A Time-Dependent Model of Dark Energy Based on Four-Dimensional Continuous Deformation Theory

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Abstract

In this article, we investigate the mechanism of cosmological expansion and inflation by modeling dark energy as a four-dimensional continuous medium, with its elastic deformation described by a four-dimensional vector field. We demonstrate that when the bulk modulus of this cosmological medium is $K = 1.64 \times 10^{109} \text{ N} \cdot \text{m}^{-2}$, the dark energy density, corresponding to the stress-energy associated with the deformation of the medium, decreases by a factor of $\sim 10^{122}$ while the scaling factor expands from $\sim 10^{-60}$ to $\sim 10^{-32}$ over approximately $10^{-42}$ seconds during cosmological inflation in the early universe. Our analysis suggests three potential new physical phenomena for future investigation: detecting longitudinal modes of elastic waves, examining discrepancies in the redshift of light from the early universe, and fitting supernova curves using the parameters introduced in our model.

Keywords: Cosmic inflation, Cosmological expansion, Dark energy

1 Introduction

Recent observations of gravitational waves by LIGO Scientific Collaboration and Virgo Collaboration have opened new frontiers in the search for invisible forms of energy and phenomena that were beyond the current observable universe using optical instruments [1–3]. In addition, modern cosmology has a plethora of precise observations, such as supernovae
luminosity distances [4] and Cosmic Microwave Background (CMB) [5–7]. These results have been used to refine the prediction of the ΛCDM model parameters [8, 9], even though there is a significant discrepancy in the value of the Hubble constant between the results from Planck mission, the Hubble Space Telescope [8, 10], and most recently the Dark Energy Spectroscopic Instrument (DES) collaboration [11, 12]. The LIGO Scientific Collaboration and the Virgo Collaboration have also predicted a value of the Hubble constant [13–15] and, most certainly, future data obtained using gravitational detectors (aLIGO, eLISA, Einstein Telescope) will give new horizon for cosmology [16–18]. However, important theoretical questions remain unsolved and must be addressed to clarify the interpretation and understanding of future observations. Consider, for instance, the puzzling cosmological constant problem [19], i.e., the 120 orders of magnitude discrepancy between the theoretical prediction and the observed value of the cosmological constant.

Different theoretical investigations are using various approaches to address this issue. The Eckart-Israel formalism viscous fluid model [20, 21] was used to develop dissipative cosmology [22] where dark energy is described through the concept of bulk viscosity [23]. Alternative gravity theories were also developed extensively, e.g., quintessence theory [24–27], k-essence theory [28], scalar field theories with higher order derivative Lagrangian density [29], f(R) theories, and scalar-tensor theory, see [30] for a comprehensive review of the subject. The possibility of the violation of energy conservation relying on an effective time-dependent cosmological constant has been explored [31, 32] and also drawn a connection with quantum gravity. An alternative model involving a negative mass fluid has been employed to simulate galaxy rotations and cosmological expansion [33]. Another approach that consists of assigning physical property to the space-time continuum, interpreted as an elastic medium [34, 35] or as a solid material [36–38]. In [39] and [40], the authors introduced a vector deformation field into the Einstein equations, coupling it with matter-energy through a higher derivative Lagrangian theory. However, cosmological inflation and expansion mechanisms have not yet been investigated using the approaches mentioned above. In this article, I propose a model of dark energy based on a four-dimensional continuous medium deformation theory and examine the mechanism of cosmological expansion and inflation using this approach.

The content of this article is twofold: we first introduce a model to describe the deformation of a four-dimensional elastic continuous medium modeling dark energy, and then, we investigate the mechanism of cosmological expansion and inflation. We first introduce the general features of the theory and then modify the Friedmann equations obtained for the Friedmann-Lemaître-Robertson-Walker (FLRW) metric. We show that the dark energy density is proportional to the product of the bulk modulus $B = -K < 0$ and the time-dependent deformation rate $\varepsilon(t)$. We study the effects of these two parameters on the luminosity curves of celestial bodies obtained for the dark energy density ratios found by Planck Collaboration and by the Hubble Space Telescope [8, 9]. We show that for $K \approx 1.64 \times 10^{109}\text{N}\cdot\text{m}^{-2}$ the dark energy density drops by a factor $\sim 10^{122}$, while the scaling factor expands from $\sim 10^{-60}$ to $\sim 10^{-32}$ during a time of the order of $10^{-42}\text{s}$. At the end of the article, we discuss a microscopic interpretation of the inflation mechanism and future investigations.
2 Dark energy as a cosmological medium - paradigm and theory.

2.1 Postulates.

We first propose generalizing to any deformations, i.e., elastic, inelastic, solid, etc. Let us reformulate the hypothesis through these three postulates:

(P1) Dark energy is considered a continuous medium with specific physical properties, including elasticity, rigidity, stiffness, solidity, shear strength, and incompressibility, characterized by physical constants such as Young’s modulus, stiffness, shear modulus, and other relevant parameters.

(P2) The stress resulting from the deformations of this medium is mathematically described by a covariant tensor that alters the internal energy of the system. Consequently, the stress tensor corresponding to the medium is incorporated into the Einstein equations to account for the cost of energy associated with the deformation of the medium.

(P3) The strain field of dark energy generates acceleration currents of the matter-energy contained within the space-time. In other words, the strain field distorts the geodesic paths of all types of matter-energy fields, including massive particles and electromagnetic fields.

The first postulate (P1) postulates that dark energy can be described by a four-dimensional continuous elastic theory, which extends beyond the traditional approach of treating dark energy as a constant field in the Einstein equation (despite the various alternative approaches outlined in the introduction). The second postulate (P2) introduces an additional stress-energy tensor (the stress energy caused by the elastic deformation of the medium) into the Einstein equation.

The novelty of our approach lies in the combination of postulates (P2) and (P3), with particular emphasis on postulate (P3), which is less intuitive compared to the second postulate. According to (P2), matter-energy induces a deformation of the continuous medium (dark energy), while the reciprocal, stated by postulate (P3) suggests that the deformation of the continuous medium influences the trajectories of matter-energy within spacetime.

We interpret postulate (P3) as analogous to Newton’s third law of action and reaction in mechanics or Faraday’s induction principle, where a change in the deformation field (strain field) leads to an acceleration (in a covariant sense). This analogy underscores the reciprocal actions between matter-energy (electric current) and the strain field (electromagnetic field).

Mathematically, we demonstrate that the standard General Relativity (GR) framework, where dark energy is modeled as a constant, can be derived as a mathematical limit of our extended GR theory. We use the term “extended” to denote that our theory not only introduces additional terms in the stress-energy tensor of the Einstein equation but also incorporates the action of the strain field on matter-energy as per postulate (P3).

In the subsequent subsection, we will present the mathematical expressions corresponding to postulates (P1)-(P3) outlined above.
2.2 Mathematical equations of the model

Strain and stress tensors. The first postulate (P1) means that the space-time continuum is a physical substance that can be deformed by stretch, compression, and shear. We assume a linear relation between the stress-energy tensor and the strain tensor $\varepsilon^{\mu\nu}$ as

$$\sigma^{\mu\nu} = C^{\mu\nu\alpha\beta} \varepsilon^{\alpha\beta},$$  \hspace{1cm} (1)

where $C^{\mu\nu\alpha\beta}$ is the elasticity tensor that characterizes the physical properties of the 3+1-dimensional continuous medium. Of course, we could extend our model to a non-linear theory, but we leave this for future research work.

By analogy with three-dimensional continuous elastic theory [39, 40], the strain tensor is the symmetrical gradient of the 4-vector deformation $G_\mu$

$$\varepsilon^{\mu\nu} = D_\mu G_\nu + D_\nu G_\mu,$$  \hspace{1cm} (2)

where $D_\mu$ is the covariant derivative.

Modified Einstein equations. Mathematically, the second postulate (P2) states that there is an additional term in the action that characterizes the stress-energy:

$$S = \int (R + \sigma + T) \sqrt{-g} \, d^4x,$$  \hspace{1cm} (3)

where $T = g_{\mu\nu} T^{\mu\nu}$ and $\sigma = g_{\mu\nu} \sigma^{\mu\nu}$. It comes that the energy corresponding to the elastic deformation of the cosmological continuous medium is encoded in a stress-energy tensor.
\( \sigma_{\mu \nu} \), which is added to Einstein’s equations

\[
R^\nu_\mu = \frac{8 \pi G}{c^4} \left( T^\nu_\mu + \sigma^\nu_\mu - \frac{1}{2} \delta^\nu_\mu (T + \sigma) \right),
\]

(4)

where \( T^\nu_\mu = g^{\nu \lambda} T_{\mu \lambda} \) is the momentum-energy tensor. We denote \( T = T^\mu_\mu \) and \( \sigma = \sigma^\mu_\mu \) the traces of the tensors. The meaning of this additional term is that the elastic deformations of the 3+1D medium cost some energy. Another consequence of this modification is that the matter-energy density is generally not conserved:

\[
D^\nu T^\mu_\nu = - D^\nu \sigma^\mu_\nu.
\]

(5)

Universal coupling between the deformations and the matter-energy within the space-time. Postulate (P3) indicates that there is a coupling between the deformation vector \( G^\mu \) and the covariant acceleration of the matter-energy, i.e., the covariant derivative of matter-energy tensor \( D^\mu T^\nu_\mu \). We posit that this coupling is linear and has the form [40]

\[
\Lambda_{int} = - G^\mu D^\nu T^\mu_\nu,
\]

(6)

where \( \Lambda_{int} \) is the Lagrangian density, \( D^\nu T^\mu_\nu \) is the 4-vector acceleration current [40], and \( T^\mu_\nu = \sum_{j=1}^N T_{(j)}^\mu_\nu \) is the sum of the matter-energy tensors. This means that the coupling with the 4-vector deformation is universal, in the sense that any type of matter-energy experiences a deflection. From equation (6), we find that the equations of motion for the \( j \)th tensor energy are modified as follows

\[
D^\mu T_{(j)}^\mu_\nu = - \epsilon^\mu_\nu \partial^\gamma T_{(j)}^\nu_\gamma + \Delta^\mu_\nu T_{(j)}^\nu_\gamma - \Delta^\gamma_\nu T_{(j)}^\mu_\gamma,
\]

(7)

with \( \Delta^\gamma_\nu \equiv \frac{1}{2} g^{\gamma \lambda} \left( \partial_\mu \epsilon_{\lambda \nu} + \partial_\nu \epsilon_{\lambda \mu} - \partial_\lambda \epsilon_{\mu \nu} \right) \). For example, for a massive particle, the modified geodesic reads \( x^\mu + T^\mu_\nu x^\nu x^\lambda = - \epsilon^\mu_\nu x^\nu + \Delta^\mu_\nu x^\nu x^\lambda \), see [39, 40]. We would like to point out that the deformation of geodesic is formally analog to the linear transformations

\[
x^\mu \mapsto x'^\mu (x) = x^\mu + G^\mu (x),
\]

(8a)

\[
g_{\mu \nu} \mapsto g_{\mu \nu} + \epsilon_{\mu \nu},
\]

(8b)

where \( \epsilon_{\mu \nu} (x) = D_\mu G_\nu (x) + D_\nu G_\mu (x) \). However, it is important to understand that these transformations are not equivalent to the procedure detailed above that leads to the system of equations (5)-(7). Indeed, it is well-known that the standard GR is invariant by diffeomorphism transformations. Using the transformations above one can formally find equation (7), however the physical meaning of the terms in the equation is completely different.

**Discussion.** Equations (5) mean that the density of the matter-energy is dissipated. This feature has been described through different settings, see [31, 32], but the covariant expressions (5) in this paper are more general as they encode potential longitudinal modes which are forbidden in [31, 32] because of the diffeomorphism symmetry. Notice that we do not break diffeomorphism symmetry in extended GR, but instead, we introduce a novel source of
energy (stress of the continuum describing dark energy) which also deforms the geodesic and trajectories of matter-energy, as described by equations (7) and by postulate (P3). Thus, the source of the violation of standard energy conservation and the possible existence of longitudinal modes are inherited from the reciprocal deformation of the medium and trajectories of the matter-energy. Notice that the combination of postulates (P2) and (P3) makes the extended GR theory consistent, complete, and constitutive, see Figure 2.

3 Cosmological expansion: revisited.

3.1 Modified $\Lambda$CDM model

*Isotropic deformation of a 3+1D medium.* Now, we consider the Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$ds^2 = -c^2 dt^2 + a(t)^2 \left( \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2 \right)$$

where $k$ is the scalar curvature. Recent observations support the hypothesis of zero curvature $\kappa = 0 [6, 9]$, while other observations favor the closed universe assumption with $k > 0 [41]$. From the cosmological principle (i.e., assuming the medium is homogeneous and isotropic),

![Diagram](image-url)
we find that the stress tensor can be expressed as
\[ \sigma_{\mu\nu} = B \varepsilon(t) g_{\mu\nu} + 3S \left( 4\varepsilon_{\mu\nu} - \frac{1}{4} \varepsilon(t) g_{\mu\nu} \right) \] (9)

where \( B \) is the bulk modulus, \( S \) is the shear modulus, and \( \varepsilon(t) \equiv \varepsilon_\mu^\mu \) is the rate of deformation.

If we assume that \( \varepsilon_{\mu\nu} \) is invariant by rotation (no shear deformations/strains), we have \( \varepsilon_{ij} = 0 \), for \( i \neq j \). In theory, we could have time-space shear deformations \( \varepsilon_{0i} \neq 0, i \neq 0 \), but we assume they are equal to zero. This means that the 4-vector deformation \( G_\mu \) is a function of time for \( \mu = 0 \) and of the variable \( r \) for \( \mu \neq 0 \). This is a reasonable assumption for the scaling factor \( a(t) \) is stretched independently of the location in space and vice versa. In this case, we obtain that \( \varepsilon = \text{Tr} (\varepsilon_{\mu\nu}) = \varepsilon_\mu^\mu = \alpha + 3\beta \). The components of the stress tensor are

\[
\begin{align*}
\sigma_0^0 &= (B + \frac{9}{4}S) \alpha + (3B - \frac{3}{2}S) \beta = Be + \frac{95}{2} \Delta , \\
\sigma_i^i &= (B - \frac{3}{2}S) \alpha + (3B + \frac{3}{2}S) \beta = Be - \frac{55}{2} \Delta, \quad i \neq 0 , \\
\sigma_\mu^\nu &= 0, \quad \mu \neq \nu ,
\end{align*}
\] (10)

where we recall that \( \varepsilon \equiv \varepsilon_\mu^\mu \) is the trace of the strain tensor and \( \Delta \equiv \alpha - \beta \) is the space-time anisotropy deformation rate. Thus, we find that the trace of the stress and strain tensors are proportional
\[ \sigma = \sigma_\mu^\mu = 4Be, \] (11)

which is similar to a three-dimensional elastic medium \( \sigma_i^i = 3Be_i \).

Finally, we find the equation of state
\[ P = w \rho c^2 , \] (12)

where \( P \equiv +\sigma_i^i, \quad i = 1, 2, 3 \) and \( \rho c^2 \equiv -\sigma_0^0 \) (because of the signature \((-+++))\), with
\[ w = \frac{(B - \frac{35}{4}) \alpha + (3B + \frac{35}{4}) \beta}{(B + \frac{95}{4}) \alpha + (3B - \frac{35}{4}) \beta} = \frac{4Be - 35 \Delta}{4Be + 95 \Delta} . \] (13)

Notice that in general the coefficient \( w \) is time-dependent and that \( w = -1 \) (dark energy-like) is either \( S = 0 \) or \( \Delta = 0 \). If \( \varepsilon = 0 \) or \( B = 0 \), we find \( w = 1/3 \) (radiation-like), which is expected as in this case \( \sigma = 0 \). If \( \beta = 0 \) and \( S = 4B/3 \), we find \( w = 0 \) (cold matter-like), i.e., the pressure is negligible. The speed of sound \( c_s \) is determined as
\[ c_s = \sqrt{\frac{|P|}{\rho}} = c \sqrt{|w|} , \] (14)

which is less or equal than \( c \) if \( |w| \leq 1 \). Interestingly, for \( w = -1 \) (dark energy-like) the speed of sound equals the speed of light. This means that the longitudinal waves propagate at the same speed as the transversal waves. For \( w = 1/3 \) the sound waves \( c_s = c/\sqrt{3} \), which is an analog to the cosmic sound [42].
Table 1 Different regimes. If \( w = 1/3 \ldots \) blah. For an isobaric model, i.e., stress is the same in all spatial directions with \( w = -1 \), and hence, \( P_i = -\rho c^2 \), \( i = 1, 2, 3 \). In this case, there are three different critical regimes: incompressible deformation (ID), isotropic auxetic deformation (IAD), and uniaxial deformation (UAD).

|                     | \( \beta \) | \( \Delta \) | \( \varepsilon \) | \( B \) | \( S \) | \( w \) |
|---------------------|-------------|-------------|------------------|-------|-------|-------|
| **Incompressible deformation** | ANY         | ANY         | 0                | \( \to \infty \) | \( \to \text{cst} \) | \( \geq -1 \) |
| **Isotropic Deformation**      | ANY         | 0           | ANY              | ANY   | ANY   | \( -1 \) |
| **Uniaxial strain**            | 0           | ANY         | ANY              | ANY   | ANY   | \( -\frac{B - 3S/4}{B + 3S/4} \) |
| **Shearless stress**           | ANY         | ANY         | ANY              | ANY   | \( \neq 0 \) | 0    | \( -1 \) |

It is clear that cases (i) and (ii) match the interpretation of the \( \Lambda \)CDM model. However, the theory proposes a different mathematical and physical description of the dark energy and shows two distinct situations depending on the sign of the bulk modulus. From Eqs. (4)-(9), we derive the set of modified Friedmann equations describing the general dynamics of the cosmological medium and of the matter-energy in the FLRW metric space:

\[
\frac{\dot{a}(t)^2}{a(t)^2} + \frac{kc^2}{a^2} = \frac{8\pi G}{3c^2} \left( \rho(t)c^2 - B\varepsilon(t) - \frac{9S}{4}\Delta(t) \right),
\]

\[
\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3c^2} \left( \rho(t)c^2 + 3p + 2B\varepsilon(t) - \frac{9S}{2}\Delta(t) \right),
\]

From these two equations, we find the following identity:

\[
\dot{\rho} + 3\frac{\dot{a}}{a} \left( \rho + \frac{P}{c^2} \right) = \frac{B}{c^2} \varepsilon + \frac{9S}{4c^2}\Delta + 9\frac{\dot{a}}{a} \frac{S}{c^2} \Delta.
\]

In our model, we consider that the strain tensor also deforms the geodesic, see equation (7). It follows that for any matter-energy contained in the space, the modified equations of motion read:

\[
\dot{\rho}^{(j)} + 3\frac{\dot{a}}{a} \left( \rho^{(j)} + \frac{P^{(j)}}{c^2} \right) = -\alpha \rho^{(j)} - \frac{3}{a^2} \left( \frac{\dot{\beta}}{2} - \frac{\dot{\varepsilon}}{2} \right) \left( \rho^{(j)} + \frac{P^{(j)}}{c^2} \right),
\]

where \( \rho^{(j)} \), \( j = 1, 2, 3, \ldots \) are the densities of matter-energy (e.g., cold matter, electromagnetic radiation, etc.) and the \( P^{(j)} \)'s are the associated pressures, \( \rho = \sum_j \rho^{(j)} \) and \( P = \sum_j P^{(j)} \), and where \( \alpha = \alpha(t) = \varepsilon_0(t) \) and \( \beta = \beta(t) = \varepsilon_i(t) \), \( i \neq 0 \).
3.2 Stress effort in the time-direction and Poisson ratio

We can interpret the 3+1D elastic deformation as follows. Consider that the 3+1D volume \( V = x^0 \prod_{i=1}^{3} x^i \) is stretched along the time-direction:

\[
x^0 \mapsto x^0 + \delta x^0 = x^0 + \varepsilon_0^0 x^0 ,
\]

where \( \varepsilon_0^0 = \frac{\delta x^0}{x^0} \) is the rate of deformation along the time-direction. Here, we consider small deformations, i.e., \( |\delta x^0| \ll |x^0| \Rightarrow \varepsilon_0^0 \ll 1 \). While stretched along the time-direction, the volume contract or expand isotropically in the other spatial directions \( x^i \mapsto x^i + \delta x^i , i = 1, 2, 3 \):

\[
\varepsilon_i^i \equiv \frac{\delta x^i}{x^i} = -\nu \frac{\delta x^0}{x^0} - \nu \varepsilon_0^0 , i = 1, 2, ,
\]

where \( \nu \) is the Poisson ratio and \( \varepsilon_i^i \) is the rate of deformation along the spatial directions. It turns out that the 3+1 volume

\[
V' = (x^0 + \delta x^0) \prod_{i=1}^{3} (x^i + \delta^i)
\]

\[
= (x^0 + \varepsilon_0^0 x^0) \prod_{i=1}^{3} (x^i + \varepsilon_i^i x^i)
\]

\[
\approx \varepsilon_0^0 x^0 \prod_{i=1}^{3} x^i + x^0 \sum_{i=1}^{3} \varepsilon_i^i x^i \prod_{j \neq i, j \neq 0} x^j ,
\]

where we only kept the first order terms in \( \varepsilon_0^0 \). It follows that the rate of change of the 3+1D volume

\[
\frac{\delta V}{V} \approx \varepsilon = (1 - 3\nu) \varepsilon_0^0
\]

as \( \varepsilon = \varepsilon_0^0 + \sum_{i=1}^{3} \varepsilon_i^i = (1 - 3\nu) \varepsilon_0^0 \). This equation is the 3+1D analog of the deformation rate for a three-dimensional continuous medium stretched in one direction (e.g., in the direction \( x \)): \( \delta V / V \approx \varepsilon_i^i (1 - 2\nu) \). For \( \beta = \varepsilon_i^i , i = 1, 2, 3 \), and \( \varepsilon_i^i = -\nu \varepsilon_0^0 \), we find that \( \Delta = \alpha - \beta = (1 + \nu) \varepsilon_0^0 = (1 + \nu) \alpha \). Hence, we can rewrite equation (10) as

\[
\begin{cases}
\sigma_0^0 = B\alpha (1 - 3\nu) + \frac{9\nu}{2} \alpha (1 + \nu) , \\
\sigma_i^i = B\alpha (1 - 3\nu) - \frac{3\nu}{2} \alpha (1 + \nu) , i \neq 0 , \\
\sigma_{ij}^\nu = 0 , \mu \neq \nu ,
\end{cases}
\]

(19)

The Young modulus is defined as the ratio between the strain and the stress in the direction of the deformation, whence

\[
E = \frac{\sigma_0^0}{\varepsilon_0^0} .
\]

(20)
Then, considering an isotropic medium and uniaxial stress in the time direction, we shall have

\[
\begin{align*}
\sigma_0^0 &= E\varepsilon_0^0 , \\
\sigma_i^0 &= 0 , \ i \neq 0 .
\end{align*}
\]  

(21)

Therefore, the traces of the strain tensor (\(\sigma\)) is proportional to the stress \(\alpha\):

\[
\sigma = 4B\varepsilon = E\varepsilon_0^0 = E\alpha .
\]  

(22)

After combining equations (11), (18) with (22), we find the relation between the Bulk modulus \(B\) and the Young modulus \(E\)

\[
B = \frac{1}{4} \frac{E}{1 - 3\nu} .
\]  

(23)

From equations (19) and (23) we can easily deduce the relation between the shear modulus \(S\) and the Young modulus \(E\)

\[
S = \frac{1}{3} \frac{E}{1 + \nu} .
\]  

(24)

Notice the analogy with the three-dimensional relations \(B = E/(3(1-2\nu))\) and \(S = E/(2(1+\nu))\).

The Poisson ratio determines the nature of the elastic deformation. From the relations (23)-(24), it is clear that \(-1 < \nu < 1/3\). There are three elastic regimes:

- **Standard (S):** \(0 < \nu \leq 1/3\). In this case, the deformation rates in the spatial directions are negative (positive, resp.) \(\varepsilon_i^t = -\nu\varepsilon_0^0\) if the deformation rate in the time direction is positive (negative, resp.). This situation is standard in material science for three-dimensional materials.

- **Auxetic (A):** \(-1 \leq \nu < 0\). This case is the exact opposite of the previous one, i.e., the deformation rates in the spatial directions are positive (negative, resp.) \(\varepsilon_i^t = -\nu\varepsilon_0^0\) if the deformation rate in the time-direction is positive (negative, resp.). This situation has been investigated in material science for three-dimensional materials since the 90’s [43].

- **Uniaxial deformation (UD):** \(\nu = 0\). In this situation, there is no deformation in the spatial directions \(\varepsilon_i^t = 0\). We say that space is rigid in the sense that stretch or contraction in the time direction does not affect space directions. The three-dimensional analog is cork material for which the Poisson ratio is about 0.5 (this makes cork well-suited as a material for wine bottle stoppers).

In particular, we have two critical regimes when \(\nu = 1/3\) (Incompressible regime), in which the 3+1D volume is unchanged (even though the spatial volume is compressed/stretched for \(\alpha > 0/\alpha < 0\), respectively), and \(\nu = -1\) (Isotropic Auxetic regime), in which the rate of deformation is the same for all directions (time and space). We summarized these particular regimes in Table 2. In what’s next, we assume that there is no deformation of the space directions, i.e., the Poisson ratio \(\nu = 0\) that corresponds to the Uniaxial Deformation regime.
(UD), see Table 2. Hence, the components of the stress tensor now read (see )

\[
\begin{align*}
\rho c^2 &= \sigma_{00} = (B + \frac{9}{4} S) \varepsilon, \\
P &= -\sigma_{i} = -(B - \frac{3}{4} S) \varepsilon, \quad i \neq 0.
\end{align*}
\]

(25)

\[w = -\frac{4B - 3S}{4B + 9S} \]

(26)

| \(v\) | \(\varepsilon\) | \(B/E\) | \(S/E\) |
|---|---|---|---|
| Incompressible deformation (ID) | 1/3 | 0 | \(\infty\) | 1/4 |
| Isotropic Auxetic Deformation (IAD) | -1 | 4\(\alpha\) | 1/16 | \(\infty\) |
| Uniaxial Deformation (UAD) | 0 | \(\alpha\) | 1/4 | 1/3 |

Table 2 Stress in the time-direction. We consider stress only in the time direction (hence \(R = 0, i = 1, 2, 3\)).

3.3 Exact solutions to the modified Friedmann equations for the isobaric pressure model

From equation (26), we find that the equation of states describes a negative pressure similar to that of vacuum energy for \(w = -1\) if \(S = 0\) (no shear modulus) or if \(\Delta = 0\) (isotropic deformation). Either way, we find that the expression of the stress tensor of the cosmological medium is

\[\sigma_{\mu\nu} = B\varepsilon(t)g_{\mu\nu}.\]

We straightforwardly notice the analogy with any time-dependent cosmological constant models with \(\Lambda(t) \equiv \frac{8\pi G}{c^2} K\varepsilon(t)\). We introduced the constant \(K = -B\) that can be interpreted as a stress-energy density that characterizes the medium for \(K > 0\). We find different scenarios depending on the signs of the bulk modulus \(B = -K\) and the deformation rate \(\varepsilon\).

**Case (i):** For \(K > 0\) \((B < 0)\) and \(\varepsilon > 0\) the term \(\sigma_{00} > 0\) (i.e., \(\Lambda(t) > 0\)) can be interpreted as the dark energy density corresponding to the stress-energy of the medium. The acceleration of the expansion of the cosmological medium is due to the presence of internal pressure (the spatial terms \(\sigma_{ii}\) are negative) with a positive deformation rate \(\varepsilon > 0\).
Case (ii): Similarly, in this case, the bulk modulus is positive ($K < 0$ or $B > 0$) while the deformation rate is negative $\varepsilon < 0$, we find that the energy density is also positive $\sigma_{00} > 0$ and that the pressure is negative.

The last two cases describe an elastic material experiencing positive pressure and correspond to a universe with decelerated expansion ($\Lambda < 0$):

Case (iii): For $K > 0$ with $\varepsilon < 0$ (negative bulk modulus and deformation rate).

Case (iv): For $K < 0$ with $\varepsilon > 0$ (positive bulk modulus and deformation rate).

In what follows, we will assume that $\beta = 0$ (homogeneity of space) and $S = 0$ (equation of state $w = -1$). From equations (15), we find that the Friedmann equations read

$$H(t)^2 \equiv \frac{\dot{a}(t)^2}{a(t)^2} = \frac{8\pi G}{3c^2} \left( \rho(t)c^2 + K\varepsilon(t) \right),$$  
(27a)

$$\sum_w (\dot{\rho}_w(t) + 3(1+w)H(t)\rho_w(t)) = -\frac{K}{c^2}\varepsilon(t),$$  
(27b)

$$\dot{\rho}_w(t) + 3(1+w)H(t)\rho_w(t) = -\varepsilon(t)\dot{\rho}_w(t),$$  
(27c)

where $H(t) \equiv \dot{a}(t)/a(t)$ is the Hubble parameter, $\rho_w c^2$ stands for energy density of the pressureless cold matter for $w = 0$, and of the radiation for $w = 1/3$. Equation (27b) has been derived from equation (5), after taking $S = 0$ in equation (9) (which means that $w = -1$). Equation (27c) is a consequence of the deformations of the equations of motion, see equation (17). Equations (27b) and (27c) lead to

$$\varepsilon(t) = \varepsilon_0 \exp \left\{ \frac{c^2}{K} (\rho(t) - \rho_0) \right\}$$

where $\varepsilon_0 \equiv \varepsilon(t_0)$ is the deformation rate at the present epoch $t_0$. This means that the deformation rate increases (decreases) with the density of matter $\rho(t)$ in case (i) (case (ii), respectively). As mentioned above, one can interpret the density of dark energy $\rho_\Lambda(t)c^2$ as the stress-energy density of the cosmological medium which is equal to the opposite bulk modulus $K = -B$ times the deformation rate $\varepsilon(t)$, see equation (27a)

$$\rho_\Lambda(t)c^2 = K\varepsilon(t) = K\varepsilon_0 \cdot \exp \left\{ \frac{c^2}{K} (\rho(t) - \rho_0) \right\}$$  
(28)

It follows that the cosmological constant $\Lambda = 10^{-52} m^{-2}$ corresponds to the stress-energy density at the present epoch multiplied by the constant $8\pi G/c^4$, i.e., $\Lambda = 8\pi G\varepsilon_0/c^4$. Assuming that the universe will expand indefinitely, we find that $\rho_\Lambda(\infty) = K\varepsilon_0 \exp \left\{ \frac{c^2}{K} \rho_0 \right\}$. This scenario corresponds to a de Sitter model of infinite expansion with a smaller or bigger rate than the one predicted by the $\Lambda CDM$ model for case (i) and case (ii) respectively. The

\[ \text{See also the analogy with the geodesic deformation (8b) with } g_{00} \rightarrow g_{00} + \varepsilon_{00} = 1 + \varepsilon(t). \]
time-variation of dark energy also changes the relation between the scaling factor and the total density of matter-energy in the matter-dominated ($w = 0$) and radiation-dominated eras ($w = 1/3$). From Eqs. (27b) and (27c), we obtain

$$a(t) = \left( \frac{\rho_0}{\rho} \right)^{\frac{1}{3(1+w)}} \cdot \exp \left\{ \frac{\varepsilon_0}{3(1 + w)} \left( \frac{\varepsilon^2}{K\rho_0} - \frac{\varepsilon^2}{K\rho} \right) \right\}, \quad (29)$$

where $\varepsilon_0 \equiv \varepsilon_0 \exp \left\{ -\frac{\varepsilon^2}{K\rho_0} \right\}$ is the value of the deformation rate in the limit $t \to +\infty$. In Eq. (29), the first algebraic term corresponds to the scaling factor for the standard $\Lambda$CDM and the term in the exponent is the modification caused by the deformation of the medium, where the exponential-integral function $\text{Ei}(x) \equiv \int_0^x du \, u^{-1} \exp(u)$ [44]. From Eq. (29) it is clear that the scaling factor can grow fast as the density decreases for case (i) while it asymptotically behaves as $\rho^{-3(1+w)}$ for case (ii). We shall now discuss the relation to the $\Lambda$CDM model. From Eqs. (27)-(29), it is now clear that we find the $\Lambda$CDM model in the limit $|K| \to +\infty$ and $|\varepsilon_0| \to 0$ and keeping the product constant $K\varepsilon_0 \to 3\Lambda c^2/(8\pi G)$, where the cosmological constant $\Lambda \sim 10^{-52} m^{-2}$. However, this limit is not valid if the total density of matter-energy is comparable with $K/c^2$ as the term in the exponential in Eq. (28) and Eq. (29) is no longer negligible. Note that the extension of the result in the non-linear regime is straightforward for the FLRW model. Indeed, consider non-linear deformations for the variable $x^0 = ct$, i.e., $ct \mapsto ct + G(t)$ where $G(t)$ is the non-zero- value of the deformation vector $G^\mu(x)$ corresponding to the time component $\mu = 0$. Following a similar approach than for the linear case, we find that $c^2 dt^2 \mapsto \left( 1 + 2\frac{G}{c^2} \right) c^2 dt^2$. This leads to the deformed metric $g_{00} \mapsto 1 + \varepsilon(t)$, with $\varepsilon(t) = \frac{2G}{c^2} + \frac{\varepsilon^2}{K\rho}$ instead of $\frac{2G}{c^2}$ for the linear case. Assuming Hooke’s law, see Eq. (1), we find the exact same equations as Eqs. (27) - (29). This yields some interesting results for case (i) which we are going to discuss next.
3.4 Cosmological expansion - Parameters estimation.

In our model, we introduced two new fundamental parameters, namely the bulk modulus $B = -K$ and the deformation rate $\varepsilon_0$ at the present epoch. We propose to examine their effects on the luminosity curves of celestial bodies. We assume that the product between these two quantities equals the density of dark energy at the present time, e.g., see the values in [9] or in [8]. We recall that in case (i) the bulk modulus is negative (i.e., $K > 0$) and the deformation rate is positive ($\varepsilon_0 > 0$) while in case (ii) the signs are the opposite. The luminosity distance of a celestial body distant with a redshift $z$ from an observer is given by [45]

$$d_L(z) = c(1+z) \int_0^z \frac{dz'}{H(z')} ,$$

where $H(z')$ is the Hubble factor depending on the redshift $z'$. In astronomy, it is useful to introduce the distance modulus [45]

$$m - M = 5 \log_{10} \left( \frac{d_L}{10 \text{pc}} \right) ,$$

where the conversion of the parsec unit is $1\text{pc} = 3.086 \times 10^{16} \text{m}$. In Fig. 3, we display relative distance modulus $\Delta(M - m)$ calculated by subtracting the distance modulus computed for the empty universe Milne model [45] from the distance modulus of the $\Lambda$CDM model and of the modified $\Lambda$CDM model. Fig. 3(a) shows $\Delta(M - m)$ for two different values of the density of dark energy ratio $\Omega_\Lambda \equiv \rho_\Lambda / \rho_c$, namely, $\Omega_\Lambda = 0.692$ obtained by the Planck Collaboration [9] and $\Omega_\Lambda = 0.62$ obtained by the Hubble Space Telescope [8], where the critical density $\rho_c \equiv 3H(0)^2 / (8\pi G)$. We can see that the relative modulus for case (i) (case (ii)) is greater (smaller, respectively) than that of the $\Lambda$CDM for fixed $\Omega_\Lambda$. In Fig. 3(b), we illustrate the variation of the relative distance modulus for case (ii) for different values of $\varepsilon_0$. Fig. 3 shows qualitatively that the additional parameter $\varepsilon_0$ offers the possibility to find a better agreement between the Planck Collaboration and the Hubble Space Telescope [8, 9]. We mention that it is possible to modify the effective equation of state for the dark energy by taking a non-zero shear modulus $S$, see Eq. (9). It is also possible to include an effective space-curvature $\kappa$ by adding a deformation rate in the radial coordinate. This is beyond the scope of this paper, but we hope this could motivate future research.

4 Cosmological inflation: revisited.

4.1 Inflation mechanism of an elastic medium

Interestingly, case (i) (negative bulk modulus $K = -B > 0$ and positive deformation rate $\varepsilon_0 > 0$) shows similar features to the inflation scenario [46–49]. The rate $c^2 / K$ which appears in the exponential term in Eqs. (28) and (29) has the unit of the inverse of a density $\rho_d \equiv K / c^2$ that characterizes the threshold between two different regimes. In the first regime, as $\rho \ll \rho_d$ the deformations are negligible and then $a \sim \rho^{-1/(3+3w)}$. In the second regime for relatively large density $\rho \gg \rho_d$, the scaling factor decreases super-exponentially fast as the density
grows

\[ a(t) \sim a(t_1) \exp \left\{ -\frac{\rho_\Lambda(t) - \rho_\Lambda(t_1)}{4\rho_t} \right\}, \quad (30) \]

see Fig. 4. It follows that the dark energy density increases exponentially fast when the density of matter-energy is saturated to an order of magnitude determined by the density \( \rho_\Lambda \sim 10^9 \rho_d \) (so that \( \rho \gg \rho_d \)). Therefore, for case (i) the parameter \( \varepsilon_0 \) has to be very small, i.e., \( \rho_d \) very large compared to \( \rho_{\text{CMB}} \sim 10^{-18} \text{kg} \cdot \text{m}^{-3} \) where \( \rho_{\text{CMB}} \) is the density of baryonic and dark matter at the CMB epoch.

Here, we consider an inflation scenario that antedates the Planck epoch. We take \( K = 1.64 \times 10^{109} \text{N} \cdot \text{m}^{-2} \) and assume that \( K\varepsilon_0 = \rho_\Lambda(0) \), where \( \rho_\Lambda(0) = 6.023 \times 10^{-27} \text{kg} \cdot \text{m}^{-3} \) is the dark energy density at the present epoch (we take the Hubble Space Telescope 2016 value, see [8]). We find that the bulk modulus is \( K \approx 1.64 \times 10^{109} \text{N} \cdot \text{m}^{-2} \), the characteristic density is \( \rho_d = 1.82 \times 10^{92} \text{kg} \cdot \text{m}^{-3} = 3.5 \times 10^{-4} \rho_{\text{Pl}}, \) and that the critical value for the energy-density \( \rho_{\text{c}} \) is of the order of \( 10^{-3} \rho_{\text{Pl}} \), where \( \rho_{\text{Pl}} = E_{\text{Pl}} / (\hbar c)^3 \approx 2 \times 10^{113} \text{J} \cdot \text{m}^{-3} = 1.3 \times 10^{123} \text{GeV} \cdot \text{m}^{-3} \) is the quantum fluctuation density (or Planck density) at the Planck epoch, with \( E_{\text{Pl}} = 1.6 \times 10^{19} \text{GeV} \). When \( \rho > \rho_d \), the density of dark energy starts to increase exponentially fast and reaches the Planck energy density (also called quantum fluctuation density) \( \rho_{\text{Pl}} \) of \( E_{\text{Pl}} / (\hbar c)^3 \approx 4.66 \times 10^{111} \text{J} \cdot \text{m}^{-3} = 2.91 \times 10^{123} \text{GeV} \cdot \text{m}^{-3} \), where the Planck energy is \( E_{\text{Pl}} = 1.96 \times 10^{19} \text{GeV} \), see Fig.4(a). The scaling factor decreases from \( \sim 10^{-32} \) to \( \sim 10^{-60} \) super-exponentially fast when \( \rho > \rho_c \), see Fig.4(b). The time taken during inflation can be estimated through this integral

\[ \Delta t = \int_{t_1}^{t_{\text{GU}}} \frac{\rho t}{\sqrt{3 \rho_{\text{c}}}} \left( \frac{da}{d\rho} \right) \left( \rho + \rho_\Lambda(\rho) \right)^{-1/2} \approx 2.2 \times 10^{-42} \text{s}, \quad (31) \]

where the relation between the scaling factor and the density \( \rho \) can be found using Eqs. (29)-(28) with \( \rho_\Lambda = 9.5 \times 10^{-3} \rho_{\text{Pl}}, \rho_t = 10^{-2} \rho_{\text{Pl}} \). A similar calculation leads to \( t_{\text{GU}} \sim 10^{-38} \text{s} \), where \( t_{\text{GU}} = \rho_{\text{GU}} c^2 = E_{\text{GU}} / (\hbar c)^3 = O(10^{101} \text{J} \cdot \text{m}^{-3}) = O(10^{111} \text{GeV} \cdot \text{m}^{-3}) \), where \( E_{\text{GU}} \approx O(10^{16} \text{GeV}) \) is the grand unification energy scale.

### 4.2 Transition between elastic state to solid state

Notice that the deformation rate \( \varepsilon(t) \) is also decreasing super-exponentially fast as is it proportional to the density \( \rho_\Lambda(t) \). This means that the continuous medium modeling dark energy transitions from being elastically stretched during the inflation period to becoming rigid-like afterward. This is due to the large value of the bulk modulus \( K \sim 10^{109} \text{N} \cdot \text{m}^{-2} \) and also due to the super-fast expansion of the scaling factor that causes the dilution of the deformation strength of the medium. By analogy with material science, we can say that during the inflation period, the cosmological medium experiences a phase transition between an elastic state (large value of the deformation rate \( \varepsilon \) despite the large value of the bulk modulus \( K \)) and a solid-state (relatively small/large value of deformation rate/bulk modulus, respectively). We interpret that medium looks stiff and non-deformable in our present epoch by this very small value of the deformation rate \( \varepsilon_0 \sim 10^{-133} \). In this limit, the dissipative terms in equations (3.1) become negligible (\( \varepsilon \sim 0 \)), and hence, the energy is conserved, as stated in the standard
Fig. 4 Inflation mechanism. In this figure we plot the dark energy density curve (a), the scaling factor (b) for the modified $\Lambda$CDM (continuous line) and the $\Lambda$CDM standard model (dashed line), as functions of time, for $K = 1.64 \times 10^{10}$ N·m$^{-2}$, where $\rho_\Lambda(0) = 6.023 \times 10^{-27}$ kg·m$^{-3}$ is the dark energy density at the present epoch (see, [8]). These two graphs show the fast decay of the dark energy density and the super-exponential growth of the scaling factor.

GR. Also, as $\varepsilon(t) = 2D \mu G^\text{R}$, in this limit, our model becomes analogous to that of [31], which states the diffeomorphism symmetry.

5 Discussion and outlook

Our model introduces two additional fundamental parameters, namely, the bulk modulus of the medium $B = -K$ and the deformation rate $\varepsilon_0$ at the present epoch. We investigated different scenarios of expansion depending on the sign of the deformation rate and of the bulk modulus, either (i) $B < 0$ and $\varepsilon_0 > 0$ or (ii) $B > 0$ and $\varepsilon_0 < 0$, see Fig. 3. This raises the question of the microscopic origin of the signs. We think that the values of the bulk modulus and the deformation rate could be derived from a more fundamental microscopic theory as it was done in material science, see [50]. Heuristically, we can interpret the negative sign of the deformation rate of the medium from cases (i) and (ii) as follows:

**Case (i):** the effect of repulsive interactions between particles constituting the medium. It follows the negativity of the spring constant in Hooke’s law $F = +kx$, where $x$ is a small deformation of the repulsive pairwise potential and $k$ is the strength of the locally downward parabolic potential $V(i) = -\frac{k}{2}x^2$.

**Case (ii):** the spring constant is positive and the attractive interactions are described by a locally upward parabolic potential $V(ii) = +\frac{k}{2}x^2$.

In this work, we show that an inflation scenario occurs for case (i). We predict that there is an exponential decay of dark energy density and that the scaling factor expands super-exponentially fast from $10^{-60}$ to $10^{-32}$ during a time $\sim 10^{-42}$ s, see Fig. 4. An alternative scenario would consider a transition between the two cases (i) and case (ii) during inflation. This hypothesis shows similar features as the theory of symmetry breaking between a false vacuum and a true vacuum [46–49]. The best fit of luminosity curves using the parameters $K < 0$ and $\varepsilon_0 > 0$ (see Fig. 3) could validate that the current situation corresponds to case (ii), and that a transition between cases (i) and (ii) is a feasible hypothesis. Theoretically, we
need to extend Hooke’s law (see Eq. (9)) to the non-linear regime to investigate such a transition. Searching for experimental evidence of cosmological inflation is a challenge for future research. Unlike gravitational waves, the non-linear deformation waves that occur during inflation are longitudinal as the gauge $\partial_{\mu}\epsilon^{\mu}_{\nu} = 0$ is broken, see Eq. (27b). Hence, we expect longitudinal cosmic waves to be a signature of the inflation mechanism described in this article.

As for the large order of magnitude of the bulk modulus, let us recall that classically an estimate shows that this value should be about $10^{20}$ times that of the steel, more specifically equal to $c^2 f^2 / G \sim 4.5 \times 10^{31} \text{N} \cdot \text{m}^{-2}$ for typical gravitational wave frequencies $f \sim 100\text{Hz}$ [51]. This value has been found using an analogy between the Einstein tensor with a stress tensor for shear deformation. In GR theory space-time is not a physical substance, and hence, the analogy with shear stress modulus that was drawn to derive the result is purely formal and not justified physically. However, we can compare with the values found using other approaches to the fabric of space-time. For example, the Young modulus was found to be of the order of $c^2 / (\hbar G^2) \sim 5 \times 10^{113} \text{N} \cdot \text{m}^{-2}$, see [37, 38]. This value can be understood as the Planck’s Young modulus and characterizes the stiffness of a quantum space-time. Consequently, this value does not correspond to an independent parameter in the cosmological model. On the contrary, in our present model, we introduce a new fundamental constant $K$ independent of any other constants.

Let us now comment on the differences between our model and the existing models proposed in the literature. The main difference between our approach and alternative gravity theories [24–27, 29] and dissipative cosmology [22, 23] is the idea of a universal coupling between the vector field $G_\mu$ and all kinds of matter-energy, see equations (6), (7), and postulate (P3), that has not been proposed in the literature yet. The consequence of the equation (7) is that the modified (RFW) model has an additional equation in the system (17), which leads to an expression of the scaling factor that has never been found before this present work (see equation (27c)). The idea of violation of conservation energy has already been proposed in [31, 32], however, their volume-preserving diffeomorphism condition does not allow them to explore such coupling between dark energy and matter-energy as we do in the present paper. This constraint is justified by the fact that their theory relies on the paradigm of metric description of space-time and on the strong equivalence principle. As a result of this constraint, they find a change in the order of magnitude of the cosmological constant during inflation time that does not match the expected discrepancy. This highlights the limitation of describing inflation through a rigid/undeformable spacetime, while our model interprets inflation as the deformation of a cosmological medium (dark energy) with a very large bulk modulus.

Finally, in this article, we found three possible empirical evidence of our theory: a direct one which consists of the existence of longitudinal modes in gravitational waves, and an indirect one regarding the additional parameters to fit the supernovae curves. Another optical evidence could be an effective shift of light frequency caused by the effective geodesic deformation (8b) that might not be detectable at the astronomical scale for technical reasons but could be seen at the cosmological scale (early universe photons). We also want to point out that there are many more possibilities to investigate using this model by implementing a non-zero shear modulus $\Sigma$ and scalar curvature $\kappa$. It would be interesting to best fit the parameters
to different phenomena, such as the Cosmic Microwave Background, supernovae luminosity curves, and galaxy rotation. This would require advanced numerical computations.

References

[1] Abbott, B.P., et al.: Observation of gravitational waves from a binary black hole merger. Phys. Rev. Lett. 116, 061102 (2016) https://doi.org/10.1103/PhysRevLett.116.061102

[2] Abbott, B.P., et al.: Gw150914: The advanced ligo detectors in the era of first discoveries. Phys. Rev. Lett. 116, 131103 (2016) https://doi.org/10.1103/PhysRevLett.116.131103

[3] Abbott, B.P., et al.: Gravitational waves and gamma-rays from a binary neutron star merger: Gw170817 and grb 170817a. Ap. J. Lett. 848 (2017)

[4] Riess, A.G., et al.: Observational evidence from supernovae for an accelerating universe and a cosmological constant. Astron. J. 116, 1009–1038 (1998)

[5] Smoot, G.F., et al.: Structure in the cobe differential microwave radiometer first-year maps. apjl 396, 1 (1992) https://doi.org/10.1086/186504

[6] Komatsu, E., et al.: Seven-year wilkinson microwave anisotropy probe (wmap) observations: Cosmological interpretation. Astrophys. J. Suppl. 192, 18 (2011)

[7] Ade, P.A.R., et al.: Planck 2013 results. i. overview of products and scientific results. A&A 571, 1 (2014)

[8] Bonvin, V., et al.: Holicow - v. new cosmograil time delays of he 0435-1223: H0 to 3.8 per cent precision from strong lensing in a flat $\Lambda$cdm model. MNRAS 465(4), 4914–4930 (2016)

[9] Ade, P.A.R., et al.: Planck 2015 results xiii. cosmological parameters. A&A 594, 13 (2016)

[10] Riess, A.G., et al.: New parallaxes of galactic cepheids from spatially scanning the hubble space telescope: Implications for the hubble constant. Ap. J. 855(2) (2018)

[11] Collaboration, D.: First Year Results from the Dark Energy Spectroscopic Instrument (DESI). APS April Meeting 2024, Session F01 (2024). https://april.aps.org/

[12] Collaboration, D., et al.: DESI 2024 VI: Cosmological Constraints from the Measurements of Baryon Acoustic Oscillations (2024)

[13] Abbott, B.P., et al.: A gravitational-wave standard siren measurement of the hubble constant. Nature 551(7678), 85–88 (2017) https://doi.org/10.1038/nature24471

[14] Abbott, B.P., et al.: A gravitational-wave measurement of the hubble constant following the second observing run of advanced ligo and virgo. The Astrophysical Journal 909(2),
218 (2021) https://doi.org/10.3847/1538-4357/abdcb7

[15] Abbott, B.P., *et al.*: Erratum: “a gravitational-wave measurement of the hubble constant following the second observing run of advanced ligo and virgo” (2021, apj, 909, 218). The Astrophysical Journal 923(2), 279 (2021) https://doi.org/10.3847/1538-4357/ac4267

[16] Sesana, A.: Prospects for multiband gravitational-wave astronomy after gw150914. Phys. Rev. Lett. 116, 231102 (2016) https://doi.org/10.1103/PhysRevLett.116.231102

[17] Askar, A., *et al.*: Black holes, gravitational waves and fundamental physics: a roadmap. Classical and Quantum Gravity 36(14), 143001 (2019) https://doi.org/10.1088/1361-6382/ab0587

[18] Amaro-Seoane, P., *et al.*: Astrophysics with the laser interferometer space antenna. Living Reviews in Relativity 26(1), 2 (2023) https://doi.org/10.1007/s41114-022-00041-y

[19] Weinberg, S.: The cosmological constant problem. Rev. Mod. Phys. 61, 1–23 (1989) https://doi.org/10.1103/RevModPhys.61.1

[20] Eckart, C.: The thermodynamics of irreversible processes. iii. relativistic theory of the simple fluid. Phys. Rev. 58, 919–924 (1940) https://doi.org/10.1103/PhysRev.58.919

[21] Israel, W.: Nonstationary irreversible thermodynamics: A causal relativistic theory. Annals of Physics 100(1), 310–331 (1976) https://doi.org/10.1016/0003-4916(76)90064-6

[22] Maartens, R.: Dissipative cosmology. Classical and Quantum Gravity 12(6), 1455 (1995) https://doi.org/10.1088/0264-9381/12/6/011

[23] Fabris, J.C., Gonçalves, S.V.B., Ribeiro, R.d.S.: Bulk viscosity driving the acceleration of the universe. General Relativity and Gravitation 38(3), 495–506 (2006) https://doi.org/10.1007/s10714-006-0236-y

[24] Ratra, B., Peebles, P.J.E.: Cosmological consequences of a rolling homogeneous scalar field. Phys. Rev. D 37, 3406–3427 (1988) https://doi.org/10.1103/PhysRevD.37.3406

[25] Caldwell, R.R., Dave, R., Steinhardt, P.J.: Cosmological imprint of an energy component with general equation of state. Phys. Rev. Lett. 80, 1582–1585 (1998) https://doi.org/10.1103/PhysRevLett.80.1582

[26] Brax, P., Martin, J.: Robustness of quintessence. Phys. Rev. D 61, 103502 (2000) https://doi.org/10.1103/PhysRevD.61.103502

[27] Tsujikawa, S.: Quintessence: a review. Classical and Quantum Gravity 30(21), 214003 (2013) https://doi.org/10.1088/0264-9381/30/21/214003

[28] Armendariz-Picon, C., Mukhanov, V., Steinhardt, P.J.: Essentials of k-essence. Phys.
[29] Anisimov, A., Babichev, E., Vikman, A.: B-inflation. Journal of Cosmology and Astroparticle Physics 2005(06), 006 (2005) https://doi.org/10.1088/1475-7516/2005/06/006

[30] Nojiri, S., Odintsov, S.D.: Unified cosmic history in modified gravity: From f(r) theory to lorentz non-invariant models. Physics Reports 505(2), 59–144 (2011) https://doi.org/10.1016/j.physrep.2011.04.001

[31] Josset, T., Perez, A., Sudarsky, D.: Dark energy from violation of energy conservation. Phys. Rev. Lett. 118, 021102 (2017) https://doi.org/10.1103/PhysRevLett.118.021102

[32] Perez, A., Sudarsky, D.: Dark energy from quantum gravity discreteness. Phys. Rev. Lett. 122, 221302 (2019) https://doi.org/10.1103/PhysRevLett.122.221302

[33] Farnes, J. S.: A unifying theory of dark energy and dark matter: Negative masses and matter creation within a modified ΛCDM framework. A&A 620, 92 (2017) https://doi.org/10.1051/0004-6361/201832898

[34] Tartaglia, A., Radicella, N.: A tensor theory of spacetime as a strained material continuum. Classical and Quantum Gravity 27(3), 035001 (2010) https://doi.org/10.1088/0264-9381/27/3/035001

[35] Millette, P.A.: On the decomposition of the spacetime metric tensor and of tensor fields in strained spacetime. Progress in Physics 8(4), 5–8 (2012)

[36] Pearson, J.A.: Material models of dark energy. Annalen der Physik 526(7-8), 318–339 (2014) https://doi.org/10.1002/andp.201400052 https://onlinelibrary.wiley.com/doi/pdf/10.1002/andp.201400052

[37] Tenev, T.G., Horstemeyer, M.F.: Mechanics of spacetime — a solid mechanics perspective on the theory of general relativity. International Journal of Modern Physics D 27(08), 1850083 (2018) https://doi.org/10.1142/S0218271818500839

[38] David, I.: Mechanical conversion of the gravitational einstein’s constant $\kappa$. Pramana 94(1), 119 (2020) https://doi.org/10.1007/s12043-020-01954-5

[39] Beau, M.: Théorie du champ des déformations en relativité générale et expansion cosmologique. Annales de la Fondation Louis de Broglie 40 (2015)

[40] Beau, M.R.: On generalized forces in higher derivative lagrangian theory. Acta Physica Polonica B 51(9), 1841 (2020)

[41] Di Valentino, E., Melchiorri, A., Silk, J.: Planck evidence for a closed universe and a possible crisis for cosmology. Nature Astronomy 4(2), 196–203 (2020) https://doi.org/10.1038/s41550-019-0906-9
[42] Eisenstein, D.J., et al.: Detection of the baryon acoustic peak in the large-scale correla-
tion function of sdss luminous red galaxies. The Astrophysical Journal 633(2), 560 (2005) https://doi.org/10.1086/466512

[43] Evans, K.E.: Auxetic polymers: a new range of materials. Endeavour 15(4), 170–174 (1991) https://doi.org/10.1016/0160-9327(91)90123-S

[44] Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions with Formulas,
Graphs, and Mathematical Tables, ninth dover printing, tenth gpo printing edn. Dover,
New York City (1964)

[45] Wright, E.L.: Constraints on dark energy from supernovae, gamma-ray bursts, acoustic
oscillations, nucleosynthesis, large-scale structure, and the hubble constant. The
Astrophysical Journal 664(2), 633 (2007) https://doi.org/10.1086/519274

[46] Starobinsky, A.A.: A new type of isotropic cosmological models without singularity.
Physics Letters B 91(1), 99–102 (1980) https://doi.org/10.1016/0370-2693(80)
90670-X

[47] Guth, A.H.: Inflationary universe: A possible solution to the horizon and flatness
problems. Phys. Rev. D 23, 347–356 (1981) https://doi.org/10.1103/PhysRevD.23.347

[48] Albrecht, A., Steinhardt, P.J.: Cosmology for grand unified theories with radiatively
induced symmetry breaking. Phys. Rev. Lett. 48, 1220–1223 (1982) https://doi.org/10.
1103/PhysRevLett.48.1220

[49] Linde, A.D.: A new inflationary universe scenario: A possible solution of the horizon,
flatness, homogeneity, isotropy and primordial monopole problems. Physics Letters B
108(6), 389–393 (1982) https://doi.org/10.1016/0370-2693(82)91219-9

[50] I. Goldhirsch, C. Goldenberg: On the microscopic foundations of elasticity. Eur. Phys.
J. E 9(3), 245–251 (2002) https://doi.org/10.1140/epje/i2002-10073-5

[51] McDonald, K.T.: What is the Stiffness of Spacetime? Accessed: May 30th, 2024 (2018).
http://kirkmcd.princeton.edu/examples/stiffness.pdf