Maxwell meets Korn: A new coercive inequality for tensor fields in $\mathbb{R}^{N\times N}$ with square-integrable exterior derivative

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For a bounded domain $\Omega \subset \mathbb{R}^N$ with connected Lipschitz boundary, we prove the existence of some $c > 0$, such that

$$c \| T \|_{L^2(\Omega; \mathbb{R}^{N\times N})} \leq \| \text{sym} \ T \|_{L^2(\Omega; \mathbb{R}^{N\times N})} + \| \text{Curl} \ T \|_{L^2(\Omega; \mathbb{R}^{N\times(N-1)N/2})}$$

holds for all square-integrable tensor fields $T : \Omega \mapsto \mathbb{R}^{N\times N}$, having square-integrable generalized “rotation” tensor fields Curl $T : \Omega \mapsto \mathbb{R}^{N\times(N-1)N/2}$ and vanishing tangential trace on $\partial \Omega$, where both operations are to be understood row-wise. Here, in each row, the operator curl is the vector analytical reincarnation of the exterior derivative $d$ in $\mathbb{R}^N$. For compatible tensor fields $T$, that is, $T = \nabla v$, the latter estimate reduces to a non-standard variant of Korn’s first inequality in $\mathbb{R}^N$, namely

$$c \| \nabla v \|_{L^2(\Omega; \mathbb{R}^{N\times N})} \leq \| \text{sym} \ \nabla v \|_{L^2(\Omega; \mathbb{R}^{N\times N})}$$

for all vector fields $v \in H^1(\Omega; \mathbb{R}^N)$, for which $\nabla v_n, n = 1, \ldots, N$, are normal at $\partial \Omega$. Copyright © 2012 John Wiley & Sons, Ltd.

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1. Introduction and preliminaries

We extend the results from [1, 2], which have been announced in [3], to the $N$-dimensional case following in close lines, the arguments presented there. Let $N \in \mathbb{N}$ and $\Omega$ be a bounded domain in $\mathbb{R}^N$ with connected Lipschitz boundary $\Gamma^0 : = \partial \Omega$. We prove a Korn-type inequality in $H^1(\text{Curl}; \Omega)$ for eventually non-symmetric tensor fields $T$ mapping $\Omega$ to $\mathbb{R}^{N\times N}$. More precisely, there exists a positive constant $c$, such that

$$c \| T \|_{L^2(\Omega)} \leq \| \text{sym} \ T \|_{L^2(\Omega)} + \| \text{Curl} \ T \|_{L^2(\Omega)}$$

holds for all tensor fields $T \in H^1(\text{Curl}; \Omega)$, where $T$ belongs to $H^1(\text{Curl}; \Omega)$, if $T \in H(\text{Curl}; \Omega)$ has vanishing tangential trace on $\Gamma$. Thereby, the generalized Curl and tangential trace are defined as row-wise operations. For compatible tensor fields $T = \nabla v$ with vector fields $v \in H^1(\Omega)$, for which $\nabla v_n, n = 1, \ldots, N$, are normal at $\partial \Omega$, the latter estimate reduces to a non-standard variant of the well known Korn’s first inequality in $\mathbb{R}^N$

$$c \| \nabla v \|_{L^2(\Omega)} \leq \| \text{sym} \ \nabla v \|_{L^2(\Omega)}.$$

Our proof relies on three essential tools, namely

1. Maxwell estimate (Poincaré-type estimate),
2. Helmholtz’ decomposition,
3. Korn’s first inequality.
In [1], we already pointed out the importance of the Maxwell estimate and the related question of the Maxwell compactness property. Here, we mention the papers [4–10]. Results for the Helmholtz decomposition can be found in [6, 8, 10–16]. Nowadays, differential forms find prominent applications in numerical methods like Finite Element Exterior Calculus [17, 18] or Discrete Exterior Calculus [19].

1.1. Differential forms

We may look at

\[ \forall E, H \in L^2(q)(\Omega) \quad (E, H)_{L^2(q)(\Omega)} := \int_{\Omega} E \wedge * H. \]

Here, \( C^{\infty,q}(\Omega) \) denotes the space of compactly supported and smooth \( q \)-forms on \( \Omega \). Using this duality, we can define weak versions of \( \delta \) and \( \delta \). The corresponding standard Sobolev spaces are denoted by

\[ D^q(\Omega) := \left\{ E \in L^2(q)(\Omega) : dE \in L^2(q+1)(\Omega) \right\}, \]
\[ \Delta^q(\Omega) := \left\{ H \in L^2(q)(\Omega) : \delta H \in L^2(q-1)(\Omega) \right\}. \]

The homogeneous tangential boundary condition \( \tau_{\Gamma} E = 0 \), where \( \tau_{\Gamma} \) denotes the tangential trace, is generalized in the space

\[ D^q_{\|}(\Omega) := \overline{C^{\infty,q}(\Omega)}, \]

where the closure is taken in \( D^q(\Omega) \). In classical terms, we have for smooth \( q \)-forms \( \tau_{\Gamma} = \iota^* \) with the canonical embedding \( \iota : \Gamma \hookrightarrow \overline{\Omega} \).

An index 0 at the lower right position indicates vanishing derivatives, that is,

\[ \Delta^q_{\|}(\Omega) = \left\{ H \in \Delta^q(\Omega) : \delta H = 0 \right\}. \]

By definition and density, we have

\[ \Delta^q_{\|}(\Omega) = (dD^q_{\|}(\Omega))^\perp, \quad \Delta^q_{\|}(\Omega)^\perp = d\Delta^q_{\|}(\Omega), \]

where \( ^\perp \) denotes the orthogonal complement with respect to the \( L^2(q)(\Omega) \)-scalar product and the closure is taken in \( L^2(q)(\Omega) \). Hence, we obtain the \( L^2(q)(\Omega) \)-orthogonal decomposition, usually called Hodge–Helmholtz decomposition,

\[ L^2(q)(\Omega) = \overline{D^q_{\|}(\Omega) \oplus \Delta^q_{\|}(\Omega)}, \quad \text{(1.1)} \]

where \( \oplus \) denotes the orthogonal sum with respect to the \( L^2(q)(\Omega) \)-scalar product. In [7, 10], the following crucial tool has been proved:

**Lemma 1** (Maxwell compactness property)

For all \( q \), the embeddings

\[ D^q(\Omega) \cap \Delta^q(\Omega) \hookrightarrow L^2(q)(\Omega) \]

are compact.

As the first immediate consequence, the spaces of so called "harmonic Dirichlet forms"

\[ \mathcal{H}^q(\Omega) := D^q_{\|}(\Omega) \cap \Delta^q_{\|}(\Omega) \]

are finite dimensional. In classical terms, a \( q \)-form \( E \) belongs to \( \mathcal{H}^q(\Omega) \), if

\[ dE = 0, \quad \delta E = 0, \quad \iota^* E = 0. \]
The dimension of $\mathcal{H}^q(\Omega)$ equals the $(N - q)$th Betti number of $\Omega$. Because we assume the boundary $\Gamma$ to be connected, the $(N - 1)$th Betti number of $\Omega$ vanishes and therefore there are no Dirichlet forms of rank 1 besides zero, for example,

$$\mathcal{H}^1(\Omega) = \{0\}. \tag{1.2}$$

This condition on the domain $\Omega$ respectively its boundary $\Gamma$ is satisfied, for example, for a ball or a torus.

By a usual indirect argument, we achieve another immediate consequence:

**Lemma 2** (Poincaré estimate for differential forms)

For all $q$ there exist positive constants $c_{p,q}$ such that for all $E \in \mathcal{D}^q(\Omega) \cap \Delta^q(\Omega) \cap \mathcal{H}^q(\Omega)$

$$\|E\|_{L^2(\Omega)} \leq c_{p,q} \left( \|dE\|^2_{L^{2q+1}(\Omega)} + \|\delta E\|^2_{L^{2q-1}(\Omega)} \right)^{1/2}. \tag{1.1}$$

Because

$$d\mathcal{D}^q-1(\Omega) \subset \mathcal{D}^q(\Omega)$$

(note that $dd = 0$ and $\delta\delta = 0$ hold even in the weak sense) we get by (1.1)

$$d\mathcal{D}^q-1(\Omega) = d\left(\mathcal{D}^q-1(\Omega) \cap \Delta^q-1(\Omega)\right) = d\left(\mathcal{D}^q-1(\Omega) \cap \Delta^q-1(\Omega) \cap \mathcal{H}^q-1(\Omega)\right). \tag{1.1}$$

Now, Lemma 2 shows that $d\mathcal{D}^q-1(\Omega)$ is already closed. Hence, we obtain a refinement of (1.1)

**Lemma 3** (Hodge–Helmholtz decomposition for differential forms)

The decomposition

$$L^2(\Omega) = d\mathcal{D}^q-1(\Omega) \oplus \mathcal{D}^q(\Omega)$$

holds.

### 1.2. Functions and vector fields

Let us turn to the special case $q = 1$. In this case, we choose, for example, the identity as single global chart for $\Omega$ and use the canonical identification isomorphism for 1-forms (i.e., Riesz’ representation theorem) with vector fields $dx_n \equiv e^n$, namely,

$$\sum_{n=1}^N \nu_n(x)dx_n \equiv \nu(x) = \begin{bmatrix} \nu_1(x) \\ \vdots \\ \nu_N(x) \end{bmatrix}, \quad x \in \Omega.$$

0-forms will be isomorphically identified with functions on $\Omega$. Then, $d \equiv \text{grad} = \nabla$ for 0-forms (functions) and $\delta \equiv \text{div} = \nabla \cdot$ for 1-forms (vector fields). Hence, the well known first order differential operators from vector analysis occur. Moreover, on 1-forms, we define a new operator $\text{curl} : \equiv d$, which turns into the usual curl if $N = 3$ or $N = 2$. $L^2(\Omega)$ equals the usual Lebesgue spaces of square integrable functions or vector fields on $\Omega$ with values in $\mathbb{R}^n$, $n := n_{\nu,q} := \binom{n}{q}$, which will be denoted by $L^2(\nu, \mathbb{R}^n)$. $\mathcal{D}^q(\Omega)$ and $\Delta^1(\Omega)$ are identified with the standard Sobolev spaces

$$\text{H}(<\text{grad}; \Omega>) := \left\{ u \in L^2(\nu, \mathbb{R}) : \text{grad} u \in L^2(\nu, \mathbb{R}^N) \right\} = \text{H}^1(\nu)$$

$$\text{H}(<\text{div}; \Omega>) := \left\{ v \in L^2(\nu, \mathbb{R}^N) : \text{div} v \in L^2(\nu, \mathbb{R}) \right\},$$

respectively. Moreover, we may now identify $\mathcal{D}^1(\Omega)$ with

$$\text{H}(<\text{curl}; \Omega>) := \left\{ v \in L^2(\nu, \mathbb{R}^N) : \text{curl} v \in L^2(\nu, \mathbb{R}^{(N-1)W/2}) \right\},$$

which is the well-known $\text{H}(<\text{curl}; \Omega>)$ for $N = 2, 3$. For example, for $N = 4$ we have

$$\text{curl} v = \begin{bmatrix} \partial_1 v_2 - \partial_2 v_1 \\ \partial_1 v_3 - \partial_3 v_1 \\ \partial_1 v_4 - \partial_4 v_1 \\ \partial_2 v_3 - \partial_3 v_2 \\ \partial_2 v_4 - \partial_4 v_2 \\ \partial_3 v_4 - \partial_4 v_3 \end{bmatrix} \in \mathbb{R}^6$$
and for $N = 5$, we get $\nabla v \in \mathbb{R}^{10}$. In general, the entries of the $(N-1)N/2$-vector $\nabla v$ consist of all possible combinations of

$$\partial_{n} v_{m} - \partial_{m} v_{n}, \quad 1 \leq n < m \leq N.$$ 

Similarly, we obtain the closed subspaces

$$\overset{\circ}{\mathcal{H}}(\text{grad}; \Omega) = \overset{\circ}{\mathcal{H}}^{1}(\Omega), \quad \overset{\circ}{\mathcal{H}}(\text{curl}; \Omega)$$

as reincarnations of $\overset{\circ}{\mathcal{D}}^{0}(\Omega)$ and $\overset{\circ}{\mathcal{D}}^{1}(\Omega)$, respectively. We note

$$\overset{\circ}{\mathcal{H}}(\text{grad}; \Omega) = \overset{\circ}{\mathcal{C}}^{\infty}(\Omega), \quad \overset{\circ}{\mathcal{H}}(\text{curl}; \Omega) = \overset{\circ}{\mathcal{C}}^{\infty}(\Omega),$$

where the closures are taken in the respective graph norms, and that in these Sobolev spaces the classical homogeneous scalar and tangential (compare with $N = 3$) boundary conditions

$$u|_{\Gamma} = 0, \quad v \times v|_{\Gamma} = 0$$

are generalized. Here, $v$ denotes the outward unit normal for $\Gamma$. Furthermore, we have the spaces of irrotational or solenoidal vector fields

$$\mathcal{H}(\text{curl}_{0}; \Omega) = \{ v \in \mathcal{H}(\text{curl}; \Omega) : \text{curl } v = 0 \},$$

$$\overset{\circ}{\mathcal{H}}(\text{curl}_{0}; \Omega) = \{ v \in \overset{\circ}{\mathcal{H}}(\text{curl}; \Omega) : \text{curl } v = 0 \},$$

$$\mathcal{H}(\text{div}_{0}; \Omega) = \{ v \in \mathcal{H}(\text{div}; \Omega) : \text{div } v = 0 \}. $$

Again, all these spaces are Hilbert spaces. Now, we have two compact embeddings

$$\overset{\circ}{\mathcal{H}}(\text{grad}; \Omega) \hookrightarrow L^{2}(\Omega), \quad \overset{\circ}{\mathcal{H}}(\text{curl}; \Omega) \cap \mathcal{H}(\text{div}; \Omega) \hookrightarrow L^{2}(\Omega),$$

that is, Rellich’s selection theorem and the Maxwell compactness property. Moreover, the following Poincaré and Maxwell estimates hold:

**Corollary 4** (Poincaré estimate for functions)

Let $c_{p} := c_{p,0}$. Then, for all functions $u \in \overset{\circ}{\mathcal{H}}(\text{grad}; \Omega)$

$$|u|_{L^{2}(\Omega)} \leq c_{p} |\text{grad } u|_{L^{2}(\Omega)}.$$ 

**Corollary 5** (Maxwell estimate for vector fields)

Let $c_{m} := c_{p,1}$. Then, for all vector fields $v \in \overset{\circ}{\mathcal{H}}(\text{curl}; \Omega) \cap \mathcal{H}(\text{div}; \Omega)$

$$|v|_{L^{2}(\Omega)} \leq c_{m} \left( |\text{curl } v|_{L^{2}(\Omega)}^{2} + |\text{div } v|_{L^{2}(\Omega)}^{2} \right)^{1/2}. $$

We note that generally $\mathcal{H}^{0}(\Omega) = \{0\}$ and by (1.2) also $\mathcal{H}^{1}(\Omega) = \{0\}$. The appropriate Helmholtz decomposition for our needs is

**Corollary 6** (Helmholtz decomposition for vector fields)

$$L^{2}(\Omega) = \text{grad } \overset{\circ}{\mathcal{H}}(\text{grad}; \Omega) \oplus \mathcal{H}(\text{div}_{0}; \Omega)$$

### 1.3. Tensor fields

We extend our calculus to $(N \times N)$-tensor (matrix) fields. For vector fields $v$ with components in $\mathcal{H}(\text{grad}; \Omega)$ and tensor fields $T$ with rows in $\mathcal{H}(\text{curl}; \Omega)$ resp. $\mathcal{H}(\text{div}; \Omega)$, that is,

$$v = \begin{bmatrix} v_{1} \\ \vdots \\ v_{N} \end{bmatrix}, \quad v_{n} \in \mathcal{H}(\text{grad}; \Omega), \quad T = \begin{bmatrix} T_{1}^{1} \\ \vdots \\ T_{N}^{1} \end{bmatrix}, \quad T_{n} \in \mathcal{H}(\text{curl}; \Omega) \text{ resp. } \mathcal{H}(\text{div}; \Omega)$$

for $n = 1, \ldots, N$, we define

$$\text{Grad } v := \begin{bmatrix} \text{grad } v_{1} \\ \vdots \\ \text{grad } v_{N} \end{bmatrix}, \quad \text{Curl } v := \begin{bmatrix} \text{curl } T_{1} \\ \vdots \\ \text{curl } T_{N} \end{bmatrix}, \quad \text{Div } T := \begin{bmatrix} \text{div } T_{1} \\ \vdots \\ \text{div } T_{N} \end{bmatrix}.$$
where $J_v$ denotes the Jacobian of $v$ and $^t$ the transpose. We note that $v$ and $\text{Div } T$ are $N$-vector fields, $T$ and $\text{Grad } v$ are $(N \times N)$-tensor fields, whereas $\text{Curl } T$ is a $(N \times (N - 1)N/2)$-tensor field that may also be viewed as a totally anti-symmetric third order tensor field with entries
\[
(Crul T)_{ijk} = \partial_j T_{ik} - \partial_k T_{ij}.
\]
The corresponding Sobolev spaces will be denoted by
\[
\begin{align*}
H(\text{Grad}; \Omega), & \quad \hat{H}(\text{Grad}; \Omega), & \quad H(\text{Div}; \Omega), & \quad \hat{H}(\text{Div}; \Omega), \\
H(\text{Curl}; \Omega), & \quad \hat{H}(\text{Curl}; \Omega), & \quad H(\text{Curl}; \Omega), & \quad \hat{H}(\text{Curl}; \Omega).
\end{align*}
\]
There are three crucial tools to prove our estimate. First, we have obvious consequences from Corollaries 4, 5, and 6:

**Corollary 7** (Poincaré estimate for vector fields)
For all $v \in \hat{H}(\text{Grad}; \Omega)$
\[
\|v\|_{L^2(\Omega)} \leq c_p \|\text{Grad } v\|_{L^2(\Omega)}.
\]

**Corollary 8** (Maxwell estimate for tensor fields)
The estimate
\[
\|T\|_{L^2(\Omega)} \leq c_m \left( \|\text{Curl } T\|_{L^2(\Omega)}^2 + \|\text{Div } T\|_{L^2(\Omega)}^2 \right)^{1/2}
\]
holds for all tensor fields $T \in \hat{H}(\text{Curl}; \Omega) \cap H(\text{Div}; \Omega)$.

**Corollary 9** (Helmholtz decomposition for tensor fields)
\[
L^2(\Omega) = \text{Grad } \hat{H}(\text{Grad}; \Omega) \oplus H(\text{Div}; \Omega)
\]
The last important tool is Korn’s first inequality.

**Lemma 10** (Korn’s first inequality)
For all vector fields $v \in \hat{H}(\text{Grad}; \Omega)$
\[
\|\text{Grad } v\|_{L^2(\Omega)} \leq \sqrt{2} \|\text{sym Grad } v\|_{L^2(\Omega)} - \|\text{skew } T\|_{L^2(\Omega)}.
\]
Here, we introduce the symmetric and skew-symmetric parts
\[
sym T := \frac{1}{2}(T + T^t), \quad \text{skew } T := \frac{1}{2}(T - T^t)
\]
of a $(N \times N)$-tensor $T = \text{sym } T + \text{skew } T$.

**Remark 11**
We note that the proof including the value of the constant is simple. By density, we may assume $v \in C^\infty(\Omega)$. Twofold partial integration yields
\[
\langle \partial_n v_m, \partial_m v_n \rangle_{L^2(\Omega)} = \langle \partial_m v_m, \partial_n v_n \rangle_{L^2(\Omega)}
\]
and hence
\[
2 \|\text{sym Grad } v\|_{L^2(\Omega)}^2 = \frac{1}{2} \sum_{n,m=1}^N \|\partial_n v_m + \partial_m v_n\|_{L^2(\Omega)}^2 = \sum_{n,m=1}^N \left( \|\partial_n v_m\|_{L^2(\Omega)}^2 + \|\partial_m v_m\|_{L^2(\Omega)}^2 + \langle \partial_n v_m, \partial_n v_m \rangle_{L^2(\Omega)} \right)
\]
\[
\geq \|\text{Grad } v\|_{L^2(\Omega)}^2 + \|\text{Div } v\|_{L^2(\Omega)}^2 \geq \|\text{Grad } v\|_{L^2(\Omega)}^2.
\]
More on Korn’s first inequality can be found, for example, in [20].

**2. Results**

For tensor fields $T \in H(\text{Curl}; \Omega)$, we define the semi-norm
\[
\|T\| := \left( \|\text{sym } T\|_{L^2(\Omega)}^2 + \|\text{Curl } T\|_{L^2(\Omega)}^2 \right)^{1/2}.
\]
The main step is to prove the following.
Lemma 12
Let $\hat{c} := \max \left\{ 2, \sqrt{3c_{m}} \right\}$. Then, for all $T \in \mathring{H}(\text{Curl}; \Omega)$

$$
\|T\|_{L^2(\Omega)} \leq \hat{c} \|T\|
$$

Proof
Let $T \in \mathring{H}(\text{Curl}; \Omega)$. According to Corollary 9, we orthogonally decompose

$$
T = \text{Grad} v + S \in \text{Grad} \mathring{H}(\text{Grad}; \Omega) \oplus \text{H}(\text{Div}; \Omega).
$$

Then, $\text{Curl} T = \text{Curl} S$ and we observe $S \in \mathring{H}(\text{Curl}; \Omega) \cap \text{H}(\text{Div}; \Omega)$ because

$$
\text{Grad} \mathring{H}(\text{Grad}; \Omega) \subset \mathring{H}(\text{Curl}; \Omega).
$$

By Corollary 8, we have

$$
\|S\|_{L^2(\Omega)} \leq c_m \|\text{Curl} T\|_{L^2(\Omega)}.
$$

Then, by Lemma 10 and (2.2), we obtain

$$
\|T\|^2_{L^2(\Omega)} = \|\text{Grad} v\|^2_{L^2(\Omega)} + \|S\|^2_{L^2(\Omega)} \leq 2 \|\text{sym} \text{Grad} v\|^2_{L^2(\Omega)} + \|S\|^2_{L^2(\Omega)} \leq 4 \|\text{sym} T\|^2_{L^2(\Omega)} + 5 \|S\|^2_{L^2(\Omega)},
$$

which completes the proof.

The immediate consequence is our main result.

Theorem 13
On $\mathring{H}(\text{Curl}; \Omega)$ the norms $\|\cdot\|_{\mathring{H}(\text{Curl}; \Omega)}$ and $\|\cdot\|$ are equivalent. In particular, $\|\cdot\|$ is a norm on $\mathring{H}(\text{Curl}; \Omega)$ and there exists a positive constant $c$, such that

$$
c \|T\|^2_{\mathring{H}(\text{Curl}; \Omega)} \leq \|T\|^2 = \|\text{sym} T\|^2_{L^2(\Omega)} + \|\text{Curl} T\|^2_{L^2(\Omega)}
$$

holds for all $T \in \mathring{H}(\text{Curl}; \Omega)$.

Remark 14
For a skew-symmetric tensor field $T : \Omega \rightarrow \mathfrak{s o}(N)$, our estimate reduces to a Poincaré inequality in disguise, because $\text{Curl} T$ controls all partial derivatives of $T$ (compare with [21]) and the homogeneous tangential boundary condition for $T$ is implied by $T|_{\Gamma} = 0$.

Setting $T := \text{Grad} v$, we obtain the following.

Remark 15 (Korn’s first inequality: tangential-variant)
For all $v \in \mathring{H}(\text{Grad}; \Omega)$

$$
\|\text{Grad} v\|_{L^2(\Omega)} \leq \hat{c} \|\text{sym} \text{Grad} v\|_{L^2(\Omega)}
$$

(2.3)

holds by Lemma 12 and (2.1). This is just Korn’s first inequality from Lemma 10 with a larger constant $\hat{c}$. Because $\Gamma$ is connected, that is, $\mathring{H}^1(\Omega) = \{0\}$, we even have

$$
\text{Grad} \mathring{H}(\text{Grad}; \Omega) = \mathring{H}(\text{Curl}; \Omega).
$$

Thus, (2.3) holds for all $v \in H(\text{Grad}; \Omega)$ with $\text{Grad} v \in \mathring{H}(\text{Curl}; \Omega)$, that is, with $\text{grad} v_n, n = 1, \ldots, N$, normal at $\Gamma$, which then extends Lemma 10 through the (apparently) weaker boundary condition.

The elementary arguments above apply certainly to much more general situations, for example, to not necessarily connected boundaries $\Gamma$ and to tangential boundary conditions that are imposed only on parts of $\Gamma$. These discussions are left to forthcoming papers.

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