Acoustic shock and acceleration waves in selected inhomogeneous fluids

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Abstract

Acoustic shock and acceleration waves in inhomogeneous fluids are investigated using both analytical and numerical methods. In the context of start-up signaling problems, and based on linear acoustics theory, we study the propagation of such waveforms in the atmosphere and in fluids that possess a periodic ambient density profile. It is shown that vertically-running shock and acceleration waves in the atmosphere suffer amplitude growth. In contrast, those in the periodic-density fluid have bounded amplitudes that exhibit periodic, but non-trivial, oscillations; this is illustrated via a series of numerically-generated profile-evolution plots, which were computed using the PyClaw software package.

Keywords: Linear acoustics, inhomogeneous fluids, shock and acceleration waves, Laplace transform

1. Introduction

The central focus of this paper is singular surfaces, specifically, acoustic shock and acceleration waves, in inhomogeneous fluids. The study of acoustic propagation in such media dates back more than two centuries to the works of Laplace and Poisson on propagation in the atmosphere; see, e.g., Refs. \cite{laplace1825}, \cite{poisson1830}, p. 553, and those cited therein. The modern treatment of acoustic phenomena in inhomogeneous fluids, however, can be traced back to the early 20th century and the works of Lamb \cite{lamb1932}, \cite{lamb1933}, who, like his forerunner, considered propagation in the atmosphere. The problem of acoustic waves in inhomogeneous media \cite{lamb1932} remains of significant interest due, e.g., to the importance of sound interaction with both the atmospheric boundary layer and features in the terrain \cite{laplace1825}. Yet, most of the current work remains purely computational.

On the other hand, the results of the singular surface analyses carried out below are exact. The three cases considered are based on linear acoustics and involve the simplest density profiles used to model inhomogeneous fluids. Our aims are to shed light on how shock and acceleration waves evolve in such fluids and highlight the effectiveness of singular surface theory as a tool to probe inhomogeneous media in general.

To this end, the present article is organized as follows. In Sect. 2 the system of Euler equations governing compressible flow in inhomogeneous fluids is stated and terms/quantities are defined. In Sect. 3 we revisit the issue of vertical propagation in the atmosphere, extending a number of Lamb’s studies \cite{lamb1932}, \cite{lamb1933} to include shock and acceleration waves. Then, in Sect. 4 we do the same for the case of a fluid that exhibits a periodic ambient density profile and is free of external body forces, and we also provide numerical results and details of the implementation of our model into a modern shock-capturing numerical software package. Finally, in Sect. 5 we note, and briefly discuss, three possible extensions of the present investigation.

Before stating our governing system, however, it should be noted that researchers in continuum physics have investigated singular surfaces not only in fluids, but also in solids; see, e.g., Refs. \cite{maugin1995}, \cite{maugin1996}, \cite{maugin2001}, and those cited therein. Indeed, the present study was inspired, in part, by the work of Berezovski et al. \cite{berezovski2004}, \cite{berezovski2005} on waves in inhomogeneous thermoelastic media, which extended Maugin’s \cite{maugin1995} work on material inhomogeneities in elasticity to thermoelastic media. And as we do here, Berezovski et al. employed singular surface theory and presented numerical simulations using LeVeque’s \cite{leveque2002} shock-capturing scheme; see also Maugin’s \cite{maugin2001} discussion of shock waves, singular surfaces, and phase-transition fronts.

2. Euler equations for compressible flow in inhomogeneous fluids

In the case of an inhomogeneous fluid, which we assume to be lossless,\textsuperscript{2} the (Euler) system of equations that governs compressible flow becomes \cite{leveque2002}:

\begin{align}
Dq/Dt &= -q(\nabla \cdot \mathbf{u}), \quad (1a) \\
qDu/Dt &= -\nabla p + q\mathbf{b}, \quad (1b) \\
Dp/Dt &= c^2 Dq/Dt \quad (D\eta/Dt = 0). \quad (1c)
\end{align}

Here, \(\mathbf{u} = (u, v, w)\) is the velocity vector; \(q(> 0)\) is the mass density; \(p(> 0)\) is the thermodynamic pressure; \(\eta\) is the specific entropy; \(\mathbf{b} = \mathbf{b}(x, y, z)\) is the external (per unit mass) body force vector; \(D/Dt\) is the material derivative; and the (thermo-

dynamic) variable \(c(> 0)\) denotes the sound speed.

\textsuperscript{2}That is, the flow is isentropic \cite{poisson1830} p. 60; see Eq. \cite{leveque2002}.
For fluids in general, $c^2 = A/\varrho$, where $A$ is the adiabatic bulk modulus [28 p. 30]. In the case of perfect gases\footnote{Also known as polytropic gases [33 p. 154]; see Thompson [32 §2.5].}, however, $A = \gamma p$; therefore, in such gases, $c^2 = \gamma p/\varrho$ and, moreover, $p$, $\varrho$, and $\vartheta$ satisfy the following special case of the \textit{ideal gas law}:

$$p = (c_p - c_v)\vartheta \quad (c_p, c_v: \text{const.}) \tag{2}$$

Here, $\vartheta(>0)$ is the absolute temperature; $c_p > c_v > 0$ are the specific heats at constant pressure and volume, respectively; and $\vartheta = c_p/c_v$, where $\gamma \in (1, 5/3)$ for perfect gases.

In what follows, we shall restrict our attention to propagation in 1D and investigate linearized versions of Sys. (1). Moreover, the ambient state of the fluid shall always be taken as quiescent [28 p. 14]; i.e., while $p_a$, $\varrho_a$, $\vartheta_a$, and $\eta_a$ may vary with, at most, position, $u_a = (0, 0, 0)$, where a subscript ‘a’ denotes the ambient state value of the quantity to which it is attached. And lastly, we reserve ‘$\zeta$’ for use hereafter as a ‘dummy’ variable.

3. \textit{Vertical propagation in the atmosphere}

In this section we consider the special case of Sys. (1) discussed in Ref. [17 p. 159], wherein $u = (0, 0, w(z, t))$, $p = p(z, t)$, $\varrho = \varrho(z, t)$, and $b = (0, 0, -g)$; here, $g$ denotes the acceleration due to gravity near the surface, where $g \approx 9.81 \text{m/s}^2$ in the case of Earth, and the $+z$-axis is directed vertically upwards.

And for later reference we observe that, for perfect gases,

$$\eta_a(z) - \eta_0 = c_1 \left[ \ln[p_a(z)/p_0] - \gamma \ln[\varrho_a(z)/\varrho_0] \right] \tag{3}$$

a relation easily derived from, e.g., Ref. [13 Eq. (19)]. Here, we introduce the notation $p_0 := \lim_{z \to 0} p_a(z)$, $\varrho_0 := \lim_{z \to 0} \varrho_a(z)$, $\vartheta_0 := \lim_{z \to 0} \vartheta_a(z)$, and $\eta_0 := \lim_{z \to 0} \eta_a(z)$, where $p_0$, $\varrho_0$, $\vartheta_0$, and $\eta_0$ are constants, with $p_0 \neq 0$, $\varrho_0 \neq 0$, and $\vartheta_0$ connected via Eq. (2).

3.1. Linearized system and equation of motion

We begin by eliminating $D\varrho/Dt$ between Eqs. (1) and (3). Then, on setting $\partial \vartheta(z, t) = p(z, t) - p(z, t)$ and $g\varrho(z, t) = g\varrho(z, t) - g\varrho_a(z)$ and linearizing about the ambient state, Sys. (1) becomes

$$\vartheta_t + g\varrho_a(z)w_t + \varrho_a(z)w_z = 0,$$  
\hspace{1cm} \text{(4a)}

$$\varrho_a(z)w_t + \vartheta_t = -[p_a'(z) + g\varrho(z, t)]$$,  
\hspace{1cm} \text{(4b)}

$$\vartheta_t + A(z)w_c - p_a'(z)w_c,$$  
\hspace{1cm} \text{(4c)}

where in this section a prime denotes $d/dz$.

Making use of the ‘statical’ relation (Lamb [24 p. 541])

$$p_a'(z) = -g\varrho_a(z),$$ \hspace{1cm} \text{(5)}

which stems from the fact that the field variables’ ambient state values must satisfy Sys. (4), yields the further simplification

$$\vartheta_t + g\varrho_a(z)w_t + \varrho_a(z)w_z = 0,$$  
\hspace{1cm} \text{(6a)}

$$\varrho_a(z)w_t + \vartheta_t = -g\varrho,$$  
\hspace{1cm} \text{(6b)}

$$\vartheta_t + A(z)w_c = g\varrho_a(z)w_c,$$  
\hspace{1cm} \text{(6c)}

Now eliminating $\varrho$ between Eqs. (6a) and Eq. (6c), after applying $\partial/\partial t$ to the former and $\partial/\partial z$ to the latter, yields

$$\varrho(z)w_t + [-A(z)w_c] = 0, \tag{7}$$

where we have also made use of Eq. (5a). On setting $\varrho(z, t) = -A(z)w_c$ in Eq. (7) and then integrating with respect to $t$, we are led to consider the \textit{two-equation system}

$$\varrho(z)w_t + \varphi_c = 0, \tag{8a}$$

$$\varphi_t + A(z)w_c = 0, \tag{8b}$$

where the resulting function of integration has been set equal to zero without loss of generality.

In anticipation of the singular surface analyses that shall be carried out in the next subsection, let us divide Sys. (8) by $\varrho(z)$, and set $c_0^2(z) := A(z)/\varrho(z)$. After further simplifying and then setting $\psi(z, t) := \varphi(z, t)/\varrho(z)$, Sys. (8) becomes

$$w_t + \psi_t = -[g\varrho_a(z)/\varrho(z)]\psi, \tag{9a}$$

$$\psi_t + c_0^2(z)w_c = 0. \tag{9b}$$

It is a straightforward matter to now eliminate $\psi$ between the PDEs of Sys. (9) and obtain the equation of motion governing vertical propagation in an atmosphere (and ocean), namely,

$$w_{tt} = c_0^2(z)w_z + [A'(z)/\varrho(z)]w_z. \tag{10}$$

In the case of a perfect gas, wherein $A(z) = \gamma p(z)$, Eq. (10), with the aid of Eq. (5), reduces to

$$w_{tt} = c_0^2(z)w_z - g\varrho_w. \tag{11}$$

Equations (10) and (11) are, we observe, equivalent to the first and second displayed PDEs in Ref. [37 p. 160]. Notice also that both Eqs. (10) and (11) are second-order wave equations with variable coefficients. A survey of such PDEs is given in [10], wherein it is shown that they also arise in propagation problems involving \textit{homogeneous} media with moving boundaries.

In this section we shall, for the two most common cases of $\varrho(z)$ (relating to the atmosphere), investigate the following \textit{hybrid}\footnote{Time- and Laplace-domain BCs are used to select the solution that is initially, i.e., prior to encountering any boundary that might be present, right-propagating. The idea for this hybrid formulation came from the problem treated in Ref. [82], which also exemplifies the fact that the spatial asymptotic behavior of a Laplace transform need not be reflected in its inverse.} initial-boundary value problem (hBVVP):

$$w_{tt} = c_0^2(z)w_z - g\varrho_w, \quad (z, t) \in (0, L) \times (0, \infty), \tag{12a}$$

$$w(0, t) = W_0(t)(1, t), \quad w(L, s) < \infty, \quad t, s > 0, \tag{12b}$$

$$w(z, 0) = 0, \quad w_z(0, t) = 0, \quad z \in (0, L), \tag{12c}$$

where $W_0(>0)$ is a constant, $s$ is the Laplace transform parameter, a bar over a quantity denotes the image of that quantity in the Laplace transform domain, and $l(>0)$ will either be assigned a (fixed) value or be replaced with $\infty$. Also, in this communication we let

$$f(t) := \begin{cases} 1 & \text{Shock input,} \\ \sin(\omega t) & \text{Acceleration wave input,} \end{cases} \tag{13}$$
where the angular frequency $\omega(>0)$ is a constant, and $\Theta(\zeta)$ denotes the Heaviside unit step function. This hIBVP, we observe, is known as a (acoustic) signaling problem \[13\] p. 189].

3.2. Singular surface results

As in Ref. \[3\] §4, we define the amplitude of the jump in a function $\delta = \delta(z, t)$ across a singular surface $z = \Sigma(t)$ as

$$\frac{d\Sigma(t)}{dt} = V^2(t),$$

(15)
to be solved subject to the IC $\Sigma(0) = 0$, with only the positive (i.e., ‘+’ sign case) solution retained. Here, $V(t) = \gamma_0(\Sigma(t))$ is the speed at which $\Sigma(t)$ propagates (upward) along the $z$-axis.

Applying the tools of singular surface theory (see, e.g., Refs. \[3\] §6.3, \[23\]) to Sys. \ref{e:IBVP}, we are able to determine the evolution of $[w]$, in the shock case, and Maxwell’s theorem \[33\] p. 494], in the acceleration wave case, leads us to the ODE

$$\left(\frac{d\Sigma(t)}{dt}\right)^2 = V^2(t),$$

(16)

which on carrying out the integration and simplifying becomes

$$\|w\| = \|w\|_0 \sqrt{\frac{V(0)}{V(t)}} \sqrt{\frac{\phi_0(\Sigma(0))}{\phi_0(\Sigma(t))}} V(\zeta) d\zeta,$$

(17)

In Eq. \ref{e:w}, $\|w\|_0$ denotes the value of $\|w\|$ at time $t = 0$, which, in the case of hIBVP \[12\], has the value $\|w\|_0 = W_0$, and we note that $V(0) = \gamma_0(\Sigma(0))$. (While the notation has been suppressed, the reader should keep in mind that $\|w\|$ and all other jump relations derived hereafter, are explicitly functions of $t$.)

For the acceleration wave case, $\|w\|_0 = 0$, but $\|w\|_0 \neq 0$; nevertheless, we get a similar expression:

$$\|w\| = -\|w\|_0 \sqrt{\frac{V(0)}{V(t)}} \exp \left( -\frac{1}{2} \int_0^t \frac{\phi_0'(\Sigma(\zeta))}{\phi_0(\Sigma(\zeta))} V(\zeta) d\zeta \right),$$

(18)

which we can immediately reduce to

$$\|w\| = -\|w\|_0 \sqrt{\frac{V(0)}{V(t)}} \frac{\phi_0(\Sigma(t))}{\phi_0(\Sigma(0))} V(\zeta) d\zeta,$$

(19)

Here, $\|w\|_0$ denotes the value of $\|w\|$ at time $t = 0$; in the case of hIBVP \[12\], it has the value $\|w\|_0 = \omega W_0$. Also, in obtaining Eq. \ref{e:w} we have once again made use of Maxwell’s theorem, this time in the form $\|w\| = -V(t)\|w\|_0$.

Lastly, it should be noted that the expressions derived in this subsection apply only for times prior to the time the wavefront in question encounters a boundary, assuming one is present.

3.3. Exponential density profile

We begin with the simplest case of $\rho_0(z)$, i.e., of the so-called ‘exponential atmosphere’;

$$\rho_0(z) = \rho_0 \exp(-z/\ell) \quad (z > 0).$$

(20)

Here, $H = c^0/(\gamma \gamma_p)$ is the height of the ‘homogeneous atmosphere’ under the assumption that $\theta_0(z)$ is constant, specifically, that $\theta_0(z) = \theta_0$ for all $z > 0$ \[24\] p. 542.

In the case of Eq. \ref{e:rho}, then, Eq. \ref{e:w} implies that $\rho_0(z) = p_0 \exp(-z/H)$, and Eq. \ref{e:rho} and \ref{e:w} reduce to

$$w_0 = c_0^2 w_0 = \gamma g w_0.$$  

(21)

On replacing Eq. \ref{e:rho} and $\ell$ with Eq. \ref{e:rho} and $\infty$, respectively, hIBVP \[12\] becomes

$$w_0 = c_0^2 w_0 - \gamma g w_0, \quad (z, t) \in (0, \infty) \times (0, \infty),$$

(22a)

$$w(0, t) = W_0 \Theta(t)\{f, \quad |f| < \chi, \quad t, s > 0,$$

(22b)

$$w(z, 0) = 0, \quad w(z, 0) = 0, \quad \chi \in (0, \infty),$$

(22c)

Exact solutions to hIBVP \[22\] for both cases of $f(t)$ can be determined using the Laplace transform; see Ref. \[8\] §82., wherein the exact solution for the case $f(t) = 1$ is given.

In the case of hIBVP \[22\], $\Sigma(t) = \gamma_0 t$, meaning that $V(t) = \gamma_0 t$; thus, from Eqs. \ref{e:w} and \ref{e:w}, one finds that the resulting shock and acceleration wave amplitudes are given by

$$\|w\| = W_0 \exp\{c_0 t/(2H)\} \quad (t > 0),$$

(23)

$$\|w\| = -\omega a^{-1} W_0 \exp\{c_0 t/(2H)\} \quad (t > 0),$$

(24)

respectively, with both jumps occurring across $\Sigma(t) = \gamma_0 t$. Note that, like the plane wave solution to Eq. \ref{e:w} derived by Whitham \[37\] p. 160](see also Lamb \[20\] p. 543), the solutions to which these jumps correspond are only valid for $z \ll H$ since their magnitudes also increase without bound as $t \to \infty$. [This is easily established in the case of Eq. \ref{e:rho} because $w^c(t) = \|w\|$].

Remark 3.1. It is apt to mention the nonlinear acceleration wave analysis performed by Walsh \[35\], who also considered vertically-propagating acoustic wavefronts in the atmosphere; Walsh, however, took $\rho_0(z)$ and $\theta_0(z)$ as exponential functions.

3.4. Algebraic density profile

We now consider a variant of the profile in \[20\] §310, viz.,

$$\rho_0(z) = \rho_0 (1 - z/\ell)^\chi \quad (0 < z < \ell),$$

(25)

which, with the aid of Eq. \ref{e:w}, yields

$$\rho_0(z) = p_0 (1 - z/\ell)^{\chi+1}.$$  

(26)

Here, we must have $\ell := (\chi + 1)H$, where $\chi(\geq 0)$ is a (known) constant, to ensure that $\theta'_0(z)$ is always negative and constant-valued \[24\] p. 545., and $p_0 = g_0 H$. Thus, for this class of density profiles Eq. \ref{e:w} becomes

$$w_0 = c_0^2 w_0 - c_0^2 \ell^{-1} (\chi + 1) w_0. \quad \left[c_0^2 (z) = c_0^2 (1 - z/\ell).\right]$$

(27)
Now replacing Eq. (12a) with Eq. (27) and setting \( \lambda = \ell \), IBVP (12) becomes
\[
w_2 = c_0^2 (1 - z/\ell)w_z - c_0^2 \ell^{-1} (\chi + 1) w_z, \quad (z, t) \in (0, \ell) \times (0, \infty); \tag{28a}
\]
with \( \ell = W_0 \Theta(t)f(t), \quad |\mathbb{W}(\ell, s)| < \infty, \quad t, s > 0; \tag{28b} \]
\[
w(z, 0) = 0, \quad w_z(z, 0) = 0, \quad z \in (0, \ell). \tag{28c}
\]
Introducing the following dimensionless variables:
\[
W = w/W_0, \quad Z = z/\ell, \quad T = t/(\ell/c_0), \tag{29}
\]
hIBVP (28) simplifies to
\[
W_{TT} = (1 - Z)W_{ZZ} - (\chi + 1)W, \quad (Z, T) \in (0, 1) \times (0, \infty); \tag{30a}
\]
\[
W(0, T) = \Theta(T)f(T), \quad \mathbb{W}(1, s)| < \infty, \quad T, s > 0; \tag{30b}
\]
\[
W(Z, 0) = 0, \quad W_T(Z, 0) = 0, \quad Z \in (0, 1). \tag{30c}
\]
Here, on setting \( \Omega := \omega \ell/c_0 \), we now have
\[
f(T) = \begin{cases} 
1 & \Rightarrow \text{Shock input}, \\
\sin(\Omega T) & \Rightarrow \text{Acceleration wave input}. \tag{31}
\end{cases}
\]
Applying the Laplace transform to Eq. (30a) and the left-BC, hIBVP (30) is reduced to the following boundary value problem (BVP) in the transform domain:
\[
(1 - Z)\mathbb{W}_{ZZ} - (\chi + 1)\mathbb{W} = s^2 \mathbb{W}, \quad Z \in (0, 1), \tag{32a}
\]
\[
\mathbb{W}(0, s) = \mathbb{f}(s), \quad \mathbb{W}(1, s)| < \infty, \quad s > 0. \tag{32b}
\]
This subsidiary equation can be transformed into a Bessel-type ODE (see Ref. [28], p. 546), after which it is not difficult to obtain the exact (transform-domain) solution
\[
\mathbb{W}(Z, s) = \mathbb{f}(s) \left[ I_\ell(2s\sqrt{1 - Z}) \right] / \left[ I_\ell(2s) \right], \tag{33}
\]
where it should be noted that
\[
\lim_{Z \to 1} \mathbb{W}(Z, s) = \mathbb{f}(s) \left[ I_\ell(2s) \right] / \left[ I_\ell(2s) \right]. \tag{34}
\]
Here, \( I_\ell(\zeta) \) and \( \Gamma(\zeta) \) denote the modified Bessel function of the first kind of order \( \ell \) and the gamma function, respectively. As page limitations prevent us from doing so, we leave the inversion of Eq. (33), for both cases of \( \mathbb{f}(s) \), to the reader.

Now, using Ref. [1], Eq. (9.7.1), it can be shown that Eq. (33) admits the asymptotic expansion
\[
\mathbb{W}(Z, s) \sim \mathbb{f}(s) \exp[-2s(1 - \sqrt{1 - Z})] / \sqrt{(1 - Z)^{\ell+\frac{1}{2}}} \times \left[ 1 + \frac{4\chi^2 - 1}{16s} \left( 1 - \frac{1}{\sqrt{1 - Z}} \right) + \cdots \right] \quad (s \to \infty). \tag{35}
\]
In the remainder of this section, we shall limit our focus to \( 0 < T < T_R \), where \( T_R \) is the time that the wavefront created by the signal in question first reaches the right boundary (i.e., \( Z = 1 \)).

For the shock and acceleration wave cases, then, inverting Eq. (35) term-by-term yields the small-\( T \) approximations:
\[
W(Z, T) = \frac{\Theta[T - 2(1 - \sqrt{1 - Z})]}{\sqrt{(1 - Z)^{\ell+\frac{1}{2}}} \times \left[ 1 + \frac{4\chi^2 - 1}{16s} \left( 1 - \frac{1}{\sqrt{1 - Z}} \right) + \cdots \right]} \quad (T \ll T_R), \tag{36}
\]
for which \( \mathbb{f}(s) = 1/s \) was used, and
\[
W(Z, T) = \frac{\Theta(T)[T - 2(1 - \sqrt{1 - Z})]}{\sqrt{(1 - Z)^{\ell+\frac{1}{2}}} \times \left[ 1 + \frac{4\chi^2 - 1}{16s} \left( 1 - \frac{1}{\sqrt{1 - Z}} \right) + \cdots \right]} \quad (T \ll T_R), \tag{37}
\]
for which \( \mathbb{f}(s) = \Omega/(s^2 + \Omega^2) \sim \Omega s^{-2}(1 - \Omega^2/s^2 + \cdots) \) was used, respectively.

If we now replace \( Z \) with \( \vartheta(T) \) in the argument of the Heaviside function in these approximations, set the argument expression to zero, and then solve for \( \vartheta(T) \), we find that
\[
\vartheta(T) = T - \frac{T_2}{2} \quad \left( 0 < T < 2 \right), \tag{38}
\]
from which it follows that \( V(T) = 1 - T/2 \) and \( T_R = 2 \); here, \( \vartheta \) denotes the dimensionless version of \( \Sigma \). Thus, using the general expressions given in Eqs. (17) and (19), the shock and acceleration wave amplitudes stemming from the density profile given in Eq. (25) are found to be
\[
\|W\| = \frac{1}{(1 - T/2)^{\ell+\frac{1}{2}}} \quad \left( 0 < T < 2 \right), \tag{39}
\]
\[
\|W_z\| = -\frac{\Omega}{(1 - T/2)^{\ell+\frac{1}{2}}} \quad \left( 0 < T < 2 \right), \tag{40}
\]
respectively, with both jumps occurring across \( Z = \vartheta(T) \). In the shock case we once again see amplitude blow-up, but now as \( T \to T_R(= 2) \). In contrast, the acceleration wave case is more interesting. This is because, like the nonlinear version of this case of hIBVP (30) involving a homogeneous gas (see, e.g., Ref. [1]), Eq. (40) exhibits wavefront steepening (i.e., ‘shocking-up’) as \( T \to 2 \); however, unlike that of the former, the solution profile corresponding to Eq. (40) also blows-up as \( T \to 2 \). These behaviors are illustrated below in Fig. 11, wherein snapshots in the evolution of \( \|W\| \) and \( \|W_z\| \) are presented.

**Remark 3.2.** Of particular interest are the special cases \( \chi = 0 \) and \( \chi = (\gamma - 1)^{-1} \). For these values of \( \chi \), Eq. (25) becomes
\[
\vartheta(\zeta) = \Theta \left[ \frac{1}{(1 - \zeta/H_b)^{\ell+\frac{1}{2}}}, \quad \chi = 0, \quad \chi = (\gamma - 1)^{-1}. \tag{41}
\]
where \( H_b := H/\gamma - 1 \). Here, \( \chi = 0, (\gamma - 1)^{-1} \) correspond to \( \ell = H/H_b \), respectively, where \( H_b > H \), and the subscript
\(\text{Remark 3.3.}\) The shock and acceleration wave results presented in this subsection can also be obtained by applying the theorem given in Ref. [6, §4] to Eq. (35) and then, in the case of the latter, employing Maxwell’s theorem.

3.5. Numerical results: algebraic density profile

In this subsection we compute and plot the velocity field solution, for several cases of the algebraic density profile, by numerically inverting the Laplace domain solution [Eq. (33)] using the formula

\[
W(Z, T) \approx \frac{\exp(4.7)}{T} \left( \frac{2}{M} \right)^{\frac{1}{2}} \left( \frac{Z}{4.7} \right)^{\frac{1}{2}} M^{-1} \\
+ \mathcal{R} \left[ \sum_{m=0}^{\infty} (-1)^{m} \left( \frac{Z}{4.7 + \text{im} \pi} \right)^{m} \right],
\]

(42)

where \(T > 0\). Equation (42) is a modified version of Tzou’s \(t\)-sum inversion approximation, obtained by introducing Lanczos’ \(\sigma\)-factors \(21\), i.e., the \(\text{sinc}(m\pi/M)\), where

\[
\text{sinc}(\xi) := \begin{cases} \xi^{-1} \sin(\xi), & \xi \neq 0 \\ 1, & \xi = 0 \end{cases}
\]

(43)

into the latter. This is done to reduce the Gibbs phenomenon in Fourier series, such as Eq. (42), near discontinuities in the function being approximated, as in, e.g., our shock solution [i.e., the case \(\chi = 1/s\)], without affecting the series’ convergence.

Here, we have set \(M = 1.500\), the number ‘4.7’ and the quantity to which it is assigned are discussed in Ref. [34, p. 41].

Figure 1 shows the evolution of a shock wave [panels (a) and (b)] and an acceleration wave [panels (c) and (d)] based on Eq. (35) for three choices of the algebraic exponent: \(\chi = 0, 2.5, 4\), where \(\chi = 2.5\) corresponds to air, for which \(\gamma = 1.4\), when the ambient state is homentropic (see \text{Remark 3.2}). As suggested by Eq. (39), the location of the wavefront is independent of \(\chi\). The shock and acceleration wave amplitudes grow algebraically without bound, however, as indicated by Eqs. (39) and (40). In Fig. 1 we have also included the acceleration wave’s wavefront location and amplitude, i.e., the straight lines plotted from \(W_Z(T)[Z - \sigma(T)]\), where the acceleration wave amplitude is given by Eq. (39) and the wavefront location by Eq. (38), which show excellent agreement with the solution obtained via numerical inversion.

4. Propagation in a fluid with a periodic density profile in the absence of external body forces

4.1. Linearized system and equation of motion

In this section we assume 1D planar propagation along the \(x\)-axis; i.e., we take \(u = (u(x, t), 0, 0)\) and \(b = (0, 0, 0)\), and observe that in this setting \(p\) and \(\varnothing\) are, like \(u\), both functions of \(x\) and \(t\) only. Again eliminating \(D\varnothing/Dr\) between Eq. (43) and (44), but now setting \(p(x, t) = \rho(x, t) - p_0(x)\), it is a straightforward matter to linearize this special case of Sys. (11) and express it as

\[
\varnothing_u u_t + p_x = -p'_0(x),
\]

(44a)

\[
p_t + A_\nu u_x = -p'_0(x)u_t,
\]

(44b)

where in this section a prime denotes \(d/d\lambda\). Here, we observe that if one seeks to determine only \(u\) and/or \(p\), then the \(b = (0, 0, 0)\) special case of the 1D version of Sys. (11) can always be reduced to a two-equation system.

Since the ambient state values of the field variables must also satisfy this system, it follows that \(p_\nu(x)\) is necessarily a constant, specifically, \(p_\nu(x) = p_\nu\), where \(p_\nu\) is a reference value of \(p_\nu(x)\); hence, \(p'_0(x) = 0\) and, on setting \(\varnothing(x, t) := \rho(x, t)/\rho_\nu(x)\), Sys. (44) can be recast as

\[
u_t + \varnothing_x = -[\varnothing'_0(x)/\rho_\nu(x)]\varnothing,
\]

(45a)

\[
\varnothing_t + c^2_\nu(x)u_x = 0,
\]

(45b)

where we note that \(c^2_\nu(x) = A_\nu(x)/\rho_\nu(x)\) in the present section.

On eliminating \(\varnothing\) between the equations of Sys. (45), the corresponding equation of motion is easily shown to be

\[
u_{tt} - c^2_\nu(x)u_{xx} = 0.
\]

(46)

If we now assume the periodic density profile

\[
\varnothing(x) = \varnothing_0[1 + \epsilon \cos(\pi k x/L)] \quad (0 < x < L),
\]

(47)

and that \(A_\nu(x)\) is constant\(^6\), i.e., \(A_\nu(x) = A_\nu\), then it follows that

\[
c^2_\nu(x) = c^2_0 + \epsilon \cos(\pi k x/L) \quad (k \in \mathbb{N}).
\]

(48)

\(^6\)In the case of perfect gases this need not be assumed since \(A_\nu(x) = \gamma p_\nu\) holds exactly; i.e., \(A_\nu = \gamma p_\nu\), and thus \(c^2_\nu = \gamma p_\nu/\rho_\nu\), for perfect gases.
Here, \( \varphi \) and \( A_r \) represent reference values of \( c_a(x) \) and \( A_l(x) \), respectively; \( \epsilon \in (0, 1) \) is a (dimensionless) parameter; \( c_1^2 = A_l/\varphi \); and \( L \) is a characteristic length of the domain.

4.2. Formulation of IBVP and singular surface results

We now consider the following IBVP involving Eq. (46) with \( c_s \) given by (45), and wherein \( U_0(>0) \) is a (known) constant:

\[
\begin{align*}
\frac{\partial c}{\partial t} &= c \frac{\partial c}{\partial x}, & (x, t) \in (0, L) \times (0, t_f), \\
U(0, t) &= U_0 \Theta(t) f(t), & u_s(L, t) = 0, & t \in (0, t_f), \\
U(x, 0) &= 0, & u_s(x, 0) = 0, & x \in (0, L).
\end{align*}
\]

As the shock and acceleration wave results presented below are only for such times, we have limited our focus to \( t < t_f \), where \( t_f \) is the time at which the wavefront of the input signal in question first reaches the right boundary (i.e., \( x = L \)).

Notwithstanding the fact that it involves a linear PDE and linear BCs, at present, there appears little hope of obtaining an analytical solution to this IBVP. Accordingly, we must turn to numerical methods if further progress is to be achieved.

To this end, we introduce the following dimensionless variables: \( U = u/\varphi \), \( X = x/L \), and \( T = t/(L/c_1) \). With these substitutions, our IBVP is reduced to

\[
\begin{align*}
\frac{\partial U}{\partial T} &= \frac{\partial^2 U}{\partial X^2}, & (X, T) \in (0, 1) \times (0, T_f), \\
U(0, T) &= \Theta(T) f(T), & U(1, T) = 0, & T \in (0, T_f), \\
U(X, 0) &= 0, & U_s(X, 0) = 0, & X \in (0, 1).
\end{align*}
\]

4.3. Approximations relating to IBVP

On expanding Eq. (52) for small-\( T \) we find that

\[
\Upsilon(T) \approx U(\epsilon)\frac{T}{(1 - (\epsilon/\Upsilon))^2} \quad [0 < T \ll \min(1, T^*)].
\]

Remark 4.1. To handle the case \( \epsilon \in (-1, 0) \), one must modify Eqs. (51) and (52) in accordance with Ref. [1] Eq. (17.4.18).

4.4. Numerical results: periodic density profile

Our numerical approach to acoustic propagation in a periodic fluid medium in the absence of external body forces is based on the dimensionless version of the non-conservative system (43):

\[
C_a(X)^{-1} U_T + P_X = 0, \quad P_T + U_X = 0.
\]
where $P = p/(\rho_c U_0)$. To solve Sys. (62) numerically, subject to the stated ICs and BCs, we employ the modern extensible software package PyClaw [12, 16]. PyClaw is based on LeVeque’s CLAWPACK [23]. We employed the PyClaw solver based on the second-order-accurate wave propagation algorithm [23]; see also Refs. [14, 23]. The wave propagation method is a high-resolution shock-capturing scheme capable of handling non-conservative hyperbolic systems of PDEs. The PyClaw package and its Riemann solvers (for handling discontinuous solutions) have been benchmarked against other methods and exact solutions; thus, the numerical solutions shown below are robust, reproducible, and highly accurate. Specifically, we have employed a Python implementation of the variable coefficient acoustics Riemann solver [14]. High-resolution shock-capturing schemes require limiters to resolve discontinuities; we employed the so-called monotonic central (MC) limiter [23]. The simulations discussed below were performed using a computation grid of 20,000 (for $k = 5$ and 8) and 60,000 (for $k = 15$ and 25) cells on the domain $X \in [0, 1]$. A second-order, adaptive time-stepping scheme was used, which maintained a ‘target’ Courant–Friedrichs–Lewy (CFL) number of 0.9. At $X = 0$, an inlet velocity BC was applied, while at $X = 1$ a transmissive (i.e., extrapolation) BC was employed. These BCs were implemented using two ghost cells on each side of the computational domain (see Ref. [23]) to maintain the overall second-order accuracy of the scheme.

Figures 2 and 3 show the evolution of an acceleration wave, while Figs. 4(a) and 4(b) show the evolution of a shock wave in $U(X, T)$. The numerical solutions of the IBVP are compared to the theoretical results from singular surface theory discussed in Sect. 4.2. The agreement at the wavefront $X = \gamma(T)$ is very good, however, as $k$ becomes large (in Figs. 2 and 4), the wavefront becomes highly localized, which leads to some small amount of numerical error, e.g., in Fig. 4(f). The numerical solutions reveal many more features than the theoretical discussion. Specifically, we observe that the wave profile behind the wavefront, i.e., for $0 < X < \gamma(T)$, is quite complex due to the periodic density profile, especially for the case of a shock wave in Figs. 4(d,e,f). From Eq. (62), we see that the periodic density profile necessitates a periodic sound speed. Thus, as the acceleration (or shock) wave propagates forward, acoustic disturbance emanates backwards from it as it has to slow down or speed up due to the variable sound speed. These disturbances reach the boundary $X = 0$, reflect and create this complex superposition that is particularly well illustrated in Fig. 4.

5. Closure

Lastly, we offer the following as possible, analytically tractable, extensions of the present study.

- Consider vertically-running shock and acceleration waves under Taylor’s [31] two-layer atmosphere model; viz.:

$$\phi_{\omega}(z) = \phi_0 \begin{cases} (1-z/H_1)^{\ell_i/\ell_0}, & z \in (0, z_i), \\ \exp[-(z-z_i)/H_2] (1-z_i/H_1)^{\ell_i/\ell_0}, & z \geq z_i, \end{cases}$$

(63)

with $\phi_{\omega} = \partial_0 - \beta_1 z/2$ for $z \in (0, z_i)$ and $\phi_{\omega} = \partial_i$ for $z \geq z_i$. Here, $H_1 := 2H_0$ and $H_2 := \partial_i/\beta$, the interface between the layers lies at $z = z_i$, and $\partial_i = \partial_0 - \beta_1 z_i/2$.

- The following outlines what is, perhaps, the most promising approach by which exact solutions to the simplest (i.e., full-Dirichlet) version of IBVP (50) might be derived: Apply the Laplace transform to Eq. (50) and the BCs, where the right-BC now reads $U(1, T) = 0$, and then make use of
the ICs to get the BVP
\[ \begin{align*}
U_{XX} - s^2 [1 + \epsilon \cos(k\pi X)] U &= 0, \quad X \in (0, 1), \\
U(0, s) &= \overline{T}(s), \quad U(1, s) = 0, \quad s > 0,
\end{align*} \]  
\tag{64a}
\tag{64b}

the exact solution of which is readily found to be
\[ U(X, s) = \frac{\overline{T}(s)}{C} \left[ \begin{array}{c}
\frac{4 \epsilon^2}{k^2 \pi^2} - \frac{2 \epsilon^2 s^2}{k^2 \pi^2} + \frac{1}{k\pi X} \\
\frac{4 \epsilon^2}{k^2 \pi^2} - \frac{2 \epsilon^2 s^2}{k^2 \pi^2} + \frac{1}{k\pi X}
\end{array} \right] \times S \left( \frac{4 \epsilon^2}{k^2 \pi^2} - \frac{2 \epsilon^2 s^2}{k^2 \pi^2} + \frac{1}{k\pi X} \right) \].  
\tag{65}

Here, \( C(\zeta_1, \zeta_2, \zeta) \) and \( S(\zeta_1, \zeta_2, \zeta) \) are the even and odd Mathieu functions \( R_{2, \kappa}(\zeta) \), respectively.

In principle, the exact time-domain solution, \( U(X, T) \), can be determined by applying the ‘Inversion Theorem’ \[ ] (also known as the complex inversion formula) to Eq. \( 65). \]

- Examine signaling problems wherein the present linear equations of motion are replaced by their weakly-nonlinear counterparts; e.g., re-work the (weakly-nonlinear) IBVP analyzed in Ref. \[ ] , wherein \( f(t) \propto \sin(\omega t) \) was also used, assuming an inhomogeneous gas.

Acknowledgments

The authors thank Profs. A. Rosato and M. Destrade for their kind invitation to contribute to this special issue, co-guest edited by M.D. and I.C.C., memorializing Prof. Gérard Maugin. The authors also thank the anonymous reviewer for his/her valuable comments and suggestions. R.S.K. and P.M.J. were supported by ONR funding. I.C.C. thanks Prof. Kyle Mandli for helpful discussions on PyClaw.

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