GENERALIZATION OF HARISH-CHANDRA’S BASIC THEOREM FOR
RIEMANNIAN SYMMETRIC SPACES OF NON-COMPACT TYPE

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ABSTRACT. A basic exact sequence by Harish-Chandra related to the invariant differential operators on a Riemannian symmetric space \( G/K \) is generalized for each \( K \)-type in a certain class which we call ‘single-petaled’. The argument also includes a further generalization of Broer’s generalization of the Chevalley restriction theorem.

1. Introduction and main results

Let \( \mathfrak{g} \) be a real semisimple Lie algebra and \( \theta \) a fixed Cartan involution of \( \mathfrak{g} \). In this paper the subscript \( \mathbb{C} \) is used for indicating the complexification of a real object. Denote the universal enveloping algebra of the complex Lie algebra \( \mathfrak{g}_\mathbb{C} \) by \( U(\mathfrak{g}_\mathbb{C}) \), the center of \( U(\mathfrak{g}_\mathbb{C}) \) by \( Z(\mathfrak{g}_\mathbb{C}) \), and the symmetric algebra of \( \mathfrak{g}_\mathbb{C} \) by \( S(\mathfrak{g}_\mathbb{C}) \). Similar notation is used for other complex Lie algebras or vector spaces. Let \( G_{\text{ad}} \) be the adjoint group of \( \mathfrak{g} \), \( G_\theta \) the subgroup of the adjoint group of \( \mathfrak{g}_\mathbb{C} \) consisting of all the elements that leave \( \mathfrak{g} \) stable, and \( G \) an arbitrary group such that \( G_{\text{ad}} \subset G \subset G_\theta \). The adjoint action of \( G_{\text{ad}} \) or \( G \) or \( G_\theta \) (resp. \( \mathfrak{g}_\mathbb{C} \)) is denoted by \( \text{Ad} \) (resp. \( \text{ad} \)). Let \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \) be the Cartan decomposition. Take a maximal Abelian subspace \( \mathfrak{a} \) of \( \mathfrak{p} \) and fix a basis \( \Pi \) of the restricted root system \( \Sigma \) for \((\mathfrak{g}, \mathfrak{a})\). \( \Pi \) defines the system \( \Sigma^\theta \) of positive roots. Put \( K = G^\theta \) (the subgroup of \( G \) which commutes with \( \theta \)), \( M = Z_K(\mathfrak{a}) \) (the centralizer of \( \mathfrak{a} \) in \( K \)), and \( \mathfrak{m} = \text{Lie}(M) \). Using \( N_K(\mathfrak{a}) \) (the normalizer of \( \mathfrak{a} \) in \( K \)) define the Weyl group \( W = N_K(\mathfrak{a})/M \). For each \( \alpha \in \Sigma \), let \( H_\alpha \) be the corresponding element of \( \mathfrak{k} \) via the Killing form \( B(\cdot, \cdot) \) of \( \mathfrak{g} \) and put \( |\alpha| = B(H_\alpha, H_\alpha)^\frac{1}{2} \), \( \alpha^\vee = \frac{2H_\alpha}{|\alpha|} \) (the coroot of \( \alpha \)). Denote the restricted root space for \( \alpha \in \Sigma \) by \( \mathfrak{g}_\alpha \) and put \( \mathfrak{n} = \sum_{\alpha \in \Sigma^\theta} \mathfrak{g}_\alpha \), \( \rho = \frac{1}{2} \sum_{\alpha \in \Sigma^\theta}(\dim \mathfrak{g}_\alpha)\alpha \). Let us now define the map \( \gamma \) of \( U(\mathfrak{g}_\mathbb{C}) \) into \( S(\mathfrak{a}_\mathbb{C}) \) by the projection

\[
U(\mathfrak{g}_\mathbb{C}) = U(\mathfrak{a}_\mathbb{C}) \oplus (\mathfrak{m}_\mathbb{C} U(\mathfrak{g}_\mathbb{C})) + U(\mathfrak{g}_\mathbb{C}) \mathfrak{t}_\mathbb{C} \to U(\mathfrak{a}_\mathbb{C}) \cong S(\mathfrak{a}_\mathbb{C})
\]

followed by the translation

\[
S(\mathfrak{a}_\mathbb{C}) \ni f(\lambda) \mapsto f(\lambda + \rho) \in S(\mathfrak{a}_\mathbb{C}).
\]

Here we identified \( S(\mathfrak{a}_\mathbb{C}) \) with the space of holomorphic polynomials on the dual space \( \mathfrak{a}_\mathbb{C}^* \) of \( \mathfrak{a}_\mathbb{C} \). We call \( \gamma \) the Harish-Chandra homomorphism. Let \( U(\mathfrak{g}_\mathbb{C})^K \) (resp. \( S(\mathfrak{a}_\mathbb{C})^W \)) be the subalgebra of invariants in \( U(\mathfrak{g}_\mathbb{C}) \) (resp. \( S(\mathfrak{a}_\mathbb{C}) \)) under the action of \( K \) (resp. \( W \)). In this paper the superscript of an operator domain generally indicates the subspace of invariants. Harish-Chandra showed in [HC] the following exact sequence of algebra homomorphisms:

\[
0 \to U(\mathfrak{g}_\mathbb{C})^K \cap U(\mathfrak{g}_\mathbb{C}) \mathfrak{t}_\mathbb{C} \to U(\mathfrak{g}_\mathbb{C})^K \gamma \to S(\mathfrak{a}_\mathbb{C})^W \to 0.
\]

\(^1\)Actually his proof targets only the case of \( G = G_{\text{ad}} \), but the general case follows from it since [24] is valid for any \( G \) (cf. [KR] Proposition 10).
On the other hand, let $a^+$ be the orthogonal complement of $a$ in $\mathfrak{p}$ relative to the Killing form $B(\cdot, \cdot)$ and $\gamma_0$ the projection of $S(p_{\mathbb{C}})$ onto $S(a_{\mathbb{C}})$ defined by

$$S(p_{\mathbb{C}}) = S(a_{\mathbb{C}}) \oplus S(p_{\mathbb{C}})(a^+)_{\mathbb{C}} \rightarrow S(a_{\mathbb{C}}).$$

Then the restriction of $\gamma_0$ to $S(p_{\mathbb{C}})^K$ gives the algebra isomorphism

$$(1.2) \quad \gamma_0 : S(p_{\mathbb{C}})^K \cong S(a_{\mathbb{C}})^W,$$

which is known as the Chevalley restriction theorem. Let $\text{symm} : S(\mathfrak{g}_{\mathbb{C}}) \rightarrow U(\mathfrak{g}_{\mathbb{C}})$ be the symmetrization map. Then one has the symmetrization map. Then one has the $K$-module decomposition

$$(1.3) \quad U(\mathfrak{g}_{\mathbb{C}}) = \text{symm}(S(p_{\mathbb{C}})) \oplus U(\mathfrak{g}_{\mathbb{C}})_{\mathbb{C}},$$

so that $\Gamma_1$ is considered as a non-commutative counterpart of $\Gamma_2$. Hereafter we use the same symbol $\mathcal{A}$ for the three algebras $U(\mathfrak{g}_{\mathbb{C}})^K \cap U(\mathfrak{g}_{\mathbb{C}})_{\mathbb{C}}, S(p_{\mathbb{C}})^K$ and $S(a_{\mathbb{C}})^W$ identified with one another.

Note that $\Gamma_1$ and $\Gamma_2$ are rewritten as

$$(1.4) \quad 0 \rightarrow \text{Hom}_{\mathcal{K}}(\text{triv}, U(\mathfrak{g}_{\mathbb{C}})_{\mathbb{C}}) \rightarrow \text{Hom}_{\mathcal{K}}(\text{triv}, U(\mathfrak{g}_{\mathbb{C}})) \overset{\Gamma_0^{\text{inv}}}{\longrightarrow} \text{Hom}_{\mathcal{W}}(\text{triv}, S(a_{\mathbb{C}})) \rightarrow 0$$

and

$$(1.5) \quad \Gamma_0^{\text{inv}} : \text{Hom}_{\mathcal{K}}(\text{triv}, S(p_{\mathbb{C}})) \rightarrow \text{Hom}_{\mathcal{W}}(\text{triv}, S(a_{\mathbb{C}})),$$

respectively. Here ‘triv’ denotes the trivial representation of $K$ or $W$ over $\mathbb{C}$. The definitions of $\Gamma_0^{\text{inv}}$ and $\Gamma_0^{\text{inv}}$ are clear.

First, we generalize the Chevalley restriction theorem in the form $\Gamma_5$. We say a $K$-type $(\sigma, V)$ is quasi-spherical if the $\mathfrak{r}$-isotypic component of $(\text{Ad} |_{\mathcal{K}}, S(p_{\mathbb{C}}))$ is not $0$. From $\text{[Kr]}$ we know $(\sigma, V)$ is quasi-spherical if and only if $V^M \neq 0$. Suppose $(\sigma, V)$ is quasi-spherical. Then $W$ naturally acts on $V^M$. Define the map

$$(1.6) \quad \Gamma^\sigma_0 : \text{Hom}_{\mathcal{K}}(V, S(p_{\mathbb{C}})) \ni \Phi \mapsto \varphi \in \text{Hom}_{\mathcal{W}}(V^M, S(a_{\mathbb{C}}))$$

so that the image $\varphi$ is given by the composition

$$(1.7) \quad \varphi : V^M \hookrightarrow V \overset{\Phi}{\twoheadrightarrow} S(p_{\mathbb{C}}) \overset{\gamma_0}{\twoheadrightarrow} S(a_{\mathbb{C}}).$$

It is obviously well-defined and if $(\sigma, V) = (\text{triv}, \mathbb{C})$, the map $\Gamma_0^{\text{inv}}$ coincides with $\Gamma_5$. Both $\text{Hom}_{\mathcal{K}}(V, S(p_{\mathbb{C}}))$ and $\text{Hom}_{\mathcal{W}}(V^M, S(a_{\mathbb{C}}))$ have natural $\mathcal{A}$-module structures coming from multiplication of images by elements in $S(p_{\mathbb{C}})^K$ or $S(a_{\mathbb{C}})^W$. Observe that $\Gamma^\sigma_0$ intertwines these $\mathcal{A}$-module structures. Let us introduce a new class of $K$-types:

**Definition 1.1.** Put $\Sigma_1 = \Sigma \setminus 2\Sigma$. Choose a subset $\mathcal{R}$ of $\Sigma_1$ so that $\mathcal{R}$ intersects each $W$-orbit of $\Sigma_1$. For each $\alpha \in \mathcal{R}$ fix $X_\alpha \in \mathfrak{g}_a \setminus \{0\}$. Then we call a quasi-spherical $K$-type $(\sigma, V)$ is single-petaled if and only if

$$(1.8) \quad \sigma(X_\alpha + \theta X_\alpha)(\sigma(X_\alpha + \theta X_\alpha)^2 - 2|\sigma|\theta B(X_\alpha, \theta X_\alpha))v = 0 \quad \forall v \in V^M, \forall \alpha \in \mathcal{R}.$$
Then we have

**Theorem 1.3.** For any quasi-spherical \((\sigma, V)\), \(\Gamma_0^\sigma\) is injective. On the other hand, \(\Gamma_0^\sigma\) is surjective if and only if \((\sigma, V)\) is single-petaled.

This theorem gives a generalization of Broer’s theorem for a complex semisimple Lie algebra \((Bn)\) into the case of a Riemannian symmetric space of non-compact type. As mentioned in a footnote of \([Br]\), Broer’s theorem can also be proved by using the results of \([PRV]\). Similarly, using the results of \([Ko3]\), which generalize the results of \([PRV]\) into the Riemannian symmetric case, we can show Theorem 1.3 in a purely algebraic manner. However, our proof in §3 employs an analytic method modeled on \([Da]\). This method leads to further generalizations of Theorem 1.3 in some directions (Proposition 3.1, Theorem 3.5, Theorem 3.15). In particular, for some wider class of \(K\)-types than ‘single-petaled’, which will be called ‘quasi-single-petaled’, a result close to Theorem 1.3 holds.

The generalization of (1.4), which is the main theme of this paper, requires the notion of the degenerate affine Hecke algebra \(H\) associated naturally to the data \((n, a, A)\) (Definition 4.1). We here state a few properties of \(H\). \(H\) is an algebra over \(\mathbb{C}\) including \(S(a_c)\) and the group algebra \(\mathbb{C}[W]\) of \(W\) as subalgebras with the same 1 and the center of \(H\) is \(S(a_c)^W\). The map \(S(a_c) \otimes \mathbb{C}[W] \to H\) defined by multiplication gives a \(\mathbb{C}\)-linear isomorphism. Hence the left \(H\)-module

\[
(1.9) \quad S_H(a_c) := H \left/ \sum_{w \in W(1)} H(w-1) \right. \approx S(a_c) \otimes \mathbb{C}[W] / \sum_{w \in W(1)} \mathbb{C}[W](w-1)
\]

is naturally identified with \(S(a_c)\) as a left \(S(a_c)\)-module. It is notable that although the left \(W\)-action on \(S_H(a_c)\) differs from the original \(W\)-action on \(S(a_c)\), the space of \(W\)-fixed elements in \(S_H(a_c)\) equals \(S(a_c)^W\) (Corollary 4.4). Hence we may replace \(\text{Hom}_W(\text{triv}, S(a_c))\) in (1.4) with \(\text{Hom}_W(\text{triv}, S_H(a_c))\).

Suppose \((\sigma, V)\) is a quasi-spherical \(K\)-type. Define the map

\[
(1.10) \quad \Gamma^\sigma : \text{Hom}_K(V, U(a_c)) \ni \Psi \mapsto \psi \in \text{Hom}_C(V^M, S_H(a_c))
\]

so that the image \(\psi\) is given by the composition

\[
(1.11) \quad \psi : V^M \hookrightarrow V \xrightarrow{\Psi} U(a_c) \xrightarrow{\gamma} S(a_c) \approx S_H(a_c).
\]

Note that the space in the right-hand side of (1.10) is not \(\text{Hom}_W(V^M, S_H(a_c))\). In fact the map \(\psi\) defined by (1.11) does not always commute with the \(W\)-actions. Now we state the main result of this paper:

**Theorem 1.4.** For any \((\sigma, V)\), the kernel of \(\Gamma^\sigma\) equals \(\text{Hom}_K(V, U(a_c))\). The image of \(\Gamma^\sigma\) is included in \(\text{Hom}_W(V^M, S_H(a_c))\) if and only if \((\sigma, V)\) is single-petaled. If this condition is satisfied, the image equals \(\text{Hom}_W(V^M, S_H(a_c))\) and therefore we have the exact sequence

\[
(1.12) \quad 0 \to \text{Hom}_K(V, U(a_c)) \to \text{Hom}_K(V, U(a_c)) \xrightarrow{\Gamma^\sigma} \text{Hom}_W(V^M, S_H(a_c)) \to 0.
\]

**Remark 1.5.** If \(D \in U(a_c)^K\) and \(\Psi \in \text{Hom}_K(V, U(a_c))\), then the right multiplication of the image of \(\Psi\) by \(D\) gives a new element \(\Psi \cdot D \in \text{Hom}_K(V, U(a_c))\). This \(U(a_c)^K\)-module structure of \(\text{Hom}_K(V, U(a_c))\) induces an \(\mathcal{A}\)-module structure of \(\text{Hom}_K(V, U(a_c))/U(a_c)^K\) \(\approx\)
Hom\(_K(V, U(g)) \oplus \text{Hom\(_K(V, U(g)\ell_C))\). Also, we naturally consider \(\text{Hom\(_C(V^M, S_H(a_C))\) and Hom\(_W(V^M, S_H(a_C))\) as \(\mathcal{A}\)-modules. Since it is clear that

\[
\Gamma^\sigma(\Psi \cdot D) = \Gamma^\sigma(\Psi) \cdot \gamma(D) \quad \forall D \in U(g), \forall \Psi \in \text{Hom\(_K(V, U(g)))},
\]

\(\Gamma^\sigma\) induces an \(\mathcal{A}\)-homomorphism \(\text{Hom\(_K(V, U(g)))) / U(g)\ell_C \rightarrow \text{Hom\(_C(V^M, S_H(a_C))\). Moreover, if (\(\sigma, V\)) is single-petaled, we get a natural \(\mathcal{A}\)-isomorphism

\[
\text{Hom\(_K(V, U(g)) / U(g)\ell_C \rightarrow \text{Hom\(_W(V^M, S_H(a_C))-\}}
\]

The proof of Theorem 1.4 is given in \(\S\)4 with a related result on the quasi-single-petaled \(K\)-types (Theorem 4.11).

If \(g\) is a complex semisimple Lie algebra, then a quasi-spherical \(K\)-type is naturally identified with a finite-dimensional irreducible holomorphic representation of \(G\). Under this identification, a single-petaled \(K\)-type is nothing but an irreducible small representation of \(G\) in the sense of \(\text{[KR]}\) (Corollary \(\S\).2). Moreover, in this case we can deduce from Theorem 1.4 a generalization of the celebrated \textit{Harish-Chandra isomorphism} (Theorem \(\S\).9). In \(\S\)5 we also study two topics related to the generalized Harish-Chandra isomorphism—construction of new kinds of non-commutative determinants, and a natural correspondence between the submodules of the Verma module \(M(\lambda)\) of \(U(g)\) and the submodules of a certain basic module \(A(\lambda)\) of the degenerate affine Hecke algebra \(\hat{H}\) associated to this complex case.

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\section{2. Quasi-spherical \(K\)-types}

We shall prepare some results on quasi-spherical \(K\)-types which will be used in the subsequent sections. Most of the results in this section are known.

Identify the \(K\)-module \(S(p_C)\) with the \(K\)-module \(\mathcal{P}(\mathcal{P})\) of \(C\)-valued polynomial functions on \(\mathcal{P}\) via the Killing form. Each \(X \in p\) defines the partial differential operator \(\partial(X)\) on \(p\). Extend the correspondence \(\partial : X \mapsto \partial(X)\) to the algebra homomorphism from \(S(p_C)\) to the algebra of partial differential operators on \(p\).

We say an element in \(S(p_C)\) is \(K\)-harmonic if it is killed by \(\partial(F)\) for any \(F \in S(p_C)^K \cap S(p_C)p_C\). Let \(\mathcal{H}_K(p)\) denote the set of \(K\)-harmonics. Note that \(\mathcal{H}_K(p)\) is independent of the choice of \(G\) (\(G_{ad} \subset G \subset G_\theta\)). The following is essentially due to \(\text{[KR]}\):

\textbf{Proposition 2.1.} \textit{The map}

\[
S(p_C)^K \otimes \mathcal{H}_K(p) \rightarrow S(p_C)
\]

defined by multiplication is a \(K\)-module isomorphism. Moreover, for any finite-dimensional representation (\(\sigma, V\)) of \(K\) over \(C\), \(\dim_C \text{Hom}_K(V, \mathcal{H}_K(p)) = \dim_C V^M\). Hence

\[
\text{Hom}_K(V, S(p_C)) \simeq \mathcal{A} \otimes \text{Hom}_K(V, \mathcal{H}_K(p)) \simeq \mathcal{A}^{\text{Hom}(\sigma)} \quad \text{with} \quad m(\sigma) = \dim_C V^M.
\]

\textbf{Corollary 2.2.} \textit{As \(K\)-modules,}

\[
U(g) \simeq U(g\ell_C) \oplus \text{symm}(\mathcal{H}_K(p)) \otimes \text{symm}(S(p_C)^K).
\]
Let \((\sigma, V)\) a quasi-spherical \(K\)-type and put \(m(\sigma) = \dim_{\mathbb{C}} V^M\). Let \([v_1, \ldots, v_{m(\sigma)}]\) be a basis of \(V^M\) and \([\Phi_1, \ldots, \Phi_{m(\sigma)}]\) a basis of \(\text{Hom}_K(V, \mathcal{H}_K(p))\). We put \(\Psi_j = \text{symm} \circ \Phi_j \in \text{Hom}_K(V, U(q))\) \((j = 1, \ldots, m(\sigma))\). In [Ko2, Ko3], Kostant studied the \(S(\mathbb{K})\)-valued \(m(\sigma) \times m(\sigma)\)-matrix \(P^\sigma = (\Phi_j \circ \Psi_{[i]} v[i])_{i,j = 1, \ldots, m(\sigma)}\), which is closely related to the theme of the present paper. In particular he determined the value of \(\det P^\sigma\). It is clear that \(\det P^\sigma\), up to a scalar multiple, does not differ for any choice of bases. Let \((\cdot, \cdot)\) be the \(\mathbb{C}\)-bilinear form on \(\mathfrak{a}_c^* \times \mathfrak{a}_c^*\) induced from \(B(\cdot, \cdot)\).

**Proposition 2.3.** Suppose \(\lambda \in \mathfrak{a}_c^*\) satisfies \(\Re(\lambda, \alpha) \geq 0\) for any \(\alpha \in \Sigma^+\). Then \((\det P^\sigma)(\lambda) \neq 0\) for any \((\sigma, V)\).

**Proof.** The proposition is a direct consequence of [Ko2, Ko3] if \(G = G_\theta\). We shall translate this result to the general case. Denote \(K, M\) for \(G_\theta\) by \(K_\theta, M_\theta\). Let \(F\) be the subgroup of the adjoint group of \(G_\theta\) consisting of all elements of \(\exp \mathfrak{a}_c\) with order not greater than 2. Then \(K_\theta = KF, \ M_\theta = MF, \) and \(F\) normalizes \(K\) and \(M\) ([KR Proposition 1, Lemma 20]). Since \(F\) is isomorphic to a direct product of \(\mathbb{Z}/2\mathbb{Z}\), we can choose a subgroup \(F_1\) so that \(K_\theta = K \rtimes F_1\) and \(M_\theta = M \times F_1\). Suppose \((\sigma, V)\) is a quasi-spherical \(K\)-type. Then \(V_\theta := \mathbb{C}[F_1] \otimes V\) has the natural \(K_{\theta}\)-module structure \(\sigma_\theta\) defined by \(\sigma_\theta(k \sigma) (a' \otimes v) = a a' \otimes (a' \sigma(a) v)\) for \(k \in K, a, a' \in F, v \in V\). Observe that \((V_\theta)^M = \bigoplus_{a \in F_1} a \otimes V^M\) and \((V_\theta)^{M_\theta} = (\bigoplus_{a \in F_1} a) \otimes V^M\). Let \((\sigma_\theta, V_\theta) = (\sigma(1), V_1) \oplus \cdots \oplus (\sigma(t), V_t)\) be an irreducible decomposition as a \(K_\theta\)-module such that \((\sigma_1, V_1), \ldots, (\sigma_t, V_t)\) are just all the quasi-spherical components. Since \((\frac{1}{m(\sigma)} \sum_{a \in F_1} a) \otimes V^M = (V_1)^{M_\theta} \oplus \cdots \oplus (V_t)^{M_\theta}\), we have \(m(\sigma) = m(\sigma_1) + \cdots + m(\sigma_t)\). Take a basis \([v_{1,1}^{(1)}, \ldots, v_{m(\sigma_1)}^{(1)}], \ldots, [v_{1,t}^{(t)}, \ldots, v_{m(\sigma_t)}^{(t)}]\) of \(V^M\) so that \(\left(\left(\frac{1}{m(\sigma)} \sum_{a \in F_1} a\right) \otimes v_{1,1}^{(1)} \cdots , \left(\frac{1}{m(\sigma)} \sum_{a \in F_1} a\right) \otimes v_{1,t}^{(t)}\right)\) forms a basis of \((V_{s})^{M_\theta}\) for each \(s = 1, \ldots, t\). On the other hand, since \((\sigma_\theta, V_\theta)\) is naturally considered as the induced \(K_\theta\)-module from \((\sigma, V)\), by defining \(K\)-homomorphisms \(\iota_s : V \rightarrow V_{\theta}\), we get the isomorphism

\[
\bigoplus_{s=1}^t \text{Hom}_{K_\theta}(V_s, \mathcal{H}_{K_\theta}(p)) \ni (\Phi_1^{(s)}, \ldots, \Phi_{m(\sigma_s)}^{(s)}) \mapsto \Phi_1^{(s)} \circ \iota_1 + \cdots + \Phi_{m(\sigma_s)}^{(s)} \circ \iota_t \in \text{Hom}_K(V, \mathcal{H}_K(p)).
\]

Therefore, if we take a basis \([\Phi_1^{(s)}, \ldots, \Phi_{m(\sigma_s)}^{(s)}]\) of \(\text{Hom}_{K_\theta}(V_s, \mathcal{H}_{K_\theta}(p))\) for each \(s = 1, \ldots, t\), then

\[
\left(\Phi_1^{(s)} \circ \iota_1, \ldots, \Phi_{m(\sigma_s)}^{(s)} \circ \iota_1, \ldots, \Phi_1^{(s)} \circ \iota_t, \ldots, \Phi_{m(\sigma_s)}^{(s)} \circ \iota_t\right)
\]

is a basis of \(\text{Hom}_K(V_s, \mathcal{H}_K(p))\). Let us consider \(P^\sigma\) with respect to this basis and the basis of \(V^M\) defined above. Note that \(M_\theta\) acts on \(S(\mathbb{K})\) trivially and that \(\gamma\) is an \(M_\theta\)-homomorphism because \(M_\theta\) normalizes \(\mathfrak{t}_c\) and \(\mathfrak{u}_c\). Hence, \(\gamma(D) = \gamma\left(\frac{1}{m(\sigma)} \sum_{a \in F_1} a\right) D\) for any \(D \in U(q)\). We thereby get for \(s, s' = 1, \ldots, t, i = 1, \ldots, m(\sigma_s),\) and \(j = 1, \ldots, m(\sigma_s),\)

\[
\gamma \circ \text{symm} \circ \Phi_j^{(s)} \circ \iota_k [v_{i, j}^{(s')} \gamma] = \begin{cases} \gamma \circ \text{symm} \circ \Phi_j^{(s)} \left(\frac{1}{m(\sigma)} \sum_{a \in F_1} a\right) v_{i, j}^{(s')} & \text{if } s = s', \\ 0 & \text{if } s \neq s', \end{cases}
\]

which implies the equality

\[
P^\sigma = \begin{pmatrix} P^\tau_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & P^\tau_t \end{pmatrix}.
\]
Now our claim follows from Kostant’s result. \( \square \)

Next, we consider some basic single-petaled \( K \)-types. Clearly the trivial \( K \)-type is single-petaled. Other examples are constituents of \( (\text{Ad}, p_\mathbb{C}) \).

**Lemma 2.4.** For any \( \alpha \in \Sigma \) (not \( \Sigma_1 \)), \( \alpha \in \mathfrak{g}_\alpha \), and \( H \in \mathfrak{a}_{\mathbb{C}} \),

\[
\text{ad}(X_{\alpha} + \theta X_{\alpha})(\text{ad}(X_{\alpha} + \theta X_{\alpha})^2 - 2|\alpha|^2 B(X_{\alpha}, \theta X_{\alpha}))H = 0.
\]

**Proof.** If \( \alpha(H) = 0 \), then \( \text{ad}(X_{\alpha} + \theta X_{\alpha})H = 0 \). On the other hand,

\[
(\text{ad}(X_{\alpha} + \theta X_{\alpha})^2 - 2|\alpha|^2 B(X_{\alpha}, \theta X_{\alpha}))H = 0
\]

\[
= |\alpha|^2 \text{ad}(X_{\alpha} + \theta X_{\alpha})(-X_{\alpha} + \theta X_{\alpha}) - 2|\alpha|^2 B(X_{\alpha}, \theta X_{\alpha})H
\]

\[
= 2|\alpha|^2 ([X_{\alpha}, \theta X_{\alpha}] - B(X_{\alpha}, \theta X_{\alpha})H) = 0. \quad \square
\]

**Definition 2.5.** In this paper we say \( G/K \) is of Hermitian type if and only if \( \mathfrak{p} \) has a \( K \)-invariant complex structure.

**Theorem 2.6.** Suppose \( \mathfrak{g} \) is simple.

(i) Suppose \( G/K \) is of Hermitian type and \( \mathfrak{p} \) has a \( K \)-invariant complex structure \( J \).

Extend \( J \) to the \( \mathbb{C} \)-linear endomorphism on \( p_\mathbb{C} \) and let \( p_\pm \subset p_\mathbb{C} \) be the eigenspaces of \( J \) with eigenvalues \( \pm \sqrt{-1} \). Then \( (p_\mathbb{C})^{M} = \mathfrak{a}_{\mathbb{C}} \oplus J \mathfrak{a}_{\mathbb{C}} \). Moreover, the two \( K \)-types \((\text{Ad}, p_{\pm})\) are single-petaled and \((p_{\pm})^{M} \cong \mathfrak{a}_{\mathbb{C}} \) (the reflection representation) as \( W \)-modules.

(ii) Suppose \( G/K \) is not of Hermitian type. Then the \( K \)-type \((\text{Ad}, p_{\mathbb{C}})\) is single-petaled and \((p_{\mathbb{C}})^{M} \cong \mathfrak{a}_{\mathbb{C}} \) (the reflection representation) as a \( W \)-module.

**Proof.** If \( G = G_{\text{ad}} \), then \((p_{\mathbb{C}})^{M}) \) is as stated in the proposition by [10].

(i) Suppose \( G/K \) is of Hermitian type. Then \( G_{\text{ad}}/(K \cap G_{\text{ad}}) \) is also of Hermitian type. Hence by [10], \((p_{\mathbb{C}})^{M/G_{\text{ad}}} = \mathfrak{a}_{\mathbb{C}} \oplus J \mathfrak{a}_{\mathbb{C}} \). Since \( J \) and the \( K \)-action on \( p_{\mathbb{C}} \) are commutative, \((p_{\mathbb{C}})^{M} = \mathfrak{a}_{\mathbb{C}} \oplus J \mathfrak{a}_{\mathbb{C}} \). The rest is clear from the \( K \)-isomorphism \( \frac{1 + \sqrt{-1}}{2} : p \cong p_{\pm} \) and Lemma 2.4.

(ii) Suppose \( G/K \) is not of Hermitian type. If \( G_{\text{ad}}/(K \cap G_{\text{ad}}) \) is of Hermitian type, then its complex structure \( J \) gives the \((K \cap G_{\text{ad}})\)-module decomposition \( p_{\mathbb{C}} = p_{+} \oplus p_{-}. \) In this case, from (i) and Proposition 2.1, \( \text{dim}_{\mathbb{C}} \text{Hom}_{K/(G_{\text{ad}})}(p_{+}, \mathcal{H}_{K}(p)) = \text{dim}_{\mathbb{C}} \mathfrak{a}_{\mathbb{C}} \). Since the \( K \)-action on \( p_{\mathbb{C}} \) does not commute with \( J \), \( p_{\pm} \) is an irreducible \( K \)-module. Hence we get a natural injection \( \text{Hom}_{K}(p_{\mathbb{C}}, \mathcal{H}_{K}(p)) \rightarrow \text{Hom}_{K/(G_{\text{ad}})}(p_{+}, \mathcal{H}_{K}(p)) \), which implies \( \text{dim}_{\mathbb{C}}(p_{\mathbb{C}})^{M} \leq \text{dim}_{\mathbb{C}} \mathfrak{a}_{\mathbb{C}} \) in view of Proposition 2.1. But since \((p_{\mathbb{C}})^{M} \supset \mathfrak{a}_{\mathbb{C}} \), we have \((p_{\mathbb{C}})^{M} = \mathfrak{a}_{\mathbb{C}} \).

On the other hand, if \( G_{\text{ad}}/(K \cap G_{\text{ad}}) \) is not of Hermitian type, then \((p_{\mathbb{C}})^{M} = \mathfrak{a}_{\mathbb{C}} \) since \((p_{\mathbb{C}})/(M/G_{\text{ad}}) = \mathfrak{a}_{\mathbb{C}} \) by [10]. The rest is clear from Lemma 2.4. \( \square \)

In the remainder of this section, we assume \( \mathfrak{g} \) has real rank 1 and give a close look at its quasi-spherical \( K \)-types. Let \( \sigma \) be the unique element in \( \Sigma \cap \Sigma^{\sigma} \) and choose \( X_{\sigma} \in \mathfrak{g}_{\sigma} \) so that \( B(X_{\sigma}, \theta X_{\sigma}) = -\frac{1}{2|\sigma|}. \) If we put \( Z = \sqrt{-1}X_{\sigma} + \sqrt{-1}\theta X_{\sigma} \), then the condition \( |\sigma| \) for a quasi-spherical \((\sigma, V)\) to be single-petaled is reduced to \( Z(Z^2 - 1)V = 0 \).

**Lemma 2.7.** Suppose \((\sigma, V)\) is a quasi-spherical \( K \)-type.

(i) All the eigenvalues of \( \sigma(Z) \) are integers. We denote the largest one by \( e(\sigma) \).

(ii) \( \text{dim}_{\mathbb{C}} V^{\mathbb{C}} = 1. \) Hence \( \sigma^r \in V^{\mathbb{C}} \setminus \{0\} \) and \( \Phi^{\sigma^r} \in \text{Hom}_{K}(V, \mathcal{H}_{K}(p)) \setminus \{0\} \) are uniquely determined up to scalar multiples.
(iii) Let $V_{e^r}$ be the eigenspace of $e^r(Z)$ with eigenvalue $e^r$. Let $(V^M)_b$ denote the orthogonal complement of $V^M$ in $V$ with respect to some $K$-invariant Hermitian inner product $(\cdot, \cdot)_V$ on $V$. Then $V_{e^r} \not\subset (V^M)_b$.

(iv) Put $\delta = \dim \mathfrak{g}_{\mathfrak{a}}$ and $h = \frac{a^2}{2} + \dim \mathfrak{a}_{\mathfrak{c}} \in S(\mathfrak{a}_{\mathfrak{c}})$ (recall $a^2 = \frac{2H}{\mathfrak{m}}$). Then we can choose a pair $(i, j)$ of non-negative integers with $2i + j = |e^r|$ so that $\gamma \circ \text{symm} \circ \Phi^r[e^r]$ equals

\begin{equation}
\frac{\delta(h + \delta + 2) \cdots (h + \delta + 2(i + j) - 2)}{(h + 1)(h + 3) \cdots (h + 2i - 1)}
\end{equation}

up to a scalar multiple.

(v) $(\sigma, V)$ is the trivial $K$-type $\iff e(\sigma) = 0$. $(\sigma, V)$ is a constituent of $(\text{Ad}, \mathfrak{p}_{\mathfrak{c}})$ $\iff |e(\sigma)| = 1$.

Proof. Use the same notation as in the proof of Proposition 2.3. We may assume $\mathfrak{g}$ is simple. If $G = G_0$, then all assertions of the lemma are consequences of [Ko2, Ko3].

Suppose $G \neq G_0$. Firstly, we consider the case where $\mathfrak{g} \neq \mathfrak{sl}(2, \mathbb{R})$. In this case each quasi-spherical $K_0$-type $(\sigma, V)$ is irreducible as a $K$-module and $V^M = V^M$ ([Ko3 Chapter II, §2]). Also, because a quasi-spherical $K$-type is a constituent of $S(\mathfrak{p}_{\mathfrak{c}})$, it must be the restriction of some quasi-spherical $K_0$-type. Hence the lemma follows from the case for $G_0$.

Secondly, suppose $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{t} = \mathfrak{so}(2, \mathbb{R})$. Then $G = G_{\text{ad}} = \text{Ad}(\text{SL}(2, \mathbb{R}))$, $\mathfrak{f}_1 = \{1, a\}$ with $a = \text{Ad} \left( \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} \right)$, and $Z = \pm \left( \begin{array}{cc} 0 & -\frac{1}{2} \sqrt{-1} \\ \frac{1}{2} \sqrt{-1} & 0 \end{array} \right)$. For each integer $e$, define the 1-dimensional quasi-spherical $K$-type $(\sigma_e, V_e)$ by $\sigma_e(Z) = e$. Then a quasi-spherical $K$-type equals some $(\sigma_e, V_e)$. For each $(\sigma_e, V_e)$, define the $K_0$-module $(V_e)_0 = 1 \otimes V_e + a \otimes V_e$ as in the proof of Proposition 2.3. Then $a \otimes V_e \approx V_{-e}$ as a $K$-module. If $e \neq 0$, then $(V_e)_0$ is an irreducible $K_0$-module and hence $P_{\sigma_e} = P_{(\sigma_e)_0}$ by (2.2), which assures \textbf{(iii)} for $(\sigma_e, V_e)$. If $e = 0$, then $\sigma_e$ is trivial and clearly \textbf{(i)}--\textbf{(iv)} hold. It is also clear that \textbf{(v)} follows from the case for $G_0$. \hfill $\square$

Combining Lemma 2.7, \textbf{(iii)}, \textbf{(v)} and Theorem 2.6, we can conclude

Corollary 2.8. If $\mathfrak{g}$ has real rank 1, then the trivial $K$-type and the $K$-types appearing in $(\text{Ad}, \mathfrak{p}_{\mathfrak{c}})$ exhaust all the single-petaled $K$-types.

3. The Chevalley restriction theorem

The purpose of this section is to prove Theorem 1.3. Although the method is modeled on that of [Da], in large part, some points are improved by use of the rational Dunkl operators. We note our method is applicable even to the classical case.

Under the setting of §1, suppose $(\sigma, V)$ is a quasi-spherical $K$-type. Let $\mathfrak{F}$ represent one of the following $\mathbb{C}$-valued function classes: $\mathcal{C}$ (continuous functions), $\mathcal{C}^{\infty}$ (smooth functions), or $\mathcal{P}$ (polynomial functions). Define the map

\begin{equation}
\text{Hom}_K(V, \mathfrak{F}(\mathfrak{p})) \ni \Phi \mapsto (\varphi : V^M \ni v \mapsto \Phi[\varphi]) \in \text{Hom}_W(V^M, \mathfrak{F}(\mathfrak{a}))
\end{equation}

We consider $\mathfrak{p}$ and $\mathfrak{a}$ as Euclidean spaces by the Killing form. Under the natural identifications $\mathfrak{P}(\mathfrak{p}) \simeq S(\mathfrak{p}_{\mathfrak{c}})$ and $\mathfrak{P}(\mathfrak{a}) \simeq S(\mathfrak{a}_{\mathfrak{c}})$, $\Gamma_0^r$ in §1 coincides with (3.1) for $\mathfrak{F} = \mathfrak{F}_0$. Hence we use the same symbol $\Gamma_0^r$ for (3.1) in general cases. First we shall prove
Proposition 3.1. The map $\Gamma_0$ for $Z = \mathcal{C}, \mathcal{C}^\infty$ or $\mathcal{P}$ is injective.

Proof. We may assume $Z = \mathcal{C}$. Let
\[
V = V^M \oplus \sum_{\tau \neq \text{triv}} V_{\tau}
\]
be the decomposition into isotypic components of the $M$-module $V = V|M$ and define the projection map
\[
p^\tau : V = V^M \oplus \sum_{\tau \neq \text{triv}} V_{\tau} \to V^M.
\]
Suppose $\Phi \in \text{Hom}_K(V, \mathcal{C}(v))$. Then clearly $\Phi[v](H) = \Phi[p^\tau(v)](H)$ for any $v \in V$ and $H \in a$. Let $\varphi \in \text{Hom}_K(V^M, \mathcal{C}(a))$ be the image of $\Phi$. Since each element $X \in \mathfrak{p}$ can be written as $X = \text{Ad}(k)H$ for some $k \in K$ and $H \in a$, we have for any $v \in V$
\[
\Phi[v](X) = \Phi[v](\text{Ad}(k)H) = \Phi[\sigma(k^{-1})v](H)
\]
(3.4)
\[
\quad = \Phi[p^\tau(\sigma(k^{-1})v)](H) = \varphi(p^\tau(\sigma(k^{-1})v))(H).
\]
Thus $\Phi$ can be completely reproduced by $\varphi$. \hfill \Box

To discuss the image of $\Gamma_0$ we introduce two $W$-subspaces of $V^M$.

**Definition 3.2.** Put $V^M_{\text{single}} = \{ v \in V^M; \sigma(X_\alpha + \theta X_\alpha)(\sigma(X_\alpha + \theta X_\alpha)^2 - 2|\alpha|^2 B(X_\alpha, \theta X_\alpha))v = 0 \}$
\[
\quad \forall \alpha \in \Sigma_1, \forall X_\alpha \in \mathfrak{g}_\alpha \},
\]

\[
V^M_{\text{double}} = V^M \cap \sum_{\alpha \in \Sigma_1, X_\alpha \in \mathfrak{g}_\alpha} \{ \sigma(X_\alpha + \theta X_\alpha)(\sigma(X_\alpha + \theta X_\alpha)^2 - 2|\alpha|^2 B(X_\alpha, \theta X_\alpha))V;
\]
\[
\quad \alpha \in \Sigma_1, X_\alpha \in \mathfrak{g}_\alpha \}.
\]

**Lemma 3.3.** $V^M = V^M_{\text{single}} \oplus V^M_{\text{double}}$.

Proof. Let $(\cdot, \cdot)_V$ be a $K$-invariant Hermitian inner product on $V$. Then the isotypic components $V^M$ and $V_{\tau}$ in (3.2) are orthogonal to one another. Let $(V^M_{\text{double}})^\perp$ be the orthogonal complement of $V^M_{\text{double}}$ in $V^M$. Since $\sigma(X_\alpha + \theta X_\alpha)$ is skew-Hermitian with respect to $(\cdot, \cdot)_V$, we easily get $V^M_{\text{single}} \subseteq (V^M_{\text{double}})^\perp$. Conversely, suppose $v \in (V^M_{\text{double}})^\perp$. Since $V^M_{\text{double}}$ is the image of the $M$-module
\[
\sum_{\alpha \in \Sigma_1, X_\alpha \in \mathfrak{g}_\alpha} \{ \sigma(X_\alpha + \theta X_\alpha)(\sigma(X_\alpha + \theta X_\alpha)^2 - 2|\alpha|^2 B(X_\alpha, \theta X_\alpha))v; \alpha \in \Sigma_1, X_\alpha \in \mathfrak{g}_\alpha \}
\]
under the projection map (3.3), we have for any $v' \in V$, $\alpha \in \Sigma_1$, and $X_\alpha \in \mathfrak{g}_\alpha$,
\[
(V^M_{\text{double}})^\perp = \langle \sigma(X_\alpha + \theta X_\alpha)(\sigma(X_\alpha + \theta X_\alpha)^2 - 2|\alpha|^2 B(X_\alpha, \theta X_\alpha))v, v' \rangle_V
\]
\[
= -\langle v, \sigma(X_\alpha + \theta X_\alpha)(\sigma(X_\alpha + \theta X_\alpha)^2 - 2|\alpha|^2 B(X_\alpha, \theta X_\alpha))v' \rangle_V
\]
\[
= -\langle v, p^\tau(\sigma(X_\alpha + \theta X_\alpha)(\sigma(X_\alpha + \theta X_\alpha)^2 - 2|\alpha|^2 B(X_\alpha, \theta X_\alpha))v') \rangle_V
\]
\[
= 0.
\]
It shows $v \in V^M_{\text{single}}$. Thus we get $V^M_{\text{single}} = (V^M_{\text{double}})^\perp$. \hfill \Box

From Remark 1.2 $(\sigma, V)$ is single-petaled if and only if $V^M_{\text{double}} = 0$. 
Lemma 3.4. For any $\alpha \in \Sigma$ (not $\Sigma_1$), $v \in V^M_{\text{single}}$, and $X_{a} \in \mathfrak{g}_{\alpha}$,
\[ \sigma(X_{a} + \theta X_{a})(\sigma(X_{a} + \theta X_{a})^2 - 2|\alpha|^2 B(X_{a}, \theta X_{a}))v = 0. \]

Proof. It suffices to show the equality for a root $2\alpha$ with $\alpha \in \Sigma_1$. Put $g(\alpha) = m + \mathbb{R}H_{\alpha} + \sum_{\beta \in \Sigma_2 \cap 2\alpha} \mathbb{R}g_{\beta}$ and $\mathfrak{g}_{\alpha}(\alpha) = \{g(\alpha), g(\alpha)\}$. Then $\mathfrak{g}_{\alpha}(\alpha)$ is a semisimple Lie algebra with real rank 1. Let $G_{\alpha}(\alpha) \subset G$ be the analytic subgroup of $\mathfrak{g}_{\alpha}(\alpha)$ and put $t_{\alpha}(\alpha) = 1 \cap \mathfrak{g}_{\alpha}(\alpha)$, $K_{\alpha}(\alpha) = K \cap G_{\alpha}(\alpha)$, and $M_{\alpha}(\alpha) = Z_{G_{\alpha}(\alpha)}(\mathbb{R}H_{\alpha})$. Let $U(t_{\alpha}(\alpha)_{\mathbb{C}})v = V^{(1)} \oplus \cdots \oplus V^{(t)}$ be an irreducible decomposition as a $G_{\alpha}(\alpha)$-module and $v = v^{(1)} + \cdots + v^{(t)}$ the corresponding decomposition. Since $M \cap G_{\alpha}(\alpha) = M_{\alpha}(\alpha)$, $v^{(s)} (s = 1, \ldots, t)$ is a non-zero $M_{\alpha}(\alpha)$-fixed vector of $V^{(s)}$. Moreover, since $M_{\alpha}(\alpha)$ contains the center of $G_{\alpha}(\alpha)$, we can essentially regard each $V^{(s)}$ as a ‘$K$-type’ of the adjoint group of $G_{\alpha}(\alpha)$ and apply the results of [2] to it. Choose $X_{a} \in \mathfrak{g}_{\alpha}$ so that $B(X_{a}, \theta X_{a}) = -\frac{1}{\dim}$ and put $Z = \sqrt{-1}X_{a} + \sqrt{-1}\theta X_{a}$. Since $v \in V^M_{\text{single}}, Z(Z^2 - 1)v = 0$. Then $Z(Z^2 - 1)v^{(s)} = 0$ for $s = 1, \ldots, t$ because $Z \in t_{\alpha}(\alpha)_{\mathbb{C}}$. Hence by Lemma 2.7 (ii) each $V^{(s)}$ is single-petalled as a ‘$K$-type’ of the adjoint group of $G_{\alpha}(\alpha)$. Now it follows from Corollary 2.8 that each $V^{(s)}$ is either the trivial $K_{\alpha}(\alpha)$-type or a $K_{\alpha}(\alpha)$-type appearing in $(\text{Ad}, \mathfrak{g}_{\alpha}(\alpha)_{\mathbb{C}})$. Therefore Theorem 2.6 and Lemma 2.4 imply
\[ \sigma(X_{a} + \theta X_{a})(\sigma(X_{a} + \theta X_{a})^2 - 2|\alpha|^2 B(X_{a}, \theta X_{a}))v^{(s)} = 0 \quad \text{for} \quad s = 1, \ldots, t. \]
Thus we get the lemma. \qed

For any $W$-submodule $V'$ of $V^M$, we naturally identify $\text{Hom}_W(V^M/V', \mathcal{F}(\alpha))$ with the linear space $\{\varphi \in \text{Hom}_W(V^M, \mathcal{F}(\alpha)) \mid \varphi[v] = 0 \ \forall v \in V'\}$. Hereafter in this paper, we repeatedly use similar identifications without notice. The second assertion of Theorem 1.3 can be made more precise as follows:

Theorem 3.5. Suppose $\mathcal{F} = \mathcal{C}, \mathcal{C}^{\infty}$, or $\mathcal{P}$. For any $\varphi \in \text{Hom}_W(V^M/\text{V}_{\text{double}}, \mathcal{F}(\alpha))$ there exists a unique $\Phi \in \text{Hom}_W(V, \mathcal{F}(\alpha))$ such that $\Gamma^\mathcal{F}_W(\Phi) = \varphi$.

The proof is a bit long and a large part of this section is devoted to it. Retain the notation in the proof of Proposition 3.1. We first show the theorem for $\mathcal{F} = \mathcal{C}$. Suppose $\varphi \in \text{Hom}_W(V^M/\text{V}_{\text{double}}, \mathcal{C}(\alpha))$. For each $v \in V$ we define $\Phi_v \in \mathcal{C}(K \times a)$ by
\[ (3.5) \quad \Phi_v(k, H) = \varphi[ p^\alpha(\sigma(k^{-1})v)](H) \quad \text{for} \quad (k, H) \in K \times a. \]

Lemma 3.6. Suppose $k_1, k_2 \in K$ and $H_1, H_2 \in a$ satisfy $\text{Ad}(k_1)H_1 = \text{Ad}(k_2)H_2$. Then $\Phi_v(k_1, H_1) = \Phi_v(k_2, H_2)$ for any $v \in V$.

Proof. Note that $H_1$ and $H_2$ in the lemma are conjugate by some element of $N_K(\alpha)$ ([He1 Chapter VII, Proposition 2.2]). By the definition (3.5) we see for any $v \in V, k_1 \in K, \tilde{w} \in N_K(\alpha) \text{ and } H \in a$, the following equalities hold:
\[ \Phi_v(k^{-1}_1 H, k_H) = \Phi_v(k_1, H), \]
\[ \Phi_v(k \tilde{w}, H) = \Phi_v(k, wH) \quad \text{with} \quad w \equiv \tilde{w} \mod M \in W. \]
Therefore, if we show
\[ (3.6) \quad \Phi_v(k, H) = \Phi_v(e, H) \quad \text{for} \quad H \in a, k \in K^H, \text{ and } v \in V, \]
\[ \Phi_v(k, H) = \Phi_v(e, H) \quad \text{for} \quad H \in a, k \in K^H, \text{ and } v \in V, \]
our claim follows from it. Here $e$ and $K^H$ in (3.6) are a unit element and the stabilizer of $H$ in $K$, respectively. To show (3.6), fix an arbitrary $H \in a$ and define $\lambda_H \in V^*$ by

$$\lambda_H : V \ni v \mapsto \Phi_{\sigma}(e, H).$$

Let $(\sigma^*, V^*)$ be the dual $K$-type of $(\sigma, V)$ and $(\cdot, \cdot)$ the canonical bilinear form on $V^* \times V$. For $\bar{\omega} \in N_K(a) \cap K^H$ and $v \in V$,

$$\sigma^*(\bar{\omega}) \lambda_H, v) = \langle \lambda_H, \sigma(\bar{\omega})^{-1}v \rangle = \Phi_{\sigma(\bar{\omega})^{-1}}(e, H)$$

and therefore $\lambda_H \in (V^*)^M$. Furthermore, for $\alpha \in \Sigma_1$, $X_\alpha \in \mathfrak{g}_a$, and $v \in V$,

$$\sigma^*(X_\alpha + \theta X_\alpha)(\sigma^*(X_\alpha + \theta X_\alpha)^2 - 2|\alpha|^2 B(X_\alpha, \theta X_\alpha)) \lambda_H, v)$$

$$= -\sigma(\sigma(X_\alpha + \theta X_\alpha)(\sigma(X_\alpha + \theta X_\alpha)^2 - 2|\alpha|^2 B(X_\alpha, \theta X_\alpha))\lambda_H, v)$$

$$= -\Phi_{\sigma(X_\alpha + \theta X_\alpha)}(\sigma(X_\alpha + \theta X_\alpha)^2 - 2|\alpha|^2 B(X_\alpha, \theta X_\alpha))(e, H)$$

$$= -\varphi[\rho^*(\sigma(X_\alpha + \theta X_\alpha)(\sigma(X_\alpha + \theta X_\alpha)^2 - 2|\alpha|^2 B(X_\alpha, \theta X_\alpha)))](H)$$

$$= 0$$

since $\rho^*(\sigma(X_\alpha + \theta X_\alpha)(\sigma(X_\alpha + \theta X_\alpha)^2 - 2|\alpha|^2 B(X_\alpha, \theta X_\alpha))) \in V^M_{\text{double}}$. Thus $\lambda_H \in (V^*)^M_{\text{single}}$.

Put $\Sigma^H = \{ \sigma \in \Sigma; \sigma(H) = 0 \}$ and take an arbitrary $\alpha \in \Sigma^H \cap \Sigma_1$ and $X_\alpha \in \mathfrak{g}_a$. We shall prove $\sigma^*(X_\alpha + \theta X_\alpha) \lambda_H = 0$. We may assume $B(X_\alpha, \theta X_\alpha) = -\frac{1}{2\pi i}$. Put $Z = \sqrt{-1}X_\alpha$ and $\tilde{s}_a = \exp(\pi \sqrt{-1}Z)$. Then $\tilde{s}_a \in N_K(a) \cap K^H$ and hence $\sigma^*(\tilde{s}_a) \lambda_H = \lambda_H$. Let

$$\lambda_H = \lambda_H^{(0)} + \lambda_H^{(+)} + \lambda_H^{(-)}$$

be the decomposition into $\sigma^*(Z)$-eigenvectors with eigenvalues 0, 1, and $-1$. Then we have

$$\sigma^*(\tilde{s}_a) \lambda_H = \lambda_H^{(0)} + e^{\sqrt{-1}} \lambda_H^{(+)} + e^{-\sqrt{-1}} \lambda_H^{(-)} = \lambda_H^{(0)} + (\lambda_H^{(+)} + \lambda_H^{(-)}),$$

which shows $\lambda_H = \lambda_H^{(0)}$ and hence $\sigma^*(Z) \lambda_H = 0$.

Note that $t^H := \{ \alpha \in \Sigma^H; X_\alpha \in \mathfrak{g}_a \}$ is the Lie algebra corresponding to $K^H$ and is generated by $\mathfrak{m}$ and $X_\alpha + \theta X_\alpha (\alpha \in \Sigma^H \cap \Sigma_1, X_\alpha \in \mathfrak{g}_a)$. Hence we get $\sigma^*(X) \lambda_H = 0$ for any $X \in t^H$. If we define the analytic subgroup $(K^H)_0$ with Lie algebra $t^H$, a usual argument leads us to

$$K^H = (N_K(a) \cap K^H) \cdot (K^H)_0.$$ 

It shows the $K^H$-invariance of $\lambda_H$ and therefore (3.6).

**Lemma 3.7.** The natural topology of $\mathfrak{p}$ coincides with the quotient topology of the surjective map $q : \mathfrak{k} \times a \rightarrow \mathfrak{p}$ defined by $q(k, H) = \text{Ad}(k)H$.

**Proof.** Notice that $B(\cdot, \cdot)$ is $K$-invariant. Hence if we put for any positive number $R$

$$a_R = \{ H \in a; B(H, H) \leq R \}, \quad p_R = \{ X \in \mathfrak{p}; B(X, X) \leq R \},$$

then $q(S \cap (K \times a_R)) = q(S) \cap \mathfrak{p}_R$ for any closed subset $S \subset K \times a$. Here $S \cap (K \times a_R)$ is compact and so is $q(S) \cap \mathfrak{p}_R$ with respect to the natural topology of $\mathfrak{p}$. It implies that $q(S)$ is closed by the natural topology and hence the lemma. \qed
From Lemma 3.6 and Lemma 3.7, \( \Phi_k \) induces the continuous function \( \Phi[v] \) on \( \mathfrak{p} \) for each \( v \in V \). Clearly the correspondence \( \Phi : v \mapsto \Phi[v] \) commutes with the \( K \)-actions and satisfies the relation (3.8). Therefore \( \Phi \) is a unique element of \( \text{Hom}_K(V, \mathcal{C}(\mathfrak{p})) \) such that \( \Gamma_0^\sigma(\Phi) = \varphi \).

To show Theorem 3.9 for \( \mathcal{F} = \mathcal{C}^\infty \) we need some preparation.

**Definition 3.8.** Let \( k : \Sigma \to \mathbb{C} \) be a multiplicity function, that is, a function which takes the same value on each \( W \)-orbit of \( \Sigma \). For \( \xi \in \mathfrak{a} \) we define the operator \( \mathcal{K}_k(\xi) \) acting on \( f \in \mathcal{C}^\infty(\mathfrak{a}) \) or \( \mathcal{D}(\mathfrak{a}) \) (infinite differentiable functions with compact support) by

\[
(3.7) \quad \mathcal{K}_k(\xi)f(H) = \partial(\mathcal{K}_k(\xi)f(H)) + \sum_{\alpha \in \Sigma^+} k(\alpha) \alpha(\mathcal{K}_k(\xi)f(H)) \frac{f(H) - f(s_\alpha H)}{\alpha(H)},
\]

where \( \partial(\mathcal{K}_k(\xi)) \) is the \( \xi \)-directional derivative and \( s_\alpha \in W \) is the reflection with respect to \( \alpha \).

**Remark 3.9.** The result of (3.8) belongs to the original function class and it holds that \( w\mathcal{K}_k(\xi) = \mathcal{K}_k(\xi)w \) for any \( w \in W \). The operator \( \mathcal{K}_k(\xi) \) is introduced in [Dun1] and is called the rational Dunkl operator. Is is known that \( \mathcal{K}_k(\xi) \mathcal{K}_k(\eta) = \mathcal{K}_k(\eta) \mathcal{K}_k(\xi) \) for any \( \xi, \eta \in \mathfrak{a} \). In this section we consider only one special case where \( k(\alpha) = \frac{\dim \mathfrak{a}}{2 \alpha} \). Hence hereafter we drop the suffix \( k \) in \( \mathcal{K}_k \).

**Lemma 3.10.** Let \( L_\sigma \) be the flat Euclidean Laplacian on \( v \). Let \( \xi_1, \ldots, \xi_l \) be an orthonormal basis of \( \mathfrak{a} \) and put \( \mathcal{L}_\sigma = \sum_{i=1}^l \mathcal{K}(\xi_i)^2 \). Suppose \( \Phi \in \text{Hom}_K(V, \mathcal{C}^\infty(\mathfrak{p})) \) and \( v \in \mathcal{V}_\text{single} \). Then

\[
(L_\sigma \Phi[v])|_{\mathfrak{g}_0} = \mathcal{L}_\sigma(\Phi[v])|_{\mathfrak{g}_0}.
\]

**Proof.** Note that for \( X \in \mathfrak{p} \) and \( Y \in \mathfrak{t} \)

\[
(3.8) \quad \Phi[\sigma(Y)v](X) = \frac{d}{dt} \Phi[\sigma(\exp ty)v](X) \bigg|_{t=0} = \frac{d}{dt} \Phi[v](\text{Ad}(\exp -ty)X) \bigg|_{t=0} = \partial([X, Y])\Phi[v](X).
\]

Hence for \( H \in \mathfrak{a}, \alpha \in \Sigma, \) and \( X_\alpha \in \mathfrak{g}_\alpha \) we have

\[
(3.9) \quad \Phi[\sigma(X_\alpha + \theta X_\alpha)^2v](H) = \partial([H, X_\alpha + \theta X_\alpha], \mathfrak{a})\Phi[\sigma(X_\alpha + \theta X_\alpha)v](H)
\]

\[
= \alpha(H) \partial(X_\alpha - \theta X_\alpha)\Phi[\sigma(X_\alpha + \theta X_\alpha)v](H)
\]

\[
= \alpha(H) \frac{d}{dt} \Phi[\sigma(X_\alpha + \theta X_\alpha)v](H + t(X_\alpha - \theta X_\alpha)) \bigg|_{t=0}
\]

\[
= \alpha(H) \frac{d}{dt} \partial((H + t(X_\alpha - \theta X_\alpha), X_\alpha + \theta X_\alpha))\Phi[v](H + t(X_\alpha - \theta X_\alpha)) \bigg|_{t=0}
\]

\[
= \alpha(H)^2 \frac{d}{dt} \partial(X_\alpha - \theta X_\alpha)\Phi[v](H + t(X_\alpha - \theta X_\alpha)) \bigg|_{t=0}
\]

\[
+ \alpha(H) \frac{d}{dt} 2\partial([X_\alpha, \theta X_\alpha])\Phi[v](H + t(X_\alpha - \theta X_\alpha)) \bigg|_{t=0}
\]

\[
\]
Lemma 3.11. There exists a positive constant $C$ function $F$.

Lemma 3.12. For any $\phi$.

Proof. Recall the decomposition

\begin{equation}
(3.10)
\end{equation}

From (3.9) and (3.10) we get

\[
\frac{\partial X_0 - \partial X_0^2}{-2B(X_0, \partial X_0)} \Phi[v](H) = \frac{1}{\alpha(H) \Phi[v](H)} - \frac{1}{2\alpha(H)^2}.
\]

Therefore

\[
L_v \Phi[v](H) = \sum_{i=1}^\ell \frac{\partial (\xi_i^2 \Phi[v](H))}{\alpha_i^2} + \sum_{\alpha \in \Sigma} \dim \omega_{\alpha} \left( \frac{2}{\alpha(H) \Phi[v](H)} - \frac{1}{2\alpha(H)^2} \right).
\]

which equals $\sum_{i=1}^\ell \mathcal{T}(\xi_i^2 \Phi[v](H))$ by [Dun1] Theorem 1.10. \hfill \square

Let $dX$ (resp. $dH$) be the canonical measure of the Euclidean space $v$ (resp. $\alpha$).

Lemma 3.11. There exists a positive constant $C_a$ such that for any $K$-invariant continuous function $F(X)$ on $v$ with compact support

\[
\int_v F(X) \ dX = C_a \int_a F(H) \prod_{\alpha \in \Sigma} |\alpha(H)|^{\dim \omega_{\alpha}} \ dH.
\]

Proof. See [He2] Chapter I, Theorem 5.17. \hfill \square

Lemma 3.12. For any $\phi \in C^\infty(\alpha), f \in \mathcal{D}(\alpha),$ and $\xi \in \mathcal{A},$

\[
\int_a (\mathcal{T}(\xi) \phi)(H) f(H) \prod_{\alpha \in \Sigma} |\alpha(H)|^{\dim \omega_{\alpha}} dH = -\int_a \phi(H) (\mathcal{T}(\xi) f)(H) \prod_{\alpha \in \Sigma} |\alpha(H)|^{\dim \omega_{\alpha}} dH.
\]

Proof. By a straightforward calculation (cf. [Dun2] Lemma 2.9). \hfill \square

Lemma 3.13. Recall the decomposition (3.2). Suppose $\{v_1, \ldots, v_n\}$ is a basis of $V$ such that $\{v_1, \ldots, v_m\}$ and $\{v_{m+1}, \ldots, v_n\}$ are bases of $V^{\text{single}}$, $V^{\text{double}}$ and $\Sigma_{\text{triv}} V^\tau$, respectively. Let $\{v_1^\tau, \ldots, v_m^\tau\}$ be the dual basis of $\{v_1, \ldots, v_m\}$. Then, $\{v_1^\tau, \ldots, v_m^\tau\}$ and $\{v_{m+1}^\tau, \ldots, v_n^\tau\}$ are bases of $(V^\tau)^{\text{single}}$, $(V^\tau)^{\text{double}}$ and $\Sigma_{\text{triv}} (V^\tau)^\tau$, respectively.

Proof. Let $(\cdot, \cdot)$ be the canonical bilinear form on $V^\tau \times V$. Then $((V^\tau)^\tau, V^\tau) = 0$ unless $\tau = \mu^\tau$. Hence the lemma follows if we show

\[
V^{\text{single}} = \{v \in V^\tau; (v^\tau, v) = 0 \ \forall v^\tau \in (V^\tau)^{\text{double}}\},
\]

\[
(V^\tau)^{\text{single}} = \{v^\tau \in (V^\tau)^\tau; (v^\tau, v) = 0 \ \forall v \in V^{\text{double}}\}.
\]

But this argument is quite similar to the proof of Lemma 3.8 and we omit it. \hfill \square

Now suppose $\phi \in \text{Hom}_w(V^\tau, V^{\text{double}} \circ \mathcal{C}^\infty(\alpha))$. Let $\phi^\tau$ stand for the unique element of $\text{Hom}_w(V, \mathcal{C}^\infty)$ such that $\Gamma^{c^*}(\phi^\tau) = \phi$. It follows from Remark 3.9 that the map

\begin{equation}
(3.11)
\end{equation}

belongs to $\text{Hom}_w(V^\tau, V^{\text{double}} \circ \mathcal{C}^\infty)$. Denote the map (3.11) by $\mathcal{L}_w \phi$ and let us show

\begin{equation}
(3.12)
\end{equation}

\[
L_v \phi^\tau [v] = (\mathcal{L}_w \phi)^\tau [v] \quad \forall v \in V.
\]
In the left-hand side we consider \( \varphi^{-}[v] \) an element in \( \mathcal{D}'(\mathfrak{g}) \) (the space of distributions) on which \( L_a \) is acting. On the other hand we know the right-hand side is a continuous function. Hence by successive use of Lemma 3.12 and Weyl’s lemma on regularity, we can conclude \( \varphi^{-}[v] \in \mathcal{E}'(\mathfrak{g}) \).

Let \( \{v_1, \ldots, v_n\} \) be a basis of \( V \) as in Lemma 3.13 and \( \{v_1^*, \ldots, v_n^*\} \) its dual basis. It suffices to show that for any \( n \) test functions \( F_1, \ldots, F_n \in \mathcal{D}(\mathfrak{g}) \),

\[
\sum_{i=1}^{n} \int_{\mathfrak{g}} \varphi^{-}[v_i] (L_a F_i) dX = \sum_{i=1}^{n} \int_{\mathfrak{g}} (\mathcal{L}_a \varphi)^{-}[v_i] F_i dX.
\]

To do this, using the linear map \( F : V^* \to \mathcal{D}(\mathfrak{g}) \) defined by \( v_i^* \mapsto F_i \) (\( i = 1, \ldots, n \)), put

\[
\mathcal{F} : V^* \ni \varphi^* \mapsto \int_{K} F[\sigma^*(k)](\text{Ad}(k)X) dk \in \mathcal{D}(\mathfrak{g}),
\]

where \( dk \) is a normalized measure on \( K \). Then \( \mathcal{F} \in \text{Hom}_K(V^*, \mathcal{E}'(\mathfrak{g})) \) and the left-hand side of (3.13) equals

\[
\int_{K} \sum_{i=1}^{n} \int_{\mathfrak{g}} \varphi^{-}[v_i] (L_a F[\sigma^*(k)](v_i^*)) (X) dX dk
\]

\[
= \sum_{i=1}^{n} \int_{K} \int_{\mathfrak{g}} \varphi^{-}[v_i] ((\text{Ad}(k)^{-1})X) (L_a F[\sigma^*(k)](v_i^*)) (X) dX dk
\]

\[
= \sum_{i=1}^{n} \int_{K} \int_{\mathfrak{g}} \varphi^{-}[v_i] (X) \int_{K} (L_a F[\sigma^*(k)](v_i^*)) (\text{Ad}(k)X) dk dX
\]

\[
= \int_{\mathfrak{g}} \sum_{i=1}^{n} \int_{K} \varphi^{-}[v_i] (X) (L_a \mathcal{F}[v_i^*])(X) dX
\]

\[
= C_a \int_{\mathfrak{g}} \sum_{i=1}^{n} \varphi^{-}[v_i] ((L_a \mathcal{F}[v_i^*]) (H) \prod_{\alpha \in \Sigma^*} |\alpha(H)|^{\dim_{\mathbb{A}_\alpha}} dH
\]

\[
= C_a \int_{\mathfrak{g}} \sum_{i=1}^{m'} \varphi[v_i] (H) (L_a \mathcal{F}[v_i^*]) (H) \prod_{\alpha \in \Sigma^*} |\alpha(H)|^{\dim_{\mathbb{A}_\alpha}} dH.
\]

Here the fourth equality comes from the \( K \)-invariance of the integrand and Lemma 3.11. The last equality is based on the fact that \( \varphi^{-}[v_i] \big|_{\mathfrak{a}} = 0 \) for \( i = m' + 1, \ldots, n \) (see the proof of Proposition 3.1). Similarly the right-hand side of (3.13) is changed into the form

\[
C_a \int_{\mathfrak{g}} \sum_{i=1}^{m'} (\mathcal{L}_a \varphi)[v_i] (H) \mathcal{F}[v_i^*] (H) \prod_{\alpha \in \Sigma^*} |\alpha(H)|^{\dim_{\mathbb{A}_\alpha}} dH,
\]

which equals the final form of (3.14) because from Lemma 3.12 and Lemma 3.10 we have for \( i = 1, \ldots, m' \)

\[
\int_{\mathfrak{g}} (\mathcal{L}_a \varphi)[v_i] (H) \mathcal{F}[v_i^*] (H) \prod_{\alpha \in \Sigma^*} |\alpha(H)|^{\dim_{\mathbb{A}_\alpha}} dH = \int_{\mathfrak{g}} \varphi[v_i] (H) \mathcal{L}_a(\mathcal{F}[v_i^*]) (H) \prod_{\alpha \in \Sigma^*} |\alpha(H)|^{\dim_{\mathbb{A}_\alpha}} dH
\]

\[
= \int_{\mathfrak{g}} \varphi[v_i] (H) (L_a \mathcal{F}[v_i^*]) (H) \prod_{\alpha \in \Sigma^*} |\alpha(H)|^{\dim_{\mathbb{A}_\alpha}} dH.
\]

Thus we get Theorem 3.5 for \( \mathcal{F} = \mathcal{E}' \).
Finally, to show Theorem 3.5 for \( \tilde{\mathcal{P}} = \mathcal{P} \), suppose \( \varphi \in \text{Hom}_W(V^M/V^M_{\text{double}}; \mathcal{P}(a)) \) and put \( \Phi = \varphi \). Let us prove \( \Phi[v] \in \mathcal{P}(v) \) for any \( v \in V \). We may assume for any \( v \in V^M \), \( \varphi[v] \) is a homogenous polynomial of the same degree, say \( j \). Then \( \Phi[v] \) is also a homogenous function of degree \( j \) for any \( v \in V \). It is clear from (3.8) and the relation \( \Phi[v](\text{Ad}(k)\mathcal{H}) = \Phi_k(\mathcal{H}) \) for \( k \in K \) and \( \mathcal{H} \in \mathfrak{a} \). Since a \( \mathcal{C}^\infty \) homogenous function defined around 0 is a polynomial, we get the claim. This completes the proof of Theorem 3.5.

Remark 3.14. In our proof of Theorem 3.5, the results in 3.4 are valid for any \( \mathfrak{a} \), but nowhere else. Hence if we replace the definitions of (3.8) and (3.9) is valid for any \( \mathfrak{a} \), and put

\[
\text{Lemma 3.17. Suppose } \Phi \in \text{Hom}_K(V, S(\mathfrak{p}_C)) \text{ satisfies } \Gamma_0^X(\Phi)[v] \in \mathscr{H}_W(\mathfrak{a}) \text{ for any } v \in V^M. \text{ Then } \Gamma_0^X(\Phi)[v] = 0 \text{ for any } v \in V^M_{\text{double}}.
\]

Proof of Lemma 3.17. Recall the notation in Lemma 3.10 and its proof. Put \( L_\alpha = \sum_{i=1}^r \partial(\xi_i)^2 \).

Notice that (3.8) and (3.9) are valid for any \( v \in V \). Hence for any \( v \in V \), \( \alpha \in \Sigma_1 \), and \( X_\alpha \in \mathfrak{g}_\alpha \),
a similar calculation to (3.2) implies
\[
\Phi[\sigma(X_\alpha + \theta X_\alpha)(\sigma(X_\alpha + \theta X_\alpha)^2 - 2|\alpha|^2 B(X_\alpha, \theta X_\alpha))]v(H)
\]
\[
= \alpha(H)^2 \partial(X_\alpha - \theta X_\alpha)^2 \Phi[\sigma(X_\alpha + \theta X_\alpha)v](H)
\]
\[
+ 2\alpha(H)B(X_\alpha, \theta X_\alpha)\partial(H_\alpha)\Phi[\sigma(X_\alpha + \theta X_\alpha)v](H) - 2|\alpha|^2 B(X_\alpha, \theta X_\alpha)\Phi[\sigma(X_\alpha + \theta X_\alpha)v](H)
\]
\[
= \alpha(H)^2 \partial(X_\alpha - \theta X_\alpha)^2 \Phi[\sigma(X_\alpha + \theta X_\alpha)v](H)
\]
\[
+ 2\alpha(H)B(X_\alpha, \theta X_\alpha) \frac{d}{dt} \Phi[\sigma(X_\alpha + \theta X_\alpha)v](H + tH_\alpha)\bigg|_{t=0}
\]
\[
- 2\alpha(H)|\alpha|^2 B(X_\alpha, \theta X_\alpha)\partial(X_\alpha - \theta X_\alpha)\Phi[v](H)
\]
\[
= \alpha(H)^2 \partial(X_\alpha - \theta X_\alpha)^2 \Phi[\sigma(X_\alpha + \theta X_\alpha)v](H)
\]
\[
+ 2\alpha(H)B(X_\alpha, \theta X_\alpha) \left(|\alpha|^2 B(X_\alpha, \theta X_\alpha)\Phi[v](H) + \alpha(H)\partial(H_\alpha)\partial(X_\alpha - \theta X_\alpha)\Phi[v](H)\right)
\]
\[
- 2\alpha(H)|\alpha|^2 B(X_\alpha, \theta X_\alpha)\partial(X_\alpha - \theta X_\alpha)\Phi[v](H)
\]
\[
= \alpha(H)^2 \left(\partial(X_\alpha - \theta X_\alpha)^2 \Phi[\sigma(X_\alpha + \theta X_\alpha)v](H) + 2B(X_\alpha, \theta X_\alpha)\partial(H_\alpha)\partial(X_\alpha - \theta X_\alpha)\Phi[v](H)\right).
\]

It shows
\[
\Phi[\sigma(X_\alpha + \theta X_\alpha)(\sigma(X_\alpha + \theta X_\alpha)^2 - 2|\alpha|^2 B(X_\alpha, \theta X_\alpha))]v\bigg|_a
\]
\[
= \Phi[p''(\sigma(X_\alpha + \theta X_\alpha)(\sigma(X_\alpha + \theta X_\alpha)^2 - 2|\alpha|^2 B(X_\alpha, \theta X_\alpha)))v]\bigg|_a \in \alpha^2 H(\alpha).
\]

Since no element of \(\alpha^2 H(\alpha)\) other than 0 is killed by \(L_a\), we have
\[
\Gamma_0^\alpha(\Phi)[p''(\sigma(X_\alpha + \theta X_\alpha)(\sigma(X_\alpha + \theta X_\alpha)^2 - 2|\alpha|^2 B(X_\alpha, \theta X_\alpha)))v] = 0,
\]
and the lemma. \(\square\)

We conclude this section by introducing a new class of \(K\)-types.

**Definition 3.18.** We say a \(K\)-type \((\sigma, V)\) is quasi-single-petaled when \(V^M_{\text{single}} \neq 0\).

If \(\sigma\) has real rank 1, then Lemma 3.17 assures all the quasi-single-petaled \(K\)-types are single-petaled and their number is finite by Corollary 3.16. In general we have

**Proposition 3.19.** The number of quasi-single-petaled \(K\)-types is finite.

**Proof.** Suppose \((\sigma, V)\) is quasi-single-petaled. Then it follows from Theorem 3.16 there is a nontrivial \(\Phi \in \operatorname{Hom}_q(V, S(\gamma_0))\) such that \(\Gamma_0^\alpha(\Phi)[v] \in H_W(\alpha)\) for any \(v \in V^H\). Recall that the degree of an element of \(H_W(\alpha)\) is not greater than the number of reflections in \(W\), say, \(r\) (cf. [He2, Chapter III, Theorem 3.6]). Since \(\gamma_0\) maps a homogeneous element to a homogeneous element of the same degree, the degree of \(\Phi[v]\) for each \(v \in V\) is at most \(r\). Hence \(V\) must be equivalent to an irreducible \(K\)-subspace of \(\{F \in S(\gamma_0); \deg F \leq r\}\). \(\square\)

4. The Harish-Chandra homomorphism

In this section we prove Theorem 4.1. Let us start with the definition of the degenerate affine Hecke algebra, which is due to [L1].

**Definition 4.1.** Let \(k : \Sigma_1 \to \mathbb{C}\) be a multiplicity function. Then there exists uniquely (up to equivalence) an algebra \(H_k\) over \(\mathbb{C}\) with the following properties:

(i) \(H_k \cong S(a_\mathbb{C}) \otimes \mathbb{C}[W]\) as a \(\mathbb{C}\)-linear space.
We call \( \textbf{H}_k \) the degenerate affine Hecke algebra associated to the data \((\mathfrak{a}_C, \Pi, k)\).

**Remark 4.2.** By \((\textbf{iv})\) we identify \( S(\mathfrak{a}_C) \) and \( \mathbb{C}[W] \) with subalgebras of \( \textbf{H}_k \). Then \((\textbf{iv})\) is simply written as

\[
(4.1) \quad s_\alpha \cdot \xi = s_\alpha(\xi) \cdot s_\alpha - k(\alpha) \alpha(\xi) \quad \forall \alpha \in \Pi \forall \xi \in \mathfrak{a}_C.
\]

The center of \( \textbf{H}_k \) equals \( S(\mathfrak{a}_C)^W \) ([La Theorem 6.5]) as we stated in \((\textbf{ii})\). In this section we fix

\[
(4.2) \quad k(\alpha) = \dim g_\alpha + 2 \dim g_{\alpha^2}
\]

and drop the suffix \( k \) in \( \textbf{H}_k \). Note that \( \textbf{H} \) is fully determined by the data \((n, a)\).

As in \((\textbf{1.1})\) we define the left \( \textbf{H} \)-module \( S_H(\mathfrak{a}_C) \) by \((\textbf{1.9})\).

**Lemma 4.3.** Suppose \( \alpha \in \Pi \) and put \( a(\alpha) = \{ H \in a; \alpha(H) = 0 \} \). Then

\[
S_H(\mathfrak{a}_C) = S(a(\mathfrak{a}_C)) \cdot \mathbb{C}[\{(\alpha')^2\}] \oplus S(a(\mathfrak{a}_C)) \cdot \mathbb{C}[\{(\alpha')^2\}]^{(\alpha')^2} + \dim g_\alpha + 2 \dim g_{\alpha^2}
\]

is the decomposition into the eigenspaces of \( s_\alpha \in \textbf{H} \) with eigenvalues \( 1, -1 \).

**Proof.** Using \((\textbf{3.1})\), we have

\[
s_\alpha(\alpha') + k(\alpha) = -\alpha' s_\alpha - k(\alpha) \cdot 2 + k(\alpha) s_\alpha \equiv -\alpha' + k(\alpha) \mod \sum_{w \in W_1} \textbf{H}(w - 1).
\]

Likewise, \( s_\alpha(\alpha')^2 = (\alpha')^2 s_\alpha \), and \( s_\alpha \cdot \xi = \xi' s_\alpha \) for \( \xi \in a(\alpha) \). Now the lemma is clear. \( \square \)

**Corollary 4.4.** Under the natural identification \( S_H(\mathfrak{a}_C) = S(\mathfrak{a}_C) \) (see \((\textbf{1.1})\). The space of \( W \)-fixed elements in \( S_H(\mathfrak{a}_C) \) equals \( S(\mathfrak{a}_C)^W \).

For a quasi-spherical \( K \)-type \((\sigma, V)\) we shall investigate the map \( \Gamma^\sigma \) in \((\textbf{3.1})\) By virtue of \((\textbf{1.3})\) we have

\[
\text{Hom}_K(V, U(g_{\mathfrak{c}})) = \text{Hom}_K(V, \text{symm}(S(\mathfrak{p}_{\mathfrak{c}}))) \oplus \text{Hom}_K(V, U(g_{\mathfrak{c}}) \mathbf{1}_{\mathfrak{c}}).
\]

Let \( \{ S^d(\mathfrak{p}_{\mathfrak{c}}) \}_{d=0}^{\infty} \) (resp. \( \{ S^d(\mathfrak{a}_{\mathfrak{c}}) \}_{d=0}^{\infty} \)) be the standard grading of \( S(\mathfrak{p}_{\mathfrak{c}}) \) (resp. \( S(\mathfrak{a}_{\mathfrak{c}}) \)). It is easy to see

\[
(4.3) \quad \gamma \circ \text{symm}(F) - \gamma_0(F) \in \sum_{i=0}^{d-1} S^i(\mathfrak{a}_{\mathfrak{c}}) \quad \forall F \in S^d(\mathfrak{p}_{\mathfrak{c}}).
\]

Therefore, as a corollary of Proposition 3.1 we get the following exact sequence:

\[
(4.4) \quad 0 \to \text{Hom}_K(V, U(g_{\mathfrak{c}}) \mathbf{1}_{\mathfrak{c}}) \to \text{Hom}_K(V, U(g_{\mathfrak{c}})) \overset{\Gamma^\sigma}{\to} \text{Hom}_K(V^M, S_H(\mathfrak{a}_C)).
\]

By the decomposition \( \mathscr{A} = \bigoplus_{d=0}^{\infty} S(\mathfrak{a}_{\mathfrak{c}})^K \cap S^d(\mathfrak{p}_{\mathfrak{c}}) = \bigoplus_{d=0}^{\infty} S(\mathfrak{a}_{\mathfrak{c}})^W \cap S^d(\mathfrak{a}_{\mathfrak{c}}) \), \( \mathscr{A} \) is a graded algebra. Also, by the decompositions \( \text{Hom}_K(V, S(\mathfrak{p}_{\mathfrak{c}})) = \bigoplus_{d=0}^{\infty} \text{Hom}_K(V, S^d(\mathfrak{p}_{\mathfrak{c}})) \) and \( \text{Hom}_K(V^M, S(\mathfrak{a}_{\mathfrak{c}})) = \bigoplus_{d=0}^{\infty} \text{Hom}_K(V^M, S^d(\mathfrak{a}_{\mathfrak{c}})) \), \( \text{Hom}_K(V, S(\mathfrak{p}_{\mathfrak{c}})) \) and \( \text{Hom}_K(V^M, S(\mathfrak{a}_{\mathfrak{c}})) \) are graded \( \mathscr{A} \)-modules. Homogeneity of an element of these modules is defined in the usual way.
Lemma 4.5. There is a non-zero homogeneous element \( b \in \mathcal{A} \) such that \( b \)-\( \text{Hom}_W(V^M, S(\alpha_C)) \subset \Gamma_0^\ast(\text{Hom}_K(V, S(p_C))) \).

Proof. Note that \( \mathcal{A} \) is an integral domain. In view of (2.1) and (3.16), both \( \text{Hom}_K(V, S(p_C)) \) and \( \text{Hom}_W(V^M, S(\alpha_C)) \) are free \( \mathcal{A} \)-modules of the same rank and admit bases consisting of homogeneous elements. By Proposition 3.1 \( \Gamma_0^\ast : \text{Hom}_K(V, S(p_C)) \to \text{Hom}_W(V^M, S(\alpha_C)) \) is an injective \( \mathcal{A} \)-homomorphism. Moreover, clearly \( \Gamma_0^\ast \) maps a homogeneous element to a homogeneous element of the same degree. From these facts the lemma follows easily. \( \Box \)

Let the map \( p^\ast : V \to V^M \) be as in the proof of Proposition 3.1.

Lemma 4.6. For any \( \Psi \in \text{Hom}_K(V, U(g_C)) \) and \( v \in V \), \( \gamma(\Psi[v]) = \Gamma^\ast(\Psi)[p^\ast(v)] \).

Proof. Since \( M \) normalizes \( t_C \) and \( n_C \), \( \gamma \) is an \( M \)-homomorphism from \( U(g_C) \) to a trivial \( M \)-module \( S(\alpha_C) \). Hence the map \( V \ni v \mapsto \gamma(\Psi[v]) \in S(\alpha_C) \) is an \( M \)-homomorphism and the lemma follows. \( \Box \)

The sufficiency of the second statement of Theorem 1.4 comes from

Theorem 4.7. Suppose \( v \in V^M_{\text{single}} \). Then for any \( \Psi \in \text{Hom}_K(V, U(g_C)) \) and \( w \in W 

\begin{equation}
\Gamma^\ast(\Psi)[vw] = w\Gamma^\ast(\Psi)[v],
\end{equation}

where the action of \( w \) in the right-hand side is that on \( S_{W}(a_C) \).

Proof. Suppose \( \Psi \in \text{Hom}_K(V, U(g_C)) \). For each \( \alpha \in \Pi \), we define \( g(\alpha), g_C(\alpha), t_\alpha, S(\alpha_C), G_\alpha(a), K_\alpha(a), \) and \( M_\alpha(a) \) as in the proof of Lemma 3.4. Recall \( a(\alpha) = \{ H \in \alpha; \alpha(H) = 0 \} \). Moreover, put
\n\[ y(\alpha) = \text{the center of } g(\alpha), \quad \eta_\alpha = \sum_{\beta \in \Sigma \setminus \Sigma_{\alpha}} g_\beta, \quad t_\alpha = \sum_{\beta \in \Sigma \setminus \Sigma_{\alpha}} [R(\theta_{\beta}(X_{\beta}) + \theta_{\beta}X_{\beta}); \beta \in \Sigma \setminus \Sigma_{\alpha}, X_{\beta} \in g_\beta]. \]

If we define the projection map
\n\[ \gamma_\alpha : U(g_C) = (\{ t_\alpha \} U(g_C) + U(g_C)(t_\alpha) + U(g_C)\gamma(\alpha_C)) \oplus U(\alpha(a)C) + g_C(\alpha_C) \to U(\alpha(a)C) + g_C(\alpha_C), \]

then \( \gamma_\alpha \) is a \( K_\alpha(a) \)-homomorphism and \( \gamma \circ \gamma_\alpha = \gamma \). Let \( U(t_\alpha(\alpha_C)v = V^{(1)} \oplus \cdots \oplus V^{(t)} \) be an irreducible decomposition as a \( K_\alpha(a) \)-module and \( v = \gamma_1^{(1)}(v) + \cdots + \gamma_1^{(t)}(v) \) the corresponding decomposition. Then by the same argument as in the proof of Lemma 3.4 for each \( s = 1, \ldots, t, V^{(s)} \) is considered as a single-petaled 'K-type' of the adjoint group of \( g_C(a) \) and \( v^{(s)} \) is a non-zero \( M_{ss}(a) \)-fixed vector of \( V^{(s)} \). Let \( H_a \) be the set of \( K_\alpha(a) \)-harmonic elements in \( S(p_C \cap g_C(a)C) \) and fix an arbitrary \( \Psi^{(s)} \in \text{Hom}_{K_\alpha(a)}(V^{(s)}; \text{symm}(H_a) \setminus \{0\} \). Put \( S_a = \text{symm}(S(p_C \cap g_C(a)C)) \).

Define \( \Psi_s \in \text{Hom}_{K_{\alpha}(a)}(V^{(s)}; U(\alpha(\alpha)C + g_C(\alpha_C))) \) by
\n\[ V^{(s)} \leftarrow V \xrightarrow{\Psi} U(g_C) \xrightarrow{\gamma_\alpha} U(\alpha(\alpha)C + g_C(\alpha_C)). \]

Since
\n\[ \text{Hom}_{K_\alpha(a)}(V^{(s)}, U(\alpha(\alpha)C + g_C(\alpha_C))) \cong \text{Hom}_{K_\alpha(a)}(V^{(s)}, U(g_C(a)C)) \otimes S(\alpha(a)C), \]

\[ \text{Hom}_{K_\alpha(a)}(V^{(s)}, U(g_C(a)C)) \cong \text{Hom}_{K_\alpha(a)}(V^{(s)}, U(g_C(a)C) t_\alpha(a)C) \]

\[ \oplus \text{Hom}_{K_\alpha(a)}(V^{(s)}, \text{symm}(H_a)) \otimes S_a, \]

\[ \text{Hom}_{K_\alpha(a)}(V^{(s)}, \text{symm}(H_a)) = C\Psi^{(s)}, \]
we can choose \( f_1, \ldots, f_\mu \in S(\alpha(\sigma)_C) \) and \( D_1, \ldots, D_\mu \in S_{\sigma} \) so that \( \Psi_f - \Psi^{\sigma}(D_1 f_1 + \cdots + D_\mu f_\mu) \in \text{Hom}_{K_{\sigma}(\alpha)}(V^{(\sigma)}, U(\alpha)_C + g_{sa}(\alpha)_C) t_{sa}(\alpha)_C) \). Then by Lemma 4.6,

\[
\Gamma^\sigma(\Psi)[p^\sigma(v^{(\sigma)})] = \gamma(\Psi_1(v^{(\sigma)})) = \gamma(\Psi^{\sigma}(v^{(\sigma)})) \cdot (\gamma(D_1) f_1 + \cdots + \gamma(D_\mu) f_\mu),
\]

Now by Corollary 2.8, each \( V^{(\sigma)} \) is either the trivial \( K_{\sigma}(\alpha) \)-type or a \( K_{\sigma}(\alpha) \)-type appearing in \( \text{Ad} \), \( g_{sa}(\alpha)_C \). Suppose \( V^{(\sigma)} \) is the trivial \( K_{\sigma}(\alpha) \)-type. Then it follows from Lemma 2.7 (vi) that \( \gamma(\Psi^{\sigma}(v^{(\sigma)})) \) is a scalar. Hence by Lemma 4.3, \( s_a \Gamma^\sigma(\Psi)[p^\sigma(v^{(\sigma)})] = \Gamma^\sigma(\Psi)[p^\sigma(v^{(\sigma)})] \). On the other hand, suppose \( \text{Ad}, g_{sa}(\alpha)_C \) is a \( K_{\sigma}(\alpha) \)-type appearing in \( \text{Ad}, g_{sa}(\alpha)_C \). This time it follows from Lemma 2.7 (vi) that \( \gamma(\Psi^{\sigma}(v^{(\sigma)})) \) equals \( \alpha^\vee + \text{dim } g_a + 2 \text{dim } g_{sa} \) up to a scalar multiple. Hence Lemma 4.3 implies \( s_a \Gamma^\sigma(\Psi)[p^\sigma(v^{(\sigma)})] = -\Gamma^\sigma(\Psi)[p^\sigma(v^{(\sigma)})] \). Also, in this case \( s_a v^{(\sigma)} = -v^{(\sigma)} \) by Theorem 2.6. Thus, in either case we get \( \Gamma^\sigma(\Psi)[p^\sigma(s_a v^{(\sigma)})] = s_a \Gamma^\sigma(\Psi)[p^\sigma(v^{(\sigma)})] \) for each \( s = 1, \ldots, t \). Hence \( \Gamma^\sigma(\Psi)[s_a v^{(\sigma)}] = \Gamma^\sigma(\Psi)[p^\sigma(s_a v^{(\sigma)})] = s_a \Gamma^\sigma(\Psi)[v^{(\sigma)}] \) for each \( s = 1, \ldots, t \), which assures 4.5 for any \( w \in W \).

Suppose \( \alpha \in \Pi \) and retain the notation of the proof of Theorem 4.7. We say a \( K_{\sigma}(\alpha) \)-type \((\sigma', V')\) is quasi-spherical if \( V' \) has a non-zero \( M_{\sigma}(\alpha) \)-fixed vector. A quasi-spherical \( K_{\sigma}(\alpha) \)-type is naturally identified with a quasi-spherical \( 'K' \)-type of the adjoint group of \( g_{sa}(\alpha) \). Choose \( X_\sigma \in g_a \) so that \( B(X_\sigma, \partial X_\sigma) = -\frac{1}{2m_\sigma} \) and put \( Z = \sqrt{-1}X_\sigma + \sqrt{-1}\theta X_\sigma \). According to Lemma 2.7 (v), for each quasi-spherical \( K_{\sigma}(\alpha) \)-type \((\sigma', V')\) we define the integer \( \epsilon(\sigma') \) as the largest eigenvalue of \( \alpha'(Z) \). If we decompose the \( K_{\sigma}(\alpha) \)-module \( U(t_{sa}(\alpha)_C) V^M \) into irreducible submodules, then each submodule is quasi-spherical. In fact, if the corresponding decomposition of a given \( v \in V^M \) is \( v = v^{(1)} + v^{(2)} + \cdots \), then each \( v^{(s)} \) is \( M_{\sigma}(\alpha) \)-fixed vector. Let us consider the direct sum decomposition

\[
U(t_{sa}(\alpha)_C) V^M = V^{[0]} \oplus V^{[1]} \oplus \cdots \oplus V^{[k]},
\]

where \( V^{[s]} (s = 1, \ldots, k) \) is the sum of all irreducible \( K_{\sigma}(\alpha) \)-submodules which are isomorphic to some \( K_{\sigma}(\alpha) \)-type \((\sigma', V')\) with \( |\epsilon(\sigma')| = s \).

**Lemma 4.8.** \( p^{\sigma'}(V^{M_{\sigma}(\alpha)}_{[0]}) \cap p^{\sigma'}(V^{M_{\sigma}(\alpha)}_{[1]}) = 0 \) and \( p^{\sigma'}(V^{M_{\sigma}(\alpha)}_{[2]} + \cdots + V^{M_{\sigma}(\alpha)}_{[k]}) \subset V^M \) double. Moreover,

\[
V^M_{\text{double}} = \left( V^M_{\text{single}} \cap p^{\sigma'}(V^{M_{\sigma}(\alpha)}_{[0]}) \right) \oplus V^M_{\text{double}} \cap p^{\sigma'}(V^{M_{\sigma}(\alpha)}_{[1]}),
\]

\[
P^{\sigma'}(V^{M_{\sigma}(\alpha)}_{[2]} + \cdots + V^{M_{\sigma}(\alpha)}_{[k]}).
\]

**Proof.** From Lemma 2.7 (ii), (iii) and the inclusion relation

\[
V^M \subset V^{M_{\sigma}(\alpha)}_{[0]} \oplus V^{M_{\sigma}(\alpha)}_{[1]} \oplus \cdots \oplus V^{M_{\sigma}(\alpha)}_{[k]},
\]

we have \( V^M_{\text{single}} \subset V^{M_{\sigma}(\alpha)}_{[0]} \oplus V^{M_{\sigma}(\alpha)}_{[1]} \). Let \( (\cdot, \cdot)_V \) be a \( K \)-invariant Hermitian inner product on \( V \). Then the proof of Lemma 4.3 says the orthogonal complement \( (V^M_{\text{single}})^\perp \) of \( V^M_{\text{single}} \) in \( V^M \) equals \( V^M_{\text{double}} \). Since \( (V^M_{\text{single}} \cap p^{\sigma'}(V^{M_{\sigma}(\alpha)}_{[0]}), V^M_{\text{double}} \cap p^{\sigma'}(V^{M_{\sigma}(\alpha)}_{[1]})) \subset (V_{[0]} + V_{[1]}, V_{[2]} + \cdots + V_{[k]}, 0, p^{\sigma'}(V^{M_{\sigma}(\alpha)}_{[2]} + \cdots + V_{[k]}), (V^M_{\text{single}})^\perp = V^M_{\text{double}} \). On the other hand, it follows from Lemma 2.7 (x) and Theorem 2.6 that \( s_a \) acts on \( V^{M_{\sigma}(\alpha)}_{[0]} \) and \( V^{M_{\sigma}(\alpha)}_{[1]} \) by \(+1\) and \(-1\), respectively. But since \( p^{\sigma'} : V \rightarrow V^M \) is an \( N_K(\alpha) \)-homomorphism, \( s_a \)
acts on \( p^r(\gamma_{M_0(\alpha)}) \) and \( p^r(\gamma_{M_0(\alpha)}) \) by +1 and −1, respectively. Hence the decomposition of \( V_{M_0(\alpha)} \) into the eigenspaces of \( s_\alpha \) is

\[
V_{M_0(\alpha)} = \bigoplus_{i,j} V_{(i,j)}^{\alpha, \gamma} \bigoplus_{i,j} V_{(i,j)}^{\alpha, -\gamma}
\]

We thus get

\[
V_{M_0(\alpha)} = \bigoplus_{i,j} V_{(i,j)}^{\alpha, \gamma} \bigoplus_{i,j} V_{(i,j)}^{\alpha, -\gamma}
\]

The necessity of the second statement of Theorem 4.4 follows from the next proposition:

**Proposition 4.9.** Suppose \( V_{M_0(\alpha)} \neq 0 \). Then \( \text{4.3} \) does not hold for a suitable combination of \( v \in V_{M_0(\alpha)} \), \( w \in W \) and \( \Psi \in \text{Hom}_W(V, U(\mathcal{O}_C)) \).

**Proof.** By the assumption of the proposition, there exists an \( \alpha \in \Pi \) for which \( V_{(1,\ldots,1)} \) in the above argument is not 0. Take \( s (2 \leq s \leq k) \) so that \( V_{(s)} \neq 0 \). Fix an irreducible \( K_{ss(\alpha)} \)-submodule \( V_0 \) of \( V_{(s)} \) and \( v_0 \in V_{M_0(\alpha)} \setminus \{ 0 \} \).

First, we shall show \( p^r(v_0) = 0 \). For this, let \( U(t_{ss}(\alpha))V^M = V^{(1)} \oplus V^{(2)} \oplus \cdots \) be an irreducible decomposition as a \( K_{ss(\alpha)} \)-module. Since \( V^{(1)} = V_0 \) and each component is orthogonal to the other components with respect to a \( K \)-invariant Hermitian inner product \( \langle \cdot, \cdot \rangle_V \) on \( V \), if \( v = v^{(1)} + v^{(2)} + \cdots \) is the corresponding decomposition of any \( v \in V^M \), then each \( v^{(s)} (s = 1, 2, \ldots) \) is an \( M_0(\alpha) \)-fixed vector. Since \( V^M \) generates \( U(t_{ss(\alpha)})V^M \), there exists \( v \in V^M \) such that \( v^{(1)} \neq 0 \). Since \( \dim \sigma V_{M_0(\alpha)} = 1 \), \( v^{(1)} = cv \) for some constant \( c \neq 0 \). Now \( (v, p^r(v_0))_V = (v, v_0)_V = (v^{(1)}, v_0)_V = (c(v_0, v_0)) \neq 0 \), which proves \( p^r(v_0) = 0 \).

Choose a homogeneous \( \phi \in \text{Hom}_W(V^M, S(\mathcal{A}_C)) \) so that \( \phi[p^r(v_0)] \neq 0 \). Let \( b \in \mathcal{A}_C \) be the homogeneous element in Lemma 4.3. Then \( \Gamma(\Phi) = b \cdot \phi \) for some \( \Phi \in \text{Hom}_W(V, S(\mathcal{C})) \).

Put \( \Psi = \text{symm} b \phi \in \text{Hom}_W(V, U(\mathcal{C})) \). Since \( \Phi \) is homogeneous and \( \gamma(\Phi(\mathcal{C})) = b \cdot \phi[p^r(v_0)] \neq 0 \), \( \text{4.3} \) implies \( \gamma(\Psi(\mathcal{C})) \neq 0 \).

Let \( \gamma_0, \gamma_1, \gamma_2 \) be the same as in the proof of Theorem 4.4. Fix an arbitrary \( \Psi \in \text{Hom}_{K_{ss(\alpha)}(V_0, \text{symm}(\mathcal{A}_0))) \setminus \{ 0 \} \} \) and define \( \Psi \in \text{Hom}_{K_{ss(\alpha)}(V_0, U(\mathcal{A}_C + \mathcal{B}_C))) \) by

\[
V_0 \hookrightarrow V \xrightarrow{\psi} U(\mathcal{C}) \xrightarrow{\gamma_0} U(\mathcal{A}_C + \mathcal{B}_C).
\]

Then, as in the proof of Theorem 4.4, we can choose \( f_1, \ldots, f_\delta \in S(\mathcal{A}_C) \) and \( D_1, \ldots, D_\delta \in S_\alpha \) so that \( \Psi_0 - \Psi(\mathcal{D}_1 f_1 + \cdots + \mathcal{D}_\delta f_\delta) \in \text{Hom}_{K_{ss(\alpha)}(V_0, U(\mathcal{A}_C + \mathcal{B}_C)))} \).

Hence

\[
\gamma(\Psi(\mathcal{C})) = (\gamma(\Psi_0)) = (\gamma(\mathcal{D}_1 f_1 + \cdots + \mathcal{D}_\delta f_\delta))
\]

From Lemma 4.7, there is a pair \( (i, j) \) of non-negative integers with \( 2i + j = s \) such that \( \gamma(\mathcal{D}_1 f_1 + \cdots + \mathcal{D}_\delta f_\delta) \)

up to a scalar multiple (here \( h = \frac{1}{2} + \frac{\dim \sigma}{2} \in S(\mathcal{A}_C) \) and \( \delta = \dim \mathcal{B}_C \)). Since it is clear that \( \gamma(\mathcal{D}_1 f_1 + \cdots + \mathcal{D}_\delta f_\delta) \notin S(\mathcal{A}_C) \bigcup S(\mathcal{B}_C) \), we have \( \gamma(\Psi(\mathcal{C})) = \Gamma^*(\Psi)[p^r(v_0)] \neq S(\mathcal{A}_C) \cup S(\mathcal{B}_C) \). Hence from Lemma 4.3, \( \gamma(\Psi)[p^r(v_0)] \neq \pm \Gamma^*(\Psi)[p^r(v_0)] \). On the other hand, since \( \dim \sigma V_{M_0(\alpha)} = 1 \), we have \( s_\alpha v_0 = \pm 0 \) and therefore \( s_\alpha p^r(v_0) = \pm p^r(v_0) \). □
Remark 4.10. In the above proof we can say $s_a p^r(v_0) = (-1)^r p^r(v_0)$. Indeed, if we choose $\Phi^0 \in \text{Hom}_{K_0}(V_0, \mathcal{H}_c)$ so that $\Phi^0 = \text{symm} \circ \Phi^0$, then it is a homogeneous element by Lemma 4.9. Hence from 4.8 and 4.7, $\gamma_0(\Phi^0[v_0]) = C^{(a \gamma)^r}$ for some non-zero constant $C$. With respect to the ordinary action of $s_a$ on $\mathbb{C}[a \gamma]$, we have $s_a v_0 = (\alpha)^r v_0$, which shows our claim.

In the rest of this section we shall prove the following theorem, which is considered as a non-commutative counterpart of Theorem 4.5.

**Theorem 4.11.** For any $\psi \in \text{Hom}_w(V^M/V^M_{\text{double}}, S_H(a_c))$ there exists $\Psi \in \text{Hom}_K(V, U(a_c))$ such that $\Gamma^w(\Psi) = \psi$.

Suppose $V' \subset V^M$ is an arbitrary $W$-submodule. Let $t^w_{V'}$ denote the map

$$\text{Hom}_c(V^M, S^*_H(a_c)) \ni \psi \mapsto \psi|_{V'} \in \text{Hom}_c(V', S^*_H(a_c))$$

or the map

$$\text{Hom}_w(V^M, S(a_c)) \ni \varphi \mapsto \varphi|_{V'} \in \text{Hom}_w(V', S(a_c)).$$

Under the natural identification $S^*_H(a_c) \cong S(a_c)$, put $S^*_H(a_c) = \sum_{i=0}^{d} S_i(a_c)$ ($d = 0, 1, \ldots$). Then $\text{Hom}_c(V', S^*_H(a_c))$ has the natural filtration $[\text{Hom}_c(V', S^*_H(a_c))]_{d=0}^{\infty}$ by which it is considered as a filtered $A'$-module. For each $\psi \in \text{Hom}_c(V', S^*_H(a_c))$ put

$$\deg \psi = \begin{cases} d & \text{if } \psi \in \text{Hom}_c(V', S^*_H(a_c)) \setminus \text{Hom}_c(V', S^{d-1}_H(a_c)), \\ -\infty & \text{if } \psi = 0. \end{cases}$$

Also, for $d = -\infty, 0, 1, \ldots$, define the natural map

$$q_d : \text{Hom}_c(V', S^*_H(a_c)) \rightarrow \text{Hom}_c(V', S^d_H(a_c)).$$

**Lemma 4.12.** For any $\Psi \in \text{Hom}_K(V, U(a_c))$, $t^w_{V've} \circ \Gamma^w(\Psi) \in \text{Hom}_w(V^M_{\text{single}}, S^*_H(a_c))$. For any $\psi \in \text{Hom}_w(V^M_{\text{single}}, S^*_H(a_c))$ with $d = \deg \psi$, $q_d(\psi) \in \text{Hom}_w(V^M_{\text{single}}, S^d(a_c))$.

**Proof.** The first assertion is due to Theorem 4.7. The second assertion follows from the fact that the map $S^d_H(a_c) = \sum_{i=0}^{d} S_i(a_c)$ is a $W$-homomorphism, which is easily checked by use of (4.1) and (1.9). $\square$

**Lemma 4.13.** For any $\Psi \in \text{Hom}_K(V, U(a_c))$, put $\psi := t^w_{V've} \circ \Gamma^w(\Psi) \in \text{Hom}_c(V^M_{\text{double}}, S^*_H(a_c))$ and $d = \deg \psi$. Then $q_d(\psi) \in \text{Hom}_w(V^M_{\text{double}}, S^d(a_c))$.

**Proof.** We shall check for any $a \in \Pi$ and $v \in V^M_{\text{double}}$

$$q_d(\psi)|_{s_a v} = s_a q_d(\psi)[v].$$

For this, fix $a \in \Pi$ and consider the decomposition (4.6).

Take an arbitrary irreducible $K_{\alpha}(a)$-submodule $V_0 \subset V_{[i]}$ ($s = 2, 3, \ldots$) and $v_0 \in V_{[0]}$. Then for the same reason as the proofs of Theorem 4.7 and Proposition 4.9 there are some non-negative integers $i, j$ with $2i + j = s$ and some $D_j \in S(\alpha(a)_c) \cdot \mathbb{C}[a \gamma]$ such that

$$\Gamma^w(\Psi)[p^r(v_0)] = D_j \cdot [(h+\delta)(h+\delta+2) \cdots (h+\delta+2(i+j)-2)] \cdot [(h+1)(h+2) \cdots (h+2i-1)].$$
(Recall $h = \frac{\omega}{2} + \dim_{\mathbb{C}} V$. Therefore there exists a homogeneous element $D_0$ in $S(a\alpha)\cdot \mathbb{C}[\alpha\vert^2]$ such that

$$q_d(\psi)[p^{\gamma}(\psi_0)] = D_0 \left( \frac{\alpha}{2} \right).$$

Hence $s_q q_d(\psi)[p^{\gamma}(\psi_0)] = (-1)^q s_d q_d(\psi)[p^{\gamma}(\psi_0)]$. But since $s_q p^{\gamma}(\psi_0) = (-1)^q p^{\gamma}(\psi_0)$ by Remark 4.11, (4.8) is valid for any $s = 1\ldots t$.

Secondly, let $V^{(1)} \oplus \cdots \oplus V^{(t)}$ be an irreducible decomposition of the $K\alpha$-module $V_{[1]}$. Suppose $v_{[1]} \in V_{[1]}$ satisfies $p^{\gamma}(v_{[1]}) \in V_{\text{double}}$ and let $v_{[1]} = v^{(1)} + \cdots + v^{(t)}$ be the decomposition according to the above decomposition. Since $v^{(s)} \in (V^{(s)})_{\mathfrak{M}(\alpha)}$ ($s = 1\ldots t$), as in the proof of Theorem 4.7 there exist $D_1\ldots D_t \in S(a\alpha)\cdot \mathbb{C}[\alpha\vert^2]$ such that

$$\Gamma'(\Psi)[p^{\gamma}(\psi(s))] = D_s(\alpha\vert + dim \mathfrak{g}_a + 2 dim \mathfrak{g}_2) \quad s = 1\ldots t.$$ 

Hence there exists a homogeneous element $D$ in $S(a\alpha)\cdot \mathbb{C}[\alpha\vert^2]$ such that

$$q_d(\psi)[p^{\gamma}(\psi)] = D \alpha\vert.$$ 

Since $s_q p^{\gamma}(v_{[1]}) = -p^{\gamma}(v_{[1]})$ and $s_q D \alpha\vert = -D \alpha\vert$, (4.8) is valid for $v = p^{\gamma}(v_{[1]})$.

Similarly, we can show (4.8) for any $v \in V_{\text{double}} \cap p^{\gamma}(V_{\mathfrak{M}(\alpha)})$. Hence from 4.6, 4.8 is valid for any $v \in V_{\text{double}}$.

Lemma 4.14. For any $\psi \in \text{Hom}_{\mathbb{K}}(V_{\text{single}}^M, S_H(\alpha)) \setminus \{0\}$ there exists $\Psi \in \text{Hom}_{\mathbb{K}}(V, U(\mathfrak{g}))$ such that

$$t^\#_{\text{single}} \circ \Gamma'(\Psi) = \psi, \quad \text{deg} t^\#_{\text{double}} \circ \Gamma'(\Psi) < \text{deg} \psi.$$ 

Proof. Put $d = \text{deg} \psi$. Assume that for some $i \in \{d+1, d, d-1, \ldots, 1\}$ we already have $\Psi_i \in \text{Hom}_{\mathbb{K}}(V, U(\mathfrak{g}))$ such that

$$\text{deg} \left( \psi - t^\#_{\text{single}} \circ \Gamma'(\Psi_i) \right) < i, \quad \text{deg} t^\#_{\text{double}} \circ \Gamma'(\Psi_i) < d.$$ 

Then from Lemma 4.12 we get $\varphi_{i-1} := q_{i-1} \left( \psi - t^\#_{\text{double}} \circ \Gamma'(\Psi_i) \right) \in \text{Hom}_{\mathbb{K}}(V_{\text{single}}^M, S^{i-1})(\alpha)$. Since $V_{\text{single}}^M \simeq V_{\text{double}}^M \cup V_{\text{single}}^M$, we identify $\varphi_{i-1}$ with an element of $\text{Hom}_{\mathbb{K}}(V^M, S^{i-1}(\mathfrak{a}))$. Then by Theorem 3.3, there exists a unique $\Phi_{i-1} \in \text{Hom}_{\mathbb{K}}(V, S^{i-1}(\mathfrak{g}))$ such that $\Gamma^\#(\Phi_{i-1}) = \varphi_{i-1}$. In view of (3.3) we see $\text{deg} \Gamma'(\text{symm} \circ \Phi_{i-1}) \leq i - 1$ and

$$q_{i-1} \circ t^\#_{\text{single}} \circ \Gamma'(\text{symm} \circ \Phi_{i-1}) = q_{i-1} \circ t^\#_{\text{single}} \circ \Gamma'(\Psi_i),$$

$$q_{i-1} \circ t^\#_{\text{double}} \circ \Gamma'(\text{symm} \circ \Phi_{i-1}) = t^\#_{\text{double}} \circ \Gamma'(\Psi_i).$$ 

Hence, $\text{deg} t^\#_{\text{single}} \circ \Gamma'(\text{symm} \circ \Phi_{i-1}) < i - 1$ and if we put $\Psi_{i-1} = \Psi_i + \text{symm} \circ \Phi_{i-1}$, then

$$\text{deg} \left( \psi - t^\#_{\text{single}} \circ \Gamma'(\Psi_{i-1}) \right) < i - 1.$$ 

Thus, if we start with $\Psi_{d+1} := 0$ and define $\Psi_{d}, \Psi_{d-1}, \ldots$ as above, then $\Psi := \Psi_0$ satisfies the desired properties.

Proof of Theorem 4.7. Let $m = \dim_{\mathbb{C}} V_{\text{single}}$ and $m' = \dim_{\mathbb{C}} V_{\text{single}}$. Take a basis $[\varphi_{m+1}, \ldots, \varphi_m]$ of $\text{Hom}_{\mathbb{K}}(V_{\text{single}}^M, \mathcal{H}_H(\alpha))$ so that each $\varphi_i$ is homogeneous (note that $\mathcal{H}_H(\alpha) \cong \mathbb{C}[W]$). Let $b \in \mathfrak{a}$ be the homogeneous element in Lemma 4.3. Then there exist $\Phi_{m'+1}, \ldots, \Phi_m \in \mathfrak{g}$ such that $\Gamma'(\Phi_i) = b \varphi_i$. Put $d_i = \text{deg} \varphi_i (i = m'+1, \ldots, m)$ and $d_0 = \text{deg} b$. Then
Owing to (4.3), Lemma (4.12) and Lemma (4.14) by modifying \text{symm} \circ \Phi_{m' + 1}, \ldots, \text{symm} \circ \Phi_m in lower-order terms, we can get \( \Psi_{m' + 1}, \ldots, \Psi_m \in \text{Hom}_K(V, U(\mathfrak{g}_C)) \) which satisfy for each \( i = m' + 1, \ldots, m, \)

\[
\deg \Gamma^r(\Psi_i) = \deg \tilde{d}_i + d_0, \quad q_{d_0 + d_0} \circ \tilde{t}_{V_M} \circ \Gamma^r(\Psi_i) = b \cdot \tilde{t}_{V_M} \circ \Gamma^r(\Psi_i), \quad \tilde{t}_{V_M} \circ \Gamma^r(\Psi_i) = 0.
\]

Put \( \mathcal{M} = \tilde{t}_{V_M} \circ \Gamma^r(\text{Hom}_K(V, U(\mathfrak{g}_C))). \) Then by (1.3), it is a submodule of the filtered \( \mathcal{A} \)-module \( \text{Hom}_\mathcal{A}(V_M^{\text{double}}, S \mathfrak{h}(\mathfrak{a}_C)). \) Also from Lemma (4.13) \( \text{gr} \mathcal{M} \subset \text{Hom}_\mathcal{A}(V_M^{\text{double}}, S(\mathfrak{a}_C)). \)

Since \( \text{gr} \mathcal{M} \) is finitely generated over \( \mathcal{A} \), we can take \( \tilde{\Psi}_1, \ldots, \tilde{\Psi}_k \in \text{Hom}_K(V, U(\mathfrak{g}_C)) \) so that \( \{ \tilde{q}_{d_0 + d_0} \circ \Gamma^r(\tilde{\Psi}_i); i = 1, \ldots, k \} \) generates \( \text{gr} \mathcal{M} \) over \( \mathcal{A} \) (here \( \tilde{d}_i := \deg \tilde{t}_{V_M} \circ \Gamma^r(\tilde{\Psi}_i) \)). Now, from (1.3) we have \( U(\mathfrak{g}_C)^K = \text{symm}(S(\mathfrak{v}_C)^K) \oplus U(\mathfrak{g}_C)^K \cap U(\mathfrak{g}_C)_{\mathcal{C}}. \) Hence by the exactness of (1.4), \( \gamma \) gives the isomorphism \( \text{symm}(S(\mathfrak{v}_C)^K) \cong S(\mathfrak{a}_C)^W. \) For each \( a \in \mathcal{A}' = S(\mathfrak{a}_C)^W, \) we denote by \( \hat{a} \) the unique element of \( \text{symm}(S(\mathfrak{v}_C)^K) \) such that \( \gamma(\hat{a}) = a. \) Then for any \( \Psi \in \text{Hom}_K(V, U(\mathfrak{g}_C)), \) there exist \( a_1, \ldots, a_k \in \mathcal{A}' \) with \( \deg a_i \leq \deg \tilde{t}_{V_M} \circ \Gamma^r(\tilde{\Psi}_i) \) such that \( \tilde{t}_{V_M} \circ \Gamma^r(\tilde{\Psi}_i - \tilde{\Psi}_1 \hat{a}_1 - \cdots - \tilde{\Psi}_k \hat{a}_k) = 0. \) Now since \( \{ \tilde{t}_{V_M} \circ (\tilde{\Psi}_i); i = m' + 1, \ldots, m \} \) is a basis of \( \text{Hom}_\mathcal{A}(V_M^{\text{double}}, S(\mathfrak{a}_C)) \) over \( \mathcal{A}, \) we can take homogeneous elements \( b_{i, s} \in \mathcal{A} \) \( (i = 1, \ldots, k, \ s = m' + 1, \ldots, m) \) so that

\[
q_{d_0 + d_0} \circ \tilde{t}_{V_M} \circ \Gamma^r(\tilde{\Psi}_i) = \sum_{s = m' + 1}^{m} b_{i, s} \cdot \tilde{t}_{V_M} \circ (\tilde{\Psi}_i), \quad \tilde{d}_i = \deg b_{i, s} + d_0 or b_{i, s} = 0.
\]

Put \( \tilde{\Psi}_i := \tilde{\Psi}_i \hat{b} - \sum_{s = m' + 1}^{m} \Psi_s \hat{b}_{i, s}. \) Then hence \( (4.9), \) \( \deg \tilde{t}_{V_M} \circ \Gamma^r(\tilde{\Psi}_i) < \deg b_0 + \tilde{d}_i. \) Hence there exist \( a_{ij} \in \mathcal{A}' \) \( (i, \ j = 1, \ldots, k) \) with \( \deg a_{ij} < \deg d_0 + \tilde{d}_i - \tilde{d}_j \) such that

\[
\tilde{t}_{V_M} \circ \Gamma^r(\tilde{\Psi}_i - \sum_{j = 1}^{k} \Psi_j \hat{a}_{ij}) = 0.
\]

Let us define the \( \mathcal{A} \)-valued \( k \times k \)-matrix \( \hat{A} := \text{diag}(b, \ldots, b) - (a_{ij})_{1 \leq i \leq j \leq k}. \) By estimating the degree of each coefficient of \( A, \) we can easily see \( \det A \neq 0. \) Let \( \hat{A} = (\hat{a}_{ij})_{1 \leq i \leq j \leq k} \) be the cofactor matrix of \( A. \) Observe that

\[
(4.10) \quad \tilde{t}_{V_M} \circ \Gamma^r(\tilde{\Psi}_i \cdot \det \hat{A} - \sum_{j = 1}^{k} \sum_{s = m' + 1}^{m} \Psi_s \hat{a}_{ij} \hat{b}_{js}) = 0 \quad i = 1, \ldots, k.
\]

Now let \( \psi \in \text{Hom}_\mathcal{A}(V_M^{\text{double}}, S \mathfrak{h}(\mathfrak{a}_C)). \) By Lemma (4.14) there is \( \Psi \in \text{Hom}_K(V, U(\mathfrak{g}_C)) \) such that \( \tilde{t}_{V_M} \circ \Gamma^r(\psi) = \tilde{t}_{V_M} \circ \Gamma^r(\psi). \) Then we can take \( a_1, \ldots, a_k \in \mathcal{A}' \) so that \( \tilde{t}_{V_M} \circ \Gamma^r(\Psi - \tilde{\Psi}_1 \hat{a}_1 - \cdots - \tilde{\Psi}_k \hat{a}_k) = 0. \) Hence if we put

\[
\Psi' = \Psi \cdot \det \hat{A} - \sum_{i = 1}^{k} \sum_{j = 1}^{k} \sum_{s = m' + 1}^{m} \Psi_s \hat{a}_{ij} \hat{a}_{ij} \hat{b}_{js},
\]

then from (4.9) and (4.10) we have \( \tilde{t}_{V_M} \circ \Gamma^r(\Psi') = \det A \cdot \tilde{t}_{V_M} \circ \Gamma^r(\psi) \) and \( \tilde{t}_{V_M} \circ \Gamma^r(\Psi') = 0, \) namely, \( \Gamma^r(\Psi') = \det A \cdot \psi. \) Hence if we put \( \mathcal{A} = \{ c \in \mathcal{A}; \ c \cdot \psi \in \mathcal{A} \} \) \( \text{Hom}_K(V, U(\mathfrak{g}_C))), \) then \( \mathcal{A} \ni \det a \neq 0. \) Note that \( \mathcal{A} \) is an ideal of \( \mathcal{A}. \)

In order to complete the proof, it suffices to show \( \mathcal{A} = \mathcal{A}' \). Assume \( \mathcal{A} \subseteq \mathcal{A}. \) From inside \( \mathcal{A}' \setminus \{0\}, \) take an element \( c \) so that it has the lowest degree. Then by assumption, \( c \) is
Proposition 5.1. Let $\Psi'' \in \text{Hom}_K(V, U(\mathfrak{g}_C))$ be such that

$$\Gamma''(\Psi'') = c \cdot \psi.$$  

With respect to a basis $\{\overline{V}_1, \ldots, \overline{V}_m\}$ of $\text{Hom}_K(V, \text{symm}(\mathcal{H}_K^s(\nu)))$ and a basis $\{v_1, \ldots, v_m\}$ of $V^M$, we define the matrix $P'' = (\gamma \circ \overline{V}_j[v_i])_{1 \leq i, j \leq m}$ as in [24]. By virtue of Corollary 2.2 and the exactness of $\mathcal{M}^s$, we can take $e_1, \ldots, e_m \in \mathcal{M}$ so that $\Psi'' - \overline{V}_1 e_1 - \cdots - \overline{V}_m e_m \in \text{Hom}_K(V, U(\mathfrak{g}_C))$. Then we have $\Gamma''(\overline{V}_1) e_1 + \cdots + \Gamma''(\overline{V}_m) e_m = c \cdot \psi$, which, using $P''$, are rewritten as

$$P'' \begin{pmatrix} e_1 \\ \vdots \\ e_m \end{pmatrix} = c \begin{pmatrix} \psi[v_1] \\ \vdots \\ \psi[v_m] \end{pmatrix}.$$  

(4.11)

We assert that if $\lambda \in \mathfrak{a}_C^+$ satisfies $c(\lambda) = 0$, then $e_1(\lambda) = \cdots = e_m(\lambda) = 0$. To show this, suppose $\lambda \in \mathfrak{a}_C^+$ satisfies $c(\lambda) = 0$. Then there exists $w \in W$ such that $\text{Re}(w, \lambda) \geq 0$ for any $\alpha \in \Sigma^+$. Since $c \in \mathcal{M} = S(\mathfrak{a}_C)^W$, $c(\lambda) = 0$ implies $c(w\lambda) = 0$. Evaluating both sides of (4.11) at $w\lambda$, we have

$$P''(w\lambda) \begin{pmatrix} e_1(w\lambda) \\ \vdots \\ e_m(w\lambda) \end{pmatrix} = 0.$$  

Then $c_1(w\lambda) = \cdots = c_m(w\lambda) = 0$ since $P''(w\lambda)$ is a regular matrix by Proposition 2.3. Because $e_1, \ldots, e_m \in \mathcal{M} = S(\mathfrak{a}_C)^W$, $e_1(\lambda) = \cdots = e_m(\lambda) = 0$. Thus we get the assertion.

Now, $\mathcal{M}$ is isomorphic to a polynomial ring ([He2, Ch. III, Theorem 3.1]) and a maximal ideal of $\mathcal{M}$ equals \{ $f \in \mathcal{M}$ : $f(\lambda) = 0$ for some $\lambda \in \mathfrak{a}_C^+$ \} (ibid., Ch. III, Lemma 3.11). Hence by the fact shown above, $e_j$ ($j = 1, \ldots, m$) are divisible by any irreducible factor $c_0$ of $c$. Let $c', c'_1, \ldots, c'_m \in \mathcal{M}$ be such that $c = c' c_0$, $e_j = e'_j c_0$ and put $\Psi''' = \overline{V}_1 e'_1 + \cdots + \overline{V}_m e'_m$. Then we have

$$\Gamma''(\Psi''') = c' \cdot \psi, \quad \deg c' < \deg c, \quad c' \neq 0.$$  

It contradicts the minimality of the degree of $c$. Thus we get $\mathcal{J} = \mathcal{M}$. \qed

5. Complex semisimple Lie algebras

Suppose $\mathfrak{g}$ is a complex semisimple Lie algebra with complex structure $J$. In this case one has $G = G_{ad} = G_\mathfrak{g}$. Throughout this section we use the symbols $U$, $u$, $b_{\mathbb{R}}$, and $b$ in place of $K$, $\mathfrak{k}$, $\mathfrak{a}$, and $m + a$, respectively. Then $b$ is a Cartan subalgebra. Extend each $\alpha \in \Sigma$ to a complex linear form on $b$ and put $\bar{\rho} = \frac{1}{2} \sum_{\alpha \in \Sigma} \alpha \in b^*$. By the unitary trick, a $U$-type $(\sigma, V)$ is naturally identified with a finite-dimensional irreducible holomorphic representation of $G$. Since $M = \exp(Jb_{\mathbb{R}})$, $V^M$ equals $V^b$, the space of $0$-weight vectors. Hence in this section, we denote $V^b_{\text{single}}$ and $V^b_{\text{double}}$ by $V^b_{\text{single}}$ and $V^b_{\text{double}}$, respectively. One knows each finite-dimensional irreducible holomorphic representation $(\sigma, V)$ of $G$ satisfies $V^b \neq 0$. This means all $U$-types are quasi-spherical. From now on we always assume a representation of $G$ is holomorphic.

Proposition 5.1. For any finite-dimensional irreducible representation $(\sigma, V)$ of $G$,

$$V^b_{\text{single}} = \left\{ v \in V^b ; \sigma(X_\alpha)^2 v = 0 \quad \forall \alpha \in \Sigma, \forall X_\alpha \in \mathfrak{g}_\alpha \right\},$$  

(5.1)
Corollary 5.2. in a quite similar way to the proof of Lemma 3.3. Thus we get \( (5.2) \)

\[
V^h_{\text{double}} = V^h \cap \sum \{ \sigma(\alpha)^2 V; \alpha \in \Sigma, X_\alpha \in \mathfrak{g}_\alpha \}.
\]

Proof. Suppose \( v \in V^h \) and \( \alpha \in \Sigma \). Choose \( X_\alpha \in \mathfrak{g}_\alpha \) so that \( B(X_\alpha, \theta X_\alpha) = -\frac{1}{2}\dim \). Put \( Z = \sqrt{-1}X_\alpha + \sqrt{-1}\theta X_\alpha \in \mathfrak{u}_\mathbb{C} \) and \( Z^* = JX_\alpha + J\theta X_\alpha \in JU \subset \mathfrak{g}_\alpha \). Then \( \sigma(Z) = \sigma(Z^*) \).

Denote by \( \mathfrak{s} \mathfrak{l}_2(2, \mathbb{C}) \) the three-dimensional simple subalgebra spanned by \( \{X_\alpha, \alpha^\vee, \theta X_\alpha\} \) over \( \mathbb{C}_J := \mathbb{R} \oplus \mathbb{R}J \). We identify \( \mathfrak{s} \mathfrak{l}_2(2, \mathbb{C}) \) with \( \mathfrak{sl}(2, \mathbb{C}) \) by the following correspondence:

\[
X_\alpha \leftrightarrow \begin{pmatrix} 0 & 1/2 \\ 0 & 0 \end{pmatrix}, \quad \alpha^\vee \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \theta X_\alpha \leftrightarrow \begin{pmatrix} 0 & 0 \\ 1/2 & 0 \end{pmatrix}.
\]

Let \( U(\mathfrak{s} \mathfrak{l}_2(2, \mathbb{C}))v = V^{(1)} + \ldots + V^{(t)} \) be an irreducible decomposition as an \( \mathfrak{s} \mathfrak{l}_2(2, \mathbb{C}) \)-module and \( v = v^{(1)} + \ldots + v^{(t)} \) the corresponding decomposition. Then for each \( s = 1, \ldots, t \), we have \( v^{(s)} \neq 0 \) and \( \sigma(\alpha^\vee)v^{(s)} = 0 \). If we put \( d_s = \dim_{\mathbb{C}_J} V^{(s)} \), then by the representation theory of \( \mathfrak{sl}(2, \mathbb{C}) \), we can make the following identification:

\[
V^{(s)} = \sum_{i=0}^{d_s-1} \mathbb{C}x^{d_s-i} y^i \subset \mathbb{C}[x, y],
\]

\[
\sigma(X_\alpha + \theta X_\alpha)|_{V^{(s)}} = -\frac{1}{2} \left( y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right), \quad \sigma(\alpha^\vee)|_{V^{(s)}} = -x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}.
\]

We see \( x^{d_s-1} y^i \) is a \( \sigma(\alpha^\vee) \)-eigenvector with eigenvalue \( 2i + 1 - d_s \) and because \( \sigma(\alpha^\vee)v^{(s)} = 0, d_s \) is necessarily odd. On the other hand, if we put \( z = x + \sqrt{-1}y, \bar{z} = x - \sqrt{-1}y \), then \( V^{(s)} = \sum_{i=0}^{d_s-1} \mathbb{C}z^{d_s-i} \bar{z}^i \) and \( z^{d_s-1} \bar{z}^i \) is a \( \sigma(X_\alpha + \theta X_\alpha) \)-eigenvector with eigenvalue \( \frac{1}{2}(d_s - 1 - 2i) \). Also, \( v^{(s)} \) equals

\[
(z + \bar{z})^\frac{d_s-1}{2}(z - \bar{z})^\frac{d_s-1}{2} = (z^2 - \bar{z}^2)^\frac{d_s-1}{2}
\]

up to a scalar multiple. Since \( \sigma(Z^*) z^{d_s-1} = -d_s \) \( \bar{z}^{d_s-1} \) and \( \sigma(Z) \bar{z}^{d_s-1} = \bar{z}^{d_s-1} \), it follows that

\[
\sigma(Z^*) \sigma(Z)^2 - 1)v^{(s)} = 0 \iff d_s = 1 \text{ or } d_s = 3 \iff \sigma(X_\alpha)^2 v^{(s)} = 0.
\]

Thus we get \( \sigma(Z)(\sigma(Z)^2 - 1)v = \sigma(X_\alpha)^2 v = 0 \), which proves \( 5.3 \).

To prove \( 5.2 \) it suffices to show with respect to a \( U \)-invariant Hermitian inner product \( (\cdot, \cdot)_V \) on \( V \), \( V^h_{\text{single}} \) equals the orthogonal complement of \( V^h_{\text{double}} \) in \( V^h \). But it can be checked in a quite similar way to the proof of Lemma 5.3. □

As a consequence of this proposition, the condition that \( V^h = V^h_{\text{single}} \) is equivalent to the condition that twice a root is not a weight of \( (\sigma, V) \). Hence we get

**Corollary 5.2.** A single-petaled \( U \)-type \( (\sigma, V) \) is nothing but an irreducible small representation of \( G \) in the sense of [B15].

**Definition 5.3.** We say an irreducible representation \( (\sigma, V) \) of \( G \) is quasi-small when \( V^h_{\text{single}} \neq 0 \), that is, \( (\sigma, V) \) is quasi-single-petaled as a \( U \)-type.

Since \( \theta \) is the conjugation map of \( \mathfrak{g} \) with respect to the real form \( \mathfrak{u} \), the \( U \)-homomorphism

\[
\mathfrak{g} \oplus \mathfrak{g} \xrightarrow{id \oplus \theta} \mathfrak{g} \oplus \mathfrak{g} \xrightarrow{1 - \sqrt{-1}J / 2} \mathfrak{g} \oplus \mathfrak{g} \xrightarrow{1 + \sqrt{-1}J / 2} \mathfrak{g} = \mathfrak{g}_\mathbb{C}
\]
Here we identified \( \mathfrak{g} \oplus \mathfrak{g} \cong \mathfrak{g}_C \) of complex Lie algebras. If we identify these two complex Lie algebras, then their subspaces correspond in the following way:

\[
\{(X, \theta X); X \in \mathfrak{g}\} \leftrightarrow \mathfrak{g}, \quad \Delta \mathfrak{g} := \{(X, X); X \in \mathfrak{g}\} \leftrightarrow u_C,
\]

\[
(5.3) \quad \nabla \mathfrak{g} := \{(X, -X); X \in \mathfrak{g}\} \leftrightarrow \{J \mu\mathfrak{c}, \quad \{(H, -H); H \in \mathfrak{h}\} \leftrightarrow \{(\mathfrak{h}_\mathbb{R})_C, \nabla \mathfrak{g} = \{J \mathfrak{c}\}
\]

(5.4) \quad \nabla \mathfrak{g} = \{J \mathfrak{c}\}

Here \( (\mathfrak{h}_\mathbb{R})^\perp \) is the orthogonal complement of \( \mathfrak{h}_\mathbb{R} \) in \( \mathfrak{h} \) and \( \mathfrak{n} = \theta \mathfrak{h} \). Extend the \( U \)-isomorphism \( \eta_0 : \mathfrak{g} \ni X \mapsto (-X, X) \in \nabla \mathfrak{g} = \{J \mu\mathfrak{c}, \) to the algebra isomorphism \( \eta_0 : S(\mathfrak{g}) \rightarrow S(\nabla \mathfrak{g}) \cong S(J \mathfrak{c}) \). Then the restriction of \( \eta_0 \) to \( S(\mathfrak{h}) \) gives an isomorphism \( S(\mathfrak{h}) \rightarrow S((\mathfrak{h}_\mathbb{R})_C) \). We denote its inverse by \( \chi_0 \). Clearly \( \chi_0 \) commutes with the \( W \)-actions. As a variation of the map \( \gamma_0 : S((\mathfrak{h}_\mathbb{R})_C) \rightarrow S((\mathfrak{h}_\mathbb{R})_C) \), define the map

\[
\tilde{\gamma}_0 := \chi_0 \circ \gamma_0 \circ \eta_0 : S(\mathfrak{g}) \rightarrow S(\mathfrak{h}).
\]

Then \( \tilde{\gamma}_0 \) induces the algebra isomorphism \( \hat{\gamma}_0 : S(\mathfrak{g})^G \rightarrow S(\mathfrak{h})^W \), by which we identify the two algebras and denote both of them by the same symbol \( \mathfrak{g} \). Note that by \( \text{S.3} \) \( \gamma_0 \) is reduced to the projection map

\[
S(\mathfrak{g}) = S(\mathfrak{h}) \oplus S(\mathfrak{g})(\mathfrak{n} + \mathfrak{n}) \rightarrow S(\mathfrak{h}).
\]

Now the result of \( \text{S.3} \) is generalized to the case of a quasi-small representation as follows:

**Theorem 5.4.** For a finite-dimensional irreducible representation \((\sigma, V)\) of \( G \), define the map

\[
\hat{\Gamma}_0^\sigma : \text{Hom}_G(V, S(\mathfrak{g})) \ni \Phi \mapsto \varphi \in \text{Hom}_W(V^\mathfrak{h}, S(\mathfrak{h}))
\]

so that the image \( \varphi \) is given by the composition

\[
\varphi : V^\mathfrak{h} \leftrightarrow V \Phi \rightarrow S(\mathfrak{g}) \xrightarrow{\hat{\gamma}_0} S(\mathfrak{h}).
\]

Then \( \hat{\Gamma}_0^\sigma \) is an injective \( \mathfrak{g}^\mathfrak{h} \)-homomorphism (clearly \( \text{Hom}_G(V, S(\mathfrak{g})) \) and \( \text{Hom}_W(V, S(\mathfrak{h})) \) have natural \( \mathfrak{g}^\mathfrak{h} \)-module structures). On the other hand, \( \hat{\Gamma}_0^\sigma \) is surjective if and only if \((\sigma, V)\) is small. Furthermore, for any \( \varphi \in \text{Hom}_W(V^\mathfrak{h}/V^\mathfrak{h} \text{ double}, S(\mathfrak{h})) \) there exists \( \Phi \in \text{Hom}_G(V, S(\mathfrak{g})) \) such that \( \hat{\Gamma}_0^\sigma(\Phi) = \varphi \).

**Proof.** The theorem follows immediately from the results of \( \text{S.3} \) and the fact that \( \hat{\Gamma}_0^\sigma \) coincides with the composition

\[
\text{Hom}_G(V, S(\mathfrak{g})) = \text{Hom}_G(V, (\mathfrak{g}(\mathfrak{g})) \xrightarrow{n_0^\omega} \text{Hom}_G(V, S((\mathfrak{h}_\mathbb{R})_C)) \xrightarrow{\hat{\Gamma}_0^\sigma} \text{Hom}_W(V^\mathfrak{h}, S((\mathfrak{h}_\mathbb{R})_C)) \xrightarrow{\chi_0^\omega} \text{Hom}_W(V^\mathfrak{h}, S(\mathfrak{h})). \tag{5.4}
\]

**Definition 5.5.** We define the map \( \tilde{\gamma} \) of \( U(\mathfrak{g}) \) into \( S(\mathfrak{h}) \) by the projection

\[
U(\mathfrak{g}) = (\mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}) \oplus U(\mathfrak{h}) \approx S(\mathfrak{h})
\]

followed by the translation

\[
S(\mathfrak{h}) \ni f(\lambda) \mapsto f(\lambda - \tilde{\rho}) \in S(\mathfrak{h}).
\]

Here we identified \( S(\mathfrak{h}) \) with the space of holomorphic polynomials on the (complex) dual space \( \mathfrak{h}^* \) of \( \mathfrak{h} \).

**Lemma 5.6.** For any \( D_1 \in U(\mathfrak{g}) \) and \( D_2 \in U(\mathfrak{g})^\mathfrak{h} \), \( \tilde{\gamma}(D_1D_2) = \tilde{\gamma}(D_2D_1) = \tilde{\gamma}(D_1)\tilde{\gamma}(D_2) \).
Proof. Let \( \tilde{D}_1 \) and \( \tilde{D}_2 \) be the images of \( D_1 \) and \( D_2 \) under the projection \([5,3]\), respectively. Since \( U(\varnothing) = \bar{n}U(n) + U(\varnothing) \) and \( \bar{n}U(n) \) as an \( ad(h) \)-module, we have \( \tilde{D}_2 \equiv (U(\varnothing) + (\bar{n}U(\varnothing) + U(\varnothing))^0) \) and \( \tilde{D}_2 D_1 \equiv (\bar{n}U(\varnothing) + U(\varnothing))^0 \). Hence \( \tilde{D}_2 D_1 \equiv \bar{n}U(\varnothing) \), and \( \bar{n} = \bar{n} \). Hence we get \( \tilde{D}_2 D_1 \equiv \bar{n}U(\varnothing) + U(\varnothing) \). \( \square \)

The isomorphism \( g \odot \varnothing \cong C \) induces the algebra isomorphism \( U(\varnothing) \odot U(\varnothing) \cong U(C) \), which clearly commutes with the \( U \)-actions. Define the map \( \eta : U(\varnothing) \ni D \mapsto 1 \odot D \in U(\varnothing) \odot U(\varnothing) \cong U(C) \). Then we obtain the direct sum decomposition

\[
U(C) = U(C) \odot U(\varnothing) \cong (U(\varnothing) \odot U(\varnothing) + U(C)) \cong (1 \odot U(\varnothing)) \]

as a \( U \)-module.

**Lemma 5.7.** Under the composition map

\[
U(\varnothing) \ni f \mapsto U(C) \ni f \mapsto S((b_\varnothing)_C) \ni S(h),
\]

the image of \( D = U(\varnothing) \in (\text{mod } \varnothing \odot \varnothing) \).

**Proof.** For \( D \in U(\varnothing) \) and \( X \in \varnothing \)

\[
\eta(DX) = 1 \odot DX = (1 \odot D) \cdot (1 \odot X + X \odot 1) - (X \odot 1) \cdot (1 \odot D).
\]

Hence by \([5,3]\), \( \eta(U(\varnothing)) \subset (1 \odot U(\varnothing)) \Delta \varnothing + (\n \odot \n) (1 \odot U(\varnothing)) \subset U(C) \odot U(\varnothing) \). Also, since \( \eta(\bar{n}U(n)) \equiv (\n \odot \n) (1 \odot U(\varnothing)) \subset \varnothing \subset U(C) \odot U(\varnothing) \), we have \( \gamma \circ \eta(\bar{n}U(n)) = \bar{n} \). On the other hand, if \( f \in S(h) \) and \( H \in \varnothing \), then \( \eta(\gamma \circ \eta(H)) = (\n \odot \n) \cdot (1 \odot f) \cdot (1 \odot \bar{n} \odot \n \odot \n) \cdot (1 \odot f) \equiv \eta(H) \eta(f) \) (mod \( U(C) \odot U(\varnothing) \)). Hence for any \( f(\lambda) \in S(h) \),

\[
\eta(f(\lambda)) \equiv \eta_{\varnothing}(f(\lambda)) \text{ (mod } U(\varnothing) \).
\]

In addition, by the correspondence

\[
\eta_{\varnothing}(H + \sum \varnothing) = (-H + \sum \varnothing), \quad \forall H, \sum \varnothing \in \varnothing,
\]

we have \( \eta_{\varnothing}(f(\lambda - 2\bar{n})) = (\eta_{\varnothing}(f(\lambda))) \) for \( f \in S(h) \). Therefore \([5,3]\) maps \( f \in S(h) \) in the following way:

\[
\begin{array}{c}
\eta(f(\lambda)) \mapsto \eta_{\varnothing}(f(\lambda)) \mapsto \eta_{\varnothing}(f(\lambda - 2\bar{n})) \mapsto \eta_{\varnothing}(f(\lambda - \bar{n})).
\end{array}
\]

Since this image is \( \bar{n}(f(\lambda)) \), we get the lemma. \( \square \)

By the multiplicity function \( \kappa : \Sigma \ni \alpha \mapsto -1 \in \mathbb{C} \), we define the degenerate affine Hecke algebra \( \hat{H}_\varnothing \) associated to the data \( b, \Xi, \kappa \) as in Definition \([1,4]\) and denote it simply by \( \hat{H} \).

The key relations in \( \hat{H} \) are

\[
s_\alpha : \xi = s_\alpha(\xi) \cdot s_\alpha(\alpha(\xi)) \quad \forall \alpha \in \Xi \quad \forall \xi \in b.
\]

As in the case of \( S(H((b_\varnothing)_C)) \equiv S((b_\varnothing)_C) \), the left \( \hat{H} \)-module

\[
S_{\hat{H}}(b) := \hat{H} / \sum_{w \in W \backslash (1')} \hat{H}(w - 1)
\]

is naturally identified with \( S(h) \) as a left \( S(h) \)-module and under this identification the space of \( W \)-fixed elements in \( S_{\hat{H}}(b) \) equals \( S(h)^W \).
Lemma 5.8. The map
\[ \tilde{S}_H(b_H) \approx S((b_H)_C) \xrightarrow{x_0} S(b) \ni f(\lambda) \mapsto f(2\lambda) \in S(b) \approx S_H(b) \]
commutes with the W-actions.

Proof. Let \( \alpha \in \Pi \) and put \( b_{H}(\alpha) = \{ H \in b_{R}; \alpha(H) = 0 \} \). Then from Lemma 4.5, the eigenspaces of \( S_{\alpha} \) in \( S_H((b_H)_C) \) are
\[ S(b_{H}(\alpha)_C) \cdot \mathbb{C}[\alpha^\vee (\alpha^\vee)^{2}], \quad S(b_{H}(\alpha)_C) \cdot \mathbb{C}[(\alpha^\vee)^{2}](\alpha^\vee + 2), \]
which have eigenvalues 1, \( -1 \), respectively. If we apply the map in the lemma to them, then by (5.7) their images are respectively
\[ S(b_{H}(\alpha)_C) \cdot \mathbb{C}[\alpha^\vee (\alpha^\vee)^{2}], \quad S(b_{H}(\alpha)_C) \cdot \mathbb{C}[\alpha^\vee (\alpha^\vee)^{2}](\alpha^\vee - 1), \]
where \( \mathbb{C}_J = \mathbb{R} \oplus \mathbb{R}J \). They, in turn, are shown to be the eigenspaces of \( S_{\alpha} \) in \( S_H(b) \) with eigenvalues 1, \( -1 \) by the same argument as the proof of Lemma 4.3.

Now in view of (5.8), Lemma 5.7 and Lemma 5.8, the results of §4 give the following generalization of the Harish-Chandra isomorphism:

Theorem 5.9. For a finite-dimensional irreducible representation \( (\sigma, V) \) of \( G \), define the map
\[ \tilde{\Gamma}^\sigma : \text{Hom}_{\mathbb{C}}(V, U(\mathfrak{g})) \ni \Psi \mapsto \psi \in \text{Hom}_{\mathbb{C}}(V_H^b, S_H(b)) \]
so that the image \( \psi \) is given by the composition
\[ \varphi : V_H^b \hookrightarrow V \xrightarrow{\Psi} \tilde{\gamma} \xrightarrow{\tilde{\gamma}^\sigma} S_H(b). \]
Then we have
(i) \( \tilde{\Gamma}^\sigma \) is injective.
(ii) For any \( \psi \in \text{Hom}_{\mathbb{C}}(V_H^b, S_H(b)) \) there exists \( \Psi \in \text{Hom}_{\mathbb{C}}(V, U(\mathfrak{g})) \) such that \( \tilde{\Gamma}^\sigma(\Psi) = \psi \).
(iii) \( \tilde{\Gamma}^\sigma(\text{Hom}_{\mathbb{C}}(V, U(\mathfrak{g}))) \subset \text{Hom}_{\mathbb{C}}(V_H^b, S_H(b)) \) if and only if \( (\sigma, V) \) is small. If this condition is satisfied, then from (i), (ii) we have the isomorphism
\[ \tilde{\Gamma}^\sigma : \text{Hom}_{\mathbb{C}}(V, U(\mathfrak{g})) \xrightarrow{\sim} \text{Hom}_{\mathbb{C}}(V_H^b, S_H(b)). \]
(iv) In particular, for the trivial representation \( (\text{triv}, \mathbb{C}) \), the map \( \tilde{\Gamma}^{\text{triv}} : \text{Hom}_{\mathbb{C}}(\text{triv}, U(\mathfrak{g})) \xrightarrow{\sim} \text{Hom}_{\mathbb{C}}(V_H^b, S_H(b)) \) is essentially equal to the classical Harish-Chandra isomorphism
\[ \tilde{\gamma} : U(\mathfrak{g})^G \xrightarrow{\sim} S(\mathfrak{h})^W. \]
Since it is an algebra isomorphism by Lemma 5.6, in addition to \( S(\mathfrak{g})^G \) and \( S(\mathfrak{h})^W \), we denote \( U(\mathfrak{g})^G \) also by \( \mathcal{A} \).
(v) For a general \( (\sigma, V) \), \( \text{Hom}_{\mathbb{C}}(V, U(\mathfrak{g})) \) and \( \text{Hom}_{\mathbb{C}}(V_H^b, S_H(b)) \) have the natural \( \mathcal{A} - \text{module structures which are intertwined by } \tilde{\Gamma}^\sigma \). Especially, if \( (\sigma, V) \) is small, then \( \tilde{\Gamma}^\sigma \) is an \( \mathcal{A} - \text{module isomorphism.} \)

In the rest of this section, we never refer to \( \mathfrak{g}_C \) and consider \( \mathfrak{g} \) itself to be defined over \( \mathbb{C} \) by letting \( J = \sqrt{-1} \). Let \( B(\cdot, \cdot) \) be the Killing form for the complex Lie algebra \( \mathfrak{g} \) and \( \langle \cdot, \cdot \rangle \) the bilinear form on \( b^* \times b^* \) induced by \( B(\cdot, \cdot) \). Note that \( B(\cdot, \cdot) = 2 \text{ Re } B(\cdot, \cdot) \). Clearly, each irreducible constituent of the adjoint representation \( (\text{Ad}, \mathfrak{g}) \) is small and \( g^b = \mathfrak{h} \).
Proposition 5.10. Let $(\sigma, V_\rho)$ be the finite-dimensional irreducible representation with highest weight $\rho$ and $(\sigma, V)$ an arbitrary irreducible quasi-small representation. Then $(\sigma, V)$ is isomorphic to an irreducible constituent of the $G$-module $\text{End} V_\rho \simeq V_\rho \otimes V_\rho$. Moreover, the multiplicity of $(\sigma, V)$ in $\text{End} V_\rho$ is

$$\dim_{\mathbb{C}}\{v \in V_\rho^h; \sigma(X_\alpha)v = 0 \quad \forall \alpha \in \Pi, \forall X_\alpha \in g_0\}.$$ 

Proof. For any finite-dimensional irreducible representation $(\sigma, V)$ of $G$, put $V_\rho(n) \equiv \{v \in V_\rho^h; \sigma(X_\alpha)v = 0 \quad \forall \alpha \in \Pi, \forall X_\alpha \in g_0\}$. Then the multiplicity of $(\sigma, V)$ in $\text{End} V_\rho$ equals $\dim_{\mathbb{C}} V_\rho(n)$ (Re1, 4.3). In particular, if $(\sigma, V)$ is quasi-small, then $V_\rho(n) \ni V_\rho^h \neq 0$. □

Let $\ell = \dim_{\mathbb{C}} h$. Then one has $\bigwedge g \simeq \left(\text{End} V_\rho\right)^{\bigwedge 2\ell}$ as $G$-modules (see Ko4 [Re1]). Hence each irreducible quasi-small representation appears also in $\bigwedge g$. Related to this, we have the following: For $k = 0, \ldots, \ell$, we consider the $G$-module $\bigwedge^k \text{Ad}$, $\bigwedge^k g$. Although it may be reducible, we define $V_\rho^h, V_\rho^{\text{single}}, V_\rho^{\text{double}}$, and $\check{\Gamma}$ as in the irreducible case. Observe that $\bigwedge^k h \subset V_\rho^{\text{single}}$. Let $B_k(\cdot, \cdot)$ be the unique $G$-invariant non-degenerate symmetric bilinear form on $V \times V$ such that

$$B_k(u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_k) = \det\left(B(u_i, v_j)\right)_{1 \leq i, j \leq k}.$$ 

Since $B_k(\cdot, \cdot)$ is non-degenerate both on $V^h \times V^h$ and on $\bigwedge^k h \times \bigwedge^k h$, we can define the orthogonal complement $(\bigwedge^k h)^\perp$ of $\bigwedge^k h$ in $V^h$ with respect to $B_k(\cdot, \cdot)$. It is easy to see $V_\rho^{\text{double}} \subset (\bigwedge^k h)^\perp$. Therefore it follows from Theorem 5.9 that for any $\psi \in \text{Hom}_W(V^h/\bigwedge^k h^\perp, S_R(\mathfrak{h}))$ there exists a unique $\Psi \in \text{Hom}_G(V, U(\mathfrak{g}))$ such that $\check{\Gamma}(\Psi) = \psi$. We denote the set of all such $\Psi$ by $\mathcal{M}_k$, in other words, we put

$$\mathcal{M}_k = \{\Psi \in \text{Hom}_G(V, U(\mathfrak{g})); \check{\Gamma}(\Psi)[v] = 0 \quad \forall v \in (\bigwedge^k h)^\perp\}.$$ 

Then $\mathcal{M}_k$ is an $\mathfrak{g}$-module $\text{Hom}_G(V, U(\mathfrak{g}))$.

Theorem 5.11. Define the $\mathfrak{g}$-homomorphism $\omega : \bigwedge^k \text{Hom}_G(\mathfrak{g}, U(\mathfrak{g})) \rightarrow \text{Hom}_G\left(\bigotimes^k \mathfrak{g}, U(\mathfrak{g})\right)$ so that the image of $\Psi_1 \wedge \cdots \wedge \Psi_k \in \bigwedge^k \text{Hom}_G(\mathfrak{g}, U(\mathfrak{g}))$ is given by

$$(5.9) \quad \omega(\Psi_1 \wedge \cdots \wedge \Psi_k) : \bigotimes^k \mathfrak{g} \ni X_1 \otimes \cdots \otimes X_k \mapsto \det\left(\Psi_i[X_j]\right)_{1 \leq i, j \leq k} \in U(\mathfrak{g}).$$ 

Here the symbol $\det$ in (5.9) stands for a so-called ‘column-determinant’, that is,

$$\det\left(\Psi_i[X_j]\right)_{1 \leq i, j \leq k} = \sum_{\mu \in \mathfrak{S}_k} (\text{sgn} \mu) \Psi_{\mu(1)}[X_1] \cdots \Psi_{\mu(k)}[X_k].$$ 

($\mathfrak{S}_k$ denotes the $k$-th symmetric group.) Then, for any $\Psi_1 \wedge \cdots \wedge \Psi_k \in \bigwedge^k \text{Hom}_G(\mathfrak{g}, U(\mathfrak{g}))$, $X_1, \ldots, X_k \in \mathfrak{g}$, and $\tau \in \mathfrak{S}_k$,

$$(5.10) \quad \omega(\Psi_1 \wedge \cdots \wedge \Psi_k)[X_{\tau(1)} \otimes \cdots \otimes X_{\tau(k)}] = (\text{sgn} \tau) \omega(\Psi_1 \wedge \cdots \wedge \Psi_k)[X_1 \otimes \cdots \otimes X_k].$$
By this, we consider $\omega(\Psi_1 \land \cdots \land \Psi_k) \in \text{Hom}_C(V, U(g))$. Then $\omega(\Psi_1 \land \cdots \land \Psi_k) \in M_k$ and moreover,

\begin{equation}
(5.11)
\Gamma(\omega(\Psi_1 \land \cdots \land \Psi_k))[H_1 \land \cdots \land H_k] = det\left(\Gamma(\omega(\Psi_i))[H_j]\right)_{1 \leq i,j \leq k} \quad \forall H_1, \ldots, H_k \in h.
\end{equation}

Here the right-hand side is the determinant of an $S(h)$-valued matrix. Furthermore, $\omega : \land^k \text{Hom}_C(g, U(g)) \to M_k$ is an $\mathcal{S}^\omega$-module isomorphism.

**Proof.** By an elementary argument, we can see (5.10) follows if we prove it for the special case where $k = 2$ and $\tau = (1, 2)$. Hence for a while we assume $k = 2$ and $\tau = (1, 2)$. In order to show (5.10), for any $\Psi_1, \Psi_2 \in \text{Hom}_C(g, U(g))$ define $\Psi \in \text{Hom}_C(g \otimes g, U(g))$ by

$$
\Psi[X_1 \otimes X_2] = \omega(\Psi_1 \land \Psi_2)[X_1 \otimes X_2 + X_2 \otimes X_1]
$$

and let us prove $\Psi = 0$. By Theorem 5.9, it suffices to show $\Gamma(\omega)(\Psi) = 0$. Assume $\Gamma(\omega)(\Psi) \neq 0$. First we note $(g \otimes g)^h = h \oplus \bigoplus_{\alpha \in \Lambda_0} g_{-\alpha} \oplus g_0$, and $\text{Hom}_C(V, U(g)) \cong \text{Hom}_C(V^h, U^h)$. Hence there exist $\alpha \in \Pi, X_{-\alpha} \in g_{-\alpha}$, and $X_\alpha \in g_0$ such that $\Gamma(\omega)(\Psi)[X_{-\alpha} \otimes X_\alpha] \neq 0$. Put

$$
s_{\alpha}(2, C) = g_{-\alpha} + C\alpha \gamma + g_0, \quad h(\alpha) = \{H \in h; \alpha(H) = 0\}, \quad \eta_\alpha = \sum_{\beta \in \Sigma^+ \setminus \{\alpha\}} g_\beta, \quad \bar{\eta}_\alpha = \theta \eta_\alpha
$$

and define the projection map

$$
\bar{\gamma}_\alpha : U(g) = (\bar{\eta}_\alpha U(g) + U(g)\eta_\alpha) \oplus U(h(\alpha) + s_{\alpha}(2, C)) \to U(h(\alpha) + s_{\alpha}(2, C)).
$$

Then $\gamma_\alpha$ is an $s_{\alpha}(2, C)$-homomorphism and $\bar{\gamma}_\alpha \circ \gamma_\alpha = \bar{\gamma}_\alpha$. Also, in a similar way to the proof of Lemma 5.6, we can show for any $D_1 \in U(g)$ and $D_2 \in U(g)^{h(\alpha)}$,

$$
\gamma_\alpha(D_1 D_2) = \gamma_\alpha(D_1) \gamma_\alpha(D_2), \quad \bar{\gamma}_\alpha(D_2 D_1) = \bar{\gamma}_\alpha(D_2) \bar{\gamma}_\alpha(D_1).
$$

Define the $s_{\alpha}(2, C)$-homomorphism $\Psi$ by

$$
s_{\alpha}(2, C) \otimes s_{\alpha}(2, C) \xhookrightarrow{\gamma_\alpha} U(g) \xrightarrow{\Psi} U(h(\alpha) + s_{\alpha}(2, C))
$$

and the $s_{\alpha}(2, C)$-homomorphism $\Psi$ by

$$
s_{\alpha}(2, C) \otimes s_{\alpha}(2, C) \xhookrightarrow{\gamma_\alpha} U(g) \xrightarrow{\Psi} U(h(\alpha) + s_{\alpha}(2, C)).
$$

Since $\Psi[j] \in U(g)^{h(\alpha)}$ for any $X_1, X_2 \in s_{\alpha}(2, C)$, we have

$$
(5.12) \quad \Psi[X_1 \otimes X_2] = \Psi_1[X_1] \Psi_2[X_2] - \Psi[G_1]\Psi_1^2[X_1] + \Psi_1^3[X_1][\Psi_1^2[X_1]] - \Psi_2^2[X_2]\Psi_1[X_1].
$$

Now from Kostant’s theorem (Ko1), the inclusion map $\iota^\alpha : s_{\alpha}(2, C) \xhookrightarrow{} U(h(\alpha) + s_{\alpha}(2, C))$ satisfies

$$
\text{Hom}_C(s_{\alpha}(2, C), U(h(\alpha) + s_{\alpha}(2, C))) = U(s_{\alpha}(2, C)) s_{\alpha}(2, C) \cdot U(h(\alpha)) \cdot \iota^\alpha.
$$
Hence there exist $Z_1, Z_2 \in U(\mathfrak{sl}_2(\mathbb{C}))^{\mathbb{C}(2,\mathbb{C})} \cdot U(\mathfrak{h}(\mathfrak{a}))$ such that $\Psi_1 = Z_1 \cdot t^a (i, j = 1, 2)$. Accordingly, the right-hand side of (5.12) becomes

$$Z_1 t^a[X_1] Z_2 t^a[X_2] - Z_2 t^a[X_1] Z_1 t^a[X_2] + Z_1 t^a[X_2] Z_2 t^a[X_1] - Z_2 t^a[X_2] Z_1 t^a[X_1]
= Z_1 Z_2 (t^a[X_1] t^a[X_2] - t^a[X_1] t^a[X_2] + t^a[X_2] t^a[X_1] - t^a[X_2] t^a[X_1]) = 0.$$  

In particular we have $\Gamma^{\text{Ad}^2}(\Psi)[X_{\omega} \otimes X_{\omega}] = \tilde{\gamma} \circ \Psi^a[X_{\omega} \otimes X_{\omega}] = 0$, a contradiction. Thus we get $\Gamma^{\text{Ad}^2}(\Psi) = 0$ and hence (5.10).

Suppose $k$ is general and $\Psi_1 \wedge \cdots \wedge \Psi_k \in \wedge^k \text{Hom}_G(\mathfrak{g}, U(\mathfrak{g}))$. Put $\mathfrak{g}_0 := \mathfrak{h}$. Take $a_1, \ldots, a_k \in \Sigma \cup \{0\}$ so that $a_1 + \cdots + a_k = 0$ and at least one $a_j$ is not 0. Also, take $X_{a_j} \in \mathfrak{g}_{a_j}$ for each $a_j$ and consider the element $X_{a_1} \wedge \cdots \wedge X_{a_k} \in \mathbb{V}^\Psi$. Then there exists at least one $j_0 = 1, \ldots, k$ such that $a_{j_0} \in \Sigma^+$. Since $\Psi_1[X_{a_{j_0}}] \in U(\mathfrak{g})(n)$ ($i = 1, \ldots, k$), (5.10) implies $\omega(\Psi_1 \wedge \cdots \wedge \Psi_k)[X_{a_1} \wedge \cdots \wedge X_{a_k}] \in U(\mathfrak{g})n$ and hence $\Gamma^\nu(\omega(\Psi_1 \wedge \cdots \wedge \Psi_k))[X_{a_1} \wedge \cdots \wedge X_{a_k}] = 0$. Since such $X_{a_1} \wedge \cdots \wedge X_{a_k}$ span $(\wedge^k \mathfrak{h})^\tau$, we get $\omega(\Psi_1 \wedge \cdots \wedge \Psi_k) \in \mathcal{M}_k$. Obviously (5.11) follows from Lemma 5.6.

Lastly the next lemma assures $\omega : \wedge^k \text{Hom}_G(\mathfrak{g}, U(\mathfrak{g})) \rightarrow \mathcal{M}_k$ is an isomorphism. □

Lemma 5.12. Define the $\mathcal{A}$-homomorphism $\omega_0 : \wedge^k \text{Hom}_W(\mathfrak{h}, S(\mathfrak{b})) \rightarrow \text{Hom}_W \left( \wedge^k \mathfrak{b}, S(\mathfrak{b}) \right)$ so that the image of $\varphi_1 \wedge \cdots \wedge \varphi_k$ is given by the map

$$\bigwedge^k \mathfrak{b} \ni H_1 \wedge \cdots \wedge H_k \mapsto \det \left( \varphi_i[H_j] \right)_{1 \leq i, j \leq k} \in S(\mathfrak{b}).$$

Then it is an isomorphism.

Proof. $\tilde{B}_k(\cdot, \cdot)$ induces the $W$-module isomorphism $(\wedge^k \mathfrak{b})^\ast \simeq \wedge^k \mathfrak{b}$. Hence we have the following natural $\mathcal{A}$-module isomorphisms: $\text{Hom}_W \left( \bigwedge^k \mathfrak{b}, S(\mathfrak{b}) \right) \simeq \left( \left( \wedge^k \mathfrak{b} \otimes S(\mathfrak{b}) \right)^W \right. \approx \left. \left( \wedge^k \mathfrak{b} \otimes S(\mathfrak{b}) \right)^W \approx \{W\text{-invariant polynomial coefficient } p\text{-form on } \mathfrak{b} \} \right)$. Suppose $I_1, \ldots, I_\ell$ are algebraically independent homogeneous elements of $\mathcal{A} = S(\mathfrak{b})^W$ and they constitute a generator system of $\mathcal{A}$. Then $[dI_1 \wedge \cdots \wedge dI_\ell]$ ($1 \leq i_1 < \cdots < i_\ell \leq \ell$) forms a basis of $\left( \bigwedge^k \mathfrak{b} \otimes S(\mathfrak{b}) \right)^W$ over $\mathcal{A}$ (ω). In particular, $[dI_i]$ ($i = 1, \ldots, \ell$) is a basis of $(\mathfrak{h} \otimes S(\mathfrak{b}))^W \approx \text{Hom}_W(\mathfrak{h}, S(\mathfrak{b}))$ over $\mathcal{A}$. It is easy to check under the identification $\text{Hom}_W \left( \bigwedge^k \mathfrak{b}, S(\mathfrak{b}) \right) \approx \left( \bigwedge^k \mathfrak{b} \otimes S(\mathfrak{b}) \right)^W$, the image $\omega_0(dI_1 \wedge \cdots \wedge dI_\ell) \in \text{Hom}_W \left( \bigwedge^k \mathfrak{b}, S(\mathfrak{b}) \right)$ of $dI_1 \wedge \cdots \wedge dI_\ell \in \bigwedge^k \text{Hom}_W(\mathfrak{b}, S(\mathfrak{b}))$ equals $dI_1 \wedge \cdots \wedge dI_\ell \in \left( \bigwedge^k \mathfrak{b} \otimes S(\mathfrak{b}) \right)^W$. □

To find all the equivalence classes of irreducible quasi-small representations and to determine the $W$-module structure of $V_{\text{sing}}$ for each irreducible quasi-small representation $(\sigma, V)$ are both important problems. The next lemma seems to be a help to solving them:

Lemma 5.13. Suppose $\Omega_{\mathfrak{g}} \in U(\mathfrak{g})$ is the Casimir element of $\mathfrak{g}$. That is, if we choose $X_\alpha \in \mathfrak{g}_\alpha$ for each $\alpha \in \Sigma$ so that $B(X_\alpha, X_{-\alpha}) = 1$ and if we take a basis $\{H_1, \ldots, H_\ell\}$ of $\mathfrak{h}$ so that $B(H_i, H_j) = \delta_{ij}$, then

$$\Omega_{\mathfrak{g}} = \sum_{i=1}^\ell H_i^2 + \sum_{\alpha \in \Sigma^+} (X_\alpha X_{-\alpha} + X_{-\alpha} X_\alpha).$$

Define the following central element of $\mathbb{C}[W]$:

$$\Omega_W = \sum_{\alpha \in \Sigma^+} \langle \alpha, \alpha \rangle (1 - s_\alpha).$$
Suppose $\sigma$, $V$ is an irreducible quasi-small representation of $G$ and its highest weight is $\lambda$. Then for any $v \in V_{\text{single}}$

$$\Omega_\sigma v = \langle \lambda, \lambda + 2\hat{\rho} \rangle v = \Omega_\rho v.$$  

Proof. Since $\Omega_\sigma$ acts on $V$ by a scalar, we get the first equality of (5.13) by calculating the action of $\Omega_\sigma$ on a highest weight vector of $V$. To show the second equality, take an arbitrary $\alpha \in \Sigma^+$ and define $sl_\alpha(2, \mathbb{C})$ to be the three-dimensional simple subalgebra spanned by $(X_\alpha, \alpha^\vee, X_{-\alpha})$. If we consider $U(sl_\alpha(2, \mathbb{C}))v$ as an $sl_\alpha(2, \mathbb{C})$-module, then from (5.1) each irreducible constituent of $U(sl_\alpha(2, \mathbb{C}))v$ is isomorphic either to the trivial representation or to the adjoint representation. Hence if we put $v_0 := \frac{1 + \alpha}{2}v$, $v_1 := \frac{1 - \alpha}{2}v$, then $sl_\alpha(2, \mathbb{C})$ acts trivially on $v_0$. Thus $(X_\alpha X_\alpha + X_{-\alpha} X_{-\alpha})v_0 = 0$. On the other hand, if $v_1 \neq 0$, then there exists an isomorphism from $sl_\alpha(2, \mathbb{C})v_1$ to $sl_\alpha(2, \mathbb{C})$ such that $v_1 \mapsto \alpha^\vee$. Since

$$\text{ad}(X_\alpha X_{-\alpha} + X_{-\alpha} X_\alpha)\alpha^\vee = 2\text{ad}(X_\alpha)X_{-\alpha} - 2\text{ad}(X_{-\alpha})X_\alpha = 4[X_\alpha, X_{-\alpha}] = 2\langle \alpha, \alpha \rangle \alpha^\vee,$$

we get $(X_\alpha X_{-\alpha} + X_{-\alpha} X_\alpha)v_1 = 2\langle \alpha, \alpha \rangle v_1$. Therefore the second equality of (5.13) holds. \hfill \Box

Example 5.14. Suppose $\mathfrak{g}$ is the complex simple Lie algebra of type $(B_2)$. As usual, take a basis $\{e_1, e_2\}$ of $\mathfrak{h}^\perp$ so that $\Sigma^+ = \{e_1 \pm e_2, e_1, e_2\}$ and $\langle e_i, e_j \rangle = \frac{1}{2}\delta_{ij}$. In addition to the equivalence class of the trivial representation ‘triv’, that of the reflection representation ‘ref’, and that of the sign representation ‘sgn’, we have two other equivalence classes of irreducible representations of $W$: One is the class of the one-dimensional representation $\tau$ which takes the value $\tau(w) = 1$ or $-1$ ($w \in W$) according as the number of appearances of the reflections associated to short roots $(\pm e_1, \pm e_2)$ is even or odd when we express $w$ as a product of reflections. The other is the class of $\tau \otimes \text{sgn}$, which behaves similarly for the long roots. On each irreducible representation of $W$, $\Omega_\tau$ acts by the scalar whose value is indicated in Table 5.1.

| rep. | triv | ref | sgn | $\tau$ | $\tau \otimes \text{sgn}$ |
|------|------|-----|-----|-------|-----------------|
| $\Omega_\tau$ | 0 | 1 | 2 | $\frac{1}{2}$ | $\frac{1}{2}$ |

| rep. | $\sigma_{(0,1)}$, $\sigma_{(1,0)}$, $\sigma_{(1,1)}$, $\sigma_{(2,1)}$ |
|------|----------------------------------|
| $\Omega_\sigma$ | 0 | $\frac{1}{2}$ | 1 | 2 |

The set of the highest weights of all finite-dimensional irreducible representations of $G$ is $\{(i + j)e_1 + je_2; i, j \in \mathbb{Z}_{\geq 0}\}$. Let $\sigma_{(i,j)}$ denote the finite-dimensional irreducible representation of $G$ with highest weight $ie_1 + je_2$. Note that $2\tilde{\rho} = 3e_1 + e_2$ and that $\langle \lambda, \lambda + 2\tilde{\rho} \rangle = \langle \lambda + \tilde{\rho}, \lambda + \tilde{\rho} \rangle - \langle \tilde{\rho}, \tilde{\rho} \rangle$. Then we easily recognize a representation $\sigma_{(i,j)}$ for which the value of $\sigma_{(i,j)}(\Omega_\sigma)$ coincides with one of the values of $\Omega_\rho$ in Table 5.1 is isomorphic to one of those representations in Table 5.2.

Among the irreducible representations in Table 5.2, $\sigma_{(0,0)}$, $\sigma_{(1,0)}$, and $\sigma_{(1,1)}$ are small (cf. Figure 5.1). Hence by Lemma 5.13, their 0-weight spaces have neither sgn nor $\tau \otimes \text{sgn}$ as a constituent. On the other hand, $\sigma_{(2,1)}$ is not small because it has the weight $2e_1$. But since the $W$-module $\bigwedge^2 \mathfrak{h}$ is isomorphic to sgn, the argument before Theorem 5.11 implies that there is an irreducible quasi-small submodule $V$ of $\bigwedge^2 \mathfrak{g}$ such that sgn appears in $V_{\text{single}}^0$ as a constituent. From Lemma 5.13, Table 5.1, and Table 5.2, we conclude this $V$ must be isomorphic to $\sigma_{(2,1)}$. 

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\[ \sigma_{(2,1)} \] is an example of an irreducible representation of \( G \) which is quasi-small but not small. \( \tau \otimes \text{sgn} \) is an example of an irreducible representation of \( W \) which does not appear in \( V^\text{single} \) as a constituent for any irreducible quasi-small representation \( (\sigma, V) \) of \( G \).

Suppose \( \lambda \in \mathfrak{h}^* \). In the rest of this section, we apply Theorem 5.9 to the study of a new relation we shall establish between a basic \( A \) and the Verma module \( \alpha \) of \( \mathfrak{h} \).

**Definition 5.15.** We introduce the correspondence \( \Xi_A : \{ \text{\( \tilde{H} \)}-submodule of \( A(\lambda) \) \} \rightarrow \{ \text{\( \tilde{U} \)}(\mathfrak{g})\)-submodule of \( M(\lambda) \) \} \) defined by

\[ S \mapsto \sum_{H \in \Xi} \left\{ \text{\( VM(\lambda) ; V \subset U(\mathfrak{g}) \) is \( \text{\( \tilde{u} \)}(\mathfrak{g}) \)-stable and satisfies \( \tilde{g}(V)A(\lambda) \subset S \) } \right\}, \]

and the correspondence \( \Theta_A : \{ \text{\( \tilde{U} \)}(\mathfrak{g})\)-submodule of \( M(\lambda) \) \} \rightarrow \{ \text{\( \tilde{H} \)}-submodule of \( A(\lambda) \) \} defined by

\[ T \mapsto \sum_{H \in \Theta} \left\{ \text{\( \tilde{H} \tilde{g}(V)A(\lambda) ; V \subset U(\mathfrak{g}) \) is \( \text{\( \tilde{u} \)}(\mathfrak{g}) \)-stable and satisfies \( \tilde{V}M(\lambda) \subset T \) } \right\}. \]

Obviously they are well-defined and preserve any inclusion relation.

**Proposition 5.16.** \( \Xi_A(A(\lambda)) = M(\lambda) \) and \( \Xi_A(0) = 0 \). Let \( S \) be an arbitrary \( \text{\( \tilde{H} \)}\)-submodule of \( A(\lambda) \). Then \( Y_A \circ \Xi_A(S) \subset S \). Moreover, suppose \( S \) admits a \( \text{\( \tilde{W} \)}\)-stable subspace \( E \) with the following properties: (a) \( S = \text{\( \tilde{W} \)}E \); (b) as a \( \text{\( \tilde{U} \)}(\mathfrak{g}) \)-module, each irreducible constituent of \( E \) belongs to \( \text{\( \tilde{W} \)}\text{\( \tilde{u} \)}(\mathfrak{g}) \text{-module} \), where \( \text{\( \tilde{W} \)}\text{\( \tilde{u} \)}(\mathfrak{g}) \) denotes the set of equivalence classes of those irreducible representations of \( W \) which appear in \( V^\text{single} \) for some irreducible quasi-small representation \( (\sigma, V) \) of \( G \). Then \( Y_A \circ \Xi_A(S) = S \).

**Proof.** It is clear that \( \Xi_A(A(\lambda)) = M(\lambda) \). To show \( \Xi_A(0) = 0 \), let \( T_0 \) be the unique irreducible \( U(\mathfrak{g})\)-submodule of \( M(\lambda) \). Then \( T_0 \) is isomorphic to \( M(w_0) \) for some \( w_0 \in W \). Fix a highest weight vector \( v_0 \) of \( T_0 \).

Suppose an \( \text{\( \tilde{u} \)}(\mathfrak{g}) \)-stable subspace \( V \) satisfies \( \tilde{g}(V)A(\lambda) = 0 \). Then by the definition \( (5.14) \) of \( A(\lambda) \), \( \tilde{g}(V) \subset \sum_{f \in S^\text{\( \tilde{u} \)}(\mathfrak{g})} S(b) (f - f(\lambda)) = \sum_{f \in S^\text{\( \tilde{u} \)}(\mathfrak{g})} S(b) (f - f(w_0, \lambda)) \). By using the
direct sum decomposition \( U(\mathfrak{g}) = \bar{n}U(\bar{n} + \mathfrak{h}) \oplus U(\mathfrak{h}) \oplus U(\mathfrak{g})n \), we have \( V_{T_0} \subset (\bar{\gamma}(V)(w_0) + \bar{n}U(\bar{n})w_0) = \bar{n}U(\bar{n})w_0 \) and hence \( VT_0 = VU(\bar{n})w_0 = U(\bar{n})Vw_0 \subset \bar{n}U(\bar{n})w_0 \subset T_0 \). Since \( VT_0 \) is a \( U(\mathfrak{g}) \)-submodule of \( T_0 \), we get \( VT_0 = 0 \). But from Duflo’s theorem \([Duf]\), \( \text{Ann}(M(\lambda)) = \text{Ann} M(w_0 \lambda) = \sum_{D \in U(\mathfrak{g})} U(\mathfrak{g})(D - \bar{\gamma}(D)(\lambda)) \). Hence we get \( V \subset \text{Ann} M(w_0 \lambda) = \text{Ann}(M(\lambda)) \), or equivalently \( VM(\lambda) = 0 \). It proves \( \Xi_4(0) = 0 \).

Secondly, let \( S \) be an arbitrary \( \mathfrak{H} \)-submodule of \( A(\lambda) \). Put

\[
I := \sum \{ V \subset U(\mathfrak{g}) ; V \text{ is ad}(\mathfrak{g})\text{-stable and satisfies } \bar{\gamma}(V)A(\lambda) \subset S \}.
\]

Then

\[
\Xi_4(S) = IM(\lambda).
\]

We assert \( I \) is a two-sided ideal of \( U(\mathfrak{g}) \). Indeed, \( \mathfrak{g}I \) is an ad(\( \mathfrak{g} \))-stable subspace of \( U(\mathfrak{g}) \) and satisfies \( \mathfrak{g}I \subset (\bar{n}I + \mathfrak{h}n) + \text{ad}(\mathfrak{h})I + \mathfrak{h}I \subset (\bar{n}I + \mathfrak{h}n) + I + \mathfrak{h}I \), from which we easily deduce \( \bar{\gamma}(\mathfrak{g}I) \subset \bar{\gamma}(I) + \mathfrak{h}\bar{\gamma}(I) \). Since \( (\bar{\gamma}(I) + \mathfrak{h}\bar{\gamma}(I))A(\lambda) \subset S \), \( \mathfrak{g}I \subset I \). Similarly we can show \( I \Gamma \subset I \). Thus \( I \) is a two-sided ideal. Moreover, since \( \bar{\gamma}(\text{Ann}(M(\lambda))) = \sum_{f \in U(\mathfrak{g})} \text{Ann}(M(\lambda))(f - f(\lambda)) \), \( I \subset \text{Ann}(M(\lambda)) \). Hence from \([Jos, BG]\), we get \( I = \text{Ann}(M(\lambda)/IM(\lambda)) \) and therefore

\[
(5.15) \quad Y_d(IM(\lambda)) = \mathfrak{H}\bar{\gamma}(\text{Ann}(M(\lambda)/IM(\lambda)))A(\lambda) = \mathfrak{H}\bar{\gamma}(I)A(\lambda) \subset S.
\]

Lastly, we assume the above \( S \) admits a \( W \)-stable subspace \( E \) which satisfies the conditions (a) and (b) in the proposition. Let \( E_1 \subset E \) be an irreducible \( W \)-module. Then there exists a \( W \)-stable subspace \( \bar{E}_1 \) in \( S_{\mathfrak{H}}(\mathfrak{h}) \) which is isomorphic to \( E_1 \) via the natural surjective \( W \)-homomorphism \( S_{\mathfrak{H}}(\mathfrak{h}) \to A(\lambda) \). By the condition (b), the equivalence class of \( \bar{E}_1 \) belongs to \( \bar{W}_{\text{single}} \). Hence it follows from Theorem \([Jos, BG]\) that there exists an ad(\( \mathfrak{g} \))-submodule \( V \) of \( U(\mathfrak{g}) \) which is isomorphic to an irreducible quasi-small representation of \( G \) and satisfies \( \bar{\gamma}(V) = \bar{E}_1 \). Since \( \bar{\gamma}(V)A(\lambda) = \bar{S}(\mathfrak{h})E_1 \subset S \), the above \( I \) satisfies \( I \subset V \) and therefore \( \bar{\gamma}(I)A(\lambda) \supset E_1 \). Hence by the condition (a), we can replace the last inclusion relation in \((5.15)\) with ‘=’. Thus we get \( Y_d \circ \Xi_4(S) = S \).

\[\square\]

**Corollary 5.17.** Suppose \( \mathfrak{g} \) is a complex simple Lie algebra of type \( (A) \). Then \( \Xi_4 \) is injective and it holds that \( Y_d \circ \Xi_4(S) = S \) for any \( \mathfrak{H} \)-submodule \( S \) of \( A(\lambda) \).

**Proof.** Under the assumption that \( \mathfrak{g} \) is of type \( (A) \), all the equivalence classes of irreducible representations of \( W \) belong to \( \bar{W}_{\text{single}} \) \([Br]\).

\[\square\]

**Remark 5.18.** If \( A(\alpha^\vee) \neq 1 \) for all \( \alpha \in \Sigma \), then \( A(\lambda) \) is irreducible (cf. \[Ch\]). Hence \( \Xi_4 \) is not necessarily surjective.

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