Supersymmetric quantum mechanics of the flux tube

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Abstract

The Operator Product Expansion approach to scattering amplitudes in maximally supersymmetric gauge theory operates in terms of pentagon transitions for excitations propagating on a color flux tube. These obey a set of axioms which allow one to determine them to all orders in 't Hooft coupling and confront against explicit calculations. One of the simplifying features of the formalism is the factorizability of multiparticle transitions in terms of single-particle ones. In this paper we extend an earlier consideration of a sector populated by one kind of excitations to the case of a system with fermionic as well as bosonic degrees of freedom to address the origin of the factorization. While the purely bosonic case was analyzed within an integrable noncompact open-spin chain model, the current case is solved in the framework of a supersymmetric \text{sl}(2|1) magnet. We find the eigenfunctions for the multiparticle system making use of the R-matrix approach. Constructing resulting pentagon transitions, we prove their factorized form. The discussion corresponds to leading order of perturbation theory.
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Figure 1: Tessellation of a polygon into null squares merged into pentagons (shown in different dashed contours). We picked an intermediate pentagon transition with flux-tube excitations inserted in the bottom and top portions of the contour.

1 Introduction

The Operator Product Expansion for scattering amplitudes [1 2] of planar maximally supersymmetric Yang-Mills theory in the dual language of the Wilson loop stretched on a null polygonal contour in superspace [3 4 5 6 7 8] paved a way for their weak and strong coupling analysis in a multi-collinear limit with a naturally built-in consistent scheme for inclusion of subleading corrections [10 11 12 13 14 15 16 17 18 20 21 22 23]. It is based on geometrization of the contour in terms of a sequence of null squares with adjacent ones sharing a side merged into pentagons, see Fig. 1. The bottom of the loop can be decomposed into an infinite series of excitations with the strength of contributions being exponentially suppressed with their number (or more precisely, their cumulative twist) in the collinear limit. These propagate upwards from the bottom through a series of pentagons and are absorbed at the top. Every pentagonal Wilson loop in the chain of transitions contains insertions of elementary fields of the theory with their total quantum numbers fixed by the choice of the component of the amplitude under study. These pentagons play a pivotal role in the entire construction. They obey a set of natural axioms [1] that are inherited from the integrable dynamics of the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. However, the question of their operatorial origin remains obscure.
1.1 Embedding of different multiplets

Some time ago [11], we studied the system of excitations of a single type interacting on the flux-tube. It was shown to be equivalent to solving the spectral problem for a noncompact open spin chain whose \( \text{sl}(2) \) invariance is broken by boundary Hamiltonians. Presently we will provide its generalization to the minimally supersymmetric sector of the \( \mathcal{N} = 4 \) super Yang-Mills theory in the planar limit. In the absence of a covariant superspace formulation of the theory, the light-cone formalism becomes advantageous. In this framework, all propagating fields in the maximally supersymmetric Yang-Mills theory can be accommodated into a single light-cone chiral superfield [24, 25, 26],

\[
\Phi_{\mathcal{N}=4}(x^\mu, \theta^A) = \partial_+^{-1}A(x^\mu) + \theta^A \partial_+^{-1}\bar{\psi}_A(x^\mu) + \frac{i}{2!}\theta^A \theta^B \phi_{AB}(x^\mu) \\
+ \frac{1}{3!} \varepsilon_{ABCD} \theta^A \theta^B \theta^C \psi^D(x^\mu) - \frac{1}{4!} \varepsilon_{ABCD} \theta^A \theta^B \theta^C \theta^D \partial_+ \bar{A}(x^\mu),
\]

where \( A \) and \( \bar{A} \) are the holomorphic and antiholomorphic components of the gauge field, respectively, \( \psi \) and \( \bar{\psi} \) are the dynamical “good” components of the fermion fields transforming in the \( 4 \) and \( \bar{4} \) of the internal \( SU(4) \) symmetry group and, finally, \( \phi_{AB} \) is a sextet of scalars.

There are two possible subsectors we can analyze. One them is of Wess-Zumino type. It is is composed of a scalar and a fermion

\[
\Phi_{s=1/2}(x^\mu, \theta) = \phi(x^\mu) + \theta \psi(x^\mu)
\]

and is obtained from (1.1) via the projection [27]

\[
\Phi_{\mathcal{N}=4}(x^\mu, \theta^A)|_{\theta^2=\theta^3=0} = \cdots + \theta^1 \theta^4 \Phi_{s=1/2}(x^\mu, \theta).
\]

The other one is the antiholomorphic part of the \( \mathcal{N} = 1 \) superYang-Mills multiplet,

\[
\Phi_{s=1}(x^\mu, \theta) = \psi(x^\mu) - \theta \bar{F}(x^\mu),
\]

built from a fermion and antiholomorphic field strength \( \bar{F} = \partial_+ \bar{A} \), found in the top two components of the \( \mathcal{N} = 4 \) superfield,

\[
\Phi_{\mathcal{N}=4}(x^\mu, \theta^A)|_{\theta^1=\theta^3} = \cdots + \theta^1 \theta^2 \theta^3 \Phi_{s=1}(x^\mu, \theta).
\]

In both cases, we displayed the conformal spin of the minisuperfield, which is determined by the one of its lowest field component, as a subscript.

1.2 Superlight-cone operators and Hamiltonians

As can be seen from the representation of the pentagon transition in Fig. [1] it is related to the correlation function of two \( \Pi \)-shaped Wilson loops [28, 29] with insertions of elementary fields into their bottom and top contours, schematically

\[
P(\text{bottom}|\text{top}) \sim \langle O_{\Pi_{\text{top}}}O_{\Pi_{\text{bottom}}} \rangle,
\]

where

\[
O_{\Pi}(Z) = W^\dagger(0)\Phi_s(Z_1)\Phi_s(Z_2)\cdots\Phi_s(Z_N)W(\infty),
\]
is built from superfields $\Phi$ inserted along the light-cone direction $z_n = x_n^c$ and depends on respective Grassmann variable $\theta_n$ that together can be encoded in a superspace coordinate $Z_n = (z_n, \theta_n)$. The gauge links between supercoordinates $Z = (Z_1, Z_2, \ldots, Z_N)$ can be ignored due to the choice of the light-cone gauge condition $A^+ = 0$. The two light-like Wilson lines $W$ in the direction of particles propagating along the vertical segments of the pentagon are attached at its ends.

At leading order of perturbation theory (and multicolor limit), the renormalization group evolution of these operators can be cast in the form of a Schrödinger equation with Hamiltonian given by the sum of pairwise Hamiltonians between adjacent superfields supplemented with the interaction of the first and last one with the boundary Wilson lines. The latter read

$$H_{01} W^+(0) \Phi_s(Z_1) = W^+(0) \int_0^1 \frac{d\beta}{1 - \beta} \left[ \beta^{2s-1} \Phi_s(\beta Z_1) - \Phi_s(Z_1) \right],$$

(1.8)

$$H_{N\infty} \Phi_s(Z_N) W(\infty) = \int_1^\infty \frac{d\beta}{\beta - 1} \left[ \Phi(\beta z_N, \theta_N) - \beta^{-1} \Phi_s(Z_N) \right] W(\infty).$$

(1.9)

We can use the light-cone superspace formulation of the $N = 4$ dilatation operator [26] to project out following Ref. [27] the scalar-fermion sector in question or directly get the $N = 1$ superYang-Mills [26] for the multiplet (1.4). The $N = 4$ pairwise Hamiltonian for superfields $\Phi_{N=4}$ of conformal spin $s = -\frac{1}{2}$ sitting away from the boundary Wilson lines is

$$H_{12} \Phi_{N=4}(Z_1) \Phi_{N=4}(Z_2) = \int_0^1 \frac{d\alpha}{\alpha} \left[ (1 - \alpha)^{-2} \Phi_{N=4}((1 - \alpha)Z_1 + \alpha Z_2) \Phi_{N=4}(Z_2) 
+ (1 - \alpha)^{-2} \Phi_{N=4}(Z_1) \Phi_{N=4}((1 - \alpha)Z_2 + \alpha Z_1) 
- 2 \Phi_{N=4}(Z_1) \Phi_{N=4}(Z_2) \right].$$

(1.10)

Projecting out the Wess-Zumino multiplet via (1.3) changes the power of the $\alpha$-dependent prefactor from $-2$ to 0. For the antiholomorphic Yang-Mills multiplet (1.4), the same power changes from $-2$ to 1. We can combine the two options by encoding them in the exponent $2s - 1$. Let us change the integration variables in the integrand of $H_{12}$, as well as modify the subtraction term, i.e.,

$$H'_{12} = H_{12} + \delta H_{12} \quad \text{with} \quad \delta H_{12} = \ln z_2/z_1,$$

(1.11)

such that in the limit $z_2 \gg z_1$, we get the sum of two boundary Hamiltonians (1.8). Here, the pair-wise Hamiltonian is split in two

$$H'_{12} \Phi_s(Z_1) \Phi_s(Z_2) = H_{12}^+ \Phi_s(Z_1) \Phi_s(Z_2) + H_{12}^- \Phi_s(Z_1) \Phi_s(Z_2),$$

(1.12)

that act in the following fashion on the nearest-neighbor fields

$$H_{12}^+ \Phi_s(Z_1) \Phi_s(Z_2) = \int_1^{z_2/z_1} \frac{d\beta}{\beta - 1} \left[ \left( \frac{z_2 - \beta z_1}{z_2 - z_1} \right)^{2s-1} \Phi_s \left( \beta z_1, \frac{z_2 - \beta z_1}{z_2 - z_1} \theta_1 + \frac{z_1(\beta - 1)}{z_2 - z_1} \theta_2 \right) - \beta^{-1} \Phi_s(Z_1) \right] \Phi_s(Z_2),$$

(1.13)
\[ \mathcal{H}_{12} \Phi_s(Z_1) \Phi_s(Z_2) = \Phi_s(Z_1) \int_{z_1/z_2}^1 \frac{d\beta}{1-\beta} \times \left[ \left( \frac{z_1 - \beta z_2}{z_1 - z_2} \right)^{2s-1} \Phi_s \left( \frac{\beta z_2 - z_1}{z_2 - z_1} \theta_2 + \frac{z_2(1-\beta)}{z_2 - z_1} \theta_1 \right) - \Phi_s(Z_2) \right]. \]  

Thus the Hamiltonian that we have to solve the eigensystem for is

\[ \mathcal{H}_N = \mathcal{H}_{01} + \mathcal{H}_{12} + \cdots + \mathcal{H}_{N-1,N} + \mathcal{H}_{N\infty}. \]  

Depending on the conformal spin of the superfields, it encodes both the Wess-Zumino and Yang-Mills multiplets. Since the two differ only by the value of the spin, the following discussion will be done for arbitrary \( s \). This also points out that, while the scalars and fermions carry the R-charge in the \( \mathcal{N} = 4 \) theory, this spin model will not be able to accommodate for nontrivial rational prefactors that arise in the pentagon approach, otherwise, these would arise in the fermion-gluon sectors as well. However, the latter is free from these ‘complications’ since the gluon is singlet with respect to \( \text{SU}(4) \). Thus, the leading order description within the supersymmetric lattice model will provide information on the dynamical portion of the pentagons only. The rational factors as well as helicity form factors stemming from crossing conditions will not be accounted for within the current formalism.

### 2 Supersymmetric open spin chain

The light-cone chiral superfield \( \Phi_s(Z) \) defines an infinite-dimensional chiral representation \( V_s \) of the superconformal \( \mathfrak{sl}(2|1) \) algebra labeled by the conformal spin \( s \). The generators of the algebra are realized as first order differential operators in bosonic \( z \) and fermionic \( \theta \) variables

\[ S^- = -\partial_z, \quad S^+ = z^2 \partial_z + 2zs + z\theta \partial_\theta, \quad S^0 = z\partial_z + s + \frac{1}{2} \theta \partial_\theta, \quad B = \frac{1}{2} \theta \partial_\theta - s, \quad V^- = \partial_\theta, \quad W^- = \theta \partial_z, \quad V^+ = z \partial_\theta, \quad W^+ = \theta(z \partial_z + 2s). \]  

Thus, the Hamiltonian (1.15) defines a non-periodic homogeneous open superspin chain. We will demonstrate below that it is in fact integrable.

#### 2.1 Scalar product and involution properties of generators

As will be clear from our discussion it will be indispensable to introduce an inner product on the space of functions depending on superspace variable \( Z \). While the bosonic variable lives on the real axis, it is instructive to address the spectral problem by promoting it to the upper half of the complex plane. This formulation is of paramount importance for the construction of eigenfunctions (holomorphic functions in the upper semiplane) in the representation of Separated Variables \cite{30, 31, 11} and computation of various inner product \cite{11}. The flux-tube matrix elements entering the Operator Product Expansion can be regarded as their boundary values.

The chiral scalar product on the space of superfunctions

\[ \Phi_s(Z) = \Phi_s(z) + \theta \Phi_{s+1/2}(z), \]  

holomorphic in the upper semiplane of the complex plane is defined as

\[ \langle \Phi'_s | \Phi_s \rangle = \int [DZ]_s (\Phi'_s(Z))^* \Phi_s(Z), \]
where the sl(2|1) invariant measure reads

$$\int [DZ]_s = \frac{e^{-i\pi(s-1)}}{\pi} \int d\theta^* d\theta \int_{\Im z > 0} dz^* dz (z - z^* + \theta\theta^*)^{2s-1}. \quad (2.4)$$

Notice that the phases chosen in this inner product are correlated with the integration and involution rules adopted for Grassmann variables. Throughout the paper they obey the following rules

$$\int d\theta \theta^* = 1, \quad (\theta^* \theta)^* = \theta \theta^*. \quad (2.5)$$

In the component form, we find

$$\langle \Phi'_s | \Phi_s \rangle = \int [Dz]_s (\Phi'_s(z))^* \Phi_s(z) + \frac{1}{2is} \int [Dz]_{s+1/2} (\Phi'_{s+1/2}(z))^* \Phi_{s+1/2}(z), \quad (2.6)$$

where, e.g., $\Phi_s = \phi$ and $\Phi_{s+1/2} = \psi$ for the field content of the $s = 1/2$ multiplet \( \text{(1.2)} \). Here we recognize in the first term the well-known expression for the bosonic sl(2)-invariant inner product with the measure

$$\int [Dz]_s \equiv (2s - 1)\frac{e^{-i\pi(s-1)}}{\pi} \int_{\Im z > 0} dz^* dz (z - z^*)^{2s-2}. \quad (2.7)$$

Notice an extra phase in front of the second term in Eq. \( (2.6) \) to make it real by virtue of Eq. \( (2.5) \) for fermionic fields. For $s = 1$, one has to change $\phi \to \psi$ and $\psi \to -\bar{F}$. Since the resulting superfield \( \text{(1.4)} \) is fermionic, one has to multiply the inner product $\langle \Phi'_s | \Phi_s \rangle$ by an $i$ such that this phase will migrate from the second term to the first. We will imply this convention from now on so that we could avoid repetitive formulas corresponding to each case. This nuisance will not affect any of our considerations which follow.

We conventionally define the adjoint operator with respect to the inner product \( (2.3) \) as

$$\langle \Phi' | G \Phi \rangle = \langle G^\dagger \Phi' | \Phi \rangle. \quad (2.8)$$

Then we can easily verify the following conjugation properties of the sl(2|1) generators \( (2.1) \) using integration by parts

$$\left(S^{\pm,0}\right)^\dagger = -S^{\pm,0}, \quad B^\dagger = B, \quad \left(V^\pm\right)^\dagger = -W^\pm. \quad (2.9)$$

Notice that the chirality generator is hermitian compared to antihermitian generators of the sl(2) subalgebra. From the involution rules for Grassmann variables, it follows that

$$\left(GG'\right)^\dagger = (-1)^{\text{grad}G} \text{grad}G' G'^\dagger G^\dagger. \quad (2.10)$$

The Hilbert space of the $N$-site model spanned on the light-cone operators \( \text{(1.7)} \) is formed by the tensor product of Hilbert spaces at the position of each superfield $\otimes_{k=1}^N \mathcal{V}_{s,k}$. Then, one can immediately proof the self-adjoint property of the Hamiltonian \( \text{(1.15)} \),

$$\mathcal{H}^\dagger_N = \mathcal{H}_N \quad (2.11)$$
with respect to the inner product for multivariable $Z = (Z_1, Z_2, \ldots, Z_N)$ function $\Phi_s = \Phi_s(Z)$,

$$\langle \Phi'_s | \Phi_s \rangle = \int \prod_{n=1}^{N} [DZ_n]_s \ (\Phi'_s(Z))^* \Phi_s(Z).$$

To see this more efficiently, it is convenient to recast the individual pair-wise Hamiltonians in the non-local form,

$$\mathcal{H}_{01} = \psi(1) - \psi(z_1 \partial_{z_1} + \theta_1 \partial_{\theta_1} + 2s),$$

(2.12)

$$\mathcal{H}_{N\infty} = \psi(1) - \psi(-z_N \partial_{z_N}),$$

(2.13)

for the boundary and

$$\mathcal{H}_{12} = 2\psi(1) - \psi(z_{12} \partial_{z_1} + \theta_{12} \partial_{\theta_1} + 2s) - \psi(z_{21} \partial_{z_2} + \theta_{21} \partial_{\theta_2} + 2s),$$

(2.14)

$$\delta \mathcal{H}_{12} = \ln z_2/z_1,$$

(2.15)

for bulk ones, respectively. In fact, we can rearrange different contributions entering the bulk into the boundary Hamiltonians to better match them to the ones emerging from R-matrices. Namely, splitting the logarithmic terms in $\delta \mathcal{H}_{12}$, we can identify the bulk Hamiltonian with the sl(2|1) invariant one $h_{12} = \mathcal{H}_{12}$,

$$h_{12}^- = \psi(1) - \psi(z_{12} \partial_{z_1} + \theta_{12} \partial_{\theta_1} + 2s), \quad h_{12}^+ = \psi(1) - \psi(z_{21} \partial_{z_2} + \theta_{21} \partial_{\theta_2} + 2s),$$

(2.16)

while the boundary ones now read

$$h_{01} = \mathcal{H}_{10} - \ln z_1 = -\ln (z_1^2 \partial_{z_1} + z_1 \theta_1 \partial_{\theta_1} + 2sz_1), \quad h_{N\infty} = \mathcal{H}_{N\infty} + \ln z_N = -\ln \partial_{z_N}. \quad (2.17)$$

### 2.2 Integrals of motion and hermiticity issues

Let us construct the integrals of motion of the $N$-site Hamiltonian following the standard procedure of the R-matrix approach \[32\]. The sl(2|1) Lax operator acting on the direct product of a graded three-dimensional space and the chiral space $V_{s,k}$ of the $k$-th site reads

$$\mathbb{L}_k(u) = \begin{pmatrix} u + S^0_k - B_k & -W^-_k & S^-_k \\ -V^+_k & u - 2B_k & V^-_k \\ S^+_k & -W^+_k & u - S^0_k - B_k \end{pmatrix}. \quad (2.18)$$

It depends on the complex spectral parameter $u$. Notice that for a generic case, the representation of the algebra is parametrized by the conformal spin $s$ and chirality $|b| \neq s$. Thus the Lax operator can be viewed as a function of three linear combinations of $u$, $s$ and $b$, namely,

$$u_1 = u + s - b, \quad u_2 = u - 2b, \quad u_3 = u - s - b, \quad (2.19)$$

such that $\mathbb{L}(u) = \mathbb{L}(u_1, u_2, u_3)$ is a function of $u_\alpha$. For the chiral case at hand, $b = -s$. However, we will use three distinct $u_\alpha$ parameters below to our advantage. The product of $N$ of these (with the increasing site number from left to right) defines the monodromy matrix

$$\mathbb{T}_N(u) = \mathbb{L}_1(u)\mathbb{L}_2(u) \ldots \mathbb{L}_N(u) = \begin{pmatrix} A_N^{[2] \times [2]}(u) & B_N^{[2]}(u) \\ C_N^{[2]}(u) & D_N(u) \end{pmatrix}, \quad (2.20)$$
where we displayed the dimensions of the corresponding blocks as superscripts, e.g., \( B_N^{[2]} = (B_N^1, B_N^2) \) etc. Our focus will be on the element \( D_N(u) \). As can easily be found from the Yang-Baxter equation, \( D_N(u) \) commutes with itself for arbitrary values of the spectral parameter \([D_N(u'), D_N(u)] = 0\). And as it will be established in the next section, it commutes with the Hamiltonian as well,

\[
[D_N(u), \mathcal{H}_N] = 0. \tag{2.21}
\]

\( D_N(u) \) thus generates a family of commuting charges which arise as coefficients of degree \( N \) polynomial in \( u \). However, we immediately find ourselves in a predicament, since the operator \( iN D_N(iw) \) is not self-adjoint! It is obvious already for one site, where the only charge \( d_1 \) reads

\[
D_1(u) = u + d_1, \quad d_1 = -S_1^0 - B_1, \tag{2.22}
\]

with \( S_1^0 \) and \( B_1 \) having opposite conjugation properties in light of Eqs. (2.9). This implies that the eigenvalues of the operator \( D_N \) are not real. This is not a problem by itself, however, it implies that the Hamiltonian will share only a subset of the eigenfunctions of the latter, i.e., the ones that yield its real eigenvalues. In fact, the complex nature of \( D_N \) eigenvalues will be a virtue rather than a bug explaining the incremental shift in energy eigenvalues for excitations propagating on the flux tube. One can always define a new self-adjoint operator

\[
\Omega_N(w) = iN D_N(iw) + (-i)^{N} D_N^\dagger(-iw), \tag{2.23}
\]

that will possess real eigenvalues. However, since we will be devising a procedure to calculate the eigenstates of the Hamiltonian \( \mathcal{H}_N \) based on a recursion for \( D_N \), using \( \Omega_N \) for this purpose will be a significant obstacle on this route.

### 2.3 Commutativity

For one- and two-site cases, the proof of commutativity can be done by brute force. Namely, for \( N = 1 \), the Hamiltonian (1.15) in the representation (2.12) can be rewritten in terms of generators as

\[
\mathcal{H}_1 = 2\psi(1) - \psi(S_1^0 + B_1 + 2s) - \psi(-S_1^0 + B_1 + 2s). \tag{2.24}
\]

It is obviously self-adjoint and commutes with (2.22) by virtue of the sl(2\|1) commutator algebra. For \( N = 2 \), the operator \( D_2(u) \) is a second order polynomial in spectral parameter

\[
D_2(u) = u^2 + ud_1 + d_2, \tag{2.25}
\]

with operator coefficients

\[
d_1 = -S_1^0 - S_2^0 - B_1 - B_2, \quad d_2 = S_1^+ S_2^- + (S_1^0 + B_1)(S_2^0 + B_2) - W_1^+ V_2^- . \tag{2.26}
\]

While the commutativity of \( \mathcal{H}_2 \) with \( d_1 \) is almost obvious, the same property for the second-order differential operator \( d_2 \) is far from this. In fact, the direct calculation results in the following relations for individual components of the two-site Hamiltonian,

\[
[d_2, \mathcal{H}_{12}] = -z_1 \partial_{z_1} - \theta_1 \partial_{\theta_1} + z_2 \partial_{z_2} + \theta_2 \partial_{\theta_2}, \tag{2.27}
\]
Adding these together, one recovers the anticipated result (2.21) for $N$ the $R$-matrix approach. In fact, as was demonstrated in the seminal paper [33], the $R$-operators obeying the conventional Yang-Baxter relation

\[ [d_2, \mathcal{H}] = \frac{z_1}{z_2} \left( z_1 \partial_{z_1} + 2s + \theta_1 \partial_{\theta_1} \right), \]

\[ [d_2, \delta \mathcal{H}] = \frac{z_1}{z_2} \left( z_1 \partial_{z_1} + 2s + \theta_1 \partial_{\theta_1} \right) + z_1 \partial_{z_2} + \theta_1 \partial_{\theta_2} \]

\[ + z_1 \partial_{z_1} + \theta_1 \partial_{\theta_1} - z_2 \partial_{z_2} + \theta_2 \partial_{\theta_2}. \]

Adding these together, one recovers the anticipated result (2.21) for $N = 2$.

Beyond $N = 2$, the direct proof becomes tedious and it is instructive to rely on the power of the $R$-matrix approach. In fact, as was demonstrated in the seminal paper [33], the $R$-operators obeying the conventional Yang-Baxter relation

\[ \mathcal{R}_{12}(v - u) \mathbb{L}_1(u) \mathbb{L}_2(v) = \mathbb{L}_1(v) \mathbb{L}_2(u) \mathcal{R}_{12}(v - u) \]

with $\mathcal{R}_{12} = \Pi_{12} \mathcal{R}_{12}$ having the two quantum spaces interchanged with the permutation $\Pi_{12}$, can be factorized in terms of three intertwiners

\[ \mathcal{R}_{12}(v - u) = \mathcal{R}_{12}^{(1)}(v - u_1) \mathcal{R}_{12}^{(2)}(v_2 - u_2) \mathcal{R}_{12}^{(3)}(v_3 - u_3), \]

each exchanging only a pair of combinations of spectral parameters introduced in Eq. (2.19), e.g.,

\[ \mathcal{R}_{12}^{(1)}(v_1 - u_1) \mathbb{L}_1(v_1, u_2, u_3) \mathbb{L}_2(u_1, u_2, u_3) = \mathbb{L}_1(v_1, u_2, u_3) \mathbb{L}_2(u_1, u_2, u_3) \mathcal{R}_{12}^{(1)}(v_1 - u_1), \]

\[ \mathcal{R}_{12}^{(3)}(v_3 - u_3) \mathbb{L}_1(v_1, u_2, u_3) \mathbb{L}_2(v_1, u_2, u_3) = \mathbb{L}_1(v_1, u_2, u_3) \mathbb{L}_2(v_1, u_2, u_3) \mathcal{R}_{12}^{(3)}(v_3 - u_3), \]

where\(^1\)

\[ \mathcal{R}_{12}^{(1)}(v_1 - u_1) \equiv \mathcal{R}_{12}^{(1)}(v_1|u_1, u_2, u_3), \]

\[ \mathcal{R}_{12}^{(3)}(v_1 - u_1) \equiv \mathcal{R}_{12}^{(3)}(v_1, v_2, v_3|u_3) \]

depend in a translation invariant manner on the displayed spectral parameters and are actually independent of the ones not shown. They thus solve simplified RLL relations displayed above.

For now, we will focus on $\mathcal{R}_{12}^{(3)}$ which is the generator of the bulk Hamiltonians. Namely, making use of its chiral limit $\mathcal{R}_{12}^{(3)}$ from the generic form derived in Ref. [34], we find the following integral in the upper half of the complex plane that can be easily converted into the line integral representation for the function $\Phi(Y_1, Y_2)$ of $Y_n = (y_n, \vartheta_n)$,

\[ \mathcal{R}_{12}^{(3)}(u) \Phi(Y_1, Y_2) = \int [DZ]_s(y_1 - z^* + \vartheta_1 \theta^*)^{-u} - 2s(y_2 - z^* + \vartheta_2 \theta^*)^u \Phi(Z, Y_2) \]

\[ = \frac{\Gamma(2s)}{\Gamma(-u) \Gamma(u + 2s)} \int_0^1 d\tau \tau^{-u - 1} \tau^u + 2s - 1 \Phi(\tau Y_1 + \tau Y_2, Y_2). \]

As can be easily verified expanding $\mathcal{R}_{12}^{(3)}(u)$ in the vicinity of $u = \varepsilon \to 0$, we find the Hamiltonian

\[ h_{12}^{-}, \]

\[ \mathcal{R}_{12}^{(3)}(\varepsilon) = 1 - \varepsilon h_{12}^{-} + O(\varepsilon^2), \]

\(^1\)The consideration $\mathcal{R}^{(2)}$ was also done in [34], however, it will not play any role in our construction and is thus completely disregarded.
such that the RLL relation to this order yields the commutation

\[ [h_{12}, L_1(v)L_2(v)] = M_n^{-1}L_2(v) - L_1(v)(v)M_n^-, \] (2.38)

where

\[ M_n^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -z_n & \theta_n & 1 \end{pmatrix}, \] (2.39)

Similar relations can be found for \( h_{12}^+ \) by expanding in the vicinity of \( u = -2s + \varepsilon \) as \( \varepsilon \to 0 \).

As one can see, the bulk Hamiltonians commute with the monodromy matrix \( T_N \) up to boundary terms. The latter are cancelled by the boundary Hamiltonians \( h_{01} \) and \( h_{N\infty} \) in the same fashion as in the sl(2) case analyzed in [11].

### 3 Brute force diagonalization

In this introductory section, we will perform the diagonalization of the Hamiltonian by solving the emerging differential equations for eigenfunctions of the generating function \( D_N \) of conserved charges. We define an energy eigenstate of the flux-tube \( |E(\lambda)\rangle \) with \( N \) excitations possessing rapidities \( \lambda = (\lambda_1, \ldots, \lambda_N) \). Then, the matrix element of the light-cone operator between the vacuum \( |0\rangle \) and \( |E(\lambda)\rangle \)

\[ \Phi_s(Z; \lambda) = \langle 0|O_{11}(Z)|E(\lambda)\rangle \] (3.1)

will be an eigenfunction of \( \mathcal{H}_N \).

#### 3.1 One-particle matrix element

The solution of the one-particle problem is trivial as it arises from the first-order differential equation determining the eigensystem for \( D_1 \)

\[ D_1(iw)\Phi_s(Z_1; \lambda_1) = (iw - i\lambda_1 + s)\Phi_s(z_1; \lambda_1) + (iw - i\lambda_1 + s - \frac{1}{2})\theta_1\Phi_{s+1/2}(z_1; \lambda_1). \] (3.2)

It yields for the individual eigenfunctions

\[ \Phi_s(z_1; \lambda_1) = z_1^{i\lambda_1-s}, \quad \Phi_{s+1/2}(z_1; \lambda_1) = z_1^{i\lambda_1-s-1/2}, \] (3.3)

that define the one-superparticle matrix element

\[ \Phi_s(Z_1; \lambda_1) = \Phi_s(z_1; \lambda_1) + \theta_1\Phi_{s+1/2}(z_1; \lambda_1). \] (3.4)

These are plane wave with complex wave numbers. As we alluded to above, the eigenvalues of \( D \) are complex. However, its eigenfunctions generate eigenvalues of the flux-tube Hamiltonian

\[ \mathcal{H}_s \Phi_s(Z_1; \lambda_1) = E_s(\lambda_1)\Phi_s(z_1; \lambda_1) + E_{s+1/2}(\lambda_1)\theta_1\Phi_{s+1/2}(z_1; \lambda_1), \] (3.5)

with the well-known (one-loop) energy

\[ E_s(\lambda_1) = 2\psi(1) - \psi(s - i\lambda_1) - \psi(s + i\lambda_1). \] (3.6)
3.2 Two-particle matrix element

Now, we move on to the two-particle case. We decompose the eigenfunction of $D_2$ in double Grassmann series over the two fermionic variables

$$\Phi_s(Z) = \Phi_{ss}(z) + \theta_1 \Phi_{s+1/2,s}(z) + \theta_2 \Phi_{s,s+1/2}(z) + \theta_1 \theta_2 \Phi_{s+1/2,s+1/2}(z),$$  \hspace{1cm} (3.7)

with individual components depending on the bosonic variables $z = (z_1, z_2)$. We have to solve the following equation in the component form

$$D_2(iw)\Phi_s(Z) = (iw - i\lambda_1 + s)(iw - i\lambda_2 + s)\Phi_{ss}(z) + (iw - i\lambda_1 + s - \frac{1}{2})(iw - i\lambda_2 + s) \left[ \theta_1 \Phi_{s+1/2,s}(z) + \theta_2 \Phi_{s,s+1/2}(z) \right] + (iw - i\lambda_1 + s - \frac{1}{2})(iw - i\lambda_2 + s - \frac{1}{2})\theta_1 \theta_2 \Phi_{s+1/2,s+1/2}(z).$$  \hspace{1cm} (3.8)

The first-order differential equations arising from it fix the overall plane-wave factors of various contributions. The second order differential equations determine the remaining function of the ratio $z_1/z_2$ accompanying the waves and read

$$[-z_1(z_1 - z_2)\partial_{z_1}\partial_{z_2} - 2s z_1\partial_{z_2} - (i\lambda_1 - s)(i\lambda_2 - s)]\Phi_{ss}(z) = 0, \hspace{1cm} (3.9)$$

$$[-z_1(z_1 - z_2)\partial_{z_1}\partial_{z_2} - 2s z_1\partial_{z_2} + z_1\partial_{z_1} - (i\lambda_1 - s + \frac{1}{2})(i\lambda_2 - s)] \Phi_{s,s+1/2}(z) = 0, \hspace{1cm} (3.10)$$

$$[-z_1(z_1 - z_2)\partial_{z_1}\partial_{z_2} - ((2s + 1)z_1 - z_2)\partial_{z_2} - (i\lambda_1 - s + \frac{1}{2})(i\lambda_2 - s)] \Phi_{s+1/2,s}(z) - (z_1\partial_{z_1} + 2s)\Phi_{s,s+1/2}(z) = 0, \hspace{1cm} (3.11)$$

$$[-z_1(z_1 - z_2)\partial_{z_1}\partial_{z_2} - ((2s + 1)z_1 - z_2)\partial_{z_2} + z_1\partial_{z_1} + 1 - (i\lambda_1 - s + \frac{1}{2})(i\lambda_2 - s + \frac{1}{2})] \Phi_{s+1/2,s+1/2}(z) = 0. \hspace{1cm} (3.12)$$

The solutions to these equations can be found in a straightforward fashion

$$\Phi_{ss}(z) = z_1^{i\lambda_1 - s}z_2^{i\lambda_2 - s}F_1 \left( s + i\lambda_1, s - i\lambda_2 \left| \frac{1 - z_1}{z_2} \right. \right), \hspace{1cm} (3.13)$$

$$\Phi_{s+1/2,s}(z) = (s + i\lambda_2)z_1^{i\lambda_1 - s - 1}z_2^{i\lambda_2 - s}F_1 \left( s + \frac{1}{2} + i\lambda_1, s - i\lambda_2 \left| \frac{1 - z_1}{z_2} \right. \right), \hspace{1cm} (3.14)$$

$$\Phi_{s,s+1/2}(z) = (s - i\lambda_2)z_1^{i\lambda_1 - s - 1}z_2^{i\lambda_2 - s}F_1 \left( s + \frac{1}{2} - i\lambda_1, s + 1 - i\lambda_2 \left| \frac{1 - z_1}{z_2} \right. \right), \hspace{1cm} (3.15)$$

$$\Phi_{s+1/2,s+1/2}(z) = z_1^{i\lambda_1 - s - 1}z_2^{i\lambda_2 - s - 1}F_1 \left( s + \frac{1}{2} + i\lambda_1, s + \frac{1}{2} - i\lambda_2 \left| \frac{1 - z_1}{z_2} \right. \right), \hspace{1cm} (3.16)$$

Notice that the solution to (3.11) is not unique since one can always add to it a solution of the homogeneous equation with an arbitrary coefficient! Particularly noteworthy is the following one that solves Eq. (3.11)

$$\Phi'_{s+1/2,s}(z) = -(s - i\lambda_2)z_1^{i\lambda_1 - s - 1}z_2^{i\lambda_2 - s}F_1 \left( s + 1 - i\lambda_2, s + 1/2 + i\lambda_1 \right| \frac{1 - z_1}{z_2} \right), \hspace{1cm} (3.17)$$

since it is given by a single hypergeometric function and thus can be cast in a concise “pyramid” representation to be introduced later. The difference between the two solutions $\Phi'_{s+1/2,s} - \Phi_{s+1/2,s}$ is indeed a solution to the homogeneous equation. Finally for the mixed wave functions, there is yet another (trivial) solution to the eigenvalue equation for the $D_2$-operator, i.e., $\Phi_{s,s+1/2} = 0,$
\( \Phi_{s+1/2, s} = 0 \), however, like the previous one, it does not lead to consistent eigenvalue equation for the Hamiltonian.

With above results in our hands, we can immediately verify that they yield correct eigenvalues of the Hamiltonian \( \mathcal{H}_2 \), namely, we find

\[
\mathcal{H}_2 \Phi_s(\lambda; z) = [E_s(\lambda_1) + E_s(\lambda_2)]\Phi_s(z) + [E_{s+1/2}(\lambda_1) + E_{s+1/2}(\lambda_2)]\theta_1 \theta_2 \Phi_{s+1/2}(z) + [E_{s+1/2}(\lambda_1) + E_s(\lambda_2)] \left[ \theta_1 \Phi_{s+1/2}(z) + \theta_2 \Phi_{s,s+1/2}(z) \right].
\]

(3.18)

Notice that the two eigenfunctions \( \Phi_{s+1/2, s}(z) \) and \( \Phi_{s,s+1/2}(z) \) possess the same eigenvalue!

### 4 Algebraic construction of eigenfunctions

Beyond \( N = 2 \), i.e., for three sites and more, the brute force solution of higher-order differential equations is hopeless. Therefore, we will devise a recursive algebraic procedure to find the eigenfunctions of the operator \( D_N \). It will turn out that the formalism will produce only one representative solution at a given Grassmann degree. The rest however will be generated by means of supersymmetry. The procedure will be based on the intertwiner \( R^{(1)}_{12} \) introduced earlier in Section 2.3 that will yield a closed recursion for the matrix element \( D \) of the monodromy operator. However, we have to find first its representation on the space of chiral matrix elements.

#### 4.1 Lowest component

To start with let us recall the solution for the lowest component \( \Phi_{s...s}(z) \) of the \( N \)-particle supermatrix element (3.1). It is determined by the sl(2) open spin chain that was addressed in Ref. [11]. The intertwiner that is used in the recursive procedure to solve the eigenvalue equation for the bosonic counterpart of \( D_N \) reads [11]

\[
R^{(1)}_{12}(u) \Phi(y_1, y_2) = \frac{\Gamma(2s+1)\Gamma(y_{21}\partial_{y_2} + u + 2s)}{\Gamma(y_2 + 2s + 1)\Gamma(y_{21}\partial_{y_2} + 2s)} \Phi(y_1, y_2)
\]

(4.1)

\[
= \int [Dz]_s(y_1 - z^*)^u(y_2 - z^*)^{-u-2s} \Phi(y_1, z).
\]

For instance, the two-particle eigenstate is

\[
\Phi_{s,s}(z; \lambda) = z^{i\lambda_1 - s} r^{(1)}_{12}(-i\lambda_1 - s) z_2^{i\lambda_2 - s} = z^{i\lambda_1 - s} z_2^{i\lambda_2 - s} \binom{s + i\lambda_1, s - i\lambda_2}{2s} \left( 1 - \frac{z_1}{z_2} \right),
\]

(4.2)

and agrees with Eq. (3.13) found earlier. For a generic \( N \)-particle case \( z = (z_1, z_2, \ldots, z_N) \), we found

\[
\Phi_{s...s}(z; \lambda) = z^{i\lambda_1 - s} r^{(1)}_{12...N}(-i\lambda_1 - s) z_2^{i\lambda_2 - s} r^{(1)}_{2...N}(-i\lambda_2 - s) z_3^{i\lambda_3 - s} \ldots r^{(1)}_{N-1N}(-i\lambda_N - s) z_N^{i\lambda_N - s},
\]

(4.3)

where

\[
R^{(1)}_{n...N}(u) = R^{(1)}_{N-1,N}(u) R^{(1)}_{N-2,N-1}(u) \ldots R^{(1)}_{n,n+1}(u).
\]

(4.4)

Let us now turn to further components in the Grassmann expansion. To this end we have to deduce the intertwiner that will simplify the solution for \( D_N \) in the supersymmetric case.
4.2 Chiral limit of factorized matrices

The discussion in Ref. \[34\] was done for generic representations, i.e., involving both chiral and antichiral Grassmann variables $\theta$ and $\bar{\theta}$. We start therefore with derived there integral (zig-zag) representation for the intertwiner $R^{(1)}_{12}$ and take its chiral limit. Ignoring a convention-dependent normalization factor, we define

$$ R^{(1)}_{12}(u) \Phi(Y_1, Y_2) = \int [DZ]_{jj} K_{0,-u}(Y_1, \bar{Z}^*) K_{j,j+u}(Y_2, Z^*) \Phi(Y_1, Z), \quad (4.5) $$

where $Y = (y, \vartheta, \bar{\vartheta})$, $Z = (z, \theta, \bar{\theta})$ and the measure reads

$$ \int [DZ]_{jj} = \frac{j + \bar{j}}{jj} \int_{\Im \theta > 0} \frac{d^2 z}{\pi} \int d\theta d\bar{\theta} (z_+ - z_+^*)^j (z_- - z_-^*)^\bar{j} (y - y^*)^\bar{j} \quad (4.6) $$

along with the reproducing kernel

$$ K_{jj}(Y, Z^*) = (y_+ - z_+^* - \vartheta \theta^*)^{-j} (y_- - z_-^* - \bar{\vartheta} \bar{\theta}^*)^{-\bar{j}}. \quad (4.7) $$

Here, we introduced a notation for (anti)chiral bosonic coordinates $z_\pm = z \pm \frac{1}{2} \bar{\theta} \theta$, $y_\pm = y \pm \frac{1}{2} \bar{\vartheta} \vartheta$ with conjugate ones found according to the rule (2.5).

To reach the chiral limit in the above expressions, we take into account that $\Phi$ depends on $\bar{\theta}$ only through the chiral bosonic coordinates,

$$ \Phi(Y_1, Y_2) = \Phi(Y_1, Y_2), \quad (4.8) $$

with $Y = (y_+, \vartheta)$. Then, shifting the bosonic integration variables $z_+ \to z$, we can perform the integration with respect to $\bar{\theta}$ and $\bar{\theta}^*$ and send $\bar{j} \to 0$ and $j \to 2s$ afterwards. We obtain

$$ R^{(1)}_{12}(u) \Phi(Y_1, Y_2) = \int [DZ]_{s}(y_1 - z^* - \theta \theta^*)^u (y_2 - z^* - \theta \theta^*)^{-u} (y_2 - z^* - \bar{\vartheta} \bar{\theta}^*)^{-2s} \Phi(Y_1, Z), \quad (4.9) $$

where the chiral integration measure was introduced earlier in Eq. (2.4). One can actually rewrite this operator in terms of a nonlocal differential operator using the properties of bosonic reproducing kernels such that it reads explicitly

$$ R^{(1)}_{12}(u) = \frac{\Gamma(2s + 1) \Gamma(y_{21} \partial_{y_2} + u + 2s + 1)}{\Gamma(u + 2s + 1) \Gamma(y_{21} \partial_{y_2} + 2s + 1)}, \quad (4.10) $$

In turn, it can be cast as an integral on the real line adopting the well-known integral representation for the Euler Beta function, or the bosonic integral in the upper half of the complex plane,

$$ R^{(1)}_{12}(u) \Phi(Y_1, Y_2) = \frac{\Gamma(2s + 1)}{\Gamma(-u) \Gamma(u + 2s + 1)} \int_0^1 d\tau \tau^{-u-1} \tau^{u+2s} \Phi(Y_1, \tau y_1 + \bar{\tau} y_2, \vartheta_2) $$

$$ = \int [Dz]_{s+1/2}(y_1 - z^*)^u (y_2 - z^*)^{-u-2s-1} \Phi(Y_1, z, \vartheta_2). \quad (4.11) $$

Here, the conformal spin $s$ may be understood to admit two different values depending on the component field the operator it acts on, for instance for $s \to s - 1/2$, we fall back into the bosonic case discussed in the earlier section, $R^{(1)}_{12}(u)|_{s \to s - 1/2} = \iota^{(1)}_{12}(u)$. 

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4.3 Two-particle case

Let us start applying above results to the derivation of the two-site eigenfunctions. Making use of the known right factorization property of the Lax operator,

\[ \mathbb{L}_n(v_1, u_2, u_3) = \mathbb{M}_n(u_1, u_2, u_3) \mathbb{M}_n(u_1 | v_1), \quad \text{with} \quad \mathbb{M}_n(u_1 | v_1) = \begin{pmatrix} v_1/u_1 & 0 & 0 \\ 0 & 1 & 0 \\ (v_1/u_1 - 1)z_n & 0 & 1 \end{pmatrix}, \]

which allows us to restore the same spectral parameter in \( \mathbb{L}_n \), we can write a relation between two- and one-site monodromy matrices \( \mathbb{R}_1^{(1)}(v_1 - u_1) \mathbb{L}_1(v_1, u_2, u_3) \mathbb{T}_1(u) = \mathbb{T}_2(u) \mathbb{M}_2(u_1 | v_1) \mathbb{R}_1^{(1)}(v_1 - u_1). \) (4.13)

Projecting out the 33-entry of the monodromy matrix \( D_2(u) = [\mathbb{T}_2(u)]_{33} \) in the right-hand side, we end up with the relation

\[ D_2(u) \mathbb{R}_1^{(1)}(v_1 - u_1) = \mathbb{R}_1^{(1)}(v_1 - u_1) [z_1(\partial_z + \Theta_1 + 1 - u_3) B_1^1(u) - \Theta_1(z_3 - \partial_z + 1 - u_2) B_2^2(u) + (u_3 - z_1\partial_z - \Theta_1 B_1^1(u)), \]

where the elements of the one-particle monodromy (i.e., the Lax operator itself) matrix in the right-hand side of this equation act only on the variables of the second site, i.e.,

\[ B_1^1(u) = -\partial_z, \quad B_2^2(u) = \partial_{\beta_2}, \quad D_1(u) = u_3 - z_2\partial_{z_2} - \beta_2\partial_{\beta_2}. \] (4.15)

In order to construct a recursion, the first two terms in the right-hand side of Eq. (4.14) have to vanish when acting on a state of our choice. There are two \(^2\) such choices cumulatively denoted by \( \Phi^{(0)}(\mathbf{Z}) \),

\[ \Phi^{(0)}(\mathbf{Z}) = \Theta_1 \Phi^{(0)}_{s+1/2,s}(z) + \Theta_1 \Theta_2 \Phi^{(0)}_{s+1/2,s+1/2}(z), \] (4.16)

where \( \mathbf{Z} = (Z_1, Z_2) \) and \( z = (z_1, z_2) \). Notice that the Grassmann structure \( \Theta_2 \) will necessarily involve \( B \)-operators and will not be closed under recursion. However, as will be demonstrated below, it can be found by virtue of supersymmetry.

Since different degree Grassmann components do not talk to each other, we can analyze them separately. Let us start with the \( \Theta_1 \) component and cast it in the factorized form \( \Theta_1 \Phi^{(0)}_{s+1/2,s}(z) = \Theta_1 z_{1}^{\alpha} \Phi^{(0)}_{s}(z_2) \) and fix the value of \( \alpha \) from the vanishing of the action of the first term in the brackets, \( \alpha = u_3 - v_1 - 1 \) and provides the eigenvalue of the first level of recursion \( v_1 = i\omega - i\lambda_1 - s + \frac{1}{2}. \)

\[ D_2(u) \mathbb{R}_1^{(1)}(v_1 - u_1) \Theta_1 \Phi^{(0)}_{s+1/2,s}(z) = (i\omega - i\lambda_1 - s + \frac{1}{2}) \Theta_1 z_{1}^{\alpha} \Phi^{(0)}_{s+1/2,s}(z_2) \mathbb{R}_1^{(1)}(v_1 - u_1) D_1(u) z_{2}^{\beta}, \] (4.17)

such that \( \alpha = i\lambda_1 - s - 1/2 \) and \( u_3 = i\omega, u_1 = u_2 = i\omega + s + 1/2 \). Here we took into account that \( \mathbb{R}_1^{(1)} \) acts on \( z_2 \) coordinate only such that we can move \( z_1 \)-dependent factor to its left. Next, substituting \( \beta = i\lambda_2 - s \) in \( \Phi^{(0)}_{s}(z_2) = z_{2}^{\beta} \), we immediately obtain

\[ D_2(i\omega) \mathbb{R}_1^{(1)}(-i\lambda_1 - s + \frac{1}{2}) \Phi^{(0)}_{s+1/2,s}(z) \]

\(^2\)In fact since \( B_2^2(u) = \partial_{\beta_2} \) is a derivative that annihilates a Grassmann constant, we can encode the lowest component into this bare function by adding a \( \Theta_2 \)-independent term \( \Phi^{(0)}_{s,s}(z) \). This eigenfunction was discussed in Section 4.1 already.

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\[ \begin{align*}
&= (iw - i\lambda_1 + s - \frac{1}{2})(iw - i\lambda_2 + s)z_1^{i\lambda_1 - s - 1/2}R_{12}^{(1)}(-i\lambda_1 - s - \frac{1}{2})z_2^{i\lambda_2 - s},
\end{align*} \]

with the resulting eigenfunction being

\[ \Phi_{s+\frac{1}{2},s}(z; \lambda) = z_1^{i\lambda_1 - s - 1/2}R_{12}^{(1)}(-i\lambda_1 - s - \frac{1}{2})z_2^{i\lambda_2 - s} \]

\[= z_1^{i\lambda_1 - s - 1/2}z_2^{i\lambda_2 - s}F_1 \left( \left. \begin{array}{c} s + 1/2 + i\lambda_1, s - i\lambda_2 \\ 2s + 1 \end{array} \right| 1 - \frac{z_1}{z_2} \right). \] (4.19)

This, up to an overall normalization coefficient, is the result \( \Phi_{s+\frac{1}{2},s}(z) \) of the previous section. The missing prefactor that plays a crucial role in proper diagonalization of the Hamiltonian will be fixed making of supersymmetry later in this section.

For the highest component \( \theta_1\theta_2\Phi_{s+\frac{1}{2},s+\frac{1}{2}}^{(0)}(z) \) adopting an analogous factorizable Ansatz

\[ \Phi_{s+\frac{1}{2},s+\frac{1}{2}}^{(0)}(z) = z_1^{\alpha}\Phi_{s+\frac{1}{2}}^{(0)}(z_2), \]

with one-particle wave function \( \Phi_{s+\frac{1}{2}}^{(0)}(z_2) = z_2^{\beta} \), we deduce in the same fashion

\[ \begin{align*}
D_2(iw)R_{12}^{(1)}(-i\lambda_1 - s - 1/2)\theta_1\theta_2\Phi_{s+\frac{1}{2},s+\frac{1}{2}}^{(0)}(z) \\
&= (iw - i\lambda_1 + s - \frac{1}{2})(iw - i\lambda_2 + s - \frac{1}{2})\theta_1\theta_2\Phi_{s+\frac{1}{2},s+\frac{1}{2}}^{(0)}(z),
\end{align*} \]

with the explicit eigenfunction being

\[ \Phi_{s+\frac{1}{2},s+\frac{1}{2}}^{(0)}(z; \lambda) = z_1^{i\lambda_1 - s - 1/2}R_{12}^{(1)}(-i\lambda_1 - s - 1/2)z_2^{i\lambda_2 - s - 1/2} \]

\[= z_1^{i\lambda_1 - s - 1/2}z_2^{i\lambda_2 - s - 1/2}F_1 \left( \left. \begin{array}{c} s + 1/2 + i\lambda_1, s + 1/2 - i\lambda_2 \\ 2s + 1 \end{array} \right| 1 - \frac{z_1}{z_2} \right). \] (4.21)

To summarize, the two-particle eigenfunctions constructed via the advocated algebraic procedure are (we added here the lowest component as well)

\[ \Phi_s(Z; \lambda) = \Phi_{ss}(z; \lambda) + \theta_1\Phi_{s+\frac{1}{2},s}(z; \lambda) + \theta_1\theta_2\Phi_{s+\frac{1}{2},s+\frac{1}{2}}(z; \lambda), \] (4.22)

where the individual components are given by Eqs. (4.12), (4.19) and (4.21), respectively. Though, this construction does not allow one to find all eigenfunctions in the Grassmann expansion, e.g., in front of \( \theta_2 \) for the case at hand, and endow them with correct coefficients that they enter the supereigenfunction, one can use a recipe to restore all of them as suggested below.

Before, we outline it, let us introduce another representation for eigenfunctions which will be indispensable in the proof of their orthogonality as well as analytic verification of factorizability of multiparticle pentagon transitions. It is the so-called pyramid representation which gives diagrammatic interpretation for eigenfunctions in two-dimensional space. Making use of results in Appendix B of Ref. [III], we can cast the above matrix elements in the form

\[ \Phi_{ss}(z; \lambda) = z_1^{i\lambda_1 - s} \int [Dz]_s(z_1 - z^*)(-i\lambda_1 - s)(z_2 - z^*)i\lambda_1 - s z_2^{i\lambda_2 - s}, \] (4.23)

\[ \Phi_{s+\frac{1}{2},s+\frac{1}{2}}(z; \lambda) = z_1^{i\lambda_2 - s - 1/2} \int [Dz]_{s+\frac{1}{2}}(z_1 - z^*)^{-i\lambda_2 - s - 1/2}(z_2 - z^*)i\lambda_2 - s - 1/2 z_2^{i\lambda_1 - s - 1/2}, \] (4.24)

for the same-flavor components and

\[ \Phi_{s+\frac{1}{2},s}(z; \lambda) = z_1^{i\lambda_1 - s - 1/2} \int [Dz]_{s+\frac{1}{2}}(z_1 - z^*)^{-i\lambda_1 - s - 1/2}(z_2 - z^*)i\lambda_1 - s - 1/2 z_2^{i\lambda_2 - s} \] (4.25)
for the $\theta_1$ component. Its graphical representation of (the second form of) this eigenfunction in terms of a “pyramid” is shown in Fig. 2. Now the missing eigenfunction can be simply found by promoting the internal bosonic propagators in the second representation to their supersymmetric extension

$$
(z' - z^*)^{-\alpha} \rightarrow [Z' - Z^*]^{-\alpha} \equiv (z' - z^* + \theta' \theta^*)^{-\alpha}.
$$

The Grassmann degree-one two-particle pyramid

$$
\Phi_2^{[1]}(Z; \lambda) \equiv \theta_1 \Phi_{s+1/2, s}(z; \lambda) + \theta_2 \Phi_{s, s+1/2}(z; \lambda),
$$

reads

$$
\Phi_2^{[1]}(Z; \lambda) = \int d\theta^* z_1^{i\lambda_2-s} \int [Dz]_{s+1/2} [Z_1 - Z^*]^{-i\lambda_2-s} [Z_2 - Z^*]^{i\lambda_2-s} z_1^{i\lambda_1-s-1/2}.
$$

Expanding the integrand in the fermionic variables, we uncover the missing solution $\Phi_{s, s+1/2}$ as well automatically produce the correct relative coefficients as functions of the rapidity variables.

### 4.4 Three-particle case and beyond

The one-third of the Yang-Baxter equation for the three-site case reads

$$
\mathbb{R}_{123}^{(1)}(v_1 - u_1) \mathbb{L}_1(v_1, u_2, u_3) \mathbb{L}_2(u_1, u_2, u_3) \mathbb{L}_3(u_1, u_2, u_3) = \mathbb{L}_1(u_1, v_2, v_3) \mathbb{L}_2(u_1, u_2, u_3) \mathbb{L}_3(v_1, u_2, u_3) \mathbb{R}_{123}^{(1)}(v_1 - u_1),
$$

where

$$
\mathbb{R}_{123}^{(1)}(v_1 - u_1) \equiv \mathbb{R}_{23}^{(1)}(v_1 - u_1) \mathbb{R}_{12}^{(1)}(v_1 - u_1).
$$
Making use of Eq. (4.12), this relation can be rewritten for the monodromy matrices with decreasing number of sites

\[ R_{123}^{(1)}(v_1 - u_1)M_1(v_1, u_2, u_3)T_2(u) = T_3(u)M_3(u_1|v_1)R_{123}^{(1)}(v_1 - u_1). \]  

(4.31)

Extracting the 33-matrix component from both sides and acting with the result on a test function \( \Phi^{(0)}(Z) \) of three variables \( Z = (Z_1, Z_2, Z_3) \), we find

\[ R_{123}^{(1)}(v_1 - u_1)[z_1(z_1\partial_{z_1} + \theta_1\partial_{\theta_1} + v_1 - u_3)B_2(u) - \theta_3(z_3\partial_{z_3} - \theta_3\partial_{\theta_3}) + (u_3 - z_1\partial_{z_1} - \theta_1\partial_{\theta_1})D_2(u)]\Phi^{(0)}(Z) = D_3(u)R_{123}^{(1)}(v_1 - u_1)\Phi^{(0)}(Z), \]

(4.32)

where

\[
\begin{align*}
B_2(u) &= -(u_1 + z_2\partial_{z_2} - \theta_2\partial_{\theta_2} - \theta_3\partial_{\theta_3})(u_3 - z_3\partial_{z_3} - \theta_3\partial_{\theta_3}), \\
B_3^2(u) &= (z_2\partial_{\theta_2} - (u_2 - u_1)\theta_2)\partial_{z_3} + (u_2 - \theta_2\partial_{\theta_2})\partial_{z_3} + \theta_2(u_3 - z_3\partial_{z_3} - \theta_3\partial_{\theta_3}).
\end{align*}
\]

(4.33, 4.34)

To construct a self-contained recursion, we have to choose the bare three-particle wave function \( \Phi^{(0)}(Z) \) that eliminates the first two terms in Eq. (4.32). It is achieved by the factorized Ansatz

\[
\Phi^{(0)}(Z) = \theta_1 z_1^{i\lambda_1 - s - 1/2}\Phi(Z_2, Z_3),
\]

(4.35)

where we set \( v_1 = i\omega - i\lambda_1 \) and \( \Phi(Z_2, Z_3) \) is the two-particle eigenfunction whose three components were computed in the previous subsection. So the three of the three-particle eigenfunctions are

\[
\Phi_s(Z) = \theta_1 z_1^{i\lambda_1 - s - 1/2}R_{123}^{(1)}(-i\lambda_1 - s - 1/2)\Phi_s(Z_2, Z_3),
\]

(4.36)

with \( \Phi_s(Z_2, Z_3) \) given in Eq. (4.22) with shifted labels of supercoordinates \( k \to k + 1 \). Finally, the lowest, i.e., \( \theta \)-independent component of the eigenfunction can be found by eliminating any reference to Grassmann variables in the above equations, \( \theta \to 0 \), \( \partial_\theta \to 0 \), and was quoted in Section 4.1

\[
\Phi_{sss}(z) = z_1^{i\lambda_1 - s - 3/2}R_{123}^{(1)} - i\lambda_1 - s)\Phi_{ss}(z_2, z_3),
\]

(4.37)

where the operator \( R_{123}^{(1)} \) is understood as the one with the shift in the spin \( s \to s - 1/2 \). Thus the vector of eigenfunction that can be obtained by means of the above algebraic constructions are

\[
\Phi_s(Z; \lambda) = z_1^{i\lambda_1 - s}R_{123}^{(1)}z_2^{i\lambda_2 - s}R_{23}^{(1)}z_3^{i\lambda_3 - s} + \theta_1 z_1^{i\lambda_1 - s - 1/2}R_{123}^{(1)}z_2^{i\lambda_2 - s}R_{23}^{(1)}z_3^{i\lambda_3 - s} + \theta_1\theta_2 z_1^{i\lambda_1 - s - 1/2}R_{123}^{(1)}z_2^{i\lambda_2 - s}R_{23}^{(1)}z_3^{i\lambda_3 - s - 1/2}.
\]

(4.38)

The generalization to \( N \)-particle case is now straightforward,

\[
\Phi_s(Z) = z_1^{i\lambda_1 - s}R_{123}^{(1)}z_{2-N}^{i\lambda_2 - s}R_{2-N}^{(1)}z_{3-N}^{i\lambda_3 - s} \cdots R_{N-1,N}^{(1)}z_N^{i\lambda_N - s} + \theta_1 z_1^{i\lambda_1 - s - 1/2}R_{123}^{(1)}z_{2-N}^{i\lambda_2 - s}R_{2-N}^{(1)}z_{3-N}^{i\lambda_3 - s} \cdots R_{N-1,N}^{(1)}z_N^{i\lambda_N - s} + \theta_1\theta_2 \cdots \theta_{N-1} z_1^{i\lambda_1 - s - 1/2}R_{123}^{(1)}z_{2-N}^{i\lambda_2 - s}R_{2-N}^{(1)}z_{3-N}^{i\lambda_3 - s} \cdots R_{N-1,N}^{(1)}z_N^{i\lambda_N - s - 1/2}.
\]

(4.39)

Along the same route as was done in two-particle case, one can recover all eigenfunctions by employing supersymmetry at each level of odd variables in the pyramid representation of the eigenfunctions and, thus, restore relative coefficients accompanying them.
4.5 Orthogonality

Before, we move on to using the above eigenfunctions for the calculation of pentagon transitions, let us prove their orthogonality first. In fact the technique that will be used for it here is readily adoptable for the calculation of the latter as well.

4.5.1 One site

To keep track of different components in the Grassmann expansion, it is convenient to introduce a marker variable $\varepsilon$ via $\theta \rightarrow \varepsilon \theta$ for the in-state eigenfunction and, correspondingly, $\varepsilon'$ for the out state. Then, using (2.6), we find

$$\langle \Phi_s(\lambda_1')|\Phi_s(\lambda_1) \rangle = \langle \Phi_s(\lambda_1')|\Phi_s(\lambda_1) \rangle + \frac{\varepsilon' \varepsilon}{2is} \langle \Phi_{s+1/2}(\lambda_1')|\Phi_{s+1/2}(\lambda_1) \rangle$$

(4.40)

where the component inner products

$$\langle \Phi_s(\lambda_1')|\Phi_s(\lambda_1) \rangle = 2\pi e^{-\pi\lambda_1} \mu_s^{-1}(\lambda_1) \delta(\lambda_1' - \lambda_1),$$

(4.41)

are expressed in terms of the measure

$$\mu_s(\lambda) = \frac{\Gamma(s + i\lambda_1)\Gamma(s - i\lambda_1)}{\Gamma(2s)},$$

(4.42)

for the spin-$s$ flux-tube excitation. For $s = 1/2$, these reduce to the hole and fermion excitations for $(\varepsilon\varepsilon')^0$ and $(\varepsilon\varepsilon')^1$, respectively. While for $s = 1$, they accommodate the fermion as the lowest and the gauge field as the highest component of the $\mathcal{N} = 1$ gauge supermultiplet.

4.5.2 Permutation identity in superspace

To work out the two particle case and beyond, we have to introduce an identity that will be instrumental in the concise proof of orthogonality. Namely, it is indispensable to use the permutation identity in the language of Feynman graphs lifted to the superspace. Introducing the superpropagator (4.27), from the superpoint $Z = (z, \theta)$ to $Z' = (z', \theta')$, one can show that

$$[Z_1' - Z_1']^{i\lambda - i\lambda'} X (Z; \lambda|Z'; \lambda') = X (Z; \lambda'|Z'; \lambda) [Z_2' - Z_2]^{i\lambda - i\lambda'},$$

(4.43)
where the supercross is given by

$$X(Z, \lambda | Z', \lambda') \equiv \int [DY] [Y - Z_s^1]^i \lambda - s [Y - Z_s^2]^i \lambda - s [Z_s^1 - Y^s] - i \lambda - s [Z_s^2 - Y^s] - i \lambda - s [Z_s^2 - Y^s] i \lambda - s \, . \quad (4.44)$$

It depends on four superpoints through $Z = (Z_1, Z_2)$ and $Z' = (Z'_1, Z'_2)$. Its form in terms of Feynman graphs is demonstrated in Fig. 3. The identity reduces to its known bosonic counterpart, when all external Grassmann variables are set to zero, see Appendix A.

### 4.5.3 Two sites and more

For two excitations, the eigenfunctions of the matrix elements are given in Eqs. (4.23), (4.24) for the same-flavor case and (4.29) for the mixed one. For $\Phi_{ss}$ and $\Phi_{s+1/2,s+1/2}$ eigenfunctions, the proof of the orthogonality condition repeats the steps of the bosonic consideration [11]. Namely, using a chain of transformations, exhibited in Fig. 4, which consists of using (i) the chain rule (A.3), (ii) the permutation identity (A.6), (iii) the chain rule (twice again), one reduces the inner product to the one-particle case, analyzed above, such that we immediately find

$$\langle \Phi_{ss}(\lambda') | \Phi_{ss}(\lambda) \rangle = a_s(s - i \lambda_1, s + i \lambda_2) a_s(s + i \lambda'_1, s - i \lambda_2) \langle \Phi_s(\lambda'_2) | \Phi_s(\lambda_2) \rangle \langle \Phi_s(\lambda'_1) | \Phi_s(\lambda_1) \rangle \, . \quad (4.45)$$

Here, the inner product involves the spin-$s$ component in the Grassmann expansion of the one-particle eigenfunction (3.4).

To understand what to anticipate for the mixed $\Phi_{s,s+1/2}$ and $\Phi_{s+1/2,s}$ eigenfunctions, let us point out that we are dealing with a degenerate case. Namely, the two-particle mixed sector can be cast in the following matrix form

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} |\psi_1 \rangle \\ |\psi_2 \rangle \end{pmatrix} = E \begin{pmatrix} |\psi_1 \rangle \\ |\psi_2 \rangle \end{pmatrix} \, ,$$

Figure 4: Steps in evaluation of the bosonic inner product (4.45).
where the two eigenstates $|\psi_1\rangle \to \Phi_{21}$ and $|\psi_2\rangle \to \Phi_{12}$ share the same eigenvalue $E$, see Eq. (3.18). Then multiplying this equation from the left by the conjugate two-vector of eigenfunctions, we find

$$
(E' - E) \left[ \langle \psi'_1 | \psi_1 \rangle + \langle \psi'_2 | \psi_2 \rangle \right] = 0,
$$

so that

$$
\langle \psi'_1 | \psi_1 \rangle + \langle \psi'_2 | \psi_2 \rangle = \delta(E' - E),
$$

and not separately for each eigenstate. Thus, the orthogonality has to emerge from the sum of integrals

$$
\langle \Phi_{s,s+1/2}(\lambda') | \Phi_{s,s+1/2}(\lambda) \rangle + \langle \Phi_{s+1/2,s}(\lambda') | \Phi_{s+1/2,s}(\lambda) \rangle
= \int [Dz_1]_{s+1/2} \int [Dz_2]_s \left( \Phi_{s+1/2,s}(z; \lambda') \right)^* \Phi_{s+1/2,s}(z; \lambda) + \int [Dz_1]_s \int [Dz_2]_{s+1/2} \left( \Phi_{s,s+1/2}(z; \lambda') \right)^* \Phi_{s,s+1/2}(z; \lambda).
$$

Making use of the pyramid representation for each eigenfunction

\begin{align*}
\Phi_{s+1/2,s}(z) &= (s + i\lambda_2)z^{i\lambda_2 - s} \int [Dz]_{s+1/2} (z_1 - z^*)^{-i\lambda_2 - s - 1}(z_2 - z^*)^{i\lambda_2 - s} z^{i\lambda_1 - s - 1/2}, \\
\Phi_{s,s+1/2}(z) &= (s - i\lambda_2)z^{i\lambda_2 - s} \int [Dz]_{s+1/2} (z_1 - z^*)^{-i\lambda_2 - s}(z_2 - z^*)^{i\lambda_2 - s - 1} z^{i\lambda_1 - s - 1/2},
\end{align*}

a simple-minded application of the rules used in the bosonic subsector fails at the second step. In spite of the fact that one can find a way out of this predicament by using inversion\footnote{We would like to thank Sasha Manashov for this suggestion.} as demonstrated in Appendix B, we will follow a different route that can be applied for pentagon transitions studied later in the paper.

First, we introduce a conjugate pyramid. It will be defined by the same graph as the original one but with all lines reversed and changed sign of all rapidities. It is proportional to the complex conjugate wave function up to a phase factor that can be easily established from the involution rules

$$
((z' - z^*)^{-\alpha})^* = e^{i\pi\alpha^*} (z - z'^*)^{-\alpha^*}.
$$

This way the wave function $\left( \Phi_{s+1/2,s}(z) \right)^*$ is determined by the reversed graph with overall phase factor

$$
e^{i\pi(s+i\lambda_2)}e^{i\pi(s+1/2+i\lambda_1)}e^{i\pi(2s+1)}.
$$

Here, the first two factors stem from lines connecting vertices with $w = 0$ and the rest arise from the internal lines. The same prefactor accompanies the definition of $\left( \Phi_{s,s+1/2}(z) \right)^*$.

Now we proceed with the verification of orthogonality. The sum in Eq. (4.46) is shown by the top row in Fig. 3 up to an overall phase (4.50), where all rapidities have to be dressed with primes since they emerge from the wave function in the out state. Starting the reduction from
Figure 5: Reduction steps in the evaluation of the inner product (4.46).
right to left, we integrate first with respect to the vertex $z_2$ by means of the chain rules (A.3). This will yield different $a$-factors that accompany the reduced graphs, due to the different spins of the corresponding integration measures. Pulling out the overall factor
\[ e^{-i\pi s}a_s(s - i\lambda_2, s + i\lambda_2') \]
the two contributions with corresponding rapidity-dependent coefficients are shown in the middle row in Fig. 5. The subsequent reduction is based on the use of the permutation identity in superspace, which allows us to move the right vertical propagator through the entire graph to the left. To achieve this, we choose the coordinates as
\[ Z_1 = (w, 0), \quad Z_2 = (z, \theta), \quad Z_1' = (w', 0), \quad Z_2' = (z', \theta'), \]
in Eq. (4.43), where obviously $w = w' = 0$. Collecting terms accompanying the Grassmann structure $\theta^\dagger\theta$, we find the relation
\[
(z' - z^*)^{i\lambda'-i\lambda}(s - i\lambda')(s + i\lambda)X_{s+1/2}(w, z; \lambda|w', z'; \lambda')
- 2is(i\lambda' - i\lambda)(z' - z^*)^{i\lambda'-i\lambda-1}X_s(w, z; \lambda|w', z'; \lambda')
= (w' - w^*)^{i\lambda'-i\lambda'}(s + i\lambda')(s + i\lambda)X_{s+1/2}(w, z; \lambda|w', z'; \lambda),
\]
between the crosses with the spin-$s$ and spin-$(s + \frac{1}{2})$ measures,
\[
X_s(w, z; \lambda|w', z'; \lambda') \equiv \int [Dy]_s(y - w^*)^{i\lambda-s}(w' - y^*)^{-i\lambda-s}(y - z^*)^{-i\lambda-s}(z' - y^*)^{i\lambda'-s},
\]
\[
X_{s+1/2}(w, z; \lambda|w', z'; \lambda') \equiv \int [Dy]_{s+1/2}(y - w^*)^{i\lambda-s}(w' - y^*)^{-i\lambda-s}(y - z^*)^{-i\lambda-s-1}(z' - y^*)^{i\lambda'-s-1}.
\]
These arise from the Grassmann expansion of the supercross. We can recognize right away, in the left-hand side of Eq. (4.52), the sum of contributions with correct accompanying coefficients in Fig. 5 (middle row). This allows us to use this permutation identity and pass to the leftmost graph in the bottom row of diagrams in Fig. 5, where we relied on the identity (A.5) for $w' = w = 0$ which yielded the inner product of one-particle spin-$s$ component eigenfunctions (3.4) along with the corresponding phase,
\[
(s - i\lambda_2)(s + i\lambda_2')e^{-i\pi(s+i\lambda_2)}\langle \Phi_s(\lambda_2')|\Phi_s(\lambda_2)\rangle.
\]
We also included the overall factor of rapidities that stems from the right-hand side coefficient in the permutation identity (4.52). This completes the first level in recursive reduction.

At the next step, we use the chain rule twice, at the vertices $z$ and $z'$. This procedure generates the multiplicative factors
\[
e^{-i\pi(2s+1)}a_{s+1/2}(\frac{1}{2} + s - i\lambda_1, 1 + s + i\lambda_2)\]
accompanying the integral that can be computed by means of the chain rule (rightmost graph in the last row of Fig. 5), or rather the orthogonality identity, giving
\[
e^{-i\pi(s+1/2+i\lambda_1^s)}\langle \Phi_{s+1/2}(\lambda_1^s)|\Phi_{s+1/2}(\lambda_1)\rangle.
\]
Combining everything together, we realize that all phases cancel out and we end up with the anticipated orthogonality relation
\[
\langle \Phi_{s+1/2}(\lambda')|\Phi_{s+1/2}(\lambda)\rangle + \langle \Phi_{s+1/2}(\lambda')|\Phi_{s+1/2}(\lambda)\rangle = (s - i\lambda_2) (s + i\lambda'_2) a_{s+1/2}(\lambda') a_{s+1/2}(\lambda)
\]
\[
\times a_{s+1/2}(\lambda') a_{s+1/2}(\lambda)
\]
\[
\times a_{s+1/2}(\lambda') a_{s+1/2}(\lambda)
\]
in terms of the individual one-particle component \[(3.4)\] inner products defined in Eq. (4.41).
Since the procedure is inductive, the above reduction procedure suffices in the proof of the generic \(N\)-site case.

5 From matrix elements to wave functions

In the previous sections, we were dealing with the matrix elements \[(3.1)\] of the flux-tube operators that diagonalize the light-cone Hamiltonian \[(1.15)\]. Let us pass to the flux-tube wave function \(\Psi_s(X;\lambda)\) of \(N\) excitations, — localized at supercoordinates \(X = (x_1, x_2, \ldots, x_N)\) where \(X_n = (x_n, \vartheta_n)\) with the bosonic component belonging to the real axis, i.e., \(\Im [x_n] = 0\), and having rapidities \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)\), — that underlines the physics of the flux-tube for scattering amplitudes. A flux-tube state \(|E(\lambda)\rangle\) can be represented in its terms as
\[
|E(\lambda)\rangle = \int_S d^N x \int d^N \vartheta \Psi_s(X;\lambda) \mathcal{O}_\Pi(X)|0\rangle,
\]
where the differential measures are
\[
d^N x = dx_1 dx_2 \ldots dx_N, \quad d^N \vartheta = d\vartheta_1 d\vartheta_2 \ldots d\vartheta_N
\]
and the integration with respect to the bosonic variables is performed over the simplex \(S = \{\infty > x_N \geq x_{N-1} \geq \cdots \geq x_1 \geq 0\}\). Notice that bosonic and fermionic content of corresponding components jump places in the wave function compared to the superfield operator, e.g., for one-particle \(\Psi_s(x_1;\lambda_1) = \Psi_{s+1/2}(x_1;\lambda_1) + \vartheta_1 \Psi_s(x_1;\lambda_1)\), where, for instance, for the \(s = 1/2\) case, the lowest and highest components are fermion and boson, respectively, i.e., opposite to the matrix element \[(3.4)\].

One can easily deduce the representation of the \(\text{sl}(2|1)\) generators on the space of the wave functions. Making use of
\[
\mathcal{G}|E(\lambda)\rangle = \int_S d^N x \int d^N \vartheta \Psi_s(X;\lambda) \sum_{n=1}^N G_n \mathcal{O}_\Pi(X)|0\rangle,
\]
where \(\mathcal{G}\) acts on the Hilbert space of the flux-tube states and \(\mathcal{G}\) being its representation on flux-tube superfields \([\mathcal{G}, \mathcal{O}_\Pi(X)] = \sum_{n=1}^N G_n \mathcal{O}_\Pi(X)\), and integrating by parts, we find the representation \(\tilde{\mathcal{G}}\) on wave functions
\[
\int_S d^N x \int d^N \vartheta \Psi_s(X;\lambda) \sum_{n=1}^N G_n \mathcal{O}_\Pi(X)|0\rangle = \int_S d^N x \int d^N \vartheta \left(\sum_{n=1}^N \tilde{G}_n \Psi_s(X;\lambda)\right) \mathcal{O}_\Pi(X)|0\rangle.
\]
Their explicit expressions read
\[
\begin{align*}
\hat{S}^- &= \partial_x, & \hat{S}^+ &= -x^2 \partial_x + x(2s - 1) - x \vartheta \partial_\vartheta, & \hat{S}^0 &= -x \partial_x + s - \frac{1}{2} - \frac{1}{2} \vartheta \partial_\vartheta, \\
\hat{B} &= -\frac{1}{2} \vartheta \partial_\vartheta - s + \frac{1}{2}, & \hat{V}^- &= \partial_\vartheta, & \hat{W}^- &= \vartheta \partial_x, & \hat{V}^+ &= x \partial_\vartheta, & \hat{W}^+ &= \vartheta (x \partial_x + 1 - 2s).
\end{align*}
\]
5.1 Wave function Hamiltonians

In this section, we will derive Hamiltonians acting on the space of wave functions. To start with, it is instructive to recall the bosonic case but we defer this discussion to Appendix C, which the reader should consult first. Below, we proceed directly to the sl(2|1) case and address the problem in two ways, first, by integration by parts and, then, using an intertwiner.

5.1.1 Integration by parts

Since the two-particle case contains all required elements, i.e., the bulk and boundary Hamiltonians,

\[ H_2 = H_{01} + H_{12}^- + H_{12}^+ + H_{1\infty}, \]  

as alluded to in Section 1.2, we will use it as a representative example. Following the same steps as above, we can calculate the Hamiltonian for the sl(2|1) wave function. The latter enters the definition of the two-particle state

\[ |E(\lambda)\rangle = \int_S d^2 x \int d^2 \vartheta \Psi_s(X; \lambda)|0\rangle. \]

Starting with the action of the Hamiltonian \( H \) on the Hilbert space of flux-tube excitations,

\[ \mathbb{H}|E(\lambda)\rangle = \int_S d^2 x \int d^2 \vartheta \Psi_s(X; \lambda) H \mathcal{O}_\Pi(X)|0\rangle = \int_S d^2 x \int d^2 \vartheta \left( \mathcal{H}_s(X; \lambda) \right) \mathcal{O}_\Pi(X)|0\rangle, \]

we immediately obtain its integral representation on the space of wave functions

\[ \mathcal{H}_2 = \mathcal{H}_{01} + \mathcal{H}_{12}^+ + \mathcal{H}_{12}^- + \mathcal{H}_{2\infty} \]

with individual components

\[ \mathcal{H}_{01} \Psi_s(X) = \int_{x_2/x_1}^{x_2/x_1} \frac{d\alpha}{\alpha - 1} \left[ \alpha^{1-2s} \Psi_s(\alpha X_1, X_2) - \frac{1}{\alpha} \Psi_s(X_1, X_2) \right], \]  

\[ \mathcal{H}_{2\infty} \Psi_s(X) = \int_{x_1/x_2}^{1} \frac{d\alpha}{1 - \alpha} \left[ \Psi_s(X_1, \alpha x_2, \vartheta_2) - \Psi_s(X_1, X_2) \right], \]  

\[ \mathcal{H}_{12}^+ \Psi_s(X) = \int_{1}^{\infty} \frac{d\alpha}{\alpha - 1} \left[ \left( \frac{\alpha x_2 - x_1}{x_2 - x_1} \right)^{-2s} \Psi_s \left( X_1, \alpha x_2, \frac{\alpha x_2 - x_1}{x_2 - x_1} \vartheta_2 - \frac{(\alpha - 1)x_2}{x_2 - x_1} \vartheta_1 \right) - \frac{1}{\alpha} \Psi_s(X_1, X_2) \right], \]  

\[ \mathcal{H}_{12}^- \Psi_s(X) = \int_{0}^{1} \frac{d\alpha}{1 - \alpha} \left[ \left( \frac{x_2 - \alpha x_1}{x_2 - x_1} \right)^{-2s} \Psi_s \left( \alpha x_1, \frac{x_2 - \alpha x_1}{x_2 - x_1} \vartheta_1 - \frac{(1 - \alpha)x_1}{x_2 - x_1} \vartheta_2, X_2 \right) - \Psi_s(X_1, X_2) \right], \]

where, for brevity, we did not display the dependence of \( \Psi_s \) on \( \lambda \).
5.1.2 Intertwiner

A generalization of the intertwiner to the sl(2|1) case is relatively straightforward. Rather than being multiplication by a function as in the bosonic case, see [11] and Appendix C, it becomes an operator in Grassmann variables. Namely, it admits the following form

\[ \mathcal{W}_N = (2s)^{-N} \int d\vartheta'_1 d\vartheta'_2 \ldots d\vartheta'_N \left( (x_1 + \vartheta'_1 \vartheta_1)(x_{21} + \vartheta'_{21} \vartheta_{21}) \ldots (x_{N,N-1} + \vartheta'_{N,N-1} \vartheta_{N,N-1}) \right)^{2s}, \]

and induces the change from the matrix element to the wave function representations

\[ \hat{H}_N \mathcal{W}_N = \mathcal{W}_N \mathcal{H}_N. \]  

(5.13)

More specifically, we deduce the following relations between the two-particle bulk and boundary Hamiltonians

\[ \hat{H}^{-1/2}_2 \mathcal{W}_2 = \mathcal{W}_2 \mathcal{H}_0, \quad \hat{H}_0 \mathcal{W}_2 = \mathcal{W}_2 \mathcal{H}^{-1/2}_1, \quad \hat{H}_2 \mathcal{W}_2 = \mathcal{W}_2 \mathcal{H}^{-1/2}_1. \]  

(5.14)

Though the individual components of the Hamiltonians transform differently under the operation of integration by parts and by means of the intertwiner, the total sums are obviously the same.

5.2 Wave functions

We can adopt the above intertwiner in order to find the form of wave functions from the eigenfunctions of matrix elements via the relation

\[ \Psi_s(X; \lambda) = \mathcal{W}_N \Phi_s(X; \lambda). \]  

(5.16)

However, as a cross check of the formalism that we developed here, it is instructive to solve for them explicitly diagonalizing the generator of conserved changes \( \hat{D}_N \). Below, we will limit ourselves to the case of one- and two-particle excitations.

For a single excitation, as a solution to the eigenvalue equation

\[ \hat{D}_1(iw) \Psi_s(X_1; \lambda_1) = (iw + i\lambda_1 + s - \frac{1}{2}) \Psi_{s+1/2}(x_1; \lambda_1) + (iw + i\lambda_1 + s) \vartheta_1 \Psi_s(x_1; \lambda_1), \]  

(5.17)

we find for the component wave functions

\[ \Psi_s(x_1; \lambda_1) = x_1^{2s-1} \Phi_s(x_1, \lambda) = x_1^{i\lambda_1+s-1/2}, \]  

\[ \Psi_{s+1/2}(x_1; \lambda_1) = (2s)^{-1} x_1^{2s} \Phi_{s+1/2}(x_1, \lambda) = (2s)^{-1} x_1^{i\lambda_1+s-1}. \]  

(5.18)

(5.19)

These expressions are in agreement with the intertwining relation (5.16) with the one-particle matrix element from Eq. (3.4).

Moving on to two excitations, the eigenfunction is decomposed in the component form as

\[ \Psi_s(X; \lambda) = \Psi_{s+1/2,s+1/2}(x) + \vartheta_1 \Psi_{s,s+1/2}(x) + \vartheta_2 \Psi_{s+1/2,s}(x) + \vartheta_1 \vartheta_2 \Psi_{ss}(x), \]  

(5.20)

and the solution to the eigenvalue equation for \( \hat{D}_2 \)

\[ \hat{D}_2(iw) \Psi_s(X; \lambda) = (iw + i\lambda_1 + s - \frac{1}{2})(iw + i\lambda_2 + s - \frac{1}{2}) \Psi_{s+1/2,s+1/2}(x; \lambda) \]  

(5.21)
Acting with \(\vartheta_1 \vartheta_2 \Psi_{s,s}(x; \lambda)\) on the wave function derived above, we find the expected eigenvalues.

Rewrite these expressions as a sum of incoming and outgoing waves with modulated profiles. For a simple use of well-known connection formulas for hypergeometric functions, allows one to generate the solutions.

\[
\Psi_{s+1/2,s+1/2}(x) = -(2s)^{-2} x_1^{i\lambda_1 + s - 1/2} x_2^{i\lambda_2 + s - 1/2} \left(1 - \frac{x_1}{x_2}\right)^{2s-1} \mathbf{2F}_1 \left(\begin{array}{c} s + \frac{1}{2} + i\lambda_1, s + \frac{1}{2} - i\lambda_2 \\ s \end{array} \right) \left(1 - \frac{x_1}{x_2}\right), \quad (5.22)
\]

\[
\Psi_{ss}(z) = x_1^{i\lambda_1 + s - 1} x_2^{i\lambda_2 + s - 1} \left(1 - \frac{x_1}{x_2}\right)^{2s-1} \mathbf{2F}_1 \left(\begin{array}{c} s + i\lambda_1, s - i\lambda_2 \\ s \end{array} \right) \left(1 - \frac{x_1}{x_2}\right), \quad (5.23)
\]

\[
\Psi_{s,s+1/2}(x) = x_1^{i\lambda_1 + s - 1/2} x_2^{i\lambda_2 + s - 1} \left(1 - \frac{x_1}{x_2}\right)^{2s-1} \mathbf{2F}_1 \left(\begin{array}{c} s + \frac{1}{2} + i\lambda_1, s - i\lambda_2 \\ s \end{array} \right) \left(1 - \frac{x_1}{x_2}\right), \quad (5.24)
\]

\[
\Psi_{s+1/2,s}(x) = -x_1^{i\lambda_2 + s - 3/2} x_2^{i\lambda_1 + s - 3/2} \left(1 - \frac{x_1}{x_2}\right)^{2s-1} \mathbf{2F}_1 \left(\begin{array}{c} s + \frac{1}{2} - i\lambda_1, s + i\lambda_2 \\ s \end{array} \right) \left(1 - \frac{x_1}{x_2}\right). \quad (5.25)
\]

A simple use of well-known connection formulas for hypergeometric functions, allows one to rewrite these expressions as a sum of incoming and outgoing waves with modulated profiles. For \(s = 1/2\), we reproduce results also obtained in Ref. [35] found by diagonalizing \(\Omega_2\) defined in Eq. (2.23). Acting with \(\vartheta\) on the wave function derived above, we find the expected eigenvalues.

\[
\hat{H}\Psi_{s}(X; \lambda) = (E_{s+1/2}(\lambda_1) + E_{s+1/2}(\lambda_2)) \Psi_{s+1/2,s+1/2}(x; \lambda) + \vartheta_1 \vartheta_2 (E_s(\lambda_1) + E_s(\lambda_2)) \Psi_{ss}(x; \lambda)
\]

\[
+ (E_{s+1/2}(\lambda_1) + E_s(\lambda_2)) \left[\vartheta_1 \Psi_{s,s+1/2}(x; \lambda) + \vartheta_2 \Psi_{s+1/2,s}(x; \lambda)\right]. \quad (5.26)
\]

The intertwiner involves all components at a given order in Grassmann decomposition,

\[
\mathcal{W}_2 \Phi_2^{[g]} = \Psi_2^{[g]}, \quad (5.27)
\]

where \(\Psi^{[g]}\) with \(N \geq g \geq 0\) is defined in the same fashion as for the matrix element. Using the two-particle \(\mathcal{W}_2\), we can verify the above formulas by means of well-known relations between hypergeometric functions.

\[
\Psi_{ss}(x; \lambda) = (x_1 x_2)^{2s-1} \Phi_{ss}(x; \lambda), \quad (5.28)
\]

\[
\Psi_{s+1/2,s+1/2}(x; \lambda) = -(2s)^{-2} (x_1 x_2)^{2s} \Phi_{s+1/2,s+1/2}(x; \lambda), \quad (5.29)
\]

\[
\Psi_{s,s+1/2}(x; \lambda) = (2s)^{-1} (x_1 x_2)^{2s-1} [x_1 \Phi_{s,s+1/2} + x_2 \Phi_{s+1/2,s}](x; \lambda), \quad (5.30)
\]

\[
\Psi_{s+1/2,s}(x; \lambda) = -(2s)^{-1} (x_1 x_2)^{2s-1} [x_1 \Phi_{s,s+1/2} + x_2 \Phi_{s+1/2,s}](x; \lambda). \quad (5.31)
\]

These indeed coincide with Eqs. (3.13) – (3.16).

### 6 Inner product on the line

Making use of the above properties of the intertwining operator, we can introduce the following inner product for the boundary value of the matrix element eigenfunctions on the real line.

\[
\langle \Phi' | \Phi \rangle \equiv \int_S d^N x \int d^N \vartheta (\Phi'(X))^* \mathcal{W}_N \Phi(X), \quad (6.1)
\]
where $\mathbf{X} = (X_1, \ldots, X_N)$ with $X_n = (x_n, \vartheta_n)$. Employing Eqs. (5.14) and (5.7), it is straightforward to verify that the Hamiltonian is hermitian with respect to this inner product,

$$ (\Phi'|\mathcal{H}_N\Phi) = (\mathcal{H}_N\Phi'|\Phi). \quad (6.2) $$

We can relate the above inner product to the one in the upper half-plane of the complex plane. We substitute in form replying on the defining property of the reproducing kernel

$$ \Phi(\mathbf{X}) = \int [D^N\mathbf{Z}]_s K_j(\mathbf{X}, \mathbf{Z}^*) \Phi(\mathbf{Z}), \quad (6.3) $$

where $\mathbf{X} = (X_1, \ldots, X_N)$ belongs to the real axis while $\mathbf{Z} = (Z_1, \ldots, Z_N)$ with $Z_n = (z_n, \theta_n)$ is being complex. The measure and reproducing kernels are

$$ [D^N\mathbf{Z}]_s = \prod_{n=1}^N [DZ_n]_s, \quad K_s(\mathbf{X}, \mathbf{Z}^*) = \prod_{n=1}^N K_s(X_n, Z_n^*), \quad (6.4) $$

with

$$ K_s(X, Z^*) = (x - z^* + \vartheta \theta^*)^{-2s}. \quad (6.5) $$

Then we deduce the relation

$$ (\Phi'|\Phi) = \langle \Phi'|\mathcal{X}_N|\Phi \rangle, \quad (6.6) $$

where the operator $\mathcal{X}_N$ is determined by its integral kernel

$$ \mathcal{X}_N\Phi(\mathbf{Z}) = \int [D^N\mathbf{W}]_s (K_s(\mathbf{Z})|K_s(\mathbf{W}^*)) \Phi(\mathbf{W}) \quad (6.7) $$

$$ = \int s d^N x \int d^N \vartheta K_s(Z, X) W_N \Phi(X). \quad (6.8) $$

Finally, using the properties

$$ (S^0_Z + B_Z)K_s(Z, X) = -(S^0_X - B_X)K_s(Z, X), \quad S^+_Z K_s(Z, X) = -S^+_X K_s(Z, X), \quad \quad (6.9) $$

$$ W^+_Z K_s(Z, X) = -V^+_X K_s(Z, X), \quad V^-_Z K_s(Z, X) = -W^-_X K_s(Z, X), \quad \quad (6.10) $$

one can prove commutativity with $D_N$

$$ [\mathcal{X}_N, D_N] = 0. \quad (6.11) $$

### 6.1 Eigenvalues of $\mathcal{X}$

Let us turn to evaluation of the eigenvalues of the operator $\mathcal{X}_N$ on the eigenfunctions $\Phi_s$. 

[27]
6.1.1 One excitation

For the eigenfunction of one-particle matrix element, we find

\[ \mathcal{X}_1 \Phi_s(Z_1; \lambda_1) = \int dx_1 \int d\vartheta_1 K_j(Z_1, X_1) \mathcal{W}_1 \Phi_s(X_1; \lambda_1), \] (6.12)

where, see Eq. (5.13),

\[ \mathcal{W}_1 \Phi_s(X_1) = \int d\vartheta'_1 (x_1 + \vartheta'_1 \vartheta_1) \Phi_s(x_1, \vartheta'_1). \] (6.13)

Substituting Eq. (3.4), we find

\[ \mathcal{X}_1 \Phi_s(Z_1; \lambda_1) = \mathcal{X}_s(\lambda_1) \Phi_s(x_1; \lambda_1) + \mathcal{X}_{s+1/2}(\lambda_1) \vartheta_1 \Phi_{s+1/2}(x_1; \lambda_1), \] (6.14)

with the eigenvalues arising from the evaluation of the integral

\[ \mathcal{X}_s(\lambda_1) = \int_0^\infty dy (y - 1)^{-2s} y^{\lambda_1+s-1} = e^{-\pi(\lambda_1+s)} \mu_s(\lambda_1), \] (6.15)

where the spin-s flux-tube measure was introduced in Eq. (4.42).

6.1.2 Two excitations and more

In the two-particle case, the action of \( \mathcal{X}_2 \) reads explicitly

\[ \mathcal{X}_2 \Phi_s(Z; \lambda) = \int_S d^2x \int d^2\vartheta (z_1 - x_1 + \vartheta_1 \vartheta_1)^{-2s} (z_2 - x_2 + \vartheta_2 \vartheta_2)^{-2s} \]
\[ \times \int d^2\vartheta'(x_1 + \vartheta'_1 \vartheta_1)^{2s}(x_{21} + \vartheta'_{21} \vartheta_{21})^{2s} \Phi_s(x_1, \vartheta'_1, x_2, \vartheta'_2; \lambda). \] (6.16)

For the lowest and highest Grassmann components, i.e.,

\[ \Phi_2^{[0]}(X; \lambda) = \Phi_{ss}(X; \lambda), \quad \Phi_2^{[2]}(X; \lambda) = \vartheta_1 \vartheta_2 \Phi_{s+1/2,s+1/2}(X; \lambda), \] (6.17)

according to the terminology of Section 4 we get the anticipated result as in the purely bosonic model \[ \Pi \] for different values of the conformal spin,

\[ \mathcal{X}_2 \Phi_2^{[0]}(Z; \lambda) = \mathcal{X}_s(\lambda_1) \mathcal{X}_s(\lambda_2) \Phi_2^{[0]}(Z; \lambda), \quad \mathcal{X}_2 \Phi_2^{[2]}(Z; \lambda) = \mathcal{X}_{s+1/2}(\lambda_1) \mathcal{X}_{s+1/2}(\lambda_2) \Phi_2^{[2]}(Z; \lambda). \] (6.18)

These results can be easily found going to the asymptotic region \( x_2 \gg x_1 \) and making use of the asymptotic form of the eigenfunctions \( \Phi_{ss}(x; \lambda) \simeq x_1^{i\lambda_1-s} x_2^{i\lambda_2-s} \) and the same for \( \Phi_{s+1/2,s+1/2} \) with an obvious shift of the spin.

For the mixed components,

\[ \Phi_2^{[1]}(X; \lambda) = \vartheta_1 \Phi_{s+1/2,s}(x; \lambda) + \vartheta_2 \Phi_{s,s+1/2}(x; \lambda), \] (6.19)

the situation is trickier and we want to perform the diagonalization exactly. Namely, after performing the Grassmann integration we obtain

\[ \mathcal{X}_2 \Phi_2^{[1]}(Z; \lambda) = \vartheta_1 \int_S d^2x (z_1 - x_1)^{-2s-1} (z_2 - x_2)^{-2s} \Psi_{s+1/2,s}(x; \lambda) \] (6.20)
where the integrand is given in terms of two-particle wave functions (5.30) and (5.31). A calculation, following the steps outlined in Appendix C.2 of Ref. [11], demonstrates that

$$\int_S d^2x (z_1 - x_1)^{-2s-1}(z_2 - x_2)^{-2s-1}\Psi_{s,s+1/2}(x; \lambda) = \mathcal{X}_{s+1/2,s}^{(1)}(\lambda)\Phi_{s,s+1/2}(z_1, z_2),$$

(6.22)

and

$$\int_S d^2x (z_1 - x_1)^{-2s-1}(z_2 - x_2)^{-2s-1}\Psi_{s,s+1/2}(x_1, x_2) = \mathcal{X}_{s+1/2,s}^{(2)}(\lambda)\Phi_{s,s+1/2}(z_1, z_2),$$

(6.23)

so that $\Phi_2^{[g]}(Z; \lambda)$ is an eigenfunction of $\mathcal{X}_2$

$$\mathcal{X}_2\Phi_2^{[g]}(Z; \lambda) = \mathcal{X}_{s+1/2,s}^{(1)}(\lambda)\Phi_2^{[g]}(Z; \lambda),$$

(6.24)

with the same eigenvalue for its both components

$$\mathcal{X}_{s+1/2,s}(\lambda) = \mathcal{X}_{s+1/2}(\lambda_1)\mathcal{X}_s(\lambda_2).$$

(6.25)

It is expressed in terms of the one-particle eigenvalues (6.15).

This result immediately generalizes to any $N$. For the Grassmann degree-$g$ $N$-particle case, we have

$$\mathcal{X}_N^{[g]} \Phi_N^{[g]}(Z; \lambda) = \left(\prod_{n=1}^g \mathcal{X}_{s+1/2}(\lambda_n)\right) \left(\prod_{n=g+1}^N \mathcal{X}_s(\lambda_n)\right) \Phi_N^{[g]}(Z; \lambda),$$

(6.26)

again observing factorization of multiparticle eigenvalues.

### 7 Square transitions

The $N$-particle wave functions have to be orthogonal with respect to the so-called square transitions, i.e., when both the incoming and outgoing states are in the same conformal frame. Namely, we define

$$B(\lambda|\lambda') \equiv \langle E(\lambda')|E(\lambda) \rangle$$

$$= \int d^N X' \int d^N X (\Psi_s(X'; \lambda'))^* G(X', X) \Psi_s(X; \lambda),$$

(7.1)

where $G$ is a product

$$G(X', X) = \prod_{n=1}^N G(X'_n, X_n)$$

(7.2)

of supersymmetric propagators on the real axis

$$G(X'_n, X_n) = (x_n + x'_n + \theta_n\theta'_n)^{-2s}.$$
We will demonstrate its relation to the inner product in the upper half of the complex plane of the matrix element eigenfunctions for the two-to-two transition. As before, since components of different Grassmann degree do not talk to each other, we would like to keep track of these by using a marker variable as in Section 4.5. So the decomposition of the supersquare transition into three independent Grassmann components is

\[
B(\lambda|\lambda') = B_{s+1/2,s+1/2|s+1/2,s+1/2}(\lambda|\lambda') + \varepsilon\varepsilon'B_{s,s+1/2|s+1/2,ss+1/2}(\lambda|\lambda') + (\varepsilon\varepsilon')^2 B_{ss|ss}(\lambda|\lambda').
\] (7.4)

These are related via the equations

\[
B_{s+1/2,s+1/2|s+1/2,s+1/2} = B_{s+1/2,s+1/2}, \quad B_{s,s+1/2|s+1/2,ss+1/2} = B_{s,s+1/2} + B_{s+1/2,s}, \quad B_{ss|ss} = B_{ss},
\] (7.5)

to individual integrals \( B_{s1,s2} \) involving spin-\((s_1, s_2)\) wave functions connected with propagators from top to bottom of the square

\[
B_{s1,s2}(\lambda|\lambda') = \int_{S'} d^2x' \int_S d^2x \frac{\langle \Psi_{s1,s2}(x';\lambda') | \Psi_{s1,s2}(x;\lambda) \rangle}{(x_1 + x_1)^{2s_1}(x_2' + x_2)^{2s_2}}.
\] (7.6)

These integrals were computed in Section 6.1.2 with the result

\[
\int_S d^2x \Psi_{s1,s2}(x;\lambda)(x_1' + x_1)^{-2s_1}(x_2' + x_2)^{-2s_2} = \mathcal{X}_{s1,s2}(\lambda)\Phi_{s1,s2}(x';\lambda),
\] (7.7)

where \( \mathcal{X}_{s1,s2}(\lambda) \) is the eigenvalue of the operator \( \mathcal{X}_2 \), such that

\[
B_{s1,s2}(\lambda|\lambda') \equiv \langle \Phi_{s1,s2}(\lambda') | \Phi_{s1,s2}(\lambda) \rangle = \mathcal{X}_{s1,s2}(\lambda) \int_{S'} d^2x' \langle \Psi_{s1,s2}(x';\lambda') | \Phi_{s1,s2}(x';\lambda) \rangle.
\] (7.8)

Since the wave functions \( \Psi \) of the top and bottom components are related to the matrix elements \( \Phi \) by means of a multiplicative factor of bosonic coordinates, we recognize in the right-hand side of the above relation, the inner product in the upper half plane for \( \Phi \). Their orthogonality was demonstrated in Ref. [11], as well as was recapitulated above in Section 4.5.3. Only the mixed component require special attention. Starting from the relation between the inner products (6.6), we can extract the mixed components and find

\[
\int_S d^2x \left[ \langle \Phi_{s+1/2,s}(x;\lambda') | \Phi_{s+1/2,s}(x;\lambda) \rangle + \langle \Phi_{s,s+1/2}(x;\lambda') | \Phi_{s,s+1/2}(x;\lambda) \rangle \right]
= \int_S d^2x \left[ \langle \Psi_{s+1/2,s}(x;\lambda') | \Phi_{s+1/2,s}(x;\lambda) \rangle + \langle \Psi_{s,s+1/2}(x;\lambda') | \Phi_{s,s+1/2}(x;\lambda) \rangle \right]
= \mathcal{X}_{s+1/2,s}(\lambda) \left\{ \langle \Phi_{s+1/2,s}(\lambda') | \Phi_{s+1/2,s}(\lambda) \rangle + \langle \Phi_{s,s+1/2}(\lambda') | \Phi_{s,s+1/2}(\lambda) \rangle \right\}.
\] (7.9)

In the right-hand side of the above equation, we used the eigenvalue equation (6.24), while in the left-hand side, we employed the relation between the matrix elements and wave functions (5.30) and (5.31). The right-hand side of the above equation was calculated in Section 4.5.3 and shows orthogonality of wave functions with respect to the square transitions.

Generalization to \( N \)-particle square transitions goes along the same lines making use of results of the previous section.
8 Pentagon transitions

The $N$-particle super-wave functions define the pentagon transitions, i.e., the building blocks of the Operator Product Expansion for scattering amplitudes as was reviewed in the Introduction. Namely

$$
P(\lambda|\lambda') \equiv \langle E(\lambda')|P|E(\lambda)\rangle = \int_{S'} d^N X' \int_S d^N X \left( \Psi'_s(X'; \lambda') \right)^* G(X', X) \Psi_s(X; \lambda),
$$

(8.1)

where compared to the just discussed box transitions, the wave function in the final state is in a different conformal frame (to be specified later) compared to the initial one. The reduction of the $N$-particle pentagon to $N-1$ pentagon goes through the same chain of transformation as the inductive proof for the orthogonality condition. Thus we will demonstrate it for first non-trivial case, i.e., two-site wave functions.

In complete analogy with the above consideration, one finds the following component expansion for the two-to-two pentagon transition

$$
P(\lambda|\lambda') = P_{s+1/2,s+1/2|s+1/2,s+1/2}(\lambda|\lambda') + \varepsilon \varepsilon' P_{s,s+1/2|s,s+1/2}(\lambda|\lambda') + (\varepsilon \varepsilon')^2 P_{ss|ss}(\lambda|\lambda'),
$$

(8.2)

where we adopted a notation $P_{s_1 s_2 | s'_1 s'_2}$ used in the pentagon approach [10, 11, 12, 13, 14, 15, 16, 17, 18, 20, 21, 22, 23] for particles with spins $(s_1, s_2)$ undergoing a transition to particles with spins $(s'_1, s'_2)$. These are related via the equations

$$
P_{s+1/2,s+1/2|s+1/2,s+1/2} = P_{s+1/2,s+1/2}, \quad P_{s+1/2,s|s+1/2,s} = P_{s,s+1/2} + P_{s+1/2,s}, \quad P_{ss|ss} = P_{ss},
$$

(8.3)

to integrals involving an overlap of wave functions in different conformal frames

$$
P_{s_1 s_2}(\lambda|\lambda') = \int_{S'} d^2 x' \int_S d^2 x \frac{\left( \Psi'_{s_1 s_2}(x'; \lambda') \right)^* \Psi_{s_1 s_2}(x; \lambda)}{(x_1 + x'_1)^{2s_1}(x_2 + x'_2)^{2s_2}}.
$$

(8.4)

Before we move on to its calculation, we will take a detour by calculating the inverse wave functions first.

8.1 Inversion of wave functions

As we will show in the next subsection, the pentagon transitions can be reduced to the inner product of matrix elements inverted with respect to the origin and one of them shifted away from it. Thus, we will introduce the operation of inversion

$$
z \rightarrow z' = 1/z,
$$

(8.5)

and construct the resulting wave functions. To start with, the spin-$s$ measure changes according to the rule

$$
[Dz]_s \rightarrow [Dz']_s = (zz^*)^{-2s}[Dz]_s.
$$

(8.6)
Since the same-flavor wave functions and corresponding pentagons were already discussed in Ref. [11], we will not repeat it here. Thus, we address only the mixed-flavor case. For \( \Phi_{s+1/2,s} \) in Eq. (4.47), we have

\[
\Phi_{s+1/2,s}^{I}(z; \lambda) \equiv e^{i\pi(2s+1)}z_1^{-2s-1}z_2^{-2s} \Phi_{s+1/2,s}(z'; \lambda) \\
= (s + i\lambda_2) \int [Dz]_{s+1/2} z^{-i\lambda_1-s-1/2}(z_1 - z^*)^{-i\lambda_2-s-1}(z_2 - z^*)^{i\lambda_2-s}z_2^{-i\lambda_2-s}, \quad (8.7)
\]

where \( z_1^{-2s-1}z_2^{-2s} \) is the scaling factor with exponents proportional to the conformal weights of the points \( z_1 \) and \( z_2 \) and overall phase factor was introduced to get rid of the one emerging from the inversion. Similarly we find for (4.48),

\[
\Phi_{s,s+1/2}^{s}(z; \lambda) \equiv e^{i\pi(2s+1)}z_1^{-2s-1}z_2^{-2s-1} \Phi_{s,s+1/2}(z'; \lambda) \\
= (s - i\lambda_2) \int [Dz]_{s+1/2} z^{-i\lambda_1-s-1/2}(z_1 - z^*)^{-i\lambda_2-s-1}(z_2 - z^*)^{i\lambda_2-s}z_2^{-i\lambda_2-s}. \quad (8.8)
\]

The graphical representation for \( \Phi_{s+1/2,s}^{I} \) is shown in Fig. 2 (on the right panel).

### 8.2 Mixed pentagons

Let us calculate the pentagon transitions corresponding to the wave function \( \Psi_{s+1/2,s} \) and \( \Psi_{s,s+1/2} \). They are

\[
P_{s,s+1/2} = \int_{S'} d^2x' \int_{S} d^2x \frac{\Psi_{s,s+1/2}(x'; \lambda')^* \Psi_{s,s+1/2}(x; \lambda)}{(x_1 + x_1')^{2s}(x_2 + x_2')^{2s+1}}, \quad (8.9)
\]

\[
P_{s+1/2,s} = \int_{S'} d^2x' \int_{S} d^2x \frac{\Psi_{s+1/2,s}(x'; \lambda')^* \Psi_{s+1/2,s}(x; \lambda)}{(x_1 + x_1')^{2s+1}(x_2 + x_2')^{2s}}, \quad (8.10)
\]

where the wave functions are connected point-by-point with spin-s propagators. They are related to the matrix elements via Eqs. (5.30) and (5.31). Notice that the wave function in the out-state is in a different conformal frame with respect to the incoming ones, i.e.,

\[
\Psi_{s_1s_2}^{s}(x'; \lambda) = \left( \frac{\partial x''}{\partial x'} \right)^{1-s_1} \left( \frac{\partial x''}{\partial x'} \right)^{1-s_2} \Psi_{s_1s_2}(x''; \lambda), \quad x'' = \frac{x'}{1 - x'}, \quad (8.11)
\]

Making use of the integrals displayed in Eqs. (6.22) and (6.23), we can rewrite the above pentagons as

\[
P_{s,s+1/2}(\lambda'| \lambda) = \chi_{s+1/2,s}(\lambda) \int_{S'} d^2x' \frac{(\Psi_{s,s+1/2}^{s}(x'; \lambda'))^* \Phi_{s,s+1/2}(x; \lambda)}{(x_1 + x_1')^{2s}(x_2 + x_2')^{2s+1}}, \quad (8.12)
\]

\[
P_{s+1/2,s}(\lambda'| \lambda) = \chi_{s+1/2,s}(\lambda) \int_{S'} d^2x' \frac{(\Psi_{s+1/2,s}(x'; \lambda'))^* \Phi_{s+1/2,s}(x; \lambda)}{(x_1 + x_1')^{2s+1}(x_2 + x_2')^{2s}}, \quad (8.13)
\]

It is important to realize that individually we cannot relate these integrals to the inner product on the line since the intertwiner \( \mathcal{W} \) (see Eq. (5.13)) acts on the superwave function \( \Phi_2 = \theta_1 \Phi_{s+1/2,s} + \theta_2 \Phi_{s,s+1/2} \), not its separate components \( \Phi_{s_1s_2} \) as shown in Eq. (5.27).

Further, to relate the product on the line to the one in upper half-plane, one has to take into account that the operator \( \lambda \) has well-defined eigenvalue again only on the total mixed

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superfunction \([6.24]\), not on its components in the Grassmann decomposition. This immediately implies that (as shown in Section 7)

\[
\int_S d^2x \left[ (\Psi'_{s+1/2,s}(x; \lambda'))^* \Phi_{s+1/2,s}(x; \lambda) + (\Psi'_{s,s+1/2}(x; \lambda'))^* \Phi_{s,s+1/2}(x; \lambda) \right]
\]

\[
= \mathcal{X}_{s+1/2} \left[ (\Phi'_{s+1/2,s}(x; \lambda'))\Phi_{s+1/2,s}(x; \lambda) + (\Phi'_{s,s+1/2}(x; \lambda'))\Phi_{s,s+1/2}(x; \lambda) \right].
\]

(8.14)

Therefore, we can relate the mixed pentagon to the sum of the inner products of the matrix element eigenfunctions in the upper half-plane of the complex plane, i.e.,

\[
P_{s+1/2,s}(\lambda'|\lambda') + P_{s,s+1/2}(\lambda'|\lambda') = \langle \Phi'_{s+1/2,s}(\lambda')|\Phi_{s+1/2,s}(\lambda) \rangle + \langle \Phi'_{s,s+1/2}(\lambda')|\Phi_{s,s+1/2}(\lambda) \rangle.
\]

(8.15)

The latter can be rewritten as

\[
\langle \Phi'_{s+1/2,s}(\lambda')|\Phi_{s+1/2,s}(\lambda) \rangle + \langle \Phi'_{s,s+1/2}(\lambda')|\Phi_{s,s+1/2}(\lambda) \rangle = \langle \Phi_{s+1/2,s}(\lambda'; 0)|\Phi_{s+1/2,s}(\lambda; 1) \rangle + \langle \Phi_{s,s+1/2}(\lambda'; 0)|\Phi_{s,s+1/2}(\lambda; 1) \rangle,
\]

in terms of the inverted eigenfunctions at position \(-\gamma\),

\[
\Phi'_{s+1/2,s}(\lambda; \gamma) = (s + i\lambda_2)
\]

\[
\times \int [Dz]_{s+1/2} (z + \gamma)^{-i\lambda_1 - s - 1/2}(z_1 - z^*)^{-i\lambda_2 - s - 1}(z_2 - z^*)^{i\lambda_2 - s}(z_2 + \gamma)^{-i\lambda_2 - s},
\]

(8.17)

\[
\Phi_{s,s+1/2}(\lambda; \gamma) = (s - i\lambda_2)
\]

\[
\times \int [Dz]_{s+1/2} (z + \gamma)^{-i\lambda_1 - s - 1/2}(z_1 - z^*)^{-i\lambda_2 - s - 1}(z_2 - z^*)^{i\lambda_2 - s - 1}(z_2 + \gamma)^{-i\lambda_2 - s}.
\]

(8.18)

Let us calculate the transition

\[
T_{s+1/2,s}^{i}(\lambda'|\lambda) = \langle \Phi_{s+1/2,s}(\lambda'; 0)|\Phi_{s+1/2,s}(\lambda; \gamma) \rangle + \langle \Phi_{s,s+1/2}(\lambda'; 0)|\Phi_{s,s+1/2}(\lambda; \gamma) \rangle,
\]

(8.19)

for a generic values of \(\gamma\). The calculation of the inner product in the right-hand side of this equation goes along the same lines as the one for the evaluation of the inner product discussed in Section 4.5.3 this time, only from left to right. The first step, shown in the first line of Fig. 6 consists in the integration over the vertex at \(z_1\) making use of the chain rule \((\Lambda.3)\) and pulling out the overall factor

\[
e^{-i\pi s}a_s(s + i\lambda_2, s - i\lambda'_2)
\]

from both contributions, which yields the relative coefficient for the first graph

\[
e^{-i\pi/2}(s + i\lambda_2)(s - i\lambda'_2)a_{s+1/2}(1 + s + i\lambda_2, 1 + s - i\lambda'_2) = 2s(\lambda_2 - \lambda'_2).
\]

(8.20)

as shown in Fig. 6 (middle row diagrams).

The subsequent step requires the application of the superpermutation identity. For the case at hand, we introduce the following supercoordinates in Eq. \((4.43)\) to do the job

\[
Z_1 = (z, \theta), \quad Z_2 = (-\gamma, 0), \quad Z'_1 = (z', \theta'), \quad Z'_2 = (0, 0),
\]

(8.21)
Figure 6: Procedure for the calculation of the mixed pentagons (8.16).
and rapidities \( \lambda' = \lambda_2', \lambda = \lambda_2 \). Keeping the \( \theta \theta^* \) term in the Grassmann expansion on both sides of the permutation identity, we find the relation

\[
(z' - z^*)^{i\lambda_2 - i\lambda_2}(s - i\lambda_2)(s + i\lambda_2')X_{s+1/2}(z, -\gamma; \lambda_2|z', 0; \lambda_2') - 2is(i\lambda_2' - i\lambda_2)(z' - z^*)^{i\lambda_2 - i\lambda_2 - 1}X_s(z, -\gamma; \lambda_2|z', 0; \lambda_2)
= \gamma^{i\lambda_2 - i\lambda_2}(s - i\lambda_2')(s + i\lambda_2)X_{s+1/2}(z, -\gamma; \lambda_2'|z', 0; \lambda_2),
\] (8.22)

between crosses with spin-\( s \) and spin-(\( s + \frac{1}{2} \)) measures,

\[
X_s(z, \gamma; \lambda_2|z', 0; \lambda_2') = \int [Dy]_s(y - z^*)^{i\lambda_2 - s}(y + \gamma)^{-i\lambda_2 - s}(z' - y^*)^{-i\lambda_2 - s}(-y^*)^{i\lambda_2 - s},
\] (8.23)

\[
X_{s+1/2}(z, \gamma; \lambda_2|z', 0; \lambda_2') = \int [Dy]_{s+1/2}(y - z^*)^{i\lambda_2 - s}(y + \gamma)^{-i\lambda_2 - s}(z' - y^*)^{-i\lambda_2 - s}(-y^*)^{i\lambda_2 - s}.
\] (8.24)

Finally, using the chain rule three times, we acquire factors

\[
e^{-i\pi(2s+1)}a_{s+1/2}(\frac{1}{2} + s + i\lambda_1, 1 + s - i\lambda_2')a_{s+1/2}(\frac{1}{2} + s - i\lambda_1', 1 + s + i\lambda_2)
\] (8.25)

and

\[
e^{-i\pi(s+1/2)}a_{s+1/2}(\frac{1}{2} + s + i\lambda_1, \frac{1}{2} + s - i\lambda_1'),
\] (8.26)

from the steps shown in Fig. 6 in the left and right panels of the last row graphs, respectively.

Assembling everything together (along with phases that emerge from conjugated pyramid), we find

\[
T_{s+1/2,s}(\lambda|\lambda') = e^{-\pi\sum_{n=1}^{2}\lambda_n}e^{i\sum_{n=1}^{2}(\lambda_n - \lambda_n')}(s + i\lambda_2)(s - i\lambda_2')a_{s+1/2}(\frac{1}{2} + s + i\lambda_1, \frac{1}{2} + s - i\lambda_1')
\times a_s(s + i\lambda_2, s - i\lambda_2')a_{s+1/2}(\frac{1}{2} + s + i\lambda_1, 1 + s - i\lambda_2')a_{s+1/2}(\frac{1}{2} + s - i\lambda_1', 1 + s + i\lambda_2).
\] (8.27)

This relation implies factorizable structure of multiparticle pentagons, i.e., two-to-two in the current case,

\[
P_{s+1/2,s|s+1/2,s}(\lambda|\lambda') = e^{-\pi\sum_{n=1}^{2}\lambda_n}
\times P_{s+1/2,s|s+1/2,s}(\lambda_1|\lambda_1')P_{s+1/2,s}(\lambda_1|\lambda_2')P_{s+1/2,s}(\lambda_2|\lambda_1')P_{s|s}(\lambda_2|\lambda_2'),
\] (8.28)

in terms of one-particle pentagon transitions

\[
P_{s|s}(\lambda|\lambda') = \frac{\Gamma(i\lambda - i\lambda')\Gamma(2s)}{\Gamma(s + i\lambda)\Gamma(s - i\lambda')},
\] (8.29)

\[
P_{s+1/2|s+1/2}(\lambda|\lambda') = \frac{\Gamma(i\lambda - i\lambda')\Gamma(2s + 1)}{\Gamma(s + \frac{1}{2} + i\lambda)\Gamma(s + \frac{1}{2} - i\lambda')},
\] (8.30)

\[
P_{s|s+1/2}(\lambda|\lambda') = \frac{\Gamma(\frac{1}{2} + i\lambda - i\lambda')\Gamma(2s + 1)}{\Gamma(s + i\lambda)\Gamma(s + \frac{1}{2} - i\lambda')}.
\] (8.31)

Finally \( P_{s+1/2,s}(\lambda|\lambda') = P_{s+1/2,s}(-\lambda'|-\lambda) \). Had we chosen the normalization of the wave functions \( \Psi_{s,s+1/2} \) and \( \Psi_{s+1/2,s} \) according to the coordinate Bethe Anzats, such that the asymptotic incoming wave come with a unit amplitude, we would cancel the prefactor \( e^{-\pi\sum_{n=1}^{2}\lambda_n} \) in the right-hand
side of the above relation as well as recover pentagons $P_{s+1/2}(s)\langle \lambda'_1 | \lambda'_2 \rangle P_{s+1/2}(\lambda_2 | \lambda_1)$ entering the denominator of the transitional factorized form of the pentagons, as was shown for bosonic case in Ref. [11]. We will not do it here though, since the above form already proves the factorized form of multiparticle pentagons [1,16]. Generalization to arbitrary $N$ is straightforward since the procedure is inductive.

9 Conclusions

In this paper we solved an open superspin chain model that describes minimally supersymmetric sectors of the $\mathcal{N} = 4$ flux tube. Depending on the conformal spin assignment for the lowest component of the supermultiplet of flux-tube fields, it encodes either hole-fermion or fermion-gluon excitations. The bulk interactions between the adjacent superfields building up the light-cone operators inherit the sl(2|1) invariance of four-dimensional theory, however, the presence of the boundary breaks it down to the diagonal subgroup. Using the factorized R-matrix structure of the sl(2|1) symmetric spin chain, we constructed the eigenfunctions of the model, analytically continued to the upper half of the complex plane, in the form of multiple integrals which admit an intuitive Feynman graph representation. The latter was indispensable for analytical proof of their orthogonality. The same framework was applied to calculate the so-called pentagon transitions between the states of the flux-tube in different conformal planes. The latter serve as building blocks in the framework of the Operator Product Expansion to null polygonal Wilson superloops. The outcome of this analysis revealed factorizable structure of the dynamical part of multiparticle pentagons in terms of single-particle ones as was already extensively used in the past.

Our consideration can be extended to include all propagating modes of the maximally supersymmetric Yang-Mills theory and encode them in the noncompact sl(2|4) superchain. From the technical point of view the changes would appear to be minimal: one would have to replace single Grassmann variables $\theta$ by an SU(4) vector $\theta^A$ and any product of conjugate ones by the sum, $\theta\theta^* \rightarrow \theta^A\theta^*_A$. The question remains open however whether this construction will be able to encode and unravel the matrix part of the pentagon transitions.

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A Feynman graph primer

In this appendix we provide a reminder of Feynman graphs used in the representation of eigenfunctions and some elementary operations on them. The main building block in the construction is the propagator that receives Schwinger parametrization valid in the upper half of the complex plane.
Figure 7: Graphical representation of the propagator (A.1) (left panel) and the chain rule (A.3) (right panel).

The propagators obey the following set of rules useful in proofs of various relations.

- Chain rule:

\[
\int [Dz]s(w - z^*)^{-2s}e^{-i\pi s}\Phi(z) = e^{-i\pi s}\Phi(w). \tag{A.2}
\]

This is easily verified by applying the above chain rules from right to left and Eq. (A.1).

- Orthogonality identity:

\[
a_s(s - i\lambda, s + i\lambda') = \frac{2\pi \Gamma(2s)}{\Gamma(s - i\lambda)\Gamma(s + i\lambda')}\delta(\lambda - \lambda'). \tag{A.5}
\]

This is easily verified by applying the above chain rules from right to left and Eq. (A.1).

- Permutation identity:

\[
(z_1' - z_1^*)^{i\lambda'} - i\lambda X(z; \lambda|z'; \lambda') = X(z; \lambda'|z'; \lambda) (z_2' - z_2^*)^{i\lambda - i\lambda'}, \tag{A.6}
\]

with the bosonic cross

\[
X(z, \lambda|z', \lambda') \equiv \int [Dw]s(w - z_1^*)^{i\lambda - s}(w - z_2^*)^{-i\lambda - s}(z_1' - w^*)^{-i\lambda' - s}(z_2' - w^*)^{i\lambda' - s}, \tag{A.7}
\]

where \(z = (z_1, z_2)\) and \(z' = (z_1', z_2')\). The proof for this relation can be found in Appendix B of [11] and references cited therein.
Figure 8: Proof of the orthogonality of the mixed inner product \((B.1)\) using inversion.

B Alternative proof of orthogonality

As we pointed out in the main text, instead of using the permutation identity for superpropagators, one can prove the orthogonality of mixed eigenfunctions using a more ‘down-to-earth’ procedure. Though, it is not generalizable to pentagon transitions, we think it is worthwhile to demonstrate it by showing that

\[
\left[ \langle \Phi_{s+1/2,s}(\lambda') | \Phi_{s+1/2,s}(\lambda) \rangle + \langle \Phi_{s,s+1/2}(\lambda') | \Phi_{s,s+1/2}(\lambda) \rangle \right]_{\lambda' \neq \lambda} = 0. \tag{B.1}
\]

We start with \(\Phi_{s+1/2,s}\) and perform the integration with respect to \(z_2\). This yields according to the chain rule \((A.3)\) the propagator connecting the points \(z'\) and \(z\) with the exponent \(i\lambda_2 - i\lambda_2'\) multiplied by the factor of rapidities \(a_s(s - i\lambda_2, s + i\lambda_2')\), as shown in Fig. 8 (second panel). Next we perform the variable transformation \(z \rightarrow 1/z\) that transforms the graph into the one in Fig. 8 (third panel). The subsequent integration with respect to the vertex \(z_1\) becomes trivial via the chain rule and we end up with the integral \(\mathcal{I}\)

\[
\mathcal{I} \equiv \int [Dz]_{s+1/2} \int [Dz']_{s+1/2} (z' - z)^{-1} i\lambda_2' - s(z') i\lambda_1 - s - 1/2(-z^*) - i\lambda_2 - s - 1/2(-z^*) - i\lambda_2' - s .
\]

represented by the rightmost graph in Fig. 8 along with the factor \(a_{s+1/2}(s + 1 + i\lambda_2, s + 1 - i\lambda_2')\). So combining everything together, we obtain (up to an inessential overall phase)

\[
\langle \Phi_{s+1/2,s} | \Phi_{s+1/2,s} \rangle = (\frac{1}{2} + i\lambda_2)(\frac{1}{2} - i\lambda_2')a_s(s - i\lambda_2, s + i\lambda_2')a_{s+1/2}(s + 1 + i\lambda_2, s + 1 - i\lambda_2') \mathcal{I} . \tag{B.2}
\]

Analogous reduction applies to the matrix element of \(\Phi_{s,s+1/2}\) and we find (up to the same phase factor)

\[
\langle \Phi_{s,s+1/2} | \Phi_{s,s+1/2} \rangle = (\frac{1}{2} - i\lambda_2)(\frac{1}{2} + i\lambda_2')a_s(s + i\lambda_2, s - i\lambda_2')a_{s+1/2}(s + 1 - i\lambda_2, s + 1 + i\lambda_2') \mathcal{I} . \tag{B.3}
\]
with the very same integral $I$! Summing up the two contributions (B.2) and (B.3) together one observes that the sum of the coefficients accompanying $I$ cancels out between the two, so we verify the validity of (B.1) even without knowing the explicit form of $I$.

C Wave-function Hamiltonian in sl(2) sector

In this Appendix, we recall the derivation of the Hamiltonians on the space of wave functions starting with the ones acting on matrix elements. We start by performing this transformation for the one-particle state and then move on to two particles which involves all bulk and boundary interactions intrinsic to an $N$-particle case.

The one-particle flux-tube state with energy $E(\lambda)$ is given by
\[
|E(\lambda)\rangle = \int_0^\infty dx_1 \psi_s(x_1; \lambda)O_\Pi(x_1)|0\rangle, \tag{C.1}
\]
where the field $O_\Pi(x_1) = W^\dagger(0)\phi_s(x_1)W(\infty)$ creates an excitation out of the vacuum with $\psi_s(x_1; \lambda)$ being the eigenfunction of the Hamiltonian with the eigenvalue $E(\lambda)$. Our goal here, knowing the from of $H$ acting on $O_\Pi$, find $\hat{H}$ for $\psi_s$. The action of $\hat{H}$ on $|E(\lambda)\rangle$ is
\[
\hat{H}|E(\lambda)\rangle = \int_0^\infty dx_1 \psi_s(x_1; \lambda)(H_{01} + H_{1\infty})O_\Pi(x_1)|0\rangle, \tag{C.2}
\]
with
\[
H_{01}O_\Pi(x_1) = \int_0^1 \frac{d\beta}{1-\beta} \left[\beta^{2s-1}O_\Pi(\beta x_1) - O_\Pi(x_1)\right], \tag{C.3}
\]
\[
H_{1\infty}O_\Pi(x_1) = \int_1^\infty \frac{d\beta}{\beta - 1} \left[O_\Pi(\beta x_1) - \beta^{-1}O_\Pi(x_1)\right]. \tag{C.4}
\]
Changing the integration variable in the right-hand side of Eq. (C.2) as $x_1 \to x_1/\beta$ and $\beta = 1/\alpha$, we immediately find
\[
\int_0^\infty dx_1 \psi_s(x_1; \lambda)HO_\Pi(x_1) = \int_0^{\infty} dx_1 \left(\hat{H}\psi_s(x_1; \lambda)\right)O_\Pi(x_1), \tag{C.5}
\]
with
\[
\hat{H}_{01}\psi_s(x_1) = \int_1^\infty \frac{d\alpha}{\alpha - 1} \left[\alpha^{1-2s}\psi_s(\alpha x_1) - \alpha^{-1}\psi_s(x_1)\right], \tag{C.6}
\]
\[
\hat{H}_{1\infty}\psi_s(x_1) = \int_0^1 \frac{d\alpha}{1-\alpha} \left[\psi_s(\alpha x_1) - \psi_s(x_1)\right], \tag{C.7}
\]
where, for brevity, we dropped the dependence of the wave function on the rapidity $\lambda$.

Next we turn to the case of two flux-tube excitations with rapidities $\lambda = (\lambda_1, \lambda_2)$ that will unravel two questions: (i) how the strong ordering of coordinates in multiparticle wave functions

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4Which can be easily calculated in fact using Feynman parametrization for the propagator $(\hat{z'} - \hat{z'})^{-1}$ and one of the adjacent factors involving $\hat{z'}$ to $\hat{z}$ and subsequent use of the chain rule. This demonstrates that $I \sim \delta(\lambda_1 + \lambda_2 - \lambda'_1 - \lambda'_2)$. 

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enters the game as we as (ii) the form of the bulk Hamiltonians. As we will see from our
calculation, the bulk and boundary Hamiltonians jump places as we pass from matrix elements
to wave functions. The state is
\[ |E(\lambda)\rangle = \int_0^\infty \! d^2x \ \theta(x_2 - x_1)|\psi_s(x; \lambda)O_\Pi(x)\rangle|0\rangle , \]  
(C.8)where \( O_\Pi(x) = W^\dagger(0)|\phi_s(x_1)\phi_s(x_2)W(\infty) \) creates two particles localized at \( x = (x_1, x_2) \) integrated with the measure \( d^2x = dx_1 \, dx_2 \). The Hamiltonian splits up into the bulk \( H_{12}^+ \) and boundary \( H_{01/1\infty} \) terms,
\[ \mathbb{H}|E(\lambda)\rangle = \int_0^\infty \! d^2x \ \theta(x_2 - x_1)|\psi_s(x; \lambda)(H_{01} + H_{12}^+ + H_{12}^- + H_{1\infty})O_\Pi(x)\rangle|0\rangle , \]  
(C.9)with
\[ H_{01}O_\Pi(x) = \int_0^1 \! \frac{d\beta}{1 - \beta} \left[ \beta^{2s-1}O_\Pi(\beta x_1, x_2) - O_\Pi(x_1, x_2) \right] , \]  
(C.10)\[ H_{12}^+O_\Pi(x) = \int_{x_1/x_2}^1 \! \frac{d\beta}{1 - \beta} \left[ \left( \frac{\beta x_2 - x_1}{x_2 - x_1} \right)^{2s-1}O_\Pi(x_1, \beta x_2) - O_\Pi(x_1, x_2) \right] , \]  
(C.11)\[ H_{12}^-O_\Pi(x) = \int_{1/x_2/x_1}^1 \! \frac{d\beta}{\beta - 1} \left[ \left( \frac{x_2 - \beta x_1}{x_2 - x_1} \right)^{2s-1}O_\Pi(\beta x_1, x_2) - \beta^{-1}O_\Pi(x_1, x_2) \right] , \]  
(C.12)\[ H_{2\infty}O_\Pi(x) = \int_0^1 \! \frac{d\beta}{1 - \beta} \left[ O_\Pi(x_1, \beta x_2) - \beta^{-1}O_\Pi(x_1, x_2) \right] . \]  
(C.13)Again, the Hamiltonian on the space of wave functions is found by means of the integration by parts
\[ \int_0^\infty \! d^2x \ \theta(x_2 - x_1)|\psi_s(x; \lambda)HO_\Pi(x)\rangle = \int_0^\infty \! d^2x \ \theta(x_2 - x_1)\left( \hat{H}\psi_s(x; \lambda) \right)O_\Pi(x) , \]  
(C.14)with particular attention paid to strong ordering. In the \( H_{01} \) term, we change the integration variable as \( x_1 \rightarrow x_1/\beta \) and immediately find that the step-function imposes a lower limit on the range of \( \beta, \beta > x_1/x_2 \). Thus, substituting \( \beta = 1/\alpha \), we find
\[ \hat{H}_{01}\psi_s(x) = \int_{1/x_2/x_1}^{x_2/x_1} \! \frac{d\alpha}{\alpha - 1} \left[ \alpha^{1-2s}\psi_s(\alpha x_1, x_2) - \alpha^{-1}\psi_s(x_1, x_2) \right] , \]  
(C.15)Similarly for \( H_{2\infty} \), we substitute \( x_2 \rightarrow x_2/\beta \) and get the constraint on \( \beta, x_2/x_1 > \beta \). Inverting \( \beta, \beta = 1/\alpha \), we eventually obtain
\[ \hat{H}_{2\infty}\psi_s(x) = \int_{x_1/x_2}^1 \! \frac{d\alpha}{1 - \alpha} \left[ \psi_s(x_1, \alpha x_2) - \psi_s(x_1, x_2) \right] . \]  
(C.16)As we can see, the boundary Hamiltonians takes on the form of the bulk one. In an analogous manner, we derive \( \hat{H}_{12}^\pm \) following the same route as above
\[ \hat{H}_{12}^\pm\psi_s(x) = \int_1^\infty \! \frac{d\alpha}{\alpha - 1} \left[ \left( \frac{\alpha x_2 - x_1}{x_2 - x_1} \right)^{1-2s}\psi_s(x_1, \alpha x_2) - \alpha^{-1}\psi_s(x_1, x_2) \right] , \]  
(C.17)
\[ \hat{H}_{12} \psi_s(x) = \int_0^1 \frac{d\alpha}{1-\alpha} \left[ \left( \frac{x_2 - \alpha x_1}{x_2 - x_1} \right)^{1-2s} \psi_s(\alpha x_1, x_2) - \psi_s(x_1, x_2) \right], \quad (C.18) \]

taking on the form more resembling the boundary interactions due to the form of integration limits.

The same results can be obtained by using an intertwiner \[ \Pi \]
\[ W_N = (x_1 x_2 \ldots x_{N,N-1})^{2s-1} \quad (C.19) \]
that exchanges the index of the representation from \( s \) to \( 1-s \). Namely,
\[ \hat{H} = W H W^{-1}. \quad (C.20) \]

We get
\[ \hat{H}_{01} \psi_s(x) = \int_0^1 \frac{d\alpha}{1-\alpha} \left[ \left( \frac{x_2 - \alpha x_1}{x_2 - x_1} \right)^{1-2s} \psi_s(\alpha x_1, x_2) - \psi_s(x_1, x_2) \right], \quad (C.21) \]
\[ \hat{H}_{2,\infty} \psi_s(x) = \int_1^\infty \frac{d\alpha}{\alpha-1} \left[ \left( \frac{\alpha x_2 - x_1}{x_2 - x_1} \right)^{1-2s} \psi_s(x_1, \alpha x_2) - \alpha^{-1} \psi_s(x_1, x_2) \right], \quad (C.22) \]
\[ [\hat{H}^+_{12} + \hat{H}^-_{12}] \psi_s(x) = \int_{x_1/x_2}^1 \frac{d\alpha}{1-\alpha} \left[ \psi_s(x_1, \alpha x_2) - \psi_s(x_1, x_2) \right] \]
\[ + \int_{x_2/x_1}^1 \frac{d\alpha}{\alpha-1} \left[ \alpha^{1-2s} \psi_s(\alpha x_1, x_2) - \alpha^{-1} \psi_s(x_1, x_2) \right]. \quad (C.23) \]

As we can see, the individual Hamiltonians do not jump places from bulk to boundary and back since the transformation mechanism is different compared to the integration by parts technique. The final cumulative answer is however the same as it has to be.

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