On Holomorphic Contractibility of Teichmüller Spaces

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Abstract: The problem of the holomorphic contractibility of Teichmüller spaces $T(0, n)$ of the punctured spheres ($n > 4$) arose in the 1970s in connection with solving algebraic equations in Banach algebras. Recently it was solved by the author. In the present paper, we give a refined proof of the holomorphic contractibility for all spaces $T(0, n)$, $n > 4$ and provide two independent proofs of holomorphic contractibility for low-dimensional Teichmüller spaces, which has intrinsic interest.

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1. Preamble
1.1. Holomorphic Contractibility

A complex Banach manifold $X$ is contractible to its point $x_0$ if there exists a continuous map $F : X \times [0, 1] \rightarrow X$ with $F(x, 0) = x$ and $F(x, 1) = x_0$ for all $x \in X$. If map $F$ can be chosen so that for every $t \in [0, 1]$ the map $F_t : x \mapsto F(x, t)$ of $X$ is holomorphic to itself and $F_1(x_0) = x_0$, then $X$ is called holomorphically contractible to $x_0$.

The problem of holomorphic contractibility of Teichmüller spaces $T(0, n)$ of the punctured spheres ($n > 4$) arose in the 1970s in connection with solving algebraic equations in Banach algebras. It was caused by the fact that in the space $\mathbb{C}^m$, $m > 1$, there are domains (even bounded) that are only topologically but not holomorphically contractible (see [1–5]).

The simplest example of holomorphically contractible domains in complex Banach spaces is given by starlike domains. However, all Teichmüller spaces of sufficiently great dimensions are not starlike (see [6,7]).

Earle [8] established the holomorphic contractibility of two modified Teichmüller spaces related to asymptotically conformal maps.

Recently, this problem was solved positively in [9]. It was established that all spaces $T(0, n)$, $n > 4$, are holomorphically contractible.

Theorem 1. Any space $T(0, n)$ with $n > 4$ is holomorphically contractible.

The proof of Lemma 3 in that paper contains a wrong assertion (which is not used here) that the map $s_m$, including the space $T(0, n)$ into $T(0, n)$, is a section of the forgetful map $\chi_m : T(0, n) \rightarrow T(0, n)$. Such sections do not exist if $n > 6$.

In fact, $s_m$ as an open holomorphic map (of a domain onto a manifold) was only used in the proof of Lemma 3 (and of Theorem 1), and the openness is preserved for the limit map $s = \lim_{m \rightarrow \infty} s_m$, which determines an $(n - 3)$-dimensional complex submanifold $s T(0, n)$ in the universal Teichmüller space $T$.

In the present paper, we give a refined proof of holomorphic contractibility for all spaces $T(0, n)$, $n > 4$ and provide two independent proofs of holomorphic contractibility for low-dimensional Teichmüller spaces (of dimensions two and three). The second proof
has its own interest in view of the importance of the problem. Its underlying idea is different; the arguments do not extend to higher dimensions.

1.2. Teichmüller Spaces of Low Dimensions

There are two Teichmüller spaces of dimension two: the space $T(0,5)$ of the spheres with five punctures and the space $T(1,2)$ of twice-punctured tori; these spaces are biholomorphically equivalent. Such spheres and tori are uniformized by the corresponding Fuchsian groups $\Gamma$ and $\Gamma'$ so that $\Gamma$ is a subgroup of index two in $\Gamma'$; letting $T(0,5) = T(\Gamma)$, $T(1,2) = T(\Gamma')$, we have $T(\Gamma') = T(\Gamma)$.

In a similar way, the Teichmüller spaces $T(0,6)$ of spheres with six punctures and $T(2,0)$ of closed Riemann surfaces of genus 2 also are biholomorphically equivalent, and in terms of the corresponding Fuchsian groups $\Gamma$ and $\Gamma'$ we have the same relationship $T(\Gamma') = T(\Gamma)$. We state:

**Theorem 2.** The spaces $T(0,5)$, $T(1,2)$, $T(0,6)$, $T(2,0)$ are holomorphically contractible.

The Teichmüller space $T(1,3)$ of tori with three punctures also has three dimensions; it will not be involved here.

2. Underlying Results

2.1. Teichmüller Spaces of Punctured Spheres

Consider the ordered $n$-tuples of points

$$a = (0, 1, a_1, \ldots, a_{n-3}, \infty), \quad n > 4,$$

with distinct $a_j \in \hat{C} \setminus \{1, -1, i\}$ and the corresponding punctured spheres

$$X_a = \hat{C} \setminus \{0, 1, a_1, \ldots, a_{n-3}, \infty\}, \quad \hat{C} = \mathbb{C} \cup \{\infty\},$$

regarded as Riemann surfaces of genus zero. Fix a collection $a^0 = (0, 1, a^0_1, \ldots, a^0_{n-3}, 1, \infty)$ defining the base point $X_{a^0}$ of Teichmüller space $T(0,n) = T(X_{a^0})$ of $n$-punctured spheres. Its points are the equivalence classes $[\mu]$ of Beltrami coefficients from the ball

$$\text{Belt}(\mathbb{C})_1 = \{\mu \in L_\infty(\mathbb{C}) : \|\mu\|_\infty < 1\},$$

under the relationship that $\mu_1 \sim \mu_2$ if the corresponding quasiconformal homeomorphisms $w^{\mu_1}, w^{\mu_2} : X_{a^0} \to X_a$ (the solutions of the Beltrami equation $\overline{\partial} w = \mu \partial w$ with $\mu = \mu_1, \mu_2$) are homotopic on $X_{a^0}$ (and hence coincide at the points $0, 1, a^0_1, \ldots, a^0_{n-3}, \infty$). This models $T(0,n)$ as the quotient space

$$T(0,n) = \text{Belt}(\mathbb{C})_1 / \sim$$

with a complex Banach structure of dimension $n - 3$ inherited from the ball $\text{Belt}(\mathbb{C})_1$.

Another canonical model of $T(0,n) = T(X_{a^0})$ is obtained using the uniformization of Riemann surfaces and the holomorphic Bers embedding of Teichmüller spaces. Consider the upper and lower half-planes

$$U = \{z = x + iy \in \mathbb{C} : y > 0\}, \quad U^* = \{z = x + iy \in \mathbb{C} : y < 0\}$$

The holomorphic universal covering map $h : U \to X_{a^0}$ provides a torsion-free Fuchsian group $\Gamma_0$ of the first kind acting discontinuously on $U \cup U^*$, and the surface $X_{a^0}$ is represented (up to conformal equivalence) as the quotient space $U/\Gamma_0$. The functions $\mu \in L_\infty(X_{a^0}) = L_\infty(\mathbb{C})$ are lifted to $U$ as the Beltrami $(-1, 1)$-measurable forms $\bar{\mu}d\bar{z}/dz$ on $U$ with respect to $\Gamma_0$ satisfying $(\bar{\mu} \circ \gamma)^\prime/\gamma^\prime = \bar{\mu}, \gamma \in \Gamma_0$ and forming the corresponding Banach space $L_\infty(U, \Gamma_0)$. We extend these $\bar{\mu}$ by zero to $U^*$ and consider the unit ball.
Axioms 2022, 11, 548

with a uniform estimate of the ratio of quasiconformal maps of $X$ to $\mathbb{C}$, locally, in the space $\mathcal{B}(\Gamma_0)$. The corresponding quasiconformal maps $w^\mu$ are conformal on the half-plane $U^{*}$, and their Schwarzian derivatives,

$$S_w(z) = \left( \frac{w''(z)}{w'(z)} \right)' - \frac{1}{2} \left( \frac{w''(z)}{w'(z)} \right)^2, \quad w = w^\mu,$$

fill a bounded domain in the complex $(n - 3)$-dimensional space $\mathcal{B}(\Gamma_0) = \mathcal{B}(U^*, \Gamma_0)$ of hyperbolically bounded holomorphic $\Gamma_0$-automorphic forms of degree $-4$ on $U^*$ (i.e., satisfy $(\varphi \circ \gamma)(\gamma')^2 = \varphi$, $\gamma \in \Gamma_0$), with norm

$$\|\varphi\| = \sup_{U^*} 4y^2|\varphi(z)|.$$

This domain models the Teichmüller space $T(\Gamma_0)$ of the group $\Gamma_0$. It is canonically isomorphic to the space $T(X_{\varphi})$ (and to a bounded domain in the complex Euclidean space $\mathbb{C}^{n-3}$).

The indicated map $\hat{\mu} \to S_{w^\mu}$ determines a holomorphic map $\phi_T : \text{Belt}(U, \Gamma_0)_1 \to \mathcal{B}(\Gamma_0)$; it has only local holomorphic sections.

Note also that $T(\Gamma_0) = T \cap \mathcal{B}(\Gamma_0)$, where $T$ is the universal Teichmüller space (modeled as a domain of the Schwarzian derivatives of all univalent functions on $U^*$ admitting quasiconformal extension to $U$).

2.2. Connection with Interpolation of Univalent Functions

The collections (1) fills a domain $D_n$ in $\mathbb{C}^{n-3}$ obtained by deleting from this space the hyperplanes $\{z = (z_1, \ldots, z_{n-3}) : z_j = z_l, j \neq l\}$, and with $z_1 = 0, z_2 = 1$. This domain represents the Teorelli space of the spheres $X_a$ and is covered by $T(0, n)$, which is given by the following lemma (cf., e.g., [10]; [11], Section 2.8).

**Lemma 1.** The holomorphic universal covering space of $D_n$ is the Teichmüller space $T(0, n)$. This means that for each punctured sphere $X_a$, there is a holomorphic universal covering

$$\pi_a : T(0, n) = T(X_a) \to D_n.$$

The covering map $\pi_a$ is well defined by

$$\pi_a \circ \phi_a(\mu) = (0, 1, w^\mu(a_1), \ldots, w^\mu(a_{n-3}), \infty),$$

where $\phi_a$ denotes the canonical projection of the ball Belt($\mathbb{C}$) onto the space $T(X_a)$.

Lemma 1 also yields that the truncated collections $a_a = (a_1, \ldots, a_{n-3})$ provide the local complex coordinates on the space $T(0, n)$ and define its complex structure.

These coordinates are simply connected with the Bers local complex coordinates on $T(0, n)$ (related to basis of the tangent spaces to $T(0, n)$ at its points, see [12]) via standard variation of quasiconformal maps of $X_a = U/\Gamma_a$

$$w^\mu(z) = z - \frac{z(z-1)}{\pi} \int_{\mathbb{C}} \frac{\mu(\zeta)}{\zeta(z-1)(\zeta - z)} d\zeta d\eta + O(||\mu||^2_{\infty})$$

$$= z - \frac{z(z-1)}{\pi} \sum_{\gamma \in \Gamma_a} \int_{\Gamma \Gamma_a} \frac{\gamma(\zeta) |\gamma'(\zeta)|^2}{\gamma(\zeta - 1)(\gamma - z)} d\zeta d\eta + O(||\mu||^2_{\infty}).$$

with a uniform estimate of the ratio $O(||\mu||^2_{\infty}) / ||\mu||^2_{\infty}$ on compacts in $\mathbb{C}$ (see, e.g., [13]).

It turns out that one can obtain the whole space $T(X_{\varphi})$ using only the similar equivalence classes $[\mu]$ of the Beltrami coefficients from the ball $\mu \in \text{Belt}(U)_1$ (vanishing on $U^*$),
requiring that the corresponding quasiconformal homeomorphisms \( w^\mu \) are homotopic on the punctured sphere \( X_{w^\mu} \). Surjectivity of this holomorphic quotient map

\[
\chi : \text{Belt}(U)_1 \to T(0,n),
\]

is a consequence of the following interpolation result from [14].

**Lemma 2.** Given two cyclically ordered collections of points \( (z_1, \ldots, z_m) \) and \( (\xi_1, \ldots, \xi_m) \) on the unit circle \( S^1 \), there exists a holomorphic univalent function \( f \) in the closure of the unit disk \( \mathbb{D} = \{ |z| < 1 \} \) such that \( f(z) < 1 \) for \( z \in \overline{\mathbb{D}} \) distinct from \( z_1, \ldots, z_m \), and \( f(z_k) = \xi_k \) for all \( k = 1, \ldots, m \). Moreover, there exist univalent polynomials \( f \) with such an interpolation property.

It follows that the function \( f \) given by Lemma 2 is actually holomorphic and univalent (hence, maps conformally) in a broader disk \( \mathbb{D}_r, r > 1 \).

First of all, \( f'(z) \neq 0 \) on the unit circle \( S^1 \). Indeed, if \( f'(z_0) = 0 \) at some point \( z_0 \in S^1 \), then in its neighborhood \( f(z) = c_s(z - z_0)^s + O((z - z_0)^{s+1}) = c_s z^s \), where \( c_s \neq 0 \) for some \( s > 1 \), which contradicts the injectivity of \( f(z) \) on \( S^1 \). Therefore, \( f \) is univalent in some disk \( \mathbb{D}_r = \{ |z| < r \}, r > 1 \).

Assuming, on the contrary, that \( f \) is not globally univalent in any admissible disk \( \mathbb{D}_r \) with \( r > 1 \), one obtains the distinct sequences \( \{ z'_n \}, \{ z''_n \} \subset \mathbb{D}_r \) with \( f(z'_n) = f(z''_n) \) for any \( n \), whose limit points \( z'_0, z''_0 \) lie on \( S^1 \). Then, in the limit, we have \( f(z'_0) = f(z''_0) \), which in the case \( z'_0 \neq z''_0 \), contradicts the univalence of \( f \) on \( S^1 \) given by Lemma 2, and in the case \( z'_0 = z''_0 = z_0 \), contradicts the local univalence of \( f \) in a neighborhood of \( z_0 \).

The interpolating function \( f \) given by Lemma 2 is extended quasiconformally to the whole sphere \( \hat{\mathbb{C}} \) across any circle \( \{ |z| = r \} \) with \( r > 1 \) indicated above. Hence, given a cyclically ordered collection \( (z_1, \ldots, z_m) \) of points on \( S^1 \), then for any ordered collection \( (\xi_1, \ldots, \xi_m) \) in \( \hat{\mathbb{C}} \), there exists a quasiconformal homeomorphism \( \hat{f} \) of the sphere \( \hat{\mathbb{C}} \) carrying the points \( z_j \) to \( \xi_j, j = 1, \ldots, m \), and such that its restriction to the closed disk \( \overline{\mathbb{D}} \) is biholomorphic on \( \overline{\mathbb{D}} \).

Taking the quasicircles \( L \) passing through the points \( \xi_1, \ldots, \xi_m \) and such that each \( \xi_j \) belongs to an analytic subarc of \( L \), one obtains quasiconformal extensions of the interpolating function \( f \), which are biholomorphic on the union of \( \overline{\mathbb{D}} \) and some neighborhoods of the initial points \( z_1, \ldots, z_m \in S^1 \). Now Lemma 1 provides quasiconformal extensions of \( f \) lying in prescribed homotopy classes of homeomorphisms \( X_z \to X_{w^\mu} \).

**2.3. The Bers fiber space**

Pick a space \( T(0,n) = T(X_{w^\mu}) \) with \( n \geq 5 \) and let

\[
X_{w^\mu} = X_{w^\mu} \setminus \{ a_{n-3}^0 \} = U/\Gamma_0.
\]

Due to the Bers isomorphism theorem, the space \( T(X_{w^\mu}) \) is biholomorphically isomorphic to the Bers fiber space

\[
\mathcal{F}(0,n) := \mathcal{F}(T(X_{w^\mu})) = \{(\phi T(\mu), z) \in T(X_{w^\mu}) \times \hat{\mathbb{C}} : \mu \in \text{Belt}(U, \Gamma_0)_1, z \in w^\mu(\overline{\mathbb{D}}) \}
\]

over \( T(X_{w^\mu}) \) with holomorphic projection \( \pi(\varphi, z) = \varphi \) (\( \varphi \in T(X_{w^\mu}) \) (see [15]).

This fiber space is a bounded hyperbolic domain in \( B(\Gamma_0) \times \hat{\mathbb{C}} \) and represents the collection of domains \( D_\mu = w^\mu(U) \) (the universal covers of the surfaces \( X_{w^\mu} \)) as a holomorphic family over the space \( T(0,n-1) = T(X_{w^\mu}) \).

The indicated isomorphism between \( T(0,n+1) \) and \( \mathcal{F}(0,n) \) is induced by the inclusion map \( j : \mathbb{D} \to \hat{\mathbb{D}} \) forgetting the puncture at \( a_{n-3}^0 \), via

\[
\mu \mapsto (S_{w^\mu}, w^\mu(\overline{a_{n-3}^0})) \quad \text{with} \quad \mu_1 = j_\mu := (\mu \circ j_0)^2 / j_0^2,
\]

(2)
where \( \widehat{\gamma} \) is the lift of \( j \) to \( U \) and \( \tilde{a}_{n-3}^0 \) is one of the points from the fiber over \( a_n^0 \) under the quotient map \( U \rightarrow U/\Gamma_0 \).

Note also that the holomorphic universal covering maps \( h : U^* \rightarrow U^*/\Gamma_0 \) and \( h' : U^* \rightarrow U^*/\Gamma_0' \) (and similarly, the corresponding covering maps in \( U \)) are related by

\[
j \circ h' = h \circ \widehat{j},
\]

where \( \widehat{j} \) is the lift of \( j \). This induces a surjective homomorphism of the covering groups \( \theta : \Gamma_0 \rightarrow \Gamma_0' \) by

\[
\widehat{j} \circ \gamma = \theta(\gamma) \circ \gamma, \quad \gamma \in \Gamma_0',
\]

and the norm preserving isomorphism \( \widehat{j}_* : B(\Gamma_0) \rightarrow B(\Gamma_0') \) by

\[
\widehat{j}_* \varphi = (\varphi \circ \widehat{j}) (\widehat{j})^2,
\]

which projects to the surfaces \( X_{\varphi} \) and \( X_{\varphi}' \) as the inclusion of the space \( Q(X_{\varphi}) \) of holomorphic quadratic differentials corresponding to \( B(\Gamma_0) \) in the space \( Q(X_{\varphi}') \) (cf. [16]).

The Bers theorem is valid for the Teichmüller space \( T(X_0 \setminus \{x_0\}) \) of any punctured hyperbolic Riemann surface \( X_0 \setminus \{x_0\} \) and implies that \( T(X_0 \setminus \{x_0\}) \) is biholomorphically isomorphic to the Bers fiber space \( F(T(X_0)) \) over \( T(X_0) \).

2.4. Holomorphic Curves and Holomorphic Sections

The group \( \Gamma_0 \) uniformizing the surface \( X_{\varphi} \) acts discontinuously on the fiber space \( F(\Gamma_0) \) as a group of biholomorphic maps by

\[
\gamma(\phi_T(\mu), z) = (\phi_T(\mu), \gamma^\mu z),
\]

where \( \mu \in \text{Belt}(U, \Gamma_0), \; z \in w^\mu(U), \; \gamma \in \Gamma_0, \) and

\[
\gamma^\mu \circ w^\mu = w^\mu \circ \gamma
\]

(see [15]). The quotient space

\[
V(0, n) := V(\Gamma_0) = T(0, n + 1)/\Gamma_0
\]

is called the \( n \)-punctured Teichmüller curve and depends only on the analytic type of the \( \Gamma_0 \) group. The projection \( \pi : F(0, n) \rightarrow T(0, n) \) induces a holomorphic projection

\[
\pi_1 : V(0, n) \rightarrow T(0, n).
\]

This curve is also a complex manifold with fibers \( \pi^{-1}(x) = X_{\varphi} \).

Due to the deep Hubbard–Earle–Kra theorem [16,17], the projections \( V(0, n) \rightarrow T(0, n) \) and \( (4) \) have no holomorphic sections for any \( n \geq 7 \) (more generally, for all spaces \( T(\Gamma) \) corresponding to groups \( \Gamma \) without elliptic elements, excluding the spaces \( T(1, 2) \simeq T(0, 5) \) and \( T(2, 0) \simeq T(0, 6) \)). Such sections exist for \( \Gamma \) groups containing elliptic elements.

In the exceptional cases of \( T(1, 2) \) and \( T(2, 0) \), there is a group \( \Gamma' \) that contains \( \Gamma \) as a subgroup of index two. Then, \( T(\Gamma') = T(\Gamma), \; F(\Gamma') = F(\Gamma), \) and the elliptic elements \( \gamma \in \Gamma' \) produce the indicated holomorphic sections \( s \) as the maps

\[
\phi_T(\mu) \mapsto (\phi_T(\mu), w^\mu(z_0)),
\]

where \( z_0 \) is a fixed point of \( \gamma \) in the half-plane \( U \). These sections are called the Weierstrass sections (in view of their connection with the Weierstrass points of the hyperelliptic surface \( U/\Gamma \)). We describe these sections following [16].
We also consider the punctured fiber space $\mathcal{F}_0(\Gamma)$ to be the largest open dense subset of $\mathcal{F}(\Gamma)$ on which $\Gamma$ acts freely and let

$$\mathcal{V}'(\Gamma) = \mathcal{F}_0(\Gamma)/\Gamma.$$  

For $\Gamma$ with no elliptic elements, the universal covering space for $\mathcal{V}'(g, n) = \mathcal{V}'(\Gamma)$ is $T(g, n + 1)$.

If $\Gamma$ contains elliptic elements $\gamma$, then any holomorphic section $T(\Gamma) \to \mathcal{F}(\Gamma)$ is determined by map (6) so that $w^\mu(z_0)$ is exactly one fixed point of corresponding map (4) in the fiber $w^\mu(U)$. These holomorphic sections take their values in the set $\mathcal{V}(\Gamma) \setminus \mathcal{V}'(\Gamma)$ and do not provide, in general, sections of projection (5).

In the case of spaces $T(1, 2)$ and $T(2, 0)$, either of the corresponding curves $\mathcal{V}(1, 2)$ or $\mathcal{V}(2, 0)$ has a unique biholomorphic self-map $\gamma$ of order two, which maps each fiber into itself. The fixed-point locus of $\gamma$ is a finite set of connected closed complex submanifolds of $\mathcal{V}'(g, n)$, and the restriction of map (5) to one of these submanifolds is a holomorphic map onto $T(0, n)$; its inverse is a Weierstrass section. The restriction of $\gamma$ to each fiber is a conformal involution of the corresponding hyperelliptic Riemann surface interchanging its sheets, and the fixed points of $\gamma$ are the Weierstrass points on this surface.

In dimension one, there are three biholomorphically isomorphic Teichmüller spaces $T(1, 0)$, $T(1, 1)$ and $T(0, 4)$ (see, e.g., [15,18]). We shall use the last two spaces. Their fiber space $\mathcal{F}(0, 4)$ is isomorphic to $T(0, 5)$.

As a special case of the Hubbard–Earle–Kra theorem [16,17], we have:

**Lemma 3.** (a) The curve $\mathcal{V}(0, 4)$ has, for any of its points $x$, a unique holomorphic section $s$ with $s(\pi_1(x)) = x$.

(b) If $\dim \mathcal{V}(g, n) > 1$, only curves $\mathcal{V}(1, 2)'$ and $\mathcal{V}(2, 0)'$ have holomorphic sections, which are their Weierstrass sections.

In particular, curve $\mathcal{V}(2, 0)$ has six disjoint holomorphic sections corresponding to the Weierstrass points of hyperelliptic surfaces of genus two.

3. Holomorphic Maps of $T(0, n)$ into Universal Teichmüller Space and Holomorphic Contractibility

3.1. Equivalence Relations

The universal Teichmüller space $T = \text{Teich}(U)$ is the space of quasisymmetric homeomorphisms of the unit circle factorized by Möbius maps; all Teichmüller spaces have their isometric copies in $T$.

The canonical complex Banach structure on $T$ is defined by the factorization of the ball of the Beltrami coefficients

$$\text{Belt}(U)_1 = \{ \mu \in L_\infty(\mathbb{C}) : \mu|U^* = 0, \|\mu\|_\infty < 1 \}$$

(i.e., supported in the upper-half plane), letting $\mu_1, \mu_2 \in \text{Belt}(U)_1$ be equivalent if the corresponding quasiconformal maps $w^{\mu_1}, w^{\mu_2}$ coincide on $\mathbb{R} = \mathbb{R} \cup \{\infty\} = \partial U^*$ (hence, on $U^*$). Such $\mu$ and the corresponding maps $w^\mu$ are called $T$-equivalent. The equivalence classes $[w^\mu]_T$ are in one-to-one correspondence with the Schwarzian derivatives $S_\phi$ in $U^*$, which fill a bounded domain in the space $B = B(U^*)$ (see Section 2.1).

The map $\phi_T : \mu \to S_\phi$ is holomorphic and descends to a biholomorphic map of the space $T$ onto this domain, which we will identify with $T$. As was mentioned above, it contains the Teichmüller spaces of all hyperbolic Riemann surfaces and of Fuchsian groups as complex submanifolds.

On this ball, we also define another equivalence relationship, letting $\mu, \nu \in \text{Belt}(U)_1$ be equivalent if $w^\mu(a_j) = w^\nu(a_j)$ for all $j$ and the homeomorphisms $w^\mu, w^\nu$ are homotopic on the punctured sphere $X_2$. Let us call such $\mu$ and $\nu$ strongly $n$-equivalent.
Lemma 4. If the coefficients $\mu, \nu \in \text{Belt}(U)_1$ are $T$-equivalent, then they are also strongly $n$-equivalent.

The proof of this lemma is given in [19].

In view of Lemmas 1 and 4, the above factorizations of the ball Belt$(U)_1$ generate (by descending to the equivalence classes) a holomorphic map $\chi$ of the underlying space $T$ into $T(0, n) = T(X_{\phi})$.

This map is a split immersion (has local holomorphic sections), which is a consequence, for example, of the following existence theorem from [13], which we present here as

Lemma 5. Let $D$ be a finitely connected domain on the Riemann sphere $\hat{\mathbb{C}}$. Assume that there are a set $E$ of positive, two-dimensional Lebesgue measures and a finite number of points $z_1, \ldots, z_m$ distinguished in $D$. Let $\alpha_1, \ldots, \alpha_m$ be non-negative integers assigned to $z_1, \ldots, z_m$, respectively, so that $\alpha_j = 0$ if $z_j \in E$.

Then, for a sufficiently small $\epsilon_0 > 0$ and $\epsilon \in (0, \epsilon_0)$, and for any given collection of numbers $w_{0j}, s = 0, 1, \ldots, \alpha_j, j = 1, 2, \ldots, m$, which satisfy the conditions $w_{0j} \in D$,

$$|w_{0j} - z_j| \leq \epsilon, \quad |w_{1j} - 1| \leq \epsilon, \quad |w_{sj}| \leq \epsilon (s = 0, 1, \ldots, \alpha_j, j = 1, \ldots, m),$$

there exists a quasiconformal automorphism $h$ of domain $D$, which is conformal on $D \setminus E$ and satisfies

$$h^{(s)}(z_j) = w_{sj} \quad \text{for all } s = 0, 1, \ldots, \alpha_j, j = 1, \ldots, m.$$

Moreover, the Beltrami coefficient $\mu_h$ of $h$ on $E$ satisfies $\|\mu_h\|_{\infty} \leq M$. The constants $\epsilon_0$ and $M$ depends only upon the sets $D, E$ and the vectors $(z_1, \ldots, z_m)$ and $(\alpha_1, \ldots, \alpha_m)$.

3.2. Surjectivity

In fact, we have more, given by the following theorem.

Theorem 3. Map $\chi$ is surjective and generates an open holomorphic map $s$ of the space $T(0, n) = T(X_{\phi})$ into the universal Teichmüller space $T$, embedding $T(0, n)$ into $T$ as a $(n - 3)$-dimensional submanifold.

In particular, this theorem corrects the assertion of Lemma 3 in [9] (mentioned in the preamble).

Proof of Theorem 3. The surjectivity of $\chi$ is a consequence of Lemma 2. To construct $s$, we take a dense subset $e = \{x_1, x_2, \ldots\} \subset X_{\phi} \cap \mathbb{R}$ accumulating to all points of $\mathbb{R}$ and considering the punctured spheres $X^{m}_{\phi'} = X_{\phi'} \setminus \{x_1, \ldots, x_m\}$ with $m > 1$. The equivalence relations on Belt$(\mathbb{C})_1$ for $X^{m}_{\phi'}$ and $X_{\phi'}$ generate the corresponding holomorphic map $\chi_m : T(X^{m}_{\phi'}) \to T(X_{\phi'})$. □

Uniformizing the surfaces $X_{\phi}$ and $X^{m}_{\phi}$ by the corresponding torsion-free Fuchsian groups $\Gamma_0$ and $\Gamma_0^m$ of the first kind acting discontinuously on $U \cup U'$ and applying the construction from Section 2.3 to $U' / \Gamma_0$ and $U' / \Gamma_0^m$ (forgetting the additional punctures), one obtains, similar to (3), the norm-preserving isomorphism $\tilde{j}_{m, *}: B(\Gamma_0) \to B(\Gamma_0^m)$ by

$$\tilde{j}_{m, *} \varphi = (\varphi \circ \tilde{j})(\tilde{j})^2,$$

which projects to the surfaces $X_{\phi}$ and $X^{m}_{\phi}$ as the inclusion of the space $Q(X_{\phi})$ of quadratic differentials corresponding to $B(\Gamma_0)$ into the space $Q(X^{m}_{\phi})$, and (since projection $\eta_m$ has local holomorphic sections) geometrically, this relation yields a holomorphic embedding of the space $T(\Gamma_0)$ into $T(\Gamma_0^m)$ as a $(n - 3)$-dimensional submanifold. We denote this embedding by $s_m$.  

Axioms 2022, 11, 548
7 of 12
To investigate the limit function for \( m \to \infty \), we compose the maps \( s_m \) with the canonical biholomorphic isomorphisms

\[
\eta_m : T(X_{a_0}^m) \to T(\Gamma_0^m) = T \cap B(\Gamma_0^m) \quad (m = 1, 2, \ldots).
\]

Then the elements of \( T(\Gamma_0^m) \) are given by

\[
\mathcal{S}_m(X_a) = \eta_m \circ s_m(X_a),
\]

and this is a collection of the Schwarzians \( S_{f_m}(z) \) corresponding to the points \( X_a \) of \( T(X_{a_0}) \).

Therefore, for any surface \( X_a \), we have

\[
\mathcal{S}_m(X_a) = S_{f_m}(z). \tag{7}
\]

Each \( \Gamma_0^m \) is the covering group of the universal cover \( h_m : U^* \to X_{a_0}^m \), which can be normalized (conjugating appropriately \( \Gamma_0^m \)) by \( h_m(-i) = -i, h_m'(-i) > 0 \). Take the fundamental polygon \( P_m \) obtained as the union of the circular \( m \)-gon in \( U^* \) centered at \( z = -i \) with zero angles at the vertices and its reflection with respect to one of the boundary arcs. These polygons increasingly exhaust the half-plane \( U^* \); hence, by the Carathéodory kernel theorem, the maps \( h_m \) converge to the identity map locally uniformly in \( U^* \).

Since the set of punctures \( \epsilon \) is dense on \( \mathbb{R} \), it completely determines the equivalence classes \( [w^p] \) and \( S_{wp^p} \) as the points of the universal space \( T \). The limit function \( \mathcal{S} = \lim_{m \to \infty} \mathcal{S}_m \) maps \( T(X_{a_0}) = T(0, n) \) into the space \( T \) and also is canonically defined by the marked spheres \( X_a \).

Similar to (7), the function \( \mathcal{S} \) is represented as the Schwarzian of some univalent function \( f^m \) on \( U^* \) with a quasiconformal extension to \( \mathbb{C} \) determined by \( X_a \). Then, by the well-known property of elements in the functional spaces with sup norms, \( \mathcal{S} \) is also holomorphic in the \( B \)-norm on \( T \).

Lemma 5 yields that \( \mathcal{S} \) is a locally open map from \( T(X_{a_0}) \) to \( T \). Therefore, the image \( \mathcal{S}(T(X_{a_0})) \) is an \( (n-3) \)-dimensional complex submanifold in \( T \), biholomorphically equivalent to \( T(\Gamma_0) \). The proof of Theorem 2 is completed.

The holomorphy property indicated above is based on the following lemma of Earle [20].

**Lemma 6.** Let \( E, T \) be open subsets of complex Banach spaces \( X, Y \) and \( B(E) \) be the Banach space of holomorphic functions on \( E \) with sup norm. If \( \varphi(x, t) \) is a bounded map \( E \times T \to B(E) \) such that \( t \mapsto \varphi(x, t) \) is holomorphic for each \( x \in E \), then map \( \varphi \) is holomorphic.

The holomorphy of \( \varphi(x, t) \) in \( t \) for fixed \( x \) implies the existence of complex directional derivatives

\[
\varphi_t(x, t) = \lim_{\xi \to 0} \frac{\varphi(x, t + \xi v) - \varphi(x, t)}{\xi} = \frac{1}{2\pi i} \int_{|\xi| = 1} \frac{\varphi(x, t + \xi v)}{\xi^2} d\xi,
\]

while the boundedness of \( \varphi \) in the sup norm provides the uniform estimate

\[
\|\varphi(x, t + c\xi v) - \varphi(x, t) - \varphi_t(x, t)cv\|_{B(E)} \leq M|c|^2,
\]

for sufficiently small \( |c| \) and \( ||v||_Y \).

### 3.3. Explicit Construction of Holomorphic Homotopy

Now we may construct the desired holomorphic homotopy of \( T(0, n) = T(X_{a_0}) \) into its base point and establish the general result:

Pick a collection \( a^0 = (0, 1, a^0_1, \ldots, a^0_{n-3}, \infty) \) and the marked surface \( X_{a_0} \) as indicated above, and consider its Teichmüller spaces \( T(X_{a_0}) \) and \( T(\Gamma_0) \).
Using the canonical embedding of $T(0, n)$ in $T$, we define on the space $T(\Gamma_0)$ a holomorphic homotopy applying the maps

\[ W^\mu = \sigma^{-1} \circ w^\mu \circ \sigma, \quad \mu \in \text{Belt}(U)_1; \quad \sigma(\bar{z}) = i(1 + \bar{z})/(1 - \bar{z}), \quad \bar{z} \in \mathbb{D}, \]

and $w^\mu_t(z) := w^\mu(z, t) = \sigma \circ W^\mu_t \circ \sigma^{-1}(z)$; then,

\[ S_{w^\mu}(\cdot, t) = t^2 S_{w^\mu}(\cdot) = t^{-2}(S_{w^\mu} \circ \sigma^{-1})(\sigma')^{-2}. \tag{8} \]

This point-wise equality determines a bounded holomorphic map by Lemma 6 $\eta(\varphi_t) = S_{w^\mu} : T \times \mathbb{D} \rightarrow T$ with $\eta(0, t) = 0$, $\eta(\varphi, 0) = 0$, $\eta(\varphi, 1) = \varphi$; its boundedness follows from the estimate

\[ S_{w^\mu}(\xi) < 6t^2/(|\xi|^2 - 1)^2, \quad \xi \in \mathbb{U}^\ast. \]

We apply homotopy (8) to $\varphi = S_{w^\mu} \in T(\Gamma_0)$. Since it is not compatible with the group $\Gamma_0$, there are images $\varphi_t := \eta(\varphi_t) = S_{w^\mu}$ that are located in $T$ outside of $T(\Gamma_0)$. Map $\chi \circ \eta(\varphi_t)$ carries these images to the points of the space $T(0, n) = T(X_\varphi)$. We compose this map with the holomorphic map $s$ given by Lemma 3 and with a biholomorphism $\zeta : s(T(X_\varphi)) \rightarrow T(\Gamma_0)$, getting the function

\[ \Theta(\varphi, t) = \zeta \circ s \circ \chi \circ \eta(\varphi_t) \tag{9} \]

which maps holomorphically $T(\Gamma_0) \times \mathbb{D}$ into $T(\Gamma_0)$ with $\Theta(\varphi, 0) = 0$.

A crucial step in constructing is to establish that function (9) extends holomorphically to the limit points $(\varphi, 1)$ representing the initial Schwarzians $S_{w^\mu}$. This property does not extend (in the B-norm) to all points of $T$.

To prove the limit holomorphy, fix a point $\varphi_0 \in T(\Gamma_0)$ and consider, in its small neighborhood $V_0$, the local coordinates $a_0 = (a_1, \ldots, a_{n-3})$ introduced above.

Both maps $\eta$ and $\Theta$ are holomorphic in the points $(\varphi_0, t)$ of this neighborhood for all $t$ with $|t| < 1$. On the other hand, coordinates $a_0$ are determined by the corresponding quasiconformal maps $\overline{w}^\mu_t$ and, together with these maps, are uniformly continuous in $t$ in the closed disk $\{|t| \leq 1\}$. This follows from the uniform boundedness of dilatations given by the estimate

\[ k(w^\mu_t) = ||\mu_t||_{\infty} \leq |t||\mu||_{\infty} < 1 \tag{10} \]

(which holds for generic holomorphic motions) and from non-increasing the Kobayashi metric $d_X(\cdot, \cdot)$ under holomorphic maps. Since this metric on Teichmüller spaces equals their intrinsic Teichmüller metric $\tau_T(\Gamma_0)$, one gets from (10),

\[ \tau_T(\Gamma_0)(0, \Theta(\varphi, t)) = d_T(\Gamma_0)(0, \Theta(\varphi, t)) \leq \tanh^{-1}(|t||\mu||_{\infty}). \]

Hence, function $\Theta(\varphi, t)$ determines a normal family on $V_0 \cap T(\Gamma_0)$.

Applying the classical Weierstrass theorem about the locally uniform convergent sequences of holomorphic functions in finite-dimensional domains, one derives that the limit function

\[ \Theta(\varphi, 1) = \lim_{t \rightarrow 1} \Theta(\varphi, t) \]

is also holomorphic on $V_0 \cap T(\Gamma_0)$ and then on $T(\Gamma_0)$, which completes the construction of the desired holomorphic homotopy on $T(0, n)$.

4. Second Proof of Holomorphic Contractibility for Low-Dimensional Teichmüller Spaces

The previous section implies the proof of holomorphic contractibility for all spaces $T(0, n)$ with $n \geq 5$, which also yields, in particular, Theorem 2. In this section, we provide
another proof of this important theorem; it relies on the intrinsic features of the two and three-dimensional Teichmüller spaces mentioned in Section 2.4.

(a) **Case n = 5 (dimension two).** It is enough to establish holomorphic contractibility of the space $T(0, 5) \simeq F(0, 4)$ for the spheres

$$X_\mathbf{a} = \hat{C} \setminus \{0, 1, a_1, a_2, \infty\}.$$ 

The fibers of $T(0, 5)$ are the spheres with quadruples of punctures $\{0, 1, a_1, \infty\}$.

We start with the construction of the needed holomorphic homotopy of the space $T(0, 5)$ to its base point $X_\mathbf{a}$ and first apply the assertion (a) of Lemma 3 of holomorphic sections over $T(0, 4)$. It implies that for any point

$$x = (S_{w^{0,1}}, w^{0,1}(\tilde{a}_2)) \in T(0, 5)$$

a unique holomorphic section $s : T(0, 5) \to T(0, 4)$ with $s(\pi_1(x)) = x$. This section has a common point with each fiber $\pi^{-1}(x)$ over $T(0, 4)$.

Since $T(0, 4)$ is (up to a biholomorphic equivalence) a simply connected bounded Jordan domain $D \subset \mathbb{C}$ containing the origin, there is a holomorphic isotopy $h(\zeta, t) : D \times [0, 1] \to D$ with $h(\zeta, 0) = \zeta$, $h(z, 1) = 0$. Using this isotopy, we define a homotopy $h_2(\varphi, t)$ on $T(0, 5)$, which carries each point $x = (S_{w^{0,1}}, w^{0,1}(\tilde{a}_2)) \in T(0, 5)$ to its image on the section $s$ passing from $x$; that is,

$$h_2(\varphi, w^{0,1}(\tilde{a}_2)) = (h(\varphi), \tilde{a}_2), \quad \varphi = S_{w^{0,1}}, \quad \mu \in \text{Belt}(\mathbb{C})_1,$$  

(11)

where $\tilde{a}_2$ is the common point of the fiber $h(\varphi)$ and the selected section $s$. The homomorphy of this homotopy in variables $x = (S_{w^{0,1}}, w^{0,1}(\tilde{a}_2))$, for any fixed $t \in [0, 1]$ follows from Lemmas 1, 2, and the Bers isomorphism theorem. The limit map

$$h_1^*(x) = \lim_{t \to 1} h_1(x, t),$$

carries each fiber $w^{0,1}(U)$ to the initial half-plane $U$.

There is a canonical holomorphic isotopy

$$h_2(\zeta, t) : U \times [0, 1] \to U$$

(12)

of $U$ into its point corresponding to the origin of $T(0, 5)$. Now make $h(x, t)$ equal to $h_1(x, 2t)$ for $t \leq 1/2$ and equal to $h_2(x, 2t - 1)$ for $x \in U$ and $1/2 \leq t \leq 1$.

This function is holomorphic at $x \in T(0, 5)$ for any fixed $t \in [0, 1]$ and hence provides the desired holomorphic homotopy of the space $T(0, 5)$ into its base point.

(b) **Case n = 6 (dimension three).** This case is more complicated.

We prescribe to each ordered sextuple $\mathbf{a} = \{0, 1, a_1, a_2, a_3, \infty\}$ of distinct points the corresponding punctured sphere

$$X_\mathbf{a} = \hat{C} \setminus \{0, 1, a_1, a_2, a_3, \infty\}$$

(13)

and the two-sheeted closed hyperelliptic surface $\hat{X}_\mathbf{a}$ of genus two with the branch points $0, 1, a_1, a_2, a_3, \infty$. The corresponding Teichmüller spaces $T(0, 6)$ and $T(2, 0)$ coincide up to a natural biholomorphic isomorphism. Note also that the collection $\mathbf{a} = \{0, 1, a_1, a_2, a_3, \infty\}$ provides the local complex coordinates on spaces $T(0, 6)$ and $T(2, 0)$.

In view of the symmetry of hyperelliptic surfaces, it suffices to deal with the Beltrami differentials $\mu d\bar{z} / dz$ on $\hat{X}_\mathbf{a}$, which are compatible with a conformal involution $f_\mathbf{a}$ of $\hat{X}_\mathbf{a}$, hence, satisfying $\mu(f_\mathbf{a} z) = \mu(z)' / T'$. In other words, these $\mu$ are obtained by lifting to $\hat{X}_\mathbf{a}$ of the Beltrami coefficients on $X_\mathbf{a}$. This extends Lemma 2 and its consequences on holomorphy in the neighborhoods of the boundary interpolation points to the corresponding two-sheeted disks on hyperelliptic surfaces.
We fix a base point of $T(2, 0)$, determining a Fuchsian group $\Gamma$ for which $T(\Gamma) = T(2, 0)$. The corresponding Teichmüller curve $\mathcal{V}(2, 0)$ is a 4-dimensional, complex analytic manifold with projection $\pi_1: \mathcal{V}(2, 0) \to T(2, 0)$ onto $T(2, 0)$ such that for every $\varphi \in T(2, 0)$ the fiber $\pi_1^{-1}(\varphi)$ is a hyperelliptic surface, determined by $\varphi$ (see Section 2.4).

Due to assertion (b) of Lemma 3, this curve has, for any point

$$\tilde{X}_a = (S_{\omega^1}, w^h(\tilde{a}_{n-3})) \in T(2, 0)$$

six distinct holomorphic sections $\tilde{s}_1, \ldots, \tilde{s}_6$, corresponding to the Weierstrass points of the surface $X_a$, with $\tilde{s}_j(\pi_1(X_a)) = X_a$, and either from these sections has one common point with every fiber over $T(2, 0)$. We lift these sections to the Bers fiber space $\mathcal{F}(\Gamma)$ covering $\mathcal{V}(2, 0)$.

As mentioned in Section 2.4, these sections are generated by the space $\mathcal{F}(\Gamma') = \mathcal{F}(\Gamma)$ corresponding to the extension $\Gamma'$ of $\Gamma$, for which $\Gamma'$ is a subgroup of index two. Every section $\tilde{s}_j$ acts on $T(\Gamma')$ via (6), where $z_0$ is now the corresponding Weierstrass point of hyperelliptic surface $\tilde{X}_a$, and $\tilde{s}_j$ is compatible with action (2) of the Bers isomorphism.

Thus each $\tilde{s}_j$ descends to a holomorphic map $s_j: T(0, 6) \to \mathcal{V}(0, 6)$ of the underlying space $T(0, 6)$ for the punctured spheres (10). We choose one from these maps and denote it by $s$.

The features of sections $\tilde{s}_j$ provide that the descended map $s$ also determines, for each point $z_0 \in X_a$, its unique image on every fiber $w^h(X_a)$ with $\mu \in \text{Belt}(X_a)_1$, and this image is the point $w^h(z_0)$.

The next preliminary construction consists of embedding space $T(0, 5)$ into $T(0, 6)$, using the forgetting map (3). Its image $j_* T(0, 5)$ is a connected submanifold in $T(0, 6)$, and the corresponding fibers of the curve $\mathcal{V}(0, 6)$ over the points $j_*$ $\varphi \in j_* T(0, 5)$ are the surfaces $w^{h^\mu}(X_a)$ with $j_* \mu(z) = \mu(\tilde{j}(z))\tilde{j}'(z)$. The covering domains $w^{h^\mu}(U)$ over these surfaces fill a submanifold $\mathcal{F}(0, 7) \subset T(0, 7)$, which is biholomorphically equivalent to the space $T(0, 6)$.

Using the biholomorphic equivalence of space $T(0, 5)$ to its image $j_* T(0, 5)$ in $T(0, 6)$, we carry over to $j_* T(0, 5)$ the result of the previous step (a) on the holomorphic contractibility of $T(0, 5)$, which provides a holomorphic homotopy

$$h(j_*, \varphi, t): j_* T(0, 5) \times [0, 1] \to j_* T(0, 5) \quad \text{with} \quad h(j_*, \varphi, 0) = j_* \varphi, \quad h(j_*, \varphi, 1) = 0$$

(14)

(here, $0$ stands for the origin of $j_* T(0, 5)$).

Now we may construct the desired holomorphic homotopy of $T(0, 6)$, contracting this space to its origin.

First, regarding $T(0, 6)$ as the Bers fiber space $\mathcal{F}(0, 5)$ over $T(0, 5)$ (whose fibers are the covers of surfaces $X_a$ with collections of five punctures $a' = (0, 1, a_1, a_2, \infty)$), we apply homotopy (11) and define, for any pair $x = (j_*, \varphi, z)$ with $\varphi \in T(0, 5)$ and $z \in X_a$, the map

$$\tilde{h}_1((j_*, \varphi, z), t) = (h(j_*, \varphi), t), w^{h_{j_*}}_t(z), \quad \varphi \in T(0, 5),$$

(15)

noting that the image point $w^{h_{j_*}}_t(z)$ is uniquely determined on surface $w^h(j_*)X_a$ by map $s$, as indicated above.

The pairs $(j_*, \varphi, z)$ are located in the space $\mathcal{F}(0, 6)$ and fill a three-dimensional submanifold $\mathcal{F}(0, 6)$ biholomorphically equivalent to $T(0, 6)$.

Homotopy (15) is well defined on $\mathcal{F}(0, 6) \times [0, 1]$ and contracts the set $\tilde{T}(0, 6)$ into fiber $\tilde{U}$ over the base point. It is holomorphic with respect to the space variable $x = (j_*, \varphi, z)$ for any fixed $t \in [0, 1]$ and continuous in both variables.

In view of the biholomorphic equivalence of $\tilde{T}(0, 6)$ to $T(0, 6)$, (15) generates a holomorphic homotopy $h_1(x, t)$ of the space $T(0, 6)$ onto the initial fiber (half-plane) $U$ over the origin of $T(0, 5)$.
It remains to combine this homotopy \( h_1 \) with the additional homotopy (12) of \( U \) into its point corresponding to the origin of \( T(0,6) \). This provides the desired homotopy \( h \) and completes the proof of Theorem 1.

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**References**

1. Gorin, E.A. Zapiski Nauchn. Seminarov Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 1978, 81, 58–61.
2. Hirschowitz, A. A propos de principe d’Oka. C.R. Acad. Sci. Paris 1971, 272, A792–A794.
3. Zaidenberg, M.G. Holomorphic rigidity of polynomial polyhedrons and quasihomogeneity. In Global Analysis—Studies and Applications; Borisovich, Y.G., Gliklikheds, Y.E., Eds.; Lect. Notes Math. 1453; Springer: Berlin/Heidelberg, Germany, 1990; pp. 291–307.
4. Zaidenberg, M.G.; Lin, V.Y. On bounded domains of holomorphy that are not holomorphically contractible. Soviet Math. Dokl. 1979, 20, 1262–1266.
5. Zaidenberg, M.G.; Lin, V.Y. The Finiteness Theorems for Holomorphic Maps; Current Problems in Mathematics. Fundamental Directions, Vol. 9. Itogi Nauki i Techniki; Vsesoyuzn. Inst. Nauchn. i Technich. Inform.: Moscow, Russia, 1986; pp. 127–193. (In Russian)
6. Krushkal, S.L. Teichmüller spaces are not starlike. Ann. Acad. Sci. Fenn. Ser. A.I. Math. 1995, 20, 167–173.
7. Toki, M. On non-starlikeness of Teichmüller spaces. Proc. Japan Acad. Ser. A Math. Sci. 1993, 69, 58–60.
8. Earle, C.J. The holomorphic contractibility of two generalized Teichmüller spaces. Publ. Inst. Math. 2004, 75, 109–117. [CrossRef]
9. Krushkal, S.L. Holomorphic contractibility of Teichmüller spaces. Complex Anal. Oper. Theory 2019, 13, 2829–2838. [CrossRef]
10. Kaliman, I. Holomorphic universal covering spaces of polynomials without multiple roots. Funktional Anal. Prilozh. 1975, 71, 71. (In Russian) [CrossRef]
11. Nag, S. The Complex Analytic Theory of Teichmüller Spaces; Wiley: New York, NY, USA, 1988. [CrossRef]
12. Bers, L. Spaces of Riemann surfaces. In Proceedings of the International Congress of Mathematicians, Edinburgh, UK, 14–21 August 1958; Cambridge University Press: New York, NY, USA, 1960; pp. 349–361.
13. Clunie, J.G.; Hallenbeck, D.J.; MacGregor, T.H. A peaking and interpolation problem for univalent functions. J. Math. Anal. Appl. 1985, 111, 559–570.
14. Bers, L. Fiber space over Teichmüller spaces. Acta Math. 1973, 130, 89–126. [CrossRef]
15. Earle, C.J.; Kra, I. On holomorphic mappings between Teichmüller spaces. In Contributions to Analysis; Academic Press: New York, NY, USA, 1974; pp. 107–124. [CrossRef]
16. Hubbard, J. Sur le non-existence de sections analytiques a la courbe universelle de Teichmüller. C.R. Acad. Sci. Paris Ser. A-B 1972, 274, A978–A979.
17. Patterson, D.B. The Teichmüller spaces are distinct. Proc. Amer. Math. Soc. 1972, 35, 179–182.
18. Gardiner, F.P. Teichmüller Theory and Quadratic Differentials; Wiley: New York, NY, USA, 1987.
19. Earle, C.J. On Quasiconformal Extensions of the Beurling-Ahlfors Type. In Contribution to Analysis; Academic Press: New York, NY, USA, 1974; pp. 99–105.