Unexpected Effects of Online K-means Clustering

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Abstract
In this paper we study k-means clustering in the online setting. In the offline setting the main parameters are number of centers, \( k \), and size of the dataset, \( n \). Performance guarantees are given as a function of these parameters. In the online setting new factors come into place: the ordering of the dataset and whether \( n \) is known in advance or not. One of the main results of this paper is the discovery that these new factors have dramatic effects on the quality of the clustering algorithms. For example, for constant \( k \): (1) \( \Omega(n) \) centers are needed if the order is arbitrary, (2) if the order is random and \( n \) is unknown in advance, the number of centers reduces to \( \Theta(\log n) \), and (3) if \( n \) is known, then the number of centers reduces to a constant. For different values of the new factors, we show upper and lower bounds that are exactly the same up to a constant, thus achieving optimal bounds.

1 Introduction
Clustering is an unsupervised learning problem where the goal is to group data into few clusters. It is an important exploratory data analysis step and is used in different domains like bioinformatics, image analysis, and information retrieval. In the literature there are many algorithms for clustering in the offline setting, where all the points in the dataset are given in advance. The output of a clustering algorithm is a set of centers in the dataset, where each center is a “representative” of one cluster.

In the online setting, points in the dataset arrive one after another, and a decision whether to take the current point as a center needs to be made before observing the next point. Studying the online setting is more important these days as enormous amounts of new data is created every second [1]. In the online setting, new factors come into place: the order of the input points, and whether the number of points in the dataset is known in advance or not. It makes sense that the order of the points would impact performance, but how much? Can the apriori knowledge of the size of the dataset improve performance? In this paper we answer these questions and find that these two new factors have unexpected effects on online clustering. Specifically, we show that the ordering of the dataset can exponentially increase the number of centers. We also prove that merely knowing the size of the dataset can reduce the number of centers logarithmically.

1.1 The online framework
For any dataset \( D = \{x_1, \ldots, x_n\} \subseteq \mathbb{R}^d \) and desired number of clusters \( k \), the k-means cost is defined as the sum of squared \( \ell_2 \)-distances of each point in the dataset to its closest center:

\[
\text{cost}(c_1, \ldots, c_k) = \sum_{t=1}^{n} \|x_t - c(x_t)\|^2,
\]

where \( c(x) \) is the closest center to \( x \), i.e., \( c(x) = \arg\min_{c_i \in \{c_1, \ldots, c_k\}} \|x - c_i\| \). We denote by \( \text{cost}(\text{opt}_k) \) the optimal cost using \( k \) centers:

\[
\text{cost}(\text{opt}_k) := \min_{c_1, \ldots, c_k \in D} \text{cost}(c_1, \ldots, c_k).
\]

In the offline setting an algorithm receives a dataset \( D \) and a desired number of clusters \( k \), and in \( \text{poly}(n) \) time returns a set of centers

\[1\]More generally, one can ease the requirement, and allow the centers to be in \( \mathbb{R}^d \) and not necessarily in \( D \).
c_1, \ldots, c_\ell \in D$ such that (1) the number of centers, $\ell$, is small compared to $k$ and (2) $\text{cost}(c_1, \ldots, c_\ell)$ is small compared to $\text{cost}(\text{opt}_k)$.

Our setting is online k-means which was used previously in [12]. At each time step when a new point in the dataset arrives, the algorithm needs to decide whether to take it as a center or not. In this paper we consider constant approximation algorithms, meaning $\text{cost}(c_1, \ldots, c_\ell) \leq a \cdot \text{cost}(\text{opt}_k)$, where $a$ is some constant. We refer to such a clustering algorithm as a $\Theta(1)$-approximation. The goal of the online algorithm is to minimize the number of centers $\ell$ and make it as close as possible to $k$.

### 1.2 Results summary

Henceforth we focus on the case of constant $k$. Note that in the online setting even the case of $k = 1$ is not trivial and in fact quite interesting, as will be discussed below. A conceptual contribution of this paper is the observation that the new factors in online clustering (namely, the ordering and whether $n$ is known in advance) significantly influence the performance of the optimal algorithms. We are able to pinpoint the exact number of centers needed and sufficient to achieve a constant approximation for different values of the new factors.

In the offline setting, an efficient algorithm that returns $\Theta(1)$ centers is known (see [6], [3]). The case of $k = 1$ is simpler than that; the optimal center is clearly the average point. That’s the end of the story for offline clustering. But in the online setting the story only begins.

We next give a sample of the results we prove in the paper. In the case of $k = 1$ when $n$, the number of points, is unknown and the order can be the arbitrary, then $\Theta(\log n)$ centers are needed, and also sufficient. In the case that $k = 2$ we prove that if the order is arbitrary then any learning algorithm must take as a center basically all the examples, regardless of whether the algorithm knows what $n$ is or not. If the order is random and $n$ is unknown we show an algorithm that uses $O(\log n)$ centers. This algorithm is optimal up to a constant, as we show a matching lower bound of $\Omega(\log n)$, i.e., we show that any algorithm must use $\Omega(\log n)$ centers.

To summarize, the ordering of the dataset can exponentially increase the number of centers. If $n$ is known, an optimal algorithm can use only $\Theta(1)$ centers and still be a $\Theta(1)$-approximation. This means that the mere knowledge of the size of $n$ can give the clustering algorithm the ability to dramatically improve its performance. Our results are summarized in the following list and in Figure 1.

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2 If the average point is not in the dataset, take the closest point to it.
• For \( k = 1 \):
  1. If \( n \) is known or the order of examples is random, \( \Theta(1) \) centers is sufficient (and obviously necessarily). See Lemma 2 and Theorem 3.
  2. If \( n \) is unknown and the order is arbitrary, then \( \Theta(\log n) \) centers is needed and sufficient. See Theorem 5 and Theorem 6.

• For constant \( k \geq 2 \):
  1. If the order is arbitrary, then \( \Theta(n) \) centers is needed (and obviously sufficient). See Theorem 7.
  2. If the order is random and \( n \) is known then \( \Theta(1) \) centers is sufficient (and obviously necessarily). See Theorem 8.
  3. If the order is random and \( n \) is unknown we show that \( \Theta(\log n) \) centers is needed and sufficient. See Theorems 9, 10, 11, 13.

1.3 Related Work

Liberty, Sriharsha, and Sviridenko [12] described an algorithm for online k-means that use the same definition of online clustering as this paper. Their algorithm is an adaptation of the k-means++ algorithm [6] to the online case. Inherently, their algorithm cannot get the optimal bound, as will be explained in Section 4.3.

In this paper, we improve both the quality of the approximation and the number of centers to the optimal values (see Algorithm 5).

The work [11] designed a sublinear time algorithm for k-median, a problem similar to the k-means problem. At a high level, one of the algorithms described in this paper (see Algorithm 2) seems similar to [11]. But, there are a few significant differences between the two algorithm as discussed thoroughly in Section 4.2.

A recent work [10] also used the same online setting described in this paper. Their algorithm bears some similarity to Algorithm 2. However, they [10] considered the statistical question where there is an underlying distribution, as in [7]. In this statistical setting the ordering is not a factor. Also they have to assume that the example space is bounded. This nonexistence of outliers simplifies the solution.

In [2] the goal is to recover structure in the data, i.e., they assume that the data has some structure (“nice” or “perfect”) which they wish to discover. We, however, do not have any assumptions on the data and our algorithms function correctly under any dataset. Our goal is to design algorithms that are competitive with the optimal algorithm.

In the streaming model [4, 9] points arrive one after another, as in our online setting. However, there is an important difference between the two models. In the streaming model, it is allowed to choose a center after new points were observed, while we do not allow it. Many works [13, 6, 8] designed algorithms in this setting (in other words, they proved upper bounds) for k-means clustering. We give a full picture by designing lower and upper bounds that are optimal in the setting described in Section 2.

2 Preliminaries

In this paper we fix the desired number of clusters to be some constant \( k \). We want to design algorithms that minimize the k-means cost. When the algorithm is understood from the context we denote its cost by \( \text{cost}(\text{alg}) \). We focus on \( \Theta(1) \)-approximation algorithm which are formally defined next.

**Definition 1** (\( a \)-approximation). We say that a clustering algorithm is a \( a \)-approximation if for every series of \( n \) data points with probability at least 0.9 the following ratio is bounded by \( a > 1 \)

\[
\frac{\text{cost}(\text{alg})}{\text{cost}(\text{opt}_k)} \leq a.
\]
In the paper we focus on the case that \( a \) is some constant and the goal is to minimize the number of centers. We focus either on a fixed order of examples or random (uniform) order. Note that there are two possible sources of randomness: the randomness of the algorithm and the randomness of the order. The algorithm should succeed with probability 0.9 (this is some arbitrary constant close to 1) when considering the two sources together.

3 The Curious Case of \( k = 1 \)

In this section we focus on the case that there is only one center, i.e., \( k = 1 \). In the offline setting this problem is trivial, simply take \( \frac{1}{n} \sum_{i=1}^{n} x_i \), or a point that is closest to it as center. So it is surprising that in the online case there is a complex behavior.

1. If the order of the examples is random, \( O(1) \) centers suffices.
2. If the learner knows \( n \) in advance, then again \( O(1) \) centers suffices.
3. If none of the above cases hold, then \( \Theta(\log n) \) centers is both required and sufficient.

3.1 Random order

If the order of the points is random, then there is a simple algorithm that uses only one center while preserving a constant approximation: simply taking the first point.

Lemma 2. If the data points appear in a random order, there is an online algorithm that uses only one center and with probability at least 0.9 it holds that cost(alg) \( \leq 20 \cdot \text{cost(opt)} \).

The main tool in proving the theorem, which will also be useful in cases where \( k > 1 \), is the following known lemma that basically means that a random point in a cluster can be the center of this cluster.

Lemma 3. Let \( x_1, \ldots, x_n \in \mathbb{R}^d \) it holds that

\[
E_{j \in [n]} \left[ \sum_{i=1}^{n} \|x_i - x_j\|^2 \right] = 2 \sum_{i=1}^{n} \|x_i - \mu\|^2,
\]

where \( \mu = \frac{1}{n} \sum_{i=1}^{n} x_i \) is the optimal center and \( j \) is chosen uniformly at random from \([n] \).

For completeness, the proof of the lemma appears in the Appendix. To prove Lemma 2 we use Lemma 3 and Markov’s inequality.

3.2 Arbitrary order

We proceed to the case where the order of the data points is not random but it can appear in the worst order possible. In this case we witness another surprising result — it matters whether \( n \) is known or not. If \( n \) is known in advance, then the the algorithm can take one random point as center.

Theorem 4. There is an online algorithm that receives as input \( n \), the size of the dataset and \( n \) data points such that the following holds. For any order of the data points, the algorithm uses only one center and with probability at least 0.9 it holds that cost(alg) \( \leq 20 \cdot \text{cost(opt)} \).

The correctness of the algorithm is similar to Lemma 2. What if \( n \) is unknown in advance? We will prove that for any \( c > 1 \), the learner must use \( \lceil \log_c(n) \rceil \) centers for the algorithm to be a \( c \)-approximation, and we will prove that this is tight.

Theorem 5. For any clustering algorithm that is a \( c \)-approximation, for any integer \( n \) and any \( c > 1 \), there are \( n \) data points and an ordering of them such that the algorithm must take \( \Omega(\log_c(n)) \) centers with probability at least 0.8.
We remark that the constant 0.8 is just a number smaller than 0.9, which appeared in in the definition of a $c$-approximation, Section 2. The idea of the proof is to construct a dataset and an order on them such that the number of centers taken is $\Omega(\log_c n)$ for any $c$-approximation algorithm. The dataset is composed of $\Omega(\log_c n)$ groups. The groups are evenly spaced on the line (see Figure 2). The number of points in each group is exponential increasing. The points are given, group by group from smallest to largest. Since the learner does not know $n$, the current group can be the last. Thus, the algorithm must take a center from each group. The formal proof is in the appendix.

The lower bound we just proved is tight, as the next theorem proves.

**Theorem 6.** For any $c > 1$ there is an algorithm that obtains $O(c)$-approximation with $O(\log_c n)$ centers, no matter what the order is and even if $n$ is unknown.

Intuitively, since the algorithm does not know the value of $n$ it guess it and applies the algorithm from Theorem 4. The algorithm starts by assuming that $n$ is small ($n = 1$). Once more data is arrived, it increases the value of $n$ to $c$, and then to $c^2$ and so on. For each value of $n$, it applies the algorithm from Theorem 4, i.e., picks one random point in the next $n$ data points and takes only this point as a center. This algorithm uses only $O(\log_c n)$ centers. Intuitively, it is $O(c)$-approximation because in the last iterations most of points are still not received and the algorithm chooses a random point among them which yields a good center by Lemma 3.

**Algorithm 1** On line clustering with $k = 1$, $n$ unknown, arbitrary order

```
last = 0, n' = 1, i* = 1
for t = 1, \ldots do
    if t == last + i* then
        take $x_t$ as a center
    end if
    if t == last + n' then
        pick $i* \in [n']$ at random
        last = last + n', n' := c \cdot n'
    end if
end for
```

In the first line of Algorithm 1 it initialize the parameters. $last$ is the number of points we encountered before the current round, $n'$ is our current guess of the number of points, and $i^*$ is the point the algorithm will choose as center in the current round. In the loop, after the algorithm encounters $n'$ points, it increases $n'$ by a factor of $c$, we signal that we have encountered more examples by increasing last and a new point, $i^*$ to be our next center is picked.
4 The Case of Constant \( k \geq 2 \)

In this section we explore the case where the optimal clustering contains \( k \) centers, where \( k > 1 \) is any constant. The results in this sections are:

1. (arbitrary order) There is an order of the dataset such that any clustering algorithm must choose \( \Omega(n) \) points as centers, which is tight.

2. (random order) For a random order and known \( n \), \( \Theta(1) \) centers are sufficient, which is tight.

3. (random case) For a random order and unknown \( n \), \( \Theta(k \log \frac{n}{k}) \) centers are needed and sufficient. Again, this is a tight result.

4.1 Arbitrary order

The next theorem shows that if the order of the dataset can be arbitrary, then any \( c \)-approximation algorithm must basically take all the points in the dataset.

**Theorem 7.** For any integers \( k \geq 2 \) and \( n \), any scalar \( c > 1 \), and for any clustering algorithm that is a \( c \)-approximation (even if \( n \) is known) there are \( n \) points and an ordering of them such that the algorithm must take \( \Omega(n) \) centers with probability at least 0.8.

The idea of the proof is the following. We present \( n \) non-negative points on the line, i.e., \( x_t \in \mathbb{R} \) in increasing order. Each point is much further than the previous one. Since it is so further away it has to be taken, otherwise the rest of the points will be set to 0. In this case the maximal point has to be taken or the ratio \( \frac{\text{cost}(\text{alg})}{\text{cost}(\text{opt})} \) is arbitrarily large. Thus, all the points need to be taken if the algorithm is deterministic. If the algorithm is stochastic \( \Omega(n) \) of the points need to be taken with probability at least 0.8. The details appear in the appendix. Any algorithm can take \( n \) data points as centers and achieve minimal cost of 0. Thus, we conclude that the upper and lower bounds coincide when the order is arbitrary.

4.2 Random order and known \( n \)

Now let’s assume that the data arrives in a random order. If \( n \) is known we show an algorithm that uses \( \Theta(1) \) centers and is a \( \Theta(1) \)-approximation, for any constant \( k \). The idea is to use a small number of points to find \( k \) centers that are optimal with respect to the current points. This points cannot be taken as centers in retrospect in our framework, so in the second phase, the algorithm takes, among the remaining points, centers that are close to the centers found in the first phase. Since the order is random the centers chosen in the first phase are good centers for the entire dataset.

The first \( \alpha n \) of the points, \( \alpha \in (0, 1) \) is a constant to be chosen later, are saved in memory and the algorithm does not take any of them as centers. We denote this set by \( M_1 \). Since \( n \) is known, the algorithm can decide not to take \( \alpha n \) of the points as centers and still the cost will not increase by much. After the first \( \alpha n \) points arrive the algorithm use them to find \( k \) centers \( c_1^{M_1}, \ldots, c_k^{M_1} \) that are optimal centers for them. Then, then algorithm takes the following points:

1. Point that is close to \( c_1^{M_1} \), a point that is close to \( c_2^{M_1} \) and so on. At most \( k \) points of this type.

2. Points that are very far away, these points can form a cluster of their own. To define what are far points, we take another \( \alpha n \) points. A far point is one that is further than the \( 2\alpha n \) points.

More formally, the algorithm is described in Algorithm [2]

**Theorem 8.** For any constant integer \( k \geq 2 \), there is an algorithm that given \( n \), the size of the dataset, \( k \), and the dataset is given in a random order, with probability at least 0.9 uses \( \Theta(1) \) centers and \( \text{cost}(\text{alg}) \leq \Theta(1) \cdot \text{cost}(\text{opt}_k) \).
Algorithm 2 On line clustering with $k > 1$, $n$ known, random order

1: **phase 1: collect data**
2: $M_1 =$ save the first $\lfloor \frac{0.01}{k}n \rfloor$ points
3: find optimal offline clustering for $M_1$ with clusters $(C^M_1)_{i=1}^k$ and centers $(c^M_1)_{i=1}^k$
4: **phase 2: collect more data to define “far” points**
5: $M_2 =$ save the next $\lfloor \frac{0.001}{k}n \rfloor$ points
6: $R_{max}[i] = \max_{y \in C^M_1 \cap (M_1 \cup M_2)} \| y - c^M_1 \|$ (for each cluster in $C^M_1$ save max distance)
7: $flag = [False] * k$ (for each cluster denote if we took a center close to $(c^M_1)$ in phase 3)
8: **phase 3: take centers**
9: for $t = 2 \lfloor \frac{0.001}{k}n \rfloor + 1, \ldots, n$ do
10: $i^* = \arg \min_{i} \| x_t - c^M_1 \|$ (find closest center)
11: if $\| x_t - c^M_1 \| > R_{max}[i^*]$ then
12: (take points that are far away)
13: take $x_t$ as a center
14: end if
15: if $flag[i^*] == False$ then
16: (no point close to $c^M_1$ was taken yet ⇒ take a close point)
17: take $x_t$ as center
18: $flag[i^*] = True$
19: end if
20: end for

The proof idea is that the points collected in the first phase gives enough information to find all clusters that are not of constant size. Since the order is random, the points collected in the first phase are representative of the entire dataset. Since $n$ is known, the algorithm is able to take a small portion of the points and be sure that more points that can be good centers will be received. Clusters of constant size will not be chosen in the first phase, since $\alpha$ it is too small. These small clusters must be very far away, otherwise they will not be of constant size. Thus they will be taken in the third phase.

The work [11] designed a sublinear time algorithm for $k$-median, which is a similar to the $k$-means problem discussed in this paper. Their algorithm has some similarities Algorithm 2. At a very high level, both algorithms have two phases (i) solve the problem for large enough clusters using a few random points (ii) take as centers points that are far. But, there are a few significant differences between their algorithm and ours:

1. Parameter regime: “large” cluster in inherently different in the two algorithms. In [11], large means $O(\sqrt{n})$, as they cannot take more points for the algorithm to be with sublinear time. On the other hand, for Algorithm 2 “large” means some constant fraction because the algorithm is allowed to take only a constant number of centers in the second phase.

2. Centers from phase 1: [11] simply takes the centers that were chosen in phase 1. In our framework this is not allowed since once a center was observed the algorithm cannot retake it. To overcome this obstacle we take centers that are close to the centers chosen in phase 1. But then we need to decide what is close which complicates the analysis.

3. Far points: in [11], far points are the furthest points from the cluster defined in phase 1. We cannot use this definition in our framework. To resolve this issue, we add an intermediate step where the algorithm saves a constant fraction of number of points to set a bar that defines far. In the last phase, only points that are above the bar, are taken as centers. Note that the points in the first phase cannot be used to define the bar, as they were used to define the cluster.
4. Different analysis and algorithm: the outcome of all the above differences is that our correctness proof is distinct from the one in [11] and that we added an additional step besides the two phases used in [11].

4.3 Random order and unknown n

To simplify the presentation we start with the case where $k = 2$. Many of the ideas in the algorithm and the lower bound in the case of $k = 2$ are also applicable in the case of $k > 2$.

4.3.1 The case of $k = 2$

We prove that $\Theta(\log n)$ centers are needed and sufficient for $\Theta(1)$-approximation, when $n$ is unknown. We show a simple, bounded-memory algorithm that learns $\Theta(1)$-approximation and choose $\log(n) + 1$ centers. Our results are tight when $n$ is unknown.

**Theorem 9.** For any integer $n$, any scalar $c > 1$, and for any clustering algorithm that does not know what $n$ is and is a $c$-approximation, there are $n$ points such that the algorithm must take $\Omega(\log n)$ centers with probability at least $0.7$.

To prove the theorem we take the same dataset as in the proof of Theorem 7. In this construction, at each iteration the point with the maximal value has to be taken, otherwise the examples can stop and the algorithm will not be a $c$-approximation. If the order is random, there will be $\Omega(\log n)$ points that are maximal, as the probability that the $i$-th point to be maximal is $1/i$. Thus $\Omega(\log n)$ have to be taken as centers.

We now move on to prove a matching upper bound. We show a simple algorithm that saves only $\Theta(1)$ bits of memory (more specifically, it saves the first example and just one more number) and by choosing only $O(\log n)$ centers it is able to achieve $\Theta(1)$-approximation.

**Algorithm 3** On line clustering with $k = 2$, $n$ unknown, random order

```
| take \(x_1\) as a center |
| \(x := x_1\) (save first data point) |
| max_dis := 0 |
| for \(t = 2, \ldots, n\) do |
| \(\text{if } \|x_t - x\| > \text{max_dis} \text{ then} \) |
| \(\text{take } x_t \text{ as a center} \) |
| \(\text{max_dis := } \|x_t - x\| \) |
| end if |
| end for |
```

**Theorem 10.** There is an online algorithm such that if the examples are received with random order ($n$ does not have to be known) then with probability at least 0.9 it holds that number of centers is $O(\log n)$ and $\text{cost}(alg) \leq 201 \cdot \text{cost}(opt_2)$.

To prove the theorem we show two things. First, that not too many centers will be chosen, the reason is that the $i$-th example is chosen as a center only if it is the furthest from $x$ (recall that $x$ is the first example). This will happen with probability $\frac{1}{i-1}$. Thus, the expected number of centers is the harmonic series which is about $\log n$. The second thing that the theorem shows is that with probability at least 0.9 the algorithm is a $\Theta(1)$-approximation. To prove that we consider two separate cases: either the two clusters are close to each other or not. It the two clusters are close we can treat them as one cluster. Using Lemma the first point is a good center. If the two clusters are far apart then the most probably the first point from the second cluster is the furthest away from $x$ among all points reached so far.
4.3.2 The case of constant \( k > 2 \)

For the case of random order, unknown \( n \) and general \( k \), we will show a lower bound of \( \Omega(k \log \frac{n}{k}) \) on the number of centers.

**Theorem 11.** *For any scalar \( c > 1 \), integers \( k \geq 2 \) and \( n \), and for any clustering algorithm that does not know what \( n \) is and is a \( c \)-approximation, there are \( n \) points that arrive uniformly at random and the algorithm must take \( \Omega(k \log \frac{n}{k}) \) centers with probability at least 0.7.*

The proof idea is similar to Theorem 9, where we had a series of points in \( \mathbb{R}_{\geq 0} \) with distance that is keep increasing. To adapt this lower bounds for \( k > 2 \) we create \( k \) such serieses each with \( n/k \) points. Each group \( i \) will correspond to one dimension \( i \). The points are zero everywhere except dimension \( i \). In the dimension \( i \) the values are the same as the series defined in Theorem 9. The same argument as in Theorem 9 applies here as points from different groups are far away from each other.

We now move on to the upper bound when \( k \) is a constant. We present an algorithm that uses \( O(\log n) \) centers and is \( \Theta(1) \)-approximation, Algorithm 5. The algorithm uses the known farthest-first-traversal algorithm as a subroutine, for completeness it appears as Algorithm 4.

**Algorithm 4** Farthest-first-traversal\((M, s, k)\)

1: \( S = \{s\} \)
2: \( \textbf{for } t = 2, \ldots, k \textbf{ do} \)
3: \( v = \arg \max_{x \in M - S} \min_{y \in S} ||x - y|| \)
4: \( S = S \cup \{v\} \)
5: \( \textbf{end for} \)
6: \( \textbf{return } S \)

The farthest-first-traversal algorithm takes \( k \) (a parameter) points that are far away from each other in a given dataset \( M \). Specifically, it starts with some point \( s \in M \) that is given as an input. Then it takes the point \( x_2 \in M \) that it furthest away from \( s \). Then a point \( x_3 \) that maximizes the distance to \( S \), where the distance is equal to \( \text{dis}(S, x) = \min_{y \in S} ||y - x|| \).

For our purposes, the main claim we need from farthest-first-traversal is that if \( S \) was returned from farthest-first-traversal, then the distance of any point \( x \) to \( S \) is smaller than the distance between any two points inside \( S \), as the next lemma proves.

**Lemma 12.** *Suppose \( S = \text{Farthest-first-traversal}(M, s, k) \) and \( x \in M - S \), then*

\[
\min_{y \in S} ||x - y|| \leq \min_{y_1, y_2 \in S} ||y_1 - y_2||.
\]

The online clustering algorithm, Algorithm 5, saves in memory all the points encountered so far. To decide whether to take the current point as a center or not, it uses the farthest-first-traversal algorithm that picks \( k \) far away points. The last point is taken as a center if it is chosen as one of the \( k \) points.

**Algorithm 5** Online clustering with \( k \geq 2 \), \( n \) unknown, random order

1: Take \( x_1, \ldots, x_k \) as centers and save them in \( M \)
2: \( \textbf{for } t = k + 1, \ldots \textbf{ do} \)
3: \( S = \text{Farthest-first-traversal}(M, x_1, k) \)
4: \( \textbf{if } x_t \in S \textbf{ then} \)
5: \( \text{take } x_t \text{ as a center} \)
6: \( \textbf{end if} \)
7: \( M = \text{append}(M, x_t) \)
8: \( \textbf{end for} \)
Theorem 13. There is an online algorithm such that for any constant $k$, if the examples are received with random order ($n$ does not have to be known) then with probability at least 0.9 the algorithm uses $O(\log n)$ centers and achieves $\Theta(1)$-approximation.

To prove the theorem we show two things. Both are similar to the proof of Theorem 10. First, that not too many centers will be chosen as centers, the reason is that the $i$-th example, $i > k$, is chosen as a center only if it is one of the $k$ points defining the unique farthest-first-traversal for the first $i$ points. This will happen with probability $\frac{k}{i}$. Thus, the expected number of centers can be calculated using the harmonic series which is about $k \log \frac{n}{k}$.

The second thing that the theorem shows is that with high probability the algorithm finds a $\Theta(1)$-approximation. Focus on a cluster $C'$. We want to show that the first point from this cluster $x \in C'$ will be chosen as a center. Using Lemma 12, we know that $x$ will be chosen as a center if it far from points from other clusters. So, if $C'$ is far from all other clusters, $x$ will be chosen as a center, otherwise we can add $C'$ to another cluster.

A previous work [12] described an algorithm for online k-means that used the same online framework as this paper. The algorithm suggested is an adaptation of the k-means++ algorithm [6] to the online case. Inherently, their algorithm cannot get the optimal bound, as will be explained next, in contrast to our algorithm that is able to achieve optimal results. In the k-means++ algorithm a point is chosen with probability $\frac{\text{distance to points chosen so far}}{Z}$, where $Z$ is the normalization factor. In the online case $Z$ is unknown as it depends on future points and thus cannot be calculated. Instead [12] suggests to start with some small $Z$ and to keep increasing it by a factor of 2. This implies that the number of centers taken by the algorithm depends on the scale of the data. As an illustration, if the distances are about $s$, then $\log s$ centers are needed till the “probability” to take a center will be less than 1. The scale of the data is summarized in the aspect ratio parameter $\gamma = \frac{\max_{v, v' \in D} \|v - v'\|}{\min_{v, v' \in D} \|v - v'\|}$. They [12] take $O(\log n(\log \gamma + \log n))$ centers and achieve $O(\log n)$-approximation. Inherently this algorithm depends on $\gamma$ as it must reach the scale of the data to be able to take only a small subset of points as centers. In this paper, we improve both the quality of the approximation and the number of centers by the algorithm to the optimal values (see Algorithm 5).

5 Conclusions and Open Problems

In this paper we showed optimal bounds for online clustering when the number of centers, $k$, is a constant, i.e., we showed matching upper and lower bounds. We uncovered a complex behavior of the online setting compared to the offline setting. Specifically in the former, new factors arise: the order of the dataset and knowing in advance the size of the dataset. This factors have enormous effects on online algorithms as illustrated in Figure 1. In the paper, we designed new algorithms that are able to learn under different circumstances. Among them are algorithms that are able to use only $\Theta(1)$ centers if $n$ is known, and $O(\log(n))$ centers if $n$ is unknown, both are optimal bounds.

We have reached optimal bounds for constant $k$. Are the algorithms presented in this work also optimal for non constant $k$? We analyzed the case where the order of the examples can be arbitrary and we also analyzed the case where the order is uniform. What if the order of the examples is defined using other distributions? Is there a simple characterization?

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6 Proofs

This section includes the proofs for the theorems and claims that appear in the paper.

Proof of Lemma 3

Proof.

\[ E_{j \in [n]} \left[ \sum_{i=1}^{n} \| x_i - x_j \|^2 \right] = \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{n} \| x_i - x_j \|^2 \]

\[ = \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{n} \left( \| x_i \|^2 + \| x_j \|^2 - 2 \langle x_i, x_j \rangle \right) \]

\[ = \sum_{i=1}^{n} \| x_i \|^2 + \sum_{j=1}^{n} \| x_j \|^2 - \frac{2}{n} \sum_{i,j} \langle x_i, x_j \rangle \]

\[ = 2 \sum_{i=1}^{n} \| x_i \|^2 - \frac{2}{n} \sum_{i,j} \langle x_i, x_j \rangle \]

\[ \sum_{i=1}^{n} \| x_i - \mu \|^2 = \sum_{i=1}^{n} \left( \| x_i \|^2 + \| \mu \|^2 - 2 \langle x_i, \mu \rangle \right) \]

\[ = \left( \sum_{i=1}^{n} \| x_i \|^2 \right) + n \| \mu \|^2 - 2 \sum_{i=1}^{n} \langle x_i, \mu \rangle \]

\[ = \left( \sum_{i=1}^{n} \| x_i \|^2 \right) + n \| \mu \|^2 - 2 \langle n\mu, \mu \rangle \]

\[ = \left( \sum_{i=1}^{n} \| x_i \|^2 \right) + n \| \mu \|^2 - 2n \| \mu \|^2 \]

\[ = \left( \sum_{i=1}^{n} \| x_i \|^2 \right) - n \| \mu \|^2 \]

\[ = \left( \sum_{i=1}^{n} \| x_i \|^2 \right) - n \left( \frac{1}{n} \sum_{i=1}^{n} x_i \right)^2 \]

\[ = \left( \sum_{i=1}^{n} \| x_i \|^2 \right) - \frac{1}{n} \sum_{i,j} \langle x_i, x_j \rangle \]

Proof of Theorem 5

Proof. The examples are 1, 2, 3, . . . and so on. Example with value i appears \((7c)^i\) times (except the last value that might appear less than that). The points are given, group by group from smallest to largest. Note that there are \(\Omega(\log_c n)\) different examples. We will show that any \(c\)-approximation algorithm must take \(\Omega(\log_c n)\) examples as centers with probability at least 0.8. This will finish the proof.

There are two cases: either for each example (out of the \(\Omega(\log_c n)\) different examples) the probability it will be taken as a center (at least one of the \((7c)^i\) instances of this example) is at least 0.9 or not. First
case implies, using Claim 15 that with probability at least 0.8 the algorithm takes as centers at least 0.5 of the examples, i.e., it takes \( \Omega(\log n) \) examples as centers. Second case implies that there is an example \( t^* \) such that probability that the algorithm will not take as centers is more than 0.1. We will show that the algorithm is not a \( c \)-approximation and we will reach a contradiction.

We focus on the series \( 1, \ldots, t^* \). Note that since \( n \) is unknown the algorithm works the same up until \( t^* \), no matter if there are more examples after \( t^* \) or not. This means that also for the shorter series that includes only examples \( 1, \ldots, t^* \), the probability that \( t^* \) is taken as a center is smaller than 0.1. We will prove that if \( t^* \) is not taken, which happens with probability more than 0.1, it holds that \( c \cdot \text{cost}(\text{opt}_1) \leq \text{cost}(\text{alg}) \), which is a contradiction to the assumption that the algorithm is a \( c \)-approximation (i.e., that \( \text{cost}(\text{alg}) \leq c \cdot \text{cost}(\text{opt}_1) \)) with probability at least 0.9.

The cost of the algorithm is at least \( \text{cost}(\text{alg}) \geq (7c)^{t^*} \), as there are \((7c)^{t^*}\) examples with value \( t^* \) and the distance to the closest center is at least \( c \). We will prove that if \( t^* \) is not taken, which happens with probability more than 0.1, it holds that \( c \cdot \text{cost}(\text{opt}_1) \leq \sum_{i=1}^{t^*} (7c)^i(t^* - i)^2 \). Using Claim 14 we have that

\[
\text{cost}(\text{opt}_1) \leq 6(7c)^{t^* - 1}
\]

This implies that

\[
c \cdot \text{cost}(\text{opt}_1) \leq 6c(7c)^{t^* - 1} < (7c)^{t^*} \leq \text{cost}(\text{alg}),
\]

which is a contradiction to the \( c \)-approximation of the algorithm. \( \square \)

**Claim 14.** For any scalar \( q \geq 6 \) and an integer \( n \geq 1 \) it holds that

\[
\sum_{i=1}^{n} q^i(n - i)^2 \leq 6 \cdot q^{n-1}.
\]

**Proof.**

\[
\sum_{i=1}^{n} q^i(n - i)^2 = q^n \sum_{i=1}^{n} \left(\frac{1}{q}\right)^i(n - i)^2
\]

\[
= q^n \sum_{j=0}^{n-1} \left(\frac{1}{q}\right)^j j^2
\]

\[
\leq q^n \sum_{j=1}^{n-1} \left(\frac{3}{q}\right)^j
\]

\[
\leq q^n \cdot \frac{3q}{1 - 3/q}
\]

\[
\leq q^n \cdot \frac{6}{q}
\]

where in the first equality we multiply and divide by \( q^n \), in the second equality we reverse the order of summation, in the first inequality we use the bound

\[
\left(\frac{1}{q}\right)^j j^2 \leq \left(\frac{3}{q}\right)^j
\]

which is true for any \( j \geq 0 \), the second inequality uses the known bound for sum of a geometric series, and in the last inequality we use the bound \( 1 \leq 2(1 - 3/q) \), which is true for \( q \geq 6 \). \( \square \)

**Claim 15.** Suppose there are \( n \) events \( A_1, \ldots, A_n \) such that for every \( i \in [n] \) it holds that \( \Pr(A_i) \geq 0.9 \). Denote by \( G \) the event that there is \( I \subseteq [n] \) with \( |I| \geq 0.5n \) and \( \cap_{i \in I} A_i \) happens. It holds that \( \Pr(G) \geq 0.8 \).
We now focus on iteration \( i^* \) and in this iteration \( n' = c^i = 1 \). The last iteration \( i^* \) is the first where \( \sum_{i=0}^{i'} c^i > n \) for the first time. This means that

\[
\frac{c^{i^*} - 1}{c - 1} \leq n < \frac{c^{i^*+1} - 1}{c - 1}.
\]

We now focus on iteration \( i^* - 1 \) where the value of \( n' \) is \( c^{i^*-1} \). From Equation 2 we know that \( n \leq \frac{n' \cdot c^{i+1}}{c-1} \Rightarrow \frac{n' \cdot c^{i+1}}{c-1} \leq n' \Rightarrow \frac{c^{i+1}}{c-1} \leq n' \) (in the last equation we used that w.l.o.g \( c \geq 2 \)). In words, in the one before the last iteration \( n' \) is big compared to \( n \), it’s at least a fraction \( 1/2c \) of \( n \).

From Lemma 3 and Markov’s inequality we get that for at most \( \frac{1}{20c} \) of the data points \( x \) it holds that \( \text{cost}(x) > 40 \cdot c \cdot \text{cost}(\text{opt}_1) \). Thus with probability at least 0.9 the learner chooses a data point \( x \) out of the \( n' \) points at iteration \( i^* - 1 \) with \( \text{cost}(x) \leq 40c \cdot \text{cost}(\text{opt}_1) \).}

Proof of Theorem 6

Proof. It is easy to bound the number of centers Algorithm 1 uses, it’s bounded by the number of times \( n' \) is increased. The latter is bounded by \( \lceil \log_c(n) \rceil \).

Next we prove that the algorithm is an \( O(\cdot) \)-approximation. For ease of notation, iteration 0 is the first iteration and in this iteration \( n' = c^0 = 1 \). The last iteration \( i^* \) is the one where \( \sum_{i=0}^{i'} c^i > n \) for the first time. This means that

\[
\frac{c^{i^*} - 1}{c - 1} \leq n < \frac{c^{i^*+1} - 1}{c - 1}.
\]

We now focus on iteration \( i^* - 1 \) where the value of \( n' \) is \( c^{i^*-1} \). From Equation 2 we know that \( n \leq \frac{n' \cdot c^{i+1}}{c-1} \Rightarrow \frac{n' \cdot c^{i+1}}{c-1} \leq n' \Rightarrow \frac{c^{i+1}}{c-1} \leq n' \) (in the last equation we used that w.l.o.g \( c \geq 2 \)). In words, in the one before the last iteration \( n' \) is big compared to \( n \), it’s at least a fraction \( 1/2c \) of \( n \).

From Lemma 3 and Markov’s inequality we get that for at most \( \frac{1}{20c} \) of the data points \( x \) it holds that \( \text{cost}(x) > 40 \cdot c \cdot \text{cost}(\text{opt}_1) \). Thus with probability at least 0.9 the learner chooses a data point \( x \) out of the \( n' \) points at iteration \( i^* - 1 \) with \( \text{cost}(x) \leq 40c \cdot \text{cost}(\text{opt}_1) \).

Proof of Theorem 7

Proof. We will define a series of points in \( \mathbb{R} \), \( 0 = x_1 < x_2 < \ldots < x_n \) such that any \( c \)-approximation algorithm must take \( \Omega(n) \) points as centers with probability at least 0.8. This will finish the proof.

If the probability that each example is taken as center is at least 0.9, then with probability at least 0.8 the algorithm takes 0.5\( n = \Omega(n) \) points as centers, see Claim 16, and the theorem follows. Focus on the first example \( x_{i^*} \) such that the probability that the algorithm will not take it as a center is more than 0.1. We will show that the algorithm is not a \( c \)-approximation. We focus on the series of points \( x_1, \ldots, x_{i^*}, 0, 0, \ldots, 0 \). Note that up until example \( x_{i^*} \) the two Series \( (x_1, x_2, \ldots, x_n) \) and \( (x_1, \ldots, x_{i^*}, 0, 0, \ldots, 0) \) are the same, thus the probability that \( x_{i^*} \) will not taken as a center is more than 0.1.
One can take \( x_{t^*} \) and 0 as centers, the optimal cost can only be smaller, thus

\[
cost(\text{opt}_k) \leq \sum_{t=2}^{t^*-1} \|x_t - x_1\|^2 = \sum_{t=2}^{t^*-1} x_t^2.
\]

Since the algorithm did not take \( x_{t^*} \) as a center we have that

\[
cost(\text{alg}) \geq \|x_{t^*} - x_{t^*-1}\|^2.
\]

For this lower bound to work, we take the series of examples such the following strict inequality holds

\[
cost(\text{alg}) \geq (x_{t^*} - x_{t^*-1})^2 > c \cdot \sum_{t=2}^{t^*-1} x_t^2 \geq c \cdot cost(\text{opt}_k).
\]

In different words,

\[
x_{t^*} > x_{t^*-1} + \sqrt{c \cdot \sum_{t=2}^{t^*-1} x_t^2}
\]

And we are done since with probability more than 0.1 it holds that \( cost(\text{alg}) > c \cdot cost(\text{opt}_k) \), i.e., the algorithm is not a \( c \)-approximation.

**Claim 17.** For any \( u, v \in \mathbb{R}^d \) it holds that

\[
\|v + u\|^2 \leq 2 \|v\|^2 + 2 \|u\|^2.
\]

**Proof.**

\[
\|v + u\|^2 = \sum_i (v_i + u_i)^2 = \sum_i v_i^2 + \sum_i u_i^2 + 2 \sum_i v_i \cdot u_i
\]

\[
= \|v\|^2 + \|u\|^2 + 2 \langle v, u \rangle
\]

\[
\leq \|v\|^2 + \|u\|^2 + 2 \|v\| \|u\|
\]

\[
\leq 2 \|v\|^2 + 2 \|u\|^2
\]

where the first inequality follows from Cauchy–Schwarz inequality and the second inequality follows from the following inequality

\[
0 \leq (\|u\| - \|v\|)^2 = \|v\|^2 + \|u\|^2 - 2 \|v\| \|u\|
\]

**Proof of Theorem 8**

**Proof.** Number of example is \( n \), number of examples the algorithm saves in the first phase is equal to 

\[|M_1| = \alpha n, \text{ where } \alpha := \left\lfloor \frac{k \cdot \nu \cdot \alpha}{n} \right\rfloor, \text{ and } |M_2| = \alpha n \text{ is the number of points received in phase 2.}
\]

In the proof we consider three clusterings. The first \( C_{M_1} = (C^i_{M_1})_{i=1}^k \) is the optimal clustering with respect to the points \( M_1 \) the algorithm saves in the first phase. The second is the optimal clustering \( C^* = (C^*_i)_{i=1}^k \) induced by the entire dataset. And the third \( C = (C_i)_{i=1}^k \) is induced by all the centers taken by the algorithm in the third phase. We prove that with probability at least 0.9 the following two claims hold for any constant \( k \)

1. The number of centers the algorithm takes is \( \Theta(1) \)
2. \( cost(C) \leq \Theta(1) \cdot cost(C^*) \),
Bounding number of centers: Let us start with the first argument, bounding the number of centers the algorithm takes. The algorithm takes as a center two types of points. Either close (Line 17 in Algorithm 2) or far (Line 13). Bounding the number of close points by \( k \) is easy, as it follows immediately from the definition of the algorithm — the algorithm only takes one close point per cluster. The interesting claim is bounding the number of centers that are far.

The key idea is that \( M_2 \) is a good representation of the clustering \( C^{M_1} \) in the sense that the algorithm received a constant fraction of points from each large enough cluster in \( C^{M_1} \) (this is true using Claim 22). Thus, the probability that a point will be taken as a far point, i.e., being further from the center than the first point from this cluster will be a good representative of this cluster (2) does not contain points from other clusters and then the first point from this cluster will be a good representative of this cluster (2) there are many points from different cluster \( C^* \). But then we prove that

More formally, we separate the analysis depending on whether the points left from the cluster after phase 1 is small, \(|C_i^{M_1} - M_1| \leq \frac{400k}{\alpha} \), or large. If it’s small, i.e., \(|C_i^{M_1} - M_1| \leq \frac{400k}{\alpha} \), then the algorithm takes at most all the points remaining in this cluster, \( \frac{400k}{\alpha} = O(k^2) \). Since there are at most \( k \) small clusters (because there are \( k \) clusters in total), the total number of centers taken because of small clusters is \( O(k^3) \).

For large clusters, \(|C_i^{M_1} - M_1| \geq \frac{400k}{\alpha} \), we will use Claim 22 and union bound to show that, with probability at least 0.99, for all large clusters, \( C_i^{M_1} - M_1 \) the following holds

\[
\frac{\alpha}{2} |C_i^{M_1} - M_1| \leq |(C_i^{M_1} - M_1) \cap M_2|.
\]

A point is taken as a far point if it’s further away from all points in this cluster compared to points received in the first and second phase. In different words, a point is taken as a far center if it is farthest away from \( C_i^{M_1} \) compared to all points in \( C_i^{M_1} \cap (M_1 \cup M_2) \). Thus, from Inequality 2 the probability of taking a point as far, happens with probability at most

\[
\alpha |C_i^{M_1} - M_1|.
\]

There are at most \( (1 - \frac{\alpha}{2})|C_i^{M_1} - M_1| \) points in \( C_i^{M_1} \) that are not saved in the first phase and not in the second phase. So in total, the expected number of points for cluster \( C_i^{M_1} \) taken as far points is bounded by

\[
\frac{2}{\alpha |C_i^{M_1} - M_1|} \cdot \left(1 - \frac{\alpha}{2}\right) |C_i^{M_1} - M_1| = \frac{2 - \alpha}{\alpha}.
\]

Use Markov’s inequality to show that with probability at least 0.99 all the large clusters cause at most \( O(\frac{2 - \alpha}{\alpha} \cdot k^2) = O(k^3) \) far centers.

Summing the two cases, the number of centers taken as centers by the algorithm is \( O(k^3) \) with probability at least 0.99.

Bounding cost: Now we move on to the second step in the proof, bounding the cost of clustering defined by the algorithm, \( C \). We want to prove that \( cost(C) \) is at most some function of \( k^* \) times \( cost(opt_k) \). We will prove a stronger result. We will prove that when the algorithm is given \( k^* \) and a dataset it performs well for any integer \( k \leq k^* \). Namely, we prove that for any integer \( k \leq k^* \)

\[
\text{cost(alg)} \leq \left( \frac{808808}{\alpha} \right)^k (k!)^4 \cdot \text{cost(opt)_k}.
\]

We prove this claim by induction on \( k \). For \( k = 1 \) the claim follows immediately, similarly to Algorithm 1. We separate the analysis into two cases depending on whether \( C^*_i \) is large or not. If it small, we will show the algorithm takes a member from \( C^*_i \) and it is good enough. We focus now on an optimal clustering \((C^*_1, \ldots, C^*_k)\). We start with the case that \( C^*_i \) is a large cluster.

\( C^*_i \) is large: The high level idea is that a cluster in \( C^{M_1} \) that contains many of the points of \( C^*_i \) either (1) does not contain points from other clusters and then the first point from this cluster will be a good representative of this cluster (2) there are many points from different cluster \( C_j^* \), but then we prove that
$C_i^*$ can be merged into $C_j^*$ without harming the cost by much, and then the claim follows by the induction hypothesis.

We first define the set of good points for the cluster $C_i^*$ as the set of points that taking them as a center will not increase the cost of the cluster by much. More formally, 

$$\text{Good}_i = \left\{ x \mid \sum_{y \in C_i^*} \| x - y \|^2 \leq \frac{100}{\alpha} \sum_{y \in C_i^*} \| y - \mu^*(i) \|^2 \right\},$$

where $\mu^*(i) = \frac{1}{|C_i^*|} \sum_{y \in C_i^*} y$ is the optimal center for cluster $C_i^*$. From Lemma 8 and Markov's inequality we know that $\text{Good}_i$ is large. Specifically,

$$\frac{|\text{Good}_i|}{|C_i^*|} \geq 1 - \frac{\alpha}{50k}.$$ 

The first auxiliary claim we prove states that if $C_i^*$ is large enough, then any cluster in $C^{M_1}$ with center $c_{M_1}$ that contains at least $b|C_i^*|$ members, its center $c_{M_1}$ is a good center for the entire cluster $C_i^*$.

**Claim 18.** For any cluster $|C_i^*| \geq \frac{800k}{\alpha}$ and any cluster $C_i^{M_1}$ with center $c_{M_1}$ that contains at least $b|C_i^*|$ points from $C_i^*$ the following holds. With probability at least $\frac{1}{100k}$ it holds that

$$\sum_{y \in C_i^*} \| y - c_{M_1} \|^2 \leq \frac{4000k}{\alpha \cdot \text{cost}(\text{opt}_k)}.$$ 

**Proof.** There are two steps in the proof: 1. there is a center in $C_i^{M_1}$ that is a good representative to $C_i^*$ 2. this implies that also $c_{M_1}$ is a good representative, because many points in $C_i^*$ are in $C_i^{M_1}$ and not in the other good cluster.

We know, from Claim 22, that with probability at least $\frac{1}{100k}$, there are at least $\frac{\alpha}{2k} |\text{Good}_i|$ points in $\text{Good}_i$ that are received in phase 1 and they are all in the same cluster $C_i^{M_1}$ with center $c_{M_1}$. We will show that this center is a good representative for the entire cluster $C_i^*$.

Among all points in $\text{Good}_i \cap M_1 \cap C_i^{M_1}$ denote by $x^{\text{good}}$ the one that it’s closest to $c_{M_1}$. By Claim 17

$$\sum_{x \in C_i^*} \| x - c_{M_1} \|^2 \leq 2 \sum_{x \in C_i^*} \| x - x^{\text{good}} \|^2 + 2 \sum_{x \in C_i^*} \| x^{\text{good}} - c_{M_1} \|^2$$

The first term is bounded by $\frac{200k}{\alpha} \cdot \text{cost}(\text{opt}_k)$ by the definition of $\text{Good}_i$. The second term is equal to

$$2|C_i^*| \| x^{\text{good}} - c_{M_1} \|^2 = \frac{4k}{\alpha} \sum_{x \in \text{Good}_i \cap C_i^{M_1}} \| x^{\text{good}} - c_{M_1} \|^2 \leq \frac{4k}{\alpha} \sum_{x \in \text{Good}_i \cap M_1 \cap C_i^{M_1}} \| x - c_{M_1} \|^2 \leq \frac{4k}{\alpha} \cdot \text{cost}(\text{opt}_k),$$

where in the last inequality we used the fact that $C_i^{M_1}$ is optimal to the points in $M_1$ and in particular its cost is better than the cost of the entire data set.

So in total

$$\sum_{x \in C_i^*} \| x - c_{M_1} \|^2 \leq \frac{204k}{\alpha} \cdot \text{cost}(\text{opt}_k) \tag{4}$$

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Let’s move on to the second step in the proof. We will show that since many point are closer to $c_i^{M_1}$ than $c_i^{M_1}$ and $c_i^{M_1}$ is a good representative of $C_i^*$, then $c_i^{M_1}$ is also a good representative of $C_i^*$. By Claim 17

$$\sum_{x \in C_i^*} \left\| x - c_i^{M_1} \right\|^2 \leq 2 \sum_{x \in C_i^*} \left\| x - c_i^{M_1} \right\|^2 + 2 \sum_{x \in C_i^*} \left\| c_i^{M_1} - c_i^{M_1} \right\|^2$$

We use the fact that for each $x \in C_i^{M_1}$ it holds that $\left\| x - c_i^{M_1} \right\| \leq \left\| x - c_i^{M_1} \right\|$ to bound the second sum, which is equal to

$$2|C_i^*| \left\| c_i^{M_1} - c_i^{M_1} \right\|^2 \leq \frac{4}{b} \sum_{x \in C_i^* \cap C_i^{M_1}} \left\| c_i^{M_1} - c_i^{M_1} \right\|^2$$

$$\leq \frac{8}{b} \sum_{x \in C_i^* \cap C_i^{M_1}} \left\| c_i^{M_1} - x \right\|^2 + \left\| x - c_i^{M_1} \right\|^2$$

$$\leq \frac{16}{b} \sum_{x \in C_i^* \cap C_i^{M_1}} \left\| c_i^{M_1} - x \right\|^2$$

So in total, using Equation [4], we have that $c_i^{M_1}$ is a good center for $C_i^*$

$$\sum_{x \in C_i^*} \left\| x - c_i^{M_1} \right\|^2 \leq \frac{18}{b} \sum_{x \in C_i^* \cap C_i^{M_1}} \left\| c_i^{M_1} - x \right\|^2$$

$$\leq \frac{18}{b} \cdot \frac{204k}{\alpha} \cdot \text{cost}(\text{opt}_k) \leq \frac{400k}{b\alpha} \cdot \text{cost}(\text{opt}_k)$$

In the next auxiliary claim we prove that the last claim implies that $c_i^{M_1}$ is close to $c_i^*$.

**Claim 19.** For any cluster $|C_i^*| \geq \frac{800k}{\alpha}$ and any cluster $C_i^{M_1}$ with center $c_i^{M_1}$ that contains at least $b|C_i^*|$ points, with probability at least $\frac{1}{100k}$ it holds that

$$|C_i^*| \left\| c_i^{M_1} - c_i^* \right\|^2 \leq \frac{4002k}{b\alpha} \cdot \text{cost}(\text{opt}_k).$$

**Proof.** Use Claim 13

$$|C_i^*| \left\| c_i^* - c_i^{M_1} \right\|^2 \leq \frac{2}{b} \sum_{x \in C_i^{M_1} \cap C_i^*} \left\| c_i^* - x \right\|^2 + \left\| x - c_i^{M_1} \right\|^2$$

$$\leq \frac{2}{b} \cdot \text{cost}(\text{opt}_k) + \frac{4000k}{b\alpha} \cdot \text{cost}(\text{opt}_k)$$

The third and last auxiliary claim shows that if there is a cluster $C_i^{M_1}$ that contains many points from two different optimal clusters, then these clusters can be merged, without harming the cost by much.

**Claim 20.** For any clusters $C_i^*$, $C_j^*$ with $|C_j^*| \geq |C_i^*| \geq \frac{800k}{\alpha}$ if there is a cluster in $C^{M_1}$ that contains at least $a|C_i^*|$ points from $C_i^*$ and at least $b|C_j^*|$ from $C_j^*$, then with probability at least $\frac{1}{100k}$ it holds that

$$\text{cost}(\text{opt}_{k-1}) \leq \left( 4 + \frac{8004k}{b\alpha} + \frac{8004k}{a\alpha} \right) \text{cost}(\text{opt}_k)$$
Proof. We want to bound the cost of the following clustering with \( k - 1 \) centers: \( c_1^*, \ldots , c_k^* \) without \( c_i^* \) and all points in \( C_i^* \) will be assigned to \( c_i^* \). The cost is equal to

\[
\sum_{x \in C_i^*} \|x - c_j^*\|^2 + \sum_{r \neq i} \sum_{x \in C_i^*} \|x - c_i^*\|^2
\]

Let us bound the first sum using Claim 19 and Claim 17

\[
\sum_{x \in C_i^*} \|x - c_j^*\|^2 \leq 4 \sum_{x \in C_i^*} \|x - c_i^*\|^2 + 2 \sum_{x \in C_i^*} \|c_i^* - c_M^*\|^2 + 2 \sum_{x \in C_i^*} \|c_M^* - c_j^*\|^2
\]

The first sum is bounded by \( 4 \cdot \text{cost}(\text{opt}_k) \). The second and third terms are bounded by \((\frac{8004k^4}{\alpha} + \frac{8004k^4}{\alpha}) \cdot \text{cost}(\text{opt}_k)\).

Now are are ready to bound the cost for large clusters, \( |C_i^*| \geq \frac{8000k^4}{\alpha} \). There is a cluster \( C_i^M \) that contains at least \( \frac{|C_i^*|}{k} \) points from \( C_i^* \). If this cluster contains at least \( \frac{|C_i^*|}{100k^2} \) points from a different cluster \( C_j^* \), then Claim 20 implies that

\[
\text{cost}(\text{opt}_{k-1}) \leq \frac{80080k^4}{\alpha} \cdot \text{cost}(\text{opt}_k).
\]

Using the induction hypothesis

\[
\text{cost}(\text{alg}) \leq \left(\frac{80080}{\alpha}\right)^{k-1} ((k-1)!)^4 \cdot \text{cost}(\text{opt}_{k-1}).
\]

Together we have

\[
\text{cost}(\text{alg}) \leq \left(\frac{80080k^4}{\alpha}\right)^{k} (k!)^4 \cdot \text{cost}(\text{opt}_k).
\]

In case that there are at most \( \frac{|C_i^*|}{100k^2} \) point from different clusters, there are at most \( \frac{|C_i^*|}{100k^2} \) points that that are not in \( C_i^* \). Thus, the probability that the first point will not from \( C_i^* \) is at most \( \frac{1}{100k^2} \). The probability that the first point will not be in \( \text{Good}_i \) is at most \( \frac{1}{50k} \). And we are done.

\( C_i^* \) is small: Now let’s focus on a small cluster \( C_i^* \), where \( |C_i^*| < \frac{8000k^4}{\alpha} \). Even if \( |C_i^*| \) only contains 1 point, the probability that the algorithm takes all members in \( C_i^* \) in the first phase is small, it’s at most \( \alpha < \frac{1}{100k} \). Denote by \( \text{dis}_{\text{outer}} \) the distance between \( C_i^* \) to any cluster other than \( C_i^* \), i.e.,

\[
\text{dis}_{\text{outer}} = \min_{x \in C_i^*} \min_{y \in \text{clusters} \neq C_i^*} \|x, y\|.
\]

Denote the cluster that is closest to \( C_i^* \) by \( C_j^* \). Denote by \( \text{max}_{\text{dis}_{\text{inner}}} \) the maximal distance between any two point in the same cluster not in \( C_i^* \), i.e.,

\[
\text{max}_{\text{dis}_{\text{inner}}} = \max_{C_j^* \neq C_i^*} \max_{x, y \in C_j^*} \|x, y\|.
\]

There are two cases, depending on the relation between \( \text{dis}_{\text{outer}} \) and \( \text{max}_{\text{dis}_{\text{inner}}} \). In the first case,

\[
\text{dis}_{\text{outer}} \leq \text{max}_{\text{dis}_{\text{inner}}}.
\]

In this case we can add \( C_i^* \) to \( C_j^* \) and the cost will increase by a factor of at most \( O(|C_i^*|) \) and recall that \( |C_i^*| < \frac{8000k^4}{\alpha} \). On the other hand, if

\[
\text{dis}_{\text{outer}} > \text{max}_{\text{dis}_{\text{inner}}}.
\]
Figure 3: Bounding the cost for a small cluster $C_i^*$. (a) merge $C_i^*$ into its closest cluster (b) the first point in $C_i^*$ will be taken

then the algorithm will certainly take a member in $C_i^*$ in Line 13 Algorithm 2. The cost can increase by a factor of at most $O(|C_i^*|)$. So over all the small size clusters, the cost will increase by a factor of at most $80000k^5\alpha$. Also note that $80000k^5\alpha \leq \left(\frac{808808}{\alpha}\right)^k(k!)^4$.

And the claim by induction is proven.

**Claim 21.** Fix a dataset $D$, a subset $S \subseteq D$ and a scalar $0 \leq \beta \leq 1$. Assume a subset $S' \subseteq D$ with $\frac{|S'|}{|D|} = \beta$ is chosen uniformly at random. Then, for any $a > 0$,

$$\Pr\left(\left| |S' \cap S| - \frac{|S'|}{|S|} \right| \geq \sqrt{\frac{|S'||S|}{a|D|}}\right) \leq a$$

**Proof.** Denote by $X$ the random variable that is equal to $|S \cap S'|$. Order the members in $S$ in some arbitrary order. Denote by $X_i$, $i = 1, \ldots, |S|$, the binary variable that is equal to 1 if the $i$'th member in $S$ is also in $S'$, and $X_i = 0$ otherwise. Note two basic properties of the random variables $X_i$'s:

$$E[X_i] = \beta,$$  \hspace{1cm} (5)

and for every $i \neq j$ it holds that

$$E[X_iX_j] = \Pr(X_i = 1 \land X_j = 1) = \frac{|S'||S'|-1(|D|-2)!}{|D|!} = \frac{|S'||S'|-1}{|D|(|D|-1)} \leq \beta^2,$$  \hspace{1cm} (6)

where the second equality holds since we can view the process of taking $S'$ as if we order all the points in $D$ and then take the first $|S'|$ members; with this view, the $i$th member has $|S'|$ places and the $j$th member has $|S'| - 1$ places and then arrange all the other members, $(|D|-2)!$ options.

We now analyze the expectation and the variance of $X$.

$$E[X] = \sum_{i=1}^{|S|} E[X_i] = \beta|S|.$$
Bounding the variance of $X$

$$\text{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$= \mathbb{E} \left[ \left( \sum_{i=1}^{[S]} X_i \right)^2 \right] - \left( \sum_{i=1}^{[S]} \mathbb{E}[X_i] \right)^2$$

$$= \mathbb{E} \left[ \sum_{i \neq j} X_i X_j \right] + \mathbb{E} \left[ \sum_{i=1}^{[S]} X_i^2 \right] - \sum_{i,j} \mathbb{E}[X_i] \mathbb{E}[X_j]$$

$$\leq \beta^2 |S|^2 + \beta |S| - \beta^2 |S|^2 = \beta |S|$$

where the fourth equality follows from the fact that $X_i^2 = X_i$ as $X_i$ is a binary random variable and the inequality follows from Equation 5 and Equation 6. Next we use Chebyshev’s inequality which is the following bound for any $C > 0$:

$$\Pr(|X - \mathbb{E}[X]| \geq C) \leq \frac{\text{Var}[X]}{C^2}.$$ 

In our case, take

$$C = \sqrt{\frac{\text{Var}[X]}{a}} \leq \sqrt{\frac{\beta |S|}{a}}$$

and Chebyshev’s inequality implies that

$$\Pr \left( |X - \beta |S| \right) \geq \sqrt{\frac{\beta |S|}{a}} \leq a.$$ 

Claim 22. Fix a dataset $D$ and a subset $S \subseteq D$ and a scalar $0 \leq \beta \leq 1$. Assume a subset $S' \subseteq D$ of size $|S| = \beta |D|$ is chosen uniformly at random. For any $a > 0$, if $|S| \geq \frac{4}{\beta a}$ then

$$\Pr \left( \frac{\beta}{2} |S| \leq |S \cap S'| \leq 2\beta |S| \right) \geq 1 - a$$

Proof. From Claim 21 we know that

$$\Pr \left( ||S \cap S'| - \beta |S| \right) \leq \sqrt{\frac{\beta |S|}{a}} \leq a.$$ 

To prove the claim, it is enough to prove that

$$\sqrt{\frac{\beta |S|}{a}} \leq \frac{\beta}{2} |S|$$

$$\iff \frac{2}{\sqrt{a}} \leq \sqrt{\beta |S|}$$

$$\iff \frac{4}{a} \leq \beta |S|$$

$$\iff \frac{4}{\beta a} \leq |S|$$

$\square$
Proof of Theorem 9

Proof. We will use the same series of points in $\mathbb{R}$ that was used in the proof of Theorem 7. We say that a point in the $i$-th iteration is maximal if it’s the largest value so far. We prove the following two claims:

1. For any $c$-approximation it must take at least 0.5 of the maximal points with probability at least 0.8.

2. With probability at least 0.99 there are $\Omega(\log n)$ maximal points.

Once we prove the two steps we are done.

Claim 1 - There are two cases (i) for each maximal point, the probability the algorithm takes is as a center is at least 0.9. (ii) there is a maximal point such that the probability the algorithm takes is as a center is less than 0.9. In case (i), using Claim 15, we know that with probability at least 0.8, the algorithm takes half of the maximal points. In case (ii), from the same argument as the proof of Theorem 7 we have a contradiction to the assumption that the learner is a $c$-approximation.

Claim 2 - denote by $X$ the random variable that is equal to the number of maximal points and denote by $X_i$ the binary random variable that is equal to 1 if the $i$-th example is maximal, and otherwise $X_i = 0$. For any $i$, $E[X_i]$ is equal to the probability that the $i$-th example is the largest than all previous examples. This probability is equal to $E[X_i] = 1/i$. Thus

$$E[X] = E\left[\sum_{i=1}^{n} X_i \right] = \sum_{i=1}^{n} \frac{1}{i} = \log n.$$  

Note that $X_i$’s are independent and thus we can use Hoeffding’s inequality.

Proof of Theorem 10

Proof. We start with bounding the number of expected centers the algorithm uses. Denote by $X$ the random variable that is equal to the number of centers chosen. Denote by $X_i$ the random variable that is 1 if the $i$-th example is taken as center and 0 otherwise. Then, the expected number of centers the algorithm chooses is equal to

$$E[X] = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \Pr(i\text{-th point is a center})$$

By the definition of the algorithm $X_1$ is always 1. For $1 < i \leq n$ it holds that $X_i = 1$ only if the $i$-th example is the furthest away from the first example among examples 1, . . . , $i - 1$. The probability that the $i$-th point is the furthest is equal to $\frac{1}{i-1}$. Thus

$$E[X] = 1 + \sum_{i=2}^{n} \frac{1}{i-1} = 1 + \sum_{i=1}^{n-1} \frac{1}{i} = 1 + \log(n - 1).$$

From Markov’s inequality with probability at least $1/20$ number of centers is at most

$$20(1 + \log(n - 1)) = O(\log n).$$

We are now left with proving that Algorithm 3 is $\Theta(1)$-approximation. Focus on the optimal clustering. Denote by $C_1^*$ the points in the first cluster and by $C_2^*$ the points in the second cluster. We define the set of good points for a cluster $r$ ($r = 1, 2$) as the set of points that taking them as a center will not increase the cost by much. More formally,

$$Good_r = \{ y_r \in C_r^* | \sum_{x_i \in C_2^*} \| x_i - y_r \|^2 \leq 100 \sum_{x_i \in C_2^*} \| x_i - \mu^*(r) \|^2 \},$$

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where $\mu^*(r) = \frac{1}{|C^*_r|} \sum_{x_i \in C^*_r} x_i$ is the optimal center for cluster $C^*_r$. From Lemma 3 and Markov’s inequality we know that $Good_r$ is large. Specifically,

$$\frac{|Good_r|}{|C^*_r|} \geq 1 - \frac{1}{50}.$$ 

Thus, using union bound, with probability at least $1 - \frac{2}{50}$ the first point the algorithm encounters from each cluster $r$ is good.

Fix the first point the algorithm encounters by $x$, w.l.o.g $x \in C^*_1$. Define by $y^*_2 \in Good_2$ the closest point to $x$ in $Good_2$, i.e.,

$$y^*_2 = \arg\min_{y_2 \in Good_2} \|y_2 - x\|.$$

Denote by $B$ all the points in $C^*_1$ that are further from $x$ than $y^*_2$, i.e.,

$$B = \{y_1 \in C^*_1 : \|y_1 - x\| \geq \|y^*_2 - x\|\}.$$

There are two cases

1. $|B| \leq 0.01|C^*_2|$: we will show that most probability, the first point the algorithm encounters in $C^*_2$ will be chosen as a center. We know that

$$|B| \leq 0.01|C^*_2| \leq 0.02|Good_2|.$$

Thus, with probability at least $1 - 0.02$, the first point in $Good_2 \cup B$ is in $Good_2$.

2. $|B| > 0.01|C^*_2|$: we will show that $C^*_1$ and $C^*_2$ can be viewed as one cluster with $x$ as its center without harming the cost by much.

$$\sum_{y \in C^*_1 \cup C^*_2} \|y - x\|^2 = \sum_{y_1 \in C^*_1} \|y_1 - x\|^2 + \sum_{y_2 \in C^*_2} \|y_2 - x\|^2$$

$$= \sum_{y_1 \in C^*_1} \|y_1 - x\|^2 + \sum_{y_2 \in C^*_2} \|(y^*_2 - x) + (y_2 - y^*_2)\|^2$$

$$\leq \sum_{y_1 \in C^*_1} \|y_1 - x\|^2 + 2|C^*_2| \|y^*_2 - x\|^2 + 2 \sum_{y_2 \in C^*_2} \|y_2 - y^*_2\|^2$$

$$\leq \sum_{y_1 \in C^*_1} \|y_1 - x\|^2 + 2 \cdot 100 \sum_{y_1 \in B} \|y_1 - x\|^2 + 2 \sum_{y_2 \in C^*_2} \|y_2 - y^*_2\|^2$$

$$\leq 201 \cdot \text{cost}(\text{opt}_2),$$

where the first inequality follows from Claim 17 the second from the definition of $B$, and the third from the definition of $Good$. \hfill \Box

**Proof of Theorem 11**

*Proof.* We will take $k$ series of points in $\mathbb{R}^k$. The $i$-th series will contain $\left\lfloor \frac{n}{k} \right\rfloor$ points each will be zeros everywhere except index $i$. Projecting them into the $i$-th coordinate will form exactly the series described in Theorem 7.

We say that a point in the $i$-th iteration is maximal if it’s the largest value so far comparing to the examples with a similar non-zero coordinate. There theorem follows from the following two claims:

1. For any $c$-approximation it must take at least 0.5 of the maximal points with probability at least 0.8.

2. With probability at least 0.99 there are $\Omega(k \log \frac{n}{k})$ maximal points.

The two claims follow from the proof of Theorem 9. More specifically, apply Theorem 9 for each group, which implies that there are $\Omega(\log \frac{n}{k})$ maximal for each of the $k$ coordinates. \hfill \Box
Proof of Lemma 12

Proof. Take \( y = \arg \min_{y \in S} \| x - y \| \) and any \( y_1, y_2 \in S \) we will show that
\[
\| y_1 - y_2 \| \geq \| x - y \| .
\]

W.l.o.g \( y_2 \) was added to \( S \), after \( y_1 \) did. Focus at the time \( y_2 \) was added to \( S \). Denote by \( l \in S \) the closest point in \( S \) at the time to \( x \). Then, since \( y_2 \) was added to \( S \) and not \( x \) we know that
\[
\| y_2 - y_1 \| \geq \| x - l \| \geq \| x - y \| .
\]
\[\square\]

Proof of Theorem 13

Proof. We prove that for any constant \( k \)

1. Number of centers is bounded by \( O(k \log \frac{n}{k}) \), with probability at least 0.96.

2. \( \text{cost}(\text{alg}) \leq \Theta(1) \cdot \text{cost}(\text{opt}_k) \), with probability at least 0.96.

To bound the number of centers the algorithm uses, we use a similar argument as in Theorem 10. The algorithm takes the first \( k \) points. For \( i > k \) we want to find the probability that the \( i \)-th point is selected as a center. This happens if among the \( i \) points read so far, this point is one of the \( k \) members in the Farthest-first-traversal. The probability for this is
\[
\frac{k}{i}.
\]
Thus the expected number of centers taken by the algorithm throughout its entire run is
\[
k + \sum_{j=k+1}^{n} \frac{k}{j} = k + k \left( \sum_{i=1}^{n} \frac{1}{i} - \sum_{i=1}^{k} \frac{1}{i} \right) = k + k \log \frac{n}{k}.
\]
Use Markov’s inequality to prove that with probability 0.99 number of centers takes by the algorithm is at most
\[
100k \cdot 100k \log \frac{n}{k} = \Theta \left( k \log \frac{n}{k} \right),
\]
The last equality holds since w.l.o.g \( k \leq \frac{n}{2} \) (otherwise the algorithm can simply take all the points as centers and the number of center is optimal, \( \Theta(k) \)) which implies that \( \log \frac{n}{k} \geq 1 \).

Next we will prove that the algorithm is a \( \Theta(1) \)-approximation. We will prove something stronger: For any \( 1 \leq k \leq k^* \) with probability at least 0.96
\[
\text{cost}(\text{alg}) \leq 80200^2 k^2 \cdot \text{cost}(\text{opt}_k).
\]
We prove this claim by induction on \( k \). For \( k = 1 \) the claim follows immediately, similarly to Algorithm 11

Denote the \( k \) optimal clusters by \( C_1^*, \ldots, C_k^* \). Focus on a cluster \( C_i^* \). The idea of the proof is that either the first point in \( C_i^* \), most probably, is chosen as a center or the entire cluster can be added to another cluster. When the first point \( x \in C_i^* \) arrives, its closest point is \( c \notin C_i^* \). If \( x \) is not chosen as a center, then there are \( k \) centers from \( k - 1 \) clusters that the distance between any two is larger than \( \| x - c \| \). There are two cases, as in Theorem 10 either this is a common scenario, and then \( C_i^* \) can be added to another cluster, or it’s rear case and this means that with high probability \( x \) will be chosen as a center.

We define the set of \( \text{good} \) points for a cluster \( r \in [k] \) as the set of points that taking them as a center will not increase the cost by much. More formally,
\[
\text{Good}_r = \left\{ y_r \in C_i^* \mid \sum_{x_i \in C_i^*} \| x_i - y_r \|^2 \leq 100k \sum_{x_i \in C_i^*} \| x_i - \mu^*(r) \|^2 \right\},
\]

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where \( \mu^*(r) = \frac{1}{|C_i^*|} \sum_{x_i \in C_i^*} x_i \) is the optimal center. From Lemma 9 and Markov’s inequality we know that \( \text{Good}_r \) is large. Specifically,
\[
|\text{Good}_r| \geq 1 - \frac{1}{50k}.
\]
Thus, using union bound, we know that with probability at least \( 1 - \frac{1}{50} \), the first point the algorithm encounters from each cluster \( r \) is in \( \text{Good}_r \).

Fix a cluster \( C_i^* \) for some \( i \in [k] \). The distance between \( \text{Good}_i \) and a point \( y \) is defined as
\[
\text{dis}(\text{Good}_i, y) = \min_{y' \in \text{Good}_i} \| y' - y \|.
\]
Denote by \( N \) the set of points that are not in the cluster \( C_i^* \) (i.e., in \( C_i^* - C_i^* \)) and are one of the \( |C_i^*|/100k \) closest points to \( \text{Good}_i \). Denote the max distance in \( N \) by \( \text{dis} \), i.e., the distance that is \( |C_i^*|/100k \)-closest to \( \text{Good}_i \):
\[
\text{dis} = \max_{y' \in N} \min_{y \in \text{Good}_i} \| y' - y \|.
\]

Denote the point the achieves this minimum in \( \text{Good}_i \) by \( x_i \). We want to define bad points that can cause the algorithm not to take the first point from \( C_i^* \) as a center. For that we first take for each cluster \( r \neq i \) an arbitrary point in \( x_r \in \text{Good}_r \). Now we are ready to define the bad points:
\[
B = \bigcup_{r \neq i} \left\{ y \in C_r^* : \| y - x_r \|^2 \geq \frac{\text{dis}^2}{4} \right\}.
\]

There are two cases, either \( B \) is large compared to \( C_i^* \), or it is not.
Figure 5: Bounding the cost of $C^*_i$. (a) In the case that $B$ is large compared to $C^*_i$, the cluster $C^*_i$ can be merged into another cluster without harming the cost too much (b) In the case that $B$ is small compared to $C^*_i$, when the first point $y_i$ from $C^*_i$ is received, the algorithm did not encounter, with high probability, a point from (i) $N$, this is why the distance of $y_i$ to other points is at least $\text{dis}$ (ii) $B$, this is why all distances inside clusters is smaller than $\text{dis}$. The last two observations imply that $y_i$ will be taken as a center.

If $|B| \geq \frac{1}{100k} |C^*_i|$: we will show that taking the $k-1$ centers — all the centers $x_r$'s without $x_i$ — is a good enough clustering and then the claim follows by the induction assumption. To prove that, first take, among all points in $N$, one $y_j \in N$ that it's closest to its representative $x_j$:

$$y_j = \arg\min_{j \in [k]-i, y \in N \cap C^*_j} \|y - x_j\|$$

$$\text{cost}(\text{opt}_{k-1}) \leq \sum_{y \in C^*_i} \|y - x_j\|^2 + \sum_{r \neq i} \sum_{y \in C^*_r} \|y - x_r\|^2$$

$$\leq 2 \sum_{y \in C^*_i} \|y - x_j\|^2 + 4 \sum_{y \in C^*_j} \|x_i - y_j\|^2 + 4 \sum_{y \in C^*_i} \|y_j - x_j\|^2 + \sum_{r \neq i} \sum_{y \in C^*_r} \|y - x_r\|^2$$

$$\leq 4 \sum_{y \in C^*_i} \|x_i - y_j\|^2 + 4 \sum_{y \in C^*_j} \|y_j - x_j\|^2 + 2 \sum_{r} \sum_{y \in C^*_r} \|y - x_r\|^2$$

$$= 4|C^*_i| \|x_i - y_j\|^2 + 4|C^*_i| \|y_j - x_j\|^2 + 2 \sum_{r} \sum_{y \in C^*_r} \|y - x_r\|^2$$

$$\leq 4|C^*_i| \text{dis}^2 + 4|C^*_i| \|y_j - x_j\|^2 + 2 \sum_{r} \sum_{y \in C^*_r} \|y - x_r\|^2$$

$$\leq 400k|B| \text{dis}^2 + 400k \frac{|C^*_i|}{100k} \|y_j - x_j\|^2 + 2 \sum_{r} \sum_{y \in C^*_r} \|y - x_r\|^2$$

$$\leq 802k \sum_{r} \sum_{y \in C^*_r} \|y - x_r\|^2$$

$$\leq 80200k^2 \cdot \text{cost}(\text{opt}_k)$$
where the second inequality follows from Claim 17, the forth since $y_j \in N$, the sixth because $y_j$ is closest to its representative from all $N$, and the last inequality follows from the definition of good.

From the induction assumption we know that
\[
\text{cost}(\text{alg}) \leq 80200^{k-1}(k-1)!^2 \cdot \text{cost}(\text{opt}_{k-1})
\leq 80200^{k-1}(k-1)!^2 \cdot 80200k^2 \cdot \text{cost}(\text{opt}_k)
= 80200^k k!^2 \cdot \text{cost}(\text{opt}_k)
\]

And this proves the claim in this case. Let’s move on to the next case.

If $|B| < \frac{1}{100k} |C^*_i|$: We will show that with probability at least $1 - \frac{4}{100k}$, the first point from $C^*_i$ will be chosen as a center and is in $\text{Good}_i$. Then using union bound over all the $k$ centers will finish the proof.

Focus on the time where the first point in $C^*_i$, $y_i \in C^*_i$ was given. With probability at least $1 - \frac{1}{50k}$ it is in $\text{Good}_i$. With probability at least $1 - \frac{1}{100k}$ a point from $|B|$ was not chosen yet (as $B$ is much smaller than $C^*_i$). With probability at least $1 - \frac{1}{100k}$ a point from $N$ was not chosen yet (as $N$ is much smaller than $C^*_i$), thus minimal distance from a point in $\text{Good}_i$ to another is at least $\text{dis}$. For the sake of contradiction, let us assume that $y_i$, the first point from $C^*_i$, was not chosen as a center. Then all points $S$ returned by the farthest-first-traversal algorithm must came from at most $k-1$ optimal clusters. From the Pigeonhole principle, there are two points $y_1, y_2 \in S$ that are in the same cluster $C^*_j$. From Lemma 12 we have that
\[
\text{dis} \leq \|y_1 - y_2\|.
\]

From Claim 17 we have that $\|y_1 - y_2\|^2 \leq 2 \|y_1 - x_j\|^2 + 2 \|x_j - y_2\|^2$. This implies that $\|y_1 - x_j\|^2 \geq \text{dis}^2/4$ (or $\|y_2 - x_j\|^2 \geq \text{dis}^2/4$). This means that $y_1 \in B$. Which is a contradiction to the assumption that no point in $B$ was chosen yet.