DOUBLE MINIMALITY, ENTROPY AND DISJOINTNESS WITH ALL MINIMAL SYSTEMS

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Abstract. In this paper we propose a new sufficient condition for disjointness with all minimal systems. Using proposed approach we construct a transitive dynamical system \((X, T)\) disjoint with every minimal system and such that the set of transfer times \(N(x, U)\) is not an \(IP^*\)-set for some nonempty open set \(U \subset X\) and every \(x \in X\). This example shows that the new condition sharpens sufficient conditions for disjointness below previous bounds. In particular our approach does not rely on distality of points or sets.

1. Introduction. In 1967 in [6] Furstenberg introduced the notion of topological disjointness and stated the following (see [6, Problem G, p. 34]):

**Problem 1.1.** Describe the class of systems disjoint with all distal systems and with all minimal systems.

He also proved that both classes are nonempty. The first class was very quickly fully characterized in short and elegant paper of Petersen [25]. After almost 50 years the second problem remains open, even when restricted to the class of transitive systems. Still, some “upper” and “lower” bounds on the shape of this class are known. For example, if \((X, T)\) is a transitive dynamical system disjoint with all minimal systems then it has to be a weakly mixing M-system [13]. It was also known from real beginning (see [6]) that every weakly mixing system with dense periodic points is disjoint with any minimal system. This condition was further generalized (e.g. see [4, Theorem 7.14] or [22, Theorem 4.4]) and probably the most general known version was obtained in [18] (an example satisfying these assumptions but not those from [4, 22] was recently presented in [19]):

**Theorem 1.2.** If \((X, T)\) is weakly mixing and \((K(X), T_K)\) has dense distal points then \((X, T)\) is disjoint with all minimal systems.

In the above theorem \(K(X)\) is the space of all nonempty closed subsets of \(X\) endowed with the Hausdorff metric, and \(T_K : K(X) \to K(X)\) is induced map defined by \(T_K(A) = T(A)\) for each \(A \in K(X)\). It is immediate to see, having at
hand Furstenberg’s characterization of distality [6], that when assumptions of Theorem 1.2 are satisfied, then for every nonempty open set $U$ there exists $x \in U$ such that $N(x, U)$ is an IP*-set. Knowing recent advances in topological dynamics (e.g. see proofs in [18]) it is also relatively easy to prove the following two theorems (proofs are presented in Section 3):

**Theorem 1.3.** If $(X, T)$ is weakly mixing and for every minimal system $(Y, S)$ there exists a countable set $D \subset X$ such that for every nonempty open set $U \subset X$ the following condition holds:

1. for any $y \in Y$ and any open neighborhood $V \ni y$ there is $x \in U \cap D$ such that the set of transfer times $N_{T \times S}((x, y), U \times V)$ is syndetic,

then $(X, T)$ is disjoint with every minimal system.

**Theorem 1.4.** If $(X, T)$ is transitive with at least two points and $(\omega_T(x), T)$ is weakly mixing for each minimal point $x$, then for every transitive point $q$ there exists an open set $U \ni q$ such that $N_T(z, U)$ is not an IP*-set for any $z \in X$.

Before we proceed, let us make an important observation. We claim that if $N_T(x, U)$ is an IP*-set then for any minimal dynamical system $(Y, S)$, any $y \in Y$ and any open set $V \ni y$, the set $N_T(x, U) \cap N_S(y, V)$ is syndetic. Otherwise there is a thick set $A$ such that $A \cap N_T(x, U) \cap N_S(y, V) = \emptyset$. But since $y$ is a minimal point, it is not hard to construct an IP-set in $A \cap N_S(y, V)$ which leads to a contradiction. Indeed, the claim holds. This shows that if a point $x$ is such that $N(x, U)$ is an IP*-set then condition (1) is satisfied with this particular $x$. Therefore, if we fix a countable base of topology of $X$ and for each $U$ in this base find a point $x_U \in U$ such that $N_T(x_U, U)$ is an IP*-set, then assumptions of Theorem 1.3 are satisfied.

By the above considerations (see also Remark 3.2) Theorem 1.3 generalizes conditions presented in [18] on transitive $(X, T)$, sufficient for disjointness with all minimal systems. To the best author knowledge, conditions from [18] are the most general method for describing the class of systems disjoint with all minimal systems, which at the same time can be quite easily applied in practice. We will show that our Theorem 1.3 goes beyond results of [18], providing a system which satisfies assumptions of this theorem but not of [18].

Our aim for completing this goal is to show that there exists a dynamical system satisfying assumptions of both Theorems 1.3 and 1.4. This way we will prove the following.

**Theorem 1.5.** There exists a weakly mixing M-system $(X, T)$ disjoint with all minimal systems and a nonempty open set $U \subset X$ such that $N_T(x, U)$ is not an IP*-set for any $x \in X$.

Therefore Theorem 1.3 makes a step towards answering question of Furstenberg in Problem 1.1, since as we claimed before, dynamical system from Theorem 1.4 does not satisfy sufficient conditions proposed in [18]. This also motivates the following question.

**Question 1.6.** Are assumptions of Theorem 1.3 also a necessary condition for M-system to be disjoint with all minimal systems?

2. **Preliminaries.** We denote by $\mathbb{Z}$ the set of all integers and by $\mathbb{N}$ the set of all positive integers. If $A$ is a set, then its cardinality is denoted by $|A|$.
2.1. Topological dynamics. A dynamical system is a pair \((X,T)\) where \((X,d)\) is a compact metric space and \(T: X \to X\) is a continuous surjection. Let \((X,T)\) be a dynamical system. For \(\varepsilon > 0\) and a set \(A \subset X\) we denote \(B_\varepsilon(A) = \{x : \text{dist}(x,A) < \varepsilon\}\). A set \(J \subset X\) is invariant if \(T(J) = J\). When \(J\) is invariant then we simply write \((J,T)\) instead of \((J,T_J)\) to describe associated dynamical system. For \(x \in X\) denote its forward orbit by \(O^+_T(x) = \{T^n(x) : n \geq 0\}\). If \(T\) is invertible, we can define orbit of \(x\) by \(O_T(x) = \{T^n(x) : n \in \mathbb{Z}\}\). The \(\omega\)-limit set of \(x\) is \(\omega_T(x) = \bigcap_{n=1}^{\infty} O^+_T(T^n(x))\). A set \(M \subset X\) is minimal if \(O^+_T(x) = M\) for every \(x \in M\). Equivalently it means that \(M\) is a nonempty, closed and invariant subset of \(X\) without proper subsets with these three properties. A point \(x \in X\) is minimal if \(O^+_T(x)\) is a minimal set; recurrent if \(x \in \omega_T(x)\); transitive if \(\omega_T(x) = X\). For \(x \in X\) and \(U,V \subset X\) define sets of transfer times:

\[
N_T(x,U) = \{n > 0 : T^n(x) \in U\}, \quad N_T(U,V) = \{n > 0 : T^n(U) \cap V \neq \emptyset\}.
\]

When the acting map \(T\) is clear from the context we will simply write \(\omega(x), O(x), N(x,U), \) etc.

A dynamical system \((X,T)\) is transitive if \(N(U,V) \neq \emptyset\) for any two nonempty open sets \(U,V \subset X\); weakly mixing when the product system \((X \times X, T \times T)\) is transitive; M-system when it is transitive and minimal points are dense in \(X\). It is well known that \((X,T)\) is transitive iif the set of transitive points is residual in \(X\).

It is a classical result that a transitive dynamical system \((X,T)\) is weakly mixing provided that for any nonempty open set \(U\) there is \(n > 0\) such that \(n,n+1 \in N_T(U,U)\), see [13, Lemma 5.1]. A simple modification of that condition was presented later in [23, Lemma 3.8]: if there is \(x \in X\) which has dense forward orbit and for any open set \(U \ni x\) there is \(n > 0\) such that \(n,n+1 \in N_T(U,U)\), then \((X,T)\) is weakly mixing. We will use this condition in proofs without any further reference.

A pair of points \(x,y\) is proximal if \(\liminf_{n \to \infty} d(T^n(x),T^n(y)) = 0\). A point \(x \in X\) is distal if \(O^+(x)\) does not contain point \(y \neq x\) such that the pair \((x,y)\) is proximal. By result of Auslander and Ellis (see [7, Theorem 8.7]) every point is proximal to some minimal point, in particular distal point has to be minimal.

If \(A \subset \mathbb{N}\) is such that for some \(k > 0\) and every \(i > 0\) we have \([i,i+k] \cap A \neq \emptyset\) then we say that \(A\) is syndetic. A set \(A \subset \mathbb{N}\) is thick if \(A \cap Q \neq \emptyset\) for any syndetic set \(Q\).

A set \(A \subset \mathbb{N}\) is an IP-set if there exists a sequence \(\{p_i\}_{i=1}^\infty\) such that \(p_i + \ldots + p_{i_n} \in A\) for any \(n > 0\) and any finite sequence of indexes \(i_1 < i_2 < \ldots < i_n\). If \(A \subset \mathbb{N}\) has the property that \(A \cap Q \neq \emptyset\) for any IP-set \(Q\) then we say that \(A\) is an IP*-set. It is well known that if \(x\) is distal then \(N(x,U)\) is an IP*-set for any open neighborhood \(U \ni x\); see [7, Theorem 9.11] and [24, Theorem 5] for more details on conditions equivalent to distality.

A joining of dynamical systems \((X,T), (Y,S)\) is any closed and \(T \times S\)-invariant set \(J \subset X \times Y\) whose projections on the first and second coordinate are \(X\) and \(Y\), respectively. Clearly \(X \times Y\) is always a joining, and if it is the only joining, then we say that \((X,T)\) and \((Y,S)\) are disjoint. It is not hard too see that if two dynamical systems are disjoint then one of them must be minimal and it also not hard to see that when two minimal systems are disjoint, then \(X \times Y\) is a minimal set.

The reader is referred to monographs [7, 17, 26] for other basic notions and facts on dynamical systems, including definition of topological entropy.

2.2. Symbolic dynamics. Let \(A\) be any finite set (an alphabet). We endow \(A\) with discrete topology and \(A^\mathbb{Z}\) with the product topology. Clearly \(A^\mathbb{Z}\) is a compact
metrizable space. For any \( x \in \mathcal{A}^\mathbb{Z} \) and \( i \leq j \) we write \( x_{[i,j]} = x_i \ldots x_j \). Similarly we denote \( x_{[i,j]} = x_{i,j-1} \), provided that \( i < j \). By word (over \( \mathcal{A} \)) we mean any finite sequence \( w = w_0 \ldots w_{n-1} \) and its length is \( |w| = n \). Following standard notation, we do not separate symbols in \( w \) by \( , \). If \( u, w \) are words of length \( m, n \) then their concatenation is word \( uw = u_0 \ldots u_{m-1}w_0 \ldots w_{n-1} \). If \( w \) is a word and \( n > 0 \) is an integer then \( w^n = w w \ldots w \) is concatenation of \( n \) copies of \( w \). As usual, \( \mathcal{A}^\mathbb{Z} \) is equipped with the standard shift map \( \sigma \) given by \( (\sigma(x))_n = x_{n+1} \) for every \( n \in \mathbb{Z} \). It is well known that \( \sigma \) is continuous, hence any closed and \( \sigma \)-invariant set \( X \subset \mathcal{A}^\mathbb{Z} \) (so-called subshift) defines a dynamical system \( (X, \sigma) \). Language of shift \( X \subset \mathcal{A}^\mathbb{Z} \) is denoted by \( \mathcal{L}(X) = \{ x_{[i,j]} : x \in X, i \leq j \} \) and \( \mathcal{L}_n(X) = \mathcal{L}(X) \cap \mathcal{A}^n \).

The reader is referred to [20] or [17] for more information on subshifts and their properties, in particular such standard classes of shift spaces as shifts of finite type or sofic shifts.

3. Proofs of preliminary results.

**Proof of Theorem 1.3.** Fix any minimal dynamical system \((Y, S)\) and accompanying set \( D \). It is well known (e.g. see [1]) that the proximal cell

\[
\text{Prox}(T)(z) = \{ y \in X : \liminf_{n \to \infty} d(T^n(z), T^n(y)) = 0 \}
\]

is residual in \( X \) for any \( z \in X \), therefore since \( D \) is countable, there exists a transitive point \( q \in X \) such that \((q, p)\) is a proximal pair for every \( p \in D \).

Take any joining \( J \subset X \times Y \) of \((X, T)\) and \((Y, S)\). By the definition, there exists \( z \in Y \) such that \((q, z) \in J \). Fix any nonempty open set \( U \subset X \) and any open neighborhood \( V \) of \( z \). Let \( W \) be an open set such that \( \overline{W} \subset U \). There is \( \varepsilon > 0 \) such that \( B_z(\varepsilon \overline{W}) \subset U \). Fix any point \( p \in W \cap D \) such that \( N_{T \times S}((p, z), V \times V) \) is syndetic. By the definition \((q, p)\) is proximal pair, hence the set \( \{ n > 0 : d(T^n(q), T^n(p)) < \varepsilon \} \) is thick. This shows that \( N_{T \times S}((q, z), U \times V) \neq \emptyset \) and so \( J \cap (U \times V) \neq \emptyset \). But \( U \) is an arbitrary open set, \( V \) is an arbitrarily small neighborhood of \( z \), \( J \) is closed and \( T \times S\)-invariant. We immediately obtain that \( X \times \{ z \} \subset J \) which, since \( z \) has dense forward orbit in \( Y \), gives \( J = X \times Y \).

The following lemma is a simple modification of [24, Corollary 17] (see also [1]).

**Lemma 3.1.** Let \( M \) be a minimal weakly mixing set and let \( x \) be such that \( M \subset \omega(x) \). Then \( \text{Prox}(T)(x) \cap M \) is residual in \( M \), where \( \text{Prox}(T)(x) \) is the set of points proximal to \( x \).

**Proof.** First we claim that for every open set \( U \) intersecting \( M \) and each \( \varepsilon > 0 \) there are \( n \in \mathbb{N} \) and \( y \in U \cap M \) with \( d(T^n(x), T^n(y)) \leq \varepsilon \). Fix any \( \varepsilon > 0 \), let \( U \) be an open set intersecting \( M \) and let \( \{ V_1, \ldots, V_k \} \) be a cover of \( M \) consisting of open sets (in \( X \)) with diameters less than \( \varepsilon \).

By weak mixing of \( M \) there are open set \( U_1, \ldots, U_k \subset U \) intersecting \( M \) and an integer \( n \) such that \( T^n(U_j) \subset V_j \) for \( j = 1, \ldots, k \). There is \( z \in M \), its open neighborhood \( W \) and positive integers \( s_1, \ldots, s_k \) such that \( T^{s_j}(W) \subset U_j \). But \( N(z, W) \) is syndetic and therefore there is a syndetic set \( S \subset N(U_j \cap M, V_j) \) for each \( j \). Observe that \( M \subset \omega(x) \) implies that \( N(x, \bigcup_{j=1}^k V_j) \) is thick, in particular it intersects \( S \). Take any \( m \in S \cap N(x, \bigcup_{j=1}^k V_j) \) and let \( i \) be such that \( T^m(x) \in V_i \). There is also \( y \in U_i \cap M \subset U \cap M \) such that \( T^m(y) \in V_i \). This proves the claim.
By the above claim, if we consider the set $M_\varepsilon$ consisting of all points $y \in M$ such that $d(T^n(x), T^n(y)) < \varepsilon$ for some $n > 0$, then $M_\varepsilon$ is a dense subset of $M$. But it is also easy to verify that $M_\varepsilon$ is an open subset of $M$. Clearly we have

$$\text{Prox}(T)(x) \cap M \supset \bigcap_{n=1}^{\infty} M_{\frac{1}{n}},$$

which shows that $\text{Prox}(T)(x) \cap M$ is residual in $M$. The proof is completed.

**Proof of Theorem 1.4.** Fix any point $q \in X$ with dense forward orbit and let $U$ be a neighborhood of $q$ such that $U \cap T(U) = \emptyset$. It is enough to show that for any $z \in X$ the set $N(z, U)$ is not IP*. Take any minimal set $M \subset \omega(z)$. By assumptions $M$ is weakly mixing. Denote $V = X \setminus \overline{U}$. Then either $M \subset V$, or $M \cap T(V) \neq \emptyset$ which also implies that $V \cap M \neq \emptyset$. By Lemma 3.1 there is $y \in M \cap V$ such that the pair $(z, y)$ is proximal. Since $y$ is a minimal point, the set $N(z, V)$ is a central set, therefore contains an IP-set (e.g. see [7, Prop. 8.10]). Indeed $N(z, U)$ is not IP*, which completes the proof.

Since for any open set $U \subset X$ the set $\langle U \rangle = \{ E \in \mathcal{K}(X) : E \subset U \}$ is open in $\mathcal{K}(X)$, the following is obvious:

**Remark 3.2.** If $(K(X), T_K)$ has dense distal points then for every nonempty open set $U$ there exists a point $x \in U$ such that $N(x, U)$ is an IP*-set.

4. **Doubly minimal systems.** An invertible dynamical system $(X, T)$ is doubly minimal (see [27]) or has topological minimal self-joinings in the sense of del Junco (see [16]) if $(X, T^k)$ is minimal for every $k \in \mathbb{Z} \setminus \{0\}$ and for any $x, y \in X$ not in the same orbit (i.e. $x \neq T^k(y)$ for every $k \in \mathbb{Z}$) we have $\overline{O_{T^kX}(x, y)} = X \times X$.

**Remark 4.1.** If $(X, T)$ is doubly minimal with $X$ infinite, then it is weakly mixing.

It is known for a while that infinite doubly minimal systems exist. Explicit examples were provided in [16] and later in [27] by a different argument. It is also worth mentioning that every doubly minimal system must be a subshift [15], so these constructions are in a sense the only possible. In this section we will follow approach of King from [16]. While results of [27] can give us stronger statements on constructed doubly minimal systems than zero entropy, the technique is much more complicated to deal with (and for our purposes we do not need to fully characterize the constructed zero entropy measure).

In the original construction in [16] the initial point with dense forward orbit is constructed in a subshift over the alphabet $\{a, b, s\}$ with spacers “s” properly placed between one letter words $a, b$. In our approach we will need longer words in place of $a$ and $b$, so we have to present a small extension of the construction from [16]. A careful reader will notice, however, that the core of the construction is exactly the same as in [16].

Before we start the construction, let us present an important property of doubly minimal systems used later in the text. When solving an open problem from [12], Glasner and Weiss used (and reproved) a results of [8] (see [10, Theorem 0.2]).

**Lemma 4.2.** Let $(X, T)$ be doubly minimal and let $(Y, S)$ be a minimal system which is not an extension of $(X, T)$. Then $(X, T)$ and $(Y, S)$ are disjoint.
Another important property of doubly minimal system \((X,T)\) is that for any \(x,y \in X\) the pair \((x,y)\) is recurrent (but not necessarily positively recurrent). In fact it was proved in [15] that doubly minimal systems are never 2-rigid. This prevent us from applying directly results of [3]) when proving that doubly minimal systems have zero entropy, since these subshifts can have asymptotic pairs.

However it is possible to use other argument\(^1\). Suppose that \(\pi: (X,T) \to (Y,S)\) is a factor map and that \(\pi^{-1}(y)\) for some \(y \in Y\) has at least two points. If \(\pi^{-1}(y)\) contains two points from the same orbit, then \((Y,S)\) is a single periodic orbit, so in fact \(Y\) is a single point because \((X,T^n)\) is minimal for every \(n\). But if \(\pi^{-1}(y)\) contains two points from different orbits, then it is also not hard to see that \(\pi^{-1}(y) = X\). In other words, \((X,T)\) has only trivial topological factors. By [21, Theorem 4.6] every zero dimensional system has a nontrivial factor with arbitrarily small entropy. This immediately implies that entropy of \((X,T)\) is zero (see also comments on page 232 of [21]).\(^2\)

The main result of this section is the following.

**Theorem 4.3.** For any two distinct words \(A,B \in \{0,1\}^s\), where \(s \geq 1\), and for any \(K > 0\) there exists a minimal subshift \(X \subset \{0,1\}^s\) and \(N\) such that:

1. \((X,\sigma)\) is doubly minimal, weakly mixing and has zero topological entropy,
2. \(A,B \in \mathcal{L}(X)\),
3. \(\frac{1}{n}|\{j : x_{i+j} = 1, j < n\}| \leq 1/K\) for every \(x \in X\), \(i \in \mathbb{Z}\) and \(n \geq N\),
4. each \(x \in X\) is a concatenation of words \(A,B\) separated by sequences of zeros.

The rest of this section is devoted to construction proving Theorem 4.3. We will fist define a subshift \(X\) over an extended alphabet \(\{0,1,a\}\), where \(a\) is a special spacer to ensure proper shifts between cylinder sets.

Extending both words \(A,B\) if necessary by adding suffix \(0^n\) with some \(m > 0\), we may assume that \(|A| > 2K\) and if \(w \in \{A,B\}\) then

\[
\frac{1}{|w|}|\{1 \leq i \leq n : w_i = 1\}| < 1/2K.
\]

We will define inductively a sequence of words \(H_n^{(0)},H_n^{(1)}\) of equal length. We put \(H_0^{(0)} = A\) and \(H_0^{(1)} = B\) and denote \(h_0 = |H_0^{(0)}|\). Assume that words of equal length \(H_{n-1}^{(0)}, H_{n-1}^{(1)}\) have already been defined for some \(n > 0\). Denote \(h_{n-1} = |H_{n-1}^{(0)}| = |H_{n-1}^{(1)}|\) and take \(k_n > h_{n-1}(2h_{n-1} + 1)\) sufficiently large to ensure that for every \(0 \leq |j| \leq 2h_{n-1} + 1\) there are \(1 \leq q_1,q_2 < k_n/6\) such that \(2q_1h_{n-1} + j = q_2(2h_{n-1} + 1)\). Since \(2h_{n-1} + 2h_{n-1} + 1\) is co-prime, numbers \(q_1,q_2\) solving this equation always exist provided that \(k_n\) is large enough. Furthermore we may assume that \(6\) divides \(k_n\).

Denote \(U_n^{(k)} = H_{n-1}^{(1)}a(H_{n-1}^{(0)})^kH_{n-1}^{(1)}a\) and then define

\[
H_n^{(0)} = (H_0^{(0)})^{k_n}U_{n-1}^{(3)}(H_{n-1}^{(1)}a)^{k_n}U_{n-1}^{(4)}(H_0^{(0)})^{k_n}U_{n-1}^{(5)}(H_{n-1}^{(1)}a)^{k_n}H_{n-1}^{(0)},
\]

\[
H_n^{(1)} = (H_0^{(0)})^{k_n}U_{n-1}^{(3)}(H_{n-1}^{(1)}a)^{k_n/2}U_{n-1}^{(4)}(H_0^{(0)})^{k_n}U_{n-1}^{(5)}(H_{n-1}^{(1)}a)^{k_n/2}H_{n-1}^{(1)}.
\]

Observe that \(|H_n^{(0)}| = |H_n^{(1)}|\) and that each of the words \(H_n^{(0)}\) and \(H_n^{(1)}a\) can be presented as a concatenation of words \(H_n^{(0)}\) and \(H_n^{(1)}\). Then by induction, for

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\(^1\)It was brought to our attention by Song Shao

\(^2\)Another way to see this is to apply variational principle together with [9, Theorem 13.1].
any \( m > n \) both \( H_m^{(0)} \) and \( H_m^{(1)} a \) can be presented as a concatenation of words \( H_n^{(0)} \) and \( H_n^{(1)} a \).

Let \( x \in \{0, 1, a\}^\mathbb{Z} \) be a point defined by \( x_{[-h_n, h_n]} = H_n^{(0)} H_n^{(0)} \). Since \( H_n^{(0)} \) is both suffix and prefix of \( H_{n+1}^{(0)} \), point \( x \) is well defined. Put
\[
X = \overline{O(x)}.
\]

**Lemma 4.4.** Subshift \((X, \sigma)\) is minimal, weakly mixing and \(|X| = +\infty\).

**Proof.** Since each \( H_m^{(0)} \) for \( m > n + 1 \) is a concatenation of words \( H_{n+1}^{(0)}, H_{n+1}^{(1)} a \), word \( H_n^{(0)} H_n^{(0)} \) appear syndetically in \( x \). This shows that \( x \) is minimal, and so is \( X \) as the closure of its orbit. Weak mixing follows from the fact, that \( x \) contains \( H_n^{(0)} H_n^{(1)} a H_n^{(0)} \) and \((H_n^{(0)})^3\) as subwords, hence if \( W \ni x \) is cylinder set defined by word \( H_{n-1}^{(0)} H_n^{(0)} \) then \( 2h_n, 2h_n + 1 \in N_\sigma(x, W) \). This proves weak mixing. Since \( X \) is not a single point, we have \(|X| = +\infty\).

Now we will state analogs of “opposite types lemma” and “any types lemma” from [16]. First lemma ensures synchronization of words of different types, provided that their relative shift is not greater than \( h_n/2 \).

**Lemma 4.5.** Let “\( b \)” be an auxiliary symbol \( b \not\in \{0, 1, a\} \) and let \( v = b^k \) for some \( k \leq h_n/2 \). Assume additionally that \( \alpha \in \{0, 1\} \). If \( u = H_n^{(\alpha)} v \) and \( w = v H_n^{(\alpha)\circ} \) then there exists \( k \leq i \leq h_n - h_{n-1} \) such that \( u_{[i, i+h_{n-1}]} = w_{[i, i+h_{n-1}]} = H_n^{(0)} \).

**Proof.** Assume that \( \alpha = 0 \). The proof for the case \( \alpha = 1 \) is symmetric. Picture \( u \) written horizontally above \( w \). Note that the second occurrence of \((H_n^{(0)})^k\) appears over a part of \( H_n^{(1)} \), that is word \( v \) finished in \( w \) before \((H_n^{(0)})^k\) started in \( u \). Then under this word we will see in \( w \) a word of the form \( v'(H_{n-1}^{(0)} H_n^{(1)} a)^p v''(H_n^{(0)} H_n^{(1)} a)^q \) where \( \max\{p, q\} \geq k\alpha/6 \). But then there are words \( w', w'' \), both of length not exceeding \( 2h_{n-1} + 1 \) and \( r > 0 \) such that in \( u \) we see \((H_n^{(0)})^{k\alpha/3 + r}\) and under it in \( w \) we have \( w'(H_{n-1}^{(0)} H_n^{(1)} a)^{k\alpha/6} w'' \). By the definition of \( k_n \) there are \( q_1, q_2 < k_n/6 \) such that \( 2q_1 h_{n-1} - |w'| = q_2(2h_{n-1} + 1) \) and so the words \((H_{n-1}^{(0)})^{2q_1+1} \) and \( w'(H_{n-1}^{(0)} H_n^{(1)} a)^{q_2} H_n^{(0)} \) are of equal length and can be visualized as written one over another in \( u \) and \( w \). This completes the proof.

**Lemma 4.6.** Let “\( b \)” be an auxiliary symbol \( b \not\in \{0, 1, a\} \), let \( v = b^k \) for some \( h_n/2 < k \leq h_{n+1}/2 \) and let \( \alpha, \beta \in \{0, 1\} \). If \( u = H_n^{(\alpha)} v \) and \( w = v H_n^{(\beta)\circ} \) then there exists \( k \leq i \leq h_{n+1} - h_{n-1} \) such that \( u_{[i, i+h_{n-1}]} = w_{[i, i+h_{n-1}]} = H_n^{(0)} \).

**Proof.** If \( \alpha \neq \beta \) then by Lemma 4.5 there is \( k \leq i \leq h_{n+1} - h_n \) such that \( u_{[i, i+h_n]} = w_{[i, i+h_n]} = H_n^{(0)} \) and then \( u_{[i, i+h_{n-1}]} = w_{[i, i+h_{n-1}]} = H_n^{(0)} \) which ends the proof. Therefore we may assume that \( \alpha = \beta \). Similarly as in the proof of Lemma 4.5 picture \( u \) written horizontally above \( w \).

**Case 1.** Assume that \( \alpha = \beta = 1 \). Word \( H_{n+1}^{(1)} \) in \( u \) contains \( U_n^{(5)} = H_n^{(1)} a (H_n^{(0)})^5 H_n^{(1)} a \) and clearly it starts in \( u \) at position further than \( k \). Therefore under \( w \) we can see only a subword of \( H_{n+1}^{(1)} \). If under \((H_n^{(0)})^5 \) appears \( u'H_n^{(1)} w'' \) then we are done by Lemma 4.5. Therefore the only remaining possibility is that under \((H_n^{(0)})^5 \) we see \( v'(H_n^{(0)})^4 v'' \). If it comes from spacer \( H_n^{(1)} a (H_n^{(0)})^4 H_n^{(1)} a \) then under \( U_n^{(5)} \) in
\(w\) we see the word \(w' H_n^{(1)} \alpha (H_n^{(0)})^a H_n^{(1)} aw''\) with \(|w'| + |w''| = h_n\). But then relative shift of some \(H_n^{(0)}\) in \(u\) with respect to \(H_n^{(1)}\) in \(w\) does not exceed \(h_n/2\) and so again we may apply Lemma 4.5. The final case is that there are words \(z', z''\) with lengths \(|z'|, |z''| < h_n\) and such that \(U_n^{(5)} z'\) is written over \(z'' U_n^{(5)}\). But by assumptions \(|z''| \geq k > h_n/2\) and so again relative shift between \(H_n^{(1)}\) and \(H_n^{(0)}\) allows us to apply Lemma 4.5.

**Case 2.** Assume that \(\alpha = \beta = 0\). Note that if \((H_n^{(0)})_{z_n}\) in \(u\) is written above a word in \(w\) containing \(H_n^{(1)}\) then we may apply Lemma 4.5 and the proof is finished. Therefore we may assume that \(1 \leq k < h_n\), while by assumptions \(k > h_n/2\). But then the first occurrence of \(H_n^{(0)} H_n^{(1)} a\) in \(u\) is written over \(q' H_n^{(0)} q''\) with \(h_n/2 < |q'| = k < h_n\). Observe that \(h_n - |q'| \leq h_n/2\) and so again relative shift between \(H_n^{(1)}\) in \(u\) and \(H_n^{(0)}\) in \(w\) does not exceed \(h_n/2\), so Lemma 4.5 can be applied. The proof is completed.

It was first observed by King in [16] that to prove double minimality it is enough to prove that for \(y, z\) from different orbits, the pair \((x, y)\) is proximal for \(\sigma\) or \(\sigma^{-1}\).

**Lemma 4.7.** For \(y, z \in \mathcal{X}\) from distinct orbits, pair \((y, z)\) is proximal under \(\sigma\) or \(\sigma^{-1}\). In particular subshift \((\mathcal{X}, \sigma)\) is doubly minimal.

**Proof.** Fix any \(y, z \in \mathcal{X}\) from different orbits and assume on the contrary that they are not proximal neither for \(\sigma\) nor \(\sigma^{-1}\). Fix any \(n\) and observe that since \(y\) is a bi-infinite concatenation of copies of \(H_n^{(0)}\) and \(H_n^{(1)} a\) there is index \(-3h_n/2 \leq i \leq -h_n/2\) such that \(x_{i, i+h_n} = H_n^{(\alpha)}\) for some \(\alpha \in \{0, 1\}\). By the same argument, there is \(i - h_n/2 \leq j \leq i + h_n/2\) such that \(z_{i, j+h_n} = H_n^{(\beta)}\) for some \(\beta \in \{0, 1\}\). Denote their relative shift by \(s_n = i - j\) and observe that \(|s_n| \leq h_n/2\). Let \(m_n \leq n\) be a unique integer such that \(m_n - 1/2 < |s_n| \leq m_n/2\). Note that for \(s < n\) the word \(H_n'(0)\) is a prefix of both \(H_n^{(0)}\) and \(H_n^{(1)}\), so either \(m_n \leq 5\) or by Lemma 4.6 there is \(t_n\) such that

\[y_{[t_n, t_n+h_{m_n-2}]} = z_{[t_n, t_n+h_{m_n-2}]} = H_n^{(0)}_{m_n-2} = H_{m_n-2}^{(0)} \]

If \(\lim_{n \to \infty} m_n = +\infty\) then \(x, y\) are proximal for \(\sigma\) or \(\sigma^{-1}\) because when written horizontally one over another, they share common sub-block of length \(h_{m_n} \to \infty\). In the latter case, the sequence \(\{s_n\}^n_{n=1}\) is uniformly bounded, hence there is \(k \in \mathbb{Z}\) such that \(s_n = k\) for infinitely many \(n\). If we can see \(H_n^{(0)}\) written one over another in \(u\) and \(w\) for arbitrarily large \(m\) then \(x, y\) are proximal for \(\sigma\) or \(\sigma^{-1}\) hence assume that \(N\) is such that we will not see \(H_n^{(0)}\) simultaneously at a position \(i \in \mathbb{Z}\) in both \(u\) and \(w\) with \(m \geq N\). Take \(n > N-2\) such that \(k < h_n/2\) and \(s_n = k\). There is \(i\) such that \(y_{[i, i+h_n]} = H_n^{(\alpha)}\) and \(z_{[i+k, i+k+h_n]} = H_n^{(\beta)}\). Since \(k < h_n/2\), by Lemma 4.5 pair \(y, z\) share \(H_n^{(0)}\) provided \(\alpha \neq \beta\), hence \(\alpha = \beta\). If \(\alpha = 1\) then \(y_{[i, i+h_n]} = z_{[i+k, i+k+h_n]} = a\) and then \(y_{[i+h_n+1, i+2h_n]} = H_n^{(\alpha')}\) and \(z_{[i+h_n+k+1, i+2h_n+k]} = H_n^{(\beta')}\) which by Lemma 4.5 again and definition of \(N\) leads to \(\alpha' = \beta'\). By the same argument for \(\alpha = 0\) we have \(y_{[i+h_n, i+2h_n]} = z_{[i+h_n+k, i+2h_n+k]} = H_n^{(\alpha')}\) for some \(\alpha'\) and thus by induction we obtain that \(y_{[i, \infty]} = z_{[i+k, \infty]}\). Repeating the same methodology for left half of \(y\) and \(z\) we easily obtain that \(\sigma^k(y) = z\) which is a contradiction.

As we mentioned before, results of [21] (or [9]) imply that every doubly minimal system has zero entropy. In our context zero entropy can be proved almost effortlessly, without referring to advanced tools from [21]. We present a simple calculation below.
Lemma 4.8. Subshift \((X, \sigma)\) has zero entropy.

Proof. Let \(h_n = |H_n^{(0)}|\). By the construction each \(x \in X\) can be presented as a concatenation of words \(H_n^{(0)}\) and \(H_n^{(1)}a\). Therefore, each word \(w \in \mathcal{L}_{h_n}(X)\) is a subword of one of four possible arrangements of these concatenations \(H_n^{(0)}H_n^{(0)}\), \(H_n^{(0)}H_n^{(1)}a\), \(H_n^{(1)}aH_n^{(0)}\) and \(H_n^{(1)}aH_n^{(1)}a\). This leads to estimate (for large \(n\))

\[
\mathcal{L}_{h_n}(X) \leq 4(|H_n^{(1)}| + 1) \leq 4(h_n + 1) \leq h_n^2
\]

and then

\[
h_{\text{top}}(X) \leq \inf_{n} \frac{1}{n} \log |\mathcal{L}_n(X)| \leq \inf_{n} \frac{2}{h_n} \log h_n = 0.
\]

The proof is completed.

Proof of Theorem 4.3. Recall that \((X, \sigma)\) is doubly minimal, hence by previous discussion any factor of \((X, \sigma)\) is either conjugate dynamical system or a single point. Let \(\pi: X \to \{0, 1\}^\mathbb{Z}\) be a map given by \(\pi(z)_i = 1\) if \(z_i = 1\) and \(\pi(z)_i = 0\) if \(z_i \in \{0, a\}\). Denote \(X = \pi(X)\). Clearly \(\pi: (X, \sigma) \to (X, \sigma)\) is a continuous factor map. By the definition \(X\) is not a single point, therefore \(\pi\) is a injective and \((X, \sigma)\) and \((X, \sigma)\) are conjugate.

Finally, any word \(w \in \mathcal{L}(X)\) is a subword of a word which is concatenation of \(A\) and \(B0\). Let \(N = 4|B|\) and assume that \(|w| \geq N\). In the construction we assumed that each of \(A, B\) has at most \(1/2K\) occurrences of symbol 1. There is \(j \geq 4\) such that \(j|A| \leq |w| < (j + 1)|A|\) and then

\[
\frac{1}{|w|}|\{i : w_i = 1\}| \leq \frac{1}{j} \cdot \frac{j + 2}{2K} \leq \frac{1}{K}
\]

which completes the proof.

5. Uniform positive entropy minimal systems. In our proof of Theorem 1.5 we will need another class of weakly mixing minimal systems. We are going to consider the class of u.p.e. systems (in fact K-systems) because there is well documented methodology to construct them. Later we will see that it would be enough if these systems were diagonal. There are also many good papers on constructions of diagonal systems (e.g. see \([14]\)), but apparently it would be less automatic to prove this weaker condition. In what follows, we will make use of method developed in \([11]\).

Theorem 5.1. For any two distinct words \(A, B \in \{0, 1\}^s\), where \(s \geq 1\), and for any \(K > 0\) there exists a minimal subshift \(X \subset \{0, 1\}^\mathbb{Z}\) and \(N\) such that:

1. \((X, \sigma)\) is minimal, topologically mixing with positive entropy and disjoint with every minimal system of zero topological entropy (in fact, has uniform positive entropy of all orders),
2. \(A, B \in \mathcal{L}(X)\),
3. \(\frac{1}{n}|\{j : x_{i+j} = 1, j < n\}| \leq 1/K\) for every \(x \in X\), \(i \in \mathbb{Z}\) and \(n \geq N\),
4. each \(x \in X\) is a concatenation of words \(A, B\) separated by sequences of zeros.

Proof. Put \(\hat{A} = A0^{2K|A|}\) and \(\hat{B} = B0^{2K|A|+1}\) and note that \(|\hat{A}| + 1 = |\hat{B}|\) and

\[
\frac{1}{|\hat{A}|}|\{j : \hat{A}_j = 1, j < |\hat{A}|\}| < 1/2K, \quad \frac{1}{|\hat{B}|}|\{j : \hat{B}_j = 1, j < |\hat{B}|\}| < 1/2K.
\]

Let \(G\) be a graph composed by two cycles \(c_A, c_B\) starting and terminating in the same vertex and of lengths, respectively, \(|c_A| = |\hat{A}|\) and \(|c_B| = |\hat{B}|\). Let \(X_G\) be a
shift of finite type defined by bi-infinite paths on graph $G$ and let $Y_G$ be a sofic shift obtained by labeling cycle $c_A$ by $\hat{A}$ and $c_B$ by $\hat{B}$. This defines also a natural factor map $\pi: (X_G, \sigma) \rightarrow (Y_G, \sigma)$. Furthermore, since $c_A, c_B$ are of co-prime length, $(X_G, \sigma)$ is topologically mixing; the reader is referred to [20] or [17] for more details on dynamics of shifts of finite type and sofic shifts. Clearly both subshifts have positive topological entropy.

By construction in [11] (see Theorem 13 there), there exists a strictly ergodic K-system $Z \subset X_G$ with entropy arbitrarily close to that of $X_G$. Denote $X = \pi(Z)$. By results of [14] subshift $(Z, \sigma)$ has uniform positive entropy of all orders (and is mixing), so $(X, \sigma)$ has the same properties. It is also clear that $X$ is not a singleton, therefore (2) holds. To see that (1) holds (we need to verify statement about disjointness) it is enough to apply results of [2].

Let $N = 18|B|$, $r = 3|A| < 3|B|$. By the construction, every $y \in X$ can be decomposed into a sequence of words $\hat{A}, \hat{B}$. For each $n > N$ there is $j \geq 6$ such that $jr \leq n < (j + 1)r$. Let

$$\xi = \max \{|\{i : \hat{A}_i = 1\}|, |\{i : \hat{B}_i = 1\}|\}$$

and note that $\xi/r < 1/4K$. Then for any $w \in \mathcal{L}_n(X)$ we have

$$\frac{1}{n} |\{i : w_i = 1\}| \leq 3\xi^j + 2 \leq \frac{3\xi}{jr} + \frac{6\xi}{jr} \leq \frac{3}{4K}(1 + \frac{2}{j}) \leq \frac{1}{K}$$

hence (3) holds, and by the definition of words $\hat{A}, \hat{B}$, also (4) holds, completing the proof. 

6. **Main example.** In this section we will combine construction from [19] (see also [5]) with doubly minimal systems obtained with help of Theorem 4.3 and systems with positive topological entropy provided by Theorem 5.1.

**Theorem 6.1.** There exists a weakly mixing subshift $(X, \sigma)$ and a countable collection of minimal sets $\{D_n\}_{n=0}^\infty$ and $\{P_n\}_{n=1}^\infty$ such that:

1. if $M \subset X$ is minimal, then $M = D_n$ or $M = P_n$ for some $n$,
2. $\bigcup_{n=1}^\infty D_n = X$ and $\bigcup_{n=1}^\infty P_n = X$,
3. each $(D_n, \sigma)$ is doubly minimal (with zero entropy) and each $(P_n, \sigma)$ has uniform positive topological entropy,
4. all $(D_n, \sigma)$ and $(P_n, \sigma)$ are weakly mixing.

**Proof.** Start with $A_1 = 010^{10}, B_1 = 01^{12}, K_1 = 9$ and let $D_1$ be a doubly minimal weakly mixing system provided by Theorem 4.3 together with a constant $N_1$ for $A_1, B_1, K_1$. Let

$$\alpha_1 = \inf_{x \in D_1} \lim_{n \rightarrow \infty} \frac{1}{n} \sup_{i \in \mathbb{Z}} |\{j : x_{i+j} = 1, 0 \leq j < n\}|.$$ 

Note that $\alpha_1 > 0$ because $D_1$ is minimal and has at least two points. Take $L_1 \geq 9$ such that $1/L_1 < \alpha_1/2$ and let $P_1$ be a mixing minimal system with uniform positive topological entropy provided by Theorem 5.1 together with a constant $M_1$ for $A_1, B_1, L_1$ and denote

$$\beta_1 = \inf_{x \in P_1} \lim_{n \rightarrow \infty} \frac{1}{n} \sup_{i \in \mathbb{Z}} |\{j : x_{i+j} = 1, 0 \leq j < n\}|.$$ 

Observe that $\beta_1 \leq 1/L_1 < \alpha_1$ hence $D_1 \cap P_1 = \emptyset$. 


Next suppose that pairwise disjoint minimal sets $D_i, P_i$ have been constructed for $i = 1, \ldots, n$, together with constants $\alpha_i, \beta_i, N_i, M_i$, etc. Let $t_n = \sum_{i=1}^{n}(N_i + M_i)$. Enumerate words $L_{t_n}(\bigcup_{i=1}^{n} D_i \cup P_i) = \{u_1, \ldots, u_{s_n}\}$ and let
\[ A_{n+1} = u_1 \theta \theta^{n} u_2 \theta \theta^{n} \cdots u_{s_n} \theta \theta^{n}, \quad B_{n+1} = 0 \setminus A_{n+1}. \]
Take $K_{n+1} > 9^{n+1}$ such that
\[ 1/K_{n+1} \leq \min\{\alpha_1/2, \beta_1/2, \ldots, \alpha_n/2, \beta_n/2\} \]
and let $D_{n+1}$ be a doubly minimal weakly mixing system provided by Theorem 4.3 together with a constant $N_{n+1}$ for $A_{n+1}, B_{n+1}, K_{n+1}$. Let
\[ \alpha_{n+1} = \inf_{x \in D_{n+1}} \lim_{n \to \infty} \frac{1}{n} \sup_{i \in \mathbb{Z}} \{|j : x_{i+j} = 1, 0 \leq j < n\}|. \]
Take $L_{n+1} \geq 9^{n+1}$ such that $1/L_{n+1} < \alpha_{n+1}/2$ and let $P_{n+1}$ be a weakly mixing minimal system with uniform positive topological entropy provided by Theorem 5.1 together with a constant $M_{n+1}$ for $A_{n+1}, B_{n+1}, L_{n+1}$ and denote
\[ \beta_{n+1} = \inf_{x \in P_{n+1}} \lim_{n \to \infty} \frac{1}{n} \sup_{i \in \mathbb{Z}} \{|j : x_{i+j} = 1, 0 \leq j < n\}|. \]
Each dynamical system $(D_i, \sigma)$ is (doubly) minimal with zero topological entropy and each $(P_j, \sigma)$ is minimal with positive entropy, hence $P_i \cap D_j = \emptyset$. Furthermore, by the construction $\beta_{n+1} \leq \frac{1}{L_{n+1}} < \alpha_{n+1} \leq \frac{1}{K_{n+1}} < \beta_n$, thus also $P_i \cap P_j = \emptyset$, $D_i \cap D_j = \emptyset$ for any $1 \leq i < j \leq n+1$. In other words all constructed minimal systems are pairwise disjoint and nontrivial. As the last element we put $D_0 = \{0^{\infty}\}$, i.e. singleton consisting of constant sequence $x_i = 0$ for all $i \in \mathbb{Z}$. Finally denote $X = \bigcup_{n=1}^{\infty}(D_n \cup P_n)$ and observe that $X$ is a subshift, since it is closed and $\sigma$-invariant.

We claim that there is no minimal subset $Z \subset X$ other than $D_n$ or $P_n$ for some $n$. Assume on the contrary that such $Z$ exists. Since $Z \neq D_0$, for any $z \in Z$ the set \( \{i : z_i = 1\} \) is syndetic. Pick $z \in Z$ with $z_0 = 1$, for each $k > 0$ take $j_k$ such that $z_{j_k} = 1$ and $|\{i : z_i = 1, 0 \leq i \leq j_k\}| = k + 1$. Denote $v_k = z_{[0,j_k]}$ and observe that there is $\alpha > 0$ such that
\[ 0 < \alpha < \liminf_{n \to \infty} \frac{1}{n} \{|i : z_i = 1, 0 \leq i < n\}| = \liminf_{k \to \infty} \frac{k}{j_k}. \]
Since $Z$ is distinct from any of the sets $P_n, D_n$, for each $n \geq 1$ there exists $K_n > 0$ such that $v_k \notin \mathcal{L}(\bigcup_{i=1}^{n} D_i \cup P_i)$ for any $k > K_n$.

Fix any $m > 1$, take any $k > K_m$ and let $j > m$ be the minimal integer such that $v_k \in \mathcal{L}(D_j \cup P_j)$. Since $v_k$ starts and ends by symbol 1 and does not belong to $\mathcal{L}(\bigcup_{i=1}^{m} D_i \cup P_i)$, it contains at least one occurrence of the word $0^j \theta^i$. By the structure of $A_j, B_j$ provided by Theorem 4.3 and Theorem 5.1 we see that there is $s \geq 1$ such that
\[ \frac{k+1}{j_k+1} = \frac{1}{|v_k|} |\{i : (v_k)_i = 1\}| \leq s + \frac{2}{j_k \theta^i} \sup_{w \in \mathcal{L}_j(\bigcup_{i=1}^{m} D_i \cup P_i)} |\{i : w_i = 1\}| \leq \frac{3}{j_k} \max\{\alpha_1, \beta_1, \ldots, \alpha_{j-1}, \beta_{j-1}\} \leq \frac{1}{9^m}. \]
Since we can take $m$ arbitrarily large (provided that $k$ is sufficiently large as well), we see that
\[ 0 < \alpha \leq \liminf_{k \to \infty} \frac{k+1}{j_k+1} \leq \lim_{m \to \infty} 9^{-m} = 0. \]
which is a contradiction. Indeed, the claim holds. We proved so far that (1), (3) and (4) are satisfied. Now fix any $P_m$ and any $w \in \mathcal{L}(P_m)$. Then $w$ is a subword of $A_n$ for some sufficiently large $n > 0$ and so $w \in \mathcal{L}(\bigcup_{i=0}^{\infty} D_i)$. This shows that $P_m \subset \bigcup_{i=0}^{\infty} D_i$ and by symmetric argument $D_m \subset \bigcup_{i=0}^{\infty} P_i$. This proves (2).

Finally, fix any words $u, v \in \mathcal{L}(X)$. By the definition, there exists $n$ such that both $u, v$ are subwords of $A_n$. But $D_n$ is weakly mixing, so there exist words $w, w'$ with $|w| = |w'| + 1$ such that $A_n w A_n, A_n w' A_n \in \mathcal{L}(D_n) \subset \mathcal{L}(X)$. Indeed $(X, \sigma)$ is weakly mixing which completes the proof. \hfill \Box

7. Proof of Theorem 1.5. Dynamical system necessary for proving Theorem 1.5 is already provided by Theorem 6.1. The above statement is a consequence of the following result.

**Theorem 7.1.** Let $(X, T)$ be a weakly mixing dynamical system, such that for every nonempty open set $U$ there are minimal sets $M, M' \subset X$ intersecting $U$ such that:

1. $(M, T)$ is doubly minimal with zero entropy;
2. $(M', T)$ is diagonal.

Then $(X, T)$ is disjoint with any minimal system.

**Proof.** Fix any minimal system $(Y, S)$. For every open set $U$ there are minimal sets $M, M'$ like in the assumptions and points $y_U \in M \cap U$, $y'_U \in M' \cap U$. Let $U$ be a countable base of the topology of $X$ and let $D = \{y_U, y'_U : U \in U\}$.

Fix any $p \in Y$ and any nonempty open sets $U \subset X$, $V \subset Y$, $p \in V$. Fix any doubly minimal $M$ such that there exists $y \in U \cap M \cap D$. If $(M, T)$, $(Y, S)$ are disjoint, then $(y, p)$ is minimal for $T \times S$ and so $N_{T \times S}((y, p), U \times V)$ is syndetic.

Next, let us assume that the other possibility takes place, that is $(M, T)$ and $(Y, S)$ are not disjoint. Then, by Lemma 4.2 there exists a factor map $\pi : (Y, S) \to (M, T)$. Denote $z = \pi(p) \in M$. There are $k > 0$ and $p' \in Y$ such that $S^k(p') = p$ and $\pi(p') \in U$. Denote $z' = \pi(p')$ and observe that $T^k(z') = z$. Fix an $\varepsilon > 0$ and an open set $Q \ni z'$ such that $B_\varepsilon(Q) \subset U$. There exists an open neighborhood $W$ of $p'$, $W \subset Y$ such that $S^k(W) \subset V$ and therefore $N(p', W) \subset N(p, V)$. Assume additionally, decreasing $W$ when necessary, that $N(W) \subset Q$. Note that $N(p', W) \subset N(z', Q)$ and $N(p', W)$ is syndetic. Let $r$ be such that $N(p', W) \cap [i, i + r] \neq \emptyset$ for every $i \geq 0$. Finally, let $R \subset Q$ be an open set such that $z' \in R$ and if $u, v \in R$ then $d(T^i(u), T^i(v)) < \varepsilon$ for $i = 0, \ldots, r$. Let $(M', T)$ be a diagonal minimal system such that there is $y' \in M' \cap R \cap D$. By disjointness of diagonal systems with minimal zero entropy systems [2, Proposition 6], $(y', z')$ is a minimal point for $T \times T$ on $M' \times M$, in particular the set of return times $A = N_{T \times T}((y', z'), R \times R)$ is syndetic. Fix any $j \in A$ and observe that by the definition of $r$ there is $s \in [0, r]$ such that $S^{i+s}(p') \in W$. For this particular integer $s$ we have $T^{j+s}(z') \in Q$ and $S^{i+s}(p) \in V$. But by the fact that $T^j(z'), T^j(y') \in R$ we obtain that

$$d(T^{i+s}(z'), T^{i+s}(y')) < \varepsilon$$

which gives $T^{j+s}(y') \in B_\varepsilon(Q) \subset U$. We obtained that $N_{T \times S}((y', p), U \times V)$ is syndetic.

The proof is completed by direct application of Theorem 1.3. \hfill \Box

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