DESCRIPTONS OF THE CRYSTAL $B(\infty)$ FOR $G_2$

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Abstract. We study the crystal base of the negative part of a quantum group. Two explicit descriptions of the crystal $B(\infty)$ for types $G_2$ are given. The first is given in terms of extended Nakajima monomials and the second realization follows a similar result given for other finite types by Cliff.

1. Introduction

Quantum group $U_q(g)$ is a $q$-deformation of the universal enveloping algebra over a Lie algebra $g$, and crystal bases reveal the structure of $U_q(g)$-modules in a very simplified form. As these $U_q(g)$-modules are known to be $q$-deformations of modules over the original Lie algebras, knowledge of these structures also affects the study of Lie algebras.

The crystal $B(\infty)$, which is the crystal base of the negative part $U_q^-(g)$ of a quantum group, has received attention since the very birth of crystal base theory [8, 9]. This is not only because it is an essential part of the grand loop argument proving the existence of crystal bases, but because it gives insight into the structure of quantum group itself.

Much effort has been made [1, 3, 12, 14, 15, 18, 19] to give explicit description of the crystals $B(\infty)$ over various Kac-Moody algebras. A related known result is that it is possible to characterize the highest weight crystal $B(\lambda)$ over symmetrizable Kac-Moody algebras, in terms of Nakajima monomials [5, 6, 11, 13, 20], an object which was introduced by Nakajima [16, 17]. This has lead to the belief that it should be possible to give a similar description for $B(\infty)$ also. Starting from a theorem of Kashiwara and Nakajima on the crystal structure of monomials [11], we can argue that it is not possible to find the crystal $B(\infty)$ within the set of Nakajima monomials with their given crystal structure. Hence, in our recent work [15], we constructed the set of extended monomials and developed a crystal structure on it, conjecturing that a certain connected component of the crystal would be isomorphic to $B(\infty)$. Actually, the set of Nakajima monomials can be embedded as a subcrystal in this set of extended Nakajima monomials. Thus, the monomial theory developed for irreducible highest weight crystal can easily be transferred to the extended monomial set.

In the current work, we restrict ourselves to the $G_2$-type finite simple Lie algebra. For this case, we give an explicit set of extended Nakajima monomials and show it to be isomorphic to $B(\infty)$. The previous work [3], giving a Young tableaux realization of $B(\infty)$ for the finite simple types, is used in doing this. We also extend Cliff’s [1] realization of $B(\infty)$ for classical finite types, given in terms of a completely different object, to the $G_2$-type.

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The paper is organized as follows. We start by reviewing the notion of extended Nakajima monomials and the crystal structure given on the set of such monomials. Also, we cite Young tableau expression of crystal $B(\infty)$ for type $G_2$ which play a crucial role in our work. We then proceed to give a monomial realization of the crystal $B(\infty)$. In the process of obtaining these results, we give new expressions for the Kashiwara operators acting on the extended Nakajima monomials, more appropriate for the situation in hand. In the last section, we deal with Cliff’s approach of realizing $B(\infty)$.

2. Extended Nakajima monomials and Young tableaux

In this section, we introduce notation and cite facts that are crucial for our work. Please refer to the references cited in the introduction or books on quantum groups [2, 4] for the basic concepts on quantum groups and crystal bases.

Let us first fix the basic notation.

- $I = \{1, 2\}$ : index set for $G_2$-type.
- $A = (a_{ij})_{i,j \in I}$ : Cartan matrix of type $G_2$.
- $\alpha_i, \Lambda_i (i \in I)$ : simple root, fundamental weight.
- $\Pi = \{\alpha_i | i \in I\}$ : the set of simple root.
- $P = \bigoplus_{i \in I} \mathbb{Z} \alpha_i$ : weight lattice.
- $U_q(G_2) :$ quantum group for $G_2$.
- $U_q^{-}(G_2) :$ subalgebra of $U_q(G_2)$ generated by $f_i \ (i \in I)$.
- $\tilde{f}_i, \tilde{e}_i :$ Kashiwara operators.
- $B(\infty) :$ crystal base of $U_q^{-}(G_2)$.

Throughout this paper, a $U_q(G_2)$-crystal will refer to a (abstract) crystal associated with the Cartan datum $(A, \Pi, P)$.

2.1. Nakajima monomials. We now recall the set of monomials and its crystal structure discovered by Nakajima [17] and also recall their extension introduced in [15]. Both of these sets were defined for all symmetrizable Kac-Moody algebras, but we shall restrict ourselves to the $G_2$ case in this paper.

Let $\mathcal{M}^E$ be a certain set of formal monomials in the variables $Y_i(m) \ (i \in I, m \in \mathbb{Z})$. More explicitly,

$$\mathcal{M}^E = \left\{ \prod_{(i,m) \in I \times \mathbb{Z}} Y_i(m)^{y_i(m)} \bigg| y_i(m) = (y^0_i(m), y^1_i(m)) \in \mathbb{Z} \times \mathbb{Z} \text{ vanishes except at finitely many } (i,m) \right\}.$$  

We give the lexicographic order to the set $\mathbb{Z} \times \mathbb{Z}$ of variable exponents. Fix any set of integers $c = (c_{ij})_{i \neq j \in I}$ such that

$$c_{ij} + c_{ji} = 1,$$

and set

$$A_i(m) = Y_i(m)^{(0,1)} Y_i(m+1)^{(0,1)} \prod_{j \neq i} Y_j(m + c_{ji})^{(0,\langle h_j, \alpha_i \rangle)}.$$
The crystal structure on $\mathcal{M}^E$ is defined as follows. For every monomial $M = \prod_{(i,m) \in I \times Z} Y_i(m)^{y_i(m)}$, we set
\begin{align}
\text{wt}(M) &= \sum_i (\sum_m y_i(m)) \Lambda_i, \\
\varphi_i(M) &= \max \{ \sum_{k \leq m} y_i(k) \mid m \in \mathbb{Z} \}, \\
\epsilon_i(M) &= \max \{ -\sum_{k > m} y_i(k) \mid m \in \mathbb{Z} \}.
\end{align}

Notice that the coefficients of $\tilde{\text{wt}}(M)$ are pairs of integers. In this setting, we have $\varphi_i(M) \geq (0,0)$, $\epsilon_i(M) \geq (0,0)$, and $\text{wt}(M) = \sum_i (\varphi_i(M) - \epsilon_i(M)) \Lambda_i$. Set
\begin{align}
\tilde{f}_i(M) &= \begin{cases} 
0 & \text{if } \varphi_i(M) = (0,0), \\
A_i(m_f)^{-1} M & \text{if } \varphi_i(M) > (0,0),
\end{cases} \\
\tilde{e}_i(M) &= \begin{cases} 
0 & \text{if } \epsilon_i(M) = (0,0), \\
A_i(m_e) M & \text{if } \epsilon_i(M) > (0,0).
\end{cases}
\end{align}

Here,
\begin{align}
m_f &= \min \{ m \mid \varphi_i(M) = \sum_{k \leq m} y_i(k) \} = \min \{ m \mid \epsilon_i(M) = -\sum_{k > m} y_i(k) \}, \\
m_e &= \max \{ m \mid \varphi_i(M) = \sum_{k \leq m} y_i(k) \} = \max \{ m \mid \epsilon_i(M) = -\sum_{k > m} y_i(k) \}.
\end{align}

Note that $y_i(m_f) > (0,0)$, $y_i(m_f + 1) \leq (0,0)$, $y_i(m_e + 1) < (0,0)$, and $y_i(m_e) \geq (0,0)$.

The Kashiwara operators, together with the maps $\varphi_i$, $\epsilon_i$ ($i \in I$), wt, define a crystal structure on the set $\mathcal{M}^E$ [15].

The set of monomials $\prod_{(i,m) \in I \times Z} Y_i(m)^{y_i^0(m)} y_i^1(m)$ of $\mathcal{M}^E$ with $y_i^0(m) = 0$ is exactly the Nakajima monomial set $\mathcal{M}$ if we identify $Y_i(m)^{(0,y_i^1(m))}$ in $\mathcal{M}^E$ with $Y_i(m)^{y_i^1(m)} \in \mathcal{M}$. The crystal structure on $\mathcal{M}$, introduced in [11], is compatible with that on $\mathcal{M}^E$ under this identification. The crystal $\mathcal{M}$ is a subcrystal of $\mathcal{M}^E$.

Restriction of the following theorem to just the monomials of $\mathcal{M}$ appears in [11].

**Theorem 2.1.** ([15]) If a monomial $M \in \mathcal{M}^E$ of $\tilde{\text{wt}}(M) = \sum_i (0, p_i) \Lambda_i$, where each $p_i$ is a nonnegative integer, satisfies $\tilde{e}_i(M) = 0$ for all $i \in I$, then the connected component of $\mathcal{M}^E$ containing $M$ is isomorphic to $\mathcal{B}(\sum_i p_i \Lambda_i)$ as a $U_q(\mathfrak{g})$-crystal.
Conversely, given any subset of $\mathcal{M}^E$ isomorphic to $B(\sum_i p_i \Lambda_i)$, there exists an element $M$ in the subset such that $\tilde{w}(M) = \sum_i (0, p_i) \Lambda_i$, and $\tilde{e}_i(M) = 0$ for all $i \in I$.

The following is a conjecture on the crystal $B(\infty)$ introduced in [15] and stated for all symmetrizable Kac-Moody algebras. Its converse is known to be true [15].

If a monomial $M \in \mathcal{M}^E$ of $\tilde{w}(M) = \sum_i (p_i, 0) \Lambda_i$, where each $p_i$ is a positive integer, satisfies $\tilde{e}_i(M) = 0$ for all $i \in I$, then the connected component of $\mathcal{M}^E$ containing $M$ is isomorphic to $B(\infty)$ as a $U_q(g)$-crystal.

In [14, 15], result for $B(\infty)$ of type $A_n$ and $A_n^{(1)}$ was given as evidence supporting this conjecture. In the next section, we will give a concrete listing of elements containing a weight zero vector $M \in \mathcal{M}^E$ mentioned in the above theorem, for the case of $G_2$. We shall show this set to be a crystal and give an isomorphism between the crystal and crystal $B(\infty)$. This is another result supporting the above conjecture.

2.2. Young tableaux. In this section, we recall a Young tableaux description for the crystal $B(\infty)$ over type $G_2$ introduced in [3].

For the $G_2$-type, we shall take the Young tableau realization of highest weight crystal $B(\lambda)$ given in [7] as the definition of semi-standard tableaux. Since the work is a rather well known result, we refer readers to the original papers and shall not repeat the complicated definition here. The alphabet to be used inside the boxes constituting the Young tableaux will be denoted by $J$, and it will be equipped with an ordering $\prec$, as given in [7].

$$J = \{1 \prec 2 \prec 3 \prec 0 \prec 3 \prec 2 \prec 1\}.$$

Definition 2.2.

1. A semi-standard tableau $T$ of shape $\lambda \in P^+$, equivalently, an element of an irreducible highest weight crystal $B(\lambda)$ for the $G_2$ type, is large if it consists of 2 non-empty rows, and if the number of 1-boxes in the first row is strictly greater than the number of all boxes in the second row and the second row contains at least one 2-box.

2. A large tableau $T$ is marginally large, if the number of 1-boxes in the first row of $T$ is greater than the number of all boxes in the second row by exactly one and the second row of $T$ contain one 2-box.

In Figure 1, we give examples of semi-standard tableaux. The one on the left is large, the one on the middle is marginally large, and the one on the right is not large.

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & 2 & 3 & 0 & 3 \ 1 & 2 & 0 & 3 & 2 & 1 \ 2 & 2 & 3 & 1 & 0 & 3 & 2 \ \hline
\end{tabular}
\end{table}
Definition 2.3. We denote by $\mathcal{T}(\infty)$ the set of all marginally large tableaux. The marginally large tableau whose $i$-th row consists only of $i$-boxes ($i \in I$) is denoted by $T_\infty$.

The set $\mathcal{T}(\infty)$, consists of all tableaux of the following form. The unshaded part must exist, whereas the shaded part is optional with variable size.

$$T = \begin{array}{ccccccccc}
1 & \cdots & 1 & 1 & 2 & 2 & 3 & 3 & 0 & 3 & 3 & 3 & 2 & 2 & 1 & 1 \\
2 & 3 & 3 \\
\end{array}$$

The element $T_\infty$ is

$$T_\infty = \begin{array}{c}
1 \\
1 \\
2 \\
\end{array}.$$ 

We recall the action of Kashiwara operators $\tilde{f}_i$, $\tilde{e}_i$ ($i \in I$) on marginally large tableaux $T \in \mathcal{T}(\infty)$.

1. We first read the boxes in the tableau $T$ through the far eastern reading and write down the boxes in tensor product form. That is, we read through each column from top to bottom starting from the rightmost column, continuing to the left, and lay down the read boxes from left to right in tensor product form.

2. Under each tensor component $x$ of $T$, write down $\varepsilon(x)$-many 1s followed by $\varphi(x)$-many 0s. Then, from the long sequence of mixed 0s and 1s, successively cancel out every occurrence of $(0,1)$ pair until we arrive at a sequence of $1s$ followed by 0s, reading from left to right. This is called the $i$-signature of $T$.

3. Denote by $T'$, the tableau obtained from $T$, by replacing the box $x$ corresponding to the leftmost 0 in the $i$-signature of $T$ with the box $\tilde{f}_i x$.
   - If $T'$ is a large tableau, it is automatically marginally large. We define $\tilde{f}_i T$ to be $T'$.
   - If $T'$ is not large, then we define $\tilde{f}_i T$ to be the large tableau obtained by inserting one column consisting of $i$ rows to the left of the box $\tilde{f}_i x$ acted upon. The added column should have a $k$-box at the $k$-th row for $1 \leq k \leq i$.

4. Denote by $T'$, the tableau obtained from $T$, by replacing the box $x$ corresponding to the rightmost 1 in the $i$-signature of $T$ with the box $\tilde{e}_i x$.
   - If $T'$ is a marginally large tableau, then we define $\tilde{e}_i T$ to be $T'$.
   - If $T'$ is large but not marginally large, then we define $\tilde{e}_i T$ to be the large tableau obtained by removing the column containing the changed box. It will be of $i$ rows and have a $k$-box at the $k$-th row for $1 \leq k \leq i$.

5. If there is no 1 in the $i$-signature of $T$, we define $\tilde{e}_i T = 0$.

Let $T$ be a tableau in $\mathcal{T}(\infty)$ with the second row consisting of $b_3^2$-many 3s, one 2 and the first row consisting of $b_2^1$-many $j$s ($1 < j \leq 1$), $(b_3^2 + 2)$-many 1s. We define the maps $w_t: \mathcal{T}(\infty) \to P$, $\varphi_i, \varepsilon_i: \mathcal{T}(\infty) \to \mathbb{Z}$ by setting

(2.14) \hspace{1cm} wt(T) = (-b_2^1 - b_3^1 - 2b_0^1 - 3b_3^1 - 3b_2^1 - 4b_1^1)\alpha_1 \\
+ (-b_3^1 - b_0^1 - b_3^1 - 2b_2^1 - 2b_1^1 - 2b_3^1)\alpha_2,

(2.15) \hspace{1cm} \varepsilon_i(T) = \text{the number of 1s in the $i$-signature of $T$},

(2.16) \hspace{1cm} \varphi_i(T) = \varepsilon_i(T) + \langle h_i, wt(T) \rangle.$
\textbf{Theorem 2.4.} ([3]) The operator given by equations (2.14) to (2.16), with Kashiwara operators define a crystal structure on $T(\infty)$. And the crystal $T(\infty)$ is isomorphic to $B(\infty)$ as a $U_q(G_2)$-crystal.

3. Monomial Description of $B(\infty)$

We give a new realization of the crystal $B(\infty)$, for $G_2$-type, in terms of extended monomials.

For simplicity, from now on, we take the set $C = (c_{ij})_{i \neq j \in I}$ to be $c_{12} = 1$ and $c_{21} = 0$. Then for $m \in \mathbb{Z}$, we have

$$
\begin{align*}
A_1(m) &= Y_1(m) Y_1(m + 1) Y_2(m) Y_1(m + 1), \\
A_2(m) &= Y_2(m) Y_1(m + 1) Y_1(m + 1).
\end{align*}
$$

The set we define below was originally obtained by applying Kashiwara actions $\tilde{f}_i$ continuously on the single element $Y_1(-1) Y_2(-2) \in M^E$. This choice of starting monomial will allow us to relate monomials of the set defined below to tableaux in $T(\infty)$ naturally.

\textbf{Definition 3.1.} Consider elements of $M^E$ having the form

$$
M = Y_1(-1) Y_1(0) Y_2(-1) Y_2(0),
$$

with conditions

\begin{enumerate}
\item $(a_2^8 - a_2^{-1})$, $a_2^4$, $a_2^2 \leq 0$,
\item $(a_2^{-1} - a_2^{-1}) + (2a_2 - a_2^2 - a_2^{-1} - a_2^2) = 0$ and $a_2^2 + a_2^2 + a_2^2 = 0$,
\item $(a_2^0 + a_2^{-1} - a_2^{-1})$, $a_2^{-1} - a_2^{-1} \in 2\mathbb{Z}$ or $a_2^{-1} - a_2^{-1} \in \mathbb{Z}$.
\end{enumerate}

Specifically, in case of $a_2^2 = 0$ for all $i, j$, we have

$$
M = Y_1(-1) Y_2(-2).
$$

We denote by $M(\infty)$ the set of all monomials of this form and by $M_\infty$ the monomial of (3.3).

Actually, as we will become apparent later, this set $M(\infty)$ is closed and connected under Kashiwara operators. Figure 2 is the top part of monomial set $M(\infty)$.

We now introduce new expressions for elements of $M(\infty)$. First, we introduce the following notation.

\textbf{Definition 3.2.} For $u \in \mathbb{Z}_{\geq 0}$, $v \in \mathbb{Z}$, and $m \in \mathbb{Z}$, we use the notation

\begin{enumerate}
\item $X_j(m)^{(u,v)} = \begin{cases} Y_j(m)^{(u,v)} Y_{j-1}(m + 1)^{-1} & \text{for } j = 1, 2, \\
Y_1(m + 1)^{2u} Y_2(m + 1)^{-1} & \text{for } j = 3, \end{cases}$
\item $X_0(m)^{(u,v)} = Y_1(m + 1)^{(u,v)} Y_1(m + 2)^{-1}$,
\item $X_j(m)^{(u,v)} = \begin{cases} Y_j(m + 1)^{(u,v)} Y_j(m + 1)^{-1} & \text{for } j = 1, 2, \\
Y_2(m + 1)^{(u,v)} Y_1(m + 2)^{2u} & \text{for } j = 3. \end{cases}$
\end{enumerate}

Here, we set $Y_0(k)^{(u,v)} = 1$. 
Remark 3.3. Using the above notation, we may write

\[ A_1(m) = X_1(m)^{(0,1)} X_2(m)^{(0,-1)} = X_3(m-1)^{(0,1)} X_0(m-1)^{(0,-1)} = X_0(m-1)^{(0,1)} X_3(m-1)^{(0,-1)} \]

\[ A_2(m) = X_2(m)^{(0,1)} X_3(m)^{(0,-1)} = X_3(m-1)^{(0,1)} X_2(m-1)^{(0,-1)} \]

This is very useful when computing Kashiwara action on monomials written in terms of \( X_j(m)^{(u,v)} \) or \( X_j^*(m)^{(u,v)} \).

Proposition 3.4. Consider elements of \( \mathcal{M}^E \) having the form

\[
M = X_1(-1,2,-b_i^2 b_{i-1}^{-1} b_{i-2}^{-1} b_{i-3}^{-1}) X_2(-1,0,b_i^{-1}) X_3(-1,0,b_i^{-1})
\]

\[
\cdot X_0(-1,0,b_i^{-1}) X_3(-1,0,b_i^{-1}) X_2(-1,0,b_i^{-1}) X_1(-1,0,b_i^{-1})
\]

\[
\cdot X_2(-2,1,-b_i^2) X_3(-2,0,b_i^{-2})
\]

(3.7)

where \( b_i^2 \geq 0 \) for all \( i,j \) and \( b_i^{-1} \leq 1 \). Each element of \( \mathcal{M}(\infty) \) may be written uniquely in this form. Conversely, any element of this form is an element of \( \mathcal{M}(\infty) \).

Proof. Given any monomial

\[
M = Y_1(-1,1, a_i^{-1}) Y_1(0, a_i^2) Y_1(1, a_i) Y_1(2, a_i^2)
\]

\[
\cdot Y_2(-2,1, a_i^{-2}) Y_2(-1, a_i^2) Y_2(0, a_i^2) Y_2(1, a_i^2) \in \mathcal{M}(\infty),
\]
through simple computation, we can obtain the expression

\[
M = X_1(-1)^{2,a_1^{-1}+3a_2^{-2}} X_2(-1)^{0,a_2^{-1}a_2^{-2}} Y_4(1)^{0,a_2^{-1}Y_2(2)^{1,0}} Y_3(1)^{0,a_2^{-1}Y_2(2)^{1,0}}
\]

(3.8)

where either

- \( t_0^{-1} = 0, \ 2t_3^{-1} = a_0 + a_2^{-1} - a_2^{-2}, \) and \( 2t_3^{-1} = -a_1^2 - a_2^2, \)

or

- \( t_0^{-1} = 1, \ 2t_3^{-1} = a_0 + a_2^{-1} - a_2^{-2} - 1, \) and \( 2t_3^{-1} = -a_1^2 - a_2^2 - 1. \)

Since \( M \in \mathcal{M}(\infty) \), from the conditions given in (3.2), we obtain the form given in (3.7). Specifically, the element \( M_\infty = Y_1(-1)^{1,0} Y_2(-2)^{1,0} \) corresponds to \( X_1(-1)^{2,0} X_2(-2)^{1,0}. \)

Conversely, given any monomial of the form (3.7), we have

\[
M = Y_1(-1)^{1,-b_1^{-1}-b_2^{-1}-b_3^{-1}+3b_2^{-2}} Y_1(0)^{0,b_1^{-1}+2b_3^{-1}} Y_2(1)^{0,-b_2^{-1}+b_2^{-1}} Y_2(2)^{0,-b_2^{-1}}
\]

(3.9)

It is now straightforward to check that \( M \in \mathcal{M}(\infty) \). We have thus shown that \( \mathcal{M}(\infty) \) consists of elements of the form (3.7).

The uniqueness part may be proved through simple computation. \( \square \)

**Remark 3.5.** There are other ways to write each element of \( \mathcal{M}(\infty) \) as products of the terms \( X_j(m)^{(a,v)} \) and \( X_j(m)^{(a,v)} \). The product form (3.7) was chosen because it allows us to relate monomials of the set \( \mathcal{M}(\infty) \) to tableaux in \( T(\infty) \) directly.

Now, we translate the Kashiwara actions (2.10), (2.11) into a form suitable for the new monomial expression of \( \mathcal{M}(\infty) \).

**Lemma 3.6.** The Kashiwara operator action on \( \mathcal{M}^E \) may be rewritten as given below for elements

\[
M = X_1(-1)^{2,-b_1^{-1}-b_2^{-1}-b_3^{-1}+b_2^{-1}} X_2(-1)^{0,b_2^{-1}Y_3(1)^{0,b_2^{-1}} Y_2(2)^{0,b_2^{-1}}}
\]

(3.10)

of \( \mathcal{M}(\infty) \). Elements of the above form constitutes \( \mathcal{M}(\infty) \) and this set is closed under Kashiwara operator actions.

1. Consider the following ordered sequence of some components of \( M \):

\[
X_1(-1)^{0,b_2^{-1}} X_2(-1)^{0,b_2^{-1}} X_3(-1)^{0,b_2^{-1}} X_4(-1)^{0,b_2^{-1}} X_5(-1)^{0,b_2^{-1}} X_6(-1)^{0,b_2^{-1}}.
\]

2. Under each of the components

\[
X_1(-1)^{0,b_2^{-1}}, X_2(-1)^{0,b_2^{-1}}, X_2(-1)^{0,b_2^{-1}}.
\]
given in the above sequence, write $b_j^{-1}$-many 1’s and under $X_3(-1)(0,b_5^{-1})$, write $(2b_3^{-1})$-many 1’s. Also, under each of the components

\[ X_2(-1)(0,b_5^{-1}), \ X_0(-1)(0,b_6^{-1}), \]

write $b_j^{-1}$-many 0’s and under $X_3(-1)(0,b_5^{-1})$, write $(2b_3^{-1})$-many 0’s.

- From this sequence of 1’s and 0’s, successively cancel out each (0,1)-pair to obtain a sequence of 1’s followed by 0’s (reading from left to right). This remaining 1 and 0 sequence is called the 1-signature of $M$.
- Depending on the component $X$ corresponding to the leftmost 0 of the 1-signature of $M$, we define $\tilde{f}_1 M$ as follows:

\[
\tilde{f}_1 M = \begin{cases} 
MX_2(-1)(0^{-1})X_1(-1)(0,1) = MA_1(1)^{-1} & \text{if } X = X_2(-1)(0,b_5^{-1}) \\
MX_0(-1)(0^{-1})X_3(-1)(0,1) = MA_1(0)^{-1} & \text{if } X = X_0(-1)(0,b_6^{-1}) \\
MX_3(-1)(0^{-1})X_0(-1)(0,1) = MA_1(0)^{-1} & \text{if } X = X_3(-1)(0,b_7^{-1}) 
\end{cases}
\]

We define

\[
\tilde{f}_1 M = MX_1(-1)(0^{-1})X_2(-1)(0,1) = MA_1(-1)^{-1}
\]

if no 0 remains.
- Depending on the component $X$ corresponding to the rightmost 1 of the 1-signature of $M$, we define $\tilde{e}_1 M$ as follows:

\[
\tilde{e}_1 M = \begin{cases} 
MX_2(-1)(0,1)X_1(-1)(0^{-1}) = MA_1(1) & \text{if } X = X_1(-1)(0,b_5^{-1}) \\
MX_0(-1)(0,1)X_3(-1)(0^{-1}) = MA_1(0) & \text{if } X = X_3(-1)(0,b_6^{-1}) \\
MX_3(-1)(0,1)X_0(-1)(0^{-1}) = MA_1(0) & \text{if } X = X_0(-1)(0,b_7^{-1}) \\
MX_1(-1)(0,1)X_2(-1)(0^{-1}) = MA_1(-1) & \text{if } X = X_2(-1)(0,b_7^{-1}) 
\end{cases}
\]

We define $\tilde{e}_1 M = 0$ if no 1 remains.

(2) Kashiwara actions $\tilde{f}_2$ and $\tilde{e}_2$ :
- Consider the following finite ordered sequence of some components of $M$.

\[ X_2(-1)(0,b_5^{-1})X_3(-1)(0,b_5^{-1})X_3(-1)(0,b_7^{-1})X_2(-1)(0,b_5^{-1})X_3(-2)(0,b_7^{-2}) \]

- Under each of the components

\[ X_2(-1)(0,b_5^{-1}), X_3(-1)(0,b_7^{-1}), X_3(-2)(0,b_7^{-2}), \]

from the above sequence, write $b_j^k$-many 1’s, and under each

\[ X_3(-1)(0,b_7^{-1}), X_2(-1)(0,b_5^{-1}) \]

write $b_j^{-1}$-many 0’s.
- From this sequence of 1’s and 0’s, successively cancel out each (0,1)-pair to obtain a sequence of 1’s followed by 0’s. This remaining 1 and 0 sequence is called the 2-signature of $M$.
- Depending on the component $X$ corresponding to the leftmost 0 of the 2-signature of $M$, we define $\tilde{f}_2 M$ as follows:

\[
\tilde{f}_2 M = \begin{cases} 
MX_3(-1)(0^{-1})X_2(-1)(0,1) = MA_2(0)^{-1} & \text{if } X = X_3(-1)(0,b_7^{-1}) \\
MX_2(-1)(0^{-1})X_3(-1)(0,1) = MA_2(-1)^{-1} & \text{if } X = X_2(-1)(0,b_7^{-1}) 
\end{cases}
\]
We define

\[ f_2 M = MX_2(-2)^{(0,-1)}X_3(-2)^{(0,1)} = MA_2(-2)^{-1} \]

if no 0 remains.

- Depending on the component \( X \) corresponding to the rightmost 1 of the 2-signature of \( M \), we define \( \tilde{e}_2 M \) as follows:

\[
\tilde{e}_2 M = \begin{cases} 
MX_3(-1)^{(0,1)}X_2(-1)^{(0,-1)} = MA_2(0) & \text{if } X = X_2(-1)^{(0,b_2^{-1})}, \\
MX_2(-1)^{(0,1)}X_3(-1)^{(0,-1)} = MA_2(-1) & \text{if } X = X_3(-1)^{(0,b_3^{-1})}, \\
MX_2(-2)^{(0,1)}X_3(-2)^{(0,-1)} = MA_2(-2) & \text{if } X = X_3(-2)^{(0,b_3^{-2})}.
\end{cases}
\]

We define \( \tilde{e}_2 M = 0 \) if no 1 remains.

**Proof.** We first show that the action of these operators is closed on \( \mathcal{M}(\infty) \).

For \( M \in \mathcal{M}(\infty) \), if the \( i \)-signature of \( M \) contains at least one 0, then the exponent of component \( X_i(-1)^{(0,b_i^{-1})} \) corresponding to the left-most 0 shows the property \( (0,b_i^{-1}) \geq (0,1) \). In particular, in dealing with the action of \( \tilde{f}_i \), if the left-most 0 corresponds to \( X_3(-1)^{(0,b_3^{-1})} \), it means that the exponent \( b_3^{-1} \) of \( X_3(-1)^{(0,b_3^{-1})} \), a component of \( M \), is 0, due to the \((0,1)\)-pair cancellation rule. Thus the monomial \( \tilde{f}_i M \) defined in (3.11) and (3.13) is contained in \( \mathcal{M}(\infty) \). In the case where \( i \)-signature of \( M \) contains no 0, the exponents of the components \( X_i(-i) \) of \( M \) show the property \( \geq (0,1) \). Thus \( \tilde{f}_i M \) given in (3.12) and (3.14) also are in \( \mathcal{M}(\infty) \). So the set \( \mathcal{M}(\infty) \) is closed under the above operator \( \tilde{f}_i \).

As we can see in equations (3.11) to (3.14), for each \( M \in \mathcal{M}(\infty) \), \( \tilde{f}_i M \) can also be expressed in form \( MA_i(m^{-1}) \). To show that this operation is just another interpretation of the Kashiwara operator \( f_i \) given on \( \mathcal{L} \), restricted to \( \mathcal{M}(\infty) \), it is enough to show that \( m_f \) defined in (2.12) for each \( M \) is equal to \( m \) of \( MA_i(m^{-1}) \) given in equations (3.11) to (3.14).

Given a monomial \( M \in \mathcal{M}(\infty) \), we can express it in the following two forms.

\[
M = X_1(-1)^{(2,-b_2^{-1}b_3^{-1}b_3^{-1}b_2^{-1}b_3^{-1}-b_2^{-1}b_3^{-1})}X_2(-1)^{(0,b_2^{-1})}X_3(-1)^{(0,b_3^{-1})}X_0(-1)^{(0,b_0^{-1})}X_3(-1)^{(0,b_3^{-1})}X_2(-1)^{(0,b_2^{-1})}X_1(-1)^{(0,b_1^{-1})}X_2(-2)^{(1,-b_2^{-2})}X_3(-2)^{(0,b_3^{-2})}
\]

\[
Y_1(-1)^{(1,-b_3^{-1}-b_2^{-1}+b_2^{-1}+3b_3^{-2})}Y_1(0)^{(0,b_0^{-1}-b_2^{-1}+b_2^{-1}+3b_3^{-2})}Y_1(1)^{(0,-b_0^{-1}+b_2^{-1}+b_2^{-1}+3b_3^{-2})}Y_1(2)^{(0,-b_1^{-1})}Y_2(-2)^{(1,-b_3^{-2})}Y_2(-1)^{(0,b_2^{-1}-b_3^{-2})}Y_2(0)^{(0,b_3^{-1}-b_5^{-2})}Y_2(1)^{(0,-b_2^{-1})}.
\]

If the 1-signature of \( M \) contains at least one 0 and \( X \) is the component corresponding to the left-most 0 in the 1-signature of \( M \), then we can obtain

\[
m_f = \min \{ j \in \mathbb{Z} \mid \max \{ \sum_{k \leq j} y_1(k) \} \} = \begin{cases} 
1 & \text{if } X = X_2(-1)^{(0,b_2^{-1})}, \\
0 & \text{if } X = X_0(-1)^{(0,b_0^{-1})}, \\
0 & \text{if } X = X_3(-1)^{(0,b_3^{-1})},
\end{cases}
\]

where \( y_1(k) \) is the exponent of \( Y_1(k) \) appearing in \( M \) given by expression (3.16). If the 2-signature of \( M \) contains at least one 0 and \( X \) is the component corresponding
to the left-most 0 in the 2-signature of $M$, then we can obtain

$$m_f = \min\{j \in \mathbb{Z} \mid \max\{\sum_{k \leq j} y_2(k)\}\} = \begin{cases} 0 & \text{for } X = X_3(-1)^{0, b_3^{-1}}, \\ -1 & \text{for } X = X_2(-1)^{0, b_2^{-1}}, \end{cases}$$

where $y_2(k)$ is the exponent of $Y_2(k)$ appearing in $M$ given by expression (3.16).

If the $i$-signature of $M$ contains no 0,

$$m_f = \min\{j \in \mathbb{Z} \mid \max\{\sum_{k \leq j} y_i(k)\}\} = -i,$n

where $y_i(k)$ is the exponent of $Y_i(k)$ appearing in $M$ given by expression (3.16).

In all cases, we can confirm that $m_f = m$, where $m$ is given through equations (3.11) to (3.14) stating $\tilde{f}_i M = MA_i(m)^{-1}$. Proof for the statements concerning $\tilde{e}_i$ may be done in a similar manner.

In the above lemma, we showed that the action of Kashiwara operators (2.10) and (2.11) on $M^\xi$ satisfy the following properties:

$$\tilde{f}_i M(\infty) \subset M(\infty), \quad \tilde{e}_i M(\infty) \subset M(\infty) \cup \{0\} \quad \text{for all } i \in I.$$

Thus we obtain the following result.

**Proposition 3.7.** The set $M(\infty)$ forms a $U_q(G_2)$-subcrystal of $M^\xi$.

Figures 3 illustrates the top part of crystal $M(\infty)$ for finite type $G_2$. It was obtained by applying the Kashiwara actions introduced in Lemma 3.6 on the new expression for elements of $M(\infty)$. Readers may want to compare this with Figure 2.

Actually, from property of the crystal structure of $M^\xi$ we can obtain more general results.
**Definition 3.8.** Fix any set of positive integers \( p_i \) and any integer \( r \). Consider elements of \( \mathcal{M}^E \) having the form
\[
M = Y_1(r - 1)^{(p_1, a_1^{-1})} Y_1(r)^{(0, a_0^0)} Y_1(r + 1)^{(0, a_1^1)} Y_1(r + 2)^{(0, a_2^2)}
\]
\[
\cdot Y_2(r - 2)^{(p_2, a_2^{-2})} Y_2(r - 1)^{(0, a_1^{-1})} Y_2(r)^{(0, a_0^2)} Y_2(r + 1)^{(0, a_2^1)}
\]
(3.21)
satisfying the same condition given to (3.2). When \( a_i^j = 0 \) for all \( i, j \), this reduces to
\[
M = Y_1(r - 1)^{(p_1, 0)} Y_2(r - 2)^{(p_2, 0)}.
\]
(3.22)
We denote by \( \mathcal{M}(p_1, p_2; r; \infty) \) the set of all monomials of this form and write the monomial of (3.22) as \( M(p_1, p_2; r; \infty) \).

A result similar to Proposition 3.4 may be obtained for \( M(p_1, p_2; r; \infty) \).

**Proposition 3.9.** Each element of \( \mathcal{M}(p_1, p_2; r; \infty) \) may be written uniquely in the form
\[
M = X_1(r - 1)^{(p_1, a_1^{-1})} X_1(r)^{(0, a_0^0)} X_1(r + 1)^{(0, a_1^1)} X_1(r + 2)^{(0, a_2^2)}
\]
\[
\cdot X_0(r - 1)^{(0, a_0^0)} X_0(r)^{(0, a_1^1)} X_0(r - 1)^{(0, b_0^0)} X_0(r - 2)^{(0, b_0^0)}
\]
\[
\cdot X_3(r - 2)^{(0, b_0^0)} X_3(r - 1)^{(0, b_0^1)} X_3(r)^{(0, b_0^2)}
\]
(3.23)
where \( b_i^j \geq 0 \) for all \( i, j \) and \( b_i^0 \leq 1 \). Conversely, any element in \( \mathcal{M}^E \) of this form is an element of \( \mathcal{M}(p_1, p_2; r; \infty) \).

We believe the readers can easily write down the process for change of variable similar to that given by (3.8) and (3.9) for \( \mathcal{M}(p_1, p_2; r; \infty) \).

The set \( \mathcal{M}(\infty) \) is a special case of this set \( \mathcal{M}(p_1, p_2; r; \infty) \) corresponding to \( r = 0 \) and \( p_i = 1 \) for all \( i \in I \).

**Remark 3.10.** It is possible to obtain the result of Lemma 3.6 also for the case \( \mathcal{M}(p_1, p_2; r; \infty) \). Thus we can state that the set \( \mathcal{M}(p_1, p_2; r; \infty) \) forms a \( U_q(G_2) \)-subcrystal of \( \mathcal{M}^E \).

**Proposition 3.11.** The set \( \mathcal{M}(p_1, p_2; r; \infty) \) forms a subcrystal of \( \mathcal{M}^E \) isomorphic to \( \mathcal{M}(\infty) \) as a \( U_q(G_2) \)-crystal.

**Proof.** As mentioned in Remark 3.10, we can show that the set \( \mathcal{M}(p_1, p_2; r; \infty) \) forms a \( U_q(G_2) \)-subcrystal of \( \mathcal{M}^E \). Let us show that the crystal \( \mathcal{M}(p_1, p_2; r; \infty) \) is isomorphic to \( \mathcal{M}(\infty) \) as a \( U_q(G_2) \)-crystal.

First, we define a canonical map \( \phi : \mathcal{M}(\infty) \rightarrow \mathcal{M}(p_1, p_2; r; \infty) \) by setting
\[
\phi(M) = X_1(r - 1)^{(p_1, a_1^{-1})} X_1(r)^{(0, a_0^0)} X_1(r + 1)^{(0, a_1^1)} X_1(r + 2)^{(0, a_2^2)}
\]
\[
\cdot X_0(r - 1)^{(0, a_0^0)} X_0(r)^{(0, a_1^1)} X_0(r - 1)^{(0, b_0^0)} X_0(r - 2)^{(0, b_0^0)}
\]
\[
\cdot X_3(r - 2)^{(0, b_0^0)} X_3(r - 1)^{(0, b_0^1)} X_3(r)^{(0, b_0^2)}
\]
(3.24)
for \( M \) of the form (3.7). Note that the monomial \( M_\infty \) of \( \mathcal{M}(\infty) \) is mapped onto the vector \( M(p_1, p_2; r; \infty) \). It is obvious that this map \( \phi \) is well-defined and that it is actually bijective.

The outputs of the functions \( \text{wt}, \varphi_i, \) and \( \varepsilon_i \), defined in (2.7), (2.8), and (2.9) do not depend on \( r \) or on any fixed positive integers \( p_i \). Using Lemma 3.6 and its counterpart for \( \mathcal{M}(p_1, p_2; r; \infty) \), we can easily show that the map \( \phi \) commutes with
the Kashiwara operators. Hence, the set $\mathcal{M}(p_1, p_2; r; \infty)$ forms a subcrystal of $\mathcal{M}^\xi$ isomorphic to $\mathcal{M}(\infty)$ as a $U_q(G_2)$-crystal.

**Remark 3.12.** The monomial of (3.21) is the element of $\mathcal{M}(p_1, p_2; r; \infty)$ corresponding to the monomial (3.2) of $\mathcal{M}(\infty)$ under the natural isomorphism $\phi$ mentioned in the proof of Proposition 3.11.

**Remark 3.13.** Since $\hat{e}_i(M_\infty) = (0, 0)$ for all $i \in I$, we have $\hat{e}_i(M_\infty) = 0$ for all $i \in I$. And $\text{wt}(M_\infty) = \sum_{i \in I}(1, 0)\Lambda_i$, so we have $\text{wt}(M_\infty) = 0$. Note that $M_\infty$ satisfies the condition for a monomial $M$ mentioned in the conjecture introduced in Section 2.1.

The monomial $M_{(p_1, p_2; r; \infty)}$ with $\text{wt}(M_{(p_1, p_2; r; \infty)}) = \sum_{i \in I}(p_i, 0)\Lambda_i$, also satisfies the condition for a monomial $M$ mentioned in the conjecture.

**Remark 3.14.** It should be clear from the proof of Proposition 3.11, that in developing any theory for $\mathcal{M}(p_1, p_2; r; \infty)$ the actual values of integer $r$ or $(p_1, p_2)$ will not be very important. Arguments made for any set of such values can easily be adapted to applied to other set of such values. Hence, we shall concentrate on the theory for $\mathcal{M}(\infty)$ only.

Now, we will show that $\mathcal{M}(\infty)$ is a new realization of $\mathcal{B}(\infty)$ by giving a crystal isomorphism. Recall from Theorem 2.4 that the set $\mathcal{T}(\infty)$ gives a realization of the crystal $\mathcal{B}(\infty)$.

Here is one of our two main realization theorems.

**Theorem 3.15.** There exists a $U_q(G_2)$-crystal isomorphism

\[(3.25) \quad \mathcal{B}(\infty) \xrightarrow{\sim} \mathcal{T}(\infty) \xrightarrow{\sim} \mathcal{M}(\infty)\]

which maps $T_\infty$ to $M_\infty$.

**Proof.** We define a canonical map $\Theta : \mathcal{T}(\infty) \to \mathcal{M}(\infty)$ by setting, for each tableau $T \in \mathcal{T}(\infty)$ with second row consists of $b_3^2$-many 3-boxes and just one 2-box, and with first row consists of $b_j^1$-many $j$-boxes, for each $j > 1$, and $(b_3^2 + 2)$-many 1-boxes, $\Theta(T) = M$, where

\[
M = X_1(-1)^{(2, -b_3^1, -b_3^2, -b_3^1, -b_3^2, -b_3^1, -b_3^2)} X_2(-1)^{(0, b_3^1)} X_3(-1)^{(0, b_3^2)} \\
\cdot X_0(-1)^{(0, b_3^1)} X_3(-1)^{(0, b_3^2)} X_2(-1)^{(0, b_3^1)} X_3(-1)^{(0, b_3^2)} \\
\cdot X_2(-2)^{(1, -b_3^2)} X_3(-2)^{(0, b_3^2)} \in \mathcal{M}(\infty).
\]

It is obvious that this map $\Theta$ is well-defined and that it is actually bijective.

The action of Kashiwara operators on $\mathcal{M}(\infty)$ given in Lemma 3.6 follows the process for defining it on $\mathcal{T}(\infty)$. Hence, the map $\Theta$ naturally commutes with the Kashiwara operators $\hat{f}_i$ and $\hat{e}_i$. Other parts of the proof are similar or easy. □

**Remark 3.16.** From Proposition 3.11 and Theorem 3.15, we can conclude that the crystal $\mathcal{M}(p_1, p_2; r; \infty)$ is also $U_q(G_2)$-crystal isomorphic to $\mathcal{B}(\infty)$.

**Example 3.17.** We illustrate the correspondence between $\mathcal{T}(\infty)$ and $\mathcal{M}(\infty)$. A monomial of $\mathcal{M}(\infty)$

\[
M = Y_1(-1)^{(1, 1)} Y_1(1)^{(0, -1)} Y_1(2)^{(0, -1)} \\
\cdot Y_2(-2)^{(1, -2)} Y_2(-1)^{(0, -1)} Y_2(0)^{(0, 2)}
\]
can be expressed as
\[ M = X_1(-1)^{(2,-5)}X_2(-1)^{(0,1)}X_0(-1)^{(0,1)}X_3(-1)^{(0,2)}X_1(-1)^{(0,1)} \]
\[ \cdot X_2(-2)^{(1,-2)}X_3(-2)^{(0,2)} \]
by (3.8). Hence we have the following marginally large tableau as the image of \( M \) under \( \Theta^{-1} \).
\[ \Theta^{-1}(M) = \begin{array}{ccccccc}
1 & 1 & 1 & 1 & 2 & 0 & 1 \\
2 & 2 & 2 & 2 & 0 & 1 & 1
\end{array} \in T(\infty). \]

**Remark 3.18.** Note that Remark 3.13 and Remark 3.16 provide evidence supporting the conjecture introduced in Section 2.1, for the \( G_2 \) case.

### 4. Cliff’s Description of \( B(\infty) \)

Let us recall the abstract crystal \( B_i = \{ b_i(k) | k \in \mathbb{Z} \} \) introduced in [10] for each \( i \in I \). It has the following maps defining the crystal structure.
- \( \varphi_i(b_i(k)) = k \), \( \varepsilon_i(b_i(k)) = -k \),
- \( \varphi_i(b_j(k)) = -\infty \), \( \varepsilon_i(b_j(k)) = -\infty \), for \( i \neq j \),
- \( \tilde{f}_i(b_i(k)) = b_i(k-1) \), \( \tilde{e}_i(b_i(k)) = b_i(k+1) \),
- \( \tilde{f}_i(b_j(k)) = 0 \), \( \tilde{e}_i(b_j(k)) = 0 \), for \( i \neq j \).

From now on, we will denote the element \( b_i(0) \) by \( b_i \). We next cite the tensor product rule on crystals.

**Proposition 4.1.** ([10]) Let \( B^k(1 \leq k \leq n) \) be crystals with \( b^k \in B^k \). We set
\[ a_k = \varepsilon_i(b^k) - \sum_{1 \leq v < k} \langle h_i, \mathrm{wt}(b^v) \rangle. \]

Then we have
1. \( \tilde{e}_i(b^1 \otimes \cdots \otimes b^n) = b^1 \otimes \cdots \otimes b^{k-1} \otimes \tilde{e}_i b^k \otimes b^{k+1} \otimes \cdots \otimes b^n \)
   if \( a_k > a_v \) for \( 1 \leq v < k \) and \( a_k \geq a_v \) for \( k < v \leq n \),
2. \( \tilde{f}_i(b^1 \otimes \cdots \otimes b^n) = b^1 \otimes \cdots \otimes b^{k-1} \otimes \tilde{f}_i b^k \otimes b^{k+1} \otimes \cdots \otimes b^n \)
   if \( a_k \geq a_v \) for \( 1 \leq v < k \) and \( a_k > a_v \) for \( k < v \leq n \).

Kashiwara has shown [10] the existence of an injective strict crystal morphism
\[ \Psi : B(\infty) \to B(\infty) \otimes B_{i_k} \otimes B_{i_{k-1}} \otimes \cdots \otimes B_{i_1} \]
which sends the highest weight element \( u_\infty \) to \( u_\infty \otimes b_{i_k} \otimes \cdots \otimes b_{i_1} \), for any sequence \( S = i_1, i_2, \ldots, i_k \) of numbers in the index set \( I \) of simple roots. In [1], Cliff uses this to give a combinatorial description of \( B(\infty) \) for all finite classical types, with a specific choice of sequence \( S \). It is our goal to do this for type \( G_2 \).

**Proposition 4.2.** We define
\[ B(1) = B_3 \otimes B_2 \otimes B_1 \otimes B_2 \otimes B_1 \text{ and } B(2) = B_2. \]

Consider the subset of crystal \( B(\infty) \otimes B(1) \otimes B(2) \) given by
\[ \mathcal{I}(\infty) = \{ u_\infty \otimes \beta_1 \otimes \beta_2 \}, \]
where

\begin{align*}
\beta_1 &= b_1(-k_{1,2}) \otimes b_2(-k_{1,3}) \otimes b_1(-k_{1,3}) \otimes b_2(-k_{1,2}) \otimes b_1(-k_{1,1}) \in \mathcal{B}(1), \\
\beta_2 &= b_2(-k_{2,2}) \in \mathcal{B}(2),
\end{align*}

and where \( k_{u,v} \) are any nonnegative integers such that

\[ 0 \leq k_{1,\hat{u}} \leq k_{1,\hat{v}} \leq k_{1,3} / 2 \leq k_{1,2} \leq k_{1,1}. \]

The set \( \mathcal{I}(\infty) \) forms a \( U_q(G_2) \)-subcrystal of \( \mathcal{B}(\infty) \otimes \mathcal{B}(1) \otimes \mathcal{B}(2) \).

Proof. It suffices to show that the action of Kashiwara operators satisfy the following properties:

\[ \tilde{f}_i \mathcal{I}(\infty) \subset \mathcal{I}(\infty), \quad \tilde{e}_i \mathcal{I}(\infty) \subset \mathcal{I}(\infty) \cup \{0\}, \]

for all \( i \in I \).

We will compute the value \( \tilde{f}_i \) on each element of \( \mathcal{I}(\infty) \), using the tensor product rule given in Proposition 4.1. First, we compute the finite sequence \( \{a_k\} \) set by (4.1) for

\[ b = u_\infty \otimes \beta_1 \otimes \beta_2 \]

\[ = u_\infty \otimes b_1(-k_{1,2}) \otimes b_2(-k_{1,3}) \otimes b_1(-k_{1,3}) \otimes b_2(-k_{1,2}) \otimes b_1(-k_{1,1}) \otimes b_2(-k_{2,2}). \]

In the \( i = 1 \) case, we have

\[ a_1 = 0, \quad a_3 = a_5 = a_7 = -\infty, \]

\[ a_2 = k_{1,2}, \quad a_4 = k_{1,3} + 2k_{1,2} - 3k_{1,3}, \]

\[ a_6 = k_{1,1} + 2k_{1,\hat{u}} - 3k_{1,\hat{v}} + 2k_{1,3} - 3k_{1,2}, \]

and for \( i = 2 \) case,

\[ a_1 = 0, \quad a_2 = a_4 = a_6 = -\infty, \]

\[ a_3 = k_{1,3} - k_{1,2}, \quad a_5 = k_{1,2} - k_{1,2} + 2k_{1,3} - k_{1,3}, \]

\[ a_7 = k_{2,2} - k_{1,\hat{u}} + 2k_{1,\hat{v}} - k_{1,3} + 2k_{1,2} - k_{1,1}. \]

By Proposition 4.1, we obtain the following three candidates of \( \tilde{f}_i(b) \) for each \( i \):

\[ \tilde{f}_1(b) = u_\infty \otimes \tilde{f}_1(b_1(-k_{1,\hat{u}})) \otimes b_2(-k_{1,\hat{v}}) \otimes b_1(-k_{1,\hat{v}}) \otimes b_2(-k_{1,2}) \otimes b_1(-k_{1,1}) \]

\[ \otimes b_2(-k_{2,2}) \]

\[ = u_\infty \otimes (b_1(-k_{1,\hat{u}} - 1) \otimes b_2(-k_{1,\hat{v}}) \otimes b_1(-k_{1,\hat{v}}) \otimes b_2(-k_{1,2}) \otimes b_1(-k_{1,1})) \]

\[ \otimes b_2(-k_{2,2}) \in u_\infty \otimes \mathcal{B}(1) \otimes \mathcal{B}(2), \]

when \( a_2 \geq a_k \) for \( 1 \leq k < 2 \) and \( a_2 > a_k \) for \( 2 < k \leq 7 \),

\[ \tilde{f}_1(b) = u_\infty \otimes b_1(-k_{1,\hat{u}}) \otimes b_2(-k_{1,\hat{v}}) \otimes \tilde{f}_1(b_1(-k_{1,\hat{v}})) \otimes b_2(-k_{1,2}) \otimes b_1(-k_{1,1}) \]

\[ \otimes b_2(-k_{2,2}) \]

\[ = u_\infty \otimes (b_1(-k_{1,\hat{u}}) \otimes b_2(-k_{1,\hat{v}}) \otimes b_1(-k_{1,\hat{v}}) - 1) \otimes b_2(-k_{1,2}) \otimes b_1(-k_{1,1})) \]

\[ \otimes b_2(-k_{2,2}) \in u_\infty \otimes \mathcal{B}(1) \otimes \mathcal{B}(2), \]
when \( a_4 \geq a_k \) for \( 1 \leq k < 4 \) and \( a_4 > a_k \) for \( 4 < k \leq 7 \),
\[
\hat{f}_1(b) = u_\infty \otimes b_1(-k_{1,2}) \otimes b_2(-k_{1,3}) \otimes b_1(-k_{1,3}) \otimes b_2(-k_{1,2}) \otimes \hat{f}_1(b_1(-k_{1,1}))
\otimes b_2(-k_{2,2})
\]
\[
= u_\infty \otimes (b_1(-k_{1,2}) \otimes b_2(-k_{1,3}) \otimes b_1(-k_{1,3}) \otimes b_2(-k_{1,2}) \otimes b_1(-k_{1,1} - 1))
\otimes b_2(-k_{2,2}) \in u_\infty \otimes B(1) \otimes B(2),
\]
when \( a_6 \geq a_k \) for \( 1 \leq k < 6 \) and \( a_6 > a_k \) for \( 6 < k \leq 7 \),
\[
\hat{f}_2(b) = u_\infty \otimes b_1(-k_{1,2}) \otimes b_2(-k_{1,3}) \otimes b_1(-k_{1,3}) \otimes b_2(-k_{1,2}) \otimes b_1(-k_{1,1})
\otimes b_2(-k_{2,2})
\]
\[
= u_\infty \otimes (b_1(-k_{1,2}) \otimes b_2(-k_{1,3}) \otimes b_1(-k_{1,3}) \otimes b_2(-k_{1,2}) \otimes b_1(-k_{1,1}))
\otimes b_2(-k_{2,2}) \in u_\infty \otimes B(1) \otimes B(2),
\]
when \( a_3 \geq a_k \) for \( 1 \leq k < 3 \) and \( a_3 > a_k \) for \( 3 < k \leq 7 \),
\[
\hat{f}_2(b) = u_\infty \otimes b_1(-k_{1,2}) \otimes b_2(-k_{1,3}) \otimes b_1(-k_{1,3}) \otimes \hat{f}_2(b_2(-k_{2,2})) \otimes b_1(-k_{1,1})
\otimes b_2(-k_{2,2})
\]
\[
= u_\infty \otimes (b_1(-k_{1,2}) \otimes b_2(-k_{1,3}) \otimes b_1(-k_{1,3}) \otimes b_2(-k_{1,2} - 1) \otimes b_1(-k_{1,1}))
\otimes b_2(-k_{2,2}) \in u_\infty \otimes B(1) \otimes B(2),
\]
when \( a_5 \geq a_k \) for \( 1 \leq k < 5 \) and \( a_5 > a_k \) for \( 5 < k \leq 7 \),
\[
\hat{f}_2(b) = u_\infty \otimes b_1(-k_{1,2}) \otimes b_2(-k_{1,3}) \otimes b_1(-k_{1,3}) \otimes b_2(-k_{1,2}) \otimes b_1(-k_{1,1})
\otimes \hat{f}_2(b_2(-k_{2,2}))
\]
\[
= u_\infty \otimes (b_1(-k_{1,2}) \otimes b_2(-k_{1,3}) \otimes b_1(-k_{1,3}) \otimes b_2(-k_{1,2}) \otimes b_1(-k_{1,1}))
\otimes b_2(-k_{2,2} - 1) \in u_\infty \otimes B(1) \otimes B(2),
\]
when \( a_7 \geq a_k \) for \( 1 \leq k < 7 \). And for each case given above, we obtain the following result from conditions for the sequence \( a_k \). In the \( i = 1 \) case, \( k_{u,v} \) values, appearing in the above expression for \( \hat{f}_b \), are nonnegative integers satisfying

- \( k_{1,3} + 1 \leq k_{1,3} \leq k_{1,3}/2 \leq k_{1,2} \leq k_{1,1} \),
- when \( a_2 \geq a_k \) for \( 1 \leq k < 2 \) and \( a_2 > a_k \) for \( 2 < k \leq 7 \),
- \( k_{1,2} \leq k_{1,3} \leq (k_{1,3} + 1)/2 \leq k_{1,2} \leq k_{1,1} \),
- when \( a_4 \geq a_k \) for \( 1 \leq k < 4 \) and \( a_4 > a_k \) for \( 4 < k \leq 7 \),
- \( k_{1,2} \leq k_{1,3} \leq k_{1,3}/2 \leq k_{1,2} \leq k_{1,1} + 1 \),
- when \( a_6 \geq a_k \) for \( 1 \leq k < 6 \) and \( a_6 > a_k \) for \( 6 < k \leq 7 \),

and in the \( i = 2 \) case,

- \( 0 \leq k_{1,2} \leq k_{1,3} + 1 \leq k_{1,3}/2 \leq k_{1,2} \leq k_{1,1} \),
- when \( a_3 \geq a_k \) for \( 1 \leq k < 3 \) and \( a_3 > a_k \) for \( 3 < k \leq 7 \),
- \( 0 \leq k_{1,2} \leq k_{1,3} \leq k_{1,3}/2 \leq k_{1,2} + 1 \leq k_{1,1} \),
- when \( a_5 \geq a_k \) for \( 1 \leq k < 5 \) and \( a_5 > a_k \) for \( 5 < k \leq 7 \),
- \( 0 \leq k_{2,2} + 1 \),
- when \( a_7 \geq a_k \) for \( 1 \leq k < 7 \).

Thus the action of Kashiwara operator \( \hat{f}_i \) is closed on \( \mathcal{I}(\infty) \).

Proof for the statements concerning \( \hat{e}_i \) may be done in a similar manner. \( \square \)

The notation \( \beta_1 \) and \( \beta_2 \) appearing in this proposition will be used a few more times in this section.
**Theorem 4.3.** There exists a $U_q(G_2)$-crystal isomorphism

\[
B(\infty) \cong T(\infty) \cong T(\infty) \subset B(\infty) \otimes B(1) \otimes B(2),
\]

which maps $T(\infty)$ to $u_\infty \otimes (b_1 \otimes b_2 \otimes b_1 \otimes b_2) \otimes (b_2)$.

**Proof.** With the help of tensor product rules, it is easy to check the compatibility of this map with Kashiwara operators. Other parts of the proof are similar or easy. Hence we shall only write out the maps and give no proofs.

For each tableau with the second row consisting of $b_3^2$-many 3-boxes and just one 2-box, and with the first row consisting of $b_j^1$-many $j$-boxes, for each $j > 1$, and $(b_3^3 + 2)$-many 1-boxes, we may map it to the element $u_\infty \otimes \beta_1 \otimes \beta_2$ where

\[
k_{1,1} = \sum_{j=2}^{1} b_j^1, \quad k_{1,2} = \sum_{j=3}^{1} b_j^1, \quad k_{1,3} = 2\left(\sum_{j=3}^{1} b_j^1\right) + b_0^1,
\]

\[
k_{1,3} = b_2^2 + b_1^1, \quad k_{2,2} = b_2^2.
\]

Conversely, an element $u_\infty \otimes \beta_1 \otimes \beta_2$ is sent to the tableau whose shape we describe below row-by-row.

- The first row consists of

\[
\text{(k}_{1,3}^2/2 - k_{1,3})\text{-many } 2s, \quad (k_{1,3} - k_{1,2})\text{-many } 2s, \\
\text{[k}_{1,3}^2/2 - k_{1,3})\text{-many } 3s, \quad ((A + B) - (A' + B'))\text{-many } 0s, \\
\text{(k}_{1,2} - k_{1,3})/2\text{-many } 3s, \quad (k_{1,1} - k_{1,2})\text{-many } 2s, \text{ and } \\
\text{(k}_{2,2} + 2)\text{-many } 1s.
\]

- The second row consists of

\[
\text{(k}_{2,2} + 2)\text{-many } 3s \quad \text{and} \quad \text{one } 2.
\]

Here, $A = k_{1,2} - k_{1,3}/2$, $B = k_{1,3}/2 - k_{1,3}$, $A' = [k_{1,2} - k_{1,3}/2]$, and $B' = [k_{1,3}/2 - k_{1,3}]$.

Since the above theorem has shown $B(\infty) \cong T(\infty)$ as crystals, image of the injective crystal morphism

\[
\Psi: B(\infty) \rightarrow B(\infty) \otimes B(1) \otimes B(2) = B(\infty) \otimes (B_1 \otimes B_2 \otimes B_1 \otimes B_2 \otimes B_1) \otimes (B_2),
\]

which maps $u_\infty$ to $u_\infty \otimes (b_1 \otimes b_2 \otimes b_1 \otimes b_2) \otimes (b_2)$ is $T(\infty)$.

In the following corollary, a combinatorial description of $B(\infty)$ for $G_2$-type is given following Cliff’s method. A specific choice for the index sequence of crystals $S = (1, 2, 1, 2, 1, 2)$ corresponding to a longest word $w_0 = s_1 s_2 s_1 s_2 s_1 s_2$ of the Weyl group is used.

**Corollary 4.4.** Image of the injective strict crystal morphism

\[
\Psi: B(\infty) \rightarrow B(\infty) \otimes (B_1 \otimes B_2 \otimes B_1 \otimes B_2 \otimes B_1) \otimes (B_2),
\]

which maps $u_\infty$ to $u_\infty \otimes (b_1 \otimes b_2 \otimes b_1 \otimes b_2) \otimes (b_2)$ is given by

\[
\Psi(B(\infty)) = T(\infty) = \{u_\infty \otimes \beta_1 \otimes \beta_2\}.
\]

We illustrate the correspondence between $T(\infty)$ and $\Psi(B(\infty))$ for type $G_2$.

**Example 4.5.** The marginally large tableau

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 \\
2 & 3 & 3 & 3
\end{array}
\]
of $\mathcal{T}(\infty)$ corresponds to the element
\[ u_\infty \otimes b_1(-1) \otimes b_2(-1) \otimes b_1(-7) \otimes b_2(-4) \otimes b_1(-5) \otimes b_2(-2) \]
of $\Psi(\mathcal{B}(\infty))$ under the map given in Theorem 4.3.

Remark 4.6. We gave two new explicit descriptions of the crystal $\mathcal{B}(\infty)$ in this paper, namely, $\Psi(\mathcal{B}(\infty))$ and $\mathcal{M}(\infty)$ for the $G_2$ case. We can provide maps between the two giving crystal isomorphisms
\[ \mathcal{M}(\infty) \overset{\sim}{\rightarrow} \Psi(\mathcal{B}(\infty)) \]
in both directions. The maps can easily be drawn from Theorem 3.15 and Theorem 4.3.

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