Continuous approximation of breathers in one- and two-dimensional DNLS lattices

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Abstract

In this paper we construct and approximate breathers in the DNLS model starting from the continuous limit: such periodic solutions are obtained as perturbations of the ground state of the NLS model in $H^1(\mathbb{R}^n)$, with $n = 1, 2$. In both the dimensions we recover the Sievers–Takeno and the Page (P) modes; furthermore, in $\mathbb{R}^2$ also the two hybrid (H) modes are constructed. The proof is based on the interpolation of the lattice using the finite element method (FEM).

Mathematics Subject Classification: 37K60

1. Introduction

In this paper we study the problem of constructing breathers in the one- and two-dimensional discrete nonlinear Schrödinger (DNLS) equation starting from the continuous limit.

The breathers we construct are critical points of the Hamiltonian function constrained to the surface of constant $\ell^2$ norm. Such critical points are obtained by continuation from the continuous model constituted by the nonlinear Schrödinger (NLS) equation. The connection between the discrete and the continuous system is obtained by using the finite elements (finite element method—FEM). This allows us to identify the phase space of the discrete system with a subspace of the phase space of the continuous system.

For example, consider the one-dimensional case. The space of the finite elements is constructed as follows: first we associate with the $j^{th}$ point of the discrete lattice a continuous piecewise linear function $s_j(x)$, whose value is 1 at $x = j$ and which vanishes for $|x - j| \geq 1$ (see figure 1). With a sequence $\psi_j$ we associate the function $\psi(x) := \sum_j \psi_j s_j(x/\mu)$, where $\mu > 0$ is a small parameter representing the mesh of the lattice. The space generated by the functions $s_j(x/\mu)$ will be denoted by $E_\mu$.

Once this is done one can compare the functionals of the continuous system and those of the discrete one. In order to do this, denote by $H_c$ and $N_c$ the Hamiltonian and the norm of the continuous system, and consider the restriction of such functions to the space of the discrete
system $E_\mu$. By the standard theory of integration one can say that the restricted functionals are close to the Hamiltonian $H_d$ and the norm $N_d$ of the discrete system. So the idea is to consider a non-degenerate critical point of the functional of the continuous system, a critical point lying close to the manifold of the finite elements and to continue such a critical point to a critical point of the discrete functional.

However, there is a delicate point in the game: namely that the difference between the discrete functional and the continuous one should be small when the phase space is endowed with the energy norm. This turns out to be true thanks to a special property of the finite elements: the fact that one has

$$\int_{\mathbb{R}^n} |\nabla \psi|^2 \, dx = \mu^n \sum_{|j-l|=1} |\psi_j - \psi_l|^2 / \mu^2,$$

with no error. Due to this property the difference between the continuous and the discrete functional turns out to be a functional which is small and smooth on the energy space. This allows us to apply the implicit function theorem and to continue critical points of the continuous system to critical points of the discrete one.

In order to be concrete we study in detail a one-dimensional and a two-dimensional model. We use known results on existence and non-degeneracy of the ground state of the continuous system in order to apply the above theory. In this paper we construct two (respectively four) kinds of discrete breathers in the 1- (respectively 2)-dimensional case, which are the continuation of the continuous breather. In order to avoid problems related to the translational invariance of the continuous system we work here in spaces of reflection invariant sequences. Thus, the breathers we find for the discrete system are reflection invariant too.

In dimension one, the breather of the first kind is centred at a lattice site and corresponds to the so-called Sievers–Takeno mode (ST), while the breather of the second kind is centred in the middle of a cell of the lattice and corresponds to the so-called Page mode (P). In dimension two, besides the ST and P modes, we have two other localized solutions, usually called hybrid (H) modes since they are centred in the middle of one of the two faces of the cell.

As far as we know, the result of this paper is the first one in which the continuous approximation is used in order to construct exact breathers of a lattice model. In dimension 1 the method of spatial dynamics also allows us to construct and approximate breathers (see [Jam03]). However, such a method is strictly one dimensional, while our method in principle applies to any dimension. Existence of breathers was also proved variationally in [Wei99] and in [AKK01], but such methods do not allow us to approximate the breathers and only allow us to find one breather for each model. Breathers in DNLS have also been widely studied numerically (see for example [KRB01, CJK+08, FW98].
The main advantage of our method is that it is quite flexible and allows us to directly deduce information on the shape of the breather starting from the continuous limit.

We recall that the possibility of using the continuous limit in order to approximate the dynamics of discrete systems has been widely investigated; in particular we recall the papers [BCP02, Sch98, KSM92, SW00, BP06, BCP09] in which an approximation valid for long but finite times and the papers [FP99, FP02, FP04a, FP04b, HW08, MP08] where an infinite time approximation have been obtained.

The plan of the paper is as follows. In section 2 we present the result and motivate our continuum limit approach. In section 3 we formulate in theorem 3.1 the implicit function theorem applied to our problem and in section 4 we construct the FEM to interpolate the discrete model and verify the hypothesis of theorem 3.1.

2. Main result

We study here the discrete focusing nonlinear Schrödinger equation (DNLS) in $\mathbb{R}^n$ with $n = 1, 2$:

$$i\dot{\psi}_l = -\frac{1}{\mu^2} (\Delta_1 \psi)_l - |\psi_l|^{2p} \psi_l, \quad l \in \mathbb{Z}^n, \quad \frac{1}{2} \leq p < \frac{2}{n},$$

where $\Delta_1$ is the $n$-dimensional discrete Laplacian defined by

$$(\Delta_1 \psi)_j := \psi_{j+1} + \psi_{j-1} - 2\psi_j,$$

and $\mu$ is the lattice mesh. In particular, we look for solutions of the form

$$\psi_l(t) = e^{-i\lambda t} \psi_l.$$

Then, the sequence $\psi_l$ fulfills

$$\lambda \psi_l = -\frac{1}{\mu^2} (\Delta_1 \psi)_l - |\psi_l|^{2p} \psi_l,$$

and thus it is a critical point of the Hamiltonian function

$$H_d := \mu^n \left[ \frac{1}{2} \sum_{|l-j|=1} |\psi_j - \psi_l|^2 - \frac{1}{p+1} \sum_{k \in \mathbb{Z}^n} |\psi_l|^{2p+2} \right]$$

constrained to a surface of constant value of the norm

$$N_d := \mu^n \sum_{l \in \mathbb{Z}^n} |\psi_l|^2,$$

where the factors $\mu^n$ have been inserted for future convenience. The main result of this paper consists of showing that such a solution can be constructed and approximated starting from the continuous model constituted by the NLS, namely,

$$i\dot{\psi} = -\Delta \psi - |\psi|^{2p} \psi.$$

More precisely, consider the Hamiltonian $H_c$ and (the square of) the $L^2$ norm $N_c$, given by

$$H_c := \int_{\mathbb{R}^n} \left[ |\nabla \psi|^2 - \frac{1}{p+1} |\psi|^{2p+2} \right], \quad N_c := \int_{\mathbb{R}^n} |\psi|^2,$$

then a periodic solution $\psi(x, t) = e^{-i\lambda t} \psi(x)$ of (6) fulfills the following continuous approximation of (3):

$$\lambda \psi = -\Delta \psi - |\psi|^{2p} \psi.$$
According to classical results on (8) (see [BL83, BLP81, CGM78]), there exists a unique real valued, positive, radially symmetric and exponentially decaying function $\psi_c$ which realizes the minimum of $H_c|_{N c=1}$. For example, in the case $n=1$ and $p=1$ it can be computed explicitly

$$\psi_c(x) := \frac{1}{\sqrt{2}} \sech \left( \frac{x}{2} \right).$$  \hspace{1cm} (9)

If we interpret the discrete functionals $H_d, N_d$ as $\mu$-perturbations of $H_c, N_c$ and we restrict to a class of ‘even’ functions in order to remove any possible degeneracy of the minimum $\psi_c$, then we can continue the solution $\psi_c$ of (8) to a solution $\psi(\mu)$ of (3).

In order to state the precise result we are going to prove, we first need to define the configuration space $Q_\mu$ for $\psi$:

**Definition 2.1.** The space $C^2(\mathbb{Z}^n, \mathbb{R})$ will be denoted by $Q_\mu$ when endowed with the norm

$$\| \psi \|^2_{Q_\mu} := \mu^n \| \psi \|^2_{L^2} + \frac{1}{\mu^{2-n}} \| \psi, -\Delta_1 \psi \|^2_{L^2}.$$  \hspace{1cm} (10)

**Theorem 2.1.** For any $\mu$ small enough and $\frac{1}{2} \leq p < \frac{3}{4}$ there exist $2^n$ distinct real valued sequences $\psi^i(\mu)$ which are solutions of (3). Such solutions are even sequences $\psi_{-\lambda} = \psi_\lambda$ lying on the surface $N_d = 1$. One has

$$\| \psi^i - \Psi^i \|^2_{Q_\mu} \leq C_1 \mu, \quad \sup_{l \in \mathbb{Z}^n} | \psi^i(t) - \Psi^i(t) | \leq C_2 \mu^{3/2}$$  \hspace{1cm} (11)

where $\Psi^i$ is defined by

$$\begin{cases}
\Psi^1_1 := \psi_c (\mu j), \\
\Psi^2_1 := \psi_c (\mu j + \frac{\mu}{2}), & n = 1, \\
\Psi^1_2 := \psi_c (\mu j, \mu k), \\
\Psi^2_2 := \psi_c (\mu j, \mu k + \frac{\mu}{2}), & n = 2, \\
\Psi^3_1 := \psi_c (\mu j + \frac{\mu}{2}, \mu k), \\
\Psi^4_1 := \psi_c (\mu j + \frac{\mu}{2}, \mu k + \frac{\mu}{2}).
\end{cases}$$

### 2.1. Comments

1. the first of (11) is not empty since by using (47) we get

$$\| \Psi^i \|^2_{Q_\mu} \sim 1.$$  

Moreover, we stress that by its definition the approximating sequence $\Psi^i$ is bounded uniformly in $\mu$

$$\Psi^i \leq \| \psi_c \|_{L^\infty}$$

but is localized\(^1\) on an increasing interval $[-k, k]^n$ with $k \sim 1/\mu$.

2. the first of (11) immediately implies

$$\| \psi^i - \Psi^i \|^2_{C^1} \leq C_1 \mu^{1/2} \implies | \psi^i(t) - \Psi^i(t) | \leq C_2 \mu^{3/2} \quad \forall l,$$

an estimate which is empty in the case $n = 2$. Lemma 4.7 in section 4.3 is necessary to improve the above result. We do not know whether the exponent $\frac{3}{2} = (n/2)$ is optimal or not.

\(^1\) We can fix the set where $\Psi^i$ is localized as $\Omega := \{ l \in \mathbb{Z}^n \text{ s.t. } \Psi^i \geq \frac{1}{2} \Psi_0 \}$. 

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3. We stress that problem (3) is equivalent to the $\mu$-independent one
\[ \tilde{\lambda}\phi_l = -(\Delta_1\phi_l - |\phi_l|^2\phi_l) \quad (12) \]
with the constraint
\[ \sum_{l \in \mathbb{Z}} |\phi_l|^2 = E, \quad E \ll 1. \]
This can be seen by the scaling
\[ \phi_l = \mu^{\frac{1}{p}} \psi_l, \quad \tilde{\lambda} = \mu^2 \lambda, \quad \mu^{\frac{1}{p} - n} = E, \]
and observing that
\[ \frac{2}{p} - n > 0 \iff p < \frac{2}{n}. \]

3. The implicit function theorem

The situation we will meet is summarized in the following abstract scheme. Let $\mathcal{H}$ be a Hilbert space, and for any $\mu$, let $\mathcal{E}_\mu$ be a subspace of $\mathcal{H}$. Let $H_{\mu} \in C^2(\mathcal{H})$ and $N_{\mu} \in C^\infty(\mathcal{H})$ be two functionals, with $N_{\mu}$ being a submersion. Correspondingly, we define
\[ S := \{ \psi \in \mathcal{H} : N_{\mu}(\psi) = 1 \}. \]
Then we define the ‘discrete’ objects: let $H_{\epsilon, \mu} := H_{\epsilon_1, \mu} \in C^2(\mathcal{E}_\mu)$ and $N_{\epsilon, \mu} := N_{\epsilon_2, \mu} \in C^\infty(\mathcal{E}_\mu)$ be functionals depending smoothly on two additional parameters $\epsilon_1, \epsilon_2$. Define
\[ S_{\epsilon_1, \mu} := \{ \psi \in \mathcal{E}_\mu : N_{\epsilon_1, \mu}(\psi) = 1 \}. \]
We make some assumptions.

i. There exists $\psi_c \in \mathcal{H}$ which is a coercive minimum of $H_{\mu}|_{S}$, namely, it is a minimum and fulfills
\[ \frac{d^2}{d}\big|_{\phi_c}(\psi_c)(h, h) \geq C \|h\|^2, \quad h \in T_{\psi_c} S; \quad (13) \]
moreover,
\[ d(\mathcal{E}_\mu, \psi_c) \leq C_{\mu}, \quad (14) \]
for all $\mu$ small enough.
Let $\psi_0 \in \mathcal{E}_\mu$ be such that $\|\psi_c - \psi_0\| \leq C_{\mu}$ and let $\mathcal{U} \subset \mathcal{E}_\mu$ be an open neighbourhood of $\psi_0$; then we assume

ii. \[ \|H_{\epsilon_1, \mu} - H_{0, \mu}\|_{C^2(\mathcal{U})} \leq C(\epsilon_1 + \mu), \quad H_{0, \mu} := H_{\mu}|_{\mathcal{E}_\mu} \]
\[ \|N_{\epsilon_2, \mu} - N_{0, \mu}\|_{C^0(\mathcal{U})} \leq C(\epsilon_2 + \mu), \quad N_{0, \mu} := N_{\mu}|_{\mathcal{E}_\mu} \]
for some large enough $k$.

**Theorem 3.1.** Under the above assumptions, for any $\epsilon_1, \epsilon_2, \mu$ small enough, there exists a unique $\psi_{\epsilon_1, \epsilon_2, \mu}$, which is a coercive minimum of $H_{\epsilon_1, \mu}|_{S_{\epsilon_1, \mu}=1}$. Moreover, one has
\[ \|\psi_{\epsilon_1, \epsilon_2, \mu} - \psi_c\| \leq C(\mu + \epsilon_1 + \epsilon_2). \quad (16) \]
Proof. The result is local, so we restrict to a neighbourhood of \( \psi_c \). Define
\[
S_{0,\mu} := \{ \psi \in E_{\mu} : N_{0,\mu}(\psi) = 1 \} = S \cap E_{\mu}
\] (17)
and take \( \psi_0 \in S_{0,\mu} \). Note that, due to smoothness of \( H_c \) one has
\[
\|d(H_c_{|S_{0,\mu}})(\psi_0)\| \leq C\mu.
\] (18)
By coercivity (13) and the Lax–Milgum lemma, the second differential
\[
d^2(H_c_{|S_{0,\mu}})(\psi_0) : T_{\psi_0}S_{0,\mu} \to T_{\psi_0}S_{0,\mu}
\]
defines an isomorphism bounded together with its inverse uniformly with respect to all the parameters.

From assumption ii, there exists a local isomorphism
\[
I_{\epsilon_2,\mu} : S_{0,\mu} \to S_{\epsilon_2,\mu},
\]
which satisfies
\[
\|I_{\epsilon_2,\mu} - Id\|_{C^1} \leq C(\epsilon_2 + \mu).
\]
The statement is then equivalent to the existence of a coercive minimum of \( H_{\epsilon_1,\mu} \circ I_{\epsilon_2,\mu} \). To get it note that
\[
H_{\epsilon_1,\mu} \circ I_{\epsilon_2,\mu} = H_{c} _{|S_{0,\mu}} + O(\epsilon_1 + \epsilon_2 + \mu).
\] (19)
Due to (18) and (13), the implicit function theorem applies and gives the result. \( \square \)

4. Applications to breathers

In order to avoid gauge and the translational invariance of the problem, in particular of the continuous system, we will work in a space of real valued functions ‘invariant’ under the involution
\[
S : \psi(z) \mapsto \psi(-z), \quad z \in \mathbb{R}^n.
\] (20)
More precisely, in \( H^1(\mathbb{R}^2, \mathbb{R}) \) we will consider functions fulfilling
\[
\int_{\mathbb{R}^n} |\psi(z) - \psi(-z)|^2 = \|\psi - S\psi\|^2_{L^2} = 0,
\] (21)
which is equivalent to (20) almost everywhere and is a condition well defined in \( H^1(\mathbb{R}^2, \mathbb{R}) \).

Lemma 4.1. Let \( \psi_c \) be a solution of (8) with \( p < 2/n \) and let
\[
\mathcal{H} = \{ \psi \in H^1(\mathbb{R}^n, \mathbb{R}) : \|\psi - S\psi\|^2_{L^2} = 0 \} =: H^1_{\mu},
\]
then assumption (13) of theorem 3.1 holds.

Proof. This lemma directly follows from proposition D.1 of [FGJS04] by noting that \( T_{\psi}S \subset X \), with \( X \) defined in the statement of proposition D.1. \( \square \)

Remark 4.1. We stress that the constraint (21) is ‘natural’ for problem (3), since (20) is a symmetry for both the Hamiltonian (4) and the norm (5). Hence, a critical point for the restricted problem is also a critical point for the original problem.

In the following subsections we construct the linear manifold \( \mathcal{E}_\mu \) of the finite elements, and prove estimates (14) and (15) for the two considered applications. We deal with the ST-breather, since the other ones follow by small changes in the definition of \( \mathcal{E}_\mu \).
4.1. The case \( n = 1 \)

Let \( \ell = j \) and define the sequence of functions \( s_j(x) \) by

\[
s_j(x) = \begin{cases} 
0, & \text{if } |x - j| > 1 \\
-x + j + 1, & \text{if } j - 1 \leq x \leq j \\
x - j + 1, & \text{if } j \leq x \leq j + 1 
\end{cases}
\]  

and, with a sequence \( \psi_j \in Q\mu \), we associate a function

\[
\Psi(x) := \sum_j \psi_j s_j(x/\mu).
\]

On the interval \( T_j := [\mu j, \mu(j + 1)] \) the above function reads

\[
\Psi(x) = (x - \mu j) \left( \frac{\psi_{j+1} - \psi_j}{\mu} + \psi_j \right).
\]

**Definition 4.1.** We denote by \( E_\mu \) the linear space composed of the functions of the form (23) with \( \psi_j \in Q\mu \).

The following lemma gives the equivalence between the function space \( E_\mu \) and the sequence space \( Q\mu \).

**Lemma 4.2.** Let \( \Psi \in E_\mu \) then

\[
\int_\mathbb{R} \Psi^2 = \frac{1}{\mu} \sum_{j \in \mathbb{Z}} (\psi_{j+1} - \psi_j)^2.
\]

Moreover,

\[
\mu \sum_j \psi_j^2 = \int_\mathbb{R} \Psi^2 + \frac{\mu}{3} \int_\mathbb{R} \Psi_x^2 \, d x.
\]

**Proof.** Let us first decompose \( \mathbb{R} = \bigcup_{j \in \mathbb{Z}} T_j \). The weak derivative of \( \Psi \) is

\[
\Psi_x(x) = \sum_{j \in \mathbb{Z}} \left( \frac{\psi_{j+1} - \psi_j}{\mu} \right) \chi_{(\mu j, \mu(j+1))}(x)
\]

which gives immediately

\[
\Psi_x^2(x) = \left[ \sum_{j \in \mathbb{Z}} \left( \frac{\psi_{j+1} - \psi_j}{\mu} \right) \chi_{T_j}(x) \right]^2 = \sum_{j \in \mathbb{Z}} \frac{(\psi_{j+1} - \psi_j)^2}{\mu^2} \chi_{T_j}(x),
\]

since \( \chi_{T_i}(x) \chi_{T_j}(x) = 0 \) for \( i \neq j \). From \( \int_\mathbb{R} \chi_{T_j}(x) \, d x = \mu \) one gets

\[
\int_\mathbb{R} \Psi_x^2 \, d x = \mu \sum_{j \in \mathbb{Z}} \frac{(\psi_{j+1} - \psi_j)^2}{\mu^2} = \frac{1}{\mu} \sum_{j \in \mathbb{Z}} (\psi_{j+1} - \psi_j)^2.
\]

If we plug (24) in the integral \( \| \Psi^2 \|_{L^2} \), a direct computation gives estimate (26). \( \square \)

**Proposition 4.1.** Let \( \Psi \in E_\mu \) be as in (23) and let us define

\[
G_c(\Psi) := \int_\mathbb{R} |\Psi|^q, \\
G_d(\Psi) := \mu \sum_{j \in \mathbb{Z}} |\psi_j|^q, \\
RG(\Psi) := G_c(\Psi) - G_d(\Psi);
\]
if \( q \geq 1 \) then \( R_G \in C^2(\mathcal{E}_\mu) \) and for any bounded open set \( \mathcal{U} \subset \mathcal{E}_\mu \), there exists \( C(\mathcal{U}) \) such that
\[
\| R_G \|_{C^1(\mathcal{U})} \leq C\mu. \tag{29}
\]

**Proof.** The term \( R_G \) can be represented through the Euler–MacLaurin formula
\[
\sum_{j \in \mathbb{Z}} f(j) = \int_{\mathbb{R}} f(y) \, dy + \int_{\mathbb{R}} f'_1(y) P_1(y) \, dy, \quad P_1(s) = s - [s] - \frac{1}{2}.
\]
Indeed, if we set \( f(y) = |\Psi(\mu y)|^{q+2} \), we have
\[
\mu \sum_{j \in \mathbb{Z}} |\Psi(j)|^{q+2} = \mu \sum_{j \in \mathbb{Z}} |\Psi(\mu j)|^{q+2} = \mu \sum_{j \in \mathbb{Z}} f(j) = \mu \int_{\mathbb{R}} |\Psi(\mu y)|^{q+2} \, dy
\]
\[
+ \mu^2 \int_{\mathbb{R}} \Psi(\mu y)|\Psi(\mu y)|^q |\Psi'_1(\mu y)| P_1(y) \, dy.
\]
Hence
\[
R_H = \mu \int_{\mathbb{R}} |\Psi|^q |\Psi'_1| P_1(x/\mu) \, dx.
\]
A direct computation of the first and second differential shows that
\[
dR_H(\Psi)[h] = \mu \int_{\mathbb{R}} (|\Psi|^q)_x h P_1(x/\mu) \, dx + \mu \int_{\mathbb{R}} |\Psi|^q h_x P_1(x/\mu) \, dx,
\]
\[
d^2R_H(\Psi)[h, h] = \mu \int_{\mathbb{R}} (|\Psi|^q)_{xx} h^2 P_1(x/\mu) \, dx + \mu \int_{\mathbb{R}} |\Psi|^q (h^2)_x P_1(x/\mu) \, dx.
\]
The smallness is represented by the prefactor \( \mu \): so Sobolev embedding theorems and \( |\Psi'_1(\mu y)| \leq 1/2 \) yield (29). □

We have thus verified the assumptions of theorem 3.1 which implies the existence and the estimate of the ST-mode for the case \( n = 1 \). The same statement for the P-mode follows by a translation of the basis of \( \mathcal{E}_\mu \)
\[
\Psi(x) := \sum_{j \in \mathbb{Z}} \psi_j s_j(x/\mu + 1/2), \quad \psi_{-j+1} = \psi_j, \tag{31}
\]
with \( s_j \) defined in (22).

### 4.2. The case \( n = 2 \)

Let us take \( \psi_{j,k} \in Q_\mu \). For each multindex \( I = (j, k) \), let us consider the function \( s_{j,k}(x, y) \) which represents the exagonal pyramid of height one centred in \( (j, k) \) whose support is the union of the six triangles of figure 2. More precisely we define \( T^+_{j,k} \), the triangle whose vertexes are \( (j, k), (j+1, k), (j, k+1) \) and \( T^-_{j,k} \) the one whose vertexes are \( (j, k), (j-1, k), (j, k-1) \). Hence, for example, on \( T^+_{j,k} \) the function \( s_{j,k} \) represents the plane in \( \mathbb{R}^3 \)
\[
s_{j,k}(x, y) = -x - y + j + k + 1.
\]

The set of functions \( \{ s_{j,k}(x/\mu, y/\mu) \}_{(j,k) \in \mathbb{Z}} \) is a basis which generates a piecewise linear function \( \Psi(x, y) \) interpolating \( \psi_{j,k} \)
\[
\Psi(x, y) := \sum_{(j,k) \in \mathbb{Z}} \psi_{j,k}s_{j,k}(x/\mu, y/\mu). \tag{32}
\]
Figure 2. Triangulation and finite element in the bidimensional lattice.

Note that on the triangle $T_{j,k}$ the function $\Psi$ is the plane

$$\Psi(x, y) = \psi_{j,k} + (x - \mu j) \psi_x + (y - \mu k) \psi_y.$$  \hfill (33)

**Definition 4.2.** We denote by $E_\mu$ the linear space composed by the functions of the form (32) with $\psi_{j,k} \in Q_\mu$.

The following lemma gives the equivalence between the function space $E_\mu \subset H^1$ and the sequence space $Q_\mu$.

**Lemma 4.3.** Let $\Psi \in E_\mu$ then it holds true

$$\int_{\mathbb{R}^2} \Psi_x^2 = \sum_{(j,k) \in \mathbb{Z}^2} (\psi_{j+1,k} - \psi_{j,k})^2, \quad \int_{\mathbb{R}^2} \Psi_y^2 = \sum_{(j,k) \in \mathbb{Z}^2} (\psi_{j,k+1} - \psi_{j,k})^2.$$ \hfill (34)

Moreover,

$$\mu^2 \sum_{j,k} \psi_{j,k}^2 = \int_{\mathbb{R}^2} \Psi^2 + \frac{\mu^2}{2} \int_{\mathbb{R}^2} (\Psi_x^2 + \Psi_y^2 - \Psi_x \Psi_y).$$ \hfill (35)

**Proof.** From (33) we have that on each triangle $T_{j,k}$ it holds

$$\Psi_x = \pm \frac{(\psi_{j+1,k} - \psi_{j,k})}{\mu}, \quad \Psi_y = \pm \frac{(\psi_{j,k+1} - \psi_{j,k})}{\mu}.$$  

Formula (35) follows from a direct computation as in lemma 4.2. \hfill $\square$

The next three lemmas provide the proof of the following main proposition.

**Proposition 4.2.** Let $\Psi \in E_\mu$ be as in (32) and let us define

$$G_c(\Psi) := \int_{\mathbb{R}^2} |\Psi|^{q+2},$$

$$G_d(\Psi) := \mu^2 \sum_{j,k \in \mathbb{Z}^2} |\psi_{j,k}|^{q+2},$$

$$R_G(\Psi) := G_c(\Psi) - G_d(\Psi);$$

if $q \geq 1$ then $R_G \in C^2(E_\mu)$ and in any open set $U \subset E_\mu$ one has

$$\|R_G\|_{C^0(U)} \leq C_U \mu.$$  


Lemma 4.4. Under the assumptions of proposition 4.2 one has
\[ |R_G(\Psi)| \leq C\mu \|\Psi\|_{H^1}^{q+2}. \]

Proof. Let us set \( f(x, y) = |\Psi(x, y)|^{q+2} \) and let us take \((x, y) \in T_{j,k}^\pm\) then we can use a Taylor expansion with integral remainder
\[ f(x, y) = f(\mu_j, \mu_k) + (x - \mu_j) \int_0^1 f_x(\gamma(t, x, y)) \, dt + (y - \mu_k) \int_0^1 f_y(\gamma(t, x, y)) \, dt, \]
where
\[ \gamma(t, x, y) = (tx + (1-t)\mu_j, ty + (1-t)\mu_k) \quad t \in [0, 1] \]
is the segment connecting \((\mu_j, \mu_k)\) with \((x, y)\) and lies in the triangle \( T_{j,k}^\pm \). Hence
\[ \int_{T_{j,k}^\pm} f(x, y) = \frac{\mu_j^2}{2} f(\mu_j, \mu_k) + \int_{T_{j,k}^\pm} (x - \mu_j) \int_0^1 f_x(\gamma(t, x, y)) \, dt \, dx \, dy \]
\[ + \int_{T_{j,k}^\pm} (y - \mu_k) \int_0^1 f_y(\gamma(t, x, y)) \, dt \, dx \, dy. \]
By the initial definition of \( f(x, y) \) one has
\[ |\partial_x f| = (q + 2)|\Psi|^{q+1} \|\Psi_x\| = (q + 2)|\Psi|^{q+1} \|\Psi_{j,k}^{j+1,k} - \Psi_{j,k}\|, \]
\[ |\partial_y f| = (q + 2)|\Psi|^{q+1} \|\Psi_y\| = (q + 2)|\Psi|^{q+1} \|\Psi_{j,k}^{j,k+1} - \Psi_{j,k}\|. \]

Since
\[ \int_{\mathbb{R}^2} f(x, y) - \mu^2 \sum_{j,k \in \mathbb{Z}^2} f(j, k) = \sum_{j,k \in \mathbb{Z}^2} \left( \int_{T_{j,k}^\pm} f(x, y) - \frac{\mu_j^2}{2} f(j, k) \right) \]
we can use (37) and (38) to estimate
\[ \left| \int_{\mathbb{R}^2} f(x, y) - \mu^2 \sum_{j,k \in \mathbb{Z}^2} f(j, k) \right| \leq \sum_{j,k \in \mathbb{Z}^2} \left| \int_{T_{j,k}^\pm} f(x, y) - \frac{\mu_j^2}{2} f(j, k) \right| \]
\[ \leq \mu \sum_{j,k \in \mathbb{Z}^2} \int_{T_{j,k}^\pm} \left| f_x(\gamma(t, x, y)) \right| + \left| f_y(\gamma(t, x, y)) \right| \, dt \, dx \, dy. \]
By inserting (39) and observing that
\[ \Psi \circ \gamma = \Psi(\gamma(t, x, x)) = \psi_{j,k} + t(x - \mu_j)\psi_x + t(y - \mu_k)\psi_y \]
(41) one may remove the integration along the segment
\[ \int_{T_{j,k}^\pm} \int_0^1 |f_x(\gamma(t, x, y))| \leq \frac{|\psi_{j+1,k} - \psi_{j,k}|}{\mu} \int_{T_{j,k}^\pm} \left[ |\psi_{j,k}| + |(\nabla \Psi, \delta_{j,k})| \right]^{q+1} \]
where we set \( \delta_{j,k}(x, y) := (x - \mu_j, y - \mu_k) \) and
\[ (\nabla \Psi, \delta_{j,k}) = (x - \mu_j)\psi_x + (y - \mu_k)\psi_y. \]
Note that from (33) one has
\[ |\psi_{j,k}| \leq |\Psi(x, y)| + \mu \left(|\Psi_x| + |\Psi_y|\right) \]
thus it is possible to estimate the argument of the integral as follows:\(^2\)

\[
|\psi_{j,k}| + |\langle \nabla \Psi, \delta_{j,k} \rangle| \leq 4^q \left[ |\Psi(x, y)|^{q+1} + \mu^{q+1} \left( |\Psi_x| + |\Psi_y| \right)^{q+1} \right]
\]

hence recalling also that \( |\Psi_x| + |\Psi_y| \leq \sqrt{2} |\nabla \Psi| \) one has

\[
\mu \sum_{j,k \in \mathbb{Z}^2} \int_{T_{j,k}^+} \int_{T_{j,k}^-} \left| f_\gamma(y(t, x, y)) \right| \, dt \, dx \, dy
\]

\[
= \mu C(q) \sum_{j,k \in \mathbb{Z}^2} \int_{T_{j,k}^+} \left| \nabla \Psi \right| \left( |\Psi(x, y)|^{q+1} + \mu^{q+1} |\nabla \Psi|^{q+1} \right)
\]

\[
\leq \mu C(q) \sum_{j,k \in \mathbb{Z}^2} \left[ \int_{T_{j,k}^+} \left| \nabla \Psi \right| |\Psi(x, y)|^{q+1} + \mu^{q+1} \int_{T_{j,k}^+} |\nabla \Psi|^{q+2} \right].
\]

The first sum gives

\[
\sum_{j,k \in \mathbb{Z}^2} \int_{T_{j,k}^+} \left| \nabla \Psi \right| |\Psi(x, y)|^{q+1} = \int_{\mathbb{R}^2} \left| \nabla \Psi \right| |\Psi(x, y)|^{q+1}
\]

\[
\leq \| \nabla \Psi \|_{L^2} \| \Psi \|_{L^{q+2}(\mathbb{R}^2)}^{q+1}
\]

\[
\leq C_1 \| \nabla \Psi \|_{L^2} \| \Psi \|_{H^{q+1}}^{q+1}.
\]

Using lemma 4.3 the second instead gives

\[
\mu^{q+1} \sum_{j,k \in \mathbb{Z}^2} \int_{T_{j,k}^+} |\nabla \Psi|^{q+2} = \mu^{q+3} \sum_{j,k \in \mathbb{Z}^2} |\nabla \Psi|^{q+2}
\]

\[
= \mu \sum_{j,k \in \mathbb{Z}^2} \left[ (\psi_{j+1,k} - \psi_{j,k})^2 + (\psi_{j,k+1} - \psi_{j,k})^2 \right]^{1/2}
\]

\[
\leq C_2 \mu \| \nabla \Psi \|_{L^2}^{q+2}.
\]

Collecting the above estimates we obtain

\[
\left| \int_{\mathbb{R}^2} |\Psi(x, y)|^{q+2} - \mu^2 \sum_{j,k \in \mathbb{Z}^2} |\psi_{j,k}|^{q+2} \right| \leq \mu C_1 \| \nabla \Psi \|_{L^2} \| \Psi \|_{H^{q+1}} + \mu^2 C_2 \| \nabla \Psi \|_{L^2}^{q+2}
\]

which finally gives

\[
\left| \int_{\mathbb{R}^2} |\Psi(x, y)|^{q+2} - \mu^2 \sum_{j,k \in \mathbb{Z}^2} |\psi_{j,k}|^{q+2} \right| \leq \mu C \| \Psi \|_{H^{q+2}}^{q+2}.
\]

\(^2\) We here remind the inequalities for \( a, b > 0 \)

\[
(a + b)^s \leq 2^s (a^s + b^s), \quad 0 < s < 1
\]

and

\[
(a + b)^s \leq 2^{s-1} (a^s + b^s), \quad s \geq 1;
\]

the second follows easily from the convexity of the function \( g(x) = x^s, x \in \mathbb{R}^s \). The first is a direct consequence:

\[
(a + b)^s = \left( \frac{(a + b)^{s+1}}{(a + b)} \right) \leq 2^s \left( \frac{a^{s+1}}{a + b} + \frac{b^{s+1}}{a + b} \right) \leq 2^s (a^s + b^s).
\]
Lemma 4.5. Under the assumptions of proposition 4.2 one has also
\[ \| R_G' (\Psi) \|_{L(\mathbb{E}_\mu, \mathbb{R})} \leq C \mu \| \Psi \|_{H^1}^{q+1}. \] (42)

Proof. A direct computation easily gives for any \( h \in \mathbb{E}_\mu \)
\[ G'_c(\Psi)[h] = \int_{\mathbb{R}^2} |\Psi|^q \Psi h, \quad G'_d(\Psi)[h] = \mu^2 \sum_{j,k} |\psi_{j,k}|^q \psi_{j,k} h_{j,k} \]
with obviously \( h_{j,k} = h(\mu j, \mu k) \). In order to estimate
\[ \| G'_c(\Psi) - G'_d(\Psi) \|_{L(\mathbb{E}_\mu, \mathbb{R})} = \sup_{\| h \|_{H^1} = 1} \frac{|G'_c(\Psi)[h] - G'_d(\Psi)[h]|}{\| h \|_{H^1}} \]
we need to control
\[ |G'_c(\Psi)[h] - G'_d(\Psi)[h]| = \int_{\mathbb{R}^2} |\Psi|^q \Psi h - \mu^2 \sum_{j,k} |\psi_{j,k}|^q \psi_{j,k} h_{j,k} |. \] (43)

We proceed in the same way as in lemma 4.4, exploiting the fact that also \( h(x, y) \in \mathbb{E}_\mu \). We thus define in this case \( f(x, y) = |\Psi(x, y)|^q \Psi(x, y) h(x, y) \), so that
\[ |f_x| + |f_y| \leq q |\Psi|^q (|\Psi_x| + |\Psi_y|) |h| + |\Psi|^{q+1} |\nabla h|. \]

We recall that
\[ |\Psi \circ \gamma| \leq |\Psi| + 2 \mu (|\Psi_x| + |\Psi_y|), \]
\[ |h \circ \gamma| \leq |h| + 2 \mu |\nabla h|, \]
hence (43) can be split into four terms
\[ \sum_{2^n} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |f_x \circ \gamma| + |f_y \circ \gamma| \leq \int_{\mathbb{R}^2} |\Psi|^q |\nabla \Psi| ((|h| + \mu |\nabla h|) \]
\[ + \mu^2 \int_{\mathbb{R}^2} |\nabla \Psi|^{q+1} (|h| + \mu |\nabla h|) \]
\[ + \int_{\mathbb{R}^2} |\Psi|^{q+1} |\nabla h| \]
\[ + \mu \int_{\mathbb{R}^2} |\Psi|^{q+1} |\nabla h|. \]

1. using Schwarz and \( |\nabla| \Psi|^{q+1} |^2 = C(q) |\nabla \Psi|^2 |\Psi|^{2q} \) we get
\[ \int_{\mathbb{R}^2} |\Psi|^q |\nabla \Psi| ((|h| + \mu |\nabla h|) \leq \| \Psi \|_{H^1}^{q+1} \| h \|_{H^1}; \]

2. using Schwarz and \( \ell^2 \rightarrow \ell^s, s > 2 \)
\[ \mu \int_{\mathbb{R}^2} |\nabla \Psi|^{q+1} (|h| + \mu |\nabla h|) \leq C_1 \mu^q \int_{\mathbb{R}^2} |\nabla \Psi|^{q+2} \| h \|_{H^1} \]
\[ \leq C_2 |\nabla \Psi|_{L^2}^{q+1} \| h \|_{H^1}; \]

3. as in 1.
\[ \int_{\mathbb{R}^2} |\Psi|^{q+1} |\nabla h| \leq \| \Psi \|_{L^{q+1}}^{q+1} \| \nabla h \|_{L^2} \leq C_1 \| \Psi \|_{H^1}^{q+1} \| h \|_{H^1}; \]
4. as in 2.

\[ \mu^{q+1} \int_{\mathbb{R}^2} |\nabla \Psi|^q |\nabla h| \leq \mu^{q+1} \sqrt{\int_{\mathbb{R}^2} |\nabla \Psi|^{2q+2} |\nabla h|_{L^2}^2} \]

\[ \leq C_1 \|\nabla \Psi\|^q_{L^2} \|h\|_{H^1}. \]

Collecting we get

\[ |R'_G(\Psi)[h]| \leq C(q) \mu \|\Psi\|^q_{H^1} \|h\|_{H^1}, \]

hence the thesis. \(\square\)

**Lemma 4.6.** Under the assumptions of propositions 4.2 one has also

\[ \|R''_G(\Psi)\|_{L^2(\mathcal{E}_\mu, \mathbb{R})} \leq C \mu \|\Psi\|^q_{H^1}. \] (44)

**Proof.** Also in this case, a direct computation easily gives for any \(h \in \mathcal{E}_\mu\)

\[ G''_c(\Psi)[h, h] = \int_{\mathbb{R}^2} |\Psi|^q h^2, \quad G''_d(\Psi)[h, h] = \mu^2 \sum_{j,k} |\psi_{j,k}|^q h_{j,k}^2. \]

In order to estimate

\[ \|G''_c(\Psi) - G''_d(\Psi)\|_{L^2(\mathcal{E}_\mu, \mathbb{R})} = \sup_{h \neq 0} \frac{|G''_c(\Psi)[h, h] - G''_d(\Psi)[h, h]|}{\|h\|^2_{H^1}} \]

we need to control

\[ |G''_c(\Psi)[h, h] - G''_d(\Psi)[h, h]| = \int_{\mathbb{R}^2} |\Psi|^{q} h^2 - \mu^2 \sum_{j,k} |\psi_{j,k}|^q h_{j,k}^2. \] (45)

We proceed as in the previous lemmas, by defining

\[ f(x, y) := |\Psi(x, y)|^q h^2(x, y), \]

so that

\[ |f_x| + |f_y| \leq q |\Psi|^{q-1} |\nabla \Psi| |h^2 + 2 |\Psi|^q |h \nabla h|. \]

We distinguish the case \(q = 3\), which is easier, and \(q > 3\).

\(q = 1\) In this case we have

\[ \sum_{Z_n} \int_{T_{j,k}} \int_{0}^{\tau_t} |f_x \circ \gamma| + |f_y \circ \gamma| \leq 2 \int_{\mathbb{R}^2} |\nabla \Psi| |h| \]

\[ + 4 \mu^2 \int_{\mathbb{R}^2} |\nabla \Psi| |h| \]

\[ + \int_{\mathbb{R}^2} |\Psi| |\nabla h|(|h| + \mu |\nabla h|) \]

\[ + \mu \int_{\mathbb{R}^2} |\nabla \Psi| |\nabla h|(|h| + \mu |\nabla h|). \]

The thesis can be obtained using Schwarz and observing that

\[ \int_{\mathbb{R}^2} |\nabla h|^4 \leq \frac{c_1}{\mu^2} \sum_{j,k} \left( (h_{j+1,k} - h_{j,k})^2 + (h_{j,k+1} - h_{j,k})^2 \right)^2 \leq \frac{c_2}{\mu^2} \|\nabla h\|_{L^2}^4. \]
The steps are the same as usual; the only difference is that we have to deal with
\[
\int_{\mathbb{R}^2} |\Psi|^2 \sigma |\nabla \Psi|^2, \quad \sigma > 0,
\]
but it is enough to note again that the integral above is the (square) \( L^2 \) norm of \( \nabla |\Psi|^\sigma \), thus
\[
\int_{\mathbb{R}^2} |\Psi|^2 \sigma |\nabla \Psi|^2 \leq C \|\Psi\|_{H^{2\sigma}}^2.
\]
□

This concludes the case related to the construction and approximation of the ST-mode. The other three modes (the P-mode and the two H-mode) are obtained by translation of the basis \( s_{j,k} \) either in one or in both the directions.

### 4.3. Proof of theorem 2.1

We begin with the following

**Definition 4.3.** Let \( n = 1, 2 \) and consider \( \psi \in H^2(\mathbb{R}^n) \hookrightarrow C^0 \) on \( \mathcal{E}_\mu \). We define
\[
\Pi_\mu : \psi \mapsto \Pi_\mu \psi = \sum_{l \in \mathbb{Z}^n} \psi(\mu l) s_l(x/\mu), \quad x \in \mathbb{R}^n
\]
the projection of \( H^2(\mathbb{R}^n) \) on \( \mathcal{E}_\mu \). By classical results on polynomial approximation in Sobolev spaces (chapter 4 of [BS08]) one has
\[
\|\Pi_\mu \psi - \psi\|_{H^2} \leq C \mu \|\psi\|_{H^2}.
\]

We need also a simple lemma to obtain the second estimate of (11).

**Lemma 4.7.** For any \( l \in \mathbb{Z}^n \) we have
\[
|\psi_l| \leq 2 \mu^{1 - \frac{n}{2}} \|\psi\|_{Q_\mu}.
\]

**Proof.** We write the proof for the case \( n = 2 \). The case \( n = 1 \) is simpler. Denote \( l = (j, k) \); one has
\[
\psi_{j,k}^2 = \sum_{m = -\infty}^h (\psi_{m,k}^2 - \psi_{m-1,k}^2) = \sum_{m = -\infty}^h (\psi_{m,k} - \psi_{m-1,k})(\psi_{m,k} + \psi_{m-1,k})
\]
which gives
\[
\sup_{(j,k) \in \mathbb{Z}^2} \psi_{j,k}^2 \leq 4 \sqrt{\sum_{m \in \mathbb{Z}} \psi_{m,k}^2} \sqrt{\sum_{m \in \mathbb{Z}} (\psi_{m+1,k} - \psi_{m,k})^2}
\]
\[
\leq \left( 2 \sqrt{\|\psi\|_{C^1} \mu^{1 - \frac{n}{2}}} \left[ \mu^0 \left( \psi; -\Delta \psi \right)_{C^1} \right]^{1/4} \right)^2
\]
and (48). □

Now we easily verify the hypothesis of the abstract theorem 3.1. First, we define \( \psi_c \) as the (smooth) solution of (8) and \( \psi_0 = \nu \Pi_\mu \psi_c \), with \( \nu \) such that \( N_{0,\mu}(\psi_0) = 1 \). Then, condition (13) follows from lemma 4.1 while condition (14) comes from the above (47). Finally, requirement ii is given by lemmas 4.2 and 4.3 and by propositions 4.1 and 4.2. This directly gives the first of (11). The second of (11) is a byproduct of either the first and lemma 4.7, indeed
\[
|\psi_1(t) - \Psi_1(t)| \leq 2 \mu^{1 - \frac{n}{2}} \|\psi_1 - \Psi_1\|_{Q_\mu} \leq C \mu^{\frac{3}{2} - \frac{n}{2}}.
\]
□
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