Transport in networks with multiple sources and sinks

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Abstract – We investigate the electrical current and flow (number of parallel paths) between two sets of \(n\) sources and \(n\) sinks in complex networks. We derive analytical formulas for the average current and flow as a function of \(n\). We show that for small \(n\), increasing \(n\) improves the total transport in the network, while for large \(n\) bottlenecks begin to form. For the case of flow, this leads to an optimal \(n^*\) above which the transport is less efficient. For current, the typical decrease in the length of the connecting paths for large \(n\) compensates for the effect of the bottlenecks. We also derive an expression for the average flow as a function of \(n\) under the common limitation that transport takes place between specific pairs of sources and sinks.

Transport processes, such as electrical current, diffusion, and flow, are fundamental in physics, chemistry, and biology. Transport properties depend critically on the structure of the medium, and have been studied for a large variety of geometries [1]. Of current interest are situations where the transport occurs on a network. For example, information is transferred over computer or social networks, vehicles traverse transportation networks, and electrical current flows in power-grid networks. Therefore, understanding of mechanisms to increase transport effectiveness is of great importance.

Transport properties usually have been investigated in the context of transport between a single pair comprising one source and one sink [2–8]. The quality of transport strongly depends on the degree (number of connections) of the source and the sink, whereas the rest of the network serves as an approximately resistance-free substrate for the transport process. Consequently, the transport was found to strongly depend on the degree distribution.

In more realistic situations, transport takes place between many nodes simultaneously. For example, in peer-to-peer and other computer networks users exchange files in parallel over the network links. In transportation networks, vehicles travel between many sources and many destinations through the network infrastructure. The presence of many parallel transport processes on the same underlying network leads to interactions between the different deliveries and a change in network efficiency. In this article we will quantify this phenomenon analytically and numerically, and show how different network usage leads to different behaviors. We reported some preliminary results in [9].

We focus on the class of non-directed, non-weighted model networks and we also study a real network. The first model is the Erdős-Rényi (ER) network, in which each link exists with independent small probability \(p\). This leads to a Poisson degree distribution [10,11]. The second model is the scale-free (SF) network, characterized by a broad, power law degree distribution and was recently found to describe many natural systems [12–14]. The ensemble of SF networks we treat is the “configuration model”, in which node degrees are drawn from a power law distribution (see below), and then open links are connected [15]. We also compute the distribution of flows in a real network, the internet [16].

We consider a transport process between two randomly chosen, non-overlapping sets (sources and sinks) of nodes of size \(n\) each, where \(1 \leq n \leq N/2\), \(N\) is the total number of nodes. We focus on three explicit forms of transport, described below.

i) Maximum flow (henceforth denoted flow) between the sources and sinks, when each link has unit capacity [8,9,16–19]. For non-weighted networks the flow is equivalent to the total number of disjoint paths (i.e. paths that do not share any edge) that connect the sources and sinks. Therefore, it quantifies any flow which does not
deteriorate with distance, such as flow of frictionless fluids, traffic flow and information flow in communication networks.

ii) **Electrical current** in the network when the sources are short circuited into a unit electrical potential, and the sinks to the ground (assuming each link is a unit resistor). Electrical current has special significance since it also describes any general transport process in which the transport efficiency decreases with the length of the path (due to increase of resistance). In addition, the current is equal to the probability of a random walker starting at any of the sources to escape to any of the sinks [20].

iii) **Maximum multi-commodity (MC) flow.** Same as i), but where the sources and sinks form ordered pairs, so the flow is directed from a given source to a specific sink, and not to any other sink [17] (here the network is directed and the sources and sinks may overlap). This describes direct communication between users in computer networks, or traffic of supplies over road networks.

Our goal is to study the dependence of the total transport on the number of sources/sinks in the three transport forms. We denote the flow, electrical current, and MC flow by $F$, $I$, and $F_{MC}$, respectively. In the case of a single pair, it was shown [3,9] that the transport between a source and a sink with degrees $k_1$ and $k_2$ is approximately: $F_{n=1}(k_1, k_2) \approx F_{MC} (k_1, k_2) \approx \min \{k_1, k_2\}$ for dense enough networks, and $I_{n=1}(k_1, k_2) \approx \frac{c k_1 k_2}{z_1 + z_2}$, where $c \lesssim 1$ is a fitting parameter. These equations state that the transport is dominated by the degrees of the involved nodes, and the rest of the network is practically a perfect conductor. Using the degree distribution: $P(k) = e^{-\langle k \rangle} k^k / k!$ for ER networks with average degree $\langle k \rangle$, and $P(k) \sim k^{-\gamma}$, $k \geq m$ for SF networks with degree exponent $\gamma$ and minimum degree $m$, we can find the distribution of flow or current.

Do transport properties change in the case of more than one pair? For a small number of sources/sinks $n$, we expect no significant difference. The backbone of the network remains an almost perfect conductor, but the degree $k_1$ has to be replaced with the total number of links emanating from the sources, and similarly for the sinks [9]. Because it is assumed that $n$ is small, we neglect the possibility of internal links inside each set. Define the sum of $n$ degrees $z \equiv \sum_{i=1}^{n} k_i$. For ER networks, $P_Z(z) = e^{-n \langle k \rangle} (n \langle k \rangle)^z / z!$.

1Formally, the maximum multi-commodity flow problem is defined as follows [17]. There are $n$ commodities defined by the set: $\{s_i, t_i\}, i = 1, \ldots, n$, where $s_i$ and $t_i$ are the source and sink of commodity $i$, respectively. The flow of commodity $i$ along the edge $\{u, v\}$ is $f_i(u, v)$. The total flow in an edge is limited by its capacity $\sum_{i=1}^{n} f_i(u, v) \leq c(u, v)$ (if $c(u, v) = 1$ for all edges). The maximum multi-commodity flow maximizes $\sum_{i=1}^{n} \sum_{w} f_i(s_i, w)$, where the second sum is over all neighbors $w$ of the source. The MC flow problem can be recast into a linear programming problem, and thus can be solved in polynomial time. However, in practice, only small instances (up to about hundred nodes) could be solved using the GLPK (http://www.gnu.org/software/glpk/) linear programming (interior point) solver that we used. Fast approximation algorithms exist [21], which guarantee a solution close to the optimal (however, we used only the exact solution).

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Fig. 1: Optimal number of sources/sinks. Average flow per source/sink $F/(n(k))$ vs. the number of sources/sinks $n$. Symbols represent simulation results ($N = 4096$, $k = 8, 16$; average is taken over many realizations of the network and many randomly chosen sets), while lines represent the theory based on the small-$n$ assumption (eq. (5)). For $n ≤ √N/k$ (indicated with arrows), the flow per source/sink always increases, as predicted by the theory. However, there is an optimal point beyond which the flow decreases.

Fig. 2: Flow in SF networks. (a) Probability distribution of the flow $Φ_n(F)$ vs. $F$, for $n = 1, 3, 5, 10$ (symbols), for the network of the Internet as of 2007 [16], which is approximately scale-free with degree exponent $γ ∼ 2.5$. Since $F$ is at least mm, a normalized power law takes the form $Φ_n(F) ∼ N^{2γ−2} F^{−(2γ−1)}$. Thus, we divide the vertical axis by $n^{2γ−2}$ to make all curves collapse. Theoretical slope for all $n$’s, $−2γ + 1 = −4$ is indicated by the straight line. (b) Average flow $F/(k)$ vs. the number of sources/sinks $n$ for random SF networks. Simulation results ($N = 4096$, $γ = 2.5$, $m = 2$), are shown in circles, the line represents theoretical curves. For the theory we used the upper bound for paths of lengths up to three (see main text and appendix).

Fig. 3: Transport in ER networks. (a) Average flow $F/(k)$ vs. the number of sources/sinks $n$. Simulation results ($N = 4096$, $k = 8, 16$) are shown in symbols, lines represent theoretical curves. For the theory we used the upper bound for paths of lengths up to three (see main text and appendix). (b) Same as (a) for the average electrical current $T$ ($N = 1024$, $k = 4, 8$).

Paths of length two involve one intermediate node. If the intermediate node $i$ is connected to $n_s(i)$ sources and $n_t(i)$ sinks, the flow it can channel is $n_{min}(i) = \min\{n_s(i), n_t(i)\}$. Since each edge exists with independent probability $p$, $n_s$ and $n_t$ are binomial variables with parameters $(n, p)$. The probability for $n_{min}$ to take the value $m$ is given by

$$P_{n_{min}}(m) = 2 \left( \sum_{j=m}^{n} P_h(j) \right) - \left[ P_h(m) \right]^2,$$

where $P_h(j) = (\binom{n}{j} p^j (1-p)^{n-j}$ since $n_s$ and $n_t$ are binomials. For large $n$ and small $p$ the binomial variable can be approximated by a Poisson distribution, for which eq. (7) reduces to (cf. eq. (5))

$$P_{n_{min}}(m) = 2 \left( np e^{-np} \right) \frac{\gamma(m, np)}{\Gamma(m)} - \frac{(np)^m e^{-np}}{2m!}.$$  

The average of $n_{min}$ is $\langle n_{min} \rangle = \sum_{m=0}^{n} m P_{n_{min}}(m)$. Since there are $(N − 2n)$ possibilities of choosing the intermediate node, $\frac{T^2}{F_1 + F_2}$ becomes accurate as $n \rightarrow N/2$, where there are only very few intermediate nodes and thus a very small probability for a longer path.

Paths of length three involve two intermediate nodes and their average number is more difficult to compute. However, by a mapping onto a matching problem we are able to provide lower and upper bounds for $\frac{T^2}{F_1}$ (see appendix). In fig. 3(a), we plot $F(n)$ obtained with the upper bound, neglecting flows of higher order $F_3, F_5, \ldots$, and find agreement with the simulations.

An interesting outcome of the above calculation is an immediate result for the electrical current. Since all links have unit resistance, a path of length $\ell$ has total resistance $\ell$ so $T = \frac{F_1}{1} + \frac{F_2}{2} + \frac{F_3}{3} + \ldots$. Our simulation results agree with our theory, as shown in fig. 3(b). While it is not in general correct that in SF networks each edge exists with independent probability $p$, applying the same approach for SF networks results in a good qualitative agreement with the simulations —but not as good as for ER networks (fig. 2(b)).
Focusing attention on the transport per source/sink, \( \bar{F}/n \) and \( \bar{T}/n \), we observe that while the flow \( \bar{F}/n \) decreases with \( n \), the current \( \bar{T}/n \) increases (fig. 4). The decrease in the case of the flow is intuitively clear. As more paths become “bottlenecks”, the total number of paths connecting the sources and sinks decreases. However, when increasing \( n \), the paths that exist have shorter lengths since the probability for direct or almost direct link between the sources and sinks is higher. While this effect does not influence the flow, in the case of electrical current it reduces the resistivity of the paths and increases the total current [9]. This is a fundamental difference between the two forms of transport, which our analysis reveals.

For MC flow, paths cannot become shorter once a pair of a source and a sink is added, since the transport takes place between specific pairs (see footnote 1) (fig. 5(a)). Therefore, we expect that for some \( n^* \), the network will saturate and will not be able to carry any more flow. In other words, not only the MC flow \( F_{MC}/n \) will decrease, but the MC itself will not grow.

To develop a theory for the average MC flow in ER networks, we look at the pairs of sources and sinks as if they were added one at a time. As the \( n \)-th pair is added, we assume the paths connecting the previous \( n-1 \) pairs remain unchanged, such that the new pair can connect only through these network links not already in use. We also assume the used links are randomly spread over the network and denote the average effective degree of the unused part as \( k_n \). Under these assumptions, the new pair will be connected, on average, with a number of paths that is the minimum of two degrees of average \( k_n \) (see above). Thus, the total MC flow can be approximated as

\[
F_{MC}(n) \approx \sum_{n'=0}^{n-1} \mu(k_{n'}),
\]

where \( \mu(k) = \sum_{j=0}^{N} \frac{2j^2 e^{-j}}{(j!)^2} \left( \frac{2}{j} \right)^{k} \) is the average of the minimum of two degrees drawn from a Poisson distribution with average \( k \), as in eqs. (5) and (8).

To complete the derivation, we need to find the average effective degree \( k_n \). This is straightforward to calculate, if we assume that to optimize the total flow (since we look at maximum flow), the transport between each new pair uses only shortest paths. Recalling that in ER networks with average degree \( \langle k \rangle \) the average shortest path is of length \( \log N/\log \langle k \rangle \) [11], we can write a recursion equation for the evolution of \( k_n \)

\[
k_{n+1} = k_n - \mu(k_n) \frac{\log N}{N \log k_n}, \quad k_0 = \langle k \rangle.
\]

Evaluating (9) and (10), we find agreement with simulations of the exact MC flow (see footnote 1) (fig. 5(b)). Note also, that this formula for the MC flow is an improvement over the result obtained with the small-\( n \) assumption, eq. (2), as shown in the inset of fig. 5(b).

For large \( n \) such that \( k_n \) is small, the new pair is no longer guaranteed to be connected. To find the value of \( n^* \), above which additional sources and sinks cannot communicate, we use the result of percolation theory that the network becomes fragmented when the average degree decreases below one [11]. Thus \( n^* \) satisfies \( k_n^* \geq 1 \). Since substituting \( k_n^* = 1 \) in eq. (10) does not yield a closed form formula, we bound \( n^* \) by assuming the number of paths connecting the \( n \)-th pair \( \mu(k_n) \) satisfies \( 1 \leq \mu(k_n) \leq k_n \).

Equation (10) splits into two inequalities

\[
k_{n+1} \geq k_n - \frac{\log N}{N \log k_n},
\]

\[
k_{n+1} \leq k_n - \frac{\log N}{N \log k_n},
\]
Approximating $k_{n+1} - k_n \approx \frac{\partial k_n}{\partial n}$, setting $k_{n=0} = \langle k \rangle$ and solving the two differential inequalities, we obtain

\[
(\log k_n)^2 \geq (\log \langle k \rangle)^2 - 2 \frac{\log N}{N} n,
\]

\[
(k_n \log k_n - k_n) \leq (\langle k \rangle \log \langle k \rangle - \langle k \rangle) - \frac{\log N}{N} n.
\]

(12)

Substituting $k_n = 1$, the resulting bounds for $n^*$ are

\[
\frac{\log^2 \langle k \rangle}{2} \leq n^* \cdot \frac{\log N}{N} \leq \langle k \rangle \log \langle k \rangle - \langle k \rangle + 1.
\]

(13)

From (13), $n^* = O(N/\log N)$, i.e., the maximal number of sources/sinks pairs that can communicate is of the order of $N/\log N$.

Due to the long computation time of the exact MC flow, only small system sizes could be considered in the simulations, thereby leading to a finite-size effect that obscures the percolation transition in a way that the precise point where the network saturates cannot be observed (fig. 5(b)). The continuous increase in the MC flow for $n > n^*$ in our simulations is expected since for finite networks, even when $k_n < 1$ some nodes are still connected [23]. Moreover, the probability of a pair of nodes to belong to the same small (non-giant) cluster cannot be neglected for finite systems.

For SF networks, the degree exponent $\gamma$ plays a significant role. For $2 < \gamma < 3$, there is no percolation threshold [24], and we expect more sources/sinks to be able to communicate in comparison to ER networks. For $\gamma > 3$, a percolation threshold exists, and we expect behavior qualitatively similar to that of ER networks. However because of the limitation on computation time, we leave it as a conjecture.

In conclusion, we investigate the efficiency of transport in complex networks when many sources and sinks are communicating simultaneously. We obtain analytical results for the total electrical current and flow when the number of sources/sinks is very small. For a large number of sources/sinks, we derive approximate expressions for the total transport in ER networks by looking at the lengths of the paths used for the transport, and identify a fundamental difference between the behavior of flow and current. For multi-commodity flow, we calculate the mean value of the flow in ER networks. We also argue that in scale-free networks more sources/sinks can communicate because of the lack of a percolation threshold.

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to as many other nodes in $I_2$, such that the number of connections of a node in $I_1$ will not exceed its value of $n_s - n_t$ and that the number of connections of a node in $I_2$ will not exceed its value of $n_s - n_t$. For a given node in $I_1$, denote the number of its spare links $n_s - n_t$ as $s_1$, and define $s_2$ similarly for a node in $I_2$. The maximum flow problem is thus reduced to a generalized bipartite matching problem. This is illustrated in fig. 6. Due of symmetry, $\langle |I_1| \rangle = \langle |I_2| \rangle = \langle |I| \rangle$ and $\langle s_1 \rangle = \langle s_2 \rangle = \langle s \rangle$. In addition, these quantities can be calculated. The probability for an intermediate node to have $n_s > n_t$ is

$$ P(n_s > n_t) = P_b(1)P_b(0) + P_b(2)[P_b(0) + P_b(1)] + \ldots = \frac{1 - \sum_{i=0}^{n} [P_b(i)]^2}{2} = P(n_t > n_s). \quad (A.2) $$

and therefore $\langle |I_2| \rangle = (N - 2n)P(n_s > n_t) = (N - 2n) \times \frac{1 - \sum_{i=0}^{n} [P_b(i)]^2}{2} / \langle |I| \rangle$. The average number of spare links $\langle s_1 \rangle = \langle s_2 \rangle = \langle s \rangle$ is

$$ \langle s \rangle = \langle s_1 \rangle = \sum_{i=1}^{n} i \cdot P\{ (n_s - n_t) = i, \text{ given } n_s > n_t \} = \frac{\sum_{i=1}^{n} i \cdot P\{ (n_s - n_t) = i \}}{P(n_s > n_t)} = \frac{2}{1 - \sum_{i=0}^{n} [P_b(i)]^2} \cdot \sum_{i=1}^{n} \sum_{j=1}^{i} i \cdot P_b(j)P_b(j-i). \quad (A.3) $$

As a first approximation, we assume $\langle s \rangle = 1$, to return to a regular bipartite matching problem. A recent theorem for ER bipartite networks has proved that a network with minimal degree at least 2 has a perfect matching with high probability [25]. Therefore, a lower bound for the matching size will be the 2-core of the bipartite network (consisting of the nodes in $I_1$ and $I_2$), which is given by $x (\langle |I| \rangle)$ where $x = 1 - e^{-\beta} (1 + \beta)$, and $\beta$ is the solution of $\frac{\alpha^2}{\alpha} = 1 - e^{-\beta} \quad [26]$. Since each such matching contributes one unit of flow of length three, the total matching size $x (\langle |I| \rangle)$ is our lower bound for $F_3$. Neglecting flows of higher orders $F_4, F_5, \ldots$, we obtain a lower bound for the average flow: $\bar{F} = n^2 p + (N - 2n) (n_{\text{min}}) + x (\langle |I| \rangle)$. This bound becomes exact in the limit of large $n$, where there are very few intermediate nodes and thus a very small probability for a long path.

An upper bound for the flow can be obtained by assuming that each node in the bipartite graph is able to match (on average) the minimum between its degree (in the bipartite network) and its number of spare links $s$. This overestimation of the flow of length three happens to compensate for neglecting flows of longer lengths, and in most cases agrees with simulations (see fig. 3).

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