Construction of the surfaces with horizontal tangential planes at given points

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Abstract

The problem of construction of the surfaces with given sets of the points with horizontal tangential planes is considered. Such considerations are of interest in the problem of computer simulations of the waved ocean surfaces.

1 Statement of the problem

1.1 Physical statement

Various problems of optical/acoustical monitoring of wavy water surface are related with simulations of the surfaces with given properties [1–3]. The example of it is the problem of construction of two-dimension surface having horizontal target planes at the given points (Luchinin A.G., 2003). Variants of solutions of this problem are presented in this article.

1.2 Mathematical formulation

The main problem A. Let \( \mathcal{M} = \{ x_i, \ i = 0, N \} \) be the finite subset of the given open set \( \Omega \subset \mathbb{R}^n, n \geq 1 \). Construct a function \( F(x) \) of the given class \( \mathcal{F} \in C^1(\Omega) \) possessing zero gradient \( \nabla F(x) = 0 \) at all points of \( \mathcal{M} \).

We study also the next specifications of the main problem:

B. the values of the function at the points \( \mathcal{M} \) are given (\( F(x_i) = z_i, \ i = 0, N \)).
gradient of the function takes zero values only in \( \mathcal{M} \).

It is clear that sets of the solutions of \( A-C \) are infinite. In general, solutions of the main problem \( A \) do not solve problems \( B,C \). The main goal of the next research is to determine the existence of the solution of a problem in a given class of functions \( \mathcal{F} \).

2 Analytical approach

Let \( \mathcal{F} = C^k(\Omega), 1 \leq k \leq \infty \). For this class of functions we can easily construct an analytic solution of the main problem. One of possible constructions is described below. Let \( |\cdot| \) be Euclidean norm in \( \mathbb{R}^n \). Consider for every point \( x_i \in \mathcal{M} \) the open sphere

\[
S_i \equiv \{ x : |x - x_i| < r_i \} \subset \Omega
\]

with the radius \( r_i > 0 \) and center point \( x_i \). Let \( a_i > 1, c_i \) be arbitrary values, \( \varphi_i(\cdot) \in C^k(\mathbb{R}^n) \) be arbitrary functions such that \( \varphi_i(0) = 1, \nabla \varphi_i(0) = 0 \).

Let

\[
\chi_i(x) \equiv \begin{cases} a_i \frac{|x|^2}{|x|^2 - r_i^2}, & |x| < r_i, \\ 0, & |x| \geq r_i, \end{cases}
\]

\[
F(x) = \sum_{i=0}^{N} c_i \chi_i(x - x_i) \varphi_i(x - x_i).
\] (1)

It easy to prove the following statements.

**Proposition 1.** For every set of sufficiently small values \( r_i \) (such that \( x_i \notin S_j, j \neq i \)) and for any values \( c_i \) the function (1) is a solution of the problem \( A \). In addition, if

1. \( c_i = z_i, i = 0,N \), then (1) gives a solution of the problem \( B \);

2. all spheres \( S_j \) do not intersect, then (1) gives a solution of the problem \( C \).

**Proposition 2.** Let \( n = 1, x_0 < x_1 < \ldots < x_N \). Then under the conditions

\[
r_i \leq |x_i - x_{i-1}|, c_{i-1}c_i < 0, i = 1,N,
\]
The function (1) gives a solution of the problem $C$. The peculiarity of the described construction is the provided by multipliers $\chi_i(x)$ finiteness of the terms of the sum (1); this provides zero values of gradient at the points of $\mathfrak{M}$. Note that using the corresponding values of gradients $\nabla \varphi_i(0)$ we can assume these gradients not only zero but also any given, so we can use the construction (1) for solving of more complicated problems to construct of surfaces with given slopes at the given points.

From physical point of view the solutions (1) can simulate only oscillation of too viscous liquid. An example of the numerical simulation by means of formula (1) for $a_i = e$, $\varphi_i(x) \equiv 1$, 10 random points $\mathfrak{M}$ and random coefficients $c_i$, is presented on Fig. 1.

3 Spline approach

In this section we use splines of various types to solve problems A-C for $n = 1, 2$. Polynomial splines of the 4-th order (see sec. 3.1.1, 3.1.2) make it possible to solve problems A and B. As for the problem C then in the case

Figure 1: The surface (1) for the set of 10 random points marked on the graph by stars.
\( n = 2 \) spline approach is useless if only the set \( \mathcal{M} \) does not have the form of the product
\[
\mathcal{M} = X \times Y, \quad \begin{cases} X = \{x_k\}, \ k = 0, K, \\ Y = \{y_l\}, \ l = 0, L. \end{cases}
\] (2)

But in the case \( n = 1 \) the solution of the problem \( \mathcal{C} \) can be found in the class of trigonometric splines (sec. 3.2.1).

### 3.1 Polynomial splines

#### 3.1.1 One-dimensional problem \((n = 1)\)

Consider the following problem \( \mathcal{B} \): for given sets \( a = x_0 < x_1 < \ldots < x_N = b \) and \( \{z_i, i = 0, N, \} \) build a spline of the 4-th order \( F(\cdot) \in C^1[a,b] \) such that

\[
\begin{align*}
F(x) &= s_i(x), \ x \in [x_{i-1}, x_i], \ i = 1, N, \\
s_i(x) &= a_i + b_i(x-x_i) + c_i(x-x_i)^2/2! + d_i(x-x_i)^3/3! + e_i(x-x_i)^4/4!, \\
F(x_{i-1}) &= z_{i-1}, \ F(x_i) = z_i, \\
F'(x_{i-1}) &= 0, \ F'(x_i) = 0. \\
\end{align*}
\] (3)

Now we prove that such splines exist. We have for the coefficients of the spline

\[
\begin{align*}
F(x_i) &= a_i = z_i, \\
F'(x_i) &= b_i = 0, \\
F(x_{i-1}) &= a_i + b_i \Delta x_i + c_i \Delta x_i^2/2 + d_i \Delta x_i^3/6 + e_i \Delta x_i^4/24 = z_{i-1}, \\
F'(x_{i-1}) &= b_i + c_i \Delta x_i + d_i \Delta x_i^2/2 + e_i \Delta x_i^3/6 = 0, \\
\Delta x_i &\equiv x_{i-1} - x_i, \ i = 1, N. \\
\end{align*}
\]

So we obtain the equations

\[
\begin{align*}
c_i &= -d_i \Delta x_i/2 - e_i \Delta x_i^2/6, \\
-d_i \Delta x_i^3/12 - e_i \Delta x_i^4/24 &= \Delta z_i \equiv z_{i-1} - z_i, \\
d_i + e_i \Delta x_i/3 &= -2c_i/\Delta x_i, \\
d_i + e_i \Delta x_i/2 &= -12\Delta z_i/\Delta x_i^3, \\
\end{align*}
\]

and we can evaluate all coefficient by \( c_i \),

\[
\begin{align*}
a_i &= z_i, \\
b_i &= 0, \\
c_i &= c_i, \\
d_i &= -6 \left( c_i - 4\Delta z_i \Delta x_i^{-2} \right)/\Delta x_i, \\
e_i &= 12 \left( c_i - 6\Delta z_i \Delta x_i^{-2} \right)/\Delta x_i^2.
\end{align*}
\] (4)
Spline and its derivative are continuous both. In general, for arbitrary coefficients \( c_i \) we obtain smooth splines with discontinuous second derivative. The examples of such splines are presented on Fig. 2.

If we do not fix parameters \( z_i \) in (4) then splines solve only the problem A (see Fig. 3).

**\( C^2 \)-spline** Let us now construct a spline with the continuous second derivative. For realization of it we must require

\[
s''_{i-1}(x_{i-1}) = s''_i(x_{i-1}), \ i = 1, N,
\]

that tends to the equations

\[
c_{i-1} = c_i + d_i \Delta x_i + \frac{\epsilon_i}{2} \Delta x_i^2 = c_i - 12 \Delta z_i \Delta x_i^{-2}.
\]

This provides the recurrence

\[
c_i = c_{i-1} + 12 \Delta z_i \Delta x_i^{-2}, \quad (5)
\]
Figure 3: Examples of polynomial splines of the fourth order from $C^2 [a, b]$ solving the problem A.

so

$$c_i = c_0 + 12 \sum_{j=1}^{i} \Delta z_j \Delta x_j^{-2}, \ i = 1, N.$$

Hence we obtain a one-parameter aggregate of splines of the class $C^2 [a, b]$. The parameter of it is $c_0$, the second derivative of the spline at the initial point $a$. By choosing of $c_0$, we can construct splines with different properties. For example, when

$$c_0 = -12 \min_{i} \sum_{j=1}^{i} \Delta z_j \Delta x_j^{-2},$$

all points of $\mathcal{M}$ are points of local minima (see Fig. 4 a), for

$$c_0 = -12 \max_{i} \sum_{j=1}^{i} \Delta z_j \Delta x_j^{-2},$$

all points $\mathcal{M}$ are points of local maxima (see Fig. 4 b), and for

$$c_0 = -\frac{12}{n} \sum_{j=1}^{i} \Delta z_j \Delta x_j^{-2}.$$
Figure 4: Polynomial splines of the forth order with additional properties: a) all points of $\mathcal{M}$ are points of local minima, b) all points of $\mathcal{M}$ are points of local maxima, c) the mean value of the second derivative for over all points of $\mathcal{M}$ takes its minimum.

the mean value of the second derivative for over all points of $\mathcal{M}$ takes its minimum (see Fig. 4, c)

3.1.2 Two-dimensional problem ($n = 2$)

Consider the following two-dimensional problem $\mathbf{B}$: for given set of points $(x_k, y_l, z_{kl})$, $a = x_0 < x_1 < \ldots < x_K = b$, $c = y_0 < y_1 < \ldots < y_L = d$ build a spline of the 4-th order $F(\cdot) \in C^1([a, b] \times [c, d])$ such that

$$
\begin{align*}
F(x, y) &= s_{kl}(x, y), \quad (x, y) \in \mathbf{p}_{kl} \equiv [x_{k-1}, x_k] \times [y_{l-1}, y_l] \quad k = 1, K, \quad l = 1, L, \\
F(x_k, y_l) &= z_{kl}, \\
\nabla F(x_k, y_l) &= 0.
\end{align*}
$$

(6)
By decomposition of the function into one-dimensional splines (3) onto grid lines \((x_k, y_l), k = 1, K, l = 1, L\), we receive the following equations for the coefficients of the two-dimensional spline,

\[
\sum_{i=0}^{4} \left( \sum_{j=0}^{4} a_{ij}^{(kl)} \Delta y_j^l \right) (x - x_k)^i = \sum_{i=0}^{4} \gamma_i^{(k_l-1)} (x - x_k)^i, \quad y = y_l-1, \\
\sum_{i=0}^{4} a_{i0}^{(kl)} (x - x_k)^i = \sum_{i=0}^{4} \gamma_i^{(kl)} (x - x_k)^i, \quad y = y_l, \\
\sum_{j=0}^{4} \left( \sum_{i=0}^{4} a_{ij}^{(kl)} \Delta x_k^i \right) (y - y_l)^j = \sum_{j=0}^{4} \beta_j^{(k_l-1)} (y - y_l)^j, \quad x = x_{k-1}, \\
\sum_{j=0}^{4} a_{0j}^{(kl)} (y - y_l)^j = \sum_{j=0}^{4} \beta_j^{(kl)} (y - y_l)^j, \quad x = x_k,
\]

where we know values \(\gamma_i^{(kl)}, \beta_j^{(kl)}\) from formulae (4); so we know the first column and the first row of the matrix \(A^{(kl)} \equiv \{a_{ij}^{(kl)}\}_{i,j=0}^{4}\). Last 16 coefficients

\[
A^{(kl)} = \begin{pmatrix}
  z_{kl} & 0 & \beta_2^{(kl)} & \beta_3^{(kl)} & \beta_4^{(kl)} \\
  0 & a_{11}^{(kl)} & a_{12}^{(kl)} & a_{13}^{(kl)} & a_{14}^{(kl)} \\
  \gamma_2^{(kl)} & a_{21}^{(kl)} & a_{22}^{(kl)} & a_{23}^{(kl)} & a_{24}^{(kl)} \\
  \gamma_3^{(kl)} & a_{31}^{(kl)} & a_{32}^{(kl)} & a_{33}^{(kl)} & a_{34}^{(kl)} \\
  \gamma_4^{(kl)} & a_{41}^{(kl)} & a_{42}^{(kl)} & a_{43}^{(kl)} & a_{44}^{(kl)}
\end{pmatrix}
\]

are unknown. For them we have from (7) the equations

\[
\begin{align*}
\sum_{j=1}^{4} a_{ij}^{(kl)} \Delta y_j^l = \gamma_i^{(k_l-1)} - \gamma_i^{(kl)} \equiv \Delta \gamma_i^{(kl)}, \quad i = 1,4, \\
\sum_{i=1}^{4} a_{ij}^{(kl)} \Delta x_k^i = \beta_j^{(k_l-1)} - \beta_j^{(kl)} \equiv \Delta \beta_j^{(kl)}, \quad j = 1,4.
\end{align*}
\]
The matrix of this system is

\[
D = \begin{bmatrix}
\Delta y_l & 0 & 0 & 0 & \Delta x_k & 0 & 0 & 0 \\
\Delta y_l^2 & 0 & 0 & 0 & 0 & \Delta x_k & 0 & 0 \\
\Delta y_l^3 & 0 & 0 & 0 & 0 & 0 & \Delta x_k & 0 \\
\Delta y_l^4 & 0 & 0 & 0 & 0 & 0 & 0 & \Delta x_k \\
0 & \Delta y_l & 0 & 0 & \Delta x_k^2 & 0 & 0 & 0 \\
0 & \Delta y_l^2 & 0 & 0 & 0 & \Delta x_k^2 & 0 & 0 \\
0 & \Delta y_l^3 & 0 & 0 & 0 & 0 & \Delta x_k^2 & 0 \\
0 & \Delta y_l^4 & 0 & 0 & 0 & 0 & 0 & \Delta x_k^2 \\
0 & 0 & \Delta y_l & 0 & \Delta x_k^3 & 0 & 0 & 0 \\
0 & 0 & \Delta y_l^2 & 0 & 0 & \Delta x_k^3 & 0 & 0 \\
0 & 0 & \Delta y_l^3 & 0 & 0 & 0 & \Delta x_k^3 & 0 \\
0 & 0 & \Delta y_l^4 & 0 & 0 & 0 & 0 & \Delta x_k^3 \\
0 & 0 & 0 & \Delta y_l & \Delta x_k^4 & 0 & 0 & 0 \\
0 & 0 & 0 & y^2 & 0 & x^4 & 0 & 0 \\
0 & 0 & 0 & y^3 & 0 & 0 & x^4 & 0 \\
0 & 0 & 0 & y^4 & 0 & 0 & 0 & x^4
\end{bmatrix},
\]

and its rank is equal 7. If we choose

\[
a_{11}^{(kl)}, a_{12}^{(kl)}, a_{13}^{(kl)}, a_{14}^{(kl)}, a_{21}^{(kl)}, a_{31}^{(kl)}, a_{41}^{(kl)},
\]

as known values (they correspond to nonzero minor of the order 7) and choose other 9 elements as free, then we obtain from (8) the system

\[
\begin{align*}
a_{11}^{(kl)} \Delta y_l & + a_{12}^{(kl)} \Delta y_l^2 + a_{13}^{(kl)} \Delta y_l^3 + a_{14}^{(kl)} \Delta y_l^4 = \Delta \gamma_1^{(kl)} = 0, \\
a_{21}^{(kl)} \Delta y_l & = \Delta \gamma_2^{(kl)} - a_{22}^{(kl)} \Delta y_l^2 + a_{23}^{(kl)} \Delta y_l^3 + a_{24}^{(kl)} \Delta y_l^4, \\
a_{31}^{(kl)} \Delta y_l & = \Delta \gamma_3^{(kl)} - a_{32}^{(kl)} \Delta y_l^2 + a_{33}^{(kl)} \Delta y_l^3 + a_{34}^{(kl)} \Delta y_l^4, \\
a_{41}^{(kl)} \Delta y_l & = \Delta \gamma_4^{(kl)} - a_{42}^{(kl)} \Delta y_l^2 + a_{43}^{(kl)} \Delta y_l^3 + a_{44}^{(kl)} \Delta y_l^4, \\
a_{11}^{(kl)} \Delta x_k & + a_{21}^{(kl)} \Delta x_k^2 + a_{31}^{(kl)} \Delta x_k^3 + a_{41}^{(kl)} \Delta x_k^4 = \Delta \beta_1^{(kl)} = 0, \\
a_{12}^{(kl)} \Delta x_k & = \Delta \beta_2^{(kl)} - a_{22}^{(kl)} \Delta x_k^2 - a_{23}^{(kl)} \Delta x_k^3 - a_{24}^{(kl)} \Delta x_k^4, \\
a_{13}^{(kl)} \Delta x_k & = \Delta \beta_3^{(kl)} - a_{23}^{(kl)} \Delta x_k^2 - a_{33}^{(kl)} \Delta x_k^3 - a_{34}^{(kl)} \Delta x_k^4,
\end{align*}
\]
with the solution

\[
\begin{aligned}
& a_{13}^{(kl)} = \Delta \beta_3^{(kl)} \Delta x_{k-1} - a_{23}^{(kl)} \Delta x_k - a_{33}^{(kl)} \Delta x_k^2 - a_{43}^{(kl)} \Delta x_k^3, \\
& a_{12}^{(kl)} = \Delta \beta_2^{(kl)} \Delta x_{k-1} - a_{22}^{(kl)} \Delta x_k - a_{32}^{(kl)} \Delta x_k^2 - a_{42}^{(kl)} \Delta x_k^3, \\
& a_{11}^{(kl)} = \Delta \gamma_1^{(kl)} \Delta y_{l-1} - a_{21}^{(kl)} \Delta y_l - a_{31}^{(kl)} \Delta y_l^2 - a_{41}^{(kl)} \Delta y_l^3, \\
& a_{10}^{(kl)} = \Delta \gamma_0^{(kl)} \Delta y_{l-1} - a_{20}^{(kl)} \Delta y_l - a_{30}^{(kl)} \Delta y_l^2 - a_{40}^{(kl)} \Delta y_l^3, \\
& a_{43}^{(kl)} = \Delta \beta_3^{(kl)} \Delta x_{k-1} - a_{33}^{(kl)} \Delta x_k - a_{43}^{(kl)} \Delta x_k^2 - a_{53}^{(kl)} \Delta x_k^3, \\
& a_{42}^{(kl)} = \Delta \beta_2^{(kl)} \Delta x_{k-1} - a_{32}^{(kl)} \Delta x_k - a_{42}^{(kl)} \Delta x_k^2 - a_{52}^{(kl)} \Delta x_k^3, \\
& a_{41}^{(kl)} = \Delta \gamma_1^{(kl)} \Delta y_{l-1} - a_{31}^{(kl)} \Delta y_l - a_{41}^{(kl)} \Delta y_l^2 - a_{51}^{(kl)} \Delta y_l^3, \\
& a_{40}^{(kl)} = \Delta \gamma_0^{(kl)} \Delta y_{l-1} - a_{30}^{(kl)} \Delta y_l - a_{40}^{(kl)} \Delta y_l^2 - a_{50}^{(kl)} \Delta y_l^3.
\end{aligned}
\]

(10)

\[C^0\text{-spline} \ \text{Let us set, for example, in (10)} \]

\[a_{ij}^{(kl)} = 0, \ i = \frac{1}{2}, \frac{3}{4}, \ j = \frac{1}{2}, \frac{3}{4}.\]

Then we obtain from (10)

\[
\begin{aligned}
& a_{13}^{(kl)} = \Delta \beta_3^{(kl)} \Delta x_k, \\
& a_{12}^{(kl)} = \Delta \beta_2^{(kl)} \Delta x_k, \\
& a_{11}^{(kl)} = \Delta \gamma_1^{(kl)} \Delta y_l, \\
& a_{10}^{(kl)} = \Delta \gamma_0^{(kl)} \Delta y_l, \\
& a_{21}^{(kl)} = \Delta \gamma_2^{(kl)} \Delta y_l, \\
& a_{20}^{(kl)} = \Delta \gamma_1^{(kl)} \Delta y_l, \\
& a_{21}^{(kl)} = \Delta \gamma_2^{(kl)} \Delta y_l, \\
& a_{20}^{(kl)} = \Delta \gamma_1^{(kl)} \Delta y_l.
\end{aligned}
\]

(11)

The spline with such coefficients is continuous but, in general, its first derivatives have jumps on the boundaries of the cells \(P_{kl}\). The example of such spline see on Fig. [5]

\[C^1\text{-spline} \ \text{Let us try to choose such free parameters that spline (8) becomes } C^1\text{-function. It is sufficiently to require that the first derivative } \partial F/\partial x \text{ is continuous at all vertical lines } x = x_k \text{ and the } \partial F/\partial y \text{ is continuous at horizontal lines } y = y_l \text{ of the grid. This tends to the equations}
\]

\[
\begin{aligned}
a_{ij}^{(kl)} &= \sum_{i=1}^{4} a_{ij}^{(k+1,l)} \Delta x_{k+1}^{i-1}, \\
a_{i1}^{(kl)} &= \sum_{j=1}^{4} a_{ij}^{(k,l+1)} \Delta y_{l+1}^{j-1},
\end{aligned}
\]

(12)

\[j = \frac{1}{4}, \ i = \frac{1}{2}, \frac{3}{4}, \ k = \frac{1}{4}, \frac{3}{4}, \ l = \frac{1}{4}, \frac{3}{4}.\]

10
Figure 5: The example of polynomial $C^0$-spline with the coefficients (11). The points of $\mathcal{M}$ are marked by stars, $K = L = 10$.

We see that the second row and the second column of the matrix are completely determined by the coefficients of the matrix from contiguous cells (that lie to right and above from a given cell). On the other hand due to the continuity of the spline and the equations (10) these elements can be found by nine coefficients of the matrix

$$
\begin{bmatrix}
  a_{22}^{(kl)} & a_{23}^{(kl)} & a_{24}^{(kl)} \\
  a_{32}^{(kl)} & a_{33}^{(kl)} & a_{34}^{(kl)} \\
  a_{42}^{(kl)} & a_{43}^{(kl)} & a_{44}^{(kl)}
\end{bmatrix}
$$

(13)

From the first and the fifth equations of the (9) and from the (12) we receive

$$
\sum_{j=1}^{4} a_{ij}^{(kl)} \Delta y_j^{i-1} = \sum_{j=1}^{4} \sum_{i=1}^{4} i a_{ij}^{(k+1 \ell)} \Delta x_{k+1}^{i-1} \Delta y_j^{i-1} = 0,
$$

$$
\sum_{i=1}^{4} a_{i1}^{(kl)} \Delta x_k^{i-1} = \sum_{i=1}^{4} \sum_{j=1}^{4} j a_{ij}^{(k \ell+1)} \Delta y_{i+1}^{j-1} \Delta x_k^{i-1} = 0.
$$
So we obtain for $a_{ij}^{(kl)}$

$$
\begin{align*}
\sum_{i=1}^{4} \sum_{j=1}^{4} j a_{ij}^{(kl)} \Delta y_i^{j-1} \Delta x_k^{i-1} &= 0, \\
\sum_{j=1}^{4} \sum_{i=1}^{4} i a_{ij}^{(kl)} \Delta x_k^{i-1} \Delta y_i^{j-1} &= 0.
\end{align*}
$$

(14)

Five additional equations we obtain from the system (9),

$$
\begin{align*}
\sum_{i=1}^{4} & a_{2i}^{(kl)} \Delta x_k + a_{42}^{(kl)} \Delta x_k^2 = \Delta \beta_2^{(kl)} \Delta x_k^{-2} - a_{12}^{(kl)} \Delta x_k^{-1}, \\
\sum_{i=1}^{4} & a_{3i}^{(kl)} \Delta x_k + a_{43}^{(kl)} \Delta x_k^2 = \Delta \beta_3^{(kl)} \Delta x_k^{-2} - a_{13}^{(kl)} \Delta x_k^{-1}, \\
\sum_{i=1}^{4} & a_{2i}^{(kl)} \Delta y_i + a_{24}^{(kl)} \Delta y_i^2 = \Delta \gamma_2^{(kl)} \Delta y_i^{-2} - a_{21}^{(kl)} \Delta y_i^{-1}, \\
\sum_{i=1}^{4} & a_{3i}^{(kl)} \Delta y_i + a_{34}^{(kl)} \Delta y_i^2 = \Delta \gamma_3^{(kl)} \Delta y_i^{-2} - a_{31}^{(kl)} \Delta y_i^{-1}, \\
\sum_{i=1}^{4} & a_{4i}^{(kl)} \Delta y_i + a_{44}^{(kl)} \Delta y_i^2 = \Delta \gamma_4^{(kl)} \Delta y_i^{-2} - a_{41}^{(kl)} \Delta y_i^{-1}.
\end{align*}
$$

(15)

From systems (15) and (14) we receive the system of seven equations for nine coefficients (13)

$$
\begin{align*}
\sum_{i=2}^{4} \sum_{j=2}^{4} j a_{ij}^{(kl)} \Delta y_i^{j-1} \Delta x_k^{i-1} &= - \sum_{i=2}^{4} j a_{ij}^{(kl)} \Delta y_i^{j-1} - \sum_{i=2}^{4} a_{i1}^{(kl)} \Delta x_k^{i-1}, \\
\sum_{j=2}^{4} \sum_{i=2}^{4} i a_{ij}^{(kl)} \Delta y_i^{j-1} \Delta x_k^{i-1} &= - \sum_{j=1}^{4} a_{ij}^{(kl)} \Delta y_j^{j-1} - \sum_{i=2}^{4} a_{i1}^{(kl)} \Delta x_k^{i-1}.
\end{align*}
$$

(16)
However the matrix of this system

$$
\begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 2\Delta y_l \Delta x_k & 2\Delta x_k \Delta y_l \\
\Delta x_k & 0 & 0 & 1 & 0 & 2\Delta y_l \Delta x_k^2 & 3\Delta x_k^2 \Delta y_l \\
\Delta x_k^2 & 0 & 0 & 0 & 1 & 2\Delta y_l \Delta x_k^3 & 4\Delta x_k^3 \Delta y_l \\
0 & 1 & \Delta y_l & 0 & 0 & 3\Delta y_l^2 \Delta x_k & 2\Delta x_k \Delta y_l^2 \\
0 & \Delta x_k & 0 & \Delta y_l & 0 & 3\Delta y_l^2 \Delta x_k^2 & 3\Delta x_k^2 \Delta y_l^2 \\
0 & \Delta x_k^2 & 0 & 0 & \Delta y_l & 3\Delta y_l^2 \Delta x_k^3 & 4\Delta x_k^3 \Delta y_l^2 \\
0 & 0 & \Delta y_l^2 & 0 & 0 & 4\Delta y_l^3 \Delta x_k & 2\Delta x_k \Delta y_l^3 \\
0 & 0 & 0 & \Delta y_l^2 & 0 & 4\Delta y_l^3 \Delta x_k^2 & 3\Delta x_k^2 \Delta y_l^3 \\
0 & 0 & 0 & 0 & \Delta y_l^2 & 4\Delta y_l^3 \Delta x_k^3 & 4\Delta x_k^3 \Delta y_l^3 \\
\end{pmatrix}
$$

has the rank 5. This means that system (16) has no solution. Hence two-dimensional problem $B$ has no solution in the class of $C^1$-splines.

Besides the problem of smoothness the next problem is the appearance of false stationary points that do not belong to $\mathcal{M}$. The above considerations show that class of polynomial splines is not convenient for solutions of the problem $B,C$.

### 3.2 Trigonometric splines

#### 3.2.1 One-dimensional problem

Consider the following problem $B$: for given sets $(x_i, z_i)$, $a = x_0 < x_1 < \ldots < x_N = b$ build a trigonometric spline $F(\cdot) \in C^1[a, b]$ such that

$$
\begin{align*}
F(x) &= t_i(x), \quad x \in [x_{i-1}, x_i], \quad i = 1, N, \\
t_i(x) &\equiv a_i + b_i \cos (c_i x + d_i), \\
F(x_i) &= z_i, \\
F'(x_i) &= 0.
\end{align*}
$$

Conditions (17) do not identify the spline. For example, we can take the following coefficients

$$
\begin{align*}
a_i &= (z_{i-1} + z_i) / 2, \\
b_i &= (z_{i-1} - z_i) / 2, \\
c_i &= \pi / \Delta x_i, \\
d_i &= -\pi x_{i-1} / \Delta x_i, \\
\Delta x_i &\equiv x_i - x_{i-1}.
\end{align*}
$$
The corresponding spline has the form

\[ t_i(x) = \frac{z_{i-1} + z_i}{2} + \frac{z_{i-1} - z_i}{2} \cos \frac{\pi}{2} \frac{x - x_{i-1}}{\Delta x_i}. \]

For such coefficients false stationary points are absent; this means that really
the spline solves the problem \( \textbf{C} \).

In general, for arbitrary values \( z_i \) the spline has jumps at points \( x_i \). If to
forbid the jumps,

\[ t''_{i-1}(x_{i-1} + 0) = -\pi^2 (z_{i-2} - z_{i-1}) \frac{\Delta x_{i-2}}{2} = \]

\[ = \pi^2 (z_{i-1} - z_i) \frac{\Delta x_{i-2}}{2} = t''_i(x_{i-1} - 0), \]

then we receive the relations

\[ \frac{z_i - z_{i-1}}{\Delta x_i^2} = \mu (-1)^i, \quad i = 1, n, \]

where \( \mu \) is an arbitrary value. Such kind 2-splines form two-parametrical set
as for fixed value \( z_0 \) other values \( z_i \) can be calculated by the formula

\[ z_i = z_0 + \mu \sum_{l=1}^{i} (-1)^l \Delta x_i^2 \quad (18) \]

which contains two free parameters \( z_0 \) and \( \mu \). For fixed set \( \{x_i\} \) the parameter
\( \mu \) evaluates mean deviation of the spline from the initial value \( z_0 \):

\[ \sum_{i=1}^{n} (z_i - z_0) / n = \mu \sum_{l=1}^{n} (-1)^l (1 - l/n) \Delta x_i^2. \]

Examples of trigonometric splines (17) are presented on Fig. 6.

### 3.2.2 Two-dimensional problem

Consider the following problem \( \textbf{B} \): for given sets \( (x_k, y_l, z_{kl}) \), \( a = x_0 < x_1 < \ldots < x_K = b \), \( c = y_0 < y_1 < \ldots < y_L = d \) build a spline \( F(\cdot) \in C^1([a, b] \times [c, d]) \) such that

\[
\left\{ \begin{array}{l}
F(x, y) = t_{kl}(x, y), \quad (x, y) \in \mathbf{P}_{kl}, \quad k = 1, K, \quad l = 1, L, \\
t_{kl}(x, y) = [a_{kl} + b_{kl} \cos (c_{kl}x + d_{kl})] \left[ A_{kl} + B_{kl} \cos (C_{kl}y + D_{kl}) \right], \\
F(x_k, y_l) = z_{kl}, \\
\nabla F(x_k, y_l) = 0.
\end{array} \right. \quad (19)
\]
Figure 6: A beam of smooth trigonometric splines. The values \( \{z_i\} \) were taken randomly from a given interval.

In analogues to one-dimensional case the conditions (19) do not identify the spline. For example, we can take

\[
\begin{align*}
    c_{kl} &= \frac{\pi}{\Delta x_k}, \\
    d_{kl} &= -\frac{\pi}{\Delta x_k}, \\
    C_{kl} &= \frac{\pi}{\Delta y_l}, \\
    D_{kl} &= -\frac{\pi}{\Delta y_l},
\end{align*}
\]

(20)

where \( \Delta x_k \equiv x_k - x_{k-1}, \Delta y_l \equiv y_l - y_{l-1} \). In this case \( \nabla F(x_k, y_l) = 0 \) at the vertex of \( P_{kl} \) (and only at these points; so, actually the spline solves the problem \( C \)). For other coefficients of the spline we have the equations

\[
\begin{align*}
    t_{kl} (x_{k-1}, y_{l-1}) &= [a_{kl} + b_{kl}][A_{kl} + B_{kl}] = z_{k-1l-1}, \\
    t_{kl} (x_k, y_{l-1}) &= [a_{kl} - b_{kl}][A_{kl} + B_{kl}] = z_{kl-1}, \\
    t_{kl} (x_{k-1}, y_l) &= [a_{kl} + b_{kl}][A_{kl} - B_{kl}] = z_{k-1l}, \\
    t_{kl} (x_k, y_l) &= [a_{kl} - b_{kl}][A_{kl} - B_{kl}] = z_{kl}
\end{align*}
\]

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which give
\[
\begin{align*}
    a_{kl}A_{kl} &= (z_{k-1l-1} + z_{k-1l} + z_{kl-1} + z_{kl}) / 4, \\
b_{kl}A_{kl} &= (z_{k-1l-1} + z_{k-1l} - z_{kl-1} - z_{kl}) / 4, \\
a_{kl}B_{kl} &= (z_{k-1l-1} + z_{kl-1} - z_{k-1l} - z_{kl}) / 4, \\
b_{kl}B_{kl} &= (z_{k-1l-1} + z_{kl-1} + z_{k-1l} - z_{kl}) / 4,
\end{align*}
\]

that is sufficient for the spline determination. As coefficients of \( t_{kl} (x, y) \) depend only on the spline values at vertexes of \( P_{kl} \), then we must restrict these values in order to provide the smoothness of the spline.

It is easy to prove that we need to provide the smoothness of the spline on the boundaries of \( P_{kl} \) only. Take for example the common boundary of \( P_{kl} \) and \( P_{kl-1} \), the segment \([x_{k-1}, x_k] \times \{y_{l-1}\}\). The spline is continuous on this segment if and only if for all \( x \in [x_{k-1}, x_k] \)
\[
[a_{kl} + b_{kl} \cos \pi (x - x_{k-1}) / \Delta x_k] [A_{kl} + B_{kl}] =
[a_{kl-1} + b_{kl-1} \cos \pi (x - x_{k-1}) / \Delta x_k] [A_{kl-1} - B_{kl-1}].
\]

So we have the equations
\[
\begin{align*}
a_{kl-1} (A_{kl-1} - B_{kl-1}) &= a_{kl} (A_{kl} + B_{kl}), \\
b_{kl-1} (A_{kl-1} - B_{kl-1}) &= b_{kl} (A_{kl} + B_{kl}).
\end{align*}
\]  

Substitute formulae \( 21 \) to these equations we receive the identities for \( z_{kl} \). Also we get the identities when check the continuity of the spline on the vertical boundary \( \{x_{k-1}\} \times [y_{l-1}, y_l] \). Hence the spline is continuous for all values \( \{z_{kl}\} \). Actually the spline belongs even to the class \( C^1 ([a, b] \times [c, d]) \).

To prove it we must prove the continuity of the gradient on the boundaries of \( P_{kl} \). For example, using \( 22 \), we have on the segment \([x_{k-1}, x_k] \times \{y_{l-1}\}\)
\[
F_y' (x, y_{l-1} \pm 0) \equiv 0, \\
F_x' (x, y_{l-1} \pm 0) \equiv \pi b_{kl} \sin \left( \pi (x - x_{k-1}) / \Delta x_k \right) [A_{kl} + B_{kl}] / \Delta x_k \equiv \\
\equiv \pi b_{kl-1} \sin \left( \pi (x - x_{k-1}) / \Delta x_k \right) [A_{kl-1} - B_{kl-1}] / \Delta x_k.
\]

If we want to construct the more smooth spline then we must to impose its values \( \{z_{kl}\} \). For example the condition of continuity of the second derivative \( F_{yy}'' \) on the boundary \([x_{k-1}, x_k] \times \{y_{l-1}\}\) tends to the equations
\[
[a_{kl} + b_{kl} \cos \pi (x - x_{k-1}) / \Delta x_k] B_{kl} C_{kl}^2 =
[a_{kl-1} + b_{kl-1} \cos \pi (x - x_{k-1}) / \Delta x_k] B_{kl-1} C_{kl-1}^2,
\]

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hence,

\[ a_{kl} B_{kl} C_{kl}^2 = -a_{kl-1} B_{kl-1} C_{kl-1}^2, \quad b_{kl} B_{kl} C_{kl}^2 = -b_{kl-1} B_{kl-1} C_{kl-1}^2. \]

If we put (21) to these equalities, then we get

\[
(z_{k-1l-1} + z_{kl-1} - z_{kl}) \beta_l = (z_{k-1l-1} + z_{kl-1} - z_{k-1l-2} - z_{k-2l}), \\
(z_{k-1l-1} + z_{k-1l} - z_{kl-1}) \beta_l = (z_{k-1l-1} + z_{kl-2} - z_{k-1l-2} - z_{k-1l-1}),
\]

where \( \beta_l \equiv (\Delta y_{l-1}/\Delta y_l)^2 \). In analogues the condition of continuity of the second derivative \( F_{x_2}'' \) on the boundary \( \{x_{k-1}\} \times \{y_{l-1}, y_l\} \) tends to the equations

\[
[A_{kl} + B_{kl} \cos \pi (y - y_{l-1}) / \Delta y_l] b_{kl} C_{kl}^2 = -[A_{k-1l} + B_{k-1l} \cos \pi (y - y_{l-1}) / \Delta y_l] b_{k-1l} C_{k-1l}^2,
\]

hence,

\[
A_{kl} b_{kl} C_{kl}^2 = -A_{k-1l} b_{k-1l} C_{k-1l}^2, \quad B_{kl} b_{kl} C_{kl}^2 = -B_{k-1l} b_{k-1l} C_{k-1l}^2,
\]

and

\[
(z_{k-1l-1} + z_{kl} - z_{kl-1} - z_{k-1l-2} - z_{k-2l}) \delta_k = z_{k-1l-1} + z_{k-1l} - z_{k-2l-1} - z_{k-2l}, \\
(z_{k-1l-1} + z_{k-1l} - z_{kl-1} - z_{k-1l-2} - z_{k-2l}) \delta_l = z_{k-2l} + z_{k-1l-1} - z_{k-2l-1} - z_{k-1l},
\]

where \( \delta_k \equiv (\Delta x_{k-1}/\Delta x_k)^2 \). It is easy to prove that other derivatives of the second order are also continuous in this case.

From (23) and (24) we have

\[
\begin{cases}
  z_{kl-2} = \beta_l z_{kl} + z_{kl-1} (1 - \beta_l), \\
  z_{k-1l-2} = \beta_l z_{k-1l} + z_{k-1l-1} (1 - \beta_l), \\
  z_{k-2l} = \delta_k z_{kl} + z_{k-1l} (1 - \delta_k), \\
  z_{k-2l-1} = \delta_k z_{kl-1} + z_{k-1l-1} (1 - \delta_k).
\end{cases}
\]

Differently these formulas can be written in the form

\[
(z_{k-1l} - z_{kl-2}) / \Delta y_{l-1}^2 = - (z_{kl} - z_{kl-1}) / \Delta y_l^2, \\
(z_{k-1l-1} - z_{k-2l}) / \Delta x_{k-1}^2 = - (z_{kl-1} - z_{k-1l}) / \Delta x_k^2,
\]

and this means that

\[
(z_{kl} - z_{kl-1}) / \Delta y_l^2 = (-1)^l \mu_k, \quad l = 1, L, \quad k = 0, K \]
\[
(z_{kl} - z_{k-1l}) / \Delta x_k^2 = (-1)^k \nu_l, \quad k = 1, K, \quad l = 0, L
\]
for some sets of values \( \{ \mu_k \} \), \( \{ \nu_l \} \). So,

\[
z_{kl} = z_{k0} + \mu_k \sum_{j=1}^{l} (-1)^j \Delta y_j^2 = z_{0l} + \nu_l \sum_{i=1}^{k} (-1)^i \Delta x_i^2,
\]

\[
z_{kl} - z_{k-1l} = z_{k0} - z_{k-10} + (\mu_k - \mu_{k-1}) \sum_{j=1}^{l} (-1)^j \Delta y_j^2 = \nu_l (-1)^k \Delta x_k^2,
\]

\[
(-1)^k \frac{(\mu_k - \mu_{k-1})}{\Delta x_k^2} \sum_{j=1}^{l} (-1)^j \Delta y_j^2 = \nu_l - \nu_0.
\]

For fulfilment of the last equalities for all \( k, l \) one can take for any value \( \lambda \)

\[
\mu_k = \mu_0 + \lambda \sum_{j=1}^{k} (-1)^j \Delta x_j^2,
\]

In this case

\[
\nu_l = \nu_0 + \lambda \sum_{j=1}^{l} (-1)^j \Delta y_j^2,
\]

\[
z_{kl} = z_{00} + \nu_0 \sum_{i=1}^{k} (-1)^i \Delta x_i^2 + \mu_0 \sum_{j=1}^{l} (-1)^j \Delta y_j^2 + \\
+ \lambda \sum_{i=1}^{k} (-1)^i \Delta x_i^2 \sum_{j=1}^{l} (-1)^j \Delta y_j^2.
\]

We see that this family of \( 2 \)-splines depends only on the next four free parameters \( z_{00}, \nu_0, \mu_0, \lambda \).

Examples of two-dimensional trigonometric splines see in Figs. 7, 8.

4 Superpositions of splines

In the last section we have established that for two-dimensional sets \( \mathfrak{M} \) with matrix structure (2) solutions of the problems \( \mathbf{A} - \mathbf{C} \) can be found in the form of splines (polynomial or trigonometric with different types of smoothness). If \( \mathfrak{M} \) does not have the matrix form any spline has an additional (false) points where its gradient equals to zero; so it can not solve the problem \( \mathbf{C} \). To overcome this obstacle some constructions may be proposed.

For example, consider the following procedure consisting of the steps
Figure 7: An example of two-dimensional trigonometric spline \([19]\). Values \(\{z_{ij}\}, i, j = \overline{1, 10}\) were taken randomly from a fixed segment.

**Step 1.** Complete \(M\) to a set \(M_{XY}\) of the matrix form and build any spline \(F(x, y; M_{XY})\) that solves the corresponding problem \(C\);

**Step 2.** Turn the coordinate system \(XY\) for some ("sufficiently small") angle \(\phi_1\) and build an analogous spline in the coordinates \(X_1Y_1,\)

\[
F(x_1, y_1; M_{X_1Y_1}) \equiv F(x \cos \phi_1 - y \sin \phi_1, x \sin \phi_1 + y \cos \phi_1; M_{X_1Y_1});
\]

**Step 3.** Repeat the above steps several times for different angles \(\phi_k\); then build the function

\[
\tilde{F}(x, y) = \sum_{k \geq 1} F(x \cos \phi_k - y \sin \phi_k, x \sin \phi_k + y \cos \phi_k; M_{X_kY_k}). \quad (26)
\]

As gradients of all summands are zero at points \(M\), the function \(\tilde{F}(x, y)\) (superposition of splines) solves the problem \(A\). At the same time the additional points for matrixes \(M_{X_kY_k}\) are not the same (they are removed from their initial places); so, the probability that gradients of the summary function (26) at additional points are not zero is high. Test calculations verify these assumption.
Figure 8: An example of two-dimensional trigonometric $C^2$-spline (19). Values $\{z_{ij}\}, i, j = 1, 10$ were calculated by the formula (25).

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