A Galerkin approach to optimization in the space of convex and compact subsets of $\mathbb{R}^d$

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Abstract
The aim of this paper is to open up a new perspective on set and shape optimization by establishing a theory of Galerkin approximations to the space of convex and compact subsets of $\mathbb{R}^d$ with favorable properties, both from a theoretical and from a computational perspective. Galerkin spaces consisting of polytopes with fixed facet normals are first explored in depth and then used to solve optimization problems in the space of convex and compact subsets of $\mathbb{R}^d$ approximately.

Keywords Set optimization · Galerkin approximation · Convex sets · Polytopes

Mathematics Subject Classification 65K10 · 52A20 · 52B11

1 Introduction

Optimization in the space of convex sets is a subject with a long history. While hard theoretical problems such as Mahler’s conjecture are still under investigation today, see [11,12], more and more researchers who are inspired by practical and semi-practical problems are developing numerical methods for the approximation of optimizers. Some approaches, see [3] and the references therein, explore the concepts of shape calculus and the associated discretizations, see [4,5], in the particular setting of convex sets, while others focus on the discretization of the space of convex sets and study the optimizers of the resulting finite-dimensional problems, see e.g. [2,8].

The present paper attempts to make and support progress on both, the theoretical understanding of optimization problems in the space of convex sets, as well as their numerical solution, by transferring the concept of Galerkin approximations to this setting. In the context of partial differential equations, Galerkin approximations are the conceptual link between their analysis and modern numerical methods for their solution. Once the regularity of a solution is established, finite-dimensional Galerkin approximations to the corresponding function space are designed, and efficient numerical methods compute approximate solutions to the
discretized problems. This simple, but very powerful idea has been driving the evolution of major parts of applied mathematics for more than half a century.

The aim of this article is to create an analog of Galerkin approximations for problems posed in the space $\mathcal{K}_c(\mathbb{R}^d)$ of all nonempty, convex and compact subsets of $\mathbb{R}^d$. Spaces of polytopes with prescribed facet orientations, which have been described, e.g., in [1], are natural candidates for this purpose.

Our analysis in Sect. 2 shows that these spaces are useful objects, both from a theoretical as well as from a computational perspective. They possess a natural system of coordinates, which can be characterized as the set of all vectors in $\mathbb{R}^N$ satisfying a linear inequality, and they have approximation properties, which are very similar to those of finite element spaces. A substantial part of Sect. 2 is devoted to the classification and geometry of polytopes corresponding to interior and boundary points of our coordinate spaces, which is important for the practical solution of optimisation problems in the space $\mathcal{K}_c(\mathbb{R}^d)$ as well as for the interpretation of the result.

In Sect. 3, we demonstrate that there exist nested Galerkin sequences, which are the equivalents of nested conforming finite element schemes in the world of sets, and that minimizers of auxiliary optimization problems posed in the Galerkin spaces approximate minimizers of optimization problems posed in $\mathcal{K}_c(\mathbb{R}^d)$.

2 The space $\mathcal{G}_A$

In this section, we characterize and analyze the space $\mathcal{G}_A$ of all polyhedra with prescribed facet orientations, which are encoded in terms of a matrix $A$ containing the corresponding outer normals. The space $\mathcal{G}_A$ is the equivalent of a Galerkin approximation in $\mathcal{K}_c(\mathbb{R}^d)$. It possesses a finite-dimensional parametrization, it inherits some useful properties from $\mathcal{K}_c(\mathbb{R}^d)$, and it approximates the entire space $\mathcal{K}_c(\mathbb{R}^d)$ in a certain sense uniformly from within. In this section, we will verify the above assertions, and in Sect. 3, we will study how the optimizers of an optimization problem in $\mathcal{K}_c(\mathbb{R}^d)$ behave when the problem is restricted to $\mathcal{G}_A$, where it can be solved as a finite-dimensional optimization problem over the parametrization.

After collecting some preliminaries in Sect. 2.1 and introducing a natural space $\mathcal{C}_A$ of coordinates in Sect. 2.2, we investigate in Sect. 2.3, under which circumstances the spaces $\mathcal{G}_A$ consist of bounded polytopes. In Sect. 2.4, we characterize $\mathcal{C}_A$ in terms of a system of linear inequalities, and after collecting some information on vertices of certain polyhedra in Sect. 2.5, we eliminate some redundancies in the characterizing system of inequalities in Sect. 2.6. In Sect. 2.7 we discuss the geometry of polyhedra corresponding to coordinate vectors in the interior and in different parts of the boundary of $\mathcal{C}_A$, and in Sect. 2.8, we quantify how well the space $\mathcal{G}_A$ approximates $\mathcal{K}_c(\mathbb{R}^d)$.

2.1 Setting and preliminaries

We denote the unit sphere in $\mathbb{R}^d$ by $S^{d-1} := \{x \in \mathbb{R}^d : \|x\|_2 = 1\}$. Throughout this section, we will assume that $A \in \mathbb{R}^{N \times d}$ is a fixed matrix which consists of pairwise distinct rows $a_1^T, \ldots, a_N^T$ satisfying $a_i \in S^{d-1}$ for $i = 1, \ldots, N$. Denoting $\mathbb{R}_+ := \{s \in \mathbb{R} : s \geq 0\}$, for every $b \in \mathbb{R}^N$ and $c \in \mathbb{R}^d$, we define

$$Q_{A,b} := \{x \in \mathbb{R}^d : Ax \leq b\}, \quad Q_{A,c}^* := \{p \in \mathbb{R}_+^N : A^T p = c\}.$$
and for \( x \in \mathbb{R}^d \) and \( p \in \mathbb{R}^N_+ \), we set
\[
I_x^b := \{ i : a_i^T x = b_i \}, \quad I_p^* := \{ i : p_i > 0 \}.
\]

We define a space of polyhedra by setting
\[
G_A := \{ Q_{A,b} : b \in \mathbb{R}^N \}\setminus\{\emptyset\}.
\]
The vectors \( b \in \mathbb{R}^N \) provide a natural system of coordinates on \( G_A \). We denote \( \emptyset = (0, \ldots, 0)^T \in \mathbb{R}^N \) and \( 1 = (1, \ldots, 1)^T \in \mathbb{R}^N \). For any closed convex set \( C \subset \mathbb{R}^d \), we denote its set of extremal points by \( \text{ext}(C) \).

Let \( K_c(\mathbb{R}^d) \) denote the space of all nonempty, convex and compact subsets of \( \mathbb{R}^d \), and consider the Hausdorff semi-distance \( \text{dist} : K_c(\mathbb{R}^d) \times K_c(\mathbb{R}^d) \to \mathbb{R}_+ \) and the Hausdorff-distance \( \text{dist}_H : K_c(\mathbb{R}^d) \times K_c(\mathbb{R}^d) \to \mathbb{R}_+ \) defined by
\[
\text{dist}(C, \hat{C}) := \sup_{c \in C} \inf_{\hat{c} \in \hat{C}} \| c - \hat{c} \|_2, \\
\text{dist}_H(C, \hat{C}) := \max\{\text{dist}(C, \hat{C}), \text{dist}(\hat{C}, C)\}.
\]

In addition, we introduce the size and the support function
\[
\| \cdot \|_2 : K_c(\mathbb{R}^d) \to \mathbb{R}_+, \quad \| C \|_2 := \sup_{c \in C} \| c \|_2, \\
\sigma : K_c(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}, \quad \sigma_C(x) := \sup_{c \in C} x^T c.
\]

The following statements are Corollary 4.5 in [9] and the first theorem in Section 2.5 of [10].

**Lemma 1** (Vertices of polyhedra) Let \( b \in \mathbb{R}^N \) and \( c \in \mathbb{R}^d \).

(a) A point \( x \in Q_{A,b} \) satisfies \( x \in \text{ext}(Q_{A,b}) \) if and only if \( x \in Q_{A,b} \) and \( \text{rank}([a_i : i \in I_x^b]) = d \).

(b) A point \( p \in Q_{A,c}^* \) satisfies \( p \in \text{ext}(Q_{A,c}^*) \) if and only if the vectors \( \{a_i : i \in I_p^*\} \) are linearly independent.

The following facts are Propositions 1.7 and 1.9 in [16].

**Proposition 2** (Farkas’ Lemma) Let \( b \in \mathbb{R}^N \), \( c \in \mathbb{R}^d \) and \( \beta \in \mathbb{R} \).

(a) Either \( Q_{A,b} \neq \emptyset \), or there exists \( p \in Q_{A,0}^* \) with \( b^T p < 0 \).

(b) Assume that \( Q_{A,b} \neq \emptyset \). Then the inequality \( c^T x \leq \beta \) is true for all \( x \in Q_{A,b} \) if and only if there exists \( p \in Q_{A,c}^* \) with \( p^T b \leq \beta \).

Finitely generated convex cones will play a major role in this paper.

**Definition 3** Given vectors \( v_1, \ldots, v_k \in \mathbb{R}^m \), we define the cone generated by these vectors by \( \text{cone}((v_1, \ldots, v_k)) := \{ \sum_{j=1}^k \lambda_j v_k : \lambda \in \mathbb{R}^k_+ \} \).

Carathéodory’s theorem for cones is taken from Theorem 3.14 in [9].

**Proposition 4** (Carathéodory’s theorem for cones) Let \( I \subset \{1, \ldots, N\} \), and let \( c \in \text{cone}([a_i : i \in I]) \). Then there exists a subset \( J \subset I \) such that \( c \in \text{cone}([a_i : i \in J]) \) and the vectors \( \{a_i : i \in J\} \) are linearly independent.
The following version of the strong duality theorem for linear programming can be found, e.g., in [7, Theorem 4.13].

**Theorem 5** (Strong duality for LP) Let \( b \in \mathbb{R}^N \) and \( c \in \mathbb{R}^d \). For the LPs

\[
\max \{ c^T x : x \in Q_{A,b} \} \quad \text{and} \quad \min \{ b^T p : p \in Q_{A,c}^* \},
\]

precisely one of the following alternatives holds:

(a) We have \( Q_{A,b} \neq \emptyset \) and \( Q_{A,c}^* \neq \emptyset \), and

\[
\max \{ c^T x : x \in Q_{A,b} \} = \min \{ b^T p : p \in Q_{A,c}^* \}. \]

(b) We have either \( Q_{A,b} = \emptyset \) and \( \inf \{ b^T p : p \in Q_{A,c}^* \} = -\infty \), or \( Q_{A,c}^* = \emptyset \) and \( \sup \{ c^T x : x \in Q_{A,b} \} = \infty \).

(c) We have \( Q_{A,b} = \emptyset \) and \( Q_{A,c}^* = \emptyset \).

Hoffman’s error bound was first given in the paper [6].

**Theorem 6** (Hoffman’s error bound) There exists a constant \( L_A \geq 0 \) such that for all \( x \in \mathbb{R}^d \) and all \( b \in \mathbb{R}^N \) with \( Q_{A,b} \neq \emptyset \), we have

\[
\text{dist}(x, Q_{A,b}) \leq L_A \| \max \{ 0, Ax - b \} \|_\infty.
\]

where the maximum is to be interpreted component-wise.

The following facts are proved in Lemmas 1.8.12 and 1.8.14 in [15].

**Lemma 7** For all \( x, \tilde{x} \in \mathbb{R}^d \) and \( C, \tilde{C} \in K_c(\mathbb{R}^d) \), we have

\[
|\sigma_C(x) - \sigma_{\tilde{C}}(\tilde{x})| \leq \max \{ \| C \|_2, \| \tilde{C} \|_2 \} \| x - \tilde{x} \|_2
+ \max \{ \| x \|_2, \| \tilde{x} \|_2 \} \text{dist}_H(C, \tilde{C}),
\]

as well as

\[
\text{dist}_H(C, \tilde{C}) = \sup_{x \in S^{d-1}} |\sigma_C(x) - \sigma_{\tilde{C}}(x)|.
\]

### 2.2 Coordinates on \( G_A \)

In this section, we check that the mapping \( \varphi : C_A \to G_A, \varphi(b) := Q_{A,b} \),

is a bijection between the coordinate space

\[
C_A := \{ b \in \mathbb{R}^N : \forall i \in \{1, \ldots, N\} \exists x_i \in Q_{A,b} \text{ with } a_i^T x_i = b_i \}
\]

and the space of polyhedra \( G_A \). It is immediate that \( Q_{A,b} \neq \emptyset \) whenever \( b \in C_A \). The following lemmas state that vectors in \( C_A \) are, in a sense, minimal descriptions of polyhedra in \( G_A \).

**Lemma 8** If \( b \in C_A \) and \( \tilde{b} \in \mathbb{R}^N \) with \( Q_{A,b} = Q_{A,\tilde{b}} \), then \( b \leq \tilde{b} \).

**Proof** Let \( i \in \{1, \ldots, N\} \). By definition of \( C_A \), there exists \( x_i \in Q_{A,b} \) with \( b_i = a_i^T x_i \), and since \( Q_{A,b} = Q_{A,\tilde{b}} \), we have \( a_i^T x_i \leq \tilde{b}_i \). \( \square \)
In the next proof, we use Farkas’ lemma instead of compactness arguments, because we are currently not excluding unbounded polyhedra $Q_{A,b}$.

**Lemma 9** Let $\bar{b} \in \mathbb{R}^N$ with $Q_{A,\bar{b}} \neq \emptyset$, and define

$$b_i := \sup\{a_i^T x : x \in Q_{A,\bar{b}}\}, \quad i = 1, \ldots, N. \quad (1)$$

Then $b \in C_A$, and we have $b \leq \bar{b}$ and $Q_{A,b} = Q_{A,\bar{b}}$.

**Proof** Because of $Q_{A,\bar{b}} \neq \emptyset$ and by statement (1), we have $-\infty < b_i \leq \bar{b}_i$ for every $i \in \{1, \ldots, N\}$, and hence $b \in \mathbb{R}^N$ and $Q_{A,b} \subset Q_{A,\bar{b}}$. On the other hand, for every $x \in Q_{A,\bar{b}}$ and every $i \in \{1, \ldots, N\}$, statement (1) yields $a_i^T x \leq b_i$, so $Q_{A,b} \subset Q_{A,\bar{b}}$. All in all, we have $Q_{A,b} = Q_{A,\bar{b}} \neq \emptyset$.

Now fix $i \in \{1, \ldots, N\}$, and let us check that there exists $x_i \in Q_{A,b}$ with $a_i^T x_i = b_i$. According to statement (1), for every $\varepsilon > 0$, there exists $x_\varepsilon \in Q_{A,b}$ with $a_i^T x_\varepsilon > b_i - \varepsilon$. By Proposition 2b, this implies that

$$p^T b \geq b_i \quad \forall \ p \in Q_{A,a_i}^*, \quad (2)$$

Now consider $(p, s) \in \mathbb{R}_+^N \times \mathbb{R}_+$ with $A^T p - s a_i = 0$. If $s > 0$, then we have $s^{-1} p \in Q_{A,a_i}^*$, so by (2), we obtain $b^T p - b_i s \geq 0$. If $s = 0$, then $A^T p = 0$, and since $Q_{A,b} \neq \emptyset$, Proposition 2a yields $b^T p \geq 0$. Thus, in both cases we find

$$(b^T, -b_i) \begin{pmatrix} p \\ s \end{pmatrix} \geq 0,$$

and according to Proposition 2a, this implies

$$\{x \in \mathbb{R}^d : Ax \leq b, -a_i^T x \leq -b_i\} \neq \emptyset,$$

so there exists $x_i \in Q_{A,b}$ with $a_i^T x_i = b_i$. \hfill \Box

Now we establish the converse of Lemma 8.

**Lemma 10** If $b \in \mathbb{R}^N$ and $b \leq \bar{b}$ for all $\bar{b} \in \mathbb{R}^N$ with $Q_{A,b} = Q_{A,\bar{b}}$, then we have $b \in C_A$.

**Proof** If $Q_{A,b} = \emptyset$, then $\bar{b} := b - 1 < b$ and $Q_{A,b-1} = \emptyset = Q_{A,b}$, which contradicts the assumption. Hence $Q_{A,b} \neq \emptyset$, and by Lemma 9, there exists some $\bar{b} \in C_A$ with $\bar{b} \leq b$ and $Q_{A,b} = Q_{A,\bar{b}}$. By assumption, we have $b \leq \bar{b}$, so $b = \bar{b}$ and $b \in C_A$. \hfill \Box

Let us sum up the preceding discussion.

**Proposition 11** For $b \in \mathbb{R}^N$, we have $b \in C_A$ if and only if $b \leq \bar{b}$ for all $\bar{b} \in \mathbb{R}^N$ with $Q_{A,b} = Q_{A,\bar{b}}$.

**Proof** Combine Lemmas 8 and 10. \hfill \Box

Finally, we conclude that $\varphi$ is a nice parametrization of $\mathcal{G}_A$.

**Theorem 12** The mapping $\varphi : \mathcal{C}_A \rightarrow \mathcal{G}_A$, $\varphi(b) = Q_{A,b}$, is a homeomorphism between $(\mathcal{C}_A, \| \cdot \|_\infty)$ and $(\mathcal{G}_A, \text{dist}_H)$. The mapping $\varphi$ is $L_A$-Lipschitz, and its inverse $\varphi^{-1}$ is $1$-Lipschitz.
Proof By definition of \( C_A \), it is clear that \( \varphi(C_A) \subset G_A \).

Let \( \tilde{b} \in \mathbb{R}^N \) with \( Q_A \tilde{b} \in G_A \). By Lemma 9, there exists \( b \in C_A \) such that \( Q_A b = Q_A \tilde{b} \), so \( \varphi \) is surjective. Assume that there exist \( b, \tilde{b} \in C_A \) with \( Q_A b = \varphi(b) = \varphi(\tilde{b}) = Q_A \tilde{b} \). By Proposition 11, we have \( b \leq \tilde{b} \) and \( \tilde{b} \leq b \), so \( b = \tilde{b} \). Hence \( \varphi \) is injective.

The Lipschitz property of \( \varphi \) is a consequence of Hoffman’s error bound, see Theorem 6. To check the Lipschitz property of the inverse, let \( b, \tilde{b} \in C_A \) and fix \( i \in \{1, \ldots, N\} \). By definition of \( C_A \), there exists \( x \in \varphi(b) = Q_A b \) with \( a_i^T x = b_i \), and there exists \( \tilde{x} \in \varphi(\tilde{b}) = Q_A \tilde{b} \) with \( \|x - \tilde{x}\|_2 \leq \text{dist}(Q_A b, Q_A \tilde{b}) \).

By symmetry, and since the above argument holds for any \( i \), we obtain
\[
\|b - \tilde{b}\|_\infty \leq \text{dist}(Q_A b, Q_A \tilde{b}).
\]

\( \square \)

2.3 Spaces of polytopes or unbounded polyhedra

Now we will see that \( G_A \) consists either entirely of polytopes, or entirely of unbounded polyhedra.

The recession cone of a convex set describes its behavior at infinity.

Definition 13 The recession cone of a closed convex set \( C \subset \mathbb{R}^d \) is the set
\[
\text{rec}(C) := \{ c \in \mathbb{R}^d : x + \lambda c \in C \ \forall \lambda \geq 0, \ \forall x \in C \}.
\]

We use this notion to prove a theorem of the alternative for the space \( G_A \).

Theorem 14 Either \( G_A \) is a collection of unbounded polyhedra, or it is a collection of bounded polytopes. The latter alternative is equivalent with the statement \( Q_{A,0} = \{0\} \), and it is also equivalent with the condition
\[
Q^*_A c \neq \emptyset \ \forall c \in \mathbb{R}^d.
\]

Proof Let \( Q_A, b \in G_A \) be arbitrary. According to Theorem 8.4 in [14], the set \( Q_A b \) is bounded if and only if \( \text{rec}(Q_A b) = \{0\} \), and by Proposition 1.12 from [16], we have \( \text{rec}(Q_A b) = Q_{A,0} \).

By definition, we have \( 0 \in \text{rec}(Q_A b) \). By Theorem 5, the statement \( Q_{A,0} = \{0\} \) implies that \( Q^*_A c \neq \emptyset \) for all \( c \in \mathbb{R}^d \). Conversely, if there exists \( c \in Q_{A,0} \setminus \{0\} \), then \( \lambda c \in Q_{A,0} \) for all \( \lambda \geq 0 \), and \( \max \{|c^T x : x \in Q_{A,0}\} \) is unbounded. But then, Theorem 5 yields \( Q^*_A c = \emptyset \). \( \square \)

The following statement gives some insight into the structure of \( G_A \).

Corollary 15 The space \( G_A \) is a cone if and only if \( G_A \) consists of polytopes.

Proof For any \( \lambda > 0 \) and \( b \in C_A \), we have \( \lambda Q_A b = Q_{A,\lambda b} \in G_A \). If \( G_A \) consists of polytopes, then Theorem 14 yields \( 0 \cdot Q_{A,b} = \{0\} = Q_{A,0} \in G_A \), and \( G_A \) is a cone. If \( G_A \) consists of unbounded polyhedra, then \( 0 \cdot Q_{A,b} = \{0\} \notin G_A \). \( \square \)
2.4 An explicit characterization of $C_A$

We characterize the set $C_A$ in terms of a system of linear inequalities, which allows to convert any problem formulated in $G_A$ into a linearly constrained problem formulated in $\mathbb{R}^N$. To maintain readability, we will denote

$$Q_{A,0}^\circ = \{ q \in \mathbb{R}_+^N : A^T q = 0, \ 1^T q = 1 \}.$$  

For the interpretation of Theorem 16, note that $\text{ext}(Q_{A,0}^\circ)$ can be empty, while $e_i \in \text{ext}(Q_{A,a_i}^*)$ for all $i \in \{1, \ldots, N\}$.

**Theorem 16** Let $b \in \mathbb{R}^N$. Then we have $b \in C_A$ if and only if

$$0 \leq b^T p \ \forall \ p \in Q_{A,0}^\circ, \tag{3a}$$

$$b_i \leq b^T p \ \forall \ p \in Q_{A,a_i}^*, \ \forall i \in \{1, \ldots, N\}, \tag{3b}$$

which is equivalent with

$$0 \leq b^T p \ \forall \ p \in \text{ext}(Q_{A,0}^\circ), \tag{4a}$$

$$b_i \leq b^T p \ \forall \ p \in \text{ext}(Q_{A,a_i}^*), \ \forall i \in \{1, \ldots, N\}. \tag{4b}$$

**Remark 17** In fact, the conditions (4a) are equivalent with $Q_{A,b} \neq \emptyset$, and when $Q_{A,b} \neq \emptyset$, the conditions (4b) guarantee that every inequality $a_i^T x \leq b_i$ is attained.

**Proof of Theorem 16** By definition, we have $b \in C_A$ if and only if

$$\max\{a_i^T x : x \in Q_{A,b}\} = b_i \ \forall i \in \{1, \ldots, N\}. \tag{5}$$

Let us show that (5) is equivalent with conditions (3a, 3b).

Assume that statement (5) holds. Then we have $Q_{A,b} \neq \emptyset$, so Proposition 2a yields condition (3a). Applying Theorem 5 to statement (5) gives

$$\min\{b^T p : p \in Q_{A,a_i}^*\} = b_i \ \forall i \in \{1, \ldots, N\}, \tag{6}$$

which implies condition (3b).

Conversely, assume that conditions (3a) and (3b) hold. Statement (3a) and Proposition 2a imply $Q_{A,b} \neq \emptyset$. Statement (3b) and Theorem 5 imply

$$b_i \leq \min\{b^T p : p \in Q_{A,a_i}^*\} = \max\{a_i^T x : x \in Q_{A,b}\} \leq b_i \ \forall i \in \{1, \ldots, N\},$$

so statement (5) holds.

Since $Q_{A,0}^\circ$ is a bounded polytope, conditions (3a) and (4a) are equivalent. Statement (3b) clearly implies (4b). Assume that (4a) and (4b) hold. By Lemma 1b, we have $e_i \in \text{ext}(Q_{A,a_i}^*)$, so by Theorem 4.24 in [9], we have

$$Q_{A,a_i}^* = \text{conv}(\text{ext}(Q_{A,a_i}^*)) + \text{rec}(Q_{A,a_i}^*) = \text{conv}(\text{ext}(Q_{A,a_i}^*)) + Q_{A,0}^\circ.$$ 

Let $p \in Q_{A,a_i}^*$, and let $q \in \text{conv}(\text{ext}(Q_{A,a_i}^*))$ and $r \in Q_{A,0}^\circ$ with $p = q + r$. Statement (4b) gives $b_i \leq b^T q$, and statement (4a) implies $0 \leq b^T r$, so

$$b_i \leq b^T (q + r) = b^T p.$$  

$\square$
We use Theorem 16 to characterize $C_A$. Note that ext($Q_{A,0}^\circ$) as well as ext($Q_{A,a_i}$) \{e_i\}, with $i \in \{1, \ldots, N\}$ can be the empty set. In this case, the corresponding matrices are to be interpreted as the empty matrix.

**Corollary 18** If we enumerate
\[
\text{ext} (Q_{A,0}^\circ) = \{ f^{0,1}, \ldots, f^{0,m_0} \},
\]
\[
\text{ext} (Q_{A,a_i}) \{ e_i \} = \{ f^{i,1}, \ldots, f^{i,m_i} \}, \quad i \in \{1, \ldots, N\},
\]
and form the matrix $F := (F_0, \ldots, F_N)$ with
\[
F_0 := (f^{0,1}, \ldots, f^{0,m_0}) \in \mathbb{R}^{N \times m_0},
\]
\[
F_i := (f^{i,1} - e_i, \ldots, f^{i,m_i} - e_i) \in \mathbb{R}^{N \times m_i}, \quad i \in \{1, \ldots, N\},
\]
then $b \in C_A$ is equivalent with $F^T b \geq 0$.

This representation is the key for the practical applicability of the theory laid out in this paper. In addition, it has nice theoretical consequences.

**Corollary 19** The set $C_A$ is a closed convex subcone of $(\mathbb{R}^N, \| \cdot \|_\infty)$.

We can immediately draw the following conclusion.

**Corollary 20** The metric space $(G_A, \text{dist}_H)$ is complete.

**Proof** According to Theorem 12, the mapping $\varphi : C_A \to G_A, \varphi(b) = Q_{A,b}$, is a bi-Lipschitz homeomorphism between $(C_A, \| \cdot \|_\infty)$ and $(G_A, \text{dist}_H)$, and by Corollary 19, the space $(C_A, \| \cdot \|_\infty)$ is complete. \hfill \Box

### 2.5 Vertices of $Q^{A,0}_A$ and $Q^{A,c}$

In this section, we gather information on the representation of vertices that will be used later in the paper.

**Lemma 21** Let $p \in Q^{A,0}_A$. Then $p \in \text{ext}(Q^{A,0}_A)$ if and only if the vectors $\{(a_i^T, 1)^T : i \in I_p^*\}$ are linearly independent.

**Remark 22** An elementary, but lengthy proof shows that the following statement holds: If $p \in Q^{A,0}_A$, then $p \in \text{ext}(Q^{A,0}_A)$ if and only if for every $i_0 \in I_p^*$, the vectors $\{a_i : i \in I_p^* \backslash \{i_0\}\}$ are linearly independent.

**Proof of Lemma 21** Apply Lemma 1b to $Q^{A,0}_A = Q^{a(T,1)^T, (0_{\mathbb{R}^d},1)^T}_A$. \hfill \Box

The following statement is an immediate consequence of the above lemma.

**Corollary 23** If $p \in Q^{A,0}_A$ and $\tilde{p} \in \text{ext}(Q^{A,0}_A)$ satisfy $I_p^* \subset I_{\tilde{p}}^*$, then $p = \tilde{p}$.

**Proof** Since $\tilde{p} \in \text{ext}(Q^{A,0}_A)$, the vectors $\{(a_i^T, 1)^T : i \in I_{\tilde{p}}^*\}$ are linearly independent by Lemma 21. The vector $\hat{p} := \frac{1}{2}(p + \tilde{p}) \in Q^{A,0}_A$ clearly satisfies $I_{\hat{p}}^* = I_{\tilde{p}}^*$. Hence Lemma 21 yields $\hat{p} \in \text{ext}(Q^{A,0}_A)$, which forces $p = \tilde{p} = \hat{p}$. \hfill \Box

Let us check that ext($Q^{A,c}_A$) $\neq \emptyset$ whenever $Q^{A,c}_A \neq \emptyset$.
Lemma 24 If \( c \in \mathbb{R}^d \setminus \{0\} \), then for every \( p \in Q_{A,c}^* \), there exists some \( \tilde{p} \in \text{ext}(Q_{A,c}^*) \) with \( I_p^* \subset I_{\tilde{p}}^* \).

**Proof** If \( p \in Q_{A,c}^* \), then \( c \in \text{cone}(\{a_i : i \in I_p^*\}) \). By Proposition 4, there exists \( J \subset I_p^* \) such that \( c \in \text{cone}(\{a_j : j \in J\}) \) and \( \{a_j : j \in J\} \) are linearly independent. Since \( c \neq 0 \), we have \( J \neq \emptyset \), and there is \( \tilde{p} \in Q_{A,c}^* \) with \( I_{\tilde{p}}^* \subset J \). By Lemma 1b, we have \( \tilde{p} \in \text{ext}(Q_{A,c}^*) \). \( \square \)

A statement similar with Corollary 23 holds for vertices of \( Q_{A,c}^* \).

Lemma 25 Let \( c \in \mathbb{R}^d \). If \( p \in Q_{A,c}^* \) and \( \tilde{p} \in \text{ext}(Q_{A,c}^*) \) satisfy \( I_p^* \subset I_{\tilde{p}}^* \), then we have \( p = \tilde{p} \). In particular, if \( p \in \text{ext}(Q_{A,a_i}^*) \), then \( p = 0 \).

**Proof** By assumption, we can represent \( \sum_{i \in I_{\tilde{p}}^*} p_i a_i = c = \sum_{i \in I_p^*} \tilde{p}_i a_i \). Since \( \tilde{p} \in \text{ext}(Q_{A,c}^*) \), Lemma 1b yields that the vectors \( \{a_i : i \in I_{\tilde{p}}^*\} \) are linearly independent, which forces \( p = \tilde{p} \). Now let \( p \in \text{ext}(Q_{A,a_i}^*) \) with \( p_i > 0 \). Then the inclusions \( e_i \subset I_{p_{j}}^* \) imply \( e_i = p \). \( \square \)

### 2.6 Redundancy in the characterisation of \( C_A \)

The system \((4a, 4b)\) is, in general, highly redundant, which means that it is a suboptimal characterization of the parameter space \( C_A \). In practice, these redundancies can be eliminated in an offline computation. The results in this section are useful for this elimination process, and they provide some intuition for the origin of the redundancy.

**Theorem 26** The system of inequalities \((4a)\) does not contain redundant conditions.

**Proof** Let \( \text{ext}(Q_{A,0}^*) = \{p^1, \ldots, p^m\} \) with pairwise distinct \( p^j \in \mathbb{R}^N_+ \). Assume that for some \( k \in \{1, \ldots, m\} \), the condition \((p^k)^T b \geq 0 \) is redundant in system \((4a)\). By Proposition 2b, there exists \( \lambda \in \mathbb{R}^m_+ \setminus \{0\} \) with \( \lambda_k = 0 \) and \( p^k = \sum_{j=1}^m \lambda_j p^j \). Since \( p^j \in \mathbb{R}^N_+ \) for \( j \in \{1, \ldots, m\} \), it follows that

\[
I_{p^j}^* \subset I_{p^k}^* \quad \forall j \in \{1, \ldots, m\},
\]

so Corollary 23 gives \( p^j = p^k \) for all \( j \in \{1, \ldots, m\} \) with \( \lambda_j > 0 \). This is a contradiction. \( \square \)

The way in which the geometry of the matrix \( A \) determines the redundancies in conditions \((4b)\) is vaguely related to Haar’s lemma. It is currently not clear whether the complete system \((4a, 4b)\) of linear inequalities contains redundancies other than those identified in the following theorem.

**Theorem 27** Let \( k \in \{1, \ldots, N\} \), and let \( p, \tilde{p} \in \text{ext}(Q_{A,a_k}^*) \setminus \{e_k\} \) with

\[
\text{cone}(\{a_i : i \in I_{p}^*\}) \subsetneq \text{cone}(\{a_i : i \in I_{\tilde{p}}^*\}).
\]

Then the condition \( b_k \leq b^T \tilde{p} \) is redundant in the system of inequalities \((4b)\).

**Proof** First note that the condition \( b_k \leq b^T p \) is one of the conditions in statement \((4b)\), and by (7), we have \( p \neq \tilde{p} \).

Again by (7), for all \( i \in I_{\tilde{p}}^* \), there exist \( p^i \in Q_{A,a_i}^* \) with \( I_{p^i}^* \subset I_{\tilde{p}}^* \). Lemma 1b and \( \tilde{p} \in \text{ext}(Q_{A,a_i}^*) \) imply linear independence of the vectors \( \{a_i : i \in I_{p^i}^*\} \). Lemma 1b and linear independence of \( \{a_i : i \in I_{p^i}^*\} \) imply \( p^i \in \text{ext}(Q_{A,a_i}^*) \), so the conditions

\[
b_i \leq b^T p^i \quad \forall i \in I_{p^i}^*,
\]

\( \square \)
occur in the system of inequalities (4b) as well. By Lemma 25 and since \( p \neq e_k \), we have \( p_k = 0 \). Hence \( a_i \neq a_k \) for all \( i \in I_p^* \), which implies \( A^T p^i = a_i \neq a_k = A^T \tilde{p} \) for all \( i \in I_p^* \), so that \( p^i \neq \tilde{p} \) for all \( i \in I_p^* \).

Now we show that the condition \( b_k \leq b^T \tilde{p} \) is a consequence of the inequalities \( b_k \leq b^T p \) and (8). We compute

\[
\sum_{j \in I_p^*} \tilde{p}_j a_j = a_k = \sum_{i \in I_p^*} p_i a_i = \sum_{i \in I_p^*} p_i \sum_{j \in I_p^*} p_j^i a_j = \sum_{j \in I_p^*} \left( \sum_{i \in I_p^*} p_i p_j^i \right) a_j,
\]

and since \( \{a_j : j \in I_p^*\} \) are linearly independent, it follows that

\[
\tilde{p}_j = \sum_{i \in I_p^*} p_i p_j^i \quad \forall \ j \in I_p^*.
\]

Using (8) and (9), we arrive at the estimate

\[
b_k \leq b^T p = \sum_{i \in I_p^*} b_i p_i \leq \sum_{i \in I_p^*} p_i \sum_{j \in I_p^*} b_j p_j^i = \sum_{j \in I_p^*} b_j \sum_{i \in I_p^*} p_i p_j^i = \sum_{j \in I_p^*} b_j \tilde{p}_j = b^T \tilde{p},
\]

which proves that the condition \( b_k \leq b^T \tilde{p} \) is indeed redundant.

The following immediate consequence of Theorem 27 explains the small number of redundant constraints in (4b) when \( d = 2 \).

**Corollary 28** Let \( t_1 < \cdots < t_N \in [0, 2\pi) \), and let \( a_i^T = (\sin t, \cos t) \). Then every condition \( b_i \leq p^T b \) with \( p \in \text{ext}(Q_{A,a_i}^*) \) is redundant in (4b) unless

\[
I_p^* = \begin{cases} \{N, 2\}, & i = 1, \\ \{i - 1, i + 1\}, & i \in \{2, \ldots, N - 1\}, \\ \{N - 1, 1\}, & i = N. \end{cases}
\]

### 2.7 Interior and boundary points of \( C_A \)

In this section, we will characterize interior and boundary points of \( C_A \). The results will help to identify interior points of \( C_A \), to perturb arbitrary \( b \in C_A \) into the interior of \( C_A \), and to understand the geometry of a set \( Q_{A,b} \) in terms of the vector \( F^T b \). The results will justify the use of interior-point methods for optimization on \( G_A \), and they are useful in theory, because they allow to characterize the solution of an optimization problem as a polyhedron of a particular type.

For any \( b \in C_A \) and \( I \subset \{1, \ldots, N\} \), we define affine subspaces and facets by

\[
H(A,b,I) := \{x \in \mathbb{R}^d : a_i^T x = b_i, \ i \in I\}, \quad Q_{A,b}^I := Q_{A,b} \cap H(A,b,I).
\]

The set \( C_A \) has nonempty interior, which can be characterized easily in the setting of Corollary 18.

**Theorem 29** The topological interior \( \text{int}(C_A) \) of \( C_A \) in \( \mathbb{R}^N \) coincides with the set \( \{b \in \mathbb{R}^N : F^T b > 0\} \), and we have \( \mathbb{1} \in \text{int}(C_A) \).
Since applied to the inequalities \( \sum_{j=1}^{N} f_{j}^{i,k} - \sum_{j=1}^{N} f_{j}^{i,k}a_{j}a_{i} = a_{i}^{T}a_{i} = 1, \)
which, after division by \(-a_{i}^{T}a_{i} < 1\) for \(i \neq j\), we find
\[
1^{T}f_{i,k} = \sum_{j=1}^{N} f_{j}^{i,k} > \sum_{j=1}^{N} f_{j}^{i,k}a_{i}^{T}a_{j} = a_{i}^{T}\sum_{j=1}^{N} f_{j}^{i,k}a_{j} = a_{i}^{T}a_{i} = 1,
\]
which shows that \((f_{i,k} - e_{i})^{T}1 > 0\), as desired.

Let us take a closer look at the boundary of \(C_{A}\). When one of the inequalities in (4a) is an equality, then \(Q_{A,b}\) is flat.

**Proposition 30** Let \(b \in C_{A}\), and let \(p \in \text{ext}(Q_{A,0}^{*})\). Then \(b^{T}p = 0\) holds if and only if \(Q_{A,b} \subset Q_{A,b}^{I_{p}^{*}}\).

**Proof** Let \(b^{T}p = 0\). For any \(i \in I_{p}^{*}\) and \(x \in Q_{A,b}\), we compute
\[
-p_{i}a_{i}^{T}x = p^{T}Ax - p_{i}a_{i}^{T}x = \sum_{j \in I_{p}^{*}} p_{j}a_{j}^{T}x - p_{i}a_{i}^{T}x = \sum_{j \in I_{p}^{*} \setminus \{i\}} p_{j}a_{j}^{T}x \leq \sum_{j \in I_{p}^{*} \setminus \{i\}} p_{j}b_{j} = p^{T}b - p_{i}b_{i} = -p_{i}b_{i},
\]
which, after division by \(-p_{i}\), gives \(a_{i}^{T}x \geq b_{i}\) and hence \(a_{i}^{T}x = b_{i}\) for all \(i \in I_{p}^{*}\).

Conversely, assume that \(Q_{A,b} \subset Q_{A,b}^{I_{p}^{*}}\) holds. Since \(b \in C_{A}\), there exists \(x \in Q_{A,b}\), and we obtain
\[
p^{T}b = \sum_{i \in I_{p}^{*}} p_{i}b_{i} = \sum_{i \in I_{p}^{*}} p_{i}a_{i}^{T}x = (A^{T}p)^{T}x = 0.
\]

In terms of dimension, this means the following.

**Proposition 31** Let \(b \in C_{A}\). Then \(\dim(Q_{A,b}) \leq d - 1\) if and only if there exists \(p \in \text{ext}(Q_{A,0}^{*})\) with \(b^{T}p = 0\).

**Proof** Assume that there exists \(p \in \text{ext}(Q_{A,0}^{*})\) with \(b^{T}p = 0\). Since \(1^{T}p = 1\), we have \(#I_{p}^{*} > 0\). Now Proposition 30 yields \(\dim(Q_{A,b}) \leq \dim(Q_{A,b}^{I_{p}^{*}}) \leq d - 1\).

Conversely, assume that \(\dim(Q_{A,b}) \leq d - 1\). Then there exist \(\alpha \in \mathbb{R}\) and \(c \in \mathbb{R}^{d} \setminus \{0\}\) such that \(c^{T}x = \alpha\) for all \(x \in Q_{A,b}\). By assumption, we have \(Q_{A,b} \neq \emptyset\), and by Proposition 2b applied to the inequalities \(c^{T}x \leq \alpha\) and \((-c)^{T}x \leq -\alpha\), we conclude that there exist \(q \in Q_{A,c}^{*}\) with \(b^{T}q \leq \alpha\) and \(\tilde{q} \in Q_{A,-\tilde{c}}^{*}\) with \(b^{T}\tilde{q} \leq -\alpha\). Then
\[
\hat{p} := \frac{q + \tilde{q}}{1^{T}q + 1^{T}\tilde{q}} \in Q_{A,0}^{*}, \quad b^{T}\hat{p} = \frac{b^{T}q + b^{T}\tilde{q}}{1^{T}q + 1^{T}\tilde{q}} \leq 0.
\]
Since $Q^k_{A,0}$ is a bounded polytope, this implies that there exists $p \in \text{ext}(Q^k_{A,0})$ with $b^T p \leq 0$. By Theorem 16, we have $b^T p = 0$.

If one of the inequalities in (4b) is attained, the corresponding facet is degenerated.

**Proposition 32** Let $b \in C_A$, and let $p \in \text{ext}(Q^*_{A,a_k})\{e_k\}$. Then $k \notin I^*_p$, and $b^T p = b_k$ holds if and only if $Q^k_{A,b} \subset Q^I_p$.

**Proof** We have $k \notin I^*_p$ by Lemma 25.

Assume that $p^T b = b_k$ holds. Then for any $i \in I^*_p$ and $x \in Q^k_{A,b}$, we get

$$-p_ia_i^T x = p^T Ax - a_k^T x - p_ia_i^T x = \sum_{j \in I^*_p} p_ja_j^T x - b_k - p_ia_i^T x = \sum_{j \in I^*_p \setminus \{i\}} p_ja_j^T x - b_k \leq \sum_{j \in I^*_p \setminus \{i\}} p_jb_j - b_k = p^T b - p_ib_i - b_k = -p_ib_i,$$

so $a_i^T x \geq b_i$. Conversely, let $Q^k_{A,b} \subset Q^I_p$. Since $b \in C_A$, there exists some $x \in Q^k_{A,b}$, and we find

$$p^T b = \sum_{i \in I^*_p} p_ib_i = \sum_{i \in I^*_p} p_ia_i^T x = (A^T p)^T x = a_k^T x = b_k.$$

Similarly, we can describe this situation from an algebraic perspective: If one of the inequalities in (4b) is an equality, then the corresponding condition $a_k^T x = b_k$ is redundant in the definition of $Q_{A,b}$.

**Proposition 33** Let $b \in C_A$, and let $p \in \text{ext}(Q^*_{A,a_k})\{e_k\}$. Then $k \notin I^*_p$, and $b^T p = b_k$ holds if and only if $a_i^T x \leq b_i$ for all $i \in I^*_p$ implies $a_k^T x \leq b_k$.

**Proof** Again, we have $k \notin I_p$ by Lemma 25.

If we assume that $b^T p = b_k$ holds and that $a_i^T x \leq b_i$ for all $i \in I^*_p$, then $a_k^T x \leq b_k$ follows directly from $A^T p = a_k$ and Proposition 2b.

Conversely, assume that $a_i^T x \leq b_i$ for all $i \in I^*_p$ implies $a_k^T x \leq b_k$. By Proposition 2b, there exists $\tilde{p} \in Q^*_{A,a_k}$ with $I^*_p \subset I^*_p$ and $b^T \tilde{p} \leq b_k$. Corollary 23 yields $\tilde{p} = p$, so $b^T p \leq b_k$, and $b \in C_A$ implies $b^T p \geq b_k$ by Theorem 16.

The correspondence between dimensionality and algebraic inequalities is more complicated for facets $Q^k_{A,b}$ than for the entire polyhedron $Q_{A,b}$.

**Corollary 34** Let $b \in C_A$, and let $p \in \text{ext}(Q^*_{A,a_k})\{e_k\}$. Then $b^T p = b_k$ implies $\dim(Q^k_{A,b}) \leq d - 2$.

**Proof** By Proposition 32, the identity $b^T p = b_k$ implies $Q^k_{A,b} \subset Q^I_p$. Since $p \in \text{ext}(Q^*_{A,a_k})\{e_k\}$, Lemma 25 yields $p_k = 0$, so $\#I_p \geq 2$. By Lemma 1b, the vectors $\{a_i : i \in I^*_p\}$ are linearly independent. Hence we conclude $\dim(Q^k_{A,b}) \leq \dim(H(A, b, I^*_p)) \leq d - 2$.

The following statement is a semi-converse of Corollary 34.
Proposition 35  Let $b \in C_A$. If $\dim(Q_A^k) \leq d - 2$, then $\dim(Q_{A,b}) \leq d - 1$ or there exists $p \in \text{ext}(Q_A^*_{a_k}) \setminus \{e_k\}$ with $b^T p = b_k$.

Proof  If $\dim(Q_A^k) \leq d - 2$, then there exist $\alpha \in \mathbb{R}$ and $c \in \mathbb{R}^d$ with $c \neq 0$ such that $\{a_k, c\}$ are linearly independent and $c^T x = \alpha$ holds for all $x \in Q_A^k$. Applying Proposition 2b to the inequalities

$$c^T x \leq \alpha \quad \forall x \in Q_A^k, \quad (-c)^T x \leq -\alpha \quad \forall x \in Q_A^k,$$

yields $q, \tilde{q} \in \mathbb{R}_+^N$ and $t, \tilde{t} \in \mathbb{R}_+$ with

$$A^T q - ta_k = c, \quad b^T q - tb_k \leq \alpha,$$

$$A^T \tilde{q} - \tilde{t}a_k = -c, \quad b^T \tilde{q} - \tilde{t}b_k \leq -\alpha.$$

Since $\{a_k, c\}$ are linearly independent, there exist $\ell, \tilde{\ell} \in \{1, \ldots, N\} \setminus \{k\}$ with $q_{\ell} > 0$ and $\tilde{q}_{\tilde{\ell}} > 0$. Setting $\hat{p} := q + \tilde{q}$ and $s := t + \tilde{t}$, we obtain $\hat{p} \in \mathbb{R}_+^N$, $\hat{p}_{\ell}, \hat{p}_{\tilde{\ell}} > 0$, $s \geq 0$ and

$$A^T \hat{p} = sa_k, \quad b^T \hat{p} \leq sb_k. \quad \text{(10)}$$

Case 1  If $\hat{p} \geq s$, define $\bar{\hat{p}} := \hat{p} - se_k$. Then $\bar{\hat{p}} \in \mathbb{R}_+^N \setminus \{0\}$, and from statement (10) we have $A^T \bar{\hat{p}} = 0$ and $b^T \bar{\hat{p}} \leq 0$. In particular, we have $\frac{\bar{\hat{p}}}{1^T \bar{\hat{p}}} \in Q_{A,0}^\circ$ and Theorem 16 yields $b^T \frac{\bar{\hat{p}}}{1^T \bar{\hat{p}}} = 0$. Since $Q_{A,0}^\circ$ is a bounded polytope, Proposition 31 yields $\dim(Q_{A,b}) \leq d - 1$.

Case 2  If $\hat{p} < s$, define $\hat{p} := \hat{p} - \tilde{p}ek$. Then $\tilde{p} \in \mathbb{R}_+^N \setminus \{0\}$ and $\hat{p} = 0$, and statement (10) yields

$$A^T \tilde{p} = ak, \quad b^T \tilde{p} \leq b_k. \quad \text{(11)}$$

In particular, we have $\frac{\tilde{p}}{s - \hat{p}} \in Q_A^*_{a_k}$. As in the proof of Theorem 16, we can write $\frac{\tilde{p}}{s - \hat{p}} = v + w$ with $v \in \text{conv}(\text{ext}(Q_A^*_{a_k}))$ and $w \in Q_A^*_{a_k}$. Since $\tilde{p} = 0$ and $v, w \geq 0$, we have $v_k = 0$. Denote $\text{ext}(Q_A^*_{a_k}) \setminus \{e_k\} = \{f^{k,1}, \ldots, f^{k,m_k}\}$, and let $\lambda \in \mathbb{R}_+^{m_k}$ and $\mu \geq 0$ with $1^T \lambda + \mu = 1$ and $v = \sum_{j=1}^{m_k} \lambda_j f^{k,j} + \mu e_k$. Then $v = 0$ and $f^{k,j} \geq 0$ for all $j \in \{1, \ldots, m_k\}$ force $\mu = 0$, so we have

$$\sum_{j=1}^{m_k} \lambda_j = 1, \quad v = \sum_{j=1}^{m_k} \lambda_j f^{k,j}. \quad \text{(12)}$$

By Theorem 16 and by statement (11), we have

$$\sum_{j=1}^{m_k} \lambda_j b^T f^{k,j} = b^T v \leq b^T v + b^T w = b^T (v + w) \leq b_k.$$

But Theorem 16 also guarantees $b^T f^{k,j} \geq b_k$ for all $j \in \{1, \ldots, m_k\}$, which implies $b^T f^{k,j} = b_k$ for every $j \in \{1, \ldots, m_k\}$ with $\lambda_j > 0$. By statement (12), there exists at least one such $j$, which completes the proof. \hfill \Box

Now we characterize the polyhedra $Q_{A,b}$, which correspond to interior points of $C_A$. Recall the definition of the matrix $F$ from Corollary 18.

Theorem 36  We have $b \in \text{int} C_A$ if and only if

$$\dim(Q_{A,b}) = d, \quad \dim(Q_A^k_{A,b}) = d - 1 \quad \forall k \in \{1, \ldots, N\}. \quad \text{(13)}$$
**Proof** According to Theorem 29, we have \( b \in \text{int} \, C_A \) if and only if \( F^T b > 0 \). If \( F^T b > 0 \), then Propositions 31 and 35 imply (13). Conversely, if condition (13) holds, then Proposition 31 and Corollary 34 imply \( F^T b > 0 \). \( \square \)

### 2.8 Approximation properties

The overall purpose of this paper is to replace a problem in \( K_c(\mathbb{R}^d) \) with a simpler problem in \( G_A \), having optimizers of a similar quality. For this approach to be meaningful, we must guarantee that for any given precision, we can choose the matrix \( A \) in such a way that the Galerkin space \( G_A \) approximates the space \( K_c(\mathbb{R}^d) \) from within with this desired precision.

First, we introduce a projector from \( K_c(\mathbb{R}^d) \) to \( G_A \).

**Proposition 37** Let \( \varphi \) and \( L_A \) as in Sect. 2.2. The mapping

\[
P_{C_A} : K_c(\mathbb{R}^d) \to C_A, \quad P_{C_A}(C) := (\sigma_C(a_1), \ldots, \sigma_C(a_N))
\]

is well-defined and \( 1 \)-Lipschitz from \( (K_c(\mathbb{R}^d), \text{dist}_H) \) to \( (\mathbb{R}^N, \| \cdot \|_\infty) \) with

\[
P_{C_A}(C) \leq P_{C_A}(\tilde{C}) \quad \forall \, C, \tilde{C} \in K_c(\mathbb{R}^d) \text{ with } C \subset \tilde{C},
\]

and the mapping

\[
P_{G_A} : K_c(\mathbb{R}^d) \to G_A, \quad P_{G_A}(C) := \varphi(P_{C_A}(C))
\]

is an \( L_A \)-Lipschitz projector from \( (K_c(\mathbb{R}^d), \text{dist}_H) \) onto \( (G_A, \text{dist}_H) \) with

\[
P_{G_A}(C) \subset P_{G_A}(\tilde{C}) \quad \forall \, C, \tilde{C} \in K_c(\mathbb{R}^d) \text{ with } C \subset \tilde{C}.
\]

**Proof** If \( C \in K_c(\mathbb{R}^d) \), then for all \( k \in \{1, \ldots, N\} \), there exists \( x_k \in C \) with

\[
a_k^T x_k = \sup_{x \in C} a_k^T x = \sigma_C(a_k), \quad a_k^T x_k \leq \sigma_C(a_k) \quad \forall \ell \in \{1, \ldots, N\}.
\]

In particular \( x_k \in Q_A(P_{C_A}(C)) \) for all \( k \in \{1, \ldots, N\} \), so \( P_{C_A}(C) \in C_A \), and the mapping \( P_{C_A} \) is well-defined.

It follows from Lemma 7 and \( \|a_k\|_2 = 1 \) for \( k \in \{1, \ldots, N\} \) that

\[
\|P_{C_A}(C) - P_{C_A}(\tilde{C})\|_\infty \leq \text{dist}_H(C, \tilde{C}) \quad \forall \, C, \tilde{C} \in K_c(\mathbb{R}^d).
\]

Since \( \varphi \) is \( L_A \)-Lipschitz according to Theorem 12, so is \( P_{G_A} \). By construction,

\[
P_{G_A}(Q_{A,b}) = \varphi(P_{C_A}(Q_{A,b})) = \varphi(b) = Q_{A,b} \quad \forall \, b \in C_A,
\]

so \( P_{G_A} \) is indeed a projector from \( K_c(\mathbb{R}^d) \) onto \( G_A \). \( \square \)

Now we investigate the quality of the approximation of \( K_c(\mathbb{R}^d) \) by \( G_A \). Theorem 38, originally proved in [13], provides a measure in terms of the metric density

\[
\delta_A := \sup_{c \in S^{d-1}} \min\{\|c - a_k\|_2 : k = 1, \ldots, N\}
\]

of the vectors \( \{a_k : k = 1, \ldots, N\} \) in the sphere \( S^{d-1} \). The assumption \( \delta_A \in (0, 1) \) is only restrictive when working with very coarse spaces \( G_A \) in high-dimensional ambient spaces \( \mathbb{R}^d \).
Theorem 38 Let $G_A$ be a space of polytopes (see Theorem 14), and let $\delta_A \in (0, 1)$. Then for every $C \in K_c(\mathbb{R}^d)$, we have $C \subseteq P_{G_A}(C)$ and

$$\text{dist}(P_{G_A}(C), C) \leq \frac{2 - \delta_A}{1 - \delta_A} \delta_A \|C\|_2.$$ 

Proof The inclusion $C \subseteq P_{G_A}(C)$ holds by construction of $P_{G_A}$, and since $G_A$ is a space of polytopes, Lemma 7 applied with $\tilde{C} = \{0\}$ yields

$$\|\sigma_{P_{G_A}(C)}\|_\infty = \|P_{G_A}(C)\|_2 < \infty.$$ 

Let $x \in P_{G_A}(C)$ and $c \in S^{d-1}$. By assumption, there exists $k \in \{1, \ldots, N\}$ with $\|c - a_k\|_2 \leq \delta_A$, so, again by Lemma 7, we have

$$c^T x = (c - a_k)^T x + a_k^T x$$

$$\leq \|c - a_k\|_2 \|x\|_2 + \|\sigma_C\|_\infty \leq \delta_A \|\sigma_{P_{G_A}(C)}\|_\infty + \|\sigma_C\|_\infty.$$ 

Since $x$ and $c$ were arbitrary, we have

$$\|\sigma_{P_{G_A}(C)}\|_\infty \leq \delta_A \|\sigma_{P_{G_A}(C)}\|_\infty + \|\sigma_C\|_\infty,$$

so that

$$\|\sigma_{P_{G_A}(C)}\|_\infty \leq \frac{1}{1 - \delta_A} \|\sigma_C\|_\infty. \quad (14)$$

Again, let $x \in P_{G_A}(C)$, $c \in S^{d-1}$ and $k \in \{1, \ldots, N\}$ with $\|c - a_k\|_2 \leq \delta_A$. Using inequality (14), Lemma 7 and $\sigma_C(a_k) = \sigma_{P_{G_A}(C)}(a_k)$, we obtain

$$c^T x - \sigma_C(c) = (c - a_k)^T x + a_k^T x - \sigma_C(c)$$

$$\leq \|c - a_k\|_2 \|x\|_2 + \|\sigma_C(a_k) - \sigma_C(c)\|_\infty \leq \delta_A \|\sigma_{P_{G_A}(C)}\|_\infty + \delta_A \|C\|_2$$

$$\leq \frac{\delta_A}{1 - \delta_A} \|\sigma_C\|_\infty + \delta_A \|C\|_2 = \frac{2 - \delta_A}{1 - \delta_A} \delta_A \|C\|_2.$$ 

Since $x$ and $c$ were arbitrary, it follows from Lemma 7 that

$$\text{dist}_H(P_{G_A}(C), C) = \|\sigma_{P_{G_A}(C)} - \sigma_C\|_\infty \leq \frac{2 - \delta_A}{1 - \delta_A} \delta_A \|C\|_2.$$ 

While the number $\delta_A$ only measures metric density, the quantity

$$\kappa_A := \sup_{c \in S^{d-1}} \inf \left\{ \sum_{k \in I_p} p_k \|a_k - \frac{c}{\|p\|_1}\|_2 : p \in Q^*_A, c \right\}$$

encodes the geometry of the matrix $A$. Let us first check that it is well-defined when $G_A$ is a space of bounded polytopes and the geometry of $A$ is sufficiently rich. For the interpretation of the following proposition, note that $\lim_{\rho \nearrow 1} \sqrt{(2 - 2\rho)/\rho} = 0$.

Proposition 39 Assume that there exists $\rho \in (0, 1)$ such that for every $c \in S^{d-1}$, there is $p \in Q^*_A, c$ with $\min_{i,j \in I_p} a_i^T a_j \geq \rho$. Then

$$\kappa_A \in \left[0, \sqrt{\frac{2 - 2\rho}{\rho}} \right].$$
Proof Let $c \in S^{d-1}$. By assumption, there exists $p \in \mathbb{R}^N_+$ with $A^T p = c$ and $\min_{i,j \in I_p^*} a_i^T a_j \geq \rho$, so

$$\rho \| p \|_2^2 = \rho \sum_{i,j \in I_p^*} p_i p_j \leq \sum_{i,j \in I_p^*} p_i a_i^T a_j p_j, \quad (15)$$

$$\| p \|_1^2 = \sum_{i,j \in I_p^*} p_i p_j \geq \sum_{i,j \in I_p^*} p_i a_i^T a_j p_j, \quad (16)$$

$$1 = \| c \|_2^2 = \| A^T p \|_2^2 = \sum_{i,j \in I_p^*} p_i a_i^T a_j p_j, \quad (17)$$

and combining statements (16) and (17), we obtain

$$\| a_k - \frac{c}{\| p \|_1} \|_2^2 = \| a_k \|_2^2 - \frac{2}{\| p \|_1^2} a_k^T c + \frac{\| c \|_2^2}{\| p \|_1^2} = 1 - \frac{2}{\| p \|_1^2} \sum_{i \in I_p^*} p_i a_i^T a_i + \frac{1}{\| p \|_1^2} \leq 2 - 2 \rho.$$

Using this and combining statements (15) and (17), we arrive at

$$\sum_{k \in I_p^*} p_k \| a_k - \frac{c}{\| p \|_1} \|_2 \leq \sqrt{\frac{2 - 2 \rho}{\rho}}.$$

Now we estimate the quality of the approximation of $K_c(\mathbb{R}^d)$ by $G_A$.

Theorem 40 Let $G_A$ be a space of polytopes. Then for every $C \in K_c(\mathbb{R}^d)$, we have $C \subset P_{G_A}(C)$ and

$$\text{dist}(P_{G_A}(C), C) \leq \kappa_A \| C \|_2.$$

Proof The definition of $P_{G_A}$ implies $C \subset P_{G_A}(C)$. Let us fix $C \in K_c(\mathbb{R}^d)$ and $z \in P_{G_A}(C)$. Then for every $c \in S^{d-1}$ and every $p \in \text{ext}(Q^*_{A,c})$, we obtain, using Lemma 7, that

$$c^T z - \sigma_C(c) = \sum_{k=1}^N p_k a_k^T z - \left( \sum_{k=1}^N \frac{p_k}{\| p \|_1} a_k \right) \sigma_C(c) \leq \sum_{k=1}^N p_k \sigma_C(a_k) - \sum_{k=1}^N \frac{p_k \sigma_C(c)}{\| p \|_1}$$

$$= \sum_{k=1}^N p_k \left( \sigma_C(a_k) - \sigma_C \left( \frac{c}{\| p \|_1} \right) \right) \leq \| C \|_2 \sum_{k=1}^N p_k \| a_k - \frac{c}{\| p \|_1} \|_2.$$

It follows from $C \subset P_{G_A}(C)$, Lemma 7 and the above computation that

$$\text{dist}(P_{G_A}(C), C) \leq \sup_{c \in S^{d-1}} | \sigma_{P_{G_A}(C)}(c) - \sigma_C(c) | \leq \kappa_A \| C \|_2.$$

$\square$
3 Galerkin optimization in $K_c(\mathbb{R}^d)$

In this section, we use the spaces analyzed in Sect. 2 to solve optimization problems in $K_c(\mathbb{R}^d)$ approximately. After gathering a few preliminaries in Sect. 3.1 we prove a convergence result for an abstract set optimization problem and suitable auxiliary problems in Sect. 3.2. In Sect. 3.3, we introduce the concept of Galerkin approximations to $K_c(\mathbb{R}^d)$, and in Sect. 3.4, we show in detail that an important class of optimization problems in $K_c(\mathbb{R}^d)$ and their Galerkin approximations are a special case of the abstract framework discussed in Sect. 3.2.

3.1 Preliminaries

All notions of convergence, continuity and compactness are to be understood in terms of the Hausdorff distance $\text{dist}_H$. We equip the space of all compact subsets of $(K_c(\mathbb{R}^d), \text{dist}_H)$ with the Hausdorff semi-distance and the Hausdorff-distance given by

$$D : 2^{K_c(\mathbb{R}^d)} \times 2^{K_c(\mathbb{R}^d)} \to \mathbb{R}_+, \quad D(M, \tilde{M}) := \sup_{C \in M} \inf_{\tilde{C} \in \tilde{M}} \text{dist}_H(M, \tilde{M}),$$

$$D_H : 2^{K_c(\mathbb{R}^d)} \times 2^{K_c(\mathbb{R}^d)} \to \mathbb{R}_+, \quad D_H(M, \tilde{M}) := \max\{D(M, \tilde{M}), D(\tilde{M}, M)\}.$$

Let us fix some notation for the objective function.

Definition 41 Consider a functional $\Phi : K_c(\mathbb{R}^d) \to \mathbb{R}$.

(a) The function $\Phi$ is called lower semicontinuous if for every $C \in K_c(\mathbb{R}^d)$ and every sequence $(C_k)_{k=0}^\infty \subset K_c(\mathbb{R}^d)$ with $\lim_{k \to \infty} \text{dist}_H(C_k, C) = 0$, we have $\liminf_{k \to \infty} \Phi(C_k) \geq \Phi(C)$.

(b) For every $\beta \in \mathbb{R}$, we denote $S(\Phi, \beta) := \{C \in K_c(\mathbb{R}^d) : \Phi(C) \leq \beta\}$.

The following result is Theorem 1.8.7 in [15].

Theorem 42 (Blaschke selection theorem) For every $R > 0$, the collection $\{C \in K_c(\mathbb{R}^d) : \|C\|_2 \leq R\}$ is compact.

3.2 An abstract framework

In this section, we develop an abstract framework, which guarantees the convergence of the minimizers of auxiliary problems to minimizers of an original problem in $K_c(\mathbb{R}^d)$. In the following sections, we will see that many optimization problems in $K_c(\mathbb{R}^d)$ are particular cases of this framework when discretized appropriately.

The following statements are variations of well-known facts.

Lemma 43 Let $\Phi : K_c(\mathbb{R}^d) \to \mathbb{R}$ be lower semicontinuous, let $\beta \in \mathbb{R}$, and let $M \subset K_c(\mathbb{R}^d)$ be a closed set. Then the following statements hold.

(a) The set $S(\Phi, \beta)$ is closed.

(b) The set $\arg\min_{C \in M} \Phi(C)$ is closed.

(c) If $M \cap S(\Phi, \beta)$ is nonempty and compact, then $\arg\min_{C \in M} \Phi(C) \neq \emptyset$.

The sets of global minima of suitable auxiliary problems converge to the set of global minima of the original optimization problem.
Theorem 44  Let \( \mathcal{M} \subset \mathcal{K}_c(\mathbb{R}^d) \) be nonempty and compact, and let \( (\mathcal{M}_k)_{k=0}^\infty \) be a sequence of nonempty and compact subsets \( \mathcal{M}_k \subset \mathcal{K}_c(\mathbb{R}^d) \) with
\[
\lim_{k \to \infty} \mathcal{D}_H(\mathcal{M}, \mathcal{M}_k) = 0. \tag{18}
\]
Let \( \Phi : \mathcal{K}_c(\mathbb{R}^d) \to \mathbb{R} \) be continuous, let \( (\Phi_k)_{k=0}^\infty \) be a sequence of mappings \( \Phi_k : \mathcal{M}_k \to \mathbb{R} \) satisfying
\[
\lim_{k \to \infty} \sup_{C \in \mathcal{M}_k} |\Phi(C) - \Phi_k(C)| = 0, \tag{19}
\]
and let \( \arg\min_{C \in \mathcal{M}_k} \Phi_k(C) \neq \emptyset \). Then \( \arg\min_{C \in \mathcal{M}} \Phi(C) \neq \emptyset \), and
\[
\lim_{k \to \infty} \mathcal{D}(\arg\min_{C \in \mathcal{M}_k} \Phi_k(C), \arg\min_{C \in \mathcal{M}} \Phi(C)) = 0. \tag{20}
\]
Proof  Consider a subsequence \( \mathbb{N}' \subset \mathbb{N} \) and sets \( C^*_k \in \arg\min_{C \in \mathcal{M}_k} \Phi_k(C) \) for all \( k \in \mathbb{N}' \). By statement (18), there exists a sequence \( (C_k)_{k \in \mathbb{N}'} \subset \mathcal{M} \) with
\[
\lim_{\mathbb{N}' \ni k \to \infty} \text{dist}_H(C_k, C^*_k) = 0.
\]
Since \( \mathcal{M} \) is compact, there exist \( C^* \in \mathcal{M} \) and a subsequence \( \mathbb{N}'' \subset \mathbb{N}' \) with \( \lim_{\mathbb{N}'' \ni k \to \infty} \text{dist}_H(C_k, C^*) = 0 \), so all in all, we have
\[
\lim_{\mathbb{N}'' \ni k \to \infty} \text{dist}_H(C^*_k, C^*) = 0. \tag{21}
\]
Continuity of \( \Phi \) and statements (19) and (21) yield
\[
\begin{align*}
\lim_{\mathbb{N}'' \ni k \to \infty} |\Phi(C^*) - \Phi_k(C^*_k)| & \leq \lim_{\mathbb{N}'' \ni k \to \infty} |\Phi(C^*) - \Phi(C^*_k)| + \lim_{\mathbb{N}'' \ni k \to \infty} |\Phi(C^*_k) - \Phi_k(C^*_k)| = 0.
\end{align*}
\]
\[
\begin{align*}
\lim_{\mathbb{N}'' \ni k \to \infty} \text{dist}_H(\tilde{C}_k, C) = 0. \tag{23}
\end{align*}
\]
Let \( C \in \mathcal{M} \). By statement (18), there exists \( (\tilde{C}_k)_{k \in \mathbb{N}''} \) with \( \tilde{C}_k \in \mathcal{M}_k \) and
\[
\lim_{\mathbb{N}'' \ni k \to \infty} \text{dist}_H(\tilde{C}_k, C) = 0.
\]
Again, statements (19) and (23) yield
\[
\begin{align*}
\lim_{\mathbb{N}'' \ni k \to \infty} |\Phi(C) - \Phi_k(\tilde{C}_k)| & \leq \lim_{\mathbb{N}'' \ni k \to \infty} |\Phi(C) - \Phi(\tilde{C}_k)| + \lim_{\mathbb{N}'' \ni k \to \infty} |\Phi(\tilde{C}_k) - \Phi_k(\tilde{C}_k)| = 0,
\end{align*}
\]
and because of statements (22) and (24), we have
\[
\Phi(C) = \lim_{\mathbb{N}'' \ni k \to \infty} \Phi_k(\tilde{C}_k) \geq \lim_{\mathbb{N}'' \ni k \to \infty} \Phi_k(C^*_k) = \Phi(C^*).
\]
All in all, we have \( C^* \in \arg\min_{C \in \mathcal{M}} \Phi(C) \).

Now assume that statement (20) is false. Then there exist \( \epsilon > 0 \), a subsequence \( \mathbb{N}' \subset \mathbb{N} \) and sets \( C^*_k \in \arg\min_{C \in \mathcal{M}_k} \Phi_k(C) \) for all \( k \in \mathbb{N}' \) with
\[
\mathcal{D}(C^*_k, \arg\min_{C \in \mathcal{M}} \Phi(C)) \geq \epsilon \ \forall k \in \mathbb{N}'. \tag{25}
\]
But the first part of the proof shows that there exists \( C^* \in \arg\min_{C \in \mathcal{M}} \Phi(C) \) such that statement (21) holds. This contradicts statement (25). \( \Box \)
3.3 Galerkin sequences

Now we introduce the equivalent to Galerkin schemes from the realm of partial differential equations.

**Definition 45** A sequence \((A_k)_{k=0}^{\infty}\) of matrices \(A_k \in \mathbb{R}^{N_k \times d}\) with \(N_k \in \mathbb{N}, k \in \mathbb{N}\), is called a Galerkin sequence if there exists a sequence \((\alpha_k)_{k=0}^{\infty} \in \mathbb{R}^+\) with \(\lim_{k \to \infty} \alpha_k = 0\) such that

\[
\inf_{C \in \mathcal{G}_{A_k}} \text{dist}_H(C, \tilde{C}) \leq \alpha_k \|C\|_2 \quad \forall C \in \mathcal{K}_c(\mathbb{R}^d).
\]

If, in addition, \(A_k\) is a submatrix of \(A_{k+1}\) for all \(k \in \mathbb{N}\), then we call \((A_k)_{k=0}^{\infty}\) a nested Galerkin sequence.

Let us draw some immediate conclusions from Definition 45.

**Lemma 46** If \((A_k)_{k=0}^{\infty}\) is a Galerkin sequence, the following statements hold.

(a) The spaces \(G_{A_k}\) consist of polytopes.

(b) If \((A_k)_{k=0}^{\infty}\) is nested, then \(G_{A_k} \subset G_{A_{k+1}}\) for all \(k \in \mathbb{N}\).

**Proof** (a) Fix \(k \in \mathbb{N}\). Since \(\{0\} \in \mathcal{K}_c(\mathbb{R}^d)\), we have

\[
\inf_{C \in \mathcal{G}_{A_k}} \text{dist}_H([0], C) \leq \alpha_k \cdot 0 = 0.
\]

In particular, there exists \(C \in \mathcal{G}_{A_k}\) with \(\|C\|_2 \leq 1\), and hence, by Theorem 14, the entire space \(G_{A_k}\) consist of polytopes.

Statement (b) is trivial. \(\square\)

Let us check that the concept of Galerkin sequences makes sense.

**Theorem 47** For every \(d \geq 2\), there exists a nested Galerkin sequence \((A_k)_{k=0}^{\infty}\) of matrices \(A_k \in \mathbb{R}^{N_k \times d}\).

**Proof** Consider spherical coordinates \(\zeta : [0, \pi)^{d-2} \times [0, 2\pi] \to S^{d-1}\) given by \(\zeta_i(\theta) = \cos(\theta_i) \prod_{j=1}^{i-1} \sin(\theta_j)\) for \(i \in \{1, \ldots, d-1\}\) and \(\zeta_d(\theta) = \prod_{j=1}^{d-1} \sin(\theta_j)\). For every \(k \in \mathbb{N}_1\), we consider the grid

\[
\Delta_k := \frac{\pi}{2^k} \mathbb{Z}^{d-1} \cap ([0, \pi)^{d-2} \times [0, 2\pi]),
\]

we define \(\{a_1^k, \ldots, a_{N_k}^k\} := \zeta(\Delta_k)\), and we let \(A_k\) be the matrix consisting of the rows \((a_1^k)^T, \ldots, (a_{N_k}^k)^T\). Since the grids \((\Delta_k)_{k=0}^{\infty}\) are nested, so are the matrices \((A_k)_{k=0}^{\infty}\).

For every \(c \in S^{d-1}\), there exists \(\theta \in [0, \pi)^{d-2} \times [0, 2\pi]\) with \(\zeta(\theta) = c\). By definition, there exists \(\tilde{\theta} \in \Delta_k\) with \(\|\theta - \tilde{\theta}\|_\infty \leq 2^{-k-1}\pi\). An elementary computation shows

\[
\|\zeta(\theta) - \zeta(\tilde{\theta})\|_2 \leq \sqrt{d} \|\zeta(\theta) - \zeta(\tilde{\theta})\|_\infty \leq d\sqrt{d}\|\theta - \tilde{\theta}\|_\infty \leq 2^{-k-1}\pi d\sqrt{d},
\]

so by Theorem 38, the spaces \(G_{A_k}\) have the desired approximation properties for sufficiently large \(k\). \(\square\)

The following proposition reveals additional details of the relationship between two polytope spaces from a nested Galerkin sequence, which may be of some interest for numerical computations with adaptive refinement.
Proposition 48 Let \( N_1, N_2 \in \mathbb{N} \) with \( N_1 < N_2 \), let \( a_1, \ldots, a_{N_2} \in S^{d-1} \) be pairwise distinct, and let \( A_1 \in \mathbb{R}^{N_1 \times d} \) and \( A_2 \in \mathbb{R}^{N_2 \times d} \) be the matrices consisting of the rows \( a_1^T, \ldots, a_{N_1}^T \) and \( a_{N_1+1}^T, \ldots, a_{N_2}^T, \) respectively. If the space \( G_{A_1} \) consists of polytopes, then the following statements hold:

(a) The space \( G_{A_2} \) consists of polytopes.
(b) We have \( P_{G_{A_2}}(C) \subset P_{G_{A_1}}(C) \) for any \( C \in \mathcal{K}_c(\mathbb{R}^d) \).
(c) We have \( P_{G_{A_2}}(G_{A_1}) \subset \text{bd}(G_{A_2}) \).

**Proof** Statement (a) follows from Theorem 14 and the fact that \( p \in Q_{A_1,c}^* \) implies \((p^T, 0_{\mathbb{R}^{N_2-N_1}})^T \in Q_{A_2,c}^* \). Statement (b) is obvious.

Let \( b_1^1 \in C_{A_1} \), and let \( b_2 := P_{C_{A_2}}(A_{1,b_1}^1). \) By the definitions of \( C_{A} \) and \( P_{C_{A}} \), we have \( b_2^i = b_1^i \) for all \( i \in \{1, \ldots, N_1\} \), and for every \( i \in \{N_1+1, \ldots, N_2\} \), Theorem 5 gives

\[
 b_2^i = \max \{ a_i^T x : x \in Q_{A_1,b_1} \} = \min \{ (b_1^i)^T p : p \in Q_{A_1,ai}^* \}.
\]

In particular, we have \( Q_{A_1,ai}^* \neq \emptyset \), and by Lemma 24, we have \( \text{ext}(Q_{A_1,ai}^*) \neq \emptyset \), so there exists \( p \in \text{ext}(Q_{A_1,ai}^*) \) with \( b_2^i = p^T b_1 \). By Lemma 1b, we have \((p^T, 0_{N_2-N_1}) \in \text{ext}(Q_{A_2,ai}^*) \).

Since \((p^T, 0_{N_2-N_1}) \neq e_i \) and

\[
 (p^T, 0_{N_2-N_1})b_2 = p^T b_1 = b_2^i,
\]

it follows from Theorem 29 that \( b_2^i \notin \text{int}(C_{A_2}) \).

Property (d) above may be undesirable. In particular, interior point methods require an initial guess in \( \text{int}(C_{A_2}) \). A simple solution to this problem is provided in the following proposition.

Proposition 49 Let \( A \in \mathbb{R}^{N \times d} \) such that \( G_{A} \) consists of polytopes, and let \( \lambda \in (0, 1) \). Then the mappings

\[
P_{\lambda C_{A}} : \mathcal{K}_c(\mathbb{R}^d) \rightarrow \text{int}(C_{A}), \quad P_{\lambda C_{A}}(C) := (1-\lambda)P_{C_{A}}(C) + \lambda \| C \|_2 \mathbb{1},
\]
\[
P_{\lambda G_{A}} : \mathcal{K}_c(\mathbb{R}^d) \rightarrow G_{A}, \quad P_{\lambda G_{A}}(C) := \varphi(P_{\lambda C_{A}}(C))
\]

with \( \varphi \) as in Theorem 12 satisfy

\[
 \| P_{\lambda C_{A}}(C) - P_{C_{A}}(C) \|_{\infty} \leq 2\lambda \| C \|_2, \quad \forall C \in \mathcal{K}_c(\mathbb{R}^d),
\]
\[
 \text{dist}_H(P_{\lambda C_{A}}(C), P_{\lambda G_{A}}(C)) \leq 2\lambda L_A \| C \|_2, \quad \forall C \in \mathcal{K}_c(\mathbb{R}^d).
\]

**Proof** According to Theorem 29, we have \( 1 \in \text{int}(C_{A}) \), and by Corollary 15, the coordinate space \( C_{A} \) is a convex cone. Since \( P_{C_{A}}(C) \in C_{A} \), it follows that

\[
 (1-\lambda)P_{C_{A}}(C) + \lambda \| C \|_2 \mathbb{1} \in \text{int}(C_{A}).
\]

The estimates follow from Lemma 7, the computation

\[
 \| P_{\lambda C_{A}}(C) - P_{C_{A}}(C) \|_{\infty} \leq \lambda \| P_{C_{A}}(C) - \| C \|_2 \mathbb{1} \|_{\infty} \leq \lambda(\| \sigma_C \|_{\infty} + \| C \|_2) \leq 2\lambda \| C \|_2,
\]

and the fact that \( \varphi \) is \( L_A \)-Lipschitz.

\[ \square \]
3.4 A concrete optimization problem in \( \mathcal{K}_c(\mathbb{R}^d) \)

Throughout this section, we fix a continuous functional \( \Phi : \mathcal{K}_c(\mathbb{R}^d) \to \mathbb{R} \), an \( L \)-Lipschitz constraint \( \Psi : (\mathcal{K}_c(\mathbb{R}^d), \text{dist}_H) \to (\mathbb{R}^m, \| \cdot \|_\infty) \) as well as sets \( \hat{C}, \hat{\hat{C}} \in \mathcal{K}_c(\mathbb{R}^d) \), and we consider the model problem

\[
\min_{C \in \mathcal{K}_c(\mathbb{R}^d)} \Phi(C) \quad \text{subject to} \quad \Psi(C) \leq 0, \quad \hat{C} \subset C \subset \hat{\hat{C}}. \tag{26}
\]

We fix a nested Galerkin sequence \((A_k)_{k=0}^\infty\) with \( A \in \mathbb{R}^{N_k \times d} \) and approximate this problem with a sequence

\[
\min_{C \in \mathcal{G}_{A_k}} \Phi_k(C) \quad \text{subject to} \quad \Psi_k(C) \leq 0, \quad P_{\mathcal{G}_{A_k}}(\hat{C}) \subset C \subset P_{\mathcal{G}_{A_k}}(\hat{\hat{C}}) \tag{27}
\]

of finite-dimensional problems with suitable mappings \( \Psi_k \), which become

\[
\min_{b \in \mathcal{C}_{A_k}} \Phi_k(Q_{A_k,b}) \quad \text{subject to} \quad \Psi_k(Q_{A_k,b}) \leq 0, \quad P_{\mathcal{C}_{A_k}}(\hat{\hat{C}}) \leq b \leq P_{\mathcal{C}_{A_k}}(\hat{C}) \tag{28}
\]

when expressed in coordinates. By Corollary 18, the constraint \( b \in \mathcal{C}_{A_k} \) can be represented as a linear inequality. Let us denote

\[
\mathcal{M} := \{ C \in \mathcal{K}_c(\mathbb{R}^d) : \Psi(C) \leq 0, \quad \hat{C} \subset C \subset \hat{\hat{C}} \}, \quad \mathcal{M}_k := \{ C \in \mathcal{G}_{A_k} : \Psi_k(C) \leq 0, \quad P_{\mathcal{G}_{A_k}}(\hat{C}) \subset C \subset P_{\mathcal{G}_{A_k}}(\hat{\hat{C}}) \}.
\]

Conditions \((4a)\) are redundant in the characterization of \( \mathcal{M}_k \), which is convenient from a computational perspective.

**Lemma 50** For any \( b \in \mathbb{R}^{N_k} \) with \( P_{\mathcal{G}_{A_k}}(\hat{\hat{C}}) \subset Q_{A_k,b} \) and any \( a \in \text{ext}(Q_{A_k,b}^o) \), we have \( 0 \leq a^T p \).

**Proof** Since \( \emptyset \neq P_{\mathcal{G}_{A_k}}(\hat{\hat{C}}) \subset Q_{A_k,b} \) this follows from Proposition 2a.

The constraints \( \Psi_k \) can be designed in such a way that the sets \( \mathcal{M}_k \) converge to \( \mathcal{M} \). Recall the definition of the constant \( \kappa_A \) from Sect. 2.8.

**Proposition 51** The set \( \mathcal{M} \) is compact. Assume that \( \mathcal{M} \neq \emptyset \), and define

\[
\Psi_k : \mathcal{K}_c(\mathbb{R}^d) \to \mathbb{R}^m, \quad \Psi_k(C) := \Psi(C) - L\kappa_{A_k} \| \hat{C} \|_2 \mathbb{1}_{\mathbb{R}^m}.
\]

Then the sets \( \mathcal{M}_k \) are nonempty and compact for all \( k \in \mathbb{N} \). If, in addition, we have \( \lim_{k \to \infty} \kappa_{A_k} = 0 \), then we have

\[
D_H(\mathcal{M}, \mathcal{M}_k) \to 0 \quad \text{as} \quad k \to \infty.
\]

**Proof** By Theorem 42, the set \( \{ C \in \mathcal{K}_c(\mathbb{R}^d) : \hat{C} \subset C \subset \hat{\hat{C}} \} \) is relatively compact. Since \( \hat{\hat{C}} \subset C \subset \hat{C} \) holds if and only if we have \( \text{dist}(\hat{\hat{C}}, C) = 0 \) and \( \text{dist}(C, \hat{C}) = 0 \), and since \( C \mapsto \text{dist}(\hat{C}, C) \) and \( C \mapsto \text{dist}(C, \hat{\hat{C}}) \) are continuous, the set \( \{ C \in \mathcal{K}_c(\mathbb{R}^d) : \hat{C} \subset C \subset \hat{\hat{C}} \} \) is closed. By continuity of \( \Psi \), the set \( \{ C \in \mathcal{K}_c(\mathbb{R}^d) : \Psi(C) \leq 0 \} \) is closed as well. All in all, the set \( \mathcal{M} \) is compact, and the sets \( \mathcal{M}_k, k \in \mathbb{N} \), are compact for the same reasons.

Let \( \mathcal{M} \neq \emptyset \). All \( C \in \mathcal{M} \) satisfy \( C \in \mathcal{K}_c(\mathbb{R}^d) \) and \( \Psi(C) \leq 0 \), as well as \( \hat{\hat{C}} \subset C \subset \hat{C} \). By Proposition 37, we have \( P_{\mathcal{G}_{A_k}}(\hat{\hat{C}}) \subset P_{\mathcal{G}_{A_k}}(C) \subset P_{\mathcal{G}_{A_k}}(\hat{C}) \), and according to Theorem 40, we have

\[
\text{dist}_H(C, P_{\mathcal{G}_{A_k}}(C)) \leq \kappa_{A_k} \| C \|_2 \leq \kappa_{A_k} \| \hat{\hat{C}} \|_2.
\]
But then
\[
\Psi_k(P_{\mathcal{G}A_k}(C)) = \Psi(P_{\mathcal{G}A_k}(C)) - L \kappa_{A_k} \| \hat{C} \|_2 \leq \Psi(C) + \| \Psi(P_{\mathcal{G}A_k}(C)) - \Psi(C) \|_\infty - L \kappa_{A_k} \| \hat{C} \|_2 \leq 0
\]
shows \( P_{\mathcal{G}A_k}(C) \in \mathcal{M}_k \). Hence \( \mathcal{M}_k \neq \emptyset \) and \( D(\mathcal{M}, \mathcal{M}_k) \to 0 \) as \( k \to \infty \).

Assume that \( D(\mathcal{M}_k, \mathcal{M}) \not\to 0 \) as \( k \to \infty \). Then there exist a number \( \epsilon > 0 \), a subsequence \( N'' \subset N \) and \( (C_k)_{k \in N'} \) with \( C_k \in \mathcal{M}_k \) for all \( k \in N' \) and
\[
D(\mathcal{M}_k, \mathcal{M}) \geq \epsilon \quad \forall k \in N'.
\]
By Lemma 48b, we have \( C_k \subset P_{\mathcal{G}A_k}(\hat{C}) \subset P_{\mathcal{G}A_0}(\hat{C}) \), and by Lemma 46a, the set \( P_{\mathcal{G}A_0}(\hat{C}) \) bounded, so according to Theorem 42, there exist a subsequence \( N'' \subset N \) and \( C \in \mathcal{K}_c(\mathbb{R}^d) \) such that \( \lim_{N'' \ni k \to \infty} \text{dist}_H(C, C_k) = 0 \). By continuity of \( \Psi \) and by the definition of \( \Psi_k \), we have
\[
\Psi(C) = \lim_{N'' \ni k \to \infty} \Psi(C_k) = \lim_{N'' \ni k \to \infty} \left( \Psi_k(C_k) + L \kappa_{A_k} \| \hat{C} \|_2 \right) = 0.
\]
Moreover, since \( C_k \subset P_{\mathcal{G}A_k}(\hat{C}) \) and by Proposition 40, we have
\[
\text{dist}(C, \hat{C}) \leq \text{dist}(C, C_k) + \text{dist}(C_k, P_{\mathcal{G}A_k}(\hat{C})) + \text{dist}(P_{\mathcal{G}A_k}(\hat{C}), \hat{C}) \to 0 \quad \text{as} \quad N'' \ni k \to \infty,
\]
so \( C \subset \hat{C} \). But then \( C \in \mathcal{M} \), which contradicts statement (30). All in all, we proved that \( D(\mathcal{M}_k, \mathcal{M}) \to 0 \) as \( k \to \infty \).

Now we gather the results from this section in a final statement.

**Theorem 52** For \( k \in \mathbb{N} \), let \( \Phi_k : \mathcal{K}_c(\mathbb{R}^d) \to \mathbb{R} \) be lower semicontinuous mappings which satisfy condition (19), let constraints \( \Psi_k : \mathcal{K}_c(\mathbb{R}^d) \to \mathbb{R}^m \) be defined by (29), and assume that \( \mathcal{M} \neq \emptyset \). Then \( \arg \min_{C \in \mathcal{M}} \Phi_k(C) \neq \emptyset \), and
\[
\lim_{k \to \infty} D(\arg \min_{C \in \mathcal{M}_k} \Phi_k(C), \arg \min_{C \in \mathcal{M}} \Phi(C)) = 0.
\]

**Proof** By Proposition 51, the set \( \mathcal{M} \) is compact, and for every \( k \in \mathbb{N} \), there exists \( C_k \in \mathcal{M}_k \), and the set \( \mathcal{M}_k \) is compact. As \( \Phi_k \) is lower semicontinuous, the nonempty sets \( S(\Phi_k, \Phi_k(C_k)) \) are closed by Lemma 43a. Lemma 43c yields that \( \arg \min_{C \in \mathcal{M}_k} \Phi_k(C) \neq \emptyset \). In addition, Proposition 51 ensures that condition (18) is satisfied. Thus Theorem 44 applies and yields the desired statement.

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