A geometric approach to free variable loop equations in discretized theories of 2D gravity

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Abstract

We present a self-contained analysis of theories of discrete 2D gravity coupled to matter, using geometric methods to derive equations for generating functions in terms of free (noncommuting) variables. For the class of discrete gravity theories which correspond to matrix models, our method is a generalization of the technique of Schwinger-Dyson equations and is closely related to recent work describing the master field in terms of noncommuting variables; the important differences are that we derive a single equation for the generating function using purely graphical arguments, and that the approach is applicable to a broader class of theories than those described by matrix models. Several example applications are given here, including theories of gravity coupled to a single Ising spin (\(c = 1/2\)), multiple Ising spins (\(c = k/2\)), a general class of two-matrix models which includes the Ising theory and its dual, the three-state Potts model, and a dually weighted graph model which does not admit a simple description in terms of matrix models.
1 Introduction

A major goal of theoretical physics is the construction of a self-consistent quantum theory of gravity coupled to matter in four dimensions. Because such a theory has proven to be very difficult to construct, it is interesting to consider toy models of gravity coupled to matter in lower dimensions, in the hope that such simplified theories can provide information about the nature of quantum gravity. In particular, it has been possible to construct consistent quantum theories of (Euclidean) gravity in two dimensions coupled to conformal matter. These theories have been the subject of considerable interest over the last decade, partly because they are the simplest available theories of gravity coupled to matter and partly because of the relevance of these models to string theory. Two dimensional gravity theories of this type can be defined either through the continuum Liouville theory approach [1, 2, 3], or through discrete models of dynamically triangulated gravity (for reviews see [4, 5, 6]).

The study of discretized two-dimensional gravity theories has a long history [7, 8]. A major breakthrough occurred when it was realized that a summation over “triangulations” of a 2-manifold by equilateral polygons (not necessarily triangles) can be described by a zero-dimensional field theory, or matrix model [9-14]. By writing a discretized 2D gravity theory as an integral over hermitian matrices, it becomes possible to use analytic methods to calculate correlation functions in these models, and to show that in an appropriate continuum limit these models appear to correspond to continuum theories of Liouville gravity coupled to conformal matter field (for a review see [15]). The matrix model technology which was developed also made it possible to sum the partition function for these models over Riemann surfaces of all genera, giving nonperturbative results for many theories of this type [16, 17, 18].

One approach which has been used to analyze matrix model theories involves the use of Schwinger-Dyson equations [19, 20, 21]. These equations, derived by demanding the invariance of the matrix model correlation functions under infinitesimal changes of variables, may be interpreted geometrically as “loop equations” describing the deformation of two-dimensional geometries. The Schwinger-Dyson/loop equations are a powerful tool for analyzing the algebraic structure of correlation functions, although it has proven difficult to find solutions in more complex theories. Recently, the Schwinger-Dyson equations for matrix models have been used to discuss $c < 1$ string field theories [22-31].

In this paper we generalize the Schwinger-Dyson technique, developing a systematic method for deriving a single equation satisfied by the generating function in a very general class of discrete 2D gravity theories. Our approach is based on the combinatorial methods pioneered by Tutte [32], who found an algebraic description of one class of triangulations by writing a discrete recursion relation for the number of triangulations of a 2-manifold with a fixed number of boundary edges (see also [33, 34]). We show how this type of recursion relation can be derived in the presence of discrete data defined on the triangulation, representing matter fields in the continuum. Using this technique we are able to describe a variety of theories, including theories such as the recently described dually weighted graph models [35, 36] which do not always admit a simple description in terms of matrix models. For matrix model theories, this formalism can be used to describe generating functions for
correlation functions with a general class of boundary conditions, which can lead to results which cannot be easily obtained through the matrix model formalism.

In theories with matter, the generating function for the disk amplitude is a formal sum over all possible matter configurations on the boundary of the triangulations. We encode boundary data for the matter fields as a string of free (noncommuting) variables associated with the matter configuration on the edges or vertices of the boundary. In this way the generating function is an element of the free algebra generated by the variables representing different matter states. One of the principal results presented in this paper is the derivation of a “generating equation,” satisfied by the generating function. This generating equation is closely related to the master equation which has recently been discussed by a variety of authors [37-42]. The generating equation describes the effect of removing a marked edge from a triangulation, which results in one of two basic geometric “moves” – the removal of the triangle attached to the edge or, if the edge is connected directly to another boundary edge, the removal of that pair of boundary edges. In fact, the correspondence between these moves and the generating equation is sufficiently precise that the equation may be written down directly upon consideration of the possible effects of the moves, without going through the intermediate step of an explicit recursion equation.

Once we have a single generating equation in terms of noncommuting variables for the correlation functions in the theory, the next step is to try to find a subset of correlation functions for which a closed system of equations can be written in terms of commuting variables. This is analogous to the technique used in Ref. [43] to find the cubic equation satisfied by the generating function of homogeneous correlation functions in the $c = 1/2$ matrix model which describes an Ising matter field coupled to 2D gravity; a description of this calculation in terms of noncommuting variables was given in Ref. [42]. From a polynomial equation of this type it is straightforward if somewhat tedious to derive information about the scaling behaviour of correlation functions in the continuum limit.

A principal focus of this paper is the development of a systematic approach to finding and solving closed sets of equations in commuting variables starting from the noncommuting generating equation. We give several examples of polynomial equations which can be derived in this fashion. In particular, we explicitly solve a 1-parameter family of two-matrix models which may be thought of as the Ising model (in which Ising spins are located on the faces of the triangles) and its dual model (in which the spins are located on the vertices of the triangulation). The solution reproduces the cubic equations for the Ising and dual theories, and shows that the intermediate models satisfy a quartic. We also describe the generating equations for the 3-state Potts model and the $2^k$-matrix model with $c = k/2$ which describes $k$ Ising spins coupled to 2D gravity; in both of these examples we are unable to find a closed system of equations in commuting variables, and we discuss the obstacles encountered in this approach.

We do not discuss the continuum limit of any of the polynomial equations that are derived in this paper. All the equations are at most quartic in a single commuting generating function, and contain a finite number of functions that correspond to boundary data for the
generating function. The continuum limit is given by the expansion of the parameters of the theory around their critical values, as governed by the appropriate polynomial equation; for an example see \[22\]. The continuum limit of some of the more interesting models considered in this paper will be discussed in Refs. \[14, 15\].

The approach we have taken in this paper is to develop the general formalism in a piecewise fashion, introducing each additional complication in the context of a particular model where extra structure is present. The outline of the paper is as follows: in Section 2 we describe the geometric approach used throughout the paper by examining in detail the generating equation for pure triangulated 2D gravity, which can be written in terms of a single variable that encodes boundary information. In Section 3, we consider gravity coupled to a single Ising spin, and associate boundary configurations with words in a free algebra generated by noncommuting variables. We derive the cubic equation governing homogeneous boundary configurations in this model. We also give a generating equation for a more general class of boundary variables in this model which incorporates information about nearest-neighbor spin pairs. In Section 4 we study the Ising theory on the dual lattice, and derive the cubic equation for the homogeneous disk amplitude in this formulation of the model. In Section 5 we find the generating equation for a general two-matrix model with cubic interactions, which includes the Ising theory, dual Ising theory, and the 1-parameter family of models describing the Ising theory with boundary magnetic field. We also outline the derivation of a quartic equation for the homogeneous disk amplitude in these models. Some of the results mentioned in this section will be described in greater detail in a separate paper \[15\]. In Section 6 we examine the 3-state Potts model, which serves as an example of the obstacles faced by this kind of approach. In Section 7, we describe the generating equation for the theory of triangulated gravity coupled to multiple Ising spins \(c = k/2\). In Section 8, we consider a more general class of discrete gravity models, and derive the generating equation for a simple dually weighted graph model. Section 9 contains concluding remarks.

2 Pure Gravity

In this section we consider pure 2D gravity in the absence of matter fields; this corresponds to the generic one-matrix model. We use this simple model to present an introduction to the geometrical techniques that are applied to solve more complex models in subsequent sections. We derive the well known equation for the disk amplitude generating function, in the case of cubic interactions \[10\]. Although the generating function in this model is written in terms of a single commuting variable, we show how this result may be written in a form that generalizes to free variables. Finally, we derive a more elaborate system of generating equations for higher-genus amplitudes.
2.1 Review of model

The partition function for pure 2D gravity is formally defined as

\[ Z = \sum_h \left( \int \frac{\mathcal{D}M}{\text{Diff}_M} e^{-S_E[g_{\mu\nu}]} \right), \]  

(1)

where the genus \( h \) of the orientable 2-manifold \( M \) is summed over and

\[ S_E[g_{\mu\nu}] = \frac{1}{16\pi G} \int d^2x \sqrt{g} \left( -R + 2\Lambda \right). \]  

(2)

A discretized version of this theory can be constructed by summing over all triangulated 2-manifolds. This discrete theory has a partition function

\[ Z = \sum_\Delta \frac{1}{S(\Delta)} N^{\chi(\Delta)} g^n(\Delta), \]  

(3)

where \( g \) and \( N \) are coupling constants that replace \( \Lambda \) and \( G \), \( \chi(\Delta) \) is the Euler character of the triangulation \( \Delta \), \( S(\Delta) \) is the symmetry factor of \( \Delta \), and \( n(\Delta) \) is the number of triangles in \( \Delta \). The matrix model expression for this partition function is given by

\[ Z = \int DU \exp \left( -N \left[ \frac{1}{2} \text{Tr} U^2 - \frac{g}{3} \text{Tr} U^3 \right] \right), \]  

(4)

where \( U \) is an hermitian \( N \times N \) matrix.

We are interested in computing the generating function \( \Phi(u, g) \) for the disk amplitude, defined by

\[ \Phi(u, g) = \sum_{k=0}^{\infty} p_k(g) u^k = \sum_{k, n=0}^{\infty} N(k; n) g^n u^k, \]  

(5)

where \( p_k \) is equal to the partition function summed over all triangulations having \( k \) boundary edges. The correspondence with the matrix model language is that \( N(k; n) \) is equal to the number of planar diagrams (up to symmetry factors) with \( n \) trivalent vertices and \( k \) external legs; these may be thought of as the dual graphs to the triangulations. Thus, \( p_k \) is given by the matrix model expectation value

\[ p_k = \frac{1}{N} \left\langle \text{Tr} (U^k) \right\rangle. \]  

(6)

so that

\[ \Phi(u, g) = \frac{1}{N} \left\langle \text{Tr} \frac{1}{1 - uU} \right\rangle. \]  

(7)

(Note: in this paper all matrix model expectation values are evaluated in the large \( N \) limit.)

There are two simple ways to solve for the function \( \Phi(u, g) \). In Ref. [32], a recursion relation is derived for the number of triangulations in which no two edges join the same pair of vertices, no internal edge is connected to two external vertices, and no two triangles have
three vertices in common. From the point of view of the dual graphs, this is equivalent to
demanding that the graphs be built from connected, one-particle irreducible diagrams with
no self-energy corrections to the propagator. Similar recursion relations are discussed in Ref.
[33, 34]. Alternatively, \( \Phi(u, g) \) can be evaluated by computing the functional integral over
matrices. In this case, all Feynman graphs are summed over automatically, and the coeffi-
cients of \( \Phi(u, g) \) can be obtained by saddle point methods or by the method of orthogonal
polynomials [9, 10]. Although the numerical values of the coefficients \( \mathcal{N}(k; n) \) are different
in the Tutte and matrix model schemes, the universal behaviour of \( \Phi(u, g) \) (which governs
the continuum limit) is the same. We shall follow Tutte in deriving recursion equations for
the number of triangulations, but we adopt the matrix model conventions for counting: our
dual graphs need not be connected nor one-particle irreducible, and we allow self-energy
corrections to the propagator.

\[ \text{2.2 Generating equation} \]

We now proceed to derive a recursion relation for the coefficients \( \mathcal{N}(k; n) \), which leads to a
single generating equation satisfied by \( \Phi(u, g) \). We consider a triangulation of a disk \( D \) with
\( n \) triangles and \( k \) external edges, one of which is marked. The given triangulation can be
disassembled step by step, by repeatedly removing the marked edge and marking a new edge
on the resulting, smaller triangulation. (A similar approach has been used recently to derive
a discrete Hamiltonian operator for string field theory in the proper time gauge [22, 46, 47].)
Each time a marked edge is removed, this has the effect of either removing a triangle, or
removing a pair of boundary edges. In this way, for each triangulation there is a unique
sequence of “moves” reducing to the trivial triangulation, a single dot. By considering all
possible sequences of moves, we can recursively count the number of triangulations \( \mathcal{N}(k; n) \).

Some conventions are necessary to make the recursion relations well-defined. When the
marked edge is a boundary of a triangle, one removes the triangle attached to the edge; the
resulting triangulation has one less triangle, but one additional edge. We use a convention
that marks the edge which was counterclockwise from the original marked edge in the triangle
that was removed (see Fig. 1). In the case where the marked boundary edge is connected
to another boundary edge, one should remove the pair of edges, leaving two triangulations.
On each of the remaining triangulations, we choose the convention of marking the edge next
to edge which was just removed (see Fig. 2). In Fig. 3, we illustrate the results that can be
obtained upon removing various possible marked edges from a simple triangulation.

The number of triangulations with \( n \) triangles and \( k \) boundary edges is therefore equal
to the number with one fewer triangle and one additional edge, plus the sum of all possible
combinations of two triangulations with the same number of triangles but a total of two
fewer boundary edges:

\[
\mathcal{N}(k; n) = \mathcal{N}(k + 1; n - 1) + \sum_{i=0}^{k-2} \sum_{n'=0}^{n} \mathcal{N}(i; n') \mathcal{N}(k - 2 - i; n - n') .
\] (8)

In order that this recursion relation completely determine \( \mathcal{N}(k; n) \), it is necessary to impose
boundary conditions: there is one trivial triangulation, so that \( \mathcal{N}(0,0) = 1 \), and in addition we specify that \( \mathcal{N}(0;n) = 0 \) for all \( n \geq 1 \). These conditions suffice to recover the counting of the matrix model.

Using (5), it is straightforward to rewrite (8) as an equation for the generating function \( \Phi(u, g) \):

\[
\Phi(u, g) = 1 + \frac{g}{u}(\Phi(u, g) - p_1(g)u - 1) + u^2\Phi^2(u, g)
\]

The first term on the right hand side of (9) fixes the boundary condition \( \mathcal{N}(0;n) = \delta_{n,0} \). The second term corresponds to the first term in (8), and represents the removal of a triangle, as in Fig. 1; the negative terms are subtracted from \( \Phi(u, g) \) since no move that removes a triangle can result in a triangulation with fewer than two edges. The third term corresponds to the second term in (8).

It is important to note that (9) cannot be solved for \( \Phi \) directly, since it contains the function \( p_1(g) \) which is a boundary condition for \( \Phi(u, g) \) that must be computed separately. Equations of this general form have been studied extensively in the literature, and it is possible to show that the requirement that \( \Phi(u, g) \) have a well defined power series expansion in \( u \) and \( g \) around the origin is sufficient to completely determine the power series \( p_1(g) \) from
Figure 3: A simple triangulation is shown at top, with numbers representing different edges which could be marked. The six diagrams below are the results of removing the appropriate marked edge, and show which edge of the new triangulation becomes marked.
We refer the reader to Refs. [32, 33, 34] for details of this procedure for the present example, and for the more complicated cases that we shall encounter in later sections.

## 2.3 Graphical interpretation

The generating equation (9) may be written more compactly by introducing a special derivative operator, which will turn out to be useful in more complicated matrix models where \( \Phi \) will be a function of free variables. We define the operation of \( \partial_u \) on a polynomial in \( u \) so that it removes one power of \( u \) from terms proportional to \( u^k \) (without introducing a factor of \( k \)), and annihilates terms which are independent of \( u \) [39, 42]. Thus,

\[
\partial_u \sum_{k=0}^{\infty} c_k u^k = \sum_{k=1}^{\infty} c_k u^{k-1}.
\]

Such an operator does not obey the usual Leibniz rule; rather,

\[
\partial_u (\Phi \Psi) = (\partial_u \Phi) \Psi + \Phi(u = 0) \partial_u \Psi.
\]

Using this definition, we have

\[
\partial_u \Phi = \frac{1}{u} (\Phi - 1), \quad \partial_u^2 \Phi = \frac{1}{u^2} (\Phi - 1 - up_1),
\]

and (9) reduces to

\[
\Phi(u, g) = 1 + gu\partial_u^2 \Phi(u, g) + u^2 \Phi^2(u, g).
\]

In this notation, the correspondence between the terms in the generating equation and the graphical representation of the basic moves becomes precise, in the sense that every symbol appearing in eq. (13) has a direct interpretation. On the right hand side there are three terms, corresponding to a boundary condition, the move which removes a triangle, and the move which removes two identified boundary edges; each of these three types of terms will appear in the more elaborate generating equations we present in the remainder of the paper. The right hand side of each such equation will begin with a one, which sets the number of triangulations with no boundary edges equal to one. The terms corresponding to removal of a triangle will have a factor of the weight for a single triangle (in this case, \( g \)) and an operation in free variables (in this case, \( u\partial_u^2 \)) which represents replacing the single marked edge with two new edges. The terms corresponding to removing two identified boundary edges will contain a variable for each boundary edge (in this case, two factors of \( u \)) and two functions (in this case, two factors of \( \Phi \)) representing the two triangulations that remain after the removal of the boundary edges. In more complicated models these terms may be multiplied by coupling constants representing the interaction between matter fields, as we shall see in the next section.

\[\text{The notation } u\partial_u^2 \text{ may seem to suggest replacing two edges with a single one, rather than the other way around. This is merely because we have been using language in which the triangulations represented by } \Phi \text{ are disassembled step by step, rather than being built up from nothing; but either sense is equally legitimate.}\]
Eq. (13) is closely related to the Schwinger-Dyson equation for the 1-matrix model, which can be written as

\[ \partial_u \Phi = u \Phi^2 + g \partial_u^2 \Phi. \]  

(14)

It is easy to see that acting by \( \partial_u \) on (13) yields (14). The generating equation includes the information contained within the Schwinger-Dyson equation, as well as additional boundary conditions. This will continue to be the case for more complicated theories described by multi-matrix models, where we shall derive a single equation equivalent to the entire set of Schwinger-Dyson equations plus boundary conditions.

2.4 Higher genus

It is reasonably straightforward to extend the recursion relation (8) from the disk amplitude to an amplitude for a higher genus surface with an arbitrary number of boundaries. We imagine that the set of boundaries is ordered, and that each one has a marked edge, and derive a recursion equation by removing the marked edge on the first boundary loop. The number of triangulations of a surface with genus \( G \) and with \( b \) boundaries, represented by

\[ N^{(b)}_G(k_1, k_2, \ldots, k_b; n), \]  

(15)

with all \( k_a > 0 \), is given by the equation

\[
N^{(b)}_G(k_1, k_2, \ldots, k_b; n) = N^{(b)}_G(k_1 + 1, k_2, \ldots, k_b; n - 1) \\
+ \sum_{a=0}^{b-1} \sum_{\{\sigma_i\}} H_{i=0}^{G-1} \sum_{m=0}^{k_1-2} n \sum_{i=0}^{k_a-1} (i, k_{\sigma_1}, \ldots, k_{\sigma_a}; m) N^{(b-a)}_{G-H}(k_{1-i}, k_{\sigma_{a+1}}, \ldots, k_{\sigma_{a-1}}; n - m) \\
+ \sum_{a=2}^{b} \sum_{k_1^{a-2}}^{k_a} \sum_{i=0}^{k_a^{b-1}} (i, k_1 - i - 2, k_2, \ldots, k_b; n)
\]

(16)

where \( \{\sigma_i\} \) in the second term represents the set of all permutations of \((2, 3, \ldots, b)\) such that \( \sigma_1 < \sigma_2 < \ldots < \sigma_a \) and \( \sigma_{a+1} < \sigma_{a+2} < \ldots < \sigma_{b-1} \), and we set \( N^{(b)}_G(k_1, k_2, \ldots, k_b; n) = 0 \) when any of the \( k_a \)'s are zero (except that, as before, \( N^{(1)}_0(0; 0) = 0 \)).

Once again, each term in this expression has a geometric interpretation (see Fig 4). The first term on the right hand side corresponds to the removal of a single triangle (Fig 4a), while the remaining three correspond to the removal of an identified pair of boundary edges. The second term is related to the second term in (8), derived from the splitting of a single triangulation into all possible combinations of two (Fig 4b). The third term corresponds to the removal of two identified edges which belong to different boundary components; the result is to decrease the number of boundaries by one (Fig 4c). The last term comes from the removal of two identified edges belonging to the same boundary component, such that the resulting triangulation does not split into two; the result is to increase the number of boundaries by one, while decreasing the genus by one (Fig 4d).
Figure 4: For a higher genus surface, there are two new moves (c) and (d), as well as the two moves (a) and (b) needed for the disk amplitude.

One can define generating functions

$$\Phi^{(b)}(u_1, u_2, \ldots, u_b, g, h) = \sum_{G=0}^{\infty} \sum_{\{k_j\}, n=0}^{\infty} h^G N_{G}^{(b)}(k_1, k_2, \ldots, k_b; n) g^n u_1^{k_1} u_2^{k_2} u_3^{k_3} \cdots u_b^{k_b}$$  \hspace{1cm} (17)

for triangulations of a surface with $b$ boundaries of arbitrary genus. It is now possible to convert the recurrence relation (16) into an equation that relates amplitudes for surfaces of arbitrary genus with different numbers of boundaries:

$$\Phi^{(b)}(u_1, \ldots, u_b, g, h) =$$

$$\delta_{b1} + g u_1 \partial^2_{u_1} \Phi^{(b)}(u_1, \ldots, u_b, g, h)$$

$$+ u_1^2 \sum_{a=0}^{b-1} \sum_{\{\sigma_i\}} \Phi^{(a+1)}(u_1, u_{\sigma_1}, \ldots, u_{\sigma_a}, g, h) \Phi^{(b-a)}(u_1, u_{\sigma_a+1}, \ldots, u_b, g, h)$$

$$+ \sum_{j=2}^{b} u_j \frac{d}{du_j} \left[ u_1 u_j D_{u_1, u_j} \Phi^{(b-1)}(u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_b, g, h) \right]$$

$$+ h u_1^2 \sum_{j=2}^{b} \Phi^{(b+1)}(u_1, \ldots, u_{j-1}, u_1, u_j, \ldots, u_b, g, h),$$  \hspace{1cm} (18)

where $\{\sigma_i\}$ is defined as before, the operator $D_{u_1, u_j}$ is defined by

$$D_{u_1, u_j} \Phi^{(b-1)}(u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_b, g, h) =$$

$$\frac{u_1 \Phi^{(b-1)}(u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_b, g, h) - u_j \Phi^{(b-1)}(u_2, \ldots, u_b, g, h)}{u_1 - u_j},$$  \hspace{1cm} (19)
and \( d/du \) is an ordinary derivative operator.

Equation (18) is useful in part because it contains a set of linear relations that permit an iterative solution for generating functions of arbitrary genus and with an arbitrary number of surfaces starting from the solution of Eq. (9) for the disk. For example, the simplest relation contained in that equation is obtained by projecting out the genus zero component \( h^0 \) and taking \( b = 2 \). This gives a relation between the disk and cylinder generating functions (the \( h^0 \) components \( \Phi^{(1)}_0 \) and \( \Phi^{(2)}_0 \) of \( \Phi^{(1)} \) and \( \Phi^{(2)} \)):

\[
\Phi^{(2)}_0(u_1, u_2, g) = 2u_1^2 \Phi^{(1)}_0(u_1, g) \Phi^{(2)}_0(u_1, u_2, g) + gu_1 \partial_{u_1}^2 \Phi^{(2)}_0(u_1, u_2, g) + u_2 \sum_{n=1}^{\infty} \frac{d}{du_2} \left( u_1^2 u_2 \Phi^{(1)}_0(u_1, g) - u_2^2 u_1 \Phi^{(1)}_0(u_2, g) \right).
\]

(20)

Another simple relation exists between the generating functions for the disk, the cylinder and the single-boundary torus (the \( h^1 \) component taking \( b = 1 \)):

\[
\Phi^{(1)}_1(u_1, g) = 2u_1^2 \Phi^{(1)}_0(u_1, g) \Phi^{(1)}_1(u_1, g) + gu_1 \partial_{u_1}^2 \Phi^{(1)}_1(u_1, g) + u_2^2 \Phi^{(2)}_0(u_1, u_1, g).
\]

(21)

This procedure may be continued to obtain equations for arbitrary genus and number of boundary components.

### 3 Ising matter

We now consider 2D gravity coupled to the Ising model. This model corresponds to the \( c = 1/2 \) minimal model CFT coupled to Liouville gravity, and has a simple description as a two-matrix model. As in the previous section, most of the results we describe here have long been understood in terms of the matrix model description of the theory; the novelty of our presentation lies in the purely geometric derivation of a single generating equation written in terms of noncommuting variables.

#### 3.1 Review of model

The coupling of the Ising model to gravity is described by summing over all triangulations of a closed 2-manifold just as in the pure gravity model. Now, however, each triangle carries a label \( U \) or \( V \), corresponding to Ising spins up or down respectively. The weight of a given triangulation \( \Delta \) is defined to be

\[
W(\Delta) = \frac{1}{S(\Delta)} N^{x(\Delta)} g^{n(\Delta)} \left( \frac{1}{1 - c^2} \right)^{uu(\Delta)+vv(\Delta)} \left( \frac{c}{1 - c^2} \right)^{uv(\Delta)}
\]

(22)

where the notation is the same as in (3), with the addition of a new coupling constant \( c \); \( uu(\Delta) \) is the number of edges in \( \Delta \) separating two \( U \) triangles, and similarly for \( uv(\Delta) \) and \( vv(\Delta) \). The coupling \( c \) is related to the usual Ising model coupling \( J \) by

\[
c = e^{-2J}.
\]

(23)
We use $c$ rather than $J$ because this is the natural notation in the description of the theory as a 2-matrix model, with partition function given by [48, 49]

$$Z = \sum_{\Delta} W(\Delta) = \int DU\,DV \, \exp (-NS(U,V)) ,$$

(24)

with

$$S(U,V) = \frac{1}{2} \text{Tr} U^2 + \frac{1}{2} \text{Tr} V^2 - c\text{Tr} UV - \frac{g}{3}(\text{Tr} U^3 + \text{Tr} V^3) ,$$

(25)

where $U$ and $V$ are $N \times N$ hermitian matrices.

### 3.2 Generating equation

We would like to analyze the generating function associated with a disk, as in the pure gravity theory discussed in Section 2. However, in this model it is necessary to specify the configuration of Ising spins on the boundary rather than just the number of edges. Thus, we must associate with each boundary a length $l$, and a string of $l$ labels, each of which is either a $u$ or a $v$. These labels may be thought of as spins lying outside the boundary of the triangulation, meaning that each boundary edge contributes a weight $1/(1 - c^2)$ or $c/(1 - c^2)$ depending on whether the spin on the boundary edge is the same as or different from the spin on the triangle adjacent to that edge; although this may seem to be a counterintuitive convention, it will simplify the following analysis. By choosing an orientation and a marked edge on the boundary, we have a natural ordering to the boundary labels; this ordering is given by beginning with the marked edge and proceeding around the boundary. It is now natural to consider $u$ and $v$ as noncommuting variables; the generating function for the disk can then be written as (throughout this section we suppress the dependence of $\Phi$ on the couplings $g, c$ and the noncommuting variables $u, v$)

$$\Phi = 1 + p_u(u + v) + p_{uu}(u^2 + v^2) + p_{uv}(uv + vu)$$

$$+ p_{uuu}(u^3 + v^3) + p_{uvw}(uvw + uvu + vuv + uvv + vvu) + \ldots$$

(26)

The coefficients $p_{w(u,v)}$ represent the disk amplitude for triangulations with boundary condition specified by $w(u,v)$, a word in the free algebra generated by $u$ and $v$ (i.e., an ordered product of $u$’s and $v$’s). Notice that, due to the symmetry between plus and minus spins, the expectation values are related by $p_{w(u,v)} = p_{w(v,u)}$. (In the models we shall study in subsequent sections, this will no longer be the case.) There are also symmetries under cyclic permutation and reversal of $w(u,v)$. In matrix model language, $p_{w(u,v)}$ is just the usual expectation value of a trace of a product of hermitian matrices (in the large $N$ limit):

$$p_{w(u,v)} = \frac{1}{N} \langle \text{Tr} w(U,V) \rangle ,$$

(27)
Figure 5: A diagram contributing a factor of $N g^2 c^3 / (1 - c^2)^5$ to the expectation value $\langle \text{Tr } U^4 \rangle = N p_{uuuu}$. 

and $\Phi$ is the generating function of planar Green’s functions [37, 41, 42]:

$$\Phi = \frac{1}{N} \sum_{n=0}^{\infty} \langle \text{Tr } (uU + vV)^n \rangle$$

$$= \frac{1}{N} \sum_{w(u,v)} w(u, v) \langle \text{Tr } w(U, V) \rangle .$$

(28)

For example, Figure 5 illustrates a diagram contributing a factor of $N g^2 c^3 / (1 - c^2)^5$ to the expectation value $\langle \text{Tr } U^4 \rangle = N p_{uuuu}$.

We can now, as in the previous section, use purely geometrical arguments to derive a single generating equation satisfied by $\Phi$, taking advantage of the interpretation discussed after Eq. (13) to write down the generating equation directly, without first deriving a recursion relation. We have associated the first symbol in a word $w(u, v)$ with a marked edge on the boundary of a triangulation. If we remove this edge, then as in the pure gravity case there are two possibilities: either the edge is identified with another edge of the boundary (which must also be removed), or the edge is attached to a triangle which can be removed. Unlike the pure gravity case, however, in each of these cases there are multiple possible spin configurations which must be included.

It is useful to define derivative operators $\partial_u$ and $\partial_v$ that are similar to the derivative operator defined in the previous section. (Operators of this form were introduced in Refs. [39, 42].) If $\partial_u$ ($\partial_v$) acts on a word with a $u$ ($v$) at the leftmost position, that $u$ ($v$) is removed; if a $v$ ($u$) is at the leftmost then the term is annihilated. For example:

$$\partial_u(uuuuv + vvu) = uuuv, \quad \partial_v(uuuuv + vv) = vu .$$

(29)

In terms of these operators, the generating equation is

$$\Phi = 1 + \frac{1}{1 - c^2} \left[ u\Phi u\Phi + v\Phi v\Phi + cu\Phi v\Phi + cv\Phi u\Phi + gu\partial_u^2 \Phi + gv\partial_v^2 \Phi + cgv\partial_u^2 \Phi + cgud\partial_v^2 \Phi \right] .$$

(30)
Again, each term corresponds to a possible outcome of removing a boundary edge. The first four terms in the square brackets are associated with the removal of two identified edges: a marked $u$ identified with another $u$, a marked $v$ identified with another $v$, etc. (Note the factor of $c$ appearing when a $u$ is identified with a $v$.) The other four terms correspond to removing a triangle (and therefore have a factor of $g$); they represent the four possibilities of a $u$ or $v$ marked edge being identified with a $u$ or $v$ triangle. A diagrammatic description of the fourth and eighth terms inside the brackets is given in Figure 6.

The generating equation (30) is closely related to the Schwinger-Dyson equations for the corresponding matrix model. The Schwinger-Dyson equations are given by (31):

\[
\begin{align*}
\partial_u \Phi &= \Phi u \Phi + g \partial_u^2 \Phi + c \partial_v \Phi \\
\partial_v \Phi &= \Phi v \Phi + g \partial_v^2 \Phi + c \partial_u \Phi.
\end{align*}
\]

(31)

It is fairly easy to check that these equations are equivalent to (30) except that they do not determine the leading constant in $\Phi$. Thus, the single generating equation contains all the information contained in the pair of Schwinger-Dyson equations. Having a single equation for $\Phi$ in terms of noncommuting variables simplifies the subsequent analysis of such a system, as we shall see in the following sections. Finally, note that it is straightforward to write down the generalization of the higher genus generating equation (18) in the presence of Ising matter.

Figure 6: The removal of a pair of edges of opposite spin results in a term $c u \Phi v \Phi$ on the right hand side of the generating equation. The removal of a triangle with a spin at its center not equal to the spin on its edge results in a term $c g u \partial^2 \Phi$. Note that the two new edges carry the spin of the triangle that was removed.
3.3 Solving the generating equation

Now that we have a single generating equation describing the set of all disk amplitudes, we would like to find a closed form solution for $\Phi$ in terms of noncommuting variables. Unfortunately, however, finding a complete solution to such a nonlinear equation in noncommuting variables seems to be a difficult problem [41, 42]. It is possible to directly compute any specific disk amplitude with fixed boundary length $l$ and a finite power of $g$ by repeated application of (30). However, to understand the continuum limit of the theory we need to have information about amplitudes with arbitrarily large boundaries. In certain cases it has been found possible to find a closed set of loop equations for matrix models which allow the computation of generating functions with certain constraints on boundary configurations [50, 51, 43]. We shall now demonstrate in the generating function approach a simple systematic procedure by which such closed systems of equations can be found and solved. The following analysis closely parallels that of [42].

By factoring out leading terms which depend on $v$, the generating function $\Phi$ can be expanded in terms of functions $\phi_{w(u,v)}$ which only depend on $u$:

$$
\Phi(u, v) = \phi(u) + v\phi_v(u) + v^2\phi_{vv}(u) + uv\partial_u\phi_v(u) + \ldots
$$

(32)

The functions $\phi_{w(u,v)}(u)$ are defined by

$$
\phi_{w(u,v)} = [w(\partial_u, \partial_v)\Phi]_{v=0},
$$

(33)

where $w(u, v)$ begins and ends with $v$. (If $w(u, v)$ begins with a $u$, we can use the cyclic symmetry of the generating function to replace the corresponding function $\phi_{w(u,v)}$ by a derivative of a lower-order function, as we have done in (32).) Thus, $w(u, v)\phi_{w(u,v)}$ is that piece of $\Phi$ containing words consisting of $w(u, v)$ multiplied on the right by a string of $u$'s. (Note that (33) actually extracts the term with the $u$'s and $v$'s in the opposite order, but the functions are equal by symmetry.)

Considering only those terms on each side of (30) with a fixed initial string followed by $u$'s, we can write a closed system of equations for a subset of the functions $\phi_{w(u,v)}(u)$. For example, taking only terms with $u$'s, we have

$$
\phi = 1 + \frac{1}{1 - c^2} \left( u^2\phi^2 + gu\partial_u\phi + cg\phi_{uv} \right). \quad (34)
$$

Taking terms of the form $uv^i$, $uvu^j$, $v^2u^k$ give respectively

$$
\phi_v = \frac{1}{1 - c^2} \left( cu\phi^2 + g\phi_{vv} + cg\partial_u\phi \right),
$$

$$
\partial_u\phi_v = \frac{1}{1 - c^2} \left( u\phi_v + c\phi + g\partial_u^2\phi_v + cg\phi_{vvv} \right),
$$

$$
\phi_{vv} = \frac{1}{1 - c^2} \left( \phi + cu\phi_v + g\phi_{vvv} + cg\partial_u^2\phi_v \right). \quad (35)
$$
This gives us 4 equations in 4 unknowns, each of which is a function of the single commuting variable $u$. These 4 equations can be combined to give a single equation for $\phi$ and its derivatives:

$$0 = gu^3 \phi^3 + (cu^2 - gu + g^2) \phi^2 - g^2 + (c^2u - cu - gu^2 \phi) \partial_u \phi$$

$$+ (gu + cgu + g^2u^2 \phi) \partial^2_u \phi - 2g^2u \partial^3_u \phi + g^3u \partial^4_u \phi.$$  

(36)

As was done in Eq. (12) for pure gravity, we can write the derivatives of $\phi$ in terms of $\phi$, $u$, and a finite number of the individual coefficients $p_u$, $p_{uu}$, and so on. This allows us to rewrite (36) as a pure cubic in $\phi$:

$$f_3 \phi^3 + f_2 \phi^2 + f_1 \phi + f_0 = 0,$$

where

$$f_0 = -g^3 + g^2 (2 - gp_1) u - g (1 + c - 2gp_1 + g^2p_2) u^2$$

$$- (g^2 + g(1+c)p_1 - 2g^2p_2 + g^3p_3 - c(1-c^2)) u^3$$

$$f_1 = g^3 - 2g^2u + g (1 + c) u^2 - (c(1 - c^2) + g^2) u^3 + g (1 - gp_1) u^4$$

$$f_2 = u^3(2g^2 - 2gu + cu^2)$$

$$f_3 = gu^6$$

(38)

Notice that the coefficients $p_u$ and $p_{uu}$ appearing here are not independent; the order $u$ terms in (34) give

$$p_{uu} = (1 - c) p_u.$$  

(39)

The cubic (37) describing homogeneous (fixed spin) disk amplitudes for the Ising model coupled to gravity is well known and has been derived using a variety of matrix model methods [50, 51, 43, 42]. The derivation we have given is completely independent of the matrix model formalism, and depends only on the underlying geometric theory. The existence of a single equation for the noncommuting generating function (30) rather than a pair of Schwinger-Dyson equations has simplified our analysis.

It is straightforward given the disk amplitude to obtain expressions for generating functions in commuting variables for boundary data with multiple domains of like spins. One proceeds iteratively, deriving an expression for the two domain amplitude in terms of the homogeneous amplitudes and so on [43]. Defining $\tau(u, v)$ as the two-domain generating function, including all terms homogeneous in $u$ and $v$, it follows from (30) that

$$\tau - \phi(v) = \frac{1}{1 - c^2} \left[ u\phi(u)u\tau(u, v) + cu\tau(u, v)\phi(v) \right.$$  

$$+ \frac{g}{u}(\tau(u, v) - u\phi_v(v) - \phi(v)) + \frac{cgu}{v^2}(\tau(u, v) - v\phi_v(u) - \phi(u)) \right]$$

(40)

which is linear in $\tau(u, v)$. (Note that in this equation $u$ and $v$ act as ordinary commuting variables, and we have converted the derivative operators into their algebraic expressions.) Similar relations can easily be derived for the higher domain generating functions.
3.4 Alternate boundary conditions

We have shown how noncommuting variables can be used to encode the boundary data used in a sum over triangulations coupled to Ising matter. However, the decision to describe the boundary as a sequence of plus and minus spins is not unique. In this subsection we briefly describe an alternative description of the Ising theory, in which the boundary condition is specified by labelling the vertices between boundary edges as either “stick” or “flip,” depending on whether the spins on either side are like or unlike. We outline the derivation of the cubic equation corresponding to all-stick boundary conditions, which is equivalent to (37). Although we do not derive any results here which cannot be found using standard matrix model methods, there are certain correlation functions which can be derived using this alternative formulation which appear to be inaccessible using the matrix model language. An example of such a calculation appears in [44].

The generating equation in the flip/stick variables may be derived using the kind of geometric argument used to find (30). For example, using the symbols $f$ and $s$ for flip and stick, the term $cgf^2\partial_3^2\Phi/(1-c^2)$ will arise when a marked edge bounded on each side by a flip is connected to a triangle, which when removed reveals two edges (and thus three vertices) labelled by $sss$. This corresponds in the $u, v$ variables to removing a plus spin which is surrounded by two minuses and connected to a minus triangle, or the same operation with plus and minus interchanged. Taking all possible such moves into account, we obtain

$$\Phi = 1 + \frac{1}{1-c^2}\left\{g[(s+cf^2)\partial_s]+(cs+s f)\partial_f]\partial_3^2\Phi + g[(fs+csf)\partial_s+(c^2+f^2)\partial_f]\partial_s\partial_f\Phi \right.$$ 

$$+\frac{1}{2}\left[(s^2+cf^2)(\partial_s\Phi f+\partial_f\Phi s)+(fs+csf)\partial_s^2\Phi f+\partial_f^2\Phi s\right]$$

$$+\frac{1}{2}\left[(s^2+cf^2)(2+\partial_s\Phi f+\partial_f\Phi s)+(fs+sf)(\partial_s\Phi f+\partial_f\Phi s)\right](2+s\partial_s\Phi + f\partial_f\Phi)$$

$$+g(1+c)spss\right\}$$ (41)

The leading term of $\Phi$ is 1, but all other terms appear twice corresponding to the symmetry induced by interchanging pluses and minuses on the boundary.

From here, we can define as before a set of functions $\phi_{w(s,f)}(s)$ and look for a system of equations which close. One such set of four equations is given by

$$\phi_- = 1 + \frac{1}{1-c^2}\left[\frac{1}{2}s^2(2+s\partial_s\phi_-)+g\left(s^2\partial_3^2\phi_- + spss\right) + cgs\left(s\phi_{fsf} + pss\right)\right]$$

$$\phi_{ff} = \frac{1}{1-c^2}\left[\frac{1}{2}(2+s\partial_s\phi_-)^2 + g\phi_{fsf} + c\partial_3^2\phi_-\right]$$

$$\phi_{ff} = \frac{1}{1-c^2}\left[\frac{1}{2}s(2+s\partial_s\phi_-)(s\phi_{ff} + ps)\right.$$

$$+c(2+s\partial_s\phi_-) + cg\left(s\phi_{fsf} + pss\right) + g\left(s\partial_3^2\phi_{ff} + p_{fsf}\right)\right]$$

$$\phi_{fsf} = \frac{1}{1-c^2}\left[\partial_s\phi_- + \frac{1}{2}c(2+s\partial_s\phi_-)(p_s + s\phi_{ff})\right.$$
\[ + g \phi_{ssf} + cg \partial^2 \phi_{ff} \]

where the homogeneous disk amplitude with all sticks is represented by \( \phi_-(s) \). These equations are exactly equivalent to the four equations in (34) and (35) as can be seen by noting the relations

\[ \phi_-=2\phi-1, \quad s\phi_{ff}+p_s=2\phi_v, \quad s\phi_{ssf}+p_{ss}=2\phi_{vv}, \quad s\phi_{ssf}+p_{sss}=2\phi_{vvv} \]

where the factors of 2 arise because of the interchange symmetry. The cubic equation obtained by solving (42) for \( \phi_-(s) \) is therefore exactly equivalent to the equation for the homogeneous disk amplitude \( \phi(u) \).

Again, it is straightforward to extend the generating equation (41) to arbitrary genus. This extension turns out to be useful for computing the correlation functions of operators on the sphere [44].

### 4 The dual Ising theory

One of the most remarkable properties of the Ising model on a fixed lattice is the Kramers-Wannier duality symmetry, which states that the partition function of the Ising model on a lattice \( L \) at a fixed temperature \( T \) is equal to the partition function on the dual lattice \( \hat{L} \) at a temperature \( T' \sim 1/T \). By putting spins on the vertices of a random triangulation, rather than on the triangles, it is possible to construct a dual version of the random-surface Ising theory discussed in the previous section. The resulting model can be written as a two-matrix model, and is a special case of a more general O(\( n \)) loop model [52, 53, 54, 55, 56, 57]. In this section we consider this dual model, and show that the homogeneous disk amplitude of the dual model satisfies a cubic equation very similar to (36); an equivalent cubic was derived in [56] using different methods. This homogeneous disk amplitude for the dual model is precisely equal to the disk amplitude of the original Ising theory with free boundary conditions.

#### 4.1 Dual model

In the dual model, we associate a spin with each polygon in the dual triangulation. The model can be described in terms of combinatorial data on the original triangulation by taking advantage of the fact that any dual spin configuration is determined (up to an overall sign) by a loop graph on the original triangulation – that is, by a set of edges which divide regions of opposite spin (see Figure 7). Each edge of the triangulation (or the corresponding edge of the dual triangulation) is labeled with a \( Y \) if it separates different spins (i.e., if a domain boundary passes through it), and with an \( X \) if it connects identical spins. We then sum over all loop graphs on triangulations by summing over all triangulations and labelings such that the boundary edges of each triangle either have 3 \( X \) labels, or 1 \( X \) label and 2 \( Y \) labels. The
Figure 7: A sample triangulation contributing to the dual model. The dual graph, consisting of polygons which meet at trivalent vertices, is shown with doubled lines. Putting the spins on the dual graph rather than the original triangulation results in a different coupling between the matter and gravity in the two models. The edges labeled Y (denoted by a thick line inside the double lines) define the boundaries between domains of like spin, and thus determine the matter configuration up to a sign. Note that a boundary condition with all X’s in the dual theory is equivalent to a homogeneous boundary configuration for the dual spins.
weight given to each triangulation is
\[ W(\Delta) = \frac{1}{S(\Delta)^N} \chi^n(\Delta) \left( \frac{1}{1-c} \right)^{x(\Delta)} \left( \frac{1}{1+c} \right)^{y(\Delta)}, \] (44)
where \( x(\Delta), y(\Delta) \) are the number of \( X, Y \) edges in \( \Delta \) respectively. Putting the spins on
the vertices of the triangulation leads to a different coupling of the Ising model to gravity
than that considered in the previous section (see Figure [F]). We shall continue to refer to the
model previously introduced as the Ising theory, and we will refer to the theory defined by
Eq. (44) as the dual theory.

A matrix model description of the partition function of the dual theory is \[ 54, 56 \]
\[ Z = \int D X D Y \exp \left( -N \left[ \frac{1-c}{2} \text{Tr} \ X^2 + \frac{1+c}{2} \text{Tr} \ Y^2 - \frac{\hat{g}}{3}(\text{Tr} \ X^3 + 3\text{Tr} \ XY^2) \right] \right). \] (45)
It is a remarkable fact that the matrix model (45) is precisely equivalent to the original
model \[ 24, 54, 57 \]. This equivalence can be explicitly seen by replacing
\[ X \rightarrow \frac{1}{\sqrt{2}}(U + V) \]
\[ Y \rightarrow \frac{1}{\sqrt{2}}(U - V) \]
\[ \hat{g} \rightarrow g/\sqrt{2}. \] (46)
Because of this correspondence, the partition functions on a closed manifold for the two
theories are precisely the same. Nevertheless, this does not by itself imply that Kramers-
Wannier duality is preserved in the presence of gravity; the transformation (46) exchanges
the disorder and spin operators, and takes homogeneous \( X \) boundary conditions in the dual
model to free boundary conditions in the original model (and vice versa). A true duality
would not exchange operators and would relate amplitudes with the same types of boundary
conditions. In \[ 44 \] we examine the issue of duality between these models more closely.

4.2 Generating equation
We again study the generating function for a disk. Boundary conditions on the disk are
specified by a string of labels \( x \) or \( y \), whereas in the Ising theory the labels are taken to live
outside the boundary of the disk. Taking into account that there must be an even number
of \( y \)'s on the boundary (since each \( y \) on the boundary is the endpoint of a loop), we can
expand the generating function as
\[ \Phi(x, y) = 1 + px + pxx^2 + pyy^2 + pxxx^3 + pxyy(xy + yxy + yyy) + \cdots, \] (47)
where the coefficients may be thought of as expectation values:
\[ p_{w(x,y)} = \frac{1}{N} \langle \text{Tr} \ w(X,Y) \rangle. \] (48)
We can now derive the generating equation for $\Phi$ using the geometric techniques of the previous sections. We obtain

$$\Phi = 1 + \frac{1}{1 - c} \left[ \hat{g} x \partial_x^2 \Phi + \hat{g} x \partial_y^2 \Phi + x \Phi x \Phi \right] + \frac{1}{1 + c} \left[ \hat{g} y \partial_y \partial_x \Phi + \hat{g} y \partial_x \partial_y \Phi + y \Phi y \Phi \right].$$

(49)

The terms in (49) correspond to moves that are analogous to those for the Ising theory. Note, however, that the equation is not symmetric in $x$ and $y$. Again there is a direct relation between (49) and the Schwinger-Dyson equations

$$\partial_x \Phi = 1 + \frac{1}{1 - c} \left[ \hat{g} \partial_x^2 \Phi + \hat{g} \partial_y^2 \Phi + \Phi x \Phi \right]$$

$$\partial_y \Phi = 1 + \frac{1}{1 + c} \left[ \hat{g} \partial_y \partial_x \Phi + \hat{g} \partial_x \partial_y \Phi + \Phi y \Phi \right]$$

(50)

for the matrix model.

As we did in the Ising theory, we may define functions which depend only on the single variable $x$:

$$\phi_{w(x,y)}(x) = [w(\partial_x, \partial_y) \Phi]_{y=0},$$

(51)

where $w(x, y)$ begins and ends with $y$. It is then possible to expand $\Phi(x, y)$ as

$$\Phi(x, y) = \phi(x) + x y \phi_{yy}(x) + y x y \phi_{xxy}(x) + y y y \phi_{yyyy}(x) + \cdots$$

(52)

Bearing in mind the asymmetry between $X$ and $Y$, we can use (49) to compute the generating function $\phi(x)$ for homogeneous $X$ boundary conditions. As mentioned above, $\phi$ has two interpretations: it is identically equal to the generating function for free boundary conditions for the Ising theory, while it can also be interpreted as a generating function for homogeneous boundary conditions for the dual Ising theory of (45).

It is straightforward to derive the closed set of five equations

$$\phi = 1 + \frac{1}{1 - c} \left[ \phi^2 x^2 + \hat{g} x \partial_x^2 \phi + \hat{g} x \phi_{yy} \right]$$

$$\phi_{yy} = 1 + \frac{1}{1 + c} \left[ \phi + \hat{g} \phi_{xxy} + \hat{g} \partial_x \phi_{yy} \right]$$

$$\phi_{xxy} = 1 + \frac{1}{1 + c} \left[ \phi \partial_x \phi + \hat{g} \phi_{xxy} + \hat{g} \partial_x \phi_{xy} \right]$$

$$\phi_{xyx} = 1 + \frac{1}{1 - c} \left[ \phi \partial_x \phi_{yy} + \hat{g} \phi_{xxy} + \hat{g} \partial_x \phi_{yy} \right]$$

$$\partial_x \phi_{yy} = 1 + \frac{1}{1 - c} \left[ x \phi \phi_{yy} + \hat{g} \phi_{yyyy} + \hat{g} \partial_x^2 \phi_{yy} \right]$$

(53)

from (49). These equations can be combined to solve for $\phi$. Once again replacing the derivatives of $\Phi$ with algebraic expressions, we obtain a cubic equation for $\phi$:

$$\hat{f}_3 \phi^3 + \hat{f}_2 \phi^2 + \hat{f}_1 \phi + \hat{f}_0 = 0$$

(54)

22
with coefficients
\[
\hat{f}_0 = -4c\hat{g}^2(1 + xp_x + x^2p_{xx}) + 2c\hat{g}x(3 - c)(1 + p_xx) - 2cx^2(1 - c^2) - \hat{g}^2x^2
\]
\[
\hat{f}_1 = 4c\hat{g}^2 - 2c\hat{g}x(3 - c) + 2cx^2(1 - c^2) + \hat{g}x^3(1 - 3c) - 2\hat{g}^2p_xx^3
\]
\[
\hat{f}_2 = x^2(\hat{g}^2 - \hat{g}x(1 - 5c) - 2cx^2(1 + c))
\]
\[
\hat{f}_3 = \hat{g}x^5
\]  
(55)

As in the case of the Ising theory, it is straightforward, given an expression for the homogeneous disk amplitude, to derive expressions for amplitudes with nontrivial domain structure on the boundary through an iterative procedure. In this case, however, the objects of interest are the generating functions with a finite (even) number of Y’s on the boundary. For example, the two-domain generating function \(\tau(x_1, x_2)\) is defined to be the generating function for all boundary configurations with a Y (the starting point of the domain wall) followed by a series of X’s followed by another Y (the end point of the domain wall) followed by another series of X’s. It follows from (49) that \(\tau(x_1, x_2)\) is given by the equation
\[
\tau(x_1, x_2) = \frac{1}{1 + c} \left[ \phi(x_1)\phi(x_2) + \hat{g}x_1^{-1}(\tau(x_1, x_2) - \phi_{yy}(x_2)) + \hat{g}x_2^{-1}(\tau(x_1, x_2) - \phi_{yy}(x_1)) \right]
\]  
(56)

An analysis of the continuum limit of the homogeneous and higher domain amplitudes for the dual model is given in Ref. 14.

5 General cubic two-matrix model

The previous two sections have described the Ising and dual Ising theories, which correspond in the continuum limit to two distinct descriptions of the \(c = 1/2\) CFT coupled to quantum gravity. In this section we investigate the most general two-matrix model with cubic interactions. This general class of matrix models cannot be solved by saddle point or orthogonal polynomial methods, since the form of the interaction terms prevents the angular degrees of freedom from being integrated out using the methods of 58, 59. However, by using the techniques of this paper it is possible to find the homogeneous disk amplitude for any model in this class.

After a description of the general model and a discussion of the quartic equation satisfied by the homogeneous disk amplitude in the general case, we look at a simple one-parameter family of models which interpolate between the Ising and dual Ising theories. Every model in this family has a homogeneous disk amplitude which satisfies a quartic equation. These homogeneous disk amplitudes can be interpreted as the disk amplitude for the regular Ising theory coupled to gravity in the presence of a boundary magnetic field. This set of models is of considerable intrinsic interest, and we shall study it in more detail in a subsequent paper 45.
5.1 Definition and generating equation

The general two-matrix model with cubic interactions is specified by its action, which we may write in the form

\[
S = \frac{1}{2(\alpha \beta - \gamma^2)} \text{Tr} (\alpha Q^2 + \beta R^2 - 2\gamma QR) - \frac{1}{3} \text{Tr}(aQ^3 + 3bQ^2R + 3dQR^2 + eR^3) .
\] (57)

The coefficients of the quadratic terms have been chosen to yield a simple form for the propagator, which in the \((Q, R)\) basis is

\[
\begin{pmatrix}
\beta & \gamma \\
\gamma & \alpha
\end{pmatrix} .
\] (58)

Note that \(\alpha\), which multiplies \(Q^2\) in the action, is what propagates \(R\) to \(R\), and vice-versa.

From the point of view of triangulations, this theory may be thought of as one in which the edges of each triangle are labeled either \(Q\) or \(R\); weights are assigned to each triangle according to the labels on its three edges and to each pair of neighboring edges of triangles or boundaries. Thus, the weight of a given triangulation is

\[
W(\Delta) = \frac{1}{S(\Delta)} N^{\chi(\Delta)} a^{qqq(\Delta)} b^{qqr(\Delta)} d^{rrr(\Delta)} e^{rrr(\Delta)} \alpha^{rr(\Delta)} \beta^{qq(\Delta)} \gamma^{qr(\Delta)} ,
\] (59)

where \(qqq(\Delta)\) is the number of triangles with three \(Q\) edges, \(qq(\Delta)\) is the number of times an \(Q\) edge is identified with another \(Q\) edge, and so on. The partition function is given by

\[
Z = \sum_{\Delta} W(\Delta) = \int DQ DR \exp [-NS(Q, R)] .
\] (60)

As in the previous sections, we can use geometric arguments of the kind leading to (30) and (49) in order to derive an equation satisfied by the generating function associated with a disk. A boundary of length \(l\) is specified by a string of \(l\) labels \(q\) or \(r\); however, these labels do not correspond to “spins” in any simple way. Once again, a marked edge on the boundary is either attached to a triangle or to another boundary edge. Taking into account all possible combinations of labelings for the marked edge and the neighboring triangle or boundary edge, we find that the generating function \(\Phi(q, r)\) satisfies

\[
\Phi(q, r) = 1 + \beta q \Phi q \Phi + \gamma (q \Phi r \Phi + r \Phi q \Phi) + \alpha r \Phi r \Phi
\]

\[
+ (a'q + b'r)\partial_q^2 \Phi + (b''q + d''r)(\partial_q \partial_r \Phi + \partial_r \partial_q \Phi) + (d'q + e'r)\partial_r^2 \Phi ,
\] (61)

where we have defined the new variables

\[
a' = \beta a + \gamma b ,
\]

\[
b' = \alpha b + \gamma a ,
\]

\[
b'' = \beta b + \gamma d ,
\]

\[
d' = \beta d + \gamma e ,
\]

\[
d'' = \alpha d + \gamma b ,
\]

\[
e' = \alpha e + \gamma d .
\] (62)
The form of these new variables has a geometric interpretation. As an example consider $d'$, the coefficient of $q\partial^2\Phi$. This term originates from a marked edge labeled $q$ being attached to a triangle, which when removed leaves two edges each labeled by $r$. In such a situation the triangle to be removed could have been labeled either $qrr$ or $rrr$. In the first case the weight is given by multiplying the weight $d$ associated with an $qrr$ triangle by the weight $\beta$ which propagates $q$ to $q$, while in the second case we multiply $e$ (for a $rrr$ triangle) by $\gamma$ (for an $q-r$ propagator). It is therefore natural to define $d' = \beta d + \gamma e$, and similarly for the rest of (62).

It is straightforward to check that special cases of the generating equation (61) yield the previous equations (30) for the Ising theory and (49) for the dual theory.

5.2 Homogeneous boundary conditions

As with the models studied in the previous sections, there are no general techniques for solving the generating equation (61), but we can derive a polynomial equation for the generating function $\phi$ restricted to homogeneous boundary conditions (all $Q$’s on the boundary). Unfortunately, in the general 2-matrix model the resulting equation for $\phi$ is a quartic with extremely complicated coefficients, which it is impractical to present explicitly. Here we will describe the steps followed in deriving the quartic.

We begin by expanding $\Phi$ as

$$\Phi(q, r) = 1 + p_q q + p_r r + p_{qq} q^2 + p_{rr} r^2 + p_{qr} (q r + r q) + p_{qqq} q^3 + p_{rrr} r^3 + p_{qqr} (q qr + q r q + r qq) + p_{qrr} (q rr + r qr + r qr) + \ldots \quad (63)$$

We then define generating functions which depend on $q$ alone, as in the previous sections:

$$\phi_{w(q, r)}(q) = [w(\partial_q, \partial_r)\Phi]_{r=0} \quad (64)$$

To solve for $\phi$, we would like to find a set of equations which closes on a set of these generating functions. It turns out that there are 10 equations which close on the quantities
\[(\phi, \phi_r, \phi_{rr}, \phi_{rrr}, \phi_{qq}, \phi_{qqq}, \phi_{qqrr}, \phi_{qqrr}):\]

\[
\begin{align*}
\partial_q \phi &= \beta q\phi^2 + d'\phi_{rr} + a'\partial_q^2 \phi + 2b''\partial_q \phi_r \\
\phi_r &= \gamma q\phi^2 + c'\phi_{rr} + b'q^2 \phi + 2d''\partial_q \phi_r \\
\partial_q \phi_r &= \gamma \phi + \beta q\phi \phi_r + b'q^2 \phi_{qrr} + d'\phi_{rrr} + a'\partial_q^2 \phi_r + 2b''\partial_q \phi_{rr} \\
\partial_q \phi_{rr} &= \gamma p_q + \beta \phi_r + \beta q\phi \partial_q \phi_r + b'q^2 \phi_{qrr} + d'\phi_{rrr} + a'\partial_q^2 \phi_r + 2b''\partial_q \phi_{rr} \\
\phi_{rr} &= \alpha \phi + \gamma q\phi \phi_{rr} + d''q^2 \phi_{qr} + e'\phi_{rrr} + b'\partial_q^2 \phi_r + d''\partial_q \phi_{rr} \\
\phi_{qr} &= p_q \alpha \phi + \gamma \phi_r + b'q^2 \phi_{qrr} + e'\phi_{rrr} + \gamma q\phi \partial_q \phi_r + b'\partial_q^2 \phi_r + d''\partial_q \phi_{qrr} \\
\phi_{rr} &= p_q \alpha \phi + \alpha \phi_r + \gamma q\phi \phi_{rr} + d''q^2 \phi_{qr} + e'\phi_{rrr} + b'\partial_q^2 \phi_r + d''\partial_q \phi_{rr} \\
\phi_{qrr} &= 2 \gamma \phi_r + \beta q\phi^2 + a'\phi_{qq} + 2b''\phi_{qrr} + d'\phi_{rrr} \\
\phi_{qq} &= 2 \alpha \phi_r + \gamma q\phi^2 + b'\phi_{qq} + 2d''\phi_{qrr} + e'\phi_{rrr} \\
\end{align*}
\] (65)

The first eight of these equations come from considering terms in (61) which begin with a certain string; for example the third equation (for \(\partial_q \phi_r\)) comes from terms of the form \(qrrq\ldots\). The last two equations come from terms of the form \(qrrq\ldots qqrr\) and \(rrqq\ldots qqqr\), which are related by cyclic symmetry to the functions \(\phi_{qq}\) and \(\phi_{qqr}\), respectively.

These equations may be transformed into algebraic relations by replacing the derivatives as in the previous two sections; the resulting system of equations can be used to derive a single quartic equation involving only \(\phi\).

Note that we can cut down somewhat on the number of undetermined variables by looking at equations (65) order by order. These give the following relations between the constants:

\[
\begin{align*}
p_q &= d'p_{rr} + a'p_{qq} + 2b''p_{qr} \\
p_{qq} &= \beta + d'p_{qrr} + a'p_{qq} + 2b''p_{qqr} \\
p_r &= e'p_{rr} + b'p_{qq} + 2d''p_{qr} \\
p_{qr} &= \gamma + e'p_{qrr} + b'p_{qq} + 2d''p_{qqr} \\
p_{qq} &= \gamma + b''p_{qrr} + d'p_{rrr} + a'p_{qqr} \\
p_{qrr} &= \alpha + 2d''p_{qrr} + e'p_{rrr} + b'p_{qqr} \\
\end{align*}
\] (66)

These allow \((p_{qq}, p_{rr}, p_{qq}, p_{qqr}, p_{qqr}, p_{qrr}, p_{rrr})\) to be expressed in terms of the three constants \((p_q, p_r, p_{qr})\), although we shall not present the explicit results here.

### 5.3 Models interpolating between Ising and dual models

In Sections 3 and 4 we discussed models in which Ising spins were coupled directly to triangulations and their duals. These two geometrically distinct formulations of the \(c = 1/2\)
theory are related by a simple coordinate transformation, as was discussed in section 4. In this section, we explore a continuous 1-parameter family of models which interpolate between the Ising and dual Ising theories. This entire family of theories are related to the original Ising theory by linear transformations on the defining hermitian matrices. The homogeneous disk amplitude in these interpolating theories can be interpreted physically as the disk amplitude in the original Ising theory where an external magnetic field is imposed on the boundary. Recently, there has been much work on understanding the effects of boundary magnetic fields on the Ising CFT in flat space [60, 61, 62]; the discussion here provides a formalism by which similar questions can be asked of the theory after coupling to 2D gravity. A more detailed investigation of the physical properties of this theory will be carried out in a separate publication [45].

The original Ising theory is defined by the action

$$S = \frac{1}{2} \text{Tr}(U^2 + V^2) - c \text{Tr} UV - \frac{g}{3} \text{Tr}(U^3 + V^3),$$

(67)

while the dual theory is defined by

$$S = \frac{1 - c}{2} \text{Tr} X^2 + \frac{1 + c}{2} \text{Tr} Y^2 - \frac{\hat{g}}{3} \text{Tr}(X^3 + 3XY^2).$$

(68)

The two models are related by the transformation (46). We now consider the more general continuous family of theories defined by the following transformation of the variables of the original Ising theory:

$$Q = sU + tV,$$

$$R = s'U + t'V.$$

(69)

We obtain a theory of the two matrices $Q$ and $R$, with action given by (57), where the parameters are expressed in terms of $s, t, s', t'$ and the couplings $c$ and $g$ of the original model:

$$\alpha = \frac{1}{1 - c^2}(s'^2 + t'^2 + 2cs't')$$

$$\beta = \frac{1}{1 - c^2}(s^2 + t^2 + 2cst)$$

$$\gamma = \frac{1}{1 - c^2}(ss' + tt' + cst + cs't')$$

$$a = \frac{g}{(st' - ts')^3}(t'^3 - s'^3)$$

$$b = \frac{g}{(st' - ts')^3}(ss'^2 - tt'^2)$$

$$d = \frac{g}{(st' - ts')^3}(t'^2t' - s'^2s')$$

$$e = \frac{g}{(st' - ts')^3}(s^3 - t'^3)$$

(70)
The original Ising theory corresponds to \( s = t' = 1, t = s' = 0 \), while the dual Ising theory corresponds to \( s = t = -s' = t' = 1/\sqrt{2} \), with the identification \( \hat{g} = g/\sqrt{2} \).

The generating equation for this set of models may be derived from the geometric arguments used in previous sections, or directly from substitution of (70) into (61):

\[
\Phi(q, r) = 1 + \frac{1}{1-c^2} \left[ (s^2 + t^2 + 2cst)q\Phi q\Phi + (s's' + ctst' + ctt' + 2cs^2t')q\partial_q^2\Phi + (ss' + t't')r\partial_r^2\Phi \right] + \frac{g}{1-c^2} \left[ (css'^2 + ts'^2 + ts'^2 + ctt'^2)q\partial_q^2\Phi + (s' + t')(cs^2 + s't' - ct'^2)r\partial_r^2\Phi \right] + \frac{g}{1-c^2} \left[ (css'^2 + t's'^2 + st't' + ctt'^2)r\partial_r^2\Phi \right] + \frac{g}{1-c^2} \left[ (css'^2 + ts'^2 + t's'^2 + ctt'^2)q\partial_q^2\Phi + (s + t)(cs^2 + st - c^2)t^2)q\partial_q^2\Phi \right] + \frac{g}{1-c^2} \left[ (css'^2 + t's'^2 + s'^2t' + ctt'^2)r\partial_r^2\Phi \right].
\]

In a similar fashion, we may substitute (70) into the quartic satisfied by the generating function \( \phi \) for homogeneous boundary conditions in the general 2-matrix model, to obtain a new quartic which is somewhat simpler. Since \( \phi \) describes boundary conditions which are all \( Q \)'s, we expect that the quartic will be independent of \( s' \) and \( t' \) (which merely specify \( R \) in terms of the original variables \( U \) and \( V \)), and this turns out to be the case. We do not present the quartic here, but will examine it as part of a discussion of the interpolating models in Ref. [45].

The generating function \( \phi(q) \) describes the amplitude for a disk with all \( Q \)'s on the boundary. Since \( Q = sU + tV \), the coefficient of \( q^n \) in the expansion for \( \phi \) is the correlation function

\[
\left\langle \frac{1}{N} \text{Tr}(sU + tV)^n \right\rangle.
\]

We can therefore calculate the disk amplitude for the original Ising theory with a family of different boundary conditions, ranging from homogeneous \( (s = 1, t = 0) \) to completely free \( (s = t) \). Writing

\[
\begin{align*}
  s &= \alpha e^h \\
  t &= \alpha e^{-h}
\end{align*}
\]

where \( \alpha \) is a normalization constant, we see that each choice of \( s, t \) corresponds to a choice of boundary magnetic field. Homogeneous boundary conditions correspond to an infinite boundary field, while free boundary conditions correspond to a vanishing boundary field. We shall discuss the continuum limit and interpretations of these boundary conditions in Ref. [45].

28
6 3-state Potts model

As mentioned previously, the approach to discretized gravity theories via generating equations in free variables, when applied to matrix model theories, may be thought of as an extension of the technique of Schwinger-Dyson equations. Although such equations are in principle applicable to models where conventional methods (such as orthogonal polynomials) are inadequate, in practice it has proven difficult to take advantage of this greater generality. One set of models for which orthogonal polynomials seem to fail is those in which the target space graph (defining the matter fields) contains a nontrivial cycle, since the nontrivial cycle prevents simultaneous diagonalization of the matrices [58, 59]. There is some evidence that loop equation methods also fail for these models [2], although to our knowledge a clear understanding of why this failure occurs is still lacking.

The simplest theory with a nontrivial cycle is the 3-state Potts model. In fact this model is solvable by conventional matrix model techniques, as there exists a change of variables which allows the matrices to be diagonalized [63, 64, 65]; presumably the same change of variables would allow solution by the methods considered here. Nevertheless, in order to illustrate the difficulties encountered when a nontrivial cycle is present, in this section we examine the 3-state Potts model in its original form, and discuss the obstacles to finding a solution for the homogeneous disk amplitude.

In the 3-state Potts model we associate to each triangle one of three “spins,” labelled by the free variables $x_0$, $x_1$, and $x_2$. There is a nearest-neighbor interaction which is the same for any two neighboring unequal spins. (The target graph may therefore be thought of as a triangle with the three free variables labelling the vertices.) The weight associated with each triangulation $\Delta$ is

$$W(\Delta) = \frac{1}{S(\Delta)} N^\chi(\Delta) g^n(\Delta) e^{\nu(\Delta)},$$  \hspace{1cm} (74)

where $\nu(\Delta)$ is the total number of edges which connect unlike spins. The partition function may be cast in matrix model form, in which case it becomes an integral over three matrices $X_0, X_1, X_2$, with an action given by

$$S = \frac{1}{2(1+c-2c^2)} \text{Tr} \left[ (1+c)(X_0^2 + X_1^2 + X_2^2) - 2c(X_0X_1 + X_0X_2 + X_1X_2) \right] + \frac{2}{3} \text{Tr}(X_0^3 + X_1^3 + X_2^3).$$  \hspace{1cm} (75)

Once again we have chosen the quadratic terms in the action such that the propagator takes on a simple form.

From the geometric arguments familiar from previous sections, we derive a generating

\[ \text{We would like to thank Matthias Staudacher for explaining this issue to us.} \]
The equation governing the disk amplitude in this model:

\[
\Phi(x_0, x_1, x_2) = 1 + x_0 \Phi(x_0 + c x_1 + c x_2) + x_1 \Phi(c x_0 + x_1 + c x_2) + x_2 \Phi(c x_0 + c x_1 + x_2) + g(x_0 + c x_1 + c x_2) \partial_0^2 \Phi + g(c x_0 + x_1 + c x_2) \partial_1^2 \Phi + g(c x_0 + c x_1 + x_2) \partial_2^2 \Phi.
\]

(76)

Continuing to follow the procedure outlined in previous sections, from (76) we derive a set of seven equations in seven functions (\(\phi, \phi_1, \phi_{11}, \phi_{122}, \phi_{1111}, \phi_{1122}\)) of the single variable \(x_0\):

\[
\begin{align*}
\phi &= 1 + 2gc x_0 \phi_{11} + gx_0 \partial_0^2 \phi + x_0^2 \phi^2 \\
\phi_1 &= g(1 + c) \phi_{11} + gc \partial_0^2 \phi + cx_0 \phi^2 \\
\phi_{11} &= gc \phi_{122} + g \phi_{111} + gc \partial_0^2 \phi_1 + cx_0 \phi_1 + \phi \\
\partial_0 \phi_1 &= gc \phi_{122} + gc \phi_{111} + g \partial_0^2 \phi_1 + x_0 \phi_1 + c \phi \\
\phi_{111} &= gc \phi_{1122} + g \phi_{1111} + gc \partial_0^2 \phi_{11} + \phi_{11} + p_1 \phi + cx_0 \phi_{11} \\
\phi_{122} &= g \phi_{1122} + gc \phi_{1111} + gc \partial_0^2 \phi_{11} + c \phi_1 + cp_1 \phi + cx_0 \phi_{11} \\
\partial_0 \phi_{11} &= gc \phi_{1122} + gc \phi_{1111} + g \partial_0^2 \phi_{11} + c \phi_1 + cp_1 \phi + x_0 \phi_{11}
\end{align*}
\]

(77)

We might expect to be able to solve these equations to get a single equation for \(\phi(x_0)\). However, closer inspection reveals that they are not independent, with one redundancy hidden among them.

It is possible to uncover a geometric understanding of why these equations are redundant. Two of the equations – the one for \(\phi_1\) and the one for \(\partial_0 \phi_1\) – can be thought of as governing what happens when an edge is removed from a disk with a single \(x_1\) and a number of \(x_0\)'s; the first comes from removing the \(x_1\) edge, and the second from removing the \(x_0\) edge immediately next to the \(x_1\). We could imagine acting on such a disk by removing both the \(x_1\) edge and the \(x_0\) next to it, in either order; the resulting expressions must be compatible, and must be equivalent to performing the appropriate operations on the right hand sides of the two equations. But the result of performing such operations on the right hand sides is already contained in the other equations in this set (77), so there must be a consistency condition, reducing the number of independent equations by one. On the other hand, the two equations for \(\phi_{11}\) and \(\partial_0 \phi_{11}\) do not share this problem, since the right hand side of the equation for \(\partial_0 \phi_{11}\) contains \(\phi_{1111}\) and \(\phi_{1122}\), and the act of removing an edge from the triangulations represented by these quantities is not contained in the seven equations. We therefore are left with six equations in seven unknowns, insufficient to find a single equation for \(\phi\).

By expanding the set of unknowns to include the three additional quantities (\(\phi_{11111}, \phi_{11112}, \phi_{11122}\))...
\( \phi_{11211} \), \( \phi_{11222} \)), we can derive four additional equations:

\[
\begin{align*}
\phi_{1111} &= g c \phi_{11222} + g \phi_{11111} + g c \phi_{1111} + \phi_{11} + p_1 \phi_1 + p_{11} \phi + c x_0 \phi \phi_{111} \\
\phi_{1122} &= g c \phi_{11211} + g \phi_{11222} + g c \phi_{1122} + \phi_{11} + c p_1 \phi_1 + c p_{12} \phi + c x_0 \phi \phi_{122} \\
\partial_0 \phi_{111} &= g c \phi_{11222} + g c \phi_{11111} + g c \phi_{111} + c \phi_{11} + c p_1 \phi_1 + c p_{11} \phi + c x_0 \phi \phi_{111} \\
\partial_0 \phi_{122} &= g c \phi_{11211} + g c \phi_{11222} + g c \phi_{1122} + c \phi_{11} + c p_1 \phi_1 + c p_{12} \phi + c x_0 \phi \phi_{122}
\end{align*}
\]  

(78)

But once again, there is an additional consistency condition which reduces the number of independent equations by one. In this case, we have introduced equations with \( \phi_{1111} \) and \( \phi_{1122} \) on the left hand sides, which leads to another condition on the moves represented by the \( \phi_1 \) and \( \partial_0 \phi_{11} \) equations, as discussed above.

We are therefore still left with one fewer equation than unknowns. As far as we can determine, this will continue to be the case no matter how many additional equations we introduce, although we have not been able to find a general proof of this fact. It remains an open question whether some other method of solution can yield information about this model from the generating equation (76).

\section{Multiple Ising spins}

The major limitation on the usefulness of matrix models is the fact that no known methods for solving matrix models apply to theories with central charge \( c > 1 \). Since the systems of the greatest physical interest fall into this category, the “\( c = 1 \) wall” has always been a major obstacle in the further understanding of 2D gravity and string theories. However, the fact that it is difficult to integrate out angular degrees of freedom in matrix models with \( c > 1 \) has always seemed to be a rather technical problem, and it has been unclear whether some other method might make it possible to find analytic results or solutions for systems with larger central charge.

In this section we shall discuss a class of matrix models formed by taking \( k \) independent Ising spins coupled to quantum gravity. The model with \( k \) spins corresponds to a CFT with \( c = k/2 \) coupled to gravity, and can be described as a \( 2^k \)-matrix model. We analyze the 4-matrix model corresponding to the 2-spin \( c = 1 \) CFT coupled to gravity. As was the case with the 3-state Potts model, we find that it seems to be impossible to find a closed system of equations describing the disk amplitude.

\subsection{Multiple spin model}

The multiple spin Ising theory is defined in a manner precisely analogous to the Ising theory in Section 3, except that each triangle carries \( k \) independent Ising spins. Thus, in the language of triangulations, we sum over all triangulations \( \Delta \) where each triangle carries \( k \) spins \( \sigma_1, \ldots, \sigma_k \). We set the interaction energy to be identical for each type of spin, so that
the weight of a given triangulation $\Delta$ is given by

$$W(\Delta) = \frac{1}{S(\Delta)} N^{x(\Delta)} g^{n(\Delta)} e^{\nu(\Delta)}$$  \hfill (79)

where $\nu(\Delta)$ is the total number of adjacent pairs of unequal spins with the same index, and $c$ is the interaction between unequal spins (note that we the normalization factor $1 - c^2$ is moved into the matrix model action for convenience in this section).

The matrix model description of this partition function is given by an integral over $2^k$ matrices $X_{\sigma_1 \cdots \sigma_k}$, where $\sigma_i \in \{+1, -1\}$:

$$Z = \sum_{\Delta} W(\Delta) = \int \prod_{\{\sigma_1, \ldots, \sigma_k\}} D X_{\sigma_1 \cdots \sigma_k} \exp (-NS)$$  \hfill (80)

with

$$S = \sum_{\{\sigma_1, \ldots, \sigma_k\}} \left[ \frac{1}{2(1 - c^2)^k} \sum_{\{\tau_1, \ldots, \tau_k\}} (-c)^{\frac{1}{2}(k - \sum_{i=1}^{k} \sigma_i \tau_i)} \Tr(X_{\sigma_1 \cdots \sigma_k} X_{\tau_1 \cdots \tau_k}) - \frac{g}{3} \Tr(X^3_{\sigma_1 \cdots \sigma_k}) \right].$$  \hfill (81)

By the same geometric argument as has been used throughout the paper, we can write a generating equation for the disk amplitude as a function of free noncommuting variables $x_{\sigma_1 \cdots \sigma_k}$ which takes the very simple form

$$\Phi = 1 + \sum_{\sigma_1, \ldots, \sigma_k, \tau_1, \ldots, \tau_k} c^{\frac{1}{2}(k - \sum_{i=1}^{k} \sigma_i \tau_i)} x_{\sigma_1 \cdots \sigma_k} (\Phi x_{\tau_1 \cdots \tau_k} \Phi + g \partial^2_{\tau_1 \cdots \tau_k} \Phi)$$  \hfill (82)

7.2 2-spin model

For simplicity, we shall restrict attention for the remainder of this section to the 2-spin model with $c = 1$. This 4-matrix model has a continuum limit which can be identified with the orbifold $c = 1$ bosonic string at radius $r = 1$ [66, 67]. Labeling the 4 matrices according to their binary representations for shorthand ($i.e., X_3 = X_{++}$, $X_2 = X_{+-}$, $X_1 = X_{-+}$, $X_0 = X_{--}$), the matrix model action is given by

$$S = \frac{1}{2} \sum_{i,j=0}^{3} [C_{ij} \Tr X_i X_j] - \frac{g}{3} \sum_{i=0}^{3} \Tr X_i^3,$$  \hfill (83)

with

$$C_{ij} = \frac{1}{(1 - c^2)^2} \begin{pmatrix} 1 & -c & -c & c^2 \\ -c & 1 & c^2 & -c \\ -c & c^2 & 1 & -c \\ c^2 & -c & -c & 1 \end{pmatrix}.$$  \hfill (84)
The generating equation (82) becomes

\[ \Phi = 1 + x_3 \Phi x_3 \Phi + x_2 \Phi x_2 \Phi + x_1 \Phi x_1 \Phi + x_0 \Phi x_0 \Phi + c \left( (x_2 + x_1) \Phi (x_3 + x_0) \Phi + (x_3 + x_0) \Phi (x_2 + x_1) \Phi \right) + c^2 \left( x_3 \Phi x_0 \Phi + x_2 \Phi x_1 \Phi + x_1 \Phi x_2 \Phi + x_0 \Phi x_3 \Phi \right) + g \left( x_3 \partial^2_3 \Phi + x_2 \partial^2_2 \Phi + x_1 \partial^2_1 \Phi + x_0 \partial^2_0 \Phi \right) \]

\[ + gc \left( (x_3 + x_0) (\partial^2_3 \Phi + \partial^2_0 \Phi) + (x_2 + x_1) (\partial^2_2 \Phi + \partial^2_1 \Phi) \right) + gc^2 \left( x_3 \partial^2_3 \Phi + x_2 \partial^2_2 \Phi + x_1 \partial^2_1 \Phi + x_0 \partial^2_0 \Phi \right). \] (85)

As for the 3-state Potts model, it appears not to be possible to derive a single algebraic equation for the homogeneous disk amplitude from this equation. Once again, however, we do not have any proof that this must be the case. However, it seems unlikely that any algebraic equation for the homogeneous disk amplitude exists in view of the logarithmic behavior of other previously studied \(c = 1\) models.

It is nevertheless conceivable that some other means of analyzing the generating equation (85) could lead to a single equation characterizing \(\phi\). Furthermore, we mention the promising result that all coefficients \(p_s\) can be determined in terms of the three coefficients \(p_0, p_{000}, p_{0000}\), just as in the Ising theory where only two such coefficients were necessary. Presumably, however, any method for solving the \(c = 1\) model would not generalize to the \(c = k/2\) models with \(k > 2\) since the number of unknowns in the higher \(k\) models grows much faster than when \(k = 2\).

8 Dually weighted models

So far in this paper, all of the specific models which we have considered have simple descriptions in terms of the matrix model formalism (although we described a class of boundary conditions for the Ising theory in Section 3.4 which cannot easily be described using matrix model methods). However, the methodology we have developed is applicable to a much wider class of discrete gravity models. In general, we can construct discrete matter fields by endowing a dynamical triangulation with an arbitrary set of local discrete data. For example, we can simultaneously give discrete labels \((f_i, e_i, v_i)\) to faces (polygons), edges, and vertices of a triangulation. If we make the weight function local, in the sense that the only interactions are between adjacent faces, edges, and vertices, then the partition function corresponding to such a theory is of the form

\[ Z = \sum_{\Delta, \Lambda} \frac{1}{S(\Delta)} N^{\chi(\Delta)} \left[ \prod_{f \in \text{faces}} \exp(\rho(f, \{ e \in \partial f \}, \{ v \in \partial f \})) \right] \left[ \prod_{e \in \text{edges}} \exp(\eta(\{ f : e \in \partial f \}, e, \{ v \in \partial e \})) \right] \] (86)
where $\Lambda$ is a set of combinatorial data $(f_i, e_i, v_i)$ on the simplices in a triangulation $\Delta$; $\rho$ is a function of the geometry and labels of a fixed face $f$ and the edges and vertices on its boundary; similarly $\eta$ is a function of a fixed edge $e$, the faces of which $e$ is a boundary, and the vertices on the boundary of $v$; and $\nu$ is a function of a vertex $v$ and the edges and faces of which it is a boundary.

A generating equation can be derived for a model of the general form (86) by associating independent noncommuting variables with the discrete labels on edges, vertices, and faces. Although the complexity of such an equation will increase significantly as the set of discrete matter fields increases, in principle the methods of this paper can provide an approach to the analysis of any such theory.

A particular set of theories in this more general class are the “dually weighted graph” models recently studied by Di Francesco and Itzykson [35] and by Kazakov, Staudacher, and Wynter [36]. In these models, weights are associated with vertices of a triangulation according to the coordination number, and weights are associated with the faces of a triangulation according to the number of incident edges as in a general one-matrix model. The dually weighted graph models are of particular interest because they contain continuous families of theories which interpolate between conformal field theories on flat space and conformal field theories completely coupled to Liouville gravity. Although it is possible to construct most dually weighted graph models as matrix models by introducing external matrices encoding the vertex weights, the resulting models are rather unwieldy and difficult to solve by standard matrix model techniques. An example of a model which can be analyzed using matrix model methods is the model where vertices of both the original and dual graph are restricted to have even coordination number [36].

We now derive the generating equation for a general dually weighted graph model. The most general dually weighted graph model has a weight of $g_i$ for every face with $i$ edges, and a weight of $w_i$ for every vertex with coordination number $i$. Thus, the partition function for a general dually weighted graph model is given by

$$Z = \sum_{\Delta} \frac{1}{S(\Delta)} N^{\chi(\Delta)} \prod_i g_i^{F_i(\Delta)} w_i^{V_i(\Delta)}$$

(87)

where $F_i(\Delta)$ is the number of $i$-gon faces in $\Delta$ and $V_i(\Delta)$ is the number of vertices with coordination number $i$.

To describe the generating function for the disk amplitude of this model, we must associate noncommuting variables with the vertices on the boundary of the disk. These noncommuting variables must contain information about the coordination number of the vertex, as well as the number of faces inside the disk which are incident on that vertex. Thus, we choose variables $x_{i,j}$, with $i < j$ to denote a vertex with $i$ interior faces and a coordination number of $j$. (An example is shown in Figure 8.)
The generating equation for the general dually weighted graph model is

$$\Phi = 1 + \left( \sum_{i<j,i',j'<j',k} g_k x_{i+1|j} x_{i'j'1} \partial_{i'j'} \partial_i^{k-2} \partial_{ij} \Phi \right)$$

$$+ \left( \sum_{r+j+k,i'+j'+k'} x_{i|k} x_{i'|k'} \left( \partial_{i'+j'-k'|k} \Phi \right) x_{j'|k'} x_{j|k} \left( \partial_{i+j-k} \Phi \right) \right)$$

(88)

(89)

where we have defined

$$\partial_s = \sum_{i>2} w_i \partial_{i-1|i}.$$  

(90)

As an extremely simple example, let us consider a “flat space” dually weighted graph model where only hexagonal faces and vertices with coordination number 3 are allowed (i.e., $g_i = \delta_i,6$; $w_i = \delta_i,3$). This is simply a theory of “triangulations” of flat surfaces by hexagons; each term in the disk amplitude for this theory corresponds to a closed path on the honeycomb lattice which bounds an embedded oriented surface composed of hexagons. The generating equation for this model reduces to (writing $x_i = x_{i|3}$)

$$\Phi = 1 + x_2 x_2 \partial_1 \partial_2^1 \partial_1 \Phi + x_2 x_1 \partial_0 \partial_2^1 \partial_0 \Phi + x_1 x_2 \partial_1 \partial_2^1 \partial_0 \Phi + x_1 x_1 \partial_0 \partial_2^1 \partial_0 \Phi
$$

$$+ x_2 \left[ x_2 (\partial_1 \Phi) x_2 + x_1 (\partial_0 \Phi) x_2 + x_2 (\partial_0 \Phi) x_1 \right] x_2 (\partial_1 \Phi)
$$

$$+ x_2 \left[ x_2 (\partial_1 \Phi) x_2 + x_1 (\partial_0 \Phi) x_2 + x_2 (\partial_0 \Phi) x_1 \right] x_1 (\partial_0 \Phi)
$$

$$+ x_1 \left[ x_2 (\partial_1 \Phi) x_2 + x_1 (\partial_0 \Phi) x_2 + x_2 (\partial_0 \Phi) x_1 \right] x_2 (\partial_0 \Phi).$$

35
There are a number of slightly more interesting models which are very simple to describe, and which connect flat space theories such as the above with fluctuating 2D gravity theories which presumably correspond to coupling in some form of matter fields. For example, consider the theory with only square faces, and vertices of coordination number between 3 and 5 defined by

\[
\begin{align*}
g_i &= \delta_i,4 \\
\omega_i &= \delta_i,4 + a\delta_i,3 + b\delta_i,5.
\end{align*}
\]  

(91)

When \(a = b = 0\), this is simply another flat space theory. As \(a\) and \(b\) become nonzero, fluctuations are introduced. It should be possible by tuning the ratio \(a/b\) to find a 1-parameter family of theories which correspond to a continuous limit; any detailed understanding of the behavior of this class of theories would be extremely interesting for understanding the effects of coupling dynamical gravity to a flat theory. The generating equation for this model is simply the restriction of (86) to the theory with weights (91).

It would be interesting to see whether for these simple models there is any reasonable algebraic method for extracting information from the generating equation.

9 Conclusion

We have developed a systematic method for analyzing discretized theories of 2D gravity with matter. Our method combines a geometric/combinatorial approach with the tools of free variables and loop equations which have previously been used to study matrix models. Although this approach gives rise to equations which are essentially equivalent to the Schwinger-Dyson loop equations for standard matrix model theories, our results are more general in several ways. First, because our methodology is completely based on a geometric definition of the discrete gravity theories, the analysis applies to models which cannot be described in terms of the matrix model language. Second, we can describe correlation functions in standard matrix model theories using variables which are apparently inaccessible using the matrix model formalism. For example, in [44] we describe the calculation using our techniques of the exact correlation function between disorder operators in the Ising theory. As far as we have been able to ascertain, this correlation function cannot be computed in any simple way using matrix model methods. Finally, we believe that the principal result presented in this paper, the general single generating equation for the generating function of all disk amplitudes, represents a conceptual and algebraic simplification and synthesis of previous work in this direction.

The methods we have presented here have enabled us to analyze in a systematic fashion a variety of models describing 2D gravity with matter fields. One question which has never been fully resolved is the question of which matrix models allow solution by loop equation methods. To date, the only theories which have proven tractable using this approach have been the pure gravity and \(c = 1/2\) models. We have found that it is possible to use this approach to derive polynomial equations satisfied by the generating function of disk amplitudes not only in the
standard Ising formulation of the \( c = 1/2 \) theory, but also in the dual formulation, and even in a continuous family of models connecting these two formulations. In addition, we have found that a very general class of two-matrix models, which cannot be solved by orthogonal polynomials, have uniform boundary condition disk amplitude generating functions which satisfy a quartic.

We have attempted to solve several other models using these methods, including the 3-state Potts model and the \( c = 1 \) model defined by two Ising spins coupled to gravity. We have found that these models resist solution, in accordance with the empirical belief that models whose target graphs contain loops somehow have more intrinsic degrees of freedom, and are not soluble by loop equations or algebraic means. However, in our attempt to solve these equations, we uncovered algebraic structure to the obstruction, characterized by degeneracies in subsets of the loop equations. This algebraic structure may give some hint of how to use other approaches to solve these models.

In the discrete theories that we have considered, the existence of a single equation describing the effect of all possible moves that reduce a triangulation suggests that this equation should be related to a Hamiltonian operator for the theory. In this context it would be interesting to consider the continuum limit of the equation in terms of noncommuting variables and see if can be related to recent attempts to construct string field hamiltonians for theories with \( c < 1 \) \cite{22-30}.

The approach we have detailed here simplifies the calculation of correlation functions in a variety of interesting 2D theories. Some particular results along these lines will be described in \cite{44,45}. However, the major conceptual issue which needs to be dealt with for these methods to be applicable in a truly general context is the development of some more sophisticated and powerful methods for dealing with algebraic equations in noncommuting variables such as \cite{30}. This is a very difficult problem, as has been emphasized in a number of recent papers \cite{39,40,41,42}; however, in these works and others some progress has been made, giving hope that in the future these equations may seem more tractable.

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