SEMISTABILITY OF CERTAIN BUNDLES ON A QUINTIC CALABI-YAU THREEFOLD.

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Abstract. In the paper “Chirality change in string theory”, by Douglas and Zhou, the authors give a list of bundles on a quintic Calabi-Yau threefold. Here we prove the semistability of most of these bundles. This provides examples of string theory compactifications which have a different number of generations and can be connected.

1. Introduction

In [2] Douglas and Zhou study string theory compactification and illustrate the chirality change with different examples.

In particular, in [2] Section 3 they consider heterotic string theory with gauge group $E_8 \times E_8$ compactified on a simply-connected compact Calabi-Yau manifold $M$. In order to show that there exist compactifications on the same Calabi-Yau with different number of generations which can be connected, they need to find examples of semistable holomorphic vector bundles on the Calabi-Yau, whose Chern classes differ only in $c_3$.

In [2] Section 3.3 Douglas and Zhou provide a list of bundles $V$ on a quintic Calabi-Yau $M \subset \mathbb{P}^4$, which satisfy the following conditions:

$$c_1(V) = 0, \quad c_2(V) = c_2(TM), \quad c_3 \text{ arbitrary.}$$

We recall this list in Table 3 above.

Furthermore in order to get supersymmetric vacua it is necessary to require the semistability of these bundle. In [2] Appendix A the authors check one interesting example ($V_8$ in Table 3) proving that it is stable against subsheaves that have a similar monad description.

We recall that a holomorphic vector bundle $V$ on a projective manifold $X$ with $\text{Pic}(X) \cong \mathbb{Z}$ is called stable if for any coherent subsheaf $S$ of $V$ with $0 < \text{rk } S < \text{rk } V$ we have $\mu(S) < \mu(V)$, where $\mu = \frac{c_1}{\text{rk}}$, and semistable if for any coherent subsheaf $S$ we have $\mu(S) \leq \mu(V)$. This notion of semistability is also called slope-semistability.

Here we complete the proof of the semistability for most of the bundles in the list. In particular in Proposition 3.1 we prove the semistability of the bundles with rank 4 on a generic smooth quintic in $\mathbb{P}^4$. The proof is based on standard computations and on Flenner’s theorem.
In Proposition 3.3, we prove that all the sheaves with rank 3 in the table, when restricted to a generic smooth quintic hypersurface in $\mathbb{P}^4$, are stable bundles. In this case we do computation directly on the threefold, in order to have locally free sheaves.

Finally, in Proposition 3.4, we prove the stability for some of the bundles with higher rank in the table. In this case we can prove the stability of a generic bundle with given resolution on a generic smooth quintic in $\mathbb{P}^4$. We prove this result by restricting to a generic plane and using the Drézet-Le Potier criterion for the existence of a stable bundle on $\mathbb{P}^2$. From this argument we can deduce the stability of our bundles but only when the resolution is generic in $\mathbb{P}^2$ (that is only for bundles $V_{10}, V_{12}, V_{14}, V_{16}$ in Table 3).

2. Preliminaries

Here we collect some useful results on vector bundles on projective varieties without giving the proofs. For more details see e.g. [5].

Let $\mathbb{P}^n$ denote the complex projective space of dimension $n$. Let $X$ be a complex projective manifold with Pic($X$) $\cong \mathbb{Z}$. A bundle $E$ on $X$ is called normalized if $c_1(E) \in \{-r+1, \ldots, -1, 0\}$, i.e. if $-1 < \mu \leq 0$. We denote by $E_{\text{norm}}$ the unique twist of $E$ which is normalized.

The following criterion for stability of bundles is a consequence of the definition:

**Proposition 2.1.** Let $V$ be a vector bundle on a projective manifold $X$ with Pic($X$) $\cong \mathbb{Z}$. If $H^0(X, (\wedge^q V)_{\text{norm}}) = 0$ for any $1 \leq q \leq \text{rk}(V) - 1$, then $V$ is stable.

**Remark 2.2.** Any exact sequence of vector bundles

$$0 \to A \to B \to C \to 0$$

induces the following exact sequence for any $q \geq 1$

$$0 \to S^q A \to S^{q-1} A \otimes B \to \cdots \to A \otimes \wedge^{q-1} B \to \wedge^q B \to \wedge^q C \to 0$$

We state Flenner’s theorem in the particular case of hypersurfaces in $\mathbb{P}^n$:

**Theorem 2.3** (Flenner). Assume

$$\binom{d+n}{d} - d - 1 > d \max \left\{ \frac{r^2 - 1}{4}, 1 \right\}.$$ 

If $E$ is a semistable sheaf of rank $r$ on $\mathbb{P}^n$, then the restriction $E|_X$ on a generic smooth hypersurface $X$ of degree $d$ in $\mathbb{P}^n$ is semistable.

The following criterion is a particular case (for $c_1 = 0$) of the Drézet-Le Potier theorem (see [3, Theorem C]):
Theorem 2.4 (Drézet-Le Potier). Given $r, c \in \mathbb{Z}$ such that

$$c \geq r > 0,$$

then there exists a stable bundle on $\mathbb{P}^2$ with rank $r$ and Chern classes $c_1 = 0$ and $c_2 = c$.

Let us denote by $M(r, c_1, c_2)$ the moduli space of semistable sheaves on $\mathbb{P}^2$ of rank $r$ and Chern classes $c_1, c_2$. It is known that $M(r, c_1, c_2)$ is irreducible and moreover we have the following useful result (see [4] and [1]).

Proposition 2.5. A generic bundle in the space $M(r, c_1, c_2)$ has resolution either of the form

$$0 \to \mathcal{O}(k-2)^a \oplus \mathcal{O}(k-1)^b \to \mathcal{O}(k)^c \to F \to 0,$$

or

$$0 \to \mathcal{O}(k-2)^a \to \mathcal{O}(k-1)^b \oplus \mathcal{O}(k)^c \to F \to 0,$$

for some $k \in \mathbb{Z}$, $a, b \geq 0$ and $c > 0$.

Finally we recall the following

Proposition 2.6. Let $\phi : E \to F$ be a morphism of vector bundles on a variety of dimension $N$, with $e = \text{rk}(E)$, $f = \text{rk}(F)$ and $e \leq f$. If $E^* \otimes F$ is globally generated and $f - e + 1 > N$, then for a generic $\phi$ the sheaf $\text{Coker}(\phi)$ is locally free, i.e. is a vector bundle.

3. Results

In Table 3 the list of sheaves on $\mathbb{P}^4$ contained in Table 1 of [2] is recalled. To every entry $(n_i, m_j)$ of the table we associate a sheaf with the following resolution

$$0 \to V \to \bigoplus_{i=1}^{r+m} \mathcal{O}_{\mathbb{P}^4}(n_i) \to \bigoplus_{j=1}^{m} \mathcal{O}_{\mathbb{P}^4}(m_j) \to 0.$$

It is easy to check that all these sheaves on $\mathbb{P}^4$ have Chern classes $c_1 = 0$ and $c_2 = 10$.

Moreover if the map in the resolution is generic, by Proposition 2.6 these sheaves are locally free on $\mathbb{P}^4$ only if the rank is bigger or equal than 4. Nevertheless, by restricting these sheaves to a generic quintic threefold $M$ in $\mathbb{P}^4$, we obtain locally free sheaves also in the case of rank 3.

We are interested in proving the (semi)stability of the restriction of these sheaves to a quintic $M$ in $\mathbb{P}^4$. First of all we can prove the following result.

Proposition 3.1. Let $V$ be a generic bundle with rank 4 in Table 3, i.e. with resolution of type $V_k$, for $6 \leq k \leq 9$. Then $V$ is semistable on a generic smooth quintic hypersurface in $\mathbb{P}^4$. 

Since (semi)stability is invariant up to duality, we will check the (semi)-stability of the dual bundles $V^*$. Let $E$ be the dual of a bundle of rank 4 in Table 3. To check the stability of $E$ we need to show that $H^0(\mathbb{P}^4, (\wedge^q E)_\text{norm}) = 0$ for any $1 \leq q \leq 3$. Since $c_1(E) = 0$, it is obvious that $(\wedge^q E)_\text{norm} = \wedge^q E$ for any $q$.

Before giving the proof of the previous proposition, let us consider in detail the example of a bundle of rank 4 corresponding to $V_8$.

**Example 3.2.** Let $E$ be a bundle with the following resolution on $\mathbb{P}^4$:

\[
(3.1) \quad 0 \to \mathcal{O}(-4)^2 \to \mathcal{O}(-2)^2 \oplus \mathcal{O}(-1)^4 \to E \to 0.
\]

In order to apply Proposition 2.1 we need to check the following conditions

\[
H^0(\mathbb{P}^4, E) = 0, \quad H^0(\mathbb{P}^4, \wedge^2 E) = 0, \quad H^0(\mathbb{P}^4, \wedge^3 E) = 0.
\]

By the cohomology sequence associated to (3.1) we immediately get the first vanishing. Indeed by Remark 2.2 we can compute the following resolution for $\wedge^2 E$:

\[
0 \to \mathcal{O}(-8)^3 \to \mathcal{O}(-6)^4 \oplus \mathcal{O}(-5)^8 \to \mathcal{O}(-4)^\oplus \mathcal{O}(-3)^8 \oplus \mathcal{O}(-2)^6 \to \wedge^2 E \to 0,
\]

and for $\wedge^3 E$:

\[
0 \to \mathcal{O}(-12)^4 \to \mathcal{O}(-10)^6 \oplus \mathcal{O}(-9)^{12} \to \mathcal{O}(-8)^2 \oplus \mathcal{O}(-7)^{16} \oplus \mathcal{O}(-6)^{12} \to \]

**Table 1.** The list of Douglas and Zhou.

| rank | $(n_i)$               | $(m_j)$               |
|------|-----------------------|-----------------------|
| $V_1$ | 3         | (2222222222)   | (33334)   |
| $V_2$ | 3         | (122222)       | (344)     |
| $V_3$ | 3         | (112233)       | (444)     |
| $V_4$ | 3         | (11222)        | (35)      |
| $V_5$ | 3         | (11133)        | (45)      |
| $V_6$ | 4         | (1122222222)   | (333333)  |
| $V_7$ | 4         | (11122222)     | (3334)    |
| $V_8$ | 4         | (111122)       | (44)      |
| $V_9$ | 4         | (11111)        | (5)       |
| $V_{10}$ | 5      | (1111122222)  | (33333)   |
| $V_{11}$ | 5       | (11111122)    | (334)     |
| $V_{12}$ | 6      | (1111111122)  | (3333)    |
| $V_{13}$ | 6       | (1111111111)  | (234)     |
| $V_{14}$ | 7      | (111111111111) | (2333)    |
| $V_{15}$ | 7       | (111111111111) | (22224)   |
| $V_{16}$ | 8      | (11111111111111) | (222233) |
\[ O(-5)^4 \oplus O(-4)^{12} \oplus O(-3)^4 \rightarrow \wedge^3 E \rightarrow 0. \]

From the resolution of \( \wedge^2 E \) we get the following two short exact sequences:

\[ 0 \rightarrow O(-8)^3 \rightarrow O(-6)^4 \oplus O(-5)^8 \rightarrow K_0 \rightarrow 0 \]
\[ 0 \rightarrow K_0 \rightarrow O(-4) \oplus O(-3)^8 \oplus O(-2)^6 \rightarrow \wedge^2 E \rightarrow 0, \]

and since \( H^1(\mathbb{P}^4, K_0) = 0 \), we get \( H^0(\mathbb{P}^4, \wedge^2 E) = 0 \). Analogously from the resolution of \( \wedge^3 E \) we get

\[ 0 \rightarrow O(-12)^4 \rightarrow O(-10)^6 \oplus O(-9)^{12} \rightarrow K_1 \rightarrow 0 \]
\[ 0 \rightarrow K_1 \rightarrow O(-8)^2 \oplus O(-7)^{16} \oplus O(-6)^{12} \rightarrow K_2 \rightarrow 0 \]
\[ 0 \rightarrow K_2 \rightarrow O(-5)^4 \oplus O(-4)^{12} \oplus O(-3)^4 \rightarrow \wedge^3 E \rightarrow 0, \]

from which we obtain \( H^2(\mathbb{P}^4, K_1) = 0 \), \( H^1(\mathbb{P}^4, K_2) = 0 \), and \( H^0(\mathbb{P}^4, \wedge^3 E) = 0 \). Hence we get

\[ H^0(\mathbb{P}^4, E) = H^0(\mathbb{P}^4, \wedge^2 E) = H^0(\mathbb{P}^4, \wedge^3 E) = 0, \]

which implies that \( E \) is stable on \( \mathbb{P}^4 \).

Now from Flenner’s theorem it follows that if \( E \) is a semistable bundle of rank 4 on \( \mathbb{P}^4 \), then its restriction on a generic hypersurface of degree \( d \geq 2 \) is semistable. Hence we conclude that the restriction of \( E \) on a generic smooth quintic hypersurface in \( \mathbb{P}^4 \) is semistable.

**Proof of Proposition 3.2.** It is easy to check that the argument used in Example 3.2 holds for all bundles in Table 3 with rank 4. Hence all the bundles with rank 4 are semistable on a generic smooth quintic. \( \square \)

**Proposition 3.3.** Let \( V \) be a generic sheaf of rank 3, i.e. with resolution of type \( V_k \) for \( 1 \leq k \leq 5 \). Then the restriction of \( V \) to a generic smooth quintic hypersurface in \( \mathbb{P}^4 \) is a stable bundle.

**Proof.** Let \( V \) be any sheaf of rank 3 of Table 3. Since \( V \) can be not locally free on \( \mathbb{P}^4 \), the argument used in Example 3.2 does not hold. Nevertheless, for a generic quintic \( M \) in \( \mathbb{P}^4 \), the restriction \( V|_M \) is locally free, by Proposition 2.6.

Let \( E|_M \) denote the dual of \( V|_M \). For example in the case \( V_5 \) we have the following resolution

\[ 0 \rightarrow O(-5)|_M \oplus O(-4)|_M \rightarrow (O(-3)|_M)^2 \oplus (O(-1)|_M)^3 \rightarrow E|_M \rightarrow 0 \]

and we want to apply Proposition 2.4 to \( E|_M \).

First of all, computing the cohomology of \( O(-k)|_M \) from the following exact sequence

\[ 0 \rightarrow O_{\mathbb{P}^4}(-5) \rightarrow O_{\mathbb{P}^4} \rightarrow O_M \rightarrow 0 \]
we easily obtain $H^0(M, E|_M) = 0$. On the other hand, by Remark 2.2 we can compute the resolution of $\wedge^2(E|_M)$ and we get $H^0(M, \wedge^2(E|_M)) = 0$. Analogously we can check that

$$H^0(M, E|_M) = H^0(M, \wedge^2(E|_M)) = 0$$

for all cases $V_1, \ldots, V_5$. Hence by Proposition 2.1 the restriction $E|_M$ is stable on $M$. We conclude that every sheaf with rank 3 of Table 3 restricted to a generic smooth quintic in $\mathbb{P}^4$ is a stable bundle. □

Furthermore we can prove that some of the bundles with higher rank in the table are semistable.

**Proposition 3.4.** Let $V$ be a generic bundle with resolution of the form $V_{10}, V_{12}, V_{14},$ or $V_{16}$ in Table 3. Then $V$ is semistable on a generic smooth quintic in $\mathbb{P}^4$.

**Proof.** Let us show first that $E = V^*$ is stable on $\mathbb{P}^4$. In order to do this, it suffices to prove that the restriction of $E$ on a plane is stable.

If $V$ is a generic bundle with resolution of the form $V_{10}, V_{12}, V_{14},$ or $V_{16}$, then the restriction $E|_\Pi$ on a generic plane $\Pi$ has a resolution of the form

$$0 \to O(-3)^a \oplus O(-2)^b \xrightarrow{\phi} O(-1)^c \to E|_\Pi \to 0$$

or

$$0 \to O(-3)^a \xrightarrow{\phi} O(-2)^b \oplus O(-1)^c \to E|_\Pi \to 0,$$

where $a, b, c \in \mathbb{N}$ are given and $\phi$ is a generic map.

On the other hand, we know from Theorem 2.4 that there exists a stable bundle on $\mathbb{P}^2$ with $c_1 = 0$, $c_2 = 10$ and $r \leq 10$. Hence the space $M(r, 0, 10)$ is not empty for $5 \leq r \leq 8$ and irreducible. Hence, by Proposition 2.3 it follows that a generic bundle in $M(5, 0, 10)$ (or $M(6, 0, 10)$, $M(7, 0, 10)$, $M(8, 0, 10)$ respectively) has resolution of the form $V_{10}$ (or $V_{12}$, $V_{14}$, $V_{16}$ respectively). Therefore the corresponding bundle $E|_\Pi$ is stable on the plane and $E$ is stable on $\mathbb{P}^4$.

Finally from Flenner’s theorem it follows that if $E$ is a semistable bundle of rank $5 \leq r \leq 8$ on $\mathbb{P}^4$, then its restriction to a generic hypersurface of degree 5 is semistable. Hence we conclude that the restriction of $E$ on a generic smooth quintic hypersurface in $\mathbb{P}^4$ is semistable. □

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