L²-RIEMANN–ROCH FOR SINGULAR COMPLEX CURVES

JEAN RUPPENTHAL AND MARTIN SERA

Abstract. We present a comprehensive L²-theory for the ∂-operator on singular complex curves, including L²-versions of the Riemann-Roch theorem and some applications.

1. Introduction

The L²-theory for the ∂-operator is one of the central tools in complex analysis on complex manifolds, but still not very well developed on singular complex spaces. Just recently, considerable progress has been made in understanding the L²-cohomology of singular complex spaces with isolated singularities. Let X be a Hermitian complex space of pure dimension n and with isolated singularities only. For simplicity, we assume that X is compact. Let H_{w}^{p,q}(X^*) be the L²-Dolbeault cohomology on the level of (p,q)-forms with respect to the ∂-operator in the sense of distributions, denoted by ∂_{w} in the following, computed on X^* = Reg X. Let π : M → X be a resolution of singularities with snc exceptional divisor, Z := π−¹(Sing X) the unreduced exceptional divisor. Then it has been shown by Øvrelid, Vassiliadou [ØV12] and the first author [Rup10, Rup11] by different approaches that there exists an effective divisor D ≥ Z − |Z| on M such that there are natural isomorphisms

\[ H_{w}^{n,q}(X^*) \cong H^{n,q}(M) , \]
\[ H_{w}^{0,q}(X^*) \cong \frac{H^{q}(M, O(D))}{H_{|Z|}^{q}(M, O(D))} \]

for all 0 ≤ q ≤ n. Here, H_{|Z|}^{q} denotes the cohomology with support on |Z|. If dim X ≤ 2, then (1.1) holds with the divisor D = Z − |Z| and H_{|Z|}^{q}(M, O(Z − |Z|)) = {0}, so that (1.1) gives a nice smooth representation of the L²-cohomology groups H_{w}^{0,q}(X^*). In case dim X > 2, it is conjectured that (1.1) holds with D = Z − |Z| (see [Rup11]).

However, the L²-theory for the ∂-operator developed in [ØV12] and [Rup10, Rup11] applies only to dim X ≥ 2 (for dim X = 1, (1.1) has been known before, see [Par89, PS91]). The purpose of the present paper is to give a complete L²-theory for the ∂-operator on a singular complex curve, including L²-versions of the Riemann-Roch theorem, and to understand the appearance of the divisor Z − |Z| in the case dim X = 1.

Let us explain some of our results in detail. Let X be a Hermitian singular complex space\(^1\) of dimension 1, i.e., a Hermitian complex curve, and L → X a Hermitian metric in G whose restriction to U is \(g|_{U}\).
holomorphic line bundle. Let \( \overline{\partial}_w : L^{p,q}(X^*, L) \to L^{p,q+1}(X^*, L) \) denote the weak extension of the Cauchy-Riemann operator \( \overline{\partial} : \mathcal{D}^{p,q}(X^*, L) \to \mathcal{D}^{p,q+1}(X^*, L) \), i.e., the \( \overline{\partial} \)-operator in the sense of distributions. Here, \( \mathcal{D}^{p,q}(X^*, L) := \mathcal{C}_\infty^0(X^*, \Lambda^{p,q}T^*X^* \otimes L) \) denotes the set of differential forms with compact support in \( X^* \) and values in \( L \) and \( L^{p,q}(X^*, L) \) is the set of square-integrable forms with values in \( L \) and respect to the Hermitian metrics on \( X^* \) and \( L \).

Let \( H^{p,q}_w(X^*, L) \) denote the \( L^2 \)-Dolbeault cohomology on \( X^* \) with respect to \( \overline{\partial}_w \) and \( h^{p,q}_w(X^*, L) := \dim H^{p,q}_w(X^*, L) \). Note that the genus \( g = g(X) \) of \( X \) and the degree \( \deg(L) \) of \( L \) are well-defined, even in the presence of singularities (see Section 2.2). For a singular point \( x \in \text{Sing} \, X \), we define its modified multiplicity \( \text{mult}'_x X \) as follows: Let \( X_j, j = 1, \ldots, m, \) be the irreducible components of \( X \) in the singular point \( x \). Then

\[
\text{mult}'_x X := \sum_{j=1}^{m} (\text{mult}_x X_j - 1).
\]

Note that regular irreducible components do not contribute to \( \text{mult}'_x X \). In Section 2.2 we recall the definition of the multiplicity \( \text{mult}_x X_j \) and present different ways to compute it.

**Theorem 1.1** (\( \overline{\partial}_w \)-Riemann-Roch). Let \( X \) be a compact Hermitian complex curve with \( m \) irreducible components and \( L \to X \) a holomorphic line bundle. Then

\[
h^{0,0}_w(X^*, L) - h^{0,1}_w(X^*, L) = m - g + \deg(L) + \sum_{x \in \text{Sing} \, X} \text{mult}'_x X,
\]

and

\[
h^{1,1}_w(X^*, L) - h^{1,0}_w(X^*, L) = m - g - \deg(L).
\]

Theorem 1.1 is a corollary of Theorem 1.2 which we prove in Section 3. We also consider an \( L^2 \)-dual version there, i.e., an \( L^2 \)-Riemann-Roch theorem for the minimal closed \( L^2 \)-extension of the \( \overline{\partial}_w \)-operator which we denote by \( \overline{\partial}_s \) (see Section 2.1).

On singular complex curves, the \( \overline{\partial}_s \)-operator is of particular importance because of its relation to weakly holomorphic functions. Namely, the weakly holomorphic functions are precisely the \( \overline{\partial}_s \)-holomorphic \( L^2_{\text{loc}} \)-functions (for a localized version of the \( \overline{\partial}_s \)-operator, see Section 5). Let \( H^{p,q}_{s,\text{loc}}(X^*) \) denote the \( L^2_{\text{loc}} \)-Dolbeault cohomology on \( X^* \) with respect to \( \overline{\partial}_s \), and \( \widehat{O}_X \) the sheaf of germs of weakly holomorphic functions on \( X \). Then:

**Theorem 1.2.** Let \( X \) be a Hermitian complex curve. Then

\[
H^0(X, \widehat{O}_X) = H^{0,0}_{s,\text{loc}}(X^*), \quad H^1(X, \widehat{O}_X) \cong H^{1,1}_{s,\text{loc}}(X^*).
\]

If \( X \) is irreducible and compact, then \( \dim H^0(X, \widehat{O}_X) = 1, \dim H^1(X, \widehat{O}_X) = g(X) \). We prove Theorem 1.2 in Section 5.

To exemplify the use of \( L^2 \)-theory for the \( \overline{\partial}_s \)-operator on a singular complex space, in particular the \( L^2 \)-Riemann-Roch theorem, we give in Section 6 two applications. There, we use our \( L^2 \)-theory to give alternative proofs of two well-known facts. First, we show that each compact complex curve can be realized as a ramified covering of \( \mathbb{C}P^1 \). Second, we show that a positive holomorphic line bundle over a compact complex curve is ample, yielding that any compact complex curve is projective.
Let us clarify the relation to previous work of others. In the case of complex curves, (1.1) was in essence discovered by Pardon [Par89], and one can deduce parts of Theorem 1.2 and the second statement of Corollary 1.4 from Pardon’s work by some additional arguments on the regularity of the $\overline{\partial}$-operator. The first part of Corollary 1.3 was discovered by Haskell [Has89], and from that one can deduce the second statement of Theorem 1.1 by use of $L^2$-Serre duality. Moreover, Theorem 1.1 was proved in essence by Bruning, Peyerimhoff and Schröder in [BPS90] and [Sch89] by computing the indices of the $\overline{\partial}_w$- and the $\overline{\partial}_s$-operator.

The new point in the present work is that we can put all the partial results mentioned above in the general framework of a comprehensive $L^2$-theory. From that, we draw also a new understanding of weakly holomorphic functions (Theorem 1.2) and by computing the indices of the $\overline{\partial}_w$-operator. This is essential for applications as we illustrate by the examples mentioned above.

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2. Preliminaries

2.1. Closed extensions of the Cauchy-Riemann operator. Let $X$ be a complex curve and $X^* := \text{Reg} X$ the set of regular points. We assume that $X$ is a Hermitian complex space in the sense that $X^*$ carries a Hermitian metric $\gamma$ which is locally given as the restriction of the metric of the ambient space when $X$ is embedded holomorphically into some complex number space.

We denote by $\mathcal{D}^{p,q}(X^*)$ the smooth differential forms of degree $(p, q)$ with compact support in $X^*$ (test forms) and by $L^{p,q}(X^*)$ the set of square-integrable forms with respect to the metric $\gamma$ on $X^*$.

Let $\overline{\partial}_s : L^{p,q}(X^*) \to L^{p,q+1}(X^*)$ be the minimal (strong) closed $L^2$-extension of the Cauchy-Riemann operator $\overline{\partial} : \mathcal{D}^{p,q}(X^*) \to \mathcal{D}^{p,q+1}(X^*)$, i.e., $\overline{\partial}_s$ is defined by the closure of the graph of $\overline{\partial}$ in $L^{p,q}(X^*) \times L^{p,q+1}(X^*)$. $\overline{\partial}_w : L^{p,q}(X^*) \to L^{p,q+1}(X^*)$ is the maximal (weak) closed $L^2$-extension of $\overline{\partial}$, i.e., $\overline{\partial}_w$ is defined in sense of distributions. We denote by $H^{p,q}_{w/s}(X^*)$ the Dolbeault cohomology with respect to $\overline{\partial}_w$ or $\overline{\partial}_s$, respectively, and by $\dim H^{p,q}_{w/s}(X^*)$ the dimension of $H^{p,q}_{w/s}(X^*)$.

Let $\overline{\partial} : \mathcal{D}^{p,q+1}(X^*) \to \mathcal{D}^{p,q}(X^*)$ be the formal adjoint of $\overline{\partial}$ and $\overline{\partial}_{s/w} := \overline{\partial}_{w/s}$ the Hilbert-space adjoint of $\overline{\partial}_{w/s}$. This notation makes sense as $\overline{\partial}_{w/s}$ is in fact the maximal (weak) or minimal (strong), respectively, $L^2$-extension of $\overline{\partial}$. Let $\overline{\partial} : L^{p,q}(X^*) \to L^{1-p,1-q}(X^*)$ be the conjugated Hodge-$*$-operator with respect to the metric $\gamma$. Then we have $\overline{\partial}_{w/s} = - \overline{\partial} \overline{\partial}_{w/s} \overline{\partial}$.

Let $L \to X$ be a Hermitian holomorphic line bundle on $X$ with an (arbitrary) metric on $L$ which is smooth on the whole of $X$. We define $\mathcal{D}^{p,q}(X^*, L) := \xi_{\text{cpt}}^{p,q}(X^*, \Lambda^{p,q}T^* X^* \otimes L)$ as the smooth $(p, q)$-forms with compact support and values in $L$, and $L^{p,q}(X^*, L)$ as the Hilbert space of square-integrable forms with values in $L$. We consider the Cauchy-Riemann operator $\overline{\partial} : \mathcal{D}^{p,q}(X^*, L) \to \mathcal{D}^{p,q+1}(X^*, L)$ locally given by $\overline{\partial} : \mathcal{D}^{p,q}(X^*) \to \mathcal{D}^{p,q+1}(X^*)$. Since $\overline{\partial}$ commutes with the trivializations of
the holomorphic line bundle, \( \overline{\mathcal{O}} \) is well defined. We get the weak and strong extensions \( \overline{\mathcal{O}}_{w}, \overline{\mathcal{O}}_{s} : L^{p,q}(X^*, L) \to L^{p,q+1}(X^*, L) \) and the cohomology \( H^{p,q}_{w/s}(X^*, L) \) as above.

In Section 2 we will study following other closed extensions of \( \overline{\mathcal{O}} \) than the minimal \( \overline{\mathcal{O}}_{s} \) and the maximal \( \overline{\mathcal{O}}_{w} \). Let \( D \subset \mathbb{C}^{n} \) be a domain, and \( X \subset D \) an analytic set of dimension one with \( \text{Sing} X = \{0\} \). We can interpret \( \overline{\mathcal{O}}_{s} \) as \( \overline{\mathcal{O}}_{w} \) with certain boundary conditions. The boundary of \( X^* \) consists of two parts, the singular point \( \{0\} \) and the boundary at \( \partial D : \partial X = \partial X^* \setminus \{0\} \). Therefore, there are two boundary conditions. Let \( \overline{\mathcal{O}}_{s,w} \) denote the closed \( L^{2} \)-extension which satisfies the boundary condition at \( \{0\} \), i.e., \( f \in \text{dom} \overline{\mathcal{O}}_{s,w} \) iff \( f \in \text{dom} \overline{\mathcal{O}}_{w} \) and there is a sequence \( \{f_j\} \) in dom \( \overline{\mathcal{O}}_{w} \) such that \( \text{supp} f_j \cap \{0\} = \emptyset \), \( f_j \to f \), and \( \overline{\mathcal{O}}_{w} f_j \to \overline{\mathcal{O}}_{w} f \) in \( L^{2} \). \( \overline{\mathcal{O}}_{s,w} \) denotes the extension which satisfies the boundary condition at \( \partial X \), i.e., \( f \in \text{dom} \overline{\mathcal{O}}_{s,w} \) iff \( f \in \text{dom} \overline{\mathcal{O}}_{w} \) and there is a sequence \( \{f_j\} \) in dom \( \overline{\mathcal{O}}_{w} \) such that \( \text{supp} f_j \cap \partial X = \emptyset \), \( f_j \to f \), and \( \overline{\mathcal{O}}_{w} f_j \to \overline{\mathcal{O}}_{w} f \) in \( L^{2} \). We define the adjoint operators

\[
\theta_{s,w} := -\overline{\pi} \overline{\mathcal{O}}_{s,w} \overline{\pi} \quad \text{and} \quad \theta_{w,s} := -\overline{\pi} \overline{\mathcal{O}}_{w,s} \overline{\pi},
\]

which we can realize as Hilbert-space adjoint operators (see [Rup10, Lem. 5.1]):

**Lemma 2.1.** The Hilbert-space adjoints \( \overline{\mathcal{O}}_{s,w} \) and \( \overline{\mathcal{O}}_{w,s} \) satisfy the representations

\[
\overline{\mathcal{O}}_{s,w} \theta_{w,s} = -\overline{\pi} \overline{\mathcal{O}}_{s,w} \overline{\pi} \quad \text{and} \quad \overline{\mathcal{O}}_{w,s} \theta_{s,w} = -\overline{\pi} \overline{\mathcal{O}}_{w,s} \overline{\pi},
\]

respectively.

### 2.2. Resolution of complex curves, divisors, line bundles.

Every (reduced) complex space \( X \) (which is countable at infinity) has a resolution of singularities \( \pi : M \to X \), i.e., there are a complex manifold \( M \), a proper complex subspace \( S \) of \( X \) which contains the singular locus of \( X \) and a proper holomorphic map \( \pi : M \to X \) such that the restriction \( M \setminus \pi^{-1}(S) \to X \setminus S \) of \( \pi \) is biholomorphic, and \( \pi^{-1}(S) \) is the locally finite union of smooth hypersurfaces (see [Hir64] and [Hir77, Thm. 7.1]).

If \( X \) is a compact complex curve, then such a resolution is given just by the normalization of the curve, and it is unique up to biholomorphism: Let \( \pi_1 : M_1 \to X \) and \( \pi_2 : M_2 \to X \) be two resolutions of \( X \). Then \( \psi := \pi_2^{-1} \circ \pi_1 : M_1 \setminus \pi_1^{-1}(\text{Sing} X) \to M_2 \setminus \pi_2^{-1}(\text{Sing} X) \) is biholomorphic and bounded in the singular locus. Yet, \( \pi_1^{-1}(\text{Sing} X) \) consist of isolated points. Therefore, \( \psi \) has a (bi-) holomorphic extension.

Let \( \pi : M \to X \) be a resolution of a compact complex curve \( X \). We define the genus of \( X \) by the genus of the resolution

\[
g := h^1(M) = \dim H^1(M, \mathcal{O}).
\]

If \( X \) has more then one irreducible component then \( M \) is not connected and \( h^1(M) \) is the sum of the genera of the connected components. Since the resolution is unique, this is well-defined.

Throughout the article (except of Section 6.1), we will work with divisors on compact Riemann surfaces only. Therefore, there is no difference between Cartier and Weil divisors, and we can associate to each line bundle a divisor.

Let \( L \to X \) be a holomorphic line bundle. Then the pull-back \( \pi^* L \to M \) is well-defined by the pull-back of the transition functions of the line bundle. There is a divisor \( D \) on \( M \) associated to \( \pi^* L \) such that \( \mathcal{O}(\pi^* L) \cong \mathcal{O}(D) \) and \( \deg \pi^* L = \deg D \). The uniqueness of the resolution (up to biholomorphism) implies the independence of \( \deg \pi^* L \) from \( \pi \), so that

\[
\deg L := \deg \pi^* L
\]

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2 We denote by \( \mathcal{O}(D) \) the sheaf of germs of holomorphic functions \( f \) such that \( \text{div}(f) + D \geq 0 \).
is also well-defined.

For any divisor $D$ on $M$, there exists a holomorphic line bundle $L_D \to M$ associated to $D$ such that $\mathcal{O}(L_D) \cong \mathcal{O}(D)$. The constant function $f = 1$ induces a meromorphic section $s_D$ of $L_D$ such that $\text{div}(s_D) = D$. One can then identify sections in $\mathcal{O}(D)$ with sections in $\mathcal{O}(L_D)$ by $g \mapsto g \otimes s_D$, and we denote the inverse mapping by $s \mapsto s \cdot s_D^{-1}$. If $Y$ is an effective divisor, then $s_Y$ is a holomorphic section of $L_Y$ and $\mathcal{O} \subset \mathcal{O}(Y)$. Hence, there is the natural inclusion $\mathcal{O}(L_D) \subset \mathcal{O}(L_{D+Y})$ which induces the inclusion $\mathcal{O}(L_D) \subset \mathcal{O}(L_{D+Y})$ given by $s \mapsto (s \cdot s_D^{-1}) \otimes s_{D+Y}$. For $U \subset M$, we obtain the inclusion

$$L^p,q_{\text{loc}}(U, L_D) \hookrightarrow L^p,q_{\text{loc}}(U, L_{D+Y}), s \mapsto (s \cdot s_D^{-1}) \otimes s_{D+Y}. \quad (2.1)$$

Here, $L^p,q_{\text{loc}}(U, L_D)$ denotes the locally square-integrable forms with values in $L_D$. This definition is independent of the chosen Hermitian metric on $L_D$. If $M$ is compact, all metrics are equivalent and we get the inclusion

$$L^p,q(M, L_D) \subset L^p,q(M, L_{D+Y}). \quad (2.2)$$

Let $Z := \pi^{-1}(\text{Sing } X)$ be the unreduced exceptional divisor and $|Z|$ the underlying reduced divisor. Then $\deg(Z - |Z|)$ is independent of the resolution as well. We will discuss some alternative ways to compute $\deg Z$.

Locally, the resolution is given by the Puiseux parametrization: Let $A$ be an analytic set of dimension 1 in $D \subset \mathbb{C}^n$ with $\text{Sing } A = \{0\}$ which is irreducible at 0. Shrinking $D$, there are coordinates $z, w_1, ..., w_{n-1}$ around 0 such that $A$ is contained in the cone $||w|| \leq C |z|$, $w = (w_1, ..., w_{n-1})$. The projection $\text{pr}_z : A \to \mathbb{C}_z$ on the $z$-coordinate is a finite ramified covering. Let $s$ be the number of the sheets of $\text{pr}_z$. Generic choice of the coordinates gives the same number of sheets $s$, called the multiplicity $\text{mult}_0 A$ of $A$ in $\{0\}$. There exists a parametrization $\pi : \Delta \to A, t \mapsto (t^s, w_1(t), ..., w_{n-1}(t))$, where $\Delta := \{t \in \mathbb{C} : |t| < 1\}$; cf. e.g. [Chi89, Sect. 6.1]. $\pi$ is called the Puiseux parametrization. The unreduced exceptional divisor is just $Z = (\pi^{-1}(z)) = (t^s)$, and so $\deg Z = s$.

The number of sheets of the covering $\text{pr}_z$ is also equal to the Lelong number $\nu([A], 0)$ of the positive current $[A]$ given by the integration over $A$ (see [Chi89, Prop. 2 in § 3.15], [Dem12, Thm. 7.7] or [GH78, § 3.2]).

The tangent cone gives another way to compute $\text{mult}_0 A$. For a holomorphic function $f$ on $D$, let $f = \sum_{k=0}^{\infty} f_k$ be the decomposition in homogeneous polynomials $f_k$ of degree $k$ with $f_{k_0} \neq 0$ (choosing a smaller $D$) and $f^* := f_{k_0} \neq 0$ be the initial homogeneous polynomial of $f$. If $A$ is given by the ideal sheaf $\mathcal{I}_A$, then

$$C_0(A) = \{ \alpha \in \mathbb{C}^n : f^*(\alpha) = 0 \forall f \in \mathcal{I}_A, 0 \} \subset T_0\mathbb{C}^n$$

is called the tangent cone of $A$ in 0 (cf. [Chi89, Sect. 8.4]). The natural projection $\mathbb{C}^n \setminus 0 \to \mathbb{C}^p \setminus 0$ maps $C_0(A)$ on a projective variety $C_0(A)$. The degree $\deg Y$ of a projective variety $Y$ in $\mathbb{C}^p \setminus 0$ of dimension $p$ is defined as the class of $Y$ in $H_{2p}(\mathbb{C}^p \setminus 0, Z) \cong Z$, and $\text{mult}_0 A = \deg C_0(A)$ (see Sect. 2 of [GH78, § 1.3]). In the case of a complex curve $A$, note that $C_0(A)$ is just a point of multiplicity $\text{mult}_0 A$.

All in all, we have

$$\deg Z = \text{mult}_0 A = \nu([A], 0) = \deg C_0(A).$$
2.3. Extension theorems. We need the following extension theorem. Let $\Delta$ be the unit disc in $\mathbb{C}$ and $\Delta^* := \Delta \setminus \{0\}$.

**Theorem 2.2 ($L^2$-extension).** If $u \in L^p_{\text{loc}}(\Delta)$ and $v \in L^p_{\text{loc}}(\Delta^*)$ satisfy $\overline{\partial} u = v$ on $\Delta^*$ in the sense of distributions, then $\overline{\partial} u = v$ on $\Delta$.

A more general statement is true for domains in $\mathbb{C}^n$ and proper analytic subsets of arbitrary codimension, cf. e.g. [Rup09, Thm. 3.2].

If $A \subset D \subset \mathbb{C}^n$ is a pure dimensional analytic set, let $\hat{\mathcal{O}} = \hat{\mathcal{O}}_A$ be the normalization sheaf of $\mathcal{O}_A$ which is defined stalk-wise by the integral closure of $\mathcal{O}_{A,x}$ in the sheaf $\mathcal{M}_{A,x}$ of meromorphic functions for all $x \in A$. A function in $\hat{\mathcal{O}}(U)$, $U \subset A$ open, is called *weakly holomorphic*. Weakly holomorphic functions are holomorphic in regular points of $A$ and bounded in singular points. If $A$ is locally irreducible, then weakly holomorphic functions are continuous in $\text{Sing} \, A$ (cf. e.g. [GR84, § VI.4].)

The classical Riemann extension theorem generalizes to the following result (see e.g. [GR84, Sect. VII.4.1]).

**Theorem 2.3** (Riemann extension). Let $A \subset D \subset \mathbb{C}^n$ be a pure dimensional analytic set. Every holomorphic function on $A^* := \text{Reg} \, A$ which is bounded at $\text{Sing} \, A$ is weakly holomorphic on $A$.

3. Local $L^2$-theory of complex curves

In this section, we study the local $L^2$-theory of (locally) irreducible analytic curves in $\mathbb{C}^n$. By the remarks on the local structure of singular complex curves in Section 2.2 and 4, it follows that the studied situation is general enough. We will compute the $L^2$-Dolbeault cohomology by use of the Puiseux parametrization and will see why the term $\sum_{x \in \text{Sing} \, X} \text{mult}_{x}^{t} \, X$ occurs in (1.2).

Let $A$ be an irreducible analytic curve in $\Delta^n \subset \mathbb{C}^{n_{w_1} \ldots w_{n-1}}$ given by the Puiseux parametrization

$$\pi: \Delta \rightarrow \mathbb{C}^n, \quad \pi(t) := (t^s, w(t)), $$

where $w = (w_1, \ldots, w_{n-1}): \Delta \rightarrow \Delta^{n-1}$ is a holomorphic map such that each component $w_i$ vanishes at least of the order $s+1$ in the origin. Here, $\Delta$ is the unit disk $\{t \in \mathbb{C} : |t| < 1\}$. We can assume that $\pi$ is bijective, in particular, a resolution/normalization of $A$ such that $\text{mult}_0 A = s$. Further, we can assume that 0 is the only singular point of $A$.

For a regular point $(z_0, w_0) \in A^* := \text{Reg} \, A$, let $t_0 \in \Delta^*$ be the preimage under $\pi$. Since $\pi$ is biholomorphic on $\Delta^* := \Delta \setminus \{0\}$, $d\pi_{t_0}(\frac{\partial}{\partial t}) = st_0^{s-1} \frac{\partial}{\partial z} + \sum_{k=1}^{n-1} w'_k(t_0) \frac{\partial}{\partial w_k}$ is a non-vanishing tangent vector of $A^*$ in $(z_0, w_0)$, i.e.,

$$1 + \|st_0^{1-s} w'(t_0)\|^2 = \left(1 + \sum_{k=1}^{n-1} \frac{1}{s} t_0^{1-s} w'_k(t_0) \frac{\partial}{\partial w_k}\right)$$

is a normalized generator of $T_{(z_0, w_0)} A^*$ and $(1 + \|st_0^{1-s} w'(t_0)\|^2)^{1/2} dz$ is a normalized generator of $T_{(z_0, w_0)} A^*$. Since $w'_k$ vanishes at least of order $s$ in the origin, we obtain $1 + \|\frac{1}{s} t_0^{1-s} w'(t_0)\|^2 \sim 1$ on $\Delta$ and $dV_{A^*} \sim dz \wedge d\overline{z}$, where $dV_{A^*}$ denotes the volume form on $A^*$ induced by the standard Euclidean metric of $\mathbb{C}^n$. Using $\pi^* dz = d(\pi^* z) = dt^s = st^{s-1} dt$ and $\pi^*(dz \wedge d\overline{z}) = s^2 |t|^{2(s-1)} dt \wedge d\overline{t}$, we get

$$\pi^* dV_{A^*} \sim |t|^{2(s-1)} dV_{\Delta}.$$
Let \( \iota: A^* \rightarrow \Delta^* \) be the inverse of \( \pi \). Then, \( \iota(z, w) = \sqrt{z} \) with \( w = w(t) \).
We get \( \iota^*(dt) = \frac{1}{2}z^{1/2}dz \) and \( \iota^*(dt \wedge d\overline{t}) = \frac{1}{z^{1/2}}|z|^{2(1/2) - 1}dz \wedge d\overline{t} \), i.e.,
\[
\iota^*dV_\Delta \sim |z|^{2(1/2) - 1}dV_{A^*}.
\]

If \( g \) is a measurable function on \( A^* \), we obtain
\[
\int_{A^*} |g|^2dV_{A^*} = \int_{\Delta} |\pi^*g|^2\pi^*dV_{A^*} \sim \int_{\Delta} |\pi^*g|^2 \cdot |t|^{2(s-1)}dV_\Delta.
\]
Hence,
\[
g \in L^{0,0}(A^*) \leftrightarrow t^{s-1}\pi^*g \in L^{0,0}(\Delta).
\]

For (0, 1)-forms and (1, 1)-forms, we have
\[
\pi^*(gd\overline{z}) = \pi^*g \cdot \pi^*(d\overline{z}) = t^{-1}\pi^*(g)d\overline{t},
\]
\[
\pi^*(gdz \wedge d\overline{z}) = |t|^{2(s-1)}\pi^*(g)dt \wedge d\overline{t}.
\]

Thus
\[
f \in L^{0,0}(A^*) \Leftrightarrow t^{s-1} \pi^*f \in L^{0,0}(\Delta),
\]
\[
f \in L^{1,0}(A^*) \Leftrightarrow \pi^*f \in L^{1,0}(\Delta),
\]
\[
f \in L^{0,1}(A^*) \Leftrightarrow \pi^*f \in L^{0,1}(\Delta), \quad \text{and}
\]
\[
f \in L^{1,1}(A^*) \Leftrightarrow t^{1-s} \pi^*f \in L^{1,1}(\Delta).
\]

On the other hand, if \( v \in L^{0,0}(\Delta) \), we get
\[
\infty > \int_{\Delta} |v|^2dV_\Delta = \int_{A^*} |\iota^*v|^2\iota^*dV_\Delta \sim \int_{A^*} |\iota^*v|^2 \cdot |z|^{2(s-1)}dV_{A^*}.
\]
Thus, \( |z|^{1/2}t^{s-1}v \) is square-integrable on \( A^* \). For each (0, 1)-form \( vd\overline{t} \in L^{0,1}(\Delta) \), we get \( s\iota^*(vd\overline{t}) = \overline{z}^{1/2}t^s(v)d\overline{z} \in L^{0,1}(A^*) \), and for each (1, 1)-form \( vdt \wedge d\overline{t} \in L^{1,1}(\Delta) \), we get \( |z|^{-1/2}t^{s-1}(vdt \wedge d\overline{t}) \in L^{1,1}(A^*) \).

So, if \( f \in L^{0,1}(A^*) \), then \( u := \pi^*f \) is in \( L^2 \), too. Since \( \dim \Delta = 1 \), there exists \( v \in L^{0,0}(\Delta) \) with \( \overline{\partial}_wv = u \). We set \( g := \iota^*v \). Since \( |z|^{1/2}g \) is in \( L^2 \) and \( |z|^{2(1-s)/s} \) is bounded,
\[
\|g\|^2_{L^2} = \int_{A^*} |z^{1/2}g|^2 \cdot |z|^{2(1-s)}dV_{A^*} \leq \|z^{1/2}g\|_{L^2} \cdot \|z^{2(1-s)/s}\|_{L^\infty} < \infty.
\]

Hence, we get an \( L^2 \)-solution for \( \overline{\partial}_w^*g = f \) and
\[
H_w^{0,1}(A^*) = L^{0,1}(A^*)/\mathcal{R}(\overline{\partial}_w) = 0.
\]

In the same way, it is easy to compute
\[
H_w^{1,1}(A^*) = 0.
\]

We will now determine \( H_w^{0,0}(A^*) = \ker(\overline{\partial}_w : L^{p,0} \rightarrow L^{p,1}) \) by use of the \( L^2 \)-extension theorem (Theorem 22). For this, let \( \mathcal{O}_{L^2}(\Delta) \) be the square-integrable holomorphic functions on \( \Delta \), and let \( \Omega^1_{L^2}(\Delta) \) be the holomorphic 1-forms with square-integrable coefficient. If \( g \in L^{0,0}(A^*) \) and \( \overline{\partial}_w g = 0 \), then \( v := \pi^*g \in |t|^{1-s}L^{0,0}(\Delta) \) and \( \overline{\partial}_w v \equiv 0 \) on \( \Delta^* \). Therefore, \( \overline{\partial}(t^{s-1}v) = 0 \) on \( \Delta^* \) and \( t^{s-1}v \in L^{0,0}(\Delta) \). The extension theorem implies \( \overline{\partial}(t^{s-1}v) = 0 \) on \( \Delta \), i.e., \( v \) is a meromorphic function with a pole of order \( s - 1 \) or less at the origin. We say \( v \in t^{1-s}\mathcal{O}_{L^2}(\Delta) \). Since, on the other hand, \( \iota^*(t^{1-s}\mathcal{O}_{L^2}(\Delta)) \subset \ker \overline{\partial}_w \), we conclude
\[
H_w^{0,0}(A^*) \cong t^{1-s}\mathcal{O}_{L^2}(\Delta).
\]
If \( f \in L^{1,0}(A^*) \) and \( \bar{\partial}_w f = 0 \), then \( u := \pi^* f \in L^{1,0}(\Delta) \) and \( \bar{\partial}_w u = 0 \) on \( \Delta \) (using the extension theorem again). Hence, \( u \) is holomorphic on \( \Delta \) and

\[
H^{1,0}_w(A^*) \cong \Omega_{1,0}^1(\Delta).
\]

To compute the cohomology groups \( H^{s,*}_\bar{\partial}(A^*) \), we use \( L^2 \)-duality:

**Lemma 3.1.** Let \( \bar{\partial}_z \) denote either the weak or the strong closed extension of \( \bar{\partial} \), and \( \bar{\partial}_w \) the other one. For \( p \in \{0, 1\} \), let the range \( \mathcal{R}(\bar{\partial}_z) \) of \( \bar{\partial}_z : L^{p,0} \to L^{p,1} \) be closed. Then

\[
H^{p,1}_z(A^*) \cong H^{1-p,0}_w(A^*).
\]

For the proof see e.g. [Rup10 Thm. 2.3].

**Lemma 3.2.** For \( p \in \{0, 1\} \),

\[
H^{p,0}_w(A^*) \cong H^{1-p,1}_w(A^*) = 0 \quad \text{and} \quad H^{s,1}_w(A^*) \cong H^{1-s,0}_w(A^*).
\]

**Proof.** Recall that \( H^{1-p,1}_w(A^*) = 0 \). This implies \( L^{1-p,1}(A^*) = \mathcal{R}(\bar{\partial}_w) \) and, particularly, that the range of \( \bar{\partial}_w : L^{1-p,0} \to L^{1-p,1} \) is closed. As \( \bar{\partial}_w = -\bar{\pi} \bar{\partial}_w \bar{\pi} \) and \( \bar{\pi} \) is an isometric isomorphism, we conclude that the range of \( \bar{\partial}_w : L^p \to L^p \) is closed as well. This is equivalent to the range of \( \bar{\partial}_s = \bar{\partial}_w^* : L^{p,0} \to L^{p,1} \) being closed (standard functional analysis). Lemma 3.1 implies both isomorphisms.

To get the complete picture, we also need to understand the Dolbeault cohomology groups of the closed extensions \( \bar{\partial}_{s,w} \) and \( \bar{\partial}_{w,s} \), respectively.

**Lemma 3.3.** For \( p \in \{0, 1\} \),

\[
H^{p,0}_{w,s}(A^*) = 0.
\]

**Proof.** Let \( f \in \ker \bar{\partial}_{w,s} = H^{p,0}_{w,s}(A^*) \). We have showed \( \omega \cdot u := \omega \cdot \pi^* f \in L^{p,0}(\Delta) \) with \( \omega(t) = t^{p-1} \) if \( p = 0 \) and \( \omega(t) \equiv 1 \) if \( p = 1 \). By the extension theorem, we conclude \( \bar{\partial}_{s}(\omega \cdot u) = 0 \) on \( \Delta \), where \( \bar{\partial}_{s} \) denotes the (strong) closure of \( \bar{\partial}_{c} : \mathcal{C}_{c}^{\infty}(\Delta) \to \mathcal{C}_{c}^{\infty,1}(\Delta) \). The generalized Cauchy condition implies that the trivial extension of \( \omega u \) to the complex plane is a holomorphic \( p \)-form with compact support (cf. [LM02 § V.3]). We deduce that \( \omega u = 0 \) and, hence, \( f = 0 \).

**Lemma 3.4.**

\[
H^{0,0}_{s,w}(A^*) \cong \mathcal{O}_{L^2}(\Delta) \quad \text{and} \quad H^{1,0}_{s,w}(A^*) \cong t^{s-1} \Omega_{0,1}^1(\Delta).
\]

As \( \mathcal{O}_{L^2}(\Delta) \cong \bar{\partial}_{L^2}(A) \), the first isomorphism implies that the \( \bar{\partial}_{s,w} \)-holomorphic functions on a singular complex curve are precisely the square-integrable weakly holomorphic functions.

**Proof.** First, we prove that \( \mathcal{O}_{L^2}(\Delta) = \pi^*(\ker \bar{\partial}_{s,w} : L^{0,0}(A^*) \to L^{1,1}(A^*)) \).

1) For \( v \in \mathcal{O}_{L^2}(\Delta) \), we claim that \( g := i^* v \in \ker \bar{\partial}_{s,w} \). To see that, choose smooth functions \( \tilde{\chi}_k : \mathbb{R} \to [0, 1] \) with \( \tilde{\chi}_k(-\infty) = 0 \), \( \tilde{\chi}_k((k+1, \infty) = 1 \) and \( |\chi_k'| \leq 2 \). We get \( \chi_k \circ \log \circ \log |\rho| \circ (\rho) = \chi_k(\log |\log |z||) \) (which is inspired by [PS91 p. 617]) and get supp \( \bar{\partial} \chi_k \subset A^* \cap \Delta_k^0 \), where \( \epsilon_k := \exp(-\exp(k)) \to 0 \) if \( k \to \infty \). As \( v \in L^{0,0}(\Delta) \), we have \( g \in z^{1-2} L^{0,0}(A^*) \subset L^{0,0}(A^*) \).
Then \( g \cdot \chi_k \to g \) in \( L^2 \). As a holomorphic function, \( v \) is bounded in a neighborhood of 0. Therefore,

\[
\|g \bar{\partial} \chi_k\|_{A^s}^2 = \left\| g \cdot \frac{\chi_k(\log(\log |z|))}{|z| \log |z|} \bar{\partial} z \right\|_{A^s \cap \Delta_k^0}^2 \lesssim \left\| g \cdot \frac{1}{|z| \log |z|} \right\|_{A^s \cap \Delta_k^0}^2
\]

\[
\sim \left\| v \cdot \frac{|t|^{s-1}}{|t| \log |t|} \right\|_{\Delta_k}^2 \lesssim \left\| \frac{1}{|t| \log |t|} \right\|_{\Delta_k}^2 = \int_{\Delta_k} \frac{1}{|t|^2 \log^2 |t|} dV
\]

\[
= 2\pi \int_0^{e_k} \frac{\rho}{\rho^2 \log^2 \rho} d\rho \sim \left[-\frac{1}{\log \rho}\right]_0^{e_k} \to 0, \text{ if } k \to \infty.
\]

Hence, \( \bar{\partial}(g \chi_k) = g \bar{\partial} \chi_k \to 0 = \bar{\partial}_u g \) in \( L^2 \). So, \( g \in \text{dom} \bar{\partial}_{s,w} \).

ii) \( \pi^* (\text{ker} \bar{\partial}_{s,w}) \subset \mathcal{O}_{L^2}(\Delta) \) (cf. the proof of Lem. 6.2 in [Rup10]): Let \( g \) be in ker \( \bar{\partial}_{s,w} \), i.e., there are \( g_j \in L^2(A^s) \) with \( g_j \to g \), \( \bar{\partial} g_j \to 0 \) in \( L^2(A^s) \) and \( 0 \notin \text{supp} g_j \). Let \( \chi \in \mathcal{C}_{\text{cpt}}(\Delta, [0, 1]) \) be identically 1 on \( \Delta_{j_k} \). We define \( u := \chi \pi^* g \) and \( u_j := \chi \pi^* g_j \). It follows that \( t^{s-1} u_j \to t^{s-1} u \) and \( \bar{\partial} u_j \to \bar{\partial} u \) in \( L^2(\Delta) \). Let \( P : L^2(\Delta) \to L^2(\Delta) \) be the Cauchy-operator on the punctured disc, i.e.,

\[
[P(h)](t) := \frac{1}{2\pi i} \int_{\Delta^*} \frac{h(\zeta)}{\zeta - t} d\zeta d\bar{\zeta}.
\]

Since the support of \( u_j \) is away from 0 and \( \partial \Delta \), we get \( u_j = P \left( \frac{\partial u}{\partial \zeta} \right) \). The \( L^2 \)-continuity of \( P \) and \( \bar{\partial} u_j \to \bar{\partial} u \) in \( L^2 \) imply that

\[
u_j = P \left( \frac{\partial u_j}{\partial \zeta} \right) \to P \left( \frac{\partial u}{\partial \zeta} \right)
\]

in \( L^2 \). Since \( t^{s-1} \) is bounded, we obtain \( t^{s-1} u_j \to t^{s-1} P \left( \frac{\partial u}{\partial \zeta} \right) \) and, hence, \( u = P \left( \frac{\partial u}{\partial \zeta} \right) \) in \( L^2 \). That yields \( \pi^* g \in L^2(\Delta) \). With \( \pi^* g \) in \( t^{1-s} \mathcal{O}_{L^2}(\Delta) \) and the extension theorem, we conclude \( \pi^* g \in \mathcal{O}_{L^2}(\Delta) \).

Second, we claim that ker \( \bar{\partial}_{s,w} : L^{1,1}(A^s) \to L^{1,1}(A^s) \) \( \cong \) \( t^{s-1} \Omega_{L^2}(\Delta) \). \( f = gdz \) is in ker \( \bar{\partial}_{s,w} \) if \( g \in \text{ker} \bar{\partial}_{s,w} \). This is equivalent to \( \pi^* g \in \mathcal{O}_{L^2}(\Delta) \). Since \( \pi^*(dz) = t^{s-1} dt \), we infer that \( \pi^* : H^{1,1}(A^s) \to t^{s-1} \Omega_{L^2}(\Delta) \), \( \pi^* f = t^{s-1} \pi^* f dt \) is an isomorphism.

The Riemann extension theorem (Theorem 2.3) implies \( \mathcal{O}_{L^2}(\Delta) \cong \hat{\mathcal{O}}_{L^2}(A) \) and the last statement.

\[\square\]

**Lemma 3.5.** For \( p \in \{0, 1\} \), \( \mathcal{R}(\bar{\partial}_{w,s} : L^{p,0}(A^s) \to L^{p,1}(A^s)) \) and \( \mathcal{R}(\bar{\partial}_{s,w} : L^{p,0}(A^s) \to L^{p,1}(A^s)) \) both are closed, and

\[
H^{p,1-q}_{s,w}(A^s) \cong H^{1-p,q}_{s,w}(A^s) \quad \text{for } q \in \{0, 1\}.
\]

**Proof.** Since \( \bar{\partial}_{w,s} = \partial_{w,s} \), \( \mathcal{R}(\bar{\partial}_{s,w}) \) is closed if \( \mathcal{R}(\partial_{w,s}) \) is closed.

Therefore, it is enough to show that \( \mathcal{R}(\bar{\partial}_{w,s}) \) is closed.
Let \( \varphi : \Delta^* \to \mathbb{R} \) be the smooth function defined by \( \varphi(t) := (1 - s) \log |t|^2 \). Then \( L^{p,q}(\Delta, \varphi) = t^{1-s}L^{p,q}(\Delta) \) for the \( L^2 \)-space \( L^{p,q}(\Delta, \varphi) \) with weight \( e^{-\varphi} \).

We set \( T_1 := \pi^* \bar{\Omega}_{u,s} \tau^* : L_{0,0}(\Delta, \varphi) \to L_{0,1}(\Delta) \). The extension theorem implies that \( T_1 \) is the (strong) closure of \( \bar{\Omega}_{u,s} \tau^* : L_{0,0}(\Delta, \varphi) \to L_{0,1}(\Delta) \). Therefore, \( T_1^* \) is the weak closed extension of \( \delta_{\text{cpt}}^\varphi : L_{0,1}(\Delta) \to L_{0,0}(\Delta, \varphi) \) which is defined by

\[
(\delta_{\text{cpt}}^\varphi \alpha, \beta) = (\alpha, \delta_{\text{cpt}}^\varphi \beta) = \int \langle \alpha, \delta_{\text{cpt}}^\varphi \beta \rangle e^{-\varphi} dV.
\]

We set \( \tau_\varphi := e^{-\varphi} \tau \). Then \( T_2 := -\tau_\varphi T_1^* \tau \) is the weak closed extension of \( \bar{\Omega}_{u,s} \tau^* : L_{0,0}(\Delta) \to L_{0,1}(\Delta, -\varphi) \) because integration by parts implies \( \delta_{\text{cpt}}^\varphi = -\tau_\varphi \delta_{\text{cpt}} \tau \).

\[
(\alpha, \tau_\varphi \delta_{\text{cpt}} \tau \beta) = \int \alpha \wedge \tau_\varphi \tau_\varphi \delta_{\text{cpt}} \tau \beta = (-1)^{1-p} \int \alpha \wedge \delta_{\text{cpt}} \tau \beta
\]

\[
= -\int \delta_{\text{cpt}} \alpha \wedge \tau_\varphi \tau \beta = (\delta_{\text{cpt}} \alpha, \tau \beta).
\]

Hence, \( T_2 \) is \( \bar{\Omega}_{u} : L_{0,1}(\Delta) \to L_{0,1}(\Delta, -\varphi) \) in the sense of distributions. Since for all \( u \in L_{0,1}(\Delta, -\varphi) = t^{1-s}L_{0,1}(\Delta) \subset L_{0,1}(\Delta) \) there is a \( v \in L_{0,0}(\Delta) \) with \( T_2 v = \bar{\Omega}_{u} v = u \), the range of \( T_2 \) is closed. Thus, the range of \( T_1^* \) and the range of \( \bar{\Omega}_{u,s} = i^* T_1^* \tau^* : L_{0,0}(\Delta^*) \to L_{0,1}(\Delta^*) \) are closed as well.

We set \( S_1 := \pi \bar{\Omega}_{u,s} \tau^* : L_{0,0}(\Delta) \to L_{0,1}(\Delta, -\varphi) \) and \( S_2 := -\tau_\varphi S_1^* \tau_\varphi \). Then \( S_2 \) is the weak closure of \( \delta_{\text{cpt}} : L_{0,0}(\Delta, \varphi) \to L_{0,1}(\Delta) \).

\[
\mathcal{R}(S_2) = \{ u \in L_{0,1}(\Delta) : \exists v \in L_{0,0}(\Delta, \varphi) = t^{1-s}L_{0,0}(\Delta) \text{ with } S_2 v = u \}
\]

\[
\supset \{ u \in L_{0,1}(\Delta) : \exists v \in L_{0,0}(\Delta) \text{ with } \bar{\Omega}_{u} v = u \} = L_{0,1}(\Delta).
\]

Therefore, \( \mathcal{R}(S_2) = L_{0,1}(\Delta) \) is closed. This implies the claim.

\[ \square \]

4. \( L^2 \)-cohomology of complex curves

We will prove Theorem 4.1 in this section. As a preparation, we consider the following local situation: Let \( A \) be a locally irreducible analytic set of dimension one in a domain \( D \subset \mathbb{C}^n \) with \( \text{Sing} A = \{0\} \), let \( dV \) denote the volume form on \( A^* := \text{Reg} A \) which is induced by the Euclidean metric and let \( z : A \to \mathbb{C} \) be the projection on the first coordinate. Let us mention (cf. e.g. Prop. in [Chi89 Sect. 8.1]):

**Theorem 4.1.** The set of all tangent vectors at a point of a one-dimensional irreducible analytic set in \( \mathbb{C}^n \) is a complex line.
Thus, we can assume that $C_0(A) = \mathbb{C}_z \times \{0\} \subset \mathbb{C}_z \times \mathbb{C}_w^{n-1}$, and, therefore, $dV = dz \wedge dw$ (by shrinking $D$ if necessary).

Let $\pi : M \to A$ be a resolution of $A$, $x_0 := \pi^{-1}(0)$. Then $Z = (\pi^*(z))$ is the unreduced exceptional divisor of the resolution. After shrinking $A$ and $M$ again, we can assume that $M$ is covered by a single chart $\psi : M \to \mathbb{C}$ with $x_0 \in M$ and $\psi(x_0) = 0$. We set $\zeta := \pi^*(z)$ and get $Z = (\zeta)$. $|Z| = (\psi)$ implies $Z - |Z| = (\frac{\zeta}{\psi})$. We obtain

$$\pi^*(dz) = d(\pi^*z) = \frac{\partial \zeta}{\partial \psi} d\psi \sim \frac{\zeta}{\psi} d\psi.$$ 

Therefore, $\pi^*(dV) \sim \left|\frac{\zeta}{\psi}\right|^2 d\psi \wedge d\psi$, and we conclude (recall the definition of line bundles $L_D$ from Section 2.2):

$$f \in L^{p,q}(A^*) \iff \left| \frac{\zeta}{\psi} \right|^{1-p-q} \cdot \pi^*f \in L^{p,q}(M) \iff \pi^*f \in L^{p,q}(M, L_{(1-p)(Z-|Z|)}),$$

(4.1)

Nagase stated this equivalence already in Lem. 5.1 of [Nag90]. By use of the extension Theorem 2.2, we get:

$$f \in \text{dom} \left( \overline{\partial}_w : L^{p,0}(A^*) \to L^{p,1}(A^*) \right)$$

$$\iff \pi^*f \in \text{dom} \left( \overline{\partial}_w : L^{p,0}(M, L_{(1-p)(Z-|Z|)}) \to L^{p,1}(M, L_{p(|Z|-Z)}) \right).$$

(4.2)

The essential observation for the proof of Theorem 1.1 is the following:

**Theorem 4.2.** Let $X$ be a compact complex curve and $L \to X$ a holomorphic line bundle. Let $\pi : M \to X$ be a resolution of $X$ with exceptional divisor $Z$, and $D$ a divisor on $M$ such that $\pi^*L \cong L_D$, i.e., $\mathcal{O}(\pi^*L) \cong \mathcal{O}(D)$. Then

$$H^{0,0}_w(X^*, L) \cong H^0(M, \mathcal{O}(Z - |Z| + D)),$$

$$H^{0,1}_w(X^*, L) \cong H^1(M, \mathcal{O}(Z - |Z| + D)),$$

$$H^{1,0}_w(X^*, L) \cong H^0(M, \Omega^1(D)) \cong H^1(M, \mathcal{O}(-D)), \text{ and}$$

$$H^{1,1}_w(X^*, L) \cong H^1(M, \Omega^1(D)) \cong H^0(M, \mathcal{O}(-D)).$$

In [Par89, §5], Pardon proved that $H^{0,q}_{(2),sm}(X^*) \cong H^q(M, \mathcal{O}(Z - |Z|))$, where $H^{p,q}_{(2),sm}(X^*)$ denotes the $\overline{\partial}$-cohomology with respect to smooth $L^2$-forms. We will use similar arguments here.

**Proof.** Let $x_0$ be in Sing $X$, and let $A$ be an open neighborhood of $x_0 = 0$ in $X$ embedded locally in $\mathbb{C}^n$. We assume that $A = A_1 \cup \ldots \cup A_m$ with at $x_0$ irreducible analytic sets $A_i$. We obtain resolutions $\pi_i := \pi|_{\pi^{-1}(A_i)} : M_i \to A_i$ of $A_i$. The sets $M_i$ are pairwise disjoint in $M$ and, also, the support of the exceptional divisors $Z_i$ of the resolution $\pi_i$. We get $Z|_{\pi^{-1}(A)} = \sum_{i=1}^m Z_i$ and $|Z||_{\pi^{-1}(A)} = \sum_{i=1}^m |Z_i|$. Therefore, the consideration in the local case (see (4.2)) implies that $\overline{\partial}_w : L^{p,0}(X^*, L) \to L^{p,1}(X^*, L)$ can be identified with

$$\overline{\partial}_w : L^{p,0}(M, L_{(1-p)(Z-|Z|)+D}) \to L^{p,1}(M, L_{p(|Z|-Z)+D}).$$

Hence,

$$H^{0,0}_w(X^*, L) \cong \ker(\overline{\partial}_w : L^{0,0}(M, L_{Z-|Z|+D}) \to L^{0,1}(M, L_D)) \cong H^0(M, \mathcal{O}(Z - |Z| + D))$$
and
\[ H_{w}^{1,0}(X^{*}, L) \cong \ker (\overline{\partial}_{w} : L^{1,0}(M, L_{D}) \to L^{1,1}(M, L_{|Z+D|})) \cong H^{0}(M, \Omega^{1}(D)). \]

Serre duality (see Theorem 2 in [Ser55, § 3.10]) implies
\[ H_{w}^{1,0}(X^{*}, L) \cong H^{0}(M, \Omega^{1}(D)) \cong H^{1}(M, \mathcal{O}(-D)). \]

To prove the other two isomorphisms, consider the following general situation: Let \( E \) be a divisor on \( M \), \( L_{E} \) the associated bundle, and let \( L_{E}^{p,q} = (U) := L_{\text{loc}}^{p,q}(U, L_{E}) \) for each open set \( U \subset M \). Let \( E' \leq E \) be another divisor. Consider the \( \overline{\partial} \)-operator in the sense of distributions \( \overline{\partial}_{w} : L_{E}^{p,0} \to L_{E}^{p,1} \). Let \( C_{E,E'}^{p,0} \) denote the sheaf defined by
\[
C_{E,E'}^{p,0}(U) := \text{dom} (\overline{\partial}_{w} : L_{\text{loc}}^{p,0}(U, L_{E}) \to L_{\text{loc}}^{p,1}(U, L_{E}')).
\]
Then \( C_{E,E'}^{p,0} \) is fine and, in particular, \( H^{1}(M, C_{E,E'}^{p,0}) = 0 \). We get the sequence
\[
0 \to \Omega^{p}(E) \to C_{E,E'}^{p,0} \xrightarrow{\partial_{w}} L_{E'}^{p,1} \to 0 \tag{4.3}
\]
which is exact by the usual Grothendieck-Dolbeault lemma because there is an embedding \( L_{E'}^{p,q} \subset L_{E}^{p,q} \) (induced by the natural inclusion \( \mathcal{O}(E') \subset \mathcal{O}(E) \), see (2.1)).

This induces the long exact sequence of cohomology groups
\[
0 \to \Gamma(M, \Omega^{p}(E)) \to C_{E,E'}^{p,0}(M) \xrightarrow{\partial_{w}} L_{E'}^{p,1}(M) \to H^{1}(M, \Omega^{p}(E)) \to H^{1}(M, C_{E,E'}^{p,0}) = 0.
\]
Hence, \( L_{E'}^{p,1}(M) / \overline{\partial_{w}} C_{E,E'}^{p,0}(M) \cong H^{1}(M, \Omega^{p}(E)) \). We conclude
\[
H_{w}^{1,1}(X^{*}, L) \cong L_{D}^{1,1}(M) / \overline{\partial}_{w} C_{D,1,|Z|+D,D}(M) \cong H^{1}(M, \mathcal{O}(Z - |Z| + D))
\]
and, using the Serre duality again,
\[
H_{w}^{1,1}(X^{*}, L) \cong L_{D,1,|Z|+D}(M) / \overline{\partial}_{w} C_{D,1,|Z|+D}(M) \cong H^{1}(M, \Omega^{1}(D)) \cong H^{0}(M, \mathcal{O}(-D)).
\]

Theorem 4.1 follows now as a simple corollary by use of the classical Riemann-Roch theorem for each connected component of the Riemann surface \( M \), keeping in mind that by definition \( g(M) = g (X) \), \( \deg L = \deg \pi^{*}L = \deg D \) and \( \text{mult}_{x}X = \sum_{p \in \pi^{-1}(x)} \deg_{p}(Z - |Z|) \).

To deduce also a Riemann-Roch theorem for the \( \overline{\partial}_{w} \)-cohomology, we can use the following \( L^{2} \)-version of Serre duality:

**Theorem 4.3.** For each \( p \in \{0,1\} \), the range of \( \overline{\partial}_{w} : L^{p,0}(X^{*}, L) \to L^{p,1}(X^{*}, L) \) is closed. In particular, we get
\[
H_{w}^{p,q}(X^{*}, L) \cong H_{s}^{1,p-1,q}(X^{*}, L^{-1}).
\]

**Proof.** Recall the following well-known fact. If \( P : H_{1} \to H_{2} \) is a densely defined closed operator between Hilbert spaces with range \( \mathcal{R}(P) \) of finite codimension, then the range \( \mathcal{R}(P) \) is closed in \( H_{2} \) (see e.g. [HL84], Appendix 2.4).

As \( M \) is compact, Theorem 4.2 implies particularly that the range of \( \overline{\partial}_{w} \) is finite codimensional and, therefore, closed. Since \( \overline{\partial}_{s} \) is the adjoint of \( -\overline{\partial}_{w} \), the range of \( \overline{\partial}_{s} : L^{1-p,0}(X^{*}, L^{-1}) \to L^{p,1}(X^{*}, L^{-1}) \) is closed as well. That both ranges are closed implies the \( L^{2} \)-duality (cf. Lemma 3.1)
\[
H_{w}^{p,q}(X^{*}, L) \cong \mathcal{K}_{w}^{p,q}(X^{*}, L) \cong \mathcal{K}_{s}^{1-p,1-q}(X^{*}, L^{-1}) \cong H_{s}^{1-p,1-q}(X^{*}, L^{-1}),
\]
where $\mathcal{H}^{p,q}_{w/s}(X^*, L) := \ker \overline{\partial}_{w/s} \cap \ker \overline{\partial}_{w/s}$ denotes the space of $\overline{\partial}$-harmonic forms with values in $L$.

Therefore, Theorem 1.2 yields:

\begin{align*}
H^{0,0}_{w}(X^*, L) &\cong H^{1,0}_{w}(X^*, L^{-1}) \cong H^{0}(M, \mathcal{O}(D)), \\
H^{1,0}_{w}(X^*, L) &\cong H^{0,1}_{w}(X^*, L^{-1}) \cong H^{1}(M, \mathcal{O}(D)), \\
H^{0,1}_{w}(X^*, L) &\cong H^{1,0}_{w}(X^*, L^{-1}) \cong H^{1}(M, \mathcal{O}(Z - |Z| - D)), \\
H^{1,1}_{w}(X^*, L) &\cong H^{0,0}_{w}(X^*, L^{-1}) \cong H^{0}(M, \mathcal{O}(Z - |Z| - D)).
\end{align*}

(4.4)

Haskell computed $H^{0,q}_{\text{cpt}}(X^*) \cong H^{q}(M, \mathcal{O}_M)$, where $H^{p,q}_{\text{cpt}}(X^*)$ denotes the $\overline{\partial}$-cohomology with respect to smooth forms with compact support (see Thm. 3.1 in [Has89]). From (4.4), we obtain the dual version of Theorem 1.1 i.e., the Riemann-Roch theorem for the $\overline{\partial}_s$-cohomology:

**Corollary 4.4 ($\overline{\partial}_s$-Riemann-Roch).** Let $X$ be a compact complex curve with $m$ irreducible components, $L \to X$ be a holomorphic line bundle, and $\pi: M \to X$ be a resolution of $X$. Then,

\[ h_s^{0,0}(X, L) - h_s^{0,1}(X, L) = m - g + \deg L, \text{ and} \]

\[ h_s^{1,0}(X, L) - h_s^{1,1}(X, L) = m - g + \deg(\pi - |Z|) - \deg L, \]

where $Z$ is the exceptional divisor of the resolution.

In [BPS90], Brüning, Peyerimhoff and Schröder proved that $h_s^{0,0}(X^*) - h_s^{0,1}(X^*) = m - g$ and $h_s^{0,0}(X^*) - h_s^{1,0}(X^*) = m - g + \deg Z - |Z|$ by computing the indices of the differential operators $\overline{\partial}_s$ and $\overline{\partial}_w$. Schröder generalized this result for vector bundles in [Sch89].

### 5. Weakly holomorphic functions

In this section, we will prove Theorem 1.2 by studying weakly holomorphic functions and a localized version of the $\overline{\partial}_s$-operator.

Recalling the arguments at the beginning of Section 4, it is easy to see that the results of Section 3 generalize to arbitrary complex curves. In particular, the $\overline{\partial}_{s,w}$-holomorphic functions on a singular complex curve are precisely the square-integrable weakly holomorphic functions (cf. Lemma 3.3), and the $\overline{\partial}_{s,w}$-equation is locally solvable in the $L^2$-sense (combine Lemma 3.4 and 3.5).

Let $X$ be a singular complex curve, $\mathcal{L}^{\omega,q}_X$ the sheaf of locally square-integrable forms, and let $\overline{\partial}_w: \mathcal{L}^{\omega,0}_X \to \mathcal{L}^{\omega,1}_X$ be the $\overline{\partial}$-operator in the sense of distributions. For each open set $U \subset X$, we define $\overline{\partial}_{s,\text{loc}}$ on $L^{p,0}_{\text{loc}}(U)$ by $f \in \text{dom } \overline{\partial}_{s,\text{loc}}$ iff $f \in \text{dom } \overline{\partial}_w$ and $f \in \text{dom } (\overline{\partial}_{s,w}: L^{p,0}(V) \to L^{p,1}(V))$ for all $V \subset U$ (for more details, see [Rup10] Sect. 6). Let $\mathcal{F}_X^{p,0}$ be the sheaf of germs defined by $\mathcal{F}_X^{p,0}(U) := \text{dom } (\overline{\partial}_{s,\text{loc}}: L^{p,0}_{\text{loc}}(U) \to L^{p,1}_{\text{loc}}(U))$, and let $\mathcal{O}_X$ denote the sheaf of germs of weakly holomorphic functions on $X$. Then our considerations above yield an exact sequence

\[ 0 \to \mathcal{O}_X = \ker \overline{\partial}_{s,\text{loc}} \hookrightarrow \mathcal{F}_X^{p,0} \xrightarrow{\overline{\partial}_{s,\text{loc}}} \mathcal{L}^{0,1}_X \to 0. \]
The sheaves $\mathcal{F}_X^{0,0}$ and $\mathcal{L}_X^{0,1}$ are fine and so (5.1) is a fine resolution of $\hat{\mathcal{O}}_X$. Let $H^{0,0}_{s,\text{loc}}(X^*)$ denote the $L^2_{\text{loc}}$ Dolbeault cohomology on $X^*$ with respect to the $\bar{\partial}_{s,\text{loc}}$ operator. Using $\hat{\mathcal{O}}_X = \pi_*\mathcal{O}_M$, we deduce from (5.1):

$$H^0(M, \mathcal{O}_M) \cong H^0(X, \hat{\mathcal{O}}_X) = H^{0,0}_{s,\text{loc}}(X^*),$$

$$H^1(M, \mathcal{O}_M) \cong H^1(X, \hat{\mathcal{O}}_X) = H^{0,1}_{s,\text{loc}}(X^*),$$

where $\pi : M \to X$ is a resolution of $X$. That proves Theorem 1.2.

### 6. Applications

There are many applications of the classical Riemann-Roch theorem; we will transfer two of them to our situation to exemplify how the $L^2$-Riemann-Roch theorem can substitute the classical one on singular spaces.

#### 6.1. Compact complex curves as covering spaces of $\mathbb{C}P^1$

Let $X$ be a compact irreducible complex curve with $\text{Sing} X = \{x_1, ..., x_k\}$, let $(h_i)_{x_i} \in \mathcal{O}_{x_i}$ be chosen such that $(h_i)_{x_i} \hat{\mathcal{O}}_{x_i} \subset \mathcal{O}_{x_i}$ and let $U_i \subset X$ be a (Stein) neighborhood of $x_i$ with $h_i \cdot \hat{\mathcal{O}}(U_i) \subset \hat{\mathcal{O}}(U_i)$ (for the existence of the $h_i$, see e.g. Thm. 6 and its Cor. in [Nar66 §III.2]). Choose an $x_0 \in X^*$ and a (Stein) neighborhood $U_0$ of $x_0$. We can assume that $U_0$, ..., $U_k$ are pairwise disjoint.

We define a line bundle $L \to X$ as follows. Let $U_{k+1} = X^* \setminus \{x_0\}$ and choose $f_0 \in \mathcal{O}(U_0)$ such that $f_0$ is vanishing to the order $r := \text{ord}_{x_0} f_0 \geq 1$ in $x_0$, which we will determine later, but has no other zeros. We also set $f_i := 1/h_i$ for $i = 1, ..., k$ and $f_{k+1} = 1$, and consider the Cartier divisor $\{ (U_i, f_i) \}_{i=0, ..., k+1}$ on $X$. Let $L \to X$ be the line bundle associated to this divisor. As the $f_i$ have no zeros for $i > 0$, there exists a non-negative integer $\delta$ such that $\deg L = r - \delta$. Now choose $r := g(X) + \delta + 1$. It follows that $\deg L = g(X) + 1$. Give $L$ an arbitrary smooth Hermitian metric.

There is a canonical way to identify holomorphic sections of $L$ with meromorphic functions on $X$. A holomorphic section $s \in \mathcal{O}(L)$ is represented by a tuple $\{s_i\}_i$ where $s_j/f_j = s_i/f_i$ on $U_j \cap U_i$. This gives a meromorphic function $\Psi(s)$ by setting $\Psi(s) := s_j/f_j$ on $U_j$. Note that $\Psi(s)$ has zeros in the singular points $x_1, ..., x_k$ and may have a pole of order $r$ at $x_0 \notin \text{Sing} X$.

We can now apply our $L^2$-Riemann-Roch theorem. The $\bar{\partial}_s$-Riemann-Roch theorem, Corollary 4.4 implies $\dim H^{0,0}_s(X^*, L) \geq 1 - g(X) + \deg L = 2$. Therefore, there is a section $\tau \in L^{0,0}(X^*, L)$ with $\bar{\partial}_s \tau = 0$ and $\tau_{k+1}$ is non-constant, where $\tau = \{\tau_i\}_{i=0, ..., k+1}$ is written in the trivialization as above. This means that $\tau_i \in L^{0,0}(X^* \cap U_i)$, $\bar{\partial}_s \tau_i = 0$, and $\tau_j/f_j = \tau_{k+1}$ is non-constant on $U_j \cap U_{k+1}$. Theorem 1.2 implies that $\tau_i \in \hat{\mathcal{O}}(U_i)$, $i = 1, ..., k+1$. Now consider $\Psi(\tau)$ as defined above, i.e., $\Psi(\tau) = \tau_i/f_i$ on $U_i$. We conclude that $\Psi(\tau)/h_i \in \hat{\mathcal{O}}(U_i)$, thus $\Psi(\tau) \in \mathcal{O}(U_i)$ for $i = 1, ..., k$. Moreover, $\Psi(\tau)$ is non-constant, so it cannot be holomorphic on the whole compact space $X$, thus must have a pole of some order $\leq r$ in $x_0$.

$$\Psi(\tau) : X \setminus \{x_0\} \to \mathbb{C},$$

and

$$\tilde{\Psi}(\tau) : X \to \mathbb{C}P^1, x \mapsto \begin{cases} [\Psi(\tau)(x) : 1], & x \neq x_0 \\ [1 : \frac{1}{\Psi(\tau)(x)}], & x \in U_0 \end{cases}.$$
are finite, open and, hence, analytic ramified coverings (Covering Lemma, see [GR84 Sect. VII.2.2]). In particular, $X \setminus \{x_0\}$ is Stein (use e.g. Thm. 1 in [GR79 §V.1]).

6.2. Projectivity of complex curves. A line bundle $L \to X$ on a compact complex space is called very ample if its global holomorphic sections induce a holomorphic embedding into the projective space $\mathbb{C}P^N$, i.e., if $s_0, \ldots, s_N$ is a basis of the space of holomorphic sections $\Gamma(X, \mathcal{O}(L))$, then the map

$$\Phi : X \to \mathbb{C}P^N, \ x \mapsto [s_0(x) : \ldots : s_N(x)],$$

given in local trivializations of the $s_j$, defines a holomorphic embedding of $X$ in $\mathbb{C}P^N$. If some positive power of the line bundle has this property, then we say that it is ample. A compact complex space is called projective if there is an ample (and, hence, a very ample) line bundle on it.

A classical application of the Riemann-Roch theorem is that any compact Riemann surface is projective, and a line bundle on a Riemann surface is ample if its degree is positive (cf. e.g. [Nar92 Sect. 10]). This generalizes to singular complex curves:

**Theorem 6.1.** Let $X$ be a compact locally irreducible complex curve. If $L \to X$ is a holomorphic line bundle with $\deg L > 0$, then $L$ is very ample. In particular, $X$ is projective and each holomorphic line bundle on $X$ is ample if its degree is positive.

Clearly, this result is well-known and follows from more general sheaf-theoretical methods (vanishing theorems) once one knows that $L$ is positive iff $\deg L > 0$ (cf. e.g. Thm. 4.4 in [Pet94 Sect. V.4.3] or Satz 2 in [Gra62 §3]). Nevertheless, it seems interesting to us to present another proof of Theorem 6.1 which is based on the $L^2$-Riemann-Roch of singular complex curves. The assumption that $X$ must be locally irreducible in Theorem 6.1 is not necessary. One can prove the result without this assumption easily by the same technique. Yet, to keep the notation simple, we present here only the locally irreducible case.

Let us make some preparations for the proof of Theorem 6.1. Let $X$ be a connected complex curve and $\pi : M \to X$ a resolution of $X$. We choose a point $x_0 \in \text{Sing} X$ and a small neighborhood $U \subset X$ of $x_0$ with $U^* := U \setminus \{x_0\} \subset \text{Reg} X$. Assume $U$ is irreducible at $x_0$. We define $p_0 := \pi^{-1}(x_0)$, $V := \pi^{-1}(U)$, and $V^* := V \setminus \{p_0\}$. We can assume that there is a chart $t : V \to \mathbb{C}$ such that the image of $t$ is bounded.

The Riemann extension theorem implies that $\pi^{-1} : U \to V$ is weakly holomorphic or, briefly, $\tau := t \circ \pi^{-1} \in \hat{\mathcal{O}}(U)$ (see Theorem 2.3). We show that $\tau$ generates the weakly holomorphic functions at $x_0$ in the following sense: Let $f \in \hat{\mathcal{O}}(U)$. Then $f \circ \pi$ is holomorphic on $V^*$ and bounded in $p_0$. This implies that $f \circ \pi$ is holomorphic on $V$, $f \circ \pi(t) = \sum_{i=0}^{\infty} a_i t^i$, and $f(x) = \sum a_i \tau(x)^i$ (by shrinking $U$ and $V$ if necessary). This allows to define the order $\text{ord}_{x_0} f$ of vanishing of $f$ in $x_0$ by $r \in \mathbb{N}_0$ if $a_r \neq 0$ and $a_r = 0$ for $i < r$. In particular,

$$\text{ord}_{x_0} f = \text{ord}_{p_0}(f \circ \pi).$$

Note that this definition does not depend on the resolution as different resolutions are biholomorphically equivalent.

The $L^2$-extension theorem (see Theorem 2.2 and (4.2)) imply

$$f \in H^0_{L^2}(U) \iff r_0 \cdot \pi^* f \in \mathcal{O}_{L^2}(V) \iff (\tau^{r_0} \cdot f \in \hat{\mathcal{O}}(U) \text{ and } f \in L^2(U)), \quad (6.2)$$

where $Z$ denotes the exceptional divisor of the resolution and $r_0 := \deg_{p_0}(Z - |Z|)$. In particular, we get the representation $f(x) = \sum_{i \geq -r_0} a_i \tau(x)^i$ and $\text{ord}_{x_0} f := \text{ord}_{p_0} \pi^* f \geq -r_0$ is again well-defined. $f$ is weakly holomorphic iff $\text{ord}_{x_0} f \geq 0$. 

We denote by $L_{x_0}$ the holomorphic line bundle on $X$ which is trivial on $X \setminus \{x_0\}$ and is given by $\tau$ on $U$, i.e., the line bundle on $X$ given by the open covering $U_i := X \setminus \{x_0\}, U_0 := U$ and the transition function $g_{01} := \tau : U_0 \cap U_1 \to \mathbb{C}$. Then $\pi^* L_{x_0} \cong L_{p_0}$, where $L_{p_0}$ is the holomorphic line bundle $L_{p_0} \to M$ associated to the divisor $\{p_0\}$.

Let $L \to X$ be any holomorphic line bundle, $L' := L \otimes L_{x_0}^{-1}$, and let $s'$ be a section in $H^{0,0}_w(X^*, L')$. We can assume that $L$ and $L'$ are given by divisors $\{(U_j, f_j)\}$ and $\{(U_j, f'_j)\}$, respectively, where $\{U_j\}$ is an open covering of $X$ with $U_0 = U$ and $x_0 \notin U_j$ for $j \neq 0$ and where $f_j, f'_j \in \mathcal{M}(U_j)$ with $g_{jk} := f_j/f_k$ and $g'_{jk} := f'_j/f'_k$ in $\mathcal{O}(U_j \cap U_k)$ ($g_{jk}$ and $g'_{jk}$ are the transition functions of $L$ and $L'$, respectively).

We get $f_0 = f'_0 : \tau$ and $f_j = f'_j$ for $j \neq 0$. There is a meromorphic function $s := \Psi(s') \in \mathcal{M}(X)$ representing $s'$. This meromorphic function is defined by $s = s'_j/f'_j$ on $U_j$, where $s'_j$ is the trivialization of $s'$ on $U_j$. We can define a section $s = (s_j) \in H^{0,0}_w(X^*, L)$ by $s_j = s \cdot f_j$. Thus $s_0 = s'_0 : \tau$ and $s_j = s'_j$ for $j \neq 0$. Hence, $\text{ord}_{x_0} s_0 = \text{ord}_{x_0} s'_0 + 1$. Summarizing, we get an injective linear map

$$T : H^{0,0}_w(X^*, L \otimes L_{x_0}^{-1}) \to H^{0,0}_w(X^*, L), \ s' \mapsto s,$$

which we call the \textit{natural inclusion}. It follows from the construction above and by use of (6.2) that each section $s \in H^{0,0}_w(X^*, L)$ with $\text{ord}_{x_0} s_0 > -r_0$ is in the image of $T$.

As $H^1(M, \mathcal{O}(D')) = 0$ for a Divisor $D'$ with $\deg D' > 2g - 2$ by the classical Riemann-Roch theorem (cf. e.g. \cite[Sect. 10]{Nar92}), Theorem 4.2 (more precisely, $H^{0,1}_w(X^*, L) \cong H^1(M, \mathcal{O}(Z - |Z| + D))$) implies the following vanishing theorem.

**Theorem 6.2.** If $L \to X$ is a holomorphic line bundle on an irreducible compact complex curve $X$ with $\deg L > 2g - 2 - \sum_{x \in \text{Sing} X} \text{mult}_x X$, then $H^{0,1}_w(X^*, L) = 0$.

As a preparation for the proof of Theorem 6.1 we get our main ingredient:

**Lemma 6.3.** Let $L \to X$ be a holomorphic line bundle on a connected compact locally irreducible complex curve $X$ with $\deg L > 2g - 1 - \sum_{x \in \text{Sing} X} \text{mult}_x X$. Then the natural inclusion

$$T : H^{0,0}_w(X^*, L \otimes L_{x_0}^{-1}) \to H^{0,0}_w(X^*, L)$$

is not surjective. If $\deg L > 2g + r_0 - 1 - \sum_{x \in \text{Sing} X} \text{mult}_x X$, then there is a section $s \in H^{0,0}_w(X^*, L)$ which is weakly holomorphic on $U(x_0)$ and does not vanish in $x_0$.

Recall that $r_0 = \text{mult}_{x_0} X - 1 = \deg_{p_0} (Z - |Z|).

**Proof.** i) As $\pi^*(L \otimes L_{x_0}^{-1}) \cong \pi^* L \otimes L_{p_0}^{-1}$, we get $\deg L \otimes L_{x_0}^{-1} = \deg L - 1 > 2g - 2 - \deg(Z - |Z|)$. The $\overline{\partial}_w$-Riemann-Roch theorem and $h^{0,1}_w(X^*, L) = 0 = h^{0,1}_w(X^*, L \otimes L_{x_0}^{-1})$ (using Theorem 6.2) yield

$$h^{0,0}_w(X^*, L \otimes L_{x_0}^{-1}) = 1 - g + \deg(Z - |Z|) + \deg L \otimes L_{x_0}^{-1} + 1 - g + \deg(Z - |Z|) + \deg L = h^{0,0}_w(X^*, L).$$

Therefore, the natural inclusion $T$ cannot be surjective.

ii) The image of $T^{r_0} : H^{0,0}_w(X^*, L \otimes L_{x_0}^{-r_0}) \to H^{0,0}_w(X^*, L)$ are the sections $s$ with $\text{ord}_{x_0} s_0 \geq 0$, where $s_0$ is the trivialization of $s$ over $U(x_0)$, i.e., the sections where $s_0$ is weakly holomorphic on $U(x_0)$. As $H^{0,0}_w(X^*, L \otimes L_{x_0}^{-r_0}) \to H^{0,0}_w(X^*, L \otimes L_{x_0}^{-r_0})$ is not surjective (use $\deg L \otimes L_{x_0}^{-r_0} = \deg L - r_0 > 2g - 1 - \deg(Z - |Z|)$ and part (i)),
there is a section \( s' \in H_{w}^{0,0}(X^{*}, L \otimes L_{x_{0}}^{-r_{0}}) \) with \( \text{ord}_{x_{0}} s' = -r_{0} \) and \( \text{ord}_{x_{0}} (T'^{0}(s'))_{0} = 0 \). So, \( s := T'^{0}(s') \) is the section of \( H_{w}^{0,0}(X^{*}, L) \) we were looking for. \( \square \)

**Proof of Theorem 6.1.** Let \( X \) be a connected compact locally irreducible complex curve with \( \text{Sing \ } X = \{ x_{1}, ..., x_{k} \} \), and \( L \rightarrow X \) a line bundle with \( \text{deg } L \gg 0 \). Following the classical arguments to show that the map \( \Phi \) in (6.1) is a well-defined holomorphic embedding (see e.g. [Pet94, V.4, Thm. 4.4]), we have to prove: i) For \( x \in X \), there exists \( s \in \Gamma(X, \mathcal{O}(L)) \) such that \( s(x) \neq 0 \).

ii) For \( x, y \in X \), there exists \( s \in \Gamma(X, \mathcal{O}(L)) \) such that \( s(x) \neq s(y) \).

iii) For \( x \in X \), there exists \( s \in \Gamma(X, \mathcal{O}(L)) \) with differential \( ds_{x} \neq 0 \).

We will prove the statements (i) and (iii) for singular points \( x \in \text{Sing } X \). The case of regular points is simpler and follows easily with the natural inclusion and Lemma 6.3. The statement (ii) can be seen just as (i) by imposing the additional condition that \( s(y) = 0 \) in what we do to prove the statement (i).

Let \( \pi : M \rightarrow X \) be a resolution of singularities. Set \( X^{*} = \text{Reg } X, \ M^{*} = \pi^{-1}(X^{*}) \), \( p_{j} := \pi^{-1}(x_{j}) \), and \( r_{j} := \text{deg}_{p_{j}}(Z - |Z|) \), where \( Z \) is the unreduced exceptional divisor of the resolution. Fix a \( \mu \in \{ 1, ..., k \} \) and choose a neighborhood \( U_{\mu} \) of \( x_{\mu} \) such that \( U_{\mu} \) is isomorphic to an analytic set \( A \subset D \in \mathbb{C}^{w_{1}, ..., w_{m-1}}_{w_{0}} \) such that \( x_{\mu} \) is identified with \( 0 \in \mathbb{C}^{n} \) and with tangent cone \( C_{0}(A) = \mathbb{C}_{z} \times 0 \). We can assume furthermore that there exists a chart \( t : V_{\mu} \rightarrow \mathbb{C} \) with \( t_{\mu}(p_{\mu}) = 0 \) on \( V_{\mu} := \pi^{-1}(U_{\mu}) \). The projection \( z : U_{\mu} \cong A \rightarrow \mathbb{C}_{z} \) is of interest to us because it is a holomorphic map with injective tangent map. Note that \( r_{\mu} = \text{ord}_{x_{\mu}} z - 1 \) since \( \pi^{*}(Z|U_{\mu}) = (z) \).

For each singularity \( x_{j} \), there is a function \( h_{j} \in \mathcal{O}(U_{j}) \) such that \( h_{j} \hat{\mathcal{O}}(U_{j}) \subset \mathcal{O}(U_{j}) \) for a neighborhood \( U_{j} \) of \( x_{j} \) small enough (see [Nar60, §III.2]). The number

\[
\eta_{j} := \text{ord}_{x_{j}} h_{j}
\]

is important for our considerations because of the following fact. If \( f \) is a function on \( U_{j} \) with \( \text{ord}_{x_{j}} f \geq \eta_{j} \), then \( f/h_{j} \) is bounded at \( x_{j} \) (\( \text{ord}_{x_{j}} f/h_{j} \geq 0 \)); this implies \( f/h_{j} \in \hat{\mathcal{O}}(U_{j}) \) and, hence, \( f \in \mathcal{O}(U_{j}) \).

We can choose a weakly holomorphic section \( \sigma \in H_{w}^{0,0}(X^{*}, L) \) such that \( \sigma \) does not vanish in \( x_{\mu} \) and \( \text{ord}_{x_{j}} \sigma \geq \eta_{j} \) for \( j \neq \mu \). This section \( \sigma \) exists as we have the natural inclusion (see the construction above)

\[
H_{w}^{0,0}(X^{*}, L \otimes L_{x_{\mu}}^{-r_{\mu}} \otimes \bigotimes_{j \neq \mu} L_{x_{j}}^{-\eta_{j} - r_{j}}) \rightarrow H_{w}^{0,0}(X^{*}, L)
\]

and \( \text{deg } L \gg 0 \) implies by Lemma 6.3 that the natural inclusion

\[
H_{w}^{0,0}(X^{*}, L \otimes L_{x_{\mu}}^{-r_{\mu} - 1} \otimes \bigotimes_{j \neq \mu} L_{x_{j}}^{-\eta_{j} - r_{j}}) \rightarrow H_{w}^{0,0}(X^{*}, L \otimes L_{x_{\mu}}^{-r_{\mu}} \otimes \bigotimes_{j \neq \mu} L_{x_{j}}^{-\eta_{j} - r_{j}})
\]

is not surjective.

Note that \( \sigma \) is holomorphic on \( X - \{ x_{\mu} \} \) but just weakly holomorphic in \( x_{\mu} \). We will now modify \( \sigma \) so that it becomes holomorphic and non-vanishing in \( x_{\mu} \). Shrink \( U_{\mu} \) such that \( \sigma = \sum_{a_{0} \geq 0} a_{0}^{r_{0}} \) on \( U_{\mu} \) with \( a_{0} \neq 0 \). Let \( \sigma' := \sigma/a_{0} \) so that \( \text{ord}_{x_{\mu}} (\sigma' - 1) \geq 1 \), i.e., \( \sigma' - 1 = \sum_{a_{0} \geq 1} a_{0}^{r_{0}} \) on \( U_{\mu} \). Choose as above a \( \tilde{\sigma} \in H_{w}^{0,0}(X^{*}, L) \) with \( \text{ord}_{x_{\mu}} \tilde{\sigma} = 1 \) and \( \text{ord}_{x_{j}} \tilde{\sigma} \geq \eta_{j} \) for \( j \neq \mu \). Let \( \tilde{\sigma} = \sum_{a_{0} \geq 1} a_{0}^{r_{0}} \) close to \( x_{\mu} \) with \( a_{1} \neq 0 \). We define \( \sigma'' := \sigma' - a_{1}^{r_{0}} \tilde{\sigma} \). Then, \( \text{ord}_{x_{\mu}} (\sigma'' - 1) \geq 2 \) and \( \text{ord}_{x_{j}} \sigma'' \geq \eta_{j} \) for \( j \neq \mu \). We repeat this procedure recursively to get a section \( \xi = \{ \xi_{j} \} \in H_{w}^{0,0}(X^{*}, L) \) with \( \text{ord}_{x_{\mu}} (\xi_{\mu} - 1) \geq \eta_{\mu} \).
and \( \text{ord}_x \xi_j \geq \eta_j \) for \( j \neq \mu \). Thus, \( \xi \) is a holomorphic section on \( X \), non-vanishing in \( x_\mu \). That shows (i) for \( x = x_\mu \).

We can use the same procedure to construct a holomorphic section \( s = \{ s_j \} \in H^{0,0}_w(X^*, L) \) with \( \text{ord}_x(s_\mu - z) \geq \eta_\mu \) and \( \text{ord}_x s_j \geq \eta_j \) for \( j \neq \mu \). Since \( s_\mu \) behaves like \( z \) in \( x_\mu \) (i.e., \( s_\mu = z \cdot h \) for an \( h \in \mathcal{O}(U_\mu) \) with \( h(x_\mu) \neq 0 \)), we claim that the tangential map of \( s_\mu \) is injective in \( x_\mu \). For each tangent vector \( v \in C_{x_\mu}(X) \) there is a path \( \alpha : [-1, 1] \to X \) with \( \alpha(0) = x_\mu \) and \( \alpha'(0) = v \). Let us assume \( v \neq 0 \), i.e., \((z \circ \alpha)'(0) \neq 0\). Then

\[
((z \cdot h) \circ \alpha)'(0) = \lim_{\theta \to 0} \frac{(z \cdot h) \circ \alpha(\theta) - 0}{\theta} = \lim_{\theta \to 0} \left( \frac{z \circ \alpha(\theta)}{\theta} \cdot h(\alpha(\theta)) \right),
\]

\[
= (z \circ \alpha)'(0) \cdot h(\alpha(0)) \neq 0. \]

That proves the claim and thus statement (iii) for \( x = x_\mu \). \( \square \)

For this proof, \( L \) has to satisfy

\[
\deg L > 2g + \sum_{j=1}^{k} (\eta_j + r_j) - \deg(Z - |Z|) = 2g + k + \sum_{j=1}^{k} \eta_j.
\]

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WUPPERTAL, GAUSSSTR. 20, 42119 WUPPERTAL, GERMANY.

E-mail address: ruppenthal@uni-wuppertal.de, sera@math.uni-wuppertal.de