REEDY CATEGORIES AND THE Θ-CONSTRUCTION

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Abstract. We use the notion of multi-Reedy category to prove that, if \( C \) is a Reedy category, then \( \Theta C \) is also a Reedy category. This result gives a new proof that the categories \( \Theta_n \) are Reedy categories. We then define elegant Reedy categories, for which we prove that the Reedy and injective model structures coincide.

1. Introduction

In this note, we generalize two known facts about the category \( \Delta \), which has the structure of a Reedy category. The first is that the categories \( \Theta_k \), obtained from \( \Delta \) via iterations of the \( \Theta \) construction, are also Reedy categories. The second is that, on the category of simplicial presheaves on \( \Delta \), or functors \( \Delta^{op} \to SSets \), the Reedy and injective model structures agree.

For the first generalization, we use the notion of multi-Reedy category to prove that for any Reedy category \( C \), we get that \( \Theta C \) is also a Reedy category. For the second, we give a sufficient condition for the Reedy and injective model structures to coincide; such a Reedy category we call elegant.

A Reedy category is defined by two subcategories, the direct and inverse subcategories, and a degree function. (A precise definition is given in Section 2.) A consequence of the results of this paper is that the Reedy structure on \( \Theta_k \) is characterized by:

1. A map \( \alpha : \theta \to \theta' \) is in \( \Theta_k^- \) if and only if \( F \alpha : F \theta \to F \theta' \) is an epimorphism in \( \text{Psh}(\Theta_k) \).
2. A map \( \alpha : \theta \to \theta' \) is in \( \Theta_k^+ \) if and only if \( F \alpha : F \theta \to F \theta' \) is a monomorphism in \( \text{Psh}(\Theta_k) \).
3. There is a degree function \( \text{deg} : \text{ob}(\Theta_k) \to \mathbb{N} \), defined inductively by

\[
\text{deg}([m](\theta_1, \ldots, \theta_m)) = m + \sum_{i=1}^{m} \text{deg}(\theta_i).
\]

Here, \( \text{Psh}(\Theta_k) \) denotes the category of presheaves on \( \Theta_k \) and \( F \) denotes the Yoneda functor. In itself, this result is not new; \( \Theta_k \) was shown to be a Reedy category by Berger [3].

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Terminology 1.1. We note two differences in terms from other work. First, by “multicategory” we mean a generalization of a category in which a function has a single input but possibly multiple (or no) outputs. This notion is dual to the usual definition of multicategory, in which a function has multiple inputs but a single output, equivalently defined as a colored operad. Perhaps the structure we use would better be called a co-multicategory, but we do not because it would further complicate already cumbersome terminology.

Second, some of the ideas in this work are related to similar ones used by Berger and Moerdijk in [4]. For example, their definition of EZ-category is more general than ours, in that some of their examples fit into their framework of generalized Reedy categories.

2. Reedy categories and multi-Reedy categories

2.1. Presheaf categories. Given a small category $\mathcal{C}$, we write $\text{Psh}(\mathcal{C})$ for the category of functors $\mathcal{C}^{\text{op}} \to \text{Set}$. We write $\text{Psh}(\mathcal{C}, \mathcal{M})$ for the category of functors $\mathcal{C}^{\text{op}} \to \mathcal{M}$, where $\mathcal{M}$ is any category, and $F_C: \mathcal{C} \to \text{Psh}(\mathcal{C})$ for the Yoneda functor, defined by $(F_C(c))(d) = \mathcal{C}(d, c)$. When clear from the context, we usually write $F$ for $F_C$.

We use the following terminology. Given an object $c$ of $\mathcal{C}$ and a presheaf $X: \mathcal{C} \to \text{Set}$, a $c$-point of $X$ is an element of the set $X(c)$. Given a $c$-point $x \in X(c)$, we write $\bar{x}: Fc \to X$ for the map which classifies the element in $X(c)$.

2.2. Reedy categories. Recall that a Reedy category is a small category $\mathcal{C}$ equipped with two wide subcategories (i.e., subcategories with all objects of $\mathcal{C}$), denoted $\mathcal{C}^+$ and $\mathcal{C}^-$ and called the direct and inverse subcategories, respectively, together with a degree function $\deg: \text{ob}(\mathcal{C}) \to \mathbb{N}$ such that the following hold.

1. Every morphism $\alpha$ in $\mathcal{C}$ admits a unique factorization of the form $\alpha = \alpha^+ \alpha^-$, where $\alpha^+$ is in $\mathcal{C}^+$ and $\alpha^-$ is in $\mathcal{C}^-$.
2. For every morphism $\alpha: c \to d$ in $\mathcal{C}^+$ we have $\deg(c) \leq \deg(d)$, and for every morphism $\alpha: c \to d$ in $\mathcal{C}^-$, we have $\deg(c) \geq \deg(d)$. In either case, equality holds if and only if $\alpha$ is an identity map.

Note that, as a consequence, $\mathcal{C}^+ \cap \mathcal{C}^-$ consists exactly of the identity maps of all the objects, and that identity maps are the only isomorphisms in $\mathcal{C}$. Furthermore, for all objects $c$ of $\mathcal{C}$, the slice categories $(c \downarrow \mathcal{C}^+)$ and $(\mathcal{C}^+ \downarrow c)$ have finite-dimensional nerve.

2.3. Multi-Reedy categories. Associated to a Reedy category is a structure which looks much like that of a multicategory, which has morphisms with one input object but possibly multiple output objects.

Let $\mathcal{C}$ be a small category. For any finite sequence of objects $c, d_1, \ldots, d_m$ in $\mathcal{C}$, with $m \geq 0$, define

$$\mathcal{C}(c; d_1, \ldots, d_m) \overset{\text{def}}{=} \mathcal{C}(c, d_1) \times \cdots \times \mathcal{C}(c, d_m).$$
This notation also extends to empty sequences; \( C(c; ) \) denotes a one-point set. We refer to elements \( \alpha = (\alpha_s : c \to d_s)_{s=1,\ldots,m} \) as multimorphisms of \( C \), and we sometimes use the notation \( \alpha : c \to d_1, \ldots, d_m \) for such a multimorphism. Let \( C(*) \) denote the symmetric multicategory whose objects are those of \( C \), and whose multimorphisms \( c \to d_1, \ldots, d_m \) are as indicated above. Note that \( C(c; d) = C(c,d) \), and that \( C \) may be viewed as a subcategory of the multicategory \( C(*) \).

**Definition 2.4.** A multi-Reedy category is a small category \( C \) equipped with a wide subcategory \( C^- \subseteq C \), and a wide sub-multicategory \( C^+(*) \subseteq C(*) \), together with a function \( \deg : \text{ob}(C) \to \mathbb{N} \) such that the following hold:

1. Every multimorphism
   \[ \alpha = (\alpha_s : c \to d_s)_{s=1,\ldots,m} \]
   in \( C(*) \) admits a unique factorization of the form \( \alpha = \alpha^+ \alpha^- \), where \( \alpha^- : c \to x \) is a morphism in \( C^- \) and \( \alpha^+ : x \to d_1, \ldots, d_m \) is a multimorphism in \( C^+(*) \).

2. For every multimorphism \( \alpha : c \to d_1, \ldots, d_m \) in \( C^+(*) \) we have
   \[ \deg(c) \leq \sum_{i=1}^{m} \deg(d_i). \]

   If \( \alpha : c \to d \) is a morphism in \( C^+ = C \cap C^+(*) \), then \( \deg(c) = \deg(d) \)
   if and only if \( \alpha \) is an identity map. For every morphism \( \alpha : c \to d \)
   in \( C^- \), we have \( \deg(c) \geq \deg(d) \), with equality if and only if \( \alpha \) is an
   identity map.

Note that for degree reasons, \( C^+(c; ) = \emptyset \) if \( \deg(c) > 0 \), while \( C^+(c; ) \) is
non-empty if \( \deg(c) = 0 \). In particular, if \( c \) is any object in \( C \), there exists a
unique map \( \sigma : c \to c_0 \) in \( C^- \) where \( c_0 \) is an object of degree 0.

The proof of the following proposition follows from the above constructions.

**Proposition 2.5.** If \( C \) is a multi-Reedy category, then \( C \) is a Reedy category
with inverse category \( C^- \), direct category \( C^+ = C^+(*) \cap C \), and degree function \( \deg \).

**Example 2.6.** The terminal category \( C = 1 \), with the subcategory \( C^- = C \)
and the sub-multicategory \( C^+(*) = C(*) \), and degree function \( \deg : \text{ob}(C) \to \mathbb{N} \), defined by \( d(1) = 0 \), is a multi-Reedy category.

**Example 2.7.** Let \( \Delta \) be the skeletal category of non-empty finite totally
ordered sets. Let \( \Delta^- \subseteq \Delta \) be the subcategory of \( \Delta \) consisting of surjective
maps, and let \( \Delta^+(*) \subseteq \Delta(*) \) be the submulticategory consisting of sequences
of maps
\[ \alpha_s : [m] \to [n_s], \quad s = 1, \ldots, u \]
which form a monomorphic family; i.e., if \( \beta, \beta' : [k] \to [m] \) satisfy \( \alpha_s \beta = \alpha_s \beta' \)
for all \( s = 1, \ldots, u \), then \( \beta = \beta' \). Let \( \deg : \text{ob}(\Delta) \to \mathbb{N} \) be defined by \( \deg([m]) = m \). Then \( \Delta \) is a multi-Reedy category.
Note that the set $\Delta^+([m]; [n_1], \ldots, [n_r])$ corresponds to the set of non-degenerate $m$-simplices in the prism $\Delta^{n_1} \times \cdots \times \Delta^{n_r}$.

**Remark 2.8.** Note that the notion of multi-Reedy category, while having the structure of a multicategory, is being associated to an ordinary Reedy category. This definition can be extended to an arbitrary multicategory, thus giving rise to the notion of a “Reedy multicategory”, as we investigate briefly in Section 5.

2.9. The $\Theta$ construction. Given a small category $C$, we define $\Theta C$ to be the category whose objects are $[m](c_1, \ldots, c_m)$ where $m \geq 0$, and $c_1, \ldots, c_m \in \text{ob}(C)$, and such that morphisms $[m](c_1, \ldots, c_m) \to [n](d_1, \ldots, d_n)$ correspond to $(\alpha, \{f_i\})$, where $\alpha: [m] \to [n]$ is a morphism of $\Delta$, and for each $i = 1, \ldots, m$,

$$f_i: c_i \to d_{\alpha(i)}$$

is a multimorphism in $C(\ast)$, which is to say $f_i = (f_{ij})$ where $f_{ij}: c_i \to d_j$ for $\delta(i-1) < j \leq \delta(i)$ is a morphism of $C$.

2.10. Multi-Reedy categories preserved under applying $\Theta$. Let $C$ be a multi-Reedy category, and consider the category $\Theta C$. We make the following definitions.

- Let $(\Theta C)^- \subseteq \Theta C$ be the collection of morphisms $f = (\alpha, \{f_i\}): [m](c_1, \ldots, c_m) \to [n](d_1, \ldots, d_n)$ such that $\alpha: [m] \to [n]$ is in $\Delta^-$, and for each $i = 1, \ldots, m$ such that $\alpha(i-1) < \alpha(i)$, the map $f_i: c_i \to d_{\alpha(i)}$ is in $C^-$.
- Let $(\Theta C)^+(\ast) \subseteq (\Theta C)(\ast)$ be the collection of multimorphisms $f = (f_s)_{s=1,\ldots,u}$, where $f_s = (\alpha_s, \{f_{si}\}): [m](c_1, \ldots, c_m) \to [n_s](d_{s1}, \ldots, d_{sn_s})$ such that the multimap $\alpha = (\alpha_s): [m] \to [n_1], \ldots, [n_u]$ is in $\Delta^+(\ast)$ and for each $i$, the multimap $(f_{sij}: c_i \to d_{sj})_{s=1,\ldots,u, j=\alpha_s(i-1)+1,\ldots,\alpha_s(i)}$ is in $C^+(\ast)$.
- Let $\text{deg}: \text{ob}(\Theta C) \to \mathbb{N}$ be defined by

$$\text{deg}([m](c_1, \ldots, c_m)) = m + \sum_{i=1}^m \text{deg}(c_i).$$

**Proposition 2.11.** Let $C$ be a multi-Reedy category. Then $\Theta C$ is a multi-Reedy category, with $(\Theta C)^-, (\Theta C)^+(\ast)$ and $\text{deg}: \text{ob}(\Theta C) \to \mathbb{N}$ defined as above. In particular, $\Theta C$ admits the structure of a Reedy category.
We assure the reader that the proof is entirely formal; however, we will do our best to obscure the point by presenting a proof full of tedious multiple subscripts.

Proof. First we observe that \((\Theta C)^-\) is closed under composition and contains identity maps; i.e., it is a subcategory of \(\Theta C\). Notice that \((\Theta C)^-\) contains all identity maps. Suppose we have two morphisms in \((\Theta C)^-\) of the form

\[
[m](c_1, \ldots, c_m) \xrightarrow{f=\langle \sigma, f_s \rangle} [n](d_1, \ldots, d_n) \xrightarrow{g=\langle \tau, g_t \rangle} [p](e_1, \ldots, e_p).
\]

The composite has the form \(h = (\tau \sigma, h_i)\), where \(h_i\) is defined exactly if \(\tau \sigma(i - 1) < \tau \sigma(i)\), in which case \(h_i = g_{\sigma(i)} f_i: c_i \to e_{\tau \sigma(i)}\). Since \(\tau \sigma \in \Delta^-\) and \(h_i = g_{\sigma(i)} f_i\) is in \(C^-\), we see that \(h\) is in \((\Theta C)^-\).

Next we observe that \((\Theta C)^+(\cdot)\) is closed under multi-composition and contains identity maps; i.e., it is a sub-multicategory of \((\Theta C)(\cdot)\). Again, note that \((\Theta C)^+(\cdot)\) contains all identity maps. Suppose we have a multimorphism \(f\) in \((\Theta C)^+(\cdot)\) of the form

\[
f = (f_s: [m](c_1, \ldots, c_m) \to [n](d_{s1}, \ldots, d_{sn}))_{s=1,\ldots,u}
\]

with \(f_s = (\delta_s, f_{si})\) where \(\delta_s: [m] \to [n_s]\) and \(f_{si} = (f_{sij}: c_i \to d_j)_{\delta_s(i) < j \leq \delta_s(i)}\), and suppose we have a sequence of multimorphisms \(g_1, \ldots, g_u\) in \((\Theta C)^-\) of the form

\[
g_s = (g_{st}: [n_s](d_{s1}, \ldots, d_{sn}) \to [p_s](e_{st1}, \ldots, e_{stp}))_{t=1,\ldots,v_s},
\]

where \(g_{st} = (\varepsilon_{st}, g_{stj})\), with \(\varepsilon_{st}: [n_s] \to [p_s]\) and

\[
g_{stj} = (g_{stjk}: d_{sj} \to e_{st\kappa})_{\varepsilon_{st}(j-1) < k \leq \varepsilon_{st}(j)}.
\]

The composite multimorphism

\[
h = (h_{st}: [m](c_1, \ldots, c_m) \to [p_s](e_{st1}, \ldots, e_{stp}))_{s=1,\ldots,u, t=1,\ldots,v_u}
\]

is such that for each \(s = 1, \ldots, u\) and \(t = 1, \ldots, v_s\), the map \(h_{st} = (\varepsilon_{st} \delta_s, h_{sti})\) in \(\Theta C\) is defined so that the multimap \(h_{sti}\) in \(\mathcal{C}(\cdot)\) is given by

\[
h_{sti} = (h_{stijk} = g_{stjk} f_{sj}: c_i \to e_{st\kappa})_{\delta_s(i) < j \leq \delta_s(i), \varepsilon_{st}(j-1) < k \leq \varepsilon_{st}(j)}.
\]

Since \(\Delta^+(\cdot)\) is a sub-multicategory of \(\Delta(\cdot)\), we get that \(\varepsilon_{st} \delta_s: [n] \to [p_s]\) is a multimorphism in \(\Delta^+(\cdot)\), while since \(\mathcal{C}^+(\cdot)\) is a sub-multicategory of \(\mathcal{C}(\cdot)\), we have that for each \(s, t,\) and \(i\), the multimap \(h_{sti}\) is in \(\mathcal{C}^+(\cdot)\). Thus, the multimap \(h\) is in \((\Theta C)^+(\cdot)\) as desired.

Next, suppose we are given a multimorphism \(f = (f_s)_{s=1,\ldots,u}\) in \((\Theta C)(\cdot)\), where

\[
f_s = (\alpha_s, f_{si}): [m](c_1, \ldots, c_m) \to [p_s](e_{s1}, \ldots, e_{sp}).
\]

We will show that there is a unique factorization of \(f\) into a morphism \(g\) of \((\Theta C)^-\) followed by a multimorphism \(h\) of \((\Theta C)^+(\cdot)\). Since \(\alpha = (\alpha_s)_{s=1,\ldots,u}\) is a multimorphism in \(\Delta(\cdot)\) it admits a unique factorization \(\alpha = \delta \sigma\), where \(\sigma: [m] \to [n]\) is in \(\Delta^-\) and

\[
\delta = (\delta_s: [n] \to [p_s])_{s=1,\ldots,u}
\]
is in $\Delta^+(\ast)$. Thus, any factorization $f = hg$ of the kind we want must be such that

$$g = (\sigma, g_i), \quad g_i : c_i \to d_{\sigma(i)} \text{ defined when } \sigma(i - 1) < \sigma(i),$$

and $h = (h_s)_{s=1,\ldots,u}$ such that

$$h_s = (\delta_s, h_{sj}), \quad h_{sj} : d_j \to e_{\delta(j-1)+1}, \ldots, e_{\delta(j)},$$

and so that for each $i = 1, \ldots, m$ such that $\sigma(i - 1) < \sigma(i)$, the composite of the morphism $g_i$ of $C$ with the multimorphism $h_{s\sigma(i)} = (h_{s\sigma(i)})_{s=1,\ldots,u}$ of $C(\ast)$ must be equal to the multimorphism $f_{si} = (f_{si})_{s=1,\ldots,m}$ of $C(\ast)$. In fact, since $C$ is a multi-Reedy category, there is a unique way to produce a factorization $f_{si} = h_{s\sigma(i)}g_i$ with the property that $g_i$ is in $C^-$ and $h_{s\sigma(i)}$ is in $C^+(\ast)$.

Suppose that $f = (\sigma, f_i) : [m](c_1, \ldots, c_m) \to [n](d_1, \ldots, d_n)$ is a morphism in $(\Theta C)^-$. Then

$$\deg([m](c_1, \ldots, c_m)) = m + \sum_{i=1}^m \deg(c_i)$$

$$\geq n + \sum_{j=1}^n \deg(d_j)$$

$$= \deg([n](d_1, \ldots, d_n)).$$

The inequality in the second line follows from the fact that $m \geq n$ since $\sigma \in \Delta^-$, and the fact that for each $j = 1, \ldots, n$, there is exactly one $i$ such that $\sigma(i - 1) < j \leq \sigma(i)$, for which the map $f_i : c_i \to d_j$ in $C^-$, whence $\deg(c_i) \geq \deg(d_j)$.

If equality of degrees hold, then we must have $m = n$, whence $\sigma$ is the identity map of $[m]$, and thus we must have $\deg(c_i) = \deg(d_i)$ for all $i = 1, \ldots, m$, whence each $f_i$ is the identity map of $c_i$.

Suppose that $f = (f_s)_{s=1,\ldots,u}$ is a multimorphism in $(\Theta C)^+(\ast)$, where

$$f_s = (\delta_s, f_{si}) : [m](c_1, \ldots, c_m) \to [n_s](d_{s1}, \ldots, d_{sn_s}).$$

Since $(\delta_s) \in \Delta^+(\ast)$, we have $m \leq \sum_{s=1}^u n_s$. For each $i = 1, \ldots, m$, the multimorphism

$$f_{sis} = (f_{sij} : c_i \to d_{sj})_{s=1,\ldots,u, \ j=\delta_s(i-1)+1,\ldots,\delta_s(i)}$$

is in $C^+(\ast)$, and thus

$$\deg(c_i) \leq \sum_{s=1}^u \sum_{j=\delta_s(i-1)+1}^{\delta_s(i)} \deg(d_{sj}).$$
For each \( s = 1, \ldots, u \) and \( j = 1, \ldots, n_s \), there is at most one \( i \) such that \( \delta_s(i-1) < j \leq \delta_s(i) \). Thus
\[
\sum_{i=1}^{m} \deg(c_i) \leq \sum_{i=1}^{m} \sum_{s=1}^{u} \delta_s(i) \deg(d_{sj}) \leq \sum_{s=1}^{u} \sum_{j=1}^{n_s} \deg(d_{sj}),
\]
and thus
\[
\deg([m](c_1, \ldots, c_m)) = m + \sum_{i=1}^{m} \deg(c_i)
\leq \sum_{s=1}^{u} n_s + \sum_{s=1}^{u} \sum_{j=1}^{n_s} \deg(d_{sj})
= \sum_{s=1}^{u} \deg([n_s](d_1, \ldots, d_{n_s})).
\]

If \( u = 1 \) and if equality of degrees holds, then we must have \( m = n \), whence \( \delta_1 \) is the identity map, and then we must have \( \deg(c_i) = \deg(d_i) \) for \( i = 1, \ldots, m \), whence each \( f_i \) is an identity map. \( \square \)

**Remark 2.12.** The \( \Theta \) construction can be applied to an arbitrary multicategory \( \mathcal{M} \); when the multicategory \( \mathcal{M} = \mathcal{C}(\ast) \) for some category \( \mathcal{C} \), then the construction specializes to the one we have used. Given a suitable notion of “Reedy multicategory”, it seems that the above proof can be generalized to show that \( \Theta \mathcal{M} \) is a Reedy multicategory whenever \( \mathcal{M} \) is; we state this result in Section 5. These ideas generalize Angeltveit’s work on enriched Reedy categories constructed from operads [1].

### 2.13. The direct sub-multicategory of \( \Theta \mathcal{C} \)

We give a criterion which can be useful for identifying the morphisms of \( (\Theta \mathcal{C})^+ \), and more generally the multimorphisms of \( (\Theta \mathcal{C})^+(\ast) \).

Given a multimorphism \( f = (f_s: c \to d_s)_{s=1, \ldots, u} \) in the multicategory \( \mathcal{C}(\ast) \) associated to a category \( \mathcal{C} \), let \( Ff \) denote the induced map of of \( \mathcal{C} \)-presheaves
\[
(Ff_1, \ldots, Ff_u): Fc \to Fd_1 \times \cdots \times Fd_u.
\]

**Proposition 2.14.** Let \( \mathcal{C} \) be a multi-Reedy category, and suppose that for every \( f \) in \( \mathcal{C}^+(\ast) \), the map \( Ff \) is a monomorphism in \( \text{Psh}(\mathcal{C}) \). Then for every \( g \) in \( (\Theta \mathcal{C})^+(\ast) \), the map \( Fg \) is a monomorphism in \( \text{Psh}(\Theta \mathcal{C}) \).

**Proof.** Let \( g = (g_s)_{s=1, \ldots, u} \) be a multimorphism in \( (\Theta \mathcal{C})^+(\ast) \), where
\[
g_s = (\beta_s, g_{sj}): [n](d_1, \ldots, d_n) \to [p_s](e_{s1}, \ldots, e_{sp_s}).
\]
We need to show that if
\[
f, f': [m](c_1, \ldots, c_m) \to [n](d_1, \ldots, d_n)
\]
are maps in \( \Theta \mathcal{C} \) such that \( g_s f = g_s f' \) for all \( s = 1, \ldots, u \), then \( f = f' \). Write \( f = (\alpha, f_i) \) and \( f' = (\alpha', f'_i) \). Then \( g_s f = g_s f' \) implies \( \beta_s \alpha = \beta_s \alpha' \) for all \( s \),
whence $\alpha = \alpha'$ since 

$$(F\beta_s): F[n] \to F[p_1] \times \cdots \times F[p_u]$$

is a monomorphism in $\text{Psh}(\Delta)$. Thus for each $i = 1, \ldots, m$ and $\alpha(i-1) < j \leq \alpha(i)$ we have $f_i, f'_i: c_i \to d_j$, which satisfy $g_{sj}f_i = g_{sj}f'_i$ for all $s = 1, \ldots, u$. By hypothesis on $C$, it follows that $f_i = f'_i$. \hfill \Box

3. Elegant Reedy categories

In this section, we give sufficient conditions on a Reedy category to ensure that the Reedy and injective model structures agree. The categories of degeneracies and inclusions considered by Baues in [2] are similar.

3.1. Degenerate and non-degenerate points. Let $C$ be a Reedy category, and suppose that $X$ is an object of $\text{Psh}(C)$.

**Definition 3.2.** A $c$-point $x \in X(c)$ is **degenerate** if there exist $\alpha: c \to d$ in $C^-$ and $y \in X(d)$ such that

1. $(X\alpha)(y) = x$, and
2. $\alpha$ is not an identity map (or equivalently, $\deg(c) > \deg(d)$).

A $c$-point $x \in X(c)$ is **non-degenerate** if it is not degenerate.

We write $X_{\text{dg}}(c), X_{\text{nd}}(c) \subseteq X(c)$ for the subsets of degenerate and non-degenerate $c$-points of $X$, respectively; thus

$$X(c) = X_{\text{dg}}(c) \amalg X_{\text{nd}}(c).$$

If $f: X \to Y$ in $\text{Psh}(C)$ is a map, then $f(X_{\text{dg}}(c)) \subseteq Y_{\text{dg}}(c)$, while $f^{-1}(Y_{\text{nd}}(c)) \subseteq X_{\text{nd}}(c)$.

**Definition 3.3.** A $c$-point $x \in X(c)$ is a **degeneracy** of $y \in X(d)$ if there exists $\alpha: c \to d$ in $C^-$ such that $x = (X\alpha)(y)$. Thus, every point is a degeneracy of itself; a point is non-degenerate if and only if it is a degeneracy of only itself.

Because the slice category $(c \downarrow C^-)$ is finite dimensional, every point in $X$ is the degeneracy of at least one non-degenerate point.

For an object $c \in C$, a point $\alpha \in (Fc)(d)$ is non-degenerate if and only if $\alpha: c \to d$ is in $C^+$. **Warning:** It is not the case that $\alpha: c \to d$ in $C^+$ implies that $F\alpha: Fc \toFd$ is injective.

3.4. Elegant Reedy categories.

**Definition 3.5.** A Reedy category $C$ is **elegant** if

(E) for every presheaf $X$ in $\text{Psh}(C)$, every object $c \in C$, and every $c$-point $x \in X(c)$, there exists a unique pair $(\sigma: c \to d \in C^- \text{ and } y \in X_{\text{nd}}(d))$ such that $(X\sigma)(y) = x$. 
In other words, elegant Reedy categories have the feature that every point of every presheaf is uniquely a degeneracy of unique non-degenerate point. The standard example of an elegant Reedy category is the category $\Delta$, as we will see in the next section.

Condition (E) admits the following equivalent reformulation.

(E') For every presheaf $X$ in $\text{Psh}(C)$ and every object $c$ in $C$, the map

$$\prod_{d \in \text{ob}(C)} \prod_{x \in X_{nd}(d)} C^-(c, d) \rightarrow X(c),$$

$$(d, x, \alpha) \mapsto (X\alpha)(x)$$

is a bijection.

3.6. Characterization of elegant Reedy categories. The material in this section is prefigured in Gabriel-Zisman [5, §II.3].

**Definition 3.7.** A strong pushout in a category $C$ is a commutative square in $C$ such that its image under the Yoneda functor $F: C \rightarrow \text{Psh}(C)$ is a pushout square.

Note that every strong pushout is actually a pushout in $C$.

**Proposition 3.8.** Let $C$ be a Reedy category. Then $C$ is elegant if and only if the following property (SP) holds.

(SP) Every pair of maps $\sigma_s: c \rightarrow d_s$, $s = 1, 2$, in $C^-$, extends to a commutative square in $C^-$ which is a strong pushout in $C$. That is, there exist $\tau_s: d_s \rightarrow e$ in $C^-$ such that $\tau_1\sigma_1 = \tau_2\sigma_2$ and such that

$\begin{array}{ccc}
F_c & F_d_1 \\
F_{\sigma_2} \downarrow & \downarrow F_{\tau_1} \\
F_{d_2} & F_e \\
F_{\sigma_1} \downarrow & \downarrow F_{\tau_2} \\
F_d_1 & \\
\end{array}$

in a pushout square in $\text{Psh}(C)$.

We note some immediate consequences of property (SP).

1. In a Reedy category, all isomorphisms are identity maps, and thus colimits are unique up to identity if they exist. Thus, the strong pushout guaranteed by property (SP) is unique up to identity.

2. If $\sigma: c \rightarrow d$ is in $C^-$, then $F\sigma: Fc \rightarrow Fd$ is a surjective map of presheaves. That is,

$$\text{colim}(Fd \leftarrow Fc \xrightarrow{F\sigma} Fd) \xrightarrow{(F1_d, F1_d)} Fd$$

is an isomorphism. By condition (SP), there are maps $\tau_s: j \rightarrow k$ for $s = 1, 2$ such that $\tau_1\sigma = \tau_2\sigma$ fitting into a strong pushout square.
Then there is a unique $\gamma: e \to d$ in $C$ making the diagram commute. Since $\gamma \tau_s = 1_d$ and $\tau_s \in C^-$ for $s = 1, 2$, we must have that $\gamma$ is an identity map, since $C$ is a Reedy category.

(3) The preceding remark implies that each $\sigma: c \to d$ in $C^-$ is a split epimorphism. That is, $\sigma$ is a map such that there exists $\delta: d \to c$ in $C$ such that $\sigma \delta = 1_d$. Furthermore, a morphism $\alpha: c \to d$ in $C$ is in $C^-$ if and only if $F\alpha$ is surjective; to prove the if part, note that any split epimorphism in $C$ is necessarily in $C^-$.

(4) The slice category $(c \downarrow C^{-})$ is cocomplete. Since all morphisms are epimorphisms, $(c \downarrow C^{-})$ is a poset. It has an initial object $1_c: c \to c$, and has finite coproducts by property (SP), and so has a finite colimits. Since $(c \downarrow C^{-})$ has finite dimensional nerve, it trivially has all filtered colimits.

To prove the proposition, we use the following lemma, suggested by the referee.

**Lemma 3.9.** All idempotents in a Reedy category are split, and thus all retracts of representable presheaves are representable. In particular, if $C$ is a Reedy category and $e: c \to c$ is such that $ee = e$, then there exists $\sigma: d \to c$ in $C^+$ and $\rho: c \to d$ in $C^-$ such that $1_d = \rho \sigma$ and $e = \sigma \rho$; if $X$ is a retract of $Fc$ with associated idempotent $F\epsilon$, then $X \cong Fd$.

**Proof.** First, we factor $\epsilon = \sigma \rho$ with $\rho: c \to d$ in $C^-$ and $\sigma: d \to c$ in $C^+$. Then we factor $\rho \sigma = \sigma' \rho'$ with $\rho'$ in $C^-$ and $\sigma'$ in $C^+$. By the unique factorization property of Reedy categories, the identity $\sigma \rho = \sigma \rho \sigma \rho = \sigma \sigma' \rho' \rho$ implies that $\rho'$ and $\sigma'$ are identity maps, whence $\rho \sigma = 1_d$ as desired. The statement about retracts follows easily. $\square$

**Proof of Proposition 3.8.** Suppose $C$ is a Reedy category which satisfies property (SP). To prove (E), suppose that $x \in X(c)$, and suppose that we are given $\sigma_s: c \to d_s$ in $C^-$ and $y_s \in X_{nd}(d_s)$ for $s = 1, 2$, such that
\((X\sigma_s)(y_s) = x\). Then there is a unique dotted arrow \(\bar{z}\) making the diagram commute, where the pair of maps \(\tau_s: d_s \to e\) in \(\mathcal{C}^-\) forms the strong pushout in \(\mathcal{C}\) of the original pair of maps \(\sigma_s\). But since \(y_1\) and \(y_2\) are non-degenerate, we must have that \(\tau_1\) and \(\tau_2\) are identity maps, whence \(y_1 = y_2\) and \(\sigma_1 = \sigma_2\).

Next we prove that if \(\mathcal{C}\) is an elegant Reedy category, then property (SP) holds. Suppose that \(\sigma_s: c \to d_s\), \(s = 1, 2\), is a pair of maps in \(\mathcal{C}^-\). Let \(X\) denote the pushout of \(F\sigma_1\) and \(F\sigma_2\) in \(\text{Psh}(\mathcal{C})\), with maps \(\bar{y}_s: Fd_s \to X\) such that \(\bar{y}_1(F\sigma_1) = \bar{y}_2(F\sigma_2)\). We write \(y_s \in X(d_s)\) for the point corresponding to the map \(\bar{y}_s\). Recalling the fact given immediately after Definition 3.3, there exist \(\tau_s: d_s \to e_s\) in \(\mathcal{C}^-\) and \(z_s \in X_{nd}(e_s)\) for \(s = 1, 2\) such that \((X\tau_s)(z_s) = y_s\). Since \((X\tau_1\sigma_1)(z_1) = (X\tau_2\sigma_2)(z_2)\), the uniqueness statement of (E) implies that \(e_1 = e_2\), \(z_1 = z_2\), and \(\tau_1\sigma_1 = \tau_2\sigma_2\). Write \(z = z_1\) and \(e = e_1\), and consider the commutative diagram

\[
\begin{array}{ccc}
F\sigma_1 & \xrightarrow{F\tau_1} & F\sigma_2 \\
\downarrow{\bar{y}_1} & \quad & \downarrow{\bar{y}_2} \\
F\tau_1 & \xrightarrow{Fd_1} & F\tau_2 \\
\downarrow{X} & \quad & \downarrow{X} \\
F & \xrightarrow{f} & Fe \\
\downarrow{\bar{y}_1} & \quad & \downarrow{\bar{y}_2} \\
\bar{z} & \xrightarrow{\bar{z}} & \bar{z} \\
\end{array}
\]

The arrow \(f\) exists because \(X\) is a pushout, and we have \(\bar{zf} = 1_X\). Therefore \(X\) is a retract of \(Fe\), and hence is representable by (3.9), and thus provides the desired strong pushout. \(\square\)

As a consequence, we obtain the following.

**Proposition 3.10.** If \(\mathcal{C}\) is an elegant Reedy category, and if \(f: X \to Y\) is a monomorphism between presheaves of sets on \(\mathcal{C}\), then \(X_{nd}(c) = f^{-1}(Y_{nd}(c))\) as subsets of \(X(c)\).

**Proof.** It is clear that \(f^{-1}(Y_{nd}(c)) \subseteq X_{nd}(c)\) from the definition of non-degeneracy. To show that \(X_{nd}(c) \subseteq f^{-1}(Y_{nd}(c))\), note that if \(x \in X_{nd}(c)\) but \(f(x) \in Y(c)\) is degenerate, then there exists \(\alpha: c \to d\) in \(\mathcal{C}^-\) fitting into
a commutative square

\[ \begin{array}{ccc}
Fc & \xrightarrow{\alpha} & X \\
\downarrow & & \downarrow f(c) \\
F \bar{d} & \xrightarrow{y} & Y
\end{array} \]

As noted in remark (2) after (3.8) above, property (SP) implies that if \( \sigma: c \to d \) is in \( \mathcal{C}^- \), then \( F \sigma \) is an epimorphism of presheaves. Since \( f(c) \) is a monomorphism there must exist a dotted arrow making the diagram commute, contradicting the hypothesis that \( x \) is non-degenerate. \( \square \)

3.11. **Equivalence of Reedy and injective model structures.** Let \( \mathcal{C} \) be a Reedy category. Given a presheaf \( X \) in \( \text{Psh}(\mathcal{C}, \mathcal{M}) \) on \( \mathcal{C} \) taking values in some cocomplete category \( \mathcal{M} \), for each object \( c \) in \( \mathcal{C} \) the latching object at \( c \) is an object \( L_c X \) of \( \mathcal{M} \) together with a map \( p_c = p_c^X: L_c X \to X(c) \), defined by

\[
L_c X \overset{\text{def}}{=} \text{colim}_{(\alpha: c \to d) \in \partial(c \downarrow \mathcal{C}^-)} X(d) \xrightarrow{(X \alpha)} X(c),
\]

where \( \partial(c \downarrow \mathcal{C}) \) denotes the full subcategory of the slice category \( (c \downarrow \mathcal{C}) \) whose objects are morphisms \( \alpha: c \to d \) which are not in \( \mathcal{C}^+ \). It is straightforward to show that the inclusion functor \( \partial(c \downarrow \mathcal{C}^-) \to \partial(c \downarrow \mathcal{C}) \) is final, so that the natural map

\[
L_c X \to \text{colim}_{(\alpha: c \to d) \in \partial(c \downarrow \mathcal{C}^-)} X(d)
\]

is an isomorphism.

In the case that \( X \) is a set-valued presheaf (i.e., an object of \( \text{Psh}(\mathcal{C}) \)), it is clear that for each object \( c \) in \( \mathcal{C} \) the map \( p_c \) factors through a surjection \( q_c = q_c^X: L_c X \to X_{\text{dg}}(c) \).

**Proposition 3.12.** Let \( \mathcal{C} \) be an elegant Reedy category. Then for each set-valued presheaf \( X \) on \( \mathcal{C} \) and each object \( c \) of \( \mathcal{C} \), the map \( q_c^X: L_c X \to X_{\text{dg}}(c) \) is a bijection.

**Proof.** We have already noticed that \( q_c^X \) is surjective, so it suffices to prove injectivity. Given \( x \in X_{\text{dg}}(c) \), observe that the preimage of \( x \) in \( L_c X \) may be identified with the colimit of the functor \( F_x: \partial(c \downarrow \mathcal{C}^-) \to \text{Set} \) which sends \( \alpha: c \to d \) in \( \partial(c \downarrow \mathcal{C}^-) \) to the set of \( y \in X(d) \) such that \( X(\alpha)(y) = x \). The colimit of \( F_x \) is thus isomorphic to the set of path components of the category \( \mathcal{F}_x \), having

- objects the triples \((d, \alpha, y)\) where \( d \) is an object of \( \mathcal{C} \), \( \alpha: c \to d \) in \( \partial(c \downarrow \mathcal{C}^-) \), and \( y \in X(d) \) such that \( X(\alpha)(y) = x \), and
- morphisms \((d, \alpha, y) \to (d', \alpha', y')\) the maps \( \beta: d \to d' \) in \( \mathcal{C}^- \) such that \( \alpha' = \beta \alpha \) and \( X(\beta)(y') = y \).

The condition that \( \mathcal{C} \) is elegant says that for each \( x \in X_{\text{dg}}(c) \), the category \( \mathcal{F}_x \) has an initial object (namely, the unique nondegenerate point associated to \( x \)), and therefore the colimit of \( F_x \) (equal to \( q_X^{-1}(x) \)) must be a single point, as desired. \( \square \)
Remark 3.13. In a preprint version of this paper, we made a stronger statement in place of (3.12): namely, that $\mathcal{C}$ is elegant if and only if all $q^X_c: L_c X \to X_{dg}(c)$ are bijections. This assertion is presumably not true, since for $q^X_c$ to be a bijection it suffices for each category $\mathcal{F}_x$ (as in the proof of (3.12)) to have have connected nerve, whereas elegance imposes the stronger condition that each $\mathcal{F}_x$ have an initial object.

Proposition 3.14. Let $\mathcal{C}$ be an elegant Reedy category. Then for every monomorphism $f: X \to Y$ in $\text{Psh}(\mathcal{C})$, and every object $c$ of $\mathcal{C}$, the induced map

$$g_c: X(c) \coprod_{L_c X} L_c Y \to Y(c)$$

is a monomorphism.

Proof. Given a map $f: X \to Y$ in $\text{Psh}(\mathcal{C})$, and an object $c$ of $\mathcal{C}$, we consider the following commutative diagram.

The map $u$ in the diagram is the evident isomorphism produced using the disjoint union decomposition $X(c) \cong X_{nd}(c) \coprod X_{dg}(c)$. Recall from the discussion above that there is a tautological map $p^X_c: L_c X \to X(c)$, which factors through a surjection $q^X_c: L_c X \to X_{dg}(c)$. The map $g_c$ in the diagram is induced by the tautological maps $p^X_c$ and $p^Y_c$, while the map $\bar{q}$ in the diagram is induced by the surjections $q^X_c$ and $q^Y_c$, and therefore is itself a surjection. The map $g'_c$ is the unique one such that $g'_c \bar{q} = g_c$, and $g'_c = g_c u$.

We first prove that (1) implies (2), i.e., if $\mathcal{C}$ is elegant and $f$ is a monomorphism, then the map $g_c$ is a monomorphism. Given an injective map $f: X \to Y$, and an object $c$ in $\mathcal{C}$, Lemma 3.12 implies that $q^X_c$ and $q^Y_c$ are isomorphisms, and hence the map $\bar{q}$ in the above diagram is an isomorphism. Therefore, it will suffice to show that $g''_c$ is a monomorphism. The restriction $g''_c|_{X_{nd}(c)}$ is the inclusion of $X_{dg}(c)$ in $Y(c)$, and so is injective. The restriction $g''_c|_{X_{nd}(c)}$ is equal to $f|_{X_{nd}(c)}$. Thus, to show that $g''_c$ is injective it suffices to show that

(i) $f|_{X_{nd}(c)}$ is injective, and
(ii) $f(X_{nd}(c)) \subseteq Y_{nd}(c)$.

Statement (i) follows since $f$ is injective, and statement (ii) is (3.10). Thus $g''_c$ is injective, and thus $g_c$ is injective.

Next we show that (2) implies (1). If $X = \emptyset$ and we consider a map $f: \emptyset \to Y$ and an object $c$ in $\mathcal{C}$, then condition (2) implies that $g_c: L_c Y \to Y(c)$ is injective, which implies that the surjection $\bar{q} = q_c: L_c Y \to Y_{dg}(c)$ is actually an isomorphism. Thus we have proved (E').
Proposition 3.15. Let $C$ be an elegant Reedy category, and let $\mathcal{M} = \operatorname{Psh}(\mathcal{D}, \mathbf{SSets})$ be the category of simplicial set-valued presheaves on $\mathcal{D}$, equipped with the injective model category structure (in which cofibrations are precisely the monomorphisms, and weak equivalences are pointwise). Then the injective and Reedy model structures on $\operatorname{Psh}(C, \mathcal{M})$ coincide.

Proof. Reedy cofibrations are always monomorphisms in $\operatorname{Psh}(C, \mathcal{M})$ by [6, 15.7.2]. The converse statement, that monomorphisms in $\operatorname{Psh}(C, \mathcal{M})$ are Reedy cofibrations, was proved for elegant Reedy categories $C$ in the previous proposition. □

4. The Eilenberg-Zilber Lemma

We describe a way to prove that certain Reedy categories are elegant, using an observation of Eilenberg and Zilber. The notion of “EZ-Reedy category” has also been described in [4] and [7], with a somewhat different formulation.

Let $C$ be a Reedy category. Given a map $\alpha : c \to d$ in $C$, let $\Gamma(\alpha)$ denote the set of sections of $\alpha$; that is,

$$\Gamma(\alpha) = \{ \beta : d \to c \in C \mid \alpha \beta = 1_d \}.$$ 

Note that if $\sigma : c \to d$ is a map in $C^-$, then $\Gamma(\sigma) \subseteq C^+(d, c)$.

Definition 4.1. A Reedy category $C$ is an EZ-Reedy category if the following two conditions hold.

1. (EZ1) For all $\sigma : c \to d$ in $C^-$, the set $\Gamma(\sigma)$ is non-empty.
2. (EZ2) For all pairs of maps $\sigma, \sigma' : c \to d$ in $C^-$, if $\Gamma(\sigma) = \Gamma(\sigma')$, then $\sigma = \sigma'$.

Note that if $C$ is EZ-Reedy, then every $\sigma : c \to d$ in $C^-$ is a split epimorphism, and therefore $F\sigma : Fc \toFd$ is a surjection in $\operatorname{Psh}(C)$.

The following argument is due to Eilenberg and Zilber; it is proved in [5, §II.3].

Proposition 4.2. If $C$ is an EZ-Reedy category, then $C$ is elegant.

Proof. Suppose that $C$ is an EZ-Reedy category. Let $f : X \to Y$ be a monomorphism in $\operatorname{Psh}(C)$, and suppose that $x \in X(c)$ and $f(x) \in Y_d(c)$. Thus, there exist $\sigma : c \to d$ in $C^-$ and $y \in Y(d)$ such that $(Y\sigma)(y) = f(x)$ and $\sigma$ is not an identity map. Since $\Gamma(\sigma)$ is non-empty by (EZ1), $\sigma$ is a split epimorphism, and thus $F\sigma$ is a surjection in $\operatorname{Psh}(C)$. Therefore, a dotted arrow exists in the diagram

$$\begin{array}{ccc}
\text{Fc} & \xrightarrow{\bar{x}} & X \\
\downarrow F\sigma & & \downarrow f \\
Fd & \xrightarrow{\bar{y}} & Y
\end{array}$$

thus showing that $x \in X(c)$ is also degenerate. This proves property (E1).
Now suppose \( x \in X(c) \), and that there are \( \sigma_s : c \to d_s \) in \( C^- \) and \( y_s \in X_{ud}(d_s) \) such that \((X\sigma_s)(y_s) = x\), for \( s = 1, 2 \). For any choices of \( \delta_s \in \Gamma(\sigma_s) \), we have a diagram

\[
\begin{array}{ccc}
F\delta_1 & \xrightarrow{F\delta_2} & Fd_1 \\
\downarrow & & \downarrow \\
Fc & \xrightarrow{F\sigma_1} & Fd_2 \\
\downarrow & & \downarrow \\
F\sigma_2 & \xrightarrow{F\sigma_2} & Fd_2 \\
\end{array}
\]

in which \( \bar{y}_1 = y_1(F\sigma_1)(F\delta_1) = \bar{x}(F\delta_1) = y_2(F(\sigma_2\delta_1)) \) and \( \bar{y}_2 = y_2(F\sigma_2)(F\delta_2) = \bar{x}(F\delta_2) = y_1(F(\sigma_1\delta_2)) \). Since \( \bar{y}_1 \) and \( \bar{y}_2 \) are non-degenerate points with a common degeneracy, it follows that \( d_1 = d_2, \bar{y}_1 = \bar{y}_2, \) and \( \sigma_2\delta_1 = 1 = \sigma_1\delta_2 \). Since \( \delta_1 \) and \( \delta_2 \) were arbitrary choices of sections, we see that \( \Gamma(\sigma_1) = \Gamma(\sigma_2) \), and thus \( \sigma_1 = \sigma_2 \). Thus we have proved property (E2). \( \square \)

4.3. The category \( \Theta_k \) is EZ-Reedy, and so is elegant.

**Proposition 4.4.** Let \( C \) be a multi-Reedy category such that \( C(c, d) \) is nonempty for all objects \( c, d \) of \( C \). If \( C \) is an EZ-Reedy category, then so is \( \Theta C \).

**Proof.** Fix a morphism

\[
f = (\sigma, f_i) : [m](c_1, \ldots, c_m) \to [n](d_1, \ldots, d_n)
\]

in \( (\Theta C)^- \). We first determine the structure of the set of sections \( \Gamma(f) \). Observe that the functor \( \Theta C \to \Delta \) induces a natural map \( \varphi_f : \Gamma(f) \to \Gamma(\sigma) \). For each \( \delta \in \Gamma(\sigma) \), we write \( \Gamma_\delta(f) \) for the fiber of \( \varphi \) over \( \delta \). It is straightforward to check that \( \Gamma_\delta(f) \) consists of all maps of the form \( g = (\bar{\delta}, g_j) \), where each \( g_j = (g_{ji} : d_{j} \to c_{i})_{\delta(j - 1) \leq i \leq \delta(j)} \) is a multimorphism in \( C \), with the following property: for \( \delta \) such that \( \sigma(i - 1) < j = \sigma(i) \), we have \( g_{ji} \in \Gamma(f_i) \).

Thus, \( \Gamma_\delta(f) \) is in bijective correspondence with a subset of

\[
\prod_{j = 1}^{n} \prod_{i = \delta(j - 1) + 1}^{\delta(j)} C(d_{j}, c_{i})
\]

namely the set \( G_\delta(f) = \prod_{j = 1}^{n} \prod_{i = \delta(j - 1) + 1}^{\delta(j)} G_{ij}(f) \), where

\[
G_{ij}(f) = \begin{cases} 
\Gamma(f_i) & \text{if } \sigma(i - 1) < j = \sigma(i), \\
C(d_{j}, c_{i}) & \text{otherwise}.
\end{cases}
\]

Because every set \( C(d_{j}, c_{i}) \) is non-empty, and since \( \Gamma(f_i) \) is non-empty by hypothesis, we have that \( \Gamma_\delta(f) \) is non-empty for each \( \delta \in \Gamma(\sigma) \). Thus \( \varphi_f \) is surjective, and since \( \Gamma(\sigma) \) is non-empty, proving (EZ1).

Now suppose \( f = (\sigma, f_i) \) and \( f' = (\sigma', f'_i) \) are two maps \([m](c_1, \ldots, c_m) \to [n](d_1, \ldots, d_n)\) in \((\Theta C)^-\), and that \( \Gamma(f) = \Gamma(f') \). Since \( \phi_f : \Gamma(f) \to \Gamma(\sigma) \) and \( \phi_{f'} : \Gamma(f') \to \Gamma(\sigma') \) are surjective, we must have \( \Gamma(\sigma) = \Gamma(\sigma') \), and thus
\[ \sigma = \sigma'. \] Thus, for each \( i \) and \( j \) such that \( \sigma(i-1) < j = \sigma(i) \), we have maps \( f_{ij}, f'_{ij}: c_i \to d_j \). For each \( \delta \in \Gamma(\sigma) \) we therefore have \( \Gamma_\delta(f) = \Gamma_\delta(f') \), which must therefore both correspond to the same subset of
\[
\prod_{j=1}^n \prod_{\delta(j-1)+1}^{\delta(j)} C(d_j, c_i).
\]
Therefore \( G_{ij}(f) = G_{ij}(f') \) for all \( \delta(j-1) < i \leq \delta(j) \). In particular, for every \( i = 1, \ldots, n \) such that \( \sigma(i-1) < \sigma(i) \), we have that \( \Gamma(f) = G_{i\sigma(i)}(f') = \Gamma(f'_i) \), and hence \( f_i = f'_i \) since \( C \) is EZ-Reedy.

**Corollary 4.5.** For all \( k \geq 0 \), \( \Theta_k \) an elegant Reedy category, with direct and inverse subcategories as described in the introduction.

**Proof.** Proposition 2.11 allows us to put a Reedy model category structure on each \( \Theta_k \), and we have shown that with this structure, every \( \alpha: \theta \to \theta' \) in \( \Theta_k^+ \) induces a monomorphism \( F\alpha: F\theta \to F\theta' \) of presheaves.

Induction on \( k \), together with the fact that each \( \Theta_k \) has a terminal object, shows that each \( \Theta_k \) with the multi-Reedy structure is an EZ-Reedy category. It follows that \( \Theta_k \) is elegant, and also that every \( \alpha: \theta \to \theta' \) in \( \Theta_k^- \) induces an epimorphism \( F\alpha: F\theta \to F\theta' \) of presheaves.

Since any map of presheaves factors uniquely, up to isomorphism, into an epimorphism followed by a monomorphism, this argument shows that \( \Theta_k^- \) and \( \Theta_k^+ \) must be exactly the classes of maps described in the introduction. \( \square \)

## 5. Reedy multicategories

With the definition of multi-Reedy category, which is a multicategory arising from a Reedy category, one might ask whether the notion of Reedy category can be extended to that of a Reedy multicategory. In this section, we propose a definition and give examples.

**Definition 5.1.** A Reedy multicategory is a symmetric multicategory \( \mathcal{A} \) equipped with the following structure: wide submulticategories \( \mathcal{A}^- \) and \( \mathcal{A}^+ \), where \( \mathcal{A}^- \) only has multimorphisms with valence at most 1, and \( \mathcal{A}^+ \) only has multimorphisms with valence at least 1, together with a function \( \text{deg}: \text{ob}(\mathcal{A}) \to \mathbb{N} \) such that the following properties hold.

1. For all \( f: a \to b_1, \ldots, b_m \) in \( \mathcal{A} \) with \( m \geq 1 \), there exists a unique factorization in \( \mathcal{A} \) of the form \( f = f^- f^+ \), where \( f^- \) is in \( \mathcal{A}^- \) and \( f^+ \) is in \( \mathcal{A}^+ \).

2. For all \( f: a \to b \) in \( \mathcal{A}^- \), we have \( \text{deg}(a) \geq \text{deg}(b) \), with equality if and only if \( f \) is an identity map. For all \( f: a \to b_1, \ldots, b_m \) in \( \mathcal{A}^+ \), we have \( \text{deg}(a) \leq \sum_{i=1}^m \text{deg}(b_i) \); when \( m = 1 \), equality holds if and only if \( f \) is an identity map.

Note that the underlying category of \( \mathcal{A} \) is a Reedy category.
Example 5.2. Let $A$ be a symmetric multicategory with one object, i.e., an operad. Write $A_n$ for $A(a; a, \ldots, a)$, where $a$ is the unique object. Suppose that $A_0 = A_1 = \ast$. Then $A$ can be given the structure of a Reedy multicategory, with $A_k^- = A_k$ for $k \leq 1$ and $A_k^+ = A_k$ for $k \geq 1$, and with $\deg(a) = 0$.

Given a symmetric multicategory $A$, we define a category $\Theta A$ as follows. The objects of $\Theta A$ are $[m](a_1, \ldots, a_m)$, $m \geq 0$, $a_1, \ldots, a_m \in \text{ob}(A)$. The morphisms $f : [m](a_1, \ldots, a_m) \to [n](b_1, \ldots, b_n)$ are given by data $(\alpha, \{f_i\})$, where $\alpha : [m] \to [n]$ in $\Delta$, and for each $i = 1, \ldots, m$, $f_i : a_i \to b_{\delta(i)-1+1}^{\delta(i)}$, $\ldots$, $b_{\delta(i)}$ in $A$.

The category $\Theta A$ can be extended to a multicategory. A multimorphism $f : [m](a_1, \ldots, a_m) \to [n_1](b_{11}, \ldots, b_{1n_1}), \ldots, [n_u](b_{u1}, \ldots, b_{un_u})$ is given by $(\alpha, \{f_i\})$ consisting of a multimorphism $\alpha : [m] \to [n_1], \ldots, [n_u]$ in $\Delta$, i.e., a sequence of morphisms $\alpha_s : [m] \to [n_s]$ in $\Delta$, and for each $i = 1, \ldots, m$ a multimorphism $f_i : a_i \to b_{1,\alpha_1(i-1)+1}, \ldots, b_{1,\alpha_1(i)}, \ldots, b_{u,\alpha_u(i-1)+1}, \ldots, b_{u,\alpha_u(i)}$; i.e., a multimorphism with target $(b_{s,j})_{1 \leq s \leq u, \delta_s(i-1) < j \leq \delta_s(i)}$.

As remarked above, the same sort of argument used to prove Proposition 2.11 can be used to establish the following result.

Proposition 5.3. $\Theta A$ is a Reedy multicategory.

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