ON WICK POWER SERIES CONVERGENT TO
NONLOCAL FIELDS

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Abstract

The infinite series in Wick powers of a generalized free field are considered that are convergent under smearing with analytic test functions and realize a nonlocal extension of the Borchers equivalence classes. The nonlocal fields to which they converge are proved to be asymptotically commuting, which serves as a natural generalization of the relative locality of the Wick polynomials. The proposed proof is based on exploiting the analytic properties of the vacuum expectation values in $x$-space and applying the Cauchy–Poincaré theorem.

1 Introduction

In this paper we continue the investigation \cite{1, 2} of the infinite series of the form

$$
\sum_{k=0}^{\infty} d_k \phi^k(x)
$$

(1)

in Wick powers of a neutral scalar field $\phi$, whose basic point is the systematic use of the analytic properties of the vacuum expectation values in $x$-space. The developed approach is primarily aimed at applications to gauge field theory, where the two-point function $w(x - x') = \langle \Psi_0, \phi(x)\phi(x')\Psi_0 \rangle$ does not necessarily satisfy the positivity condition $w(f^* \otimes f) \geq 0$ (where $f$ is a test function), and it not only allows easily finding the test function class on which a given series is convergent, but also enables one to establish the properties of the limiting field $\varphi$. In the positive metric case, it is customary to use another approach \cite{3} based on estimating the terms of the series representing the vacuum expectation value $\langle \Psi_0, \varphi(x_1) \ldots \varphi(x_n)\Psi_0 \rangle$ in momentum space, where they are expressible through positive measures. It is commonly assumed that this is the

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only possible way of handling the problem if the sum $\varphi$ of a series is a nonlocal field because in this case the analyticity domain of its vacuum expectation values in $x$-space is empty. Nevertheless, our approach is applicable to such series as well if the analyticity properties of each their particular term are duly taken into account, see [1]. It is essential that this approach covers the fields $\phi$ of zero mass, which were not considered in [3], and, moreover, the generalized free fields in a space-time of arbitrary dimension $d$. Here we shall prove the relative asymptotic commutativity of the nonlocal fields to which the series in Wick powers of a generalized free field converge. The role of the asymptotic commutativity condition in the theory of nonlocal interactions was analyzed in [4], where it was proved that this condition ensures the normal connection between spin and statistics and the CPT-invariance. Within the framework of the traditional approach, the properties of nonlocal Wick series of a free field with nonzero mass were considered earlier in [5], whereas the relation of the essential locality condition used in [3] to the asymptotic commutativity is discussed in [6]. The general construction of Wick powers of generalized free fields was considered in [7]. Other important motivations for a deeper analysis of nonlocal Wick series (in addition to the fact that they form an extension of the Borchers equivalence class of the field $\phi$) are the connection of nonlocal quantum field models exhibiting singular ultraviolet behavior with string theory and M-theory [8], especially in the context of AdS/CFT correspondence [9], and the use of nonlocal formfactors for removing ultraviolet divergences in phenomenological models proposed as an alternative to string theory [10]. In particular, the developed technique may be useful for the treatment of the problem of a possible CPT-invariance breaking in such models, which is discussed in [10].

2 Analytic properties of two-point function

From this point on, we shall assume $\phi$ to be a tempered distribution generalized free field [11]. The $n$-point vacuum expectation values are expressible in terms of the two-point one by the same recurrence relation as in the case of a free field, which makes it possible to define the Wick ordered powers $:\phi^k(x)$. The analytic function whose boundary value is the distribution $w(x)$ will be denoted by $w(z)$. As is shown in [11], if the positivity condition is satisfied, the test function class on which series (1) is conver-
gent is determined by the behavior of \( w(z) \) in the imaginary directions, its growth for \( \Im z \to 0 \) (resp. for \( \Im z \to \infty \)) being essential in the localizable (resp. nonlocalizable) case. As a characteristic of this behavior one can take the restriction of the function \( w(z) \) to the semi-axis \( \Re z = 0 \), \( \Im z = (-\tau, 0, \ldots, 0) \), \( \tau > 0 \), and we shall denote this restriction by \( u(\tau) \). The formula for the Laplace transformation

\[
w(z) = (2\pi)^{-d} \int \exp(-ipz)\hat{w}(p) \, dp
\]

where \( \hat{w}(p) \) is a positive polynomially bounded measure supported by the closed upper light cone \( \bar{V}_+ \), shows that \( u(\tau) \) is a strictly positive nondecreasing function which majorizes \( |w(z)| \) for \( \Im z \) of the specified form and for any \( \Re z \). Hence, by the Lorentz invariance of \( \hat{w}(p) \), we have

\[
|w(x + iy)| \leq u(\sqrt{y^2}), \quad \text{for all} \quad y \in V_-, \quad (2)
\]

where \( y^2 \) is the Lorentz square of \( y \). Thus, \( u(\tau) \) indeed can serve as an indicator function according to the definition \[1\], where it was denoted by \( w_{UV}(\tau) \) in order to distinguish it from the function characterizing the infrared behavior of \( w \), which is necessary in theories with an indefinite metric. Moreover, the function \( u(\tau) \) is the least one among all functions satisfying (2) and, therefore, is the best characteristic of the behavior of \( w(z) \). It should also be noted that \( u(\tau) \) is infinitely differentiable and increases indefinitely with decreasing argument.

As usual, we denote by \( L_+(\mathbb{C}) \) the component of identity of the complex Lorentz group and \( T^\text{ext} \) the extended analyticity domain of \( w(z) \) which is generated from the primitive domain \( \mathbb{R}^d + iV_- \) by applying arbitrary transformations in \( L_+(\mathbb{C}) \) according to the Bargmann–Hall–Wightman theorem, whose proof for an arbitrary space-time dimension can be found in \[12\]. The domain \( T^\text{ext} \) is invariant under the full reflection \( z \to -z \). Indeed, for an even \( d \) the reflection belongs to \( L_+(\mathbb{C}) \), and the general case can be treated as follows. If \( z \) is in \( T^\text{ext} \), then there exists a transformation \( \Lambda \in L_+(\mathbb{C}) \) which takes \( z \) to a point with an imaginary part belonging to the negative \( y^0 \)-semi-axis. Now the statement follows if we note that the composition of \( \Lambda \) with the partial reflection \((z^0, z^1, z^2, \ldots, z^{d-1}) \to (-z^0, -z^1, z^2, \ldots, z^{d-1})\), which also belongs to \( L_+(\mathbb{C}) \), takes \(-z \) to a point with the same imaginary part. In particular, the inclusion \( \mathbb{R}^d + iV \subset T^\text{ext} \)
is valid. The following simple lemma allows us to estimate the function $w(z)$ for real arguments provided we know its behavior in the imaginary directions.

**Lemma 1.** Let $0 \leq \tau' < \tau$, $z = x + iy$, and let $x^2 \leq -\tau^2$, $y^2 > -\tau'^2$. Then there exists $\Lambda \in L_+ (\mathbb{C})$ such that $\Lambda z \in \mathbb{R}^d + i\mathbb{V}_-$ and $(\text{Im} \, \Lambda z)^2 \geq \tau^2 - \tau'^2$.

**Proof.** Suppose first that $y^2 > 0$ and let $\Lambda_1$ be a real Lorentz transformation taking $y$ to a vector of the form $(\tilde{y}^0, 0, \ldots, 0)$. Let $\Lambda_2$ be a pure rotation which takes $\Lambda_1x$ to a vector $(\tilde{x}^0, \tilde{x}^1, 0, \ldots, 0)$. Set $\Lambda = \Lambda_3 \Lambda_2 \Lambda_1$, where $\Lambda_3(z^0, \ldots, z^{d-1}) = (\pm i z^1, \pm i z^0, z^2, \ldots, z^{d-1})$. Then $\text{Im} \, \Lambda z = (\pm \tilde{x}^1, \pm \tilde{x}^0, 0, \ldots, 0)$ belongs to $\mathbb{V}_-$ under the proper choice of the sign, and $(\text{Im} \, \Lambda z)^2 = -x^2 \geq \tau^2$. Now suppose $-\tau'^2 < y^2 \leq 0$. Then there exists a Lorentz transformation $\Lambda_1$ taking $y$ to a vector $y_1$ such that $\|y_1\| \leq \tau'$. Let $\Lambda_2, \Lambda_3$, and $\Lambda$ be defined as before, and let $\tilde{x} = \Lambda_2 \Lambda_1 x$ and $\tilde{y} = \Lambda_2 \Lambda_1 y = \Lambda_2 y_1$. Then $\text{Im} \, \Lambda z = (\pm \tilde{x}^1, \pm \tilde{x}^0, \tilde{y}^2, \ldots, \tilde{y}^{d-1})$, $(\text{Im} \, \Lambda z)^2 \geq -x^2 - \|\tilde{y}\|^2 = -x^2 - \|y_1\|^2 \geq \tau^2 - \tau'^2$, and the proper choice of the sign in the definition of $\Lambda_3$ ensures that $\text{Im} \, \Lambda z \in \mathbb{V}_-$. The lemma is proved.

It is well known that all spacelike vectors belong to $T^{\text{ext}}$, which also follows from Lemma 1. For any such vector $x$ one can find $\Lambda \in L_+ (\mathbb{C})$ such that $\Lambda x = -x$, and hence $w(x) = w(-x)$. Therefore, by the uniqueness theorem,

$$w(z) = w(-z), \quad z \in T^{\text{ext}}.$$  \hspace{1cm} (3)

In particular, at the level of the two-point vacuum expectation values, the locality is a consequence of the other Wightman axioms. Using the notation

$$G^\tau = \{ x \in \mathbb{R}^d | x^2 \leq -\tau^2 \}, \quad V^\tau = \{ y \in \mathbb{R}^d | y^2 > -\tau'^2 \}$$  \hspace{1cm} (4)

and combining (3) with Lemma 1, we obtain the estimate

$$|w(z)| \leq u(\sqrt{\tau^2 - \tau'^2}), \quad z \in G^\tau + iV^\tau' \subset T^{\text{ext}},$$  \hspace{1cm} (5)

which holds for $0 \leq \tau' < \tau$.

### 3 Wightman functions of Wick power series

Let us denote the series $\{s\}$ by $s$. We shall also consider its subordinate series and use the notation $s' \prec s$ which means that for all indices $k$, with the possible exception of a
finite subset the inequality $|d'_k| \leq C_s |d_k|$ holds, where $C_s'$ is a positive constant. It is reasonable to impose the following conditions on the coefficients of series (1):

$$d_k \geq 0, \quad d_k d_l \leq C h^{k+l} d_{k+l},$$

where $C, h$ are constants whose role is explained in [1]; the subordinate series need not satisfy them. We use the Gelfand–Shilov spaces $S^b$ as test function spaces, see [13]. The defining index $b$ can be regarded as an indicator function characterizing the momentum space behavior of the test functions. More precisely, the derivatives of the Fourier transform of $f \in S^b(\mathbb{R}^n)$ satisfy the inequalities

$$|\partial^\kappa \hat{f}(p)| b\left(\frac{|p|}{B}\right) \leq C_\kappa,$$

with $C_\kappa$ and $B$ positive constants depending on $f$. For definiteness, we assume the norm $|\cdot|$ in $\mathbb{R}^n$ to be uniform. In the context of QFT, the function $b$ characterizes the high energy behavior of the fields defined over $S^b$.

Let $\varphi_{s_1}, \ldots, \varphi_{s_n}$ be the fields determined by the series $s_j \triangleleft s$. Consider the $n$-point vacuum expectation value $W_{s_1, \ldots, s_n}(x_1, \ldots, x_n) = \langle \Psi_0, \varphi_{s_1}(x_1) \ldots \varphi_{s_n}(x_n) \Psi_0 \rangle$. Applying the Wick theorem gives the well-known formal representation

$$W_{s_1, \ldots, s_n} = \sum_K D_K W^K,$$

where $K$ is an integer-valued multi-index with nonnegative components $k_{j,m}, 1 \leq j < m \leq n$, which have the sense of the number of pairings between the terms of the series $s_j$ and $s_m$, and the designation

$$W^K = \prod_{1 \leq j < m \leq n} w(x_j - x_m)^{k_{j,m}}$$

is used. The numerical coefficients $D_K$ are expressible in terms of the coefficients of the series $s_j$ in the following way:

$$D_K = \frac{\kappa!}{K!} \prod_{1 \leq j \leq n} d^{(j)}_{\kappa_j}, \quad \text{where} \quad \kappa_j = k_{1j} + \ldots + k_{j-1,j} + k_{j,j+1} + \ldots + k_{jn}.$$
to the strong topology of its dual space because the latter is a Montel space. If this is the case for any set of series subordinate to \(s\), then, as is shown in [1], the fields \(\varphi_{s'}, s' \subset s\) are well defined as operator-valued generalized functions over \(S^b(\mathbb{R}^d)\) acting in the Hilbert space \(\mathcal{H}\) of the initial field \(\phi\). In particular, the vector series that define the repeated action of these operators on the vacuum \(\Psi_0\) are unconditionally convergent, and the linear span of all vectors of the form \(\varphi_{s_1}(f_1) \ldots \varphi_{s_n}(f_n)\Psi_0, f_i \in S^b(\mathbb{R}^d)\) serves as a common dense invariant domain of definition for the family of fields \(\{\varphi_{s'}\}_{s' \subset s}\) in \(\mathcal{H}\).

We shall denote this domain by \(D(s)\).

The distribution (11) is the boundary value of the analytic function

\[
W^K(z) = \prod_{1 \leq j < m \leq n} w(z_j - z_m)^{k_{jm}}
\]

from the cone

\[
V_{n-} = \{ y \in \mathbb{R}^{nd} | y_j - y_m \in \mathbb{V}_-, 1 \leq j < m \leq n \},
\]

and the representation (8) can be rewritten in the following more precise form:

\[
\mathcal{W}_{s_1, \ldots, s_n}(x_1, \ldots, x_n) = \sum_K D_K b_{\pi V_{n-}} W^K,
\]

where \(b_{\pi V_{n-}}\) is the boundary value operator. It is worth noting that the function (11) is defined and analytic in the open set \(T_n = \{ z \in \mathbb{C}^{nd} | z_j - z_m \in \mathbb{T}^\text{ext}, j \neq m \}\) and, in particular, in the tube \(\mathbb{R}^{nd} + iV_n\), where \(V_n = \{ y \in \mathbb{R}^{nd} | y_j - y_m \in \mathbb{V}, j \neq m \}\). For what follows, it is essential to know the transformation law for representation (12) under the rearrangements of the operators \(\varphi_{s_j}(x_j)\) entering into the vacuum expectation value.

**Lemma 2.** Suppose the distribution series on the right-hand side of (12) is unconditionally convergent in \(S^b(\mathbb{R}^{nd})\) for any \(s_j \subset s, j \leq n\). Let \(\pi\) be a permutation of the indices \((1, \ldots, n)\). Then

\[
\mathcal{W}_{s_{\pi_1}, \ldots, s_{\pi_n}}(x_{\pi_1}, \ldots, x_{\pi_n}) = \sum_K D_K b_{\pi V_{n-}} W^K,
\]

where \(\pi V_{n-} = \{ y \in \mathbb{R}^{nd} | y_{\pi j} - y_{\pi m} \in \mathbb{V}_-, 1 \leq j < m \leq n \}\).

**Proof.** Because of (12) we have

\[
\mathcal{W}_{s_{\pi_1}, \ldots, s_{\pi_n}}(x_{\pi_1}, \ldots, x_{\pi_n}) = \sum_K D_K' (b_{V_{n-}} W^K)(x_{\pi_1}, \ldots, x_{\pi_n}),
\]

(14)
where \( D'_K \) is the coefficient corresponding, by (10), to the permuted set \( s_{\pi_1}, \ldots, s_{\pi_n} \). Let \( K' \) be the multi-index whose components \( k'_{jm} \) are equal to \( k_{\pi_j,\pi_m} \) for \( \pi j < \pi m \) and \( k_{\pi_m,\pi_j} \) for \( \pi j > \pi m \). From (11), it follows that \( W^{K'}(z_{\pi_1}, \ldots, z_{\pi_n}) \) coincides with the product defining \( W^K(z) \) to within the signs of the arguments of some factors, and in view of (3) we conclude that \( W^K(z_{\pi_1}, \ldots, z_{\pi_n}) = W^{K'}(z_{\pi_1}, \ldots, z_{\pi_n}) \) in \( T_n \). Passing in this equality to the boundary values from the cone \( \pi V_{n-} \), we obtain

\[
(b_{\pi V_{n-}} W^K)(x_{1}, \ldots, x_{n}) = (b_{V_{n-}} W^{K'})(x_{\pi_1}, \ldots, x_{\pi_n}),
\]

wheras the relation \( \kappa'_i = \kappa_{\pi i} \), which follows from (10), implies that \( D'_K = D_K \). Making the change \( K \rightarrow K' \) of the summation indices in the unconditionally convergent series (14) and applying the above identities, we arrive at (13). The lemma is proved.

4 Generalization of the locality axiom

If the space \( S^b \) on which series (1) converges contains functions of compact support, i.e., the field \( \varphi_s \) is an (operator-valued) ultradistribution, then the fulfilment of the Wightman axioms for this field is easily established by the same arguments as in [2], where even the more general case of an indefinite metric was considered. In particular, the locality of \( \varphi_s \) and, moreover, the relative locality of the fields \( \varphi_{s'}, s' < s \), immediately follow from the relative locality of the Wick monomials :\( \phi^k(x) \). This property can be also derived from Lemma 2 if \( \pi \) is taken to be the transposition \( \tau_j \) of the neighbouring indices \( j \) and \( j + 1 \). The distribution \( b_{V_{n-}} W^K - b_{\tau_j V_{n-}} W^K \) is supported by the closed cone

\[
V_{j,j+1} = \{ x \in \mathbb{R}^n | (x_j - x_{j+1})^2 \geq 0 \}. \tag{15}
\]

Therefore, the support of the functional

\[
W_{g_1,\ldots,g_n}(x_1, \ldots, x_j, x_{j+1}, \ldots, x_n) - W_{g_1,\ldots,g_{j+1},g_j,\ldots,g_n}(x_1, \ldots, x_{j+1}, x_j, \ldots, x_n) \tag{16}
\]

is also contained in this cone, whence \([\varphi_{s_j}(f_j), \varphi_{s_{j+1}}(f_{j+1})]\Psi = 0 \) for spacelike separated supports of the test functions \( f_j, f_{j+1} \in S^b \) and for any \( \Psi \in D(s) \) because \( n \) and the rest of \( s_i \) in (16) can be taken arbitrary.

The locality condition implies that the vacuum expectation values in momentum space have less than exponential growth, see, e.g., [11]. For this reason, the Gelfand–Shilov space determined by the indicator function \( b(s) = e^s \), which is customarily
denoted by $S^1$, is universal for local fields. The elements of $S^1$ allow analytic continuation into a complex neighbourhood of the real space and never have compact support. Nevertheless, the methods of the hyperfunction theory make it possible to give a correct definition of support for the functionals belonging to the dual space $S^1'$. We refer the reader to [14] for details and confine ourselves to saying that the support of $v \in S^1'$ is contained in a closed cone $K$ if and only if $v$ has a continuous extension to each space $S^1(U) = \bigcup_{B > 0} S^{1, B}(U)$, where $U$ is an open cone containing $K \setminus \{0\}$ and $S^{1, B}(U)$ consists of analytic functions on the complex $1/B$-neighbourhood $\tilde{U}^{1/B}$ of the set $U$ with the property that the norms

$$
\|f\|_{U, N, B} = \sup_{z \in \tilde{U}^{1/B}} |f(z)|(1 + |x|)^N
$$

are finite. The corresponding generalization of local commutativity for the fields $\varphi, \varphi'$ defined over $S^1(\mathbb{R}^d)$ means that the matrix elements $\langle \Phi, [\varphi(x), \varphi'(x')]\Psi \rangle$ have support in the closed cone

$$
\tilde{V}^{(2d)} = \{(x, x') \in \mathbb{R}^{2d} | (x - x')^2 \geq 0\}
$$

for any $\Phi$ and $\Psi$ belonging to the common domain of definition of these fields. Within the framework of hyperfunction theory, Wick power series of the free field of nonzero mass were studied earlier in the work [15]. In our notation, the restriction on the series coefficients found there takes the form

$$
\lim_{k \to \infty} (d_{2k} k!)^{1/k} = 0.
$$

The fulfilment of the Wightman axioms for the (hyper)fields defined by Wick series was established in [15] indirectly, by means of an equivalence theorem of Osterwalder–Schrader type for a properly modified Euclidean field theory and the Minkowski quantum field theory formulated in terms of Fourier-hyperfunctions. Below we shall obtain, as a by product, a simple direct proof showing that in this case the limiting fields satisfy the generalized locality condition.

If $\ln b(s)$ grows faster than linearly, then the elements of $S^b$ are entire functions and the functionals belonging to $S^b$ are nonlocal. However, under suitable restrictions on $b$, they inherit an important part of the properties of hyperfunctions which has the sense of
angular localizability. In [14] a corresponding theory has been developed for the spaces whose indicator functions are exponentials of order $\sigma > 1$, and in [2] it has been extended to a more general case. In the work [2], another scale of spaces $S^b$ was considered which is required for the indefinite metric field theory, but the construction proposed there is applicable to $S^b$ as well. Namely, let $\beta(\tau)$ be a nonnegative, convex, differentiable, and indefinitely increasing function on the half-axis $\tau \geq 0$ and let $U$ be an open cone in $\mathbb{R}^n$.

We denote by $\delta_U(x)$ the distance from the point $x$ to the cone $U$ and consider the space $E_\beta(U) = \bigcup_{B>0} E_{\beta,B}(U)$, where $E_{\beta,B}(U)$ consists of entire analytic functions on $\mathbb{C}^n$ such that the norms

$$\|f\|_{U,N,B} = \sup_{x,y} |f(x+iy)| (1+|x|)^N \exp\{-\beta(B|y|) - \beta \circ \delta_U(Bx)\}$$

are finite for any $N = 0, 1, \ldots$. The topology of $E_\beta(U)$ is defined to be that of the inductive limit of the countably normed spaces $E_{\beta,B}(U)$ with the index $B \to \infty$.

**Lemma 3.** The space $E_\beta(\mathbb{R}^n)$ coincides with the space $S^b$ defined by the indicator function $b(s) = e^{\beta_*(s)}$, where $\beta_*(s) = \sup_{\tau>0} (s\tau - \beta(s))$.

**Proof.** Taking into account the elementary inequalities $1+|z| \leq (1+|x|)(1+|y|)$ and $\beta((1+\epsilon)\tau) - \beta(\tau) \geq C_\epsilon + h_\epsilon \tau$, where $\epsilon > 0$ is arbitrarily small and $h_\epsilon$ is a positive constant, we see that replacing the factor $(1+|x|)^N$ in (19) by $\max_{|\kappa| \leq \kappa} |z^{\kappa}|$ leads to an equivalent definition of $E_\beta$. Let $f \in E_\beta(\mathbb{R}^n)$. Then

$$|\partial^{\kappa} \hat{f}(p)| = \left| \int_{\text{Im}z=y} z^\kappa e^{ipz} f(z) \, dz \right| \leq \kappa \int (1+|x|)^{-(n+1)} e^{-py + \beta(B|y|)} \, dx.$$ 

Making use of the freedom in the choice of the plane of integration, we set $y = \tau p/|p|$ and take the infimum with respect to $\tau > 0$. As a result, we obtain an estimate of the form (19) with $b = e^{\beta_*}$. Conversely, if (19) holds for such an indicator function, then taking the inverse Fourier–Laplace transformation, we find that

$$|z^{\kappa} f(z)| \leq (2\pi)^{-n} \int e^{n|p||y|} |\partial^{\kappa} \hat{f}(p)| \, dp \leq C_{\kappa,\epsilon} \sup_p e^{n|p||y| - \beta_*(|1-\epsilon|p'/B)},$$

and so $f \in E_\beta(\mathbb{R}^n)$ because the Legendre transformation is involutory. The lemma is proved.

The spaces $S^b$ of the specified type will be called the Gelfand–Shilov–Gurevich spaces because they also belong to the class of spaces of type $W$ introduced by B. L. Gurevich.
Among such spaces, a special role is played by that defined by $\beta(\tau) = \tau$. This space customarily denoted by $S^0$ is nothing but the Fourier transformed Schwartz’s space $\mathcal{D}$. It is universal for nonlocal fields because in this case $b(s) = 1$ for $0 \leq s \leq 1$ and $b(s) = \infty$ for $s > 1$, i.e., the test functions have compact support in momentum space, and so fields with an arbitrarily singular ultraviolet behavior can be smeared with them.

**Definition 1.** A closed cone $K$ is called a carrier cone of a functional $v \in \mathcal{E}^{\beta}$ if $v$ has a continuous extension to each space $\mathcal{E}^\beta(U)$, $U \supset K \setminus \{0\}$.

For any $v \in \mathcal{E}^{\beta}$ there exists a unique minimal closed carrier cone. This has been proved in [14] for the spaces $S^b$ defined by the exponentials of order $> 1$, and just the properties of them that were used in this proof are included into the definition of $\mathcal{E}^\beta$.

**Definition 2.** We say that the fields $\varphi$, $\varphi'$ defined on the test function space $\mathcal{E}^{\beta}(\mathbb{R}^d)$ asymptotically commute for large spacelike separations of their arguments if the matrix elements $\langle \Phi, [\varphi(x), \varphi'(x')] \Psi \rangle$ are carried by cone (17) for any $\Phi, \Psi$ in the common domain of definition of these fields.

5 Conditions of convergence of Wick power series on analytic test functions

A general convergence criterion for Wick series can be formulated in terms of the above-mentioned characteristic $u(\tau)$ of the two-point function $w(z)$ and the indicator function $b(s)$ as follows. The series (1) is convergent under smearing with test functions in $S^b$ if

$$
\sum_{k=0}^{\infty} L^k k! d_{2k} \inf_{\tau > 0} \frac{u(\tau)^k e^{\epsilon \tau} \leq C_{L, \epsilon} b(\epsilon s)}{2k}
$$

for any $L > 0$ and $\epsilon > 0$. The proof of this criterion is the same as that of Theorem 4 in [1], where it has been established in the case of the free field of mass $m$ and the explicit form of the corresponding function $u(\tau)$ has been used. For the Gelfand–Shilov–Gurevich spaces, an alternative formulation is possible in terms of the function $\beta$, which will be useful below. Its derivation is much simpler than that in the general case considered in [1], where test functions are not necessarily analytic.

**Theorem 1.** Let $\phi$ be a scalar neutral generalized free field and let

$$
u(\tau) = (2\pi)^{-d} \int e^{-\tau p^\beta} \hat{w}(p) \, dp,$$
where \( \hat{w} \) is the Fourier transform of its two-point function. Suppose restrictions (3) on the coefficients of the Wick power series \( s \) hold. If the function \( \beta \) defining the space \( \mathcal{E}^\beta \) satisfies the condition
\[
\sum_{k=0}^{\infty} L^k k! d_{2k} \inf_{\tau > 0} u(\tau)^k e^{\beta(B\tau)} < \infty
\]
(21)
for arbitrarily large \( L, B > 0 \), then the field \( \varphi_s \) and all fields \( \varphi_{s'}, s' < s \), are well defined as operator-valued generalized functions over \( \mathcal{E}^\beta(\mathbb{R}^d) \) acting in the Hilbert space \( \mathcal{H} \) of the initial field \( \phi \).

**Proof.** According to what has been said above, it is sufficient to show that (21) implies the absolute convergence of the series on the right-hand side of (8) on every test function \( f \in \mathcal{E}^\beta \). Let \( \eta = (\tau, 0, \ldots, 0) \in V_+ \) and let \( y = (\eta, 2\eta, \ldots, n\eta) \in V_{n-} \). Then
\[
W^K(f) = \int W^K(x + iy) f(x + iy) \, dx.
\]
(22)
Making use of (2), (19), the monotonicity of \( u \), and the equality \( |\eta| = \sqrt{\eta^2} = \tau \), we obtain the estimate
\[
|W^K(x + iy) f(x + iy)| \leq \frac{\|f\|_{2nd,B}}{(1 + |x|)^{2nd}} u(\tau)^{|K|} e^{\beta(B\tau)}, \quad f \in \mathcal{E}^\beta,B(\mathbb{R}^d).
\]
In view of the freedom in the choice of \( \tau \) we have
\[
|W^K(f)| \leq C(f) \inf_{\tau > 0} u(\tau)^{|K|} e^{\beta(B\tau)}.
\]
Thus, the required convergence of the series (8) is ensured by the convergence of the number series
\[
\sum_K |D_K| \inf_{\tau > 0} u(\tau)^{|K|} e^{\beta(B\tau)}
\]
(23)
with arbitrarily large \( B > 0 \). From the conditions (3) and the properties of the polynomial coefficients, it follows that \( |D_K| \leq h^{|K|} |K|! d_{2|K|} \), see [1]. Since the number of multi-indices with the norm \( |K| = k \) does not exceed \( k^{n(n-1)/2} \), we conclude that series (23) is majorized by series (21) for sufficiently large \( L \). The theorem is thus proved.

**Lemma 4.** Criterion (20) is equivalent to condition (18) for \( b(s) = e^s \) and condition (21) for \( b(s) = e^{\beta_* s} \).
Proof. Let \( b(s) = e^s \). Under condition (18) the series \( \sum d_{2k} k! z^k \) is convergent everywhere. Majorizing the infimum on the left-hand side of inequality (20) by the value of the function at \( \tau = \epsilon \), we see that it is valid with the constant
\[
C_{L, \epsilon} = \sum d_{2k} k! (Lu(\epsilon))^k.
\]
To prove the inverse implication (20) \( \Rightarrow \) (18), choose \( m > 0 \) such that
\[
\int_{\mathcal{V}_+ \cap \{p : p^0 \leq m\}} \hat{w}(p) \, dp > 0.
\]
Then \( u(\tau) \geq C e^{-m\tau} \) and condition (20) ensures that
\[
(CL)^k |d_{2k}| k! \leq C_{L, \epsilon} e^{\epsilon s}
\]
for \( mk \leq s \). Setting \( s = mk \) and making use of the arbitrariness of \( L \), we obtain (18).

Now let \( b(s) = e^{\beta_s(s)} \). If \( \beta \) grows linearly, then the statement of the lemma is verified immediately. So we assume that
\[
\lim_{\tau \to \infty} \beta(\tau)/\tau = \infty.
\]
Let us demonstrate that
\[
\inf_{\tau > 0} u(\tau)^{k e^{\beta(\tau)}} = \sup_{s \geq 0} \inf_{\tau > 0} u(\tau)^{k e^{s \tau - \beta_s(s)}}.
\]
(This relation generalizes the conclusion of Lemma 1 in [2], where \( u(\tau) = 1/\tau \).) Since the Legendre transformation is involutory, the equality (24) holds for \( k = 0 \). Let \( k > 0 \). Since \( u(\tau) \geq C e^{-m\tau} \), the infimum on the left-hand side occurs at some finite point \( \tau_k > 0 \) satisfying the equation
\[
\beta'(\tau_k) = -ku'((\tau_k)/u(\tau_k)).
\]
Set \( s_k = \beta'(\tau_k) \). Making use of the relation
\[
\frac{d^2 \ln u(\tau)}{d\tau^2} = \frac{1}{2u(\tau)^2} \int_{\mathbb{R}^d \times \mathbb{R}^d} (p_0 - p'^0)^2 e^{-(p_0 + p'^0)\tau} \hat{w}(p) \hat{w}(p') \, dp \, dp' \geq 0
\]
and taking into account that, for convex functions, every stationary point is the point of absolute minimum, we conclude that the extrema \( \inf_{\tau > 0} e^{-s_k\tau + \beta(\tau)} = e^{-\beta_s(s_k)} \) and \( \inf_{\tau > 0} u(\tau)^{k e^{s_k\tau}} \) are attained at \( \tau = \tau_k \). Estimating from below the supremum with respect to \( s \) by the value of the function at the point \( s_k \), we obtain
\[
\sup_{s \geq 0} \inf_{\tau > 0} u(\tau)^{k e^{s \tau - \beta(s)}} \geq e^{-s_k\tau_k + \beta(\tau_k)} \inf_{\tau > 0} u(\tau)^{k e^{s_k\tau}} = e^{\beta(\tau_k) u(\tau_k)^k}.
\]
Since \( \inf_{s} \sup_{\tau} G(\tau, s) \geq \sup_{s} \inf_{\tau} G(\tau, s) \) for any function \( G \), the inverse inequality also holds, and so (24) is proved. Supposing (20) is valid and applying (24), we have

\[
L^k d_k k! \inf_{\tau>0} u(\tau)^k e^{\beta(B(\tau))} \leq C_{2L,1/B} 2^{-k},
\]

whence (21) immediately follows. Conversely, setting \( B = 1/\epsilon \) in (21), using (24) and estimating from below the suprema with respect to \( s \) by the value of the function at a fixed point, we arrive at (20). The lemma is proved.

6 Proof of asymptotic commutativity

We proceed to show that the generalizations of local commutativity considered in Section 4 are fulfilled for the fields determined by the Wick series convergent on analytic test functions.

**Theorem 2.** Let \( s \) be a series in the Wick powers of a generalized free field \( \phi \) and let its coefficients satisfy assumption (1). Suppose the indicator function \( b(s) \) of the test function space satisfies condition (20). If \( S^b \) is a Gelfand–Shilov–Gurevich space, then the fields \( \varphi_s \) determined by the series \( s' \prec s \) commute asymptotically. If \( S^b = S^1 \), then they are relatively local in the sense of hyperfunction theory.

**Lemma 5.** Let the condition (21) be satisfied. If for any set of series \( s_j \prec s \), \( 1 \leq j \leq n \), functional (16) has a continuous extension to the space \( E^\beta(V_{j,j+1}) \), where \( V_{j,j+1} \) is the open cone \( \{ x \in \mathbb{R}^{2d} | (x_j - x_{j+1})^2 > 0 \} \), then all the fields determined by series subordinate to \( s \) commute asymptotically. If condition (18) holds and the functional (16) allows a continuous extension to \( S^1(V_{j,j+1}) \), then they are relatively local in the sense of hyperfunction theory.

**Proof.** Let \( \Phi = \varphi_{s_1}(f_1) \ldots \varphi_{s_l}(f_l) \Psi_0 \) and \( \Psi = \varphi_{s'_1}(g_1) \ldots \varphi_{s'_m}(g_m) \Psi_0 \), where \( s_i, s'_k \prec s \), \( f_i, g_k \in E^\beta(\mathbb{R}^d) \). Then the value of the functional \( \langle \Phi, [\varphi_s(x') \varphi_s(x'')] \Psi \rangle \) on a test function \( f \in E^\beta(\mathbb{R}^{2d}) \) coincides with that of the functional of the form (16) with a suitable set of indices on the test function \( f_1 \otimes \ldots \otimes f_1 \otimes f \otimes g_1 \otimes \ldots \otimes g_m \). If \( f \in E^\beta(V^{(2d)}) \), then this tensor product is the element of \( E^\beta(V_{j,j+1}) \) which depends continuously on \( f \). Thus, the existence of a continuous extension to \( E^\beta(V_{j,j+1}) \) for the functionals of the form (16) ensures that the matrix elements in question have continuous extensions to \( E^\beta(V^{(2d)}) \) and all the more to \( E^\beta(U) \), where \( U \supset V^{(2d)} \setminus \{0\} \). Analogous statements,
with proper changes in notation, are valid for the space $S^1$. Since $D(s)$ is the linear span of the vectors $\Phi, \Psi$ of the specified form, the lemma is proved.

In the above derivation of the convergence criterion [21] the key role is played by the variation of the plane of integration in the representation [22] which enables us to obtain the best estimate for each term of the series on the right-hand side of (8). We shall apply the same idea to prove the asymptotic commutativity of the sums of Wick series. However, this will require integrating the corresponding analytic functions over surfaces of a more complicated form. The variation of such surfaces in the analyticity domain is admissible by the Cauchy–Poincaré theorem [16], but for obtaining concrete estimates it will be convenient to use directly the Stokes theorem which lies at the basis of its derivation.

According to Lemma 2 functional (16) is represented by the series

$$\sum_K D_K(b_{V_{n-1}} W^K - b_{j_{V_{n-1}}} W^K).$$

(25)

In view of the barrelledness of $E_{\beta}(V_{j,j+1})$ and $S^1(V_{j,j+1})$, in order to extend continuously this functional to these spaces it is sufficient to construct a continuous extension of each term of series (25) and show that it is absolutely convergent on every element of the corresponding space. We shall consider in detail such a procedure for $E_{\beta}(V_{j,j+1})$ and then explain how the proof should be modified for the case of hyperfunctions.

For $\tau \geq 0$ and $1 \leq j \leq n$, we define the regions

$$\Gamma^\tau_j = \{ x \in \mathbb{R}^n \mid (x_j - x_{j+1})^2 \leq -\tau^2 \} = \{ x \in \mathbb{R}^n \mid x_j - x_{j+1} \in G^\tau \},$$

$$U^\tau_j = \{ y \in \mathbb{R}^n \mid y_i - y_l \in \mathcal{V} \text{ for } i < l, (i, l) \neq (j, j+1); \ y_j - y_{j+1} \in V^\tau \}.$$ 

The relations (2)–(5) show that, for $0 \leq \tau' < \tau$, the set $\Gamma^\tau_j + iU^\tau_j$ lies in the analyticity domain of $W^K(z)$, and for $z = x + iy \in \Gamma^\tau_j + iU^\tau_j$, the following inequality holds:

$$|W^K(z)| \leq u \left( \frac{\tau^2 - \tau'^2}{\sqrt{\tau^2 - \tau'^2}} \right)^{K_{j,j+1}} \prod_{i<l, (i,l)\neq (j,j+1)} u \left( \frac{(y_i - y_l)^2}{(y_j - y_{j+1})^2} \right)^{K_{il}}.$$ 

(26)

**Lemma 6.** Let $\tau > 0$, $\eta = (\tau, 0, \ldots, 0) \in \mathcal{V}_+$, and $f \in \mathcal{E}^\beta(\mathbb{R}^n)$. With the notation

$$y(t) = (\eta, \ldots, (j + (1-t)/2)\eta, (j + (1 + t)/2)\eta, (j + 2)\eta, \ldots, n\eta),$$

$$F(x, t) = W^K(x + iy(t)) f(x + iy(t)),$$

(27)
for any \( K \) and \( 1 \leq j \leq n \), the identity
\[
(b_{V_{n-}} W^K - b_{\tau_j V_{n-}} W^K)(f) = I_1 + I_2
\]
holds, where
\[
I_1 = \int_{\partial G^\tau} (F(x, 1) - F(x, -1)) \, dx,
\]
\[
I_2 = i \tau \int_{-1}^{1} dt \int_{R^{(n-1)d}} d\xi_1 ... d\xi_{j-1} d\xi_{j+1} ... d\xi_n \int_{\partial G^\tau} F(P^{-1} \xi, t) \nu^0 \, dS_\tau,
\]
\( P \) is the linear operator taking \( x = (x_1, ..., x_n) \) to \( (x_1 - x_2, ..., x_{n-1} - x_n, x_n) \), \( \nu \) is the unit inward normal to \( \partial G^\tau \), and \( dS_\tau \) is the surface measure on \( \partial G^\tau \).

**Proof.** Note that \( y(1) \in V_{n-} \) and \( y(-1) \in \tau_j V_{n-} \) and hence
\[
(b_{V_{n-}} W^K - b_{\tau_j V_{n-}} W^K)(f) = \int_{\mathbb{R}^n} (F(x, 1) - F(x, -1)) \, dx.
\]
We split the integration domain into \( \Gamma^\tau_j \) and \( C \Gamma^\tau_j \), rewrite the right-hand side of the last equality as \( I_1 + J \), where
\[
J = \int_{\Gamma^\tau_j} (F(x, 1) - F(x, -1)) \, dx,
\]
and shall show that \( J = I_2 \). Let \( 0 < \tau' < \tau \). From (19), (26), it follows that, for any compactum \( Q \subset U^\tau_{j'} \), the estimate
\[
|W^K(z)f(z)| \leq C_{N,Q}(1 + |x|)^{-N}
\]
holds if \( z \in \Gamma^\tau_j + iQ \). Making use of Cauchy’s integral formula, one can easily show that analogous inequalities are satisfied for the derivatives as well. Therefore, the function
\[
\Phi(y) = \int_{\Gamma^\tau_j} W^K(x + iy)f(x + iy) \, dx
\]
is differentiable and, applying the Cauchy–Riemann equations, we have
\[
\frac{\partial \Phi(y)}{\partial y_l^0} = i \int_{\Gamma^\tau_j} \frac{\partial [W^K(x + iy)f(x + iy)]}{\partial x_l^0} \, dx, \quad y \in U^\tau_{j'}.
\]
Since \( y(t) \in U^\tau_{j'} \) for \(-1 \leq t \leq 1\) and
\[
\int_{\Gamma^\tau_j} F(x, t) \, dx = \Phi(y(t)),
\]
we obtain
\[ J = \int_{-1}^{1} dt \frac{d}{dt} \Phi(y(t)) = \frac{i\tau}{2} \int_{-1}^{1} dt \int_{\Gamma_j^\tau} \left( \frac{\partial F}{\partial x_{j+1}}(x, t) - \frac{\partial F}{\partial x_j}(x, t) \right) dx. \]

Let us denote \( \tilde{F}(\xi, t) = F(P^{-1}\xi, t) \) and make the change of variables \( x \to P^{-1}x \). Then, observing that \( P(\Gamma_j^\tau) = \mathbb{R}^{(n-1)d} \times G^\tau \), we get the equality
\[ J = \frac{i\tau}{2} \int_{-1}^{1} dt \int_{\mathbb{R}^{(n-1)d} \times G^\tau} \left( \frac{\partial \tilde{F}}{\partial \xi_{j+1}}(\xi, t) + \frac{\partial \tilde{F}}{\partial \xi_{j-1}}(\xi, t) - 2\frac{\partial \tilde{F}}{\partial \xi_j}(\xi, t) \right) d\xi. \]

From (19), (26), it follows that \( \tilde{F}(\xi, t) \to 0 \) as \( |\xi| \to \infty \), \( \xi \in \mathbb{R}^{d(n-1)} \times G^\tau \), and therefore the integration of \( \partial \tilde{F}/\partial \xi_{j+1} \) and \( \partial \tilde{F}/\partial \xi_{j-1} \) yields zero. Reducing the multiple integral to the iterated one, we obtain
\[ J = -i\tau \int_{-1}^{1} dt \int_{\mathbb{R}^{(n-1)d}} \hat{d} \xi \int_{G^\tau} d\xi_j \frac{\partial \tilde{F}}{\partial \xi_j}(\xi, t), \]

where \( \hat{d} \xi = d\xi_1 \ldots d\xi_{j-1}d\xi_{j+1} \ldots d\xi_n \). The last integrand can be regarded as the divergence of the vector field \( (\tilde{F}, 0) \). Applying the Stokes theorem and replacing \( \tilde{F}(\xi, t) \) by \( F(P^{-1}\xi, t) \), we conclude that \( J = I_2 \).

**Lemma 7.** The functional \( b_{V_n} W^K - b_{\tau_j V_n} W^K \) can be continuously extended to \( \mathcal{E}^\beta(V_{j,j+1}) \) and, for any \( B > 0 \), \( N \geq (n + 1)d \) there exist \( B', C \) such that
\[ ||(b_{V_n} W^K - b_{\tau_j V_n} W^K)(f)|| \leq C \| f \|_{V_{j,j+1}, N, B} \inf_{\tau > 0} u(\tau)^{K} e^{\beta(B'\tau)} \]  for all \( f \in \mathcal{E}^{B}(V_{j,j+1}) \)

**Proof.** Let us show that formulas (28)-(30) determine the desired extension if \( f \) is assumed to be an element of the space \( \mathcal{E}^\beta(V_{j,j+1}) \). We first estimate \( I_1 \) supposing \( f \in \mathcal{E}^{\beta,B}(V_{j,j+1}) \). It is easy to see that \( \delta_{V_{j,j+1}}(x) \leq \tau \) for \( x \in \mathbb{C} \Gamma_j^\tau \). Besides, \( |y(t)| \leq n|\eta| = n\tau \) for \(-1 \leq t \leq 1 \) and, in view of (19) and the monotonicity of \( \beta(s) \), we have
\[ |f(x + iy(t))| \leq || f \|_{V_{j,j+1}, N, B} (1 + |x|)^{-N} e^{2\beta(B\tau)}, \quad x \in \mathbb{C} \Gamma_{n,j}^\tau. \]

Taking into account that \( y(\pm 1) \in V_n \) and using relations (2), (3), the monotonicity of \( u \), and the equality \( \sqrt{n^2} = \tau \), we find that
\[ |F(x, \pm 1)| \leq || f \|_{V_{j,j+1}, N, B} (1 + |x|)^{-N} u(\tau)^{K} e^{2\beta(B\tau)} \]
for $x \in \mathbb{C} \Gamma_j^\tau$. Thus, we have
\[ |I_1| \leq C_1 \| f \|_{V_j,j+1,N,B} u(\tau)^{|K|} e^{2\beta(Bn\tau)}. \] (33)

Now let us estimate $I_2$. Observe that if $\xi_j \in \partial G^\tau$, then $P^{-1}\xi \in \partial \Gamma_j^\tau$. Furthermore, $(1 + |P|)(1 + |P^{-1}\xi|) \geq (1 + |\xi|)$, where $|P| = \sup_{|x| \leq 1} |Px|$. Using (32) and the definition $|\xi| = \max(|\xi_j|, |\hat{\xi}|)$, we obtain
\[ |f(P^{-1}\xi + iy(t))| \leq \frac{\| f \|_{V_j,j+1,N,B}(1 + |P|)^N}{(1 + |\xi|)^{N-d}(1 + |\xi_j|)^d} e^{2\beta(Bn\tau)}, \quad \xi_j \in \partial G^\tau. \] (34)

Next we apply (26), taking into account that $y(t) \in U_j^0$ for $t \neq 0$ and using the monotonicity of $u$. As a result, we get
\[ |W_K(P^{-1}\xi + iy(t))| \leq u(\tau)|K|, \quad \xi_j \in \partial G^\tau. \] (35)

Further, there exists a constant $C$ independent on $\tau$ and such that
\[ \int (1 + |x|)^{-d}|\nu^0| \, dS_\tau \leq C. \]

Indeed,
\[ |\nu^0| \, dS_\tau = r^{d-2} \, dr \, d\Omega, \] (36)

where $r = \|x\|$ and $d\Omega$ is the area element for the surface of the unit sphere in $\mathbb{R}^{d-1}$.

Let us denote
\[ C_2 = C(1 + |P|)^N \int \frac{d\hat{\xi}}{(1 + |\hat{\xi}|)^{N-d}} \]
and substitute relations (34), (35) into (27), (30). As a result, we obtain
\[ |I_2| \leq C_2 \tau \| f \|_{V_j,j+1,N,B} u(\tau)^{|K|} e^{2\beta(Bn\tau)}. \] (37)

The presence of the factor $\| f \|_{V_j,j+1,N,B}$ in estimates (33), (37) ensures that formulas (28)-(30) define a continuous extension of the distribution $b_{V_n-} W^K - b_{\tau V_n-} W^K$ to $E^\beta(V_{j+1})$ and, by the Cauchy–Poincaré theorem, the extensions corresponding to different $\tau$ coincide with each other. Since $\beta$ is unbounded from above and convex, we have $C_1 + C_2 \tau \leq C' \exp(\beta(Bn\tau))$, where $C' > 0$. Besides, $3\beta(nB\tau) \leq \beta(3nB\tau) + 2\beta(0)$
because of the convexity of $\beta$. Combining these inequalities with (33), (37) and passing to the infimum with respect to $\tau$, we obtain (31) with $B' = 3nB$. The lemma is proved.

From Lemma 7, it immediately follows that series (25) is absolutely convergent on every test function in $E^\beta(V_{j,j+1})$ provided number series (23) converges for any $B > 0$. As was established in proving Theorem 1, this is ensured by condition (21) which is equivalent to the criterion (20) by Lemma 4. Thus, Theorem 2 is proved for the case of spaces $E^\beta$.

The proof of Lemma 6 is extended immediately to the case $f \in S^{1,B}(\mathbb{R}^n)$, if we assume $\tau < 1/(nB)$. Repeating, with appropriate changes, the derivation of Lemma 7, we make sure that, for all $\tau < 1/(nB)$, formulas (28)-(30) define the same continuous extension of the functional $b_{V_n - W^K} - b_{\tauVV_n - W^K}$ to $S^{1,B}(V_{j,j+1})$ and

$$|(b_{V_n - W^K} - b_{\tauVV_n - W^K})(f)| \leq C\|f\|_{V_{j,j+1},N,B}u\left(\frac{1}{nB}\right)^{|K|}. \quad (38)$$

The extensions corresponding to different $B$ are obviously compatible and so define a continuous extension to $S^1(V_{j,j+1})$. Because of (38), series (25) is absolutely convergent on every $f \in S^1(V_{j,j+1})$ if $\sum_K L^{|K|} |D_K| < \infty$ for all $L > 0$. The estimate of the coefficients $D_K$ mentioned in the proof of Theorem 1 shows that the latter is ensured by the condition (18), and it remains to apply Lemma 4 to complete the proof.

7 Concluding remarks

Together with the results of work [2] Theorem 2 shows that the nonlocal fields determined by series in Wick powers of a generalized free field satisfy all requirements of Wightman’s formulation if the locality axiom is replaced by the asymptotic commutativity condition. It is noteworthy that using the generalized Gelfand–Shilov spaces enables one to consider also the series convergent on quasianalytic test function classes defined by the indicator functions $b$ that grow slower than any linear exponential but do not satisfy the strict localizability condition

$$\int_1^{\infty} \frac{\beta(s)}{s^2} ds < \infty$$

ensuring that $S^b$ contains functions of compact support. In fact, Theorem 2 is applicable to this case as well because the generalized functions defined on such spaces have the same supports as their restrictions to $S^1$, see [17].
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