Classical limit of entangled states of two angular momenta

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We consider the classical limit of a system of two particles, each with arbitrary angular momentum $j$, in a state with zero total angular momentum. The state is maximally entangled and therefore exhibits non-classical features. To compare the quantum system with its classical counterpart the probabilities of finding projections of the angular momenta on selected axes are determined. Quantum probabilities are typically used for the study of generalized Bell’s inequalities. Violation of these inequalities is a proof that the system differs significantly from a classical probabilistic system with hidden variables. We use statistical methods to show that in the case of very large $j$, most of Bell’s inequalities are not violated. Further, it is almost impossible to find an inequality that is violated. In practice, the quantum system cannot be distinguished from its classical counterpart.

I. INTRODUCTION

Large quantum system should exhibit classical features. This is the main idea of the correspondence principle formulated by Bohr in the first years of quantum mechanics. There still is a question how this classical behavior is reached. This applies in particular to systems in entangled states. Quantum counterparts in classical physics. Examples of such quantities are superposition of special “pointer” states [1–3].

The classical limit has been studied via different routes. One universal mechanism of reaching the classical limit by large quantum system is through interactions with environment. The interactions lead to “dephasing” of the quantum system – during time evolution the density matrix evolves from describing a pure state to a mixed state, an incoherent superposition of special “pointer” states [1–3].

Another possibility of discussing the classical limit is to consider quantum mechanical quantities that have their counterparts in classical physics. Examples of such quantities are expectation values of position, momentum, energy etc. Moreover, off-diagonal elements of position or momentum operators between states of different energy have their classical counterparts as well.

Yet another aspect of classicality to notice is that very precise measurements have to be performed in order to detect quantum effects in systems close to the classical limit.

Entangled states of quantum systems, in turn, exhibit correlations that do not have classical counterparts. Question regarding classical limit of entangled states remains, however, to be discussed. This paper addresses this question using example of spinning tops.

John Bell [4] found a set of inequalities that are fulfilled by probabilities, obtained within any hidden variable theory with local realism. Violation of Bell’s inequalities is a proof that quantum mechanics cannot be replaced by any form of probabilistic theory with local realism. This assertion was confirmed by experiments with two state systems (qubits).

Original and early versions of Bell’s inequalities [4,5] involved two observers, each one having a choice of two mutually incompatible experiments. The inequalities have been generalized and can involve many observers and multi-state systems. From a geometric point of view, Bell’s inequalities describe a bound convex set – intersection of a finite number of half spaces. Finding all Bell’s inequalities gives a necessary and sufficient condition for deciding whether a given state can be viewed as representing a local theory with hidden variables. This is, however, a computationally demanding NP problem [6]. A complete list of Bell’s inequalities exists only for simplest cases [5, 7, 10].

In this paper we study an example of a system in an entangled state that has a well defined classical limit. The system consists of two subsystems, each with angular momentum $j$. Classical limit is reached when the value of both angular momenta are much larger than $\hbar$ – the quantum unit of angular momenta. We will discuss generalized Bell’s inequalities for this system. It is known that for each entangled system an inequality exists that is valid in case of a classical system, but is violated by the quantum nature of the state. In case of the system studied here, it is very hard to find such inequality and prove that it is violated. This illustrates the fact that the system cannot be distinguished, or at least it would be very hard to distinguish, from a classical system of two classical correlated angular momenta. This approach to the classical limit sheds more light on the structure of entanglement of systems close to classical limit.

II. THE SYSTEM - QUANTUM AND CLASSICAL DESCRIPTION

We consider a system consisting of two particles (tops), each characterized by angular momentum quantum number $j$. The square of the angular momentum is thus $\hbar^2 j(j + 1)$. We are interested in case of large $j$, hence consider states close to the classical limit. We assume that angular momenta are expressed in units of $\hbar$. The system is in the state with the total
angular momentum equal to zero. This state is given by

$$\Psi = \sum_{m=-j}^{j} (-1)^{j-m} \frac{1}{\sqrt{2j+1}} |m\rangle |m\rangle - m \rangle \tag{1}$$

This is definitely an entangled state with maximal entanglement.

Two observers, say A and B, measure components of angular momenta along arbitrarily chosen axes in the state $|\Psi\rangle$. Let us name these axes $a$ for observer A and $b$ for observer B. The result of such measurement is a number $m_1$ in case of observer A and $m_2$ in case of observer B, with $-j \leq m_1, m_2 \leq j$. Thus there are $2j+1$ possible outcomes of measurement for each observer. In case of the state $|\Psi\rangle$ with zero total angular momentum the distribution of the results is flat, the probability of finding the value $m$ of the angular momentum by each observer along each axis is $1/2j+1$.

We will now find the probability amplitude $a(a, b, m_1, m_2)$ of detecting a value $m_1$ of the angular momentum of the first particle in the direction $a$ and a value of $m_2$ of the angular momentum of the second particle in the direction $b$. We make use of the fact that the state $|\Psi\rangle$ is rotationally invariant, i.e. has the same form in all coordinate systems. We choose, therefore, the system of coordinates where vector $a$ is along the $z$ axis and vector $b$ lies in the $x-z$ plane. Thus the $y$ axis is perpendicular to the plane spanned by the two vectors. The probability amplitude $a(a, b, m_1, m_2)$ of detecting values $m_1$ and $m_2$ in the chosen coordinate system is:

$$a(a, b, m_1, m_2) = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{j} \sum_{m'=-j}^{j} (-1)^{j-m_1} \langle m_1 | m \rangle \langle m_2 | m' \rangle d^j_{m, m'}(\beta), \tag{2}$$

where $\beta$ is the angle between $a$ and $b$. After short manipulations one can write:

$$a(a, b, m_1, m_2) = \frac{1}{\sqrt{2j+1}} (-1)^{j-m_1} d^j_{-m_1,-m_2}(\beta). \tag{3}$$

where $d^j_{m, m'}(\beta)$ denotes the Wigner rotation function. [11]

Probability of detecting angular momenta equal to $m_1$ and $m_2$ along appropriate directions is plotted in Fig.1 for several values of $j$. Let us note that this function depends only on the angle between vectors $a$ and $b$. Because of rotational invariance of the state $|\Psi\rangle$, the same formula is valid in all coordinate systems.

In addition to the probability distribution $a(a, b, m_1, m_2)$ given by (3), we will consider lowest order correlation functions of two angular momentum components along an arbitrary axis $a$ of one angular momentum and the second angular momentum component along a different axis $b$. This correlation function $E(aJ_1, bJ_2)$ is obtained by taking expectation value of the operator $aJ_1 \cdot bJ_2$ in the state $|\Psi\rangle$. We choose the $a$ vector along the $z$ axis and get:

$$E(aJ_1, bJ_2) = \sum_{m=-j}^{j} \frac{1}{2j+1} m (-m) b_z. \tag{4}$$

Note that (5) is consistent with the correlation function found in the case of spin $\frac{1}{2}$ system, see [12].

We will proceed with the study of a purely classical model of two correlated system having angular momentum $J$ each. The systems are spatially separated, however their total angular momentum is zero. The system may consists of two spinning tops, with opposite axes of rotation. The direction of the axes is random with uniform probability distribution, therefore the average value of angular momentum of each top is equal to zero. Random uniform distribution of the rotation axis means that the direction $n$ is a unit vector, while its polar angle $\Theta$ and azimuthal angle $\Phi$ are random numbers with distribution $\frac{1}{2\pi} \sin \Theta$.

We will determine the probability distribution $\rho(a, b, K, L)$ of finding the value $K$ of the component of the first angular momentum $J_1$, and the value $L$ of the second
angular momentum \( J_2 \) along the \( b \) axis. This is the classical counterpart of the quantum expression for \( |a(a, b, m_1, m_2)|^2 \) in (3). We will use the fact that the vector \( J_1 \) of the first top has direction along vector \( n \) and the vector \( \mathbf{J}_2 \) of the second top has direction \(-n: J_1 = n/J, J_2 = -nJ. \) The classical probability distribution is then given by the formula:

\[
R(a, b, K, L) = \frac{1}{4\pi} \int d\Phi \sin \Theta d\Theta \delta(K - Jn) - (L + Jbn).
\]

(6)

Explicit evaluation of this integral leads to:

\[
R(a, b, K, L) = \frac{1}{2\pi J} (-K^2 - L^2 - 2KL \cos \theta + J^2 \sin^2 \theta)^{-\frac{1}{2}},
\]

(7)

where \( \theta \) is the angle between vectors \( a \) and \( b \). Let us introduce the scaled angular momenta \( k = K/J \) and \( l = L/J \). The probability for finding the quantities \( k \) and \( l \) within the range of \( dk \) and \( dl \) is simply expressed by \( p_c(a, b, k, l) = \rho(a, b, k, l) dk dl \). The probability distribution therefore reads as:

\[
\rho(a, b, k, l) = \frac{1}{2\pi} (-k^2 - l^2 - 2k l \cos \theta + \sin^2 \theta)^{-\frac{1}{2}}.
\]

(8)

In order to compare quantum probabilities with their classical counterparts, we choose the classical angular momentum \( J \) equal to the quantum angular momentum \( \hbar \sqrt{j(j+1)} \). Also the classical projections \( k \) and \( l \) are identified with \( m_1/j \) and \( m_2/j \). The probability \( p_c(a, b, k, l) \), with \( dk = 1/j = dl \) is plotted in Fig. 2.

Now we will discuss the classical correlation \( E(aJ_1, bJ_2) \) between a component of one angular momentum along an arbitrary axis \( a \) and of the second angular momentum along a different axis \( b \). The correlation is defined as the average value of the product \( (aJ_1 \cdot bJ_2) \), according to:

\[
E(aJ_1, bJ_2) = \frac{L^2}{4\pi} \int d\Theta \sin \Theta d\Phi (a \cdot n)(b \cdot n).
\]

(9)

Calculation of the integral gives:

\[
E(aJ_1, bJ_2) = -\frac{1}{3} J^2 a \cdot b.
\]

(10)

Correlation (10) found in this purely classical limit is consistent with the quantum correlation (5), the only difference is that there is \( J^2 \) in the classical case as opposed to \( \hbar^2 j(j+1) \) in the quantum case.

III. SEMICLASSICAL APPROXIMATION

Semiclassical approximation to the quantum description of provides a link between the quantum and semiclassical approaches. We will find now the semiclassical approximation to the probability amplitude \( a(a, b, m_1, m_2) \) and discuss its relation to classical probability. We will use the WKB method treating \( \frac{1}{J} \) as a small parameter.

We will start with the full Wigner rotation function \( D_{m_1, m_2}^{j}(\alpha, \beta, \gamma) \). It satisfies differential equation:

\[
\left[ -\frac{1}{\sin^2 \beta} \left( \frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \gamma^2} 2 \cos \beta \frac{\partial^2}{\partial \alpha \partial \gamma} \right) - \frac{\partial^2}{\partial \beta^2} - \frac{1}{\sin \beta} \frac{\partial}{\partial \beta} \right] D^{j}_{m, m}(\alpha, \beta, \gamma) = 0
\]

(11)

Standard semiclassical (WKB) method allows to find the asymptotic behavior of the \( D_{m, m}^{j}(\beta) = d_{m, m}^{j}(\beta) \), where \( d_{m, m}^{j}(\beta) \) is the matrix element of the rotation around the \( y \)-axis:

\[
d_{m, m}^{j}(\beta) = (-1)^{j} \sqrt{\frac{2J}{\pi}} \frac{\partial S_{0}}{\partial m} \cos \left( \frac{J S_{0}(\beta) - \pi}{4} \right).
\]

(12)

where \( j = \frac{J}{2} \) and \( S_{0} \) is the generating function of the classical rotation in canonical variables \( (J \cos \theta, \phi) \), \( (\theta, \phi) - \) the spherical coordinates).

\[
S_{0}(m', m) = \frac{m}{J} \text{arc} \cos \left( \frac{m \cos \beta - m'}{\sin \beta \sqrt{J^2 - m^2}} \right) - \frac{m'}{J} \text{arc} \cos \left( \frac{m - m' \cos \beta}{\sin \beta \sqrt{J^2 - m'^2}} \right) + \text{arc} \cos \left( \frac{m m' - J^2 \cos \beta}{\sqrt{J^2 - m'^2} \sqrt{J^2 - m^2}} \right)
\]

(13)

Hence,

\[
d_{m, m}^{j}(\beta) = (-1)^{j} \cos \left( \frac{J S_{0}(\beta) - \pi}{4} \right) \times \left( \frac{2}{\pi \sin^2 \beta - \frac{1}{J^2} (m^2 + m'^2 - 2 m m' \cos \beta)} \right)^{\frac{1}{2}}.
\]

(14)

These approximate formulas are not valid in the vicinity of the classical turning points, i.e points where the denominator in (13) is close to zero.

Comparison of exact values of the \( d_{m, m}^{j}(\beta) \) functions and their semiclassical approximations is illustrated in Fig. 2. Observe, that the probability distribution \( \rho(a, b, k, l) \) found in the previous section is equal to the square of the envelope of the \( d_{m, m}^{j}(\beta) \) function in the semiclassical approximation. The difference between the studied probability of the quantum system and the corresponding probability distribution for the classical system lies in small high frequency oscillations seen in Fig. 2.

IV. STATISTICAL APPROACH TO GENERALIZED BELL’S INEQUALITIES

In this section we will concentrate on entanglement of the system and possible experimental tests that can verify existence of the entanglement. Entanglement is a non-classical
To our best knowledge the case of large angular momentum are also known [9, 10], see also [14–20].

Quantum nature of probabilities that stems from quantum variables. Violation of these inequalities is a clear sign of the quantum probabilities in case of entangled states. Linear combinations of joined probabilities are equalities. These are inequalities that should be satisfied by variables or local realism exist in quantum physics. This means that the probabilities satisfy a set of inequalities known as (generalized) Bell’s inequalities.

Bell’s inequalities and their generalizations provide a powerful tool to prove existence of entanglement. The idea of Bell’s inequalities can be formulated as follows. One considers a set of probabilities: \( p(a, m) \) and \( p(a, b, m_1, m_2) \) where the measurement axes are chosen in various directions defined by vectors \( a \) and \( b \). Two sets of vectors \( a_r \) and \( b_s \), where \( r = 1, 2 \) and \( s = 1, 2 \) should be considered. Notice that probabilities \( p(a_r, b_s, m_1, m_2) \) depend on vectors \( a_r \) and \( b_s \) by the angle between them. If hidden variables exist and local realism is valid then probabilities \( p(a_r, m) \) and \( p(a_r, b_s, m_1, m_2) \) are within the convex hull spanned by vertices defined above. This means that the probabilities satisfy a set of inequalities known as (generalized) Bell’s inequalities.

Numerous experiments showed that these inequalities are violated by quantum probabilities in case of entangled states of two \( j = 1/2 \) states, known as qubits, [12] and references therein. This provides a strong argument that no hidden variables or local realism exist in quantum physics.

Let us now discuss in more detail the generalized Bell’s inequalities. These are inequalities that should be satisfied by linear combinations of joined probabilities \( p(a_r, b_s, m_1, m_2) \) under the assumption of local realism and existence of hidden variables. Violation of these inequalities is a clear sign of the quantum nature of probabilities that stems from quantum entanglement. Analagous of Bell’s inequalities in case of large angular momentum are also known [9, 10], see also [14–20]. To our best knowledge the case of large \( j \) and the limit \( j \to \infty \) has not been considered yet.

In case of arbitrary \( j \) there are \( S = 4 \times (2j + 1)^2 \) joined probabilities \( p(a_r, b_s, m_1, m_2) \). According to [21], all Bell’s inequalities are of the form

\[
M_1 \leq \sum_{r,s,\mu,\nu} p(a_r, b_s, m_\mu, m_\nu) c(r,s, m_\mu, m_\nu) \leq M_2,
\]

where \( M_1 \) and \( M_2 \) are fixed numbers and \( c(r,s, m_\mu, m_\nu) \) are coefficients. The coefficients are equal to 0 or to natural numbers with plus or minus sign, [21]. As mentioned earlier, the number of inequalities grows rapidly with \( j \).

We are interested in the case of large \( j \), hence the number of Bell’s inequalities is huge. It is not realistic to consider them all, so we will turn to statistical approach. Inspiration to use statistical methods stems from a somewhat similar approach to spectra of complex systems, where the Hamilton operator is replaced by a matrix with elements being random numbers with a statistical distribution. The eigenvalues are random numbers and in fact are studied using statistical methods. To deal with Bell’s inequalities, we will treat the probabilities of measuring given values of angular momenta by the two observers as random numbers.

Quantum probabilities \( p(a_r, b_s, m_1, m_2) \) have their classical analogue \( p_c(a_r, b_s, k, l) \), as we saw in Sec[11]. It is helpful to consider quantum corrections, i.e. differences between the quantum and classical probabilities. The quantum corrections \( \delta p(a_r, b_s, m_1, m_2) \) to the classical probabilities can be defined in the following way:

\[
\delta p(a_r, b_s, m_1, m_2) = p(a_r, b_s, m_1, m_2) - p_c(a_r, b_s, k, l) dk dl,
\]

where \( k = m_1/j \), \( l = m_2/j \) and increments are chosen to fit the increments of quantum numbers \( m_1 \) and \( m_2 \), therefore \( dk = 1/j = dl \).

Classical probabilities satisfy Bell’s inequalities and violation of these inequalities by quantum systems can be due to the quantum corrections only.

Now, we will study how the corrections scale with the value of angular momentum \( j \). First we will show that the corre-
we can clearly see that as a function of angle between vectors \( \mathbf{a}_r \) and \( \mathbf{b}_s \) for various angle between \( \mathbf{a}_r \) and \( \mathbf{b}_s \) are not correlated. This

Finally we will show that probabilities \( p(\mathbf{a}_r, \mathbf{b}_s, m_1, m_2) \) for various angle between \( \mathbf{a}_r \) and \( \mathbf{b}_s \) are not correlated. This

Figure 5: Histograms of \( \delta p(\mathbf{a}_r, \mathbf{b}_s, m_1, m_2) \times N \) for many values of projections of momentum \( m_{1,2} \) and angle \( \beta \) between vectors \( \mathbf{a}_r \) and \( \mathbf{b}_s \). Values \( m_{1,2} \) vary from \(-j\) to \(j\) by one. Values \( j \) and \( \beta \) are placed on the plot.

Figure 4: The scaling of average squares of \( \delta p(\mathbf{a}_r, \mathbf{b}_s, m_1, m_2) \) versus momentum \( j \). The scaling factor is proportional to \( j^2N \), where \( N = (2j+1)^2 \).

\[ \delta p_{av}^2 = \frac{1}{(2j+1)^2} \sum_{m_1,m_2} \delta p(\mathbf{a}_r, \mathbf{b}_s, m_1, m_2). \]

The behavior of this quantity is illustrated in Fig.4 as a function of \( j \). Notice that \( \delta p_{av} \) tends to zero when \( j \to \infty \). The next figure, Fig.4, shows the average squares of \( \delta p_{av} \), i.e. \( \delta p_{av}^2 = \frac{1}{(2j+1)^2} \sum_{m_1,m_2} \delta p(\mathbf{a}_r, \mathbf{b}_s, m_1, m_2)^2 \) as functions of \( j \). It is clear that the larger the \( j \) the smaller the average difference between the classical and quantum probability. Notice from the Fig.4 that \( \delta p_{av}^2 \) goes to zero as \( j^{-4} \).

Fig.5 presents histograms of differences between classical and quantum probabilities for various \( j \) and various angles \( \beta \) between vectors \( \mathbf{a}_r \) and \( \mathbf{b}_s \). We see that the differences are localized around zero value and the distribution is close to a gaussian. Thus we may say that the differences between classical and quantum probabilities are random numbers with a narrow distribution.

Therefore the linear combination can either tend to zero or to a constant, independent of \( j \). Since \( \delta p(\mathbf{a}_r, \mathbf{b}_s, m_1, m_2) \) can be positive or negative the linear combination of them in any of Bell’s inequalities contains terms with alternating signs and thus the total sum tends to zero with \( j \to \infty \). The only exception is the case when the signs of \( \delta p(\mathbf{a}_r, \mathbf{b}_s, m_1, m_2) \) are correlated with the signs of \( c(r, s, m_1, m_2) \), i.e. when all or at least most products \( c(r, s, m_1, m_2) \times \delta p(\mathbf{a}_r, \mathbf{b}_s, m_1, m_2) \) are of the same sign.

We should stress also that probabilities \( p(\mathbf{a}_r, \mathbf{b}_s, m_1, m_2) \) depend on the choice of measurement performed by the two observers. A measurement is defined by the directions \( \mathbf{a}_r \) and \( \mathbf{b}_s \) along which the angular momenta are measured. From Fig.2 we can clearly see that the probabilities \( p(\mathbf{a}_r, \mathbf{b}_s, m_1, m_2) \) are quite sensitive to the angle between the two vectors. Even a small change of the angle leads to a large change of \( p(\mathbf{a}_r, \mathbf{b}_s, m_1, m_2) \). Suppose one of Bell’s inequalities is violated by a set of probabilities for given vectors \( \mathbf{a}_r \) and \( \mathbf{b}_s \). A small change of these vectors, or rather the angle between them, by about \( \pi/j \) leads to a substantial change of the probabilities. In fact, instead of minima of \( p \) (as functions of angle) we may have maxima after a small change of angles and the inequality is no longer violated. Thus verification if a Bell inequality is violated or not depends on
the choice of angles, and therefore on the accuracy of angle determination.

In order to violate a Bell inequality the coefficients $c(r, s, m_1, m_2)$ have to be correlated with the probabilities $p(a_r, b_s, m_1, m_2)$. It is only if the values of $c(r, s, m_1, m_2)$ are negative and large (in absolute values) and the corresponding probabilities $p(a_r, b_s, m_1, m_2)$ are large when the negative terms can dominate over the positive terms and the inequality can be violated.

These "specially designed" conditions do not indicate that Bell’s inequalities cannot be violated by probabilities obtained from the state $Ψ$. They show, however, that possible violation is very small, at most. Only very few inequalities, out of a huge number of them, can be violated. A small number of violated inequalities indicate that the degree of violation does not grow with $j$. In this way the quantum state, $|Ψ⟩$, being a quantum maximally entangled state, reproduces classical angular momentum.

Observe that the classical limit proved here is not reached in a uniform way. It is not said that each of Bell’s inequalities is satisfied in the limit of large $j$. What is said is that out of a huge number of inequalities only a small number of them can be violated and not to a high degree.

A completely different approach to Bell’s inequalities is to consider correlation functions rather than the probabilities, see [1]. The correlation function $E$ can be written as a sum $E = E_{\text{classical}} + E_{\text{quantum}}$, where $E_{\text{classical}} = \frac{1}{2}a \cdot b$, and $E_{\text{quantum}} = \frac{1}{2^j}a \cdot b$. The quantum part is by a factor $j$ smaller than the classical one. Then, for large $j$ values $E_{\text{quantum}}$ is a small correction to $E_{\text{classical}}$. In the classical limit $j \to \infty$ the correlation functions $E$ tend to their classical counterparts and no violation of Bell’s inequalities is possible.

V. CONCLUSIONS

There are many approaches to study of the classical limit in case of large quantum system. Some most popular ones were mentioned in the Introduction and have been illustrated in this paper using example of two entangled tops. One kind of approach is based on measurements and finite resolution of any measuring device. If the resolution of angular momentum measurement is below the quantum unit $\hbar$ cannot measure the probability $p(a_r, m)$ nor the joined probability $p(a_r, b_s, m_1, m_2)$. Each such measurement gives an average over many $m_1$ and $m_2$. It is clear from the plots that such averaging leads directly to classical probabilities. This is a simple observation.

Another possibility of looking at the classical limit is to consider interaction of the system with the environment. This approach was formulated by Zurek [1] and used in many subsequent papers. Going along this line we see that the state $|Ψ⟩$ is entangled, any change of phase between the terms in the sum defining the state destroys entanglement. We have not discussed this point any deeper, one would have to introduce time scale of dephasing and hence the entanglement lifetime. This can be done only in the framework of a model environment.

This last approach stresses the importance of the environment and damping of the quantum coherence by mutual interaction between the system and the environment. Our approach to the classical limit is, however, very different from the one mentioned above. There is no need for coherence damping in the present approach. The probabilities of measurement and also correlation functions of the quantum theory are sufficiently close to their classical counterparts to make the quantum case look very much the same as the classical system.

Obviously, the quantum states differ from the classical distribution function. The difference is entanglement, that has no classical counterpart. In this case the relation between classical and quantum description is much more subtle than in previously mentioned cases. Existence of entanglement can be verified on the basis of Bell’s inequalities and their generalizations. While a lot is known about Bell’s inequalities for small systems, two state systems in particular, the case of large entangled system is less known. We formulated a statistical approach to the problem based on the fact the number of Bell’s inequalities grows rapidly with the size of the system and that probabilities of measuring a single state is very difficult to determine. Using the example of two large but entangled angular momenta we showed that joined probabilities $p(a_r, b_s, m_1, m_2)$ behave more or less like random numbers in case of large $j$.

Our statistical approach showed that it is very difficult to find an inequality that is violated by the entangled state of two large angular momenta. This does not prove that such an inequality does not exist. On the contrary, it has been shown, [19], that, in case of a similar system, an inequality is violated by a system considered, even in the case of large systems. Our result shows that there are very few of such violated inequalities and therefore it is very difficult to find them.

In addition to Bell’s inequalities we studied quantum correlation functions and showed that they tend to the classical limit of two correlated classical tops like $1/j$. Because of this rapid approach to the classical limit the correlation functions cannot be used efficiently to discriminate between classical and quantum states.

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[1] W. Zurek, Phys. Rev. D. 24, 1516 (1981).
[2] D. A. R. Dalvit, J. Dziarmaga, and W. H. Zurek, Phys. Rev. A 72, 062101 (2005).
[3] A. Venugopalan, Phys. Rev. A. 61, 012102 (1999).
[4] J. S. Bell, Physics 1, 195 (1964).
[5] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. 23, 880 (1969).
[6] D. Avis, H. Imai, T. Ito, and Y. Sasaki, arXiv:quant-ph/0404014v3 (2004).
[7] D. Rosset, J.-D. Bancal, and N. Gisin, J. Phys. A: Math. Theor. 47, 424022 (2014).
[8] S. Pironio, J. Phys. A: Math. Theor. 47, 424020 (2014).
[9] I. Pitowsky and K. Svozil, Phys. Rev. A 64, 014102 (2001).
[10] C. Sliwa, Phys. Lett. A 317 (2003).
[11] E. P. Wigner (Academic Press, Princeton, New Jersey, 1959).
[12] A. Peres (Kluwer Academic, Dordrecht, 1993).
[13] P. A. Braun, P. Gerwinski, F. Haake, and H. Schomerus, Z. Phys. B 100, 115 (1996).
[14] A. Garg and N. D. Mermin, Phys. Rev. Lett. 49, 1220 (1982).
[15] N. D. Mermin, Philos. Sci. 50, 359 (1983).
[16] P. Horodecki and R. Horodecki, Phys. Rev. Lett. 76, 2196 (1996).
[17] N. D. Mermin and A. Garg, Phys. Rev. Lett. 76, 2197 (1996).
[18] N. D. Mermin, Phys. Rev. Lett. 65, 1838 (1990).
[19] N. Gisin and A. Perez, Phys. Lett. A 162, 15 (1992).
[20] S. Popescu and D. Rohrlich, Phys. Lett. A 166, 293 (1992).
[21] A. Peres, Foundations of Physics 29, 4 (1999).