NON-ASSOCIATIVE MAGNETIC TRANSLATIONS: A QFT CONSTRUCTION

JOUKO MICKELSSON

ABSTRACT. The non-associativity of translations in a quantum system with magnetic field background has received renewed interest in association with topologically trivial gerbes over \( \mathbb{R}^n \). The non-associativity is described by a 3-cocycle of the group \( \mathbb{R}^n \) with values in the unit circle \( S^1 \). The gerbes over a space \( M \) are topologically classified by the Dixmier-Douady class which is an element of \( \text{H}^3(M, \mathbb{Z}) \). However, there is a finer description in terms of local differential forms of degrees \( d = 0, 1, 2, 3 \) and the case of the magnetic translations for \( n = 3 \) the 2-form part is the magnetic field \( B \) with non zero divergence. In this paper we study a quantum field theoretic construction in terms of \( n \)-component fermions on a real line or a unit circle. The non associativity arises when trying to lift the translation group action on the 1-particle system to the second quantized system.

MSC classification: 81T50 (primary); 22E67, 81R15, 22E70 (secondary)

1. Introduction

The motivation for the present short note is to understand the recent paper by Bunk, Müller and Szabo \[1\] in terms of quantization of Dirac operators on a real line or on the circle coupled to an abelian vector potential with gauge group \( \mathbb{R}^n \) or the torus \( T^n = \mathbb{R}^n/\mathbb{Z}^n \). The central topic in \[1\] is a 3-cocycle on \( \mathbb{R}^n \) arising from composing certain functors coming from translations acting on differential data of a topologically trivial gerbe on \( \mathbb{R}^n \). The non-associativity in the case of a magnetic field with sources in the case \( n = 3 \) was suggested already long ago in \[5\]. An interpretation of the 3-cocycle in terms of representations of canonical anticommutation relations was then proposed in \[2\].

In this paper we interpret the magnetic translations as (non periodic) gauge transformations on real line acting on fermions with \( n \) complex components. They actually define true operators on the level of 1-particle Dirac operators. However, they cannot be lifted to unitary operators in the fermionic Fock space; if they could, there would be no 3-cocycle since the composition of linear operators is associative. Nevertheless, these gauge transformations define functors acting on certain categories of representations of canonical anticommutation relations. The composition of functors respects the group law in \( \mathbb{R}^n \) only modulo the action of automorphisms in the Fock space; these automorphisms come from a projective representation of an abelian gauge group.

Let \( G \) be a simply connected Lie group. Let \( H \) be the space of square integrable functions on the real line taking values in the complex vector space \( \mathbb{C}^n \) with a unitary \( G \) action through a representation \( \rho \) of \( G \). The group \( LG \) of smooth \( G \) valued functions \( f \) on \( \mathbb{R} \) such that \( f(t) \) is constant \( g \) outside of a compact set acts unitarily on \( H \). Using the stereographic projection from the unit circle to the real
axis we can actually identify $LG$ as a subgroup of the smooth loop group on the unit circle.

Let $C_g$ be the category of smooth paths $f$, parametrized by a closed interval of the real axis, in $G$ starting from the unit element $e$ and with the end point $g$ with vanishing derivatives at the points $e,g$. Morphisms in the category $C_g$ are smooth homotopies of paths with fixed end points. Next we choose a representation of the canonical anticommutation relations in a fermionic Fock space $\mathcal{F}_f$ with a Fock vacuum defined by a polarization $H = H_+(f) \oplus H_-(f)$ of $H$. Starting from the polarization $H = H_+ \oplus H_-$ defined by the Fourier decomposition to non negative and negative Fourier modes we set $H_+(f) = f \cdot H_+$ and $H_-(f)$ its orthogonal complement. [Alternatively, for the purposes of the present note, we could consider polarizations of the one dimensional Dirac operator $D_f = i \frac{d}{dx} + if^{-1}df$].

The fixed polarization $H_+ \oplus H_-$ defines a representation of the canonical commutation relations generated by the elements $a^*(v), a(v)$ for $v \in H$ with nonzero anticommutation relations

$$a^*(u)a(v) + a(v)a^*(u) = 2 < u, v >_H$$

and a Fock vacuum $\psi$ with $a^*(u)\psi = 0 = a(v)\psi$ for $u \in H_-$ and $v \in H_+$.

We have a functor from the category of paths $C_g$ to the category of CAR algebra representations $\mathcal{F}(g)$ sending $f$ to $\mathcal{F}_f$. The CAR representations in the category $\mathcal{F}(g)$ are all equivalent. Two paths $f, f' \in C_g$ are related by a point-wise multiplication by an element $h$ of the loop group $LG$. An element of the loop group is represented as an unitary operator $T(h)$ in the Fock space and $T(h)a^*(v)T(h)^{-1} = a^*(hv)$. However, the operator $T(h)$ is fixed only up to a phase due to the fact that the loop group is projectively represented, through a central extension $\hat{LG}$. For this reason the functor $F$ is projective in the sense that morphisms in the category $C_g$ go over to morphisms (unitary equivalences) in $\mathcal{F}(g)$ respecting the composition only up to a phase.

Each element $g_1 \in G$ defines a functor $F_{g_1} : C_g \to C_{gg_1}$ as follows. Fix a path $f_1$ joining $e$ to $g_1$. Take any $f \in C_g$, parametrized by an interval $[a, b]$. Parametrize $f_1$ by an interval $[b, c]$. Then joining the two paths gives a new path $f \ast f_1$ as follows: First travel $f$ until the end point $g = f(b)$. Then continue with $t \mapsto f(b)f_1(t)$ for $b \leq t \leq c$ ending at $gg_1$. The functor $F_{g_1}$ from $C_g$ to $C_{gg_1}$ defines naturally also a functor from $\mathcal{F}(g)$ to $\mathcal{F}(gg_1)$. Namely, the path $f_1$ joining $e$ to $g_1$ defines an automorphism of the CAR algebra by $a(v) \mapsto a(gf_1v)$ taking a representation in the category $\mathcal{F}(g)$ to a representation in the category $\mathcal{F}(gg_1)$.

Next fix a pair $g_1, g_2 \in G$. Choose as above a pair of paths $f_1, f_2$ parametrized by the intervals $[b, c]$ and $[c, d]$ correspondingly. On the other hand, we have a path $f_{12}$ joining $e$ to $g_1g_2$. Finally, we have a loop $\ell(g_1,g_2)$ by composing $f_1 \ast f_2 \ast f_{12}^{-1}$; recall that functions on the real line constant outside of a compact set can be identified as elements of the loop group. The last factor involves the point-wise inverse of the function travelled in the opposite direction. This loop construction is similar but different from the construction in $\mathcal{F}$ where a functorial approach to group 3-cocycles was discussed.

The functor $F(g)$ can be represented as an operator by a point-wise multiplication in the 1-particle Hilbert space $H$. However, it does not define an operator in a Fock space. The reason is that the off-diagonal blocks of the 1-particle operator
are not Hilbert-Schmidt with respect to energy polarization due to the non-periodicity of the path; this is seen by a simple Fourier analysis using the polarization defined by $D = i \frac{d}{dt}$.

It follows directly from the definition that we have the 2-cocycle property
\begin{equation}
\ell(g_1, g_2)\ell(g_1 g_2, g_3) = [f_1 \ell(g_1, g_2 g_3) f_1^{-1}] \ell(g_2, g_3).
\end{equation}

Using the construction in [7] for any loop $\ell \in LG$ we can fix an element in the standard central extension $\hat{LG}$ of $LG$ by $S^1$ by choosing an extension $\tilde{\ell}$ to the unit disk; on the boundary $\tilde{\ell}$ is equal to $\ell$. As a circle bundle, the central extension consists of equivalence classes of pairs $(\tilde{\ell}, \lambda)$ with $\lambda \in S^1$ with the equivalence relation
\[(\tilde{\ell}, \lambda) \sim (\tilde{\ell}', \lambda')\]

with $\lambda' = \lambda e^{2\pi i \int_V \Omega}$ where $V$ is a volume in $G$ with boundary obtained by gluing the surfaces $\tilde{\ell}, \tilde{\ell}'$ along the common boundary $\ell$ and $\Omega$ is a representative of a class in $H^3(G, \mathbb{Z})$.

The 2-cocycle property above fails for the lifts of the loop group elements to the central extension $\hat{LG}$. A triple $g_1, g_2, g_3$ determines through the choices of the loops $\ell$ in (1.1) and their extensions $\tilde{\ell}$ a tetraed with faces given by the four extensions $\tilde{\ell}(g_1, g_2), \tilde{\ell}(g_1 g_2, g_3), \tilde{\ell}(g_2, g_3), \tilde{\ell}(g_1, g_2 g_3)$. This closed 2-surface $\Sigma$ in $G$ is then equivalent to the phase
\[c(g_1, g_2, g_3) = \exp 2\pi i \int_V \Omega\]

where $V$ is the volume in $G$ with boundary $\Sigma$. This is the 3-cocycle which comes from the extensions of the loops in (1.1) to the central extension,

\begin{equation}
\tilde{\ell}(g_1, g_2)\tilde{\ell}(g_1 g_2) = [f_1 \tilde{\ell}(g_1, g_2 g_3) f_1^{-1}] \tilde{\ell}(g_2, g_3) \times c(g_1, g_2, g_3)
\end{equation}

Because of the different choices made in the construction of [2] the cocycle $c$ is smooth only in an open neighborhood of the unit element in $G$.

2. THE CASE OF $G = \mathbb{R}^n$

The group $G = \mathbb{R}^n$ has an unitary representation in the Hilbert space $H$ of square integrable functions on $\mathbb{R}$ with values in $\mathbb{C}^n$ through multiplication $z_k \mapsto e^{iz_k} z_k$ for $k = 1, 2, \ldots, n$. This defines also an action of the loop group $LG$ in $H$ through point-wise multiplication by the phase $e^{iz_k(t)}$.

**Lemma 2.1.** The group of continuous piecewise smooth loops satisfies the Hilbert-Schmidt condition on off-diagonal blocks for the energy polarization $H = H_+ \oplus H_-$. 

**Proof.** We set $n = 1$; the multicomponent case is proven in a similar way. The Fourier components of the multiplication operator $\ell(t)$ can be estimated by integration by parts: restricting to any interval $[a, b]$ where $\ell$ is smooth we get for momenta $p, q$ of opposite sign
\[<p|\ell|q> = \int_a^b e^{i(p-q)t} \ell(t) dt = \int_a^b \frac{1}{i(p-q)} e^{i(p-q)t} \ell'(t) dt + \int_a^b \frac{1}{i(p-q)} e^{i(p-q)t} \ell(t)|_a^b.
\]

The valuations at the end points cancel when summing over all intervals for a periodic continuous function $\ell$. In the first term on the right we can repeat the integration by parts. Now since the derivative $\ell'$ might be discontinuous at the end
points of the intervals the insertion terms do not cancel. However, they involve the same factor \((p - q)^{-2}\) as in the integration term involving \(\ell''(t)\). The Hilbert-Schmidt condition follows taking the square and observing that

\[
\int_{p > \gamma, q < -\gamma} \frac{1}{(p - q)^{2}} \, dp dq < \infty
\]

for any positive \(\gamma\), and likewise for \(p < -\gamma, q > \gamma\). \(\square\)

For any \(x \in \mathbb{R}^n\) choose the path \(f_x\) as the straight line from the origin to the point \(x\). Then proceeding as the general case above for any pair of vectors \(x, y \in \mathbb{R}^n\) we have the closed loop \(\ell(x, y)\) as the triangle with vertices at \(0, x, x + y\). According to the Lemma this piece-wise smooth loop is represented as a unitary operator in the Fock space defined by the energy polarization of the free Dirac operator \(i \frac{d}{dt}\) on the real line.

Next fix a a closed 3-form on \(\mathbb{R}^n\) by \(\Omega = \sum a_{ijk} dx^i \wedge dx^j \wedge dx^k\) where \(a\) is any antisymmetric tensor. This closed form is exact, \(\Omega = dB\) with \(B = \sum a_{ijk} x^i dx^j \wedge dx^k\). The forms \(\Omega, B\) define a topologically trivial gerbe over \(\mathbb{R}^n\).

For a pair \(x, y\) of vectors the loop \(\ell(x, y)\) is the boundary of a triangle \(\tilde{\ell}(x, y)\) and for a triple \(x, y, z \in \mathbb{R}^n\) we have a tetraed \(V(x, y, z)\) with faces consisting of the triangles \(\tilde{\ell}(x, y), \tilde{\ell}(y, z), \tilde{\ell}(x + y, z), \tilde{\ell}(x + y, z)\) where \(\ell_x\) denotes the triangle \(\ell\) translated by the vector \(x\). Thus the vertices of the tetraed \(V(x, y, z)\) are located at the points \(0, x, x + y, x + y + z\). We observe

\[
\int_{V(x,y,z)} \Omega = \sum a_{ijk} x^i y^j z^k.
\]

In particular, when \(n = 3\) and \(a_{ijk} = \epsilon_{ijk}\) the value of the integral is equal to the volume of the tetraed \(V(x, y, z)\). As before, the corresponding 3-cocycle is

\[
c(x, y, z) = e^{2\pi i \sum a_{ijk} x^i y^j z^k}.
\]

Although this cocycle for (nonzero \(a\)) is nontrivial as a group cocycle it is however trivial as a transformation groupoid cocycle: The group \(\mathbb{R}^n\) acts on itself by translations and \(c = \delta b\) for the the 2-cochain \(b(u; x, y) = c(u, x, y)\) with

\[
(\delta b)(x, y, z) = b(u; x, y)^{-1} b(u; x + y, z)^{-1} b(u; x, y + z) b(u + x; y, z)
\]

The cocycle \(c\) is equal to the identity if all the vectors belong to the subgroup \(\mathbb{Z}^n \subset \mathbb{R}^n\). In that case all the functors corresponding to the edges of the tetraed are actually loops in \(T^n = \mathbb{R}^n / \mathbb{Z}^n\) and are represented by unitary operators in the Fock space.

Following the rules of the canonical quantization of bounded operators in the 1-particle Hilbert space satisfying the Hilbert-Schmidt condition on the off-diagonal blocks with respect to the energy polarization the Lie algebra is represented projectively in the fermionic Fock space. The projective action is characterized by the 2-cocycle \([6]\)

\[
c_2(f, g) = \frac{1}{2} \text{TR} f[\epsilon, g].
\]

The trace is computed in the 1-particle Hilbert space. In particular, when the Lie algebra consist of multiplication operators by smooth functions (on a circle or on
the real line, constant outside a compact set) we have
\[ \frac{1}{2} \text{TR} f[\epsilon, g] = \frac{1}{2\pi i} \int \text{tr} f dg \]
where the second trace is evaluated in the representation \( \rho \) of \( G \) in \( \mathbb{C}^n \).

In the present setting the loops take values in \( \mathbb{R}^n \) and \( \rho(x)z_k = e^{ix_k} z_k \) for \( k = 1, 2, \ldots, n \). Each component defines a circle value function \( e^{if(t)} \) acting as a multiplication operator in the 1-particle Hilbert space \( H = L_2(\mathbb{R}, \mathbb{C}^n) \). The cocycle \( c_2 \) is nontrivial on the abelian loop group. However, in the case of a family of Dirac operators \( D_A = i\frac{d}{dx} + A \) coupled to an abelian vector potential \( A \) (with values in \( \mathbb{R}^n \)) the cocycle becomes trivial: we have \( c_2 = \delta b_1 \) where
\[ b_1(A; X) = \frac{1}{4\pi i} \int \sum_k A_k X_k dt \]
and the loop algebra element \( X \) acts on \( A \) through the gauge transformation \( A \mapsto A + dX \). For this reason the bundle of Fock spaces parametrized by the vector potentials becomes equivariant with respect to the gauge action and can be pushed forward to a bundle over the flat moduli space \( \mathbb{R}^n = A/G \) of gauge potentials; here \( G \) is the group of periodic functions \( f : \mathbb{R} \to \mathbb{R}^n \) (that is, \( f \) is a constant outside of a compact set) acting on potentials as \( A \mapsto A + df \).

3. The case of a torus

If we replace the gauge group \( \mathbb{R}^n \) by the torus \( T^n \) the situation becomes different. All the maps \( f : \mathbb{R} \to \mathbb{R}^n \) which are periodic modulo \( \mathbb{Z} \) (that is, the asymptotic values of \( f \) on the right in \( \mathbb{R} \) are related to the values on the left by a shift in \( \mathbb{Z}^n \)) satisfy the Hilbert-Schmidt condition on off-diagonal blocks with respect to the energy polarization; again, a function \( f \) defines a multiplication operator in the one-particle space through \( z_k \mapsto e^{2\pi i f_k} z_k \). These functions \( f \) can be viewed as loops \( S^1 \to T^n \). Now the group of gauge transformations \( G \) factorizes as a product of the group \( G_0 \) contractible maps to \( T^n \) (represented as loops on \( \mathbb{R}^n \)) and a group \( \mathbb{Z}^n \) of maps of the form \( f(t) = 0 \) for \( t \leq 0 \), \( f(t) = tv \) for \( 0 \leq t \leq 1 \) with \( v \in \mathbb{Z}^n \) and \( f(t) = v \) for \( t \geq 1 \).

The moduli space of gauge potentials \( \mathcal{A}/\mathcal{G} \) is now the torus \( T^n \); we have \( \mathcal{A}/\mathcal{G}_0 = \mathbb{R}^n \) and the second factor in \( \mathcal{G} \) is isomorphic to the subgroup \( \mathbb{Z}^n \subset \mathbb{R}^n \). In the case of \( \mathbb{R}^n \) there was no restriction on the normalization of the 3-cocycle [as a group cocycle or as a 3-form on \( \mathbb{R}^n \)] but in the case of the torus the 3-cocycle must satisfy an integrality constraint in order that the gerbe over \( T^n \) is well-defined.

As explained in [1], [9] (see also [10] Section 7) the 1-particle Dirac Hamiltonians can be twisted in such a way that their K-theory class over the moduli space \( T^n \) is nontrivial: the Chern character has a nonzero component \( \omega_3 \) in \( H^3(T^n, \mathbb{Z}) \). The basis in \( H^3(T^n, \mathbb{Z}) \) is given by the 3-forms \( \omega_3 = \sum a_{ijk} dx_i \wedge dx_j \wedge dx_k \) where the \( a \)'s form a basis of totally antisymmetric tensors of rank 3 with integral coefficients, The pull-back with respect to the projection \( \mathbb{R}^n \to T^n \) is the form \( \sum a_{ijk} x_i dx_j \wedge dx_k \). The quantum field theoretic construction of a gerbe over the torus from a non zero class \( \omega_3 \) is recalled in the Appendix.

The 3-form part \( \omega_3 \) of the Chern character is the Dixmier-Douady class of the projective vector bundle over \( T^n \) obtained by canonical quantization of the family of 1-particle Dirac operators. The pull-back of this bundle over \( \mathbb{R}^n \) comes by
projectivization of a vector bundle (the bundle of fermionic Fock spaces). The group $\mathbb{Z}^n$ acts through an abelian extension of the Fock spaces. The extension is defined by the 2-cocycle

$$c_2(u; x, y) = e^{2\pi i \sum a_{ijk} u_i x_j y_k}$$

where $x, y \in \mathbb{Z}^n$ and $u \in \mathbb{R}^n$ and $\mathbb{Z}^n$ acts on the functions of the vector $u$ as translations.

The 3-cocycle is identically = 1 when the arguments are in $\mathbb{Z}^n \subset \mathbb{R}^n$ in conformity with the (projective) action of $\mathbb{Z}^n$ on the Fock spaces.

**Remark** The cocycle $c_2$ is also a group cocycle even in the case of constant coefficients (no group action on $u$) but since the coboundary operator is different the cohomology with variable coefficients is different from the cohomology with constant coefficients.

4. Appendix

In this appendix we briefly recall the quantum field theoretic construction of a gerbe over the torus using a twisted family of CAR algebra representations, [4], [9].

A hermitean complex line bundle $L$ over the torus $T^n$ is characterized by a class $\omega$ in $H^2(T^n, \mathbb{Z})$. Parametrizing the circles in the torus by the interval $[0, 1]$ the 2-cohomology is spanned by antisymmetric bilinear forms on $\mathbb{R}^n$ such that $\omega(x, y) \in \mathbb{Z}$ for $x, y \in \mathbb{Z}^n$. The pull-back of $L$ over $\mathbb{R}^n$ is trivial and the sections of that line bundle are complex valued functions $\psi$ such that

$$\psi(x + z) = \psi(x)e^{2\pi i \omega(x, z)}$$

for $z \in \mathbb{Z}^n$.

Next we construct a family of fermionic Fock spaces parametrized by vectors in $\mathbb{R}^n$. For each $k = 1, 2, \ldots, n$ and $u, v \in H$ let $a_k(v), a_k^*(u)$ be generators of a CAR algebra with nonzero anticommutators

$$a_k^*(u)a_k(v) + a_k(v)a_k^*(u) = 2 < u, v > .$$

The generators for different lower indices are assumed to commute. It will be convenient to compactify the real line to the unit circle so we can take $H = L_2(S^1, \mathbb{C}^n)$ and we can work with the orthonormal basis of Fourier modes in each of the $n$ directions.

We twist the CAR algebra by the line bundle $L$. This means that the families of creation and annihilation operators are sections of the tensor products of $L$ or its dual and the CAR algebra. The sections are $\mathbb{Z}^n$ equivariant functions on $\mathbb{R}^n$, that is, for $x \in \mathbb{R}^n$ and for $z \in \mathbb{Z}^n$

$$a_k^*(u, x + z) = a_k^*(u, x)e^{2\pi i \omega(x, z)}, \ a_k(u, x + z) = a_k(u, x)e^{-2\pi i \omega(x, z)} .$$

The right-hand-side of the canonical anticommutation relations, when evaluated at a point $x \in \mathbb{R}^n$, is multiplied by the pairing of sections of $L, L^*$ involved in the construction of $a_k^*(u, x) = a_k^*(u) \otimes \psi(x)$ and of $a_k(v) \otimes \xi$. The Fock vacuum is again annihilated by $a^*(u, x)$ and $a(v, x)$ for $u \in H_-$ and $v \in H_+$. (One could generalize this construction by allowing the modes for different lower index $k$ be twisted by different line bundles.)
Thus the states with net particle number $N$ in the Fock space are twisted by
the $N$:th tensor power of $L$.

The group $\mathbb{Z}^n$ acts as automorphisms of the twisted CAR algebra by

$$g(p)a^*(u, x)g(p)^{-1} = a^*(p \cdot u, x + p) = e^{2\pi i \omega(x, p)}a^*(p \cdot u, x)$$

where $p$ acts on a function $u(\zeta)$ by multiplication by a phase, $u_k \mapsto e^{2\pi i p_k}u_k$, that is, the Fourier modes are shifted by $p$ units. Likewise, for the annihilation operators

$$g(p)a(u, x)g(p)^{-1} = e^{-2\pi i \omega(x, p)}a((-p) \cdot u, x).$$

The action of $g(p)$ in the Fock spaces parametrized by $x$ is now completely defined by fixing the action on the vacuum vector $\psi$. This is easiest done thinking the vectors as elements in the semi-infinite cohomology (in physics terms, the 'Dirac sea'). For $n = 1$ the vacuum is symbolically the semi-infinite product

$$\psi = a_0^*a_{-1}^*a_{-2}^* \cdots$$

where the lower index refers to the Fourier modes in $L_2(S^1)$. For general $n$ the vacuum is defined in a similar way inserting the non-negative Fourier modes for each of the $n$ components. The CAR generators are labelled by a double index $(k, j)$ with $k \in \mathbb{Z}$ and $j = 1, 2, \ldots, n$. The action of $g(p)$ on the vacuum is now defined as a shift operator: The index $k$ of the element $a_{k, j}$ is shifted by the integer $p_j$ for $j = 1, 2, \ldots, n$, $k \mapsto k + p_j$.

Because of the phase shifts when the CAR algebra generators are conjugated by $g(p)$ the product $g(p)g(q)$ is not equal to $g(p + q)$ but they differ by a $x$ dependent phase,

$$g(p)g(q) = C(x; p, q)g(p + q) = e^{2\pi i N\omega(x, p)}g(p + q)$$

where $N$ is the particle number of the state $g(q)\psi$, that is, $N = \sum_{j=1}^{n} q_j$.

**Remark** The projective vector bundles over $T^n$ are classified by elements of $H^3(T^n, \mathbb{Z})$. Representatives of these elements can be written as de Rham forms $\Omega = \sum a_{ijk}dx_i \wedge dx_j \wedge dx_k$ where the coefficients $a_{ijk}$ are integers. The pull-back of $\Omega$ with respect to the projection $\pi: \mathbb{R}^n \to T^n$ is $\pi^*\Omega = d\theta = d\sum a_{ijk}x_i dx_j \wedge dx_k$. Evaluating $\theta$ for tangent vectors $u, v$ in the integral lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ and exponentiating gives the 2-group cocycle

$$C'(x; u, v) = e^{2\pi i \sum a_{ijk}x_i u_j v_k}$$

where the group $\mathbb{Z}^n$ acts on the vector $x$ by $x \mapsto x + u$. According to the discussion in [10], Section 7.1, there is a 1-1 correspondence between the group cohomology $H^2_{grp}(\mathbb{Z}^n, A)$ and the de Rham cohomology $H^3(T^n, \mathbb{Z})$ where $A$ is the $\mathbb{Z}^n$ module of (smooth) functions $T^n \to S^1$. The above map $\{c_{ijk}\} \to C'$ realizes this isomorphism.

**Example:** When $n = 3$ the cocycle $C$ is equivalent to $C'$ for the choice $a_{ijk} = \alpha \epsilon_{ijk}$ where $\epsilon$ is totally antisymmetric tensor with $\epsilon_{123} = 1$ and $\alpha = 2(\omega_1 + \omega_2 + \omega_3)$ where $\omega = \omega_{12}dx_1 \wedge dx_2 + \omega_{31}dx_3 \wedge dx_1 + \omega_{23}dx_2 \wedge dx_3$. This is seen by projecting the exponent in $C$ to its totally antisymmetric component.

**References**

[1] Severin Bunk, Lukas Müller and Richard J. Szabo: Geometry and 2-Hilbert Space for Nonassociative Magnetic Translations. arXiv:1804.08953
[2] Alan Carey: *The origin of three-cocycles in quantum field theory*. Phys. Lett. B 194 (1987), no. 2, 267 - 270.

[3] Alan Carey, Jouko Mickelsson and Michael Murray: *Index theory, gerbes, and Hamiltonian quantization*. Comm. Math. Phys. 183 (1997), no. 3, 707 - 722.

[4] Antti J. Harju and Jouko Mickelsson: *Twisted K-theory constructions in the case of a decomposable Dixmier-Douady class*. J. K-Theory 14 (2014), no. 2, 247 - 272.

[5] Roman Jackiw: *3-cocycle in mathematics and physics*. Phys. Rev. Lett. 54 (1985), 159 - 162.

[6] Lars-Erik Lundberg: *Quasi-free ”second quantization”.* Comm. Math. Phys. 50 (1976), no. 2, 103 - 112.

[7] Jouko Mickelsson: *Kac-Moody groups, topology of the Dirac determinant bundle, and fermionization*. Comm. Math. Phys. 110 (1987), no. 2, 173 - 183.

[8] Jouko Mickelsson: *From gauge anomalies to gerbes and gerbal actions*. Motives, quantum field theory, and pseudodifferential operators, 211 - 220, Clay Math. Proc., 12, Amer. Math. Soc., Providence, RI, (2010).

[9] Jouko Mickelsson: *Extensions of lattice groups, gerbes and chiral fermions on a torus*. J. Geom. Phys. 121 (2017), 378 - 385.

[10] Jouko Mickelsson and Stefan Wagner: *Third group cohomology and gerbes over Lie groups*. J. Geom. Phys. 108 (2016), 49 - 70.

Department of Mathematics and Statistics, University of Helsinki

*E-mail address: jouko@kth.se*