Metric dimension of lexicographic product of some known graphs

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Abstract

For an ordered set \( W = \{w_1, w_2, \ldots, w_k\} \) of vertices and a vertex \( v \) in a connected graph \( G \), the ordered \( k \)-vector \( r(v|W) := (d(v, w_1), d(v, w_2), \ldots, d(v, w_k)) \) is called the (metric) representation of \( v \) with respect to \( W \), where \( d(x, y) \) is the distance between the vertices \( x \) and \( y \). The set \( W \) is called a resolving set for \( G \) if distinct vertices of \( G \) have distinct representations with respect to \( W \). The minimum cardinality of a resolving set for \( G \) is its metric dimension. In this paper, we investigate the metric dimension of the lexicographic product of graphs \( G \) and \( H \), \( G[H] \) for some known graphs.

Keywords: Lexicographic product; Resolving set; Metric dimension; Basis; Adjacency dimension.

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1 Introduction

In this section, we present some definitions and known results which are necessary to prove our main results. Throughout this paper, \( G \) is a simple graph with vertex set \( V(G) \), edge set \( E(G) \) and order \( n(G) \). We use \( \overline{G} \) for the complement of graph \( G \). The distance between two vertices \( u \) and \( v \), denoted by \( d_G(u, v) \), is the length of a shortest path between \( u \) and \( v \) in \( G \). Also, \( N_G(v) \) is the set of all neighbors of vertex \( v \) in \( G \). We write these simply \( d(u, v) \) and \( N(v) \), when no confusion can arise. The diameter of a connected graph \( G \) is \( \text{diam}(G) = \max_{u,v \in V(G)} d(u, v) \).

The symbols \((v_1, v_2, \ldots, v_n)\) and \((v_1, v_2, \ldots, v_n, v_1)\) represent a path of order \( n \), \( P_n \), and a cycle of order \( n \), \( C_n \), respectively. We also use notation \( 1 \) for the vector \((1,1,\ldots,1)\) and \( 2 \) for \((2,2,\ldots,2)\).

For an ordered subset \( W = \{w_1, \ldots, w_k\} \) of \( V(G) \) and a vertex \( v \) of a connected graph \( G \), the metric representation of \( v \) with respect to \( W \) is

\[
r(v|W) = (d(v, w_1), \ldots, d(v, w_k)).
\]
The set $W$ is a resolving set for $G$ if the distinct vertices of $G$ have different metric representations, with respect to $W$. A resolving set $W$ for $G$ with minimum cardinality is a metric basis of $G$, and its cardinality is the metric dimension of $G$, denoted by $\dim(G)$.

The concepts of resolving sets and metric dimension of a graph were introduced independently by Slater [12] and by Harary and Melter [6]. Resolving sets have several applications in diverse areas such as coin weighing problems [11], network discovery and verification [2], robot navigation [9], mastermind game [4], problems of pattern recognition and image processing [10], and combinatorial search and optimization [11]. For more results about resolving sets and metric dimension see [1, 3, 4, 5, 7].

The lexicographic product of graphs $G$ and $H$, denoted by $G[H]$, is a graph with vertex set $V(G) \times V(H) := \{(v, u) \mid v \in V(G), u \in V(H)\}$, where two vertices $(v, u)$ and $(v', u')$ are adjacent whenever, $v$ is adjacent to $v'$, or $v = v'$ and $u$ is adjacent to $u'$. When the order of $G$ is at least 2, it is easy to see that $G[H]$ is a connected graph if and only if $G$ is a connected graph.

Jannesari and Omoomi [8] studied the metric dimension of the lexicographic product of graphs using adjacency dimension of graphs.

Let $G$ be a graph, and $W = \{w_1, \ldots, w_k\} \subseteq V(G)$. For each vertex $v \in V(G)$, the adjacency representation of $v$ with respect to $W$ is the $k$-vector

$$r_2(v|W) = (a_G(v, w_1), \ldots, a_G(v, w_k)),$$

where $a_G(v, w_i) = \min\{2, d_G(v, w_i)\}; 1 \leq i \leq k$. The set $W$ is an adjacency resolving set for $G$ if the vectors $r_2(v|W)$ for $v \in V(G)$ are distinct. The minimum cardinality of an adjacency resolving set is the adjacency dimension of $G$, denoted by $\dim_2(G)$. An adjacency resolving set of cardinality $\dim_2(G)$ is an adjacency basis of $G$.

We say that a set $W$ (adjacency) resolves a set $T$ of vertices in $G$ if the (adjacency) metric representations of vertices in $T$ with respect to $W$ are distinct. To determine whether a given set $W$ is a (an adjacency) resolving set for $G$, it is sufficient to look at the (adjacency) metric representations of vertices in $V(G) \setminus W$, because $w \in W$ is the unique vertex of $G$ for which $d(w, w) = 0$.

In this paper we investigate metric dimension of the lexicographic product, for some known graphs. Jannesari and Omoomi [8] studied the metric dimension of the lexicographic product of graphs. To express their results we need some definitions.

Two distinct vertices $u, v$ are said twins if $N(v) \setminus \{u\} = N(u) \setminus \{v\}$. It is called that $u \equiv v$ if and only if $u = v$ or $u, v$ are twins. In [7], it is proved that “$\equiv$” is an equivalent relation. The equivalence class of a vertex $v$ is denoted by $v^*$. Hernando et al. [7] proved that $v^*$ is a clique or an independent set in $G$. As in [7], we say $v^*$ is of type (1), (K), or (N) if $v^*$ is a class of size 1, a clique of size at least 2, or an independent set of size at least 2. We denote the number of equivalence classes of $G$ with respect to “$\equiv$” by $\iota(G)$. We mean by $\iota_K(G)$ and $\iota_N(G)$, the number of classes of type (K) and type (N) in $G$, respectively. We also use $a(G)$ and $b(G)$ for the number of all vertices in $G$ which have at least an adjacent twin and a none-adjacent twin vertex in $G$, respectively. On the other way, $a(G)$ is the number of all vertices in the classes of type (K) and $b(G)$ is the number of all vertices in the classes of type (N). Clearly, $\iota(G) = n(G) - a(G) - b(G) + \iota_K(G) + \iota_N(G)$. 

Observation 1.1 Suppose that $u, v$ are twins in a graph $G$ and $W$ resolves $G$. Then $u$ or $v$ is in $W$. Moreover, if $u \in W$ and $v \notin W$, then $(W \setminus \{u\}) \cup \{v\}$ also resolves $G$.

Theorem 1.2 Let $G$ be a connected graph of order $n$. Then,

(i) $\dim(G) = 1$ if and only if $G = P_n$,

(ii) $\dim(G) = n - 1$ if and only if $G = K_n$.

Proposition 1.3 For every connected graph $G$, $\dim(G) \leq \dim_2(G)$.

Proposition 1.4 For every graph $G$, $\dim_2(G) = \dim_2(\overline{G})$.

Let $G$ be a graph of order $n$. It is easy to see that, $1 \leq \dim_2(G) \leq n - 1$. In the following proposition, all graphs $G$ with $\dim_2(G) = 1$ and all graphs $G$ of order $n$ and $\dim_2(G) = n - 1$ are characterized.

Proposition 1.5 If $G$ is a graph of order $n$, then

(i) $\dim_2(G) = 1$ if and only if $G \in \{P_1, P_2, P_3, \overline{P_2}, \overline{P_3}\}$.

(ii) $\dim_2(G) = n - 1$ if and only if $G = K_n$ or $G = \overline{K_n}$.

Proposition 1.6 If $n \geq 4$, then $\dim_2(C_n) = \dim_2(P_n) = \left\lfloor \frac{2n+2}{5} \right\rfloor$.

Proposition 1.7 If $K_{m_1,m_2,...,m_t}$ is the complete $t$-partite graph, then

\[
\dim_2(K_{m_1,m_2,...,m_t}) = \dim(K_{m_1,m_2,...,m_t}) = \begin{cases} 
  m - r - 1 & \text{if } r \neq t, \\
  m - r & \text{if } r = t,
\end{cases}
\]

where $m_1, m_2, \ldots, m_r$ are at least 2, $m_{r+1} = \cdots = m_t = 1$, and $\sum_{i=1}^{t} m_i = m$.

Jannesari and Omoomi obtained the metric dimension of lexicographic product of graphs through the following four theorems.

Theorem 1.8 Let $G$ be a connected graph of order $n$ and $H$ be an arbitrary graph. If there exist two adjacency bases $W_1$ and $W_2$ of $H$ such that, there is no vertex with adjacency representation 1 with respect to $W_1$ and no vertex with adjacency representation 2 with respect to $W_2$, then $\dim(G[H]) = \dim(G[\overline{H}]) = n \dim_2(H)$.

Theorem 1.9 Let $G$ be a connected graph of order $n$ and $H$ be an arbitrary graph. If for each adjacency basis $W$ of $H$ there exist vertices with adjacency representations 1 and 2 with respect to $W$, then $\dim(G[H]) = \dim(G[\overline{H}]) = n(\dim_2(H) + 1) - \iota(G)$.

Theorem 1.10 Let $G$ be a connected graph of order $n$ and $H$ be an arbitrary graph. If $H$ has the following properties
(i) for each adjacency basis of $H$ there exists a vertex with adjacency representation 1,

(ii) there exists an adjacency basis $W$ of $H$ such that there is no vertex with adjacency representation 2 with respect to $W$,

then $\dim(G[H]) = n \dim_2(H) + a(G) - \iota_n(G)$.

**Theorem 1.11**  
Let $G$ be a connected graph of order $n$ and $H$ be an arbitrary graph. If $H$ has the following properties

(i) for each adjacency basis of $H$ there exists a vertex with adjacency representation 2,

(ii) there exists an adjacency basis $W$ of $H$ such that there is no vertex with adjacency representation 1 with respect to $W$,

then $\dim(G[H]) = n \dim_2(H) + b(G) - \iota_n(G)$.

**Corollary 1.12**  
If $G$ has no pair of twin vertices, then $\dim(G[H]) = n \dim_2(H)$.

### 2 Main results

In this section we investigate metric dimension of the lexicographic product of graphs for some families of graphs. Theorems 1.8, 1.9, 1.10 and 1.11 imply that to find the exact value of $\dim(G[H])$, we need to know all twin vertices in $G$ and adjacency resolving sets for $H$.

By Corollary 1.12 to compute the $\dim(G[H])$, where $G$ has no any pair of twin vertices, it is enough to obtain the value of $\dim_2(H)$.

**Lemma 2.1**  
If $KG(k, r)$, $k \geq 2r + 1$ be the Kneser graph, then $G$ have no any pair of twin vertices.

**Proof.**  
If $A$ and $B$ are distinct twin vertices in $G$, then $A \cap C = \emptyset$ if and only if $B \cap C = \emptyset$, for each $C \in V(G)$. Now let $C \in V(G) \setminus \{A, B\}$, $A \cap C = \emptyset$, $x \in A \setminus B$, and $y \in C$. Let $D = C \cup \{x\} \setminus \{y\}$. Therefore, $A \cap D \neq \emptyset$ and $B \cap D = \emptyset$, which is a contradiction.  

Note that the line graph of the complete graph $K_n$ is the complement of $KG(n, 2)$. Since all twin vertices of a graph are twins in its complement, as well; by Lemma 2.1, $L(K_n)$, $n \geq 5$, have no any pair of twin vertices. Also, the path $P_n$, $P_n$, $n \geq 4$, and the cycle $C_n$, $C_n$, $n \geq 5$, have no any pair of twin vertices. Thus, by Theorems 1.9 the exact values when $G \in \{P_n \ (n \geq 4), C_n \ (n \geq 5), L(K_n) \ (n \geq 5), K(k, r)\}$ and $H \in \{P_m, C_m, P_m, C_m, K_m, K_m, P, K_{m_1},...,m_r\}$ are obtained.

To study the adjacency basis of a graph $H$, we need the following definitions. Let $S$ be a subset of vertices of $H$, where $|S| \geq 2$. The set of vertices of a nonempty connected component of the induced subgraph by $V(H)\setminus S$ of $H$ is called a gap of $H$. This definition agrees with the one in [3] which is given for the cycle $C_m$. If $Q_1, Q_2$ are two gaps of $S$ for which there exists a vertex $x \in S$ such that the induced subgraph by $Q_1 \cup Q_2 \cup \{x\}$ is connected, then $Q_1$ and $Q_2$ are called neighboring gaps. In [3], the following observation is expressed for the gaps of a basis of $C_m \vee K_1$. Particularly, it is true for an adjacency basis of $C_m$.  


Observation 2.2 If $B$ is an adjacency basis of $C_m$, then

1. Every gap of $B$ contains at most three vertices.
2. At most one gap of $B$ contains three vertices.
3. If a gap of $B$ contains at least two vertices, then any neighboring gaps of which contain one vertex.

It is easy to see the following observation for $P_m$.

Observation 2.3 Let $B$ be an adjacency basis of $P_m = \{w_1, w_2, \ldots, w_m\}$. If $R_1$ and $R_2$ are gaps of $P_n$ with $w_1 \in R_1$ and $w_m \in R_2$, then

1. Every gap of $B$ contains at most three vertices and $|R_i| \leq 2$, where $1 \leq i \leq 2$.
2. At most one gap of $B$ contains three vertices and at most one of the gaps $R_1$ and $R_2$ contains two vertices.
3. If $|R_i| = 2$ for some $i$, $1 \leq i \leq 2$, then all gaps of $B$ contains at most two vertices.
4. If a gap of $B$ contains at least two vertices, then any neighboring gaps of which is neither $R_1$ nor $R_2$ and contain one vertex.

Proposition 2.4 Let $G$ be a connected graph of order $n$ and $H \in \{P_m, C_m\}$, $m = 5k + r$, where $m \not\in \{2, 3\}$.

(i) If $r$ is even, then $\dim(G[H]) = \dim(G[\overline{H}]) = n\lfloor \frac{2m+2}{5} \rfloor$.

(ii) If $m = 6$, then $\dim(G[H]) = \dim(G[\overline{H}]) = n\lfloor \frac{2m+2}{5} \rfloor + a(G) + b(G) - \nu(K(G)) - \nu(N(G))$.

(iii) If $r$ is odd and $m \not= 6$, then $\dim(G[H]) = n\lfloor \frac{2m+2}{5} \rfloor + b(G) - \nu(N(G))$ and $\dim(G[\overline{H}]) = n\lfloor \frac{2m+2}{5} \rfloor + a(G) - \nu(K(G))$.

Proof. Let $P_m = \{w_1, w_2, \ldots, w_m\}$ and $C_m = \{w_1, w_2, \ldots, w_m, w_1\}$. If $m = 4$, then the set $B = \{w_1, w_4\} \subseteq V(H)$ is an adjacency basis of $H$ and $r_2(w_i|B)$ is neither 1 nor 2, for each $i$, $1 \leq i \leq 4$. Therefore, by Proposition 1.6 and Theorem 1.8, $\dim(G[H]) = \dim(G[\overline{H}]) = n\lfloor \frac{2m+2}{5} \rfloor$. If $m = 5$, then the sets $B_1 = \{w_1, w_5\}$ and $B_1 = \{w_2, w_4\}$ are adjacency bases of $H$ and for each $i$, $1 \leq i \leq 5$, $r_2(w_i|B_1)$ is not 1 and $r_2(w_i|B_2)$ is not 2. Hence, by Lemma 1.6 and Theorem 1.8, $\dim(G[H]) = \dim(G[\overline{H}]) = n\lfloor \frac{2m+2}{5} \rfloor$. If $m = 6$, then it is easy to check that for each adjacency basis $A$ of $H$ there exist vertices $x_A, y_A \in V(H)$ such that $x_A \sim w$ and $y_A \sim w$ for each $w \in A$. Consequently, by Theorem 1.9, $\dim(G[H]) = \dim(G[\overline{H}]) = n\lfloor \frac{2m+2}{5} \rfloor + a(G) + b(G) - \nu(K(G)) - \nu(N(G))$.

Now, let $m \geq 7$. By Proposition 1.6, $\dim_2(H) \geq 3$. Let $B$ be an adjacency basis of $H$. Since each vertex of $H$ has at most two neighbors, $r_2(w|B)$ is not 1, for each $w \in V(H)$. If $r$ is even, then, let $S_0 = \{w_5q+2, w_5q+4 | 0 \leq q \leq k-1\}$, $S_2 = S \cup \{w_{5k+1}\}$, and $S_4 = S \cup \{w_{5k+1}, w_{5k+3}\}$. Thus, the set $S_t$, is an adjacency basis of $H$ when $r = t$, $t \in \{0, 2, 4\}$. Also, $r_2(w|S_t)$ is neither 1 nor 2, for each $w \in V(H)$ and $t \in \{0, 2, 4\}$. Hence, by Lemma 1.6 and Theorem 1.8, $\dim(G[H]) = \dim(G[\overline{H}]) = n\lfloor \frac{2m+2}{5} \rfloor$. 


If $r$ is odd, then Observations 2.2 and 2.3 imply that for each adjacency basis $A$ of $H$ there exists a vertex $y_A \in V(H)$ such that $y_A \sim w$ for each $w \in A$. Therefore, by Theorem 1.11, $\dim(G[H]) = n\lfloor \frac{2m+2}{5} \rfloor + b(G) - t_k(G)$. Since the adjacency bases of $H$ and $\overline{H}$ are the same, for each adjacency basis $Q$ of $\overline{H}$ there exists a vertex $x_Q \in V(\overline{H})$ such that $x_Q \sim u$ for each $u \in Q$. Hence, by Theorem 1.10, $\dim(G[\overline{H}]) = n\lfloor \frac{2m+2}{5} \rfloor + a(G) - t_k(G)$. 

**Corollary 2.5** Let $m = 5k + r$. If $H \in \{P_m, C_m\}$, then for all $n \geq 2$,

1. $\dim(K_n[H]) = \begin{cases} 2n - 1 & \text{if } H = P_2 \text{ or } H = P_3, \\ 3n - 1 & \text{if } H \in \{C_3, P_6, C_6\}, \\ n\lfloor \frac{2m+2}{5} \rfloor & \text{otherwise}. \end{cases}$

2. $\dim(P_n[H]) = \begin{cases} 5 & \text{if } n = 2 \text{ and } H = C_3, \\ 2n & \text{if } n \neq 2 \text{ and } H = C_3, \\ n\lfloor \frac{2m+2}{5} \rfloor + 1 & \text{if } n = 2 \text{ and } H \in \{P_2, P_3, P_6, C_6\}, \\ n\lfloor \frac{2m+2}{5} \rfloor + 1 & \text{if } n = 3, r \text{ is odd, and } H \neq C_3, \\ n\lfloor \frac{2m+2}{5} \rfloor & \text{otherwise}. \end{cases}$

3. $\dim(C_n[H]) = \begin{cases} 8 & \text{if } n = 3 \text{ and } H = C_3, \\ 2n & \text{if } n \neq 3 \text{ and } H = C_3, \\ n\lfloor \frac{2m+2}{5} \rfloor + 2 & \text{if } n = 3 \text{ and } H \in \{P_2, P_3, P_6, C_6\}, \\ n\lfloor \frac{2m+2}{5} \rfloor + 2 & \text{if } n = 4, r \text{ is odd, and } H \neq C_3, \\ n\lfloor \frac{2m+2}{5} \rfloor & \text{otherwise}. \end{cases}$

4. $\dim(K_{n_1, n_2, \ldots, n_t}[H]) = \begin{cases} n\lfloor \frac{2m+2}{5} \rfloor + t - j - 1 & \text{if } H = P_2 \text{ and } j \neq t, \\ n(m - 1) + t - j - 1 & \text{if } H = C_3 \text{ and } j \neq t, \\ n(m - 1) & \text{if } H = C_3 \text{ and } j = t, \\ n\lfloor \frac{2m+2}{5} \rfloor + n - j - 1 & \text{if } H \in \{P_3, P_6, C_6\} \text{ and } j \neq t, \\ n\lfloor \frac{2m+2}{5} \rfloor + n - t & \text{if } H \in \{P_3, P_6, C_6\} \text{ and } j = t, \\ n\lfloor \frac{2m+2}{5} \rfloor + n - t & \text{if } m \geq 7 \text{ and } r \text{ is odd,} \\ n\lfloor \frac{2m+2}{5} \rfloor & \text{otherwise}. \end{cases}$

where $n_1, n_2, \ldots, n_j$ are at least 2, $n_{j+1} = \ldots = n_t = 1$, and $\sum_{i=1}^{t} n_i = n$.

**Proof.** Since $K_n$ does not have any pair of none-adjacent twin vertices, by Proposition 2.4, $\dim(K_n[H]) = n\lfloor \frac{2m+2}{5} \rfloor$ for $m \notin \{2, 3, 6\}$. If $H = P_2$ or $H = C_3$, then $K_n[H]$ is the complete graph and hence, $\dim(K_n[P_2]) = 2n - 1$ and $\dim(K_n[C_3]) = 3n - 1$.

Now let $H \in \{P_3, P_6, C_6\}$. Also, let $P_m = (w_1, w_2, \ldots, w_m)$, $C_m = (w_1, \ldots, w_m, w_1)$, and $B$ is a basis of $K_n[H]$. By the proof of Lemma 2.2, $B$ contains at least $\dim_2(H)$ vertices from each set $R_i = \{v_r \in V(K_n[H]) | r = i\}$, and $B \cap R_i$ resolves $R_i$, $1 \leq i \leq n$. Let $J = \{i | \dim_2(H) = |B \cap R_i|\}$. If $|J| \geq 2$, then there exist $i, j, 1 \leq i, j \leq n$, such that $|B \cap R_i| = |B \cap R_j| = \dim_2(H)$. Let $A_i = \{w_s | v_is \in B \cap R_i\}$ and $A_j = \{w_s | v_js \in B \cap R_j\}$. Since $d_{K_n[H]}(v_{rs}, v_{rq}) = a_H(w_s, w_q)$ for each $r, s, q$, $1 \leq r \leq n$, $1 \leq s, q \leq m$, we conclude that $A_i$ and $A_j$ are adjacency bases of $H$. On the other hand, for each adjacency basis $A$ of $H$ there exist a vertex $w \in V(H)$ such that $r_2(w|A) = (1, 1, \ldots, 1)$. Therefore, there exist vertices $v_1, v_2 \in V(H)$ such that $r_2(v_1|A_i) = r_2(v_2|A_j) = (1, 1, \ldots, 1)$. Consequently, $r(v_1|B \cap R_i) = r(v_2|B \cap R_j) = (1, 1, \ldots, 1)$. Also, we have $r(v_1|B \setminus R_i) = r(v_2|B \setminus R_j) = (1, 1, \ldots, 1)$. Hence, $r(v_1|B) = r(v_2|B)$, which is a contradiction. Thus, $|J| \leq 1$. Therefore, $\dim(K_n[H]) \geq \dim_2(H) + n - 1$. On the other
hand, the set \( \{ v_{rs} \in V(K_n[P_3]) | s \neq 3 \} \setminus \{ v_{12} \} \) is a resolving set for \( K_n[P_3] \) with cardinality \( n \dim_2(H) + n - 1 = 2n - 1 \). Also, the set \( \{ v_{rs} \in V(K_n[H]) | 2 \leq s \leq 4 \} \setminus \{ v_{13} \} \) is a resolving set for \( K_n[H] \), for \( H \in \{ P_6, C_6 \} \). Consequently, \( \dim(K_n[P_6]) = \dim(K_n[C_6]) = 3n - 1 \).

\[ \left\{ \begin{array}{ll}
  n\left\lfloor \frac{2m+2}{5} \right\rfloor + n - 1 & \text{if } H \neq C_3 \text{ and } r \text{ is odd}, \\
  2n & \text{if } H = C_3, \\
  n\left\lfloor \frac{2m+2}{5} \right\rfloor & \text{otherwise.}
\end{array} \right. \]

\[ \dim(P_n[H]) = \left\{ \begin{array}{ll}
  4 & \text{if } n = 2 \text{ and } H = \overline{C}_3, \\
  n\left\lfloor \frac{2m+2}{5} \right\rfloor + 1 & \text{if } n = 2, \text{ } r \text{ is odd, and } H \neq \overline{C}_3, \\
  n\left\lfloor \frac{2m+2}{5} \right\rfloor + 1 & \text{if } n = 3 \text{ and } H \in \{ P_2, P_3, P_6, C_6 \}, \\
  7 & \text{if } n = 3, \text{ and } H = \overline{C}_3, \\
  n(m - 1) & \text{if } n \geq 4 \text{ and } H = \overline{C}_3, \\
  n\left\lfloor \frac{2m+2}{5} \right\rfloor & \text{otherwise.}
\end{array} \right. \]

\[ \dim(C_n[H]) = \left\{ \begin{array}{ll}
  n\left\lfloor \frac{2m+2}{5} \right\rfloor + n - t & \text{if } H = P_2, \\
  n(m - 1) + n - t & \text{if } H = \overline{C}_3, \\
  n\left\lfloor \frac{2m+2}{5} \right\rfloor + n - j - 1 & \text{if } H \in \{ P_3, P_6, \overline{C}_6 \} \text{ and } j \neq t, \\
  n\left\lfloor \frac{2m+2}{5} \right\rfloor + n - t & \text{if } H \in \{ P_3, P_6, \overline{C}_6 \} \text{ and } j = t, \\
  n\left\lfloor \frac{2m+2}{5} \right\rfloor + t - j - 1 & \text{if } m \geq 7, \text{ } r \text{ is odd, and } j \neq t, \\
  n\left\lfloor \frac{2m+2}{5} \right\rfloor & \text{otherwise.}
\end{array} \right. \]

where \( n_1, n_2, \ldots, n_j \) are at least 2, \( n_{j+1} = \ldots = n_t = 1 \), and \( \sum_{i=1}^t n_i = n \).

**Corollary 2.7** For \( n \geq 2 \),

(1) \( \dim(K_n[K_m]) = nm - 1 \)

(2) \( \dim(P_n[K_m]) = \left\{ \begin{array}{ll}
  n(m - 1) & \text{if } n \geq 3, \\
  n(m - 1) + 1 & \text{if } n = 2.
\end{array} \right. \)

(3) \( \dim(C_n[K_m]) = \left\{ \begin{array}{ll}
  n(m - 1) & \text{if } n \geq 4, \\
  n(m - 1) + 2 & \text{if } n = 3.
\end{array} \right. \)

(4) \( \dim(K_{n_1,n_2,\ldots,n_t}[K_m]) = \left\{ \begin{array}{ll}
  n(m - 1) + t - j - 1 & \text{if } j \neq t, \\
  n(m - 1) & \text{if } j = t,
\end{array} \right. \)

where \( n_1, n_2, \ldots, n_j \) are at least 2, \( n_{j+1} = \ldots = n_t = 1 \), and \( \sum_{i=1}^t n_i = n \).

(5) \( \dim(K_n[K_m]) = n(m - 1) \)
Corollary 2.8 Let $m_1, \ldots, m_q \geq 2$, $m_{q+1} = \ldots = m_s$, and $m = \sum_{i=1}^{s} m_i$. Then for $n \geq 2$,

1. \[ \dim(K_n[K_{m_1, \ldots, m_s}]) = \begin{cases} n(m-q) & \text{if } q = s, \\ n(m-q) - 1 & \text{otherwise}. \end{cases} \]
2. \[ \dim(P_n[K_{m_1, \ldots, m_s}]) = \begin{cases} n(m-q) & \text{if } q = s, \\ n(m-q - 1) & \text{if } q \neq s \text{ and } n \geq 3, \\ n(m-q - 1) + 1 & \text{otherwise}. \end{cases} \]
3. \[ \dim(C_n[K_{m_1, \ldots, m_s}]) = \begin{cases} n(m-q) & \text{if } q = s, \\ n(m-q - 1) & \text{if } q \neq s \text{ and } n \geq 4, \\ n(m-q - 1) + 1 & \text{otherwise}. \end{cases} \]

4. \[ \dim(K_{n_1, n_2, \ldots, n_t}[K_{m_1, \ldots, m_s}]) = \begin{cases} n(m-q) & \text{if } q = s, \\ n(m-q - 1) & \text{if } q \neq s \text{ and } j = t, \\ n(m-q - 1) + t - j - 1 & \text{otherwise}, \end{cases} \]
where $n_1, n_2, \ldots, n_j$ are at least 2, $n_{j+1} = \ldots = n_t = 1$, and $\sum_{i=1}^{t} n_i = n$.

5. \[ \dim(K_n[\overline{K}_{m_1, \ldots, m_s}]) = \begin{cases} n(m-q) & \text{if } q = s, \\ n(m-q - 1) & \text{otherwise}. \end{cases} \]
6. \[ \dim(P_n[\overline{K}_{m_1, \ldots, m_s}]) = \begin{cases} n(m-q) & \text{if } q = s, \\ n(m-q - 1) & \text{if } q \neq s \text{ and } n \neq 3, \\ n(m-q - 1) + 1 & \text{otherwise}. \end{cases} \]
7. \[ \dim(C_n[\overline{K}_{m_1, \ldots, m_s}]) = \begin{cases} n(m-q) & \text{if } q = s, \\ n(m-q - 1) & \text{if } q \neq s \text{ and } n \neq 4, \\ n(m-q - 1) + 2 & \text{otherwise}. \end{cases} \]

8. \[ \dim(K_{n_1, n_2, \ldots, n_t}[\overline{K}_{m_1, \ldots, m_s}]) = \begin{cases} n(m-q) & \text{if } q = s, \\ n(m-q) - t & \text{otherwise}, \end{cases} \]
where $\sum_{i=1}^{t} n_i = n$.

Corollary 2.9 Let $P$ be the Petersen graph. Then for $n \geq 2$,

1. \[ \dim(K_n[P]) = 3n \]
2. \[ \dim(P_n[P]) = \begin{cases} 3n & \text{if } n \neq 3, \\ 3n + 1 & \text{if } n = 3. \end{cases} \]
3. \[ \dim(C_n[P]) = \begin{cases} 3n & \text{if } n \neq 4, \\ 3n + 2 & \text{if } n = 4. \end{cases} \]
(4) \( \dim(K_{n_1, n_2, \ldots, n_t}[P]) = 4n - t \), where \( \sum_{i=1}^{t} n_i = n \).

(5) \( \dim(K_n[P]) = 4n - 1 \)

(6) \( \dim(P_n[P]) = \begin{cases} 3n & \text{if } n \geq 3, \\ 7 & \text{if } n = 2. \end{cases} \)

(7) \( \dim(C_n[P]) = \begin{cases} 3n & \text{if } n \geq 4, \\ 11 & \text{if } n = 3. \end{cases} \)

(8) \( \dim(K_{n_1, n_2, \ldots, n_t}[P]) = \begin{cases} 3n + t - j - 1 & \text{if } j \neq t, \\ 3n & \text{if } j = t, \end{cases} \)

where \( n_1, n_2, \ldots, n_j \) are at least 2, \( n_{j+1} = \ldots = n_t = 1 \), and \( \sum_{i=1}^{t} n_i = n \).

It is easy to see that if \( G \) is a bipartite graph of order at least 3, then it does not have any pair of adjacent twins. Therefore, by Theorem 1.11, \( \dim(G[H]) = n \dim_2(H) + b(G) - \iota_N(G) \).

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