \) is bounded below by a linear function of \( n \).

**Constructing a disc diagram:** Let \( A_n \) be the subpath of \( A \) joining \( a \) to \( g^n a \). Let \( D \to \tilde{X}^* \) be a minimal disc diagram with boundary path \( A_n Q^{-1} \). Moreover, choose the \( C_j \), and \( Q \), subject to the above constraints, so that \( D \) has minimal complexity among all diagrams constructed in the preceding manner. See Figure 12.

Since \( Q \) has no backtracks and \( A_n \) is embedded, \( D \) has no spurs along \( Q \) or \( A_n \). (So there are at most two spurs.)

**The first square homotopy:** If \( D \) contains an exposed square (or generalised corner of a square) corresponding to a length–2 subpath \( ef \) (with \( e, f \) single 1–cubes) that lies on \( A_n \), then we can perform a square homotopy (shuffling first if necessary, see Remark 3.20) to replace \( A_n \) by a square-homotopic embedded path \( P \) with the same length and endpoints.

Thus: \( D = D' \cup_P D'' \), where \( D' \) is a square diagram bounded by the paths \( P, A_n \) and \( D'' \) is a diagram bounded by \( P \) and \( Q \), and there are no generalised corners or spurs along \( P \).

---

\(^1\)Since there is no uniform bound on the systoles of the relators, the second part of Claim 9 is insufficient to give the bound \( q_4 \), because \( \tilde{A} \) can pass through infinitely many orbits of elevations of relators. We use the bound \( ||Y_i||/2 \) to prove that \( g \) is loxodromic, and the bound \( \frac{35}{72} ||Y_i|| \) for proving that \( g \) is fast.
Figure 12. The diagram $D$ between $A_n$ and $Q$, along with the surrounding relators and carriers $C_1, \ldots, C_N$ along $Q$. Various illegal features are shown. The exposed square along $\alpha_3$ belongs to $C_3$ by convexity, so we could have homotoped $\alpha_3$ across the square to obtain a lower-complexity diagram. The dual curve carrier traveling from $\alpha_{N-3}$ to $\alpha_{N-1}$ maps to a hyperplane carrier intersecting $C_{N-3}$ and $C_{N-1}$, so we could have replaced $C_{N-2}$ by this carrier to yield a lower-complexity diagram. The cone-cell intersecting $\alpha_2, \alpha_N$ contradicts minimality of $N$. If the cone-cell with outer path on $A_n$ is a positively curved shell, then asystolicity of $g$ is contradicted.

Bounding subpaths of $P$ in relators and carriers: Since $D'$ is a square diagram, it lifts to a square diagram $\tilde{D}' \to \tilde{X}$ bounded by the subpath of $\tilde{A}$ joining $\tilde{a}$ to $\tilde{g^n a}$ (since $\tilde{A}$ is a lift of $A$) and a lift $\tilde{P}$ of the embedded path $P$.

The path $\tilde{A}$ is a geodesic, so every dual curve in $D'$ starting on $A_n$ ends on $P$. Now, since $|P| = |A_n|$, we also have that $\tilde{P}$ is a geodesic of $\tilde{X}$, so dual curves in $D'$ starting on $P$ end on $A_n$. Hence, if $P_1$ is a subpath of $P$ lying in a cone-cell $\Upsilon$, we have $|P_1| < \|Y_i\|/2$, where $Y_i$ is the relator to which $\Upsilon$ maps. If $P_2$ is a subpath of $P$ lying in a hyperplane carrier, $|P_2| \leq p_g$.

Analysis of $D''$: Now we prove some claims about $D''$.

Claim 10. Let $K$ be a dual curve or cone-cell in $D''$. Suppose that $K$ intersects $\alpha_i, \alpha_j$. Then $|i - j| \leq 1$.

Proof of Claim 10. Minimality of $N$ implies that $|i - j| \leq 2$, for otherwise the hyperplane or relator to which $K$ maps would provide a shortcut between $C_i$ and $C_j$, contradicting minimality of $N$.

Now suppose that $i - j = 2$. Let $Y$ be the relator or hyperplane to which $K$ maps. By replacing $C_{j+1}$ by $Y$, replacing $\alpha_i, \alpha_j$ by appropriate subpaths, and replacing $\alpha_{j+1}$ by part of the boundary path of $K$, we have found a new choice of $Q$ so that $Q, A_n$ bound a proper subdiagram of $D$ from which the subdiagram $K$ has been removed. Chopping off any spurs doesn’t increase the complexity. So we have a new choice of $Q$ leading to a lower-complexity diagram, contradicting minimality of $D$. Hence, $|i - j| \leq 1$. ■

Claim 11 (Ruling out shells in $D''$, part I). The diagram $D''$ contains no positively-curved shell whose outer path is a subpath of $P$.

Proof of Claim 11. Suppose that $K$ is a positively-curved shell in $D''$ with boundary path $OI$, where the outer path $\tilde{O}$ is a subpath of $P$. Let $Y$ be the relator to which $K$ maps, and let $I'$ be a shortest path in $Y$ that is square-homotopic in $Y$ rel endpoints to $I$. Theorem 3.24 implies
$|I'| < |O|$. By Claim 9 and the fact that $P$ and $A_n$ cross the same hyperplanes, $|O| < ||Y||/2$, so $|I'O| < Y$. Hence $I'O$, and thus $IO$, is inessential in $Y$ and thus $K$ can be replaced by a square diagram, contradicting minimal complexity of $D$. Thus there are no positively-curved shells along $P$.

Since $P$ is embedded, there are also no spurs, and there are no generalised corners in $D''$ along $P$. Hence $D''$, by Claim 11, $D''$ has no feature of positive curvature whose outer path is a subpath of $P$.

**Claim 12** (Ruling out shells in $D''$, part II). The diagram $D''$ contains no positively-curved shell whose outer path is a subpath of $Q$.

**Proof of Claim 12.** Suppose that $K$ is a positively-curved shell with boundary path $OI$, where the outer path $O$ is a subpath of $Q$. Then the relator $Y$ to which $K$ maps must intersect $C_i, C_j$ for some $i, j$. This can only happen if $|i - j| \leq 1$, by Claim 10.

By minimal complexity of $D$, it follows that $O$ is square-homotopic to the concatenation of at most 2 pieces. Hence, letting $O'$ be a shortest path in $Y$ that is square-homotopic rel endpoints to $O$, we have $|O'| \leq 2M$. By Theorem 3.24, we have $|O| > 72M$, a contradiction. Hence there is no positively-curved shell whose outer path is a subpath of $Q$. ■

**The second square homotopy:** We are working toward an application of the ladder theorem — in view of the diagram trichotomy, we now just need to remove generalised corners along $Q$ using square homotopies, as follows.

Let $K$ be a dual curve with one end on $Q$, i.e. one end on some $\alpha_i$. We have seen already that $K$ cannot have its other end on $\alpha_j$ for $|i - j| \geq 2$, because of Claim 10.

We claim that $K$ cannot end on $\alpha_{i+1}$. Indeed, otherwise $D$ would have a subdiagram $E$ bounded by the subpath of $\alpha_i \alpha_{i+1}$ (say) subtended by the 1–cubes dual to $K$, along with a path on $N(K)$. Minimal complexity of $D$ and Theorems 3.23, 3.24 imply that $E$ is a square diagram. Choosing $K$ to be innermost, every dual curve in $E$ travels from $N(K)$ to $\alpha_i \alpha_{i+1}$. Now, no two dual curves emanating from $\alpha_i$ (or $\alpha_{i+1}$) can cross, because an innermost such pair would give an exposed square in $D$ along $\alpha_i$; convexity of $C_i$ would then yield a contradiction with minimal complexity by absorbing the square into $C_i$. By shuffling, we can also assume that no two dual curves in $E$ crossing $K$ can cross (if a square has two consecutive edges along the carrier of $K$, we can push it out of $E$ without changing $\partial_p E$). Hence either $\alpha_i$ has its terminal edge equal to the initial edge of $\alpha_{i+1}$ (contradicting that $Q$ has no backtracks), or $\partial_p E$ has an exposed square along $\alpha_i$ (contradicting minimal complexity). See Figure 13.

![Figure 13](image_url)

**Figure 13.** If a dual curve $K$ in $D$ intersects $\alpha_i, \alpha_{i+1}$, then $D$ has a subdiagram $E$ as shown. (Also shown are $C_i, C_{i+1}$). This must be a square diagram, and convexity of $C_i, C_{i+1}$ then allow us to conclude, from minimality of complexity, that the 1–cubes dual to the first and last points of $K$ coincide. So no dual curve starts and ends on $Q$.

A simpler version of this argument also shows that $K$ cannot have two ends on $\alpha_i$. Indeed, if both ends of $K$ are on $\alpha_i$, then $D$ would have a subdiagram $E$ bounded by the subpath of
\[ \alpha_i \] between (and including) the 1–cubes dual to \( K \) and a path in \( \mathcal{N}(K) \). As in the preceding argument, \( E \) must be a square diagram, so its boundary path lifts to a closed path in \( \tilde{X} \), whence \( \alpha_i \) contains two edges dual to the same hyperplane, contradicting that \( \alpha_i \) is a geodesic.

Let \( Q_1 \) be an embedded path in \( D'' \) such that:

1. \( Q_1 \) and \( Q \) have the same endpoints;
2. \( |Q_1| \leq |Q| \);
3. \( Q_1 \) has no spurs;
4. the subdiagram \( F \) of \( D'' \) bounded by \( Q \) and \( Q_1 \) is a square diagram, and has as many squares as possible subject to the above constraints.

**Claim 13.** Every dual curve in \( F \) with an end on \( Q \) has an end on \( Q_1 \), and every dual curve with an end on \( Q_1 \) has an end on \( Q \).

**Proof of Claim 13**. The preceding discussion shows that no dual curve in \( D'' \) (of which \( F \) is a subdiagram) starts and ends on \( Q \). So, since \( F \) is a square diagram, any dual curve starting on \( Q \) ends on \( Q_1 \). Since \( |Q_1| \leq |Q| \), it follows that every dual curve in \( F \) with one end on \( Q_1 \) has an end on \( Q \).

**The subdiagram** \( D''_1 \): Let \( D''_1 \) be the subdiagram of \( D'' \) bounded by \( Q_1 \) and \( P \).

**Claim 14.** There are no generalised corners of squares of \( D''_1 \) lying along \( Q_1 \).

**Proof of Claim 14**. Suppose that \( ef \) is a length–2 subpath of \( Q_1 \) corresponding to a generalised corner of a square in \( D''_1 \). By shuffling, we can assume that \( ef \) is a subpath of the boundary path of a square \( s \) in \( D''_1 \). Then \( \partial_p s = ef e'f' \), and we can replace \( ef \) by \( e'f' \) (and remove spurs if necessary) in \( Q_1 \) to obtain a new path \( Q'_1 \), of length at most \( |Q_1| \), such that the subdiagram bounded by \( Q \) and \( Q'_1 \) has more squares than \( F \), a contradiction. See Figure 14.

![Figure 14](image)

**Figure 14.** The diagram \( D''_1 \) has no generalised corner of a square along \( Q_1 \), for otherwise we could enlarge the subdiagram \( F \) with a square homotopy.

We found a ladder:

**Claim 15.** The diagram \( D''_1 \) is a ladder.

**Proof of Claim 15**. We have already seen that \( D'' \), and hence \( D''_1 \), has no positively-curved feature (shell, spur, generalised corner) along \( P \). We have also seen already that \( Q_1 \) has no spur and \( D''_1 \) has no generalised corner on \( Q_1 \).

Suppose that \( K \) is a positively-curved shell in \( D''_1 \) with boundary path \( OI \), where the outer path \( O \) is a subpath of \( Q_1 \). Then \( O \) is also a subpath of \( \partial_p F \), and every dual curve in \( F \)
emanating from $O$ ends on some $\alpha_i$. Let $i_0, i_1$ be the minimal and maximal values of $i$ such that $F$ contains a dual curve $K'$ that travels from $\alpha_i$ to $O$. Let $Y$ be the relator to which $K$ maps. Then there is a sequence $C_{i_0}, H, Y, H', C_{i_1}$, where $H, H'$ are hyperplanes and consecutive terms intersect. Hence, by minimality of $N$, we have $i_1 - i_0 \leq 4$. See Figure 15.

We’d now like to write $O$ as a concatenation of at most 5 pieces.

The square diagram $F$ lifts to a disc diagram $F \to \tilde{X}$ with boundary path $\tilde{Q}\tilde{Q}_1^{-1}$, where $\tilde{Q}$ is a lift of $Q$ and $\tilde{Q}_1$ is a lift of $Q_1$. Moreover, $\tilde{Q}_1$ has a subpath $\tilde{O}$ lying on an elevation $\tilde{Y}$ of $Y$, where $\tilde{O}$ is a lift of $O$. Also, $\tilde{Q} = \tilde{\alpha}_0 \cdots \tilde{\alpha}_N$, where each $\tilde{\alpha}_i$ is a lift of $\alpha_i$ to an elevation $\tilde{C}_i$ of $C_i$. (So, either $\tilde{C}_i$ is a hyperplane-carrier or is the universal cover of a relator.)

Note that, if two dual curves emanating from $\tilde{O}$ cross, then by shuffling, we find a square in $F$ with two consecutive boundary edges on $\tilde{O}$. This square maps to $\tilde{Y}$, and its image in $D$ can thus be absorbed into $K$, contradicting minimality of the complexity. So no two dual curves emanating from $\tilde{O}$ can cross in $F$.

Thus, we can write $\tilde{O} = \theta_0\theta_{i_0+1} \cdots \theta_{i_1}$, where each $\theta_j$ has the property that all dual curves starting on $\theta_j$ end on $\tilde{\alpha}_j$. For each $j$, if $\tilde{C}_j \neq \tilde{Y}$ and $\tilde{C}_j$ is not the carrier of a hyperplane crossing $\tilde{Y}$, we therefore have that $\theta_j$ is a path in an abstract piece.

Suppose that $\tilde{C}_j = \tilde{Y}$ or $\tilde{C}_j$ is the carrier of a hyperplane crossing $\tilde{Y}$. Let $\kappa$ be a dual curve from $\tilde{O}$ to $\tilde{\alpha}_j$, and let $H$ be a hyperplane crossing the image of $\kappa$ in $\tilde{X}$. Then $F$ contains a dual curve $\omega$ mapping to $H$ and crossing $\kappa$. The dual curve $\omega$ must cross all dual curves emanating from $\tilde{O}$. Hence either $H$ crosses $\tilde{Y}$, or $\theta_j$ is a (wall) piece in $\tilde{Y}$.

Take $H$ to be the hyperplane crossing the image of $\kappa$ inside the square $s$ of $F$ containing the first edge of $\kappa$. If $H$ crosses $\tilde{Y}$, then convexity of $\tilde{Y}$ implies that $s$ is in $\tilde{Y}$. Hence we could have absorbed $s$ into $K$ to reduce the complexity of $D$. Thus either $\theta_j$ is a path in an abstract piece, or all such $\kappa$ have length 0, so $\theta_j$ is a subpath of $\alpha_j$.

Let $j_0$ and $j_1$ be the largest and smallest $j$ for which the latter holds. Let $U^{-1}$ be the subpath of $O^{-1}$ from the initial point of $\theta_{j_0}$ to the initial point of $O$, and let $V$ be the subpath of $O$ from

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**Figure 15.** The diagram $D$ is the union of square diagrams $D', F$ and a ladder $D''_i$. Dual curves in $F$ travel from $Q$ to $Q_1$, and dual curves in $D'$ travel from $P$ to $A_n$. The cone-cells in $D''_i$ have small projection to $A_n$ because of asystolicity of $g$. They have bounded projection to $Q$ because of minimality of $N$: any dual curve in $F$ emanating from a common cone-cell of $D''_i$ must end on $\alpha_i$ for at most one of 5 values of $i$. This is ultimately used to show that $D''_i$, and hence all of $D$, is a square diagram.
the terminal point of $\theta_{j_1}$ to the terminal point of $O$. (Here we regard $\theta_0, \theta_1$ as paths in $\tilde{X}^*$.) Then $UV \to \tilde{X}^*$ is a path factoring through $Y$ and $D$, and joining $C_{j_0}$ to $C_{j_1}$. So we can replace $C_{j_0+1}, \ldots, C_{j_1-1}$ by $Y$ and modify $Q$ to obtain a new diagram which is a proper subdiagram of $D$ (the cone-cell $K$ is removed). This contradicts minimal complexity. The exception is when $j_0 = j_1$ and $\theta_{j_0}$ is a trivial path (because then removing $K$ from $D$ would give a singular disc diagram, not a disc diagram). However, in this case, each $\theta_j, j \neq j_0$ is a piece and $O$ is the concatenation of at most 4 pieces.

We have shown that $O$ is a concatenation of at most 5 paths, each of which is square-homotopic to a piece. Thus $O$ is square-homotopic to a path of length at most $5M$. By Theorem 3.24, $\partial_p K$ is therefore square-homotopic to a path of length at most $10M$, so, by the small-cancellation condition, $\partial_p K$ is not essential in $Y$, and thus $K$ could have been replaced by a square subdiagram, lowering complexity.

It then follows from Theorem 3.22 that $D''_i$ is a ladder.

The ladder is a square diagram: We now analyse the ladder $D''_i$ more carefully.

Claim 16. $D$ is a square diagram.

Proof of Claim 16. Let $\Upsilon$ be a cone-cell of $D''_i$. Then $\partial_p \Upsilon = e_{Q_1}f_1e_PF_2$, where $f_1, f_2$ are trivial or are subpaths in the incident pseudorectangles, and $e_{Q_1}$ is a maximal subpath of $\partial_p \Upsilon$ lying on $Q_1$, and $e_P$ is a maximal subpath on $P$. Let $Y_\Upsilon$ be the relator to which $\Upsilon$ maps. Then $|e_P| < \|Y_\Upsilon\|/2$, because of the bound on the diameter of projections of $\tilde{A}$ to elevations of relators (Claim 9). On the other hand, $e_{Q_1}$ is square-homotopic to a path of length at most $5M$, by the exact same argument as was used in Claim 15 to rule out shells with outer path on $Q_1$. Finally, $f_1, f_2$ are either trivial or pieces.

Thus, up to square-homotopy, $|\partial_p \Upsilon| < 7M + \frac{1}{2}\|Y_\Upsilon\|$. Minimal complexity of $D$ requires that $|\partial_p \Upsilon| > \|Y_\Upsilon\|$, so $7M > \frac{1}{2}\|Y_\Upsilon\|$, contradicting the uniform $C''(\frac{1}{13})$ condition. Thus $D''_i$ cannot contain any cone-cells. In other words, $D''_i$ is a square diagram. Since $D$ consists of $D''_i$ together with the square subdiagrams $F$ and $D'$, it follows that $D$ is a square diagram.

At this point, we are ready to prove:

Claim 17. The element $g$ is loxodromic on $\mathcal{H}$.

Proof of Claim 17. Claim 16 showed that $D$ is a square diagram. So $D$ lifts to a diagram $D \to \tilde{X}$, bounded by a lift $Q$ of $Q$ and a lift $\tilde{A}_n$ of $A_n$ joining $\tilde{a}$ to $\tilde{g}^na$. Now, each $\alpha_i$ lifts to a subpath $\tilde{\alpha}_i$ of $Q$, and $\tilde{\alpha}_i$ lies in an elevation $\tilde{C}_i$ of the hyperplane carrier or relator $C_i$. Hence $d_{\tilde{H}}(\tilde{a}, \tilde{g}^na)$ is bounded above by a linear function of $N$, by Lemma 6.2.

If $N$ grows sublinearly in $n$, this means that $\tilde{g}$ is not loxodromic on $\tilde{H}$. But recall (from the very beginning of the proof of the lemma) that $\tilde{g}$ is loxodromic on $\tilde{H}$ because it is embeddable and loxodromic on $\hat{H}$. Hence, $N$ grows linearly in $n$, so $g$ is loxodromic on $\mathcal{H}$.

To conclude, we have to prove that $g$ is fast. This is similar to the proof that $g$ is loxodromic, except we use the stronger estimates from Claim 9 that require $\frac{17}{36}$-asystolicity.

Claim 18. There exists $\Delta$ such that $g$ is $\Delta$-fast.

Proof of Claim 18. Let $n \geq 0$. Let $S$ be a geodesic in $\text{Cay}(X^*)$ from $a$ to $g^n a$. Write $S = S_0 \cdots S_N$, with each $S_i$ lying in a relator or hyperplane carrier. Let $D \to \tilde{X}^*$ be a minimal-complexity diagram bounded by $A_n$ and $S$. The proof of Claim 11 implies that $D$ contains no positively curved shell with outer path along $A_n$. Also, $D$ has no positively curved shell with outer path along $S$, by Theorem 3.24 since $S$ is a geodesic.

There are no spurs in $A_n$, since $A$ is an embedded path, and no spurs in $S$ since $S$ is a geodesic. So, any feature of positive curvature in $D$ occurring along $A_n$ or $S$ is a generalised
corner of a square. Hence, performing square homotopies and applying Theorem\footnote{It was much simpler to see that $D_2^n$ is a ladder than it was for $D_2^n$ above because $S$ is a Cay($X^*$)-geodesic.} 3.22 we have the following. The diagram $D$ is square-homotopic to a ladder $D_2^n$ with boundary path $A_nT^{-1}$, where $T$ is a geodesic square-homotopic to $S$ and $A_n$ is square-homotopic to, and has the same length as, $A_n$.

Let $K$ be a cone-cell of $D_2^n$. Let $\gamma$ be the part of $\partial_pK$ lying on $T$ and let $\sigma$ be the part of $\partial_pK$ lying on $A_n$. All dual curves in $D$ starting on $A_n$ end on $A_n'$, because $A_n'$ was obtained from $A_n$ by square homotopies across generalised corners. Thus, $\frac{17}{36}$–asystolicity and Claim\footnote{The diagram $D_2^n$ is a ladder.} 6 imply that $|\sigma| < \frac{35}{72} \|Y\|$, where $Y$ is the relator to which $K$ maps.

Now, since $T$ is a geodesic and $D_2^n$ is a ladder, we have $|\gamma| \leq |\sigma| + 2M$. Hence $|\partial_pK| < \|Y\|(\frac{1}{72} + \frac{35}{72}) < \|Y\|$. Thus $\partial_pK$ bounds a square diagram in $X$, contradicting minimal complexity. Hence $D$ contains no cone-cells. Hence $D$ is a square diagram, and dual curves starting on $S$ end on $A_n$. We thus have $|S_i| \leq q_g$ for all $i$, by Claim\footnote{This is used to show that $\bar{g}_i$ acts loxodromically on $\bar{H}$.} 6. Thus there exists $\Delta$ (depending on $g$) so that $g$ is $\Delta$–fast.

To conclude, Claim\footnote{This is used to show that $\bar{g}_i$ acts loxodromically on $\bar{H}$.} 17 and Claim\footnote{This is used to show that $\bar{g}_i$ acts loxodromically on $\bar{H}$.} 18 combine with Lemma 4.5 to show that $g$ acts on $H$ as a WPD element. \hfill $\blacksquare$

6.5. Achieving asystolicity in the essential case. In this subsection, we additionally assume that $\pi_1X$ acts essentially on $\bar{X}$, in the sense of \cite{CS11}. This means that for each hyperplane $H$ of $\bar{X}$, each of the components of $\bar{X} - H$ contains points in any (fixed) $\pi_1X$–orbit arbitrarily far from $H$. In particular, there is no proper $\pi_1X$–invariant convex subcomplex.

By [\cite{Hag13}, Theorem 5.4] (which in turn relies on results in \cite{CS11}), and Lemma 6.2 one of the following holds:

1. $\bar{X} \approx A \times B$, where $A,B$ are unbounded CAT(0) cube complexes. In this case, either $\pi_1X^{\ast}$ is finite or $\pi_1X^{\ast} = \pi_1X$. Indeed, if $\bar{Y}_i = \bar{X}$, then $Y_i \to X$ is a covering map. Since $Y_i$ is compact, this implies that $Y_i$ is a finite cover, whence $\pi_1X^{\ast}$ is finite. Otherwise, since $\bar{Y}_i \subseteq \bar{X}$ is convex, we have $\bar{Y}_i = A' \times B'$, where $A' \subset A$ and $B' \subset B$ are convex subcomplexes, and one of the two containments is proper. Choose a hyperplane $H$ disjoint from $\bar{Y}_i$. Without loss of generality, $H = H' \times B$, where $H'$ is a hyperplane of $A$ disjoint from $A'$. Then $B'$ is an abstract wall-piece, so $Y'$ has bounded diameter. Hence there is a hyperplane $V'$ of $B$ disjoint from $B'$, so $\bar{Y}_i$ is disjoint from the hyperplane $A \times V'$. Thus $A'$ is an abstract wall-piece and thus bounded. Hence $\bar{Y}_i$ is bounded and therefore has trivial stabiliser, i.e. $\bar{Y}_i$ is contractible.

2. There exists $\tilde{g} \in \pi_1X$ acting loxodromically on $\bar{H}$.

We now restrict to case (2). So, we are assuming that some $\tilde{g} \in \pi_1X$ acts loxodromically on $H$. By Lemma 6.4, either $\tilde{g}$ acts loxodromically on $\bar{H}$, or stabilises some $\bar{Y}_i$. The next lemma arranges for the former to hold, in a slightly more general setting.

Lemma 6.8. Let $X$ be a compact connected nonpositively curved cube complex such that $\pi_1X$ acts on $\bar{X}$ essentially. Let $\{\bar{Y}_i\}_{i \in I}$ be a $\pi_1X$–invariant collection of convex subcomplexes such that $\text{Stab}_{\pi_1X}(\bar{Y}_i)$ acts cocompactly on $\bar{Y}_i$ for all $i$. Suppose that there exists $M$ such that $\text{diam}(\tilde{g}_{\bar{Y}_i}(\bar{Y}_j)) \leq M$ whenever $\bar{Y}_i \neq \bar{Y}_j$. Suppose also that for all $\bar{Y}_i$ and hyperplanes $H$ disjoint from $\bar{Y}_i$, we have $\text{diam}(\tilde{g}_{\bar{Y}_i}(N(H))) \leq M$.

Let $\bar{H}$ be obtained from $\bar{X}$ by coning off each hyperplane carrier and each $\bar{Y}_i$. Then one of the following holds:

- There exists $i$ such that $\bar{X} = \bar{Y}_i$.
- There exists $\tilde{g} \in \pi_1X$ acting loxodromically on $\bar{H}$.}
\begin{itemize}
  \item $\tilde{X}$ decomposes as a product with unbounded factors.
\end{itemize}

\textit{Proof.} If $\tilde{X}$ is a product with unbounded factors, we are done, so we can assume that $\pi_1 X$ contains elements acting loxodromically on $\hat{\mathcal{H}}$. Let $\tilde{g} \in \pi_1 X$ be such an element.

If $\tilde{g}$ acts loxodromically on $\hat{\mathcal{H}}$, we are done. If not, then by the proof of Lemma \ref{lem:double-skewering}, $\tilde{g}$ stabilises some $\tilde{Y}_i$.

Suppose that there exists a hyperplane $U$ of $\tilde{X}$ that is disjoint from $\tilde{Y}_i$. Let $\ell = d_\tilde{X}(\mathcal{N}(U), \tilde{Y}_i)$. Let $n \gg 0$ be chosen sufficiently large in terms of $\ell$ and $M$.

Then $d_\tilde{X}(U, \tilde{g}^n U) > 10^9 (\ell + M)$ and $d_{\hat{\mathcal{H}}}(U, \tilde{g}^n U) > 100$. Moreover, $\tilde{g}^n U$ is also disjoint from $\tilde{Y}_i$, since $\tilde{g}$ stabilises $\tilde{Y}_i$. (So $\tilde{Y}_i$ lies “between” the hyperplanes $U, \tilde{g}^n U$.)

Apply the Double-Skewering Lemma \cite{CS11} to obtain $\tilde{k} \in \pi_1 X$ such that some (hence any) combinatorial geodesic axis $\tilde{B}$ for $\tilde{k}$ is cut by both $U$ and $\tilde{g}^n U$. Since $U, \tilde{g}^n U$ are 100–far in $\hat{\mathcal{H}}$, Lemma 6.2 and \cite{Hag13} Proposition 5.3 imply that $\tilde{k}$ acts loxodromically on $\hat{\mathcal{H}}$.

Now, any hyperplane $H$ separating $U$ from $\tilde{g}^n U$ must cross $\tilde{B}$. Such an $H$ either crosses $\tilde{Y}_i$, or separates either $U$ or $\tilde{g}^n U$ from $\tilde{Y}_i$. So there are at least $10^9 (\ell + M) - 2\ell$ hyperplanes $H$ that cross both $\tilde{B}$ and $\tilde{Y}_i$.

Suppose $\tilde{k}$ stabilises some $\tilde{Y}_j$. Then we could have chosen $\tilde{B}$ to lie in $\tilde{Y}_j$. So, there are $> M$ hyperplanes that cross $\tilde{Y}_i, \tilde{Y}_j$, and the geodesic $\tilde{B}$. It follows that $\text{diam}(\tilde{g}_i(\tilde{Y}_j)) > M$, which is a contradiction unless $\tilde{Y}_i = \tilde{Y}_j$.

But $\tilde{k}$ cannot stabilise $\tilde{Y}_i$. Indeed, $\tilde{Y}_i$ lies in the $U$–halfspace containing $\tilde{g}^n U$ and in the $\tilde{g}^n U$–halfspace containing $U$. Since $\tilde{k}$ skews $U, \tilde{g}^n U$, we thus have that $\tilde{k} \tilde{Y}_i$ and $\tilde{Y}_i$ are separated by $U$ or $\tilde{g}^n U$, and are thus distinct.

Now Lemma \ref{lem:double-skewering} implies that $\tilde{k}$ is loxodromic on $\hat{\mathcal{H}}$, as required.

The above works provided $\tilde{Y}_i$ is disjoint from some hyperplane. But suppose $\tilde{Y}_i$ intersects every hyperplane. Then for any $\tilde{Y}_i \in \pi_1 X$, infinitely many hyperplanes intersect $\tilde{Y}_i$ and $\tilde{\ell} \tilde{Y}_i$, implying that $\pi_1 X = \text{Stab}_{\pi_1 X}(\tilde{Y}_i)$. By essentiality, $\tilde{X} = \tilde{Y}_i$. \hfill $\Box$

Now we find asymptotic elements, under the two different situations in Theorem \cite{B}

Given $\tilde{g} \in \pi_1 X$, let $\tau_{\tilde{g}} = \inf_{\hat{\mathcal{H}}} d_\tilde{X}(\tilde{u}, g \tilde{u})$ be the combinatorial translation length of $\tilde{g}$ on $\tilde{X}$. We have $\tau_{\tilde{g}} \geq 1$ whenever $\tilde{g} \neq 1$.

\begin{definition}[The constant $L_0$] Let $L_0$ be the (nonempty) set of $\tilde{g} \in \pi_1 X$ such that $\tilde{g}$ is loxodromic on $\hat{\mathcal{H}}$. Let $L_0 = L_0(\tilde{X}) = \inf_{\tilde{g} \in L_0} \tau_{\tilde{g}}$.
\end{definition}

\begin{remark}[Conditions enabling asystolicity] Let $\langle X \mid \{Y_i \rightarrow X\}_{i \in \mathcal{I}} \rangle$ be a cubical presentation such that one of the following holds:

\begin{enumerate}
  \item $\langle X \mid \{Y_i \rightarrow X\}_{i \in \mathcal{I}} \rangle$ satisfies the uniform $C''(\frac{1}{L_0})$ condition. Moreover, the cubical presentation is \textit{minimal} (in the sense of Definition \ref{def:asystolic}), i.e. for all $\tilde{Y}_i \in \mathcal{I}$ and some (hence any) elevation $\tilde{Y}_i \rightarrow \tilde{X}$ of $Y_i \rightarrow X$, the subgroups $\pi_1 Y_i$ and $\text{Stab}_{\pi_1 X} (\tilde{Y}_i)$ are conjugate. Let $M = \inf_{\tilde{Y}_i \in \mathcal{I}} ||Y_i||$ be the bound on the diameters of abstract pieces provided by the small-cancellation condition.
  \item $\langle X \mid \{Y_i \rightarrow X\}_{i \in \mathcal{I}} \rangle$ is a cubical presentation with a uniform bound $M$ on the diameters of abstract cone-pieces and abstract wall-pieces, and each $\text{Stab}_{\pi_1 X} (\tilde{Y}_i)$ has infinite index in $\pi_1 X$. In this case, Lemma \ref{lem:acylindrical} provides a nonempty set $\mathcal{L}_1 \subseteq \pi_1 X$ such that each $\tilde{g} \in \mathcal{L}_1$ acts loxodromically on $\hat{\mathcal{H}}$. Let $L_1 = \inf_{\tilde{g} \in \mathcal{L}_1} \tau_{\tilde{g}}$
\end{enumerate}

For each $i \in \mathcal{I}$, let $\tilde{Y}_i \rightarrow Y_i$ be a finite connected regular cover so that the following holds:

\begin{enumerate}
  \item If (I) holds, then $||\tilde{Y}_i|| > \max\{144, 7L_0\} \cdot M$ for all $i \in \mathcal{I}$.
  \item If (II) holds, then $||\tilde{Y}_i|| > \max\{144, 7L_1\} \cdot M$ for all $i \in \mathcal{I}$.
\end{enumerate}
Note that $L_0$ does not depend on the $Y_i$. The constant $L_1$ depends on the $Y_i$, but not on the further finite covers.

**Lemma 6.11** (Finding fast $H$–loxodromics). Let $\langle X \mid \{Y_i\}_{i \in I}\rangle$ satisfy (I) [resp. (II)] and let the finite covers $\tilde Y_i \to Y_i$ satisfy (I) [resp. (II)]. Let $G$ be the group with cubical presentation $\langle X \mid \{\tilde Y_i\}_{i \in I}\rangle$. Suppose that $\tilde X$ contains a nontrivial wall-piece, or a pair of distinct $\tilde Y_i, \tilde Y_j$ with $\text{diam}(\tilde g_{\tilde Y_i}(\tilde Y_j)) \geq 1$. Then $G$ is either virtually cyclic or acylindrically hyperbolic.

**Proof.** Choose $\tilde g \in \mathcal{L}_0$ as follows. If (I) holds, then choose $\tilde g$ with $\tau_g = L_0$. If (II) holds, choose $\tilde g$ to be in $\mathcal{L}_1$ and satisfy $\tau_g = L_1$.

If (I) holds, $\tilde g$ cannot be conjugate into any $\pi_1 Y_i$, since $\|\tilde Y_i\| > 7L_0$. The minimality hypothesis thus implies that $\tilde g$ cannot stabilise any $\tilde Y_i$.

We now argue that $\tilde g$ is $\frac{17}{36}$–asytotic (recall that this means that every axis of $\tilde g$ is $\frac{17}{36}$–asytotic). Suppose not, then, by definition, for some combinatorial geodesic axis $\tilde A$ of $\tilde g$, there is a subpath $P$ of $\tilde A$ such that $\tilde P$ lies in some $\tilde Y_i$ and $\|\tilde P\| \geq \frac{17}{36}\|\tilde Y_i\|/36$.

Let $\sigma = \min \|\tilde Y_i\|$. If (II) holds, we have $\sigma > 7L_0$, since there is a nontrivial piece. Likewise, if (I) holds, we have $\sigma > 7L_1$. So, in either case, $\sigma > 7\tau_g$.

Now, $\tilde P$ contains a 0–cube $\tilde a$ such that $\tilde a, \tilde g\tilde a, \tilde g^2\tilde a \in \tilde P$, since $\|\tilde P\| > 7 \cdot 17 \cdot \tau_g/36$. Thus $Q = \tilde P \cap \tilde g\tilde P$ is a geodesic that lies in $\tilde Y_i \cap \tilde g\tilde Y_i$. Since $\tilde g\tilde Y_i \neq \tilde Y_i$, the geodesic $Q$ is a cone-piece.

Hence $\|Q\| < \frac{1}{144}\|\tilde Y_i\|$. On the other hand, $\|Q\| \geq \|\tilde P\| - 2\tau_g$. So, $\|Q\| > \frac{17}{36}\|\tilde Y_i\| - 2\tau_g$. So, $\tau_g > \frac{17}{36}\sigma$, while on the other hand, $\tau_g < \sigma/7$. This is a contradiction, so $\tilde g$ is $\frac{17}{36}$–asytotic.

Let $X^*$ be the presentation complex formed from $\langle X \mid \{\tilde Y_i\}_{i \in I}\rangle$ and let $H$ be obtained from $\tilde X^*$ by coming off the various hyperplane carriers and $\tilde Y_i$. Then Lemma 6.7 implies that $\tilde g$ acts on $H$ as a WPD element. Applying [Osi16] Theorem 1.1 shows that $G$ is virtually cyclic or acylindrically hyperbolic. 

6.6. **Proving Theorem (E)**. To summarise the essential case, we have:

**Proposition 6.12** (Acylindrical hyperbolicity when $\pi_1 X$ acts essentially). Let $X$ be a compact nonpositively curved cube complex such that $\pi_1 X$ acts essentially on $\tilde X$.

Let $\{Y_i \to X\}_{i \in I}$ be a (possibly infinite) set of local isometries of nonpositively curved cube complexes with each $Y_i$ compact. Suppose that $\tilde X$ does not split as a nontrivial product.

(1) Suppose that $\langle X \mid \{Y_i\}_{i \in I}\rangle$ satisfies (I). Then there exists $\alpha_0 = \alpha_0(X)$ such that the following holds. For each $i \in I$, let $Y_i \to Y_i$ be a finite regular cover such that $\langle X \mid \{\tilde Y_i\}_{i \in I}\rangle$ is a $C^\infty(\alpha)$ presentation for some $\alpha \in [0, \alpha_0]$ and let $G = \pi_1(\langle X \mid \{\tilde Y_i\}_{i \in I}\rangle)$. Then $G$ is finite, two-ended, or acylindrically hyperbolic.

(2) Suppose that $\langle X \mid \{Y_i\}_{i \in I}\rangle$ satisfies (II). Then there exists $\alpha_1 = \alpha_1(X, \{Y_i\}_{i \in I})$ such that the following holds. For each $i \in I$, let $\tilde Y_i \to Y_i$ be a finite regular cover such that $\langle X \mid \{\tilde Y_i\}_{i \in I}\rangle$ is a $C^\infty(\alpha)$ presentation for some $\alpha \in [0, \alpha_1]$ and let $G = \pi_1(\langle X \mid \{\tilde Y_i\}_{i \in I}\rangle)$. Then $G$ is finite, two-ended, or acylindrically hyperbolic.

**Remark 6.13.** The distinction between (I) and (II) is subtle. The first says that if our initial (minimal) cubical presentation already satisfies a cubical small-cancellation condition depending only on $X$, then replacing each $Y_i$ by a suitable finite cover (of degree at least a constant independent of $X$, say 144), we achieve acylindrical hyperbolicity in the quotient. Condition (II) only requires our initial cubical presentation to satisfy some uniform small-cancellation condition (not depending on $X$); the price is that to achieve acylindrical hyperbolicity, we have to pass to covers $\tilde Y_i \to Y_i$ of degree at least a constant depending on our initial cubical presentation.
Proof of Proposition 6.12. We can assume that each $Y_i$ is non-contractible, by removing contractible relators, without affecting anything. Note that the graph $\tilde{\mathcal{H}}$ depends only on $X$, and $\tilde{\mathcal{H}}$ depends on $X$ and the $Y_i$ but not on the further finite covers $\tilde{Y}_i$. In fact, the collection of subgroups $\text{Stab}_{\pi_1 X}(\tilde{Y}_i)$ depends only on the initial data $\{Y_i \to X\}_{i \in I}$.

We can also assume $\tilde{X}$ is not a product, so $\pi_1 X$ has elements acting loxodromically on $\tilde{\mathcal{H}}$.

Assume that (I) [resp. (II)] holds and let $\{\tilde{Y}_i \to Y_i\}_{i \in I}$ be as in the statement, where $\alpha_{0}$ [resp. $\alpha_1$] comes from condition (I) [resp. (II)]. Suppose that there exists $i$ such that for some elevation $\tilde{Y}_i \subseteq \tilde{X}$, there is a subcomplex $\tilde{B}$ such that $g_{\tilde{Y}_i}(\tilde{B})$ has diameter at least 1 and either $B = \tilde{Y}_j \neq \tilde{Y}_i$ or $B$ is the carrier of a hyperplane not crossing $\tilde{Y}_i$. Then Lemma 6.8 implies the proposition in this case.

Otherwise, whenever $\tilde{Y}_i, \tilde{Y}_j$ are distinct, no hyperplane crosses both, and whenever $H$ is a hyperplane disjoint from $\tilde{Y}_i$, no hyperplane crosses both $H$ and $\tilde{Y}_i$.

If $H$ crosses $\tilde{Y}_i$, and $v \in N(H)$ is a vertex not in $\tilde{Y}_i$, then some hyperplane $V$ separates $v$ from $\tilde{Y}_i$. So $g_{\tilde{Y}_i}(N(V))$ is nontrivial (since $H$ crosses $\tilde{Y}_i$ and $V$), leading to a contradiction. Hence $H \subseteq \tilde{Y}_i$. Thus, $\tilde{X}$ decomposes as a tree of spaces with each edge space a 0–cube; each elevation of each relator is a vertex space.

Hence, $X$ is a finite graph of spaces with trivial edge spaces and a vertex space $Z$ which is a compact nonpositively-curved cube complex having trivial intersection with the image of each $Y_i$; the various images of the local isometries $\tilde{Y}_i \to Y_i \to X$ are also vertex spaces. Thus, either $G$ splits over the trivial group, or $G = \pi_1 Z$. In the former case, we are done. In the latter case, the result follows because $G$ is the fundamental group of a nonpositively curved cube complex.

We are now ready to prove Theorem B.

Proof of Theorem B. Let $\langle X \mid \{Y_i\}_{i \in I}\rangle$ be as in the statement.

By [CS11] Proposition 3.5, there is a convex $\pi_1 X$–invariant subcomplex $\tilde{Z} \subseteq \tilde{X}$ on which $\pi_1 X$ acts cocompactly and essentially. (Although it is not made explicit in [CS11], $\tilde{Z}$ is in general only a subcomplex of the first cubical subdivision of $\tilde{X}$, because the action may have inversions across hyperplanes. But, we can pass to the cubical subdivision without affecting the argument, and thus assume that $\tilde{Z}$ is a subcomplex.)

Let $Z = \pi_1 X \setminus \tilde{Z}$, which is a compact nonpositively curved cube complex. Each $\tilde{g} \in \pi_1 Z = \pi_1 X$ has the same translation length on $Z$ as on $\tilde{X}$, since $\tilde{Z}$ is a $\pi_1 X$–equivariant 1–lipschitz retract of $\tilde{X}$ (use the gate map to $\tilde{Z}$). The inclusion $\tilde{Z} \to \tilde{X}$ descends to a local isometry $Z \to X$ (inducing an isomorphism on fundamental group).

Form a collection $\{U_j \to Z\}_{j}$ of local isometries as follows. For each $Y_i$ and each lift $\tilde{Y}_i \subseteq \tilde{X}$, let $\tilde{U}_i = g_{Z}(\tilde{Y}_i)$, which is a convex subcomplex. Since $\pi_1 X$ stabilises $Z$, we have $\text{Stab}_{\pi_1 X}(\tilde{Y}_i) \subseteq \text{Stab}_{\pi_1 X}(\tilde{U}_i)$. The reverse inclusion also holds. Indeed, suppose $\tilde{g}\tilde{U}_i = \tilde{U}_i$. We can assume that $\tilde{U}_i$ is unbounded since we can assume $\pi_1 Y_i$ is infinite (otherwise we would have discarded the simply connected relator $Y_i$). So the small-cancellation conditions imply $\tilde{g}\tilde{Y}_i = \tilde{Y}_i$. Hence $\tilde{Y}_i$ and $\tilde{U}_i$ have the same stabiliser. Letting $U_i = \pi_1 Y_i \setminus \tilde{U}_i$, the inclusion $\tilde{U}_i \to \tilde{Z}$ induces a local isometry $U_i \to Z$. The cubical presentation $\langle Z \mid \{U_i\}_{i}\rangle$ presents the same group as $\langle X \mid \{Y_i\}_{i \in I}\rangle$. (For each $Y_i$, there is a nonempty family of $U_j \to Z$ representing subgroups conjugate to $\pi_1 Y_i$.)

Any wall-piece or cone-piece in $\tilde{U}_i$ is the image under $g_{Z}$ of a piece in some $\tilde{Y}_i$, so lengths of pieces have not increased, while systoles have not decreased. So any small-cancellation condition satisfied by the original cubical presentation persists. The theorem now follows from Proposition 6.12. □
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