Quadratic metric-affine gravity

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Abstract

We consider spacetime to be a connected real 4-manifold equipped with a Lorentzian metric and an affine connection. The 10 independent components of the (symmetric) metric tensor and the 64 connection coefficients are the unknowns of our theory. We introduce an action which is quadratic in curvature and study the resulting system of Euler–Lagrange equations. In the first part of the paper we look for Riemannian solutions, i.e. solutions whose connection is Levi-Civita. We find two classes of Riemannian solutions: 1) Einstein spaces, and 2) spacetimes with metric of a pp-wave and parallel Ricci curvature. We prove that for a generic quadratic action these are the only Riemannian solutions. In the second part of the paper we look for non-Riemannian solutions. We define the notion of a “Weyl pseudoinstanton” (metric compatible spacetime whose curvature is purely Weyl) and prove that a Weyl pseudoinstanton is a solution of our field equations. Using the pseudoinstanton approach we construct explicitly a non-Riemannian solution which is a wave of torsion in Minkowski space. We discuss the possibility of using this non-Riemannian solution as a mathematical model for the graviton or the neutrino.

Key words: Yang–Mills equation, instanton, gravity, torsion

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1 Mathematical model

We consider spacetime to be a connected real 4-manifold $M$ equipped with a Lorentzian metric $g$ and an affine connection $\Gamma$. The 10 independent components of the (symmetric) metric tensor $g_{\mu\nu}$ and the 64 connection coefficients $\Gamma^\lambda_{\mu\nu}$ are the unknowns of our theory. This approach is known as metric-affine gravity. Its origins lie in the works of authors such as É. Cartan, A.S. Eddington, A. Einstein, T. Levi-Civita, E. Schrödinger and H. Weyl. A review of the more recent work in this area can be found in [8].

We define our action as

$$ S := \int q(R) $$

where $q$ is an $O(1, 3)$-invariant quadratic form on curvature $R$. Independent variation of the metric $g$ and the connection $\Gamma$ produces Euler–Lagrange equations which we will write symbolically as

$$ \frac{\partial S}{\partial g} = 0, $$

$$ \frac{\partial S}{\partial \Gamma} = 0. $$

Our objective is the study of the combined system of field equations (2), (3). This is a system of $10 + 64$ real nonlinear partial differential equations with $10 + 64$ real unknowns.

Our motivation comes from Yang–Mills theory. The Yang–Mills action for the affine connection is a special case of (1) with

$$ q(R) = q_{YM}(R) := R^\kappa_{\lambda\mu\nu} R^\lambda_{\kappa\mu\nu}. $$

With this choice of $q$ equation (3) is the Yang–Mills equation for the affine connection. There is a substantial bibliography devoted to the study of the system (2), (3) in the special case (4); see, for example, references in [17]. Without going into the details of the historical development of the subject let us mention the contributions of C.N. Yang [19] and E.W. Mielke [11] who showed, respectively, that Einstein spaces satisfy equations (3) and (2).

The idea of using a quadratic action in General Relativity goes back to H. Weyl, see end of his paper [18]. Weyl also pointed out that such an action should contain all possible invariant quadratic combinations of curvature, say, the square of Ricci curvature, the square of scalar curvature, etc. It turns out (see Appendix A) that in the metric-affine setting curvature has 11 irreducible pieces. There are (see Appendix B) 16 ways of squaring these irreducible pieces to a scalar. The reason why the number of different quadratic combinations is greater than the number of irreducible pieces is that some of the irreducible
pieces are isomorphic. The general formula for an O(1, 3)-invariant quadratic form on curvature is given in Lemma B.1.

**Definition 1.1** We call a spacetime \( \{M, g, \Gamma\} \) Riemannian if the connection is Levi-Civita (i.e. \( \Gamma^\lambda_{\mu\nu} = \{^\lambda_{\mu\nu}\} \)), and non-Riemannian otherwise.

**Remark 1.1** The word “Riemannian” has a different meaning in mathematics and theoretical physics. In mathematical literature the connection is usually Levi-Civita by default and “Riemannian” indicates that the metric is definite, whereas in theoretical physics literature the metric is usually Lorentzian by default (as it is in our paper) and “Riemannian” indicates that the connection is Levi-Civita. In Definition 1.1 we adopt the theoretical physics terminology.

**Remark 1.2** We call definite metrics “Euclidean”, and non-degenerate metrics of arbitrary signature “pseudo-Euclidean”.

The aim of this paper is to study the field equations (2), (3), so as to find

- all Riemannian solutions, and
- some non-Riemannian solutions.

The paper has the following structure. In Section 3 we write down explicitly the field equations (2), (3) in the Riemannian case. In Sections 4–6 we construct three types of Riemannian solutions. In Section 7 we prove a uniqueness theorem stating that for a generic quadratic action solutions from Sections 4–6 are the only Riemannian solutions; this uniqueness theorem is the main result of our paper. In subsequent sections we look for non-Riemannian solutions, and succeed (Section 10) in constructing explicitly one particular non-Riemannian solution. We discuss our results in Section 11. Finally, Appendices A–C contain statements and proofs of some auxiliary mathematical facts.

## 2 Notation

Our notation follows [9,17]. In particular, we denote local coordinates by \( x^\mu, \mu = 0, 1, 2, 3 \), and write \( \partial_\mu := \partial/\partial x^\mu \). We define the covariant derivative of a vector function as \( \nabla_\mu v^\lambda := \partial_\mu v^\lambda + \Gamma^\lambda_{\mu\nu} v^\nu \), torsion as \( T^\lambda_{\mu\nu} := \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \), curvature as \( R^\kappa_{\lambda\mu\nu} := \partial_\mu \Gamma^\kappa_{\nu\lambda} - \partial_\nu \Gamma^\kappa_{\mu\lambda} + \Gamma^\kappa_{\mu\eta} \Gamma^\eta_{\nu\lambda} - \Gamma^\kappa_{\nu\eta} \Gamma^\eta_{\mu\lambda} \), Ricci curvature as \( \text{Ric}_{\lambda\nu} := R^\lambda_{\kappa\lambda\nu} \), scalar curvature as \( \mathcal{R} := \text{Ric}^\lambda_{\lambda} \), and trace-free Ricci curvature as \( \mathcal{R}_{\text{ic}} := \text{Ric}_{\lambda\nu} - \frac{1}{2} g_{\lambda\nu} \mathcal{R} \). We denote Weyl curvature by \( \mathcal{W} = R^{(10)} \) (see also Appendix A). Given a scalar function \( f : M \to \mathbb{R} \) we write for brevity \( \int f := \int_M f \sqrt{|\det g|} \, dx^0 \, dx^1 \, dx^2 \, dx^3 \) where \( \det g := \det(g_{\mu\nu}) \). The totally antisymmetric quantity is denoted by \( \varepsilon_{\kappa\lambda\mu\nu} \). The Christoffel symbol is \( \{^\lambda_{\mu\nu}\} := \frac{1}{2} g^{\lambda\kappa} (\partial_\mu g_{\kappa\nu} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu}) \).
3 Field equations in the Riemannian case

When looking for Riemannian solutions we need to specialise our field equations (2), (3) to the Levi-Civita connection. We will write the resulting equations symbolically as

$$\left. \frac{\partial S}{\partial g} \right|_{L-C} = 0, \quad \left. \frac{\partial S}{\partial \Gamma} \right|_{L-C} = 0.$$  \hspace{1cm} (5)

It is important to understand the logical sequence involved in the derivation of equations (5), (6): we set $\Gamma^\lambda_{\mu \nu} = \{^\lambda_{\mu \nu}\}$ after the variations of the metric and the connection have been carried out.

Equations (5), (6) are equations for the unknown metric in the usual, Riemannian, setting. In the Riemannian case curvature has only 3 irreducible pieces, so the LHS’s of (5), (6) can be expressed via scalar curvature $\mathcal{R}$, trace-free Ricci curvature $\mathcal{Ric}$, and Weyl curvature $\mathcal{W}$. Lengthy but straightforward calculations give the following explicit representation for equations (5), (6):

$$d_1 \mathcal{W}^{\kappa \lambda \mu \nu} \mathcal{R}^{\kappa \lambda \mu \nu} + d_2 \mathcal{R} \mathcal{R}^{\lambda \nu} + d_3 \left( \mathcal{R}^{\lambda \kappa} \mathcal{R}^{\kappa \nu} - \frac{1}{4} g^{\lambda \nu} \mathcal{R}^{\kappa \mu} \mathcal{R}^{\kappa \mu} \right) = 0, \quad \text{(7)}$$

$$d_4 g_{\kappa \mu} \partial_\lambda \mathcal{R} - d_5 g_{\lambda \mu} \partial_\kappa \mathcal{R} + d_6 \nabla_\lambda \mathcal{R}^{\kappa \mu} - d_7 \nabla_\kappa \mathcal{R}^{\lambda \mu} = 0, \quad \text{(8)}$$

where

$$d_1 = b_{912} - b_{922} + b_{10}, \quad d_2 = -b_1 - \frac{b_{911}}{4} + \frac{b_{912}}{6} + \frac{b_{922}}{12}, \quad d_3 = b_{922} - b_{911},$$

$$d_4 = -b_1 + \frac{b_{912} - b_{922}}{4} + \frac{b_{10}}{12}, \quad d_5 = -b_1 + \frac{b_{912} - b_{911}}{4} + \frac{b_{10}}{12},$$

$$d_6 = b_{912} - b_{911} + b_{10}, \quad d_7 = b_{912} - b_{922} + b_{10}, \quad \text{(9)}$$

the $b$’s being the coefficients from formula (B.3). Observe that the LHS of (7) is trace-free. This is a consequence of the conformal invariance of our action (1), see also Remark 1.1 in [17].

The LHS’s of equations (7) and (8) are the components of the tensors $A$ and $B$ from the formula $\delta S = \int (2A^{\lambda \nu} \delta g_{\lambda \nu} + 2B^{\kappa \mu \lambda} \delta \Gamma^\lambda_{\mu \nu})$. Here $\delta g$ and $\delta \Gamma$ are the (independent) variations of the metric and the connection, and $\delta S$ is the resulting variation of the action. In (8) we lowered the first two indices of $B$ to make the expression easier to read. The same conventions were used in equations (25) and (26) of [17].

In deriving explicit formulae for tensors $A$ and $B$ we simplified our calculations.
by adopting the following argument. Formula (B.3) can be rewritten as

\[ q(R) = b_1 R^2 + \sum_{l,m=1}^2 b_{9lm}(S^{(l)}, S^{(m)}) + b_{10}(R^{(10)}, R^{(10)})_{YM} + \ldots \]

\[ = b_1 R^2 + \sum_{l,m=1}^2 b_{9lm}(Ric^{(l)}, Ric^{(m)}) + b_{10}(R^{(10)}, R^{(10)})_{YM} + \ldots \]

where by \ldots we denote terms which do not contribute to \( \delta S \) when we start our variation from a Riemannian spacetime. Recall that the \( Ric^{(l)} \) are defined in accordance with (A.5). Put

\[ Ric_\pm = \frac{Ric^{(1)} \pm Ric^{(2)}}{2}, \quad Ric_\pm = Ric_\pm - \frac{1}{4} g \text{tr} Ric_\pm = \frac{Ric^{(1)} \pm Ric^{(2)}}{2}. \]

Note that the tensor \( Ric_\pm \) is trace-free, and that in the Riemannian case we get \( Ric_+ = 0, Ric_- = Ric \). Our quadratic form can now be rewritten as

\[ q(R) = b_1 R^2 + b_{10}(R^{(10)}, R^{(10)})_{YM} + (b_{911} - 2b_{912} + b_{922})(Ric_-, Ric_-) \]

\[ + 2(b_{911} - b_{922})(Ric, Ric_+) + \ldots \]

\[ = \sum_{j=1}^3 c_j(R^{(j)}, R^{(j)})_{YM} + 2(b_{911} - b_{922})(Ric, Ric_+) + \ldots \]

where

\[ c_1 = -\frac{1}{2}(b_{911} - 2b_{912} + b_{922}), \quad c_2 = -6b_1, \quad c_3 = b_{10}, \quad (10) \]

and the \( R^{(j)} \)'s are the irreducible pieces of curvature labelled in accordance with [17]; note that the labelling of irreducible pieces in [17] differs from that in the current paper. The variation of \( \int \sum_{j=1}^3 c_j(R^{(j)}, R^{(j)})_{YM} \) is given by the LHS’s of formulae (25), (26) of [17]. Thus, the problem reduces to computing the variation of \( \int (Ric, Ric_+) \). The latter turns out to be

\[ \delta \int (Ric, Ric_+) = \int \left( \frac{1}{2} W^\kappa_\lambda_\mu_\nu \text{Ric}_{\kappa_\mu} - \frac{1}{6} R \text{Ric}^\lambda_\nu \right) \left( \text{Ric}^\lambda_\kappa \text{Ric}^{\lambda_\kappa_\nu} - \frac{1}{4} g^{\lambda_\nu} \text{Ric}_{\kappa_\mu} \text{Ric}_{\kappa_\mu} \right) \delta g_{\lambda_\nu} \]

\[ + \int \left( -\frac{1}{8} (g_{\kappa_\mu} \partial_\lambda \text{Ric} + g_{\lambda_\mu} \partial_\kappa \text{Ric}) - \frac{1}{2} (\nabla_\lambda \text{Ric}_{\kappa_\mu} + \nabla_\kappa \text{Ric}_{\lambda_\mu}) \right) \delta \Gamma^{\lambda_\mu_\kappa}. \]

Consequently, the constants \( d_1, \ldots, d_7 \) appearing in (7), (8) are expressed via
the constants $c_1, c_2, c_3$ and $b_{911} - b_{922}$ as

$$d_1 = c_1 + c_3 + \frac{b_{911} - b_{922}}{2}, \quad d_2 = \frac{c_1 + c_2 - b_{911} + b_{922}}{6}, \quad d_3 = b_{922} - b_{911},$$

$$d_4 = \frac{c_1}{4} + \frac{c_2}{6} + \frac{c_3}{12} + \frac{b_{911} - b_{922}}{8}, \quad d_5 = \frac{c_1}{4} + \frac{c_2}{6} + \frac{c_3}{12} - \frac{b_{911} - b_{922}}{8},$$

$$d_6 = c_1 + c_3 - \frac{b_{911} - b_{922}}{2}, \quad d_7 = c_1 + c_3 + \frac{b_{911} - b_{922}}{2}. \quad (11)$$

Substituting (10) into (11) we arrive at (9).

4 Riemannian solutions of type 1

**Definition 4.1** An Einstein space is a Riemannian spacetime with $\text{Ric} = \Lambda g$ where $\Lambda$ is some real “cosmological” constant.

For an Einstein space $\partial \mathcal{R} \equiv 0$ and $\mathcal{R}ic \equiv 0$, so equations (7), (8) are clearly satisfied. We call Einstein spaces *Riemannian solutions of type 1*.

5 Riemannian solutions of type 2

A metric of the form

$$g_{\mu\nu} \, dx^\mu \, dx^\nu = 2 \, dx^0 \, dx^3 - (dx^1)^2 - (dx^2)^2 + f(x^1, x^2, x^3) \, (dx^3)^2 \quad (12)$$

is called a *metric of a pp-wave*, see Section 21.5 in [10]. Such metrics were introduced by Peres [13] and have since been widely used in General Relativity. The remarkable property of the metric (12) is that the corresponding curvature tensor $R$ is linear in $f$.

There are differing views on what the “pp” stands for. According to [10] “pp” is an abbreviation for “plane-fronted gravitational waves with parallel rays”. According to Peres himself [14] “pp” is an abbreviation for “plane polarized gravitational waves”.

**Definition 5.1** A pp-space is a Riemannian spacetime whose metric can be written locally in the form (12).

The advantage of Definition 5.1 is that it gives an explicit formula for the metric of a pp-space. Its disadvantage is that it relies on a particular choice of local coordinates in each coordinate patch. We give now an alternative definition of a pp-space which is much more geometrical.
Definition 5.2  A pp-space is a Riemannian spacetime which admits a non-
vanishing parallel rank 1 spinor field.

We use the term “parallel” to describe the situation when the covariant deriva-
tive of some tensor or spinor field is identically zero.

It is known, see Section 4 in [2] or Section 3.2.2 in [4], that Definitions 5.1 and
5.2 are equivalent.

Remark 5.1  We do not assume that our spacetime admits a (global) spin
structure, cf. Section 11.6 in [12]. In fact, our only topological assumption is
connectedness. This does not prevent us from defining and parallel transporting
spinors locally.

Remark 5.2  Whenever we deal with a parallel tensor or spinor field we allo-
we this field to be multivalued. Multivaluedness may arise as a global phenome-
on when we combine the results of local parallel transport along sections of a loop.

Remark 5.3  In theoretical physics literature the metric of a pp-wave is often
characterised by the condition that the spacetime admits a nonvanishing paral-
lel real null vector field, see, for example, Section 21.5 in [10]. Here the authors
assume implicitly the fulfillment of the vacuum Einstein equation $\nabla \text{Ric} = 0$. It
turns out, see proof of Lemma C.1, that under the condition $\nabla \text{Ric} = 0$ the
existence of a nonvanishing parallel real null vector field is equivalent to the
existence of a nonvanishing parallel rank 1 spinor field.

Yet another way of characterising a pp-space is by its restricted holonomy
group $\text{Hol}^0$. Elementary calculations show that Definition 5.2 is equivalent to

Definition 5.3  A pp-space is a Riemannian spacetime whose holonomy $\text{Hol}^0$
is, up to conjugation, a subgroup of the group

$$B^2 := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \bigg| b \in \mathbb{C} \right\}. \quad (13)$$

Here we use the standard identification of the proper orthochronous Lorentz
group with $\text{SL}(2, \mathbb{C})$, see Example 5.57(c) in [12]. Our notation for subgroups
of the proper Lorentz group follows that of Section 10.122 of [3]; note that
some care is required because the text in [3] contains numerous mistakes.

It is interesting that the group (13) is, up to conjugation, the unique nontrivi-
abian Lie subgroup of $\text{SL}(2, \mathbb{C})$. In this statement “nontrivial” is understood
as “not 1-dimensional and not a product of 1-dimensional subgroups”, with
dimension understood as real dimension.
Having stated the basic facts concerning pp-spaces, let us now return to our
analysis of the field equations (7), (8). We claim that pp-spaces with parallel
Ricci curvature are solutions of (7), (8). The fact that such spacetimes satisfy
(8) is trivial, so we need only to explain why they satisfy (7). The Ricci
curvature of a pp-space is, up to a scalar factor $s$, the tensor square of a
nonvanishing parallel real null vector field $l$,

$$\text{Ric}_{\alpha\beta} = s l_{\alpha} l_{\beta}. \quad (14)$$

In view of (14) checking (7) reduces to checking

$$\nabla^{\kappa\lambda\mu\nu}_{\kappa\lambda\mu\nu} l_{\kappa} l_{\mu} = 0. \quad (15)$$

In order to establish (15) it is sufficient to establish

$$\nabla^{\kappa\lambda\mu\nu}_{\kappa\lambda\mu\nu} l_{\lambda} = 0. \quad (16)$$

As the vector field $l$ is parallel we have

$$R^{\kappa}_{\lambda\mu\nu} l_{\lambda} = 0. \quad (17)$$

It remains to observe that formulae (17) and (14) imply (16).

We call pp-spaces with parallel Ricci curvature Riemannian solutions of type 2.

In local coordinates the function $s$ from (14) is expressed via the function $f$
from (12) as $s = c(f_{11} + f_{22})$ where $f_{\alpha\beta} := \frac{\partial^2 f}{\partial x^\alpha \partial x^\beta}$ and $c \neq 0$ is some constant.
This observation gives us a simple algorithm for determining whether the Ricci
curvature of a pp-space is parallel. Namely, the Ricci curvature of a pp-space is parallel if and only if $f_{11} + f_{22} = $ const, and identically zero if and only if $f_{11} + f_{22} = 0$. Note that in the latter case the full (rank 4) curvature tensor $R$ is not necessarily zero because it is a linear function of the full Hessian
$(f_{\alpha\beta})_{\alpha,\beta=1}^2$, and not only its trace.

6 Riemannian solutions of type 3

Consider a Riemannian spacetime which has zero scalar curvature and is lo-
cally a product of a pair of Einstein 2-manifolds. (Of course, a 2-manifold is
Einstein if and only if it has constant curvature.) Clearly, such a spacetime is
a solution of the field equation (8). Straightforward calculations show that it
is also a solution of the field equation (7).

We call Riemannian spacetimes which have zero scalar curvature and are lo-
cally a product of Einstein 2-manifolds Riemannian solutions of type 3.
The underlying reason why spacetimes described in this section are indeed solutions of our field equations is as follows: if we change the sign of the metric of the Lorentzian 2-manifold then the product becomes a 4-dimensional Einstein space. Changing the sign of the metric of the Lorentzian 2-manifold is equivalent to interchanging the roles of the time and space coordinates. This means that a Riemannian solution of type 3 is a Riemannian solution of type 1 with the wrong choice of the time coordinate. In other words, for all practical purposes Riemannian solutions of type 3 are a special case of Riemannian solutions of type 1. We have to distinguish them only for the sake of mathematical bookkeeping.

7 Uniqueness of Riemannian solutions

The following uniqueness theorem is the main result of this paper.

**Theorem 7.1** Suppose that our coupling constants satisfy the inequalities

\[ b_{911} \neq b_{922}, \quad (18) \]
\[ c_1 + c_2 \neq 0, \quad (19) \]
\[ c_1 + c_3 \neq 0, \quad (20) \]
\[ 3c_1 + c_2 + 2c_3 \neq 0. \quad (21) \]

Then solutions of types 1, 2 and 3 described in Sections 4, 5 and 6 respectively are the only Riemannian solutions of our field equations (2), (3).

Here the \( b \)'s are the original coupling constants appearing in formula (B.3) whereas the \( c \)'s are defined in accordance with formulae (10).

Note that conditions (19)–(21) appeared previously in [17]. Namely, conditions (19), (20) coincide with condition (38) of [17], whereas condition (21) is equivalent to the condition \( c \neq -\frac{1}{3} \) mentioned in the very end of Section 11 of [17]. Thus, the new condition which enables us to establish uniqueness is condition (18). This matter will be discussed in greater detail in Section 11.

**Proof of Theorem 7.1** The crucial observation is that under the conditions (18) and (20) the field equation (8) is equivalent to

\[ \nabla \text{Ric} = 0. \quad (22) \]

This fact is established by a sequence of elementary manipulations with (8): separate (8) into equations symmetric and antisymmetric in the pair of indices \( \kappa, \lambda \), then contract \( \kappa \) with \( \mu \) in the symmetric equation which gives \( \partial \mathcal{R} = 0 \),

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etc. In performing these manipulations it is convenient to express the constants $d_4, \ldots, d_7$ via the constants $c_1, c_2, c_3$ and $b_{911} - b_{922}$ in accordance with (11).

Condition (22) allows us to apply the powerful Lemma C.1. The proof of Theorem 7.1 is therefore reduced to the analysis of the situation when our spacetime is locally a nontrivial product of Einstein manifolds, with “nontrivial” meaning that the spacetime itself is not Einstein. We have to examine which nontrivial products of Einstein manifolds satisfy the field equation (7), and show that the only ones that do are solutions of type 3 introduced in Section 6.

The possible decompositions into a nontrivial product are 3+1 and 2+2 where the numbers are the dimensions of Einstein manifolds. Below we analyze each of these cases. In doing this we use local coordinates which are a concatenation of local coordinates on our Einstein manifolds; consequently, our metric and curvature have block diagonal structure. As usual, Greek letters in tensor indices run through four possible values. Note also that the 3+1 case actually splits into two subcases, depending on whether the metric of the 3-manifold is Euclidean or Lorentzian; this distinction turns out to be unimportant because the arguments presented below are insensitive to the signatures of the metrics.

**Case 3+1.** In this case

$$g_{\mu\nu} = h_{\mu\nu} + k_{\mu\nu}$$

where $h$ and $k$ are the metrics of the 3- and 1-manifolds respectively, and

$$R_{\kappa\lambda\mu\nu} = \frac{1}{6} (h_{\kappa\mu} h_{\lambda\nu} - h_{\lambda\mu} h_{\kappa\nu}) r$$

where $r \neq 0$ is the (constant) scalar curvature of the 3-manifold. Straightforward calculations show that in this case equation (7) takes the form

$$\frac{6d_2 - d_3}{72} (h^{\lambda\nu} - 3k^{\lambda\nu}) r^2 = 0.$$  

(Note the absence of the coefficient $d_1$ in this equation. This is because in the 3+1 case Weyl curvature is zero.) In view of (11) we have $6d_2 - d_3 = c_1 + c_2$, so under the condition (19) the above equation cannot be satisfied.

**Case 2+2.** In this case the metric is given by formula (23) where $h$ and $k$ are the metrics of the two 2-manifolds, and

$$R_{\kappa\lambda\mu\nu} = \frac{1}{2} (h_{\kappa\mu} h_{\lambda\nu} - h_{\lambda\mu} h_{\kappa\nu}) r + \frac{1}{2} (k_{\kappa\mu} k_{\lambda\nu} - k_{\lambda\mu} k_{\kappa\nu}) s$$

where $r \neq s$ are the two corresponding (constant) scalar curvatures. Straightforward calculations show that in this case equation (7) takes the form

$$\frac{d_1 + 3d_2}{12} (h^{\lambda\nu} - k^{\lambda\nu}) (r^2 - s^2) = 0.$$
In view of (11) we have $d_1 + 3d_2 = \frac{1}{2}(3c_1 + c_2 + 2c_3)$, so under the condition (21) the above equation is equivalent to $r + s = 0$ which means that we are looking at a solution of type 3, see Section 6. \(\square\)

8 The pseudoinstanton construction

We are about to proceed to the study of non-Riemannian solutions of our field equations (2), (3). It would be unrealistic to expect to find all non-Riemannian solutions, so we need a method for finding at least some non-Riemannian solutions. The following construction provides such a method.

**Definition 8.1** We call a spacetime $\{M, g, \Gamma\}$ a pseudoinstanton if the connection is metric compatible and curvature is irreducible and simple.

Here irreducibility of curvature means that all irreducible pieces but one are identically zero. Simplicity means that the given irreducible subspace is not isomorphic to any other irreducible subspace. Metric compatibility means, as usual, that $\nabla g \equiv 0$.

The irreducible decomposition of curvature is described in Appendix A. It is easy to see that there are only three possible types of pseudoinstantons:

- *scalar* pseudoinstanton (all pieces of curvature apart from the scalar piece $R^{(1)}$ are identically zero),
- *pseudoscalar* pseudoinstanton (all pieces of curvature apart from the pseudoscalar piece $R^{(1)*}_s$ are identically zero), and
- *Weyl* pseudoinstanton (all pieces of curvature apart from the Weyl piece $R^{(10)}$ are identically zero).

**Theorem 8.1** A pseudoinstanton is a solution of the field equations (2), (3).

**PROOF.** Put $R_{\text{pseudo}} := R^{(1)}$ or $R_{\text{pseudo}} := R^{(1)*}_s$ or $R_{\text{pseudo}} := R^{(10)}$, depending on the type of our pseudoinstanton (see above). Then

$$q(R) = q(R_{\text{pseudo}}) + q(R - R_{\text{pseudo}}).$$

Note that we used here the fact that the piece $R_{\text{pseudo}}$ is simple; if not, then we would have cross-over terms of the type $R_{\text{pseudo}} \times (R - R_{\text{pseudo}})$.

When we start our variation from a spacetime with $R - R_{\text{pseudo}} \equiv 0$ the resulting variation of $\int q(R - R_{\text{pseudo}})$ is zero. Thus, the proof of Theorem 8.1 reduces to proving that our pseudoinstanton is a stationary point of the action $S_{\text{pseudo}} := \int q(R_{\text{pseudo}})$. But, according to Lemma B.1, $q(R_{\text{pseudo}}) = c(R_{\text{pseudo}}, R_{\text{pseudo}})_{YM}$
where $c$ is some constant, so the action $S_{\text{pseudo}}$ is of the type studied in [17] and the result follows from Theorem 2.1 of that paper. \qed

Further on we will be dealing only with the Weyl pseudoinstanton as it is the most interesting of the three possible types (the subspace of Weyl curvatures has dimension 10, whereas the subspaces of scalar and pseudoscalar curvatures have dimension 1). It is useful to rewrite Definition 8.1 for this particular case.

**Definition 8.2** A Weyl pseudoinstanton is a spacetime \( \{M, g, \Gamma\} \) whose connection is metric compatible and curvature purely Weyl.

The advantage of Definition 8.2 is that it can be used without knowledge of the full irreducible decomposition of curvature (material from Appendix A).

In particular, Weyl curvature can be understood as curvature satisfying

\[
R_{\kappa\lambda\mu\nu} = R_{\mu\nu\kappa\lambda}, \quad \text{Ric} = 0, \quad \varepsilon^{\kappa\lambda\mu\nu} R_{\kappa\lambda\mu\nu} = 0,
\]

which is the traditional definition. It is, of course, equivalent to the definition given in Appendix A.

9 Riemannian pseudoinstantons

As mentioned in the beginning of Section 8, the pseudoinstanton construction was developed primarily for the purpose of finding non-Riemannian solutions to our field equations (2), (3). We already know the Riemannian solutions (see Sections 4–7 for details) and these were found by means of a meticulous straightforward analysis of the field equations. Nevertheless, it is interesting to revisit the Riemannian case using the pseudoinstanton technique.

A Riemannian spacetime is a Weyl pseudoinstanton if and only if

\[
\text{Ric} = 0. \quad (24)
\]

Therefore, according to Theorem 8.1, Riemannian spacetimes satisfying the vacuum Einstein equation (24) are solutions to our field equations (2), (3).

The above argument demonstrates both the power and the limitations of the pseudoinstanton technique. This technique allowed us to obtain an important class of solutions without having to write down explicitly the field equations. On the other hand, it did not give us all the Riemannian solutions: an Einstein space is not necessarily a pseudoinstanton, and neither is a pp-space.
10 Non-Riemannian pseudoinstantons

We know only one non-Riemannian solution, and it is constructed as follows. Let us define Minkowski space $\mathbb{M}^4$ as a real 4-manifold equipped with global coordinates $(x^0, x^1, x^2, x^3)$ and metric $g_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$. Let 
\[ u(x) = w e^{-il\cdot x} \]  
be a plane wave solution of the equation $\ast du = \pm i du$ in $\mathbb{M}^4$. Define torsion $T = \frac{1}{2} \text{Re}(u \otimes du)$, and let $\Gamma$ be the corresponding metric compatible connection. Then, as shown in [17], the spacetime $\{\mathbb{M}^4, \Gamma\}$ is a Weyl pseudoinstanton, hence, by Theorem 8.1, a solution of our field equations (2), (3).

For the Yang–Mills case (4) the “torsion wave” solution described above was first obtained by Singh and Griffiths: see last paragraph of Section 5 in [15] and put $k = 0, N = e^{-il\cdot x}$. Our contribution is the observation that this torsion wave remains a solution for a general quadratic action (1) and that this fact can be established without having to write down explicitly the field equations.

The curvature of our spacetime $\{\mathbb{M}^4, \Gamma\}$ is 
\[ R = \text{Re}(du \otimes du). \]  
(26)

The necessary and sufficient conditions for non-flatness are $l \neq 0$ and $w \not\in \text{span} l$, and further on we assume that these conditions are fulfilled.

We claim that our spacetime $\{\mathbb{M}^4, \Gamma\}$ has holonomy $B^2$ (see formula (13)). Indeed, it is easy to check that the tensor field $F := l \wedge w$ is parallel. But $\ast F = \pm i F$ and $\det F = 0$, and the existence of a nonvanishing parallel tensor field with such properties is equivalent to the statement that the holonomy is a subgroup of the group $B^2$. (In fact, a tensor $F$ with such properties is equivalent to a rank 1 spinor: this spinor is the “square root” of $F$, see [9,16] for details.) Finally, examination of formula (26) shows that the holonomy is at least 2-dimensional, hence it coincides with $B^2$. Comparing this result with Definitions 5.2, 5.3 we conclude that our torsion wave is a non-Riemannian analogue of a pp-space.

11 Discussion

The main result of our paper is Theorem 7.1 which is a uniqueness theorem for Riemannian solutions of our metric-affine field equations (2), (3).

The problem of uniqueness of Riemannian solutions in quadratic metric-affine gravity has a long history. One of the first attempts at establishing uniqueness
was Fairchild’s paper [6] in which the author considered equations (2), (3) in the Yang–Mills case (4) and “proved” that Einstein spaces are the only solutions; the mistake was acknowledged in [7]. In [17] the Yang–Mills action was replaced by a more general quadratic action, the hope being that this would break certain symmetries and lead to a uniqueness result; unfortunately, the action in [17] still possessed a substantial degree of symmetry and the uniqueness problem was not resolved. The basic difficulty with [17] was the fact that the quadratic action was not general enough, namely, it did not contain cross-over terms from pairs of isomorphic pieces of curvature whereas the isomorphic pieces themselves were chosen in an arbitrary (traditional) fashion. The reason why in the current paper we succeeded in obtaining a uniqueness result for Riemannian solutions is that the quadratic form \( q \) appearing in (1) is a most general \( O(1,3) \)-invariant quadratic form on curvature.

An example of a quadratic form satisfying the conditions of Theorem 7.1 is

\[
q(R) = Ric_{\lambda\nu} Ric^{\lambda\nu}.
\]  

(27)

In the representation (B.3) the nonzero \( b \)'s for this quadratic form are

\[
b_1 = 1/4, \quad b_{611} = 1, \quad b_{911} = 1,
\]

hence the \( c \)'s defined in accordance with formulae (10) are

\[
c_1 = -1/2, \quad c_2 = -3/2, \quad c_3 = 0.
\]

With these \( b \)'s and \( c \)'s all four conditions of Theorem 7.1 are satisfied.

Quadratic forms considered in [17] do not satisfy the conditions of Theorem 7.1 because for such forms condition (18) fails. In particular, the Yang–Mills quadratic form (4) does not satisfy the conditions of Theorem 7.1.

The uniqueness Theorem 7.1 still leaves us with the problem of providing a physical interpretation for Riemannian solutions which are not Einstein spaces. At the moment all we can say is that the set of such “extra” solutions is very meagre: each is described locally by a real-valued function \( f(x^1, x^2, x^3) \) (see (12)) satisfying, upon appropriate rescaling, the equation

\[
\frac{\partial^2 f}{\partial (x^1)^2} + \frac{\partial^2 f}{\partial (x^2)^2} = 1.
\]

The second major result of our paper is the non-Riemannian solution (“torsion wave”) described in the beginning of Section 10. It is tempting to view this torsion wave solution as a very basic model for some elementary particle. We nominate two candidates: the graviton and the neutrino. Our (highly speculative) arguments go as follows.

Torsion is not an accepted physical observable but curvature is, so we base our interpretation on the analysis of the curvature generated by our torsion wave. Examination of the explicit formula (26) indicates that it is more convenient to
deal with the complexified curvature $d\mathbf{u} \otimes d\mathbf{u}$; note also that complexification is in line with the traditions of quantum mechanics. Our complex curvature is polarized, $^\ast(d\mathbf{u} \otimes d\mathbf{u}) = (d\mathbf{u} \otimes d\mathbf{u})^\ast = \pm i(d\mathbf{u} \otimes d\mathbf{u})$, and purely Weyl, hence it is equivalent to a (symmetric) rank 4 spinor $\zeta$; see subsection 1.2.3 in [5] or Appendix C in [17] for details. A rank 4 spinor corresponds to a spin 2 particle, and one naturally thinks of the graviton.

However, a closer examination reveals that our rank 4 spinor has additional algebraic structure: it is the 4th tensor power of a rank 1 spinor, $\zeta = \xi \otimes \xi \otimes \xi \otimes \xi$. Direct calculations [9,16] show that the rank 1 spinor field $\xi$ satisfies Weyl’s equation, which is the accepted mathematical model for the neutrino.

It is worth pointing out that we actually have two nonvanishing rank 1 spinor fields associated with our torsion wave solution: one is the spinor field $\xi$ from the previous paragraph, and the other is the parallel spinor field $\xi^\parallel$. The latter exists because our torsion wave solution has holonomy $B^2$, see end of Section 10. The two spinor fields differ by a scalar factor, namely,

$$\xi = c e^{-\frac{i}{2} \int l \cdot dx \xi^\parallel}$$  \hspace{1cm} (28)

where $c \neq 0$ is some constant and $l$ is the nonvanishing parallel real null (co)vector field from formula (25). In Alekseevsky’s terminology [2], $\xi$ and $\xi^\parallel$ are conformally equivalent and $\xi$ is almost parallel. Given a standard choice of coordinates and Pauli matrices in Minkowski space, it is easy to see that the components of $\xi^\parallel$ are constant, so formula (28) implies $\nabla \xi = \partial \xi$ (covariant derivative coincides with partial derivative). Thus, in our non-Riemannian spacetime the covariant derivative of the spinor field $\xi$ is the same as in flat Minkowski space.

It is also worth pointing out that in our non-Riemannian spacetime Weyl’s equation is the same as in flat Minkowski space: this is a consequence of the fact that our torsion is purely tensor, see [1] and [17] for details.

The above arguments indicate that observation of our torsion wave solution may lead the observer to believe that they are in a Riemannian spacetime.

A Irreducible decomposition of curvature

A curvature generated by a general affine connection has only one (anti)symmetry, namely,

$$R^\kappa_{\lambda\mu\nu} = -R^\kappa_{\lambda\nu\mu}.$$  \hspace{1cm} (A.1)

For a fixed $x \in M$ we denote by $\mathbf{R}$ the 96-dimensional vector space of real rank 4 tensors $R^\kappa_{\lambda\mu\nu}$ satisfying condition (A.1).
Let \( g \) be the Lorentzian metric at the point \( x \in M \) and let \( O(1, 3) \) be the corresponding full Lorentz group, i.e. the group of linear transformations of coordinates in the tangent space \( T_x M \) which preserve the metric. It is known, see Appendix B.4 from [8], that the vector space \( \mathbb{R} \) decomposes into a direct sum of 11 subspaces which are invariant and irreducible under the action of \( O(1, 3) \). These subspaces are listed in Table A.1. Note that our notation differs from that of [8]: we want to emphasize the fact that there are 3 groups of isomorphic subspaces, namely,

\[
\{ \mathbb{R}^{(6,l)}, l = 1, 2, 3 \}, \quad \{ \mathbb{R}^{(9,l)}, l = 1, 2 \}, \quad \{ \mathbb{R}_*^{(9,l)}, l = 1, 2 \}. \tag{A.2}
\]

Two subspaces are said to be isomorphic is there is a linear bijection between them which commutes with the action of \( O(1, 3) \).

In order to give an explicit description of irreducible subspaces of curvature we introduce the following conventions. We lower and raise tensor indices using the metric, and we also denote

\[
(R^*)_{\kappa\lambda\mu\nu} := \frac{1}{2} \sqrt{|\det g|} \, R_{\kappa\lambda\mu'\nu'} \varepsilon^{\mu'\nu'}. \tag{A.3}
\]

The map \( R \to R^* \) is an endomorphism in \( \mathbb{R} \) which we call the right Hodge star. Note that as we are working in the real Lorentzian setting the Hodge star has no eigenvalues.

The explicit description of irreducible subspaces of dimension \(< 10\) is given in Table A.2. Here \( \mathcal{R}, \mathcal{R}_* \) are arbitrary scalars, \( \mathcal{A}^{(l)} \) are arbitrary rank 2 antisymmetric tensors, and \( \mathcal{S}^{(l)}, \mathcal{S}_*^{(l)} \) are arbitrary rank 2 symmetric trace-free tensors, with “arbitrary” meaning that the quantity in question spans its vector space. The \( a \)'s in Table A.2 are some fixed real constants, the only condition being that \( a_1, a_1^*, \det (a_{6lm})_{l,m=1}^3, \det (a_{9lm})_{l,m=1}^2 \) and \( \det (a_{9lm}^*)_{l,m=1}^2 \) are nonzero. The freedom in choosing irreducible subspaces of dimension 6 and 9 is due to the fact that we have groups of isomorphic subspaces (A.2).
Table A.2
Explicit description of irreducible subspaces of dimension < 10

| Subspace | Formula for curvature $R$ |
|----------|--------------------------|
| $R^{(1)}$ | $R_{\kappa\lambda\mu\nu} = a_1 (g_{\kappa\mu} g_{\lambda\nu} - g_{\kappa\nu} g_{\lambda\mu})$ |
| $R^*_s(1)$ | $(R^*_s)_{\kappa\lambda\mu\nu} = a_1^* (g_{\kappa\mu} g_{\lambda\nu} - g_{\kappa\nu} g_{\lambda\mu})$ |
| $R^{(6,\ell)}$ | $R_{\kappa\lambda\mu\nu} = a_{6\ell 1} (g_{\kappa\mu} A^{(\ell)}_{\lambda\nu} - g_{\kappa\nu} A^{(\ell)}_{\lambda\mu}) + a_{6\ell 2} (g_{\lambda\mu} A^{(\ell)}_{\kappa\nu} - g_{\lambda\nu} A^{(\ell)}_{\kappa\mu})$ $+ a_{6\ell 3} g_{\kappa\lambda} A^{(\ell)}_{\mu\nu}$ |
| $R^{(9,\ell)}$ | $R_{\kappa\lambda\mu\nu} = a_{9\ell 1} (g_{\kappa\mu} S^{(\ell)}_{\lambda\nu} - g_{\kappa\nu} S^{(\ell)}_{\lambda\mu}) + a_{9\ell 2} (g_{\lambda\mu} S^{(\ell)}_{\kappa\nu} - g_{\lambda\nu} S^{(\ell)}_{\kappa\mu})$ |
| $R^*_s(9,\ell)$ | $(R^*_s)_{\kappa\lambda\mu\nu} = a^*_{9\ell 1} (g_{\kappa\mu} S^{(\ell)}_{\lambda\nu} - g_{\kappa\nu} S^{(\ell)}_{\lambda\mu}) + a^*_{9\ell 2} (g_{\lambda\mu} S^{(\ell)}_{\kappa\nu} - g_{\lambda\nu} S^{(\ell)}_{\kappa\mu})$ |

It is convenient to choose the following $a$’s:

$$a_1 = a_1^* = \frac{1}{12}$$

$$(a_{6\ell m}) = \begin{pmatrix} 5/12 & -1/12 & -1/6 \\ -1/12 & 5/12 & -1/6 \\ -1/12 & -1/12 & 1/3 \end{pmatrix}, \quad (a_{9\ell m}) = (a^*_{6\ell m}) = \begin{pmatrix} 3/8 & -1/8 \\ -1/8 & 3/8 \\ -1/8 & -3/8 \end{pmatrix}.$$

Then the lower rank tensors $\mathcal{R}$, $\mathcal{R}_s$, $A^{(l)}$, $S^{(l)}$, $S^*_s$ appearing in Table A.2 are expressed via the full (rank 4) curvature tensor $R$ according to the following simple formulae:

$$\mathcal{R} := R^{\kappa\lambda}_{\kappa\lambda}, \quad Ric^{(1)}_{\lambda\nu} := R^{\kappa}_{\lambda\kappa\nu}, \quad Ric^{(2)}_{\kappa\nu} := R^{\kappa}_{\kappa\lambda\nu},$$

$$\mathcal{R}ic^{(1)} := Ric^{(1)} - \frac{1}{4} \mathcal{R}g, \quad \mathcal{R}ic^{(2)} := Ric^{(2)} + \frac{1}{4} \mathcal{R}g,$$

$$S^{(l)}_{\mu\nu} := \frac{\mathcal{R}ic^{(l)}_{\mu\nu} + \mathcal{R}ic^{(l)}_{\nu\mu}}{2}, \quad A^{(l)}_{\mu\nu} := \frac{\mathcal{R}ic^{(l)}_{\mu\nu} - \mathcal{R}ic^{(l)}_{\nu\mu}}{2}, \quad l = 1, 2,$$

$$A^{(3)}_{\mu\nu} := R^{\kappa}_{\kappa\mu\nu}.$$ (A.5)

and

$$\mathcal{R}_s := (R^*_s)^{\kappa\lambda}_{\kappa\lambda}, \quad Ric^*_s^{(1)}_{\lambda\nu} := (R^*_s)^{\kappa}_{\kappa\lambda\nu}, \quad Ric^*_s^{(2)}_{\kappa\nu} := (R^*_s)^{\kappa}_{\kappa\lambda\nu},$$

$$\mathcal{R}ic^*_s^{(1)} := Ric^*_s^{(1)} - \frac{1}{4} \mathcal{R}_s g, \quad \mathcal{R}ic^*_s^{(2)} := Ric^*_s^{(2)} + \frac{1}{4} \mathcal{R}_s g,$$

$$S^*_s^{(l)}_{\mu\nu} := \frac{\mathcal{R}ic^*_s^{(l)}_{\mu\nu} + \mathcal{R}ic^*_s^{(l)}_{\nu\mu}}{2}, \quad A^*_s^{(l)}_{\mu\nu} := \frac{\mathcal{R}ic^*_s^{(l)}_{\mu\nu} - \mathcal{R}ic^*_s^{(l)}_{\nu\mu}}{2}, \quad l = 1, 2,$$

$$A^*_s^{(3)}_{\mu\nu} := (R^*_s)^{\kappa}_{\kappa\mu\nu}.$$ (A.6)

Note that the tensors $A^{(l)}_s$ are not used in Table A.2. This is not surprising as
the tensors $\mathcal{A}^{(l)}$ and $\mathcal{A}^{* (l)}$ are not independent: the $\mathcal{A}^{(l)}$ are linear combinations of the Hodge duals of $\mathcal{A}^{* (l)}$ and vice versa.

All calculations in the main text of the paper use the (A.4) choice of $a$’s.

Finally, let us give an explicit description of the 10- and 30-dimensional irreducible subspaces. $R^{(10)}$ is the subspace of curvatures $R$ such that

$$
R^\kappa_{\lambda\nu} = (R^*)^\kappa_{\lambda\nu} = 0, \quad R^\lambda_{\kappa\lambda\nu} = (R^*)^\lambda_{\kappa\lambda\nu} = 0, \quad R^\kappa_{\kappa\mu\nu} = 0 \quad \text{(A.7)}
$$

(all possible traces are zero) and $R^\kappa_\lambda = -R^\lambda_\kappa$. $R^{(30)}$ is the subspace of curvatures $R$ satisfying (A.7) and $R^\kappa_\lambda = R^\lambda_\kappa$.

Given a decomposition

$$
R = R^{(1)} \oplus R^{* (1)} \oplus \sum_{l=1}^{3} R^{(6,l)} \oplus \sum_{l=1}^{2} R^{(9,l)} \oplus R^{(10)} \oplus R^{(30)}
$$

any $R \in R$ can be uniquely written as

$$
R = R^{(1)} + R^{* (1)} + \sum_{l=1}^{3} R^{(6,l)} + \sum_{l=1}^{2} R^{(9,l)} + \sum_{l=1}^{2} R^{(9,l)} + R^{(10)} + R^{(30)}
$$

where the $R$’s in the RHS are from the corresponding irreducible subspaces. We will call these $R$’s the irreducible pieces of curvature. We will call the irreducible pieces $R^{(1)}$, $R^{* (1)}$, $R^{(10)}$, $R^{(30)}$ simple because their subspaces are not isomorphic to any other subspaces.

**Remark A.1** “Starred” and “unstarred” subspaces of same dimension are not isomorphic under the action of the group $O(1,3)$ because the Hodge star is a linear map which depends on the choice of the element of the group. This dependence is encoded in the normalisation of the totally antisymmetric quantity: $\varepsilon_{0123} = +1$ or $\varepsilon_{0123} = -1$ depending on whether the orientation of the coordinate system is positive or negative.

**Remark A.2** The global definition of the Hodge star requires the orientability of our manifold $M$. However, for the purpose of decomposing curvatures orientability is not needed: any abstract vector subspace is preserved under inversion (vector $\mapsto -$vector), so when writing explicit formulae for subspaces it does not matter whether $\varepsilon_{0123} = +1$ or $\varepsilon_{0123} = -1$. The delicate features of the Hodge star come to light only when we examine the relationship between pairs of different subspaces, see Remark A.1.

**Remark A.3** If we complexify our problem then our 11 subspaces will still remain irreducible under the action of $O(1,3)$. In order to justify this claim we argue as follows. Replace the full Lorentz group $O(1,3)$ by the proper orthochronous Lorentz group $SO(1,3)^+$. Then we have the standard algorithm (see, for example, Section 1.2 in [5]) for finding irreducible subspaces in terms of
spinors. Applying this algorithm we see that the complexified subspaces $\mathbb{R}^{(6,l)}$, $\mathbb{R}^{(10)}$ and $\mathbb{R}^{(30)}$ split into eigenspaces of the right Hodge star (A.3), and these “halves” are the only proper $SO(1,3)^\dagger$-invariant subspaces of the original subspaces. However, the “halves” are not invariant under change of orientation.

B Quadratic forms on curvature

Let us define an inner product on rank 2 tensors

$$(K, L) := K_{\mu\nu} L^{\mu\nu},$$

(B.1)

and a Yang–Mills inner product on curvatures

$$(R, Q)_{YM} := R^\kappa_{\lambda\mu\nu} Q^\lambda_{\kappa\mu\nu}.$$  

(B.2)

**Lemma B.1** Let $q : \mathbb{R} \to \mathbb{R}$ be an $O(1,3)$-invariant quadratic form on curvature. Then

$$q(R) = b_1 R^2 + b_1^* R^2_*$$

$$+ \sum_{l,m=1}^3 b_{6lm}(A^{(l)}, A^{(m)}) + \sum_{l,m=1}^2 b_{6lm}(S^{(l)}, S^{(m)}) + \sum_{l,m=1}^2 b_{6lm}^*(S^{(l)}_*, S^{(m)}_*)$$

$$+ b_{10}(R^{(10)}, R^{(10)})_{YM} + b_{30}(R^{(30)}, R^{(30)})_{YM}$$

(B.3)

with some real constants $b_1$, $b_1^*$, $b_{6lm}$, $b_{6lm}^*$, $b_{9ml}$, $b_{9ml}^*$, $b_{10}$, $b_{30}$. Here $\mathcal{R}$, $\mathcal{R}_*$, $A^{(l)}$, $S^{(l)}$, $S^{(l)}_*$, $R^{(10)}$, $R^{(30)}$ are tensors defined in Appendix A.

**PROOF.** Let us equip each of the 11 irreducible subspaces of curvature (see Table A.1) with an $O(1,3)$-invariant non-degenerate inner product. For 1-dimensional subspaces we employ the usual multiplication of scalars, and for 6- and 9-dimensional subspaces we employ (B.1); here scalars and rank 2 tensors are related to irreducible pieces of curvature in accordance with Table A.2. Note that these inner products are well defined even if the manifold $M$ is non-orientable: the fact that in a “starred” subspace our scalar or rank 2 tensor may be defined up to sign has no bearing on the inner product because both entries in the inner product would simultaneously retain or change sign upon continuation over a loop in $M$. See also Remark A.2. Our inner products on 1-, 6- and 9-dimensional subspaces are clearly non-degenerate.

For 10- and 30-dimensional subspaces we employ the inner product (B.2). It is not *a priori* clear that this inner product is non-degenerate on these subspaces. We establish non-degeneracy as follows. The inner product (B.2) is clearly non-degenerate on the whole vector space $\mathbb{R}$. It is easy to check that
any pair of non-isomorphic subspaces is orthogonal with respect to the inner product (B.2), so \( \mathbb{R} \) decomposes into a direct sum of 7 orthogonal subspaces

\[
\mathbb{R}^{(1)}, \quad \mathbb{R}^{(1)*}, \quad \oplus_{l=1}^{3} \mathbb{R}^{(6,l)}, \quad \oplus_{l=1}^{2} \mathbb{R}^{(9,l)}, \quad \oplus_{l=1}^{2} \mathbb{R}^{(9,l)*}, \quad \mathbb{R}^{(10)}, \quad \mathbb{R}^{(30)}.
\]

Hence, the inner product (B.2) is non-degenerate on each of these 7 subspaces. In particular, it is non-degenerate on \( \mathbb{R}^{(10)} \) and \( \mathbb{R}^{(30)} \).

Further on in the proof we deal with the bilinear form \( b : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) associated with the quadratic form \( q \), i.e. \( q(R) = b(R,R) \).

Let \( V \) and \( W \) be irreducible subspaces of \( \mathbb{R} \) and let \( (\cdot,\cdot)_V \) and \( (\cdot,\cdot)_W \) be their \( O(1,3) \)-invariant non-degenerate inner products. Here \( V \) and \( W \) are not necessarily distinct. Consider the \( O(1,3) \)-invariant bilinear form \( b_{VW} := b|_{V \times W} \). Then there is a unique linear map \( B_{VW} : V \to W \) such that

\[
(B_{VW}v,w)_W = b_{VW}(v,w), \quad \forall v \in V, \quad \forall w \in W;
\]

and this map commutes with the action of \( O(1,3) \). By Schur’s lemma \( B_{VW} \) is either zero or a bijection, in which case \( V \) and \( W \) are isomorphic. Thus, only pairs of isomorphic irreducible subspaces can give nonzero contributions to the bilinear form \( b \).

The proof of Lemma B.1 has been reduced to proving the following fact: if \( V \) is an irreducible subspace of \( \mathbb{R} \) and \( B_V : V \to V \) is a linear operator which commutes with the action of \( O(1,3) \) then \( B_V \) is a multiple of the identity map. In order to prove this fact we complexify our problem, noting that by Remark A.3 this does not affect the irreducibility of \( V \). After complexification the fact we are proving becomes a special case of a well known abstract result. \( \square \)

### C Spacetimes with parallel Ricci curvature

The purpose of this appendix is to state and prove the following

**Lemma C.1** A Riemannian spacetime has parallel Ricci curvature if and only if

(a) it is locally a product of Einstein manifolds, or  
(b) it is a pp-space with parallel Ricci curvature (see Section 5).

Before proceeding to the proof of Lemma C.1 let us recall that throughout this paper our spacetime is assumed to be 4-dimensional, real, connected and equipped with Lorentzian metric; see also Remark 1.1 on the meaning of “Riemannian”. All these assumptions are important in Lemma C.1.
The notion of an Einstein manifold is understood as in Definition 1.95 of [3]: a real manifold of arbitrary dimension equipped with a pseudo-Euclidean metric and a Levi-Civita connection, and such that the Ricci tensor is proportional to the metric with a constant proportionality factor.

Note that Lemma C.1 has a well-known Euclidean analogue. Namely, in the Euclidean case Ricci curvature is parallel if and only if the manifold is locally a product of Einstein manifolds; see Theorem 1.100 and Section 16.A in [3].

**Proof of Lemma C.1** The fact that assertion (a) or (b) implies (22) is obvious, so we only need to prove the converse statement.

It is known, see [2] or Section 10.119 in [3], that our spacetime \((M, g)\) is, at least locally, a product of pseudo-Euclidean manifolds \((M_j, g_j)\), \(j = 1, \ldots, k\), whose holonomies are weakly irreducible. Here “weak irreducibility” means that the only non-degenerate (with respect to the metric) invariant subspaces of the tangent space are \(\{0\}\) and the tangent space itself. Condition (22) implies

\[
\nabla \text{Ric}_j = 0, \quad (C.1)
\]

\(j = 1, \ldots, k\), where \(\text{Ric}_j\) is the Ricci curvature of \((M_j, g_j)\).

Let us examine a given manifold \((M_j, g_j)\). If \(\dim M_j = 1\) then \((M_j, g_j)\) is clearly Einstein. If \(\dim M_j = 2\) then (C.1) implies that \((M_j, g_j)\) is Einstein. If \(\dim M_j = 3\) or \(\dim M_j = 4\) then \((M_j, g_j)\) may not be Einstein, in which case, in view of (C.1), it admits a nonvanishing parallel symmetric rank 2 trace-free tensor field. But all such manifolds have been classified, see Table 2 in [2]. Analysis of the latter shows that if our spacetime is not a product of Einstein manifolds then we have one of the following three cases:

\[
\text{Hol}^0 = A^1 \times \{1\}, \quad (C.2)
\]

\[
\text{Hol}^0 = B^2, \quad (C.3)
\]

\[
\text{Hol}^0 = B^3_i. \quad (C.4)
\]

Here

\[
A^1 := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \bigg| \ b \in \mathbb{R} \right\}, \quad B^3_i := \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \bigg| \ a, b \in \mathbb{C}, \ |a| = 1 \right\},
\]

and \(B^2\) is defined in accordance with (13); note that we continue using the notation from Section 10.122 of [3]. Cases (C.2) and (C.3) correspond to pp-spaces (see Definition 5.3), whereas (C.4) does not. It remains to show that the case (C.4) is impossible.

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In the remainder of the proof we assume that we have (C.4). We will show that this leads to a contradiction.

Condition (C.4) implies the existence of a nonvanishing parallel real null vector field \( l \). This condition also restricts the possible structure of the full (rank 4) curvature tensor \( R \). To understand the latter let us fix an arbitrary point \( x \in M \) and choose a pair of real vectors \( v_1, v_2 \) such that \( l \cdot v_1 = l \cdot v_2 = v_1 \cdot v_2 = 0 \) and \( v_1 \cdot v_1 = v_2 \cdot v_2 = -1 \) where the dot denotes the standard inner product on \( T_x M \). Put \( A_j := l \wedge v_j, j = 1, 2, A_3 := v_1 \wedge v_2 \). It is easy to see that \( \{ A_1, A_2, A_3 \} \) is a basis for \( b_3^i \), the Lie algebra of the group \( B_3^i \). Also, \( \{ A_1, A_2 \} \) is a basis for \( b_2^i \), the Lie algebra of the group \( B_2^i \). Condition (C.4) implies that at the point \( x \) the curvature tensor has the structure

\[
R = \sum_{j,k=1}^{3} c_{jk} A_j \otimes A_k \tag{C.5}
\]

where \( c_{jk} = c_{kj} \) are some real numbers.

Further on we denote by \( u^2 := u \otimes u \) the tensor square of a vector, and by \( u \vee v := u \otimes v + v \otimes u \) the symmetric product of a pair of vectors.

We have (22) and, therefore, \( \nabla R \text{ic} = 0 \). According to Table 2 in [2], under the condition (C.4) the only (up to rescaling) nonvanishing parallel symmetric trace-free rank 2 tensor field is \( l^2 \), hence \( R \text{ic} \) is a multiple of \( l^2 \). But formula (C.5) implies

\[
R \text{ic} = - (c_{11} + c_{22}) l^2 + c_{13} l \vee v_2 - c_{23} l \vee v_1 - c_{33} \left( \frac{1}{2} g + v_1^2 + v_2^2 \right),
\]

so \( R \text{ic} \) is a multiple of \( l^2 \) if and only if \( c_{13} = c_{23} = c_{33} = 0 \). Formula (C.5) now becomes

\[
R = \sum_{j,k=1}^{2} c_{jk} (l \wedge v_j) \otimes (l \wedge v_k). \tag{C.6}
\]

Denote \( L := \text{span} \ l \subset T_x M, L^\perp := \{ u \mid u \perp l \} \subset T_x M \), and let \( R_{\mu \nu} : T_x M \to T_x M \) be the linear operator defined by \( (R_{\mu \nu} u)^\kappa := R^\kappa{\lambda \mu \nu} u^\lambda \). Inspection of formula (C.6) shows that

\[
R_{\mu \nu}(L^\perp) \subset L. \tag{C.7}
\]

A convenient way of interpreting this result is to think of a connection on \( L^\perp / L \) : then (C.7) is the statement that the curvature of such a connection is zero. (The connection on \( L^\perp / L \) is, in fact, equivalent to a \( U(1) \)-connection.)

Let us now fix a point \( x_0 \in M \). Put \( l_0 := l \mid_{x=x_0}, L_0 := L \mid_{x=x_0}, L^\perp_0 := L^\perp \mid_{x=x_0} \). Let \( \Lambda \) be an arbitrary loop based at \( x_0 \) which is homotopic to the constant loop at \( x_0 \). Denote by \( h_\Lambda : T_{x_0} M \to T_{x_0} M \) the linear operator describing the
result of parallel transport of a vector along this loop. As the vector field $l$ is parallel we have

$$(h_A - \text{id})(l_0) = 0. \quad (C.8)$$

In view of (C.7) we also have

$$(h_A - \text{id})(L_{0}^\perp) \subset L_0. \quad (C.9)$$

It is easy to see that properties (C.8) and (C.9) imply $\text{Hol}_0^0 \leq B^2$, which contradicts (C.4). Thus, the case (C.4) is impossible.  □

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