Knotted strings and leptonic flavor structure

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Tight knots and links arising in the infrared limit of string theories may provide an interesting alternative to flavor symmetries for explaining the observed flavor patterns in the leptonic sector. As an example we consider a type I seesaw model where the Majorana mass structure is based on the discrete length spectrum of tight knots and links. It is shown that such a model is able to provide an excellent fit to current neutrino data and that it predicts a normal neutrino mass hierarchy as well as a small mixing angle $\theta_{13}$.

PACS numbers: 14.60.Pq, 14.60.St, 02.10.Kn

I. INTRODUCTION

One of the most profound problems in contemporary particle physics is the flavor puzzle or the question whether the observed flavor structure is governed by anarchy (essentially random numbers) [1–4] or a (typically discrete) flavor symmetry (see e.g. [5–8]). In this paper we will propose a third idea, namely that the leptonic flavor structure arises from the topological configurations of closed strings.

Closed strings are a fundamental ingredient of string theory, including in particular the graviton and its AdS/QCD dual, the glueball, as well as dilaton superfields with fermionic degrees of freedom having the correct quantum numbers of a right-handed neutrino (a fact used extensively e.g. in neutrino mass models with large extra dimensions, see e.g. Ref. [9]).

It thus follows naturally that topologically nontrivial string configurations such as knots and links can contribute to the mass of closed string states, and may even dominate it. As string tension tends to minimize the string length, the knot or link length can be assumed to be directly proportional to the string spectrum. For example, it has been shown in Ref. [10–12] that the experimental spectrum of glueball candidates can be fitted very nicely by knot and link energies. An application of a knot model to flavor physics has recently been discussed in Ref. [13].

Here we exploit another interesting feature of the knot and link spectrum. Typically there exist different close-to-degenerate states with very small energy gaps. If right-handed neutrino masses are dominated by the knots and links of closed strings in a seesaw framework, large and maximal leptonic mixing may result naturally from the knot and link spectrum without the need for any flavor symmetry.

II. DESCRIPTION OF THE MODEL AND PREDICTIONS

We assume a generic type I seesaw model, in which the flavor structure originates from the mass matrix of the right-handed sector, which is generated by a GUT scale mass spectrum of knots and links. Consequently the Dirac masses are assumed to be diagonal:

$$m_D = \begin{pmatrix} m_1^D & 0 & 0 \\ 0 & m_2^D & 0 \\ 0 & 0 & m_3^D \end{pmatrix},$$

while the symmetric Majorana mass matrix has the following structure:

$$M = \begin{pmatrix} m_1^K & m_1^L & m_1^R \\ m_1^L & m_2^K & m_2^L \\ m_1^R & m_2^L & m_3^K \end{pmatrix}. $$

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This way the choice of the Dirac masses is mainly responsible for the absolute mass scale of neutrinos, while the choice of the structure in the right-handed sector implies the leptonic mixing matrix. This is the general structure of mass matrices analyzed in Ref. [14].

We assume that the heavy masses \( m_i^{K/L} \) assume values according to the spectrum of characteristic lengths of knots and links \([15]\) multiplied by a common GUT scale factor \( m_{\text{GUT}} \):

\[
 m_i^{K/L} = \ell_i^{K/L} \cdot m_{\text{GUT}},
\]

where the \( \ell_i^{K/L} \) refer to the characteristic lengths of Table [11]. The diagonal entries of the mass matrix are generated by the knots’ lengths, while the off-diagonal entries are related to the characteristic lengths of the links.

This spectrum has the particular property that it provides many entries lying close to each other thus making predictions similar to those obtained by a discrete flavor symmetry.

The mass matrix of the three light left-handed neutrinos in the flavor basis is then obtained using the usual formula \( M\nu^L = m_D^L M^{-1} m_D^R \).

\[
 M\nu^L = \frac{1}{\Delta^3} \begin{pmatrix}
 (m_2^K m_3^K - (m_3^L)^2) (m_3^D)^2 & (m_2^K m_3^K - m_3^K m_1^K) m_1^D m_2^D & (m_1^K m_2^K - m_2^K m_1^K) m_1^D m_2^D \\
 (m_2^K m_3^K - m_3^K m_1^K) m_1^D m_2^D & (m_2^K m_3^K - (m_3^L)^2) (m_2^D)^2 & (m_1^K m_2^K - m_1^K m_3^K) m_2^D m_3^D \\
 (m_1^K m_2^K - m_2^K m_1^K) m_2^D m_3^D & (m_1^K m_2^K - m_1^K m_3^K) m_2^D m_3^D & (m_1^K m_3^K - (m_3^L)^2) (m_3^D)^2
\end{pmatrix},
\]

where the common factor of mass dimension three is given by \( \Delta^3 = -m_2^K (m_3^L)^2 + 2m_1^K m_2^K m_3^L - m_2^K (m_3^L)^2 - m_1^K (m_3^L)^2 + m_1^K m_2^K m_3^K \). Note that the structure of this matrix is sensitive to the relative differences between the lengths of the knots and links.

The (1,1) element of Eq. [4] is the effective mass observed in double-beta decays \( m_{\beta\beta} \). In order to make statements about the neutrino mixing angles and squared mass differences, the mass matrix has to be diagonalized using an eigenvalue decomposition, yielding the mixing matrix \( U \). As in the previous work Ref. [14], we use the ordering scheme of Ref. [10] in which the the labels \( m_1 \) and \( m_2 \) are assigned to the pair of eigenvalues whose absolute squared mass difference is minimal. Out of these two the eigenvalue whose corresponding eigenvector has the smaller modulus in the first component is labeled \( m_2 \). The hierarchy of the neutrino masses is then given by the sign of the squared mass differences \( \Delta m_{31}^2 \) or \( \Delta m_{22}^2 \). We call a mass hierarchy normal, if it features two small masses and one large mass. An inverted hierarchy thus has one small mass eigenvalue and two large ones.

It is instructive to examine the structure of the effective mass matrix analytically to determine some general features of the model. As the tri-bimaximal mixing pattern [17] is a good fit for the experimental data, we look at the general structure needed to generate such mixing angles. In general, a mass matrix that leads to tri-bimaximal mixing can be parametrized as [18]

\[
 M\nu^{\text{TBM}} = \begin{pmatrix}
 x & y & y \\
 y & x + v & y - v \\
 y & y - v & x + v
\end{pmatrix},
\]

where \( x, y \) and \( z \) are real numbers.

Assuming that a normal mass hierarchy can be approximated as two vanishing neutrino masses and one neutrino mass at a higher scale \( \tilde{m} \)—i.e. a diagonal mass matrix of \( \text{diag}(0, 0, \tilde{m}) \), this leads to a mass matrix of the form

\[
 \tilde{m} \cdot \begin{pmatrix}
 0 & 0 & 0 \\
 0 & \frac{1}{2} & -\frac{1}{2} \\
 0 & -\frac{1}{2} & \frac{1}{2}
\end{pmatrix},
\]

which can be thought of as setting \( x = 0, y = 0, v = 1/2 \) in Eq. [4]. This matrix is then compared to the mass matrix of Eq. [3]. The comparison yields a set of relations between the characteristic lengths and the other parameters:

\[
 \frac{K_3}{K_2} = \frac{(m_2^D)^2}{(m_3^D)^2},
\]

\[
 \frac{L_2}{L_1} = \frac{m_3^D}{m_2^D},
\]

\[
 K_2 L_2 = L_1 L_3,
\]

\[
 K_1 K_2 \neq L_2^2,
\]

\[
 \tilde{m} = 2 (m_3^D)^2 \left( L_1^2 - K_1 K_2 \right).
\]
If the Dirac masses $m_D^i$ are assumed to be roughly equal, the conditions of Eqs. 7–9 can be fulfilled if the selected lengths $K_i$ and $L_i$ are close to each other. In general, as the order (crossing number) of the knots increase, the spacing decreases since the length grows roughly linearly with crossing number, but the number of knots grows faster than exponentially with crossing number.

In order to achieve a low neutrino mass scale in Eq. 11, the electro-weak scale $m_D^3$ factor needs to be compensated by making the expression in the parenthesis small; this can again be achieved by having an almost degenerate spectrum for $K_i$ and $L_i$. Since the spectrum of knots and links features almost degenerate lengths, it is thus expected that it will provide a much better fit to the leptonic flavor structure than random numbers.

The corresponding condition for an inverted hierarchy, which is approximated as two neutrino masses at a higher scale $\tilde{m}$ and one neutrino mass set to zero—i.e. the diagonal mass matrix $\text{diag}(\tilde{m}, \tilde{m}, 0)$, gives a mass matrix

$$
\tilde{m} \cdot \begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{1}{2} & \frac{1}{2} \\
0 & \frac{1}{2} & \frac{1}{2}
\end{pmatrix}.
$$

(12)

Comparing this to the mass matrix of Eq. 4 gives a system of equations that can only be solved when $\tilde{m} = 0$. This means that in this approximation it is not possible to generate an inverted neutrino mass hierarchy. In a realistic scenario where the facts that tribimaximal mixing is only an approximation and that the smaller mass difference is not zero are taken into account, one would expect that in this model the inverted mass hierarchy should be strongly suppressed.

Finally, we analyze the compatibility of the model with a degenerate neutrino mass spectrum. Assuming a diagonal mass matrix $\text{diag}(\tilde{m}, \tilde{m}, \tilde{m})$ the conditions that follow from Eq. 4 read:

$$
m_1^D m_3^D \neq 0,
$$

$$
K_2 = K_1 \frac{(m_2^D)^2}{(m_1^D)^2},
$$

$$
K_3 = K_1 \frac{(m_3^D)^2}{(m_1^D)^2},
$$

$$
K_3 \neq 0,
$$

$$
K_1 K_2 \neq 0,
$$

$$
K_1 \neq 0,
$$

$$
L_1 = L_2 = L_3 = 0,
$$

$$
K_1 K_2 \left(\frac{m_3^D}{m_1^D}\right)^2 + \tilde{m} = 0.
$$

(13)

Out of these conditions, the last two are in contradiction with the framework of the model: The $L_i$ and $K_i$ parameters cannot be zero or close to zero. The model investigated in this paper thus cannot be used to explain a degenerate neutrino mass hierarchy.

### III. NUMERICAL ANALYSIS

In order to investigate the viability of the models, every possible combination of characteristic lengths up to a given knot order is sampled using a computer code. No duplicate lengths of knots or links are allowed.

The parameters $m_i^D$ for $i = 1 \ldots 3$ as well as the overall scale of the Majorana masses are not fixed by the model. As the scope of this analysis is the viability of the choice of knots and links as a source of Majorana masses, the Dirac masses are chosen in a way as to minimize the $\chi^2$ value of the squared mass differences of the neutrinos compared to experimental data[19]. This way, no potentially viable combinations of knots and links are discarded due to a wrong choice for the Dirac masses.

For the subset of models that have acceptable squared mass differences, the mixing angles are calculated and also compared to the experimental values. All models with a $\chi^2 < 16.8$ for the mixing angles are considered viable. This corresponds to a $P$ value of 0.01 and six degrees of freedom.

The scan covering 10692864 possible combinations of knots and links’ lengths results in 321952 models with normal neutrino mass hierarchy and 400 models with an inverted neutrino mass hierarchy that fall below the $\chi^2$ limit of 16.8. This means that about 3% of all possible combinations yield phenomenologically acceptable results. The best fit lies in the regime of normal hierarchy with a $\chi^2_{\text{best}} = 0.002$. The best fit model is described by parameters in Table 1. This has to be compared with models based on random numbers.
In order to assess the viability of these models we compare them with an approach based on random numbers: The same model structure is assumed, but the list of characteristic lengths is replaced by a list of random numbers between 0 and the largest knot length of the actual list of knot lengths. This is repeated for several sets of random numbers. The results of this comparison can be found in Tab. III In all cases, the best fit model is in the normal hierarchy regime and the total number of viable models with a normal neutrino mass hierarchy is much larger than the number of models with an inverted mass hierarchy. There are two effects that explain this discrepancy: In our models with an inverted mass hierarchy, θ₁₃ is usually predicted to be larger than the current bound, while the models with normal mass hierarchy predict a naturally small θ₁₃. This explains the discrepancy observed within each set of random numbers, as can be seen in Tab. III

In addition to this, the relative number of models with a normal neutrino mass hierarchy is even larger in the case of knots and links. This can be explained by the conditions of Eqs. (7)–(11) which lead to the spectrum of knots and links being able to fit the requirements for a normal mass hierarchy easier than random numbers.

For all random cases considered the total number of acceptable models is lower than the total number of acceptable models in the case of knots and links. This means that the models using the characteristic lengths of knots and links are more suitable to fit the neutrino data than a fit using random numbers.

| hierarchy | χ²  | K₁  | K₂  | K₃  | L₁  | L₂  | L₃  | m₀[GeV] | m₁[GeV] | m₂[GeV] | m₃[GeV] | Scale factor [GeV] |
|-----------|-----|-----|-----|-----|-----|-----|-----|---------|---------|---------|---------|-------------------|
| normal    | 0.002 | 01  | 05  | 11  | 11  | 12  | 06  | 100.391 | 104.155 | 101.971 | 5.9·10¹³ |
| inverted  | 7.262 | 01  | 11  | 06  | 12  | 00  | 13  | 202.484 | 126.717 | 80.652  | 1.6·10¹³ |

TABLE I. The model parameters giving the best fit for normal and inverted hierarchies. The knots and links indices refer to table III

To determine the phenomenological consequences of the allowed models, the following observables are calculated:

- the double beta decay parameter m_ββ, given by the 11-entry of Eq. [1]
- the lightest neutrino mass m₀,
- the neutrino mixing angle θ₁₃.

In the normal hierarchy case, m_ββ tends to be small, i.e. between 0.001 eV and 0.01 eV. Note that the best fits yield values for m_ββ between 0.001 eV and 0.007 eV. In the case of an inverted neutrino mass hierarchy, m_ββ takes on values between 0.01 eV and 0.02 eV. As is to be expected for a model with inverted hierarchy, the double beta decay parameter m_ββ is larger than in the normal case.

As the angle θ₁₃ is small and the contribution from m₀ is negligible, this is in line with the results from Ref. [14], where the double-beta parameter m_ββ is given as:

\[
m_{\beta\beta} \approx \begin{cases} \sqrt{\Delta m_{12}^2} \sin^2 (\theta_{12}) & \text{for normal hierarchy} \\ \sqrt{\Delta m_{23}^2} \cos (2\theta_{12}) & \text{for inverted hierarchy} \end{cases}
\]

The lightest neutrino mass in the normal hierarchy case is below 0.003 eV. This is illustrated in Fig. 1. The corresponding plot for an inverted neutrino mass hierarchy shows that a slightly larger lightest mass of up to 0.007 eV is possible in that case. In both cases, the neutrino masses are well below the current bound on the sum of the lightest neutrino masses \( \sum m_i \lesssim 0.5 \text{ eV} \).\(^{[20,29]} \)

The neutrino mixing angle θ₁₃ has not been measured directly yet and only reliable upper bounds are available, although a recent measurement from the T2K experiment\(^{[30,31]} \) suggests a non-zero lower bound for θ₁₃ at 2.5σ. Within the framework of the global fit of experimental data used here\(^{[19]} \), a best fit value of the angle can be calculated. That best fit value has been used to fit the model parameters of this work. If one takes only the angles and squared mass differences into account that have actually been observed and fits the model parameters to those values, it is possible to predict the mixing angle θ₁₃. This has been done for normal and inverted neutrino mass hierarchies and the results are shown in Figs. 3 and 4. It can be seen that in the case of a normal neutrino mass hierarchy, a small angle θ₁₃ close to zero is strongly preferred. In the case of inverted mass hierarchy, a large angle θ₁₃ is preferred, although some results are still below the applicable bound.

Taking the results from the recent T2K publication\(^{[30]} \) into account a range for \( \sin^2 2\theta_{13} \) is given as 0.03(0.04) < \( \sin^2 2\theta_{13} < 0.28(0.34) \) for a normal (inverted) neutrino mass hierarchy at 90% CL. Within the allowed \( \chi^2 \) range assumed here, the results of this paper are compatible with these bounds.
IV. POSSIBLE UV COMPLETIONS

The effective model we are pursuing here may result from various ultraviolet completions. First, fundamental closed strings could be considered, but it is not clear here if these can have tight knots because of their vanishingly small cross section. Another option to generate massive knots near the GUT scale are cosmic strings. If a collapsing loop of nontrivial topology \( K \) tightens before it decays, then the tight knot configuration will have mass

\[
M_K \sim L_K \langle \phi \rangle
\]

near the symmetry breaking scale \( \langle \phi \rangle \), where the \( U(1) \) is broken that gives rise to the cosmic string. Here \( L_K \) is the dimensionless length of the knot \( K \), i.e., the length of the knot divided by the radius of the cosmic string.

If a knot is bosonic above the SUSY breaking scale, then it will also have a fermionic partner of the same mass. Furthermore, If the fermionic knots are gauge singlets, then they can serve as the heavy right-handed neutrinos needed for the seesaw mechanism to generate the very light observed neutrino states.

The stability of various knot types will be model dependent, hence the lightest knots may not be stable and so may not be the ones that mix with the light neutrinos. To arrive at yet a different possibility to generate the spectrum of knots and links we use in this work, consider a ten dimensional \( E_8 \otimes E_8' \) heterotic superstring theory and compactify it on a Calabi–Yau manifold \( K \) with \( SU(3) \) holonomy. With a proper choice of \( K \) and identification of its holonomy group with a subgroup of \( E_8 \) we can arrive at a four dimensional \( E_6 \otimes E_8' \) theory with three chiral \( E_6 \) families \([32]\), i.e., three \( 27s \) of \( E_6 \). The three family \( SU(3) \times SU(2) \times U(1) \) standard model is embedded in the \( E_8 \) sector and has been studied. But we will focus our attention on the \( E_8' \) hidden sector, which has only gravitational interactions with standard model particles.

Let the \( E_6 \) and \( E_8' \) gauge couplings be unified a string scale \( M_{\text{string}} \sim 5 \times 10^{17} \) GeV. Then we expect \( \alpha_{\text{string}}(M_{\text{string}}) \sim 1/50 \) in order that the SM couplings agree with experiment. Since the hidden sector has no chiral fermions, \( \alpha_{E_8} \) runs quickly to \( O(1) \) at the mass scale approaches \( M_{E_8} \sim 10^{13} \) GeV.

The hidden sector has been assumed to generate supersymmetry breaking via a gluino condensation when the theory becomes nonperturbative \([33, 34]\) at \( \alpha_{E_8}(M_{E_8}) \). In addition, the \( E_8' \) theory becomes confining near this energy scale, but since there are no fundamental chiral fermions in the hidden sector, there are no “light” \( \sim 10^{13} \) GeV mesonic or baryonic \( E_8' \) states. The lightest particles in the hidden sector will be closed \( E_8 \) flux loops. These solitons will be glueball like and we expect their spectrum to scale like the tight knot/link spectrum \([10, 11]\). Furthermore, supersymmetry will either be unbroken, or an approximate symmetry at the \( M_{E_8} \) scale, so these solitons will have equal mass fermionic partners. We expect bosonic glueballs to have \( J^{PC} = 0^{++} \) quantum numbers, and their fermionic partners to be \( 1^{++} \) states with no standard model quantum numbers. The lightest \( E_8' \) glueball is stable because there are no lighter hidden sector states into which it can decay. The entire \( E_8' \) knot spectrum is metastable since each knot-soliton carries a different topological charge derivable from its individual unique knot invariants.

This analysis leads us to suggest that the knotted fermionic hidden sector solitons provide a natural source of neutral heavy singlet fermions for the seesaw mechanism in order to give mass to light neutrinos, and that our observation that the low energy neutrino data is better fit by a knot spectrum than by a random mass spectrum may indicate that we have found evidence for hidden sector dynamics, i.e., hidden sector confinement and tightly knotted flux tube formation.

V. CONCLUSIONS

In this paper we have proposed an alternative to both flavor anarchy and flavor symmetry: a seesaw type I model whose Majorana mass structure is governed by the discrete spectrum of tight knots and links. A possible UV completion of this model may result from supersymmetric cosmic strings arising from the GUT scale breaking of an additional \( U(1) \) gauge symmetry, or as fermionic partners of glueball-like states originating as hidden sector flux tubes in ten dimensional \( E_8 \otimes E_8' \) heterotic superstring theory. Based on the general structure of the mass matrices of Ref. \([14]\), we have shown that the model fits the current experimental neutrino data on squared mass differences and mixing angles. It has also been shown that the spectrum of knots and links produces a larger number of viable models than a spectrum of random numbers. The model favors a normal neutrino mass hierarchy and predicts a small mixing angle \( \theta_{13} \).

[1] L. J. Hall, H. Murayama, and N. Weiner, Phys. Rev. Lett. 84, 2572 (2000) [arXiv:hep-ph/9911341 [hep-ph]]
TABLE II. Comparison of the number of accepted mode in the case of knots and links vs. ten cases where the list of knots and links is replaced by sets of random numbers.

| set               | number of accepted samples |
|-------------------|-----------------------------|
|                   | normal hierarchy | inverted hierarchy |
| knots and links   | 321952       | 400 |
| random 0          | 120619       | 3685 |
| random 1          | 231748       | 2957 |
| random 2          | 266868       | 1941 |
| random 3          | 142333       | 3598 |
| random 4          | 145388       | 3427 |
| random 5          | 167163       | 3376 |
| random 6          | 244459       | 1283 |
| random 7          | 189467       | 5200 |
| random 8          | 256586       | 4078 |
| random 9          | 229931       | 1649 |

TABLE III. Characteristic lengths of knots and links up to knot order 7, taken from Ref. [15]
FIG. 1. The lightest neutrino mass $m_0$ (models with $\chi^2 < 16.8$) for the subset of models that generate a normal neutrino mass hierarchy. The plot has been divided into 117 bins along the $x$-axis and 35 bins along the $y$-axis. The shade of the rectangles represents the number of models found in that particular area. Outside of the boundary line, less than ten hits per rectangle were recorded.

FIG. 2. The lightest neutrino mass $m_0$ (models with $\chi^2 < 16.8$) for the subset of models that generate an inverted neutrino mass hierarchy. The plot has been divided into 256 bins along the $x$-axis and 35 bins along the $y$-axis. The shade of the rectangles represents the number of models found in that particular area.
FIG. 3. The quantity $\sin^2 (2\theta_{13})$ (models with $\chi^2 < 16.8$) for the subset of models that generate a normal mass hierarchy. For this plot, a fit of the model to the squared mass differences and all angles except for $\theta_{13}$ was performed. The plot has been divided into 334 bins along the $x$-axis and 35 bins along the $y$-axis. The shade of the rectangles represents the number of models found in that particular area. The 90% confidence level bound on $\theta_{13}$ given by the Particle Data Group is given as a dashed line.

FIG. 4. The quantity $\sin^2 (2\theta_{13})$ (models with $\chi^2 < 16.8$) for the subset of models that generate an inverted mass hierarchy. For this plot, a fit of the model to the squared mass differences and all angles except for $\theta_{13}$ was performed. The plot has been divided into 334 bins along the $x$-axis and 35 bins along the $y$-axis. The shade of the rectangles represents the number of models found in that particular area. The 90% confidence level bound on $\theta_{13}$ given by the Particle Data Group is given as a dashed line.