ON INDICATORS OF HOPF ALGEBRAS

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Abstract. Recently, Kashina, Montgomery and Ng introduced the $n$-th indicator $\nu_n(H)$ of a finite-dimensional Hopf algebra $H$ and showed that it has several interesting properties such as the gauge invariance. The aim of this paper is to investigate properties of indicators finite-dimensional Hopf algebras, especially those of non-semisimple one.

We develop some techniques to compute indicators and apply them to study relations between indicators and the quasi-exponent. Also relations between indicators and the length of the coradical filtration are discussed.

Our results are also applied to the finite-dimensional pointed Hopf algebra $u(D, \lambda, \mu)$ introduced by Andruskiewitsch and Schneider. The Taft algebra and the small quantum group associated with $\mathfrak{sl}_2$ are examples of $u(D, \lambda, \mu)$. It turns out that indicators of them can be expressed by special values of the generating function of certain type of partitions at roots of unity.

1. Introduction

A monoidal category is a category endowed with an associative and unital binary operation between its objects (see [25] or [20] for the formal definition). They arise many contexts in mathematics and have many applications to, for example, representation theory and low-dimensional topology.

We are especially interested in monoidal categories coming from Hopf algebras [39]. We say that two Hopf algebras over the same field $k$ are monoidally Morita equivalent if their module categories are equivalent as $k$-linear monoidal categories.

From the viewpoint of the theory of monoidal categories, it is important to study properties of Hopf algebras invariant under monoidal Morita equivalence.

To study this equivalence relation, it is inevitable to introduce gauge transformations and twisting deformation due to Drinfeld (see [22] and references therein). By the result of Schauenburg [35], two finite-dimensional Hopf algebras $H$ and $H'$ are monoidally Morita equivalent if and only if they are gauge equivalent, that is, there exists a gauge transformation $F$ on $H$ such that $H'$ is isomorphic to the twisting deformation $H^F$ of $H$ by $F$. However, in general, it is difficult to determine whether such $F$ exists. Thus we are lead to study gauge invariants, that is, invariants of Hopf algebras under gauge equivalence.

Several gauge invariants have been introduced and studied, see, e.g., [11], [12], [18], [36], [37], [40], [41]. The main subject of this paper is indicators of finite-dimensional Hopf algebras, which is a gauge invariant recently introduced by Kashina, Montgomery and Ng in [18]. Motivated by the formula of the $n$-th Frobenius-Schur indicator of the regular representation of semisimple Hopf algebras [19], they defined the $n$-th indicator $\nu_n(H)$ of a finite-dimensional Hopf algebra $H$ as the trace of the linear map

$$h \mapsto S(h_{(1)} \cdots h_{(n-1)}) \quad (h \in H),$$
where \( h_{(1)} \otimes \ldots \otimes h_{(n-1)} \in H^{\otimes(n-1)} \) is the comultiplication of \( h \in H \) in Sweedler notation and \( S \) is the antipode of \( H \). In particular, the second indicator of \( H \) is the small quantum group \( \text{sl}_2 \).

If \( H = kG \) is the group algebra of a finite group \( G \), then the above map reduces to the linear map \( kG \to kG \) given by \( g \mapsto g^{-n+1} \) (\( g \in G \)). Hence, we have
\[
\nu_n(kG) = \# \{ g \in G \mid g^n = 1 \}
\]
for every \( n \geq 1 \). The right-hand side is a very basic subject in the theory of finite groups. Therefore, indicators are expected to play an important role in the theory of Hopf algebras. The aim of this paper is to study basic properties of indicators of finite-dimensional Hopf algebras, especially those of non-semisimple ones.

This paper is organized as follows: In Section 2, we briefly recall basic definitions and notations. We also discuss the exponent and the quasi-exponent of Hopf algebras. The aim of this paper is to study basic properties of indicators of finite-dimensional Hopf algebras, especially those of non-semisimple ones. In Section 3, we study basic properties of indicators and provide some lemmas to compute them. Let \( H \) be a finite-dimensional Hopf algebra over the field \( \mathbb{C} \) of complex numbers. In [18], they raised a question whether the equation
\[
\nu_n(D(H)) = |\nu_n(H)|^2
\]
holds for every \( n \geq 1 \) and showed that the equation holds if \( H \) is semisimple. We study indicators of the Drinfeld double and show that the above equation holds if \( n = 2 \) (Theorem 3.0). We also show that it holds for every \( n \geq 1 \) if \( H \) is pointed such that the group of grouplike elements is abelian and is generated by grouplike and skew-primitive elements (Corollary 3.11).

Note that the right-hand side of (1.1) is periodic in \( n \). It is known that the sequence \( \{\nu_n(H)\}_{n \geq 1} \) is periodic if \( H \) is a finite-dimensional semisimple Hopf algebra over a field of characteristic zero [19]. Thus it is natural to ask how this sequence behaves in general. In Section 4, we show that, given a finite-dimensional Hopf algebra \( H \), there exists a finite number of periodic sequences \( \{c_j(n)\}_{n \geq 1} \) such that
\[
\nu_n(H) = c_0(n) + \binom{n}{1} c_1(n) + \cdots + \binom{n}{d} c_d(n) \quad (n = 1, 2, \ldots)
\]
(see Theorem 4.5 for the precise statement). In particular, if the base field is of positive characteristic, then \( \{\nu_n(H)\}_{n \geq 1} \) is periodic. If the base field is of characteristic zero (or large enough), then the above can be written as
\[
\nu_n(H) = c_0(n) + c'_1(n)n + \cdots + c'_d(n)n^d \quad (n = 1, 2, \ldots)
\]
for some periodic sequences \( \{c'_j(n)\}_{n \geq 1} \). Relations between the period of \( \{c_j(n)\}_{n \geq 1} \) and the quasi-exponent of \( H \) is discussed. Also relations between the “degree” \( d \) and the length of the coradical filtration is discussed in the case where \( H \) has the dual Chevalley property (see, e.g., Theorem 4.12).

In [5], Andruskiewitsch and Schneider introduced a large class of finite-dimensional pointed Hopf algebras, denoted by \( u(D, \lambda, \mu) \). In Section 5 we demonstrate how our results can be applied to \( u(D, \lambda, \mu) \) and show that the \( n \)-th indicator of \( u(D, \lambda, \mu) \) depends only on \( D \) (Theorem 5.3). An important example of \( u(D, \lambda, \mu) \) is the small quantum group \( u_q(\text{sl}_2) \). By applying results in Section 5 we show that indicators of \( u_q(\text{sl}_2) \) (over \( \mathbb{C} \)) are real numbers. We also give an application to the representation theory of \( u_q(\text{sl}_2) \).
It is interesting to compute explicitly the values of the indicator. In Section 6, we derive closed, but complicated formulas of the n-th indicator of the Taft algebra and $u_q(\mathfrak{sl}_2)$ (Theorems 6.2 and 6.6). These formulas relate to the generating function of certain type of partitions studied in [42]. By evaluating that function, we obtain, if $q$ is a primitive third root of unity, then $\nu_n(u_q(\mathfrak{sl}_2)) = n^2$.

2. Preliminaries

2.1. Conventions. Throughout this paper, $k$ denotes a fixed base field. Unless otherwise noted, a vector space, an algebra, a coalgebra and a Hopf algebra are those over $k$. Given two vector spaces $V$ and $W$, the tensor product $V \otimes_k W$ is often denoted by $V \otimes W$. Given an algebra $A$, the category of finite-dimensional left $A$-modules is denoted by $\text{Rep}(A)$.

For the theory of Hopf algebras, the reader is referred to [6], [29] and [39]. Given a Hopf algebra $H$, the unit, the comultiplication, the counit, and the antipode of $H$ are denoted by $1_H$, $\Delta_H$, $\varepsilon_H$, and $S_H$, respectively. The subscript $H$ is often omitted if it is obvious from the context. We use the Sweedler notation to denote the iterated comultiplication, as

$$\Delta^{(n)} = \Delta_H^{(n)} : H \to H^{\otimes n}, \quad h \mapsto h_{(1)} \otimes \cdots \otimes h_{(n)} \quad (n = 1, 2, \ldots).$$

The multiplication map will be denoted by

$$\nabla^{(n)} = \nabla_H^{(n)} : H^{\otimes n} \to H, \quad h_1 \otimes \cdots \otimes h_n \mapsto h_1 \cdots h_n \quad (n = 1, 2, \ldots).$$

By convention, we set $H^{\otimes 0} = k$, $\Delta^{(0)} = \varepsilon$ and $\nabla^{(0)}(a) = a1_H$ ($a \in k$).

The basic theory of monoidal categories and related topics are freely used. The reader is referred to [20] and [25] for these topics.

2.2. Gauge equivalence. The notion of twisting deformation of Hopf algebras is due to Drinfeld [9]. Following [20], a gauge transformation on a Hopf algebra $H$ is an invertible element $F \in H \otimes H$ satisfying

$$(\varepsilon \otimes \text{id})(F) = 1 = (\text{id} \otimes \varepsilon)(F).$$

Given such $F$, one can define the twisted comultiplication by

$$\Delta^F(h) = F\Delta(h)F^{-1} \quad (h \in H).$$

The algebra $H$ together with the comultiplication $\Delta^F$ is a quasi-Hopf algebra in general. We denote it by $H^F$ and call it the twisting deformation of $H$ by $F$. If $F$ satisfies the dual 2-cocycle condition

$$(F \otimes 1) \cdot (\Delta \otimes \text{id})(F) = (1 \otimes F) \cdot (\text{id} \otimes \Delta)(F),$$

then $H^F$ is a Hopf algebra (see, e.g., [1] for the description of the antipode). We say that two Hopf algebras $H$ and $H'$ are gauge equivalent if $H'$ is isomorphic to $H^F$ for some $F$.

Drinfeld [9] showed that if two Hopf algebras $H$ and $H'$ are gauge equivalent, then $\text{Rep}(H)$ and $\text{Rep}(H')$ are equivalent as $k$-linear monoidal categories. Schauenburg showed that the converse holds under the assumption of finite-dimensionality:

**Theorem 2.1** (Schauenburg [35]). Let $H$ and $H'$ be two finite-dimensional Hopf algebras. Then the following conditions are equivalent:

1. $\text{Rep}(H)$ and $\text{Rep}(H')$ are equivalent as $k$-linear monoidal categories.
2. $H$ and $H'$ are gauge equivalent.
A quantity \( \nu(H) \) defined for all (finite-dimensional) Hopf algebra \( H \) is called a 
\textit{gauge invariant} if \( \nu(H) = \nu(H') \) whenever \( H \) and \( H' \) are gauge equivalent.

2.3. \textbf{Filtered and graded Hopf algebras}. Given a monoidal category \( C \), we can 
define an algebra object and a coalgebra object in \( C \). If, moreover, \( C \) is braided, then 
we can define a Hopf algebra object in \( C \). We note that an ordinary Hopf algebra 
is nothing but a Hopf algebra object in the symmetric monoidal category \( \text{Vect}(k) \) 
of vector spaces over \( k \). Here we introduce two symmetric monoidal categories to 
treat filtered and graded Hopf algebras.

A \textit{filtered vector space} is a vector space \( V \) endowed with a 
filtration, that is, a 
sequence \( V_0 \subset V_1 \subset \cdots \) of subspaces of \( V \) such that \( V = \bigcup_{i=0}^{\infty} V_i \). Let \( \text{FiltVect}(k) \) 
denote the category of filtered vector spaces with morphisms being linear maps 
preserving the filtrations. If \( V, W \in \text{FiltVect}(k) \), then the tensor product \( V \otimes W \) 
is filtered with filtration 
\[
(V \otimes W)_n = \sum_{i+j \leq n} V_i \otimes W_j \quad (n = 0, 1, \ldots).
\]

\( \text{FiltVect}(k) \) is a symmetric monoidal category with this tensor product. Now we 
can define a \textit{filtered Hopf algebra} to be a Hopf algebra object in \( \text{FiltVect}(k) \).

A \textit{graded vector space} is a vector space \( V \) endowed with a 
grading, that is, a 
direct sum decomposition \( V = \bigoplus_{i=0}^{\infty} V^i \). Let \( \text{GrVect}(k) \) 
denote the category of graded vector spaces with morphisms being linear maps 
preserving the grading. If \( V, W \in \text{GrVect}(k) \), then \( V \otimes W \) is graded with grading 
\[
(V \otimes W)(n) = \bigoplus_{i=0}^{n} V^i \otimes W_{n-i} \quad (n = 0, 1, \ldots).
\]

\( \text{GrVect}(k) \) is a symmetric monoidal category with this tensor product. We define 
a \textit{graded Hopf algebra} to be a Hopf algebra object in \( \text{GrVect}(k) \).

Let \( V \) be a filtered vector space. For convention, we set \( V_{-1} = 0 \). The associated 
graded vector space is the vector space 
\[
\text{gr } V = \bigoplus_{n=0}^{\infty} V^n/V_{n-1} \quad \text{with grading} \quad (\text{gr } V)(n) = V^n/V_{n-1}.
\]
The assignment \( V \mapsto \text{gr } V \) gives rise to a symmetric monoidal functor 
\[
\text{gr} : \text{FiltVect}(k) \to \text{GrVect}(k).
\]
Hence, if \( H \) is a filtered Hopf algebra, then \( \text{gr } H \) is naturally a graded Hopf algebra 
as the image of a Hopf algebra object under a symmetric monoidal functor.

2.4. \textbf{Coradical filtration}. Let \( C \) be a coalgebra with comultiplication \( \Delta \). The 
\textit{coradical} of \( C \), denoted by \( C_0 \), is the sum of all simple subcoalgebras of \( C \). For 
each \( n \geq 1 \), we define \( C_n \subset C \) inductively by 
\[
C_n = \{ c \in C \mid \Delta(c) \in C \otimes C_0 + C_{n-1} \otimes C \}.
\]
The sequence \( C_0 \subset C_1 \subset \cdots \) is called the \textit{coradical filtration} \( \text{[29, \S 5]} \). We note that 
\( C \) turns into a filtered coalgebra with respect to the coradical filtration.

The \textit{Loewy length}, \( \text{Lw}(C) \), is the smallest \( n \geq 0 \) such that \( C_{n-1} = C \); if such \( n \) 
does not exist, we set \( \text{Lw}(C) = \infty \). If it is finite, then we have 
\[
(2.1) \quad \text{Lw}(C) = \min \{ n \geq 0 \mid C_n = C_{n-1} \}.
\]
Let $C$ be a coalgebra, and let $A$ be an algebra. Then the set $\text{Hom}_k(C, A)$ is an algebra with respect to the convolution product $\star$ defined by

$$(f \star g)(c) = f(c(1))g(c(2)) \quad (f, g \in \text{Hom}_k(C, A), c \in C).$$

Let $f \in \text{Hom}_k(C, A)$. We denote by $f^n$ the $n$-th power of $f$ with respect to $\star$. The following lemma is well-known and can be proved by induction on $n$.

**Lemma 2.2.** If $f(C_0) = 0$, then $f^n(C_{n-1}) = 0$ for every $n \geq 1$.

Let $H$ be a Hopf algebra. We say that $H$ has the dual Chevalley property if the coradical $H_0$ is a Hopf subalgebra. If this is the case, the coradical filtration for $H$ is a Hopf algebra filtration, and therefore the associated graded vector space $\text{gr } H$ is naturally a graded Hopf algebra.

### 2.5. Bosonization

**Bosonization.** Let $H$ be a Hopf algebra with bijective antipode. A left Yetter-Drinfeld module over $H$ is a vector space $V$ equipped with a left $H$-module structure $\rightarrow$ and a left $H$-comodule structure $\rho : V \rightarrow H \otimes V$, $v \mapsto v_{(-1)} \otimes v_{(0)}$, such that

$$\rho(h \cdot v) = h_{(1)}v_{(0)}S_H(h_{(3)}) \otimes (h_{(2)} \rightarrow v_{(0)}) \quad (h \in H, v \in V).$$

We denote by $^H_H \mathcal{YD}$ the $k$-linear braided monoidal category of left Yetter-Drinfeld modules over $H$ (see [29 §10.6]); the braiding of $^H_H \mathcal{YD}$ is given by

$$(2.2) \quad c_{V, W}(v \otimes w) = (v_{(-1)} \rightarrow w) \otimes v_{(0)} \quad (v \in V, w \in W).$$

A braided Hopf algebra over $H$ is a Hopf algebra object in $^H_H \mathcal{YD}$. Given a braided Hopf algebra $B$, we denote the comultiplication of $b \in B$ by $\Delta(b) = b^{(1)} \otimes b^{(2)}$ in order not to confuse with the coaction of $H$. The **bosonization**, or the **biproduct**, of $B$ (see Radford [32] and Majid [26]) is an ordinary Hopf algebra $B \# H$ with the underlying vector space $B \otimes H$, the multiplication

$$(2.3) \quad (b \# h) \cdot (b' \# h') = b(h_{(1)} \rightarrow b') \# h_{(2)}h' \quad (b, b' \in B, h, h' \in H),$$

and the comultiplication

$$(2.4) \quad \Delta(b \# h) = b^{(1)} \# b^{(2)}(\cdot_{(-1)}h(1)) \otimes b^{(2)}(\cdot_{(0)}h(2)) \quad (b \in B, h \in H).$$

Here we denoted $b \otimes h \in B \# H$ by $b \# h$. The antipode of $B \# H$ is given by

$$(2.5) \quad S_{B \# H}(b \# h) = (1 \# S_H(b_{(-1)}h)) \cdot (S_B(b_{(0)}) \# 1).$$

One can define the braided monoidal category of graded Yetter-Drinfeld modules over $H$ in a similar way as $\text{GrVect}(k)$. A graded braided Hopf algebra over $H$ is a Hopf algebra object in this category. If a braided Hopf algebra $B$ over $H$ is graded, then also $B \# H$ is graded with grading $(B \# H)(n) = B(n) \otimes H$.

The bosonization naturally arises from Hopf algebras with a projection; let $A$ be a Hopf algebra having $H$ as a Hopf subalgebra, and let $\pi : A \rightarrow H$ be a Hopf algebra map such that $\pi_H = \text{id}_H$. Then the set

$$B := A^{co \pi} = \{b \in A \mid (\text{id} \otimes \pi)\Delta(b) = b \otimes 1\}$$

has a structure of braided Hopf algebra over $H$, and the map

$$(2.6) \quad B \# H \rightarrow A, \quad b \# h \mapsto bh \quad (b \in B, h \in H)$$

is an isomorphism of Hopf algebras.
Now we consider the case where $A = \bigoplus_{n=0}^{\infty} A(n)$ is a graded Hopf algebra such that $A(0) = H$, and $\pi : A \to H$ is the projection. Then $B$ is a graded braided Hopf algebra over $H$ with grading $B(n) = A(n) \cap B$. Moreover, if we identify $A$ with $B \# H$ via (2.6), then $A(n) = B \otimes H$ (see, e.g., [2, §2]).

**Lemma 2.3.** Let $H$ be a finite-dimensional Hopf algebra having the dual Chevalley property. Then $\text{Lw}(H) \leq [H : H_0]$.

**Proof.** We consider the coradical filtration for $H$. By applying the above arguments to the associated graded Hopf algebra $A = \text{gr} H$, we may assume that there exists a graded braided Hopf algebra $B = \bigoplus_{n=0}^{\infty} B(n)$ over $H$ such that

$$\text{gr} H = B \# H_0 \quad \text{and} \quad A(n) = H_n / H_{n-1} = B(n) \otimes H_0.$$ By (2.7), $B(n) \neq 0$ if and only if $0 \leq n < \text{Lw}(H)$. Therefore

$$\text{Lw}(H) = \min \{ n \geq 0 \mid B(n) = 0 \} \leq \dim_k B = [H : H_0],$$

and thus our claim is proved. $\square$

### 2.6. Nichols algebras.

Let $C$ be a coalgebra. The elements of the set

$$G(C) = \{ g \in C \mid \Delta(g) = g \otimes g \text{ and } \varepsilon(g) = 1 \}$$

are called **grouplike elements**. Given $g, h \in G(C)$, the elements of the set

$$\mathcal{P}_{g,h}(C) = \{ x \in C \mid \Delta(x) = x \otimes g + h \otimes x \}$$

are called $(g,h)$-skew-primitive elements. If $B$ is a (braided) Hopf algebra, then the elements of $\mathcal{P}(B) := \mathcal{P}_{1,1}(B)$ are called **primitive elements**.

Let $H$ be a Hopf algebra with bijective antipode. A **Nichols algebra** [3, Definition 2.1] is a graded braided Hopf algebra $B = \bigoplus_{n=0}^{\infty} B(n)$ over $H$ such that $B(0) = k$, $B(1) = \mathcal{P}(B)$ and $\mathcal{P}(B)$ generates $B$ as an algebra. For each $V \in H^H \mathcal{YD}$, there exists a Nichols algebra $\mathfrak{B}(V)$, which is unique up to isomorphism, such that $\mathcal{P}(\mathfrak{B}(V)) \cong V$.

Let $A = \bigoplus_{n=0}^{\infty} A(n)$ be a graded pointed Hopf algebra such that

$$A_n = A(0) \oplus A(1) \oplus \cdots \oplus A(n)$$

for every $n \geq 1$, where $A_0 \subset A_1 \subset \ldots$ is the coradical filtration of $A$, and let $\pi : A \to A(0)$ be the projection. If $A$ is generated by $G = G(A)$ and skew-primitive elements, then $B = A^{\text{co} \pi}$ is a Nichols algebra, and hence there exists $V \in k_G \mathcal{YD}$ such that

$$A \cong \mathfrak{B}(V) \# kG$$

as graded Hopf algebras. We see that, as a vector space,

$$V = A(1) \cap A^{\text{co} \pi} = \bigoplus_{g \in G} \mathcal{P}_{1,g}(A),$$

and the action $\to$ and the coaction $\rho$ on $V$ are given respectively by

$$g \to x = gxg^{-1} \quad (g \in G, x \in V),$$

$$\rho(x) = h \otimes x \quad (x \in \mathcal{P}_{1,h}(A), h \in G).$$

(2.7)
2.7. The exponent and the quasi-exponent. The following definitions are due to Etingof and Gelaki ([11], [12]).

Definition 2.4. Let $H$ be a finite-dimensional Hopf algebra, and let $u$ be the Drinfeld element of the Drinfeld double $D(H)$.

(1) The exponent of $H$, $\exp(H)$, is the order of $u$.
(2) The quasi-exponent of $H$, $\exp(H)$, is the smallest integer $n \geq 1$ such that $u^n - 1$ is nilpotent.

We shall recall characterizations of the exponent and the quasi-exponent. Let $H$ and $u$ be as above. Define

\[ T_n : H \to H \quad (n = 0, 1, \ldots) \]

\[
T_0(h) = \varepsilon(h)1 \quad \text{and} \quad T_n(h) = h_{(1)}S^{-2}(h_{(2)}) \cdots S^{-2n+2}(h_{(n)}) \quad (n \geq 1).
\]

Let $f(X) = \sum_{i=0}^{m} c_i X^i \in k[X]$ be a polynomial. Etingof and Gelaki showed that $f(u) = 0$ if and only if $c_0 T_0 + \ldots + c_m T_m = 0$ [12, Proposition 2.3]. Therefore $\exp(H)$ is the smallest integer $n \geq 1$ such that

\[ T_n = T_0, \]

and $\exp(H)$ is the smallest integer $n \geq 1$ such that

\[ \sum_{i=0}^{m} (-1)^i \binom{m}{i} T_{ni} = 0 \]

for some integer $m \geq 1$. These characterizations allow us to compute the exponent of finite group algebras. If $G$ is a finite group, then

\[ \exp(kG) = \min \{ n \geq 1 \mid g^n = 1 \text{ for all } g \in G \}. \]

In group theory, the right-hand side has been called the exponent of $G$. Hence we denote it by $\exp(G)$ by abuse of notation.

The quasi-exponent is always finite (see Corollary 4.10 of [11]). For a while, we assume that the characteristic of $k$ is zero. Then the exponent of $H$ may be infinite. If it is finite, then there holds

\[ \exp(H) = \exp(H). \]

Suppose that $H$ is semisimple. Kashina ([16], [17]) conjectured that $\exp(H)$ is finite and divides $\dim_k(H)$. This is partially solved; Etingof and Gelaki [11] showed that $\exp(H)$ divides the algebraic integer $\dim_k(H)^3$. Natale [30] gave several estimations for the exponent of group-theoretical fusion categories and showed that $\exp(H)$ divides $\dim_k(H)^2$ if $H$ is group-theoretical.

Now we suppose that $k$ is of characteristic $p > 0$. Then also the exponent is always finite. However, equation (2.12) does not hold in general; indeed, by the following lemma, we see that $\exp(k\mathbb{Z}_p) = p$ while $\exp(k\mathbb{Z}_p) = 1$.

Lemma 2.5. Let $H$ be a finite-dimensional Hopf algebra over a field $k$ of characteristic $p > 0$, and let $p^m$ be the largest power of $p$ which divides $\exp(H)$. Then we have $\exp(H) = p^m \exp(H)$.

Proof. We may assume that $k$ is algebraically closed. Let $q = p^{-m} \exp(H)$, and let $u$ be the Drinfeld element of $D(H)$. By the definition of the exponent, we have

\[ (u^q - 1)^{p^m} = u^{qp^m} - 1 = u^{\exp(H)} - 1 = 0, \]

and hence $\exp(H)$ divides $q$. 

We next show that \( q \) divides \( \text{qexp}(H) \). Let \( \omega_1, \ldots, \omega_n \) be eigenvalues of the action of \( u \) on \( D(H) \). By the definition of the quasi-exponent, \( \text{qexp}(H) \) is the least common multiple of the orders of \( \omega_i \)'s. By (2.13), we see that \( \omega_i^q = 1 \) for every \( i \). Therefore \( q \) divides \( \text{qexp}(H) \), and hence \( q = \text{qexp}(H) \). \( \square \)

**Remark 2.6.** Etingof and Gelaki [11] showed that the exponent is a gauge invariant. In [12], they showed that the quasi-exponent is a gauge invariant when the base field is of characteristic zero. If the base field is of positive characteristic, their proof do not work. However, even in this case, the quasi-exponent can be shown to be a gauge-invariant by Lemma 2.5 and the gauge invariance of the exponent.

### 3. Indicators of Hopf algebras

#### 3.1. Indicators of Hopf algebras

Throughout this section, we work over a fixed base field \( k \). Let \( H \) be a finite-dimensional Hopf algebra. For \( n \geq 1 \), we define

\[
P_H^{(n)} = \text{id}_H^n = \underbrace{\text{id}_H \ast \cdots \ast \text{id}_H}_n,
\]

where \( \ast \) is the convolution product in \( \text{End}_k(H) \). This map is called the \( n \)-th Sweedler power map, or the \( n \)-th Hopf power map on \( H \). Note that there holds

\[
P_H^{(n)} = \nabla_H^{(n)} \circ \Delta_H^{(n)}.
\]

Following [18], for an integer \( n \geq 1 \), we define the \( n \)-th indicator of \( H \) by

\[
\nu_n(H) = \text{Tr} \left( S_H \circ P_H^{(n-1)} \right).
\]

They showed that \( \nu_n(H) \) is a gauge invariant [18, Theorem 2.2]. They also showed that indicators can be written in terms of integrals of \( H \). If \( \Lambda \in H \) and \( \lambda \in H^* \) are both left or both right integrals such that \( \langle \lambda, \Lambda \rangle = 1 \), then

\[
\nu_n(H) = \langle \lambda, \Lambda^{[n]} \rangle
\]

for every \( n \geq 1 \) [18, Corollary 2.6]

We prove several supplemental lemmas for computing indicators. Let \( H \) and \( L \) be two finite-dimensional Hopf algebras. The tensor product \( H \otimes L \) is naturally a Hopf algebra. One can check that the equality

\[
S_{H \otimes L} \circ P_{H \otimes L}^{(n-1)} = (S_H \circ P_H^{(n-1)}) \otimes (S_L \circ P_L^{(n-1)})
\]

holds for every \( n \). Taking the trace, we get the following lemma:

**Lemma 3.1.** \( \nu_n(H \otimes L) = \nu_n(H) \nu_n(L) \) for every \( n \geq 1 \).

Let \( \Lambda \in H \) and \( \lambda \in H^* \) be both right integrals such that \( \langle \lambda, \Lambda \rangle = 1 \). By (3.2) and the definition of the dual Hopf algebra, we have

\[
\nu_n(H) = \langle \lambda, \Lambda^{[n]} \rangle = \langle \Lambda_{(1)}, \Lambda_{(1)} \rangle \cdots \langle \Lambda_{(n)}, \Lambda_{(n)} \rangle = \langle \lambda^{[n]}, \Lambda \rangle = \langle \Lambda^{**}, \lambda^{[n]} \rangle,
\]

where \( \Lambda^{**} \in H^{**} \) is the element corresponding to \( \Lambda \) under the canonical isomorphism \( H \cong H^{**} \). Since \( \Lambda^{**} \) is a right integral such that \( \langle \Lambda^{**}, \lambda \rangle = 1 \), the right hand side is the \( n \)-th indicator of \( H^* \). Therefore:

**Lemma 3.2.** \( \nu_n(H^*) = \nu_n(H) \).
Of course, this also follows from

\[ S_H \circ P_{H}^{(n)} = \left( S_H \circ P_{H}^{(n)} \right)^*. \]

Doi \cite{7} introduced the dual notion to twisting transformation, called 2-cocycle deformation. Lemma 3.2 yields the following important consequence:

**Lemma 3.3.** Indicators are invariant under 2-cocycle deformation.

**Proof.** Let \( H \) be a finite-dimensional Hopf algebra, and let \( L \) be a Hopf algebra obtained by 2-cocycle deformation from \( H \). Then \( H^* \) and \( L^* \) are gauge equivalent. By the previous lemma and the gauge invariance of \( \nu_n \), we have

\[ \nu_n(H) = \nu_n(H^*) = \nu_n(L^*) = \nu_n(L) \]

for every \( n \geq 1 \). \(\Box\)

### 3.2. Indicators of filtered Hopf algebras.

Let \( H \) be a finite-dimensional filtered Hopf algebra. Here we prove the following lemma:

**Lemma 3.4.** \( \nu_n(H) = \nu_n(gr H) \) for every \( n \geq 1 \).

**Proof.** In \cite{32} we have introduced two symmetric monoidal categories \( \text{FiltVect}(k) \) and \( \text{GrVect}(k) \) and a symmetric monoidal functor

\[ gr : \text{FiltVect}(k) \to \text{GrVect}(k). \]

Let \( V \) be a finite-dimensional filtered vector space, and let \( f : V \to V \) be a linear map that preserves the filtration. By linear algebra, we have

\[ \text{Tr}(\text{gr}(f)) = \text{Tr}(f). \]

Our claim is obvious when \( n = 1 \), and hence we assume \( n \geq 2 \). For filtered vector spaces \( X_1, \ldots, X_m \), we denote by

\[ \varphi_{X_1, \ldots, X_m} : \text{gr}(X_1) \otimes \cdots \otimes \text{gr}(X_m) \to \text{gr}(X_1 \otimes \cdots \otimes X_m) \]

the canonical isomorphism coming from the monoidal structure of \( gr \). By the definition of \( gr H \), we have

\[ \Delta_{gr H}^{(m)} = \varphi_{H, \ldots, H}^{-1} \circ \text{gr}(\Delta_{H}^{(m)}) \quad \text{and} \quad \nabla_{gr H}^{(m)} = \text{gr}(\nabla_{H}^{(m)}) \circ \varphi_{H, \ldots, H} \]

for every \( m \geq 1 \). Letting \( m = n - 1 \), we obtain

\[ \text{gr}(S_{gr H} \circ P_{gr H}^{(n-1)}) = S_{gr H} \circ P_{gr H}^{(n-1)}. \]

By \cite{33} and the definition of the \( n \)-th indicator, we have

\[ \nu_n(gr H) = \text{Tr}(S_{gr H} \circ P_{gr H}^{(n-1)}) = \text{Tr}(S_H \circ P_{H}^{(n-1)}) = \nu_n(H). \]

The proof is completed. \(\Box\)

### 3.3. Indicators of the Drinfeld double.

Let \( H \) be a finite-dimensional Hopf algebra. Doi and Takeuchi \cite{8} showed that the Drinfeld double \( D(H) \) can be obtained by 2-cocycle deformation from \( (H^{op})^* \otimes H \). By Lemmas 3.1, 3.2 and 3.3 we immediately have the following consequence:

**Corollary 3.5.** \( \nu_n(D(H)) = \nu_n(H) \nu_n(H^{op}) \) for every \( n \geq 1 \).
For a while, we work over the field $\mathbb{C}$ of complex numbers. In [18], they raised a question whether the equation
\begin{equation}
(3.4) \quad \nu_n(D(H)) = |\nu_n(H)|^2
\end{equation}
holds for every finite-dimensional Hopf algebra $H$ and $n \geq 1$. They also showed that (3.4) holds if $H$ is semisimple. In view of their question and the above corollary, it is interesting to know when the following equation, which implies (3.4), holds:
\begin{equation}
(3.5) \quad \nu_n(H^{op}) = \nu_n(H).
\end{equation}

The author showed that (3.5) holds for every $n$ if $H$ is semisimple [36]. We will show that (3.5) holds for every $n$ if $H$ is pointed such that the group $G(H)$ of grouplike elements is abelian (Corollary 3.11).

Aside from these results, (3.5) holds for $n = 2$ without any assumptions on $H$.

**Theorem 3.6.** $\nu_2(H^{op}) = \nu_2(H)$. Hence, $\nu_2(D(H)) = |\nu_2(H)|^2$.

**Proof.** Recall that the antipode of $H^{op}$ is $S^{-1}_H$. Let $\omega_1, \ldots, \omega_m$ be eigenvalues of $S_H$, counting multiplicities. By the theorem of Radford [31], the order of $S_H$ is finite, and hence $\omega_i$’s are roots of unity. Therefore
\[
\frac{\nu_2(H)}{\text{Tr}(S_H)} = \sum_{i=1}^m \omega_i = \sum_{i=1}^m \omega_i^{-1} = \text{Tr}(S_H^{-1}) = \nu_2(H^{op}).
\]
The claim is proved. $\square$

This yields the following non-quasitriangularity criterion:

**Corollary 3.7.** If $H$ is quasitriangular, then $\nu_2(H)$ is a real number.

**Proof.** $\text{Rep}(H)$ and $\text{Rep}(H^{op})$ are equivalent as $\mathbb{C}$-linear monoidal categories since they are braided. By the gauge invariance and Theorem 3.6, we have
\[
\nu_2(H) = \nu_2(H^{op}) = \nu_2(H^{op}) = \nu_2(H).
\]
Therefore $\nu_2(H)$ is a real number. $\square$

It is interesting to apply Corollary 3.7 to the Taft algebra. Let $N > 1$ be an integer and let $\omega$ be a primitive $N$-th root of unity. The Taft algebra $T_{N^2}(\omega)$ is an algebra generated by $x$ and $g$ with relations
\[
x^N = 0, \quad g^N = 1, \quad \text{and} \quad gx = \omega xg.
\]
$T_{N^2}(\omega)$ is a Hopf algebra with the coalgebra structure determined by
\begin{equation}
(3.6) \quad \Delta(x) = x \otimes 1 + g \otimes x, \quad \varepsilon(x) = 0, \quad \Delta(g) = g \otimes g, \quad \text{and} \quad \varepsilon(g) = 1,
\end{equation}
and the antipode determined by $S(x) = -g^{-1}x$ and $S(g) = g^{-1}$.

The second indicator of $T_{N^2}(\omega)$ is computed in [18] as follows (remark that our definition of $T_{N^2}(\omega)$ is slightly different from that in [18]):
\begin{equation}
(3.7) \quad \nu_2(T_{N^2}(\omega)) = \begin{cases} 2(1 + \omega - \frac{1}{2}(N+1))^{-1} & \text{if } N \text{ is odd}, \\ 4(1 - \omega^{-1})^{-1} & \text{otherwise}. \end{cases}
\end{equation}
Corollary 3.7 implies that $T_{N^2}(\omega)$ is not quasitriangular if $N > 2$. On the other hand, $\text{Tr}(S) \in \mathbb{Z}$ if $N = 2$. It is known that $T_4(-1)$ has infinitely many universal $R$-matrices [33].
3.4. Complex conjugate. To study when equation (3.5) holds, we introduce the complex conjugate of complex Hopf algebras.

We work in general situation. Let $K/k$ be an extension of fields, and let $\sigma$ be an automorphism of $K$. Define a $K$-bimodule $K_\sigma$ by $K_\sigma = K$ with the left and the right actions given respectively by

$$a \cdot x = ax \quad \text{and} \quad x \cdot a = x\sigma(a) \quad (a \in K, x \in K_\sigma).$$

Let $V \in \text{Vect}(k)$. Given $v \in V$, we write $[v]$ for $1 \otimes v \in K_\sigma \otimes_k V$. Observe

$$(3.8) \quad [av] = \sigma(a)[v] \quad (a \in k, v \in V).$$

There exists a natural isomorphism

$$\varphi_{V,W} : (K_\sigma \otimes_k V) \otimes_k (K_\sigma \otimes_k W) \to K_\sigma \otimes_k (V \otimes_k W), \quad [v] \otimes [w] \to [v \otimes w]$$

and an isomorphism $\varphi_0 : K \to K_\sigma \otimes_k k$, $a \mapsto a \otimes 1$ ($a \in K$). We see that the functor $K_\sigma \otimes_k (-)$ actually gives rise to a symmetric monoidal functor

$$K_\sigma = (K_\sigma \otimes_k (-), \varphi, \varphi_0) : \text{Vect}(k) \to \text{Vect}(K),$$

which may not be linear in general. Now we provide the following lemma:

**Lemma 3.8.** Let $V$ be a finite-dimensional vector space over $k$, and let $f : V \to V$ be a linear map. Then $\text{Tr}(K_\sigma \otimes f) = \sigma(\text{Tr}(f))$ in $K$.

**Proof.** Let $e_1, \ldots, e_n$ be a $k$-basis of $V$. Then $[e_1], \ldots, [e_n]$ is a $K$-basis of $K_\sigma \otimes_k V$. We define $a_{ij} \in k$ by $f(e_i) = \sum_{j=1}^{n} a_{ij} e_j$. Then, by (3.8), we have

$$(K_\sigma \otimes f)([e_i]) = [f(e_i)] = \sum_{j=1}^{n} [a_{ij} e_j] = \sum_{j=1}^{n} \sigma(a_{ij})[e_j].$$

Therefore, $\text{Tr}(K_\sigma \otimes f) = \sum_{i=1}^{n} \sigma(a_{ii}) = \sigma(\sum_{i=1}^{n} a_{ii}) = \sigma(\text{Tr}(f))$. \hfill \Box

Let $H$ be a Hopf algebra over $k$. Then $K_\sigma \otimes_k H$ is naturally a Hopf algebra over $K$ as the image of a Hopf algebra under a symmetric monoidal functor. In the case where $\sigma = \text{id}_K$, $K_\sigma \otimes_k H$ is nothing but the coefficient extension of $H$ to $K$.

Suppose that $H$ is finite-dimensional. Then the $n$-th indicator of $K_\sigma \otimes_k H$ is, by definition, the trace of the linear map given by

$$K_\sigma \otimes_k H \to K_\sigma \otimes_k H, \quad [h] \mapsto [S_H \circ \rho^{(n-1)}_H(h)] \quad (h \in H).$$

Hence, by the previous lemma, we have:

**Lemma 3.9.** $\nu_n(K_\sigma \otimes_k H) = (\nu_n(H))$ for every $n \geq 1$.

In what follows, we consider the case where $K = k = \mathbb{C}$ and $\sigma(z) = \overline{z}$ is the complex conjugate. We write $\overline{V}$ for $K_\sigma \otimes_k V$ by convention and call it the complex conjugate of $V$. Note that if $H$ is a complex Hopf algebra, then $\overline{H}$ is naturally a complex Hopf algebra.

**Theorem 3.10.** Let $\Gamma$ be a finite abelian group, and let $V \in \text{Vect}(\mathbb{C})$. Then

$$(\mathfrak{B}(V) \# \mathbb{C} \Gamma)^{\text{op}} \cong \overline{\mathfrak{B}(V) \# \mathbb{C} \Gamma}$$

as graded Hopf algebras.

Therefore, by Lemma 3.9, equation (3.5) holds for every finite-dimensional Hopf algebra of the form $H = \mathfrak{B}(V) \# \mathbb{C} \Gamma$, where $\Gamma$ is abelian, and every $n \geq 1$. More strongly, we obtain the following result as a corollary of Theorem 3.10.
Lemma 3.12. Let $H$ be a finite-dimensional complex pointed Hopf algebra generated by grouplike elements and skew-primitive elements. Suppose that $G(H)$ is abelian. Then (3.7) holds for every $n \geq 1$, and hence $\nu_n(D(H)) = |\nu_n(H)|^2$ for every $n \geq 1$.

Proof. Consider $H$ as a filtered Hopf algebra with the coradical filtration. Then, by the assumptions, there exists $V \in \mathcal{CYD}$ such that $\text{gr } H \cong \mathcal{B}(V)\# \mathcal{G}$. Hence

$$\nu_n(H^{\text{op}}) = \nu_n((\text{gr } H)^{\text{op}}) = \nu_n((\mathcal{B}(V)\# \mathcal{G})^{\text{op}}) = \nu_n(\mathcal{G}) = \nu_n(H)$$

by Lemma 3.3, Lemma 3.9 and Theorem 3.10.

Corollary 3.11. Let $H$ be a finite-dimensional complex pointed Hopf algebra generated by grouplike elements and skew-primitive elements. Suppose that $G(H)$ is abelian. Then (3.7) holds for every $n \geq 1$, and hence $\nu_n(D(H)) = |\nu_n(H)|^2$ for every $n \geq 1$.

Proof. Consider $H$ as a filtered Hopf algebra with the coradical filtration. Then, by the assumptions, there exists $V \in \mathcal{CYD}$ such that $\text{gr } H \cong \mathcal{B}(V)\# \mathcal{G}$. Hence

$$\nu_n(H^{\text{op}}) = \nu_n((\text{gr } H)^{\text{op}}) = \nu_n((\mathcal{B}(V)\# \mathcal{G})^{\text{op}}) = \nu_n(\mathcal{G}) = \nu_n(H)$$

by Lemma 3.3, Lemma 3.9 and Theorem 3.10.

To prove Theorem 3.10 we introduce several notations. Let $\hat{\mathcal{G}}$ denote the group of (C-valued) characters of $\mathcal{G}$. For $V \in \mathcal{CYD}$, $\chi \in \hat{\mathcal{G}}$ and $g \in \mathcal{G}$, we set

$$V_{\chi} = \{ v \in V \mid \gamma \to v = \chi(\gamma)v \text{ for all } \gamma \in \mathcal{G} \},$$

$$V_g = \{ v \in V \mid \rho_V(v) = g \otimes v \}, \text{ and } V_{g, \chi} = V_g \cap V_{\chi},$$

where $\to$ and $\rho_V$ are the action and the coaction of $\mathcal{G}$ on $V$. Since $\mathcal{C}$ is algebraically closed and of characteristic zero, we have a decomposition

$$V = \bigoplus_{g \in \mathcal{G}, \chi \in \hat{\mathcal{G}}} V_{g, \chi}.$$

Given $V \in \mathcal{CYD}$, we define a left Yetter-Drinfeld module $V^{\text{op}}$ over $\mathcal{G}$ to be the vector space $V$ endowed with the $\mathcal{C}$-action $-^{\text{op}}$ given by

$$g^{\text{op}} v := g^{-1} \to v \quad (g \in \mathcal{G}, v \in V)$$

and the same $\mathcal{C}$-$\mathcal{G}$-coaction $\rho^{\text{op}} = \rho_v$ as $V$. By (2.7), we have the following lemma, which is a special case of results of Radford and Schneider [34].

Lemma 3.12. $(\mathcal{B}(V)\# \mathcal{G})^{\text{op}} \cong \mathcal{B}(V^{\text{op}})\# \mathcal{G}$.

The complex conjugate $\overline{\mathcal{G}}$ turns into a left Yetter-Drinfeld module over $\mathcal{G}$ with the $\mathcal{C}$-action $-^c$ and the $\mathcal{C}$-$\mathcal{G}$-coaction $\rho_V^c$ given respectively by

$$g^c [v] = [g \to v], \quad \rho_V^c([v]) = h \otimes v \quad (g, h \in \mathcal{G}, v \in V_h).$$

Again by (2.7), we have the following lemma:

Lemma 3.13. $(\mathcal{B}(V)\# \mathcal{G})^{c} \cong \mathcal{B}(\overline{V})\# \mathcal{G}$.

Proof of Theorem 3.10. In view of previous two lemmas, it is sufficient to show that left Yetter-Drinfeld modules $V^{\text{op}}$ and $\overline{V}$ are isomorphic. To see this, we fix a basis $e_1, \ldots, e_n$ of $V$ such that $e_i \in V_{g_i, \chi_i}$ for some $g_i \in \mathcal{G}$ and $\chi_i \in \hat{\mathcal{G}}$. By using this basis, we define a linear map $f : V^{\text{op}} \to \overline{V}$ by

$$f(e_1, \ldots, e_n) = \chi_1(e_1) + \ldots + \chi_n(e_n) \quad (e_i \in \mathcal{C}).$$

Obviously, $f$ is bijective. Note that $z^{-1} = \overline{z}$ if $|z| = 1$. By (3.8), we have

$$g^c f(e_i) = [g \to e_i] = [\chi_i(g)e_i] = \chi_i(g)[e_i] = f(\chi_i(g^{-1})e_i) = f(g^{-\text{op}} e_i).$$

Hence $f$ is $\mathcal{C}$-$\mathcal{G}$-linear. We also have

$$\rho_V^c(f(e_i)) = \rho_V^c([e_i]) = g_i \otimes [e_i] = (\text{id}_{\mathcal{C}} \otimes f)\rho_V(e_i).$$

This shows that $f$ is $\mathcal{C}$-$\mathcal{G}$-colinear. Therefore $f$ is an isomorphism of Yetter-Drinfeld modules over $\mathcal{C}$-$\mathcal{G}$, and hence the result follows.
4. Indicators and periodic sequences

4.1. The minimal polynomial. Let \( a = \{a_n\}_{n \geq 1} \) be a sequence of elements of a vector space. We say that \( a \) is linearly recursive if there exists a polynomial \( f(X) = \sum_{i=0}^{m} c_i X^i \in k[X] \) such that

\[
(4.1) \quad c_0 a_n + c_1 a_{n+1} + \ldots + c_m a_{n+m} = 0
\]

for all \( n \geq 1 \). If this is the case, such polynomials form a non-zero ideal of \( k[X] \); the minimal polynomial of \( a \) is the monic polynomial generating this ideal.

Let \( H \) be a finite-dimensional Hopf algebra. In [18], they showed that the sequence \( P_H := \{P_H^{(n-1)}\}_{n \geq 1} \) is linearly recursive in \( \text{End}_k(H) \). Since the sequence \( \nu_H := \{\nu_n(H)\}_{n \geq 1} \) is obtained as the image of \( P_H \) under the linear map

\[
\text{End}_k(H) \to k, \quad f \mapsto \text{Tr}(S_H \circ f),
\]

also \( \nu_H \) is linearly recursive (see [18] Proposition 2.7).

Let \( \Phi_H(X) \) and \( \phi_H(X) \) be the minimal polynomials of the sequences \( P_H \) and \( \nu_H \), respectively. By the above arguments, \( \Phi_H(X) \) divides \( \phi_H(X) \). Since the sequence \( \nu_H \) is a gauge invariant, as remarked in [18], also \( \phi_H(X) \) is a gauge invariant. We list some elementary properties of \( \Phi_H(X) \) and \( \phi_H(X) \).

**Lemma 4.1.** Let \( H \) be a finite-dimensional Hopf algebra. Then:

1. \( \deg \phi_H(X) \leq \deg \Phi_H(X) \leq \dim_k(H)^2 \).
2. \( \Phi_H(0) \neq 0 \) and \( \phi_H(0) \neq 0 \).
3. \( \Phi_H(1)(X) = \Phi_H(X) \) and \( \phi_H(1)(X) = \phi_H(X) \).
4. If \( L \) is either a quotient Hopf algebra or a Hopf subalgebra of \( H \), then \( \Phi_L(X) \) divides \( \Phi_H(X) \).

**Proof.** (1) was proved in the proof of [18] Proposition 2.7. We prove (2). Suppose to the contrary that \( \Phi_H(0) = 0 \). Then \( \Phi_H(X) = 0 \). By the definition of \( \Phi_H(X) \), \( \sum_{i=0}^{m} c_i P_H^{(i+n)} = 0 \) for every \( n \geq 1 \). Recall \( P_H^{(n)} = \text{id}_H^n \). Since \( S_H \) is the inverse of \( \text{id}_H \) with respect to \( \ast \), we have

\[
\sum_{i=0}^{m} c_i P_H^{(i-1+n)} = S_H \ast \sum_{i=0}^{m} c_i P_H^{(i+n)} = 0.
\]

This contradicts to the minimality of \( \Phi_H(X) \), and thus \( \Phi_H(0) \neq 0 \). Since \( \phi_H(X) \) divides \( \Phi_H(X) \), we also have \( \phi_H(0) \neq 0 \).

The former equation of (3) follows from the equation \( P_H^{(n)} = (P_H^{(n)})^* \), and the latter follows from Lemma 3.2. The proof of (4) is obvious. \( \square \)

Motivated by the question whether equation \( \Phi_H(1)(X) = \Phi_H(X) \) holds, we prove:

**Proposition 4.2.** Let \( H \) be a finite-dimensional Hopf algebra. Then

\[
\Phi_{H^\circ}(X) = \Phi_H(0)^{-1} \Phi_H(X^{-1})^m,
\]

where \( m = \deg \Phi_H(X) \). More precisely, if \( \Phi_H(X) = \sum_{i=0}^{m} c_i X^i \), then

\[
(4.2) \quad \Phi_{H^\circ}(X) = c_0^{-1}(c_m + c_{m-1}X + \cdots + c_1 X^{m-1} + c_0 X^m).
\]

This proposition shows that \( \alpha \mapsto \alpha^{-1} \) gives a one-to-one correspondence between roots of \( \Phi_H(X) \) (counting multiplicities) and those of \( \Phi_{H^\circ}(X) \). We will see that
every root of $\Phi_H(X)$ is a root of unity (Theorem 1.3). Hence, if $H$ is a finite-dimensional complex Hopf algebra, then $\alpha \mapsto \overline{\alpha}$ gives a one-to-one correspondence between roots of $\Phi_H(X)$ and those of $\Phi_{H^\oplus}(X)$.

If (3.5) holds for every $n$, then $\alpha \mapsto \overline{\alpha}$ must give a one-to-one correspondence between roots of $\phi_H(X)$ and those of $\phi_{H^\oplus}(X)$. We might think Proposition 4.2 as a supporting evidence for (3.5).

Proof of Proposition 4.2. By the definition of $\Phi_H(X)$, we have $\sum_{i=0}^{m} c_i P_H^{(i)} = 0$. Let $n \geq 1$. By multiplying $S_H^{(m+n)}$ to this equation, we have

\[(4.3) \quad c_m S_H^n + \cdots + c_0 S_H^{(n+m)} = 0.\]

Since $S_H$ is an anti-algebra map, we compute

\[S_H^f(h) = S_H(h(1)) \cdots S_H(h(t)) = S_H(h(1) \cdots h(t)) = S_H \circ P_H^{(f)}(h) \quad (h \in H).\]

Combining this and equation (4.3), we have

\[c_m P_H^{(n)} + \cdots + c_0 P_H^{(n+m)} = S_H^{-1} \circ (c_m S_H^n + \cdots + c_0 S_H^{(n+m)}) = 0.\]

Let $\Psi(X)$ be the right hand side of (4.2). The above equation implies that $\Phi_{H^\oplus}(X)$ divides $\Psi(X)$. In particular, $\deg \Phi_{H^\oplus}(X) \leq \deg \Psi(X) = m$. By the same argument, we also have $\deg \Phi_{H^\oplus}(X) \geq m$. Hence $\Phi_{H^\oplus}(X) = \Psi(X)$. \qed

4.2. Roots of the minimal polynomial. Let $H$ be a finite-dimensional Hopf algebra. Here we study roots of $\Phi_H(X)$ and prove the following theorem:

**Theorem 4.3.** Every root of $\Phi_H(X)$ is a root of unity.

Since $\phi_H(X)$ divides $\Phi_H(X)$, also every root of $\phi_H(X)$ is a root of unity.

We first prove this theorem in the case where $H$ is pivotal [18, Definition 4.4], that is, there exists a grouplike element $g \in H$ such that $S_H^2(h) = ghg^{-1}$ for every $h \in H$. Such $g$ is called a pivotal element of $H$. The consequence is slightly stronger than Theorem 1.3 and relates to the quasi-exponent:

**Lemma 4.4.** If $H$ is a finite-dimensional pivotal Hopf algebra, then every root of $\Phi_H(X)$ is a root of unity whose order divides $\exp(H)$.

**Proof.** Let $a$ and $e$ be elements of a Hopf algebra. If $a$ is grouplike, then

\[(xa)^n = (x(1))(a \triangleright x(2))(a^2 \triangleright x(3)) \cdots (a^{n-1} \triangleright x(n)) \cdot g^n\]

for every $n \geq 1$, where $a^i \triangleright x = a^i xa^{-i}$.

Now let $g \in H$ be a pivotal element, and let $R_g : H \to H$ be the linear map given by $R_g(h) = h g$ ($h \in H$). We have $g^{-1} h g = S^{-2}(h)$ ($h \in H$) by definition. By using (4.4), we compute

\[P_H^n \circ R_g^{-1}(h) = T_n(h) g^{-n} \quad (h \in H),\]

where $T_n : H \to H$ is the map given by (2.8).

Let $e$ be the least common multiple of $\exp(H)$ and the order of $g$. Then, by the characterization (2.10) of the quasi-exponent, there exists $m \geq 0$ such that

\[\sum_{i=0}^{m} (-1)^i \binom{m}{i} T_{ei} = 0.\]
Hence, for every \( n \geq 0 \), we have
\[
\sum_{i=0}^{m} (-1)^i \binom{m}{i} P_H^{(n+ei)} = P_H^{(n)} \ast \left( \sum_{i=0}^{m} (-1)^i \binom{m}{i} T_{ei} \circ R_g \right) = 0.
\]
This means that \( \Phi_H(X) \) divides \((X^e - 1)^n\). The rest of the proof depends on the characteristic of \( k \).

(1) If \( k \) is of characteristic zero, then the order of \( g \) divides \( \text{qexp}(H) \) \([2]\) Proposition 2.5], and hence \( e = \text{qexp}(H) \). Since \( \Phi_H(X) \) divides \((X^e - 1)^n\), our claim is proved.

(2) If \( k \) is of characteristic \( p > 0 \), then the order of \( g \) divides \( \text{exp}(H) \) \([11]\) Proposition 2.2]. By Lemma 2.5, \( e \) divides \( p^i \text{qexp}(H) \) for some \( i \geq 0 \). Let \( \alpha \) be a root of \( \Phi_H(X) \). Since \( \Phi_H(X) \) divides \((X^e - 1)^n\), we have \( \alpha^e = 1 \), and hence we have
\[
(\alpha^{\text{qexp}(H)} - 1)^{p^i} = \alpha^{p^i \text{qexp}(H)} - 1 = 0.
\]
This implies \( \alpha^{\text{qexp}(H)} = 1 \).

\( \square \)

Proof of Theorem 4.3. Let \( G \) be the subgroup of \( \text{Aut}_{\text{hopf}}(H) \) generated by \( S^G_H \), and let \( L = H \rtimes G \) be the semidirect product of \( H \) and \( (\text{the group algebra of}) \ G \). \( L \) is pivotal with pivotal element \( S^G_H \in L \). Thus, by Lemma 4.4, every root of \( \Phi_L(X) \) is a root of unity. As \( H \) is a Hopf subalgebra of \( L \), \( \Phi_H(X) \) divides \( \Phi_L(X) \), and hence also every root of \( \Phi_H(X) \) is a root of unity.

By the above proof, we see that the order of every root of \( \Phi_H(X) \) divides the quasi-exponent of \( L = H \rtimes G \). It seems to be difficult to give an efficient bound for \( \text{qexp}(L) \), and hence we do not discuss it in this work.

4.3. Indicators and periodic sequences.

Theorem 4.5. Let \( H \) be a finite-dimensional Hopf algebra over a field \( k \). There uniquely exists a sequence \( \{ c_j = \{ c_j(n) \}_{n \geq 1} \}_{j \geq 0} \) of periodic sequences in \( k \) satisfying the following three conditions:

1. The period of every \( c_j \) is nonzero in \( k \).
2. \( c_j = 0 \) for all sufficiently large \( j \).
3. For every \( n \geq 1 \), the following equality holds:
\[
\nu_n(H) = \sum_{j=0}^{\infty} \binom{n}{j} c_j(n).
\]

By the condition (2), the above sum is actually a finite sum. Let \( c_H := \{ c_j \}_{j \geq 0} \) be the sequence of periodic sequences characterized by (1), (2) and (3). Our proof will show that \( c_H \) is determined only from the sequence \( \{ \nu_n(H) \}_{n \geq 1} \). Therefore \( c_H \) is a gauge invariant of \( H \), which has the same information about \( H \).

Let \( \bar{k} \) be the algebraic closure of \( k \). Factorize \( \phi_H(X) \) as
\[
\phi_H(X) = \prod_{i=1}^{m} (X - \omega_i)^{\ell_i} \quad (\omega_i \neq \omega_j \text{ whenever } i \neq j)
\]
in \( \bar{k}[X] \). By Theorem 4.3, every \( \omega_i \) is a root of unity. So let \( e_H \) be the least common multiple of the orders of \( \omega_1, \ldots, \omega_m \). Our proof will show also that \( c_H \) satisfies the following conditions:

(1) The period of every \( c_j \) divides \( e_H \).
Lemma 4.6. Let $a$, $b$ and $m$ be non-negative integers. Then
\[
\det_{0 \leq i, j \leq m} \left( \begin{array}{c}
(a + bi) \\
j
\end{array} \right) = b^{m(m-1)}.
\]

Proof. Since both sides are integers, we may work over the field of rational numbers. Let $P_0, \ldots, P_m$ be polynomials of the form $P_j(X) = a_j X^j + \ldots$ (lower terms). We use the following generalization of the Vandermonde determinant [21 Proposition 1]:
\[
\det_{0 \leq i, j \leq m} (P_j(X_i)) = a_0 a_1 \ldots a_m \prod_{0 \leq i < j \leq m} (X_j - X_i).
\]

Substituting $P_0(X) = 1$,
\[
P_j(X) = \frac{X(X-1) \cdots (X-j+1)}{j!} (j = 1, \ldots, m),
\]
and $X_i = a + bi (i = 1, \ldots, m)$, we obtain the desired formula. \hfill \Box

Proof of Theorem 4.5. We first show the existence. Let us use the same notations as above. Let $e = e_H$ and $\ell = \max\{\ell_1, \ldots, \ell_m\}$. Since $\phi_H(X)$ divides $(X^e - 1)^\ell$, $(\nu_n(H))_{n \geq 1}$ satisfies the linear recurrence equation
\[
\sum_{i=0}^\ell (-1)^i \binom{\ell}{i} \nu_{n+ei}(H) = 0 \quad (n = 1, 2, \ldots).
\]

Let $\omega \in \overline{k}$ be a primitive $e$-th root of unity. The above recurrence equation can be solved in $\overline{k}$ (see, e.g., [13 §1.5]); $\nu_n(H)$ is written in the form
\[
\nu_n(H) = \sum_{i=0}^{e-1} \sum_{j=0}^{\ell-1} \binom{n}{j} \lambda_{ij} \omega^{ni} \quad (n = 1, 2, \ldots)
\]
for some $\lambda_{ij} \in \overline{k}$. Put $c_j(n) = \sum_{i=0}^{e-1} \lambda_{ij} \omega^{ni}$ ($j = 0, \ldots, \ell$). Then we have
\[
\nu_n(H) = c_0(n) + \binom{n}{1} c_1(n) + \binom{n}{2} c_2(n) + \cdots + \binom{n}{\ell-1} c_{\ell-1}(n) \quad (n = 1, 2, \ldots)
\]
and that $c_j = \{c_j(n)\}_{n \geq 1}$ is a periodic sequence in $\overline{k}$ with period dividing $e$. We need to show that $c_j$ is a sequence of elements of $k$. By the periodicity of $c_j$, for each $n \geq 1$, we have a system of linear equations
\[
\sum_{j=0}^{\ell-1} \binom{n+ei}{j} c_j(n) = \nu_{n+ei}(H) \quad (i = 0, \ldots, \ell - 1)
\]
in variables $c_j(n)$ ($j = 0, \ldots, \ell - 1$). Let $A$ be the matrix of coefficients. Note that $e$ is nonzero in $k$. By Lemma 4.6 we have $\det A = e^{\frac{1}{2} \ell(\ell-1)(\ell-2)} \neq 0$. Hence $A$ is invertible. We can solve the system of linear equations as follows:
\[
\begin{pmatrix} c_0(n) \\
\vdots \\
c_{\ell-1}(n) \end{pmatrix} = A^{-1} \begin{pmatrix} \nu_n(H) \\
\vdots \\
\nu_{n+(\ell-1)e}(H) \end{pmatrix} \in k^\ell.
\]

Now we define $c_j = 0$ for every $j \geq \ell$. Then $(c_j)_{j \geq 0}$ satisfies (1), (2) and (3). It follows from the definition of $c_j$ that also (1)' and (2)' are satisfied.
Next we prove the uniqueness. Suppose that sequences \( \{ c'_j \}_{j \geq 0} \) and \( \{ c''_j \}_{j \geq 0} \) satisfy the conditions. By the conditions (2) and (3), there exists \( N \geq 0 \) such that

\[
\nu_n(H) = \sum_{j=0}^{N} \binom{n}{j} c'_j(n) = \sum_{j=0}^{N} \binom{n}{j} c''_j(n) \quad (n = 1, 2, \ldots).
\]

Let \( E \) be the least common multiple of the periods of \( c'_j \) and \( c''_j \) (\( j \geq 0 \)). For each \( n \geq 1 \), we have a system of linear equation

\[
\sum_{j=0}^{N} \left( \binom{n + i E}{j} \right) (c'_j(n) - c''_j(n)) = 0 \quad (i = 0, 1, \ldots, N).
\]

Again by Lemma 4.6, the matrix of coefficients has non-zero determinant, and hence it is invertible. Therefore \( c'_j = c''_j \) for all \( j \). \( \square \)

If the characteristic of \( k \) is positive, then the sequence \( \{ \binom{n}{j} \}_{n \geq 1} \) is periodic by Lucas’ theorem. Hence we have the following corollary:

**Corollary 4.7.** If \( H \) is a finite-dimensional Hopf algebra over a field of positive characteristic, then the sequence \( \{ \nu_n(H) \}_{n \geq 1} \) is periodic.

If the characteristic of \( k \) is zero (or sufficiently large enough), then

\[
\binom{n}{j} = \frac{n(n-1)\cdots(n-j+1)}{j!} = \sum_{i=0}^{j} \frac{s(j,i)}{j!} n^i,
\]

where \( s(j,i) \) is the Stirling number of the first kind. Hence we can write \( \nu_n(H) \) in the following “polynomial form”:

\[
\nu_n(H) = \sum_{j=0}^{\infty} c'_j(n)n^j, \quad \text{where} \quad c'_j(n) = \sum_{i=j}^{\infty} \frac{s(i,j)}{i!} c_i(n).
\]

Suppose that the base field \( k \) is the field \( \mathbb{C} \) of complex numbers. Then, given a finite-dimensional Hopf algebra, we can discuss the asymptotic behavior of the sequence \( \{ \nu_n(H) \}_{n \geq 1} \). By Theorem 4.5 we immediately have the following:

**Corollary 4.8.** There exists a positive integer \( \ell \) such that

\[
\lim_{n \to \infty} \nu_n(H) n^{-\ell} = 0.
\]

Let \( d(H) \) be the smallest positive integer \( \ell \) satisfying the above equation. Then there uniquely exists a periodic sequence \( \{ a_n(H) \}_{n \geq 1} \) such that

\[
\lim_{n \to \infty} \left| \nu_n(H)n^{1-d(H)} - a_n(H) \right| = 0.
\]

Also \( d(H) \) and \( \{ a_n(H) \}_{n \geq 1} \) are gauge invariants of \( H \). It would be interesting to study indicators of finite-dimensional complex Hopf algebra from such an analytical point of view.
4.4. Refinements for pointed Hopf algebras. We give some refinements of previous results for pointed Hopf algebras. Recall that we denote by $C_0$ the coradical of a coalgebra $C$. We begin with the following lemma:

**Lemma 4.9.** Let $H$ be a finite-dimensional Hopf algebra over $k$ satisfying the following conditions:

1. $H$ has the dual Chevalley property.
2. $H_0$ is involutive, that is, the square of the antipode of $H_0$ is the identity.

Then $\Phi_H(X)$ divides $(X^e - 1)^{\ell}$, where $e = \exp(H_0)$ and $\ell = \text{Lw}(H)$.

The proof is different from that of Lemma 4.4 and rather simpler.

Note that, by definition, $H_0$ is cosemisimple. It is conjectured that every finite-dimensional cosemisimple Hopf algebra is involutive (Kaplansky’s fifth conjecture [15]). This was proved by Larson and Radford ([22], [23]) when $k$ is of characteristic zero, and by Sommerh"{a}user [38] and Etingof and Gelaki [10] when the characteristic of $k$ is large. Hence, in these cases, the assumption (2) is unnecessarily.

**Proof of Lemma 4.9.** By the characterization (2.9) of the exponent, we have

$$x^{[\ell]} = x(1)x(2)\cdots x(\ell) = x(1)S_H^{-2}(x(2))\cdots S_H^{-2(\ell-1)}(x(\ell)) = \varepsilon(x)1 \quad (x \in H_0).$$

Thus the restriction of $P_H^{(e)} - P_H^{(0)}$ to $H_0$ is zero. By Lemma 2.2, we have

$$\sum_{i=0}^{\ell} (-1)^i \binom{\ell}{i} P_H^{(e+n)} = P_H^{(n)} \ast (P_H^{(e)} - P_H^{(0)})^* = 0$$

for every $n \geq 1$. This means that $\Phi_H(X)$ divides $(X^e - 1)^{\ell}$. \hfill \Box

This lemma gives a better bound of the degree of $\phi_H(X)$ for pointed Hopf algebras. Indeed, if $H$ satisfies the assumptions, then, by Lemma 2.3, we have

$$\deg \phi_H(X) \leq \deg \Phi_H(X) \leq \exp(H_0) \cdot \text{Lw}(H) \leq \frac{\exp(H_0) \dim_k(H)}{\dim_k(H_0)}.$$

If $H$ is pointed, then $\exp(H_0) = \exp(G(H))$ divides $\dim_k(H_0) = |G(H)|$. Thus we have a bound

(4.5) \quad \deg \phi_H(X) \leq \dim_k(H).

**Question 4.10.** When does inequality (4.5) hold?

Kashina’s conjecture, referred in [2.7] would give a partial answer to this question. More precisely, we can prove the following Theorem 4.11 by the same argument as above. In the rest of this subsection, $H$ denotes a finite-dimensional Hopf algebra satisfying the assumptions (1) and (2) of Lemma 4.9.

**Theorem 4.11.** If $\exp(H_0)$ divides $\dim_k(H_0)$, then inequality (4.5) holds.

Now let $e = \exp(H_0)$, and let $\ell = \text{Lw}(H)$. We give some applications of Theorem 4.11 and Lemma 4.9.

**Theorem 4.12.** Suppose that $k$ is of characteristic zero. Then there exist periodic sequences $c_j = \{c_j(n)\}_{n \geq 0}$ $(j = 0, 1, \ldots, \ell - 1)$ in $k$ with period dividing $e$ such that

$$\nu_n(H) = c_0(n) + \binom{n}{1} c_1(n) + \cdots + \binom{n}{\ell - 1} c_{\ell - 1}(n)$$

for every $n \geq 1$. 

Theorem 4.13. Suppose that \( k \) is of characteristic \( p > 0 \).

(1) The period of the sequence \( \{\nu_n(H)\}_{n \geq 1} \) divides \( p^j \), where \( j \) is the smallest integer such that \( p^j \geq \ell \).

(2) If \( e \) is a power of \( p \), then there exist \( c_0, \ldots, c_{m-1} \in k \), where \( m = e\ell \), such that

\[
\nu_n(H) = \sum_{i=0}^{m-1} c_i \binom{n}{i}
\]

for every \( n \geq 1 \).

Proof. (1) By Lemma 4.9, \( \phi_H(X) \) divides \( (X^e - 1)^\ell \). Our claim follows from

\[
(X^e - 1)^\ell \cdot (X^e - 1)^{p^j - \ell} = (X^e - 1)^{p^j} = X^{p^j e} - 1.
\]  

(2) We have \( (X^e - 1)^\ell = (X - 1)^m \) since \( e \) is a power of \( p \). Thus our claim follows from Theorem 4.5. \( \square \)

The assumption (2) of the above theorem is satisfied if, for example, \( H \) is a pointed Hopf algebra such that \( G(H) \) is a \( p \)-group. In particular, it is satisfied if \( H \) is pointed irreducible.

5. Applications to a family of pointed Hopf algebras

5.1. Pointed Hopf algebras of Andruskiewitsch and Schneider. In this section, we apply our results to the family of finite-dimensional pointed Hopf algebras introduced by Andruskiewitsch and Schneider. Throughout, the base field \( k \) is assumed to be algebraically closed and of characteristic zero.

In [5], Andruskiewitsch and Schneider classified all finite-dimensional pointed Hopf algebras \( H \) such that \( G(H) \) is abelian and all prime divisors of the order of \( G(H) \) are greater than 7; as a result, such Hopf algebras are parameterized by certain parameters \( D, \lambda \) and \( \mu \). Following [5], we roughly recall the construction of the corresponding Hopf algebra \( u(D, \lambda, \mu) \).

Fix a finite abelian group \( \Gamma \). A datum \( D \) of finite Cartan type for \( \Gamma \),

\[
D = (\Gamma, (g_i)_{i \in I}, (\chi_i)_{i \in I}, A = (a_{ij})_{i,j \in I}),
\]

consists of elements \( g_i \in \Gamma \), \( \chi_i \in \Gamma^\vee \) (\( i \in I \)), indexed by a totally-ordered finite set \( I \), and a Cartan matrix \( A \) of finite type indexed by \( I \) such that

\[
q_{ij}q_{ji} = q_{ii}^{a_{ij}} \text{ and } q_{ii} \neq 1 \text{ for all } i, j \in I, \text{ with } q_{ij} = \chi_j(g_i).
\]

Let \( \Phi \) be the root system of \( A \), let \( \Phi^+ \) be a system of positive roots, and let \( \{\alpha_i\}_{i \in I} \) be a system of simple roots. We write \( i \sim j \) if \( \alpha_i \) and \( \alpha_j \) are in the same connected component of the Dynkin diagram of \( \Phi \). For simplicity, we always assume that \( D \) satisfies the following two conditions:

- The order of \( q_{ii} \) is odd for all \( i \in I \).
- If \( i \in I \) lies in a connected component of type \( G_2 \), then the order of \( q_{ii} \) is relatively prime to three.

We note that number of results in [5] are proved under these assumptions. Now fix a datum \( D \) of finite Cartan type. Then a left Yetter-Drinfeld module \( V_D \) over \( k\Gamma \) is defined by

\[
V_D = \bigoplus_{i \in I} kx_i, \quad g \rightarrow x_i = \chi_i(g)x_i, \quad \rho(x_i) = g_i \otimes x_i \quad (g \in \Gamma, i \in I).
\]
The Hopf algebra \( u(D, \lambda, \mu) \) will be defined as a quotient of the bosonization of the braided tensor algebra \( T(V_D) \) by a certain Hopf ideal depending on parameters \( \lambda \) and \( \mu \). To describe parameters \( \lambda \) and \( \mu \), we introduce some notations. Let \( \alpha \in \Phi^+ \) be a positive root, and write \( \alpha = \sum_{i \in I} n_i \alpha_i \) (\( n_i \in \mathbb{Z}_{\geq 0} \)). We put

\[
g_\alpha = \prod_{i \in I} g_{n_i}^{n_i} \quad \text{and} \quad \chi_\alpha = \prod_{i \in I} \chi_i^{n_i}.
\]

There exists unique \( J_\alpha \in I/\sim \) such that \( n_i \neq 0 \) implies \( i \in J_\alpha \). Fix \( i \in J_\alpha \) and put \( N_\alpha = N_i \), where \( N_i \) is the order of \( q_{n_i} \). This does not depend on the choice of \( i \in J_\alpha \) since if \( i \sim j \), then \( N_i = N_j \) follows from the assumptions on \( D \).

For each \( \alpha \in \Phi^+ \), an element \( x_\alpha \in T(V_D) \) is defined. If \( \alpha = \alpha_i \) is a simple root, then \( x_\alpha = x_i \). For general \( \alpha \in \Phi^+ \), \( x_\alpha \) is defined by a certain iterated braided commutators of \( x_i \)'s (see [2, §4]).

We say that \( i, j \in I \) are linkable if \( i \neq j \), \( g_i g_j \neq 1 \) and \( \chi_i \chi_j = 1 \). The second parameter \( \lambda \), called a linking parameter, is a family \( \lambda = (\lambda_{ij})_{i,j \in I; i < j} \) of elements of \( k \) such that \( \lambda_{ij} = 0 \) if \( i \sim j \) and \( j \) are not linkable.

The third parameter \( \mu \), called a root vector parameter, is a family \( \mu = (\mu_\alpha)_{\alpha \in \Phi^+} \) of elements of \( k \) such that \( \mu_\alpha = 0 \) if \( g_{N_\alpha}^{N_\alpha} = 1 \) or \( \chi_{N_\alpha}^{N_\alpha} \neq 1 \). For each \( \alpha \in \Phi^+ \), an element \( u_{\alpha}(\mu) \) of the augmentation ideal of \( k(g_{N_\alpha}^{N_\alpha} \mid i \in I) \) is defined in a certain way; we omit the details of the construction but note that if \( \mu_\alpha = 0 \) for all \( \alpha \), then \( u_{\alpha}(\mu) = 0 \) for all \( \alpha \).

Now we consider the bosonization \( U = T(V_D) \# k\Gamma \). We regard both \( T(V_D) \) and \( k\Gamma \) as subalgebras of \( U \). The Hopf algebra \( u(D, \lambda, \mu) \) is defined to be the quotient Hopf algebra of \( U \) by the ideal generated by

\[
\begin{align*}
ad_c(x_i)^{1-a_{ij}}(x_j) & \quad (i, j \in I; i \neq j, i \sim j), \\
ad_c(x_i)(x_j) - \lambda_{ij}(1 - g_i g_j) & \quad (i, j \in I; i < j, i \neq j), \\
u_{\alpha}(\mu)^{N_\alpha} - u_{\alpha}(\mu) & \quad (\alpha \in \Phi^+),
\end{align*}
\]

where \( \text{ad}_c \) is the braided adjoint action [5, (1.14)]. \( u(D, \lambda, \mu) \) is a finite-dimensional Hopf algebra of dimension

\[
\dim_k (u(D, \lambda, \mu)) = |\Gamma| \cdot \prod_{J \in I/\sim} N_{\mu}(\Phi^+) = |\Phi^+|,
\]

where \( |\Phi^+| \) is the number of positive roots of the root system of the connected component \( J \subset I \), such that \( G(u(D, \lambda, \mu)) = \Gamma \).

Abusing notation, the images of \( x_i \in T(V_D) \) and \( g \in \Gamma \) in \( u(D, \lambda, \mu) \) under the quotient map are denoted by the same symbols \( x_i \) and \( g \), respectively. By the construction, the coalgebra structure is given by

\[
\Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i \quad (i \in I), \quad \Delta(g) = g \otimes g \quad (g \in \Gamma).
\]

The following is one of the most important examples of \( u(D, \lambda, \mu) \):

**Example 5.1.** Let \( g \) be a finite-dimensional semisimple Lie algebra, and let \( q \) be a root of unity of odd order \( N > 1 \). We suppose that \( N \) is relatively prime to 3 if \( g \) has a component of type \( G_2 \). Given such \( g \) and \( q \), a finite-dimensional Hopf algebra \( u_q(g) \), so-called the small quantum group, is defined (Lusztig [23]).
Let $n$ be the rank of $\mathfrak{g}$, and let $A = (a_{ij})_{i,j=1,\ldots,n}$ be the Cartan matrix of $\mathfrak{g}$ with symmetrizing matrix $D = \text{diag}(d_1,\ldots,d_n)$. We set
$$\bar{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$
and $I = \{1,2,\ldots,2n\}$.

Let $\Gamma$ be the group generated by $g_i$ $(i=1,\ldots,n)$ with relations $g_i g_j = g_j g_i$ and $g_i^N = 1$ $(i,j = 1,\ldots,n)$. Obviously, $\Gamma$ is the direct product of $n$ copies of a cyclic group of order $N$. Now we define $\chi_i \in \hat{\Gamma}$ $(i \in I)$ by
$$\chi_i(g_j) = q^l(a_{ij}) \quad \text{and} \quad \chi_{i+n} = \chi_i^{-1} \quad (i,j=1,\ldots,n).$$

Set $g_{i+n} = g_i$ $(i=1,\ldots,n)$. Then $D = (\Gamma,(g_i)_{i \in I},(\chi_i)_{i \in I},\bar{A})$ is a datum of finite Cartan type. Define a linking parameter $\lambda$ by
$$\lambda_{ij} = \frac{1}{q^j-q^i} \quad (1 \leq i < j \leq n).$$

Recall that $u_q(\mathfrak{g})$ has standard generators $E_i$, $F_i$ and $K_i$ $(i=1,\ldots,n)$. $u_q(\mathfrak{g})$ is isomorphic to $u(D,\lambda,0)$ as a Hopf algebra via the algebra map given by
$$K_i \mapsto g_i, \quad E_i \mapsto x_i, \quad \text{and} \quad F_i \mapsto x_{i+n} g_i^{-1} \quad (i=1,\ldots,n).$$

**Example 5.2.** Keep the notation as above. Let $u_q(\mathfrak{g})^{\geq 0}$ be the Hopf subalgebra of $u_q(\mathfrak{g})$ generated by $E_i, K_i$ $(i=1,\ldots,n)$. It is isomorphic to $u(D_+,0,0)$ as a Hopf algebra, where $D_+ = (\Gamma,(g_i)_{i=1}^n,(\chi_i)_{i=1}^n,A)$.

We go back to the general situation. Fix a datum $D$ of finite Cartan type, a linking parameter $\lambda$ and a root vector parameter $\mu$. Consider the coradical filtration for $u(D,\lambda,\mu)$. Then, by [13 Corollary 5.2], we have
$$\text{gr}(u(D,\lambda,\mu)) \cong u(D,0,0) \cong \mathfrak{B}(V_D) \# k\Gamma.$$

Hence, by Lemma 5.3, we have the following:

**Theorem 5.3.** $\nu_n(u(D,\lambda,\mu)) = \nu_n(\mathfrak{B}(V_D) \# k\Gamma)$.

This also follows from Lemma 5.3. Indeed, Grunenfelder and Mastnak showed that $u(D,\lambda,\mu)$ can be obtained from $\text{gr}(u(D,\lambda,\mu)) \cong \mathfrak{B}(V_D) \# k\Gamma$ by 2-cocycle deformation [13 Theorem 3.3] (see also [27 Appendix]).

**Corollary 5.4.** Suppose $k = \mathbb{C}$. Let $u = u(D,\lambda,\mu)$.

1. $\nu_n((u^{\text{op}}) = \nu_n(u)$ for every $n \geq 1$.
2. $\nu_n(u) \in \mathbb{R}$ for every $n \geq 1$ if $V_D^{\text{op}} \cong V_D$.

See Lemma 5.12 for the definition of $V_D^{\text{op}}$.

**Proof.** (1) This is a special case of Theorem 3.10.

(2) If this is the case, by Lemma 3.12 we have
$$\nu_n((u^{\text{op}}) = \nu_n(\mathfrak{B}(V_D^{\text{op}}) \# \mathbb{C} \Gamma) = \nu_n(\mathfrak{B}(V_D) \# \mathbb{C} \Gamma) = \nu_n(u).$$

Combining with (1), we have $\nu_n(u^{\text{op}}) = \nu_n(u)$, and hence $\nu_n(u) \in \mathbb{R}$. \hfill \Box

Let $\mathfrak{g}$ and $q$ be as in Example 5.1. By this corollary, we have:

**Corollary 5.5.** $\nu_n(u_q(\mathfrak{g})) \in \mathbb{R}$ for every $n \geq 1$. 


5.2. **Factorization of the second indicator.** Fix a finite abelian group \( \Gamma \). We suppose that the order of \( \Gamma \) is odd. We say that \( V, W \in \mathbb{k}^\Gamma_1 \text{YD} \) are symmetric if
\[
eq = \text{id}_{V \otimes W},
\]
where \( c \) is the braiding of \( \mathbb{k}^\Gamma_1 \text{YD} \). Let \( B_1 \) and \( B_2 \) are braided Hopf algebras over \( k\Gamma \). If they are symmetric, then also \( B_1 \otimes B_2 \) is naturally a braided Hopf algebra over \( k\Gamma \) (the braided tensor product). Here we prove the following:

**Theorem 5.6.** Let \( B_1 \) and \( B_2 \) be two finite-dimensional braided Hopf algebras over \( k\Gamma \). If they are symmetric, then we have
\[
\nu_2((B_1 \otimes B_2) \# k\Gamma) = \nu_2(B_1 \# k\Gamma) \cdot \nu_2(B_2 \# k\Gamma).
\]

To prove this theorem, we first recall how the antipode of the bosonization is given. Let \( B \) be a braided Hopf algebra over \( k\Gamma \) and let \( b \in B_g \). By (2.5),
\[
(5.2) \quad S_{B \# k\Gamma}(b \# h) = (h^{-1} g^{-1} \rightarrow S_B(b)) \# (h^{-1} g^{-1}).
\]
Since \( S_B \) preserves the coaction, \( S_B(b) \in B_g \). Therefore we have
\[
(5.3) \quad S_{B \# k\Gamma}(B_g \# h) \subset B_g \# (h^{-1} g^{-1}) \quad (g, h \in \Gamma).
\]

Given \( V \in \mathbb{k}^\Gamma_1 \text{YD} \), we define \( \theta_V : V \rightarrow V \) by
\[
\theta_V(v) = v(-1) \rightarrow v(0) \quad (v \in V).
\]
We have \( \theta_{V \otimes W} = (\theta_V \otimes \theta_W)c_{V,W}c_{W,V} \). In particular, if \( V \) and \( W \) are symmetric, then \( \theta_{V \otimes W} = \theta_V \otimes \theta_W \). The order of \( \theta \) is equal to \( e = \exp(\Gamma) \), which is odd by our assumption. We write
\[
\theta_V^{\frac{1}{2}} = \theta_V^{\frac{1}{2}(e-1)}
\]
as if it were a square root of the inverse of \( \theta_V \).

**Lemma 5.7.** If \( B \) is finite-dimensional, then we have
\[
\nu_2(B \# k\Gamma) = \text{Tr}(S_{B \# k\Gamma}) = \text{Tr}(\theta_B^{-\frac{1}{2}} S_B).
\]

**Proof.** The first equality is definition. We prove the second one. Let \( g \in \Gamma \). Since the order of \( \Gamma \) is odd, there uniquely exists \( h \in \Gamma \) such that \( h^2 = g^{-1} \), which we denote by \( g^{-\frac{1}{2}} \). By (5.3), we can define a linear map
\[
T_g : B_g \# g^{-\frac{1}{2}} \rightarrow B_g \# g^{-\frac{1}{2}} \quad (g \in \Gamma)
\]
to be the restriction of \( S_{B \# k\Gamma} \). By (5.2),
\[
T_g(b \# g^{-\frac{1}{2}}) = (g^{-\frac{1}{2}} \rightarrow S_B(b)) \# g^{-\frac{1}{2}} = \theta_B^{-\frac{1}{2}} S_B(b) \# g^{-\frac{1}{2}}
\]
for all \( b \in B_g \). Again by (5.3), \( \text{Tr}(S_{B \# k\Gamma}) = \sum_{g \in \Gamma} \text{Tr}(T_g) = \text{Tr}(\theta_B^{-\frac{1}{2}} S_B) \). \( \Box \)

**Proof of Theorem 5.6.** Let \( B = B_1 \otimes B_2 \) be the braided tensor product of \( B_1 \) and \( B_2 \). Then, since \( B_1 \) and \( B_2 \) are symmetric, \( \theta_B = \theta_{B_1} \otimes \theta_{B_2} \). Hence,
\[
\theta_B^{-\frac{1}{2}} S_B = (\theta_{B_1}^{-\frac{1}{2}} \otimes \theta_{B_2}^{-\frac{1}{2}})(S_{B_1} \otimes S_{B_2}) = (\theta_{B_1}^{-\frac{1}{2}} S_{B_1}) \otimes (\theta_{B_2}^{-\frac{1}{2}} S_{B_2}).
\]
Taking the trace and applying the previous lemma, we have the result. \( \Box \)
We apply Theorem 5.6 to the Hopf algebra $u(D, \mu, \lambda)$. Let
\[ D_j = (\Gamma, (g_i)_{i \in I(j)}, (\chi_i)_{i \in I(j)}, A_j) \quad (j = 1, 2) \]
be data of finite Cartan type. We say that they are disjoint if $\chi_i(g_j)\chi_j(g_i) = 1$ for all $i \in I(1)$ and $j \in I(2)$. If this is the case, then also the disjoint union
\[ D_1 \sqcup D_2 = (\Gamma, (g_i)_{i \in I(1) \sqcup I(2)}, (\chi_i)_{i \in I(1) \sqcup I(2)}, A_1 \oplus A_2) \]
is a datum of finite Cartan type.

**Theorem 5.8.** Let $D_1$ and $D_2$ be disjoint data of finite Cartan type for $\Gamma$. For any linking parameter $\lambda$ and root vector parameter $\mu$ for $D_1 \sqcup D_2$, we have
\[ \nu_2(u(D_1 \sqcup D_2, \lambda, \mu)) = \nu_2(\mathcal{B}(V_{D_1} \# k\Gamma)) \cdot \nu_2(\mathcal{B}(V_{D_2} \# k\Gamma)). \]

**Proof.** Since $V_{D_1}$ and $V_{D_2}$ are symmetric, their Nichols algebras are symmetric, and thus there exists a surjective map of braided Hopf algebras
\[ \mathcal{B}(V_{D_1}) \otimes \mathcal{B}(V_{D_2}) \to \mathcal{B}(V_{D_1 \sqcup D_2}) = \mathcal{B}(V_{D_1 \sqcup D_2}) \]
by [28 Proposition 3.4] and [4 Proposition 2.2 (iii)]. Comparing the dimensions, we conclude that the above map is an isomorphism. Thus the result follows from Theorems 5.3 and 5.6. \qed

5.3. **Application to the small quantum group.** In what follows, we suppose $k = \mathbb{C}$. Let $g$ and $q$ be as in Example 5.1. We give an application of our results to the small quantum group $u_q(g)$.

**Theorem 5.9.** $\nu_2(u_q(g)) = |\nu_2(u_q(g)^{\geq 0})|^2$.

**Proof.** Let $D$ be as in Example 5.1 and let
\[ D_+ = (\Gamma, (g_i)_{i=1}^n, (\chi_i)_{i=1}^n, A), \quad D_- = (\Gamma, (g_i)_{i=n+1}^{2n}, (\chi_i)_{i=n+1}^{2n}, A). \]
Note that $V_{D_-}$ is isomorphic to $V_{D_+}^{op}$. Since $D = D_+ \sqcup D_-$, we have
\[ \nu_2(u_q(g)) = \nu_2(\mathcal{B}(V_{D_+} \# \Gamma)) \cdot \nu_2(\mathcal{B}(V_{D_-} \# \Gamma)) \]
\[ = \nu_2(\mathcal{B}(V_{D_+} \# \Gamma)) \cdot \nu_2(\mathcal{B}(V_{D_-} \# \Gamma)) = \nu_2(u_q(g)^{\geq 0}) \]
by Lemma 3.12, Corollary 5.4 and Theorem 5.8. \qed

Let $q$ be a root of unity of odd order $N > 1$. $u_q(\mathfrak{sl}_2)$ is generated by $E$, $F$ and $K$ with relations $E^N = F^N = 0$, $K^N = 1$, $KE = q^2EK$, $KF = q^{-2}FK$, and
\[ EF - FE = \frac{K - K^{-1}}{q - q^{-1}}. \]
The coalgebra structure is given by $\Delta(K) = K \otimes K$, $\varepsilon(K) = 1$,
\[ \Delta(E) = E \otimes 1 + K \otimes E, \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F, \quad \varepsilon(E) = \varepsilon(F) = 0 \]
and the antipode $S$ is given by
\[ S(K) = K^{-1}, \quad S(E) = -K^{-1}E, \quad S(F) = -FK. \]
$u_q(\mathfrak{sl}_2)^{\geq 0}$, which is the Hopf subalgebra of $u_q(\mathfrak{sl}_2)$ generated by $E$ and $K$, is isomorphic to the Taft algebra $T_{N^2}(\omega)$ at $\omega = q^2$. Combining Theorem 5.9 with (5.7), we have
\[ \nu_2(u_q(\mathfrak{sl}_2)) = 4|1 + q|^2. \]
Theorem 5.10. Let \( p \) and \( q \) be roots of unity of the same odd order. Then the following two conditions are equivalent:

1. \( u_p(\mathfrak{sl}_2) \) and \( u_q(\mathfrak{sl}_2) \) are isomorphic as Hopf algebras;
2. \( u_p(\mathfrak{sl}_2) \) and \( u_q(\mathfrak{sl}_2) \) are gauge equivalent;
3. \( p = q \) or \( p = q^{-1} \).

Proof. It is obvious that (1) implies (2). Let \( \alpha \) and \( \beta \) be the arguments of \( p \) and \( q \), respectively. By \( (5.5) \), we have

\[
\nu_2(u_p(\mathfrak{sl}_2)) = (1 + \cos \alpha)^{-1} \quad \text{and} \quad \nu_2(u_q(\mathfrak{sl}_2)) = (1 + \cos \beta)^{-1}.
\]

Therefore, they are equal if and only if \( \cos \alpha = \cos \beta \), if and only if \( p = q^{\pm 1} \). By the gauge invariance of the second indicator, we have that (2) implies (3). Now we suppose (3). If \( p = q \), then (1) is obvious. If \( p = q^{-1} \), then we can directly check that the algebra map \( u_q(\mathfrak{sl}_2) \to u_p(\mathfrak{sl}_2) \) determined by

\[
E \mapsto FK, \quad F \mapsto K^{-1}E \quad K \mapsto K
\]
is an isomorphism of Hopf algebra. Thus (3) implies (1). \( \square \)

Remark 5.11. (1) Theorem 5.10 holds over an arbitrary algebraically closed field of characteristic zero; indeed, similar arguments apply if we replace \( (5.5) \) with \( (5.4) \) replaced by

\[
\nu_\alpha(u_q(\mathfrak{sl}_2)) = \frac{4}{(1 + q)(1 + q^{-1})}.
\]

(2) \( u'_q := \text{gr } u_q(\mathfrak{sl}_2) \) is generated by \( E, F, K \) with the same relations as \( u_q(\mathfrak{sl}_2) \) but with \( (5.4) \) replaced by \( EF - FE = 0 \). By Lemma 3.4, we have

\[
\nu_\alpha(u_q(\mathfrak{sl}_2)) = \nu_\alpha(u'_q)
\]

for every \( n \geq 1 \). We can derive \( (5.4) \) in a more direct approach. It is easy to see that \( u'_q \) has the set \( \{E^r F^s K^i \mid r, s, i = 0, \ldots, N - 1\} \) as a basis. Let us compute the antipode \( S \) with respect to this basis. Since \( E \) and \( F \) commute in \( u'_q \), we have

\[
S(E^r F^s K^i) = (-1)^{s-r} q^{2((s-r)-(r+1)-s(s-1)+2rs)} E^r F^s K^{(s-r)-i}
\]

without much difficulty. Therefore the second indicator, that is, the trace of the antipode, is computed as follows:

\[
\text{Tr}(S) = \sum_{r,s=0}^{N-1} (-1)^{s-r} q^{s-r} = \sum_{r,s=0}^{N-1} (-1)^{s-r} q^{s-r} = \frac{1 - (-q)^N}{1 - (-q)} \cdot \frac{1 - (-q)^{-N}}{1 - (-q^{-1})} = \frac{4}{|1 + q|^2}.
\]

6. Higher indicators of the Taft algebra

6.1. Computing the Sweedler power maps. In this section, we derive a closed formula of indicators of the Taft algebra and \( u_q(\mathfrak{sl}_2) \) and discuss properties of indicators.

Suppose that the base field \( k \) is an algebraically closed field of characteristic zero. Fix \( q \in k^* \). Let \( U(q) \) be the algebra over \( k \) generated by \( g \), \( g^{-1} \) and \( x \) with defining relations \( gg^{-1} = 1 \) and \( qxq^{-1} = qx \). \( U(q) \) admits a Hopf algebra structure determined by the same formula as \( (3.6) \) and has the Taft algebra \( T_{N^2}(q) \) as a quotient when \( q \) is a primitive \( N \)-th root of unity.
We introduce several notations. For an integer \( n \geq 0 \), we set
\[
(0)_q = 0 \quad \text{and} \quad (n)_q = 1 + q + \cdots + q^{n-1}.
\]
The \( q \)-factorial is defined by \( (n)_q! = (n-1)_q! \cdot (n)_q \) (\( n \geq 1 \)) and \( (0)_q! = 1 \). For non-negative integers \( m \) and \( a \), the \( q \)-binomial coefficient is defined by
\[
\binom{m}{a}_q = \frac{(m)_q!}{(a)_q!(m-a)_q} \quad \text{if} \ m \geq a. \quad \text{Otherwise, we set} \ \binom{m}{a}_q = 0.
\]

For non-negative integers \( L, a \) and \( m \), we set
\[
\left( L \atop a, m \right)_q = \sum_{j_1 + \cdots + j_m = a} q^{j_1^2 + \cdots + j_m^2} \binom{L}{j_1}_q \binom{j_1}{j_2}_q \cdots \binom{j_m-1}{j_m}_q,
\]
where the sum is taken over all non-negative integers \( j_1, \ldots, j_m \) satisfying \( j_1 + \cdots + j_m = a \). This is the generating function of \((m, m+1; L, a)\)-partitions in the sense of Warnaar [42, Definition 7]. Note that the summand is zero unless the condition
\[
L \geq j_1 \geq j_2 \geq \ldots \geq j_m \geq 0
\]
is satisfied. Thus in fact the right-hand side of (6.1) is a sum taken over partitions of \( a \).

**Lemma 6.1.** Let \( n \) and \( r \) be non-negative integers, and let \( \ell \) be an integer. Then the \( n \)-th Sweedler power of \( x^r g^\ell \in U(q) \) is given by the formula
\[
(x^r g^\ell)^{[n]} = \sum_{a=0}^{r(n-1)} \binom{r}{a, n-1}_q q^{a \cdot x} x^a g^{a + n \cdot \ell}.
\]

**Proof.** Define \( T : U(q) \to U(q) \) by \( u \mapsto g u g^{-1} \). By (4.3), we have
\[
(x^r g^\ell)^{[n]} = \left( \nabla^{(n)} \circ (id \otimes T \otimes \cdots \otimes T^{n-1})^{\ell} \circ \Delta^{(n)} \right) (x^r) \cdot g^{n \cdot \ell}.
\]
In what follows, we use multi-index notation; an \( n \)-dimensional multi-index is an \( n \)-tuple \( \mathbf{i} = (i_1, \ldots, i_n) \) of non-negative integers. Given such \( \mathbf{i} \), we set
\[
|\mathbf{i}| = i_1 + \cdots + i_n, \quad \text{and if} \ |\mathbf{i}| = m, \quad \begin{pmatrix} m \\ 1 \end{pmatrix}_q = \frac{[m]_q!}{[i_1]_q! \cdots [i_n]_q!}.
\]
Observe \( \Delta^{(n)}(x) = x_1 + \cdots + x_n \), where
\[
x_i = g^\otimes i_1 \otimes g \otimes x \otimes \cdots \otimes x_{n-i} \in U(q)^{\otimes n} \quad (i = 1, \ldots, n).
\]
Given a multi-index \( \mathbf{i} = (i_1, \ldots, i_n) \), we set \( \mathbf{x}^\mathbf{i} = x_1^{i_1} \cdots x_n^{i_n} \). Note that \( x_i \)'s do not commute; if \( i < j \), then \( x_j x_i = q x_i x_j \). We denote by \( I_{n, r} \) the set of all \( n \)-dimensional multi-indices \( \mathbf{i} \) such that \( |\mathbf{i}| = r \). The \( q \)-multinomial formula yields
\[
\Delta^{(n)}(x^r) = \sum_{\mathbf{i} \in I_{n, r}} \binom{r}{\mathbf{i}}_q \mathbf{x}^\mathbf{i}.
\]
For \( \mathbf{i} = (i_1, \ldots, i_n) \), we set \( S_m(\mathbf{i}) = \sum_{c=1}^{n} (i_{c+1} + \cdots + i_n)^m \). After a slightly tedious computation, we obtain
\[
(id \otimes T \otimes \cdots \otimes T^{n-1})(\mathbf{x}^\mathbf{i}) = q^{S_m(\mathbf{i})} \mathbf{x}^\mathbf{i} \quad \text{and} \quad \nabla^{(n)}(\mathbf{x}^\mathbf{i}) = q^{S_m(\mathbf{i})} x^r g^{S_m(\mathbf{i})}.
\]
Observe $0 \leq S_1(i) \leq r(n - 1)$ if $i \in I_{n,r}$. Hence, by (6.3), we have

$$
(x^r g^a)^{[n]} = \sum_{a=0}^{r(n-1)} \sum_{i \in I_{n,r}, S_1(i) = a} \binom{r}{i} q^{a+2} g(a, i) x^r g^a + n i.
$$

Let $J_{n,r}$ be the set of all $(n - 1)$-tuple $(j_1, \ldots, j_{n-1})$ of non-negative integers satisfying (6.2) with $L = r$ and $m = n - 1$. The map

$$f : I_{n,r} \to J_{n,r}, \quad (i_1, \ldots, i_r) \mapsto (i_1 + \cdots + i_r, i_3 + \cdots + i_n, \ldots, i_{n-1} + i_n, i_n)
$$
gives a bijection between them. If $(j_1, \ldots, j_{n-1}) = f(i)$, then we get

$$S_m(i) = j_1^m + j_2^m + \cdots + j_{n-1}^m \quad \text{and} \quad \binom{r}{i} = \binom{r}{j_1} \binom{r}{j_2} \cdots \binom{r}{j_{n-1}}.
$$

Now the desired formula is obtained by rewriting (6.5) as the sum taken over $0 \leq a \leq r(n - 1)$ and $(j_1, \ldots, j_{n-1}) \in J_{n,r}$ such that $j_1 + \cdots + j_{n-1} = a$.

6.2. Higher indicators of the Taft algebra. Let $\omega$ be a primitive $N$-th root of unity. Then the Taft algebra $T_{N^2}(\omega)$ is defined. The element

$$\Lambda = \sum_{i=0}^{N-1} x^{N-1} g_i$$

is a right integral. $T_{N^2}(\omega)$ has the set $\{x^i g^j \mid i, j = 0, \ldots, N - 1\}$ as a basis. With respect to this basis, we define a linear map $\lambda : T_{N^2}(\omega) \to k$ by

$$\langle \lambda, x^i g^j \rangle = \delta_{i,N-1} \delta_{j,0} \quad (i, j = 0, \ldots, N - 1).
$$

$\lambda$ is a right integral such that $\langle \lambda, \Lambda \rangle = 1$.

**Theorem 6.2.** The $n$-th indicator of $T = T_{N^2}(\omega)$ is given by

$$\nu_n(T) = \sum_{a,i} \left\{ \begin{array}{c} N - 1 \\ a, n - 1 \end{array} \right\} \omega^{-ai},
$$

where the sum is taken over all integers $a$ and $i$ satisfying

$$0 \leq a \leq (n - 1)(N - 1), \quad 0 \leq i < N, \quad \text{and} \quad a \equiv ni \pmod{N}.
$$

**Proof.** By the formula $\nu_n(T) = \langle \lambda, \Lambda^{[n]} \rangle$ and Lemma 6.1 we compute

$$\nu_n(T) = \sum_{i=0}^{N-1} \langle \lambda, (x^{N-1} g^i)^{[n]} \rangle = \sum_{i=0}^{N-1} \sum_{a=0}^{(N-1)(n-1)} \left\{ \begin{array}{c} N - 1 \\ a, n - 1 \end{array} \right\} \omega^{-ai} \langle \lambda, x^{N-1} g^{a-ni} \rangle.
$$

By the definition of $\lambda$, the summand vanishes unless (6.6) is satisfied. Hence we have the desired formula.

Therefore computations of indicators of the Taft algebra $T_{N^2}(\omega)$ reduce to the evaluation of (6.1) at $L = N - 1$ and $q = \omega$. This seems to be a quite difficult problem when the order $N$ of $q = \omega$ is large. In this paper, we consider the cases where $N = 2$ and $N = 3$. 

First we consider the case where \( N = 2 \). Let \( a \) and \( m \) be integers satisfying \( 0 \leq a \leq m \). Then the summand of the right-hand side of (6.1) is non-zero if and only if \( j_1 = \ldots = j_a = 1 \) and \( j_{a+1} = \ldots = j_m = 0 \). Hence we have
\[
\left\{ \frac{1}{a, m} \right\}_{-1} = (-1)^a.
\]
Now we have the following proposition, which has been obtained in [18] in a different and more direct way.

**Proposition 6.3.** \( \nu_n(T_4(-1)) = n \) for every \( n \geq 1 \).

**Proof.** By Theorem 6.2 and the above observation, we have
\[
\nu_n(T) = \sum_{a,i} (-1)^a (1-1)
\]
where the sum is taken over all \( i = 0, 1 \) and \( a = 0, \ldots, n-1 \) satisfying \( ni \equiv a \) (mod 2). If \( n \) is odd, then \( ni \equiv a \) (mod 2) holds if and only if \( a \equiv i \) (mod 2). Hence we compute
\[
\nu_n(T) = \sum_{a=0}^{n-1} (-1)^a (1-a) = \sum_{a=0}^{n-1} 1 = n.
\]
Otherwise, if \( n \) is even, then \( ni \equiv a \) (mod 2) holds if and only if \( a \) is even. By writing \( a = 2b \), we have
\[
\nu_n(T) = \sum_{i=0}^{\frac{n}{2}-1} \sum_{b=0}^{n-1} (-1)^{2b} (1-i) = \sum_{i=0}^{\frac{n}{2}} \frac{1}{2} n = n.
\]

Next we consider the case where \( N = 3 \). Let \( \omega \) be a primitive third root of unity, and let \( a \) and \( m \) be integers satisfying \( 0 \leq a \leq 2m \). Our aim is to compute
\[
(6.7) \quad \left\{ \frac{2}{a, m} \right\}_\omega = \sum_{j_1, \ldots, j_m=a} \omega^{j_1^2 + \cdots + j_m^2} \left( \frac{2}{j_1} \right)_{\omega} \cdots \left( \frac{j_m-1}{j_m} \right)_{\omega}.
\]
Let \( j_1, \ldots, j_m \) be integers satisfying \( j_1 + \cdots + j_m = a \) and \( \omega^2 \) with \( L = 2 \). Suppose \( j_1 = \cdots = j_b = 2, \ j_{b+1} = \cdots = j_{b+c} = 1, \ j_{b+c+1} = \cdots = j_m = 0 \). Then we have \( 0 \leq b + c \leq m \), \( a = 2b + c \), and
\[
\omega^{j_1^2 + \cdots + j_m^2} \left( \frac{L}{j_1} \right)_{\omega} \cdots \left( \frac{j_m-1}{j_m} \right)_{\omega} = \omega^{b+c} \times \begin{cases} 1 & \text{if } c = 0, \\ 1 + \omega & \text{otherwise}. \end{cases}
\]
If \( \ell = b + c \), then \( 0 \leq \ell \leq m \), \( b = a - \ell \), and \( c = a + 2\ell \). Since \( b \) and \( c \) are non-negative, we have \( a/2 \leq \ell \leq \min\{a, m\} \). (Note that \( a/2 \) may not be an integer.) Let us rewrite the right-hand side of (6.7) as the sum taken over \( \ell \). If \( a \) is odd, then
\[
\left\{ \frac{2}{a, m} \right\}_\omega = (1 + \omega) \sum_{\ell = (a+1)/2}^{\min\{a, m\}} \omega^\ell = 1 + \omega \left( \omega^{\frac{a+1}{2}} - \omega^{\min\{a, m\}+1} \right).
\]
If \( a \) is even, then

\[
\left\{ \frac{2}{a,m} \right\}_\omega = (1 + \omega) \sum_{\ell = a/2}^{\min\{a,m\}} \omega^\ell - \omega^{\frac{a}{2}n + 1}
\]

\[
= 1 + \omega \left( \omega^{\frac{a}{2}n} - \omega^{\min\{a,m\} + 1} \right) - \omega^{\frac{a}{2}n + 1}.
\]

By using these formula and a case-by-case analysis, we obtain Table 1. (In particular, the value depends only on the congruence classes of \( a \) and \( m \) modulo 3 and whether \( a \) is greater than \( m \).)

We prove the following proposition, which has been announced in [18] without proof. (Remark that our definition of \( T_{Nz}(\omega) \) is slight different from that in [18].)

**Proposition 6.4.**

\[
\nu_n(T_9(\omega)) = n \times \begin{cases} 
1 - \omega & \text{if } n \equiv 0 \pmod{3}, \\
1 & \text{if } n \equiv 1 \pmod{3}, \\
1 + \omega^2 & \text{if } n \equiv 2 \pmod{3}.
\end{cases}
\]

**Proof.** By Theorem 6.2, we have

\[
\nu_n(T_9(\omega)) = \sum_{a,i} \left\{ \frac{2}{a,m} \right\}_\omega \omega^{-ai},
\]

where the sum is taken over all integers \( a \) and \( i \) satisfying

\[
0 \leq a \leq 2(n - 1), \quad 0 \leq i \leq 2, \quad \text{and} \quad ni \equiv a \pmod{3}.
\]

1. Suppose \( n = 3\ell \) for some \( \ell > 0 \). Then \( ni \equiv a \pmod{3} \) if and only if \( a \) is divisible by 3. By using Table 1, we obtain

\[
\nu_n(T_9(\omega)) = 3 \sum_{b=0}^{2\ell - 1} \left\{ \frac{2}{3b,3\ell - 1} \right\}_\omega = 3(\ell + \ell \cdot (-\omega)) = n(1 - \omega).
\]

2. Suppose \( n = 3\ell + 1 \) for some \( \ell \geq 0 \). Then \( ni \equiv a \pmod{3} \) if and only if \( i \equiv a \pmod{3} \). By using Table 1, we obtain

\[
\nu_n(T_9(\omega)) = 2\ell - \ell \omega^{-1} - \ell \omega = 3\ell + 1 = n.
\]
(3) Suppose \( n = 3\ell + 2 \) for some \( \ell \geq 0 \). Then \( ni \equiv a \mod 3 \) if and only if \( i \equiv -a \mod 3 \). By using Table \( \text{I} \) we obtain

\[
\nu_n(Tq(\omega)) = \sum_{b=0}^{2\ell} \left( \begin{array}{c} 2 \\ 3b, 3\ell + 1 \end{array} \right) + \left( \begin{array}{c} 2 \\ 3b + 1, 3\ell + 1 \end{array} \right) \omega + \left( \begin{array}{c} 2 \\ 3b + 2, 3\ell + 1 \end{array} \right) \omega
= \ell + 1 - (2\ell + 1)\omega + (\ell + 1)\omega^2 = (3\ell + 2)(1 + \omega^2) = n(1 + \omega^2).
\]

By combining (1), (2) and (3), we have the result. \( \square \)

6.3. Higher indicators of \( u_q(sl_2) \). Let \( N > 1 \) be an odd integer, and let \( q \) be a primitive \( N \)-th root of unity. Then the Hopf algebra \( u_q(sl_2) \) is defined. Consider the coradical filtration for \( u_q(sl_2) \) and let \( u'_q = \text{gr} u_q(sl_2) \) (see Remark \( \text{5.11} \)). By Lemma \( \text{5.3} \) we have \( \nu_n(u_q(sl_2)) = \nu_n(u'_q) \) for every \( n \geq 1 \).

As we have remarked in \( \text{5.3} \) as an algebra, \( u'_q \) is generated by \( K, E \) and \( F \) with the same relations as \( u_q(sl_2) \) but with \( \text{(5.4)} \) replaced by \( EF - FE = 0 \). Now we define elements \( g, x, y \in u'_q \) by \( g = K, x = E \) and \( y = FK \). It is easy to see that \( u'_q \) is generated by \( g, x \) and \( y \) with defining relations

\[
x^N = y^N = 0, \quad g^N = 1, \quad gx = q^2 gx, \quad gy = q^{-2} yg, \quad yx = q^2 xy.
\]

With respect to these generators, the coalgebra structure is given by

\[
\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g \otimes x, \quad \Delta(y) = y \otimes 1 + g \otimes y.
\]

Lemma 6.6. Let \( n, r \) and \( s \) be non-negative integers, and let \( \ell \) be an integer. Then the \( n \)-th Sweedler power map of \( x^r y^s g^\ell \in u'_q \) is given by the formula

\[
(x^r y^s g^\ell)^[n] = \sum_{a=0}^{s(n-1)} \sum_{b=0}^{r(n-1)} \left( \begin{array}{c} r \\ a, n-1 \end{array} \right) q^2 \left( \begin{array}{c} s \\ b, n-1 \end{array} \right) q^{-2} 2^{(\ell-b)} x^r y^s g^{a+b+\ell n}.
\]

Proof. Define \( T : U(g) \rightarrow U(q) \) by \( u \mapsto gu^{-1} \). By \( \text{4.4} \), we have

\[
(x^r y^s g^\ell)^[n] = \left( \nabla^{(n)} \circ (\text{id} \otimes T \otimes \cdots \otimes T^{n-1})^\ell \circ \Delta^{(n)} \right)(x^r y^s) \cdot g^{n\ell}.
\]

We use the same notation as the proof of Lemma \( \text{6.6} \). In particular, \( I_{n,r} \) means the set of \( n \)-dimensional multi-indices \( i \) satisfying \( |i| = r \). Given such \( i \), we set

\[
x^i = x_1^{i_1} \cdots x_n^{i_n} \quad \text{and} \quad y^i = y_1^{i_1} \cdots y_n^{i_n},
\]

where \( x_i \) is given by \( \text{6.4} \) and \( y_i \) is given by the same formula but with \( x \) replaced by \( y \). By the \( g \)-multinomial formula, we have

\[
\Delta^{(n)}(x^r y^s) = \sum_{i \in I_{n,r}} \sum_{j \in I_{n,s}} \left( \begin{array}{c} r \\ i \end{array} \right) q^2 \left( \begin{array}{c} s \\ j \end{array} \right) q^{-2} x^i y^j.
\]

For \( i = (i_1, \ldots, i_n) \), we set \( S_m(i) = \sum_{c=1}^n (i_{c+1} + \cdots + i_n)^m \) and obtain

\[
(id \otimes T \otimes \cdots \otimes T^{n-1})(x^i y^j) = q^{2(S_i(\bar{i}) - S_i(\bar{\bar{j}}))} x^i y^j.
\]
Given elements $X_c$ ($c = 1, \ldots, n$) of an algebra, we denote $X_1 \cdots X_n$ by $\prod_{c=1}^n X_c$.

We compute $\nabla^{(n)}(x^i y^j)$ as follows:

\[
\nabla^{(n)}(x^i y^j) = \prod_{c=1}^n x^{i_c} g^{j_{c+1}} \cdots y^{j_n} g^{j_{c+1}} \cdots j_n
\]

\[
= \prod_{c=1}^n q^{-2j_c(i_{c+1} + \cdots + i_n)} x^{i_c} y^{j_c} g^{i_{c+1} + \cdots + i_n + j_{c+1} + \cdots + j_n}
\]

\[
= \left( \prod_{c=1}^n q^{-2j_c(i_{c+1} + \cdots + i_n)} x^{i_c} y^{j_c} \right) \cdot q^{2(S_2(i) - S_2(j))} g^{S_1(i) + S_1(j)}
\]

\[
= q^{2(S_2(i) - S_2(j))} x^i y^j g^{S_1(i) + S_1(j)}.
\]

Set $m(i) = \ell S_1(i) + S_2(i)$ for a multi-index $i$. By the above computations, we have

\[
(x^r y^s g^\ell)^{[n]} = \sum_{i \in I_{n,r}} \sum_{j \in I_{n,s}} \binom{r}{i} \binom{s}{j} \cdot q^{2m(i) - 2m(j)} x^r y^s g^{S_1(i) + S_1(j) + n\ell}.
\]

The desired formula is obtained by rewriting the sum in a similar way as the proof of Lemma 6.1.

The element $\Lambda = \sum_{c=0}^{N-1} x^{N-1} y^{N-1} g^\ell$ is both a left and a right integral. $u'_q$ has the set $\{x^r y^s g^\ell \mid r, s, \ell = 0, 1, \ldots, N-1\}$ as a basis. With respect to this basis, we define a linear map $\lambda : u'_q \to k$ by

\[
(\lambda, x^r y^s g^\ell) = \delta_r, N-1 \delta_s, N-1 \delta_{\ell,0}.
\]

Then $\lambda$ is a right integral such that $\langle \lambda, \Lambda \rangle = 1$. The following description of the $n$-th indicator of $u_q(sl_2)$ follows immediately from the previous lemma.

**Theorem 6.6.** The $n$-th indicator of $u_q(sl_2)$ is given by

\[
\nu_n(u_q(sl_2)) = \sum_{a, b, i} \left\{ \begin{array}{c} N - 1 \\ a, n - 1 \end{array} \right\} q^{2a_i} \cdot \left\{ \begin{array}{c} N - 1 \\ b, n - 1 \end{array} \right\} q^{-2a_i},
\]

where the sum is taken over all integers $a, b$ and $i$ satisfying

\[
0 \leq i \leq N - 1, \quad 0 \leq a, b \leq (N - 1)(n - 1) \quad \text{and} \quad a + b \equiv ni \pmod{N}.
\]

This formula is similar to that of the Taft algebra. In some restricted cases, the indicator of $u_q(sl_2)$ relates directly to the indicator of the Taft algebra.

**Proposition 6.7.** Let $T = T_{N^2}(q^2)$, $\overline{T} = T_{N^2}(q^{-2})$, and let $n \geq 1$ be an integer.

1. Suppose that $n$ is relatively prime to $N$. Then

\[
\nu_n(u_q(sl_2)) = \nu_n(T) \cdot \nu_n(\overline{T}).
\]

2. Suppose that $n$ is divisible by $N$. Then

\[
\nu_n(u_q(sl_2)) = \frac{\nu_n(T) \cdot \nu_n(\overline{T})}{N}.
\]

**Proof:** (1) If this is the case, there exists an integer $n'$ such that $n \cdot n' \equiv 1 \pmod{N}$.

By Theorem 6.2, we have

\[
\nu_n(T) = \sum_{a=0}^{(N-1)(N-1)} \left\{ \begin{array}{c} N - 1 \\ a, n - 1 \end{array} \right\} q^{-n'a^2}.
\]

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If \( a + b \equiv ni \pmod{N} \) holds, then we have \( q^{-2ai} \cdot q^{2bi} = q^{-2n'a^2} \cdot q^{-2n'b^2} \). Therefore, by Theorem 6.6 we compute

\[
\nu_n(u_q(\mathfrak{sl}_2)) = \sum_{a,b=0}^{(n-1)/(N-1)} \left\{ \begin{array}{c} N-1 \\ a, n-1 \end{array} \right\} q^{-2n'a^2} \cdot \left\{ \begin{array}{c} N-1 \\ b, n-1 \end{array} \right\} q^{2n'b^2} = \nu_n(T) \cdot \nu_n(\mathfrak{T}).
\]

(2) If this is the case, by Theorem 6.2, we have

\[
(6.8) \quad \nu_n(T) = N \sum_{a} \left\{ \begin{array}{c} N-1 \\ a, n-1 \end{array} \right\} q^{2},
\]

where the sum is taken over all integer \( a \) satisfying

\[
(6.9) \quad 0 \leq a \leq (N-1)(n-1) \quad \text{and} \quad a \equiv 0 \pmod{N}.
\]

On the other hand, by Theorem 6.6 we compute

\[
\nu_n(u_q(\mathfrak{sl}_2)) = \sum_{a,b}^{N-1} \sum_{i=0}^{N-1} \left\{ \begin{array}{c} N-1 \\ a, n-1 \end{array} \right\} \left\{ \begin{array}{c} N-1 \\ b, n-1 \end{array} \right\} q^{2i},
\]

where \( a \) and \( b \) run over all integers from 0 to \( (N-1)(n-1) \) satisfying

\[
(6.10) \quad a + b \equiv 0 \pmod{N}.
\]

Observe \( \sum_{i=0}^{N-1} q^{2(b-a)i} \) is equal to \( N \) if

\[
(6.11) \quad a - b \equiv 0 \pmod{N},
\]

and it is equal to 0 otherwise. Since \( N \) is odd, we have that both (6.10) and (6.11) hold if and only if \( a \equiv b \equiv 0 \pmod{N} \). Therefore

\[
\nu_n(u_q(\mathfrak{sl}_2)) = N \cdot \left( \sum_{a} \left\{ \begin{array}{c} N-1 \\ a, n-1 \end{array} \right\} q^{2} \right) \left( \sum_{a} \left\{ \begin{array}{c} N-1 \\ a, n-1 \end{array} \right\} q^{-2} \right)
\]

where, in each sums, \( a \) runs over all integers satisfying (6.9). By (6.8), we obtain the result.

The following proposition follows from Propositions 6.4 and 6.7.

**Proposition 6.8.** Let \( q \) be a primitive third root of unity. Then we have

\[
\nu_n(u_q(\mathfrak{sl}_2)) = n^2,
\]

for every \( n \geq 1 \).
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