INJECTIVITY RADIUS BOUNDS IN HYPERBOLIC CONVEX CORES I

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Abstract. A version of a conjecture of McMullen is as follows: Given a hyperbolizable 3-manifold $M$ with incompressible boundary, there exists a uniform constant $K$ such that if $N$ is a hyperbolic 3-manifold homeomorphic to the interior of $M$, then the injectivity radius based at points in the convex core of $N$ is bounded above by $K$. This conjecture suggests that convex cores are uniformly congested. In previous work, the author has proven the conjecture for $I$-bundles over a closed surface, taking into account the possibility of cusps. In this paper, we establish the conjecture in the case that $M$ is a book of $I$-bundles or an acylindrical, hyperbolizable 3-manifold. In particular, we show that if $M$ is a book of $I$-bundles, then the bound on injectivity radius depends on the number of generators in the fundamental group of $M$.

1. Introduction

In this paper, we investigate the geometry of convex cores of hyperbolic 3-manifolds which are homeomorphic to the interior of a book of $I$-bundles or an acylindrical, hyperbolizable 3-manifold. Specifically, we show that if $M$ is a book of $I$-bundles or an acylindrical, hyperbolizable 3-manifold, then there exists a uniform upper bound on injectivity radius for points in the convex core of any hyperbolic 3-manifold homeomorphic to the interior of $M$.

The main result relies on a theorem of Kerckhoff-Thurston [KT] which established the existence of an upper bound on injectivity radius for points in the convex core of hyperbolic 3-manifolds without cusps where the manifolds are homotopy equivalent to a closed surface. The main theorem also makes use of an extension of their theorem [F1] which includes the possibility of cusps. Our main result is:

**Theorem 1.1.** Let $M$ be a book of $I$-bundles or an acylindrical, hyperbolizable 3-manifold. Then there exists a constant $K$ such that if $N$ is a hyperbolic 3-manifold homeomorphic to the interior of $M$ and $x \in C(N)$, then $\text{inj}_N(x) \leq K$.

The main theorem is also related to a conjecture of McMullen:

**Conjecture 1.2.** (McMullen, [Bi]) Let $N$ be a hyperbolic 3-manifold homotopy equivalent to a compact 3-manifold $M$. Then $C(N)$ does not contain an embedded ball of radius $L$, where $L$ depends on the number of generators of $\pi_1(N)$.

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For books of $I$-bundles, we have result which is slightly stronger than what is predicted by the conjecture, because the injectivity radius measures the radius of balls embedded in $N$, not in the convex core of $N$:

**Corollary 1.3.** Let $N$ be a hyperbolic 3-manifold homotopy equivalent to a book of $I$-bundles. Then there exists a constant $L$ such that for $x \in C(N)$, $\text{inj}_N(x) \leq L$, where $L$ depends on the number of generators of $\pi_1(N)$.

The upper bound on injectivity radius, combined with a result of McMullen’s [McM], can be used to show that the limit set varies continuously over the space of hyperbolic 3-manifolds homeomorphic to the interior of a fixed hyperbolizable 3-manifold $M$ under the geometric topology.

**Corollary 1.4.** Let $M$ be a book of $I$-bundles or an acylindrical, hyperbolizable 3-manifold. Let $\{N_i = \mathbb{H}^3/\Gamma_i\}$ be a sequence of hyperbolic 3-manifolds with base point in $C(N_i)$ such that each $N_i$ is homeomorphic to the interior of $M$. If $\{N_i\}$ converges geometrically to $N = \mathbb{H}^3/\Gamma$, then $\{\Lambda_{\Gamma_i}\}$ converges to $\Lambda_{\Gamma}$ in the Hausdorff topology.

Another corollary, which does not involve base point considerations, is:

**Corollary 1.5.** Let $M$ be a book of $I$-bundles or an acylindrical, hyperbolizable 3-manifold. Let $\{N_i = \mathbb{H}^3/\Gamma_i\}$ be a sequence of hyperbolic 3-manifolds homeomorphic to the interior of $M$. If the $\{N_i\}$ converge geometrically to $N = \mathbb{H}^3/\Gamma$, and $\Gamma$ is nonabelian, then $\{\Lambda_{\Gamma_i}\}$ converges to $\Lambda_{\Gamma}$ in the Hausdorff topology.

In a future paper [F2], the main result of this paper will be used to show that for hyperbolic 3-manifolds homeomorphic to the interior of a fixed hyperbolizable 3-manifold with incompressible boundary, the injectivity radius in the convex cores is uniformly bounded above.

In §2 of this paper, we will introduce some background material in hyperbolic geometry and spaces of hyperbolic 3-manifolds, and review the relevant lemmas from [F1]. In §4, we will prove the main theorem in the case of a book of $I$-bundles. In §5, we will prove the main theorem in the case of an acylindrical, hyperbolizable 3-manifold. In the last section, we present some corollaries.

## 2. Background Material

### 2.1. Hyperbolic Geometry and Kleinian Groups.** In this section, we review the relevant lemmas from [F1] used in the proof of the main theorem. For the sake of brevity, we will assume that the reader is familiar with the background material given in Section 2 of [F1]. More details about hyperbolic geometry and Kleinian groups can be found in Beardon [Be], Benedetti and Petronio [BP], Canary-Epstein-Green [CEG], and Maskit [Ma].

First we note that because hyperbolic space is negatively curved, injectivity radius strictly increases while travelling out the product structure of a geometrically finite end.

**Lemma 2.1.** (Lem 2.1 in [F1]) Let $N = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold. Let $U$ be a component of $N - C(N)$, and let $S$ be the component of $\partial C_1(N)$ associated to $U$. Then $U$ has a product structure $S \times [0, \infty)$ with nearest point coordinates. For $(x, t) \in S \times (0, \infty)$, the function $\text{inj}_N(x, t)$ is strictly increasing in $t.$
Next, we see that given an upper bound on injectivity radius in a covering space, one can also deduce an upper bound on injectivity radius in the base manifold.

**Lemma 2.2.** (Lem 2.2 in [F1]) Let M be a cover of N, with covering map p. Then for $x \in M$, $\text{inj}_N(p(x)) \leq \text{inj}_M(x)$.

For a cover associated to a boundary component of the convex core of a manifold, a component of the complement of the convex core of the base space lifts homeomorphically to a component of the complement of the convex core of the cover.

**Lemma 2.3.** (Lem 2.3 in [F1]) Let $\delta > 0$. Let $N$ be a hyperbolic 3-manifold. Let $U$ be a component of $N - \text{int} C(N)$, and let $S$ be the component of $\partial C(N)$ associated to $U$. Let $M = \mathbb{H}^3/\pi_1(S)$ be a cover of $N$ with projection map $p$. Then there exists a lift $\tilde{U}$ of $U$ such that $p|\tilde{U} : \tilde{U} \rightarrow U$ is a homeomorphism, $(p|\tilde{U})^{-1}(S)$ is a component of $\partial C(M)$, and $\tilde{U}$ is a component of $M - \text{int} C(M)$. Furthermore, if $T \subset U$ is a component of $\partial C_\delta(N)$, then $(p|\tilde{U})^{-1}(T)$ is a component of $\partial C_\delta(M)$.

We can also show that if a neighborhood of a geometrically finite end of a covering space $M$ embeds as a neighborhood of a geometrically finite end in the base manifold, then the neighborhood associated to $M - \text{int} C(M)$ embeds in the base manifold as well.

**Lemma 2.4.** (Lem 2.4 in [F1]) Let $0 < \epsilon < \epsilon_3$, and let $\delta \geq 0$. Let $M = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold which covers another hyperbolic 3-manifold $N = \mathbb{H}^3/\Gamma$ with projection map $p$. Let $E$ be a geometrically finite end of $M_\epsilon^\circ$, and let $\tilde{V}$ be a component of $M - \text{int} C_\delta(M)$ such that $\tilde{V} \cap M_\epsilon^\circ$ is a neighborhood of $E$. Let $\tilde{S} = \partial C_\delta(M) \cap \tilde{V}$. Suppose there exists a neighborhood $\tilde{U}$ of $E$ such that $\tilde{U} \subset \tilde{V}$ and $\tilde{U}$ embeds in $N$. Then $p(\tilde{V})$ is a component of $N_{\epsilon}^\circ - C_\delta(N)$, and $S = p(\tilde{S})$ is a boundary component of $\partial C_\delta(N)$.

A nice property of topologically tame manifolds is that given a compact set in a topologically tame manifold, it is possible to find a compact core that contains the compact set.

**Lemma 2.5.** (Lem 2.6 in [F1]) Let $N$ be a topologically tame hyperbolic 3-manifold. Let $\mathcal{P}$ be a collection of Type I and Type II components of $N_{\text{thin}}(\epsilon)$. Let $K$ be a compact set in $N - \mathcal{P}$. Then there exists a relative compact core $R$ of $N - \mathcal{P}$ such that $K \subset R$ and the components of $(N - \mathcal{P}) - R$ are topologically a product.

Another property of topologically tame manifolds is that for any relative compact core $R$ of a topologically tame manifold $N$ where the components of $\partial R - P$ are incompressible, the components of $(N - \mathcal{P}) - \text{int} R$ possess a product structure.

**Lemma 2.6.** (Lem 2.7 in [F1]) Let $N$ be a topologically tame hyperbolic 3-manifold. Let $\mathcal{P}$ be a collection of Type I and Type II components of $N_{\text{thin}}(\epsilon)$. Let $R$ be a relative compact core of $N - \mathcal{P}$ with associated parabolic locus $P$. Let $\{S_j\}$ be the components of $\partial R - P$, and let $U_j$ be the component of $(N - \mathcal{P}) - \text{int} R$ associated to $S_j$. If the $\{S_j\}$ are incompressible, then the $\{U_j\}$ are homeomorphic to $S_j \times [0, \infty)$.

Now we present a general fact about ends of hyperbolic manifolds and their covers which will be used throughout the paper.
Lemma 2.7. (Lem 2.8 in [F1]) Let $N$ be a topologically tame hyperbolic 3-manifold. Let $0 < \epsilon < \epsilon_3$, and let $\delta \geq 0$. Let $\mathcal{P}$ be a collection of Type I components of $N_{\text{thin}}(\epsilon)$. Let $S$ be an incompressible separating surface of $N - \mathcal{P}$ such that $\partial S \subset \partial \mathcal{P}$. Let $M = \mathbb{H}^3/\pi_1(S)$ be a cover of $N$ with covering map $p$. Let $U$ be the closure of a component of $(N - \mathcal{P}) - S$ such that there exists $\tilde{U} \subset M$ such that $p|_{\tilde{U}} : \tilde{U} \to U$ is a homeomorphism and $(p|_{\tilde{U}})^{-1}(S) \subset C_{\delta}(M)$. Then

$$(p|_{\tilde{U}})^{-1}[(N - C_{\delta}(N)) \cap U] = (M - C_{\delta}(M)) \cap \tilde{U}$$

2.2. Simplicial Hyperbolic Surfaces. In this section, we briefly review simplicial hyperbolic surfaces. For more details, we refer the reader to Section 3.2 of [F1], or [Cl].

Let $f : S \to N$ be a proper map from $S$ into $N$ where $S$ has triangulation $T$. We will use the notation $f : (S,T) \to N$ to denote such a map. Suppose $f$ has the following properties:

1. $f$ weakly preserves parabolicity,
2. $f$ maps every edge $e$ in $T$ to a geodesic arc, and
3. $f$ maps each face of $T$ to a non-degenerate, totally geodesic triangle in $N$.

Then $f$ is a simplicial pre-hyperbolic surface. Let $\text{ang} f(v)$ be the total angle about a vertex $f(v)$, that is, the sum of the angles based at $f(v)$ in each of the geodesic triangles which share $f(v)$ as a vertex. If $\text{ang} f(v) \geq 2\pi$ for every internal vertex $v$, then $f$ is a simplicial hyperbolic surface. This angle condition guarantees that the intrinsic geometry of $f(S)$ is like that of a surface of curvature $\leq -1$.

The map $f$ induces a piecewise Riemannian metric on $S$ called the simplicial hyperbolic structure, denoted $\tau$. The surface $(S,\tau)$ has curvature $\leq -1$ at every point except at the vertices which have “concentrated” negative curvature because the total angle about each vertex is $\geq 2\pi$. By the Gauss-Bonnet Theorem for hyperbolic triangles, the area of $(S,\tau)$ is

$$\text{area}(S,\tau) = 2\pi|\chi(S)| - \sum_{v \in T} (\text{ang} f(v) - 2\pi)$$

(Lem 1.13, Bonahon [B2]) In particular, $\text{area}(S,\tau) \leq 2\pi|\chi(S)|$.

Using the area bound of $(S,\tau)$, we can deduce a bound on injectivity radius for points in the image of a $\pi_1$-injective simplicial hyperbolic surface.

Lemma 2.8. (Lem 3.1 in [F1]) Let $f : S \to N$ be a $\pi_1$-injective simplicial hyperbolic surface. Then there exists a constant $K_S$ that depends on the Euler characteristic of $S$ such that for $x \in f(S)$, $\text{inj}_N(x) \leq K_S$.

Now let us explicitly describe a construction due to Bonahon [B2] of a simplicial pre-hyperbolic surface which will be used in the proof of the book of I-bundles case. Let $T$ be a triangulation of $S$ with no doubly ideal edges, and let $f : S \to N$ be a proper map that weakly preserves parabolicity; maps every edge $e$ in $T$ with both endpoints at the same internal vertex to a homotopically non-trivial loop in $N$, and maps no two vertices of $T$ to the same point. First, we homotop $f$, keeping $f(V)$ fixed, to a map $f_1 : S \to N$ such that if $e$ is a finite edge in $T$, then $f_1(e)$ is the unique geodesic arc in its homotopy class. Then we properly homotop $f_1$, fixing $\bigcup f_1(e)$ over all finite edges $e$ to a map $f_2 : S \to N$ such that if $e'$ is a half infinite edge, then $f_2(e')$ is the half infinite geodesic ray which has the same endpoint and is properly homotopic to $f(e')$. Note that in the universal cover, a lift of $f_2(e')$ connects a lift of $f(v)$ and a fixed point of a parabolic element of $\pi_1(N)$. Recall
that we are assuming that there are no doubly ideal edges, so there are no edges in
the triangulation with endpoints at two distinct ideal vertices. Finally, we properly
homotop $f_2$, fixing $f_2(T)$, to a map $F(f,T) : S \to N$ such that each face of $T$
is taken to the totally geodesic triangle spanned by the images of its edges. This
homotopy results in a well-defined map $F(f,T)$ which is a simplicial pre-hyperbolic
surface.

A simplicial hyperbolic surface $f : (S,T) \to N$ is useful if the triangulation $T$
contains exactly one internal vertex, and $T$ contains a distinguished edge $e$ which
passes through $v$ and is mapped to a closed geodesic. It is relatively easy to check
if a simplicial hyperbolic surface is mapped into the convex core.

**Lemma 2.9.** (Lem 3.7 in [F1]) Let $f : (S,T) \to N$ be a simplicial hyperbolic
surface. If $f$ maps every internal vertex of $T$ into the convex core, then
$f(S) \subset C(N)$. In particular, if $f$ is a useful simplicial hyperbolic surface, then $f(S) \subset C(N)$.

2.3. Injectivity Radius Bounds for Hyperbolic $I$-bundle Convex Cores.

In this section, we state the theorems on injectivity radius bounds for hyperbolic
$I$-bundle convex cores. The following theorem of Kerckhoff-Thurston [KT] states
that given a Kleinian group that is type-preserving isomorphic to a Fuchsian group,
then there exists an upper bound on injectivity radius for points in the convex
core of the associated hyperbolic 3-manifold.

**Theorem 2.10.** (Kerckhoff-Thurston [KT]) Let $\Theta$ be a cofinite area torsion-free
Fuchsian group, and let $S = \mathbb{H}^2/\Theta$. Then there exists a constant $K_S$ such that for
any Kleinian group $\Gamma$ such that there exists a type-preserving isomorphism between
$\Theta$ and $\Gamma$ and for $x \in C(N)$ where $N = \mathbb{H}^3/\Gamma$, inj$_N(x) \leq K_S$.

A proof of their theorem also appears in Canary [C1]. A special case of the above
theorem is as follows: given an $I$-bundle $M$ over a closed surface $S$ and a hyperbolic
3-manifold $N$ without cusps such that $N$ is homeomorphic to the interior of $M$,
then there exists an upper bound on injectivity radius for points in the convex core
of $N$.

The following extension of their theorem will be the main tool with which we
prove our main theorem.

**Theorem 2.11.** (Thm 5.2 in [F1]) Let $\Theta$ be a cofinite area torsion-free Fuchsian
group, and let $S = \mathbb{H}^2/\Theta$. Then there exists a constant $L_S$ such that for any
Kleinian group $\Gamma$ such that there exists an isomorphism between $\Theta$ and $\Gamma$ that
weakly preserves parabolicity, and for $x \in C(N)$ where $N = \mathbb{H}^3/\Gamma$, inj$_N(x) \leq L_S$.

2.4. Spaces of Hyperbolic Manifolds. In this section, we introduce the space of
discrete, faithful representations of a Kleinian group $\Gamma$. Given an orientable,
irreducible 3-manifold $M$, we say that $M$ is hyperbolizable if there exists a discrete
faithful representation, $\rho : \pi_1(M) \to Isom^+(\mathbb{H}^3)$ such that $N = \mathbb{H}^3/\rho(\pi_1(M))$ is a
hyperbolic 3-manifold homeomorphic to the interior of $M$, where $N$ has a geometric
structure given by the representation. In this case, $N$ is a hyperbolization of $M$.

A Kleinian group is Fuchsian if $\rho : \Gamma \to Isom^+(\mathbb{H}^2) \subset Isom^+(\mathbb{H}^3)$ so that $\Gamma$
acts properly discontinuously on $\mathbb{H}^2 \subset \mathbb{H}^3$, and $\mathbb{H}^2/\Gamma$ is a hyperbolic orbifold. If $\Gamma$
is torsion-free, then $\mathbb{H}^2/\Gamma$ is a hyperbolic surface.

Let $D(\pi_1(M)) = \{\text{discrete faithful representations of } \pi_1(M) \text{ into } PSL_2(\mathbb{C})\}$,
where $D(\pi_1(M)) \subset Hom(\pi_1(M), PSL_2(\mathbb{C}))$. We can give $D(\pi_1(M))$ the compact-open
topology, i.e., $\rho_i \to \rho$ if and only if $\rho_i(g) \to \rho(g)$ for every $g \in \pi_1(M)$. Given
a representation $\rho : \pi_1(M) \to \text{PSL}_2(\mathbb{C})$, $N_\rho = \mathbb{H}^3/\rho(\pi_1(M))$ is a hyperbolic 3-manifold that is homotopy equivalent to $M$ via a homotopy equivalence induced by the map $\rho$. It is convenient to consider two manifolds to be equivalent if their representations in $\mathcal{D}(\pi_1(M))$ differ by an element of $\text{PSL}_2(\mathbb{C})$. Under this equivalence relation, we say that $N_\rho$ is a marked hyperbolic manifold, and we consider $\mathcal{AH}(M) = \mathcal{D}(\pi_1(M))/\text{PSL}_2(\mathbb{C})$ to be the space of all marked hyperbolic 3-manifolds homotopy equivalent to $M$. We give $\mathcal{AH}(M)$ the induced topology which we call the algebraic topology. If a sequence $\{\rho_i\}$ of representations converges to $\rho$ under the algebraic topology, then we say that the sequence $\{\rho_i\}$ converges algebraically to the algebraic limit $\rho$.

We will be exclusively interested in orientable, irreducible 3-manifolds with incompressible boundary. An orientable manifold is irreducible if every embedded sphere bounds a ball. An orientable manifold $M$ has incompressible boundary if $i_* : \pi_1(S) \to \pi_1(M)$ is injective for each component $S$ of $\partial M$. In fact, an orientable, irreducible 3-manifold $M$ has incompressible boundary if and only if its fundamental group is freely indecomposable, i.e., if $\pi_1(M) = G * H$, then $G$ or $H = \{1\}$. (see Thm 7.1, Hempel [He]). Also, $M$ has incompressible boundary if and only if $M$ contain no compressing disks. If $i : (D, \partial D) \to (M, \partial M)$ is an embedding and $i(\partial D)$ is a homotopically non-trivial loop in $\partial M$, then we say $i(D)$ is a compressing disk for $M$. More generally, a compact surface $S$ that is not $D^2$ or $S^2$ is incompressible in $M$ if $S$ is properly embedded in $M$ and $i_* : \pi_1(S) \to \pi_1(M)$ is injective. An orientable, irreducible 3-manifold is Haken if it contains an incompressible surface. Note that every compact, orientable, irreducible 3-manifold with non-empty boundary is Haken.

Furthermore, we only consider manifolds that are atoroidal. An embedding of a torus $i : T^2 \to M$ is essential if $i_*$ is injective and $i$ is not homotopic to a map $j : T^2 \to \partial M$. A manifold $M$ is atoroidal if it contains no essential tori. Thurston’s Geometrization Theorem states that the interior of any compact, oriented, irreducible, atoroidal 3-manifold with nonempty boundary admits a hyperbolic structure.

Let $M$ be a compact, orientable, irreducible 3-manifold with incompressible boundary. Then $M$ is acylindrical if every properly embedded incompressible annulus can be properly homotoped into the boundary of $M$. For an acylindrical manifold, Thurston [Th2] has shown that its representation space is compact:

**Theorem 2.12.** (Thurston’s Compactness Thm [Th2]) Let $M$ be an atoroidal, acylindrical 3-manifold. Then $\mathcal{AH}(M)$ is compact.

We can also consider the space of Kleinian groups with the geometric topology or Chabauty topology, that is, the topology of closed subgroups. A sequence $\{\Gamma_i\}$ of Kleinian groups converges geometrically to a group $\Gamma$ if the following conditions are satisfied:

1. if $g \in \text{PSL}_2(\mathbb{C})$ is an accumulation point of a sequence $\{g_i\} \in \{\Gamma_i\}$, then $g \in \Gamma$, and
2. if $g \in \Gamma$, then there exists a sequence $\{g_i\}$ such that each $g_i \in \Gamma_i$ and $g_i \to g$.

One can also think of geometric convergence in terms of hyperbolic 3-manifolds with base frame. Choose a base point $p$ in $\mathbb{H}^3$ and an orthonormal frame $\omega$ in the tangent space at $p$. Given a manifold with base frame $(M, \omega)$, there exists a unique Kleinian group $\Gamma$ with the property that $\Gamma$ is the group of covering transformations.
acting on \( \mathbb{H}^3 \) such that \( (\mathbb{H}^3/G, \omega) = (M, e) \), where \( e \) is the image of the standard frame \( \omega \) at \( p \).

We can now give the space of hyperbolic 3-manifolds with base frame the Chabauty or geometric topology. We say \( f \) is a framed \((K, r)\)-approximate isometry between two manifolds with base frame \((M_1, e_1)\) and \((M_2, e_2)\) if \( f : (X_1, e_1) \rightarrow (X_2, e_2) \) is a diffeomorphism such that \( B_{M_1}(x_1, r) \subseteq (X_1, x_1) \subseteq (M_1, x_1) \), \( B_{M_2}(x_2, r) \subseteq (X_2, x_2) \subseteq (M_2, x_2) \), \( Df(e_1) = e_2 \), and
\[
\frac{d(x, y)}{K} \leq d(f(x), f(y)) \leq Kd(x, y)
\]
for all \( x, y \in X_1 \). So \( \{(M_i, e_i)\} \) converges geometrically to \((M, e)\) if there exists a sequence of \((K_i, r_i)\)-approximate isometries \( \{f_i : (M_i, e_i) \rightarrow (M, e)\} \) such that \( K_i \rightarrow 1 \) and \( r_i \rightarrow \infty \). The topology induced by framed \((K, r)\)-approximate isometries coincides with the geometric topology. (Cor 3.2.11, Canary-Epstein-Green [CEG])

Thus, a sequence of manifolds with base frame \((M_i, e_i)\) converges to \((M, e)\) in the geometric topology if and only if their corresponding Kleinian groups \( \Gamma_i \) converge to \( \Gamma \) in the geometric topology.

3. The Book of I-Bundles Case

In this section, we will prove the main theorem in the case that \( M \) is a book of I-bundles. First we will give a sketch of the proof of the theorem for a simpler example of a book of I-bundles. Then we will prove the main theorem for a general book of I-bundles.

3.1. The Motivating Example. Let us consider a simpler example of a book of I-bundles called a plain book of I-bundles. To construct a plain book of I-bundles, let \( \{B_i\} \) be a collection of surfaces, each of which is a closed orientable surface minus an open disk. For each \( i \), let \( E_i = B_i \times [0, 1] \), and let \( \partial E_i \) be the annulus \( \partial B_i \times [0, 1] \). Consider a solid torus \( V = D^2 \times S^1 \) whose boundary is decomposed into a union of disjoint parallel annuli, \( A_1, A_1', A_2, A_2', \ldots, A_n, A_n' \), such that each annulus is homotopy equivalent to a \((1, 0)\)-curve on the boundary of the torus. We can order the annuli \textit{mod} \( n \). Let \( M \) be the union of \( V \) and the \( \{E_i\} \), where each \( \partial E_i \) is glued to \( A_i \) by a homeomorphism. We can think of \( V \) as the binding and the \( \{E_i\} \) as the pages in this book of I-bundles.

**Theorem 3.1.** Let \( M \) be a plain book of I-bundles. Then there exists a constant \( L \) such that if \( N \) is a hyperbolic 3-manifold homeomorphic to the interior of \( M \) and \( x \in C(N) \), then \( \text{inj}_N(x) \leq L \), where \( L \) depends on the maximum genus of the boundary components of \( M \).

**Proof.** For the sake of simplicity, let us assume that \( N \) is convex co-compact. Let us start with an outline of the sketch of the proof in this case. The idea is to choose a 2-complex \( D \subset \text{int} \ M \) which will be a union of surfaces \( \{T_i\} \) which are isotopic to the boundary components \( \{S_i\} \) of \( M \). We will then consider a map \( f : D \rightarrow N \) such that \( f(D) \subset C(N) \) and such that \( f \) is in the appropriate homotopy class so that each of the components of \( C(N) - f(D) \) will lie in the image of the convex core of a manifold whose convex core has bounded injectivity radius, thus proving the theorem.

Now we will give a more complete sketch of the proof. First we construct a 2-complex \( D \) in \( \text{int} \ M \). For each \( i \), let \( A_i \subset V \) be an annulus whose boundary components are the core curve of \( V \) and the core curve of \( A_i \). Let \( D \) be the
2-complex formed by gluing the boundaries of \{ (B_i, \frac{1}{2}) \subset E_i \} to those of \{ A_i \}. Then by construction, the inclusion of \( D \) into \( M \) is a homotopy equivalence. Let \( S_i = (B_i, 0) \cup A_i' \cup (B_{i+1}, 0) \) be the \( i \)-th boundary component of \( M \), and let \( T_i = (B_i, \frac{1}{2}) \cup A_i \cup A_{i+1} \cup (B_{i+1}, \frac{1}{2}) \), glued along their boundaries. Then \( S_i \) is isotopic to \( T_i \), and the component of \( M - D \) containing \( S_i \) is homeomorphic to \( S_i \times [0, 1] \).

Let \( M \) be our plain book of \( I \)-bundles described above. Let \( \rho \in \mathcal{D}(\pi_1(M)) \), and let \( N = \mathbb{H}^3/\rho(\pi_1(M)) \) be a hyperbolic 3-manifold homeomorphic to \( \text{int} \, M \), via a map \( h : \text{int} \, M \to N \). Let \( M_i = \mathbb{H}^3/\rho(\pi_1(S_i)) \) be a cover of \( N \) with covering map \( p_i \).

Let the vertex set \( W \) on \( D \) consist of exactly one internal vertex on the core curve of \( V \). Triangulate \((D, W)\). Let \( f : D \to N \) be a map such that for each \( i \), \( f|_{T_i} : T_i \to N \) is a useful simplicial hyperbolic surface so that \( f(D) \subset C(N) \).

Moreover, construct \( f \) so that it is homotopic to \( h|_D \) and hence \( \pi_1 \)-injective. We further require that for each \( i \), the map \( f|_{T_i} \) lifts to \( \tilde{f}_i : T_i \to M_i \) such that \( \tilde{f}_i(T_i) \subset C(M_i) \).

By Lemma 2.3, a boundary component \( A \) of \( C(N) \) lifts to a boundary component \( \tilde{A} \) of \( C(M_i) \). We can construct a homotopy \( H_i : S_i \times [0, 1] \to M_i \) between \( \tilde{A} \) and \( \tilde{f}_i(T_i) \) such that \( H_i(S_i, [0, 1]) \subset C(M_i) \). Because the image of the homotopy lies in the convex core of a surface group, by Theorem 2.11 for \( x \in H_i(S_i, [0, 1]) \), \( \text{inj}_M(x) \leq L_{S_i} \), where \( L_{S_i} \) depends on \( S_i \). So by Lemma 2.2 for \( x \in \bigcup p_i(H_i(S_i, [0, 1])) \), \( \text{inj}_N(x) \leq L_{S_i} \). With some work, we can show that

\[
C(N) \subset \bigcup p_i(H_i(S_i, [0, 1])) \subset \bigcup p_i(C(M_i))
\]

Thus, for \( x \in C(N) \), \( \text{inj}_N(x) \leq \max \{ L_{S_i} \} \).

**3.2. The General Case.** Now we will prove the main theorem in the case of a general book of \( I \)-bundles. A book of \( I \)-bundles is a compact, connected, irreducible 3-manifold with boundary \( M = E \cup V \) such that

1. \( E \) is an \( I \)-bundle over \( B \), a non-empty compact 2-manifold with boundary,
2. each component of \( V \) is homeomorphic to \( D^2 \times S^1 \),
3. the set \( A = E \cap V \) is the inverse image of \( \partial B \) under the bundle projection \( b : E \to B \), and
4. each component of \( A \) is an annulus in \( \partial V \) which is homotopically non-trivial in \( V \).

Equivalently, a compact, connected, irreducible 3-manifold \( M \) is a book of \( I \)-bundles if there exists a disjoint collection \( A \) of incompressible annuli such that each component of the manifold obtained by cutting \( M \) along \( A \) is either a solid torus, or an \( I \)-bundle \( R \) over a surface of negative Euler characteristic such that \( \partial R \cap \partial M \) is the associated \( \partial I \)-bundle.

The proof of the motivating example was simpler for several reasons: \( N \) was convex co-compact; the annuli in \( \partial V \) were homotopic to \((1, 0)\)-curves, not \((p, q)\)-curves; \( M \) did not contain multiple bindings, i.e., \( V \) had only one component; and there were no twisted \( I \)-bundles. These are all considerations that we will have to take into account when we prove the main theorem in the case of a general book of \( I \)-bundles.

**Theorem 3.2.** Let \( M \) be a book of \( I \)-bundles. Then there exists a constant \( L \) such that if \( N \) is a hyperbolic 3-manifold homeomorphic to the interior of \( M \) and if
$x \in C(N)$, then $\text{inj}_N(x) \leq L$, where $L$ depends on the maximum genus of the boundary components of $M$.

**Proof.** The outline of the proof is the similar to that for a plain book of $I$-bundles. The idea is to use the image of a 2-complex $D$ to divide $C(N)$ into portions, each of which lies in the image of the convex core of a manifold whose convex core has bounded injectivity radius.

In the following lemma we will choose a 2-complex $D \subset \text{int} \, M$ such that the inclusion of $D$ into $M$ is a homotopy equivalence, and we will construct maps \( \{g_i : S_i \times [0, 1] \to M\} \) whose images can be glued along $D$ to form a 3-manifold homeomorphic to $M$.

**Lemma 3.3.** Let \( \{S_i\} \) be the boundary components of $M$. Then there exists a 2-complex $D \subset \text{int} \, M$ such that $D$ is a deformation retract of $M$. Furthermore, there exist maps \( \{g_i : S_i \times [0, 1] \to M\} \) with the following properties:

1. for each $i$, the image $g_i(S_i, 0) = S_i$,
2. the map $g_i|_{S_i \times [0, 1]} : S_i \times [0, 1] \to M$ is a homeomorphism onto its image,
3. the image $g_i(S_i, 1) \subset D$, and
4. the manifold $M$ is homeomorphic to the quotient space $\prod(S_i, [0, 1]) / \sim$, where $(x_i, 1) \sim (y_j, 1)$ if and only if $(g_i(x_i), 1) = (g_j(y_j), 1)$.

**Proof.** Recall that $M = E \cup V$, where $E$ is an $I$-bundle over $B$, and $B$ is a disjoint collection \( \{B_s\} \) of surfaces with boundary. Here, $V$ is a disjoint collection $V_1, \ldots, V_m$ of solid tori. The boundary of each $V_j$ consists of a collection of $2n_j$ parallel annuli $(A_{jn_1}, A_{jn_2}, A_{jn_3}, \ldots, A_{jn_1})$, which we can order cyclically modulo $n_j$.

Let $b : E \to B$ be the bundle projection map. Given a component $B_s$ of $B$, for each component $Q$ of $\partial B_s$, $b^{-1}(Q)$ is identified with some annulus $A_{jk}$ on the boundary of some $V_j$. In fact, there is a one-to-one correspondence between the set \( \{A_{jk}\} \) and the set \( \{b^{-1}(Q) : Q \text{ is a boundary component of some } B_s\} \).

We define two annuli $A_{jk}$ and $A_{jl}$ to be adjacent on $\partial V_j$ if they are ordered consecutively on $\partial V_j$. Note that if $A_{jk}$ and $A_{jk(k+1)}$ are adjacent annuli on $\partial V_j$, then they will be physically separated by the parallel annulus $A_{jk}$ to which no $I$-bundle is identified. Given an $I$-bundle $E_s$ over $B_s$, either $E_s = B_s \times [0, 1]$, or $E_s$ is a twisted $I$-bundle over $B_s$. In the latter case, $E_s = \tilde{B}_s \times [0, 1]/(x, t) \sim (\tau(x), 1 - t)$, where $\tau : \tilde{B}_s \to \tilde{B}_s$ is a free involution and $\tilde{B}_s/\tau = B_s$. For a component $B_s$ of $B$, let $\tilde{B}_s$ denote the middle surface of the $I$-bundle over $B_s$, where $\tilde{B}_s = (B_s, \frac{1}{2})$ if $E_s = B_s \times [0, 1]$, and $\tilde{B}_s = (\tilde{B}_s, \frac{1}{2})/\tau$ if $E_s$ is a twisted $I$-bundle over $B_s$. Let $\tilde{B}$ denote the set of all the middle surfaces \( \{\tilde{B}_s\} \).

Let $M = E \cup (\partial V \times [0, 1])$ be a compact 3-manifold, where $E$ is an $I$-bundle over $B$ and $V$ is a disjoint collection of solid tori as before. For each $j$, $(\partial V_j, 0)$ is decomposed into parallel annuli $(A_{j1}, 0), (A_{j2}, 0), \ldots, (A_{jn_j}, 0), (A_{jn_j}, 0)$. Given a component $B_s$ of $B$ with boundary component $Q$, if $b^{-1}(Q)$ is identified with $A_{jk}$ on $\partial V_j$ in $M$, then identify $b^{-1}(Q)$ with $(A_{jk}, 0)$ on $(\partial V_j, 0)$ in $M$. Consider the quotient map $q : M \to M$ defined such that $q|_B$ is the identity map, $q|_{\partial V \times [0, 1]}$ is a homeomorphism, and $q$ identifies $\partial V_j$ to the core curve $\gamma_j$ of $V_j$.

Now we will construct a disjoint union of surfaces, $\tilde{D}$, such that $q(\tilde{D})$ will be the desired 2-complex $D$. For a fixed annulus $(A_{jk}, 0)$ in $(\partial V_j, 0)$, construct the annulus $\tilde{A}_{jk} = (\partial \tilde{B}_s \cap A_{jk}) \times [0, 1] \subset (\partial V_j \times [0, 1])$. Then for fixed $s$, let $\tilde{A}_s = \bigcup \tilde{A}_{jk}$ where the union is taken over all components of $(\partial \tilde{B}_s \cap \partial V, 0)$. Then $\tilde{F}_s = \tilde{B}_s \cup \tilde{A}_s$.
is a surface with boundary \{ (\partial \bar{B} \cap A_{jk}, 1) \}. Then we define \( \bar{D} = \bigcup \bar{F}_s \), and 
\( D = q(D) \subset \text{int } M \).

There is a natural product structure on each component of \( E - \bar{B} \). For adjacent
annuli \((A_{jk}, 0)\) and \((A_{j(k+1)}, 0)\), we can extend this product structure so that the
component of \((\partial V_j \times [0, 1]) - [A_{jk} \cup A_{j(k+1)} \cup (\partial V_j, 1)]\) containing \((A_{jk}', 0)\) also has a
product structure as shown in Figure 1. This product structure induces a product
structure on all of \( M - D \). Note that as a result, we can deformation retract
\( M \) along the product structure of \( M - D \) onto \( D \).

![Figure 1. Cross Section of Product Structure on \( \partial V_j \times [0, 1] \) in \( \bar{M} \)](image)

Now let us construct the maps \( \{ g_i : S_i \times [0, 1] \to M \} \). Consider a boundary
component \( S_i \) of \( M \). Then for each \( i \), let \( g_i : S_i \times [0, 1] \to M - D \) be a homeo-
morphism along the product structure of \( M - D \) where \( g_i(S_i, 0) = S_i \). We can let
\( g_i(x_i, 1) = \lim_{t \to 1} g_i(x_i, t) \) for \( x_i \in S_i \). Then for all \( i \), \( g_i(S_i, 1) \subset D \). By identifying
the images of the maps in \( D \), we see that \( M \) is homeomorphic to the quotient space
\( \coprod (S_i, [0, 1]) / \sim \), where \((x_i, 1) \sim (y_j, 1)\) if and only if \((g_i(x_i), 1) = (g_j(y_j), 1)\). \[\square\]

Let \( N \) be a hyperbolic 3-manifold homeomorphic to \( \text{int } M \) via a homeomorphism
\( h : \text{int } M \to N \). For ease of exposition, let \( \tilde{g}_i : S_i \to D \subset M \) be defined by
\( \tilde{g}_i(x_i) = g_i(x_i, 1) \) for \( x_i \in S_i \). In the next lemma, we will construct a map \( f : D \to N \)
such that for each \( i \), the image \( f \circ \tilde{g}_i(S_i) \) lifts to the \( \epsilon \)-neighborhood of the convex
core of a cover of \( N \).

**Lemma 3.4.** Let \( N \) be a hyperbolic 3-manifold homeomorphic to \( \text{int } M \) via a
homeomorphism \( h : \text{int } M \to N \). Let \( i : D \to \text{int } M \) be the inclusion map,
and let \( k = h \circ i : D \to N \). Let \( \Gamma_i = (k \circ \tilde{g}_i)_* [\pi_1(S_i)] \), and let \( M_i = \mathbb{H}^3 / \Gamma_i \).
Then there exists a map \( f : D \to N \) such that for each \( i \), \( f \circ \tilde{g}_i \) lifts to a map
\( \tilde{f} \circ \tilde{g}_i : S_i \to M_i \) such that \( \tilde{f} \circ \tilde{g}_i(S_i) \subset C_\epsilon(M_i) \).
Proof. Recall that $D = q(D)$ where $D = \bigcup \tilde{F}_s$, a disjoint union of surfaces with boundary. We will define the map $f : D \to N$ by constructing a map $\tilde{f} : D \to N$, where for each $\tilde{F}_s \subset D$, the map $\tilde{f}|_{\tilde{F}_s}$ is a simplicial hyperbolic surface that factors through the quotient map $q$.

Let us construct a triangulation on $\bar{D}$ that will induce a “triangulation” on $D$. We put “triangulation” in quotes, because $D$ is a 2-complex, not a surface. First choose a vertex set $W$ on $D$ to be exactly one internal vertex $v_j$ on each $\gamma_j$. Let $W = q^{-1}(W)$ be the vertex set on $\bar{D}$. Include the arcs $\{(\partial B_s \cap A_{jk}, 1) - \bar{W}\}$ in the edge set of the triangulation on $(\bar{D}, \bar{W})$. For a fixed $s$ and $j$, if $(\partial B_s \cap A_{jk}, 0)$ is homotopic to a $(p_j, q_j)$-curve on $\partial V_j$, then there are $p_j$ vertices on $(\partial B_s \cap A_{jk}, 1)$, and the edge set $\{(\partial B_s \cap A_{jk}, 1) - \bar{W}\}$ in $\bar{D}$ is a $p_j$-to-1 cover of $(\gamma_j - W)$ in $D$. Triangulate the remainder of $(\bar{D}, \bar{W})$, and call it $T$. Then $T = q(T)$ is a “triangulation” on $(D, W)$.

Because $D$ is a deformation retract of $M$, the inclusion map $i : D \to \text{int} M$ is a homotopy equivalence. Then $k = h \circ i : D \to N$ is $\pi_1$-injective. Now we will construct a map $\tilde{k} : D \to N$ such that $\tilde{k}$ will be homotopic to $k$, and $\tilde{f}$ will be a simplicial hyperbolic surface which is a “straightening” of $\tilde{k} \circ q$.

For each $j$, $k(\gamma_j)$ represents either a hyperbolic element or a parabolic element in $\pi_1(N)$. If $k(\gamma_j)$ represents a hyperbolic element, then its geodesic representative, $\hat{k}(\gamma_j)^*$, lies in $C(N)$. Let $\hat{k} : D \to N$ be a map that is homotopic to $k : D \to N$ such that $\hat{k}(v_j)$ is a point on $k(\gamma_j)^*$.

If $\hat{k}(\gamma_j)$ represents a parabolic element, then $k(\gamma_j)$ has no geodesic representative in $N$. Normalize so that the fixed point of the parabolic element which $k(\gamma_j)$ represents is at infinity in the upper half space model of $\mathbb{H}^3$. Recall that $\gamma_j$ is the core curve of $V_j$. Consider the collection $\{S_i\}$ of boundary components of $M$ such that $S_i \cap V_j \neq \emptyset$. Consider the collection of covers $\{M_i = \mathbb{H}^3 / \Gamma_i\}$ of $N$ associated to the $\{S_i\}$, where each $\Gamma_i = (k \circ \hat{g}_i)^*[\pi_1(S_i)]$. Because each $C(M_i)$ is non-empty, the convex hull of each $\Lambda_{\Gamma_i}$ must also contain a geodesic ray $Y_i$ with endpoint at infinity. Consider a horoball $L_j = \{(z, t) : t \geq c_j\}$ about infinity. Under the action of the normalized parabolic element $z \mapsto z + 1$, the horoball $L_j$ induces a rank one cusp in $N$, the boundary of which is an infinite annulus. Then for large enough $c_j$, there exists an arc $\alpha_j$ on $\partial L_j$ such that $\alpha_j = \hat{\alpha}_j / (z \mapsto z + 1)$ is a closed curve in $N$ on the infinite annulus $\partial L_j / (z \mapsto z + 1)$, and such that $\hat{\alpha}_j$ is contained in the $\epsilon$-neighborhood of each $Y_i$ so that $\hat{\alpha}_j \subset \text{CH}(\Lambda_{\Gamma_i})$ for each $\Gamma_i$. Let $\hat{k} : D \to N$ be a map that is homotopic to $k : D \to N$ such that $\hat{k}(v_j)$ is a point on $\alpha_j$.

Using Bonahon’s construction, let $\tilde{f} = F(\hat{k} \circ q, T) : D \to N$. Then for each $\tilde{F}_s \subset D$, $\tilde{f}|_{\tilde{F}_s}$ is a simplicial pre-hyperbolic surface. Because $\tilde{f}$ is in the homotopy class of $k \circ q$, $\tilde{f}$ is a $\pi_1$-injective map. By construction, the map $\tilde{f}$ respects the quotient map $q$, so there exists a map $f : D \to N$ which is a simplicial pre-hyperbolic 2-complex, i.e., $f$ weakly preserves parabolicity, maps every edge in $T$ to a geodesic arc, and maps each face of $T$ to a non-degenerate totally geodesic triangle in $N$. Furthermore, by construction, $f$ is homotopic to $k : D \to N$ and hence is $\pi_1$-injective.

Now we will show that $f \circ \hat{g}_i(S_i)$ lifts to $C_e(M_i)$. Because $f \circ \hat{g}_i(S_i)$ is homotopic to $k \circ \hat{g}_i(S_i)$, by the Lifting Theorem, $f \circ \hat{g}_i(S_i)$ lifts to a map $\tilde{f} \circ \hat{g}_i : S_i \to M_i$.

Because $\hat{g}_i(S_i)$ is a subset of $D$, $f \circ \hat{g}_i$ is a simplicial pre-hyperbolic surface. Then $\tilde{f} \circ \hat{g}_i : S_i \to M_i$ is also a simplicial pre-hyperbolic surface.
convex, by an argument similar to that in the proof of Lemma 2.3 [KS], in order to show that \( f \circ \tilde{g}_i(S_i) \subset C_\epsilon(M_i) \), it suffices to show that for each internal vertex \( v_j \) in the triangulation on \( \tilde{g}_i(S_i) \), \( f(v_j) \) lifts to \( C_\epsilon(M_i) \).

Let \( v_j \in \gamma_j \) be an internal vertex in the triangulation on \( \tilde{g}_i(S_i) \subset D \). For each \( \gamma_j \), \( k(\gamma_j) \) represents either a hyperbolic element or a parabolic element in \( \Gamma_i \). If \( k(\gamma_j) \) represents a hyperbolic element in \( \Gamma_i \), then \( f(v_j) = \tilde{k}(v_j) \in k(\gamma_j)^* \). The closed geodesic \( k(\gamma_j)^* \) lifts to \( \mathbb{H}^3 \) to an axis of a hyperbolic element of \( \Gamma_i \) which lies in \( CH(\Lambda \Gamma_i) \). Then \( f(v_j) \) lifts to \( C(M_i) \). If \( k(\gamma_j) \) represents a parabolic element in \( \Gamma_i \), then, by construction, \( f(v_j) \) lifts to \( \mathbb{H}^3 \) to a point on \( \alpha_j \) which lies in \( CH_\epsilon(\Lambda \Gamma_i) \). Therefore, \( f(v_j) \) lifts to \( C_\epsilon(M_i) \), and we have shown that \( f \circ \tilde{g}_i(S_i) \subset C_\epsilon(M_i) \). \( \square \)

The next step in the proof of the book of \( I \)-bundles case involves finding a compact core \( R \) of \( N \), such that the points in \( R \) have bounded injectivity radius.

Given \( 0 < \epsilon < \epsilon_3 \). By the results of McCullough [McC] and Kulkarni-Shalen [KS] and Lemma 2.3, there exists a compact core \( R^* \) of \( C_\epsilon(N) \) such that \( \partial C_\epsilon(N) \cap C_\epsilon^\circ(N) \subset \partial R^* \), and such that \( f(D) \subset R^* \).

In the next lemma, we will choose a compact core \( R \) of \( N \) such that \( R \) contains \( R^* \), \( R \) lies in \( C_2^\epsilon(N) \), and such that points in \( R \) have bounded injectivity radius.

**Lemma 3.5.** Let \( \Gamma_i = (k \circ \tilde{g}_i)_\ast[\pi_1(S_i)] \), and let \( M_i = \mathbb{H}^3/\Gamma_i \) be a cover of \( N \) with covering map \( p_i \). Let \( \{U_i\} \) be the components of \( C_2^\epsilon(N) \setminus \text{int } R^* \). For all \( 0 < \epsilon < \epsilon_3 \), there exists a compact core \( R \) of \( N \) bounded by the surfaces \( \{T_i\} \) such that the following are true:

1. \( R \subset C_2^\epsilon(N) \),
2. if \( x \in R \), then \( \text{inj}_N(x) \leq \max\{L_{S_i} + 2\epsilon\} \), and
3. for each \( i \), there exists \( \tilde{U}_i \subset M_i \) such that \( p_i|_{\tilde{U}_i} : \tilde{U}_i \to U_i \) is a homeomorphism, and \( (p_i|_{\tilde{U}_i})^{-1}(T_i) \subset C_2^\epsilon(M_i) \).

**Proof.** Let us start with a sketch of the proof. First we will construct surfaces \( \{T_i \subset C_2^\epsilon(N)\} \) such that for each \( i \), there exists a homotopy between \( f \circ \tilde{g}_i : S_i \to N \) and the inclusion map \( i|_{T_i} : T_i \to N \). We will construct each homotopy so that its image lies in the image of the convex core of a manifold whose convex core has bounded injectivity radius. Then we will construct surfaces \( \{T_i \subset C_2^\epsilon(N)\} \) such that there exists a homotopy between the inclusion maps of \( T_i \) and \( T_i \). Again, we will construct each homotopy so that its image lies in the image of the convex core of a manifold whose convex core has bounded injectivity radius. Finally, we will show that the \( \{T_i\} \) bound a compact core \( R \) and that \( R \) is contained in the union of the images of these homotopies. Thus, by construction, all points in \( R \) will have bounded injectivity radius.

Note that \( \pi_1(N) = \pi_1(C_\epsilon(N)) \), so that \( R^* \) is also a compact core of \( N \). Because \( N \) is homeomorphic to \( \text{int } M \), and \( R^* \) is a compact core of \( N \), we can conclude that \( M \) is homeomorphic to \( R^* \). (Thm 1, McCullough-Miller-Swarup [MMS].) Then because \( M \) has incompressible boundary, \( R^* \) also has incompressible boundary. Note that \( R^* \) is also a compact core for \( C_2^\epsilon(N) \) where the components of \( \partial R^* \) are incompressible in \( C_2^\epsilon(N) \). Let \( \{Q_i\} \) be the components of \( \partial R^* \). Then by Lemma 2.4, each component \( U_i \) of \( C_2^\epsilon(N) \setminus \text{int } R^* \) possesses a product structure \( Q_i \times [0, \infty) \). Because \( i_\ast(\pi_1(Q_i)) = \Gamma_i \), by the Lifting Theorem, the inclusion map \( i : Q_i \times [0, \infty) \to U_i \) lifts to a map \( \tilde{i} : Q_i \times [0, \infty) \to M_i \). Let \( \tilde{U}_i = \tilde{i}(Q_i, [0, \infty)) \).
Then $\tilde{U}_i$ is also possesses a product structure $Q_i \times [0, \infty)$, and $p_{i|\tilde{U}_i} : \tilde{U}_i \to U_i$ is a homeomorphism.

Now we will show that for each $i$, $\tilde{U}_i \subset C_{2e}(M_i)$. Suppose not. Then there exists $\tilde{x} \in \tilde{U}_i - C_{2e}(M_i)$ and a component $\tilde{G}_i$ of $\partial C_{2e}(M_i)$ such that $\tilde{G}_i$ separates $\tilde{x}$ from $C_{2e}(M_i)$. Consider a geodesic ray $\tilde{g}_x$ in $M_i - C_{2e}(M_i)$ that is perpendicular to $\tilde{G}_i$ and passes through $\tilde{x}$. Let $g_x$ be the portion of $\tilde{g}_x$ beginning at $\tilde{x}$. There exists a constant $\eta > 0$ such that $\tilde{x} \in (M_i)^0_\eta$. Recall that by Lemma 2.4, the injectivity radius strictly increases out a geometrically finite end, therefore the ray $g_x$ is contained in $(M_i)^0_\eta$. Without loss of generality, let us assume that $g_x$ intersects $(Q_i, 0)$ transversely. Then one of the following three cases occurs: $g_x$ intersects $(Q_i, 0)$ an odd number of times, $g_x$ intersects a different boundary component of the closure of $\tilde{U}_i$, or all but a compact portion of $g_x$ is contained in $\tilde{U}_i$.

Suppose $g_x$ intersects $(Q_i, 0) \subset U_i$ an odd number of times. Because $(Q_i, 0) \subset \tilde{U}_i$ is an embedded surface separating $M_i$, $(Q_i, 0)$ has two sides. Let the component of $M_i - (Q_i, 0)$ containing $\tilde{U}_i$ be the positive side of $(Q_i, 0)$. Then the last time $g_x$ intersects $(Q_i, 0) \subset \tilde{U}_i$, $g_x$ passes from the positive to the negative side of $(Q_i, 0)$.

We can also let the side of $C_{2e}(N) - (Q_i, 0)$ containing $\tilde{U}_i$ be the positive side of $(Q_i, 0) \subset U_i$. Recall that because $f \circ g_i(S_i, 1) \subset R^*$ and $U_i$ is a component of $C_{2e}(N) - \text{int} R^*, f \circ g_i(S_i, 1)$ lies on the negative side of $(Q_i, 0) \subset U_i$.

Because $p_{i|\tilde{U}_i}$ is a homeomorphism and $(Q_i, 0) \subset U_i$ lifts to $(Q_i, 0) \subset \tilde{U}_i$, $f \circ g_i(S_i, 1)$ lies to the negative side of $(Q_i, 0) \subset \tilde{U}_i$. Note that $f \circ g_i(S_i, 1)$ is an incompressible separating surface of $M_i$, and that $f \circ g_i : (S_i, 1) \to M_i$ is homotopic to the inclusion map $i : (Q_i, 0) \to M_i$. Because $g_x$ leaves every compact set of $M_i$ and all but a compact portion of $g_x$ lies in $M_i - \tilde{U}_i$, if $g_x$ intersects $(Q_i, 0) \subset \tilde{U}_i$, then $g_x$ must also intersect $f \circ g_i(S_i, 1)$. But $g_x \subset M_i - C_{2e}(M_i)$ and $f \circ g_i(S_i, 1) \subset C_i(M_i)$. So this is a contradiction.

Suppose $g_x$ intersects a different boundary component of the closure of $\tilde{U}_i$. Then $p_i(g_x)$ intersects some component $H_i$ of $\partial C_{2e}(N)$ such that $H_i$ is a boundary component of the closure of $U_i$ in $N$. By Lemma 2.4, $H_i$ lifts to a component $\tilde{H}_i$ of $\partial C_{2e}(M_i)$. Because $g_x$ intersects $\tilde{H}_i$, there is a portion of $\tilde{g}_x$ that contains $\tilde{x}$ and joins two components of $\partial C_{2e}(M_i)$. Because $C_{2e}(M_i)$ is convex, $\tilde{x} \in g_x$ lies in $C_{2e}(M_i)$, which is a contradiction.

Then all but a compact portion of $g_x$ is contained in $\tilde{U}_i$. Let $\tilde{W}_i$ be the closure of the component of $M_i - \tilde{G}_i$ that contains $g_x$. Let $X_\delta = [M_i - C_\delta(M_i)] \cap \tilde{W}_i$. Let $\tilde{U}_i$ be the closure of $\tilde{U}_i$ in $M_i$. Then because $\partial \tilde{U}_i$ is compact and $g_x \cap X_\delta \neq \emptyset$ for every $\delta > 2e$, we can guarantee that $X_\delta \subset \tilde{U}_i$ for large enough $\delta$. Then $V_i = X_\delta$ is a component of $M_i - C_\delta(M_i)$ which embeds in $N$. Then by Lemma 2.4, $\tilde{W}_i = p_i(\tilde{W}_i)$ is a component of $N - C_{2e}(N)$ with boundary $p_i(\tilde{G}_i)$. Because $\tilde{x} \in M_i - C_{2e}(M_i)$, $\tilde{x} \in \text{int} \tilde{W}_i$. Then $p_i(\tilde{x}) \in \text{int} W_i \subset N - C_{2e}(N)$. But by hypothesis, $x = p_i(\tilde{x}) \in C_{2e}(N)$ so this is a contradiction. Thus, $\tilde{U}_i \subset C_{2e}(M_i)$ for each $i$.

For each $i$, choose $\tilde{T}_i$ to be a level surface in $\tilde{U}_i$. Recall that $\tilde{f} \circ g_i(S_i, 1)$ does not intersect $\tilde{U}_i$ so that $\tilde{f} \circ g_i(S_i, 1)$ lies to one side of $\tilde{T}_i$. By construction, $\tilde{T}_i$ is embedded, and the inclusion map of $\tilde{T}_i$ into $N$ is homotopic to the map $f \circ g_i : (S_i, 1) \to N$.

Let $\alpha^*_i : S_i \times [0, 1] \to M_i$ be a homotopy between the inclusion map $i_{\tilde{T}_i} : \tilde{T}_i \to N$ and the map $\tilde{f} \circ g_i : (S_i, 1) \to N$, where $\alpha^*_i|_{S_i, 0} = i_{\tilde{T}_i}$ and $\alpha^*_i|_{S_i, 1} = \tilde{f} \circ g_i|_{S_i, 1}$. 


Let \( \alpha_i : S_i \times [0, 1] \to M_i \) be the ruled homotopy constructed from \( \alpha_i^* \) such that for \( x \in S_i \), \( \alpha_i(x, [0, 1]) \) is the geodesic arc with the same endpoints and in the same homotopy class as \( \alpha_i^*(x, [0, 1]) \). Because \( C_{2\epsilon}(M_i) \) is convex, we know that
\[
\alpha(S_i, [0, 1]) \subset C_{2\epsilon}(M_i).
\]
Because \( \Gamma_i \) is a surface group, by Theorem 2.11, for \( x \in \alpha_i(S_i, [0, 1]) \), \( inj_M(x) \leq L_{S_i} + 2\epsilon \).

For each \( i \), let \( T_i = p_i(\tilde{T}_i) \) be a level surface in \( U_i \). Consider the homotopy \( \beta_i = p_i \circ \alpha_i : S_i \times [0, 1] \to N \). Here \( \beta_i|_{(S_i, 0)} : S_i \to N \) is the inclusion map \( i_{\tilde{T}_i} : \tilde{T}_i \to N \) which is the projection of the inclusion map \( i_{\tilde{T}_i} : \tilde{T}_i \to M_i \); \( \beta_i|_{(S_i, 1)} = f \circ g_i|_{(S_i, 1)} \); and \( \beta_i(S_i, [0, 1]) \subset C_{2\epsilon}(N) \). By Lemma 2.2, for \( x \in \beta_i(S_i, [0, 1]) \), \( inj_N(x) \leq L_{S_i} + 2\epsilon \).

Now for each \( i \), choose \( \tilde{T}_i \) to be a level surface in \( U_i \) such that \( (p_i|_{\tilde{T}_i})^{-1}[\bigcup \beta_i(S_i, [0, 1])] \) lies to one side. Let \( T_i = p_i(\tilde{T}_i) \) be a level surface in \( U_i \). By construction \( (p_i|_{\tilde{T}_i})^{-1}(T_i) = \tilde{T}_i \subset C_{2\epsilon}(M_i) \).

Because each of the \( T_i \) is incompressible, disjoint, and homotopic to the boundary components of \( R^* \), the \( \{T_i\} \) and \( \{(Q_i, 0)\} \) span a product structure in \( C_{2\epsilon}(N) \). (see Thm 10.5, Hempel [He] Hence, the \( \{T_i\} \) are the boundary components of a new compact core \( R \) of \( N \). In particular, since the \( \{T_i\} \) lie in \( C_{2\epsilon}(N) \), we can conclude that \( R \subset C_{2\epsilon}(N) \).

For each \( i \), let \( \zeta_i : S_i \times [-1, 0] \to M_i \) be homotopy along the product structure of \( \tilde{U}_i \) between the inclusion maps \( i_{\tilde{T}_i} : \tilde{T}_i \to N \) and \( i_{\tilde{T}_i} : \tilde{T}_i \to N \), where \( \zeta_i|_{(S_i, -1)} = i_{\tilde{T}_i} \) and \( \zeta_i|_{(S_i, 0)} = i_{\tilde{T}_i} \). Because \( \zeta_i(S_i, [-1, 0]) \subset C_{2\epsilon}(M_i) \), by Theorem 2.11, for \( x \in \zeta_i(S_i, [-1, 0]) \), \( inj_{M_i}(x) \leq L_{S_i} + 2\epsilon \). Then \( \nu_i = p_i \circ \zeta_i : S_i \times [-1, 0] \) is a product homotopy in \( U_i \) between the inclusion maps \( i_{\tilde{T}_i} : \tilde{T}_i \to N \) and \( i_{\tilde{T}_i} : \tilde{T}_i \to N \), where \( \nu_i|_{(S_i, -1)} = i_{\tilde{T}_i} \) and \( \nu_i|_{(S_i, 0)} = i_{\tilde{T}_i} \). Furthermore, \( \nu_i(S_i, [-1, 0]) \subset C_{2\epsilon}(N) \). By Lemma 2.2, for \( x \in \nu_i(S_i, [-1, 0]) \), \( inj_N(x) \leq L_{S_i} + 2\epsilon \).

Finally we will show that
\[
R \subset \bigcup \beta_i(S_i \times [0, 1]) \cup \bigcup \nu_i(S_i \times [-1, 0])
\]
By Lemma 3.2, we know that \( M \cong \bigsqcup (S_i \times [-1, 1]) / \sim \), where \( (x_i, 1) \sim (y_j, 1) \) if and only if \( (g_i(x_i), 1) = (g_j(y_j), 1) \). Let \( \psi : M \to R \) be a map such that \( \psi|_{(S_i, -1, 0]} = \nu_i \) and \( \psi|_{(S_i, [0, 1])} = \beta_i \). Then by construction, if \( (g_i(x_i), 1) = (g_j(y_j), 1) \), then \( \psi(x_i, 1) = \psi(y_j, 1) \). Hence we have a well-defined map \( \psi : M \to R \). Using standard degree arguments, \( \psi(M) \) is contained in the image of any proper homotopy between \( M \) and \( R \). (see Thm 2.14, Lloyd [L]) In particular, \( R \subset \psi(M) \subset \bigcup \beta_i(S_i \times [0, 1]) \cup \bigcup \nu_i(S_i \times [-1, 0]) \).

So for \( x \in R \), \( inj_N(x) \leq max\{L_{S_i} + 2\epsilon\} \). This completes the proof of Lemma 3.2.

Now we have bounded the injectivity radius for points in \( R \). Now consider points in \( C(N) - R \). If \( x \in C(N) - R \), then there exists an \( i \), such that \( x \in U_i \). Recall that in the previous lemma we showed that \( T_i \) was a closed incompressible separating surface and that \( (p_i|_{\tilde{T}_i})^{-1}(T_i) \subset C_{2\epsilon}(M_i) \). By Lemma 2.7, we know that if \( x \in C(N) \cap U_i \), then \( x \in p_i(C(M_i)) \). So \( inj_N(x) \leq L_{S_i} \) for some \( i \).

Then we can conclude that for \( x \in C(N) \), \( inj_N(x) \leq max\{L_{S_i} + 2\epsilon\} \). We can do this for all \( \epsilon > 0 \) so that for \( x \in C(N) \), \( inj_N(x) \leq max\{L_{S_i}\} \). This completes the proof of Theorem 3.2.
4. The Acylindrical Case

In this section, we will prove our main theorem in the case that \( M \) is an acylindrical, hyperbolizable 3-manifold. Our theorem is:

**Theorem 4.1.** Let \( M \) be an acylindrical, hyperbolizable 3-manifold. Then there exists a constant \( K \) such that if \( N \) is a hyperbolic 3-manifold homeomorphic to the interior of \( M \) and if \( x \in C(N) \), then \( \text{inj}_N(x) \leq K \).

**Proof.** We begin with a sketch of the proof of the acylindrical case. The proof will be by contradiction. Suppose there exists a sequence of points in the convex cores of manifolds such that the injectivity radius based at these points goes to infinity. We will find a compact core \( R \) in the algebraic limit which embeds in the geometric limit as \( \pi(R) \). The compact core \( \pi(R) \) will pull back to a compact core \( R_i \) in each manifold \( N_i \) in the sequence such that points in \( R_i \) will have uniformly bounded injectivity radius. The complement of \( R_i \) in \( N_i \) will either be covered by the convex cores of manifolds whose convex cores have bounded injectivity radius or will have injectivity radius uniformly bounded by the injectivity radius of a fixed compact subset of the geometric limit of the sequence of manifolds. Thus, we will have found a uniform bound on the injectivity radius for points in the convex core of each manifold in the sequence, which is a contradiction.

Suppose for contradiction that there does not exist an upper bound on the injectivity radius for points in the convex cores of hyperbolic 3-manifolds homeomorphic to \( \text{int} \ M \). Then there exists a sequence of representations \( \{ \rho_i : \pi_1(M) \to \text{Isom}^+(\mathbb{H}^3) \} \) together with its corresponding sequence of manifolds \( \{ N_i = \mathbb{H}^3/\rho_i(\pi_1(M)) \} \), and a sequence of points \( \{ x_i \in C(N_i) \} \), such that \( \{ \text{inj}_{N_i}(x_i) \} \) diverges to infinity.

By Thurston’s Compactness Theorem 2.12, a subsequence of \( \{ \rho_i \} \) converges algebraically, up to conjugation, to a representation \( \rho \). Let the algebraic limit manifold be \( N = \mathbb{H}^3/\rho(\pi_1(M)) \). Using a result of Jørgensen-Marden (Prop 4.2, [JM]), we can take a further subsequence, again called \( \{ \rho_i \} \), such that \( \{ \rho_i(\pi_1(M)) \} \) converges to \( \hat{\Gamma} \) geometrically. Let the geometric limit manifold be \( \hat{N} = \mathbb{H}^3/\hat{\Gamma} \). By definition of geometric convergence, there exists a sequence of \( (K_i, r_i) \)-approximate isometries \( f_i : B_{r_i}(0) \subset N_i \to \hat{N} \) such that \( K_i \to 1 \) and \( r_i \to \infty \). Furthermore, because \( \rho(\pi_1(M)) \) is a subgroup of \( \hat{\Gamma} \), there is a natural covering map from the algebraic limit to the geometric limit \( \pi : N \to \hat{N} \).

Let \( N^* = (\mathbb{H}^3 \cup \Omega_{\rho(\pi_1(M))}/\rho(\pi_1(M))) \) be the conformal extension of \( N \). Anderson and Canary [AC] have an alternate definition of an accidental parabolic which, to avoid confusion, we will call an *unexpected parabolic*. We say that \( \rho(\pi_1(M)) \) has connected limit set and no unexpected parabolics if and only if every closed curve \( \gamma \) in \( \partial N^* \) which is homotopic to a curve of arbitrarily small length in \( N^* \) is homotopic to a curve of arbitrarily small length in \( \partial N^* \).

The following lemma shows that the fundamental group of an acylindrical, hyperbolizable manifold has connected limit set and no unexpected parabolics. These properties will be useful in showing the existence of a compact core in the algebraic limit which will embed in the geometric limit.

**Lemma 4.2.** Let \( M \) be an acylindrical, hyperbolizable 3-manifold. Let \( \rho \in \mathcal{D}(\pi_1(M)) \), \( N = \mathbb{H}^3/\rho(\pi_1(M)) \), and \( N^* = (\mathbb{H}^3 \cup \Omega_{\rho(\pi_1(M))}/\rho(\pi_1(M))) \). Then every closed curve \( \gamma \) in \( \partial N^* \) which is homotopic to a curve of arbitrarily small length in \( N^* \) is homotopic to a curve of arbitrarily small length in \( \partial N^* \). Therefore, \( \rho(\pi_1(M)) \) has connected limit set and no unexpected parabolics.
Proof. The proof will be by contradiction. For $0 < \epsilon < \epsilon_3$, McCullough [McC] and Kulkarni-Shalen [KS] guarantee the existence of a relative compact core $R^*$ of $C_0(\hat{\gamma})$ with associated parabolic locus $P = \partial R^* \cap \partial N_{\text{thin}}(\epsilon)$ such that $\partial C(\hat{\gamma}) \cap C_0^*(\hat{\gamma}) \subset \partial R^*$. Note that $\pi_1(N) = \pi_1(C_0(\hat{\gamma}))$, so that $R^*$ is also a compact core of $N$. Because $N$ is homeomorphic to the interior of $M$ and $R^*$ is a compact core of $N$, we can conclude that $R^*$ is homeomorphic to $M$. (Thm 1, McCullough-Miller-Swarup [MMS]) Therefore, because $N$ is acylindrical, $R^*$ has incompressible boundary and is acylindrical.

Suppose for contradiction that there exists a homotopically non-trivial, closed curve $\gamma$ in $\partial N^*$ which is homotopic to a curve of arbitrarily small length in $N^*$, but that $\gamma$ is not homotopic to a curve of arbitrarily small length in $\partial N^*$.

Recall that Sullivan [Su] has shown that there exists a homeomorphism $g : N^* \to C(N)$ which is homotopic to the canonical nearest point retraction $r : N^* \to C(N)$ (see Sec 1.3, Epstein-Marden [EM]), and which is $K$-bilipschitz on $\partial N^*$ where $K$ is independent of $\Gamma$.

Suppose $\gamma$ is homotopically trivial in $N^*$, then $g(\gamma)$ is a homotopically non-trivial, closed curve in $\partial C(N)$ which is homotopically trivial in $C(N)$. Here, $\partial C(N) \cap C_0^*(\gamma)$ is a compact core of $\partial C(N)$ so that there exists a homotopically non-trivial curve in $\partial C(N) \cap C_0^*(\gamma)$ which is homotopically trivial in $C(N)$. But $\partial C(N) \cap C_0^*(\gamma)$ is an incompressible subsurface of $\partial R^*$, and $R^*$ has incompressible boundary, so we can conclude that $\partial C(N) \cap C_0^*(\gamma)$ is incompressible in $C(N)$. But this is a contradiction.

Therefore $\gamma$ is homotopically non-trivial in $N^*$. Because $g$ is $K$-bilipschitz on $\partial N^*$, we know that $g(\gamma)$ is a closed curve in $\partial C(N)$ which is homotopic to a curve of arbitrarily small length in $C(N)$, but that $g(\gamma)$ is not homotopic to a curve of arbitrarily small length in $\partial C(N)$. Because $\partial C(N) \cap C_0^*(\gamma)$ is a compact core of $\partial C(N)$, we know that $g(\gamma)$ is homotopic to a curve on $\partial C(N) \cap C_0^*(\gamma) \subset \partial R^*$. Therefore, without loss of generality, we can consider $g(\gamma)$ to be a closed curve on $\partial R^*$. Because $g(\gamma)$ is homotopic to a curve of arbitrarily small length in $C(N)$, there exists a closed curve $\alpha \subset P$ such that $g(\gamma)$ is homotopic to $\alpha$ in $C(N)$. Then since $P \subset \partial R^*$ and $R^*$ is a compact core, we can conclude that $g(\gamma)$ is homotopic to $\alpha$ in $R^*$.

Suppose $g(\gamma)$ is homotopic to $\alpha$ in $\partial R^*$. Then $g(\gamma)$ is homotopic into the parabolic locus of $R^*$ in $\partial R^*$. Then $g(\gamma)$ is a peripheral curve in $\partial C(N)$, and hence is homotopic to a curve of arbitrarily small length in $\partial C(N)$. Then $\gamma$ must have been homotopic to a curve of arbitrarily small length in $\partial N^*$. But this contradicts our initial assumptions about $\gamma$. So $g(\gamma)$ is not homotopic to $\alpha$ in $\partial R^*$.

Using the Homotopy Annuuls Theorem (Thm VIII.10, Jaco [Ja]), we can construct a $\pi_1$-injective, proper embedding $f : S^1 \times [0, 1] \to R^*$ such that $f(S^1, 0) = g(\gamma)$ and $f(S^1, 1) = \beta$ where $f(S^1, [0, 1])$ cannot be properly homotoped into $\partial R^*$. Therefore, $R^*$ is not acylindrical. But this is also a contradiction. Thus, every closed curve $\gamma$ in $\partial N^*$ which is homotopic to a curve of arbitrarily small length in $N^*$ is homotopic to a curve of arbitrarily small length in $\partial N^*$. Therefore, $\rho(\pi_1(M))$ has connected limit set and no unexpected parabolics. This completes the proof of Lemma 4.2.

Next, we will show that there exists a compact core $R$ in the algebraic limit $N = \mathbb{H}^3 / \rho(\Gamma)$ which embeds in the geometric limit $\hat{N} = \mathbb{H}^3 / \hat{\Gamma}$ as $\pi(R)$. 


 Lemma 4.3. Let $M$ be an acylindrical, hyperbolizable 3-manifold. Let $\pi : N \to \hat{N}$ be the covering map between the algebraic limit $N = \mathbb{H}^3/\rho(\pi_1(M))$ and the geometric limit $\hat{N} = \mathbb{H}^3/\hat{\Gamma}$. Then there exists a compact core $R \subset N$ such that $\pi(R)$ embeds in $\hat{N}$.

Proof. There are two cases—either the limit set of $\rho(\pi_1(M))$ is the entire sphere or not.

Suppose the limit set of $\rho(\pi_1(M))$ is the entire sphere. Because $\rho(\pi_1(M))$ satisfies Bonahon’s Condition (B), $N = \mathbb{H}^3/\rho(\pi_1(M))$ is topologically tame. Then the following theorem of Canary states that the algebraic limit and the geometric limit agree, and, in this case, any compact core of the algebraic limit is also a compact core of geometric limit by default.

Theorem 4.4. (Thm 9.2, Canary [C1]) Let $\{\rho_i : \pi_1(M) \to \text{Isom}^+(\mathbb{H}^3)\}$ be a sequence of discrete faithful representations converging algebraically to $\rho : \pi_1(M) \to \text{Isom}^+(\mathbb{H}^3)$. If the limit set of $\rho(\pi_1(M))$ is all of $\mathbb{S}^2_{\infty}$ and $N = \mathbb{H}^3/\rho(\pi_1(M))$ is topologically tame, then $\{\rho_i\}$ converges strongly to $\rho$.

If the limit set of $\rho(\pi_1(M))$ is not the entire sphere, then the domain of discontinuity of $\rho(\pi_1(M))$ has nonempty domain of discontinuity. Recall that by Lemma 4.2, $\rho(\pi_1(M))$ has connected limit set and no unexpected parabolics. In this case, the following theorem of Anderson-Canary guarantees that given an algebraically convergent sequence such that its associated image groups converge geometrically, we can find a compact core in the algebraic limit which embeds in the geometric limit.

Theorem 4.5. (Cor B, Anderson-Canary [AC1]) Let $\pi_1(M)$ be a finitely generated, torsion-free, nonabelian group, and let $\{\rho_i\}$ be a sequence in $\mathcal{D}(\pi_1(M))$ converging algebraically to $\rho$. Suppose that $\{\rho_i(\pi_1(M))\}$ converges geometrically to $\hat{\Gamma}$. Let $N = \mathbb{H}^3/\rho(\pi_1(M))$, $\hat{N} = \mathbb{H}^3/\hat{\Gamma}$, and let $\pi : N \to \hat{N}$ be the covering map. If $\rho(\pi_1(M))$ has nonempty domain of discontinuity, connected limit set, and contains no unexpected parabolics, then there exists a compact core $R$ of $N$ such that $\pi : N \to \hat{N}$ is an embedding restricted to $R$.

Thus, in either case, we can find a compact core in the algebraic limit which embeds in the geometric limit. This completes the proof of Lemma 4.3. \(\square\)

By Lemma 4.2, we know that $\rho(\pi_1(M))$ has connected limit set and contains no unexpected parabolics. The next lemma due to Canary-Minsky and Anderson-Canary shows that for large enough $i$, the pull back of $\pi(R)$ to $N_i$ is a compact core of $N_i$.

Lemma 4.6. (Lem 7.2, Anderson-Canary [AC1]) Let $\pi_1(M)$ be a finitely generated, torsion-free, nonabelian group, and let $\{\rho_i\}$ be a sequence in $\mathcal{D}(\pi_1(M))$ converging algebraically to $\rho$. Suppose that $\{\rho_i(\pi_1(M))\}$ converges geometrically to $\hat{\Gamma}$. Let $N = \mathbb{H}^3/\rho(\pi_1(M))$, $\hat{N} = \mathbb{H}^3/\hat{\Gamma}$, and let $\pi : N \to \hat{N}$ be the covering map. Suppose $\rho(\pi_1(M))$ has connected limit set and contains no unexpected parabolics. Let $R$ be a compact core of $N$ such that $\pi$ is an embedding restricted to $R$. Then for large enough $i$, $R_i = f_i^{-1}(\pi(R))$ is a compact core for $N_i$.

Now let us show that points in $R_i$ have uniformly bounded injectivity radius, where the bound depends on the compact set $\pi(R)$ in $\hat{N}$. 
Lemma 4.7. For large enough $i$, and for $x \in R_i = f_i^{-1}(\pi(R)) \subset N_i$, we have $\text{inj}_{N_i}(x) \leq 2\kappa_R$.

Proof. Note that because $\pi(R)$ is a compact set in $\tilde{N}$, there exists a constant $\kappa_R$ such that for $x \in \pi(R)$, we have $\text{inj}_{\tilde{N}}(x) \leq \kappa_R$. Then, because $R$ is compact and $f_i : B_{r_i}(0) \to \tilde{N}$ is a $(K_i, r_i)$-approximate isometry such that $K_i \to 1$ and $r_i \to \infty$, it is possible to choose $I > 0$ such that for $i > I$, we guarantee that $K_i < 2$ and that the closure of the $4\kappa_R$-neighborhood of $R_i$ lies in $B_{r_i}(0)$.

Let $\gamma_x$ be a homotopically non-trivial loop in $\tilde{N}$ that is based at $x \in \pi(R)$ and is of length $\leq 2\kappa_R$. If $f_i^{-1}(\gamma_x)$ is a homotopically trivial loop, then $f_i^{-1}(\gamma_x)$ bounds a disk $D_x$ in $N_i$. Consider the immersion $g : D^2 \to D_x$. For $y, z \in \partial D^2$, let $yz$ be the line segment in $D^2$ joining $y$ and $z$. Let $g(yz)^*$ be the geodesic arc in $N_i$ that is properly homotopic to the segment $g(yz)$. Fix $z \in D^2$. Then the new disk $D'_x = \bigcup_y g(yz)^*$ has diameter $\leq 2K_i\kappa_R$ in $N_i$. For $i > I$, we know that $K_i < 2$ so that $D'_x \subset B_{r_i}(0)$. Then $f_i(D'_x)$ is a disk in $\tilde{N}$, so $\gamma_x$ is homotopically trivial in $\tilde{N}$. But this is a contradiction. Therefore, $f_i^{-1}(\gamma_x)$ is a homotopically non-trivial loop based at $f_i^{-1}(x)$ of length $\leq 2K_i\kappa_R$. Then for $x \in R_i$, we know that $\text{inj}_{N_i}(x) \leq K_i\kappa_R$. In particular, for $i > I$, we know that $K_i < 2$, so for $x \in R_i$, we have $\text{inj}_{N_i}(x) \leq 2\kappa_R$.

The following lemma shows that points in $C(N_i) - R_i$ also have bounded injectivity radius, because they are either covered by manifolds whose convex cores have bounded injectivity radius or have injectivity radius bounded by the injectivity radius of a fixed compact subset of the geometric limit.

Lemma 4.8. Let $0 < \epsilon < \epsilon_3$, and let $\{S_j\}$ be the boundary components of $\pi(R) \subset \tilde{N}$. Then for large enough $i$ and $x \in C(N_i) - R_i$, we have $\text{inj}_{N_i}(x) \leq \max\{L_{S_j}, 2\kappa_{S_j}, \epsilon\}$.

Proof. Temporarily fix $i$. Because $N_i$ is homeomorphic to the interior of $M$ where $M$ has incompressible boundary, the compact core $R_i$ of $N_i$ is homeomorphic to $M$. (Thm 1, McCullough-Miller-Swarup) Since $M$ has incompressible boundary, so does $R_i$. Let $\{T_{ij}\}$ be the components of $\partial R_i$. Then each $T_{ij} = f_i^{-1}(S_j)$ is homeomorphic to $S_j$ and is an incompressible separating surface of $N_i$.

Let $U_j$ be the component of $N_i - \text{int} R_i$ with boundary component $T_{ij}$. Because $R_i$ is a compact core of a topologically tame 3-manifold with incompressible boundary, by Lemma 2.4, $U_j$ possesses a product structure $T_{ij} \times [0, \infty)$ for each $j$. Because $i_*(\pi_1(T_{ij})) = \pi_1(M_j)$, by the Lifting Theorem, the inclusion map $i : T_{ij} \times [0, \infty) \to U_j$ lifts to a map $\tilde{i} : T_{ij} \times [0, \infty) \to M_j$. Let $\tilde{U}_j = \tilde{i}(T_{ij} \times [0, \infty))$. Then the projection map $p_j|_{\tilde{U}_j} : \tilde{U}_j \to U_j$ is a homeomorphism. Let $\tilde{T}_{ij} = (p_j|_{\tilde{U}_j})^{-1}(T_{ij})$.

If $x \in C(N_i) - R_i$, then there exists $j$ such that $x \in U_j$. By Lemma 2.4, it suffices to bound the injectivity radius based at $\tilde{x} = (p_j|_{\tilde{U}_j})^{-1}(x)$ in $M_j$. There are two possibilities: either $\tilde{x} \in C(M_j)$ or not. If $\tilde{x} \in C(M_j)$, then by Theorem 2.11, $\text{inj}_{N_i}(x) \leq L_{S_j}$.

Suppose $\tilde{x} \in M_j - C(M_j)$. There are two possibilities here also: either $\tilde{x} \in (M_j)_\infty$ or $\tilde{x} \in (M_j)_\infty - C(M_j)$. If $\tilde{x} \in (M_j)_\infty$, then $\text{inj}_{M_j}(\tilde{x}) \leq \epsilon$.

Suppose $\tilde{x} \in (M_j)_\infty - C(M_j)$. Then there exists a geodesic ray $\tilde{g}_x$ that is perpendicular to a component $\tilde{A}_j$ of $\partial C(M_j)$ and that passes through $\tilde{x}$. Let $g_x$ be the portion of $\tilde{g}_x$ beginning at $\tilde{x}$. Note that by Lemma 2.4, the injectivity radius strictly increases out a geodesically finite end. Because $\tilde{x} \in (M_j)_\infty$, the ray $g_x$ is entirely contained in $(M_j)_\infty$. Then either $g_x$ is contained in $\tilde{U}_j$, or $g_x$ intersects $\tilde{T}_{ij}$.
Suppose \( g_\mathbf{x} \) is contained in \( \mathbf{U}_j \). Let \( \mathbf{W}_j \) be the closure of the component of \( M_j - \mathbf{A}_j \) that contains \( g_\mathbf{x} \). Let \( X_\delta = [M_j - C_\delta(M_j)] \cap \mathbf{W}_j \). Because \( \mathbf{T}_{ij} \) is compact and \( g_\mathbf{x} \cap X_\delta \neq \emptyset \) for all \( \delta > 0 \), we can guarantee that \( X_\delta \subset \mathbf{U}_j \) for large enough \( \delta \). Then \( \mathbf{V}_j = X_j \) is a component of \( M_j - C_\delta(M_j) \) that embeds in \( N_j \). By Lemma

\[ \mathbf{W}_j = p_j(\mathbf{W}_j) \]

is the component of \( N_j - C(N_j) \) with boundary \( p_j(\mathbf{A}_j) \). Because \( x \in M_j - C(M_j) \), \( \mathbf{x} \in \operatorname{int} \mathbf{W}_j \). Then \( p_j(\mathbf{x}) \in \operatorname{int} W_j \subset N_j - C(N_j) \). But by hypothesis, \( x = p_j(\mathbf{x}) \in C(N_j) \) so this is a contradiction.

Thus \( g_\mathbf{x} \) must intersect \( \mathbf{T}_{ij} \). By Lemma \[ \text{[2.4]} \]

we know that the injectivity radius strictly increases out a geometrically finite end. Thus, it suffices to bound the injectivity radius for points \( \mathbf{y} \in \mathbf{T}_{ij} \).

Because \( S_j \) is in the image of a compact set in \( \mathbf{N} \), there exists a constant \( \kappa_{S_j} \) such that for \( z \in S_j \), \( \inj_{S_j}(z) \leq \kappa_{S_j} \). Because \( T_{ij} = f_{i}^{-1}(S_j) \) and \( f_i \) is a \((K_i, r_i)\)-approximate isometry, by the argument in the proof of Lemma \[ \text{[4.7]} \]

for \( y \in T_{ij} \), we know \( \inj_{T_{ij}}(y) \leq K_i \kappa_{S_j} \). Furthermore, because \( p_j|_{\mathbf{U}_j} \) is an isometry and \( \mathbf{T}_{ij} = (p_j|_{\mathbf{U}_j})^{-1}(T_{ij}) \), for \( \mathbf{y} \in \mathbf{T}_{ij} \), we know \( \inj_{T_{ij}}(\mathbf{y}) \leq K_i \kappa_{S_j} \). Because \( T_{ij} \) is incompressible in \( N_j \) and \( \mathbf{T}_{ij} \) is a lift of \( T_{ij} \), \( \mathbf{T}_{ij} \) is incompressible in \( M_j \). So for \( \mathbf{y} \in \mathbf{T}_{ij} \), we know that \( \inj_{M_j}(\mathbf{y}) \leq \inj_{T_{ij}}(\mathbf{y}) \).

Then by Lemmas \[ \text{[2.3]} \]

and \[ \text{[2.1]} \]

for \( x \in C(N_j) - R_i \), \( \mathbf{x} \in (M_j)^0 - C(M_j) \), and \( g_\mathbf{x} \) intersecting \( \mathbf{T}_{ij} \) at \( \mathbf{y} \), we have \( \inj_{N_i}(x) \leq \inj_{M_j}(\mathbf{x}) \leq \inj_{M_j}(\mathbf{y}) \leq K_i \kappa_{S_j} \). Because \( K_i \to 1 \), for large \( i \), we have \( \inj_{N_i}(x) \leq 2 \kappa_{S_j} \).

Thus, for large \( i \) and \( x \in C(N_j) - R_i \), we can conclude that \( \inj_{N_i}(x) \leq \max\{L_j, 2 \kappa_{S_j}, \epsilon\} \). This completes the proof of Lemma \[ \text{[4.8]} \]

Therefore, by Lemmas \[ \text{[4.7]} \]

and \[ \text{[4.3]} \]

for \( x \in C(N_i) \), we know that \( \inj_{N_i}(x) \leq \max\{2 \kappa_{R_i}, L_j, 2 \kappa_{S_j}, \epsilon\} \). We can do this for all \( 0 < \epsilon < \epsilon_3 \), so that for \( x \in C(N_i) \), we have \( \inj_{N_i}(x) \leq \max\{2 \kappa_{R_i}, L_j, 2 \kappa_{S_j}\} \). This uniform bound contradicts the assumption that there exists a sequence of points \( \{x_i \in C(N_i)\} \) such that \( \{\inj_{N_i}(x_i)\} \) converges to infinity. This completes the proof of Theorem \[ \text{[4.3]} \]

5. Some Consequences

In this section, we present some consequences of the main theorem. First we present a slightly stronger answer to McMullen’s conjecture in the case of a book of \( I \)-bundles. We will show that if \( N \) is a hyperbolic 3-manifold homotopy equivalent to a book of \( I \)-bundles, then there exists an upper bound on injectivity radius for points in the convex core of \( N \), where the bound depends on the number of generators in \( \pi_1(N) \).

Corollary 5.1. Let \( N \) be a hyperbolic 3-manifold homotopy equivalent to a book of \( I \)-bundles. Then there exists a constant \( L' \) such that for \( x \in C(N_i) \), \( \inj_{N_i}(x) \leq L' \), where \( L' \) depends on the number of generators of \( \pi_1(N) \).

Proof. First let us show that if \( N \) is homotopy equivalent to a book of \( I \)-bundles, then \( N \) is homeomorphic to the interior of a book of \( I \)-bundles. Let \( R \) be a compact core for \( N \). Because \( \pi_1(N) \) satisfies Bonahon’s Condition (B), \( N \) is topologically tame, and hence \( N \) is homeomorphic to the interior of its compact core \( R \). Thus, it suffices to show that \( R \) is a book of \( I \)-bundles.

We will use the characteristic submanifold theory developed by Johannson \[ \text{[J]} \]

and Jaco-Shalen \[ \text{[JS]} \]. First let us introduce some definitions. A map \( f : (V, \partial V) \to (M, \partial M) \) of an annulus or torus, \( V \), into \( M \) is essential if \( f \) is \( \pi_1 \)-injective and \( f(V) \)
where $\beta$ component of $R$ Shalen (Prop 4.3, [CS]), homotopy equivalence. Because $x$ $\chi$ $R$ $\{\}$ where the maximum is taken over all boundary components

$\{\}$ such that $\partial R \cap \partial M$ is the associated $\partial I$-bundle. Thus, we know that $\Sigma_M$ is a collection of I-bundles and solid tori, and $M - \Sigma_M$ is a collection of solid tori.

Because $R$ is a hyperbolizable 3-manifold and $\pi_1(R)$ contains no $\mathbb{Z} \oplus \mathbb{Z}$ subgroup, then (Sec 11, Morgan [Mo]) $\Sigma_R$ is a collection of I-bundles and solid tori. Let $H_0 : M \to R$ be a homotopy equivalence. By Johannson (Thm 24.2, [Jo]), the map $H_0 : M \to R$ is homotopic to a map $H_1 : M \to R$ such that $H_1^{-1}(\Sigma_R) = \Sigma_M$, $H_1|_{M - \Sigma_M} : M - \Sigma_M \to R - \Sigma_R$ is a homeomorphism, and $H_1|_{\Sigma_M} : \Sigma_M \to \Sigma_R$ is a homotopy equivalence. Because $H_1 : M - \Sigma_M \to R - \Sigma_R$ is a homeomorphism, each component of $R - \Sigma_R$ is a solid torus. Then as a consequence of a result of Culler-Shalen (Prop 4.3, [CS]), $R$ is also a book of I-bundles. Thus, $N$ is homeomorphic to the interior of a book of I-bundles $R$.

Now we will show a relationship between the Euler characteristic of the boundary components of $R$ and the number of generators of $\pi_1(R)$. Let $DR$ be the double of $R$. Then $DR$ is a closed 3-manifold which has Euler characteristic $\chi(DR) = 0$. So $\chi(DR) = 2\chi(R) - \chi(\partial R)$ or $\chi(\partial R) = 2\chi(R)$. Recall $\chi(R) = \beta_0 - \beta_1 + \beta_2 - \beta_3$ where $\beta_i$ is the rank of $H_i(R)$. Because $R$ is a connected 3-manifold with boundary, $\chi(R) = 1 - \beta_1 + \beta_2 \geq 1 - \beta_1$. Because $H_1(R)$ is the abelianization of $\pi_1(R)$, $\beta_1 \leq n$ where $n$ is the number of generators in $\pi_1(R)$. So $\chi(R) \geq 1 - n$ or $\chi(\partial R) \geq 2 - 2n$.

By applying Theorem 5.2 to $R$ and $N$, we obtain an upper bound $L$ such that for $x \in C(N)$, $\text{inj}_N(x) \leq L$. Recall from the proof of Theorem 5.2, $L = \max\{L_{S_j}\}$, where the maximum is taken over all boundary components $\{S_j\}$ of $R$ and $L_{S_j}$ is the bound obtained in Theorem 2.11. Let $L' = \max\{L_{S_j}\}$ where the maximum is taken over all surfaces $\{S_j\}$ such that $|\chi(S_j)| \leq 2n - 2$. Note that this is a finite set of surfaces. Then $L \leq L'$, and hence for $x \in C(N)$, $\text{inj}_N(x) \leq L'$, where the bound $L'$ depends only on the number of generators of $\pi_1(R) = \pi_1(N)$. This completes the proof of Corollary 5.3.

Remark 5.2. Note that McMullen’s conjecture concerns the radius of balls embedded in $C(N)$ rather than injectivity radius which involves the radius of balls embedded in $N$. We can see the necessity for this by considering the case when $M$ is a handlebody of genus 2. Let $\Gamma_i$ be a free group on two generators constructed as follows: in the ball model of hyperbolic 3-space, let $\Gamma_i$ be generated by hyperbolic isometries which identify two pairs of disjoint hemispheres which are perpendicular to $S^2_{\infty}$ and whose fixed points are antipodally situated on $S^2_{\infty}$. Then $N_i = \mathbb{H}^3/\Gamma_i$. 


is a hyperbolic 3-manifold homotopy equivalent to $M$. In this case, a fundamental domain in $\mathbb{H}^3$ for the action of $\Gamma_i$ is the portion of $\mathbb{H}^3$ lying “outside” the hemispheres. Because the fixed points are antipodally situated on $S^2_\infty$, the origin is in $CH(\Lambda_{\Gamma_i})$ and hence its projection $p_i(0)$ will lie in $C(N_i)$. The injectivity radius based at $p_i(0)$ in $N_i$ is greater than or equal to the radius of the largest ball in $\mathbb{H}^3$ based at the origin that can be embedded in the fundamental domain. As $i \to \infty$, let the Euclidean radius of the hemispheres shrink to 0. Then as $i \to \infty$, $inj_{N_i}(p_i(0)) \to \infty$ so that a uniform upper bound does not exist over all hyperbolic 3-manifolds homotopy equivalent to $M$.

When $i$ is large, however, $CH(\Lambda_{\Gamma_i})$ is very “thin and long” so that large balls based at the origin cannot be embedded in $C(N_i)$, and hence large balls based at the origin cannot be embedded in $C(N)$. Because of this example, McMullen only considered the radius of balls embedded in $C(N)$, rather than the injectivity radius which involves the radius of balls embedded in $N$.

Despite this example, in some cases it is still possible to find a uniform upper bound on injectivity radius. Note that in the example which failed to have uniformly bounded injectivity radius, $M$ had compressible boundary. In this paper, we only considered the case that $M$ had incompressible boundary.

The Main Theorem, along with a result of McMullen, shows that the limit set varies continuously over the space of hyperbolic 3-manifolds of the same topological type.

**Corollary 5.3.** Let $M$ be a book of I-bundles or an acylindrical, hyperbolizable 3-manifold. Let $\{N_i = \mathbb{H}^3/\Gamma_i\}$ be a sequence of hyperbolic 3-manifolds with base frame $\omega_i$ in $C(N_i)$ such that each $N_i$ is homeomorphic to the interior of $M$ and the injectivity radius at $\omega_i$ is bounded away from 0. If $\{N_i\}$ converges geometrically to $N = \mathbb{H}^3/\Gamma$, then $\{\Lambda_{\Gamma_i}\}$ converges to $\Lambda_{\Gamma}$ in the Hausdorff topology.

**Proof.** The proof is a direct corollary of the Main Theorem and Prop 2.4, McMullen [McM] which relates upper and lower bounds on injectivity radius in the convex core to convergence of limit sets.

Now we will present another corollary of the main theorem which does not involve base frame considerations:

**Corollary 5.4.** Let $M$ be a book of I-bundles or an acylindrical, hyperbolizable 3-manifold. Let $\{N_i = \mathbb{H}^3/\Gamma_i\}$ be a sequence of hyperbolic 3-manifolds homeomorphic to the interior of $M$. If $\{N_i\}$ converges geometrically to $N = \mathbb{H}^3/\Gamma$ and $\Gamma$ is nonabelian, then $\{\Lambda_{\Gamma_i}\}$ converges to $\Lambda_{\Gamma}$ in the Hausdorff topology.

**Proof.** Let us first state and outline the proof of the result of McMullen cited in the proof of the previous theorem.

**Theorem 5.5.** (Prop 2.4, McMullen [McM]) For $0 < r < R$, let $\{N_i = \mathbb{H}^3/\Gamma_i\}$ be a sequence of hyperbolic 3-manifolds with base frame $\omega_i$ such that:
1. the baseframe $\omega_i$ lies in $C(N_i)$,
2. the injectivity radius at $\omega_i$ is greater than $r$, and
3. for $x \in C(N_i)$, we require that $inj_{N_i}(x)$ be bounded above by $R$.
Suppose $\{N_i\}$ converges geometrically to a limit manifold $N = \mathbb{H}^3/\Gamma$. Then $\{\Lambda_{\Gamma_i}\}$ converges to $\Lambda_{\Gamma}$ in the Hausdorff topology.
Proof. Let \( T(\Gamma, R) = \{ x \in \mathbb{H}^3 : inj_N(x) \leq R \} \). Because each point in \( T(\Gamma, R) \) is a bounded distance (depending only on \( R \)) away from \( C(N) \) or \( N_{\text{thin}}(x) \), we know that a limit point of \( T(\Gamma_i, R) \) in \( S_2^\infty \) must be a point in \( \Lambda_{\Gamma_i} \), that is, \( T(\Gamma_i, R) \cap S_2^\infty \subset \Lambda_{\Gamma_i} \).

By geometric convergence, \( inj_N(x) \) converges uniformly to \( inj_N(x) \) on compact subsets of \( \mathbb{H}^3 \). Therefore \( \lim \sup T(\Gamma_i, R) \subset T(\Gamma, R) \).

Because injectivity radius for points in the convex core is bounded above by \( K \), where \( CH(\Lambda_{\Gamma_i}) \subset T(\Gamma_i, R) \). Without loss of generality, suppose the origin 0 is the basepoint of \( \Gamma_i \) for all \( i \). By hypothesis, we know that 0 \( \in CH(\Lambda_{\Gamma_i}) \) for all \( i \), where \( CH \) denotes convex hull. Then \( T(\Gamma, R) \) contains all limits of rays from 0 to \( \Lambda_{\Gamma_i} \). Therefore,

\[
\lim \sup \Lambda_{\Gamma_i} \subset T(\Gamma, R) \cap S_2^\infty \subset \Lambda_{\Gamma_i}.
\]

Since \( \Lambda_{\Gamma_i} \subset \lim \inf \Lambda_{\Gamma_i} \) for all \( i \), the result follows.

From the proof of Theorem 5.3, we can see that if for large \( i \), the basepoint of \( \Gamma_i \) lies in the \( K \)-neighborhood of \( CH(\Lambda_{\Gamma_i}) \), then \( T(\Gamma, R + K) \) contains all limits of rays from the basepoint of \( \Gamma_i \) to \( \Lambda_{\Gamma_i} \). Then we know

\[
\lim \sup \Lambda_{\Gamma_i} \subset T(\Gamma, R + K) \cap S_2^\infty \subset \Lambda_{\Gamma_i},
\]

and hence \( \Lambda_{\Gamma_i} \) converges to \( \Lambda_{\Gamma_i} \) in the Hausdorff topology.

Without loss of generality, suppose the origin 0 is the basepoint of \( \Gamma_i \). We will show that for large \( i \), the origin 0 lies within a uniformly bounded neighborhood of \( CH(\Lambda_{\Gamma_i}) \).

Because \( \Gamma \) is a limit of torsion-free Kleinian groups, \( \Gamma \) itself is torsion-free. (Lem 3.1.4, Canary-Epstein-Green CEG) Then since \( \Gamma \) is also nonabelian, we can conclude that \( \Gamma \) is nonelementary. Therefore, \( \Gamma \) contains a hyperbolic element \( \gamma \). (Prop E.1, Maskit Ma) Let \( A_{\gamma} \) denote the axis of \( \gamma \) in \( \mathbb{H}^3 \). Then there exists a constant \( K \) such that \( d(0, A_{\gamma}) \leq K \). As an element of the geometric limit, there exists a sequence \( \{ \gamma_i \in \Gamma_i \} \) such that \( \gamma_i \to \gamma \). For large \( i \), \( \gamma_i \) is a hyperbolic element, and therefore, \( A_{\gamma_i} \to A_{\gamma} \). Then for large \( i \), \( d(0, A_{\gamma_i}) \leq K + 1 \). Because \( A_{\gamma_i} \subset CH(\Lambda_{\Gamma_i}) \) for all \( i \), we can conclude that \( d(0, CH(\Lambda_{\Gamma_i})) \leq K + 1 \) for large \( i \). This concludes the proof of Corollary 5.4.

\[\square\]

References

AC1. J.W. Anderson and R.D. Canary, “Cores of hyperbolic 3-manifolds and limits of Kleinian groups,” Amer. J. of Math. 118(1996), 745–779.
Be. A. Beardon, The Geometry of Discrete Groups, Graduate Texts in Mathematics 91, Springer-Verlag, 1983.
BP. R. Benedetti and C. Petronio, Lectures on Hyperbolic Geometry, Universitext, Springer-Verlag, 1992.
Bi. B. Bielefeld, “Conformal Dynamics Problems Set,” Institute of Mathematical Sciences–Stony Brook preprint.
Bo. F. Bonahon, “Bouts des variétés hyperboliques de dimension 3,” Ann. of Math. 124(1986), 71–158.
C1. R.D. Canary, “A Covering Theorem for Hyperbolic 3-Manifolds,” Topology 35(1996), 751–778.
CEG. R.D. Canary, D.B.A. Epstein, and P. Green, “Notes on notes of Thurston,” in Analytical and Geometrical Aspects of Hyperbolic Spaces, Cambridge University Press, 1987, 3–92.
CM. R.D. Canary and Y.N. Minsky, “On limits of tame hyperbolic 3-manifolds,” J. Diff. Geom. 43(1996), 1–41.
CS. M. Culler and P. Shalen, “Volumes of hyperbolic Haken manifolds, I,” Invent. Math. 118(1994), 285–329.
F1. C. Fan, “Injectivity Radius Bounds in Hyperbolic I-bundle Convex Cores,” preprint.
F2. C. Fan, “Injectivity Radius Bounds in Hyperbolic Convex Cores II,” in preparation.
He. J. Hempel, 3-manifolds, Annals of Mathematics Studies 86, Princeton University Press, 1976.
Ja. W. Jaco, Lectures on Three-manifold Topology, Regional Conference Series in Mathematics 43, American Mathematical Society, 1980.
JS. W. Jaco and P. Shalen, “Selfert fibered spaces in 3-manifolds,” Mem. Amer. Math. Soc. 21(1979).
Jo. K. Johannson, Homotopy Equivalences of 3-manifolds, Lecture Notes in Mathematics 761, Springer-Verlag, 1979.
JM. T. Jorgensen and A. Marden, “Algebraic and geometric convergence of Kleinian groups,” Math. Scan. 66(1990), 47–72, convergence of limit sets stuff
KT. S.P. Kerckhoff and W.P. Thurston, “Non-continuity of the action of the mapping class group at Bers’ boundary of Teichmuller space,” Invent. Math. 100(1990), 25–47.
KS. R.S. Kulkarni and P.B. Shalen, “On Ahlfors’ Finiteness Theorem,” Adv. Math. 76(1989), 155–169.
L. N.G. Lloyd, Degree Theory, Cambridge Tracts in Mathematics 73, Cambridge University Press, 1978.
Ma. B. Maskit, Kleinian Groups, Grundlehren der mathematischen Wissenschaften 287, Springer-Verlag, 1988.
McC. D. McCullough, “Compact submanifolds of 3-manifolds with boundary,” Quart. J. Math. Oxford Ser. (2) 37(1986), 299–307.
MMS. D. McCullough, A. Miller, and G.A. Swarup, “Uniqueness of cores of non-compact 3-manifolds,” J. London Math. Soc. 61(1985), 548–556.
McM. C.T. McMullen, Renormalization and 3-Manifolds which Fiber over a Circle, Annals of Mathematics Studies 142, Princeton University Press, 1996.
Mor. J.W. Morgan, “On Thurston’s uniformization theorem for three-dimensional manifolds,” in The Smith Conjecture, edited by J. Morgan and H. Bass, Academic Press, 1984, 37–125.
Su2. D. Sullivan, “Travaux de Thurston sur les groupes quasi-fuchsiens et les variétés hyperboliques de dimension 3 fibrés sur $S^1$,” in Bourbaki Seminar, Vol. 1979/80, Lecture Notes in Mathematics 842, Springer-Verlag, 1981, 196–214.
Th1. W.P. Thurston, The Geometry and Topology of 3-manifolds, lecture notes.
Th2. W.P. Thurston, “Hyperbolic Structures on 3-manifolds, I: Deformations of Acylindrical manifolds,” Ann. of Math. 124(1986), 203–246.

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