Relativistic potential of a hydrogen-like system in Poincaré invariant quantum mechanics

V.V. Andreev

Francisk Skorina Gomel State University, Gomel, Belarus

To describe a relativistic hydrogen atom we used the Poincaré-covariant model of a two particle system with gauge invariant potential. The kernel of the radial integral equation is obtained which describes a system of two fermions with electromagnetic interaction.

PACS numbers: 13.60.F, 11.10.E, 11.80
Keywords: hydrogen atom, Poincaré-covariant model
I. INTRODUCTION

The investigation of the energy spectra of hydrogenic atoms is of great importance for high accuracy verification of the Standard Model and derivation of more correct values of fundamental physical constants (the fine structure constant, the masses of the muon and electron, the proton charge radius, etc.) [1–3].

The experiment to measure the energy interval \( E(2P_{3/2}^{F=2}) - E(2S_{1/2}^{F=1}) \) of the muonic hydrogen atom [4] led to a significant difference with theoretical calculations. This circumstance stimulated a new round of research on the simplest atomic systems and their parameters (see, for example, [5–8]).

The important stimulus for these evaluation is provided by the spectacular experimental progress in measurements of two fermion system energy levels. The relative uncertainty of the atomic energy levels’ frequency measurement was reduced to \( 4.5 \times 10^{-15} \) [9].

The basis for calculating the energy structure of coupled systems is the procedure for constructing the relativistic potential of particle interaction. The construction of the operator for the interaction potential of a particles system is carried out, as a rule, with the help of the corresponding amplitude \( T_{fi} \) of elastic scattering [10–12].

The most common technique is to calculate spinor structures in terms of Pauli matrices and momenta using explicit bispinors [12–15]. Such a calculation, as a rule, is done approximately, using the expansion in the velocities \( \nu/c \) of the particles of the system. Next, the potential \( V(r) \) in the coordinate space is calculated as the Fourier transformation of the above scattering amplitude \( T_{fi} \).

In contrast to the above-mentioned technique, the work plans to make an accurate calculation (without the expansion in the velocities of the system particles) of the kernel of the radial equation of the relativistic system as a scalar function of the momenta of the particles. In this situation, it is advisable to use methods of direct calculation of the corresponding matrix elements as explicitly scalar functions.

The aim of this work is to calculate the kernel of the radial equation of a two-fermion relativistic system in momentum space based on an accurate calculation of the amplitude of one-boson exchange.
This calculation is not straightforward and therefore represents an independent task. Calculation of such a kernel will be possible in the future to refine the contributions of higher-order relativistic effects to the energy spectra of hydrogen-like systems.

II. TWO-PARTICLE SYSTEM IN POINCARÉ INvariant QUANTUM MECHANICS

For calculation of fermion-fermion systems energy levels with electromagnetic interaction there exist a number of various models: Bethe-Salpeter equation [16, 17], model based on effective Dirac equation [1], quasipotential approach [18–20], variational Hamiltonian formalism [15] and others.

In our article we use the description of bound states with the help of the Poincaré-invariant quantum mechanics [21] (or relativistic Hamiltonian dynamics (RHD) [22]). In this approach, the Hamiltonian $\hat{H}$ is assumed to be the sum of a relativistic kinetic energy operator $T(k)$ that represents the invariant mass of two noninteracting particles plus phenomenological interaction $\hat{V}$. The kinetic energy operator has the form

$$T(k) \equiv M_0 = \sqrt{m_1^2 + k^2} + \sqrt{m_2^2 + k^2},$$

(1)

where $k$ is the relative momentum. The total momentum of the free-system $P$ is

$$P = p_1 + p_2,$$

(2)

and $\omega_m(p) = \sqrt{m^2 + p^2}$, $k = |k|$.

In this approach the bound system with the momentum $Q$, eigenvalues $E$ and spin $J$ is described by the wave function $\Phi^{J\mu}_{Q;\sigma_1\sigma_2}(k)$ of the two-particle state which satisfies the equation

$$\sum_{\lambda_1, \lambda_2} \int <k, \sigma_1, \sigma_2 \parallel \hat{V} \parallel k', \lambda_1, \lambda_2 > \Phi^{J\mu}_{Q;\lambda_1\lambda_2}(k') \, dk' = (E - M_0) \Phi^{J\mu}_{Q;\sigma_1\sigma_2}(k).$$

(3)

The radial equation for the two-particle bound state in the center-momentum system has the following form

$$\sum_{\ell, S, \ell', S'} \int_0^{\infty} V^{J}_{\ell, S; \ell', S'}(k, k') \Phi^{J\mu}_{\ell', S'}(k') k'^2 \, dk' = (M - M_0) \Phi^{J}_{\ell, S}(k).$$

(4)
The equation (4) can be obtained with the help of the Clebsh-Gordan coefficients of Poincaré group (see, [22]). The corresponding potential operator $V^J_{\ell, S', \ell, S}(k', k) = \langle k', J, \mu, \ell', S' \parallel \hat{V} \parallel k, J, \mu, \ell, S \rangle$ is obtained from the equation

$$V^J_{\ell, S', \ell, S}(k', k) = \frac{\sqrt{\binom{2\ell + 1}{2\ell' + 1}}}{2J + 1} \sum_{\lambda_1, \lambda_2, \lambda'_1, \lambda'_2} \mathbf{C} \left\{ \frac{1}{2}, \frac{1}{2}, S \right\} \mathbf{C} \left\{ \ell, S, J \right\} \times$$

$$\times \mathbf{C} \left\{ \ell', S', J' \right\} \mathbf{C} \left\{ \frac{1}{2}, \frac{1}{2}, S \right\} \langle k', J, \mu, \lambda_1, \lambda_2 \parallel \hat{V} \parallel k, J, \mu, \lambda_1, \lambda_2 \rangle,$$

(5)

The matrix element $V^J_{\lambda'_1, \lambda'_2, \lambda_1, \lambda_2}(k', k) = \langle k', J', \mu', \lambda'_1, \lambda'_2 \parallel \hat{V} \parallel k, J, \mu, \lambda_1, \lambda_2 \rangle$ is related with $\langle k', \lambda'_1, \lambda'_2 \parallel \hat{V} \parallel k, \lambda_1, \lambda_2 \rangle$ by the means of the Jacob-Wick decomposition (see, for example, [23]) and has the form

$$V^J_{\lambda'_1, \lambda'_2, \lambda_1, \lambda_2}(k', k) = \frac{\sqrt{\binom{2J + 1}{2J' + 1}}}{4\pi} \times$$

$$\times \int d^2k' d^2\hat{k}' D_{\mu', \lambda'}^{J'}(\varphi_{k'}, \theta_{k'}, -\varphi_{k'}) D_{\mu, \lambda}^{J}(\varphi_k, \theta_k, -\varphi_k) \times$$

$$\times \langle k', \lambda'_1, \lambda'_2 \parallel \hat{V} \parallel k, \lambda_1, \lambda_2 \rangle,$$

(6)

with $\lambda = (\lambda_1 - \lambda_2) / 2$. Functions $\mathbf{C} \left\{ s_1, s_2, S \right\}$, are Clebsh-Gordan coefficients of the $SU(2)$-group and the function $D_{\mu, \lambda}^{J}(\varphi_k, \theta_k, -\varphi_k)$ is the Wigner $D$-function with the angle of vector $\hat{k} = k / |k|$.

Using the Wigner-Eckart theorem we have the Eq.(6) transformed into

$$V^J_{\lambda'_1, \lambda'_2, \lambda_1, \lambda_2}(k', k) = \delta_{J,J'} \delta_{\mu, \mu'} V^J_{\lambda'_1, \lambda'_2, \lambda_1, \lambda_2}(k', k),$$

$$V^J_{\lambda'_1, \lambda'_2, \lambda_1, \lambda_2}(k', k) =$$

$$= \int_1^{2\pi} d(\cos \beta) \int_{-1}^{0} d\varphi D_{\lambda, \lambda'}^{J}(\varphi, \beta, -\varphi) \langle k', \lambda'_1, \lambda'_2 \parallel \hat{V} \parallel k, \lambda_1, \lambda_2 \rangle,$$

(7)

where

$$\cos \beta = \frac{(kk')}{(|k||k'|)} = \cos \theta_{k'} \cos \theta_k + \cos (\varphi_{k'} - \varphi_k) \sin \theta_{k'} \sin \theta_k.$$  

(8)

III. RELATIVISTIC POTENTIAL OF HYDROGEN-LIKE SYSTEMS

The interaction potential is constructed with the help of the scattering amplitude according to the below prescription [12]:
1.: Compute the scattering amplitude $R_{fi}$, which is defined in terms of the $S$-matrix element by the decomposition

$$S_{fi} = \delta_{fi} - i (2\pi) \delta (E_f - E_i) R_{fi},$$

(9)

where $i$ and $f$ denote initial and final states respectively.

2.: The potential $\hat{V}$ can be extracted with the help of the relation

$$R_{fi} = \langle f|\hat{V}|i \rangle + \sum_{n \neq i} \frac{\langle f|\hat{V}|n \rangle R_{ni}}{E_i - E_n + i\epsilon}.$$  

(10)

or the operator form

$$\hat{V} = \frac{R}{1 + G_{\text{red}} R},$$

(11)

where $G_{\text{red}}$ is the reduced Green function

$$G_{\text{red}} = \sum_{n \neq i}^{\infty} \frac{|n\rangle \langle n|}{E_i - E_n + i\epsilon}.$$  

(12)

Therefore, by investigating the corresponding scattering problem of bound-state constituents the potential (or a part of the potential) may be derived according to the above-mentioned procedure. Let us illustrate the recipe of a potential calculation by applying it to the electron-proton bound system (Hydrogen atom).

We start with the process

$$e^{-}(k_1, \lambda_{k_1}) + p(k_2, \lambda_{k_2}) \rightarrow e^{-}(p_1, \lambda_{p_1}) + p(p_2, \lambda_{p_2}),$$

(13)

where the momenta of the particles and spin numbers ($\lambda_{k_1} = \pm 1, \lambda_{p_1} = \pm 1$) are given between the parentheses.

According to the perturbation theory, the matrix element of the fermion-fermion system potential will be represented by a series of matrix elements with respect to the fine structure constant $\alpha$, where the main contribution determines the one-photon exchange between $e^-$ and $p$.

The initial approximation of the potential $V$ for a bound system was selected in the form of the potential which corresponds to the tree-level diagram, as depicted in Fig. [1].
Using the Feynman rules we write down the matrix element corresponding to the one-photon exchange diagram 1:

\[
< k', \lambda_{p_1}, \lambda_{p_2} \parallel \hat{V} \parallel k, \lambda_{k_1}, \lambda_{k_2} >= V_{1\gamma}(k, k') = (-1) N_{k,k'} \frac{Z \alpha}{8\pi^2 q^2} j^\mu_{\lambda_{p_1}, \lambda_{k_1}} (p_1, k_1) D_{\mu\nu} (q) j^\nu_{\lambda_{p_2}, \lambda_{k_2}} (p_2, k_2) ,
\]

where

\[
j^\mu_{\lambda_{p_i}, \lambda_{k_i}} (p_i, k_i) = \bar{u}_{\lambda_{p_i}} (p_i) \gamma^\mu u_{\lambda_{k_i}} (k_i) \ (i = 1, 2) .
\]

Four-momenta of particles in the center of mass frame have components

\[
k_1 = (\omega_{m_1} (k), \ k) \ , \ p_1 = (\omega_{m_1} (k'), \ k') ,
\]

\[
k_2 = (\omega_{m_2} (k), -k) \ , \ p_2 = (\omega_{m_2} (k'), -k') .
\]

The parameter \( Z \) specifies the value of the electric charge of the second fermion (for a hydrogen atom \( Z = 1 \)), while

\[
N_{k,k'} = 1/\sqrt{\omega_{m_1} (k) \omega_{m_1} (k') \omega_{m_2} (k) \omega_{m_2} (k')} .
\]

The function \( D_{\mu\nu} (q) \) is associated with the photon propagator (without \( q^2 \) and normalization factors)

\[
D_{\mu\nu} (q) = \left( g_{\mu\nu} - \frac{q_{\mu} q_{\nu}}{q^2} + \xi A \frac{q_{\mu} q_{\nu}}{q^2} \right) ,
\]
where $\xi_A$ is the gauge parameter and $q$ is the photon momentum in the center of mass system:

$$q = \{ q_0 = 0, \ k - k' \} .$$

For the potential with $V_1 \gamma \left( \mathbf{k}, \mathbf{k}' \right)$ it must be modified in order to satisfy the gauge invariance condition. The current conservation requirement (see [11]) leads to the fact that

$$q_{\mu} \, j^\mu_{\lambda_{p_1}, \lambda_{k_1}} (p_1, k_1) = q_{\mu} j^\mu_{\lambda_{p_2}, \lambda_{k_2}} (p_2, k_2) = 0 .$$

(18)

However, as follows from the definition of (15), the relations

$$q_{1\mu} \, j^\mu_{\lambda_{p_1}, \lambda_{k_1}} (p_1, k_1) = 0 , \quad q_{2\mu} \, j^\mu_{\lambda_{p_2}, \lambda_{k_2}} (p_2, k_2) = 0 ,$$

with

$$q_1 = \{ \omega_{m_1} (k) - \omega_{m_1} (k') , \ k - k' \} , \quad q_2 = \{ \omega_{m_2} (k') - \omega_{m_2} (k) , \ k - k' \} ,$$

are satisfied, but not the requirement (19).

The reason for this is the difference between the components $q_0 = 0, q_{10}$ and $q_{20}$. This difference is due to the need to take into account the effects of the virtuality of particles of the linked system [11]. Therefore, the relation $k = k'$ which follows from the conservation laws of the 4-momentum in the case of elastic scattering cannot be applied to eliminate the mismatch of the zero components of the 4-vectors $q$ and $q_1, q_2$.

The requirement of gauge invariance (19) can be met by modifying the currents $j^\mu_{\lambda_{p_1}, \lambda_{k_1}} (p_1, k_1)$ by overriding them (see, for example, [24]):

$$j^\mu_{\lambda_{p_1}, \lambda_{k_1}} (p_i, k_i) \rightarrow j^\mu_{\lambda_{p_1}, \lambda_{k_1}} (i) = \left( g^\mu_{\nu} - \frac{q^\mu q_\nu}{q^2} \right) j^\nu_{\lambda_{p_1}, \lambda_{k_1}} (p_i, k_i) .$$

(20)

As a result, the potential of one-boson exchange for the Feynman gauge of photon, taking into account the conservation of currents, will be written in the form

$$V_1 \gamma \left( \mathbf{k}, \mathbf{k}' \right) = N_{k,k'} \frac{Z \alpha}{8 \pi^2} \frac{1}{q^2} \, j^\mu_{\lambda_{p_1}, \lambda_{k_1}} (1) \left( - \frac{1}{q^2} \frac{q^\mu q_\nu}{q^2} \right) j^\nu_{\lambda_{p_2}, \lambda_{k_2}} (2) .$$

(21)

(22)

Taking into account the contributions of higher-order diagrams in the interaction constant $\alpha$ (vacuum polarization and photon exchange between electrons) we then obtain the following expression for the potential

$$< \mathbf{k}', \lambda_{p_1}, \lambda_{p_2} \parallel \hat{V} \parallel \mathbf{k}, \lambda_{k_1}, \lambda_{k_2} >= -N_{k,k'} \frac{Z \alpha}{8 \pi^2} \frac{\Pi (\alpha, q^2)}{q^2} \times$$

$$\times j^\mu_{\lambda_{p_1}, \lambda_{k_1}} (p_1, k_1) \left( g^\mu_{\rho} - \frac{q^\mu q_\rho}{q^2} \right) j^\rho_{\lambda_{p_2}, \lambda_{k_2}} (p_2, k_2) .$$

(23)
where single-particle fermion currents are written in the form:

\[
J^\mu_{\lambda_1, \lambda_1} (p_1, k_1) = \bar{u}_{\lambda_1} (p_1) \left( F_1^e (q_1^2) \gamma^\mu + \frac{F_2^e (q_1^2)}{2 m_1} i \sigma^{\mu \nu} q_{1, \nu} \right) u_{\lambda_1} (k_1) , \tag{25}
\]

\[
J^\mu_{\lambda_2, \lambda_2} (p_2, k_2) = \bar{u}_{\lambda_2} (p_2) \left( F_1^p (q_2^2) \gamma^\mu + \frac{F_2^p (q_2^2)}{2 m_2} i \sigma^{\mu \tau} q_{2, \tau} \right) u_{\lambda_2} (k_2) . \tag{26}
\]

In equations (25) and (26), the function \( \Pi (\alpha, q^2) \) determines the contribution from the vacuum polarization due to fermionic loops, and \( F_{1,2}^{e,p} (q^2) \) are the form factors of the electron and proton. Explicit forms \( \Pi (\alpha, q^2) \) and \( F_{1,2}^{e,p} (q^2) \) can be found in e.g. [10, 25].

Proton electromagnetic form factors \( F_{1,2}^p (q^2) \) are related to Sachs form factors by means of the equation

\[
G_E^p (q^2) = F_1^p (q^2) + \frac{q^2}{4 m_p^2} F_2^p (q^2) , \quad G_M^p (q^2) = F_1^p (q^2) + F_2^p (q^2) . \tag{27}
\]

The square of transfer momentum \( q_{1,2}^2 \)

\[
q_1^2 = (p_1 - k_1)^2 , \quad q_2^2 = (k_2 - p_2)^2
\]

can be represented by

\[
q_{1,2}^2 = (1 + \delta q_{1,2}^2) q^2 , \quad \text{where}
\]

\[
|\delta q_{1,2}^2| = \left| \frac{(\omega_{m_{1,2}} (k) - \omega_{m_{1,2}} (k'))}{q^2} \right| < 1 . \tag{30}
\]

Hence the contributions \( \sim \delta q_{1,2}^2 \) can be considered as corrections for the leading contribution (see, [26]) where

\[
q_{1,2}^2 = q^2 . \tag{31}
\]

**IV. CALCULATION OF THE SPINOR PART OF THE POTENTIAL**

To simplify the calculations, we transform the currents [26] using the Gordon identity. Then, the first part of the potential [24] that is proportional to the scalar product of fermion
currents is reduced to the sum:

\[ V^{(A)}_{\lambda p_1, \lambda p_2, \lambda k_1, \lambda k_2} (k', k) = - \frac{Z \alpha}{8 \pi^2 q^2} N_{k,k'} \left\{ K^{(I)} (q^2) \times \right. \]

\[ \times \bar{u}_{\lambda p_1} (p_1) \gamma_\mu u_{\lambda k_1} (k_1) \bar{u}_{\lambda p_2} (p_2) \gamma_\mu u_{\lambda k_2} (k_2) + \frac{K^{(IV)} (q^2)}{4 m_1 m_2} \times \]

\[ \times (p_1 + k_1) \nu (p_2 + k_2) \nu \bar{u}_{\lambda p_1} (p_1) u_{\lambda k_1} (k_1) \bar{u}_{\lambda p_2} (p_2) u_{\lambda k_2} (k_2) - \]

\[ - \frac{K^{(II)} (q^2)}{2 m_1} \bar{u}_{\lambda p_1} (p_1) u_{\lambda k_1} (k_1) \bar{u}_{\lambda p_2} (p_2) \left( \hat{p}_1 + \hat{k}_1 \right) u_{\lambda k_2} (k_2) - \]

\[ - \frac{K^{(III)} (q^2)}{2 m_2} \bar{u}_{\lambda p_1} (p_1) \left( \hat{p}_2 + \hat{k}_2 \right) u_{\lambda k_1} (k_1) \bar{u}_{\lambda p_2} (p_2) u_{\lambda k_2} (k_2) \right\} \]  \( (32) \)

with functions depending on \( q^2 = -(k - k')^2 \) and \( k, k' \):

\[ K^{(I,II)} (q^2) = \Pi (\alpha, q^2) G_M^e (q^2) \left\{ G_M^e (q^2) , F_2^e (q^2) \right\} , \]  \( (33) \)

\[ K^{(III,IV)} (q^2) = \Pi (\alpha, q^2) F_2^e (q^2) \left\{ G_M^e (q^2) , F_2^e (q^2) \right\} , \]  \( (34) \)

and the magnetic form factor of fermions \( G_M^e (q^2) \).

To convert spinor structures into explicitly scalar functions we use the method of basis spinors \( \text{(MBS)} \) \( [27, 28] \). A detailed description of the \( \text{MBS} \) elements can be found in the appendix \( A \).

With its help fermionic “chains” with the operator \( \gamma_\mu \) are represented as:

\[ \bar{u}_{\lambda p} (p, s_p) \gamma_\mu u_{\lambda k} (k, s_k) = \sum_{\sigma, \rho = -1}^{1} \sum_{A, C = -1}^{1} \tilde{s}_{s p, \sigma}^{(C)} (p) \Gamma_{\sigma, \rho}^{C, A} [\gamma_\mu] s_{s, \lambda k}^{(A)} (k) , \]  \( (35) \)

where the expansion coefficients in basis spinors \( s_{s, \lambda k}^{(A)} (k) \) for helicity states are defined by the relations

\[ s_{s, \lambda k}^{(A)} (p) = \bar{u}_p (b_A) u_{\lambda k} (p) = - \lambda \tilde{W}_{m p} (-\lambda \rho p) D^{1/2}_{A\rho/2, -\lambda/2} (\varphi_p , \theta_p , -\varphi_p) \]  \( (36) \)

with

\[ \tilde{W}_{m p} (\pm p) = \sqrt{\omega_{m p} (p) \pm p} . \]  \( (37) \)

The \( \Gamma \) function is calculated in terms of the 4-vectors of the isotropic tetrad \( b_A \) and \( n_\lambda \ (A, \lambda = \pm 1) \) using the basic relations \( \text{MBS} \) (see appendix \( A \)):

\[ \Gamma_{\rho, \sigma}^{C, A} [\gamma_\mu] = 2 \delta_{\sigma, -\rho} (\delta_{C, -A} b^\rho_{-A} + A \delta_{C, A} n^\mu_{-A \times \rho}) . \]  \( (38) \)
Since the scalar products of the isotropic tetrad vectors satisfy the relations

\[(b_{\rho}b_{-\lambda}) = \delta_{\lambda, \rho}/2, \quad (n_{\lambda}n_{-\rho}) = \delta_{\lambda, \rho}/2, \quad (b_{\rho}n_{\lambda}) = 0,\]

(39)

using the expansion coefficients (36), we find the product of fermionic currents in Eq. (32) in the form

\[N_{k,k'} \bar{u}_{\lambda_{p_1}} (p_1) \gamma^{\mu} u_{\lambda_{k_1}} (k_1) \bar{u}_{\lambda_{p_2}} (p_2) \gamma_{\mu} u_{\lambda_{k_2}} (k_2) =
\]

\[= 2 \sum_{\sigma, \rho = -1}^{1} \sqrt{(1 - \sigma \lambda_{k_1} u_{k_1})(1 - \rho \lambda_{k_2} u_{k_2})} \sqrt{(1 - \sigma \lambda_{p_1} u_{p_1})(1 - \rho \lambda_{p_2} u_{p_2})} \times
\]

\[\times \left[ \delta_{\lambda_{k_1}, \lambda_{k_2}} \rho \sigma D^{*1/2}_{\lambda_{k_1}, \lambda_{p_2}/2}(\varphi, \beta, -\varphi) D^{*1/2}_{\lambda_{p_1}, \lambda_{k_2}/2}(\varphi, \beta, -\varphi) +
\]

\[+ \delta_{\rho \lambda_{k_1}, \sigma \lambda_{k_2}} D^{*1/2}_{\lambda_{k_1}, \lambda_{p_2}/2}(\varphi, \beta, -\varphi) D^{*1/2}_{\lambda_{p_1}, \lambda_{k_2}/2}(\varphi, \beta, -\varphi) \right].\]

(40)

The following notation is introduced in the equation (40):

\[v_{k_1} = \frac{k}{\omega_{m_1}(k)}, \quad v_{p_1} = \frac{k'}{\omega_{m_1}(k')}, \quad v_{k_2} = \frac{k}{\omega_{m_2}(k)}, \quad v_{p_2} = \frac{k'}{\omega_{m_2}(k')}.\]

(41)

Further, according to (7), for the integration over angular variables the spinor structures of the potential \(\text{ref vector1}\) multiplied by the function \(D_{\lambda,\lambda'}(\varphi, \beta, -\varphi)\) with \(\lambda = (\lambda_{k_1} - \lambda_{k_2})/2\) and \(\lambda' = (\lambda_{p_1} - \lambda_{p_2})/2\).

The representation of the spinor part in the form (40) and the Clebsch-Gordan expansion for \(D\)-matrices allow us to write the formula (7) into the integrand, as a linear combination of the Legendre polynomials \(P_{\ell} (\cos \beta)\), and thus separate the angular variables and the \(k = |k|, k' = |k'|\). This construction greatly simplifies the next stage of integration. To shorten the notation of the calculated structures, we introduce auxiliary functions

\[G_{\lambda_{k_1}, \lambda_{k_2}; \lambda_{p_1}, \lambda_{p_2}}^{J_{\lambda_{k_1}, \lambda_{k_2}}} [\Phi (x)] =
\]

\[= \sum_{s_1 + s_2 = J_{\lambda_{k_1}, \lambda_{k_2}}}^{J_{\lambda_{p_1}, \lambda_{p_2}}} \sum_{s = |s_1 - s_2|}^{s_1 + s_2} \frac{(2\ell + 1)}{(2J + 1)} \mathcal{C} \left\{ \lambda_{k_1}/2, -\lambda_{k_2}/2, \lambda \right\} \mathcal{C} \left\{ \ell_{0}, \lambda, \lambda \right\} \times
\]

\[\times \mathcal{C} \left\{ \lambda_{p_1}/2, -\lambda_{p_2}/2, \lambda' \right\} \mathcal{C} \left\{ \ell_{0}, \lambda', \lambda' \right\} \Phi_{\ell} (x)
\]

(42)

and

\[W_{\lambda, \rho} (k) = \sqrt{1 + \lambda v_{k_1} \sqrt{1 + \rho v_{k_2}}}, \quad W_{\lambda, \rho} (k') = \sqrt{1 + \lambda v_{p_1} \sqrt{1 + \rho v_{p_2}}}.\]
Other spinor structures of the potential (32) have a similar (40) structure. Let us present the final expressions for the spinor structures multiplied by the function $D^J_{\lambda', \lambda} (\varphi, \beta, -\varphi)$:

$$D^J_{\lambda', \lambda} (\varphi, \beta, -\varphi) \ N_{k, k'} \bar{u}_{\lambda p_1} (p_1) \gamma^\mu u_{\lambda k_1} (k_1) \bar{u}_{\lambda p_2} (p_2) \gamma_\mu u_{\lambda k_2} (k_2) =$$

$$= 2 \sum_{\sigma, \rho = -1}^{1} W_{-\sigma \lambda k_1, -\rho \lambda k_2} (k) W_{-\sigma \lambda p_1, -\rho \lambda p_2} (k') \times$$

$$\times \left[ \delta_{k_1, k_2} \rho \sigma G^J_{-\lambda k_1, -\lambda k_1; \lambda p_1, \lambda p_2} [P_\ell (x)] + \delta_{\rho \lambda k_1, \sigma \lambda k_2} G^{J,1/2,1/2}_{\lambda k_1, \lambda k_1; \lambda p_1, \lambda p_2} [P_\ell (x)] \right], \quad (43)$$

$$D^J_{\lambda', \lambda} (\varphi, \beta, -\varphi) \ N_{k, k'} \bar{u}_{\lambda p_1} (p_1) u_{\lambda k_1} (k_1) \bar{u}_{\lambda p_2} (p_2) \left( \hat{p}_1 + \hat{k}_1 \right) u_{\lambda k_2} (k_2) =$$

$$= \sum_{\sigma, \rho = -1}^{1} W_{-\sigma \lambda k_1, -\rho \lambda k_2} (k) W_{\sigma \lambda p_1, -\rho \lambda p_2} (k') \times$$

$$\times \left[ k' \rho \lambda k_2 \left( 3 G^{J,1/2,1/2}_{\lambda k_1, \lambda k_2; \lambda p_1, \lambda p_2} [x P_\ell (x)] - 2 G^{J,1/2,3/2}_{\lambda k_1, \lambda k_2; \lambda p_1, \lambda p_2} [P_\ell (x)] \right) +$$

$$+ G^{J,1/2,1/2}_{\lambda k_1, \lambda k_2; \lambda p_1, \lambda p_2} [P_\ell (x)] \left\{ \rho (3 \lambda k_2 k - 2 \lambda p_2 k') - 3 \left( \omega m_1 (k) + \omega m_1 (k') \right) \right\} \right], \quad (44)$$

$$D^J_{\lambda', \lambda} (\varphi, \beta, -\varphi) \ N_{k, k'} \bar{u}_{\lambda p_1} (p_1) \left( \hat{p}_2 + \hat{k}_2 \right) u_{\lambda k_1} (k_1) \bar{u}_{\lambda p_2} (p_2) u_{\lambda k_2} (k_2) =$$

$$= \sum_{\sigma, \rho = -1}^{1} W_{-\sigma \lambda k_1, -\rho \lambda k_2} (k) W_{-\sigma \lambda p_1, \rho \lambda p_2} (k') \times$$

$$\times \left[ k' \sigma \lambda k_1 \left( 3 G^{J,1/2,1/2}_{\lambda k_1, \lambda k_2; \lambda p_1, \lambda p_2} [x P_\ell (x)] - 2 G^{J,3/2,1/2}_{\lambda k_1, \lambda k_2; \lambda p_1, \lambda p_2} [P_\ell (x)] \right) +$$

$$+ G^{J,1/2,1/2}_{\lambda k_1, \lambda k_2; \lambda p_1, \lambda p_2} [P_\ell (x)] \left\{ \sigma (3 \lambda k_2 k - 2 \lambda p_1 k') - 3 \left( \omega m_2 (k) + \omega m_2 (k') \right) \right\} \right], \quad (45)$$

where $x = \cos \beta$. 
The second part of the potential, using the Eqs. (18), (20) and (21), is converted into a product that contains zero current components:

\[
V^{(B)}_{\lambda_1, \lambda_2, \lambda_1, \lambda_2} (k', k) = \frac{Z \alpha \Pi (\alpha, q^2)}{8 \pi^2 q^4} (\omega_{m_1} (k') - \omega_{m_1} (k)) \times \\
(\omega_{m_2} (k') - \omega_{m_2} (k)) N_{k, k'} J^{(0)}_{\lambda_1, \lambda_2} (p_1, k_1) J^{(0)}_{\lambda_2, \lambda_2} (p_2, k_2) .
\]

(47)

Applying the method of basis spinors for calculating the spinor part (47) and the method described above, we find that

\[
N_{k, k'} D_{\lambda, \lambda'}^J (\varphi, \beta, -\varphi) N_{k, k'} J^{(0)}_{\lambda_1, \lambda_1} (p_1, k_1) J^{(0)}_{\lambda_2, \lambda_2} (p_2, k_2) = \\
\frac{1}{4m_1 m_2} \sum_{\sigma, \rho = -1} W_{-\sigma \lambda_1, -\rho \lambda_2} (k) G^{J_{1/2, 1/2}}_{\lambda_1, \lambda_2, \lambda_1, \lambda_2} [p_\ell (x)] \times \\
(\omega_{m_1} (k) + \omega_{m_1} (k')) - 2m_2 K^{(II)} (\tilde{q}^2) W_{-\sigma \lambda_1, -\rho \lambda_2} (k') \times \\
(\omega_{m_2} (k') + \omega_{m_2} (k)) + K^{(IV)} (\tilde{q}^2) W_{-\sigma \lambda_1, -\rho \lambda_2} (k') \times \\
(\omega_{m_2} (k') + \omega_{m_2} (k)) \right],
\]

(48)

where the functions \( K (\tilde{q}^2) \) are defined by equations (33)-(34).

V. RADIAL EQUATION KERNEL STRUCTURE

The main characteristic of our calculation method is to use the momentum space and accurate relativistic evaluation of radial kernel \( V_{\ell, S; \ell', S'} (k, k') \). After accurate analytic calculation of potential spinor part (21) with the help of the method of basis spinors radial kernel of the relativistic fermion-fermion system \( V_{\ell, S; \ell', S} (k', k) \) with arbitrary angular momentum \( J \) and spin total momentum \( S = 0, 1 \) is obtained using the equation (27).

\[
V^J_{\ell', S'; \ell, S} (k', k) = \frac{\sqrt{(2 \ell + 1) (2 \ell' + 1)}}{2J + 1} (-1)^J \frac{Z \alpha}{4 \pi} \sum_{\lambda_1, \lambda_2 = -1} C \left\{ \frac{1/2}{\lambda_1, 2}, \frac{1/2}{-\lambda_2, 2}, S \right\} \times \\
C \left\{ \frac{0}{\ell}, \frac{0}{S}, \frac{J}{J} \right\} C \left\{ \frac{1/2}{\lambda_1, 2}, \frac{1/2}{-\lambda_2, 2}, S \right\} \left( V^I_{\lambda_1, \lambda_2, \lambda_1, \lambda_2} + V^{II}_{\lambda_1, \lambda_2, \lambda_1, \lambda_2} + \\
V^{III}_{\lambda_1, \lambda_2, \lambda_1, \lambda_2} + V^{IV}_{\lambda_1, \lambda_2, \lambda_1, \lambda_2} + V^{(B)}_{\lambda_1, \lambda_2, \lambda_1, \lambda_2} \right),
\]

(49)
where

\[
V_{\lambda k_1, \lambda k_2, \lambda p_1, \lambda p_2}^I = \frac{1}{2} \sum_{\sigma, \rho=-1} W_{-\sigma \lambda k_1, -\rho \lambda k_2} (k) W_{-\sigma \lambda p_1, -\rho \lambda p_2} (k') \left[ \delta_{\lambda k_1, \lambda k_2} \rho \sigma \times G_{\lambda k_1, \lambda k_2, \lambda p_1, \lambda p_2}^{I/2,1/2} \left[ \tilde{R}_\ell (I) (k, k') \right] \right] + \delta_{\rho \lambda k_1, \sigma \lambda k_2} G_{\lambda k_1, \lambda k_2, \lambda p_1, \lambda p_2}^{I/2,1/2} \left[ \tilde{R}_\ell (I) (k, k') \right],
\]

(50)

\[
V_{\lambda k_1, \lambda k_2, \lambda p_1, \lambda p_2}^\prime I = -\frac{1}{2 m_1} \sum_{\sigma, \rho=-1} W_{-\sigma \lambda k_1, -\rho \lambda k_2} (k) W_{\sigma \lambda p_1, -\rho \lambda p_2} (k') \left[ \left( 3 G_{\lambda k_1, \lambda k_2, \lambda p_1, \lambda p_2}^{I/2,1/2} \left[ \tilde{Z}_{\lambda}^{(I)} (k', k) \right] - 2 G_{\lambda k_1, \lambda k_2, \lambda p_1, \lambda p_2}^{I/2,3/2} \left[ \tilde{R}_{\lambda}^{(I)} (k', k) \right] \right) \right] +
\]

(51)

\[
V_{\lambda k_1, \lambda k_2, \lambda p_1, \lambda p_2}^\prime III = -\frac{1}{2 m_2} \sum_{\sigma, \rho=-1} W_{-\sigma \lambda k_1, -\rho \lambda k_2} (k) W_{-\sigma \lambda p_1, -\rho \lambda p_2} (k') \left[ \left( 3 G_{\lambda k_1, \lambda k_2, \lambda p_1, \lambda p_2}^{I/2,1/2} \left[ \tilde{Z}_{\lambda}^{(III)} (k', k) \right] - 2 G_{\lambda k_1, \lambda k_2, \lambda p_1, \lambda p_2}^{I/2,3/2} \left[ \tilde{R}_{\lambda}^{(III)} (k', k) \right] \right) \right] +
\]

(52)

\[
V_{\lambda k_1, \lambda k_2, \lambda p_1, \lambda p_2}^IV = \frac{1}{4 m_1 m_2} \sum_{\sigma, \rho=-1} W_{-\sigma \lambda k_1, -\rho \lambda k_2} (k) W_{\sigma \lambda p_1, -\rho \lambda p_2} (k') \times
\]

(53)

\[
W_{\lambda, \rho} (k) = \sqrt{1 + k' v_{k_1} \sqrt{1 + \rho v_{k_2}}} \cdot W_{\lambda, \rho} (k') = \sqrt{1 + k v_{p_1} \sqrt{1 + \rho v_{p_2}}}.
\]

(54)

with

\[
u_{k_1} = \frac{k}{\omega_{m_1} (k)}, \quad v_{p_1} = \frac{k'}{\omega_{m_1} (k')}, \quad v_{k_2} = \frac{k}{\omega_{m_2} (k)}, \quad v_{p_2} = \frac{k'}{\omega_{m_2} (k')}.
\]

(55)
The analytic expression of the last potential part is determined by

\[ V_{\lambda_1, \lambda_2, \lambda_{p1}, \lambda_{p2}}^B = \sum_{\sigma, \rho = -1}^1 W_{-\sigma \lambda_1, -\rho \lambda_2} (k) \left[ G_{\lambda_1, \lambda_2; \lambda_{p1}, \lambda_{p2}}^{J,1/2,1/2} \left[ \tilde{U}^{(I)} (k', k) \right] \times \right. \]
\[ \times W_{-\sigma \lambda_1, -\rho \lambda_2} (k') - \frac{1}{2m_2} G_{\lambda_{k_1}, \lambda_{k_2}; \lambda_{p1}, \lambda_{p2}}^{J,1/2,1/2} \left[ \tilde{U}^{(II)} (k', k) \right] \times \]
\[ \left. \times W_{-\sigma \lambda_1, -\rho \lambda_2} (k') (\omega_{m_2} (k') + \omega_{m_2} (k)) \right]. \]

(56)

The functions \( \tilde{R}_\ell (k', k) \) and \( \tilde{U}_\ell (k', k) \) in equations (50)-(56) are represented as the integrals

\[ \tilde{R}_\ell (k', k) = \int_{-1}^1 \frac{K(q^2) P_\ell (x)}{q^2} \, dx , \]

(57)

\[ \tilde{U}_\ell (k', k) = g_{12} (k', k) \int_{-1}^1 \frac{K(q^2) P_\ell (x)}{q^4} \, dx , \]

(58)

where the dimension factor \( g_{12} (k', k) \) is

\[ g_{12} (k', k) = (\omega_{m_1} (k') - \omega_{m_1} (k)) (\omega_{m_2} (k') - \omega_{m_2} (k)) \]

(59)

and

\[ q^2 = -2k k' (y - x) , \]

(60)

\[ y = \frac{k^2 + k'^2}{2k k'} . \]

(61)

In the case when \( m_1 = m_2 \) the Lorentz structures \( \gamma^\mu \otimes \gamma_\mu \) and \( I \otimes I \) of the potential (49) coincide with similar structures obtained in \[23\].

Expanding in fermion velocities, it can be shown that the potential (49) transforms into both the nonrelativistic Schrödinger equation and the Breit equation in the momentum representation.

The potential in the form (49) with the terms (50)-(56) allows one to estimate the contributions of both the proton structure and the higher order electromagnetic corrections. To calculate this or that correction it is enough to define the explicit form of the function \( \tilde{R}_\ell \) and \( \tilde{U}_\ell \), while the rest of the structure remains unchanged.
VI. SUMMARY

In this model we have the gauge invariant effective potential and the exact calculation of the relativistic kernel of the two-fermion equation with electromagnetic interaction that was performed using the method of basis spinors.

The resulting kernel of the radial equation (4) for an arbitrary total angular momentum $J$ (total spin momentum $S = 0, 1$) automatically takes into account recoil effects and allows one to take into account higher-order relativistic effects caused by the motion of fermions when calculating the energy contributions.

The proposed technique can also be applied to build the potential of one-gluon exchange without significant additional calculations.
Appendix A: The Method of Basic Spinors

When evaluating a Feynman amplitude involving fermions, the amplitude is expressed as
sum of terms which have the form
\[ \mathcal{M}_{\lambda_p, \lambda_k} (p, s_p, k, s_k ; Q) = \mathcal{M}_{\lambda_p, \lambda_k} ([p], [k] ; Q) = \]
\[ = \bar{w}_{\lambda_p}^A (p, s_p) Q w_{\lambda_k}^B (k, s_k) , \]  
(A1)
where \( \lambda_p \) and \( \lambda_k \) are spin indices of the external fermions with four-momenta \( p, k \) and arbitrary
polarization vectors \( s_p, s_k \). The operator \( Q \) is a sum of products of Dirac \( \gamma \)-matrices. The
notation \( w_{\lambda_p}^A (p, s_p) \) stands for either \( u_{\lambda_p} (p, s_p) \) (bispinor of fermion; \( A = +1 \)) or \( \nu_{\lambda_p} (p, s_p) \)
(bispinor of antifermion; \( A = -1 \)).

The main aim of the calculation is to transform (A1) into and explicitly scalar form (scalar
products of four-vectors, Lorentz tensors, and so on). The main approach which has gained
popularity in the past decades is to calculate Feynman amplitudes directly. Many different
methods of calculating reaction amplitudes with fermions have been developed [29–32] et.al. In
the paper we describe an approach to Feynman diagrams which is based on the utilization of an
isotropic tetrad in the Minkowski space and massless basis spinors connected with it and which
we call the Method of Basis Spinors (MBS) [27, 33]) Let us briefly describe the main relationships
of the MBS.

1. The isotropic tetrad and massless basis spinors

Let us introduce the orthonormal four-vector basis in the Minkowski space which satisfies
the relations:
\[ l_0^\mu l_0^\nu - \sum_{j=1}^{3} l_j^\mu l_j^\nu = g^{\mu\nu} , \quad (l_A \cdot l_B) = g_{AB} , \]  
(A2)
where \( g \) is the Lorentz metric tensor.

With the help of vectors \( l_A \) we can define lightlike vectors which form the isotropic tetrad
in the Minkowski space
\[ b_\rho = (l_0 + \rho l_3)/2 , \quad n_\lambda = (\lambda l_1 + i l_2)/2 , \quad (\lambda, \rho = \pm 1) . \]  
(A3)
From Eqs. (A2), (A3) it follows that

\[ (b_\rho \cdot b_{-\lambda}) = (n_\rho \cdot n_{-\lambda}) = \frac{\delta_{\lambda,\rho}}{2}, \quad (b_\rho \cdot n_\lambda) = 0, \]

(A4)

\[
 g^{\mu\nu} = \sum_{\lambda=-1}^{1} \left[ \tilde{b}_\lambda^\mu \cdot b_{-\lambda}^\nu + \tilde{n}_\lambda^\mu \cdot n_{-\lambda}^\nu \right],
\]

(A5)

\[
 \tilde{b}_\lambda^\mu = 2 b_\lambda^\mu, \quad \tilde{n}_\lambda^\mu = 2 n_\lambda^\mu.
\]

(A6)

It is always possible to construct the basis of the isotropic tetrad (A3) as numerical four-vectors

\[ (b_{\pm 1})_\mu = (1/2) \{1, 0, 0, \pm 1\}, \quad (n_{\pm 1})_\mu = (1/2) \{0, \pm 1, i, 0\} \]

(A7)

or by means of physical vectors for reaction.

By means of the isotropic tetrad (A3) we define basis spinors \( u_\lambda (b_{-1}) \) and \( u_\lambda (b_{1}) \):

\[ \slashed{b}_{-1} u_\lambda (b_{-1}) = 0, \quad u_\lambda (b_{1}) \equiv \slashed{b}_1 u_{-\lambda} (b_{-1}), \]

(A8)

\[ \omega_{\lambda} u_\lambda (b_A) = u_\lambda (b_A), \quad (A = \pm 1) \]

(A9)

with matrix \( \omega_\lambda = 1/2 (1 + \lambda \gamma_5) \) and normalization condition

\[ u_\lambda (b_A) \bar{u}_\lambda (b_A) = \omega_\lambda \slashed{b}_A. \]

(A10)

The relative phase between basis spinors with different helicity is given by

\[ \varphi_{\lambda \rho} (b_{-1}) = \delta_{\lambda,\rho} u_\lambda (b_{-1}). \]

(A11)

The important property of basis spinors (A8) is the completeness relation:

\[ \sum_{\lambda, A = -1}^{1} u_\lambda (b_A) \bar{u}_{-\lambda} (b_{-A}) = 1, \]

(A12)

which follows from Eqs.(A8)–A11). Thus, an arbitrary bispinor can be decomposed in terms of basis spinors \( u_\lambda (b_A) \).
2. Main equations of the MBS and Dirac spinors

An arbitrary Dirac spinor can be determined through the basis spinor \( \Lambda \) with the help of projection operators \( \tau_\lambda (p, s_p) = u_\lambda p (p, s_p) \bar{u}_\lambda p (p, s_p) \). Dirac spinors \( w^A_\lambda (p, s_p) \) for massive fermion and antifermion with four-momentum \( p \) (\( p^2 = m^2 \)), arbitrary polarization vector \( s_p \) and spin number \( \lambda = \pm 1 \) can be obtained with the help of basis spinors by means of equation:

\[
w^A_\lambda (p, s_p) = (A\lambda) \frac{(p + A m_p) (1 + \lambda \gamma_5 s_p)}{2\sqrt{(b_1 \cdot (p + m_p s_p))}} u_{-A \times \lambda} (b_1) \quad (A13)
\]

Spinor products of basis spinors are simple and similar to scalar products of isotropic tetrad vectors

\[
\bar{u}_\lambda (b_C) u_\rho (b_A) = \delta_{\lambda, -\rho} \delta_{C, -A} . \quad (A14)
\]

With the help of Eq.(A5) the Dirac matrix \( \gamma^\mu \) can be rewritten as

\[
\gamma^\mu = \sum_{\lambda = -1}^1 \left[ \gamma^- \tilde{b}_\lambda^\mu + \gamma^- \tilde{n}_\lambda^\mu \right] \quad (A15)
\]

and using Eqs.(A9), (A11) and (A15) we can obtain that

\[
\gamma^\mu u_\lambda (b_A) = \tilde{b}_A^\mu u_{-\lambda} (b_{-A}) - A \tilde{n}_{-A \times \lambda}^\mu u_{-\lambda} (b_A) , \quad (A16)
\]

which allows to transform the Dirac matrix into some combination of isotropic tetrad vectors in the basis spinor space and

\[
\gamma_5 u_\rho (b_A) = \rho u_\rho (b_A) . \quad (A17)
\]

Eqs. (A14), (A16) and (A17) underly the method of basis spinors (MBS).

3. The MBS and the technique of “building” blocks

The basic idea of the Method of Basis Spinors is to replace Dirac spinors in Eq.(A1) by massless basis spinors \( u_\lambda (b_{\pm 1}) \) (Eq.(A13)) and to use only three Eqs. (A14), (A16) and (A17) to calculate the matrix element (A1) in terms of scalar functions.

Let us consider an important type of the matrix element (A1) when \( p = b_{-C} \) and \( k = b_A \), i.e.

\[
\mathcal{M}_{\sigma, -\rho} (b_C, b_A; Q) \equiv \Gamma^C_{\sigma, A} [Q] = \bar{u}_\sigma (b_C) Q u_{-\rho} (b_{-A}) . \quad (A18)
\]
We call this type of matrix element as the **basic matrix element**. By means of MBS relations \((A14), (A16)\) and \((A17)\) it is easy to calculate \(\Gamma_{\sigma,\rho}^{C,A}\) in terms of isotropic tetrad vectors.

With the help of the completeness relation \((A12)\) the amplitude \((A1)\) can be expressed as combinations of the lower-order matrix elements ("building" blocks)

\[
M_{\lambda_\rho,\lambda_k} (p, s_p \, k, s_k; Q) = \sum_{A, C, \sigma, \rho = -1}^1 \left\{ \bar{w}_{\lambda_\rho}^D (p, s_p) \bar{u}_{-\sigma} (b_{-C}) \right\} \times \left\{ \bar{u}_{\sigma} (b_C) Q \bar{u}_{-\rho} (b_{-A}) \right\} \left\{ \bar{u}_\rho (b_A) \bar{w}_{\lambda_k}^F (k, s_k) \right\} = \sum_{\sigma, \rho = -1}^1 \sum_{A, C = -1}^1 \bar{s}_{\sigma, \lambda_\rho}^{(C, D)} (p, s_p) \Gamma_{\sigma,\rho}^{C,A} [Q] s_{\rho,\lambda_k}^{(A, F)} (k, s_k). \tag{A19}
\]

Decomposition coefficients for the helicity states of fermions can be easily calculated:

\[
s^{(A,D)}_{\rho,\lambda} (p, s_{\text{hel}}) = D\lambda \, W_m (-\lambda \rho Dp) f(\rho \lambda, D) D^{*1/2}_{\lambda\rho/2,-\lambda\rho/2} (\phi, \theta, -\phi) \tag{A20}
\]

where

\[
W_m(\pm p) = \sqrt{\omega_m(p) \pm p}, \quad \omega_m(p) = \sqrt{p^2 + m^2}, \quad p = |p |
\]

\[
f(A, D) = \delta_{A,-1} + D \delta_{A,1} \tag{A21}
\]

and \(D^{1/2}_{\sigma_1,\sigma_2}(\phi, \theta, -\varphi) = \exp(-i\phi) d^{1/2}_{\sigma_1,\sigma_2}(\theta) \exp(-i\varphi)\) is the Wigner function.
[1] M. I. Eides, H. Grotch, and V. A. Shelyuto, Phys. Rept. 342, 63 (2001), hep-ph/0002158.
[2] S. G. Karstenboim, Int. J. Mod. Phys. A19, 3879 (2004), hep-ph/0310099.
[3] S. G. Karstenboim et al., Nucl. Phys. Proc. Suppl. 162, 260 (2006), hep-ph/0608236.
[4] R. Pohl et al., Nature 466, 231 (2010).
[5] A. De Rujula, Phys. Lett. B693, 555 (2010), 1008.3861.
[6] U. Jentschura, Annals of Physics 326, 500 (2011), ISSN 0003-4916, 1011.5275, URL http://www.sciencedirect.com/science/article/pii/S0003491610002010.
[7] U. Jentschura, Annals of Physics 326, 516 (2011), ISSN 0003-4916, 1011.5453, URL http://www.sciencedirect.com/science/article/pii/S0003491610002009.
[8] A. Dorokhov, R. Faustov, A. Martynenko, and F. Martynenko (2020), 2010.07380.
[9] A. Matveev, C. G. Parthey, K. Predehl, J. Alnis, A. Beyer, R. Holzwarth, T. Udem, T. Wilken, N. Kolachevsky, M. Abgrall, et al., Phys. Rev. Lett. 110, 230801 (2013), URL https://link.aps.org/doi/10.1103/PhysRevLett.110.230801.
[10] A. I. Akhiezer and V. B. Berestetskii, *Quantum Electrodynamics* (Interscience Publishers, New York, 1965).
[11] H. M. Pilkuhn, *Relativistic Particle Physics* (Springer Berlin Heidelberg, New York, 1979), 1st ed.
[12] W. Lucha, H. Rupprecht, and F. F. Schoberl, Phys. Rev. D44, 242 (1991).
[13] V. O. Galkin, A. Y. Mishurov, and R. N. Faustov, Sov. J. Nucl. Phys. 55, 1207 (1992).
[14] H. W. Crater, C. W. Wong, and C.-Y. Wong, Int. J. Mod. Phys. E5, 589 (1996), hep-ph/9603402.
[15] A. G. Terekidi and J. W. Darewych, J. Math. Phys. 46, 032302 (2005), hep-ph/0311132.
[16] H. A. Bete and E. E. Salpeter, Phys. Rev. 84, 1232 (1951).
[17] E. E. Salpeter, Phys. Rev. 87, 328 (1952).
[18] R. N. Faustov, A. Karimkhodzhaev, and A. P. Martynenko, Phys. Atom. Nucl. 62, 2103 (1999), hep-ph/9808365.
[19] R. N. Faustov and A. P. Martynenko, Phys. Atom. Nucl. 61, 471 (1998), hep-ph/9709374.
[20] A. P. Martynenko (2006).
[21] W. N. Polyzou, Y. Huang, C. Elster, W. Glockle, J. Golak, R. Skibinski, H. Witala, and H. Ka-
[22] B. D. Keister and W. N. Polyzou, Adv. Nucl. Phys. 20, 225 (1991).

[23] G. E. Brown and A. D. Jackson, *The Nucleon-nucleon interaction* (North-Holland publishing, New York, 1976).

[24] W. H. Klink, Few Body Syst. 33, 99 (2003).

[25] J. D. Bjorken and S. Drell, *Relativistic Quantum Mechanics*, vol. 1 (McGraw-Hill, Berlin-Göttingen-Heidelberg, 1964).

[26] G. T. Bodwin and D. R. Yennie, Phys. Rev. D37, 498 (1988).

[27] V. V. Andreev, Physics of Atomic Nuclei 66, 383 (2003), ISSN 1063-7788, 10.1134/1.1553511, URL [http://dx.doi.org/10.1134/1.1553511](http://dx.doi.org/10.1134/1.1553511).

[28] V. Andreev, in *18th International Workshop on High-Energy Physics and Quantum Field Theory* (2004), pp. 148–153, hep-ph/0407055.

[29] E. Bellomo, Il Nuovo Cimento 21, 730 (1961).

[30] A. A. Bogush and F. I. Fedorov, Vesti AN BSSR ser. fiz.-m. n., 26 (1962), in Russian.

[31] R. Gastmans and T. T. Wu, *The Ubiquitous photon: Helicity method for QED and QCD* (Oxford, UK, 1990).

[32] S. Dittmaier, Phys. Rev. D59, 016007 (1999), hep-ph/9805445.

[33] V. V. Andreev, *Nonlinear phenomena in complex systems* 12, 338 (2009).