ON HYPERBOLIC KNOTS REALIZING THE MAXIMAL DISTANCE BETWEEN TOROIDAL SURGERIES

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Abstract. For a hyperbolic knot in the 3-sphere, the distance between toroidal surgeries is at most 5, except the figure eight knot. In this paper, we determine all hyperbolic knots that admit two toroidal surgeries with distance 5.

1. Introduction

Let $K$ be a hyperbolic knot in the 3-sphere $S^3$ with exterior $E(K) = S^3 – \text{Int} N(K)$. For a slope $\gamma$ on $\partial E(K)$, $K(\gamma)$ denotes the manifold obtained by $\gamma$-surgery. That is, $K(\gamma) = E(K) \cup V$, where $V$ is a solid torus attached to $E(K)$ along $\partial E(K)$ in such a way that $\gamma$ bounds a meridian disk in $V$. By Thurston’s hyperbolic Dehn surgery theorem [17], $K(\gamma)$ is a hyperbolic 3-manifold, except at most finitely many slopes $r$. If $K(\gamma)$ fails to be hyperbolic, then it is conjectured that $K(\gamma)$ is a Seifert fibered manifold or a toroidal manifold [6]. A 3-manifold is said to be toroidal if it contains an essential torus. If $K(\gamma)$ is toroidal, then the slope $\gamma$ (or the surgery) is said to be toroidal. By [7], a toroidal slope corresponds to an integer or a half-integer under the standard parameterization of slopes by the set $\mathbb{Q} \cup \{1/0\}$ (see [14]). In other words, a toroidal slope runs at most twice along the knot. Eudave-Muñoz [2] constructed an infinite family of hyperbolic knots, denoted by $k(\ell, m, n, p)$ in his notation, each of which admits a half-integral toroidal surgery. In fact, Gordon and Luecke [9] proved that these are the only knots that admit half-integral toroidal surgery.

Let $\Delta(\alpha, \beta)$ denote the distance between slopes $\alpha$ and $\beta$. That is, it is the minimal geometric intersection number between $\alpha$ and $\beta$. As it is well known [3], the figure eight knot admits exactly three toroidal slopes $-4, 0$ and $4$. Notice that $\Delta(-4, 4) = 8$. For convenience, assume that $K$ is not the figure eight knot. Then the distance between toroidal slopes of $K$ is at most $5$ by [5], and one between integral toroidal slopes is at most $4$ by [10]. Thus we see that if $K$ admits two toroidal slopes at distance $5$ then one slope is half-integral, and therefore $K$ is one of Eudave-Muñoz knots $k(\ell, m, n, p)$. In fact, any $k(2, -1, n, 0)$, for $n \neq 1$, admits two toroidal slopes $25n - 16$ and $25n - 37/2$ of distance $5$ [3]. We remark that $k(2, -1, n, 0)$ is obtained from the trefoil component of the Whitehead sister link (the $(-2, 3, 8)$-pretzel link) by doing $-1/(n - 1)$-surgery along the unknotted component, and that $k(2, -1, 1, 0)$ is the trefoil and $k(2, -1, 0, 0)$ is the mirror image of the $(-2, 3, 7)$-pretzel knot. The purpose of this paper is to prove that these are the only hyperbolic knots that realize distance $5$ between toroidal slopes.

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Theorem 1.1. Let $K$ be a hyperbolic knot in $S^3$. If $K$ admits two toroidal slopes $\alpha$ and $\beta$ with $\Delta(\alpha, \beta) = 5$, then $K$ is the Eudave-Muñoz knot $k(2, -1, n, 0)$ for some integer $n \neq 1$, and $\{\alpha, \beta\} = \{25n - 16, 25n - 37/2\}$.

In Figure 1, the knot $k(2, -1, n, 0)$, expressed in a braid form, is obtained after $-1/n$-surgery along the unknotted circle. This is directly obtained from [2, Figure 24]. Notice that $k(2, -1, n_1, 0)$ and $k(2, -1, n_2, 0)$ are not equivalent if $n_1 \neq n_2$, since any knot admits at most one half-integral toroidal slope [11].

Among the toroidal slopes listed in [2], the distance 5 is realized only by the knots in our Theorem 1.1. If the list of [2] is complete, then our Theorem would be its consequence. But it seems to be hard to confirm the completeness, and hence we will take another approach. As in [5, 16], we consider the labelled intersection graphs on two essential tori in the surgered manifolds.

It is conjectured that a hyperbolic knot in $S^3$ has at most 3 toroidal slopes [2] (see also [12, Problem 1.77 A(5)]). In [16], we gave an upper bound 5 for the number of toroidal slopes. Theorem 1.1 implies that the conjecture holds for Eudave-Muñoz knots.

Corollary 1.2. Any Eudave-Muñoz knot admits at most 3 toroidal slopes.

Proof. Let $K = k(\ell, m, n, p)$ be an Eudave-Muñoz knot. Then $K$ has the unique non-integral toroidal slope $r$ by [11]. Let $s = r + 1/2$. Then the slopes $s - 1$ and $s$ yield atoroidal Seifert fibered manifolds [2, 9].

If $K$ is not equivalent to $k(2, -1, n, 0)$, then the distance between two toroidal slopes is at most 4 by Theorem 1.1. Hence $\{s - 2, r, s + 1\}$ gives the set of possible toroidal slopes. If $K = k(2, -1, n, 0)$, then $s = 25n - 18$ and $s + 1$ yields an atoroidal Seifert fibered manifold [3]. Hence $\{s - 2, r, s + 2\} = \{25n - 20, 25n - 37/2, 25n - 16\}$ is the set of possible toroidal slopes. □

In Section 2, we collect basic facts for the analysis of the pair of graphs. In Sections 3 and 4, we will show that there is only one possible pair for the graphs realizing the distance 5. Finally, we prove that the pair determines the knot by using the construction of [9] in Section 5.

2. Preliminaries

Let $K$ be a hyperbolic knot. Assume that $K$ admits two toroidal slopes $\alpha$ and $\beta$ with distance 5. Then one of the slopes is half-integral by [16]. Hence we assume that $\beta$ is half-integral. By [9], we know that $K$ is one of Eudave-Muñoz knots. Notice that $K(\alpha)$ and $K(\beta)$ are irreducible by [13, 18]. For the attached solid torus $V_\gamma$ of $K(\gamma)$ where $\gamma \in \{\alpha, \beta\}$, $K_\gamma$ denotes the core of $V_\gamma$.
Let $\hat{S}$ be an essential torus in $K(\alpha)$.

**Lemma 2.1.** $\hat{S}$ is separating in $K(\alpha)$.

*Proof.* If not, $\alpha = 0$ by homological reason. Thus $K(0)$ contains a non-separating torus, and then $K$ has genus one by [4]. But a genus one knot does not have half-integral toroidal surgery [15].

We can assume that $\hat{S}$ intersects the attached solid torus $V_\alpha$ of $K(\alpha)$ in a disjoint union of $s$ meridian disks, $u_1, u_2, \ldots, u_s$ numbered successively along $V_\alpha$, and that $s$ is minimal among all essential tori in $K(\alpha)$. By Lemma [2.1] $s$ is even. Let $S = \hat{S} \cap E(K)$. Then $S$ is an incompressible and boundary-incompressible, punctured torus properly embedded in $E(K)$, each of whose boundary components has slope $\alpha$ on $\partial E(K)$.

**Lemma 2.2.** $K(\beta)$ does not contain a Klein bottle.

*Proof.* Since $\beta$ is not integral, this follows from [7] Theorem 1.3].

Let $\hat{T}$ be an essential torus in $K(\beta)$. Then $\hat{T}$ is separating and we can assume that $\hat{T}$ intersects the attached solid torus $V_\beta$ in just two meridian disks $v_1$ and $v_2$. Let $T = \hat{T} \cap E(K)$. Then $T$ is an incompressible and boundary-incompressible, twice-punctured torus properly embedded in $E(K)$, where each component of $\partial T$ has slope $\beta$.

We can assume that $S$ and $T$ intersect transversely, and that no arc component of $S \cap T$ is parallel to $\partial S$ in $S$ or $\partial T$ in $T$. Also, we can assume that no circle component of $S \cap T$ bounds a disk in $S$ or $T$ by the incompressibility of $S$ and $T$, and that $\partial u_i$ meets $\partial v_j$ in 5 points for any pair of $i$ and $j$.

If we choose a basis $\{\mu, \alpha\}$ for $H_1(\partial E(K))$, where $\mu$ is the meridian of $K$, then $\beta$ can be represented by $2\alpha \pm 5\mu$, since $\Delta(\mu, \beta) = 2$ and $\Delta(\alpha, \beta) = 5$. (This fact is called that the jumping number between $\alpha$ and $\beta$ is two in the literature [5 [10].)

**Lemma 2.3.** Let $a_1, a_2, a_3, a_4, a_5$ be the points in $\partial u_i \cap \partial v_j$, numbered successively along $\partial u_i$. Then these points appear in the order of $a_1, a_3, a_5, a_2, a_4$ in some direction on $\partial v_j$. In other words, two points in $\partial u_i \cap \partial v_j$ are adjacent on $\partial u_i$ if and only if they are not adjacent on $\partial v_j$, and vice-versa.

*Proof.* This follows from [10] Lemma 2.10].

In the usual way [15 [5], we obtain two graphs $G_S$ on $\hat{S}$ and $G_T$ on $\hat{T}$. More precisely, $G_S$ has $s$ vertices $u_1, u_2, \ldots, u_s$, and $G_T$ has two vertices $v_1$ and $v_2$. The edges are the arc components of $S \cap T$. Note that neither graph has a trivial loop.

An endpoint of an edge $e$ in $G_S$ at $u_i$ has label $j$ if it is in $\partial u_i \cap \partial v_j$. Thus the pair of labels 1 and 2 appears 5 times around $u_i$. Similarly, the endpoints of edges in $G_T$ are labelled, and then the sequence of labels 1, 2, $\ldots$, $s$ appears 5 times around $v_j$. An edge is called a $\{i, j\}$-edge if it has labels $i$ and $j$ at its endpoints.

Let $G = G_S$ or $G_T$. An edge of $G$ is said to be **positive** if it connects two vertices (possibly, the same vertex) with the same parity. Otherwise, it is said to be **negative**. We denote by $G^+$ the subgraph of $G$ consisting of all vertices and positive edges of $G$.

A subgraph $H$ of $G$ on a torus $F$ is said to have an **annulus support** if there is an annulus $A$ on $F$ such that $H \subset \text{Int} A$ and a core of $A$ is essential on $F$, and there is no disk on $F$ containing $H$.
A cycle in $G$ is called a Scharlemann cycle if it bounds a disk face of $G$ and all the edges are positive $\{i, i+1\}$-edges for some $i$. The pair of labels $\{i, i+1\}$ is called the label pair of the Scharlemann cycle. In particular, a Scharlemann cycle consisting of two edges is called an $S$-cycle for short.

**Lemma 2.4.**

1. No two edges can be parallel in both graphs.
2. (The parity rule) An edge is positive in one graph if and only if it is negative in the other.
3. If $\rho$ is a Scharlemann cycle in $G_S$ (resp. $G_T$), then its edges cannot lie in a disk on $\hat{T}$ (resp. $\hat{S}$). In particular, if $\rho$ is an $S$-cycle, then its edges form a cycle with an annulus support in the other graph.

**Proof.** (1) is [5, Lemma 2.1]. (2) can be found in [1, p.279]. (3) is [7, Lemma 3.1] (Recall that $K(\alpha)$ and $K(\beta)$ are irreducible.) \hfill $\square$

The reduced graph $\overline{G}$ of $G$ is obtained from $G$ by amalgamating each family of parallel edges into a single edge. For an edge $e$ of $\overline{G}$, the weight of $e$ is the number of edges in the corresponding family of parallel edges in $G$. In particular, $\overline{G}_T$ is a subgraph of the graph in Figure 2 where the sides of the rectangle are identified to form the torus $\hat{T}$ in the usual way ([5, Lemma 5.2]). Also, $q_i$ denotes the weight of an edge. Notice that $v_1$ and $v_2$ are incident to the same number of loops in $G_T$. Since $G_T$ is determined by non-negative integers $q_i$, we say $G_T \cong G(q_1, q_2, q_3, q_4, q_5)$. Let $Q_i$ denote the family of parallel edges in $G_T$ with weight $q_i$ for $i = 2, 3, 4, 5$.

**Lemma 2.5.** $G_S$ satisfies the following.

1. Any family of parallel positive edges contains at most 3 edges.
2. Any family of parallel negative edges contains at most two edges.

**Proof.** (1) If there are 4 parallel positive edges in $G_S$, then there are two bigons among them which lie on the same side of $\hat{T}$. By Lemma 2.4(1), these 4 edges belong to mutually distinct families of parallel negative edges in $G_T$. But this implies that $K(\beta)$ contains a Klein bottle (see the proof of [8, Lemma 5.2]), which contradicts Lemma 2.2.

(2) Any negative edge in $G_S$ corresponds to a loop at $v_1$ or $v_2$ in $G_T$ by the parity rule. The result follows from that any two loops at $v_i$ are parallel and Lemma 2.4(1). \hfill $\square$

**Lemma 2.6.** $G_T$ satisfies the following.

1. If $s = 4$, then there are no two $S$-cycles with disjoint label pairs.
(2) If $s \geq 4$, then any family of parallel positive edges contains at most $s/2 + 1$ edges. Moreover, if it contains $s/2 + 1$ edges, then two adjacent edges on one end form an $S$-cycle.

(3) Any family of parallel negative edges contains at most $s$ edges.

(4) The faces of two $S$-cycles with disjoint label pairs lie on the same side of $\hat{S}$.

Proof. (1) Let $\rho_1$ and $\rho_2$ be $S$-cycles with disk faces $f_1$ and $f_2$, and label pairs $\{1, 2\}$ and $\{3, 4\}$, say. Let $H_{12}$ and $H_{34}$ be the parts of $V_{\alpha}$ between $u_1$ and $u_2$, $u_3$ and $u_4$, respectively. Then shrinking $V_{12}$ radially to its core in $V_{12} \cup f_1$ gives a Möbius band $B_1$ such that $\partial B_1$ is the loop on $\hat{S}$ formed by the edges of $\rho_1$. Similarly, we obtain another Möbius band $B_2$ whose boundary is disjoint from $\partial B_1$. Let $A$ be an annulus between $\partial B_1$ and $\partial B_2$ on $\hat{S}$. Then $B_1 \cup A \cup B_2$ is a Klein bottle $\hat{F}$ in $K(\alpha)$, which meets the core of $V_{\alpha}$ in two points (after a perturbation). Then $F = \hat{F} \cap E(K)$ gives a twice-punctured Klein bottle in $E(K)$. By attaching a suitable annulus on $\partial E(K)$ to $F$ along their boundaries, we obtain a closed non-orientable surface in $S^3$, a contradiction.

(2) By [13] Lemma 1.4, such family contains at most $s/2 + 2$ edges. Furthermore, if it contains $s/2 + 2$ edges, then we can assume that there are two $S$-cycles with disjoint label pairs $\{1, 2\}$ and $\{s/2 + 1, s/2 + 2\}$ in the family (see [13] Figure 1). By the same construction as in (1), we have two Möbius bands $B_1$ and $B_2$ and an annulus $A$ on $\hat{S}$. In $\hat{S}$, two vertices $u_i$ and $u_{s-i+3}$ are connected by an edge in the family for $i = 3, 4, \ldots, s/2$. Hence $\text{Int} A$ contains an even number of vertices. Then $B_1 \cup A \cup B_2$ is a Klein bottle which meets $V_{\alpha}$ in an even number of meridian disks. This leads to a contradiction as before.

If the family contains $s/2 + 1$ edges, then it contains an $S$-cycle at one end [13] Lemma 1.4.

(3) If $s \geq 4$, then this is [10] Lemma 2.3(1)]. If $s = 2$, then any negative edge in $G_T$ has the same label at its endpoints. Thus if a family of parallel negative edges contains 3 edges, then there would be two edges which are parallel in both graphs. This contradicts Lemma 2.4.

(4) is [13] Lemma 1.7.

Suppose $q_i = s$. Let $e_1, e_2, \ldots, e_s$ be the edges of $Q_i$, numbered successively. We may assume that $e_j$ has label $j$ at $v_1$ for $1 \leq j \leq s$. Then these edges define a permutation $\sigma$ of the set $\{1, 2, \ldots, s\}$ such that $e_j$ has label $\sigma(j)$ at $v_2$. In fact, $\sigma(j) \equiv j + k \pmod{s}$ for some even $k$. This $\sigma$ is called the associated permutation to $Q_i$. By the parity rule, $\sigma$ has at least two orbits, and all orbits have the same length. According to the orbits of $\sigma$, the edges of $Q_i$ form disjoint cycles on $\hat{S}$.

Lemma 2.7. Each of these cycles is essential on $\hat{S}$.

Proof. This is [5] Lemma 2.3.

3. THE CASE THAT $s \geq 4$

In this section, we will show that the case where $s \geq 4$ is impossible. Recall that $q_1 \leq s/2 + 1$ and $q_i \leq s$ for $i \geq 2$ by Lemma 2.6.

Lemma 3.1. $q_1 = s/2$ or $s/2 + 1$.

Proof. Since $2q_1 + q_2 + q_3 + q_4 + q_5 = 5s$, $q_1 \geq s/2$. The result follows from this.
We distinguish two cases.

3.1. \( q_1 = s/2 \). Then \( G_T \cong G(s/2, s, s, s, s) \). We can assume that \( G_T \) has labels as in Figure 3.

Let \( \sigma \) be the associated permutation to \( Q_2 \) such that an edge in \( Q_2 \) has label \( i \) at \( v_1 \) and label \( \sigma(i) \) at \( v_2 \). Notice that the associated permutations to \( Q_3, Q_4 \) and \( Q_5 \) are equal to \( \sigma \).

Lemma 3.2. \( \sigma \) is not the identity.

Proof. Suppose that \( \sigma \) is the identity. Then each vertex of \( G_5 \) is incident to 4 loops. Let \( G(1, s) \) be the subgraph of \( G_5 \) spanned by \( u_1 \) and \( u_s \). Since \( s \geq 4 \), \( G(1, s) \) has an annulus support on \( \hat{S} \), and there are only two possibilities for it as in Figure 4, where the top and bottom edges are identified to form an annulus.

Let \( a \) be the \( \{1, 1\} \)-edge in \( Q_2 \), and let \( a_i \) be its endpoint at \( v_i \) for \( i = 1, 2 \). Let \( e \) (resp. \( f \)) be the \( \{1, s\} \)-loop at \( v_1 \) (\( v_2 \)), and let \( e_1 \) and \( f_1 \) be their endpoints with label 1. Around \( v_1 \), \( a_1 \) and \( e_1 \) are not successive among 5 occurrences of label 1, but \( a_2 \) and \( f_1 \) are successive among 5 occurrences of label 1 around \( v_2 \). By Lemma 2.3 \( a_1 \) and \( e_1 \) are successive among 5 occurrences of label 1 around \( u_1 \), but \( a_2 \) and \( f_1 \) are not successive among 5 occurrences of label 2 around \( u_1 \). But this is not satisfied in both configurations of Figure 4. \( \square \)

Lemma 3.3. \( \sigma^2 \) is the identity. In particular, each orbit of \( \sigma \) has length two, and \( \sigma(i) = i + s/2 \).

Proof. The edges of \( Q_2 \) form disjoint cycles in \( G_5 \) according to the orbits of \( \sigma \), and such cycle is essential on \( \hat{S} \) by Lemma 2.3. Recall that there are at least two such
cycles. Let \( L \) be the cycle corresponding to the orbit of \( \sigma \) containing 1. Let \( a \) and \( b \) be the edges of \( Q_2 \) and \( Q_3 \) with label 1 at \( v_1 \), respectively (see Figure 3). Then \( L \cup b \) has an annulus support on \( \hat{S} \). Thus there are two possibilities for \( L \cup b \) as in Figure 5(1) and (2), where \( r = \sigma(1) \). Note that \( a \) and \( b \) have label 1 at \( u_1 \).

For Figure 5(1), there is another edge \( e \) between \( a \) and \( b \). Then \( e \) is a negative \( \{1, r\} \)-edge in \( G_T \) with label \( r \) at \( v_1 \) and label 1 at \( v_2 \). Although \( e \) need not be in \( Q_2 \), this implies \( \sigma(r) = 1 \), because any family of negative parallel edges has the same permutation \( \sigma \). Hence \( \sigma^2 \) is the identity.

For Figure 5(2), let \( c \) and \( d \) be the edges of \( Q_4 \) and \( Q_5 \) with label 1 at \( v_1 \), respectively. Then there would be a pair of parallel edges among \( a, b, c, d \) on \( \hat{S} \). Thus the above argument gives the conclusion. \( \square \)

3.4. \( q_1 = s/2 \) is impossible.

Proof. By Lemma 3.3, \( G_T \) consists of \( s/2 \) components, each of which has an annulus support. In fact, any component consists of two vertices and 8 edges. Thus there are at least 4 mutually parallel positive edges. But this is impossible by Lemma 2.5(1). \( \square \)

3.5. \( G(s/2 + 1, s, s, s - 1, s - 1) \) is impossible.

Proof. Assume \( G_T \cong G(s/2 + 1, s, s, s - 1, s - 1) \). By the parity rule, each edge of \( Q_2 \) has labels of the same parity. But then each edge of \( Q_4 \) has labels of opposite parities at its ends. This contradicts the parity rule. \( \square \)

Thus \( G_T \cong G(s/2 + 1, s, s, s - 2) \). We can assume that the edges of \( Q_2 \) have labels 1, 2, \ldots, \( s \) at \( v_1 \) as in Figure 4. Then there is an \( S \)-cycle with label pair \( \{s/2, s/2 + 1\} \) among loops at \( v_1 \). Let \( \sigma \) be the associated permutation to \( Q_2 \) as before. Notice that \( Q_3 \) and \( Q_5 \) also associate to \( \sigma \). (Although \( q_5 = s - 2 \), the associated permutation is obviously defined.)

3.6. \( \sigma \) is the identity.
Proof. Among loops at $v_2$, there is an $S$-cycle with label pair \{\(r + s/2 - 1, r + s/2\}\}, where $r = \sigma(1)$. Assume that $\sigma$ is not the identity. Then $r$ is odd by the parity rule, and $r \geq 3$.

If $s = 4$, then $r = 3$. Hence $G_T$ contains two $S$-cycles with label pairs \{2, 3\} and \{4, 1\}. But this contradicts Lemma 2.4(1). Hence $s > 4$.

We claim that $\sigma^2$ is the identity. To see this, assume not. As in the proof of Lemma 3.3, the edges of $Q_2$ form disjoint cycles according to the orbits of $\sigma$. Let $L$ be the cycle through $u_1$. Notice that $L$ contains at least three vertices. Let $a$ and $b$ be the edges of $Q_2$ and $Q_3$ with label 1 at $v_1$. Then $L \cup b$ has an annulus support. If $b$ is parallel to $a$, then $\sigma^2$ would be the identity. Hence $L \cup b$ is as shown in Figure 5(2). Let $c$ be the edge of $Q_5$ with label 1 at $v_1$. If $c$ is parallel to $a$ or $b$ on $\hat{S}$, then $\sigma^2$ would be the identity again. Hence $c$ is located as in Figure 5(3). Let $d$ and $e$ be the edges of $Q_2$ and $Q_3$ with label $r$ at $v_1$. Of course, $d$ is in $L$. But $e$ must be parallel to $d$. Hence $\sigma^2$ would be the identity, a contradiction.

Therefore $r = s/2 + 1$. Since $r$ is odd, $s \equiv 0 \pmod{4}$. The edges of $Q_2$ form cycles of length two on $\hat{S}$, and there are at least 4 such cycles. In particular, $u_1$ and $u_{s/2 + 1}$ lie on the same cycle, and so do $u_{s/2}$ and $u_s$.

Also, $G_T$ contain two $S$-cycles with label pairs \{\(s/2, s/2 + 1\)\} and \{\(s, 1\)\}. But we cannot locate the edges of these $S$-cycles to form essential cycles on $\hat{S}$ simultaneously. This contradicts Lemma 2.4(3).

\[\square\]

Lemma 3.7. $s = 4$.

Proof. Assume $s > 4$. Notice that the associated permutation $\tau$ to $Q_4$ is defined as $\tau(i) = i - 2$. Hence the edges of $Q_4$ form two cycles on $\hat{S}$, each of which contains at least three vertices. By Lemma 3.10 each vertex of $G_S$ is incident to a loop. Hence there would be a trivial loop. \[\square\]

Lemma 3.8. The case $q_1 = s/2 + 1$ is impossible.

Proof. In $G_S$, $u_1$ and $u_4$ are incident to 3 loops respectively, and $u_2$ and $u_3$ are incident to two loops. Also, $G_T$ contains two $S$-cycles with label pair \{2, 3\}. The edges of them give 4 edges between $u_2$ and $u_3$ in $G_S$. There are two edges between $u_1$ and $u_3$, and $u_2$ and $u_4$, which belong to $Q_4$ in $G_T$. Hence they are not parallel in $G_S$ by Lemma 2.2(1). Thus $G_S$ is as shown in Figure 6.

Let $a, b, d, e$ be the edges of $Q_2, Q_3, Q_4, Q_5$ with label 1 at $v_1$. Also, let $c$ be the loop at $v_1$ with label 1. By Lemma 2.2 these edges appear in the order $a, c, e, b, d$ around $u_1$ in some direction. Let $f$ be the edge of $Q_4$ with label 3 at $v_1$. Then the endpoints of $b$ and $f$ at $v_2$ are consecutive among 5 occurrences of label 1.
4. The case that \( s = 2 \)

In this section, we show that there is only one possible pair of the graphs, which will lead to the determination of knots in the next section.

The reduced graphs \( \overline{G}_S \) and \( \overline{G}_T \) are subgraphs of the graph as shown in Figure \ref{fig:graphs}. We use \( p_i \) (resp. \( q_i \)) to denote the weight of edge in \( G_S \) (resp. \( G_T \)). In Figure \ref{fig:graphs} \( q_i \) is indicated, but \( p_i \) is assigned at the same place as \( q_i \). Also, we say \( G_S \sim G_T \) or \( G_T \sim G_S \) similarly to \( G_T = G(4,2,0,0,0) \). Since \( 2p_1 + p_2 + p_3 + p_4 + p_5 = 10 \), \( p_1 \geq 1 \).

Lemma 4.1. \( p_1 = 1 \) is impossible.

Proof. If \( p_1 = 1 \), then \( p_i = 2 \) for any \( i \neq 1 \). Then \( G_T = G(4,2,0,0,0) \) or \( G(4,1,1,0,0) \). But these are eliminated as in the proof of Lemma \ref{lem:case1}.

Thus \( p_1 = 2 \) or 3.

Lemma 4.2. If \( p_1 = 2 \), then the graphs are as shown in Figure \ref{fig:graphs}. The correspondence between the edges in \( G_S \) and \( G_T \) is uniquely determined up to symmetry.

Proof. Suppose \( p_1 = 2 \). By the parity rule, \( p_2 + p_3 \) and \( p_4 + p_5 \) is even. Hence we can assume that \( p_2 + p_3 = 4 \) and \( p_4 + p_5 = 2 \). Then \( p_2 = p_3 = 2 \) by Lemma \ref{lem:case1}. If \( p_4 = p_5 = 1 \), then the labels in \( G_S \) contradicts the parity rule. Thus we can assume that \( G_S \cong G(2,2,2,2,0) \), and then there are 4 possibilities for \( G_T \) up to symmetry as in Figure \ref{fig:graphs}.

If \( G_T \) is (1), then the endpoints of two negative \{1, 1\}-edges at \( v_1 \) are not successive among 5 occurrences of label 1. But these points are also not successive at \( u_1 \). This contradicts Lemma \ref{lem:case1}.

To eliminate (2), notice that \( G_S \) contains two \( S \)-cycles \( \rho_1 \) and \( \rho_2 \) whose faces lie on the same side of \( \hat{T} \). From the labeling of \( G_S \), we can determine the edges of \( \rho_i \) in \( G_T \) as in Figure \ref{fig:graphs}. Then it is impossible to connect these edges of \( \rho_1 \) and \( \rho_2 \) on \( \partial V_\beta \) simultaneously.

Clearly, (3) contradicts the parity rule.
For (4), the edge correspondence is determined in the same way as [10, Lemma 5.3] by using the fact that the jumping number is two. Notice that this pair coincides with that of [10, Figures 5.2(e) and 5.3(b)], when $G_T$ is regarded as a graph in an annulus. □

Lemma 4.3. If $p_1 = 3$, then the graphs are the same as in Figure 7 with exchanging $G_S$ and $G_T$.

Proof. We may assume that $(p_2 + p_3, p_4 + p_5) = (2, 2)$ or $(4, 0)$ by symmetry. In the former case, there are three possibilities for $G_S$, up to equivalence, as shown in Figure 8 (1), (2) and (3).

(3) is impossible by the parity rule. If $G_S$ is (2), then the labeling of $G_S$ implies that $G_T$ contains an $S$-cycle at each vertex. It is easy to see that their faces lie on the same side of $\hat{S}$. In fact, $G_T \cong G(2, 2, 2, 2, 0)$. Hence the argument in the proof of Lemma 4.2 works again (with an exchange of roles between $G_S$ and $G_T$). If $G_S$ is (1), then the endpoints of two $\{1, 1\}$ negative edges at $u_1$ are not successive among 5 occurrences of label 1. Hence Lemma 2.3 implies that $(q_2 + q_3, q_4 + q_5) = (6, 0)$, up to symmetry. But this is impossible since $q_i \leq 2$ for $i \neq 1$. 

Figure 8.

Figure 9.
Thus \((p_2 + p_3, p_4 + p_5) = (4, 0)\), giving \(p_2 = p_3 = 2\). Hence \(q_1 = 2\), and so we can assume that \((q_2 + q_3, q_4 + q_5) = (6, 0)\) or \((4, 2)\). But \((6, 0)\) is impossible again. By using the parity rule, it is easy to see that \(G_T\) is as in Figure 7 (with exchanging \(G_S\) and \(G_T\)).

We have thus shown that there is only one possibility for the pair \{\(G_S, G_T\)\} realizing the distance 5, as shown in Figure 7 (or with an exchange of \(G_S\) and \(G_T\)).

5. Determining the knot

In this section, we will show that the only possibility for \{\(G_S, G_T\)\} corresponds to the Eudave-Muñoz knot \(k(2, -1, n, 0)\). We follow the construction in [9] Section 5, and use the same notation as far as possible. We assume that the pair is as in Figure 7 with exchanging \(G_S\) and \(G_T\) to fit the notation to [9]. As shown in Lemmas 4.2 and 4.3, \(G_S\) and \(G_T\) can be exchanged. Hence we cannot say which of \(\alpha\) and \(\beta\) is non-integral with respect to the original framing of \(K\) in the following argument.

Let \(f_1\) and \(f_2\) be the bigons in \(G_S\) bounded by the edges \(c\) and \(j\), \(a\) and \(c\), respectively. Also, let \(f_3\) be the 3-gon bounded by \(a\), \(g\) and \(f\). Up to homeomorphism, we can assume that these edges are as in Figure 10 on \(\hat{T}\), For \(i = 1, 2\), let \(A_i\) be the annulus support of the edges of \(f_i\), and let \(A'_i = \text{cl}(\hat{T} - A_i)\).

Let \(K(\beta) = M_1 \cup M_2\), and let \(H_i = V_\beta \cap M_i\). We may assume that \(f_i\) lies in \(M_i\). Then \(W_i = N(A_i \cup H_i \cup f_i) \subset M_i\) is a solid torus [7] Lemma 3.7], and moreover \(M_i = W_i \cup W'_i\), where \(W'_i\) is a solid torus, is a Seifert fibered manifold over the disk with two exceptional fibers, one of which has index two [7] Lemma 3.8]. Let \(\partial W_i = A_i \cup C_i\). Take a disk (rectangle) \(R_i\) in \(\text{cl}(W_i - H_i)\) as in [9]. See Figures 11 and 12.

Now, regard \(\hat{T} \cup C_1 \cup C_2 \cup R_1 \cup R_2 \cup V_\beta\) as a subset of \(S^3\) as follows. Embed \(\hat{T}\) as a standard torus in \(S^3\), which splits \(S^3\) into two solid tori \(U_1\) and \(U_2\). We assume that the components of \(\partial A_1\) are longitudes of \(U_1\), and those of \(\partial A_2\) are meridians of \(U_1\). Then \(C_i\) is identified with the obvious annulus in \(U_i\), separating \(U_i\) into two solid tori \(V_i\) and \(V'_i\), where \(\partial V_i = A_i \cup C_i\) and \(\partial V'_i = A'_i \cup C_i\). Finally, embed \(V_\beta\) in the obvious way, and \(R_i\) as in Figures 11 and 12.
Figure 11.

Figure 12.

Let \( K_i \) be a core of \( V_i \) disjoint from \( H_i \cup R_i \), and let \( K'_i \) be a core of \( V'_i \). Notice that \( N(\hat{T} \cup C_1 \cup C_2) \) is the exterior of the link \( L_0 = K_1 \cup K'_1 \cup K_2 \cup K'_2 \). Since \( W'_i \) is a solid torus, it is obtained from \( V'_i \) by some Dehn surgery on \( K'_i \). Similarly, \( (W_i; H_i, R_i) \) is obtained from \( (V_i; H_i, R_i) \) by some Dehn surgery on \( K_i \). Let \( K_0 \) be a core of \( V_\beta \subset S^3 \). Thus \( (K(\beta), K_\beta) \) is obtained from \( (S^3, K_0) \) by some Dehn surgery on \( L_0 \). In other words, \( E(K) \) is obtained from \( E(K_0) \) by Dehn surgery on \( L_0 \). See Figure 13.

**Lemma 5.1.** In the surgery description of \( E(K) \) in Figure 13, \( \alpha = -3/5 \) and \( \beta = 1/0 \).

**Proof.** This is obvious for \( \beta \). For \( \alpha \), we draw \( \partial u_1 \) on \( V_\beta \). See Figure 14. Thus \( \alpha = -3/5 \). \( \square \)

**Lemma 5.2.** The filling slopes for \( (K_1, K'_1, K_2, K'_2) \) are \( (-2, 3, -2, p/q) \), where \( p/q \neq 0/1 \).

**Proof.** In Figure 14 it is easy to see that there is an annulus between \( \partial f_1 \) and the curve of slope \(-2/1\) on \( \partial N(K_1) \). This is similar for \( K_2 \). Although we cannot determine the filling slope on \( K'_2 \), it cannot be \( 0/1 \), since a regular fiber of the Seifert fibration in \( M_2 \) is isotopic to a core of \( A_2 \) (see [7, Lemma 3.8]).
To determine the slope on $K_1'$, we need the 3-gon $f_3$. Notice that $\partial f_3$ is non-separating on $\partial W_1'$. Hence $f_3$ gives a meridian disk of $W_1'$. Let $f$ be the disk obtained by band summing $f_3$ and $f_1$ along the band on $\hat{T}$ as shown in Figure 15. Then it is easy to see that there is an annulus between $\partial f$ and the curve of slope $3/1$ on $\partial N(K_1')$. □

Lemma 5.3. $E(K)$ has the surgery description as in Figure 16 where the filling slopes for $(K_1', K_2')$ are $(4, (p+q)/q)$. Also, $\alpha = 1/2$ and $\beta = 3$ on $K_0$.

Proof. In the surgery description in Figure 13 perform a Kirby-Rolfsen calculus [14] in the following way (keeping the same symbols for the knots): a 1-twist on $K_0$, a 1-twist on $K_2$, and finally a 1-twist on $K_1$. This sequence of twisting gives the surgery description of Figure 16. □

Let $\mu$ be the meridian of $K$ in $S^3$.

Lemma 5.4. If $\alpha \notin \mathbb{Z}$ and $\beta \in \mathbb{Z}$ with respect to the original framing of $K$, then $\mu = 1/0$ on $K_0$ in the surgery description of Figure 16. If $\alpha \in \mathbb{Z}$ and $\beta \notin \mathbb{Z}$, then $\mu = 1/1$.
Proof. Let \( \mu = m/n \) on \( K_0 \) in Figure 16. By Lemma 5.3, \( \alpha = 1/2 \) and \( \beta = 3 \) on \( K_0 \). If \( \alpha \notin \mathbb{Z} \) and \( \beta \in \mathbb{Z} \) with respect to the original framing of \( K \), then \( \Delta(\mu, \alpha) = |2m - n| = 2 \) and \( \Delta(\mu, \beta) = |3n - m| = 1 \). (Recall that \( \alpha \) is half-integral.) This gives \((m, n) = (1, 0) \) or \((-1, 0) \). Hence \( \mu = 1/0 \).

If \( \alpha \in \mathbb{Z} \) and \( \beta \notin \mathbb{Z} \) with respect to the original framing of \( K \), then a similar calculation shows that \((m, n) = (1, 1) \) or \((-1, -1) \). Thus \( \mu = 1/1 \). \( \square \)

Therefore, \( K \) can be identified with \( K_0 \) in Figure 16 when \( \alpha \notin \mathbb{Z} \) and \( \beta \in \mathbb{Z} \). Otherwise, \( K \) is the image of \( K_0 \) after a \((-1)\)-twisting along \( K_0 \).

**Lemma 5.5.** If \( \alpha \notin \mathbb{Z} \) and \( \beta \in \mathbb{Z} \) with respect to the original framing of \( K \), then \( 4p + 3q = \pm 1 \). Otherwise, \( 4q - 3p = \pm 1 \).

**Proof.** Assume that \( \alpha \notin \mathbb{Z} \) and \( \beta \in \mathbb{Z} \). By Lemma 5.3, the surgery description of Figure 16 deleting \( K_0 \), gives \( S^3 \). A Kirby-Rolfsen calculus shows that this is equivalent to the trivial knot with framing \(-4p + 3q)/(3p + 2q)\) as in Figure 17. Hence \( 4p + 3q = \pm 1 \).

When \( \alpha \in \mathbb{Z} \) and \( \beta \notin \mathbb{Z} \), first do a \((-1)\)-twist on \( K_0 \) in Figure 16 and eliminate \( K_0 \). This gives \( S^3 \), and Figure 18 shows that this is equivalent to the trivial knot with framing \((4q - 3p)/(q - p)\). Hence \( 4q - 3p = \pm 1 \). \( \square \)

**Proof of Theorem 1.1.** First assume that \( \alpha \notin \mathbb{Z} \) and \( \beta \in \mathbb{Z} \). We modify the surgery link of Figure 16 furthermore. After a \((-3)\)-twisting along \( K' \), do a \((-1)\)-twist
along $K'_1$. See Figure 17, where $K = K_0$ is expressed in a braid form. The framing on $K'_2$ is $-(4p + 3q)/(3p + 2q) = \pm 1/(3p + 2q)$ by Lemma 5.5 and $\alpha = -37/2$ and $\beta = -16$.

Put $n = (4p + 3q)(3p + 2q) = \pm (3p + 2q)$. After a 1-twisting along $K'_2$, we have the Whitehead sister link, where the framing on the unknotted component is $-1/(n - 1)$. Hence $K = k(2, -1, n, 0)$, and $\alpha = 25n - 37/2$, $\beta = 25n - 16$. It is easy to see that $n \neq 1$ and $n$ can be any other value.

Next, assume that $\alpha \in \mathbb{Z}$ and $\beta \notin \mathbb{Z}$. In this case, we perform a $(-1)$-twist along $K_0$ in Figure 19. Then $\mu = 1/0$ in the resulting surgery link, and so $K$ can be identified with $K_0$. Figure 20 starts from this link. At this point, $\alpha = 1$ and $\beta = -3/2$. After a $(-1)$-twist along $K'_2$, do a 1-twist along $K'_1$. Figure 20 shows only the last result, where the framing on $K'_2$ is $(4q - 3p)/(q - p) = \pm 1/(q - p)$ by Lemma 5.5 and $\alpha = 9$, $\beta = 13/2$. Again, this is the Whitehead sister link with the framing $\pm 1/(q - p)$ on the unknotted component. Hence $K = k(2, -1, n, 0)$, where $n = -(4q - 3p)(q - p) + 1 = \pm (q - p) + 1$, and $\alpha = 25n - 16$, $\beta = 25n - 37/2$. □

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