Lateral-torsional Buckling Moment of Simply Supported Unrestrained Monosymmetric Beams

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Abstract. The stability equations for simply supported beams were solved approximately using the Bubnov–Galerkin method. We omit the influence of displacements in the plane of bending. The analytical solutions were used for checking the stability of laterally unrestrained monosymmetric beams. The lateral-torsional buckling (LTB) moment depends on bending distribution and on the load height effect. Each of applied concentrated and distributed loads, may have arbitrary direction and optional coordinate for the applied force along the cross section’s height. Derived equations allow for simple, yet fast control of lateral buckling moment. Numerical examples show good consistency of analytic results with those obtained from Finite Element Method (FEM) software.

1. Introduction
The lateral-buckling resistance assessment is usually based on buckling curves and requires the computation of the elastic critical moment. To determine LTB moment of the monosymmetric I-shaped beams the LTBeam [1] programme is commonly used. In this paper, a contribution to the LTB of unrestrained thin-walled elements with monosymmetric open sections is investigated. According to the classical paper [2] an analytical solution is used for checking the lateral stability of beams. It depends on bending distribution, on load height effect and on degree of the monosymmetry of the section called Wagner’s parameter. Coefficients $C_1$, $C_2$ and $C_3$ are respectively affected to these parameters. In presented work simply supported beams is analysed – the most common case in design practice. Non-linear distributions of the bending moment along the element are considered in the $C_1$ coefficient. It is determined on the basis of the absolute values of the bending moment in 1/4, 1/2, 3/4 of the beam span and its maximum absolute value. There are several $C_1$ approximation methods based on these parameters in the literature [3, 4, 5].

Coefficient $C_2$ take into account of effect of position of load application with respect to height of the element cross-section for arbitrary loads [5, 6, 7, 8]. It was found, in available literature, only two general approaches of solving this problem [5, 9]. Formula presented in work [5] is correct only in some cases (Table 4). The second method [9] is the modification of the $C_1$ coefficient on the basis of numerical simulations. There, the case was considered, when the load was applied at the same height of the cross section. In this publication arbitrary load may be applied at different heights of the cross section. The method presented here is the significant extension existing methods. This is a development of the author’s previous work [10].

The elastic LTB moment of mono-symmetric thin-walled beams may be computed using the expression [2, 6]:

$$M_{LTB} = C_1 M_{b} + C_2 M_{h} + C_3$$
\[ M_0^b = C_1 N_{cr,z} (B + \sqrt{D + B^2}) \]  

(1)

where: \( N_{cr,z} = \pi^2 E I_z / L^2 \), \( D = I_w / I_z + G I_T / N_{cr,z} \) (\( E \) – Young’s modulus, \( I_z \) – second moment of area about the \( z \)-\( z \) axis, \( L \) – beam span, \( I_w \) – warping moment of inertia, \( G \) – shear modulus, \( I_T \) – St. Venant torsion constant).

Coefficient \( B \) we calculate from the formula:

\[ B = C_2 e_g + C_3 \beta_z \]  

(2)

where: \( e_g \) – ordinate of the load application position relative to the shear centre \( M \) (Figure 2), \( \beta_z \) – parameter of asymmetry.

The LTB moment \( M_0^b \) is designated from the stability equation as:

\[ M_0^b = \mu_b M_0 \]  

(3)

where: \( M_0 \) – max. absolute value of bending moment \( (M_0 = \max \left| M_y(x) \right| \text{ for } 0 \leq x \leq L) \), \( \mu_b \) – the smallest absolute value of critical load multiplier.

Table 1 shows the coefficients \( C_1, C_2 \) and \( C_3 \) for the two basic load schemes. If we use this table, the \( z \) axis must coincide with the load direction (Figure 2) and pay attention to the correct sign of the asymmetry parameter \( \beta_z \) (Figure 3).

| Static scheme | Dist. of the bending moment | \( M_0 \) | \( C_1 \) | \( C_2 \) | \( C_3 \) |
|---------------|-----------------------------|----------|----------|----------|----------|
| qL/8          | qL/8                        | 1.13     | 0.46     | 0.53     |
| QL/4          | QL/4                        | 1.36     | 0.55     | 0.41     |

In order to take into account position of the load with respect to height of the cross-section [5] proposed the following relationship: \( C_2 = 0.4 C_1 \). As the example in Table 4 shows, this is incorrect for some static schemes. The error in calculating the LTB moment in the case of load applied to the upper flange is important (Table 4). Using the same as [5] approach to determining the coefficient \( C_1 \), based on [10], correct formulas for the coefficient \( C_2 \) can be obtained. The results of these analyses are presented in Tables 2 and 3.

Table 2. Coefficients \( C_1, C_2 \) for special case

| Static scheme | \( \psi \) | \( C_1 \) | \( C_2 \) |
|---------------|----------|----------|----------|
| \( 0 \leq \psi \leq 0.75 \) | 1.13 +0.12\( \psi \) | 0.41\( C_1/(1-2\psi/3) \) |
| \( 0.75 \leq \psi \leq 1 \) | -2.8 +5.4\( \psi \) | 0.61\( C_1/\psi \) |

In columns 5, 6 Table 3 the LTB moment is presented for beams with the span \( L = 8 \) and 12 m together with the relative error of approximation in relation to the results of FEM [1]. The \( z \) axis and load direction they are pointing down. The load is applied to the upper flange I-section IPE 500.
Table 3. Coefficients $C_1$, $C_2$ and LTB moment for different values of the coefficient $\psi$ obtained based on the basis of Table 2

| Moment distribution | (1) $\psi M$ | (2) $\psi M$ | (3) $1.5M(1-2\psi/3)$ | (4) $L=8m$ | (5) $L=12m$ |
|---------------------|--------------|--------------|-------------------------|------------|------------|
| $\psi$              | $C_1$       | $C_2$       | $L=8m$                 | $L=12m$   |
| 0                   | 1.130       | 0.463       | 238.0/−0.4             | 152.3/−0.3|
| 0.4                 | 1.178       | 0.659       | 221.2/−0.5             | 145.4/−0.3|
| 0.8                 | 1.520       | 1.159       | 217.9/ 2.1             | 151.7/ 2.8|
| 1                   | 2.600       | 1.586       | 304.6/−0.4             | 219.6/−0.4|

Table 4. Comparison of values derived from [5, 10]

|          | [10] | [5] |
|----------|------|-----|
| $C_1$    | 2.6  | 2.42|
| $C_2$    | 1.6  | 0.97|
| $C_3$    | 0    | 0   |
| $M_{cr}$ | 304.1 (err. -0.4%) | 382.5 (err. -25.1%) |

A major advantage of some codes, such as the American AISC LRFD [4] is that they provide closed-form expressions to compute the $C_1$ coefficient for any moment distribution. In this code the $C_1$ coefficient can be estimated based on the absolute values of bending moments:

$$C_1 = \frac{12.5 M_0}{2.5 M_0 + 3 M_2 + 4 M_3 + 3 M_4} \leq 2.27$$

(4)

where: $M_2$, $M_3$ and $M_4$ – absolute bending moments values at $x=L/4$, $L/2$ and $3L/4$ (Figure 1a).

Equation (4) is some modification of formula shown by [3]. Analogous dependencies can be found in the works [5, 10].

![Geometrical interpretation of data in formulas (4) and (5), respectively](image1.png)

Figure 1. Geometrical interpretation of data in formulas (4) and (5), respectively

However, in the case of a linear distribution of the bending moment coefficients $C_1$, $C_3$ approximated by the following relations:

$$C_1 = 1.77 - 1.04 \psi + 0.27 \psi^2 \leq 2.60,$$

$$C_3 = 0,$$

$$C_3 = c_1 \frac{1+\psi}{2}$$

(5a-c)

where: $\psi$ is the ratio of the moments at the ends of the beam ($-1 \leq \psi \leq 1$, Figure 1b ).

2. Analytical solution for checking the stability of laterally unrestrained monosymmetric beams

2.1. Differential equation of lateral-torsional buckling of monosymmetric beams

Let’s consider the simply supported beam. It is loaded by the moments, which are concentrated at its ends and the lateral load at span (Figure 1). Any distributed and concentrated loads may have an arbitrary coordinate for the force applied at the height of the cross section. The centre of gravity of the
cross section is denoted as $G$ and the centre of shear as $M$. Figure 2 shows the positive support moments for the adopted coordinate system.

![Figure 2. Static scheme: a) positive directions of load b) cross section’s geometry.](image)

Considering the boundary conditions for the simply supported beam, the differential equation of the flexural-torsional loss of stability, as the function of the torsion angle $\varphi(x)$ can be written as [7]:

$$ -\frac{[M_b^2(x)]^2}{EI_z} \varphi - [2\beta_z M_y^b(x)\varphi^{(i)}] + \bar{q}_z^b(x) \varphi + EI_{\beta} \varphi^{(4)} - GI_T \varphi^{(2)} = 0 $$

(6)

where: $\varphi^{(k)} = \frac{\partial^k \varphi}{\partial x^k}$ – denotation of derivative.

The upper index $b$ refers to the critical state. In this differential equation, $\varphi(x)$ describes buckling mode angle of the cross-section. The remaining parameters in the critical state are determined from the dependence:

$$ M_y^b(x) = \mu_b \cdot M_y(x), \quad \bar{q}_y^b(x) = \mu_b \cdot \bar{q}_y(x), \quad \bar{q}_z^b(x) = q_z e_d + \sum_{k=1}^{N} Q_z e_k \delta(x - x_k) $$

(7a-c)

where: $M_y(x)$ – distribution of the bending moment along the beam, $q_z(x)$ – distributed load in the $z$ direction, $N$ – the number of concentrated forces, $Q_z$ – $k^{th}$ concentrated load in the $z$ direction, $x_k$ – coordinate of applied $k^{th}$ concentrated load, $\Delta$ – Dirac’s function. The place where the load is applied at the cross-section height in the case of distributed $e_d$ and concentrated $e_k$ load is determined in relation to the shear centre $M$ (Figure 2).

The only difference in the differential equation (6) in relation to the work [7] is shown in formula (7c) and refers to the place where the load is applied at the cross-sectional height.

The distribution of the bending moment along the $x$-axis can be written using formula:

$$ M_y(x) = M_L + (M_R - M_L) \frac{x}{L}\left(1 - \frac{x}{L}\right) + \sum_{k=1}^{N} M_y^k(x), \quad M_y^k(x) = \begin{cases} \frac{Q_z e_k \delta(x - x_k) / L}{1 - x_k / L} & 0 \leq x \leq x_k \\ Q_z e_k \delta(x - x_k) / L & x_k \leq x \leq L \end{cases} $$

(8a-b)

where: $M_L, M_R$ – concentrated moments applied to the left and right end of the beam at the supports respectively (Figure 2).

2.2. The Bubnov–Galerkin’s orthogonalization method

The stability equations (6) were solved approximately using the Bubnov–Galerkin method. In case of simply supported beams with free warping, sinusoidal mode is assumed for the torsion angle. The angle is approximated using only the first term of series. After integration, the lateral buckling moment is given by the roots of the quadratic equation, with regard to the absolute of the maximum bending moment $M_0$ (Figure 2), where:

$$ a_1 (M_0^h)^2 + N_{\varphi,z}^2 (a_{zz} + a_z \beta_z) M_0^h + N_{\varphi,z}^2 D = 0, $$

(9)
where: $a_1$ – coefficient describes the influence of distribution of bending moment along the length of an element, $a_2$ – the influence load height effect while $a_3$ – degree of the monosymmetry of the cross-section:

$$a_1 = \frac{2}{M_0^2 L_0} \int_0^L M_y^2(x) \sin^2(\pi x/L) \, dx,$$

$$a_{2z} = \frac{2L}{M_0 \pi^2} \left[ \frac{q_2 L}{2} \xi + \sum_{k=1}^N Q_z k \sin^2(\pi \xi_k) \right]$$

$$a_3 = \frac{M_L + M_R}{M_0} + \frac{L}{M_0} \left\{ 0, 116 q_2 L + \sum_{k=1}^N Q_z k \left[ \xi_k (1 - \xi_k) - 0, 1 \sin^2(\pi \xi_k) \right] \right\}$$

where: $\xi_k = \frac{x_k}{L}$.

Equation (9) can be further converted into:

$$(M_0^b)^2 - 2C_1 N_{cr,z} B \cdot M_0^b - C_1^2 N_{cr,z} D = 0$$

where:

$$C_1 = \frac{1}{\sqrt{a_1}} = \frac{M_0}{\sqrt{\int_0^L M_y^2(x) \sin^2(\pi x/L) \, dx}}$$

$$\bar{B} = C_{2z} + C_3 \beta_z \quad C_{2z} = C_1 \frac{a_{2z}}{2}, \quad C_3 = C_1 \frac{a_3}{2}$$

Derived from the solution of quadratic equation (13) the value of critical moment is expressed as:

$$M_0^b = C_1 N_{cr,z} \left( \bar{B} \pm \sqrt{\bar{D} + \bar{B}^2} \right) > 0$$

Because $M_0 > 0$, the LTB buckling is determined by the positive root of the equation (16). If each of the transverse loads is applied at the same cross-section height ($e_z = e_A = e_k$), the $a_{2z}$ and $B$ factors are determined from formulas (17, 18). Then equation (16) coincides with formula (2).

$$a_{2z} = a_2 e_z, \quad a_2 = \frac{2L}{M_0 \pi^2} \left[ \frac{q_2 L}{2} + \sum_{k=1}^N Q_z k \sin^2(\pi \xi_k) \right]$$

$$\bar{B} = B = C_2 e_z + C_3 \beta_z, \quad C_2 = C_1 \frac{a_2}{2}$$

3. Example

Let’s consider an I-beam with a monosymmetrical cross-section (Figure 3) in both examples below. For simplicity, it was neglected dead weight of beam. Spacing of beam supports is equal to $L=7.5$ m.

![Figure 3](image-url)  

**Figure 3.** a) cross section’s geometry, b) positive directions of concentrated moments and the $\beta_z$ asymmetry parameter.
For monosymmetric, welded I-section we have:

\[ z_M = h_2 - \frac{I_{z1}}{I_z} h_0, \quad I_w = \frac{I_{z1} I_{z2}}{I_z} h_0^2, \quad \beta_z = \frac{p_0 + p_1 + p_2}{2 I_w}, \quad (19a-c) \]

\[ p_0 = t_w (\bar{h}_1^3 - \bar{h}_1^4) / 4, \quad p_1 = -I_{z1} h_1 - A_1 h_1^3, \quad p_2 = I_{z2} h_2 + A_2 h_2^3 \quad (20a-c) \]

where: \( A_1 = t_1 b_1, \quad I_{z1} = t_1 b_1^3 / 12, \quad A_2 = t_2 b_2, \quad I_{z2} = t_2 b_2^3 / 12, \quad t_1, t_2 - \) thickness of the bottom and top flange, respectively, \( t_w - \) web thickness, \( \bar{h}_1 = h_1 - t_1 / 2, \quad \bar{h}_2 = h_2 - t_2 / 2. \)

Substituting: \( h_2 = 20.9 \text{ cm}; \quad A_2 = 120 \text{ cm}^2; \quad I_{z2} = 16000 \text{ cm}^4; \quad h_1 = 26.1 \text{ cm}; \quad A_1 = 90 \text{ cm}^2; \quad I_{z1} = 6750 \text{ cm}^4; \quad L = 22762 \text{ cm}^4, \quad I = 124820 \text{ cm}^4. \) (flange of an I-section beam denoted as 2 is considered at the positive \( z \)-axis) in the equation \((19, 21)\) we’ll get: \( z_M = 6.95 \text{ cm}, \quad \beta_z = -8.68 \text{ cm}, \quad I_w = 10487 \times 10^3 \text{ cm}^6, \quad I_T = 679.5 \text{ cm}^4. \) So, for the beam of \( L = 7.5 \text{ m} \) span: \( N_{cr,z} = 8387 \text{ kN}, \quad D = 1117 \text{ cm}^2. \)

Geometrical characteristics determined with the FEM \([1]\) program are respectively: \( z_M = 6.99 \text{ cm}, \quad \beta_z = -8.64 \text{ cm}, \quad I_T = 651 \text{ cm}^4, \quad I_w = 10500 \times 10^3 \text{ cm}^6. \) The effect of this difference on the critical moment of lateral torsional buckling is negligible.

Let’s consider simple supporting beam shown on Figure 4. Substituting (in kNm): \( M_0 = 250; \quad M_2 = 144.3; \quad M_3 = 200.8; \quad M_4 = 19.3 \) into equation \((4)\), we obtain:

\[ C_1 = \frac{12.5 \cdot 250}{2.5 \cdot 250 + 3 \cdot 144.3 + 4 \cdot 200.8 + 3 \cdot 19.3} = 1.628 \quad (22) \]

Then let's substitute (in kN, m, Figure 4) \( M_0 = 250, \quad q_z = -25, \quad Q_{z1} = -80, \quad x_1 = 3.75 \) and \( M_R = 250 \) into equation \((12)\):

\[ a_3 = \frac{250}{250} + \frac{7.5}{250} \left[ 0.116 \cdot (-25) \cdot 7.5 + (-80) \cdot \left( 0.5 \cdot (1 - 0.5) - 0.1 \cdot \sin(0.5\pi) \right) \right] = -0.009 \quad (23) \]

In the equation \((11)\) we’ll substitute: \( M_0 = 250, \quad q_z = -25, \quad Q_{z1} = -80, \quad x_1 = 3.75 \) (in kN and m). In case of distributed and concentrated load shown at Figure 4c \( (e_A = e_1 = 19.44 \text{ cm}) \) using equation \((11)\) we’ll get \( a_{2z} = -20.53 \text{ cm}. \) For the load shown in Figure 2d \( (e_A = 19.44 \text{ cm}, \quad e_1 = -34.56 \text{ cm}) \) \( a_{2z} = 5.70 \text{ cm}. \) Next, for both types of loads, we’ll calculate the coefficient \( C_{2z} \) from equation \((15b)\) and \( C_3 \) from equation \((15c)\), respectively. The comparison of LTB moments obtained with equation \((16)\) and FEM \([1]\) program is presented at the Table 4. In both cases the estimation error is less 1%.

![Figure 4](image-url)
Table 5. Comparison of the lateral buckling moment as function of placement of the lateral load (\(e_x = 194.4 \text{ mm}\))

| No. | Position of concentrated force Figure 4c,d | \(C_1\) | \(C_2\) [cm] | \(C_3\) | \(M_0^b\)/ err.% |
|-----|------------------------------------------|--------|-------------|--------|-----------------|
| 1.  | \(e_1=194.4 \text{ mm (Figure 4c)}\)    | 1.628  | -16.71      | -0.007 | 2824.9/-0.4     |
| 2.  | \(e_1=-345.6 \text{ mm (Figure 4d)}\)  | 1.628  | 4.640       | -0.007 | 5250.1/0.7      |

Let's consider simple supporting beam shown on Figure 5.

**Figure 5.** Static scheme: a) positive directions of load b) cross section’s geometry

Let us assume the direction of the \(z\) axis as in Figure 5a. In this case, \(\beta_z = 8.679 \text{ cm}\) and \(e_g = -19.44 \text{ cm}\). Substituting (in kNm): \(M_0=312; M_z=195\) \(M_2=292.5; M_t=243.8\) into equation (4), we obtain:

\[
C_1 = \frac{12.5 \cdot 312}{2.5 \cdot 312 + 3 \cdot 195 + 4 \cdot 292.5 + 3 \cdot 243.8} = 1.194
\]  

(24)

Then let's substitute (in kN, m, Figure 4): \(M_0=312, Q_z=130, x_1=3, Q_z=130, x_2=250\) into equation (17b,12) we get:

\[
a_2 = \frac{2 \cdot 7.5}{312 \cdot \pi^2} \left[ 130 \cdot \sin^2(\pi \cdot 0.4) + 130 \cdot \sin^2(\pi \cdot 0.7) \right] = 0.792
\]  

(25)

\[
a_3 = \frac{7.5}{312} \left[ 130 \cdot 0.4(1 - 0.4) - 0.1 \cdot \sin^2(0.4\pi) + 130 \cdot 0.8(1 - 0.8) - 0.1 \cdot \sin^2(0.8\pi) \right] = 0.884
\]  

(26)

Next, we’ll calculate the coefficient \(C_2=0.194\cdot0.792/2=0.473\) and \(C_3=1.194\cdot0.885/2=0.510\) from equation (18b) and (15c) respectively.

The LTB moment determined from formula (1) is \(M_0^b = 2903.2 \text{ kNm}\) (in comparison to the results obtained in the FEM [1] program, error is \(0.3\%\)).

4. Conclusions

Proposed method allows for determination of critical moment in case, where each load is applied in different point, with respect to height of the cross-section and has any (upward or downward) direction. Shown examples prove that it allows for estimation of critical moment with enough accuracy from practical point of view.

In order to take into account position of the load with respect to height of the cross-section, Trahair proposed the estimation of coefficient \(C_2=0.4C_1\). It has been shown that this estimation is incorrect.

It should be emphasized that the obtained formulas give a very good LTB moment estimate for bisymmetrical beams. In the case of monosymmetric sections, the errors are also small, but only if the
second moment of inertia of the compression flange is larger than the tensioning flange on almost the entire beam span.

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