Stochastic Saddle Point Problems with Decision-Dependent Distributions

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Abstract

This paper focuses on stochastic saddle point problems with decision-dependent distributions in both the static and time-varying settings. These are problems whose objective is the expected value of a stochastic payoff function, where random variables are drawn from a distribution induced by a distributional map. For general distributional maps, the problem of finding saddle points is in general computationally burdensome, even if the distribution is known. To enable a tractable solution approach, we introduce the notion of equilibrium points – which are saddle points for the stationary stochastic minimax problem that they induce – and provide conditions for their existence and uniqueness. We demonstrate that the distance between the two classes of solutions is bounded provided that the objective has a strongly-convex-strongly-concave payoff and Lipschitz continuous distributional map. We develop deterministic and stochastic primal-dual algorithms and demonstrate their convergence to the equilibrium point. In particular, by modeling errors emerging from a stochastic gradient estimator as sub-Weibull random variables, we provide error bounds in expectation and in high probability that hold for each iteration; moreover, we show convergence to a neighborhood in expectation and almost surely. Finally, we investigate a condition on the distributional map—which we call opposing mixture dominance—that ensures the objective is strongly-convex-strongly-concave. Under this assumption, we show that primal-dual algorithms converge to the saddle points in a similar fashion.

1 Introduction

The broad goal of stochastic optimization is to find an optimal decision for an objective with uncertainty in some parameters [39, 49, 57]. Uncertainty may be due to several factors, depending on the particular application domain and problem formulations. For example, in statistical learning, parameters may be taken to be data-label pairs in large data-sets [15, 23]; in the context of optimization of physical and dynamical systems, they may model externalities and random exogenous inputs, or system parameters that are predicted from data and are accompanied by given error statistics [9]. A key assumption that is typically leveraged for providing theoretical guarantees for stochastic optimization algorithms is that the distributions of random parameters are stationary [10]. However, in contemporary machine learning and cyber-physical systems applications, data may be subject to temporal shift, where the distribution may slowly change in time, or decision-dependent shift, whereby the distribution is inextricably tied to the decision variables. This is the case in several applications that span finance, energy systems, transportation networks, and healthcare just to mention a few. For example, in the energy systems context, the problem of finding an optimal charging policy for a fleet of electric vehicles involves uncertainty in the price of energy; indeed, real-time prices are subject to change due to varying demand (which is the decision variable), as well on external factors such as spot market behavior [21].

In this work, we are interested in solving a stochastic saddle point problem where: (i) the distribution of random parameters is decision dependent; (ii) the distribution of the parameters may change in time; and, (iii) the minimax function may be time-varying as it may include some deterministic parameters that shift in time. Accordingly, combining these features yields the problem:

$$\min_{x \in X} \max_{y \in Y} \Phi_t(x, y) := \mathbb{E}_{w \sim D_t(x,y)} [\phi_t(x,y,w)].$$

(1)
where \( t \in \mathbb{N} \) is a time index, \( \mathcal{X}_t \subset \mathbb{R}^n \) and \( \mathcal{Y}_t \subset \mathbb{R}^m \) are compact constraint sets, \( \phi_t : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R} \) is a scalar-valued function of the decision variables \((x, y)\) and it is parameterized a random vector \( w \), \( D_t \) is a distribution inducing map, and \( w \) is supported on a complete and separable metric space \( M \) with metric \( d \) (technical assumptions will be outlined shortly). We remark that the distribution of \( w \) depends on the decision variables \((x, y)\). When solutions to the problem \([\textbf{1}]\) exist, we will denote these solutions as \((x_t^\star, y_t^\star)\).

Problem \([\textbf{1}]\) is hereafter referred to as time-varying minimax with decision-dependent distributions (or simply time-varying problem) to emphasize the three features (i)-(ii) listed above. Furthermore, we refer to \( \Phi_t \) as the objective and the function \( \phi_t \) as the minimax function. One of our main efforts in this paper is in developing first-order algorithms to identify saddle points \((x_t^\star, y_t^\star)\), in cases where the distributional map \( D_t \) and the minimax function \( \phi_t(x, y, w) \) may change at each iteration of the algorithm and information arrives sequentially (i.e., no predictions are available). As a sub-case, our framework includes the setup where both the distribution and the minimax function do not change during the steps of an iterative algorithm, although the distribution of the random parameters is decision dependent; this setup will be referred to as static minimax with decision-dependent distributions.

Another main effort in this paper is in developing deterministic and stochastic first-order algorithms with varying degrees of explicit access to the distributional map \( D_t \). We are concerned with two primary settings: (1) the map \( D_t \) is known; and, (2) the map \( D_t \) is unknown and we receive samples from the distribution that \( D_t \) induces after querying with a decision. The former is prevalent in the literature where \( D_t \) is taken to be a qualitative model of decision inducing performative effects. By contrast, the latter requires access to a sampling mechanism by which data in given in response to a queried points.

For general distributional maps \( D_t \), the objective \( \Phi_t \) in \([\textbf{1}]\) may be non-convex-non-concave. Moreover, since the distribution of \( w \) depends on the decision variables \((x, y)\), the problem of finding \((x_t^\star, y_t^\star)\) is computationally burdensome even when the distributional map \( D_t \) is in known. For these reasons, we introduce an equilibrium problem associated with \([\textbf{1}]\) for which solutions will be the saddle points of the stationary problem that they induce. These can be seen as the counterparts of the so-called performatively stable points in \([13,42,56]\) in our stochastic minimax setup \([\textbf{1}]\). We first introduce the equilibrium problem in the static (or time invariant) setting. We then provide conditions for the existence of such equilibrium points. In particular, existence and compactness of the set equilibrium points is shown when the minimax function is convex in \( x \) and concave in \( y \) for a given \( w \), and under continuity of the distributional map. Building on these results, and focusing on strongly-convex-strongly-concave functions \( \phi_t \), we then develop deterministic and stochastic projected primal-dual algorithms that can determine equilibrium points. First we do so in the static setting, and later in the online setting—in contexts where the minimax function and constraint set are revealed sequentially.

Additionally, in the spirit of \([33]\), we investigate a stochastic dominance condition, which we call opposing mixture dominance, on the distributional map that allows us to guarantee strong-convexity-strong-concavity of the function \( \Phi_t \). In our minimax setup, the opposing mixture dominance condition amounts to the distributional map being convex-concave in the decision variables. We then develop deterministic and stochastic primal-dual algorithms that determine the saddle points.

Before describing prior works in context and outlining our contributions, we introduce some notation and present three examples for the problem \([\textbf{1}]\).

**Notation.** We let \( \mathbb{R} \) denote the set of real numbers. For given column vectors \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^n \), we let \((x, y)\) denote their concatenation; that is, \([x^T, y^T]^T\), with \( ^T \) denoting transposition; for \( x, y \in \mathbb{R}^n \), we let \((x, y) \) denote the inner product. Given the functions \( \{f_i : \mathbb{R}^n \to \mathbb{R}\}_{i=1}^N \), \((f_1, \ldots, f_N)\) denotes a vector valued function from \( \mathbb{R}^n \) to \( \mathbb{R}^N \). For a given column vector \( x \in \mathbb{R}^n \), \( \|x\| \) is the Euclidean norm; for a matrix \( X \in \mathbb{R}^{n \times m} \), \( \|X\|_F \) denotes the Frobenious norm and \( \|X\|_* \) the nuclear norm.

Given a differentiable function \( f : \mathbb{R}^n \to \mathbb{R} \), \( \nabla f(x) \) denotes the gradient of \( f \) at \( x \) (taken to be a column vector). For a function \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \), \( \nabla_x f(x, x) \) denotes the partial derivatives of \( f \) with respect to \( x \). Given a closed convex set \( C \subset \mathbb{R}^n \), \( \Pi_C : \mathbb{R}^n \to \mathbb{R}^n \) denotes the Euclidean projection of \( y \) onto \( C \), namely \( \Pi_C(y) := \arg \min_{v \in C} \|y - v\| \). Given a set \( C \subset \mathbb{R}^n \), \( P(C) \) denotes its power set.

For a given random variable \( w \in \mathbb{R} \), we write \( w \sim \mu \) to mean that \( w \) is a random variable with law \( \mu \), a probability measure supported on \( \mathbb{R} \). Hence \( \mu(A) = \mathbb{P}(w \in A) \) for all \( A \subset \mathbb{R} \). Furthermore, \( \mathbb{E}[w] \) denotes the expected value of \( w \), and \( \mathbb{P}(w \leq c) \) denotes the probability of \( w \) taking values smaller than or equal to \( c \): \( \|w\|_p := \mathbb{E}[|w|^p]^{1/p} \), for any \( p \geq 1 \). Finally, \( e \) denotes Euler’s number.
1.1 Motivating Examples

1.1.1 Noncooperative Minimax Game

As a first example, we consider a class of noncooperative games [20,31] where each player minimizes a stochastic minimax function (we drop the time index $t$ again for notational convenience). Consider a setting with $N$ players, and let $x_i \in \mathbb{R}^{n_i}$ be the decision variable of the $i$th player (with $n = \sum_{i=1}^{N} n_i$); furthermore, let $x_{-i}$ denote the vectors of variables $\{x_j, j \neq i\}$, and assume that $\mathcal{X}$ can be written as $\mathcal{X} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_N$. Provided $x_{-i}$, each player solves the following optimization problem

$$\min_{x_i \in \mathcal{X}_i} f_i(x_i, x_{-i}) := J_i(x_i, x_{-i}) + \max_{y \in \mathcal{Y}} \mathbb{E}_{w \sim D_i(x,y)} [G(x, y, w)]$$

(2)

where $J_i(x_i, \cdot)$ is a continuous convex function and $G_i(x, y, w)$ is a continuous function that is convex in $x$ and concave in $y$ for each given realization of $w$. The tuple $\mathcal{K} := \{\{1, \ldots, N\}, \{f_i\}, \mathcal{X}, \mathcal{Y}\}$ including the set of players, the set of strategies, functions, and the set of feasible $y$ defines a noncooperative minimax game. Defining a gradient map associated with (2) and adopting a variational inequality representation [20], the deterministic and stochastic primal-dual methods presented in this paper can be utilized to identify Nash equilibria of the game $\mathcal{K}$; see also the primal-dual method used in [41] in the context of constrained aggregative games.

As an example of application of the game $\mathcal{K}$, consider an electric vehicle (EV) charging problem similar to, e.g., [21]. Assume that the $i$th player is an operator of a fleet of EVs (or example, for car sharing services), and $x_i$ represents the charging rate over a temporal horizon of $n_i = \tau \in \mathbb{N}$ times slots (e.g., intervals of 5 minutes) for the EVs. Moreover, $\theta$ represents here the power demand from EVs owned by third parties or independent drivers. The charging price over $\tau$ slots can be decomposed as $p(x) = p_0(x) + w$, where $p_0(x) \in \mathbb{R}^{\tau}$ is a nominal price (that depends on the overall demand) and $w \in \mathbb{R}^{\tau}$ represents an uncertain component of the price. In particular, the uncertain component of the price may be influenced by externalities (for example, on the energy spot market) and by the total demand $\sum_i x_i + y$. Accordingly, we can model each entry $w_i$ of the vector $w$ as a random variable whose distribution depends on both the fleets’ charging demand $\{x_i\}$ and the charging demand of third-party EVs $y$. For example, one can model $w_i$ as $w_i \sim \mathcal{N}(\mu_i(x_i + y_i), \sigma_i^2)$, for some $\mu_i, \sigma_i > 0$. In this example, one can set the function in (2) to $J_i(x_i, x_{-i}) = x_i^T p_0(x)$ and $G(x, y, w) = \sum_{i=1}^{n_i} x_i^T z + y^T z$. The overall problem represents here a robust EV charging approach where the EV fleet operators see to minimize the charging cost under decision-dependent uncertainty on the prices and in a worst-case setup induced by third-party EVs.

1.1.2 Multitask Learning

The second example is a supervised multitask learning problem [60]; in this sub-section, we consider a static problem and, thus, we drop the subscript $t$. Supervised multitask learning is the problem of learning a collection of $m$ parameters or classifiers simultaneously, while imposing constraints to promote a commonality in these learning tasks. As an example, consider the school data regression problem, which is concerned with predicting the exam scores of 15,362 students from 139 secondary schools in London between 1985-1987 [23,25]. Each school has a data set $\mathcal{D}_i = \{(a^j_i, b^j_i)\}_{j=1}^{n_i}$ where features $\{a^j_i\}$ correspond to individual students and include information about the year, school, and students themselves, while labels $\{b^j_i\}$ are the corresponding exam scores. The goal is to learn a collection of $m$ functions $\{f_i\}_{i=1}^m$ such that $f_i(a^j_i) \approx b^j_i$ for all $i$ and $j$.

Existing modeling approaches for this problem include feature learning [2] and relation learning [61]. The feature learning problem is typically non-convex; however, under some assumptions, this problem is equivalent to the convex minimization problem

$$\min_{X \in \mathbb{R}^{d \times m}} \sum_{i=1}^{m} \sum_{j=1}^{n_i} \ell(b^j_i, \langle x^j, a^j_i \rangle) + \mu \|X\|_*$$

(3)

where $X = [x_1, \ldots, x_m] \in \mathbb{R}^{d \times m}$, $\ell : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the least squares loss function defined by $\ell(s, r) = \frac{1}{2}(s - r)^2$, and $\|\cdot\|_* : \mathbb{R}^{d \times m} \to \mathbb{R}$ is the nuclear norm defined by $\|A\|_* = \sum_{k=1}^{\min\{d, m\}} \sigma_k(A)$ for all $A \in \mathbb{R}^{d \times m}$ [23,13] (and where $\sigma_k(A)$ denotes the $k$th singular value of $A$). The term $\mu \|X\|_*$ encourages the matrix $X$ to be low-rank.
A challenge in solving (3) is that the nuclear norm is not differentiable. To avoid using proximal methods, we can introduce dual variables by observing that the nuclear norm is the dual of the operator norm and hence

$$\|X\|_* = \max_{Y: Y \preceq X} -\langle X, Y \rangle_F$$

where $\langle \cdot, \cdot \rangle_F : \mathbb{R}^{d \times m} \times \mathbb{R}^{d \times m} \to \mathbb{R}$ is the Frobenius inner product defined by $\langle X, Y \rangle_F = \text{trace}(X^TY)$ [43].

To introduce decision-dependence, we assume that after observing the regression model, each school solves a best response problem to modify their features so as to increase their exam scores. In this way, the exam score prediction model will trigger each school to take actions that will influence the outcome they are attempting to predict; this is precisely the notion of performative prediction [34, 42]. The goal in this decision-dependent problem is to learn a regression function for each school that performs well even after the modification of features is made. This feature modification takes the form:

$$a_i \in \arg \min_{a'} [u_i(a'_i, x) - c_i(a'_i, a^0_i)]$$

for all $i = 1 : m$, where $a^0_i$ is the feature originally provided to the regressor. Functions $u_i$ and $c_i$ are the utility and cost associated with modifying features for task $i$ respectively. If we assume each task has linear utility $u_i(a, x) = -\langle a, x \rangle$ and quadratic cost $c_i(a, a') = 1/2\|a - a'\|^2$, then the best response is given by $a_i = a^0_i + \varepsilon_i x_i$. This is equivalent to overall best response $A = A_0 + X \varepsilon$, where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m) \in \mathbb{R}^m$ and features matrices $A, A_0 \in \mathbb{R}^{d \times m}$, resulting from the cross-task response problem

$$A \in \arg \min_{A'} \left[ -\langle A', X \rangle_F + \frac{1}{2} \|A' - A_0\|_F^{-1}\|\varepsilon\|^2 \right]$$

where $E = \text{diag}(\varepsilon) \in \mathbb{R}^{d \times d}$. By including the dual variables as well as the decision-dependence of the features, the decision-dependent multi-task regression problem can be written as

$$\min_{X \in \mathbb{R}^{d \times m}} \max_{Y: Y \preceq X} \mathbb{E}_{W \sim D(X)} \left[ L(X, W) - \mu(X, Y) \right]$$

where the total loss is given by $L(X, W) = \sum_{i=1}^m \ell(b_i, \langle x^i, a_i \rangle)$. To ensure that our strong-concavity assumption is satisfied, we can regularize the objective over the dual variable. Since the matrix of weights cannot be arbitrarily far from the origin, imposing the artificial constraint that $X$ lie in some convex compact subset $X \subset \mathbb{R}^{d \times m}$ will not change the performance of the resulting model.

### 1.1.3 Constrained Stochastic Minimization Problems

We start by considering a constrained optimization problem of the form

$$\min_{x \in X, \ w \sim D_0(x)} \mathbb{E} [F_t(x, w)], \quad \text{s.t.: } g(x) \leq 0$$

where $X \subset \mathbb{R}^n$ is a convex and compact set, $g(x) = (g_1(x), \ldots, g_m(x))$ with each function $g_i(x) : \mathbb{R}^n \to \mathbb{R}$ convex, closed, and proper, and where the cost function is convex. In particular, each constraint function $g_i(x)$ can be either deterministic or of the form $g_i(x) = \mathbb{E}_{w \sim D(x)} [G_i(x, w)]$.

When the projection onto the convex set $X \cap \{x \in \mathbb{R}^n : g(x) \leq 0\}$ exists in closed form or is cheap to compute, projected gradient methods have been proposed for computing optimizers of [7, 13, 42, 56]. When the projection is computationally heavy, typical approaches for stationary problems include primal-dual methods and interior point methods [10]. Focusing on primal-dual methods, we can construct the Lagrangian function [13, 10]

$$L_t(x, y) = \mathbb{E}_{z \sim D_t(x)} [F_t(x, w)] + \langle y, g(x) \rangle$$

where $y \in \mathbb{R}^m$ is the vector of Lagrange multipliers associated with the constraints $g(x) \leq 0$. We can further formulate the associated dual problem

$$\max_{y \geq 0} \min_{x \in X} L_t(x, y).$$

4
Under Slater’s condition, strong duality holds dual optimal solution $y^*_t$ exists. The former implies that the primal problem in (4) and the dual problem in (5) have the same optimal value \[15\] Theorem 11.50. The primal and dual problems can be equivalently formulated as coupled variational inequalities as in \[29\]. Furthermore, the fact that $X_t$ is compact and $L_t$ is convex-concave is sufficient for the minimax equality to hold \[50\]. Thus,

$$\max_{y \geq 0} \min_{x \in X_t} L_t(x, y) = \min_{x \in X_t} \max_{y \geq 0} L_t(x, y).$$

To align with our formulation (4), we need to (artificially) constrain the dual variables $y$ to a compact convex subset $\mathcal{Y}_t$ of the non-negative orthant of $\mathbb{R}^m$ containing the optimal dual variables $y^*_t$. The compact set $\mathcal{Y}_t$ can be constructed based on a Slater’s point as described in, e.g., \[29\]. This leads to the problem

$$\max_{y \in \mathcal{Y}_t} \min_{x \in X_t} L_t(x, y)$$

which is shown to have the same set of solutions as (9) in \[29\], and is in line with (1).

In this paper, we primarily focus on the development of algorithms for saddle functions that are strongly-convex-strongly-concave. In this case, when the cost $\mathbb{E}_{z \sim D_t(x)} [F_t(x, w)]$ is strongly-convex in $x$ but only concave in $y$, we can utilize the regularization approach proposed in \[29\] and consider the modified problem:

$$\min_{x \in X_t} \max_{y \in \mathcal{Y}_t} L_t(x, y) - \frac{\beta}{2} \|y\|^2$$

where $\beta > 0$. In this case, the regularized Lagrangian $L_t(x, y) - \frac{\beta}{2} \|y\|^2$ is strongly-convex-strongly-concave; the deviation from the saddle-points of $L_t(x, y)$ induced by the Tikhonov regularization can be bounded as shown in \[29\]. When the cost $\mathbb{E}_{z \sim D_t(x)} [F_t(x, w)]$ is convex, an additional Tikhonov regularization term in $x$ can be added.

### 1.2 Related Works

In this subsection, we provide a sample of works in the context of saddle-point problems, stochastic optimization with decision-dependent distributions, and online optimization, and discuss how these work are related to this paper. The contributions of this paper are then outlined in Section 1.3.

**Saddle Point Problems.** Time-invariant saddle point problems have been well investigated, with theoretical guarantees in distance metrics (squared euclidean metric) and dual-gap metrics. Common approaches for these problems are either proximal methods or primal-dual methods, with version existing for both deterministic and stochastic problems. Proximal methods include Mirror-Prox \[38\] and Extragradient \[35\] for deterministic objectives and Stochastic Mirror-Prox \[39\] for stochastic objectives; primal-dual methods include Primal-Dual or Gradient Descent-Ascent \[29\] and Optimistic Gradient Descent-Ascent \[35\]. Stochastic Primal-Dual methods and their accelerated varieties are studied in \[38\]. Convergence guarantees for stochastic optimization algorithms typically come in the form of convergence with a diminishing step-size policy \[37–39\], or convergence to a neighborhood with fixed step-size \[29\].

**Decision-Dependent Distributions.** Our work is most closely related to the body of work on stochastic optimization with decision-dependent distributions and performative prediction. These are two related paradigms for stochastic programs where the random variables are parameterized by the decision variables. Deterministic and stochastic algorithms for solving the equilibrium problem in the batch setting are considered in \[18, 33, 42\]. In particular: the equilibrium problem is presented and solved via conceptual deterministic algorithms in \[42\]; second moment analysis for stochastic algorithms with access to a sampling oracle is provided in \[33\]; and proximal first-order algorithms for regularized objectives are studied in \[18\]. An extension of the decision dependent framework was proposed in \[13\] as a game between the institution and an adversary that chooses a distribution dependent on the last available classifier. In \[56\], first moment and high probability tracking analysis are provided under a sub-Weibull gradient error for a time-varying optimization problem.

Conditions for strong-convexity of the minimization objective are established in \[54\]. Additionally, they demonstrate convergence to the unique optimizer for location scale families using a two-stage algorithm that coarsely approximates the distributional map.

**Online Optimization.** The form of our objective is the mean of a sequence of stochastic payoff functions, where random variables are drawn from a distribution that also shifts in time. Hence, our work is...
related to online optimization as well as problems that incorporate temporal distributional shifts. Online optimization is a time-varying optimization paradigm where functions are revealed sequentially in time \( t \). The goal is typically to minimize the regret (for convex objectives) or the tracking error (for strongly-convex objectives) by tracking the changing sequence of optimizers. First-order methods for online minimization problems have been well investigated. We refer the reader to the representative references \([16, 32, 36, 44, 47]\), as well as pertinent references therein. First-order methods to solve online problems make some finite number of updates each time a new objective is revealed with either diminishing step-sizes for finite time-horizons or fixed step-sizes for infinite time-horizons \([6, 26]\).

While there is a rich literature on saddle point problems in the time-invariant setting, works on time-varying problems are lacking. Most works consider the saddle point problem arising from a dual-problem \([17, 59]\). These approaches typically opt for a tracking error metric. In the more general online saddle point problem, one seeks to find a sequence of strategy pairs that minimize a saddle point regret \([28, 46]\).

The problem of learning with temporal distributional shifts has been well studied within the machine learning literature \([3, 4, 7, 8, 14, 22, 31]\). A prevailing theme within these works is framing the learning problem as a time-varying optimization problem in which the objective is to track solutions as new information arrives. The adaptive sequential learning problem is similar, but makes the assumption that the data distribution changes slowly. In these problems, the payoff is assumed to either be fixed or slowly changing due to some sequence of deterministic parameters. In their analysis, these works are particularly interested in estimating the drift in the solution to establish a sampling strategy \([3, 53, 54]\).

Sub-Weibull Error Models. Sub-Gaussian and Sub-exponential gradient error models are common in the literature on stochastic gradient methods. Empirical and theoretical results have demonstrated that heavier tailed distributions arise naturally in deep learning. The class of sub-Weibull random variables subsumes the sub-Gaussian and sub-Exponential classes of distributions, while also including heavier tailed distributions \([30, 52, 55]\). It also includes random variables whose distribution has a bounded support. Given their broad application, this model has been receiving increasing use in the literature. Convergence rates of Lasso estimates using Sub-Weibull covariates and additive noise assumptions are provided in \([30]\). High probability bounds for the tracking error of time-varying minimization problems have been established using sub-Weibull distributions \([5, 56]\).

Connection to Related Works. Relative to the referenced work on stochastic saddle point problems, we consider the case where the function \( \phi(x, y, w) \) is strongly-convex-strongly-concave in the decisions \((x, y)\) and it is time-varying; additionally, we consider the case where the distribution of \( w \) is decision dependent. Without any additional assumptions on the distributional map, the decision-dependence makes the objective non-convex-non-concave. Hence, metrics involving the objective value are not an appropriate measure of performance. Contrary to most of the work on the stochastic saddle point problems that use the distance metric, we analyze using the first moment, rather than the second moment. In this way, we alleviate the need for stochastic filtrations to analyze dependent stochastic-processes. We also provide bounds in high probability that hold for each iteration.

Relative to the body of work on performative prediction and stochastic optimization with decision dependent distribution, we consider the saddle point problems in the time-varying setting. This is a unique contribution of this paper.

Finally, relative to the literature on online convex optimization and online saddle point problems, our objectives are stochastic and decision dependent and our analysis is done in the tracking error metric. Relative to the work on concept drift and adaptive learning, we allow the payoff \( \phi \) to vary in time.

1.3 Contributions

In this paper, we offer the following main contributions.

C1) We consider the stochastic minimax problem \([1]\) and propose a notion of equilibrium points for general decision-dependent distributional maps; we then provide conditions for their existence and uniqueness. The conditions apply to both the static minimax problem and the time-varying minimax problem.

C2) When the minimax function is strongly-convex-strongly-concave, and under additional mild assumptions on the distributional map, we provide bounds for the distance between the unique equilibrium point and the saddle-point of the problem \([1]\).

C3) We develop primal-dual type algorithms to solve the equilibrium problem in both the batch and the online settings. We prove linear convergence of the primal-dual algorithm when the minimax function is
strongly-convex-strongly-concave.

C4) When the distributional map is unknown, we propose a stochastic primal-dual method where the gradient is estimated from samples. We model the gradient errors with sub-Weibull random variables, and derive error bounds and neighborhood convergence. We provide bounds in expectation and in high probability that hold for each iteration; moreover, we show convergence to a neighborhood in expectation and almost surely. The convergence results are provided for both the static and online settings.

C5) We investigate conditions on the distributional map that imply strong-convexity-strong–concavity of the minimax objective, and provide deterministic and stochastic primal dual algorithms to determine the unique saddle point.

The paper is organized as follows. In Section 2 we define and solve the the equilibrium point problem in the static setting. Section 3 investigates the time-varying problem via online optimization methods. Section 4 re-visits the saddle point problem for special classes of distributions.

2 Minimax Problem with Decision-dependent Distributions

We first consider the static minimax with decision-dependent distributions, which is the time invariant modification of \( \text{1} \) given as follows:

\[
\begin{aligned}
\min_{x \in X} \max_{y \in Y} \Phi(x, y) &:= \mathbb{E}_{w \sim D(x, y)} [\phi(x, y, w)], \\
\phi &\in C(\mathbb{R}^n, \mathbb{R}),
\end{aligned}
\]  

where the sets \( X \subseteq \mathbb{R}^n \) and \( Y \subseteq \mathbb{R}^m \) are convex and compact. Let \( \mathcal{P}(M) \) be the set of Radon probability measures on a complete and separable metric space \( M \) with finite first moment, and observe that the objective function can be written in integral form as

\[
\Phi(x, y) = \int_M \phi(x, y, w) \mu(x, y) (dw)
\]  

where \( \mu(x, y) \in \mathcal{P}(X \times Y) \) is given as the output of the distributional map \( D \) for each \( (x, y) \in X \times Y \). In most applications, explicit knowledge of the distributional map \( D \) may not be available. Even if \( D \) is known, computing the integral in closed form is computationally intractable and approximating the integral is expensive in higher-dimensions. Furthermore, one may make assumptions of convexity and concavity on \( \phi \), but this does not guarantee that the same property hold for \( \Phi \). For these reasons, and in the spirit of \([18, 42, 56]\), in the following we turn the attention to a class of solutions that are saddle points for the stationary problem that they induce.

**Definition 1.** (Equilibrium Points) A pair \((\bar{x}, \bar{y}) \in X \times Y\) is an equilibrium point if:

\[
\begin{aligned}
\bar{x} &\in \arg\min_{x \in X} \left( \max_{y \in Y} \mathbb{E}_{w \sim D(x, y)} [\phi(x, y, w)] \right), \\
\bar{y} &\in \arg\max_{y \in Y} \left( \min_{x \in X} \mathbb{E}_{w \sim D(x, y)} [\phi(x, y, w)] \right).
\end{aligned}
\]

Our first objective in this work will be to provide conditions for the existence and uniqueness of these equilibrium points. Later, we develop first order algorithms and demonstrate their convergence to equilibrium points. Crucial to our analysis will be the properties of the “decoupled objective,” which is defined as

\[
\Phi(x, y; x', y') = \mathbb{E}_{w \sim D(x', y')} [\phi(x, y, w)]
\]

for \( x, x' \in X \) and \( y, y' \in Y \). Here, the distribution is fixed, for given points \((x', y')\). With these definitions, we consider a correspondence \( H : X \times Y \to P(X \times Y) \), defined by

\[
H(x, y) = \left( \arg\min_{x' \in X} \max_{y' \in Y} \Phi(x', y'; x, y), \ \arg\max_{y'' \in Y} \min_{x'' \in X} \Phi(x'', y''; x, y) \right)
\]

which maps pairs in the product space to its power set \( P(X \times Y) \). In light of Definition 1, the equilibrium points are fixed points of the map \( H \); that is,

\[(\bar{x}, \bar{y}) \in H(\bar{x}, \bar{y}).\]
For notional convenience, we will introduce the stacked vector in the product space \( z = (x, y) \in \mathcal{X} \times \mathcal{Y} \) (consequently, we can identify \( H(z) \) and \( \Phi(z'; z) \) with the above functions whenever convenient). In the following section, we provide sufficient conditions for the existence of equilibrium points by leveraging results from fixed point theory.

### 2.1 Existence of Equilibrium Points

Our goal is to demonstrate the existence and uniqueness of equilibrium points. First, we demonstrate the existence of equilibrium points by showing that the fixed point set of \( H \), defined as \( \text{Fix}(H) := \{ z \in \mathcal{X} \times \mathcal{Y} | z \in H(z) \} \), is nonempty. The crux of our proof is showing that, under appropriate assumptions, \( H \) is an upper hemicontinuous function. Next, we provide this definition as well as the notion of a topological neighborhood.

**Definition 2. (Neighborhood)** ([1] Sec. 17.2) If \( A \) is a topological space and \( x \in A \), then a neighborhood of \( x \) is a set \( V \subset A \) such that there exists an open set \( U \) with \( x \in U \subset V \). If the set \( V \) is open, then we say that \( V \) is an open neighborhood.

**Definition 3. (Upper Hemicontinuity)** ([1] Sec. 17.2) If \( A \) and \( B \) are two topological metric spaces, then a set valued function \( \varphi : A \mapsto P(B) \) is upper hemicontinuous (uhc) at \( x \in A \) provided that for every neighborhood \( U \) of \( \varphi(x) \subset B \), the upper inverse set \( \varphi^n(U) = \{ x : \varphi(x) \subset U \} \) is a neighborhood of \( x \). If \( \varphi \) is uhc at every \( x \in A \), then we say that \( \varphi \) is uhc on \( A \).

We next state our result for the existence of equilibrium points.

**Theorem 2.1. (Existence of Equilibrium Points)** Suppose that the following assumptions hold:

1. \( x \mapsto \phi(x, y, w) \) is convex in \( x \) for all \( y \in \mathcal{Y} \) and for all realizations of \( w \);
2. \( y \mapsto \phi(x, y, w) \) is concave in \( y \) for all \( x \in \mathcal{X} \) and for all realizations of \( w \);
3. \( \phi \) is continuous on \( \mathcal{X} \times \mathcal{Y} \) for all \( w \);
4. \( \mathcal{X} \subset \mathbb{R}^d, \mathcal{Y} \subset \mathbb{R}^n \) are convex compact subset;
5. the distributional map \( D : \mathcal{X} \times \mathcal{Y} \to P(M) \) is continuous.

Then the fixed point set \( \text{Fix}(H) \) is nonempty and compact.

**Proof.** The proof amounts to showing that \( H \) satisfies the hypotheses of Kakutani’s Fixed Point Theorem ([1] Corollary 17.55) for correspondences (set-valued functions). Since the domain \( \mathcal{X} \times \mathcal{Y} \) is convex and compact by hypothesis, we show that \( H \) has a closed graph and non-empty convex and compact set values in \( P(\mathcal{X} \times \mathcal{Y}) \). Following the Closed Graph Theorem ([1] Theorem 17.11), compactness of \( \mathcal{X} \times \mathcal{Y} \) implies that \( H \) has closed graph if and only if it is closed valued and upper hemicontinuous. Hence our proof reduces to showing that (i) \( H \) has non-empty closed values, (ii) \( H \) is upper hemicontinuous, and (iii) \( H \) has convex values.

Define the intermediate functions

\[
f(x'; z) = \max_{y' \in \mathcal{Y}} \Phi(x', y'; z) \quad \text{and} \quad g(y'; z) = \min_{x \in \mathcal{X}} \Phi(x', y'; z)
\]

as well as the realization functions

\[
F(z) = \arg \min_{x \in \mathcal{X}} f(x'; z) \quad \text{and} \quad G(z) = \arg \max_{y' \in \mathcal{Y}} g(y'; z).
\]

for all \( x' \in \mathcal{X}, y' \in \mathcal{Y}, \) and \( z \in \mathcal{X} \times \mathcal{Y} \). Using this convention, \( H \) can be written compactly as \( H(z) = (F(z), G(z)) \). It follows from continuity of \( \phi \) and \( D \) on \( \mathcal{X} \times \mathcal{Y} \), as well as compactness of \( \mathcal{X} \) and \( \mathcal{Y} \) that \( f \) and \( g \) are continuous ([1] Theorem 17.31). The Maximum Theorem applied to \( F \) and \( G \) implies that \( F \) and \( G \) are upper hemicontinuous and have nonempty compact set values. Here, compactness implies closed-ness. Thus the values of \( H \) are closed since the Cartesian product of closed sets is closed. This proves (i).

To see that \( H \) is upper hemicontinuous, fix \( z \in \mathcal{X} \times \mathcal{Y} \) and let \( U \) be an open set such that \( H(z) \subset U \). Then \( H \) will be upper hemicontinuous provided that we can show that there exists an open neighborhood \( W \) of \( z \) such that \( H(W) \subset U \). Given that \( H(z) \) is a compact subset of \( U \), ([1] Theorem 2.62) guarantees the existence of open sets \( V_x \subset \mathcal{X} \) and \( V_y \subset \mathcal{Y} \) such that \( H(z) \subset V_x \times V_y \subset U \). Since \( F \) and \( G \) are upper hemicontinuous, then the upper inverse sets \( F^u(V_x) = \{ z : F(z) \subset V_x \} \) and \( G^u(V_y) = \{ z : G(z) \subset V_y \} \).
are open in $\mathcal{X} \times \mathcal{Y}$. Let $W = F^u(V_x) \cap G^u(V_y)$. Then $z \in W$ by construction, so $W, H(W) \neq \emptyset$. Furthermore, $W$ is an open neighborhood of $z$ and $H(W) \subset V_x \times V_y \subset U$. Thus condition (ii) holds.

Observe that since $x' \mapsto f(x'; z)$ is convex for all $z$ and $\mathcal{X}$ is convex, then $F(z)$ is convex for all $z \in \mathcal{X} \times \mathcal{Y}$. Similarly, $G(z)$ is convex for all $z$. Since the Cartesian product of convex sets is convex, then condition (iii) follows.

In the previous results, the distributional map $D$ depends on both $x$ and $y$. However, as shown in some of our motivating examples, the distributional map $D$ may not always depend on both $x$ and $y$ simultaneously. In the following corollary, we show that equilibrium points exist for the sub-class of problems where $D$ depends only on $x$.

**Corollary 2.2.** Suppose that the assumptions for Theorem 2.1 hold, but the distributional map given by $D : \mathcal{X} \to \mathcal{P}(M)$ is continuous and a function of $x$ alone. Then, there exist points $(\tilde{x}, \tilde{y}) \in \mathcal{X} \times \mathcal{Y}$ such that

\[
\tilde{x} \in \arg\min \max_{x \in \mathcal{X}} \mathbb{E}_{y \sim D(x)} [\phi(x, y, w)],
\]

\[
\tilde{y} \in \arg\max \min_{y \in \mathcal{Y}} \mathbb{E}_{x \sim D(y)} [\phi(x, y, w)].
\]

**Proof.** We demonstrate that this is special case for which the map $H$ defined in (17) is constant in $y$. Define the intermediate functions $f : \mathcal{X} \times (\mathcal{X} \times \mathcal{Y}) \to \mathbb{R}$ and $g : \mathcal{Y} \times (\mathcal{X} \times \mathcal{Y}) \to \mathbb{R}$ by

\[
f(x'; x, y) = \max_{y' \in \mathcal{Y}} \mathbb{E}_{w \sim D(x')} [\phi(x', y', w)], \quad \text{and} \quad g(y'; x, y) = \min_{x \in \mathcal{X}} \mathbb{E}_{w \sim D(x')} [\phi(x', y', w)].
\]

Observe that $\phi$ is continuous over $(\mathcal{X} \times \mathcal{Y}) \times \mathcal{M}$, and $D$ is continuous over $\mathcal{X}$; so, by Berge’s Maximum Theorem, $f$ and $g$ are continuous over $\mathcal{X} \times \mathcal{X}$ and $\mathcal{Y} \times \mathcal{X}$, respectively, with $y \in \mathcal{Y}$ fixed. Since the maps $y \mapsto f(x', x, y)$ for any $x, x' \in \mathcal{X}$ and $y \mapsto g(y', x, y)$ for any $y' \in \mathcal{X}$, $x \in \mathcal{X}$ are constant, both functions are continuous over their respective domain. Thus, the maps $F$ and $G$ are upper-hemicontinuous by Berge’s Maximum Theorem; the remainder of the proof proceeds exactly as in the Theorem 2.1.

In the following, we show that the equilibrium points are in fact saddle points for the stationary problem that they induce.

**Proposition 2.3. (Saddle point and Equilibrium Equivalence)** Suppose that the following assumptions hold:

i) $x \mapsto \phi(x, y, w)$ is convex in $x$ for all $y \in \mathcal{Y}$ and for all realizations of $w$;

ii) $y \mapsto \phi(x, y, w)$ is concave in $y$ for all $x \in \mathcal{X}$ and for all realizations of $w$;

iii) $\phi$ is continuous on $\mathcal{X} \times \mathcal{Y}$ for all $w$; and,

iv) $\mathcal{X} \subset \mathbb{R}^d, \mathcal{Y} \subset \mathbb{R}^n$ are convex compact subset.

Then, $(\tilde{x}, \tilde{y})$ is an equilibrium point if and only if

\[
\Phi(\tilde{x}, \tilde{y}; \tilde{x}, \tilde{y}) \leq \Phi(\tilde{x}, \tilde{y}; \tilde{x}, \tilde{y}) \leq \Phi(\tilde{x}, \tilde{y}; \tilde{x}, \tilde{y})
\]

for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$.

**Proof.** Based on the assumptions i)-iv), the equilibrium points exist and $(x, y) \mapsto \Phi(x, y; \tilde{x}, \tilde{y})$ satisfies the hypothesis of the Minimax Theorem. Thus, $\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} \Phi(x, y; \tilde{x}, \tilde{y}) = \max_{y \in \mathcal{Y}} \min_{x \in \mathcal{X}} \Phi(x, y; \tilde{x}, \tilde{y})$. From the definition of equilibrium points, we get that

\[
\max_{y \in \mathcal{Y}} \min_{x \in \mathcal{X}} \Phi(x, y; \tilde{x}, \tilde{y}) = \min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} \Phi(x, y; \tilde{x}, \tilde{y}) \leq \Phi(\tilde{x}, \tilde{y}; \tilde{x}, \tilde{y})
\]

and

\[
\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} \Phi(x, y; \tilde{x}, \tilde{y}) = \max_{y \in \mathcal{Y}} \min_{x \in \mathcal{X}} \Phi(x, y; \tilde{x}, \tilde{y}) \geq \Phi(\tilde{x}, \tilde{y}; \tilde{x}, \tilde{y}).
\]

Combining these relations with the Minimax Theorem yields

\[
\Phi(\tilde{x}, \tilde{y}; \tilde{x}, \tilde{y}) \leq \max_{y \in \mathcal{Y}} \Phi(\tilde{x}, \tilde{y}; \tilde{x}, \tilde{y}) = \min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} \Phi(x, y; \tilde{x}, \tilde{y}) = \max_{y \in \mathcal{Y}} \min_{x \in \mathcal{X}} \Phi(x, y; \tilde{x}, \tilde{y}) \leq \Phi(\tilde{x}, \tilde{y}; \tilde{x}, \tilde{y}),
\]
and
\[ \Phi(x, y; \bar{x}, \bar{y}) \geq \min_{x \in X} \Phi(x, y; \bar{x}, \bar{y}) = \min_{x \in X} \min_{y \in Y} \Phi(x, y; \bar{x}, \bar{y}) = \min_{y \in Y} \max_{x \in X} \Phi(x, y; \bar{x}, \bar{y}) = \Phi(\bar{x}, \bar{y}; \bar{x}, \bar{y}). \]

Conversely, if \((\bar{x}, \bar{y})\) satisfy the saddle inequality in (21), notice that
\[ \min_{x \in X} \max_{y \in Y} \Phi(x, y; \bar{x}, \bar{y}) \leq \max_{y \in Y} \Phi(\bar{x}, y; \bar{x}, \bar{y}) \leq \Phi(\bar{x}, \bar{y}; \bar{x}, \bar{y}) \leq \min_{y \in Y} \max_{x \in X} \Phi(x, y; \bar{x}, \bar{y}). \]
Hence, (21) and the minimax equality imply that \((\bar{x}, \bar{y})\) satisfy Definition 1.

We will leverage the results of this section in the analysis of first-order methods that will be utilized to solve the stochastic minmax problem. In the following, we outline some working assumptions used in the algorithmic synthesis and analysis, and provide additional intermediate results.

### 2.2 Equilibrium Points for Strongly Monotone Gradient Maps

In what follows we, we outline relevant assumptions that we use in this paper for the synthesis and analysis of first-order deterministic and stochastic algorithms to identify equilibrium points.

**Assumption 1. (Strong-Convexity-Strong-Concavity)** The function \( \phi \) is is continuously differentiable over \( X \times Y \) for any realization of \( w \). The function \( \phi \gamma \)-strongly-convex-\( \gamma \)-strongly-concave, for any realization of \( w \); that is, \( \phi \) is \( \gamma \)-strongly-convex in \( x \) for all \( y \in Y \) and \( \gamma \)-strongly-concave in \( y \) for all \( x \in X \).

**Assumption 2. (Joint Smoothness)** The gradient map \( \psi(z, w) := (\nabla_x \phi(z, w), -\nabla_y \phi(z, w)) \) is \( L \)-Lipschitz in \( z \) and \( w \). Namely,
\[ \| \psi(z, w) - \psi(z', w) \| \leq L \| z - z' \|, \]
\[ \| \psi(z, w) - \psi(z, w') \| \leq L d(w, w'). \]
for any \( z, z' \in X \times Y \) and \( w, w' \) supported on \( M \). Here \( d : M \times M \to \mathbb{R} \) denotes the metric on \( M \).

**Assumption 3. (Distributional Sensitivity)** The distributional map \( D : X \times Y \to \mathcal{P}(M) \) is \( \varepsilon \)-Lipschitz. Namely,
\[ W_1(D(z), D(z')) \leq \varepsilon \| z - z' \| \]
for any \( z, z' \in X \times Y \), where \( W_1 \) is the Wasserstein-1 distance.

**Assumption 4. (Compact Convex Sets)** The sets \( X \subset \mathbb{R}^d \) and \( Y \subset \mathbb{R}^n \) are compact and convex.

Typically, the assumption of strong-convexity-strong-concavity enables unique convergence to saddle-points in standard primal-dual methods [29]. Furthermore, strong-convexity-strong-concavity implies uniqueness of saddle point solutions; this allows to derive convergence results to the unique saddle-point in the static case, and tracking results in the context of time-varying minmax problems [17]. We also note that this assumption is useful in this paper in order to characterize the intrinsic relationship between optimal solutions to (1) and equilibrium points. We also note that, for simplicity, the assumption imposes a common geometry parameter in \( \phi \) for both the \( x \) and \( y \) values; however, our analysis is the same for functions \( \phi \) being \( \gamma_1 \)-strongly-convex in \( x \) and \( \gamma_2 \)-strongly-concave in \( y \) (as we can take \( \gamma = \min\{\gamma_1, \gamma_2\} \)).

Assuming that the distributional map \( \varepsilon \)-Lipschitz and the gradient is Lipschitz in the random variable is commonplace in the literature on decision-dependent distributions to characterize the overall effects of the distributional maps on the random variables [13][12][56]. Since we assume the support of our random variables \( w \) reside in a complete and separable metric space (Polish space), then a natural way to relate the resulting distributions is via the Wasserstein-1 metric. Following Kantorovich-Rubenstein Duality [11][27], this metric can be written as
\[ W_1(\mu, \nu) = \sup \left\{ \mathbb{E} \left[ g(w) \right] - \mathbb{E} \left[ g(w) \right] \mid g : M \to \mathbb{R}, \text{Lip}(g) \leq 1 \right\} \]
for all \( \mu, \nu \in \mathcal{P}(M) \). Here the supremum is taken over all Lipschitz-continuous functionals on \( M \) with Lipschitz constant less than or equal to one.
Closed and convex constraint sets are common in the literature on primal-dual methods, which are the main algorithms that will be considered shortly [24][29]. Due to Heine-Borel, compactness of \( \mathcal{X} \) and \( \mathcal{Y} \) simply means closed and bounded. The addition of boundedness here is not restrictive; one can assume boundedness while the underlying sets can still be made arbitrarily large to include the saddle-points. As an illustration, consider the closed rectangles \( \mathcal{X} = [-r, r]^d \) and \( \mathcal{Y} = [-r, r]^n \) for some \( r > 0 \). Then \( \mathcal{X} \) and \( \mathcal{Y} \) are compact and convex for any \( r > 0 \), and \( r \) can be made an arbitrarily large positive number. See, e.g., [29] for an example in the context of constrained optimization problems.

In what follows, we demonstrate uniqueness of the equilibrium points for saddle point problems that satisfy the above assumptions. To proceed, we cast the equilibrium point problem into the variational inequality framework. Recall that in Assumption 2, we introduce the stochastic gradient map \( \psi(z, w) = (\nabla_x \phi(z, w), -\nabla_y \phi(z, w)) \).

Using this convention, we denote the decoupled gradient map as

\[
\Psi(z; z') = \mathbb{E}_{w \sim D(z')} [\psi(z, w)].
\]

In the following, we show that when \( z \mapsto \Psi(z; z') \) is strongly monotone for all \( z' \in \mathcal{X} \times \mathcal{Y} \), a unique equilibrium point exists. Furthermore, under this assumption, we can show that the distance between the saddle points for the original problem in (1) and the unique equilibrium point is bounded.

**Proposition 2.4.** Suppose that Assumption 2 holds. Then, for any \( w \in M \), \( z \mapsto \psi(z, w) \) is \( \gamma \)-strongly-monotone. Furthermore, for any \( z' \in \mathcal{X} \times \mathcal{Y} \), \( z \mapsto \Psi(z; z') \) is \( \gamma \)-strongly-monotone.

Proof of this result follows naturally from the Proposition A.3. Below we provide a Lemma that allows us to characterize the changes in the distributional argument of the decoupled gradient map \( \Psi \).

This amounts to the decoupled gradient map being Lipschitz continuous in the distributional argument.

**Lemma 2.5.** *(Gradient Deviations)* Suppose that Assumptions 1, 3, and 4 hold. Then, for any \( \hat{z} \in \mathcal{X} \times \mathcal{Y} \), the decoupled gradient map satisfies:

\[
\|\Psi(\hat{z}; z) - \Psi(\hat{z}; z')\| \leq \epsilon L \|z - z'\| \leq \epsilon LD \|z\|
\]  (25)

for all \( z, z' \in \mathcal{X} \times \mathcal{Y} \) where \( D_z = \text{diam}(\mathcal{X} \times \mathcal{Y}) < \infty \).

**Proof.** Let \( v \in \mathcal{X} \times \mathcal{Y} \) be an arbitrary unit vector and fix \( \hat{z}, z, z' \in \mathcal{X} \times \mathcal{Y} \). It follows that

\[
\langle v, \Psi(\hat{z}, z) - \Psi(\hat{z}, z') \rangle = \mathbb{E}_{w \sim D(z)} [\langle v, \psi(\hat{z}, w) \rangle] - \mathbb{E}_{w \sim D(z')} [\langle v, \psi(\hat{z}, w) \rangle].
\]

By our assumption, we have that \( w \mapsto \langle v, \psi(\hat{z}, w) \rangle \) is Lipschitz with constant \( L \|v\| = L \). Thus, from Kantorovich and Rubenstein, we have that

\[
\mathbb{E}_{w \sim D(z)} [\langle v, \psi(\hat{z}, w) \rangle] - \mathbb{E}_{w \sim D(z')} [\langle v, \psi(\hat{z}, w) \rangle] \leq LW_1(D(z), D(z')) \leq \epsilon L \|z - z'\|,
\]

where that last inequality follows from \( \epsilon \)-sensitivity of \( D \). Thus we have that for any unit vector \( v \),

\[
\langle v, (\Psi(\hat{z}, z) - \Psi(\hat{z}, z')) \rangle \leq \epsilon L \|z - z'\|.
\]

Since this is true for any unit vector \( v \), choosing

\[
v = (\Psi(\hat{z}, z) - \Psi(\hat{z}, z'))/\|\Psi(\hat{z}, z) - \Psi(\hat{z}, z')\|
\]

yields the result. ❄

In what follows, we demonstrate existence and uniqueness of equilibrium points. Similar to the statement of existence, we show that \( H \) satisfies the Banach-Picard Fixed Point Theorem by providing conditions for which \( H \) is a strict contraction.

**Theorem 2.6.** *(Existence and Uniqueness of Equilibrium Points)* Suppose that Assumptions 2, 4 hold. Then:
1. For all \( z, z' \in X \times Y \), \( \| H(z) - H(z') \| \leq \frac{\varepsilon L}{\gamma} \| z - z' \| \),

2. If \( \frac{\varepsilon L}{\gamma} < 1 \), then there exists a unique equilibrium point \((\bar{x}, \bar{y}) \in X \times Y\).

Proof. Let \( \tilde{x}, \tilde{z} \in X \times Y \) be fixed. Then the maps \( z \to \Psi(z; \tilde{z}) \) and \( z \to \Psi(z; \tilde{z}) \) are \( \gamma \)-strongly-strictly monotone. Furthermore, our strong-convexity and strong-concavity assumptions on \( \phi \) imply that \( H(\tilde{z}) \) and \( H(\tilde{z}) \) are single valued in \( X \times Y \). Recall from our definition of \( H \) that \( H(\tilde{z}) \) and \( H(\tilde{z}) \) are solutions to the variational inequalities induced by \( \tilde{z} \) and \( \tilde{z} \) respectively. That is, for all \( z, \bar{z} \in X \times Y \),

\[
\langle H(\tilde{z}), \Psi(\tilde{z}; \tilde{z}) \rangle \geq 0 \quad \text{and} \quad \langle z - H(\tilde{z}), \Psi(\tilde{z}; \tilde{z}) \rangle \geq 0.
\]

(26)

It follows from strong monotonicity that

\[
\langle H(\tilde{z}) - H(\tilde{z}), \Psi(\tilde{z}; \tilde{z}) - \Psi(\tilde{z}; \tilde{z}) \rangle \geq \gamma \| H(\tilde{z}) - H(\tilde{z}) \|^2
\]

and the variational inequalities in (26) imply that \( \langle H(\tilde{z}) - H(\tilde{z}), \Psi(\tilde{z}; \tilde{z}) \rangle \geq 0 \). Hence,

\[
\langle H(\tilde{z}) - H(\tilde{z}), \Psi(\tilde{z}; \tilde{z}) \rangle \leq -\gamma \| H(\tilde{z}) - H(\tilde{z}) \|^2.
\]

(27)

To proceed, we provide a lower bound for the quantity on the left-hand side. By applying Cauchy-Schwartz and Lemma 2.5, we get that

\[
\langle H(\tilde{z}) - H(\tilde{z}), \Psi(\tilde{z}; \tilde{z}) - \Psi(\tilde{z}; \tilde{z}) \rangle \leq \varepsilon L \| H(\tilde{z}) - H(\tilde{z}) \| \| \tilde{z} - \tilde{z} \|. \quad (28)
\]

Since (26) implies that \( \langle H(\tilde{z}) - H(\tilde{z}), \Psi(\tilde{z}; \tilde{z}) \rangle \geq 0 \), then we get that

\[
\langle H(\tilde{z}) - H(\tilde{z}), \Psi(\tilde{z}; \tilde{z}) \rangle \geq -\varepsilon L \| H(\tilde{z}) - H(\tilde{z}) \| \| \tilde{z} - \tilde{z} \|. \quad (28)
\]

Combining inequalities (27) and (28) yields

\[
-\gamma \| H(\tilde{z}) - H(\tilde{z}) \|^2 \geq -\varepsilon L \| H(\tilde{z}) - H(\tilde{z}) \| \| \tilde{z} - \tilde{z} \|,
\]

and we conclude that

\[
\| H(\tilde{z}) - H(\tilde{z}) \| \leq \frac{\varepsilon L}{\gamma} \| \tilde{z} - \tilde{z} \|.
\]

Since \( H \) is Lipschitz continuous, then it is a strict contraction provided that \( \varepsilon L/\gamma < 1 \). Uniqueness of the fixed point follows from the Banach-Picard Fixed-Point Theorem.

We have demonstrated existence and uniqueness of equilibrium points for some classes problems; next, we characterize the relationship between equilibrium points and solutions of the original problem in (1). First, an important observation is that when \( z = 0 \), the problem statement in (12) has a stationary probability distribution with respect to the decisions. Hence, saddle points coincide with equilibrium points. When \( \varepsilon > 0 \), we provide a guarantee on the distance between solutions of the two problems.

Proposition 2.7. (Bounded Distance) If Assumptions 1-4 hold; let \( z^* \) be the optimal solution of (12), and let \( \bar{z} \) be the equilibrium point. Then,

\[
\| z^* - \bar{z} \| \leq \frac{\varepsilon L}{\gamma} D_Z.
\]

(29)

Proof. From the optimality conditions, we have that the decoupled gradient map satisfies

\[
\langle \bar{z} - z^*, \Psi(\bar{z}; z^*) \rangle \geq 0 \quad (30)
\]

\[
\langle z^* - \bar{z}, \Psi(\bar{z}; \bar{z}) \rangle \geq 0 \quad (31)
\]

By combining these results with results with our gradient deviation bound in Lemma 2.5, we obtain the following:

\[
\langle \bar{z} - z^*, \Psi(\bar{z}; \bar{z}) - \Psi(\tilde{z}; \tilde{z}) \rangle = \langle \bar{z} - z^*, \Psi(\bar{z}; \bar{z}) \rangle - \langle \bar{z} - z^*, \Psi(\tilde{z}; \tilde{z}) \rangle \leq \langle \bar{z} - z^*, \Psi(z^*; z^*) - \Psi(z^*; \tilde{z}) \rangle \leq \| \bar{z} - z^* \| \| \Psi(z^*; z^*) - \Psi(z^*; \tilde{z}) \| \leq \varepsilon L D_Z \| \bar{z} - z^* \|,
\]

(32)
where the second to last step follows from the Cauchy-Schwartz inequality. It follows from $\gamma$-strong-monotonicity that
\[
\gamma \| \bar{z} - z^* \|^2 \leq \langle \bar{z} - z^*, \Psi(z; \bar{z}) - \Psi(z^*; \bar{z}) \rangle \leq \varepsilon LD \| \bar{z} - z^* \|
\]
so that canceling terms and dividing by $\gamma$ yields the result. \qed

Remark 1. We note that relative to the decision-dependent minimization problem in [18,42], our result does not require the stochastic objective $\phi$ to be Lipschitz in the random variable $w$. On the other hand, we enforce compactness of the constraint sets and include the set diameter in the upper bound in Proposition 2.3. Intuitively, this is a sensitivity result relating the saddle point problem and the equilibrium problem. The distance between these two solutions, then, is bounded by the product of the $\varepsilon L/\gamma$ condition of the objective (represented by $\varepsilon L/\gamma$) and the size of the constraints imposed on the decision variables (represented by the set diameter $D Z$). The set diameter here is a stand-in for the fact that the objective values $\Phi(z^*; z^*)$ and $\Phi(\bar{z}; \bar{z})$ do not have an ordering that arises immediately from the definition or any assumptions we impose.

2.3 Finding the Equilibrium Point via Primal-Dual Algorithm

In this section, we focus on a primal-dual method for finding the equilibrium point, and we provide results in terms of convergence at a linear rate. In particular, we focus on the Equilibrium Primal-Dual (EPD) algorithm, which is based on the algorithmic map:
\[
G(z; z') := \Pi_{X \times Y}(z - \eta \Psi(z; z'))
\]
for all $z, z' \in X \times Y$, where we recall that $\Pi_{X \times Y}(z) = \arg\min_{z' \in X \times Y} \frac{1}{2} \| z - z' \|^2$ is the projection map and $\eta > 0$ is a positive step size. Given an initial point $z_0 \in X \times Y$, the algorithm then generates a sequence via the Banach-Picard iteration:
\[
z_{t+1} = G(z_t; z_t) = \Pi_{X \times Y}(z_t - \eta \Psi(z_t; z_t)), \quad t = 1, 2, \ldots
\]
where we recall that $\Psi(z_t; z_t) = \mathbb{E}_{w \sim D(z_t)} [\psi(z_t, w)]$, with $D(z_t)$ the distribution induced by $z_t$. A key feature of this method is that each step projects onto the constraint sets, and hence $z_t \in X \times Y$ for all $t \geq 1$ for any initial condition $z_0$. In the following we demonstrate that the equilibrium points are fixed points of the algorithmic map. We then provide linear convergence results for the case when our assumptions hold on just the constraint sets $X \times Y$, and later globally.

Proposition 2.8. Let Assumptions 14 hold and suppose that $\varepsilon L/\gamma < 1$. A point $\bar{z} \in X \times Y$ is an equilibrium point if and only if $\bar{z} = G(\bar{z}; \bar{z})$.

Proof. From [19, Theorem 1.5.5] a property of the projection map is that, for any $\bar{z} \in X \times Y$
\[
\langle z - \Pi_{X \times Y}(\bar{z}), \Pi_{X \times Y}(\bar{z}) - \bar{z} \rangle \geq 0
\]
for all $z \in X \times Y$. This follows from the fact that the above is the variational inequality associated with the quadratic optimization problem given by the projection map. Observe then that $\bar{z} \in X \times Y$ is an equilibrium point if and only if
\[
\langle z - \bar{z}, \bar{z} - (\bar{z} - \eta \Psi(\bar{z}; \bar{z})) \rangle \geq 0
\]
for $\eta > 0$. This is equivalent to the original inequality by multiplying by $\eta$ and adding and subtracting $\bar{z}$. By setting $\bar{z} = \bar{z} - \eta \Psi(\bar{z}; \bar{z})$ in [19, Theorem 1.5.5], we have that the inequality above is equivalent to $\bar{z} = \Pi_{X \times Y}(\bar{z} - \eta \Psi(\bar{z}; \bar{z}))$. \qed

Theorem 2.9. (EPD Convergence) Suppose that Assumptions 14 hold. Suppose further that $\varepsilon L/\gamma < 1$. Then:

1. If $\eta \leq \gamma / L^2$, then
\[
\| G(z; \bar{z}) - G(z'; \bar{z}) \| \leq \beta \| z - z' \|
\]
for any $z, z', \bar{z} \in X \times Y$, where $\beta := \sqrt{1 - \eta}$. 

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2. For all \( t \geq 0 \), the sequence \( z_{t+1} = G(z_t; z_t) \) satisfies:
\[
\|z_{t+1} - \bar{z}\| \leq (\beta + \varepsilon \eta L)\|z_t - \bar{z}\|
\]
provided that \( \eta \leq \gamma / L^2 \).

3. The sequence \( z_{t+1} = G(z_t; z_t) \) converges linearly to the unique equilibrium point \( \bar{z} \) provided that
\[
\max \left\{ 0, \frac{2\varepsilon L - \gamma}{\varepsilon^2 L^2} \right\} < \eta \leq \frac{\gamma}{L^2}.
\]

**Proof.** Proving 1. amounts to showing that the stationary projection method is Lipschitz continuous. Fix \( \hat{z}, z, z' \in X \times Y \). It follows from the non-expansiveness of the projection map that
\[
\|G(z; \hat{z}) - G(z'; \hat{z})\|^2 \leq \|z - z'\|^2 + \eta^2\|G(z; \hat{z}) - G(z'; \hat{z})\|^2 - 2\eta(z - z', \Psi(z; \hat{z}) - \Psi(z'; \hat{z}))
\]
where the last inequality follows from applying the strong-monotonicity and Lipschitz inequalities. We conclude that
\[
\|G(z; \hat{z}) - G(z'; \hat{z})\|^2 \leq (1 + \eta^2 L^2 - 2\eta\gamma)\|z - z'\|^2.
\]
observe that \( \eta L^2 - 2\gamma \leq -\gamma \) if and only if \( \eta \leq \gamma / L^2 \). Hence \( \eta \leq \gamma / L^2 \) yields
\[
\|G(z; \hat{z}) - G(z'; \hat{z})\| \leq \sqrt{1 - \eta\gamma}\|z - z'\|.
\]

We prove the result by comparing the \( G(z_t; z_t) \) to the stationary update \( G(z_t; \hat{z}) \). By the triangle inequality, we have that
\[
\|z_{t+1} - \bar{z}\| = \|G(z_t; z_t) - G(z_t; \hat{z})\| \leq \|G(z_t; z_t) - G(z_t; \hat{z})\| + \|G(z_t; \hat{z}) - \bar{z}\|.
\]
To bound the deviation in the distribution argument, we apply our result of Lemma 2.5. This result implies that
\[
\|G(z_t; z_t) - G(z_t; \hat{z})\| \leq \eta\|\Psi(z_t; z_t) - \Psi(z_t; \hat{z})\| \leq \eta \varepsilon L\|z_t - \bar{z}\|,
\]
where the intermediate step follows from the non-expansiveness of the projection map. Furthermore, from 1. we have that
\[
\|G(z_t; \hat{z}) - G(z_t; \bar{z})\| \leq \beta\|z_t - \bar{z}\|
\]
where \( \beta = \sqrt{1 - \eta\gamma} \) for \( \eta \leq \gamma / L^2 \). Hence adding yields \( \|z_{t+1} - \bar{z}\| \leq (\beta + \eta \varepsilon L)\|z_t - \bar{z}\| \). Linear convergence will follow if \( \beta + \eta \varepsilon L < 1 \). This condition holds if and only if
\[
1 - \eta\gamma < 1 - \eta \varepsilon L + \eta^2 \varepsilon^2 L^2.
\]
This inequality is equivalent to requiring that \( \eta \) satisfy
\[
0 < \eta \left( \eta \varepsilon^2 L^2 + \gamma - 2\varepsilon L \right).
\]
We conclude that \( \beta + \eta \varepsilon L < 1 \) provided that \( \eta > 0 \) and \( \eta > \frac{2\varepsilon L - \gamma}{\varepsilon^2 L^2} \), so the result follows.

**Remark 2.** One can show that \( \frac{2\varepsilon L - \gamma}{\varepsilon^2 L^2} < 0 \) if and only if \( \frac{\varepsilon L}{\gamma} < \frac{1}{2} \). Enforcing this condition shows that the tracking error error is contractive, \( \sqrt{1 - \eta\gamma} + \eta \varepsilon L < 1 \), provided that \( \eta \) is upper bounded by the co- coercivity constant of the gradient map. This matches the result in IS for the decision dependent minimization problem. Namely, IS shows a specific result for which convex L-smooth function give rise to 1/L-co-coercive gradient maps. In general, a \( \gamma \)-strongly-monotone and L-Lipschitz map \( \Psi : \mathbb{R}^m \rightarrow \mathbb{R}^m \) is \( \gamma / L^2 \)-co-coercive since
\[
\frac{\gamma}{L^2} \|\Psi(z) - \Psi(z')\| \leq \gamma\|z - z'\| \leq (z - z', \Psi(z) - \Psi(z'))
\]
for all \( z, z' \in \mathbb{R}^m \).
2.4 Stochastic Projected Primal-Dual Method

In the previous section, we showed convergence of a primal-dual algorithm. However, this algorithm requires the computation of the gradient; that is, one can either compute the function \( E_{\tilde{w} \sim D(z')} [\psi(z, w)] \) in closed form, or we have access to a black box evaluator that can compute this exactly for any \( z' \in \mathcal{X} \times \mathcal{Y} \). In the first case, this means that we have explicit knowledge of the distributional map \( D \). The latter assumes that the integral expression can be computed exactly or approximated to machine precision. However, this integral approximation problem is known to suffer from the curse of dimensionality.

When the computation of the gradient is not possible, a common approach is to utilize a stochastic gradient estimate. Accordingly, in this work we assume access to an oracle \( \Omega : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}^{d+n} \) that returns a gradient estimator at a given point in the domain. Conceptually, \( \Omega \) is a stochastic process indexed over \( \mathcal{X} \times \mathcal{Y} \) and may take the form of common estimators used in the stochastic optimization literature:

\[
\Omega(z) = \begin{cases} 
\psi(z, w_1), & w_1 \sim D(z) \\
\frac{1}{N} \sum_{i=1}^{N} \psi(z, w_i), & w_1, \ldots, w_N \overset{i.i.d.}{\sim} D(z) \\
\Psi(z; z) + \epsilon, & \epsilon \sim \mathcal{N}(0, \sigma^2 I)
\end{cases}
\]

where \( \{w_i\}_{i=1}^{N} \) are random samples drawn from \( D(z) \). The first two represent the stochastic gradient and mini-batch gradient estimator, respectively. The third one is a noisy gradient evaluation that models noisy black-box evaluations of the true gradient map \( \Psi \). Assuming access to \( \Omega \), we define the Stochastic Equilibrium Primal-Dual (SEPD) method via the algorithmic map:

\[
\hat{G}(z) = \Pi_{\mathcal{X} \times \mathcal{Y}} (z - \eta \Omega(z))
\]

so that the SEPD algorithm generates the sequence

\[
z_{t+1} = \hat{G}(z_t), \quad t \geq 0
\]

starting from a point \( z_0 \in \mathcal{X} \times \mathcal{Y} \). In our analysis, we associate the gradient error with a class of potentially heavy tailed probability distributions. As in [52], we consider class of sub-Weibull random variables as defined next.

**Definition 4. (Sub-Weibull Random Variable)** A random variable \( \xi \) is sub-Weibull, denoted \( \xi \sim \text{sub} W(\theta, \nu) \), if there exists \( \theta > 0, \nu > 0 \) such that

\[
P(|z| \geq \epsilon) \leq 2 \exp(- (\epsilon/\nu)^{1/\theta})
\]

for all \( \epsilon > 0 \).

In this definition, \( \theta \) measures the heaviness of the tail (higher values of \( \theta \) correspond to heavier tails); for example, \( \theta = 1 \) and \( \theta = 1/2 \) correspond to sub-exponential and sub-Gaussian random variables respectively. The parameter \( \nu \) represents a proxy for the variance of \( \xi \). This variance proxy is typically omitted in the definition of a sub-Weibull class, where \( \theta \) alone is considered sufficient to identify a family of distributions [52, 55]. We include it here as it allows to better describe the behavior of distributions generated from the closure properties. This approach has been used in the study of stochastic gradient methods in [5, 56].

Our definition of sub-Weibull random variables as heavy-tailed distributions can also be translated into information about the moments of these random variables. In the following, we restate the relevant portions of equivalence conditions found in [52, 55]. Proof of this result may also be found in these works.

**Proposition 2.10. (Equivalent Characterizations)** If \( \xi \) is a sub-Weibull random variable with tail parameter \( \theta > 0 \), then the following characterizations are equivalent (we recall that \( \|z\|_k = E[|z|^k]^{1/k} \)):

1. **Tail Probability Characterization.** \( \exists \nu_1 > 0 \) such that \( P(|z| \geq \epsilon) \leq 2 \exp(- (\epsilon/\nu_1)^{1/\theta}) \) for all \( \epsilon > 0 \).

2. **Moment Characterization.** \( \exists \nu_2 > 0 \) such that \( \|z\|_k \leq \nu_2 k^\theta \) for all \( k \geq 1 \).
Equivalence here implies a relationship between the proxy variance parameters employed in the two conditions above. If \( \xi \sim \text{subW}(\theta, \nu_1) \), then the condition on the moments holds with \( \nu_2 = (2e(\frac{1}{\theta^1})^\theta \nu_1 \). By employing the moment characterization, one can demonstrate that sub-Weibull random variables enjoy inclusion and closure properties.

**Proposition 2.11. (Sub-Weibull Inclusion)** If \( \xi \sim \text{subW}(\theta, \nu) \) and \( \theta', \nu' > 0 \) such that \( \theta \leq \theta' \) and \( \nu \leq \nu' \) then \( \xi \sim \text{subW}(\theta', \nu') \).

**Proposition 2.12. (Sub-Weibull Closure)** Let \( \xi_1 \sim \text{subW}(\theta_1, \nu_1) \) and \( \xi_2 \sim \text{subW}(\theta_2, \nu_2) \) be (possibly coupled) sub-Weibull random variables and let \( c \in \mathbb{R} \). Then, the following hold:

1. \( \xi_1 + \xi_2 \sim \text{subW}(\max\{\theta_1, \theta_2\}, \nu_1 + \nu_2) \);
2. \( \xi_1 \xi_2 \sim \text{subW}(\theta_1 + \theta_2, \psi(\theta_1, \theta_2) \nu_1 \nu_2), \psi(\theta_1, \theta_2) := (\theta_1 + \theta_2)^{\theta_1 + \theta_2} / (\theta_1^{\theta_1} \theta_2^{\theta_2}) \);
3. \( c \xi_1 \sim \text{subW}(\theta_1, c \nu_1) \).

In addition to the fact that heavy tailed distributions are well-motivated in the literature on stochastic gradient methods \[13,24\], the inclusion and closure properties offer useful theoretical tools. In particular, leveraging these facts will allow us to establish high-probability bounds for the error in our algorithm—provided that our gradient estimator follows a sub-Weibull error model.

**Assumption 5. (Sub-Weibull Gradient Error Process)** Let \( \Omega \) be such that there exist tail parameter \( \theta > 0 \) and bounded variance proxy function \( \nu : \mathbb{R}^d \times \mathbb{R}^n \mapsto \mathbb{R}_{\geq 0} \) such that

\[
\xi(z) = \|\Omega(z) - \mathbb{E}_{w \sim D(z)}[\psi(z, w)]\| \sim \text{subW}(\theta, \nu(z))
\]  

(39)

for all \( z \in X \times Y \). Denote \( \tilde{\nu} \in (0, \infty) \) such that \( \nu(z) \leq \tilde{\nu} \) for all \( z \in X \times Y \).

A typically assumption in the literature is that the norm of \( \xi \) is uniformly bounded in expectation. Here, we assume that the norm of the gradient error is distributed according to a heavy-tailed distribution. If the variance proxy function \( \nu \) is continuous over the compact set \( X \times Y \), we retrieve the uniform boundedness property. By assuming \( \theta \) is fixed for any \( z \in X \times Y \), we assume that all realizations of the process belong to the same sub-Weibull class.

The goal in analyzing algorithms in the static setting is typically to demonstrate convergence of the sequence of iterates to the unique fixed point, or convergence in the objective value. A weaker, yet related notion is that of convergence to a neighborhood.

**Definition 5. (Convergence to a Neighborhood)** A sequence \( \{z_t\}_{t \geq 0} \subseteq \mathbb{R}^m \) converges to an \( R \)-neighborhood of its limit \( \bar{z} \) if there exists \( R > 0 \) such that

\[
\limsup_{t \to \infty} \|z_t - \bar{z}\| \leq R.
\]  

(40)

An immediate consequence is that for any \( \epsilon > 0 \), there exists \( T > 0 \) such that \( |s_t - s| < \epsilon \) for any \( t \geq T \), where \( s_t = \sup_{s \geq t} \|z_t - \bar{z}\| \) and \( s = \lim_{t \to \infty} s_t \). Thus, for any \( t \geq T \),

\[
\|z_t \| \leq \sup_{T \leq t} \|z_t - \bar{z}\| \leq \epsilon + R.
\]  

(41)

This is equivalent to saying that for any choice of \( \epsilon > 0 \), there exists a \( T > 0 \) such that \( t \geq T \) implies

\[
z_t \in B_{R + \epsilon}(\bar{z}) = \{ z \mid \|z - \bar{z}\| \leq R + \epsilon \}.
\]  

(42)

Before demonstrating convergence to a neighborhood of the stochastic primal-dual algorithm, we characterize the error between the sequence generated by the algorithm and the equilibrium points.

**Theorem 2.13. (SEPD Error Bounds)** Let Assumptions \[13\] hold. Suppose that \( \frac{\alpha}{\gamma} < 1 \). Then, the following hold:
1. The sequence of iterates \( z_{t+1} = \tilde{G}(z_t) \) satisfies the stochastic recursion

\[
\| z_{t+1} - \bar{z} \| \leq \alpha \| z_t - \bar{z} \| + \eta \xi_t \tag{43}
\]

where \( \xi_t = \xi(z_t) \) and \( \alpha = \beta + \eta \varepsilon L \).

2. The sequence \( \{ z_t \}_{t \geq 0} \) satisfies the bound in expectation

\[
\mathbb{E} \| z_{t+1} - \bar{z} \| \leq \alpha^{t+1} \| z_0 - \bar{z} \| + \sum_{i=0}^{t} \alpha^i \mathbb{E}[\xi_{t-i}].
\]

for all \( t \geq 0 \).

3. If \( \delta \in (0, 1) \), then

\[
\mathbb{P} \left( \| z_{t+1} - \bar{z} \| \leq \alpha^{t+1} \| z_0 - \bar{z} \| + c(\theta) \log^\theta \left( \frac{2}{\delta} \right) \sum_{i=0}^{t} \alpha^i \nu_{t-i} \right) \geq 1 - \delta
\]

for all \( t \geq 0 \), with \( c(\theta) := \left( \frac{2}{\theta} \right)^\theta \).

Proof. We proceed by treating the stochastic gradient step as an inexact step using the triangle inequality:

\[
\| z_{t+1} - \bar{z} \| = \| \tilde{G}(z_t) - G(\bar{z}; \bar{z}) \| + \| G(z_t; z_t) - G(\bar{z}; \bar{z}) \|.
\]

Then, using the non-expansiveness of the projection map and Assumption 5, we find that

\[
\| \tilde{G}(z_t) - G(z_t; z_t) \| \leq \eta \| \Omega(z_t) - \Psi(z_t; z_t) \| = \eta \xi(z_t).
\]

It follows from Theorem 2.9 that

\[
\| G(z_t; z_t) - G(\bar{z}; \bar{z}) \| \leq \alpha \| z_t - \bar{z} \|
\]

where \( \alpha = \beta + \eta \varepsilon L \) so that adding these inequalities yields the result in (43).

Repeated application of the recursion yields

\[
\| z_{t+1} - \bar{z} \| \leq \alpha^{t+1} \| z_0 - \bar{z} \| + \eta \sum_{i=0}^{t} \alpha^i \xi_{t-i} \tag{45}
\]

where \( \alpha = \beta + \eta \varepsilon L \). By taking the expectation of both sides, we get the expectation bound:

\[
\mathbb{E} \| z_{t+1} - \bar{z} \| \leq \alpha^{t+1} \| z_0 - \bar{z} \| + \sum_{i=0}^{t} \alpha^i \mathbb{E}[\xi_{t-i}].
\]

For notational convenience, we denote the equilibrium error as \( \epsilon_t = \| z_t - \bar{z} \| \). Then, denote the left term in the stochastic recursion in (45) as \( \ell_t = \alpha^{t+1} \epsilon_0 \) and the right term as \( \kappa = \eta \sum_{i=0}^{t} \alpha^i \xi_{t-i} \) so that \( e_{t+1} \leq \ell_t + \kappa \). From our sub-Weibull assumption, we have that \( \xi_{t-i} \sim \text{subW}(\theta, \nu_{t-i}) \), where \( \nu_{t-i} = \nu(z_{t-i}) \). It follows from the closure under product and addition in Proposition 2.12 that \( \kappa \sim \text{subW}(\theta, \nu_t) \) with \( \nu_t = \eta \sum_{i=1}^{t} \alpha^i \nu_{t-i} \). Hence,

\[
\mathbb{P} \left( \kappa \geq \epsilon \right) \leq 2 \exp \left( -\frac{\theta}{2\epsilon} \left( \frac{\epsilon}{\nu_t} \right)^\frac{1}{\theta} \right) \tag{46}
\]

By setting the right-hand side equal to \( \delta \), we find that

\[
\epsilon = c(\theta) \log^\theta \left( \frac{2}{\delta} \right) \nu_t \tag{47}
\]
where \( c(\theta) = (\frac{2\epsilon}{\alpha})^\theta \). Now, observe that our stochastic recursion implies that for any \( c > 0 \),
\[
P(\ell_t + \epsilon_t \geq c) \geq P(\epsilon_t \geq c).
\]
It follows by using the compliment relationship that
\[
P(\epsilon_{t+1} \leq \ell_t + \epsilon) \geq P(\ell_t + \epsilon_t \leq \ell_t + \epsilon) = P(\epsilon_t \leq \epsilon) \geq 1 - \delta,
\]
and the result follows by substituting the expressions for \( \ell_t \) and \( \epsilon \) accordingly.

The bounds naturally translate to convergence results by considering the limit supremum. Now we demonstrate that the algorithm converges to a neighborhood of the equilibrium in expectation and almost surely.

**Theorem 2.14.** Let Assumptions 1-5 hold. Assume that \( \eta \) satisfies the condition (36). Then, the sequence of iterates \( \{z_t\}_{t \geq 0} \) converges to a neighborhood of the equilibrium in expectation and almost surely. In particular,
\[
\limsup_{t \to \infty} E \|z_t - \bar{z}\| \leq \frac{\eta \xi}{1 - \alpha},
\]
and
\[
P \left( \limsup_{t \to \infty} \|z_t - \bar{z}\| \leq \frac{\eta \xi}{1 - \alpha} \right) = 1.
\]

**Proof.** For notational convenience, we define the error as \( e_t := \|z_t - \bar{z}\| \). Then we can write the expectation bound in 2.13 compactly as
\[
E[e_t + 1] \leq \alpha e_0 + \eta \sum_{i=0}^{t} \alpha^i E[\xi_{t-i}].
\]
If we denote \( \tilde{\xi} = \sup_{t \geq 0} E[\xi_t] \), then by Assumption 5 it follows that \( \tilde{\xi} \leq \tilde{\nu} < \infty \). From (36), it follows that \( \alpha < 1 \). We conclude that
\[
\eta \sum_{i=0}^{t} \alpha^i E[\xi_{t-i}] \leq \frac{\eta \tilde{\xi}}{1 - \alpha}
\]
and hence
\[
\limsup_{t \to \infty} E[\epsilon_{t+1}] \leq \frac{\eta \tilde{\xi}}{1 - \alpha}.
\]

Now if we define the random variable \( E_t = \max\{0, e_{t+1} - \frac{\eta \tilde{\xi}}{1 - \alpha}\} \), then \( E_t \leq \alpha e_0 \). By Markov’s inequality,
\[
P(E_t \leq \epsilon) \leq \frac{E[|E_t|]}{\epsilon} \leq \frac{\alpha e_0}{\epsilon}
\]
for all \( \epsilon > 0 \). Summing over \( t \) yields
\[
\sum_{t=0}^{\infty} P(E_t \geq \epsilon) \leq \frac{e_0}{\epsilon(1 - \alpha)} < \infty.
\]
It follows from the Borel-Cantelli Lemma that, since the sum of tail probabilities is finite, then
\[
P \left( \limsup_{t \to \infty} E_t \leq \epsilon \right) = 1.
\]
Since this is true for any \( \epsilon > 0 \), then
\[
P \left( \limsup_{t \to \infty} E_t \leq 0 \right) = 1.
\]
and hence
\[
\limsup_{t \to \infty} e_{t+1} \leq \frac{\eta \tilde{\xi}}{1 - \alpha}
\]
almost surely.

This concludes our analysis of the time-invariant equilibrium problem. In the next section, we focus on the time-varying problem (1).
3 Time-Varying Stochastic Minimax Problems

In this section, we focus on the problem \( \Pi \) where the function \( \phi_t \) and the distributional map \( D_t \) are time-varying. This leads to a setup where information is received and processed sequentially and \( \phi_t \) and \( D_t \) may change at each iteration of the algorithm \([17, 20]\). Similar to the static setting of the previous section, we will assume that \( \phi_t \) is strongly-convex-strongly-concave for each time \( t \) a is unique at each time \( t \).

Our definition of equilibrium points in Definition 3 remains the same, and it is applied to the points \((x_t, y_t)\) at each \( t \). Therefore, our objective will be to track the sequence equilibrium points \( \{(x_t, y_t)\}_{t \geq 0} \) and demonstrate that the tracking error converges to a neighborhood of zero.

To obtain concise results, our asymptotic analysis relies on the assumption that the problem parameters are uniformly bounded. In the appendix, we demonstrate how relaxing this assumption yields the same result.

Assumption 6. (Time-Varying Assumptions)

1. (Strong-convexity-strong-concavity) The family of time-varying functions \( \{\phi_t\}_{t \geq 0} \) shares a strongly-convex-strongly-concave parameter \( \gamma \). Namely, for all \( t \geq 0 \), \( \phi_t \) is \( \gamma_t \)-strongly-convex-strongly-concave and there exists \( \gamma > 0 \) such that \( \gamma \leq \gamma_t \).

2. (Joint Smoothness) For all \( t \geq 0 \), \( \psi(z, w) = (\nabla_x \phi(z, w), -\nabla_x \phi(z, w)) \) is \( L_t \)-Lipschitz in \( z \) and \( w \); there exists \( L \in (0, \infty) \) such that \( L_t \leq L \) for all \( t \geq 0 \).

3. (Lipschitz Distributional Map) For all \( t \geq 0 \), \( D_t \) is \( \epsilon_t \)-Lipschitz on the metric space \((\mathcal{P}(M), W_1)\) and there exists \( \epsilon > 0 \) such that \( \epsilon_t \leq \epsilon \) for all \( t \geq 0 \).

4. (Compact-Convex Sets) For all \( t \geq 0 \), the sets \( X_t \subseteq \mathbb{R}^d \) and \( Y_t \subseteq \mathbb{R}^n \) are convex and compact.

5. (Sub-Weibull Gradient Error) Let \( \Omega_t \) be such that there exist \( \theta > 0 \) and proxy-variance function \( \nu_t : X \times Y \mapsto \mathbb{R}_{\geq 0} \) such that

\[
\xi_t(z) = ||\Omega_t(z) - \mathbb{E}_{w \sim D_t(z)}[\psi_t(z, w)]|| \sim \text{subW}(\theta, \nu_t(z)) \tag{48}
\]

for all \( z \in X_t \times Y_t \), where \( \theta_t \) and \( \nu_t \) are uniformly bounded. Let \( \tilde{\theta}, \tilde{\nu} \in (0, \infty) \) be such that \( \theta_t \leq \tilde{\theta} \) and \( \nu_t(z) \leq \tilde{\nu} \) for all \( z \in X_t \times Y_t \) and \( t \in \mathbb{N}_0 \).

We also define the equilibrium drift as \([17, 47]\):

\[
\Delta_t := ||z_{t+1} - z_t||
\]

and we denote \( \Delta \) as a uniform bound; that is, there exists \( \Delta \in [0, \infty) \) such that \( \Delta_t \leq \Delta \) for all \( t \geq 0 \).

3.1 Online Equilibrium Points

The assumptions introduced in the previous section are analogs of the assumptions introduction in Section 2 for the static equilibrium point problem. Similarly, the time-varying decoupled objective is given by

\[
\Phi_t(x, y; x', y') := \mathbb{E}_{w \sim D_t(x', y')}[\phi(x, y, w)]
\]

so that that the sequence of equilibrium points \( \{z_t\}_{t \geq 0} = \{(\tilde{x}_t, \tilde{y}_t)\}_{t \geq 0} \) satisfy

\[
x_t \in \arg\min_{x \in X_t} \max_{y \in Y_t} \Phi_t(x, y; x_t, y_t) \quad \text{and} \quad y_t \in \arg\min_{x \in X_t} \max_{y \in Y_t} \Phi_t(x, y; x_t, y_t), \quad \forall \ t \geq 0.
\]

To track equilibrium points in the presence of full gradient information, we generate the sequence

\[
z_{t+1} = \mathcal{G}_t(z_t; z_t)
\]

where the online equilibrium primal-dual (OEPD) map is given by

\[
\mathcal{G}_t(z; z') = \Pi_{X_t \times Y_t} \left( z - \eta \Psi_t(z; z') \right), \quad \Psi_t(z; z') = \left( \mathbb{E}_{w \sim D_t(z')}[\nabla_x \phi_t(z, w)], \mathbb{E}_{w \sim D_t(z')}[-\nabla_y \phi_t(z, w)] \right)
\]
for all $z, z' \in X_t \times Y_t$.

In the time-varying setting, error bounds are a valuable result in their own right; they allow us to demonstrate that the tracking error has a recursive relationship despite the fact that the solution is changing. To demonstrate the efficiency of this tracking in the long run, we provide an asymptotic result. This shows that the sequence of equilibrium points and the sequence generated by the algorithm are bounded within a ball whose radius is prescribed by the upper bound.

**Theorem 3.1. (OSEPD Tracking)** Let Assumption \textup{6} hold and let $\frac{\eta}{\gamma} < 1$. Then the sequence of iterates $z_{t+1} = \hat{G}_t(z_t; z_t)$ satisfies the tracking error recursion
\begin{equation}
\|z_{t+1} - \bar{z}_{t+1}\| \leq \alpha \|z_t - \bar{z}_t\| + \Delta
\end{equation}
where $\alpha = \sqrt{1 - \eta \gamma} + \eta L$. Moreover, if $\eta$ satisfies the condition in \textup{36}, the asymptotic is bounded as
\begin{equation}
\limsup_{t \to \infty} \|z_t - \bar{z}_t\| \leq \frac{\Delta}{1 - \alpha}.
\end{equation}

**Proof.** By applying the triangle inequality, we find that
\begin{equation}
\|z_{t+1} - \bar{z}_{t+1}\| \leq \|z_{t+1} - \bar{z}_t\| + \|\bar{z}_t - \bar{z}_{t+1}\| = \|\hat{G}_t(z_t; z_t) - \hat{G}_t(\bar{z}_t; \bar{z}_t)\| + \Delta.
\end{equation}

Similar to the batch setting case, if $\eta \leq \gamma / L^2$, then
\begin{equation}
\|\hat{G}_t(z_t; z_t) - \hat{G}_t(\bar{z}_t; \bar{z}_t)\| \leq \beta_t \|z_t - \bar{z}_t\|
\end{equation}
where $\beta_t = \sqrt{1 - \eta \gamma}$. Furthermore, we have that
\begin{equation}
\|\hat{G}_t(\bar{z}_t; z_t) - \hat{G}_t(\bar{z}_t; \bar{z}_t)\| \leq \eta \bar{z}_t L \|z_t - \bar{z}_t\|
\end{equation}
due to the gradient deviations result in Lemma \textup{25}. Thus adding yields
\begin{equation}
\|\hat{G}_t(z_t; z_t) - \hat{G}_t(\bar{z}_t; \bar{z}_t)\| \leq (\beta_t + \eta \bar{z}_t L) \|z_t - \bar{z}_t\|.
\end{equation}

It follows from our uniform boundedness assumptions in Assumption \textup{6} that if we denote $\alpha_t := \beta_t + \eta \bar{z}_t L_t$, then $\alpha_t \leq \alpha$ for all $t \geq 0$ where $\alpha = \sqrt{1 - \eta \gamma} + \eta L$. Hence the tracking error satisfies the recursion $\|z_{t+1} - \bar{z}_{t+1}\| \leq \alpha \|z_t - \bar{z}_t\| + \Delta$. If we enforce that $\eta$ satisfies the condition in \textup{36} then $\alpha < 1$. Repeated application of the error recursion then yields
\begin{equation}
\|z_{t+1} - \bar{z}_{t+1}\| \leq \alpha^{t+1} \|z_0 - \bar{z}_0\| + \Delta \frac{1 - \alpha^{t+1}}{1 - \alpha}.
\end{equation}

We conclude that
\begin{equation}
\limsup_{t \to \infty} \|z_t - \bar{z}_t\| \leq \frac{\Delta}{1 - \alpha}.
\end{equation}

Similarly to the time-invariant case, computing the integral expression for the gradient in closed form or with high accuracy is generally not possible. In the following, we consider the case where we have access to a stochastic gradient estimator, with a sub-Weibull model for the gradient error as in Assumption \textup{6}. The online stochastic equilibrium primal-dual (OSEPD) algorithm generates a sequence
\begin{equation}
z_{t+1} = \hat{G}_t(z_t), \quad \hat{G}_t(z) = \Pi_{X_t \times Y_t} (z - \eta \Omega_t(z))
\end{equation}
where $\Omega_t(z)$ is again a stochastic estimate of the gradient (obtained through a single-sample approximation or a mini-batch gradient estimation). The performance of the OSPD method is analyzed next.

**Lemma 3.2. (OSEPD Error Bounds)** Let assumptions \textup{6} hold and suppose that $\frac{\eta}{\gamma} < 1$. Furthermore, suppose that the norm of the error of the gradient $\xi_t$ is distributed according to a sub-Weibull rv as in Assumption \textup{6} for some tail parameter $\theta$ and parameters $\{\nu_t\}_{t \geq 0}$. Then, the following holds.
1. The sequence of iterates \( z_{t+1} = \hat{G}(z_t) \) satisfies the stochastic recursion
\[
\|z_{t+1} - \bar{z}_{t+1}\| \leq \alpha \|z_t - \bar{z}_t\| + \Delta + \eta \xi_t
\] (54)
where \( \xi_t = \xi(z_t) \) and \( \alpha = \beta + \eta \epsilon L \).

2. The sequence \( \{z_t\}_{t \geq 0} \) satisfies the expectation bound
\[
\mathbb{E}\|z_{t+1} - \bar{z}_{t+1}\| \leq \alpha^{t+1}\|z_0 - \bar{z}_0\| + \sum_{i=0}^{t} \alpha^i (\Delta + \eta \mathbb{E}[\xi_{t-i}]).
\]
for all \( t \geq 0 \).

3. If \( \delta \in (0, 1) \), then
\[
\mathbb{P}\left( \|z_{t+1} - \bar{z}_{t+1}\| \leq \alpha^{t+1}\|z_0 - \bar{z}_0\| + \Delta + \sum_{i=0}^{t} \eta \alpha^i \log(\frac{2}{\delta}) \right) \geq 1 - \delta
\]
for all \( t \geq 0 \).

**Proof.** The tracking error satisfies
\[
\|z_{t+1} - \bar{z}_{t+1}\| \leq \|z_{t+1} - \bar{z}_t\| + \|\bar{z}_t - \bar{z}_{t+1}\| = \|\hat{G}_t(z_t) - \hat{G}_t(\bar{z}_t; \bar{z}_t)\| + \Delta_t.
\]
Another application of the triangle inequality yields
\[
\|\hat{G}_t(z_t) - \hat{G}_t(\bar{z}_t; \bar{z}_t)\| \leq \|\hat{G}_t(z_t) - \hat{G}_t(z_t; z_t)\| + \|\hat{G}_t(z_t; z_t) - \hat{G}_t(\bar{z}_t; \bar{z}_t)\|,
\]
So that by applying Assumption 6.6 and the inequality in (51), we get that the tracking error satisfies
\[
\|z_{t+1} - \bar{z}_{t+1}\| \leq (\beta_t + \eta \epsilon L_t) \|z_t - \bar{z}_t\| + \eta \xi_t + \Delta_t.
\]
where \( \xi_t = \xi_t(z_t) \) is the stochastic sequence of gradient errors. Applying our boundedness assumption yields the simplified recursion \( \|z_{t+1} - \bar{z}_{t+1}\| \leq \alpha \|z_t - \bar{z}_t\| + \Delta + \eta \xi_t \). Repeated application yields
\[
\|z_{t+1} - \bar{z}_{t+1}\| \leq \alpha^{t+1}\|z_0 - \bar{z}_0\| + \sum_{i=0}^{t} \alpha^i (\Delta + \eta \xi_{t-i}).
\]
(55)

To obtain the expectation bound, we simply take the expectation of both sides. To obtain the high probability bound, fix \( \delta \in (0, 1) \) and set
\[
\ell_t = \alpha^{t+1}\|z_0 - \bar{z}_0\| + \Delta \sum_{i=0}^{t} \alpha^i \text{ and } \xi_t = \eta \sum_{i=0}^{t} \alpha^i \xi_{t-i}.
\]
The remainder of the proof follows as in Lemma 2.13.

**Theorem 3.3.** *(OSEPD Tracking)* Let Assumptions 1-5 hold and suppose \( \eta \) satisfies the condition in (36). Then the sequence of iterates \( \{z_t\}_{t \geq 0} \) converges to a neighborhood of \( \bar{z} \) in expectation and almost surely. That is,
\[
\lim_{t \to \infty} \mathbb{E}[\|z_t - \bar{z}_t\|] \leq \frac{\Delta + \eta \xi}{1 - \alpha}
\]
and
\[
\mathbb{P}\left( \lim_{t \to \infty} \|z_t - \bar{z}_t\| \leq \frac{\Delta + \eta \xi}{1 - \alpha} \right) = 1.
\]
Proof. The proof proceeds in a similar fashion to that of Theorem 2.14. From the expectation bound in Lemma 3.2 and the step size condition, have that
\[ \mathbb{E}[\|z_{t+1} - \tilde{z}_{t+1}\|] \leq \alpha^{t+1} \|z_0 - \tilde{z}_0\| + (\Delta + \eta \xi) \frac{1 - \alpha^{t+1}}{1 - \alpha}. \]
By taking the limit supremum of both sides, we obtain the first result. To proceed, we define a new sequence of random variables given by
\[ E_t = \max \left\{ 0, \|z_{t+1} - \tilde{z}_{t+1}\| - (\Delta + \eta \xi) \frac{1 - \alpha^{t+1}}{1 - \alpha} \right\} \]
so that the above becomes \( \mathbb{E}[E_t] \leq \alpha^{t+1} \|z_0 - \tilde{z}_0\| \). The remained of the proof follows as in Theorem 2.14.

4 Saddle Points and Mixture Dominance

By introducing the equilibrium point problem, we have shifted the attention to a class of solutions that are less computationally burdensome to obtain while still serving as meaningful solutions within the context of decision-dependent stochastic problems. In this section, we demonstrate that finding saddle points is still possible for some well-behaved distributional maps. In particular, we consider a condition which we call opposing mixture dominance (we use this term because we require the distribution induced by a convex combination to dominate the convex mixture in the convex variable and the opposite to be true for the concave variable).

To outline the main arguments, we focus on the static saddle-point problem (12) (the technical arguments straightforwardly extends to the time-varying setting). We show that if one has knowledge of the probability density associated with the distributional map or access to a suitable gradient oracle, arguments straightforwardly extends to the time-varying setting. We show that if one has knowledge of the probability density associated with the distributional map or access to a suitable gradient oracle, then we can obtain convergence results similar to those in Theorem 2.9 and 2.14.

In following, we define the notion of opposing mixture dominance.

**Assumption 7. (Opposing Mixture Dominance)** For any \( x, x', x_0 \in \mathcal{X}, y, y', y_0 \in \mathcal{Y} \) and \( \tau \in [0, 1] \),

distributional map satisfies a convex shift in \( x \)
\[ \mathbb{E}_{w \sim D(\tau x + (1 - \tau)x', y)} [\phi(x_0, y_0, w)] \leq \mathbb{E}_{w \sim \tau D(x, y) + (1 - \tau)D(x', y)} [\phi(x_0, y_0, w)], \tag{57} \]
and concave shift in \( y \)
\[ \mathbb{E}_{w \sim \tau D(x, y) + (1 - \tau)D(x', y)} [\phi(x_0, y_0, w)] \leq \mathbb{E}_{w \sim D(x, \tau y + (1 - \tau)y')} [\phi(x_0, y_0, w)]. \tag{58} \]

As an example, we show how a class of Bernoulli mixtures satisfies this assumption.

**Example 1. (Bernoulli Mixtures)** If the distributional map \( D : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{P}(M) \) is given by \( D(x, y) = \text{Bernoulli}(p(x, y)) \) where \( p : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R} \) is convex-concave and \( 0 \leq p(x, y) \leq 1 \) for all \( (x, y) \in \mathcal{X} \times \mathcal{Y} \), then Assumption 7 is satisfied for any cost \( \phi \). Indeed, if \( \tau \in (0, 1) \) and \( x, x' \in \mathcal{X} \), then
\[ \mathbb{E}_{w \sim D(\tau x + (1 - \tau)x', y)} [\phi(x_0, y_0, w)] = \phi(x_0, y_0, w) p(\tau x + (1 - \tau)x', y) \]
\[ \leq \tau \phi(x_0, y_0, w) p(x, y) + (1 - \tau) \phi(x_0, y_0, w) p(x', y) \]
\[ = \tau \mathbb{E}_{w \sim D(x, y)} [\phi(x_0, y_0, w)] + (1 - \tau) \mathbb{E}_{w \sim D(x', y)} [\phi(x_0, y_0, w)]. \]
A similar argument holds for the shift in the \( y \) argument. Furthermore, the inequality in (57) and (58) is tight for \( p(x, y) = c + a^T x + b^T y + x^T A y \) where \( c \in \mathbb{R}, a \in \mathbb{R}^d, b \in \mathbb{R}^n \), and \( A \in \mathbb{R}^{d \times n} \).

In the previous section, we made the assumption that our random variables are supported on some general Polish space and are induced by a Radon probability measure parameterized by \( z = (x, y) \in \mathcal{X} \times \mathcal{Y} \). Here, we assume without loss of generality that the distributional map induces a probability density function \( p(w; x, y) \) and write the objective as
\[ \Phi(x, y) = \int_M \phi(x, y, w)p(w; x, y)dw. \tag{59} \]
Indeed, the analysis that follows is identical for the case when the density \( p(w, x, y) \) corresponds to discrete probability distribution parameterized by \((x, y)\) and the proofs follow mutatis mutandis.

Below, we demonstrate that the opposing mixed dominance assumption is sufficient to guarantee that the objective is convex-concave in the distribution inducing arguments. The crux of this proof is observing that convex combinations of probability distributions have a density function defined by the convex combination of the underlying density functions.

**Lemma 4.1.** Let Assumptions [ hold. Then, for any \( z_0 \in \mathcal{X} \times \mathcal{Y} \), the function \((x, y) \mapsto \mathbb{E}_{w \sim D(x, y)}[\phi(z_0, w)] \) is convex-concave over \( \mathcal{X} \times \mathcal{Y} \).

**Proof.** Fix \( z_0 \in \mathcal{X} \times \mathcal{Y}, x, x' \in \mathcal{X}, \) and \( y, y' \in \mathcal{Y} \) and let \( \tau \in [0, 1] \). Observe that since the distribution \( \tau D(x, y) + (1 - \tau)D(x', y) \) is a convex mixture, then its probability density function is convex sum of the probability density functions for \( D(x, y) \) and \( D(x', y) \). That is, if \( p_\tau \) is the density function for the convex mixture, and \( p_1 \) and \( p_2 \) are the density functions for \( D(x, y) \) and \( D(x', y) \), respectively, then \( p_\tau(w) = \tau p_1(w) + (1 - \tau)p_2(w) \). From this, we conclude that

\[
\mathbb{E}_{w \sim \tau D(x, y) + (1 - \tau)D(x', y)}[\phi(z_0, w)] \leq \tau \mathbb{E}_{w \sim D(x, y)}[\phi(z_0, w)] + (1 - \tau) \mathbb{E}_{w \sim D(x', y)}[\phi(z_0, w)].
\]

Combining this with Assumption [ we get that

\[
\mathbb{E}_{w \sim D(x, y)}[\phi(z_0, w)] \leq \tau \mathbb{E}_{w \sim D(x, y)}[\phi(z_0, w)] + (1 - \tau) \mathbb{E}_{w \sim D(x', y)}[\phi(z_0, w)].
\]

This proves convexity of \( x \mapsto \mathbb{E}_{w \sim D(x, y)}[\phi(z_0, w)] \) for any \( y \). The concavity in \( y \) can be shown using similar steps. \( \square \)

We can then utilize this result in conjunction with our previous assumptions to get strong-convexity-strong-concavity of the objective \( \Phi \).

**Theorem 4.2. (Strong-Convexity-Strong-Concavity)** Suppose that Assumptions [ hold. Then, \((x, y) \mapsto \Phi(x, y)\) is \((\gamma - 2\varepsilon L)\)-strong-convex-strongly-concave over \( \mathcal{X} \times \mathcal{Y} \).

**Proof.** We prove the assertion by first demonstration that strong-convexity holds in \( x \) for \( y \) fixed. Strong-concavity will follow similarly. By applying \( \gamma \)-strong-convexity of \( \phi \) in \( x \), we get that

\[
\mathbb{E}_{w \sim D(x', y)}[\phi(x', y, w)] - \mathbb{E}_{w \sim D(x, y)}[\phi(x, y, w)] \geq \langle x' - x, \mathbb{E}_{w \sim D(x', y)}[\nabla_x \phi(x, y, w)] \rangle + \frac{\gamma}{2} \|x - x'\|^2. \tag{60}
\]

Following \( L \) smoothness of the gradient, we get that

\[
\langle x' - x, \mathbb{E}_{w \sim D(x, y)}[\nabla_x \phi(x, y, w)] \rangle - \mathbb{E}_{w \sim D(x', y)}[\nabla_x \phi(x', y, w)] \leq \varepsilon L \|x - x'\|^2
\]

which is equivalent to

\[
0 \geq \langle x' - x, \mathbb{E}_{w \sim D(x, y)}[\nabla_x \phi(x, y, w)] \rangle - \mathbb{E}_{w \sim D(x', y)}[\nabla_x \phi(x, y, w)] - \frac{2\varepsilon L}{2} \|x - x'\|^2. \tag{61}
\]

Since for any \( z_0 \in \mathcal{X} \times \mathcal{Y} \) the function \((x, y) \mapsto \mathbb{E}_{w \sim D(x, y)}[\phi(z_0, w)] \) is convex-concave, we have that

\[
\mathbb{E}_{w \sim D(x, y)}[\phi(x, y, w)] - \mathbb{E}_{w \sim D(x', y)}[\phi(x, y, w)] \geq \langle x' - x, \mathbb{E}_{w \sim D(x', y)}[\phi(x, y, w) \nabla_x \log p(w; x, y)] \rangle \tag{62}
\]

by setting \( z_0 = (x, y) \). By adding inequalities \(60-62\) we obtain

\[
\mathbb{E}_{w \sim D(x', y)}[\phi(x', y, w)] - \mathbb{E}_{w \sim D(x, y)}[\phi(x, y, w)] \geq \langle x' - x, \mathbb{E}_{w \sim D(x, y)}[\nabla_x \phi(x, y, w)] \rangle + \frac{\gamma - 2\varepsilon L}{2} \|x - x'\|^2,
\]

which is equivalent to strong-convexity in \( x \). Proof of strong-concavity in \( y \) follows similarly. \( \square \)
4.1 Algorithms

In the case where full gradient information is available, the problem reduces to the classic saddle point problem; namely, we are able to compute $\hat{\Psi}(z) = (\nabla_x \Phi(z), -\nabla_y \Phi(z))$. If we denote

$$
\Psi(z) = \left( \mathbb{E}_{w \sim D(z)} [\nabla_x \phi(z, w)], \mathbb{E}_{w \sim D(z)} [-\nabla_y \phi(z, w)] \right) = \mathbb{E}_{w \sim D(z)} [\psi(z, w)]
$$

and

$$
\Lambda(z) = \left( \mathbb{E}_{w \sim D(z)} [\phi(z, w) \nabla_x \log p(w; z)], \mathbb{E}_{w \sim D(z)} [-\phi(z, w) \nabla_y \log p(w; z)] \right) = \mathbb{E}_{w \sim D(z)} [\lambda(z, w)],
$$

then, the full gradient can be written as $\hat{\Psi}(z) = \Psi(z) + \Lambda(z)$. Note that the map $\Lambda$ captures the change in the distribution due to $x$ and $y$.

If we can compute $\hat{\Psi}$, we can use the projected primal-dual algorithm, with algorithmic map given by

$$
\mathcal{F}(z) = \Pi_{\mathcal{X} \times \mathcal{Y}}(z - \eta \hat{\Psi}(z)),
$$

for all $z \in \mathbb{R}^d \times \mathbb{R}^n$. The algorithm converges provided that $\Lambda$ is Lipschitz continuous and the step-size is chosen appropriately. We present this in the following (the proof is omitted, since it follows similar steps as in the proof of Theorem 4.3).

**Theorem 4.3. (Primal-Dual Convergence under Mixture Dominance)** Suppose that Assumptions 4.3 and 4.4 hold. Furthermore, let $\frac{\epsilon_L}{\gamma} < \frac{1}{2}$ and let $\Lambda$ be $\ell$-Lipschitz. Then:

1. If $\eta \leq \frac{\gamma - 2\epsilon_L}{(L + \ell)^2}$, then

$$
\| \mathcal{F}(z) - \mathcal{F}(z') \| \leq \sqrt{1 - \eta(\gamma - 2\epsilon_L)} \| z - z' \|
$$

for any $z, z' \in \mathcal{X} \times \mathcal{Y}$.

2. The sequence $z_{t+1} = \mathcal{F}(z_t)$ converges linearly to the unique equilibrium point $\bar{z}$ provided that

$$
0 < \eta \leq \frac{\gamma - 2\epsilon_L}{(L + \ell)^2}.
$$

While having access to $\Psi$ or $\psi$ is a realistic assumption, explicit knowledge of $\Lambda$ may not be available. For this reason, we consider a stochastic gradient oracle that is a function of both $\psi$ and $\lambda$. We then consider an oracle $\Xi : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^d \times \mathbb{R}^n$, which is assumed to be some function of $\psi$ and $\lambda$. Examples include, but are not limited to:

$$
\Xi(z) = \begin{cases} 
\psi(z, w_1) + \lambda(z, w_1), & w_1 \sim D(z) \\
\frac{1}{N} \sum_{i=1}^N \psi(z, w_i) + \lambda(z, w_i), & w_1, \ldots, w_N \sim D(z) \\
\hat{\Psi}(z; \epsilon) + \epsilon \sim \mathcal{N}(0, \sigma^2 I)
\end{cases}
$$

which represent single-point feedback estimate, mini-batch estimate, and generic black-box estimates. Then, we can formulate a stochastic gradient algorithm with map $\hat{\mathcal{F}} : \mathbb{R}^d \times \mathbb{R}^n \to \mathcal{X} \times \mathcal{Y}$ defined by

$$
\hat{\mathcal{F}}(z) = \Pi_{\mathcal{X} \times \mathcal{Y}}(z - \eta \Xi(z))
$$

for step-size $\eta > 0$. Following our analysis of the stochastic primal-dual method in Section 2.5, we assume that the gradient error follows a sub-Weibull error model of the form:

$$
\tilde{\xi}(z) = \| \Xi(z) - \hat{\Psi}(z) \| \sim \text{subW}(\theta, \nu(z))
$$

for tail-parameter $\theta > 0$ and bounded variance proxy function $\nu : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}$. Since this is a natural extension of our work on the stochastic primal-dual for the equilibrium point problem, two analogous results can be provided.

**Lemma 4.4. (SPD Error Bounds)** Let assumptions 4.3 hold. Suppose that $\frac{\epsilon_L}{\gamma} < \frac{1}{2}$ and that the step size $\eta$ satisfies the condition in (67). Then:
1. The sequence of iterates $z_{t+1} = \bar{F}(z_t)$ satisfies the stochastic recursion
\[ \|z_{t+1} - z^*\| \leq \rho \|z_t - z^*\| + \eta \xi_t \]
where $\xi_t = \xi(z_t)$ and $\rho = \sqrt{1 - \eta (\gamma - 2\epsilon L)}$.

2. The sequence $\{z_t\}_{t \geq 0}$ satisfies the expectation bound
\[ E \|z_{t+1} - z^*\| \leq \rho^{t+1} \|z_0 - z^*\| + \sum_{i=0}^{t} \rho^i E[\xi_{t-i}]. \]
for all $t \geq 0$.

3. If $\delta \in (0, 1)$, then
\[ P \left( \|z_{t+1} - z^*\| \leq \rho^{t+1} \|z_0 - z^*\| + c(\theta) \log^\theta \left( \frac{2}{\delta} \right) \sum_{i=0}^{t} \rho^i \nu_{t-i} \right) \geq 1 - \delta \]
for all $t \geq 0$.

**Theorem 4.5.** Let Assumptions 1-5 hold. Then the sequence of iterates $\{z_t\}_{t \geq 0}$ converges to a neighborhood of $\bar{z}$ in expectation and almost surely. That is,
\[ \limsup_{t \to \infty} E \|z_t - z^*\| \leq \frac{\eta \bar{\xi}}{1 - \rho}, \]
and
\[ P \left( \limsup_{t \to \infty} \|z_t - z^*\| \leq \frac{\eta \bar{\xi}}{1 - \rho} \right) = 1. \]

We conclude by noting that one can extend the results of Lemma 4.4 and Theorem 4.5 to the online stochastic primal-dual for the time-varying problem in (1). We omit the statement of these results as they are similar to Theorems 3.1 and 3.3.

5 Concluding Remarks

The paper focused on stochastic saddle point problems with decision-dependent distributions. We introduced the notion of equilibrium points and provide conditions for their existence and uniqueness. We showed that the distance between the two classes of solutions is bounded provided that the objective has a strongly-convex-strongly-concave payoff and Lipschitz continuous distributional map. We developed and analyzed deterministic and stochastic primal-dual algorithms. In particular, using a sub-Weibull model for the errors emerging in the gradient computation, we provided error bounds in expectation and in high probability that hold for each iteration; we also showed convergence to a neighborhood in expectation and almost surely. Finally, we investigate an opposing mixture dominance condition that ensures the objective is strongly-convex-strongly-concave. In this context, we provided convergence results for primal-dual methods.

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A Background

A.1 Saddle Point Problems

The connection between minimax problems and variational inequalities is well known, however a comprehensive review of this fact is not well documented outside the context of the more general variational analysis theory. Our goal in this section is to demonstrate this connection as well as its importance for convex-concave minimax problems over compact sets. The results provided here will then be critical in the analysis of our specific stochastic optimization problem.

The power between this connection is that, under suitable differentiability and minimax equality assumptions, characterizing solutions of the minimax problem can be replaced with showing properties of the associated gradient map. Then the fixed point theory that is ubiquitous throughout optimization can be applied to study this problem.

Throughout this section, we will shift our attention away from the problem statement in (1), and instead consider the more general minimax problem given by

\[
\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} \Phi(x, y) \tag{70}
\]

where the function \( \Phi : \mathcal{X} \times \mathcal{Y} \mapsto \mathbb{R} \) is a jointly continuous and differentiable concave-convex function. That is, \( x \mapsto \Phi(x, y) \) is convex for all \( y \in \mathcal{Y} \) and \( y \mapsto \Phi(x, y) \) is concave for all \( x \in \mathcal{X} \). The sets \( \mathcal{X} \subset \mathbb{R}^d \) and \( \mathcal{Y} \subset \mathbb{R}^n \) are convex and compact.

Definition 6. A saddle point \((x^*, y^*) \in \mathcal{X} \times \mathcal{Y}\) for \( \Phi \) is a point for which

\[
\Phi(x^*, y) \leq \Phi(x^*, y^*) \leq \Phi(x, y^*) \tag{71}
\]

for all \( x \in \mathcal{X} \) and \( y \in \mathcal{Y} \).

This contrasts the typical notion of optimizers for a convex function since here saddle points merely constitute relative optimizers. That is, \( x^* \) is a minimizer for the function \( x \mapsto \Phi(x, y^*) \) and \( y^* \) is a maximizer for \( y \mapsto \Phi(x^*, y) \). A common way to parse this fact in the literature is to introduce the functions

\[
f(x) = \max_{y \in \mathcal{Y}} \Phi(x, y) \quad \text{and} \quad g(y) = \min_{x \in \mathcal{X}} \Phi(x, y). \tag{72}
\]

where \( f \) is referred to as the primal function and \( g \) the dual. The minimax inequality then states that, in general \( \max_{y \in \mathcal{Y}} g(y) \leq \min_{x \in \mathcal{X}} f(x) \). Conditions for which equality holds are well studied in the Minimax Theorems. The conditions we placed on \( \Phi \) and \( \mathcal{X} \times \mathcal{Y} \) in (70) are sufficient for minimax equality to hold, and in fact are stronger than what is required \[50\]. Under this condition, we can characterize the saddle points via \( x^* \in \arg \min_{x \in \mathcal{X}} f(x) \) and \( y^* \in \arg \max_{y \in \mathcal{Y}} g(y) \). We summarize this fact in the following theorem.

Theorem A.1. (Saddle Point Characterization) Suppose that minimax equality holds. A pair \((x^*, y^*) \in \mathcal{X} \times \mathcal{Y}\) is a saddle point for \( \Phi \) if and only if and

\[
x^* \in \arg \min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} \Phi(x, y)
\]

\[
y^* \in \arg \min_{y \in \mathcal{Y}} \max_{x \in \mathcal{X}} \Phi(x, y).
\]
Proof of this result can be found in [45, Theorem 7.6]. This Theorem provides us with our primary means of characterizing saddle point for convex-concave functions. In what follows, we demonstrate that saddle points for Φ are solutions to a variational inequality. Making this connection allows us to cast (76) as a variational inequality problem for which existence and uniqueness conditions can be readily prescribed. Furthermore, we can use the framework to inspire natural algorithms such as the projection method, which is essentially the analog of projected gradient descent in this setting.

**Theorem A.2. (Variational Inequality Equivalence) Suppose that Φ : X × Y → R is jointly continuous and differentiable, and convex-concave on the compact convex domain X × Y. Denote the gradient map of Φ as the function Ψ : X × Y → X × Y defined by

\[ Ψ(z) = (\nabla_x Φ(z), -\nabla_y Φ(z)) \]  

for all z = (x, y) ∈ X × Y. Then, \( z^* = (x^*, y^*) \in X \times Y \) is a saddle point of Φ if and only if

\[ \langle z - z^*, Ψ(z^*) \rangle \geq 0. \]  

for all \( z \in X \times Y \).

For proof of this result, see [45, Example 12.50]. The proof amounts to observing that the set-valued map \( T(z) = Ψ(z) + N_{X \times Y}(z) \) is monotone and that \( z^* = (x^*, y^*) \in X \times Y \) is a saddle point if and only if \( 0 \in T(z^*) \). Here \( N_{X \times Y}(z^*) = \{ g ∈ X \times Y | \langle z - z^*, g \rangle ≤ 0, \forall z ∈ X \times Y \} \) denotes the normal cone of \( X \times Y \) at \( z \).

### A.2 Strongly-Monotone Maps

Necessary to the remainder of our analysis is the notion of strongly monotone maps. These maps are crucial in the study of variational inequalities, where strong-monotonicity of the map for a variational inequality implies existence and uniqueness of solutions. Hence, they constitute the best case scenario for our analysis.

**Definition 7. (Strongly-Monotone Map) A map Ψ : Z → Z where Z ⊆ R^m is \( γ \)-strongly-monotone provided that

\[ \langle z - z', Ψ(z) - Ψ(z') \rangle \geq γ\|z - z'\|^2 \]  

for all \( z, z' \in Z \), where \( γ > 0 \).

The relationship between strong-monotonicity and strong-convexity (and hence strong-concavity) is well known within the optimization literature. Namely, that strongly-convex differentiable functions yields strongly-monotone gradient maps. In what follows, we demonstrate a similar relationship for our setting.

**Proposition A.3. (Strongly-Monotone Gradient Map Characterization) Suppose that Φ : X × Y → R is continuously differentiable with associated gradient map Ψ : X × Y → X × Y defined by Ψ(z) = (\nabla_x Φ(z), -\nabla_y Φ(z)) for all \( z = (x, y) \in X \times Y \). Then, \( Φ \) is \( γ \)-strongly-convex-\( γ \)-strongly-concave if and only if Ψ is \( γ \)-strongly-monotone.

**Proof.** Suppose that Φ is \( γ \)-strongly-convex-\( γ \)-strongly-concave and let \( x, x' \in X, y, y' \in Y \). By \( γ \)-strong-convexity of Φ over \( X \), we have the following inequalities:

\[ Φ(x', y) ≥ Φ(x, y) + \langle x' - x, ∇_x Φ(x, y) \rangle + \frac{γ}{2}\|x - x'\|^2, \]

\[ Φ(x, y') ≥ Φ(x', y') + \langle x - x', ∇_x Φ(x', y') \rangle + \frac{γ}{2}\|x - x'\|^2. \]

Adding yields

\[ \langle x - x', ∇_x Φ(x, y) - ∇_x Φ(x', y') \rangle ≥ Φ(x, y) - Φ(x', y) + Φ(x', y') - Φ(x, y') + γ\|x - x'\|^2. \]  

(76)

Similarly,

\[ -Φ(x, y') ≥ -Φ(x, y) + \langle y' - y, -∇_y Φ(x, y) \rangle + \frac{γ}{2}\|y - y'\|^2, \]

\[ -Φ(x', y) ≥ -Φ(x', y') + \langle y' - y, -∇_y Φ(x', y') \rangle + \frac{γ}{2}\|y - y'\|^2. \]
and hence
\[-\langle y - y', \nabla_y \Phi(x, y) - \nabla_y \Phi(x', y') \rangle \geq -\Phi(x, y) + \Phi(x', y') + \Phi(x, y') + \gamma \|y - y'\|^2. \tag{77}\]

Finally, adding (76) and (77) yields
\[
\langle z - z', (\Psi(z) - \Psi(z')) \rangle = \langle x - x', \nabla_x \Phi(x, y) - \nabla_x \Phi(x', y') \rangle - \langle y - y', \nabla_y \Phi(x, y) - \nabla_y \Phi(x', y') \rangle \\
\geq \gamma \left( \|x - x'\|^2 + \|y - y'\|^2 \right) \\
= \gamma \|z - z'\|^2.
\]

where \(z = (x, y)\) and \(z' = (x', y')\).

Conversely, suppose that \(\Psi\) is \(\gamma\) strongly monotone and let \(z_1 = (x, y), z_2 = (x', y)\), and \(z_3 = (x, y')\). It follows that
\[
\langle x - x', \nabla_x \Phi(x, y) - \nabla_x \Phi(x', y) \rangle = \langle z_1 - z_2, \Psi(z_1) - \Psi(z_2) \rangle \geq \gamma \|z_1 - z_2\|^2 = \gamma \|x - x'\|^2,
\]
and
\[
-\langle y - y', \nabla_y \Phi(x, y) - \nabla_y \Phi(x, y') \rangle = \langle z_1 - z_3, \Psi(z_2) - \Psi(z_3) \rangle \geq \gamma \|z_2 - z_3\|^2 = \gamma \|y - y'\|^2.
\]

\(\square\)