Stochastic Quantization of Matrix Models
and Field Theory of Non-Orientable Strings

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Abstract

In quantizing gravity based on stochastic quantization method, the stochastic time
plays a role of the proper time. We study 2D and 4D Euclidean quantum gravity in this
context. By applying stochastic quantization method to real symmetric matrix models,
it is shown that the stochastic process defined by the Langevin equation in loop space
describes the time evolution of the non-orientable loops which defines non-orientable 2D
surfaces. The corresponding Fokker-Planck hamiltonian deduces a non-orientable string
field theory at the continuum limit. The strategy, which we have learned in the example
of 2D quantum gravity, is applied to 4D case. Especially, the Langevin equation for the
stochastic process of 3-geometries is proposed to describe the ( Euclidean ) time evolution
in 4D quantum gravity with Ashtekar’s canonical variables. We present it in both lattice
regularized version and the naive continuum limit.

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Introduction

String field theory [1] is believed to be the most promising approach to investigate non-perturbative effect in string theories. Recently, non-critical string field theories have been proposed for \( c = 0 \) [2][3][4][5] and for \( 0 < c < 1 \) [6][7]. Among these, some [2][4][6][7] are based on the transfer-matrix formalism [8] in dynamical triangulation of random surfaces [9]. While some [3][5] are derived by using stochastic quantization method [10].

In the approach by stochastic quantization of matrix models, one can interpret the stochastic (fictitious) time as a Euclidean time coordinate in 2D quantum gravity. There is also another example of this particular observation, the proper time interpretation of the stochastic time, which was found in the course to study QCD in terms of the Nicolis-Langevin maps [11], that stochastic quantization of 3D Chern-Simons theory recovers the time evolution in 4D Euclidean Yang-Mills theory [12]. These facts motivate us to interpret the stochastic time as the time coordinate in Euclidean 4D quantum gravity [14].

In this paper, we illustrate how to apply stochastic quantization method to real symmetric matrix models [13] and show that it leads to a field theory of non-orientable (non-critical) strings [5]. The stochastic process defined by the Langevin equation in loop space describes the time evolution of the non-orientable loops on non-orientable 2D surfaces. The corresponding Fokker-Planck hamiltonian is a loop space hamiltonian of non-orientable string field theories. At the equilibrium limit, it deduces the Virasoro constraint equation for the probability distribution functional. The continuum limit of the field theory of discretized non-orientable loops is taken for the simplest one-matrix case (\( c = 0 \)) and deduces the continuum field theory of non-orientable strings.

Then we apply the strategy we have learned in 2D case to 4D Euclidean quantum gravity [14]. The Langevin equation for 3-geometries is proposed in the Ashtekar’s canonical variables to describe the time evolution in 4D Euclidean quantum gravity in the sense, that the corresponding Fokker-Planck hamiltonian recovers the hamiltonian of 4D quantum gravity exactly. The stochastic time corresponds to Euclidean time in the temporal gauge, \( N = 1 \) and \( N^i = 0 \). In this context, 4D quantum gravity is understood as a stochastic process where the scale of the fluctuation of “triad” is characterized by the
curvature at one unit time step before. The lattice regularization of the approach in 4D Euclidean spacetime is also presented to play the same game as what we have done in 2D case with matrix models.

Stochastic Quantization of Real Symmetric Matrix Models in Loop Space

Let us start with the Langevin equation for one matrix model,

\[
\Delta M_{ij}(\tau) = -\frac{\partial}{\partial M} S(M)_{ij}(\tau) \Delta \tau + \Delta \xi_{ij}(\tau),
\]

\[
S(M) = -\sum_{\alpha=0} g_{\alpha} \alpha + 2N^{-\alpha/2} \text{tr}M^{\alpha+2},
\]

(1)

\(M_{ij}\) denotes a real symmetric matrix. The stochastic time \(\tau\) is discretized with the unit time step \(\Delta \tau\). We consider the discretized version of time evolution \(M_{ij}(\tau + \Delta \tau) \equiv M_{ij}(\tau) + \Delta M_{ij}(\tau)\), with the Langevin equation for convenience of stochastic calculus and for understanding the corresponding stochastic process precisely. The discretized stochastic time development with \(\Delta \tau\) precisely corresponds to the one step deformation in dynamical triangulation in random surfaces. In the following argument, the specific form of the action of the matrix model is not relevant. The correlation of the white noise \(\Delta \xi_{ij}\) is defined by

\[
<\Delta \xi_{ij}(\tau) \Delta \xi_{kl}(\tau)>_\xi = \Delta \tau (\delta_{il} \delta_{jk} + \delta_{ik} \delta_{jl}) .
\]

(2)

It is uniquely determined\(^1\) from the requirement that (1) is transformed covariantly preserving the white noise correlation (2) invariant under the transformation \(M \rightarrow UMU^{-1}\), where \(U\) denotes orthogonal matrices for the real symmetric matrix models.

The basic field variables are loop variables \(\phi_n = \text{tr}(M^n)N^{-1-\frac{n}{2}}\). Following to Ito’s stochastic calculus \([13]\), we evaluate

\[
\Delta \phi_n \equiv \phi_n(\tau + \Delta \tau) - \phi_n(\tau),
\]

\[
= n\text{tr}(\Delta MM^{n-1})N^{-1-\frac{n}{2}} + \frac{1}{2} n \sum_{k=0}^{n-2} \text{tr}(\Delta MM\Delta MM^{n-k-2})N^{-1-\frac{n}{2}} + O(\Delta \tau^{3/2}) .
\]

(3)

\(^1\) For an hermitian matrix \(M_{ij}\) in (1), the nose correlation is \(<\Delta \xi_{ij}(\tau) \Delta \xi_{kl}(\tau)>_\xi = 2\Delta \tau \delta_{il} \delta_{jk}\).
The terms in R.H.S. should be of the order $\Delta \tau$, thus we obtain

$$
\Delta \phi_n = \Delta \tau \frac{n-2}{2} \left\{ \sum_{k=0}^{n-2} \phi_k \phi_{n-k-2} + (n-1) \frac{1}{N} \phi_{n-2} \right\} + \Delta \tau n \sum_{\alpha=0} g_\alpha \phi_{n+\alpha} + \Delta \zeta_{n-1},
$$

$$
\Delta \zeta_{n-1} \equiv \text{tr}(\Delta \xi M^{n-1}) N^{-1-\frac{2}{N}}.
$$

(4)

The correlation of the new noise variables appeared in (4) is given by

$$
< \Delta \zeta_{m-1}(\tau) \Delta \zeta_{n-1}(\tau) >_\xi = \Delta \tau \frac{2}{N^2} nm < \phi_{m+n-2}(\tau) >_\xi,
$$

(5)

The new noise is not a simple white noise but includes the value of the loop variable itself. In a practical sense, it might be tedious to generate the noise variable. We notice that $\phi_{m+n-2}(\tau)$ in R.H.S. of eq.(5) does not include the white noise at $\tau$ but the series of noises up to the one step (stochastic time unit $\Delta \tau$) before. This means that the expectation value in R.H.S. should be defined with respect to the white noise correlation up to the stochastic time $\tau - \Delta \tau$.

We also notice $< \Delta \phi_n(\tau) >_\xi = 0$ by means of Ito’s stochastic calculus. In the context of SQM approach, the property of the noise yields the Schwinger-Dyson equation by assuming the existence of the equilibrium limit at the infinite stochastic time, or equivalently, $\lim_{\tau \rightarrow \infty} < \Delta \phi_n(\tau) >_\xi = 0$. We have,

$$
< \frac{n}{2} \sum_{k=0}^{n-2} \phi_k \phi_{n-k-2} + \frac{1}{2} (n-1) \frac{1}{N} \phi_{n-2} + n \sum_{\alpha=0} g_\alpha \phi_{n+\alpha} >_\xi = 0.
$$

(6)

The order of the noise correlation (5), $1/N^2$, realizes the factorization condition in the large N limit. Therefore we obtain the S-D equation at large N limit for discretized non-orientable strings.

$$
\frac{1}{2} \sum_{k=0}^{n-2} < \phi_k >_\xi < \phi_{n-k-2} >_\xi + \sum_{\alpha=0} g_\alpha < \phi_{n+\alpha} >_\xi = 0.
$$

(7)

This shows that the S-D equation for non-orientable strings takes the same form as that for orientable strings at large N limit. The correspondence at the large N limit is exact if we define the corresponding hermitian matrix model by replacing all the couplings ,
$g_\alpha \rightarrow 2g_\alpha$ in (1). As a consequence, the disc amplitude in non-orientable strings is exactly the same as that in orientable strings.

The geometrical meaning of the stochastic process described by the Langevin equation (4) is the following. The one step stochastic time evolution of a discretized loop, \( \phi_n(\tau) \rightarrow \phi_n(\tau) + \Delta \phi_n(\tau) \), generates the splitting of the loop into two smaller pieces, \( \phi_k \) and \( \phi_{n-k-2} \). The process is described by the first term in R.H.S. of (4). In a field theoretical sense, it is interpreted as the annihilation of the loop \( \phi_n \) and the simultaneous pair creation of loops, \( \phi_k \) and \( \phi_{n-k-2} \). The first term in R.H.S. of (4) preserves the orientation of these loops, while the second term, which is the characteristic term of the order of \( \frac{1}{N} \) for non-orientable interaction, does not preserve the orientation. Since the new noise variables in (5), \( \Delta \zeta_{n-1} \)'s, are translated to “annihilation” operators in the corresponding Fokker-Planck hamiltonian, the factor 2 in the correlation (5) for the new noise variables comes from the sum of the orientation preserving and non-preserving merging interactions. Namely, (5) describes the simultaneous annihilation of two loops \( \phi_m \) and \( \phi_n \) and the creation of a loop \( \phi_{m+n-2} \). The geometrical picture allows us to identify the power “n” of matrices in \( \phi_n \) to the length of the discretized non-orientable loop \( \phi_n \). We notice that, in each time step, the interaction process decreases the discretized loop length by the unit “2”. The process which comes from the original action of matrix models extends the length of discretized loops.

The definition of the F-P hamiltonian operator gives the precise definition of a field theory of second quantized non-orientable strings. In terms of the expectation value of an observable \( O(\phi) \), a function of \( \phi_n \)'s, the F-P hamiltonian operator \( \hat{H}_{FP} \) is defined by,

\[
\langle \phi(0)|e^{-\tau\hat{H}_{FP}}O(\phi(\tau))|0 \rangle \equiv < O(\phi_{\xi}(\tau)) >_{\xi} .
\]

In R.H.S., \( \phi_{\xi}(\tau) \) denotes the solution of the Langevin equation (4) with the initial configuration \( \phi(0) \neq 0 \). The time evolution of R.H.S. is given by,

\[
< \Delta O(\phi(\tau)) >_{\xi} = \sum_m \partial_m O(\phi(\tau)) \Delta \phi_m + \frac{1}{2} \sum_{m,n} \partial_m \partial_n O(\phi(\tau)) \Delta \phi_m \Delta \phi_n >_{\xi} + O(\Delta \tau^{3/2}) ,
\]

\[
\equiv -\Delta \tau < H_{FP}(\tau)O(\phi(\tau)) >_{\xi} ,
\]

(9)
where $\partial_n \equiv \frac{\partial}{\partial \phi_n}$. By substituting the Langevin equation (4) and the noise correlation (5) into (9), we obtain

$$H_{FP}(\tau) = - \sum_{n>0} X_n n \pi_n ,$$

$$X_n \equiv \frac{1}{N^2} \sum_{m} m \phi_{m-n-2} \pi_n + \frac{1}{2} \sum_{k=0}^{n-2} \phi_k \phi_{n-k-2} + \frac{1}{2} (n-1) \frac{1}{N} \phi_{n-2} + \sum_{\alpha=0} g_{\alpha} \phi_{n+\alpha} ,$$

where $\pi_n \equiv \frac{\partial}{\partial \phi_n}$. To define the operator formalism corresponding to eq.(8), we introduce $\hat{\phi}_m$ and $\hat{\pi}_m$ as the creation and the annihilation operators for the loop with the length $n$, respectively. Then we assume the commutation relation $[\hat{\pi}_m, \hat{\phi}_n] = \delta_{mn} , \text{ and the existence of the vacuum, } |0> , \text{ with } \hat{\pi}_m |0> = 0 \text{ for } m > 0.$ In the representation, $<Q> \equiv <0| e^{\sum_{m} \xi_m \hat{\pi}_m} |Q> \equiv \Pi_m \delta(\hat{\phi}_m - Q_m) |0>$, the F-P hamiltonian operator $\hat{H}_{FP}$ in (8) is given by replacing $\phi_m \to \hat{\phi}_m$, and $\pi_m \to \hat{\pi}_m$ in $H_{FP}$ in (10) with the same operator ordering.

From the equality (8), the probability distribution function $P(\phi, \tau)$, which is defined by $<O(\phi(\tau))>_{\xi} \equiv \int \Pi_n d\phi_n O(\phi) P(\phi, \tau)$, is given by,

$$P(\phi, \tau) = <\phi(0)| e^{-\tau \hat{H}_{FP}} |\phi> .$$

The initial distribution, $P(\phi, 0) = \Pi_m \delta(\phi_m - \phi_m(0))$, represents the initial value of the solution of the Langevin equation (4). Eq.(11) follows the Fokker-Planck equation for the probability distribution,

$$\Delta P(\phi, \tau) = +\Delta \tau \hat{H}_{FP} P(\phi, \tau) ,$$

where $\hat{H}_{FP}$ is the adjoint of $H_{FP}$ in (10),

$$\hat{H}_{FP} = - \sum_{n>0} n \pi_n \hat{X}_n ,$$

$$\hat{X}_n \equiv - \frac{1}{N^2} \sum_{m} m \pi_m \phi_{m+n-2} + \frac{1}{2} \sum_{k=0}^{n-2} \phi_k \phi_{n-k-2} + \frac{1}{2} (n-1) \frac{1}{N} \phi_{n-2} + \sum_{\alpha=0} g_{\alpha} \phi_{n+\alpha} .$$
The remarkable observation is that it includes the Virasoro constraint \([\mathcal{F}]\). Since the stochastic time evolution is generated by the noise essentially, the emergence of Virasoro constraint is traced to the noise correlations in eq.(5) which are equivalent to the insertion of matrices into the loop variable, \(M \to M + \Delta \tau M^{m-1}\), in \(\phi_n\). It generates the transformation \([-\Delta \tau L_{m-2}, \phi_n] = n\Delta \tau \phi_{m+n-2}\), which corresponds to the noise correlation (5).

In fact, for real symmetric matrix models (non-orientable strings), \(L_n \equiv -N^2 X_{n+2}\) satisfies the Virasoro algebra without central extension,

\[ [L_m, L_n] = (m - n)L_{m+n}. \quad (14) \]

It is also worthwhile to note that the F-P equation (12) realizes the Virasoro constraint for the probability distribution. Namely, \(\tilde{L}_n \equiv N^2 \tilde{X}_{n+2}\) also satisfies the Virasoro algebra without central extension (14). Therefore, the F-P equation deduces a constraint equation for the distribution function even at the discretized level, justifying the generation of the partition function which satisfies the Virasoro constraint at the infinite stochastic time.

\[ \tilde{L}_n \lim_{\tau \to \infty} P(\phi, \tau) = 0, \quad \text{for } n = -1, 0, 1, ... . \quad (15) \]

For hermitian matrix models, the Virasoro constraint for the partition function (15) was found as the S-D equation \([16]\). In the continuum limit, it deduces the continuum version of the Virasoro constraints \([17]\). The expressions (8) and (11) also give a constraint on possible initial condition dependence of the expectation value and the partition function at the infinite stochastic time limit, such as, \(\lim_{\tau \to \infty} H_{FP}[\phi(0), \frac{\partial}{\partial \phi(0)}]P[\phi, \tau] = 0\). This implies that these quantities may have the initial value dependence up to the solution of the Virasoro constraint.

**Continuum Limit and Continuum Field Theory of Non-Orientable Strings**

Now we take the continuum limit. First we introduce a length scale “\(a\)” and define the physical length of the loop created by \(\phi_n\) with \(l = na\). Then we may redefine field variables and the stochastic time at the continuum limit as follows.

\[ G_{st} \equiv N^{-2} a^{-D}, \]
\[ d\tau \equiv a^{-2+D/2}\Delta \tau , \]
\[ \Phi(l) \equiv a^{-D/2}\phi_n , \]
\[ \Pi(l) \equiv a^{-1+D/2}\pi_n , \]

(16)

where we would like to keep the string coupling constant, \( G_{st} \), finite at the double scaling limit. For the existence of the smooth limit from the discretized stochastic time evolution to the “continuum” one, we require the condition, \( \frac{D}{2} - 2 > 0 \). The scaling dimensions of all the quantities in (16) have been determined except the scaling dimension of the string coupling constant, \( D \), by assuming \[ \Delta \tau = d\tau H_{FP} \] 

\[ [\Pi(l), \Phi(l)] = \delta(l - l') . \]

(17)

Then we obtain the continuum F-P hamiltonian, \( H_{FP} \), from \( H_{FP} \) at the continuum limit,

\[ H_{FP}^{\text{non-or.}} = -\frac{1}{2} \int_0^\infty dl \{2G_{st} \int_0^\infty dl' \Phi(l + l')\Pi(l')l\Pi(l) + \int_0^l dl' \Phi(l - l')\Phi(l')l\Pi(l) + \sqrt{G_{st}}l\Phi(l)\Pi(l) + \rho(l)\Pi(l) \} , \]

(18)

for non-orientable strings. By the redefinition (16), the F-P hamiltonian (18) is uniquely fixed at the continuum limit except the cosmological term. To specify the explicit form of the cosmological term \( \rho(l) \) in (18), we have to carefully evaluate the contribution which comes from the 3-point splitting interaction term and the matrix model potential, \( a^{1-D} \sum_{\alpha=0} g_\alpha \phi_{n+\alpha} \), at the double scaling limit of the real symmetric matrix model. Here we remember that the S-D equation for non-orientable string at large N limit takes exactly the same form as the orientable one under the suitable choice of the matrix model coupling constants. Since the continuum limit is taken by using the universal part of the disc amplitude, we naively expect \( \rho(l) \) takes the same form as that for orientable strings.
To show explicitly this is indeed the case, we consider the simplest matrix potential given by, \( g_0 = -1/2, g_1 = g/2, g_2 = g_3 = ... = 0 \) in (1), which corresponds to \( c = 0 \). Let us introduce the string field variable

\[
\phi(z) \equiv \sum_n z^{-1-n} \phi_n = \frac{1}{N} \text{tr} \frac{1}{z - MN^{-1/2}},
\]

\[
\Delta \zeta(z) \equiv \sum_n z^{-1-n} \Delta \zeta_n.
\]

To take the continuum limit, we redefine the field variable,

\[
\phi(z) \equiv \frac{1}{2}(z - gz^2) + c_0 z_c^{-1} a^{3/2} \Phi(u),
\]

\[
\Delta \zeta(z) \equiv c_0 z_c^{-1} a^{3/2} \tilde{d} \zeta(u),
\]

where we have introduced the “renormalized” parameters, \( z \equiv z_c e^{au} \), and \( g \equiv g_c e^{-c_1 a^2 t} \), and the “continuum” stochastic time \( d\tau \equiv c_0 z_c^{-2} a^{1/2} \Delta \tau \), where \( z_c = 3^{1/4} (3^{1/4} + 1) \) and \( g_c = \frac{1}{2^{3^{1/4}}} \) are the critical values and \( t \) denotes the cosmological constant. The constants \( c_0 \) and \( c_1 \) are chosen for convenience. The scaling dimension of all the quantities have been determined so that the string coupling, \( 1/G_{st} \equiv c_0^2 N^2 a^5 \), is fixed at the double scaling limit \[18\].

By using the Laplace transformation,

\[
\Phi(u) = \int_0^\infty dl e^{-ul} \Phi(l),
\]

in the Langevin equation derived for the redefined field variable \( \Phi(u) \) in (20) with the same procedure as shown in (4), we obtain the following Langevin equation which is equivalent to the continuum F-P Hamiltonian (18) \[3\],

\[
d\Phi(l) = d\zeta(l) + \frac{1}{2} d\tau \{ l \int_0^l dl' \Phi(l') \Phi(l - l') + \rho(l) + \sqrt{G_{st}} l^2 \Phi(l) \},
\]

\[
<d\zeta(l) d\zeta(l')> = 2d\tau G_{st} l l' < \Phi(l + l') > ,
\]

\[
\rho(l) = 3 \delta''(l) - \frac{3t}{4} \delta(l),
\]

(22)
for non-orientable string. It is consistent with the naive continuum limit of its discretized version (4) except the term $\rho(l)$. As we have shown explicitly, the cosmological term $\rho(l)$ takes the same form both for orientable and non-orientable strings. The field theory of non-orientable strings is also consistent with ref. [4] in transfer matrix formalism. The double scaling limit of the real symmetric matrix model has been studied in a quartic potential [13], while our result shows that it happens in the cubic potential as well. We notice that the continuum F-P hamiltonian includes the continuum Virasoro generator $L(l)$,
\[
L(l) = -\left\{ \int_0^\infty dl' \Phi(l + l') l'\Pi(l') + \frac{1}{2G_{st}} \int_0^l dl' \Phi(l - l') \Phi(l') \right. \\
+ \frac{1}{2\sqrt{G_{st}}} l \Phi(t) + \frac{1}{2G_{st}} \rho(l) \left\}.
\]
These generators satisfy the continuum Virasoro algebra, $[L(l), L(l')] = (l - l')L(l + l')$. In the stochastic quantization viewpoint, since the stochastic time scaling dimension is given by $D/2 - 2 = \frac{1}{2} > 0$ for $c = 0$, we expect that the discretized version of the loop space Langevin equation for real symmetric matrix models may provide a possible method for numerical calculation of non-orientable 2D random surfaces to sum up the topologies of surfaces. In the next section, we extend the idea which we have learned in 2D case to 4D Euclidean quantum gravity.

**4D Quantum Gravity from Stochastic 3-Geometries**

Here we point out that the time evolution in 4D quantum gravity is described by a Langevin equation for 3-geometries in terms of the Ashtekar’s canonical field variables by showing that the corresponding Fokker-Planck hamiltonian operator exactly recovers the hamiltonian of 4D Euclidean quantum gravity in the gauge $N = 1$ and $N^i = 0$.

The Hartle-Hawking type boundary condition is naturally imposed in this scheme by specifying the initial probability distribution functional. We also present the lattice reg-

\footnote{In 2D quantum gravity, it is pointed out that this gauge fixing recovers the time evolution in non-critical string field theories[?].}
ularization of this approach which defines a lattice regularization of Ashtekar’s canonical formalism \cite{14}.

At first, We propose the basic Langevin equation for 3-geometry in terms of the Ashtekar’s canonical variables \cite{19} to recover the hamiltonian of 4D quantum gravity with the corresponding F-P hamiltonian defined latter. The simplest form of the Langevin equation is defined by,

\[
\Delta A^a_i(x, \tau) = \Delta \zeta^a_i(x, \tau),
\]

\[
< \Delta \zeta^a_i(x, \tau) \Delta \zeta^b_j(y, \tau) >_\zeta = \frac{\kappa}{2} \Delta \tau \epsilon^{abc} < F^c_{ij}(x, \tau) >_\zeta \delta^3(x - y), \tag{24}
\]

where $A^a_i(x, \tau)$ is a SU(2) gauge field in Euclidean Ashtekar’s canonical formalism\textsuperscript{3}. In this note, Latin indices “i,j,k,...”denote the spatial part of the spacetime coordinate indices. While the Latin letters “a,b,c,...”denote the spatial part of the internal indices. $x, y, ...$ denote spatial spacetime coordinates. The one step time evolution is defined by $\Delta A^a_i(x, \tau) \equiv A^a_i(x, \tau + \Delta \tau) - A^a_i(x, \tau)$ in (24). The coupling constant $\kappa$ is defined by $\kappa \equiv 16\pi G$ with $G$, the gravitational (Newton’s) constant in the natural unit $\hbar = c = 1$.

The noise variable in (24) is not a simple white noise. The expectation value of the R.H.S. of the noise correlation is understood to be taken with respect to the noises up to the one unit time step before, $\tau - \Delta \tau$, in the sense of Ito’s stochastic calculus. It is also equivalent to require $< \Delta \zeta^a_i(\tau) >= 0$ in Ito’s calculus.

The invariant property of (24) is not apparent even if we introduce algebra-valued 1-form, $A(x, \tau) \equiv A^a_i T^a dx^i$, $\Delta \zeta^a_i(x, \tau) \equiv \Delta \zeta^a_i T^a dx^i$, and algebra-valued curvature 2-form, $F(x, \tau) = dA + A \wedge A$. Then (24) is rewritten by,

\[
\Delta A(x, \tau) = \Delta \zeta(x, \tau),
\]

\[
< \Delta \zeta(x, \tau) \wedge \Delta \zeta(x, \tau) > = \kappa \Delta \tau \delta^3(0) F(x, \tau). \tag{25}
\]

The R.H.S. of the noise correlation should be regularized in a gauge invariant way. The basic Langevin equation is manifestly covariant under the SU(2) local gauge transformation, $A^a_i(x) \rightarrow A^a_i(x) + D_i^{ab} \omega^b(x)$. While it is not covariant under the spatial general

\textsuperscript{3} We use the notation, $F^a_{ij} = \partial_i A_{ij}^a - \partial_j A_{ij}^a + \epsilon^{abc} A_{ij}^b A_{ij}^c$, and $D_{ij}^{ac} = \delta^{ac} \partial_i + \epsilon^{abc} A_{ij}^b$. The field variable $A^a_i(x)$ is \textit{real} in the Euclidean Ashtekar’s formalism.
coordinate transformation, \( A_i^a(x) \rightarrow A_i^a(x) + F_{ij}^a(x) \), due to the appearance of the divergent term, \( \kappa \Delta \tau \delta^3(0) F_{ji}^a \), in the transformation of the R.H.S of (24). It is formally cancelled by adding a term in the R.H.S. of the Langevin equation (24) which comes from the invariant path-integral measure we discuss latter.

In terms of the solution of the Langevin equation, the following equality holds.

\[
\langle \Pi_{x,i,a} \delta(A_i^a(x, \tau) - A_i^a(x_{\text{final}})) \rangle_{\zeta} = \langle A(x_{\text{final}}) | e^{-\tau H_{FP}[\pi, \hat{A}] - H_{FP}[\hat{A}, \pi]} | A(x_{\text{initial}}) \rangle.
\] (26)

In the L.H.S., \( A_i^a(x, \tau) \) denotes the solution of the Langevin equation with the initial condition, \( A_i^a(x, 0) = A_i^a(x_{\text{initial}}) \). In the R.H.S., \( \tilde{H}_{FP}[\tilde{\pi}, \hat{A}] \) and \( H_{FP}[\hat{A}, \tilde{\pi}] \) are defined by,

\[
\tilde{H}_{FP}[\tilde{\pi}, \hat{A}] = \frac{G_0}{4} \int d^3 x \epsilon^{abc} \tilde{\pi}_a^i(x) \tilde{\pi}_b^j(x) \tilde{F}_{ij}^c(x),
\]
\[
H_{FP}[\hat{A}, \tilde{\pi}] = \frac{G_0}{4} \int d^3 x \epsilon^{abc} \hat{F}_{ij}^c(x) \tilde{\pi}_a^i(x) \tilde{\pi}_b^j(x).
\] (27)

To show the equality (26), the commutation relation,

\[
[\tilde{\pi}_a^i(x), \hat{A}_b^j(y)] = \delta_a^b \delta^3(x-y),
\] (28)

and the vacuum, \( |0> \) with \( \tilde{\pi}_a^i|0> = \langle 0|\hat{A}_b^j = 0 \), have been assumed (see also eq.(8)).[6]

The Fokker-Planck hamiltonians (27) are just the hamiltonian for 4D quantum gravity without cosmological term in Ashtekar’s variables[15] with different operator orderings. The operator \( \tilde{\pi}_a^i \) is interpreted as the “triad” in the Euclidean Ashtekar’s formalism. Namely, \( \tilde{\pi}_a^i(x) = 2 \tilde{\pi}_a^i(x) \). Stochastic time evolution with the Langevin equation (24) corresponds to the Euclidean time evolution in 4D quantum gravity with gauge fixing, \( N = 1 \) and \( N^i = 0 \). In the stochastic process (24), the “triad” plays the role of the noise variable and the scale of the fluctuation is characterized by the curvature. Though the two expressions in (26) are precisely equivalent, the second definition of the F-P hamiltonian

\[
\hat{e}_a^i \equiv q^{1/2} e_a^i, \text{ where } q = \det(q_{ij}) \text{ with spatial metric } q^{ij} = e_a^i e_a^j.
\]

4
in (27) has an advantage for further discussion. This is because the vacuum satisfies the (local) hamiltonian constraint with the operator ordering in $H_{FP}$, $<0|\mathcal{H}(\hat{A}(x), \hat{\pi}(x)) = \mathcal{H}(\hat{A}(x), \hat{\pi}(x))|0> = 0$, where $H_{FP}[\hat{A}, \hat{\pi}] \equiv \int d^3x \mathcal{H}(\hat{A}(x), \hat{\pi}(x))$. It should be noticed that the operator ordering of the F-P hamiltonian operator $H_{FP}$ is different from that appeared in the F-P equation.

Let us consider the initial distribution dependence of the probability distribution functional by averaging the expectation value (26) with respect to the initial probability distribution. It is defined by integrating out the initial configuration $A^{a(initial)}_i(x)$, on which the solution of the Langevin equation $A^{a}_{\xi}(x, \tau)$ depends, with the distribution, $P[A^{initial}, 0]$, in the L.H.S. of (26). It gives a generalized form of the distribution functional $P[A, \tau]$, which is defined by $<O[A_\zeta](\tau)>_\zeta \equiv \int \mathcal{D}AO[A]P[A, \tau]$, as follows

$$P[A, \tau] = \int \mathcal{D}A^{initial} < A^{initial} | P[\hat{A}, 0] e^{-\tau H_{FP}[\hat{A}, \hat{\pi}]} | A > . \quad (29)$$

For an arbitrary observable $O[A]$, the average with respect to the initial value distribution also gives,

$$<O[A_\zeta(\tau)]>_\zeta = \int \mathcal{D}A^{initial} < A^{initial} | P[\hat{A}, 0] e^{-\tau H_{FP}[\hat{A}, \hat{\pi}]} O[A] | 0 > . \quad (30)$$

In the definition of the expectation value in the L.H.S. of (30), the average is also taken with respect to the initial values with the distribution $P[A^{initial}, 0]$. For example, eq.(26) is given for $O[A] = \Pi_x, a_i, \delta \left( A^a_i(x) - A^a_{final}(x) \right)$, with the initial distribution, $P[A, 0] = \Pi_x, a_i, \delta \left( A^a_i(x) - A^a_{initial}(x) \right)$. From (30), the time evolution equation for the expectation values of observables is given by,

$$\frac{d}{d\tau} <O[A_\zeta(x, \tau)]>_\zeta = <H_{FP}[A^b_j(x), \frac{\delta}{\delta A^a_i(x)}]O[A^a_i(x)]|A^a_i(x) = A^a_{\zeta}(x, \tau)>_\zeta . \quad (31)$$

The initial condition dependence in the amplitude (30) leads a restriction on the boundary condition with respect to the Euclidean time. We notice that eq.(30) also gives a constraint for initial distribution,

$$\frac{d}{d\tau} <O[A_\zeta(x, \tau)]>_\zeta = \int \mathcal{D}A^{initial} P[A^{initial}, 0] H_{FP}[A^{initial}, \frac{\delta}{\delta A^{initial}}] < A^{initial} e^{-\tau H_{FP}[\hat{A}, \hat{\pi}]} O[A] | A > . \quad (32)$$
The existence of the equilibrium limit requires that the R.H.S. in (32) should be zero at the infinite stochastic time. A trivial solution of this constraint is the vanishing curvature at any spatial points, \( P[A, 0] = \Pi_{x,a,ij} \delta \left( F^a_{ij}(x) \right) \). This initial distribution, however, should be excluded because the R.H.S. of (31) is identically zero even at finite stochastic time and there is no time development. In general, if we choose a solution of the hamiltonian constraint as the initial value distribution, obviously there is no time evolution in the corresponding F-P equation. Especially, the constraint does not allow us to solve the Langevin equation with the initial condition \( A_a^{(initial)}(x) = 0 \).

To specify the physical boundary condition and define a class of solutions for the local hamiltonian constraint of 4D quantum gravity in this context, we may choose the following initial condition which generates a nontrivial time evolution as an analogue in 2D case,

\[
P_{H-H}[A, 0] = \Pi_{x \neq z_0, a, i} \delta \left( F^a_{ij}(x) \right).
\] (33)

The spatial coordinates, \( x = z_0 \), is identified to the point where the 3D manifold is absorbed into nothing in the sense of Hartle-Hawking type boundary condition. The local hamiltonian constraint is broken at this point but it may be recovered at an equilibrium limit.

Let us consider the gauge invariant path-integral measure in the Ashtekar’s variable, which are implicitly assumed in the expectation value (30). One way to specify the path-integral measure is to introduce a regularization for the noise correlation (24). In the following, we show that an extra term is necessary in a lattice regularization of the Langevin equation (24) to define an invariant and well-defined measure by identifying the noise correlation in (24) to a “superspace” metric with Ashtekar’s variable.

By using the lattice regularization of the Langevin equation and the noise correlation (24), the invariant property of the Langevin equation in the sense of Ito’s calculus naturally introduces the path-integral measure in the configuration space with Ashtekar variables. It is given by

\[
\Delta U(x, i)_{a\beta}(\tau) = \Delta W(x, i)_{a\beta}(\tau) + \Delta \tau |G|^{1/2} \frac{\partial}{\partial U(y, j)} \left( |G|^{-1/2} G[x, i; y, j]_{a\beta; \gamma\delta} \right). \tag{34}
\]
The regularized Langevin equation describes the one step time evolution of the link variables \( U(x, i)(\tau + \Delta \tau) = U(x, i)(\tau) + \Delta U(x, i)(\tau) \). The dynamical variable and the noise variable, \( U(x, i) \) and \( \Delta W(x, i) \) respectively, have been assigned on the link of 3-dimensional lattice, which is specified by the site \( x \) and its nearest neighbor in the \( i \)-th direction denoted by \( x + \hat{i} \). \( U(x, i) \) is an element of SU(2) group in the adjoint representation, while noise \( \Delta W(x, i) \) is algebra valued. The quantity \( G[x, i; y, j]_{\alpha\beta;\gamma\delta} \) is interpreted as the inverse of the “superspace” metric and \( |G| \) denotes its determinant. The superspace is spanned by the configuration \( \{ U_{\alpha\beta}(x, i) \} \). The inverse of the superspace metric is given in the following regularized noise correlation.

\[
< \Delta W(x, i)_{\alpha\beta}(\tau) \Delta W(y, j)_{\gamma\delta}(\tau) > = \Delta \tau < G[x, i; y, j]_{\alpha\beta;\gamma\delta}(\tau) > ,
\]

where,

\[
G[x, i; y, j]_{\alpha\beta;\gamma\delta} = \delta_{x+i, y+j} \delta_{\beta\delta} \{ U(x, i)U(x + \hat{i}, -j) - U(x, -j)U(x - \hat{j}, i) \}_{\alpha\gamma} \\
+ \delta_{x+i, y+j} \delta_{\beta\gamma} \{ U(x, i)U(x + \hat{i}, j) - U(x, j)U(x + \hat{j}, i) \}_{\alpha\delta} \\
+ \delta_{x+y, i+j} \delta_{\alpha\delta} \{ U(x + \hat{i}, -i)U(x, -j) - U(x + \hat{i}, -j)U(x + \hat{i} - \hat{j}, -i) \}_{\beta\gamma} \\
+ \delta_{x+y, i+j} \delta_{\alpha\gamma} \{ U(x + \hat{i}, -i)U(x, j) - U(x + \hat{i}, j)U(x + \hat{i} + \hat{j}, -i) \}_{\beta\delta} .
\]

The regularized Langevin equation (34) is invariant under the “general coordinate transformation ”in superspace, \( \{ U(x, i) \} \).

\[
U(x, i)_{\alpha\beta} \rightarrow V(x, i)_{\alpha\beta}[U] ,
\]

where \( V_{\alpha\beta}(x, i) \) is also an element of SU(2) group in the adjoint representation and an arbitrary functional of \( U_{\gamma\delta}(y, j) \). The second term in the R.H.S. in (34) is necessary for the invariance in Ito’s calculus \[14\]. The role of the contribution from the invariant measure was first clarified in the case of the Langevin equation for a particle moving on a Riemann surface \[23\] ( see also Ref. \[25\] and the Appendix in Ref. \[26\] ). The similar argument

\[\text{\footnote{For the detailed discussion to keep the dynamical variable within the element of SU(2) group along the time development and for the derivation of the corresponding lattice regularized F-P Hamiltonian, see Ref.\[14\]}}\]
is formally possible even in the non-regularized version (24) if we identify the R.H.S. of noise correlation in (24) to the continuum superspace metric. Therefore, the naive continuum limit of these equations, (34) (35) and (36), coincides with those in eq.(24) except a divergent term with $\delta^3(0)$ which comes from the second term in the R.H.S. of the Langevin equation (34). Though the divergent term actually represents the contribution from the path-integral measure and it is necessary for the invariant property of the Langevin equation in a formal sense, it is not well-defined without regularization. Thus the regularized Langevin equation (34) provides a possible basis for numerical simulation on 4D quantum gravity.

We comment on the two important consequences of the lattice regularized Langevin equation [14]. One is the corresponding equilibrium distribution,

$$
\lim_{\tau \to \infty} P[U, \tau] = \mathcal{G}^{-1/2},
$$

(38)

which defines a measure invariant under the superspace general coordinate transformation (37),

$$
\mathcal{D}U \equiv \mathcal{G}^{-1/2} \Pi_{\text{link}} dU,
$$

(39)

in the regularized version of the expectation values (29) and (30). The other is the Schwinger-Dyson equation in this context. It is given by,

$$
< |\mathcal{G}|^{1/2} \frac{\partial}{\partial U(y, j)^{\gamma\delta}} (|\mathcal{G}|^{-1/2} \mathcal{G}[x, i; y, j^{\alpha\beta; \gamma\delta}] ) >= 0,
$$

(40)

at an equilibrium limit.

**Conclusion**

In conclusion, we have shown that the Langevin equation for real symmetric matrix models written by the loop variables defines the time evolution of non-orientable strings which defines non-orientable 2D surfaces at both discretized and continuum levels. The partition function in loop space satisfies the Virasoro constraint at the equilibrium limit in both discretized and continuum level.
The idea has been extended to 4D Euclidean quantum gravity. In the Langevin equation (24) and the regularized one (34), the “triad” in Ashtekar’s variables are realized as noise variables, which presumably represents the stochastic 3-geometries. The corresponding F-P hamiltonian is equivalent to the hamiltonian of 4D Euclidean quantum gravity in the temporal gauge. It allows us to interpret the stochastic time as the Euclidean proper time. The strategy we would like to adopt here is to characterize the 3-boundaries in the 4D spacetime by using the solution for both momentum constraint and Gauss law constraint as the initial distribution and the observables. As it is clear from (30), initial distributions and observables are the key quantities to specify 3-boundaries on a 4D spacetime in this approach. The method developed in 2D quantum gravity in loop space indicates that the expectation values of such quantities satisfy all the constraints in 4D quantum gravity at the equilibrium limit. There are some candidates useful to characterize 3-boundaries, such as the extrinsic curvature term, 3D Chern-Simons term, topological invariants [21] and loop variables [22] for which the present formalism can be applied. For these observables, the gauge fixing terms for the spatial general coordinate invariance and local Lorentz invariance, which are introduced as a drift force in the Langevin equation (24) following a standard method [24][25], do not change the expectation value of observables at the equilibrium limit [14].

Though there is no well-defined drift force in the Langevin equation (24) without regularization of path-integral measure, there is another machinery to introduce a drift force. In the Langevin equation for observables which is equivalent to the F-P equation for the expectation value of observables (30), a drift force appears effectively as a direct consequence of the fact that Ito’s stochastic calculus pick up the Jacobian factor which comes from the change of variables from the gauge fields to observables[1][1][25]. In the

6 The simplest example is 2D Euclidean Yang-Mills theory. It can be defined by the same Langevin equation for SU(N) as (24) with 1-dimensional field variables $A^a(x, \tau)$ and the white noise correlation, $\langle \Delta \zeta^a(x, \tau) \Delta \zeta^b(y, \tau) \rangle = \frac{2}{g^2} \Delta \tau \delta^{ab} \delta(x-y)$. The noise is translated as the canonical momentum variable. Then the Langevin equation for the expectation value of Wilson loops defines the collective field theory for 2D Yang-Mills field.
lattice regularized Langevin equation (34), the contribution from the invariant measure introduces a well-defined drift force.

The problem of the present scheme is that the noise correlation in the basic Langevin equation (24) is not positive definite. It would force the Langevin equation (24) to be complex, although the field variables are real in the Euclidean Ashtekar’s formalism. The point would be main difficulty for the numerical analysis in this scheme. One way to deal with the problem may be to extend the Langevin equation (24) to a class of more general gauge fixing. It is always possible by multiplying the noise correlation in (24) with lapse function. Then one may chose the lapse function so that the noise correlation keeps the value to be positive definite. It is an open question if this gauge fixing procedure, a choice of non-trivial lapse function, make sense in numerical simulation. Another way may be to consider the Wick rotation of the conformal mode. Apart from these questions, the description with the Langevin equation has a topological feature in the sense of Nicoli-Langevin map. Such a topological feature would relax another difficulty, renormalizability of quantum gravity. We hope that the approach is useful for deeper understanding of quantum gravity.

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