I. INTRODUCTION

Spin qubits are a promising platform for quantum information processing machines [1]. Recent progress includes the development of high-fidelity single-qubit operations [2], coupling to resonators [3], a programmable quantum processor [4], universal quantum logic at 1.5K [5], and capacitive coupling of spin qubits [6]. The main candidates, tracking down the exact sources has proved problematic. On the positive side, the qubits themselves are very sensitive noise detectors. They can be used to detect the spatial locations of these sources and determine which is the most important. In this paper, we propose that anisotropy in T₁ and T₂ with respect to the direction of the applied magnetic field can reveal much about these aspects of the noises. We investigate the anisotropy patterns of charge noise, evanescent-wave Johnson noise, and hyperfine noise. It is necessary to have a rather well-characterized sample to get the maximum benefit from this technique. The general anisotropy patterns are elucidated and then we calculate the expected anisotropy in a particular Si/SiGe quantum dot device.

Electron spin qubits are a promising platform for quantum computation. Environmental noise impedes coherent operations by limiting the qubit relaxation (T₁) and dephasing (T₂) times. There are multiple sources of such noise, which makes it important to devise experimental techniques that can detect the spatial locations of these sources and determine which is the most important. In this work, we consider noise from phonons [14–16], which is usually a measure of a weighted average of the noise strength on the qubit operating frequency. Here the angle brackets denote a quantum and thermal average and the subscripts are Cartesian components. We define the noise power tensor by

\[
\langle B_1^{(\text{eff})} B_2^{(\text{eff})} \rangle_\omega = \int dt \ e^{i\omega t} \langle B_1^{(\text{eff})}(t) B_2^{(\text{eff})}(0) \rangle.
\]  

Here the angle brackets denote a quantum and thermal average and the subscripts are Cartesian components. For the moment, let us assume that \( B_0 = B_0 \hat{z} \). T₁ is determined by the transverse noise components: 1/T₁ ∝ \( \langle B_x^{(\text{eff})} B_x^{(\text{eff})} \rangle_{\omega_{\text{op}}} + \langle B_y^{(\text{eff})} B_y^{(\text{eff})} \rangle_{\omega_{\text{op}}} \). The dephasing rate 1/T₂, on the other hand, is determined by a weighted average of the longitudinal noise strength \( \langle B_z^{(\text{eff})} B_z^{(\text{eff})} \rangle_\omega \). Since we have 1/T₂ = 1/T₁ + 1/T₂, all of the diagonal components of the noise tensor are accessible to experiment. For most spin qubits T₁ ≫ T₂, so we have the simpler equation T₂ = T₂. Henceforth we shall assume this to be the case and will refer only to T₂.

This vector character of the coherence time equations opens up the possibility of a novel experimental tool to investigate noise in spin qubit systems: anisotropy in T₁ and T₂ as a function of the direction of the applied field. Simply put, if the applied field is in the direction \( \mathbf{R} \hat{z} \), where \( \mathbf{R} \) is a rotation operator that takes \( \hat{z} \) into the direction with polar angles (θ, φ), then 1/T₁(θ, φ) ∝ \( \langle B_x^{(\text{eff})} B_x^{(\text{eff})} \rangle_{\omega_{\text{op}}} + \langle B_y^{(\text{eff})} B_y^{(\text{eff})} \rangle_{\omega_{\text{op}}} \), while T₂(θ, φ) depends on \( \langle B_z^{(\text{eff})} B_z^{(\text{eff})} \rangle_\omega \). The pattern in (θ, φ) gives information about the nature of the noise sources and their positions relative to the qubit.

For charge qubits, the analog of the direction of the external field is the direction of the line in space that connects the two qubits. It is normally not possible to adjust this over a wide range and even in a narrow range it is unlikely to be possible in a controllable way. As a result, the experiments we propose are only possible for spin qubits. For hybrid qubits that operate sometimes in the charge regime and sometimes in the spin regime, it may be possible to disentangle the sources of noise using the techniques presented here along with careful analysis of the experimental data. However, we will not attempt that here, since we are mainly attempting to establish the basic principles involved.
In this paper we focus on the anisotropy of three different types of noise sources: charge noise, hyperfine noise, and evanescent-wave Johnson noise (EWJN) in silicon devices. Charge noise is the most important at low frequencies and generally determines $T_2$. EWJN is important at higher frequencies and low magnetic fields, and in many cases may determine $T_1$. Hyperfine noise also is important, particularly in GaAs systems. It is expected to be isotropic in $\vec{B}_0$, which from the viewpoint of this paper is a key experimental signature of this type of noise [17], as we shall discuss below. Phonon relaxation, in contrast, is highly anisotropic, as has been shown previously [18, 19]. This mechanism is important at higher magnetic fields $B$. Since we do not include it, the results we present here hold only for $B \leq 3.4$ T [19]. At these lower fields $T_1$ saturates. It’s important to note that the anisotropy due to phonon effects is determined by the orientation of $\vec{B}_0$ relative to the crystal axes, while the anisotropies considered in this paper are relative to directions determined by the geometry of the device.

In nearly all experiments on spin qubits, the relative orientation of the sample and the applied magnetic field is not allowed to vary. However, rotatable sample holders can give some variation in the angle between the growth direction and the applied field. See, for example, Ref. 20. Full coverage of the whole solid angle can be obtained from vector magnetic arrangements with appropriate parameters of the magnets. Indeed, experiments to optimize qubit operation by so changing the direction of the field using a vector magnet have been carried out [21]. The present paper can be viewed as pointing a direction for these kinds of efforts.

In Sec. [H] we give the basic theory of the anisotropy of the coherence times. In Sec. [II] the noise sources under consideration are reviewed with an eye towards the anisotropy problem. In Sec. [IV] we show how to make quantitative predictions for the anisotropy in a specific quantum dot device. In Sec. [V] we summarize and discuss our results and indicate some promising future directions for research.

II. ANISOTROPY IN COHERENCE TIMES OF A SPIN QUBIT

A. Relaxation time

The relaxation rate of a spin qubit in the noise magnetic field depends on the noise correlation function

$$\langle B_i^{(\text{eff})}(t)B_j^{(\text{eff})}(0) \rangle_{\omega_{op}}$$

where $\vec{B}^{(\text{eff})}$ is the effective noise magnetic field, $\omega_{op}$ is the operating frequency of the qubit. Below, the time arguments and the subscript $\omega$ or $\omega_{op}$ may be suppressed for brevity when there is no possibility of confusion. The effective noise magnetic field is any time-dependent field that couples to the spin in the usual way. Hence this could be a physical magnetic field, a field that comes from the motion of the qubit in an inhomogeneous field, a field that results from phonons mediated by spin-orbit coupling, etc.

$T_1$, the relaxation time, depends only on the transverse components of the correlation function. If we define $T_1^{(i)}$ as the relaxation time when the applied field is in the $i$-direction, then:

$$\frac{1}{T_1^{(x)}} = \left( \frac{qq}{4m_e c^2} \right)^2 \left[ \langle B_x^{(\text{eff})}B_x^{(\text{eff})} \rangle + \langle B_y^{(\text{eff})}B_y^{(\text{eff})} \rangle \right]$$

$$\frac{1}{T_1^{(y)}} = \left( \frac{qq}{4m_e c^2} \right)^2 \left[ \langle B_y^{(\text{eff})}B_y^{(\text{eff})} \rangle + \langle B_z^{(\text{eff})}B_z^{(\text{eff})} \rangle \right]$$

$$\frac{1}{T_1^{(z)}} = \left( \frac{qq}{4m_e c^2} \right)^2 \left[ \langle B_z^{(\text{eff})}B_z^{(\text{eff})} \rangle + \langle B_y^{(\text{eff})}B_y^{(\text{eff})} \rangle \right]$$

In general $q$ is the charge and $g$ is the g-factor of the qubit. In what follows We will take $q$ to be the electron charge and $g = 2$.

If the applied field is in an arbitrary direction $\hat{n} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z}$, where $\theta$ is polar angle and $\phi$ is azimuthal angle, then the relaxation rate becomes

$$\frac{1}{T_1(\theta, \phi)} = \left( \frac{qq}{4m_e c^2} \right)^2 \sum_{ij} Q_{ij}^{(1)} \langle B_i^{(\text{eff})}B_j^{(\text{eff})} \rangle$$

where

$$Q_{ij}^{(1)} = \begin{bmatrix} \cos^2 \phi \cos^2 \theta + \sin^2 \phi & -\cos \phi \sin \phi \sin^2 \theta & -\cos \phi \cos \theta \sin \theta \\ -\cos \phi \sin \phi \sin^2 \theta & \sin^2 \phi \cos^2 \theta + \cos^2 \phi & -\sin \phi \cos \theta \sin \theta \\ -\cos \phi \cos \theta \sin \theta & -\sin \phi \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix}$$

with $\{x, y, z\}$ as the basis for the matrix $Q_{ij}^{(1)}$. Note that if there are any nonzero off-diagonal correlation functions
\( \langle B_i^{(\text{eff})} B_j^{(\text{eff})} \rangle \) \((i \neq j)\), they are also needed in the expression for the relaxation time.

### B. Dephasing time

The calculation of the dephasing time is more complicated than that for the relaxation time, since it depends more sensitively on the full frequency spectrum of the noise and higher-level correlation functions. For the purposes of this paper only ratios of \(T_2\) for different applied field angles are important. Hence the specific approximation used to compute \(T_2\) is not so crucial. It will be sufficient to assume that the field fluctuation obey Gaussian statistics. Then if the applied field is in the \(z\)-direction, the off-diagonal components of the density matrix of the qubit decay according to the expression \(\exp[-\Gamma(t)]\) with

\[
\Gamma(t) = \frac{t^2}{2} \left( \frac{qq^*}{2m_e c^2} \right) \int_{-\infty}^{\infty} d\omega \langle B_z^{(\text{eff})} B_z^{(\text{eff})}\rangle \sin^2(\omega t/2).
\]

The dephasing time \(T_\phi\) of the qubit is obtained by solving the transcendental equation \(\Gamma(T_\phi) = 1\). Here the sinc function is defined by \(\text{sinc}(x) = \sin x/x\). If the applied field is in the \((\theta, \phi)\) direction, then

\[
\Gamma(t) = \frac{t^2}{2} \left( \frac{qq^*}{2m_e c^2} \right) \sum_{ij} Q_{ij}^{(2)}
\]

\[
\times \int_{-\infty}^{\infty} d\omega \langle B_i^{(\text{eff})} B_j^{(\text{eff})}\rangle \sin^2(\omega t/2)
\]

where

\[
Q^{(2)} = \begin{bmatrix}
\cos^2 \phi \sin^2 \theta & \cos \phi \sin \phi \sin^2 \theta & \cos \phi \cos \theta \sin \theta \\
\cos \phi \sin \phi \sin^2 \theta & \sin^2 \phi \sin^2 \theta & \sin \phi \cos \theta \sin \theta \\
\cos \phi \cos \theta \sin \theta & \sin \phi \cos \theta \sin \theta & \cos^2 \theta
\end{bmatrix}
\]

in the same basis as used for \(Q^{(1)}\).

### III. NOISE SOURCES

In this section we show how electric noise decohers a spin qubit and review the theory of the two types of noise we will consider in this paper.

#### A. Effective Magnetic Field Noise in the Presence of a Micromagnet

Single-qubit logic gates in spin systems are often implemented using a micromagnet to set up magnetic field gradients. This has the unwanted complication that electric field noise moves the spin up and down the gradient, causing a time-dependent magnetic field \(\vec{B}^{(E)}\) that can decohere the qubit. Here we outline how this plays into the anisotropy effect.

We will take a simple model of a quantum dot in a harmonic potential. The Hamiltonian is

\[
H = - \sum_i \frac{\hbar^2}{2m_i} \frac{\partial^2}{\partial x_i^2} + \sum_i \frac{1}{2} k_i x_i^2 - q \sum_i x_i E_i(t)
\]

where \(i\) is a Cartesian index, \(m_i\) and \(k_i\) are the effective mass and spring constant of the electron in the \(i\)-direction, \(q\) is the electric charge of the electron and \(E_i(t)\) is the noise electric field component in the \(i\)-direction. The frequency of the electric noise is much smaller than the natural frequencies of the harmonic motion, so the Born-Oppenheimer approximation applies and the effect of \(E_i(t)\) is to shift the position of the minimum of the potential by \(\Delta x_i(t) = qE_i(t)/2k_i\). Actually, the harmonic approximation is not very good for the \(z\)-direction perpendicular to the layer but this does not matter, since the confinement in the \(z\)-direction is much stronger and we can simply drop the \(z\) term in Eq\[7\] in the potential and treat the dot as two-dimensional.

The micromagnet sets up a static field \(\vec{B}^m\) that varies strongly in space. The associated effective noise field that acts on the spin is

\[
B_i^{(E)}(t) = \frac{\partial B_i^m}{\partial x_j} \Delta x_j(t) = q \sum_j \frac{1}{2k_j} \frac{\partial B_i^m}{\partial x_j} E_j(t).
\]

The field gradient and the spring constant are device parameters. This equation already shows that considerable device modeling is necessary to extract any interesting information about the noise field \(E_i(t)\).

#### B. Evanescent-wave Johnson Noise

Evanescent-wave Johnson Noise (EWJN) is due to the random motion of charges in the metallic elements of the device. This motion produces random electric and magnetic fields on the qubits in the vicinity. In most Si/SiGe heterostructure and Si MOS devices the gates form a layer of metal that can be approximated as a uniform layer from the standpoint of noise production. We neglect the thickness of the gate layers, which may detract from our main point. The results are well-known: again since ratios are what we are after, this does not detract from our main point. The results are well-known: we need only a few formulas from the theory of EWJN.

Let us take the growth direction for the device to be the \(z\)-direction, the distance of the qubit from the gate layer as \(d\), and the dielectric constant of the dielectric as \(\epsilon_d\). Then the noise correlation functions for the electric field are

\[
\langle E_z E_z \rangle = \frac{\hbar \omega d}{8\pi \sigma d^3} \coth \frac{\hbar \omega}{2k_B T}
\]
The other elements of the noise tensor are \( \langle E_x E_z \rangle = \langle E_y E_y \rangle = (1/2) \langle E_x E_z \rangle \), while the off-diagonal elements of the tensor vanish. This electric noise is converted into effective magnetic field noise using the techniques of the previous section.

The magnetic EWJN noise correlation function is given by

\[
\langle B_z B_z \rangle = \frac{\pi \hbar \sigma \omega}{2 e^2 d} \coth \frac{\hbar \omega}{2 k_B T}. \tag{9}
\]

The other elements of this noise tensor are \( \langle B_z B_x \rangle = \langle B_y B_y \rangle = (1/2) \langle B_z B_z \rangle \), while the off-diagonal elements of the tensor vanish. Unlike the electric noise, magnetic EWJN acts directly on the qubit spin to produce decoherence.

### C. Charge Noise

The exact nature of the charge noise remains controversial. We will show that different hypotheses about these sources can be tested by doing anisotropy experiments.

Charge noise comes from the random motion of electric charge in the system. We will investigate two types of motion. The first is the classic two-level system that is found in insulators, where localized defects create two nearby potential minima that an electric randomly fluctuates between. Thus we have a collection of dipoles, each with a fixed direction \( \vec{p} \) in space that switches back and forth between the directions \( \pm \vec{p} \). We will assume that the orientation of these dipoles is uniformly random on the unit sphere, and that they are distributed uniformly in space in the oxide layer above the qubits. We refer to this model as the “random dipole” model. The second type of noise is more particular to a system that has metal-insulator interfaces. Here there are traps in the insulator whose energy is close to the Fermi energy of the conductor and whose location is close enough to the interface that electrons can tunnel in and out of the trap. Here we expect the motion to be mainly perpendicular to the interface and again assume that they are distributed uniformly on the interface. We shall refer to this as the “trap” model.

The noise correlation function for both models may be calculated as follows.

Let the qubit be at the point \( \vec{r} \), so we are interested in the correlation function \( \langle E_i(\vec{r}, t) E_j(\vec{r}, 0) \rangle \) and let there be a dipole at \( \vec{r}' \) and the root-mean-square dipole strength \( p_0 \). Define \( \vec{R} = \vec{r} - \vec{r}' \). Then

\[
E_i(\vec{r}, t) = \sum_k \frac{3 R_i' p_k(t) R_k' - R_k' R_i' p_k(t)}{|\vec{R}|^5} \tag{10}
\]

while

\[
E_j(\vec{r}, 0) = \sum_m \frac{3 R_j' p_m(0) R_m' - R_m' R_j' p_m(0)}{|\vec{R}|^5}. \tag{11}
\]

The dipoles are assumed to be statistically independent, so their correlation function is

\[
\langle p_i(\vec{r}', t) p_j(\vec{r}', 0) \rangle = \delta_{ij} p_0^2 g(t). \tag{12}
\]

where model dependent factor \( \delta_{mod} = \delta_{ij}/3 \) for random dipole model and \( \delta_{mod} = \delta_{ij}/3 \) for trap model. Here \( g(t) \) is the time correlation function for a single dipole. We substitute Eqs. 10 and 12 into the definition of the electric field correlation function and, after some calculation and a time Fourier transform, find

\[
\langle E_i(\vec{r}, t) E_j(\vec{r}, 0) \rangle = \frac{1}{3} \rho_e p_0^2 g(\omega) \int d^3 r' \frac{3 R_i' R_j' + \delta_{ij} |\vec{R}'|^2}{|\vec{R}'|^8} \tag{13}
\]

for the random dipole model and

\[
\langle E_i(\vec{r}, t) E_j(\vec{r}, 0) \rangle = \rho_e p_0^2 g(\omega) \int d^3 r' |\vec{R}'|^{-10} \times \left[ 9 R_i^2 R_j' R_j' - 3 R_i R_j' |\vec{R}'|^2 \delta_{ij} \right.

\[ - 3 R_i R_j' |\vec{R}'|^2 \delta_{ij} \| + |\vec{R}'|^4 \delta_{ij} \delta_{jj} \right] \tag{14}
\]

for the trap model. Here \( \rho_e \) is the volume density of dipoles, \( \rho_d \) is the areal density of dipoles, and \( g(\omega) \) is the Fourier transform of \( g(t) \). For charge noise \( g(\omega) \) is often of the \( 1/\omega \) type. However, one of the important advantages of the experiments described in this paper is that we can investigate the sources of noise using spatial and geometric information alone, and the frequency spectrum of the noise is less important, a point we will return to below.

### IV. CASE STUDY

#### A. Device Description

In this section we show how to use the above theory to make predictions for the coherence-time anisotropy for an actual device. We have chosen the device used by Kawakami et al. [25] because it has been particularly well-characterized: \( T_1 \) and \( T_2 \) were measured, the distance from the qubits to gates is accurately known as \( d = 137 \text{ nm} \), and, most importantly, the field created by the micromagnet was simulated in detail. The results for the field gradients at the qubit are \( \partial B_x / \partial x = -0.20 \text{ mT/nm} \), \( \partial B_y / \partial x = -0.05 \text{ mT/nm} \), \( \partial B_x / \partial y = -0.27 \text{ mT/nm} \), \( \partial B_y / \partial y = -0.03 \text{ mT/nm} \), \( \partial B_y / \partial y = 0.18 \text{ mT/nm} \), and \( \partial B_y / \partial y = -0.02 \text{ mT/nm} \). The \( z \)-direction of the device is taken to be the growth direction. The variation in the \( z \)-direction is not needed in the two-dimension approximation we are using. Other important parameters are the thickness of the aluminum oxide layer \( l = 100 \text{ nm} \), the dielectric constant \( \epsilon_d = 13.05 \) for SiO\(_2\)/Ge\(_{0.7}\)Ge\(_{0.3}\) [26], and the transverse effective mass \( m = 0.19 m_e = 1.73 \times 10^{-31} \text{ kg} \). The lowest orbital excitation frequency is taken as \( \omega_{orb} = 6.84 \times 10^{11} \text{ s}^{-1} \) and it is related to the spring constants
by the equations $k_x = k_y = m \omega_{\text{orb}}^2$. As mentioned above, we take $k_z \to \infty$ since confinement is strong along the growth direction. $T = 150 \text{ mK}$ is the base temperature. While the actual temperature at the sample is probably somewhat higher, we will use this value. In addition, the magnitude of the applied field is always less than 1T, so phonon relaxation should not be important. The only other parameter needed as input to the theory is the conductivity $\sigma$ of the Au films. This was not measured in this device, but under similar growth conditions for Au films a value of $\sigma = 2 \times 10^8 \text{ S/m}$ was obtained at the temperatures of the experiment [25]. We should regard this as a probably somewhat high order-of-magnitude estimate of $\sigma$ in the actual device, and below we will investigate a range of values for $\sigma$. The qubit operating frequency is $\omega_{\text{op}} = 2\pi \times 12.9 \text{ GHz} = 8.11 \times 10^{10} \text{ s}^{-1}$. Two micromagnets made of cobalt are defined on top of the growth direction. Those equations are obtained by converting electric field noise correlations (Eq. 7) to the gradient in $x$- and $y$-direction respectively. They are related to experimental parameters as follows. For the UD model we have:

$$
\gamma_{2x}(t) = \gamma_{2y}(t) = (\frac{qq}{2m_e c})^2 (\frac{q}{2m \omega_{\text{orb}}^2})^2 \frac{2\pi \rho_e \rho_n \gamma_0^2}{32 d^4} \times 2\tau (t + (e^{-\tau} - 1)\tau).
$$

(16)

For the UT model we find

$$
\gamma_{2x}(t) = (\frac{qq}{2m_e c})^2 (\frac{q}{2m \omega_{\text{orb}}^2})^2 \frac{2\pi \rho_e \rho_n \gamma_0^2}{32 d^4} \times 2\tau (t + (e^{-\tau} - 1)\tau),
$$

$$
\gamma_{2y}(t) = (\frac{qq}{2m_e c})^2 (\frac{q}{2m \omega_{\text{orb}}^2})^2 \frac{2\pi \rho_e \rho_n \gamma_0^2}{32 d^4} \times 2\tau (t + (e^{-\tau} - 1)\tau).
$$

(17)

Here $\rho_e$ and $\rho_n$ are the volume and areal density and $\tau$ is the characteristic time constant of the dipoles. The temporal part of $\gamma_{2x}$ and $\gamma_{2y}$ results from the integration of the product of Lorentzian, $g(\omega) = 2\pi / (1 + (\omega / \omega_{\text{op}})^2)$, and $\sin^2(\omega t / 2)$. The choice of Lorentzian as $g(\omega)$ is an example, which does not affect the anisotropy because it depends on the ratios of $T_2$ with variable applied field direction. Those equations are obtained by converting electric field noise correlations (Eq. 13 or Eq. 14) into effective magnetic field correlation using Eq. 7 (See details in Supplemental Material).

$\rho_e$ and $\rho_n$ are poorly known, so we use them as fitting parameters. $T_2^y = 840 \text{ ns}$ was measured for only a single direction of the field, indicated by the red dots in Fig. 4. This yields $\rho_v = 2.39 \times 10^{13} \text{ cm}^{-3}$ and $\rho_n = 5.33 \times 10^9 \text{ cm}^{-2}$ for Fig. 4a and 4b, respectively.

The anisotropy maps of $T_2^y$ for the various models are shown in Fig. 5. There is some redundancy in the maps since they are symmetric under the transformation $\theta \to -\theta$ and $\phi \to \pi + \phi$, stemming from $T_2^y(B_0) = T_2^y(-B_0)$. This same redundancy also arises in the anisotropy maps of $T_1$ in Fig. 2.

We have chosen to show the full angular ranges since in some instances the topology of the function is clearer this way. The number of maxima ($N_{\text{max}}$), minima ($N_{\text{min}}$), and saddle points ($N_s$) in each map are shown in Table 1. When the dephasing and relaxation times in the maps are continuous functions without higher-order critical points, Morse theory can be applied. In particular for such functions on a 2D sphere (genus zero), they follow the relation: $N_{\text{max}} + N_{\text{min}} = N_s + 2$. Table 1 shows how the topology of the function can change when parameters affecting the noise are varied. For the maps of Fig. 4e...
and (f), the theory cannot be applied because the ridges in the maps correspond to a line of higher-order critical points.

Fig. 1(a) shows the results for the UD model and Fig. 1(b) for the UT model. The horizontal (vertical) axis denotes polar (azimuthal) angle with respect to the device’s z-direction. The key feature of these two models is that anisotropy of $T_2$ results only from the magnetic field gradients. The patterns are not too dissimilar, with the ratio between maximum and minimum values being both around 2. The main difference between the UD and UT models is that the peaks and valleys are broader in the UD model. In the UD model, the dipoles are oriented randomly, while in the UT model they are in the z-direction. The differences in the anisotropy maps between UD and UT can be traced back to the different behavior of electric field lines from these two different types of sources.

The anisotropy maps for the CD model, a localized dipole cluster, are shown in Fig. 1(c) and in Fig. 1(d). The cluster is located at $(x, y, z) = (37, 0, 37)$ nm and $(x, y, z) = (0, 37, 37)$ nm respectively. The maps for the CT model, a localized trap cluster, are shown in Fig. 1(e) and Fig. 1(f). The trap is located at $(x, y, z) = (37, 0, 137)$ nm and $(x, y, z) = (0, 37, 137)$ nm respectively. In both CD and CT model, the qubit is located at the origin. Thus Figs. 1(c) is directly comparable to Fig. 1(e) and Fig. 1(d) is directly comparable to Fig. 1(f). The overall dipole strength $p_0$ is used for the fitting parameter of these single cluster models, once more by using the experimental value measured at the red point. Maps 1(c)-(f) exhibit more anisotropy relative to the uniform distribution models. This is expected since the localization of the source itself introduces anisotropy.

The difference between the CD and CT models lies in the dipole orientation. In the CD model, it is assumed
that the cluster contains dipoles of all orientations and the noise electric field is averaged over the solid angle. This washes out the anisotropy to some extent, but the pattern still depends on the direction of the line connecting the dipole and the qubit. The distance between them just changes the overall magnitude of $T_2$. In the CT model, however, the trap generates a noise electric field with more directionality, so the overall anisotropy patterns are sharper and both the direction and the distance are important.

Comparing (1c) to (1d) and (1e) to (1f) indicates that the source position has a large effect. To understand this in more detail, let us focus on the CT model in Fig. (1c) and 1(f). Note that $\partial B_z/\partial x$, $\partial B_z/\partial x$, and $\partial B_y/\partial y$ are an order of magnitude greater than the other gradient terms. In the Fig. (1e), the electric field at the qubit has only $x$ and $z$ components. The $x$ component contributes to $1/T_2$ through the products with $\partial B_z/\partial x$ and $\partial B_z/\partial x$. Thus small $T_2$ is expected when the applied field is in the $x$- and $z$-direction, which can be identified on the map with $(\theta, \phi) = (\pi/2, 0)$ and $(0, 0)$, respectively. On the other hand, in the Fig. (1f), the electric field at the qubit has only $y$ and $z$ components. The leading contribution to $1/T_2$ is then the product with $\partial B_y/\partial y$. Small $T_2$ is expected with the applied field in the $y$-direction, which is the case $(\theta, \phi) = (\pi/2, \pi/2)$ on the map.

It is important to point out that if relatively few two-level systems contribute to the dephasing of the qubit, as is often hypothesized based on deviations for power-law spectra [13, 24], the anisotropy can be used to determine the source position and to distinguish between random dipole and trap models for the charge noise. The present method can be extended to models with very few sources simply by eliminating the averaging we have performed.

C. Anisotropy of $T_1$

In the experiment, $T_1$ of the device is in the order of 1 s. To estimate the contribution of charge noise, we use the results from [11A] together with the determination of densities from $T_2$. This leads immediately to an estimate in the range of 10$^3$s so we conclude that charge noise is not important for spin relaxation in the single-qubit system considered here. To exclude phonon relaxation we need to stipulate for the moment that the external field strength is less than about 1 T. This leaves EWJN as the dominant mechanism.

The relaxation rate with applied field direction in $(\theta, \phi)$ for EWJN can be written as

$$\frac{1}{T_1(\theta, \phi)} = \alpha + \beta \sum_{ij} Q_{ij}^{(1)} \left[ \frac{\partial B_i}{\partial x} \frac{\partial B_j}{\partial x} + \frac{\partial B_i}{\partial y} \frac{\partial B_j}{\partial y} \right]$$

where

$$\alpha = \frac{\pi \hbar \sigma_{\text{op}}}{e^2 d} \coth \frac{\hbar \omega_{\text{op}}}{2k_B T},$$

$$\beta = \frac{q}{2m \omega_{\text{orb}}^2} \frac{\hbar \omega_{\text{op}} c d}{16 \pi c d} \coth \frac{\hbar \omega_{\text{op}}}{2k_B T}. \quad (20)$$

The first term represents the direct effect of magnetic field noise while the second term is due to the electric field noise that is converted to effective magnetic field noise by the field gradient created by the micromagnet. The angular variation in $Q_{ij}^{(1)}$ and the rather anisotropic character of the magnetic field gradients imply that the anisotropy of $T_1$ will be intensified when the electric term is bigger than the direct magnetic term. Noting that $\alpha \sim \sigma$ and $\beta \sim 1/\sigma$, we see that varying $\sigma$ will change the anisotropy pattern of $T_1$. As noted above, $\sigma$ was not measured in the experiment and it makes sense to vary it to investigate the angular dependence of $T_1$.

The anisotropy map for $T_1$ is shown in Fig. 2 with (a) $\sigma = 2 \times 10^8$ S/m, (b) $\sigma = 2 \times 10^7$ S/m, and (c) $\sigma = 2 \times 10^6$ S/m. The anisotropy pattern in Fig. 2(a) is fairly simple because the magnetic noise is dominant and the direct magnetic EWJN itself is not very anisotropic, as can be seen from Eq. (9) and the text following it. The anisotropy is increased as shown in Fig. 2(b) where the magnetic noise is somewhat more comparable to the electric noise. The anisotropy becomes even larger in Fig. 2(c) where the magnetic noise is one order of magnitude smaller than the electric noise. This pattern looks like the reversal of the anisotropy map for $T_2$ in Fig. 1(a) and 1(b). This is natural since $T_2$ of a spin qubit is due to longitudinal noise while $T_1$ is due to transverse noise. From practical point of view, the qubit performance would be improved when the applied field direction is set to the angles that give maximal $T_2$ (in the case of $T_2 \ll T_1$).

When the strength of the applied magnetic field is increased, there is a crossover from EWJN- to phonon-dominated spin relaxation. The anisotropy maps for phonons was worked out in Ref. [13]. They are determined by the orientation of the field relative to the crystal axes, not axes coming from the the device geometry. Hence we expect sharp changes in the anisotropy map as the field is increased.

V. CONCLUSION

The anisotropy pattern of dephasing time of a spin qubit comes from the combination of magnetic field gradient setting and noise electric field determined by the configuration of noise dipoles. By introducing a vector magnet in a quantum dot device, noise characteristics such as noise dipole type and/or spatial distribution of noise dipoles can be experimentally investigated. Another way is to exploit controllable magnetic field gradient for a spin qubit of nitrogen vacancy center in a
diamond \cite{30, 31}. In this case, the gradient can be varied instead of the direction of applied magnetic field to study noise characteristics.

The difference in anisotropy maps of the relaxation times can be explained with the direct magnetic noise and indirect electric noise. The magnetic noise part resulting from EWJN has isotropic property in $xy$ plane due to half-space model structure. However, the electric noise affects the qubit through magnetic field gradients, by which anisotropy is entirely determined. As a result, the anisotropy gets bigger as the influence of electric noise part increases.

In conclusion, we have shown anisotropy in relaxation time and dephasing time with the case study of the quantum dot device of Kawakami et al. To our knowledge, this has not been experimentally studied in any physical system. Making the anisotropy map will be helpful to understand the noise mechanism playing behind the scene. Specifically, this will benefit the solid state quantum processors where the causes of noise are still being searched. Our work contributes to the noisy intermediate-scale quantum era by suggesting a new experimental method for noise characterization of spin qubit devices.

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Supplemental Material for: Anisotropy with respect to the applied magnetic field in relaxation and dephasing times of spin qubits

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I. EVANESCENT-WAVE JOHNSON NOISE

A. Relaxation time

This section gives further details for the calculation of $T_1$ of a spin qubit affected by evanescent-wave Johnson noise (EWJN) is given as follows. There is both direct magnetic noise $\vec{B}(t)$ and indirect magnetic noise $\vec{B}^{(E)}$ due to the electrically-induced motion of the qubit in the magnetic field gradient.

The correlation function of total noise field $\vec{B}^{(eff)}$ can be expanded as

$$\langle B^{(eff)}_i B^{(eff)}_j \rangle = \langle (B_i + B^{(E)}_i)(B_j + B^{(E)}_j) \rangle$$

$$= \langle B_i B_j \rangle + \langle B^{(E)}_i B_j \rangle + \langle B_i^{(E)} B^{(E)}_j \rangle + \langle B^{(E)}_i B^{(E)}_j \rangle$$

$$= \langle B_i B_j \rangle + \sum_{mn} \left[ \frac{q^2}{4k_m k_n} \frac{\partial B_i}{\partial x_m} \frac{\partial B_j}{\partial x_n} \langle E_m E_n \rangle \right]$$

$$+ \frac{q}{2k_n} \frac{\partial B_i}{\partial x_n} \langle B_i E_n \rangle + \frac{q}{2k_n} \frac{\partial B_j}{\partial x_n} \langle E_n B_j \rangle \right]. \tag{1}$$

Using Eqs. (6) and (7) of the main text for the dynamics of $B_i^{(E)}(t)$ and assuming that there is no correlation between $B_i(t)$ and $B_i^{(E)}(t)$, we arrive at the correlation functions for the effective magnetic field.

The correlation functions become

$$\langle B^{(eff)}_x B^{(eff)}_x \rangle = \langle B_x B_x \rangle + \left( \frac{q}{2m \omega^2_{orb}} \right)^2 \left\{ \left( \frac{\partial B_x}{\partial x} \right)^2 \langle E_x E_x \rangle + \left( \frac{\partial B_x}{\partial y} \right)^2 \langle E_y E_y \rangle \right\}$$

$$= \left[ \frac{\pi \hbar \sigma \omega_{op}}{4c^2 d} + \left( \frac{q}{2m \omega^2_{orb}} \right)^2 \frac{\hbar \omega_{op} \epsilon_d}{16 \pi \sigma d^3} \left\{ \left( \frac{\partial B_x}{\partial x} \right)^2 + \left( \frac{\partial B_x}{\partial y} \right)^2 \right\} \right] \coth \frac{\hbar \omega_{op}}{2k_B T}, \tag{2}$$

$$\langle B^{(eff)}_y B^{(eff)}_y \rangle = \langle B_y B_y \rangle + \left( \frac{q}{2m \omega^2_{orb}} \right)^2 \left\{ \left( \frac{\partial B_y}{\partial x} \right)^2 \langle E_x E_x \rangle + \left( \frac{\partial B_y}{\partial y} \right)^2 \langle E_y E_y \rangle \right\}$$

$$= \left[ \frac{\pi \hbar \sigma \omega_{op}}{4c^2 d} + \left( \frac{q}{2m \omega^2_{orb}} \right)^2 \frac{\hbar \omega_{op} \epsilon_d}{16 \pi \sigma d^3} \left\{ \left( \frac{\partial B_y}{\partial x} \right)^2 + \left( \frac{\partial B_y}{\partial y} \right)^2 \right\} \right] \coth \frac{\hbar \omega_{op}}{2k_B T},$$

$$\langle B^{(eff)}_z B^{(eff)}_z \rangle = \langle B_z B_z \rangle + \left( \frac{q}{2m \omega^2_{orb}} \right)^2 \left\{ \left( \frac{\partial B_z}{\partial x} \right)^2 \langle E_x E_x \rangle + \left( \frac{\partial B_z}{\partial y} \right)^2 \langle E_y E_y \rangle \right\}$$

$$= \left[ \frac{\pi \hbar \sigma \omega_{op}}{2c^2 d} + \left( \frac{q}{2m \omega^2_{orb}} \right)^2 \frac{\hbar \omega_{op} \epsilon_d}{16 \pi \sigma d^3} \left\{ \left( \frac{\partial B_z}{\partial x} \right)^2 + \left( \frac{\partial B_z}{\partial y} \right)^2 \right\} \right] \coth \frac{\hbar \omega_{op}}{2k_B T},$$

while the terms $\langle B^{(eff)}_i B^{(eff)}_j \rangle$ with $i \neq j$ are

$$\langle B^{(eff)}_i B^{(eff)}_j \rangle = \left( \frac{q}{2m \omega^2_{orb}} \right)^2 \frac{\hbar \omega_{op} \epsilon_d}{16 \pi \sigma d^3} \left\{ \frac{\partial B_i}{\partial x} \frac{\partial B_j}{\partial x} + \frac{\partial B_i}{\partial y} \frac{\partial B_j}{\partial y} \right\} \coth \frac{\hbar \omega_{op}}{2k_B T}. \tag{3}$$
Substituting Eqs. (8) and (9) for the EWJN into equation of this kind, and applying Eq. (3) to rotate the results into a general direction, we arrive at Eqs. (19) and (20) from which the results in Fig. 2 are calculated.

Eqs. (8) and (9) themselves are the half-space local (point qubit) correlation functions for EWJN [1]. They are valid only in the \( d \ll \delta \) regime. The distance \( d \) between the qubit and accumulation gates is about 137 nm while skin depth \( \delta = \frac{c}{\sqrt{2\pi\sigma\omega_{op}}} = 313 \text{ nm} \) with the conductivity of gates estimated very roughly as \( \sigma = 2 \times 10^8 \text{ S/m} \) [2] and the operating frequency \( \omega_{op} = 2\pi \times 12.9 \text{ GHz} = 8.11 \times 10^{10} \text{ s}^{-1} \). Thus the device in the case study satisfies the \( d \ll \delta \) requirement.

The dephasing time \( T_\phi \) is calculated in a similar fashion, with the main difference being that it only depends on a single correlation function.

The gaussian approximation for \( T_\phi \) should be valid for EWJN since the noise comes from a large number of modes in the solid. The decay in the off-diagonal components of the density matrix is initially gaussian in time and then crosses over to exponential. The detailed time dependence is given by \( \exp[-\Gamma(t)] \) with \( \Gamma(t) \) given by Eq. (4) in the main text. \( T_\phi \) is then obtained by solving the transcendental equation \( \Gamma(T_\phi) = 1 \). When rotated into the \((\theta, \phi)\) direction we obtain Eq. (5).

II. CHARGE NOISE

A. Relaxation time

In this section we give further details of calculations in Sec. III C of the main text. The relaxation time due to charge noise can be obtained by following the same way as that of EWJN with Eqs.(2) of the main text. The difference is that the correlation function of effective noise field now consists only of the electric noise part such that

\[
\langle B_{i}^{(\text{eff})} B_{j}^{(\text{eff})} \rangle = \sum_{mn} \frac{q^2}{4k_n k_m} \frac{\partial B_i}{\partial x_m} \frac{\partial B_j}{\partial x_n} \langle E_m E_n \rangle. \tag{4}
\]

The electric fields from a point dipole are given as

\[
E_i(\vec{r}, t) = \sum_k \frac{3R'_k p_k(t) R'_i - R'_k R'_k p_i(t)}{|\vec{R}'|^5},
\]

\[
E_j(\vec{r}, 0) = \sum_m \frac{3R'_m p_m(0) R'_j - R'_m R'_m p_j(0)}{|\vec{R}'|^5}. \tag{5}
\]
where \( \vec{R}' = \vec{r} - \vec{r} \) is the displacement vector from the position of a dipole \( \vec{r} \) to that of qubit \( \vec{r} \). The dipoles are assumed to be statistically independent, so their correlation function is

\[
\langle p_i(\vec{r}', t)p_j(\vec{r}', 0) \rangle = \delta_{mod} \rho_0^2 g(t). \tag{6}
\]

where model dependent factor \( \delta_{mod} = \delta_{ij}/3 \) for random dipole model and \( \delta_{mod} = \delta_{iz}\delta_{jz} \) for trap model. The time correlation function is assumed to be exponential: \( g(t) = e^{-|t|/\tau} \) where \( \tau \) is the characteristic time of the dipoles. This corresponds to random telegraph noise, but the precise model for the time correlations is not that important, since the anisotropy maps depend only on ratios of specific averages of the correlation functions.

Combining all the equations above, the electric field correlation function is

\[
\langle E_i(\vec{r}, t)E_j(\vec{r}, 0) \rangle = \sum_{km} |\vec{R}|^{-10} \langle 9R_k' R_m' R_i' R_j' p_k(t)p_m(0) - 3R_m' R_i' R_k' p_i(t)p_m(0) \\
- 3R_k' R_m' R_i' p_k(0) + R_i' R_k' R_m' p_i(t)p_j(0) \rangle. \tag{7}
\]

which simplifies to

\[
\langle E_i(\vec{r}, t)E_j(\vec{r}, 0) \rangle = \frac{1}{3} \rho_\nu \rho_0^2 g(\omega) \int d^2 r' \frac{3 R_i' R_j' + \delta_{ij} |\vec{R}'|^2}{|\vec{R}'|^8} \tag{8}
\]

for the random dipole model and

\[
\langle E_i(\vec{r}, t)E_j(\vec{r}, 0) \rangle = \rho_a \rho_0^2 g(\omega) \int d^2 r' \frac{9 R_k^2 R_i' R_j' - 3 R_z R_j' |\vec{R}'|^2 \delta_{iz} - 3 R_z R_i' |\vec{R}'|^2 \delta_{jz} + |\vec{R}'|^4 \delta_{iz} \delta_{jz}}{|\vec{R}'|^10} \tag{9}
\]

for the trap model. Here \( \rho_\nu \) is the volume density of dipoles and \( \rho_a \) is the areal density of traps, which are assumed to be uniformly distributed in the region of integration and used as fitting parameters to make \( T_2^{(x)} = 840 \) ns. \( g(\omega) = 2\tau/(1 + (\omega_{op}\tau)^2) \) is the Fourier transform of \( g(t) \) and \( \tau \) is taken as the inverse of maximum attempt frequency in \([3]\). The integration region for the uniformly distributed random dipole model (UD) is the aluminum oxide layer which is infinite in \( xy \) plane, about 37 nm above the qubit, and whose thickness is about \( l = 100 \) nm, namely \( \rho \in [0, \infty), \phi \in [0, 2\pi), z \in [37, 137] \) nm with cylindrical coordinate system. That for uniformly distributed trap model (UT) is the interface between the oxide layer and accumulation gates that is about \( d = 137 \) nm above the qubit, \( i.e. \rho \in [0, \infty), \phi \in [-\pi/4, 5\pi/4], \) at \( z = d \). The domain of \( \phi \in [-\pi/4, 5\pi/4] \) takes into account the actual gate geometry of the device in the case study \([4]\).
The relaxation rate with applied field in \((\theta, \phi)\) direction is then
\[
\frac{1}{T_1(\theta, \phi)} = \sum_{ij} Q_{ij}^{(1)} \left[ \gamma_{1x} \frac{\partial B_i}{\partial x} \frac{\partial B_j}{\partial x} + \gamma_{1y} \frac{\partial B_i}{\partial y} \frac{\partial B_j}{\partial y} \right]
\]
where \(\gamma_{1x}\) and \(\gamma_{1y}\) are the prefactors related to the gradient in \(x\)- and \(y\)-direction respectively, and determined by experimental parameters as follows: with the UD model
\[
\gamma_{1x} = \gamma_{1y} = \left( \frac{q g}{4 m_e c} \right)^2 \left( \frac{q}{2 m w_{orb}^2} \right)^2 \frac{\pi \rho_v p_0^2}{12} \left( \frac{1}{l^3} - \frac{1}{d^3} \right) \frac{2\tau}{1 + \omega_{op} \tau},
\]
and with the UT model
\[
\gamma_{1x} = \left( \frac{q g}{4 m_e c} \right)^2 \left( \frac{q}{2 m w_{orb}^2} \right)^2 \frac{9\pi + 6}{32d^4} \frac{\rho_a p_0^2}{1 + \omega_{op} \tau},
\]
\[
\gamma_{1y} = \left( \frac{q g}{4 m_e c} \right)^2 \left( \frac{q}{2 m w_{orb}^2} \right)^2 \frac{9\pi - 6}{32d^4} \frac{\rho_a p_0^2}{1 + \omega_{op} \tau}.
\]
The simulation results of \(T_1^{(x)}\) are \(6.50 \times 10^9\) s for the UD model and \(1.72 \times 10^{10}\) s for the UT model with the fitted volume and areal densities respectively.

**B. Dephasing time**

Using the above Eqs. 4, 8, and 9, \(\Gamma(t)\) for the applied field in the \((\theta, \phi)\) direction can be written as
\[
\Gamma(t; \theta, \phi) = \sum_{ij} Q_{ij}^{(2)} \left[ \gamma_{2x}(t) \frac{\partial B_i}{\partial x} \frac{\partial B_j}{\partial x} + \gamma_{2y}(t) \frac{\partial B_i}{\partial y} \frac{\partial B_j}{\partial y} \right]
\]
where \(\gamma_{2x}(t)\) and \(\gamma_{2y}(t)\) are the prefactors related to the gradients in \(x\)- and \(y\)-direction respectively. They are calculated as follows: for the UD model
\[
\gamma_{2x}(t) = \gamma_{2y}(t) = \left( \frac{q g}{2 m_e c} \right)^2 \left( \frac{q}{2 m w_{orb}^2} \right)^2 \frac{\pi \rho_v p_0^2}{12} \left( \frac{1}{l^3} - \frac{1}{d^3} \right) 2\pi \tau (t + (e^{-t/\tau} - 1)\tau),
\]
and for the UT model
\[
\gamma_{2x}(t) = \left( \frac{q g}{2 m_e c} \right)^2 \left( \frac{q}{2 m w_{orb}^2} \right)^2 \frac{9\pi + 6}{32d^4} \frac{\rho_a p_0^2}{2\pi \tau (t + (e^{-t/\tau} - 1)\tau)},
\]
\[
\gamma_{2y}(t) = \left( \frac{q g}{2 m_e c} \right)^2 \left( \frac{q}{2 m w_{orb}^2} \right)^2 \frac{9\pi - 6}{32d^4} \frac{\rho_a p_0^2}{2\pi \tau (t + (e^{-t/\tau} - 1)\tau)}.
\]

Here \(p_0 = 2.0 \times 10^{-29}\) C·m is the root-mean-square strength of the dipole that is assumed to be the product of elementary charge and the size of an aluminum atom. \(\rho_v\) and \(\rho_a\) are the volume and areal density of the dipoles and used as fitting parameters for the experimental
value $T^x_2 = T_2(\pi/2, 0) = 840$ ns, yielding $v = 2.39 \times 10^{13}$ cm$^{-3}$ and $a = 5.33 \times 10^7$ cm$^{-2}$. Since $T_\phi \ll T_1$, we assume $T_2 = T_\phi$ by the relation $1/T_2 = 1/2T_1 + 1/T_\phi$.

For localized cluster dipole model (CD) and localized cluster trap model (CT), the integration region and the volume and areal densities are removed such that the electric field correlation function is

$$\langle E_i(\vec{r}, t)E_j(\vec{r}, 0) \rangle = \frac{1}{3}p_0^2 g(\omega) \frac{3R_i'R_j' + \delta_{ij}||\vec{R}'||^2}{||\vec{R}'||^8}$$  \hspace{1cm} (16)

for the CD model and

$$\langle E_i(\vec{r}, t)E_j(\vec{r}, 0) \rangle = p_0^2 g(\omega) \frac{9R_i'^2R_j' - 3R_i'R_j'|\vec{R}'|^2\delta_{i2} - 3R_i'R_i'|\vec{R}'|^2\delta_{j2} + |\vec{R}'|^4\delta_{i2}\delta_{j2}}{||\vec{R}'||^10}$$  \hspace{1cm} (17)

for the CT model. In these single cluster models, $p_0$ is used as fitting parameter because there are no densities.

The hyperfine noise contribution for dephasing rate is calculated: $1/T_2^{hyper} = (1.83 \mu s)^{-1} - (20.4 \mu s)^{-1} = (2.01 \mu s)^{-1}$. This comes from the difference in decoherence rates of natural silicon device and isotopically purified silicon device [5, 6]. Since it is isotropic, it should be added to all the $T_2(\theta, \phi)$. Hence, $1/T_2(\pi/2, 0) + 1/T_2^{hyper}$ is set to (840 ns)$^{-1}$ by fitting the densities or dipole strength.

The equations in this section are used to generate Fig. 1 of the main text.

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