Non-unitarity in quantum affine Toda theory
and perturbed conformal field theory

Gábor Takács\(^1\) and Gérard Watts\(^2\)

\(^1\) INFN – Sezione di Bologna and Dip. di Fisica – Università di Bologna
Via Irnerio 46, 40126 Bologna, Italy

\(^2\) Department of Mathematics, King’s College London,
Strand, London, WC2R 2LS, U.K.

ABSTRACT

There has been some debate about the validity of quantum affine Toda field theory
at imaginary coupling, owing to the non-unitarity of the action, and consequently
of its usefulness as a model of perturbed conformal field theory. Drawing on our
recent work, we investigate the two simplest affine Toda theories for which this is
an issue – \(a_2^{(1)}\) and \(a_2^{(2)}\). By investigating the \(S\)-matrices of these theories before
RSOS restriction, we show that quantum Toda theory, (with or without RSOS
restriction), indeed has some fundamental problems, but that these problems are
of two different sorts. For \(a_2^{(1)}\), the scattering of solitons and breathers is flawed
in both classical and quantum theories, and RSOS restriction cannot solve this
problem. For \(a_2^{(2)}\), however, while there are no problems with breather–soliton
scattering there are instead difficulties with soliton-excited soliton scattering in
the unrestricted theory. After RSOS restriction, the problems with kink–excited
kink may be cured or may remain, depending in part on the choice of gradation, as
we found in \([12]\). We comment on the importance of regradations, and also on the
survival of \(R\)-matrix unitarity and the \(S\)-matrix bootstrap in these circumstances.

\[^{1}\text{e-mail: takacs@bo.infn.it}\]
\[^{2}\text{e-mail: gmtw@mth.kcl.ac.uk}\]
1 Introduction

Affine Toda field theory has proved to be an extremely useful description of a wide class of two dimensional scattering theories. The main impetus behind the quantum treatments of these models has been the description of massive scattering theories as perturbed conformal field theories [1], and the fact that when treated in the free-field formulation the actions of such perturbed conformal field theories have the form of affine Toda theory actions [2].

These theories depend on a coupling constant $\beta$, and for real values of $\beta$ describe a set of particles with purely elastic $S$-matrices [3]. For imaginary values of $\beta$, the picture is more involved. The action is not real for real values of the fields, and typically when one finds a reality condition on the fields which will leave the action real, this will have a potential unbounded below. Clearly this makes the quantum theory problematic.

However, in the classical theory the potential has discrete vacua at real values of the fields, and there are solitonic solutions interpolating these vacua and breather solutions of zero topological charge; despite not taking real values, they have real energy and momentum [4], and indeed all the conserved quantities take real values [5]. Given these properties of the solitons and breathers, it is not surprising that much effort has been spent trying to quantise these systems in spite of the other unpleasant properties. One suggestion that has been made is that one simply needs to restrict to some well–chosen subspace of well-behaved classical solutions.

There have been several attempts [6] to follow this scheme and attempt a semi-classical quantisation of the soliton and breather sectors in the manner of [8]. This has not been entirely successful. While correctly ‘predicting’ the quantum mass corrections in some theories [8], it has failed in some other cases [9] – but neither of these papers actually applied the method properly. The semi-classical calculation of mass corrections consists of finding the spectrum of small oscillations around a background soliton, but since the Hamiltonian cannot be shown to be Hermitian, there is no guarantee that the results of [7, 8] will have any relation to the ‘real’ answer.

Other problems have been highlighted by Khastgir and Sasaki [10] who have found classical solutions with complex energy which become singular in finite time, and solutions with negative real energy, which would lead to problems if the desired quantum vacuum could tunnel to these lower energy states, creating particles in the process. However one might be able to argue that these are not in the ‘proper’ class of initial ‘breather and soliton’ configurations.

The second route to quantise Toda theories has been through the use of their quantum group symmetry. This has been carried out (at least in part) for a number of Toda theories and has been successful in some ways – although this quantum–group procedure cannot be guaranteed to give the correct $S$-matrices for the quantum solitons, the fact that in every case so far investigated [6, 8–17] the $S$-matrix of the lowest mass zero-topological charge bound states, or breathers, is identical (after analytic continuation in the coupling constant) to the conjectured $S$-matrix of the lightest mass particles in the real coupling Toda theory argues very strongly that they are right.

\footnote{There is a wide literature on the spectra of non-self-adjoint operators – for an instructive example see e.g. [10] and the references therein.}
Apart from Sine-Gordon theory \(a_1^{(1)}\), it is not expected that the \(S\)-matrices for the fundamental solitons which result from the quantum group method are unitary matrices, and the only known way to correct this is quantum group or RSOS restriction \([13, 18]\).

However, in our recent paper \([12]\) on \(\Phi_{1.5}\) perturbations of minimal models, we found examples of RSOS restricted affine Toda field theories with ‘unitary’ \(S\)-matrices for the fundamental particle but nevertheless with a complex finite-size spectrum.

This result motivated us to reexamine these issues in the two simplest affine Toda field theories, \(a_2^{(1)}\) and \(a_2^{(2)}\). These two theories have already been studied a lot; for \(a_2^{(1)}\) there are papers on their classical and quantum integrability, classical soliton solutions, classical breathers, semi-classical mass corrections, quantum \(S\)-matrices, \(S\)-matrix bootstrap and RSOS restrictions; for \(a_2^{(2)}\) there is almost an equal amount of work done.

One of the objects of this paper is to highlight problems that can already be found in the results in the literature but which have been overlooked to date. As we argue in section 2.1, it is more important that the eigenvalues of the \(S\)-matrices and higher transfer matrices are phases than that the matrix is itself unitary, since this leads to a real finite-size spectrum (to leading order); unfortunately these eigenvalues are not preserved under RSOS restriction, where the relevant matrices are the transfer matrices of \([19, 20]\). Furthermore, the choice of ‘gradation’ is also important; while this leaves the eigenvalues of the unrestricted transfer matrices invariant, it has an effect on the form of the \(S\)-matrix bootstrap and is an essential factor in the possible RSOS restrictions, and hence in the eigenvalues of the restricted transfer matrices.

We first investigate the unrestricted \(S\)-matrices of \(a_2^{(1)}\) and find that while the fundamental soliton–soliton \(S\)-matrix has phase eigenvalues, both the three-particle soliton transfer matrix and the fundamental soliton–antisoliton transfer matrix do not have phase eigenvalues for generic coupling. While this problem may or may not be cured by any particular RSOS restriction, we then show that the soliton–breather scattering in \(a_2^{(1)}\) also has intrinsic problems in both the quantum (section 3) and classical theory (section 4) which cannot be removed by RSOS restriction, and that any theory containing both solitons and breathers will have a complex spectrum.

For \(a_2^{(2)}\) we find rather different results, in that the unrestricted fundamental soliton–soliton scattering is well behaved, but that the \(S\)-matrix of the fundamental soliton with the first excited soliton is badly behaved. For \(a_2^{(2)}\) however, the classical theory (section 6) does not seem to reflect these difficulties.

First we start in section 2 with some generalities on \(S\)-matrices, their properties and constructions, and treat \(a_2^{(1)}\) and \(a_2^{(2)}\) in turn in sections 3–6. In section 7 we discuss a particular example which shows how RSOS restriction can fail in rendering the scattering theory consistent and finish with our conclusions in section 8.

2 Generalities
2.1 S-matrices and unitarity

A quantum field theory is unitary if the inner product on the space of states is positive definite and the S-matrix is unitary with respect to this inner product. However, in statistical field theory this is too strong a requirement, and a more natural one is that the spectrum is real.

We will consider integrable field theories which have factorised S-matrices. We consider two sorts of theories: those where the particles – solitons and breathers – are classified by their topological charge; and later another where it is instead the vacua interpolated by the particles which are labelled.

2.1.1 S-matrices of particle theories

The particles are divided into different ‘species’, all the particles of the same species having the same non-zero mass, and each species having one or more ‘flavours’. We denote a particle of species $A$, flavour $i$ and rapidity $\theta$ by $A_i(\theta)$.

If the in state is a 2-particle state $|A_i(\theta) B_j(\phi)\rangle_{in}$ with $\theta > \phi$, then the S-matrix is

$$|A_i(\theta) B_j(\phi)\rangle_{in} = S_{AB}(\theta - \phi)_{ij}^{kl} |A_k(\theta) B_l(\phi)\rangle_{out},$$

(2.1)

where $\theta - \phi$ is real and positive for physical scattering. $S_{AB}$ is a unitary matrix if

$$\sum_{k,l} S_{AB}(\theta)_{ij}^{kl} (S_{AB}(\theta)_{i'j'}^{kl})^* = \delta_{ii'} \delta_{jj'},$$

(2.2)

or in matrix notation

$$S_{AB}(\theta) S_{AB}(\theta)^\dagger = 1.$$  

(2.3)

The S-matrix of a unitary field theory possesses several properties, such as real analyticity (A), and may possess other discrete symmetries such as parity (P), time reversal (T) and charge-conjugation (C). These are given in [21] as

$$S_{AB}(\theta) = \begin{bmatrix} A & C & P & T \end{bmatrix} \begin{bmatrix} S_{AB}(\theta)_{ij}^{kl} \\ -(S_{AB}(\theta)_{ij}^{kl})^* \\ S_{BA}(\theta)_{ji}^{lk} \\ S_{BA}(\theta)_{ji}^{lk} \end{bmatrix}$$

Using (A) and (PT) it is possible to rewrite (2.2) as

$$\sum_{k,l} S_{AB}(\theta)_{ij}^{kl} S_{AB}(\theta)_{ji}^{mn} = \delta_{ij} \delta_{ji},$$

(2.5)

for all values of $\theta$, which is often simply written in matrix notation

$$S_{AB}(\theta) S_{AB}(\theta) = 1.$$  

(2.6)

This is usually known as ‘unitarity’, but the only Toda theory for which this is true seems to be Sine-Gordon theory, and it has no direct relation to either unitarity (2.2) nor to the ‘$R$-matrix unitarity’ of the underlying quantum group structure.
An alternative notation for the $S$-matrix uses the matrix $\tilde{S}_{AB}$ which is simply related to $S_{AB}$ by

$$\tilde{S}_{AB}(\theta)_{ij}^{kl} = S_{AB}(\theta)_{ij}^{ik},$$

(2.7)

If $S$ satisfies (A) and (T) we can express unitarity (2.2) as the matrix equation

$$\tilde{S}_{AB}(\theta) \tilde{S}_{BA}(-\theta) = 1,$$

(2.8)

which is again also known as ‘unitarity’. Since this is indeed the equation satisfied by the $S$-matrices derived by the quantum group construction, we shall refer to this as $R$-matrix unitarity, or RU for short.

Note that RU is equivalent to (2.6) if the $S$-matrix is (P) invariant, which is the case in Sine-Gordon theory.

2.1.2 Reality of the spectrum

When we try to describe a two-dimensional statistical model in terms of an effective scattering theory, it is not necessary for the space of states to have a positive definite inner product, and indeed many interesting and physically relevant models are of this sort, for example the scaling Yang–Lee model [22]. However, whether or not the $S$-matrix satisfies unitarity (2.2) for a particular choice of the Hilbert space inner product, it is certainly necessary that the eigenvalues of the $S$-matrix are phases for real rapidities if the spectrum is to be real. While we do not know the exact equations for the finite-size spectrum in the general case\footnote{Note exact equations for the finite size spectrum are now known for several non-trivial interacting field theories, see e.g. [23].}, we know that the leading order finite-size effects are given by the Bethe Ansatz.

The Bethe Ansatz equations express the condition that the wave-function of a multi-particle state on a circle is single-valued under the operation of taking a particle around the circle. For a state of two particles $A_a(\theta_A)$ and $B_b(\theta_B)$ on a circle of circumference $R$, the wavefunctions $\psi_{ab}(\theta_A, \theta_B)$ must satisfy the two simultaneous equations

$$S_{AB}(\theta_A - \theta_B)_{a'b'}^{a'b} \psi_{a'b'} = \eta \exp(-iRm_A \sinh \theta_A) \psi_{ab},$$

$$S_{BA}(\theta_B - \theta_A)_{b'a'}^{b'a} \psi_{b'a'} = (1/\eta) \exp(-iRm_B \sinh \theta_B) \psi_{ab},$$

(2.9)

where $\eta$ is a phase depending on the relative statistics of particles $A$ and $B$. These equations are only consistent if the two matrices commute, but in our situations they obey the stronger condition RU and so consistency is guaranteed. Diagonalising these equations, they reduce to

$$s_{AB}(\theta_A - \theta_B) \exp(iRm_A \sinh \theta_A) = \eta, \quad \exp(iR(m_A \sinh \theta_A + m_B \sinh \theta_B)) = 1,$$

(2.10)

where $s_{AB}$ is an eigenvalues of the $S$-matrix $S_{AB}(\theta)$. For this equation to be satisfied with real rapidities, $s_{AB}(\theta_A - \theta_B)$ should be a pure phase for real $\theta_A, \theta_B$; if it is not a pure phase, there can be no solutions with real rapidities, and consequently the two-particle states satisfying this quantisation condition will have complex energy and momentum.
The generalisation of (2.10) to multi-particle states of particles $A, B \ldots N$ of masses $m_A, \ldots$ and rapidities $\theta_A \ldots$ is
\[
t_A(\theta_A, \ldots, \theta_N) \exp(iRm_A \sinh \theta_A) = \eta_{AB} \eta_{AC} \ldots \eta_{AN},
\]
\[
t_B(\theta_A, \ldots, \theta_N) \exp(iRm_B \sinh \theta_B) = \eta_{BC} \eta_{BD} \ldots \eta_{BA}, \quad \text{etc}
\] (2.11)
where $\eta_{AB}$ are phases depending on the relative statistics of the particles, and $t_A, t_B$ etc are simultaneous eigenvalues of the commuting transfer matrices $T_A, T_B$ etc which are shown graphically in figure 1 and which are given by

\[
T_A(\theta_A, \ldots, \theta_N)_{a' b' c' \ldots n'} = S_{AB}(\theta_A - \theta_B)_{a b} S_{AC}(\theta_A - \theta_C)_{a c} \ldots S_{AN}(\theta_A - \theta_N)_{a (n-2)},
\]
\[
T_B(\theta_A, \ldots, \theta_N)_{a' b' c' \ldots n'} = S_{BC}(\theta_B - \theta_C)_{b c} S_{BD}(\theta_B - \theta_D)_{b d} \ldots S_{BA}(\theta_B - \theta_A)_{b (n-2)}, \quad \text{etc}
\] (2.12)

Figure 1: The transfer matrices $T_A$ and $T_B$.

\[
T_A(\theta_A, \ldots, \theta_N)_{a' b' c' \ldots n'} = S_{AB}(\theta_A - \theta_B)_{a' b} S_{AC}(\theta_A - \theta_C)_{a' c} \ldots S_{AN}(\theta_A - \theta_N)_{a' (n-2)};
\]
\[
T_B(\theta_A, \ldots, \theta_N)_{a' b' c' \ldots n'} = S_{BC}(\theta_B - \theta_C)_{b' c} S_{BD}(\theta_B - \theta_D)_{b' d} \ldots S_{BA}(\theta_B - \theta_A)_{b' (n-2)}, \quad \text{etc}
\]

Just as in the two-particle case, we need the eigenvalues $t_A \ldots$ to be pure phases for real rapidities if the multi-particle states are to have real energy.

If (2.2) holds, i.e. if $S_{AB}(\theta)$ is a unitary matrix for real $\theta$, then it is easy to see that all the transfer matrices $T_A$ are also unitary and hence all their eigenvalues are pure phases. However, (2.2) is not a necessary condition – if $S_{AB}$ is conjugate to a unitary matrix by a change of basis of the one particle states, then the transfer matrices $T_A$ are also conjugate to unitary matrices and hence will have phase eigenvalues. This distinction that the change of basis must only depend on the one particle states is important: if $S_{AB}$ has phase eigenvalues, then it is given by a unitary matrix in some basis, but it is only if this basis is related to the original basis by a change of the one-particle states that all the higher transfer matrices will have phase eigenvalues.

The point of this discussion is that it is very hard to determine whether a given $S$-matrix is conjugate to a unitary matrix by a change of the inner products of one-particle states, since this change of basis may be very complicated and involve rapidity-dependent changes of inner products, but if the eigenvalues of any transfer matrix are not phases then we can be sure that this is not the case and that the finite size spectrum will not be real.

---

3 The transfer matrices $T_A, T_B$ etc commute by virtue of the Yang-Baxter relation for the $S$-matrices.
2.2 Quantisation of affine Toda theory: Quantum group method

As shown by Bernard and LeClair [24], the ‘imaginary-coupling’ quantum affine Toda field theory based on the affine algebra $g$ has a $U_q(g^\vee)$ symmetry. Since this algebra includes the topological charge as one of its generators, we must assume that the solitons transform non-trivially under this algebra, and this leads to the requirement that the soliton-soliton $S$-matrices are proportional to the $R$-matrices of the symmetry algebra.

The $S$-matrix of the fundamental excitation is derived from the universal $R$-matrix of an affine quantum group in its fundamental representation. There are again two different notations $R$ and $\tilde{R}$ for the $R$-matrix which are related by $\tilde{R} = P \cdot R$ where again $P$ is the permutation operator.

Such a matrix $\tilde{R}(x,q)$ has two parameters - $q$ the deformation parameter which is a pure phase, and $x$ the spectral parameter. $\tilde{R}$ satisfies the matrix equation

$$\tilde{R}(x,q) \tilde{R}(1/x,q) \propto 1.$$ \hspace{1cm} (2.13)

In the theory of $R$-matrices an $R$-matrix is also said to be unitary if (2.13) holds [13].

The $S$-matrix $\tilde{S}_{AA}$ of the fundamental excitation is then obtained as

$$\tilde{S}_{AA}(\theta)^{kl}_{ij} = f(\theta) \tilde{R}(\eta AA x(\theta), q)^{kl}_{ij},$$ \hspace{1cm} (2.14)

where $x(\theta) = \exp(\alpha \theta)$ for some constant $\alpha$, $\eta AA = \pm 1$ (the sign to be decided on the basis of the bootstrap) and where $f(\theta)$ is a suitable scalar function ensuring (RU) holds:

$$\tilde{S}_{AA}(\theta) \tilde{S}_{AA}(-\theta) = 1,$$ \hspace{1cm} (2.15)

and which can be fixed by requiring crossing symmetry and a suitable number of CDD poles. (n.b. different physical models may differ exactly in the number and position of these CDD poles – see [25]).

The $S$-matrices for the higher particles are to obtained using bootstrap fusion. It is important to note here that since $\tilde{S}_{AB}$ and $\tilde{S}_{BA}$ are derived independently using the $S$-matrix bootstrap, it is not free for us to impose Parity invariance on a $S$-matrix, rather we have to check whether or not it holds. As noted by Delius, RU will also hold for all higher particle $S$-matrices, assuming that one can associate a representation of the quantum group to each particle species [26].

As stated before, RU is only equivalent to (2.6) if the $S$-matrices are (P) invariant, and this is usually not the case in affine Toda field theories – the only known exception being Sine–Gordon theory. If we really want the $S$-matrix to be a unitary matrix, then as noted by Smirnov [13], the only known method is quantum group restriction to an RSOS scattering theory of kinks (and breathers).

Quantum group restriction relies on a choice of gradation, and so we discuss this first.

2.2.1 Gradations

The symmetry algebra of $g$–affine Toda field theory is $U_q(g^\vee)$ which is generated by non-local charges with non-trivial Lorentz spin, and the Lorentz spins of these charges may be changed
by redefining the energy momentum tensor, or, equivalently, by the addition to the action of a Feigin-Fuchs term as we discuss in the appendix [3].

Furthermore the charges act on multi-particle states by a representation which is rapidity dependent, the dependence being determined by the spins of the charges. This may be made clearer by considering the charges $e_{\alpha}$ associated to the simple roots $\alpha$ of $g^{\vee}$, which have spins

$$s_{\alpha} = \frac{4\pi}{\beta^2} |\alpha|^2 - 1 + \gamma \cdot \alpha,$$  

(2.16)

where $\gamma$ is an arbitrary vector corresponding to the ambiguity in the spins; a choice of $\gamma$ is called a choice of gradation. There are two distinguished classes of gradations:

- The spin gradation
  In this gradation the charges have their canonical spins, and is given by $\gamma = 0$.
- Homogeneous gradations
  A gradation is called homogeneous if all spins $s_{\alpha}$ are zero apart from one.

The $S$-matrices depend on the gradation, and a change of gradation is a rapidity dependent similarity transformation on $S$.

For $U_q(a^{(1)}_2)$ all homogeneous gradations are equivalent, and the algebra of zero-spin charges is $U_q(a_2)$; there are two inequivalent homogeneous gradations for $U_q(a^{(2)}_2)$, for which the algebras of zero-spin charges are $U_q(a_1)$ and $U_{q^4}(a_1)$ and which we call the (1, 2) and (1, 5) gradations respectively.

### 2.2.2 Quantum group restriction

Quantum group restriction relies on a choice of gradation for which a subset of the non-local conserved quantities are Lorentz scalars; this means that the action of the corresponding quantum subalgebra on multi-particle states commutes with $\mathcal{R}$. Given this situation it then makes sense to restrict to highest weight states of this subalgebra. As a result, $N$–particle states may be labelled by sequences of $N+1$ highest weights (obeying certain relations). For $q$ a root of unity, one can then consistently restrict the Hilbert space further so that the highest weights of a sequence are chosen from a finite subset (depending on $q$), and these theories are exactly the quantum group restricted RSOS theories. In this formulation we label $S$-matrix for particles of species $A$ and $B$ in terms of the vacua by

$$d \quad a \quad c \quad b \quad \theta \quad \varphi \quad \leftrightarrow \quad S_{AB} \left(\begin{array}{cc} a & c \\ b & \end{array} \right) (\theta - \phi).$$  

(2.17)

This procedure preserves several properties of the unrestricted theory, including the Yang-Baxter relation, $R$-matrix unitarity and crossing symmetry, and furthermore the eigenvalues of the matrices $t_{AB}^{(ac)}$ whose elements are

$$\left( t_{AB}^{(ac)} \right)_{bd} = S_{AB} \left(\begin{array}{cc} a & c \\ b & \end{array} \right)$$  

(2.18)
are a subset of those of $\mathcal{S}$. However, what makes this procedure useful and interesting is that the physically relevant data, i.e. the eigenvalues of the transfer matrices (see [19, 20])

$$T_{a_1 a_2 \ldots a_{k-1} b_k}^{b_1 b_2 \ldots b_{k-1} b_k} (\theta_1, \ldots, \theta_k) = g_{a_1}^{b_1} \prod_{j=2}^{k} S_{A_1 A_j} (b_{j-1} a_j, a_{j+1}) (\theta_1 - \theta_j), \quad (2.19)$$

have little relation to those of the unrestricted $S$-matrix $\mathcal{S}$. This means that it is possible to obtain a unitary theory from a nonunitary theory by RSOS restriction, and the resulting theories can have a very rich finite-size spectrum.

### 3 Quantum $S$-matrices of the $a_2^{(1)}$ theory

The first affine Toda theory to be investigated after Sine-Gordon theory, and the first in which the problems mentioned above may arise, is the $a_2^{(1)}$ theory. In the real-coupling case, this leads to a theory of two real scalar fields which may be thought of as charge conjugates of each other. The first author to treat the imaginary coupling theory was Hollowood, who found a spectrum of solitons and breathers, in different species and different degrees of excitation, with the exact details depending on the coupling constant $\beta$.

In order to state his results, we give our conventions (which in this case follow sections 2 and 5 of [16])

$$x = \exp(3\lambda \theta), \quad q = -\exp(i\pi \lambda), \quad \lambda = \frac{4\pi}{\beta^2} - 1. \quad (3.1)$$

Hollowood found that the solitons come in two conjugate species $(a) = 1, 2$ each with three flavours (corresponding to the $3$ and $\overline{3}$ representations of $a_2$), and in various excitation levels $k = 0, 1, 2, \ldots, [\lambda]$; such solitons are denoted $A_k^{(a)}$ and have mass

$$M_k^{(A)} = 2M \cos \left( \frac{\pi}{3} \left( 1 - \frac{k}{\lambda} \right) \right). \quad (3.2)$$

The fundamental solitons are then $A_0^{(1)}$ and $A_0^{(2)} = A_0^{(1)}$. Hollowood also found that there were two conjugate breather species $(a) = 1, 2$ which have excitation levels $p = 1, 2, \ldots, \lfloor \frac{3\lambda}{2} \rfloor$, are denoted by $B_p^{(a)}$ and have mass

$$M_p^{(B)} = 2M \sin \left( \frac{\pi p}{3\lambda} \right). \quad (3.3)$$

Again we have $B_p^{(2)} = B_p^{(1)}$.

Hollowood only considered the $S$-matrices for the scattering of fundamental solitons and anti-solitons, and the first author to treat the remaining $S$-matrices was Gandenberger [27]. We give their results and comment upon them in the next two sections.

#### 3.1 The unrestricted $S$-matrices for soliton–soliton scattering in $a_2^{(1)}$

The unrestricted $S$-matrices for $a_2^{(1)}$ soliton scattering are given in [1], but the eigenvalues were not investigated, nor was the matrix-unitarity of the $S$-matrix. The relevant $S$-matrices are the $9 \times 9$ matrices $S_{A_k A_l} (\theta)$ and $S_{A_k A_l} (\theta)$, which take the form

$$S_{A_k A_l} (\theta) = f_{kl} (\theta) R_{33} (\eta_{kl} x, q), \quad S_{A_k A_l} (\theta) = \tilde{f}_{kl} (\theta) R_{33} (\overline{\eta}_{kl} x, q), \quad (3.4)$$
where the prefactors $f_{kl}, \tilde{f}_{kl}$ are always phases for real $\theta$ and $\eta_{kl}$ and $\tilde{\eta}_{kl}$ are phases which may depend on the gradation. In the homogeneous and spin gradations these phases take values $\eta_{kl} = -\tilde{\eta}_{kl} = (-1)^{k+l}$.

The eigenvalues of $R_{33}(x, q)$ are each triply degenerate, and are

$$1, \left(\frac{1 + q\sqrt{x}}{q + \sqrt{x}}\right), \left(\frac{1 - q\sqrt{x}}{q - \sqrt{x}}\right)$$

which are clearly phases for non-negative real $x$ and $q$ a phase. However, for $q$ a phase and $x$ real and negative these are not phases, and so we immediately see that the $S$-matrix $S_{A_0A_1}$ has non-physical eigenvalues. Hence for the unrestricted theory to be unitary, we need to be in the regime $\lambda < 1$ for which there are no excited solitons (Hollowood also reached this conclusions but from different reasonings).

Furthermore, when we investigated the Bethe-Ansatz equations for three particles of type $3$ numerically, we found that the transfer matrices did not in general have phase eigenvalues. Hence, while $S_{33}$ is conjugate to a unitary matrix, this does not simply involve a change of inner product on the one particle states, and even the fundamental $S_{A_0A_0} S$-matrix is not well behaved.

The problems with $a_2^{(1)}$ are also clearly seen in the eigenvalues of $R_{3\bar{3}}(x, q)$, which are 1 (six times degenerate) and the three eigenvalues

$$\left(\frac{1-qx^{1/3}}{q-x^{1/3}}\right), \left(\frac{1-q^{x^{1/3}}e^{2\pi i/3}}{q-x^{1/3}e^{2\pi i/3}}\right), \left(\frac{1-q^{x^{1/3}}e^{4\pi i/3}}{q-x^{1/3}e^{4\pi i/3}}\right).$$

Thus the eigenvalues of $R_{3\bar{3}}(x, q)$ are not all phases for $q$ a generic phase and $x$ a positive or negative real number, and there is no need to examine the three-particle states.

This certainly suggests (contrary to the comments in [6]) that the unrestricted $a_2^{(1)}$ theory is not a unitary theory (in any sense other than RU), even in the repulsive (no bound state) regime, apart from possible discrete values of the coupling constant.

### 3.2 The $S$-matrices for soliton–breather scattering in $a_2^{(1)}$

While Hollowood discussed the presence of breathers in the spectrum, Gandenberger was the first to compute their $S$-matrices; his results for the $S$-matrices of the fundamental soliton ($A=A_0^{(1)}$), anti–soliton ($\bar{A}$), fundamental breather ($B=B_1^{(1)}$) and conjugate breather ($\bar{B}$) are

$$S_{AB} = S_{BA} = S_{\bar{A}B} = S_{\bar{B}A} = \frac{\langle 3 + B \rangle \langle 3 - B \rangle}{\langle -1 + B \rangle \langle 7 - B \rangle}.$$  

$$S_{BA} = S_{\bar{A}B} = S_{\bar{B}A} = S_{AB} = \frac{\langle -7 + B \rangle \langle 1 - B \rangle}{\langle 3 - B \rangle \langle -3 + B \rangle},$$

where

$$\langle x \rangle = \sinh \left(\frac{\theta}{2} + \frac{i\pi x}{12}\right), \quad B = -\frac{1}{2\pi} \frac{\beta^2}{1 - \beta^2/4\pi} = -\frac{2}{\lambda}.\quad (3.8)$$

As can be readily seen, this does not satisfy the full range of properties, breaking (P) and (T), but preserving (A), (C) and (PT) while also being crossing symmetric. Also, since it is
derived by the quantum–group method, it obeys RU (2.8)

\[ S_{AB}(\theta) S_{BA}(-\theta) = 1 , \]  

for all \( \theta \), but since it breaks \((\mathcal{P})\), it does not satisfy (2.6), that is for real \( \theta \),

\[ S_{AB}(\theta) (S_{AB}(-\theta)) \neq 1 , \]  

Hence, in this case, we can attribute the failure of unitarity (2.2) to the failure of parity, since unitarity is equivalent to (A), RU and (P), and of these only the last fails. As a result, we expect that this model will have a complex spectrum whenever it contains both breathers and solitons.

4 Classical scattering in \( a_2^{(1)} \)

In the previous section we have just seen that the quantum scattering of solitons with breathers in \( a_2^{(1)} \) has severe problems – but how can this be reconciled with the supposedly consistent scattering in the classical theory? To answer this question we first have to recall some elements of the construction of soliton and breather solutions.

All the known soliton solutions of the \( a_1^{(1)} \) Toda theories were found by Hollowood in [28], the breathers by Harder et al in [29], and the scattering of solitons by Olive et al. in [30]. We take the action of \( a_2^{(1)} \) to be

\[ S = \frac{1}{2} (\dot{\phi}^2 - \phi'^2) - \frac{\mu^2}{\beta^2} \sum_{i=0}^{2} (\exp(i \beta \alpha_i \cdot \phi) - 1) , \]  

where \( \beta \) is imaginary, and \( \alpha_i \) are the simple roots of \( a_2^{(1)} \), with \( \alpha_1^2 = 2 \). The stationary points of the potential in (4.1) are

\[ \phi = \frac{2\pi}{\beta} (m_1 \lambda_1 + m_2 \lambda_2) , \]  

where \( \lambda_i \) are the fundamental weights of \( a_2^{(1)} \). We consider only the solutions which tend to one of these vacua as \( |x| \to \infty \), and denote the topological charge of such a solution by \((m_1, m_2)\), where

\[ \phi|_{x\to\infty} - \phi|_{x\to-\infty} = \frac{2\pi}{\beta} (m_1 \lambda_1 + m_2 \lambda_2) . \]  

Hollowood gives the generic multi-soliton solution of \( a_2^{(1)} \) in terms of tau-functions \( \tau_j \) as

\[ -\beta \phi = \sum_{j=0}^{2} \alpha_j \log \tau_j = \frac{1}{\sqrt{2}} \left( \log(\tau_1^2/(\tau_0 \tau_2)), \log(\tau_2/\tau_0) \right) , \]

\[ \tau_j = \sum_{k=0}^{N} \sum_{1 \leq j_1 < \ldots < j_k \leq N} \left( \prod_{i=1}^{k} \omega^j \alpha_i p_{j_i} X(\theta_{j_i}, x, t) \right) \left( \prod_{1 \leq i < \ell \leq k} X_{\alpha_i, \alpha_{j_\ell}}(\theta_{j_i} - \theta_{j_\ell}) \right) , \]

where

\[ \omega = \exp(2\pi i/3) , \quad X(\theta, x, t) = \exp(m(x \cosh \theta - t \sinh \theta)) , \]

\[ X_{1,1}(\theta) = X_{2,2}(\theta) = \frac{\cosh \theta - 1}{\cosh \theta + 1/2} , \quad X_{1,2}(\theta) = X_{2,1}(\theta) = \frac{\cosh \theta - 1/2}{\cosh \theta + 1} . \]
where \( m = \sqrt{3} \mu \), and a particular solution is given by a choice of \( N \) and the \( 3N \) parameters \( \{ a_i, \theta_i, p_i \} \) with \( a_i \in \{1, 2\} \), \( \theta_i \in \mathbb{C} \) and \( p_i \in \mathbb{C}^* \); such a solution has total energy \( E \) and momentum \( P \) given by

\[
E = M \sum_i \cosh \theta_i, \quad P = M \sum_i \sinh \theta_i, \tag{4.6}
\]

where \( M = (8m)/\beta^2 \). To find the allowed values of \( p_i, \theta_i, a_i \) for which this gives a solution regular for all \( x, t \), is in general a difficult problem, as it is to find the overall topological charge of a solution, but this can be fully analysed for the cases \( N=1 \) and \( N=2 \).

### 4.1 The fundamental soliton solutions of \( a_2^{(1)} \)

The simplest such solutions are the single solitons, which are parametrised by a rapidity \( \theta \), a ‘species’ label \( a \in \{1, 2\} \), and a complex number \( p \) of which the magnitude determines the initial position and the phase determines the ‘shape’ and (together with \( a \)) the topological charge. For the single solitons, (4.4) becomes

\[
\tau_j = 1 + \omega^{aj} p X(\theta, x, t). \tag{4.7}
\]

This solution will be singular for any values of \( x, t \) for which one of \( \tau_j \) is zero. Since \( X(\theta, x, t) \) can take any real positive values, each \( \tau_j \) will be zero for some (particular) values of \( x \) and \( t \) if the phase of \( p \) is \( \pi + 2n\pi/3 \) for any integer \( n \). As a result, the only restriction is on \( p \) which is that its phase must not be equal to \( (2n + 1)\pi/3 \) for any integer \( n \).

For each choice of \( a = 1, 2 \), the topological charge of the soliton is constant on each region of allowed \( p \) values. By inspection, we find that the topological charges (4.3) take the values \( (m_1, m_2) \) as shown in figure 3. Note that \( a = 1 \) gives topological charges in the fundamental representation \((1, 0) \equiv 3 \) of \( a_2 \), and \( a = 2 \) in the fundamental representation \((0, 1) \equiv \bar{3} \).

![Figure 2: The dependence on \( p \) of the topological charges of the fundamental solitons of \( a_2^{(1)} \) for \( a = 1 \) and \( a = 2 \).](image)

### 4.2 Solutions with \( N = 2 \)

For a solution (4.4) of \( a_2^{(1)} \) with \( N = 2 \) to have real energy and momentum, we must either take \( \theta_1, \theta_2 \) to be both real, or to be a complex conjugate pair. The solutions with two real rapidities correspond to scatterings of two fundamental solitons and we discuss these in section 4.3, and now we consider only the solutions with complex conjugate rapidities.
We also can simplify further our analysis by a Lorentz transformation so that \( \theta_1 = -\theta_2 = i \alpha \) is pure imaginary and the solutions are now stationary and periodic in time. The mass of such a solution is \( 2M \cos \alpha \), where \( M \) is the mass of a single soliton, and consequently the fundamental region of \( \alpha \) is \([-\pi/2, \pi/2]\).

By the following redefinitions,

\[
\begin{align*}
p_1 &= r_1 \exp(i\phi_1), & p_2 &= r_2 \exp(i\phi_2), \\
u &= \sqrt{r_1/r_2}, & \delta_j &= (\phi_1 + \phi_2)/2 + \pi j (a + b)/3, \\
\end{align*}
\]

the general two soliton solution with parameters \( \{a, p_1, i\alpha\} \) and \( \{b, p_2, -i\alpha\} \) can be cast in the form

\[
\tau_j = 1 + e^{i\delta_j} (u z_j(x, t) + z_j(x, t)^* / u) + e^{2i\delta_j} X_{ab}(2i\alpha) |z_j(x, t)|^2.
\]

The \( x \) and \( t \) dependence of \((4.10)\) is contained in \( z_j(x, t) \), which will take any value in \( C^* \) for suitable \( x \) and \( t \). Hence, the two-soliton solution will be singular if

\[
\tau_j = 1 + e^{i\delta_j} (u z + z^* / u) + e^{2i\delta_j} X_{ab}(2i\alpha) |z|^2,
\]

is zero for any complex number \( z \). This is amenable to complete analysis, which we relegate to appendix A, and the result is that the solution is singular for some value of \( x \) and \( t \) provided that, for each \( j \),

\[
\left( \tau - (2X_{ab}(2i\alpha)(\cos(2\delta_j) + 1) - 2) \right) \left( \tau - (2X_{ab}(2i\alpha)(\cos(2\delta_j) - 1) + 2) \right) \geq 0,
\]

\[
2 \left( 2X_{ab}(2i\alpha) - 1 \right) \cos(2\delta_j) \leq \tau,
\]

where \( \tau = u^2 + u^{-2} \).

To find zero-topological charge breathers we must take \( a = 1 \) and \( b = 2 \), and find \( X_{12}(2i\alpha) = 1 - 3/(4 \cos^2 \alpha) < 1 \). Since we require \( X < 0 \) or \( X > 1 \), we must take \( \pi/2 > |\alpha| > \pi/6 \), so that \( 0 < M(\alpha) < \sqrt{3}M \). We note here that the solutions with \( \alpha > 0 \) and \( \alpha < 0 \) can be regarded as conjugate to each other, in the manner of Gandenberger. It is also interesting to note that the maximum mass of a regular classical breather solution is \( \sqrt{3}M \), whereas quantum considerations suggest the breathers have masses up to \( 2M \). Whether this signals that the quantum \( S \)-matrices obtained from quantum groups methods are not the quantisations of the classical theories, or that there are classical breather solutions to be found, or whether there is some other explanation, we cannot say.

Next, since \( \delta_j(x, t) \) is independent of \( j \), we need only consider \( j = 0 \) with the result that for each value of \( \pi/6 < \alpha < \pi/2 \) there is a single finite region of \( \{2 < \tau, 0 < \delta < \pi\} \) with regular solutions, bounded by the curves \( \tau = 2 \) and \( \tau = 2 + 2X_{12}(2i\alpha)(\cos(2\delta) - 1) \). For example, figure B shows the allowed region for \( \alpha = \pi/3 \). By inspection, the topological charge of a regular breather solution is always zero.

### 4.3 Scattering of two solitons in \( a_2^{(1)} \)

The scattering of solitons has already been investigated in [30]. The scattering process itself is complicated, but it is quite straightforward to examine the final state, and it was shown that the only result of solitons scattering is a change in trajectory of the participating solitons.
Figure 3: Allowed ranges of $\tau, \delta$ (shaded) for a breather of $a_2^{(1)}$ with $\alpha = \pi/3$

If we want to describe the scattering of a fundamental soliton of parameters $\{a_1, \theta_1, p_1\}$ with another fundamental soliton of parameters $\{a_2, \theta_2, p_2\}$, this is not simply the solution \[\text{(4.4)}\] with parameters $\{a_1, \theta_1, p_1; a_2, \theta_2, p_2\}$. To find the correct $N = 2$ soliton solution we must consider the $N = 2$ solution with parameters $\{a_1, \theta_1, q_1; a_2, \theta_2, q_2\}$ so that $\tau_j$ takes the form

$$
\tau_j = 1 + \omega^{a_1j} q_1 Y + \omega^{a_2j} q_2 \Upsilon + \omega^{(a_1 + a_2)j} X_{a_1, a_2}(\theta_{12})q_1q_2 Y \Upsilon ,
$$

(4.14)

where

$$
Y = X(\theta_1, x, t) , \quad \Upsilon = X(\theta_2, x, t)
$$

(4.15)

We assume that $\theta_{12}$ is positive, so that in the initial state for $t \ll 0$, $|Y| \gg |\Upsilon|$. As a result, for the two ranges of $x$ such that $|Y q_1| \sim 1$ and $|Y q_2| \sim 1$, the solution describes two well separated solitons. In the first case, $|q_2 \Upsilon| \ll 1$, so that for this range of $x$ the tau function is dominated by

$$
\tau_j \sim 1 + \omega^{a_1j} q_1 Y ,
$$

(4.16)

which is a single soliton of parameters $(a_1, \theta_1, q_1)$. However, in the range of $x$ such that $|Y q_2| \sim 1$, we see that $|Y q_1| \gg 1$ so that the tau function is dominated by

$$
\tau_j \sim q_1 Y \omega^{a_1j} \left(1 + \omega^{a_2j} (q_2 X_{a_1, a_2}(\theta_{12})) \Upsilon\right) ,
$$

(4.17)

which is a single soliton with parameters $(a_2, \theta_2, q_2 X_{a_1, a_2}(\theta_{12}))$. Hence, in the initial configuration, we must take the parameters $q_1$ and $q_2$ to be

$$
q_1 = p_1 , \quad q_2 = \frac{p_1}{X_{a_1, a_2}(\theta_{12})} .
$$

(4.18)

We can repeat this exercise for $t \gg 0$ and find that the solitons in the final state have parameters

$$
(a_1, \theta_1, q_1 X_{a_1, a_2}(\theta_{12})) , \quad (a_2, \theta_2, q_2)
$$

(4.19)

Thus it appears that the net effect of the scattering process has been to change the parameters of the two solitons in the following way:

$$
(a_1, \theta_1, p_1) \mapsto (a_1, \theta_1, p_1 X_{a_1, a_2}(\theta_{12}))
$$

$$
(a_2, \theta_2, p_2) \mapsto (a_2, \theta_2, p_2 / X_{a_1, a_2}(\theta_{12}))
$$

(4.20)
However, since $X_{a_1,a_2}(\theta_{12})$ is real for real rapidities $\theta_i$, the net effect can be simply absorbed into a time delay or advance for the trajectory of each soliton, as can be seen from

$$\tau_j(a_1, \theta_1, p_1 X_{a_1,a_2}(\theta_{12}))(x,t) = \tau_j(a_1, \theta_1, p_1)(x,t + \Delta t_1), \quad (4.21)$$

$$\tau_j(a_2, \theta_2, p_2 / X_{a_1,a_2}(\theta_{12}))(x,t) = \tau_j(a_2, \theta_2, p_2)(x,t + \Delta t_2), \quad (4.22)$$

where

$$\exp(m \sinh \theta_1 \Delta t_1) = X_{a_1,a_2}(\theta_{12}), \quad \exp(m \sinh \theta_2 \Delta t_2) = X_{a_1,a_2}(\theta_{12})^{-1}. \quad (4.23)$$

There is perhaps some slight sign of potential problems in the quantum theory in that for each choice of $\theta_1, \theta_2, p_1$, then for any $x, t$ one can find a value of $p_2$ such that the solution is singular at that point – but in general these singularities are isolated in spacetime, and one might argue that they present no real problems.

### 4.4 The results of general scatterings in $a_2^{(1)}$

It should be clear from the previous calculation how to generalise this result to the arbitrary scatterings of solitons, breathers and breathing solitons of $a_2^{(1)}$, simply by considering the scattering of their constituent solitons.

The situation of interest to us is the scattering of a breather with a single soliton. Consider a soliton of parameters $(a, \theta, p)$ scattering from a breather of parameters $(\phi, \alpha, q_1, q_2)$, which in terms of its constituent solitons is formed from two solitons of parameters $(1, \phi + i\alpha, q_1)$ and $(2, \phi - i\alpha, q_2)$, with $\theta > \phi$ as before.

After scattering, the labels and rapidities are unchanged but the ‘shape’ parameters have changed as follows:

$$p \mapsto p X_{a_1}(\theta - \phi - i\alpha) X_{a_2}(\theta - \phi + i\alpha) \quad (4.24)$$

$$q_1 \mapsto q_1 / X_{a_1}(\theta - \phi - i\alpha) \quad (4.25)$$

$$q_2 \mapsto q_2 / X_{a_2}(\theta - \phi + i\alpha) \quad (4.26)$$

If we confine our attention to the breather, we see that the result of the scattering is not simply a change in trajectory. The ‘shape’ parameter of the breather has changed by

$$u^2 \exp(2i\delta) = q_1/q_2^* \mapsto u^2 \exp(2i\delta') = u^2 \exp(2i\delta) \frac{X_{a_2}(\theta - \phi + i\alpha)^*}{X_{a_1}(\theta - \phi - i\alpha)} \quad (4.27)$$

For $a = 1, 2$ this factor takes the values

$$X_{a_2}(\theta - \phi + i\alpha)^*/X_{a_1}(\theta - \phi - i\alpha) = \begin{cases} 
\frac{(\cosh(\theta - \phi - i\alpha) - 1/2)(\cosh(\theta - \phi - i\alpha) + 1/2)}{(\cosh(\theta - \phi - i\alpha) + 1)(\cosh(\theta - \phi - i\alpha) - 1)}, & a = 1 \\
\frac{(\cosh(\theta - \phi - i\alpha) - 1)(\cosh(\theta - \phi - i\alpha) + 1)}{(\cosh(\theta - \phi - i\alpha) + 1/2)(\cosh(\theta - \phi - i\alpha) - 1/2)}, & a = 2 
\end{cases} \quad (4.28)$$

For generic values of $\theta, \phi$ and $\alpha$ this is a generic complex number, so that in a typical scattering both $\delta$ and $u$ will be altered. Note however that this factor tends to one as $|\theta - \phi| \to \infty.$
We can now try as before to absorb all the effects of the scattering into an effective change of trajectory, corresponding to a time displacement $\delta t$ and space displacement $\delta x$. From eqn. (4.10) this leads to the two equations

\[
\begin{align*}
(q_1/X_{a1}(\theta - \phi - i\alpha)) &= q_1 \exp(m(\delta x \cosh(\phi + i\alpha) - \delta t \sinh(\phi + i\alpha))) \\
(q_2/X_{a2}(\theta - \phi + i\alpha)) &= q_2 \exp(m(\delta x \cosh(\phi - i\alpha) - \delta t \sinh(\phi - i\alpha)))
\end{align*}
\] (4.29)

The result is that both $\delta x$ and $\delta t$ are complex for a generic scattering, as is the effective time delay

\[\Delta T = \frac{\delta x}{v} - \delta t.\] (4.30)

While this may appear a strange procedure from the classical point of view, it agrees perfectly with the results of Gandenberger, as the time delay $\Delta T$ for a light breather scattering off a heavy soliton this can be compared with the quantum scattering using the formalism of Faddeev and Korepin [31], and we find perfect agreement, as we now show.

Let us recall how the correspondence between the phase-shift and the classical time delay is made. We consider the scattering of the lowest breather with mass $m_{B_1}$ in the $a_2^{(1)}$ theory on a soliton of mass $M$ at rest. The energy in this frame is $E = M + m_{B_1} \cosh \theta$, where $\theta$ is the rapidity of the incoming breather. Given the phase-shift $\delta(E)$ as a function of the incident energy $E$ the classical time delay can be computed as

\[\Delta T(E) = \frac{d\delta(E)}{dE},\] (4.31)

assuming that $M \gg m_{B_1}$ which means that we treat the soliton as a static potential well. In terms of the rapidity we can write

\[\Delta T(E) = \frac{1}{\sinh \theta} \frac{d}{d\theta} \left(-i \log S_{AB}(\theta)\right),\] (4.32)

where $S_{AB}(\theta)$ is given by eqn. (3.7). The semi-classical limit is given by taking $\beta \to 0$. Performing the calculation we find

\[\Delta T_{\text{limit}}(E) = \frac{\cosh \theta}{m \sinh \theta} \left(\frac{1}{i \sinh \theta - 1} - \frac{1}{i \sinh \theta + 1/2}\right).\] (4.33)

We only need to check now that in the semi-classical limit the breather mass is much less than the soliton mass: the semi-classical limit is $\beta \to 0$, i.e. $\lambda \to \infty$, so that $m_{B_1} \sim \frac{2M}{3\lambda} \sim \frac{\beta^2M}{6} = m$, and indeed $m_{B_1} \to 0$ and the approximation $M \gg m_{B_1}$ is valid in the semi-classical limit.

If we now use the results (4.29) and (4.31) in (3.31), we can compute $\Delta T$ for a breather of internal momentum

\[\alpha = \pi/2 - \epsilon, \epsilon \to 0,\] (4.34)

which corresponds to the semi-classical limit of the first breather $B_1$. The result is

\[\Delta T_{\text{classical}}(E) = \frac{\cosh \theta}{m \sinh \theta} \left(\frac{1}{i \sinh \theta - 1} - \frac{1}{i \sinh \theta + 1/2}\right),\] (4.35)
which coincides with (1.33). If instead we had taken \( \alpha = -\pi/2 + \epsilon \), we would have instead found agreement with \( S_{AB} \).

Therefore one concludes that the pathological behaviour of the classical breather-soliton scattering, i.e. the existence of complex time delays precisely corresponds to the non-unitarity of the phase shift observed at the quantum level and that the quantum scattering of solitons and breathers in \( a_2^{(1)} \) is fundamentally flawed.

Moreover, on closer examination we find that the scattering is also fundamentally flawed at the classical level. Since the effect of breather–soliton scattering is to change the shape parameters of both solutions, there is no guarantee that the final shape parameters will be in the allowed ranges for the solutions to be regular. While it is not hard to find explicit values which lead to singular final states from regular initial states, the following argument is quite general and illustrates the point well.

Consider the scattering of a fast soliton off a breather. This will change the parameter \( \delta \) of the breather by a small amount \( \Delta \delta \). Then, \( n \) repeated scatterings of the same species of soliton with very similar rapidities (exactly equal rapidities are not allowed) off the breather will change \( \delta \) by \( n \Delta \delta \), and will eventually be forced into the singular region. Thus, the final state obtained starting from an initial state of regular solitons and a regular breather will be solitons and a singular breather.

5 Quantum S-matrices of the \( a_2^{(2)} \) theory

The \( a_2^{(2)} \) action has a single scalar field, and the unreduced S-matrix was first given by Smirnov [13]. This S-matrix describes the scattering of a soliton transforming in the triplet representation of \( U_q(a_2^{(2)}) \), that is three particles of topological charge \(-1, 0\) and \(1\) units respectively.

As we mentioned before, \( U_q(a_2^{(2)}) \) has two inequivalent subalgebras, \( U_q(a_1) \) and \( U_q(\mathfrak{a}_1) \), which means that there are two inequivalent ways of performing the RSOS restriction, which we refer to as the \((1,2)\) and \((1,5)\) restrictions, the RSOS reduced S-matrices corresponding to perturbations of Virasoro minimal models by these two primary fields.

5.1 Unrestricted S-matrices of \( a_2^{(2)} \)

In any gradation, the unrestricted S-matrix of the fundamental soliton \( K_0 \) is given by

\[
S_{K_0 K_0}(\theta)^{kl}_{ij} = S_0(\theta) \ R(x,q)^{kl}_{ij} \quad (5.1)
\]

where \( q \) and the spectral parameter \( x \) are given in terms of a variable \( \xi \) parametrising the coupling constant dependence of the theory by\footnote{The parameter \( q \) appearing in this \( R \)-matrix does not agree with the conventions of [16]. Depending on the definition of the quantum group relations used, the definition of \( q \) depends on the lengths of the roots of the Lie algebra or on the numerical values of the symmetrised Cartan matrix. For an explanation of the conventions used in this section, see [32].}

\[
x = \exp (2\pi \theta/\xi) \quad , \quad q = i \exp \left( \frac{i \pi^2}{3 \xi} \right) . \quad (5.2)
\]
In the $(1,2)$ gradation, if we define the matrix $\tilde{R} = \mathbf{P}R$ where $\mathbf{P}$ is the permutation matrix on the two spaces, then $\tilde{R}$ commutes with the $(1,2)$ subalgebra of $U_q(a_2^{(1)})$ and can be expanded in terms of projectors $\tilde{P}$ on the irreducible representations of the $U_q(a_1)$ algebra in the tensor product $3 \otimes 3 = 1 \oplus 3 \oplus 5$ as
\[
\tilde{R}(x,q) = \tilde{P}_5 + \frac{xq^4 - 1}{x-q^4} \tilde{P}_3 + \frac{xq^6 + 1}{x+q^6} \tilde{P}_1.
\]
(5.3)
While it is obvious that the eigenvalues of $\tilde{R}$ are phases for real $x$ and $q$ a phase, the physically relevant eigenvalues are those of $R$, which are only phases for $x > 0$ and $q$ a phase; for $x < 0$ and $q$ a phase they are not.

To be explicit, the eigenvalues of $R(x,q)$ are three pairs of doubly degenerate eigenvalues,
\[
1, \quad \left(\frac{1 - q^2\sqrt{x}}{q^2 - \sqrt{x}}\right), \quad \left(\frac{1 + q^2\sqrt{x}}{q^2 + \sqrt{x}}\right)
\]
and three eigenvalues $\lambda$ satisfying
\[
(x - q^4)(x + q^6)\lambda^3 + q^6(2 + q^2)(\lambda^2 + x^2\lambda) \\
+ x(q^2 - 1)(1 - 3q^4 + q^8)(\lambda^2 + \lambda) \\
- q^2(1 + 2q^2)(x^2\lambda^2 + \lambda) + (1 - q^4)\lambda(1 + q^6 x) = 0.
\]
(5.4)
The $R$-matrix in the $(1,5)$ gradation differs by a similarity transformation from the previous one:
\[
\tilde{R}(y,q) = y^{H_2} R(y^2,q) y^{-H_2},
\]
(5.6)
where for convenience we introduced a new spectral parameter $y = \exp(\pi\theta/\xi)$, an element $H$ in the Cartan subalgebra of $U_q(a_2^{(1)})$, and where the subscript 2 indicates that this matrix acts in the second space. Explicitly,
\[
H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]
(5.7)
The eigenvalues of $\mathbf{P}\tilde{R}$ are different from those of $\tilde{R}$, but the eigenvalues of the $S$-matrices are unaffected by the regradation as this is just a similarity transformation on $R$.

From (5.3) it is clear that there are poles in $S_{K_0K_0}$ for $x = q^4$ and $x = -q^6$. The latter correspond to the possibility to have scalar (breather) bound states $B_p$ and the former to excited solitons $K_m$. The exact spectrum may be very complicated (see [13]) but at least for these particles we have $k = 0, 1, 2, \ldots, [\lambda]$, of mass
\[
M(K_m) = 2M \cos\left(\frac{\pi}{3} - m\frac{\xi}{2}\right), \quad m = 0, 1, \ldots [2\pi/3\xi],
\]
(5.8)
\[
M(B_p) = 2M \sin\left(\frac{p\xi}{2}\right), \quad p = 1, 2, \ldots [\pi/\xi].
\]
(5.9)
We now consider the $S$-matrix bootstrap to find the $S$-matrices of the excited kinks.

\footnote{It is clear that these are exactly the same masses as in the $a_2^{(1)}$ theory with $\lambda = 2\pi/(3\xi)$, but there is no other obvious relation between $S$-matrices of the two theories.}
5.1.1 Fusion of the unrestricted $S$-matrices in the (1,2) gradation

The equation which describes the bootstrap procedure in the (1,5) gradation is

$$S_{K_0 K_1|14} \propto C_{4|23} R_{13} (\varepsilon x^2) R_{12} (\varepsilon x/q^2) C_{23|4} = \frac{\varepsilon x^2 - 1}{\varepsilon x - q^2} R_{14} (-\varepsilon x, q), \quad (5.10)$$

$$S_{K_4 K_0|41} \propto C_{4|12} R_{13} (\varepsilon x/q^2) R_{12} (\varepsilon x^2) C_{12|4} = \frac{\varepsilon x^2 - 1}{\varepsilon x - q^2} R_{41} (-\varepsilon x, q) \quad (5.11)$$

where the indices 1, 2, 3, 4 indicate in which spaces the matrices act, $\epsilon = \pm 1$ and $C_{4|23}$ and $C_{23|4}$ are the Clebsh-Gordan coefficients specifying the embedding of the triplet representation space 4 into the product of the two triplet representation spaces 2 and 3.

In fact, as noted by Smirnov, these relations describe all the bootstraps in the (1,2) gradation, and that all higher kink–higher kink $S$-matrices have the same group structure,

$$S_{K_n K_0} \propto R((−1)^{m+n} x, q). \quad (5.12)$$

Since we have already seen that $R(−x, q)$ doesn’t have phase eigenvalues for positive real $x$ and $q$ a phase, we immediately see that, as for $a_2^{(1)}$, it is only possible for the unrestricted theory to have unitary $S$-matrices if we must again restrict the coupling constant, this time to $\xi > 2\pi/3$.

As we have noted, it is not sufficient for the two-particle $S$-matrix to have phase eigenvalues, but every $n$–particle transfer matrix must also have phase eigenvalues; this is guaranteed if the two-particle $S$-matrix is conjugate to a unitary matrix up to a change of inner products on the one-particle states, but this is too hard for us to check. Equally it is impossible for us to check all transfer matrix eigenvalues, but we have investigated the three-particle case numerically and find that for a large set of values of $q$ and rapidities, the three-particle transfer matrix indeed has phase eigenvalues. Thus we conjecture that this is true for all multi-particle states and that (to leading order) the finite size spectrum is real.

5.1.2 Bootstrap fusion for the ZMS model in the (1,5) gradation

The bootstrap relation looks a little different in the (1,5) gradation. If $H$ were to have had only even integer eigenvalues, then this would make no difference to the bootstrap, but that fact that it has also odd eigenvalues means that the bootstrap equations resulting from the two poles $y = \pm q^2$ corresponding to $x = q^4$ are different.

After including the effects of $S_0$, we have two series of poles in $\theta$, which we call ‘even’ and ‘odd’, according to

$$y = q^2 \quad \theta = 2i\pi/3 - n\xi, \quad n \text{ odd},$$

$$y = -q^2 \quad \theta = 2i\pi/3 - n\xi, \quad n \text{ even}, \quad (5.13)$$

Denoting the fundamental kink by $K_0$ and the kinks arising from the poles in (5.13) by $K_n$, we see that we have to distinguish between $S_{K_0 K_{even}}$ and $S_{K_0 K_{odd}}$. The $R$-matrix parts of
these $S$-matrices are now given by

\begin{align}
y = q^2 : & \quad A_4^{-1} D_{4|23} \tilde{R}_{13}(eyq) \tilde{R}_{12}(ey/q) D_{23|4} A_4 = \frac{y^2 q^2 - 1}{y^2 - q^2} \tilde{R}_{14}(-ei \gamma), \\
y = -q^2 : & \quad A_4 \tilde{D}_{4|23} \tilde{R}_{13}(i\gamma y/q) \tilde{R}_{12}(-i\gamma y/q) \tilde{D}_{23|4} A_4^{-1} = \frac{y^2 q^2 + 1}{y^2 + q^2} \tilde{R}_{14}(\epsilon y),
\end{align}

(5.14)

where again $\epsilon = \pm 1$, $D_{23|4}$ etc are Clebsh-Gordan coefficients for the $(1,5)$ algebra and $A$ is a matrix which rescales the eigenvectors since $\mathbf{P} \tilde{R}$ no longer degenerates to a pure projector at $y = \pm q^2$.

Thus we find that

\begin{align}
S_{K_0 K_{\text{even}}} \propto \tilde{R}(\pm y, q) \sim R(y^2, q), & \quad S_{K_0 K_{\text{odd}}} \propto \tilde{R}(\pm i \gamma y, q) \sim R(-y^2, q),
\end{align}

(5.15)

and although there are now four different $\tilde{R}$ matrices appearing in the soliton $S$-matrices and e.g. $S_{K_0 K_2}$ is no longer proportional to $S_{K_0 K_0}$, the eigenvalues of the unrestricted $S$-matrices in the $(1,5)$ gradation are the same as in the $(1,2)$ gradation, and the same comments about the restriction to $\xi > 2\pi/3$ still apply.

6 Classical scattering in $a_2^{(2)}$

The classical solitons and breathers in $a_2^{(2)}$ are not as well understood as those in $a_2^{(1)}$. In particular, quantum considerations suggest that there should be classical breather solutions with all masses up to twice $M_K$, the kink mass, but these are hard to find. The quantum spectrum also suggests that there might be a class of classical configurations of zero topological charge with masses lying between $M_K$ and $2M_K$, but which are somehow to be regarded as distinct from the breathers, rather behaving like the zero topological charge component of the kink triplet. The main difficulty is that there is no consistent way to find classical solutions of any particular topological charge; secondly, the $a_2^{(2)}$ classical solutions can be found as a subset of the $a_2^{(1)}$ classical solutions, but with typically twice as many component 'kinks' in the tau functions. One possibility is that the new methods of Beggs and Johnson [33] may be adapted to find such solutions, but at present this is an open problem.

As a result it is very hard to compare classical and quantum scattering in $a_2^{(2)}$ since so many of the classical configurations are missing. We have been able to compare the scattering of non-zero charge solitons and low mass breathers, and there are no discernible problems – all the effects of scatterings seem to be able to be described by a simple time delay, and the resulting time delays agree with the semi-classical limit of the quantum transition amplitudes according to the formalism of Faddeev and Korepin [31].

However, the specific problems we encountered in the quantum $a_2^{(2)}$ scattering were a result of regradations, which do not affect transition amplitudes. To see the full effect and to compare these with classical results, we need to compute the reflection amplitudes in the semi-classical formalism, something we have not yet done.

We would like to note here that a discussion (we believe new) of how different gradations can be understood as arising from different classical actions in the path integral formalism can be found in appendix B.
7 An explicit example of both $a_2^{(1)}$ and $a_2^{(2)}$ Toda theories: $M_{3,14} + \Phi_{1,5}$

What initially stimulated our interest in this problem was the strange behaviour of $M_{3,14} + \Phi_{(1,5)}$, and since this is instructive in many ways, we briefly summarise the situation in this model. The main point of interest with this model is that there are a pair of particles $A, \overline{A}$ of mass $M$ and a particle $B$, or pair of particles $B, \overline{B}$, of mass $(1+\sqrt{3})M/\sqrt{2}$, for which the $S$-matrix $S_{AB}(\theta)$ is not a phase for real $\theta$ [32].

When we try to identify this model with an RSOS restriction of a Toda theory, we find that it is somewhat special in that conformal field theory of the ‘D’ modular invariant of the Virasoro minimal model $M_{3,14}$ is the minimal model $M_{A2}(3,7)$ of the $W_3$ algebra; correspondingly the primary field of weight $-5/7$ is both the $\Phi_{(1,5)}$ primary field of the Virasoro algebra and the $\Phi_{(11;22)}$ primary field of the $W_3$ algebra. Since perturbing a Virasoro model by the $\Phi_{(1,5)}$ field gives the $a_2^{(2)}$ Toda theory, and perturbing a $W_3$ minimal model by the $\Phi_{(11;22)}$ field gives the $a_2^{(1)}$ Toda theory, this single model gives an example of RSOS restrictions of the two theories we have discussed in this paper.

From the point of view of $a_2^{(1)}$ Toda theory, this is the RSOS restriction of a model with $\lambda = 4/3$, and the solitons are entirely eliminated by the RSOS restriction; the fundamental particles $A$ and $\overline{A}$ of the RSOS restriction discussed in [12, 32] are the first breathers $B_1$ and $\overline{B}_1$. However the particle(s) $B$ are neither breathers nor solitons in the generic unrestricted model, and do not appear in the spectra of Gandenberger or Hollowood. From the point of view of $a_2^{(2)}$ theory, this is the $(1,5)$ restriction with $\xi = \pi/2$; $A$ and $\overline{A}$ are the only remnants of the fundamental soliton, and $B$ and $\overline{B}$ are the remnants of the first excited soliton.

This model is instructive in two ways:

Firstly it shows that a restriction of $a_2^{(1)}$ to a regime where there are no kinks does not automatically imply that the theory is unitary. Even if there is only one RSOS restriction, problems can arise when one attempts to complete the bootstrap. As we have seen, there is a particle $B$ arising in the course of the bootstrap whose $S$-matrices with $A$ are not phases. The reason why Hollowood and Gandenberger miss this particle is that in nonunitary theories it is difficult to decide which poles to invite in the bootstrap. From the point of view of $a_2^{(2)}$ it was very natural to conjecture the complete spectrum, and we found a set of $S$-matrix amplitudes which is closed under the bootstrap and is proven to be correct by TCSA and TBA analysis. The problem of the existence and uniqueness of a closed system of scattering amplitudes given the $S$-matrix of some fundamental particle is very nontrivial and unsolved in general.

Secondly it shows that whether the RSOS restriction of $a_2^{(2)}$ is unitary or not depends in part on whether we mean the $(1,2)$ or $(1,5)$ RSOS restrictions – since, as we discuss in [12], the $(1,2)$ RSOS restriction of $a_2^{(2)}$ at the same value $\xi = \pi/2$ gives the perfectly satisfactory model $M_{6,7} + \Phi_{(1,2)}$ already treated in [13]. We shall treat this and other problems connected with RSOS restrictions in [34].
8 Conclusion

First we review the results for $a_2^{(1)}$ and $a_2^{(2)}$ Toda theories separately; we then consider their implications and discuss whether one can find a coherent picture for Toda theories into which they fit.

8.1 $a_2^{(1)}$ Toda theory

We first investigated the unrestricted $S$-matrices of the fundamental and excited solitons in both the homogenous and spin gradations; we found that the $S$-matrices describing the scattering of solitons of the same species had phase eigenvalues, and hence are unitary with an appropriate inner product, but that this cannot simply be achieved by changing the inner products of the one particle states. This leads to non-phase eigenvalues for the three-particle transfer matrix and a complex finite size spectrum. Even more clearly, the $S$-matrices describing the scattering of conjugate species did not have phase eigenvalues and hence cannot be made unitary by any change of inner product. The only known way to obtain a unitary theory with both solitons and antisolitons is by quantum group restriction to an RSOS theory.

We next investigated the scattering of solitons and breathers and again found that the amplitudes were not phases for real rapidity differences. The difference now is that since breathers are scalars under the quantum group, these $S$-matrices are unaltered by RSOS restriction.

However, de Vega and Fateev have presented a set of $S$-matrices which they claim correspond to unitary theories, including $a_2^{(1)}$ Toda theory in particular. The answer lies in the fact that their $S$-matrices correspond to particular values of the coupling constant for which, as Hollowood pointed out, it is possible to remove the breathers consistently. According to Hollowood, there are no poles corresponding to breathers in the physical strip provided

$$0 \leq 3\lambda \leq 1. \tag{8.1}$$

The $S$-matrices of de Vega and Fateev correspond to perturbations of the unitary minimal models of the $W_3$ algebra with

$$\frac{\beta^2}{4\pi} = \frac{m}{m+1}, \tag{8.2}$$

and as a result, the unitarity-violating breather-soliton $S$-matrix element does not occur as the breathers are entirely decoupled.

If $\beta^2/(4\pi) < 3/4$ then breathers do arise as poles in the physical strip and cannot be ignored, and in this case the only possibility is to remove the entire set of solitons from the theory by choosing the values of $\beta$ appropriately. Whenever $q^3 = \pm 1$, the RSOS restriction ensures that there is only a single topological charge possible, and hence the kinks are entirely removed from the spectrum. We see that this corresponds to

$$\frac{\beta^2}{4\pi} = \frac{3}{q}, \tag{8.3}$$

i.e. corresponding to the non-unitary minimal models $M_{3,q}$ of the $W_3$ algebra. However, we have already seen in section 5 that this does not automatically ensure that the spectrum is real.
For any other value of \( \beta \), the breather arises as a bound state from a pole in the physical strip, and the RSOS procedure, while changing restricting the soliton vacua to a finite set and changing the \( S \)-matrices of the kinks, cannot affect the kink-breather \( S \)-matrix as the breather is a quantum group singlet. As a result, we expect that all these theories will have a complex spectrum.

Since this is such a strange result, that all quantum soliton–breather scatterings are nonunitary, we also investigated the classical theory. We found that the classical results exactly mirror the quantum results, in that the time delay (the analogue of the \( S \)-matrix) is also complex in soliton–breather scatterings. We also found that the soliton–breather solutions are frequently singular, and that it does not appear possible to construct a consistent theory of breathers and solitons which will lead to purely regular solutions.

We believe that this is a general property of affine Toda theories which contain non-self-conjugate particles, and is related to the classical scattering properties of breathers and solitons, which we describe in section 8.3.

(The soliton–soliton scattering in the classical theory is slightly problematic in that there are solutions with isolated singularities, but the time delays are real and the semi-classical limits of the quantum \( S \)-matrices are unitary.)

### 8.2 \( a_2^{(2)} \) Toda theory

First of all we investigated the unrestricted \( S \)-matrices for the fundamental soliton and the excited solitons in the homogeneous and spin gradations. We found that the \( S \)-matrices of the fundamental solitons had phase eigenvalues, but the \( S \)-matrices of the fundamental soliton-first excited soliton never had phase eigenvalues.

This means that for an unrestricted theory to have a real finite size spectrum, one has to go to a repulsive regime, in which there are no excited solitons in the spectrum.

It is possible that RSOS restriction can cure the problems in soliton–excited soliton scattering, but as shown clearly in section 4, the details will depend on the gradation, and we shall turn our attention to these and other issues of RSOS theories in [34].

We found no evidence of any problems in the classical theory, apart from the absence of certain solutions one might expect to exist from quantum considerations – a problem we return to in section 8.3.

### 8.3 Quantum vs. classical Toda theory

There is a general issue, which is to what extent classical and quantum Toda theories with imaginary couplings are sensible physical theories. While it was recognised that the unrestricted theories were apparently non-Hermitian, it was thought that these problems could be cured by restriction – to a suitable space of classical solutions in one case, and by a quantum group restriction in the other – and that the resulting models would be a good description of perturbed conformal field theories.

All the models which had been investigated prior to [12] had been found to have an entirely real spectrum, as they did indeed correspond to unitary \( S \)-matrices found by one method or
another. However, as we found in [12], this is not necessarily the case— as indeed one should expect. There is no reason why the formal Hamiltonian of a perturbation of a non-unitary conformal field theory should have a real spectrum, even if the theory appears integrable.

What is most remarkable about the result of [12] is that the spectrum of the perturbed Virasoro minimal model $M_{3,14} + \Phi_{(1,5)}$ is in excellent agreement with the nonunitary $S$-matrices obtained by RSOS restriction and the bootstrap, even including the complex eigenvalues of the Hamiltonian which result from $S$-matrix elements which are not phases. Furthermore, the complex $S$-matrices describing soliton–breather scattering in $a_2^{(1)}$ are in perfect agreement with the (complex) classical time delays, even though the corresponding classical solutions may be singular.

Thus we can conclude that the integrable structure and quantum group symmetry which is generically present in affine Toda theories and in perturbed conformal field theories is still present even when the spectrum is no longer real, and that a formal $S$-matrix description still remains valid even when (2.2) or (2.5) no longer hold.

As noted by Gandenberger and MacKay in [36], there is a close correspondence between the transmission coefficients $X_{ab}(k)$ which play an important role in the semi-classical method, and the soliton–a–breather-b $S$-matrix $S_{ab}$, namely that

$$X_{ab}(m_a \sinh(\theta)) = \lim_{\beta \to 0} S_{ab}(\theta).$$

This can be easily checked for the case of $a_2^{(1)}$, as the transmission coefficients for $a_n^{(1)}$ are

$$X_{ab}(k) = \frac{ik - m_a \cos((a - b)\pi/(n + 1))}{ik - m_a \cos((a + b)\pi/(n + 1))}, \quad m_a = 2m \cos(a\pi/(n + 1)).$$

(8.5)

For $a_2^{(1)}$, we find as required that

$$X_{11}(k) = \frac{\sinh \theta - i}{\sinh \theta - i/2} = \lim_{B \to 0} S_{AB_1}(\theta).$$

(8.6)

The transmission coefficients for all simply-laced and twisted affine Toda theories have been found in [37,38], and they do not satisfy (2.5) exactly when both species are not-self-conjugate; such particles occur for $a_n^{(1)}$, $a_{2n+1}^{(1)}$ and $e_6^{(1)}$ theories. Since this property will be inherited by the soliton–breather $S$-matrices, we expect that in all these cases there is no unitary $S$-matrix possible, even after RSOS restriction, whenever both the relevant solitons and particles remain in the spectrum.

There is also the problem of missing solutions: it is well known that for a general affine Toda theory the simplest ansatz for soliton solutions do not fill out the representations and there some topological charges missing, but we also mention that it is possible that this problem also applies to breather solutions and that not all the classical solutions expected on quantum grounds are known at present. It is possible that this will be cured by the new methods of Beggs and Johnson [33].

It is indeed interesting that the unrestricted $S$-matrices of the fundamental soliton–soliton scattering in both $a_2^{(2)}$ and $a_2^{(1)}$ in the repulsive regimes have only phase eigenvalues, and we reinvestigate RSOS theories in this light in [34].
Finally, we should like to note that $S$-matrices do not only appear in scattering theories and statistical field theory, but other physical applications such as diffusion–annihilation problems (see e.g. [39]) and it may well be that a non–real spectrum is even necessary in such applications.

9 Acknowledgments

GMTW would like to thank H.W. Braden, E. Corrigan, G. Delius, P.E. Dorey, E.B. Davies, M. Freeman, G. Gandenberger, U. Harder, P.R. Johnson, H.G. Kausch, N.J. MacKay and R. Sasaki for enlightening discussions and INFN Bologna for hospitality and support from INFN (Italy) grant *Iniziativa Specifica* TO12. We would also like to thank F. Ravanini for discussions, SISSA for hospitality during several stages of this work, and the EU TMR network FMRX-CT96-0012 for support, and especially the organisers of the Miramare98 meeting where this work was completed.

GMTW thanks EPSRC for an advanced fellowship and support under grant GR/K30667. GT thanks INFN for a postdoctoral fellowship, and Hungarian grants FKFP 0125/1997 and OTKA T016251 for partial support.

A Analysis of singularities

To determine for which parameters $a_2^{(1)}$ and $a_2^{(2)}$ breathers and breathing kinks are singular, it is necessary to decide for which values of $\delta, u, X \in \mathbb{R}$ the following equation

$$1 + \exp(i\delta)(zu + z^*/u) + \exp(2i\delta)Xzz^* = 0,$$

has a solution for some value of $z \in \mathbb{C}$. We first turn this complex equation into two real equations for the real variables $x, y$ where $z = x + iy$. These two equations are

$$0 = 1 + x \cos \delta (u + 1/u) + y \sin \delta (-u + 1/u) + X(x^2 + y^2) \cos(2\delta) \quad (A.2)$$
$$0 = y \cos \delta (u - 1/u) + x \sin \delta (u + 1/u) + X(x^2 + y^2) \sin(2\delta) \quad (A.3)$$

These are the equations of two circles in the plane, with radii $r_1, r_2$ and distance $d$ between the centres given by

$$(r_1)^2 = \frac{\tau + (2 - 4X)X}{4X^2c^2}$$
$$(r_2)^2 = \frac{\tau - 2c}{4X^2s^2}$$
$$d^2 = \frac{\tau - 2c}{4X^2s^2c^2} \quad (A.4)$$

where

$$\tau = u^2 + 1/u^2, \quad c = \cos(2\delta).$$

For eq (A.1) to have a solution we need these two circles to have real solutions and to intersect, that is

$$(r_1)^2 > 0 \quad (r_2)^2 > 0$$
$$(r_1^2 - 2r_1r_2 + r_2^2 - d^2)(r_1^2 + 2r_1r_2 + r_2^2 - d^2) < 0 \quad (A.5)$$
These three equations, after much manipulation, can be put in the form

$$\tau > (4X - 2)c \quad \tau > 2c \quad (\tau - (2X(1 + c) - 2))(\tau - (2X(c - 1) + 2)) > 0$$  \hspace{1cm} (A.6)

which, together with the obvious equation

$$\tau \geq 2$$  \hspace{1cm} (A.7)

are the full set of equations giving the limits on $\tau$ in terms of $\delta$.

**B  Regrading in quantum and classical pictures**

Let us finally comment on some properties of the regrading transformations in the context of quantised affine Toda theories.

In a general affine Toda theory, the minima of the potential are labelled by weights $\lambda_a$ of some Lie algebra (which for untwisted affine Toda theories is the dual of the corresponding finite dimensional algebra), and so typical $S$-matrix elements are labelled.

$$S \left( \lambda_a, \lambda^d, \lambda^c \right).$$  \hspace{1cm} (B.1)

The consequence of a changing the gradation $\gamma$ in eqn. (2.16) is to change this $S$-matrix element by a factor

$$\exp(\theta \delta \gamma \cdot (\lambda_a + \lambda^d - \lambda_b - \lambda_c)).$$  \hspace{1cm} (B.2)

This can be thought of as a change in the action $I$, and such a term will change the scattering amplitude by a factor $\exp(i \delta I)$. The correct form of $\delta I$ is

$$\delta I = \int d^2x \sqrt{-g} R(\delta \gamma \cdot \Phi)/2,$$  \hspace{1cm} (B.3)

where the integral is over all spacetime, $g$ is the metric, $R$ the scalar curvature and $\Phi$ the Toda field appropriately normalised. This is zero in the interior of space time, but Minkowski space has singularities at infinity and this integral can give a finite answer by the following prescription.

Consider the following variables $\rho, \theta, \phi$ given in terms of $x$ and $t$ by

$$t = \rho \cosh \phi, \quad x = \rho \sinh \phi, \quad \tanh \theta/2 = 1/\rho.$$  \hspace{1cm} (B.4)

These variables do not cover the whole of the forward light-cone, but a subset which does, however, include the whole of the forward time-like infinity, which is the line $\theta = 0, -\infty < \phi < \infty$, and for which the metric is

$$ds^2 = d\rho^2 - \rho^2 d\phi^2 = ds^2 = \Omega^2 \left( d\theta^2 - \sin^2 \theta d\phi^2 \right), \quad \Omega = \rho/\sinh \theta.$$  \hspace{1cm} (B.5)

In these variables we can calculate the integral (B.3) using a test function, using the result

$$\frac{1}{\sqrt{-g}} R = -2 \frac{\partial}{\partial \theta} \left( \sinh \theta \frac{\partial}{\partial \theta} \log (\Omega \sinh \theta) \right),$$  \hspace{1cm} (B.6)
so that
\[
\int_{-\infty}^{\infty} d\phi \int_0^{\infty} d\theta \; \sqrt{-gR} \; f(\phi) = 2 \int_{-\infty}^{\infty} d\phi \; f(\phi, \theta = 0) = 2 \int_{-\infty}^{\infty} d\phi \; f(\phi, t = \infty) . \tag{B.7}
\]

There is a similar contribution from backward time-like infinity which gives us the total contribution from the two time-like infinities to the additional terms in our action \(\delta S\)
\[
\delta I = \int d^2x \; \sqrt{-gR} (\delta \gamma \cdot \Phi) / 2 = \int_{-\infty}^{\infty} d\phi \; \delta \gamma \cdot (\Phi(\phi, t = \infty) - \Phi(\phi, t = -\infty)) . \tag{B.8}
\]

To evaluate this for a two-particle scattering amplitude as in figure 4, we first note that a particle of rapidity \(\phi_0\) intersects time-like infinity at \(\phi = \phi_0\), and evaluating (B.8) with a suitable cutoff to regulate the \(\phi\) integral gives (the cutoff disappears as we would like)
\[
\delta I = (\theta_1 - \theta_2) \; \delta \gamma \cdot (\lambda_a + \lambda_c - \lambda_b - \lambda_d) , \tag{B.9}
\]

which is exactly the result of a regradation in the quantum group picture.

Figure 4: A two–particle scattering showing the cutoffs in \(\phi\) required to give a finite answer for \(\delta S\)
References

[1] A.B. Zamolodchikov, Integrable Field Theory from Conformal Field Theory, in ‘Advances in Pure Mathematics 19: Integrable Systems in Quantum Field Theory and Statistical Mechanics’, eds. M. Jimbo, T. Miwa and A. Tsuchiya, (1989) 641–674.

[2] T. Hollowood and P. Mansfield, Rational conformal field theory at, and away from criticality, as Toda field theories, Phys. Lett. B226 (1989) 73.

[3] H.W. Braden, E.F. Corrigan, P.E. Dorey and R. Sasaki, Extended Toda field theory and exact S-matrices, Phys. Lett. B227 (1989) 411; Affine Toda field theory and exact S matrices, Nucl. Phys. B338 (1990) 689–746, G.W. Delius, M.T. Grisaru and D. Zanon, Exact S-matrices for non-simply-laced affine Toda theories, Nucl. Phys. B382 (1992) 365–408.

[4] D.I. Olive, N. Turok and J.W.R. Underwood, Solitons and the energy momentum tensor for affine Toda theory, Nucl Phys. B401 (1993) 663–697.

[5] M. Freeman, Conserved charges and soliton solutions in affine Toda theory, Nucl. Phys. 433 (1995) 657–670, [hep-th/9408092].

[6] T.J. Hollowood, Quantizing SL(N) solitons and the Hecke algebra, Int. J. Mod. Phys. A8 (1993) 947–982, [hep-th/9203078].

[7] G.M.T. Watts, Phys. Lett. B338 (1994) 40–46. N.J. MacKay and G.M.T. Watts, Nucl. Phys. B441 (1995) 277–309. G.W. Delius, Nucl. Phys. B441 (1995) 259–276.

[8] R.F. Dashen, B. Hasslacher and A. Neveu, Phys. Rev. D10 (1974) 4114; Phys. Rev. D11 (1975) 3424; Phys. Rev. D12 (1975) 2443.

[9] T. J. Hollowood, Quantum soliton mass corrections in SL(N) affine Toda field theory, Phys. Lett. B300 (1993) 73–83, [hep-th/9209024].

[10] E.B. Davies, Pseudospectra, the harmonic oscillator and complex resonances, King’s College London preprint KCL-MTH-98-03, http://www.mth.kcl.ac.uk/research/preprints/ebd_darmon10.ps.

[11] S.P. Khastgir and R. Sasaki, Instability of Solitons in imaginary coupling affine Toda Field Theory, Prog. Theor. Phys. 95 (1996) 485, [hep-th/9507001].

[12] G. Takács, H.G. Kausch and G.M.T. Watts, On the relation between Φ(1,2) and Φ(1,5) perturbed minimal models, Nucl. Phys. B489 [FS] (1997) 557–579, [hep-th/9605104].

[13] F.A. Smirnov, Exact S matrices for φ1,2–perturbed minimal models of conformal field theory, Int. J. Mod. Phys. A6 (1991) 1407–1428.

[14] C.J. Efthimiou, Quantum group symmetry for the Φ(1,2) perturbed and Φ(2,1) perturbed minimal models of conformal field theory Nucl. Phys. B398 (1993) 697–740.

[15] G.M. Gandenberger and N.J. MacKay, Exact S matrices for D(2)N+1 affine Toda solitons and their bound states Nucl. Phys. B457 (1995) 240–272.

[16] G.M. Gandenberger, N.J. MacKay and G.M.T. Watts, Twisted algebra R matrices and S matrices for h(1)N affine Toda solitons and their bound states Nucl. Phys. B465 (1996) 329–349.

[17] G. Takács, ‘The R-matrix of the Uq(d(3)4) algebra and g(1)2 affine Toda field theory’, Nucl. Phys. B501 (1997) 711–727, [hep-th/9702196].

‘Quantum Affine Symmetry and Scattering Amplitudes of the Imaginary Coupled d(3)4 Affine Toda Field Theory’, Nucl. Phys. B502 (1997) 629–648, [hep-th/9701113].

[18] N.Yu. Reshetikhin and F.A. Smirnov, Hidden quantum group symmetry and integrable perturbations of conformal field theories Commun. Math. Phys. 131 (1990) 157–178.

[19] Al.B. Zamolodchikov, Thermodynamic Bethe Ansatz for RSOS scattering theories, Nucl. Phys. B358 (1991) 497–523.
[20] T.R. Klassen and E. Melzer, *Kinks in Finite Volume*, Nucl. Phys. B382 (1992) 441-485, [hep-th/9202034].
[21] A.B. Zamolodchikov, *Factorized S-matrices and lattice statistical systems*, Sov. Sci. Rev. Physics 2 (1980) 1.
[22] J.L. Cardy and G. Mussardo, *S matrix of the Yang-Lee edge singularity in two-dimensions*, Phys. Lett. B225 (1989) 275.
[23] V.V. Bazhanov, S.I. Lukyanov and Al.B. Zamolodchikov, *Integrable quantum field theories in finite volume: Excited state energies*, Nucl. Phys. B482 (1997) 487–531, [hep-th/9607099]; P.E. Dorey and R. Tateo, *Excited states by analytic continuation of TBA equations*, Nucl. Phys. B482 (1996) 639–659, [hep-th/9607167]; G. Feverati, F. Ravanini and G. Takács, *Nonlinear integral equation and finite volume spectrum of Sine-Gordon theory*, [hep-th/9805117]; D. Fioravanti, A. Mariotti, E. Quattrini and F. Ravanini, *Excited state Destri-De Vega equation for Sine-Gordon and restricted Sine-Gordon models*, Phys. Lett. B390 (1997) 243–251, [hep-th/9608091].
[24] D. Bernard and A. LeClair, *Quantum Group Symmetries and non-local Currents in 2D QFT*, Commun. Math. Phys. 142 (1991) 99–138.
[25] A. Koubek, *S matrices of \( \phi \)–perturbed minimal models: IRF formulation and bootstrap program*, Int. J. Mod. Phys. A9 (1994) 1909–1927, [hep-th/9211134].
[26] G.W. Delius, *Exact S-matrices with affine quantum group symmetry*, Nucl. Phys. B451 (1995) 445, [hep-th/9503079].
[27] G.M. Gandenberger, *Exact S-Matrices for Bound States of \( a_2^{(1)} \) Affine Toda Solitons*, Nucl. Phys. B449 (1995) 375-405, [hep-th/9501136]; *Trigonometric \( S \) matrices, affine Toda solitons and supersymmetry*, [hep-th/9703158].
[28] T.J. Hollowood, *Solitons in affine Toda field theories*, Nucl. Phys. B384 (1992) 523–540.
[29] U. Harder, A.A. Iskandar and W.A. McGhee, *On the breathers of \( a_n \)–affine Toda field theory*, Int. J. Mod. Phys. A10 (1995) 1879–1904, [hep-th/9409035].
[30] A. Fring, P.R. Johnson, M.A.C. Kneipp and D.I. Olive, *Vertex operators and soliton time delays in affine Toda field theory*, Nucl. Phys. B430 (1994) 597-614, [hep-th/9405034].
[31] L.D. Faddeev and V.E. Korepin, *Quantum theory of solitons*, Phys. Rep. 42 (1976) 1-87.
[32] G. Takács, *A new RSOS restriction of the Zhiber-Mikhailov-Shabat model and \( \Phi_{(1,5)} \) perturbations of nonunitary minimal models*, Nucl. Phys. B489 [FS] (1997) 532-556, [hep-th/9604098].
[33] G. Takács and G.M.T. Watts, *RSOS revisited*, in preparation.
[34] H.J. de Vega and V.A. Fateev, *Factorizable S-matrices for perturbed W-invariant theories*, Int. J. Mod. Phys. A, Vol 6. No 18 (1991) 3221–3234.
[35] F.C. Alcaraz, M. Droz, M. Henkel and V. Rittenberg, *Reaction–diffusion processes, critical dynamics and quantum chains*, Ann. Phys. 230 (1994) 250–302, [hep-th/9302112].