ON CERTAIN RECURRENT AND AUTOMATIC SEQUENCES IN F infinite fields

ALAIN LASJAUNIAS AND JIA-YAN YAO

Abstract. In this work we extend our study on a link between automaticity and certain algebraic power series over finite fields. Our starting point is a family of sequences in a finite field of characteristic 2, recently introduced by the first author in connection with algebraic continued fractions. By including it in a large family of recurrent sequences in an arbitrary finite field, we prove its automaticity. Then we give a criterion on automatic sequences, generalizing a previous result and this allows us to present new families of automatic sequences in an arbitrary finite field.

1. Introduction

The present work is a continuation of our article [15] in which we have addressed a question concerning the automaticity of the sequence of leading coefficients of partial quotients for certain algebraic power series. To know more about the motivation and the history, the reader can consult the introduction of [15] and the references given there.

Let \( \mathbb{F}_q \) be the finite field containing \( q \) elements, with \( q = p^s \) where \( p \) is a prime number and \( s \geq 1 \) is an integer. We denote by \( \mathbb{F}(q) \) the field of power series in \( 1/T \), with coefficients in \( \mathbb{F}_q \), where \( T \) is a formal indeterminate. Hence, an element in \( \mathbb{F}(q) \) can be written as \( \alpha = \sum_{k \leq k_0} u(k)T^k \) with \( k_0 \in \mathbb{Z} \), and \( u(k) \in \mathbb{F}_q \) for all integers \( k \leq k_0 \). These fields of power series are analogues of the field of real numbers. As in the real case, it is well known that the sequence of coefficients of this power series \( \alpha \), \( (u(k))_{k \leq k_0} \), is ultimately periodic if and only if \( \alpha \) is rational, i.e., \( \alpha \in \mathbb{F}_q(T) \). Moreover, and remarkably, due to the rigidity of the formal case, this sequence of coefficients, for all the elements in \( \mathbb{F}(q) \) which are algebraic over \( \mathbb{F}_q(T) \), belongs to a class of particular sequences introduced by computer scientists. The origin of the following theorem can be found in the work of Christol [8] (see also the article of Christol, Kamae, Mendès France, and Rauzy [9]).

Theorem 1 (Christol). Let \( \alpha \in \mathbb{F}(q) \) with \( q = p^s \). Let \( (u(k))_{k \leq k_0} \) be the sequence of digits of \( \alpha \) and \( v(n) = u(-n) \) for all integers \( n \geq 0 \). Then \( \alpha \) is algebraic over \( \mathbb{F}_q(T) \) if and only if the following set of subsequences of \( (v(n))_{n \geq 0} \)

\[ K(v) = \left\{ (v(p^i n + j))_{n \geq 0} \middle| i \geq 0, 0 \leq j < p^i \right\} \]

is finite.
The sequences having the finiteness property stated in this theorem are called $p$-automatic sequences. A full account on this topic and a very complete list of references can be found in the book [3] of Allouche and Shallit.

Concerning algebraic elements in $\mathbb{F}(q)$, a particular subset need to be considered. An irrational element $\alpha$ in $\mathbb{F}(q)$ is called hyperquadratic, if $\alpha^{r+1}$, $\alpha^r$, $\alpha$, and 1 are linked over $\mathbb{F}_q(T)$, with $r = p^t$ and $t \geq 0$ an integer. The subset of all these elements, noted $\mathcal{H}(q)$, contains the quadratic ($r = 1$) and the cubic power series ($r = p$), but also algebraic elements of arbitrary large degree. For different reasons, $\mathcal{H}(q)$ could be regarded as the analogue of the subset of quadratic real numbers, particularly when considering the continued fraction algorithm. See [4] for more information on this notion. An irrational element $\alpha$ in $\mathbb{F}(q)$ can be expanded as an infinite continued fraction $\alpha = [a_1, a_2, \ldots, a_n, \ldots]$, where the partial quotients $a_n$ are polynomials in $\mathbb{F}_q[T]$, all of positive degree, except perhaps for the first one. The explicit description of continued fractions for algebraic power series over a finite field goes back to Baum and Sweet [5, 6], and was carried on ten years later by Mills and Robbins [16]. It happens that this continued fraction expansion can be explicitly given for various elements in $\mathcal{H}(q)$. This is certainly the case for quadratic power series, where the sequence of partial quotients is simply ultimately periodic (as it is for quadratic real numbers). It was first observed by Mills and Robbins [16] that other hyperquadratic elements have also partial quotients of bounded degrees, with an explicit continued fraction expansion, as a famous cubic over $\mathbb{F}_2$ introduced by Baum and Sweet [5]. Some of these examples, belonging to $\mathcal{H}(p)$ with $p \geq 5$, are such that $a_n = \lambda_n T$, for $n \geq 1$, with $\lambda_n \in \mathbb{F}_p^*$. Then Allouche [1] showed that for each example given in [16], with $p \geq 5$, the corresponding sequence of partial quotients is automatic. Another case, in $\mathcal{H}(3)$ also given in [16], having $a_n = \lambda_n T + \mu_n$, with $\lambda_n, \mu_n \in \mathbb{F}_3$ for $n \geq 1$, was treated by Allouche et al. in [2]. Recently we have investigated the existence of such hyperquadratic power series, having partial quotients of degree 1, in the largest setting with odd characteristic (see [14] and particularly the comments in the last section). However, concerning the cubic power series introduced by Baum and Sweet in [5], Mkaouar [17] showed that the sequence of partial quotients (which takes only finitely many values) is not automatic (see also [18]). Besides, we know that most of the elements in $\mathcal{H}(q)$ have partial quotients of unbounded degrees (see the introduction in [15]). Hence, it appears that the link between automaticity and the sequence of partial quotients is not straight.

With each infinite continued fraction in $\mathbb{F}(q)$, we can associate a sequence in $\mathbb{F}_q^*$ as follows: if $\alpha = [a_1, a_2, \ldots, a_n, \ldots]$ with $a_n \in \mathbb{F}_q[T]$, then for all integers $n \geq 1$, we define $u(n)$ as the leading coefficient of the polynomial $a_n$. For several examples in $\mathcal{H}(q)$, we have observed that this sequence $(u(n))_{n \geq 1}$ is automatic. Indeed, a first observation in this area is the result of Allouche [1] cited above. Very recently we have described in [15] three other families of hyperquadratic continued fractions and have shown that the associated sequences as indicated above are automatic. For an algebraic (even hyperquadratic) power series, the possibility of describing explicitly the continued fraction expansion and consequently the sequence $(u(n))_{n \geq 1}$ is sometimes a difficult problem. In this work we start with such a description given by the first author in [12] in characteristic 2. In the next section, we show that this sequence belongs to a large family of automatic sequences in a finite field. More precisely, we give the explicit algebraic equation satisfied by the generating function
attached to each such sequence. In the last section, we generalize an automaticity criterion introduced in our previous work [15] and this allows us, as an application, to present other recurrent and automatic sequences in a finite field, more general than the preceding ones.

2. A First Family of Automatic Sequences

The starting-point of the present work is a family of sequences, defined in a finite field of characteristic 2, which are derived from an algebraic continued fraction in power series fields. The proposition stated below is a simplified version of a theorem proved recently by the first author in [12], improving an earlier result [13] Proposition 5, p. 556]. For the effective coefficients of the algebraic equation appearing in this proposition, the reader is referred to [12].

**Proposition 1.** Let \( q = 2^s \) and \( r = 2^t \) with \( s, t \geq 1 \) integers. Let \( \ell \geq 1 \) be an integer, and \( \Lambda_{\ell+2} = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell}, \varepsilon_1, \varepsilon_2) \in (\mathbb{F}_q^*)^{\ell+2} \). We define the sequence \((\lambda_n)_{n \geq 1}\) in \( \mathbb{F}_q^* \), recursively from the \( \ell \)-tuple \((\lambda_1, \lambda_2, \ldots, \lambda_{\ell})\) as follows. For \( m \geq 0 \),

\[
\begin{align*}
\lambda_{\ell + rm + 1} &= (\varepsilon_2/\varepsilon_1)\varepsilon_2^{(-1)^m+1} \lambda_{\ell m + 1}, \\
\lambda_{\ell + rm + i} &= (\varepsilon_1/\varepsilon_2)^{(-1)^i} \quad \text{for} \quad 2 \leq i \leq r.
\end{align*}
\]

Then there exist \((u, v, w, z, (\mathbb{F}_q[T])^4\), depending on \( \Lambda_{\ell+2} \), such that the continued fraction \( \alpha = [\lambda_1 T, \lambda_2 T, \ldots, \lambda_{\ell} T, \ldots, \lambda_n T, \ldots] \in \mathbb{F}(q) \) satisfies the following algebraic equation

\[uX^r + vX^r + wX + z = 0.\]

We shall prove that the sequence \((\lambda_n)_{n \geq 1}\), introduced in this proposition, is \( 2 \)-automatic. Here again, this underlines the existence of a link between automaticity and certain algebraic continued fractions, mentioned in the introduction. Indeed we are going to prove the automaticity, via Christol theorem, for a larger class of sequences in a finite field including these introduced above. We prove the following theorem.

**Theorem 2.** Let \( \ell \geq 1 \) be an integer, \( p \geq 2 \) a prime number, \( q = p^s \) and \( r = p^t \) with \( s, t \geq 1 \) integers. Let \( \beta \geq 1 \) be an integer dividing \( r \). Let \((\lambda_1, \lambda_2, \ldots, \lambda_{\ell})\) be a given \( \ell \)-tuple in \( (\mathbb{F}_q^*)^\ell \). We define recursively in \( \mathbb{F}_q \) the sequence \((\lambda_n)_{n \geq 1}\) as follows. For \( m \geq 0 \),

\[
\begin{align*}
\lambda_{\ell+1+r(\ell+1)} &= \alpha_{i+1}\lambda_{\ell+1}, \\
\lambda_{\ell+1+r(\ell+2)+j} &= \beta_j, \quad \text{for} \quad 1 \leq j < r,
\end{align*}
\]

where \( \alpha_{i+1} (0 \leq i < k) \) in \( \mathbb{F}_q^* \) and \( \beta_j (0 \leq j < r) \) in \( \mathbb{F}_q \) are fixed elements. Set

\[
\theta = \sum_{n \geq 1} \lambda_n T^{-n}.
\]

Then there exist \( A, B, C \) in \( \mathbb{F}_q(T) \), with \( C \neq 0 \), and \( \rho \) in \( \mathbb{F}(q) \) such that

\[
\theta = A + \rho \quad \text{and} \quad \rho = B + C\rho^r.
\]

Hence \( \theta \) is algebraic over \( \mathbb{F}_q(T) \), and then the sequence \((\lambda_n)_{n \geq 1}\) is \( p \)-automatic.

**Remark.** Note that the sequences in (1) correspond, in (2), to the case:

\[p = 2, k = 2, \alpha_1 = \varepsilon_1^{-1}, \alpha_2 = \varepsilon_2^2 \varepsilon_1^{-1} \quad \text{and} \quad \beta_j = (\varepsilon_2/\varepsilon_1)^{(-1)^j} \quad \text{for} \quad j = 1, \ldots, r-1.\]
Proof. According to Christol’s theorem, the sequence \((\lambda_n)_{n \geq 1}\) is \(p\)-automatic if \(\theta\) is algebraic. Let us prove that \(\theta\) satisfies an algebraic equation of hyperquadratic type. We define two subset of positive integers: \(E = \{\ell + rn + 1 \mid n \geq 0\}\) and \(F = \{\ell + rn + i \mid n \geq 0, \quad 2 \leq i \leq r\}\). Hence, we have the following partition \(\mathbb{N}^* = \{1, \ldots, \ell\} \cup F \cup E\). We define

\[
\rho = \sum_{n \in E} \lambda_n T^{-n} \quad \text{and} \quad \rho_1 = \sum_{n \in F} \lambda_n T^{-n}.
\]

Hence, we have \(\theta = \sum_{m=1}^{\ell} \lambda_m T^{-m} + \rho_1 + \rho\). By the recursive relations (2), we obtain

\[
\rho_1 = \sum_{1 \leq j < r} \sum_{m \geq 0} \lambda_{\ell+1+rm+j} T^{-(\ell+1+rm+j)} = \sum_{1 \leq j < r} \sum_{m \geq 0} \beta_j T^{-(\ell+1+rm+j)}
\]
\[
= \sum_{m \geq 0} \left( \sum_{1 \leq j < r} \beta_j T^{-(\ell+1-j)} \right) T^{-rm}
\]
\[
= (1 - T^{-1})^{-r} \sum_{1 \leq j < r} \beta_j T^{-(\ell+1-j)},
\]

since we have \(\sum_{m \geq 0} T^{-m} = (1 - T^{-1})^{-1}\) in \(\mathbb{F}(q)\). Hence we can write

\[
\theta = \sum_{m=1}^{\ell} \lambda_m T^{-m} + (1 - T^{-1})^{-r} \sum_{1 \leq j < r} \beta_j T^{-(\ell+1-j)} + \rho = A + \rho,
\]

with \(A \in \mathbb{F}_q(T)\).

To simply the notation, we extend the finite sequence \((\alpha_i)_{1 \leq i \leq k}\) into a purely periodic sequence of period length \(k\), also denoted by \((\alpha_n)_{n \geq 1}\). Similarly from the recursive relations (2), noting that \(E = \{\ell + 1 + r(km + i) \mid m \geq 0, \quad 0 \leq i < k\}\) and since \(\alpha_{i+1} = \alpha_{km+i+1}\), we obtain

\[
\rho = \sum_{n \in E} \lambda_n T^{-n} = \sum_{0 \leq i < k} \sum_{m \geq 0} \lambda_{\ell+1+rm+i} T^{-(\ell+1+rm+i)}
\]
\[
= \sum_{0 \leq i < k} \sum_{m \geq 0} \alpha_{km+i+1} \lambda_{\ell+1+rm+i} T^{-(\ell+1+rm+i)} = T^{r-\ell-1} \sum_{n \geq 0} \alpha_{n+1} \lambda_{n+1} T^{-r(n+1)}.\]

Consequently, and using our partition of \(\mathbb{N}^*\), we can write

\[
T^{\ell+1-r}\rho = \sum_{n \geq 1} \alpha_n \lambda_n^r T^{-rn}
\]
\[
= \sum_{m=1}^{\ell} \alpha_m \lambda_m^r T^{-rm} + \sum_{n \in E} \alpha_n \lambda_n^r T^{-rn} + \sum_{n \in F} \alpha_n \lambda_n^r T^{-nr}.
\]
Since $k$ divides $r$, again by periodicity, we have $\alpha_{\ell+1+rm+j} = \alpha_{\ell+1+j}$. Applying (2), we get
\[
\sum_{n \in \mathbb{F}} a_n \lambda_n T^{-nr} = \sum_{1 \leq j < r} \alpha_{\ell+1+rm+j} \sum_{m \geq 0}^{} \lambda_{\ell+1+rm+j} T^{-r(\ell+1+rm+j)} \\
= \sum_{1 \leq j < r} \alpha_{\ell+1+j} \sum_{m \geq 0}^{} \beta_j T^{-r(\ell+1+rm+j)} \\
= \sum_{m \geq 0}^{} (\sum_{1 \leq j < r} \alpha_{\ell+1+j} \beta_j T^{-r(\ell+1+j)}) T^{-mr^2} \\
= (1 - T^{-1})^{-r^2} \sum_{1 \leq j < r}^{} \alpha_{\ell+1+j} \beta_j T^{-r(\ell+1+j)}. \quad (4)
\]

By periodicity, we also have $\alpha_{\ell+1+(km+i)} = \alpha_{\ell+1}$. Hence, we obtain
\[
\sum_{n \in \mathbb{E}} a_n \lambda_n T^{-nr} = \sum_{0 \leq i < k} \sum_{m \geq 0}^{} \alpha_{\ell+1+(km+i)} \lambda_{\ell+1+(km+i)} T^{-r(\ell+1+(km+i))} \\
= \alpha_{\ell+1} \sum_{0 \leq i < k} \sum_{m \geq 0}^{} \lambda_{\ell+1+(km+i)} T^{-r(\ell+1+(km+i))} = \alpha_{\ell+1} \rho^r. \quad (5)
\]

Combining (3), (4) and (5), we obtain $\rho = B + C \rho^r$ with $B, C$ in $\mathbb{F}_q(T)$, where
\[
C = \alpha_{\ell+1} T^{r-\ell-1} \quad \text{and} \quad B = T^{r-\ell-1} \left( \sum_{m=1}^{\ell} \alpha_m \lambda_m T^{-rm} + (1 - T^{-1})^{-r^2} \sum_{1 \leq j < r}^{} \alpha_{\ell+1+j} \beta_j T^{-r(\ell+1+j)} \right).
\]

Thus $\rho$ and also $\theta$ are algebraic over $\mathbb{F}_q(T)$ and the proof is complete. \hfill \Box

**Remark.** From a number-theoretic point of view, the sequences described in Proposition 1 are most important because they are associated with an algebraic continued fraction. This association is not relevant for the more general sequences of Theorem 2 as well as for others of a similar type, even more general, considered in the next section. Hence coming back to the sequences $(\lambda_n)_{n \geq 1}$ in a finite field of characteristic 2, defined by (1), a natural question arises: what can be said about the algebraic degree over $\mathbb{F}_q(T)$ of the continued fraction $\alpha = [\lambda_1 T, \lambda_2 T, \ldots, \lambda_n T, \ldots]$? According to Proposition 1, this degree is in the range $[2, r+1]$ since $\alpha$ is irrational and satisfies an algebraic equation of degree $r + 1$. It is a classical fact that $\alpha$ is quadratic if and only if the sequence $(\lambda_n)_{n \geq 1}$ is ultimately periodic. Hence $\alpha$ is quadratic if and only if $\theta$ (the generating function of the sequence introduced in Theorem 2) is rational.

In the particular and simplest case $r = 2$, we are able to give a necessary and sufficient condition to have this rationality. We prove the following.

**Proposition 2.** Let $(\lambda_n)_{n \geq 1}$ be the sequence defined in Proposition 1, by (1), assuming that we have $r = 2$. Then this sequence is periodic (and purely periodic of period length less or equal to 2) if and only if we have
\[
\lambda_m = (\varepsilon_1/\varepsilon_2) \epsilon_2^{(-1)^{m-1}(-1)^m}/2 \quad \text{for} \quad 1 \leq m \leq \ell.
\]

**Proof.** We will apply Theorem 2, in the particular case $p = 2, k = 2$ and $r = 2$. Hence we have $\theta = A + \rho$ and $\rho = B + C \rho^2$. Let $V \in \mathbb{F}_q(T)$ be given. We have
\[ V + \rho = V + B + CV^2 + C(V + \rho)^2. \] Setting \( \sigma = C(V + \rho) \), multiplying by \( C \) this last equality we obtain
\[ \sigma = CV + CB + (CV)^2 + \sigma^2 = U + \sigma^2. \] (6)

Applying the formulas in Theorem 2, in our particular case, we have \( C = \alpha_{\ell+1}T^{1-\ell} \) and
\[ B = T^{1-\ell}(\sum_{m=1}^{\ell} \alpha_m \lambda_m^2 T^{-2m} + \alpha_{\ell+2}\beta_1^2 (1 + T)^{-4}T^{-2\ell}). \]

The sequence \( (\alpha_m)_{m \geq 1} \) is 2-periodic. Indeed, we have \( \alpha_m = (\varepsilon_1/\varepsilon_2)\varepsilon_2^{(-1)^m} \) for \( m \geq 1 \) and \( \beta_1 = \varepsilon_1/\varepsilon_2 \).

Hence, we get \( \alpha_{\ell+1}\alpha_{\ell+2}\beta_1^2 = 1 \). Now we choose \( V = \alpha_{\ell+1}^{-1}T^{1-\ell}(T + 1)^{-2} \). Accordingly, a straightforward computation shows that
\[ U = CV + CB + (CV)^2 = T^{2-2\ell} \sum_{m=1}^{\ell} (\alpha_{\ell+1}\alpha_m \lambda_m^2 + 1)T^{-2m}. \]

We set \( u_m = \alpha_{\ell+1}\alpha_m \lambda_m^2 + 1 \) and we have
\[ U = u_1T^{-2} + u_2T^{-2\ell/2} + \ldots + u_{l}T^{-4l/2}. \]

Note that, for \( m \geq 0 \), between \( U^{2m} \) and \( U^{2m+1} \), we have a gap of length \( 2m+1 \). Consequently \( \sum_{m \geq 0} U^{2m} \) is irrational in \( \mathbb{F}(q) \), since it has arbitrarily long blocks of zeros in the \((1/T)\) power series expansion unless \( U = 0 \). By (6), we have \( \sigma = \sum_{m \geq 0} U^{2m} \). We also have \( \theta = A + V + C^{-1}\sigma \). Therefore \( \theta \in \mathbb{F}(q) \) if and only if \( u_m = 0 \) for \( 1 \leq m \leq l \) or equivalently if and only if
\[ \lambda_m^2 = (\alpha_{\ell+1}\alpha_m)^{-1} = (\varepsilon_1/\varepsilon_2)^2\varepsilon_2^{(-1)^l-(-1)^m} \] for \( 1 \leq m \leq \ell \).

It can be easily verified that the sequence \( (\lambda_n)_{n \geq 1} \) is then 2-periodic : \( \varepsilon_1, \varepsilon_1/\varepsilon_2, \ldots \) or \( \varepsilon_1/\varepsilon_2, \varepsilon_1/\varepsilon_2, \ldots \) according to the parity of \( l \). So the proof is complete. 

**Remark.** The statement \( \theta = A + V + C^{-1}\sigma \) and \( \sigma = U + \sigma^2 \) was given in [12], without proof. Moreover we can observe the following: \( \theta \) is rational if and only if \( \alpha \) is quadratic or if and only if \( \alpha \) is purely 2-periodic. Indeed, if \( (\lambda_n)_{n \geq 1} \) is not purely 2-periodic then \( \alpha \) is cubic over \( \mathbb{F}(T) \). Furthermore, if we define \( \omega(T) = [T, T, \ldots, T, \ldots] \) (which is the analogue in the formal case of the golden number \( (1 + \sqrt{5})/2 = [1, 1, 1, \ldots] \)), then \( \alpha \) is quadratic if and only if we have \( \alpha(T) = (\lambda_1/\lambda_2)^{q/2}\omega((\lambda_1\lambda_2)^{q/2}/T) \).

Inspired by the form of the sequences presented in Theorem 2, we shall give below a criterion for automatic sequences.

### 3. A criterion for automatic sequences

In this work, we consider sequences of the form \( v = (v(n))_{n \geq 1} \). Let \( r \geq 2 \) be an integer. Equivalently, the sequence \( v \) is \( r \)-automatic if its \( r \)-kernel
\[ K_r(v) = \{ (v(rj + n))_{n \geq 1} \mid i \geq 0, 0 \leq j < r^i \} \]
is a finite set (see Cobham [17], p. 170, Theorem 1, see also Eilenberg [14], p. 107, Proposition 3.3)). For more details on automatic sequences, see the book [13] of Allouche and Shallit. Recall that all ultimately periodic sequences are \( r \)-automatic for all integers \( r \geq 2 \), adding or chopping off a prefix to a sequence does not change
its automaticity (see [3, p. 165]), and that a sequence is $r$-automatic if and only if it is $r^m$-automatic for all integers $m \geq 1$ (see [3, Theorem 6.6.4, p. 187]).

For all integers $j, n$ ($0 \leq j < r$, and $n \geq 1$), define
\[
(T_j v)(n) = v(rn + j).
\]
Then for all integers $n, a \geq 1$, and $0 \leq b < r^a$ with $r$-adic expansion
\[
b = \sum_{i=0}^{a-1} b_i r^i \quad (0 \leq b_i < r),
\]
with the help of the operators $T_j$ ($0 \leq j < r$), we obtain
\[
v(r^a n + b) = (T_{b_{a-1}} \circ T_{b_{a-2}} \circ \cdots \circ T_{b_0} v)(n).
\]
In particular, we obtain that $v$ is $r$-automatic if and only if all $T_j v$ ($0 \leq j < r$) are $r$-automatic, for we have $K_r(v) = \{v\} \cup \bigcup_{j=0}^{r-1} K_r(T_j v)$.

The following theorem generalizes Theorem 2 in [14], and can be compared with a result of Allouche and Shallit (see [4, Theorem 2.2]).

**Theorem 3.** Let $r \geq 2$ be an integer. Let $v = (v(n))_{n \geq 1}$ be a sequence in a finite set $A$, and $\sigma$ a bijection on $A$. Fix an integer $i$ with $0 \leq i < r$. Then, for all integer $m \geq 0$, we have the following statement.

(i) If $(T_i v)(n + m) = \sigma(v(n))$ for all integers $n \geq 1$, and $T_j v$ is $r$-automatic for all integers $j$ ($0 \leq j < r$) with $j \neq i$, then $v$ is $r$-automatic.

**Proof.** Since $A$ is finite and $\sigma$ is a bijection on $A$, there exists an integer $l \geq 1$ such that $\sigma^l = \text{id}_A$, the identity mapping on $A$. In the following we shall show $(i_m)$ by induction on $m$. For this, we need only show that $T_i v$ is $r$-automatic under the conditions of $(i_m)$.

If $m = 0$, then under the conditions of $(i_0)$, we have $T_i v = \sigma(v)$, and then
\[
K_r(T_i v) = \{\sigma^a(T_i v) | 0 \leq a < l\} \cup \bigcup_{0 \leq b < l} \sigma^b(K_r(T_j v)),
\]
so $K_r(T_i v)$ is finite, as $T_j v$ is $r$-automatic for all integers $j$ ($0 \leq j < r$) with $j \neq i$.

If $m = 1$, then under the conditions of $(i_1)$, we have $(T_i v)(n + 1) = \sigma(v(n))$ for all integers $n \geq 1$, and $T_j v$ is $r$-automatic for all integers $j$ ($0 \leq j < r$) with $j \neq i$.

Below we distinguish two cases:

**Case I:** $0 \leq i \leq r - 2$. Then for all integers $n \geq 1$, we have
\[
(T_0 T_i)(n + 1) = (T_i v)(rn + r) = \sigma(v(rn + r - 1)) = \sigma((T_{r-1} v)(n)),
\]
hence $T_0(T_i v)$ is $r$-automatic, since it is obtained from $\sigma(T_{r-1} v)$ by adding a letter before, and $T_{r-1} v$ is $r$-automatic by hypothesis, for $r - 1 \neq i$.

Let $j$ be an integer such that $1 \leq j < r$. Then for all integers $n \geq 1$,
\[
(T_j T_i)(n) = (T_i v)(rn + j) = \sigma(v(rn + j - 1)) = \sigma((T_{j-1} v)(n)).
\]
Hence if $j \neq i + 1$, then $T_j v$ is $r$-automatic, for $j - 1 \neq i$, and thus $T_{j-1} v$ is $k$-automatic by hypothesis. Moreover for $j = i + 1$, we have $T_{i+1}(T_i v) = \sigma(T_i v)$. Note that $T_j(T_i v)$ is $r$-automatic for all integers $j$ ($0 \leq j < r$) with $j \neq i + 1$, hence we can apply $((i+1)_0)$ with $T_i v$, and we obtain that $T_i v$ is $r$-automatic.

**Case II:** $i = r - 1$. Then for all integers $j, n$ ($1 \leq j < r$ and $n \geq 1$), we have
\[
(T_j T_{r-1})(n) = (T_{r-1} v)(rn + j) = \sigma(v(rn + j - 1)) = \sigma((T_{j-1} v)(n)).
\]
So $T_j(T_{r-1}v)$ is $r$-automatic, for $j - 1 \neq i$, and thus $T_{j-1}v$ is $r$-automatic by hypothesis. Moreover for all integers $n \geq 1$, we have

$$(T_0T_{r-1}v)(n + 1) = (T_{r-1}v)(rn + r) = \sigma(v(rn + r - 1)) = \sigma((T_{r-1}v)(n)).$$

Since $T_j(T_{r-1}v)$ is $r$-automatic for all integers $j$ ($1 \leq j < r$), we can apply (01) proved above with $T_{r-1}v$, and we obtain that $T_{r-1}v$ is $r$-automatic.

Now let $m \geq 1$ be an integer, and assume that $(i_j)$ holds for all integers $i,j$ ($0 \leq i < r$ and $0 \leq j \leq m$). We shall show that $(i_{m+1})$ holds for all integers $i$ ($0 \leq i < r$). Namely, under the conditions that $(T_v(n + m + 1) = \sigma(v(n)))$ for all integers $n \geq 1$, and $T_v$ is $r$-automatic for all integers $j$ ($0 \leq j < r$) with $j \neq i$, we shall show that $T_v$ is $r$-automatic. For this, we distinguish two cases below.

Write $m = r[\frac{m}{r}] + a$, with $0 \leq a < r$ an integer.

**Case I:** $0 \leq i < r - a - 1$. Let $j$ ($0 \leq j < r$) be an integer. If $j < a + 1$, then for all integers $n \geq 1$, we have

$$(T_jT_i(v)(n + [\frac{m}{r}] + 1) = (T_i(v)(rn + r[\frac{m}{r}] + k + j) = (T_i(v)(rn + m + r + j - a)
\sigma(v(rn + r + j - a - 1)) = \sigma((T_{r+j-a}v)(n)),$$

hence $T_j(T_i(v)$ is $r$-automatic, since it is obtained from $\sigma(T_{r+j-a}v)$ by adding a prefix of length $[\frac{m}{r}] + 1$, and the latter is $r$-automatic by hypothesis, for we have $j \geq 0 > i + a + 1$. Now assume $j \geq a + 1$. Then for all integers $n \geq 1$, we have

$$(T_jT_i(v)(n + [\frac{m}{r}] + 1) = (T_i(v)(rn + r[\frac{m}{r}] + j) = (T_i(v)(rn + m + j - a)
\sigma(v(rn + j - a - 1)) = \sigma((T_{j-a}v)(n)).$$

If $j \neq i + a + 1$, then $T_j(T_i(v)$ is $r$-automatic, since it is obtained from $\sigma(T_{j-a}v)$ by adding a prefix of length $[\frac{m}{r}]$, and the latter is $r$-automatic by hypothesis, for we have $j - a - 1 \neq i$. If $j = i + a + 1$, then $(T_jT_i(v)(n + [\frac{m}{r}] = \sigma((T_i(v)(n)))$ for all integers $n \geq 1$. Note here that we have $[\frac{m}{r}] \leq m$ and $T_j(T_i(v)$ is $r$-automatic for all integers $j$ ($0 \leq j < r$) with $j \neq i + a + 1$, hence we can apply $((i + a + 1)[\frac{m}{r}])$ with $T_i(v)$, and we obtain at once that $T_i(v)$ is $r$-automatic.

**Case II:** $r - a - 1 \leq i < r$. Let $j$ ($0 \leq j < r$) be an integer. If $j \geq a + 1$, then for all integers $n \geq 1$, we have

$$(T_jT_i(v)(n + [\frac{m}{r}] + 1) = (T_i(v)(rn + r[\frac{m}{r}] + j) = (T_i(v)(rn + m + j - a)
\sigma(v(rn + j - a - 1)) = \sigma((T_{j-a}v)(n)),$$

hence $T_j(T_i(v)$ is $r$-automatic, since it is obtained from $\sigma(T_{j-a}v)$ by adding a prefix of length $[\frac{m}{r}]$, and the latter is $r$-automatic by hypothesis, for we have $i \geq r - a - 1 > j - a - 1$. If $j < a + 1$, then for all integers $n \geq 1$, we have

$$(T_jT_i(v)(n + [\frac{m}{r}] + 1) = (T_i(v)(rn + r[\frac{m}{r}] + r + j) = (T_i(v)(rn + m + r + j - a)
\sigma(v(rn + r + j - a - 1)) = \sigma((T_{r+j-a}v)(n)).$$

If $j \neq i + a + 1 - r$, then $T_j(T_i(v)$ is $r$-automatic, since it is obtained from $\sigma(T_{r+j-a}v)$ by adding a prefix of length $[\frac{m}{r}] + 1$, and the latter is $r$-automatic by hypothesis, for we have $r + j - a - 1 \neq i$. If $j = i + a + 1 - r$, then $(T_jT_i(v)(n + [\frac{m}{r}] + 1) = \sigma((T_i(v)(n)))$, for all integers $n \geq 1$. Now that $[\frac{m}{r}] + 1 \leq m$ and $T_j(T_i(v)$ is $r$-automatic for all integers $j$ ($0 \leq j < r$) with $j \neq i + a + 1$, hence we can apply $((i + a + 1 - r)[\frac{m}{r}])$ with $T_i(v)$, and we obtain that $T_i(v)$ is $r$-automatic.
ultimately periodic except for $j$ we have $\alpha$ where $u$ is $q$ an integer coprime with $1$ we have $\alpha$, we deduce at once that all the sequences are $r$-automatic.

Proof. For all integers $i, n (0 \leq i < r, n \geq 1)$, set $u_i(n) = u(rn + i)$, and we only need to show that all the $u_i (0 \leq i < r)$ are $r$-automatic.

Write $\ell + 1 = ra + b$, with $a, b$ integers such that $a \geq 0, 0 \leq b < r$. Then $a+b \geq 1$. From the recursive relations (3), we deduce at once that all the $u_j (0 \leq j < r)$ are ultimately periodic except for $j = b$, and for all integers $m \geq 0$ and $0 \leq i < k$, we have $u_b(km + i + a) = \alpha_{i+1}(u(km + i + 1))^{\gamma}$. Since all the ultimately periodic sequences are $r$-automatic, there remains for us to show that $u_b$ is $r$-automatic.

Extend $(\alpha_i)_{1 \leq i \leq k}$ to be a periodic sequence of period $k$, denoted by $(\alpha_n)_{n \geq 0}$. Then for all integers $n \geq 1$, we have $u_b(n - 1 + a) = \alpha_n(u(n))^{\gamma}$, from which, by noting that $k$ divides $r$, we obtain, for all integers $m \geq 1$,

\[
\begin{align*}
&u_b(rm + a + b - 1) = \alpha_{rm+b}(u_b(m))^{\gamma} = \alpha_b(u_b(m))^{\gamma}, \\
u_b(rm + a + j - 1) = \alpha_{rm+j}(u_j(m))^{\gamma} = \alpha_j(u_j(m))^{\gamma} (0 \leq j < r, j \neq b).
\end{align*}
\]

Write $a+b-1 = rc + d$, with $c, d$ integers such that $c \geq 0$, and $0 \leq d < r$. Then all the $T_iu_b (0 \leq i < r)$ are ultimately periodic (thus $r$-automatic) except for $i = d$, as all the $u_j (0 \leq j < r, j \neq b)$ are ultimately periodic. Moreover for all integers $n \geq 1$, we have $(T_iu_b)(m + c) = \alpha_b(u_b(m))^{\gamma} = \sigma(u_b(m))$, where $\sigma(x) = \alpha_x^{\gamma}$, and $\sigma$ is bijective on $\mathbb{F}_q$, since $\gamma$ is coprime with $q - 1$. To conclude, it suffices to apply Theorem 3 with $u_b$ to obtain that $u_b$ is also $r$-automatic.

Remark. In this theorem, if $\gamma$ is a power of $p$ then we are in the case of Theorem 2 and the automaticity follows directly from Christol theorem, as we have seen. In all cases the generating functions of the sequences defined in Theorem 4 are algebraic, due to Christol theorem, but the algebraic equation is not given in the general case and it may not be as simple (hyperquadratic type) as it is in Theorem 2.

Acknowledgments. Part of the work was done while Jia-Yan Yao visited the Institut de Mathématiques de Jussieu-PRG (CNRS), and he would like to thank his colleagues, in particular Jean-Paul Allouche, for their generous hospitality and interesting discussions. He would also like to thank the National Natural Science Foundation of China (Grants no. 10990012 and 11371210) for partial financial support.
References

[1] J.-P. Allouche, *Sur le développement en fraction continue de certaines séries formelles*, C. R. Acad. Sci. Paris 307 (1988), 631–633.
[2] J.-P. Allouche, J. Betrema, J. O. Shallit, *Sur des points fixes de morphismes d’un monoïde libre*, RAIRO, Inf. Théor. Appl. 23 (1989), 235–249.
[3] J.-P. Allouche and J. Shallit, *Automatic sequences. Theory, applications, generalizations*. Cambridge University Press, Cambridge (2003).
[4] J.-P. Allouche and J. Shallit, *A variant of Hofstadter’s sequence and finite automata*. J. Aust. Math. Soc. 93 (2012), 1–8.
[5] L. E. Baum and M. M. Sweet, *Continued fractions of algebraic power series in characteristic 2*, Ann. of Math. 103 (1976), 593–610.
[6] L. E. Baum and M. M. Sweet, *Badly approximable power series in characteristic 2*, Ann. of Math. 105 (1977), 573–580.
[7] A. Bluher and A. Lasjaunias, *Hyperquadratic power series of degree four*, Acta Arith. 124 (2006), 257–268.
[8] G. Christol, *Ensembles presques périodiques k-reconnaissables*. Theoret. Comput. Sci. 9 (1979), 141-145.
[9] G. Christol, T. Kamae, M. Mendès France and G. Rauzy, *Suites algébriques, automates et substitutions*. Bull. Soc. Math. France 108 (1980), 401–419.
[10] A. Cobham, *Uniform tag sequences*. Math. Systems Theory 6 (1972), 164–192.
[11] S. Eilenberg, *Automata, Languages and Machines*. Vol. A. Academic Press (1974).
[12] A. Lasjaunias, *A note on hyperquadratic continued fractions in characteristic 2 with partial quotients of degree 1*, [http://arxiv.org/abs/1511.08353](http://arxiv.org/abs/1511.08353) 2015, 7 pages.
[13] A. Lasjaunais and and J.-J. Ruch, *On a family of sequences defined recursively in $F_q$ (II)*. Finite Fields Appl. 10 (2004), 551–565.
[14] A. Lasjaunias and J.-Y. Yao, *Hyperquadratic continued fractions in odd characteristic with partial quotients of degree one*. J. Number Theory 149 (2015), 259–284.
[15] A. Lasjaunias and J.-Y. Yao, *Hyperquadratic continued fractions and automatic sequences*. Finite Fields Appl. 40 (2016), 46–60.
[16] W. Mills and D. P. Robbins, *Continued fractions for certain algebraic power series*. J. Number Theory 23 (1986), 388–404.
[17] M. Mkaouar, *Sur le développement en fraction continue de la série de Baum et Sweet*, Bull. Soc. Math. France 123 (1995), 361–374.
[18] J.-Y. Yao, *Critères de non-automaticité et leurs applications*, Acta Arith. 80 (1997), 237–248.

Alain LASJAUNIAS
Institut de Mathématiques de Bordeaux
CNRS-UMR 5251
Talence 33405
France
E-mail: Alain.Lasjaunias@math.u-bordeaux.fr

Jia-Yan YAO
Department of Mathematics
Tsinghua University
Beijing 100084
People’s Republic of China
E-mail: jyyao@math.tsinghua.edu.cn