CHSH Inequality on a single probability space

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Abstract
Khrennikov and co-workers have suggested in a series of papers that it is inappropriate to combine data from different experiments when undertaking experimental tests of Bell’s inequalities. They suggest that a correct analysis, using a single probability space, leads to inequalities which are not violated by experiment. If correct, this would be contrary to the normal interpretation of such experimental data.

However, in this note, a generalised Clauser-Horne-Shimony-Holt (CHSH) inequality is derived for a system of four experiments constructed on a single probability space which combines the data from the four experiments. In contradiction to Khrennikov et al, it is shown that this leads to the standard CHSH inequality which is normally used to interpret experimental data. Thus the commonly accepted conclusion that experimental violations of Bell’s inequality imply that local realistic models are inconsistent with the predictions of quantum mechanics has not been challenged.

1 Introduction
In a series of papers Khrennikov and co-workers [1–4] have made strong claims that the experimental violation of the CHSH inequalities is a consequence of an erroneous use of experimental data. It is suggested that the experimental violations of the CHSH inequality arise through incorrectly combining data from a number of experiments and suggest that mixing the data from different experimental contexts is an improper use of statistical data.

In this paper we dispute these conclusions. We use the single probability space proposed by Khrennikov et al and demonstrate that the central assumption they make about the CHSH inequality for quantum systems is too weak. The analysis presented below shows that a correct analysis of a model based on a single probability space yields the standard CHSH inequalities which are violated experimentally and this violation is not a consequence of incorrect use of the experimental data.

In more recent papers Khrennikov asserts that the error arises because of the confusion between conditional and absolute correlations (eg page 772 of [5] and page 7
of [4]). Once again the analysis presented below shows that the conditional correlations obey the standard CHSH inequality and the absolute correlations obey a stronger version of the CHSH inequality. Since there is a simple mathematical relationship between the conditional and absolute correlations and the inequalities which they obey, any circumstances which lead to one inequality being violated will also lead to the other being violated.

Avis et al [1] give a detailed analysis of a EPR-Bohm-Bell experiments and suggests a single probability space for the analysis of such experiments; they describe this as a proper probability space. It is also described as a common Kolmogorov probability space in Khrennikov [4]. In [1, 4] they assume a CHSH inequality of the form

\[ |< A^{(1)}, B^{(1)} > - < A^{(1)}, B^{(2)} > + < A^{(2)}, B^{(2)} > + < A^{(2)}, B^{(1)} >| \leq 2 \]  

(1)

for random variables \( A' \) and \( B' \) defined on a single Kolmogorov probability space which they construct to describe four separate experiments; \( A' \) and \( B' \) and are defined below. They then claim that the “Bellian” variables \( a_i \) and \( b_i \) used to describe a single experiment obey the modified CHSH inequality (eg equation (2) in [1])

\[ |< a^{(1)}, b^{(1)} > - < a^{(1)}, b^{(2)} > + < a^{(2)}, b^{(2)} > + < a^{(2)}, b^{(1)} >| \leq 8 \]  

(2)

and that this inequality is not violated experimentally.

In this paper we show that the CHSH inequalities (1) and (2) are too weak and therefore lead to erroneous conclusions. We show that a correct analysis modifies the expression (1) to make it more restrictive, giving the modified form of the CHSH inequality shown in equations (11) or (13), for the correlations of the random variables \( A' \) and \( B' \). It follows from this new expression that the standard CHSH inequality, equation (12), is recovered for the correlation functions \( < a_i, b_i > \) which are normally considered in single experiments and this inequality is violated experimentally. The correlation function \( < a_i, b_i > \) is equivalent to the conditional correlations defined by Khrennikov [4, 5].

\section{2 Background}

Bell [6], and subsequently other workers, established a number of inequalities which should be obeyed by local hidden variable models of quantum mechanics; see [7] for a collection of Bell’s publications and also Clauser et al [8]. The inequality derived in the latter paper will be referred to here as the CHSH inequality. Bell type inequalities have been extensively tested experimentally and it is found that the inequalities are normally violated by quantum mechanical systems. Thus, although there remain some possible but unlikely loopholes, there is now a consensus that local realistic models are inconsistent with quantum mechanics. Two recent reviews of Bell’s theorem and the associated experimental work are to be found in [9] and [10].

The canonical system to which Bell’s result is typically applied is a modification suggested by Bohm [11] of the EPR experiment proposed by Einstein et al [12] which measures the spin of two entangled particles rather than their position; we refer to such experiments as EPR-Bohm-Bell experiments.
3 Hidden variables on a single probability space

The work in this paper is based directly on the arguments Avis et al. and Khrennikov used [1, 4] to construct a single Kolmogorov probability space for the CHSH experiments. However, we come to different conclusions. We use the random variables set out in Section 3, *Proper random experiment*, of [1] and derive results which show that the inequality (2) in [1] (ie equation (2) above) is in error by being too weak.

We consider an EPR-Bohm-Bell experiment in which pairs of spin $1/2$ particles are created and subsequently separate in opposite directions $A$ and $B$. In each direction there are assumed to be two spin measurement devices aligned in orientations $a_i; i=1,2$ and $b_j; j=1,2$ on sides $A$ and $B$ respectively. These devices measure the spin orientation of the particle in direction $a_i$ or $b_j$. We assume that the measurement devices are idealised and only yield outputs of ±1 representing the possible outcomes of the measurements for a spin $1/2$ particle. Once the particles are separated it is supposed that there is a device which randomly determines which one measurement device will be used on side $A$, $a_1$ and $a_2$. Similarly a device randomly determines which device is used on side $B$, $b_1$ or $b_2$. It is supposed that the events of determining which measurement devices are used on each side are space-like separated. Such a choice will limit the interaction between the experiments on sides $A$ and $B$. With this arrangement there are therefore four possible experiments which are undertaken depending on the random choice of $(a_1$ or $a_2)$ and $(b_1$ or $b_2)$.

The central problem is to understand the mechanism by which the space like separated particles give correlated responses to the measurement devices. At its simplest, is found that if $a_1$ and $b_1$ have the same orientation they will always yield anti-correlated results when those two devices are selected. If the devices have non-parallel orientation, the response is more complicated but successfully described by quantum mechanics. We follow Bell’s argument to seek a local realistic explanation of this phenomenon. We assume that at the time of creation of the pair of particles they are each invested with information in the form of “hidden variables” which they carry to the detectors; these variables determine their response to the detectors they encounter and must ensure the observed correlations. This assumption provides a “local” mechanism by which we might understand how the subsequent space-like separated measurements are correlated in the absence of faster than light signalling.

It is reasonable to suppose that the hidden variables must be stochastic since the observed responses to the measurement devices are stochastic and the response of the detectors is determined by the hidden variables which are created at the time of the pair creation. Indeed it would be difficult to introduce a stochastic element once the particles are sufficiently separated and still retain the observed correlations. Following Bell [13] we will assume that there is a random variable $\lambda$ which determines the outcomes of the possible measurements. It is assumed that $\lambda$ is described by a probability distribution $\rho(\lambda)$ normalised such that

$$\int \rho(\lambda)d\lambda = 1$$

although it is not necessary to specify the precise mathematical nature of $\lambda$. We introduce random variables $\mu_A = 1,2$ and $\mu_B = 1,2$ to govern the process by which the
detectors $a_i$ and $b_j$ are connected to the particle source. Thus, for example, when $\mu_A = 1$ the detector $a_1$ is connected to the source. We assume a probability measure $p_{\mu_A, \mu_B}
abla_1 \sum_{\mu_A=1}^2 \sum_{\mu_B=1}^2 p_{\mu_A, \mu_B} = 1$

where, for example, $p_{1,2}$ is the probability that $\mu_A = 1$ and $\mu_B = 2$. Avis et al.[1] assume

$$p_{\mu_A, \mu_B} = \frac{1}{4} \quad \forall \mu_A, \mu_B$$

and we will use this value initially in the following argument.

The random variable $\mu_A$ and $\mu_B$ determine which detectors are used in any given experiment and $\lambda$ determines the outcome of the measurement. Thus we have defined a two dimensional sample space $\Omega = \{\lambda, (\mu_A, \mu_B)\}$ which can be considered to be an element of a single Kolmogorov probability space

$$P = (\Omega, \mathcal{F}, \mathbf{P})$$

where $\mathcal{F}$ is the $\sigma$ algebra associated with $\Omega$ and $\mathbf{P}$ is the probability measure on $\mathcal{F}$. We note that the hidden variable $\lambda$ is defined on a sample subspace $\Omega_0 = \{\lambda\} \in \Omega$.

In order to set the problem in its physical context we assume that we are able to generate a large ensemble of pairs of particles. Each particle pair is assumed to sample a value for $\lambda$ from the distribution $\rho(\lambda)$ and this choice of $\lambda$ specifies the outcome of the possible measurements which we represent by the four random variables $a_i(\lambda); i = 1, 2$ and $b_j(\lambda); j = 1, 2$. These are defined on the sample space $\Omega_0$ where, for example, $a_i(\lambda)$ corresponds to the output for the detector $a_i$ for the given value of the hidden variable $\lambda$; $a_i(\lambda)$ and $b_j(\lambda)$ are the Bellian variables introduced by Avis. The variables $a_i(\lambda)$ and $b_j(\lambda)$ take only the values $\pm 1$ corresponding to the outputs of the detectors with orientation $a_i$ or $b_j$ for any given value of $\lambda$. We have used the notation $\lambda$ to be consistent with the notation used by Bell rather than the notation $\omega$ Avis used to represent an equivalent random variable.

In order to make contact with the definitions used in Section 3 of Avis et al.[1] we define four random variables $A^l(\lambda, \mu_A)$ and $B^l(\lambda, \mu_B)$ by

$$A^l(\lambda, \mu_A) = \delta_{\mu_A, i} a_i(\lambda) \quad i = 1, 2$$
$$B^l(\lambda, \mu_B) = \delta_{\mu_B, j} b_j(\lambda) \quad j = 1, 2$$

(3)

where $\delta_{\mu, \nu}$ is the Kronecker delta function. Thus the variables $A^l(\lambda, \mu_A)$ has the following properties:

1. $A^l(\lambda, \mu_A) = \pm 1$, if $\mu_A = 1$
2. $A^l(\lambda, \mu_A) = 0$, if $\mu_A = 2$

with similar definitions for $A^j(\lambda, \mu_A)$ and $B^j(\lambda, \mu_B)$ for $j = 1, 2$. The variables $A^l(\lambda, \mu_A)$ and $B^l(\lambda, \mu_B)$ are equivalent to the random variables $A^{0l}(\omega)$ and $B^{0l}(\omega)$ introduced by Avis.
4 Derivation of the CHSH inequality

We now explore the derivation of the CHSH inequality (1) given above. The expression makes use of the correlation function
\[
< A^i(i), B^j(j) > = \sum_{\mu_A = 1}^{2} \sum_{\mu_B = 1}^{2} p_{\mu_A, \mu_B} \int A^i(i, \mu_A) B^j(j, \mu_B) \rho(\lambda) d\lambda
\]
where \( \rho(\lambda) \) is the probability distribution for the hidden variables \( \lambda \) and we sum over the possible values of \( \mu_A \) and of \( \mu_B \); thus in equation (4) we have integrated and summed over the complete sample space \( \Omega_{CMC} = \{ \lambda, (\mu_A, \mu_B) \} \). We now attempt to recover the CHSH inequality by following the standard arguments; see eg page 37 of [7].

\[
< A^{(1)}, B^{(1)} > - < A^{(1)}, B^{(2)} > = \sum_{\mu_A = 1}^{2} \sum_{\mu_B = 1}^{2} p_{\mu_A, \mu_B} \int [A^{(1)}(\lambda, \mu_A) B^{(1)}(\lambda, \mu_B) - A^{(1)}(\lambda, \mu_A) B^{(2)}(\lambda, \mu_B)] \rho(\lambda) d\lambda
\]

In the analysis on page 37 of [7] the following modification of the RHS is made:

\[
= \sum_{\mu_A = 1}^{2} \sum_{\mu_B = 1}^{2} p_{\mu_A, \mu_B} \{ 
\int [A^{(1)}(\lambda, \mu_A) B^{(1)}(\lambda, \mu_B) \pm A^{(2)}(\lambda, \mu_A) B^{(2)}(\lambda, \mu_B)] \rho(\lambda) d\lambda 
- \int [A^{(1)}(\lambda, \mu_A) B^{(2)}(\lambda, \mu_B) \pm A^{(2)}(\lambda, \mu_A) B^{(1)}(\lambda, \mu_B)] \rho(\lambda) d\lambda \}
\]

(5)

However this introduces products of the form
\[ A^{(1)}(\lambda, \mu_A) A^{(2)}(\lambda, \mu_A) \quad \text{and} \quad B^{(1)}(\lambda, \mu_B) B^{(2)}(\lambda, \mu_B) \]

which are always zero and therefore the subsequent analysis in [7] is not valid. This arises because, for example, if \( \mu_A = 1 \), \( A^{(2)}(\lambda, \mu_A) = 0 \) and if \( \mu_A = 2 \), \( A^{(1)}(\lambda, \mu_A) = 0 \); these represent the experimental arrangement in which \( A^{(1)}(\lambda, \mu_A) \) and \( A^{(2)}(\lambda, \mu_A) \) correspond to different experiments with different detectors \( a_1 \) and \( a_2 \) and these therefore never operate simultaneously. Similar arguments apply to side B. Thus the derivation [7] of the CHSH inequalities must be modified when considering the single probability space and cannot be undertaken directly in terms of the random variables \( A^i(\lambda, \mu_A) \) and \( B^i(\lambda, \mu_B) \).
In order to proceed, we use the definitions (3) to find

\[
< A^{(i)}, B^{(j)} > = \sum_{\mu_A, \mu_B} p_{\mu_A, \mu_B} \int A^{(i)}(\lambda, \mu_A) B^{(j)}(\lambda, \mu_B) \rho(\lambda) d\lambda
\]

\[
= \sum_{\mu_A=1}^{2} \sum_{\mu_B=1}^{2} p_{\mu_A, \mu_B} \delta_{\mu_A,i} \delta_{\mu_B,j} \int a_i(\lambda) b_j(\lambda) \rho(\lambda) d\lambda
\]

\[
= p_{i,j} \int a_i(\lambda) b_j(\lambda) \rho(\lambda) d\lambda
\]

\[
= p_{i,j} < a_i, b_j >
\]

(6)

The quantity \(< A^{(i)}, B^{(j)} >\) is less than \(< a_i, b_j >\) because the variables \(A^{(i)}\) and \(B^{(j)}\) can be zero whereas \(a_i\) and \(b_j\) can only have values of \(\pm 1\).

In order to make contact with Khrennikov et al (4-5), we note that equation (6) is consistent with the quantity \(< A^{(i)}, B^{(j)} >\) being identified as an unconditional or absolute correlation and \(< a_i, b_j >\) being identified as a conditional correlation where the condition is the choice of experiment \(\{i, j\}\). This is in agreement with equation (4) in (1) and the result (5) is also identical with the equation:-

\[
C_{i,j} = p(a = i, b = j) Q_{i,j}
\]

(7)

given on page 721 of (3), where \(C_{i,j} \equiv < A^{(i)}, B^{(j)} >\) is termed a classical correlation and \(Q_{i,j} \equiv < a_i, b_j >\) is termed a quantum correlation. Further, if we assume \(p_{\mu_A, \mu_B} = 1/4\), equation (5) becomes the result obtained in Section 5 of (1).

We note that \(a_i(\lambda)\) and \(b_j(\lambda)\) are defined on the subspace \(\Omega_0\) of \(\Omega\) and hence the correlation function

\[
< a_i, b_j > = \int a_i(\lambda) b_j(\lambda) \rho(\lambda) d\lambda
\]

is well defined since they are each non-zero functions of \(\lambda\). We are now able to make the expansion

\[
< A^{(1)}, B^{(1)} > - < A^{(1)}, B^{(2)} >
\]

\[
= \sum_{\mu_A} \sum_{\mu_B} p_{\mu_A, \mu_B} \int [(A^{(1)}(\lambda, \mu_A) B^{(1)}(\lambda, \mu_B) - A^{(1)}(\lambda, \mu_A) B^{(2)}(\lambda, \mu_B)] \rho(\lambda) d\lambda
\]

\[
= \frac{1}{4} \int [(a_1(\lambda) b_1(\lambda) - a_1(\lambda) b_2(\lambda)] \rho(\lambda) d\lambda
\]

(8)

If we now follow the expansion in equation (5)

\[
< A^{(1)}, B^{(1)} > - < A^{(1)}, B^{(2)} >
\]

\[
= \frac{1}{4} \int (a_1(\lambda) b_1(\lambda)[1 \pm (a_2(\lambda) b_2(\lambda)] \rho(\lambda) d\lambda
\]

\[
- \frac{1}{4} \int (a_1(\lambda) b_2(\lambda)[1 \pm (a_2(\lambda) b_1(\lambda)] \rho(\lambda) d\lambda
\]

(9)
the products implicit in equation (9) are now well defined and non-zero. We may follow the standard argument and take the absolute value of both sides and then apply the triangle inequality to find:

$$|<A^{(1)}, B^{(1)}> - <A^{(1)}, B^{(2)}>| \leq \frac{1}{4} \int |(a_1(\lambda)b_1(\lambda))[1 \pm (a_2(\lambda)b_2(\lambda))]\rho(\lambda)d\lambda|$$

$$+ \frac{1}{4} \int |(a_1(\lambda)b_2(\lambda))[1 \pm (a_2(\lambda)b_1(\lambda))]\rho(\lambda)d\lambda|$$

We know that $[1 \pm (a_2(\lambda)b_2(\lambda))]\rho(\lambda)$ and $[1 \pm (a_2(\lambda)b_1(\lambda))]\rho(\lambda)$ are both non-negative and hence

$$|<A^{(1)}, B^{(1)}> - <A^{(1)}, B^{(2)}>| \leq \frac{1}{4} \int |(a_1(\lambda)b_1(\lambda))[1 \pm (a_2(\lambda)b_2(\lambda))]\rho(\lambda)d\lambda$$

$$+ \frac{1}{4} \int |(a_1(\lambda)b_2(\lambda))[1 \pm (a_2(\lambda)b_1(\lambda))]\rho(\lambda)d\lambda|$$

But we have that $|a_i(\lambda)| \leq 1$ and $|b_i(\lambda)| \leq 1$ so that the right hand side must be less than or equal to

$$\frac{1}{4} \int [1 \pm (a_2(\lambda)b_2(\lambda))]\rho(\lambda)d\lambda + \frac{1}{4} \int [1 \pm (a_2(\lambda)b_1(\lambda))]\rho(\lambda)d\lambda$$

and we may rewrite this as

$$\frac{1}{2} + \frac{1}{4} \int (a_2(\lambda)b_2(\lambda))\rho(\lambda)d\lambda + \int (a_2(\lambda)b_1(\lambda))\rho(\lambda)d\lambda]$$

$$= \frac{1}{2} + \frac{1}{4}[<a_2, b_2> + <a_2, b_1>]$$

Hence we have

$$|<A^{(1)}, B^{(1)}> - <A^{(1)}, B^{(2)}>| \leq \frac{1}{2} + \frac{1}{4}[<a_2, b_2> + <a_2, b_1>] \quad (10)$$

We make use of the relation $<A^{(i)}, B^{(j)}> = \frac{1}{4} <a_i, b_j>$ between the Avis and Bellian variables from equation (6) to write

$$|<A^{(1)}, B^{(1)}> - <A^{(1)}, B^{(2)}>| \leq \frac{1}{2} + \frac{1}{4}[<A^{(2)}, B^{(2)}> + <A^{(2)}, B^{(1)}>]$$

or

$$|<A^{(1)}, B^{(1)}> - <A^{(1)}, B^{(2)}>| \leq \frac{1}{2} - |<A^{(2)}, B^{(2)}> + <A^{(2)}, B^{(1)}>|$$
rearranging and using the triangle inequality gives the result

\[ |< A^{(1)}, B^{(1)} > - < A^{(1)}, B^{(2)} > + < A^{(2)}, B^{(2)} > + < A^{(2)}, B^{(1)} >| \leq \frac{1}{2} \] (11)

This is a stronger condition than equation (11) cited by Avis and Khrennikov [1, 4]. If we use the result \( < A^{(i)}, B^{(j)} > = \frac{1}{4} < a_i, b_j > \) it also leads to a standard form of the CHSH inequality used to analyse experiments. ie

\[ |< a_1, b_1 > - < a_1, b_2 > + < a_2, b_2 > + < a_2, b_1 >| \leq 2 \] (12)

This is in direct contradiction to the claims in [1, 4].

In a more recent paper, Khrennikov [5] gives in his equation (33) a modified version of equation (2) in which the factor 8 on the right hand side is replaced by 4. As has been shown above this result is also too weak; the factor should neither 8 nor 4 but 2 giving the standard CHSH result used in experimental analysis of EPR experiments.

5 Generalised CHSH inequality

We may generalise the result (11) for the case when \( p_{\mu_A, \mu_B} \neq \frac{1}{4} \) by modifying equation (8) as follows:

\[ < \frac{1}{p_{1,1}} A^{(1)}, B^{(1)} > - \frac{1}{p_{1,2}} < A^{(1)}, B^{(2)} > = \sum_{\mu_A} \sum_{\mu_B} p_{\mu_A, \mu_B} \int \left[ \frac{1}{p_{1,1}} (A^{(1)}(\lambda, \mu_A) B^{(1)}(\lambda, \mu_B)) \right. \]

\[ - \frac{1}{p_{1,2}} (A^{(1)}(\lambda, \mu_A) B^{(2)}(\lambda, \mu_B)) \rho(\lambda) d\lambda \]

\[ = \int ([a_1(\lambda)b_1(\lambda) - a_1(\lambda)b_2(\lambda)] \rho(\lambda) d\lambda \]

We may then follow the argument above in a straightforward way until we find a modified version of equation (10)

\[ | \frac{1}{p_{1,1}} < A^{(1)}, B^{(1)} > - \frac{1}{p_{1,2}} < A^{(1)}, B^{(2)} > | \leq 2 \pm |< a_2, b_2 > + < a_2, b_1 >| \]

We now use the result (6) to write

\[ | \frac{1}{p_{1,1}} < A^{(1)}, B^{(1)} > - \frac{1}{p_{1,2}} < A^{(1)}, B^{(2)} > | \leq \]

\[ 2 \pm \left( \frac{1}{p_{2,2}} < A^{(2)}, B^{(2)} > + \frac{1}{p_{2,1}} < A^{(2)}, B^{(1)} > \right) \]
and hence equation (11) becomes

\[
\begin{align*}
\left| \frac{1}{p_{1,1}} < A^{(1)}, B^{(1)} > - \frac{1}{p_{1,2}} < A^{(1)}, B^{(2)} > \\
+ \frac{1}{p_{2,2}} < A^{(2)}, B^{(2)} > + \frac{1}{p_{2,1}} < A^{(2)}, B^{(1)} > \right| \leq 2
\end{align*}
\]

(13)

In the case that all the probabilities are equal to \(1/4\) this result becomes equation (11). As can be seen from equation (6), the quantities \(<A_i,B_j>/p_{i,j}\) are simply the value of the expectation value \(<A^{(i)}B^{(j)}>/p_{i,j}\) conditional on the choice of the experiment \((i,j)\); this conditional probability is equal to \(<a_i,b_j>\) and hence this result is identical to the standard CHSH result (12).

The result (13) is a modified CHSH inequality applicable when four separate experiments each with probability \(p_{i,j}\) are analysed on a single probability space.

6 Conclusions

We can conclude therefore that the correct treatment of a single Kolmogorov probability space for an EPR-Bohm-Bell experiment leads to a modified form of the CHSH inequality, equation (13). This result is equivalent to the standard CHSH inequality, equation (12) in contradiction to the claims in [1–5].

The CHSH inequality (12) has been shown experimentally to be violated (eg [14]) and this latter result is commonly interpreted to mean that it is not possible to construct a local hidden variable model which is consistent with quantum mechanics, or more importantly, experiment.

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