Segmented strings in flat space are piecewise linear classical string solutions. Kinks between the segments move with the speed of light and their worldlines form a lattice on the worldsheet. This idea can be generalized to AdS$_3$ where the embedding is built from AdS$_2$ patches. The construction provides an exact discretization of the non-linear string equations of motion.

This paper computes the area of segmented strings using cross-ratios constructed from the kink vectors. The cross-ratios can be expressed in terms of either left-handed or right-handed variables. The string equation of motion in AdS$_3$ is reduced to that of an integrable time-discretized relativistic Toda-type lattice. Positive solutions yield string embeddings that are unique up to global transformations. In the appendix, the Poincaré target space energy is computed by integrating the worldsheet current along kink worldlines and a formula is derived for the integrated scalar curvature of the embedding.

I. INTRODUCTION

Strings moving in anti-de Sitter spacetime are interesting for many reasons. Strings lie at the heart of the AdS/CFT correspondence. Understanding string theory enables us to study the correspondence beyond the gravity approximation. An open string ending on the boundary is dual to a Wilson loop in the boundary theory. Strings moving on a fixed asymptotically AdS background are among the simplest holographic non-equilibrium systems. Finally, the string worldsheet with the induced metric can be thought of as a two-dimensional toy model for gravity.

In this paper, we consider classical strings in AdS$_3$ described by the Nambu-Goto action. The theory has the remarkable property of being integrable. It can be reduced to the two-dimensional generalized sinh-Gordon theory. The string embedding can be reconstructed by solving an auxiliary linear problem.

The analog of a straight string in flat space is an embedding AdS$_2 \subset$ AdS$_3$ with a constant surface normal vector. More complex string solutions can be constructed by gluing AdS$_2$ patches with different normal vectors. At the joints between adjacent segments, the string embedding contains kinks that move with the speed of light. On the worldsheet, their worldlines form a quad lattice as seen in FIG. 1. Each square in the figure is an AdS$_2$ patch with a constant normal vector. The kink collision events will be called kink vertices. Note that in the sinh-Gordon picture segmented strings are special solutions, because the generalized sinh-Gordon equation degenerates into the Liouville equation.

In [12, 13], the basic motion of segmented strings has been analyzed. The technique is ideally suited for numerical simulations, because the discretization is exact. This means that there are no numerical errors that otherwise may accumulate over time.

The present paper computes various properties of the string, including its area, energy, and scalar curvature. The area can be expressed in terms of left (or right) variables $a_{ij}$ (or $\tilde{a}_{ij}$) where $i$ and $j$ label the edges of the kink lattice. The field $a_{ij}$ is in some sense holographic. As we will see, its value is related to the (retarded or advanced) Poincaré time when the kink corresponding to the $(ij)$ edge would reach the AdS boundary if there were no other kinks on the worldsheet.

In the next section, we discuss the basics of segmented strings in AdS$_3$. Section III computes the area of a sin-
gle patch that is bounded by four kink lines. Section IV computes the total area and the new equations of motion. Finally, reconstruction of the string from the Toda solution is discussed. In the appendix, the string energy on the Poincaré patch and the scalar curvature of the worldsheet are computed.

II. SEGMENTED STRINGS

Let us recall that the (universal cover of the) surface

\[ \bar{Y}^i \cdot \bar{Y} = -Y_{21}^2 - Y_0^2 + Y_1^2 + Y_2^2 = -L^2 \]  

(1)
gives an embedding of AdS\(_3\) into \(\mathbb{R}^{2,2}\). \(L\) denotes the AdS\(_3\) radius that we henceforth set to one. A function \(\bar{Y}(z, \bar{z})\) maps the string worldsheet into this target space. The equations of motion supplemented by the Virasoro constraints are\(^1\)

\[ \partial \bar{\partial} \bar{Y} - (\partial \bar{Y} \cdot \partial \bar{Y}) \bar{Y} = 0 \]  

(2)

\[ \partial \bar{Y} \cdot \partial \bar{Y} = \partial \bar{Y} \cdot \partial \bar{Y} = 0 \]

where the scalar product is again that of \(\mathbb{R}^{2,2}\). A normal vector to the string can be defined by

\[ N_a = \epsilon_{abcd} Y^b \partial Y^c \partial Y^d \]

It satisfies \(\bar{N} \cdot \bar{Y} = \bar{N} \cdot \partial \bar{Y} = \bar{N} \cdot \partial \bar{Y} = 0\) and \(\bar{N} \cdot \bar{N} = 1\). The simplest solution of (2) has a constant normal vector. It is the AdS\(_3\) analog of an infinitely long straight string.

Segmented strings are obtained by gluing worldsheet patches that have constant normal vectors. \(^1\) At the edges of the patches the string “breaks”: on a fixed timeslice the embedding contains a kink that moves with the speed of light.

Normal vectors change whenever two kinks collide. The collision on the worldsheet is shown in FIG. 2. Worldsheet time increases towards the top. The kink worldlines are indicated by two intersecting lines. Before the collision, the string consists of three segments \(A, B, C\) that are characterized by three normal vectors: \(\bar{A}, \bar{B}, \bar{C}\). We require \(\bar{A} \cdot \bar{B} = \bar{B} \cdot \bar{C} = 1\). This ensures that the kinks move on null geodesics.

After the collision, the new normal vector for the middle string piece is given by the collision formula \(^1\)

\[ \bar{B}' = -\bar{B} + 4 \frac{\bar{A} + \bar{C}}{(A + C)^2} \]  

(3)

One can check that \(\bar{A} \cdot \bar{B} = \bar{B} \cdot \bar{C} = 1\). This means that after the collision the kinks still move with the speed of light in AdS\(_3\), only the directions change.

Note that \(\bar{A} + \bar{C} \propto \bar{B} + \bar{B}'\). The collision formula computes any one of the four vectors from the other three by an appropriate relabeling. Further collisions between other pairs of kinks can be computed by repeated applications of the formula.

A. Dual description

There is an internal \(SO(2,2)\) symmetry that acts on the variables (see [13])

\[ q_1 = \bar{Y}, \quad q_2 = e^{-\alpha} \partial \bar{Y}, \quad q_3 = e^{-\alpha} \bar{Y}, \quad q_4 = \bar{N} \]

Here \(\bar{Y} \in \mathbb{R}^{2,2}\) is a point in target space, \(\bar{N}\) is the normal vector, and the sinh-Gordon field is defined by

\[ e^{2\alpha} = -2 \partial \bar{Y} \cdot \partial \bar{Y} \]

The symmetry treats spacetime points and normal vectors on the same footing. Therefore, we expect to have a dual description of segmented strings in terms of points in target space instead of normal vectors. Without proof we present here the evolution equation directly in terms of the kink vertices

\[ \bar{Y}_3 = -\bar{Y}_1 - 4 \frac{\bar{Y}_2 + \bar{Y}_4}{(\bar{Y}_2 + \bar{Y}_4)^2} \]

Here \(\bar{Y}_i\) are the four vertices of a single patch as in the right of FIG. 2. This equation is dual to (3). Note the sign change in the equation. Products of vertices that are connected by a kink line are constrained, similarly to adjacent normal vectors. For instance,

\[ \bar{Y}_1 \cdot \bar{Y}_2 = -1 \]

This ensures that the kink vertices \(\bar{Y}_1\) and \(\bar{Y}_2\) are connected by a null vector in \(\mathbb{R}^{2,2}\).

---

1 In the \(SL(2)\) WZW model, spacetime points are \(g \in SL(2, \mathbb{R})\) group elements. Classical solutions are given by \(g = g_1(z)g_2(\bar{z})\). In this paper, we consider an ordinary sigma model and set the NS-NS fields (other than the metric) to zero. Thus, classical solutions will not have such a simple product structure.
B. Equivalent descriptions

A segmented string solution can be given by assigning normal vectors to faces in a square lattice. (In the dual picture, position vectors are assigned to the vertices.) The map is not one-to-one. In fact, a physical string embedding can be described by different vector lattices. There are two basic operations that preserve the physical string, but modify the lattice of vectors:

- **Adding zero strength kinks.** One can always add an extra kink line to the lattice, see FIG. 4. This replaces a kink by a “composite” kink, see FIG. 5. The new patches in between have zero area and thus the string embedding is still the same.

- **Splitting kinks.** This operation replaces a kink by a “composite” kink, see FIG. 4. The new patches in between have zero area and thus the string embedding is still the same.

The two operations are dual under the $SL(2)$ transformation of the previous section. At the graphical level this can be seen by placing the dual vertices in the middle of the faces and rotating the edges by $90^\circ$.

Smooth string solutions can be obtained by considering a continuum limit with weaker and weaker kinks. Even though segmented strings have no diffeomorphism or Weyl symmetries, the redundancies discussed above will form the basis of the worldsheet conformal symmetry.

III. AREA OF A SINGLE PATCH

What is the string area in terms of the discrete data that defines segmented strings? Let us first focus on a single patch with a constant normal vector, see FIG. 5. The boundary of the worldsheet patch consists of four kink lines. In the target space, these are mapped to straight null lines (a consequence of the Virasoro constraints). Let us denote the four vertices of the patch by $\vec{V}_i \in \mathbb{R}^{2,2}$. We have $(\vec{V}_i)^2 = -1$. Let us define the lightlike direction vectors as in the figure:

$$
\vec{p}_1 = \vec{V}_2 - \vec{V}_1 \quad \vec{p}_2 = \vec{V}_3 - \vec{V}_2 \\
\vec{p}_3 = \vec{V}_3 - \vec{V}_4 \quad \vec{p}_4 = \vec{V}_4 - \vec{V}_1
$$

We have

$$(\vec{p}_i)^2 = 0 \quad \text{and} \quad \vec{p}_1 + \vec{p}_2 = \vec{p}_3 + \vec{p}_4$$

The latter equation can be interpreted as “momentum conservation” during the scattering of two massless scalar particles with initial and final momenta $\vec{p}_{1,2}$ and $\vec{p}_{3,4}$, respectively. The area of the patch is analogous to a scattering amplitude that is invariant under the $SO(2,2)$ isometry group of $AdS_3$. The only independent invariants are the Mandelstam variables $s = (\vec{p}_1 + \vec{p}_2)^2$ and $u = (\vec{p}_1 - \vec{p}_4)^2$. The patch area is then

$$A_{\text{patch}} = L^2 F \left( \frac{u}{s} \right)$$

where $L$ is the $AdS_3$ radius (henceforth set to one) and $F(x)$ is a dimensionless function to be determined.

Let us consider an $AdS_2 \subset AdS_3$ patch with normal vector $N = (0, 0, 0, 1)$. Points on the surface are of the form $X = (x_{-1}, x_0, x_1, 0)$ with $x_{-1}^2 + x_0^2 - x_1^2 = 1$. Let us fix a parameter $a \in (0, 1)$ and consider four points:

$$
\begin{align*}
\vec{V}_1 &= (a, -\sqrt{1-a^2}, 0, 0) \\
\vec{V}_2 &= (a^{-1}, 0, -\sqrt{1-a^{-2}}, 0) \\
\vec{V}_3 &= (a, \sqrt{1-a^2}, 0, 0) \\
\vec{V}_4 &= (a^{-1}, 0, \sqrt{1-a^{-2}}, 0)
\end{align*}
$$
and we get the covariant formula
\[ g_{\text{induced metric}} = \sigma, \tau \]
the patch if \( \sigma, \tau \in (0, 1) \). After a lengthy calculation, the induced metric \( g \) gives
\[ \sqrt{-g} = \frac{2(1-a^2)(1-\tau)}{[1-4(1-a^2)(1-\sigma)(1-\tau)]^{\frac{3}{2}}} \]
Integrating with respect to \( \tau \) and \( \sigma \) gives the area of half of the patch. From this we get
\[ A_{\text{patch}} = -4 \log a \]
For our patch, the ratio of the Mandelstam variables is given by \( s/u = -a^2 \). Combining these results fixes \( \mathcal{F}(x) \) and we get the covariant formula
\[ A_{\text{patch}} = \log \left[ \frac{(p_1 - p_4)^2}{(p_1 + p_2)^2} \right]^2 \]
Positiveness of the area requires
\[ |p_1 - p_4| > |p_1 + p_2| \]
This constrains the possible values of \( p_1 \).

A. Helicity spinors

In the spinor helicity formalism, one exhibits lightlike momentum vectors as products of spinors. We define
\[ \sigma^\mu = (1, -i\sigma_2, \sigma_1, \sigma_3) \]
\[ p_{a\dot{a}} = \sigma^\mu p_{\mu} \]
Since \( p^2 = \det(p_{a\dot{a}}) = 0 \), we can write
\[ p_{a\dot{a}} = \lambda_a \dot{\lambda}_{\dot{a}} \]
Since \( \lambda_a \rightarrow s\lambda_a, \dot{\lambda}_{\dot{a}} \rightarrow \frac{1}{s} \dot{\lambda}_{\dot{a}} \) does not change \( p_{a\dot{a}} \), there is a new gauge redundancy in this description. The spinors can be chosen to be real in \( \mathbb{R}^{2,2} \).

Let us now define the \( SO(2,2) \) invariants,
\[ \langle \lambda_i, \lambda_j \rangle = \epsilon_{ab} \lambda^b_i \lambda^b_j \]
\[ [\lambda_i, \lambda_j] = \epsilon_{a\dot{a}} \lambda^{a}_i \dot{\lambda}^{\dot{a}}_j \]
and consider two vectors \( \vec{p} \) and \( \vec{q} \) with the decomposition
\[ p_{a\dot{a}} = \lambda_a \dot{\lambda}_{\dot{a}} \]
\[ q_{a\dot{a}} = \kappa_a \dot{\kappa}_{\dot{a}} \]

Their product is expressed as
\[ \vec{p} \cdot \vec{q} = \langle \lambda, \kappa \rangle [\tilde{\lambda}, \tilde{\kappa}] \]
This allows us to write (5) in the form
\[ A_{\text{patch}} = \log \left( \frac{\langle \lambda_1, \lambda_4 \rangle [\tilde{\lambda}_1, \tilde{\kappa}_4]}{\langle \lambda_1, \lambda_2 \rangle [\tilde{\lambda}_1, \tilde{\kappa}_2]} \right)^2 \]
"Momentum conservation" can be written as
\[ \sum_{i=1}^4 \lambda_i^a \dot{\lambda}^b_i = 0 \]
where \( i \) runs over the four edges around the patch. This can be used to cast the area formula in the form
\[ A_{\text{patch}} = 2 \log \left| \frac{\langle \lambda_1, \lambda_4 \rangle [\tilde{\lambda}_1, \tilde{\kappa}_4]}{\langle \lambda_1, \lambda_2 \rangle [\tilde{\lambda}_1, \tilde{\kappa}_2]} \right| \]
Note that this expression contains only "left-handed" variables. There is a similar formula with only "right-handed" \( \tilde{\lambda} \) spinors. Let us stress that the spinors cannot take on arbitrary values because the area must be non-negative.

B. Global variables

Clearly, formula (6) does not depend on the modulus of \( \lambda_i \), i.e., it is gauge-invariant. By defining \( |\lambda_i|e^{i\alpha_i} := \lambda^1_i + i\lambda^2_i \) we get
\[ A_{\text{patch}} = 2 \log \left| \frac{\sin(\alpha_1 - \alpha_4) \sin(\alpha_2 - \alpha_3)}{\sin(\alpha_1 - \alpha_2) \sin(\alpha_3 - \alpha_4)} \right| \]
For a given AdS2 with a fixed normal vector, the four angles fully specify the four (infinite) straight kink lines in \( \mathbb{R}^{2,2} \). Changing an angle means that we move the corresponding line on AdS2. Note that the area diverges whenever two adjacent angles are equal. This corresponds to a configuration where a kink collision takes place on the UV boundary of AdS.
When two kinks cross each other, the angles generically change, see FIG. 7. In a special case, e.g. \( \alpha \) and \( \beta \) are “zero kinks”; they can be removed from the kink lattice without changing the string embedding. Such redundancies have been discussed in section II.

In order to have a better understanding of the \( \alpha \) angles, let us compute them in Poincaré coordinates for a patch on a particular \( \text{AdS}_2 \subset \text{AdS}_3 \) whose normal vector is \( \vec{N} = (0, 0, 0, 1) \). Using the Poincaré parametrization of \( \text{AdS}_3 \)

\[
\vec{Y} = \left( \frac{t^2 - z^2 - x^2 - 1}{2z}, \frac{t}{z}, \frac{t^2 - z^2 - x^2 + 1}{2z}, \frac{x}{z} \right)
\]

and setting \( x = 0 \), we obtain a parametrization of our \( \text{AdS}_2 \). The induced metric is

\[
ds^2 = \frac{-dt^2 + dz^2}{z^2}
\]

The patch in \( \text{AdS}_2 \) is bounded by \( \pm 45^\circ \) lines on the \( t - z \) plane. This is shown in FIG. 8. The lines intersect the \( \text{AdS}_2 \) boundary at \( t = a_1, a_2, b_1, b_2 \) as in the figure. The four vertices are \( V_1 = v_{11}, V_2 = v_{12}, V_3 = v_{22}, V_4 = v_{21} \), where

\[
v_{ij} = \left( \frac{1 + a_i b_j}{a_i - b_j}, \frac{a_i + b_j}{a_i - b_j}, \frac{1 + a_i b_j}{a_i - b_j}, \frac{1}{a_i - b_j} \right).
\]

From this, the lightlike boundary vectors \( p_i \) can be computed. The corresponding left angles \( \alpha_i \) are (after a \( \frac{\pi}{4} \) shift in order to have simpler expressions)

\[
\tan \alpha_1 = \frac{a_1}{a_2}, \quad \tan \alpha_2 = \frac{b_2}{b_1}, \quad \tan \alpha_3 = \frac{b_1}{a_2}, \quad \tan \alpha_4 = \frac{a_2}{b_1}
\]

The patch area is given by the formula (8) that yields

\[
A_{\text{patch}} = 2 \log \left| \frac{(a_1 - b_1)(a_2 - b_2)}{(a_2 - b_1)(a_1 - b_2)} \right|
\]

This expression is equal to (7).

C. Poincaré and Schwarzschild variables

Although the patch normal vectors are generically different from \( (0, 0, 0, 1) \), motivated by (8) we define the Poincaré variables

\[
a_i := \tan \alpha_i
\]

and this equation defines the Schwarzschild variables \( \tau \) in the general case.

Let us summarize the results in this section. The global \( \alpha \) variables are the angles of the left helicity spinors \( \lambda \). The Poincaré and Schwarzschild fields are simply computed via (9) and (11), respectively. Different variables are related to different coordinate systems on \( \text{AdS}_3 \). This is shown in FIG. 8. Similarly, one defines right-handed fields starting from the spinors \( \lambda \). These variables will be denoted \( \tilde{\alpha}, \tilde{\alpha}, \) and \( \tilde{\tau} \). We note that the map between the left and right fields is non-local. Finally, the area of the string can be expressed in either left or right variables, see eqn. (7). In the next section, we will compute the string action and show how the string embedding can be reconstructed from the angle variables.
IV. TOTAL AREA

The total area of the string is the sum of individual patch areas. From [13], we have

\[ A = \sum_{f \in \text{patches}} \log \left( \frac{\sin(\alpha_{f_2} - \alpha_{f_3}) \sin(\alpha_{f_2} - \alpha_{f_4})}{\sin(\alpha_{f_1} - \alpha_{f_2}) \sin(\alpha_{f_1} - \alpha_{f_4})} \right)^2 \]

where \( f_{1,2,3,4} \) label the four edges around the patch \( f \) and \( \alpha_i \) are the left angles. The action can be cast in the form,

\[ A = 2 \sum_{i,j} \log \left| \frac{a_{i,j} - a_{i+1,j}}{a_{i,j} - a_{i,j+1}} \right| \tag{12} \]

where \( i \) and \( j \) are coordinates on a square lattice, see FIG. 1. There is a similar formula that involves only the right-handed angles. The total area can be expressed in terms of Poincaré variables as well (\( a_{ij} := \tan \alpha_{ij} \))

\[ A = 2 \sum_{i,j} \log \left| \frac{a_{i,j} - a_{i+1,j}}{a_{i,j} - a_{i,j+1}} \right| \tag{13} \]

Finally, in Schwarzschild variables (\( \tanh \frac{\tau}{2} := \tan \alpha_{ij} \)) we have

\[ A = 2 \sum_{i,j} \log \left| \frac{\tanh \frac{\tau_{i,j}}{2} - \tanh \frac{\tau_{i,j+1}}{2}}{\tanh \frac{\tau_{i,j}}{2} - \tanh \frac{\tau_{i,j+1}}{2}} \right| \tag{14} \]

Note that patches are assumed to be located entirely in the bulk. Whenever a patch intersects the boundary of AdS_3, the area must be regularized.

A. Equation of motion

The Nambu-Goto action is equal to the area of the string which can be extremized by varying the left fields. The expression for the total area defines an action for a new Toda-type theory. Classical segmented string solutions yield solutions to this theory.

Let us first consider the action in Poincaré variables. The equation of motion is \( \frac{\partial A}{\partial a_{ij}} = 0 \). From [13] we have

\[ \frac{1}{a_{i,j} - a_{i,j+1}} + \frac{1}{a_{i,j} - a_{i,j-1}} - \frac{1}{a_{i,j} - a_{i+1,j}} - \frac{1}{a_{i,j} - a_{i-1,j}} = 0 \tag{15} \]

The same equation is satisfied by the \( \bar{a} \) variables of the right-handed theory. The field computed from a string solution will satisfy this equation. We recognize this equation of motion as that of a time-discretized relativistic Toda lattice with a cubic Poisson bracket, see (10.10.6) on page 442 in [13]. Note that (15) can also be thought of as a local version of the equation of motion of the discrete-time Caloger-Moser model [21].

From (14) we have another local equation

\[ \frac{1}{\tanh(\tau_{i,j} - \tau_{i,j+1})} + \frac{1}{\tanh(\tau_{i,j} - \tau_{i,j-1})} = \frac{1}{\tanh(\tau_{i,j} - \tau_{i+1,j})} + \frac{1}{\tanh(\tau_{i,j} - \tau_{i-1,j})} \]

which is the same as (10.8.7) on page 440 in [13]. Similar equation follows from (12) with \( \tan(x) \) functions in the denominators.

Initial conditions can be specified by setting two rows in the lattice (e.g. \( a_{i0} \) and \( a_{i1} \)).

A trivial solution is given by considering a lattice of zero kinks. For such a lattice, any two angles are the same if they lie on the same kink line,

\[ a_{i,j} = \begin{cases} u(i + j) & \text{if } i + j \text{ is odd} \\ v(i - j) & \text{if } i + j \text{ is even} \end{cases} \]

This solution describes a single AdS2 with a constant normal vector. The physical string embedding does not depend on \( u \) and \( v \).

B. Reconstructing the embedding

The string embedding can be reconstructed from a solution of (15). There are some caveats, however.

- The symmetry group of AdS3 is \( SO(2,2) = SL(2)_L \times SL(2)_R \). The two \( SL(2)\)s act on the \( \lambda \) left and \( \bar{\lambda} \) right spinors, respectively. Since left fields do not change under \( SL(2)_R \), \( a_{ij} \) only determines the string embedding up to such global transformations. Similarly, \( a_{ij} \) determines only an orbit of \( SL(2)_L \).

- Note that there is a \( \mathbb{Z}_2 \) ambiguity in assigning a kink lattice to the lattice of black-and-white dots in FIG. 1 if their color is unknown. Thus, the symmetry between kink collision points in spacetime and AdS2 patch normal vectors is manifest in the Toda description. As a consequence, two different embeddings may be constructed from \( a_{ij} \).

- Not all Toda solutions correspond to string embeddings. The area of the string patches must be non-negative. This is only true for solutions satisfying

\[ (a_{i+1,j} - a_{i,j-1})(a_{i,j} - a_{i+1,j-1}) > 0 \]

for the four angles around any patch (i.e. \( ij \) is a white dot in FIG. 1).

In the following we sketch the procedure for rebuilding the string solution. Let us fix a spacetime point \( \hat{X} \in \mathbb{R}^{2,2} \) that will correspond to a particular kink collision event in FIG. 1. The four angles around the vertex are

\[ a_{ij}, \ a_{i+1,j}, \ a_{i,j+1}, \ a_{i+1,j+1} \tag{16} \]

For any one of these angles, the corresponding kink vector is computed to be

\[ \hat{p} \propto \begin{pmatrix} -X_0 + X_2 \sin 2\alpha + X_1 \cos 2\alpha \\ -X_1 - X_0 \sin 2\alpha + X_2 \cos 2\alpha \\ -X_2 + X_0 \sin 2\alpha + X_1 \cos 2\alpha \\ -X_1 - X_0 \sin 2\alpha + X_2 \cos 2\alpha \end{pmatrix} \tag{17} \]
for which \( p^2 = 0 \). Two adjacent kink vectors, e.g. \( \vec{p}_{i,j} \) and \( \vec{p}_{i+1,j} \), define an AdS\(_2\) with a constant normal vector that contains the points \( \vec{X} + \lambda \vec{p}_{i,j} \) and \( \vec{X} + \lambda' \vec{p}_{i+1,j} \) for any \( \lambda \) and \( \lambda' \). Let us pick two adjacent angles \( \alpha \) and \( \beta \) from \([13]\). The corresponding kink vectors span an AdS\(_2\) patch with normal vector

\[
\vec{N}(\alpha, \beta) = \frac{1}{\sin(\alpha - \beta)} \times \left( \begin{array}{c} X_0 \cos(\alpha - \beta) - X_1 \cos(\alpha + \beta) - X_2 \sin(\alpha + \beta) \\ -X_0 \cos(\alpha - \beta) - X_1 \cos(\alpha + \beta) + X_2 \sin(\alpha + \beta) \\ X_0 \cos(\alpha - \beta) + X_1 \cos(\alpha + \beta) - X_2 \sin(\alpha + \beta) \end{array} \right)
\]

We have seen that once the location of a kink vertex is fixed in spacetime, the four angles around it fully specify the four kink vectors. Similarly, if a normal vector is known, four angles fully specify the four boundaries of the corresponding patch. The boundary edges then intersect each other at new kink vertices and the kink vectors around those can also be computed. This procedure can be repeated until the entire worldsheet embedding is recovered.

V. DISCUSSION

Let us consider space-like string embeddings in anti-de Sitter spacetime. A smooth open string that ends on a curve \( \mathcal{C} \) in the boundary can be approximated by another string that ends on a zigzag line in the boundary whose segments are lightlike and which itself is sufficiently close to \( \mathcal{C} \) \([13]\). In the case of Lorentzian embeddings, a Lorentzian zigzag worldline constitutes a singular limit, because the boundary quark sitting on the endpoint of the string would radiate off an infinite amount of energy at the turning points. If the quark velocity cannot jump, how can smooth strings be approximated by strings that are described by discrete data? A solution is provided by segmented strings \([13], [16]\). In this case only the acceleration of the quark jumps whenever kinks enter or leave the string. Kinks between the segments move with the speed of light and (between collisions) their velocities are constant vectors in the embedding \( \mathbb{R}^{2,2} \) spacetime. When kinks collide, the new normal vector to the string is given by the collision formula \([3]\).

In this paper, we have computed the area of segmented strings in terms of cross-ratios of helicity spinors. These spinors arise from the decomposition of the kink vectors. The string area equals the Nambu-Goto action which we have expressed purely in terms of left (or right) angle variables. We have argued that the time evolution of the segmented string can be described by the evolution equation of a discrete-time Toda-type lattice.

\[\sum_{i=1}^{4} \lambda_i^a \lambda_i^b = 0\] (18)

The meaning of this equation is that the boundary of a patch is a closed loop in spacetime. (There is a similar constraint for every kink collision vertex as well.) The left and right variables are therefore not independent: they are “classically entangled”. It would be interesting to relate the Toda-type lattice to a matrix model perhaps via a (relativistic) Calogero-Moser theory.

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Appendix

A. String energy

In this section, we compute the energy of the string on the Poincaré patch of AdS\(_3\). The metric is

\[ds^2 = -dt^2 + dx^2 + dz^2 \]

Let us consider the action

\[S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}\]

where \( X^\mu \) are embedding coordinates, \( h_{ab} \) is the worldsheet metric, and \( a, b \in \{\tau, \sigma\} \). One defines the worldsheet currents of target space energy-momentum

\[P_\mu^a = -\frac{1}{2\pi\alpha'} \sqrt{-h} h^{ab} G_{\mu\nu} \partial_b X^\nu\]

From the equation of motion it follows that

\[\partial_\mu P_\mu^a - \Gamma^a_{\mu\lambda} \partial_\alpha X^\lambda P_\mu^a = 0\]

Defining \( P_\mu^a = \frac{P_\mu^a}{\sqrt{g}} \) and substituting the induced metric \( g_{ab} = \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \) for \( h_{ab} \), this can be written as

\[\nabla_a P_\mu^a - \Gamma^a_{\mu\lambda} \partial_\alpha X^\lambda P_\mu^a = 0\]
where \( \nabla \) is the covariant derivative with respect to \( g_{ab} \). Note that the target space index \( \mu \) is only a spectator when the derivative is taken. The second term is
\[
\Gamma^\alpha_{\mu \lambda} \partial_\mu X^\lambda p^a_\alpha = \frac{1}{2} G_{\nu \lambda} \partial_\mu X^\lambda \partial_\mu X^\nu g^{ab}
\]

If \( G_{\nu \lambda} \) is independent of \( X^\mu \) for some \( \mu \), then \( \zeta^\alpha = \delta^\alpha_\mu \) is a Killing vector. Then \( \zeta^\alpha \nabla_\alpha p^a_\alpha = 0 \) and one can define the conserved quantity
\[
E_\zeta = - \int d\sigma \zeta^\alpha P^\tau_\alpha
\]
that satisfies \( \partial_\tau E_\zeta = 0 \). We are going to use \( \zeta = \partial_t \) in the following.

Energy expressions are typically complicated (see [16]). In order to simplify the results, we perform the integration on the worldsheet along a path that consists of lightlike patch boundaries (instead of constant \( \tau \) slices).

In terms of the Poincaré coordinates, a single AdS2 patch is a contracting and expanding semi-circle. Let \( x_1 \) denote the path of the quark in the boundary
\[
x_1(t) = X_1 + \sqrt{R_1^2 + (t - T_1)^2}
\]
This is a hyperbola, parametrized by \( X_1, T_1, \) and \( R_1 \). The subscript indicates the patch, see FIG. 9. Using Mikhailov’s result, the string embedding is given by
\[
t(\tau, z) = \tau + \frac{z}{\sqrt{1 - x_1'(\tau)^2}}
\]
\[
z(\tau, z) = z
\]
\[
x(\tau, z) = x_1(\tau) + \frac{zz_1'(\tau)}{\sqrt{1 - x_1'(\tau)^2}}
\]
(19)

Here \( \tau \) plays the role of retarded time. The relationship between the normal vector \( \hat{N} \) and the parameters of the hyperbola are
\[
(T_1, X_1, R_1) = \left( \frac{-N_0}{N_1 + N_2}, \frac{-N_1}{N_1 + N_2}, \frac{1}{|N_1 + N_2|} \right).
\]
The induced metric on the worldsheet is
\[
g = \frac{1}{z^2} \left( \frac{z^2 - R_1^2}{\sqrt{R_1^2 + (T_1 - t)^2}} \right)
\]
\[
\begin{pmatrix}
-R_1 \\
0
\end{pmatrix}
\]
\[
\begin{pmatrix}
R_1^2 + (T_1 - t)^2 \\
R_1^2 + (T_1 - t)^2
\end{pmatrix}
\]

Finally, the contribution of the kink to the total energy is given by the integral
\[
E_{1,2} = \int_{z_a}^{z_b} dz \sqrt{-g} p_\mu^\tau = \frac{1}{2 \pi \alpha'} \left( \frac{1}{z_b - z_a} \right) \sqrt{1 + \left( \frac{T_1 - T_2}{R_1 - R_2} \right)^2}
\]
where \( z_a \) and \( z_b \) are the \( z \) coordinates of the points where the kink is created and annihilated, respectively. Note that the formula is symmetric under \( 1 \leftrightarrow 2 \) as it should be. Furthermore, \( X_1 \) and \( X_2 \) have dropped out. Note that for the correct null cusp limit one takes \( T_1 \rightarrow T_2 \) before \( R_1 \rightarrow R_2 \). The total energy of the string is given by the sum of all \( E_{i,j} \) along a zigzag path on the patch boundaries, see FIG. 9.
B. Scalar curvature

Let us consider a static string that hangs from the boundary of AdS$_3$. The induced metric on the worldsheet is that of AdS$_2$ and the Ricci scalar is $R = -2$. This is the only intrinsic curvature invariant that one can compute in two dimensions. If the string endpoint is perturbed, kinks (or waves in the continuum case) will travel down the string. However, the scalar curvature does not change. The reason for this is simple and best understood through a flat space analogy. Consider, for instance, a cube and cut out the top and bottom faces. What’s left is the four adjacent side faces that can be unwrapped and arranged on a plane. The four edges are the analogs of the kinks that move in the same direction.

The induced metric on the four faces is clearly flat. Only when kinks collide can the curvature differ from the constant value. In this section, we compute the integrated Ricci scalar at collision points.

Since kink collisions happen at single points, the background curvature can be neglected. Thus, one can analyze the problem in 2+1 dimensional Minkowski space with coordinates $x^{0,1,2}$. In order to handle the divergence in the curvature at the collision point, we smoothen the step functions corresponding to kinks.

A string with two smooth kinks colliding on it is given by the embedding

\[
\begin{align*}
x^0 &= \frac{(2 + A^2)(\sigma^+ + \sigma^-)}{2\sqrt{2}} - \frac{A^2(\tanh \epsilon \sigma^- + \tanh \epsilon \sigma^+)}{2\sqrt{2}} \\
x^1 &= \frac{(2 - A^2)(\sigma^+ - \sigma^-)}{2\sqrt{2}} - \frac{A^2(\tanh \epsilon \sigma^- - \tanh \epsilon \sigma^+)}{2\sqrt{2}} \\
x^2 &= -\frac{A(\log \cosh \epsilon \sigma^- + \log \cosh \epsilon \sigma^+)}{\epsilon}
\end{align*}
\]

Here $\sigma^+$ and $\sigma^-$ are lightcone coordinates on the worldsheet. $A$ parametrizes the kink strengths and $\epsilon$ is related to the smoothness of the step functions.

The induced metric on the worldsheet has components

\[
\begin{align*}
g_{\pm} &= -\left(1 - \frac{A^2}{2}\tanh \epsilon \sigma^- \tanh \epsilon \sigma^+\right)^2 \\
g_{++} = g_{--} &= 0
\end{align*}
\]

The Ricci scalar of the induced metric is

\[
R = -\frac{2A^2\epsilon^2}{(1 - \frac{A^2}{2}\tanh \epsilon \sigma^- \tanh \epsilon \sigma^+)^2(\cosh \epsilon \sigma^-)^2(\cosh \epsilon \sigma^+)^2}
\]

Integrating this with respect to $\sigma^-$ and $\sigma^+$ over the entire worldsheet, we get

\[
\int d^2\sigma \sqrt{-g} R = -16 \tanh^{-1} \frac{A^2}{2}
\]

Note that $\epsilon$ has dropped out and thus the result is finite in the $\epsilon \to \infty$ limit that corresponds to sharp kinks. In this limit, $R = 0$ away from the collision vertex at $\sigma^+ = \sigma^- = 0$. Thus, it is enough to integrate in an infinitesimally small neighborhood of the origin. The result is then a local feature which generalizes to AdS$_3$.

We would like to eliminate $A$ from the expression and replace it with a more natural quantity. The angle between the two static string pieces can be computed

\[
\tan \frac{\phi}{2} = \frac{2\sqrt{2}A}{A^2 - 2}
\]

Going back to AdS$_3$, we can set up a similar collision using the three patches $N_1$, $N_2$, $N_3$

\[
\begin{align*}
N_1 &= \left(0, 0, \cos \frac{\phi}{2}, \sin \frac{\phi}{2}\right) \\
N_2 &= \left(-\tan \frac{\phi}{2}, 0, \left(\cos \frac{\phi}{2}\right)^{-1}, 0\right) \\
N_3 &= \left(0, 0, \cos \frac{\phi}{2} - \sin \frac{\phi}{2}\right)
\end{align*}
\]
The angle $\phi$ can be computed using the scalar product between normal vectors

$$\cos \phi = \vec{N}_1 \cdot \vec{N}_3$$

and from this, $A$ can be determined. The final result

$$\int_{\text{vertex}} \sqrt{-g} R = 8 \log \cos \frac{\phi}{2}$$

This is the integrated Ricci scalar around a kink collision point where the string piece corresponding to the middle patch $N_2$ vanishes. Note that the formula does not depend on $N_2$. Generic normal vectors form a three-dimensional space. However, the middle patch $N_2$ is constrained by $\vec{N}_1 \cdot \vec{N}_2 = \vec{N}_2 \cdot \vec{N}_3 = 1$ and thus the allowed values form a one-dimensional subspace. Motion on this subspace corresponds to global AdS$_3$ time translations. This is a symmetry of the system that preserves the curvature.

The four angle variables around the vertex in the kink lattice are:

$$\alpha_1 = -\alpha_3 = \frac{\phi}{4}, \quad \alpha_2 = \frac{\pi}{2} - \frac{\phi}{4}, \quad \alpha_4 = \frac{\pi}{2} + \frac{\phi}{4}$$

If $\phi = 0$, then the angles do not change at the vertex (i.e. $\alpha_1 = \alpha_3$ and $\alpha_2 = \alpha_4$) and it’s clear that they describe the collision of two zero kinks.

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