Virasoro hair and entropy for axisymmetric Killing horizons

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We show that the gravitational phase space for the near-horizon region of a bifurcate, axisymmetric Killing horizon in any dimension admits a 2D conformal symmetry algebra with central charges proportional to the area. This extends the construction of [Haco et. al., JHEP 12, 098 (2018)] to generic Killing horizons appearing in solutions of Einstein’s equations, and motivates a holographic description in terms of a 2D conformal field theory. The Cardy entropy in such a field theory agrees with the Bekenstein-Hawking entropy of the horizon, suggesting a microscopic interpretation.

I. INTRODUCTION

The Bekenstein-Hawking black hole entropy $S_{\text{BH}} = A/4G$ [1–3] presents a challenge to quantum gravity to provide a microscopic explanation. One proposal is that the entropy counts edge degrees of freedom living on the horizon, and is controlled by boundary symmetries [4–6]. This idea was strikingly realized in Strominger’s derivation of the BTZ black hole entropy, using the Cardy formula for a conformal field theory (CFT) with the Brown-Henneaux central charge $\frac{c}{2}$. Much subsequent work has been devoted to generalizing this construction to other contexts.

Carlip in particular demonstrated that the conformal symmetries were not special to AdS$_3$ black holes; rather, they arise for generic Killing horizons. In all cases, postulating a CFT description led to agreement between the Cardy entropy and $S_{\text{BH}}$ [11, 12]. Although very insightful, certain aspects of Carlip’s construction raised additional questions. The symmetry generators had to satisfy periodicity conditions whose justifications were obscure, and only a single copy of the Virasoro algebra was found, whereas the 2D conformal algebra consists of two copies, $\text{Vir}_R \times \text{Vir}_L$ [13, 14]. The Kerr/CFT correspondence [15, 16] provided some clarity, by making the connection between near-horizon symmetries and holographic duality more explicit, allowing intuition from AdS/CFT to be applied. As a byproduct, it also motivated a different choice of near-horizon symmetry generators whose periodicities followed from the rotational symmetry of the Kerr black hole, thereby resolving one issue in Carlip’s original construction [17, 18]. Another significant advancement came from Haco, Hawking, Perry, and Strominger (HHPS) [19], who exhibited a full set of $\text{Vir}_R \times \text{Vir}_L$ symmetries for Kerr black holes of arbitrary nonzero spin. This work was generalized to Schwarzschild black holes using a different collection of symmetry generators in [14].

The present work will demonstrate that arbitrary, bifurcate, axisymmetric Killing horizons possess a full set of conformal symmetries, which act on edge degrees of freedom, or “hairs” [20, 21], on the horizon. By constructing a set of conformal coordinates which foliate the near-horizon region by locally AdS$_3$ geometries, we show that vector fields satisfying a Witt$_R \times$ Witt$_L$ algebra arise in the vicinity of the horizon. The coordinates depend on two free parameters, $\alpha$ and $\beta$, which are related to the CFT temperatures $T_R, T_L$ using properties of the near horizon vacuum. When the symmetries generated by the vector fields are implemented canonically on the gravitational phase space, the algebra is extended to $\text{Vir}_R \times \text{Vir}_L$, with central charges $c_R$ and $c_L$ determined in terms of the area and angular momentum of the horizon according to (31) and (32). Imposing a constraint on $\alpha$ and $\beta$, motivated by a condition related to the surface gravity variation, sets the central charges equal to each other and proportional to the horizon area according to (36). More generally, (36) applies for any choice of $\alpha$ and $\beta$ when appropriate Wald-Zoupas terms are used to define the quasilocal charges [22]. The Cardy formula [9] with the central charges (36) reproduces the entropy of the horizon, suggesting a dual description in terms of a CFT. This result therefore motivates investigations into holography for arbitrary Killing horizons, including the de Sitter cosmological horizon, and nonrotating and higher dimensional black holes.

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II. NEAR-HORIZON EXPANSION

We are interested in the form of the metric near a bifurcate, axisymmetric Killing horizon in a solution to Einstein’s equations in dimension $d \geq 3$. Axisymmetry means that, in addition to the horizon-generating Killing vector $\chi^a$, there is a commuting, rotational Killing vector $\psi^a$ with closed orbits. Axisymmetric horizons are of interest since, by the rigidity theorems, all black hole solutions are of this form [23–26]. In situations where the horizon possesses more than one rotational Killing vector, we simply single out one and proceed with the construction.

The conformal symmetries of the horizon are found by first constructing a system of “conformal coordinates”, designed to exhibit a locally AdS$_3$ factor in the metric when expanded near the bifurcation surface. The asymptotic symmetries of this AdS$_3$ factor comprise the conformal symmetries of the horizon. We first define Rindler coordinates near the bifurcation surface using a construction of Carlip [12], suitably modified to incorporate the additional rotational symmetry. The gradient of $\chi^2$ defines a radial vector

$$\rho^a = -\frac{1}{2\kappa} \nabla^a \chi^2,$$  \hspace{1cm} (1)

where $\kappa$ is the surface gravity of $\chi^a$. On the horizon, $\rho^a$ and $\chi^a$ coincide, but off the horizon, $\rho^a$ is independent. The vectors $(\chi^a, \psi^a, \rho^a)$ mutually commute, and hence can form part of a coordinate basis with coordinates $(t, \phi, r_*)$. The remaining transverse coordinates are denoted $\theta^A$. It is convenient to reparameterize the radial coordinate

$$x = \frac{1}{\kappa} e^{\kappa r_*},$$  \hspace{1cm} (2)

which has the interpretation of proper geodesic distance to the bifurcation surface to leading order near the horizon.

In these coordinates, the near-horizon metric takes on Rindler form,

$$ds^2 = -\kappa^2 x^2 dt^2 + dx^2 + \psi^2 d\phi^2 + q_{AB} d\theta^A d\theta^B - 2x^2 dt (\kappa \psi d\phi + \kappa N_A d\theta^A) + \mathcal{O}(x^2)$$  \hspace{1cm} (3)

where the higher order $\mathcal{O}(x^2)$ terms do not enter the remainder of the calculation (see appendix A for additional details on this expansion). Except for $\kappa$, all coefficients appearing in the above expansion are functions of $\theta^A$.

III. CONFORMAL COORDINATES

The conformal coordinates are now defined, in analogy to similar constructions in [19, 27], as

$$w^+ = xe^{\alpha \phi + \kappa t}$$ \hspace{1cm} (4)

$$w^- = xe^{\beta \phi - \kappa t}$$ \hspace{1cm} (5)

$$y = e^{\frac{\alpha + \beta}{2} \phi}$$ \hspace{1cm} (6)

(see appendix B for a concrete realization in the example of de Sitter space)

Here, $\alpha$ and $\beta$ are free parameters that will later be related to the left and right temperatures of the system. Because $xe^{\kappa t}$ and $xe^{-\kappa t}$ are simply the Kruskal coordinates $V, U$ near the bifurcation surface, the future horizon is at $w^- = 0$ and the past horizon is $w^+ = 0$, see Figure 1 for a visualization of the conformal coordinates in the near horizon region. Due to the periodicity $\phi \sim \phi + 2\pi$, the conformal coordinates must be identified according to

$$(w^+, w^-, y) \sim \left(e^{2\pi \alpha} w^+, e^{2\pi \beta} w^-, e^{\pi(\alpha + \beta)} y\right).$$ \hspace{1cm} (7)

The near-horizon expansion in these coordinates becomes

$$ds^2 = \frac{dw^+ dw^-}{y^2} + \frac{4\psi^2}{(\alpha + \beta)^2} \frac{dy^2}{y^2} + q_{AB} d\theta^A d\theta^B - \frac{2dy}{(\alpha + \beta) y} \left((\beta + N_\phi) w^- dw^+ + (\alpha - N_\phi) w^+ dw^-\right)$$

$$- \left(\frac{w^- dw^+}{y^2} - \frac{w^+ dw^-}{y^2}\right) \kappa N_A d\theta^A + \ldots$$ \hspace{1cm} (8)

up to higher order terms in $w^+, w^-$. The first line takes the form of a locally AdS$_3$ metric with a $\theta^A$-dependent radius of curvature $\ell = \frac{2\psi}{\alpha + \beta}$, times a transverse metric. In intuitive words, the conformal coordinates zoom in on the near horizon region through the lens of $e^{\alpha \phi}, e^{\beta \phi}$ conjoined the Kruskal coordinate, and bring out the AdS$_3$ folia explicitly.

These coordinates allow for a straightforward determination of the near horizon symmetry generators. They are the asymptotic symmetry vectors of the AdS$_3$ factor in (8), where asymptotic refers to $y \to 0$. The vectors are defined as in HHPS [19]

$$\zeta^a_r = \varepsilon(w^+) \partial^a_+ + \frac{1}{2} \varepsilon'(w^+) y \partial^a_0$$ \hspace{1cm} (9)

$$\zeta^a_\phi = \varepsilon(w^-) \partial^a_\phi + \frac{1}{2} \varepsilon'(w^-) y \partial^a_0.$$ \hspace{1cm} (10)

1 In the case of Kerr, these coordinates are rescaled by a function of $\theta$ from the coordinates used by HHPS; however, doing so does not substantially change the construction. Also, $\phi$ is the comoving angular variable, as opposed to the standard Boyer-Lindquist $\phi$. 
and one can readily verify that the Lie derivative of the first line of (8) with respect to these vectors vanishes up to \( O(y^{-2}) \) terms.\(^2\) A priori, \( \varepsilon(w^+) \) and \( \bar{\varepsilon}(w^-) \) are arbitrary functions, but in light of the periodicity condition (7), the vector fields are single-valued only when \( \varepsilon(w^+)e^{2\pi \alpha} = \varepsilon(w^+)e^{2\pi \alpha} \), \( \bar{\varepsilon}(w^-)e^{2\pi \beta} = \bar{\varepsilon}(w^-)e^{2\pi \beta} \). A basis for such functions is
\[
\varepsilon_n(w^+) = \alpha(w^+)^{1+\frac{in}{\pi}}, \quad (11)
\bar{\varepsilon}_n(w^-) = -\beta(w^-)^{1-\frac{in}{\pi}}, \quad (12)
\]
and their corresponding generators will be labeled as \( \zeta^a_n \), \( \bar{\zeta}^a_n \). The algebra satisfied by these vector fields upon taking Lie brackets is two commuting copies of the Witt algebra,
\[
[\zeta_m, \zeta_n] = i(n-m)\zeta_{m+n}, \quad (13)
\]
\[
[\bar{\zeta}_m, \bar{\zeta}_n] = i(n-m)\bar{\zeta}_{m+n}. \quad (14)
\]
The generators are defined in a neighborhood of the bifurcation surface, but oscillate wildly as it is approached. The \( \zeta^a_n \) generators are regular on the future horizon but not the past, and similarly \( \bar{\zeta}^a_n \) are regular at the past horizon, but not the future. The zero mode generators \( \zeta^a_0 \) and \( \bar{\zeta}^a_0 \) are regular everywhere, given by two helical Killing vectors
\[
\zeta^a_0 = \frac{\alpha}{\alpha+\beta} \left( \frac{\beta}{\kappa} \chi^a + \psi^a \right), \quad (15)
\]
\[
\bar{\zeta}^a_0 = \frac{\beta}{\alpha+\beta} \left( \frac{\alpha}{\kappa} \chi^a - \psi^a \right). \quad (16)
\]

The expressions (15) lead to the interpretation of \( \alpha \) and \( \beta \) in terms of the right and left temperatures. The analog of the Frolov-Thorne vacuum [31] for quantum fields near the bifurcation surface is thermal with respect to the \( \chi^a \) Killing vector. The density matrix is therefore of the form
\[
\rho \sim \exp \left( -\frac{2\pi}{\alpha} \omega_\chi \right), \quad \text{where} \quad \omega_\chi = -k_\alpha \chi^a \text{ is the frequency with respect to } \chi^a \text{ for a wavevector } k_\alpha. \quad \text{Reexpressing it in terms of } \zeta^a_n, \bar{\zeta}^a_n \text{ frequencies via}
\]
\[
\omega_\chi = -k_\alpha \left( \frac{\kappa}{\alpha} \zeta^a_0 + \frac{\kappa}{\beta} \bar{\zeta}^a_0 \right) = \frac{\kappa}{\alpha} \omega_R + \frac{\kappa}{\beta} \omega_L \quad (17)
\]
shows that \( \rho \sim \exp \left( -\frac{2\pi}{\alpha} \omega_R - \frac{2\pi}{\beta} \omega_L \right) \), allowing us to read off the temperatures \( (T_R, T_L) = (\frac{\alpha}{2\pi}, \frac{\beta}{2\pi}) \) as the thermodynamic potentials conjugate to \( \zeta^a_0, \bar{\zeta}^a_0 \).

IV. CENTRAL CHARGES

Having identified the near-horizon symmetry generators (9) and (10), the next step is to implement them on the gravitational phase space. This involves identifying Hamiltonians \( H_n, \bar{H}_n \) that generate the symmetries associated with \( \zeta^a_n, \bar{\zeta}^a_n \), meaning
\[
\delta H_n = \Omega(\delta g_{ab}, \Lambda_{\zeta_n} g_{ab}) \quad (18)
\]
where \( \Omega \) is the symplectic form of the phase space.

Assuming integrable Hamiltonians can be found, their Poisson brackets automatically reproduce the algebra satisfied by the vector fields (14), up to central extensions,
\[
\{H_m, H_n\} = -i \left( (n-m) H_{m+n} + K_R(m,n) \right) \quad (19)
\]
\[
\{\bar{H}_m, \bar{H}_n\} = -i \left( (n-m) \bar{H}_{m+n} + K_L(m,n) \right) \quad (20)
\]
\[
\{H_m, \bar{H}_n\} = 0. \quad (21)
\]
Since the Witt algebra has a unique central extension to Virasoro, the central terms in the above expression must be of the form
\[
K_{R,L}(m,n) = \frac{c_{R,L}}{12} (m^3 - m) \delta_{m+n,0} \quad (22)
\]
where the constants \( c_{R,L} \) are the central charges.
Using the covariant phase space formalism and standard Iyer-Wald identities [32–35], the right hand side of (18) can be expressed on-shell as

$$\Omega(\delta g_{ab}, \mathcal{L}_{\zeta_n} g_{ab}) = \int_{\partial \Sigma} (\delta Q_{\zeta_n} - i_{\zeta_n} \theta)$$

(23)

where the integral is over the boundary of a Cauchy surface $\Sigma$ for the exterior region of the Killing horizon. The other quantities appearing in (23) are the Noether potential $(d - 2)$-form

$$Q_{\zeta_n} = -\frac{1}{16\pi G} \xi^a_b \nabla_a \xi^b_n$$

(24)

and the symplectic potential $(d - 1)$-form,

$$\theta = \frac{1}{16\pi G} \epsilon^a (\nabla_b \delta g_{ab} - g^{bc} \nabla_a \delta g_{bc}) .$$

(25)

In these expressions, $\epsilon$ denotes the spacetime volume form, with uncontracted indices not displayed.

The zero mode generators $\zeta^a_0, \xi^a_0$ are Killing vectors whose corresponding Hamiltonians are

$$H_0 = \frac{\alpha}{\alpha + \beta} \left( \frac{\beta A}{8\pi G} + J_H \right)$$

$$\bar{H}_0 = \frac{\beta}{\alpha + \beta} \left( \frac{\alpha A}{8\pi G} - J_H \right) ,$$

(26)

(27)

where $A$ is the area of the bifurcation surface $\mathcal{B}$, and

$$J_H \equiv \int_{\mathcal{B}} Q_\psi = \frac{1}{4G} \int d\theta^A \sqrt{q} |\psi| N_\phi$$

(28)

is the angular momentum of the Killing horizon. $J_H$ agrees with the total angular momentum $J$ in an asymptotically flat vacuum solution, but generally differs when matter is present outside the horizon [36].

The Poisson bracket only involves variations of the Hamiltonians, and hence can be computed directly from (23),

$$\{H_m, H_n\} = \delta_{\zeta_m} H_n = \int_{\partial \Sigma} \left( \delta_{\zeta_m} Q_{\zeta_n} - i_{\zeta_m} \theta [\mathcal{L}_{\zeta_n} g] \right) .$$

(29)

According to (19) and (22), the central charge appears as the coefficient of $m^3$ term in $\{H_m, H_{-m}\}$. Because of the singular limit in the generators $c^a_m$, the integral cannot be evaluated directly on the bifurcation surface. Instead, we work on a cutoff surface at constant $x$ and $t$, and perform the $\phi$ integration before taking the limit $x \to 0$. This results in

$$c_R = \frac{6}{G(\alpha + \beta)^2} \int d\theta^A \sqrt{q} |\psi| (\beta + N_\phi) .$$

(30)

The first term is simply proportional to the area of the horizon, $A = 2\pi \int d\theta^A \sqrt{q} |\psi|$, while the second term is proportional to $J_H$ (28). In terms of these familiar quantities, $c_R$ may be expressed,

$$c_R = \frac{24}{(\alpha + \beta)^2} \left( \frac{\beta A}{8\pi G} + J_H \right) = \frac{24}{\alpha + \beta} \frac{H_0}{\alpha} .$$

(31)

An analogous calculation for $\{\bar{H}_m, \bar{H}_{-m}\}$ yields

$$c_L = \frac{24}{(\alpha + \beta)^2} \left( \alpha A \frac{8\pi G}{8\pi G} - J_H \right) = \frac{24}{\alpha + \beta} \frac{H_0}{\beta} .$$

(32)

V. TEMPERATURE CONDITION

Up to now, we have carried out the full calculation with arbitrary temperatures. However, different choices of $(\alpha, \beta)$ preserve different horizon boundary conditions for the phase space. Earlier work on killing horizon symmetries [12, 37] employed a fixed $\kappa$ boundary condition, which, if relaxed in a controlled way, for example in [38], can result in extended symmetries. The variation of the surface gravity is determined by (1), holding $\chi^a$ and $\rho^a$ fixed, which gives

$$\delta \kappa = \rho^a \nabla_c (\chi^a \chi^b \delta g_{ab}) .$$

(33)

By the nature of the vector fields (9), the diffeomorphisms generated by $\zeta_n$ lead to $\delta_{\zeta_n} \kappa \neq 0$ pointwise. However, a modest relaxation is to demand the average of $\delta_{\zeta_n} \kappa$ vanishes when integrated over any transverse dimension $\theta^A$ for each angle $\phi$:

$$\int d\theta^A \sqrt{q} |\psi| (\delta_{\zeta_n} \kappa) \propto \frac{\beta - \alpha}{8\pi G} A + 2J_H = 0$$

(34)

uniquely determines

$$\alpha - \beta = \frac{16\pi G J_H}{A} .$$

(35)

In a subsequent work we will show that such temperature relation is a necessary condition to have an integrable Hamiltonian; essentially, it measures a particular type of symplectic flux from the transverse dimension at each angle $\phi$ to the AdS$_3$ bulk, which possibly relates to holographic gravitational anomalies.

For static spacetimes, Eq.(35) simply sets the conformal coordinates symmetrically in the angular variable; while for rotating black holes, the chirality of spin breaks such symmetry, as expected. Interestingly, it differs from the choice of CFT temperatures proposed for Kerr as in [19, 27].

Remarkably, the condition (35), sets the two central charges equal, and proportional to the horizon area:

$$c_R = c_L = \frac{3A}{2\pi G(\alpha + \beta)} .$$

(36)
VI. ENTROPY

Since the near-horizon gravitational phase space exhibits Vir$_R \times$ Vir$_L$ symmetry, considerations from holographic duality suggest that its quantum description is given by a 2D CFT. In such a theory, unitarity and modular invariance determines the asymptotic density of states via the Cardy formula. Using the temperatures $(T_R, T_L) = (\frac{\alpha}{2\pi}, \frac{\beta}{2\pi})$ derived from properties of the Frolov-Thorne vacuum, and central charges (36), the Cardy formula for the canonical ensemble yields an entropy [9, 39]

$$S = \frac{\pi^2}{3}(c_RT_R + c_LT_L) = \frac{A}{4G},$$

(37)

which agrees with the Bekenstein-Hawking entropy. This therefore motivates an interpretation of the black hole microstates in terms of a dual CFT.

VII. WALD-ZOUPAS TERM

In writing equation (29), we implicitly assumed that the Hamiltonians were integrable. However, when evaluated on the cutoff surface, the second term in (23) may not vanish, thereby preventing $\delta H_n$ from being a total variation. In that case, one can use the Wald-Zoupas prescription for defining quasilocal charges which may not be conserved due to loss of symplectic flux from a spacetime subregion [22]. This prescription adds a correction to the definition of $\delta H_n$ in equation (18) to remove a nonintegrable piece. The bracket (29) of the charges must then be modified, in which case the the Barnich-Troessaert bracket provides a suitable definition [40]. Doing so shifts the central charges, and we note that using the same Wald-Zoupas term as HHPS yields

$$\Delta c_R = -\frac{12}{(\alpha + \beta)^2} \left( (\beta - \alpha) \frac{A}{8\pi G} + 2J_H \right)$$

(38)

$$\Delta c_L = -\Delta c_R$$

and see appendix D for details). Adding these to (31) and (32) sets the two central charges equal, and given by (36), now with any choice of $\alpha$ and $\beta$. Hence, the choice of $\alpha - \beta$ described in (35) is also the unique choice which sets the Wald-Zoupas corrections to the central charges to zero. We are currently investigating a more detailed derivation of this Wald-Zoupas prescription, and will report these findings in subsequent work.

The central charges (36) can be compared to those found by HHPS [19], whose choice of temperatures for the Kerr black hole set $\alpha + \beta = \frac{A}{8\pi GJ}$. Substituting this into (36) reproduces their result $c_R = c_L = 12J$. Our results are therefore consistent with theirs, although we have demonstrated that once the Wald-Zoupas terms are included, the construction does not appear to rely on any specific choice of temperature.

VIII. DISCUSSION

The agreement between the horizon entropy and the Cardy formula with central charges (36) suggests that the quantum description of the horizon involves a CFT. A conservative interpretation of this result is that the presence of the horizon breaks some gauge symmetry of the theory, giving rise to edge degrees of freedom [4–6]. The Vir$_R \times$ Vir$_L$ algebra then provides a symmetry principle that constrains the quantization of these edge modes, which is strong enough to determine the asymptotic density of states accounting for the entropy. This argument holds even if the conformal symmetries are only a subset of the full horizon symmetry algebra. Other horizon symmetries can include additional rotational symmetries, supertranslations, and diffeomorphisms of the bifurcation surface [13, 20, 21, 37–39, 41–48], and determining how they interact with the conformal symmetries of this paper would be an interesting direction to pursue. It is also possible that a slightly different symmetry algebra can be used to fix the entropy; in particular, [49] showed that the HHPS construction can be modified to produce a Virasoro-Kac-Moody symmetry characteristic of a warped CFT. A straightforward alteration of our construction should demonstrate how to realize warped conformal symmetries on arbitrary axisymmetric Killing horizons.

A more ambitious proposal is that the near-horizon region is holographically dual to a CFT. This is in line with the Kerr/CFT correspondence [15, 16], and raises the exciting possibility of producing new interesting examples of holography for a variety of different Killing horizons. In this picture, the expression (36) can be interpreted as determining the horizon area in terms of the temperatures $\left( \frac{\alpha}{2\pi}, \frac{\beta}{2\pi} \right)$ and the central charge. A rather nontrivial aspect of such a proposed duality, however, is the lack of a decoupling limit for the near-horizon region, due to nonextremality, $\kappa \neq 0$. The anticipated need for Wald-Zoupas terms in defining integrable charges can be viewed as one indicator of this lack of decoupling, since they imply a loss of symplectic flux from the subregion under consideration. The CFT should therefore be an open quantum system, deformed by an operator coupling it to an auxiliary system describing the far away region. This is quite reminiscent of recent models of black hole evaporation in holography [50–53], and hence studying the holographic description of Killing horizons may lead to new insights on the black hole information problem.

These results open a number of directions for further investigation. The parameters $\alpha$ and $\beta$ were not fully fixed by the arguments in this paper; even the constraint (35) does not determine the sum $\alpha + \beta$. With the Wald-Zoupas terms, any choice of $\alpha$ and $\beta$ leads to the correct Cardy entropy, and so it remains to be seen what other consistency conditions fix their value. It may be that any choice is valid, which has some advantages because...
it can be used to ensure $1 \ll c_{R,L} \ll H_0, \bar{H}_0$, which is the regime in which the Cardy formula is valid. The temperatures used by BHPS were determined using the hidden conformal symmetry of scalar scattering amplitudes in the near region of Kerr [27], and we are currently investigating its implication in our construction.

A natural question is whether this construction works for other types of horizons or subregions. With mild modifications, we expect it to work for degenerate Killing horizons with $\kappa = 0$. One could also consider noncompact horizons, such as Rindler space, in which the vector field $\psi^a$ does not have closed orbits. In these cases, one could quotient by a finite translation along $\psi^a$, which serves to both regulate the horizon area and to impose periodicity conditions on the generators. This should lead to a sensible notion of entropy density following from the Cardy formula. Other possible subregions to consider are cuts of a Killing horizon or more generic null surfaces [44, 45, 47, 48], conformal Killing horizons [54], causal diamonds [55], Ryu-Takayanagi surfaces [56, 57], and generic subregions [41, 42].

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Appendix A: Near-horizon coordinates

This appendix provides additional details leading to the Rindler expansion of the near-horizon metric (3). We set $\gamma^a = \partial_x^a$ to be the radial coordinate vector in the $x$ coordinate syste; using (2) it is related to $\rho^a$ by

$$\gamma^a = \frac{1}{\kappa x} \rho^a. \quad (A1)$$

To see that $x$ agrees with the proper distance to the bifurcation surface at leading order, first note that the norms of $\rho^a$ and $\chi^a$ are related by $\rho^2 = -\chi^2 + O(\chi^4)$ [12], which implies that $\partial_x^2(\chi^2) = \rho^a \nabla_a x^2 = 2\kappa x^2 + O(\chi^4)$, and hence $\chi^2 = -e^{2\kappa x^2}$ to leading order near the horizon. Then we find that

$$\chi^2 = -\kappa^2 x^2 + O(x^4) \quad (A2)$$

and

$$\gamma^2 = 1 + O(x^2) \quad (A3)$$

showing that $\gamma^a$ is unit normalized near the horizon, which gives its parameter $x$ the interpretation of the proper distance.

Equations (A1) and (A2) can be used to obtain more information about the near horizon expansion of the metric. Expressing the $O(x^4)$ term as $-2x^4\kappa^2 A(\theta^4)$, where $\theta^A$ are transverse coordinates on constant-$\{t, x, \phi\}$ surfaces, we find that

$$\gamma_a = -\frac{1}{2\kappa^2 x_2} \nabla_a \chi^2 = \nabla_a x \left(1 + 4x^2 A\right) + x^3 \nabla_a A + O(x^4). \quad (A4)$$

This expression determines the expansion of the $g_{\mu\nu}$ components of the metric. Note that by definition, $\chi^a \gamma_a = \psi^a \gamma_a = 0$, and so $g_{xx}$ and $g_{\phi\phi}$ identically vanish in this coordinate system. The coordinates $(t, x, \phi)$ are defined up to shifts by functions of $\theta^A$, although demanding that $x = 0$ coincides with the bifurcation surface eliminates the shift freedom for $x$. By shifting $\phi \rightarrow \phi + G(\theta^4)$, we can arrange for $g_{\phi\phi}$ to vanish on the bifurcation surface. Note that this may spoil manifest invariance with respect to other symmetries, for example, in a Myers-Perry black hole in with angular momentum in other directions besides the $\psi^a$ rotation [58, 59]. Furthermore, a bifurcate Killing horizon exhibits a discrete reflection symmetry through the bifurcation surface, which to leading order sends $x \rightarrow -x$ [60]. This can be used to rule out any terms appearing at $O(x)$ in the near horizon expansion of the metric. These considerations result in the following form of the near horizon metric,

$$ds^2 = -\kappa^2 x^2 dt^2 + \left(1 + 4x^2 A\right) dx^2 + \psi^2 d\phi^2 + q_{BC} d\theta^B d\theta^C + 2x^3 \partial_A A d\theta^B dx + x^2 \left(\psi_{(1)}^2 d\phi^2 + 2\Psi_B d\phi d\theta^B + q_{BC}^{(1)} d\theta^B d\theta^C\right) + O(x^4) \quad (A5)$$

where all the coefficients are functions of $\theta^4$, with the exception of $\kappa$, which is constant by the zeroth law of black hole mechanics. Note that $\psi^2$ coincides with the squared norm of the rotation Killing vector $\psi^\theta$ on the bifurcation surface.
The transformation between Rindler and conformal coordinates (4-6) can be inverted to obtain

$$t = \frac{1}{2\kappa} \log \left[ \frac{w^{+}}{w^{-} y^{2}(\beta - \alpha)} \right],$$  \hspace{1cm} (A6)

$$x = \left( \frac{w^{+} w^{-}}{y^{2}} \right)^{1/2},$$  \hspace{1cm} (A7)

$$\phi = \frac{2}{\alpha + \beta} \log y.$$  \hspace{1cm} (A8)

These inverse transformations are useful when expressing the generators $\xi^{a}_{n}$, $\zeta^{a}_{n}$ in terms of $\chi^{a}$, $\psi^{a}$, and $\rho^{a}$ in (A9), (A10), and are helpful when trying to understand how the AdS$_{3}$ folia embed into the near-horizon region. Using this inverse transformation, it is straightforward to express the symmetry generators (9) and (10) in terms of $\rho^{a}$ and the Killing vectors $\chi^{a}$, $\psi^{a}$,

$$\zeta^{a}_{n} = (w^{+})^{-\frac{n}{2\kappa}} \left[ \alpha^{a}_{\beta} + \frac{in\rho^{a}}{2\kappa} + \frac{in}{\alpha + \beta} \left( \frac{\beta - \alpha}{2\kappa} \chi^{a} + \psi^{a} \right) \right],$$  \hspace{1cm} (A9)

$$\xi^{a}_{n} = (w^{-})^{-\frac{n}{2\kappa}} \left[ \alpha^{a}_{\beta} + \frac{in\rho^{a}}{2\kappa} + \frac{in}{\alpha + \beta} \left( \frac{\beta - \alpha}{2\kappa} \chi^{a} + \psi^{a} \right) \right],$$  \hspace{1cm} (A10)

with $\xi^{a}_{0}$, $\zeta^{a}_{0}$ given in (15) and (16).

**Appendix B: de Sitter example**

In the course of this work, the abstract construction described in II was largely inspired by first working out the case of 4D de Sitter space. It was later realized that a similar strategy could also be applied to Kerr, and only then was the generalization to any bifurcate axisymmetric Killing horizon found. In static coordinates $(t, r, \theta, \phi)$, the de Sitter metric reads

$$ds^{2} = -\left( 1 - \frac{r^{2}}{\ell^{2}} \right) dt^{2} + \frac{dr^{2}}{1 - r^{2}/\ell^{2}} + r^{2} d\Omega^{2}_{2},$$  \hspace{1cm} (B1)

where $d\Omega^{2}_{2} = d\theta^{2} + \sin^{2} \theta d\phi^{2}$. In these coordinates, one makes explicit the presence of a cosmological horizon at $r = \ell$, which is a Killing horizon with generator $\chi^{a} = (\partial_{\theta})^{a}$, and surface gravity $\kappa = \ell^{-1}$. The azimuthal Killing vector is $\psi^{a} = (\partial_{\phi})^{a}$, and the vector $\rho^{a}$ is

$$\rho^{a} = \frac{r}{\ell} \left( 1 - \frac{r^{2}}{\ell^{2}} \right) \partial_{r} = \partial_{r^{*}},$$  \hspace{1cm} (B2)

where we can define the tortoise coordinate $r_{*}$ at lowest order near $r = \ell$ to be

$$r_{*} \approx \ell \log \sqrt{2 \left( 1 - \frac{r}{\ell} \right)},$$  \hspace{1cm} (B3)

This then implies, from the definition in Equation (2),

$$x \approx \sqrt{2\ell (\ell - r)},$$  \hspace{1cm} (B4)

and that is precisely the proper radial distance to the horizon, up to lowest nontrivial order.

Expressing the metric (B1) with the tortoise coordinate $r_{*}$, we have near the horizon

$$ds^{2} \simeq \left( 1 - \frac{r^{2}}{\ell^{2}} \right) (-dt^{2} + dr^{2}) + r^{2} d\Omega^{2}_{2},$$  \hspace{1cm} (B5)

and therefore radial null geodesics will approach the horizon with $t - r_{*} = \text{const}$ (outgoing geodesics) or $t + r_{*} = \text{const}$ (incoming geodesics). We can then parametrize the approach to the horizon with finite coordinate values by defining $w^{\pm}$ as functions of the form $f(r_{*} \pm t)$, respectively; further imposing that $w^{\pm}$ is proportional to $x$ for constant $t$ fixes the function $f(r_{*} \pm t) \propto e^{\pm \sqrt{\ell} r_{*}/\ell}$, and adding the periodicity condition via an exponential dependence on the azimuthal angle $\phi$ finally leaves us with

$$w^{+} = \sqrt{2\ell (\ell - r)} e^{\alpha + \beta t/\ell},$$

$$w^{-} = \sqrt{2\ell (\ell - r)} e^{\beta - \alpha t/\ell},$$

$$y = e^{\frac{\alpha - \beta}{2\kappa} \phi}.$$  \hspace{1cm} (B6)

**Appendix C: Computation of central charge**

The Poisson bracket to evaluate in computing the central charge is

$$\{H_{m}, H_{n}\} = \int_{\partial \Sigma} (Q_{\zeta^{a}_{n}} - i_{\zeta^{a}_{n}} \theta [L_{\zeta^{a}_{n}}, g]) = \int_{\partial \Sigma} (i_{\zeta^{a}_{n}} \theta [L_{\zeta^{a}_{n}}, g] - i_{\zeta^{a}_{n}} i_{\zeta^{a}_{n}} L - Q_{(\zeta^{a}_{n}, \zeta^{a}_{n})}),$$  \hspace{1cm} (C1)

where $L$ is the Einstein-Hilbert Lagrangian. To arrive at the second line, we use that the generators are field-independent, $\delta \zeta^{a} = 0$, that $dQ_{\zeta} = \theta [L_{\zeta}, g] - i_{\zeta} L$ on shell [34], and we drop a term $d_{\zeta} Q_{\zeta}$ that integrates to zero. Then we note from (19) and (22) that the central charge just appears as the coefficient of the $m^{3}$ term, with all other terms depending linearly on $m, n$. Examining (C1), we find that only the $\theta$ terms contain enough derivatives to produce an $m^{3}$ contribution. Evaluation of these terms is aided by noting that

$$\theta [L_{\zeta}, g] = \frac{1}{16\pi G} \epsilon^{a}_{b} \left( \nabla_{a} (\nabla_{b} \zeta^{c} - \nabla_{b} \zeta^{c}) + 2R^{a}_{b c} \zeta^{c} \right).$$  \hspace{1cm} (C2)
and the Ricci tensor term can be dropped because it will not scale as $m^3$. The remaining terms involving two derivatives can be evaluated in a near horizon expansion, about $w^+ = w^- = 0$. Note that the expression cannot be evaluated directly on the bifurcation surface because the vector fields $\zeta^a_n$ are singular there for $n \neq 0$, as discussed below equation (14). Going through the calculation using the near-horizon expansion of the metric (8), we arrive at

$$i_{\zeta_n} \theta [\mathcal{L}_{\zeta_n} g] = \frac{-i m (m^2 + \alpha^2)}{8 \pi G \alpha (\alpha + \beta)^2} \left( \beta + N_\phi \right) \frac{\psi}{w^+} \frac{d w^+}{\mu},$$

where $\mu = \sqrt{q} d\theta^1 \wedge \ldots \wedge d\theta^{(d-3)}$ is the volume form for the transverse $\theta^A$ directions. The expression for $-i_{\zeta_n} \theta [\mathcal{L}_{\zeta_n} g]$ is the same. The central charge is obtained by integrating the coefficient of $m^3$ over a cutoff surface at fixed value of $x$ and $t$, and then taking the limit to the bifurcation surface, which drops all terms regular in $w^+, w^-$. On such a cutoff surface, as $\phi$ varies over its range $(0, 2\pi)$, $w^+$ goes from $w^+_0$ to $w^+_0 e^{2\pi \alpha}$. Since

$$\int_{w^+_0}^{w^+_0} e^{2\pi \alpha} \frac{d w^+}{w^+} = 2\pi \alpha,$$

we find that

$$c_R = \frac{6}{G (\alpha + \beta)^2} \int d\theta^A \sqrt{q} \psi \left( \beta + N_\phi \right)$$

which is equation (31).

### Appendix D: Wald-Zoupas term

Here, we provide some details on how to arrive at (38) using the HHPS prescription for the Wald-Zoupas term. Their proposed definition of the Hamiltonian is

$$\delta H_n = \Omega (\delta g_{ab}, \mathcal{L}_{\zeta_n} g_{ab}) + \int_{\partial \Sigma} i \zeta_n \ast X,$$

where $\partial \Sigma$ lies on a cut of the future horizon, and $X_a$ is a one-form constructed from the metric variation and the Hajicek one-form $F_a$ according to

$$X_b = \frac{1}{8 \pi G} F^a \delta g_{ab}.$$ 

$F^a$ is related to the twist of the constant $w^+, w^-$ foliation [61], and can be computed via

$$F^a = \frac{1}{2} h^a_b [k, l]_b,$$

where $h^a_b$ is a projector onto $\partial \Sigma$, and $k^a$ and $l^a$ are two null normals with $k \cdot l = 1$. At the bifurcation surface, they can be chosen to be

$$k_a = \frac{y^{\pm 2}}{\sqrt{2}} \nabla_a w^+,$$

$$l_a = \frac{y^{\pm 2}}{\sqrt{2}} \nabla_a w^-,$$

where the $y$-dependence is chosen to ensure they are single-valued on the bifurcation surface. Raising the index on each and keeping terms to first order in $w^+$ gives

$$k^a = \sqrt{2} \left( y^{\pm 2} \partial^a \alpha + \frac{\alpha + \beta}{4 y^2} (\alpha + N_\phi) \partial^a \phi \right),$$

$$l^a = \sqrt{2} \left( y^{\pm 2} \partial^a \alpha + \frac{\alpha + \beta}{4 y^2} (\beta - N_\phi) \partial^a \phi \right),$$

where we have dropped terms proportional to $\partial^a \phi$, which do not contribute to the shift in central charges. The commutator is now straightforwardly computed near $w^+ = 0$, and yields

$$F^a = \frac{(\alpha + \beta)}{4 y^2} (\beta - \alpha + 2N_\phi) y \partial^a \phi.$$ 

This twist for is computed in a specific gauge adapted to the $w^+$ foliation; in a subsequent work, we will comment on the gauge dependence of this term.

Finally, the correction to the central charge is found by computing the brackets of the Hamiltonians defined by this prescription. This requires use of the Barnich-Troessaert bracket [40], which is defined for these Hamiltonians to be

$$\{ H_m, H_n \} = \Omega (\mathcal{L}_{\zeta_m} g, \mathcal{L}_{\zeta_n} g) + \int_{\partial \Sigma} \left( i \zeta_n \ast X \mathcal{L}_{\zeta_m} g - i \zeta_m \ast X \mathcal{L}_{\zeta_n} g \right).$$

where the second line determines the correction to the central charges. Evaluating these terms then produces the expression (38).

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