A VARIANT OF CAUCHY’S ARGUMENT PRINCIPLE FOR ANALYTIC FUNCTIONS WHICH APPLIES TO CURVES CONTAINING ZEROS

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Abstract
The standard version of Cauchy’s argument principle, applied to a holomorphic function $f$, requires that $f$ has no zeros on the curve of integration. In this note, we give a generalisation of such a principle which covers the case when $f$ has zeros on the curve, as well as an application.

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1. Introduction, statement of results and an application
The argument principle, one of the fundamental results in complex analysis, can be formulated as follows (see [1, 3, 4]).

**Theorem 1.1.** Suppose that $\gamma$ is a smooth Jordan curve in a domain $U$ and a function $f$ is analytic on $U$ with no zeros on $\gamma$. Then the number of zeros of $f$ inside $\gamma$ is equal to

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, dz. \quad (1.1)$$

This expression is also equal to the winding number of the curve $f(\gamma)$ around 0.

Equation (1.1) shows the importance of the condition that $f$ does not vanish on $\gamma$, since otherwise the integral would diverge, but it is the final statement that we will focus on. We have found the following variant of this statement which holds without the requirement that $f$ be nonzero on $\gamma$.

**Theorem 1.2.** Let $f$ be a nonzero holomorphic function defined on an open domain $U$ and let $\gamma$ be a smooth Jordan curve lying inside $U$. Then for any line $L$ passing through the origin there exist at least $2m + \lambda$ distinct points on $\gamma$ mapped to $L$ by $f$. 

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where $m$ is the number of zeros of $f$ inside $\gamma$ and $\lambda$ is the number of zeros of $f$ on $\gamma$, all counted according to their multiplicities.

We note that Theorem 1.2 is an immediate consequence of Theorem 1.1 when there are no zeros on $\gamma$, since any line will be intersected at least twice by $f(\gamma)$ each time $f(\gamma)$ winds around 0. Naturally the difficulty arises when there are zeros on $\gamma$. A first attempt may be to factor out the zeros, for instance writing $f(z) = g(z)h(z)$, where $g(z)$ is a polynomial with zeros only on $\gamma$ and $h$ does not vanish on $\gamma$, and then to try to apply Theorem 1.1 to $h$ and add the zeros from $g$. However, this approach fails, as the intersections of $f(\gamma)$ with $L$ are in general not respected by the factorisation and in any event zeros of $g$ on $\gamma$ of high order must contribute many intersections of $f(\gamma)$ with $L$.

We will prove Theorem 1.2 in the next section, but first we discuss the motivation for this result, in particular the application which inspired it. We thank Mohammed Zerrak for bringing this problem to our attention.

**Problem 1.3.** Let $a_0, \ldots, a_n$ be a sequence of real numbers and for simplicity assume that $a_0, a_n \neq 0$. (We note that this requirement can easily be removed if required.) Let $P(\theta) = \sum_{j=0}^{n} a_j \cos(j\theta)$ and $Q(\theta) = \sum_{j=0}^{n} a_{n-j} \cos(j\theta)$. Let $Z_P$ be the number of $\theta \in [0, 2\pi)$ such that $P(\theta) = 0$, and $Z_Q$ defined analogously. Show that $Z_P + Z_Q \geq 2n$.

A quick solution using complex analysis can be provided as follows. Let $f(z) = \sum_{j=0}^{n} a_j z^j$. Then $P(\theta) = \Re(f(e^{i\theta}))$ and it may be checked that $Q(\theta) = \Re(g(e^{i\theta}))$, where $g(z) = z^n f(1/z)$. If there are no zeros of $f$ on the unit circle $\{|z| = 1\}$, then by Theorem 1.1 the curve $\Re(f(e^{i\theta}))$ will intersect the imaginary axis at least $2m_f$ times, where $m_f$ is the number of zeros (counting multiplicities) of $f$ inside the unit disk $\{|z| < 1\}$, and the analogous statement holds for $g$. We see that $Z_P + Z_Q \geq 2(m_f + m_g)$. However, the number of zeros of $g$ in $\{|z| > 1\}$ is the same as the number of zeros of $f$ in $\{|z| < 1\}$, and, since $f$ has no zeros on the unit circle, we see that $m_f + m_g = n$. The result therefore follows in this case. (We should note that this exact argument, with appropriate adjustments, is used in the solution of Problem 1.8.5 of [1].)

The role of Theorem 1.2 is to extend this solution in the case that $f$ has zeros on the unit circle. Let $\lambda$ denote the sum of the multiplicities of the zeros of $f$ on $\{|z| = 1\}$, and note that the conjugates of these zeros must be zeros of $g$ with the same multiplicities. Applying Theorem 1.2, we see that $Z_P + Z_Q = 2m_f + \lambda + 2m_g + \lambda$ and, as $m_f + m_g + \lambda = n$, the result follows.

As a final comment before proving Theorem 1.2, we point out that the number $2m + \lambda$ is the best possible, as the following examples show. Let $\gamma$ be the unit circle and let $f(z) = (z + 1)^n$. Then, taking $z = e^{i\theta}$, we can check that $(e^{i\theta} + 1)^n = 2^n e^{in\theta/2} \cos^n(\theta/2)$, so the number of intersections that $f(\gamma)$ has with the real axis is the same as the number of zeros of $\Im(e^{in\theta/2}) = \sin(n\theta/2)$ in $[0, 2\pi)$, and this is $n$. This shows essentially that $\lambda$ in the expression $2m + \lambda$ is sharp. The $2m$ term is even easier, as we may take $\gamma$ again as the unit circle and $f(z) = z^n$, and $f(e^{i\theta})$ will intersect any line exactly $2n$ times as $\theta$ ranges from 0 to $2\pi$. 

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A variant of Cauchy’s argument principle
2. Proof of Theorem 1.2

The number of zeros of $f$ on and inside $\gamma$ is finite, as the latter is a Jordan curve. Let $z_1, \ldots, z_k$ be the roots of $f$ on $\gamma$ and denote by $\lambda_j$ the multiplicity of each $z_j$ and by $D_j = D_j(\varepsilon)$ the disk of radius $\varepsilon$ centred at $z_j$. The radius $\varepsilon$ is chosen small enough so that $D_j$ remains inside $U$ and contains no zeros of $f$ other than $z_j$. Now consider the Jordan curve $\gamma_\varepsilon$ constructed from $\gamma$ by replacing each part $\gamma \cap D_j$ by the arc of the circle $\partial D_j$ lying outside $\gamma$; in order to guarantee that $\gamma_\varepsilon$ is itself a Jordan curve, we may need to decrease $\varepsilon$ further. Figure 1 illustrates $\gamma_\varepsilon$.

No zeros of $f$ lie on $\gamma_\varepsilon$ and there are $m + \lambda$ zeros of $f$ inside $\gamma_\varepsilon$, so by Theorem 1.1 there are at least $2m + 2\lambda$ points on $\gamma_\varepsilon$ which are preimages of points in $L$. The trick is to show that not too many of them can be on the components of $\gamma_\varepsilon$ which lie on the boundary of some $\partial D_j$. The intuition behind this is easy: if we shrink $\varepsilon$ sufficiently, then the points on $\partial D_j$ which are preimages of points on $L$ should be approximately equidistributed on $\partial D_j$, so that about half of them lie inside and half lie outside $\gamma$. Only the ones on the outside of $\gamma$ lie on $\gamma_\varepsilon$, and any other points on $\gamma_\varepsilon$ which are preimages of points in $L$ must lie on $\gamma$ proper. The trick is making this rigorous.

Inside $\overline{D_j}$, $f$ is of the form $f(z) = (z - z_j)^{\lambda_j}g_j(z)$, where $g_j(z_j) \neq 0$. If $z = z_j + \varepsilon e^{i\theta}$, then

$$\arg(f(z_j + \varepsilon e^{i\theta})) = \lambda_j \theta + \arg(g_j(z)).$$

We claim that

$$\left| \frac{d \arg(g_j)}{d\theta}(z_j + \varepsilon e^{i\theta}) \right| \varepsilon = o(1). \quad (2.1)$$

From the previous equation,

$$\left| \frac{d \arg(f)}{d\theta}(z_j + \varepsilon e^{i\theta}) \right| \varepsilon = \lambda_j + o(1). \quad (2.2)$$
We shall use the approximation (2.2) and we prove (2.1) later in a separate lemma.

Now, we may shrink $\varepsilon$ if necessary so that $\left(\frac{d\arg(f)}{d\theta}(z_j + \varepsilon e^{i\theta})\right)$ is positive on all $\partial D_j$. From this fact, combined with Theorem 1.1, $\partial D_j$ contains exactly $2\lambda_j$ preimages of points in $L$, since the argument of the curve may not change direction in order to create extra preimages. In other words,

$$\#\{z \in \partial D_j \mid f(z) \in L\} = 2\lambda_j.$$ 

Denote by $u_0 = z_j + re^{0i}, \ldots, u_\ell = z_j + re^{\theta_i}$ the points of $\gamma_\varepsilon \cap \partial D_j$ with the property that $f(u_t) \in L$, arranged in anti-clockwise order as illustrated in Figure 2. Note that $\arg(f(u_t)) - \arg(f(u_{t-1})) = \pi$ for all $t$, since $f$ is continuous and orientation preserving. Using the mean value theorem and (2.2),

$$\left| \frac{\arg(f(u_t)) - \arg(f(u_{t-1}))}{\theta_t - \theta_{t-1}} \right| = \frac{\pi}{\theta_t - \theta_{t-1}} \leq \lambda_j + o(1),$$

whence

$$\ell \leq \sum_{t=1}^{\ell} \frac{(\theta_t - \theta_{t-1})(\lambda_j + o(1))}{\pi} = \frac{\lambda_j + o(1)}{\pi}(\theta_\ell - \theta_0) \leq \frac{\lambda_j + o(1)}{\pi} = \frac{\lambda_j + o(1)}{\pi}(\pi + \eta_\varepsilon),$$

where $\pi + \eta_\varepsilon$ denotes the angular length of the circular arc $\partial D_j \cap \gamma_\varepsilon$. Note that the smoothness of $\gamma$ implies that $\eta_\varepsilon \to 0$ as $\varepsilon \to 0$. Thus, by again shrinking $\varepsilon$ if necessary,

$$\ell \leq \lambda_j.$$ 

There are $\ell + 1$ points on $\partial D_j \cap \gamma_\varepsilon$ which are preimages of points in $L$ and applying this around every $z_j$ and combining these estimates yields at least

$$2m + 2\lambda - \sum_{j=1}^{k} (\lambda_j + 1).$$
points on $\gamma \cap \gamma_{e}$ which are preimages of points in $L$. Recalling that $\lambda = \sum_{j=1}^{k} \lambda_j$ gives at least

$$2m + \sum_{j=1}^{k}(\lambda_j - 1)$$

points on $\gamma \cap \gamma_{e}$ which are preimages of points in $L$. Finally, the points $z_1, \ldots, z_k$ are all mapped to $L$ by $f$, since $L$ passes through the origin, and taking these into account completes the proof of the theorem.

It remains only to prove the lemma used earlier on the derivative of the argument of $g$.

**Lemma 2.1.** We have

$$\left| \frac{d\arg(g_j)}{d\theta}(z_j + \varepsilon e^{i\theta}) \right| = o(1).$$

**Proof.** The Taylor expansion of $g(z)$ about $z_j$ gives

$$g_j(z_j + \varepsilon e^{i\theta}) = g_j(z_j) + \sum_{r=1}^{\infty} \frac{g_j^{(r)}(z_j)}{r!} \varepsilon^r e^{i\theta} = g_j(z_j) + \varepsilon \varphi_j(\theta).$$

Here, the function $\varphi_j$ is differentiable (with respect to $\theta$) with bounded derivative. Set $g_j(z_j) = a + bi$ and $\varphi_j(\theta) = \varepsilon \alpha(\theta) + \varepsilon \beta(\theta) i$. Without loss of generality, we may assume that $a, b > 0$ and hence for small $\varepsilon$ we get $a + \varepsilon \alpha(\theta), b + \varepsilon \beta(\theta) > 0$. In particular, we can express $g_j(z_j + \varepsilon e^{i\theta})$ as

$$g_j(z_j + \varepsilon e^{i\theta}) = \arctan\left(\frac{b + \varepsilon \beta(\theta)}{a + \varepsilon \alpha(\theta)}\right).$$

Therefore,

$$\frac{d\arg(g_j)}{d\theta}(z) = \frac{\varepsilon \beta'(\theta)(a + \varepsilon \alpha(\theta)) - \varepsilon \alpha'(\theta)(b + \varepsilon \beta(\theta))}{(a + \varepsilon \alpha(\theta))^2} \times \frac{1}{1 + (b + \varepsilon \beta(\theta)/a + \varepsilon \alpha(\theta))^2} = O(\varepsilon) = o(1). \quad \Box$$

### 3. Concluding remarks

We remark first that the method of excising a gap from $\gamma$ and replacing it with a sub-arc of a circle in order to avoid a troublesome point (in this case, a zero of $f$) is similar to, and inspired by, a similar method for calculating integrals using the residue theorem. A good example is the evaluation of the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx$$

by integrating $e^{iz}/z$ along a contour along the real line from $-R$ to $R$ and then back along a semicircular arc in the upper half-plane. The problem is that the function $e^{iz}/z$
has a singularity at \( z = 0 \). However, we may deform the contour so that it includes another semicircle of small radius \( \varepsilon \) and, as \( \varepsilon \to 0 \), we essentially pick up half of the residue at 0 (for details, see [5, Section 7.18]). This idea also appears in the study of Hilbert transforms (see [2, Ch. 3]) and the Plemelj formulae (see [6]).

The condition that \( \gamma \) be smooth can be weakened to piecewise smooth if required. In that case we can associate an interior angle \( \alpha_j \) to each of the zeros \( z_j \), as shown in Figure 3. The same reasoning applies, except that an upper bound on the number of preimages of points on \( L \) on each \( \partial D_j \) is given by \( \lfloor (2\pi - \alpha_j)\lambda_j/\pi \rfloor + 1 \), where \( \lfloor \cdot \rfloor \) is the floor function. A bit of algebra yields the following generalisation of Theorem 1.2.

**Theorem 3.1.** Let \( f \) be a nonzero holomorphic function defined on an open domain \( U \) and let \( \gamma \) be a piecewise-smooth Jordan curve lying inside \( U \). Suppose that the zeros of \( f \) lying on \( \gamma \) are \( z_1, \ldots, z_k \). Let \( \lambda_j \) be the multiplicity of the zero at \( z_j \) and let \( \alpha_j \) be the interior angle of \( \gamma \) at \( z_j \). Then for any line \( L \) passing through the origin there exist at least

\[
2m + \sum_{j=1}^{k} \lceil \lambda_j \alpha_j/\pi \rceil
\]

distinct points on \( \gamma \) mapped to \( L \) by \( f \), where \( m \) is the number of zeros of \( f \) inside \( \gamma \) counted according to multiplicities and \( \lceil \cdot \rceil \) denotes the ceiling function.

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