LIFTINGS AND QUASI-LIFTINGS OF DG MODULES

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Abstract. We prove lifting results for DG modules that are akin to Auslander, Ding, and Solberg’s famous lifting results for modules.

Introduction

Convention. Throughout this paper, let $R$ be a commutative noetherian ring.

Hochster famously wrote that “life is really worth living” in a Cohen-Macaulay ring. For instance, if $R$ is Cohen-Macaulay and local with maximal regular sequence $t$, then $R/(t)$ is artinian and the natural epimorphism $R \to R/(t)$ is nice enough to allow for transfer of properties between the two rings. Thus, if one can prove a result for artinian local rings, then one can (often) prove a similar result for Cohen-Macaulay local rings by showing that the desired conclusion descends from $R/(t)$ to $R$. When $R$ is complete, then this is aided sometimes by the lifting result of Auslander, Ding, and Solberg [2, Propositions 1.7 and 2.6].

Theorem. Let $t \in R$ be an $R$-regular sequence, and let $M$ be a finitely generated $R/(t)$-module. Assume that $R$ is local and $(t)$-adically complete.

(a) If $\text{Ext}^2_R(R/(t), M) = 0$, then $M$ is “liftable” to $R$, that is, there is a finitely generated $R$-module $N$ such that $R/(t) \otimes_R N \cong M$ and $\text{Tor}^R_i(R/(t), N) = 0$ for all $i \geq 1$.

(b) If $\text{Ext}^1_R(R/(t), M) = 0$, then $M$ has at most one lift to $R$.

In this paper, we are concerned with what happens when the sequence $t$ is not $R$-regular. One would like a similar mechanism for reducing questions about arbitrary local rings to the artinian case.

It is well known that the map $R \to R/(t)$ is not nice enough in general to guarantee good descent/lifting behavior. Our perspective in this matter is that this is not the right map to consider in general: the correct one is the natural map from $R$ to the Koszul complex $K = K^R(R/(t))$. This perspective requires one to make some adjustments. For instance, $K$ is a differential graded $R$-algebra, so not a commutative ring in the traditional sense. This may cause some consternation, but the payoff can be handsome. For instance, in [8] we use this perspective to answer

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1We know of this quote from [6].

2This perspective is not original to our work. We learned of it from Avramov and Iyengar.
a question of Vasconcelos [9]. One of the tools for the proof of this result is the following version of Auslander, Ding, and Solberg’s lifting result. Note that we do not assume that $R$ is local in part (a) of this result.

**Main Theorem.** Let $t = t_1, \cdots, t_n$ be a sequence of elements of $R$, and assume that $R$ is $tR$-adically complete. Let $D$ be a DG $K^R(\mathfrak{t})$-module that is homologically bounded below and homologically degreewise finite.

(a) If $\text{Ext}^2_{K^R(\mathfrak{t})}(D, D) = 0$, then $D$ is quasi-liftable to $R$, that is, there is a semi-free $R$-complex $D'$ such that $D \simeq K^R(\mathfrak{t}) \otimes_R D'$.

(b) Assume that $R$ is local. If $D$ is quasi-liftable to $R$ and $\text{Ext}^1_{K^R(\mathfrak{t})}(D, D) = 0$, then any two homologically degreewise finite quasi-lifts of $D$ to $R$ are quasiisomorphic over $R$.

This result is proved in Corollaries 3.7 and 3.12 which follow from more general results on liftings along morphisms of DG algebras. Note that it is similar to, but quite different from, some results of Yoshino [10].

We briefly describe the contents of the paper. Section 1 contains some background material on DG algebras and DG modules. Section 2 contains some structural results for DG modules and homomorphisms between them. Finally, Section 3 is where we prove our Main Theorem.

1. DG Modules

We assume that the reader is familiar with the category of $R$-complexes. For clarity, we include a few definitions.

**Definition 1.1.** In this paper, complexes of $R$-modules (“$R$-complexes” for short) are indexed homologically:

$$M = \cdots \xrightarrow{\partial_n^{M^2}} M_{n+1} \xrightarrow{\partial_n^{M^1}} M_n \xrightarrow{\partial_n^M} M_{n-1} \xrightarrow{\partial_n^{-1}} \cdots.$$  

The degree of an element $m \in M$ is denoted $|m|$. The tensor product of two $R$-complexes $M, N$ is denoted $M \otimes_R N$, and the Hom complex is denoted $\text{Hom}_R(M, N)$. A chain map $M \to N$ is a cycle of degree 0 in $\text{Hom}_R(M, N)$. An $R$-complex $M$ is homologically bounded below if $H_i(M) = 0$ for $i > 0$; it is bounded below if $M_i = 0$ for $i > 0$.

Next, we begin our background material on DG algebras; see [1] [3] [4].

**Definition 1.2.** A commutative differential graded $R$-algebra (DG $R$-algebra for short) is an $R$-complex $A$ equipped with a chain map $\mu^A: A \otimes_R A \to A$ with $ab := \mu^A(a \otimes b)$ that is:

- **associative:** for all $a, b, c \in A$ we have $(ab)c = a(bc)$;
- **unital:** there is an element $1 \in A_0$ such that for all $a \in A$ we have $1a = a$;
- **graded commutative:** for all $a, b \in A$ we have $ab = (-1)^{|a||b|}ba$ and $a^2 = 0$ when $|a|$ is odd; and
- **positively graded:** $A_i = 0$ for $i < 0$.

The map $\mu^A$ is the product on $A$. Given a DG $R$-algebra $A$, the underlying algebra is the graded commutative $R$-algebra $A^2 = \oplus_{i \geq 0} A_i$.

A morphism of DG $R$-algebras is a chain map $f: A \to B$ between DG $R$-algebras respecting products and multiplicative identities: $f(aa') = f(a)f(a')$ and $f(1) = 1$. 

Example 1.3. The ring $R$, considered as a complex concentrated in degree 0, is a DG $R$-algebra. Given a DG $R$-algebra $A$, the map $R \to A$ given by $r \mapsto r \cdot 1$ is a morphism of DG $R$-algebras.

Fact 1.4. Let $A$ be a DG $R$-algebra. The fact that the product on $A$ is a chain map says that $\partial^A$ satisfies the Leibniz rule:

$$\partial^A_{|a|+|b|}(ab) = \partial^A_a(a)b + (-1)^{|a|}a\partial^A_b(b).$$

It is straightforward to show that the $R$-module $A_0$ is an $R$-algebra. Moreover, the natural map $A_0 \to A$ is a morphism of DG $R$-algebras. The condition $A_{-1} = 0$ implies that $A_0$ surjects onto $H_0(A)$ and that $H_0(A)$ is an $A_0$-algebra. Furthermore, the $R$-module $A_i$ is an $A_0$-module, and $H_i(A)$ is an $H_0(A)$-module for each $i$.

Given a second DG $R$-algebra $K$, the tensor product $K \otimes_R A$ is also a DG $R$-algebra with multiplication $(x \otimes a)(x' \otimes a') := (-1)^{|a||x'|}(xx') \otimes (aa')$.

Definition 1.5. Let $A$ be a DG $R$-algebra. We say that $A$ is noetherian if $H_0(A)$ is noetherian and the $H_0(A)$-module $H_i(A)$ is finitely generated for all $i \geq 0$. When $(R, m)$ is local, we say that $A$ is local if it is noetherian and the ring $H_0(A)$ is a local $R$-algebra with maximal ideal $m_{H_0(A)}$.

Fact 1.6. Assume that $(R, m)$ is local, and let $A$ be a local DG $R$-algebra. The composition $A \to H_0(A) \to H_0(A)/m_{H_0(A)}$ is a surjective morphism of DG $R$-algebras with kernel of the form $m_A = \cdots \xrightarrow{\partial^A_2} A_1 \xrightarrow{\partial^A_1} m_0 \xrightarrow{0} 0$ for some maximal ideal $m_0 \subset A_0$. The quotient complex $A/m_A$ is $A$-isomorphic to $H_0(A)/m_{H_0(A)}$. Since $H_0(A)$ is a local $R$-algebra, we have $mA_0 \subset m_0$.

Definition 1.7. If $R$ is local and $A$ is a local DG $R$-algebra, then the subcomplex $m_A$ is the augmentation ideal of $A$.

The following example is key for this investigation.

Example 1.8. Given a sequence $t_1, \cdots, t_n \in R$, the Koszul complex $K = K^R(t)$ is a DG $R$-algebra with product given by the wedge product. If $(R, m)$ is local and $t \in m$, then $K$ is a local DG $R$-algebra with augmentation ideal $m_K = (0 \to R \to \cdots \to R^n \to m \to 0)$.

Definition 1.9. Let $A$ be a DG $R$-algebra. A DG $A$-module is an $R$-complex $M$ with a chain map $\mu^M : A \otimes_R M \to M$ such that the rule $am := \mu^M(a \otimes m)$ is associative and unital. The map $\mu^M$ is the scalar multiplication on $M$. A morphism of DG $A$-modules is a chain map $f : M \to N$ between DG $A$-modules that respects scalar multiplication: $f(am) = af(m)$. Isomorphisms in the category of DG $A$-modules are identified by the symbol $\cong$. Quasiisomorphisms in the category of DG $A$-modules are identified by the symbol $\simeq$; these are the morphisms that induce bijections on all homology modules. Two DG $A$-modules $M$ and $N$ are quasiisomorphic, written $M \simeq N$ if there is a finite sequence of quasiisomorphisms $M \xrightarrow{\simeq} \cdots \xrightarrow{\simeq} N$.

Example 1.10. Consider the ring $R$ as a DG $R$-algebra. A DG $R$-module is just an $R$-complex, and a morphism of DG $R$-modules is simply a chain map.\footnote{This means that $H_0(A)$ is a local ring whose maximal ideal contains the ideal $m_{H_0(A)}$.}
Fact 1.11. Let $A$ be a DG $R$-algebra, and let $M$ be a DG $A$-module. The fact that the scalar multiplication on $M$ is a chain map says that $\partial^A$ satisfies the Leibniz rule: $\partial^A_{[a]+[m]}(am) = \partial^A_{[a]}(a)m + (-1)^{|a|}\partial^M_{[m]}(m)$. The $R$-module $M_i$ is an $A_0$-module, and $H_i(M)$ is an $H_0(A)$-module for each $i$.

Definition 1.12. Let $A$ be a DG $R$-algebra, and let $i$ be an integer. The $i$th suspension of a DG $A$-module $M$ is the DG $A$-module $\Sigma^i M$ defined by $(\Sigma^i M)_n := M_{n-i}$ and $\partial^{\Sigma^i M} := (-1)^i\partial^M$. The scalar multiplication on $\Sigma^i M$ is defined by the formula $\mu^{\Sigma^i M}(a \otimes m) := (-1)^{|a|i}\mu^M(a \otimes m)$.

Definition 1.13. Let $A$ be a DG $R$-algebra. A DG $A$-module $M$ is homologically degreewise finite if $H_i(M)$ is finitely generated over $H_0(A)$ for all $i$; it is homologically finite if it is homologically degreewise finite and $H_i(M) = 0$ for $|i| \gg 0$.

Definition 1.14. Let $A$ be a DG $R$-algebra, and let $M$, $N$ be DG $A$-modules. The tensor product $M \otimes_A N$ is the quotient $(M \otimes_R N)/U$ where $U$ is the subcomplex generated by all elements of the form $(am) \otimes n - (-1)^{|a|m}m \otimes (an)$. Given an element $m \otimes n \in M \otimes_R N$, we denote the image in $M \otimes_A N$ as $m \otimes n$.

Fact 1.15. Let $A$ be a DG $R$-algebra, and let $M$, $N$ be DG $A$-modules. The tensor product $M \otimes_A N$ is a DG $A$-module via the action $a(m \otimes n) := (am) \otimes n = (-1)^{|a|m}m \otimes (an)$.

Fact 1.16. Let $A \to B$ be a morphism of DG $R$-algebras. Given a DG $A$-module $M$, the “base changed” complex $B \otimes_A M$ has the structure of a DG $B$-module by the action $b(b' \otimes m) := (bb') \otimes m$. This structure is compatible with the DG $A$-module structure via restriction of scalars.

Definition 1.17. Let $A$ be a DG $R$-algebra, and let $M$ be a DG $A$-module. The underlying $A^2$-module associated to $M$ is the $A^2$-module $M^2 = \bigoplus_{i=-\infty}^{\infty} M_i$. A subset $E$ of $M$ is called a semi-basis if it is a basis of the underlying $A^2$-module $M^2$. If $M$ is bounded below, then $M$ is called semi-free if it has a semi-basis. A semi-free resolution of a DG $A$-module $N$ is a quasiisomorphism $F \xrightarrow{\sim} N$ of DG $A$-modules such that $F$ is semi-free.

Assume that $R$ and $A$ are local. A minimal semi-free resolution of $M$ is a semi-free resolution $F \xrightarrow{\sim} M$ such that $F$ is minimal, i.e., each (equivalently, some) semi-basis of $F$ is finite in each degree and the differential on $(A/m_A) \otimes_A F$ is 0.

Fact 1.18. Let $A$ be a DG $R$-algebra. Let $M$ be a homologically bounded below DG $A$-module. Then $M$ has a semi-free resolution over $A$ by [4] Theorem 2.7.4.2.

Assume that $A$ is noetherian, and let $j$ be an integer. If $H_i(A)$ is finitely generated over $H_0(A)$ for all $i$, and $H_i(M) = 0$ for $i < j$, then $M$ has a semi-free resolution $F \xrightarrow{\sim} M$ such that $F^2 \cong \bigoplus_{i=j}^{\infty} \Sigma^i(A^2)^{\beta_i}$ with $\beta_i \in \mathbb{Z}$ for all $i$ and $F_i = 0$ for all $i < j$; see [1] Proposition 1]. In particular, homologically finite DG $A$-modules admit such “degree-wise finite, bounded below” semi-free resolutions. Note that the condition $F^2 \cong \bigoplus_{i=j}^{\infty} \Sigma^i(A^2)^{\beta_i}$ says that the degree-$i$ piece of the semi-basis $E_i = E \cap F_i$ is finite for each $i$, and $E_i = \emptyset$ for $i < j$.

Assume that $R$ and $A$ are local. If $H_i(A)$ is finitely generated over $H_0(A)$ for all $i$, and $H_i(M) = 0$ for $i < j$, then $M$ has a minimal semi-free resolution $F \xrightarrow{\sim} M$.

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\[4\] As is noted in [4], when $M$ is not bounded below, the definition of “semi-free” is significantly more technical.
such that $F_i = 0$ for all $i < j$; see [I Proposition 2]. In particular, homologically finite DG $A$-modules admit minimal semi-free resolutions.

**Definition 1.19.** Let $A$ be a DG $R$-algebra, and let $M, N$ be DG $A$-modules. Given an integer $i$, a DG $A$-module homomorphism of degree $i$ is a homomorphism $f: M \to N$ of the underlying $R$-complexes such that $f(am) = (-1)^{|a||f|} f(m)$ for all $a \in A$ and $m \in M$. We write $|f| = i$. The (graded) submodule of $\text{Hom}_A(M, N)$ consisting of all DG $A$-module homomorphisms $M \to N$ is denoted $\text{Hom}_A(M, N)$.

A homomorphism $f \in \text{Hom}_A(M, N)_i$ is null-homotopic if it is in $\text{Im}(d^i)$; see [I, Proposition 2]. Two homomorphisms $M \to N$ are homotopic if their difference is null-homotopic.

**Fact 1.20.** Let $A$ be a DG $R$-algebra, and let $M, N$ be DG $A$-modules. The complex $\text{Hom}_A(M, N)$ is a DG $A$-module via the action

$$(af)(m) := a(f(m)) = (-1)^{|a||f|} f(am).$$

**Definition 1.21.** Let $A \to B$ be a morphism of DG $R$-algebras, and let $M, M'$ be DG $A$-modules. Given $f \in \text{Hom}_A(M, M')$, define $B \otimes_A f: B \otimes_A M \to B \otimes_B M'$ by the formula $(B \otimes_A f)(b \otimes m) = (-1)^{|b||f|} b \otimes f(m)$.

**Fact 1.22.** Let $A \to B$ be a morphism of DG $R$-algebras, and let $M, M'$ be DG $A$-modules. Given a homomorphism $f \in \text{Hom}_A(M, M')_i$, the function $B \otimes_A f$ is $B$-linear, that is, an element of $\text{Hom}_B(B \otimes_A M, B \otimes_A M')_i$. Furthermore, if $f$ is a cycle in $\text{Hom}_A(M, M')_i$, then $B \otimes_A f$ is a cycle in $\text{Hom}_B(B \otimes_A M, B \otimes_A M')_i$.

**Definition 1.23.** Let $A$ be a DG $R$-algebra, and let $M, N$ be DG $A$-modules. Given a semi-free resolution $F \xrightarrow{\sim} M$, we set $\text{Ext}^i_A(M, N) = H_{-i}(\text{Hom}_A(F, N))$ for each integer $i$.

**Fact 1.24.** Let $A$ be a DG $R$-algebra, and let $M, N$ be DG $A$-modules. For each $i$, the module $\text{Ext}^i_A(M, N)$ is independent of the choice of semi-free resolution of $M$, and we have $\text{Ext}^i_A(M, N) \cong \text{Ext}^i_A(M', N')$ whenever $M \cong M'$ and $N \cong N'$; see [3, Propositions 1.3.1–1.3.3]. Given a semi-free resolution $F \cong M$ and an integer $i$, the elements of $\text{Ext}^i_A(M, N)$ are by definition the homotopy equivalence classes of morphisms of DG $A$-modules $F \to \Sigma^{-i}N$.

2. Structure of Semi-free DG Modules and DG Homomorphisms

The proof of our Main Theorem involves the manipulation of the differentials on certain DG modules to construct isomorphisms that are amenable to lifting. For this, we need a concrete understanding of these differentials and the homomorphisms between these DG modules. This concrete understanding is the goal of this section. We begin by establishing some notation to be used for much of the paper.

**Notation 2.1.** Let $A$ be a DG $R$-algebra such that each $A_i$ is free over $R$ of finite rank. Given an element $t \in R$, let $K = K^R(t)$ denote the Koszul complex $0 \to K_1 \xrightarrow{t} K_0 \to 0$ with $K_1 \cong R \cong K_0$ and basis elements $1 \in K_0$ and $e \in K_1$. We fix a basis $\{\gamma_{i,1}, \ldots, \gamma_{i,r_i}\}$ for $A_i$. Let $B$ denote the DG $R$-algebra $K^R(t) \otimes_R A$, which has the following form

$$B \cong \cdots \xrightarrow{\partial^B_{i-1}} A_{i-1} \oplus A_1 \xrightarrow{\partial^B_1} A_{i-2} \oplus A_{i-1} \xrightarrow{\partial^B_1} \cdots \xrightarrow{\partial^B_{i-1}} A_0 \oplus A_1 \xrightarrow{\partial^B_1} 0 \oplus A_0 \to 0.$$ 

This uses the isomorphism $B_i = (K_1 \otimes_R A_{i-1}) \oplus (K_0 \otimes_R A_i) \cong A_{i-1} \oplus A_i$. We identify $B_i$ with $A_{i-1} \oplus A_i$ for the remainder of this paper. Under this identification,
the sum $e \otimes a_{i-1} + 1 \otimes a_i \in B_i$ corresponds to the column vector $[a_{i-1}^j] \in A_{i-1} \oplus A_i$.

The use of column vectors allows us to identify the differential of $B$ as the matrix

$$\partial^B_i = \begin{bmatrix} -\partial^A_{i-1} & 0 \\ t & \partial^A_i \end{bmatrix}.$$ 

**Remark 2.2.** In Notation 2.1 the algebra structure on $B$ translates to the formula

$$\begin{bmatrix} a_{i-1} \\ a_i \end{bmatrix} \begin{bmatrix} c_{j-1} \\ c_j \end{bmatrix} = \begin{bmatrix} a_{i-1}c_j + (-1)^ia_ic_{j-1} \\ a_ic_j \end{bmatrix}$$

where $[a_{i-1}^j] \in B_i$ and $[c_{j-1}^j] \in B_j$. This uses the fact that $e^2 = 0$ in $K$.

Note that a basis of $B_i$ is

$$\left\{ \begin{bmatrix} \gamma_{i-1,1} \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} \gamma_{i-1,r_i-1} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \gamma_{i,1} \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ \gamma_{i,r_i} \end{bmatrix} \right\}.$$ 

Also, note that the assumptions on $B$ that the augmentation ideal of $A$ is a direct sum of copies of $j$ indexed by $\beta_i$ over $A_0$, and set $\beta = \bigcup_j \beta_i$ considered as a subset of the disjoint union $\bigcup_i M_i$. Define scalar multiplication on $M$ over $A$ using the scalar multiplication on each $A^{(\beta)}$.

Consider $R$-module homomorphisms

$$\xi_i : M_i \to M_{i-1}, \quad \tau_i : M_i \to M_i, \quad \delta_i : M_i \to M_{i-2}, \quad \text{and} \quad \alpha_i : M_i \to M_{i-1}.$$ 

For each $i$, set

$$M_i = \bigoplus_{j=0}^\infty A_j^{(\beta_i-j)}$$

where $A_j^{(\beta_i-j)}$ is a direct sum of copies of $A_j$ indexed by $\beta_i-j$. Identify each $\beta_i$ with a basis of $A_j^{(\beta_i)}$ over $A_0$, and set $\beta = \bigcup_j \beta_i$ considered as a subset of the disjoint union $\bigcup_i M_i$. Define scalar multiplication on $M$ over $A$ using the scalar multiplication on each $A^{(\beta)}$.

Consider $R$-module homomorphisms

$$\xi_i : M_i \to M_{i-1}, \quad \tau_i : M_i \to M_i, \quad \delta_i : M_i \to M_{i-2}, \quad \text{and} \quad \alpha_i : M_i \to M_{i-1}.$$ 

For each $i$, set

$$N_i = M_{i-1} \oplus M_i \quad \text{and} \quad \partial^N_i = \begin{bmatrix} \xi_{i-1} & \delta_i \\ \tau_{i-1} & \alpha_i \end{bmatrix} : N_i \to N_{i-1}.$$ 

We consider the sequences

$$M = \cdots \xrightarrow{a_{i+1}} M_i \xrightarrow{a_i} M_{i-1} \xrightarrow{a_{i-1}} \cdots$$

and

$$N = \cdots \xrightarrow{\partial^N_{i+1}} N_i \xrightarrow{\partial^N_i} N_{i-1} \xrightarrow{\partial^N_{i-1}} \cdots.$$ 

Given elements $[a_{i-1}^j] \in B_i$ and $[m_j^m_{j-1}] \in N_j$, we define

$$\begin{bmatrix} a_{i-1} \\ a_i \end{bmatrix} \begin{bmatrix} m_{j-1} \\ m_j \end{bmatrix} = \begin{bmatrix} a_{i-1}m_j + (-1)^ia_im_{j-1} \\ a_im_j \end{bmatrix}.$$
For each $\beta_{i,j} \in \beta_i$ we set $e_{i,j} = [\beta_{i,j}^0] \in N_i$. For each $i$, set $E_i = \{e_{i,j}\}$. Let $E = \bigcup_i E_i$ considered as a subset of the disjoint union $\bigsqcup_i N_i$.

**Remark 2.4.** In Notation 2.3 the sequences $M$ and $N$ may not be complexes. Note that the scalar multiplications defined on $M$ and $N$ make $\oplus_i M_i$ and $\oplus_i N_i$ into graded free modules over $A^\delta$ and $B^\delta$, respectively.

The next result is a straightforward consequence of the definitions in 2.3.

**Lemma 2.5.** We work in the setting of Notations 2.1 and 2.3. The following conditions are equivalent.

(i) The sequence $M$ is a semi-free DG $A$-module;
(ii) The sequence $M$ is a DG $A$-module; and
(iii) For all integers $i$ and $j$ we have

$$\alpha_{t-1}\alpha_i = 0 \quad \alpha_{i+1}(\gamma_{i,s}m_j) = \partial_i^A(\gamma_{i,s})m_j + (-1)^i\gamma_{i,s}\alpha_j(m_j)$$

for $s = 1, \ldots, r_i$ and for all $m_j \in M_j$.

Next, we give a similar result for the sequence $N$.

**Lemma 2.6.** We work in the setting of Notations 2.1 and 2.3. The following conditions are equivalent.

(i) The sequence $N$ is a semi-free DG $B$-module;
(ii) The sequence $N$ is a DG $B$-module; and
(iii) For all integers $i$ and $j$ we have

$$\xi_i = -\alpha_i \quad \tau_i = t \quad \alpha_{i-1}\alpha_i = -t\delta_i \quad \delta_i\alpha_{i+1} = \alpha_{i-1}\delta_{i+1}$$

$$\delta_{i+1}(\gamma_{i,s}m_j) = \gamma_{i,s}\delta_j(m_j) \quad \alpha_{i+1}(\gamma_{i,s}m_j) = \partial_i^A(\gamma_{i,s})m_j + (-1)^i\gamma_{i,s}\alpha_j(m_j)$$

for $s = 1, \ldots, r_i$ and for all $m_j \in M_j$.

In particular, if $N$ is a DG $B$-module, then

$$\partial_i^N = \begin{bmatrix} -\alpha_{i-1} & \delta_i \\ t & \alpha_i \end{bmatrix}.$$
The Leibniz rule for products of the form $[\gamma_i, s] [m_j] = 0$ is equivalent to the following:

$$\tau_{i+j} (\gamma_i, s m_j) = \gamma_i, s \tau_j (m_j)$$

(2.6.10)

$$\xi_{i+j} (\gamma_i, s m_j) = -\partial^i (\gamma_i, s) m_j + (-1)^i \gamma_i, s \xi_j (m_j).$$

(2.6.11)

The Leibniz rule for $[\gamma_i, s] [m_j] = 0$ is equivalent to the following:

$$(-1)^j \gamma_i, s m_j + (-1)^{i+1} \gamma_i, s \tau_j (m_j) = 0.$$  

(2.6.12)

Equation (2.6.3) is the same as (2.6.6), and equation (2.6.4) is the same as (2.6.7). Comparing equations (2.6.7) and (2.6.9) with $\gamma_{0,1} = 1$, we find $\xi_i = -\alpha_i$. Using equation (2.6.8) also with $\gamma_{0,1} = 1$, we see that $\tau_i = t$. This explains (2.6.2) and (2.6.5). It also shows that (2.6.12) is trivial. Since $N$ is an $R$-complex, we have $\partial^i \partial^j_{i+1} = 0$ which gives the equations in (2.6.2). This completes the proof of the implication.

The implication (ii) $\iff$ (iii) is handled similarly, and the equivalence (i) $\iff$ (iii) is straightforward. □

Our next two results characterize semi-free DG modules over $A$ and $B$. The first one is straightforward.

**Lemma 2.7.** We work in the setting of Notations 2.1 and 2.3. If $F$ is a bounded below semi-free DG $A$-module with semi-basis $G$, then $F \cong M$ for some appropriate choice of $\alpha_i$ satisfying (2.6.1) for all $i$ and $j$ where $\beta_i = [G \cap F_i]$. 

**Lemma 2.8.** We work in the setting of Notations 2.1 and 2.3. If $F$ is a bounded below semi-free DG $B$-module with semi-basis $G$, then $F \cong N$ for some appropriate choices of $\xi_i$, $\tau_i$, $\alpha_i$, and $\delta_i$ satisfying (2.6.1)–(2.6.4) for all $i$ and $j$ where $\beta_i = [G \cap F_i]$. 

**Proof.** Let $F$ be a bounded below semi-free DG $B$-module with semi-basis $G$. For each $i$, set $\beta_i = [G \cap F_i]$. Since $F$ is semi-free, it is straightforward to show that $F_i \cong \bigoplus_{j=0}^{\infty} B_j^{(\beta_{i-j})}$. Decomposing $B_j$ as $A_{j-1} \bigoplus A_j$, we see that $F_j \cong N_j$ for each $j$. Since the $R$-module homomorphisms $N_j \to N_{j-1}$ are necessarily of the form $[\xi_i, \tau_i, \alpha_i, \delta_i]$, it follows that there are appropriate choices of $\xi_i$, $\tau_i$, $\alpha_i$, and $\delta_i$ such that $F \cong N$. Finally, the fact that $F$ is a DG $B$-module implies that the maps $\xi_i$, $\tau_i$, $\alpha_i$, and $\delta_i$ satisfy (2.6.1)–(2.6.4), by Lemma 2.6. □

The next result indicates how a semi-free DG $B$-module should look in order to be liftable to $A$. See Section 3 for more about this.

**Lemma 2.9.** We work in the setting of Notations 2.1 and 2.3. If $M$ is a semi-free DG $A$-module, then $B \otimes_A M$ is a semi-free DG $B$-module, identified with a DG $B$-module $N$ with

$$\partial^N = \begin{bmatrix} -\alpha_{i-1} & 0 \\ t & \alpha_i \end{bmatrix}.$$

**Proof.** Using the isomorphisms

$$B \otimes_A M \cong (K^R(t) \otimes_R A) \otimes_A M \cong K^R(t) \otimes_R M$$

the result follows directly from the definitions in (2.3) with Lemmas 2.5 and 2.6. □

Next, we describe DG module homomorphisms over $A$ and $B$. Again, the proof of the first of these results is straightforward.
Lemma 2.10. We work in the setting of Notations 2.1 and 2.3. Assume that $M$ is a semi-free DG $A$-module, and let $M'$ be a second semi-free DG $A$-module. Fix an integer $p$. A sequence of $R$-module homomorphisms $\{u_i : M_i \to M'_{i+p}\}$ is a DG $A$-module homomorphism $M \to M'$ of degree $p$ if and only if it is a degree-$p$ homomorphism $M \to M'$ of the underlying $R$-complexes and

$$u_{i+j}(\gamma_i, sm_j) = (-1)^p \gamma_i, u_j(m_j)$$

for $s = 1, \ldots, r_i$ and for all $m_j \in M_j$ for each integer $j$.

Lemma 2.11. We work in the setting of Notations 2.1 and 2.3. Assume that $N$ is a semi-free DG $B$-module. Let $N'$ be a second semi-free DG $B$-module built from modules $M'_i$ and maps $\xi_i, \tau'_i, \delta'_i, \nu'_i$ as in 2.3. Fix an integer $p$. A sequence of $R$-module homomorphisms $\{S_i : N_i \to N'_{i+p}\}$ is a DG $B$-module homomorphism $N \to N'$ of degree $p$ if and only if it is a degree-$p$ homomorphism $N \to N'$ of the underlying $R$-complexes such that for all integers $i$ we have $S_i = \left[ (-1)^p z_{i-1} v_i \right]_{0 \to z_i}$ for some $z_i : M_i \to M'_{i+p}$ and $v_i : M_i \to M'_{i+p-1}$ and

$$v_{i+j}(\gamma_i, s m_j) = (-1)^{(p+1)} \gamma_i, v_j(m_j)$$

$$z_{i+j}(\gamma_i, s m_j) = (-1)^p \gamma_i, z_j(m_j)$$

for $s = 1, \ldots, r_i$ and for all $m_j \in M_j$ for each integer $j$.

Proof. Fix a sequence of $R$-module homomorphisms $S = \{S_i : N_i \to N'_{i+p}\}$. By assumption, we have $N_i = M_{i-1} \oplus M_i$ and $N'_i = M'_{i-1} \oplus M'_i$, so the maps $S_i$ have the form

$$S_i = \left[ \begin{array}{ccc} u_{i-1} & v_i & 0 \\ y_{i-1} & 0 & z_i \end{array} \right] : M_{i-1} \oplus M_i \to M'_{i+p-1} \oplus M'_{i+p}$$

where $u_{i-1} : M_{i-1} \to M'_{i+p-1}$, $v_i : M_i \to M'_{i+p-1}$, $y_{i-1} : M_{i-1} \to M'_{i+p}$, and $z_i : M_i \to M'_{i+p}$.

Assume that $S$ is a DG $B$-module homomorphism $N \to N'$ of degree $p$. For each $b_q \in B_q$ and $n_d \in N_d$, we have

$$S_{q+d}(b_q n_d) = (-1)^p b_q S_d(n_d). \quad (2.11.3)$$

Therefore, given integers $i$ and $j$, for $s = 1, \ldots, r_i$ and for all $m_j \in M_j$, by writing equation 2.11.3 for the elements $[\gamma_i, s] \in B_{i+1}$ and $[0, m_j] \in N_j$ we have

$$y_{i+j}(\gamma_i, s m_j) = 0$$

$$u_{i+j}(\gamma_i, s m_j) = (-1)^{(i+1)p} \gamma_i, z_j(m_j). \quad (2.11.4)$$

Using the elements $[0, \gamma_i, s] \in B_i$ and $[0, m_j] \in N_j$ we have

$$v_{i+j}(\gamma_i, s m_j) = (-1)^{(p+1)} \gamma_i, v_j(m_j)$$

$$z_{i+j}(\gamma_i, s m_j) = (-1)^p \gamma_i, z_j(m_j). \quad (2.11.5)$$

Similar equations arise using the elements $[\gamma_i, s] \in B_{i+1}$ and $[0, m_j] \in N_j+1$ and the elements $[0, \gamma_i, s] \in B_i$ and $[m_j, 0] \in N_{j+1}$.

By comparing equations 2.11.4 and 2.11.5 we conclude that $z_i = (-1)^p u_i$.

Equation 2.11.1 with $\gamma_{0,1} = 1$ implies that $y_i = 0$ for all $i$. Therefore, $S_i$ has the desired form $S_i = \left[ (-1)^p z_{i-1} v_i \right]_{0 \to z_i}$. Also, equations 2.11.6 and 2.11.7 are exactly 2.11.1 and 2.11.2. This completes the proof of the forward implication. The converse is established similarly.

□
The last result of this section describes some homomorphisms between semi-free DG $B$-modules that are liftable to $A$.

Lemma 2.12. We work in the setting of Notations 2.7 and 2.8. Let $M$ and $M'$ be semi-free DG $A$-modules, and let $f \in \text{Hom}_A(M, M')_j$. If $(B \otimes_A M)_i$ is identified with $M_{i-1} \oplus M_i$ and similarly for $(B \otimes_A M')_i$, then the map $(B \otimes_A f)_i: (B \otimes_A M)_i \to (B \otimes_A M')_{i+j}$ is identified with the matrix \[\begin{pmatrix} (-1)^{i} f_{i-1} & 0 \\ 0 & f_j \end{pmatrix}.\]

Proof. This follows directly from Definition 1.19. \qed

3. Liftings and Quasi-liftings of DG Modules

In this section we prove our Main Theorem, starting with the definitions of our notions of liftings in the DG arena.

Definition 3.1. Let $T \to S$ be a morphism of DG $R$-algebras, and let $D$ be a DG $S$-module. Then $D$ is quasi-liftable to $T$ if there is a semi-free DG $T$-module $D'$ such that $D \simeq S \otimes_T D'$; in this case $D'$ is called a quasi-lifting of $D$ to $T$. We say that $D$ is liftable to $T$ if there is a DG $T$-module $D'$ such that $D \simeq S \otimes_T D'$; in this case $D'$ is called a lifting of $D$ to $T$.

Remark 3.2. In the definition of “quasi-liftable” we require that $D'$ is semi-free in order to avoid the need for derived categories. If one prefers, one can remove the semi-free assumption and require that $D \simeq S \otimes_T D'$ instead.

Our next result is a technical lemma for use in the proof of our Main Theorem.

Lemma 3.3. We work in the setting of Notations 2.7 and 2.8. Let $n \geq 1$, and let $N^{(n-1)}$ be a semi-free DG $B$-module

\[N^{(n-1)} = \cdots \to M_{i-1} \oplus M_i \xrightarrow{\begin{pmatrix} -\alpha^{(n-1)}_{i-1} t^{n-1} \delta^{(n-1)}_{i-1} \\ t \end{pmatrix}} M_{i-2} \oplus M_{i-1} \to \cdots\]

whose semi-basis over $B$ is finite in each degree. In the case $n \geq 2$, assume that for each index $i$ there are $R$-module homomorphisms $v_i^{(n-2)}: M_i \to M_{i-2}$ and $z_i^{(n-2)}: M_i \to M_{i-1}$ such that

\begin{align*}
\alpha_i^{(n-1)} &= \alpha_i^{(n-2)} + t^{n-1} z_i^{(n-2)} \\
\delta_i^{(n-1)} &= v_i^{(n-2)} - t^{n-2} z_i^{(n-2)} z_i^{(n-2)} \\
v_{i+1}^{(n-2)}(\gamma_{i,s} m_j) &= \gamma_{i,s} z_j^{(n-2)}(m_j) \\
z_{i+1}^{(n-2)}(\gamma_{i,s} m_j) &= (-1)^{i} \gamma_{i,s} z_j^{(n-2)}(m_j). \\
\alpha_{i-1}^{(n-2)} &= \alpha_{i-1}^{(n-2)} + z_{i-2}^{(n-2)} \alpha_i^{(n-2)} + t v_{i-1}^{(n-2)} = \alpha_i^{(n-2)} \\
-\alpha_{i-2}^{(n-2)} v_i^{(n-2)} + t^{n-2} \delta_i^{(n-2)} z_i^{(n-2)} - t^{n-2} \delta_i^{(n-2)} z_i^{(n-2)} - v_{i-2}^{(n-2)} \delta_i^{(n-2)} + v_{i-1}^{(n-2)} \alpha_i^{(n-2)} = 0.
\end{align*}

for $s = 1, \cdots, r_i$, for all $m_j \in M_j$, and for each integer $j$. 
If $\Ext_B^2(N^{n-1}, N^{n-1}) = 0$, then there is a semi-free DG $B$-module $N^{(n)}$ and
an isomorphism of DG $B$-modules

$$N^{(n-1)} = \cdots \rightarrow M_{i-1} \oplus M_i \rightarrow M_{i-2} \oplus M_{i-1} \rightarrow \cdots$$

such that for each index $i$ there are $R$-module homomorphisms $v^{(n-1)}_i : M_i \rightarrow M_{i-2}$
and $z^{(n-1)}_i : M_i \rightarrow M_{i-1}$ such that

$$\alpha^{(n)}_i = \alpha^{(n-1)}_i + t^n z^{(n-1)}_i$$

(3.3.7)

$$\delta^{(n)}_i = v^{(n-1)}_i - t^{n-1} z^{(n-1)}_i$$

(3.3.8)

$$v^{(n-1)}_{i+j}(\gamma_i s m_j) = \gamma_i s v^{(n-1)}_j(m_j)$$

(3.3.9)

$$z^{(n-1)}_{i+j}(\gamma_i s m_j) = (-1)^i \gamma_i s z^{(n-1)}_j(m_j).$$

(3.3.10)

$$\alpha^{(n-1)}_{i-2} z^{(n-1)}_{i-1} + z^{(n-1)}_{i-2} \alpha^{(n-1)}_{i-1} + t v^{(n-1)}_{i-1} = \delta^{(n-1)}_{i-1}$$

(3.3.11)

$$-\alpha^{(n-1)}_{i-2} v^{(n-1)}_i - t^{n-1} \delta^{(n-1)}_{i-1} z^{(n-1)}_{i-1} - t^n z^{(n-1)}_{i-2} \delta^{(n-1)}_{i-1} + \alpha^{(n-1)}_{i-1} \alpha^{(n-1)}_i = 0.$$  

(3.3.12)

for $s = 1, \cdots, r_i$, for all $m_j \in M_j$, and for each integer $j$.

Proof. By Lemma 2.6 we conclude that for all integers $i, j$ we have

$$\alpha^{(n-1)}_{i-1} \alpha^{(n)}_i = -t^n z^{(n-1)}_i$$

$$t^{n-1} \alpha^{(n)}_{i+1} \alpha^{(n-1)}_i = \alpha^{(n-1)}_i t^{n-1} \delta^{(n-1)}_{i+1}$$

(3.3.13)

$$t^{n-1} \delta^{(n)}_{i+1}(\gamma_i s m_j) = \gamma_i s t^{n-1} \delta^{(n-1)}_j(m_j)$$

(3.3.14)

$$\alpha^{(n-1)}_{i+j} \gamma_i s m_j = \alpha^{(n-1)}_i \gamma_i s m_j + (-1)^i \gamma_i s \alpha^{(n-1)}_j(m_j)$$

(3.3.15)

for $s = 1, \cdots, r_i$ and for all $m_j \in M_j$.

Note that the sequence

$$\begin{bmatrix} \delta^{(n-1)}_{i-1} & 0 \\ 0 & \delta^{(n-1)}_i \end{bmatrix} : M_{i-1} \oplus M_i \rightarrow M_{i-3} \oplus M_{i-2}$$

is a cycle of degree $-2$ in the complex $\Hom_B(N^{n-1}, N^{n-1})$. Indeed, in the case $n = 1$, this follows from equations (3.3.13), (3.3.14); in the case $n \geq 2$, this follows from equations (3.3.11)−(3.3.6). The assumption $\Ext_B^2(N^{n-1}, N^{n-1}) = 0$ implies that this cycle is null-homotopic, that is, there is a DG $B$-module homomorphism $S^{(n-1)} = \{S^{(n-1)}_i : N^{(n)} \rightarrow N^{(n-1)}\}$ of degree $-1$ such that

$$\begin{bmatrix} \delta^{(n-1)}_{i-1} & 0 \\ 0 & \delta^{(n-1)}_i \end{bmatrix} = \begin{bmatrix} -\alpha^{(n-1)}_{i-2} t^{n-1} z^{(n-1)}_i \\ t^{n-1} \alpha^{(n-1)}_{i-1} \end{bmatrix} S^{(n-1)}_{i-1} + S^{(n)}_{i+1} \begin{bmatrix} -\alpha^{(n-1)}_{i-1} t^{n-1} z^{(n-1)}_i \\ t \alpha^{(n-1)}_{i-1} \end{bmatrix}.$$

(3.3.16)

Lemma 2.11 implies that each $S^{(n-1)}_i$ is of the form

$$S^{(n-1)}_i = \begin{bmatrix} -z^{(n-1)}_{i-1} v^{(n-1)}_i \\ 0 & z^{(n-1)}_i \end{bmatrix}$$
where \( v^{(n-1)}_i : M_i \to M_{i-2} \) and \( z^{(n-1)}_i : M_i \to M_{i-1} \), and for \( s = 1, \ldots, r_i \), and for all \( m_j \in M_j \) for each integer \( j \) we have

\[
\begin{align*}
v^{(n-1)}_{i+j}(\gamma_{i,s} m_j) &= \gamma_{i,s} v^{(n-1)}_j(m_j) \quad (3.3.17) \\
z^{(n-1)}_{i+j}(\gamma_{i,s} m_j) &= (-1)^i \gamma_{i,s} z^{(n-1)}_j(m_j). \quad (3.3.18)
\end{align*}
\]

Hence for every \( i \) the equality (3.3.16) implies that we have

\[
\begin{align*}
\alpha^{(n-1)}_{i-2} z^{(n-1)}_{i-1} + z^{(n-1)}_{i-2} \alpha^{(n-1)}_{i-1} + tv^{(n-1)}_{i-1} &= \delta^{(n-1)}_{i-1} \quad (3.3.19) \\
-\alpha^{(n-1)}_{i-2} v^{(n-1)}_i + t^n z^{(n-1)}_1 z^{(n-1)}_i - t^n z^{(n-1)}_1 z^{(1-1)}_1 z^{(n-1)}_i + \alpha^{(n-1)}_{i-1} \alpha^{(n-1)}_{i-1} &= 0. \quad (3.3.20)
\end{align*}
\]

Now let

\[
\alpha^{(n)}_i = \alpha^{(n-1)}_i + t^n z^{(n-1)}_i \\
\delta^{(n)}_i = \delta^{(n-1)}_i - t^n z^{(n-1)}_i z^{(n-1)}_i
\]

and

\[
N^{(n)} = \cdots \to M_{i-1} \oplus M_i \to M_{i-2} \oplus M_{i-1} \to \cdots.
\]

Note that the conclusions (3.3.7)–(3.3.12) follow directly from (3.3.17)–(3.3.21).

Since \( N^{(n)} \) is a DG \( B \)-module, equations (3.3.13), (3.3.19), and (3.3.21) give the following equation for all \( i \):

\[
\alpha^{(n)}_{i-1} \alpha^{(n)}_i + t^{n+1} \delta^{(n)}_i = 0. \quad (3.3.22)
\]

By equations (3.3.13), (3.3.19), and (3.3.21), for all \( i \) we have

\[
-\alpha^{(n)}_{i-1} t^n \delta^{(n)}_i + t^n \delta^{(n)}_i \alpha^{(n)}_{i-1} = 0. \quad (3.3.23)
\]

For \( s = 1, \ldots, r_i \), and for all \( m_j \in M_j \), equations (3.3.14), (3.3.17), (3.3.18), and (3.3.21) give the following equality:

\[
t^n \delta^{(n)}_{i+j}(\gamma_{i,s} m_j) = \gamma_{i,s} t^n \delta^{(n)}_j(m_j). \quad (3.3.24)
\]

Also, by equations (3.3.15), (3.3.18), and (3.3.21) we conclude that

\[
\alpha^{(n)}_{i+j}(\gamma_{i,s} m_j) = \delta^{(n)}_{i+j}(\gamma_{i,s} m_j) + (-1)^i \gamma_{i,s} \alpha^{(n)}_{i+j}(m_j). \quad (3.3.25)
\]

Therefore, equations (3.3.22)–(3.3.25) imply that \( N^{(n)} \) is a semi-free DG \( B \)-module. Equations (3.3.18)–(3.3.19) and (3.3.21) provide the next morphism of DG \( B \)-modules:

\[
\begin{align*}
N^{(n-1)} &= \cdots \to M_{i-1} \oplus M_i \to M_{i-2} \oplus M_{i-1} \to \cdots \\
&\quad \downarrow \begin{bmatrix} 1 & -t^n z^{(n-1)}_j \\ 1 & 1 \end{bmatrix} \quad \downarrow \begin{bmatrix} -\alpha^{(n)}_{i-1} t^n \delta^{(n-1)}_i \\ t \alpha^{(n)}_i \end{bmatrix} \quad \downarrow \begin{bmatrix} 1 & -t^n z^{(n-1)}_j \\ 1 & 1 \end{bmatrix} \\
N^{(n)} &= \cdots \to M_{i-1} \oplus M_i \to M_{i-2} \oplus M_{i-1} \to \cdots.
\end{align*}
\]

Similar reasoning shows that the sequence \( \left\{ \begin{bmatrix} 1 & t^n z^{(n-1)}_j \\ 0 & 1 \end{bmatrix} \right\} \) is a morphism of DG \( B \)-modules, and it is straightforward to show that these sequences are inverse isomorphisms. \( \square \)

Part (v) of our Main Theorem is a consequence of the next result.
Theorem 3.4. We work in the setting of Notations 2.1 and 2.3. Assume that $R$ is $tR$-adically complete and that $N$ is semi-free such that its semi-basis over $B$ is finite in each degree. If $\Ext^2_B(N, N) = 0$, then $N$ is liftable to $A$ with semi-free lifting.

Proof. Set $N^{(0)} = N$. Here, we use a natural variation of Notation 2.3 see equation (2.6.5):

$$N^{(0)} = \cdots \to M_{i-1} \oplus M_i \to M_{i-2} \oplus M_{i-1} \to \cdots.$$  

Because of our assumptions, each $M_i$ is a finitely generated free $R$-module. Lemma 3.3 implies that for each $n \geq 1$ there is a semi-free DG $B$-module $N^{(n)}$ and an isomorphism of DG $B$-modules

$$N^{(n-1)} = \cdots \to M_{i-1} \oplus M_i \to M_{i-2} \oplus M_{i-1} \to \cdots$$

such that for each index $i$ there are $R$-module homomorphisms $v_i^{(n-1)}: M_i \to M_{i-2}$ and $z_i^{(n-1)}: M_i \to M_{i-1}$ such that

$$\alpha_i^{(n)} = \alpha_i^{(n-1)} + t^n z_i^{(n-1)} \quad (3.4.1)$$

$$\delta_i^{(n)} = v_i^{(n-1)} - t^{-1} z_i^{(n-1)} z_i^{(n-1)} \quad (3.4.2)$$

$$v_i^{(n-1)}(\gamma_{i,s} m_j) = \gamma_{i,s} v_j^{(n-1)}(m_j) \quad (3.4.3)$$

$$z_i^{(n-1)}(\gamma_{i,s} m_j) = \gamma_{i,s} z_j^{(n-1)}(m_j) \quad (3.4.4)$$

$$\alpha_i^{(n-1)} z_i^{(n-1)} + z_i^{(n-1)} \alpha_i^{(n-1)} + t v_i^{(n-1)} = \delta_i^{(n-1)} \quad (3.4.5)$$

$$-\alpha_i^{(n-1)} v_i^{(n-1)} + t^{-1} \delta_i^{(n-1)} z_i^{(n-1)} - t^{-1} z_i^{(n-1)} \delta_i^{(n-1)} + v_i^{(n-1)} \alpha_i^{(n-1)} = 0. \quad (3.4.6)$$

for $s = 1, \cdots, r_i$, for all $m_j \in M_j$, and for each integer $j$. This follows by induction on $n$; note that this uses the isomorphism $N^{(n-1)} \cong N$ in the induction step to conclude that $\Ext^2_B(N^{(n-1)}, N^{(n-1)}) \cong \Ext^2_B(N, N) = 0$.

A straightforward calculation shows that

$$\prod_{j=0}^{n-1} \begin{bmatrix} 1 - t v_i^{(j)} & \cdot \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 - \sum_{j=0}^{n-1} t v_i^{(j)} & \cdot \\ 0 & 1 \end{bmatrix}.$$
Hence, the composite isomorphism $N^{(0)} \to N^{(n)}$ has the following form:

$$N^{(0)} = \cdots \to M_{i-1} \oplus M_i \xrightarrow{\left[ \begin{array}{cc} \alpha_{i-1}^{(0)} & \delta_i^{(0)} \\ \delta_i^{(0)} & \alpha_i^{(0)} \end{array} \right]} M_{i-2} \oplus M_{i-1} \cdots$$

$$N^{(n)} = \cdots \to M_{i-1} \oplus M_i \xrightarrow{\left[ \begin{array}{cc} -\alpha_{i-1}^{(n)} & t^n \delta_i^{(n)} \\ t^n \delta_i^{(n)} & \alpha_i^{(n)} \end{array} \right]} M_{i-2} \oplus M_{i-1} \cdots$$

Furthermore, equation (3.4.11) shows that

$$\alpha_i^{(n)} = \alpha_i^{(0)} + \sum_{j=0}^{n-1} t^{j+1} z_i^{(j)}.$$

We now define $N^{(\infty)}$ as follows:

$$N^{(\infty)} = \cdots \to M_{i-1} \oplus M_i \xrightarrow{\left[ \begin{array}{cc} -\alpha_{i-1}^{(\infty)} & 0 \\ 0 & \alpha_i^{(\infty)} \end{array} \right]} M_{i-2} \oplus M_{i-1} \to \cdots$$

where

$$\alpha_i^{(\infty)} = \alpha_i^{(0)} + \sum_{j=0}^{\infty} t^{j+1} z_i^{(j)}.$$

Note that $\alpha_i^{(\infty)}$ is well-defined because $R$ is $tR$-adically complete and the modules $M_i$ and $M_{i-1}$ are finitely generated free $R$-modules.

We claim that $N^{(\infty)}$ is a semi-free DG $B$-module. For all indices $i$ and $n$, set

$$\zeta_i^{(n)} = \sum_{j=0}^{\infty} t^j z_i^{(j+n)}$$

and notice that

$$\alpha_i^{(\infty)} = \alpha_i^{(n)} + t^{n+1} \zeta_i^{(n)}.$$  \hspace{1cm} (3.4.7)

Using (3.3.22), it follows that

$$\alpha_i^{(\infty)} \alpha_i^{(\infty)} = t^{n+1} (-\delta_i^{(n)}) + \alpha_i^{(n)} \alpha_i^{(n)} + \zeta_i^{(n)} \alpha_i^{(n)} + t^{n+1} \zeta_i^{(n)} \zeta_i^{(n)}.$$  \hspace{1cm} (3.4.8)

Using (3.3.22), it follows that

$$\alpha_i^{(\infty)} \alpha_i^{(\infty)} = t^{n+1} (-\delta_i^{(n)}) + \alpha_i^{(n)} \alpha_i^{(n)} + \zeta_i^{(n)} \alpha_i^{(n)} + t^{n+1} \zeta_i^{(n)} \zeta_i^{(n)}.$$  \hspace{1cm} (3.4.9)

It follows that $\alpha_i^{(\infty)} \alpha_i^{(\infty)} = \bigcap_{n=1}^{\infty} t^{n+1} \text{Hom}_R(M_i, M_{i-1}) = 0$, by Krull’s Intersection Theorem, so we have

$$\alpha_i^{(\infty)} \alpha_i^{(\infty)} = 0.$$  \hspace{1cm} (3.4.10)

Furthermore, for $s = 1, \ldots, r_i$ and for all $m_j \in M_j$ equations (3.3.25), (3.4.4), (3.4.7), and (3.4.8) imply that

$$\alpha_i^{(\infty)} \alpha_i^{(\infty)} = \alpha_i^{(\infty)} \alpha_i^{(\infty)} = 0.$$  \hspace{1cm} (3.4.11)

Thus by equations (3.4.10), (3.4.11) and Lemma 2.6 we conclude that $N^{(\infty)}$ is a DG $B$-module.

Now consider the chain map $\varphi = \{ \varphi_i \} : N^{(0)} \to N^{(\infty)}$ defined by

$$\varphi_i = \prod_{j=0}^{\infty} \left[ \begin{array}{cc} 1 - t^j z_i^{(j)} \\ 0 \end{array} \right] = \left[ \begin{array}{cc} 1 - \sum_{j=0}^{\infty} t^j z_i^{(j)} \\ 0 \end{array} \right].$$
This map is well-defined because $R$ is $tR$-adically complete and $N_i$ is finitely generated over $R$. Using these assumptions with equation (3.4.4) we conclude that

$$\left( - \sum_{i=0}^{\infty} t^i z_{i+1}^{(l)} \right) (\gamma_i, s m_j) = (-1)^i \gamma_i, s \left( - \sum_{i=0}^{\infty} t^i z_{i}^{(l)} \right) (m_j)$$

for $s = 1, \ldots, r_i$, and for all $m_j \in M_j$ for each integer $j$. Thus $\phi$ is $B$-linear and satisfies the assumptions of Lemma 2.11 so $\phi$ is a morphism of DG $B$-modules. Similar reasoning shows that the sequence $\{ \left[ \sum_{j=0}^{\infty} t^j z_{j+1}^{(l)} \right] \}$ is a morphism of DG $B$-modules, and it is easy to show that these sequences are inverse isomorphisms.

On the other hand, by equations (3.4.10), (3.4.11) and Lemma 2.3 the sequence

$$M^{(\infty)} = \cdots \rightarrow M_i \xrightarrow{\alpha_i^{(\infty)}} M_{i-1} \rightarrow \cdots$$

is a semi-free DG $A$-module. Now Lemma 2.9 implies that $M^{(\infty)}$ is a lifting of $N^{(\infty)}$ to $A$. Since $N \cong N^{(\infty)}$ by definition, we conclude that $M^{(\infty)}$ is a lifting of $N$ to $A$, so $N$ is liftable to $A$. \hfill \Box

**Corollary 3.5.** We work in the setting of Notation 2.1. Assume that $R$ is $tR$-adically complete. Let $D$ be a DG $B$-module that is homologically both bounded below and degreewise finite. If $\Ext^2_B(D, D) = 0$, then $D$ is quasi-liftable to $A$.

**Proof.** Fact 1.19 implies that $D$ has a semi-free resolution $N \simeq D$ over $B$ such that the semi-basis for $N$ is finite in each degree. Since $\Ext^2_B(N, N) \cong \Ext^2_B(D, D) = 0$, Theorem 3.4 implies that $N$ is liftable to $A$, with semi-free lifting $M$. Thus, we have $B \otimes_A M \cong N \simeq D$, so $D$ is quasi-liftable to $A$. \hfill \Box

The proof of our Main Theorem uses induction on $n$, the length of the sequence $t$. The next result is useful for the induction step in this proof.

**Proposition 3.6.** We work in the setting of Notation 2.1. Assume that $R$ is $tR$-adically complete, and let $D$ be a DG $B$-module that is homologically bounded below and homologically degreewise finite such that $\Ext^d_B(D, D) = 0$ for some integer $d$. If $M$ is a quasi-lifting of $D$ to $A$, then $\Ext^d_A(M, M) = 0$.

**Proof.** Assume without loss of generality that $M$ is degreewise finite and bounded below; see Fact 1.19. Lemma 2.24 shows that $M$ has the shape dictated by Notation 2.3. Since $M$ is a quasi-lifting of $D$ to $A$, we see that $N = B \otimes_A M \cong R^R(t) \otimes_R M$ is a semi-free resolution of $D$ over $B$; see Lemma 2.9. To show that $\Ext^d_A(M, M) = 0$, let $f = \{ f_i : M_i \rightarrow M_{i-d} \}$ be a cycle in $\Hom_A(M, M)_{-d}$; we need to show that $f$ is null-homotopic. The fact that $f$ is a cycle says that for every $i$ we have $f_i \alpha_i+1 = (-1)^d \alpha_{i+1} - f_{i+1}$. For each $i$ set $v_i^{(-1)} = f_i$.

Claim: For all $n \geq 0$ and for all $i \in \mathbb{Z}$, there are maps $v_i^{(n)} : M_i \rightarrow M_{i-d}$ and $z_i^{(n)} : M_i \rightarrow M_{i+1-d}$ such that for $s = 1, \cdots, r_i$, and for all $m_j \in M_j$ for each $j$

$$v_i^{(n)}(\gamma_i, s m_j) = (-1)^{-id} \gamma_i, s v_j^{(n)} (m_j)$$
$$z_i^{(n)}(\gamma_i, s m_j) = (-1)^{(1-d)} \gamma_i, s z_j^{(n)} (m_j)$$

$$(-1)^{d} z_i^{(n)} \alpha_i - tv_i^{(n)} + \alpha_i - d z_i^{(n)} v_i^{(n)} = v_i^{(n-1)}$$

$$(-1)^{d} v_i^{(n)} \alpha_i - \alpha_i - d v_i^{(n)} = 0.$$ (3.6.4)

To prove the claim, we proceed by induction on $n$. We verify the base case and the inductive step simultaneously. Let $n \geq 0$ and assume that for each $i$ there
exists \( v_i^{(n-1)} : M_i \to M_{i-d} \) such that for \( s = 1, \cdots, r_i \), and for all \( m_j \in M_j \) for each integer \( j \), we have

\[
v_{i+j}^{(n-1)}(\gamma_{i,s}m_j) = (-1)^{id_i}s\gamma_{i,s}v_j^{(n-1)}(m_j)
\]

Thus, the sequence \( \{ (-1)^{d_i}s\gamma_{i,s}v_j^{(n-1)}(m_j) \} : M_{i-1} \oplus M_i \to M_{i-d-1} \oplus M_{i-d} \) is a cycle in \( \text{Hom}_B(N, N)_{-d} \), by Lemma 2.11. As \( \text{Ext}^d_B(D, D) = 0 \), this morphism is null-homotopic. Thus there exists a DG \( B \)-module homomorphism \( S^{(n)} = \{ S_i^{(n)} \} \in \text{Hom}_B(N, N)_{1-d} \) such that for every \( i \) we have

\[
\begin{pmatrix}
-\alpha_{i-d} & 0 \\
0 & -\alpha_{i-d+1}
\end{pmatrix} \sum_{i} S_i^{(n)} = (-1)^{1-d} \begin{pmatrix}
-\alpha_{i-1} & 0 \\
0 & -\alpha_{i}
\end{pmatrix} = \begin{pmatrix}
(-1)^{d_i}s\gamma_{i,s}v_{i+1}^{(n-1)} & 0 \\
0 & v_i^{(n-1)}
\end{pmatrix}.
\]

Lemma 2.11 implies that each \( S_i^{(n)} \) is of the form

\[
S_i^{(n)} = \begin{pmatrix}
(1)^{-d_i}z_{i-1}^{(n)} & 0 \\
0 & z_i^{(n)}
\end{pmatrix} : N_i \to N_{i-d+1}
\]

where \( v_i^{(n)} : M_i \to M_{i-d} \) and \( z_i^{(n)} : M_i \to M_{i+1-d} \), and for \( s = 1, \cdots, r_i \), and for all \( m_j \in M_j \) for each integer \( j \) we have

\[
v_{i+j}^{(n)}(\gamma_{i,s}m_j) = (-1)^{id_i}(-\alpha_{i-1}+\alpha_{i-d})z_{i-1}^{(n)}(m_j)
\]

\[
z_{i+j}^{(n)}(\gamma_{i,s}m_j) = (-1)^{i(1-d)}z_{i-1}^{(n)}(m_j).
\]

Hence for each \( i \), equation (3.6.7) implies that we have

\[
(-1)^{d_i}z_{i-1}^{(n)}(\gamma_{i,s}m_j) = v_{i+1}^{(n-1)} - (-1)^{d_i}z_{i-1}^{(n)}(\gamma_{i,s}m_j) = v_{i-1}^{(n-1)} + (-1)^{d_i}z_{i-1}^{(n)}(\gamma_{i,s}m_j) = 0.
\]

This completes the proof of the claim.

Equation (3.6.3) implies the following equality for each \( i \):

\[
f_i = (-1)^{d_i} \sum_{j=0}^{n} t_j z_{i-1}^{(j)} + (-1)^{d_i} \sum_{j=0}^{n} t_j z_{i}^{(j)} + t^{n+1} v_i^{(n)}.
\]

Since \( R \) is \( IR \)-adically complete and each \( M_i \) is finitely generated over \( R \), each series \( \eta_i = \sum_{j=0}^{\infty} t_j z_{i-1}^{(j)} \) converges in \( \text{Hom}_R(M_i, M_{i+1-d}) \), and for every \( i \) we have

\[
f_i = (-1)^{d_i} \sum_{j=0}^{\infty} t_j z_{i-1}^{(j)} + (-1)^{d_i} \sum_{j=0}^{\infty} t_j z_{i}^{(j)} + t^{n+1} v_i^{(n)}.
\]

By equation (3.6.2), we conclude that \( \eta_i + \gamma_j(\gamma_{i,s}m_j) = (-1)^{i(1-d)}\gamma_i, s\eta_j(m_j) \) for all \( i, j, s \). Thus, Lemma 2.11 implies that \( \eta = \{ \eta_i \} \in \text{Hom}_A(M, M) \) is a DG \( A \)-module homomorphism of degree \(-d \). Equation (3.6.12) implies that \( f = \{ f_i \} \) is null-homotopic, as desired.

The next result contains part (iii) of our Main Theorem.

**Corollary 3.7.** Let \( \underline{t} = t_1, \cdots, t_n \) be a sequence of elements of \( R \), and assume that \( R \) is \( IR \)-adically complete. Let \( D \) be a DG \( KR(\underline{t}) \)-module that is homologically bounded below and homologically degreewise finite. If \( \text{Ext}^2_{K_R(\underline{t})}(D, D) = 0 \), then \( D \) is quasi-liftable to \( R \).

**Proof.** By induction on \( n \), using Corollary 3.5 and Proposition 3.6. \( \square \)
The version of our Main Theorem used in [8] requires a bit more terminology.

**Definition 3.8.** Let $A$ be a DG $R$-algebra, and let $M$ be a DG $A$-module. For each $a \in A$ the homothety (i.e., multiplication map) $\mu^{M,a}: M \to M$ given by $m \mapsto am$ is a homomorphism of degree $|a|$. The homothety morphism $X^A_M: A \to \text{Hom}_A(M, M)$ is given by $X^A_M(a) = \mu^{M,a}$, i.e., $X^A_M(a)(m) = am$.

If $A$ is noetherian and $M$ is semi-free, then $M$ is a semidualizing DG $A$-module if $M$ is homologically finite and the homothety morphism $X^A_M: A \to \text{Hom}_A(M, M)$ is a quasiisomorphism. We say that an $R$-complex is semidualizing provided that it is semidualizing as a DG $R$-module. Let $\mathcal{G}(A)$ be the set of shift-quasiisomorphism classes of semidualizing DG $A$-modules, that is, the set of equivalence classes of semidualizing DG $A$-modules where $C \sim C'$ if there is an integer $n$ such that $C' \simeq \Sigma^nC$.

**Remark 3.9.** Let $A$ be a DG $R$-algebra, and let $M$, $M'$ be semi-free DG $A$-modules. It is straightforward to show that if $M \sim M'$, then $M$ is semidualizing if and only if $M'$ is semidualizing. Note that our semi-free assumption in the definition of semidualizing is only made in order to avoid the need for the derived category $D(A)$. If one prefers to work in $D(A)$, then a homologically finite DG $A$-module $N$ is semidualizing if (and only if) the induced map $\chi^A_N: A \to \text{RHom}_A(N, N)$ is an isomorphism in $D(A)$.

**Corollary 3.10.** Let $t = t_1, \ldots, t_n$ be a sequence of elements of $R$, and assume that $R$ is $tR$-adically complete. If $D$ is a semidualizing DG $K^R(t)$-module, then there exists a semidualizing $R$-complex $C$ which is a quasi-lifting of $D$ to $R$. Moreover, the base-change operation $C \mapsto K^R(t) \otimes_R C$ induces a bijection $\mathcal{G}(R) \xrightarrow{\cong} \mathcal{G}(K^R(t))$.

**Proof.** Note that the fact that $R$ is $tR$-adically complete implies that $tR$ is contained in the Jacobson radical of $R$. Using this, one checks readily that the conclusions of [6, Lemma A.3] hold in our setting. The existence of an $R$-complex $C$ that is a quasi-lifting of $D$ to $R$ follows from Corollary 3.8 and $C$ is semidualizing over $R$ by [6, Lemma A.3(a)]. This says that the base-change map $\mathcal{G}(R) \to \mathcal{G}(K^R(t))$ is surjective; it is injective by [6, Lemma A.3(b)].

Part 3 of our Main Theorem is a consequence of the next result.

**Theorem 3.11.** We work in the setting of Notation 2.7. Assume that $R$ is $tR$-adically complete, and assume that $R$, $A$, and $A_0$ are local, and let $D$ be a DG $B$-module that is homologically bounded below and homologically degreewise finite. If $D$ is quasi-liftable to $A$ and $\text{Ext}^1_B(D, D) = 0$, then any two homologically degreewise finite quasi-lifts of $D$ to $A$ are quasiisomorphic over $A$.

**Proof.** The assumption that $R$ is $tR$-adically complete and local implies that $t$ is in the maximal ideal $m \subset R$.

Let $C$ and $C'$ be two homologically degreewise finite semi-free $A$-modules such that $B \otimes_A C \simeq D \simeq B \otimes_A C'$. Let $M \xrightarrow{\sim} C$ and $M' \xrightarrow{\sim} C'$ be minimal semi-free resolutions of $C$ and $C'$ over $A$. Lemma 2.7 shows that $M$ and $M'$ have the shape dictated by Notation 2.3. Since $C$ and $C'$ are quasi-liftings of $D$ to $A$, we see that $N := B \otimes_A M \cong K^R(t) \otimes_R M$ and $N' := B \otimes_A M' \cong K^R(t) \otimes_R M'$ are semi-free resolutions of $D$ over $B$; see Lemma 2.9. Furthermore, from Remark 2.2 we have the isomorphism $B/m_B \otimes_B (B \otimes_A M) \cong A/m_A \otimes_A M$. 


Since $M$ is minimal over $A$, the differential on this complex is 0, so $B \otimes_A M$ is minimal over $B$, and similarly for $B \otimes_A M'$.

From [14, Theorem 2.12.5.2 and Example 2.12.5.4] there exists an isomorphism $\Upsilon: N \cong N'$. Lemma \[2.11\] implies that $\Upsilon$ has the following form

$$N = \cdots \rightarrow M_{i-1} \oplus M_i \rightarrow \begin{bmatrix} -\alpha_{i-1} & 0 \\ \alpha_i & 1 \end{bmatrix} M_{i-2} \oplus M_{i-1} \rightarrow \cdots \quad (3.11.1)$$

$$N' = \cdots \rightarrow M'_{i-1} \oplus M'_i \rightarrow \begin{bmatrix} -\alpha'_{i-1} & 0 \\ \alpha'_i & 1 \end{bmatrix} M'_{i-2} \oplus M'_{i-1} \rightarrow \cdots$$

and that we have

$$v_{i+j}(\gamma_{i,s}m_j) = (-1)^j \gamma_{i,s}v_j(m_j) \quad (3.11.2)$$

$$z_{i+j}(\gamma_{i,s}m_j) = \gamma_{i,s}z_j(m_j) \quad (3.11.3)$$

for all $i$, for $s = 1, \ldots, r_i$ and for all $m_j \in M_J$ for each integer $j$. As the diagram $\[3.11.1\]$ commutes, we have

$$z_{i-1}\alpha_i = tv_i + \alpha'_i z_i \quad (3.11.4)$$

for all $i$. Since $\Upsilon$ is an isomorphism, it follows that the $z_i$’s are isomorphisms.

The condition $\[3.11.2\]$ implies that $v = \{v_i\} \in \text{Hom}_A(M, M')_{-1}$; cf. Lemma \[2.10\]. As the diagram $\[3.11.1\]$ commutes, we have $v_{i-1}\gamma_i = -\alpha'_{i-1}v_i$ for all $i$, so $v$ is a cycle in $\text{Hom}_A(M, M')_{-1}$. This yields a cycle $B \otimes_A v \in \text{Hom}_B(N, N')_{-1}$ which has the form $\{v_i\}_{i=1}^{r_i}$. Lemma \[2.11\] implies that $\Upsilon$ has the following form

$$\Upsilon = \begin{bmatrix} u_i^{(0)} & p_i^{(0)} \\ 0 & u_i^{(0)} \end{bmatrix} \in \text{Hom}_B(N, N')_0$$

such that for every $i$ we have

$$\begin{bmatrix} -\alpha'_{i-1} & 0 \\ \alpha'_i & 1 \end{bmatrix} T_i^{(0)} - T_{i-1}^{(0)} \begin{bmatrix} -\alpha_{i-1} & 0 \\ \alpha_i & 1 \end{bmatrix} = \begin{bmatrix} -v_{i-1} & 0 \\ 0 & v_i \end{bmatrix}. \quad \text{Therefore for all } i \in \mathbb{Z} \text{ we obtain the following equations:}$$

$$-u_i^{(0)}\alpha_i + t p_i^{(0)} + \alpha'_i u_i^{(0)} = v_i$$

$$u_i^{(0)}(\gamma_{i,s}m_j) = \gamma_{i,s}u_i^{(0)}(m_j)$$

$$p_i^{(0)}(\gamma_{i,s}m_j) = (-1)^j \gamma_{i,s}p_i^{(0)}(m_j)$$

$$p_i^{(0)}\alpha_i = -\alpha'_{i-1}p_i^{(0)}.$$

The process repeats using $p^{(0)} = \{p_i^{(0)}\}$ in place of $p^{(-1)} = v$. Inductively, for each $n \geq 0$ one can construct a DG $B$-module homomorphism

$$T^{(n)} = \begin{bmatrix} u_i^{(n)} & p_i^{(n)} \\ 0 & u_i^{(n)} \end{bmatrix} \in \text{Hom}_B(N, N')_0$$

such that for every $i$ we have

$$\begin{bmatrix} -\alpha'_{i-1} & 0 \\ \alpha'_i & 1 \end{bmatrix} T_i^{(n)} - T_{i-1}^{(n)} \begin{bmatrix} -\alpha_{i-1} & 0 \\ \alpha_i & 1 \end{bmatrix} = \begin{bmatrix} -p_{i-1}^{(n-1)} & 0 \\ 0 & p_i^{(n-1)} \end{bmatrix}. \quad \text{Therefore for all } i \in \mathbb{Z} \text{ we obtain the following equations:}$$

$$-u_i^{(n)}\alpha_i + t p_i^{(n)} + \alpha'_i u_i^{(n)} = v_i$$

$$u_i^{(n)}(\gamma_{i,s}m_j) = \gamma_{i,s}u_i^{(n)}(m_j)$$

$$p_i^{(n)}(\gamma_{i,s}m_j) = (-1)^j \gamma_{i,s}p_i^{(n)}(m_j)$$

$$p_i^{(n)}\alpha_i = -\alpha'_{i-1}p_i^{(n)}.$$
Therefore for all \( i \in \mathbb{Z} \) and \( n \geq 0 \) we get the following equations:

\[
-u_{i-1}^{(n)}\alpha_t + \beta_i^{(n)} + \gamma_i^{(n)} = p_i^{(n)}
\]

and hence

\[
v_i = p_i^{(-1)} = \alpha_t \left[ \sum_{j=0}^{n} t^j u_i^{(j)} \right] + t^{n+1} p_i^{(n)} - \left[ \sum_{j=0}^{n} t^j u_i^{(j)} \right] \alpha_t. \tag{3.11.6}
\]

Since \( R \) is \( tR \)-adically complete, the next series converges for each \( i \)

\[
\xi_i = \sum_{j=0}^{\infty} t^j u_i^{(j)}
\]

and equation \((3.11.6)\) implies that

\[
v_i = \alpha_t \xi_i - \xi_{i-1} \alpha_t. \tag{3.11.7}
\]

Combining equations \((3.11.4)\) and \((3.11.7)\), for each \( i \) we have

\[
(z_{i-1} + t\xi_{i-1}) \alpha_t = \alpha_t (z_i + t\xi_i). \tag{3.11.8}
\]

This shows that the sequence \( z + t\xi : M \rightarrow M' \) is a degree-0 homomorphism of the underlying \( R \)-complexes. Combining equations \((3.11.3)\) and \((3.11.5)\), we see that

\[
(z + t\xi)_{i+j} (\gamma_{i,s} m_j) = (z + t\xi) (z + t\xi) (m_j)
\]

for all \( i \), for \( s = 1, \ldots, r_i \) and for all \( m_j \in M_j \) for each \( j \). So, Lemma \(2.10\) implies that \( z + t\xi \) is a cycle in \( \text{Hom}_A(M,M')_0 \). Since each \( z_i \) is bijective and \( t \in \mathfrak{m} \), Nakayama’s Lemma implies that for every \( i \), the map \( z_i + t\xi_i \) is also bijective. Hence \( z + t\xi \) is an isomorphism \( M \xrightarrow{\sim} M' \), so \( C \simeq M \cong M' \cong C' \), as desired. \( \square \)

Here is Main Theorem \(3\) from the introduction.

**Corollary 3.12.** Let \( t = t_1, \ldots, t_n \) be a sequence of elements of \( R \), and assume that \( R \) is local and \( tR \)-adically complete. Let \( D \) be a DG \( K^\mathfrak{m}(\mathbb{Z}) \)-module that is homologically bounded below and homologically degreewise finite. If \( D \) is quasi-liftable to \( R \) and \( \text{Ext}_{K^\mathfrak{m}(\mathbb{Z})}^1(D,D) = 0 \), then any two homologically degreewise finite quasi-lifts of \( D \) to \( R \) are quasiisomorphic over \( R \).

**Proof.** By induction on \( n \), using Theorem \(3.11\) and Proposition \(3.6\) \( \square \)

We conclude the paper with an example showing that the quasi-lifts in the previous two results must be homologically degreewise finite.

**Example 3.13.** Let \( (R, \mathfrak{m}) \) be a local integral domain that is not a field. Let \( Q(R) \) be the field of fractions of \( R \), and let \( 0 \neq t \in \mathfrak{m} \). If \( F \) is an \( R \)-free resolution of \( Q(R) \), then \( F \) and \( 0 \) are both quasi-lifts of \( 0 \) from \( K^R(t) \) to \( R \).

**References**

1. D. Apassov, *Homological dimensions over differential graded rings*, Complexes and Differential Graded Modules, Ph.D. thesis, Lund University, 1999, pp. 25–39.
2. M. Auslander, S. Ding, and Ø. Solberg, *Liftings and weak liftings of modules*, J. Algebra 156 (1993), 273–397. MR 94d:16007
3. L. L. Avramov, *Infinite free resolutions*, Six lectures on commutative algebra (Bellaterra, 1996), Progr. Math., vol. 166, Birkhäuser, Basel, 1998, pp. 1–118. MR 99m:13022
4. L. L. Avramov, H.-B. Foxby, and S. Halperin, *Differential graded homological algebra*, in preparation.

5. W. Bruns and J. Herzog, *Cohen-Macaulay rings*, revised ed., Studies in Advanced Mathematics, vol. 39, University Press, Cambridge, 1998. MR 1251956 (95h:13020)

6. L. W. Christensen and S. Sather-Wagstaff, *Descent via Koszul extensions*, J. Algebra 322 (2009), no. 9, 3026–3046. MR 2567408

7. M. Hochster, *Some applications of the Frobenius in characteristic 0*, Bull. Amer. Math. Soc. 84 (1978), no. 5, 886–912. MR 0485848 (58 #5650)

8. S. Nasseh and S. Sather-Wagstaff, *A local ring has only finitely many semidualizing complexes up to shift-isomorphism*, preprint (2012) arXiv:1201.0037.

9. W. V. Vasconcelos, *Divisor theory in module categories*, North-Holland Publishing Co., Amsterdam, 1974, North-Holland Mathematics Studies, No. 14, Notas de Matemática No. 53. [Notes on Mathematics, No. 53]. MR 0498530 (58 #16637)

10. Y. Yoshino, *The theory of L-complexes and weak liftings of complexes*, J. Algebra 188 (1997), no. 1, 144–183. MR 98i:13024

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