CONVERGENCES OF ASYMPTOTICALLY AUTONOMOUS PULLBACK ATTRACTORS TOWARDS SEMIGROUP ATTRACTORS

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Dedicated to Professor Peter Kloeden on his 70th birthday

Abstract. For pullback attractors of asymptotically autonomous dynamical systems we study the convergences of their components towards the global attractors of the limiting semigroups. We use some conditions of uniform boundedness of pullback attractors, instead of uniform compactness conditions used in the literature. Both forward convergence and backward convergence are studied.

1. Introduction. The theory of pullback attractors is a useful tool to study the long time behavior of evolution systems with non-autonomous forcings. Unlike autonomous systems, long time behavior in non-autonomous systems have interpretations pullback and forward.

In dynamical system theory, such a non-autonomous evolution system is often formulated as a process, i.e., a mapping \( U : \mathbb{R}^2 \times X \to X \) satisfying \( U(\tau, \tau, x) = x \) and \( U(t, s, U(s, \tau, x)) = U(t, \tau, x) \) for all \( t \geq s \geq \tau \) and \( x \in X \), where \( \mathbb{R}^2 := \{(t, \tau) \in \mathbb{R}^2 : t \geq \tau\} \) and \( X \) a complete metric space, while the pullback attractor \( \mathfrak{A} \) of a process \( U \) is defined as a compact non-autonomous set in the form \( \mathfrak{A} = \{A(t)\}_{t \in \mathbb{R}} \) which is the minimal among those that are invariant and pullback attract non-empty bounded sets in \( X \). The pullback attractor gives rich information of the asymptotic dynamics of the system from the past, and under certain conditions can have close relationship to other kinds of attractors, such as uniform attractors, cocycle attractors, etc., see [1, 7, 3]. Its time-dependence is directly related to the non-autonomous characteristic of the system.

Intuitively, when the non-autonomous forcing of a dynamical system becomes more and more autonomous, i.e., mathematically, converges in time to a time-independent forcing in some sense, the non-autonomous nature of the pullback attractor should correspondingly become weaker and weaker. Hence, if the limiting system is an autonomous semigroup with a global attractor, the pullback attractor should, in some sense, converge to that global attractor in time. This motivates to the asymptotically autonomous study of pullback attractors, see, e.g. [2, 14, 15, 17].
Most recently, Li et al. [17] showed that

**Theorem 1 ([17]).** Let $\mathcal{A}$ be the pullback attractor of a process $U$ and $\mathcal{A}$ the global attractor of a semigroup $S$. Suppose that

(i) for any $\{x_\tau\}$ with $\lim_{\tau \to \infty} d_X(x_\tau, x_0) = 0$,

$$
\lim_{\tau \to \infty} \text{dist}(U(\tau + t, \tau, x_\tau), S(t, x_0)) = 0, \quad \forall t \geq 0,
$$

where \(\text{dist}\) denotes the Hausdorff semi-metric introduced latter in Section 2;

(ii) the pullback attractor $\mathcal{A}$ is forward compact, i.e., the union $\bigcup_{s \geq 0} A(s)$ is pre-compact.

Then

$$
\lim_{t \to \infty} \text{dist}(A(t), \mathcal{A}) = 0.
$$

The above theorem improves the corresponding results of Kloeden and Simsen [14] who used similar compactness conditions but with more uniformity. However, in view of applications, proving the compactness of the attractor itself is often where the real difficulty lies, especially in cases in which Sobolev compact embeddings cannot help.

In this paper we give an alternative theorem, making use of a forward boundedness condition instead of the forward compactness condition and with condition (1) slightly modified. By an example on an unbounded domain presented in the last section, we show that the modified version of condition (1) can be verified quite easily and that the forward boundedness condition can be obtained directly by estimates of solutions, without any particular techniques needed as previously to show forward compactness.

We also study the backwards convergence problem, i.e., of $A(t)$ converging to $\mathcal{A}$ as $t \to -\infty$. We first give a theorem using compactness conditions analogously to Theorem 1, and then improve it with boundedness conditions. Remarkably, the boundedness conditions together with the pullback attraction of the attractor also enable us to show the backwards convergence to be in the full Hausdorff metric sense, not only in the semi-metric sense. In other words, the global attractor can be the $\alpha$-limit set of the pullback attractor.

Note that similar topics to asymptotic equivalence have also been studied for skew-product flows and ordinary differential equations, see, e.g., [20, 13, 22, 3]. Our results do not require a compact phase space, and we obtain convergences in the full Hausdorff metric sense rather than only in the semi-metric sense. In addition, no continuity conditions of the dynamical systems are assumed throughout this paper.

2. **Sufficient conditions.** In this section we formulate our theoretical results on sufficient conditions ensuring the convergences of pullback attractors towards global attractors. For a complete metric space $(X, d_X)$ we denote by $\text{dist}$ the Hausdorff semi-metric between nonempty sets, i.e.,

$$
\text{dist}(A, B) := \sup_{a \in A} \inf_{b \in B} d_X(a, b),
$$

and denote the Hausdorff metric by $\text{dist}_H(A, B) := \max\{\text{dist}(A, B), \text{dist}(B, A)\}$.

**Definition 2.** A family $\{E(t)\}_{t \in \mathbb{R}}$ of nonempty sets is said to be

(i) **forward bounded/compact,** if there exists a bounded/compact set $B$ such that

$$
\bigcup_{t \geq 0} E(t) \subset B;
$$
(ii) *backwards bounded/compact*, if there exists a bounded/compact set $K$ such that
\[ \bigcup_{t \leq 0} E(t) \subset K. \]

### 2.1. Forward convergence in distant future.

Now we establish an alternative theorem for Theorem 1, using forward boundedness condition instead of the forward compactness condition.

**Theorem 3.** Suppose that $U$ is a process with pullback attractor $\mathfrak{A} = \{A(t)\}_{t \in \mathbb{R}}$ and $S$ is a semigroup with global attractor $\mathscr{A}$. If

(i) $\mathfrak{A}$ is forward bounded, i.e., there is a bounded set $B$ such that $\bigcup_{t \geq 0} A(t) \subset B$;

(ii) the following asymptotically autonomous condition holds
\[ \lim_{t \to \infty} \sup_{x \in B} d_X(U(t + T, t, x), S(T, x)) = 0, \quad \forall T > 0, \quad (2) \]

then
\[ \lim_{t \to \infty} \text{dist}(A(t), \mathscr{A}) = 0. \quad (3) \]

**Proof.** If it is not the case, then there exist $\delta > 0$ and $t_n \to \infty$ such that
\[ \text{dist}(A(t_n), \mathscr{A}) > \delta, \quad \forall n \in \mathbb{N}. \]

By the compactness of the attractor $\mathfrak{A}$, i.e., each section $A(t)$ being compact, for any $n \in \mathbb{N}$ there exists an $x_n \in A(t_n)$ such that
\[ \text{dist}(x_n, \mathscr{A}) = \text{dist}(A(t_n), \mathscr{A}) > \delta. \quad (4) \]

Since $B$ is attracted by $\mathscr{A}$ under $S$, there exists a $T > 0$ such that
\[ \text{dist}(S(T, B), \mathscr{A}) < \delta/2. \quad (5) \]

In addition, by the invariance of $\mathfrak{A}$, for each $n \in \mathbb{N}$ there exists $y_n \in A(t_n - T) \subset B$ such that $x_n = U(t_n, t_n - T, y_n)$, which along with (4) implies
\[ \text{dist}(U(t_n, t_n - T, y_n), \mathscr{A}) = \text{dist}(x_n, \mathscr{A}) > \delta, \quad \forall n \in \mathbb{N}. \quad (6) \]

On the other hand, by condition (2), there exists an $N = N(\delta) > 0$ such that
\[ d_X(U(t_N, t_N - T, y_N), S(T, y_N)) \leq \sup_{x \in B} d_X(U(t_N, t_N - T, x), S(T, x)) < \delta/2, \]

so, by (5),
\[ \text{dist}(U(t_N, t_N - T, y_N), \mathscr{A}) \leq d_X(U(t_N, t_N - T, y_N), S(T, y_N)) + \text{dist}(S(T, y_N), \mathscr{A}) < \delta, \]

which contradicts (6). Hence the theorem holds.

### 2.2. Backwards convergence in distant past.

Now we turn to the backwards case. Firstly, analogously to Theorem 1 we have the following backwards theorem.

**Theorem 4.** Let $\mathfrak{A}$ be the pullback attractor of a process $U$ and $\mathscr{A}$ the global attractor of a semigroup $S$. Suppose that

(i) for any $\{x_t\}$ with $\lim_{t \to -\infty} d_X(x_t, x_0) = 0$,
\[ \lim_{t \to -\infty} d_X(U(t, t - T, x_t), S(T, x_0)) = 0, \quad \forall T \in \mathbb{R}^+; \quad (7) \]

(ii) $\mathfrak{A}$ is backwards compact.
Then
\[ \lim_{t \to -\infty} \text{dist} (A(t), \mathcal{A}) = 0. \]

Proof. We prove by contradiction. Suppose that for some \( \delta > 0 \) there exists a sequence \( t_n \to \infty \) such that
\[ \text{dist} (A(-t_n), \mathcal{A}) \geq \delta, \quad \forall n \in \mathbb{N}. \]
Then there exists a sequence \( x_n \in A(-t_n) \) such that
\[ \text{dist} (x_n, \mathcal{A}) = \text{dist} (A(-t_n), \mathcal{A}) \geq \delta, \quad \forall n \in \mathbb{N}. \] (8)
Since \( \mathfrak{A} \) is backwards compact, the set \( B := \bigcup_{t \leq 0} A(t) \) is compact. By the forward attraction of \( \mathcal{A} \) under \( S \), there exists a \( T_0 > 0 \) such that
\[ \text{dist} (S(T_0, B), \mathcal{A}) < \delta/2. \] (9)
Besides, by the invariance of \( \mathfrak{A} \), for every \( x_n \) there is \( b_n \in A(-t_n - T_0) \subset B \) such that
\[ x_n = U(-t_n, -t_n - T_0, b_n), \]
and \( b_n \to b_0 \) as \( n \to \infty \) for some \( b_0 \in B \). Hence, by condition (7), there exists an \( N = N(\delta) > 0 \) such that
\[ d_X (x_N, S(T_0, b_0)) = d_X (U(-t_N, -t_N - T_0, b_N), S(T_0, b_0)) < \delta/2. \] (10)
Therefore, from (10) and (9) it follows that,
\[ \text{dist} (x_N, \mathcal{A}) \leq d_X (x_N, S(T_0, b_0)) + \text{dist} (S(T_0, b_0), \mathcal{A}) < \delta, \]
which contradicts (8). Hence we have the theorem.

Next, we establish an alternative theorem using different conditions thanks to which we can further obtain the convergence in the full Hausdorff metric sense, not only in the semi-metric sense. Note that this is a major difference between pullback and forward convergences, since for the forward convergence we have only the semi-convergence in Hausdorff semi-metric sense due to the fact that a pullback attractor is not forward attracting in general (see [6, 9, 11, 10, 12, 16] for conditions ensuring a pullback attractor to be forward attracting) so that the method we will use below for pullback convergences does not apply to forward convergences.

It is convenient to begin with a more general convergence, that of pullback attractors to pullback attractors as \( t \to -\infty \).

**Proposition 5.** Suppose that \( \mathfrak{A} = \{A(t)\}_{t \in \mathbb{R}} \) and \( \mathfrak{A}_\infty = \{A_\infty(t)\}_{t \in \mathbb{R}} \) are pullback attractors of processes \( U \) and \( U_\infty \), respectively. If
(i) \( \mathfrak{A} \) is backwards bounded, i.e. there is a bounded set \( B \) such that \( \bigcup_{t \leq 0} A(t) \subset B \);
(ii) the following convergence holds
\[ \sup_{x \in B, \tau \in \mathbb{R}^+} d_X \left( U(t, t - \tau, x), U_\infty(t, t - \tau, x) \right) \to 0, \quad \text{as } t \to -\infty, \] (11)
then
\[ \lim_{t \to -\infty} \text{dist} (A(t), A_\infty(t)) = 0. \]
If, moreover, \( \mathfrak{A}_\infty \) is also backwards bounded in \( B \), then the two attractors \( \mathfrak{A} \) and \( \mathfrak{A}_\infty \) are asymptotically identical in distant past, i.e.,
\[ \lim_{t \to -\infty} \text{dist}_H (A(t), A_\infty(t)) = 0. \]
Theorem 6. Suppose that for some $\delta > 0$ there exists a sequence $t_n \to \infty$ such that
\[
\text{dist} \left( A(-t_n), A_\infty(-t_n) \right) \geq \delta, \quad \forall n \in \mathbb{N}.
\]
Then by the compactness of $\mathfrak{A}$ there exists a sequence $x_n \in A(-t_n)$ such that
\[
\text{dist} \left( x_n, A_\infty(-t_n) \right) = \text{dist} \left( A(-t_n), A_\infty(-t_n) \right) \geq \delta, \quad \forall n \in \mathbb{N}. \tag{12}
\]
Since $\mathfrak{A}$ is invariant, for every $m, n \in \mathbb{N}$ we have a $b_{n,m} \in A(-t_n - m) \subset B$ such that
\[
x_n = U(-t_n, -t_n - m, b_{n,m}), \quad \forall n, m \in \mathbb{N}.
\]
Hence, by condition (11), there exists an $N = N(\delta) > 0$ such that for all $m \in \mathbb{N}$
\[
d_X \left( x_N, U(-t_N, -t_N - m, b_{N,m}) \right)
= d_X \left( U(-t_N, -t_N - m, b_{N,m}), U_\infty(-t_N, -t_N - m, b_{N,m}) \right)
\leq \sup_{x \in B, \tau \in \mathbb{R}^+} d_X \left( U(-t_N, -t_N - \tau, x), U_\infty(-t_N, -t_N - \tau, x) \right) < \delta/2. \tag{13}
\]
In addition, since $\{b_{n,m}\} \subset B$ is pullback attracted by $\mathfrak{A}_\infty$ under $U_\infty$, there is an $M = M(N, \delta) > 0$ such that
\[
\text{dist} \left( U_\infty(-t_N, -t_N - m, b_{N,m}), A_\infty(-t_N) \right)
\leq \text{dist} \left( U_\infty(-t_N, -t_N - m, B), A_\infty(-t_N) \right) < \delta/2, \quad \forall m \geq M. \tag{14}
\]
Therefore, from (13) and (14) it follows that, for all $m \geq M$,
\[
\text{dist} \left( x_N, A_\infty(-t_N) \right) \leq d_X \left( x_N, U_\infty(-t_N, -t_N - m, b_{N,m}) \right)
+ \text{dist} \left( U_\infty(-t_N, -t_N - m, b_{N,m}), A_\infty(-t_N) \right) < \delta,
\]
which contradicts (12).

As a corollary of Proposition 5, we have

\textbf{Theorem 6.} Suppose that $U$ is a process with pullback attractor $\mathfrak{A} = \{ A(t) \}_{t \in \mathbb{R}}$ and that $S$ is a semigroup with global attractor $\mathcal{A}$. If

(i) $\mathfrak{A}$ is backwards bounded, i.e. there is a bounded set $B$ such that $\cup_{t \leq 0} A(t) \subset B$;
(ii) the following backwards asymptotically autonomous condition holds
\[
\sup_{x \in B, \tau \in \mathbb{R}^+} d_X \left( U(t, t - \tau, x), S(\tau, x) \right) \to 0, \quad \text{as } t \to -\infty, \tag{15}
\]

then the global attractor $\mathcal{A}$ is the $\alpha$-limit set of the pullback attractor $\mathfrak{A}$, namely,
\[
\lim_{t \to -\infty} \text{dist}_H \left( A(t), \mathcal{A} \right) = 0.
\]

\textbf{Proof.} Define $U_\infty(t, s, x) := S(t - s, x)$ for $t \geq s$ and $x \in X$, then $U_\infty$ is a process with pullback attractor $\mathfrak{A}_\infty$ with $A_\infty(t) \equiv A$. Hence, by Proposition 5 the theorem follows. \qed
3. **Necessary conditions.** In the previous section we established sufficient conditions ensuring the convergence of pullback attractors towards global attractors. In this section, we study necessary conditions. We first recall a locally uniform compactness.

**Definition 7.** ([5]) A family $\mathcal{E} = \{E_t\}_{t \in \mathbb{R}}$ of nonempty compact sets is said to be *locally uniformly compact*, if for any bounded interval $I \subset \mathbb{R}$ the union $\cup_{t \in I} E_t$ is precompact.

For the pullback attractor $\mathcal{A}$ of a process $U$, the locally uniform compactness of $\mathcal{A}$ is often trivial since the mapping $t \mapsto A_t$ is often continuous in the full Hausdorff metric sense, provided that $s \to U(s, \tau, x)$ is continuous, see [13, p31], also [18]. In the framework of random attractors Cui et al. [5] studied the case without the continuity in $s$.

The following proposition indicates that, with the locally uniform compactness, a pullback attractor $A$ forward converges to a global attractor $A$ implies that $A$ is forward compact. Analogously, the backwards convergence and backwards compactness have the same relationship. Hence, roughly, forward and backward compactnesses of pullback attractors can be necessary conditions of the corresponding convergences towards global attractors. Forward and backward boundednesses can be necessary conditions as well.

**Proposition 8.** Suppose that $\{E_t\}_{t \in \mathbb{R}}$ is a locally uniformly compact family of nonempty compact sets in $X$. Then there is a nonempty compact set $E$ such that

$$\lim_{t \to \infty} \text{dist}(E_t, E) = 0$$

if and only if $\{E_t\}_{t \in \mathbb{R}}$ is forward compact.

**Proof.** Sufficiency. The sufficiency is clear taking $E := \bigcup_{t \geq 0} E_t$.

Necessity. To prove the forward compactness, for any a sequence $\{x_n\}_{n \in \mathbb{N}} \subset \bigcup_{t \geq 0} E_t$ we need to prove that $\{x_n\}$ has a convergent subsequence. Since each $E_t$ is compact, without loss of generality we suppose that there exists increasing sequence $\{t_n\}_{n \in \mathbb{N}} \subset [0, \infty)$ such that $x_n \in E_{t_n}$ for each $n \in \mathbb{N}$. Then two possibilities occur. If $t_n \to \infty$, then as $\text{dist}(x_n, E) \leq \text{dist}(E_{t_n}, E) \to 0$ and $E$ is compact, $\{x_n\}$ has indeed a convergent subsequence; if $\sup_{n \in \mathbb{N}} t_n < a < \infty$, then because of $\{x_n\} \subset \bigcup_{t \in [0, a]} E_t$ being precompact, $\{x_n\}$ has convergent subsequences as well. Hence we have the proposition. 

4. **An example of a reaction-diffusion equation on unbounded domain.**

Consider the following non-autonomous reaction-diffusion equation on $\mathbb{R}$

$$\frac{du}{dt} - u_{xx} + \lambda u + f(u) = g(x, t), \quad (16)$$

with initial condition

$$u(\tau) = u_0, \quad (17)$$

where $\lambda > 0$ and the nonlinearity $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfying

$$f(u)u \geq 0, \quad f(0) = 0, \quad f'(u) \geq -c, \quad |f'(u)| \leq c(1 + |u|^p) \quad (18)$$

with $p > 0$. The non-autonomous forcing $g \in L^2_{\text{loc}}(\mathbb{R}, L^2(\mathbb{R}))$ satisfies the tempered condition

$$\int_{-\infty}^{0} e^{\lambda s} \|g(s)\|^2 \, ds < \infty. \quad (19)$$
Consider also the autonomous case with the same conditions
\[
\frac{du}{dt} - u_{xx} + \lambda u + f(u) = g_0(x),
\]
where \(g_0(x) \in L^2(\mathbb{R})\).

It is well-known that the systems (16) and (20) have unique solutions, see, e.g., [19, 23, 21, 24]. Besides, under some conditions of \(f\) and \(g\), the proof of the existence of the pullback attractor is quite standard, see for instance [3]. For brevity we will not pursue the details here, but assume it and focus on the asymptotically autonomous properties of the pullback attractor.

**Assumption 1.** Assume that the systems (16) and (20) both have unique solutions, and have a pullback attractor \(A = \{A(t)\}_{t \in \mathbb{R}}\) and a global attractor \(\mathcal{A}\) in \(L^2(\mathbb{R})\), respectively.

The unique existence of solutions implies that the mapping \(U(t, \tau, u_0) := u(t, \tau, u_0), t \geq \tau\), corresponding to the solutions \(u\) of (16) defines a process, and \(S(s, u_{2,0}) : = u_2(s, 0, u_{2,0}) = u_2(s + \tau, \tau, u_{2,0}), s \geq 0, \tau \in \mathbb{R}\), corresponding to the solutions \(u_2\) of (20) defines a semigroup.

**Lemma 9.** Any solution \(u(t, \tau, u_0)\) of problem (16) satisfies
\[
\|u(t, \tau, u_0)\|^2 \leq e^{\lambda(\tau-t)}\|u_0\|^2 + c \int_{\tau}^{t} e^{\lambda(s-t)}\|g(s)\|^2 \, ds, \quad \forall t \geq \tau.
\]

**Proof.** Taking the inner product of (16) with \(u\) in \(L^2(\mathbb{R})\) we have
\[
\frac{1}{2} \frac{d}{dt} \|u\|^2 + \|u_x\|^2 + \|u\|^2 + (f(u), u) = (g(t), u),
\]
which follows that
\[
\frac{d}{dt} \|u\|^2 + \lambda \|u\|^2 \leq c\|g(t)\|^2.
\]

By Gronwall’s inequality we have the result. \(\square\)

Hence, define \(\mathfrak{B} = \{B(t)\}_{t \in \mathbb{R}}\) with
\[
B(t) := \left\{ u \in L^2(\mathbb{R}) : \|u\|^2 \leq c \int_{-\infty}^{t} e^{\lambda(s-t)}\|g(s)\|^2 \, ds + 1 \right\}, \quad \forall t \in \mathbb{R}. \tag{21}
\]

Then \(\mathfrak{B}\) is a pullback absorbing set containing the pullback attractor, i.e., \(A(t) \subset B(t)\).

Now we study the forward and backward convergences of the pullback attractor \(\mathfrak{A}\) towards the global attractor \(\mathcal{A}\). Naturally, different conditions of the non-autonomous forcing \(g(x,t)\) will be required, but we do not need particular techniques to obtain further compactness of the pullback attractor since our theorems only need boundedness conditions.

**Theorem 10** (Forward convergence). If
\[
\lim_{\tau \to \infty} \int_{\tau}^{\infty} \|g(s) - g_0\|^2 \, ds = 0, \tag{22}
\]
then
\[
\lim_{t \to \infty} \text{dist}(A(t), \mathcal{A}) = 0.
\]
Proof. Let $B$ be any nonempty bounded set, and let $u_1(t, \tau, x)$ and $u_2(t, \tau, x)$ be the solutions of (16) and (20), respectively, with the same initial value $x_0 \in B$ at $\tau$. Then the difference $w(t, \tau, 0) := u_1(t, \tau, x) - u_2(t, \tau, x)$ satisfies

$$\frac{dw}{dt} - w_{xx} + \lambda w + f(u_1) - f(u_2) = g(t) - g_0,$$

with initial value $w(\tau) = 0$. Taking the inner product with $w$ in $L^2(\mathbb{R})$ we have

$$\frac{1}{2} \frac{d}{dt} \|w\|^2 + \|w_x\|^2 + \lambda \|w\|^2 + (f(u_1) - f(u_2), w) = (g(t) - g_0, w).$$

Hence, by (18) and Young’s inequality,

$$\frac{d}{dt} \|w\|^2 \leq c \|w\|^2 + \|g(t) - g_0\|^2.$$

Notice that condition (22) implies

$$\lim_{t \to \infty} \int_{t-T}^t \|g(s) - g_0\|^2 \ ds = 0, \ \forall T > 0,$$

so by Gronwall’s inequality we have, for any $T > 0$,

$$\|w(t, t-T, 0)\|^2 \leq c \int_{t-T}^t e^{c(t-s)} \|g(s) - g_0\|^2 \ ds$$

$$\leq ce^{cT} \int_{t-T}^t \|g(s) - g_0\|^2 \ ds \to 0, \ \text{as} \ t \to \infty,$$

i.e., condition (ii) in Theorem 3 is satisfied.

In the following, by Theorem 3, we only need to show the forward boundedness of the pullback attractor. By condition (22), there exists an $N \in \mathbb{N}$ such that

$$\int_N^\infty \|g(s) - g_0\|^2 \ ds \leq 1.$$ 

Therefore,

$$\sup_{t \geq N} \int_{-\infty}^t e^{\lambda(s-t)} \|g(s)\|^2 \ ds \leq \sup_{t \geq N} \int_{-\infty}^t e^{\lambda(s-t)} \left(\|g(s) - g_0\|^2 + \|g_0\|^2\right) \ ds$$

$$\leq \sup_{t \geq N} \int_{-\infty}^t e^{\lambda(s-t)} \|g(s) - g_0\|^2 \ ds + \frac{\|g_0\|^2}{\lambda}$$

$$\leq \sup_{t \geq N} \left(\int_{-\infty}^N e^{\lambda(s-t)} \|g(s) - g_0\|^2 \ ds + \int_N^t \|g(s) - g_0\|^2 \ ds\right) + \frac{\|g_0\|^2}{\lambda}$$

$$\leq \int_{-\infty}^N e^{\lambda s} \|g(s) - g_0\|^2 \ ds + 1 + \frac{\|g_0\|^2}{\lambda} < \infty.$$ 

This implies that the pullback absorbing set $\mathcal{B}$ given by (21) is forward bounded, and so is the pullback attractor.

\[\square\]

**Theorem 11** (Backwards convergence). If

$$\lim_{\tau \to -\infty} \int_{-\infty}^\tau \|g(s) - g_0\|^2 \ ds = 0,$$

then the global attractor $\mathcal{A}$ is the $\alpha$-limit set of the pullback attractor $\mathfrak{A}$, i.e.,

$$\lim_{t \to -\infty} \text{dist}_H (A(t), \mathcal{A}) = 0.$$
Proof. Let \( B \) be any nonempty bounded set, and let \( u_1(t, \tau, x) \) and \( u_2(t, \tau, x) \) be the solutions of (16) and (20), respectively, with the same initial value \( x \in B \) at \( \tau \). Then the difference \( w(t, \tau, 0) := u_1(t, \tau, x) - u_2(t, \tau, x) \) satisfies (24), i.e.,

\[
\frac{d}{dt} \|w\|^2 \leq c\|w\|^2 + \|g(t) - g_0\|^2.
\]

Since the initial value of \( w \) is always zero, by Gronwall’s inequality we have

\[
\|w(t, t - T, 0)\|^2 \leq \int_{t-T}^t e^{-c(t-s)} \|g(s) - g_0\|^2 \, ds \\
\leq \int_{-\infty}^t \|g(s) - g_0\|^2 \, ds, \quad \forall T > 0.
\]

Thus, \( \sup_{T>0} \|w(t, t - T, 0)\| \to 0 \) as \( t \to -\infty \), i.e., condition (ii) of Theorem 6 holds.

Next, we need to establish the backward boundedness of \( \mathfrak{A} \), i.e., condition (i) of Theorem 6. Since by (27) there exists an \( M \in \mathbb{N} \) such that

\[
\int_{-\infty}^{-M} \|g(s) - g_0\|^2 \, ds \leq 1,
\]

we have

\[
\sup_{t < -M} \int_{-\infty}^t e^{\lambda(s-t)} \|g(s)\|^2 \, ds \leq \sup_{t \leq -M} \int_{-\infty}^t e^{\lambda(s-t)} \left(\|g(s) - g_0\|^2 + \|g_0\|^2\right) \, ds \\
\leq \sup_{t \leq -M} \int_{-\infty}^t e^{\lambda(s-t)} \|g(s) - g_0\|^2 \, ds + \sup_{t \leq -M} \int_{-\infty}^t e^{\lambda(s-t)} \|g_0\|^2 \, ds \\
\leq \int_{-\infty}^{-M} \|g(s) - g_0\|^2 \, ds + \frac{1}{\lambda} \int_{-\infty}^0 e^{\lambda s} \|g_0\|^2 \, ds < \infty.
\]

Hence, the pullback absorbing set \( \mathfrak{A} \) given by (21) is backward bounded, and so is the pullback attractor.

5. Endnotes. In this paper we studied an asymptotically autonomous problem of non-autonomous dynamical systems in terms of the convergences of the pullback attractor towards the global attractor of the limiting semigroup. By boundedness conditions we established criteria Theorems 3 and 6, which weakened the corresponding compactness conditions in Theorems 1 and 4, with asymptotic autonomy conditions (1) and (7) also modified to conditions (2) and (15), respectively. In addition, as an advantage of (15) over (7), in Theorem 6 we obtained backward convergence in the full Hausdorff metric sense. All the analysis in this paper does not require any continuity conditions of the dynamical systems.

In view of applications, Theorems 3 and 6 can be much easier to apply than Theorems 1 and 4. First, boundedness conditions are clearly much easier to establish than compactness conditions; second, conditions (2) and (15) in Theorems 3 and 6 can also be easier to verify than conditions (1) and (7) in Theorems 1 and 4, since in (1) and (7) the process and the limiting semigroup are considered for different initial data which, however, can sometimes cause big difficulties. A multi-valued ODE example that condition (1) does not hold while condition (2) holds easily was given in the work [4]; see also [8, Section 5.2].
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