Entanglement Entropy for Singular Surfaces

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Abstract: We study entanglement entropy for regions with a singular boundary in higher dimensions using the AdS/CFT correspondence and find that various singularities make new universal contributions. When the boundary CFT has an even spacetime dimension, we find that the entanglement entropy of a conical surface contains a term quadratic in the logarithm of the UV cut-off. In four dimensions, the coefficient of this contribution is proportional to the central charge $c$. A conical singularity in an odd number of spacetime dimensions contributes a term proportional to the logarithm of the UV cut-off. We also study the entanglement entropy for various boundary surfaces with extended singularities. In these cases, similar universal terms may appear depending on the dimension and curvature of the singular locus.
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1. Introduction

Entanglement entropy has emerged as an interesting theoretical quantity for studies of quantum matter. For example, it allows one to distinguish new topological phases or different critical points [1, 2, 3]. Entanglement entropy has also been considered in discussions of holographic descriptions of quantum gravity, in particular, for the AdS/CFT correspondence [4, 5, 6]. In this context, as well as characterizing new properties of holographic field theories [7], it has been suggested that entanglement entropy may play a fundamental role in the quantum structure of spacetime, e.g., [8, 9].

In quantum field theory (QFT), the typical calculation of entanglement entropy begins by choosing a particular spatial region \( V \) and integrating out the degrees of freedom in the complement \( \bar{V} \). Then with the resulting density matrix \( \rho_V \), one calculates:

\[
S_{\text{EE}} = -\text{Tr} \left( \rho_V \log \rho_V \right).
\]

Unfortunately, this calculation generically yields a UV divergent answer because the result is dominated by short distance correlations in the vicinity of the boundary \( \Sigma = \partial V \). However, if we regulate the calculation by introducing a short distance cut-off \( \delta \), the entanglement entropy exhibits an interesting geometric structure [10, 11]. For example, with a QFT in \( d \) spacetime dimensions, this allows us to organize the results as follows:

\[
S_{\text{EE}} = \frac{c_2}{\delta^{d-2}} + \frac{c_4}{\delta^{d-4}} + \cdots ,
\]

where each of the coefficients \( c_i \) involves an integration over the boundary \( \Sigma \). In particular, the first two coefficients may be written as

\[
c_2 = \int_{\Sigma} d^{d-2}y \sqrt{h} h_2 = h_2 \mathcal{A}_\Sigma ,
\]

\[
c_4 = \int_{\Sigma} d^{d-2}y \sqrt{h} \left[ h_{4,1} \mathcal{R} + h_{4,2} K^{i\alpha}_{a} K^{i\beta}_{b} \right].
\]

Of course, the leading term then yields the famous ‘area law’ result [12, 13]. The geometry of the boundary becomes even more evident in the second coefficient \( c_4 \) with the appearance of \( \mathcal{R} \) and \( K^{i\alpha}_{a} \), the intrinsic Ricci scalar and the extrinsic curvature of this surface.\(^1\) The coefficients \( h_{k,a} \) above will depend on the detailed structure of the underlying QFT. In particular, we note that they may become functions of the cut-off through their dependence on mass scales \( \mu_i \) in the QFT since the latter will only appear in the dimensionless combination \( \mu_i \delta \), i.e.,

\[
h_{k,a} = h_{k,a}(\mu_i \delta) \quad [11].
\]

The geometric character of the entanglement entropy illustrated here naturally follows from the implicit choice of a covariant regulator in the QFT and the fact that the UV divergences are local.

Unfortunately, the coefficients appearing in the expansion above are scheme dependent. Clearly, if we shift \( \delta \rightarrow \alpha \delta \), we find \( h_{k,a} \rightarrow \hat{h}_{k,a} = \alpha^{k-d} h_{k,a}(\alpha \mu_i \delta) \). Hence the regulator dependence here comes both from the implicit dependence on mass scales in the QFT and the ‘classical’ engineering dimension of the individual coefficients. Of course, the latter can

\(^1\)We are assuming here that the background geometry is simply \( d \)-dimensional flat space. Otherwise additional contributions could appear in \( c_4 \) involving the background curvature.
be avoided by carrying the expansion in eq. (1.1) to sufficiently high orders. In particular, one may find a logarithmic contribution to the entanglement entropy

\[ S_{\text{univ}} = c_d \log(\delta/L) \]  

where \( L \) is some (macroscopic) scale in the geometry of \( \Sigma \). At this order in the expansion, the coefficient \( c_d \) is dimensionless. Further it may be natural to eliminate any intrinsic scales in the QFT by focussing on fixed point theories of the RG flow.\(^2\) In this case, shifting \( \delta \) as above makes a finite shift in the entanglement entropy but \( c_d \) is left unchanged. Hence one expects that this coefficient contains universal information that characterizes the underlying CFT. For example, in four dimensions, this universal coefficient is simply given by \[ \frac{1}{16\pi} \int_{\Sigma} d^2 y \sqrt{h} \left[ c \left( K^i_a K^j_b - \frac{1}{2} K^i_a K^i_b \right) + a \mathcal{R} \right], \]  

where \( a \) and \( c \) are the two central charges of the CFT. In principle then, this provides an approach to determine these central charges. For example, calculating the entanglement entropy for a sphere yields \( c_d \propto a \), while only \( c \) appears for a cylinder.\(^1\)

The preceding discussion implicitly assumes that the boundary of the region in question is smooth. The purpose of this paper is to explore the effects of geometric singularities in this ‘entangling surface’ \( \Sigma \). In part, our motivation comes from the observation that in three dimensions, if the boundary contains corners or kinks, the corresponding entanglement entropy will contain additional logarithmic contributions\[^{17,18,19,20}\]

\[ S_{\text{kink}} = q_3(\Delta\theta) \log(\delta/L), \]  

where \( \Delta\theta \) is the opening angle of the kink – see figure 1. As a function of \( \Delta\theta \), the coefficient \( q_3(\Delta\theta) \) satisfies certain simple properties\[^{17,18,19}\]. In particular, \( q_3(\Delta\theta = \pi) = 0 \) since the \( \Sigma \) becomes a smooth surface when the angle is set to \( \pi \). Strong subadditivity can be used to argue that in general \( q_3(\Delta\theta) \) must satisfy certain inequalities, e.g., \( q_3(\Delta\theta) \geq 0 \) and \( q_3'(\Delta\theta) \leq 0 \)\[^{18,19}\]. If the QFT is in a pure state, we have \( S_{\text{EE}}(V) = S_{\text{EE}}(\bar{V}) \) and so \( q_3(\Delta\theta) = q_3(2\pi - \Delta\theta) \) in this case. Further, in examples\[^{17,18,20,19}\], one finds for a small opening angle: \( q_3(\Delta\theta \to 0) \propto 1/\Delta\theta \). However, we must add that the precise universal information contained in \( q_3(\Delta\theta) \) remains to be understood.

A natural question is to ask whether similar contributions arise for singular entangling surfaces in higher dimensions. If yes, we can ask what the geometric dependence of these new terms is. Further, if we focus on CFT’s, we might ask if the coefficients in these contributions have a simple dependence on the central charges, analogous to that in eq. (1.5). The AdS/CFT correspondence\[^{21}\] provides a simple framework with which we may begin to address these questions. In fact, using the standard calculation of holographic entanglement entropy\[^{4,5}\],

\[^2\]Universal terms may also appear in other circumstances, either as a finite contribution, e.g.,\[^{1,3}\] or even when explicit mass scales are present\[^{14,15}\].

\[^3\]Again, we are assuming that the background geometry is flat.
the logarithmic contribution (1.6) associated with a kink in three dimensions has already been identified in [19]. More generally, this approach allows us to easily study boundary CFT’s in a variety of dimensions and further the geometric structure of the entanglement entropy becomes readily evident in holographic calculations [15, 22]. In the following then, we use holography to study some simple singular entangling surfaces in higher dimensions and we find a variety of new universal contributions. While these are just first steps towards a full understanding of these universal terms, our results indicate a rich geometric structure for the entanglement entropy of singular surfaces.

The remainder of this paper is organized as follows: In the next section, we give a brief overview of our calculations and summarize our main results. In section 3, we consider entangling surfaces with a conical singularity for boundary CFT’s with \( d = 4, 5 \) and 6. In these cases, the singularity in the geometry of the entangling surface is confined to a single point and so we broaden our calculations to consider extended singularities in section 4. There we find that the appearance of universal terms in the entanglement entropy depends on the dimension and the curvature of the singular locus. Section 5 presents calculations of holographic entanglement entropy for singular surfaces in boundary CFTs which are dual to the Gauss-Bonnet gravity. These calculations allow us to examine the dependence of the universal terms on the central charges of the underlying CFT. In section 6, we briefly discuss our results and consider future directions. Appendix A describes an alternate calculation of the entanglement entropy associated with a conical singularity. In particular, we make a conformal transformation from \( R^d \) to \( R \times S^{d-1} \) and so the conical entangling surface becomes a cylinder in the latter background. In appendix B, we give certain details for lengthy calculations presented in sections 3, 4 and 5.

While we were in the final stages of preparing this paper, ref. [23] appeared which contains some results which overlap with ours. Particularly, authors have calculated the holographic entanglement entropy for a cone and a crease for a four-dimensional CFT. We also learned of an upcoming paper [24] where the entanglement entropy for a cone in a six-dimensional CFT is studied.

2. Singular entangling surfaces and summary of results

In the sections 3, 4 and 5, we will describe in detail various holographic calculations of the entanglement entropy for certain singular surfaces. Each of these calculations is quite lengthy and individually they are not very enlightening. Hence in this section, we provide an overview of these calculations and a summary of our results. We begin by describing the kinds of singular entangling surfaces which we will consider.

Let us go back to eq. (1.6) for three dimensions. In this case, the entangling surface is a one-dimensional curve and the ‘singular surface’ would be one containing a kink or a corner where the direction of the tangent vector changes discontinuously at a point. We can characterize this behaviour by saying that the geodesic curvature of the curve contains a \( \delta \)-function singularity. In higher dimensions, the entangling surface is a \((d-2)\)-dimensional submanifold.
Figure 1: (Colour Online) Panel (a) shows a kink in constant Euclidean time \( t_E \) slice in \( d = 3 \). Panel (b) shows the cone \( c_1 \). Panel (c) shows the crease \( k \times R^1 \), which divides into two a time slice of the \( d \)-dimensional background spacetime. In this case, the natural extension of geodesic curvature is the extrinsic curvature of these surfaces. However, a distinct feature characterizing the geometry of these higher dimensional surfaces is now their intrinsic curvature. Of course, for a fixed background, these two curvatures will be related to each other (and the background curvature) through the Gauss-Codazzi equations. However, it is worth noting that in discussing singular surfaces, we might consider singularities in either the extrinsic curvature or the intrinsic curvature. In particular, as we show with simple examples below, it is possible to construct surfaces where the intrinsic curvature is everywhere smooth while the extrinsic curvature is singular. The other possibility which we consider is when both the extrinsic and intrinsic curvatures have singularities. In either case, the examples which we consider below contain \( \delta \)-function singularities. That is, in all of our examples, the curvatures characterizing the entangling surface are finite and smooth everywhere, except for a particular locus or subset of points.

In considering such singular entangling surfaces, we introduce an intuitive nomenclature to simplify the discussion: *kink*, *cone* and *crease*. Examples of each are illustrated in figure 1. To explain these terms, it suffices to consider evaluating the entanglement entropy in flat \( d \)-dimensional background \( R^d \). With Euclidean signature, the metric can then be written in ‘cylindrical’ coordinates as

\[
ds^2 = dt_E^2 + d\rho^2 + \rho^2 \left( d\theta^2 + \sin^2 \theta \, d\Omega_n^2 \right) + \sum_{i=1}^{m} (dx^i)^2
\]

(2.1)

\(^4\)We will assume the geometry of the background is everywhere smooth in the following. Combined with the Gauss-Codazzi equations, this assumption rules out the possibility that the extrinsic curvature may be smooth while the intrinsic curvature is singular.
where $d\Omega_n^2$ denotes the round metric on a unit $n$-sphere. Hence we have $d = 3 + n + m$. The entangling surface will be a $(d - 2)$-dimensional geometry embedded in this background (on a surface of constant $t_E$).

The natural model of a kink is then given in $d = 3$ ($n = 0 = m$) as the union of two rays: $k = \{ t_E = 0, \rho = [0, \infty), \theta = \pm \Omega \}$. With this choice of angles, the opening angle between the two rays is $\Delta \theta = 2\Omega$. Similarly, our prototype for a cone is given in $d = 3 + n$ as the surface: $c_n = \{ t_E = 0, \rho = [0, \infty), \theta = \Omega \}$. Hence with this construction, we are only considering cones with a spherical cross-section $S^n$. We define a crease to be the higher dimensional extension of either of these singular surfaces where we take the direct product of a kink $k$ or a cone $c_n$ with some other manifold. With the flat background (2.1) above, the natural extension of the previous constructions gives the crease $k \times R^m$ in $d = 3 + m$ or a conical crease $c_n \times R^m$ in $d = 3 + n + m$.

In a certain sense the kink is simply a lower dimensional version of a cone, i.e., $k = c_0$. However, we prefer to distinguish these two classes to highlight the difference in the singularities noted above. If we consider the crease $k \times R^m$, the intrinsic geometry of this submanifold is smooth everywhere including $\rho = 0$ and in fact for the construction above, the intrinsic curvature vanishes everywhere. On the other hand, the extrinsic curvature contains a $\delta$-function singularity at the tip of the kink, i.e., $\rho = 0$. Again for the above construction, the extrinsic curvature vanishes everywhere away from this singularity. Now considering the conical crease $c_n \times R^m$, one finds that both the intrinsic and extrinsic curvatures have $\delta$-function singularities at the tip of the cone, i.e., $\rho = 0$. For the prototype constructed above, the intrinsic curvature vanishes away from this singular point but the extrinsic curvature is nonvanishing.

In our calculations of the entanglement entropy, it will also be useful to consider the CFT in a background of the form $R^{3+n} \times S^m$, for which we write the metric as

$$ds^2 = dt_E^2 + d\rho^2 + \rho^2 \left( d\theta^2 + \sin^2 \theta d\Omega_n^2 \right) + R^2 d\Omega_m^2. \quad (2.2)$$

This allows us to consider creases of the form $k \times S^m$ or $c_n \times S^m$. With these geometries, the intrinsic curvature of the entangling surface is nonvanishing everywhere but in particular, the geometry of the singular locus is now $S^m$.

Now after this description of our singular surfaces, we briefly explain our holographic calculations and the results. We use the AdS/CFT correspondence to calculate the holographic entanglement entropy for a boundary CFT with the above singular entangling surfaces. In different calculations, we use the following bulk geometries:

$$ds^2 = \frac{L^2}{z^2} \left( dz^2 + dt_E^2 + d\rho^2 + \rho^2 \left( d\theta^2 + \sin^2 \theta d\Omega_n^2 \right) + \sum_{i=1}^m (dx_i)^2 \right), \quad (2.3)$$

$$ds^2 = \frac{L^2}{z^2} \left( dz^2 + f_1(z) \left[ dt_E^2 + d\rho^2 + \rho^2 \left( d\theta^2 + \sin^2 \theta d\Omega_n^2 \right) \right] + f_2(z) R^2 d\Omega_m^2 \right). \quad (2.4)$$

Here $L$ is the AdS curvature scale and $z$ is the radial coordinate with the asymptotic boundary at $z = 0$. Of course, eq. (2.3) is just $AdS_{d+1}$ where $d = 3 + n + m$ as above and with a
flat boundary metric written in the coordinates given in eq. (2.1). The geometry in eq. (2.4) approaches AdS\(_{d+1}\) asymptotically with the curved boundary metric given in eq. (2.2). However, because this boundary geometry is not conformally flat, the bulk geometry is not purely AdS\(_{d+1}\) and the functions, \(f_1\) and \(f_2\), must be determined by evaluating the gravitational equations of motion. With these bulk metrics, (2.3) and (2.4), we can readily calculate the holographic entanglement entropy for the kink, cone or crease geometries described above.

Recall the proposal for calculating holographic entanglement entropy \([4, 5]\): Having selected a particular entangling surface \(\Sigma\) in the \(d\)-dimensional boundary theory, we must find the minimal area surface \(m\) (of dimension \(d-1\)) which extends in the bulk geometry such that the boundary of \(m\) matches \(\Sigma\) at the asymptotic AdS\(_{d+1}\) boundary. The entanglement entropy is then given by\(^5\)

\[
S_{\text{EE}} = \frac{2\pi}{\ell_p^{d-1}} \min_{\partial m \sim \Sigma} [A(m)]. \quad (2.5)
\]

This conjecture passes a variety of consistency tests, \(e.g.,\) see \([5, 6, 25, 22]\) and in fact, for spherical entangling surfaces, there is a derivation of holographic entanglement entropy which yields this result (2.5) \([26]\). An implicit assumption in eq. (2.5) is that the bulk physics is described by (classical) Einstein gravity. However, this proposal was extended \([22, 27]\) to certain dual gravity theories with higher derivative interactions, namely, Lovelock theories \([28]\). In this extension, which we use in section 5, one finds the surface which minimizes a new functional of the geometry of \(m\). In particular, the latter was chosen to match an expression for the black hole entropy in the Lovelock theories \([29]\).

The details of our holographic calculations of the entanglement entropy for kinks, cones and creases are given in the following sections. As expected, these calculations contain a variety of divergences that must be regulated. However, we pay particular attention to the question of whether there are new singular contributions associated with geometric singularity in the entangling surface. Further, in certain cases, we find that the singularity produces new universal contributions. Table 1 summarizes our results.

As described above, if the entangling surface has a kink in \(d = 3\), there is a new universal contribution (1.6) to the entanglement entropy. We review the holographic calculation of \([19]\) which reveals this result in section 3. These calculations are extended to creases of the form \(k \times R^m\) in section 4 and there we find no universal contribution, \(i.e.,\) no log \(\delta\) term, for these cases. However, we note that in this case the locus of the singularity, \(i.e.,\) the tip of the crease, is flat. The results become more interesting if this locus is curved. If the singular locus is curved but odd dimensional, we again find no logarithmic contribution. However, if the singular locus is both curved and even dimensional, we find that the entanglement entropy of the crease contains a log \(\delta\) contribution. While we have a limited number of explicit examples of this behaviour, \(i.e.,\) the case of \(k \times S^2\), it suggests to us that generally for creases of the

\(^{5}\)All of our calculations use Euclidean signature for which the \textit{minimal} area surface is appropriate. If the calculation is done in a Minkowski signature background, one seeks to extremize the area but this surface will only be a saddle point. In either case, the area must be suitably regulated to produce a finite answer.
| $d$ | Background spacetime | Geometry of entangling surface | Crease dimension | Crease curvature | Expected Divergences | New Divergences |
|-----|----------------------|-------------------------------|-----------------|-----------------|----------------------|----------------|
| 3   | $R^3$                | $k$                           | 0               | –               | $1/\delta$           | log($\delta$) * |
| 4   | $R^4$                | $c_1$                         | 0               | –               | $1/\delta^2$, log($\delta$) | log($\delta^2$) * |
| 5   | $R^5$                | $c_2$                         | 0               | –               | $1/\delta^3$, $1/\delta$ | log($\delta$) * |
| 6   | $R^6$                | $c_3$                         | 0               | –               | $1/\delta^4$, $1/\delta^2$, log($\delta$) | log($\delta^2$) * |
| $>3$| $R^d$                | $k \times R^{d-3}$            | $d - 3$         | flat            | $1/\delta^{d-2}$     | $1/\delta^{d-3}$ |
| 4   | $R^3 \times S^1$    | $k \times S^1$               | 1               | flat            | $1/\delta^2$         | $1/\delta$ |
| 5   | $R^3 \times S^2$    | $k \times S^2$               | 2               | curved          | $1/\delta^3$, $1/\delta$ | $1/\delta^2$, log($\delta$) * |
| 6   | $R^3 \times S^3$    | $k \times S^3$               | 3               | curved          | $1/\delta^3$, $1/\delta^2$ | $1/\delta^3$, $1/\delta$ |
| 6   | $R^4 \times S^2$    | $k \times (R^1 \times S^2)$  | 3               | curved          | $1/\delta^4$, $1/\delta^2$ | $1/\delta^3$, $1/\delta$ |
| 5   | $R^5$                | $c_1 \times R^1$             | 1               | flat            | $1/\delta^3$, $1/\delta$ | log($\delta$)/$\delta$ |
| 6   | $R^6$                | $c_1 \times R^2$             | 2               | flat            | $1/\delta^3$, $1/\delta^2$, log($\delta$) * | log($\delta$)/$\delta$ |
| 7   | $R^7$                | $c_1 \times R^4$             | 3               | flat            | $1/\delta^3$, $1/\delta$ | log($\delta)/\delta^3$ |
| 5   | $R^4 \times S^1$    | $c_1 \times S^1$             | 1               | flat            | $1/\delta^4$, $1/\delta$ | log($\delta$)/$\delta^2$ |
| 6   | $R^4 \times S^2$    | $c_1 \times S^2$             | 2               | curved          | $1/\delta^4$, $1/\delta^2$, log($\delta$) | log($\delta$)/$\delta^3$ |
| 7   | $R^4 \times S^3$    | $c_1 \times S^3$             | 3               | curved          | $1/\delta^5$, $1/\delta^3$, $1/\delta$ | log($\delta$)/$\delta^4$ |
| 6   | $R^6$                | $c_2 \times R^1$             | 1               | flat            | $1/\delta^3$, $1/\delta^2$, log($\delta$) * | $1/\delta$ |
| 7   | $R^7$                | $c_2 \times R^2$             | 2               | flat            | $1/\delta^4$, $1/\delta^3$, $1/\delta$ | $1/\delta^2$ |
| 6   | $R^5 \times S^1$    | $c_2 \times S^1$             | 1               | flat            | $1/\delta^3$, $1/\delta^2$, log($\delta$) * | $1/\delta$ |
| 7   | $R^5 \times S^2$    | $c_2 \times S^2$             | 2               | curved          | $1/\delta^3$, $1/\delta^3$, $1/\delta$ | $1/\delta^2$, log($\delta$) * |

Table 1: Summary of the divergent terms for various singular surfaces from our holographic calculations. Here $d$ is the spacetime dimension of the CFT background, which can be both flat or curved. The ‘expected’ divergences are those which can arise with a smooth entangling surface – see discussion in the introduction. The ‘new’ divergences are produced by the singularity in the surface and vanish when the surface is smooth, i.e., $\Omega = \pi/2$. Any universal terms are marked with a ‘*’ – see the discussion in the main text.

form $k \times \Sigma_{2m}$, there are new universal terms of the form

$$S_{\text{univ}} \sim \int_{\sigma} d^2m y \sqrt{h} [R^m] \log \delta$$

(2.6)

where $\sigma$ is the singular locus on the entangling surface and $[R^m]$ represents some curvature invariant containing $m$ powers of the curvature on this submanifold.

We also consider entangling surfaces with conical singularities in section 3. In this case, if the boundary CFT lives in an odd number of spacetime dimensions, i.e., $c_n$ with $n$ even, we find that the singularity contributes a log $\delta$ term to the entanglement entropy. However, for an even dimensional boundary theory, i.e., $c_n$ with $n$ odd, we find that the new universal contribution actually diverges as $\log^2 \delta$. However, we note, and explain in detail in section 6, that part of this log $^2 \delta$ contribution can be identified with the ‘smooth’ contribution given in eq. (1.5). That is, part of this divergence should be associated with correlations away from the
singularity and so depends on details of the smooth part of the geometry away from the tip of the cone. However, we also argue part of the contribution is associated with the singularity itself and so should still be expected to arise for more general situations, independent of this smooth geometry. Given that our boundary field theory is a CFT, we might ask if the coefficients of these new universal contributions are simple functions of the central charges. As a step in this direction, we work with Gauss-Bonnet gravity in the bulk in section 5, as this allows us to begin to distinguish the boundary central charges, e.g., see [30, 31]. In the case of even dimensions, we see that the coefficient of the \( \log^2 \delta \) contribution is proportional to a particular central charge, i.e., for \( d = 4 \), it is \( c \). However, for odd dimensions, the \( \log \delta \) term does not yield any such simple result.

The holographic calculations are also extended to consider conical creases of the form \( c_n \times R^m \) or \( c_n \times S^m \) in section 4. For these cases, we find that the nature of universal contributions again depends on the dimension of both the full spacetime and the singular locus, as well as the curvature of the latter. In particular, if both of these dimensions is even and the locus is curved, e.g., \( c_1 \times S^2 \), then a \( \log^2 \delta \) term arises. Alternatively, if the background is odd dimensional but the locus is even dimensional and curved, e.g., \( c_2 \times S^2 \), then a \( \log \delta \) contribution appears. In any other cases, the singularity does not contribute any universal terms of this form to the entanglement entropy. Our results again suggest the appearance of universal contributions of the form given in eq. (2.6) for odd dimensional theories, while similar terms with a \( \log^2 \delta \) divergence seem to be present for even dimensional CFT’s.

In discussing these results, we need to be careful about an important point. As we have illustrated with eq. (1.5), in even dimensional CFT’s, the smooth part of the entangling surface will already produce a universal term proportional to \( \log \delta \). Hence, as noted in our description of the results for cones, we must distinguish this term from universal contributions associated with the singularity. Other cases where this issue arises include: \( c_1 \times R^2 \), \( c_2 \times R^1 \) and \( c_2 \times S^1 \). All of these examples are in six dimensions where we do not have the analog of the \( d = 4 \) expression in eq. (1.5). So while a detailed comparison is not possible, in considering the corresponding holographic calculations in detail, we see that the coefficient of the \( \log \delta \) receives contributions at all values of the radius \( \rho \) and that there is no singularity at \( \rho = 0 \). Hence we can clearly infer that this contribution is coming from the smooth part of the geometry and the singularity is not making a universal contribution to the entanglement entropy.

To close this section, let us note that for many of the examples in table 1, there were no logarithmic terms in the entanglement entropy. In some of those cases, it may still be that the finite contribution exhibits some universal behaviour but we did not investigate this possibility here.

3. EE for singular embeddings

In this section, we will study entanglement entropy (EE) with singular entangling surfaces in a flat background for holographic CFTs which are dual to Einstein gravity. The simplest case
in this category is the kink in $d = 3$. It is already known that EE for a kink has a logarithmic divergence [17, 18, 20]. This calculation for holographic EE was first done by Hirata and Takayanagi in [19]. So before calculating EE for cones in higher dimensions, we briefly review this case.

We begin with $\{d, n, m\} = \{3, 0, 0\}$ in metric (2.3). The kink in the boundary is defined by $\rho \in [0, H]$ and $\theta \in [-\Omega, \Omega]$, where $H$ is an IR cut-off. The holographic entanglement entropy for this geometry is given by (2.5), that is the area of the minimal area surface which hangs in the bulk and is homologous to the kink on the boundary. We assume that the induced coordinate for the minimal area surface are $(\rho, \theta)$ and the radial coordinate $z = z(\rho, \theta)$. Now we can find the induced metric $h_{\mu\nu}$ over the surface and the entanglement entropy is given by

$$S_{3}|_k = \frac{2\pi}{L_p^2} \int d\rho d\theta \sqrt{h} = \frac{2\pi}{L_p^2} \int d\rho d\theta \frac{L^2}{z^2} \sqrt{\rho^2 + \rho^2 z'^2 + \dot{z}^2},$$

where $z' = \partial_{\rho} z$ and $\dot{z} = \partial_{\theta} z$. Here we point out that the EE for entangling surface $\Sigma$ in $d$-dimensional field theory will be represented by $S_{d}|_{\Sigma}$. Now we can easily find the equation of motion for $z(\rho, \theta)$. From scaling symmetry of the AdS space and the fact that there is no other scale in the problem, we can say that

$$z = \rho h(\theta),$$

where $h(\theta)$ is such that $h \to 0$ as $\theta \to \pm \Omega$. After using this ansatz, the entropy functional reduces to

$$S_{3}|_k = \frac{4\pi L^2}{\ell_p^2} \int_{\delta/h_0}^{H} \frac{d\rho}{\rho} \int_{0}^{\Omega-\epsilon} d\theta \frac{\sqrt{1 + h^2 + \dot{h}^2}}{h^2}.$$

where we have introduced the UV cut-off at $z = \delta$, $\dot{h} = dh/d\theta$ and defined $h_0$ such that at $\theta = 0$, $h(0) = 0$. Note that $h(0) = 0$ and $h_0$ is the maximum value of $h(\theta)$. Also, $\epsilon$ is a function of $\rho$ defined using (3.2), such that at $z = \delta$, $h(\Omega-\epsilon) = \delta/\rho$. Further, the substitution of ansatz (3.2) in equation of motion for $z(\rho, \theta)$ gives

$$h(1 + h^2)\ddot{h} + 2h^2 + (h^2 + 1)(h^2 + 2) = 0.$$  \hspace{1cm} (3.4)

For this equation of motion, we can easily see that there exists a quantity $K$ which is conserved along $\theta$ translation and is given by

$$K = \frac{1 + h^2}{h^2 \sqrt{1 + h^2 + \dot{h}^2}} = \frac{\sqrt{1 + h_0^2}}{h_0^2},$$

where we have used $h(0) = 0$ and $h(0) = h_0$ to get the expression on the right hand side. Now plan is to convert $\theta$ integral in (3.3) to integral over $h$ and then separate the divergent part in the integral. We also make the coordinate transformation to $y = \sqrt{1/h^2 - 1/h_0^2}$, where $y \to \infty$ as we approach the boundary. After this coordinate transformation, the integrand has following divergence in the limit $y \to \infty$:

$$\sqrt{\frac{1 + h_0^2(1 + y^2)}{2 + h_0^2(1 + y^2)}} \sim 1 + O\left(\frac{1}{y^3}\right).$$

\hspace{1cm} (3.6)
So now we can write EE as

\[ S_3 \big|_k = \frac{4\pi L^2}{\ell_p^2} \int_{\delta/h_0}^H d\rho \int_0^{(\rho/\delta)^2 - 1/h_0^2} dy \left( \frac{1 + h_0^2(1 + y^2)}{2 + h_0^2(1 + y^2)} - 1 \right) \]

\[ + \frac{4\pi L^2}{\ell_p^2} \int_{\delta/h_0}^H d\rho \sqrt{\frac{\rho^2}{\delta^2 - \frac{1}{h_0^2}}} , \]

which can be simplified to give

\[ S_3 \big|_k = \frac{4\pi L^2}{\ell_p^2} \left( \frac{H}{\delta} + q_3(\Omega) \log \left( \frac{\delta}{H} \right) + O(\delta) \right) , \tag{3.7} \]

where \( q_3(\Omega) \) is

\[ q_3(\Omega) = \int_0^\infty dy \left[ 1 - \sqrt{\frac{1 + h_0^2(1 + y^2)}{2 + h_0^2(1 + y^2)}} \right] . \tag{3.8} \]

Note that \( q_3(\Omega) \) is the cut-off independent term in the EE for the kink. After this quick review of the calculations by Hirata and Takaynagi, we turn towards the cone in higher dimensions where we will see \( \log^2 \delta \) divergence for even \( d \).

### 3.1 Cone in \( d = 4, 5 \) and 6 CFT

In this section, we will calculate EE for cone \( c_n \) in some even and odd dimensional spacetime. We will give detailed calculations for \( c_1 \) in \( d = 4 \) and discuss the final results for \( c_2 \) and \( c_3 \) in \( d = 5, 6 \) dimensional CFTs\(^6\). However, in the beginning of the calculations, we will keep the discussion general for arbitrary \( d \).

With \( m = 0 \) in the bulk metric (2.3), we can define the cone geometry by \( \theta \in [0, \Omega] \) and \( \rho \in [0, H] \), where \( H \) is the IR cut-off for the geometry. Now, we define the minimal area surface, that gives the entanglement entropy, by coordinates \((\rho, \theta, \xi_i)\) where \( \xi_i \)'s are coordinates on sphere \( S^n \) in (2.3). As for the cone \( c_n \), we have a rotational symmetry \( SO(n+1) \) along the sphere \( S^n \), the radial coordinate \( z \) will depend only on \( \theta \), i.e., \( z = z(\rho, \theta) \). Then, the induced metric is given by

\[ h = \left[ \begin{array}{ccc}
\frac{L^2}{\ell_p^2} (1 + z')^2 & \frac{L^2}{\ell_p^2} (\rho') (\rho' + z') \\
\frac{L^2}{\ell_p^2} (\rho') (\rho' + z') & \frac{L^2}{\ell_p^2} \rho^2 \sin^2 \theta g_{ab}(S^n) \\
\end{array} \right] , \tag{3.10} \]

where \( \hat{z} = \partial_\theta z \), \( z' = \partial_\rho z \) and \( g_{ab}(S^n) \) represents the metric over unit \( S^n \). We also regulate the minimal area surface by stopping it at the UV cut-off \( z = \delta \). Now the entropy functional becomes

\[ S_d \big|_{c_{d-3}} = \frac{2\pi L^{d-1}}{\ell_p^{d-1}} \Omega_{d-3} \int d\rho d\theta \frac{\rho^{d-3} \sin^{d-3}(\theta)}{z^{d-1}} \sqrt{\rho^2 + (\partial_\theta z)^2 + \rho^2 (\partial_\rho z)^2} , \tag{3.11} \]

\(^6\)The cone in \( d = 4 \) CFT is also discussed in [23].
where \( \Omega_{d-3} \) is the surface area of the unit \((d - 3)\)-sphere. Note that here we have used \( n = d - 3 \). From this entropy functional, we can find the equation of motion for \( z(\rho, \theta) \).

\[
0 = \rho^2 \sin(\theta)z \left( \rho^2 + z^2 \right) z'' + \rho^2 \sin(\theta)z \left( 1 + z'^2 \right) z' + 2 \rho^2 \sin(\theta) z z' z' + (d - 1) \rho^2 \sin(\theta) d \left( z^2 + \rho^2 \left( 1 + z'^2 \right) \right) + z(d - 3) \cos(\theta) \dot{z} \left( z^2 + \rho^2 \left( 1 + z'^2 \right) \right)
\]

\[
+ \rho z \sin(\theta) z' \left( (d - 1) z^2 + (d - 2) \rho^2 \left( 1 + z'^2 \right) \right)
\]  \( (3.12) \)

To proceed further, we use the scaling symmetry of the AdS background. As the background geometry is scale invariant and the only scale in our discussion is \( \rho \), solution of \( z(\rho, \theta) \) should be of the following form

\[
z(\rho, \theta) = \rho \, h(\theta),
\]  \( (3.13) \)

where \( h(\theta) \) is a function of \( \theta \) such that as \( \theta \to \Omega, \) \( h \to 0 \). Also at \( \theta = 0, \) \( z \) achieves its maximum value and we have \( \dot{h}(0) = \frac{dh}{d\theta|_{\theta=0}} = 0 \). We also define \( h_0 \) such that \( h_0 = h(\theta = 0) \). Now to extract the cut-off independent term, first we change the integration over \( \theta \) to integration over \( h \):

\[
S_d|_{c_{d-3}} = \frac{2\pi L_{d-1} \Omega_{d-3}}{\ell_{d-1}^d} \int_{\delta/h_0}^{H} dp \int_{h_0}^{\delta/p} dh \, \frac{\sin(d-3)(\theta)}{\rho h^{d-1}} \sqrt{1 + h^2 + \dot{h}^2}.
\]  \( (3.14) \)

Now the plan is to make the integral over \( h \) finite. To do so, we find the solution of \( \sin(\theta) \) and \( \dot{h} \) near the boundary and then see how the integrand diverges in the limit \( h \to 0 \). Then, we subtract the terms up to order \( 1/h \) in the integrand of \( (3.14) \) to make it finite. At this step, we will find that it is convenient to write the asymptotic solution for \( \sin(\theta) \) for each \( d \geq 4 \) separately. Hence, now we work case by case.

First, we consider \( d = 4 \) and find \( \dot{h} \) and \( \sin(\theta) \) in terms of \( h \) in the asymptotic limit, where \( h \to 0 \). To do so, we make a change of variable \( y = \sin(\theta) \) and invert \( (3.12) \) into equation of motion for \( y = y(h) \). Using \( \dot{h} = \sqrt{1 - y^2/y'(h)} \) and \( \ddot{h} = -(yy'' + (1 - y^2)y')/y'^3 \), we find that

\[
0 = h \left( 1 + h^2 \right) y \left( 1 - y^2 \right) y'' - yy' \left( 3 + h^2 + (3 + 5h^2 + 2h^4) y'^2 \right) + 2hy^2 \left( 1 + (1 + h^2) y'^2 \right) - h \left( 1 + (1 + h^2) y'^2 \right) + (3 + h^2) y^3 y' - hy^4,
\]  \( (3.15) \)

where \( y' = dy/dh \) and \( y'' = d^2 y/dh^2 \). We can solve this equation of motion perturbatively near the boundary and find that

\[
y = \sin(\Omega) - \frac{1}{4} \cos(\Omega) \cot(\Omega) \dot{h}^2 + \left( \frac{1}{64} (3 - \cos(2\Omega)) \cot^2(\Omega) \csc(\Omega) \log(\dot{h}) + y_0 \right) \dot{h}^4 + O(h^6),
\]  \( (3.16) \)

where we have used \( y(\theta = \Omega) = y(h = 0) = \sin(\Omega) \). In the above expression, \( y_0 \) is a constant and its value is fixed by using the fact that \( y(h) \) has an extremum at \( h = h_0 \). Further, this solution also contains \( \log(\dot{h}) \), which will appear in solutions for even \( d \). As a next step, we find \( \dot{h}(\theta) \) near the boundary in terms of \( h \). For that, we define \( f(h) = \dot{h}(\theta) \) and write \( (3.15) \)
as

$$0 = h (1 + h^2) y f f' + h \sqrt{1 - y^2} f'^2 + h (3 + h^2) y f^2 + h (1 + h^2) \sqrt{1 - y^2} f + (3 + 5h^2 + 2h^4) y. \quad (3.17)$$

Now using (3.16) we can solve this equation near the boundary and find

$$f(h) = \frac{-2 \tan(\Omega)}{h} - \frac{1}{2} h (3 - \cos(2\Omega)) \csc(2\Omega) \log(h) + f_0 h + \ldots, \quad (3.18)$$

where $f_0$ is a constant. Now using (3.16), (3.18) and $d = 4$ in integrand of (3.14), we find that for small $h$

$$\frac{\sin \theta}{h h^3} \sqrt{1 + h^2} \sim -\frac{\sin(\Omega)}{h^3} + \frac{\cos(\Omega) \cot(\Omega)}{8h} + \mathcal{O}(h). \quad (3.19)$$

This implies that the $h$ integral in the entropy functional has only divergences coming from first two terms in the limit $h \to \delta/\rho$. We can separate these divergent terms and for $d = 4$, write the entropy functional (3.14) in the following form

$$S_4\big|_{c_1} = \frac{4 \pi^2 L^3}{\ell_p^3} (I_1 + I_2). \quad (3.20)$$

where

$$I_1 = \int_{\delta/h_0}^{H} \frac{d\rho}{\rho} \int_{h_0}^{\delta/\rho} dh \left[ \frac{\sin \theta}{h h^3} \sqrt{1 + h^2} + \frac{\sin(\Omega)}{h^3} - \frac{\cos(\Omega) \cot(\Omega)}{8h} \right],$$

$$I_2 = -\int_{\delta/h_0}^{H} \frac{d\rho}{\rho} \int_{h_0}^{\delta/\rho} dh \left( \frac{\sin(\Omega)}{h^3} - \frac{\cos(\Omega) \cot(\Omega)}{8h} \right).$$

If we series expand $I_1$ in terms of UV cut-off $\delta$, we find that leading term is of order $\log(\delta)$. To see that, we use (3.19) and find that in $I_1$, integration over $h$ is actually finite if we set the upper limit $h = 0$. As all the subleading terms will be of higher order in powers of $\delta$, we find that

$$I_1 = -\log(\delta) \int_{h_0}^{0} dh \left[ \frac{\sin \theta}{h h^3} \sqrt{1 + h^2 + \hat{h}^2} + \frac{\sin(\Omega)}{h^3} - \frac{\cos(\Omega) \cot(\Omega)}{8h} \right] + \mathcal{O}(\delta^0). \quad (3.21)$$

Simultaneously, we can evaluate $I_2$ and find that

$$I_2 = -\frac{H^2 \sin(\Omega)}{4\delta^2} - \frac{1}{16} \cos(\Omega) \cot(\Omega) \log(\delta/H)^2$$

$$+ \left( \frac{1}{8} \cos(\Omega) \cot(\Omega) \log(h_0) + \frac{\sin(\Omega)}{2h_0^2} \right) \log(\delta/H) + \mathcal{O}(\delta^0). \quad (3.22)$$

Now using (3.21) and (3.22) in (3.20), we find the complete structure of divergences in the entanglement entropy for cone:

$$S_4\big|_{c_1} = \frac{4 \pi^2 L^3}{\ell_p^3} \left[ -\frac{H^2 \sin(\Omega)}{4\delta^2} - \frac{1}{16} \cos(\Omega) \cot(\Omega) \log(\delta/H)^2 + q_4 \log(\delta/H) + \mathcal{O}(\delta^0) + \ldots \right]. \quad (3.23)$$
where
\[ q_4 = \frac{1}{8} \cos(\Omega) \cot(\Omega) \log(h_0) + \frac{\sin(\Omega)}{2h_0^2} \]
\[ + \int_0^{h_0} dh \left[ \frac{\sin \theta}{h h^3} \sqrt{1 + h^2 + \dot{h}^2} + \frac{\sin(\Omega)}{h^3} - \frac{\cos(\Omega) \cot(\Omega)}{8h} \right]. \] 
(3.24)

So we find that EE for a cone in \( d = 4 \) CFT has a double logarithmic term. We can notice from expression of \( I_2 \) that one of the log terms comes from integration over \( h \) and then second from integration over \( \rho \). Here, first integration over \( h \) or \( \theta \) actually brings us close to the cut-off on the smooth part of the entangling surface. Further, when second integration over \( \rho \) is performed, we approach to the singularity. This idea is consistent with the fact that in EE for even dimensions, we get a logarithmically divergent term according to [16].

Now, as a next step, we generalize our discussion to cones in higher dimensions. First, we calculate EE for cone in \( d = 5 \) CFT. In this case the calculations proceeds similar to the \( d = 4 \) and the complete expression for the EE is given in the appendix B. However, we find that that cut-off independent term is the coefficient of \( \log \delta \) divergence and it is given by
\[ S_{d=5}^{\log} \big|_{c_2} = \frac{8 \pi^2 L^4}{\ell_p^4} q_5(\Omega) \log(\delta/H), \] 
(3.25)
where
\[ q_5 = -\frac{4 \cos^2(\Omega)}{9 h_0} + \frac{\sin^2(\Omega)}{3 h_0^3} \]
\[ + \int_0^{h_0} dh \left[ \frac{\sin^2(\theta)}{h h^3} \sqrt{1 + h^2 + \dot{h}^2} + \frac{\sin^2(\Omega)}{h^3} - \frac{4 \cos^2(\Omega)}{9 h^2} \right]. \] 
(3.26)

We further draw this universal term in figure 2. There, \( \log |q_d| \) is plotted as a function of \( \log(\sin \Omega) \) for \( d = 3 \) and \( 5 \). In the limit \( \Omega \to 0 \), we see that \( \log |q_5| \) asymptotes to a straight line with slope \(-1\). This implies that for small \( \Omega \),
\[ q_5 \propto \frac{1}{\Omega}. \] 
(3.27)

Finally for \( d = 6 \), we find that the cut-off independent term is
\[ S_{d=6}^{\log} \big|_{c_3} = \frac{4 \pi^3 L^5}{\ell_p^5} \frac{9 \cos(\Omega) \cot(\Omega)(31 - \cos(2\Omega))}{8192} \log(\delta/H)^2. \] 
(3.28)

It is straightforward to see from (3.14) that all the even dimensions will produce a \( \log^2 \delta \) divergence. For even dimensions, the number of powers of \( h \) in the denominator is odd. When separating the divergences, similar to (3.19), it will produce \( 1/h \). This term integrated over \( h \) and then over \( \rho \), similar to \( I_2 \), will produce a \( \log^2 \delta \) divergence. Of course these results will persist in field theories in curved backgrounds and for dual gravities with higher derivative curvatures. We will discuss these examples in sections 4 and 5.
Figure 2: (Color online) We have drawn $\log |q_d|$ as a function of $\log(\sin \Omega)$ for $d = 3$ and 5. In the limit $\Omega \to 0$, it is known that $\log |q_3| = -\log(\sin \Omega) + \ldots$ [19]. For $d = 5$, we observe similar behavior and find that for small $\Omega$, the linear fit is $\log |q_5| = -1.0 \log(\sin \Omega) - 2.1$.

4. EE for extended singularities

In this section, we will calculate the entanglement entropy for various creases. Obviously the crease can have two types of locus of singularities: flat or curved. By studying these examples, we will argue that the contribution from the singularity will be non-zero only if the locus is even dimensional and curved.

In the first subsection, we will study creases with a flat locus of singularity. We will see that in this case, we don’t find any new logarithmic divergence coming from the singularity. The creases we will be considering are $k \times R^m$ with arbitrary $m$ and, $c_n \times R^m$ with $\{n, m\} \in \{\{1, 1\}, \{1, 2\}, \{1, 3\}, \{2, 1\}, \{2, 2\}\}$.

In second subsection, we will mainly consider the singular geometries of the form $k \times S^m$ and $c_n \times S^m$. In these cases, a generic calculation for arbitrary curvature is tedious. So we will work in certain limits where curvature of the locus of singularity is very small and show how the leading order terms behave in these cases. In the calculations, we will use following approach: If the curvature of the locus is related to $1/R_1$, then in the limit $R_1 \to \infty$, the boundary geometry will become flat. Now we make $R_1$ finite but keep it very large; and calculate the leading order corrections to EE in this limit. In this process, we will find that leading correction to EE, in a proper normalization, will be of order $O(1/R_1^2)$ but not $O(1/R_1)$. Now, this approach has one more advantage. As we are going to argue now, for some simple background geometries like $R^3 \times S^2$, the coefficient of the logarithmic term at order $1/R_1^2$ will be the complete contribution from the kink. We will note in our calculations that the only dimensionful quantities in the problem will be $R_1$, the UV cut-off $\delta$ and the IR cut-off $H$. As the coefficient of the logarithmic divergence should be dimensionless and if we series expand it in terms of $1/R_1$, the numerator can either be $\delta$ or $H$. So the new logarithmic
contribution will be of the form
\[ S_{\text{new}} = R_1^2 \left( C_0 \frac{\delta^2}{R_1^2} + \mathcal{O} \left( \frac{H^2}{R_1^4} \right) + \ldots \right) \log(\delta). \] (4.1)

In this case, the terms with $\delta$ in the numerator are not UV cut-off invariant and terms with $H$ in the numerator are not really the contribution from the kink. That is because $H$ is the scale of the bulk part of the entangling surface and a term arising from the singularity should be independent of it. So we only need to focus on the leading order correction which is precisely the contribution from the singularity. This issue will again be discussed when we rigorously calculate EE in section 4.2.

Let us summarize our strategy for the calculations here: for cases where boundary theory is on curved background, we use the Fefferman-Graham expansion to find the dual gravity near the boundary. For flat boundary, we don’t need to go through this step. Then, we calculate the entanglement entropy for the kinks and use the Fefferman-Graham expansion to separate the logarithmic divergence from the singularities. We will see that first few terms of the Fefferman-Graham expansion will be sufficient to find the leading order corrections to the logarithmic divergences in the limit $R_1 \to \infty$. To begin with, we will calculate EE for some singular geometries with flat locus in the next subsection.

4.1 EE for singularity with a flat locus

In this section, we will calculate entanglement entropy for following two types of geometries\(^7\): $k \times R^m$ and $c_n \times R^m$. We will particularly see EE for cases where $n = 1, 2$ and $m = 1, 2, 3$.

4.1.1 Crease $k \times R^m$

In this section, we will work in the dual Einstein’s gravity with bulk metric (2.3) and $n = 0$. The crease $k \times R^m$ geometry is given by $\theta \in [-\Omega, \Omega]$, $\rho \in [0, \infty]$ and $x^i \in [-\infty, \infty]$. We pick $(\rho, \theta, x^i)$ as the induced coordinates over the minimal area surface and the radial coordinate is $z = z(\rho, \theta)$. Now the induced metric becomes
\[ h = \begin{bmatrix}
\frac{L^2}{z^2} (1 + z'^2) & \frac{L^2}{z^2} z' \hat{z} & \frac{L^2}{z^2} \left( \rho^2 + \hat{z}^2 \right) \\
\frac{L^2}{z^2} z' \hat{z} & \frac{L^2}{z^2} (\rho^2 + \hat{z}^2) & \\
\frac{L^2}{z^2} & & \ddots \\
& & & \frac{L^2}{z^2}
\end{bmatrix}, \] (4.2)

and the EE is given by
\[ S_d \big|_{k \times R^{d-3}} = \frac{2\pi L^{d-1} \tilde{H}^{d-3}}{\ell_P^{d-1}} \int d\rho d\theta \frac{1}{\ell^{d-1}} \sqrt{(\hat{z}^2 + \rho^2(1 + z'^2))}, \] (4.3)
\(^7\)The EE for wedge $k \times R^1$ is also calculated in [23].
where \( \dot{z} = \partial_{\theta} z \), \( z' = \partial_{\rho} z \) and we have substituted \( m = d - 3 \). We have also imposed an IR cut-off such that \( x^i \in [-\tilde{H}/2, \tilde{H}/2] \). From this functional, we can find the equation of motion for \( z(\rho, \theta) \) and it turns out to be

\[
0 = \rho z (\rho^2 + z'^2) z'' + \rho z (1 + z'^2) \ddot{z} - 2 \rho z z' z' + z^2 ((d - 1) \rho + 2 \rho z z') + \rho^2 ((d - 1) \rho + z z') (1 + z'^2) \quad (4.4)
\]

Now, in the limit \( h \to \infty \), \( \rho \) is the only scale in the problem. Hence, using scaling symmetry of AdS, we can argue that

\[
z = \rho h(\theta) . \quad (4.5)
\]

Using this in (4.3), we find that

\[
S_d|_{k \times R^{d-3}} = \frac{4\pi L^{d-1} \tilde{H}^{d-3}}{\ell_p^{d-1}} \int_0^H \frac{d\rho}{\rho^{d-2}} \int_{\delta h_0}^{\delta/\rho h} d\rho \frac{1}{h h^{d-1}} \sqrt{\dot{h}^2 + h^2 + 1} , \quad (4.6)
\]

where we have put the UV cut-off at \( z = \delta \) and defined \( h_0 = h(0) \). Further, we have used (4.5) to find the limits of the integrations and the extra factor of two comes from changing the range of the integration from \( \theta \in [-\Omega, \Omega] \) to \( \theta \in [0, \Omega] \). Using (4.5) in (4.4), we find the equation of motion for \( h \)

\[
h(1 + h^2) \dot{h} + (d - 1) \dot{h}^2 + h^4 + dh^2 + (d - 1) = 0 . \quad (4.7)
\]

In this case, although it is not explicit, we can find a constant along the \( \theta \) translation from (4.7). It is straightforward to see that

\[
K_d = \frac{(1 + h^2)(d-1)/2}{h^{(d-1)}(d-1) \sqrt{\dot{h}^2 + h^2 + 1}} = \frac{(1 + h_0^2)(d-2)/2}{h_0^{(d-1)}(d-1)} \quad (4.8)
\]

is a conserved quantity and it satisfies the equation of motion. Now we can use (4.8) to separate the divergences in the EE (4.6). Using the fact that \( h \) decreases near the boundary and hence \( \dot{h} \) should be negative, we find

\[
\dot{h} = -\frac{\sqrt{1 + h^2} \sqrt{(1 + h^2)^{d-2} - K_d^2 h^{2(d-1)}}}{K_d h^{d-1}} . \quad (4.9)
\]

Using this in integrand of (4.6), in the limit \( h \to 0 \), we find that

\[
\frac{\sqrt{\dot{h}^2 + h^2 + 1}}{h h^{d-1}} \sim -\frac{1}{h^{d-1}} - \frac{1}{2} K_d^2 h^{d-1} + O(h^{d+1}) . \quad (4.10)
\]

Now in (4.6), we can add and subtract \( 1/h^{d-1} \) in the integrand and write

\[
S_d|_{k \times R^{d-3}} = \frac{4\pi L^{d-1} \tilde{H}^{d-3}}{\ell_p^{d-1}} \left[ \frac{H}{(d - 2) \delta^{d-2}} - \frac{1}{(d - 3) h_0 \delta^{d-3}} + O(\delta^0) + I_1 \right] , \quad (4.11)
\]
where

\[ I_1 = \int_{\delta/h_0}^{H} \frac{d\rho}{\rho^{d-2}} \int_{h_0}^{\delta/\rho} dh \left( \frac{\sqrt{h^2 + h^2 + 1}}{hh^{d-1}} + \frac{1}{h^{d-1}} \right). \]

In (4.11), first few term are coming from $h$ and $\rho$ integration of $1/h^{d-1}$. To find the divergences in $I_1$, first we represent the integrand by $J(h)$ and from (4.10), $J(h) \sim h^{d-1}$ as $h \to 0$. We perform the integration by parts and write $I_1$ as

\[
I_1 = -\frac{1}{(d-3)H^{d-3}} \int_{h_0}^{\delta/H} dh J(h) - \frac{\delta}{(d-3)} \int_{\delta/h_0}^{H} \frac{d\rho}{\rho^{d-1}} J(h)\bigg|_{h=\delta/\rho}.
\]

(4.12)

Now to separate the divergences in $I_2$, we make a change of variable from $\rho$ to $q = \delta/\rho$ and then Taylor expand the terms around $\delta = 0$:

\[
I_2 = -\frac{1}{\delta^{d-2}} \int_{h_0}^{\delta/H} dq q^{d-3} J(q) = -\frac{1}{\delta^{d-2}} \left[ \int_{h_0}^{0} dq q^{d-3} J(q) + \frac{\delta}{H} (q^{d-3} J(q))_{q=\delta/H} + \ldots \right]
\]

\[
= -\frac{1}{\delta^{d-2}} \int_{h_0}^{0} dq q^{d-3} J(q) + (O)(\delta^3).
\]

(4.13)

In the above expression, the integral over $q$ is finite because from (4.10), $q^{d-3} J(q) \sim q^{2d-4}$ for small $q$. Combining (4.11)-(4.13), we can write

\[
S_{d|k \times R^{d-3}} = 4\pi L^{d-1} \frac{\tilde{H}^{d-3}}{\epsilon_{10}^{d-1}} \left[ \frac{H}{(d-2)\delta^{d-2}} + \left( \int_{h_0}^{0} dq q^{d-3} J(q) - \frac{1}{h_0} \right) \frac{1}{(d-3)\delta^{d-3}} + O(\delta^0) \right].
\]

(4.14)

Remarkably, we find that as soon as we add a flat locus to the kink, the log divergence disappears. However, there is a new divergent term of order $1/\delta^{d-3}$ which does not arise in smooth entangling surfaces. This clearly shows that logarithmic contribution from the crease depends on the curvature of the locus of the singularity. Further, there can be a logarithmic term from the smooth part of the entangling surface in even $d$. However, the entangling surface in $k \times R^m$ is flat everywhere apart from the singularity, and hence such contributions vanish. This example clearly shows that when the locus is flat, there is no contribution from the singularity. However, we will see ahead that as soon as we turn on the curvature of the locus, the logarithmic divergence will appear only for the cases where locus is even dimensional.

4.1.2 Crease $c_n \times R^m$

In this section, we will calculate EE for the geometries $c_n \times R^m$. Although such a calculation can be extended to arbitrary $n$ and $m$, we will particularly focus on $n = 1, 2$ and $m = 1, 2$. To
begin with, we consider the metric (2.3) for the dual gravity with arbitrary $m$. The conically singular geometry $c_n \times R^m$ is given by $\rho \in [0, \infty]$ and $\theta \in [0, \Omega]$ in gravitational dual (2.3). We assume that the induced coordinates over the minimal area surface are $(\rho, \theta, \xi^i, x^j)$, where $\xi^i$'s are coordinates over unit sphere $S^n$ and $x^j$'s are on $R^m$. As we have rotational symmetry over the sphere $S^n$, the radial coordinate for the minimal area surface will be given by $z = z(\rho, \theta)$.

Now, the induced metric over the surface is

$$
\begin{align*}
\begin{bmatrix}
\frac{L^2}{z^2} (1 + z'^2) & \frac{L^2}{z^2} z' \dot{z} \\
\frac{L^2}{z^2} z' \dot{z} & \frac{L^2}{z^2} (\rho^2 + \dot{z}^2) \\
\frac{L^2 \rho^2 \sin^2(\theta)}{z^2} g_{ab}(S^n) & \frac{L^2}{z^2} \\
\frac{L^2}{z^2} & \cdots \\
& \frac{L^2}{z^2}
\end{bmatrix},
\end{align*}
$$

(4.15)

where $g_{ab}(S^n)$ represents the metric over the unit sphere $S^n$ and the EE is given by

$$
S_d|_{c_n \times R^m} = \frac{2\pi L^{d-1} \tilde{H}^m \Omega_n}{L^{d-1}} \int d\rho \, d\theta \, \frac{\rho^n \sin^n(\theta)}{z^{d-1}} \sqrt{(\dot{z}^2 + \rho^2 (1 + z'^2))},
$$

(4.16)

where $\Omega_n$ is the area of the unit $n$-sphere, $\dot{z} = \partial_\rho z$ and $z' = \partial_\theta z$. Note that we have integrated over the $x^i$'s and used the IR cut-off $x^i \in [-\tilde{H}/2, \tilde{H}/2]$. Now for this case, the equation of motion becomes

$$
0 = \rho^2 z (\rho^2 + \dot{z}^2) z'' + \rho^2 z (1 + z'^2) \dot{z} - 2\rho^2 z z' \dot{z}' + (d - 1) \rho^2 (\dot{z}^2 + \rho^2 (1 + z'^2)) + \rho^2 \cot(\theta) \dot{z}^3 + (d' + 2) \rho^2 z' \dot{z}' + d' \rho^2 \cot(\theta) \dot{z} (1 + z'^2) + (d' + 1) \rho^3 z' (1 + z'^2). 
$$

(4.17)

Once again, here we can use the scaling arguments and say that

$$
z(\rho, \theta) = \rho \, h(\theta).
$$

(4.18)

Then, EE reduces to

$$
S_d|_{c_n \times R^m} = \frac{2\pi L^{d-1} \tilde{H}^m \Omega_n}{L^{d-1}} \int_0^\delta \int_0^{h_0} \frac{d\rho}{\rho^{d-n-2}} \int_0^{\delta/\rho} \frac{d\sin^n(\theta) \sqrt{\tilde{h}^2 + h^2 + 1}}{h \, h^{d-1}},
$$

(4.19)

and equation of motion for $h$ becomes

$$
0 = h(1 + h^2) \ddot{h} + n \cot(\theta) h \dot{h}^3 + (d + nh^2 - 1) \dot{h}^2 + n \cot(\theta) h (1 + h^2) \dot{h} + (n + 1) \dot{h}^4 + (d + n) \dot{h}^2 + d - 1.
$$

(4.20)

In (4.19), we have changed the integration from $\theta$ to over $h$. We have also introduced the UV cut-off $z = \delta$ and defined $h_0 = h(0)$. 

Now we set $n = 1$ and $d = 5$ (that also mean that $m = 1$), and calculate EE for the singular surface $c_1 \times R^4$. First, we need to find $y = \sin(\theta)$ near the boundary in terms of $h$. For that, we invert the equation of motion (4.17) and get

$$0 = h\left(1 + h^2\right) y (1 - y^2) y'' - 2 \left(2 + 3h^2 + h^4\right) yy'^3 - h\left(1 + h^2\right) (1 - 2y^2) y'^2 - (4 + h^2) y (1 - y^2) y' - h \left(1 + y^2\right)^2.$$  

(4.21)

Now solving this equation perturbatively near the boundary, we get the solution

$$y = \sin(\Omega) - \frac{1}{6} h^2 \cos(\Omega) \cot(\Omega) - \frac{1}{432} h^4 \left(19 - 5 \cos(2\Omega)\right) \cot^2(\Omega) \csc(\Omega) + \mathcal{O}(h^5),$$  

(4.22)

where we have used that at $h = 0$, $y(0) = \sin(\Omega)$. Further, we can define $\dot{h}(\theta) = f(h)$ and write the equation (4.20) in the form

$$0 = h\left(1 + h^2\right) y f' + h\sqrt{1 - y^2} f' + (4 + h^2) y f'' - h\left(1 + h^2\right) \sqrt{1 - y^2} f + 2 \left(2 + 3h^2 + h^4\right) y,$$  

(4.23)

where $y' = dy/dh$. Using $y$ from (4.22) and solving this equation near the asymptotic boundary, we find

$$f = -\frac{3 \tan(\Omega)}{h} + \frac{1}{3} h (8 - \cos(2\Omega)) \csc(2\Omega) + f_0 h^2$$  

$$-\frac{1}{216} h^3 \left(435 - 404 \cos(2\Omega) + 52 \cos(4\Omega)\right) \csc^3(\Omega) \sec(\Omega) + \mathcal{O}(h^4),$$  

(4.24)

where $f_0$ is a constant which is fixed by the condition $f(h_0) = 0$. Now it is straightforward to see that near the boundary

$$\frac{\sin(\theta) \sqrt{1 + h^2 + h^2}}{h h^4} \sim -\frac{\sin(\Omega)}{h^4} + \frac{\cos(\Omega) \cot(\Omega)}{9h^2} - \frac{1}{18} \cos(\Omega) \cot(\Omega) + \ldots,$$  

(4.25)

and we can use it to make $h$ integral in entropy functional finite. So we write EE as

$$S_5 |_{c_1 \times R^4} = \frac{4 \pi^2 L^4 \tilde{H}}{\tilde{\ell}_p^4} \left[ \frac{H^2 \sin(\Omega)}{6 \delta^3} + \frac{\cos(\Omega) \cot(\Omega)}{9 \delta} \log(\delta/H) \right.$$  

$$+ \frac{2h_0^2 \cos(\Omega) \cot(\Omega) (1 - \log(h_0)) - 9 \sin(\Omega)}{18 h_0^2 \delta}$$  

$$+ \int_{\delta/h_0}^{H} \frac{d\rho}{\rho^2} \int_{h_0}^{\delta/\rho} dh \left( \frac{\sin(\theta) \sqrt{h^2 + h^2 + 1}}{hh^{d-1}} + \frac{\sin(\Omega)}{h^4} - \frac{\cos(\Omega) \cot(\Omega)}{9h^2} \right) + \mathcal{O}(h_0) \right] \]$$

where we have already performed the integrations over some terms. Now let us call the term with integration $I_1$ and the integrand $J_5(h)$. Then, near the boundary $J_5(h) \sim \mathcal{O}(h_0^0)$. Now using integration by parts, we can write

$$I_1 = \int_{\delta/h_0}^{H} \frac{d\rho}{\rho^2} \int_{h_0}^{\delta/\rho} dh J_5(h)$$

$$= -\frac{1}{H} \int_{h_0}^{\delta/H} dh J_5(h) - \delta \int_{\delta/h_0}^{H} \frac{d\rho}{\rho^2} J_5(h)|_{h = \delta/\rho}.$$
We further make the coordinate transformation $\rho = \delta / q$ and Taylor expand the second term in $\delta$:

$$I_1 = - \frac{1}{H} \int_{\delta_0}^{\delta/H} dh J_5(h) + \frac{1}{\delta} \int_{\delta_0}^{\delta/H} dq J_5(q)$$

$$= - \frac{1}{H} \int_{\delta_0}^{0} dh J_5(h) + \frac{1}{\delta} \int_{\delta_0}^{0} dq q^2 J_5(q) - \frac{\delta}{18H^3} \cos(\Omega) \cot(\Omega) + \mathcal{O}(\delta),$$

(4.27)

where we have used (4.25) to get the final expression. Combining (4.26) and (4.27), we can write

$$S_5|_{c_1 \times R^1} = 4 \pi^2 L^4 \tilde{H} \left[ \frac{H^2 \sin(\Omega)}{6\delta^3} + \frac{1}{\delta} \int_{\delta_0}^{0} dq J_5(q) + \frac{\cos(\Omega) \cot(\Omega) \log(\delta/H)}{9} \right.$$  

$$+ \frac{2H^2 \cos(\Omega) \cot(\Omega) (1 - \log(h_0)) - 9 \sin(\Omega)}{18h_0^2 \delta} + \mathcal{O}(\delta^0) \left] ,

(4.28)

Note that in the above expression, we have a new divergence of the form $\log(\delta/H)/\delta$ which does not arise in EE for smooth entangling surfaces. This term should be a contribution from the singularity. Further, we note that as soon as we add a one-dimensional locus to the conical singularity, both double log and logarithmic divergences disappear. Recall that in the previous case for cone $c_1$ in $d = 4$, i.e., eqn. (3.23), we got a double log and log terms.

We can easily generalize the above calculations to the crease $c_1 \times R^2$. For this case, the integrand near the boundary and complete expression for EE are given by (B.5) and (B.6). Once again we find a divergent term of the order $\log(\delta/H)/\delta$ in EE. Also, now the cut-off independent term is

$$S_6^{\log}|_{c_1 \times R^2} = 4 \pi^2 L^5 \tilde{H}^2 \frac{3(13 - 19 \cos(2\Omega)) \cot^2(\Omega) \csc(\Omega)}{8192H^2} \log(\delta/H).$$

(4.29)

Note that for singular geometry $c_1 \times R^2$ in $d = 6$, we only have log divergence instead of $\log^2\delta$ as compared to the cone $c_1$ in equations (3.23). Also, recall that in (3.28), we saw that a cone without a locus, $c_3$ in $d = 6$ gives a $\log^2\delta$ divergence. In (4.29), this $\log^2\delta$ term disappears because of flat locus we have added. We will see in section 4.2.2 that as soon as the curvature of the locus turned on, the $\log^2\delta$ divergence will reappear in (4.147) in $d = 6$.

In (4.29), the logarithmic contribution can be attributed to the fact that in even dimensions, the entangling surface has a logarithmic divergence. We can explicitly verify that logarithmic term in (4.29) comes from integration over $1/h$ in expansion (B.5). As $\rho$ need not to be near the singularity, this contribution is from smooth part of the entangling surface. Further, it is straightforward to find that for cone $c_1 \times R^3$ in $d = 7$ dimensional CFT, the EE has no double log or log divergence. Which is expected because for entangling surface in odd $d$, there is no logarithmic divergence and as the locus is odd dimensional and flat, there should not be any logarithmic contribution from the singularity either.
We can further extend these calculations for the case \( n = 2 \) and \( m = 1, 2 \). For crease \( c_2 \times R^1 \), we find that the universal term in EE is a logarithmic divergence and it is given by

\[
S^\text{log}_6 \big|_{c^2 \times R^1} = \frac{8 \pi^2 L^2 \hat{H} (7 - 9 \cos(2\Omega)) \cot^2(\Omega)}{256 \hat{H}^3} \log(\delta/H). \tag{4.30}
\]

We also find a new contribution from the singularity at the order \( 1/\delta \). In \( c^2 \times R^1 \), the locus of the singularity is odd dimensional and hence singularity doesn’t contribute in the universal term. However, smooth part of the entangling surface contributes through a log in \( d = 6 \) and this is the contribution (4.30). It can be verified that this log term arise from integration over \( h \) away from the singularity. Further, we can also calculate EE for crease \( c_2 \times R^2 \) and find that there is a new divergence from singularity of order \( 1/\delta^2 \). However, there is neither a log nor a \( \log^2 \) term in EE. Note that for crease \( c_2 \times R^2 \), the locus of the singularity is even dimensional but flat. So singularity does not contribute through a log term. However, as we will see in (4.152) that for a curved locus, we get a log contribution.

### 4.2 EE for singularity with curved locus

In this subsection, we will consider several singular embeddings which have a curved locus. As soon as the curvature of the locus is turned on, the double log and log terms from singularity will make appearance. In this section, we will consider the creases \( k \times \Sigma \) and \( c_n \times \Sigma \), where locus \( \Sigma \) will take the form \( S^m \) or \( S^{m-p} \times R^p \). These cases will be slightly more involved and hence we will do two things: first, we will always work in the limit where curvature of the locus is very small and we will do the calculations perturbatively. In certain cases, we will see that these perturbative calculations are sufficient to pick the complete contributions from the singularity. Second, for calculations in this section, we will foliate our minimal area surface in different way. In all the previous cases, the induced coordinates on the minimal area surface were \((\rho, \theta, \ldots)\) and we assumed that the radial coordinate \( z = z(\rho, \theta) \). However, we will find that it is more convenient to do the calculations in a coordinate system where bulk radial coordinate \( z \) is one of the induced coordinates on the minimal area surface and we have \( \rho = \rho(z, \theta) \). We have shown in figure 3, how these different induced coordinates foliate the minimal area surface in the bulk. Note that the new set of induced coordinate \((z, \theta, \ldots)\) are not the well-defined coordinates on the boundary as \( \rho = \rho(z, \theta) \) will be multivalued for \( z = 0 \) and \( \theta = \pm \Omega \). However, until unless we put the UV cut-off, we find that \( \rho = \rho(z, \theta) \) is a well-behaved function and we can work with it. Now, in the next subsection we will begin with the geometries \( k \times \Sigma \) and then we will move on to conical singularities \( c_n \times \Sigma \).

#### 4.2.1 Crease \( k \times \Sigma \)

In this section, we will mainly consider the geometries \( k \times S^2 \), \( k \times R \times S^2 \) and \( k \times S^3 \). We will see that singularities with even dimensional locus will contribute through a logarithmic term.

To begin with, let us consider \( d = 5 \) CFT on background \( R^3 \times S^2 \). Before we construct the singular entangling surface in this geometry and calculate holographic EE, we need to
Figure 3: Panel (a) shows how minimal area surface is foliated when we have the induced coordinates $(\rho, \theta, \ldots)$ and minimal area surface is given by $z = z(\rho, \theta)$. Panel (b) shows the foliation of the minimal area surface when induced coordinates are $(z, \theta, \ldots)$ and we have $\rho = \rho(z, \theta)$. Note that $z = z_m$ is the maximum value of $z$ on the surface such that $\rho(0, z_m) = H$.

find the dual gravity. So we begin with the action for six-dimensional dual Einstein gravity

$$I_6 = \frac{1}{\ell_p^4} \int d^6x \sqrt{-g} \left[ \frac{20}{L^2} + R \right],$$

for which, the equation of motion is given by

$$R_{\mu\nu} - \frac{10}{L^2} g_{\mu\nu} = 0.$$  

(4.32)

Now we ansatz that the bulk metric, which has the boundary $R^3 \times S^2$, is of the form

$$ds^2 = \frac{L^2}{z^2} \left( dz^2 + f_1(z) \left( dt^2 + d\rho^2 + \rho^2 d\theta^2 \right) + f_2(z) R_1^2 d\Omega_2^2 \right),$$

(4.33)

where $d\Omega_2^2 = d\zeta_0^2 + \sin^2(\zeta_0) d\zeta_1^2$ represents the metric over two-sphere and, $f_1$ and $f_2$ are functions of the radial coordinate. Here $R_1$ is the radius of the sphere on the boundary and in the limit $R_1 \to \infty$, we recover the flat boundary. To find $f_1$ and $f_2$, we use the Fefferman-Graham expansion near the boundary $z = 0$. The idea is to insert the ansatz (4.33) in the equation of motion (4.32) and then solve it near $z = 0$ to find

$$f_1 = 1 + \frac{z^2}{12R_1^2} + \frac{17z^4}{576R_1^4} + \ldots \quad \text{and}$$

$$f_2 = 1 - \frac{z^2}{4R_1^2} - \frac{5z^4}{192R_1^4} + \ldots.$$  

(4.34)

Knowing the dual geometry near the boundary, we can calculate EE for the geometry $k \times S^2$. On the boundary, this geometry is defined by $\rho \in [0, H]$ and $\theta \in [-\Omega, \Omega]$, where $H$ is the
IR cut-off. Now to define the minimal area surface, we use slightly different coordinates compared to the previous sections. We choose the induced coordinates to be \((z, \theta, \xi_0, \xi_1)\) and because of the rotational symmetry on the sphere, \(\rho = \rho(z, \theta)\). Although, \(\rho(z, \theta)\) is not a well defined function at the boundary \(z = 0\) but we can definitely work with it until we impose the UV cut-off and assume that the boundary is at some finite \(z = \delta\). With this coordinate choice, the induced metric becomes

\[
h = \begin{bmatrix}
\frac{L^2}{z^2} (f_1 \rho'^2 + 1) & \frac{L^2 f_1}{z^2} \hat{\rho} \rho' \\
\frac{L^2 f_1}{z^2} \hat{\rho} \rho' & \frac{L^2 f_2 R_1^2}{z^2} \sin^2(\xi_0)
\end{bmatrix},
\]

(4.35)

where \(\hat{\rho} = \partial_\theta \rho\), \(\rho' = \partial_z \rho\) and hence, the entanglement entropy is given by

\[
S_5|_{k \times S^2} = \frac{8 \pi^2 L^4 R_1^2}{\ell^4} \int dz \, d\theta \sqrt{f_1 f_2} \frac{\sqrt{ho^2 + \rho'^2}}{z^4} (1 + f_1 \rho'^2).
\]

(4.36)

We can easily find the equation of motion of \(\rho(z, \theta)\) to be

\[
0 = 2zf_1 f_2 \rho \left( \rho'^2 + \rho'' \right) + 2zf_2 \rho \left( 1 + f_1 \rho'^2 \right) \rho' - 4zf_1 f_2 \rho \rho' \rho' + 2zf_2 \rho \rho' \left( \rho'' + \rho'^2 (1 + f_1 \rho'^2) \right) + f_2 (-4 \rho'^2 + \rho (-8f_1 + 3zf_1') \rho'^2 - 2z\rho'^2 (1 + f_1 \rho'^2) + \rho^3 \rho' (3zf_1' - 8f_1^2 \rho'^2 + 2f_1 (-4 + zf_1') \rho'^2))
\]

(4.37)

where \(\hat{\rho} = \partial_\theta \rho\), \(\rho'' = \partial_z^2 \rho\) and \(\rho' = \partial_z \rho\).

As discussed earlier, we will be working in the approximation \(R_1 \to \infty\). In this approximation, we will find that leading order correction to the EE will be of the order \(O(1/R_1^3)\). To show this, first we need to find the form of the solution \(\rho(z, \theta)\) in this approximation. For that, we make following ansatz:

\[
\rho(z, \theta) = \frac{z}{h(\theta)} + \frac{z^2}{R_1} g_2(\theta) + \frac{z^3}{R_1^2} g_3(\theta) + \ldots.
\]

(4.38)

This ansatz is made on the basis of the available dimensionful quantities in the problem. Now we are going to argue that in the following expansion, the \(\theta\) dependent functions \(g_{2n}\) are zero. To see that, we insert the ansatz (4.38) in the equation of motion for \(\rho(z, \theta)\) and find following equations of motion for \(h, g_2\) and \(g_3\):

\[
h \left( 1 + h^2 \right) \ddot{h} + 4h^2 + \left( 1 + h^2 \right) \left( 4 + h^2 \right) = 0,
\]

(4.39)

\[
h^2 \left( 1 + h^2 \right)^2 g_2'' + 2h \left( 7 + 2h^2 \right) \left( 1 + h^2 \right) \dot{h} g_2' + g_2 \left( 14h^2 - 2h^4 \right) = 0,
\]

(4.40)

\[
h^2 \left( 1 + h^2 \right)^2 \ddot{g}_3 + 4h \left( 4 + h^2 \right) \left( 1 + h^2 \right) \dot{h} g_3 + g_3 \left( 35 - 2h^2 \right) + 4h^4 \left( 8 - h^2 \right) = S_1,
\]

(4.41)
where $S_1$ is given by

$$
S_1 = \frac{1}{12\hbar} \left( 8 + 7h^4 + 3h^2 + h^2 \left( 15 + 7h^2 \right) \right) + \frac{2h}{(1 + h^2)} \left( h^2 (1 + h^2)^2 (4 + h^2) g_2'^2 + 2h (8 + 7h^2 + h^2) \dot{g}_2 g_2' \right)
$$

$$
+ \left( h^2 \left( 32 - 15h^2 \right) + h^6 \left( 8 + h^2 \right) + 2h^4 \left( 15 + h^2 \right) + 2 \left( 5 + 8h^2 \right) \right) g_2^2 . \quad (4.42)
$$

Note that here we have arranged the equations of motion such that on the left hand side, we have the homogeneous part of the equation and on the right hand side, we have the source terms. So $S_1$ is source for $g_3$ and in (4.42), terms in first line come from the corrections to $f_1$ and $f_2$, and terms in last two lines are second order in $g_2$. We also note that in (4.40) and (4.41), we have used equations of motion to eliminate the second derivatives of $h$ and $g_2$. Now we notice following points: first, the corrections to $f_1$ and $f_2$ are of even powers of $1/R_1$ and hence they only source $g_{2n+1}$’s. Second, $g_{2n}$ are sourced only by $g_2$’s, where $i < n$. Third, in flat boundary, precisely in the limit $R_1 \to \infty$, we have $h \neq 0$ and $g_2 = 0$. So above arguments conclude that as we make $R_1$ finite, only $g_{2n+1}(\theta)$’s will be turned on and all the $g_{2n} = 0$, which clearly are solution of these equations of motion.

Now to separate the logarithmic divergence in entanglement entropy, first let us write $\rho = \rho_0(z, \theta) + \rho_1(z, \theta)/R_1^2$, where $\rho_0 = z/h(\theta)$ and $\rho_1$ is higher order corrections for large $R_1$. Using this and (4.34) in (4.36), and keeping only the leading order terms in $R_1$, we find that

$$
S_5 \big|_{k \times S^2} = \frac{8\pi^2 L^4 R_1^2}{\ell_P^4} \int_{z_m}^{\delta} dz \int_{-\Omega + \epsilon}^{\Omega - \epsilon} d\theta \left[ \frac{\sqrt{\rho_0^2 + \rho_0^2 (1 + \rho_0^2)}}{z^4} \right. \left. \left( \dot{\rho}_0 (5z^2 \dot{\rho}_0 - 24 \dot{\rho}_1) - 24 \rho_0 \rho_1 (1 + \rho_0^2) + \rho_0^2 (5z^2 + 4z^2 \rho_0^2 - 24 \rho_0' \rho_1') \right) \right]
$$

$$
\cdot \frac{1}{24z^4 R_1^2 \sqrt{\rho_0^2 + \rho_0^2 (1 + \rho_0^2)}} . \quad (4.43)
$$

where $\delta$ is the UV cut-off and $\epsilon = \epsilon(z)$ is defined such that $\rho(z, \Omega - \epsilon) = H$ and further, $z_m$ is defined such that $\rho(z_m, 0) = H$ or $\epsilon(z_m) = \Omega$. We can also insert this ansatz in (4.37) and find the equation of motion for $\rho_0$ and $\rho_1$ by series expanding it in terms of $R_1$. In the second term of (4.43), we can convert $\rho_1'$ and $\rho_1$ into $\rho_1$ using the integration by parts and then use equation of motion for $\rho_0$ to simplify the coefficient of $\rho_1$ to zero. This process will leave us with some boundary terms and we find that

$$
S_5 \big|_{k \times S^2} = \frac{4\pi^2 L^4 R_1^2}{\ell_P^4} \int_{z_m}^{\delta} dz \int_{-\Omega + \epsilon}^{\Omega - \epsilon} d\theta \left[ \frac{\sqrt{\rho_0^2 + \rho_0^2 (1 + \rho_0^2)}}{z^4} \right. \left. \frac{\left( 5 \rho_0^2 + \rho_0^2 (5 + 4 \rho_0^2) \right)}{24 R_1^2 z^2 \sqrt{\rho_0^2 + \rho_0^2 (1 + \rho_0^2)}} \right]
$$

$$
+ \frac{\partial}{\partial \theta} \left( \frac{\dot{\rho}_0 \rho_1}{R_1^2 z^4 \sqrt{\rho_0^2 + \rho_0^2 (1 + \rho_0^2)}} \right) + \frac{\partial}{\partial z} \left( \frac{\rho_0^2 \rho_1'}{R_1^2 z^4 \sqrt{\rho_0^2 + \rho_0^2 (1 + \rho_0^2)}} \right) . \quad (4.44)
$$
Now we can insert the ansatz \( \rho_0 = z/h(\theta) \) and \( \rho_1 = z^3 g_3(\theta) \) and simplify the functional to

\[
S_{5|k\times S^2} = \frac{16 \pi^2 L^4 R_1^2}{\ell_p^4} \int_{z_m}^\delta \frac{dh}{h} \left[ \frac{\sqrt{1 + h^2 + h^2}}{z^2 h^2} - \frac{4 + 5h^2 + 5\dot{h}^2}{24 R_1^2 z^2 h^2 \sqrt{1 + h^2 + h^2}} \right] - \frac{14 \pi^2 L^4}{\ell_p^4} \int_{z_m}^\delta \frac{dz}{z} \left. \frac{g_3 \dot{h}}{\sqrt{1 + h^2 + h^2}} \right|_{\theta = \Omega - \epsilon},
\]

where we have changed the integration limits from \((-\Omega, \Omega)\) to \((0, \Omega)\) and also changed the integration variable to \(h(\theta)\). Note that the boundary term with derivative with respect to \(z\) turns out to zero up to leading order. We have also defined \( h_0 = h(0) \) and \( h_1(\rho) = h(\Omega - \epsilon) \) and used \( h'_1(0) = 0 \) in getting the boundary terms.

Now to separate the logarithmic divergences, we consider all the contributing factors one by one. We will see that the first term in the first line of (4.45) will not contain any logarithmic divergence. The second term and the boundary terms will contain logarithmic divergence. Note that we also need to be careful and consider the divergences coming from the limits of the integrals. To begin with, we first study the behavior of \( h \) and \( g_3 \) near the asymptotic boundary. The equations of motion for \( h \) and \( g_3 \) are given by (4.39) and (4.41) with \( g_2 = 0 \). Now similar to (4.8), we find that equation of motion for \( h \) can be integrated once to get

\[
K_5 = \frac{(1 + h^2)^2}{h^4 \sqrt{1 + h^2 + h^2}},
\]

where \( K_5 \) is a constant and it can be further related to \( h(0) \) using \( \dot{h}(0) = 0 \).

To extract the logarithmic divergence, we will only need the asymptotic behavior of \( h \) and \( g_3 \). Hence we solve \( g_3 \) in terms of \( h \) in asymptotic limit, where \( h \) is small. To do so, we use (4.41) and change the variable from \( \theta \) to \( h \) by expressing \( \tilde{g}_3(\theta) = d^2 g_3/d\theta^2 \) and \( \tilde{g}_3(\theta) = dg_3/d\theta \) in terms of \( \tilde{g}_3(h) = d^2 g_3/dh^2 \) and \( \tilde{g}_3(h1) = dg_3/dh \). We further use (4.46) to express \( h1'(\theta) \) in terms of \( K_5 \) and \( h \). Finally, the equation of motion for \( g_3(h) \) becomes

\[
0 = 12h^3 \frac{1 + h^2}{2} \left( h^8 K_5^2 - (1 + h^2)^3 \right) \tilde{g}_3 + 12h^2 \left( h^8 (1 + h^2) (16 + 5h^2) K_5^2 - 4 (1 + h^2)^4 (3 + h^2) \right) \tilde{g}_3 - 12h \left( 2 (1 + h^2)^3 (12 + h^2 + h^4) - h^8 (44 + 17h^2 + 3h^4) K_5^2 \right) g_3 + (1 + h^2)^3 (3 + 7h^2) + 5h^8 K_5.
\]

Now this equation has two solutions when we solve it perturbatively in the limit \( h \to 0 \). The leading terms of these solutions go like \( 1/h^3 \) and \( 1/h^8 \). However, \( g_3 \) must be such that \( \rho \) is finite in the limit \( h \to 0 \) and \( \delta \to 0 \). As \( g_3 \) appears at order \( \delta^3 \) in (4.34), it can only go like \( 1/h^3 \). Hence, one of the constants, which is the coefficient of \( 1/h^8 \), is fixed to zero. As a result, the asymptotic solution turns out to be

\[
g_3 = \frac{b_3}{h^3} + \frac{1 + 88b_3}{56h} + \frac{4 + 72b_3}{189} h - \frac{4 + 72b_3}{693} h^3 + \ldots,
\]

where \( b_3 \) is a constant such that \( g_3(\theta) \) has an extrema at \( \theta = 0 \).
Before we begin discussing divergences of various terms, we find the series expansion of $h_{1c}$ in terms of the UV cut-off $\delta$. As $h_{1c} = h(\Omega - \epsilon)$ and $z = \delta$ at the cut-off, we can use (4.38) and perturbative solution (4.48) to find following series expansion for $h_{1c}$

$$h_{1c}(\delta) = \left( \frac{1}{H} + \frac{b_3 H}{R_1^2} \right) \delta + \frac{(1 + 88b_3) \delta^3}{56HR_1^2} + \frac{(4 + 72b_3) \delta^5}{189H^3R_1^4} + \mathcal{O}(\delta^6).$$  

(4.49)

Note that we have kept only leading corrections in $R_1$ at any order in $\delta$.

Now we return to (4.45) and analyze the divergences for each term. First, we use (4.46) in the integrand of first two terms of (4.45) and find that in the asymptotic limit

$$\frac{\sqrt{1 + h^2 + h^2}}{h h^2} \sim -\frac{1}{h^2} - \frac{1}{2} K_5^2 h^6 + \mathcal{O}(h^8)$$  

(4.50)

$$\frac{5h^2 + 5h^2 + 4}{24h h^2 \sqrt{1 + h^2 + h^2}} \sim -\frac{5}{24h^2} - \frac{K_5^2 h^6}{16} + \mathcal{O}(h^8).$$  

(4.51)

So we can make the integrands finite by organizing the terms in following form

$$I_1 = \int_{z_m}^{\delta} \frac{dz}{z^3} \int_{h_0}^{h_{1c}} dh \frac{\sqrt{1 + h^2 + h^2}}{h h^2}$$

$$= \int_{z_m}^{\delta} \frac{dz}{z^3} \int_{h_0}^{h_{1c}} dh \left[ \frac{\sqrt{1 + h^2 + h^2}}{h h^2} + \frac{1}{h^2} \right] + \int_{z_m}^{\delta} \frac{dz}{z^3} \left( \frac{1}{h_{1c}} - \frac{1}{h_0} \right)$$

(4.52)

and

$$I_2 = \int_{z_m}^{\delta} \frac{dz}{z} \int_{h_0}^{h_{1c}} dh \frac{5h^2 + 5h^2 + 4}{24h h^2 \sqrt{1 + h^2 + h^2}}$$

$$= \int_{z_m}^{\delta} \frac{dz}{z} \int_{h_0}^{h_{1c}} dh \left[ \frac{5h^2 + 5h^2 + 4}{24h h^2 \sqrt{1 + h^2 + h^2}} + \frac{5}{24h^2} \right] + \int_{z_m}^{\delta} \frac{dz}{z} \left( \frac{1}{h_{1c}} - \frac{1}{h_0} \right)$$

(4.54)

In (4.53), $I'_1$ and $I'_2$ represent the first and second integrals in (4.52). Similarly in (4.55), $I'_3$ and $I'_4$ are the first and second integrals in (4.54). Now first we consider $I'_1$. We differentiate it with respect to the UV cut-off $\delta$ and look for $1/\delta$ divergent terms. After taking the derivative, we find

$$\frac{dI'_1}{d\delta} = \int_{h_0}^{h_{1c}(\delta)} dh \left[ \frac{\sqrt{1 + h^2 + h^2}}{h h^2} + \frac{1}{h^2} \right]$$

$$= \int_{h_0}^{h_{1c}(\delta)} dh \left[ \frac{\sqrt{1 + h^2 + h^2}}{h h^2} + \frac{1}{h^2} \right] + \int_{h_0}^{h_{1c}(\delta)} dh \left[ \frac{\sqrt{1 + h^2 + h^2}}{h h^2} + \frac{1}{h^2} \right]_{h=h_{1c}(\delta)}$$

$$- \frac{K_5^2}{2H^5} \left( \frac{1}{2H^2} + \frac{7b_3}{R_1^2} \right) \delta^4 + \mathcal{O}(\delta^6),$$  

(4.56)
where we have Taylor expanded the integrand in the second line and in the third line, we have used (4.46) and (4.49) to find the leading order divergence. Here we don’t have any term of order $1/\delta$ and of $\log(\delta)/\delta$, which come consecutively from divergences of order $\log(\delta)$ and $\log(\delta)^2$ in $I_1'$.

Similar to $I_1'$, we can take a derivative of $I_1'$ with respect to $\delta$ and use (4.49) to find

$$\frac{dI_1'}{d\delta} = \left(1 - \frac{b_3H^2}{R_1^2}\right) \frac{H}{\delta^4} - \frac{1}{\delta^3} \frac{1 + 88b_3H}{56R_1^2} \frac{H}{\delta^2} - \frac{1}{189R_1^4} \frac{H}{H} + \mathcal{O}(\delta).$$

So we find that $I_1$ doesn’t have any logarithmic divergence. We can use similar steps to find

$$\frac{dI_2'}{d\delta} = \frac{1}{\delta} \int_{\rho_0}^\rho dh \left[ \frac{5h^2 + 5h^2 + 4}{24h^2\sqrt{1 + h^2 + h^2}} + \frac{5}{24h^2} \right] - \frac{K_2^2}{H^2} \left( \frac{1}{16H^2} + \frac{7b_3}{16R_1^4} \right) \delta^5 + \mathcal{O}(\delta^6)$$

and

$$\frac{dI_4'}{d\delta} = \frac{5H}{24\delta^2} \left(1 - \frac{b_3H^2}{R_1^2}\right) - \frac{5}{24h_0} \frac{1}{\delta} + \mathcal{O}(\delta^0).$$

We can also use the same procedure on the boundary term and find that

$$\frac{d}{d\delta} \int_{z_m}^{z_0} \frac{g_3\delta}{\sqrt{1 + h^2 + h^2}} \bigg|_{\theta = \Omega - \epsilon} = \frac{b_3H^3}{\delta^4} \left(1 - \frac{3b_3H^2}{R_1^2}\right) + \frac{(1 + 88b_3)H(-4b_3H^2 + R_1^2)}{56R_1^4\delta^2}$$

$$\quad + \left(\frac{4 + 72b_3}{189H^2} - \frac{(27 + 16b_3(521 + 17100b_3))H}{84672R_1^2}\right) + \mathcal{O}(\delta^0).$$

So from (4.56) - (4.60), we can find the logarithmic divergence in the EE for $k \times S^2$ geometry:

$$S_5^{\text{log}} \bigg|_{k \times S^2} = \frac{16\pi^2 L^4}{\ell_s^4} \left( - \int_{\rho_0}^\rho dh \left[ \frac{5h^2 + 5h^2 + 4}{24h^2\sqrt{1 + h^2 + h^2}} + \frac{5}{24h^2} \right] + \frac{5}{24h_0} \right) \log(\delta).$$

From (4.56) - (4.60), we also notice a divergence of order $1/\delta^2$ in EE which does not appear in EE for smooth entangling surfaces. Note that such a term also appeared in EE for $k \times R^n$ in (4.14). We further notice that logarithmic contribution in (4.61) is of next to the leading order for large $R_1$. As we are working in odd dimensional spacetime, there is no logarithmic contribution from the surface itself.

Now we turn to our next example. Having seen the logarithmic contribution from a even dimensional curved locus, now we consider odd dimensional locus. We will calculate the entanglement entropy for $k \times S^3$ geometry in CFT on $R^3 \times S^3$. For this case, the locus is $S^3$ and we will see that there will be no log contribution from the singularity. However, as the CFT is in even dimensional spacetime, we should be getting a logarithmic contribution coming from the entangling surface. In this case, the metric for the dual geometry is given by

$$ds^2 = \frac{L^2}{z^2} \left( dz^2 + f_1(z) \left( dt^2 + dr^2 + \rho^2 d\theta^2 \right) + f_2(z) R_1^2 d\Omega_3 \right),$$

where $z, r, \rho, \theta, \Omega$ are the coordinates of $R^3 \times S^3$. We will use the methods developed above to calculate the entanglement entropy.

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where $d\Omega^2_3 = d\xi_0^2 + \sin^2(\xi_0) d\xi_1^2 + \sin^2(\xi_0) \sin^2(\xi_1) d\xi_2^2$ is the unit three-sphere and $f_1$ and $f_2$ are following

$$
\begin{align*}
  f_1 & = 1 + \frac{3z^2}{20R_1^2} + \frac{69z^4}{1600R_1^4} + \ldots, \\
  f_2 & = 1 - \frac{7z^2}{20R_1^2} - \frac{11z^4}{1600R_1^4} + \ldots.
\end{align*}
$$

(4.63)

Once again we choose the induced coordinates on the minimal area surface to be $(z, \theta, \xi_0, \xi_1, \xi_2)$ and assume $\rho = \rho(z, \theta)$. Then the induced metric over the minimal area surface will be given by

$$
\begin{align*}
  h &= \left[ \frac{L^2}{z^2} \left( f_1 \rho'^2 + 1 \right) \quad \frac{L^2 f_1}{z^2} \frac{\dot{\rho}^2}{\rho'} \quad \frac{L^2 f_1}{z^2} \frac{\dot{\rho}^2}{\rho'} \quad \frac{L^2 f_2 R_1^3}{z^2} g_{ab}(S^3) \right],
\end{align*}
$$

(4.64)

where $\dot{\rho} = \frac{\partial}{\partial z} \rho$, $\rho' = \frac{\partial}{\partial z} \rho$ and the entanglement entropy is given by

$$
S_0|_{k \times S^3} = \frac{4\pi^3 L^5 R_1^3}{\ell_P^3} \int dz d\theta \left( \sqrt{f_1^2 + \rho^2 (1 + f_1 \rho'^2)} \right).
$$

(4.65)

We can easily find the equation of motion for $\rho(z, \theta)$ to be

$$
\begin{align*}
  0 &= 2zf_1 f_2 \rho \left( \rho'^2 + \dot{\rho}^2 \right) \rho'' + 2f_2 \rho \left( z + z f_1 \rho'^2 \right) \dot{\rho} - 4zf_1 f_2 \rho \dot{\rho} \rho' \rho' \\
  &\quad + 3zf_1 \rho f_2 \rho' \left( \rho' + \rho' \left( 1 + f_1 \rho'^2 \right) \right) - f_2 \left( 4z \rho^2 + \rho \left( 10f_1 - 3zf_1 \right) \rho'^2 + 2\rho^2 \left( z + zf_1 \rho'^2 \right) \right) \\
  &\quad - \rho^3 \rho' \left( 3z f_1' - 10f_2 \rho'^2 + 2f_1 \left( -5 + zf_1 \rho'^2 \right) \right),
\end{align*}
$$

(4.66)

where $\dot{\rho} = \frac{\partial}{\partial z} \rho$, $\rho'' = \frac{\partial}{\partial \theta} \rho$ and $\rho' = \frac{\partial}{\partial \theta} \rho$. Once again, we can write $\rho = \rho_0 + \rho_1/R_1^2$ and keep only the leading order correction to the entanglement entropy. We can further series expand the equation of motion (4.66) to get equation of motion for $\rho_0$ and $\rho_1$. We find that, using these equations of motion and integration by parts, the entropy functional can be simplified to

$$
S_0|_{k \times S^3} = \frac{4\pi^3 L^5 R_1^3}{\ell_P^3} \int_{z_m}^\delta dz \int_{-\Omega+\epsilon}^{\Omega-\epsilon} d\theta \left[ \sqrt{\rho_0^2 + \rho_1^2 (1 + \rho_0^2)} \left( 1 + \rho_0^2 \right) \right] - \frac{3 \left( 6\rho_0^2 + \rho_0^2 (6 + 5\rho_0^2) \right)}{40R_1^2 \sqrt{\rho_0^2 + \rho_0^2 (1 + \rho_0^2)}}
$$

(4.67)

Further, we can substitute $\rho_0 = z/h(\theta)$ and $\rho_1 = z^3 g_3(\theta)$. The equations of motion for $h$ and $g_3$ from (4.66) becomes

$$
\begin{align*}
  &h (1 + h^2) \ddot{h} + (1 + h^2) (5 + h^2) = 0, \\
  &h^2 (1 + h^2) \ddot{g}_3 + 2h \left( 9 + 11h^2 + 2h^4 \right) \dot{h} \dot{g}_3 \\
  &\quad - g_3 \left( 25 + 45h^2 + 21h^4 + h^6 - (27 - h^2 + 2h^4) h^2 \right) = S_1,
\end{align*}
$$

(4.68)

(4.69)
where source terms for $g_3$ are

$$S_1 = \frac{3 \left(10 + 19h^2 + 9h^4 + (4 + 9h^2) \dot{h}^2\right)}{20h}.$$  \hspace{1cm} (4.70)

Here, as in (4.8), we can integrate the equation of motion for $h$ once and write it as

$$K_6 = \frac{(1 + h^2)^{5/2}}{h^5 \sqrt{1 + h^2 + h^2}},$$  \hspace{1cm} (4.71)

where $K_6$ is a constant and we can relate it to $h_0$ by using $\dot{h}(0) = 0$. We can further simplify (4.67) to

$$S_6 \big|_{k \times S^3} = \frac{8 \pi^3 L_5 R_1^3}{\ell_p^6} \left(I_1 + \frac{I_2}{R_1^2} + \frac{I_3}{R_1^4} + \frac{I_4}{R_1^8}\right),$$  \hspace{1cm} (4.72)

where

$$I_1 = \int_{h_m}^{h_c} \int_{h_0}^{h_1c} dh \frac{\sqrt{1 + h^2 + h^2}}{\dot{h}} dz,$$  \hspace{1cm} (4.73)

$$I_2 = -\int_{h_m}^{h_c} \int_{h_0}^{h_1c} dh \frac{3 \left(5 + 6h^2 + 6\dot{h}^2\right)}{40R_1^2 z^2 h^2 \sqrt{1 + h^2 + h^2}},$$  \hspace{1cm} (4.74)

$$I_3 = -\int_{z_m}^{z_c} \left| \frac{g_3 \dot{h}}{\sqrt{1 + h^2 + h^2}} \right| \Omega - \epsilon dz,$$  \hspace{1cm} (4.75)

$$I_4 = -\int_{z_m}^{z_c} \int_{h_0}^{h_1c} dh \frac{g_3}{\dot{h}} \frac{\sqrt{1 + h^2 + h^2}}{\sqrt{1 + h^2 + h^2}},$$  \hspace{1cm} (4.76)

Note that the last term $I_4$ is not being integrated over $z$. That’s because $h_{1c}(z)$ appears in the limit of $\theta$ integral. Now, we can use (4.71) in (4.69) to find following equation of motion for $g_3$ in terms of $h$:

$$0 = 20h^3 \left(1 + h^2\right)^2 \left(h^{10} K_6^2 - (1 + h^2)^4\right) \ddot{g}_3 + 20h^2 (1 + h^2) \left(h^{10} (18 + 5h^2) K_6^2 - (1 + h^2)^4 (27 - h^2 + 2h^4)\right) g_3$$

$$+ 3 \left(1 + h^2\right)^4 \left(4 + 9h^2\right) + 18h^{10} K_6^2.$$  \hspace{1cm} (4.77)

This equation can be solved perturbatively for small $h$ near the asymptotic boundary. We find that this second order equation of motion has two different solutions which go like $1/h^3$ and $1/h^9$. As $\rho$ is finite at the boundary, the solution can go only with power $1/h^3$. So the solution near the boundary becomes

$$g_3(h) = \frac{b_3}{h^3} + \frac{3 + 140b_3}{80h} + \frac{3 + 36b_3}{64h} - \frac{1 + 12b_3}{96} h^3 + \frac{5 + 60b_3}{1536} h^5 + \ldots,$$  \hspace{1cm} (4.78)
where constant \( b_3 \) is fixed by the condition that \( g_3(h_0) = 0 \). We use this in \( \rho = z/h + z^3g_3/R_1^2 \) and evaluate it at \( z = \delta, \rho = H \) and \( h = h_{1c}(\delta) \). By inverting the relation and keeping only leading order terms in \( R_1 \), we find

\[
h_{1c}(\delta) = \left( \frac{1}{H} + \frac{b_3H}{R_1^2} \right) \delta + \frac{(3 + 140b_3)\delta^3}{80HR_1^2} + \frac{(3 + 36b_3)\delta^5}{64H^3R_1^2} + \mathcal{O}(\delta^6). \tag{4.79}
\]

Now we use this to separate logarithmic divergence. Using (4.71), we find that the integrands in \( I_1, I_2, I_3 \) and \( I_4 \) are of following form for small \( h \):

\[
\sqrt{1 + h^2 + \frac{\dot{h}^2}{h^2}} \sim -\frac{1}{h^2} - \frac{1}{2} K_6 h^8 + \mathcal{O}(h^9),
\]

\[
-3 \left( 5 + 6 h^2 + 6 \frac{\dot{h}^2}{h^2} \right) \sim \frac{9}{20h^2} + \frac{3}{20} K_6 h^8 + \mathcal{O}(h^9)
\]

\[
-\frac{g_3\dot{h}}{\sqrt{1 + h^2 + \left( \frac{\dot{h}}{h} \right)^2}} \sim \frac{b_3 h^3}{h^3} + \frac{3 + 140b_3}{80h} + \frac{3}{64} (1 + 12b_3) h - \frac{1}{96} (1 + 12b_3) h^3 + \mathcal{O}(h^5)
\]

\[
\frac{g_3}{hh\sqrt{1 + h^2 + \frac{\dot{h}^2}{h^2}}} \sim b_3 K_6^2 h^6 + \mathcal{O}(h^7). \tag{4.80}
\]

Using these relations, now we can make the integral over \( h \) finite and organize the terms in following form

\[
I_1 = \int_{z_m}^{\delta} \frac{dz}{z^4} \int_{h_0}^{h_{1c}} dh \left( \frac{\sqrt{1 + h^2 + \frac{\dot{h}^2}{h^2}}}{hh^2} + \frac{1}{h^2} \right) + \int_{z_m}^{\delta} \frac{dz}{z^4} \left( \frac{1}{h_{1c}(z)} - \frac{1}{h_0} \right)
= I'_1 + I'_2, \tag{4.81}
\]

\[
I_2 = -\int_{z_m}^{\delta} \frac{dz}{z^2} \int_{h_0}^{h_{1c}} dh \left( \frac{3 \left( 5 + 6 h^2 + 6 \frac{\dot{h}^2}{h^2} \right)}{40h^2\sqrt{1 + h^2 + \frac{\dot{h}^2}{h^2}}} + \frac{9}{20h^2} \right) - \frac{9}{20} \int_{z_m}^{\delta} \frac{dz}{z^2} \left( \frac{1}{h_{1c}(z)} - \frac{1}{h_0} \right)
= I'_3 + I'_4. \tag{4.82}
\]

In (4.81), \( I'_1 \) and \( I'_2 \) represent the first and second integral in the expression of \( I_1 \). Similarly in (4.82), \( I'_3 \) and \( I'_4 \) are the first and second integral in expression for \( I_2 \). We can further take derivative of \( I' \)’s with respect to \( \delta \) and then Taylor expand the integrals to get

\[
\frac{dI'_1}{d\delta} = \frac{1}{h^4} \int_{h_0}^{0} dh \frac{\sqrt{1 + h^2 + \frac{\dot{h}^2}{h^2}}}{h^2} - \frac{K_6^2 (9b_3H^2 + R_1^2)}{2H^3R_1^2} \delta^5 + \mathcal{O}(\delta^6), \tag{4.83}
\]

\[
\frac{dI'_2}{d\delta} = \frac{H}{\delta^5} \left( 1 - \frac{b_3 H^2}{R_1^2} \right) - \frac{1}{h_0 \delta^2} - \frac{(3 + 140b_3)H}{80R_1^2 \delta^3} - \frac{3}{64} H \frac{R_1^2}{R_1^2 \delta^3} + \mathcal{O}(\delta^6), \tag{4.84}
\]

\[
\frac{dI'_3}{d\delta} = -\frac{1}{\delta^2} \int_{h_0}^{0} dh \left( \frac{3 \left( 5 + 6 h^2 + 6 \frac{\dot{h}^2}{h^2} \right)}{40h^2\sqrt{1 + h^2 + \frac{\dot{h}^2}{h^2}}} \right) + \frac{3 K_6 (9b_3H^2 + R_1^2)}{20H^3R_1^2} \delta^7 + \mathcal{O}(\delta^{10}), \tag{4.85}
\]

\[
\frac{dI'_4}{d\delta} = -\frac{9H}{20 \delta^3} \left( 1 - \frac{b_3 H^2}{R_1^2} \right) + \frac{9}{20 h_0 \delta^2} + \frac{9 (3 + 140b_3) H}{1600 H^3R_1^2} \delta^8 + \mathcal{O}(\delta). \tag{4.86}
\]
Also, we can simplify the boundary terms in a similar fashion and find that
\[
\frac{dI_3}{d\delta} = \left(1 - \frac{3b_3H^2}{R_1^2}\right)\frac{b_3H^3}{\delta^5} + \left(\frac{7b_3}{20} - \frac{b_3(3 + 140b_3)H^2}{5R_1^2}\right)\frac{H}{4\delta^3} + \left(\frac{3 + 6b_3}{64H} - \frac{(9 + 1440b_3 + 26800b_3^2)H}{6400R_1^2}\right)\frac{1}{\delta} + \mathcal{O}(\delta^0),
\]
\[
(4.87)
\]
\[
\frac{dI_4}{d\delta} = -\frac{1}{\delta^2}\int_0^\infty dh \frac{g_3}{h \sqrt{1 + h^2 + h^2}} + \mathcal{O}(\delta^5).
\]
\[
(4.88)
\]
Now using (4.72) and (4.81)-(4.88), first we notice that there are new divergent terms of order \(1/\delta^3\) and \(1/\delta\) which does not arise in EE for smooth entangling surfaces in \(d = 6\). We also find that the logarithmic divergence of the entanglement entropy are
\[
S_{6}^{log} |_{k \times S^3} = \frac{8 \pi^3 L_5^5 R_1}{\ell_p^6} \left(0 + \mathcal{O}\left(\frac{1}{R_1^4}\right)\right) \log(\delta).
\]
\[
(4.89)
\]
Remarkably, the contribution to the logarithmic divergence at the leading order is zero. At higher order, i.e., at order \(\mathcal{O}\left(\frac{1}{R_1^4}\right)\), there should be logarithmic contribution coming from the bulk part of the entangling surface. According to [22], the logarithmic contribution from four-dimensional entangling surface will have the coefficient of the form \(\int dx^4 C\), where \(C\) is a combination of various curvature squared terms. These terms will contribute with \(1/R_1^4\) and so does the smooth part of the entangling surface. So up to leading order, there is no logarithmic contribution from the singularity because the locus is odd dimensional. We might ask if there can be any contribution from the singularity alone at higher order in \(1/R_1\). However, we discard such a possibility. It is easy to see that all the higher order terms will be of even powers of \(1/R_1\) and to make the coefficient dimensionless, the only other available scale will be \(H\). As \(H\) is related to the size of the surface, all such contributions actually result from the contribution from the bulk surface and not the singularity alone. Hence, we expect that there is no contribution from the singularity in this case.

Next, we see one more example where odd dimensional locus doesn’t contribute to logarithmic dimensions, in spite of non-zero curvature. We are going to consider \(k \times R^1 \times S^2\) geometry. We consider the background geometry for CFT to be \(R^4 \times S^2\). Then the bulk metric for boundary \(R^4 \times S^2\) is given by
\[
ds^2 = \frac{L^2}{z^2} \left( dz^2 + f_1(z) \left( dt^2 + d\rho^2 + \rho^2 d\theta^2 + dx^2 \right) + f_2(z) R_1^2 \, d\Omega_2^2 \right),
\]
\[
(4.90)
\]
where \(d\Omega_2\) is line element over \(S^2\) and now \(f_1, f_2\) become
\[
f_1 = 1 + \frac{z^2}{20R_1^2} + \frac{z^4}{100R_1^4} + \ldots,
\]
\[
f_2 = 1 - \frac{z^2}{5R_1^2} - \frac{7z^4}{800R_1^4} + \ldots.
\]
\[
(4.91)
\]
The $k \times R^1 \times S^2$ geometry is given by $\theta \in [-\Omega, \Omega], x \in [-\infty, \infty]$ and $\rho \in [0, \infty]$. We put IR cut-offs on $x$ and $\rho$ directions such that $x \in [-\tilde{H}/2, \tilde{H}/2]$ and $\rho \in [\rho_m, H]$, where $\rho_m$ is related to the UV cut-off $\delta$. We choose $(z, \theta, x, \xi_0)$ as induced coordinates on the minimal area surface with $\rho = \rho(z, \theta)$. Then, the induced metric over the minimal area surface will become

$$h = \begin{bmatrix}
\frac{L^2}{z^2} (f_1 \rho^2 + 1) & \frac{L^2 f_1}{z^2} \dot{\rho} \dot{\rho}' & \frac{L^2 f_1}{z^2} (\dot{\rho}' + \rho^2) \\
\frac{L^2 f_1}{z^2} \dot{\rho} \dot{\rho}' & \frac{L^2 f_1}{z^2} (\dot{\rho}' + \rho^2) & \frac{L^2 f_1}{z^2} \\
\frac{L^2 f_1}{z^2} (\dot{\rho}' + \rho^2) & \frac{L^2 f_1}{z^2} & \frac{L^2 f_2 R^2_1}{z^2} \sin^2(\xi_0)
\end{bmatrix}, \quad (4.92)
$$

where $\dot{\rho} = \partial_\theta \rho$, $\rho' = \partial_z \rho$ and now the entanglement entropy is given by

$$S_6|_{k \times R^1 \times S^2} = \frac{8 \pi^2 L^5 R^2_1 \tilde{H}}{\ell^5_p} \int d\rho \int d\theta \frac{f_1 f_2}{z^5} \sqrt{\rho^2 + \rho'^2 (1 + f_4 \rho^2)}.$$  

(4.93)

Note that here we have already performed integration over $x$. In this case, the equation of motion for $\rho(z, \theta)$ becomes

$$0 = 2z f_1 f_2 (\rho^2 + \rho'^2) \rho'' + 2z f_2 (1 + f_1 \rho^2) \rho' - 2z f_2 (\rho^2 + 2 \rho'^2) \rho' + 2 \rho (2z f_2 f_1 + f_1 (5 f_2 + z f_2')) (\rho^2 + \rho'^2) \rho' - 2z f_1 f_2 \rho^2 \rho'^2 + f_1 \rho^3 (3 z f_2 f_1' + 2 f_1 (5 f_2 + z f_2')) \rho'^3.$$  

(4.94)

where $\rho' = \partial_\rho \rho$, $\rho'' = \partial^2_\rho \rho$ and $\rho' = \partial_z \rho \partial_\theta$ $\rho$. Once again, we can write $\rho = \rho_0 + \rho_1/R^2_1$ and keep only the leading order correction to the entanglement entropy. We can further series expand the equation of motion (4.94) to get equation of motion for $\rho_0$ and $\rho_1$. We find that, using these equations of motion and integration by parts, the entropy functional can be simplified to

$$S_6|_{k \times R^1 \times S^2} = \frac{8 \pi^2 L^5 R^2_1 \tilde{H}}{\ell^5_p} \int_{z_m}^\delta dz \int_{-\Omega}^{\Omega-\epsilon} d\theta \left[ \sqrt{\rho_0^2 + \rho_0'^2} \frac{(1 + \rho_0^2)}{z^5} - \frac{3 \left( \rho_0^3 + \rho_0' \rho_0 '' (6 + 5 \rho_0^2) \right)}{40 R^2_1 z^5 \sqrt{\rho_0^2 + \rho_0'^2 (1 + \rho_0^2)}} \right]$$

$$+ \frac{\partial}{\partial \theta} \left( \rho_0 \rho_1 \frac{z^2 \sqrt{\rho_0^2 + \rho_0'^2 (1 + \rho_0^2)}}{R^2_1} \right) + \frac{\partial}{\partial z} \left( \rho_0^2 \rho_0 \rho_1 \frac{z^2 \sqrt{\rho_0^2 + \rho_0'^2 (1 + \rho_0^2)}}{R^2_1} \right).$$  

(4.95)

Further, we can substitute $\rho_0 = z/h(\theta)$ and $\rho_1 = z^3 g_3(\theta)$. The equations of motion for $h$ and $g_3$ from (4.94) becomes

$$h \left(1 + h^2 \right) \ddot{h} + 5 h^2 \left(1 + h^2 \right) \left(5 + h^2 \right) = 0, \quad (4.96)$$

$$h^2 \left(1 + h^2 \right) 2 \ddot{g}_3 + 2h \left(9 + 11 h^2 + 2 h^4 \right) \dot{h} \dot{g}_3$$

$$- g_3 \left(25 + 45 h^2 + 21 h^4 + h^6 - \left(27 - h^2 + 2 h^4 \right) \dot{h}^2 \right) = S_1, \quad (4.97)$$

where source terms for $g_3$ are

$$S_1 = \frac{\left(10 + 19 h^2 + 9 h^4 + \left(4 + 9 h^2 \right) \dot{h}^2 \right)}{20 h}.$$

(4.98)
Note that homogeneous part of the equations of motion for $h$ and $g_3$ are same as the case of kink in $R^3 \times S^3$. Similar to (4.8), we can further integrate the equation of motion for $h$ to get a conserved quantity

$$K_6 = \frac{(1 + h^2)^{5/2}}{h^5 \sqrt{1 + h^2 + \hat{h}^2}}. \quad (4.99)$$

The simplified expression for the entanglement entropy then becomes

$$S_0|_{k \times R^1 \times S^2} = \frac{16 \pi^2 L^5 \hat{H} R_1^2}{\ell_p^5} \left( I_1 + \frac{I_2}{R_1^2} + \frac{I_3}{R_1^2} + \frac{I_4}{R_1^2} \right), \quad (4.100)$$

where

$$I_1 = \int_{z_m}^{\delta} dz \int_{h_0}^{h_{1c}} \frac{dh}{h} \sqrt{1 + h^2 + \hat{h}^2}, \quad (4.101)$$

$$I_2 = - \int_{z_m}^{\delta} dz \int_{h_0}^{h_{1c}} \frac{dh}{h} \frac{(5 + 6h^2 + 6\hat{h}^2)}{40h^2 \sqrt{1 + h^2 + \hat{h}^2}}, \quad (4.102)$$

$$I_3 = - \int_{z_m}^{\delta} dz \int_{h_0}^{h_{1c}} \frac{d\hat{h}}{\sqrt{1 + h^2 + \hat{h}^2}} + \frac{g_3 h}{h \sqrt{1 + h^2 + \hat{h}^2}}, \quad (4.103)$$

$$I_4 = - \int_{z_m}^{\delta} dz \int_{h_0}^{h_{1c}} \frac{d\hat{h}}{h} \frac{g_3}{h \sqrt{1 + h^2 + \hat{h}^2}}. \quad (4.104)$$

Now we can convert the equation of motion for $g_3(\theta)$ into equation of motion for $g_3(h)$ and solve it perturbatively near the boundary in the limit $h \to 0$. We find that

$$g_3 = \frac{b_3}{h^3} + \frac{1 + 140b_3}{80h} + \frac{1}{64} (1 + 36b_3) h - \frac{1}{288} (1 + 36b_3) h^3 + \frac{5 + 180b_3}{4608} h^5 + O(h^7), \quad (4.105)$$

where constant $b_3$ is fixed by the condition $g_3(h_0) = 0$. Using this in $\rho = z/h + z^3 g_3/R_1^2$ and evaluating at $z = \delta$, $\rho = H$ and $h = h_{1c}(\delta)$, we find

$$h_{1c}(\delta) = \frac{1}{H} + \frac{b_3 H}{R_1^2} \delta + \frac{1 + 140b_3}{80HR_1^2} \delta^2 + \frac{(1 + 36b_3) \delta^5}{64HR_1^2} + O(\delta^6). \quad (4.106)$$

Note that we have only kept the terms of order $1/R_1^2$ in the above expansion. Now for small $h$, the integrands in $I_i$’s behave as

$$\frac{\sqrt{1 + h^2 + \hat{h}^2}}{h^2} \sim - \frac{1}{h^2} - \frac{1}{2} K_6^2 h^8 + O(h^9),$$

$$\frac{5 + 6h^2 + 6\hat{h}^2}{40h^2 \sqrt{1 + h^2 + \hat{h}^2}} \sim \frac{3}{20h^2} + \frac{1}{20} K_6^2 h^8 + O(h^9),$$

$$\frac{g_3 \hat{h}}{\sqrt{1 + h^2 + \hat{h}^2}} \sim \frac{b_3}{h^3} + \frac{1 + 140b_3}{80h} + \frac{1 + 36b_3}{64} h - \frac{(1 + 36b_3) h^3}{288} + \frac{(1 + 36b_3) h^3}{64} + O(h^5),$$

$$\frac{g_3}{hh \sqrt{1 + h^2 + \hat{h}^2}} \sim b_3 K_6^2 h^6 + O(h^7). \quad (4.107)$$
Using these, we can make $\theta$ integrals in $I_1$ and $I_2$ finite by separating the divergences:

$$I_1 = \int_{z_m}^{h_{1c}} \frac{dz}{z^2} \int_{h_0}^{h_{1c}} dh \left( \frac{\sqrt{1 + h^2 + \tilde{h}^2}}{hh^2} + \frac{1}{h^2} \right) + \int_{z_m}^{h_{1c}} \frac{dz}{z^2} \left( \frac{1}{h_{1c}(z)} - \frac{1}{h_0} \right)$$

$$= I'_1 + I'_2, \quad (4.108)$$

$$I_2 = -\int_{z_m}^{h_{1c}} \frac{dz}{z^2} \int_{h_0}^{h_{1c}} dh \left( \frac{5 + 6h^2 + 6\tilde{h}^2}{40h^2 \sqrt{1 + h^2 + \tilde{h}^2}} + \frac{3}{20h^2} \right) - \frac{3}{20} \int_{z_m}^{h_{1c}} \frac{dz}{z^2} \left( \frac{1}{h_{1c}(z)} - \frac{1}{h_0} \right)$$

$$= I'_3 + I'_4. \quad (4.109)$$

In (4.108), $I'_1$ and $I'_2$ are first and second integrals in $I_1$ and similarly in (4.109), $I'_3$ and $I'_4$ represent first and second integrals in $I_2$. Now we can further differentiate $I_i$'s with respect to $\delta$ and Taylor expand the expressions to separate the divergences:

$$\frac{dI'_1}{d\delta} = \frac{1}{\delta^4} \int_{h_0}^{h_1} dh \frac{\sqrt{1 + h^2 + \tilde{h}^2}}{hh^2} - \frac{K_6^2 (9b_3H^2 + R_1^2)}{2H^9R_1^2} \delta^5 + O(\delta^6), \quad (4.110)$$

$$\frac{dI'_2}{d\delta} = \frac{H}{\delta^5} \left( 1 - \frac{b_3H^2}{R_1^2} \right) - \frac{1}{h_0^4 \delta^4} - \frac{(1 + 140b_3) H}{80R_1^3 \delta^5} - \frac{1 + 36b_3}{64H R_1^2 \delta} + O(\delta^6), \quad (4.111)$$

$$\frac{dI'_3}{d\delta} = -\frac{1}{\delta^2} \int_{h_0}^{h_1} dh \left( \frac{5 + 6h^2 + 6\tilde{h}^2}{40h^2 \sqrt{1 + h^2 + \tilde{h}^2}} \right) + \frac{K_6^2 (9b_3H^2 + R_1^2)}{20H^9R_1^2} \delta^5 + O(\delta^{10}), \quad (4.112)$$

$$\frac{dI'_4}{d\delta} = -\frac{3H}{20\delta^5} \left( 1 - \frac{b_3H^2}{R_1^2} \right) + \frac{3}{20h_0^5 \delta^2} + \frac{3 (3 + 140b_3) H}{1600R_1^5 \delta} + O(\delta). \quad (4.113)$$

Also, we can simplify the boundary terms in a similar fashion and find that

$$\frac{dI_3}{d\delta} = \left( 1 - \frac{3b_3H^2}{R_1^2} \right) \frac{b_3H^3}{\delta^5} + \left( 1 - \frac{4b_3H^2}{R_1^2} \right) H (1 + 140b_3) \left( \frac{h_0}{64H} - \frac{(1 + 480b_3 + 26800b_3^2) H}{6400R_1^2} \right) \frac{1}{\delta} + O(\delta^0), \quad (4.114)$$

$$\frac{dI_4}{d\delta} = -\frac{1}{\delta^2} \int_{h_0}^{h_1} dh \frac{g_3}{hh^2 \sqrt{1 + h^2 + \tilde{h}^2}} + O(\delta^5). \quad (4.115)$$

Using (4.100) and (4.108)-(4.115), we find new divergence of order $1/\delta^3$ and $1/\delta$, and the universal term in the entanglement entropy

$$S_6^{\log} = \left|_{k \times R^1 \times S^2} \frac{16 \pi^2 L^5 \tilde{H}}{\ell_s^6} \right( 0 + O \left( \frac{1}{R_1^2} \right) ) \log(\delta). \quad (4.116)$$

Once again, we find that leading contribution in logarithmic divergence disappears, which is consistent with the fact that there is no contribution from the singularity with odd dimensional locus. In the next subsection, we will consider some more geometries with conical singularities to push our hypothesis.
4.2.2 Crease $c_n \times \Sigma$

In this section, we will calculate EE for conical singularities of the form $c_n \times S^m$. We will mainly consider following singular geometries: $c_1 \times S^1$, $c_1 \times S^2$, $c_1 \times S^3$, $c_2 \times S^1$ and $c_2 \times S^2$.

The case with $\{n, m\} = \{1, 1\}$ will turn out to be trivial and it will be straightforward to see that there is no new contribution to the log$^2$ contribution. For $\{n, m\} = \{1, 2\}$, we will find that there is a log$^2$ contribution in EE which had disappeared when the locus was taken to be flat in (4.29). For $\{n, m\} = \{1, 3\}$, we will see that there is no log$^2$ contribution from the singularity as the locus is odd dimensional. Finally for $\{n, m\} = \{2, 1\}$ and $\{2, 2\}$, we will find that entanglement entropy contains a logarithmic divergence. In $\{n, m\} = \{2, 1\}$, this contribution is actually coming from the smooth part of the surface as $d = 6$ and logarithmic contribution from the trace anomaly is non-zero. In $\{n, m\} = \{2, 2\}$, the logarithmic contribution comes from the singularity and this is consistent with the idea that for even dimensional curved locus, the singularity in odd $d$ will contribute through a logarithmic divergence.

To begin with, we consider the simplest case with $m = 1$. In this case, the background geometry for the CFT is $R^4 \times S^1$. Then, the dual bulk geometry is given by

$$
\frac{ds^2}{z^2} = \frac{L^2}{z^2} (dz^2 + f_1(z) (dt^2 + \rho^2 d\theta^2 + \rho^2 \sin^2(\theta) d\phi^2) + f_2(z) R_1^2 d\xi^2) ,
$$

(4.117)

where $f_1 = 1 + O(1/R_1^6)$ and $f_2 = 1 + O(1/R_1^6)$. For this bulk, we consider the singular surface given by $\theta \in [0, \Omega]$, $\phi \in [0, 2\pi]$, $\rho \in [0, H]$ and $\xi^0 \in [0, 2\pi]$. For the minimal area surface which gives us the entanglement entropy, we assume that $\rho = \rho(z, \theta)$. Then, EE is given by

$$
S_{5 \mid c_1 \times S^1} = \frac{4 \pi^2 L^4 R_1}{\ell_\text{pl}^4} \int dz \ d\theta \ \frac{\sin(\theta) \rho f_1 \sqrt{f_2}}{z^4} \sqrt{\rho^2 + \rho^2(1 + f_1 \rho^2)} ,
$$

(4.118)

where $\dot{\rho} = \partial_\theta \rho$ and $\rho' = \partial_\rho \rho$. Using this, we can find the equation of motion for $\rho(z, \theta)$ to be

$$
0 = 2z f_1 f_2 \sin(\theta) \rho^2 (\rho^2 + \rho') \rho'' + 2z f_2 \sin(\theta) \rho^2 (1 + f_1 \rho^2) \dot{\rho} - 4z f_1 f_2 \sin(\theta) \rho^3 \rho' \rho' \rho' \\
+ 2z f_1 f_2 \rho^2 (-2 \sin(\theta) \rho + \cos(\theta) \rho) \rho' \rho^2 + f_1 \sin(\theta) \rho^4 \left(3 z f_2 f_1' + f_1 (-8 f_2 + z f_2') \right) \rho^3 \\
+ 2z f_2 \left(\cos(\theta) \rho (\rho' \rho^2) - \sin(\theta) \rho (2 \rho^2 + 3 \rho') \right) \\
+ \sin(\theta) \rho^2 \left(4 z f_2 f_1' + f_1 (-8 f_2 + z f_2') \right) \rho (\rho' \rho^2) \rho' .
$$

(4.119)

As the conical singularity has a one dimensional locus, we expect that there will be no logarithmic contribution from the singularity. Now we substitute $\rho = \rho_0 + \rho_1/R_1$ in (4.118) and find the leading order correction to the entanglement entropy. However, when we use the equation of motion for $\rho_0$ and simplify the entropy functional, the contribution depending on the coefficients of leading order terms of $f_1$ and $f_2$ will vanish because $f_1 = 1 + O(1/R_1^6)$ and
For this case, the bulk metric is given by

\[ f_2 = 1 + O(1/R_1^6). \]

So we find that simplified expression has following form

\[
S_5|_{c_1 \times S^1} = \frac{4 \pi^2 L^4 R_1}{\ell_p^4} \int d\theta \left[ \frac{\sin(\theta) \rho_0}{z^4} \sqrt{\rho_0^2 + \rho_1^2(1 + \rho_1^2)} + \frac{1}{R_1^2} \frac{\partial}{\partial \theta} \left( \frac{\rho_0^3 \rho_0 \rho_1}{z^4 \sqrt{\rho_0^2 + \rho_1^2(1 + \rho_1^2)}} \right) \right].
\]

Now we can insert the ansatz \( \rho_0 = z/h(\theta) \) and \( \rho_1 = z^3 g_3(\theta) \) in the functional and find that

\[
S_5|_{c_1 \times S^1} = \frac{4 \pi^2 L^4 R_1}{\ell_p^4} \left[ \int_{z_m}^\delta \frac{dz}{z^2} \int_{h_0}^{h_{1c}(z)} \frac{d\theta}{h} \sin(\theta) \sqrt{1 + h^2 + \frac{h^2}{h^3}} - \frac{1}{R_1^2} \int_{z_m}^\delta \frac{dz}{h} \frac{g_3 \sin(\theta) h}{h \sqrt{1 + h^2 + \frac{h^2}{h^3}} \h_{1c}(z)} \right].
\]

Also from (4.119), the equations of motion for \( h \) and \( g_3 \) reduces to

\[
0 = \ddot{h} \left( 1 + h^2 \right) \sin(\theta) + \cos(\theta) h \dot{h} h^3 + \left( 4 + h^2 \right) \sin(\theta) h^2 + \cos(\theta) \left( 1 + h^2 \right) h \dot{h} + 2 \left( 2 + 3h^2 + h^4 \right) \sin(\theta)
\]

\[
0 = h^2 \left( 1 + h^2 \right)^2 \sin(\theta) \dot{g}_3 + h \left( 1 + h^2 \right) \left( 2 \left( 8 + 3h^2 \right) \sin(\theta) \dot{h} + \cos(\theta) h \left( 1 + h^2 + 3\dot{h}^2 \right) \right) \dot{g}_3 - \left( 2 \left( 1 + h^2 \right) \left( 10 + 8h^2 + h^4 \right) \sin(\theta) + 3 \left( 8 + 3h^2 + h^4 \right) \sin(\theta) h^2 + 2 \cos(\theta) h \left( 4 + h^2 \right) \dot{h}^3 \right) g_3.
\]

Now, we find that \( g_3 \) is not sourced by \( h \) and hence, a homogeneous solution \( g_3 = 0 \) will be the exact solution for this case. This implies that excluding the first term in the above expression, all the other terms will vanish. As \( g_3 \) is zero, there will not be any new contribution to the limits of the integrations either. This result is consistent with the idea that singularity will contribute in EE only if the locus is curved and even dimensional.

As a next example, we consider the singular geometry \( c_1 \times S^2 \) in CFT background \( R^4 \times S^2 \). For this case, the bulk metric is given by

\[
d s^2 = \frac{L^2}{z^2} \left( dz^2 + f_1(z) \left( dt^2 + d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2(\theta) d\phi^2 \right) + f_2(z) R_1^2 d\Omega_2^2 \right),
\]

where \( d\Omega_2^2 = d\xi_0^2 + \sin^2(\xi_0) d\xi_0^2 \) is metric over unit two-sphere and

\[
f_1 = 1 + \frac{z^2}{20 R_1^2} + \frac{z^4}{100 R_1^4} + \ldots , \quad f_2 = 1 - \frac{z^2}{5 R_1^2} - \frac{7 z^4}{800 R_1^4} + \ldots .
\]

Now the entanglement entropy is given by

\[
S_6|_{c_1 \times S^2} = \frac{16 \pi^3 L^5 R_1^2}{\ell_p^5} \int dz d\theta \frac{f_1 f_2 \sin(\theta) \rho \sqrt{\rho^2 + \rho^2(1 + f_1 \rho^2)}}{z^5},
\]

\[ -37 - \]
and equation of motion for $\rho(z, \theta)$ becomes

\[
0 = 2z f_1 f_2 \sin(\theta) \rho^2 (\rho^2 + \rho'') \rho'' + 2z f_2 \sin(\theta) \rho^2 (1 + f_1 \rho^2) \dot{\rho} - 4 z f_1 f_2 \sin(\theta) \rho^2 \dot{\rho} \rho' \rho' \\
+2z \cos(\theta) f_2 \dot{\rho} (\rho^2 + \rho^2 (1 + f_1 \rho^2)) + \sin(\theta) \rho \left( -2 z f_2 (2 \rho^2 + 3 \rho^2) \\
+2 \rho (2 z f_2 f_1' + f_1 (-5 f_2 + z f_2')) (\rho^2 + \rho^2) \rho' - 4 z f_1 f_2 \rho^2 \rho^2 \\
+f_1 \rho^3 (3 z f_2 f_1' + 2 f_1 (-5 f_2 + z f_2')) \rho^3 \right). 
\] (4.127)

Now we can plug in the ansatz $\rho = \rho_0 + \rho_1 / R_1^2$ in the entropy functional and equation of motion. We can Taylor expand the entropy functional in $R_1$ and keep the leading correction.

In this leading correction, we can use the integration by parts and the equation of motion for $\rho_0$ to simplify the entropy functional to

\[
S_6|_{c_1 \times S^2} = \frac{16 \pi^3 L^5 R_1^2}{\ell_p^5} \int_{z_m}^{z} dz \int_{0}^{\Omega - \epsilon} d\theta \left[ \frac{\sin(\theta) \rho_0}{z^5} \sqrt{\rho_0^2 + \rho_0^2 (1 + \rho_0^2)} - \frac{\sin(\theta) \rho_0 (6 \rho_0^2 + \rho_0^2 (6 + 5 \rho_0^2))}{40 R_1^2 z^3 \sqrt{\rho_0^2 + \rho_0^2 (1 + \rho_0^2)}} \right] \\
- \frac{\partial}{\partial z} \left( \frac{\sin(\theta) \rho_0^3 \rho_0 \rho_1}{z^5 R_1 \sqrt{\rho_0^2 + \rho_0^2 (1 + \rho_0^2)}} \right) \bigg|_{\rho = \rho_0}
\] (4.128)

We can further use $\rho_0 = z / h(\theta)$ and $\rho_1 = z^2 g_3(\theta)$ and find that

\[
S_6|_{c_1 \times S^2} = \frac{16 \pi^3 L^5 R_1^2}{\ell_p^5} \left[ I_1 + \frac{I_2}{R_1^2} + \frac{I_3}{R_1^2} \right],
\] (4.129)

where

\[
I_1 = \int_{z_m}^{z} \frac{dz}{z^3} \int_{h_0}^{h_{z}(z)} d\tilde{h} \frac{\sin(\theta) \sqrt{1 + h^2 + \tilde{h}^2}}{h^3}, 
\] (4.130)

\[
I_2 = -\int_{z_m}^{z} \frac{dz}{z} \int_{h_0}^{h_{z}(z)} d\tilde{h} \frac{\sin(\theta) \left( 5 + 6 h^2 + 6 \tilde{h}^2 \right)}{40 \dot{h} h^3 \sqrt{1 + h^2 + \tilde{h}^2}}, 
\] (4.131)

\[
I_3 = -\int_{z_m}^{z} \frac{dz}{z} \frac{g_3(\theta) \dot{h}}{h \sqrt{1 + h^2 + \tilde{h}^2}} \bigg|_{h = h_{z}(z)}. 
\] (4.132)

Note that term with derivative with respect to $z$ in (4.128) vanishes. Now using the above ansatz in (4.127), we can also find the equations of motion for $h$ and $g_3$. For these equations, we can make a change of variable from $\theta$ to $y = \sin(\theta)$ and find

\[
y(1 - y^2) h(1 + h^2) \ddot{h} + (1 - y^2) h^3 + y(1 - y^2)(5 + h^2) \dot{h}^2 \\
+(1 - 2 y^2) h(1 + h^2) \dot{h} + y(1 + h^2)(5 + 2 h^2) = 0
\] (4.133)

\[
y(1 - y^2) h^2(1 + h^2) \ddot{g}_3 + h \left( -(-1 + 2 y^2) h (1 + h^2)^2 - 6 y (-1 + y^2) (3 + 4 h^2 + h^4) \right) \dot{h} \\
+3 (-1 + y^2)^2 h (1 + h^2) \dot{h}^2 \ddot{g}_3 + \left(-y (1 + h^2) (25 + 21 h^2 + 2 h^4) \right) \dot{h} \\
-3 y (-1 + y^2) (9 + 2 h^2 + h^4) \dot{h}^2 + 2 (-1 + y^2)^2 h (4 + h^2) \dot{h}^3 \ddot{g}_3 = S_1,
\] (4.134)
where $S_1$ is the source terms and it is given by

$$S_1 = \frac{y (1 + h^2) (10 + 9h^2) + 4y (1 - y^2) (1 + 2h^2) h^2 + (1 - y^2)^2 hh^3}{20h}. \quad (4.135)$$

Now to separate the logarithmic divergence, we want to find the asymptotic behavior of integrand in terms of $h$, where $h \to 0$. So we invert equation (4.133) using $\dot{h}(y) = -(1 - y^2) y'' + yy'^3$ and $\dot{y}(y) = \sqrt{1 - y^2/y'}$, where on the right hand side we have $y = y(h)$ and $y' = dy/dh$. We can also change the independent variable in (4.70) from $y$ to $h$. Apart from previous two relations, we also use

$$\dot{g}_3(\theta) = \frac{\sqrt{1 - y'^2}}{y'} \quad \dot{y}_3(\theta) = \frac{\sqrt{1 - y'^2}}{y'} \frac{dy}{dh} \left( \frac{\dot{g}_3(\theta)}{\sqrt{1 - y'^2}} \right). \quad (4.136)$$

Finally, we can rewrite (4.133) and (4.134) as

$$h (1 + h^2) y (1 - y^2) y'' - (5 + 7h^2 + 2h^4) yy'^3 \quad (4.137)$$

$$-h (1 + h^2) (1 - 2y^2) y' - (5 + h^2) y (1 - y^2) y' - h (1 - y^2)^2 = 0$$

$$h^2 (1 + h^2) (1 - y^2) y' g_3 + h (1 + h^2) (2h (1 - y^2) + (13 + 5h^2) y (1 - y^2) y'$$

$$- (1 + h^2) (5 + 2h^2) yy'^3) \dot{g}_3 + (2h (4 + h^2) (1 - y^2)^2 + 3 (9 + 2h^2 + h^4) y (1 - y^2) y'$$

$$- (1 + h^2) (25 + 21h^2 + 2h^4) yy'^3) g_3 = S_2,$$

where now $\dot{g}_3 = dg_3/dh$ and

$$S_2 = \frac{h (1 - y^2)^2 - 4 (1 + 2h^2) y (1 - y^2) y' - (1 + h^2) (10 + 9h^2) yy'^3}{20h}. \quad (4.139)$$

Now we can try to solve these equations perturbatively in terms of $h$ near the boundary, where $h$ is small. As $y = \sin(\Omega)$ at $h = 0$, we find the solution

$$y = \sin(\Omega) - \frac{\cos(\Omega) \cot(\Omega)}{8} h^2 + \frac{1}{512} (2 \csc(\Omega) - 7 \csc^2(\Omega) + 5 \sin(\Omega)) H^4 + \ldots. \quad (4.140)$$

Using this in (4.138) and solving it perturbatively, we find that

$$g_3 = -\frac{1}{20H} + \frac{1 + \csc^2(\Omega)}{96} h \log(h) + b_3 h + \ldots, \quad (4.141)$$

where $b_3$ is a constant and it is fixed by the condition $\dot{g}_3(0) = 0$. Now, we evaluate the expression $\rho = z/h + z^3 g_3/R_1^2$ at $z = \delta$, $\rho = H$ and $h(z) = h_{1_c}$. In this relation, we use (4.141) and invert it to find

$$h_{1_c} = \frac{\delta}{H} - \frac{\delta^3}{20HR_1^2} + \frac{(96b_3 + (1 + \csc^2(\Omega)) \log(\delta/H)) \delta^5}{96H^3R_1^2} + O(\delta^6). \quad (4.142)$$
Now, we can use (4.140) and (4.141) in integrands of $I_1$ and $I_2$ to find their behavior near the boundary:

\[
\frac{\sin(\theta) \sqrt{1 + h^2 + \dot{h}^2}}{hh^3} \sim -\frac{\sin(\Omega)}{h^3} + \frac{3 \cos(\Omega) \cot(\Omega)}{32h} - \frac{3h(13 - 19 \cos(2\Omega)) \cot^2(\Omega) \csc(\Omega)}{4096} + O(h^3)
\]

\[
\frac{\sin(\theta) \left( 5 + 6h^2 + 6\dot{h}^2 \right)}{40hh^3\sqrt{1 + h^2 + \dot{h}^2}} \sim -\frac{3 \sin(\Omega)}{20h^3} + \frac{\cos(\Omega) \cot(\Omega)}{64h} - \frac{h(67 - 157 \cos(2\Omega)) \cot^2(\Omega) \csc(\Omega)}{81920} + O(h^3)
\]

Using these, we can make the integrations in $I_1$ and $I_2$ finite and write in the form

\[
I_1 = \int_{z_m}^{\delta} \int_{h_0}^{h_{1c}(z)} dh \left[ \frac{\sin(\theta) \sqrt{1 + h^2 + \dot{h}^2}}{hh^3} + \frac{\sin(\Omega)}{h^3} - \frac{3 \cos(\Omega) \cot(\Omega)}{32h} + \frac{3h(13 - 19 \cos(2\Omega)) \cot^2(\Omega) \csc(\Omega)}{4096} \right]
\]

\[
+ \int_{z_m}^{\delta} \int_{h_0}^{h_{1c}(z)} dh \left( -\frac{1}{h_{1c}^2} + \frac{1}{h_0^2} \right) + \frac{3 \cos(\Omega) \cot(\Omega) \log(h_{1c}/h_0)}{8192} + \frac{3 \left( h_0^2 - h_{1c}^2 \right) (13 - 19 \cos(2\Omega)) \cot^2(\Omega) \csc(\Omega)}{8192}
\]

\[
= I'_1 + I'_2 \tag{4.144}
\]

\[
I_2 = -\int_{z_m}^{\delta} \int_{h_0}^{h_{1c}(z)} dh \left[ \frac{\sin(\theta) \left( 5 + 6h^2 + 6\dot{h}^2 \right)}{40R_1^2 h^3 \sqrt{1 + h^2 + \dot{h}^2}} + \frac{3 \sin(\Omega)}{20h^3} - \frac{\cos(\Omega) \cot(\Omega)}{64h} + \frac{(67 - 157 \cos(2\Omega)) \cot^2(\Omega) \csc(\Omega)h}{81920} \right]
\]

\[
+ \int_{z_m}^{\delta} \int_{h_0}^{h_{1c}(z)} dh \left( -\frac{1}{h_0^2} + \frac{1}{h_{1c}^2} \right) + \frac{\cos(\Omega) \cot(\Omega) \log(h_0/h_{1c})}{64} - \frac{(h_0^2 + h_{1c}^2) (67 - 157 \cos(2\Omega)) \cot^2(\Omega) \csc(\Omega)}{163840}
\]

\[
= I'_3 + I'_4 \tag{4.145}
\]

In (4.144) and (4.145), $I'_1$ and $I'_2$ are first and second integrals in $I_1$ and $I'_3$ and $I'_4$ are first and second integrals in $I_2$. Now we can take derivatives of $I'_i$’s with respect to $\delta$, and then
Taylor expand the terms to find:

\[
\frac{d I_1'}{d \delta} = \frac{1}{\delta^3} \int_{h_0}^{0} dh \left[ \frac{\sin(\theta)}{\delta} \sqrt{1 + h^2 + \dot{h}^2} + \frac{\sin(\Omega)}{h^3} - \frac{3 \cos(\Omega) \cot(\Omega)}{32h} + \frac{3h(13 - 19 \cos(2\Omega) \cot(\Omega) \csc(\Omega))}{4096} \right. \\
\left. + \frac{3 \left( \csc(\Omega) + 11 \csc^3(\Omega) + 3 \csc^5(\Omega) - 15 \sin(\Omega) \right)}{8192H^4} \right] \delta \log(\delta/H) + \ldots ,
\]

\[
\frac{d I_2'}{d \delta} = \frac{H^2 \sin(\Omega)}{2\delta^5} + \ldots - \frac{\csc(\Omega) + \sin(\Omega) \log(\delta/H)}{96 R_1^2} + O(1/\delta) ,
\]

\[
\frac{d I_3'}{d \delta} = -\frac{1}{\delta} \int_{h_0}^{0} dh \left[ \frac{\sin(\theta)}{40R_1^2 h^3 \sqrt{1 + h^2 + \dot{h}^2}} \left( 5 + 6h^2 + 6\dot{h}^2 \right) + \frac{3 \sin(\Omega)}{20h^3} - \frac{\cos(\Omega) \cot(\Omega)}{64h} \right. \\
\left. + \frac{(67 - 157 \cos(2\Omega)) \cot^2(\Omega) \csc(\Omega) h}{81920} \right] + O(\delta^3) ,
\]

\[
\frac{d I_4'}{d \delta} = -\frac{3H^2 \sin(\Omega)}{40\delta^3} - \frac{\cos(\Omega) \cot(\Omega) \log(\delta)}{64} + O(1/\delta) ,
\]

\[
\frac{d I_3}{d \delta} = -\frac{H^2 \sin(\Omega)}{20\delta^3} + \frac{(3 - 2 \cos(2\Omega)) \csc(\Omega) \log(\delta)}{192} + O(1/\delta) .
\]

(4.146)

Now using (4.144)-(4.146) in (4.129), we find that the double log contribution in the EE becomes

\[
S_{\log}^6 \bigg|_{c_1 \times S^2} = -\frac{\pi^3 L^5 \cos(\Omega) \cot(\Omega)}{8 \ell_p^5} \log(\delta)^2 .
\]

(4.147)

Note that there are other contribution to the double log term but they are of higher order in $R_1$. As the only other dimensionful quantity in the problem is the IR cut-off $H$, these terms will be of the form $O(H^2/R_1^2)$. Interestingly, all such terms, which scale with $H$ are contributions from the smooth part of the entangling surface. Hence (4.147) is the complete contribution from the singularity alone. In section 4.1.2, we saw that there was no such double logarithmic term when the locus of the singularity was either flat or it is odd dimensional. Hence, (4.147) is also consistent with the idea that similar to (2.6), generically the contribution from the singularity should be of the following form

\[
S_{\text{univ}} \sim \int d^{2m} y \sqrt{\mathcal{R}^m} \log(\delta)^2 .
\]

(4.148)

Here $\sigma$ is the $2m$-dimensional locus of the singularity and $[\mathcal{R}^m]$ is the curvature invariants with $m$ powers of the curvatures.

Having seen the appearance of the $\log^2 \delta$ divergence for even dimensional curved locus, now we turn towards odd dimensional locus. So we consider EE for geometry $c_1 \times S^3$. For this case, the calculations proceed in a similar fashion and we find that, near the boundary
$y = \sin(\theta)$ and $g_3$ in terms of $h$ are given by

\[
y(h) = \sin(\Omega) - \frac{1}{10} \cos(\Omega) \cot(\Omega) h^2 - \frac{(63 - 17 \cos(2\Omega)) \cot^2(\Omega) \csc(\Omega) h^4}{6000} + \frac{(373 \csc(\Omega) - 1853 \csc^3(\Omega) - 889 \csc^5(\Omega) + 2369 \sin(\Omega)) h^6}{450000} + \mathcal{O}(h^7),
\]

\[
g_3(h) = -\frac{1}{12h} - \frac{h(39 - 11 \cos(2\Omega)) \csc^2(\Omega)}{2250} + b_3 h^2 + \mathcal{O}(h^3),
\]

where $b_3$ is a constant and it is fixed by the condition that $g_3$ has a minimum at $\theta = 0$. However, we will see that this constant is zero. If we calculate EE, we find that there is no $\log^2 \delta$ term but there is a log term in EE at the order next to the leading order in $R_1$:

\[
S_7^{\log} |_{c_1 \times S^3} = -\frac{8 \pi^4 L_6^6 R_1}{H \ell_6^6} b_3 \sin(\Omega) \log(\delta).
\]

(4.150)

Now, we are going to argue that this logarithmic term is not coming from the singularity. We can see that, if we set $\Omega = \pi/2$, the conical singularity in the entangling surface disappears. However, the logarithmic term in (4.150) is still non-zero. Hence, this contribution is actually coming from the non-singular part of the entangling surface. Further, as we are considering the EE in odd dimensional CFTs, there should not be any log term from the smooth part of the entangling surface. Hence, the contribution (4.150) should be zero and this is only if $b_3 = 0$. This result further assures that a singularity with odd dimensional locus does not contribute through a log or $\log^2 \delta$ term.

We can further calculate EE for singularity $c_2 \times S^1$ in $d = 6$. In this case, the singularity has an odd dimensional, curved locus which does not contribute in the universal term. However, as $d$ is even, we find that the smooth part of the surface contribute through a log term and it is given by

\[
S_6^{\log} |_{c_2 \times S^1} = \frac{\pi^4 L_5^5 R_1}{\ell_5^6} \frac{(7 - 9 \cos(2\Omega)) \cot^2(\Omega)}{16 H} \log(\delta/H).
\]

(4.151)

Further, there is a new divergent term of the order $1/\delta$ in this case. Note that this term matches with the logarithmic term in EE for $c_2 \times R^1$ in (4.30). Finally, we give results for the case where a conical singularity in odd dimensions have a curved, even dimensional locus. In background $R^5 \times S^2$, we can consider the singular geometry $c_2 \times S^2$ which contains a new divergence of order $1/\delta^2$. Further, in odd $d$ the smooth part of the entangling geometry does not contribute through a logarithmic term. However, a singularity will contribute through a log($\delta$) term if the locus of the singularity is even dimensional and curved. Precisely this is what we see in EE for $c_2 \times S^2$ and find following universal term in EE

\[
S_7^{\log} |_{c_2 \times S^2} = \frac{32 \pi^3 L_6^6}{\ell_6^6} \log(\delta) \left[ - \int_0^{h_0} dh \left( \frac{L_6}{h} - \frac{7 \sin^2 \Omega}{60 h^4} + \frac{\cos^2 \Omega}{25 h^2} \right) + \frac{7 \sin^2 \Omega}{180 h_0^3} - \frac{\cos^2 \Omega}{25 h_0} \right],
\]

(4.152)
where $\dot{h} = dh/d\theta$ and
\[
\mathcal{L}_0 = -\frac{\sin^2(\theta) \left(7 \dot{h}^2 + 7 h^2 + 6\right)}{60 h^4 \sqrt{\dot{h}^2 + h^2 + 1}}.
\] (4.153)

Note that similar to previous cases, $h$ is defined such that $h = z/h(\theta) + z g_3(\theta)/R_1^2$ and it is the solution of following equation of motion
\[
h(1 + h^2)\ddot{h} + 2 \cot^2(\theta) h \dot{h}^3 + 2 (h^2 + 3) \dot{h}^2 + 2 \cot(\theta) h_1 (1 + h_1^2) \dot{h} + 3 (2 + 3h^2 + h^4) = 0,
\] (4.154)

with $h_0 = h(0)$. The log divergence (4.152) is a contribution from the singularity and it is non-zero because the locus of the singularity is even dimensional and curved. So the examples in this section reaffirm that an extended singularity contributes in the cut-off independent terms through log or $\log^2 \delta$ terms only if its locus is even dimensional and curved.

5. Universal terms and the central charges

In previous sections, we calculated the EE for various surfaces and found that singularity produces new log $\delta$ and $\log^2 \delta$ terms in the EE. As it has been seen that the regulator independent coefficients contain central charges when considering smooth surfaces, it is natural to ponder what function of central charges appears in these new universal contributions arising from the singularities. In the calculations above, we have been working with CFTs which are dual to the Einstein gravity. For these CFTs, all the central charges are equal and there is no way to distinguish these in the universal term of the EE. It has long been known that to construct a holographic model where the various central charges are distinct from one another, the gravity action must include higher curvature interactions [32]. In part, this motivated the recent holographic studies of Gauss-Bonnet gravity [28] — for example, see [30, 31]. Hence we apply this approach here as a first step towards determining the dependence of the new universal contributions on the central charges of the dual CFT. In particular, we will calculate EE for some singular geometries in Gauss-Bonnet gravities below.

5.1 Singular embedding

In this section, we will discuss EE for cone geometry in $d = 4, 5, 6$ dimensional CFTs. We will first calculate the EE for $d = 4$ systematically and discuss the results for other cases.

For the Gauss-Bonnet gravity in $d + 1$ dimensions, the action is given by
\[
I_d = \frac{1}{2\ell_p^{d-2}} \int d^{d+1}x \sqrt{-g} \left[ R + \frac{d(d-1)}{L^2} + \frac{\lambda L^2}{(d-2)(d-3)} X_4 \right],
\] (5.1)

where
\[
X_4 = R_{abcd}R^{abcd} - 4 R_{ab} R^{ab} + R^2
\] (5.2)
corresponds to the Euler density on a four-dimensional manifold (and hence we must have $d \geq 4$ here). We have introduced $L$ as a canonical scale in the curvature-squared interaction
so that strength of this term is controlled by $\lambda$, a dimensionless coupling constant. We may write metric for the AdS vacuum solution as

$$ds^2 = \frac{\tilde{L}^2}{z^2} (dz^2 + dt^2 + d\rho^2 + \rho^2 (d\theta^2 + \rho^2 \sin^2(\theta) d\Omega_{d-3}^2)) .$$

Here the AdS curvature scale $\tilde{L}$ is related to canonical scale $L$ by the following relation

$$\tilde{L}^2 = L^2 / f_\infty$$

where $f_\infty = 1 - \sqrt{1 - 4\lambda^2 / \lambda}$.

Now in Gauss-Bonnet gravity, the holographic EE is again determined by a minimization problem over surfaces $m$ in the bulk geometry which match the entangling surface $\Sigma$ at the asymptotic AdS$_{d+1}$ boundary. However, the functional that is minimized is now given by [22, 27]

$$S_d = \frac{2\pi}{\ell_p^{d-1}} \int d^{d-1}x \sqrt{h} \left[ 1 + \frac{2\lambda L^2}{(d-2)(d-3)} R \right] ,$$

where $R$ is the Ricci scalar for the induced metric $h$.

The bulk theory can now be characterized in terms of two dimensionless couplings, the ratio $\tilde{L}/\ell_p$ and the new coupling constant $\lambda$. As a result, there are two distinct central charges which characterize the boundary CFT dual to Gauss-Bonnet gravity. A convenient choice for these which applies for any value of $d$ is, e.g., see [30]:

$$\tilde{C}_T = \frac{\pi^{d/2}}{\Gamma(d/2)} \left( \frac{\tilde{L}}{\ell_p} \right)^{d-1} \left[ 1 - 2\lambda f_\infty \right] ,$$

$$a_d^* = \frac{\pi^{d/2}}{\Gamma(d/2)} \left( \frac{\tilde{L}}{\ell_p} \right)^{d-1} \left[ 1 - 2\frac{d-1}{d-3} \lambda f_\infty \right] .$$

The physical role of these charges in the boundary theory is as follows: The central charge $\tilde{C}_T$ controls the leading singularity of the two-point function of the stress tensor and $a_d^*$ appears as the universal coefficient in the entanglement entropy of a sphere $S^{d-2}$ [33, 34]. The latter has also been shown to satisfy a holographic c-theorem in arbitrary dimensions. Below we will calculate the EE for various singular surfaces and would like to see if the universal terms have some simple dependence on these central charges. If we consider the case of $d = 4$, the above expressions simplify to

$$c = \tilde{C}_T = \frac{\pi^2}{\ell_p^3} (1 - 2\lambda f_\infty) \quad \text{and} \quad a = a_4^* = \frac{\pi^2}{\ell_p^3} (1 - 6\lambda f_\infty) .$$

Here, $c$ and $a$ are the standard central charges that appear in the trace anomaly or entanglement entropy, e.g., eq. (1.5), of the four-dimensional boundary CFT.

Now for calculation of the holographic EE, the cone geometry is defined by $\rho \in [0, H]$, $\theta \in [0, \Omega]$ and $\phi \in [0, 2\pi]$. We consider the induced coordinates to be $(\rho, \theta, \phi)$ and radial
coordinate \( z = z(\rho, \theta) \). For this case, the induced metric becomes

\[
L = \begin{pmatrix}
\frac{L^2}{z^2} (1 + (\partial_\rho z)^2) & \frac{L^2}{z^2} \partial_\rho z \partial_\theta z & 0 \\
\frac{L^2}{z^2} \partial_\rho z \partial_\theta z & \frac{L^2}{z^2} (\rho^2 + (\partial_\theta z)^2) & 0 \\
0 & 0 & \frac{L^2}{z^2} \rho^2 \sin^2(\theta)
\end{pmatrix}.
\]  
(5.9)

For this metric, the expression for Ricci scalar \( \mathcal{R} \) in (5.5) contains the terms like \( \partial_\rho^2 z \) and \( \partial_\theta^2 z \). However, it is straightforward to see that the equation of motion is still second order. This is because Gauss-Bonnet term is topological in nature. Further, we impose the UV cutoff at \( z = \delta \) and define \( \epsilon(\rho) \) such that at \( \theta = \Omega - \epsilon, z(\rho, \Omega - \epsilon) = \delta \). As the background geometry has scaling symmetry and apart from \( \rho \), there are no dimensionful quantities in the problem, the solution for \( z \) should be of the following form

\[ z = \rho h(\theta). \]  
(5.10)

Here \( h(\theta) \) is a function such that \( h(\Omega) = 0 \) and \( \dot{h}(0) = 0 \). Also, the maximum value of \( h(\theta) \) is \( h(0) = h_0 \). By plugging this ansatz in equation of motion for \( z(\rho, \theta) \), which we get by applying the variational principle on entropy functional (5.5) with \( d = 4 \), the equation of motion for \( h(\theta) \) turns out to be

\[
0 = h \left( 1 + h^2 \right) \left( 1 + h^2 \right) \sin(\theta) (1 + 4\lambda f_\infty) + 6\lambda \cos(\theta) h f_\infty \dot{h} + \sin(\theta) (1 - 2\lambda f_\infty) \dot{h}^2 + \cos(\theta) h (1 - 2\lambda f_\infty) \dot{h}^5 - \sin(\theta) (1 + 2\lambda f_\infty) \dot{h}\right)
\]

\[
+ h^2 \cos(\theta) h (1 + h^2 + \lambda (1 - 2h^2) f_\infty) \dot{h}^3 - 3 \sin(\theta) (1 + h^2) \left( 2 - h^2 + 2\lambda (1 + h^2) f_\infty \right) \dot{h}\right)
\]

\[
+ h^2 \cos(\theta) h (1 + h^2)^3 (1 + 4\lambda f_\infty) \dot{h} + (1 + h^2)^2 \sin(\theta) (3 + 2h^2 (1 + \lambda f_\infty))
\]  
(5.11)

Further, we can simplify the entropy functional using this equation of motion and find that

\[
S_4|_{c_1} = \frac{4\pi^2 L^3}{\ell_p^3} \int_0^H d\rho \int_{\delta/h_0}^{\delta/\rho} dh \frac{\sin(\theta) \mathcal{L}_1}{\mathcal{L}_2},
\]  
(5.12)

where we have changed the integration from \( \theta \) to \( h \) and

\[
\mathcal{L}_1 = \left( 1 + h^2 + \dot{h}^2 \right) \left( \sin(\theta) \left( 1 + h^2 + \dot{h}^2 \right) - 2\lambda f_\infty \left( h^2 \cos(\theta) \cot(\theta) + 4 \sin(\theta) \right) \dot{h}\right)
\]

\[
+ 2h^3 \cos(\theta) \dot{h} + \left( 1 + 2h^2 + 2 \sin(\theta) \right) + 4\lambda^2 f_\infty^2 \left( h^2 \cos(\theta) \cot(\theta) + 2 \sin(\theta) \right) \dot{h}\right)
\]

\[
- 2h \left( 3 - h^2 \right) \cos(\theta) \dot{h} + 2h^2 (1 + h^2 + \lambda f_\infty) \sin(\theta) \dot{h} + h^2 \left( 4 + 2h^2 + \lambda f_\infty \right) \dot{h}\right),
\]  
(5.13)

\[
\mathcal{L}_2 = h^3 \dot{h} \left( 1 + h^2 + \dot{h}^2 \right) \left( \sin(\theta) \left( 1 + h^2 + \dot{h}^2 \right) + 2\lambda f_\infty \left( h \cos(\theta) \dot{h} + \sin(\theta) \left( 2 + 2h^2 - \dot{h}^2 \right) \right) \right).
\]

Now we want to make \( h \) integrand in (5.12) finite. For that, we define \( y = \sin(\theta) \) and find \( y \) and \( \dot{h} \) in terms of \( h \) near the asymptotic boundary. For that, we convert (5.11) into
the equation of motion for $y$ with independent variable $h$. Solving this equation of motion perturbatively, we find that near the boundary

$$y(h) = \sin(\Omega) - \frac{1}{4} h^2 \cos(\Omega) \cot(\Omega) + \frac{1}{64} h^4 \log(h)(3 - \cos(2\Omega)) \cot^2(\Omega) \csc(\Omega) + c_0 h^4 + (5.14)$$

where we have used the condition $y(0) = \sin(\Omega)$ and $c_0$ is a constant which is fixed by the condition that $y'(h_0) = 0$. We can now assume $y = y(\theta)$ and $h = h(\theta)$ in (5.14) and then invert it to find

$$\dot{h}(\theta) = -\frac{2 \tan(\theta)}{h} - \frac{(3 - \cos(2\Omega)) \csc(2\Omega)}{2} h \log(h) - \frac{\sec(\Omega)(256c_0 \tan^2(\theta) - 5 \cos(2\Omega) \csc(\Omega) + 7)}{16} h \ldots. \quad (5.15)$$

Now using (5.14) and (5.15), we can find that near the boundary

$$\sin(\theta) L_1 \sim -\frac{\sin(\Omega) (1 - 6 \lambda f_\infty)}{h^3} + \frac{\cos(\Omega) \cot(\Omega) (1 - 2 \lambda f_\infty)}{8 h} + \mathcal{O}(h), \quad (5.16)$$

which can be used to write the EE as

$$S_4 \bigg|_{c_1} = \frac{4 \pi^2 \tilde{L}^3}{\tilde{L}_p^3} \left[ H^2 \sin(\Omega) (1 - 6 \lambda f_\infty) - \frac{1}{16} \cos(\Omega) \cot(\Omega) (1 - 2 \lambda f_\infty) \log^2(\delta/H) \right. \quad (5.17)
+ \left. \left( \frac{\sin(\Omega) (1 - 6 \lambda f_\infty)}{2 h_0^2} + \frac{1}{8} \log(h_0) \cos(\Omega) \cot(\Omega) (1 - 2 \lambda f_\infty) \right) \log(\delta/H) \right.
+ \left. \int_{h_0}^H d\rho \int_{h_0}^{\delta/\rho} dh \left( \frac{\sin(\theta) L_1}{\mathcal{L}_2} + \frac{\sin(\Omega) (1 - 6 \lambda f_\infty)}{h^3} - \frac{\cos(\Omega) \cot(\Omega) (1 - 2 \lambda f_\infty)}{8 h} \right) \right].$$

In the last term, the $h$ integration is finite in the limit $\delta \to 0$ and hence the leading order divergence is logarithmic. Hence, we write

$$S_4 \bigg|_{c_1} = \frac{4 \pi^2 \tilde{L}^3}{\tilde{L}_p^3} \left[ H^2 \sin(\Omega) (1 - 6 \lambda f_\infty) - \frac{1}{16} \cos(\Omega) \cot(\Omega) (1 - 2 \lambda f_\infty) \log^2(\delta/H) \right. \quad (5.18)
+ \left. \left( \frac{\sin(\Omega) (1 - 6 \lambda f_\infty)}{2 h_0^2} + \frac{1}{8} \log(h_0) \cos(\Omega) \cot(\Omega) (1 - 2 \lambda f_\infty) \right) \log(\delta/H) \right.
+ \left. \log(\delta/H) \int_0^{h_0} dh \left( \frac{\sin(\theta) L_1}{\mathcal{L}_2} + \frac{\sin(\Omega) (1 - 6 \lambda f_\infty)}{h^3} - \frac{\cos(\Omega) \cot(\Omega) (1 - 2 \lambda f_\infty)}{8 h} \right) + \mathcal{O}(\delta^0) \right].$$

Now we can compare the coefficient of the $\log^2 \delta$ divergence with the central charge (5.6) and find that

$$S_4^{\log^2} \bigg|_{c_1} = -\frac{\tilde{C}_T}{4} \cos(\Omega) \cot(\Omega) \log^2(\delta/H) \quad (5.19)
= -\frac{c}{4} \frac{\cos^2(\Omega)}{\sin(\Omega)} \log^2(\delta/H),$$

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where the second equation follow from (5.8). So we find that for EE of the cone $c_1$ in $d = 4$, the new universal term arising from the singularity is directly proportional to the central charge $\tilde{C}_T$ and has a relatively simple dependence on the opening angle $\Omega$. We will return to discussing this result in more detail in section 6.

We can also calculate the EE for five-dimensional cone in Gauss-Bonnet gravity. For this case, the action for bulk geometry is (5.1) with $d = 5$ and canonical scale $L$ is related to $\tilde{L}$ by relation (5.4). Now the complete expression of entanglement entropy is given by (B.8) and the universal term is

$$S_{5}^{\log} \Big|_{c_2} = - \frac{8 \pi^2 \tilde{L}^4}{\ell_5^4} \left( \frac{(2 \cos^2(\Omega) (2 - 7 \lambda f_{\infty}) h_0^2 - 3 \sin^2(\Omega) (1 - 4 \lambda f_{\infty}))}{9 h_0^3} + \int_{0}^{h_0} \frac{\mathcal{L}_3}{\mathcal{L}_4} \, dh \right) \log(\delta/H),$$

where $\mathcal{L}_3$ and $\mathcal{L}_4$ are given by (B.9). In this case, we can further compare the universal term with the central charges (5.6) and (5.7). However, we observe from expression of $\mathcal{L}_3$ and $\mathcal{L}_4$ in (B.9) that there are terms of order $O(\lambda^2)$ in the universal term. This implies that the expression (5.20) is not a simple function of the central charges $\tilde{C}_T$ and $a_5^*$ and in particular, it is not a linear function.

Further, we calculate the EE for cone in $d = 6$ and find that the regulator independent term is $\log^2 \delta$ and it is given by

$$S_{6}^{\log^2} \Big|_{c_3} = \frac{12 \pi^3 \tilde{L}^5 \cos(\Omega) \cot(\Omega)}{\ell_6^5} \left( \frac{93 - 190 \lambda f_{\infty}}{8192} - \frac{3 - 2 \lambda f_{\infty}}{8192} \cos(2\Omega) \right) \log^2(\delta/H).$$

We have given the complete expression of EE in the appendix B in eqn. (B.10). In this expression, we would like to compare the $\Omega$ independent coefficients with the central charges of the CFT. So there are two terms, $(93 - 190 \lambda f_{\infty})$ and $(3 - 2 \lambda f_{\infty})$, which we can express in terms of $\tilde{C}_T$ and $a_6^*$. Using equations (5.6) and (5.7), we find that the universal terms in EE for $c_3$ can be written as

$$S_{6}^{\log^2} \Big|_{c_3} = \frac{3 \cos(\Omega) \cot(\Omega)}{1024} \left[ \left( 90 \tilde{C}_T + 3 a_6^* \right) - \left( 6 \tilde{C}_T - 3 a_6^* \right) \cos(2\Omega) \right] \log^2(\delta/H).$$

Hence we see here that in $d = 6$, the new universal term for $c_3$ depends linearly on both of the central charges, $\tilde{C}_T$ and $a_6^*$, but it still has a relatively simple dependence on the opening angle $\Omega$.

5.2 Singularity with a curved locus

In this section, we will repeat the calculation of EE for geometry $k \times S^2$ in $d = 5$ CFT, which is dual to the Gauss-Bonnet gravity. For this case, the bulk action and EE are given by (5.1) and (5.5). We further consider the metric ansatz

$$ds^2 = \frac{\tilde{L}^2}{z^2} \left( dz^2 + f_1(z) \left( dt^2 + d\rho^2 + \rho^2 \sin^2(\theta) \right) + R_1^2 f_2(z) d\Omega_2^2 \right),$$

(5.23)
where $d\Omega_2$ is line element over the two-sphere. Further, similar to (5.4), $\tilde{L}^2 = L^2/f_\infty$ where $f_\infty = (1 - \sqrt{1 - 4\lambda})/2\lambda$. Here $f_1$ and $f_2$ are functions of $z$ and one can use the Fefferman-Grahm expansion to find their values near the boundary.

Now, to calculate EE, we first use Fefferman-Grahm expansion and find that

$$f_1(z) = 1 + \frac{z^2}{12R_1^2} + \frac{z^4(51 - 58\lambda f_\infty)}{1728R_1^4(1 - 2\lambda f_\infty)} + \ldots,$$

$$f_2(z) = 1 - \frac{z^2}{4R_1^2} - \frac{z^4(15 - 2\lambda f_\infty)}{576R_1^4(1 - 2\lambda f_\infty)} + \ldots \quad (5.24)$$

Once again, we choose the parametrization $\rho = \rho(z, \theta)$ and similar to (4.35), we can find the induced metric for the entropy functional. Here, we restrain to give the complete expression for the entropy functional but the important point to note is that now there are terms with higher derivatives, like $\rho''$, $\rho'$ and $\rho''$ in the entropy functional. However, in spite of this, the equation of motion for $\rho(z, \theta)$ is still second order and that is because of the topological nature of the Gauss-Bonnet terms. Now we can simplify the entropy function by inserting the ansatz $\rho = \rho_0 + \rho_1/R_1^2$ and then series expending it for large $R_1$. We find that up to the leading order, the EE is given by

$$S_5|_{k \times S^2} = \frac{16\pi^2 \tilde{L}_P^4 R_1^2}{f_\lambda^4} \int_{z_m}^\delta dz \int_0^{\Omega-\epsilon} d\theta \left( \mathcal{L}_0(\rho_0) + \frac{1}{R_1} \left( \mathcal{L}_N(\rho_0) + \mathcal{L}_f(\rho_0, c_1, c_2) + \rho_1^L \mathcal{L}_1(\rho_0) \right. \right.$$  

$$\left. + \rho_1 \mathcal{L}_2(\rho_0) + \rho_1 \mathcal{L}_3(\rho_0) + \rho_1 \mathcal{L}_4(\rho_0) + \rho_1 \mathcal{L}_5(\rho_0) + \rho_1 \mathcal{L}_6(\rho_0) \right) \right), \quad (5.25)$$

where $\mathcal{L}_0(\rho_0)$ is the term which comes from the limit $R_1 \to \infty$ in the original lagrangian. The term $\mathcal{L}_N(\rho_0)$ is a new term which doesn’t appear for $\lambda = 0$ case. This term comes from the contribution of sphere in the Ricci scalar in (5.5). The term $\mathcal{L}_f(\rho_0, c_1, c_2)$ is the term which is independent of $\rho_1$ and comes from the leading order corrections in $f_1 = 1 + c_1 z^2/R_1^2$ and $f_2 = 1 + c_2 z^2/R_1^2$, where $c_1 = 1/12$ and $c_2 = -1/4$. Further, terms with $\mathcal{L}_1(\rho_0) - \mathcal{L}_6(\rho_0)$ are independent of $c_1$ and $c_2$ and are linear in $\rho_1$, as it is written. Now, we can write

$$\rho_1^L \mathcal{L}_1 = \partial_z (\mathcal{L}_1 \rho_1 - \mathcal{L}_1' \rho_1) + \mathcal{L}_1'' \rho_1,$$

$$\rho_1^L \mathcal{L}_2 = \partial_z (\mathcal{L}_2 \rho_1) - \partial_\theta (\mathcal{L}_2 \rho_1) + \mathcal{L}_2' \rho_1,$$

$$\rho_1^L \mathcal{L}_3 = \partial_\theta (\mathcal{L}_3 \rho_1) - \partial_\theta (\mathcal{L}_3 \rho_1) + \mathcal{L}_3' \rho_1,$$

$$\rho_1^L \mathcal{L}_4 = \partial_z (\mathcal{L}_4 \rho_1) - \mathcal{L}_4' \rho_1,$$

$$\rho_1^L \mathcal{L}_5 = \partial_\theta (\mathcal{L}_5 \rho_1) - \mathcal{L}_5' \rho_1,$$

where prime and upper dot denote the partial derivative with respect to $z$ and $\theta$. Using these
in (5.25), we write

\[ S_5|_{k \times S^2} = \frac{16 \pi^2 \tilde{L}^4 R_1^2}{\ell_p^4} \int_{z_m}^{\delta} dz \int_{0}^{\Omega - \epsilon} d\theta \left( \mathcal{L}_0(\rho_0) + \frac{1}{R_1^2} (\mathcal{L}_N(\rho_0) + \mathcal{L}_f(\rho_0, c_1, c_2)) + \rho_1(\mathcal{L}_2'' + \mathcal{L}_2' + \mathcal{L}_3 - \mathcal{L}_4 - \mathcal{L}_5) + \partial_z(\mathcal{L}_1\rho_1 - \mathcal{L}_2\rho_1 + \mathcal{L}_2\rho_1 + \mathcal{L}_4\rho_1) \right) \]

\[ + \frac{16 \pi^2 \tilde{L}^4}{\ell_p^4} \int_{z_m}^{\delta} dz \left( -\mathcal{L}_2\rho_1 + \mathcal{L}_3\rho_1 - \mathcal{L}_3\rho_1 + \mathcal{L}_5\rho_1 \right)_{\theta=\Omega - \epsilon} , \tag{5.27} \]

where the coefficient of the \( \rho_1 \) vanishes from the equation of motion of \( \rho_0 \) and we have performed the integration over \( \theta \) in terms in the last line. Note that we can not integrate over \( z \) in any term as \( \epsilon = \epsilon(z) \) and both \( \theta \) and \( z \) integrations don’t commute. Now we further insert the ansatz \( \rho_0 = z/h(\theta) \) and \( \rho_1 = z^3 g_3(\theta) \) in the above entropy functional and in the equations of motion for \( \rho_0 \) and \( \rho_1 \) to find the equations of motion for \( h \) and \( g_3 \). Once again, we restrain to give the complete expression of the equations of motion as they are not very illuminating. Similar to (4.45), at this stage we find that the terms with the partial derivative with respect to \( z \) vanish. Finally, the entropy functional reduces to

\[ S_5|_{k \times S^2} = \frac{16 \pi^2 \tilde{L}^4 R_1^2}{\ell_p^4} \int_{z_m}^{\delta} dz \int_{h_0}^{h_{1c}(z)} dh \frac{\hat{L}_0(h)}{z^3} + \frac{1}{R_1^2} \hat{L}_N(h) + \hat{L}_f(h, c_1, c_2) \]

\[ + \frac{16 \pi^2 \tilde{L}^4}{\ell_p^4} \int_{z_m}^{\delta} dz \frac{\hat{L}_B |_{\theta=\Omega - \epsilon}}{z} , \tag{5.28} \]

where we have defined \( \mathcal{L}_0 = \hat{L}_0(h)/z^3 \), \( \mathcal{L}_N = \hat{L}_N(h)/z \), \( \mathcal{L}_f = \hat{L}_f(h, c_1, c_2)/z \) and expressions for \( \hat{L} \)’s are given by (B.13) in appendix B. Now we can solve the equations of motion for \( h \) and \( g_3 \) near the asymptotic boundary to find

\[ \dot{h} = -\frac{a_1}{h^4} - \frac{2a_1}{h^2} - a_1 + \frac{h^4(1 - 6f_{\infty}\lambda)}{2a_1(1 - 2f_{\infty}\lambda)} - \frac{h^6(1 - 10f_{\infty}\lambda)}{2a_1(1 - 2f_{\infty}\lambda)} + \cdots , \tag{5.29} \]

\[ g_3 = \frac{b_3}{h^3} + \frac{9 - 26f_{\infty}\lambda + 792b_3(1 - 2f_{\infty}\lambda)}{504h(1 - 2f_{\infty}\lambda)} + \frac{4h(3 - 11f_{\infty}\lambda + 54b_3(1 - 2f_{\infty}\lambda))}{567(1 - 2f_{\infty}\lambda)} + \cdots , \]

where \( a_1 \) and \( b_3 \) are constants which are fixed by ensuring that both \( h \) and \( g_3 \) have extrema at \( \theta = 0 \). Note that here \( a_1 \) is related to a quantity which is conserved along the \( \theta \) translation similar to (4.46). Using these solutions near the boundary, we first find the value of \( h_{1c}(z) \) at the UV cut-off \( z = \delta \). To do that, we use the above solutions in the ansatz \( \rho = z/h + z^3 g_3/R_1^2 \) and inverting the relations iteratively, we find

\[ h_{1c}(\delta) = \left( \frac{1}{H} + \frac{b_3 H}{R_1^2} \right) \delta + \frac{(9 - 26f_{\infty}\lambda + 792b_3(1 - 2f_{\infty}\lambda)) \delta^3}{504HR_1^2(1 - 2f_{\infty}\lambda)} + \frac{4(3 - 11f_{\infty}\lambda + 54b_3(1 - 2f_{\infty}\lambda)) \delta^5}{567H^3R_1^2(1 - 2f_{\infty}\lambda)} + \cdots . \tag{5.30} \]
Note that above relation reduces to (4.49) for \( \lambda = 0 \). Now we use (5.29) to study the behavior of integrands in (5.28) near the asymptotic boundary, where we have \( h \to 0 \):

\[
\frac{\hat{L}_0}{h} \sim -\frac{1 - 4f_{\infty} \lambda}{h^2} + \mathcal{O}(h^4),
\]

\[
\frac{\hat{L}_N + \hat{L}_f}{h} \sim \frac{15 - 68f_{\infty} \lambda}{72h^2} + \mathcal{O}(h^2).
\]  

Using these, similar to (4.52) and (4.54), we can make the integrands in (B.13) finite. We can break the terms in following components

\[
I_1 = \int_{z_m}^{\delta} \frac{dz}{z^3} \int_{h_0}^{h_1(c(z)} dh \left( \frac{\hat{L}_0}{h} + \frac{1 - 4f_{\infty} \lambda}{h^2} \right),
\]

\[
I_2 = -\int_{z_m}^{\delta} \frac{dz}{z^3} \int_{h_0}^{h_1(c(z)} dh \frac{1 - 4f_{\infty} \lambda}{h^2},
\]

\[
I_3 = -\int_{z_m}^{\delta} \frac{dz}{z} \int_{h_0}^{h_1(c(z)} dh \left( \frac{\hat{L}_N + \hat{L}_f}{h} - \frac{15 - 68f_{\infty} \lambda}{72h^2} \right),
\]

\[
I_4 = -\int_{z_m}^{\delta} \frac{dz}{z} \int_{h_0}^{h_1(c(z)} dh \frac{15 - 68f_{\infty} \lambda}{72h^2},
\]

and then take a derivative with respect to \( \delta \) and find following series expansion:

\[
\frac{dI_1}{d\delta} = \frac{1}{\delta^3} \int_{h_0}^{0} dh \left( \frac{\hat{L}_0}{h} + \frac{1 - 4f_{\infty} \lambda}{h^2} \right) + \mathcal{O}(\delta^4)
\]

\[
\frac{dI_2}{d\delta} = \left( 1 - \frac{b_3 H^2}{R_1^2} \right) \frac{H (1 - 4f_{\infty} \lambda)}{\delta^4} - \frac{1 - 4f_{\infty} \lambda}{h_0 \delta^3}
\]

\[
- \frac{H (1 - 4f_{\infty} \lambda)(9 - 26f_{\infty} \lambda + 792b_3 (1 - 2f_{\infty} \lambda))}{504 (1 - 2f_{\infty} \lambda) R_1^2 \delta^2} + \mathcal{O}(\delta^0)
\]

\[
\frac{dI_3}{d\delta} = -\frac{1}{\delta} \int_{h_0}^{0} dh \left( \frac{\hat{L}_N + \hat{L}_f}{h} - \frac{15 - 68f_{\infty} \lambda}{72h^2} \right) + \mathcal{O}(\delta^5)
\]

\[
\frac{dI_4}{d\delta} = -\left( 1 - \frac{b_3 H^2}{R_1^2} \right) H (15 - 68f_{\infty} \lambda) + \frac{15 - 68f_{\infty} \lambda}{72h_0 \delta} + \mathcal{O}(\delta^0)
\]

\[
\frac{d}{d\delta} \int_{z_m}^{\delta} \frac{dz}{z} \frac{\hat{L}_B|_{\theta=\Omega-\epsilon}}{z} = \left( 1 - \frac{3b_3 H^2}{R_1^2} \right) \frac{b_3 H^3 (1 - 4f_{\infty} \lambda)}{\delta^4}
\]

\[
+ \left( 1 - \frac{4b_3 H^2}{R_1^2} \right) \frac{H (1 - 4f_{\infty} \lambda)(9 - 26f_{\infty} \lambda + 792b_3 (1 - 2f_{\infty} \lambda))}{504 (1 - 2f_{\infty} \lambda) \delta^2} + \mathcal{O}(\delta^0).
\]

Note that we have done the same with the boundary term in (5.28) too. Using these relations, we can read off the logarithmic term in the EE, which is given by

\[
S_{S_5}^{\log}|_{k \times S^2} = \frac{16 \pi^2 \hat{L}_B^4}{\ell_P^4} \left( \int_{h_0}^{0} dh \left( \frac{\hat{L}_N + \hat{L}_f}{h} - \frac{15 - 68f_{\infty} \lambda}{72h^2} \right) + \frac{15 - 68f_{\infty} \lambda}{72h_0} \right) \log(\delta/H),
\]  

(5.37)
where $\hat{L}_N$ and $\hat{L}_f$ are given by (B.13). Now we can try to compare this cut-off independent term with the central charges (5.6) and (5.7). However, by looking at the expression of $\hat{L}_N$ and $\hat{L}_f$, we find that there are terms of order $\mathcal{O}(\lambda^2)$. Hence the universal term is not a simple (and particularly linear) function of the central charges.

Now, it would have been interesting to further investigate the universal term for the geometry $c_1 \times S^2$. In this case, the singularity has an even dimensional, curved locus and we will get a $\log^2 \delta$ divergence. So it would have been very interesting to see if this contribution from the locus is a specific central charge. However, this calculate is tedious and demands more patience and ingenious simplifications.

6. Discussion

In this paper, we have used holography to study entanglement entropy for various singular surfaces in higher dimensions and our results are summarized in table 1. In particular, in section 3.1, we considered cones in various dimensions. This analysis suggests that for CFT’s in an odd number of spacetime dimensions, an additional universal contribution appears:

$$S_{\text{univ}} = q_d(\Omega) \log(\delta/L),$$

(6.1)

where $\Omega$ is the opening angle of the cone, as shown in figure 1. In particular, for $c_2 = R^+ \times S^2$ in $d = 5$, eq. (3.25) provides an expression for the coefficient $q_5(\Omega)$ in the holographic CFT dual to Einstein gravity. We showed that this coefficient satisfies the required properties: (a) $q_d(\Omega = \pi/2) = 0$ since the entangling surface is actually a flat plane for this angle and (b) $q_d(\Omega) = q_d(\pi - \Omega)$ since the entanglement entropy of the CFT ground state is identical for the density matrix describing the degrees of freedom inside or outside of the cone. Further we found that $q_d(\Omega \to 0) \propto 1/\Omega$ as in eq. (3.27). Again our expectation is that this behaviour extends generally to cones for odd dimensional theories. That is, the entanglement entropy acquires a universal contribution of the form given in eq. (6.1) for a cone $c_{d-3} = R^+ \times S^{d-3}$ in any odd $d$.

Here we might recall that in three dimensions, strong subadditivity was used to derive further constraints on the form of $q_3(\Omega)$ [18, 19]. In particular, one shows that this function must satisfy $q_3(\Omega) \geq 0$ and $q_3(\Omega) \leq 0$. Of course, our holographic result (3.9) satisfies both of these inequalities. It is noteworthy then that in five dimensions, our holographic result satisfies $q_5(\Omega) \leq 0$ and $q_5(\Omega) \geq 0$. It would be interesting to understand if these inequalities are again general properties originating from strong subadditivity.

Further we add that in eq. (3.25), the coefficient $q_5$ is proportional to $L^4/\ell_5^4$, which can be identified with a central charge in the dual boundary CFT. However, with Einstein gravity in the bulk, all of the central charges in the five-dimensional boundary theory are identical and so in section 5.1, we extended the calculation to Gauss-Bonnet gravity in the bulk, which allows us to distinguish at least two such central charges, as described there. However, given the result for $q_5(\Omega)$ implicit in eq. (5.20) and (B.9), it appears that this coefficient is a complicated nonlinear function of both central charges. It may in fact be that this coefficient
is not determined by the central charges alone – this would then be similar to the results found for Renyi entropies in [30].

The case of even dimensions was particularly interesting. In section 3.1 with \( d = 4 \) and 6, we found that a cone yields a universal contribution of the form:

\[
S_{\text{univ}} = \hat{q}_d(\Omega) \log^2(\delta/L),
\]

where \( \Omega \) is again the opening angle, as shown in figure 1. Again we believe this is a generic result for even dimensional theories. In our holographic examples, the coefficient functions have a relatively simple form, as shown in eqs. (3.23) and (3.28). Hence for these examples, it is straightforward to verify that \( \hat{q}_d(\Omega) \) satisfies the same properties as described for \( q_d(\Omega) \) in the first paragraph above. From these results, we also see that \( \hat{q}_d(\Omega \to 0) \propto 1/\Omega \), which parallels the behaviour seen for \( q_3 \) and \( q_5 \). Further, in section 5.1, using Gauss-Bonnet gravity, we were able to verify that this coefficient is proportional to the central charge (5.6) which controls the leading singularity of the two-point function of the stress tensor for \( d = 4 \). In particular, we found the simple result (5.19) which yields

\[
\hat{q}_4 = -\frac{c \cos^2 \Omega}{4 \sin \Omega}.
\]

For \( d = 6 \), we also found a simple expression (5.22) for the new universal term for \( c_3 \). However, in this case, \( \hat{q}_6 \) depends linearly on both of the central charges, \( \hat{C}_T \) and \( a^*_6 \), and it is still a relatively simple function of the opening angle \( \Omega \). Given the simplicity of these results, particularly in eq. (6.3), one might conjecture that the same form may arise for CFT’s in general beyond our holographic calculations.

We note here that with even \( d \), there are also contributions proportional to a single power of \( \log \delta/L \), however, these terms are no longer universal. Rather the corresponding coefficient will vary if the details of the cut-off (or the macroscopic scale \( L \)) are changed because of the presence of the \( \log^2 \delta/L \) term – see further discussion below.

Recall that the trace anomaly in an even dimensional CFT gives rise a universal contribution in the entanglement entropy with a smooth entangling surface \([5, 16, 35, 33]\). In particular, if we consider a four-dimensional CFT in a flat background, this contribution takes the form given in eq. (1.5). Given this explicit expression, it is interesting to compare this contribution for a conical entangling surface \( c_1 \) in \( d = 4 \) to the universal term (5.19) found in our holographic calculation.

Now in \( d = 6 \), we do not know the precise expression of the contribution of the trace anomaly to EE, i.e., \( d = 6 \) generalization of (1.5). Hence, now we can not distinguish the contribution from the trace anomaly with the contribution from the singularity in (5.22). Hence, we can not say what part of the universal term comes from the singularity and particularly, we can not confirm if singularity contributes only through the central charge \( \hat{C}_T \).

Of course, eq. (1.5) is only expected to apply for a smooth entangling surface and so cannot be applied directly to the cone \( c_1 \). Hence our approach is to construct a ‘regulated cone’ \( \tilde{c}_1 \) by cutting off the cone at some \( \rho_{\text{min}} \) and replacing the tip with a spherical cap of
radius $r = \rho_{\text{min}} \tan \Omega$, as shown in figure 4. Hence $\tilde{c}_1$ provides a smooth model of the desired conical entangling surface to which we can apply eq. (1.5) and we can consider the limit $\rho_{\text{min}} \to 0$ to recover the result for the singular surface $c_1$. Hence working first with finite $\rho_{\text{min}}$, it is relatively straightforward to show that eq. (1.5) yields

$$S_{\text{univ}} = -\frac{c}{2} \frac{\cos^2 \Omega}{\sin \Omega} \log(\delta/L) \log(\rho_{\text{min}}/L) + 2a (1 - \sin \Omega) \log(\delta/L).$$

Unfortunately the first term will diverge if we take the limit $\rho_{\text{min}} \to 0$. However, if the underlying CFT has been regulated with the cut-off $\delta$, it should not be able to resolve any features in geometry at shorter distances. Hence it is natural to consider the limit $\rho_{\text{min}} \to \delta$ which yields

$$S_{\text{univ}} = -\frac{c}{2} \frac{\cos^2 \Omega}{\sin \Omega} \log^2(\delta/L) + 2a (1 - \sin \Omega) \log(\delta/L).$$

Hence with this construction, the universal contribution (1.5) contains terms proportional to both $\log^2(\delta/L)$ and $\log(\delta/L)$. The most surprising aspect of this result is that the coefficient of the leading term is almost identical to the holographic result in eq. (5.19). However, there is a mismatch by a factor of two. Our interpretation of this result is that with the conical entangling surface $c_1$, part of the $\log^2(\delta/L)$ divergence should be associated with correlations across the smooth part of the entangling surface away from the singularity. The above calculation then suggests a pile up of short distance correlations in the vicinity of the tip. However, the full accumulation of correlations near the tip of the cone does not have (precisely) the form expected from the ‘smooth’ expression (1.5). Hence a part of this universal contribution should be thought of as intrinsic to the singularity at the tip of the cone itself.

As already noted, if we examine our holographic entanglement entropy for $c_1$ in eq. (3.23), we also find a term proportional to $\log(\delta/L)$. The angular dependence of the corresponding coefficient is different from that for the $\log^2(\delta/L)$ term and so one might envision that certain universal aspects can still be extracted from this coefficient. For example, in eq. (6.5), the second central charge $a$ appears in the $\log(\delta/L)$ term and so one might hope to extract this charge by studying the entanglement entropy for cones with various opening angles $\Omega$. However, we wish to emphasize that this $\log(\delta/L)$ contribution is simply not universal in the presence of the $\log^2(\delta/L)$ term. For example, let us return to our regulated model of cone above where we argued it was natural to take the limit $\rho_{\text{min}} \to \delta$ for the radius of the spherical cap. The latter was motivated by observing that $\delta$ is a short distance cut-off in the underlying CFT and so the latter can not resolve any geometric features involving shorter distance scales. However, let us note that the radius of the spherical cap was $r = \rho_{\text{min}} \tan \Omega$ and so even if we set $\rho_{\text{min}} = \delta$, for small $\Omega$, the cap is effectively much smaller than the short distance cut-off. Hence one might instead choose

$$\rho_{\text{min}} = \begin{cases} 
\delta & \text{for } \Omega > \pi/4, \\
\delta \tan \Omega & \text{for } \Omega \leq \pi/4.
\end{cases}$$

---

8On the conical portion of $\tilde{c}_1$, $R = 0$ while the only nontrivial component of the extrinsic curvature is $K_{\theta\phi} = \rho \sin \Omega \cos \Omega$. On the spherical cap, $R = 2/r^2$ while the combination of extrinsic curvatures in eq. (1.5) vanishes.

9This observation was also made in [23]. We also note that the same result appears in an alternate calculation of the holographic entanglement entropy presented in appendix A. There the mismatch can be regarded as an anomaly arising from a singular conformal transformation.
While such a choice leaves the log\(^2(\delta/L)\) term unchanged, the coefficient in the log(\(\delta/L\)) contribution acquires a complicated new angular dependence. We present this discussion here simply to illustrate that all of the details (including the angular dependence) of the coefficient of the log(\(\delta/L\)) can be expected to depend on the precise choice of the regulator in the calculation of the entanglement entropy. While eq. (6.6) gives an illustrative example, we might observe that the same regulator in the holographic calculations is far more subtle. The analog of eq. (6.6) would be \(\rho_{\text{min}} = \delta/h_0\) in section 3.1.

We also extended our holographic analysis to consider creases or extended singularities in section 4. Examining the examples summarized in table 1, we find that the crease of the form \(k \times R^m\) or \(c_n \times R^m\) creates no new universal contributions.\(^{10}\) However, in general, we found that creases can contribute additional universal terms, but singular locus must have an even dimension and must be curved. These results suggest that these new universal contributions to the entanglement entropy take the form given in eq. (2.6) with a log \(\delta\) divergence in odd dimensions and log\(^2\delta\) in even dimensions. These results indicate that there is a rich variety of new geometric contributions to entanglement entropy that can be associated with singular entangling surfaces. However, our analysis only considered simple families of singularities and does not suffice to reveal the full geometric structure of these universal terms. It would, of course, be interesting to consider more general singularities, e.g., a crease of the form \(k \times R^1\) but where the opening angle varies along \(R^1\) or where \(R^1\) was not entirely straight. Another step towards a clearer picture of these geometric coefficients would be to carry out the holographic calculations using the Fefferman-Graham expansion [36] to compute entanglement entropy along the lines discussed in [15].

In part, our motivation for these studies was the possibility that these new universal contributions may be used to identify the central charges of the underlying CFT. Recently there has been a great deal of interest in using entanglement to identify central charges that obey a c-theorem, in particular for odd spacetime dimensions, e.g., see [11, 23, 33, 34, 37, 38]. In this regard, our results only point towards a clear result for even dimensions, namely, that the coefficient of the log\(^2\delta\) contribution is proportional to a particular central

\(^{10}\)Let us re-iterate that we are ignoring certain cases here, e.g., \(c_1 \times R^2\), where a log \(\delta\) term appears but it can be attributed to the smooth part of the entangling surface. We also do not consider the possibility that the finite contributions may exhibit some new universal behaviour.
charge. Unfortunately, this central charge $\tilde{C}_T$ is not the one expected to satisfy a c-theorem. However, our holographic result seems particularly simple and so it may be that there is a general derivation for any CFT, perhaps connected to the trace anomaly as for the universal terms identified for smooth surfaces [5, 16, 33, 35]. It is also worthwhile to investigate if other central charges appear if we consider, e.g., cones with a more general cross-section than simply $S^{d-3}$.

In odd dimensions, our results in section 5.1 indicate that the universal terms associated with singularities in the entangling surface will be complicated nonlinear functions of many parameters in the underlying CFT. Here it must be said that we refer to these contributions as universal since they will be independent of the details of the UV regulator and so should characterize properties of the underlying CFT (or QFT more generally). However, the precise nature of the information contained in these terms remains to be understood. A similar result was found for Renyi entropies of spherical entangling surfaces in [30].

It would also be interesting to study these universal contributions to entanglement entropy in non-holographic theories. In particular, one might consider heat kernel methods for free field theories for simple surfaces, e.g., along the lines of [14, 40]. We have been informed by Brian Swingle that he found similar $\log^2 \delta$ terms in the entanglement entropy of surfaces with a conical singularity using a twist field calculation in the Gaussian approximation, as discussed in [41]. It is also of interest to extend our holographic calculations to consider entangling surfaces with flat faces and edges, e.g., a cube. In particular, one would like to confirm that the new $\log^2 \delta$ contributions persist for the ‘corners’ in such cases. This would have practical implications for lattice calculations in higher dimensions, where it is difficult to avoid such corners. It would also be interesting to further investigate the Renyi entropy for singular entangling surfaces [17].

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11 Setting holography aside, it was shown that for four-dimensional QFT’s, there is no possible (linear) combination of the two central charges, $c = \tilde{C}_T$ and $a = a_d$, that can satisfy a c-theorem apart from $a$ alone [39].

12 For example, in four dimensions, we can regard such a corner on the surface of a cube as a ‘cone’ with a triangular cross-section.
A. Conformal transformations and EE for $c_{d-3}$

In this section, we consider an alternate approach to calculating the entanglement entropy associated with a conical singularity. In particular, we begin by performing a conformal transformation which takes $R^d$ to $R \times S^{d-1}$. Under such a conformal transformation, the conical entangling surface $c_{d-3}$, considered in sections 2 and 3, becomes simply a uniform cylinder $R \times S^{d-3}$ in the latter background. The analogous mapping was applied to calculating the cusp anomaly for a Wilson with a sharp corner in $N=4$ SYM, e.g., see [42]. In this case, calculating the cusp anomaly becomes problem of determining the quark-antiquark potential on a three-sphere. In fact, the holographic calculation for the leading result corresponds to determining an extremal surface in the bulk with a ‘kink’ boundary condition and so it precisely matches the calculation of the holographic EE for a kink in $d = 3$ [42, 43].

We would like to calculate EE using the cylindrical background geometry $R \times S^{d-1}$ and in particular, to see if we get the same log $\delta$ and log $2 \delta$ terms that appeared in our previous calculations for the cones $c_{d-3}$. The key feature in these calculations is that the conformal mapping takes the conical entangling surface in $R^d$ to a new surface $R \times S^{d-3}$ in the cylindrical background. Now this new surface has an infinite length along the $R$ direction and so this length must be regulated to properly account for the logarithmic divergences, as we will discuss below in section A.1 when we calculate the holographic EE. We first describe desired conformal transformation in the CFT and then we consider how this transformation is implemented with a coordinate transformation in the dual bulk spacetime.

In the flat background (2.2) with $\{n, m\} = \{d - 3, 0\}$, we can make the coordinate transformations $t_E = r \cos \xi$ and $\rho = r \sin \xi$ and find following metric

$$ds^2 = dr^2 + r^2(d\xi^2 + \sin^2 \xi (d\theta^2 + \sin^2 \theta d\Omega_{d-3}^2)). \quad \text{(A.1)}$$

In these coordinates, the cone geometry discussed in section 2 translates to the surface: $c_{d-3} = \{r = [0, \infty), \xi = \pi/2, \theta = \Omega\}$. We can further perform the coordinate transformation $r = L e^{Y/L}$, and make a Weyl transformation to remove the overall factor $e^{2Y/L}$ from the resulting metric. After these transformations, we find background geometry becomes $R \times S^{d-1}$ with the metric

$$ds^2 = dY^2 + L^2(d\xi^2 + \sin^2 \xi (d\theta^2 + \sin^2 \theta d\Omega_{d-3}^2)),$$

where $Y \in [-\infty, \infty]$. Further, the conical entangling surface above is now mapped to a cylinder of infinite length, i.e.,

$$c_{d-3} = \{Y = (-\infty, \infty), \xi = \pi/2, \theta = \Omega\}. \quad \text{(A.3)}$$

Next we discuss the coordinate transformation in the bulk geometry which implements the conformal transformation between the two boundary metrics in eqs. (A.1) and (A.2). We begin with the standard description of (Euclidean) AdS$_{d+1}$ as a hyperbola embedded in the following $(d + 2)$-dimensional Minkowski space

$$ds^2 = -dU^2 + dV^2 + (dX^1)^2 + \cdots + (dX^d)^2 = -dU^2 + dV^2 + dR^2 + R^2d\Omega_{d-1}^2, \quad \text{(A.4)}$$
where in the second expression, we introduced polar coordinates on the space spanned by the \(X^i\). Now the AdS geometry is defined by the hyperbola

\[-U^2 + V^2 + (\vec{X})^2 = -U^2 + V^2 + R^2 = -L^2. \tag{A.5}\]

We can solve this constraint by writing \(U = \sqrt{R^2 + L^2} \cosh(Y/L)\) and \(V = \sqrt{R^2 + L^2} \sinh(Y/L)\), in which case the induced metric on this surface becomes

\[
ds^2 = \frac{1}{1 + R^2/L^2} dR^2 + \left(1 + \frac{R^2}{L^2}\right) dY^2 + R^2 \left[d\xi^2 + \sin^2 \xi (d\theta^2 + \sin^2 \theta d\Omega^2_{d-3})\right]. \tag{A.6}\]

Here we have written \(d\Omega^2_{d-1} = d\xi^2 + \sin^2 \xi (d\theta^2 + \sin^2 \theta d\Omega^2_{d-3})\). Of course, we recognize this metric as (Euclidean) AdS\(_{d+1}\) in global coordinates and this geometry is dual to the boundary CFT with the background metric (A.2). Alternatively, to get the bulk metric dual to eq. (A.1), we write \(R = rL/z\) and \(U + V = L^2/z\) and eq. (A.5) yields \(U - V = z + r^2/z\). With these coordinates, the induced metric on the hyperbola becomes

\[
ds^2 = \frac{L^2}{z^2} (dz^2 + dr^2 + r^2 d\Omega^2_{d-1}), \tag{A.7}\]

which matches the ‘Poincaré patch’ metric (2.3) with \(r^2 = t^2_R + \rho^2\). Clearly, the boundary metric matches the desired form given in eq. (A.1). This bulk metric (A.7) was used in our calculation of the holographic EE for the cone in section 3. So now we use the metric (A.6) in calculating the EE for the cylindrical surface produced by the conformal mapping.

**A.1 EE for cylinders**

In this section, we will calculate the holographic EE for cylindrical entangling surface \(R \times S^{d-3}\) on the background \(R \times S^{d-1}\) for \(d = 3, 4\) and 5. As we will see the calculations here are closely related to those already presented in section 3. We find that the coefficient of the universal \(\log \delta\) term matches our previous results for \(d = 3\) and 5. In \(d = 4\), the universal term is a \(\log^2 \delta\) contribution but we find the coefficient in the following does not give a precise match with that in eq. (5.19) from our previous calculation. In fact, the \(\log^2 \delta\) term here shows the same mismatch by a factor of two that we found in eq. (6.5) from considering the contribution of the trace anomaly on a regulated conical surface. We will show that this difference in the holographic results comes from choosing different UV cut-offs in the different coordinate systems. Although these cut-offs are natural in the particular coordinate system in which they are chosen, they produce different coefficients of the \(\log^2 \delta\) divergence.

Now we begin with a general discussion for a \(d\)-dimensional CFT on the background \(R \times S^{d-1}\) and later focus on the specific cases with \(d = 3, 4\) and 5. The cylindrical entangling surface on the boundary is given in eq. (A.3). However, to produce a finite result, the length the cylinder must be regulated which we do by restricting \(Y \in [Y_-, Y_+]\), where \(Y_-\) and \(Y_+\) are cut-offs to be fixed below. Now the holographic EE is calculated by first finding the minimal area surface hanging into the bulk geometry with metric (A.6) and if we define this surface
by coordinates \((Y, \theta, \Omega_{d-3})\), we will have a radial profile \(R(\theta) = Lg(\theta)\). In particular, note that \(R(\theta)\) is independent of \(Y\) because we have translational symmetry along this direction. The only scale in the geometry is \(L\), which coincides with the radius of the sphere \(S^{d-1}\) on the boundary. With this ansatz, the holographic EE is given by

\[
S_d|_{\text{cylinder}} = \frac{2\pi L^{d-2}}{\ell_p^{d-1}} \int_{\delta/h_0}^{H} d\rho \int_{h_0}^{\delta/L} dh \frac{\sin^{d-3}(\theta)}{h^{d-1}} \sqrt{\hat{g}^2 + \hat{g}^2 + \hat{g}^4},
\]

where \(\hat{g} = dg/d\theta\) and \(\Omega_{d-3}\) represents the area of the unit \((d-3)\)-sphere. Further, \(\epsilon\) is related to the UV cut-off in the theory. Note that the natural way to choose this UV cut-off for cylinder should keep \(\epsilon\) independent of \(Y\), the coordinate along the length of the cylinder. So a natural UV cut-off for the minimal area surface can be \(R = L^2/\delta\), which will fix \(\epsilon\) such that \(g(\Omega - \epsilon) = L/\delta\).

To compare the above expression (A.8) with our previous results, we need to relate the quantities for the cylinder with the quantities for the cone discussed in section 3. First we note that from the coordinate transformation taking us from eq. (A.6) to eq. (A.7), we have \(\rho = r\) and \(R = rL/z = \rho L/z\). In our previous calculation of the holographic EE in eq. (3.3) or (3.14), we used the ansatz \(z = \rho h(\theta)\). Hence, above relation implies that \(R = L/h(\theta)\). Comparing with our ansatz above, we see that \(g(\theta) = 1/h(\theta)\) and \(h(\Omega - \epsilon) = \delta/L\). Now we turn to consider the cut-offs in the coordinate \(Y\). First note that these are independent of our UV cut-off on the radial coordinate, \(i.e., R = L^2/\delta\). As the conformal transformation has mapped the cone in \(R^d\) to the present cylinder, \(Y_\pm\) and \(Y_+\) should be related to the cut-offs in \(\rho\) appearing in eq. (3.14). We use the coordinate transformation \(Y = L \log(\rho/L)\), as given above eq. (A.2), and from the range \(\rho \in [\delta/h_0, H]\) in eq. (3.14), we find that \(Y_\pm = L \log(\delta/h_0 L)\) and \(Y_+ = L \log(H/L)\). Using all these results, we can now rewrite eq. (A.8) as

\[
S_d|_{\text{cylinder}} = \frac{2\pi L^{d-1}\Omega_{d-3}}{\ell_p^{d-1}} \int_{\delta/h_0}^{H} d\rho \int_{h_0}^{\delta/L} dh \frac{\sin^{d-3}(\theta)}{h^{d-1}} \sqrt{\hat{g}^2 + \hat{g}^2 + 1}.
\]

At first sight, this expression is identical to eq. (3.14) found in the previous calculation. However, we should notice that there is crucial difference. Namely, the upper limit of integration over \(h\) here is a fixed constant while in eq. (3.14), it is a function of \(\rho\).

Now we consider cylinder in different dimensions one by one. In eq. (A.8), we first take \(d = 3\) to consider a cylinder dual to a kink in \(R^3\). Note that in this case \(\theta \in [-\Omega, \Omega]\) and we will also have an extra factor of two coming from the change in the limits of integration. We can further use the variable \(y = \sqrt{1/\hat{h}^2 - 1/\hat{h}_0^2}\) as around eq. (3.6) and follow the steps in section 3. Finally, we find that for \(d = 3\), eq. (A.9) will produce

\[
S_3|_{\text{cylinder}} = \frac{4\pi L^2}{\ell_p^2} \left[ \frac{L}{\delta} \log(H/\delta) + \frac{L}{\delta} \log(h_0) - q_3(\Omega) \log(H/\delta) + \ldots \right],
\]

where \(q_3\) is given by (3.9). We note that the universal term in EE for the cylinder here precisely matches with that for the kink. However, now the leading order divergence is different as it

\[\text{---} 58\text{---}\]
contains an extra factor of $\log(h_0H/\delta)$. Of course, these divergent terms are not expected to be universal.

In part, the aim of the present calculations is to show that our ‘universal’ terms are independent of the details of the choice of regulator. Here we chose the natural cut-off adapted to the new coordinates after the conformal transformation in the boundary geometry (or a coordinate transformation in the bulk geometry). While the entire structure of the divergent contributions was not unchanged, our universal logarithmic term matched the previous calculations. Note that in eq. (A.9), if we would have used $z = \delta$ as the UV cut-off, using $R = \rho L/z$, we would have had $g(\Omega - \epsilon) = \rho/\delta = 1/h(\Omega - \epsilon)$. In this case, eq. (A.9) (with an extra factor of two) would have become

$$
\tilde{S}_3|_{\text{cylinder}} = \frac{4\pi L^2}{\ell_p^3} \int_{\delta/h_0}^{H} \frac{d\rho}{\rho} \int_{h_0}^{\delta/\rho} \frac{dh}{h} \sqrt{\frac{h^2 + h^2 + 1}{h^2 h}},
$$

(A.11)

precisely matching with eq. (3.3). However, choosing the new radial coordinate $R$ to be a function of $\rho$ (or more clearly $Y$) would not have been a natural UV cut-off in this case. Although we have found that for $d = 3$, the universal term in eqs. (3.3) and (A.10) match, this will not be the case in $d = 4$ which we discuss next.

For a cylinder in a four-dimensional CFT on $R \times S^3$, the EE is given by eq. (A.9) with $d = 4$. Now following the calculations and steps in section 3.1, we find that

$$
S_4|_{\text{cylinder}} = \frac{4\pi^2 L^3}{\ell_p^3} \left[ \frac{L^2 \sin^2 \Omega}{2 \delta^2} \log(H/\delta) + \frac{L^2 \sin^2 \Omega}{2 \delta^2} \log(h_0) - \cos \Omega \cot \Omega \frac{\log^2(\delta/H)}{8} \log(h_0) + \mathcal{O}(\log(\delta)) \right].
$$

(A.12)

In comparing eq. (A.12) with eq. (3.24), we see that the universal term is off by a factor of two. However, this $\log^2 \delta$ contribution precisely matches with the contribution from the trace anomaly as given in eq. (6.5). Here we can see that the new universal contribution from the singularity discussed in eq. (5.19) is not invariant under the conformal transformation considered here. In eq. (A.9), if we had chosen the UV cut-off on $R = \rho L/z = L g(\theta)$ to be $z = \delta$, the upper limit of integration would have been $\delta/\rho$, instead of $\delta/L$. In that case, the results for the holographic EE here would have precisely matched our previous results for the cone $c_1$.

Here we would note that singularities appear in two places here. Of course, there is the geometric singularity in the entangling surface $c_1$. However, the conformal transformation taking us from the flat metric (A.1) to the cylindrical metric (A.2) is also singular at precisely the same point, i.e., this transformation maps the origin in $R^d$ to $Y \to -\infty$ in $R \times S^{d-1}$. Hence it seems this transformation is ‘anomalous’ in that it does not preserve the coefficient of the universal contribution to the entanglement entropy. This effect is somewhat reminiscent of the anomaly that arises in mapping a straight Wilson line in N=4 SYM to a circular Wilson loop, both in $R^4$. While the former has a vanishing expectation value, the expectation of the latter yields a nontrivial result [43, 44].

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As a final example, we consider the EE for a cylinder in \( R \times S^4 \). In this case, we insert \( d = 5 \) into eq. (A.9) and follow the steps in section 3.1 to find that

\[
S_5|_{\text{cylinder}} = \frac{8\pi^2 L^4}{\ell_p^4} \left[ \frac{L^3 \sin^2 \Omega}{3 \delta^3} \log(H/\delta) + \frac{L^3 \sin^2 \Omega}{3 \delta^3} \log(h_0) - \frac{4L \cos^2 \Omega}{9 \delta} \log(h_0) + q_5 \log(h_0) + O(\delta^0) \right],
\]

where \( q_5 \) precisely the same as given in eq. (3.26). Hence in comparing eqs. (A.13) and (3.25), we clearly see that the universal logarithmic term for a cylinder in \( R \times S^4 \) matches with that for cone \( c_2 \). However, as compared to EE for \( c_2 \) in (B.1), there are again new non-universal divergences, which take the form \( \log(h_0)/\delta^3 \) and \( \log(h_0)/\delta^2 \).

We conclude this section by saying that although we have studied examples in \( d = 3, 4 \) and 5, we expect that the \( \log^2 \delta \) term for cone \( c_{d-3} \) in even \( d \) is not invariant under the conformal mapping from flat space to a cylinder. However, for odd \( d \), the \( \log \delta \) term will be invariant under the corresponding conformal transformation.

**B. Intermediate quantities for calculation of EE**

In this section, we mention various intermediatory steps in calculation of EE.

**B.1 EE for cone \( c_2 \) in \( d = 5 \)**

\[
S_6|_{c_2} = \frac{8\pi^2 L^4}{\ell_p^4} \left[ \frac{H^3 \sin^2(\Omega)}{9 \delta^3} - \frac{4H \cos^2(\Omega)}{9 \delta} + q_5 \log(H/\delta) + O(\delta^0) \right],
\]

where \( q_5 \) is given by (3.26). The \( h(\theta) \) in (3.26) is solution of following equation of motion

\[
h(1 + h^2) \sin(\theta) \ddot{h} + 2h \cos(\theta) \dot{h} + 2(2 + h^2) \sin(\theta) \dot{h}^2 + 2h(1 + h^2) \cos(\theta) \dot{h} + (4 + 7h^2 + 3h^4) \sin(\theta) = 0.
\]

**B.2 EE for cone \( c_3 \) in \( d = 6 \)**

\[
S_6|_{c_3} = \frac{8\pi^3 L^5}{\ell_p^5} \left[ \frac{H^4 \sin^3(\Omega)}{16 \delta^4} - \frac{27H^2 \cos^2(\Omega) \sin(\Omega)}{128 \delta^2} + \frac{9 \cos(\Omega) \cot(\Omega)(31 - \cos(2\Omega))}{8192} \log(\delta/H)^2 \right. \\
\left. + \left( q_6 + \frac{\sin^3(\Omega)}{4h_0^3} - \frac{27 \cos^2(\Omega) \sin(\Omega)}{64h_0^2} \right) \frac{\log(\delta/H)}{4096} \right],
\]

where \( q_6 \) is given by

\[
q_6 = \int_0^{h_0} dh \left[ \frac{\sin^3(\theta)}{hh^5} \sqrt{1 + h^2 + h^2} + \frac{\sin^2(\Omega)}{h^5} - \frac{27 \cos^2(\Omega) \sin(\Omega)}{32h^3} \\
+ \frac{9 \cos(\Omega)(31 - \cos(2\Omega)) \cot(\Omega)}{4096h} \right].
\]
\section*{B.3 EE for conical singularity \( c_1 \times R^2 \) in \( d = 6 \)}

For this case, the integrand in EE behaves as

\[
\frac{\sin(\theta) \sqrt{1 + h^2 + h^2}}{hh^5} \sim \frac{\sin(\Omega)}{h^5} + \frac{3 \cos(\Omega) \cot(\Omega)}{32h^3} - \frac{3(13 - 19 \cos(2\Omega)) \cot^2(\Omega) \csc(\Omega)}{4096h} + \frac{\cot(\Omega) \csc^3(\Omega)(-375 + 348 \cos(2\Omega) - 45 \cos(4\Omega) - 128b_3 \sin(2\Omega) + 64b_4 \sin(4\Omega))}{32768} \]

\[
+ \mathcal{O}(h), \tag{B.5}
\]

near the boundary. Using this, we find that EE is given by

\[
S_6|_{c_1 \times R^2} = \frac{4 \pi^2 L^5 h^2}{\ell_p^4} \left[ \frac{H^2 \sin(\Omega)}{8\delta^4} + \frac{3 \cos(\Omega) \cot(\Omega) \log(\delta)}{64\delta^2} \right. \\
+ \left. \frac{3h_0^3(13 - 19 \cos(2\Omega)) \cot(\Omega) \csc(\Omega) + 384h_0^6 \cos(\Omega) \cot(\Omega) (1 - 2 \log(h_0)) - 4096 \sin(\Omega)}{16384h_0^3 \delta^2} \right. \\
- \frac{1}{2\delta^2} \int_0^{h_0} dq \, q^2 J_6(q) + \frac{3(13 - 19 \cos(2\Omega)) \cot^2(\Omega) \csc(\Omega) \log(\delta)}{8192H^2} + \mathcal{O}(\delta^0) \right], \tag{B.6}
\]

where

\[
J_6(h) = \left( \frac{\sin(\theta) \sqrt{1 + h^2 + h^2}}{hh^5} + \frac{\sin(\Omega)}{h^5} - \frac{3 \cos(\Omega) \cot(\Omega)}{32h^3} + \frac{3(13 - 19 \cos(2\Omega)) \cot^2(\Omega) \csc(\Omega)}{4096h} \right). \tag{B.7}
\]

\section*{B.4 EE for cone \( c_2 \) in CFT dual to Gauss-Bonnet gravity}

For five-dimensional cone \( R^+ \times S^2 \) in CFT dual to the Gauss-Bonnet gravity, the entanglement entropy turns out to be

\[
S_5|_{c_2} = \frac{8 \pi^2 L^4}{\ell_p^4} \left[ \frac{H^3 \sin^2(\Omega)(1 - 4\lambda f_\infty)}{9\delta^3} - \frac{2H \cos^2(\Omega)(2 - 7\lambda f_\infty)}{9\delta} \right. \\
- \left. \left( \frac{2 \cos^2(\Omega)(2 - 7\lambda f_\infty) h_0^3 - 3 \sin^2(\Omega)(1 - 4\lambda f_\infty)}{9h_0^3} \right) \right. \\
+ \left. \int_0^{h_0} dh \, \mathcal{L}_3 \mathcal{L}_4 \right) \log(\delta/H) + \mathcal{O}(\delta^0), \tag{B.8}
\]

where

\[
\mathcal{L}_3 = 3 \sin^4(\theta) \left( 1 + h^2 + h^2 \right)^2 + 4 \lambda^2 f_\infty \left( h^4 (2 + 3h^2 + h^4) \sin^4(\theta) \\
+ 4h \left( 2 + 3h^2 + h^4 \right) \cos(\theta) \sin^3(\theta) \dot{h} \right. \\
+ \left. (h^4 (4 - 4h^2 + 3h^4) \cos(2\theta) \sin(2\theta) \dot{h}^2 + (h^4 \cos^4(\theta) - 3h^2 \cos^2(\theta) \sin^2(\theta) + 6 \sin^4(\theta)) \dot{h}^4 \right) \\
+ \left. 2 \lambda \sin^3(\theta) f_\infty \left( 1 + h^2 + h^2 \right) \right) \left( -4h \cos(\theta) \dot{h} + \sin(\theta) \left( 6 + 5h^2 + 2h^2 \right) \right) \\
\mathcal{L}_4 = 3h^4 \sqrt{1 + h^2 + h^2} \left( \sin^2(\theta) \left( 1 + h^2 + \dot{h}^2 \right) + 2 \lambda f_\infty \left( 2 + 3h^2 + h^4 \right) \sin^2(\theta) \\
+ \left. \left. h \left( 2 + h^2 \right) \sin(2\theta) \dot{h} + (h^2 \cos^2(\theta) - \sin^2(\theta) \dot{h}^2 \right) \right) \right). \tag{B.9}
\]
B.5 EE for cone $c_3$ in CFT dual to Gauss-Bonnet gravity

For cone $R^+ \times S^3$ in $d = 6$ dimensional CFT, which is dual to Gauss-Bonnet gravity, the EE is given by

\[
S_0|_{c_3} = 4 \frac{\pi^3 \tilde{L}_5^3}{\ell_\text{p}^6} \left[ \frac{H^4 \sin^3(\Omega) (3 - 10 \lambda f_\infty)}{48 \delta^4} - \frac{H^2 \cos^2(\Omega) \sin(\Omega) (27 - 86 \lambda f_\infty)}{128 \delta^2} \right.
\]
\[+ \frac{3 \cos(\Omega) \cot(\Omega) ((93 - 190 \lambda f_\infty) - (3 - 2 \lambda f_\infty) \cos(2\Omega)) \log(\delta/H)^2}{8192} \]
\[+ \frac{1}{12288 h_0^4} \left[ -27 h_0^4 \cos(\Omega)(1 - \cos(2\Omega)) \cot(\Omega) \log(h_0) - 5184 h_0^2 \cos^2(\Omega) \sin(\Omega) \right.
\]
\[+ 3072 \sin^3(\Omega) - 2 \lambda f_\infty (9 h_0^4 \cos(\Omega) (-95 + \cos(2\Omega)) \cot(\Omega) \log(h_0)) \]
\[+ 8256 h_0^2 \cos^2(\Omega) \sin(\Omega) + 5120 \sin^3(\Omega) \right] + \int_0^{h_0} dh \left( \mathcal{L}_5 - \mathcal{L}_5' \right) \log(\delta/H) + \mathcal{O}(\delta^0). \]  

(B.10)

Here $\mathcal{L}_5$ and $\mathcal{L}_5'$ are terms which come from making the $h$ integrand in original expression of EE finite and these are given by

\[
\mathcal{L}_5 = \frac{\mathcal{L}_{5N}}{\mathcal{L}_{5D}} \quad \text{with} \quad \mathcal{L}_{5N} = \sin^2 \theta \left( 20 \lambda^2 f_\infty^2 h^4 - 9 \cos^2(\theta) - 8 \lambda f_\infty (1 + 3 \hat{h}^2 + 2 \hat{h}^4) \right) + h^2 \sin^2 \theta \left( 12 \cos^2(\theta) \sin^2 \theta (1 + \hat{h}^2) \right) + 4 \lambda^2 f_\infty^2 h^4 (7 + 17 \cos(2\theta) - 9 \cos^2(\theta) \hat{h}^2) - \lambda f_\infty (24 \sin^2(\theta) + (17 - 23 \cos(2\theta)) \hat{h}^2 + 6 \cos^2(\theta) \hat{h}^4) \right)
\]
\[+ h^4 \left( 24 \cos^2(\theta) + 17 + 23 \cos(2\theta) + 6 \cos^2(\theta) \hat{h}^4 \right) \right),
\]
\[
\mathcal{L}_{5D} = 6 h^5 h \sqrt{1 + h^2 + \hat{h}^2} \left( 3 \lambda f_\infty h^4 \sin^2 \theta + 9 \lambda f_\infty \cos \theta \sin \theta h \hat{h} + 6 \lambda f_\infty \cos \theta \sin \theta h \hat{h} \right.
\]
\[+ \sin^2 \theta \left( 1 + h^2 + 2 \lambda f_\infty (2 - \hat{h}^2) \right) + h^2 \left( \sin^2 \theta + \lambda f_\infty \left( 7 \sin^2 \theta + 3 \cos^2(\theta) \hat{h}^2 \right) \right) \right)
\]

and

\[
\mathcal{L}_5' = - \frac{\sin^3 \Omega (3 - 10 \lambda f_\infty)}{3 h^5} + \frac{\cos^2 \Omega \sin \Omega (27 - 86 \lambda f_\infty)}{32 h^3}
\]
\[+ \frac{3 \cos \Omega \cot \Omega ((93 - 3 \cos(2\Omega)) + 2 \lambda (-95 + \cos(2\Omega)) f_\infty)}{4096 h}. \]  

(B.12)

Note that in the asymptotic limit, $h \to 0$ and $\mathcal{L}_5 = \mathcal{L}_5' + \mathcal{O}(h)$.
B.6 EE for $k \times S^2$ in CFT dual to Gauss-Bonnet gravity

The values of $\hat{\mathcal{L}}$'s in eqn. (5.28) are following:

$$
\hat{\mathcal{L}}_0 = \frac{1}{\hbar^2 \sqrt{1 + h^2 + \hat{h}^2}} \left( 1 - 2 f_{\infty} \lambda \right) \frac{\hat{h}}{h^2} (1 - 2 f_{\infty} \lambda) h^2 + 4 f_{\infty} \lambda + 1 \right) \left( 1 - 6 f_{\infty} \lambda + 8 f_{\infty}^2 \lambda^2 \right) h^4 
\hat{\mathcal{L}}_N = \frac{2 f_{\infty} \lambda \sqrt{1 + h^2 + \hat{h}^2}}{3 h^2} \cdot \frac{-1}{72 h^2 (1 + h^2) \left( 1 + h^2 + \hat{h}^2 \right)^{3/2}} \left( (1 - 2 f_{\infty} \lambda) \hat{h}^2 + 4 f_{\infty} \lambda + 1 \right) \left( 1 - 6 f_{\infty} \lambda + 8 f_{\infty}^2 \lambda^2 \right) h^4 + \left( 4 - 26 f_{\infty} \lambda - 56 f_{\infty}^2 \lambda^2 + 3 (29 - 80 f_{\infty} \lambda + 64 f_{\infty}^2 \lambda^2) h^2 \right) \frac{\hat{f}}{h^2}
\hat{\mathcal{L}}_f = \frac{-1}{16 \hbar^2 (1 + h^2) \left( 1 + h^2 + \hat{h}^2 \right)^{3/2}} \left( (1 - 2 f_{\infty} \lambda) \hat{h}^2 + 4 f_{\infty} \lambda + 1 \right) \left( 1 - 6 f_{\infty} \lambda + 8 f_{\infty}^2 \lambda^2 \right) h^4 + \left( 4 - 26 f_{\infty} \lambda - 56 f_{\infty}^2 \lambda^2 + 3 (29 - 80 f_{\infty} \lambda + 64 f_{\infty}^2 \lambda^2) h^2 \right) \frac{\hat{f}}{h^2}
\hat{\mathcal{L}}_B = z \left( - \mathcal{L}_2 \rho + \mathcal{L}_3 \hat{\rho} - \mathcal{L}_3 \hat{\rho} + \mathcal{L}_5 \rho \right)
= \frac{2 f_{\infty} \lambda \hat{g}_3 + f_{\infty} \lambda h^3 \hat{g}_3 + (1 - 2 f_{\infty} \lambda) g_3 h^2 \hat{h} + g_3 \hat{h} \left( 1 + 6 f_{\infty} \lambda + (1 - 4 f_{\infty} \lambda) \hat{h}^2 \right)}{\left( 1 + h^2 + \hat{h}^2 \right)^{3/2}}.
$$

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