Locking and unlocking nonlocality without entanglement by postmeasurement information

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Nonlocality without entanglement (NLWE) is a nonlocal quantum phenomenon that arises in separable state discrimination. We show that the availability of the post-measurement information about the prepared subensemble can affect the occurrence of NLWE in discriminating non-orthogonal non-entangled states. We provide a two-qubit state ensemble consisting of four non-orthogonal separable pure states and show that the post-measurement information about the prepared subensemble can lock NLWE. We also provide another two-qubit state ensemble consisting of four non-orthogonal separable states and show that the post-measurement information can unlock NLWE. Our result can provide a useful method to share or hide information using non-orthogonal separable states.

RESULTS

Quantum state discrimination in two-qubit systems

In two-qubit \( (2 \otimes 2) \) systems, a state is described by a density operator \( \rho \), that is, a positive-semidefinite operator \( \rho \succeq 0 \) having unit trace \( \text{Tr}\rho = 1 \), acting on a bipartite Hilbert space, \( \mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \). A measurement with \( m \) outcomes is expressed by a positive oper-

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ator valued measure (POVM) \( \{ M_i \}_i \) that consists of \( m \) positive-semidefinite operators \( M_i \geq 0 \) on \( \mathcal{H} \) satisfying \( \sum_i M_i = 1 \), where \( \mathbb{I} \) is the identity operator on \( \mathcal{H} \). When \( \{ M_i \}_i \) is performed on a quantum system prepared with \( \rho \), the probability that \( M_i \) is detected is \( \text{Tr}(\rho M_i) \) due to Born rule.

A positive-semidefinite operator is called separable if it is a sum of positive-semidefinite product operators. Similarly, a measurement \( \{ M_i \}_i \) is called separable if \( M_i \) is separable for all \( i \). In particular, a LOCC measurement is a separable measurement that can be implemented by LOCC [24].

An operator \( E \) on \( \mathcal{H} \) is called positive partial transpose (PPT) [25, 26] if

\[
\text{PT}(E) \geq 0,
\]

where \( \text{PT}(\cdot) \) is the parital transposition taken in the standard basis \( \{|0\rangle, |1\rangle\} \) on the second subsystem (Although PPT property does not depend on the choice of sub-system to be transposed, we take the second subsystem throughout this paper for simplicity). In two-qubit systems, PPT is a necessary and sufficient condition for a positive-semidefinite operator to be separable [26]. Thus, the set of all positive-semidefinite separable operators on \( \mathcal{H} \) can be represented as

\[
\text{SEP} = \{ E \mid E \geq 0, \text{PT}(E) \geq 0 \}. \tag{2}
\]

The dual set to \( \text{SEP} \) is defined as

\[
\text{SEP}^* = \{ A \mid \text{Tr}(AB) \geq 0 \forall B \in \text{SEP} \}. \tag{3}
\]

Since all elements of \( \text{SEP} \) are positive semidefinite, all positive semidefinite operators are in \( \text{SEP}^* \). We also note that all PPT operators are in \( \text{SEP}^* \) because all elements of \( \text{SEP} \) are PPT and \( \text{Tr}(AB) = \text{Tr}[\text{PT}(A)\text{PT}(B)] \) for any two operators \( A \) and \( B \).

Throughout this paper, we only consider the situation of discriminating states from the state ensemble,

\[
\mathcal{E} = \{ \eta_i, \rho_i \}_{i \in \Lambda}, \quad \Lambda = \{ 0, 1, +, - \}, \tag{4}
\]

where \( \rho_i \) is a \( 2 \otimes 2 \) separable state and \( \eta_i \) is the probability that the state \( \rho_i \) is prepared.

The ensemble \( \mathcal{E} \) can be seen as an ensemble consisting of two subensembles,

\[
\begin{align*}
\mathcal{E}_0 &= \{ \eta_i / \sum_{j \in \mathcal{A}_0} \eta_j, \rho_i \}_{i \in \mathcal{A}_0}, \quad \mathcal{A}_0 = \{ 0, 1 \}, \\
\mathcal{E}_1 &= \{ \eta_i / \sum_{j \in \mathcal{A}_1} \eta_j, \rho_i \}_{i \in \mathcal{A}_1}, \quad \mathcal{A}_1 = \{ +, - \},
\end{align*}
\tag{5}
\]

where \( \mathcal{E}_0 \) and \( \mathcal{E}_1 \) are prepared with probabilities \( \sum_{j \in \mathcal{A}_0} \eta_j \) and \( \sum_{j \in \mathcal{A}_1} \eta_j \), respectively.

**Minimum-error discrimination**

Let us consider the state discrimination of \( \mathcal{E} \) in Eq. (4) using a measurement \( \{ M_i \}_{i \in \Lambda} \), where each measurement outcome from \( M_i \) means that the prepared state is guessed to be \( \rho_i \). ME of \( \mathcal{E} \) is to minimize the average probability of errors that occur in guessing the prepared state. Equivalently, ME of \( \mathcal{E} \) is to maximize the average probability of correctly guessing the prepared state where the optimal success probability is defined as

\[
p_{\text{GC}}(\mathcal{E}) = \max_{\{ M_i \}_i \in \Lambda} \sum_{i \in \Lambda} \eta_i \text{Tr}(\rho_i M_i) \tag{6}
\]

over all possible POVMs. The optimality of the POVMs in Eq. (6) can be confirmed by the following necessary and sufficient condition [21, 22, 27]:

\[
\sum_{i \in \Lambda} \eta_i \rho_i M_i - \eta_j \rho_j \geq 0 \quad \forall j \in \Lambda. \tag{7}
\]

When the available measurements are limited to LOCC measurements, we denote the maximum success probability by

\[
p_{\text{L}}(\mathcal{E}) = \max_{\text{LOCC}} \sum_{i \in \Lambda} \eta_i \text{Tr}(\rho_i M_i). \tag{8}
\]

Since the states of \( \mathcal{E} \) are non-entangled, NLIWE occurs in terms of ME if and only if ME of \( \mathcal{E} \) cannot be achieved only by LOCC, that is,

\[
p_{\text{L}}(\mathcal{E}) < p_{\text{GC}}(\mathcal{E}). \tag{9}
\]

The following proposition provides an upper bound of \( p_{\text{L}}(\mathcal{E}) \).

**Proposition 1** ([28]). If \( H \) is a Hermitian operator with

\[
H - \eta_i \rho_i \in \text{SEP}^* \quad \forall i \in \Lambda,
\]

then \( \text{Tr} H \) is an upper bound of \( p_{\text{L}}(\mathcal{E}) \).

**Quantum state discrimination with postmeasurement information**

In the subsection, we consider ME of \( \mathcal{E} \) in Eq. (4) when the classical information \( b \in \{ 0, 1 \} \) about the prepared subensemble \( \mathcal{E}_b \) defined in Eq. (5) is given after performing a measurement. In this situation, it is known that a measurement can be expressed by a POVM \( \{ M_\omega \}_{\omega \in \Omega} \) with the Cartesian product outcome space

\[
\Omega = \mathcal{A}_0 \times \mathcal{A}_1, \tag{11}
\]

where each \( M_{(\omega_0, \omega_1)} \) indicates the detection of \( \rho_{\omega_0} \) or \( \rho_{\omega_1} \) according to PI \( b = 0 \) or 1, respectively [17, 18].

ME of \( \mathcal{E} \) with PI is to minimize the average error probability. Equivalently, ME of \( \mathcal{E} \) with PI is to maximize the average probability of correct guessing where the optimal success probability is defined as

\[
p_{\text{PI}}(\mathcal{E}) = \max_{\{ M_\omega \}_\omega \in \Omega} \sum_{b \in \{ 0, 1 \}} \sum_{i \in \mathcal{A}_0} \eta_i \text{Tr}[\rho_i \sum_{\omega \in \Omega} M_\omega]. \tag{12}
\]
over all possible POVMs.

When the available measurements are limited to LOCC measurements, we denote the maximum success probability by

$$p_{L}^{\text{PI}}(\mathcal{E}) = \max_{\text{LOCC}} \sum_{b \in \{0,1\}} \sum_{i \in A_b} \eta_i \text{Tr}\left[\rho_i \sum_{\vec{\omega} \in \Omega} M_{\vec{\omega}}\right]$$

$$= 2 \max_{\text{LOCC}} \sum_{\vec{\omega} \in \Omega} \tau_{\vec{\omega}} \text{Tr}(\hat{\rho}_{\vec{\omega}} M_{\vec{\omega}}),$$

(13)

where

$$\tau_{\vec{\omega}} = \frac{1}{2} \sum_{b \in \{0,1\}} \eta_{wb}, \quad \hat{\rho}_{\vec{\omega}} = \frac{\sum_{b \in \{0,1\}} \eta_{wb} \rho_{wb}}{\sum_{b' \in \{0,1\}} \eta_{wb'}}.$$

Because the states in $\mathcal{E}$ are non-entangled, NLWE occurs in terms of ME with PI if and only if ME of $\mathcal{E}$ with PI cannot be achieved only by LOCC, that is,

$$p_{L}^{\text{PI}}(\mathcal{E}) < p_{G}^{\text{PI}}(\mathcal{E}).$$

(15)

Here, we note that $\{\tau_{\vec{\omega}}\}_{\vec{\omega} \in \Omega}$ is a set of positive numbers satisfying $\sum_{\vec{\omega} \in \Omega} \tau_{\vec{\omega}} = 1$ and $\{\hat{\rho}_{\vec{\omega}}\}_{\vec{\omega} \in \Omega}$ is a set of density operators. Thus, Eq. (13) implies that $p_{L}^{\text{PI}}(\mathcal{E})$ is twice the maximum success probability for ME of $\hat{\mathcal{E}}$,

$$p_{L}^{\text{PI}}(\mathcal{E}) = 2p_{L}(\hat{\mathcal{E}}),$$

(16)

where $\hat{\mathcal{E}}$ is the ensemble consisting of the average states $\hat{\rho}_{\vec{\omega}}$ prepared with the nonzero probabilities $\tau_{\vec{\omega}}$ in Eq. (14),

$$\hat{\mathcal{E}} = \{\tau_{\vec{\omega}}, \hat{\rho}_{\vec{\omega}}\}_{\vec{\omega} \in \Omega}.$$

(17)

In the following lemma, we provide an upper bound of $p_{L}^{\text{PI}}(\mathcal{E})$.

**Lemma 1.** If $\hat{H}$ is a Hermitian operator satisfying

$$\hat{H} - \eta_{\vec{\omega}} \hat{\rho}_{\vec{\omega}} \in \text{SEP} \quad \forall \vec{\omega} \in \Omega,$$

then $2\text{Tr}\hat{H}$ is an upper bound of $p_{L}^{\text{PI}}(\mathcal{E})$.

**Proof.** For the ensemble $\hat{\mathcal{E}}$ in Eq. (17), Proposition 1 implies that $\text{Tr}\hat{H}$ is an upper bound of $p_{L}(\hat{\mathcal{E}})$. Thus, $2\text{Tr}\hat{H}$ is an upper bound of $p_{L}^{\text{PI}}(\mathcal{E})$ due to Eq. (16).

**Locking NLWE by postmeasurement information**

In this section, we consider a situation where the PI about the prepared subensemble $\mathcal{E}_{b}$ in Eq. (5) locks NLWE. We first provide a specific example of a state ensemble $\mathcal{E}$ in Eq. (4) and show that NLWE occurs in discriminating the states in the ensemble. With the same ensemble, we further show that the occurrence of NLWE in the state discrimination can be vanished if the PI about the prepared subensemble is available, thus locking NLWE by PI.

**Example 1.** Let us consider the ensemble $\mathcal{E}$ in Eq. (4) with

$$\eta_0 = \frac{\gamma}{2(1+\gamma)}, \quad \rho_0 = |0\rangle\langle 0| \otimes |0\rangle\langle 0|,$$

$$\eta_1 = \frac{\gamma}{2(1+\gamma)}, \quad \rho_1 = |0\rangle\langle 0| \otimes |1\rangle\langle 1|,$$

$$\eta_+ = \frac{1}{2(1+\gamma)}, \quad \rho_+ = |+\rangle\langle +| \otimes |+\rangle\langle +|,$$

$$\eta_- = \frac{1}{2(1+\gamma)}, \quad \rho_- = |\rangle\langle \rangle \otimes |\rangle\langle \rangle,$$

(19)

where $2 \leq \gamma < \infty$, $\{0,1\}$ is the standard basis in one-qubit system, and $|\rangle\langle \rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. In this case, the subensembles in Eq. (5) become

$$\mathcal{E}_0 = \{\frac{1}{2}, |0\rangle\langle 0| \otimes |0\rangle, \frac{1}{2}, |0\rangle\langle 0| \otimes |1\rangle\langle 1|\},$$

(20)

$$\mathcal{E}_1 = \{\frac{1}{2}, |+\rangle\langle +| \otimes |+\rangle\langle +|, \frac{1}{2}, |\rangle\langle \rangle \otimes |\rangle\langle \rangle\},$$

with the probabilities of preparation $\gamma/(1+\gamma)$ and $\gamma/(1+\gamma)$, respectively.

To show the occurrence of NLWE in terms of ME about the stat ensemble $\mathcal{E}$ in Example 1, we first evaluate the optimal success probability $p_{G}(\mathcal{E})$ defined in Eq. (6). From the optimality condition in Eq. (7) together with a straightforward calculation, we can easily verify that the following POVM $\{M_{i}\}_{i \in \Lambda}$ is optimal for $p_{G}(\mathcal{E})$:

$$M_0 = |\Gamma_0\rangle\langle \Gamma_0|, \quad M_+ = |\mu_+\rangle\langle \mu_+| \otimes |+\rangle\langle +|,$$

(21)

$$M_1 = |\Gamma_1\rangle\langle \Gamma_1|, \quad M_- = |\mu_-\rangle\langle \mu_-| \otimes |-\rangle\langle -|,$$

where

$$|\mu_\pm\rangle = \sqrt{\frac{1}{2} - \frac{\gamma}{2(1+\gamma)^2}} |0\rangle \pm \sqrt{\frac{1}{2} + \frac{\gamma}{2(1+\gamma)^2}} |1\rangle,$$

$$|\Gamma_0\rangle = \frac{1}{\sqrt{2}} + \frac{\gamma}{2(1+\gamma)^2} |00\rangle - \frac{1}{\sqrt{2}} - \frac{\gamma}{2(1+\gamma)^2} |11\rangle,$$

$$|\Gamma_1\rangle = \frac{1}{\sqrt{2}} + \frac{1}{2(1+\gamma)^2} |01\rangle - \frac{1}{2} - \frac{\gamma}{2(1+\gamma)^2} |10\rangle.$$

Thus, the optimality of the POVM $\{M_{i}\}_{i \in \Lambda}$ in Eq. (21) and the definition of $p_{G}(\mathcal{E})$ lead us to

$$p_{G}(\mathcal{E}) = \frac{1}{2} \left(1 + \sqrt{\frac{1+\gamma^2}{1+\gamma}}\right).$$

(23)

In order to obtain the maximum success probability $p_{L}(\mathcal{E})$ in Eq. (8), we consider lower and upper bounds of $p_{L}(\mathcal{E})$. A lower bound of $p_{L}(\mathcal{E})$ can be obtained from the following POVM $\{M_{i}\}_{i \in \Lambda}$,

$$M_0 = |0\rangle\langle 0| \otimes |0\rangle\langle 0|, \quad M_+ = |1\rangle\langle 1| \otimes |+\rangle\langle +|,$$

(24)

$$M_1 = |0\rangle\langle 0| \otimes |1\rangle\langle 1|, \quad M_- = |1\rangle\langle 1| \otimes |-\rangle\langle -|,$$

which gives $\frac{1}{2}(1 + \frac{\gamma}{1+\gamma})$ as the success probability in discriminating the states of the ensemble $\mathcal{E}$ in Example 1. We also note that the measurement given in Eq. (24) can be achieved with finite-round LOCC: we perform a measurement $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ on first subsystem and measure $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ or $\{|+\rangle\langle +|, |-\rangle\langle -|\}$ on second subsystem depending on the first measurement result $|0\rangle\langle 0|$ or $|1\rangle\langle 1|$. Thus, the success probability for the LOCC measurement in Eq. (24) is a lower bound of $p_{L}(\mathcal{E})$, and

$$p_{L}(\mathcal{E}) \geq \frac{1}{2} \left(1 + \sqrt{\frac{1+\gamma^2}{1+\gamma}}\right).$$

(25)
To obtain an upper bound of $p_L(\mathcal{E})$, let us consider a Hermitian operator,

$$H = \frac{1}{4(1+\gamma)} (2\gamma |0\rangle\langle 0| \otimes \sigma_0 + |1\rangle\langle 1| \otimes \sigma_0 + \sigma_1 \otimes \sigma_1),$$

(26)

where $\sigma_0$ and $\sigma_1$ are the Pauli operators,

$$\sigma_0 = |0\rangle\langle 0| + |1\rangle\langle 1|,$$

$$\sigma_1 = |0\rangle\langle 1| + |1\rangle\langle 0|.$$  

(27)

We will show that $H - \eta_i \rho_i \in \text{SEP}^*$ for any $i \in \Lambda$, therefore $\text{Tr} H$ is an upper bound of $p_L(\mathcal{E})$ by Proposition 1.

For each $i \in \Lambda$, $H - \eta_i \rho_i$ can be rewritten as

$$H - \eta_0 \rho_0 = \frac{1}{4(1+\gamma)} \left[ T_0 + |11\rangle\langle 11| + \text{PT}(T_0) \right],$$

$$H - \eta_1 \rho_1 = \frac{1}{4(1+\gamma)} \left[ T_1 + |10\rangle\langle 10| + \text{PT}(T_1) \right],$$

$$H - \eta_+ \rho_+ = \frac{1}{4(1+\gamma)} \left[ \rho_0 + \rho_1 + \frac{1}{2(1+\gamma)} \rho_-, \right],$$

$$H - \eta_- \rho_- = \frac{1}{4(1+\gamma)} \left[ \rho_0 + \rho_1 + \frac{1}{2(1+\gamma)} \rho_+, \right],$$

(28)

where $T_0$ and $T_1$ are positive-semidefinite operators,

$$T_0 = \gamma |01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| + \frac{1}{4} |10\rangle\langle 10|,$$

$$T_1 = \gamma |00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + \frac{1}{4} |11\rangle\langle 11|,$$

for $2 \leq \gamma < \infty$. In other words, each $H - \eta_i \rho_i$ in Eq. (28) is a sum of positive-semidefinite operators and PPT operators. From the argument after Eq. (3), $H - \eta_i \rho_i$ is in $\text{SEP}^*$ for each $i \in \Lambda$, thus Proposition 1 leads us to

$$p_L(\mathcal{E}) \leq \text{Tr} H = \frac{1}{2} \left( 1 + \frac{\gamma}{1+\gamma} \right).$$

(30)

Inequalities (25) and (30) imply

$$p_L(\mathcal{E}) = \frac{1}{2} \left( 1 + \frac{\gamma}{1+\gamma} \right).$$

(31)

From Eqs. (23) and (31), we note that there exists a nonzero gap between $p_G(\mathcal{E})$ and $p_L(\mathcal{E})$, $p_L(\mathcal{E}) = \frac{1}{2} \left( 1 + \frac{\gamma}{1+\gamma} \right) < \frac{1}{2} \left( 1 + \frac{\sqrt{1+\gamma^2}}{1+\gamma^2} \right) = p_G(\mathcal{E})$, for $2 \leq \gamma < \infty$, thus NLWE occurs in terms of ME in discriminating the states of the ensemble $\mathcal{E}$ in Example 1.

Now, we show that the occurrence of NLWE in Inequality (32) can be vanished when the PI about the prepared subensemble is available. Let us consider the following POVM $\{M_{b}\}_{b \in \Omega}$,

$$M_{(0,+)} = |+\rangle\langle +| \otimes |0\rangle\langle 0|,$$

$$M_{(1,+)} = |+\rangle\langle +| \otimes |1\rangle\langle 1|,$$

$$M_{(0,-)} = |-\rangle\langle -| \otimes |0\rangle\langle 0|,$$

$$M_{(1,-)} = |-\rangle\langle -| \otimes |1\rangle\langle 1|,$$

(33)

which can be done using finite-round LOCC: two local measurements $\{|+\rangle, |\rangle\}$ and $\{|0\rangle, |1\rangle\}$ are performed on first and second subsystems, respectively. Moreover, it is a straightforward calculation to show that the success probability for the LOCC measurement of Eq. (33) in discriminating the states in the ensemble $\mathcal{E}$ with PI is one. That is, the states in $\mathcal{E}$ can be perfectly discriminated when PI is available.

FIG. 1. Locking NLWE by PI in terms of ME. For all $\eta_0 \in [\frac{1}{3}, \frac{1}{2})$, $p_L(\mathcal{E})$(dashed blue) is less than $p_G(\mathcal{E})$(solid blue), but $p_G(\mathcal{E})(\text{red})$ is equal to $p_L(\mathcal{E})(\text{red})$.

We note that the success probability obtained from the LOCC measurement in Eq. (33) is a lower bound of $p_L(\mathcal{E})$ in Eq. (13), therefore

$$p_L(\mathcal{E}) \equiv 1$$

(34)

for the ensemble $\mathcal{E}$ in Example 1. Moreover, from the definitions of $p_G(\mathcal{E})$ and $p_L(\mathcal{E})$ in Eqs. (12) and (13), respectively, we have

$$p_G(\mathcal{E}) \equiv p_L(\mathcal{E}).$$

(35)

As both $p_G(\mathcal{E})$ and $p_L(\mathcal{E})$ are bounded above by 1, we have

$$p_G(\mathcal{E}) = p_L(\mathcal{E}) = 1.$$

(36)

Thus, NLWE does not occur in terms ME in discriminating the states of the ensemble $\mathcal{E}$ in Example 1 when the PI about the prepared subensemble is available.

Inequality (32) shows that NLWE occurs in terms of ME about the ensemble $\mathcal{E}$ in Example 1, whereas Eq. (36) shows that NLWE does not occur when PI is available. Figure 1 illustrates the relative order of $p_G(\mathcal{E})$, $p_L(\mathcal{E})$, $p_G^\text{PI}(\mathcal{E})$, and $p_L^\text{PI}(\mathcal{E})$ for the range of $\frac{1}{3} \leq \eta_0 < \frac{1}{2}$.

**Theorem 1.** For ME of the ensemble $\mathcal{E}$ in Example 1, the PI about the prepared subensemble locks NLWE.

**Unlocking NLWE by postmeasurement information**

In this section, we consider the opposite situation to Sec. 3; the PI about the prepared subensemble $\mathcal{E}_b$ in Eq. (5) unlocks NLWE. After providing an example of a state ensemble $\mathcal{E}$ in Eq. (4), we first show that NLWE
does not occur in discriminating the states of the ensemble. With the same ensemble, we further show the occurrence of NLWE in the state discrimination with the help of PI, thus unlocking NLWE by PI.

**Example 2.** Let us consider the ensemble $\mathcal{E}$ in Eq. (4) with

$$
\begin{align*}
\eta_0 &= \frac{1}{2}(1+\sqrt{1+\frac{\gamma}{2}}), \\
\eta_1 &= \frac{1}{2}(1+\sqrt{1+\frac{\gamma}{2}}), \\
\eta_+ &= \frac{1}{2}(1+\sqrt{1+\frac{\gamma}{2}}), \\
\eta_- &= \frac{1}{2}(1+\sqrt{1+\frac{\gamma}{2}}),
\end{align*}
$$

where $2 \leq \gamma < \infty$. In this case, the subensembles in Eq. (5) become

$$
\mathcal{E}_0 = \left\{ \frac{1}{2} |0\rangle|0\rangle \otimes |0\rangle, \frac{1}{2} |0\rangle|0\rangle \otimes |1\rangle \right\},
\mathcal{E}_1 = \left\{ \frac{1}{2} |+\rangle|+\rangle \otimes |1\rangle, \frac{1}{2} |+\rangle|+\rangle \otimes |+\rangle \right\},
$$

with the probabilities of preparation $\frac{1}{1+\gamma}$ and $\frac{1}{1+\gamma}$, respectively.

To show the non-occurrence of NLWE in terms of ME about the ensemble $\mathcal{E}$ in Example 2, we first evaluate the optimal success probability $p_G(\mathcal{E})$ defined in Eq. (6). From the optimality condition in Eq. (7) together with a straightforward calculation, we can easily verify that the following POVM $\{M_i\}_{i \in I}$ is optimal for $p_G(\mathcal{E})$:

$$
\begin{align*}
M_0 &= |\nu_-\rangle|\nu_-\rangle \otimes |0\rangle|0\rangle, \\
M_+ &= |\nu_+\rangle|\nu_+\rangle \otimes |+\rangle|+\rangle, \\
M_1 &= |\nu_-\rangle|\nu_-\rangle \otimes |1\rangle|1\rangle, \\
M_- &= |\nu_+\rangle|\nu_+\rangle \otimes |-\rangle|-\rangle,
\end{align*}
$$

where

$$
|\nu_\pm\rangle = \frac{1}{\sqrt{2}} \left[ \frac{1}{\sqrt{1+\frac{\gamma}{2}}} |0\rangle \pm \frac{1}{\sqrt{1+\frac{\gamma}{2}}} |1\rangle \right].
$$

Thus, the optimality of the POVM $\{M_i\}_{i \in I}$ in Eq. (39) and the definition of $p_G(\mathcal{E})$ lead us to

$$
p_G(\mathcal{E}) = \frac{1}{2} \left( 1 + \frac{\sqrt{1+\gamma}}{1+\gamma} \right).
$$

The measurement given in Eq. (39) can be achieved with finite-round LOCC: first, a local measurement $\{|\nu_\pm\rangle, |\nu_-\rangle\}$ is performed on first subsystem, and then according to $|\nu_\pm\rangle$ or $|\nu_-\rangle$, a local measurement $\{|+\rangle|+\rangle, |-\rangle|-\rangle\}$ or $\{|0\rangle|0\rangle, |1\rangle|1\rangle\}$ is performed on second subsystem. Thus, the success probability for the LOCC measurement in Eq. (39) is a lower bound of $p_L(\mathcal{E})$ in Eq. (8), therefore

$$
p_L(\mathcal{E}) \geq \frac{1}{2} \left( 1 + \frac{\sqrt{1+\gamma}}{1+\gamma} \right) \tag{42}
$$

for the ensemble $\mathcal{E}$ in Example 2. Moreover, from the definitions of $p_G(\mathcal{E})$ and $p_L(\mathcal{E})$ in Eqs. (6) and (8), respectively, we have

$$
p_G(\mathcal{E}) \geq p_L(\mathcal{E}). \tag{43}
$$

Inequalities (42) and (43) lead us to

$$
p_L(\mathcal{E}) = p_G(\mathcal{E}) = \frac{1}{2} \left( 1 + \frac{\sqrt{1+\gamma}}{1+\gamma} \right). \tag{44}
$$

Thus, NLWE does not occur in terms of ME in discriminating the states of the ensemble $\mathcal{E}$ in Example 2.

Now, we show that NLWE occurs when the PI about the prepared subensemble is available. Let us consider the following POVM $\{M_i\}_{i \in \Omega}$,

$$
\begin{align*}
M_{0,+} &= |\Phi_+\rangle|\Phi_+\rangle, \\
M_{0,-} &= |\Phi_-\rangle|\Phi_-\rangle, \\
M_{1,+} &= |\Psi_+\rangle|\Psi_+\rangle, \\
M_{1,-} &= |\Psi_-\rangle|\Psi_-\rangle,
\end{align*}
$$

where $|\Phi_\pm\rangle$ and $|\Psi_\pm\rangle$ are Bell states,

$$
|\Phi_\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \\
|\Psi_\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle). \tag{46}
$$

From a straightforward calculation, we can easily see that the success probability obtained from the measurement of Eq. (45) in discriminating the states in the ensemble $\mathcal{E}$ with PI is one,

$$
p_{G_{PI}}(\mathcal{E}) = 1. \tag{47}
$$

That is, the states in of $\mathcal{E}$ can be perfectly discriminated when PI is available.

In order to obtain the maximum success probability $p_{G_{PI}}(\mathcal{E})$ in Eq. (13), we consider lower and upper bounds of $p_{G_{PI}}(\mathcal{E})$. For a lower bound of $p_{G_{PI}}(\mathcal{E})$, let us first consider the average state ensemble $\mathcal{E}$ defined in Eqs. (14) and (17) with respect to Example 2:

$$
\bar{\eta}_z = \frac{1}{2} \forall \bar{\omega} \in \Omega,
\bar{\rho}_{0,z} = \frac{n_0}{\eta_0 + \eta_z} \rho_0 + \frac{\eta_z}{\eta_0 + \eta_z} \rho_z
= \frac{1}{1+\gamma} |0\rangle\langle 0| \otimes |0\rangle\langle 0| + \frac{1}{1+\gamma} |+\rangle\langle +| \otimes |+\rangle\langle +|, \tag{48}
\bar{\rho}_{1,z} = \frac{n_1}{n_0 + n_2} \rho_1 + \frac{n_2}{n_0 + n_2} \rho_2
= \frac{1}{1+\gamma} |0\rangle\langle 1| \otimes |1\rangle\langle 1| + \frac{1}{1+\gamma} |+\rangle\langle +| \otimes |+\rangle\langle +|,
$$

which satisfy

$$
(\sigma_0 \otimes \sigma_2) \bar{\rho}_{0,z} (\sigma_0 \otimes \sigma_2) = \bar{\rho}_{1,z}, \tag{49a}
(\sigma_0 \otimes \sigma_1) \bar{\rho}_{0,z} (\sigma_0 \otimes \sigma_1) = \bar{\rho}_{1,z}, \tag{49b}
$$

with the Pauli operators $\sigma_0$ and $\sigma_1$ in Eq. (27) and

$$
s_2 = -i |0\rangle\langle 1| + i |1\rangle\langle 0|. \tag{50}
$$

We further consider the following Hermitian operators,

$$
\bar{\rho}_{0,+} - \bar{\rho}_{0,-}, \bar{\rho}_{1,+} - \bar{\rho}_{0,-}, \tag{51}
$$

where both of them have the same four eigenvalues: two positive eigenvalues $\lambda_+$ and $\lambda_-$, and two negative eigenvalues $-\lambda_+$ and $-\lambda_-$ with

$$
\lambda_\pm = \frac{\sqrt{1+\gamma-\gamma^2} \pm \sqrt{1-\gamma+\gamma^2}}{2(1+\gamma)} \tag{52}
$$
for $2 \leq \gamma < \infty$. We denote $\Pi_{(0,+)}$ and $\Pi_{(1,-)}$ as the projection operators onto the positive and negative eigenspaces of $\hat{\rho}_{(0,+)} - \hat{\rho}_{(1,-)}$, respectively. Similarly, we denote $\Pi_{(1,+)}$ and $\Pi_{(0,-)}$ as the projection operators onto the positive and negative eigenspaces of $\hat{\rho}_{(1,+)} - \hat{\rho}_{(0,-)}$, respectively.

Now, we consider the following POVM $\{M_\omega\}_{\omega \in \Omega}$.

$$M_{(0,+)} = \frac{1}{2} \Pi_{(0,+)}, \quad M_{(0,-)} = \frac{1}{2} \Pi_{(0,-)},$$
$$M_{(1,+)} = \frac{1}{2} \Pi_{(1,+)}, \quad M_{(1,-)} = \frac{1}{2} \Pi_{(1,-)}. \quad (53)$$

From the property of (49a) and the definition of $\Pi_\omega$, we can see that

$$(\sigma_0 \otimes \sigma_2) \Pi_{(0,+)} (\sigma_0 \otimes \sigma_2) = \Pi_{(1,-)},$$
$$(\sigma_0 \otimes \sigma_2) \Pi_{(1,+)} (\sigma_0 \otimes \sigma_2) = \Pi_{(0,-)},$$
$$\Pi_{(0,+)} + \Pi_{(1,-)} = 1,$$
$$\Pi_{(1,+)} + \Pi_{(0,-)} = 1. \quad (54)$$

Here we note that for any Hermitian operator $A$ satisfying

$$A + (\sigma_0 \otimes \sigma_2) A (\sigma_0 \otimes \sigma_2) = 1,$$

it holds that

$$\langle i0|A|j1 \rangle = \langle i1|A|j0 \rangle$$

for any $i, j \in \{0, 1\}$. From Eqs. (54), (55), and (56), we have

$$PT(\Pi_\omega) = \Pi_\omega \quad \forall \omega \in \Omega, \quad (57)$$

which implies that $\Pi_\omega$ is in SEP for any $\omega \in \Omega$. Thus, two POVMs $\{\Pi_{(0,+)}, \Pi_{(1,-)}\}$ and $\{\Pi_{(1,+)}, \Pi_{(0,-)}\}$ are separable. Moreover, both of them can be performed using finite-round LOCC because each of them consists of two orthogonal rank-2 projection operators. The measurement given in Eq. (53) can be realized with finite-round LOCC by performing two LOCC measurements $\{\Pi_{(0,+)}\}$ and $\{\Pi_{(1,+)}\}$ with the equal probability $\frac{1}{2}$.

The success probability of the LOCC measurement in Eq. (53) for the average state ensemble $\bar{\mathcal{E}}$ in Eq. (48) is

$$\sum_{\omega \in \Omega} \hat{n}_\omega \text{Tr}(\hat{\rho}_\omega M_\omega) = \frac{1}{4} \left(1 + \frac{\sqrt{1 + \gamma^2}}{1 + \gamma}\right). \quad (58)$$

This probability is upper bounded by $p_{L1}(\mathcal{E})$ which is the maximum success probability for ME of $\mathcal{E}$ when the available measurements are limited to LOCC measurements,

$$p_{L}(\mathcal{E}) \geq \frac{1}{4} \left(1 + \frac{\sqrt{1 + \gamma^2}}{1 + \gamma}\right). \quad (59)$$

Since any lower bound of $2p_{L}(\mathcal{E})$ becomes a lower bound of $p_{L1}^{PL}(\mathcal{E})$ due to Eq. (16), we have

$$p_{L1}^{PL}(\mathcal{E}) \geq \frac{1}{2} \left(1 + \frac{\sqrt{1 + \gamma^2}}{1 + \gamma}\right). \quad (60)$$

To obtain an upper bound of $p_{L1}^{PL}(\mathcal{E})$, let us first consider the following two operators,

$$K_0 = \frac{1}{2} \hat{\rho}_{(0,+)} \Pi_{(0,+)} + \frac{1}{2} \hat{\rho}_{(1,-)} \Pi_{(1,-)},$$
$$K_1 = \frac{1}{2} \hat{\rho}_{(1,+)} \Pi_{(1,+)} + \frac{1}{2} \hat{\rho}_{(0,-)} \Pi_{(0,-)}. \quad (61)$$

Since the projective measurement $\{\Pi_{(0,+)}, \Pi_{(1,-)}\}$ is optimal in ME between two states $\hat{\rho}_{(0,+)}$ and $\hat{\rho}_{(1,-)}$ with equal prior probability [20], it satisfies a necessary and sufficient condition for a measurement to be optimal in ME between two states $\hat{\rho}_{(0,+)}$ and $\hat{\rho}_{(1,-)}$ with equal prior probability $\frac{1}{2}$ [21, 22, 27],

$$K_0 - \frac{1}{2} \hat{\rho}_{(0,+)} \geq 0, \quad K_0 - \frac{1}{2} \hat{\rho}_{(1,-)} \geq 0. \quad (62)$$

Similarly, $\{\Pi_{(1,+)}, \Pi_{(0,-)}\}$ is the optimal measurement in ME between two states $\hat{\rho}_{(1,+)}$ and $\hat{\rho}_{(0,-)}$ with equal prior probability $\frac{1}{2}$, thus

$$K_1 - \frac{1}{2} \hat{\rho}_{(1,+)} \geq 0, \quad K_1 - \frac{1}{2} \hat{\rho}_{(0,-)} \geq 0. \quad (63)$$

We further note that $K_0$ and $K_1$ are Hermitian operators due to the positive semidefiniteness of (62) and (63).

Now, we consider a Hermitian operator,

$$\hat{H} = \frac{1}{2} K_0 + \frac{1}{2} K_1. \quad (64)$$

We will show that $\hat{H} - \hat{n}_\omega \hat{\rho}_\omega \in \text{SEP}^*$ for all $\omega \in \Omega$, therefore $2\text{Tr} H$ is the upper bound of $p_{L1}^{PL}(\mathcal{E})$ by Lemma 1. From Eqs. (49) and (54), we can see that

$$(\sigma_0 \otimes \sigma_1) K_0 (\sigma_0 \otimes \sigma_1) = K_1,$$
$$(\sigma_0 \otimes \sigma_1) K_1 (\sigma_0 \otimes \sigma_1) = K_0,$$
$$(\sigma_0 \otimes \sigma_2) K_0 (\sigma_0 \otimes \sigma_2) = K_0,$$
$$(\sigma_0 \otimes \sigma_2) K_1 (\sigma_0 \otimes \sigma_2) = K_1. \quad (65)$$

Moreover, for any Hermitian operator $A$ with

$$(\sigma_0 \otimes \sigma_2) A (\sigma_0 \otimes \sigma_2) = A,$$

it holds that

$$\langle i0|A|j0 \rangle = \langle i1|A|j1 \rangle, \quad \langle i0|A|j1 \rangle = -\langle i1|A|j0 \rangle \quad (66)$$

for any $i, j \in \{0, 1\}$. From Eqs. (65), (66), and (67), we have

$$PT(K_0) = (\sigma_0 \otimes \sigma_1) K_0 (\sigma_0 \otimes \sigma_1) = K_1,$$
$$PT(K_1) = (\sigma_0 \otimes \sigma_1) K_1 (\sigma_0 \otimes \sigma_1) = K_0. \quad (68)$$

Thus, for each $\omega \in \Omega$, $\hat{H} - \hat{n}_\omega \hat{\rho}_\omega$ can be rewritten as

$$\hat{H} - \hat{n}_\omega \hat{\rho}_\omega = \frac{1}{4} \left(K_0 - \frac{1}{2} \hat{\rho}_{(0,+)}\right)$$
$$+ \frac{1}{4} \text{PT}(K_0 - \frac{1}{2} \hat{\rho}_{(0,+)}),$$
$$\hat{H} - \hat{n}_\omega \hat{\rho}_\omega = \frac{1}{4} \left(K_0 - \frac{1}{2} \hat{\rho}_{(0,+)}\right)$$
$$+ \frac{1}{4} \text{PT}(K_0 - \frac{1}{2} \hat{\rho}_{(0,+)}),$$
$$\hat{H} - \hat{n}_\omega \hat{\rho}_\omega = \frac{1}{4} \left(K_1 - \frac{1}{2} \hat{\rho}_{(1,+)}\right)$$
$$+ \frac{1}{4} \text{PT}(K_1 - \frac{1}{2} \hat{\rho}_{(1,+)}),$$
$$\hat{H} - \hat{n}_\omega \hat{\rho}_\omega = \frac{1}{4} \left(K_1 - \frac{1}{2} \hat{\rho}_{(1,+)}\right)$$
$$+ \frac{1}{4} \text{PT}(K_1 - \frac{1}{2} \hat{\rho}_{(1,+)}). \quad (69)$$
For all $\eta_0 \in [\frac{1}{3}, \frac{1}{2})$, $p_L^\eta(\mathcal{E})$(blue) is equal to $p_C(\mathcal{E})$(blue), but $p_L^\eta(\mathcal{E})$(dashed red) is less than $p_G^\eta(\mathcal{E})$(solid red).

From the argument after Eq. (3) together with the positive semidefiniteness of (62) and (63), each $H - \tilde{\eta}_3 \tilde{\rho}_3$ in Eq. (69) is in SEP*, therefore Lemma 1 leads us to

$$p_L^\Pi(\mathcal{E}) \leq 2\text{Tr}H = \frac{1}{2} \left( 1 + \frac{\sqrt{1 + \gamma + \gamma^2}}{1 + \gamma} \right).$$  

(70)

Inequalities (60) and (70) imply

$$p_L^\Pi(\mathcal{E}) = \frac{1}{2} \left( 1 + \frac{\sqrt{1 + \gamma + \gamma^2}}{1 + \gamma} \right).$$  

(71)

From Eqs. (47) and (71), we note that there exists a nonzero gap between $p_G^\Pi(\mathcal{E})$ and $p_L^\Pi(\mathcal{E})$,

$$p_L^\Pi(\mathcal{E}) = \frac{1}{2} \left( 1 + \frac{\sqrt{1 + \gamma + \gamma^2}}{1 + \gamma} \right) < 1 = p_G^\Pi(\mathcal{E}),$$  

(72)

for $2 \leq \gamma < \infty$. Thus NLWE occurs in terms of ME when the PI about the prepared subensemble is available.

Equation (44) shows that NLWE does not occur in terms of ME about the ensemble $\mathcal{E}$ in Example 2, whereas Inequality (72) shows that NLWE occurs when PI is available. Figure 2 illustrates the relative order of $p_G(\mathcal{E})$, $p_L(\mathcal{E})$, $p_G^\Pi(\mathcal{E})$, and $p_L^\Pi(\mathcal{E})$ for the range of $\frac{1}{3} \leq \eta_0 < \frac{1}{2}$.

**Theorem 2.** For ME of the ensemble $\mathcal{E}$ in Example 2, the PI about the prepared subensemble unlocks NLWE.

**DISCUSSION**

We have shown that the PI about the prepared subensemble can lock or unlock NLWE in discriminating multi-party non-orthogonal non-entangled quantum states. We have first provided a two-qubit state ensemble consisting of four non-orthogonal separable states (Example 1) and shown that NLWE occurs in discriminating the states in the ensemble. With the same ensemble, we have further shown that the occurrence of NLWE in the state discrimination can be vanished when the PI about the prepared subensemble is available, thus locking NLWE by PI (Theorem 1). Moreover, we have provided another two-qubit state ensemble consisting of four non-orthogonal separable states (Example 2) and shown that NLWE does not occur in discriminating the states of the ensemble. With the same ensemble, we have further shown the occurrence of NLWE in the state discrimination with the PI about the prepared subensemble, thus unlocking NLWE by PI (Theorem 2).

We note that in both Examples 1 and 2, the prepared state can be perfectly identified by a global measurement when the PI about the prepared subensemble is provided. In Example 1, the prepared state can be perfectly identified by a LOCC measurement when the PI about the prepared subensemble is available. However, in Example 2, the prepared state cannot be perfectly discriminated by a LOCC measurement even if the PI about the prepared subensemble is available. As far as we known, the latter is the first example exhibiting NLWE in terms of perfect discrimination with the help of PI.

Our result can provide a useful method to share or hide information using non-orthogonal separable states [30–34]. In Example 1, the PI about the prepared subensemble makes the information locally accessible, and the information can be locally shared between parties. On the other hand, in Example 2, the PI about the prepared subensemble makes the information globally accessible but not locally, and the globally accessible information can be locally hidden to some extent. We finally remark that it would be an interesting future task to investigate if the availability of PI affects the occurrence of NLWE in terms of other optimal discrimination strategies besides ME.

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[1] Chefles, A. Quantum state discrimination, Contemporary Physics 41, 401 (2000).

[2] Barnett, S. M. & Croke, S. Quantum state discrimination, Adv. Opt. Photon. 1, 238 (2009).
Bhattacharya, S. S., Saha, S., Guha, T. & Banik, M.

Akibue, S., Kato, G. & Marumo, N. Perfect discrimination of nonorthogonal quantum states with posterior classical partial information, Phys. Rev. A 99, 020102 (2019).

Ballester, M. A., Wehner, S. & Winter, A. Quantum nonlocality without entanglement, J. Mod. Opt. 57, 160 (2010).

Bennett, C. H., DiVincenzo, D. P., Fuchs, C. A., Mor, T., Rains, E., Shor, P. W., Smolin, J. A. & Wootters, W. K. Entanglement of quantum states, Phys. Rev. A 59, 1070 (1999).

Peres, A. & Wootters, W. K. Optimal detection of quantum information, Phys. Rev. Lett. 66, 1119 (1991).

Chitambar, E. & Hsieh, M.-H. Revisiting the optimal detection of quantum information, Phys. Rev. A 88, 020302(R) (2013).

Chitambar, E., Leung, D., Manˇ cinska, L., Ozols, M. & Winter, A. Everything you always wanted to know about LOCC (but were afraid to ask), Commun. Math. Phys. 328, 303 (2014).

Peres, A. Separability criterion for density matrices, Phys. Rev. Lett. 77, 1413 (1996).

Horodecki, M., Horodecki, P. & Horodecki, R. Separability of mixed states: necessary and sufficient conditions, Phys. Lett. A 223, 1 (1996).

Barnett, S. M. & Croke, S. On the conditions for discrimination between quantum states with minimum error, J. Phys. A: Math. and Theor. 42, 062001 (2009).

Bandyopadhyay, S., Cosentino, A., Johnston, N., Liu, S. & Winter, A. Quantum-secret-sharing scheme based on local distinguishability of orthogonal multiqudit entangled states, Phys. Rev. A 95, 022320 (2017).

Bennett, C. H., DiVincenzo, D. P., Fuchs, C. A., Mor, T., Rains, E., Shor, P. W., Smolin, J. A. & Wootters, W. K. Quantum nonlocality without entanglement, Phys. Rev. A 59, 1070 (1999).

Peres, A. & Wootters, W. K. Optimal detection of quantum information, Phys. Rev. Lett. 66, 1119 (1991).

Chitambar, E. & Hsieh, M.-H. Revisiting the optimal detection of quantum information, Phys. Rev. A 88, 020302(R) (2013).

Chitambar, E., Leung, D., Manˇ cinska, L., Ozols, M. & Winter, A. Everything you always wanted to know about LOCC (but were afraid to ask), Commun. Math. Phys. 328, 303 (2014).

Peres, A. Separability criterion for density matrices, Phys. Rev. Lett. 77, 1413 (1996).

Horodecki, M., Horodecki, P. & Horodecki, R. Separability of mixed states: necessary and sufficient conditions, Phys. Lett. A 223, 1 (1996).

Barnett, S. M. & Croke, S. On the conditions for discrimination between quantum states with minimum error, J. Phys. A: Math. and Theor. 42, 062001 (2009).

Bandyopadhyay, S., Cosentino, A., Johnston, N., Liu, S. & Winter, A. Quantum-secret-sharing scheme based on local distinguishability of orthogonal multiqudit entangled states, Phys. Rev. A 95, 022320 (2017).

Gopal, D. & Wehner, S. Using postmeasurement information in state discrimination, Phys. Rev. A 82, 022326 (2010).

Carmeli, C., Heinosaari, T. & Toigo, A. State discrimination with postmeasurement information and incompatibility of quantum measurements, Phys. Rev. A 98, 012126 (2018).

Helstrom, C. W. Quantum Detection and Estimation Theory (Academic Press, New York, 1976).

Holevo, A. S. Probabilistic and Statistical Aspects of Quantum Theory (North-Holland, 1979).

Yuen, H., Kennedy, R. & Lax, M. Optimum testing of multiple hypotheses in quantum detection theory, IEEE Trans. Inf. Theory 21, 125 (1975).

Bae, J. Structure of minimum-error quantum state discrimination, New J. Phys. 15, 073037 (2013).

Chitambar, E., Leung, D., Manˇ cinska, L., Ozols, M. & Winter, A. Everything you always wanted to know about LOCC (but were afraid to ask), Commun. Math. Phys. 328, 303 (2014).

Peres, A. Separability criterion for density matrices, Phys. Rev. Lett. 77, 1413 (1996).

Horodecki, M., Horodecki, P. & Horodecki, R. Separability of mixed states: necessary and sufficient conditions, Phys. Lett. A 223, 1 (1996).

Barnett, S. M. & Croke, S. On the conditions for discrimination between quantum states with minimum error, J. Phys. A: Math. and Theor. 42, 062001 (2009).

Bandyopadhyay, S., Cosentino, A., Johnston, N., Russo, V., Watrous, J. & Yu, N. Limitations on separable measurements by convex optimization, IEEE Trans. Inf. Theory 61, 3593 (2015).

Chitambar, E., Duan, R. & Hsieh, M.-H. When do local operations and classical communication suffice for two-qubit state discrimination?, IEEE Trans. Inf. Theory 60, 1549 (2014).

Terhal, B. M., DiVincenzo, D. P. & Leung, D. W. Hiding bits in bell states, Phys. Rev. Lett. 86, 5807 (2001).

DiVincenzo, D. P., Leung, D. W. & Terhal, B. M. Quantum data hiding, IEEE Trans. Inf. Theory 48, 580 (2002).

Eggeling, T. & Werner, R. F. Hiding classical data in multipartite quantum states, Phys. Rev. Lett. 89, 097905 (2002).

Rahaman, R. & Parker, M. G. Quantum scheme for secret sharing based on local distinguishability, Phys. Rev. A 91, 022330 (2015).

Wang, J., Li, L., Peng, H. & Yang, Y. Quantum-secret-sharing scheme based on local distinguishability of orthogonal multiqudit entangled states, Phys. Rev. A 95, 022320 (2017).