Noether charge astronomy

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Noether’s theorem identifies fundamental conserved quantities, called Noether charges, from a Hamiltonian. To-date Noether charges remain largely elusive within theories of gravity: We do not know how to directly measure them, and their physical interpretation remains unsettled in general spacetimes. Here we show that the surface gravity as naturally defined for a family of observers in arbitrarily dynamical spacetimes is a directly measurable Noether charge. This Noether charge reduces to the accepted value on stationary horizons, and, when integrated over a closed surface, yields an energy with the characteristics of gravitating mass. Stokes’ theorem then identifies the gravitating energy density as the time-component of a locally conserved Noether current in general spacetimes. Our conclusion, that this Noether charge is extractable from astronomical observations, holds the potential for determining the detailed distribution of the gravitating mass in galaxies, galaxy clusters and beyond.

In classical physics, Noether’s theorem shows that if the evolution generated by the Hamiltonian of a system leaves some quantity invariant then conversely the evolution generated by that quantity will leave the Hamiltonian invariant.\textsuperscript{[3]} Such conserved quantities are called Noether charges and their conservation makes them fundamental quantities of that system; notable examples include energy, momentum and electric charge. The proof of Noether’s result in the Hamiltonian formulation is especially elegant\textsuperscript{[2]} (see Appendix 1 for summary). Noether charges were first discovered in the context of general relativity\textsuperscript{[1]} and gravity is also the focus of this paper.

Already in 1959, Komar\textsuperscript{[3]} found that $J^\mu \equiv \xi_{\ [\mu}d^\nu \xi_{\nu]}$ is a locally conserved 4-current for an arbitrary vector field $\xi^\mu$. Indeed, its conservation law $J^\mu_{\ ;\mu} = 0$, which holds for general dynamical spacetimes, follows purely from geometric considerations (see Appendix 1). Integrating the current $J^\mu$ on any hypersurface $\Sigma$ and applying Stokes’ theorem yields

\begin{equation}
\frac{1}{4\pi} \int_\Sigma J^\mu d\Sigma_\mu = \frac{1}{4\pi} \int_{\partial \Sigma} \xi^{[\mu} d\Sigma_{\nu]},
\end{equation}

where $d\Sigma_\mu$ and $d\Sigma_{\mu\nu}$ are the tensorial volume and surface elements on the hypersurface $\Sigma$ and its boundary $\partial \Sigma$, respectively. (See the Appendix 1 for a more detailed discussion about the conservation laws associated with $J^\mu$.) Thus, Eq.\textsuperscript{[1]} immediately implies that

\begin{equation}
\xi^{[\mu;\nu]} d\Sigma_{\mu\nu},
\end{equation}

is a local Noether charge, up to the addition of an exact 2-form, for any theory of gravity with a torsion-free metric-compatible connection (see Appendix 1). Using the Lagrangian formalism,\textsuperscript{[2]} has been confirmed as a Noether charge specifically for general relativity\textsuperscript{[3,3]} and additional Noether charges have been identified for various diffeomorphically invariant theories of gravity.\textsuperscript{[2,3]}

One road block in the application of Komar’s result has been in identifying a choice of $\xi^\mu$ for which the Noether charge\textsuperscript{[2]} is directly observable.\textsuperscript{[2]} For stationary spacetimes, the inherent anti-symmetry of the Killing condition makes the Killing vector a natural choice of this vector field. Substituting $\xi^\mu = \partial_t$ in Eq.\textsuperscript{[1]} leads to the Komar mass.\textsuperscript{[3]} On asymptotically-flat stationary spacetimes the Komar mass is equivalent to the ADM mass\textsuperscript{[3]} and can be interpreted as the force-per-unit-mass exerted at spatial infinity needed to hold objects stationary. However, beyond the stationary setting, the Komar mass has eluded interpretation.\textsuperscript{[3]}

Here we find a choice for $\xi^\mu$ even for arbitrarily dynamical spacetimes, which yields a physically meaningful Noether charge. We prove the utility of this choice of $\xi^\mu$ by showing: that\textsuperscript{[2]} reduces to the surface gravity experienced by a family of observers on some surface of interest; that the corresponding integral, Eq.\textsuperscript{[1]}, corresponds to the gravitating mass; and that its density, $J^\mu \xi^\mu / 4\pi$, is the gravitating energy density. Finally, we illustrate how the detailed distribution of gravitating mass in astronomical systems may be determined from red-shift factors.

Surface gravity as a Noether charge

Consider a congruence of (in general non-geodesic) timelike observers moving tangent to some world tube $W$. We label the observers’ 4-velocities as $v^\mu$ and their red-shift factors by $\Phi \equiv d\tau / dt$, where $\tau$ is their proper time and $t$ is the coordinate time. Throughout the paper we work in geometric units where $G = c = 1$. Although not necessary for this formalism, we envision $t$ as the proper time of some, possibly distant, reference observer. For each ‘time slice’ there is a 2-surface of intersection $S$ described by the pointwise intersection of the world tube and hypersurfaces $\Sigma_t$ of constant $t$, $S = W \cap \Sigma_t$. See Fig.\textsuperscript{[1]} To measure a surface gravity at points on the 2-surface $S$ we take our observers’ 4-velocities $v^\mu$ to be orthogonal to the tangent space of this surface.
Theorem: Let $a_\mu \equiv v_\mu v^\nu \nu$ be the 4-acceleration of the observer and $\dot{R}^\mu$ is normal to $W$ at this point (see Fig. 1). Note, that $a_\mu \dot{R}^\mu$ is the component of acceleration normal to the 2-surface in the observer’s instantaneous rest frame. We shall now show that (3) is a Noether charge in the following sense:

Proof: We start by noting that

$$\kappa_{\text{Noether}} \equiv \Phi a_\mu \dot{R}^\mu = \Phi v_\mu v^\nu \nu \dot{R}^\mu$$

$$= \Phi v_\mu v^\nu \nu \dot{R}^\mu - \Phi v_\mu v^\nu \nu \dot{R}^\mu$$

$$= \Phi v_\mu v^\nu \nu \dot{R}^\mu - (\Phi v_\mu) v^\nu \nu \dot{R}^\mu$$

$$= 2(\Phi v_\mu) v^\nu \nu \dot{R}^\mu$$

where to obtain the second line we use the facts that $v_\mu \dot{R}^\mu = 0$, $v_\mu v_\mu = -1$ and hence $v_\mu v^\nu v^\nu = 0$. Noting that

$$\xi_\mu \equiv \frac{dx^\mu(\tau)}{d\tau} = \frac{dx^\mu(\tau)}{d\tau}$$

we easily see that $\xi_\mu \xi_\mu = -\Phi^2$. It follows that

$$(\xi_\mu \xi_\mu) v_\mu \dot{R}^\mu = -2(\Phi^2) v_\mu \dot{R}^\mu$$

$$\Rightarrow 2 \xi_\mu v_\mu \dot{R}^\mu = 2(\Phi^2) v_\mu \dot{R}^\mu$$

$$\Rightarrow \xi_\mu v_\mu \dot{R}^\mu = -\Phi^2 v_\mu \dot{R}^\mu.$$

Next, recalling that the Lie derivative of a vector is given by $\mathcal{L}_\xi (R^\mu) \equiv R^\mu_{\nu \rho} S^\nu - \xi_\mu R^\nu$, Eq. (6) becomes

$$\Phi a_\mu \dot{R}^\mu = -\xi_\mu v_\mu \dot{R}^\mu$$

$$= \mathcal{L}_\xi (\dot{R}^\mu) v_\mu - \dot{R}^\mu v_\mu$$

$$= \mathcal{L}_\xi (\dot{R}^\mu) v_\mu - (\dot{R}^\mu v_\mu) v^\nu \nu$$

$$= \mathcal{L}_\xi (\dot{R}^\mu) v_\mu + v_\mu \nu \dot{R}^\mu$$

$$= \Phi a_\mu \dot{R}^\mu + \mathcal{L}_\xi (\dot{R}^\mu) v_\mu,$$

(7)

where we used $\dot{R}^\mu v_\mu = 0$ and $\xi_\mu = \Phi v_\mu$ to obtain line three. To remove the Lie derivative term one might expect to have to apply a Killing condition. However, this is unnecessary since we may always describe $\dot{R}^\mu$ as the tangent vector to some spacelike path $x^\mu$ parameterized by proper length $s$, so

$$\dot{R}^\mu = \frac{dx^\mu(s)}{ds}.$$

In particular, recalling that by construction $\dot{x}^\mu = dx^\mu / dt$, we find

$$\mathcal{L}_\xi (\dot{R}^\mu) \equiv \dot{R}^\mu v^\nu \nu - \xi_\mu v^\nu \nu \dot{R}^\mu = \dot{R}^\mu v^\nu \nu - \xi_\mu v^\nu \nu \dot{R}^\mu$$

$$= \frac{\partial}{\partial x^\nu} \left( \frac{dx^\mu}{ds} \right) \frac{dx^\nu}{dt} - \frac{\partial}{\partial x^\nu} \left( \frac{dx^\mu}{dt} \right) \frac{dx^\nu}{ds}$$

$$= \frac{d^2 x^\mu}{dt^2} - \frac{d^2 x^\mu}{ds dt} \equiv 0,$$

(9)

where in the final step, the fact that the coordinates $x^\mu$ are scalar functions allows us to exchange the order of differentiation. Thus Eq. (7) reduces to

$$\Phi a_\mu \dot{R}^\mu = \kappa_{\text{Noether}}$$

and hence Eq. (4) becomes

$$\kappa_{\text{Noether}} g^\mu_\nu dA = \left( \Phi v_\mu \right) v^\nu \nu \dot{R}^\mu g^\mu_\nu dA$$

$$= \xi_\mu v^\nu \nu \dot{R}^\mu g^\mu_\nu dA$$

$$= \xi_\mu \dot{\Sigma} g^\mu_\nu dA = \xi_\mu \dot{\Sigma} g^\mu_\nu dA.$$

(11)

Here to obtain the second line we rely on the identification of $\xi_\mu = \Phi v_\mu$ and the anti-symmetry already present in the indices $(\mu, \nu)$. To obtain the third line note that the 4-velocity $v_\mu$ is orthogonal to the spacelike tangent space of $\Sigma$. Thus, the pair of unit vectors $\{\nu, R^\mu\}$ span the same 2-dimensional tangent space as the pair of unit vectors $\{\dot{R}^\mu, \dot{R}^\nu\}$. Finally, for the 2-surface $\Sigma$ in the hypersurface $\Sigma_t$, we have $d\Sigma_{\mu \nu} = \dot{R}^\nu \dot{N}_\mu dA$.

We have now derived the naturally defined surface gravity for a family of observers and shown that it is the Noether charge, $\kappa_{\text{Noether}}$. The construction of the Noether charge from (3) is not limited to stationary spacetimes, nor does this Theorem require the existence of a Killing vector. The generality of this result and its physical interpretation both derive from the scenario we considered: on the one hand, a general timelike world tube, and on the other, an observer on the 2-surface of...
that world tube, who therefore feels a physical surface gravity. Together, the generality and physical grounding of our result provide the in-principle possibility of direct observational access to a gravitational Noether charge in dynamical spacetimes.

By choosing differing world tubes one can address different physical questions. To conclude this section, we consider the surface gravity of spacetime horizons, where the choice of world tube is unambiguous. Timelike horizons are a special case of the timelike world tubes considered above, and may be treated in the same way. To derive the surface gravity, one need only choose the world tube $W$ as the horizon world tube and apply the expression for $\kappa_{\text{Noether}}$. The case of null horizons is considered next (spacelike horizons are not considered in this paper).

Surface gravity for null horizons: For world tubes of null horizons, one may take the local surface gravity to be the limit as the congruence of observers approaches the null world tube on the hypersurface of interest. In particular, we now show that in the stationary case, this surface gravity reduces to the standard result.

Consider a non-degenerate Killing horizon with Killing vector

$$K^\mu = (\partial_t)^\mu + \Omega_H (\partial_\phi)^\mu,$$

where $t$ is the coordinate time at spatial infinity, $\phi$ is the azimuthal angular coordinate, and the constant $\Omega_H$ is the angular velocity of the horizon for a Kerr black hole. According to the conventional definition, the surface gravity, $\kappa_{\text{Killing}}$, for such a Killing horizon satisfies

$$K^{\mu\nu} K_{\nu} = \kappa_{\text{Killing}} K^\mu,$$

on the horizon.

**Lemma 1:** The Noether surface gravity, $\kappa_{\text{Noether}}$, reduces to the standard result, $\kappa_{\text{Killing}}$, for non-degenerate Killing horizons.

The detailed proof is provided in the Appendix 1.

**Gravitating mass as an integrated Noether charge**

Consider an observer, Albert, on a 2-surface $S$ of a general timelike world tube. As already mentioned, $a_{\mu} \hat{R}^\mu$ is the component of the acceleration normal to $S$ in Albert’s own rest frame. It can be thought of as the force-per-unit-mass required to keep Albert on his world tube. Viewed by the reference observer, Emmy, the Noether charge, $\Phi a_{\mu} \hat{R}^\mu$, is the force-per-unit-mass that Emmy must exert on Albert to keep him on his world tube. Thus, Emmy’s Noether charge provides a common reference for measuring the normal component of force for any accelerating observer on the world tube. Integrating these forces over $S$ yields the flux of force normal to $S$, which we interpret as the ‘gravitating mass’ responsible for the acceleration felt by a congruence of such observers.

**Corollary 1:** The ‘gravitating mass’ is defined by

$$M_{\text{Grav}} = \frac{1}{4\pi} \int_S \Phi a_\mu \hat{R}^\mu dA = \frac{1}{4\pi} \int_S \kappa_{\text{Noether}} dA.$$  \(14\)

For example, for ‘stationary’ observers anywhere outside of the horizon of a Schwarzschild black hole of mass $M$, Eq. \(14\) reduces to $M_{\text{Grav}} = M$.

Quasi-local energies take the form of integrals over a 2-surface, e.g., the Brown-York mass (see Appendix 2). While $M_{\text{Grav}}$ therefore appears to be a quasi-local energy, Eq. \(1\) allows it to be recast as a volume integral of

$$\rho_{\text{Grav}} = \frac{1}{4\pi} J^\mu \hat{T}_\mu,$$  \(15\)

i.e., the time-component of our locally conserved current $J^\mu$. We therefore dub $\rho_{\text{Grav}}$ the ‘gravitating energy density.’ To simplify Eq. \(15\), it is convenient to first restate Eq. \(10\) as a general result:

**Corollary 2:** For our family of observers

$$\kappa_{\text{Noether}} = \Phi_\mu \hat{R}^\mu.$$  \(16\)

The gravitating energy density reduces to a particularly simple form when we consider world tubes $\{W\}$, each of whose observers are at ‘rest’ with respect to Emmy’s preferred foliation of the spacetime $\Sigma$. In curved spacetime, we might consider an observer, Leonhard, to be at rest when the nearby events he classifies as simultaneous are likewise classified by Emmy (and so lie on $\Sigma$). Such observers are called Eulerian observers and their 4-velocities must be normal to the foliation\[6\] so $\nu^\mu = \bar{T}^\mu$. In Leonhard’s rest frame, the vector orthogonal to the 2-surface $S$ will therefore also be tangent to $\Sigma_t$, so that $\bar{R}^\mu = N^\mu$. Then, Corollary 2 yields

$$\kappa_{\text{Noether}} = \bar{N}^\mu \nabla_\mu \Phi_{\text{Euler}} = \bar{N}^a D_a \Phi_{\text{Euler}},$$  \(17\)

where we explicitly label the red-shift factors of our Eulerian observers as $\Phi_{\text{Euler}}$, and $D_a$ and $\bar{N}^a$ denote the covariant derivative $\nabla_\mu$ and hypersurface-tangent 4-vector $\bar{N}^\mu$, respectively, projected onto the 3-dimensional manifold of the hypersurface $\Sigma_t$ with metric $h_{ab}$. Stokes’ theorem on this 3-manifold then yields

$$\rho_{\text{Grav}} = \frac{1}{4\pi} D^a D_a \Phi_{\text{Euler}},$$  \(18\)

where $D^a \equiv h^{ab} D_b$ and $h^{ab}$ is just the inverse matrix of $h_{ab}$. We see, therefore, that given access to the red-shift factors alone of distributed families of Eulerian observers, we could determine the gravitational energy density from Eq. \(18\). Curiously, this relationship is reminiscent of Poisson’s law for Newtonian gravitation on a curved 3-geometry, where the role of the gravitational potential is played by the scalar field of red-shift factors, $\Phi_{\text{Euler}}$, for families of Eulerian observers. Note that Eq. \(18\) holds in arbitrary dynamical spacetimes and is
an exact relation rather than a slow-motion weak-field approximation of general relativity\textsuperscript{19}. Note that the gravitating mass depends on the choice of the family of observers. In fact such behavior is required by consistency with the equivalence principle for such an observable quantity. For example, for families of geodesic observers the acceleration is zero and hence for them the gravitating mass vanishes.

One might worry that although fixing the observers’ world tube to be orthogonal to the foliating hypersurfaces we have merely shifted the arbitrariness into the choice of how we foliate the spacetime. In practice, this choice of foliation is made naturally by the astronomer herself — by the reference observer Emmy. The Noether charge formulation allows Emmy, in-principle, to directly access the detailed distribution of gravitating energy density, whenever she has access to the red-shift factors of non-geodesic Eulerian observers, whose description of simultaneity agrees with her own. Unfortunately, as no such observers likely exist. To make use of the above formalism, Emmy must rely on information from geodesic observers as surrogates to extract the data needed. We now describe an approach Emmy might use to do this.

**FIG. 2**: The hypersurface $\Sigma$, world tube $\mathcal{W}$ and their intersection $\mathcal{S}$ are all in faint gray. The non-geodesic observer (not shown) follows the world tube. The co-moving geodesic observer is on a parabolic-like arc kissing the world tube (and tangent to the non-geodesic observer’s trajectory) at one spacetime point. In this illustration, the orbiting geodesic observer is hugging the outside of the world tube as it spirals around the central region. This orbital geodesic crosses the other two trajectories at their spacetime point of intersection.

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At first sight Emmy’s task seems hopeless. As noted above, geodesic observers by their very nature are in free fall and hence feel no acceleration and consequently have vanishing surface gravity. Nevertheless, according to Corollary 2, it is sufficient for Emmy’s purposes to determine what would be the red-shift factors of a set of fictitious non-geodesic Eulerian observers. To achieve this she may rely on the following:

**Observation**: The red-shift factor is solely a function of the observer’s instantaneous 4-velocity $v^{\mu}$. In particular,

$$\Phi = \frac{1}{v^t}.$$  \hspace{1cm} (19)

**Proof**: Within a proper time $\delta \tau$ this observer has moved by $\delta x^{\mu} = v^{\mu} \delta \tau$. The $t$-component of this expression is $\delta t = v^t \delta \tau$, from which the observation follows.

Since the red-shift factors only depend on the instantaneous velocity, Emmy may replace the non-geodesic observer with a co-moving geodesic observer with instantaneous common 4-velocity (see Fig. 2). Next, she may relate the red-shift factor of this co-moving observer with that of an orbiting observer crossing the same spacetime point. She can achieve this by performing a Lorentz transformation on the red-shift factor for the orbiting geodesic to instantaneously (de)boost it to that of the co-moving observer (see Fig. 2). The former replacement does not change the red-shift factor while the latter involves a Lorentz transformation factor (see Appendix 1).

In this way, Emmy may extract the red-shift factor of a fictitious non-geodesic observer from that of an orbiting observer at the same spacetime point. The orbiting observer is presumed to be emitting light with a well-characterized spectrum, e.g., it might be a star. In principle then every star could provide a point in a distribution of red-shift factors, from which Emmy may determine the local Noether-charge surface gravity for her fictitious observers, or equivalently the gravitating energy density, Eq. (18), or net gravitating mass, Eq. (14), across the population of stars consistent with how she sees the Universe.

**Discussion**

We derive a naturally defined surface gravity using a congruence of timelike observers following a world tube. We show that this surface gravity is a Noether charge. The associated Noether current is conserved in arbitrarily dynamical spacetimes and does not require the Einstein field equations to hold true (see Appendix 1). Thus, this Noether charge has a fundamental character which likely survives even in quantum gravity where quantum fluctuations are expected to make the classical field equations only a lowest-order approximation.

This Noether-charge surface gravity may be extended to null world tubes by considering the limit as the time-like world tubes locally approach such null hypersurfaces, in which case the surface gravity reduces to the standard result on Killing horizons. Its fundamental nature makes this Noether-charge formulation a natural candidate for the surface gravity of dynamical horizons (whether time-like or null). There are a number of alternative candidates for this quantity (see Appendix 2). However, there is as yet no consensus even for the simplest dynamical horizons of spherically-symmetric black holes\textsuperscript{19}.

We show that the surface gravity is in-principle observable, in this way addressing the long standing challenge to identify a generally observable gravitational Noether charge.\textsuperscript{24} We associate this surface gravity with the gravitating mass (its integrated form). We have shown that for asymptotically stationary observers (with $\xi^{\mu} = \partial_t$),
the gravitating mass reduces to the Komar energy. Thus, for the first time we may provide a physically measurable interpretation to the Komar energy even in dynamical spacetimes. The existence of such an interpretation has been long questioned. We also note that the integrated Noether charge has been previously conjectured to be the entropy of dynamical spacetime horizons even for generalized theories of gravity, rather than as the energy found here. However, this conjecture is based on scaling away the surface gravity from the Noether charge (see Appendix 2).

Finally, we have proposed a scheme by which this Noether charge formalism could be used to probe the detailed distribution of gravitating mass within galaxies, galaxy clusters etc., from the red-shift factors of well-characterized sources following orbits in such systems. Such a new and independent probe may lead to novel insights into the nature of dark matter in our universe.

APPENDIX

This supplementary information is separated into two appendices. In Appendix 1, we give detailed proofs for claims in the main manuscript. In Appendix 2, we look at previous work with connections to our results.

APPENDIX 1: DETAILED PROOFS

Hamiltonian formulation of Noether’s theorem

In the classical domain, Noether’s theorem shows that if evolution generated (in the sense of the Poisson bracket formalism) by the system Hamiltonian leaves some quantity invariant then the evolution generated by that quantity will leave the system Hamiltonian invariant and vice versa. In order make this result accessible we repeat Baez’s proof here.

Proof: Poisson brackets allow us to construct the one-parameter evolution generated by a ‘Hamiltonian’ \( H \) on some quantity \( a \) via

\[
\frac{da(t)}{dt} = \{H, a(t)\},
\]

Suppose some quantity \( a \) is invariant under this evolution, then

\[
\frac{da(t)}{dt} = 0
\]

\[
\Rightarrow \{H, a(t)\} = 0
\]

\[
\Rightarrow \{a, H\} = 0.
\]

Therefore the one-parameter action generated by \( a \) on the system Hamiltonian also vanishes, leaving \( H \) invariant under this action. \( \square \)

Conservation of the Komar current density

In 1959, within the context of general relativity, Komar showed for an arbitrary vector field \( \xi^\mu \), that the 4-current \( J^{\mu}(\xi) \) is locally conserved, where

\[
J^{\mu}(\xi) \equiv \xi^{[\nu}J^{\mu]}_{\nu}.
\]
Here, we will show that

**Lemma 0:** $J^\mu(\xi)_\mu = 0$ for any theory with a torsion-free metric-compatible connection.

**Proof:** To simplify $J^\mu(\xi)$, we first introduce the general covariant derivative formula with a non-zero torsion tensor, defined as $T^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu}$. Indeed, the torsion tensor comes from the presumed non-commutativity of the covariant derivative of a scalar function

$$f_{;\nu} - f_{;\nu} = \nabla_\nu f - \nabla_\nu f = f_{;\nu} - \Gamma^\lambda_{\nu\mu} f_{;\lambda} = \Gamma^\lambda_{\nu\mu} f_{;\lambda}$$

where we have assumed ordinary derivatives are commutative to obtain the forth line. Based on this definition of torsion tensor, it is known that $T_{\mu\nu\alpha} = \Gamma_{\mu\nu\alpha} - \Gamma_{\mu\alpha\nu}$. The Rieman tensor are anti-symmetric.

Let us first generalize the Bianchi symmetry to a geometry with a non-vanishing torsion tensor.

To derive this relation, let us first expand $2f_{[\mu\alpha]}$ with the first two indices kept anti-symmetric, yielding

$$2f_{[\mu\alpha]} = \frac{2}{3}f_{[\mu\alpha]} + \frac{2}{3}f_{[\mu\alpha]} + \frac{2}{3}f_{[\mu\alpha]}$$

where we have used Eq. (29) to obtain line two. On the other hand, we may expand $2f_{[\mu\alpha]}$ keeping the last two indices anti-symmetric to yield

$$2f_{[\mu\alpha]} = \frac{2}{3}f_{[\mu\alpha]} + \frac{2}{3}f_{[\mu\alpha]} + \frac{2}{3}f_{[\mu\alpha]}$$

where we have applied Eq. (29) to obtain line two.

Since Eqs. (29) and (30) are equal, we have

$$R_{[\mu\alpha]\mu\lambda} f_{;\lambda} - T_{[\mu\alpha]\mu\lambda} f_{;\lambda} = \frac{1}{3}(R_{[\mu\alpha]\mu\lambda} f_{;\lambda} + T_{[\mu\alpha]\mu\lambda} f_{;\lambda} - \frac{1}{3}(R_{[\mu\alpha]\mu\lambda} f_{;\lambda} + T_{[\mu\alpha]\mu\lambda} f_{;\lambda}) = 0.$$
Contracting the indices $\lambda$ and $\nu$ in Eq. (30) then yields

$$R^\lambda_{\mu\alpha\nu} + R^\lambda_{\nu\mu\alpha} = 3T^\lambda_{[\mu,\nu]} - 3T^\lambda_{[\nu,\mu]} T^\lambda_{\tau},$$

$$R^\lambda_{\mu\alpha\nu} + R^\lambda_{\nu\alpha\mu} - R_{\alpha\mu\nu} = 3T^\lambda_{[\mu,\nu]} - 3T^\lambda_{[\nu,\mu]} T^\lambda_{\tau},$$

$$R_{\mu\alpha\nu} - R_{\nu\alpha\mu} + R^\nu_{\alpha\lambda\mu} = 3T^\lambda_{[\mu,\nu]} - 3T^\lambda_{[\nu,\mu]} T^\lambda_{\tau}. \quad (31)$$

Although $R^\lambda_{\lambda\alpha\mu} = 0$ in general relativity because the first two indices are anti-symmetric, this is not necessary to be true for a generic geometry.

Inserting Eq. (31) into Eq. (25) yields

$$2J^\mu(\xi)_{;\mu} = T^\alpha_{\nu\mu} \xi^\nu_{;\mu} + 3(T^\lambda_{[\mu,\nu]} - T^\tau_{[\nu,\mu]} T^\lambda_{\tau}) \xi^{\nu\alpha} - R_{\lambda\alpha\nu} \xi^{\nu\alpha}, \quad (32)$$

To understand the geometric behavior of $R^\lambda_{\lambda\alpha\nu}$, we now consider $2g_{\mu\nu;[\alpha\beta]}$

$$2g_{\mu\nu;[\alpha\beta]} = T^\lambda_{\alpha\beta} g_{\mu\nu;\lambda} + R^\lambda_{\mu\alpha\beta} g_{\lambda\mu} + R^\lambda_{\nu\alpha\beta} g_{\lambda\mu} = T^\lambda_{\alpha\beta} g_{\mu\nu;\lambda} + R^\nu_{\beta\alpha\mu} + R_{\mu\alpha\beta}. \quad (33)$$

Contracting Eq. (33) with $g^{\mu\nu}$ then yields

$$2g^{\mu\nu} g_{\mu\nu;[\alpha\beta]} = T^\lambda_{\alpha\beta} g^{\mu\nu} g_{\mu\nu;\lambda} + 2R^{\mu}_{\nu\alpha\beta}. \quad (34)$$

Finally, inserting Eq. (34) into Eq. (32) gives

$$J^\mu(\xi)_{;\mu} = \frac{1}{2} T^\alpha_{\nu\mu} \xi^\nu_{;\mu} + 3(T^\lambda_{[\mu,\nu]} - T^\tau_{[\nu,\mu]} T^\lambda_{\tau}) \xi^{\nu\alpha} + \frac{1}{2} T^\lambda_{\nu\mu} g^{\mu\nu} \xi^{\nu\alpha} - \frac{1}{2} g^{\mu\nu} g_{\mu\nu;[\alpha\beta]} \xi^{\nu\alpha}. \quad (35)$$

Since $\xi^\mu$ is an arbitrary vector and these terms are in different orders of derivatives of $\xi^\mu$ or $T^\tau_{[\nu,\mu]}$, they are independent of each other. Therefore, the most natural and straightforward conditions to ensure that

$$J^\mu(\xi)_{;\mu} = 0, \quad (36)$$

are to require the torsion tensor vanish and the metric is compatible with the covariant derivative (or connection of the covariant derivative).

This completes the proof of Lemma 0. □

The local conservation of $J^\mu(\xi)$ relies on the geometry of the spacetime manifold; nothing else. It therefore allows us to construct a family (one for each choice of vector field $\xi^\mu$) of locally conserved quantities for any theory of gravity obeying these geometric requirements. Furthermore, these conservation laws are preserved by any Hamiltonian which respects the assumptions of being torsion free and metric compatible. Therefore by Noether’s theorem, we have an infinite family of conserved currents and Noether charges for any torsion-free metric-compatible theory of gravity.

**Lagrangian formulation of Noether’s theorem**

Consider a Lagrangian $\mathcal{L}(\phi, \phi_{,\mu})$ depending on the field $\phi$ and its first-order derivative, then the variation of $\mathcal{L}$ with respect to $\phi \rightarrow \phi + \delta \phi$ may be written

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial (\phi_{,\mu})} \delta (\phi_{,\mu}) = \left( \frac{\partial \mathcal{L}}{\partial (\phi_{,\mu})} \right)_{,\mu} \delta \phi.$$

Here the second term vanishes, ‘on-shell,’ due to the equations of motion for the system. The first term is usually neglected because $\delta \phi$ vanishes on the boundary. However, the expression itself is generally non-zero. In fact Noether’s theorem relies on this term to define a locally conserved quantity when some symmetries exist. In general, let us suppose that these symmetries are generated by variations in some generalized coordinates $\theta^A$.

Rewriting the variation of $\mathcal{L}$ in terms of these coordinates we find

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \theta^A} \delta \theta^A,$$

Since Eqs. (37) and (38) both describe the variation of the Lagrangian we have

$$\left( \frac{\partial \mathcal{L}}{\partial (\phi_{,\mu})} \right)_{,\mu} = (\mathcal{L} \delta \theta^A)_{,\mu}. \quad (39)$$

where we have assumed that the system is on-shell and hence obeys the equations of motion so that the second term in Eq. (37) vanishes.

From Eq. (38) we can see that here are two main possibilities for the Noether’s theorem. The first scenario is when the Lagrangian is invariant with respect to the generalized coordinate $\theta^A$, so that the right-hand-side of Eq. (39) vanishes. In this case, the Noether current is defined simply as

$$J^\mu = \frac{\partial \mathcal{L}}{\partial (\phi_{,\mu})} \delta \phi. \quad (40)$$

The Lagrangian of a complex scalar field describes such a scenario which we shall illustrate below.

The other main scenario we consider is when the generalized coordinates $\theta^A$ reduce to the spacetime coordinates
In this case, Eq. (39) reduces to the on-shell relation
\[ 0 = \left( \frac{\partial L}{\partial (\phi^\mu)} \right)_{\phi^\mu} - (L \delta x^\mu)_{\phi^\mu} = \left( \frac{\partial L}{\partial (\phi^\mu)} \right)_{\phi^\mu} - L \delta x^\mu, \]
\[ = J^\mu_{\phi^\mu}. \] (41)

Here, the Noether current \( J^\mu \) equals
\[ J^\mu = \frac{\partial L}{\partial (\phi^\mu)} \delta \phi - L \delta x^\mu. \] (42)

Note that the current defined in this way will always be locally conserved on-shell due to Eq. (41).

Now, let us consider the first of these scenarios in the case of the Lagrangian of a complex scalar field
\[ \mathcal{L} = \phi^* \mu \phi^\mu - m^2 \phi^* \phi. \] (43)

For the variation \( \phi \to \phi' = e^{i\theta} \phi \) the Lagrangian does not change so there is a symmetry with respect to the 'gauge' parameter \( \theta \). The variation of the Lagrangian with respect to this parameter vanishes:
\[ \delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \theta} \delta \theta = 0. \] (44)

At the same time, the variations of the scalar fields with respect to this gauge change are
\[ \delta \phi = \phi' - \phi = (e^{i\theta} - 1) \phi \]
\[ \delta \phi^* = (e^{-i\theta} - 1) \phi^*. \] (45)

Thus, based on Eq. (40), the Noether current is
\[ J^\mu = \phi^* (e^{-i\theta} - 1) \phi^* + \phi^* (e^{i\theta} - 1) \phi. \] (46)

In the limit of small \( \theta \) we have \( e^{i\theta} - 1 = i\theta + O(\theta^2) \) and \( e^{-i\theta} - 1 = -i\theta + O(\theta^2) \). In this limit, Eq. (46) may be further simplified to
\[ J^\mu = i\theta (\phi^* \phi - \phi^* \phi). \] (47)

Using the Klein-Gordon equation, it is easy to check that this current \( J^\mu \) is locally conserved, having \( J^\mu_{\phi^\mu} = 0 \). Thus, we see that the current of Eq. (47) is conserved on-shell.

All the above analysis and our example of the complex scalar field seem to imply that a Noether current must be generically an on-shell conserved current because we require the equations of motion to remove the extra terms in \( \delta \mathcal{L} \), e.g., the second term in Eqs. (37). However, we shall see below that the Noether current in a spacetime may be off-shell (meaning independent of the validity of the equations of motion) because the vanishing of these extra terms will be shown instead to be purely geometric. In this way, the conservation of the Noether current is found not to be limited to solutions of the Einstein field equations of general relativity. Indeed, this appeared to be the natural implication of the Hamiltonian formulation of the Noether current described above. We shall now show that this result appears to hold true explicitly within the Lagrangian formulation of the Noether current for general relativity.

_Lagrangian formulation of Noether’s theorem for general relativity_

After having introduced the Noether current based on the Lagrangian formulation above, we now derive the Noether current for general relativity in this formulation.\(^{18,19}\) In this section and henceforth, we directly assume the spacetime geometry is torsion-free and the covariant derivative is metric compatible. The Lagrangian for general relativity is taken to be given by the Einstein-Hilbert gravitational action with density \( R \sqrt{-g} \). The variation of this Lagrangian with respect to the metric \( g_{\mu\nu} \) may be calculated\(^{20}\)

\[ \delta(R \sqrt{-g}) = -\left( (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) h^{\mu\nu} - (g^{\mu\nu} \delta \Gamma^\alpha_{\mu\nu} - g^{\mu\alpha} \delta \Gamma^\lambda_{\mu\lambda})_{;\alpha} \right) \sqrt{-g}, \] (48)

where \( h_{\mu\nu} \equiv \delta g_{\mu\nu} = -g_{\mu\rho} g_{\nu\sigma} \delta g^{\rho\sigma} \), i.e., \( \delta g^{\sigma\tau} = -h^{\sigma\tau} \).

Now, we show that \(- (g^{\mu\nu} \delta \Gamma^\alpha_{\mu\nu} - g^{\mu\alpha} \delta \Gamma^\lambda_{\mu\lambda})_{;\alpha} = 2 h_{\mu\nu} [\mu;\nu]^{\nu} \), a result quoted in Ref.\(^{21}\), there without proof.
Since \( \delta \Gamma_{\mu \nu}^{\alpha} = \frac{1}{2} g^{\alpha \rho} (h_{\mu \rho ; \nu} + h_{\nu \rho ; \mu} - h_{\mu \nu ; \rho}) \),

\[
-(g^{\mu \nu} \delta \Gamma_{\mu \nu}^{\alpha} - g^{\mu \alpha} \delta \Gamma_{\mu \lambda}^{\lambda})_{; \alpha} = \left( g^{\mu \alpha} \frac{1}{2} g^{\lambda \rho} (h_{\mu \rho ; \lambda} + h_{\lambda \rho ; \mu} - h_{\mu \lambda ; \rho}) - g^{\mu \nu} \frac{1}{2} g^{\alpha \rho} (h_{\mu \rho ; \nu} + h_{\nu \rho ; \mu} - h_{\mu \nu ; \rho}) \right)_{; \alpha}
\]

\[
= \frac{1}{2} \left( g^{\mu \alpha} h_{\mu \rho ; \alpha} - g^{\mu \nu} (h_{\mu \nu ; \nu} + h^{\alpha \nu} ; \nu - h_{\mu \nu ; \nu}^{\alpha}) \right)_{; \alpha}
\]

\[
= \frac{1}{2} \left( h_{\mu \rho ; \alpha} - (h_{\alpha \nu} ; \nu + h^{\alpha \nu} ; \nu - h_{\mu \nu ; \nu}^{\alpha}) \right)_{; \alpha}
\]

\[
= \frac{1}{2} \left( 2 h_{\rho ; \alpha} - 2 h_{\alpha ; \nu}^{\mu} \right)_{; \alpha}
\]

\[
= 2 h_{\nu [\mu ; \nu]}^{\mu}.
\]  

(49)

Inserting Eq. (49) into Eq. (48) then yields

\[
\delta (R \sqrt{-g}) = \left( \left( R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R \right) h^{\mu \nu} + 2 h^{\mu [\mu ; \nu]} \right) \sqrt{-g}.
\]  

(50)

Now consider the coordinate change: \( x^\mu \rightarrow x^\mu + \xi^\mu \), so \( \xi^\mu = \delta x^\mu \). It is not difficult to see that the variation of the metric under such a coordinate change is nothing but the Lie derivative of the metric along \( \xi^\mu \), yielding

\[ h_{\mu \nu} = \delta g_{\mu \nu} = 2 \xi g_{\mu \nu} = 2 \xi_{(\mu ; \nu)} \].

(51)

Inserting Eq. (51) into the first term on the right-hand-side of Eq. (50) yields

\[
(R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R) h^{\mu \nu} = (R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R) 2 \xi^{(\mu ; \nu)}
\]

\[
= (R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R) 2 \xi^{\mu ; \nu}
\]

\[
= 2 (R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R) \xi^{\mu ; \nu} - 2 (R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R) \xi^{\nu ; \mu}.
\]  

(52)

Similarly, inserting Eq. (51) into the second term on the right-hand-side of Eq. (50) yields

\[
2 h^{\mu [\mu ; \nu]} = h^{\mu [\mu ; \nu]} - h^{\mu \nu ; \nu}
\]

\[
= 2 \xi^{\mu ; \nu} - \xi_{\mu ; \nu}^{\mu ; \nu} - \xi_{\nu ; \mu}^{\mu ; \nu}
\]

\[
= 2 \xi^{\mu ; \nu} - 2 \xi_{\mu ; \nu}^{\mu ; \nu} + \xi_{\mu ; \nu}^{\mu ; \nu} - \xi_{\nu ; \mu}^{\mu ; \nu}
\]

\[
= 2 (\xi^{\mu ; \nu} - \xi_{\mu ; \nu}^{\mu ; \nu}) + 2 \xi_{\nu ; \mu}^{\mu ; \nu}
\]

\[
= 2 (-R_{\mu \nu} \xi^{\mu ; \nu} + 2 \xi_{\nu ; \mu}^{\mu ; \nu},
\]  

(53)

where we have used Eq. (24) in moving from the fourth to the fifth line.

Combining the results of Eqs. (52) and (53) into Eq. (50) gives us

\[
\delta (R \sqrt{-g}) = \left( 2 (R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R) \xi^{\mu ; \nu} - 2 (R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R) \xi^{\nu ; \mu} \right) \sqrt{-g}
\]

\[
= \left( -(g_{\mu \nu} R \xi^{\mu ; \nu} - 2 (R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R) \xi^{\nu ; \mu} + 2 \xi_{\nu ; \mu}^{\mu ; \nu}) \right) \sqrt{-g}
\]

\[
= (R \xi^{\mu ; \nu} \sqrt{-g} + 2 \xi_{\nu ; \mu}^{\mu ; \nu} \sqrt{-g} + 2 (R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R) \xi^{\nu ; \mu} \sqrt{-g})
\]

\[
= (R \xi^{\mu ; \nu} \sqrt{-g})_{, \nu} + 2 (\xi_{\nu ; \mu}^{\mu ; \nu} \sqrt{-g})_{, \nu} + 2 (R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R) \xi^{\nu ; \mu} \sqrt{-g},
\]  

(54)

where we have used \( A_{\mu ; \nu} \sqrt{-g} = (A_{\nu ; \mu} - A_{\mu ; \nu}) \), for the first two terms in the final step.\(^{23}\)

Recall now Noether’s theorem discussed above, the variation of the Lagrangian may be independently obtained as \( \delta (R \sqrt{-g} \delta x^{\mu})_{, \mu} \), where \( \xi^\mu = \delta x^\mu \). Therefore, we may consider

\[
0 = \delta (R \sqrt{-g}) = \delta (R \sqrt{-g} \delta x^{\mu})_{, \mu}
\]

\[
= (R \xi^{\mu ; \nu} \sqrt{-g})_{, \mu} + 2 (\xi_{\nu ; \mu}^{\mu ; \nu} \sqrt{-g})_{, \nu} + 2 (R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R) \xi^{\nu ; \mu} \sqrt{-g} - (R \sqrt{-g} \xi^{\mu})_{, \mu}
\]

\[
= 2 (\xi_{\nu ; \mu}^{\mu ; \nu} \sqrt{-g})_{, \nu} + 2 (R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R) \xi^{\nu ; \mu} \sqrt{-g}.
\]  

(55)

Note, that up to this point we have still not used the equations of motion of general relativity to obtain our
Noether current. As noted in the previous section on the general formulation of Noether currents, their conservation generically requires imposing the equations of motion. Indeed, this was presumed to be the case in early work on the gravitational Noether current. Later it was shown that in the absence of matter the conservation of the gravitational Noether current does not require the field equations (i.e., no matter). Recall that the Lagrangian of a matter current which relies solely on the geometry of spacetime in the case of pure gravity (i.e., no matter).

The variation of this may therefore be written on the one hand as

\[
\delta L_m \sqrt{-g} = (\xi^\mu),\mu \sqrt{-g} = (L_m)^{\mu},\mu \sqrt{-g} + L_m \xi_{\mu,\mu} \sqrt{-g}. \tag{63}
\]

Combining the results of Eqs. (62) and (63) yields

\[
\delta L_m \sqrt{-g} - \delta (L_m \sqrt{-g}) = (\partial L_m / \partial x^\mu, \mu) \sqrt{-g} - (L_m)^{\mu},\mu \sqrt{-g} = 0. \tag{64}
\]

Here the difference of the variations of \(\delta (L_m \sqrt{-g})\) based on the two different approaches to computing it vanishes. Hence no contribution is made to the gravitational Noether current from the matter-part of the Lagrangian.

We therefore conclude that the off-shell character of the gravitational Noether charge appears to be a general result, applying even in the presence of matter fields. Of course this result appears trivial when the Hamiltonian formalism for Noether charges is used, but as we have seen a detailed analysis using the Lagrangian formalism comes to the same conclusion.

**Influence of the cosmological constant**

The term in the Lagrangian for the cosmological constant may be written \(\alpha \Lambda \sqrt{-g}\) where \(\alpha\) is some constant. The variation of this may therefore be written on the one hand as

\[
\delta (\alpha \Lambda \sqrt{-g}) = \frac{1}{2} \alpha \Lambda g_{\mu \nu} h^{\mu \nu} \sqrt{-g} = \alpha \Lambda g_{\mu \nu} \xi^\nu,\nu \sqrt{-g}. \tag{65}
\]

On the other hand, this variation may be equally written

\[
\delta (\alpha \Lambda \sqrt{-g}) = \delta (\alpha \Lambda \sqrt{-g} \xi^\mu),\mu = \alpha \Lambda \xi_{\mu,\mu} \sqrt{-g}. \tag{66}
\]

where we have used \(h^{\mu \nu} = 2 \xi^\nu \xi^\mu\) in the last step.

The term in the Einstein-Hilbert action density yields a vanishing contribution to the Noether current.

We shall now show that the off-shell nature of the Noether current conservation remains true even in the presence of matter. Recall that the Lagrangian of a matter current may be written \(L_m \sqrt{-g}\). The variation of this Lagrangian may then be calculated on the one hand via

\[
\delta (L_m \sqrt{-g}) = \frac{\partial L_m}{\partial x^\mu} \delta x^\mu \sqrt{-g} + \frac{\partial L_m}{\partial g^{\mu \nu}} \delta g^{\mu \nu} \sqrt{-g} - \frac{1}{2} L_m g_{\mu \nu} \delta g^{\mu \nu} \sqrt{-g} = 0. \tag{62}
\]

where we have used \(h^{\mu \nu} = 2 \xi^\nu \xi^\mu\) in the last step.

On the other hand, the variation of \(L_m \sqrt{-g}\) may be equally written as

\[
\delta (L_m \sqrt{-g}) = (L_m \sqrt{-g} \xi^\mu),\mu = (L_m)^{\mu},\mu \sqrt{-g} + L_m \xi_{\mu,\mu} \sqrt{-g}. \tag{63}
\]

We conclude that the detailed Lagrangian formulation of Noether’s theorem agrees with the result from the Hamiltonian formulation that Eq. (61) is an off-shell Noether current which relies solely on the geometry of spacetime in the case of pure gravity (i.e., no matter).
Integrated conservation laws

After proving the generalized Komar current is locally conserved, we now study the corresponding integrated conservation laws of this current. Integrating Eq. (66) (or Eq. (61)) over a 4-volume, a subvolume, \( V \subset M \) of the entire manifold, yields
\[
\int_V J^\mu_{\nu} \sqrt{-g} \, d^4z = 0,
\]
and applying Stokes’ theorem we find
\[
\int_{\partial V} J^\mu_{\nu} \hat{n}_{\nu} \sqrt{\gamma(\partial V)} \, d^3x = \int_{\partial V} \xi^{[\nu;\rho]}_{\mu} \hat{n}_{\nu} \sqrt{\gamma(\partial V)} \, d^3x = 0,
\]
where \( \partial V \) is the boundary of \( V \) and \( \hat{n}^\mu \) is the unit vector normal to \( \partial V \), see Fig. 3 and \( \gamma(S) \) is the determinant of the induced metric on (sub-)manifold \( S \). This means that the current flux into the 4-volume is the same as the current flux out. This is a local conservation law for an arbitrary vector field in an arbitrary dynamical spacetime.

Next, if we use the 3+1 split method to foliate the spacetime into a family of non-intersecting spacelike hypersurfaces labeled by \( f \) with net flux \( J^\mu_{\nu} \hat{N}_{\nu} \) out through the side timelike boundary \( \Sigma_3 \) (\( \hat{N}^\mu \) is the spacelike unit vector normal to the boundary). Consider a volume \( V \) consisting of the region between a pair of such hypersurfaces \( \Sigma_1, \Sigma_2 \) and side boundary \( \Sigma_3 \) (see Fig. 4) then from Eq. (69) we find
\[
\int_{\Sigma_1} \xi^{[\nu;\rho]}_{\mu} \hat{T}_{\mu} \sqrt{\gamma(\Sigma_1)} \, d^3x = \int_{\Sigma_2} \xi^{[\nu;\rho]}_{\mu} \hat{\Sigma}_{\mu} \sqrt{\gamma(\Sigma_2)} \, d^3x + \int_{\Sigma_3} \xi^{[\nu;\rho]}_{\mu} \hat{N}_{\mu} \sqrt{\gamma(\Sigma_3)} \, d^3x,
\]
where \( \hat{T}^\mu \) is the future directed timelike unit normal to the hypersurfaces. Note that this split formalism is unnecessary to have any connection with the coordinates system although using the coordinate to label the hypersurface sometimes may simplify the calculation. When the Noether current (some kind of ‘energy’ current) \( J^\mu_{\nu} \hat{N}_{\nu} \) vanishes on the side boundary \( \Sigma_3 \), this integral over the three-dimensional hypersurface \( \Sigma_i \) will be independent of the hypersurface.

![FIG. 3: This 4-volume \( V \) is a subset of the entire spacetime manifold \( M \). Here \( \partial V \) is the boundary of \( V \), and \( \hat{n}^\mu \) is the outgoing unit vector normal to the boundary \( \partial V \).](image)

![FIG. 4: This 4-volume \( V \) is a region between two three hypersurface \( \Sigma_1, \Sigma_2 \), and \( \Sigma_1, \Sigma_2, \Sigma_3 \) together make up of its boundary. Here \( \hat{T}^\mu \) is the timelike unit normal vector pointing to the future, and \( \hat{N}^\mu \) is the spacelike outgoing unit vector normal to the side timelike boundary.](image)

Therefore, on arbitrary spacelike three hypersurface, we can always have such a well-defined Noether current
\[
\int_{\Sigma} \xi^{[\nu;\rho]}_{\mu} \hat{T}_{\mu} \sqrt{\gamma(\Sigma)} \, d^3x,
\]
Recalling Stokes’s theorem for an anti-symmetric tensor \( F_{\mu\nu} \)
\[
\int_{\Sigma} \hat{T}_{\mu} F_{\mu\nu} \sqrt{\gamma(\Sigma)} \, d^3x = \int_{\partial\Sigma} \hat{T}_{\mu} F_{\mu\nu} \hat{N}_{\nu} \sqrt{\gamma(\partial\Sigma)} \, d^{n-2}y,
\]
where \( \hat{N}_{\mu} \) is the outgoing spacelike unit vector normal to \( \partial\Sigma \). Applying this Stokes’s theorem to Eq. (71) yields
\[
\int_{\partial\Sigma} \xi^{[\nu;\rho]}_{\mu} \hat{N}_{\nu} \hat{T}_{\mu} \sqrt{\gamma(\partial\Sigma)} \, d^2y = \int_{\partial\Sigma} \xi^{[\nu;\rho]}_{\mu} d\Sigma_{\mu\nu},
\]
where \( d\Sigma_{\mu\nu} = \hat{N}_{[\nu} \hat{T}_{\mu]} \sqrt{\gamma(\partial\Sigma)} \, d^2y = \hat{N}_{[\nu} \hat{T}_{\mu]} dA \).

**Proof that** \( \kappa_{\text{Noether}} = \kappa_{\text{Killing}} \) **on Killing horizons**

Recall that in the manuscript, we define a natural surface gravity for each point on the world tube \( W \), rescaled to the rate-of-change of coordinate time, via
\[
\kappa_{\text{Noether}} \equiv \Phi a_{\mu} \hat{R}^\mu,
\]
where \( a_{\mu} \equiv v_{\mu;\nu} v^\nu \) is the 4-acceleration of an observer from our congruence passing though this point, and \( \hat{R}^\mu \) is normal to \( W \) at this point.

Now let us consider a non-degenerate Killing horizon with Killing vector
\[
K^\mu = \partial_{\rho} + \Omega_{\rho} \partial_{\phi},
\]
where \( t \) is the coordinate time at spatial infinity, \( \phi \) is the azimuthal angular coordinate, and the constant \( \Omega_{\rho} \) is the angular velocity of the horizon for a Kerr black hole. According to the conventional definition, the surface gravity, \( \kappa_{\text{Killing}} \), for such a Killing horizon satisfies
\[
K_{\mu;\nu} K^\nu = \kappa_{\text{Killing}} K^\mu,
\]
on the horizon.

**Lemma 1:** The Noether surface gravity, $\kappa_{\text{Noether}}$, reduces to the standard result for Killing horizons, $\kappa_{\text{Killing}}$, for non-degenerate, non-bifurcate Killing horizons.

**Proof:** We start by noting that the definition of Eq. (76) is not suitable to be treated as a limit, since the two sides of this equation can only be parallel for $K^\mu$ null. Instead, we use an alternative characterization of this surface gravity. In order to formulate this alternative, consider a timelike ‘Killing observer’ with 4-velocity

$$v^\mu = K^\mu / \sqrt{-K^\mu K_{\mu}},$$

situated outside the horizon. After a proper time, $\delta \tau$, such an observer will have moved by

$$\delta x^\mu = v^\mu \delta \tau.$$  \hspace{1cm} (78)

Taking the inner product of both sides of this equation with $\nabla_\mu t$ yields

$$\delta t = \delta \tau / \sqrt{-K^\mu K_{\mu}}.$$  \hspace{1cm} (79)

Thus the red-shift factor associated with this observer satisfies $\Phi^2 = -K^\mu K_{\mu}$. We are now in a position to give the alternate formulation for the surface gravity as

$$\kappa_{\text{Killing}} = \lim \Phi a_i,$$ \hspace{1cm} (80)

where $a^2 = a^\mu a_\mu$ is the square of the 4-length of the 4-acceleration $a^\mu = v^{\mu;j}v^j$ and the limit corresponds to considering Killing observers ever closer to the horizon.

We note that in the limit of approaching the horizon $K^\mu$ is tangent to the Killing horizon world tube, so the Killing observers studied above may be considered as a suitable congruence of observers for the purposes of our Noether charge surface gravity of Eq. (74). Recalling that $\Phi^2 = -K^\mu K_{\mu} = 0$ on the Killing horizon. It follows that derivatives of any non-trivial function of $\Phi$ must therefore be normal to the horizon world tube, along $\hat{R}^\mu$. We may now write

$$-(K^\mu K_{\mu}j)^j \propto \hat{R}^j$$

$$\Rightarrow -K_{\mu ;\nu}K^{\mu} \propto \hat{R}^\nu$$

$$\Rightarrow K_{\mu ;\nu}K^\mu \propto \hat{R}^\nu$$

$$\Rightarrow a_{\nu} \propto \hat{R}_{\nu},$$ \hspace{1cm} (81)

where the Killing condition $K_{\mu ;\nu} = -K_{\nu ;\mu}$ is used in the third line. It follows that $a = a_\mu \hat{R}^\mu$ and we find that for non-degenerate non-bifurcate Killing horizons that

$$\kappa_{\text{Noether}} = \kappa_{\text{Killing}}.$$ \hspace{1cm} (82)

If the horizons are either degenerate or bifurcate the conventional definition from Eq. (76) yields $\kappa_{\text{Killing}} = 0$, whereas neither the alternative formulation of Eq. (80) for Killing horizons nor our Noether charge surface gravity of Eq. (74) suffer from this unphysical behavior.

**Local Lorentz transformation of the redshift**

We now consider how the red-shift factor for a non-geodesic, (or the co-moving) observer may be obtained by observation of the redshift of an orbiting (geodesic) observer (see Fig. 5).

![FIG. 5: On the left is Figure 2 with its non-geodesic observer moving tangent to the world tube $W$; the instantaneously co-moving geodesic observer in a parabolic-like trajectory; and the orbiting geodesic observer, in this example, spiraling around $W$. A small patch around the intersection point is expanded on the right. This shows the co-moving and orbiting observers’ world lines on a flat spacetime diagram. This shows that the relation between the red shift factors for these observers is determined by a local Lorentz transformation.](image)

The key point is that we assume that this latter red shift factor, $\Phi_{\text{orbiting}}$, is known through observation. To effect the local Lorentz transformation we also need to know the velocity, $v$, of these latter two observers relative to each other. In special relativity the velocities of either observer relative to the other are identical (up to a minus sign) even though measured in very different ways

$$v = ds_{\text{orbiting}} / d\tau_{\text{orbiting}} = -ds_{\text{co-moving}} / d\tau_{\text{co-moving}},$$ \hspace{1cm} (83)

where $s_i$ is the (proper length) location of observer $j \neq i$ in the frame of observer $i$, $\tau_i$ is that observer’s proper time and $i, j \in \{\text{orbiting, co-moving}\}$.

For simplicity, we assume that the reference observer, who is making the astronomical observation, is in the instantaneous rest frame of the non-geodesic observer who is accelerating to remain stationary on the world tube, we therefore use the latter expression of Eq. (83). The time-component of the Lorentz transformation then yields

$$\Phi_{\text{non-geodesic}} = \Phi_{\text{co-moving}} = \frac{1}{\sqrt{1 - v^2}} \Phi_{\text{orbiting}}.$$ \hspace{1cm} (84)

Let us consider this in a simple example involving circular equatorial orbits in the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2.$$ \hspace{1cm} (85)

Assuming orbits of the form $v^\mu \propto (1, 0, 0, f(r))$, and using the conditions $v^{\mu ;\nu}v^\nu = 0$ and $v^\mu v_\mu = -1$, we find

$$v^\mu = \frac{ds_{\text{orbiting}}}{d\tau_{\text{orbiting}}} = \sqrt{1 - \frac{3M}{r}} \left(1, 0, 0, \frac{1}{r} \sqrt{\frac{M}{r}}\right).$$ \hspace{1cm} (86)
From Observation 1, the orbital red shift factor is
\[ \Phi_{\text{orbiting}} = \sqrt{1 - \frac{3M}{r}}, \] (87)
which we assume to be observationally accessible.

With regard to coordinate time, the orbit follows
\[ \frac{dx^\mu_{\text{orbiting}}}{dt} = (1, 0, 0, \frac{1}{r} \sqrt{\frac{M}{r}}), \] (88)
Thus, in the rest frame the velocity is \( ds_{\text{co-moving}}/dt = r \dot{\phi}/dt = \sqrt{M/r} \), still measured with regard to coordinate time. In terms of the proper time for the co-moving observer
\[ v = \frac{ds}{d\tau_{\text{co-moving}}} = \frac{1}{\Phi_{\text{co-moving}}} \sqrt{\frac{M}{r}}, \] (89)
From Eq. (84) we then obtain
\[ \Rightarrow \Phi_{\text{co-moving}}^2 (1 - v^2) = \Phi_{\text{orbiting}}^2, \]
\[ \Rightarrow \Phi_{\text{co-moving}}^2 \left(1 - \frac{\frac{3M}{r}}{1 - \frac{3M}{r}} \right) = 1 - \frac{3M}{r}, \]
\[ \Rightarrow \Phi_{\text{co-moving}}^2 = 1 - \frac{2M}{r}, \]
\[ \Rightarrow \Phi_{\text{co-moving}} = \sqrt{1 - \frac{2M}{r}}, \] (90)
where we used Eq. (87) to obtain the second line.

Note that we only used the Lorentz transformation summarized in Eq. (84), and in principle observationally accessible values for \( \Phi_{\text{orbiting}} \) and the orbital velocity \( v \). Of course, direct access to the metric, Eq. (85) would have come to the same conclusion, but this is not directly observable.

As a final observation, we note that given a sufficient number of similar observations at differing locations, we would have been able to determine, for example, that
\[ M_{\text{Grav}} = M, \] (91)
all concentrated within a central region within \( r < 3M \) (assuming access to circular orbits only).

**APPENDIX 2: RELATED LITERATURE**

**Brown-York quasi-local mass**

In 1993, Brown and York proposed a quasi-local mass/energy definition\(^{22}\) Their proposal is based on the 3+1 split formalism and assumes the 3-dimensional boundary (that we call the observers’ world tube) is orthogonal with the 3-dimensional foliated hypersurfaces. Thus in their construction, \( v^\mu = \hat{T}^\mu \) and \( R^\mu = \bar{N}^\mu \). Therefore, the projector onto the two-surface \( S \) may be written \( \sigma_{\mu\nu} = g_{\mu\nu} + \hat{T}_\mu \hat{T}_\nu - \bar{N}_\mu \bar{N}_\nu \). The extrinsic curvature of the two surface \( S \) is hence given by \( k = \bar{N}_{\mu\nu} \sigma^{\mu\nu} \) with respect to the hypersurface that \( S \) is embedded in. Finally, Brown and York define a quasi-local mass
\[ M_{\text{B-Y}} = -\frac{1}{8\pi} \int_S (k - k_0) \, dA, \] (92)
where \( k_0 \) is the extrinsic curvature of \( S \) on a background flat spacetime.

Although Brown and York claim this proposal to be a quasi-local mass, it seems it only works at spatial infinity as a global mass definition\(^{23}\). In particular, though it reduces to the ADM mass at infinity for asymptotically-flat spacetimes, it does not yield the standard results over finite surfaces, even for a spacetime with a single black hole. For the Schwarzschild metric, it is easy to compute \( k = \frac{2}{r} \sqrt{1 - \frac{2M}{r}} \) and \( k_0 = \frac{2}{r} \) by taking \( M \to 0 \). Thus
\[ M_{\text{B-Y}} = -\frac{1}{8\pi} \int (k - k_0) \, dA, \]
\[ = -\frac{1}{8\pi} \int \left[ - \frac{2M}{r^2} - M^2 + O\left(\frac{1}{r^4}\right) \right] \, dA, \]
\[ = M + \frac{M^2}{2r} + O\left(\frac{1}{r^2}\right). \] (93)
Therefore the Brown-York mass only yields the standard result when \( r \to \infty \). Indeed, some textbooks only describe this mass definition as applying at spatial infinity, see Ref.\(^{23}\).

**Alternative dynamical surface gravities**

Traditionally, surface gravity is defined for stationary spacetime on a Killing horizon. However, a real physical black hole in our Universe must interact with other gravitating body and hence be dynamical. So several different surface gravity definitions for dynamical spacetimes have been proposed and they do not agree with each other even for the most simple spherically-symmetric dynamical scenarios\(^{28}\). We give a short review to some of these definitions here.

The first definition for a dynamical, non-Killing horizon, surface gravity proposed by Hayward\(^{29}\) is independent of the chosen normalization on the horizon \( \kappa^H = 1/\sqrt{-g^{(4)}_{\mu\nu} \theta^{(4)}_{\mu\nu}} \). However, it is known that this does not give the correct answer in the Reissner–Nordström case even for spherically symmetric spacetimes.

Since the Killing vector used in the traditional surface gravity definition is null on the Killing horizon, a generalized surface gravity may be proposed based on the outward null vector \( l^\mu \) that \( l^\mu \theta_{\mu\nu} = \kappa^{\text{null}}\theta_{\mu\nu} \). However, there are two problems for this definition: (i) There is a freedom in the normalization of such a null vector and hence the surface gravity is not uniquely defined. (ii) The horizon tube of a truly dynamical horizon may not be null and hence this null vector may not be tangent to the horizon tube (while in Killing horizon \( \xi^\mu \) is tangent to the horizon tube).
To fix the parametrization problem in the surface gravity based on outward null vector, Fodor et al.\cite{Fodor} proposed a non-local choice of normalization based on the ingoing null geodesic \( n^\mu \). They require that \( l^\mu n_\mu = -1 \) and \( l^\mu n_\mu = -1 \) where \( l^\mu \) is the asymptotically time-translational Killing vector for an asymptotically-flat spacetime. Then the surface gravity is defined as \( \kappa^F = -l_\mu l^\nu n_\nu \). However, this is only proposed for spherically symmetric spacetimes and requires the spacetime to be asymptotically flat.

Hayward also proposed a dynamical surface gravity for spherically symmetric spacetimes in terms of the Kodama vector. However, it is also only designed for spherically symmetric spacetimes\cite{Hayward}. It seems this only work for some special kind of coordinates system even for spherically symmetric spacetimes.

For an isolated horizon\cite{Ashtekar}, Ashtekar and colleagues also propose \( \kappa^B = -l_\mu e^\nu n_\nu \). To solve the normalization problem, they fix the surface gravity as a unique function of the horizon area and energy of the black hole, in terms of the known thermal relation in the static case. However, this is an effective surface gravity which means it is an average of the real local surface gravity and it does not is hard to deal with for scenarios with several horizon parameters, like the Einstein-Yang-Mills case.

Booth and Fairhurst generalized the suggestion of Ashtekar et al when they try to generalize the isolated horizon to the so-called slowly evolving horizon\cite{Booth}. They suggest \( \kappa^B = -B l_\mu e^\nu n_\nu - C n_\mu n^\nu l^\mu \) with the normal of the horizon equals \( B l^\mu + C n^\mu \) for the slowly evolving horizon. The normalization of this surface gravity also needs the help of the horizon parameters and the hold of the first law of black hole thermodynamics. So this definition is not self-consistent by itself too.

For dynamical horizon\cite{Ashtekar2}, an effective surface gravity \( \kappa = \frac{1}{\pi} \frac{d(\kappa)}{dt} \) is used in the study of black hole thermodynamics by Ashtekar and Krishnan. Once again there is freedom in the normalization that can usually be fixed by appeal to the stationary Kerr solution. Moreover, since it is derived based from an area balanced law, it more like an average of the real surface gravity for truly dynamical system.

For these proposed surface gravity definitions for dynamical spacetimes, only \( \kappa^F \) perfect agrees with the results calculated by the conventional surface gravity\cite{Fodor} but it only defines for spherically symmetric scenarios. All the other proposals either need some specific normalization or only works for some special coordinates system even for the spherically symmetric spacetimes.

### Entropy as a Noether charge?

In 1993, Wald\cite{Wald1, Wald2} considered generally diffeomorphic theories of gravity and found a ‘first law’ for black holes perturbed form stationarity of the form

\[
\delta \int Q = \delta \mathcal{E} - \text{angular momentum terms}, \quad (94)
\]

where the integral is taken over the horizon surface (a 2-surface in 3+1 dimensions) and where \( \delta \) denotes a diffeomorphic perturbation, \( Q \) is the Noether charge for the theory (here being integrated over the black hole’s horizon), and \( \mathcal{E} \) is the spacetime’s ADM energy. He argued that this corresponded precisely to the first law of black hole mechanics from which he concludes

\[
\delta \int Q = \frac{\kappa}{2\pi} \delta S, \quad (95)
\]

where \( \kappa \) is the unperturbed Killing surface gravity and \( S \) the presumed black hole entropy which he identifies as

\[
S = \frac{2\pi}{\kappa} \int Q. \quad (96)
\]

(Wald\cite{Wald1} writes this as \( S = 2\pi \int \hat{Q} \), in terms of the Noether charge, \( \hat{Q} = Q/\kappa \), obtained from a Killing field normalized to have unit surface gravity, however, this is simply Eq. \(96\). It is in this normalized form that Wald calls the entropy as an integrated Noether charge\cite{Wald2}.

A problem immediately arises from this analysis if one combines Eqs. \(95\) and \(96\) to yield

\[
\delta(\kappa S) = \kappa \delta S. \quad (97)
\]

It is easy to see that this relation fails to hold, for example, reducing to \( 1 = 2 \) for Schwarzschild black holes in ordinary general relativity, under a diffeomorphism that infinitesimally changes a black hole’s mass. This leads to the likely conclusion that Eq. \(94\) was incorrectly identified as the first law of black hole mechanics.

Indeed, Wald’s Eq. \(96\) has received support for equilibrium black holes. For example, Garfinkle and Mann\cite{Garfinkle} consider the so-called generalized gravitational entropy \( S_{\text{gen}} \), in the Euclidean domain. Adding terms to the gravitational action which preserve the equations of motion they find

\[
S_{\text{gen}} = \beta \left( \int Q + \int_\infty Q_0 \right), \quad (98)
\]

where the temperature \( 1/\beta \) is given by the periodicity of Euclidean time and the final integral denotes evaluating \( Q \) on a background spacetime at spatial infinity.

Wald’s original analysis\cite{Wald1} was generalized in Ref.\cite{Hayward} with the entropy defined as an integral of only part of the Noether current, though still keeping the claim that Eq. \(94\) is key to a first law of black hole mechanics. However, as we shall now see, this modification still leads to an inconsistency when applied to the simplest case of a Schwarzschild black hole in 3+1 spacetime dimensions.

Following Ref.\cite{Hayward} we take \( \xi^\mu = \partial_t \) for the Schwarzschild spacetime and the integral of \( Q = Q[\xi^\mu] \) at spatial infinity (denoted as \( \infty \)) is identified in Ref.\cite{Hayward} as precisely one-half of the expression for the Komar mass (see Eq. \(85\) of that reference), i.e., we must have

\[
\int_\infty Q[\partial_t] = \frac{1}{2} M. \quad (99)
\]
Now in a vacuum spacetime the Komar integral is well known to be independent of the boundary of integration,
so Eq. (99) may be rewritten as an integral over the horizon 2-surface \( H \) (labeled \( \Sigma \) in Ref. [18]) as
\[
\int_{\infty} \mathcal{Q}[\partial_t] = \int_{H} Q[\partial_t] = \int_{H} Q[\xi^\mu] = \frac{1}{2} M. \tag{100}
\]

Thus, we may explicitly evaluate the left-hand-side of Eq. (99) for a Schwarzschild black hole as
\[
\delta \int_{H} Q[\xi^\mu] = \frac{1}{2} \delta M. \tag{101}
\]

Since the horizon entropy and surface gravity of a Schwarzschild black hole are \( S = 4\pi M^2 \) and \( \kappa = 1/(4M) \), respectively, the right-hand-side of Eq. (95) reduces to
\[
\frac{\kappa}{2\pi} \delta S = \frac{1}{8\pi M} 8\pi M \delta M = \delta M. \tag{102}
\]

We now easily see that Eq. (95) leads to an explicit contradiction for even the simplest scenario.

In any of the analyses above, and consistent with what we find for the surface gravity constrained to ordinary general relativity, the integrated Noether charge is better described as proportional to the product \( \kappa S/(2\pi) \), i.e., physically as an energy, instead of an entropy. We note that Komar had already made this observation, although without explicitly recognizing his analysis as being related to a Noether charge.

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