Anomalous localization in the aperiodic Kronig–Penney model

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Abstract
We analyse the anomalous properties of specific electronic states in the Kronig–Penney model with weak compositional and structural disorder. Using the Hamiltonian map approach, we show that the localization length of the electronic states exhibits a resonant effect close to the band centre and anomalous scaling at the band edges. These anomalies are akin to the corresponding ones found in the Anderson model with diagonal disorder. We also discuss how specific cross-correlations between compositional and structural disorder can generate an anomalously localized state near the middle of the energy band. The tails of this state decay with the same stretched-exponential law which characterizes the band-centre state in the Anderson model with purely off-diagonal disorder.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The interest in one-dimensional random models has been constantly increasing since the discovery that, contrary to previous beliefs, systems of this class can undergo a sort of localization–delocalization transition when disorder exhibits specific long-range correlations [1]. Nowadays, one-dimensional models with correlated disorder are used in many fields of physics, including semiconductor superlattices [2], bilayered media [3], Bose–Einstein condensates [4], transmission in waveguides [5] and structures with corrugated surfaces [6] and DNA modelling [7].

In many of these problems, the system under study is represented in terms of an aperiodic Kronig–Penney model. The original Kronig–Penney model was introduced in the early 1930s to analyse the band structure of crystalline materials [8] and found new applications in the 1980s when it was used to study electronic states in semiconductor superlattices [9]. More
recently, the interest in random one-dimensional systems has spurred investigations of the aperiodic variants of the Kronig–Penney model, which have been analysed both theoretically and experimentally [2, 5, 10].

Because of the simplicity and versatility of the aperiodic Kronig–Penney model, it is important to obtain as much analytical information as possible on the structure of its electronic states. A detailed study of the Kronig–Penney model with weak compositional and structural disorder was done in [11], where an analytical expression for the localization length of the electronic states was derived. The formula works very well for most energy values, but fails in a neighbourhood of the band centre and of the band edges, where localization anomalies appear. The first goal of the present work is to understand these anomalies: using the Hamiltonian map approach [12], we show that they have the same form as the corresponding anomalies which exist in the standard Anderson model with diagonal disorder at the band centre [13, 14] and at the band edge [15]. These anomalies have recently attracted attention because of the related violations of the single-parameter scaling hypothesis [16] at the edge [17] and at the centre [18] of the energy band. It is therefore of some interest to see whether similar anomalies exist in the random Kronig–Penney model.

After explaining the nature of the anomalies of localization length at the band centre and band edges, we focus our attention on a particular variant of the random Kronig–Penney model, characterized by specific cross-correlations between the compositional disorder and the structural one. We show that such a Kronig–Penney model corresponds to an Anderson model with energy-dependent diagonal and off-diagonal disorder. For a specific energy value, the diagonal disorder vanishes and the Kronig–Penney model becomes equivalent to the Anderson model with purely off-diagonal disorder. As is known [19], at the band centre the Anderson model with purely diagonal disorder has anomalously localized states, whose existence is another violation of the single-parameter scaling hypothesis [20]. Because of the identity of the mathematical equations, similar states with stretched-exponential tails also exist in the considered variant of the Kronig–Penney model.

This paper is organized as follows. In section 2 we define the model under study and summarize the main results concerning the localization of the electronic states in the standard case. In section 3 we discuss the anomalies of the localization length emerging for energy values close to the band centre and in the neighbourhood of the band edge. In section 4 we analyse the anomalously localized states created by particular cross-correlations of the disorder. Conclusions are drawn in section 5.

2. Localization of the electronic states: the standard case

In this section we define the model under study and briefly derive the expression for the localization length valid in the general case.

2.1. Definition of the model

We consider a Kronig–Penney model with weak compositional and structural disorder. The model is defined by the Schrödinger equation

\[-\psi''(x) + \sum_{n=-\infty}^{\infty} (U + u_n)\delta(x - an - a_n)\psi(x) = q^2\psi(x), \]  

which describes the motion of an electron of energy $q^2$ in a potential formed by a succession of aperiodically positioned delta-barriers of random strengths. To simplify the form of the equations, here and in what follows, we use energy units such that $\hbar^2/2m = 1$. We introduce
compositional disorder in model (1) by assuming that the barrier strengths display random fluctuations $u_n$ around their mean value $U$. The structural disorder, on the other hand, is present because the positions of the barriers are displaced by a random amount $a_n$ with respect to the lattice sites $na$.

For weak disorder, it is enough to specify the statistical properties of the model in terms of the first two moments of the strength fluctuations $u_n$ and of the relative displacements $\Delta_n = a_{n+1} - a_n$ (which are more relevant than the absolute displacements $a_n$ themselves for the description of the electronic states). We assume that both variables have zero average, $\langle u_n \rangle = 0$ and $\langle \Delta_n \rangle = 0$, and that their variances satisfy the conditions of weak compositional disorder $\langle u_n^2 \rangle \ll U^2$, and of weak structural disorder $\langle \Delta_n^2 \rangle q^2 \ll 1$. (2)

We also consider the normalized binary correlators

$$
\chi_1(l) = \frac{\langle u_n u_{n+l} \rangle}{\langle u_n^2 \rangle}
\chi_2(l) = \frac{\langle \Delta_n \Delta_{n+l} \rangle}{\langle \Delta_n^2 \rangle}
\chi_3(l) = \frac{\langle u_n \Delta_{n+l} \rangle}{\langle u_n \Delta_n \rangle}
$$

as given functions. We do not attribute any specific form to the correlators (3); we only assume that, because of the spatial homogeneity in the mean of the model, they are (decreasing) functions of the difference $l$ of the site indices. For the sake of simplicity, we restrict our attention to the case in which the self-correlators $\chi_1(l)$ and $\chi_2(l)$ are even functions of $l$, but we do not make the same assumption about the cross-correlator $\chi_3(l)$ (in other words, we suppose that the model is only partially isotropic).

2.2. The Hamiltonian map approach

The Schrödinger equation (1) has the same form of the dynamical equation of a stochastic oscillator with frequency $q$ perturbed by a random succession of delta-shaped impulses (‘kicks’). The Hamiltonian of such an oscillator is

$$
H = \frac{p^2}{2} + \frac{1}{2} \left[ q^2 - \sum_{n=-\infty}^{\infty} (U + u_n) \delta(t - a_n - a_n) \right] x^2.
$$

The mathematical identity of the dynamical equation of the kicked oscillator (4) with equation (1) allows one to analyse the properties of the electronic states of the Kronig–Penney model (1) in terms of the trajectories of the kicked oscillator (4): this is the key idea of the Hamiltonian map approach [12, 21]. After integrating the Hamiltonian equations of the stochastic oscillator over the interval between two successive kicks, one obtains the Hamiltonian map

$$
\begin{pmatrix}
  x_{n+1} \\
  p_{n+1}
\end{pmatrix} = \mathbf{T}_n
\begin{pmatrix}
  x_n \\
  p_n
\end{pmatrix}
$$

with the transfer matrix

$$
\mathbf{T}_n = \begin{pmatrix}
  \cos[q(a + \Delta_n)] + (U + u_n) \frac{1}{2} \sin[q(a + \Delta_n)] & \frac{1}{2} \sin[q(a + \Delta_n)] \\
  -q \sin[q(a + \Delta_n)] + (U + u_n) \cos[q(a + \Delta_n)] & \cos[q(a + \Delta_n)]
\end{pmatrix}.
$$
To analyse the trajectories of the Hamiltonian map (5), we follow the approach proposed in [10] (see [11] for details) and we perform the canonical transformation \((x_n, p_n) \rightarrow (X_n, P_n)\) of the form
\[
(x_n, p_n) = \left( \alpha \cos \frac{q a}{2}, -q a \sin \frac{q a}{2}, \frac{1}{q a} \sin \frac{q a}{2}, \frac{1}{q a} \cos \frac{q a}{2} \right) (X_n, P_n).
\]

The parameter \(\alpha\) in equation (7) is defined by the identity
\[
\alpha^4 = \frac{1}{q^2} \frac{\sin(q a) - \frac{U}{q} \cos(q a) - 1}{\frac{U}{q} \cos(q a) + 1}.
\]

The utility of the transformation (7) lies in the fact that the Hamiltonian map (5), when expressed in terms of the new variables \((X_n, P_n)\), is reduced to a simple phase-space rotation in the absence of disorder. The rotation angle \(ka\) of the unperturbed map is defined by the identity
\[
\cos(ka) = \cos(q a) + \frac{U}{2q} \sin(q a).
\]

Equation (8) determines the band structure of the Kronig–Penney model (1) and reveals that the parameter \(k\) is the Bloch wavenumber.

To simplify the form of the Hamiltonian map (5) further, we switch from the Cartesian coordinates \((X_n, P_n)\) to action-angle variables \((J_n, \theta_n)\) via the standard relations
\[
X_n = \sqrt{2J_n} \sin \theta_n,
\]
\[
P_n = \sqrt{2J_n} \cos \theta_n.
\]

Within the second-order approximation, the Hamiltonian map (5) can be written in terms of the action-angle variables in the form [11]
\[
J_{n+1} = D_n^2 J_n
\]
\[
\theta_{n+1} = \theta_n + ka - \frac{1}{2} [1 - \cos (2\theta_n + ka)] \tilde{u}_n + \frac{1}{2} [\zeta - \cos (2\theta_n + 2ka)] \tilde{\Delta}_n
\]
\[
+ \frac{1}{4} [2 \sin (2\theta_n + ka) - \sin (4\theta_n + 2ka)] \tilde{u}_n^2
\]
\[
+ \frac{1}{4} [2 \zeta \sin (2\theta_n + 2ka) - \sin (4\theta_n + 4ka)] \tilde{\Delta}_n^2
\]
\[
+ \frac{1}{4} \sin (ka) - 2 \sin (2\theta_n + 2ka) + \sin (4\theta_n + 3ka)] \tilde{u}_n \tilde{\Delta}_n
\]
\]

with the ratio of the action variables being equal to
\[
D_n^2 = 1 + \sin (2\theta_n + ka) \tilde{u}_n - \sin (2\theta_n + 2ka) \tilde{\Delta}_n + \frac{1}{4} [1 - \cos (2\theta_n + ka)] \tilde{u}_n^2
\]
\[
+ \frac{1}{4} [1 - \zeta \cos (2\theta_n + 2ka)] \tilde{\Delta}_n^2 + [\cos (2\theta_n + 2ka) - \cos (ka)] \tilde{u}_n \tilde{\Delta}_n.
\]

In equations (9) and (10) we have introduced the rescaled disorder variables
\[
\tilde{u}_n = \frac{\sin (qa)}{\frac{q}{a} \sin (ka)} u_n \quad \text{and} \quad \tilde{\Delta}_n = \frac{U}{\sin (ka)} \Delta_n
\]
and the parameter
\[
\zeta = \frac{q \sin (ka)}{U} \left[ q \alpha^2 + \frac{1}{q a^2} \right].
\]
2.3. The localization length

The inverse localization length for the Kronig–Penney model (1) is defined as

\[ l_{-1}^{\text{loc}} = \lim_{N \to \infty} \frac{1}{Na} \sum_{n=1}^{N} \log \left| \frac{\psi_{n+1}}{\psi_{n}} \right|. \]

In the dynamical picture, \( l_{-1}^{\text{loc}} \) is equivalent to the Lyapunov exponent of the Hamiltonian map (5):

\[ \lambda = \lim_{N \to \infty} \frac{1}{Na} \sum_{n=1}^{N} \log \left| \frac{x_{n+1}}{x_{n}} \right| = \lim_{N \to \infty} \frac{1}{Na} \sum_{n=1}^{N} \log (D_n) + \lim_{N \to \infty} \frac{1}{Na} \sum_{n=1}^{N} \log (R_n), \tag{13} \]

with \( D_n \) being defined by equation (10) and

\[ R_n = \left| \frac{\sqrt{\zeta + 1} \sin \theta_{n+1} + \sqrt{\zeta - 1} \sin (qa) \cos \theta_{n+1}}{\sqrt{\zeta + 1} \sin \theta_n + \sqrt{\zeta - 1} \sin (qa) \cos \theta_{n+1}} \right|. \tag{14} \]

As discussed in section 3, in the weak-disorder case, the invariant distribution for the angular variable is either flat (the normal case) or presents a moderate modulation (as happens at the band centre and for the other cases in which the Bloch wavenumber is a rational multiple of \( \pi/a \)). Only at the band edge is the invariant distribution strongly modulated even for weak disorder. Therefore as a function of \( n \) the ratio \( (14) \) can be expected to oscillate around a unitary value, unlike the ratio \( D_n \) which is larger than 1 on average because of the exponential increase of the action variable. As a consequence, away from the band edge, one can drop the second term on the right-hand side of equation (13) and compute the inverse localization length as

\[ \lambda = \lim_{N \to \infty} \frac{1}{Na} \sum_{n=1}^{N} \log (D_n) = \langle \log D_n \rangle. \tag{15} \]

Within the second-order approximation one can expand the logarithm in equation (15) and obtain

\[
\lambda = \frac{1}{2a} \left[ \sin(2\theta_n + ka)\tilde{u}_n - \sin(2\theta_n + 2ka)\tilde{\Delta}_n + \frac{1}{4} \left[ 1 - 2 \cos(2\theta_n + ka) + \cos(4\theta_n + 2ka) \right] \tilde{u}_n^2 \right.
\]
\[
+ \frac{1}{4} \left[ 1 - 2 \cos(2\theta_n + 2ka) + \cos(4\theta_n + 4ka) \right] \tilde{\Delta}_n^2
\]
\[
- \frac{1}{2} \left[ \cos(ka) - 2 \cos(2\theta_n + 2ka) + \cos(4\theta_n + 3ka) \right] \tilde{u}_n \tilde{\Delta}_n \left. \right]. \tag{16}
\]

In the general case, the average over the angle variable on the right-hand side of equation (16) can be computed using a flat distribution \( \rho(\theta) = 1/(2\pi) \). In fact, as the Hamiltonian map (9) shows, the angle variable has a fast dynamic compared to the action variable and quickly assumes a uniform distribution in the interval \([0, 2\pi]\). In this way, following the steps of [11], one obtains that the inverse localization length (15) has the form

\[
\lambda = \frac{1}{8a} \left[ \langle \tilde{u}_n^2 \rangle W_1(ka) + \langle \tilde{\Delta}_n^2 \rangle W_2(ka) - 2 \langle \tilde{u}_n \tilde{\Delta}_n \rangle W_3(ka) \right]. \tag{17}
\]
with

\[
W_1(ka) = \sum_{l=-\infty}^{\infty} \chi_1(l) \cos(2kal),
\]

\[
W_2(ka) = \sum_{l=-\infty}^{\infty} \chi_2(l) \cos(2kal),
\]

\[
W_3(ka) = \sum_{l=-\infty}^{\infty} \chi_3(l) \cos[ka(2l + 1)].
\]

Equation (17) constitutes the standard expression of the inverse localization length of the electronic states in the Kronig–Penney model (1). Note that expression (17) is slightly more general than the corresponding formula given in [11] because we have dropped the assumption that the cross-correlator \( \chi_3(k) \) is an even function of \( k \). The difference between the two expressions shows up on the last term of equation (17) and in the definition of the \( W_3(ka) \) power spectrum. When \( \chi_3(k) = \chi_3(-k) \), equation (17) reduces to the inverse localization length derived in [11]. The inverse localization length (17) works well for almost all values of the energy, but fails close to the band centre and at the band edges, where anomalous effects appear. The nature of these anomalies is discussed in the next section.

3. The band-centre and band-edge anomalies of the localization length

As mentioned in the previous section, the derivation of formula (17) rests on the crucial assumption that the angle variable of the map (9) has a uniform invariant distribution. This assumption is generally justified on the grounds that the angle variable evolves much faster than the action variable and quickly sweeps the whole interval \([0, 2\pi]\). One should note, however, that this argument cannot be applied when the Bloch wavevector is a rational multiple of \( \pi/a \), because in this case the noiseless angular map has periodic orbits, whose effect persists in the form of a modulation of the invariant measure when a weak noise is switched on. For weak disorder, and within the limits of the second-order approximation, this implies that the general formula (17) cannot be applied when the Bloch vector takes values \( ka \simeq 0 \) or \( ka \simeq \pm \pi \), i.e. at the edges of the energy band, and for \( k = \pm \pi / 2a \), i.e. for the energy \([q(\pi/2a)]^2\) which lies close to the band centre. In principle, one should expect anomalies for all Bloch vectors of the form \( ka = \pi / n \) with \( |n| > 2 \). In practice, however, in these cases the invariant distribution of the angular variable is modified by a perturbative term proportional to \( \cos(2n\theta) \) or \( \sin(2n\theta) \) and this does not affect the outcome of the average in equation (16), which contains only second- and fourth-order harmonics of \( \theta \) (the same conclusion applies to the Anderson model, as discussed in [14]).

In conclusion, within the second-order approximation, anomalies are found only at the band edge (\( ka \simeq 0 \) or \( ka \simeq \pm \pi \)) or close to the middle of the band (\( ka = \pi / 2 \)). For these special values of \( k \) the assumption of a flat invariant distribution must be abandoned and the specific form of \( \rho(\theta) \) has to be determined before one can compute the localization length of the electronic states. The non-uniform distribution of the angular variable produces deviations from the standard formula (17) of the inverse localization length; this section is devoted to the analysis of these anomalies.

For the sake of simplicity, in this section we will restrict our attention to the case of uncorrelated disorder. In other words, we will consider two successions of random variables \( \{u_n\} \) and \( \{\Delta_n\} \) such that

\[
\langle u_n u_k \rangle = \langle u_n^2 \rangle \delta_{nk}, \quad \langle \Delta_n \Delta_k \rangle = \langle \Delta_n^2 \rangle \delta_{nk}, \quad \langle u_n \Delta_k \rangle = 0.
\]
In this case the standard expression (17) of the inverse localization length reduces to the simple form
\[ \lambda = \frac{1}{8a} \left[ (\tilde{\alpha}_n^2) + (\tilde{\Delta}_n^2) \right] = \frac{1}{8a \sin^2(ka)} \left[ \frac{\sin^2(qa)}{q^2} (\tilde{\alpha}_n^2) + U^2(\tilde{\Delta}_n^2) \right]. \] (18)

3.1. The anomaly near the middle of the energy band

We will first consider the anomaly for \( ka = \pm \pi/2 \). The corresponding energy lies close to middle of the band and therefore we will often speak, somewhat loosely, of band-centre anomaly. By assigning the value \( k = \pi/2a \) to the Bloch vector and by taking into account that the noise-angle correlators vanish for uncorrelated disorder, one can reduce the general expression (16) to the simpler form
\[ \lambda = \frac{1}{8a} [1 + 2\langle \sin(2\theta_n) \rangle - \langle \cos(4\theta_n) \rangle] [\tilde{\alpha}_n^2] [1 + 2\xi \langle \cos(2\theta_n) \rangle + \langle \cos(4\theta_n) \rangle] [\tilde{\Delta}_n^2]. \] (19)

To determine the form of the invariant distribution \( \rho(\theta) \), we follow the method introduced in [12] and consider the map for the angle variable in equation (9) which, for uncorrelated disorder and \( k = \pi/2a \) simplifies to
\[ \theta_{n+1} = \theta_n + \frac{\pi}{2} - \frac{1}{2} [1 + \sin(2\theta_n)] \tilde{u}_n + \frac{1}{2} [\xi + \cos(2\theta_n)] (\tilde{\Delta}_n) + \frac{1}{8} [2\cos(2\theta_n) + \sin(4\theta_n)] (\tilde{\alpha}_n^2) - \frac{1}{8} [2\xi \sin(2\theta_n) + \sin(4\theta_n)] (\tilde{\Delta}_n^2). \]

To get rid of the constant drift term, one can consider the fourth iterate of this map, i.e.
\[ \theta_{n+4} = \theta_n + \frac{1}{2} \sin(4\theta_n) \left( (\tilde{\alpha}_n^2) - \langle \tilde{\alpha}_n^2 \rangle \right) - \frac{1}{2} [1 + \sin(2\theta_n)] (\tilde{u}_n + \tilde{u}_{n+2}) - \frac{1}{2} [1 - \sin(2\theta_n)] (\tilde{u}_{n+1} + \tilde{u}_{n+3}) + \frac{1}{2} [\xi + \cos(2\theta_n)] (\tilde{\Delta}_n + \tilde{\Delta}_{n+2}) + \frac{1}{2} [\xi - \cos(2\theta_n)] (\tilde{\Delta}_{n+1} + \tilde{\Delta}_{n+3}). \]

Going to the continuum limit, one can replace this map with the Itô stochastic differential equation
\[ d\theta = \frac{1}{2} \sin(4\theta) \left( (\tilde{\alpha}_n^2) - \langle \tilde{\alpha}_n^2 \rangle \right) dt - \sqrt{\tilde{\alpha}_n^2/2} [1 + \sin(2\theta)] dW_1 - \sqrt{\tilde{\alpha}_n^2/2} [1 - \sin(2\theta)] dW_2 + \sqrt{\tilde{\Delta}_n^2/2} [\xi + \cos(2\theta)] dW_3 + \sqrt{\tilde{\Delta}_n^2/2} [\xi - \cos(2\theta)] dW_4, \] (20)

where \( W_1(t), \ldots, W_4(t) \) represent four independent Wiener processes. Given an initial condition \( \theta(t_0) = \theta_0 \), the Itô equation (20) defines a stochastic process \( \theta(t) \), whose conditional probability \( p(\theta, t | \theta_0, t_0) = p \) can be obtained by solving the associated Fokker–Planck equation
\[ \frac{\partial p}{\partial t} = \frac{1}{2} \left( (\tilde{\alpha}_n^2) - \langle \tilde{\alpha}_n^2 \rangle \right) \frac{\partial}{\partial \theta} \left[ \sin(4\theta)p \right] + \frac{1}{4} \frac{\partial^2}{\partial \theta^2} \left[ (3(\tilde{\alpha}_n^2) + (2\xi^2 + 1))(\tilde{\alpha}_n^2) \right] + \left( (\tilde{\Delta}_n^2) - \langle \tilde{\Delta}_n^2 \rangle \cos(4\theta) \right) p \right] \]

with the initial condition \( p(\theta, t_0 | \theta_0, t_0) = \delta(\theta - \theta_0) \) [22]. The stationary solution of equation (21) which satisfies the conditions of normalization and periodicity is [12]
\[ \rho(\theta) = \frac{\sqrt{A + |B|}}{4K(C)} \frac{1}{\sqrt{A - B \cos(4\theta)}}. \] (22)
In equation (22) \( K(C) \) is the complete elliptic integral of the first kind and we have introduced the constants

\[
A = 3\langle \tilde{u}_n^2 \rangle + (2\xi^2 + 1)\langle \tilde{\Delta}_n^2 \rangle, \quad B = \langle \tilde{u}_n^2 \rangle - \langle \tilde{\Delta}_n^2 \rangle
\]

and

\[
C = \frac{\sqrt{2|B|}}{A + |B|} = \sqrt{\frac{2|\langle \tilde{u}_n^2 \rangle - \langle \tilde{\Delta}_n^2 \rangle|}{\langle \tilde{u}_n^2 \rangle - \langle \tilde{\Delta}_n^2 \rangle + 3\langle \tilde{u}_n^2 \rangle + (2\xi^2 + 1)\langle \tilde{\Delta}_n^2 \rangle}}.
\]

Equation (22) shows that, as expected, when the Bloch vector takes the value \( k = \frac{\pi}{2a}, \) the invariant measure has period \( \frac{\pi}{2}. \) The numerical computations agree well with formula (22) as can be seen in figure 1. The data represented in figure 1 were obtained for the mean field \( U = 8 \) and disorder strengths \( \sqrt{\langle u_n^2 \rangle} = \sqrt{\langle \Delta_n^2 \rangle} = 0.02. \) Here, and in the rest of the paper, we present numerical data which were obtained for energy values within the first band.

The knowledge of the invariant distribution (22) makes it possible to compute the Lyapunov exponent (19). The averages of the functions of argument \( 2\theta \) vanish, but the average of \( \cos(4\theta) \) does not and gives rise to the anomaly of the localization length. After some algebra, one obtains that the inverse localization length for \( ka = \frac{\pi}{2} \) is

\[
\lambda = \frac{1}{8a} \left\{ \left[ \langle \tilde{u}_n^2 \rangle - \langle \tilde{\Delta}_n^2 \rangle \right] + 3\langle \tilde{u}_n^2 \rangle + (2\xi^2 + 1)\langle \tilde{\Delta}_n^2 \rangle \right\} \frac{E(C)}{K(C)} - 2 \left( \langle \tilde{u}_n^2 \rangle + \xi^2\langle \tilde{\Delta}_n^2 \rangle \right),
\]

where \( E(C) \) is the complete elliptic integral of the second kind and the argument \( C \) is defined by equation (23). The numerical computations confirm the existence of an anomaly for \( ka = \pi/2 \) as can be seen from the data represented in figure 2 which show a small but clear deviation from the value of the localization length predicted by the standard formula (18). The numerically computed inverse localization length for \( ka = \pi/2 \), on the other hand, matches well with the theoretical value (24). We observe that the numerical data represented in figure 2 were obtained for a specific realization of the disorder; when different disorder realizations are considered, the discrepancy between the numerical value of the Lyapunov exponent for
ka = π/2 and the predicted result (24) fluctuates slightly around zero, always assuming small values as in figure 2.

In conclusion, in the Kronig–Penney model (1) a resonance effect occurs for k = π/2a and produces an anomaly of the localization length. The effect has the same nature of the band-centre anomaly found in the Anderson model with diagonal disorder [12, 13].

We remark that, although our analytical results are restricted to the case of uncorrelated disorder, we found numerical evidence that disorder correlations can enhance the anomaly of the localization length near the middle of the energy band, in agreement with the theoretical conclusions of [23]. As an example, we can consider the case of structural and compositional disorder with self-correlations of the form

\[
\chi_1(l) = \chi_2(l) = \begin{cases} 
1 & \text{if } l = 0 \\
-\frac{5}{3\pi} \sin \left( \frac{2\pi l}{3} \right) & \text{if } |l| > 0
\end{cases}
\]

and no cross-correlations, \(\chi_3(l) = 0\). The long-range correlations of the form (25) create mobility edges at k = π/5a and k = 4π/5a (see [11] for details). Figure 3 represents the numerical data obtained for this kind of disorder with \(U = 8\) and \(\langle n_n^2 \rangle = \sqrt{\langle \Delta_n^2 \rangle} = 0.02\). The data clearly show an enhanced anomaly at k = π/2a with respect to the case of totally uncorrelated disorder. Adding cross-correlations does not introduce any significant modification to the picture.

We would like to stress that, although the previous example shows how correlations of the disorder can enhance the anomaly for k = π/2a, not all correlations produce the same effect. This can be appreciated in the case analysed in section 4, in which the resonance effect for k = π/2a is shadowed by a different kind of anomaly generated by specific cross-correlations for a value of k close, but not identical, to π/2a.
3.2. The band-edge anomaly

We now turn our attention to the anomaly for $ka = \varepsilon \to 0^+$, i.e. in the neighbourhood of the band edge. Because of the similarity with the band-edge anomaly in the Anderson model, we can apply the method used in [12] to the present case. We first derive the form taken by the map (9) at the band edge. For $ka = \varepsilon \to 0^+$ the rescaled random variables (11) can be approximated as

$$
\tilde{u}_n = \frac{\sin(q_0 a)}{q_0} u_n + \cdots \quad \text{and} \quad \tilde{\Delta}_n = \frac{\Delta_n}{\varepsilon} + \cdots,
$$

where we have introduced the symbol $q_0 = q(k = 0)$. In the same limit, the parameter (12) reduces to

$$
\zeta = \sqrt{1 + \frac{4q_0^2}{U^2} \sin^2(ka)} = 1 + \frac{2q_0^2}{U^2} \varepsilon^2 + \cdots. \quad (26)
$$

Taking into account these approximations, one can write the Hamiltonian map (9) in the form

$$
J_{n+1} = D_n^2 J_n \quad \theta_{n+1} = \theta_n + \varepsilon - \frac{\xi_n}{\varepsilon} \sin^2(\theta_n) + \frac{\sigma^2}{\varepsilon^2} \sin^3(\theta_n) \cos(\theta_n) + \cdots \quad (27)
$$

with

$$
D_n = 1 - \frac{\xi_n}{\varepsilon} \sin(\theta_n) \cos(\theta_n) + \frac{\sigma^2}{2\varepsilon^2} \sin^4(\theta_n) + \cdots. \quad (28)
$$

In equations (27) and (28), the symbol $\xi_n$ represents the linear combination of structural and compositional disorder

$$
\xi_n = U \Delta_n = \frac{\sin(q_0 a)}{q_0} u_n.
$$
with zero average, \( \langle \xi_n \rangle = 0 \), and variance

\[
\langle \xi_n^2 \rangle = \sigma^2 = U^2(\Delta_n^2) + \frac{\sin^2(q_0 t)}{q_0^2}\langle u_n^2 \rangle.
\]

Going to the continuum limit, one can replace the angular map in equation (27) with the stochastic Itô equation

\[
d\theta = \left[ \varepsilon + \frac{\sigma^2}{\varepsilon^2} \sin^3(\theta) \cos(\theta) \right] dt + \frac{\sqrt{\sigma^2}}{\varepsilon} \sin^2(\theta) dW
\]

whose associated Fokker–Planck equation

\[
\frac{\partial p}{\partial t} = -\frac{\partial}{\partial \theta} \left\{ \left[ \varepsilon + \frac{\sigma^2}{\varepsilon^2} \sin^3(\theta) \cos(\theta) \right] p \right\} + \frac{1}{2} \frac{\partial^2}{\partial \theta^2} \left[ \frac{\sigma^2}{\varepsilon^2} \sin^4(\theta) p \right]
\]

(30)
gives the conditional probability \( p(\theta, t|\theta_0, t_0) = p \) for the stochastic process \( \theta(t) \) [22]. By introducing the rescaled time

\[
\tau = \frac{\sigma^2}{\varepsilon^2} t,
\]

one can cast the Fokker–Planck equation (30) in the form

\[
\frac{\partial p}{\partial \tau} = -\frac{\partial}{\partial \theta} \left\{ \left[ \kappa + \frac{\sigma^2}{\varepsilon^2} \sin^3(\theta) \cos(\theta) \right] p \right\} + \frac{1}{2} \frac{\partial^2}{\partial \theta^2} \left[ \frac{\sigma^2}{\varepsilon^2} \sin^4(\theta) p \right],
\]

(31)

which contains the noise intensity \( \sigma^2 \) and the distance from the band edge \( \varepsilon \) combined in the single scaling parameter

\[
\kappa = \frac{\varepsilon^3}{\sigma^2}.
\]

The invariant distribution \( \rho(\theta) \) is the stationary solution of the Fokker–Planck (31) which is normalizable and satisfies the periodicity condition \( \rho(\theta + \pi) = \rho(\theta) \). The solution possessing these features is [12, 15]

\[
\rho(\theta) = \frac{1}{N(\kappa)} e^{-f(\theta)} \int_0^\pi \frac{e^{f(\phi)}}{\sin^2(\phi)} d\phi
\]

(32)

with

\[
f(\theta) = 2\kappa \left[ \frac{1}{3} \cot^3(\theta) + \cot(\theta) \right]
\]

and

\[
N(\kappa) = \sqrt{\frac{2\pi}{\kappa}} \int_0^\infty \frac{1}{\sqrt{x}} \exp \left[ -2x \left( \frac{\kappa^3}{12} + x \right) \right] dx.
\]

The integral representation (32) defines the invariant measure in the interval \([0, \pi]\); \( \rho(\theta) \) can be extended outside of this interval via the periodicity condition \( \rho(\theta + \pi) = \rho(\theta) \).

To obtain a qualitative understanding of the behaviour of the invariant distribution (32), it is useful to consider its values at the edges and at the centre of the \([0, \pi]\) interval. For \( \theta \to 0^+ \) and \( \theta \to \pi^- \) one has

\[
\rho(\theta) \sim \begin{cases} 
\frac{1}{\sqrt{\kappa x}} & \text{if } \kappa \to 0 \\
\frac{1}{2\pi} & \text{if } \kappa \to \infty
\end{cases}
\]

while for \( \theta \to \pi/2 \) the invariant distribution behaves as

\[
\rho(\theta) \sim \begin{cases} 
\kappa^{1/3} & \text{if } \kappa \to 0 \\
\frac{1}{2\pi} & \text{if } \kappa \to \infty
\end{cases}
\]
These equations show that, when the energy moves closer to the band edge on the scale defined by the disorder strength, i.e. for $\kappa \to 0$, the invariant distribution develops two pronounced maxima for $\theta \sim 0$ and $\theta \sim \pi$. This conclusion is supported by direct numerical computation of the invariant distribution, as shown by figure 4. The data in figure 4 were obtained for the mean field $U = 8$ and disorder strengths $\sqrt{\langle u_n^2 \rangle} = \sqrt{\langle \Delta_n^2 \rangle} = 0.02$.

The assumption of a uniform or slightly modulated distribution must therefore be radically dropped and this entails that one can no longer neglect the logarithm of the ratio (14) in expression (13) of the inverse localization length. Taking into account equation (26), in the neighbourhood of the band edge, the inverse localization length (13) can be reduced to the form

$$\lambda \simeq \frac{1}{a} \int_0^\infty x^{-1/2} \exp \left[ -\frac{2}{\kappa} \left( \frac{x^2}{12} + x \right) \right] \, dx$$

The average can now be computed using the invariant distribution (32); the final result is [12]

$$\lambda = \frac{\epsilon}{2a} \int_0^\infty x^{1/2} \exp \left[ -2\kappa \left( \frac{x^2}{12} + x \right) \right] \, dx.$$

---

**Figure 4.** Invariant distribution $\rho$ versus $\theta$. The legend shows the value of $\kappa$ corresponding to each line.
Away from the band edge (on the length scale set by the disorder strength), i.e. for \( \kappa_1 \to \infty \), equation (35) reduces to
\[
\lambda \simeq \frac{\sigma^2}{8 a \varepsilon^2},
\]
which coincides with the form of the standard expression (18) in the limit \( k a = \varepsilon \to 0^+ \). On the other hand, close to the band edge (on the length scale set by the disorder strength), i.e. for \( \kappa_1 \to 0 \), equation (35) gives
\[
\lambda \simeq \frac{6^{1/3} \sqrt{\pi}}{2 a^2 \Gamma(1/6)} (\sigma^2)^{1/3},
\]
which exhibits the same anomalous scaling found in the Anderson model at the band edge [12, 15]. This correspondence is a consequence of the fact that in both models at the band edge the invariant distribution for the angular variable has the form (32) and the ratio \( \psi_{n+1}/\psi_n \) reduces to the same function of \( \theta \).

4. Existence of anomalously localized states in the Kronig–Penney model

In this section we discuss how specific cross-correlations between the two kinds of disorder (structural and compositional) can endow the Kronig–Penney model (1) with electronic states whose amplitude, away from the localization centre \( n_0 \), decays like a stretched exponential. More precisely, one has
\[
|\psi_n| \sim \exp(-D \sqrt{|n-n_0|}),
\]
where \( D \) is a constant. This corresponds to a stretched exponential \( \exp(-|x|^\alpha) \) with stretching exponent \( \alpha = 1/2 \). The phenomenon has its counterpart in the band-centre anomaly which occurs in the Anderson model with purely off-diagonal disorder [19].

As mentioned in the previous section, the Kronig–Penney model (1) has an equivalent tight-binding model. The correspondence is easily established by eliminating the momenta from the map (5); in this way, with the obvious substitution \( x_n \to \psi_n \), one obtains the equation
\[
\frac{1}{\sin[q(a+\Delta_n)]} \psi_{n+1} + \frac{1}{\sin[q(a+\Delta_{n-1})]} \psi_{n-1} = \left\{ \cot[q(a+\Delta_n)] + \cot[q(a+\Delta_{n-1})] \right\} \psi_n + \frac{U + u_n}{q} \psi_n.
\]
(37)
It is convenient to express the coefficients of equation (37) as sums of their mean values and of fluctuating terms with zero average. Equation (37) then assumes the form
\[
(1 + \gamma_n) \psi_{n+1} + (1 + \gamma_{n-1}) \psi_{n-1} + \varepsilon_n \psi_n = E \psi_n
\]
(38)
with \( E \) being a deterministic function of the wavevector \( q \) defined by the identity
\[
E(q) = \frac{U/q + 2\langle \cot[q(a+\Delta_n)] \rangle}{\langle 1/\sin[q(a+\Delta_n)] \rangle},
\]
(39)
while the symbols \( \gamma_n \) and \( \varepsilon_n \) stand for the energy-dependent random variables
\[
\gamma_n(q) = \frac{1}{\langle 1/\sin[q(a+\Delta_n)] \rangle} - 1
\]
(40)
and
\[
\varepsilon_n(q) = \frac{1}{\langle 1/\sin[q(a+\Delta_n)] \rangle} \times \left\{ 2\langle \cot[q(a+\Delta_n)] \rangle - \cot[q(a+\Delta_n)] - \cot[q(a+\Delta_{n-1})] - \frac{U_n}{q} \right\}.
\]
(41)
Equation (38) shows that the tight-binding counterpart of the Kronig–Penney model (1) is an Anderson model with both diagonal and off-diagonal disorder. Note that, in the absence of structural disorder, the random variables (40) vanish and the Kronig–Penney’s analogue becomes the ordinary Anderson model with diagonal disorder. For purely compositional disorder, therefore, one could have predicted a priori the existence of the anomalies discussed in section 3.

We now focus our attention on the case in which the compositional disorder has the form

\[ u_n = 2q_c (\cot[q_c(a + \Delta_n)] - q_c \cot[q_c(a + \Delta_n)] - q_c \cot[q_c(a + \Delta_{n-1})] + \gamma_n(q_c)), \]

(42)

where \( q_c \) represents the wavevector fulfilling the condition

\[ E(q_c) = 0. \]

(43)

We stress that condition (42) introduces special cross-correlations between the two kinds of disorder in the Kronig–Penney model (1). We also remark that the weak-disorder condition (2) ensures that a solution of equation (43) exists. In fact, taking into account the band-structure relation (8), within the limits of the second-order approximation one can cast equation (43) in the form

\[ \cos(k_c a) \simeq \frac{q_c(\Delta_n^2)U}{2 \sin(q_c a)}. \]

If the structural disorder is weak enough, the right-hand side of this equation is less than 1 and this implies that a Bloch vector \( k_c \) exists such that \( q(k_c) \) is the solution of equation (43). Actually, a perturbative calculation shows that the Bloch vector \( k_c \) in the positive half of the first Brillouin zone is

\[ k_c \simeq \frac{\pi}{2a} - \frac{q(\Delta_n^2)U}{2a \sin(\alpha)}, \]

(44)

with \( \bar{q} = q(\frac{\pi}{2a}) \). For weak disorder the deviation of \( k_c \) from \( \pi/2a \) is not large and therefore \( q_c^2 \simeq \bar{q}^2 \), which is close to the centre of the energy band.

If the compositional disorder has the form (42), it is easy to see that, when the electron energy takes the critical value \( q_c^2 \) identified by condition (43), equation (38) becomes

\[ [1 + \gamma_n(q_c)]\psi_{n+1} + [1 + \gamma_{n-1}(q_c)]\psi_{n-1} = 0, \]

(45)

which has the same form of the Schrödinger equation for the Anderson model with purely off-diagonal disorder and zero energy. For zero energy, the latter model is known to have an electronic state which exhibits anomalous localization, because it is localized but decays away from the localization centre \( n_0 \) according to equation (36) [19]. We stress that the identity of equation (45) with the zero-energy Schrödinger equation for the Anderson model with only off-diagonal disorder ensures that, when the compositional disorder has the form (42) and the energy takes the critical value \( q_c^2 \), the Kronig–Penney model also has an anomalously localized state whose amplitude decays exponentially with the square root of the distance from the localization centre.

This property can be heuristically justified because equation (45) implies that

\[ \log|\psi_{2n}| = \log|\psi_0| + \sum_{l=0}^{n-1} [\log|1 + \gamma_{2l}(q_c)| - \log|1 + \gamma_{2l+1}(q_c)|]. \]

(46)

By invoking the central limit theorem, one can therefore conclude that, for large values of \( n \), the random variable \( \log|\psi_n| \) has zero average and a variance which increases linearly
Figure 5. Inverse localization length $\lambda$ versus $ka$ for the case of structural disorder without self-correlations. The dashed line (1) represents numerical data, while the solid line (2) corresponds to equation (54). The inset shows the anomaly in greater detail.

with $n$. We stress that the previous argument holds even if the disorder exhibits long-range self-correlations of the form

$$\chi(l) = \frac{1}{c_2 - c_1} \frac{1}{\pi l} [\sin(\pi c_2 l) - \sin(\pi c_1 l)],$$

where $c_1$ and $c_2$ are real numbers such that $0 < c_1 < c_2 \leq 1$. In fact, weaker forms of the central limit theorem can be applied to sums of correlated random variables, provided that the correlations decay fast enough [24, 25]. Specifically, given a succession of zero-average, correlated random variables $\{x_n\}$, let $S_N = \sum_{n=1}^{N} x_n$ be the sum of the first $N$ terms of the succession. The minimal condition for the mean square of $S_N$ to grow linearly with $N$, i.e. $\langle (S_N)^2 \rangle \sim N$, is that the power spectrum of the succession $\{x_n\}$ be finite at the origin. The power spectrum corresponding to the binary correlator (47) is

$$W(ka) = \begin{cases} \frac{1}{c_2 - c_1} & \text{if } ka \in \left[ c_1 \frac{\pi}{2}, c_2 \frac{\pi}{2} \right] \cup \left[ \pi - c_2 \frac{\pi}{2}, \pi - c_1 \frac{\pi}{2} \right] \\ 0 & \text{otherwise} \end{cases}$$

and vanishes at the origin; one can therefore conclude from equation (46) that

$$\log |\psi_n| \sim \sqrt{n}$$

even if the disorder is correlated.

The numerical data confirm the conclusion that the Kronig–Penney model (1) has an anomalously localized state for $q = q_c$ when the compositional disorder takes the special form (42). This can be seen from figures 5 and 6, which show how the Lyapunov exponent vanishes when the Bloch wavevector takes the critical value $k_c$.

The difference between figures 5 and 6 lies in the fact that in the case corresponding to figure 5, the structural disorder is not self-correlated, while the data represented in figure 6 were obtained for structural disorder with long-range self-correlations of the form (47) with $c_1 = 3/10$ and $c_2 = 1$. 

15
Figure 6. Inverse localization length $\lambda$ versus $ka$ for the case of self-correlated structural disorder. The dashed line (1) represents numerical data, while the solid line (2) corresponds to equation (54). The inset shows the anomaly in greater detail.

All the numerical data presented in this section were obtained for a Kronig–Penney model with mean field $U = 4$ and structural disorder characterized by a uniform distribution

$$p(\Delta_n) = \begin{cases} \frac{1}{W} & \text{if } \Delta_n \in [-W/2, W/2] \\ 0 & \text{otherwise} \end{cases}$$

(49)

with width $W = 0.1732 \ldots$, corresponding to a disorder strength $\sqrt{\langle \Delta_n^2 \rangle} = 0.05$. For the above-specified values of the mean field and of the disorder strength, formula (44) gives a value of the critical Bloch wavevector approximately equal to $k_c \simeq 0.4952\pi/a$, in relatively good agreement with the numerically obtained value $k_c \simeq 0.4996\pi/a$.

When the box distribution (49) is chosen for the displacements $\Delta_n$, the function (39) becomes

$$E(q) = \frac{UW + 2 \log \sin[q(a+W/2)]}{\log \frac{\sin[q(a)-\sin(qW/2)]}{\sin[q(a)-\sin(qW/2)]}},$$

(50)

while the random variables (40) and (41) take the forms

$$\gamma_n(q) = \frac{1}{\sin[q(a+\Delta_n)]} \log \frac{\sin[q(a+W/2)]}{\sin[q(a)-\sin(qW/2)]}$$

(51)

and

$$\varepsilon_n(q) = \frac{1}{\log \frac{\sin[q(a+W/2)]}{\sin[q(a)-\sin(qW/2)]}} \left[ 2 \log \frac{\sin[q(a+W/2)]}{\sin[q(a-W/2)]} - qW \cot[q(a+\Delta_n)] - qW \cot[q(a+\Delta_{n-1})] - Wu_n \right].$$

(52)

We would like to stress that, when performing numerical calculations, one should work with the exact form (50) of $E(q)$ and the exact expressions (51) and (52) of the coefficients $\gamma_n$ and $\varepsilon_n$, even if the disorder is weak. Obviously, the explicit expressions of these magnitudes...
depend on the distribution chosen for the variables $\Delta_n$ and must be modified if the box distribution is replaced with another one. Whatever distribution is adopted, however, it is important that the corresponding exact expressions of $E(q)$, $\gamma_n$ and $\varepsilon_n$ be used. Using second-order approximations for $E(q)$ and the random coefficients (41) works relatively well for most values of the energy, but fails at the critical point, because the neglected higher-order corrections produce non-zero diagonal terms in equation (38) which, in spite of being very small, prevent the electronic state from being anomalously localized and the Lyapunov exponent from vanishing completely.

As a side remark, we would like to add that conditions (42) and (43) ensure the existence of an anomalously localized state also for disorder of arbitrary strength, provided that equation (43) has a solution $q_c$ inside the allowed energy bands. This is confirmed by numerical calculations (which, incidentally, also show that when disorder is not weak the critical value of the energy need not be close to the band centre). To determine the conditions which guarantee the existence of such a critical value of the energy in the general case is not an easy task, however; for the sake of simplicity, we therefore restrict our attention to the case of weak disorder, for which the band structure is approximately given by equations (8) and (43) does have a solution.

In both figures 5 and 6 the numerical data are compared with the theoretical predictions derived from the general result (17). Because in this formula terms of order higher than the second are neglected, in its evaluation we replaced the exact expression (42) of the compositional disorder with its second-order approximation

$$u_n \simeq \frac{q^2}{\sin^2(q_n a)} \left( \Delta_n + \Delta_{n-1} \right).$$

In passing, we observe that equation (53) implies that the cross-correlator $\chi_3(l)$ is not an even function of its argument. When the compositional disorder has the form (53), the inverse localization length (17) becomes

$$\lambda = \frac{1}{8a} \left[ \frac{k^2}{\sin^2(k a)} \right] \left[ 1 - 2\frac{\sin(q a)}{\sin^2(q a)} \cos(k a) \right] \frac{W_2(k a)}{W_2}. \quad (54)$$

The power spectrum $W_2(k a)$ in the previous formula reduces to $W_2(k a) = 1$ in the case of uncorrelated structural disorder, while it is of the form (48) in the case of self-correlated structural disorder represented in figure 6. Note that in the latter case the long-range correlations (47) create two mobility edges at $k a = 3\pi/10$ and $k a = 7\pi/10$. As can be seen from figures 5 and 6, the theoretical formula (54) works reasonably well everywhere, except in a small neighbourhood of the critical value $k_c$. This failure must be ascribed to the fact that equation (54), as with its parent expression (17), is not valid for Bloch wavevectors lying close to the rational value $\pi/2a$, where resonance effects play a non-negligible role.

As a final comment on the specific features of the localization length in the special case in which the compositional disorder is related to the structural one by equation (42), we observe that equation (54) predicts the existence of a delocalized state for the Bloch wavevector $k^*$ identified by the condition

$$2\frac{q_{c}^2 \sin(q(k^*) a)}{q(k^*) \sin^2(q, a)} \cos(k^* a) = U.$$  

This is confirmed by the numerical computations, as can be seen from both figures 5 and 6.

To conclude this section, we would like to add some numerical evidence of the stretched-exponential behaviour of the tails of the anomalously localized state. We have numerically solved equations (38) and (45) as initial-value problems; this corresponds to constructing
the electronic states with the transfer-matrix technique. In the anomalous case one expects \( \log |\psi_n| \) to behave as the position of a random walker, i.e. as a random variable with constant zero average and a second moment linearly increasing with \( n \). For \( k \neq k_c \), on the other hand, the solution of equation (38) should behave as \( |\psi_n| \sim \exp(\alpha n) \), leading to an increase with \( n^l \) of the \( l \)th moment of \( \log |\psi_n| \). This is confirmed by the numerical data for the first two moments of the variable \( \log |\psi_n| \), represented in figures 7 and 8. In both figures 7 and 8 we considered the behaviour of \( \log |\psi_n| \) as a function of \( n \) for two Bloch wavevectors, i.e. the critical vector \( k = k_c \simeq 0.4996\pi \) and the vector \( k = 0.5036\pi \), which is close to the
critical value but not identical to it. The moments of \(|\psi_n|\) were computed with an average over 1000 disorder realizations. For the sake of simplicity, we considered compositional disorder without self-correlations. Both the first and the second moment of \(|\psi_n|\) behave as expected, corroborating the conclusion that the tails of the electronic state at the critical point are described by equation (36).

5. Conclusions

In this work we analyse the anomalous behaviour of specific electronic states in the Kronig–Penney model with weak compositional and structural disorder. In every case we discuss the analogies with the corresponding phenomena in the Anderson model.

We first show that the localization length deviates from the prediction of the standard formula (17) when the Bloch vector assumes the value \(k = \pi/2a\), which corresponds to an energy close to the band centre. This discrepancy is due to the same resonance effect which occurs in the standard Anderson model at the band centre; in both models this effect produces a modulation of period \(\pi/2\) of the invariant distribution of the angle variable of the associated Hamiltonian map. This modulation leads to the band-centre anomaly of the localization length.

We also make use of the Hamiltonian map approach to analyse the electronic states at the band edge. We find again that the same anomalous behaviour originally found in the Anderson model is present in the Kronig–Penney model. In both systems the most relevant feature at the band edge is the anomalous scaling of the localization length with the disorder strength.

We finally use the correspondence between the Kronig–Penney model and the Anderson model with diagonal and off-diagonal disorder to conclude that in the former system specific cross-correlations between the two kinds of disorder generate an electronic state, close to the band centre, whose tails decay as stretched exponentials. This state is the analogue of the anomalously localized state which occurs at the band-centre in the Anderson model with purely off-diagonal disorder.

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