BORDERED THEORY FOR PILLOWCASE HOMOLOGY

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Abstract. We construct an algebraic version of Lagrangian Floer homology for immersed curves inside the pillowcase. We first associate to the pillowcase an algebra $A$. Then to an immersed curve $L$ inside the pillowcase we associate an $A_\infty$ module $M(L)$ over $A$. Then we prove that Lagrangian Floer homology $HF(L, L')$ is isomorphic to a suitable algebraic pairing of modules $M(L)$ and $M(L')$. This extends the pillowcase homology construction — given a 2-stranded tangle inside a 3-ball, if one obtains an immersed unobstructed Lagrangian inside the pillowcase, one can further associate an $A_\infty$ module to that Lagrangian.

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1. Introduction

1.1. Background. Since the groundbreaking work of Donaldson, gauge theory became a powerful tool to study low-dimensional topology. Gauge theory provides invariants of diffeomorphism types of 4-manifolds, homological invariants of 3-manifolds, and homological invariants of knots inside 3-manifolds. These invariants are defined using moduli spaces of solutions to certain partial differential equations on manifolds. The following two theories emerged over the years: instanton theory, where one counts solutions to anti-self-dual Yang-Mills equations, and Seiberg-Witten theory, where one counts solutions to Seiberg-Witten monopole equations. The corresponding invariants of 3-manifolds are called instanton Floer homology [8], introduced by Floer, and monopole Floer homology [18], introduced by Kronheimer and Mrowka.

Another way to construct homological invariants for 3-manifolds and knots is to use symplectic geometry. The general strategy for 3-manifolds is as follows: one takes a Heegaard splitting \( U_1 \cup_{\Sigma_g} U_2 = Y_3 \), and associates to it two Lagrangians inside a certain moduli space (which should possess a natural symplectic structure): \( L(U_1), L(U_2) \to (M(\Sigma_g), \omega) \). Then the desired Floer homology is defined via Lagrangian Floer homology \( \text{HF}(L(U_1), L(U_2)) \).

The first homological invariant of this type is called Heegaard Floer homology, and was introduced by Ozsváth and Szabó [30], [31]. See a recent survey article [16] for an introduction and numerous applications of this theory.

It is interesting that these two methods to obtain Floer homologies run in parallel, connected by Atiyah-Floer type conjectures. The following is the original formulation for instantons. Having a Heegaard splitting \( Y_3 = U_1 \cup_{\Sigma_g} U_2 \), and associates to it two SU(2) representation varieties \( R(U_i), R(\Sigma_g) \). One then has maps \( R(U_i) \to R(\Sigma_g) \). It was conjectured in [2] that instanton Floer homology \( I(Y_3) \) should be equal to Lagrangian Floer homology \( \text{HF}(R(U_1), R(U_2)) \). Spaces \( R(U_1), R(U_2), R(\Sigma_g) \) are singular, and thus symplectic instanton Floer homology \( \text{HF}(R(U_1), R(U_2)) \) was not possible to define at that moment. The symplectic side of the isomorphism, as well as the proof of Atiyah-Floer conjecture, are still under development. There are different versions of symplectic instanton Floer homology, which should correspond to different versions of instanton Floer homology. Notably, the corresponding conjecture on the monopole side was proved in [22]: monopole Floer homology and Heegaard Floer homology are equal.

For a thorough introduction to the above Floer-theoretic invariants, connections between them, and their applications see [28], and references there. See also more recent papers [0, 14, 15] on symplectic instanton Floer homology, and [7] on Atiyah-Floer conjecture.

Now we turn our attention to knot invariants. The first Floer-theoretic invariant for knots inside 3-manifolds was knot Floer homology \( \hat{HFK}(Y^3, K) \), introduced by Ozsváth and Szabó in [32] using symplectic geometry. The special case of knots in a sphere is denoted by \( \hat{HFK}(K) = \hat{HFK}(S^3, K) \). Some properties and applications of knot Floer homology are the following: \( \hat{HFK}(K) \) categorifies Alexander polynomial, detects 3-dimensional genus of a knot and hence detects the unknot, detects fiberedness of a knot in \( S^3 \), and also provides lower bounds for the 4-ball genus of a knot. See [29] and [33] for an introduction to this invariant. Gauge theoretic counter-part of knot Floer homology was constructed in [20] and [23].

On instanton side, only gauge theoretic constructions of knot invariants are fully developed. Kronheimer and Mrowka in [20] constructed a knot invariant called sutured instanton knot homology \( KHI(K) \). It has properties similar to knot Floer homology, like detecting
the genus of a knot (non-vanishing result). In fact, $KHI(K)$ is conjectured to be isomorphic to knot Floer homology $\widehat{HF}(K)$. In [19] Kronheimer and Mrowka constructed another knot invariant called singular instanton knot homology, which is denoted by $I^2(K)$. It is, in fact, isomorphic to $KHI(K)$. In [21] they proved that there is a spectral sequence from Khovanov homology $Kh(K)$ to $I^2(K)$. This, together with the non-vanishing result for $KHI(K)$, proved that Khovanov homology detects the unknot.

The construction called pillowcase homology was developed by Hedden, Herald, and Kirk in [11] and [12], in order to better understand and compute $I^\#(K)$. This geometric construction potentially gives a knot invariant, which we denote by $H_{pill}(K)$. It should be the symplectic side of Atiyah-Floer conjecture for singular instanton knot homology $I^2(K)$, see [12, Conjecture 6.5].

Our primary motivation was to enhance the construction of pillowcase homology. We do it by associating algebraic invariants not only for knots, but also for tangles. Let us first describe pillowcase homology in more details.

1.2. Pillowcase homology. We sketch how the pillowcase homology construction works in the first two columns of Figure 1.

**Figure 1.** The pillowcase homology construction (1st and 2nd columns), and its algebraic extension (3rd column).
Now we describe the construction. Having a knot in $K \subset S^3$, find a Conway sphere, i.e. a 2-sphere that intersects the knot in 4 points (denote it by $(S^2, 4)$). The decomposition of a knot into two tangles by this 2-sphere should be such, that one of the tangles is a trivial tangle that consists of two arcs. Then proceed as follows:

1. To that Conway sphere with four marked points (denote by $\gamma_i$, $i = 1, 2, 3, 4$, loops around those points) associate a traceless character variety:

   $$ R(S^2, 4) = \{ h \in \text{hom}(\pi_1(S^2 \setminus \text{4pt}), SU(2)) \mid \text{tr}(h(\gamma_i)) = 0 \}/\text{conj}. $$

   It happens to be homeomorphic to the pillowcase — a torus factorized by hyperelliptic involution

   $$ R(S^2, 4) \cong P = S^1 \times S^1/((\gamma, \theta) \sim (\gamma, -\theta)), $$

   see [11, Proposition 3.1] for the proof.

2. To a trivial tangle, which consists of two arcs $A_1, A_2$, associate an immersed curve $L^3$ in the pillowcase by the following procedure. First, add to the tangle a circle $H$ with an arc $W$ as shown on the left of the second row of the Figure 3. Then form a space of traceless representations:

   $$ R^2(D^3, A_1 \cup A_2) = \{ h \in \text{hom}(\pi_1(D^3 \setminus (A_1 \cup A_2 \cup H \cup W)), SU(2)) \mid $$

   $$ |\text{tr}(h(\mu_{A_i})) = \text{tr}(h(\mu_H)) = 0, h(\mu_W) = -I \}/\text{conj}. $$

   Because $S \setminus \text{4pt} \subset D^3 \setminus (A_1 \cup A_2 \cup H \cup W)$, there is a map in the reversed direction $R^2(D^3, A_1 \cup A_2) \rightarrow R(S^2, 4)$. Because this map is singular, and $R^2(D^3, A_1 \cup A_2)$ is not 1-dimensional, one needs to do a holonomy perturbation of the space. After specifically defined perturbation (see [11, Section 7]), one gets an immersed circle $L^3 : R^2(D^3, A_1 \cup A_2) \leftrightarrow P$ depicted on the left of Figure 10, missing all four singular points.

3. With the tangle $K \setminus (A_1 \cup A_2)$ from the other side one does almost the same procedure. The only difference is that the circle and the arc $H \cup W$ are not added (this is why here the image will often pass through a singular point). One still needs to perturb $R(D^3, K \setminus (A_1 \cup A_2))$ in this case (see, for example, [12, Section 11.6] for the case of $(3, 4)$ torus knot). This results in the immersion $L_K : R^2(D^3, K \setminus (A_1 \cup A_2)) \leftrightarrow P$.

   Examples of such immersions for torus knots (with two arcs removed) are depicted on Figure 20.

4. Having done all that, one associates to the initial knot $K$ a vector space called pillowcase homology. It is equal to Lagrangian Floer homology $H_{pil}(K) = HF_*(L^3, L_K)$ inside $P$, where $P$ is the pillowcase with deleted small neighborhoods of 4 singular points.

   On the level of chain complexes, the vector space isomorphism $C_{pil}(K) \cong CI^3(K)$ is true by construction. In [12] the authors provided lots of examples where the homologies of these chain complexes are indeed isomorphic.

Let us list the missing ingredients for $H_{pil}(K)$ to be a knot invariant. Along the way one makes certain choices. First, there is a tangle decomposition of a knot along the Conway

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1. Let us stress that the map $L_K$ depends on the tangle decomposition of a knot. We chose this misleading notation for simplicity.

2. One actually obtains Lagrangians in $P$ and should consider Floer homology where discs do not cross singular points. But one can delete small neighborhoods of singular points to get $P$, and the corresponding Lagrangian Floer complex will be unchanged.
sphere, and the map $R(D^3, K \setminus (A_1 \cup A_2)) \cong \mathcal{P}$ depends on this decomposition. Second, there is a generic small perturbation of $R(D^3, K \setminus (A_1 \cup A_2))$ in order to obtain non-singular immersed Lagrangian $L_K : R_\pi(D^3, K \setminus (A_1 \cup A_2)) \cong \mathcal{P}$. Thus, in order to obtain a knot invariant, one needs to prove that $H_{pill}(K)$ does not depend on those two choices. Moreover, one needs to prove that, after those choices, the resulting immersion $L_K$ is unobstructed, and admissible w.r.t. to $L^{\#}$, so that $H_{pill}(K)$ is defined without difficulties.

1.3. The bordered construction and motivation. The construction in this paper is an algebraic enhancement of pillowcase homology. It answers the following question: what algebraic structures one should associate to $L^{\#}$ and $L_K$, in order to be able to recover $H_{pill}(K) = HF_\ast(L^{\#}, L_K)$ algebraically, without looking at the intersection picture on the pillowcase. The relevant objects can be seen in the third column of Figure 1. Namely, to a 4-punctured 2-sphere we have associated a pillowcase $\mathcal{P}$, and now further associate an algebra $A$. To a trivial 2-stranded tangle we associated an immersed circle $L^{\#}$ in $\mathcal{P}$, and now further associate a specific module $M(L^{\#})_A$, see Figure 9. To a tangle from the other side $K \setminus (A_1 \cup A_2)$, similarly already having $L_K$, we associate a module $M(L_K)_A$. To a union of these two tangles, i.e. to a knot $K$, we associate a homology $H_\ast(M(L_K)_A \boxtimes \mathcal{B}_r, 4 \boxtimes \mathcal{A} M(L^2))$. The fact that this algebraic pairing is equal to pillowcase homology $H_{pill}(K)$ is the main result of this paper. In the next subsection we formulate a slightly more general result, where we consider any two Lagrangian immersions.

Let us describe the motivation behind the bordered construction.

First, it provides a natural candidate for an algebraic invariant of 2-stranded tangle $T$ inside a ball $D^3$. To such tangle one can associate an immersed Lagrangian $L(T) : R_\pi(D^3, T) \cong R(\partial D^3, 4) = \mathcal{P}$, and then $A_\infty$ module $M(L(T))_A$ \footnote{Here one must be careful. Definition of $M(L(T))_A$ requires a parameterization of the pillowcase $R(\partial D^3, 4)$. Thus there needs to be additional information, for this parameterization to be fixed. Namely, the boundary of the tangle $(\partial D^3, 4)$ must be bordered, i.e. parameterized by a standard fixed $(S^2, 4)$.}. As with pillowcase homology, there are missing ingredients in this construction: it needs to be proved that $L(T)$ is unobstructed, and homotopy type of $M(L(T))_A$ does not depend on the perturbation $\pi$.

Building on this idea, one can isolate the part of $H_{pill}(K)$ which depends on $L_K$. I.e., if one changes $L_K$ in some way, it is more natural to understand how $M(L_K)_A$ changes, rather then $H_{pill}(K) = H_\ast(M(L_K)_A \boxtimes \mathcal{B}_r, 4 \boxtimes \mathcal{A} M(L^2))$.
A very interesting direction of research is to further develop bordered theory for pillowcase homology $H_{pil}(K)$ into full bordered theory. Let us briefly describe the way such theory would work. The strategy is the following:

1. To understand what algebra should be associated to $2n$ punctured sphere $(S^2, 2n)$.
2. To understand what bimodules (over the algebras from the previous step) correspond to tangles inside $S^2 \times I$, which connect $(S^2, 2k)$ to $(S^2, 2(k + 1))$.
3. To build up a chain complex $C_{alg}(K)$, and prove, that its homology $H_{alg}(K)$ does not depend on the knot $K$. The construction of $C_{alg}(K)$ should involve composing (via derived tensor product, or morphism space pairing) bimodules from the second step, and modules that correspond to trivial tangles $M(L_U), M(L^\times)$ (examples 8.1, 8.2).
4. To prove that this construction, in fact, computes singular instanton knot homology: $H_{alg}(K) \cong P(K)$.

This is a difficult project. Even completing the step (1) is hard. The desired algebra should be the algebra of Fukaya category of the smooth stratum of representation variety $R(S^2, 2n)$. After the pillowcase $R(S^2, 4)$, the next space of interest is $R(S^2, 6)$. It is already a complicated singular 6-dimensional manifold, see [17]. See also [13] for the study of $R(S^2, 2n)$. Let us note that additional structures on representation spaces could help to compute their Fukaya category. For example, in case of Heegaard Floer homology, the Fukaya category of $Sym^g(\Sigma_g \setminus 1pt)$ was computed in [3] using the structure of Lefschetz fibration over $\mathbb{C}$.

Nevertheless, if one manages to guess the algebras and bimodules, one can dismiss the underlying geometry and try to prove that the knot invariant is well defined algebraically (step (3)).

Examples of analogous bordered theories developed for other invariants are: bordered Heegaard Floer homology [24, 25]; bordered theory for knot Floer homology [33, 34, 35]; bordered theories for Khovanov homology [36, 37, 27]. Step (3) for Heegaard Floer homology was done in [40], and for knot Floer homology in [34, 35].

1.4. Main result. We construct an algebraic version of Lagrangian Floer homology for two immersed curves inside the pillowcase $P$. The construction works as follows. To the pillowcase $P$ we associate an algebra $\mathcal{A}$. To an immersed curve (circle or arc with ends on the boundary) $L$ inside $P$ we associate an $A_{\infty}$ module $M(L)$. Then, we prove the following pairing result:

**Theorem.** Let $L_0, L_1$ be two admissible unobstructed curves in the pillowcase $P$. Then their Lagrangian Floer complex is homotopy equivalent to the following algebraic pairing of curves:

$$CF_*(L_0, L_1) \simeq M(L_1)_{\mathcal{A}} \otimes \bar{\mathcal{A}} \otimes \mathcal{A} \overline{\mathcal{M}(L_0)}.$$ 

Terms “admissible” and “unobstructed” are defined in Section 2. $\overline{\mathcal{M}(L_0)}$ denotes a dual module, and $\bar{\mathcal{A}} \otimes \mathcal{A}$ is a specific type DD structure constructed in such a way that the above homotopy equivalence is true. From this homotopy equivalence it follows that

$$HF_*(L_0, L_1) \cong H_*(M(L_1)_{\mathcal{A}} \otimes \bar{\mathcal{A}} \otimes \mathcal{A} \overline{\mathcal{M}(L_0)}).$$

Let us mention that this construction of algebraic Lagrangian Floer homology can be generalized to any oriented surface with boundary $\Sigma$. In order for the process to be analogous, one has to make sure to put enough basepoints on $\partial\Sigma$ and parameterizing arcs on
\(\Sigma\), so that the algebra \(A\) becomes directed, i.e. there are no cycles in the generating graph \(\Gamma\). Though it is not absolutely necessary — in \([10]\) the authors work out the case of torus with boundary, without requiring the algebra to be directed. There the process was actually reversed: they started with a D-structure (or A-module), and from that they obtained an immersed curve.

The algebraic pairing \(H_*(M(L_1), A) \otimes A \bar{\text{bar}}_r A \otimes A M(L_0)\) gives an algorithm for computing geometric (i.e. minimal) intersection number of two curves, and geometric self-intersection number of one curve, on a surface with boundary \(\Sigma\).

1.5. **Underlying reasons.** The main object behind the scene is partially wrapped Fukaya category. This special flavor of Fukaya category was introduced by Auroux in \([3, 4]\), in order to reinterpret bordered Heegaard Floer homology via symplectic geometry.

What we really do in this paper, is computing the enlargement by immersed Lagrangians of partially wrapped Fukaya category of \(\mathcal{P}\). See \([5]\) Section 3 for the general description of the following process, i.e. what does it mean to generate a category, and what is Yoneda embedding.

Consider partially wrapped Fukaya category \(\mathcal{F}_{\text{pw}}(\mathcal{P})\), where stops are basepoints \(z_1, z_2, z_3, z_4\) (see left of Figure 5). Note that, because Auroux was considering cohomology instead of homology, we have \(\text{CF}_*(L_0, L_1) = \text{hom}_{\mathcal{F}_{\text{pw}}(\mathcal{P})}(L_1, L_0)\). The parameterization of \(\mathcal{P}\) by the red arcs in Section 4.1 corresponds to picking a set of Lagrangians \(L_1 = i_0, \ldots, L_6 = j_2 \in \mathcal{F}_{\text{pw}}(\mathcal{P})\). The algebra \(A\), which we define in Section 4.2 is the \(A_\infty\) algebra \(\bigoplus_{i,j} \text{hom}_{\mathcal{F}_{\text{pw}}(\mathcal{P})}(L_i, L_j) = \bigoplus_{i,j} \text{CF}_*(L_j, L_i)\).

In Section 5 to an immersed curve (circle or arc with ends on the boundary) \(L\) inside \(\mathcal{P}\) we associate an \(A_\infty\) module \(M(L)_A\). It is secretly a module \(\bigoplus_i \text{hom}_{\mathcal{F}_{\text{pw}}(\mathcal{P})}(L, L_i) = \bigoplus_i \text{CF}_*(L_i, L)\), the image of \(L\) under Yoneda embedding \(\mathcal{F}_{\text{pw}}(\mathcal{P}) \to \text{mod}_A\). We do not define this way, because partially wrapped Fukaya category was not defined using immersed Lagrangians.

Then, by \([11]\) Theorem 1, one knows that \(L_1, \ldots, L_6\) generate the category \(\mathcal{F}_{\text{pw}}(\mathcal{P})\), which consists of immersed Lagrangians. This implies that if \(L_0, L_1\) are embedded Lagrangians, then one has \(HF_*(L_0, L_1) = H_*(\text{hom}_{\mathcal{F}_{\text{pw}}(\mathcal{P})}(L_1, L_0)) \cong H_*(\text{Mor}_{\text{mod}}(M(L_1), M(L_0)))\). What we want is to extend this result to immersed Lagrangians, which were not part of the Fukaya category. We also want the algebraic part of the isomorphism to be easily computable, in the light of morphism spaces of \(A_\infty\) modules being infinitely generated.

Instead of extending the notion of partially wrapped Fukaya category to immersed Lagrangians (although for surfaces this is entirely possible), and then proving that \(L_1, \ldots, L_6\) still generate it, we choose a different method. We first note that the morphism complex can be described in the following way via bar resolution, see \([26]\) Proposition 2.10]: \(\text{Mor}_{\text{mod}}(M(L_1), M(L_0)) \cong M(L_1) \otimes A \bar{\text{bar}}_r A \otimes A M(L_0)\). In Section 6.2 we describe (following \([29]\)) a smaller model for the dual of bar resolution, \(\bar{A} \text{bar}_r A\). Although we do not explicitly prove it, this DD bimodule is homotopy equivalent to \(\bar{\text{bar}}_r A\), just as in the case of bordered algebra, see \([26]\) Proposition 5.13. We then describe explicitly the reduced version of dual small bar resolution \(\bar{\text{bar}}_r A\).

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4One should treat the case of \(\text{Per}(L_0, L_1) = \mathbb{Z}\) separately, and subtract 2 from \(\text{rk}(H_*)\) in order to obtain a geometric intersection number.
Suppose we have two immersed curves $L_0, L_1$ in the pillowcase $\overline{P}$. [4, Theorem 1] suggests that $HF_*(L_0, L_1) \cong H_*(Mor(M(L_1), M(L_0)))$. The way we constructed $A^{-\bar{r}}_A$ suggests that $H_*(Mor(M(L_1), M(L_0))) \cong H_*(M(L_1)_A \boxtimes_A \bar{r}^{-1} A \boxtimes_A \overline{M(L_0)})$. We now dismiss the morphism complex, and prove in Section 7 that $HF_*(L_0, L_1) \cong H_*(M(L_1)_A \boxtimes_A \bar{r}^{-1} A \boxtimes_A \overline{M(L_0)})$, by interpreting $A^{-\bar{r}}_A$ in a geometric way.

**Conventions, assumptions, prerequisites.** We will work over $\mathbb{F}_2$, and we will work with Lagrangian Floer homology, as opposed to cohomology.

By differential we will mean not only the map $d : C \to C$, s.t. $d^2 = 0$, but also the following. If, for example, $d(x) = y_1 + y_2 + y_3$, then we say that there is a differential from $x$ to $y_1$ (and from $x$ to $y_2$, and from $x$ to $y_3$). We will denote these differentials by arrows: $x \to y_1$.

In this paper we will use $dg$-algebras, $A_\infty$ modules, DD structures (DD bimodules), and box tensor product $\boxtimes$ operation. For definitions of these objects and operations we refer to [24] and [25].

**Acknowledgments.** I am thankful to my adviser Zoltán Szabó for suggesting the project, and his continuous support.

## 2. Immerged curves in the pillowcase: setup

### 2.1. Pillowcase.** Fix an oriented torus $T^2 = S^1 \times S^1 = \mathbb{R}/(2\pi \cdot \mathbb{Z}^2)$ as a product of two unit circles. The pillowcase is a quotient of the torus by hyperelliptic involution

$$P = T^2 / (\gamma, \theta) \sim (-\gamma, -\theta).$$

This quotient has four singular points (which are cones over $\mathbb{R}P^1$), we call them corners. The intersection theory we are interested in happens in the compliment of the corners, or, equivalently, in the compliment of small neighborhoods of the corners. Thus we delete small neighborhoods of the corners and denote the result by

$$\overline{P} = P \setminus U(0, 0) \cup U(0, \pi) \cup U(\pi, 0) \cup U(\pi, \pi).$$

We will be working with this space from now on (also calling it a pillowcase). Note that it is diffeomorphic to a 2-sphere with four discs deleted.

### 2.2. Immerged curves.** By a curve $L$ we mean a circle or an arc in the pillowcase: $L : S^1 \hookrightarrow \overline{P}$ or $L : [0, 1] \hookrightarrow \overline{P}$. Later we will often write $L$ instead of $Im(L)$ inside $\overline{P}$. Such a curve must satisfy the following properties:

- $L$ is smoothly immersed, i.e. the differential is injective. This implies that locally $L$ is an embedding.
- If $L$ is a circle, it is contained in interior $\text{int}(\overline{P})$. If $L$ is an arc, only the endpoints of it are mapped to the boundary $\partial \overline{P}$. Also endpoints of $L$ should be distinct on $\partial \overline{P}$, and transverse to boundary.
- All self-intersections of $L$ are transverse, and there are no triple self-intersections.
- $L$ is unobstructed, i.e. it is an image of an embedded arc (if $L$ is an arc) or properly embedded line (if $L$ is a circle) in the universal cover of $\overline{P}$. This is equivalent (see [1]) to saying that there is no fishtail (see Figure 3), and $L$ is not null-homotopic. In other words, there should be no discs with boundary on $L$ with 0 or 1 “switches” at self-intersections of $L$. 

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We will further assume that these properties are satisfied, and call such curves either “unobstructed curves”, or simply “curves”.

Lagrangian Floer homology is a homology theory for a pair of curves, denoted by $HF(L_0, L_1)$. We will need pairs to satisfy the following properties:

- All intersection points are transverse.
- There are no triple intersection points.
- If both curves $(L_0, L_1)$ are arcs then the following condition should be satisfied. First of all, our pillowcase will be equipped with 4 basepoints on every boundary component as on the left of Figure 5. Suppose now $a \in \partial L_0$, $b \in \partial L_1$, and $a$ and $b$ lie in the same component of $\partial P$ with basepoint $z_i$. Then, w.r.t. orientation of $\partial P$, the order of three points should be first $a$, then $b$, then $z_i$. For example, on the left of Figure 5, pair $(i_0, i_1)$ is admissible, and the pair $(j_1, j_2)$ is not.
- There is no essential immersed annulus with boundary on $L_0$ and $L_1$ $A : (S^1 \times [0,1], S^1 \times \{0\}, S^1 \times \{1\}) \to (\overline{P}, L_0, L_1)$ with no switches allowed. No switches allowed means that boundaries can be composed with $L_0$ and $L_1$, i.e. one has $A|_{S^1 \times \{i\}} = L_i \circ f : S^1 \to \overline{P}$ for some $f : S^1 \to S^1$ (or $f : S^1 \to [0,1]$ if $L_i$ is an arc).

We will call such pairs admissible.

**Assumption.** Later, when we state different Lagrangian boundary conditions, we will always assume that they can be composed with the corresponding $L_i$, or equivalently (and intuitively more clear), that there are no switches allowed.

### 3. Geometric pairing

#### 3.1. Outline.** Our first goal is to define Lagrangian Floer homology $HF(L_0, L_1)$ for a pair of immersed unobstructed curves $L_0, L_1$ in the pillowcase $\overline{P}$. We sketch here the construction, following [10], [1], [38], [12]. The plan is the following:

1. For the homology to be well-defined one needs to restrict the class of curves they consider — the appropriate class for us are admissible pairs of unobstructed curves (the same setup as in [10]), see the previous chapter for the definitions. Thus, having two unobstructed curves $(L_0, L_1)$, one needs to know how to isotope $L_0$ to $L_0'$ so that $(L_0', L_1)$ is admissible.
2. A chain complex $CF(L_0', L_1)$ is generated over $\mathbb{F}_2$ by intersection points $L_0' \cap L_1$. The differential $\partial : CF(L_0', L_1) \to CF(L_0', L_1)$ is defined on generators as mod 2 sum

$$\partial x = \sum_y \mathcal{M}(x, y) \cdot y,$$
where $M(x, y)$ counts the number of immersed discs from $x$ to $y$ in the pillow-case, with right boundary on $L'_0$ and left boundary on $L_1$ (no switches allowed), and convex angles at $x$ and $y$, see Figure 4. One requires these discs to be orientation preserving, and counts them up to reparameterizations, which are orientation preserving diffeomorphisms. The main difficulty in this step is to prove that $M(x, y)$ is finite for any two generators $x, y$.

![Figure 4. Immersed disc from $x$ to $y$. Note that there are no fishtails due to the presence of $\partial P$.](image)

(3) One proves that $\partial^2 = 0$, and, more generally, that $A_\infty$ relations hold. Then the Lagrangian Floer homology is defined by $HF(L_0, L_1) = H_*(CF(L'_0, L_1), \partial)$. The correctness of definition follows from the following two statements.

(4) Suppose $L_0 \sim L'_0$ as basepoint free loops. If $(L_0, L_1)$ and $(L'_0, L_1)$ are both admissible pairs, they can be connected through elementary isotopies (of both $L_0$ and $L_1$) called finger moves (see Figure 7) such that admissibility does not break down on each step.

(5) If admissible pair $(L_0, L_1)$ is connected to admissible pair $(L'_0, L_1)$ by a finger move, then $HF(L_0, L_1) = HF(L'_0, L_1)$.

**Remark.** From these two steps it also follows that Lagrangian Floer homology is invariant with respect to isotopies (isotopies of arcs are considered relative to endpoints).

3.2. **More details.** We will follow the plan, outlined in the previous section, giving more attention to admissibility condition — the only place where our setup is different from [12, Sections 2, 3].

(1) Here we need to show how to isotope $L_0$ to $L'_0$ so that $(L'_0, L_1)$ is admissible. The only problematic part is to rule out immersed annuli. Let us first understand when essential annuli exist at all.

**Definition 1.** A periodic map is a smooth annulus $A: (S^1 \times [0, 1], S^1 \times \{0\}, S^1 \times \{1\}) \to (\overline{P}, L_0, L_1)$ (as always, we assume that no switches are allowed).

Denote by $Per(L_0, L_1)$ the set of homotopy classes of periodic maps.

---

5Because at the domain of the map one has a disc with two right angles, sometimes these discs are called lunes.
Lemma 3.1. \( \text{Per}(L_0, L_1) = \mathbb{Z} \text{ or } \{0\} \). Admissibility can break down, i.e. essential annuli can exist, only if \( \text{Per}(L_0, L_1) = \mathbb{Z} \). This is equivalent to \( pL_0 \sim qL_1 \) as baspoint free loops for some co-prime integers \( p \) and \( q \) (in particular, both curves should be close immersed circles).

Proof. Assume \( L_i : S_i \leftrightarrow \tilde{P} \). No switches on the boundaries of annulus \( A \) allows one to compose boundaries with \( L_i \), i.e. \( A : S^1 \times \{i\} \to S_i \xrightarrow{L_i} \tilde{P} \). Thus, after introducing intersection point between \( L_0 \) and \( L_1 \) via isotopy, if necessary, one gets the following sequence

\[
\text{Per}(L_0, L_1) \to \pi_1(S_0 \times S_1) = \pi_1(S_0) \times \pi_1(S_1) \xrightarrow{L_0 \times L_1} \pi_1(\tilde{P}).
\]

The statement of the lemma follows from the fact that

\[
\text{Per}(L_0, L_1) \cong \text{Ker}(L_0 \times L_1).\]

Surjectivity is straightforward, while injectivity follows from the fact that \( \pi_2(\tilde{P}) = 0 \). \( \square \)

Definition 2. A shadow of \( A \in \text{Per}(L_0, L_1) \) is a two-chain

\[
\text{Sh}(A) = \sum_{\text{open } D_i \subset \tilde{P}} \text{deg}(A|_{D_i}) \cdot \mathcal{D}_i.
\]

The key observation is that for \( A \) to have immersed orientation preserving representative requires \( \text{Sh}(A) \) to have all coefficients positive. We call such shadows positive. In fact, we have:

Lemma 3.2. The following four statements are equivalent:

a) There exists an essential immersed periodic map \( A : (S^1 \times [0, 1], S^1 \times \{0\}, S^1 \times \{1\}) \to (\tilde{P}, L_0, L_1) \), and so \( (L_0, L_1) \) is not admissible.

b) There exists a periodic map \( A \) with positive shadow for a pair \( (L_0, L_1) \) in \( \tilde{P} \).

c) There exists a periodic map \( A \), such that \( \tilde{L}_0 \cap \tilde{L}_1 = \emptyset \), where \( \tilde{L}_i \) is a lift of \( L_i \) to a covering \( \tilde{P} \) corresponding to subgroup \( \text{Im}(A_i) \subset \pi_1(\tilde{P}) \).

d) There exists a periodic map \( A \), such that a pair \( (\tilde{L}_0, \tilde{L}_1) \) in \( \tilde{P} \) has a periodic map with positive shadow.

Proof. Suppose \( \text{Im}(A_i) = \langle pL_0 \rangle = \langle qL_1 \rangle \). Choose a metric so that \( \tilde{P} \) is hyperbolic. Then \( \tilde{P} = \mathbb{H}/\langle \gamma \rangle \), where \( \gamma \) is a translation along a geodesic. This geodesic is a preimage of a geodesic representing \( pL_0 = qL_1 \). Because this translation is fixed point free, it is either parabolic or hyperbolic. This implies that \( \tilde{P} \) is homeomorphic to a cylinder. It is now straightforward to see that d) is equivalent to all other statements. \( \square \)

Now we are prepared to make any pair \( (L_0, L_1) \) admissible. Suppose there is an essential immersed periodic map \( A \). Then isotope one of the curves (say \( \tilde{L}_0 \)) in the covering \( \tilde{P} \) to introduce an intersection with another curve, and then push the isotopy down to pillowcase. Note that if \( \text{Im}(A_i) = \langle pL_0 \rangle = \langle qL_1 \rangle \), then one needs to do isotopies of \( \tilde{L}_0 \) in \( p \) different points, so that it projects down to an isotopy of \( L_0 \).
(2) Here we need to show that $\mathcal{M}(x,y)$ is finite, assuming $(L_0, L_1)$ is admissible. Denote by $\pi_2(x,y)$ the space of homotopy classes of smooth discs from $x$ to $y$. We first show that there are finite number of elements $\phi \in \pi_2(x,y)$, which can possibly have immersed representatives (the relevant condition is shadow $Sh(\phi)$ being positive). Then we show that every such class $\phi$ has exactly one immersed representative from $\mathcal{M}(x,y)$.

Lemma 3.3. In case $\pi_2(x,y) \neq \emptyset$, we have a free and transitive action

$$\text{Per}(L_0, L_1) \cong \pi_2(x,x) \curvearrowright \pi_2(x,y).$$

Proof. The general definition of multiplication

$$\pi_2(x,y) \times \pi_2(y,z) \to \pi_2(x,z), (\phi, \psi) \mapsto \phi \ast \psi$$

is given by pinching an arc in the middle of the disc and considering maps $\phi$ and $\psi$ on the resulting two discs (which are connected by one point). The statement follows from this construction. □

Thus we get that $\pi_2(x,y) = \{\phi\}$, $\mathbb{Z}$ or $\emptyset$. Next, let us prove that in case $\pi_2(x,y) = \mathbb{Z}$ we have only finite number of elements with immersed representatives.

Shadow of an element $\phi \in \pi_2(x,y)$ is defined in the same way as for $A \in \text{Per}(L_0, L_1)$.

Proposition 3.4. Only finite number of elements in $\pi_2(x,y)$ have positive shadow, and thus can have an immersed representative from $\mathcal{M}(x,y)$.

Proof. Every $0 \neq \phi \in \pi_2(x,x)$ has a shadow with both negative and positive coefficients because $(L_0, L_1)$ is admissible (see Lemma 3.2). For $\psi \in \pi_2(x,x) \cong \text{Per}(L_0, L_1)$ and $\phi \in \pi_2(x,y)$ one has $Sh(\psi \ast \phi) = Sh(\psi) + Sh(\phi)$. This, along with Lemma 3.3, implies the statement of the proposition. □

Proposition 3.5. Element $\phi \in \pi_2(x,y)$ can have at most one immersed representative, up to smooth reparameterizations.

Proof. This follows from the fact that $\phi \in \mathcal{M}(x,y)$ can be reconstructed from its positive shadow, see the proof of [SS] Theorem 6.8, which applies in our case after passing to a universal cover and considering its compact submanifold containing immersed discs in question. □

(3) The main idea behind $\delta^2 = 0$ is that pairs of consecutive immersed discs with convex angles come in pairs (here one uses the absence of fishtails), and so they cancel each other. For the details here we refer to [11] Lemma 2.11.

(4) We have isotopy $L'_0$ from admissible $(L_0, L_1)$ to admissible $(L'_0, L_1)$. One has problems with keeping this isotopy admissible only if $\text{Per}(L_0, L_1) = \mathbb{Z}$. Suppose $A$ is a generator of that group. Then, passing to a covering $\tilde{P}$ corresponding to subgroup $\text{Im}(A_*) \subset \pi_1(\tilde{P})$, just like in Lemma 3.2 we can 1) isotope $\tilde{L}_1$ to $\tilde{L}'_1$ in such a way that all isotopies $\tilde{L}'_0$ intersect $\tilde{L}'_1$, and $L_0$, $\tilde{L}'_0$ intersect $\tilde{L}'_1$ at every time 2) do an isotopy $L'_0$ from $L_0$ to $L'_0$ 3) isotopy $\tilde{L}'_1$ back to $\tilde{L}_1$. Both steps 1) and 3) can be done in such a way that isotopy can be projected to $\tilde{P}$. This sequence of isotopies keeps the pair admissible all the time.

---

6In fact one can prove that $M(x,y)$ if finite for not admissible pairs too, but for our purposes we do not need this stronger statement.
One possibility here is to use $A_\infty$ relations to define chain maps between $CF(L_0, L_1)$ and $CF(L'_0, L_1)$ and prove that their composition is homotopic to identity, see [12, Lemma 4.2] (with appropriate change of argument because of the weaker notion of admissibility in our case).

Another approach is to note that a finger move (see Figure 7) on the level of Lagrangian Floer chain complex corresponds to a cancellation of the differential (see [38, Appendix C]). Here one needs to prove that there is exactly one immersed disc between two points on the left of Figure 7. There is only one other possibility, which is a disc covering lower left and lower right domains on the left of Figure 7. But if such immersed disc exists, one would have an immersed annulus on the right of Figure 7, and this would contradict admissibility.

4. Pillowcase algebra

In this section we will first parameterize the pillowcase $P$ by arcs. Then we will associate to this parameterization a dg-algebra $A$.

4.1. Parameterization of the pillowcase. Parameterization consists of basepoints on the boundary $\partial P$, and a set of non-intersecting embedded arcs with ends on $\partial P$. The following properties should be satisfied: each component of $\partial P = S^1 \cup S^1 \cup S^1 \cup S^1$ should get at least one basepoint, and cutting along the arcs one should get a set of discs each having exactly one basepoint on the boundary. We pick a parameterization of the pillowcase as on the left of Figure 5. Sometimes we will call parameterizing arcs the “red arcs”.

4.2. Pillowcase algebra. The parameterization of $P$ specifies a graph $\Gamma$ — the vertices are the arcs in the parameterization, and the edges are chords between the arcs on the boundary of $P$, which do not pass through basepoints. One can see the graph corresponding to our parameterization on the right of Figure 5.

**Definition 3.** Pillowcase algebra $A$ is a path algebra of the graph $\Gamma$. It means that it is generated over $F_2$ by paths in $\Gamma$ consisting of edges of one color (or same letters), and concatenating of paths corresponds to multiplication. When concatenating is not possible, or gives a path with edges of different colors, the multiplication results in zero. We mentioned that we want to have a dg-algebra corresponding to pillowcase — we define differential to be trivial on $A$. Subalgebra generated by vertices $I = \langle i_0, i_1, i_2, j_0, j_1, j_2 \rangle$ is called idempotent subalgebra.

**Explicit description of $A$.** Algebra $A$ is generated by the following elements (we specify here only those non-trivial multiplications which do not involve vertices):

\begin{equation}
A = \langle i_0, i_1, i_2, j_0, j_1, j_2, \rho_0, \rho_1, \xi_1, \xi_2, \xi_3, \\
\xi_{12} = \xi_1 \xi_2, \xi_{23} = \xi_2 \xi_3, \xi_{123} = \xi_1 \xi_2 \xi_3 = \xi_1 \xi_2 = \xi_1 \xi_3 = \xi_2 \xi_3 = \xi_1 \xi_2 = \xi_1 \xi_3 \\
\eta_1, \eta_2, \eta_3, \eta_{12} = \eta_1 \eta_2, \eta_{23} = \eta_2 \eta_3, \eta_{123} = \eta_1 \eta_2 \eta_3 = \eta_1 \eta_2 = \eta_1 \eta_3 = \eta_2 \eta_3 \rangle_{F_2}.
\end{equation}

Regarding multiplications which involve vertices: notice that constant paths (i.e. the vertices) are idempotents, and every path in $A$ has its own left and right idempotent. These idempotents correspond to vertices of the start and the end of the path. All other vertices annihilate the path. For example, for the path $\xi_{12}$ we have $i_1 \xi_{12} j_2 = \xi_{12}$, and multiplication by other idempotents results in zero.
Path algebra $\mathcal{A}$, which is generated over field $\mathbb{F}_2$ by the paths of one color in the graph below. Multiplication corresponds to concatenating paths.

**Figure 5.** Parameterization of the pillowcase, and the corresponding algebra.

### 5. FROM CURVES TO MODULES

To an immersed curve $L$ in $P$ we associate a right $A_\infty$ module $M(L)_A$ over the algebra $\mathcal{A}$. Before defining the module, one needs to isotope $L$ appropriately.

#### 5.1. Preliminary isotopies of $L$. First, if $L$ is an arc, one makes a perturbation of $L$ in the small neighborhood of $\partial \overline{P}$ by applying twist along the orientation of $\partial \overline{P}$, s.t. the end comes close to a basepoint passing all the parameterizing arcs on its way, see Figure 6. This ensures that all the parameterizing arcs $i$ are admissible with $L$ as a pair $(i, L)$.

**Figure 6.** Perturbation near the boundary.
In fact, this perturbation is enough to define \( M(A) \) up to homotopy, but for the further simplification of \( M(L) \), and for having a concrete \( M(L) \) rather then a homotopy type, we will do the following extra isotopy.

First, fix the notation of arcs, and discs on which they cut the pillowcase as on the Figure 10. Consider traversing along \( L \) on the pillowcase — this traversing (up to isotopies of \( L \) which do not change intersections with arcs) is encoded in the cyclic sequence \( S(L) \) of discs \( B_k \) (we call them big domains), which are visited by \( L \), as well as the connecting arcs between them. For example for the curve \( L^5 \) on the Figure 10 we have a cyclic sequence

\[
S(L^5) = B_1j_2B_4i_2B_1i_0B_2j_0B_1j_1B_3i_1.
\]

If \( L \) is an arc then the sequence is not cyclic.

We isotope a curve \( L \) further, so that the sequence \( S(L) \) it gives does not have the same arcs around one big domain, i.e. it does not have a pattern \( iB_i \). Such an isotopy exists because if one has such a pattern, there is a finger move isotopy of \( L \) removing \( iB_i \) from the sequence \( S(L) \), see Figure 7. The length of the sequence decreases, so the process of doing such finger move isotopies has to stop.

\[
\text{Figure 7. Finger move isotopy.}
\]

5.2. Definition of \( M(L) \). We now assume that \( L \) is perturbed and isotoped according to the previous section. A right \( A_\infty \) module \( M(L)_A \) is defined as follows. Over \( \mathbb{F}_2 \) it is generated by all intersection of the curve \( L \) with the red arcs. For example, for the curve \( L^5 \) on the Figure 10 we have \( M(L^5) =< z, w, t, y, x >_{\mathbb{F}_2} \).

Idempotent subalgebra \( I \) acts on \( M(L) \) from the right: every generator has a unique idempotent which preserves it, this idempotent corresponds to the arc on which this generator is sitting. Other idempotents annihilate the generator. For example for \( M(L^5) \) one gets the following idempotents for the generators: \( z_{i_0}, w_{j_0}, s_{j_1}, t_{i_1}, y_{j_2}, x_{i_2} \).

Now, let us explain how \( M(L) \) can be recovered combinatorially, by doing it on the example curve \( L^5 \) on the Figure 10. First, let us count all the “basic” discs, which are contained entirely in one of the big domains \( B_i \). Because there is one basepoint in each big domain \( B_i \), there is exactly one basic disc between two consecutive generators. For the curve \( L^5 \) from Figure 10, here are all the basic discs: \( z \xrightarrow{D_2} x, y \xrightarrow{D_0} x, t \xrightarrow{D_1} y, s \xrightarrow{D_4+D_3} w, s \xrightarrow{D_4+D_3} w \). Thus first we get a circle (if \( L \) is an arc one gets a sequence) of generators and actions between them, see Figure 8 for an example of the module \( M(L^5) \). Notice that except the basic actions that form a circle there are two extra actions: \( z \xrightarrow{D_4+D_3+D_5} s, t \xrightarrow{D_0+D_3} x \). These are the actions which correspond to discs which are formed by juxtaposing basic
discs along the arcs. These discs ensure that $d^2 = 0$ in our $A_\infty$ module. Note that every immersed disc can be decomposed into basic discs. Also every basic disc is contained in the finite number of discs — otherwise one would have a cycle of chords on the $\partial P$, and this is not possible because the graph $\Gamma$ has no cycles.

Figure 8. Immersed discs of this type define $A_\infty$ actions of the algebra $A$ on the module $M(L)$.

Figure 9. $M(L^3),_A$, where $L^3$ is from Figure 10.
Here we will describe how to compute $HF(L_0, L_1)$ in terms of $M(L_0)_A$ and $M(L_1)_A$. One can skip most of this section, and start reading from the Definition 6. The material before that definition is included to show how we arrived at that definition.

6.1. Koszul dual algebra. First, note that our algebra $A$ is a 1-strand moving algebra $A(Z, 1)$ (see [29, Definition 2.6]) of the arced diagram $Z$ drawn on Figure 11.

**Definition 4.** Let us define a new algebra $A^{5s}$ as 6-1=5-strand moving algebra $A(Z, 5)$ of the same arc diagram. We call it Koszul dual algebra to algebra $A$. 

![Figure 10. Curve $L^5$ on the pillowcase.](image)

6. Algebraic pairing

Here we will describe how to compute $HF(L_0, L_1)$ in terms of $M(L_0)_A$ and $M(L_1)_A$. One can skip most of this section, and start reading from the Definition 6. The material before that definition is included to show how we arrived at that definition.

6.1. Koszul dual algebra. First, note that our algebra $A$ is a 1-strand moving algebra $A(Z, 1)$ (see [29, Definition 2.6]) of the arced diagram $Z$ drawn on Figure 11.

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![Figure 10. Curve $L^5$ on the pillowcase.](image)
Explicit description of $A^{5s}$. First, consider a new graph $\Gamma'$ on Figure 12 consisting of the reversed paths in graph $\Gamma$ from Figure 5. Our algebra is a path algebra of graph $\Gamma'$, i.e.

$$ A^{5s} = \langle \text{paths in } \Gamma' \rangle >_{\mathbb{Z}}. $$

Notice that now all the paths are of the same color. Let us denote the non-idempotent elements by $i' a(e_1, e_2, \ldots, e_m) j'$, where $e_i$ is an edge in $\Gamma'$, and indices are the start and the end of the path. As before, multiplication corresponds to concatenating paths, and so non-zero multiplications are all of the form

$$ i' a(e_1, e_2, \ldots, e_m) j' \cdot j' a(e_{m+1}, e_{m+2}, \ldots, e_{m+l}) k' = i' a(e_1, e_2, \ldots, e_{m+l}) k'. $$

There is a natural 1-1 correspondence between idempotents of $A$ and $A^{5s}$ given by $i \leftrightarrow i'$. Also, edge $e$ in graph $\Gamma'$ naturally gives an element $a_A(e) \in A$ by “reversing” the path, for example, $\xi'_3 \mapsto \xi_3$.

Differential this time is not zero. First, we specify differential on “linear” elements, consisting of one edge (as always, we list only non-zero differentials):

$$
\begin{align*}
    d(a(\xi'_2)) &= a(\xi'_2, \xi'_1), \\
    d(a(\xi'_3)) &= a(\xi'_3, \xi'_2), \\
    d(a(\xi'_3, \xi'_2)) &= a(\xi'_3, \xi'_1) + a(\xi'_3, \xi'_2), \\
    d(a(\eta'_1)) &= a(\eta'_2, \eta'_1), \\
    d(a(\eta'_2)) &= a(\eta'_3, \eta'_2), \\
    d(a(\eta'_3)) &= a(\eta'_3, \eta'_1) + a(\eta'_3, \eta'_2).
\end{align*}
$$

These induce differential on paths that consist of more edges by Leibniz rule. For example, for 3-edge paths one has

$$ d(a(e_1, e_2, e_3)) = d(a(e_1) \cdot a(e_2) \cdot a(e_3)) = d(a(e_1)) \cdot a(e_2) \cdot a(e_3) + a(e_1) \cdot d(a(e_2)) \cdot a(e_3) + a(e_1) \cdot a(e_2) \cdot d(a(e_3)). $$

Remark. For the clarity we repeat here our notation: elements of algebra $A$ are denoted by letters as described in [4,1]. Some of those letters are edges of the graph $\Gamma$. For elements of the algebra $A^{5s}$ the notation is $a(e_1, \ldots, e_m)$, where $e_i$ are the edges of the graph $\Gamma'$. Each edge $e$ in graph $\Gamma'$ naturally gives an element $a_A(e) \in A$ by “reversing” the path.

Remark. Although we will not use it, let us note that algebra $A^{5s} = A(\mathbb{Z}, 5)$ is Koszul dual to $A = A(\mathbb{Z}, 1)$ in the sense of [26, Definition 8.5]. The proof is the same as for Koszul.

**Figure 11.** Arc diagram $Z$, whose 1-strand moving algebra is $A$. 

\[ \begin{array}{cccc}
\rho_0 & i_0 & j_0 & \rho_2 \\
\eta_1 & i_1 & j_1 & \eta_2 \\
\eta_3 & \xi_1 & \xi_2 & \xi_3 \\
\end{array} \]
duality of bordered algebras in bordered Heegaard Floer homology, see [26, Proposition 8.17]. Let us describe the rank-1 (over $I$) Koszul dualizing bimodule $\mathcal{A} K_1^{5s}$. We define

$$K = \langle (i_0, i_0'), (i_2, i_2'), (j_0, j_0'), (j_1, j_1'), (j_1, j_1') \rangle_{F_2} \cong 1 > I,$$

with differential $\delta^1 : K \to \mathcal{A} \otimes I K \otimes I \mathcal{A}^{5s}$ given by

$$\delta^1(k, k') = \sum_{s', e_{i_i'} \text{ edge in } \Gamma'} k a_A(e)_s \otimes (s, s') \otimes (a(s', e_{k'})).$$

6.2. DD bimodule, and the pairing.

**Definition 5.** Dual small bar resolution of algebra $\mathcal{A}$ is a type DD structure $\mathcal{A} \text{bar} \mathcal{A}$, whose generators correspond to elements of $\mathcal{A}^{5s}$, i.e. each element $i a(e_1, e_2, \ldots, e_l)_j \in \mathcal{A}^{5s}$ gives an element $b(e_1, e_2, \ldots, e_l)_j \in \text{bar}$. The type DD structure on $\text{bar}$ over $\mathcal{A}$ is given by:

$$\delta^1 : \text{bar} \to \mathcal{A} \otimes I \text{bar} \otimes I \mathcal{A},$$

$$\delta^1 (i b(e_1, e_2, \ldots, e_l)_j) = \sum_{e \in \text{Edges}(\Gamma'), \text{start}(e) = j'} 1 \otimes b(e_1, e_2, \ldots, e_l)_k \otimes k a_A(e)_j +$$

$$+ \sum_{e \in \text{Edges}(\Gamma'), \text{end}(e) = j'} i a_A(e)_k \otimes k b(e, e_1, e_2, \ldots, e_l)_j \otimes 1 +$$

$$+ \sum_{e_i \in \{e_1, e_2, \ldots, e_l\}} 1 \otimes b(e_1, e_2, \ldots, e_{i-1}) \cdot d(b(e_i)) \cdot b(e_{i+1}, e_2, \ldots, e_l) \otimes 1.$$

For the explicit description of elements and actions see Appendix. For convenience let us write here one example of how differential acts:
These are the actions of \( A_{\text{bar}}^{-A} \) preserving its homotopy type. For that there exists a convenient tool called “cancellation”. Suppose there are two generators in a DD bimodule \( A_{\text{bar}}^{-A} \) satisfying \( \delta^1(x) = y + \ldots \), i.e. there is only one action from \( x \) to \( y \), and it does not have any outgoing algebra elements (an example would be \( \delta^1(b(\eta, \xi, \zeta)) = 1 \otimes b(\eta, \xi, \zeta, \xi, \eta) \otimes 1 \)). Then one can cancel these two generators, i.e., first, erase \( x, y \) and the arrows involving them from the bimodule, and second, add some other arrows between the generators left in the bimodule, guided by a certain cancellation rule. The outcome is a bimodule \( A_{\text{bar}}^{-A} \) with less generators, and which is homotopy equivalent to the previous one \( A_{\text{bar}}^{-A} \simeq A_{\text{bar}}^{-A} \). See [10] Section 3.1 for the details of how cancellation works.

We want to cancel all possible differentials in \( A_{\text{bar}}^{-A} \). It turns out it does not matter which differentials and in which order one cancels. In the end one gets a bimodule \( A_{\text{bar}}^{-A} \) with no other possible cancellations. Instead of proving this, let us define \( A_{\text{bar}}^{-A} \) explicitly below. Note that below we change the notation: instead of primes we will write minuses, i.e. \( b(\xi^1) \) becomes \( b(-\xi) \), except for the constant path elements, in which case \( b(i_0) \) becomes \( b(i_0) \).

**Definition 6.** Reduced small bar resolution \( A_{\text{bar}}^{-A} \) of algebra \( A \) is a type DD structure which consists of the following 24 generators (we list them with their idempotents):

\[
\begin{align*}
  i_2(b(i_2))_{i_2,i_0} & (b(i_0))_{i_0,j_1} (b(j_1))_{j_1,j_2} (b(j_2))_{j_2,j_0} (b(j_0))_{j_0,i_1} (b(i_1))_{i_1}, \\
  j_0(b(-\rho_2))_{i_0,j_1} (b(-\eta_2))_{j_1,j_2} (b(-\xi_2))_{j_2,i_2}, \\
  j_0(b(-\eta_3))_{j_1,j_2} (b(-\xi_3))_{j_2,j_0} (b(-\rho_2))_{j_0,i_2} (b(-\xi_1))_{i_2,i_1}, \\
  j_2(b(-\rho_2,-\xi_2))_{i_2,i_1} (b(-\xi_3,-\rho_2))_{i_1,i_0} (b(-\xi_3,-\eta_3))_{i_0,i_2}, \\
  j_0(b(-\eta_3,-\xi_3,-\rho_2))_{i_2,i_1} (b(-\xi_3,-\rho_2,-\xi_1))_{i_1,i_0} (b(-\eta_3,-\xi_3,-\rho_2))_{i_0,i_2}, \\
  j_0(b(-\eta_3,-\xi_3,-\rho_2,-\xi_1))_{i_0,i_2} (b(-\eta_3,-\xi_3,-\rho_2,-\xi_1))_{i_2,i_1}.
\end{align*}
\]

These are the actions of \( A_{\text{bar}}^{-A} \):

\[
\begin{align*}
  b(-\eta_3,-\xi_3,-\rho_2,-\xi_1) & \rightarrow 1 \otimes b(-\eta_3,-\xi_3,-\rho_2,-\xi_1,-\eta_1) \otimes \eta_1, \\
  b(-\rho_2,-\xi_1,-\eta_1) & \rightarrow \xi_3 \otimes b(-\eta_3,-\xi_3,-\rho_2,-\xi_1,-\eta_1) \otimes \eta_1, \\
  b(-\xi_3,-\rho_2,-\xi_1,-\eta_1) & \otimes 1, \\
  b(-\rho_2,-\xi_1,-\eta_1) & \rightarrow \xi_3 \otimes b(-\eta_3,-\xi_3,-\rho_2,-\xi_1,-\eta_1) \otimes \eta_1, \\
  b(-\xi_3,-\rho_2,-\xi_1,-\eta_1) & \otimes 1, \\
  b(-\rho_2,-\xi_1,-\eta_1) & \rightarrow \xi_3 \otimes b(-\eta_3,-\xi_3,-\rho_2,-\xi_1,-\eta_1) \otimes \eta_1, \\
  b(-\xi_3,-\rho_2,-\xi_1,-\eta_1) & \otimes 1, \\
  b(-\rho_2,-\xi_1,-\eta_1) & \rightarrow \xi_3 \otimes b(-\eta_3,-\xi_3,-\rho_2,-\xi_1,-\eta_1) \otimes \eta_1, \\
  b(-\xi_3,-\rho_2,-\xi_1,-\eta_1) & \otimes 1, \\
  b(-\rho_2,-\xi_1,-\eta_1) & \rightarrow \xi_3 \otimes b(-\eta_3,-\xi_3,-\rho_2,-\xi_1,-\eta_1) \otimes \eta_1, \\
  b(-\xi_3,-\rho_2,-\xi_1,-\eta_1) & \otimes 1, \\
  b(-\rho_2,-\xi_1,-\eta_1) & \rightarrow \xi_3 \otimes b(-\eta_3,-\xi_3,-\rho_2,-\xi_1,-\eta_1) \otimes \eta_1, \\
  b(-\xi_3,-\rho_2,-\xi_1,-\eta_1) & \otimes 1, \\
  b(-\rho_2,-\xi_1,-\eta_1) & \rightarrow \xi_3 \otimes b(-\eta_3,-\xi_3,-\rho_2,-\xi_1,-\eta_1) \otimes \eta_1, \\
  b(-\xi_3,-\rho_2,-\xi_1,-\eta_1) & \otimes 1, \\
  b(-\rho_2,-\xi_1,-\eta_1) & \rightarrow \xi_3 \otimes b(-\eta_3,-\xi_3,-\rho_2,-\xi_1,-\eta_1) \otimes \eta_1, \\
  b(-\xi_3,-\rho_2,-\xi_1,-\eta_1) & \otimes 1, \\
  b(-\rho_2,-\xi_1,-\eta_1) & \rightarrow \xi_3 \otimes b(-\eta_3,-\xi_3,-\rho_2,-\xi_1,-\eta_1) \otimes \eta_1, \\
  b(-\xi_3,-\rho_2,-\xi_1,-\eta_1) & \otimes 1, 
\end{align*}
\]
\( \xi_3 \otimes b(-\xi_3, -\rho_2, -\xi_1) \otimes 1, b(i_0) \rightarrow \rho_0 \otimes b(-\rho_0) \otimes 1, b(i_0) \rightarrow \eta_1 \otimes b(-\eta_1) \otimes 1, b(-\xi_1) \rightarrow \rho_2 \otimes b(-\rho_2, -\xi_1) \otimes 1, b(-\xi_1) \rightarrow 1 \otimes b(-\xi_1, -\eta_1) \otimes \eta_1, b(-\eta_1) \rightarrow \xi_1 \otimes b(-\xi_1, -\eta_1) \otimes 1, b(-\xi_1) \rightarrow 1 \otimes b(-\xi_3, -\rho_2) \otimes \rho_2, b(-\xi_3) \rightarrow \eta_3 \otimes b(-\eta_3, -\xi_3) \otimes 1, b(-\rho_2) \rightarrow 1 \otimes b(-\rho_2, -\xi_1) \otimes \xi_1, b(-\rho_2) \rightarrow \xi_3 \otimes b(-\xi_3, -\rho_2) \otimes 1, b(-\eta_3, -\xi_3, -\rho_2, -\xi_1) \rightarrow 1 \otimes b(-\xi_3, -\rho_2, -\xi_1, -\eta_1) \otimes \eta_1, b(-\xi_3, -\rho_2) \rightarrow \eta_3 \otimes b(-\eta_3, -\xi_3, -\rho_2) \otimes 1, b(-\xi_3, -\rho_2) \rightarrow 1 \otimes b(-\xi_3, -\rho_2, -\xi_1) \otimes \xi_1, b(-\eta_3, -\xi_3) \rightarrow 1 \otimes b(-\eta_3, -\xi_3, -\rho_2) \otimes \rho_2, b(-\xi_1, -\eta_1) \rightarrow \rho_2 \otimes b(-\rho_2, -\xi_2, -\eta_1) \otimes 1. \)

Looking at the left of Figure 13, it is convenient to see generators of \( A_{\bar{\text{bar}}} A \) as paths (against orientation) on the boundaries of big domains \( B_2, B_3, B_4, B_1 \) and encoding algebra elements one encounters on that path. Differential then corresponds to prolonging paths by one chord. See Figure 13 for an example of this geometric interpretation of the differential.

**Figure 13.** Geometric interpretation of the differential in \( A_{\bar{\text{bar}}} A \).

**Algebraic pairing.** First, let us refer to [23] Section 2.3 for definitions of dual \( A_\infty \) modules and type D-structures. Having an \( A_\infty \) module \( M_A \), we denote its dual by \( \bar{\text{A}} M \). Now we are ready to define algebraic pairing of curves in the pillowcase.

**Definition 7.** Suppose \( L_0, L_1 \) are two admissible curves in the pillowcase \( P \). An algebraic pairing of curves is given by a complex

\[ M(L_1)_A \otimes_{\bar{\text{A}}} A \otimes_{\bar{\text{A}}} M(L_0). \]

**Remark.** The way we constructed \( A_{\bar{\text{bar}}} A \) ensures that \( M(L_1)_A \otimes_{\bar{\text{A}}} A \otimes_{\bar{\text{A}}} M(L_0) \simeq Mor(M(L_0)_A, M(L_1)_A) \), see Section 1.5.
7. Pairings are the same

**Theorem 7.1.** Let $L_0, L_1$ be two admissible unobstructed curves in the pillowcase $\overline{P}$. Then their Lagrangian Floer complex is homotopy equivalent to the algebraic pairing of curves:

$$CF_*(L_0, L_1) \simeq M(L_1)_A \otimes_A \overline{M} M(L_0).$$

*Proof.* The plan for the proof is the following:

1. We first isotope curves $L_0, L_1$ in a certain way.
2. We prove that pair $(L_0, L_1)$ is admissible.
3. We then prove that the two chain complexes we consider are isomorphic: $CF_*(L_0, L_1) \simeq M(L_1)_A \otimes_A \overline{M} M(L_0)$.

   a) We first see that generators are in 1-1 correspondence.

   b) We then prove that differentials coincide. This is done by “localizing” differential in the geometric pairing, i.e. by noting that every disc contributing to differential is contained almost entirely in one of the big domains $B_1, B_2, B_3, B_4$.

We will be illustrating each step on our running example of curves: $L_0 = L^3$ — curve on Figure 10, that corresponds to the trivial tangle $A_1 \cup A_2$, and $L_1 = L^b$ — belt around the pillowcase on the left of Figure 2. For $A_\infty$ actions on the dual module $\hat{M}(L^3)$ see Figure 19 and $A_\infty$ actions on $M(L^0)_A = \angle x, s, z, w >_{r_2}$ (see Figure 16) are as follows: $z \otimes (\xi_3, r_3) \to x$, $z \otimes r_0 \to x$, $z \otimes (\eta_1, \xi_1) \to w$, $w \otimes \xi_2 \to s$.

(1). For a short visual description of the required isotopy one may look at Figure 16. Let us describe it now.

First and foremost, one needs to isotope both curves $L_0, L_1$ in such a way, that one can see modules $M(L_0)_A, M(L_1)_A$ geometrically. For that see Section 5.1.

Let us describe further isotopies of $L_0$. Mark four points $b_1, b_2, b_3, b_4$ in the big domains $B_1, B_2, B_3, B_4$ like in Figure 16 and call them centers of the big domains. Then isotope the curve $L_0$ in the following way: first make it intersect every red arc near its center (centers of arcs are marked on the Figure 16). Then isotope $L_0$ so that in big domains it goes from the centers of big domains to the centers of red arcs straight (or, if $L_0$ is an arc, to the point on $\partial P$, see Figure 15). See Figure 16 for how the isotoped $L_0$ looks like.

Concerning curve $L_1$, we also make it intersect the red arcs near their centers. But the rest of the isotopy is different from $L_0$. First, tilt the angle in which it intersects the centers of the red arcs, so that the following is true. 1) $L_1$ is almost parallel to red arcs and intersects each nearby piece of $L_0$ exactly once. 2) Going clockwise around the center of the red arc, one encounters the rays in the following order: red arc, all the pieces of $L_1$, all the pieces of $L_0$. See Figure 14.

Next, we make the final isotopy of the $L_1$ curve, which has to do with the way it behaves inside the big domains. Divide $L_1$ on the segments by intersections with red arcs. We already specified how $L_1$ looks near those intersections. Now we will describe how each segment between those intersections is isoted, by traversing $L_1$. First, important thing to note, the whole $L_1$ will not leave the small neighborhood of $\partial P \cup \{\text{red arcs}\}$. One starts at the center of the red arc, enters one of the big domains, and then goes near the $\partial P \cup \{\text{red arcs}\}$ in that domain until it reaches a basepoint. If this is the end segment of $L_1$ being an arc, then $L_1$ is connected to $\partial P$ near that basepoint, such that $(L_0, L_1)$ is admissible, see Figure 15. Otherwise $L_1$ turns by $360^\circ$ (in the direction towards the other end of the segment, i.e. such that it does not introduce a fishtail), and goes backwards until it reaches the other end of the segment. See Figure 16 for how the isotoped $L_1$ looks like.
(2). Let us prove admissibility of \((L_0, L_1)\). All the non-smooth corners, triple intersections, non-transverse intersections are eliminated by introducing a slight perturbation. There are no immersed annuli because of intersections introduced on Figure 14. If \(pL_0 \sim qL_1\), and so \(\text{Per}(L_0, L_1) = \mathbb{Z}\), those intersections lift to the covering from the Lemma 3.2 d). If \(L_0, L_1\) are arcs, they are in admissible position relative to basepoints because we ensured it while isotoping \(L_1\), see Figure 15.
Figure 16. Isotopies of $L_0 = L^i$ and $L_1 = L^b$ so that the chain complexes of geometric and algebraic pairings become isomorphic: $CF_*(L_0, L_1) \cong M(L_1)_A \otimes A \bar{bar}_r A \otimes_A M(L_0)$.

(3a). Generators of $CF_*(L_0, L_1)$, as well as generators of $M(L_1)_A \otimes A \bar{bar}_r A \otimes_A M(L_0)$, are in 1-1 correspondence with the set of paths along $\partial B_i$, from intersections $L_1 \cap \{i_0, i_1, i_2, j_0, j_1, j_2\} = \{\text{generators of } M(L_1)\}$ to intersections $L_0 \cap \{i_0, i_1, i_2, j_0, j_1, j_2\} = \{\text{generators of } M(L_0)\}$. The paths are against natural orientations of $\partial B_i$, and consist of chords $-\gamma$ of length 1 (which are also elements of $\mathcal{A}$). Let us explain how to see those paths.

Remember that elements of $A \bar{bar}_r A$ naturally correspond to such paths, see Figure 13. For generators in $M(L_1)_A \otimes A \bar{bar}_r A \otimes_A M(L_0)$, they each have their element of $A \bar{bar}_r A$ in the center, and that describes the path from a generator in $M(L_1)$ to generator in $M(L_0)$. For generators of $CF_*(L_0, L_1)$, the desired path can be traversed along the $L_1$, see Figure 24.
Notice that we include “0 length” paths, which correspond to intersections introduced when both $L_0, L_1$ cross the same red arc, see Figure 14.

Considering example on Figure 16 we have:

$$M(L_1) =< x, s, z, w >_{L_0}, M(L_0) =< x^*, s^*, z^*, w^*, t^*, y^* >_{L_0}. $$

The intersection points $L_0 \cap L_1$ on Figure 16 correspond to the generators of $M(L_1)_A \boxtimes^A \bar{M}(L_0)$ in the following way:

$$w \boxtimes b(i_2) \boxtimes w^* \leftrightarrow w,$$
$$s \boxtimes b(j_2) \boxtimes s^* \leftrightarrow s,$$
$$x \boxtimes b(j_0) \boxtimes x^* \leftrightarrow x,$$
$$z \boxtimes b(i_0) \boxtimes z^* \leftrightarrow z,$$
$$s \boxtimes b(-\rho_2, -\xi_1, -\eta_1) \boxtimes z^* \leftrightarrow p_3,$$
$$s \boxtimes b(-\rho_2) \boxtimes w^* \leftrightarrow p_9,$$
$$s \boxtimes b(-\xi_2) \boxtimes w^* \leftrightarrow p_{11},$$
$$s \boxtimes b(-\rho_2, -\xi_1) \boxtimes t^* \leftrightarrow p_6,$$
$$w \boxtimes b(-\xi_1, -\eta_1) \boxtimes z^* \leftrightarrow p_2,$$
$$w \boxtimes b(-\xi_1) \boxtimes t^* \leftrightarrow p_7,$$
$$x \boxtimes b(-\eta_3) \boxtimes y^* \leftrightarrow p_{12},$$
$$x \boxtimes b(-\eta_3, -\xi_3, -\rho_2, -\xi_1, -\eta_1) \boxtimes z^* \leftrightarrow p_4,$$
$$x \boxtimes b(-\eta_3, -\xi_3) \boxtimes s^* \leftrightarrow p_{10},$$
$$x \boxtimes b(-\eta_3, -\xi_3, -\rho_2, -\xi_1) \boxtimes t^* \leftrightarrow p_5,$$
$$x \boxtimes b(-\eta_3, -\xi_3, -\rho_2) \boxtimes w^* \leftrightarrow p_8,$$
$$x \boxtimes b(-\rho_0) \boxtimes z^* \leftrightarrow p_1.$$

Here we want to show that differentials in $\text{CF}_*(L_0, L_1)$ and $M(L_1)_A \boxtimes^A \bar{M}(L_0)$ coincide. We will do it by partitioning both differentials into smaller groups, and showing how smaller groups correspond to each other.

**Lemma 7.2.** Every immersed disc contributing to differential in $\text{CF}_*(L_0, L_1)$ is contained inside a small neighborhood of one of the big domains $B_1, B_2, B_3, B_4$.

**Proof.** For an immersed disc to go from one big domain to another big domain, it must pass through an intersection of type $q \boxtimes b(i) \boxtimes k^*$, because the disc is not allowed to touch the $\partial\bar{P}$. Here, in the notation, we use 1-1 correspondence between generators of $\text{CF}_*(L_0, L_1)$ and generators of $M(L_1)_A \boxtimes^A \bar{M}(L_0)$. Such intersections happen when both $L_0$ and $L_1$ cross the red arc, i.e. they correspond to “0 length” paths, see Figure 14. Also only two opposite parts of the corner $q \boxtimes b(i) \boxtimes k^* \in L_0 \cap L_1$ are allowed to be filled by the disc. Thus the disc cannot pass through such intersection point, as such disc cannot be immersed. □

**Remark.** We need to consider small neighborhoods of the big domains, as intersections of type $q \boxtimes b(i) \boxtimes k^*$ are not happening exactly on the red arc, but rather somewhere close to its center, see Figure 14.
Lemma 7.3. Every differential in $M(L_1)_A \bar{\boxtimes} A \bar{\boxtimes} A \bar{\boxtimes} A \bar{\boxtimes} A M(L_0)$ contains $A_\infty$ action either on the $M(L_1)$ side (Figure 17), or on the $M(L_0)$ side (Figure 18), but not on both sides. Moreover, every such $A_\infty$ action comes from a “basic” disc in the definition of $M(L_i)_A$, i.e. a disc contained entirely in one of the big domains.

Proof. The first observation is that $A \bar{\boxtimes} A \bar{\boxtimes} A$ does not have differentials with algebra elements outgoing on both sides.

This implies that the chain complex structure does not depend on the brackets placement:

$$(M(L_1)_A \boxtimes M(L_0)) = M(L_1)_A \boxtimes (M(L_1)_A \boxtimes M(L_0)) = M(L_1)_A \boxtimes \bar{\boxtimes} A \bar{\boxtimes} A M(L_0).$$

Usually only the homotopy type of the box tensor product does not depend on the brackets placement. Also, it implies that the differential is either on the $M(L_1)$ side (Figure 17), or on the $M(L_0)$ side (Figure 18). We call them 1st and 2nd types of differentials.

The second observation is that differentials in $A \bar{\boxtimes} A \bar{\boxtimes} A$ do not contain outgoing algebra elements of chord length more then 1 (an example of chord length two algebra element is $\xi_{12}$). This observation implies the last statement of the lemma.

Now let us take one connected segment $l_0$ of $L_0$, cut out by a small neighborhood of a big domain $N(B_k)$. And also take one connected segment $l_1$ of $L_1$ cut out by the same neighborhood $N(B_k)$. These segments almost coincide with two of the segments from the division of $L_0$ and $L_1$ by the intersections with red arcs. Their behavior inside $N(B_k)$ is completely described by our isotopy in step (1). Also note, that such segments correspond to basic discs in the definition of $M(L_i)_A$. These basic discs have a chance to contribute to differential in $M(L_1)_A \boxtimes \bar{\boxtimes} A \bar{\boxtimes} A \bar{\boxtimes} A \bar{\boxtimes} A M(L_0)$.

Due to the first lemma above, the differential in $CF^*(L_0, L_1)$ is partitioned into differentials with boundaries on segments $l_0, l_1$. We will denote such groups of differentials by $CF^*_i(l_0, l_1)$. Due to the second lemma above, the differential in $M(L_i)_A \boxtimes \bar{\boxtimes} A \bar{\boxtimes} A M(L_0)$ is partitioned into differentials using different basic discs. We will denote such groups of differentials by $M(l_i)_A \boxtimes \bar{\boxtimes} A \bar{\boxtimes} A M(l_0)$, as basic discs correspond to segments.

We are left to show how differentials in $CF^*(l_0, l_1)$ correspond to differentials in $M(l_1)_A \boxtimes \bar{\boxtimes} A \bar{\boxtimes} A \bar{\boxtimes} A \bar{\boxtimes} A M(l_0)$ separately.

The 1st type. In this case the differential $M(l_1)_A \boxtimes \bar{\boxtimes} A \bar{\boxtimes} A \bar{\boxtimes} A \bar{\boxtimes} A M(l_0)$ has outgoing algebra elements on the left, see Figure 17. This corresponds to prolongation of the path in the backward direction, i.e. new length one chords are concatenated to the path from the left. We use the following notation on Figure 17: $i, k$ are the red arcs intersecting $l_1$ at $(u_1)$ and $(u_2)$, $j$ is the red arc intersecting $l_0$ at $(v)$, and $\gamma_m$ are representing chord length one elements of $A$.

Let us describe the corresponding disc differentials in $CF(l_0, l_1)$. Suppose the disc goes from $p$ to $q$. First, note that all intersections $l_0 \cap l_1$ are happening near one of the two ends of segment $l_0$. Points $p$ and $q$ can be on one end of the segment $l_0$, or on the different ends. Let us consider those pairs, which are on one end of the segment $l_0$. For this to happen, traversing the $l_1$ boundary of the disc, the $l_1$ must pass the $360^\circ$ rotation point and come back. See, for example, the disc from $p_6$ to $p_5$ on Figure 16. We say that such differentials are of the 1st type in $CF(l_0, l_1)$.
These are precisely the discs that correspond to the 1st type of differentials in $M(L_1)_A \boxtimes \text{bar}r^A \boxtimes_A \overline{M(L_0)}$. The reason is that both 1st type of differentials in $M(L_1)_A \boxtimes \text{bar}r^A \boxtimes_A \overline{M(L_0)}$, and 1st type of differentials in $CF(l_0, l_1)$ exist if and only if $(u_2)_k$ is before $(u_1)_i$ is before $j(v)$ w.r.t. the basepoint and the direction against natural orientation of $\partial B_i$. For example, on Figure 16 the disc from $p_6$ to $p_5$ corresponds to differential $s \boxtimes b(-\rho_2, -\gamma_1) \boxtimes t^* \to x \boxtimes b(-\eta_3, -\xi_3, -\rho_2, -\xi_1) \boxtimes t^*$.

The 2nd type. In this case the differential $M(L_1)_A \boxtimes \text{bar}r^A \boxtimes_A \overline{M(L_0)}$ has outgoing algebra elements on the right, see Figure 18. This corresponds to prolongation of the path in the forward direction, i.e. new length one chords are concatenated to the path from the right. We use the following notation on Figure 18: $i$ is the red arc intersecting $l_1$ at $(u)$, $j$, $o$ are the red arcs intersecting $l_0$ at $(v_1)$ and $(v_2)$, and $\gamma_m$ are representing chord length one elements of $A$. 

**Figure 17.** 1st type of differentials in $M(L_1)_A \boxtimes \text{bar}r^A \boxtimes_A \overline{M(L_0)}$.

**Figure 18.** 2nd type of differentials in $M(L_1)_A \boxtimes \text{bar}r^A \boxtimes_A \overline{M(L_0)}$. 

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The corresponding 2nd type of disc differentials in $CF(l_0, l_1)$ are those, which have their corners on two different ends of segment $l_0$. They do not pass through the $360^\circ$ rotation point of $l_1$, but instead they pass through the center of the big domain, see the disc from $p_{10}$ to $p_5$ on Figure 16.

Both 2nd type of differentials in $M(l_1)_{A} \boxtimes A \bar{\otimes} A M(l_0)$, and 2nd type of differentials in $CF(l_0, l_1)$ exist if and only if $(u_i)$ is before $j(v_1)$ is before $a(v_2)$ w.r.t. the basepoint and the direction against natural orientation of $\partial B_i$. For example, on Figure 16 the disc from $p_{10}$ to $p_5$ corresponds to differential $x \boxtimes b(-\eta_3, -\xi_3) \boxtimes s^* \rightarrow x \boxtimes b(-\eta_3, -\xi_3, -\rho_2, -\xi_1) \boxtimes t^*$.

\[\square\]

8. Modules associated to tangles

Here we list examples of modules $M(L_K)_{A}$ associated to tangles $K - A_1 - A_2$. We use calculations from [12] [Sections 7,11] to get immersed curves $L_K$ in the pillowcase, see Figure 20.

Example 8.1 (Trivial tangle to pair with). First we consider a trivial tangle $A_1 \cup A_2$ inside the Conway sphere, see the left picture of the second row on Figure 1. For that tangle (decorated with an additional arc and circle to avoid reducibles) one associates a curve $L^\natural$ on Figure 10 and to that curve one associates a module drawn on Figure 9.

Because in algebraic pairing $M(L_U)_{A} \boxtimes A \bar{\otimes} A M(L^\natural)$ there is a dual module $A M(L^\natural)$ involved, we describe it here:

![Figure 19. $A M(L^\natural)$](image)

In the next examples we also compute pillowcase homology via algebraic pairing, using computer program [11] for box tensor product of modules. Another way to see the chain complex and differentials is to isotope $L_K$ as in the proof of Theorem 7.1 and then use Lagrangian Floer homology $CF(L^\natural, L_K)$.

Example 8.2 (The unknot). The next example is a trivial knot tangle $U - A_1 - A_2$. Depending on how you pick the second tangle for the trivial knot (the first tangle is $A_1 \cup A_2$), the resulting curve on the pillowcase can be different. It is either an arc $\{\gamma = \pi\}$ (in case the second tangle looks like a crossing $\times$), or an arc $\{\theta = 0\}$ (in case the second tangle is horizontal smoothing of that crossing). Note that one cannot pick a vertical smoothing of a crossing, as it results in two circles if paired with $A_1 \cup A_2$. On the left of Figure 20 we depicted an arc $\{\theta = 0\} = L_U$ for a crossing $\times$. 28
Example 8.3 $(T_{(2,3)})$. An immersed curve for right-handed trefoil is depicted on the left of Figure 20. The corresponding module $M(L_{T_{(2,3)}})_{A}$ has generators:

$$u_{i_0}, e_{j_1}, v_{j_1}, q_{i_1},$$

and actions:

$$u \otimes (\eta_1, \xi_1, \rho_2, \xi_3) \to e, \quad q \otimes (\eta_2) \to e, \quad q \otimes (\xi_1, \rho_2, \xi_3) \to v.$$ 

The algebraic pairing chain complex $M(L_{T_{(2,3)}})_{A} \otimes_{A} \text{bar}_{r,A} M(L^2)$ has 15 generators and 10 differentials. Pillowcase homology then has rank three: $HF_*(L^2, L_{T_{(2,3)}}) = H_*(M(L_{T_{(2,3)}})_{A} \otimes_{A} \text{bar}_{r,A} M(L^2)) = (\mathbb{F}_2)^3$, which coincides with singular instanton knot homology $I^5(U)$.

In the next three examples immersed curves are unions of curves $R_0, R_1, R_3, R_4$, see the right of Figure 20. Notice that $R_3$ differs from $R_0$ by a twist around the boundary, and thus their pairings with $L^2$ are the same (because $L^2$ is not an arc). We describe the modules $M(L_{R_i})_{A}$ for $i = 0,1,4$ in the appendix. Using those modules we compute three more examples for tangles:

Example 8.4 $(T_{(3,7)})$. The corresponding immersed curve is depicted on the right of Figure 20. The corresponding module is $M(L_{T_{(3,7)}}) = M(L_{R_0}) \oplus M(L_{R_1}) \oplus M(L_{R_2})$. Pillowcase homology then has rank 9: $HF_*(L^2, L_{T_{(3,7)}}) = H_*(M(L_{T_{(3,7)}})_{A} \otimes_{A} \text{bar}_{r,A} M(L^2)) = (\mathbb{F}_2)^4 \oplus (\mathbb{F}_2)^4 = (\mathbb{F}_2)^9$, which coincides with singular instanton knot homology $I^5(T_{(3,7)})$.

Example 8.5 $(T_{(5,11)})$. The corresponding immersed curve is depicted on the right of Figure 20. The corresponding module is $M(L_{T_{(5,11)}}) = M(L_{R_0}) \oplus M(L_{R_1}) \oplus M(L_{R_2}) \oplus M(L_{R_4})$. Pillowcase homology then has rank 17: $HF_*(L^2, L_{T_{(5,11)}}) = H_*(M(L_{T_{(5,11)}})_{A} \otimes_{A} \text{bar}_{r,A} M(L^2)) = (\mathbb{F}_2)^4 \oplus (\mathbb{F}_2)^4 \oplus (\mathbb{F}_2)^4 \oplus (\mathbb{F}_2)^4 = (\mathbb{F}_2)^{17}$. Singular instanton Floer homology is not known for $T_{(5,11)}$.

Example 8.6 $(T_{(3,4)})$. The corresponding immersed curve is depicted on the right of Figure 20. This is an example where one actually needs to perturb $L_{T_{(3,4)}}$ in order to get an immersed 1-manifold.

The corresponding module is $M(L_{T_{(3,4)}}) = M(L_{R_1}) \oplus M(L_{R_3})$. Pillowcase homology then has rank 5: $HF_*(L^2, L_{T_{(3,4)}}) = H_*(M(L_{T_{(3,4)}})_{A} \otimes_{A} \text{bar}_{r,A} M(L^2)) = (\mathbb{F}_2)^4 \oplus \mathbb{F}_2 = (\mathbb{F}_2)^5$, which coincides with singular instanton knot homology $I^5(T_{(3,4)})$.

See [12] and [9] for other examples of immersed curves associated to tangles.
Figure 20. Different immersions associated to tangles. $L^5$ denotes an immersed curve associated to trivial tangle consisting of two arcs $A_1, A_2$, see Figure L. $L_K$ denotes an immersed curve associated to tangle $K \setminus (A_1 \cup A_2)$.

9. Appendix

Module. $M(L_{R_0})_A$

4 generators with their idempotents:
\[ a_{j_1}, c_{j_1}, b_{i_1}, d_{j_0} \]

Actions:
\[ a \otimes \eta_3 \rightarrow d, b \otimes \eta_{23} \rightarrow d, b \otimes (\xi_1, \rho_2, \xi_3) \rightarrow c, b \otimes \eta_2 \rightarrow a \]

Algebraic pairing with the trivial tangle module:
\[ H_4(M(L_{R_0})_A \otimes^A \text{bar}_r^A \otimes^A M(L^5)) = \mathbb{F}_2. \]

Module. $M(L_{R_1})_A$

4 generators with their idempotents:
\[ x_{j_1}, y_{j_1}, z_{i_1}, t_{j_1} \]
Algebraic pairing with the trivial tangle module:

\[ H_s(M(L_{R_1}), A \boxtimes \overline{A} \boxtimes \overline{A} M(\mathbb{T})) = (\mathbb{F}_2)^4. \]

**Module.** \( M(L_{R_4}), A \)

4 generators with their idempotents:

- \( a_{i_0}, c_{i_1}, b_{i_0}, e_{i_1}, d_{i_1}, g_{i_1}, h_{i_1}, m_{j_1}, l_{j_1}, q_{j_2}, p_{j_2}, s_{i_2}, r_{i_1}, u_{j_1}, t_{i_2}, w_{j_0}, v_{j_1}, y_{j_1}, x_{j_1}, z_{j_0} \)

Actions:

- \( a \otimes \eta_3 \rightarrow d, b \otimes \eta_3 \rightarrow d, b \otimes (\xi_1, \rho_2, \xi_3) \rightarrow c, b \otimes \eta_2 \rightarrow a \)
- \( p \otimes \xi_3 \rightarrow u, t \otimes \xi_2 \rightarrow q, d \otimes (\xi_1, \rho_2, \xi_3) \rightarrow l, y \otimes \eta_3 \rightarrow z, a \otimes \eta_1 \rightarrow g, s \otimes \xi_2 \rightarrow p \)
- \( c \otimes \eta_2 \rightarrow x, r \otimes \xi_{23} \rightarrow w, m \otimes \eta_2 \rightarrow y, b \otimes \xi_2 \rightarrow v \)
- \( c \otimes \eta_3 \rightarrow w, r \otimes \xi_{21} \rightarrow p, q \otimes \xi_3 \rightarrow v, r \otimes \xi_1 \rightarrow s, a \otimes p_0 \rightarrow w, t \otimes \xi_{23} \rightarrow v \)
- \( b \otimes \eta_1 \rightarrow h, b \otimes \eta_2 \rightarrow v, d \otimes \eta_3 \rightarrow z, x \otimes \eta_3 \rightarrow w, c \otimes (\xi_1, \rho_2, \xi_3) \rightarrow m, e \otimes \eta_1 \rightarrow t \)

Algebraic pairing with the trivial tangle module:

\[ H_s(M(L_{R_4}), A \boxtimes \overline{A} \boxtimes \overline{A} M(L^2)) = (\mathbb{F}_2)^4. \]
1 \otimes b(\eta_3, \xi_3, \xi_{21}, \eta_1) \otimes \eta_1, b(\eta_3, \xi_3, \xi_2, \xi_{21}) \rightarrow 1 \otimes b(\eta_3, \xi_3, \xi_2, \xi_1) \otimes 1, b(j_2) \rightarrow 1 \otimes b(\xi_3) \otimes 1, b(j'_2) \rightarrow 1 \otimes b(\xi_3) \otimes 1, b(j'_3) \rightarrow 1. 

\vspace{1cm}

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