Anisotropic Vortices in High-Temperature Superconductors and the Onset of Vortex-like Excitations above the Critical Temperature

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Abstract

In our recent papers we have found that a three-dimensional superconducting state with anisotropic vortices localized at the vortex-lattice points is a stable state in zero and nonzero external magnetic field for the layered high-temperature superconductivity materials. There exists a phase transition at the temperature $T^v_c$ from the normal phase to the vortex superconducting state which is of the first order. Note that the transition is in zero magnetic field. The first order phase transition shows overheating and overcooling effects. Nucleation of the superconducting phase in the normal phase thus may occur at temperatures higher than the transition temperature $T^v_c$. Then the onset of the vortex-like excitations above the transition temperature $T^v_c$ occurs in our theory. The onset of the vortex-like excitations in Nerst signal and some other experimental evidence for these excitations above the transition temperature $T^v_c$ in LSCO, YBCO and in other similar high-temperature materials may be explained thus by our theory. The vortex-like excitations above and below the transition temperature $T^v_c$ in LSCO, YBCO and in other similar high-temperature materials continuously evolve. This fact may be explained within our theory.
1 Introduction

Sine-Gordon vortices may exist in superconductors [1] and [2]. These vortices may be present in superconducting states also in zero magnetic fields. Their topological charge is preserved. We studied a magnetic-field-induced superconductive state in heavy fermion systems [3] and [4]. The strength of the coupling between the order parameter and magnetic field drives a superconductive state induced by a magnetic field, [3] and [4]. Thus such a state may be induced increasing the magnetic field. Experimentally it was reported that in CePb$_3$ there exists a ferromagnetic phase and simultaneously there is an evidence for presence of superconductivity. We studied the Kondo-lattice superconductivity in this connection, [4] and [5]. Superconducting states of different symmetry s, p and d were found to exist in the Kondo-lattice systems. A phase transition from the normal phase to the superconducting phase in CePb$_3$ type materials occurs in our phenomenological model through creation of a vortex lattice, in which vortices are superconducting regions induced by an external magnetic field. Increasing the magnetic field the density of such vortices and their shape becomes larger and the superconducting phase increases its volume in the material. In high-temperature superconductors, which are layered materials, the interlayer distance is larger than the in-layer lattice constant. The interlayer distance is comparable with the in-layer penetration length. Our results [3] - [5] lead to study of formation of a vortex in the plane, and to study of interaction of such vortices in neighbouring layers. In the presence of an in-layer vortex in-plane and out-of-plane magnetic field structures exist due to currents which are present. The vortex state which we described for a plane superconductor in 1982 may exist in zero magnetic field [6]. This fact is very important difference from usual vortices studied in superconductors, which are present due to an external magnetic field. The stability of the vortex in zero magnetic field [6] is due to the topological charge of this vortex which is preserved. The magnetic field generated by this stable vortex is influencing neighbouring planes and the plane in which it is localized. This field is coupled with the superconducting order parameter due to symmetry. As we discussed above the strength of the coupling between the magnetic field and the order parameter may drive a superconductive state with vortices induced by the magnetic field of this vortex, [3] and [4]. Thus in layered high-temperature superconductors coupling between the magnetic field generated by this stable vortex and the order parameter may lead to generation of other vortices (antivortices). This is a selfconsistent process. The field of these vortices influences this stable vortex. Thus then we may study vortices and their in-plane and out-of-plane interactions due to the inductance coupling. Thus the strength of the coupling between the order parameter and magnetic field drives a superconductive state induced by a magnetic field as mentioned above, [3] and [4], and in layered high-temperature superconductors drives a superconducting state with vortices. Note that also in the zero external magnetic field such a selfconsistent process may exists. Creation of a vortex in one plane leads to magnetic fields which influence the neighbouring planes, in these neighbouring planes these fields may induce creation of vortices the mag-
netic fields of which influence the original vortex in the plane: its existence and properties. Creation of a vortex in the plane leads also to magnetic fields which influence the same plane, in this plane these fields again may induce creation of vortices the magnetic fields of which influence the original vortex in the same plane: again its existence and properties.

Thus it follows that in layered materials one may expect that formation of such vortices may stabilize the superconducting state of different from classical symmetry in zero external magnetic field. Note that in nonzero external magnetic fields the self-consistent process may be modified but still present.

The vortex in the superconductor plane which we studied [6] is anisotropic vortex. This anisotropy (symmetry different from symmetry of isotropic vortices) then leads to coupling to other quantities of the system for which an invariant contributing to free energy expansion may exist. Integrating out these other quantities from the free energy of the superconducting material we obtain the effective free energy expansion for the superconducting order parameter with anisotropy terms present.

Recently we described and studied the mentioned mechanism and formulated a general problem: to study behaviour of such anisotropic vortices in zero and non-zero magnetic fields. The vortices of the in- and inter-layer type should be compared as concerning their theoretical properties in zero and non-zero fields with existing experiments concerning study of vortices in high-temperature superconductors. [7], [8], [9] and [10]. It was found experimentally that the anisotropy of the interplane-vortices in these materials is intrinsic. This is natural, the vortex is parallel to layers, layers are anisotropic. Planes are involved in such vortex configuration, we are speaking about the inter-plane vortex induced by an external magnetic field. As far as we know the anisotropy of such vortices is interpreted as due to anisotropy of layers.

The vortices which we were studied theoretically are intrinsically anisotropic. However this anisotropy is present for vortices which are present within an isotropic plane. Their anisotropy is not generated due to anisotropy of the material. Thus it is important to compare experimental results concerning anisotropy properties of the vortices in the layered materials with those properties of vortices which we found theoretically. This is the reason why we review different superconducting states with anisotropic vortices: the superconducting states in which single vortex exists, the superconducting states in which lines and lattices of vortices exist, the superconducting states in which pairs of vortex-antivortex type are present and form a kind of liquid. At larger distances between vortex-antivortex in the pair the energy is linearly dependent on this distance. This is a kind of confinement phenomena known from quarks forming baryons. For smaller distances there exists a minimum distance for which the vortex-antivortex pair exist in a ground state. The pair vortex-antivortex thus does not annihilate. Further we considered in our papers the superconducting states in which solitons and strings of vortices, which represent some kind of structures induced by internal mechanisms /twinning of the crystals, etc./ or by surface, may exist.

Intrinsic Josephson effect in layered systems was described in [11] and a sum-
mary of the Josephson effect in the high-temperature superconductors is given
in [12]. The time dependent equations for the gauge invariant phase differences
of the layered order parameters for superconductive states and for a general
vector potential of the electromagnetic field we discussed in [1] and [2]. The
interlayer coupling within the mentioned model equations was studied with its
influence on the properties of vortices in layers. These vortices in neighbouring
layers are coupled. They are forming a string of vortices perpendicular to the
layers in which the vortices exist. The string goes through the centers of these
vortices.

We used in our study a small parameter expansion method, and an exact
method of solution of the corresponding Lagrange-Euler equations to compare
different macroscopic superconducting states. We have found their free energy
and discussed their properties. The corresponding Lagrange-Euler equations are
nonlinear coupled sine-Gordon equations in the approximation of the constant
amplitude of the order parameter. Let us note that the nonlinearity describing
the vortex configurations in nonzero/ and in zero/ magnetic field is neglected
due to difficulties of the mathematical origin and mainly due to expectation that
the nonlinearity is negligible around the core of the vortex, which is assumed
to be large. However we have found that nonlinearities in order parameter in-
teractions are physically essential, [9]. The Lagrange-Euler equations for the
superconducting state order parameter and for the vector potential of the elec-
tromagnetic field in the constant amplitude approximation for the scalar (s-
type) superconducting order parameter reduced in the model studied in our
papers cited above to equations which are similar to the two-dimensional sine-
Gordon equation. They are generalized coupled two-dimensional sine-Gordon
equations. p and d symmetric superconducting states were not studied until
now.

The two-dimensional sine-Gordon equations occur in other physical systems,
in quantum antiferromagnets, in classical two-dimensional XY model with the
in-plane magnetic field [13] - [14], in models describing crystal growth [15]
and [16], and in models describing defects in incommensurate systems [17]
and [18]. It is useful to use analogy and mathematical properties of solutions of these
equations for study of anisotropic vortices in superconductors.

In our paper from 1982 we used a new method for solving a two-dimensional
sine-Gordon equation [6]. We transformed this equation into a set of two second
order ordinary nonlinear non-coupled equations. Multi - (Resonant-Soliton) so-
lutions and vortex-like solutions of the two- and three- dimensional sine-Gordon
equations were studied in [19]. The structure of a vortex in terms of the solitons
was studied in [20]. The vortex is formed from solitons which are intersecting.
The multi-vortex solutions of the sine-Gordon equation were studied in [21].
Exact rational-exponential solutions of this equation were studied in [22]. Vor-
tices and the boundary value problem were studied by the inverse scattering
transform method for this type of equation in [23]. The singular solutions of the
elliptic sine-Gordon equation were studied numerically in [24]. Numerical
studies of dynamical isoperimeter pattern are in [25].

Vortex-like excitations in \(La_{2-x}Sr_xCuO_4\) at temperatures significantly above
the critical temperature were found in the Nerst effect signal \[26\]. In over-doped $La_{2-x}Sr_xCuO_4$ (LSCO) the upper critical field $H_{c2}$ does not end at $T_{c0}$ but at a much higher temperature. These results imply, according to \[27\], to a loss in phase rigidity rather than in a vanishing of the pairing amplitude. Nerst measurements in $YBa_2Cu_3O_6$ (YBCO) and in $Bi_{2-x}Sr_xLa_2CuO_6$, \[28\], and in LSCO in fields up to 33 T show the existence of vortex-like excitations high above $T_{c0}$. This anomaly is related to the key question of whether the Meissner transition in zero field is caused by the collapse of long-range phase coherence or by the vanishing of the pairing amplitude. In zero magnetic field there is equal number of vortices with plus and minus topological charge. In nonzero fields there is nonequal number of vortices with plus and minus topological charge.

Preformed pairs as superconductor fluctuations may exist in Bose condensation of localized Cooper pairs with short coherence length, \[29\], \[30\] and \[31\]. Pairing correlations without phase coherence are described in \[32\] and \[33\].

In our papers \[1\] and \[2\], new results were found as concerning the influence of neighbouring planes on the vortex state in a given plane and new exact solutions of the coupled Lagrange-Euler sine-Gordon equations were found, their properties and free energy were compared. We have found that the three-dimensional superconducting state with vortex lattice, which is a stable state in zero external magnetic field for the layered materials. There exists a phase transition at the temperature $T_{c}^{\nu}$ from the normal phase to the vortex superconducting state which is of the first order. Depending on the parameters of the material the transition temperature $T_{c}^{\nu}$ may be higher than that in the case of the second order phase transition from normal phase to the homogeneous superconducting phase without vortices. The transition temperature $T_{c}^{\nu}$ would be zero for zero topological charge of vortices. Note that the transition to vortex lattice state is present also in zero magnetic field. The first order phase transition shows in experiments usually overheating and overcooling effects. Nucleation of the superconducting phase in the normal phase occurs at temperatures higher than the transition temperature $T_{c}^{\nu}$. And vice-versa. Nucleation of the normal phase in the superconducting phase occurs at temperatures lower than the transition temperature $T_{c}^{\nu}$. Thus the onset of the vortex-like excitations above the transition temperature $T_{c}^{\nu}$ occurs in our theory due to nucleation of the superconducting phase with vortex lattice (or parts of this lattice) in the normal phase at temperatures higher than the transition temperature $T_{c}^{\nu}$. The onset of the vortex-like Nerst signal above the transition temperature $T_{c}^{\nu}$ in LSCO, YBCO and in other similar high-temperature materials may be thus explained by our theory.

In this paper we review our results on vortex configurations in two-dimensional sine-Gordon systems in connection with high-temperature superconductors. We will use these results in our discussion concerning anisotropic vortices and the onset of Nerst signal and of some other signals corresponding to presence of vortex-like excitations above the critical temperature. The superconducting planes are coupled by the Josephson effect. The Lawrence-Doniach functional leads to the Lagrange-Euler equations for the order parameter. In \[1\] and \[2\]. We discussed the vortex states using exact solutions of these coupled Lagrange-
Euler sine-Gordon equations. Properties of the exact solutions and the free energy of the corresponding order parameter configurations were compared with the normal phase state and the homogeneous superconducting state. Stable vortex configurations in zero external magnetic field for the layered materials exist, the same holds for a lattice of vortices case. The first order phase transition overheating and overcooling effects are present here. Nucleation of the superconducting phase in the normal phase, which occurs at temperatures higher than the transition temperature $T_v^{c}$, exists. The onset of the vortex-like excitations above the transition temperature $T_v^{c}$, which occurs in our theory, is discussed. The onset of the vortex-like Nernst signal above the transition temperature $T_v^{c}$ in LSCO, YBCO and in other similar high-temperature materials is not well understood in experiments. The vortex-like excitations above the transition temperature $T_v^{c}$ in LSCO, YBCO and in other similar high-temperature materials continuously evolve into vortex states below the transition temperature $T_v^{c}$, this fact may be explained within our theory. Our theory may explain this fact and some of other experimentally observed properties of high-temperature superconductors described above.

2 Superconducting Planes Coupled by the Josephson Effect

Superconducting layers in layered superconductors of the high-temperature type are coupled due to intrinsic Josephson effect. The Lawrence - Doniach free energy functional for the order parameter for such a system has the form [1]

$$F(\Psi_n(r), A(R)) = \frac{H_c^2 s}{4\pi} \sum_n \int dr |\zeta_{ab}^2 | (-i\nabla + \frac{2\pi}{\Phi_0} A_n)\Psi_n |^2 -$$

$$\Psi_n|^2 + \frac{1}{2} |\Psi_n|^4 +$$

$$+ r(|\Psi_n|^2 + |\Psi_{n+1}|^2 - \Psi_n^* \Psi_{n+1} \exp(-i\chi_{n,n+1}) - \Psi_n^* \Psi_{n+1} \exp(+i\chi_{n,n+1}))$$

$$\int dR \frac{B^2}{8\pi}$$

The order parameter in the n-th layer is $\Psi_n(r) = |\Psi_n(r) | \exp(i\Phi_n(r))$, here $z=ns$, and s is an interlayer distance, $R=(r,z)$, $\nabla = \frac{\partial}{\partial r}$, $H_c$ is the bulk critical field, $\zeta_{ab}$ is the coherence length in the ab plane which is perpendicular to the c-direction. $\Phi_0 = \frac{\Phi_0}{|e|}$, $A_n(r) = (A_{nx}, A_{ny})$ is an average value of the vector potential over the distance $[(n - \frac{1}{2})s, (n + \frac{1}{2})s]$ along the c-axis, $B = \text{rot} (A)$, and $\chi_{n,n+1} = \frac{2\pi}{\Phi_0} \int_{ns}^{(n+1)s} dz A_z$.

If we denote the Josephson coupling constant between neighbouring layers order parameters as $r$, where $r << 1$, the coherence length $\zeta_c$ in the c-direction is given by
The in-plane penetration depth $\lambda_{ab}$ is given by

$$\lambda_{ab}^2 = \frac{\Phi_0^2}{8\pi^2 H_c^2 \zeta_{ab}}$$  \hspace{1cm} (3)$$

The anisotropy ratio $\frac{\lambda_{ab}}{\lambda_c}$ is given by

$$\frac{\lambda_{ab}}{\lambda_c} = \frac{\zeta_{ab}}{\zeta_c}$$  \hspace{1cm} (4)$$

The interlayer coupling is weak, $\zeta_c << s$. The general case and the limit of strong interlayer coupling $s << \zeta_c$ will not be discussed here.

Let $\phi_n(r)$ be the change of the order parameter phase around a vortex. The constant amplitude approximation $|\Psi|^2 = 1$ we consider first. The constant amplitude approximation in which the density of superconducting pairs is temperature dependent will be considered at the end of the paper. There is experimental evidence that the constant amplitude approximation is quite a good approximation to understand vortex state (phase), see discussion below.

3 Lagrange-Euler Equations for the Order Parameter

The Lagrange-Euler equations for the phase difference $\phi_{m,m+1}$ are the sine-Gordon coupled equations

$$-\sum_m L_{n,m} \nabla^2 \phi_{m,m+1} + \frac{1}{\lambda_{ab}^2} \sin(\phi_{n,n+1}) = 0$$  \hspace{1cm} (5)$$

where the summ over m is over layers.

The interlayer inductance $L_{n,m}$ between the layers n and m has the form

$$L_{n,m} = \int_0^{2\pi} dq \frac{\cos(n-m)q}{2\pi(1 - \cos(q))} + \frac{s^2}{\lambda_{ab}^2} = \frac{\lambda_{ab}}{s} \left(1 - \frac{s}{\lambda_{ab}}\right)|n-m|$$  \hspace{1cm} (6)$$

The boundary conditions for the Lagrange - Euler equations are

$$(\nabla_x \nabla_y - \nabla_y \nabla_x)\phi_{n,n+1}(r) = 2\pi \sum_n (|\delta(r - r_{nv}) - \delta(r - r_{nw})|)$$ $$\hspace{1cm} (7)$$

The equations are thus a system of the sine-Gordon equations in two dimensions (planes ab) coupled in the perpendicular direction c.
4 The Distance $s$ is Comparable with the Penetration Depth $\lambda_{ab}$

When the distance $s$ is comparable with the penetration depth $\lambda_{ab}$ as it is the case in high-temperature superconductors the mutual interlayer inductance $L_{n,n}$ (which is in this case when $m=n$ the inlayer inductance in fact) is nonzero and much larger than the other inductances (interlayer, because $m$ and $n$ are different) $L_{n,m}$

$$L_{n,n} = \frac{\lambda_{ab}}{s}$$  \hspace{1cm} (8)

Thus inductances $L_{n,m}$ with $n$ different from $m$ are not taken into account here in this section.

For the phase difference $\phi_{m,m+1}$ the equations have in this case the form

$$-L_{n,n} \nabla^2 \phi_{n,n+1} + \frac{1}{\lambda_{J}^2} \sin(\phi_{n,n+1}) = 0$$  \hspace{1cm} (9)

where the lower order terms in $\frac{L_{n,m}}{L_{n,n}}$ for $n$ and $m$ different are neglected. They would lead to multi-Sine-Gordon equations.

Let us denote by

$$\phi = \phi_{n,n+1}$$  \hspace{1cm} (10)

an angle variable. We assume that this angle is the same for every plane, which may be done due to expected homogeneity in the sense of the same states present in all layers.

Let us denote further

$$\lambda_p = \lambda_{J} L_{n,n}$$  \hspace{1cm} (11)

The two-dimensional sine-Gordon equation (9) takes the form

$$\triangle \phi = \frac{1}{\lambda_p^2} \sin(\phi)$$ \hspace{1cm} (12)

This form of the two-dimensional sine-Gordon equation was studied by the author in [6].

5 The Distance $s$ is Comparable with the Penetration Depth $\lambda_{ab}$ - Influence of Next Layers

In the previous section we described lagrange-Euler equations for the case when the distance $s$ is comparable with the penetration depth $\lambda_{ab}$. We neglected the influence of neighbouring layers. Let us consider now the effect of nearest neighbour layers. The interlayer inductance for neighbouring layers in this case $L_{n,n+1}$ is
The equation for the phase differences $\phi_{m,m+1}$ has the form

$$-L_{n,n} \nabla^2 \phi_{n,n+1} + L_{n+1,n} \nabla^2 \phi_{n+1,n+2} - L_{n-1,n} \nabla^2 \phi_{n-1,n} + \frac{1}{\lambda^2} \sin(\phi_{n,n+1}) = 0$$

(14)

Let us denote

$$\lambda_p = \lambda_J \sqrt{L_{n,n}}$$

(15)

The two-dimensional sine-Gordon equation takes now the form

$$-\nabla^2 \phi_{n,n+1} - l \nabla^2 \phi_{n+1,n+2} - l \nabla^2 \phi_{n-1,n} + \frac{1}{\lambda_p^2} \sin(\phi_{n,n+1}) = 0$$

(16)

where the parameter $l$ is small, and it is defined as

$$l = (1 - \frac{s}{\lambda_{ab}})^1$$

(17)

We left the exponent 1 in order to indicate how the next to the nearest layers will influence the Lagrange-Euler equations. In this form of the two-dimensional sine-Gordon equation three neighbouring planes are coupled explicitly. The solution of the equation (16) is again looking in the homogeneous form, in which we again introduce the variable

$$\phi = \phi_{n,n+1}$$

(18)

for every $n$. This again may be done due to expected homogeneity in the sense mentioned above as concerning the states in planes in this limit.

The equation (16) takes now the form

$$-\nabla^2 \phi + \frac{1}{\lambda_p^2} \sin(\phi) = 0$$

(19)

where we defined the renormalized Josephson length

$$\lambda_p' = \lambda_J \sqrt{1 + 2l}$$

(20)

which is larger than $\lambda_p$ due to the fact that $l$ is positive. Note that the larger $l$ the larger renormalised Josephson length.

This homogeneous solution considered in this section has the characteristic length $\lambda_p'$ larger than in the one-layer case length $\lambda_p$ due to the inter-layer coupling via interlayer inductance.

The free energy of the single vortex in every layer and for the same homogeneous state (homogeneous as concerning dependence on the the number $n$ of the layer, the vortex state is of course nonhomogeneous state) in all layers depends linearly on $L$, the system linear dimension.
where $N$ is the number of layers, the cutoff constant $a_0$ is a constant of the order of lattice constant.

We see that if the coupling between layers is taken into account then the free energy is lower due to smaller energy of the vortex. The interlayer coupling decreases the free energy and stabilizes the state with vortices the cores of which are localizes on the same line in the direction $c$ perpendicular to the planes $ab$. Taking into account even higher order inductance couplings it is expected that further lowering of the free energy will be found and that this will further stabilize the vortex state.

Let us consider the solution of the equation (16) in which the sign of the angle $\phi$ changes in neighbouring layers. This solution describes a nonhomogeneous state in the sense that the order parameter angle changes its sign in neighbouring planes, i.e. it is shifted by the angle $\pi$

$$\phi_{n,n+1} = (-1)^n \phi$$

for every $n$. The equation (16) takes in this case the form

$$-\nabla^2 \phi + \frac{1}{\lambda_p''^2} \sin(\phi) = 0$$

where the Josephson characteristic length has now the form

$$\lambda_p'' = \lambda_J \sqrt{1 - 2l}$$

This Josephson length is smaller than the Josephson length $\lambda_p$, we still assume $l$ to be a small parameter. Thus the solution in which the sign of angle $\phi$ changes in neighbouring layers has the length $\lambda_p''$ smaller than the length $\lambda_p'$ and smaller than the original in-plane Josephson length $\lambda_p$. The free energy of the state with a single vortex in a given layer and with minus sign in neighbouring layers depends again linearly on $L$, the system linear dimension

$$2^3 N\pi J_2^2 \frac{L}{\lambda_p''} - 2k_B TN \ln\left(\frac{L}{a_0}\right)$$

where again $N$ is the number of layers, the cutoff constant $a_0$ is a constant of the order of the lattice constant.

If the inductance coupling $l$ between layers is taken into account then we have found that the free energy of the state in which the sign of the angle $\phi$ changes in neighbouring layers is higher with respect to the state in which the sign of the angle $\phi$ does not change in neighbouring layers. The free energy of the state in which the sign of the angle $\phi$ changes in neighbouring layers is also higher with respect to the state in which the layers are not coupled at all.

The small parameter $l$ may be taken as an expansion parameter when solving the equation (16). One finds in that case that the homogeneous in above sense
state with a vortex in one layer and with other vortices in other layers with their core center in the same point in the plane ab /a line of vortices in the direction c/ is more stable. Single-vortex state in one layer and the same state in all layers give lower free energy due to the interlayer inductance coupling. The interlayer inductance coupling plays stabilisation role for these vortices. One can ask now that if many-vortex states in planes are realized, which distance between vortices is such that the vortex superconducting state becomes more preferable than the normal state, or than the homogeneous superconducting state without vortices (usual superconducting state in classical superconductors). The free energy of the single vortex in a given layer and the same (homogeneous case as concerning the signs of the phase variable) state with a vortex in neighbouring layers depends linearly on L, the system linear dimension and the (critical) value of the parameter l. At a given temperature it is given by the condition that the free energy with the vortex at the same place in every layer is the same as the free energy of the superconducting state without vortex

\[ F(\text{vortex}) = 2^2 N \pi J q^2 L \frac{L}{\lambda_p} - 2 k_B T N \ln \left( \frac{L}{a_0} \right) = F(\text{hom}) = 0 \]  

(26)

For a given temperature and other parameters of the system increasing the parameter l decreases the energy in the free energy, it may happen then that the vortex state becomes the most stable state. It is clear that the transition to the superconducting state occurs at higher temperature than for the state without vortex. The transition to the single-vortex state with the inter-layer coupling occurs at higher temperature than for the state in which the single-vortex superconducting state does not take into account the inter-layer inductance coupling.

If the density of superconducting pairs depends on temperature, we introduce \( \alpha = \alpha_0 (T - T_c) \), \( T_c \) is the bulk transition temperature, \( \beta \) is a constant, the free energy \( F(\text{hom}) \) is found from (1) for \( |\Psi|^2 = \frac{|\alpha|}{\beta} \). The Lawrence-Doniach functional for the order parameter with \( \Phi \) constant gives

\[ F(\Phi_{\text{hom}}) = -\frac{H_c^2 s L^2 N \alpha^2}{8 \pi \beta} \]  

(27)

where \( H_c \) is the bulk critical magnetic field. The constant J and the bulk critical magnetic field value \( H_c \) are related

\[ J = \frac{H_c^2 s \kappa_{ab} \alpha^2}{4 \pi \beta} \]  

(28)

Here we consider the constant amplitude approximation \( |\Psi|^2 = \frac{|\alpha|}{\beta} \) instead of \( |\Psi|^2 = 1 \), we assume that temperature is near the transition temperature. The gradient term in the phase in this case gives also contribution to the free energy. It is of the order of \( \frac{\Phi}{\lambda_p} \) as a contribution to the bulk critical temperature \( T_c \). Note that the case with non-zero magnetic field may be easily described and will not be considered here.
The normal state is given by \( \Psi = 0 \) and its free energy is 0. The superconducting state with the homogeneous (in the classical sense) superconducting order parameter without vortices is given by \( |\Psi|^2 = \frac{|\alpha|^2}{\beta} \) and its free energy is

\[
F(\text{hom}) = -\frac{NsL^2\zeta^2_{ab}H_c^2}{8\pi} = -\frac{JNL^2\alpha^2}{2\beta^2_{ab}} \tag{29}
\]

The vortex state with the \( N_v \) vortices cores of which are localized in the square /for simplicity/ lattice has the free energy

\[
F(N_v - \text{vortex}) = N_vN[\frac{2^2\pi Jq^2}{\lambda_p} L_v - 2k_BT \ln(\frac{L_v}{a_0})] \tag{30}
\]

The inter-vortex distance \( L_v = \frac{L}{N_v} \) can be found minimizing the free energy \( F(N_v - \text{vortex}) \) with respect to \( L_v \). We have found the minimal value \( L_v \)

\[
L_v = \frac{\sqrt{\pi}\lambda_p k_BT}{\pi Jq^2} \ln(\frac{\sqrt{\pi}\lambda_p k_BT}{\pi Jq^2 a_0}) \tag{31}
\]

The inter-vortex distance /vortex lattice constant/ is increasing with temperature \( T \) and is decreasing with temperature \( T \) depending whether temperature is above or below the critical temperature \( T_c^v \), respectively. The critical temperature \( T_c^v \) is given by

\[
T_c^v = \frac{\pi Jq^2 a_0}{\sqrt{2}\lambda_p k_BT} \tag{32}
\]

The free energy for the vortex lattice state with \( N_v \) vortices in the plane has the form

\[
F(N_v - \text{vortex}) = NL^2[\frac{2^2\pi J^2q^4}{\lambda_p^2 k_BT} \ln(\frac{\sqrt{\pi}\lambda_p k_BT}{\pi Jq^2 a_0}) - \frac{\pi^2 J^2q^4}{\lambda_p^2 k_BT \ln(\frac{\sqrt{\pi}\lambda_p k_BT}{\pi Jq^2 a_0})}] \tag{33}
\]

The free energy of the \( N_v \) - vortex state should be compared with the free energy \( F(N) = 0 \) of the normal (metal) state, and with the free energy of the homogeneous (in the classical sense) superconducting state without vortices given by

\[
F(\text{sc}) = -\frac{JNL^2\alpha^2}{2\beta^2_{ab}} \tag{34}
\]

As can be seen the transition at the transition temperature \( T_{cv} \) from the normal phase to the \( N_v \) - vortex superconducting state occurs, it is of the first order.

Depending on the parameters of the material the transition temperature \( T_{cv} \) may be higher for this transition than the transition temperature in the case of
the second order phase transition from the normal phase to the homogeneous superconducting phase without vortices. It is given by

\[ T_{cv} = \frac{\pi J q^2 a_0}{\sqrt{2 \lambda_p k_B}} \]  

Note that the transition temperature \( T_{cv} \) would be equal zero for zero topological charge \( q \). Note further that the transition temperature \( T_{cv} \) increases when the interlayer inductance increases.

We will now discuss several solutions of the two-dimensional Sine-Gordon equations.

### 6 Single-Vortex Configurations

In [6] we have found the single vortex configuration as a solution of the two-dimensional sine-Gordon equation, and thus of the order parameter. It has the form

\[ \phi = \pm \tan^{-1}\left( \frac{\sqrt{\frac{a}{2-a}} \sinh(\sqrt{\frac{2-a}{a}} \frac{x-x_0}{\lambda})}{\sinh(\sqrt{\frac{2-a}{a}} \frac{y-y_0}{\lambda})} \right) \]

here \( \lambda \) equals \( \lambda_p \). The vorticity of this vortex is plus minus four. In the limit of \( \lambda >> L \) we obtain a usual isotropic vortex configuration. The configuration breaks the rotational symmetry. The free energy of the configuration given by (36) is linear in \( L \). This is in difference to isotropic vortices. In the limit of \( \lambda >> L \) where a usual isotropic vortex appears the logarithmic dependence of the free energy of the configuration on \( L \) is found. Thus in systems in which anisotropic vortices are present and which are small, e.i. in which \( \lambda >> L \), the anisotropy of the vortex will be almost undetectable.

In the limit of \( \lambda_p \gg L \) the anisotropic vortex takes an isotropic form is the zeroth approximation.

\[ \phi = \tan^{-1}\left( \frac{(x-x_0)}{(y-y_0)} \right) \]

In isotropic material systems in which anisotropic vortices are present and which are large enough, e.i. in which \( \lambda << L \), the anisotropy of the vortex will be detectable.

### 7 Multi-Vortex Configurations

Multi - (Resonant-Soliton) - Soliton solutions and vortex-like solutions in two and three dimension for the sine-Gordon equation were studied in [19]. For the two- and three- dimensional sine-Gordon equations there exist exact multi - (resonant-soliton) - soliton solutions and vortex-like solutions, in addition to exact multi-soliton and resonant - soliton solutions. In [20] it was shown that
the quasi-vortex type solutions from [6] can be derived from the multiple soliton solutions by the proper procedure. Thus it follows from there exist multiple vortex-like solutions. Moreover an anisotropic vortex configuration far from the core of the vortex has the form of almost unchanging order parameter regions separated by soliton-like walls.

8 Static Lattice of Vortices, Multi-vortex Solutions

Let us briefly overview solutions of the sine-Gordon equations which represent static lattice of vortices and multi-vortex solutions. These solutions and space periodic vortex chains and lattice and breather solutions were studied in [21].

The boundary problem for two-dimensional sine-Gordon equation was studied by the inverse scattering method in [34]. Relation between vortex solutions of the two-dimensional sine-Gordon equation and solitons of the same equations were studied in [20]. Single vortex solution [6] can be derived from the known multiple soliton solutions by a proper procedure. This fact shows possibility to find multi-vortex solutions of the two-dimensional sine-Gordon equations due to existence of multisoliton solutions.

Vortex-type lattice solutions found by Hirota method and by Backlund transformation represent alternating vortices with \( q = \pm 4 \) and form a tetragonal or square lattice. Interaction of the spin-wave and vortex lattice are similar to the soliton-wave interactions.

Vortex-antivortex pair with the vorticity \( q = \pm 4 \) is given by

\[
\phi = \tan^{-1}\left(\frac{1 - x + \sinh(y) \exp(-x)}{\sinh(y) - (1 + x) \exp(-x)}\right)
\]

The total topological charge of the vortex-antivortex pair is zero. Such pairs may be in pre-vortex-lattice states, as a kind of liquid.

A vortex array was found with topological charges \( q = \pm 4 \). As far as the author is aware the vortex arrays were not observed directly, however several vortex arrays which are parallel may have an equilibrium distance between arrays and thus may be difficult to distinguish them from the tetragonal phase at first sight. However one may expect that stiffness of such configuration in the direction of arrays will be substantially different from that in the perpendicular direction. Numerical singular solutions of the elliptic sine-Gordon equation [24] were studied. A semi-infinite \( 2\pi \) - kink starting from a singular point of the vortex type was found. Ring-like \( 2\pi \) - kinks with a common center in the singular point were found. The singular point with a linear region of the logarithmic spiral was found. Exact solutions of the sine-Gordon equations as rational-exponential solutions are described in [35]. Exact rational-exponential solutions of the two-dimensional sine-Gordon equations are constructed by a method based on the formal perturbation method. These solutions represent the nonlinear superposition of two interacting \( 2\pi \) kinks, at the point of intersection we have a vortex with topological charge plus and minus 4. Numerical
analysis shows that the minimum distance between centers of vortices with opposite topological charges in dipole state exists and is given by approximately 1.1 in lambda units. These two vortices cannot annihilate in spite of zero total topological charge of the dipole.

9 Dynamic Properties of Two-Dimensional Sine-Gordon Vortices

The two-dimensional sine-Gordon equation with time dependence was studied in [25], and it was found that a particular dynamical pattern in a perturbed two-dimensional sine-Gordon systems exists. The stability of vortex-like solutions were checked. The main topological invariant of the vortex-like solutions is due to the total length of the $\pm 2\pi$ wave fronts entering as elementary kink-like patterns in the constitution of the whole configuration. This corresponds to the energy conservation law. In [25] it was also shown that a particular dynamical pattern in a perturbed or not two-dimensional sine-Gordon system exists. Its stability with respect to perturbations and in time was studied. Their main topological invariant is the total length of the kink wave fronts forming the pattern and the invariant is consistent with the conservation of energy in all situations.

In [19] the time dependent sine-Gordon equation without damping was studied. The procedure like that used in [20] leads to a two-soliton solution, thus intersection of these solitons corresponds with a moving vortex. On the other hand the author [19] shows that three or higher number soliton solutions are known to exist only with limited parameter range. Thus time dependence without damping here does not lead to a vortex-lattice in general, this may exist only in a limited range of parameters.

10 Summary and Discussion

In our papers [1] and [2] we studied layered superconducting materials. Vortex-like states were described and compared with normal metallic phase and superconducting state without vortices. Results were found as concerning the influence of neighbouring planes on the vortex state in a given plane and new exact solutions of the coupled Lagrange-Euler sine-Gordon equations were found, their properties and free energy were compared. Three-dimensional superconducting state with a vortex in every plane with cores localized in the same point on a line perpendicular to planes is a stable state in zero external magnetic field for the layered materials. The same holds for a lattice of vortices case. There exists a phase transition at the temperature $T^v_{c}$ between the normal phase and the vortex superconducting state which is of the first order. Depending on the parameters of the material the transition temperature $T^v_{c}$ may be higher than in the case of the second order phase transition from normal phase to the homogeneous (conventional) superconducting phase without vortices. The
transition temperature $T_v^c$ is zero for zero topological charge of vortices. Note that the transition is in zero magnetic field. The first order phase transition has its properties. It shows usually overheating and overcooling effects. Moreover nucleation of the superconducting phase in the normal phase occurs at temperatures higher than the transition temperature $T_v^c$. And vice-versa. Nucleation of the normal phase in the superconducting phase occurs at temperatures lower than the transition temperature $T_v^c$. Thus the onset of the vortex-like excitations above the transition temperature $T_v^c$ occurs in our theory due to presence of nucleation of the superconducting phase with vortices. The onset of the vortex-like Nerst signal above the transition temperature $T_v^c$ in experiments in LSCO, YBCO and in other similar high-temperature materials which we discussed in the Introduction may be explained by our theory. Vortex-like excitations in $La_{2-x}Sr_xCuO_4$ at temperatures significantly above the critical temperature were found in the Nerst effect signal [20]. Such excitations may correspond to formation of nuclei of the vortex-superconducting phase above the transition temperature. In overdoped $La_{2-x}Sr_xCuO_4$ (LSCO) the upper critical field $H_{c2}$ does not end at $T_{c0}$ but at a much higher temperature. The upper critical field $H_{c2}$ would correspond in our theory to a magnetic field at which pairs in nuclei are destroyed, however firstly vortex-antivortex dipols and prelattices should be destroyed. Nerst measurements in $YBa_2Cu_3O_6$ (YBCO) / and in $Bi_2Sr_{2-y}La_yCuO_6$ /, [28], and in LSCO, in fields up to 33 T show the existence of vortex-like excitations high above $T_{c0}$. For these materials the same as above holds as concerning nuclei of superconducting vortex phase and vortices. The high field phase diagram of cuprates derived from the Nernst effect implies that $T_{c0}$ corresponds to a loss in phase rigidity rather than a vanishing of the pairing amplitude, [27]. Phase rigidity should be present for the phase as the macroscopic order parameter. In nuclei this macroscopic character is preserved, but nuclei have smaller volume than a bulk superconducting phase. Thus there is a loss in phase rigidity in the sense that nuclei are not correlated, macroscopic phase in nuclei is not correlated too. In zero magnetic field there is equal number of vortices with plus and minus topological charge. Vortex-antivortex pairs or parts of a vortex lattice are strengthening presence of pairing amplitude due to selfconsistent process described above. In nonzero field there is nonequal number of vortices with plus and minus topological charge. Preformed pairs as superconductor fluctuations may exist in Bose condensation of localised Cooper pairs with short coherence length, [29], [30] and [31]. Pairing correlations without phase coherence are described in [32] and [33]. Our theory is different from just mentioned theories.

Some of recent reviews of the high-temperature superconductors properties can be found in [36], [37], [38], [39], [40] and [41]. The experimental spectroscopic data are more consistent. It is not quite clear what is the nature of connection between antiferromagnetism and superconductivity. Antiferromagnetic phase is the best understood part of the phase diagram. ARPES methods lead to better understanding of the quasiparticle dispersion [42]. Inelastic X-ray Raman scattering [43] give evidence for broken particle-hole symmetry. There is present an incommensurate phase in antiferromagnetic insulator phase [44].
corresponding Hamiltonian contains nearest neighbour, next-nearest neighbour
magnetic interactions and the interaction of spins of the fourth order. Magnetic
Raman scattering \[45\] shows that $A_2$ is very weak in $Gd_2CuO_4$. In the phase
diagram there is a pseudogap region (phase) in YBCO, LSCO and BSCCO (2212).
DC resistivity \[46\] in BSCCO shows contribution of the pseudo-gap to resistivity
at high temperatures (180K - 200K). Pseudogap is seen also in ARPES and
tunneling experiments. While there are several possible interpretations of
the pseudo-gap region presence, like stripes, antiferromagnetic fluctuations,
spin-charge separation and proximity to the QC point, the preformed pairs in
superconducting fluctuations above the critical temperature is also one of such
possibilities \[41\]. These contributions may be due to contribution of vortex-like
excitations which we described theoretically above. Aharonov-Bohm effect
is present in YBCO at 30K above the critical temperature. Oscillations with
flux period h/2e are seen \[47\]. This may correspond to the presence of the
vortex-like excitations (vortex-antivortex pairs, etc.) above the transition
temperature. Phase coherence in BSCCO was studied. Phase stiffness as a function
of temperature at frequencies of 100 GHz and 600 GHz was determined \[48\].
See also \[49\]. The Kosterlitz-Thouless superconducting transition is predicted
when the stiffness intersect this line increasing temperature. It is interesting
that high-frequency stiffness is non-vanishing above the transition (short-length
scales), and it is vanishing at low frequencies (long-length scales). We have
found \[2\] that the inter-vortex distance (vortex lattice constant) for the vor-
tex state is increasing with temperature T and decreasing with temperature T
depending whether the temperature T is above or below the critical transition
temperature, respectively. Fluctuations of the vortex state above the critical
transition temperature are bubbles which may contain pairs or remnants of the
vortex state. The decrease of the inter-vortex distance with increasing tempera-
ture in our theory is consistent with these phase stiffness measurements.
High-frequency superfluid stiffness is non-vanishing (short-length scales) which
would correspond to smaller inter-vortex distance in our theory. Note that we
used $\phi_n(r)$, a change of the order parameter phase around a vortex, for
description of the system. The constant amplitude approximation $|\Psi|^2 = 1$ was
considered, then we considered the constant amplitude approximation in which
the density of superconducting pairs is temperature dependent. The constant
amplitude approximation is quite a good approximation due to the fact that
phase coherence and pairing amplitude are present above the critical tempera-
ture in experiments: the high field phase diagram of cuprates implies that
$T_{c0}$ corresponds to a loss in macroscopic (bulk) phase rigidity rather than a
vanishing of the pairing amplitude. \[27\].

The inter-vortex distance is decreasing above this temperature in our theory
as it is observed in experiments with phase stiffness, see above. The properties
of $\phi_n(r)$, a change of the order parameter phase around a vortex, are more
important than the changes of the density of pairs for description of the vortex
state (phase), and thus changes of the amplitude of the order parameter are less
important.

In this paper we reviewed our results on vortex configurations in two-dimensional
sine-Gordon systems in connection with high-temperature superconductors. In [1] and [2], we discussed the vortex states using exact solutions of these coupled Lagrange-Euler sine-Gordon equations, their properties and their free energy was compared with the normal phase state and the homogeneous (conventional) superconducting state. Stable vortex configurations in zero external magnetic field for the layered materials exist, the same holds for a lattice of vortices case. The first order phase transition overheating and overcooling effects are present in the system. Nucleation of the superconducting phase in the normal phase, which occurs at temperatures higher than the transition temperature $T_{c}^{n}$, then occurs. The onset of the vortex-like excitations above the transition temperature $T_{c}^{n}$, which occurs in our theory, may correspond to the onset of the vortex-like Nernst signal above the transition temperature $T_{c}^{w}$ in LSCO, YBCO and in other similar high-temperature materials. Thus the vortex-like excitations not well understood in experiments may be compared with properties of vortices from our theory. It was found experimentally that the vortex-like excitations above the transition temperature $T_{c}^{n}$ in LSCO, YBCO and in other similar high-temperature materials continuously evolve as concerning their properties to vortex states below the transition temperature $T_{c}^{v}$. This fact may be explained within our theory. Thus our theory may explain some of the experimentally observed properties of high-temperature superconductors described above. It will be interesting to study and compare other properties of vortex-like excitations and vortex states in high-temperature materials and compare these properties with those described by anisotropic vortices.

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