Duality in Integrable Systems and Generating Functions for New Hamiltonians

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Duality in the integrable systems arising in the context of Seiberg-Witten theory shows that their tau-functions indeed can be seen as generating functions for the mutually Poisson-commuting hamiltonians of the dual systems. We demonstrate that the Θ-function coefficients of their expansion can be expressed entirely in terms of the co-ordinates of the Seiberg-Witten integrable system, being, thus, some set of hamiltonians for a dual system.

1 Introduction

It has become clear recently that rather simple and well-known finite-dimensional integrable hamiltonian systems play an important role in the formulation of effective action for low-dimensional SUSY string and Yang-Mills theories (see also and list of references therein). Remarkably enough, the nonlinear dynamics arises on moduli spaces of vacuum expectation values of scalar fields and Wilson loops of gauge fields, which play the role of dynamical variables (co-ordinates, momenta and action-angle variables) of an integrable system, which usually belongs to a relatively wide class of (complexified) "systems of particles" of Toda, Calogero-Moser and Ruijsenaars-Schneider type (the most recent "classification" of the Seiberg-Witten integrable systems can be found in ).

Speaking about this family of integrable models, Seiberg-Witten theory, on one hand, explains the physical origin of "relativization" of integrable systems of particles in spirit of S.Ruijsenaars – the contribution of the Kaluza-Klein sector in theories with extra compact dimensions. On the other hand, it considers co-ordinates and momenta on equal footing with the action-angle variables as moduli in compactified theory. The general picture of the Seiberg-Witten theory from this perspective and general ideology of duality in modern string or M-theory leads to expecting of some duality relations between co-ordinates and momenta on one side and action-angle variables on another one. It is not a coincidence thus, that such duality was found to act naturally on the phase spaces of Calogero-Moser models and their "compact" Ruijsenaars-Schneider relatives (the Toda chain systems arise as a special "double-scaling" degenerate case ).

This duality can be considered as a way of constructing new nontrivial completely integrable systems with very nontrivial properties. For example, it may lead to the systems where the hamiltonians depend on momenta (or on both momenta and co-ordinates) through the elliptic double-periodic functions (next step after the Ruijsenaars "relativization" when hamiltonians depend on momenta via trigonometric functions), an example of such double-elliptic system was proposed in . It was also conjectured in that the tau-functions of the Seiberg-Witten finite gap integrable systems (expressed through theta functions on corresponding Seiberg-Witten curves with specific period matrices – the set of couplings in low-energy effective SUSY gauge theories) may play a role of generating functions for the hamiltonians of the dual systems. Recently this conjecture was checked for the "perturbative" or "instanton" expansions by numeric MAPLE computations . Below in this note we will demonstrate that mutual Poisson commutativity of hamiltonians follows directly from duality arguments – the Θ-coefficients of the expansion of generating function are functions only of co-ordinates (or dual action variables) of the original Seiberg-Witten integrable system, or the function of moduli entering the superpotential of compactified theory .

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2 Duality in Integrable Systems

By duality for a system of free particle one usually means the Fourier transform, or exchange between co-ordinates and momenta, which are the integrals of motion themselves in free case. One may think of an exchange between the co-ordinates and action variables as of a natural generalization of the Fourier transform for a nontrivial or interacting dynamical system, if it is completely integrable, since both set of quantities look similar from the hamiltonian point of view – i.e. mutually Poisson commute. Following [9, 10], by dual systems we call integrable systems "living" on the same phase space, with the same symplectic form

\[ \Omega = dq \wedge dp = da \wedge dz \]

but with two different set of integrals of motion (action variables or hamiltonians which are the functions of each other) – for the first (the "Seiberg-Witten") system the hamiltonians are \( \{h_i(q, p)\} \) (or action variables \( \{a_i(q, p)\}, h_i = h_i(a) \)) – some nontrivial functions of co-ordinates and momenta, while for the dual system the role of action variables (or hamiltonians) is played by the original co-ordinates (or some algebraically independent functions of them) \( \{q_i(a, z)\} \). Both sets mutually Poisson commute

\[ \{q_i, q_j\} = 0 \quad \{a_i, a_j\} = 0 \]

\( i, j = 1, \ldots, \frac{1}{2} \text{dim(Phase Space)} \), with respect to the Poisson bracket inverse to (1).

For arbitrary integrable systems this very general property (defined already at the level of the Liouville theorem) does not lead necessarily to any interesting consequences. However, the Seiberg-Witten integrable systems possess very peculiar properties with respect to the duality transformation \( q \leftrightarrow a \). In the perturbative (weak coupling) limit, Seiberg-Witten models with the adjoint matter are described by trigonometric Calogero and Ruijsenaars models, the family of which is self-dual with respect to duality transformation \( \Omega \), i.e. the functions \( a_k = f_k(q, p) \) and \( q_k = f_k(a, z) \), expressing the action variables via co-ordinates and momenta and the functions expressing "new" actions ("old" co-ordinates) via "new" co-ordinates and momenta ("old" action and angle variables) coincide, i.e. the duality transformation is "symmetric". This is literally true for the trigonometric Ruijsenaars and rational Calogero models (with the hamiltonians \( h = 2 \cosh p \sqrt{1 + \frac{m^2}{\sinh^2 q}} \) and \( h = p^2 + \frac{m^2}{q^2} \) – in the simplest case – respectively; one may easily check this solving directly equations of motion in one-particle case) while the trigonometric Calogero model is dual to the rational Ruijsenaars.

This family of integrable models is nothing but a perturbative degeneration of the finite-gap or Hitchin integrable systems corresponding to the Seiberg-Witten curves \( \Sigma \) of the Lax form

\[ \det(\lambda - L(z)) = 0 \]

where the Lax operator \( L(z) \) is defined on some base curve \( \Sigma_0 \) – below (and usually) torus (elliptic) or sphere with punctures (rational) – with generating differential

\[ dS = \lambda dz \]

\[ \delta_{\text{moduli}} dS = \text{holomorphic} \]

The action variables are given by the Seiberg-Witten contour integrals over half of the independent contours

\[ a = \oint_A dS \]

or

\[ a^D = \oint_B dS \]

and the symplectic form is [15, 16, 17]

\[ \Omega = \delta dS|_\gamma = da \wedge dz(\gamma) = dq \wedge dp \]

where the variation of differential \( \Omega \) is computed in the divisor \( \gamma \) of the poles of the Baker-Akhiezer function.

In the perturbative limit the curve \( \Theta \) becomes rational, what corresponds to the solitonic limit of a finite-gap system. Then dependence on base curve (on spectral parameter \( z \)) effectively disappears and eq. (3) becomes a generating function for the hamiltonians of (perturbative) Seiberg-Witten integrable system – invariant combinations of matrix elements of \( L \)

\[ \det(\lambda - L) = \sum \lambda^k h_k \]
On the other hand, the determinant in (8) looks very similar to the solitonic or wronskian \( \tau \)-functions for the solutions to KP or Toda lattice hierarchies, if the spectral parameter \( \lambda \) would play the role of “main” (first in KP case or zero in Toda) time – always a parameter on base curve. This way the particular analytic representation of the Seiberg-Witten curve could be thought of as a \( \tau \)-function for a dual integrable system, or, better, of a solution to KP or Toda lattice hierarchy associated with a dual system in the sense of [13, 19, 20] – when the zeroes of tau function of KP/Toda hierarchy become co-ordinates of some integrable system of particles. The discussion of the parallels between tau-functions and spectral curve equations (3) is beyond the scope of this note, it only helps now to state that it could be natural to look for the generation function of dual mutually Poisson-commuting hamiltonians in terms of tau-functions of the Seiberg-Witten finite gap solutions – the theta functions on Seiberg-Witten spectral curves.

3 Generating Function

Let us turn now directly to the generating functions [10] to be discussed below. We will consider the situation when the spectral curve (3) covers 1-dimensional complex torus, or its degeneration – a cylinder or sphere with two punctures. The last case correspond to the periodic Toda chain or pure glyodynamics while the first case contains the elliptic Calogero-Moser and Ruijsenaars-Schneider models, corresponding to the Seiberg-Witten theories with adjoint matter.

Consider the decomposition of the Riemann theta-function, defined on Jacobian of a curve of genus \( g = N \)

\[
\Theta(z|T) = \sum_{n \in \mathbb{Z}^N} e^{2\pi i \sum_{i=1}^{N} n_i z_i + i \pi \sum_{i,j=1}^{N} n_i T_{ij} n_j}
\]

when period matrix satisfies the constraint

\[
\sum_{i=1}^{N} T_{ij} = \tau_j \sum_{j=1}^{N} T_{ij} \quad \forall j
\]

(10)

naturally arising in "elliptic" models with base torus. In other terms

\[
T_{ij} = \frac{\tau_j}{N} + \tilde{T}_{ij}
\]

(11)

Introducing also \( z_i = \frac{\tilde{z}_i}{N} + \tilde{z}_i \) with \( \sum_{i=1}^{N} z_i = z \) or \( \sum_{i=1}^{N} \tilde{z}_i = 0 \), one immediately gets

\[
\Theta(z|T) = \sum_{n \in \mathbb{Z}^N} e^{2\pi i \hat{T} z + i \pi \hat{T}} \sum_{k \in \mathbb{Z}^N} e^{2\pi i \sum_{i=1}^{N} n_i \tilde{z}_i + i \pi \sum_{i,j=1}^{N} n_i \tilde{T}_{ij} n_j} =
\]

\[
\sum_{k \in \mathbb{Z}^N} e^{2\pi i \sum_{i=1}^{N} n_i k} \sum_{n \in \mathbb{Z}^N} e^{2\pi i \sum_{i=1}^{N} n_i \tilde{z}_i + i \pi \sum_{i,j=1}^{N} n_i \tilde{T}_{ij} n_j}
\]

(12)

Presenting \( k = Nm + i \) with \( m \in \mathbb{Z} \) and \( i \in \mathbb{Z}_N = \mathbb{Z} \mod N \) (i.e. \( i = 0, 1, \ldots, N-1 \)), one finally gets [10]

\[
\Theta(z|T) = \sum_{i \in \mathbb{Z}_N} \sum_{m \in \mathbb{Z}} e^{2\pi i (m+\frac{z}{N}) z + i \pi N (m+\frac{z}{N})^2} \sum_{n \in \mathbb{Z}_N} e^{2\pi i \sum_{j=1}^{N} n_j \tilde{z}_j + i \pi \sum_{j,j'=1}^{N} n_j \tilde{T}_{jj'} n_{j'}} =
\]

\[
\sum_{i \in \mathbb{Z}_N} \Theta_i(z|N\tau) \Theta_i(\tilde{z}|\tilde{T})
\]

(13)

where \( \Theta_i(z|N\tau) \) is genus \( g = 1 \) or Jacobi theta-functions with specific characteristics \( \frac{z}{N} \equiv \left[ \frac{\tilde{z}}{\tilde{T}} \right] \), while \( \Theta_i(\tilde{z}|\tilde{T}) \) is defined on Jacobian of genus \( g = N - 1 \). Indeed, for example it can be rewritten as

\[
\Theta_i(\tilde{z}|\tilde{T}) = \sum_{n \in \mathbb{Z}_N} e^{2\pi i \sum_{j=1}^{N} n_j \tilde{z}_j + i \pi \sum_{j,j'=1}^{N} n_j \tilde{T}_{jj'} n_{j'}} =
\]

\[
\sum_{m \in \left( \mathbb{Z}_N - \frac{z}{N} \right)^{N-1}} e^{2\pi i \sum_{j=1}^{N-1} m_j \tilde{z}_j + i \pi \sum_{j,j'=1}^{N-1} m_j \tilde{T}_{jj'} m_{j'}} \equiv \Theta_i(\tilde{z}|\tilde{T})
\]

(14)
with

\[ \tilde{z}_j = \tilde{z}_j - \tilde{z}_N = \tilde{z}_j + \sum_{l=1}^{N-1} \tilde{z}_l \]

\[ \tilde{T}_{ij} = \tilde{T}_{ij} + \sum_{k=1}^{N-1} (\tilde{T}_{ik} + \tilde{T}_{kj}) + \sum_{k,l=1}^{N-1} \tilde{T}_{kl} \]

(15)

Our aim is to study the properties of the decomposition (13) and to demonstrate that the ratios of the coefficients \( \Theta \) are functions only of the co-ordinates of the integrable systems thus being a set of independent hamiltonians for a dual system.

3.1 Toda chain

Consider, first, a "degenerate" case of the periodic Toda chain with the base curve \( \Sigma_0 \) being a cylinder. If one substitutes into (14) the period matrix of the genus \( g = N - 1 \) periodic Toda chain spectral curve (3)

\[ w + \frac{\Lambda^{2N}}{w} = P_N(\lambda) = \lambda^N + \sum_{k=0}^{N-2} h_k \lambda^k \]

(16)

with the Seiberg-Witten integrals (on cylinder it is natural to choose \( z = \log w \))

\[ a_i = \oint_{A_i} \lambda \frac{dw}{w} \]
\[ a_i^D = \oint_{B_i} \lambda \frac{dw}{w} \]

(17)

one gets that

\[ \Theta(z | T) = \sum_{k \in \mathbb{Z}_N} e^{2\pi i \tilde{z}_k} \Theta_k \]

(18)

and

\[ \left\{ \Theta_i, \Theta_{i'} \right\} = 0 \]
\[ \forall i, j, i', j' \]

(19)

where the Poisson bracket is taken w.r.t. symplectic form

\[ \Omega^{Toda} = \sum_{i=1}^{N-1} d\tilde{z}_i \wedge da_i = \sum_{i=1}^{N-1} dq_i \wedge dp_i \]

(20)

since \( \Theta_i \equiv \Theta_i(\tilde{z}|\tilde{T}) \) describe exactly the co-ordinates of the solution to periodic Toda chain [21] [22]

\[ e^{q_i} = \frac{\Theta_i}{\Theta_{i-1}} \]
\[ \frac{\Theta_i}{\Theta_j} = \prod_{k=j+1}^{i} e^{q_k} \]

(21)

and obviously commute \( \{q_i, q_j\} = 0, \forall i, j \) w.r.t. (20) since [14] [15]

\[ \Omega^{Toda} = \sum_{i=1}^{N-1} d\tilde{z}_i \wedge da_i = \sum_{i=1}^{N-1} dq_i \wedge dp_i \]

(22)

where \( q_i \) and \( p_i \) are co-ordinates and momenta of the Toda chain particles. \( \Theta_i \) [21] could be thought of as the Toda chain tau-functions, depending on discrete time \( i \) – the number of particle. Eq. (19) gives an exact form of old expectation that the Toda chain tau-functions Poisson commute with each other. In the perturbative limit \( \Lambda \to 0 \) spectral curve (14) degenerates into rational \( w = P_N(\lambda) \), the period matrix becomes

\[ i\pi \tilde{T}_{ij} = \frac{\partial^2}{\partial a_i \partial a_j} \frac{1}{2} \sum_{k<l} (a_k - a_l)^2 \log \frac{a_i - a_l}{\lambda} \] and \( \Theta_k \), after redefining \( \tilde{z}_i \to \tilde{z}_i - \frac{N}{2\pi i} \log \Lambda \), \( \Theta_k \to \Lambda^{\epsilon} \Theta_k \) turn into expressions for the tau-functions of open Toda chain.
3.2 Calogero-Moser model

In the Calogero-Moser case the commutativity of \( \Theta_i \) can be shown in the following way. The (genus \( g = N \)) spectral curve is given by

\[
\det_{N \times N} (L^{CM}(z) - \lambda) = 0
\]

\[
L^{CM}(z) = \left( pH + \sum_{\alpha} F(qz|z)E_{\alpha} \right)
\]

\[
F(q|z) = m \frac{\theta_\tau(q + z|\tau)}{\theta_\tau(q|\tau)\theta_\tau(z|\tau)} e^{c(q|\tau)z}
\]

where \( \theta_\tau(z) \) is odd Jacobi theta-function, and

\[
a_i = \int_{A_i} \lambda dz
\]

\[
a_i^D = \int_{B_i} \lambda dz
\]

\[
T_{ij} = \frac{\partial a_i^D}{\partial a_j}
\]

\[
i, j = 1, \ldots, N
\]

so that eq. \( 13 \) can be considered literally. According to \( 19 \) equation

\[
0 = \Theta(z|T) = \sum_{i \in \mathbb{Z}_N} \theta_{\tau}(z|N\tau)\Theta_i
\]

as an equation on \( z \)-torus has exactly \( N \) zeroes \( \frac{1}{N} z = q_1, \ldots, q_k \). As a result one gets a system of linear equations

\[
\sum_{i=1}^{N} \theta_{\tau}(Nq_j|N\tau)\Theta_i = 0 \quad j = 1, \ldots, N
\]

The system should have nontrivial solutions, i.e. \( \det_{ij} \theta_{\tau}(Nq_j|N\tau) = 0 \), which effectively reduces the number of degrees of freedom from \( N \) to \( N - 1 \). Then \( 26 \) can be rewritten as

\[
\sum_{i \neq i_0}^{N-1} \theta_{\tau}(Nq_j|N\tau)\Theta_i = \theta_{\tau}(Nq_j|N\tau)
\]

\[
i.e.
\]

\[
\frac{\Theta_i}{\Theta_{i_0}} = \frac{\det_{k \neq i_0, i \rightarrow i_0; j = 1, \ldots, N-1} \theta_{\tau}(Nq_j|N\tau)}{\det_{k \neq i_0, j = 1, \ldots, N-1} \theta_{\tau}(Nq_j|N\tau)}
\]

Therefore, the ratios \( \frac{\Theta_i}{\Theta_{i_0}} \) depend only on the co-ordinate \( q_k, k = 1, \ldots, N \) of the Calogero-Moser particles and, thus, obviously Poisson commute \( \{ \Theta_i, \Theta_{i'} \} = 0, \forall i, j, i', j' \) with respect to the Calogero-Moser symplectic structure

\[
\Omega^{CM} = \sum_{i=1}^{N} dq_i \wedge dp_i = \sum_{i=1}^{N} dz_i \wedge da_i
\]

restricted to \( \sum_{j=1}^{N} q_j = const \) (the condition of vanishing the determinant \( \det_{ij} \theta_{\tau}(Nq_j|N\tau) = 0 \)) and

\[
\sum_{j=1}^{N} a_j = \oint_{A_0} \lambda dz = const \cdot \oint_{A_0} dz = const
\]

(where \( A_0 \) is A-cycle on base curve \( \Sigma_0 \)). The explicit form of the solutions \( 28 \) can be easily found using the \( \theta \)-function identities, coming from the Wick theorem \( 23 \)

\[
\det_{ij} \theta_{\tau}(Nq_j|N\tau) \sim \theta_{\Sigma} \left( N \sum_{k} q_k | N \tau \right) \prod_{i<j} \theta_\tau(Nq_i - Nq_j | N\tau)
\]

so that the (squared) solutions \( 28 \) become elliptic functions on base torus \( \Sigma_0 \).
3.3 Generalities

Similar arguments can be applied in the "relativistic" case of the elliptic Ruijsenaars-Schneider model. The spectral curve $\Sigma$ is again defined over elliptic base $\Sigma_0$ (see, for example, [3] and references therein for details) and one can study zeroes of, now, Toda lattice tau-function in double-periodic variable – the zero (or discrete) time [20]. The formulas of the previous section are generalized straightforwardly.

The situation is not so simple with hypothetic double-elliptic system for the general case of $N$ degrees of freedom. The only known example exists for the $N = 1$ case [10] and it is self-dual with respect to $q \leftrightarrow a$ exchange. There is no clear way yet to write down any reasonable Lax representation or spectral curve equation, except for the perturbative case, when it can be identified with $w = w_0 \prod_{\theta_i} \theta^{(\xi_i - \xi_j)}$ with $dS = \xi d \log w$ and is a degenerate case of the XYZ spin chain [24]. This problem deserves further investigation.

4 Discussion

In this note we have discussed the conjecture of [10] that tau-functions of the Seiberg-Witten integrable systems can play a role of generating functions for new commuting hamiltonians. As a result it is demonstrated above, that the arising quantities – the ratios of coefficients of decomposition over base curve are indeed Poisson-commuting quantities – the (functions of) hamiltonians of dual integrable systems or, in other words, the can be expressed via only the co-ordinates of the original integrable system.

Being an interesting academic question from the internal point of view of the theory of integrable systems, it is also related to more physical issues of the Seiberg-Witten theory. Co-ordinates and momenta of integrable system play the role of bare moduli in Seiberg-Witten theory so that the co-ordinates are the variables associated with superpotential in compactified theory with partially broken supersymmetry. It would be interesting to understand the role of generating function and $\{\Theta_i\}$ from general point of view of geometric description of the Seiberg-Witten M-theory vacua. We hope to return to this problems elsewhere.

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