Cauchy problem of nonlinear Schrödinger equation with initial data in Sobolev space $W^{s,p}$ for $p < 2$

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Abstract

In this paper, we consider in $\mathbb{R}^n$ the Cauchy problem for nonlinear Schrödinger equation with initial data in Sobolev space $W^{s,p}$ for $p < 2$. It is well known that this problem is ill posed. However, We show that after a linear transformation by the linear semigroup the problem becomes locally well posed in $W^{s,p}$ for $\frac{2n}{n+1} < p < 2$ and $s > n(1 - \frac{1}{p})$. Moreover, we show that in one space dimension, the problem is locally well posed in $L^p$ for any $1 < p < 2$.

Keyword: Cauchy problem, nonlinear Schrödinger equation, locally well-posedness, scaling limit.

1 Introduction

Consider the Cauchy problem for the linear Schrödinger equation

$$iu_t(t,x) - \Delta u(t,x) = 0,$$

$$u(0,x) = u_0(x),$$

where $\Delta$ is the Laplace operator in $\mathbb{R}^n$ for $n \geq 1$. It is well known that this problem is well posed for initial data $u_0 \in L^p(\mathbb{R}^n)$ if and only if $p = 2$. For this reason, it is believed that one can not solve the nonlinear Schrödinger equation with initial data in the Sobolev space $W^{s,p}$ for $p \neq 2$. However, this is not quite right.

Notice that the solution of the Cauchy problem for (1.1), (1.2) can be written as

$$u(t) = S(t)u_0 = E(t) * u_0,$$
where

\[ E(t, x) = \frac{1}{(-4\pi it)^\frac{1}{2}} e^{-i|x|^2/4t} \]  

(1.4)

is the fundamental solution and \( S(t) \) defines a semigroup. Thus

\[ S(-t)u(t) \equiv u_0, \]  

(1.5)

and for any norm \( X \) we have

\[ \|S(-t)u(t)\|_X = \|u_0\|_X. \]  

(1.6)

There are some examples in the literature that one studies the nonlinear Schrödinger equation by using the norm

\[ \|u\|_Y \triangleq \|S(-t)u(t)\|_X \]  

(1.7)

where \( X \) is the usual Sobolev or weighted Sobolev norm. Of course, we have the trivial example that when \( X = H^s \), we have \( X = Y \). The first nontrivial example is to take \( X \) to be the weighted \( L^2 \) norm. Thus, we take

\[ \|w\|_X = \sum_{|\alpha| \leq s} \|x^\alpha w\|_{L^2(R^n)} \]  

(1.8)

where \( \alpha \) is a multi-index. Then

\[ \|u(t)\|_Y = \sum_{|\alpha| \leq s} \|x^\alpha S(-t)u(t)\|_{L^2(R^n)} = \sum_{|\alpha| \leq s} \|S(t)x^\alpha S(-t)u(t)\|_{L^2(R^n)}. \]  

(1.9)

Noting that

\[ S(t)x_kS(-t) = x_k - 2it\partial x_k \triangleq L_k, \]  

(1.10)

we obtain

\[ \|u(t)\|_Y = \sum_{|\alpha| \leq s} \|L^\alpha u(t)\|_{L^2(R^n)}. \]  

(1.11)

This norm was first used by McKean and Shatah [9] and it was proved that one has the following global Sobolev inequality

\[ \|u(t)\|_{L^\infty} \leq C(1 + t)^{-\frac{n}{2}} \left( \sum_{|\alpha| \leq s} \|L^\alpha u(t)\|_{L^2(R^n)} + \|u(t)\|_{H^s(R^n)} \right), \quad s > \frac{n}{2}. \]  

(1.12)

This inequality is similar to the global Sobolev inequality for the wave equation obtained earlier by Klainerman (see [7]) and is very important in studying the nonlinear problem in their paper.

Another more recent example is to take \( X = H^{s_1}_t H^{s}_x \), then \( Y \) is the so called Bourgain space (see [2]). This space plays a very important role in the recent study of low regularity solution of nonlinear Schrödinger equations.
Therefore, why not take $X = L^p$ (or $W^{s,p}$)? It is our aim to investigate this problem in this paper.

Consider the Cauchy problem for the nonlinear Schrödinger equation

$$iu_t(t, x) - \Delta u(t, x) = \pm |u(t, x)|^2 u(t, x), \quad (1.13)$$

$$u(0, x) = u_0(x). \quad (1.14)$$

This problem can be reformulated as

$$u(t) = S(t)u_0 \pm \int_0^t S(t - \tau)(|u(\tau)|^2 u(\tau))d\tau. \quad (1.15)$$

Motivated by our above discussions, we make a linear transformation

$$v(t) = S(-t)u(t), \quad (1.16)$$

then

$$u(t) = S(t)v(t). \quad (1.17)$$

Therefore, we get

$$v(t) = u_0 \pm \int_0^t S(-\tau)[S(-\tau)\bar{v}(\tau)(S(\tau)v(\tau))^2]d\tau, \quad (1.18)$$

where we use the fact that $\bar{S}(\tau) = S(-\tau)$.

Our main result in this paper is that $(1.18)$ is locally well posed in Sobolev space $W^{s,p}$ for certain $p < 2$. More precisely, we have the following:

**Theorem 1.1.** Consider the nonlinear integral equation $(1.18)$, suppose that

$$u_0 \in W^{s,p}(\mathbb{R}^n) \quad (1.19)$$

for $s > n(1 - \frac{1}{p})$ and $\frac{2n}{n+1} < p < 2$, where $W^{s,p}(\mathbb{R}^n)$ is understood as $B^{s}_{p,p}(\mathbb{R}^n)$ and $B^{s}_{p,q}(\mathbb{R}^n)$ is the Besov space. Then there exists a time $T$ which only depends on $\|u_0\|_{W^{s,p}(\mathbb{R}^n)}$ such that the integral equation has a unique solution $v \in C([0, T], W^{s,p}(\mathbb{R}^n))$ satisfying

$$\|v(t)\|_{W^{s,p}(\mathbb{R}^n)} \leq 2\|u_0\|_{W^{s,p}(\mathbb{R}^n)}, \quad \forall t \in [0, T]. \quad (1.20)$$

Moreover, suppose that $v_1, v_2$ are two solutions with initial data $u_{01}, u_{02}$, then there holds

$$\|v_1(t) - v_2(t)\|_{W^{s,p}(\mathbb{R}^n)} \leq 2\|u_{01} - u_{02}\|_{W^{s,p}(\mathbb{R}^n)}, \quad \forall t \in [0, T]. \quad (1.21)$$

**Remark 1.2.** Our proof relays on a subtle cancellation in the nonlinearity and thus our result is not valid for the general nonlinearity $F(u, \bar{u})$. However, for nonlinear term of the form $\pm |u|^{2m}u$, where $m$ is an integer, it is not difficult to generalize our result to this case.
Remark 1.3. Similar results are expected for other nonlinear dispersive equations and nonlinear wave equations. However, no such result is presently known.

We point out that we can also slightly improve our result by using Besov spaces.

Theorem 1.4. Consider the nonlinear integral equation (1.18), suppose that

\[ u_0 \in \dot{B}^s_{p,1}(R^n) \]  

(1.22)

for \( s = n(1 - \frac{1}{p}) \) and \( \frac{2n}{n+1} < p < 2 \), where \( \dot{B}^s_{p,1}(R^n) \) is the homogenous Besov space. Then there exists a time \( T \) which only depends on \( \| u_0 \|_{\dot{B}^s_{p,1}(R^n)} \) such that the integral equation has a unique solution \( v \in C([0,T], \dot{B}^s_{p,1}(R^n)) \) satisfying

\[ \| v(t) \|_{\dot{B}^s_{p,1}(R^n)} \leq 2\| u_0 \|_{\dot{B}^s_{p,1}(R^n)}, \quad \forall t \in [0,T]. \]  

(1.23)

Moreover, suppose that \( v_1, v_2 \) are two solutions with initial data \( u_{01}, u_{02} \), then there holds

\[ \| v_1(t) - v_2(t) \|_{\dot{B}^s_{p,1}(R^n)} \leq 2\| u_{01} - u_{02} \|_{\dot{B}^s_{p,1}(R^n)}, \quad \forall t \in [0,T]. \]  

(1.24)

In the following, we will only prove Theorem 1.4 since the proof of Theorem 1.1 is similar.

We point out that Theorem 1.1 is only to show that one can solve the Cauchy problem in \( W^{s,p} \) for \( p < 2 \), the regularity assumption in Theorem 1.1 need not be optimal and can be improved. As an example, we will show that the problem is locally well posed in \( L^p \) for any \( 1 < p < 2 \) in one space dimension. It is proved by Y. Tsutsumi [10] that the problem is locally well-posed in \( L^2 \). Then it is proved by Grünrock [6] that the problem is locally well-posed in \( \dot{L}^p \), for any \( 1 < p < \infty \) (see also Cazenave et al [3] and Vargas and Vega [11].) Here

\[ \| f \|_{\dot{L}^p} = \| \hat{f} \|_{L^{p'}} , \]  

(1.25)

where \( \hat{f} \) is the Fourier transform of \( f \) and

\[ \frac{1}{p} + \frac{1}{p'} = 1. \]  

(1.26)

Noting that

\[ \| \hat{f} \|_{L^{p'}} \leq C\| f \|_{L^p}, \quad 1 \leq p \leq 2, \]  

(1.27)

\( \dot{L}^p \) space is slightly larger than \( L^p \) space. However, \( L^p \) is more commonly used space. More recently, there are even some local existence result in \( H^s \) for some \( s < 0 \), see Christ et al [3] as well as Koch and Tataru [8].

Our main result in one space dimension is as follows:
Theorem 1.5. Consider the nonlinear integral equation (1.18) in one space dimension, suppose that

\[ u_0 \in L^p(R) \]  

for \( 1 < p < 2 \). Then there exists a time \( T \) which only depends on \( \|u_0\|_{L^p(R)} \) such that the integral equation has a unique solution \( v \in C([0, T], L^p(R)) \) satisfying

\[ \|v(t)\|_{L^p(R)} \leq C_0 \|u_0\|_{L^p(R)}, \quad \forall t \in [0, T], \]  

and

\[ \left\{ \int_0^T \tau^{ap'}\|\partial_\tau v(\tau)\|_{L^p(R)}^{p'}d\tau \right\}^{\frac{1}{p'}} \leq C_1 \|u_0\|_{L^p(R)}^3. \]  

where

\[ \frac{1}{p} + \frac{1}{p'} = 1, \quad \theta = \frac{2}{p} - 1. \]  

Moreover, suppose that \( v_1, v_2 \) are two solutions with initial data \( u_{01}, u_{02} \), then there holds

\[ \|v_1(t) - v_2(t)\|_{L^p(R)} \leq C_0 \|u_{01} - u_{02}\|_{L^p(R)}, \quad \forall t \in [0, T]. \]  

Here \( C_0 \) and \( C_1 \) are positive constants independent of the initial data.

Remark 1.6. Let \( u(t, x) \) be a solution to the nonlinear Schrödinger equation (1.13) with initial data \( (1.14) \), then \( u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x) \) is also a solution with initial data \( u_{0\lambda} = \lambda u_0(\lambda x) \). If

\[ \|u_{0\lambda}\|_{L^p(R^n)} \equiv \|u_0\|_{L^p(R^n)}, \]  

then \( p \) is called a scaling limit. It is easy to see that \( p \) is a scaling limit in one space dimension if and only if \( p = 1 \). Thus, as \( p \) close to 1, we can go arbitrary close to the scaling limit.

In the following, \( C \) will denote a positive constant independent of the initial data and its meaning may change from line to line.

Finally, we refer to [1] for the definition of Besov spaces and homogenous Besov spaces.

2 A Key Lemma

A key Lemma leading to our local well posedness is the following:

Lemma 2.1. We consider a trilinear form

\[ v_0(\tau) = T(v_1(\tau), v_2(\tau), v_3(\tau)) = S(-\tau)[S(-\tau)v_1(\tau)S(\tau)v_2(\tau)S(\tau)v_3(\tau)]. \]  

Then, there holds

\[ \|v_0(\tau)\|_{L^1(R^n)} \leq C \tau^{-n}\|v_1(\tau)\|_{L^1(R^n)}\|v_2(\tau)\|_{L^1(R^n)}\|v_3(\tau)\|_{L^1(R^n)}. \]
**Proof.** We directly compute

\[ v_0(\tau, \alpha) = C \tau^{-2n} \int \int \int e^{i |\alpha| - |\beta|^2 + |\beta| - |\beta| - |\beta| - |\beta|} v_1(\tau, x) v_2(\tau, y) v_3(\tau, z) d\beta dx dy dz. \]  

(2.3)

It is easy to see that

\[ \tau^{-n} \int e^{i |\alpha| - |\beta|^2 + |\beta| - |\beta| - |\beta| - |\beta|} d\beta \]

\[ = \tau^{-n} e^{i |\alpha| - |\beta|^2 + |\beta| - |\beta| - |\beta| - |\beta|} \int e^{-i (\alpha + x - y - z)} \beta d\beta \]

\[ = 2^n \int e^{i |\alpha| - |\beta|^2 + |\beta| - |\beta| - |\beta| - |\beta|} \int e^{-i (\alpha + x - y - z)} \gamma d\gamma \]

\[ = Ce^{-i |\alpha| - |\beta|^2 + |\beta| - |\beta| - |\beta| - |\beta|} \delta(\alpha + x - y - z) \]

where \( \delta \) is the Dirac function. The rest part of the proof is obvious. \( \square \)

**Lemma 2.2.** Let \( v_l, l = 0, 1, 2, 3 \) satisfy (2.1), suppose that \( 2^{j-2} \leq |\xi| \leq 2^{j+2} \) in the support of \( \hat{v}_2(\tau, \xi) \) and \( 2^{k-2} \leq |\xi| \leq 2^{k+2} \) in the support of \( \hat{v}_3(\tau, \xi) \), where \( \hat{v}_2, \hat{v}_3 \) denote the space Fourier transform of \( v_2, v_3 \). Then there holds

\[ \|v_0(\tau)\|_{L^2} \leq C_2 \sum_{j+k} \|v_1(\tau)\|_{L^2} \|v_2(\tau)\|_{L^2} \|v_3(\tau)\|_{L^2}. \]  

(2.5)

**Proof.** Let \( u_l(\tau) = S(\tau) v_l(\tau), l = 0, 2, 3 \) and \( u_1(\tau) = S(-\tau) v_1(\tau) \), then \( \hat{u}_0(\tau, \xi) = e^{-i|\xi|^2 \tau} \hat{v}_0(\tau, \xi) \) etc. We have

\[ u_0(\tau) = u_1(\tau) u_2(\tau) u_3(\tau). \]

(2.6)

Therefore

\[ \|v_0(\tau)\|_{L^2(R^n)} = \|u_0(\tau)\|_{L^2(R^n)} \leq \|u_1(\tau)\|_{L^2(R^n)} \|u_2(\tau)\|_{L^2(R^n)} \|u_3(\tau)\|_{L^2(R^n)} \]

\[ \leq C \|v_1(\tau)\|_{L^2(R^n)} \|v_2(\tau)\|_{L^2(R^n)} \|v_3(\tau)\|_{L^2(R^n)} \]

\[ = C \|v_1(\tau)\|_{L^2(R^n)} \|v_2(\tau)\|_{L^2(R^n)} \|v_3(\tau)\|_{L^2(R^n)} \]

Noting the support property of \( \hat{v}_2(\tau) \) and \( \hat{v}_3(\tau) \), the desired conclusion follows from Schwartz inequality. \( \square \)

By the interpolation theorem on the multi-linear functionals (see [1] page 96 Theorem 4.4.1), we can interpolate the inequality in Lemma 2.1 and Lemma 2.2 to get the following:

**Lemma 2.3.** Let \( v_l, l = 0, 1, 2, 3 \) satisfy (2.1). Suppose that \( 2^{j-2} \leq |\xi| \leq 2^{j+2} \) in the support of \( \hat{v}_2(\tau, \xi) \) and \( 2^{k-2} \leq |\xi| \leq 2^{k+2} \) in the support of \( \hat{v}_3(\tau, \xi) \), where \( \hat{v}_2, \hat{v}_3 \) denote the space Fourier transform of \( v_2, v_3 \). Then there holds

\[ \|v_0(\tau)\|_{L^p} \leq C \tau^{-n(\frac{1}{p} - 1)} 2^n \|v_1(\tau)\|_{L^p} \|v_2(\tau)\|_{L^p} \|v_3(\tau)\|_{L^p}, \quad 1 \leq p \leq 2. \]  

(2.8)
3 Proof of the Theorem 1.4

In this section, we prove Theorem 1.4 by a contraction mapping principle.

Let us define the space

$$X = \{ w \in C([0, T], \dot{B}^s_{p, 1}(\mathbb{R}^n)) \mid \sup_{0 \leq t \leq T} \| w(t) \|_{\dot{B}^s_{p, 1}(\mathbb{R}^n)} \leq 2 \| u_0 \|_{\dot{B}^s_{p, 1}(\mathbb{R}^n)} \},$$

(3.1)

where $s = n(1 - \frac{1}{p})$ and $\frac{2n}{n+1} < p < 2$. For any $w \in X$, define a map $M$ by

$$(Mw)(t) \triangleq u_0 \pm \int_0^t S(-\tau)[S(-\tau)\bar{w}(\tau)(S(\tau)w(\tau))]^2 d\tau.$$  

(3.2)

We want to show that $M$ maps $X$ into itself and is a contraction provided that $T$ is sufficiently small.

Firstly let us recall the definition of homogenous Besov spaces. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ such that

$$\text{supp} \psi \subset \{ \xi \mid ||\xi|| \leq 1 \}$$

(3.3)

and

$$\psi(\xi) \equiv 1 \quad ||\xi|| \leq \frac{1}{2}.$$  

(3.4)

Let

$$\phi(\xi) = \psi(2^{-1}\xi) - \psi(\xi)$$

(3.5)

then

$$\sum_{j = -\infty}^{+\infty} \phi(2^{-j}\xi) \equiv 1,$$  

(3.6)

and we have the following dyadic decomposition

$$w = \sum_{j = -\infty}^{+\infty} w_j,$$  

(3.7)

where

$$\hat{w}_j(\xi) = \phi(2^{-j}\xi)\hat{w}(\xi).$$  

(3.8)

The Besov norm $\dot{B}^s_{p, 1}(\mathbb{R}^n)$ is defined by

$$\| w \|_{\dot{B}^s_{p, 1}(\mathbb{R}^n)} = \sum_{j = -\infty}^{+\infty} 2^{js} \| w_j \|_{L^p(\mathbb{R}^n)}. $$

(3.9)

Let $w \in X$, to show $M$ maps $X$ into itself, we need to estimate the nonlinear term

$$F(\tau) = S(-\tau)[S(-\tau)\bar{w}(\tau)(S(\tau)w(\tau))]$$

(3.10)

$$= \sum_{j, k, l} S(-\tau)[S(-\tau)\bar{w}_j(\tau)S(\tau)w_k(\tau)S(\tau)w_l(\tau)].$$

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To estimate $F$, we only need to estimate
\[ F_1(\tau) = \sum_{j \geq k \geq l} S(-\tau)[S(-\tau)\bar{w}_j(\tau)S(\tau)w_k(\tau)S(\tau)w_l(\tau)], \quad (3.11) \]
all the other terms in the summation can be estimated in a similar way.

By Lemma 2.3, we have
\[ \|F_1(\tau)\|_{B^s_{p,1}(R^n)} \]
\[ \leq \sum_{j=-\infty}^{+\infty} \sum_{k,l=-\infty}^{+\infty} 2^{ms} \|\phi(2^{-m}D)\{ \sum_{j,k,l=0}^{j+4} S(-\tau)[S(-\tau)\bar{w}_j(\tau)S(\tau)w_k(\tau)S(\tau)w_l(\tau)] \|_{L^p(R^n)} \]
\[ \leq C \sum_{j=-\infty}^{+\infty} \sum_{k,l=-\infty}^{+\infty} 2^{js} \| S(-\tau)[S(-\tau)\bar{w}_j(\tau)S(\tau)w_k(\tau)S(\tau)w_l(\tau)] \|_{L^p(R^n)} \]
\[ \leq C \sum_{j,k,l} 2^{js} ||S(-\tau)[S(-\tau)\bar{w}_j(\tau)S(\tau)w_k(\tau)S(\tau)w_l(\tau)] ||_{L^p(R^n)} \]
\[ \leq C T^{-n(\frac{2}{p} - 1)} \sum_{j,k,l} 2^{(j+k+l)s} ||w_j(\tau)||_{L^p(R^n)} ||w_k(\tau)||_{L^p(R^n)} ||w_l(\tau)||_{L^p(R^n)} \]
\[ \leq C T^{-n(\frac{2}{p} - 1)} ||w(\tau)||_{B^s_{p,1}(R^n)}^3, \]
where $s = n(1 - \frac{1}{p})$. Therefore
\[ \|F(\tau)\|_{B^s_{p,1}(R^n)} \leq C T^{-n(\frac{2}{p} - 1)} ||w(\tau)||_{B^s_{p,1}(R^n)}^3. \quad (3.13) \]

Noting that when $\frac{2m}{n+1} < p < 2$, we have $0 < n(\frac{2}{p} - 1) < 1$, it is easy to see
\[ \|(Mw)(\tau)\|_{B^s_{p,1}(R^n)} \leq \|u_0\|_{B^s_{p,1}(R^n)} + \int_0^t \|F(\tau)\|_{B^s_{p,1}(R^n)} \]
\[ \leq \|u_0\|_{B^s_{p,1}(R^n)} + C \int_0^t \tau^{-n(\frac{2}{p} - 1)} ||w(\tau)||_{B^s_{p,1}(R^n)}^3 d\tau \]
\[ \leq \|u_0\|_{B^s_{p,1}(R^n)} + C T^{1-n(\frac{2}{p} - 1)} \left( \sup_{0 \leq t \leq T} ||w(t)||_{B^s_{p,1}(R^n)} \right)^3 \]
\[ \leq \|u_0\|_{B^s_{p,1}(R^n)} + C T^{1-n(\frac{2}{p} - 1)} \|u_0\|_{B^s_{p,1}(R^n)}^3 \]
\[ \leq 2 \|u_0\|_{B^s_{p,1}(R^n)} \]
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provided that $T$ is sufficiently small.

Now we prove that $M$ is a contraction. Let $w^{(1)}, w^{(2)} \in X$, denote $w^* = w^{(1)} - w^{(2)}$ and $v^* = Mw^{(1)} - Mw^{(2)}$, then

$$v^* = \pm \int_0^t S(-\tau)[S(-\tau)\bar{w}^{(1)}(\tau)(S(\tau)w^{(1)}(\tau))^2 - S(-\tau)\bar{w}^{(2)}(\tau)(S(\tau)w^{(2)}(\tau))^2]d\tau \tag{3.15}$$

$$= \int_0^t S(-\tau)[S(-\tau)\bar{w}^*(\tau)(S(\tau)w^{(1)}(\tau))^2 - S(-\tau)\bar{w}^{(2)}(\tau)S(\tau)(w^{(1)}(\tau) + w^{(2)}(\tau))S(\tau)w^*(\tau)]d\tau$$

By a similar argument as before, we can get

$$\|v^*(t)\|_{\tilde{B}^s_{p,1}(R^n)} \leq C \int_0^t \tau^{-n(\frac{2}{p} - 1)}(\|w^{(1)}(\tau)\|_{\tilde{B}^s_{p,1}(R^n)} + \|w^{(2)}(\tau)\|_{\tilde{B}^s_{p,1}(R^n)})^2 \|w^*(\tau)\|_{\tilde{B}^s_{p,1}(R^n)}d\tau \leq CT^{-n(\frac{2}{p} - 1)}\|u_0\|^2_{\tilde{B}^s_{p,1}(R^n)} \sup_{0 \leq t \leq T} \|w^*(t)\|_{\tilde{B}^s_{p,1}(R^n)} \leq \frac{1}{2} \sup_{0 \leq t \leq T} \|w^*(t)\|_{\tilde{B}^s_{p,1}(R^n)}.$$ 

Therefore, we proved the existence and uniqueness of the solution. To prove the stability result, let $v^{(1)}$ and $v^{(2)}$ be two solutions with initial data $u_{01}$ and $u_{02}$. With a little abuse of notation, we still denote $v^* = v^{(1)} - v^{(2)}$. Then we have

$$v^* = u_{01} - u_{02} \tag{3.17}$$

$$\pm \int_0^t S(-\tau)[S(-\tau)\bar{v}^{(1)}(\tau)(S(\tau)v^{(1)}(\tau))^2 - S(-\tau)\bar{v}^{(2)}(\tau)(S(\tau)v^{(2)}(\tau))^2]d\tau = u_{01} - u_{02}$$

$$\pm \int_0^t S(-\tau)[S(-\tau)\bar{v}^*(\tau)(S(\tau)v^{(1)}(\tau))^2 - S(-\tau)\bar{v}^{(2)}(\tau)S(\tau)(v^{(1)}(\tau) + v^{(2)}(\tau))S(\tau)v^*(\tau)]d\tau.$$

Thus,

$$\|v^*(t)\|_{\tilde{B}^s_{p,1}(R^n)} \leq \|u_{01} - u_{02}\|_{\tilde{B}^s_{p,1}(R^n)} \tag{3.18}$$

$$+ C \int_0^t \tau^{-n(\frac{2}{p} - 1)}(\|v^{(1)}(\tau)\|_{\tilde{B}^s_{p,1}(R^n)} + \|v^{(2)}(\tau)\|_{\tilde{B}^s_{p,1}(R^n)})^2 \|v^*(\tau)\|_{\tilde{B}^s_{p,1}(R^n)}d\tau \leq \|u_{01} - u_{02}\|_{\tilde{B}^s_{p,1}(R^n)} + CT^{-n(\frac{2}{p} - 1)}(\|u_{01}\|_{\tilde{B}^s_{p,1}(R^n)} + \|u_{02}\|_{\tilde{B}^s_{p,1}(R^n)})^2 \sup_{0 \leq t \leq T} \|v^*(t)\|_{\tilde{B}^s_{p,1}(R^n)} \leq \|u_{01} - u_{02}\|_{\tilde{B}^s_{p,1}(R^n)} + \frac{1}{2} \sup_{0 \leq t \leq T} \|v^*(t)\|_{\tilde{B}^s_{p,1}(R^n)}.$$ 

Therefore

$$\sup_{0 \leq t \leq T} \|v^*(t)\|_{\tilde{B}^s_{p,1}(R^n)} \leq 2\|u_{01} - u_{02}\|_{\tilde{B}^s_{p,1}(R^n)}. \tag{3.19}$$

We completed the proof of Theorem 1.4.
4 Proof of the Theorem 1.5

In this section, we will prove theorem 1.5.

Lemma 4.1. Let \( n = 1 \) and \( v_l, l = 0, 1, 2, 3 \) be defined by Lemma 2.1, then there holds

\[
\sup_{0 \leq \tau \leq T} (\tau \|v_0(\tau)\|_{L^1(\mathbb{R})}) \leq C \prod_{i=1}^{3} (\|v_i(0)\|_{L^1(\mathbb{R})} + \int_0^T \|\partial_\tau v_i(\tau)\|_{L^1(\mathbb{R})} d\tau). \tag{4.1}
\]

Proof. \( \blacksquare \) follows from Lemma 2.1 by

\[
v_i(t) = v_i(0) + \int_0^t \partial_\tau v_i(\tau) d\tau. \tag{4.2}
\]

Lemma 4.2. Let \( n = 1 \) and \( v_l, l = 0, 1, 2, 3 \) be defined by Lemma 2.1, then there holds

\[
\left\{ \int_0^T \|v_0(\tau)\|^2_{L^2(\mathbb{R})} d\tau \right\}^{\frac{1}{2}} \leq C \prod_{i=1}^{3} \left\{ \int_0^T \|v_i(0)\|^6_{L^6(\mathbb{R})} d\tau \right\}^{\frac{1}{2}}. \tag{4.3}
\]

Proof. Let

\[
u_1(\tau) = S(\tau)\bar{v}_1(\tau), \quad u_2(\tau) = S(\tau)v_2(\tau), \quad u_3(\tau) = S(\tau)v_3(\tau), \tag{4.4}
\]

then it follows from Hölder’s inequality that

\[
\left\{ \int_0^T \|v_0(\tau)\|^2_{L^2(\mathbb{R})} d\tau \right\}^{\frac{1}{2}} \leq C \prod_{i=1}^{3} \left\{ \int_0^T \|u_i(\tau)\|^6_{L^6(\mathbb{R})} d\tau \right\}^{\frac{1}{2}}. \tag{4.5}
\]

Noting that

\[
iu_1(t, x) - \Delta u_1(t, x) = S(t)\partial_\tau \bar{v}_1(t), \tag{4.6}
\]

\[
u_1(0) = \bar{v}_1(0) \tag{4.7}
\]

as well as similar equations for \( u_2, u_3 \), the desired conclusion follows from Strichartz’ inequality. \( \blacksquare \)

By the interpolation theorem on the multi-linear functionals (see [1] page 96 Theorem 4.4.1), we can interpolate the inequality in Lemma 4.1 and Lemma 4.2 to get the following

Lemma 4.3. Let \( n = 1 \) and \( v_l, l = 0, 1, 2, 3 \) be defined by Lemma 2.1, then there holds

\[
\left\{ \int_0^T \tau^\theta \|v_0(\tau)\|_{L^p(\mathbb{R})} d\tau \right\}^\frac{1}{p} \leq C \prod_{i=1}^{3} \left\{ \|v_i(0)\|_{L^p(\mathbb{R})} + \int_0^T \|\partial_\tau v_i(\tau)\|_{L^p(\mathbb{R})} d\tau \right\}, \tag{4.8}
\]

where \( 1 < p < 2 \) and \( p', \theta \) satisfy (1.31).
We are now ready to prove Theorem 1.5.
Let us define the set
\[ X = \{ w | w(0) = u_0, \left\{ \int_0^T \tau^{\theta p'} \| \partial_\tau w(\tau) \|^p_{L^p(R)} d\tau \right\}^{\frac{1}{p'}} \leq C_1 \| u_0 \|^3_{L^p(R)} \} \] (4.9)
where \( \theta, p' \) are defined by (1.31) and \( C_1 \) is a positive constant independent of the initial data and will be determined later. For any \( w \in X \), define a map \( M \) by
\[
(Mw)(t) = u_0 \pm \int_0^t S(-\tau)[S(-\tau)\bar{w}(\tau)(S(\tau)w(\tau))]^2 d\tau.
\] (4.10)
We want to show that \( M \) maps \( X \) into itself and is a contraction.
For simplicity, we denote \( v = Mw \). Obviously,
\[
v(0) = u_0
\] (4.11)
and
\[
\partial_\tau v(\tau) = \pm S(-\tau)[S(-\tau)\bar{w}(\tau)(S(\tau)w(\tau))]^2.
\] (4.12)
Applying Lemma 4.3, we get,
\[
\left\{ \int_0^T \tau^{\theta p'} \| \partial_\tau v(\tau) \|^p_{L^p(R)} d\tau \right\}^{\frac{1}{p'}} \leq C (\| u_0 \|_{L^p} + \int_0^T \| \partial_\tau w(\tau) \|^p_{L^p(R)} d\tau)^3.
\] (4.13)
By Hölder’s inequality, we obtain
\[
\int_0^T \| \partial_\tau w(\tau) \|^p_{L^p(R)} d\tau \leq \left\{ \int_0^T \tau^{-\theta p} \right\} \left\{ \int_0^T \tau^{\theta p'} \| \partial_\tau w(\tau) \|^p_{L^p(R)} d\tau \right\}^{\frac{1}{p'}} \leq CT^{\frac{1}{p'}} \left\{ \int_0^T \tau^{\theta p'} \| \partial_\tau w(\tau) \|^p_{L^p(R)} d\tau \right\}^{\frac{1}{p'}} \leq CC_1 T^{\frac{1}{p'}} \| u_0 \|^3_{L^p(R)}.
\] (4.14)
It then follows that
\[
\left\{ \int_0^T \tau^{\theta p'} \| \partial_\tau v(\tau) \|^p_{L^p(R)} d\tau \right\}^{\frac{1}{p'}} \leq C (\| u_0 \|_{L^p(R)} + C_1 T^{\frac{1}{p'}} \| u_0 \|^3_{L^p(R)})^3 \] (4.15)
provided that \( T \) is sufficiently small. By a similar argument, we can show that \( M \) is a contraction. Moreover, it is not difficulty to prove (1.29) and (1.32).

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