Parameter identifiability and input-output equations

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Abstract

Structural parameter identifiability is a property of a differential model with parameters that allows for the parameters to be determined from the model equations in the absence of noise. One of the standard approaches to assessing this problem is via input-output equations and, in particular, characteristic sets of differential ideals. The precise relation between identifiability and input-output identifiability is subtle. The goal of this note is to clarify this relation. The main results are:

• identifiability implies input-output identifiability;
• these notions coincide if the model does not have rational first integrals;
• the field of input-output identifiable functions is generated by the coefficients of a “minimal” characteristic set of the corresponding differential ideal.

We expect that some of these facts may be known to the experts in the area, but we are not aware of any articles in which these facts are stated precisely and rigorously proved.

1 Introduction

Structural identifiability is a property of an ODE model with parameters that allows for the parameters to be uniquely determined from the model equations in the absence of noise. Performing identifiability analysis is an important first step in evaluating and, if needed, adjusting the model before a reliable practical parameter identification is performed. Details on different approaches to assessing identifiability can be found, for example, in [5, 10, 27].

Input-output equations have been used to assess structural identifiability for three decades already going back to [21], and several prominent software packages are based on this approach [1, 25, 17, 7, 2, 3, 24, 26, 16, 12, 18]. However, it has been known that input-output identifiability is not always the same as identifiability ([10, Example 2.16], [22, Section 5.2]). The goal of this note is to state and prove basic facts about these relations, some of which seem to be implicitly assumed in the current literature. The main results are

• identifiability implies input-output identifiability (Theorem 4.2);
• these notions coincide if the model does not have rational first integrals (Theorem 4.7);
• the field of input-output identifiable functions is generated by the coefficients of a “minimal” characteristic set of the corresponding differential ideal (Corollary 5.8).

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2 General definition of identifiability

2.1 Identifiability

Fix positive integers \( \lambda, n, m, \) and \( \kappa \) for the remainder of the paper. Let \( \boldsymbol{\mu} = (\mu_1, \ldots, \mu_\lambda), \) \( \mathbf{x} = (x_1, \ldots, x_n), \) \( y = (y_1, \ldots, y_m), \) and \( \mathbf{u} = (u_1, \ldots, u_\kappa). \) Consider a system of ODEs

\[
\Sigma = \begin{cases} 
    \mathbf{x}' = \frac{f(\mathbf{x}, \mathbf{\mu}, \mathbf{u})}{Q(\mathbf{x}, \mathbf{\mu}, \mathbf{u})}, \\
    \mathbf{y} = \frac{g(\mathbf{x}, \mathbf{\mu}, \mathbf{u})}{Q(\mathbf{x}, \mathbf{\mu}, \mathbf{u})}, \\
    \mathbf{x}(0) = \mathbf{x}^*,
\end{cases}
\]

(1)

where \( f = (f_1, \ldots, f_n) \) and \( g = (g_1, \ldots, g_m) \) are tuples of elements of \( \mathbb{C}[\mathbf{x}, \mathbf{u}] \) and \( Q \in \mathbb{C}[\mathbf{x}, \mathbf{u}] \setminus \{0\}. \)

**Notation 2.1** (Auxiliary analytic notation).

(a) Let \( \Omega = \{ (\mathbf{x}^*, \mathbf{\mu}, \mathbf{\hat{u}}) \in \mathbb{C}^n \times \mathbb{C}^\lambda \times (\mathbb{C}^m(0))^\kappa \mid Q(\mathbf{x}^*, \mathbf{\mu}, \mathbf{\hat{u}}(0)) \neq 0 \} \) and

\[
\Omega_h = \Omega \cap \{ (\mathbf{x}^*, \mathbf{\mu}, \mathbf{\hat{u}}) \in \mathbb{C}^n+\lambda \mid h(\mathbf{x}^*, \mathbf{\mu}) \text{ well-defined} \} \times (\mathbb{C}^m(0))^\kappa)
\]

for every given \( h \in \mathbb{C}(\mathbf{x}^*, \mathbf{\mu}). \)

(b) For \( (\mathbf{x}^*, \mathbf{\mu}, \mathbf{\hat{u}}) \in \Omega, \) let \( X(\mathbf{x}^*, \mathbf{\mu}, \mathbf{\hat{u}}) \) and \( Y(\mathbf{x}^*, \mathbf{\mu}, \mathbf{\hat{u}}) \) denote the unique solution over \( \mathbb{C}^m(0) \) of the instance of \( \Sigma \) with \( \mathbf{x}^* = \mathbf{x}^*, \mathbf{\mu} = \mathbf{\mu}, \) and \( \mathbf{u} = \mathbf{\hat{u}} \) (see [8, Theorem 2.2.2]).

**Definition 2.2** (Identifiability, see [10, Definition 2.5]). We say that \( h(\mathbf{x}^*, \mathbf{\mu}) \in \mathbb{C}(\mathbf{x}^*, \mathbf{\mu}) \) is identifiable if

\[
\exists \Theta \in \tau(\mathbb{C}^n \times \mathbb{C}^\lambda) \exists U \in \tau((\mathbb{C}^m(0))^\kappa) \\
\forall (\mathbf{x}^*, \mathbf{\mu}, \mathbf{\hat{u}}) \in (\Theta \times U) \cap \Omega_h \ |S_h(\mathbf{x}^*, \mathbf{\mu}, \mathbf{\hat{u}})\| = 1,
\]

where

\[
S_h(\mathbf{x}^*, \mathbf{\mu}, \mathbf{\hat{u}}) := \{ h(\mathbf{x}^*, \mathbf{\mu}) \mid (\mathbf{x}^*, \mathbf{\mu}, \mathbf{\hat{u}}) \in \Omega_h \text{ and } Y(\mathbf{x}^*, \mathbf{\mu}, \mathbf{\hat{u}}) = Y(\mathbf{x}^*, \mathbf{\mu}, \mathbf{\hat{u}}) \}.
\]

In this paper, we are interested in comparing identifiability and IO-identifiability (Definition 2.5), and the latter is defined for functions in \( \mathbf{\mu}, \) not in \( \mathbf{x}^*. \) Thus, just for the purpose of comparison, we will restrict ourselves to the field \( \{ h \in \mathbb{C}(\mathbf{\mu}) \mid h \text{ is identifiable} \}, \) which we will call the field of identifiable functions.

**Remark 2.3.** The above definition can be extended to functions \( h(\mathbf{x}^*, \mathbf{\mu}) \in \mathbb{C}(\mathbf{x}^*, \mathbf{\mu}) \) (see Definition 2.2). There are software tools that can assess identifiability of initial conditions (e.g. SIAN [9]). Any such tool can be used to assess identifiability of a given function \( h(\mathbf{x}^*, \mathbf{\mu}) \in \mathbb{C}(\mathbf{x}^*, \mathbf{\mu}) \) by means of the transformation described in (2) in the proof of Proposition 3.1.

2.2 IO-identifiability

**Notation 2.4** (Differential algebra).

(a) A differential ring \( (R, \delta) \) is a commutative ring with a derivation \( \delta : R \to R, \) that is, a map such that, for all \( a, b \in R, \) \( (a + b)' = a' + b' \) and \( (ab)' = a'b + ab'. \)

(b) The ring of differential polynomials in the variables \( x_1, \ldots, x_n \) over a field \( K \) is the ring \( K[x_j^{(i)}] \mid i \geq 0, 1 \leq j \leq n \) with a derivation defined on the ring by \( (x_j^{(i)})' := x_j^{(i+1)}. \) This differential ring is denoted by \( K\{x_1, \ldots, x_n\}. \)
(c) An ideal $I$ of a differential ring $(R, \delta)$ is called a differential ideal if, for all $a \in I$, $\delta(a) \in I$. For $F \subset R$, the smallest differential ideal containing set $F$ is denoted by $[F]$. 

(d) For an ideal $I$ and element $a$ in a ring $R$, we denote $I: a^\infty = \{ r \in R \mid \exists \ell: a^\ell r \in I \}$. This set is also an ideal in $R$.

(e) Given $\Sigma$ as in (1), we define the differential ideal of $\Sigma$ as $I_\Sigma = [Qx' - f, Qy - g] : Q^\infty \subset \mathbb{C}(\mu)\{x, y, u\}$.

**Definition 2.5 (IO-identifiability).** The smallest field $k$ such that $\Sigma \subset k \subset \mathbb{C}(\mu)$ and $I_\Sigma \cap \mathbb{C}(\mu)\{y, u\}$ is generated (as an ideal or as a differential ideal) by $I_\Sigma \cap k\{y, u\}$ is called the field of IO-identifiable functions. We call $h \in \mathbb{C}(\mu)$ IO-identifiable if $h \in k$.

### 3 Technical result: algebraic criterion for identifiability

Proposition 3.1 extends the algebraic criterion for identifiability [10, Proposition 3.4] to identifiability of functions of parameters rather than identifiability of just specific parameters themselves.

**Proposition 3.1.** For every $h \in \mathbb{C}(x^*, \mu)$, the following are equivalent:

- $h$ is identifiable;
- the image of $h$ in $\text{Frac}(\mathbb{C}(\mu)\{x, y, u\}/I_\Sigma)$ lies in the field generated by the image of $\mathbb{C}\{y, u\}$ in $\text{Frac}(\mathbb{C}(\mu)\{x, y, u\}/I_\Sigma)$.

**Proof.** Write $h = h_1/h_2$, where $h_1, h_2 \in \mathbb{C}[x^*, \mu]$. Let $F = \text{Frac}(\mathbb{C}(\mu)\{x, y, u\}/I_\Sigma)$ and $E$ the subfield generated by the image of $\mathbb{C}\{y, u\}$ in $F$. Let $\Sigma_1$ be the system of equations obtained by adding

$$
\begin{align*}
x'_{n+1} &= 0, \\
y_{m+1} &= x_{n+1} - h, \\
x_{n+1}(0) &= x_{n+1}^*,
\end{align*}
$$

(2)

to $\Sigma$, where $x_{n+1}$ is a new state variable and $y_{m+1}$ is a new output. We define

$$
F_1 = \text{Frac}(\mathbb{C}(\mu)\{x, x_{n+1}, y, y_{m+1}, u\}/I_{\Sigma_1}),
$$

and let $E_1$ be the subfield generated by the image of $\mathbb{C}\{y, y_{m+1}, u\}$ in $F_1$. We will talk about $\Sigma$-identifiability of $h$ and $\Sigma_1$-identifiability of $x_{n+1}^*$. The proof will proceed in the following three steps.

**Step 1.** $h$ is $\Sigma$-identifiable $\iff$ $x_{n+1}^*$ is $\Sigma_1$-identifiable. Assume that $h$ is $\Sigma$-identifiable. Let $\Theta$ and $U$ be the corresponding open subsets from Definition 2.2. We set

$$
\Theta_1 := \{(x^*, \hat{x}^*_{n+1}, \hat{\mu}) : (x^*, \hat{\mu}) \in \Theta \& h_2(x^*, \hat{\mu}) \neq 0\}.
$$

We will show that $x_{n+1}^*$ is identifiable with the open sets from Definition 2.2 being $\Theta_1$ and $U$. Let $\Omega_1$ be the set $\Omega$ for the model $\Sigma_1$, and consider $(x^*, \hat{x}^*_{n+1}, \hat{\mu}, \hat{u}) \in (\Theta_1 \times U) \cap \Omega_1$. Since, for a fixed known value of $y_{m+1}$, the values of $x_{n+1}^*$ and $h(x^*, \hat{\mu})$ uniquely determine each other, we have

$$
|S_{\hat{x}^*_{n+1}}(x^*, \hat{x}^*_{n+1}, \hat{\mu}, \hat{u})| = |S_h(x^*, \hat{\mu})| = 1.
$$

Thus, $x_{n+1}^*$ is $\Sigma_1$-identifiable.

For the other direction, assume that $x_{n+1}^*$ is $\Sigma_1$-identifiable, and $\Theta_1$ and $U_1$ are the corresponding open sets from Definition 2.2. Let $\Theta$ be the projection of $\Theta_1$ onto all of the coordinates except for $x_{n+1}^*$. We will show that $h$ is $\Sigma$-identifiable with the open sets being $\Theta$ and $U_1$. Consider $(x^*, \hat{\mu}, \hat{u}) \in (\Theta \times U_1) \cap \Omega_h$. Let $\hat{x}^*_{n+1} \in \mathbb{C}$ be such that $(x^*, \hat{x}^*_{n+1}, \hat{\mu}) \in \Theta_1$. Then, in the same way as above, we have

$$
|S_{\hat{x}^*_{n+1}}(x^*, \hat{x}^*_{n+1}, \hat{\mu}, \hat{u})| = |S_h(x^*, \hat{\mu})| = 1.
$$


Step 2. \( h \in E \iff x_{n+1} \in E_1 \). Observe that we have natural embeddings \( F \hookrightarrow F_1 \) and \( E \hookrightarrow E_1 \). If \( h \in E \), then \( x_{n+1} = y_{m+1} + h \in E_1 \).

Assume that \( x_{n+1} \in E_1 \). Then \( h = x_{n+1} - y_{m+1} \in E_1 \). Observe that \( F_1 = F(x_{n+1}) \), and \( x_{n+1} \) is transcendental over \( F \). Since \( x_{n+1} \) is a constant, there is a differential automorphism \( \alpha : F_1 \to F_1 \) such that \( \alpha(x_{n+1}) = x_{n+1} + 1 \) and \( \alpha|_F = id \). Since \( \alpha(y_{m+1}) = y_{m+1} + 1 \), we have \( \alpha(E_1) \subset E_1 \). Since \( E_1 = E(y_{m+1}) \) and \( \alpha(y_{m+1}) = y_{m+1} + 1 \), every \( \alpha \)-invariant element of \( E_1 \) belongs to \( E \). Since \( \alpha(h) = h \), we have \( h \in E \).

Step 3. From Step 1, \( h \) is identifiable if and only if \( x_{n+1}^* \) is \( \Sigma_1 \)-identifiable. By [10, Proposition 3.4 (a) \( \iff \) (c); Remark 2.2], \( x_{n+1}^* \) is \( \Sigma_1 \)-identifiable if and only if \( x_{n+1} \in E_1 \). Finally, Step 2. implies that \( x_{n+1} \in E_1 \) if and only if \( h \in E \).

4 Identifiability and IO-identifiability

4.1 Identifiability \( \implies \) IO-identifiability but not the other way around

Remark 4.1. For examples showing that being IO-identifiable does not always imply being identifiable, see [10, Example 2.14] (for a simple academic example) and [22, Section 5.2] (for a real-life example).

Theorem 4.2. For all \( \Sigma \) and \( h \in \C(\mu) \),

\[
h \text{ is identifiable } \implies h \text{ is IO-identifiable}
\]

Proof. Let \( h \in \C(\mu) \) be identifiable. By Proposition 3.1, there exist \( g \in \C(y, u) \setminus I_\Sigma \) and \( w \in \C(y, u) \) such that \( gh + w \in I_\Sigma \). Therefore, there exist \( m_1, \ldots, m_r \in \C(\mu) \{y, u\} \) and \( p_1, \ldots, p_r \in I_\Sigma \cap k\{y, u\} \) such that

\[
gh + w = m_1 p_1 + \ldots + m_r p_r.
\]

(3)

Suppose \( h \notin k \). By [20, Theorem 9.29, p. 117], there exists an automorphism \( \sigma \) on \( \C(\mu) \) that fixes \( k \) pointwise and such that \( \sigma(h) \neq h \). Let \( R_1 := \overline{\C(\mu)} \{x, y, u\} \). We extend \( \sigma \) to \( R_1 \) by letting \( \sigma \) fix \( x, y, \) and \( u \). Applying \( \sigma \) to (3) and subtracting the two equations yields

\[
g(h - \sigma(h)) = (m_1 - \sigma(m_1)) p_1 + \ldots + (m_r - \sigma(m_r)) p_r
\]

(4)

in \( R_1 \). Let \( P \) denote the differential ideal generated by \( \Sigma \) in \( R_1 \). Since \( P \) is a prime differential ideal and the right-hand side of (4) belongs to \( P \), it follows that either \( g \in P \) or \( h - \sigma(h) \in P \). But since \( h - \sigma(h) \) is a non-zero element of \( \overline{\C(\mu)} \) and \( P \) is a proper ideal, it cannot be that \( h - \sigma(h) \in P \). Therefore, \( g \in P \). Hence, \( g \notin R \cap R = I_\Sigma \), contradicting our assumption on \( g \).

4.2 Sufficient condition for “identifiable \( \iff \) IO-identifiable”

The aim of this section is Theorem 4.7, which gives a sufficient condition for the fields of identifiable and IO-identifiable functions to coincide.

Notation 4.3.

- For a differential ring \( (R, \delta) \), its ring of constants is \( C(R) := \{ r \in R \mid \delta(r) = 0 \} \).
- For elements \( a_1, \ldots, a_N \) of a differential ring, let \( \text{Wr}_M(a_1, \ldots, a_N) \) denote the \( M \times N \) Wronskian matrix of \( a_1, \ldots, a_N \), that is,

\[
\text{Wr}_M(a_1, \ldots, a_N)_{i,j} = a_j^{(i-1)}, \quad 1 \leq j \leq N, \ 1 \leq i \leq M.
\]
**Definition 4.4** (Field of definition). Let $L \subseteq K$ be fields and let $X$ be a (possibly infinite) set of variables. Let $I$ be an ideal of $K[X]$. We say the field of definition of $I$ over $L$ is the smallest (with respect to inclusion) field $k$, $L \subseteq k \subseteq K$, such that $I$ is generated by $I \cap k[X]$.

**Remark 4.5.** For a given $X$ and $I$, the field of definition of $K$ over $\mathbb{Q}$ is what is called the field of definition of $K$ (with no reference to a subfield) in [15, Definition and Theorem 3.4, p. 55]. By [15, Theorem 3.4], for every $K$ and $I$, there is a smallest field $k_0 \subseteq K$ such that $I$ is generated by $I \cap k_0[X]$. The smallest intermediate field $k$, $L \subseteq k \subseteq K$, such that $I$ is generated by $I \cap k[X]$ is equal to the smallest subfield of $K$ containing $L$ and $k_0$. Therefore, for every $L$, $K$, and $I$, the field of definition of $I$ over $L$ is well defined.

**Lemma 4.6** (cf. [6, Section 4.1], [19, Section 3.4], and [28, Section V.]). Let $g \in I_\Sigma$ be such that we can write $g = \sum_{i=1}^{N} a_i z_i$, where $N \geq 2$, $a_i \in \mathbb{C}(\mu) \setminus \{0\}$, $a_1 = 1$, and $z_1, \ldots, z_N$ are distinct monomials in $\mathbb{C}(y, u)$. If for some $Z \subseteq \{z_1, \ldots, z_N\}$ of size $N - 1$ it holds that $\det \text{Wr}_{N-1}(Z) \notin I_\Sigma$, then $a_i$ is identifiable for all $i = 1, \ldots, N$.

**Proof.** Suppose $\det \text{Wr}_{N-1}(z_1, \ldots, z_{i-1}, z_i, z_{i+1}, \ldots, z_N) \notin I_\Sigma$. Modulo $I_\Sigma$, we have

$$\sum_{i \neq j} \frac{a_i}{a_j} z_i = -z_i$$

Since $I_\Sigma$ is a differential ideal, the derivatives of (5) are also true. Differentiating (5) $N - 2$ times, we obtain the following linear system:

$$M \begin{pmatrix} \frac{a_1}{a_{i_1}}, & \frac{a_{i_1}}{a_{i_2}}, & \frac{a_{i_2}}{a_{i_3}}, & \ldots, & \frac{a_{i_{N-1}}}{a_{i_N}} \end{pmatrix}^T = -(z_{i_1}^{(N-2)})^T,$$

where $M = \text{Wr}_{N-1}(z_1, \ldots, z_{i-1}, z_i, z_{i+1}, \ldots, z_N)$. Since $M$ is nonsingular modulo $I_\Sigma$, in $\text{Frac}(\mathbb{C}(\mu)\{x,y\}/I_\Sigma)$, we have

$$\begin{pmatrix} \frac{a_1}{a_{i_1}}, & \frac{a_{i_1}}{a_{i_2}}, & \frac{a_{i_2}}{a_{i_3}}, & \ldots, & \frac{a_{i_{N-1}}}{a_{i_N}} \end{pmatrix} = (-z_{i_1}^{(N-2)})(M^{-1})^T.$$

Since the entries of the right-hand side belong to the subfield generated by $\mathbb{C}(y, u)$, the entries of the left-hand side are identifiable by Proposition 3.1. Since $a_1 = 1$, $a_{i_1}$ is identifiable and it follows that $a_{i_2}, \ldots, a_{i_N}$ are identifiable.

**Theorem 4.7.** Assume that model $\Sigma$ does not have rational first integrals (i.e., first integrals that are rational functions in the parameters and state variables), that is, the constants of $\text{Frac}(\mathbb{C}(\mu)\{x,y,u\}/I_\Sigma)$ coincide with $\mathbb{C}(\mu)$. Then, for every $h \in \mathbb{C}(\mu)$,

$$h \text{ is identifiable } \iff h \text{ is IO-identifiable.}$$

**Proof.** Proposition 4.2 implies that the field of all identifiable functions is contained in the field of all IO-identifiable functions.

Let $J := I_\Sigma \cap \mathbb{C}(\mu)\{y,u\}$. We fix an indexing of differential monomials in $y$ and $u$ by $\mathbb{N}$, it defines an $\mathbb{N}$-indexed basis $\mathcal{B}$ of $\mathbb{C}(\mu)\{y,u\}$. Consider an infinite matrix with each row being an element of a $\mathbb{C}(\mu)$-basis of $J$ written as a vector in basis $\mathcal{B}$. Let $M$ be the reduced row echelon form of the matrix. Notice that, since the original matrix has only finitely many nonzero entries in each row, $M$ also has only finitely many nonzero entries in each row. The field of definition of $J$ over $\mathbb{C}$ is contained in the field generated by the entries of $M$. Therefore, it is sufficient to prove that the entries of $M$ are identifiable. Consider any row of $M$. It corresponds to a differential polynomial $p \in J$. Assume that a proper subset of monomials of $p$ is linearly dependent modulo $J$ over $\mathbb{C}(\mu)$. This dependence yields a polynomial $q \in J$. The representation of $q$ in basis $\mathcal{B}$ must be reducible to zero by the rows of $M$. However, the reduction of $q$ with respect to $p$ is not zero (as they are not proportional), and the result of this reduction is not reducible by any other row
of $M$ by the definition of reduced row echelon form. Thus, there is no such $q$. Hence, the image of every proper subset of monomials of $p$ in $\text{Frac}(\mathbb{C}[\mu]\{x,y,u\}/I_2)$ is linearly independent over the constants of $\text{Frac}(\mathbb{C}[\mu]\{x,y,u\}/I_2)$. Thus, [13, Theorem 3.7, p. 21] implies that the Wronskian of every proper subset of monomials of $p$ does not belong to $I_2$. Lemma 4.6 implies that the coefficients of $p$ are identifiable.

\section{IO-identifiability via characteristic sets}

\subsection{Differential algebra preliminaries}

We will use the following notation and definitions standard in differential algebra (see, e.g., [14, Chapter I], [23, Chapter I], and [4, Section 2]):

\textbf{Definition 5.1.} A differential ranking on $K\{x_1,\ldots,x_n\}$ is a total order $>$ on $X := \{\delta^i x_j \mid i \geq 0, 1 \leq j \leq n\}$ satisfying:

\begin{itemize}
    \item for all $x \in X$, $\delta(x) > x$ and
    \item for all $x,y \in X$, if $x > y$, then $\delta(x) > \delta(y)$.
\end{itemize}

It can be shown that a differential ranking on $K\{x_1,\ldots,x_n\}$ is always a well order.

\textbf{Notation 5.2.} For $f \in K\{x_1,\ldots,x_n\}\setminus K$ and differential ranking $>$,

\begin{itemize}
    \item lead($f$) is the element of $\{\delta^i x_j \mid i \geq 0, 1 \leq j \leq n\}$ appearing in $f$ that is maximal with respect to $>$.
    \item The leading coefficient of $f$ considered as a polynomial in lead($f$) is denoted by in($f$) and called the initial of $f$.
    \item The separant of $f$ is $\partial f / \partial \text{lead}(f)$, the partial derivative of $f$ with respect to lead($f$).
    \item The rank of $f$ is rank($f$) = lead($f$)$^\text{deg}_{\text{lead}(f)}$.$f$.
    \item For $S \subset K\{x_1,\ldots,x_n\}\setminus K$, the set of initials and separants of $S$ is denoted by $H_S$.
    \item for $g \in K\{x_1,\ldots,x_n\}\setminus K$, say that $f < g$ if lead($f$) < lead($g$) or lead($f$) = lead($g$) and $\text{deg}_{\text{lead}(f)}$.$f$ < $\text{deg}_{\text{lead}(g)}$.$g$.
\end{itemize}

\textbf{Definition 5.3 (Characteristic sets).}

\begin{itemize}
    \item For $f,g \in K\{x_1,\ldots,x_n\}\setminus K$, $f$ is said to be reduced w.r.t. $g$ if no proper derivative of lead($g$) appears in $f$ and $\text{deg}_{\text{lead}(g)}$.$f$ < $\text{deg}_{\text{lead}(g)}$.$g$.
    \item A subset $A \subset K\{x_1,\ldots,x_n\}\setminus K$ is called autoreduced if, for all $p \in A$, $p$ is reduced w.r.t. every element of $A \setminus \{p\}$. One can show that every autoreduced set has at most $n$ elements (like a triangular set but unlike a Gröbner basis in a polynomial ring).
    \item Let $A = \{A_1,\ldots,A_r\}$ and $B = \{B_1,\ldots,B_s\}$ be autoreduced sets such that $A_1 < \ldots < A_r$ and $B_1 < \ldots < B_s$. We say that $A < B$ if
        \begin{itemize}
            \item $r > s$ and rank($A_i$) = rank($B_i$), $1 \leq i \leq s$, or
            \item there exists $q$ such that rank($A_q$) < rank($B_q$) and, for all $i$, $1 \leq i < q$, rank($A_i$) = rank($B_i$).
        \end{itemize}
\end{itemize}
An autoreduced subset of the smallest rank of a differential ideal \( I \subset K\{x_1, \ldots, x_n\} \) is called a characteristic set of \( I \). One can show that every non-zero differential ideal in \( K\{x_1, \ldots, x_n\} \) has a characteristic set. Note that a characteristic set does not necessarily generate the ideal.

**Definition 5.4** (Characteristic presentation).

- A polynomial is said to be monic if at least one of its coefficients is 1. Note that this is how monic is typically used in identifiability analysis and not how it is used in [4]. A set of polynomials is said to be monic if each polynomial in the set is monic.

- Let \( C \) be a characteristic set of a prime differential ideal \( P \subset K\{z_1, \ldots, z_n\} \). Let \( N(C) \) denote the set of non-leading variables of \( C \). Then \( C \) is called a characteristic presentation of \( P \) if all initials of \( C \) belong to \( K[N(C)] \) and none of the elements of \( C \) has a factor in \( K[N(C)] \setminus K \).

**Remark 5.5.** The proof of [10, Lemma 3.2] shows that \( I_{\mathbf{2}} \) is prime.

**Definition 5.6** (Monomial). Let \( K \) be a differential field and let \( X \) be a set of variables. An element of the differential polynomial ring \( K\{X\} \) is said to be a monomial if it belongs to the smallest multiplicatively closed set containing 1, \( X \), and the derivatives of \( X \). An element of the polynomial ring \( K[X] \) is said to be a monomial if it belongs to the smallest multiplicatively closed set containing 1 and \( X \).

### 5.2 IO-identifiable functions via characteristic presentations

Corollary 5.8 shows how the field of IO-identifiable functions can be computed via input-output equations.

**Proposition 5.7.** Let \( L \subseteq K \) be differential fields and let \( X \) be a finite set of variables. Let \( P \) be a prime non-zero differential ideal of \( K\{X\} \) such that the ideal generated by \( P \) in \( \overline{K}\{X\} \) is prime. If \( C \) is a monic characteristic presentation of \( P \), then the field of definition of \( P \) over \( L \) is the field extension of \( L \) generated by the coefficients of \( C \).

**Proof.** Let \( A \) be the set of coefficients of \( C \) and let \( k \) be the field of definition of \( P \) over \( L \).

Suppose \( A \subsetneq k \). Let \( P_1 \) be the ideal generated by the image of \( P \) in \( \overline{K}\{X\} \). We show that \( C \) is a monic characteristic presentation for \( P_1 \). We have that \( C \) is a characteristic set for \( P_1 \). Since the initials of \( C \) lie in \( K[N(C)] \), they also lie in \( \overline{K}[N(C)] \). The property of not having a factor in the nonleading variables does not depend on the coefficient field as well. By [11, Definition 2.6] and the paragraph thereafter, we have that \( P = [C]: H^\omega_C \subset K\{X\} \), and therefore \( [C]: H^\omega_C \subset P_1 \), where the differential ideal operation is taken over \( \overline{K}\{X\} \). Since \( C \) is a characteristic set of \( P_1 \), the paragraph following [11, Definition 2.4] implies that \( P_1 \) is contained in \( [C]: H^\omega_C \), so \( P_1 = [C]: H^\omega_C \). Hence, [4, Corollary 1, p. 42], we conclude that \( C \) is a monic characteristic presentation for \( P_1 \).

By [20, Theorem 9.29, p. 117], there is an automorphism \( \alpha \) of \( \overline{K} \) that fixes \( k \) but moves some element of \( A \). Extend \( \alpha \) to a differential ring automorphism on \( \overline{K}\{X\} \) that fixes \( X \). We show that \( \alpha(C) \) is a monic characteristic presentation of \( P_1 \). Since the initials of \( C \) lie in \( K[N(C)] \) and no element of \( C \) has a factor in \( K[N(C)] \setminus K \), it follows that the initials of \( \alpha(C) \) lie in \( \overline{K}[N(\alpha(C))] \) and no element of \( \alpha(C) \) has a factor in \( \overline{K}[N(\alpha(C))] \setminus \overline{K} \). Since the rank of \( \alpha(C) \) is the same as that of \( C \), it remains to show that \( \alpha(C) \subset P_1 \). Let \( f \in C \). Since \( P \) is defined over \( k \), it follows that \( P_1 \) is defined over \( k \). Therefore, there exist \( a_i \in k\{X\} \cap P_1 \) and \( b_i \in \overline{K}\{X\} \) such that \( f = \sum a_i b_i \). Thus,

\[
\alpha(f) = \sum_i a_i \alpha(b_i) \in P_1.
\]

We conclude that \( \alpha(C) \subset P_1 \) and thus is a characteristic set of \( P_1 \).
We have shown that \( C \) and \( \alpha(C) \) are monic characteristic presentations of \( P_1 \). By [4, Theorem 3, p. 42], \( \alpha(C) = C \). However, since \( \alpha \) moves some coefficient appearing in \( C \), we have a contradiction. We conclude that our assumption that \( A \not\subseteq k \) is false.

It remains to show that \( k \subseteq L(A) \). Let \( \{ h_i \}_{i \in B} \) be a monic generating set of \( P_1 \) as an ideal such that, for all \( i \in B \) and for all \( g \in P_1 \setminus \{ h_i \} \), the support of \( h_i - g \) is not a proper subset of the support of \( h_i \). We argue that such a generating set exists. We describe a map \( \phi : P_1 \to \mathcal{P}(P_1) \), where \( \mathcal{P}(P_1) \) denotes the power set of \( P_1 \), such that \( \forall b \in P_1 \)

- \( b \) belongs to the ideal generated by \( \phi(b) \) and
- \( \forall a \in \phi(b) \forall d \in P_1 \setminus \{ 0 \} \) the support of \( d \) is not a proper subset of the support of \( a \).

Let \( b \in P_1 \). Construct \( \phi(b) \) recursively as follows. If there is no element of \( P_1 \setminus \{ 0 \} \) whose support is a proper subset of the support of \( b \), let \( \phi(b) = \{ b \} \). If there is an \( a \in P_1 \setminus \{ 0 \} \) whose support is a proper subset of the support of \( b \), let \( \phi(b) = \phi(a) \cup \phi(b - ca) \), where \( c \in \mathbb{C} \) is such that \( b - ca \) has smaller support than \( b \). This completes the construction of \( \phi \). Note that the procedure terminates since for each non-terminal step, the support of each element of the output is smaller than the support of the input. Let \( \{ h_i \}_{i \in B_0} \) be a generating set for \( P_1 \) as an ideal. Now \( \bigcup_{i \in B_0} \phi(h_i) \), after normalization so that each element is monic, has the desired properties.

Fix \( i \) and suppose that some coefficient of \( h_i \) does not belong to \( L(A) \). Then by [20, Theorem 9.29, p. 117], there is an automorphism \( \alpha \) of \( \overline{\mathbb{C}} \) such that \( \alpha \) fixes \( L(A) \) and \( \alpha(h_i) \neq h_i \). Since \( h_i \) is monic, we have that \( h_i - \alpha(h_i) \) has smaller support than \( h_i \). Now we show that \( h_i - \alpha(h_i) \in P_1 \). Since \( h_i \in P_1 \), we have that \( h_i \in [\mathcal{C}] : H_{\mathbb{C}}^{\infty} \). Therefore, since \( \alpha \) fixes the coefficients of \( C \), we have

\[
\alpha(h_i) \in [\mathcal{C}] : H_{\mathbb{C}}^{\infty}.
\]

Hence,

\[
h_i - \alpha(h_i) \in [\mathcal{C}] : H_{\mathbb{C}}^{\infty} = P_1.
\]

This contradicts the definition of \( \{ h_i \}_{i \in B} \). Since the coefficients of \( h_i \) belong to \( L(A) \), \( \{ h_i \}_{i \in B} \) is also a generating set for \( P \). Therefore, \( P \) is generated by \( P \cap L(A) \{ x \} \). By the definition of \( k \), it follows that \( k \subseteq L(A) \).

**Corollary 5.8.** If \( C \) is a monic characteristic presentation of \( I_{\mathbb{C}} \cap \mathbb{C}(\mu) \{ y, u \} \), then the field of IO-identifiable functions (as in Definition 2.5) is generated over \( \mathbb{C} \) by the coefficients of the elements of \( C \).

**Proof.** The proof of [10, Lemma 3.2] shows that both \( I_{\mathbb{C}} \) and the ideal generated by the image of \( I_{\mathbb{C}} \) in \( \overline{\mathbb{C}}(\mu) \{ x, y, u \} \) are prime, since the argument does not depend on the coefficient field. Therefore \( I_{\mathbb{C}} \cap \mathbb{C}(\mu) \{ y, u \} \) and the ideal generated by \( I_{\mathbb{C}} \cap \mathbb{C}(\mu) \{ y, u \} \) in \( \overline{\mathbb{C}(\mu)} \{ y, u \} \) are prime. By Proposition 5.7 with \( L = \mathbb{C}, K = \mathbb{C}(\mu), \) and \( P = I_{\mathbb{C}} \cap \mathbb{C}(\mu) \{ y, u \} \), we have that the field of definition of \( P \) over \( \mathbb{C} \) is equal to the field extension of \( \mathbb{C} \) generated by the coefficients of \( C \). This is exactly the field of IO-identifiable functions.

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