THE LAMINATION CONVEX HULL OF STATIONARY IPM

LAURI HITRUHIN AND SAULI LINDBERG

ABSTRACT. We compute the lamination convex hull of the stationary IPM equations. We also show in bounded domains that for subsolutions of stationary IPM taking values in the lamination convex hull, velocity vanishes identically and density depends only on height. We relate the results to the infinite time limit of non-stationary IPM.

1. INTRODUCTION

We consider the flow of two immiscible incompressible fluids with equal viscosities and different densities in a porous medium. This can be modelled by the incompressible porous media equations (IPM) which consist of conservation of mass, incompressibility and Darcy’s law:

\begin{align}
\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) &= 0, \\
\nabla \cdot \mathbf{v} &= 0, \\
\frac{\mu}{\kappa} \mathbf{v} &= -\nabla p - \rho \mathbf{g},
\end{align}

where \( \rho(x, t) \in \mathbb{R} \) is the fluid density, \( \mathbf{v}(x, t) \in \mathbb{R}^2 \) is the fluid velocity and \( \mathbf{g} = (0, g) \) is gravity [7]. In the case of a smooth (simply connected) domain \( \Omega \subset \mathbb{R}^2 \), we assume the impermeability condition \( \mathbf{v} \cdot \nu = 0 \) on \( \partial \Omega \). Without loss of generality, we set \( \mu/\kappa = g = 1 \).

Córdoba, Faraco and Gancedo proved the non-uniqueness of spatially periodic weak solutions \( \mathbf{v} = (v_1, v_2) \in L^\infty(0, T; L^2) \) and \( \rho \in L^\infty(0, T; L^\infty) \) of IPM in [6]. The proof employs the method of convex integration which was first adapted to hydrodynamics by De Lellis and Székelyhidi in their ground-breaking paper [8]. The construction of [6], built on degenerate T4 configurations, provides a robust method of constructing bounded weak solutions in inviscid fluid dynamics without determining the exact \( \Lambda \)-convex hull; for an application to more general active scalar equations with an even multiplier see [22]. The existence and non-uniqueness of spatially periodic \( C^{1/3-\epsilon} \) solutions of IPM for smooth initial data was shown by Isett and Vicol in [15].

In [23], Székelyhidi computed the \( \Lambda \)-convex hull and showed it to be the exact relaxation of IPM equations. He also used it to construct infinitely many admissible weak solutions to the unstable Muskat problem in \( \Omega = (-1, 1)^2 \) with a flat interface as initial data. Székelyhidi also computed a differently normalised hull that leads to solutions with a bounded velocity. Admissible mixing solutions to the unstable Muskat problem with a non-flat \( H^5 \)-regular interface were constructed by Castro, Córdoba and Faraco in [2]. For further developments see [1, 4, 13, 19, 20]; here, also, the construction of admissible weak solutions relies on the exact hull and the construction of an admissible subsolution.

---

L.H. was supported by ICMAT Severo Ochoa project SEV-2015-0554 grant MTM2017-85934-C3-2-P and the ERC grant 30719-GFTIPFD and the ERC grant 834728 Quamap and by a grant from The Emil Aaltonen Foundation. S.L. was supported by the ERC grant 30719-GFTIPFD and by the AtMath Collaboration at the University of Helsinki.
In stationary IPM, in contrast, if \( \mathbf{v} \in L^2(\Omega, \mathbb{R}^2) \) with \( \mathbf{v} \cdot \nu|_{\partial \Omega} = 0 \) and \( \rho \in L^\infty(\Omega) \) form a weak solution, then \( \mathbf{v} \equiv 0 \) and \( \partial_t \mathbf{v} \equiv 0 \); a proof of this simple fact by Elgindi appears in [10]. As a main result, Elgindi showed on \( \mathbb{R}^2 \) and \( \mathbb{T}^2 \) that whenever solutions of non-stationary IPM have initial datas near certain stationary solutions, they must converge to the stationary solution in \( H^3 \) when \( t \to \infty \). The global well-posedness of non-stationary IPM is open, but Elgindi showed it around said stationary solutions. In [3], Castro, Córdoba and Lear proved structurally similar results for the \textit{confined IPM} case \( \Omega = \mathbb{T}^3 \times (-1,1) \), overcoming new difficulties to do with the boundary.

Nevertheless, in [5], Constantin, La and Vicol constructed solutions of stationary IPM that are smooth and vanish outside a strip that has finite width in the direction \( x_1 = kx_2 \), \( k \in \mathbb{R} \). The result is one example of their construction which uses Grad-Shafranov-like equations to obtain smooth, localised solutions in hydrodynamics, motivated by Gavrilov’s construction of smooth, compactly supported solutions of stationary Euler equations in [14]. The solutions of stationary IPM in [5] are functions of the variable \( z = x_1 - kx_2 \) and, as such, they are periodic (even constant) in the axial direction of the strip. If the direction of finite width of the strip is \((0,1)\), i.e., in the case \( \Omega = \mathbb{T}^3 \times (-1,1) \), an easy adaptation of Elgindi’s proof (see §5) rules out such a construction. This dichotomy highlights the role of the direction of gravity in IPM and is discussed briefly in Remark 5.1.

The problem we address is the determination of the \textit{relaxation} of stationary IPM. One of our aims is to shed light on the following question: how are the differences between stationary and non-stationary IPM as well as the somewhat surprising combination of the results of [5] and [10] reflected in the relaxation? We also wish to use information on the relaxation to better understand the infinite time limit of non-stationary IPM.

We set the stage by briefly describing convex integration in the Tartar framework; the relevant definitions are recalled in §2. One first decouples a system of non-linear constant-coefficient PDE’s into a system of first-order linear PDE’s \( \mathcal{L}(z) = 0 \) and the pointwise constraint that \( z(x) \) takes values in a constitutive set \( K \). In the case of stationary IPM, \( z = (\rho, \mathbf{v}, \mathbf{m}) \), the set of linear equations \( \mathcal{L}(z) = 0 \) is

\[
\begin{align*}
\nabla \cdot \mathbf{m} &= 0, \\
\nabla \cdot \mathbf{v} &= 0, \\
\n\nabla \cdot \mathbf{v} + (0, \rho) &= 0
\end{align*}
\]

and the constitutive set is

\[
K = \{(\rho, \mathbf{v}, \mathbf{m}) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 : |\rho| = 1, \ \mathbf{m} = \rho \mathbf{v}\},
\]

where the constraint \( \rho \in \{-1,1\} \) codifies the densities of the two immiscible fluids. Returning to the general Tartar framework, given initial/boundary datas, one attempts to construct a strict \textit{subsolution}, that is, \( z_0 \) satisfying \( \mathcal{L}(z_0) = 0 \) and taking values in a suitable subset \( \mathcal{W} \) of the \( \Lambda \)-convex hull \( K^\Lambda \); usually, \( \mathcal{W} = \text{int}(K^\Lambda) \). One then forms perturbations \( z_j \) of \( z_0 \) by adding localised plane waves where the admissible directions of oscillation are dictated by the \textit{wave cone} \( \Lambda \) and \( z_j \) take values in \( \mathcal{W} \). By a limiting argument, one intends to find infinitely many subsolutions with the prescribed initial/boundary conditions and with values in \( K \), which would then yield non-uniqueness of the original system of PDE’s for the given boundary/initial datas (see [9]).

The \textit{relaxation} of \( K \) can be given slightly different meanings, but it is defined here as the smallest set \( \hat{K} \supset K \) that is stable under weak convergence for solutions of (1.4)–(1.6), essentially following Tartar [24]. As such, it models macroscopic averages of solutions of stationary IPM. By a result of Tartar, \( \hat{K} \) contains the lamination convex hull \( K^{l,c,\Lambda} \) [24, 

...
Theorem 8]. We compute the lamination convex hull of stationary IPM in Theorem 1.1, and we believe that as in non-stationary IPM, the lamination and $\Lambda$-convex hulls and the relaxation coincide.

As emphasised in [9, 23], precise information on the hull is crucial in identifying the boundary/initial data for which one can run convex integration. As an example of this we mention that the hull of compressible Euler is notoriously difficult to compute and that to the authors’ knowledge, due to insufficient information on the hull, lack of uniqueness has so far only been shown for a set of data where one is able to reduce to an incompressible system; see [12, 18]. However, in [18], Markfelder computed the $\Lambda$-convex hull of a suitably normalised constraint set $K$.

Furthermore, the physical relevance of subsolutions was already emphasised in [23] in the case of the Muskat problem. The unstable Muskat problem with a flat interface is ill-posed, but in a pioneering work [21], Otto had used mass transport techniques to construct macroscopically averaged relaxed solutions that arise as an entropy solution of a scalar conservation law. At a certain asymptotic limit [23, p. 505], Székelyhidi’s subsolutions converge to Otto’s relaxed solution. A subsolution can be viewed as a kind of coarse-grained average; this interpretation is explored in detail e.g. in [4, 9, 23].

(Topological) smallness of the hull seems to reflect uniqueness of bounded solutions under trivial initial/boundary data and (in the case of evolutionary models) existence of robust conserved quantities. As an example, in IPM and other active scalar equations with an even Fourier multiplier, $K^\Lambda$ has a non-empty interior [17, 22] and there exist non-trivial bounded (even Hölder continuous) solutions with compact support in time [22, 15]. SQG, in contrast, has an odd multiplier and a trivial hull $K^\Lambda = K$ (defining $\Lambda$ as in [17, 22]), and the Hamiltonian is conserved by $L^3$ solutions, ruling out bounded solutions with compact support in time [15].

Quadratic $\Lambda$-affine functions are a simple and powerful tool in determining the size of $K^\Lambda$. To illustrate this, while 2D and 3D ideal MHD look superficially similar to Euler equations, both possess a non-trivial quadratic $\Lambda$-affine function which vanishes in $K$, making $\text{int}(K^\Lambda)$ empty. As a direct reflection of this, bounded solutions conserve the mean-square magnetic potential in 2D and the magnetic helicity in 3D. This rules out solutions with a non-trivial, compactly supported magnetic field in 2D but, perhaps surprisingly, not in 3D [11]. By Tartar’s Theorem (see [24, Theorem 11]), quadratic $\Lambda$-affine functions are weakly continuous, and as such, they also aid the understanding of various asymptotic regimes such as weak limits of (sub)solutions or the inviscid limit; see also [4, p. 58].

In non-stationary IPM, (1.4) is replaced by $\partial_t \rho + \nabla \cdot m = 0$ and the $\Lambda$-convex hull consists of triples $(\rho, v, m)$ such that $|\rho| \leq 1$ and $|m - \rho v + (0, (1 - \rho^2)/2)| \leq (1 - \rho^2)/2$ [23]. In particular, the $\Lambda$-convex hull has a non-empty interior.

In stationary IPM, however, $G(\rho, v, m) := m \cdot v^\perp$ vanishes in $K \cap \Lambda$, enforcing $\text{int}(K^\Lambda) = \emptyset$. Other quadratic $\Lambda$-affine functions of stationary IPM include $|v|^2 + \rho v_2$ and $m \cdot (v + (0, \rho))$—in fact, these three functions determine $\Lambda$ (see Proposition 2.1). If $(\rho, v, m) \in K^\Lambda$ with $v \neq 0$, then $m \cdot v^\perp = 0$ yields $m = kv$ for some $k \in \mathbb{R}$. The main challenge in the computation of $K^{lc.\Lambda}$ is the determination of the exact range of the constant of proportionality $k$ in $m = kv$. 
Theorem 1.1. \(K^{lc,\Lambda} = \bigcup_{j=1}^{4} X_j\), where
\[
X_1 := \left\{ (\rho, 0, \frac{1 - \rho^2}{2}(e - (0,1)) ) : |\rho| \leq 1, |e| \leq 1 \right\},
\]
\[
X_2 := \left\{ (\rho, v, k\nu) : |\rho| \leq 1, v \neq 0, 1 \leq k \leq \rho - \frac{(1 - \rho^2)v_2}{|v|^2} \right\},
\]
\[
X_3 := \left\{ (\rho, v, k\nu) : |\rho| < 1, v \neq 0, -1 < k = \rho - \frac{(1 - \rho^2)v_2}{|v|^2} < 1 \right\},
\]
\[
X_4 := \left\{ (\rho, v, k\nu) : |\rho| \leq 1, v \neq 0, \rho - \frac{(1 - \rho^2)v_2}{|v|^2} \leq k \leq -1 \right\}.
\]

We interpret \(K^{lc,\Lambda}\) geometrically. The projections of \(X_2\) and \(X_4\) into \(\mathbb{R} \times \mathbb{R}^2\) are cones where \(\rho \in [-1,1]\) and
\[
\begin{align*}
(1.8) & \quad |v + \left(0, \frac{1 + \rho}{2}\right)| \leq \frac{1 + \rho}{2} \iff |v|^2 + (\rho + 1)v_2 \leq 0, \quad (X_2) \\
(1.9) & \quad |v - \left(0, \frac{1 - \rho}{2}\right)| \leq \frac{1 - \rho}{2} \iff |v|^2 + (\rho - 1)v_2 \leq 0. \quad (X_4)
\end{align*}
\]

The power balance \(|v|^2 + \rho v_2\) can be interpreted as the balance between the density of energy per unit time consumed by friction and the density of work per unit time done by gravity \([4]\). Thus Theorem 1.1 shows that \(\{(\rho, v, m) \in K^{lc,\Lambda} : v \neq 0\}\) divides into two subsets: the flexible region where \(|v|^2 + \rho v_2 \leq |v_2|\) (the cones) and the parameter \(k\) in \(m = kv\) lies on a non-degenerate interval, and the rigid region where the power balance dominates the vertical speed \(|v_2|\) and \(k\) is uniquely determined. This rigid region is just the projection of \(X_3\) into \(\mathbb{R} \times \mathbb{R}^2\). When \((\rho, 0, m) \in K^{lc,\Lambda}\), and thus \(z = (\rho, 0, m) \in X_1\), the component \(m\) has the same range of values as in non-stationary IPM. Furthermore, the projection of \(X_1\) into \(\mathbb{R} \times \mathbb{R}^2\) is the line segment that is formed as an intersection of the cones resulting from \(X_2\) and \(X_4\).

The main technical difficulties of the proof involve the smallness of the set \(X_3\). Note that for any suitable pair \((\rho, v)\) there exists exactly one \(m \in \mathbb{R}^2\) such that \((\rho, v, m) \in X_3\). This makes it very challenging to construct \(\Lambda\)-convex functions that would show for these \((\rho, v)\) that the lamination convex and \(\Lambda\)-convex hull coincide. We nevertheless manage to show coincidence for all other points; see (4.1). The difficulties are also present in Propositions 4.7–4.10, most notably when showing the lamination convexity of \(X_3\); this is the technically most difficult part of the paper.

As another main result, we show that if a subsolution of stationary IPM takes values in \(K^{lc,\Lambda}\), it has a vanishing velocity. Recall that \(L^2_\rho(\Omega, \mathbb{R}^2) := \{ w \in L^2(\Omega, \mathbb{R}^2) : \nabla \cdot w = 0, w \cdot \nu|_{\partial \Omega} = 0 \}\).

Theorem 1.2. Suppose \(\Omega \subset \mathbb{R}^2\) is bounded, strongly Lipschitz and simply connected. Suppose \(\rho \in L^\infty(\Omega)\) and \(v, m \in L^2_\rho(\Omega, \mathbb{R}^2)\) satisfy (1.4)–(1.6). Suppose \((\rho, v, m)(x) \in K^{lc,\Lambda}\) a.e. \(x \in \Omega\). Then \(v = 0\) and \(\partial_t \rho = 0\).

The conclusion of Theorem 1.2 also holds in \(T^1 \times (-1, 1)\), extending the dichotomy on solutions vanishing outside a strip into subsolutions (see Remark 5.1).

Motivated by Elgindi’s computations in \([10]\) as well as Theorem 1.2, we also discuss the infinite time limit of non-stationary IPM. We show in Proposition 6.1 that if
\( \rho \in L^\infty(0, \infty; L^\infty) \) and \( v, m \in L^\infty(0, \infty; L^2_\Omega) \) form a subsolution in a bounded domain \( \Omega \), then \( v \in L^2(0, \infty; L^2_\Omega) \); in particular, \( \lim_{M \to \infty} \int_0^M \int_\Omega |v(x, t)|^2 \, dx \, dt = 0. \)

The structure of the paper is as follows. The relevant definitions are recalled in \( \S2 \), where we also compute the wave cone. The inclusion \( K^{lc, \Lambda} \supset \bigcup_{j=1}^4 X_j \) is proved in \( \S3 \) whereas \( K^{lc, \Lambda} \subset \bigcup_{j=1}^4 X_j \) is proved in \( \S4 \). The proof of Theorem 1.2 is presented in \( \S5 \), and the limit \( t \to \infty \) of non-stationary IPM is studied in \( \S6 \).

2. Relevant notions

We briefly recall some notions from the theory of differential inclusions; a thorough discussion of related topics can be found in \([16]\).

The wave cone \( \Lambda \) consists of directions \( z = (\rho, v, m) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \) such that for some \( \xi \in \mathbb{R}^2 \setminus \{0\} \), plane waves of the form \( x \mapsto h(x \cdot \xi)z : \mathbb{R}^2 \to \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \) satisfy (1.4)–(1.6) for all \( h \in C^\infty(\mathbb{R}) \). Denoting \( (\xi_1, \xi_2)^\perp = (-\xi_2, \xi_1) \), the wave cone conditions are thus

\[
(2.1) \quad m \cdot \xi = 0, \quad
(2.2) \quad v \cdot \xi = 0, \quad
(2.3) \quad (v + (0, \rho)) \cdot \xi^\perp = 0.
\]

An explicit form of \( \Lambda \) is given in Proposition 2.1 and Corollary 2.2.

**Proposition 2.1.** The wave cone of stationary IPM is

\[
\Lambda = \{(\rho, v, m) : |v + (0, \rho/2)| = |\rho|/2, \quad m \cdot \nu^\perp = 0 \quad \text{and} \quad m \cdot (v + (0, \rho) = 0 \}. \]

**Proof.** First assume \( (\rho, v, m) \in \Lambda \). The conditions (2.2)–(2.3) imply that \( v = k\xi^\perp \) and \( v + (0, \rho) = \ell \xi \) for some \( k, \ell \in \mathbb{R} \). Thus \( |v|^2 + \rho v_2 = k\xi^\perp \cdot \xi^\perp = 0 \), giving \( |v + (0, \rho/2)| = |\rho|/2 \). If \( \rho = 0 \), then \( v = 0 \) and so clearly \( m \cdot (v + (0, \rho)) = m \cdot \nu^\perp = 0 \). If \( \rho \neq 0 \), then (2.1)–(2.3) give \( m \cdot (v + (0, \rho)) = \ell m \cdot \xi = 0 \) and \( m \cdot \nu^\perp = -km \cdot \xi = 0 \).

Conversely, if \( |v|^2 + \rho v_2 = m \cdot (v + (0, \rho)) = m \cdot \nu^\perp = 0 \), then we get \( (\rho, v, m) \in \Lambda \) by choosing \( \xi = v + (0, \rho) \) if \( v \neq (0, -\rho) \neq 0 \), \( \xi = \nu^\perp \) if \( v = (0, -\rho) \neq 0 \), \( \xi = m^\perp \) if \( (0, \rho) = 0 = v \) and \( m \neq 0 \), and finally \( \xi = (1, 1) \) if \( v = (0, \rho) = m = 0 \). \( \Box \)

**Corollary 2.2.** The wave cone \( \Lambda \) consists of vectors \( z \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \) of the following three forms:

\[
\begin{align*}
&z = \left( \rho, \frac{\rho}{2}(e - (0, 1)), \ell(e - (0, 1)) \right), \quad \rho \neq 0, \quad e \in S^1 \setminus \{(0, 1)\}, \quad \ell \in \mathbb{R}, \\
&z = (\rho, 0, (m_1, 0)), \quad \rho \neq 0, \quad m_1 \in \mathbb{R}, \\
&z = (0, 0, m), \quad m \in \mathbb{R}^2. 
\end{align*}
\]

**Remark 2.3.** The first condition in Proposition 2.1 can be written as \( |v|^2 + \rho v_2 = 0 \).

Given any compact set \( C \subset \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \), the laminates \( C^{k, \Lambda}, k \in \mathbb{N}_0 \), of \( C \) are defined as follows:

\[
C^{0, \Lambda} := C, \\
C^{k+1, \Lambda} := \{(\lambda z_1 + (1 - \lambda)z_2) : z_1, z_2 \in C^{k, \Lambda}, \; z_1, z_2 \in \Lambda, \; \lambda \in [0, 1]\}.
\]

The lamination convex hull of \( C \) is defined as

\[
C^{lc, \Lambda} := \bigcup_{k=0}^\infty C^{k, \Lambda}. 
\]
Recall also that a function $G : \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is said to be $\Lambda$-convex if $t \mapsto G(z_0 + tz) : \mathbb{R} \to \mathbb{R}$ is convex for every $z_0 \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$ and $z \in \Lambda$. The $\Lambda$-convex hull $C^\Lambda$ consists of points $z \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$ that cannot be separated from $C$ by a $\Lambda$-convex function. More precisely, $z \not\in C^\Lambda$ if and only if there exists a $\Lambda$-convex function $G$ such that $G(z) > 0$ for all $z \in \Lambda$. We have $C^\Lambda \supset C^{lc}$.

**Remark 2.4.** Denote the wave cone of non-stationary IPM by $\Lambda_{ns}$. The constitutive set $K$ is the same in stationary and non-stationary IPM but $\Lambda \subset \Lambda_{ns}$, so that we immediately get $K^{lc,\Lambda} \subset K^{lc,\Lambda_{ns}}$ and $K^\Lambda \subset K^\Lambda_{ns}$.

If $\rho \in L^\infty(\Omega)$ and $v, m \in L^2_x(\Omega, \mathbb{R}^2)$ satisfy (1.4)–(1.6) and $z(x) = (\rho, v, m)(x) \in K^\Lambda$ a.e. $x \in \Omega$, then $z$ is called a subsolution of stationary IPM.

### 3. Estimating the Hull from Below

We wish to first show that $K^{lc,\Lambda}$ contains the set $\bigcup_{j=1}^4 X_j$ described in Theorem 1.1. We begin by computing the first laminate.

**Proposition 3.1.** We have

$$K^{1,\Lambda} = \left\{ (\rho, 0, m) : |\rho| \leq 1, \quad m = \frac{1 - \rho^2}{2} (e - (0, 1)), \quad |e| = 1 \right\} \bigcup \left\{ (\rho, v, m) : |\rho| \leq 1, \quad m = \frac{\rho - (1 - \rho^2)v_2}{|v|^2} v \right\}.$$

**Proof.** A general convex combination of two elements of $K$ is either an element of $K$ or of the form

$$\begin{align*}
(\rho, v, \rho v + (1 - \rho^2)w) &= \frac{1 + \rho}{2} (1, v + (1 - \rho)w, v + (1 - \rho)w) \\
&\quad + \frac{1 - \rho}{2} (-1, v - (1 + \rho)w, -v + (1 + \rho)w),
\end{align*}$$

where $|\rho| < 1$ and $v, w \in \mathbb{R}^2$.

The linear combination in (3.1)–(3.2) is $\Lambda$-convex if and only if $(2, 2w, 2v - 2\rho w) \in \Lambda$. By Proposition 2.1, this occurs precisely when $|w + (0, 1/2)| = 1/2$, $w \cdot v^\perp = 0$ and $v \cdot (w + (0, 1)) = 0$.

If $v = 0$, the wave cone conditions are equivalent to $w = (e - (0, 1))/|e| = 1$, whereas in the case $v \neq 0$ they are equivalent to $w = -(v_2/|v|^2)v$, which completes the proof. $\square$

By Corollary 2.2 and Proposition 3.1, $K^{lc,\Lambda} \supset X_1 \cup X_3$. The next two propositions, combined with Corollary 2.2, show that $K^{lc,\Lambda} \supset X_2 \cup X_4$.

**Proposition 3.2.** Suppose $|\rho| < 1$ and $v \neq 0$ with

$$\rho - \frac{(1 - \rho^2)v_2}{|v|^2} \geq 1.$$

Then

$$(\rho, v, v) \in K^{lc,\Lambda}.$$

**Proof.** Suppose (3.3) holds. As a consequence, $v_2 < 0$. Let us write

$$(\rho, v) = \lambda \left( \frac{1}{X}, \frac{v}{X} \right) + (1 - \lambda)(\psi, 0, 0),$$

where $X \in \mathbb{R}^2$ and $\psi \in \mathbb{R}$. By (3.3), $\lambda \geq 1$, so $\psi \geq 0$. This is incompatible with $v \neq 0$. Hence $\psi = 0$ and

$$\left( \frac{1}{X}, \frac{v}{X} \right) \in K^{lc,\Lambda}.$$
where \(0 < \lambda < 1\) and
\[
(3.4) \quad \psi = \frac{\rho - \lambda}{1 - \lambda}.
\]
We need to choose \(\lambda\) in such a way that \(-1 \leq \psi < \rho\) and \(z_1 - z_2 = (1 - \psi, v/\lambda, v/\lambda) \in \Lambda\).

By Proposition 2.1 and Remark 2.3, \(z_1 - z_2 \in \Lambda\) is equivalent to
\[
\left\langle \frac{v}{\lambda}, \left[\frac{v}{\lambda} + (0, 1 - \psi)\right] \right\rangle = 0.
\]
In conjunction with (3.4), this leads to the choices
\[
\lambda = \frac{|v|^2}{|v|^2 - (1 - \rho)v_2}, \quad \psi = \frac{|v|^2 + \rho v_2}{v_2}.
\]
Note that (3.3) holds if and only if \(|v|^2 + (1 + \rho)v_2 \leq 0\) if and only if \(\psi \geq -1\). Since \(v_2 < 0\), we also have \(0 < \lambda < 1\). Furthermore, \(\psi = (\rho - \lambda)/(1 - \lambda) < \rho\) since \(\rho < 1\). \(\square\)

**Proposition 3.3.** Suppose \(|\rho| \leq 1\) and \(v \neq 0\) with
\[
(3.5) \quad \rho - \frac{(1 - \rho^2)v_2}{|v|^2} \leq -1.
\]
Then
\[
(\rho, v, -v) \in K_{lc,\Lambda}.
\]

**Proof.** The proof is entirely analogous to that of Proposition 3.2; we write \((\rho, v, -v) = \lambda(-1, v/\lambda, -v/\lambda) + (1 - \lambda)(\psi, 0, 0)\) and set
\[
\lambda = \frac{|v|^2}{|v|^2 + (1 + \rho)v_2}, \quad \psi = \frac{|v|^2 + \rho v_2}{v_2}.
\]
Now (3.5) is equivalent to \(|v|^2 - (1 - \rho)v_2 \leq 0\), which in turn is equivalent to \(\psi \leq 1\). In addition, (3.5) implies \(v_2 > 0\), which in turn gives \(0 < \lambda < 1\). \(\square\)

4. **Estimating the hull from above**

We now intend to show that \(K_{lc,\Lambda} \subset \bigcup_{j=1}^{4} X_j\). The steps of the proof are as follows:

- when \(v = 0\), Corollary 4.2 shows that if \(z = (\rho, 0, m) \in K_{\Lambda}\), then \(z \in X_1\).
- when \((\rho, v, m) \in K_{\Lambda}\) and \(v \neq 0\), Corollary 4.4 shows that \(m = kv\) for some \(k \in \mathbb{R}\).
- When \(z = (\rho, v, kv) \in K_{\Lambda}\) and \(|\rho - (1 - \rho^2)v_2/|v|^2| \geq 1\), Corollary 4.6 yields \(z \in X_4\).
- When \(z = (\rho, v, kv) \in K_{lc,\Lambda}\) and \(|\rho - (1 - \rho^2)v_2/|v|^2| < 1\), Propositions 4.7–4.10 imply that \(z \in X_3\). This is the only result that we are not able to show for \(K_{\Lambda}\) but only for \(K_{lc,\Lambda}\).

We begin by recalling a proposition from [23] which also applies to stationary IPM in view of Remark 2.4:

**Proposition 4.1.** The function
\[
G_1(\rho, v, m):= \left| m - \rho v + \left(0, \frac{1 - \rho^2}{2}\right) \right| - \frac{1 - \rho^2}{2}
\]
is \(\Lambda\)-convex and vanishes in \(K\). Consequently,
\[
K_{\Lambda} \subset \left\{ (\rho, v, m) : |\rho| \leq 1, \left| m - \rho v + \left(0, \frac{1 - \rho^2}{2}\right) \right| \leq \frac{1 - \rho^2}{2} \right\}.
\]
Propositions 3.1 and 4.1 have the following consequence.

**Corollary 4.2.** Let \( |\rho| \leq 1 \). Then
\[
(\rho, 0, m) \in K^A \iff m = \frac{1 - \rho^2}{2}(e - (0,1)), \ |e| \leq 1.
\]

We then consider the case \( v \neq 0 \). The following result follows immediately from Proposition 2.1.

**Proposition 4.3.** The function
\[
G_2(\rho, v, m) := m \cdot v^L
\]
is \( \Lambda \)-affine and vanishes in \( K \).

**Corollary 4.4.** If \((\rho, v, m) \in K^A \) with \( v \neq 0 \), then \( m = kv \) for some \( k \in \mathbb{R} \).

In view of Proposition 4.2 and Corollary 4.4, the hull \( K^{lc,A} \) is determined by finding the exact range of the parameter \( k = k(\rho, v) \) in \( m = kv \). Proposition 4.1 implies that \( k \) lies between \( \rho \) and \( \rho - (1 - \rho^2)v_2/|v|^2 \), giving the optimal range in the case of non-stationary IPM. However, in the case of stationary IPM, the range of \( k(\rho, v) \) is smaller, as stated in Theorem 1.1.

We divide the set of points \((\rho, v) \in \mathbb{R} \times (\mathbb{R}^2 \setminus \{0\})\) into the cones described by (1.8)–(1.9) and the complement of their union. We first address the points of the two cones.

**Proposition 4.5.** The functions defined by
\[
G_3(\rho, v, m) := -[v - m] \cdot [v + (0, 1 + \rho)] + \frac{|v - m|^2}{2},
\]
\[
G_4(\rho, v, m) := -[v + m] \cdot [v - (0, 1 - \rho)] + \frac{|v + m|^2}{2},
\]
are \( \Lambda \)-convex and satisfy \( G_3|_K = G_4|_K = 0 \).

**Proof.** We prove the claims for \( G_3 \); the proofs for \( G_4 \) are analogous. Let us fix \( z_0 = (\rho_0, v_0, m_0) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \), \( z = (\rho, v, m) \in \Lambda \) and \( t \in \mathbb{R} \). Then
\[
G_3(z_0 + tz) = -[v_0 - m_0 + t(v - m)] \cdot [v_0 + (0, 1 + \rho_0) + t(v + (0, \rho))]
+ \frac{|v_0 - m_0 + t(v - m)|^2}{2}
= G_3(z_0) + C_{z_0,z}t + \frac{|v - m|^2}{2}t^2
\]
in view of Proposition 2.1. Furthermore, \( G_3(1, v, v) = 0 \) and \( G_3(-1, v, -v) = 0 \) for all \( v \in \mathbb{R}^2 \) so that \( G_3|_K = 0 \). \( \square \)

**Corollary 4.6.** Suppose \( |\rho| < 1 \), \( v \neq 0 \) and \((\rho, v, m) \in K^A \). If \( |\rho - (1 - \rho^2)v_2/|v|^2| \geq 1 \), then \( z \in X_2 \cup X_4 \).

**Proof.** Assume \( \rho - (1 - \rho^2)v_2/|v|^2 \geq 1 \); the proof of the case \( \rho - (1 - \rho^2)v_2/|v|^2 \leq -1 \) is analogous. By Corollary 4.4, \( m = kv \) for some \( k \in \mathbb{R} \). Our aim is to show that \((\rho, v, kv) \in X_2 \), i.e., \( 1 \leq k \leq \rho - (1 - \rho^2)v_2/|v|^2 \).

The inequality \( k \leq \rho - (1 - \rho^2)v_2/|v|^2 \) follows from Proposition 4.1. For the claim \( k \geq 1 \) note that \( \rho - (1 - \rho^2)v_2/|v|^2 \geq 1 \) can be written as \( (1 - \rho)|v|^2 + (1 - \rho^2)v_2 \leq 0 \).
We compute
\[ 0 \geq (1 - \rho)G_3(\rho, v, m) \]
\[ = (k - 1)v \cdot \left( (1 - \rho)v + (0, 1 - \rho^2) + \frac{(1 - \rho)(k - 1)v}{2} \right) \]
\[ = (k - 1) \left( (1 - \rho)|v|^2 + (1 - \rho^2)v_2 + \frac{(1 - \rho)(k - 1)|v|^2}{2} \right), \]
which implies the claim.

We show in Propositions 4.8–4.10 that we cannot have \( z \epsilon \Lambda \). Then, if \( z \epsilon \Lambda \) we write \( v \epsilon \Lambda \) \( \setminus \{ (\rho, v, m) : v \neq 0, -1 < \rho - \frac{(1 - \rho^2)v_2}{|v|^2} < 1 \} = X_1 \cup X_2 \cup X_4. \)

In other words, we have computed the exact range of the \( m \) component in all cases except \( v \neq 0, \rho - (1 - \rho^2)v_2/|v|^2 \in (-1, 1) \). We finish the proof of Theorem 1.1 by showing that
\[ K^{\text{lc}} \Lambda \cap \left\{ (\rho, v, m) : v \neq 0, -1 < \rho - \frac{(1 - \rho^2)v_2}{|v|^2} < 1 \right\} = X_3; \]
combining (4.1) and (4.2) yields \( K^{\text{lc}} \Lambda = \cup_{j=1}^4 X_j \).

The proof of (4.2) consists of two parts. First, Proposition 4.7 says that \( X_3^{1, \Lambda} = X_3 \).

Then, if \( z = (\rho, v, k) \in (\cup_{j=1}^4 X_j)^{1, \Lambda} \) and \( v \neq 0 \) and \( -1 < \rho - \frac{(1 - \rho^2)v_2}{|v|^2} < 1 \), we write \( z \) as a \( \Lambda \)-convex combination of \( z_1 \in X_1 \) and \( z_2 \in X_j \), where \( i, j \in \{1, 2, 3, 4\} \). We show in Propositions 4.8–4.10 that we cannot have \( i \neq j \). Now, since each \( X_1 \) is lamination convex, we get \( i = j = 3 \), so that \( z \in X_3 \), as claimed.

**Proposition 4.7.** \( X_3^{1, \Lambda} = X_3 \).

**Proof.** Suppose \( z_1, z_2 \in X_3 \) satisfy \( 0 \neq z_1 - z_2 \in \Lambda \). We already mention that by Propositions 4.8–4.9 below, for every \( (\rho, v, m) \in [z_1, z_2] \) we have \( -1 < \rho - \frac{(1 - \rho^2)v_2}{|v|^2} < 1 \). Let \( 0 < \lambda < 1 \) and \( \lambda + \mu = 1 \). We write
\[
\begin{align*}
z &= \lambda z_1 + \mu z_2 \\
&= \lambda (\rho + \mu t, v + \mu w, k_1(v + \mu w)) + \mu (\rho - \lambda t, v - \lambda w, k_2(v - \lambda w)) \\
&= (\rho, v, k v)
\end{align*}
\]
and wish to show that \( k = \rho - (1 - \rho^2)v_2/|v|^2 \). We write
\[ z_1 - z_2 = (t, w, (k_1 - k_2)v + (\mu k_1 + \lambda k_2)w) \epsilon \Lambda. \]
Corollary 2.2 and the assumption \( z_1 - z_2 \neq 0 \) imply that \( t \neq 0 \). Assume, without loss of generality, that \( t > 0 \).

We first note that if \( w = 0 \), then Corollary 2.2 yields \( (k_1 - k_2)v_2 = 0 \). First, in the case \( v_2 = 0 \), then the assumption \( z_1, z_2 \in X_3 \) yields \( k_1 = \rho + \mu t \) and \( k_2 = \rho - \lambda t \), so that \( k = \rho \) and \( z \in X_3 \).

We then treat the rest of the cases. Suppose, therefore, that either \( w \neq 0 \) or \( k_1 - k_2 = |w| = 0 \). In each case, by (4.3) and Corollary 2.2, we may write
\[
z_1 - z_2 = \left( t, \frac{t}{2}(e - (0, 1)), \ell(e - (0, 1)) \right)
\]
for some \( e \in S^1 \) and \( \ell \in \mathbb{R} \).
We intend show that $k_1 < k_2$. (In particular, this rules out the case $k_1 - k_2 = |w| = 0$.)
This reduces to showing a claim that we next specify. Suppose

$$
\zeta_1 = \left( \rho, v, \left[ \rho - \frac{(1 - \rho^2)v_2}{|v|^2} \right] v \right) =: (\rho, v, \ell_1 v) \in X_3,
$$

that is,

$$
|v|^2 + \rho v_2 - |v_2| > 0.
$$

Suppose $\epsilon > 0$ is small and $\zeta_2 = (\rho + \epsilon, v + \epsilon(e - (0,1))/2, \ell_2(v + \epsilon(e - (0,1))/2)) \in X_3$, that is,

$$
\zeta_2 = \left( \rho + \epsilon, v + \frac{\epsilon}{2}[e - (0,1)], \left[ \rho + \epsilon - \frac{1 - (\rho + \epsilon)^2}{|v + \epsilon(e - (0,1))|^2} \right] \left( v + \frac{\epsilon}{2}[e - (0,1)] \right) \right),
$$

where $|e| = 1$. We claim that $\ell_1 < \ell_2$.

We write $\ell_2 - \ell_1$ as a Taylor series:

$$
\ell_2 - \ell_1 = \rho + \epsilon - \frac{(1 - \rho^2)v_2 + \epsilon [(1 - \rho^2)(e_2 - 1)/2 - 2\rho v_2] + O(\epsilon^2)}{|v|^2 + \epsilon v \cdot [e - (0,1)] + O(\epsilon^2)} \notag
$$

$$
- \rho + \frac{(1 - \rho^2)v_2}{|v|^2} \notag
$$

$$
= \frac{\epsilon}{|v|^2} \left[ |v|^2 + 2\rho v_2 + \frac{(1 - \rho^2)v_1 v_2 e_1}{|v|^2} + \frac{(1 - \rho^2)(v_2^2 - v_1^2)(e_2 - 1)/2}{|v|^2} \right] + O(\epsilon^2) \notag
$$

Thus it suffices to show that

$$
H(\bar{e}) := |v|^2 + 2\rho v_2 + \frac{1 - \rho^2}{2} \left( \frac{2v_1 v_2}{|v|^2} \cdot \frac{v_2^2 - v_1^2}{|v|^2} \right) \cdot [e - (0,1)] > 0 \quad \text{for all } \bar{e} \in S^1.
$$

Note that $H$ is minimised when $\bar{e} \cdot (2v_1 v_2/|v|^2, (v_2^2 - v_1^2)/|v|^2)$ is minimised, that is, when $\bar{e} = -(2v_1 v_2/|v|^2, (v_2^2 - v_1^2)/|v|^2)$. The minimum value

$$
H \left( \frac{-2v_1 v_2}{|v|^2}, \frac{v_2^2 - v_1^2}{|v|^2} \right) = |v|^2 + 2\rho v_2 - \frac{1 - \rho^2}{2} - \frac{1 - \rho^2 v_2^2 - v_1^2}{|v|^2} 
$$

$$
= |v|^2 + 2\rho v_2 - \frac{(1 - \rho^2)v_2}{|v|^2} \notag
$$

$$
= \frac{|v|^2 + (\rho - 1)v_2[|v|^2 + (\rho + 1)v_2]}{|v|^2} \notag
$$

$$
> 0
$$

by (4.4). Thus $\ell_1 < \ell_2$. We conclude that $k_1 < k_2$ in (4.3), and so $w \neq 0$. 

Recall that \( w = t(e - (0, 1))/2 \) for some \( e \in S^1 \); since \( w \neq 0 \), we have \( e \neq (0, 1) \). Since we already showed that \( k_1 < k_2 \), we conclude from (4.3) that \( v \cdot w^\perp = 0 \).

Proposition 4.9 below implies that \( v \neq 0 \). Thus \( w = \ell v \), where \( |w + (0, t/2)| = |t/2| \) gives \( \ell = -tv_2/|v|^2 \), so that we can write

\[
\begin{align*}
z_1 &= \left( \rho + \mu t, \left( 1 - \mu \frac{tv_2}{|v|^2} \right) v, k_1 \left( 1 - \mu \frac{tv_2}{|v|^2} \right) v \right), \\
z_2 &= \left( \rho - \lambda t, \left( 1 + \lambda \frac{tv_2}{|v|^2} \right) v, k_2 \left( 1 + \lambda \frac{tv_2}{|v|^2} \right) v \right).
\end{align*}
\]

Since \( z_1, z_2 \in X_3 \), we have

\[
\begin{align*}
k_1 &= \rho + \mu t - \frac{[1 - (\rho + \mu t)^2]v_2}{\left( 1 - \mu \frac{tv_2}{|v|^2} \right) |v|^2}, \\
k_2 &= \rho - \lambda t - \frac{[1 - (\rho - \lambda t)^2]v_2}{\left( 1 + \lambda \frac{tv_2}{|v|^2} \right) |v|^2}
\end{align*}
\]
so that

\[
k = \lambda \left( 1 - \mu \frac{tv_2}{|v|^2} \right) k_1 + \mu \left( 1 + \lambda \frac{tv_2}{|v|^2} \right) k_2
= \rho - \lambda \mu \frac{t^2 v_2}{|v|^2} - \frac{v_2}{|v|^2} \left[ \lambda [1 - (\rho + \mu t)^2] + \mu [1 - (\rho - \lambda t)^2] \right]
\]

\[
= \rho - \frac{(1 - \rho^2)v_2}{|v|^2},
\]

as claimed. \( \square \)

**Proposition 4.8.** \([X_3 - (X_2 \cup X_4)] \cap \Lambda = 0.\)

**Proof.** Suppose \( z_1 = \left( \rho, v, \left[ \rho - \frac{(1 - \rho^2)v_2}{|v|^2} \right] \right) v =: (\rho, v, kv) \in X_3, \)

\( z_2 = (\psi, w, \ell w) \in X_2 \cup X_4, \)

so that \( v, w \neq 0 \). Seeking a contradiction, assume that

\( z_1 - z_2 = (\rho - \psi, v - w, k(v - w) + (k - \ell)w) \in \Lambda. \)

By the definitions of \( X_2 \) and \( X_4 \), we get \( k \neq \ell \), so that Proposition 2.1 gives \( w \cdot v^\perp = 0 \).

Now \( v = (1 + t)w \) for some \( t \in \mathbb{R} \setminus \{ -1, 0 \} \); if we had \( t = 0 \), then \( z_1 - z_2 \in \Lambda \) would imply \( \rho = \psi \), in contradiction with the definitions of \( X_2, X_3 \) and \( X_4 \).

Now, since \( z_1 - z_2 \in \Lambda \), we have

\[
0 = |v - w|^2 + (\rho - \psi)(v_2 - w_2) = t^2 |w|^2 + t(\rho - \psi)w_2
\]
so that \( v = (1 + t)w = \left[ 1 + (\psi - \rho)w_2/|w|^2 \right] w \) and \( \rho \neq \psi \). We therefore obtain

\[
k = \rho - \frac{(1 - \rho^2)v_2}{|v|^2} = \rho - \frac{(1 - \rho^2)w_2}{|w|^2 + (\psi - \rho)w_2}.
\]

We divide the rest of the proof into separate cases.
Suppose first \( z_2 \in X_2 \) (that is, \( |w|^2 + (1 + \psi)w_2 \leq 0 \)) and \( 1 + t > 0 \) (i.e. \( |w|^2 + (\psi - \rho)w_2 > 0 \)). By (4.5), the assumption \( k < 1 \) can be written as \( |w|^2 + (1 + \psi)w_2 > 0 \), which gives a contradiction.

Suppose next \( |w|^2 + (1 + \psi)w_2 \leq 0 \) and \( |w|^2 + (\psi - \rho)w_2 < 0 \). Thus \( w_2 < 0 \). Now \( k > -1 \) can be written as \( |w|^2 + (\psi - 1)w_2 < 0 \), yielding a contradiction.

Similarly, if \( z_2 \in X_4 \) (i.e. \( |w|^2 + (1 + \psi)w_2 \leq 0 \)) and \( 1 + t > 0 \), then \( k < 1 \) is in contradiction with the assumption \( z_2 \in X_4 \). Finally, if \( z_2 \in X_4 \) and \( 1 + t < 0 \), then \( k > -1 \) contradicts \( z_2 \in X_4 \).

**Proposition 4.9.** Suppose \( z_1 \in X_1, z_2 \in X_2 \cup X_3 \cup X_4 \) and, \( z_2 - z_1 \in \Lambda \). Then the half-open interval \( (z_1, z_2] \subset X_2 \cup X_4 \).

**Proof.** Suppose \( z_1 = (\rho, 0, m) \in X_1 \) and \( z_2 \in X_2 \cup X_3 \cup X_4 \) satisfy \( z_2 - z_1 \in \Lambda \). Let \( z = (\rho + \epsilon, \nu, m) \in (z_1, z_2] \); thus

\[(z - z_1) = (\epsilon, \nu, m - m) \in \Lambda.\]

Also note that \( z_2 \in X_2 \cup X_3 \cup X_4 \) implies that \( \nu \neq 0 \).

If \( \epsilon = 0 \), we get \( z - z_1 = (0, \nu, m - m) \in \Lambda \), which contradicts Corollary 2.2. We then assume that \( 0 < \epsilon \leq 1 - \rho \). By Proposition 2.1, \( |v|^2 + \epsilon v_2 = 0 \). Since \( v \neq 0 \), we conclude that \( v_2 < 0 \). Thus

\[|v|^2 + (\rho + \epsilon + 1)v_2 = (\rho + 1)v_2 \leq 0\]

which, combined with Corollary 4.6, yields \( z \in X_2 \). Similarly, if \( -1 - \rho \leq \epsilon < 0 \), then \( z_2 \in X_4 \).

We finish the proof of Theorem 1.1 by showing that a \( \Lambda \)-segment between \( z_1 \in X_2 \) and \( z_2 \in X_4 \) cannot contain \( (\rho, v, k\nu) \) with \( v \neq 0 \) and \(-1 < \rho - (1 - \rho^2)v_2/|v|^2 < 1\).

**Proposition 4.10.** \( (X_2 \cup X_4)^{1: \Lambda} \subset X_1 \cup X_2 \cup X_4 \).

**Proof.** Suppose

\[z_1 = (\rho, v, k\nu) \in X_2, \quad z_2 = (\psi, w, \ell w) \in X_4\]

and

\[z_1 - z_2 = (\rho - \psi, v - w, k(v - w) + (k - \ell)w) \in \Lambda.\]

Thus

\[1 \leq k \leq \rho - \frac{(1 - \rho^2)v_2}{|v|^2}, \quad \psi - \frac{(1 - \psi^2)w_2}{|w|^2} \leq \ell \leq -1,\]

giving \( |v|^2 + (\rho + 1)v_2 \leq 0 \) and \( |w|^2 + (\psi - 1)w_2 \leq 0 \), which in turn yields \( v_2 < 0 \). Now \( \rho \neq \psi \), as otherwise \( z_1 - z_2 \in \Lambda \) would give \( v - w = 0 \), contradicting \( v_2 < 0 \).

Choose the unique \( \tilde{\psi} = \lambda\psi + \mu\rho \in [\psi, \rho] \) (where \( 0 \leq \lambda \leq 1 \) and \( \lambda + \mu = 1 \)) such that \( \tilde{z} = (\tilde{\psi}, \tilde{w}, \tilde{m}) := \lambda z_1 + \mu z_2 \) satisfies \( \tilde{w} = 0 \) or

\[
\tilde{\psi} - \frac{(1 - \tilde{\psi}^2)\tilde{w}_2}{|\tilde{w}|^2} = -1.
\]

If \( \tilde{w} = 0 \), then \( \tilde{z} \in X_1 \) and we are reduced to the situation of Proposition 4.9. Assume, therefore, \( \tilde{w} \neq 0 \) and (4.6) holds. Consequently, \( |\tilde{w}|^2 + (\tilde{\psi} - 1)\tilde{w}_2 = 0 \), giving \( \tilde{w}_2 > 0 \). Note that (4.6) and Corollary 4.6 give \( \tilde{z} = (\tilde{\psi}, \tilde{w}, -\tilde{w}) \).

Now, by assumption,

\[z_1 - \tilde{z} = (\rho - \tilde{\psi}, v - \tilde{w}, k(v - \tilde{w}) + (k + 1)\tilde{w}) \in \Lambda,\]
so that \( \mathbf{w} \cdot \mathbf{v}^\perp = 0 \) since \( k \geq 1 \). Let us write \( \mathbf{v} = (1 + t)\mathbf{w} \); now \( z_1 - z_2 \in \Lambda \) gives \( t = (\psi - \rho)\mathbf{w}/|\mathbf{w}|^2 \). On the other hand, \( v_2 < 0 < \tilde{w}_2 \) and \( \mathbf{v} = (1 + t)\mathbf{w} \) yield \( 1 + t < 0 \), so that

\[
0 > \frac{|\mathbf{w}|^2 + (\psi - \rho)\tilde{w}_2}{|\mathbf{w}|^2} = \frac{(1 - \rho)\tilde{w}_2}{|\mathbf{w}|^2},
\]

giving a contradiction with \( \tilde{w}_2 > 0 \).

This finishes the proof of Theorem 1.1 and gives the exact description of the lamination convex hull of the stationary IPM equations. Furthermore, outside the 'rigid region' of \( K^{lc,\Lambda} \) where \((\rho, \mathbf{v}) \in \mathbb{R} \times (\mathbb{R}^2 \setminus \{0\}) \) with \(|\mathbf{v}|^2 + \rho v_2 > |v_2| \) we get the same description for the \( \Lambda \)-convex hull. If we could get this result for all \((\rho, \mathbf{v}) \in \mathbb{R} \times \mathbb{R}^2 \), we could formulate Theorem 1.2 for the \( \Lambda \)-convex hull instead of the lamination convex hull.

5. Non-existence of non-trivial subsolutions in bounded domains

As observed in [10] (although stated under different hypotheses), if \( \mathbf{v} \in L^2_0(\Omega, \mathbb{R}^2) \) and \( \rho \in L^\infty(\Omega) \) form a solution of stationary IPM, then

\[
(5.1) \quad \int_\Omega |\mathbf{v}|^2 = \int_\Omega \mathbf{v} \cdot [-\nabla \rho - (0, \rho)] = -\int_\Omega \rho v_2 = -\int_\Omega \rho \mathbf{v} \cdot \nabla y = 0.
\]

We adapt the proof to subsolutions with values in \( K^{lc,\Lambda} \) by using the exact form of \( K^{lc,\Lambda} \) computed in Theorem 1.1.

**Proof of Theorem 1.2.** Since \( \mathbf{m} \in L^2_0(\Omega, \mathbb{R}^2) = [\nabla W^{1,2}(\Omega)]^\perp \) and \((\rho, \mathbf{v}, \mathbf{m})(x) \in K^{lc,\Lambda} \) a.e. \( x \in \Omega \), we may write

\[
(5.2) \quad 0 = \int_\Omega \mathbf{m} \cdot \nabla y = \int_{v=0}^1 \frac{1 - \rho^2}{2} (v_2 - 1) + \sum_{j=2}^4 \int_{(\rho,v)\in\chi_j} k v_2.
\]

If \((\rho, \mathbf{v}) \in \chi_2 \), then \( 1 \leq k \leq \rho - (1 - \rho^2)v_2/|\mathbf{v}|^2 \) so that either \( \rho = 1 \) or \( v_2 < 0 \). In both cases, \( k v_2 \leq v_2 \). Thus

\[
(5.3) \quad \int_{(\rho,v)\in\chi_2} k v_2 \leq \int_{(\rho,v)\in\chi_2} v_2.
\]

Similarly, if \((\rho, \mathbf{v}) \in \chi_4 \), then \( \rho - (1 - \rho^2)v_2/|\mathbf{v}|^2 \leq k \leq -1 \) so that either \( \rho = -1 \) or \( v_2 > 0 \), giving \( k v_2 \leq -v_2 \) and

\[
(5.4) \quad \int_{(\rho,v)\in\chi_4} k v_2 \leq -\int_{(\rho,v)\in\chi_4} v_2.
\]

Furthermore, since \((\rho, \mathbf{v}, \mathbf{m})(x) \in K^{lc,\Lambda} \) a.e. \( x \in \Omega \), we get

\[
\int_{(\rho,v)\in\chi_3} k v_2 = \int_{(\rho,v)\in\chi_3} \rho v_2 - \int_{(\rho,v)\in\chi_3} \frac{(1 - \rho^2)v_2^2}{|\mathbf{v}|^2}.
\]

Using (5.2)–(5.4),

\[
-\int_{(\rho,v)\in\chi_3} \rho v_2 = -\int_{(\rho,v)\in\chi_3} k v_2 - \int_{(\rho,v)\in\chi_3} \frac{(1 - \rho^2)v_2^2}{|\mathbf{v}|^2}
\]

\[
= \int_{v=0}^1 \frac{1 - \rho^2}{2} (v_2 - 1) + \int_{(\rho,v)\in\chi_2 \cup \chi_4} k v_2 - \int_{\chi_3} \frac{(1 - \rho^2)v_2^2}{|\mathbf{v}|^2}
\]

\[
\leq \int_{(\rho,v)\in\chi_2} v_2 - \int_{(\rho,v)\in\chi_4} v_2 - \int_{(\rho,v)\in\chi_3} \frac{(1 - \rho^2)v_2^2}{|\mathbf{v}|^2}.
\]
and so, using the assumption that \( v \in L^2_\rho(\Omega, \mathbb{R}^2) \),

\[
0 \leq \int_{\Omega} |v|^2 = \int_{\Omega} v \cdot [-\nabla p - (0, \rho)] = - \int_{\Omega} \rho v_2 = - \sum_{j=2}^{4} \int_{(\rho, v) \in X_j} \rho v_2
\]

\[
\leq \int_{(\rho, v) \in X_2} v_2 - \int_{(\rho, v) \in X_4} v_2 - \int_{(\rho, v) \in X_3} \frac{(1 - \rho^2)v_2^2}{|v|^2}
\]

\[
= \int_{(\rho, v) \in X_2} (1 - \rho)v_2 - \int_{(\rho, v) \in X_4} (1 + \rho)v_2 - \int_{(\rho, v) \in X_3} \frac{(1 - \rho^2)v_2^2}{|v|^2}
\]

\[
\leq 0,
\]

where in the last inequality we have used \( v_2 \leq 0 \) in \( X_2 \) and \( v_2 \geq 0 \) in \( X_4 \). We thus conclude that \( v = 0 \). Now (1.6) gives \( \partial_t \rho = \nabla^1 \cdot (0, \rho) = 0 \).

\[\square\]

**Remark 5.1.** The proof of Theorem 1.2 also works essentially verbatim with impermeable walls in the vertical direction and periodic boundary conditions in the horizontal direction. Thus, the dichotomy on directions of strips that we mentioned in the introduction extends to subsolutions with values in \( K^{\top, \Omega} \).

Adapting (5.1) to a strip with finite width in the direction \((0, 1)\), we briefly indicate the role that the direction \((0, 1)\) plays. The second equality in (5.1) uses the boundary conditions that \( v \cdot \nu|_{\partial \Omega} = 0 \) when \( y = 0 \) and \( v \) is periodic in \( x \); this part works equally in the setting of [5]. However, the fourth equality in (5.1) uses the fact that \((x, y) \mapsto y\) is periodic in \( x \). It is here that the adaptation to all other strips breaks down, and thus there is no geometric obstruction to the solutions of [5]. In the proof of Theorem 1.2, the fourth equality of (5.1) is necessarily replaced by a weaker condition, and the proof requires the precise computation of \( K^{\top, \Omega} \) in Theorem 1.1.

### 6. Relation to the Infinite Time Limit of Non-stationary IPM

As the last topic of this paper, we show that Theorem 1.2 reflects the behaviour of subsolutions of non-stationary IPM at the limit \( t \to \infty \). The proof is a straightforward application of [10, Corollary 1.2] which states that \( \partial_t \int_\Omega \rho x_2 \, dx = 2^{-1} \partial_t \int_\Omega |\rho - x_2|^2 \, dx = - \int_\Omega |v|^2 \, dx \) for smooth solutions of non-stationary IPM.

**Proposition 6.1.** Suppose \( \rho \in L^\infty(0, \infty; L^\infty) \) and \( v, m \in L^\infty(0, \infty; L^2_\rho) \) form a subsolution of non-stationary IPM in a smooth, bounded, simply connected domain \( \Omega \subset \mathbb{R}^2 \). Then \( v \in L^2(0, \infty; L^2_\rho) \).

Proposition 6.1 and its proof work equally well in the confined IPM case \( \Omega = \mathbb{T}^1 \times (-1, 1) \). Before presenting the proof, we recall the definition of a subsolution in this context. Under the integrability assumptions of Theorem 6.1, \( z = (\rho, v, m) \) is a subsolution of non-stationary IPM if

\[
\tag{6.1} z(x) \in K^\rho = \left\{ (\rho, v, m) : |\rho| \leq 1, |m - \rho v| \leq \frac{1 - \rho^2}{2} \right\}
\]
Combining (6.5)–(6.7), we conclude that
\begin{equation}
0 = \int_0^\infty (\rho \partial_t \varphi + m \cdot \nabla \varphi) \, dx \, dt + \int_0^\infty \rho_0 \varphi(\cdot, 0) \, dx \quad \forall \varphi \in C_c^\infty(\Omega \times [0, \infty)), \tag{6.2}
\end{equation}
\begin{equation}
0 = \int_0^\infty \int_\Omega \nabla \varphi \, dx \, dt \quad \forall \varphi \in C_c^\infty(\overline{\Omega} \times [0, \infty)), \tag{6.3}
\end{equation}
\begin{equation}
0 = \int_0^\infty \int_\Omega (v + (0, \rho)) \cdot \nabla \perp \varphi \, dx \, dt \quad \forall \varphi \in C_c^\infty(\Omega \times [0, \infty)). \tag{6.4}
\end{equation}

Note that (6.2)–(6.3) incorporate the condition \( \nabla \perp \varphi \), and (6.4) gives \( v \cdot \nabla \varphi \).

**Proof of Proposition 6.1.** Let \( \eta \in C_c^\infty(0, \infty) \) and set \( \varphi(x, t) := \eta(t)x_2 \) in (6.2), so that
\[
\int_0^\infty \eta'(t) \int_\Omega \rho(x, t) x_2 \, dx \, dt + \int_0^\infty \eta \int_\Omega m_2(x, t) \, dx \, dt = 0.
\]
As a consequence, \( \partial_t \int_\Omega \rho(x, \cdot) x_2 \, dx = \int_\Omega m_2(x, \cdot) \, dx \in L^\infty(0, \infty) \) in the sense of distributions. Thus, after possibly modifying \( \rho \) on a set of measure zero, \( F(t) := \int_\Omega \rho(x, t) x_2 \, dx \) is Lipschitz continuous and
\[
F(t) = \int_\Omega \rho_0(x) x_2 \, dx + \int_0^t \int_\Omega m_2(x, \tau) \, dx \, d\tau \tag{6.5}
\]
for all \( t \in [0, \infty) \).

We use (6.1) to get \( m_2 = \rho v_2 + (1 - \rho^2)(v_2 - 1)/2 \), where \( \mathbf{e} = (e_1, e_2) \) takes values in \( B(0, 1) \), so that
\[
\int_0^t \int_\Omega m_2(x, \tau) \, dx \, d\tau \leq \int_0^t \int_\Omega \rho(x, \tau) v_2(x, \tau) \, dx \, d\tau. \tag{6.6}
\]
Now, approximating \( v \) in \( L^2(0, t; L^2_\sigma) \) by mappings \( \nabla \perp \varphi_j, \varphi_j \in C_c^\infty(\Omega \times [0, t)) \), the assumption (6.4) gives
\[
\int_0^t \int_\Omega \rho(x, \tau) v_2(x, \tau) \, dx \, d\tau = -\int_0^t \int_\Omega |v(x, \tau)|^2 \, dx \, d\tau. \tag{6.7}
\]
Combining (6.5)–(6.7), we conclude that
\[
\int_0^t \int_\Omega |v(x, \tau)|^2 \, dx \, d\tau - \int_\Omega \rho_0(x) x_2 \, dx \leq -F(t) \leq \int_\Omega |\rho(x, t) x_2| \, dx \leq \|\rho\|_{L^\infty(0, \infty; L^\infty)} \int_\Omega |x_2| \, dx
\]
for all \( t \in [0, \infty) \). The claim follows. \( \square \)

**Acknowledgments.** We express warm thanks to Ángel Castro, Daniel Faraco and Francisco Mengual for useful comments.

**References**

[1] V. Arnaiz, Á. Castro, and D. Faraco, *Semiclassical estimates for pseudodifferential operators and the Muskat problem in the unstable regime*, arXiv:2001.06361 (2020).

[2] Á. Castro, D. Córdoba, and D. Faraco, *Mixing solutions for the Muskat problem*, arXiv:1605.04822 (2016).

[3] Á. Castro, D. Córdoba, and D. Lear, *Global existence of quasi-stratified solutions for the confined IPM equation*, Arch. Ration. Mech. Anal. 232 (2019), no. 1, 437–471.

[4] Á. Castro, D. Faraco, and F. Mengual, *Degraded mixing solutions for the Muskat problem*, Calc. Var. Partial Differential Equations 58 (2019), no. 2, Paper No. 58, 29.
[5] P. Constantin, J. La, and V. Vicol, Remarks on a paper by Gavrilov: Grad-Shafranov equations, steady solutions of the three dimensional incompressible Euler equations with compactly supported velocities, and applications, Geom. Funct. Anal. 29 (2019), no. 6, 1773–1793.
[6] D. Córdoba, D. Faraco, and F. Gancedo, Lack of uniqueness for weak solutions of the incompressible porous media equation, Arch. Ration. Mech. Anal. 200 (2011), no. 3, 725–746.
[7] D. Córdoba and F. Gancedo, Contour dynamics of incompressible 3-D fluids in a porous medium with different densities, Comm. Math. Phys. 273 (2007), no. 2, 445–471.
[8] C. De Lellis and L. Székelyhidi, Jr., The Euler equations as a differential inclusion, Ann. of Math. (2) 170 (2009), no. 3, 1417–1436.
[9] The h-principle and the equations of fluid dynamics, Bull. Amer. Math. Soc. (N.S.) 49 (2012), no. 3, 347–375.
[10] T. M. Elgindi, On the asymptotic stability of stationary solutions of the inviscid incompressible porous medium equation, Arch. Ration. Mech. Anal. 225 (2017), no. 2, 573–599.
[11] D. Faraco, S. Lindberg, and L. Székelyhidi Jr., Bounded solutions of ideal MHD with compact support in space-time, arXiv:1909.08678 (2020).
[12] E. Feireisl, C. Klingenberg, and S. Markfelder, On the density of “wild” initial data for the compressible Euler system, arXiv:1812.11802v2 (2019).
[13] C. Förster and L. Székelyhidi, Jr., Piecewise constant subsolutions for the Muskat problem, Comm. Math. Phys. 363 (2018), no. 3, 1051–1080.
[14] A. V. Gavrilov, A steady Euler flow with compact support, Geom. Funct. Anal. 29 (2019), no. 1, 190–197.
[15] P. Isett and V. Vicol, Hölder continuous solutions of active scalar equations, Ann. PDE 1 (2015), no. 1, Art. 2, 77.
[16] B. Kirchheim, Rigidity and Geometry of Microstructures, Lecture notes, Max-Planck-Inst. für Mathematik in den Naturwiss., 2003.
[17] G. Knott, Oscillatory Solutions to Hyperbolic Conservation Laws and Active Scalar Equations, Ph.D. thesis, Universität Leipzig, 2013.
[18] S. Markfelder, On the \( \Lambda \)-Convex Hull for Convex Integration Applied to the Isentropic Compressible Euler System, arXiv:2001.04773 (2020).
[19] F. Mengual, H-principle for the 2D incompressible porous media equation with viscosity jump, arXiv:2004.03307 (2020).
[20] F. Noisette and L. Székelyhidi Jr., Mixing solutions for the Muskat problem with variable speed, arXiv:2005.08814 (2020).
[21] F. Otto, Evolution of microstructure in unstable porous media flow: a relaxational approach, Comm. Pure Appl. Math. 52 (1999), no. 7, 873–915.
[22] R. Shvydkoy, Convex integration for a class of active scalar equations, J. Amer. Math. Soc. 24 (2011), no. 4, 1159–1174.
[23] L. Székelyhidi, Jr., Relaxation of the incompressible porous media equation, Ann. Sci. Éc. Norm. Supér. (4) 45 (2012), no. 3, 491–509.
[24] L. Tartar, Compensated compactness and applications to partial differential equations, Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol. IV, Res. Notes in Math., vol. 39, Pitman, Boston, Mass.-London, 1979, pp. 136–212.