RANKS OF MAPS OF VECTOR BUNDLES

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Dedicated to Peter Newstead for his extraordinary mentorship role.

ABSTRACT. We generalize to vector bundles the techniques introduced for line bundles in [LOTZ1]. We then use this method to prove the injectivity of the Petri map for vector bundles and the surjectivity of a map related to deformation theory of Poincaré sheaves.

Many questions about the geometry of curves have a translation in terms of the rank of a map among spaces of sections of line bundles. The best known may be the Petri map:

\[ H^0(C, L) \otimes H^0(C, K \otimes L^{-1}) \rightarrow H^0(C, K). \]

The injectivity of this map determines the non-singularity of the locus of special divisors on a curve. Similarly, the maximal rank conjecture states that for a general curve and a general line bundle, the natural map of sections

\[ S^m(H^0(C, L)) \rightarrow H^0(C, L^m) \]

has maximal rank allowing one to count how many linearly independent hypersurfaces in the ambient space of a given degree contain a curve.

These conjectures have been proved by degenerating to a special curve and then showing that the kernel of the map gives rise to a small enough subspace in the limit and therefore it needs to be small too on the generic curve. A different approach is to look at the image of the map and show it is large enough. This amounts to proving the linear independence of a collection of sections obtained as product of sections of linear series in the domain. This method was used for line bundles in search of proofs of the maximal rank conjecture in [LOTZ1], [LOTZ2]. This is also essentially the strategy in the tropical setting (see [JP], [FJP]).

In this paper, we generalize to bundles of degree higher than one techniques that so far have only been used for line bundles (see section 1). We then show how they can be used to solve a couple of questions about vector bundles. In particular, we provide a new and simpler proof of the injectivity of the Petri map for vector bundles satisfying only mild numerical conditions on rank, degree and number of sections(see section 2). Note that injectivity of the Petri map guarantees existence of a component of the right dimension of the Brill Noether locus for vector bundles on a generic curve. In particular, this method could be used to fill gaps on the known Brill-Noether geography that keeps track of the values \( r, d, k \) for which the Brill-Noether locus \( W^r_{k, d} \) is non-empty.

We also look (see section 3) at the surjectivity of the cup product map between the sections of the canonical bundles of the curve and the sections of the traceless endomorphisms tensored with the canonical. This surjectivity has important consequences for the deformation theory of Poincaré bundles (see [BBN]).

It is worth pointing out that the applications in this paper are simple because we only try to show results for a generic vector bundle of a given rank and degree which in practice means checking the result for one vector bundle of that rank and degree. Checking for every vector bundle would lead to substantially more involved combinatorial issues.

While many proofs in the setting of Brill-Noether Theory for line bundles can be run in parallel using either techniques of limit linear series or tropical curves, no satisfactory theory for higher rank has so far been developed in the tropical setting. It would be nice to be able to use tropical techniques to solve the many open problems related to higher rank questions on curves but this does not seem possible at this time.

Acknowledgments: I would like to thank Peter Newstead for suggesting the problem dealt with in section 3 and to a careful referee for suggesting clarifications.
1. Vector bundles on reducible curves and limit linear series.

The goal of this paper is to show that on a generic curve of fixed genus, certain maps among spaces of sections of vector bundles have at least a certain rank. We will use degeneration methods. We review here the tools and techniques that we will need to deal with linear series on reducible, semistable, nodal curves.

Most of the definitions and results we recall in this section make sense for an arbitrary nodal curve of compact type (that is, one whose Jacobian is compact or equivalently, whose dual graph has no loops). We will only use them with a particular type of reducible curve, namely generic chains of elliptic curves of a fixed genus $g$, introduced by Welters in [W] and that we have been using systematically in the context of vector bundles of rank one and higher ([T4], [T5], [CT]).

**Definition 1.1** (generic chain of elliptic curves). A generic chain of elliptic curves is built as follows: Let $C_1, \ldots, C_M$ be curves that are either rational or elliptic, with precisely $g$ of them being elliptic, $P_i, Q_i$ generic points on $C_i$, for $i = 1, \ldots, M$. Glue $Q_i$ to $P_{i+1}$, $i = 1, \ldots, M - 1$ to form a nodal curve of genus $g$.

Consider a family of curves in which the generic curve is non-singular and the special curve is a chain of elliptic curves. Given a vector bundle together with a space of sections on the generic fiber, up to some base change and normalizations, we can complete it to a vector bundle on the family. If we start with a chain of elliptic curves as the central fiber, the new central fiber will still be a chain of elliptic curves. Tensoring the family with line bundles with support on the components of the central fiber, the vector bundle on the generic curve does not change but on the special curve it does. In particular, the limit on the special curve of a vector bundle in a family is not unique. In the case of rank one, this process allows to obtain an arbitrary distribution of degrees among the components of the reducible limit curve. This is the basis for the theory of limit linear series introduced by Eisenbud and Harris [EH1]. These authors took advantage of the possibility of concentrating all of the degree in one of the irreducible components of the central fiber. Then for every family $g_d^{k-1}$ (we use $k$ rather than $r + 1$ to avoid confusions with the rank) of linear series and for each components $C_i$ of the reducible limit curve, they obtained a line bundle $L_i$ of degree $d$ on $C_i$. When concentrating all the degree in the component $C_i$, the limit of the $k$-dimensional space of sections of the family is also concentrated on $C_i$. Therefore, there are distinguished $k$-dimensional spaces of sections $V_i$ of $L_i$ for each component $C_i$ of the central fiber. Compatibility among the data on the different components give rise to vanishing conditions of the sections of the linear series at the nodes:

**Definition 1.2.** [Limit Linear Series, rank one][EH1], Given a curve of compact type, a crude limit linear series of degree $d$ and dimension $k$ is the data of a $g_d^{k-1}$ on each irreducible component such that if $Y_1, Y_2$ are irreducible components meeting at a point $P$ and $a_1, \ldots, a_k$ are the orders of vanishing of the sections of these series at $P$ on $Y_i, i = 1, 2$, then $a_j + a_{k-j+1} \geq d$. The series is refined if these inequalities are equalities.

A similar approach can be taken for vector bundles, but we need to account for three differences

- For a line bundle and an $k$-dimensional space of its sections, the orders of vanishing of the sections at any given point are $k$ distinct integers $a_1 < \cdots < a_k$. For a vector bundle $E$ of rank $r$ and an $k$-dimensional space of its sections $V$, the orders of vanishing of the sections at a point do not normally take $k$ distinct values. But we can find $k$ integers $0 \leq a_1 \leq \cdots \leq a_k$ such that a particular value $\alpha$ appears $t$ times in the sequence with $t = \dim(V(-\alpha P)/V(- (\alpha + 1) P))$.

- If $E$ is a vector bundle of rank $r$ on an irreducible component $C_i$ and $P \in C_i$, then $\deg(E(- P)) = \deg(E) - r$. Modifying a vector bundle by tensoring with a line bundle with support on the central fiber changes the degrees of the restriction of the bundle to the components of the central fiber by multiples of $r$. We will not be able to assume that the limit on the central fiber of a vector bundle on the generic curve has degree 0 on all components of the central fiber except for one. But we can still assume that in the limit, all sections are concentrated in one component by taking sufficiently small degree (perhaps negative) on the remaining components.

- While a line bundle on a curve of compact type is completely determined by its restrictions to the components of the curve, vector bundles depend also on the gluing at the nodes.

Taking these issues into account, we consider the concept of Limit Linear Series for Vector Bundles (see for example [T6], [T1]).
Definition 1.3. [Limit Linear Series, arbitrary rank] A limit linear series of rank $r$, degree $d$ and dimension $k$ on a chain of elliptic curves consists of the following

(a) A vector bundle $E_i$ of rank $r$ and degree $d_i$ on each component $C_i$ and a $k$-dimensional space of sections $V_i$ of $E_i$.
(b) An isomorphism between the projectivization of the fibers $(E_i)_{Q_i}$ and $(E_{i+1})_{P_{i+1}}$, $i = 1, \ldots, M - 1$;
(c) Bases $s^t_{Q_i}$, $s^t_{P_{i+1}}$, $t = 1, \ldots, k$ of the vector spaces $V_i$ and $V_{i+1}$.
(d) A positive integer $a$.

These data satisfy the following conditions:

(1) $\sum_{i=1}^M d_i - r(M - 1)a = d$.
(2) The orders of vanishing at $Q_i$ and $P_{i+1}$ of the sections of the chosen bases satisfy $\text{ord}_{P_{i+1}} s^t_{P_{i+1}} + \text{ord}_{Q_i} s^t_{Q_i} \geq a$ for all $t$.
(3) Sections of the vector bundles $E_i(\alpha P_i)$ and $E_i(\alpha Q_i)$ are completely determined by their value at $P_i$ and $Q_i$, respectively.

This definition of Limit Linear Series amounts to, at each step, concentrating all of the degree and therefore all of the sections on one component of the central fiber. It has the advantage that one only needs to deal with one irreducible curve at a time but the disadvantage that all the sections must be dealt with simultaneously. This becomes difficult, if not impossible when trying to show their independence. To prove our results, we will need to spread out the degree among the different components. The theoretical framework for this alternative point of view was considered in rank one by Osserman [O] and can also be used for vector bundles [T7]. In fact, the data of a limit linear series as defined in 1.3 can be used to produce vector bundles and the corresponding spaces of sections of degree and sections spread out among the components of the reducible curve. As the distribution of degrees comes from tensoring the original vector bundle with line bundles with support on the fibers, the process changes the degree in multiples of $r$, so we need to keep the same remainder modulo $r$ for the degree on specific components. In fact, the restriction to a component will be modified by tensoring with a line bundle with support at the nodes. The sections from the limit linear series will give rise to sections of the modified vector bundle if they vanish at the nodes with the right order. This question was studied in [T7], [LT]. More specifically, we have:

Lemma 1.4. Given a chain of elliptic curves, a limit linear series corresponding to the data of vector bundles $E_i$ spaces of sections $V_i$ of $E_i$ and isomorphism of the projectivized fibers $(E_i)_{Q_i} \cong (E_{i+1})_{P_{i+1}}$ as in 1.3 and a distribution of degrees $d_i$ with $\sum d_i = d$ and $d_i$ congruent with $d_i$ modulo $r$, there exists a vector bundle on the whole curve whose restriction to the component $C_i$ has degree $d'_i$ and a vector space of sections of this vector bundle of dimension $k$.

Proof. Given a limit linear series on a chain of elliptic curves, condition (3) means that $d_i = ar + \tilde{d}_i$ where $0 \leq \tilde{d}_i < r$. Then,

$$d = \sum_{i=1}^M d_i - r(M - 1)a = \sum_{i=1}^M \tilde{d}_i + rMa - r(M - 1)a = ra + \sum_{i=1}^M \tilde{d}_i$$

As we assume $d'_i$ congruent with $d_i$ modulo $r$, we can write $d'_i = a_i r + \tilde{d}_i$. With this notation, the condition $\sum d'_i = d$ can be written as $\sum \tilde{d}_i + (\sum a_i)r = d$. Define now the vector bundle $E'_i$ and a space of its sections as follows

$$E'_i = E_i - (\sum_{j<i} a_j) \alpha P_i - (\sum_{j>i} a_j) \alpha Q_i), \quad V'_i = \{ s \in V_i | \text{ord}_{P_i} s \geq \sum_{j<i} a_j, \text{ord}_{Q_i} s \geq \sum_{j>i} a_j \}.$$ 

Take the gluing at the nodes induced from those given for the original linear series.

We now check that the degree of $E'_i$ is $d'_i$ as needed: From the definition of $E'_i$,

$$\deg(E'_i) = d_i - (\sum_{j \neq i} a_j) r = \tilde{d}_i + ar - (\sum_{j \neq i} a_j) r = d - \sum_{j \neq i} \tilde{d}_j - (\sum_{j \neq i} a_j) r = d - \sum_{j \neq i} d'_j = d'_i$$

as needed.
As the sections of the $V_i$ glued with each other, so do the sections of the newly defined $V'_i$, giving rise to sections of the new vector bundle. \qed

In each proof, our strategy to show independence of sections of a product will be to fix the distribution of degrees among the components. Then, from a limit linear series, we produce the sections of the new vector bundle with the prescribed degrees according to Lemma 1.4 and show that these are independent. If the distribution of degrees has been chosen judiciously, one can show that for a few or all the sections that are non-zero on the component, the coefficients in front of these sections in the linear combination are 0.

We need to build by hand vector bundles of rank $r$ and degree $d$ with certain number of sections. For reducible curves $X$, there exist moduli spaces of vector bundles of given rank and degree that are stable for a given polarization. These moduli spaces of vector bundles are themselves reducible with components corresponding to the distribution of degrees modulo $r$ among the components of $X$. Once the distribution of degrees has been fixed, the choice of a semistable bundle on each component of $X$ gives rise to a semistable vector bundle on $X$ which is stable so long as possible destabilizing subbundles do not glue with each other. For details, please see [T2], [T3]

In particular, we have the following

**Theorem 1.5.** Let $X = C_1 \cup \cdots \cup C_g$ be a chain of elliptic curves as in Definition 1.1, $r, d \in \mathbb{Z}$. Given semistable vector bundles on each component one of which is stable, gluing with arbitrary gluing at the nodes, one obtains a vector bundle on the chain stable by a suitable polarization. If all the vector bundles on the individual elliptic curves are strictly semistable, the resulting vector bundle is stable provided the destabilizing subbundles do not glue with each other.

The components of the moduli space of vector bundles of rank $r$ and degree $d$ on $X$ that are stable for a given polarization correspond with all the possible distributions of the degrees $d_i$ to $X_i$ modulo $r$. For a choice of such a component $M$ of the moduli space, and a component $C_i$ of $X$, if $h_i$ is the greatest common divisor of $r$ and the corresponding $d_i$, then the restriction to $C_i$ of a generic element $E \in M$ is a direct sum of $h_i$ indecomposable vector bundles of rank $\frac{r}{h_i}$ and degree $\frac{d}{h_i}$.

In particular, if the degree assigned to a given elliptic component of our curve is divisible by the rank, we need to consider direct sums of $r$ line bundles. So, let us start by looking at a single elliptic curve and the sections of a line bundle on it.

**Lemma 1.6.** Let $C$ be an elliptic curve, $P, Q \in C$ so that $P - Q$ is not a torsion element (in the group structure of $C$). Let $L$ be a line bundle of degree $d$ on $C$. For any $k, 0 \leq k \leq d - 1$, there exists up to a constant, a unique section $s_k$ of $L$ such that $s_k$ vanishes at $P$ with order at least $k$ and at $Q$ with order at least $d - k - 1$. The inequalities are in fact equalities unless $L = \mathcal{O}(kP + (d - k)Q)$ when $s_{k-1} = s_k$ or $L = \mathcal{O}((k + 1)P + (d - k - 1)Q)$ when $s_k = s_{k+1}$.

*Proof.* As $\deg(L(-kP - (d - k - 1)Q)) = 1$, from Riemann-Roch’s Theorem, $h^0(C, L(-kP - (d - k - 1)Q)) = 1$. Choose $s_k \in H^0(C, L(-kP - (d - k - 1)Q)), s_k \neq 0$. If $L \neq \mathcal{O}(kP + (d - k)Q), L \neq \mathcal{O}((k + 1)P + (d - k - 1)Q)$, then $h^0(C, L(-kP - (d - k)Q)) = 0, h^0(C, L(-(k + 1)P - (d - k - 1)Q)) = 0$. Therefore, unless we are in one of these two situations, $s_k$ vanishes at $P$ with order precisely $k$ and at $Q$ with order precisely $d - k - 1$.

If there were sections $s_{k_1}, s_{k_2}$ with orders of vanishing at $P, Q$ adding to $d$, then $L = \mathcal{O}(k_iP + (d - k_i)Q), i = 1, 2$. But

$$\mathcal{O}(k_1P + (d - k_1)Q) = \mathcal{O}(k_2P + (d - k_2)Q) \Rightarrow (k_2 - k_1)(P - Q) = 0$$

This contradicts our assumption that $P - Q$ is not a torsion point of the curve. \qed

**Corollary 1.7.** Let $C$ be an elliptic curve, $P, Q \in C$ so that $P - Q$ is not a torsion element, $L$ a line bundle of degree $d$ on $C$.

(a) If $L = \mathcal{O}(aP + (d - a)Q), 0 \leq a \leq u + t \leq d - 1$, there exists a uniquely determined $t$-dimensional space of sections of $L$ whose orders of vanishing at $P, Q$ are respectively $u, u+1, \ldots, a-3, a-2, a, a+1, a+2, \ldots, u+t$; $d-u-1, d-u-2, \ldots, d-a+2, d-a+1, d-a, d-a-2, d-a-3, \ldots, d-u-t-1$
(b) If \( L \neq \mathcal{O}(aP + (d - a)Q) \) for any \( a \) and \( 0 \leq u \leq u + t \leq d \), there exists a uniquely determined \( t \)-dimensional space of sections of \( L \) whose orders of vanishing at \( P, Q \) are respectively
\[
u, u + 1, \ldots, u + t - 1; \quad d - u - 1, d - u - 2, \ldots, d - u - t
\]

**Proof.** The sections \( s_k \) defined in Lemma 1.6 have different orders of vanishing at \( P \), therefore, they are linearly independent. If we take \( t \) of them, they span a \( t \)-dimensional space of sections of \( L \).

(a) If \( L = \mathcal{O}(aP + (d - a)Q) \), take the vector space spanned by \( s_u, s_{u+1}, \ldots, s_{a-2}, s_{a-1} = s_a, s_{a+1}, \ldots, s_{u+t} \). The vanishing conditions at the nodes are satisfied.

Conversely, assume that a space \( V \) of sections of dimension \( t \) of \( L \) has the given vanishing at the nodes. For \( u \leq x \leq a - 1 \), the subspace of \( V \) of sections vanishing to order at least \( x \) at \( P \) has dimension at least \( t - x + u \), while the subspace of those vanishing to order at least \( d - x - 1 \) at \( Q \) has dimension at least \( x - u + 1 \). As, by assumption, both subspaces live in a \( t \)-dimensional space, the two subspaces intersect. From Lemma 1.6, \( s_x \in V \).

Similarly, for \( a + 1 \leq x \leq u + t \), the subspace of sections vanishing to order at least \( x \) at \( P \) has dimension at least \( t - x + u + 1 \) while the subspace of sections vanishing to order at least \( d - x - 1 \) at \( Q \) has dimension at least \( x - u \), so again \( s_x \in V \). As the \( s_x \) are linearly independent, \( V \) is the span of \( s_u, s_{u+1}, \ldots, s_{a-2}, s_{a-1} = s_a, s_{a+1}, \ldots, s_{u+t} \).

(b) As in the proof of (a), the sections \( s_u, \ldots, s_{u+t-1} \) span the required subspace of \( H^0(C, L) \).

\( \square \)

**Lemma 1.8.** Let \( r, d \in \mathbb{Z} \), \( r > 0, d \geq 0, d = rd_1 + d_2, h = \gcd(r, d) \). Let \( C \) be an elliptic curve, \( P \in C \), \( E \) a vector bundle on \( C \) that can be written as direct sum of \( h \) indecomposable vector bundles of degree \( \frac{d}{h} \) and rank \( \frac{d}{h} \). Then, there exists a uniquely determined \( \rk_1 + d_2 \)-dimensional space of sections of \( E \) whose orders of vanishing at \( P \) are
\[
\begin{align*}
d_2 \times d_1, & \ldots, d_1, \\
r \times d_1 - 1, & \ldots, d_1 - 1, \\
r \times d_1 - 2, & \ldots, d_1 - 2, \ldots, \\
r \times d_1 - k_1 \cdot d_1 - k_1.
\end{align*}
\]

**Proof.** From Riemann-Roch’s Theorem, \( h^0(C, E(-\alpha P)) = \deg(E(-\alpha P)) = r(d_1 - \alpha) + d_2 \).

\( \square \)

In our applications, we will need to use the limit of the canonical linear series on a curve. For ease of reference, we recall here what it looks like on our chains:

**Proposition 1.9.** The canonical limit linear series on \( C_0 \) has line bundles on \( C_i \) equal to
\[
L_i = \mathcal{O}(2(i - 1)P_i + 2(g - i)Q_i)
\]

The space of sections on \( C_i \) is
\[
H^0(L_i(-(i - 2)P_i - (2g - 2i)Q_i)) \oplus H^0(L_i(-(2i - 1)P_i - (g - i - 1)Q_i)).
\]

The unique section whose order of vanishing at \( P_i \) and \( Q_i \) is \( 2g - 2 \) vanishes with order \( 2(i - 1) \) at \( P_i \) and \( 2g - 2i \) at \( Q_i \).

**Proof.** See [EH2] Th2.2 for a proof in a more general situation.

The result also follows from the one to one correspondence between limit linear series of degree \( d \) and projective dimension \( r \) on a general chain of elliptic curves and filling of Young Tableaux of dimensions \((r + 1) \times (g - d + r)\) with numbers among the 1, \ldots, \( g \) so that they are strictly increasing on each row and column (see [LT]). This description shows that in the case of \( W_{2g-2}^g \) there is a unique limit linear series and that the restriction of the line bundle to \( C_i \) is of the form \( \mathcal{O}(2(i - 1)P_i + 2(g - i)Q_i) \) and the space of sections on the component \( C_i \) has vanishing at \( P_i, Q_i \) respectively as
\[
\begin{align*}
i - 2 & \quad i - 1 & \quad \ldots & \quad 2i - 5 & \quad 2i - 4 & \quad 2i - 2 & \quad 2i - 1 & \quad \ldots & \quad g + i - 2 \\
2g - i - 1 & \quad 2g - i - 2 & \quad \ldots & \quad 2g - 2i + 2 & \quad 2g - 2i + 1 & \quad 2g - 2i & \quad 2g - 2i - 2 & \quad \ldots & \quad g - i - 1
\end{align*}
\]

\( \square \)
2. INJECTIVITY OF THE PETRI MAP IN HIGHER RANK

The Petri map for line bundles that we mentioned before $H^0(C, L) \otimes H^0(C, K \otimes L^{-1}) \to H^0(C, K)$, controls the geometry of the Brill Noether locus. The injectivity of the Petri map guarantees that $L$ gives rise to a non-singular point of the Brill Noether locus of $W^r_d$ for $d = \deg L, r = h^0(C, L) - 1$.

A similar Petri map can be defined in higher rank:

$$P : H^0(C, E) \otimes H^0(C, K \otimes E^*) \to H^0(C, K \otimes E \otimes E^*)$$

While for a generic curve $C$ and every line bundle $L$ on $C$, the Petri map is injective, it is no longer the case that even for the generic curve and every vector bundle, the higher rank Petri map is injective. On the other hand, the injectivity of the map for a particular vector bundles $E$ guarantees the existence of a component of the expected dimension of the Brill-Noether locus. Injectivity was proved in a fairly reduced number of cases (small slope or small number of sections) in [BP], [BGMMN1],[BGMMN2] and for rank two with additional conditions on degree, genus and number of sections in [CF]. A proof for the generic curve with fewer restrictive numerical conditions is provided in [CLT]. Here we give a different proof using the techniques of section 1.

We prove the following:

**Theorem 2.1.** Fix $r, d, k$. Write $d = rd_1 + d_2, k = rk_1 + k_2, 0 \leq d_2 < r, 0 \leq k_2 < r$. Assume that one of the following holds

- $d_2 \geq k_2$, $d_2 \neq 0, (k_1 + 1)(g + k_1 - d_1 - 1) \leq g - 1$
- $d_2 = k_2 = 0, k_1(g + k_1 - d_1 - 1) \leq g - 2$
- $d_2 < k_2, (k_1 + 1)(g + k_1 - d_1) \leq g - 1$

Then the generic curve has a component of expected dimension of the Brill-Noether locus of vector bundles of rank $r$ degree $d$ with $k$ sections for which the Petri map is generically injective.

**Proof.** The proof is similar in all cases but to simplify notations, we will assume

$$d_2 = k_2 \neq 0, (k_1 + 1)(g + k_1 - d_1 - 1) \leq g - 1.$$

We construct a limit series on a generic chain of elliptic curves.

(a) For $i \leq (k_1 + 1)(g + k_1 - d_1 - 1)$, write $i = (k_1 + 1)j_1 + j_2, 1 \leq j_2 \leq k_1 + 1$ (and therefore $0 \leq j_1 \leq g + k_1 - d_1 - 2$).

If $j_2 \neq k_1 + 1$, take the vector bundle on $C_i$ to be the direct sum

$$\mathcal{O}_{C_i}((i - j_1 + j_2 - 2)P_1 + (d_1 - i + j_1 - j_2 + 2)Q_1) \oplus r.$$

On the first $k_2$ line bundles $\mathcal{O}_{C_i}((i - j_1 + j_2 - 2)P_1 + (d_1 - i + j_1 - j_2 + 2)Q_1)$, from Corollary 1.7 (a), we can take a space of sections with vanishing orders at $P_i, Q_i$ respectively given as

$$i - j_1 - 1 \quad i - j_1 - 2 \quad \ldots \quad i - j_1 + j_2 - 2 \quad i - j_1 + j_2 - 1 \quad \ldots \quad i - j_1 + k_1 - 1 \quad d_1 - i + j_1 + 1 \quad d_1 - i + j_1 \quad \ldots \quad d_1 - i + j_1 - j_2 + 3 \quad d_1 - i + j_1 - j_2 + 2 \quad d_1 - i + j_1 - j_2 \quad \ldots \quad d_1 - i + j_1 - k_1 - 1$$

On the remaining $r - k_2$ line bundles, remove the last section.

(b) If $i = (k_1 + 1)j_1 + j_2 \leq (k_1 + 1)(g + k_1 - d_1 - 1), j_2 = k_1 + 1$, take the direct sum

$$\mathcal{O}_{C_i}((i + j_2 - j_1 - 2)P_1 + (d_1 + j_1 - j_2 - i + 2)Q_1) \oplus L_1 \oplus \cdots \oplus L_{r-k_2}$$

where the $L_j$ are generic line bundles of degree $d_1$. Take the space of sections as in case (a). Note that when $j_2 = k_1 + 1$ the last section is the one whose sum of vanishing at $P_i, Q_i$ is $d_1$. This section is omitted when the line bundle corresponds to an index larger than $k_2$. From Corollary 1.7 (b), this construction is possible.

(c) For $(k_1 + 1)(g + k_1 - d_1 - 1) < i \leq g - 1$, take the vector bundle on $C_i$ to be $L_1 \oplus \cdots \oplus L_r$ where the $L_j$ are generic line bundles of degree $d_1$. Write $\alpha = g + k_1 - d_1 - 1$. On the first $k_2$ line bundles, take a space of sections with vanishing orders at $P_i, Q_i$ respectively given as

$$i - \alpha - 1 \quad i - \alpha \quad \ldots \quad i - \alpha + k_1 - 1 \quad d_1 - i + \alpha \quad d_1 - i + \alpha - 1 \quad \ldots \quad d_1 - i + \alpha - k_1$$

on the remaining $r - k_2$ line bundles, remove the last section. Again, from Corollary 1.7 (b), this construction is possible.
(d) On $C_g$ take $E$ to be the direct sum of $h$ generic stable vector bundles of rank $\frac{1}{r}$ and degree $\frac{r}{h}$ where $h$ is the greatest common divisor of $d, r$. Take the space of sections $H^0(C_g, E(-(g - \alpha - 1)P_g))$ that is, the space of sections whose orders of vanishing at $P_g$ are

$$
gr - \alpha - 1, \cdots g - \alpha - 1, g - \alpha, \cdots, d_1 - 1, \cdots, d_1$$

Using that $\alpha = g + k_1 - d_1 - 1$, these vanishing can be written as

$$d_1 - k_1, \cdots, d_1 - k_1, d_1 - k_1, \cdots, d_1 - k_1 + 1, \cdots, d_1 - 1, \cdots, d_1 - 1, d_1, \cdots, d_1$$

From Lemma 1.8, this is possible.

The gluing is such that for $2 \leq i \leq (k_1 + 1)(g + k_1 - d_1 - 1)$, the line bundles in the decomposition glue with the corresponding line bundles in the decomposition on the previous curve. After this, the gluing is generic. At the last node, the subspace of sections of $E_g$ of dimension $d_2 = k_2$ with maximum vanishing at $P_g$ is glued with the subspace of sections chosen in $E_{g-1}$ with minimum vanishing at $Q_{g-1}$ but is otherwise generic. From Theorem 1.5, we obtain a stable vector bundle on the chain.

We construct in this way $k$ global sections of the vector bundle. For $i = (k_1 + 1)j_1 + j_2 \leq (k_1 + 1)(g + k_1 - d_1 - 1)$, the following sections on $C_i$ have order of vanishing on $P_i, Q_i$ adding up to $d_i$

$$s_{(j_2-1)r+1}, \cdots, s_{(j_2-1)r+r}, 1 \leq j_2 \leq k_1; \quad s_{(j_2-1)r+1}, \cdots, s_{(j_2-1)r+k_2}, j_2 = k_1 + 1$$

Recall that for the canonical limit linear series on a chain of elliptic curves, the line bundles on $C_i$ are (Proposition 1.9)

$$L_i = O(2(i - 1)P_i + 2(g - i)Q_i).$$

Therefore, the vector bundle on each of the elliptic components corresponding to $K \otimes E^*$ is well determined. There is then a limit linear series of rank $r$ degree $\tilde{d}$ and dimension $\tilde{k}$ with

$$\tilde{d} = r(2g - 2) - d = r(2(g - 1) + d_1 - 1) + r - d_2, \quad \tilde{k} = r(k_1 - d_1 + g - 1)$$

(a') If $j_2 \neq k_1 + 1$, the vector bundle on $C_i$ is

$$O_{C_i}((i + j_1 - j_2)P_i + (2g + j_2 - j_1 - i - 2)Q_i)^{\oplus r}.$$

(b') If $j_2 = k_1 + 1$, then $O_{C_i}((i + j_1 - j_2)P_i + (2g + j_2 - j_1 - i - 2)Q_i)^{\oplus k_2} \oplus \bar{L}_1 \oplus \cdots \oplus \bar{L}_{r-k_2}$.

(c') For $(k_1 + 1)(g + k_1 - d_1 - 1) < i \leq g - 1$, the vector bundle on $C_i$, $L_1 \oplus \cdots \oplus \bar{L}_r$.

(d') On $C_g$ we get a generic vector bundle of rank $r$ and degree $r(2g - 3 - d_1) + (r - d_2)$.

We construct in a similar way $\tilde{k}$ global sections of the vector bundle. The following sections on $C_i$ have order of vanishing on $P_i, Q_i$ adding up to $d_i$, for $i = (k_1 + 1)j_1 + j_2 \leq (k_1 + 1)(g + k_1 - d_1 - 1),$

$$\tilde{s}_{j_1r+1}, \cdots, \tilde{s}_{j_1r+r}, 1 \leq j_2 \leq k_1; \quad \tilde{s}_{j_1r+1}, \cdots, \tilde{s}_{j_1r+k_2}, j_2 = k_1 + 1$$

Consider now the limit linear series corresponding to the bundle $E \otimes K \otimes E^*$.

For $i = (k_1 + 1)j_1 + j_2, 0 \leq j_1 \leq g + k_1 - d_1 - 2, 1 \leq j_2 \leq k_1$, the aspect of the limit linear series on $C_i$ has vector bundle $(O_{C_i}(2(i - 1)P_i + 2(g - i)Q_i))^{r^d}$ while for $j_2 = k_1 + 1$, the bundle is $(O_{C_i}(2(i - 1)P_i + 2(g - i)Q_i))^{k_2} \oplus (\oplus_j L_j)$ with the $L_j$ line bundles of degree $2g - 2$. We have sections $s_t \tilde{s}_t, 1 \leq l \leq k, 1 \leq t \leq \tilde{k}$ of the limit linear series and our goal is to show that they are linearly independent.

Using Lemma 1.4, we will consider global sections of $E \otimes K \otimes E^*$ with degree $r$ on the first and last components and degree $2r$ on the intermediate ones. The potentially non-zero sections on component $C_i$ must vanish at $P_i$ with order at least $2i - 3, 2 \leq i \leq g$ and at $Q_i$ with order at least $2g - 2i - 1, 1 \leq i \leq g - 1$ (see the proof of Lemma 1.4).

By construction, on the curve $C_i, i = (k_1 + 1)j_1 + j_2, 0 \leq j_1 \leq g + k_1 - d_1 - 2$, the following sections vanish to order $2i - 2$ at $P_i$ and $2g - 2i$ at $Q_i$

$$1 \leq j_2 \leq k_1, s_t \tilde{s}_t, (j_2 - 1)r + 1 \leq t \leq j_2r, j_1r + 1 \leq l \leq (j_1r + 1)r,$$

$$j_2 = k_1 + 1, s_t \tilde{s}_t, k_1r + 1 \leq t \leq k_1r + k_2, j_1r + 1 \leq l \leq (j_1r + 1)r,$$
Any other section vanishes to order at most two less at either $P_i$ or $Q_i$. Consider then any possible linear dependence $\sum \lambda_t s_t \bar{s}_t = 0$. Restricting to the curve $C_i$ only the $s_t \bar{s}_t$ listed above are non-zero and these sections are sections of distinct line bundles in the direct sum. This forces $\lambda_t s_t = 0$, proving the independence of the $s_t \bar{s}_t$. Therefore the image of the Petri map has dimension at least $kk$. Then, the Petri map is injective for curves and linear series close to the one we constructed. 

\[ \square \]

3. Products of canonical and traceless endomorphisms

Assume that we work in characteristic zero. Let $E$ be a vector bundle. Then,

\[
E^* \otimes E \cong O \oplus Tr_0 E
\]

where $Tr_0 E$ is the set of traceless endomorphisms. The natural cup-product map

\[
\psi : H^0(C, K) \otimes H^0(C, K \otimes E^* \otimes E) \to H^0(C, K^2 \otimes E^* \otimes E)
\]

decomposes as direct sum of two maps

\[
\psi_1 : H^0(C, K) \otimes H^0(C, K) \to H^0(C, K^2), \quad \psi_2 : H^0(C, K) \otimes H^0(C, K \otimes Tr_0 E) \to H^0(C, K^2 \otimes Tr_0 E)
\]

Then $\psi$ is onto if and only if both $\psi_1, \psi_2$ are onto. The map $\psi_1$ is onto if and only if $C$ is not hyperelliptic. We prove here that $\psi_2$ is onto for the general curve and general vector bundle. It suffices to find a particular curve $X_0$ and a particular vector bundle $E_0$ on $X_0$ with $h^0(X_0, K \otimes E_0 \otimes E_0) = r^2(g - 1) + 1$ such that $\psi_2$ is onto for $X_0, E_0$.

Definition 3.1. Let $X_0 = C_1 \cup \cdots \cup C_g$ be a generic chain of $g$ elliptic curves as in Definition 1.1. Let $r, d$ be integers, with $1 \leq r, d \leq d < g + r$. Let $E_0$ be a vector bundle on $X_0$ that restricts to an indecomposable vector bundle of rank $r$ and degree 1 on $C_i, i = 1, \ldots, g - 1$, while the restriction to $C_g$ is the direct sum of $h = \text{gcd}(r, d - g + 1)$ generic indecomposable vector bundles of degree $\frac{d - g + 1}{h}$ and rank $\frac{r}{h}$. Take all the gluing at the nodes to be generic.

We show that $h^0(C_0, K \otimes E_0^* \otimes E_0) = r^2(g - 1) + 1$ or equivalently, $h^0(C_0, E_0^* \otimes E_0) = 1$:

Proposition 3.2. Let $X_0, E_0$ be as in Definition 3.1. Then $\text{Hom}(E_0, E_0) = E_0^* \otimes E_0$ is a direct sum of line bundles of degree 0 on each component, only one of which is trivial on $C_1, \ldots, C_{g - 1}$ while $h$ of them are trivial on $C_g$. The trivial line bundles glue with each other in $C_1, \ldots, C_{g - 1}$ and glue with one of the trivial line bundles on $C_g$ giving rise to a trivial line subbundle of $E_0^* \otimes E_0$ on $C_0$. The remaining gluing is generic. In particular, the limit linear series corresponding to $E_0^* \otimes E_0$ has a unique section.

Proof. We will use the additive and multiplicative structure of the moduli space of vector bundles on an elliptic curve (in characteristic zero) that was first described in [A]. Recall that if $\text{gcd}(r', d') = 1$, two indecomposable vector bundles of rank $r'$ and degree $d'$ differ in tensorization with a line bundle of degree 0. So, if $F$ is such an indecomposable vector bundle, any other vector bundle of rank $r'$ and degree $d'$ is of the form $F \otimes L$ for some line bundle of degree 0. Moreover, from the multiplicative structure for vector bundles on an elliptic curve (in characteristic zero)

\[
F^* \otimes F = \bigoplus M_j, \quad M_j \text{ line bundles of degree 0 of order dividing } r'.
\]

Therefore, on $C_i, i = 1, \ldots, g - 1$, the vector bundles

\[
E_i^* \otimes E_i = \bigoplus M_j^i, \quad M_j^i \text{ line bundles of degree 0 on } C_i \text{ of order dividing } r, \quad i = 1, \ldots, g - 1.
\]

Let $F$ be an indecomposable vector bundle of degree $d' = \frac{d - g + 1}{h}$ and rank $r'$ on $C_g$. Then, there exist $h$ generic line bundles of degree 0 such that $E_g = F \otimes L_1 \oplus \cdots \oplus F \otimes L_h$. We can then compute $E_g^* \otimes E_g$:

\[
E_g^* \otimes E_g = (F^* \otimes L_1^* \oplus \cdots \oplus F^* \otimes L_h^*) \otimes (F \otimes L_1 \oplus \cdots \oplus F \otimes L_h) = F^* \otimes F \otimes (\bigoplus_{i,j} L_i^* L_j) =
\]

\[
= \bigoplus_{i,j} M_{ij}^0 \otimes (\bigoplus_{i,j} L_i^* L_j), \quad M_{ij}^0 \text{ line bundles of degree 0 on } C_g \text{ of order dividing } r',
\]

Thus of $E_0^* \otimes E_0$ as $\text{Hom}(E_0, E_0)$, the identity morphism on $E_i, i = 1, \ldots, g - 1$ glues with the identity morphism on $E_{i+1}$. The identity morphism corresponds to the trivial subbundle of $E_i^* \otimes E_i, i = 1, \ldots, g - 1$ and
to the diagonal in \( O^h \) for \( i = g \). A non-trivial line bundle of degree 0 has no sections. Therefore, \( E^g_0 \otimes E_0 \) has only one limit linear section.

**Proposition 3.3.** With \( E_0 \) as in 3.1, there is a surjective product map of limit linear series

\[
H^0(C_0, K) \otimes H^0(C_0, K \otimes T_{0E_0}) \to H^0(C_0, K^2 \otimes T_{0E_0}).
\]

**Proof.** From Proposition 1.9, the canonical series on \( C_0 \) has line bundles \( L_i = O(2(i - 1)P_i + 2(g - i)Q_i) \) on \( C_i \) and limit sections \( s_1, \ldots, s_g \) with vanishing at \( P_i, Q_i \) given respectively as

\[
i - 2 \quad i - 1 \quad \ldots \quad 2i - 5 \quad 2i - 4 \quad 2i - 2 \quad 2i - 1 \quad \ldots \quad g + i - 2
2g - i - 1 \quad 2g - i - 2 \quad \ldots \quad 2g - 2i + 2 \quad 2g - 2i + 1 \quad 2g - 2i \quad 2g - 2i - 2 \quad \ldots \quad g - i - 1
\]

We showed in our proof of 3.2 that the bundle \( T_{0E_0} \) has restrictions to \( C_i, i = 1, \ldots, g - 1 \) of the form \( \oplus_{k=1}^{r-2} L_k \) where the \( L_k \) are line bundles of degree 0. Moreover, by the genericity of the chain of elliptic curves, \( P_i - Q_i \) is not a torsion point in the group structure of the elliptic curve. Therefore, none of the line bundles appearing in the decomposition of \( T_{0E_0} \) is of the form \( O(aP_i + bQ_i), a + b = 0 \) for any \( a, b \). On \( C_g \), we still have a direct sum of line bundles of degree 0 but there \( h - 1 \) of them are the trivial bundle while the rest are not of the form \( O(aP_i + bQ_i), a + b = 0 \) for any \( a, b \).

Therefore, the bundle \( K \otimes T_{0E_0} \) has restrictions to \( C_i, i = 1, \ldots, g - 1 \) of the form \( \oplus_{k=1}^{r-2} L_k \). Here the \( L_k \) are line bundles of degree \( 2g - 2 \) not of the form \( O(aP_i + bQ_i), a + b = 2g - 2 \). On \( C_g \), \( h - 1 \) of the line bundles are \( O((2g - 2)Q_i) \) while the rest are generic.

This allows us to find \( r^2 - 1 \) sections \( t^s_{1,j}, \ldots, t^s_{r-2-1,j} \) of the restriction of \( T_{0E_0} \) to \( C_i \) vanishing with order \( i + j - 2 \) at \( P_i \) and \( 2g - i - j - 1 \) at \( Q_i, j = 1, \ldots, g - 1 \) spanning the fibers of the bundle at both \( P_i, Q_i \) (see Lemma 1.6). On \( C_g \), for \( j = g - 1 \), \( h - 1 \) of the sections \( t^s_{1,g-1}, \ldots, t^s_{r-2-1,g-1} \) have vanishing order \( 2g - 2 \) instead of the minimal required of \( 2g - 3 \) at \( P_g \).

As \( t^s_{1,j}, \ldots, t^s_{r-2-1,j} \) span the fibers of \( K \otimes T_{0E_0} \) at \( P_i, Q_i \), we can assume that \( t^s_{i,j} \) glues with \( t^{i+1}_{k,j} \) in the identification of \( Q_i \) and \( P_{i+1} \).

We will show that in the cup product \( H^0(C_0, K) \otimes H^0(C_0, K \otimes T_{0E_0}) \to H^0(C_0, K^2 \otimes T_{0E_0}) \), the images of

\[
(1) \quad s_1 \otimes t_{k,1}, s_1 \otimes t_{k,2}, s_1 \otimes t_{k,3}, s_2 \otimes t_{k,2}, s_2 \otimes t_{k,3}, s_2 \otimes t_{k,4}, \ldots, s_g \otimes t_{k,g-3}, s_g \otimes t_{k,g-2}, s_g \otimes t_{k,g-1},
\]

are linearly independent.

Using Lemma 1.4, we will consider the distribution of degrees that assigns degree \( 3(r^2 - 1) \) to the components \( C_1, C_{g-2}, C_{g-1}, C_g \) and degree \( 4(r^2 - 1) \) to the remaining components. That is, we consider the vector bundles

\[
E_1(-(4g - 7)Q_1); \quad E_i(-(4i - 5)P_i -(4g - 4i - 3)Q_i), i = 2, \ldots, g - 3,
E_{g-2}(-(4g - 13)P_{g-2} - 6Q_{g-2}), \quad E_{g-1}(-(4g - 10)P_{g-1} - 3Q_{g-1}), \quad E_g(-(4g - 7)P_g)
\]

Assume that we had a linear combination of the chosen sections that equals 0.

On \( C_1 \), the only sections in (1) that vanish to order at most \( 4g - 7 \) at \( Q_1 \) are

\[
s_1 \otimes t_{k,1}, s_1 \otimes t_{k,2}, s_1 \otimes t_{k,3}, k = 1, \ldots, r^2 - 1
\]

vanishing to orders \( 4g - 5, 4g - 6, 4g - 7 \) respectively. The remaining sections, as sections from the series with adjusted degrees, are identically zero on \( C_1 \). Hence, the initial linear dependence condition among all the sections in (1) reduces on \( C_1 \) to a linear dependence condition among the \( s_1 \otimes t_{k,1}, s_1 \otimes t_{k,2}, s_1 \otimes t_{k,3} \) only. The orders of vanishing at \( P_1 \) for these three sets of sections are 0, 1, 2 respectively. The images of \( s_1 \otimes t_{k,j} \) for fixed \( j \) and varying \( k \) are linearly independent as they map onto the fiber of \( K \otimes E^* \otimes E \) at \( Q_1 \). Given that the three sets have different orders of vanishing at \( P_1 \) and that each set is linearly independent, \( s_1 \otimes t_{k,j} \) for fixed \( j \) and varying \( k \) have 0 coefficients on any zero linear combination of list (1).

Let us assume by induction that for the sections \( s_1 \otimes t_{k,1}, s_1 \otimes t_{k,2}, s_1 \otimes t_{k,3}, \ldots s_{l-1} \otimes t_{k,l-1}, s_{l-1} \otimes t_{k,l-1} \) the coefficients in the trivial linear combination are zero. Look now at the restriction to \( C_l \) of the sections in (1). Among the sections with potentially non-zero coefficients, the only ones that vanish at \( Q_l \) to order at least \( 4g - 4l - 3 \) are \( s_1 \otimes t_{k,l}, s_1 \otimes t_{k,l+1}, s_1 \otimes t_{k,l+2} \) that vanish to orders \( 4g - 4l - 1, 4g - 4l - 2, 4g - 4l - 3 \) respectively. The remaining
sections, as sections of the vector bundle with adjusted degrees vanish identically on \( C \). The orders of vanishing at \( P \) for each of these three sets of sections are different while the images on a stalk for a fixed \( j \) but varying the \( k \) are linearly independent. Therefore, the coefficients on the linear combination are zero.

A similar reasoning works on the last three components where only \( 2(r^2 - 1) \) of the sections in (1) are not identically zero and those have different orders of vanishing and map onto the fibers. □

4. Funding and/or Conflicts of interests/Competing interests.

The authors have no relevant financial or non-financial interests to disclose. The author received partial support from NSF Grant Account Number: 104301-0000

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