PROJECTION METHOD WITH INERTIAL STEP FOR NONLINEAR EQUATIONS: APPLICATION TO SIGNAL RECOVERY

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In this paper, using the concept of inertial extrapolation, we introduce a globally convergent inertial extrapolation method for solving nonlinear equations with convex constraints for which the underlying mapping is monotone and Lipschitz continuous. The method can be viewed as a combination of the efficient three-term derivative-free method of Gao and He [Calcolo. 55(4), 1-17, 2018] with the inertial extrapolation step. Moreover, the algorithm is designed such that at every iteration, the method is free from derivative evaluations. Under standard assumptions, we establish the global convergence results for the proposed method. Numerical implementations illustrate the performance and advantage of this new method. Moreover, we also extend this method to solve the LASSO problems to decode a sparse signal in compressive sensing. Performance comparisons illustrate the effectiveness and competitiveness of our algorithm.

1. Introduction. Let $\Omega$ be a nonempty, closed and convex subset of the $n$-dimensional real space $\mathbb{R}^n$ equipped with the scalar product $\langle \cdot, \cdot \rangle$ and Euclidean norm $\| \cdot \|$. A mapping $T: \mathbb{R}^n \to \mathbb{R}^n$ is said to be a monotone if
\[
\langle T(w) - T(z), w - z \rangle \geq 0, \quad \forall w, z \in \mathbb{R}^n.
\]
The concern of this article is on iterative method for solving nonlinear equations with convex constraints. Recall that, this problem is stated as:
\[
\text{find } w \in \Omega \text{ such that } T(w) = 0. \tag{1}
\]
The problem (1) is a general problem that unifies many well-known models in applied sciences. Over the years, the problem has received a lot of attention by many authors in both theory and algorithms, because it arises in various applications, for instance, the power flow equations [16], the economic equilibrium problems [30], the chemical equilibrium systems [51] and many others.

Several methods have been proposed for solving the problem (1), for instance, Newton method, Quasi-Newton method, Gauss-Newton method, Levenberg-Marquardt method and a series of their variants; see [27, 28, 29, 55, 61]. These methods are attractive because of their rapid convergence from a sufficiently good initial guess. However, an associated drawback with these methods is the computation of the Jacobian matrix or its approximation at each iteration, and as a consequence, they are inefficient at solving large-scale nonlinear systems of equations.

Considering the simplicity and low storage requirement of the conjugate gradient method [20, 21], several researchers combined the projection technique of Solodov and Svaiter [56] with the conjugate gradient methods to solve large-scale nonlinear equations, see [38, 13, 42, 40, 43, 1, 7, 45, 41, 12, 47, 10, 9, 46, 39, 52, 2, 3, 8, 53, 11, 37, 44, 4, 6, 36] and references therein. Based on the projection method, Gao and He [35] introduced an efficient three-term derivative-free method for solving nonlinear monotone equations with convex constraints (1) by choosing a part of the Liu-Storey (LS) conjugate parameter as a new conjugate parameter. The proposed method can be used to solve large-scale nonlinear monotone equations since it is derivative-free and requires low storage. Moreover, using some mild assumptions, the global convergence is proved when the line search fulfils the backtracking line search condition.

Let us now discuss an inertial-type algorithm. Polyak [54] introduced an inertial extrapolation as an acceleration mechanism to solve the smooth convex minimization problem based on the heavy ball methods of the two-order time dynamical
system. The inertial algorithm is a two-step iterative method that uses the previous two iterates to compute the next iterate, and it can be thought of as a procedure of speeding up the convergence properties, see [54, 17, 14, 5, 15]. An abundant literature has also been devoted on this subject with a lot of fast iterative algorithms constructed using the inertial extrapolation, for instance, inertial forward-backward splitting methods [18, 50], inertial Mann method [58], inertial Douglas-Rachford splitting method [24], inertial ADMM [25], inertial subgradient extragradient method [59], inertial forward-backward-forward method [22], inertial proximal-extragradient method [23], and inertial contraction method [32]. It is known that the inertial-type methods require a fewer number of iterations than the corresponding non-inertial version [26].

Motivated by the presented inertial-type methods for solving optimization problems, in this paper, we introduce an iterative algorithm for solving the convex constrained nonlinear equation (1) which is based on the inertial extrapolation method. The proposed method incorporates the inertial extrapolation step with the efficient three-term conjugate gradient method [35]. The main advantage of incorporating the inertial extrapolation step with the efficient three-term method [35], is to speed up the rate of convergence of the three-term method, which has been confirmed by several numerical results. The proposed method also uses a backtracking line search and projection technique. Under some moderate assumptions, we prove the convergence property of the proposed inertial projection method. Moreover, as the LASSO problem has been reformulated into a nonsmooth monotone equation, we apply the proposed method to solve the LASSO problem. We conduct numerical experiments to recover a sparse signal arising in compressive sensing, demonstrating that the proposed method is competitive and efficient.

The structure of the paper is as follows. In Section 2, we present the inertial three-term derivative-free projection algorithm for solving (1) and some technical tools needed in the sequel. In Section 3, the convergence analysis of the method is established. In the final section, Section 4, numerical examples are provided to test the performance of the proposed method.

2. Preliminaries and algorithm. This section collects some valuable definitions and Lemmas that will be needed in establishing the convergence of our proposed method. Moreover, the inertial-type algorithm is presented in this section.

Let \( \mathbb{R}^n \) be an Euclidean space and \( \Omega \) be a nonempty closed convex subset of \( \mathbb{R}^n \). For each \( w, z \in \mathbb{R}^n \), we have the following simple relation:

\[
\|w + z\|^2 \leq \|w\|^2 + 2(z, w + z).
\]  

(2)

For every element \( w \in \mathbb{R}^n \), there exists a unique nearest point in \( \Omega \), denoted by \( P_{\Omega}(w) \), such that

\[
P_{\Omega}(w) = \arg \min_{z \in \Omega} \|w - z\|.
\]

\( P_{\Omega} \) is called the orthogonal projection of \( \mathbb{R}^n \) onto \( \Omega \). The orthogonal projection \( P_{\Omega} \) has the following basic property:

\[
\|P_{\Omega}(w) - z\|^2 \leq \|w - z\|^2 - \|w - P_{\Omega}(z)\|^2, \ \forall w \in \mathbb{R}^n, \forall z \in \Omega.
\]

(3)

**Definition 2.1.** Let \( T : \mathbb{R}^n \to \mathbb{R}^n \) be a mapping. Then, \( T \) is said to be \( L \)-Lipschitz continuous, if there exists \( L > 0 \) such that

\[
\|Tw - Tz\| \leq L\|w - z\|, \ \forall w, z \in \mathbb{R}^n.
\]
Lemma 2.2. [19] Let \( \{w_k\} \) and \( \{z_k\} \) be sequences of nonnegative real numbers satisfying \( \sum_{k=1}^{\infty} z_k < \infty \) and \( w_{k+1} \leq w_k + z_k, \ k = 1, 2, \ldots \). Then, \( \{w_k\} \) is a convergent sequence.

We now introduce the following inertial extrapolation algorithm based on the efficient three-term derivative-free projection method for solving nonlinear monotone equations with convex constraints by Gao and He [35].

Algorithm 2.3. (Three-Term Derivative-free Projection Algorithm)

(Step 0) Choose an arbitrary initial point \( w_0, w_1 \in \Omega \), constants: \( \zeta \in (0, 1), \sigma > 0, Tol \in (0, 1), \gamma \in (0, 2), \alpha \in (0, 1) \). Set \( k := 0 \).

(Step 1) Choose \( \alpha_k \) such that \( 0 \leq \alpha_k \leq \bar{\alpha}_k \), where
\[
\bar{\alpha}_k := \begin{cases} 
\min \left\{ \alpha, \frac{1}{k^2 \|w_k - w_{k-1}\|^2} \right\} & \text{if } w_k \neq w_{k-1}, \\
\alpha & \text{otherwise}.
\end{cases}
\] (4)

(Step 2) Compute
\[ v_k = w_k + \alpha_k (w_k - w_{k-1}). \]

(Step 3) If \( \|T(v_k)\| < Tol \), stop. Otherwise compute the direction \( p_k \) as follows
\[ p_0 = -T(v_0), \ p_k = -T(v_k) + \beta_k p_{k-1} + \lambda_k T(v_{k-1}), \ k \geq 1 \] (5)
where
\[ \beta_k := -\frac{T(v_k), T(v_{k-1})}{T(v_{k-1}), p_{k-1}}, \ \lambda_k := \frac{T(v_k), p_{k-1}}{T(v_{k-1}), p_{k-1}}. \] (6)

(Step 4) Find \( u_k = v_k + \theta_k p_k \), where \( \theta_k = \zeta^i \) with \( i \) being the smallest nonnegative integer such that
\[ -(T(v_k + \theta_k p_k), p_k) \geq \sigma \theta_k \|T(v_k + \theta_k p_k)\| \|p_k\|^2. \] (7)

(Step 5) If \( u_k \in \Omega \) and \( \|T(u_k)\| \leq Tol \), stop. Otherwise, we determine the next iterate by
\[ w_{k+1} = P_{\Omega} [v_k - \gamma \eta_k T(u_k)], \] (8)
where
\[ \eta_k := \frac{T(u_k), v_k - u_k}{\|T(u_k)\|^2}. \] (9)

Let \( k := k + 1 \). Go to Step 1.

Remark 1. Observe from step 1 of Algorithm 2.3 that, for all \( k, \alpha_k \|w_k - w_{k-1}\|^2 \leq \frac{1}{k^2} \). This implies that
\[ \sum_{k=0}^{\infty} \alpha_k \|w_k - w_{k-1}\|^2 < \infty. \] (10)

Lemma 2.4. Let \( p_k \) be generated by Algorithm 2.3. Then, \( p_k \) always satisfies the sufficient descent condition, that is,
\[ (T(v_k), p_k) = -\|T(v_k)\|^2. \] (11)
Proof. The proof is in two parts. For \( k = 0 \), it clearly holds that
\[
\langle T(v_0), p_0 \rangle = -\|T(v_0)\|^2.
\]
For \( k \geq 1 \), pre-multiplying \( p_k \) in (5) by \( T(v_k)^T \), we have
\[
\langle T(v_k), p_k \rangle = -\|T(v_k)\|^2 - \frac{\langle T(v_k), T(v_k-1) \rangle}{\langle T(v_k-1), p_{k-1} \rangle} \langle T(v_k), p_{k-1} \rangle + \frac{\langle T(v_k), p_{k-1} \rangle}{\langle T(v_k-1), p_{k-1} \rangle} \langle T(v_k), T(v_k-1) \rangle
\]
\[
= -\|T(v_k)\|^2.
\]

\[\square\]

Remark 2. By (11) and the Cauchy-Schwarz inequality, we have
\[
\|T(v_k)\| \leq \|p_k\|. \quad (12)
\]

3. Convergence analysis. Throughout this paper, for solving the convex constrained nonlinear equation (1), we assume that the mapping \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \) satisfies the following conditions:
(A1) The solution set \( S \) of (1) is nonempty.
(A2) \( T \) is monotone and Lipschitz continuous.

Lemma 3.1. Let Assumption (A1)-(A2) hold and the sequence \( \{w_k\} \) and \( \{u_k\} \) be generated by Algorithm 2.3. If \( w^* \in S \), it holds that
\[
\|w_{k+1} - w^*\|^2 \leq \|v_k - w^*\|^2 - \gamma(2 - \gamma)\sigma^2\|v_k - u_k\|^4. \quad (13)
\]
Moreover,
\[
\sum_{k=0}^{\infty} \|v_k - u_k\|^4 < \infty. \quad (14)
\]

Proof. Let \( w^* \in S \), by (3) and (8), it follows that
\[
\|w_{k+1} - w^*\|^2 = \|P_T(v_k - \gamma\eta_k T(u_k)) - w^*\|^2
\]
\[
\leq \|v_k - \gamma\eta_k T(u_k) - w^*\|^2
\]
\[
= \|v_k - w^*\|^2 - 2\gamma\eta_k \langle v_k - w^*, T(u_k) \rangle + \gamma^2\eta_k^2 \|T(u_k)\|^2. \quad (15)
\]
Indeed, by the monotonicity of the mapping \( T \) and \( w^* \in S \) we get
\[
\langle v_k - w^*, T(u_k) \rangle = \langle v_k - u_k, T(u_k) \rangle + \langle u_k - w^*, T(u_k) \rangle
\]
\[
\geq \langle v_k - u_k, T(u_k) \rangle + \langle u_k - w^*, T(w^*) \rangle
\]
\[
= \langle v_k - u_k, T(u_k) \rangle \quad (16)
\]
\[
\geq \sigma \|T(u_k)\| \|v_k - u_k\|^2. \quad (17)
\]
Combining (9), (16) and (17) with (15), we have
\[
\|w_{k+1} - w^*\|^2 \leq \|v_k - w^*\|^2 - 2\gamma \frac{\langle v_k - u_k, T(u_k) \rangle^2}{\|T(u_k)\|^2} + \gamma^2 \frac{\langle v_k - u_k, T(u_k) \rangle^2}{\|T(u_k)\|^2}
\]
\[
\leq \|v_k - w^*\|^2 - \gamma(2 - \gamma) \frac{\langle v_k - u_k, T(u_k) \rangle^2}{\|T(u_k)\|^2}
\]
\[
\leq \|v_k - w^*\|^2 - \gamma(2 - \gamma)\sigma^2 \|v_k - u_k\|^4. \quad (18)
\]
From (18), we can deduce that
\[
\|w_{k+1} - w^*\| \leq \|v_k - w^*\|
\]
\[
= \|w_k + \alpha_k(w_k - w_{k-1}) - w^*\|
\]
\[
\leq \|w_k - w^*\| + \alpha_k\|w_k - w_{k-1}\|. \quad (19)
\]
By Lemma 2.2, we know that the sequence \(\{\|w_k - w^*\|\}\) is a convergent sequence. Therefore, we can deduce that \(\{\|w_k - w^*\|\}\) is bounded. That is, there exist a positive number \(M_0\) such that for all \(k\), \(\|w_k - w^*\| \leq M_0\) and \(\|w_k - w_{k-1}\| \leq 2M_0\).

From the definition of \(v_k\), (2) and the above bounds, we have
\[
\|v_k - w^*\|^2 = \|w_k + \alpha_k(w_k - w_{k-1}) - w^*\|^2
\]
\[
\leq \|w_k - w^*\|^2 + 2\alpha_k(w_k - w_{k-1})(w_k - w^* + \alpha_k(w_k - w_{k-1}))
\]
\[
\leq \|w_k - w^*\|^2 + 2\alpha_k\|w_k - w_{k-1}\|\|w_k - w^*\| + \alpha_k\|w_k - w_{k-1}\|
\]
\[
\leq \|w_k - w^*\|^2 + 2M_0\alpha_k\|w_k - w_{k-1}\| + 4M_0\alpha_k\|w_k - w_{k-1}\|
\]
\[
= \|w_k - w^*\|^2 + 6M_0\alpha_k\|w_k - w_{k-1}\|. \quad (20)
\]
Combining (18) with (20), we have
\[
\|w_{k+1} - w^*\|^2 \leq \|w_k - w^*\|^2 + 6M_0\alpha_k\|w_k - w_{k-1}\| - \gamma(2 - \gamma)\sigma^2\|v_k - u_k\|^4.
\]
Thus, we have
\[
\gamma(2 - \gamma)\sigma^2\|v_k - u_k\|^4 \leq \|w_k - w^*\|^2 + 6M_0\alpha_k\|w_k - w_{k-1}\| - \|w_{k+1} - w^*\|^2.
\]
By adding the above equations for \(k = 0, 1, 2, \cdots\), we obtain
\[
\sigma^2\gamma(2 - \gamma)\sum_{k=0}^{\infty}\|v_k - u_k\|^4 \leq \sum_{k=0}^{\infty} \left(\|w_k - w^*\|^2 + 6M_0\alpha_k\|w_k - w_{k-1}\| - \|w_{k+1} - w^*\|^2\right).
\]
Since the sequence \(\{\|w_{k+1} - w^*\|\}\) is convergent, \(\sum_{k=0}^{\infty} \left(\|w_k - w^*\|^2 - \|w_{k+1} - w^*\|^2\right)\) is finite. Moreover, since \(\sum_{k=0}^{\infty} \alpha_k\|w_k - w_{k-1}\| < \infty\), it follows that
\[
\sigma^2\gamma(2 - \gamma)\sum_{k=0}^{\infty}\|v_k - u_k\|^4 \leq \sum_{k=0}^{\infty} \left(\|w_k - w^*\|^2 - \|w_{k+1} - w^*\|^2 + 6M_0\alpha_k\|w_k - w_{k-1}\|\right)
\]
\[
< \infty,
\]
which means that
\[
\lim_{k \to \infty} \|v_k - u_k\| = 0. \quad (22)
\]
\[\square\]

Remark 3. By the definition of \(\{u_k\}\) and (22), we have
\[
\lim_{k \to \infty} \theta_k\|p_k\| = 0. \quad (23)
\]
The following Lemma shows that line search step is well-defined.

Lemma 3.2. Suppose that Assumption (A1) and (A2) hold. Let \(\{v_k\}\) and \(\{u_k\}\) be the sequence generated by Algorithm 2.3. Then, we have for all \(k\)
\[
\theta_k \geq \theta_* > 0,
\]
where \(\theta_* = \frac{\langle\|T(v_k)\|^2}{(L + \sigma\|T(v_k + \zeta \theta_k)\|\|p_k\|^2)}\).
Proof. If the algorithm terminates at some iteration $k$ then $\|T(v_k)\| = 0$, so $v_k$ is a solution. Assume that $T(v_k) \neq 0$ for all $k$, and consequently $p_k \neq 0$ from Lemma 2.4. In the following, we show that the line search procedure (7) always terminates in a finite number of steps. If $\theta_k \neq \zeta$, then $\zeta^{-1}\theta_k$ does not satisfy (7). So we can obtain

$$-\langle T(v_k + \zeta^{-1}\theta_k p_k), p_k \rangle < \sigma \zeta^{-1}\theta_k \|T(v_k + \zeta^{-1}\theta_k p_k)\| \|p_k\|^2.$$  \hfill (25)

Combining (A2) with (25), we have

$$\|T(v_k)\|^2 = -\langle T(v_k), p_k \rangle$$

$$(= \langle T(v_k + \zeta^{-1}\theta_k p_k), p_k \rangle - \langle T(v_k), p_k \rangle) - \langle T(v_k + \zeta^{-1}\theta_k p_k), p_k \rangle$$

$$\leq L\zeta^{-1}\theta_k \|p_k\|^2 + \sigma \zeta^{-1}\theta_k \|T(v_k + \zeta^{-1}\theta_k)\| \|p_k\|^2,$$

which means that

$$\theta_k \geq \frac{\zeta \|T(v_k)\|^2}{(L + \sigma \|T(v_k + \zeta^{-1}\theta_k)\|) \|p_k\|^2} > 0.$$ \hfill (26)

The proof is completed. \hfill \Box

**Lemma 3.3.** Let $\epsilon > 0$ be a constant such that for all $k \geq 0$, $\|T(v_k)\| \geq \epsilon$. Then, the search direction $p_k$ generated by Algorithm 2.3 is bounded.

**Proof.** Recall that, $\|w_k - w^*\| \leq M_0$ and $\|w_k - w_{k-1}\| \leq 2M_0$. By (A4), we can deduce that

$$\|T(v_k)\| = \|T(v_k) - T(w^*)\| \leq L\|v_k - w^*\| \leq 3LM_0 = M_T.$$ \hfill (27)

Since $T$ is monotone, by the Cauchy-Schwarz inequality and (7), we have

$$\|T(u_k)\| \|v_k - u_k\| \geq \langle T(u_k), v_k - u_k \rangle \geq \sigma \theta_k^2 \|T(u_k)\| \|p_k\|^2 = \sigma \|T(u_k)\| \|v_k - u_k\|^2.$$  \hfill (28)

Thus, there exists $a^* > 0$ such that

$$\|v_k - u_k\| \leq \frac{1}{\sigma} = a^*, \forall k.$$ \hfill (29)

On the other hand, from the construction of $p_k$, using (11), (24), (27) and (28) we can get

$$\|p_k\| \leq \|T(v_k)\| + |\beta_k|\|p_{k-1}\| + |\theta_k|\|T(v_{k-1})\|$$

$$= \|T(v_k)\| + \left| \frac{T(v_k), T(v_{k-1})}{T(v_{k-1}, p_{k-1})} \right| \|p_{k-1}\| + \left| \frac{T(v_k), p_{k-1}}{T(v_{k-1}, p_{k-1})} \right| \|T(v_{k-1})\|$$

$$\leq \left( 1 + 2 \frac{\|p_{k-1}\|}{\|T(v_{k-1})\|} \right) \|T(v_k)\|$$

$$\leq \left( 1 + 2 \frac{a^*}{\theta_k \epsilon} \right) M_T \triangleq b_*.$$ \hfill \Box

**Theorem 3.4.** Suppose that assumption (A1) and (A2) hold. Let $\{w_k\}$ be the sequences generated by Algorithm 2.3. Then we have

$$\liminf_{k \to \infty} \|T(w_k)\| = 0.$$ \hfill (29)

Furthermore, $\{w_k\}$ converges to a solution of (1).
Proof. We first prove that
\[
\liminf_{k \to \infty} \| T(v_k) \| = 0. \tag{30}
\]
Suppose (30) is not true, then there exist a constant \( \epsilon > 0 \) such that for all \( k \geq 0 \),
\[
\| T(v_k) \| \geq \epsilon. \tag{31}
\]
Combining (12) with (31), we have
\[
\| p_k \| \geq \| T(v_k) \| \geq \epsilon, \quad \forall k \geq 0.
\]
This shows that
\[
\lim_{k \to \infty} \theta_k = 0, \tag{32}
\]
which contradicts the conclusion of Lemma 3.2. Thus, (30) holds. Now, since we know that
\[
\| w_k - v_k \| = \| w_k - (w_k + \alpha_k(w_k - w_{k-1})) \| = \alpha_k \| w_k - w_{k-1} \| \to 0, \tag{33}
\]
by the continuity of \( T \), we have that
\[
\liminf_{k \to \infty} \| T(w_k) \| = 0. \tag{34}
\]
From the continuity of \( T \), the boundedness of \( \{w_k\} \) and (34), it implies that the sequence \( \{w_k\} \), generated by Algorithm 2.3, has an accumulation point \( w^* \) such that \( T(w^*) = 0 \). On the other hand, the sequence \( \{w_k - w^*\} \) is convergent by Lemma 2.2, which means that the whole sequence \( \{x_k\} \) globally converges to the solution \( w^* \) of the system (1).

4. Numerical results. In this section, the efficiency of the proposed method (Algorithm 2.3 (Iner.Algo)) is compared with the method proposed by Gao and He [35] (Algo) on some benchmark test problems. Recall that Algorithm 2.3 is an inertial version of the method in [35]. Number of iterations (NI), number of function evaluations (NF) and CPU time (CPU) are the metrics considered for the comparison and the method with least value of the metric is considered the most efficient.

During the experiments, the following were considered:

- **Dimensions:** 1000, 5000, 10000, 50000, 100000.
- **Parameters:** For Algorithm 2.3, we select \( \alpha = 0.8, \zeta = 0.7, \sigma = 0.0001 \). As for the compared method, all parameters are selected as in [35].
- **Terminating criterion:** When \( \| h(v_k) \| \leq 10^{-6} \).
- **Implementation software:** All methods are coded in MATLAB R2019ba and run on a PC with an intel COREi7 processor, 8GB of RAM and CPU 2.30GHz.

Below are the test problems used for the experiments with \( T = (t_1, t_2, ..., t_n)^T \).

**Problem 1:** Modified exponential function \([48]\)
\[
t_1(w) = e^{w_1} - 1
\]
\[
t_i(w) = e^{w_i} + w_i - 1, \quad i = 1, 2, \ldots, n - 1,
\]
\[
\Omega = \mathbb{R}^n_+.
\]

**Problem 2:** Logarithmic function \([48]\)
\[
t_i(w) = \log(w_i + 1) - \frac{w_i}{n}, \quad i = 1, 2, \ldots, n,
\]
\[
\Omega = \mathbb{R}^n_+.
\]
Problem 3: Nonsmooth function [49]
\[ t_i(w) = 2w_i - \sin(|w_i|), \text{ for } i = 1, 2, \ldots, n, \]
\[ \Omega = \left\{ w \in \mathbb{R}_+^n : w \geq 0, \sum_{i=1}^n w_i \leq n \right\}. \]

Problem 4: Strictly convex function I [48]
\[ t_i(x) = e^{w_i} - 1, \; i = 1, 2, \ldots, n, \]
\[ \Omega = \mathbb{R}_+^n. \]

Problem 5: Strictly convex function II [48]
\[ t_i(w) = \left( \frac{i}{n} \right) e^{w_i} - 1, \; i = 1, 2, \ldots, n, \]
\[ \Omega = \mathbb{R}_+^n. \]

Problem 6: Tridiagonal exponential function [48]
\[ t_1(w) = w_1 - e^{\cos(h(w_1 + w_2))}, \]
\[ t_i(w) = w_i - e^{\cos(h(w_{i-1} + w_i + w_{i+1}))}, \; i = 2, \ldots, n - 1, \]
\[ t_n(w) = w_n - e^{\cos(h(w_{n-1} + w_n))}, \]
\[ h = \frac{1}{n+1} \text{ and } \Omega = \mathbb{R}_+^n. \]

Problem 7: Nonsmooth function II [62]
\[ t_i(w) = w_i - \sin(|w_i - 1|), \text{ for } i = 1, 2, \ldots, n, \]
\[ \Omega = \left\{ w \in \mathbb{R}_+^n : w \geq -1, \sum_{i=1}^n w_i \leq n \right\}. \]

Problem 8: Penalty function I [42]
\[ \xi_i = \sum_{i=1}^n w_i^2, \; c = 10^{-5}, \]
\[ t_i(w) = 2c(w_i - 1) + 4(\xi_i - 0.25)w_i, \; i = 1, 2, \ldots, n, \]
\[ \Omega = \mathbb{R}_+^n. \]

Below is a list of the seven starting points used in the implementation of the Iner. Algo.

Case I: \( w_0 = (0.2, 0.2, \ldots, 0.2)^T \) and \( w_1 = (0.1, 0.1, \ldots, 0.1)^T \)

Case II: \( w_0 = (0.1, 0.1, \ldots, 0.1)^T \) and \( w_1 = (0.2, 0.2, \ldots, 0.2)^T \)

Case III: \( w_0 = (0.1, 0.1, \ldots, 0.1)^T \) and \( w_1 = (0.5, 0.5, \ldots, 0.5)^T \)

Case IV: \( w_0 = (0.1, 0.1, \ldots, 0.1)^T \) and \( w_1 = (1.2, 1.2, \ldots, 1.2)^T \)
Case V: \( w_0 = (0.1, 0.1, \cdots, 0.1)^T \) and \( w_1 = (1.5, 1.5, \cdots, 1.5)^T \)

Case VI: \( w_0 = (0.1, 0.1, \cdots, 0.1)^T \) and \( w_1 = (2, 2, \cdots, 2)^T \)

Case VII: \( w_0 = (0.1, 0.1, \cdots, 0.1)^T \) and \( w_1 = \text{rand}(n, 1) \)

The results of the experiments can be found in Table 2-10. We can deduce from the tables that Algorithm 2.3 has the least NI, NF and CPU in most of the problems and this is likely in connection with the inertial effect it possess. In order to depict what is contained in the tables of comparison, the Dolan and Moré performance profiles of [31] defined as:

\[
\rho(\tau) := \frac{1}{|T_P|} \left| \left\{ t_p \in T_P : \log_2 \left( \frac{t_{p,q}}{\min\{t_{p,q} : q \in Q\}} \right) \leq \tau \right\} \right|,
\]

is employed and the following figures are obtained. \( T_P \) is the test set, \(|T_P|\) is the number of problems in the test set \( T_P \), \( Q \) is the set of optimization solvers, and \( t_{p,q} \) is the NI (NF or CPU) for \( t_p \in T_P \) and \( q \in Q \).

From Figure 1 and 2, Algorithm 2.3 (Iner. Algo) has the least NI and NF in over 95% of the problems, respectively. From Figure 3 in terms of CPU, Algorithm 2.3 (Iner. Algo) has the least CPU in 60% of the problems. This can be seen on the \( y-axis \) of the plots. As a conclusion, it can be said that the purpose of introducing the inertial effect was achieved as the convergence of Algorithm 2.3 is faster in most of the problems.

4.1. Application to signal recovery. In this subsection, we shall apply Algorithm 1 to a famous machine learning problem: the LASSO problem, i.e.,

\[
\min_{x \in \mathbb{R}^n} \frac{1}{2} \| Ax - b \|_2^2 + \mu \| x \|_1,
\]

(35)
in which \( x \in \mathbb{R}^n \) is a sparse signal to be recovered, \( A \in \mathbb{R}^{m \times n} (m \ll n) \) is a linear operator, \( b \in \mathbb{R}^m \) is an observed signal, \( \|x\|_1 = \sum_{i=1}^{n} |x_i| \) is the \( \ell_1 \)-norm of \( x \), and \( \mu > 0 \) is a parameter. As stated by Figueiredo et al. [34], problem (35) can be reformulated to a smooth optimization by setting \( x = u - v \), in which \( u_i = (x_i)_+, v_i = (-x_i)_+ \).
for all \( i = 1, 2, \ldots, n \) with \((\cdot)_+ = \max\{0, \cdot\}\). Then, problem (35) can be rewritten as
\[
\min_{u,v} \frac{1}{2} \|b - A(u - v)\|_2^2 + \mu e_n^\top u + \mu e_n^\top v,
\]
subject to \( u \geq 0, v \geq 0 \).

whose compact form is
\[
\min_{u,v} \frac{1}{2} z^\top Hz + c^\top z,
\]
subject to \( z \geq 0 \),
in which
\[
z = \begin{bmatrix} u \\ v \end{bmatrix}, \quad y = A^\top b, c = \mu e_{2n} + \begin{bmatrix} -y \\ y \end{bmatrix}, \quad \text{and } H = \begin{bmatrix} A^\top A & -A^\top A \\ -A^\top A & A^\top A \end{bmatrix}.
\]

Later Xiao et al. [60] further transformed the above quadratic programming into a constrained nonlinear equation
\[
F(z) = \min\{z, Hz + c\} = 0, \quad z \geq 0, \tag{36}
\]
which is the mathematical model Algorithm 1 can deal with. Note that compared with the original LASSO model (35), the dimension of unknown vector in (36) has doubled, while the smooth (36) is generally easier to be solved than unsmooth (35).

In the experiment, we are going to reconstruct a length-n sparse signal from a length-m observation, and traditionally \( m \ll n \). That is to say, (36) is an underdetermined equation, and there are often many solutions but we only concern with the sparsest solution, which is just the signal we want to recover. As for the sufficient conditions which can ensure the optimal solution of (35) is just the sparsest solution of \( Ax = b \), the reader can refer to [33].

In the following, we conduct a synthesized medium-scale LASSO experiment and we use the experiment parameters in [57]. Specifically, we set \( n = 2048, m = 512 \), and the original signal contains 64 randomly placed spikes. The \( m \times n \) matrix \( A \) is obtained by first filling it with independent samples of a standard Gaussian distribution and then ortho-normalizing the rows. The observation \( b \) is generated by: \( b = A\tilde{x} + \omega \), where \( \omega \) is the Gaussian noise distributed as \( N(0, \delta^2 I) \) and \( \delta \geq 0 \) is the standard deviation. We set \( \mu = 0.01\|A^\top b\|_\infty \). The following relative error (RelErr) to the original signal \( \tilde{x} \) is used to measure the quality of restoration
\[
\text{RelErr} = \frac{\|\tilde{x} - x^*\|_2}{\|\tilde{x}\|_2},
\]
where \( x^* \) is the final recovered signal. Furthermore, we stop the experiment if \( \|F(z_k)\| \leq \text{tol} \), where \( \text{tol} = 10^{-5} \) is the precision parameter assigned in advance. The parameters used in the two tested algorithms are the same as before.
Numerical results in Table 1 indicate that both algorithms are robust to noise, and they can efficiently deal with the LASSO problem in the noisy environment. Even the noise with high standard deviation, they can recover the true signal with high precision. Furthermore, Iner. Algo performs a little better than Algo because the former need less CPU and number of iterations to get almost the same quality solution. Furthermore, the evolution process of RelErr and recovered signals by Iner. Algo and Algo are plotted in Figures 1-4, from which we can find that Iner. Algo performs always better than Algo. Though the final recovered quality of the two tested algorithms is similar, Iner. Algo reaches the saturation earlier than Algo.

Figure 4. Numerical results of Problem (36) with $\delta = 0$
Figure 5. Numerical results of Problem (36) with $\delta = 10^{-4}$

Figure 6. Numerical results of Problem (36) with $\delta = 10^{-3}$
Conclusion. In this article, an algorithm for finding the solution of the convex constraint nonlinear equation problem is presented. The algorithm combines the efficient three-term conjugate gradient method of Gao and He with the inertial extrapolation step to speed up the convergence of iterative procedures. The convergence of the presented algorithm is obtained under some standard assumptions. Several numerical experiments are performed to support the obtained theoretical results, and these results confirm the effectiveness and fast convergence of the new algorithm over existing method.

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Appendix.
| DIM | INP  | NI | NF | CPU         | NORM     | NI | NF | CPU         | NORM     |
|-----|------|----|----|-------------|----------|----|----|-------------|----------|
| 1000| Case I | 6  | 24 | 0.006537   | 4.32E-08 | 11 | 44 | 0.009092   | 5.72E-07 |
|     | Case II | 7  | 28 | 0.008529   | 1.46E-07 | 11 | 44 | 0.013711   | 9.90E-07 |
|     | Case III | 7  | 28 | 0.006432   | 1.93E-07 | 12 | 48 | 0.013874   | 2.73E-07 |
|     | Case IV  | 8  | 32 | 0.010414   | 1.79E-07 | 13 | 52 | 0.022209   | 3.00E-07 |
|     | Case V   | 6  | 24 | 0.010836   | 7.10E-07 | 11 | 44 | 0.013571   | 6.02E-07 |
|     | Case VI  | 8  | 32 | 0.00719    | 1.96E-07 | 13 | 52 | 0.010269   | 2.99E-07 |
|     | Case VII | 11 | 44 | 0.008426   | 1.12E-07 | 13 | 52 | 0.013525   | 6.09E-07 |
| 5000| Case I  | 6  | 24 | 0.22912    | 2.95E-08 | 12 | 48 | 0.10538    | 2.54E-07 |
|     | Case II  | 7  | 28 | 0.85665    | 1.13E-07 | 12 | 48 | 0.079825   | 4.39E-07 |
|     | Case III | 7  | 28 | 0.53427    | 4.32E-07 | 12 | 48 | 0.032032   | 5.68E-07 |
|     | Case IV  | 8  | 32 | 0.059319   | 3.98E-07 | 13 | 52 | 0.044075   | 4.28E-07 |
|     | Case V   | 7  | 28 | 0.038857   | 3.61E-08 | 13 | 52 | 0.044154   | 4.83E-07 |
|     | Case VI  | 8  | 32 | 0.065865   | 4.39E-07 | 13 | 52 | 0.18386    | 3.91E-07 |
|     | Case VII | 11 | 44 | 0.045807   | 3.86E-07 | 14 | 56 | 0.055375   | 2.68E-07 |
| 10000| Case I | 6  | 24 | 0.555883   | 2.62E-08 | 12 | 48 | 0.26245    | 3.59E-07 |
|     | Case II  | 7  | 28 | 0.10776    | 9.11E-08 | 12 | 48 | 0.084684   | 6.19E-07 |
|     | Case III | 7  | 28 | 0.12502    | 6.11E-07 | 12 | 48 | 0.07673    | 7.96E-07 |
|     | Case IV  | 8  | 32 | 0.060193   | 5.62E-07 | 13 | 52 | 0.082261   | 5.60E-07 |
|     | Case V   | 7  | 28 | 0.0602     | 5.18E-08 | 13 | 52 | 0.068433   | 4.87E-07 |
|     | Case VI  | 8  | 32 | 0.05542    | 6.21E-07 | 13 | 52 | 0.066654   | 5.01E-07 |
|     | Case VII | 13 | 52 | 0.085591   | 1.80E-07 | 14 | 56 | 0.13534    | 3.78E-07 |
| 50000| Case I  | 6  | 24 | 0.17095    | 2.91E-08 | 12 | 48 | 0.23645    | 8.03E-07 |
|     | Case II  | 7  | 28 | 0.15714    | 6.26E-08 | 13 | 52 | 0.25119    | 2.77E-07 |
|     | Case III | 8  | 32 | 0.25071    | 8.73E-08 | 17 | 56 | 0.42962    | 9.95E-07 |
|     | Case IV  | 9  | 36 | 0.46934    | 2.51E-08 | 14 | 56 | 0.26703    | 2.34E-07 |
|     | Case V   | 7  | 28 | 0.29169    | 1.17E-07 | 13 | 52 | 0.2287     | 7.20E-07 |
|     | Case VI  | 9  | 36 | 0.2315     | 2.78E-08 | 14 | 56 | 0.24843    | 2.05E-07 |
|     | Case VII | 13 | 52 | 0.56004    | 2.37E-08 | 14 | 56 | 0.32785    | 8.43E-07 |
| 100000| Case I | 6  | 24 | 0.98438    | 3.58E-08 | 13 | 52 | 0.43828    | 2.27E-07 |
|     | Case II  | 7  | 28 | 0.39255    | 6.17E-08 | 13 | 52 | 0.62503    | 3.91E-07 |
|     | Case III | 8  | 32 | 0.48543    | 3.86E-08 | 13 | 52 | 0.42684    | 4.99E-07 |
|     | Case IV  | 9  | 36 | 0.52322    | 3.56E-08 | 14 | 56 | 0.50903    | 3.28E-07 |
|     | Case V   | 7  | 28 | 0.43716    | 1.66E-07 | 13 | 52 | 0.62726    | 9.52E-07 |
|     | Case VI  | 9  | 36 | 0.54937    | 3.93E-08 | 14 | 56 | 0.49271    | 2.86E-07 |
|     | Case VII | 13 | 52 | 0.89929    | 4.99E-08 | 15 | 60 | 0.53354    | 2.38E-07 |
Table 3. Test results for problem 2

| DIM | INP | NI | NF | CPU   | NORM | NI | NF | CPU   | NORM |
|-----|-----|----|----|-------|------|----|----|-------|------|
| 1000|     |    |    |       |      |    |    |       |      |
| Case I | 4 | 12 | 0.087572 | 7.61E-09 | 4 | 12 | 0.008488 | 5.17E-07 |
| Case II | 5 | 15 | 0.016259 | 6.04E-09 | 5 | 15 | 0.006875 | 6.04E-09 |
| Case III | 5 | 15 | 0.011455 | 4.37E-07 | 5 | 15 | 0.006569 | 4.37E-07 |
| Case IV | 6 | 18 | 0.011513 | 1.52E-07 | 6 | 18 | 0.008092 | 1.52E-07 |
| Case V | 7 | 21 | 0.00633 | 1.10E-09 | 7 | 21 | 0.005713 | 1.10E-09 |
| Case VI | 7 | 21 | 0.011702 | 1.74E-08 | 7 | 21 | 0.004815 | 1.74E-08 |
| Case VII | 11 | 35 | 0.012359 | 2.77E-07 | 15 | 49 | 0.021261 | 3.66E-07 |
| 5000|     |    |    |       |      |    |    |       |      |
| Case I | 4 | 12 | 0.064074 | 8.56E-10 | 4 | 12 | 0.035632 | 1.75E-07 |
| Case II | 5 | 15 | 0.076987 | 6.27E-10 | 5 | 15 | 0.019409 | 6.27E-10 |
| Case III | 5 | 15 | 0.027949 | 1.42E-07 | 5 | 15 | 0.014664 | 1.42E-07 |
| Case IV | 6 | 18 | 0.036548 | 3.94E-08 | 6 | 18 | 0.026202 | 3.94E-08 |
| Case V | 6 | 18 | 0.01692 | 4.05E-07 | 6 | 18 | 0.018243 | 4.05E-07 |
| Case VI | 7 | 21 | 0.025943 | 2.36E-09 | 7 | 21 | 0.027663 | 2.36E-09 |
| Case VII | 13 | 43 | 0.094345 | 3.76E-09 | 18 | 62 | 0.19571 | 1.75E-07 |
| 10000|     |    |    |       |      |    |    |       |      |
| Case I | 4 | 12 | 0.022826 | 3.98E-10 | 4 | 12 | 0.02201 | 1.21E-07 |
| Case II | 5 | 15 | 0.029782 | 2.79E-10 | 5 | 15 | 0.03389 | 2.79E-10 |
| Case III | 5 | 15 | 0.036108 | 9.73E-08 | 5 | 15 | 0.020746 | 9.73E-08 |
| Case IV | 6 | 18 | 0.031098 | 2.56E-08 | 6 | 18 | 0.031954 | 2.56E-08 |
| Case V | 6 | 18 | 0.11664 | 2.93E-07 | 6 | 18 | 0.13272 | 2.93E-07 |
| Case VI | 7 | 21 | 0.039807 | 1.24E-09 | 7 | 21 | 0.056343 | 1.24E-09 |
| Case VII | 13 | 43 | 0.37509 | 1.17E-08 | 20 | 69 | 0.12928 | 9.06E-08 |
| 50000|     |    |    |       |      |    |    |       |      |
| Case I | 4 | 13 | 0.36887 | 1.05E-10 | 4 | 12 | 0.15651 | 6.32E-08 |
| Case II | 5 | 16 | 0.42862 | 6.75E-11 | 5 | 16 | 0.10554 | 6.75E-11 |
| Case III | 5 | 15 | 0.24321 | 4.87E-08 | 5 | 15 | 0.16968 | 4.87E-08 |
| Case IV | 6 | 18 | 0.46737 | 1.11E-08 | 6 | 18 | 0.10999 | 1.11E-08 |
| Case V | 6 | 18 | 0.13319 | 1.84E-07 | 6 | 18 | 0.11779 | 1.84E-07 |
| Case VI | 7 | 21 | 0.18161 | 4.01E-10 | 7 | 21 | 0.1203 | 4.01E-10 |
| Case VII | 16 | 54 | 0.46124 | 1.11E-09 | 23 | 81 | 0.5211 | 6.89E-07 |
| 100000|     |    |    |       |      |    |    |       |      |
| Case I | 4 | 13 | 0.26038 | 6.80E-11 | 4 | 12 | 0.13094 | 5.40E-08 |
| Case II | 5 | 16 | 0.30277 | 4.27E-11 | 5 | 16 | 0.18755 | 4.27E-11 |
| Case III | 5 | 15 | 0.4347 | 4.05E-08 | 5 | 15 | 0.27985 | 4.05E-08 |
| Case IV | 6 | 18 | 0.42556 | 8.15E-09 | 6 | 18 | 0.19207 | 8.15E-09 |
| Case V | 6 | 18 | 0.25488 | 1.80E-07 | 6 | 18 | 0.20156 | 1.80E-07 |
| Case VI | 7 | 21 | 0.43576 | 2.71E-10 | 7 | 21 | 0.21005 | 2.71E-10 |
| Case VII | 15 | 51 | 0.82806 | 1.64E-09 | 24 | 86 | 1.1405 | 4.14E-09 |
Table 4. Test results for problem 3

| DIM  | INP | NI | NF    | CPU     | NORM   | INP | NI | NF    | CPU     | NORM   |
|------|-----|----|-------|---------|--------|-----|----|-------|---------|--------|
| 1000 |     |    |       |         |        |     |    |       |         |        |
| Case I | 13  | 52 |    0.075605 | 3.36E-07 | 31     | 124  | 0.023401 | 6.98E-07 |
| Case II | 14  | 56 |    0.033438 | 9.91E-07 | 32     | 126  | 0.016722 | 2.09E-22 |
| Case III | 15  | 59 |    0.014509 | 6.78E-07 | 34     | 135  | 0.029896 | 7.24E-07 |
| Case IV | 15  | 59 |    0.022327 | 8.58E-07 | 35     | 140  | 0.033452 | 8.69E-07 |
| Case V  | 15  | 59 |    0.012348 | 4.93E-07 | 35     | 140  | 0.024649 | 9.73E-07 |
| Case VI | 16  | 64 |    0.011585 | 6.63E-07 | 36     | 142  | 0.041312 | 0        |
| Case VII | 15 | 60 |    0.019587 | 7.37E-07 | 34     | 136  | 0.018809 | 8.10E-07 |
| 5000  |     |    |       |         |        |     |    |       |         |        |
| Case I | 13  | 51 |    0.084308 | 7.51E-07 | 30     | 118  | 0.095447 | 0        |
| Case II | 15  | 60 |    0.052927 | 6.65E-07 | 31     | 122  | 0.50476  | 5.62E-21 |
| Case III | 16  | 62 |    0.08685  | 0        | 35     | 140  | 0.084126 | 9.71E-07 |
| Case IV | 16  | 62 |    0.054943 | 1.40E-21 | 36     | 143  | 0.095079 | 7.25E-21 |
| Case V  | 16  | 64 |    0.10298  | 3.31E-07 | 35     | 138  | 0.26686  | 1.40E-21 |
| Case VI | 17  | 66 |    0.121    | 2.34E-22 | 36     | 142  | 0.16331  | 0        |
| Case VII | 16 | 64 |    0.82826  | 5.03E-07 | 36     | 144  | 0.16091  | 6.28E-07 |
| 10000 |     |    |       |         |        |     |    |       |         |        |
| Case I | 14  | 55 |    0.094948 | 3.19E-07 | 31     | 122  | 0.12401  | 6.62E-22 |
| Case II | 15  | 60 |    0.17919  | 9.40E-07 | 32     | 126  | 0.14805  | 0        |
| Case III | 16  | 62 |    0.085448 | 3.31E-22 | 36     | 143  | 0.13867  | 8.24E-07 |
| Case IV | 16  | 64 |    0.12333  | 8.14E-07 | 35     | 138  | 0.15826  | 2.65E-21 |
| Case V  | 16  | 62 |    0.1041   | 3.31E-22 | 35     | 138  | 0.27449  | 6.62E-22 |
| Case VI | 17  | 66 |    0.11764  | 0        | 36     | 143  | 0.16026  | 2.65E-21 |
| Case VII | 16 | 64 |    0.081495 | 7.03E-07 | 36     | 144  | 0.19776  | 8.97E-07 |
| 50000 |     |    |       |         |        |     |    |       |         |        |
| Case I | 12  | 46 |    0.4807  | 1.15E-19 | 31     | 122  | 0.53862  | 2.96E-21 |
| Case II | 16  | 62 |    0.85283  | 0        | 31     | 122  | 0.5534   | 2.96E-21 |
| Case III | 16  | 62 |    0.43053  | 3.70E-21 | 33     | 130  | 0.66305  | 0        |
| Case IV | 15  | 58 |    0.34538  | 1.78E-20 | 34     | 134  | 0.59923  | 2.37E-20 |
| Case V  | 17  | 66 |    0.45061  | 0        | 35     | 138  | 0.82508  | 1.48E-21 |
| Case VI | 18  | 72 |    0.46072  | 4.22E-07 | 36     | 143  | 0.62746  | 2.96E-21 |
| Case VII | 17 | 68 |    0.40219  | 4.73E-07 | 38     | 152  | 0.6653   | 7.24E-07 |
| 100000 |     |    |       |         |        |     |    |       |         |        |
| Case I | 15  | 60 |    0.73965  | 3.02E-07 | 31     | 122  | 1.139    | 2.09E-21 |
| Case II | 16  | 62 |    1.1267   | 5.23E-22 | 37     | 148  | 1.361    | 6.48E-07 |
| Case III | 17  | 68 |    0.58078  | 6.10E-07 | 38     | 151  | 1.5385   | 9.38E-07 |
| Case IV | 17  | 68 |    0.64576  | 7.73E-07 | 37     | 146  | 1.3166   | 0        |
| Case V  | 17  | 68 |    0.77823  | 4.44E-07 | 40     | 158  | 1.4507   | 0        |
| Case VI | 17  | 66 |    0.67135  | 3.14E-21 | 40     | 159  | 1.4451   | 8.14E-07 |
| Case VII | 17 | 68 |    0.67963  | 6.70E-07 | 39     | 156  | 1.5115   | 6.14E-07 |
### Table 5. Test results for problem 4

| DIM | INP | NI | NF | CPU   | NORM | NI | NF | CPU   | NORM |
|-----|-----|----|----|-------|------|----|----|-------|------|
| 1000| Case I | 13 | 51 | 0.06059 | 3.25E-07 | 31 | 124 | 0.02262 | 6.43E-07 |
|     | Case II | 14 | 56 | 0.008665 | 7.01E-07 | 32 | 128 | 0.020618 | 7.10E-07 |
|     | Case III | 14 | 56 | 0.023062 | 7.14E-07 | 33 | 132 | 0.030138 | 8.25E-07 |
|     | Case IV | 14 | 56 | 0.009094 | 9.53E-07 | 33 | 132 | 0.022234 | 9.09E-07 |
|     | Case V | 15 | 60 | 0.019868 | 8.47E-07 | 32 | 128 | 0.020698 | 6.86E-07 |
|     | Case VI | 15 | 60 | 0.032655 | 7.79E-07 | 35 | 140 | 0.033957 | 6.08E-07 |
|     | Case VII | 16 | 64 | 0.016672 | 4.18E-07 | 33 | 132 | 0.030958 | 7.44E-07 |
| 5000| Case I | - | - | - | - | 32 | 127 | 0.083999 | 8.63E-07 |
|     | Case II | 15 | 59 | 0.047515 | 4.70E-07 | 33 | 132 | 0.15239 | 9.52E-07 |
|     | Case III | 15 | 59 | 0.032093 | 4.79E-07 | 35 | 140 | 0.061537 | 6.64E-07 |
|     | Case IV | 15 | 59 | 0.062007 | 6.39E-07 | 35 | 140 | 0.13051 | 7.32E-07 |
|     | Case V | 16 | 63 | 0.045188 | 5.68E-07 | 33 | 132 | 0.071242 | 9.20E-07 |
|     | Case VI | 16 | 64 | 0.037473 | 5.22E-07 | 36 | 144 | 0.060172 | 8.16E-07 |
|     | Case VII | 16 | 64 | 0.030254 | 8.70E-07 | 35 | 140 | 0.072475 | 6.05E-07 |
| 10000| Case I | - | - | - | - | 33 | 131 | 0.12064 | 7.32E-07 |
|     | Case II | 15 | 59 | 0.048278 | 6.65E-07 | 34 | 135 | 0.63105 | 8.08E-07 |
|     | Case III | 15 | 59 | 0.057155 | 6.77E-07 | 35 | 140 | 0.11101 | 9.39E-07 |
|     | Case IV | 15 | 60 | 0.060725 | 9.04E-07 | 36 | 143 | 0.1951 | 6.21E-07 |
|     | Case V | 16 | 63 | 0.053938 | 8.04E-07 | 34 | 135 | 0.12425 | 7.81E-07 |
|     | Case VI | 16 | 63 | 0.066593 | 7.39E-07 | 37 | 147 | 0.20795 | 6.92E-07 |
|     | Case VII | 17 | 68 | 0.057214 | 3.60E-07 | 35 | 140 | 0.12255 | 8.50E-07 |
| 50000| Case I | 14 | 55 | 1.0077 | 6.89E-07 | 34 | 134 | 0.42136 | 0 |
|     | Case II | 16 | 63 | 0.61724 | 4.46E-07 | 35 | 138 | 0.43246 | 0 |
|     | Case III | 16 | 63 | 0.32509 | 4.54E-07 | 37 | 147 | 1.0982 | 7.56E-07 |
|     | Case IV | 16 | 63 | 0.19589 | 6.06E-07 | 37 | 148 | 0.59668 | 8.33E-07 |
|     | Case V | 17 | 66 | 0.24309 | 0 | - | - | - | - |
|     | Case VI | - | - | - | - | - | - | - | - |
|     | Case VII | 17 | 68 | 0.2842 | 8.32E-07 | 37 | 148 | 0.47886 | 6.85E-07 |
| 100000| Case I | 14 | 54 | 0.32344 | 0 | - | - | - | - |
|     | Case II | 16 | 63 | 0.51789 | 6.31E-07 | 35 | 138 | 0.81304 | 0 |
|     | Case III | 16 | 62 | 0.5148 | 0 | 38 | 150 | 1.2176 | 0 |
|     | Case IV | - | - | - | - | 38 | 151 | 1.0824 | 7.07E-07 |
|     | Case V | 17 | 68 | 0.56565 | 7.63E-07 | 35 | 138 | 1.0007 | 0 |
|     | Case VI | 17 | 68 | 0.57103 | 7.01E-07 | - | - | - | - |
|     | Case VII | 18 | 72 | 0.53447 | 3.50E-07 | 37 | 148 | 0.88698 | 9.70E-07 |
Table 6. Test results for problem 5

| DIM | INP | NI | NF | CPU   | NORM  | NI | NF | CPU   | NORM  |
|-----|-----|----|----|-------|-------|----|----|-------|-------|
| 1000| Case I  | 25  | 95  | 0.1329 | 6.45E-07 | 36  | 139 | 0.019135 | 6.42E-07 |
|     | Case II | 19  | 72  | 0.014534 | 4.04E-07 | 36  | 140 | 0.018994 | 6.62E-07 |
|     | Case III | 22  | 86  | 0.022553 | 9.29E-07 | 47  | 175 | 0.044522 | 6.46E-07 |
|     | Case IV | 22  | 88  | 0.036427 | 4.00E-07 | 40  | 156 | 0.03131 | 6.54E-07 |
|     | Case V  | 25  | 100 | 0.037173 | 4.90E-07 | 37  | 146 | 0.023349 | 7.12E-07 |
|     | Case VI | 41  | 164 | 0.071153 | 6.01E-07 | 36  | 143 | 0.037977 | 9.25E-07 |
|     | Case VII| 42  | 167 | 0.12029 | 4.55E-07 | 49  | 184 | 0.060357 | 9.87E-07 |
| 5000| Case I  | 22  | 83  | 0.13573 | 6.53E-07 | 36  | 139 | 0.11871 | 6.40E-07 |
|     | Case II | 19  | 72  | 0.11048 | 8.54E-07 | 37  | 144 | 0.088934 | 9.52E-07 |
|     | Case III | 25  | 98  | 0.090732 | 3.03E-07 | 51  | 189 | 0.10334 | 9.00E-07 |
|     | Case IV | 24  | 96  | 0.061785 | 8.38E-07 | 44  | 169 | 0.44477 | 9.18E-07 |
|     | Case V  | 30  | 120 | 0.1862 | 4.11E-07 | 40  | 157 | 0.11434 | 6.51E-07 |
|     | Case VI | 54  | 216 | 0.33113 | 9.45E-07 | 41  | 160 | 0.10301 | 8.14E-07 |
|     | Case VII| 69  | 275 | 0.5637 | 3.29E-07 | 47  | 180 | 0.107 | 9.84E-07 |
| 10000| Case I  | 27  | 103 | 0.14553 | 4.01E-07 | 36  | 139 | 0.1347 | 7.33E-07 |
|     | Case II | 20  | 76  | 0.086196 | 3.64E-07 | 38  | 147 | 0.15397 | 8.23E-07 |
|     | Case III | 25  | 98  | 0.20935 | 8.75E-07 | 47  | 177 | 0.17983 | 7.72E-07 |
|     | Case IV | 24  | 96  | 0.19624 | 5.28E-07 | 44  | 170 | 0.19929 | 7.87E-07 |
|     | Case V  | 35  | 140 | 0.1977 | 6.41E-07 | 42  | 163 | 0.18619 | 9.45E-07 |
|     | Case VI | 60  | 240 | 0.65802 | 4.12E-07 | 49  | 185 | 0.47874 | 7.04E-07 |
|     | Case VII| 84  | 335 | 0.96443 | 8.09E-07 | 43  | 169 | 0.1934 | 7.22E-07 |
| 50000| Case I  | 55  | 215 | 2.6998 | 9.96E-07 | 65  | 255 | 1.3087 | 8.37E-07 |
|     | Case II | 20  | 76  | 0.2954 | 8.35E-07 | 48  | 179 | 0.92835 | 6.84E-07 |
|     | Case III | 27  | 106 | 0.47985 | 8.73E-07 | 67  | 240 | 0.92921 | 6.31E-07 |
|     | Case IV | 41  | 164 | 1.7421 | 9.71E-07 | 47  | 181 | 0.72302 | 6.44E-07 |
|     | Case V  | 36  | 144 | 1.5835 | 8.46E-07 | 45  | 174 | 0.86921 | 7.83E-07 |
|     | Case VI | 79  | 316 | 4.0512 | 6.61E-07 | 54  | 201 | 0.94527 | 9.68E-07 |
|     | Case VII| 103 | 411 | 6.5137 | 7.57E-07 | 52  | 204 | 0.79431 | 7.54E-07 |
| 100000| Case I | 74  | 291 | 7.1551 | 8.21E-07 | 83  | 327 | 3.7002 | 7.04E-07 |
|     | Case II | 21  | 80  | 0.671 | 3.61E-07 | 46  | 173 | 1.4825 | 9.75E-07 |
|     | Case III | 26  | 102 | 0.89984 | 6.93E-07 | 55  | 204 | 1.6728 | 8.97E-07 |
|     | Case IV | 44  | 176 | 2.6239 | 8.75E-07 | 52  | 196 | 1.5811 | 9.15E-07 |
|     | Case V  | 43  | 172 | 2.2944 | 8.30E-07 | 48  | 184 | 1.5047 | 6.71E-07 |
|     | Case VI | 99  | 396 | 12.5198 | 3.58E-07 | 47  | 181 | 1.4726 | 8.26E-07 |
|     | Case VII| 111 | 443 | 12.3882 | 5.20E-07 | 69  | 272 | 2.5982 | 7.84E-07 |
### Table 7. Test results for problem 6

| DIM  | INP | NI  | NF  | CPU    | NORM | CASE I   | NI  | NF  | CPU    | NORM |
|------|-----|-----|-----|--------|------|----------|-----|-----|--------|------|
| 1000 | 17  | 68  | 17  | 68     | 0.070165 | 3.66E-07 | 37  | 148 | 0.086194 | 8.53E-07 |
|      | 17  | 68  | 0.019243 | 3.42E-07 | 37  | 148 | 0.049573 | 8.20E-07 |
|      | 17  | 68  | 0.022663 | 3.01E-07 | 37  | 148 | 0.051928 | 7.22E-07 |
|      | 16  | 64  | 0.016058 | 6.87E-07 | 36  | 144 | 0.074625 | 8.24E-07 |
|      | 16  | 64  | 0.028711 | 5.52E-07 | 36  | 144 | 0.080704 | 6.61E-07 |
|      | 16  | 64  | 0.036264 | 3.25E-07 | 35  | 140 | 0.098138 | 6.50E-07 |
|      | 17  | 68  | 0.029906 | 3.02E-07 | 37  | 148 | 0.057573 | 7.28E-07 |
| 5000 | 17  | 68  | 0.11976 | 8.21E-07 | 39  | 156 | 0.16301 | 6.87E-07 |
|      | 17  | 68  | 0.13738 | 7.66E-07 | 39  | 156 | 0.1438  | 6.61E-07 |
|      | 17  | 68  | 0.070333 | 6.75E-07 | 38  | 152 | 0.14908 | 9.71E-07 |
|      | 17  | 68  | 0.097123 | 4.62E-07 | 38  | 152 | 0.16691 | 6.64E-07 |
|      | 16  | 64  | 0.065808 | 7.29E-07 | 38  | 152 | 0.36454 | 8.73E-07 |
|      | 17  | 68  | 0.089469 | 6.80E-07 | 38  | 152 | 0.19723 | 9.77E-07 |
| 10000| 18  | 72  | 0.1274 | 3.48E-07 | 39  | 156 | 0.35497 | 9.72E-07 |
|      | 18  | 72  | 0.12939 | 3.25E-07 | 39  | 156 | 0.50582 | 9.35E-07 |
|      | 17  | 68  | 0.13873 | 9.55E-07 | 39  | 156 | 0.46728 | 8.24E-07 |
|      | 17  | 68  | 0.13388 | 6.54E-07 | 38  | 152 | 0.2839  | 9.40E-07 |
|      | 17  | 68  | 0.20628 | 5.24E-07 | 38  | 152 | 0.27388 | 7.54E-07 |
|      | 17  | 68  | 0.16002 | 3.09E-07 | 37  | 148 | 0.56098 | 7.41E-07 |
|      | 17  | 68  | 0.16743 | 9.63E-07 | 39  | 156 | 0.46017 | 8.28E-07 |
| 5000 | 18  | 70  | 0.74334 | 0      | 40  | 158 | 1.2103  | 0      |
|      | 18  | 70  | 0.54759 | 0      | 40  | 158 | 1.4011  | 0      |
|      | 18  | 70  | 0.46758 | 0      | 39  | 154 | 1.3023  | 0      |
|      | 18  | 70  | 0.47821 | 0      | 39  | 154 | 1.2953  | 0      |
|      | 18  | 70  | 0.51731 | 0      | 38  | 150 | 1.2347  | 0      |
|      | 17  | 66  | 0.52485 | 0      | 37  | 146 | 1.2817  | 0      |
|      | 18  | 70  | 0.55512 | 0      | 40  | 158 | 1.3476  | 0      |
| 10000| 18  | 70  | 1.0765 | 0      | 40  | 158 | 2.4974  | 0      |
|      | 18  | 70  | 1.3017 | 0      | 39  | 154 | 2.5376  | 0      |
|      | 18  | 70  | 1.2128 | 0      | 38  | 150 | 2.45    | 0      |
|      | 17  | 66  | 1.2676 | 0      | 38  | 150 | 2.3559  | 0      |
|      | 17  | 66  | 1.1563 | 0      | 37  | 146 | 2.4861  | 0      |
|      | 17  | 66  | 1.0464 | 0      | 36  | 142 | 2.2564  | 0      |
|      | 18  | 70  | 1.0587 | 0      | 39  | 154 | 2.6041  | 0      |
Table 8. Test results for problem 7

| DIM  | INP | NI | NF | CPU   | NORM  | NI | NF | CPU   | NORM  |
|------|-----|----|----|-------|------|----|----|-------|------|
| 1000 | Case I | 9 | 36 | 0.029847 | 1.25E-07 | 14 | 56 | 0.010049 | 4.58E-07 |
|      | Case II | 8 | 32 | 0.01614 | 7.06E-07 | 14 | 56 | 0.009322 | 3.19E-07 |
|      | Case III | 7 | 28 | 0.036217 | 2.03E-07 | 11 | 44 | 0.014851 | 6.44E-07 |
|      | Case IV | 8 | 32 | 0.00896 | 1.76E-07 | 15 | 60 | 0.009756 | 2.52E-07 |
|      | Case V | 9 | 36 | 0.012999 | 2.91E-07 | 15 | 60 | 0.014388 | 3.91E-07 |
|      | Case VI | 9 | 35 | 0.015981 | 3.43E-07 | 14 | 56 | 0.017579 | 3.00E-07 |
|      | Case VII | 9 | 36 | 0.015981 | 3.43E-07 | 14 | 56 | 0.017579 | 3.00E-07 |
| 5000 | Case I | 9 | 36 | 0.040232 | 2.79E-07 | 15 | 60 | 0.06212 | 2.57E-07 |
|      | Case II | 9 | 36 | 0.038549 | 1.30E-07 | 14 | 56 | 0.035553 | 7.13E-07 |
|      | Case III | 7 | 28 | 0.02036 | 4.53E-07 | 12 | 48 | 0.037788 | 3.61E-07 |
|      | Case IV | 8 | 32 | 0.025654 | 4.15E-07 | 15 | 60 | 0.035425 | 5.64E-07 |
|      | Case V | 9 | 36 | 0.026546 | 6.51E-07 | 15 | 60 | 0.12771 | 8.73E-07 |
|      | Case VI | 9 | 35 | 0.096229 | 7.14E-07 | 15 | 59 | 0.044662 | 6.31E-07 |
|      | Case VII | 9 | 36 | 0.093227 | 7.84E-07 | 14 | 56 | 0.053641 | 6.71E-07 |
| 10000| Case I | 9 | 36 | 0.054105 | 3.95E-07 | 15 | 60 | 0.055328 | 3.64E-07 |
|      | Case II | 9 | 36 | 0.10633 | 1.84E-07 | 15 | 60 | 0.10212 | 2.53E-07 |
|      | Case III | 7 | 28 | 0.037857 | 6.41E-07 | 12 | 48 | 0.06777 | 5.11E-07 |
|      | Case IV | 8 | 32 | 0.082416 | 5.87E-07 | 15 | 60 | 0.071481 | 7.98E-07 |
|      | Case V | 9 | 36 | 0.18582 | 9.20E-07 | 16 | 64 | 0.059125 | 3.10E-07 |
|      | Case VI | 10 | 39 | 0.070373 | 8.34E-08 | 15 | 59 | 0.081851 | 8.93E-07 |
|      | Case VII | 10 | 36 | 0.071963 | 9.12E-08 | 14 | 56 | 0.31755 | 9.38E-07 |
| 50000| Case I | 9 | 36 | 0.36461 | 8.84E-07 | 15 | 60 | 0.32342 | 8.14E-07 |
|      | Case II | 9 | 36 | 0.27196 | 4.12E-07 | 15 | 60 | 0.3605 | 5.66E-07 |
|      | Case III | 8 | 32 | 0.25501 | 1.18E-07 | 13 | 52 | 0.22338 | 2.87E-07 |
|      | Case IV | 9 | 36 | 0.22228 | 1.08E-07 | 16 | 64 | 0.26331 | 4.48E-07 |
|      | Case V | 10 | 40 | 0.25618 | 1.70E-07 | 16 | 64 | 0.27305 | 6.94E-07 |
|      | Case VI | 10 | 39 | 0.27278 | 1.87E-07 | 16 | 63 | 0.28175 | 5.01E-07 |
|      | Case VII | 10 | 36 | 0.22836 | 2.04E-07 | 15 | 60 | 0.75345 | 5.27E-07 |
| 100000| Case I | 10 | 40 | 0.57002 | 1.03E-07 | 16 | 64 | 0.56535 | 2.89E-07 |
|      | Case II | 9 | 36 | 0.34715 | 5.83E-07 | 15 | 60 | 0.66238 | 8.01E-07 |
|      | Case III | 8 | 32 | 0.41892 | 1.67E-07 | 13 | 52 | 0.45765 | 4.06E-07 |
|      | Case IV | 9 | 36 | 0.34291 | 1.53E-07 | 16 | 64 | 0.68567 | 6.34E-07 |
|      | Case V | 10 | 40 | 0.53349 | 2.40E-07 | 16 | 64 | 0.51067 | 9.81E-07 |
|      | Case VI | 10 | 39 | 0.35763 | 2.64E-07 | 16 | 63 | 0.5497 | 7.09E-07 |
|      | Case VII | 10 | 40 | 0.55468 | 2.88E-07 | 15 | 60 | 0.6995 | 7.46E-07 |
Table 9. Test results for problem 8

| DIM | INP | NI  | NF  | CPU    | NORM   | NI  | NF  | CPU    | NORM   |
|-----|-----|-----|-----|--------|--------|-----|-----|--------|--------|
|     |     |     |     |        |        |     |     |        |        |
| 1000|     |     |     |        |        |     |     |        |        |
| Case I | 11 | 38  |   | 0.032713 | 3.88E-07 | 16 | 58  | 0.00921 | 9.66E-07 |
| Case II | 11 | 39  |   | 0.010107 | 3.88E-07 | 16 | 59  | 0.010495 | 9.66E-07 |
| Case III | 11 | 39  |   | 0.0097  | 3.88E-07 | 16 | 59  | 0.013631 | 9.66E-07 |
| Case IV | 11 | 39  |   | 0.006542 | 3.88E-07 | 16 | 59  | 0.00733  | 9.66E-07 |
| Case V  | 11 | 39  |   | 0.012047 | 3.88E-07 | 16 | 59  | 0.009248 | 9.66E-07 |
| Case VI | 11 | 39  |   | 0.014019 | 3.88E-07 | 16 | 59  | 0.01325  | 9.66E-07 |
| Case VII | 11 | 39  |   | 0.010456 | 3.88E-07 | 16 | 59  | 0.015507 | 9.66E-07 |
| 5000|     |     |     |        |        |     |     |        |        |
| Case I | 8  | 29  |   | 0.029519 | 8.46E-07 | 12 | 46  | 0.03298 | 6.18E-07 |
| Case II | 8  | 30  |   | 0.028146 | 8.46E-07 | 12 | 46  | 0.028464 | 6.18E-07 |
| Case III | 8  | 30  |   | 0.031133 | 8.46E-07 | 12 | 46  | 0.026699 | 6.18E-07 |
| Case IV | 8  | 30  |   | 0.032266 | 8.46E-07 | 12 | 46  | 0.065305 | 6.18E-07 |
| Case V  | 8  | 30  |   | 0.043717 | 8.46E-07 | 12 | 46  | 0.033667 | 6.18E-07 |
| Case VI | 8  | 30  |   | 0.075426 | 8.46E-07 | 12 | 46  | 0.032206 | 6.18E-07 |
| Case VII | 8  | 30  |   | 0.043783 | 8.46E-07 | 12 | 46  | 0.052994 | 6.18E-07 |
| 10000|     |     |     |        |        |     |     |        |        |
| Case I | 7  | 26  |   | 0.070454 | 5.16E-07 | 9  | 35  | 0.045216 | 7.70E-07 |
| Case II | 7  | 27  |   | 0.053641 | 5.16E-07 | 9  | 35  | 0.059089 | 7.70E-07 |
| Case III | 7  | 27  |   | 0.094187 | 5.16E-07 | 9  | 35  | 0.045507 | 7.70E-07 |
| Case IV | 7  | 27  |   | 0.20473 | 5.16E-07 | 9  | 35  | 0.074929 | 7.70E-07 |
| Case V  | 7  | 27  |   | 0.071282 | 5.16E-07 | 9  | 35  | 0.075532 | 7.70E-07 |
| Case VI | 7  | 27  |   | 0.064401 | 5.16E-07 | 9  | 35  | 0.065486 | 7.70E-07 |
| Case VII | 7  | 27  |   | 0.098105 | 5.16E-07 | 9  | 35  | 0.055328 | 7.70E-07 |
| 50000|     |     |     |        |        |     |     |        |        |
| Case I | 6  | 24  |   | 0.22768 | 1.49E-07 | 12 | 48  | 0.35733 | 5.55E-07 |
| Case II | 6  | 24  |   | 0.28215 | 1.49E-07 | 12 | 48  | 0.34089 | 5.55E-07 |
| Case III | 6  | 24  |   | 0.72157 | 1.49E-07 | 12 | 48  | 0.30864 | 5.55E-07 |
| Case IV | 6  | 24  |   | 0.53086 | 1.49E-07 | 12 | 48  | 0.37903 | 5.55E-07 |
| Case V  | 6  | 24  |   | 0.39711 | 1.49E-07 | 12 | 48  | 0.34614 | 5.55E-07 |
| Case VI | 6  | 24  |   | 0.48753 | 1.49E-07 | 12 | 48  | 0.41103 | 5.55E-07 |
| Case VII | 6  | 24  |   | 0.29061 | 1.49E-07 | 12 | 48  | 0.33605 | 5.55E-07 |
| 100000|     |     |     |        |        |     |     |        |        |
| Case I | 6  | 24  |   | 0.66013 | 6.95E-07 | 6  | 24  | 0.43192 | 5.77E-07 |
| Case II | 6  | 24  |   | 0.60897 | 6.95E-07 | 6  | 24  | 0.30724 | 5.77E-07 |
| Case III | 6  | 24  |   | 0.78423 | 6.95E-07 | 6  | 24  | 0.46109 | 5.77E-07 |
| Case IV | 6  | 24  |   | 0.7826  | 6.95E-07 | 6  | 24  | 0.37262 | 5.77E-07 |
| Case V  | 6  | 24  |   | 0.70311 | 6.95E-07 | 6  | 24  | 0.40775 | 5.77E-07 |
| Case VI | 6  | 24  |   | 0.9559  | 6.95E-07 | 6  | 24  | 0.38522 | 5.77E-07 |
| Case VII | 6  | 24  |   | 0.68726 | 6.95E-07 | 6  | 24  | 0.48134 | 5.77E-07 |
REFERENCES

[1] A. B. Abubakar, A. H. Ibrahim, A. B. Muhammad and C. Tammer, A modified descent Dai-Yuan conjugate gradient method for constraint nonlinear monotone operator equations, *Appl. Anal. Optim.*, 4 (2020), 1–24.

[2] A. B. Abubakar and P. Kumam, A descent Dai-Liao conjugate gradient method for nonlinear equations, *Numerical Algorithms*, 81 (2019), 197–210.

[3] A. B. Abubakar and P. Kumam, An improved three-term derivative-free method for solving nonlinear equations, *Comput. Appl. Math.*, 37 (2018), 6760–6773.

[4] A. B. Abubakar, P. Kumam and A. H. Ibrahim, Inertial Derivative-Free Projection Method for Nonlinear Monotone Operator Equations with Convex Constraints, IEEE Access. 2021.

[5] J. Abubakar, P. Kumam, A. H. Ibrahim, et al., Inertial iterative schemes with variable step sizes for variational inequality problem involving pseudomonotone operator, *Mathematics*, 8 (2020), 609.

[6] A. B. Abubakar, P. Kumam, A. H. Ibrahim, P. Chaipunya and S. A. Rano, New Hybrid Three-Term Spectral-Conjugate Gradient Method for Finding Solutions of Nonlinear Monotone Operator Equations with Applications, Mathematics and Computers in Simulation, 2021.

[7] A. B. Abubakar, P. Kumam, A. H. Ibrahim and J. Rilwan, Derivative-free HS-DY-type method for solving nonlinear equations and image restoration, *Helicon*, 6 (2020), e05480.

[8] A. B. Abubakar, P. Kumam and H. Mohammad, A note on the spectral gradient projection method for nonlinear monotone equations with applications, *Comput. Appl. Math.*, 39 (2020), Paper No. 129, 35 pp.

[9] A. B. Abubakar, P. Kumam, H. Mohammad and A. H. Ibrahim, PRP-like algorithm for monotone operator equations, *Jpn. J. Ind. Appl. Math.*, 38 (2021), 805–822.

[10] A. B. Abubakar, K. Muangchoo, A. H. Ibrahim, J. Abubakar and S. A. Rano, FR-type algorithm for finding approximate solutions to nonlinear monotone operator equations, *Arab. J. Math. (Springer)*, 10 (2021), 261–270.

[11] A. B. Abubakar, K. Muangchoo, A. H. Ibrahim, S. E. Fadugba, K. O. Aremu and L. O. Jolaoso, A modified scaled spectral-conjugate gradient-based algorithm for solving monotone operator equations, *J. Math.*, 2021 (2021), Art. ID 5549878, 9 pp.

[12] A. B. Abubakar, K. Muangchoo, A. H. Ibrahim, A. B. Muhammad, L. O. Jolaoso and K. O. Aremu, A new three-term Hestenes-Stiefel type method for nonlinear monotone operator equations and image restoration, *IEEE Access*, 9 (2021), 18262–18277.

[13] A. B. Abubakar, J. Rilwan, S. E. Yimer, A. H. Ibrahim and I. Ahmed, Spectral three-term conjugate descent method for solving nonlinear monotone equations with convex constraints, *Thai J. Math.*, 18 (2020), 501–517.

[14] J. Abubakar, P. Kumam, A. H. Ibrahim and A. Padcharoen, Relaxed inertial Tseng’s type method for solving the inclusion problem with application to image restoration, *Mathematics*, 8 (2020), 818.

[15] J. Abubakar, K. Sombut, H. ur Rehman, A. H. Ibrahim, et al., An accelerated subgradient extragradient algorithm for strongly pseudomonotone variational inequality problems, *Thai J. Math.*, 18 (2020), 166–187.

[16] W. Aj and B. Wollenberg, *Power Generation, Operation and Control*, New York: John Wiley & Sons. 1996, 592.

[17] F. Alvarez and H. Attouch, An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping, *Set-Valued Analysis*, 9 (2001), 3–11.

[18] H. Attouch, J. Peypouquet and P. Redont, A dynamical approach to an inertial forward-backward algorithm for convex minimization, *SIAM J. Optim.*, 24 (2014), 232–256.

[19] A. Auslender, M. Teboulle and S. Ben-Tiba, A logarithmic-quadratic proximal method for variational inequalities, In: *Comput. Optim. Appl.*, 12 (1999), 31–40.

[20] A. B. Abubakar, P. Kumam, M. Malik, P. Chaipunya and A. H. Ibrahim, A hybrid FR-DY conjugate gradient algorithm for unconstrained optimization with application in portfolio selection, *AIMS Math.*, 6 (2021), 6506–6527.

[21] A. B. Abubakar, P. Kumam, M. Malik and A. H. Ibrahim, A Hybrid Conjugate Gradient Based Approach for Solving Unconstrained Optimization and Motion Control Problems. Mathematics and Computers in Simulation, 2021.

[22] R. I. Boţ and E. R. Csetnek, An inertial forward-backward-forward primal-dual splitting algorithm for solving monotone inclusion problems, *Numer. Algorithms*, 71 (2016), 519–540.
(1977), 46–89.

Nonlinear Equations

cation to quasi-Newton methods

separable convex optimization

inclusion problems

conjugate gradient like method for signal reconstruction,

compressive sensing,

variational inclusion problems,

algorithms for variational inequalities

Conjugate Residual Algorithms for Convex Constraints Nonlinear Monotone Equations and Signal Restoration

Application to compressed sensing and other inverse problems, IEEE Journal of Selected Topics in Signal Processing, 1 (2007), 586–597.

An efficient three-term conjugate gradient method for nonlinear monotone equations with convex constraints, Calcolo, 55 (2018), Paper No. 53, 17 pp.

A three-term Polak-Ribiére-Polyak derivative-free method and its application to image restoration, Scientific African, 13 (2021), e00880. Available from: https://www.sciencedirect.com/science/article/pii/S2468227621001848.

A Modified Liu-Storey-Conjugate Descent Hybrid Projection Method for Convex Constrained Nonlinear Equations and Image Restoration, Numerical Algebra, Control & Optimization, 2021.

Derivative-free RMIL conjugate gradient method for convex constrained equations, Thai J. Math., 18 (2019), 212–232.

Re-modified derivative-free iterative method for nonlinear monotone equations with convex constraints, Ain Shams Engineering Journal, (2021).

Least-square-based three-term conjugate projection method for $\ell_1$-norm problems with application to compressed sensing, Mathematics, 8 (2020), 602.

An efficient gradient-free projection algorithm for constrained nonlinear equations and image restoration, AIMS Math., 6 (2021), 235–260.

A hybrid conjugate gradient algorithm for constrained monotone equations with application in compressive sensing, Heligen, 6 (2020), e03466.

Derivative-free conjugate residual algorithms for convex constraints nonlinear monotone equations and signal recovery, J. Nonlinear Convex Anal., 21 (2020), 1959–1972.

A derivative-free three-term hestenes-stiefel type method for constrained nonlinear equations and image restoration, International Journal of Computer Mathematics, 0 (2021), 1–22.

A family of derivative-free conjugate gradient methods for constrained nonlinear equations and image restoration, IEEE Access, 8 (2020), 162714–162729.

Spectral conjugate gradient like method for signal reconstruction, Thai J. Math., 18 (2020), 2019–2022.
[47] A. H. Ibrahim, K. Muangchoob, N. S. Mohamedc and A. B. Abubakar, Derivative-free SMR conjugate gradient method for con-straint nonlinear equations, *Journal of Mathematics and Computer Science*, 24 (2022), 147–164.

[48] W. La Cruz, J. M. Martínez and M. Raydan, Spectral residual method without gradient information for solving large-scale nonlinear systems: Theory and experiments, *Citeseer*, 2004; Technical Report RT-04-08 (https://www.ime.unicamp.br/~martinez/lmrreport.pdf).

[49] Q. Li and D.-H. Li, A class of derivative-free methods for large-scale nonlinear monotone equations, *IMA J. Numer. Anal.*, 31 (2011), 1625–1635.

[50] D. A. Lorenz and T. Pock, An inertial forward-backward algorithm for monotone inclusions, *J. Math. Imaging Vision*, 51 (2015), 311–325.

[51] K. Meintjes and A. P. Morgan, A methodology for solving chemical equilibrium systems, *Appl. Math. Comput.*, 22 (1987), 333–361.

[52] H. Mohammad, Barzilai-Borwein-like method for solving large-scale non-linear systems of equations, *J. Nigerian Math. Soc.*, 36 (2017), 71–83.

[53] H. Mohammad and A. B. Abubakar, A descent derivative-free algorithm for nonlinear monotone equations with convex constraints, *RAIRO Oper. Res.*, 54 (2020), 489–505.

[54] B. T. Polyak, Some methods of speeding up the convergence of iteration methods, *Ž. Vyčisl. Mat i Mat. Fiz.*, 4 (1964), 791–803.

[55] L. Qi and J. Sun, A nonsmooth version of Newton’s method, *Math. Programming*, 58 (1993), 353–367.

[56] M. V. Solodov and B. F. Svaiter, A new projection method for variational inequality problems, *SIAM J. Control Optim.*, 37 (1999), 765–776.

[57] M. Sun, J. Liu and Y. Wang, Two improved conjugate gradient methods with application in compressive sensing and motion control, *Math. Probl. Eng.*, 2020 (2020), Art. ID 9175496, 11 pp.

[58] D. V. Thong and D. V. Hieu, An inertial method for solving split common fixed point problems, *J. Fixed Point Theory Appl.*, 19 (2017), 3029–3051.

[59] D. V. Thong and D. V. Hieu, Modified subgradient extragradient method for variational inequality problems, *Numer. Algorithms*, 79 (2018), 597–610.

[60] Y. Xiao and H. Zhu, A conjugate gradient method to solve convex constrained monotone equations with applications in compressive sensing, *J. Math. Anal. Appl.*, 405 (2013), 310–319.

[61] N. Yamashita and M. Fukushima, On the rate of convergence of the Levenberg-Marquardt method, In: *Topics in Numerical Analysis*, 15, Springer, Vienna, (2001), 239–249.

[62] Z. Yu, J. Lin, J. Sun, Y. Xiao, L. Liu and Z. Li, Spectral gradient projection method for monotone nonlinear equations with convex constraints, *Appl. Numer. Math.*, 59 (2009), 2416–2423.

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