The nil-clean $2 \times 2$ integral units

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Abstract

We prove that all trace 1, $2 \times 2$ invertible matrices over $\mathbb{Z}$ are nil-clean and, up to similarity, that there are only two trace 1, $2 \times 2$ invertible matrices over $\mathbb{Z}$.

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1. Introduction

We first recall the following.

An element $a$ in a unital ring $R$ is clean (see [5]) if $a = e + u$ with an idempotent $e \in R$ and a unit $u \in R$, and, nil-clean (see [4]) if $a = e + t$ with an idempotent $e$ and a nilpotent $t$. It is strongly nil-clean if $et = te$. A nil-clean element is called trivial if $e \in \{0, 1\}$, the trivial idempotents. A unit $u$ is called unipotent if $u = 1 + t$, for some nilpotent $t$.

A ring is clean (or nil-clean) if so are all its elements. Via unipotent units, it is easy to see that nil-clean rings are clean.

Though all these notions are well-known for some time, very little is known about which clean elements of a ring are nil-clean. Actually, besides the unipotent units (indeed, a unit is strongly nil-clean if and only if it is unipotent), we do not know which units of a ring are nil-clean.

We can discard the trivial nil-clean elements. Indeed, if $e = 0$, then there is no unit which is nilpotent (unless $R = 0$), and if $e = 1$, $a = 1 + t$, are precisely the unipotent units. Over any commutative domain, such $2 \times 2$ matrices $M$, are easily characterized by $\det(M - I_2) = \text{Tr}(M - I_2) = 0$.

In this note, using an adequate (but nontrivial) Number Theory machinery, we characterize the (nontrivial) nil-clean units in the matrix ring $M_2(\mathbb{Z})$.

Notice that non-trivial nil-clean $2 \times 2$ matrices over any commutative domain have trace 1.

As our main result, conversely, we show that trace 1, $2 \times 2$ units over $\mathbb{Z}$ are nil-clean, that is, a $2 \times 2$ unit over $\mathbb{Z}$ is non-trivial nil-clean if and only if it has trace 1.

Up to similarity, we also prove that all trace 1, $2 \times 2$ units are similar to $\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$ or to $\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$.
2. Binary quadratic forms preliminaries

The proof of our main result requires some preparation. First consider a particular Diophantine equation, namely

\[(x + y)^2 + xy = m \quad (*)\]

where \(m\) is a positive integer.

**Lemma 2.1.** For any divisor \(m\) of a positive integer \(A(A+1) - 1, A > 1\), the equation \((*)\) is solvable.

**Proof.** From the general theory of quadratic binary forms, we know that the integer \(m\) is represented by a binary quadratic form of discriminant \(d\) only if the congruence \(u^2 \equiv d \pmod{4k}\) is solvable, where \(k\) is the square-free part of \(m\) (see [2], Theorem 7, p. 145). In our case, i.e. for the form \(G(x, y) = (x + y)^2 + xy\), \(d = 5\) and the class number of \(\mathbb{Q}(\sqrt{5})\) is 1, hence the above condition becomes necessary and sufficient. The solvability of the congruence \(u^2 \equiv 5 \pmod{4k}\) is equivalent to the property that all prime factors of form \(5s + 2\) or \(5s + 3\) from the factorization of \(m\) have even exponent.

Since we have to solve this equation for a divisor \(m\) of \(A(A+1) - 1\), this reduces to show that if \(m\) divides \(A(A+1) - 1\), then \(m\) has this property. But this holds because if a prime \(p\) divides \(A(A+1) - 1\), then it also divides \((2A+1)^2 - 5 = 4[A(A+1) - 1]\), so 5 must be a quadratic residue modulo \(p\).

On the other hand, denoting by \(\left(\frac{a}{p}\right)\) the Legendre symbol, according to the Gauss reciprocity law (see [1], Theorem 9.1.3), \(\left(\frac{5}{p}\right) = \left(\frac{p}{5}\right) = 1\). Because \(\left(\frac{5}{p}\right) = 1\), it follows \(\left(\frac{p}{5}\right) = 1\) and so \(p\) is a quadratic residue modulo 5, i.e., \(p\) is congruent to 0, 1 or 4 modulo 5, as desired. \(\Box\)

Next, we consider another particular Diophantine equation, namely

\[(x - y)^2 + xy = m \quad (**\)

where \(m\) is a positive integer.

**Lemma 2.2.** For any divisor \(m\) of a positive integer \(A(A+1) + 1, A > 1\), the equation \((**\)\) is solvable.

**Proof.** The proof is similar to the proof of the previous lemma. Just notice that now the discriminant is \(-3\) and the corresponding class number is also 1. Moreover, if a prime \(p\) divides \(A(A+1) + 1\), then it also divides \((2A+1)^2 + 3 = 4[A(A+1) + 1]\), \(-3\) must be a quadratic residue modulo \(p\) and so on. \(\Box\)

Secondly, we need the following

**Proposition 2.3.** Suppose \(A(A+1) + BC = 1\) for integers \(A, B, -C > 1\). We can always chose solutions \((b, d)\) and \((a, c)\) of the equation \((*)\) with \(m = B\) and \(m = -C\), respectively, such that \(ad - bc = 1\).

**Proof.** Again we use the theory of binary quadratic forms. Consider the quadratic form \(F(x, y) = Bx^2 + (2A+1)xy - Cy^2\).

Its discriminant is equal to \((2A+1)^2 + 4BC = 5\) (by our hypothesis). Using the reduction theory of quadratic forms, since the class number of \(\mathbb{Q}(\sqrt{5})\) is 1, it is well-known that (see [3]) all integer quadratic forms with discriminant 5 are \(SL(2, \mathbb{Z})\)-equivalent to
Suppose \( G(x, y) = (x + y)^2 + xy \), which has also discriminant 5. The equivalence means that there exist integers \( a, b, c, d \) with \( ad - bc = 1 \) such that \( G(ax + by, cx + dy) = F(x, y) \).

If we set \( x = 1, y = 0 \) we get \( G(a, c) = B \) and if we set \( x = 0, y = 1 \) we get \( G(b, d) = -C \) and we are done. \( \square \)

**Proposition 2.4.** Suppose \( A(A + 1) + BC = -1 \) for integers \( A, B, -C > 1 \). We can always chose solutions \( (b, d) \) and \( (a, c) \) of the equation (***) with \( m = B \) and \( m = -C \), respectively, such that \( ad - bc = 1 \).

**Proof.** We consider again the quadratic form \( F(x, y) = Bx^2 + (2A + 1)xy - Cy^2 \). Its discriminant is \( (2A + 1)^2 + 4BC = -3 \) and so is the discriminant of \( G(x, y) = (x - y)^2 + xy \). Since the corresponding class number is 1, these are \( SL(2, \mathbb{Z}) \)-equivalent, there exist integers \( a, b, c, d \) with \( ad - bc = 1 \) such that \( G(ax + by, cx + dy) = F(x, y) \) and we complete the proof as for the previous proposition. \( \square \)

## 3. The main result

By \( E_{11} \) we denote the matrix with all entries zero, excepting the NW corner, which is 1. Recall that over any principal ideal domain, every non-trivial \( 2 \times 2 \) idempotent matrix is similar to \( E_{11} \). The result holds also in a more general setting (see [6]), but this hypothesis suffices for our proof below.

We first give a characterization, up to similarity, of the non-trivial nil-clean units in \( M_2(\mathbb{Z}) \).

**Proposition 3.1.** An integral \( 2 \times 2 \) matrix \( U \) is a non-trivial nil-clean unit iff it is similar to one of the following two matrices: \( V_1 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, V_{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \). More precisely, if \( \det U = 1 \), it is similar to \( V_1 \) and if \( \det U = -1 \), it is similar to \( V_{-1} \).

**Proof.** Since nil-clean and unit are invariant (properties) to conjugation, up to similarity, owing to the previous paragraph, we can suppose the idempotent in the nil-clean decomposition being \( E_{11} \). Nilpotent matrices having zero trace and zero determinant, we deal with (nil-clean) matrices \( M = \begin{bmatrix} a + 1 & b \\ c & -a \end{bmatrix} \) such that \( a^2 + bc = 0 \). Since \( \det M = -(a + 1)a - bc = -a \in \{ \pm 1 \} \) we distinguish two cases.

**Case 1.** If \( a = -1 \) then \( bc = -1 \) which give two matrices: \( V_1 = E_{11} + \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \) and transpose (which is similar to \( V_1 \): just conjugate by \( \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \)).

**Case 2.** If \( a = 1 \) then \( bc = -1 \) which give two matrices: \( V_{-1} = E_{11} + \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \) and transpose (which is similar to \( V_{-1} \): the same conjugation). \( \square \)

**Example.** \( A = \begin{bmatrix} 8 & 5 \\ -11 & -7 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ -12 & -8 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \). Here \( U = \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix} \) and \( U^{-1}AU = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} U = V_{-1} \), as stated.

Just taking the conjugates of these two matrices we can find the form of all the non-trivial nil-clean units in \( M_2(\mathbb{Z}) \). This is

\[
\begin{bmatrix}
(a + c)(b + d) + ad & (b + d)^2 + bd \\
-(a + c)^2 - ac & -(a + c)(b + d) - bc
\end{bmatrix}
\]

for integers \( a, b, c, d \) with \( ad - bc = 1 \).
Theorem 3.2. Trace 1, $2 \times 2$ units over $\mathbb{Z}$ are nil-clean.

Proof. In the sequel $M = \begin{bmatrix} A+1 & B \\ C & -A \end{bmatrix}$ denotes a trace 1, $2 \times 2$ integral matrix.

We first discuss the det $M = -1$ case (i.e. $A(A+1) + BC = 1$) and (owing to the form of the non-trivial nil-clean units deduced above) prove that there are integers $a, b, c, d$ with $ad - bc = 1$ such that

$$M = \begin{bmatrix} (a+c)(b+d) + ad & (b+d)^2 + bd \\ -(a+c)^2 - ac & -(a+c)(b+d) - bc \end{bmatrix}.$$ 

Finding the integers $a, b, c, d$ amounts to solve the system

(i) $A = (a+c)(b+d) + bc$

(ii) $B = (b+d)^2 + bd$

(iii) $C = -(a+c)^2 - ac$

(iv) $1 = ad - bc$, with integer unknowns $a, b, c, d.$

First notice that $A(A+1) - 1 > 0$ with only two (integer) exceptions: $A = -1$ and $A = 0.$ The case $A = 0$ reduces to $A = -1,$ by conjugation with $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and the case $A = -1$ was already settled as Case 1, Proposition 3.1.

Hence we can assume $BC < 0$ and even $B > 0,$ $C < 0$ (otherwise we conjugate with $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$), together with $A \geq 1$ (the case $A \leq -2$ also reduces to $A \geq 1,$ by conjugation with $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$).

Secondly observe that (ii) and (iii) are equations of type $(x+y)^2 + xy = m,$ that is (*).

According to Proposition 2.3, the equations (ii), (iii) and (iv) have an integer solution.

Finally, we show that any solution of (ii), (iii) and (iv) (denoted again by $a, b, c, d$) also verifies (i) and we are done.

Indeed, $-BC = [(b+d)^2 + bd][(a+c)^2 + ac] = (b+d)^2(a+c)^2 + ac(b+d)^2 + bd(a+c)^2 + abcd$
and so we have to check whether the degree 2 equation $A(A+1) = 1 + (b+d)^2(a+c)^2 + ac(b+d)^2 + bd(a+c)^2 + abcd$ has $A = (a+c)(b+d) + bc$ as one root, i.e. $(b+d)^2(a+c)^2 + bc(b+c+1) + (2b+c+1)(a+c)(b+d) = 1 + (b+d)^2(a+c)^2 + ac(b+d)^2 + bd(a+c)^2 + abcd.$

Equivalently $bc(b+1-ad) + (2b+c+1)(ab+ad+bc+cd) = 1 + ab^2c + acd^2 + a^2bd + bc^2d + 4abcd$
or else $(bc+1-ad)(ab+cd+3bc-1) = 0.$ This holds since $ad - bc = 1.$

Next, we settle the det $M = 1$ case (i.e. $A(A+1) + BC = -1$) and prove that there are integers $a, b, c, d$ with $ad - bc = 1$ such that

$$M = \begin{bmatrix} (a-c)(b-d) + ad & (b-d)^2 + bd \\ -(a-c)^2 - ac & -(a-c)(b-d) - bc \end{bmatrix}.$$ 

Finding the integers $a, b, c, d$ amounts to solve the system

(i) $A = (a-c)(b-d) + bc$

(ii) $B = (b-d)^2 + bd$

(iii) $C = -(a-c)^2 - ac$

(iv) $1 = ad - bc$, with integer unknowns $a, b, c, d.$

Therefore now we deal with the equation (**). What remains for the proof is now deduced from Proposition 2.4 and a similar verification that any solution of (ii), (iii) and (iv) actually satisfies also (i). \hfill $\Box$

In closing we mention that this result fails for higher dimensions of matrices. Here is a $3 \times 3$ example:
take $U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & -1 \end{bmatrix}$ and $V = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, both with trace=determinant=1. Then $\text{Tr}(U^2) = -1 \neq 1 = \text{Tr}(V^2)$ and so the matrices $U$, $V$ have different characteristic polynomials. Consequently, $U$ and $V$ are not similar.

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