A New Approach to the Automorphism Group of a Platonic Surface

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Abstract

We borrow a classical construction from the study of rational billiards in dynamical systems known as the “unfolding construction” and show that it can be used to study the automorphism group of a Platonic surface. More precisely, the monodromy group, or deck group in this case, associated to the cover of a regular polygon or double polygon by the unfolded Platonic surface yields a normal subgroup of the rotation group of the Platonic surface. The quotient of this rotation group by the normal subgroup is always a cyclic group, where explicit bounds on the order of the cyclic group can be given entirely in terms of the Schläfli symbol of the Platonic surface. As a consequence, we provide a new derivation of the rotation groups of the dodecahedron and the Bolza surface.

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1 Introduction

Platonic surfaces are natural generalizations to higher genus of the surfaces of the classical Platonic solids. Consider a collection of regular (Euclidean) \( p \)-gons with topological identifications among pairs of edges to form an orientable closed surface with the property that the automorphism (Euclidean isometry) group acts transitively on flags of faces, edges, and vertices. The skeleton of a Platonic surface is commonly called a regular map. If the degree of a vertex in the regular map is \( q \), then the regular map has Schläfli symbol \( \{ p, q \} \). By regarding the regular \( p \)-gons with their Euclidean geometry, elementary concepts from the study of translation surfaces can be brought to bear on the subject of Platonic surfaces. In particular, we consider the deck group from topology of the unfolded Platonic surface induced by a covering map \( \pi \Pi \) to the \( p \)-gon or double \( p \)-gon depending on the parity of \( p \). Precise definitions will be given in the next section. We prove

**Theorem 1.1.** The deck group \( \Gamma_{\text{Mon}}(\pi \Pi) \) is isomorphic to a normal subgroup \( N \) of the rotation group \( \Gamma_{\text{Rot}}(D) \) of the Platonic surface \( D \). Furthermore,

\[
\Gamma_{\text{Rot}}(D)/N
\]

is a cyclic group.

We use the notation \( \Gamma_{\text{Mon}}(\pi \Pi) \) for the deck group because the real object we consider is the monodromy group from algebraic topology. The monodromy group refers to a different object in the field of regular maps and we avoid the terminology altogether in this introduction. Nevertheless, we use it in the remainder of the paper.

For a precise version of this theorem with additional information see Theorems 3.6 and 3.7. The significance of the theorem is that given a Platonic surface \( D \), \( \Gamma_{\text{Mon}}(\pi \Pi) \) is very easy to compute as a subgroup of a large symmetric group. While the theorem does not always provide sufficient information to compute the automorphism group of the regular map, it sometimes suffices to completely determine \( \Gamma_{\text{Rot}}(D) \).

The order of the cyclic group in Theorem 1.1 can be bounded in terms of the Schläfli symbol as follows:

**Proposition 1.2.** Let \( D \) be a Platonic surface with Schläfli symbol \( \{ p, q \} \). Define \( k' \) to be the smallest positive integer satisfying the equation

\[
k' \frac{q(p-2)}{p} \pi \equiv 0 \mod 2\pi.
\]

If \( k \) is the degree of the covering of the unfolded Platonic surface satisfies, then \( k' | k \). If \( p \) is even, then \( k' \leq k \leq p \), and if \( p \) is odd, then \( k' \leq k \leq 2p \).

\(^1\)The rotation group is often called the rotary group.
Proposition 1.2 will be proven in Section 4. As a corollary of the results in that section, we get

**Theorem 1.3.** Let $D$ be a Platonic surface with Schl"{a}fli symbol $\{p,q\}$. If $p$ and $q$ are relatively prime and either

- $p$ or $q$ is divisible by 4, or
- $p$ and $q$ are odd,

then

$$\Gamma_{Mon}(\pi_\Pi) \cong \Gamma_{Rot}(D).$$

This will be used in Section 5 to derive the rotation groups of the dodecahedron and the Bolza surface.

The significance of this paper is that it turns the problem of classifying Platonic surfaces or regular maps into a purely geometric problem where the group theoretic aspects should be manageable.

What is most surprising in this context is that the unfolding construction has been used in the field of dynamical systems since the 1930’s to study the dynamics of a ball in certain billiard tables. Here the unfolding construction proves to be so natural that it extracts group theoretic information from the Platonic surface with complete disregard for any dynamical system on the surface.

There have been numerous works studying this problem graph theoretically, group theoretically, and hyperbolic geometrically. From the graph theoretic perspective, a survey can be found in [Con03]. A census of regular maps in small genus was constructed in [CD01], and a far larger census can be found on the website of Marsten Conder, where group theoretic methods in Magma were used to generate the list [Con18]. From a geometric perspective, works of [KW99] give an excellent comprehensive exposition of some particular cases, and [MS02] is a recent textbook covering a wealth of material with an extensive bibliography.

The connection to hyperbolic geometry is as follows. If a Riemann surface has a sufficiently large automorphism group$^1$, then its automorphism group is a normal subgroup of a triangle group. Therefore, the surface is tiled by hyperbolic triangles and covers a sphere consisting of two isometric hyperbolic triangles. By replacing the hyperbolic triangles by appropriate Euclidean triangles, the surface is tiled by regular $p$-gons and is in fact a Platonic surface. Its skeleton is a regular map.

The key perspective here is that we regard Platonic surfaces by their almost everywhere flat geometry. For example, locally a cube looks geometrically like the plane everywhere except for its 8 vertices, which have angle $3\pi/2$ instead of $2\pi$, like the plane. We observe that the interiors of the edges of the polygons do not see this issue and therefore, are intrinsically geometrically indistinguishable from a small disc in the Euclidean plane. We encourage the reader to look at

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$^1$Automorphism groups of order strictly greater than $12(g-1)$ suffice. In the paper, [Swi18] large is taken to mean order at least $4(g-1)$. In any case, this will not be used in this paper.
Figure 2 which represents the Bolza surface in genus two with a flat geometric structure imposed on it. The flat geometry is clear from the fact that the figure is drawn on a piece of paper. We observe that the negative curvature of the surface is “hidden” in the corners of the octagons which have angle $3(3\pi/4) = 9\pi/4$.

Interestingly, the problem is addressed by turning specific Riemann surfaces into so-called flat cone surfaces by choosing a very particular flat structure on the surface that uncovers properties of the hyperbolic geometry. For the algebro-geometric minded reader, a specific $k$-differential, i.e. an element in the $k^{th}$ tensor power of the canonical line bundle, is chosen on a Riemann surface. The algebro-geometric perspective is not used in this paper.

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2 Preliminaries

In this section, we define all of the necessary terms and record all of the results needed from [AAH18a].

2.1 Geometry

2.1.1 The Primary Objects

Definition. Given a collection of regular Euclidean $p$-gons, let $D$ be a surface without boundary formed from an identification of the edges of the polygons so that the result is closed and orientable. Define a flag of $D$ to be the triple $(f,e,v)$ of a polygon $f$ with one of its boundary edges $e$, where $e$ is a vector in $\mathbb{R}^2$, and a vertex $v$ incident with $e$. If the automorphism group of $D$ is transitive on flags, then $D$ is called a Platonic surface.

Remark. For the reader unfamiliar with this concept, we emphasize that this is a topological construction and surfaces of higher genus with curvature as specified above cannot be embedded in 3-dimensions. In spite of this, these surfaces are very easy to depict by hiding the points with angle greater than $2\pi$ in the corners of the presentation. See Figure 2 for an example.

Definition. The Schlafli symbol of a Platonic surface is the data $\{p,q\}$, where every face is bounded by $p$ edges and every vertex has degree $q$.\footnote{This is also called a \textit{blade} in the graph theory literature.}
The concept of a translation surface is a very general concept applying to any choice of a 1-form on a Riemann surface. Such generality is completely unnecessary for this paper, which is only concerned with finite collections of regular polygons.

**Definition.** Given a collection of regular Euclidean $p$-gons, let $M$ be a surface without boundary given by an identification of the edges of the polygons so that the result is closed and orientable, and the angle at every point is an integral multiple of $2\pi$. We call such a surface with a choice of horizontal direction a regular $p$-gon-tiled surface. See Figure 3 for an example. We also remark that the regular $p$-gon-tiled surfaces considered in this paper are almost examples of Platonic surfaces. The subtle difference is that regular $p$-gon-tiled surfaces have a choice of horizontal direction and Platonic surfaces do not. Forgetting the additional information of the choice of direction, the regular $p$-gon-tiled surfaces considered in this paper will be examples of Platonic surfaces.

**Example 2.1.** The first examples the reader should consider for this paper are known as the $p$-gon or double $p$-gon, depending on the parity of $p$.

Observe that if $p$ is even, every side of a regular $p$-gon has a unique opposite side. Identifying these sides leads to an example of a translation surface denoted here by $\Pi_p$.

The second example to consider is when $p$ is odd. In this case, take two regular $p$-gons and again observe that every side has a unique opposite parallel side to which it is parallel. Identifying again in this case leads to a translation surface denoted $\Pi_p$. (The parity of $p$ implicitly records which one we consider.)

### 2.1.2 Unfolding

In this section we borrow a classical construction from the theory of rational billiards [FK36] to introduce a natural subgroup of $\text{Aut}(G)$. We observe that the holonomy on a Platonic surface most often will not lie in $2\pi\mathbb{Z}$. Nevertheless,
for some $k$, it is always possible to pass to a degree $k$ branched cover of the Platonic surface such that it does.

**Remark.** The unfolding construction is also known as the canonical $k$-cover \cite{BCG16} § 2.1.

**Definition.** Define the unfolding $\tilde{D}$ of a Platonic surface $D$, with branched covering map $\pi_D : \tilde{D} \to D$, to be the smallest cover such that the holonomy of the cover lies in $2\pi\mathbb{Z}$.

We will always assume that the unfolding is a regular $p$-gon tiled surface by choosing a horizontal direction. This is exactly the same concept as choosing a branch cut of a Riemann surface in complex analysis.

This exists and is discussed in more detail in \cite{AAH18a} § 2.2. The idea is that by rotating a presentation of a Platonic surface appropriately, every edge is rotated into $k$ copies with a unique orientation so that each edge has a unique parallel copy to which it can be identified. This is exactly what is depicted in Figure 3. An unfolding of a Platonic surface is a regular $p$-gon tiled surface.

We observe that there is a natural branched covering $\pi_\Pi : \tilde{D} \to \Pi_p$ that actually possesses extra structure. The covering map $\pi_\Pi$ induces a bijection on the set of unit vectors at a point away from the vertices of the Platonic surface. In other words directions are well-defined modulo $2\pi$ when passing between $\tilde{D}$ and $\Pi_p$. Observe that this is false for $\pi_D$ if $k > 1$.

### 2.2 Groups

**Definition.** Let $G$ be a Platonic surface with automorphism group $\text{Aut}(G)$. The rotation subgroup $\Gamma_{\text{Rot}}(G)$ of $\text{Aut}(G)$ is the subgroup of orientation preserving automorphisms.

**Definition.** A pair\footnote{This is dual to the concept of a dart, which is a pair $(e,v)$.} is a tuple $(f,e)$.

We will primarily be concerned with the group $\Gamma_{\text{Rot}}(G)$. Consequentially, it suffices to reduce flags to pairs because rotations are orientation preserving and the vertex would provide redundant information.

Since $\Gamma_{\text{Rot}}(G)$ is a subgroup of $\text{Aut}(G)$ of index two, it is a normal subgroup of $\text{Aut}(G)$.

Let $p : Y \to X$ be a branched covering map of degree $n$. There is a natural representation of the fundamental group into the symmetric group known as the monodromy representation.

$$\psi_p : \pi_1(X, p_0) \to S_n$$

is given by numbering the sheets of the cover $\tilde{D}$ by $\{1, \ldots, n\}$ and recording the induced permutation of the sheets by lifting loops in $\pi_1(X, p_0)$ to $\pi_1(Y, \tilde{p}_0)$. 


Warning. There is a concept known as the monodromy group in the study of regular maps. The monodromy group defined here is not that object. The monodromy group here is the one commonly found in topology [Bre97].

Definition. The image of the monodromy representation $\psi_{\pi_\Pi}$ is the monodromy group $\Gamma_{Mon}(\pi_\Pi)$.

In particular, the monodromy group associated to the covering map $\pi_\Pi$ will be of particular interest. We summarize the two covering maps that are the subject of this paper in the following diagram.

\[ \begin{array}{ccc}
\tilde{D} & \xrightarrow{\pi_D} & D \\
\downarrow{\pi_\Pi} & & \downarrow{\pi_\Pi} \\
\Pi & \xrightarrow{\pi_\Pi} & \Pi \\
\end{array} \] (1)

2.3 Previous Results

The key theorem from [AAH18a] that serves as the starting point for this paper is the following.

Theorem 2.2 ([AAH18a], Thm. 4.5). The covering $\pi_\Pi : \tilde{D} \to \Pi_p$ is normal. Furthermore, the group of orientation preserving isometries of $\tilde{D}$, denoted $\Gamma_{Rot}(\tilde{D})$, is transitive on pairs.

3 Proofs of the Main Results

If $p : X \to Y$, then let $\text{Deck}(p)$ denote the group of deck transformations of $X$ with respect to the covering map $p$.

Lemma 3.1. There is an injective homomorphism

\[ \tilde{\Psi} : \Gamma_{Mon}(\pi_\Pi) \hookrightarrow \Gamma_{Rot}(\tilde{D}). \]

Proof. Since $\pi_\Pi$ is a normal covering by Theorem 2.2, it follows from [Dyd11, Prop. 2.10] or [Mas91, Thm. 7.2] that $\Gamma_{Mon}(\pi_\Pi)$ is isomorphic to $\text{Deck}(\pi_\Pi)$. However, the deck transformations of $\pi_\Pi$ are naturally isometries of $\tilde{D}$ because they are homeomorphisms of $\tilde{D}$ that interchange sheets of $\pi_\Pi$ by definition. Hence, they form a subgroup of $\Gamma_{Rot}(\tilde{D})$. \hfill \Box

Proposition 3.2. Given an orientation preserving isometry $F : \tilde{D} \to \tilde{D}$, there exists an automorphism $\tilde{F} : \tilde{D} \to \tilde{D}$ such that the diagram below commutes.

\[ \begin{array}{ccc}
\tilde{D} & \xrightarrow{\tilde{F}} & \tilde{D} \\
\downarrow{\pi_D} & & \downarrow{\pi_D} \\
D & \xrightarrow{F} & D \\
\end{array} \]

\[ \text{In [AAH18a Thm. 4.5], this is phrased as simply saying that $\tilde{D}$ is a Platonic surface.} \]
Proof. Given \( F : D \rightarrow D \), choose a pair \((P_0, e_0)\) on \( D \) and its image \((P_1, e_1)\) under \( F \). Choose lifts of each pair on \( \tilde{D} \) so that \((P_i, e_i)\) lifts to \((\tilde{P}_i, \tilde{e}_i)\) for \( i \in \{0, 1\} \). By Theorem 2.2 there is an isometry \( \tilde{F} \) such that \( \tilde{F}(P_0, \tilde{e}_0) = (\tilde{P}_1, \tilde{e}_1) \). Therefore, \( \tilde{F} \) is a lift of \( F \).

Lemma 3.3. The covering map \( \pi_D \) induces a surjective homomorphism

\[
\phi_D : \Gamma_{\text{Rot}}(\tilde{D}) \rightarrow \Gamma_{\text{Rot}}(D),
\]

and

\[ \ker(\phi_D) = \Gamma_{\text{Mon}}(\pi_D). \]

Proof. We construct the map \( \phi_D \) explicitly. Given \( \tilde{F} \in \Gamma_{\text{Rot}}(\tilde{D}) \), choose a pair \((\tilde{f}_1, \tilde{e}_1)\) on \( \tilde{D} \) and let \( F(\tilde{f}_1, \tilde{e}_1) = (\tilde{f}_2, \tilde{e}_2) \). Now consider the pairs \((f_i, e_i) = \pi_D(\tilde{f}_i, \tilde{e}_i)\), for \( i = 1, 2 \). This defines a unique rotation of \( D \) by letting \( F \) be the unique rotation with the property that \( F(f_1, e_1) = (f_2, e_2) \). Next we use Proposition 3.2 to lift \( F \) to a rotation \( \tilde{F}_0 \in \Gamma_{\text{Rot}}(\tilde{D}) \). For convenience, let \( \Delta = \text{Deck}(\pi_D) \). Furthermore, \( \tilde{F}_0 \) maps \((\tilde{f}_1', \tilde{e}_1')\) to \((\tilde{f}_2', \tilde{e}_2')\), where \((\tilde{f}_i', \tilde{e}_i')\) is a lift of \((f_i, e_i)\) for \( i = 1, 2 \). Then because the lift is unique up to action by the deck group, and the deck group is transitive because \( \pi_D \) is normal, for \( i = 1, 2 \) there exist elements \( \delta_i \in \Delta \) such that \( \delta_i(\tilde{f}_i', \tilde{e}_i') = (\tilde{f}_i, \tilde{e}_i) \). Therefore, the map \( \delta_2 \circ \tilde{F}_0 \circ \delta_1 \) is a rotation of \( \tilde{D} \) that sends \( \tilde{F}(\tilde{f}_1, \tilde{e}_1) = (\tilde{f}_2, \tilde{e}_2) \). Since the action of a rotation on a pair uniquely determines the rotation, we have \( \delta_2 \circ \tilde{F}_0 \circ \delta_1 = \tilde{F} \). This implies that the construction of \( F \) given \( \tilde{F} \) did not depend on the initial choice of pair. Therefore, \( \phi_D \) is a well-defined map.

The fact that \( \phi_D \) is a homomorphism follows from a composition of commutative diagrams. Finally, the kernel of \( \phi_D \) consists of exactly those maps such that \( \pi_D(f_1, e_1) = \pi_D(f_2, e_2) \). However, this is exactly the condition that two maps differ by a deck transformation by definition.

Proposition 3.4. The covering map \( \pi_\Pi \) induces a surjective homomorphism

\[
\phi_\Pi : \Gamma_{\text{Rot}}(\tilde{D}) \rightarrow \Gamma_{\text{Rot}}(\Pi_p),
\]

and

\[ \ker(\phi_\Pi) = \Gamma_{\text{Mon}}(\pi_\Pi). \]

In particular, the monodromy group is a normal subgroup of \( \Gamma_{\text{Rot}}(\tilde{D}) \).

Proof. In this case we take advantage of the fact that the isometries of \( \tilde{D} \) and \( \Pi_p \) are affine linear transformations with constant derivative lying in a finite subgroup of the orthogonal group \( O(2) \). This follows because \( \tilde{D} \) and \( \Pi_p \) have holonomy in \( 2\pi \mathbb{Z} \). In fact, the derivative map \( T(\Gamma_{\text{Mon}}(\pi_\Pi)) \rightarrow O(2) \) is actually a faithful representation. For this reason the derivative map induces the desired homomorphism.

Observe that the isometries of \( \tilde{D} \) induced by the monodromy group always correspond to translation. Hence, every isometry given by an element of \( \Gamma_{\text{Mon}}(\pi_\Pi) \) has derivative equal to the identity, in which case the second claim follows.
Lemma 3.5. Let $\delta \in \ker(\phi_D)$ such that for some pair $(f,e)$, $\delta$ sends $(f,e)$ to $(f',e')$. Let $\theta$ be the angle between $e$ and $e'$ on $\bar{D}$. Then $\theta \equiv 0 \mod 2\pi$ if and only if $\delta = \text{Id}$.

Proof. This is trivial if $\delta = \text{Id}$. Observe that $\ker(\phi_D)$ is exactly the deck group of the degree $k$ covering $\pi_D : \bar{D} \to D$. By definition, this covering consists of all rotations of $D$ by angle $2\pi/k$. Hence, only the trivial element of the deck group does not rotate edges.

The main theorem now follows from elementary results from finite group theory.

Theorem 3.6. The monodromy group $\Gamma_{\text{Mon}}(\pi_H)$ is isomorphic to a normal subgroup of $\Gamma_{\text{Rot}}(D)$.

Proof. By Lemma 3.1 there is an injective homomorphism
\[ \tilde{\Psi} : \Gamma_{\text{Mon}}(\pi_H) \hookrightarrow \Gamma_{\text{Rot}}(\bar{D}). \]
By Proposition 3.4, $\tilde{\Psi}(\Gamma_{\text{Mon}}(\pi_H))$ is a normal subgroup of $\Gamma_{\text{Rot}}(\bar{D})$. Therefore, $\phi_D \left( \tilde{\Psi}(\Gamma_{\text{Mon}}(\pi_H)) \right)$ is a normal subgroup of $\phi_D(\Gamma_{\text{Rot}}(\bar{D}))$. By Lemma 3.3 $\phi_D$ is surjective, so $\phi_D \left( \tilde{\Psi}(\Gamma_{\text{Mon}}(\pi_H)) \right)$ is a normal subgroup of $\Gamma_{\text{Rot}}(D) = \phi_D(\Gamma_{\text{Rot}}(\bar{D}))$. Clearly,
\[ \Gamma_{\text{Rot}}(D) \cong \Gamma_{\text{Rot}}(\bar{D})/\ker(\phi_D). \]
Consider the induced homomorphism onto the quotient group
\[ \Psi : \Gamma_{\text{Mon}}(\pi_H) \to \Gamma_{\text{Rot}}(\bar{D})/\ker(\phi_D). \]
Since every element in the image of $\Psi$ preserves the angles between edges by Lemma 3.5 and every non-trivial element in $\ker(\phi_D)$ changes the angle of edges, $\Psi$ is an injective homomorphism.

Theorem 3.7. The group $\Gamma_{\text{Rot}}(D)/\phi_D(\Gamma_{\text{Mon}}(\pi_H))$ is cyclic.

Proof. As observed above, $\Gamma_{\text{Mon}}(\pi_H)$ is the kernel of the derivative map into $O(2)$. Since the image of the derivative map has finite image, the result follows.

4 Bounds on Quantities

Though explicit values for $k$ are hard to determine because they depend on the global geometry of the surface, bounds can be derived in terms of the Schläfli symbol. This in turn provides bounds on algorithms that perform group computations.
Convention. Throughout the remainder of the paper we set
\[ d = \text{gcd}(p, q). \]

Convention. Throughout the remainder of the paper we will refer to the number of faces of \( D \).

Proof of Prop. 1.2. By definition of a Platonic surface, the cone angle at each vertex is equal to
\[ \frac{q(p-2)}{p} \pi. \]

Therefore, the value of \( k' \) is clearly the smallest quantity necessary to guarantee that the angle at every cone point on the unfolding is an integral multiple of \( 2\pi \). Moreover, any positive integer that does not satisfy this equation cannot result in a cone point with cone angle that is a multiple of \( 2\pi \). Therefore, the divisibility claim follows.

The upper bound is clear because it allows every edge of every \( p \)-gon in the Platonic surface to realize every possible orientation. The parity difference is merely a result of the fact that the unfolded surface covers \( \Pi_p \), and \( \Pi_p \) could be one or two regular \( p \)-gons depending on the parity of \( p \).

Lemma 4.1. Let \( D \) have Schläfli symbol \( \{p,q\} \). Then the value of \( k' \) from Proposition 1.2 is given in Table 1.

Proof. A \( p \)-gon has individual angle \( \frac{p-2}{p} \pi \), which implies that the total angle at each vertex on the Platonic surface is \( \frac{2(q(p-2))}{p} \pi \). Consider
\[ \frac{p}{d} \left( \frac{q(p-2)}{p} \right) = \frac{q}{d}(p-2) = L, \]
and observe that it is always an integer because \( d|q \). If \( q \) is even, then \( L \) is even as well. On the other hand, if \( q \) is odd, then \( L \) is odd, and \( k' = 2p/d \). If \( 4|p \), then \( 2|(p-2) \) and \( 4 \nmid (p-2) \), which implies that \( k' = p/d \). Finally, if \( 4|(p-2) \), then \( k' = p/(2d) \) because \( L/2 \) is even in this case and \( 2 \nmid d \) because \( q \) is odd.

The following is well-known and can be found in [Bre97, Ch. III, Cor. 5.2].

Proposition 4.2. The order of the monodromy group \( \Gamma_{\text{Mon}}(\pi_\Pi) \) is equal to the degree of the cover \( \pi_\Pi : \tilde{D} \to \Pi_p \).
Table 2: Lower bounds for the order of $\Gamma_{\text{Mon}}(\pi_\Pi)$

| $p$ | $q$ | odd mod 4 | 2 mod 4 |
|-----|-----|-----------|--------|
| odd | $mp$ | $mp$      | $mp$   |
| 0 mod 4 | $mp$ | $mp$      | $mp$ |
| 2 mod 4 | $mp$ | $mp$      | $mp$ |

Table 3: Upper bounds for the order of $\Gamma_{\text{Rot}}(D)/\phi_D(\Gamma_{\text{Mon}}(\pi_\Pi))$

| $p$ | $q$ | odd mod 4 | 2 mod 4 |
|-----|-----|-----------|--------|
| odd | $d$ | $2d$      | $2d$   |
| 0 mod 4 | $d$ | $d$      | $d$ |
| 2 mod 4 | $2d$ | $d$      | $d$ |

**Proposition 4.3.** Let $D$ be a Platonic surface with Schläfli symbol $\{p,q\}$. The order of the rotation group $\Gamma_{\text{Rot}}(D)$ is $mp$.

*Proof.* Clearly, there are $mp$ pairs. Since the rotation group of the Platonic surface is transitive on pairs by definition of the Platonic surface and specifying how one pair maps to another uniquely determines the map, the order of the rotation group is $mp$. □

**Proposition 4.4.** Let $D$ be a Platonic surface with Schläfli symbol $\{p,q\}$. For each $\{p,q\}$, a lower bound for the order of the monodromy group is given in Table 2.

*Proof.* By Proposition 4.2, the monodromy group has order $km$, when $p$ is even and $km/2$ when $p$ is odd. By Proposition 1.2, $k'$ divides $k$ and by Lemma 4.1, $k'$ is given in Table 1. □

**Remark.** We observe that all values in Table 2 are indeed integers because if $p$ is odd, then $\Pi_p$ consists of two polygons and $nk$ is indeed an even number.

**Corollary 4.5.** Let $D$ be a Platonic surface with Schläfli symbol $\{p,q\}$. For each $\{p,q\}$, an upper bound for the order of the cyclic group $\Gamma_{\text{Rot}}(D)/\phi_D(\Gamma_{\text{Mon}}(\pi_\Pi))$ is given in Table 3.

At this point Theorem 1.3 is proven because it corresponds to the cases when $d = 1$ (which is implicitly excluded in many entries in the table) and the upper and lower bounds of Proposition 1.2 are equal.

**5 Examples**

We consider two examples. The first of which was carried out almost to completion in [AAH18a] and the second is a well-known example from genus two.
5.1 The Dodecahedron

Here we present a new derivation of the rotation group of the dodecahedron using the calculations from [AAH18a] and the theory developed above. Recall the Schläfli symbol for the dodecahedron is \( \{5, 3\} \). Therefore, Theorem 1.3 applies.

The generators of the monodromy group were produced in [AAH18a, § 6.3] and running the code results in four generators. However, only two are actually needed to generate the full group.

\[
\text{gen1}_\text{dodec} = [(0, 19, 21), (1, 18, 4), (2, 56, 28), (3, 7, 51), (5, 49, 58), (6, 14, 16), (8, 17, 46), (9, 11, 12), (10, 44, 48), (13, 22, 36), (15, 39, 43), (20, 34, 38), (23, 27, 31), (24, 29, 26), (25, 54, 33), (30, 32, 59), (35, 55, 57), (37, 53, 40), (41, 50, 52), (42, 47, 45)]
\]

\[
\text{gen2}_\text{dodec} = [(0, 13, 16), (1, 12, 3), (2, 50, 46), (4, 25, 48), (5, 43, 9), (6, 8, 11), (7, 45, 36), (10, 38, 14), (15, 33, 19), (17, 40, 31), (18, 28, 26), (20, 58, 29), (21, 24, 23), (22, 30, 56), (27, 55, 51), (32, 53, 34), (35, 49, 52), (37, 47, 39), (41, 44, 42), (54, 57, 59)]
\]

Applying the Sage functions below show that these generators result in a group of order 60.

\[
\text{GMon}_\text{dodec} = \text{PermutationGroup}([\text{gen1}_\text{dodec}, \text{gen2}_\text{dodec}])
\]

\[
\text{GMon}_\text{dodec}.\text{order}()
\]

Therefore, this construction produces the rotation group of the dodecahedron as a faithful representation into \( S_{60} \).

5.2 The Bolza Surface

It is well-known that there is a unique genus two surface, called the Bolza surface, with a rotation (orientation preserving automorphism) group of order 48. In this section we consider a surface of genus two and show that its rotation group has order 48. Therefore, it is the Bolza surface and the method employed here produces the automorphism group of the Bolza surface as a faithful representation into the symmetric group \( S_{48} \).
Figure 2: The Bolza Surface

We consider the surface depicted in Figure 2. An elementary Euler characteristic computation establishes that it has genus two. Furthermore, the Schlafli symbol is \(\{8, 3\}\). We observe that Theorem 1.3 applies to this case to prove that the monodromy group is in fact isomorphic to the rotation group.

We follow the computations in [AAH18a, §6, §7]. In particular, the cube is most relevant here because, like the cube, the Bolza surface consists of polygons with an even number of sides. Therefore, it avoids the extra computations needed for doubled odd polygons.

The function `build_adj_8_3` constructs the adjacencies of the octagons using the convention that the bottom horizontal edge of each octagon in \(\tilde{D}\) is labeled 0, the next edge is 1 going counter-clockwise, and continuing to 7. The lists are ordered so that index \(i\) in the list gives the coordinates of the octagon incident with edge \(i\) of octagon (\(\text{sheet, octagon}\)). A particular symmetry of this surface that we exploit is that the parallel edges of every octagon are incident with the same octagon. Therefore it suffices to double each list with the \(*2\) command rather than repeat them.

```python
def build_adj_8_3(sheet, octagon):
    i = sheet;
    oct_8_3_adj_base = 6*[None]
    oct_8_3_adj_base[0] = [[i,4],[i-1,3],[i-4,2],[i+2,5]]*2
    oct_8_3_adj_base[1] = [[i,2],[i+1,3],[i,4],[i,5]]*2
    oct_8_3_adj_base[2] = [[i,1],[i-1,5],[i-4,0],[i+2,3]]*2
```

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oct_8_3_adj_base[3] ==[[i-1,1],[i-2,2],[i+1,0],[i,4]]*2
oct_8_3_adj_base[4] ==[[i,0],[i+1,5],[i,1],[i,3]]*2
oct_8_3_adj_base[5] ==[[i-1,4],[i-2,0],[i+1,2],[i,1]]*2
prelim_adj = [oct_8_3_adj_base[octagon%6][(k-i)%8] for k in range(8)]
return [[item[0]%8, item[1]%6] for item in prelim_adj]

As in the case of the other Platonic solids from [AAH18a § 6], the only non-
trivial line here is the line defining prelim_adj because the index (k-i)%8 is not
obvious. However, this follows from the fact that the sheets of \( \tilde{D} \) are numbered
counter-clockwise in the presentation in Figure 3. Therefore, \( i \) rotations by \( 2\pi/8 \) move edge \( -i \) mod 8 into the lower horizontal position, whereby it becomes
edge 0. Thus, the index follows.

Next we implicitly number all 48 octagons in the cover by defining a complete
list of all of them.

def octagons():
    return list(itertools.product(*[range(8), range(6)]))

Finally, the permutations can be determined using the code that is a mod-
ification of the code for the cube from [AAH18a Ex. 6.1]. We again refer the
reader to that paper for an explanation.

def perm_oct(abcd):
    oct_list = octagons()
    total = []
    i = 0
    perm_a_sub = []
    perm_a = []
    while len(total) < 46:
        total += perm_a_sub
        total.sort()
        if len(total) != 0:
            i_list = [j for j in range(len(total)) if j != total[j]]
            if i_list == []:
                i = len(total)
            else:
                i = i_list[0]
            perm_a_sub = []
            while i not in perm_a_sub:
                perm_a_sub += [i]
                i = oct_list.index(tuple(build_adj_8_3 \( \backslash \\)
                               (oct_list[i][0], oct_list[i][1][abcd])))
            perm_a.append(tuple(perm_a_sub))
    return perm_a

Given the functions above, we follow the commands for the dodecahedron
above to produce the rotation group as a subgroup of \( S_{48} \).
Figure 3: The Unfolding of the Bolza Surface

\[
\text{GMon}_8 = \text{PermutationGroup}([\text{perm_oct}(0), \text{perm_oct}(1), \text{perm_oct}(2), \text{perm_oct}(3)])
\]

GMon_8.order()

The output of the last line is 48, which implies that the rotation group of this genus two surface is 48. Since the Bolza surface is the unique genus two surface with a rotation group of this order, the surface depicted in Figure 2 is indeed the Bolza surface. Its rotation group is generated by the permutations \text{perm_oct}(0), \text{perm_oct}(1), \text{perm_oct}(2), and \text{perm_oct}(3).

Remark. We observe that the first three generators above suffice to generate the rotation group.
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