PROJECTIVE VARIETIES
COVERED BY ISOTRIVIAL FAMILIES

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Abstract. Let $X,Y$ be projective schemes over a discrete valuation ring $R$, where $Y$ is generically smooth and $g : X \to Y$ is a surjective $R$-morphism such that $g_*\mathcal{O}_X=\mathcal{O}_Y$. We show that if the family $X \to \text{Spec}(R)$ is isotrivial, then the generic fiber of the family $Y \to \text{Spec}(R)$ is isotrivial.

1. Introduction

Let $k$ be a field of characteristic zero and $C$ a smooth projective curve defined over $k$ with function field $F$.

Definition 1.1. Let $S$ be a scheme over $k$ and $\pi : X \to S$ a flat family of schemes. The family $\pi$ is called trivial if there exists a scheme $X_0$ defined over $k$ such that $X \cong X_0 \times_k S$, and it is called isotrivial if there exists a finite surjective étale extension $S' \to S$ such that $\pi_{S'} : X \times_S S' \to S'$ is trivial.

Let $R$ be the local ring at a closed point $P \in C$. Let $X,Y$ be projective schemes over $\text{Spec}(R)$ where $Y$ is generically smooth and $g : X \to Y$ is a surjective $R$-morphism such that $g_*\mathcal{O}_X=\mathcal{O}_Y$. We show in Theorem 2.7 that if the family $X \to \text{Spec}(R)$ is isotrivial, then the generic fiber of the family $Y \to \text{Spec}(R)$ is isotrivial.

Sketch of proof. The deformations of $Y$ are controlled by its differentials. We study the deformations locally, i.e. over the discrete valuation ring $R$, and show that the fundamental exact sequence associated to the diagram

$$X \to Y \xrightarrow{p} \text{Spec}(R) \to \text{Spec}(k),$$

(1) $0 \to p^*\Omega_{\text{Spec}(R)/\text{Spec}(k)} \to \Omega_{Y/\text{Spec}(k)} \to \Omega_{Y/\text{Spec}(R)} \to 0$

is split exact and consequently the deformations of $Y$ are governed by those of $X$. Then we consider an infinitesimal deformation of $Y \to \text{Spec}(R)$ over the henselization of $R$ (denoted by $\bar{R}$) and show that the sequence above remains split exact at every level of the deformation. Finally we use a result of Greenberg to pass from $\bar{R}$ to $R$.

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The proof can probably be adapted as is to the positive characteristic case if we assume the extension $F/k$ is separable. As an application we shall use this result in a forthcoming paper with Lucien Szpiro on the parametrization of points of canonical height zero of an algebraic dynamical system. In [2], (3, Thm. 3.3, p. 702) Chatzidakis and Hrushovski answer the same question using model theoretic methods. Their methods are intrinsically birational; thus they have a slightly less precise conclusion. They assume the extension $F/k$ is regular. If the ground field $k$ is perfect and $Y$ is one dimensional, then their result extends to positive characteristic.

Notation. Throughout this paper $k$ denotes a field of characteristic zero and $C$ a smooth projective curve over $k$ with function field $F$. Given a scheme $X$ over $C$ we denote its generic fiber by $X_F$.

2. Descent

In this section we assume that $k$ is algebraically closed. The proofs work without this hypothesis with some minor modifications. Let $P$ be a closed point on the curve $C$ and $R = \mathcal{O}_{C,P}$, the local ring at $P$. For ease of notation we denote $\text{Spec}(R)$ and $\text{Spec}(k)$ respectively in the sheaves of differentials.

Proposition 2.1. Let

$$X \xrightarrow{g} Y \xrightarrow{p} \text{Spec}(R) \to \text{Spec}(k)$$

be morphisms of schemes where $X,Y$ are projective, $X \to \text{Spec}(R)$ is an isotrivial family, $Y$ is reduced and $g : X \to Y$ is a surjective $R$-morphism such that $g_*\mathcal{O}_X = \mathcal{O}_Y$. Then the sequence of differentials on $Y$,

$$\tag{2} p^*\Omega_{R/k} \to \Omega_{Y/k} \to \Omega_{Y/R} \to 0,$$

is split exact.

Proof. After a quasi-finite unramified base change $\text{Spec}(R') \to \text{Spec}(R)$ we may assume that the family $X_{R'} \to \text{Spec}(R')$ is trivial. Slightly abusing the notation we shall say the family $X \to \text{Spec}(R)$ is trivial. It follows that the sequence of differentials on $X$,

$$\tag{3} 0 \to g^*p^*\Omega_{R/k} \to \Omega_{X/k} \to \Omega_{X/R} \to 0,$$

is split exact. Since pullbacks preserve right exactness, the sequence of differentials on $Y$ pulled back to $X$ along $g$,

$$\tag{4} g^*p^*\Omega_{R/k} \to g^*\Omega_{Y/k} \to g^*\Omega_{Y/R} \to 0,$$

is exact. The morphism $g : X \to Y$ induces the following commutative diagram:

$$\begin{array}{ccc}
g^*p^*\Omega_{R/k} & \xrightarrow{Id} & g^*\Omega_{Y/k} & \xrightarrow{g_*} \Omega_{Y/R} & \xrightarrow{0} \\
0 & \downarrow & 0 & \downarrow & 0 \\
g^*p^*\Omega_{R/k} & \xrightarrow{g_*} \Omega_{X/k} & \xrightarrow{g_*} \Omega_{X/R} & \xrightarrow{0} \\
\end{array}$$

It follows that (4) is split exact. Since $g_*$ preserves direct sums, we have

$$g_*g^*\Omega_{Y/k} \cong g_*g^*\Omega_{Y/R} \oplus g_*g^*p^*\Omega_{R/k}.$$
The natural map $\Omega_{Y/k} \to g_*g^*\Omega_{Y/k}$ induces the following commutative diagram:

$$
\begin{array}{ccc}
0 & \to & g_*g^*\Omega_{R/k} \\
\downarrow & & \downarrow \\
g_*g^*\Omega_{Y/k} & \to & g_*g^*\Omega_{Y/R} \\
\downarrow & & \downarrow \\
g_*g^*\Omega_{Y/k} & \to & 0 \\
\end{array}
$$

Note that the bottom row is split exact, $p^*\Omega_{R/k} \cong \mathcal{O}_Y$ and $g_*g^*p^*\Omega_{R/k} \cong g_*\mathcal{O}_X$. By assumption $g_*\mathcal{O}_X = \mathcal{O}_Y$; thus the top row is split exact. □

Remark. The condition $g_*\mathcal{O}_X = \mathcal{O}_Y$ implies that $g$ has connected fibers ([7], Cor. 11.3, p. 279). If $g$ is finite, then the condition $g_*\mathcal{O}_X = \mathcal{O}_Y$ implies that $\deg(g) = 1$; hence $g$ is birational. Moreover if $Y$ is normal, using Zariski’s Main Theorem ([8], p. 209) we conclude that $g$ is an isomorphism.

We now consider an infinitesimal deformation of $Y$ over the henselian discrete valuation ring, denoted $\tilde{R}$, and proceed to show that the family $Y_F \to \text{Spec}(F)$ is isotrivial. Before we proceed we need the following definitions:

**Definition 2.2.** Let $S$ be a smooth scheme of finite type over $k$ and $f : X \to S$ a morphism of schemes. If $f$ is smooth, then the sequence

$$0 \to f^*\Omega_{S/k} \to \Omega_{X/k} \to \Omega_{X/S} \to 0$$

is exact. This extension is non-trivial in general and is given by a class $c \in \text{Ext}^1(\Omega_{X/S}, f^*\Omega_{S/k})$. Since $\Omega_{X/S}$ is locally free, one has

$$\text{Ext}^1(\Omega_{X/S}, f^*\Omega_{S/k}) \cong \text{Ext}^1(\mathcal{O}_X, T_{X/S} \otimes f^*\Omega_{S/k}) \cong H^1(X, T_{X/S} \otimes f^*\Omega_{S/k}).$$

The image of $c$ by the canonical map

$$H^1(X, T_{X/S} \otimes f^*\Omega_{S/k}) \to H^0(S, R^1f_*(T_{X/S} \otimes f^*\Omega_{S/k}))$$

is called the **Kodaira-Spencer class** of $X/S$. One can view this class as a morphism also, i.e. the Kodaira-Spencer morphism

$$\kappa_{X/S} : T_S \to R^1f_*T_{X/S}. $$

The fiber $(\kappa_{X/S})_s = \kappa_s : T_{S,s} \to H^1(X_s, T_{X,s})$ is the Kodaira-Spencer map at $s \in S$.

The Kodaira-Spencer map at $s$ measures how $X_s$ deforms in the family $X/S$ in the neighborhood of $s$ ([11], p. 165).

**Definition 2.3 ([11], p. 255).** A local ring $A$ is **henselian** if every finite $A$-algebra $B$ is a product of local rings. We define the **henselization** of $A$ to be a pair $(\hat{A}, i)$, where $\hat{A}$ is a local henselian ring and $i : A \to \hat{A}$ is a local homomorphism such that for any local henselian ring $B$ and any local homomorphism $u : A \to B$ there exists a unique local homomorphism $\tilde{u} : \hat{A} \to B$ such that $u = \tilde{u} \circ i$. 

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From here on we assume that $Y$ is generically smooth. Let $R$ be the local ring at $P \in C$, $\hat{R}$ its henselization, and $\mathfrak{m}$ the maximal ideal of $\hat{R}$. Define $R_n = \hat{R}/\mathfrak{m}^{n+1}$ for each $n \geq 0$. There are natural maps $\text{Spec}(\hat{R}) \to \text{Spec}(R)$ and $\text{Spec}(R_{n-1}) \to \text{Spec}(R_n)$ induced by the projections $R_n \to R_{n-1}$ for $n \geq 1$. Define $\tilde{Y} := Y \times_R \text{Spec}(R_n)$ for each $n \geq 0$. We have the following commutative diagram of schemes:

$$
\begin{array}{ccc}
Y_F & \to & Y \\
\downarrow & & \downarrow \text{p} \\
\text{Spec}(F) & \to & \text{Spec}(R)
\end{array}
\quad \quad
\begin{array}{ccc}
\tilde{Y} & \leftarrow & Y_0 \\
\downarrow & & \downarrow \text{p}_0 \\
\text{Spec}(R) & \leftarrow & \text{Spec}(R_n)
\end{array}
\quad \quad
\begin{array}{ccc}
\tilde{Y} & \leftarrow & \text{Spec}(R) \\
\downarrow & & \downarrow \text{p}_n \\
\text{Spec}(R_n) & \leftarrow & \text{Spec}(R_{n-1})
\end{array}
\quad \quad
\begin{array}{ccc}
\text{Spec}(R_{n-1}) & \leftarrow & \text{Spec}(R_n) \\
\downarrow & & \downarrow \text{p}_n \\
\text{Spec}(R_n) & \leftarrow & \text{Spec}(R)
\end{array}
$$

Proposition 2.4. For each $n \geq 0$ the sequence of differentials associated to $Y_n \to \text{Spec}(R_n)$, i.e.

$$
0 \to p_n^*\Omega_{R_n/k} \to \Omega_{Y_n/k} \to \Omega_{Y_n/R_n} \to 0,
$$

is split exact. Moreover, $Y_n \to \text{Spec}(R_n)$ is trivial.

Proof. Pulling back (2) along the natural map $\text{Spec}(R_n) \to \text{Spec}(R)$ we get the sequence (3). Since pullbacks preserve direct sums, the sequence (3) is split exact; i.e. the Kodaira-Spencer class of $Y_n/\text{Spec}(R_n)$ is trivial. In other words, $Y_n \to \text{Spec}(R_n)$ is trivial. It follows that $Y_n \cong Y_0 \times_k \text{Spec}(R_n)$ for each $n \geq 0$. \qed

Definition 2.5. If $V,W$ and $T$ are $S$-schemes, an $S$-isomorphism from $V$ to $W$ parametrized by $T$ will mean a $T$-isomorphism from $V \times_S T$ to $W \times_S T$. The set of all such isomorphisms will be denoted by $\text{Isom}_S(V,W)(T)$.

The association $T \mapsto \text{Isom}_S(V,W)(T)$ defines a contravariant functor

$$
\text{Isom}_S(V,W) : (\text{Sch}/S)^\circ \to (\text{Sets}).
$$

The functor $\text{Isom}_S(V,W)$ is representable whenever $V,W$ are flat and projective over $S$. For a proof of the representability of the $\text{Isom}$ functor we refer the reader to ([5] pp. 132-133). We denote the scheme representing the functor $\text{Isom}_S(V,W)$ by $\text{Isom}_S(V,W)$.

To conclude that the family $Y_F \to \text{Spec}(F)$ is isotrivial we need the following result of Greenberg:

Theorem 2.6. Let $\hat{R}$ be a henselian discrete valuation ring, with $t$ the generator of the maximal ideal. Let $\tilde{Z}$ be a scheme of finite type over $\hat{R}$. Then $\tilde{Z}$ has a point in $\hat{R}$ if and only if $\tilde{Z}$ has a point in $\hat{R}/t^n$ for every $n \geq 1$.

Proof. ([4], Corollary 2). \qed

Theorem 2.7. Let $X,Y$ be projective schemes over a discrete valuation ring $R$ where $X \to \text{Spec}(R)$ is an isotrivial family, $Y$ is generically smooth and $g : X \to Y$ is a surjective $R$-morphism such that $g_*\mathcal{O}_X = \mathcal{O}_Y$. Then $Y \to \text{Spec}(R)$ is generically isotrivial, i.e.

$$
Y_{F'} \cong Y_0 \times_k \text{Spec}(F'),
$$

where $F'$ is a finite extension of $F$.\qed
Proof. Observe that $\tilde{Y}$ and $Y_0 \times_k \text{Spec}(\tilde{R})$ are flat, projective over $\text{Spec}(\tilde{R})$. Let $\text{Isom}_R(\tilde{Y}, Y_0 \times_k \text{Spec}(\tilde{R}))(T)$ be the set of isomorphisms from $\tilde{Y} \times_R T \to (Y_0 \times_k \text{Spec}(\tilde{R})) \times_R T$.

Let $f \in \text{Isom}_R(\tilde{Y}, Y_0 \times_k \text{Spec}(\tilde{R}))(k)$ and $\Gamma_f$ denote the graph of $f$. Let $L$ be a very ample line bundle of $Y$ and $\tilde{L}$ the pullback of $L$ to $Y$ along the morphism $\tilde{Y} \to Y$. Let $P(t)$ be the Hilbert polynomial of $(\Gamma_f)_k$, the special fiber of $\Gamma_f$, with respect to $\tilde{L}$. Then the functor

$$\text{Isom}_R(\tilde{Y}, Y_0 \times_k \text{Spec}(\tilde{R})) \cap \text{Hilb}^{P(t)}_{\tilde{Y} \times_R (Y_0 \times_k \text{Spec}(\tilde{R}))}$$

is representable and the scheme representing it, denoted by $Z'$, is of finite type. For

$$Z = \text{Isom}_R(Y, Y_0 \times \text{Spec}(R)) \cap \text{Hilb}^{P(t)}_{\tilde{Y} \times_R (Y_0 \times_k \text{Spec}(R))}$$

we have $Z' = Z \times_R \text{Spec}(\tilde{R})$. By Proposition 2.4, $Y_n \cong Y_0 \times_k \text{Spec}(R_n)$ for each $n \geq 0$. Thus $Z'$ has an $R_n$-point for every $n \geq 0$. By the previous theorem $Z'$ has a $\tilde{R}$-point, i.e. $\tilde{Y} \cong Y_0 \times_k \text{Spec}(\tilde{R})$. Note that $\tilde{R}$ is a limit of étale covers of $\tilde{R}$ so there exists an étale cover $R'$ of $R$ such that $Y_{R'} \cong Y_0 \times_k \text{Spec}(R')$. Thus $F'$ (the quotient field of $R'$) satisfies the requirements of the theorem. 

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