DISPERSION RELATIONS IN QUANTUM CHROMODYNAMICS

Reinhard Oehme

Enrico Fermi Institute and Department of Physics
University of Chicago, Chicago, Illinois, 60637

ABSTRACT

Dispersion relations for the scattering of hadrons are considered within the framework of Quantum Chromodynamics. It is argued that the original methods of proof remain applicable. The setting and the spectral conditions are provided by an appropriate use of the BRST-cohomology. Confinement arguments are used in order to exclude quarks and gluons from the physical state-space. Local, BRST-invariant hadron fields are considered as leading terms in operator product expansions for products of fundamental fields. The hadronic amplitudes have neither ordinary nor anomalous thresholds which are directly associated with the underlying quark-gluon-structure. Proofs involving the Edge of the Wedge Theorem and analytic completion are discussed briefly.

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1. **Introduction**

Dispersion relations for the scattering of hadrons have been formulated [1-3] and proved in the Fifties [4-9]. They are by no means simple generalizations of the familiar Kramers-Kronig relations for the scattering of light [10,11]. The presence of finite masses presents a formidable problem for obtaining the required analytic continuations. Charges of various kinds give rise to non-trivial crossing properties, which lead to analytic connections of amplitudes for quite different reactions.

Even though they have been introduced a long time ago, dispersion relations have continued to play an important rôle in the analysis of hadron scattering. In a more general framework, the analytic properties of Green’s functions are the foundation for many important results and theorems in field theory. However, this analytic structure has not been discussed in detail within the framework non-Abelian gauge theories like QCD and, in particular, in the presence of confinement.

The original derivations of dispersion relations [4-9] are within the framework of the general postulates of relativistic quantum field theory of hadrons [12]. There is no need to specify the theory in detail. The essential input is locality, in the form of the existence of Heisenberg field operators, which commute or anti-commute at space-like separations, and which interpolate between asymptotic fields describing non-interacting physical hadrons. In addition, spectral conditions are very important for the proof. The difficulties with multi-particle intermediate states, and with the analytic structure of the corresponding multi-particle amplitudes, are the main reason for the limitations of general proofs in some interesting cases.

It is the purpose of this note to consider hadronic dispersion relations within the framework of Quantum Chromodynamics (QCD). As a constraint
system, this $SU(3)$ gauge field theory of color is best quantized with the help of the Becchi-Rouet-Stora-Tyutin (BRST) symmetry [13] in a state-space $V$ of indefinite metric, and in a covariant gauge like the Landau gauge, for example [14]. A priori, the space $V$ contains quanta like ghosts and longitudinal and time-like gluons, which are unphysical even without confinement. Using the nilpotent BRST operator $Q$, we can define a subspace of states which satisfy $Q\Psi = 0$. This is the kernel $kerQ = \{\Psi : Q\Psi = 0, \Psi \in V\}$ of the operator $Q$. For ghost number zero, the space $kerQ$ can provide the basis for a physical subspace, provided we have completeness of the BRST operator [15]. This notion implies that all states $\Psi \in kerQ$ with zero norm are of the form $\Psi = Q\Phi$, $\Phi \in V$. It is then easy to show that $kerQ$ contains no states with negative norm. We can define a cohomology space $H = kerQ/imQ$ with zero ghost number, containing only states with positive definite norm. All zero norm states in $kerQ$ are contained in the subspace $imQ = \{\Psi : \Psi = Q\Phi, \Phi \in V\}$.

As is well known, the cohomology space $H$ provides a Lorentz-invariant definition of a physical state-space. Without completeness, states in $kerQ$ with zero norm and zero ghost number could be made from ghosts and their conjugates (singlet pair representations of the BRST algebra) [14,16]. There are arguments for completeness [17,18], but we do not know of a general proof for four-dimensional gauge theories like QCD. Unless we have completeness, a consistent formulation of the theory seems to be impossible. In certain string theories, completeness has been proven explicitly [15,19,20].

In the Hilbert space $H$, the ghosts, as well as the longitudinal and the time-like gluons, are eliminated in a kinematical fashion, and this is all that happens in weak coupling QCD perturbation theory. But in the full theory, we expect that all quarks and gluons are confined. With certain limitations
concerning the number of quark flavors, one can give arguments that, for
dynamical reasons, transverse gluon states cannot be elements of the co-
holomogy space $\mathcal{H}$ [21,22]. Some more preliminary arguments also exclude
quark states [23]. Under these circumstances, $\mathcal{H}$ is a true physical Hilbert
space containing only hadronic states. Here we adopt this algebraic view
of confinement. It is quite consistent with more intuitive pictures for the
quark-gluon structure of hadrons. In particular, the existence of an approx-
imately linear quark-antiquark potential follows from the same arguments
in a natural fashion [24,25]. Our arguments for confinement make use of
the renormalization group, and are valid for zero temperature. At finite
temperature, a new, dimensionful parameter comes in, and there may be
deconfinement transitions.

We assume here that exact QCD exists as a quantum field theory, or that
possible embeddings in more comprehensive schemes are not of importance
for confinement and for scattering processes at energies well below the Planck
mass. If local field theory is considered as a low energy limit of string theory,
we may expect deviations from microscopic causality at very small distances,
and hence corresponding corrections to dispersion relations.

Since the S-matrix, as an observable operator, is invariant under BRST-
transformations, it follows that the unitarity relations involve only states
from the subspace $\mathcal{H}$, at least as far as matrix elements with respect to
physical states are concerned [14,16]. With the notion of confinement we
have adopted, this implies that only hadronic states play a direct rôle in
the physical S-matrix. Furthermore, intermediate state decompositions of
hadronic matrix elements of products of BRST-invariant operators with zero
ghost number require only a complete set of hadronic states which span the
space $\mathcal{H}$. For the purpose of deriving hadronic dispersion relations, this
implies that the spectral conditions remain the same as in the old hadronic field theory. Of course, we assume here that there exist composite hadron states in QCD.

As we have mentioned, the locality of the Heisenberg field operators is the basis for obtaining analytic properties of scattering amplitudes. These operators interpolate between asymptotic fields, which generate states of non-interacting particles [26]. Since we are interested in hadrons, we need to construct local operators related to these particles in terms of quark and gluon fields, which are the fundamental fields of QCD. These hadronic, composite Heisenberg fields are BRST-invariant, and they are asymptotically related to the corresponding non-interacting hadron fields. They can be obtained, under certain conditions, from the leading terms in the operator product expansion [27] for the product of quark and antiquark operators (mesons), or for three quark operators (baryons). The construction is not unique, but there are equivalence classes of interpolating fields which give rise to the same S-matrix, as in the case of fundamental fields.

Local field operators associated with the center-of-mass motion of composite particles have been discussed extensively in the literature [28-30]. It should not be surprising that such fields exist, because locality does not imply a point-like structure for the corresponding particles. In quantum electrodynamics, the electron has charge and magnetic moment distributions, which are described by the familiar form factors. Generally, in a relativistic field theory, a given particle can be considered as a composite of an appropriate set of the other particles. The composite structure manifests itself in the form of characteristic branch points for vertex functions and scattering amplitudes. These structure singularities are caused by thresholds in crossed channels of other amplitudes which are related by unitarity to the amplitude
under consideration [31]. For loosely bound systems, the structure branch points can appear as anomalous thresholds [7,31-34] in the physical sheet of the relevant variable.

In the case of hadrons in QCD, a new element comes in. We do not have ordinary composite systems where the constituents are observable particles, but we have confinement. In principle, we may consider a picture where nucleons and mesons are made up of quarks with rather large constituent masses. But the relevant quark masses appearing in QCD are the current masses, which are very small in comparison with hadron masses, at least as far as u- and d-quarks are concerned. The constituent masses would have to be viewed as generated in connection with the confinement process. In this process, gluons play an important rôle. Formally, they come into consideration via the anomaly in the trace of the energy-momentum tensor. Estimates indicate, that the gluons actually give the dominant contribution to the nucleon mass [35]. In view of the situation as described, we cannot use weak-coupling perturbation theory in order to argue for the existence of composite operators for the hadrons, but we must rely on what is known in general about operator products in local field theories [36].

As far as structure singularities and corresponding, possible anomalous thresholds are concerned, the situation in the case with confinement also differs from the conventional picture of a composite system with observable constituents. As explained above, hadronic amplitudes, as expressed in terms of local hadronic fields, have no thresholds associated with quarks or gluons. Consequently there are no non-hadronic structure singularities.

In deriving hadronic dispersion relations on the basis of local, hadronic field theory, we have generally considered the asymptotic condition as an additional postulate. We do the same in QCD, where we use the condition
essentially only in the physical subspace. Since the theory involves gauge fields, there is, a priori, no mass gap, and in its covariant form, QCD operates in a state-space of indefinite metric. The Haag-Ruelle arguments [37] for obtaining the asymptotic condition from the other postulates of the theory are not applicable under these circumstances.

In the following Section we give a brief account of the hadronic subspace and of confinement. In Section 3, we discuss some relevant aspects of composite operators and of operator product expansions. Section 4 is devoted to a very brief survey of the analytic methods used in proving dispersion relations.

This article can give only a brief sketch of the many problems involved in deriving analytic properties of hadronic amplitudes in QCD. We hope to present a more comprehensive report elsewhere.
2. Hadronic Subspace

In this Section we give a brief introduction to the construction of a physical subspace with positive definite metric, which, in view of confinement, can serve as a space spanned by hadronic states exclusively. Under these circumstances, the spectral conditions used in the derivation of dispersion relations for hadrons remain the same as in the generic, hadronic field theory used in the past.

As explained in the Introduction, we consider QCD in a covariant gauge, and quantize in a space \( \mathcal{V} \) of indefinite metric in accordance with BRST-symmetry. The self-adjoint BRST operator \( Q \), and the corresponding ghost number operator \( Q_c \), form the algebra \([13,14]\)

\[
Q^2 = 0, \quad i[Q_c, Q] = Q,
\]

which can be used to generate a decomposition of \( \mathcal{V} \) in the form

\[
\mathcal{V} = \ker Q \oplus \mathcal{V}_u, \quad \ker Q = \mathcal{V}_p \oplus \text{im} Q.
\]

Here we have introduced the subspaces

\[
\ker Q = \{ \Psi : Q\Psi = 0, \ \Psi \in \mathcal{V} \},
\]

and

\[
\text{im} Q = \{ \Psi : \Psi = Q\Phi, \ \Phi \in \mathcal{V} \}.
\]

We notice that \( \text{im} Q \perp \ker Q \) with respect to the indefinite inner product \((\Psi, \Phi)\) defined in \( \mathcal{V} \). A priori, the subspace \( \mathcal{V}_p \) is a candidate for a physical statespace, but it is not invariant under Lorentz transformations, nor under equivalence transformations, which leave the physics unchanged. As is well known, one therefore uses the cohomology space \( \mathcal{H} = \ker Q/\text{im} Q \), which is a
space of equivalence classes. It is isomorphic with $\mathcal{V}_p$. A state $\Psi \in \mathcal{H}$ may be written symbolically as $\Psi = \Psi_p + i m Q$, $\Psi_p \in \mathcal{V}_p$. We have ignored here the grading due to the ghost number operator, since we are interested in the sector $N_c = 0$ as far as $\mathcal{H}$ is concerned.

In order to assure a physical subspace $\mathcal{H} \simeq \mathcal{V}_p$ with positive definite metric, we must assume completeness of the BRST operator $Q$ [15]. This notion implies that all states with zero norm in $\ker Q$ are contained in $\text{im} Q$. Given completeness, it is easy to see that there cannot be any negative norm states in $\ker Q$. It is not enough to have zero ghost number, because the singlet pair representations of the BRST algebra (2.1), which include states of ghosts and anti-ghosts, must also be eliminated. There are arguments for the absence of singlet pairs in the dense subspace generated by Heisenberg operators, but in view of the indefinite metric, the extension to the full space $\mathcal{V}$ is delicate [17,18]. In certain string theories, completeness has been proven explicitly. In any case, without completeness of the BRST operator, a consistent formulation of QCD would seem to be impossible. From a mathematical point of view, the actual sign of the definite norm in the cohomology space is a convention.

Given completeness, we use a simple matrix notation for the zero ghost number sector of $\mathcal{V}$, with components referring to the subspaces $\mathcal{V}_p$, $i m Q$ and $\mathcal{V}_u$ respectively. We write

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (2.5)$$

where the self-adjoint involution $C$ may be viewed as a metric matrix. In terms of components, the inner product in $\mathcal{V}$ is then given by

$$(\Psi, \Phi) = (\Psi, C \Phi)_C = \psi_1^* \phi_1 + \psi_2^* \phi_3 + \psi_3^* \phi_2, \quad (2.6)$$

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where the subscript $C$ denotes an ordinary inner product. We see that for states $\Psi, \Phi \in \text{ker}Q$, which are representatives of physical states, only the first term on the right-hand side of Eq.\,(2.6) remains. Since $\mathcal{V}_p$ is a non-degenerate subspace, we can define a projection operator $P(\mathcal{V}_p)$ with $P^\dagger = P^2 = P$. For the inner product of two states $\Psi, \Phi \in \text{ker}Q$, and with a complete set of states $\{\Psi_n\}$ in $\mathcal{V}$, we obtain then the decomposition

\[
\langle \Psi, \Phi \rangle = \sum_n \langle \Psi, \Psi_n \rangle \langle \Psi_n, \Phi \rangle = \langle \Psi, P(\mathcal{V}_p) \Phi \rangle = \sum_n \langle \Psi, \Psi_{pn} \rangle \langle \Psi_{pn}, \Phi \rangle .
\]

(2.7)

We see that only a complete set of states $\{\Psi_{pn}\}$ in the Hilbert space $\mathcal{V}_p \approx H$ appears in the sum. It may be replaced by the equivalent set $\{\Psi_{\mathcal{H}n}\}$, where we can write symbolically $\Psi_{\mathcal{H}n} = \Psi_{pn} + imQ$. Although the projection operator $P(\mathcal{V}_p)$ is not Lorentz invariant by itself, the use in Eq.\,(2.7) is invariant.

In our matrix representation, a BRST-invariant operator $A$, which commutes with $Q$, and leaves $\text{ker}Q$ as well as $imQ$ invariant, is of the form

\[
A = \begin{pmatrix}
A_{11} & 0 & A_{13} \\
A_{21} & A_{22} & A_{23} \\
0 & 0 & A_{33}
\end{pmatrix} .
\]

(2.8)

Given a state $\Psi \in \text{ker}Q$, it follows that also $A\Psi \in \text{ker}Q$. With Eq. (2.7), states $\Psi, \Phi \in \text{ker}Q$, and BRST invariant operators $A$ and $B$, we have therefore the decomposition

\[
\langle \Psi, AB\Phi \rangle = \sum_n \langle \Psi, A\Psi_{pn} \rangle \langle \Psi_{pn}, B\Phi \rangle ,
\]

(2.9)

which involves only physical states [14,21,38,39].

Proper unphysical states are characterized by $Q\Psi \neq 0$, or $\psi_3 \neq 0$ in our matrix representation (2.5). They may well have components in $\mathcal{V}_p$, but one can always find an equivalence transformation which removes this component [38,39]. In general, the norm of these states is indefinite. It can be shown,
that some unphysical states with positive norm are needed for the consistency of the theory [21].

In QCD perturbation theory, the space $\mathcal{H}$ consists of states corresponding to quarks and transverse gluons. But in the general theory, if we have confinement, only colorless states like hadrons should be included. We understand here confinement in this algebraic fashion. If the number of flavors $N_F$ in QCD is limited ($N_F \leq 9$), arguments can be given that gluons cannot be in the physical subspace [21,22]. These arguments are based upon the existence of superconvergence relations for the structure function of the gluon propagator, which provide a connection between short and long distance properties. Within the same framework, we can also obtain an approximately linear quark-antiquark potential, because a dipole representation can be written for the propagator, which has a weight function of the appropriate shape [24]. While a linear potential may be a phenomenological reality for heavy quarks, we require an algebraic argument also for quark confinement. A sufficient condition for general color confinement in terms of the BRST-cohomology has been given by Kugo and Ojima [14], and discussed further by Nishijima [40]. So far, only approximate methods have been used in order to argue that this condition is also necessary [23]. But if we accept the necessity, the confinement of transverse gluons also implies the confinement of quarks.

For the purpose of deriving dispersion relations in QCD, we take it for granted that we have confinement in the sense that the physical state-space $\mathcal{H}$ contains only hadronic states. Under these circumstences, quarks and gluons do not appear as BRST singlets, but form quartet representations of the algebra, together with other unphysical states. As we have seen in Eq.(2.9), only hadronic states appear then in intermediate state decompositions involving hadronic operators, and we have the same spectral conditions
as used in the old derivations of dispersion relations within the framework of hadronic field theory. Under these circumstances, in the direct or crossed channels of amplitudes, there appear no thresholds which are associated with the quark-gluon structure. Also anomalous thresholds related to this sub-structure do not exist, neither in the physical nor in the unphysical sheets of the relevant variable, since they are generated by crossed channel thresholds of amplitudes related by unitarity to the one under consideration.\footnote{A detailed discussion of the mathematical and physical aspects of structure singularities may be found in Ref. 31.} We have a situation, which is quite different from the usual bound state system, where the constituents are physical particles which can contribute to intermediate state decompositions as in Eq.(2.7).

Although the quark-gluon structure of hadrons does not give rise to special singularities of hadronic amplitudes, it is expected to be of importance in the determination of the weight functions (discontinuities) in dispersion representations. To the extent that perturbative QCD is related to the weak coupling limit \( g^2 \to +0 \), the absence of quarks and gluons from the physical space \( \mathcal{H} \) of the full theory should be related to the required hadronization. In these regions, perturbation theory may be a reasonable tool for the approximate determination of weight functions. Since there is no confinement in perturbation theory, which is the extreme asymptotic limit \( g^2 \to +0 \), the absence of quarks and gluons from the physical space \( \mathcal{H} \) of the full theory should be related to the required hadronization.

Our algebraic description of confinement should be in accordance with other, perhaps more intuitive approaches to the problem. From the point of view of covariant field theory, the mathematical question is always whether
or not a given excitation is in the physical cohomology space $\mathcal{H}$. Only states in $\mathcal{H}$ are observable and contribute directly to the singularity structure of hadronic amplitudes.
3. Local Hadronic Operators

The problem of dealing with composite particles in quantum field theory was considered in the late Fifties [28,29]. The methods can be generalized to gauge field theories with a state space of indefinite metric. It is possible to define local field operators for stable composite systems, which interpolate between the asymptotic fields generating these bound states as incoming or outgoing particles in a scattering process.

Let $\psi(x)$ describe fundamental fields of the theory. We suppose that there exists a stable, composite system, which has a rest-mass $M$, and quantum numbers in accordance with those of a product like $\psi \psi$. Then we may associate a local operator field $B(x)$ with the composite system. The field $B(x)$ could be defined as the limit

$$B(x) = \lim_{\xi \to 0} \frac{\psi(x + \xi)\overline{\psi}(x - \xi)}{F(\xi)}, \quad -\xi^2 < 0. \quad (3.1)$$

The space-like approach is convenient, but not essential. The invariant function $F(\xi)$ is only of importance as far as its behavior for $\xi \to 0$ is concerned. Generally, the function is singular in this limit, in order to compensate for the expected singularity of the operator product. In fact, it must be as singular as the most singular matrix elements of this product. Here we make the assumption that such maximal matrix elements exist. Otherwise, we would have a situation where, for every matrix element with a given singularity, there exists a more singular one. Given the existence of maximal matrix elements, we generally have an equivalence class $\mathcal{K}_{\text{max}}$ of functions $F$ with the required maximum singularity. Possible oscillations in the limit (3.1) can be handled by an appropriate choice of the sequence of points in the approach to $\xi = 0$.

The limit (3.1) corresponds to the leading term in an operator product ex-
pansion [27]. Such expansions are known to exist in many lower-dimensional field theory models. They are expected to be a general property of local field theories. In four dimensions, the existence of operator product expansions, and of local, composite operators like $B(x)$ in particular, can be proven using perturbation theory methods of renormalizable field theories [41]. But in the corresponding exact theories, we still have to make the technical assumption concerning the maximal singularity mentioned above [36]. As explained in the introduction, we should not rely upon QCD perturbation theory for the purpose of deriving hadronic dispersion relations.

Given the existence of local, BRST-invariant operators in QCD which are associated with hadrons, we can write representations for amplitudes as Fourier transforms of time ordered or retarded products of these operators. The Fourier representations are then the starting points for obtaining analytic properties. In order to give some more details, we consider briefly an amplitude for the elastic scattering of hadrons in QCD. For simplicity, we ignore spin and other quantum numbers, concentrating on the general structure of the $S$-matrix elements. Consequently, the following formulae are rather symbolic. Let us define time-ordered products of basic fields in the form

$$B(x, \xi) = T\psi(x + \xi)\overline{\psi}(x - \xi), \quad (3.2)$$

or

$$B(x; \xi_1, \xi_2, \xi_3) = T\psi(x + \xi_1)\psi(x + \xi_2)\psi(x + \xi_3), \quad (3.3)$$

with $\xi^2 < 0$ and the distances $\xi_i - \xi_j$ kept space-like. We assume that these operators have non-trivial hadronic quantum numbers, so that their vacuum expectation value vanishes.

Considering $B(x, \xi)$, we suppose that there exists a hadron (meson) with
mass $M$ so that $\langle 0| B(x, \xi)|k \rangle \neq 0$ for $-k^2 = M^2$, where $|k\rangle$ is the single hadron state. The free retarded and advanced propagator functions $\Delta_{R,A}(x-x',M)$ can be used to define asymptotic fields $B_{in}(x,\xi)$ and $B_{out}(x,\xi)$ with the help of the Yang-Feldman representation. With

$$\langle 0| B(x, \xi)|k \rangle = \langle 0| B_{in}(x, \xi)|k \rangle = e^{ik \cdot x} F_k(\xi), \quad (3.4)$$

we introduce a function $F_k(\xi) = \langle 0| B(0,\xi)|k \rangle$. Denoting the Fourier transform of $B_{in}(x,\xi)$ by $B_{in}(k,\xi)$, we can show that there are creation and destruction operators like

$$\frac{B_{in}^*(k,\xi)}{F_k(\xi)} = b_{in}^*(k), \quad (3.5)$$

which are independent of $\xi$ and satisfy the usual commutation relations. In this derivation, the completeness of states in $\mathcal{V}$, which are generated by all asymptotic fields, including composite fields, has been assumed [28,29].

In principle, we may consider asymptotic fields for unphysical excitations in the state-space $\mathcal{V}$ of indefinite metric. The states generated by these fields are not elements of the physical space $\mathcal{H}$, the cohomology space of the BRST operator. We associate asymptotic fields with the poles of time ordered Green’s functions corresponding to non-negative eigenvalues of $-P^2$, where $P$ is the energy-momentum tensor [14]. We do not exclude here the possibility of multipole fields.

With the asymptotic fields (3.5), and the weak asymptotic condition

$$\lim_{\xi \rightarrow 0} (\Psi, B^f(x^0,\xi)\Phi) = (\Psi, B^f_{in}(\xi)\Phi) \quad (3.6)$$

for all $\Psi, \Phi \in \mathcal{V}$, where

$$B^f(x^0,\xi) = -i \int d^3x B(x,\xi) \partial^0 f^*(x), \quad (3.7)$$
for any normalizable \( f(x) \) satisfying \( K_x f = (\square - M^2) f(x) = 0 \), we can use the reduction formulae of Lehmann, Symanzik and Zimmermann [26] in order to obtain representations for hadronic amplitudes. For example, let us consider the scattering of mesons with mass \( M \). With the product of basic fields as defined in Eq. (3.2), we obtain a formula like

\[
\langle k', p' | S | k, p \rangle = \frac{1}{F_{k'}(\xi')F_k(\xi)} \frac{1}{(2\pi)^3} \int \int d^4x'd^4x \exp[-ik'x' + ikx]
\]

\[
K_x'K_x \langle p' | TB(x', \xi')B(x, \xi) | p \rangle,
\]

where \(-k^2 = -k'^2 = M^2\), and \(|p\), \(|p'\rangle\) are single hadron \( in\)-states. The right-hand side of Eq. (3.8) is independent of the relative coordinates \( \xi \) and \( \xi' \).

So far, we have not taken the limit \( \xi, \xi' \to 0 \). This limit is necessary in order to have the microscopic causality required for dispersion relations. Furthermore, the operator \( B(x, \xi) \) is not BRST-invariant for \( \xi \neq 0 \). Only a local limit like \( B(x) \) in Eq. (3.1) is invariant. As suggested by representations like Eq. (3.8), and the properties of the operators \( B(x, \xi) \), we restate the assumption made in connection with Eq. (3.1) and suppose that the limit

\[
B(x) = \lim_{\xi \to 0} \frac{B(x, \xi)}{F_k(\xi)}
\]

exists. It then defines a local hadronic Heisenberg operator, and it implies that the functions \( F_k(\xi) \) are elements of the equivalence class \( K_{\text{max}} \) mentioned above. If we now interchange the local limit and the space-time integrations in Eq. (3.8), we obtain a representation of the S-matrix element in terms of local hadron fields:

\[
\langle k', p' | S | k, p \rangle = \frac{1}{(2\pi)^3} \int \int d^4x'd^4x \exp[-ik'x' + ikx]
\]
We can make further reductions in Eq. (3.10) in order to get the formulae needed for the derivation of non-forward dispersion relations, and of forward relations for amplitudes with unphysical continuum contributions.

Instead of taking the limit $\xi \to 0$ in Eq. (3.8), we can use directly the local operator (3.9) and its asymptotic limit in order to construct the scattering amplitude (3.10) with the help of the reduction formula involving the local composite field $B(x)$. Under these circumstances, the reduction method is used only within the physical subspace, where there should be no problems resulting from the infra-red singularities of the theory. But even though the path via Eq. (3.8) appears to involve more assumptions, we think that it may be of interest for the understanding of the hadronic, local limit.

The reduction described above for the product (3.2) can be generalized to operator products like (3.3), as well as to other products of fundamental fields which can form color singlets. In this connection, it is important to note that the Heisenberg fields interpolating between given asymptotic, hadronic fields are not unique. There are equivalence classes of fields giving rise to the same S-matrix.

For field theories with a state space of positive definite metric, it can be shown that locality is a transitive property: two fields, which commute with a given local field, are local themselves and with respect to each other. We have equivalence classes of local fields (Borchers classes) [42]. The proof involves the equivalence of weak local commutativity and CPT-invariance [43], as well as the Edge of the Wedge Theorem [7]. It is then possible to show that different fields in a given class, which have the same asymptotic
fields, define the same S-matrix.

Given special rules for the transformation of ghost fields under CPT, we can define an anti-unitary CPT-operator in the state-space $V$ of QCD [14]. Together with the other postulates of indefinite metric field theory, this then leads to the existence of equivalence classes of local Heisenberg fields in QCD. In particular, the hadron fields $B(x)$, defined by different versions of the local limit, are in the same equivalence class, as are corresponding products involving basic fields as factors. This is a consequence of the locality of the basic fields in QCD. As long as the different composite fields $B(x)$ have the same quantum numbers and the same $in$-fields, they give rise to the same physical S-matrix in the subspace $\mathcal{H}$ of hadron states.
4. Methods of Proof

In previous Sections, we have explained that one can define local Heisenberg Operators for hadrons in QCD. We have seen that BRST methods allow for the definition of a physical subspace \( \mathcal{H} \) of the general state-space \( \mathcal{V} \) of QCD. Given confinement, the Hilbert space \( \mathcal{H} \) contains only hadronic states. With these features of QCD, we can proceed to derive dispersion relations using the methods developed within the general framework of local, hadronic field theory. In the following, we recall briefly some of the essential mathematical steps in the proof of dispersion relations for forward scattering, and for finite values of the momentum transfer.

Dispersion relations for the forward scattering amplitudes of reactions like pion-pion scattering and pion-nucleon scattering can be derived rather simply by using the gap method. As a simple model, let us consider the scattering of massive scalar particles. In terms of local Heisenberg fields, the forward amplitude has the representation

\[
F(\omega) = \int d^4xe^{i\omega x^0 - i\sqrt{\omega^2 - \mu^2} \cdot \vec{x}} \theta(x^0) \chi(x^0, |\vec{x}|) ,
\]

where

\[
\chi(x^0, |\vec{x}|) = \frac{i}{(2\pi)^3} \langle p | \left[ j(\frac{x}{2}), j(-\frac{x}{2}) \right] | p \rangle ,
\]

with \( j \equiv (\Box - \mu^2)\phi \).

If we write, with \( r = |\vec{x}| \),

\[
F(\omega) = \int_0^\infty dr F(\omega, r) ,
\]

we find that, for the relevant values of \( r \), the function \( F(\omega, r) \) is analytic in the upper half of the complex \( \omega \)-plane, because the integrand in Eq. (4.1)

\footnote{The gap method was introduced in Ref. 4 (see the appendix, in particular).}
has support only in the future cone. As a Fourier transform of a tempered distribution, it is bounded by a polynomial. We ignore a possible polynomial, which can be taken care of by subtractions, and write a Hilbert representation for $F(\omega, r)$. This representation involves an integral with the weight function $\text{Im} F(\omega + i0, r)$ along the real $\omega$-axis. As a consequence of the spectral conditions, the weight function vanishes in the gap $-\mu < \omega < +\mu$. But for $|\omega| \geq \mu$, we can perform the $r$-integration on both sides of the Hilbert transform. Using the crossing symmetry for the neutral, scalar model, we obtain the dispersion relation

$$F(\omega) = \frac{2\omega}{\pi} \int_{\mu^2}^{\infty} d\omega' \frac{\text{Im} F(\omega' + i0)}{\omega'^2 - \omega^2}.$$  \hspace{1cm} (4.4)

Although the arguments sketched above ignore many fine-points, they show directly how locality (microscopic causality) and simple spectral conditions translate into the analytic properties required for the validity of dispersion relations. The generalization of the gap method to cases with single particle states, like the nucleon in pion-nucleon amplitudes, is straightforward. We simply remove the one-particle contribution by the appropriate factor, and later regain it as a pole term in the once-subtracted dispersion relation. The real coefficient of the single-nucleon term can be identified with the pion-nucleon vertex function on the mass shell [5]. A proof involves applying the gap method also to this vertex function in a nucleon channel. For reactions involving charged particles, like $\pi^\pm p$-scattering, we have non-trivial crossing relations in the sense that the physical amplitudes for $\pi^+ p$ and $\pi^- p$-scattering are different boundary values of the same analytic function, which is regular in the cut $\omega$-plane, except for the single nucleon pole.

For reactions like $\pi\pi$- and $\pi N$-scattering, we can prove near-forward
dispersion relations with the help of the gap method. These relations involve the derivatives of amplitudes with respect to the momentum transfer $t$, evaluated at $t = 0$ [2]. But for amplitudes with fixed, finite momentum transfer, more sophisticated methods must be used. In these cases, we have continuous unphysical regions. The same is true for forward amplitudes for reactions like nucleon-nucleon scattering, where the crossed channel involves nucleon-antinucleon scattering, and has continuous contributions from states with two or more mesons.

The natural mathematical framework for the derivation of these dispersion relations is the theory of functions of several complex variables. Two aspects of this theory are of fundamental importance for our purpose: 1. The Edge of the Wedge Theorem, and 2. the existence of Envelopes of Holomorphy.

In order to describe the Edge of the Wedge Theorem [7], we use an example involving one complex four-vector. Suppose amplitudes, like those for $\pi^+N$- and $\pi^-N$- scattering, are given as Fourier transforms of tempered distributions with support in the future or the past light-cone respectively:

$$F_{\pm}(K) = \pm \frac{i}{(2\pi)^3} \int d^4x \ e^{-iK \cdot x} \theta(\pm x^0) \langle p' \mid j\left(\frac{x}{2}\right), j\left(-\frac{x}{2}\right) \mid p \rangle . \quad (4.5)$$

Here $2K = k + k'$, and $k + p = k' + p'$. As a consequence of locality, the functions $F_{\pm}(K)$ are analytic in the tubes $-(ImK)^2 > 0$, $ImK^0 > 0$ or $ImK^0 < 0$ respectively. For real values of the four-vector $K$, outside of the physical regions for both reactions, there is a domain $R$ where the two functions coincide. This is a consequence of the spectral conditions, as may be seen by making a decomposition of the absorptive parts with respect to a complete set of hadron states. Given the situation as described, the Edge of the Wedge Theorem implies that there exists a complex neighborhood $N(R)$
of the real domain $R$, where both functions are analytic and coincide. Hence there exists a unique analytic function $F(K)$, which is regular, at least, in the union of $N(R)$ and the region $-(ImK)^2 > 0$. It is important to note that $N(R)$ contains all points with sufficiently small, space-like imaginary part, which are not points of the original tubes. The physical amplitudes $F_\pm(K)$ are boundary values of the general analytic function $F(K)$ in the appropriate real regions. In many cases, the domain of analyticity obtained from the Edge of the Wedge Theorem is not yet large enough for dispersion relations, but it gives an analytic connection between the two physical amplitudes, and hence a meaning to the crossing relations.

The case described above is only a simple example of the Edge of the Wedge theorem. It has been generalized in many ways. For the proof of dispersion relations at fixed momentum transfer $t$, we have used it for functions of two complex four-vectors, together with an original domain of analyticity of the form $W \otimes W$, where $W$ is the tube $-(ImK)^2 > 0$ used above [7].

The other essential tool for the derivation of non-forward dispersion relations is analytic completion. For functions with two or more complex variables, we have the remarkable situation that, for many domains $D$, all functions, which are holomorphic in $D$, can be continued into a larger domain $E(D)$, the envelope of holomorphy of $D$. This envelope is a purely geometrical notion. The basic, generic tool for the construction of envelopes of holomorphy is the Continuity Theorem, which has been used in Ref. 44 to give a complete construction of $E(W \cup N(R))$, where $W \cup N(R)$ is the domain of

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4The Edge of the Wedge Theorem has many applications beyond the problem of dispersion relations. In the literature, one can find elaborate explanations concerning the origin of the name. In fact, while working on the problem in Princeton in 1956-57, we (BOT) called it Keilkanten Theorem, which was simply translated for the publication [7].
analyticity described in the example for the Edge of the Wedge theorem given above. On the other hand, in Ref.7, we have used a subdomain, which is a generalized semitube, and for which the envelope is well known. This gives a region of analyticity which is large enough for most purposes. For example, it touches the full envelope at points of interest for the nucleon-nucleon scattering amplitude. The boundary of an envelope of holomorphy can often be explored with the help of properly constructed examples of analytic functions [7].

For problems involving one complex four-vector, and domains of the form $W \cup R$ considered above, one can obtain the region of analyticity corresponding to the envelope of holomorphy of $W \cup N(R)$ with the help of methods from the theory of distributions and of partial differential equations. The resulting Jost-Lehmann-Dyson representation has been discussed widely in the literature [45,46]. It can be viewed as an elegant method to obtain the envelope of holomorphy for the example we have considered.

As is well known, an elaborate proof of dispersion relations for amplitudes with fixed values of the momentum transfer has been given by Bogoliubov, Medvedev and Polivanov [6]. This proof also makes use of distribution methods and other tools.

The actual proof of non-forward dispersion relations starts with the Fourier representations (4.5). A new variable $\zeta$ is introduced, which corresponds to the squared mass of the projectile for the actual physical amplitude [47]. For real, and sufficiently negative values of $\zeta$, the amplitude has cut-plane analyticity in the energy variable, so that we can write a Hilbert representation. The problem is then to show that both sides are analytic functions of $\zeta$, and that the domain of analyticity includes the physical point $\zeta = \mu^2$. For the left-hand side, we can use the domain $E(W \cup N(R))$ discussed before. For
the right-hand side, the required analytic properties can be obtained by a more extensive use of locality and spectral conditions for the absorptive part occurring as a weight function in the Hilbert representation. The tools are again the Edge of the Wedge Theorem, and the envelope of holomorphy of the domain $D \otimes D$, where $D = W \cup N(R)$ is the domain described above.

The methods of analytic completion make it possible to prove dispersion relations for many binary reactions and vertex functions. For processes like $\pi\pi$- and $\pi N$-scattering, for example, the proofs are valid for restricted values of the momentum transfer: $-t = \Delta^2 < \Delta^2_{\text{max}}$, with

\[
\Delta^2_{\text{max}} = 7\mu^2 \quad \text{and} \quad \Delta^2_{\text{max}} = \frac{8\mu^2}{3} \frac{2m + \mu}{2m - \mu}
\]

respectively. These limitations have no real physical meaning, as can be seen with the help of models which are unphysical, but satify all the assumptions we have made \[7,31,33\]. Nevertheless, it is difficult to incorporate the information contained in the detailed structure of the intermediate state spectrum. Of course, the missing features are naturally contained in a generic, hadronic perturbation theory, but in QCD we may not want to rely on that.

There are similar problems for forward scattering amplitudes with unphysical, continuous contributions. An important example is elastic nucleon-nucleon scattering, where the envelope of holomorphy leads to the limitation $\mu > (\sqrt{2} - 1)m$, which is not satisfied for pion ($\mu$) and nucleon ($m$) masses. The same limitation is obtained for the pion-nucleon vertex function in the pion channel, and for electromagnetic form factors of the nucleon. Using formal perturbation theory simulations, we find that the restriction is due to singularities describing the composite structure of the nucleon with respect

\footnote{Tables describing the limitations of proofs for many amplitudes, which we have prepared for the 1958 Rochester Conference at CERN, are still applicable. See Ref. 48. The limits are also listed in the appendix of Ref. 49.}
to physically non-existent particles with masses such that the simple spectral conditions are satisfied [31,33]. Again, a more exhaustive use of the unitarity condition is required, but difficult to implement, in particular for intermediate states with more than two particles. In contrast, dispersion relations for the pion-nucleon vertex function in a nucleon channel can be proven using the gap method [4,7]. As we have mentioned, they are of importance for a complete derivation of the pion-nucleon relations.

For amplitudes involving strong and electromagnetic interactions, we may consider dispersion relations involving their hadronic structure, treating the electromagnetic interaction in lowest, non-trivial order. Within this framework, we can prove dispersion relations for pion photoproduction and similar reactions [49]. The limitations in momentum transfer may be found in Refs. 49 and 48. There is also no difficulty in deriving a dispersion representation for the electromagnetic form factor of the pion.

The envelope of holomorphy $E(W \cup N)$ for the amplitudes $F(K)$ can be used in order to show that the real and imaginary parts of the corresponding amplitudes are analytic functions in momentum transfer, or in $\cos \theta = 1 - \frac{2t}{K^2}$. They are regular in the small or large Lehmann-ellipses respectively [50]. Consequently, there are convergent partial-wave expansions. For the absorptive parts of reactions like $\pi\pi$- or $\pi N$- scattering, these expansions provide a representation of the weight function in the unphysical region, which is always present in dispersion relations for finite momentum transfer.

Further discussions of pion-nucleon dispersion relations may be found in the papers [51].

An interesting proposal for the analytic structure of binary amplitudes has been made by Mandelstam [52]. The double dispersion relations are essentially based on the assumption that the singularities of the amplitudes
are restricted to those expected on the basis of physical intermediate states in the three channels s, t and u, where $s + t + u = \Sigma m^2$. As is evident from our previous discussion, these representations have not been proven in general hadronic field theory, and hence in QCD. They are known to be compatible with hadronic perturbation theory in lower orders. As mentioned before, hadronic perturbation theory may not be a valid approach as far as QCD is concerned. However, it could provide a hint for the analytic structure of hadronic amplitudes.

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