Data-Based Control of Feedback Linearizable Systems

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Abstract—In this article, we present an extension of Willems’ Fundamental Lemma to the class of multi-input–multi-output discrete-time feedback linearizable nonlinear systems, thus providing a data-based representation of their input–output trajectories. Two sources of uncertainty are considered. First, the unknown linearizing input is inexactly approximated by a set of basis functions. Second, the measured output data is contaminated by additive noise. Furthermore, we propose an approach to approximate the solution of the data-based simulation and output matching problems, and show that the difference from the true solution is bounded. Finally, the results are illustrated on an example of a fully actuated double inverted pendulum.

Index Terms—Data-driven control, feedback linearization, nonlinear systems.

I. INTRODUCTION

Over the past two decades, researchers explored designing controllers directly from data without explicitly identifying a mathematical model of the system (cf. [1] and the references therein). In contrast to model-based control techniques, direct data-based control can be useful in cases where modeling complex systems from first principles is challenging [2].

A remarkable result from behavioral control theory [3] states that for a controllable, discrete-time linear time-invariant (DT-LTI) system, the entire vector space of input–output trajectories can be spanned by a single, persistently exciting, input–output trajectory. Now known as the fundamental lemma, this result recently motivated a large number of works in the field of direct data-based system analysis and controller design. For example, it was used for data-based simulation and control of DT-LTI systems [4]. It has also been translated to the state-space framework [5], [6] and was used to design linear quadratic regulators (LQR) [7], [8], [9], as well as predictive controllers [10], [11] with stability and robustness guarantees [12], [13]. Extensions to Hammerstein–Wiener systems appeared in [5] and second-order Volterra systems in [14]. It was also used to design controllers for classes of nonlinear systems purely from data in [15], [16], [17], and [18]. For a more complete and comprehensive review, the reader is referred to the review paper [19]. Apart from the fundamental lemma, data-driven stabilization for single-input–single-output (SISO) feedback linearizable nonlinear systems appeared in [20]. There, input-affine continuous-time systems were addressed assuming constant intersampling behavior of the states under high enough sampling rate.

In this article, we build on our previous work [21], where SISO flat systems were investigated and a data-based system representation was given assuming that an exact expansion using basis functions is known. Compared to [21], here we treat general multi-input–multi-output (MIMO) full-state feedback linearizable nonlinear systems, which model a variety of physical systems, e.g., robotic manipulators (other examples can be found in [22]). Furthermore, we provide suitable error bounds for the results of the simulation and output matching control problems when (i) the given basis functions do not exactly represent the unknown nonlinearities, and (ii) the measured output data is noisy.

After reviewing notation, definitions, and existing results in Section II, we provide a data-based representation of feedback linearizable nonlinear systems in Section III by exploiting linearity in transformed coordinates along with a set of basis functions that depend only on input–output data to approximate the unknown nonlinearities. In Section IV, we provide constructive methods to approximately solve the simulation and output-matching control problems despite basis functions approximation error and only using input and noisy output data. We then show that the difference between the estimated and true outputs is upper bounded. We illustrate the results on a model of a fully actuated double inverted pendulum in Section V. Finally, Section VI concludes this article.

II. PRELIMINARIES

A. Notation

The set of integers in the interval $[a, b]$ is denoted by $\mathbb{Z}_{[a,b]}$. For a vector $w \in \mathbb{R}^n$, $p$-norms for $p = 1, 2, \infty$ are denoted by $\|w\|_p$, respectively, whereas $\|M\|_i$ for $i = 1, 2, \infty$ denotes the induced norm of a matrix $M$. We use $0$ to denote a vector or a matrix of zeros of appropriate dimensions. An $n \times n$ identity matrix is denoted by $I_n$.

For a sequence $\{z_{k}\}_{k=0}^{N-1}$ with $z_{0} \in \mathbb{R}^n$, each element is expressed as $z_k = \left[ z_{1,k}, z_{2,k}, \ldots, z_{n,k} \right]^\top$. The stacked vector of that sequence is given by $z = \left[ z_0^\top, \ldots, z_{N-1}^\top \right]^\top$, and a window of it by $z_{[a,b]} = \left[ z_a^\top, \ldots, z_b^\top \right]^\top$. The Hankel matrix of depth $L$ of this sequence is
given by
\[
H_L(z) = \begin{bmatrix}
z_0 & z_1 & \ldots & z_{N-L} \\
z_1 & z_2 & \ldots & z_{N-L+1} \\
\vdots & \vdots & \ddots & \vdots \\
z_{L-1} & z_L & \ldots & z_{N-1}
\end{bmatrix}.
\]

Throughout this article, the notion of persistence of excitation (PE) is defined as follows.

Definition 1: The sequence \(\{z_k\}_{k=0}^{N-1}\) is said to be persistently exciting of order \(L\) if rank \(H_L(z) = \eta L\).

B. Discrete-Time Feedback Linearizable Systems

Consider the following DT-MIMO square nonlinear system:
\[
x_{k+1} = f (x_k, u_k), \quad y_k = h(x_k),
\]
where \(x_k \in \mathbb{R}^n\) is the state vector and \(u_k, y_k \in \mathbb{R}^m\) are the input and output vectors, respectively. The functions \(f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n\), \(h : \mathbb{R}^n \to \mathbb{R}^m\) are analytic functions with \(f(0,0) = 0\) and \(h(0) = 0\). We define \(f_i^O\) as the \(i\)th iterated composition of the undriven dynamics \(f_i^O := f(\ldots f(x,0)))\). A sequence \(\{u_k, y_k\}_{k=0}^{N-1}\) is said to be an input–output trajectory of the nonlinear system (1) if there exists an initial condition \(x_0\) such that (1) holds for all \(k \in \mathbb{Z}_{[0,N-1]}\).

As defined in [23], each output \(y_i = h_i(x)\) of the nonlinear system (1), for \(i \in \mathbb{Z}_{[1,m]}\), is said to have a (globally) well-defined relative degree \(d_i\) if at least one of the \(m\) inputs at time \(k\) affects the \(i\)th output at time \(k + d_i\). In particular
\[
y_{i,k+d_i} = h_i \left( f^{d_i-1}_i (f (x_k, u_k)) \right).
\]

To bring the system in (1) to the DT normal form, we make the following standard assumptions (see [23]).

Assumption 1: For any \(x_k \in \mathbb{R}^n\), \(\exists u_k \in \mathbb{R}^m\) such that
\[
h_i \left( f^{d_i-1}_i (f (x_k, u_k)) \right) = 0, \quad \forall i \in \mathbb{Z}_{[1,m]}.
\]

Assumption 2: For any \(x_k \in \mathbb{R}^n\), the decoupling matrix \(D(x_k, u_k)\) evaluated at \(\tilde{u}_k\) has rank \(\text{rank}(D(x_k, \tilde{u}_k)) = m\), where
\[
D (x_k, u_k)_{(i,j)} = \partial h_i \left( f^{d_i-1}_i (f (x_k, u_k)) \right) / \partial u_{j,k}.
\]

Assumption 3: The sum of relative degrees of the outputs of system (1) is equal to the system dimension, i.e., \(\sum d_i = n\).

Assumptions 1 and 2 are standard and they are needed to invoke the implicit function theorem and show existence of an invertible coordinate transformation and a feedback linearizing control law \(u_k = \gamma(x_k, v_k)\) that results in a linear map from \(v_k\) to \(y_k\) (cf. [23, Prop. 3.1]). Assumption 3 is needed for the system to be fully-state feedback linearizable, i.e., having no internal dynamics. For globally well-defined relative degrees, this condition can be checked by perturbing the system from rest and recording the first time instants at which each output changes from zero (cf. (2)). If the sum of all these instances is \(n\), Assumption 3 is fulfilled. Feedback linearization of DT-MIMO nonlinear systems is formally stated as follows.

Theorem 1 ([23], Prop. 3.1): Let Assumptions 1–3 be satisfied, then there exists an invertible (w.r.t. \(v_k\)) feedback control law \(u_k = \gamma(x_k, v_k)\), with \(\gamma : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m\) and an invertible coordinate transformation \(\Xi_k = T(x_k)\), such that system (1) is input–output decoupled and can be written as
\[
\Xi_{k+1} = A \Xi_k + B v_k, \quad y_k = C \Xi_k,
\]
where \(\Xi_k = [\xi_{1,k} \ldots \xi_{n,k}]^\top \in \mathbb{R}^n\) is defined as
\[
\Xi_k = \begin{bmatrix} y_{1[k,k+d_1-1]} & \ldots & y_{m[k,k+d_m-1]} \end{bmatrix}^\top. \tag{6}
\]

Furthermore, \(A, B, C\) are in block-Brunovsky\(^1\) form, which is a controllable/observable triplet.

Theorem 1 shows that a nonlinear system (1) that satisfies Assumptions 1–3 has an equivalent linear representation where the input-state and input–output maps are linear and decoupled. That is, each synthetic input \(v_i\) only affects its corresponding output \(y_i\) for all \(i \in \mathbb{Z}_{[1,m]}\).

In the following section, we use Theorem 1 to provide a data-based representation of trajectories of full-state feedback linearizable systems.

III. WILLEMS’ FUNDAMENTAL LEMMA FOR FEEDBACK LINEARIZABLE SYSTEMS

In the setting of data-based control, one typically only has access to input–output data and not to the synthetic input \(v_i\) of the corresponding state transformation \(T(x_k)\). In order to come up with a data-based description of the trajectories of the nonlinear system (1) (or the equivalent system (5)), the synthetic input is expressed as (cf. [23])
\[
\check{v}_k = \begin{bmatrix} v_{1,k} \\ \vdots \\ v_{m,k} \end{bmatrix} = \Phi (u_k, x_k).
\]

By Theorem 1, it holds that \(x_k = T^{-1}(\Xi_k)\) and, hence, one can define \(\Phi (u_k, \Xi_k) := \Phi (u_k, T^{-1}(\Xi_k))\), which allows us to parameterize \(v_k\) using input–output data only since \(\Xi_k\) is given by shifted outputs (see (6)). Note that \(\Phi (u_k, \Xi_k)\) is unknown, therefore, we approximate it by a set of basis functions that depend only on input and output data. In particular,
\[
\check{v}_k = \Phi (u_k, \Xi_k) = \sum_{j=1}^r g_j (u_k, \Xi_k) \phi_j (u_k, \Xi_k) + \varepsilon_k (u_k, \Xi_k),
\]
where \(\Psi (u_k, \Xi_k)\) is the vector of \(r \in \mathbb{N}\) locally Lipschitz continuous and linearly independent basis functions \(\psi_j : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}\) for \(j \in \mathbb{Z}_{[1,r]}\) and \(\epsilon (u_k, \Xi_k)\) is the stacked vector of approximation errors \(\varepsilon_i : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}\) for \(i \in \mathbb{Z}_{[1,m]}\). The term \(\mathcal{G} \in \mathbb{R}^{m \times r}\) is the matrix of unknown coefficients of the basis functions and \(g_j^i\), for \(i \in \mathbb{Z}_{[1,m]}\), represent its rows. For the theoretical analysis presented in this article, we define, but never compute, \(\mathcal{G}\) as follows:
\[
\mathcal{G} := \arg \min_{\mathcal{G}} \langle \Phi - \mathcal{G} \Psi, \Phi - \mathcal{G} \Psi \rangle. \tag{9}
\]

The minimization problem in (9) is a least-squares problem that minimizes the average approximation error on a compact subset of the input-state space \(\Omega \subset \mathbb{R}^m \times \mathbb{R}^n\). Furthermore, if the choice of basis functions contains \(\Phi (u_k, \Xi_k)\) in their span and the data is noiseless,

\(^1\)We use the term “block-Brunovsky form” to refer to block diagonal matrices with Brunovsky canonical form matrices as diagonal elements. See Appendix A for more information on the structure of the \(A, B, C\) matrices.

\(^2\)The inner product is given by \(\langle \rho_1, \rho_2 \rangle = \int_{\Omega} \rho_1 (f_1, f_2) \rho_2 (f_1, f_2) df_1 df_2\) for some \(\Omega \subset \mathbb{R}^m \times \mathbb{R}^n\).
then a unique solution to (9) exists and the results of [21] are retrieved.
For the subsequent analysis, the following assumption is made on \( \mathcal{G} \).

**Assumption 4**: The matrix of coefficients \( \mathcal{G} \) has full row rank.

From (2), (7), and (8) (cf. also (14) below), it can be seen that Assumption 4 corresponds to the outputs being linearly independent. This is fulfilled in, e.g., robotic manipulators and other fully actuated mechanical systems (cf. [22]).

Inevitably, measured data is noisy. In what follows, we denote the collected output measurements by \( \hat{y}_i = y_i + w_i \), where \( ||w_i||_2 \leq w^* \), for all \( k \geq 0 \) is a uniformly bounded output measurement noise. As a result of using noisy data, the unknown nonlinear function \( \Phi \) in (8) is now expressed as

\[
\Phi(u_k, \Xi_k) = \mathcal{G} \psi(u_k, \Xi_k) + c(u_k, \Xi_k) + \delta(\omega_k),
\]

where \( \delta(\omega_k) := \mathcal{G} \psi(u_k, \Xi_k) - \psi(u_k, \Xi_k) - c(u_k, \Xi_k), \Xi_k = \Xi + \omega_k \) and \( \omega_k = \left[ w_{i, k}, w_{i, k+1-d_i-1} \ldots w_{i, k+d_i-m} \right]^\top \).

Now, one can substitute (10) back into (5) to obtain

\[
\Xi_{k+1} = A \Xi_k + B \mathcal{G} \Phi(u_k, \Xi_k) + D(\omega_k),
\]

where \( E(u_k, \Xi_k) := \mathcal{G}^\dagger(\psi(u_k, \Xi_k), D(\omega_k), \mathcal{G}^\dagger(\mathcal{G}^\dagger)^{-1} \text{ is a right inverse of } \mathcal{G}, \) which exists by Assumption 4. For convenience, we use the following notation throughout this article:

\[
\begin{align*}
\bar{\Psi}(u_k, \Xi_k) &:= \psi(u_k, \Xi_k), \\
\bar{\xi}_k(u_k, \Xi_k) &:= c(u_k, \Xi_k), \\
\bar{E}_k(u_k, \Xi_k) &:= E(u_k, \Xi_k), \\
\bar{D}_k(\omega_k) &:= D(\omega_k).
\end{align*}
\]

Using the notation in (12), system (11) can now be written as

\[
\Xi_{k+1} = A \Xi_k + B \bar{\Psi}(u_k, \Xi_k) + \bar{E}_k(u_k, \Xi_k) + \bar{D}_k(\omega_k),
\]

\[
\tilde{y}_k = C \Xi_k + w_k.
\]

For the system in (13), the following holds.

**Lemma 1**: The pair \((A, \bar{B} \mathcal{G})\) is controllable.

**Proof**: See Appendix B.

Systems that are in the block-Brunovsky canonical form as in (5) have two appealing properties that will be extensively used throughout this article. First, the state \( \Xi_k \), and its noisy counterpart \( \bar{\Xi}_k = \Xi_k + \omega_k \), are defined as the shifted outputs as in (6). Second, the system is input-output decoupled from \( \psi \) to \( \bar{\psi} \). This means that the \( i \)-th synthetic input at time \( k \) is equal to the \( i \)-th (noiseless) output at time \( k + d_i \), i.e.,

\[
y_{i, k+d_i} \overset{(2)}{=} y_{i, k+d_i} = \bar{\psi}(u_k, \bar{\Xi}_k) \overset{(8)}{=} \bar{y}_i \left[ \bar{\Psi}(u_k, \bar{\Xi}_k) + \bar{E}_k(u_k, \bar{\Xi}_k) + \bar{D}_k(\omega_k) \right],
\]

In the following theorem, we extend the results of [3] to the class of DT-MIMO full-state feedback linearizable nonlinear systems. Theorem 2 studies the nominal case for which the approximation error in (8) is zero and the data is noiseless. For this case, (13) reduces to

\[
\bar{\Xi}_{k+1} = A \bar{\Xi}_k + B \mathcal{G} \bar{\Psi}(u_k, \Xi_k), \quad \bar{y}_k = C \bar{\Xi}_k.
\]

The case for which the basis functions approximation errors and output noise are nonzero is studied in Section IV.

**Theorem 2**: Suppose Assumptions 1–4 are satisfied and let \( \{u_k\}_{k=0}^{N-1}, \{\bar{y}_k\}_{k=0}^{L+d_i-1} \), for \( i \in [m] \), be a trajectory of a full-state feedback linearizable system as in (1). Furthermore, let \( \{\bar{\Psi}(u_k, \Xi_k)\}_{k=0}^{N-1} \) from (15) be persistently exciting of order \( L + n \). Then, any \( \{\bar{u}_k\}_{k=0}^{N-1}, \{\bar{y}_k\}_{k=0}^{L+d_i-1} \) is a trajectory of system (1) if and only if there exists \( \alpha \in \mathbb{R}^{N-L+1} \) such that the following holds:

\[
\begin{bmatrix}
H_L(\bar{\Psi}(u, \Xi)) \\
H_{L+1}(\Xi)
\end{bmatrix} \alpha = \begin{bmatrix}
\bar{\Psi}(u, \Xi) \\
\bar{\Xi}
\end{bmatrix},
\]

where \( \bar{\Psi}(u, \Xi) \) is the stacked vector of the sequence \( \{\bar{\Psi}(u_k, \Xi_k)\}_{k=0}^{L-1} \), while \( \bar{\Xi} \) is the stacked vectors of \( \{\bar{\Xi}_k\}_{k=0}^{N-1} \) which, according to (6), are composed of \( \{\bar{y}_k\}_{k=0}^{L+d_i-1}, \{\bar{y}_k\}_{k=0}^{L+d_i-1} \), respectively.

**Proof**: According to Theorem 1, \( \{\bar{\Psi}(u_k, \Xi_k)\}_{k=0}^{L-1}, \{\bar{y}_k\}_{k=0}^{L+d_i-1} \) is an input–output trajectory of (15) if and only if \( \bar{u}_k \in \mathbb{R}^{N-L+1} \) is an input–output trajectory of (1).

Using the input–output trajectory \( \{u_k\}_{k=0}^{N-1}, \{\bar{y}_k\}_{k=0}^{L+d_i-1} \) of system (1), one can construct the following sequences \( \{\bar{\Psi}(u_k, \Xi_k)\}_{k=0}^{N-1} \) and \( \{\bar{\Xi}_k\}_{k=0}^{N-1} \) which correspond to an input-state trajectory of (15). Since \( \{\bar{\Psi}(u_k, \Xi_k)\}_{k=0}^{N-1} \) is persistently exciting of order \( L + n \) by assumption, and the pair \((A, \mathcal{G}\mathcal{B})\) is controllable by Lemma 1, then it was shown in [3] that any \( \{\bar{\Psi}(u_k, \Xi_k)\}_{k=0}^{N-1}, \{\bar{\Xi}_k\}_{k=0}^{N-1} \) is an input-state trajectory of (15) if and only if there exists \( \alpha \in \mathbb{R}^{N-L+1} \) such that

\[
\begin{bmatrix}
H_L(\bar{\Psi}(u, \Xi)) \\
H_{L+1}(\Xi)
\end{bmatrix} \alpha = \begin{bmatrix}
\bar{\Psi}(u, \Xi) \\
\bar{\Xi}
\end{bmatrix},
\]

Next, notice from (14) that for each output \( \bar{y}_i, \) for \( i \in [m] \), the following holds:

\[
\bar{y}_i_{[L, L+d_i-1]} \overset{(14)}{=} \begin{bmatrix}
I_{L_i} \otimes g_i^\top
\end{bmatrix} \bar{\Psi}_{[L-d_i-1, L-d_i-1]}(u, \Xi) \overset{(17)}{=} \begin{bmatrix}
I_{L_i} \otimes g_i^\top
\end{bmatrix} H_{d_1}(\bar{\Psi}_{[L-d_i-1, N]}(u, \Xi)) \alpha
\]

Furthermore, the state \( \bar{\Xi}_L \) can be written as

\[
\bar{\Xi}_L \overset{(6)}{=} \begin{bmatrix}
\bar{y}_{1\ldots, (L, L+d_i-1)} \\
\vdots \\
\bar{y}_{m\ldots, (L, L+d_i-1)} \\
H_{d_m}(\bar{y}_{m\ldots, (L, L+d_i-1)}(u, \Xi)) \alpha
\end{bmatrix}
\]

Finally, concatenating (19) with (17) results in (16) which completes the proof.

**Theorem 2** provides a purely data-based representation of full-state feedback linearizable systems. In particular, each input–output trajectory \( \{u_k\}_{k=0}^{N-1}, \{\bar{y}_k\}_{k=0}^{L+d_i-1} \), for \( i \in [m] \), can be parameterized via (16) using a priori collected data \( \{u_k\}_{k=0}^{N-1}, \{\bar{y}_k\}_{k=0}^{L+d_i-1} \). The setting of Theorem 2 is an idealization that may not be satisfied in practice due to nonzero errors \( \epsilon(u_k, \Xi_k) \) and \( \delta(\omega_k) \) in (10). Moreover, the persistency of excitation condition of \( \{\bar{\Psi}(u_k, \Xi_k)\}_{k=0}^{N-1} \) can only be checked after collecting the input–output data but cannot be enforced a priori by a suitable design of \( u \). This is because the basis functions depend not only on the input but on the output as well. For the abovementioned reasons, we provide in the following section constructive methods to approximately solve the simulation and output-matching control problems in a data-based fashion and without requiring persistency of excitation. Furthermore, we provide qualitative error bounds on the difference between the estimated and true outputs and show that these errors tend to zero if (i) the noise in the data as well as the basis function approximation error tend to zero and (ii) persistency of excitation condition is satisfied.
IV. DATA-BASED SIMULATION AND OUTPUT-MATCHING

In this section, we investigate the data-based simulation and output-matching control problems for the class of DT-MIMO feedback linearizable nonlinear systems (1). For the nominal setting, where the basis function expansion in (8) is exact and the output data is noiseless, the data-based simulation and output matching problems can be formulated in a similar manner as in [21, Propositions 2 and 3]. For space reasons, we skip the nominal case here and consider the practically more relevant setting where $c(t, \Xi) \neq 0$, $\delta(\omega_k) \neq 0$ in (10). To do so, in the following, we restrict our analysis to a compact subset of the input-state space $\Omega \subset \mathbb{R}^m \times \mathbb{R}^m$, i.e., we assume that on this $\Omega$ a priori collected input and output trajectories as well as the simulated/matched trajectories evolve in the set $\Omega$. Along with local Lipschitz continuity of $\bar{g}$ and the chosen basis functions, guarantees a uniform upper bound on the approximation error $\epsilon(c_k, \Xi)$ for all $(c_k, \Xi) \in \Omega$. This assumption is summarized as follows.

Assumption 5: The error in the basis function approximation $c_k(\Xi)$ is uniformly upper bounded by $\epsilon > 0$, i.e., $|c_k(\Xi)| \leq \epsilon$, for all $(c_k, \Xi) \in \Omega \subset \mathbb{R}^m \times \mathbb{R}^m$, where $\Omega$ is a compact subset of the input space.

Since $\tilde{E}_k(c_k, \Xi) = \tilde{G}^T \tilde{E}_k(c_k, \Xi)$, Assumption 5 implies

$$\|\tilde{E}_k(c_k, \Xi)\| \leq \|\tilde{G}^T\|_\infty |\tilde{E}_k(c_k, \Xi)| \leq \|\tilde{G}\|_\infty \epsilon.$$  \hfill (20)

**Remark 1:** (a) We denote the Lipschitz constant of $\bar{g}$ w.r.t. $\Xi$ in the compact set $\Omega$ by $K_{\Xi}$. (b) The function $|\delta(\omega_k)|$ in (10) satisfies $|\delta(0)| = 0$ and, by local Lipschitz continuity of $\bar{g}$ and $\Psi$ w.r.t. $\Xi$ on the compact set $\Omega$ and boundedness of $w_k$, there exists a $K_{w, \epsilon}$ such that

$$|\delta(\omega_k)| \leq K_{w, \epsilon} \epsilon$$

for all $k \geq 0$.

A. Data-Based Simulation

The data-based simulation problem is defined as follows.

**Definition 2:** Data-based simulation [4]: Given an input $u$ and initial conditions $x_0$ for the nonlinear system in (1), find the corresponding output trajectory $\hat{y}$ using only input–output data, i.e., without explicitly identifying a model of the system.

In the following theorem, we solve a minimization problem for $\alpha$ in contrast to solving of nonlinear equations in (16). Once a solution is obtained, an approximate output trajectory is found and its difference from the true simulated output is shown to be bounded.

In what follows, we use $\hat{y}(u, \Xi)$ to denote the stacked vector of $\{\hat{y}_k(u, \Xi)\}_{k=0}^{N-1}$. Moreover, for some input to be simulated $u$ and some vector $\alpha \in \mathbb{R}^N$, we use $\hat{y}(u, H_{(k+1)}(\Xi)\alpha)$ to denote the stacked vector of the sequence $\{\hat{y}_k(u, H_{(k+1)}(\Xi)\alpha)\}_{k=0}^{N-1}$ with each element defined as $\hat{y}_k(u, H_{(k+1)}(\Xi)\alpha) := \hat{y}(u, H_{(k+1)}(\Xi)\alpha)$.

**Theorem 3:** Suppose Assumptions 1–5 are satisfied and let $\{u_k\}_{k=0}^{N-1}$, $\{\hat{y}_k(u, H_{(k+1)}(\Xi)\alpha)\}_{k=0}^{N-1}$ for $i \in \mathbb{Z}[1, m]$, be input–output data sequences from (1). Furthermore, let $\{u_k\}_{k=0}^{N-1}$ be a new input to be simulated with $\Xi = \{\hat{y}_i|_{[0, m-1]}\} \cdots \hat{y}_i|_{[0, d-1]}$, $\hat{y}_i|_{[0, m-1]}$ specifying initial conditions for the state $\Xi$ in (13). Let the following optimization problem be feasible for the given $\Xi$:

$$\alpha^* \in \arg \min_{\alpha} J(\alpha) := \|Hx + \lambda \max{\epsilon(c_k, \Xi)}\|_\alpha \|\alpha\|_\alpha^2,$$  \hfill (21a)

such that $\Xi = H_{(k+1)}(\Xi)\alpha$, $\alpha > 0$, $H_{(k+1)}(\Xi)\alpha)$.

where $\lambda > 0$, $H_{(k+1)}(\Xi)\alpha)$.

Then, the estimated simulated outputs are given by $\hat{y}_i|_{[0, m-1]} = H_{(k+1)}(\Xi)\alpha)$.

\[3\] According to (7) and the discussion below it, $\Phi$ is the iterated composition of the continuously differentiable functions $f, g, h$, and $T^{-1}$ (which is continuously differentiable by (2) and (6)). Hence, $\Phi$ is locally Lipschitz continuous.

By definition of $\alpha^*$ from (21) and for some vector $c$ satisfying $c^T c = b$, it holds that

$$H_{(k+1)}(\Xi)\alpha^* = \hat{y}(u, \hat{\Xi}) + c.$$  \hfill (21b)

The constraint in (21b) fixes the initial estimated state $\hat{\Xi} := H_{(k+1)}(\Xi)\alpha^* \Xi_0$, and hence, $\hat{y}_i|_{[0, m-1]} = \hat{y}_i|_{[0, m-1]} = 0$ (cf. (6)). Furthermore, each estimated simulated output takes the form $\hat{y}_i|_{[0, m-1]} = H_{(k+1)}(\Xi)\alpha^* \Xi_0$, for all $k \in \mathbb{Z}[0, L-1]$. Therefore, one can write

$$\hat{y}_i|_{[0, m-1]} = H_{(k+1)}(\Xi)\alpha^* \Xi_0 + \hat{y}_i|_{[0, m-1]}.$$  \hfill (21c)

The expression in (25) can be upper bounded by

$$|\epsilon(i, k+1)| = K_{\Xi}|\Xi| \Xi_0|\Xi| \Xi_0 + \epsilon(i + 1|\Xi| \Xi_0) + |\Xi| \Xi_0 \sqrt{b}$$

and $w(1 + K_{\Xi}|\Xi| \Xi_0) \Xi_0$ for the first two terms in (25) were bounded by Lipschitz continuity of $\phi$, and the third and fifth terms were bounded by $\epsilon(i + 1|\Xi| \Xi_0) + |\Xi| \Xi_0 \sqrt{b}$, and the last two terms were bounded by $w(1 + K_{\Xi}|\Xi| \Xi_0) \Xi_0$ for $|\Xi| \Xi_0 \leq w$ by assumption. We continue the proof by induction. Let $k = 0$ in (26), and notice that

$$|\epsilon(1)| = \epsilon(i|\Xi| \Xi_0) + |\Xi| \Xi_0 \sqrt{b} + w(1 + K_{\Xi}|\Xi| \Xi_0) \Xi_0,$$  \hfill (27)
since $\Xi_0 = \Xi_0$ from (21b). Notice that (27) has the form (22) with $P^0(K_{\Xi}) = 1$. For the induction step, let the following hold for all $k \in \mathbb{Z}_{[1,L-1]}$, all $i \in \mathbb{Z}_{[0,k-1]}$, and $i \in \mathbb{Z}_{[0,m]}$:

$$
|e_{l,k+d_1}| \leq P^k(K_{\Xi})(e^* (1 + ||\alpha^*||) + ||G||_{\infty} \sqrt{b} + w^* (1 + K_w)||\alpha^*||).
$$

(28)

Since $P^k(K_{\Xi})$ increases with increasing $k$, this implies that the following bound on the previous error instances:

$$
\|e_{l,k+d_1}\|_{\infty} \leq P^{(k-1)}(K_{\Xi})(e^* (1 + ||\alpha^*||) + ||G||_{\infty} \sqrt{b} + w^* (1 + K_w)||\alpha^*||).
$$

(29)

Then, by (26), the definition of $\Xi_{k}$ as in (6) and the corresponding $\Xi_{k} = \{\hat{y}_{1,[k,k+d_1-1]} \cdots \hat{y}_{m,[k,k+d_1-1]}\}$, we have

$$
|e_{l,k+d_1}| \leq K_{\Xi} \left\| \begin{bmatrix}
    e_{1,[k,d_1-1]} \\
    \vdots \\
    e_{m,[k,d_1-1]}
\end{bmatrix} + e^* (1 + ||\alpha^*||) + ||G||_{\infty} \sqrt{b} + w^* (1 + K_w)||\alpha^*|| \right\|_{\infty}.
$$

(30)

Notice that the elements of the first term on the RHS of (30) are bounded by (29). Therefore, we have that

$$
|e_{l,k+d_1}| \leq K_{\Xi} \left( P^{(k-1)}(K_{\Xi})(e^* (1 + ||\alpha^*||) + ||G||_{\infty} \sqrt{b} + w^* (1 + K_w)||\alpha^*||) + e^* (1 + ||\alpha^*||) + ||G||_{\infty} \sqrt{b} + w^* (1 + K_w)||\alpha^*|| \right).
$$

(31)

Collecting the terms in the last inequality results in (22), which completes the proof.

Theorem 3 provides an approximate solution to the data-based simulation problem when the data are noisy and the basis functions approximation error is unknown but uniformly upper bounded as in Assumption 5. This was done by solving the optimization problem in (21). In particular, if (21) is feasible (i.e., the constraint (21b) can be satisfied) then an approximate solution can be found for the simulation problem. Notice that persistency of excitation condition is not necessary for feasibility of (21b) and, hence, is not necessarily required for Theorem 3. In fact, feasibility of (21) is given for any $\Xi_{0}$ if $H_{L}(\Xi_{[0,N-L]})$ has full row rank, which can be easily verified given the a priori collected data $(\hat{y}_{i,k})_{k=0}^{N-1}$ for $i \in \mathbb{Z}_{[0,m]}$. Furthermore, Theorem 3 provides a bound on how far the estimated outputs $\hat{y}_{i,[L,L+d_1-1]}$ deviate from the actual system response $y_{i,[L,L+d_1-1]}$.

Since the solution of (21) appears in the error bound, we incentivize solutions with smaller norm using a regularization term as in (21), which depends on the basis functions approximation error and noise bounds $e^*$ and $w^*$, respectively. It can be seen from (22) that the error bound also depends on the unknown matrix of coefficients $G$ in (8). Under certain assumptions, one can compute an upper bound on its norm in a model-free fashion (see [24, Lemma 2]).

In general, the error bound (22) can be conservative due to the use of Lipschitz arguments. In particular, the polynomial $P^k(K_{\Xi})$ diverges (as $k \to \infty$) for $K_{\Xi} > 1$ but converges to $\frac{1}{K_{\Xi}}$ for $K_{\Xi} < 1$, in which case the error bound in (22) is uniformly upper bounded for all $k$. In contrast, the actual error $e_{i,k}$ can potentially be (much) smaller than (22), depending on the quality of the collected data and the chosen basis functions.

The following lemma establishes the important qualitative result that if $\{\hat{y}_{i}(u,\Xi)\}_{k=0}^{N-1}$ is persistently exciting, then the error bound in (22) (and, hence, the true error) goes to zero as $\max(e^*, w^*)$ goes to zero.

Lemma 2: Let the conditions of Theorem 3 hold and let $\{\hat{y}_{i}(u,\Xi)\}_{k=0}^{N-1}$ be persistently exciting of order $L + n$. Then, as $\nu := \max(e^*, w^*) \to 0$, the error $e_{i,k+d_1} \to 0$.

Proof: It can be seen from (22) that, for each fixed $k$, the error bound goes to zero as $\nu \to 0$ if (i) $e_{i,k} ||\alpha^*|| \to 0$ and (ii) $p_{i,k} \to 0$. To show these two properties, notice that for sufficiently small $e^*$ and $w^*$, persistency of excitation of $\{\hat{y}_{i}(u,\Xi)\}_{k=0}^{N-1}$ implies that of $\{\hat{y}_{i}(u,\Xi) + \hat{E}_{i}(u,\Xi) + D_{i}(\omega)\}_{k=0}^{N-1}$ of the same order. This is the case, e.g., when $\|H_{L+1}(\hat{E}(u,\Xi) + D(\omega))\|_{2} \leq \sigma_{min}(H_{L}(\hat{E}(u,\Xi)))$. Now, we propose the following candidate solution to (21):

$$
\hat{\alpha} = H_{L}(\hat{E}(\hat{u},\hat{\Xi}) + H_{L+1}(\hat{E}(\hat{u},\hat{\Xi}) + D(\omega)))
$$

(31)

Notice that (31) is such that (21b) is satisfied since the pseudoinverse is in fact a right inverse given the implied persistency of excitation condition (i.e., $\|\omega^*\|_{2}$ is bounded) of (22), which implies the relationship (21) holds.

If the right-hand side of (32) goes to zero as $\nu \to 0$, then the left-hand side of (32), which is a sum of two norm terms, goes to zero as well, implying both (i) and (ii) above. Clearly, the term $\lambda \nu \|\hat{\alpha}\|_{2}$ on the right-hand side of (32) goes to zero as $\nu \to 0$ as long as $\|\hat{\alpha}\|_{2}$ is bounded. This is also the case for the first two terms of the definition of $\|e^*, w^*\|_{2}$ on the right-hand side of (32) goes to zero as $\nu \to 0$ as long as $\|\hat{\alpha}\|_{2}$ is bounded. This is done in the following.

Recall from (31) that $\hat{\alpha}$ satisfies the following:

$$
H_{L}(\hat{E}(\hat{u},\hat{\Xi}) + H_{L}(\hat{E}(\hat{u},\hat{\Xi}) + D(\omega)))
$$

which, for $\Delta \hat{\Psi} := \hat{\Psi}(\hat{u},\hat{\Xi}) - \hat{\Psi}(\hat{u},\hat{\Xi})$, can be rewritten as

$$
H_{L}(\hat{E}(\hat{u},\hat{\Xi}) + H_{L}(\hat{E}(\hat{u},\hat{\Xi}) + D(\omega)))
$$

(33)

According to [3], such a matrix with the lower block row of the form $H_{L}(\hat{E}(\hat{u},\hat{\Xi}) + H_{L}(\hat{E}(\hat{u},\hat{\Xi}) + D(\omega)))$ has full row rank. For sufficiently small $w^*$, the considered matrix still maintains full row rank and, hence, a right inverse exists. 

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For a fixed simulated trajectory \( \tilde{u}, \tilde{z} \), (33) represents a perturbed underdetermined system of linear equations. Therefore, by applying results from perturbation theory of underdetermined systems [25, Th. 5.6.1], it can be shown that

\[
\frac{\|\tilde{a} - \tilde{a}_0\|_2}{\|\tilde{a}_0\|_2} \leq K_2(\Theta) \left( c_0 \frac{\|\Delta \Theta\|_2}{\|\Theta\|_2} + \frac{\|\Delta \mu\|_2}{\|\mu\|_2} \right) + O(\varphi^2),
\]

(34)

where \( \Theta = \Theta(\mu) \) is the bounded minimum-norm solution of the unperturbed system of equationst\(\), and \( K_2(\Theta) = \|\Theta\|_2 \|\Theta\|_2 \) is the condition number of the matrix \( \Theta \). Moreover, \( c_0 \approx 1 \), \( \min \{ n_c, n_r, n_0, +1 \} \) and \( O(\varphi^2) \) are higher order terms with \( \varphi := \max \{ \|\Delta \Theta\|_2, \|\Delta \mu\|_2 \} \).

As \( \nu \to 0 \), the terms \( \|\Delta \Theta\|_2 \), \( \|\Delta \mu\|_2 \) to 0 due to Assumption 5 and local Lipschitz continuity of \( \Psi \) (see (33)). Hence, it can be seen from (34) that the candidate \( \tilde{u} \) approaches the bounded solution \( \tilde{a} \), which is finite for any fixed \( \tilde{u}, \tilde{z} \).

Lemma 2 shows that the error bound in (22) (and, hence, the true error) goes to zero as \( \max \{ \varepsilon^*, \varepsilon^* \} \) goes to zero if persistence of excitation of \( \{ \Psi_k(u, \tilde{z}) \}_{k=0}^{\infty} \) is satisfied. In the following section, we study the data-based output matching problem for full-state feedback linearizable systems.

### B. Data-Based Output-Matching

The data-based output matching control problem is defined as follows.

**Definition 3. Data-based output-matching control [4]**: Given a desired reference trajectory \( \bar{y} \) and a corresponding initial condition \( \bar{x}_0 \) for the nonlinear system in (1), find the required input trajectory \( \tilde{u} \) that, when applied to the system, results in an output that tracks the desired reference trajectory, using only input–output data.

Analogous to the discussion in Section IV-A, we propose a similar approach to the one shown in Theorem 3 for solving the data-based output-matching control problem despite an unknown basis functionapproximation error and using only input and noisy output data. In particular, we assume that the a priori collected data and the trajectory to be matched evolve in a compact subset of the input–state space as in Assumption 5. In the following, we solve for an approximate control input \( u_{0, L-1} \) that, when applied to the system, results in approximate output trajectories \( \tilde{y}_{k, [0, L+d_i-1]} \), which track the reference trajectories \( \bar{y}_{k, [0, L+d_i-1]} \) as closely as possible. The error between the two outputs is bounded, as shown in Theorem 4. Before presenting the result, the following assumption is made, which is needed in order to retrieve the desired input \( \tilde{u} \).

**Assumption 6**: Let \( \psi_j(\mu, \tilde{z}_k) := u_{j,k} \) for all \( j \in Z_{[1,m]} \).

**Assumption 6** is not restrictive since it can be replaced by only requiring \( u_{k} \) to lie in the span of \( \Theta \), or that for any \( \tilde{z}, \psi(\tilde{z}, \tilde{z}) \) is injective.

The following theorem is the dual result of Theorem 3. For some vector \( \alpha \in \mathbb{R}^{N-L+1} \) and given reference trajectories \( \bar{y}_{k} \) (with \( \tilde{z} \) being the corresponding transformed state (see (6))), we use \( \Psi(H_L(u)\alpha, \tilde{z}) \) to denote the stacked vector of the sequence \( \{ \psi_k(H_L(u)\alpha, \tilde{z}) \}_{k=0}^{L-1} \) with each element defined as \( \psi_k(H_L(u)\alpha, \tilde{z}) := \Psi_k(H_L(u)\alpha, \tilde{z}) \).

**Theorem 4**: Suppose Assumptions 1–6 are satisfied and let \( \{ u_{k, L+d_i-1} \}_{k=0}^{N-1} \) be the input–output data sequences collected from (1). Furthermore, let \( \{ \tilde{y}_{k, L+d_i-1} \}_{k=0}^{N-1} = \tilde{y}_{k, [d_i-1]} \) be desired reference trajectories with \( \tilde{z}_0 := \left[ \begin{array}{c} \bar{y}_{1,0} \cdots \bar{y}_{N,0} \end{array} \right] \).

V. Example

In this section, we illustrate the results of Theorem 3 on a discretized model of a fully actuated double inverted pendulum when using an inexact basis function approximation of the unknown nonlinearities and noisy output data (see (10)). Using Euler’s discretization of the continuous-time dynamics, we obtain the following discrete-time model:

\[
x_{k+1} = x_k + T_s (A x_k + B z_k), \quad y_k = C x_k, \quad (37)
\]

\[
x_k := \begin{bmatrix} \theta_{1,k} & \theta_{2,k} \end{bmatrix}^T, \quad y_k := \begin{bmatrix} \theta_{1,k} & \theta_{2,k} \end{bmatrix}^T, \quad (38)
\]

\[
Z_k := \begin{bmatrix} z_{1,k} & z_{2,k} \end{bmatrix} = M(\theta_k)^{-1}(\tau_k - C(\theta_k, \theta_k) \dot{\theta}_k - G(\theta_k)).
\]

In (37) and (38), \( \theta_k \) and \( \theta_k \) are the vectors of angular positions and angular velocities, respectively, \( \tau_k \) is the vector of joint torques and \( T_s \) is the sampling time. The terms \( M(\theta), C(\theta, \dot{\theta}), \text{and } G(\theta) \) represent the inertia, dissipative, and gravitational terms, respectively, and depend on the masses and lengths of the two links.

7Although feedback linearization may in general be destroyed under discretization [26], this is not the case for system (37).

8The following (unknown) model parameters were used \( m_1 = m_2 = 1 \text{ kg} \) and \( l_1 = l_2 = 0.5 \text{ m} \) for the masses and lengths of the two links, respectively.
in (37) have relative degrees $d_1 = d_2 = 2$ and $\sum_i d_i = 4 = n$. Thus, by Theorem 1, there exists a coordinate transformation $\Xi_k = T(x_k)$ such that the transformed system is full-state feedback linearizable. This transformation takes the form (by prior model knowledge)

$$
\Xi_k = \begin{bmatrix} x_{1,k} & x_{1,k} + T_{x2,k} & x_{3,k} & x_{3,k} + T_{x4,k} \end{bmatrix}^\top.
$$

Rewriting (37) in the transformed coordinates results in

$$
\Xi_{k+1} = A\Xi_k + Bv_k,
$$

where $\Xi_k = [\xi_1, \xi_2, \ldots, \xi_n]^\top$. To approximate (41), we use the following basis functions:

$$
\Psi(\tau_k, \Xi_k) = \begin{bmatrix} \tau_k \xi_1 \xi_2 \xi_3 \xi_4 \end{bmatrix}^\top
$$

where $\hat{M}$, $\hat{C}$, $\hat{G}$ contain user-provided estimates for the parameter values (in this example, they were obtained by randomly perturbing the unknown real values by up to 5%). This choice of basis functions is justified by the fact that (38) is ubiquitous in robotics, unlike accurate model parameter values, which can be very difficult to obtain.

We collect input–output data of length $N = 500$ (or 50 s with $T_s = 0.1$) by operating the double inverted pendulum using a prestabilizing controller in the following compact subset of the input-state space:

$$
\Omega = \left\{ (\tau_k, \Xi_k) \in \mathbb{R}^2 \times \mathbb{R}^4 \left| \begin{array}{c} \tau_{lb} \\
\Xi_{lb} \end{array} \right. \leq \begin{array}{c} \tau_k \\
\Xi_k \end{array} \leq \begin{array}{c} \tau_{ub} \\
\Xi_{ub} \end{array} \right. \right\},
$$

where the inequalities are defined element-wise and $\tau_{ub} = -\tau_{lb} = \begin{bmatrix} 20 & 20 \end{bmatrix}^\top$. The data are contaminated by a random additive noise sampled from a uniform random distribution $U(-0.01, 0.01)$ (i.e., $\nu^* = 0.01$). Fig. 1 shows the approximated output of the system as well as the true simulated trajectory when using $\lambda = 0.1$. The error between the true and estimated simulated outputs is $\|\hat{y} - \hat{y}\|_2 = 0.1278$. It can be seen from the figure that the proposed method in Theorem 3 yields good results when the offline collected data has sufficient information about the system under consideration. In contrast, the error bound in (22) can potentially be conservative due to the use of Lipschitz continuity arguments in its derivation. Finally, since Theorem 3 does not require persistency of excitation, it was observed that collecting longer but not necessarily persistently exciting data yields better results on the cost of increased computational burden to solve (21).

VI. CONCLUSION

In this article, we presented an extension of Willems’ fundamental lemma to the class of DT-MIMO feedback linearizable nonlinear systems. This was done by exploiting linearity in transformed coordinates and using a dictionary of known basis functions that depend only on input–output data. In practice, one must account for the nonzero approximation as well as noise in the output data. To that end, we presented constructive methods on how to solve the simulation and output matching problems despite unknown, but uniformly bounded, basis functions approximation errors and output noise, and provided error bounds on the difference between the estimated and actual output trajectories. These error bounds have an appealing qualitative property. In particular, we have shown that the error bounds (and, hence, the actual errors) go to zero if the basis functions approximation error as well as the noise in the data go to zero and persistency of excitation condition is satisfied.

APPENDIX

A. Block-Brunovsky Canonical Form

The matrices $(A, B, C)$ have the following form:

$$
A := \begin{bmatrix} A_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & A_m \end{bmatrix},
B := \begin{bmatrix} B_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & B_m \end{bmatrix},
C := \begin{bmatrix} 0 & \cdots & C_m \end{bmatrix},
$$

with $A_i \in \mathbb{R}^{d_i \times d_i}$, $B_i \in \mathbb{R}^{d_i \times 1}$, and $C_i \in \mathbb{R}^{1 \times d_i}$ for $i \in \{1, m\}$ defined as

$$
A_i := \begin{bmatrix} 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & 1 \end{bmatrix},
B_i := \begin{bmatrix} 0 \\
\vdots \\
0 \\
1 \end{bmatrix},
C_i := \begin{bmatrix} 1 \\
\vdots \\
0 \end{bmatrix}.
$$

B. Proof of Lemma 1

A necessary and sufficient condition for controllability of $(A, B)$ is that the following matrix has full row rank:

$$
\begin{bmatrix} BG & ABG & \cdots & A^{n-1}B \end{bmatrix} = \begin{bmatrix} B_1 g_1 & A_1 B_1 g_1 & \cdots & A_1^{n-1}B_1 g_1 \\
\vdots & \ddots & \ddots & \vdots \\
B_m g_m & A_m B_m g_m & \cdots & A_m^{n-1}B_m g_m \end{bmatrix}.
$$

By the structure of $(A_i, B_i)$ in Appendix A, each block row of (43) has the following form:

$$
\begin{bmatrix} B_i g_i & A_i B_i g_i & \cdots & A_i^{n-1}B_i g_i \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & g_i \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0 \end{bmatrix}.
$$

This was obtained by gridding $\Omega$ and numerically solving (9).
which clearly has full row rank (i.e., \( \text{rank} = d_i \)) since \( G \) is full row rank and, hence, no row \( g_i^\top \) is all zeros. Since, again, \( G \) is full row rank, then the concatenation of the block rows of the form in (44) results in a full row rank matrix (43) (i.e., \( \text{rank} = \sum_i d_i = n \)), implying that the pair \((A, BG)\) is controllable.

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