ON PRIMITIVE DIRICHLET CHARACTERS
AND THE RIEMANN HYPOTHESIS

WILLIAM D. BANKS
Department of Mathematics
University of Missouri
Columbia, MO 65211 USA
bbanks@math.missouri.edu

AHMET M. GÜLOĞLU
Department of Mathematics
University of Missouri
Columbia, MO 65211 USA
ahmet@math.missouri.edu

C. WESLEY NEVANS
Department of Mathematics
University of Missouri
Columbia, MO 65211 USA
nevans@math.missouri.edu

Abstract
For any natural number $n$, let $\mathcal{X}_n'$ be the set of primitive Dirichlet characters modulo $n$. We show that if the Riemann hypothesis is true, then the inequality $|\mathcal{X}_{2n_k}'| \leq C_2 e^{-\gamma} \varphi(2n_k) / \log \log(2n_k)$ holds for all $k \geq 1$, where $n_k$ is the product of the first $k$ primes, $\gamma$ is the Euler-Mascheroni constant, $C_2$ is the twin prime constant, and $\varphi(n)$ is the Euler function. On the other hand, if the Riemann hypothesis is false, then there are infinitely many $k$ for which the same inequality holds and infinitely many $k$ for which it fails to hold.
1 Introduction

For any natural number \( n \), let \( \mathcal{X}_n \) be the set of Dirichlet characters modulo \( n \), and let \( \mathcal{X}_n' \) be the subset of primitive characters in \( \mathcal{X}_n \).

The purpose of the present note is to establish a connection between the classical Riemann hypothesis and the collection of sets \( \{ \mathcal{X}_n' : n \in \mathbb{N} \} \). Our work is motivated by and relies on the 1983 paper of J.-L. Nicolas [2] in which a relation is established between the Riemann hypothesis and certain values of the Euler function \( \varphi(n) \); see also [3].

**Theorem 1.** For every \( k \geq 1 \), let \( n_k \) be the product of the first \( k \) primes. Let \( \gamma \) be the Euler-Mascheroni constant and \( C_2 \) the twin prime constant.

(i) If the Riemann hypothesis is true, then the inequality

\[
|\mathcal{X}_n'| \leq C_2 e^{-\gamma} \frac{\varphi(2n_k)}{\log \log(2n_k)}
\]

holds for all \( k \geq 1 \).

(ii) If the Riemann hypothesis is false, then there are infinitely many \( k \) for which (1) holds and infinitely many \( k \) for which it fails to hold.

We recall that

\[
\gamma = \lim_{n \to \infty} \left( \sum_{m=1}^{n} \frac{1}{m} - \log n \right) = 0.5772156649 \cdots,
\]

and

\[
C_2 = \prod_{p>2} \frac{p(p-2)}{(p-1)^2} = 0.6601618158 \cdots.
\]

To prove the theorem, we study the ratios

\[
\rho(n) = \frac{|\mathcal{X}_n'|}{|\mathcal{X}_n|} \quad (n \in \mathbb{N}).
\]

Note that \( \rho(n) \) is the proportion of Dirichlet characters modulo \( n \) that are primitive characters. Since \( \rho(n) \leq 1 \) for all \( n \in \mathbb{N} \), and \( \rho(p) = 1 - 1/(p-1) \) for every prime \( p \), it is clear that

\[
\limsup_{n \to \infty} \rho(n) = 1.
\]
As for the minimal order, we shall prove the following:

\[ \lim \inf_{n \rightarrow \infty, n \not\equiv 2 \pmod{4}} \rho(n) \log \log n = C_2 e^{-\gamma}. \]  
\hspace{1cm} (2)

Note that natural numbers \( n \equiv 2 \pmod{4} \) are excluded since \( \rho(n) = 0 \) for those numbers; see (5) below.

In Section 2 we show that the inequalities

\[ \rho(2n_k) \log \log(2n_k) \leq \rho(n) \log \log n \quad (n \not\equiv 2 \pmod{4}, \omega(n) = k) \]  
\hspace{1cm} (3)

hold for every fixed \( k > 1 \), where \( \omega(n) \) is the number of distinct prime divisors of \( n \), and we also show that

\[ \lim_{k \rightarrow \infty} \rho(2n_k) \log \log(2n_k) = C_2 e^{-\gamma}. \]  
\hspace{1cm} (4)

Clearly, (2) is an immediate consequence of (3) and (4).

Since \( |X_n| = \varphi(n) \) for all \( n \in \mathbb{N} \), the inequality (1) is clearly equivalent to

\[ \rho(2n_k) \log \log(2n_k) \leq C_2 e^{-\gamma}. \]  
\hspace{1cm} (5)

In Section 3 we study this inequality using techniques and results from [2], and these investigations lead to the statement of Theorem 1.

Acknowledgement. The authors wish to thank Pieter Moree for his careful reading of the manuscript and for several useful comments.

2 Small values of \( \rho(n) \)

The cardinality of \( X_n \) is \( \varphi(n) \), and that of \( X'_n \) is

\[ |X'_n| = n \prod_{p \parallel n} \left( 1 - \frac{2}{p} \right) \prod_{p^2 \parallel n} \left( 1 - \frac{1}{p} \right)^2 \]

(see, for example, [1, §9.1]); hence, it follows that

\[ \rho(n) = \frac{\varphi(n)}{n} \prod_{p \parallel n} \frac{p(p - 2)}{(p - 1)^2} \quad (n \in \mathbb{N}). \]  
\hspace{1cm} (6)
Turning to the proof of (3), let \( k > 1 \) be fixed, and denote by \( S \) the set of integers \( n \not\equiv 2 \pmod{4} \) with \( \omega(n) = k \). Let \( p_1, p_2, \ldots \) be the sequence of consecutive prime numbers. For each integer \( j \in \{0, \ldots, k\} \), let \( S_j \) be the set of numbers \( n \in S \) that have precisely \( j \) distinct prime divisors larger than \( p_k \). Since \( S \) is the union of the sets \( \{S_j\} \), to prove (3) it suffices to show that the inequalities

\[
\rho(2n_k) \log \log(2n_k) \leq \rho(n) \log \log n \quad (n \in S_j)
\]

hold for every fixed \( j \in \{0, \ldots, k\} \).

For any \( n \in S_0 \) we can write \( n = 2p_1^{\alpha_1} \cdots p_k^{\alpha_k} \) with each \( \alpha_j \geq 1 \). Using (6) and the fact that \( 2n_k = 2p_1 \cdots p_k \) we have

\[
\rho(2n_k) = \rho(n) \prod_{\substack{j=2 \\ (\alpha_j \geq 2)}}^{k} \frac{p_j(p_j - 2)}{(p_j - 1)^2} \leq \rho(n).
\]

Since \( 2n_k \leq n \) we also have \( \log \log(2n_k) \leq \log \log n \), and (7) follows for \( j = 0 \).

Proceeding by induction, let us suppose that (7) has been established for some \( j \in \{0, \ldots, k-1\} \). If \( n' \) is an arbitrary element of \( S_{j+1} \), then \( q \mid n' \) for some prime \( q > p_k \); note that \( q \geq 5 \) since \( k > 1 \). Writing \( n' = q^\alpha m \) with \( q \not\mid m \), we have \( \omega(m) = k - 1 \), hence for at least one index \( i \in \{1, \ldots, k\} \) the prime \( p_i \) does not divide \( m \). Put \( n = p_1^{\beta_1} \cdots p_k^{\beta_k} m \), where \( \beta = 2 \) if \( p_i = 2 \) and \( \beta = 1 \) otherwise. Clearly, \( n \in S_j \). Also, \( n \leq n' \) since \( q > \max\{p_i, 2^2\} \), and thus \( \log \log n \leq \log \log n' \). Finally, using (6) we see that

\[
\frac{\rho(n')}{\rho(m)} = \begin{cases} 1 - 1/(q - 1) & \text{if } \alpha = 1, \\ 1 - 1/q & \text{if } \alpha \geq 2, \end{cases}
\]

and

\[
\frac{\rho(n)}{\rho(m)} = \begin{cases} 1 - 1/(p_i - 1) & \text{if } \beta = 1, \\ 1/2 & \text{if } \beta = 2. \end{cases}
\]

As \( q > p_i \), we have \( \rho(n) \leq \rho(n') \) in all cases. Putting everything together, we see that

\[
\rho(2n_k) \log \log(2n_k) \leq \rho(n) \log \log n \leq \rho(n') \log \log n'.
\]

Since \( n' \in S_{j+1} \) is arbitrary, we obtain (7) with \( j \) replaced by \( j + 1 \), which completes the induction and finishes our proof of (3).
Next, we turn to the proof of (4). Using the Prime Number Theorem in the form
\[ \log n_k = \sum_{p \leq p_k} \log p = (1 + o(1))p_k \quad (k \to \infty) \]

together with Mertens' theorem (see [1, Theorem 2.7(e)]), it is easy to see that
\[ \lim_{k \to \infty} \left\{ \log \log(2n_k) \prod_{p \leq p_k} \left( 1 - \frac{1}{p} \right) \right\} = e^{-\gamma}. \]

Also,
\[ \lim_{k \to \infty} \prod_{2 < p \leq p_k} \frac{p(p-2)}{(p-1)^2} = \lim_{k \to \infty} C_2 \prod_{p > p_k} \left( 1 + \frac{1}{p(p-2)} \right) = C_2. \]

By (6) we have
\[ \rho(2n_k) = \prod_{p \leq p_k} \left( 1 - \frac{1}{p} \right) \prod_{2 < p \leq p_k} \frac{p(p-2)}{(p-1)^2} \quad (k \geq 1), \]

and thus (4) is an immediate consequence of (8) and (9).

3 Proof of Theorem 1

As in [2, Théorème 3] we put
\[ f(x) = e^\gamma \log \vartheta(x) \prod_{p \leq x} \left( 1 - \frac{1}{p} \right) \quad (x \geq 2), \]

where \( \vartheta(x) = \sum_{p \leq x} \log p \) is the Chebyshev \( \vartheta \)-function. For our purposes, it is convenient to define
\[ g(x) = e^\gamma \log (\vartheta(x) + \log 2) \prod_{p \leq x} \left( 1 - \frac{1}{p} \right) \prod_{p > x} \left( 1 + \frac{1}{p(p-2)} \right) \quad (x \geq 2), \]

This definition is motivated by the fact that
\[ g(p_k) = C_2^{-1} e^\gamma \rho(2n_k) \log \log(2n_k) \quad (k \geq 1). \]
As mentioned earlier, the inequalities (1) and (5) are equivalent, and (5) is clearly equivalent to
\[ \log g(p_k) \leq 0. \]

Thus, to prove Theorem 1 it suffices to study the sign of \( \log g(x) \).

By the trivial inequality \( \log(1 + t) \leq t \) for all \( t > -1 \) and the fact that \( g(x) > f(x) \) for all \( x \geq 2 \), it is easy to see that
\[ 0 < \log \frac{g(x)}{f(x)} \leq \frac{\log 2}{\vartheta(x) \log \vartheta(x)} + \frac{1}{x - 2} \quad (x > 2). \] (10)

Here, we have used the fact that
\[ \sum_{p > x} \frac{1}{p(p - 2)} \leq \sum_{n \geq [x] + 1} \frac{1}{n(n - 2)} = \frac{2}{2} \frac{[x] - 1}{([x] - 1)} < \frac{1}{x - 2} \quad (x > 2). \]

First, let us suppose that the Riemann hypothesis is true. In this case, we have from [2, p. 383]:
\[ \log f(x) \leq -\frac{0.8}{\sqrt{x} \log x} \quad (x \geq 3000). \]

Using this bound in (10) together with the inequality \( \vartheta(x) \geq 4x/5 \) (which holds unconditionally for \( x \geq 121 \) by [4, Theorems 4 and 18]), one sees that
\[ \log g(x) \leq \frac{\log 2}{(4x/5) \log(4x/5)} + \frac{1}{x - 2} - \frac{0.8}{\sqrt{x} \log x} \leq -\frac{0.6}{\sqrt{x} \log x} \]
for all \( x \geq 3000 \). This implies the desired bound (5) for all \( k \geq 431 \); for smaller values of \( k \), the bound (5) may be verified by a direct computation. This proves Theorem 1 under the Riemann hypothesis.

Next, suppose that the Riemann hypothesis is false, and let \( \theta \) denote the supremum of the real parts of the zeros of the Riemann zeta function. Then, by [2, Théorème 3c] one has
\[ \limsup_{x \to \infty} x^b \log f(x) > 0 \quad \text{and} \quad \liminf_{x \to \infty} x^b \log f(x) < 0 \]
for any fixed number \( b \) such that \( 1 - \theta < b < 1/2 \). In view of (10) and the Chebyshev bound \( \vartheta(x) \gg x \) it is clear that
\[ \log g(x) = \log f(x) + O(x^{-1}); \]
hence, we also have

$$\limsup_{x \to \infty} x^b \log g(x) > 0 \quad \text{and} \quad \liminf_{x \to \infty} x^b \log g(x) < 0.$$ 

In particular, $\log g(p_k)$ changes sign infinitely often, which implies Theorem [1] if the Riemann hypothesis is false.

**References**

[1] H. L. Montgomery and R. C. Vaughan, *Multiplicative number theory. I. Classical theory*. Cambridge Studies in Advanced Mathematics, 97. Cambridge University Press, Cambridge, 2007.

[2] J. L. Nicolas, ‘Petites valeurs de la fonction d’Euler,’ *J. Number Theory* **17** (1983), no. 3, 375–388.

[3] G. Robin, ‘Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann,’ *J. Math. Pures Appl. (9)* **63** (1984), no. 2, 187–213.

[4] J. B. Rosser and L. Schoenfeld, ‘Approximate formulas for some functions of prime numbers,’ *Illinois J. Math.* **6** (1962), 64–94.