Bounds in proton-proton elastic scattering at low momentum transfer

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Abstract

We present a bound on the imaginary part of the single helicity-flip amplitude for spin 1/2-spin 1/2 scattering at small momentum transfer. The variational method of Lagrange multipliers is employed to optimize the single-flip amplitude using the values of $\sigma_{\text{tot}}$, $\sigma_{\text{el}}$ and diffraction slope as equality constraints in addition to the inequality constraints resulting from unitarity. Such bounds provide important information related to the determination of polarization of a proton beam. In the case of elastic proton collisions the analyzing power at small scattering angles offers a method of measuring the polarization of a proton beam, the accuracy of the polarization measurement depending on the single helicity-flip amplitude. The bound obtained on the imaginary part of the single helicity-flip amplitude indicates that the analyzing power for proton-proton collisions in the Coulomb nuclear interference region should take positive nonzero values at high energies.

1 Introduction

The proton spin puzzle has intrigued experimentalists and theorists since the surprising result from the EMC experiment at CERN in 1988, which found a smaller than expected

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contribution to the spin of the proton from the component quarks. The question, “where does the spin of the proton come from?” remains unanswered [1, 2]. Recent data suggests a value of $\sim 20 - 30\%$ for the fraction of the spin carried by the up, down and strange quarks. The contribution from the gluons and from the orbital angular momentum of the quarks and gluons is not completely known. The Relativistic Heavy Ion Collider at Brookhaven National Laboratory plans to probe the proton structure using the deep inelastic scattering of protons at high center-of-mass energies ($\sqrt{s} = 50 - 500$ GeV) and momentum transfers ($p_T \geq 10$ GeV/c) [3]. To measure the contribution of the gluons to the spin of the proton, with sufficient accuracy, a polarized proton beam with a beam polarization error of 5% is necessary.

One method of measuring the polarization of a proton beam uses the analyzing power in elastic proton collisions at small scattering angles. The analyzing power $A_N$ for a proton and the transverse single spin asymmetry $A$ are related through the expression

$$A_N P = A$$

where $P$ is the beam polarization; for 100% beam polarization the asymmetry and analyzing power are equal. The beam polarization can be measured by counting the scatters with the beam polarized up ($N^\uparrow$) and then down ($N^\downarrow$) in a polarimeter with a known analyzing power $A_N$:

$$P = \frac{1}{A_N} \left[ \frac{N^\uparrow - N^\downarrow}{N^\uparrow + N^\downarrow} \right] = \frac{1}{A_N} A. \quad (1.2)$$

The analyzing power $A_N$ expressed in terms of the $s$-channel helicity amplitudes is

$$A_N \frac{d\sigma}{dt} = -\frac{4\pi}{s(s - 4m^2)} \text{Im} [\phi_5^* (\phi_1 + \phi_2 + \phi_3 - \phi_4)]. \quad (1.3)$$

The reduced ratio, $r_5$, of the helicity single-flip to imaginary non-flip amplitude is defined as

$$r_5 = \frac{m}{\sqrt{-t}} \times \frac{\phi_5}{\text{Im} \phi_+}. \quad (1.4)$$

The analyzing power $A_N$ for the CNI region can be written as follows, when the transverse total cross section spin difference is neglected [4]:

$$A_N = \frac{\sqrt{-t}}{m} \left( \frac{k - 2 \text{Im} r_5 (t_c/t) + 2(\rho \text{Im} r_5 - \text{Re} r_5)}{(t_c/t)^2 - 2(\rho + \delta)(t_c/t) + 1 + \rho^2 + \beta^2} \right) \quad (1.5)$$

where $t_c = -(8\pi\alpha)/\sigma_{\text{tot}}$, $\delta$ is the Coulomb phase, $\kappa + 1 = \mu = 2.7928$ is the magnetic moment of the proton, and $\beta^2$ represents the forward hadronic double spinflip contributions expected to be negligible at high energies. Apart from the photon pole term, the
$t$-dependences of helicity nonflip and flip electromagnetic and hadronic amplitudes due to form factor and nuclear slope effects are not expected to play a significant role in the amplitude ratios featuring in the asymmetry. Inside the CNI region interference between the electromagnetic and hadronic amplitudes is most prominent. An important contribution to the maximum of $A_N$, in the CNI region ($|t| < |t_c|$), comes from $\text{Im} r_5$ in the form of $\mu - 1 - 2 \text{Im} r_5$. At larger momentum transfers outside the CNI region ($|t| > |t_c|$) the analyzing power, containing hadronic terms only, is essentially

$$A_N = \frac{\sqrt{-t}}{m} \frac{2(\rho \text{Im} r_5 - \text{Re} r_5)}{1 + \rho^2 + \beta^2}$$  \hspace{1cm} (1.6)$$

and because the value of $\rho$ is expected to be no more than about 10\% at RHIC energies, the analyzing power provides a good indication of the value of $\text{Re} r_5$ just outside the CNI region. The contribution of $\text{Re} r_5$ to the maximum of the analyzing power would be reasonably well known. A bound on $\text{Im} r_5$ which satisfies $\mu - 1 - 2 \text{Im} r_5 > 0$ ensures that the maximum analyzing power in the CNI region is positive.

The optimization technique of Lagrange multipliers, extended by Einhorn and Blankenbecler [5, 6, 7] to include equality and inequality constraints in the context of scattering theory, is used to derive the bound

$$|\text{Im} r_5| \leq m \sqrt{g} \left( \frac{36\pi g \sigma_{el}}{\sigma_{tot}^2} - 1 \right)^{1/2} \times h(t)$$  \hspace{1cm} (1.7)$$

where

$$h(t) = \frac{(1 + gt(2 + 9gt/8))^{1/2}}{(1 + gt)}.$$  \hspace{1cm} (1.8)$$

The partial wave expansions for the observables, $\sigma_{tot}$, $\sigma_{el}$, and the hadronic slope parameter $g$ to be defined in Section 2, are included as equality constraints in the system, unitarity is presented as partial wave inequality constraints and the modified imaginary single-flip amplitude, the function to be optimized, is input as the objective function. Before proceeding to the derivation of the bound in Section 3 we first introduce the $s$-channel helicity amplitudes, expanded as partial wave series in the helicity representation. Errors on the derived bounds are discussed in Section 4 followed by concluding remarks in Section 5.
2 Helicity Amplitudes and Observables

For the elastic scattering of two protons at CM energy $\sqrt{s}$ and CM momentum $k = \sqrt{s - 4m^2}/2$, there are sixteen helicity amplitudes in general, each a function of $s$ and $t$. The number of independent amplitudes is reduced under the following relations [8, 9]:

Parity conservation

$$\langle \lambda'_1 \lambda'_2 | \phi | \lambda_1 \lambda_2 \rangle = (-1)^{\mu - \lambda} \langle -\lambda'_1 - \lambda'_2 | \phi | -\lambda_1 - \lambda_2 \rangle \quad (2.1)$$

Time reversal invariance

$$\langle \lambda'_1 \lambda'_2 | \phi | \lambda_1 \lambda_2 \rangle = (-1)^{\mu - \lambda} \langle \lambda_1 \lambda_2 | \phi | \lambda'_1 \lambda'_2 \rangle \quad (2.2)$$

Identical particle scattering

$$\langle \lambda'_1 \lambda'_2 | \phi | \lambda_1 \lambda_2 \rangle = (-1)^{\lambda - \mu} \langle \lambda'_2 \lambda'_1 | \phi | \lambda_2 \lambda_1 \rangle \quad (2.3)$$

where $\lambda = \lambda_1 - \lambda_2$, $\mu = \lambda'_1 - \lambda'_2$.

The sixteen helicity amplitudes reduce to two non helicity-flip amplitudes $\phi_1$ and $\phi_3$, two double helicity-flip amplitudes $\phi_2$ and $\phi_4$, and one single helicity-flip amplitude $\phi_5$, with partial wave expansions [8, 9]:

$$\phi_1(s, t) = \langle + + | \phi | + + \rangle = \frac{\sqrt{s}}{2k} \sum_J (2J + 1) \left( f^J_0(s) + f^J_{11}(s) \right) d^J_{00}(\theta) \quad (2.4)$$

$$\phi_3(s, t) = \langle + - | \phi | + - \rangle = \frac{\sqrt{s}}{2k} \sum_J (2J + 1) \left( f^J_1(s) + f^J_{22}(s) \right) d^J_{11}(\theta) \quad (2.5)$$

$$\phi_2(s, t) = \langle + + | \phi | - - \rangle = \frac{\sqrt{s}}{2k} \sum_J (2J + 1) \left( f^J_{11}(s) - f^J_0(s) \right) d^J_{00}(\theta) \quad (2.6)$$

$$\phi_4(s, t) = \langle + - | \phi | - + \rangle = \frac{\sqrt{s}}{2k} \sum_J (2J + 1) \left( f^J_{22}(s) - f^J_1(s) \right) d^J_{-11}(\theta) \quad (2.7)$$

$$\phi_5(s, t) = \langle + + | \phi | + - \rangle = \frac{\sqrt{s}}{2k} \sum_J (2J + 1) f^J_{21}(s) d^J_{10}(\theta) \quad (2.8)$$

where $f^J_i(i = 0, 1, 11, 22, 21)$ denote $s$-channel partial wave amplitudes, $\text{Im} f^J_i = a^J_i$, $\text{Re} f^J_i = b^J_i$ and $z = \cos \theta = 1 + t/2k^2$. In the Coulomb Nuclear Interference (CNI) region, $t \approx -0.0012(\text{GeV}/c)^2$, it is convenient to express the five helicity amplitudes in terms of Jacobi polynomials. To define the $d^J_{\lambda \mu}(\theta)$ function in terms of Jacobi polynomials
it is suitable to separate the space of $\lambda$ and $\mu$ into four regions A, B, C, D. In region A, where $\lambda + \mu \geq 0$ and $\lambda - \mu \geq 0$, the relation is

$$d_{\lambda\mu}^J(\theta) = \left( \frac{(J+\lambda)!}{(J+\mu)!} \frac{(J-\lambda)!}{(J-\mu)!} \right) \frac{(1+z)}{2} \frac{(1-z)}{2} \times P_{J-\lambda,\lambda+\mu}(z), \quad (2.9)$$

and $J - \lambda = 0, 1, 2, \cdots$. Equivalent forms in the other regions are obtained by use of symmetry relations [3, 10].

In region B, where $\lambda + \mu \geq 0$ and $\lambda - \mu \leq 0$, use

$$d_{\lambda\mu}^J(\theta) = (-1)^{\lambda-\mu} d_{\mu\lambda}^J(\theta). \quad (2.10)$$

In region C, where $\lambda + \mu \leq 0$ and $\lambda - \mu \leq 0$, use

$$d_{\lambda\mu}^J(\theta) = (-1)^{\lambda-\mu} d_{-\lambda-\mu}^J(\theta). \quad (2.11)$$

In region D, where $\lambda + \mu \leq 0$ and $\lambda - \mu \geq 0$, use

$$d_{\lambda\mu}^J(\theta) = d_{-\mu-\lambda}^J(\theta). \quad (2.12)$$

Expressing the $d_{\lambda\mu}^J(\theta)$ functions in terms of Jacobi polynomials the five independent helicity amplitudes can be written as

$$\phi_1(s, t) = \frac{\sqrt{s}}{2k} \sum_J (2J+1) \left( f_0^J(s) + f_1^J(s) \right) P_{J}^{(0,0)}(z) \quad (2.13)$$

$$\phi_2(s, t) = \frac{\sqrt{s}}{2k} \sum_J (2J+1) \left( f_0^J(s) - f_1^J(s) \right) P_{J}^{(0,0)}(z) \quad (2.14)$$

$$\phi_3(s, t) = \frac{\sqrt{s}(1+z)}{4k} \sum_J (2J+1) \left( f_1^J(s) + f_2^J(s) \right) P_{J-1}^{(0,2)}(z) \quad (2.15)$$

$$\phi_4(s, t) = \frac{\sqrt{s}(1-z)}{4k} \sum_J (2J+1) \left( f_2^J(s) - f_1^J(s) \right) P_{J-1}^{(2,0)}(z) \quad (2.16)$$

$$\phi_5(s, t) = \frac{\sqrt{s}\sqrt{1-z^2}}{4k} \sum_J (2J+1) \sqrt{\frac{J+1}{J}} f_1^J(s) P_{J-1}^{(1,1)}(z) \quad (2.17)$$

where $z = \cos\theta = 1 + t/(2k^2)$. The Jacobi polynomials have the properties [11]

$$P_n^{(\alpha,\beta)}(1) = \frac{\Gamma(\alpha + n + 1)}{\Gamma(\alpha 1) n!}, \quad (2.18)$$

$$\frac{d^m}{dz^m} P_n^{(\alpha,\beta)}(z) = 2^{-m} \frac{\Gamma(m+n+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+1)} P_{n-m}^{(\alpha+m,\beta+m)}(z). \quad (2.19)$$
2.1 Observables

In proton-proton elastic scattering the spin observables can be written in terms of the five helicity amplitudes, $\phi_1, \ldots, \phi_5$. The observables are vital to the optimization, since each observable can be included as an equality constraint in the optimized system. In the derivation of the bound we use three equality constraints, $\sigma_{\text{tot}}$, $\sigma_{\text{el}}$ and $g$, and two inequality constraints related to unitarity.

2.1.1 Total Cross Section

The first equality constraint uses the total cross section. The optical theorem,

$$\left. \Im \phi_+ (s, t) \right|_{t=0} = \frac{k}{2\pi} \sqrt{s} \sigma_{\text{tot}} (s),$$

is used to express the total cross section as a partial wave expansion given by

$$\sigma_{\text{tot}} (s) = \frac{\pi}{k^2} \sum_J (2J + 1) \left\{ (a^J_0 + a^J_{11}) P^J_{(0,0)} (1) + (a^J_1 + a^J_{22}) P^J_{(0,2)} (1) \right\}$$

where $\left. \Im \phi_+ (s, t) \right|_{t=0}$ is the imaginary spin average helicity non-flip amplitude. Using property (2.18) the normalized dimensionless total cross section can be expressed as a partial wave expansion:

$$A_0 = \frac{\pi}{k^2} \sum_J (2J + 1) \left\{ a^J_0 (s) + a^J_1 (s) + a^J_{11} (s) + a^J_{22} (s) \right\}$$

where $A_0 = (k^2 / \pi) \sigma_{\text{tot}}$.

2.1.2 Slope of the Imaginary Non-Flip Amplitude

The slope of the imaginary non-flip amplitude has been used in bounds for other spin dependent elastic collisions [12]. We find it convenient to use the imaginary part of the spin-averaged amplitude at a particular value of $t$. The second equality constraint employs the imaginary spin average non-flip amplitude at a particular $t$ inside the Coulomb Nuclear Interference region, written as a Taylor expansion:

$$\left. \Im \phi_+ (s, t) \right|_{t=0} = \frac{k}{2\pi} \sqrt{s} \sigma_{\text{tot}} (s),$$

where $|t|$ is sufficiently small so that inclusion of the linear term in the Taylor expansion is an accurate approximation. Use of properties (2.18) and (2.19) leads to the partial wave expansion for the imaginary non-flip amplitude:

$$\Im \phi_+ (s, t) = \frac{\sqrt{s}}{4k} \sum_J (2J + 1) \left\{ a^J_0 + a^J_1 + a^J_{11} + a^J_{22} \right\} \left( 1 - \frac{\zeta}{4} J (J + 1) \right)$$
where $\zeta = -t/k^2$. The logarithmic derivative of the imaginary spin average non-flip amplitude,

$$g = \frac{d}{dt} \ln \text{Im} \phi_+(s, t)|_{t=0} = \frac{1}{\text{Im} \phi_+(s, 0)} \left(\frac{d}{dt} \text{Im} \phi_+(s, t)\right)|_{t=0}, \quad (2.25)$$

can be expressed as

$$\left(\frac{d}{dt} \text{Im} \phi_+(s, t)\right)|_{t=0} = g \text{Im} \phi_+(s, 0). \quad (2.26)$$

The Taylor expansion for $\text{Im} \phi_+(s, t)$, given by Equation (2.23), can thus be written as

$$\text{Im} \phi_+(s, t) = \text{Im} \phi_+(s, 0) \{1 - (-t g)\}. \quad (2.27)$$

### 2.1.3 Elastic Cross Section

The third equality relates to the elastic cross section, expressed as a partial wave expansion by integrating the differential cross section over momentum transfer $t$:

$$\sigma_{el}(s) = \int_{-4k^2}^{0} dt \frac{d\sigma(s, t)}{dt}. \quad (2.28)$$

Expressing the differential cross section in terms of helicity amplitudes allows us to write the elastic cross section as

$$\sigma_{el}(s) = \frac{\pi}{2k^2 s} \int_{-4k^2}^{0} dt \left\{|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 + |\phi_4|^2 + 4|\phi_5|^2\right\}. \quad (2.29)$$

Using the expression

$$t = -2k^2(1 - z), \quad (2.30)$$

the $t$ variable can be replaced with the $z$ variable, where $t$ is the momentum transfer, $k$ is the center-of-mass three-momentum and $z = \cos \theta$. The elastic cross section expressed as an integral over $z$ becomes

$$\sigma_{el}(s) = \frac{\pi}{s} \int_{-1}^{+1} dz \left\{|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 + |\phi_4|^2 + 4|\phi_5|^2\right\}. \quad (2.31)$$

To express the elastic cross section as a partial wave expansion, the integrals

$$\int_{-1}^{+1} dz |\phi_i(s, t)|^2$$

are calculated, where $i = 1, \ldots, 5$. The integration formula [13]

$$\int_{-1}^{+1} (1 - z)^\alpha (1 + z)^\beta \frac{P_n^{(\alpha, \beta)}(z)}{P_m^{(\alpha, \beta)}(z)} \, dz = \begin{cases} 0 & m \neq n \\ \sum_{n' + \beta + 1} \frac{\Gamma(\alpha + n + 1) \Gamma(\beta + n + 1)}{n'! (\alpha + \beta + 2n + 1) \Gamma(\alpha + \beta + n + 1)} \delta_{m, n}, & m = n \end{cases} \quad (2.32)$$


may be used to find the integral
\[ \int_{-1}^{+1} d\chi^M_{\mu}(\theta) d\chi^M_{-\mu}(\theta) \, dz = \frac{2}{2J+1} \delta_{M,j}, \]  
leading to a partial wave expansion for the normalized dimensionless elastic cross section, defined as \( \Sigma_{el} = (k^2/\pi) \sigma_{el} \):

\[ \Sigma_{el}(s) = \sum_J (2J+1) \left\{ |f_{0J}|^2 + |f_{1J}|^2 + |f_{11J}|^2 + 2 |f_{21J}|^2 \right\}. \]  

2.2 Imaginary Single-Flip Amplitude

The imaginary single helicity-flip amplitude, modified by a kinematical factor, is the objective function in the system. Before optimization we must first express the single-flip amplitude in a suitable form. The imaginary amplitude,

\[ \text{Im} \phi_5(s,t) = \sqrt{\frac{s}{2k}} \sum_J (2J+1) a_{21J} d^{\mu}_{10}(\theta), \]  
written in terms of Jacobi polynomials and normalized by a kinematical factor, becomes

\[ \text{Im} \tilde{\phi}_5 = \frac{\text{Im} \phi_5(s,t)}{(1-z^2)^{1/2}} = \sqrt{\frac{\sqrt{s}}{4k}} \sum_J (2J+1) \sqrt{\frac{J+1}{J}} a_{21J} P^{(1,1)}_{j-1}(z). \]  

In the CNI region, the Jacobi polynomial \( P^{(1,1)}_{j-1}(z) \) expanded as a Taylor series is

\[ P^{(1,1)}_{j-1}(z) \approx P^{(1,1)}_{j-1}(z) \bigg|_{z=1} - \frac{\zeta}{2} \left( \frac{d}{dz} P^{(1,1)}_{j-1}(z) \right) \bigg|_{z=1}, \]  
where \( z = 1 + t/(2k^2) \) and \( \zeta = -t/k^2 \). Using properties (2.18) and (2.19), the Taylor series for \( P^{(1,1)}_{j-1}(z) \) about \( z = 1 \) is

\[ P^{(1,1)}_{j-1}(z) \approx J \left( 1 - \frac{\zeta}{8} [J(J+1) - 2] \right), \]  
and thus, inside the CNI region,

\[ \text{Im} \tilde{\phi}_5 \approx \frac{\sqrt{s}}{4k} \sum_J (2J+1) \sqrt{\frac{J+1}{J}} J \left( 1 - \frac{\zeta}{8} [J(J+1) - 2] \right) a_{21J}. \]  

At small collision angles,

\[ \frac{m}{\sqrt{-t}} \approx \frac{m}{k (1-z^2)^{1/2}}, \]  
the ratio \( \text{Im} r_5 = m \text{Im} \phi_5/\sqrt{-t} \text{Im} \phi_+ \) is

\[ \text{Im} r_5 = \frac{m}{k} \frac{\text{Im} \tilde{\phi}_5}{\text{Im} \phi_+(s,t)}. \]
2.3 Unitarity

The partial wave amplitudes obey the following unitarity inequalities [14]

\[ U_1^J = a_0^J - |f_0^J|^2 \geq 0 \] (2.41)
\[ U_2^J = a_1^J - |f_1^J|^2 \geq 0 \] (2.42)
\[ V_1^J = a_{11}^J - |f_{11}^J|^2 - |f_{21}^J|^2 \geq 0 \] (2.43)
\[ V_2^J = a_{22}^J - |f_{22}^J|^2 - |f_{21}^J|^2 \geq 0 \] (2.44)

where \( f_i^J (i = 0, 1, 11, 22, 21) \) denote the s-channel partial wave amplitudes, \( \text{Im} f_i^J = a_i^J \) and \( \text{Re} f_i^J = b_i^J \). Combining Equation (2.41) with Equation (2.42) leads to the inequality

\[ X^J = U_1^J + U_2^J \]

where

\[ X^J = a_0^J + a_1^J - |f_0^J|^2 - |f_1^J|^2 \geq 0 \] (2.45)

and the inequality \( W^J = V_1^J + V_2^J \) follows from the combination of Equation (2.43) and Equation (2.44) where

\[ W^J = a_{11}^J + a_{22}^J - |f_{11}^J|^2 - |f_{22}^J|^2 - 2|f_{21}^J|^2 \geq 0 \] (2.46)

For the elastic scattering of spin 0 on spin 1/2 particles there are two independent helicity amplitudes, a flip and a non-flip amplitude, with partial wave expansions whose partial wave amplitudes obey unitarity relations similar to relations (2.41) and (2.42). The unitarity relations (2.43) and (2.44) are characteristic of spin 1/2 - spin 1/2 scattering, the \( f_{21}^J \) term coming from the single helicity-flip amplitude \( \phi_5 \).

3 Optimization

Equipped with partial wave expansions for the observables and partial wave inequality relations, representing unitarity, we are in a position to optimize the modified helicity single-flip amplitude \( \text{Im} \tilde{\phi}_5 \). We follow the variational technique of Einhorn and Blankenbecler [5] by constructing a Lagrangian consisting of an objective function and a set of equality and inequality constraints. We use the full set of constraints, \( \sigma_{\text{tot}}, \sigma_{\text{el}}, g \) and unitarity, although a bound on \( \text{Im} \tilde{\phi}_5 \), with fewer constraints, can be derived [15].
3.1 Lagrange Formalism

The normalized dimensionless total cross section $A_0$, expressed as an equality constraint, is included in the Lagrange function along with the normalized dimensionless elastic cross section $\Sigma_{el}$, written as an equality constraint, the imaginary spin average non helicity-flip amplitude $\text{Im} \phi_+ (s, t)$ at a fixed small $|t|$ value, also expressed as an equality constraint, and the partial wave unitarity relations written as inequality constraints. The modified single helicity-flip amplitude $\text{Im} \tilde{\phi}_5$ is introduced as the objective function in the Lagrange function:

$$\mathcal{L} = \text{Im} \tilde{\phi}_5 + \alpha \left[ A_0 - \sum J (2J + 1) \left\{ a'_0 + a'_1 + a'_{11} + a'_{22} \right\} \right]$$

$$+ \beta \left[ \Sigma_{el} - \sum J (2J + 1) \left( |f'_0|^2 + |f'_1|^2 + |f'_{11}|^2 + |f'_{22}|^2 + 2|f'_{21}|^2 \right) \right]$$

$$+ \gamma \left[ \text{Im} \phi_+ - \frac{\sqrt{s}}{4k} \sum J (2J + 1) \left\{ a'_0 + a'_1 + a'_{11} + a'_{22} \right\} \left( 1 - \frac{\zeta}{4} J(J+1) \right) \right]$$

$$+ \sum J (2J + 1) \mu_J \left( a'_{11} + a'_{22} - |f'_{11}|^2 - |f'_{22}|^2 - 2|f'_{21}|^2 \right)$$

$$+ \sum J (2J + 1) \lambda_J \left( a'_0 + a'_1 - |f'_0|^2 - |f'_1|^2 \right)$$  \hspace{1cm} (3.1)

where $\alpha$, $\beta$ and $\gamma$ are equality multipliers. The inequality multipliers, $\lambda_J$ and $\mu_J$, are by definition non-negative and $\zeta = -t/(k^2)$. In the high energy or large $J$ limit only the leading order $J$ terms are included and the Lagrange function of Equation (3.1) becomes

$$\mathcal{L} = \text{Im} \tilde{\phi}_5 + \alpha \left[ A_0 - 2 \sum J \left\{ a'_0 + a'_1 + a'_{11} + a'_{22} \right\} \right]$$

$$+ \beta \left[ \Sigma_{el} - 2 \sum J \left( |f'_0|^2 + |f'_1|^2 + |f'_{11}|^2 + |f'_{22}|^2 + 2|f'_{21}|^2 \right) \right]$$

$$+ \gamma \left[ \text{Im} \phi_+ - \sum J \left\{ a'_0 + a'_1 + a'_{11} + a'_{22} \right\} \left( 1 - \frac{\zeta}{4} J^2 \right) \right]$$

$$+ 2 \sum J \mu_J \left( a'_{11} + a'_{22} - |f'_{11}|^2 - |f'_{22}|^2 - 2|f'_{21}|^2 \right)$$

$$+ 2 \sum J \lambda_J \left( a'_0 + a'_1 - |f'_0|^2 - |f'_1|^2 \right)$$  \hspace{1cm} (3.2)

and

$$\text{Im} \tilde{\phi}_5 \approx \sum J^2 \left( 1 - \frac{\zeta}{8} J^2 \right) a'_{21}.$$  \hspace{1cm} (3.3)

The system is optimized by taking first and second derivatives with respect to the real and imaginary partial wave amplitudes, $b'_j$ and $a'_j$. This gives the optimized set of partial waves, at some fixed $t$ in the CNI region;

$$b'_j = 0 \quad \forall i,$$  \hspace{1cm} (3.4)
\[ a_0^J = a_1^J = \frac{r_1 + r_2 \left( 1 - \frac{\zeta}{4} J^2 \right) + \tilde{\lambda}_J}{1 + 2\tilde{\lambda}_J}, \]

and

\[ a_{11}^J = a_{22}^J = \frac{r_1 + r_2 \left( 1 - \frac{\zeta}{4} J^2 \right) + \tilde{\mu}_J}{1 + 2\tilde{\mu}_J} \]

where \( \tilde{\lambda}_J = \frac{\lambda_J}{2\beta} \), \( \tilde{\mu}_J = \frac{\mu_J}{2\beta} \), \( r_1 = -\alpha/(2\beta) \), \( r_2 = -\gamma/(4\beta) \) and \( \beta > 0 \) for a maximum (or \( \beta < 0 \) for a minimum).

### 3.2 Unitarity Classes

Optimization under the four constraints imposes the following condition:

\[ b_i^J = 0 \ \forall \ i \implies f_i^J = a_i^J + b_i^J = a_i^J, \]

that is, there is no contribution from the real partial wave amplitudes. The imaginary partial wave amplitudes therefore obey the following unitarity inequalities:

\[ X^J = a_0^J - a_0^{J2} \geq 0 \]

\[ W^J = a_{11}^J - a_{11}^{J2} - a_{21}^{J2} \geq 0 \]

where \( a_0^J = a_1^J \) and \( a_{11}^J = a_{22}^J \). It is natural to divide the partial waves into two classes, one with contributions from the interior unitarity class \( I \) and the other with contributions from the boundary unitarity class \( B \). For the \( X^J \) unitarity inequality the interior and boundary unitarity classes are defined as

\[ I^X \equiv \{ J \mid X_J > 0, \tilde{\lambda}_J = 0 \}, \quad B^X \equiv \{ J \mid X_J = 0, \tilde{\lambda}_J \geq 0 \}. \]

Likewise for the \( W^J \) unitarity inequality the interior and boundary unitarity classes are

\[ I^W \equiv \{ J \mid W_J > 0, \tilde{\mu}_J = 0 \}, \quad B^W \equiv \{ J \mid W_J = 0, \tilde{\mu}_J \geq 0 \}. \]

#### 3.2.1 \( I^X \) and \( B^X \) Unitarity Classes

The interior unitarity class,

\[ I^X \equiv \{ J \mid X^J = a_0^J - a_0^{J2} > 0, \tilde{\lambda}_J = 0 \}, \]
under the four constraints, is expressed as

\[ I^X \equiv \left\{ J \mid 0 < a_0^J < 1, \tilde{\lambda}_J = 0 \right\}. \]  


Equation (3.5) with \( \tilde{\lambda}_J \) set to zero enables us to write the imaginary partial wave amplitude \( a_0^J \), in the interior unitarity class, as

\[ a_0^J = r_1 + r_2 \left( 1 - \frac{\zeta}{4} J^2 \right). \]  

(3.15)

The constraint \( 0 < a_0^J < 1 \) restricts the values of the equality multipliers, \( r_1 \) and \( r_2 \), to \( 0 < r_1 + r_2 < 1 \) and \( r_2 > 0 \). The number of partial waves \( J \) are thus limited to

\[ 0 \leq J^2 < \frac{4}{\zeta} \left( 1 + \frac{r_1}{r_2} \right). \]  

(3.16)

The boundary unitarity class \( B^X \) splits into two sub-classes, \( B^{X_0} \) and \( B^{X_1} \):

\[ B^X \equiv \left\{ J \mid X_J = a_0^J - a_0^{J^2} = 0, \tilde{\lambda}_J \geq 0 \right\} \]

\[ \rightarrow B^{X_0} \equiv \left\{ J \mid a_0^J = 0, \tilde{\lambda}_J \geq 0 \right\} \]

\[ B^{X_1} \equiv \left\{ J \mid a_0^J = 1, \tilde{\lambda}_J \geq 0 \right\}. \]  

(3.17)

In the boundary unitarity class \( B^{X_0} \) the imaginary partial wave amplitude \( a_0^J \) is equal to zero and from Equation (3.5) the inequality multiplier \( \tilde{\lambda}_J \) is given by

\[ \tilde{\lambda}_J = -(r_1 + r_2) + r_2 \frac{\zeta}{4} J^2 \geq 0. \]  

(3.18)

The \( B^{W_0} \) class begins at \( J^2 = M_1^2 = 4/\zeta (1+r_1/r_2) \), and for \( J \geq M_1 + 1 \), with \( 0 < r_1 + r_2 < 1 \) and \( r_2 > 0 \), the inequality multiplier \( \tilde{\lambda}_J \) is positive. Therefore the boundary unitarity class \( B^{X_0} \) is non-empty for \( J \geq M_1 + 1 \) but with \( a_0^J = 0 \), for all \( J \) in this unitarity class, there are no contributions to the observables from this unitarity class. The imaginary partial wave amplitude \( a_0^J \) is equal to unity in the boundary unitarity class \( B^{X_1} \) and from Equation (3.5) the inequality multiplier \( \tilde{\lambda}_J \) is given by

\[ \tilde{\lambda}_J = (r_1 + r_2) - 1 - r_2 \frac{\zeta}{4} J^2. \]  

(3.19)

By definition \( \tilde{\lambda}_J \geq 0 \) and the value of \( J \), in the boundary unitarity class, is limited to

\[ J^2 \leq \frac{4}{\zeta} \left( \frac{(r_1 + r_2) - 1}{r_2} \right) \]  

(3.20)
In summary, the unitarity classes, $I^X$ and $B^X_0$, are non-empty and the unitarity class $B^X_1$ is empty:

\[
I^X \equiv \{ J | 0 < a^J_0 < 1, \ 0 < J \leq M_1 \}, \tag{3.21}
\]

\[
B^X_0 \equiv \{ J | a^J_0 = 0, \ M_1 + 1 \leq J \leq M_2 \} \tag{3.22}
\]

where $M_1 = \text{Floor} \left[ \sqrt{\frac{4}{\zeta (1 + r_1/r_2)}} \right]$, $M_2 = \text{Floor} \left[ \sqrt{\frac{8}{\zeta}} \right]$ and $\zeta = -t/k^2$. The \text{Floor}[x] function gives the greatest integer less than or equal to $x$.

### 3.2.2 $I^W$ and $B^W$ Unitarity Classes

The interior unitarity class $I^W$ under the optimization becomes

\[
I^W \equiv \{ J \ | \ W^J = a^J_{11} - a^J_{11}^2 - a^J_{21}^2 > 0, \ \bar{\mu}_J = 0 \} \tag{3.23}
\]

Substituting Equations (3.6) and (3.7), with $\bar{\mu}_J = 0$, into the interior constraint $a^J_{11} - a^J_{11}^2 - a^J_{21}^2 > 0$ leads to the equation:

\[
f_2(J) = \tilde{a}_1 + \tilde{a}_2 J^2 + \tilde{a}_3 J^4 + \tilde{a}_4 J^6 > 0 \tag{3.24}
\]

where $\tilde{a}_1 = (r_1 + r_2) (1 - (r_1 + r_2))$, $\tilde{a}_2 = r_2 \zeta (2 (r_1 + r_2) - 1) / 4 - 1 / (64 \beta^2)$, $\tilde{a}_3 = \zeta / (256 \beta^2) - \frac{r_2^2 \zeta^2}{16}$, $\tilde{a}_4 = -\zeta^2 / (64 \beta^2)$, and only positive $J$ solutions are allowed. The solution is of the form

\[
0 < J^2 < \eta_2^2 \frac{4}{\zeta} \left(1 + \frac{r_1}{r_2}\right) = \eta_2^2 M_1^2 \tag{3.25}
\]

where $\eta_2 \sim 1$. The boundary unitarity class $B^W$ is written as

\[
B^W \equiv \{ J \ | \ W^J = a^J_{11} - a^J_{11}^2 - a^J_{21}^2 = 0, \ \bar{\mu}_J \geq 0 \} \tag{3.26}
\]

The constraint $a^J_{11} - a^J_{11}^2 - a^J_{21}^2 = 0$ can be written as a quadratic equation:

\[
\bar{\mu}_J^2 + \tilde{\mu}_J + f_2(J) = 0 \tag{3.27}
\]

where

\[
f_2(J) = \tilde{a}_1 + \tilde{a}_2 J^2 + \tilde{a}_3 J^4 + \tilde{a}_4 J^6. \tag{3.28}
\]

The solutions are

\[
\tilde{\mu}_J = \frac{1}{2} \left\{ \pm \sqrt{1 - 4 f_2(J)} - 1 \right\}. \tag{3.29}
\]

The function $f_2(J)$ is negative for $J > M_1 = \text{Floor} \left[ 4 / \zeta (1 + r_1/r_2) \right]$ and consequently $\bar{\mu}_J$ is positive for such $J$ values. By definition $\bar{\mu}_J \geq 0$, therefore the positive solution is chosen:

\[
\bar{\mu}_J = \frac{1}{2} \left\{ \sqrt{1 - 4 f_2(J)} - 1 \right\}. \tag{3.30}
\]
To summarize, both the classes, $I^W$ and $B^W$, are non-empty:

$$I^W \equiv \left\{ J | a_{11}^J - a_{12}^J - a_{21}^J > 0, \ 0 \leq J \leq M_1 \right\},$$  \hspace{1cm} (3.31)

$$B^W \equiv \left\{ J | a_{11}^J - a_{12}^J - a_{21}^J = 0, \ M_1 + 1 \leq J \leq M_2 \right\},$$  \hspace{1cm} (3.32)

with $\eta_2 = 1$, where $M_1 = \text{Floor} \left[ \sqrt{4/\zeta (1 + r_1/r_2)} \right]$, $M_2 = \text{Floor} \left[ \sqrt{8/\zeta} \right]$ and $\zeta = -t/k^2$.

It is important to notice that with $\eta_2 = 1$ both interior unitarity classes, $I^W$ and $I^X$, are non-empty over the same region, $J \in [0, M_1]$. Similarly the boundary unitarity classes, $B^X_0$ and $B^W$, are non-empty over the same region, $M_1 + 1 \leq J \leq M_2$. In other words there is no mixing of unitarity classes, all classes are either interior unitarity classes, $I \equiv I^X \cup I^W$, or boundary unitarity classes, $B \equiv B^X \cup B^W$, for a given $J$.

### 3.3 Solution of Interior Unitarity Class

Consider the set of interior unitarity classes, $I \equiv I^X \cup I^W$. The inequality multipliers, $\tilde{\lambda}_J$ and $\tilde{\mu}_J$, in the interior region are equal to zero. The imaginary partial wave amplitudes are therefore written as

$$a_k^J = r_1 + r_2 \left( 1 - \frac{\zeta}{4} J^2 \right),$$  \hspace{1cm} (3.33)

and

$$a_{21}^J = \frac{J}{8\beta} \left( 1 - \frac{\zeta}{8} J^2 \right),$$  \hspace{1cm} (3.34)

$k = 0, 1, 11, 22$, with $0 \leq J \leq M_1$, where $M_1 = \text{Floor} \left[ \sqrt{4/\zeta (1 + r_1/r_2)} \right]$ is the maximum $J$ in the interior unitarity class. In this case the contributions to the observables and to the objective function Im $\tilde{\phi}_5$ solely come from the interior unitarity class $I$; $A_0^I = A_0$, Im $\phi_+^I = \text{Im} \phi_+$, $\Sigma_{el}^I = \Sigma_{el}$ and Im $\tilde{\phi}_5^I = \text{Im} \tilde{\phi}_5$. The normalized dimensionless total cross section is reconstructed by substituting Equation (3.33) into

$$A_0 = \sum_{J=0}^{M_1} J \left( a_0^J + a_1^J + a_{11}^J + a_{22}^J \right)$$  \hspace{1cm} (3.35)

to give

$$A_0 = 8 \sum_{J=0}^{M_1} J \left[ r_1 + r_2 \left( 1 - \frac{\zeta}{4} J^2 \right) \right].$$  \hspace{1cm} (3.36)
The Euler-Maclaurin expansion [16] for large $J$ is used to write the normalized dimensionless total cross section $A_0$ as an integration over $J$: 

$$A_0 \approx 8 \int_0^{M_1} dJ \left((r_1 + r_2)J - r_2 \frac{\zeta}{4} J^2 \right)$$ \hspace{0.5cm} (3.37) 

$$\approx \frac{M_1^2}{2} \left\{ 8(r_1 + r_2) - r_2 \zeta M_1^2 \right\}.$$ \hspace{0.5cm} (3.38) 

Similarly the imaginary spin average helicity non-flip amplitude $\text{Im} \phi_+^I$ is reconstructed by substituting Equation (3.33) into 

$$\text{Im} \phi_+ = \sum_{J=0}^{M_1} J \left\{ a_J^0 + a_J^1 + a_{11}^J + a_{22}^J \right\} \left( 1 - \frac{\zeta}{4} J^2 \right)$$ \hspace{0.5cm} (3.39) 

to give 

$$\text{Im} \phi_+ \approx 4 \int_0^{M_1} dJ \left\{ (r_1 + r_2)J - (2r_2 + r_1) \frac{\zeta}{4} J^3 + r_2 \frac{\zeta^2}{16} J^5 \right\}$$ \hspace{0.5cm} (3.40) 

$$\approx M_1^2 \left\{ 2(r_1 + r_2) - (2r_2 + r_1) \frac{\zeta}{4} M_1^2 + \frac{r_2 \zeta^2}{24} M_1^4 \right\}.$$ \hspace{0.5cm} (3.41) 

The dimensionless normalized elastic cross section $\Sigma_{el}^I$, by substituting Equations (3.33) and (3.34) into 

$$\Sigma_{el} = 2 \sum_{J=0}^{M_1} J \left\{ a_J^2 + a_J^4 + a_{11}^2 + a_{22}^2 + 2a_{21}^2 \right\}$$ \hspace{0.5cm} (3.42) 

is reconstructed: 

$$\Sigma_{el} \approx \left\{ 4(r_1 + r_2)^2 M_1^2 - (r_1 + r_2)r_2 \zeta M_1^4 + \frac{r_2 \zeta^2}{12} M_1^6 \right\}$$ \hspace{0.5cm} (3.43) 

$$+ \frac{M_1^4}{64 \beta^2} \left\{ 1 - \frac{\zeta}{6} M_1^2 + \frac{\zeta^2}{128} M_1^4 \right\}.$$ 

The modified imaginary single-flip amplitude $\text{Im} \tilde{\phi}_5$ is reconstructed by substituting Equation (3.34) into 

$$\text{Im} \tilde{\phi}_5 = \sum_{J=0}^{M_1} J^2 \left( 1 - \frac{\zeta}{8} J^2 \right) a_{21}^J$$ \hspace{0.5cm} (3.44) 

leading to 

$$\text{Im} \tilde{\phi}_5 = \frac{1}{8 \beta} \sum_{J=0}^{M_1} J^3 \left( 1 - \frac{\zeta}{8} J^2 \right)^2.$$ \hspace{0.5cm} (3.45) 

For large $J$ the modified imaginary single-flip amplitude is written as 

$$\text{Im} \tilde{\phi}_5 \approx \frac{1}{8 \beta} \frac{M_1^4}{4} \left( 1 - \frac{\zeta}{6} M_1^2 + \frac{\zeta^2}{128} M_1^4 \right).$$ \hspace{0.5cm} (3.46)
An expression for the equality multiplier \( \beta \) is found by solving Equation (3.43):

\[
\beta = \frac{M_1^2 \left\{ 1 - \frac{\zeta}{6} M_1^2 + \frac{\zeta^2}{128} M_1^4 \right\}^{1/2}}{8 \left\{ \Sigma_{el} - \left( 4(r_1 + r_2)^2 M_1^2 - (r_1 + r_2) r_2 \zeta M_1^4 + \frac{12 \zeta^2}{12} M_1^6 \right) \right\}^{1/2}}. \tag{3.47}
\]

Rewriting the modified imaginary single-flip amplitude one obtains,

\[
\text{Im} \tilde{\phi}_5 \approx \left\{ \Sigma_{el} - \left( 4(r_1 + r_2)^2 M_1^2 - (r_1 + r_2) r_2 \zeta M_1^4 + \frac{12 \zeta^2}{12} M_1^6 \right) \right\}^{1/2} \\
\times \frac{M_1^2}{4} \left\{ 1 - \frac{\zeta}{6} M_1^2 + \frac{\zeta^2}{128} M_1^4 \right\}^{1/2}. \tag{3.48}
\]

The equality multipliers, \( r_1 \) and \( r_2 \), are found by solving Equations (3.38) and (3.41). The solutions are given by

\[
r_1 = \frac{A_0^3 \zeta (1 - 3 \text{Im} \phi_+/A_0)}{36 (1 - 2 \text{Im} \phi_+/A_0)^2} \tag{3.49}
\]

and

\[
r_2 = \frac{A_0^2 \zeta}{72 (1 - 2 \text{Im} \phi_+/A_0)^2} \tag{3.50}
\]

where \( \zeta = -t/k^2 \). The equality multiplier \( \beta \), with solutions for \( r_1 \) and \( r_2 \), is expressed as

\[
\beta = \frac{9(A_0 - 2 \text{Im} \phi_+) \sqrt{1 - 2 \text{Im} \phi_+/A_0 + 36 \text{Im} \phi_+^2/A_0^2}}{2A_0 \zeta \sqrt{72 \Sigma_{el} - A_0^2 \zeta/(1 - 2 \text{Im} \phi_+/A_0)}}. \tag{3.51}
\]

The optimized modified imaginary single-flip amplitude, expressed as a function of \( r_1 \), \( r_2 \) and \( \beta \), becomes

\[
\text{Im} \tilde{\phi}_5 = \frac{(A_0 - 2 \text{Im} \phi_+)}{4A_0 \zeta} \sqrt{1/2 - 2(1 - \text{Im} \phi_+/A_0) \text{Im} \phi_+} \tag{3.52}
\]

with

\[
J_{\text{max}} = \frac{12}{\zeta} \left( 1 - 2 \frac{\text{Im} \phi_+}{A_0} \right). \tag{3.53}
\]

For low momentum transfers the imaginary spin average non-flip amplitude \( \text{Im} \phi_+ \), expanded to order \( t \), is written as

\[
\text{Im} \phi_+ \approx \frac{A_0}{2} (1 + gt). \tag{3.54}
\]

Under this approximation the maximum \( J \) inside the interior unitarity class is independent of \( t \) and in the limit \( t \to 0 \), the number of partial waves is finite where

\[
J_{\text{max}} = \sqrt{12g k}. \tag{3.55}
\]
The equality multipliers in the low $t$ limit become

$$r_1 = \frac{A_0}{72g^2k^2} \left( \frac{1 + 3gt}{t} \right),$$  \hspace{1cm} (3.56)$$

$$r_2 = -\frac{A_0}{72g^2k^2} \frac{1}{t},$$  \hspace{1cm} (3.57)$$

and

$$\beta = \frac{\sqrt{2} 9gk^2}{\sqrt{72\Sigma_{el} - 2A_0^2/(gk^2)}} (1 + gt(2 + 9gt/8))^{1/2}. \hspace{1cm} (3.58)$$

The upper bound on $|\text{Im} r_5|$, where $|\text{Im} r_5| = m |\text{Im} \bar{\phi}_5|/(k \text{Im} \phi_+)$, can be expressed analytically:

$$|\text{Im} r_5| \leq \frac{\sqrt{2}mkg}{A_0} \left( \frac{18\Sigma_{el} - \frac{A_0^2}{2gk^2}}{h(t)} \right) \times h(t) \hspace{1cm} (3.59)$$

where

$$h(t) = \frac{(1 + gt(2 + 9gt/8))^{1/2}}{(1 + gt)}. \hspace{1cm} (3.60)$$

The variable $h(t)$ is finite at $t = 0$ and changes ‘slowly’ over the CNI region. Writing $A_0 = k^2\sigma_{tot}/\pi$ and $\Sigma_{el} = k^2\sigma_{el}/\pi$, enables the bound on $|\text{Im} r_5|$ to be expressed as

$$|\text{Im} r_5| \leq m \sqrt{2g} \left( \frac{36g\sigma_{el}}{g^2\sigma_{tot}} - 1 \right)^{1/2} \times h(t). \hspace{1cm} (3.61)$$

### 3.4 Results

The value of the bound on $|\text{Im} r_5|$ is given in Tables [1] and [2] with the values of the equality multipliers. The most noticeable feature of the bound is its size at low momentum transfers, having a value of 0.89 at $\sqrt{s} = 52.8$ GeV, $t = -0.001$ (GeV/c)$^2$. The optimized partial waves, at $\sqrt{s} = 52.8$ GeV and $t = -0.001$ (GeV/c)$^2$, is shown in Figure [3]. The partial wave series terminates at $J = 231$ which is the upper $J$ limit, $M_1$, for the interior unitarity class $I$. When considering both the interior and boundary unitarity classes, values of $J > M_1$ are permitted.

The bound on $|\text{Im} r_5|$, under the approximation

$$g \approx \frac{\sigma_{tot}^2}{32\pi\sigma_{el}^2}, \hspace{1cm} (3.62)$$

with momentum transfers in the CNI region is expressed as

$$|\text{Im} r_5| \leq m \frac{g}{8} \times h(t) \hspace{1cm} (3.63)$$
Table 1: $|\text{Im} r_5|^{\text{max}}$ inside the interior region at $t = -0.001$ (GeV/c)$^2$.

| $\sqrt{s}$ (GeV) | $r_1$ | $r_2$ | $\beta$ | $J_{\text{max}}$ | $|\text{Im} r_5|$ |
|----------------|------|------|--------|----------------|----------------|
| 19.4           | -12.54 | 12.77 | 90     | 81             | 0.97           |
| 23.5           | -12.56 | 12.79 | 117    | 98             | 0.92           |
| 30.7           | -12.02 | 12.24 | 158    | 131            | 0.92           |
| 44.7           | -11.22 | 11.44 | 217    | 195            | 1.05           |
| 52.8           | -11.53 | 11.76 | 293    | 231            | 0.89           |
| 62.5           | -11.56 | 11.79 | 358    | 276            | 0.86           |

Table 2: $|\text{Im} r_5|^{\text{max}}$ inside the interior region at $t = -0.01$ (GeV/c)$^2$.

| $\sqrt{s}$ (GeV) | $r_1$ | $r_2$ | $\beta$ | $J_{\text{max}}$ | $|\text{Im} r_5|$ |
|----------------|------|------|--------|----------------|----------------|
| 19.4           | -1.25    | 1.49  | 85     | 81             | 0.97           |
| 23.5           | -1.05    | 1.27  | 111    | 98             | 0.91           |
| 30.7           | -1.00    | 1.22  | 150    | 131            | 0.92           |
| 44.7           | -0.92    | 1.14  | 204    | 195            | 1.05           |
| 52.8           | -0.95    | 1.17  | 276    | 231            | 0.89           |
| 62.5           | -0.94    | 1.18  | 337    | 276            | 0.86           |

and in the zero momentum transfer limit, $t \to 0$, the bound on $|\text{Im} r_5|$ is finite and can be expressed analytically as

$$|\text{Im} r_5| \leq m \sqrt{\frac{g}{8}}. \quad (3.64)$$

This approximation generates a ‘stricter’ bound on $|\text{Im} r_5|$. The results are given in Table 3.

Figure 1: $a^I_k$ ($k = 0, 1, 11, 22$) and $a^J_{21}$ in the interior class; $\sqrt{s} = 52.8$ GeV, $t = -0.001$ (GeV/c)$^2$. 

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Table 3: $|\text{Im} r_5|^{\text{max}}$, with an approximation for $g$, over the CNI region.

| $\sqrt{s}$ (GeV) | $t = 0$ (GeV/c)$^2$ | $t = -0.001$ (GeV/c)$^2$ | $t = -0.01$ (GeV/c)$^2$ |
|-----------------|-----------------|-----------------|-----------------|
| 19.4            | 0.803           | 0.805           | 0.825           |
| 23.5            | 0.805           | 0.808           | 0.827           |
| 30.7            | 0.819           | 0.821           | 0.842           |
| 44.7            | 0.839           | 0.841           | 0.864           |
| 52.8            | 0.841           | 0.843           | 0.866           |
| 62.5            | 0.846           | 0.848           | 0.871           |

3.5 Solution of Interior and Boundary Classes

Consider the set of classes $I \cup B \equiv I^W \cup I^X \cup B^W \cup B^X$. The boundary unitarity classes are

$$B^X \equiv \{ J |a_{11}^J - a_{12}^J = 0, M_1 + 1 \leq J \leq M_2 \}$$

(3.65)

and

$$B^{W_0} \equiv \{ J |a_0^J = 0, M_1 + 1 \leq J \leq M_2 \}$$

(3.66)

where $M_1 = \text{Floor}[\sqrt{4/\zeta(1+r_1/r_2)}]$, $M_2 = \text{Floor}[\sqrt{8/\zeta}]$ and $\zeta = -t/k^2$. The contribution to $|\text{Im} r_5|$ from the boundary unitarity class $B$ can range from 0% to 100% and the contribution to $|\text{Im} r_5|$, from the boundary unitarity class, can be selected without violating any of the constraints and this contribution can be made arbitrarily small.

Consider the case with $\Sigma_{el}^B = 0.1\Sigma_{el}$, $\Sigma_{el}^I = 0.9\Sigma_{el}$, at $\sqrt{s} = 52.8$ GeV and $t = -0.001$ (GeV/c)$^2$. The maximum contribution to $|\text{Im} r_5|$ is 34.7 where $|\text{Im} r_5^I| \leq 0.5$ and $|\text{Im} r_5^B| \leq 34.2$. The case with $\Sigma_{el}^B = 0.01\Sigma_{el}$, $\Sigma_{el}^I = 0.99\Sigma_{el}$, leads to $|\text{Im} r_5| \leq 11.6$ where $|\text{Im} r_5^I| \leq 0.8$ and $|\text{Im} r_5^B| \leq 10.6$. The bound on $|\text{Im} r_5^B|$ falls when the fraction of $\Sigma_{el}$ in the boundary unitarity class is reduced. The partial wave amplitudes in this region also become smaller in amplitude and contribute less to the bound on $|\text{Im} r_5^B|$. The fraction of $\Sigma_{el}$ in the boundary unitarity class can be further reduced until the contribution from this class to $|\text{Im} r_5|$ is negligible in comparison with the contribution from the interior unitarity class. In this limit the bound is, as before,

$$|\text{Im} r_5| \leq m \sqrt{g} \left( \frac{36\pi g \sigma_{el}}{\sigma_{tot}^2} - 1 \right)^{1/2} \times h(t)$$

(3.67)
or, under the approximation $g \approx \sigma_{\text{tot}}^2/(32\pi \sigma_{\text{el}})$,

$$|\text{Im} r_5| \leq m\sqrt{\frac{g}{8}} \times h(t)$$  \hspace{1cm} (3.68)

where

$$h(t) = \frac{(1 + gt(2 + 9/8gt))^{1/2}}{(1 + gt)}.$$  \hspace{1cm} (3.69)

The bound is identical to the bound when only the interior unitarity class is considered. A finite number of partial waves at low momentum transfer ensures a finite value for the sum

$$\text{Im} \tilde{\phi}_5 = \sum_j J^2 a_{21}^J \left(1 - \frac{\zeta}{8} J^2\right)$$  \hspace{1cm} (3.70)

and consequently a finite upper bound on $|\text{Im} r_5|$ of less than unity.

### 4 Error on Bound

The upper bound on the imaginary single helicity-flip amplitude, modified by a kinematical factor at zero momentum transfer, is given by

$$|\text{Im} r_5| \leq m\sqrt{\frac{g}{8}} \left(\frac{36\pi g \sigma_{\text{el}}}{\sigma_{\text{tot}}^2} - 1\right)^{1/2}.$$  \hspace{1cm} (4.1)

There are errors on all the experimental quantities in Eqn. (4.1) and consequently the upper bound on $|\text{Im} r_5|$ has an uncertainty. The experimental quantities $g$, $\sigma_{\text{tot}}$ and $\sigma_{\text{el}}$ have a nominal value plus an uncertainty: $g \pm \Delta g$, $\sigma_{\text{tot}} \pm \Delta \sigma_{\text{tot}}$ and $\sigma_{\text{el}} \pm \Delta \sigma_{\text{el}}$. What are the values of $\Delta g$, $\Delta \sigma_{\text{tot}}$ and $\Delta \sigma_{\text{el}}$? Consider the value of $\text{Im} r_5$ at $\sqrt{s} = 52.8$ GeV; $|\text{Im} r_5| \leq 0.891$ with $g = 6.435 \pm 0.14$ GeV$^{-2}$ [17], $\sigma_{\text{tot}} = 42.906$ mb [18] and $\sigma_{\text{el}} = 7.407$ mb [18].

The value of $\Delta g$ is known but we must calculate $\Delta \sigma_{\text{tot}}$ and $\Delta \sigma_{\text{el}}$.

A parameterization for the total and elastic cross section in elastic $pp$ collisions [18] allows a value for the cross sections to be found and a value of their uncertainties to be calculated. Each cross section is parameterized as

$$\sigma(p) = A + Bp^n + C \log^2(p) + D \log(p)$$  \hspace{1cm} (4.2)

where $\sigma$ is in mb and $p$ is the laboratory momentum in GeV/c. The uncertainty in $\sigma$ is given by
\[ \Delta \sigma = \left\{ \left( \frac{\partial \sigma}{\partial A} \right)^2 (\Delta A)^2 + \left( \frac{\partial \sigma}{\partial B} \right)^2 (\Delta B)^2 + \left( \frac{\partial \sigma}{\partial n} \right)^2 (\Delta n)^2 + \left( \frac{\partial \sigma}{\partial C} \right)^2 (\Delta C)^2 + \left( \frac{\partial \sigma}{\partial D} \right)^2 (\Delta D)^2 \right\}^{1/2}. \]  

(4.3)

The fitted parameters \( A, B, n, C \) and \( D \) are given in Table 4.

| Reaction | \( A \)       | \( B \)  | \( n \)     | \( C \)     | \( D \)     |
|----------|--------------|---------|-------------|-------------|-------------|
| \( \sigma_{\text{tot}} \) | 48.0 ± 0.1   | -       | -           | 0.522 ± 0.005 | -4.51 ± 0.05 |
| \( \sigma_{\text{el}} \)  | 11.9 ± 0.8   | 26.9 ± 1.7 | -1.21 ± 0.11 | 0.169 ± 0.021 | -1.85 ± 0.26 |

Using Eqn. (4.3), and the values of the fitted parameters, the uncertainties \( \Delta \sigma_{\text{tot}} \) and \( \Delta \sigma_{\text{el}} \) can be written as

\[ \Delta \sigma_{\text{tot}} = \sqrt{0.01 + 2.5 \times 10^{-5} \log^4(p) + 2.5 \times 10^{-3} \log^2(p)} \]  

(4.4)

and

\[ \Delta \sigma_{\text{el}} = \sqrt{0.64 + 2.89p^{-2.42} + 12.819p^{-4.42} + 4.41 \times 10^{-4} \log^4(p) + 0.0676 \log^2(p)}. \]  

(4.5)

A laboratory beam momentum of \( p = k\sqrt{s}/m = 1485 \text{ GeV/c} \) at \( \sqrt{s} = 52.8 \text{ GeV} \) gives \( \Delta \sigma_{\text{tot}} = 0.463 \text{ mb} \), or 1.08\% of \( \sigma_{\text{tot}} \) and \( \Delta \sigma_{\text{el}} = 2.345 \text{ mb} \), or 31.66\% of \( \sigma_{\text{el}} \).

The uncertainty in \( \text{Im} r_5 \) is

\[ \Delta \text{Im} r_5 = \sqrt{\left( \frac{\partial \text{Im} r_5}{\partial g} \right)^2 (\Delta g)^2 + \left( \frac{\partial \text{Im} r_5}{\partial \sigma_{\text{tot}}} \right)^2 (\Delta \sigma_{\text{tot}})^2 + \left( \frac{\partial \text{Im} r_5}{\partial \sigma_{\text{el}}} \right)^2 (\Delta \sigma_{\text{el}})^2}. \]  

(4.6)

At \( \sqrt{s} = 52.8 \text{ GeV} \), \( g = 6.435 \pm 0.14 \text{GeV}^{-2} \), \( \sigma_{\text{tot}} = 42.906 \pm 0.463 \text{ mb} \) and \( \sigma_{\text{el}} = 7.407 \pm 2.345 \text{ mb} \). The uncertainty \( \Delta \text{Im} r_5 = 0.049 \) and the upper bound on \( |\text{Im} r_5| \) is \( 0.891 \pm 0.049 \).

The approximation

\[ g \approx \frac{\sigma_{\text{tot}}^2}{32\pi \sigma_{\text{el}}} \]  

(4.7)

can be used to write the bound on \( |\text{Im} r_5| \) as

\[ |\text{Im} r_5| \leq m \sqrt{\frac{g}{8}} \]  

(4.8)
at zero momentum transfer. The uncertainty $\Delta \text{Im} r_5$ is simply

$$\Delta \text{Im} r_5 = \sqrt{\left( \frac{\partial \text{Im} r_5}{\partial g} \right)^2 (\Delta g)^2}.$$  \hfill (4.9)

At $\sqrt{s} = 52.8$ GeV, $g = 6.435 \pm 0.14\text{GeV}^{-2}$, $|\text{Im} r_5| \leq 0.846$ and $\Delta \text{Im} r_5 = 0.009$ or $|\text{Im} r_5| \leq 0.846 \pm 0.009$. The error on $\sigma_{el}$ being large has relatively little effect on the uncertainty of the bound.

## 5 Conclusion

A bound on the size of the imaginary part of the single helicity-flip amplitude, in the CNI region, was obtained using the Lagrange variational method of Einhorn and Blankenbecler with $\sigma_{tot}$, $\sigma_{el}$ and diffraction slope expressed as equality constraints and unitarity expressed as two inequality constraints. An important feature associated with the bound is the number of partial waves at low momentum transfer. In the CNI region the number of waves is finite, that is, there is no singularity in $\sqrt{-t}$ as $t \to 0$. This feature ensures that a useful bound exists and, in fact, the bound limits $|\text{Im} r_5|$ to values less than unity near $t = 0$. With more constraints in the system an improved bound could be obtained although any additional constraints must be sufficiently well known experimentally and be computationally tractable. As the bound of 0.84 on the helicity flip amplitude ratio is less than $(\mu_p - 1)/2 = 0.896$ at the high energies considered, the coefficient of $(t_c/t)$ in the expression for the asymmetry is constrained to be positive and therefore the analyzing power is positive for at least a significant part of the interference region. Though the bound is not sufficient to limit the polarization error to the recommended 5%, it does permit the use of proton proton elastic collisions in the CNI region as a relative polarimeter.
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