\(L^p\)-versions of generalized Korn inequalities for incompatible tensor fields in arbitrary dimensions with \(p\)-integrable exterior derivative

Versions \(L^p\) des inégalités généralisées de Korn pour les champs de tenseurs incompatibles de dimension quelconque avec dérivée extérieure \(p\)-intégrable

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Abstract

For \(n \geq 2\) and \(1 < p < \infty\) we prove an \(L^p\)-version of the generalized Korn-type inequality for incompatible, \(p\)-integrable tensor fields \(P : \Omega \to \mathbb{R}^{n \times n}\) having \(p\)-integrable generalized \(\text{Curl}\) and generalized vanishing tangential trace \(P\tau_l = 0\) on \(\partial\Omega\), denoting by \(\{\tau_l\}_{l=1,\ldots,n-1}\) a moving tangent frame on \(\partial\Omega\), more precisely we have:

\[
\|P\|_{L^p(\Omega,\mathbb{R}^{n \times n})} \leq c \left( \|\text{sym} P\|_{L^p(\Omega,\mathbb{R}^{n \times n})} + \|\text{Curl} P\|_{L^p(\Omega,\mathbb{R}^{n \times (n-1)})} \right),
\]

where the generalized \(\text{Curl}\) is given by \((\text{Curl} P)_{ijk} := \partial_i P_{kj} - \partial_j P_{ki}\) and \(c = c(n, p, \Omega) > 0\).

Résumé

On montre pour \(n \geq 2\) et \(1 < p < \infty\) une version \(L^p\) de l’inégalité généralisée de Korn pour tous les champs de tenseurs incompatibles et \(p\)-intégrable \(P : \Omega \to \mathbb{R}^{n \times n}\) avec rotationnel généralisé \(p\)-intégrable et avec zéro trace tangentielle \(P\tau_l = 0\) sur \(\partial\Omega\) où \(\{\tau_l\}_{l=1,\ldots,n-1}\) est un repère tangent sur \(\partial\Omega\). Plus précisément on a

\[
\|P\|_{L^p(\Omega,\mathbb{R}^{n \times n})} \leq c \left( \|\text{sym} P\|_{L^p(\Omega,\mathbb{R}^{n \times n})} + \|\text{Curl} P\|_{L^p(\Omega,\mathbb{R}^{n \times (n-1)})} \right),
\]

où les composantes du rotationnel généralisé s’écrivent \((\text{Curl} P)_{ijk} := \partial_i P_{kj} - \partial_j P_{ki}\) et \(c = c(n, p, \Omega) > 0\).

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1 Introduction

On montre pour $n \geq 2$ et $1 < p < \infty$ une version $L^p$ de l’inégalité généralisée de Korn pour tous les champs de tenseurs incompatibles et $p$-intégrable $P : \Omega \to \mathbb{R}^{n \times n}$ avec rotationnel généralisé $p$-intégrable et avec zéro trace tangentielle $P \tau = 0$ sur $\partial \Omega$ où $\{ \tau_i \}_{i=1,\ldots,n-1}$ est un repère tangent sur $\partial \Omega$. Plus précisément on a

$$\|P\|_{L^p(\Omega,\mathbb{R}^{n \times n})} \leq c \left( \|\text{sym} P\|_{L^p(\Omega,\mathbb{R}^{n \times n})} + \|\text{Curl} P\|_{L^p(\Omega,\mathbb{R}^{3 \times (n-1)})} \right),$$

où les composantes du rotationnel généralisé s’écrivent $(\text{Curl} P)_{ijk} := \partial_i P_{kj} - \partial_j P_{ki}$ et $c = c(n,p,\Omega) > 0$.

2 Introduction

In [6] we have shown that there exists a constant $c = c(p,\Omega) > 0$ such that

$$\|P\|_{L^p(\Omega,\mathbb{R}^{3 \times 3})} \leq c \left( \|\text{sym} P\|_{L^p(\Omega,\mathbb{R}^{3 \times 3})} + \|\text{Curl} P\|_{L^p(\Omega,\mathbb{R}^{3 \times 3})} \right)$$

holds for all tensor fields $P \in W^{1,p}_0(\text{Curl};\Omega,\mathbb{R}^{3 \times 3})$, i.e., for all $P \in W^{1,p}(\text{Curl};\Omega,\mathbb{R}^{3 \times 3})$ with vanishing tangential trace $P \times \nu = 0$ ( $\iff P \tau = 0$ ) on $\partial \Omega$ where $\nu$ denotes the outward unit normal vector field and $\{ \tau_i \}_{i=1,2,3}$ a moving tangent frame on $\partial \Omega$ and $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain. The crucial ingredients for our proof were the Lions lemma and Nečas estimate, the compactness of $W^{1,p}_0(\Omega) \subset L^p(\Omega)$ and an algebraic identity in terms of components of the cross product of a skew-symmetric matrix with a vector. Recall, that for a bounded Lipschitz domain (i.e. bounded open connected with Lipschitz boundary) $\Omega \subset \mathbb{R}^n$, the Lions lemma states that $f \in L^p(\Omega)$ if and only if $f \in W^{1,p}(\Omega)$ and $\nabla f \in W^{-1,p}(\Omega,\mathbb{R}^n)$, which is equivalently expressed by the Nečas estimate

$$\|f\|_{L^p(\Omega)} \leq c \left( \|f\|_{W^{-1,p}(\Omega)} + \|\nabla f\|_{W^{-1,p}(\Omega,\mathbb{R}^n)} \right)$$

(2.1)

with a positive constant $c = c(p,n,\Omega)$. In fact, such an argumentation scheme is also used to prove the classical Korn inequalities, cf. e.g. [1, 2, 3, 4, 5, 6] and the discussions contained therein. However, [1, 2, 3, 4, 5] focus on the compatible case, i.e. $P = Du$, where we deal with general square matrices $P \in \mathbb{R}^{n \times n}$, thus, the incompatible case.

Here, we extend our results from [6] to the $n$-dimensional case, hence generalizing the main result from [8] to the $L^p$-setting. This is, we prove

$$\|P\|_{L^p(\Omega,\mathbb{R}^{n \times n})} \leq c \left( \|\text{sym} P\|_{L^p(\Omega,\mathbb{R}^{n \times n})} + \|\text{Curl} P\|_{L^p(\Omega,\mathbb{R}^{3 \times (n-1)})} \right) \quad \forall P \in W^{1,p}_0(\text{Curl};\Omega,\mathbb{R}^{n \times n}),$$

(2.2)

where the generalized Curl is given by $(\text{Curl} P)_{ijk} := \partial_i P_{kj} - \partial_j P_{ki}$ and the vanishing tangential trace condition reads $P \tau = 0$ on $\partial \Omega$ denoting by $\{ \tau_i \}_{i=1,\ldots,n-1}$ a moving tangent frame on $\partial \Omega$.

For a detailed motivation and definitions we refer to [6] and the references contained therein. Indeed, we follow the argumentation scheme presented in [6] closely, emphasizing only the necessary modifications coming from the generalization of the vector product. The latter then provides an adequate generalization of the Curl-operator to the $n$-dimensional setting. Especially, the generalized curl of vector fields can be seen as their exterior derivative, see also the discussion in [8].

3 Notations

Let $n \geq 2$. For vectors $a, b \in \mathbb{R}^n$, we consider the scalar product $\langle a, b \rangle := \sum_{i=1}^n a_i b_i \in \mathbb{R}$, the (squared) norm $\|a\|^2 := \langle a, a \rangle$ and the dyadic product $a \otimes b := (a_i b_j)_{i,j=1,\ldots,n} \in \mathbb{R}^{n \times n}$. Similarly, for matrices $P, Q \in \mathbb{R}^{n \times n}$ we define the scalar product $\langle P, Q \rangle := \sum_{i,j=1}^n P_{ij} Q_{ij} \in \mathbb{R}$ and the (squared) Frobenius-norm $\|P\|^2 := \langle P, P \rangle$. Moreover, $P^T := (P_{ij})_{i,j=1,\ldots,n}$ denotes the transposition of the matrix $P = (P_{ij})_{i,j=1,\ldots,n}$, which decomposes orthogonally into the symmetric part $P := \frac{1}{2} (P + P^T)$ and the skew-symmetric part $P :=$
The Lie-Algebra of skew-symmetric matrices is denoted by \( \mathfrak{so}(n) := \{ A \in \mathbb{R}^{n \times n} \mid AT = -A \} \). The identity matrix is denoted by \( I \), so that the trace of a matrix \( P \) is given by \( \text{tr} \, P := \langle P, 1 \rangle \).

The cross product for vectors \( a, b \in \mathbb{R}^n \) generalizes to

\[
a \times b := (a_i b_j - a_j b_i)_{i,j=1,\ldots,n} = a \otimes b - b \otimes a = 2 \cdot \text{skew}(a \otimes b) \in \mathfrak{so}(n) \cong \mathbb{R}^{n(n-1)}.
\]

Using the bijection \( \text{axl} : \mathfrak{so}(3) \to \mathbb{R}^3 \) we obtain back the standard cross product for \( a, b \in \mathbb{R}^3 \):

\[
a \times b = -\text{axl}(a \times b)
\]

where \( \text{axl} : \mathfrak{so}(3) \to \mathbb{R}^3 \) is given in such a way that

\[
A b = \text{axl}(A) \times b, \quad \forall A \in \mathfrak{so}(3), \quad b \in \mathbb{R}^3.
\]

Like in 3-dimensions it holds:

**Observation 3.1.** Let \( n \geq 2 \). For non-zero vectors \( a, b \in \mathbb{R}^n \) we have \( a \times b = 0 \) if and only if \( a \) and \( b \) are parallel.

**Proof.** Since the "if" part is obvious we show the "only if" direction:

\[
\begin{align*}
a \times b = 0 & \iff \text{skew}(a \otimes b) = 0 \iff a \otimes b = b \otimes a \implies (a \otimes b)b = (b \otimes a)b \\
& \iff a \|b\|^2 = b \langle a, b \rangle.
\end{align*}
\]

As in the 3-dimensional case, we understand the vector product of a square-matrix \( P \in \mathbb{R}^{n \times n} \) and a vector \( b \in \mathbb{R}^n \) row-wise, i.e.

\[
P \times b := ((P^T e_k) \times b)_{k=1,\ldots,n} = (P_{ki} b_j - P_{kj} b_i)_{i,j,k=1,\ldots,n} \in (\mathfrak{so}(n))^n.
\]

For index notations we set:

\[
(P \times b)_{ijk} := P_{ki} b_j - P_{kj} b_i.
\]

Especially, for skew-symmetric matrices \( A \in \mathfrak{so}(n) \) we note the following crucial relation for our considerations:

\[
(A \times b)_{kij} - (A \times b)_{kji} = A_{ik} b_j - A_{ji} b_k - (A_{ik} b_j - A_{ij} b_k) + A_{kj} b_i - A_{ki} b_j
\]

\[
\implies 2A_{ij} b_k \quad \forall \, i,j,k = 1, \ldots, n
\]

with the direct consequence

**Observation 3.2.** Let \( n \geq 2 \). For \( A \in \mathfrak{so}(n) \) and a non-zero vector \( b \in \mathbb{R}^n \) we have \( A \times b = 0 \) if and only if \( A = 0 \).

Let \( \Omega \subset \mathbb{R}^n \), \( n \geq 2 \), be a domain. As in \( \mathbb{R}^3 \) we formally introduce the generalized curl of a vector field \( v \in \mathcal{D}'(\Omega, \mathbb{R}^n) \) via

\[
\text{curl} \, v := v \times (-\nabla) = \nabla \times v = -2 \cdot \text{skew}(v \otimes \nabla) = -2 \cdot \text{skew}(Dv) \in \mathfrak{so}(n).
\]

Furthermore, for \( (n \times n) \)-square matrix fields we understand this operation row-wise:

\[
\text{Curl} \, P := P \times (-\nabla) = ((\text{curl} \, (P^T e_k))_{k=1,\ldots,n} = (\partial_i P_{kj} - \partial_j P_{ki})_{i,j,k=1,\ldots,n} \in (\mathfrak{so}(n))^n.
\]

For index notations we define: \( (\text{Curl} \, P)_{ijk} := \partial_i P_{kj} - \partial_j P_{ki} \). Of course, \( \text{Curl} \, \text{Div} \, P \equiv 0 \).

Moreover, we make use of the generalized divergence Div for matrix fields \( P \in \mathcal{D}'(\Omega, \mathbb{R}^{n \times n}) \) row-wise, via

\[
\text{Div} \, P := (\text{div}(P^T e_k))_{k=1,\ldots,n}.
\]

In fact, the crucial relation \((3.5)\) implies that the full gradient of a skew-symmetric matrix is already determined by its generalized Curl, cf. also [7, p. 155]:

\[
\frac{1}{2} (P - P^T).
\]
Corollary 3.3. Let $n \geq 2$. For $A \in \mathcal{D}'(\Omega, \mathfrak{so}(n))$ the entries of the gradient $D A$ are linear combinations of the entries from $\text{Curl} A$.

Proof. Replacing $b$ by $-\nabla$ in (3.5) we see that

$$(\text{Curl} A)_{kij} - (\text{Curl} A)_{kji} + (\text{Curl} A)_{jik} = -2 \partial_k A_{ij}.$$  

This control of all first partial derivatives of a skew-symmetric matrix field in terms of the generalized $\text{Curl}$ then immediately yields in all dimensions

Corollary 3.4. Let $n \geq 2$. For $A \in L^p(\Omega, \mathfrak{so}(n))$ we have $\text{Curl} A \equiv 0$ in the distributional sense if and only if $A = \text{const}$ almost everywhere in $\Omega$.

3.1 Function spaces

Having above relations at hand we can now catch up the arguments from [6]. For that purpose let us define for $n \geq 2$ and $1 < p < \infty$ the space

$$W^{1,p}(\text{Curl} \Omega, \mathbb{R}^{n \times n}) := \{ P \in L^p(\Omega, \mathbb{R}^{n \times n}) \mid \text{Curl} P \in L^p(\Omega, (\mathfrak{so}(n))^\ast) \}$$

equipped with the norm

$$\| P \|_{W^{1,p}(\text{Curl} \Omega, \mathbb{R}^{n \times n})} := \left( \| P \|_{L^p(\Omega, \mathbb{R}^{n \times n})}^{p} + \| \text{Curl} P \|_{L^p(\Omega, (\mathfrak{so}(n))^\ast)}^{p} \right)^{\frac{1}{p}}.$$  

By definition of the norm in the dual space, we have

$$P \in L^p(\Omega, \mathbb{R}^{n \times n}) \Rightarrow \text{Curl} P \in W^{-1,p}(\Omega, (\mathfrak{so}(n))^\ast)^\ast$$

with $\| \text{Curl} P \|_{W^{-1,p}(\Omega, (\mathfrak{so}(n))^\ast)^\ast} \leq c \| P \|_{L^p(\Omega, \mathbb{R}^{n \times n})}$.

Furthermore, we consider the subspace

$$W^{1,p}_0(\text{Curl} \Omega, \mathbb{R}^{n \times n}) := \{ P \in W^{1,p}(\text{Curl} \Omega, \mathbb{R}^{n \times n}) \mid P \times \nu = 0 \text{ on } \partial \Omega \}$$

$$= \{ P \in W^{1,p}(\text{Curl} \Omega, \mathbb{R}^{n \times n}) \mid P \tau_l = 0 \text{ on } \partial \Omega \text{ for all } l = 1, \ldots, n-1 \},$$

where $\nu$ stands for the outward unit normal vector field and $\{\tau_l\}_{l=1}^{n-1}$ denotes a moving tangent frame on $\partial \Omega$. Here, the generalized tangential trace $P \times \nu$ is understood in the sense of $W^{-1,\frac{p}{p-1}}(\partial \Omega, \mathbb{R}^{n \times n})$ which is justified by partial integration, so that its trace is defined by

$$\forall k = 1, \ldots, n, \forall Q \in W^{1,\frac{p}{p-1}}(\partial \Omega, \mathbb{R}^{n \times n}) :$$

$$\langle (P^T e_k) \times \nu, Q \rangle_{\partial \Omega} = \int_{\Omega} \langle \text{curl} (P^T e_k), \tilde{Q} \rangle_{\mathbb{R}^{n \times n}} + 2 \langle P^T e_k, \text{Div}(\text{skew} \tilde{Q}) \rangle_{\mathbb{R}^n} dx$$

having denoted by $\tilde{Q} \in W^{1,\frac{p}{p-1}}(\partial \Omega, \mathbb{R}^{n \times n})$ any extension of $Q$ in $\Omega$, where, $\langle \ldots \rangle_{\partial \Omega}$ indicates the duality pairing between $W^{-1,\frac{p}{p-1}}(\partial \Omega, \mathbb{R}^{n \times n})$ and $W^{1,\frac{p}{p-1}}(\partial \Omega, \mathbb{R}^{n \times n})$. Indeed, for $P, Q \in C^1(\Omega, \mathbb{R}^{n \times n}) \cap C^0(\overline{\Omega}, \mathbb{R}^{n \times n})$ we have

$$\frac{1}{2} \langle (P^T e_k) \times \nu, Q \rangle_{\mathbb{R}^{n \times n}} \equiv \langle \text{skew} ((P^T e_k) \otimes \nu), Q \rangle_{\mathbb{R}^{n \times n}} = \langle (P^T e_k) \otimes \nu, \text{skew} Q \rangle_{\mathbb{R}^{n \times n}}$$

$$= \sum_{i,j=1}^{n} P_{ki} \nu_j (\text{skew} Q)_{ij} = - \sum_{i,j=1}^{n} \nu_j (\text{skew} Q)_{ji} P_{ki}$$

$$= - \langle \nu, (\text{skew} Q) (P^T e_k) \rangle_{\mathbb{R}^n},$$
so that using the divergence-theorem, for \( k = 1, \ldots, n \) we have\(^1\)

\[
\int_{\partial \Omega} \langle (P^T e_k) \times \nu, Q \rangle_{\mathbb{R}^{n \times n}} dS = -2 \int_{\partial \Omega} \nu \cdot (\text{skew } Q \cdot (P^T e_k))_{\mathbb{R}^n} dS = -2 \int_{\Omega} \text{div}(\text{skew } Q \cdot (P^T e_k)) \, dx
\]

\[
= -2 \int_{\Omega} \langle \text{Div}(\text{skew } Q)^T, P^T e_k \rangle_{\mathbb{R}^n} + \langle (\text{skew } Q), \text{D}(P^T e_k) \rangle_{\mathbb{R}^{n \times n}} \, dx
\]

\[
= \int_{\Omega} \langle \text{curl } (P^T e_k), Q \rangle_{\mathbb{R}^{n \times n}} + 2 \langle P^T e_k, \text{Div}(\text{skew } Q) \rangle_{\mathbb{R}^n} \, dx. \tag{3.14}
\]

Further, following [6] we introduce also the space \( W^{1,p}_{\Gamma,0}(\text{Curl}; \Omega, \mathbb{R}^{n \times n}) \) of functions with vanishing tangential trace only on a relatively open (non-empty) subset \( \Gamma \subseteq \partial \Omega \) of the boundary by completion of \( C^\infty_{\Gamma,0}(\Omega, \mathbb{R}^{n \times n}) \) with respect to the \( W^{1,p}(\text{Curl}; \Omega, \mathbb{R}^{n \times n}) \)-norm.

**Remark 3.5** (Tangential trace condition). Note, that the vanishing of the tangential trace \( P \times \nu \) at some point is equivalent to \( P \tau_l = 0 \) for all \( l = 1, \ldots, n - 1 \), denoting by \( \{\tau_l\}_{l=1,\ldots,n-1} \) a frame of the corresponding tangent space. Indeed, by Observation 3.1 we have

\[
P \times \nu = 0 \iff \text{skew } ((P^T e_k) \times \nu) = 0, \quad k = 1, \ldots, n \iff (P^T e_k) \text{ parallel to } \nu \text{ for all } k = 1, \ldots, n
\]

\[
\iff \langle P^T e_k, \tau_l \rangle = 0 \quad \forall \ l = 1, \ldots, n - 1, \quad \forall \ k = 1, \ldots, n \iff P \tau_l = 0 \quad \forall \ l = 1, \ldots, n - 1.
\]

### 4 Main results

We will now state the results from [6] in the \( n \)-dimensional case, for details of the proofs we refer to the corresponding results therein:

**Lemma 4.1.** Let \( n \geq 2, \Omega \subseteq \mathbb{R}^n \) be a bounded Lipschitz domain and \( 1 < p < \infty \). Then \( P \in \mathcal{D}'(\Omega, \mathbb{R}^{n \times n}) \), \( \text{sym } P \in L^p(\Omega, \mathbb{R}^{n \times n}) \) and \( \text{Curl } P \in W^{1,p}(\Omega, (\mathfrak{so}(n))^n) \) imply \( P \in L^p(\Omega, \mathbb{R}^{n \times n}) \). Moreover, we have the estimate

\[
\|P\|_{L^p(\Omega, \mathbb{R}^{n \times n})} \leq c \left( \|\text{skew } P\|_{W^{-1,p}(\mathfrak{so}(n))^n} + \|\text{sym } P\|_{L^p(\Omega, \mathbb{R}^{n \times n})} + \|\text{Curl } P\|_{W^{-1,p}(\mathfrak{so}(n))^n} \right), \tag{4.1}
\]

with a constant \( c = c(n, p, \Omega) > 0 \).

**Proof.** Use Corollary 3.3 and apply the Lions lemma and Nečas estimate, [6, Theorem 2.6] to skew \( P \), cf. proof of [6, Lemma 3.1].

The general Korn-type inequalities then follow by eliminating the first term on the right-hand side of (4.1):

**Theorem 4.2.** Let \( n \geq 2, \Omega \subseteq \mathbb{R}^n \) be a bounded Lipschitz domain and \( 1 < p < \infty \). There exists a constant \( c = c(n, p, \Omega) > 0 \), such that for all \( P \in L^p(\Omega, \mathbb{R}^{n \times n}) \) we have

\[
\inf_{A \in \mathfrak{so}(n)} \|P - A\|_{L^p(\Omega, \mathbb{R}^{n \times n})} \leq c \left( \|\text{sym } P\|_{L^p(\Omega, \mathbb{R}^{n \times n})} + \|\text{Curl } P\|_{W^{-1,p}(\mathfrak{so}(n))^n} \right). \tag{4.2}
\]

**Proof.** By Corollary 3.4 the kernel of the right-hand side consists only of constant skew-symmetric matrices:

\[
K := \{ P \in L^p(\Omega, \mathbb{R}^{n \times n}) \mid \text{sym } P = 0 \text{ a.e. and } \text{Curl } P = 0 \text{ in the distributional sense} \}
\]

\[
= \{ P = A \text{ a.e. } \mid A \in \mathfrak{so}(n) \}. \tag{4.3}
\]

\(^1\)This partial integration formula slightly differs from the situation in \( \mathbb{R}^3 \) since the generalized \( \text{Curl} \) has image in \( (\mathfrak{so}(n))^n \) which corresponds to \( \mathbb{R}^{n \times n} \) only for \( n = 3 \).
Then there exist $M := \dim K = \frac{n(n-1)}{2}$ linear forms $\ell_\alpha$ on $L^p(\Omega, \mathbb{R}^{n \times n})$ such that $\ell_\alpha(P) = 0$ for all $\alpha = 1, \ldots, M$. Exploiting the compactness $L^p(\Omega, \mathbb{R}^{n \times n}) \subseteq W^{-1, p}(\Omega, \mathbb{R}^{n \times n})$ allows us to eliminate the first term on the right-hand side of (4.1) so that we arrive at

$$\|P\|_{L^p(\Omega, \mathbb{R}^{n \times n})} \leq c \left( \|\text{sym } P\|_{L^p(\Omega, \mathbb{R}^{n \times n})} + \|\text{Curl } P\|_{W^{-1, p}(\Omega, \mathbb{R}^{n \times n})} + \sum_{\alpha=1}^M |\ell_\alpha(P)| \right). \quad (4.4)$$

Considering $P - A_P$ in (4.4), where the skew-symmetric matrix $A_P \in K$ is chosen in such a way that $\ell_\alpha(P - A_P) = 0$ for all $\alpha = 1, \ldots, M$, then yields the conclusion, cf. proof of [6, Theorem 3.4].

Moreover, the kernel is killed by the tangential trace condition $P \times \nu \equiv 0$ (or $P \tau_1 \equiv 0$ for all $l = 1, \ldots, n-1$), cf. (4.3) together with Observation 3.2 (and also Remark 3.5), so that we arrive at

**Theorem 4.3.** Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $1 < p < \infty$. There exists a constant $c = c(n, p, \Omega) > 0$, such that for all $P \in W_0^{1, p}(\text{Curl}; \Omega, \mathbb{R}^{n \times n})$ we have

$$\|P\|_{L^p(\Omega, \mathbb{R}^{n \times n})} \leq c \left( \|\text{sym } P\|_{L^p(\Omega, \mathbb{R}^{n \times n})} + \|\text{Curl } P\|_{L^p(\Omega, (\mathfrak{so}(n))^n)} \right). \quad (4.5)$$

**Proof.** Having Observation 3.2 we can closely follow the proof of [6, Theorem 3.5].

Similar arguments show that (4.5) also holds true for functions with vanishing tangential trace only on a relatively open (non-empty) subset $\Gamma \subseteq \partial \Omega$ of the boundary, namely

**Theorem 4.4.** Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $1 < p < \infty$. There exists a constant $c = c(n, p, \Omega) > 0$, such that for all $P \in W_{\Gamma, 0}^{1, p}(\text{Curl}; \Omega, \mathbb{R}^{n \times n})$ we have

$$\|P\|_{L^p(\Omega, \mathbb{R}^{n \times n})} \leq c \left( \|\text{sym } P\|_{L^p(\Omega, \mathbb{R}^{n \times n})} + \|\text{Curl } P\|_{L^p(\Omega, (\mathfrak{so}(n))^n)} \right). \quad (4.6)$$

Furthermore, Theorem 4.4 reduces for compatible $P = Du$ to a tangential Korn inequality (Corollary 4.5) and for skew-symmetric $P = A$ to a Poincaré inequality in arbitrary dimensions (Corollary 4.7):

**Corollary 4.5.** Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $1 < p < \infty$. There exists a constant $c = c(n, p, \Omega) > 0$, such that for all $u \in W_{\Gamma, 0}^{1, p}(\Omega, \mathbb{R}^n)$ we have

$$\|Du\|_{L^p(\Omega, \mathbb{R}^{n \times n})} \leq c \|\text{sym } Du\|_{L^p(\Omega, \mathbb{R}^n)} \quad \text{with } Du \times \nu = 0 \quad \text{on } \Gamma. \quad (4.7)$$

**Remark 4.6.** On $\Gamma$ the boundary condition $Du \times \nu = 0$ is equivalent to $Du \tau_l = 0$ for all $l = 1, \ldots, n-1$ and is, e.g., fulfilled if $u\vert_\Gamma \equiv \text{const.}$, see Remark 3.5.

**Corollary 4.7.** Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $1 < p < \infty$. There exists a constant $c = c(n, p, \Omega) > 0$, such that for all $A \in W_{\Gamma, 0}^{1, p}(\text{Curl}; \Omega, \mathfrak{so}(n)) = W_{\Gamma, 0}^{1, p}(\Omega, \mathfrak{so}(n))$ we have

$$\|A\|_{L^p(\Omega, \mathfrak{so}(n))} \leq c \|\text{Curl } A\|_{L^p(\Omega, (\mathfrak{so}(n))^n)} \quad \text{with } A \times \nu = 0 \quad \Leftrightarrow \quad A = 0 \quad \text{on } \Gamma. \quad (4.8)$$

**Remark 4.8.** The equivalence of condition $\ast$ is seen in the following way: In any dimension the rank of the skew-symmetric matrix $A$ is an even number, cf. [9, p. 30], and by Remark 3.5 the rows $A^T e_k$ are all parallel ($\Leftrightarrow A \tau_l = 0$ for all $l = 1, \ldots, n-1$) such that the rank of $A$ is not greater then 1.

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