Scattering approach to parametric pumping

P. W. Brouwer
Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138, USA

(February 1, 2008)

A d.c. current can be pumped through a quantum dot by periodically varying two independent parameters $X_1$ and $X_2$, like a gate voltage or magnetic field. We present a formula that relates the pumped current to the parametric derivatives of the scattering matrix $S(X_1, X_2)$ of the system. As an application we compute the statistical distribution of the pumped current in the case of a chaotic quantum dot.

PACS numbers: 72.10.Bg, 73.23.-b, 05.45.+b

An electron pump is a device that generates a d.c. current between two electrodes that are kept at the same bias. In recent years, electron pumps consisting of small semiconductor quantum dots have received considerable experimental and theoretical attention. A quantum dot is a small metal or semiconductor island, confined by gates, and connected to the outside world via point contacts. Several different mechanisms have been proposed to pump charge through such systems, ranging from a low-frequency modulation of gate voltages in combination with the Coulomb blockade to photon-assisted transport at or near a resonance frequency of the dot. Their applicability depends on the characteristic size of the system and the operation frequency.

Most experimental realizations of electron pumps in semiconductor quantum dots made use of the principle of Coulomb blockade. If the dot is coupled to the outside world via tunneling point contacts, the charge on the dot is quantized, and (apart from degeneracy points) transport is inhibited as a result of the high energy cost of adding an extra electron to the dot. Pothier et al. constructed an electron pump that operates at arbitrarily low frequency and with a reversible pumping direction. The pump consists of two weakly coupled quantum dots in the Coulomb blockade regime and operates via a mechanism that closely resembles a peristaltic pump: Charge is pumped through the double-dot array from a left to the right and electron-by-electron as the voltage $U_1 \propto \sin(\omega t)$ of the left dot reaches its minima and maxima before the voltage $U_2 \propto \sin(\omega t - \phi)$ of the right one. The pumping direction can be reversed by reversing the phase difference $\phi$ of the two gate voltages.

A similar mechanism was proposed by Spivak, Zhou, and Beal Monod for an electron pump consisting of single quantum dot only. In this case a d.c. current is generated by adiabatic variation of two different gate voltages that determine the shape of the nanostructure, or any other pair of parameters $X_1$ and $X_2$, like magnetic field or Fermi energy, that modify the (quantummechanical) properties of the system, see Fig. 1. The magnitude of the current is proportional to the frequency $\omega$ with which $X_1$ and $X_2$ are varied and (for small variations) to the product of the amplitudes $\delta X_1$ and $\delta X_2$. The direction of the current depends on microscopic (quantum) properties of the system, and need not be known a priori from its macroscopic properties. As in the case of the double-dot Coulomb blockade electron pump of Ref. 2, the direction of the current in the single-dot parametric pump of Spivak et al. can be reversed by reversing the phases of the parameters $X_1$ and $X_2$. An important difference between the two mechanisms is that a parametric electron pump like the one of Ref. 1 does not require that the quantum dot is in the regime of Coulomb blockade; it operates if the dot is open, i.e. well coupled to the leads by means of ballistic point contacts. Experimentally, an electron pump in an open quantum dot has been realized only very recently. A measurement of the pumped current provides a promising tool to study properties of open mesoscopic systems at zero bias or at zero current.

In this paper we consider a parametric electron pump through an open system in a scattering approach. Our main result is a formula for the pumped current in terms of the scattering matrix $S(X_1, X_2)$. Such a formula is the analogue of the Landauer formula, which relates the conductance $G = \delta I/\delta V$ of a mesoscopic system with two contacts to a sum over the (squares of) matrix elements $S_{\alpha\beta}$,

$$
\frac{\delta I}{G} = \frac{2e^2}{h} \sum_{\alpha \in 1, \beta \in 2} |S_{\alpha\beta}|^2.
$$

The indices $\alpha$ and $\beta$ are summed over all channels in the left and right contacts, respectively, and $\delta V$ is the applied voltage. For the case of the parametric electron pump, where two parameters $X_1$ and $X_2$ are varied periodically, $\delta X_1(t) = \delta X_1 \sin(\omega t)$ and $\delta X_2(t) = \delta X_2 \sin(\omega t - \phi)$, we find that the d.c. component of the current $I$ depends on the derivatives $\partial S_{\alpha\beta}/\partial X$,

$$
\frac{\delta I}{\partial X} = \frac{\omega e \sin \phi}{2\pi} \sum_{\alpha \in 1} \sum_{\beta} \text{Im} \frac{\partial S_{\alpha\beta}^*}{\partial X_1} \frac{\partial S_{\alpha\beta}}{\partial X_2}.
$$

Like the Landauer formula, Eq. (3) is valid for a phase coherent system at zero temperature and to (bi)linear response in the amplitudes $\delta X_1$ and $\delta X_2$. [The nonlinear response is given by Eq. (3) below.] It captures both a classical contribution to the current and the quantum interference corrections. Quantum corrections can be important in the mesoscopic regime, especially if there is
no “classical” mechanism that dominates the pumping process. Eq. (2) is valid to first order in the frequency \( \omega \). This is sufficient if the period \( \tau = 2\pi/\omega \) is much larger than the time particles spend inside the quantum dot. For such low frequencies, we can assume that equilibrium is maintained throughout the pumping process. The scattering matrix formula does not capture effects of order \( \omega^2 \) (or higher) that rely on the existence of a non-equilibrium distribution inside the quantum dot. The existence of a scattering approach to parametric pumping allows us to borrow from the vast literature dealing with scattering matrices of disordered and chaotic microstructures and their parameter dependence and to directly relate the pumped current to other transport properties like e.g. the conductance.

The system under consideration is shown schematically in Fig. 1. It consists of a quantum dot, coupled to two electron reservoirs by ballistic point contacts. The two electron reservoirs are held at the same voltage. Two external parameters \( X_1(t) \) and \( X_2(t) \) of the dot are varied periodically, see Fig. 1. They can be e.g. the voltage of a plunger gate, parameters that characterize the shape, or a magnetic field. The two point contacts, which have \( N \) channels at the Fermi level \( E_F \), are labeled 1 and 2. The scattering matrix \( S \) of the system has dimension \( 2N \times 2N \) and is a function of the parameters \( X_1 \) and \( X_2 \). Since the system is well coupled to the leads, the charge is no longer quantized, the Coulomb blockade is lifted, and to a first approximation, we can use a picture of non-interacting electrons.

Starting point of our theory is a formula due to Büttiker, Thomas, and Prêtre for the current in the contacts 1 and 2 that results from an infinitesimal change of a parameter \( X \): For a small and slow harmonic variation \( X(t) = X_0 + \delta X_0 e^{i\omega t} \), the charge \( \delta Q(m) \) entering the cavity through contact \( m \), \( m = 1, 2 \), reads

\[
\delta Q(m, \omega) = e \frac{dn(m)}{dX} \delta X_0, \tag{3a}
\]

\[
\frac{dn(m)}{dX} = \frac{1}{2\pi} \sum_{\alpha} \sum_{\beta} \text{Im} \left[ \frac{\partial S^\ast_{\alpha\beta}}{\partial X} S_{\alpha\beta} \right]. \tag{3b}
\]

The index \( \alpha \) is summed from 1 to \( N \) for contact 1 and from \( N+1 \) to \( 2N \) for contact 2. The quantity \( dn(m)/dX \) is the emissivity into contact \( m \). Eq. (3) is valid to first order in the frequency \( \omega \) and assumes that the scattering properties follow the time-dependent potentials instantaneously. After Fourier transformation one obtains

\[
\delta Q(m, t) = e \frac{dn(m)}{dX} \delta X(t). \tag{4}
\]

Similarly, for a simultaneous infinitesimal variation of two parameters \( X_1 \) and \( X_2 \), the emitted charge \( \delta Q(m, t) \) through contact \( m \) is \( m = 1, 2 \)

\[
\delta Q(m, t) = e \frac{dn(m)}{dX_1} \delta X_1(t) + e \frac{dn(m)}{dX_2} \delta X_2(t). \tag{5}
\]

Next, we consider a finite variation of both parameters \( X_1 \) and \( X_2 \). The total charge emitted through contact \( m \) in one period \( \tau = 2\pi/\omega \) is found from integration of Eq. (5) to \( X_1 \) and \( X_2 \), bearing in mind that the scattering matrix \( S \) and hence the emissivities \( dn(m)/dX_1 \) and \( dn(m)/dX_2 \) are functions of \( X_1 \) and \( X_2 \),

\[
Q(m, \tau) = e \int_0^\tau dt \left( \frac{dn(m)}{dX_1} \frac{dX_1}{dt} + \frac{dn(m)}{dX_2} \frac{dX_2}{dt} \right). \tag{6}
\]

In one period, the pair of parameters \( X_1(t) \) and \( X_2(t) \) follows a closed path in the \((X_1, X_2)\) parameter space, see Fig. 1b. The total charge expelled from the dot through contact \( m \) can be rewritten as a surface integral over the area \( A \) enclosed by the path in parameter space using Green’s theorem,

\[
Q(m, \tau) = e \int_A dX_1 dX_2 \left( \frac{\partial}{\partial X_1} \frac{dn(m)}{dX_2} - \frac{\partial}{\partial X_2} \frac{dn(m)}{dX_1} \right). \tag{7}
\]

Note that the surface area \( A \), and hence the transported charge, vanish, if the parameters \( X_1 \) and \( X_2 \) vary in phase, or with a phase difference \( \pi \). The surface area is maximal if their phases differ by \( \pi/2 \). Substitution of Eq. (7) for the emissivities yields

\[
Q(m, \tau) = e \int_A dX_1 dX_2 \sum_{\alpha \in m} \sum_{\beta} \text{Im} \left[ \frac{\partial S^\ast_{\alpha\beta}}{\partial X_1} \frac{\partial S_{\alpha\beta}}{\partial X_2} \right]. \tag{8}
\]

Hence the d.c. current \( I_m \) through contact \( m \) is given by

\[
I_m = \frac{i\omega e}{4\pi^2} \sum_{\alpha \in m} \int_A dX_1 dX_2 \left[ R_{X_1}, R_{X_2} \right]_{\alpha\alpha}, \tag{8a}
\]

\[
R_{X} = -\frac{\partial S}{\partial X} S^\dag. \tag{8b}
\]

One verifies that \( I_1 = -I_2 \), indicating that no charge is accumulated. The response matrices \( R_{X_1} \) and \( R_{X_2} \) are hermitian \( 2N \times 2N \) matrices. For the (bi)linear response to the variations of the parameters \( X_1 \) and \( X_2 \), Eq. (8) simplifies to the result quoted in the introduction. Note that, since the parameters \( X_1 \) and \( X_2 \) are
dimensionless, the current formula contains no factor $\hbar$, unlike the Landauer formula \([1]\). Planck’s constant may however appear in the typical scales for the parameter dependence of the scattering matrix $S(X_1, X_2)$.

Eq. (8) is the main result of this paper. It establishes the link between the pumped current $I$ and the parametric derivatives of the scattering matrix $S$. Several qualitative observations can already be reached on the basis of Eq. (8). First, for a phase coherent quantum system, the out-of-phase variation of any pair of independent parameters will give rise to a d.c. current to order $\omega$. Second, $I$ is not quantized, unlike in the case of the electron pumps that operate in the regime of Coulomb blockade. Third, if the size of the variations $\delta X_1(t) = \delta X_1 \sin(\omega t)$ and $\delta X_2(t) = \delta X_2 \sin(\omega t - \phi)$ is small compared to the characteristic correlation scales $X_{1c}$ and $X_{2c}$, needed to change the scattering properties of the sample, we may neglect the $X_1$ and $X_2$ dependence of the integrand in Eq. (8), and recover the (bi)linear response formula (3). On the other hand, for $\delta X_j \gg X_{jc}$ ($j = 1, 2$), the integrand in Eq. (8) may have multiple sign changes within the integration area $A$, and the typical value of $I$ is proportional to $\langle \delta X_1 \delta X_2 \rangle \sim \sin(\phi)^{1/2}$. [Although the typical value of the current scales as $\langle \sin \phi \rangle^{1/2}$, the $\phi$-dependence of the sample-specific current may be quite random.]

Like the Landauer formula, the scattering matrix formula (8) describes both a classical contribution to the current and the quantum-mechanical corrections. Their roles are illustrated below in two examples. First, we consider a simple pump in a one-dimensional wire. The current and the quantum-mechanical corrections. Their sample-specific current may be quite random.

![FIG. 2. (a) An electron pump, consisting of a one-dimensional wire with a tunnel barrier at $x = 0$ and an adjustable electrostatic potential $U$ for $0 < x < L$. (b) Charge is pumped through the wire by varying the height $\gamma$ of the tunnel barrier and the potential $U$ in the following order: (i) $\gamma \to \infty$ (close barrier), (ii) $U \to \delta U$ (raise potential), (iii) $\gamma \to 0$ (open barrier), (iv) $U \to 0$ (lower potential).](image-url)

As a second example, we consider the case where electrons are pumped through a disordered or chaotic quantum dot. This application is relevant for the experiments of Ref. [13]. The two (dimensionless) parameters $X_1$ and $X_2$ characterize two different deformations of the shape of the dot, see Fig. 2a. Unlike in the previous example, where the pumping mechanism was of a mainly classical origin, for a chaotic quantum dot, there is no “classical” contribution to the pumped current. The current results from quantum interference and its size and direction depend on microscopic details of the system. Pumping occurs because the wave functions near the two point contacts are different and strongly parameter dependent, so that different amounts of current flow through the two contacts if the parameters $X_1$ and $X_2$ are varied.

For a disordered or chaotic quantum dot, the statistical distribution of the scattering matrix $S(X_1, X_2)$ and its dependence on microscopic details of the system. Pumping occurs because the wave functions near the two point contacts are different and strongly parameter dependent, so that different amounts of current flow through the two contacts if the parameters $X_1$ and $X_2$ are varied. The second term is the classical contribution to the pumped current. [Note that the local density of states for this one-dimensional system is $1/(2\pi k)$.] The second term is the correction due to quantum interference.

As a second example, we consider the case where electrons are pumped through a disordered or chaotic quantum dot. This application is relevant for the experiments of Ref. [13]. The two (dimensionless) parameters $X_1$ and $X_2$ characterize two different deformations of the shape of the dot, see Fig. 2a. Unlike in the previous example, where the pumping mechanism was of a mainly classical origin, for a chaotic quantum dot, there is no “classical” contribution to the pumped current. The current results from quantum interference and its size and direction depend on microscopic details of the system. Pumping occurs because the wave functions near the two point contacts are different and strongly parameter dependent, so that different amounts of current flow through the two contacts if the parameters $X_1$ and $X_2$ are varied.

For a disordered or chaotic quantum dot, the statistical distribution of the scattering matrix $S(X_1, X_2)$ and its dependence on microscopic details of the system. Pumping occurs because the wave functions near the two point contacts are different and strongly parameter dependent, so that different amounts of current flow through the two contacts if the parameters $X_1$ and $X_2$ are varied. The first term is the classical contribution to the pumped current. [Note that the local density of states for this one-dimensional system is $1/(2\pi k)$.] The second term is the correction due to quantum interference.

As a second example, we consider the case where electrons are pumped through a disordered or chaotic quantum dot. This application is relevant for the experiments of Ref. [13]. The two (dimensionless) parameters $X_1$ and $X_2$ characterize two different deformations of the shape of the dot, see Fig. 2a. Unlike in the previous example, where the pumping mechanism was of a mainly classical origin, for a chaotic quantum dot, there is no “classical” contribution to the pumped current. The current results from quantum interference and its size and direction depend on microscopic details of the system. Pumping occurs because the wave functions near the two point contacts are different and strongly parameter dependent, so that different amounts of current flow through the two contacts if the parameters $X_1$ and $X_2$ are varied.

The scattering matrix formula (8) allows us not only to find the statistical distribution of the pumped current $I$, but also to understand the typical scales for the parameter dependence of the scattering matrix $S$. The second term is the classical contribution to the pumped current. [Note that the local density of states for this one-dimensional system is $1/(2\pi k)$.] The second term is the correction due to quantum interference. As a second example, we consider the case where electrons are pumped through a disordered or chaotic quantum dot. This application is relevant for the experiments of Ref. [13]. The two (dimensionless) parameters $X_1$ and $X_2$ characterize two different deformations of the shape of the dot, see Fig. 2a. Unlike in the previous example, where the pumping mechanism was of a mainly classical origin, for a chaotic quantum dot, there is no “classical” contribution to the pumped current. The current results from quantum interference and its size and direction depend on microscopic details of the system. Pumping occurs because the wave functions near the two point contacts are different and strongly parameter dependent, so that different amounts of current flow through the two contacts if the parameters $X_1$ and $X_2$ are varied.
but also the statistical correlation between $I$ and the conductance $G$ of a chaotic quantum dot. The correlation between $I$ and $G$ shows a remarkable dependence on the presence or absence of time-reversal symmetry (TRS): In Ref. 7, it was shown that the statistical distribution of $R_{X_1}$ and $R_{X_2}$ is correlated with that of the conductance in the presence of TRS only. Therefore, without TRS, $I$ and $G$ are statistically uncorrelated for a chaotic quantum dot, while they are correlated in the presence of TRS. In the latter case, we find that the width of the current distribution at a fixed conductance $G$ is proportional to $G^{1/2}$ for single-channel leads.

Our main result, Eq. (6), is readily extended to include the effect of a capacitive interaction in the quantum dot within a self-consistent Hartree treatment. Following Ref. 4, the effect of the capacitive interaction is described by a self-consistent electric potential $U$. The potential $U$ is related to the (kinetic) energy $E$ and the Fermi energy $E_F$ via $E_F = E + U$. Variation of $X_1$ and $X_2$ will cause a change of $U$ and hence of $E = E_F - U$. (The Fermi energy $E_F$ is kept constant.) Hence we have to deal with a simultaneous variation of $E$, $X_1$, and $X_2$. These simultaneous variations can still be described by the scattering matrix formula (8) provided we replace the response matrices $R_{X_j}$ ($j = 1, 2$) by

$$R_{X_j} \rightarrow R_{X_j} + R_E \frac{\partial E}{\partial X_j}, \quad R_E = -i \frac{\partial S}{\partial E} S^\dagger.$$  \hspace{1cm} (13)

The derivative $\partial E / \partial X_j$ reads

$$\frac{\partial E}{\partial X_j} = -\frac{\text{tr} R_{X_j}}{\pi C/e^2 + \text{tr} R_E}, \quad j = 1, 2$$  \hspace{1cm} (14)

where $C$ is the geometrical capacitance of the dot. For many-channel contacts, inclusion of the interactions has no effect on the current distribution. For single-channel leads, we have computed the current distribution for the experimentally relevant case $C \ll e^2 \rho$, where $\rho$ is the (average) density of states in the dot, using the distribution of the matrices $R_{X_j}$ and $R_E$ in the presence of capacitive interactions. The result, which is not much different from the non-interacting case, is shown in Fig. 3.

It is a pleasure to acknowledge discussions with I. L. Aleiner, M. Büttiker, B. I. Halperin, C. M. Marcus, and Y. Oreg. This work was supported by the NSF under grants no. DMR 94-16910, DMR 96-30064, and DMR 97-14725.

\begin{thebibliography}{99}

1. L. P. Kouwenhoven \textit{et al.}, Phys. Rev. Lett. \textbf{67}, 1626 (1991).
2. H. Pothier \textit{et al.}, Europhys. Lett. \textbf{17}, 249 (1992).
3. C. Bruder and H. Schoeller, Phys. Rev. Lett. \textbf{72}, 1076 (1994).
4. B. Spivak, F. Zhou, and M. T. Beal Monod, Phys. Rev. B \textbf{51}, 13226 (1995).
5. T. H. Stoof and Yu. V. Nazarov, Phys. Rev. B \textbf{53}, 1050 (1996); preprint (cond-mat/9707310).
6. C. A. Stafford and N. S. Wingreen, Phys. Rev. Lett. \textbf{76}, 1916 (1996).
7. T. H. Oosterkamp \textit{et al.}, Phys. Rev. Lett. \textbf{78}, 1536 (1997).
8. Ph. Brune, C. Bruder, and H. Schoeller, Phys. Rev. B \textbf{56}, 4730 (1997).
9. K. Flensberg, Phys. Rev. B \textbf{55}, 13118 (1997).
10. M. H. Pedersen and M. Büttiker, preprint (cond-mat/9803306).
11. I. L. Aleiner and A. V. Andreev, Phys. Rev. Lett. \textbf{81}, 1286 (1998).
12. M. Switkes, C. M. Marcus, K. Campman, and A. C. Gossard (in preparation).
13. For a review, see C. W. J. Beenakker, Rev. Mod. Phys. \textbf{69}, 731 (1997).
14. The effect of a capacitive interaction for a similar electron pump in a dot with two single-channel leads is considered in Ref. 4.
15. M. Büttiker, H. Thomas, and A. Prêtre, Z. Phys. B \textbf{94}, 133 (1994).
16. For the dimensionless parameters $X_1$ and $X_2$ defined by Eq. (3) $X_{1e} = X_{2e} = N^{1/2}$, see Z. Pluhář \textit{et al.}, Phys. Rev. Lett. \textbf{73}, 2115 (1994) and K. Frahm, Europhys. Lett. \textbf{30}, 457 (1995).
17. P. W. Brouwer, K. M. Frahm, and C. W. J. Beenakker, Phys. Rev. Lett. \textbf{78}, 4737 (1997).
18. A more detailed analysis shows that in the presence of time-reversal symmetry the quantity $I / \sqrt{G}$ is statistically independent from $G$.
19. M. Büttiker, J. Phys. Condens. Matter \textbf{5}, 9361 (1993); M. Büttiker and T. Christen, in \textit{Quantum Transport in Semiconductor Submicron Structures}, B. Kramer ed., NATO ASI Ser. E, Vol. 326 (Kluwer, Dordrecht, 1996).
20. P. W. Brouwer \textit{et al.}, Phys. Rev. Lett. \textbf{79}, 913 (1997).
\end{thebibliography}