$S - r$ ways on a graph-lattice

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Abstract. Graph-lattice has vertices at points with non-negative integer coordinates. Each vertex has two outgoing edges: horizontal edge and vertical edge to the neighboring vertices (right and top). We considered the problem of reachability for $s - r$ ways. $s - r$ way consists of alternating pieces of horizontal edges or vertical edges, each piece consisting of horizontal arcs (with the exception of, perhaps, the final) has a length that is a multiple of $s$, and each piece consisting of vertical edges (with the exception of, perhaps, the final) has a length that is a multiple of $r$. We obtained formulas for the number of $s - r$ ways, leading from the vertex to the vertex. In the second part we investigate the problem of random walks via $s - r$ ways. The process of random walk on the $s - r$ paths isn’t Markov process. It is shown that it is locally reduced to the Markov process on the subgraph which determined by the starting vertex.

The theory of random processes developed as a practical application needs, and in connection with its applications in mathematics itself ([1–3]). Relatively recently been introduced in the consideration of new classes of graphical objects – a resource network, the study of which is also used by the instrament of random processes ([4–8]).

It is known that random processes are divided into two types – Markov and non-Markov. The author of this article do not know the results of reducible non-Markov process to Markov process, except obtained in [9–12]. In this work, devoted to the reachability vertices of the graph-lattice via $s - r$ ways and random walks along such paths, received an example where the non-Markov process of random walk via $s - r$ ways on graph-lattice, locally reduces to a Markov process on its subgraphs.

1. $S - r$ way to graph-lattice

We consider an infinite directed graph, which we call a graph-lattice. The set of vertices of the graph-lattice is $Z_+ \times Z_+$ ($Z_+ -$ the set of non-negative integers).

Each vertex $(p; q)$ has two outgoing edges to the neighboring vertices (right and top): a horizontal edge to the right vertex $(p + 1; q)$ and a vertical edge to the top vertex $(p; q + 1)$. We assume that all edges of the graph-lattice have a length equal to one.

Its clear that:

(i) A graph-lattice doesn’t have contours,
(ii) All ways on this graph are simple,
(iii) The way from the vertex $x = (m; n)$ to the vertex $y = (p; q)$ exists if and only if $(m \leq p)$ & $(n \leq q)$. All of these ways have a length equal to $(p - m) + (q - n)$ and located in a rectangle having two opposite vertices $x = (m; n)$ and $y = (p; q)$. 

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(iv) A graph-lattice has a fractal structure — any subgraph, generated by a vertex and the set of reachable vertices from it, is the graph-lattice.

All ways leading from the vertex \( \overline{O} = (0; 0) \) to the vertex \( \overline{A} = (m; n) \) have a length equal to \( m + n \) and located in a rectangle having two opposite vertices \( \overline{O} \) and \( \overline{A} \). This way is a polyline consisting of \( m \) horizontal edges and \( n \) vertical edges.

Each of these this way paths will be encoded \((m + n)\)-bit binary number, comprising ones and zeros. Unit standing on \( i \)-th position (numbering from left to right) means that \( i \)-th step on the way extends through a vertical edge, and zero on \( i \)-th position means that \( i \)-th step on the way extends through a horizontal edge.

In order to define the \( s - r \) way on a graph-lattice we need some additional definitions.

Definition 1 The initial 1–fragment of the way length of \( k \) we will call the initial piece of the way, consisting of the \( k \) following one after another horizontal edges, such that its coding is \((1\ldots10)\) (here the dots “hidden” units, “white space” area filled with an arbitrary collection zeros and ones).

Definition 1’ The internal 0–fragment of the way length of the \( k \) we will call it the initial piece of the way, consisting of \( k \) following one after another vertical edges, such that its coding is \((00\ldots01)\) (here the dots “hidden” units, “white space” area filled with an arbitrary collection zeros and ones).

Definition 2 The internal 1–fragment of the way with length of \( k \) we will call it the internal piece of the way, consisting of the \( k \) following one after another horizontal edges, such that its coding is \((01\ldots10)\) (here the dots “hidden” units, “white space” area is filled with an arbitrary collection zeros and ones).

Definition 2’ The internal 0–fragment of the way with length of \( k \) we will call the internal piece of the way, consisting of the \( k \) following horizontal edges one after another, such that its coding is \((10\ldots001)\) (here the dots “hidden” zeros, “white space” area is filled with an arbitrary collection zeros and ones).

Definition 3 The final 1–fragment of the way with length \( k \) we will call it the final piece of the way, consisting of the \( k \) following one after another horizontal edges, such that its coding is \((01\ldots1)\) (here the dots “hidden” units, “white space” area is filled with an arbitrary collection zeros and ones).

Definition 3’ The final 0–fragment of the way with length \( k \) we will call it the final piece of the way, consisting of the \( k \) following horizontal edges one after another, such that its coding is \((10\ldots0)\) (here the dots “hidden” zeros, “white space” area is filled with an arbitrary collection zeros and ones).

Now we define the \( s - r \) way on the graph-lattice.

Definition 4 The way on the graph-lattice will be called by \( s - r \) way, if the length of initial and all internal 1–fragments multiplied of \( s \) and the length of initial and all internal 0–fragments multiplied of \( r \).

Example 1. \( \blacktriangle \) Coding \((0001100000011110001)\) corresponds to a way from the vertex \( \overline{O} \) to the vertex \( \overline{A} = (m; n) \). This way is a 2–3 way. \( \blacktriangle \)

Now we investigate the problem of the quantity of \( s - r \) ways leading from vertex \( \overline{O} = (0; 0) \) to vertex \( \overline{A} = (m; n) \). We consider four cases:

(i) \( m = s \cdot k; n = r \cdot l, k, l \in \mathbb{Z} - + \). This type of point is called “\( s; r \)”;
(ii) \( m \neq s \cdot k; n = rl, k, l \in \mathbb{Z} - + \). This type of point is called “\( -s; r \)”;
(iii) \( m = s \cdot k; n \neq rl, k, l \in \mathbb{Z} - + \). This type of point is called “\( s; -r \)”;
(iv) \( m = s \cdot k; n = rl, k, l \in \mathbb{Z} - + \). This type of point is called “\( -s; -r \)”;

\( \blacktriangle \) The symbol \( \blacktriangle \) we use to denote the beginning and end of the definition, example, etc.
In the first case, coding of any $s - r$ way consist of 1–fragments length of each of this multiplied of $s$ and 0–fragments fragments length of each of this multiplied of $r$. Transform its binary encoding as follows: every $s$ adjacent ones (moving along the coding from left to right) will replace one unit, similarly, each $r$ adjacent zeros replace one zero. It is clear that this transition is a one-to-one. As a result, we obtain $(k + l)$-bit binary number, comprising $(k + l)$ – ones and zeros. The quantity of such binary numbers is equal to $C_{k+l}^{(k+l)}$. Thus, the quantity of $s - r$ ways leading from vertex $\bar{O} = (0; 0)$ to vertex $\bar{A} = (s \cdot k; r \cdot l)$ is equal to $(k+1)
choose{k} + l
choose{l}$.

In the second case $(m = s \cdot k + \delta, k \in N, \delta \in \{1, 2, \ldots, s - 1\})$ any $s - r$ way necessarily has a final 1–fragment and its length is equal to $s \cdot k_1 + \delta$. Replace each $s$ subsequent units with one without touching the last $\delta$ and replaced each $r$ subsequent zeros with one. It is clear that this transition is a one-to-one. As a result, we get a $(k + l + \delta)$–bit binary number that contains 1 zeros and $l + \delta$ ones, whose $\delta$ last places are worth unit. The quantity of such binary numbers is equal to $(k+1)
choose{k} + l
choose{l}$.

Thus, the quantity of $s - r$ ways leading from vertex $\bar{O} = (0; 0)$ to vertex $\bar{A} = (2k + \delta; 2l)$ is equal to $(k+1)
choose{k} + l
choose{l}$.

The third case (“$s$, $r$” $(m = s \cdot k, n = r \cdot l + \delta, k, l \in N, \delta \in \{1, 2, \ldots, r - 1\}$) is similar to the second. Thus, the quantity of $s - r$ ways leading from vertex $\bar{O} = (0; 0)$ to vertex $\bar{A} = (s \cdot k; r \cdot l + \delta)$ is equal to $(k+1)
choose{k} + l
choose{l}$.

In the fourth case $s - r$ ways from vertex $\bar{O}$ to vertex $\bar{A}$ do not exist.

Due to the above–noted fractal structure of the graph-lattice, namely, the fact that any subgraph is generated by its arbitrary vertex and the vertices are reachable from it, is also a graph-lattice, the problem of the quantity of the $s - r$ ways leading from the vertex vertex $\bar{A} = (m; n)$ to the vertex vertex $\bar{B} = (p; q), p \geq m, q \geq n$, is reduced to the problem of the quantity of the $s - r$ ways leading from the vertex $\bar{O} = (0; 0)$ to the vertex $\bar{B} = (p - m; q - n)$.

2. Random walks on $s - r$ ways

We assume that all of the horizontal edges of the graph-lattice have the transition probabilities equal to $\alpha$ and that all of the vertical edges of the graph–lattice have the transition probabilities equal to $\beta$, $0 < \alpha, \beta < 1, \alpha + \beta = 1$. Consider a discrete random walk process on the vertices of the graph-lattice particles at the initial moment at the vertex $\bar{O} = (0; 0)$. Believe that per quantum (time slice) particle located at the vertex $s; t$ goes in to one of the edges into the vertex $(s + 1; t)$ or $(s; t + 1)$. The transitions are made in accordance with the specified probabilities, if passed before that $s - r$ way, does not impose restrictions on the promotion. Otherwise, the transition is accompanied only by vertical or only horizontal edge. Random process in this case is not a Markov process (see eg. [1,2]). It is clear that through the $r$ steps the particle will be in one of the vertices $(m; n), m + n = r$.

Find the probability to go by $r = m + n$ steps from vertex $\bar{O}$ to the vertex $(m; n), m + n = r$ along the $s - r$ ways. Four cases are possible, specified in $1^9$.

In the first case, the passage of each second, third, $\ldots$, $s$-th horizontal edge (from left to right, if you look at the encoding of the path) and each of the second, third, $\ldots$, $r$-th vertical edge (left to right if you look at the encoding of the path) is mandatory, so we have

$$p_{s-k-r}(\bar{O}; (s \cdot k; r \cdot l)) = \binom{k+l}{k} \cdot \alpha^k \cdot \beta^l.$$  \hspace{1cm} (1)

In the second case $(m = s \cdot k + \delta, k, l \in N, \delta \in \{1, 2, \ldots, s - 1\}, n = r \cdot l)$, we have

$$p_{s-k+\delta-r-l}(\bar{O}; (s \cdot k + \delta; r \cdot l)) = \binom{k+l}{k} \cdot \alpha^{k+1} \cdot \beta^l.$$ \hspace{1cm} (2)
The third case \((m = s \cdot k, n = rl + \delta, k, l \in N, \delta \in \{1; 2; \ldots, r - 1\})\) is similar to the second, so
\[
p_{s \cdot k + rl + \delta}((O); (s \cdot k, \delta; r \cdot l + \delta)) = \binom{k + l}{l} \cdot \alpha^k \cdot \beta^{l+1}.
\]
In the fourth case we have
\[
p_r((O); (s \cdot k + \delta; r \cdot l + \delta_1)) = 0, r = s \cdot k + r \cdot l + \delta + \delta_1.
\]
Obviously, in the case when we consider the problem of the random walk from the vertex \(O\) via \(s - r\) ways, we can replace it with consideration of Markov process of a random walk on the subgraph obtained from graph-lattice removing the edges incident to the vertices of type “\(\sim s; \sim r\)”, assuming that the probability of transition along the edges emerging from each vertex, from which comes out two edges are equal to the initial probabilities, while the remaining edges are equal to one.

For the problem of the probability of getting from an arbitrary vertex along the paths it is necessary to build the subgraph described above, taking the starting vertex as a new beginning of the coordinates.

Thus, we have shown that non-Markov process of a random walk on a \(s - r\) ways on a graph-lattice is reduced to the Markov process of a random walk on the subgraph, corresponding to the starting vertex. This property of the process we have called the local Markov.

In [9–13] considered the problem of random walks on the vertices of graphs with different types of restrictions on the reachability. For the restrictions on reachability (mixed, barrier, valve, magnetic, etc.) this process was not a Markov process. However, its transfer to the scan-graph (the auxiliary graph having a greater size than the original) made it possible to reduce it to a Markov process on the scan-graph.

In the above case, this work could be applied to the “ideology” as consideration is only \(2 - 2\) ways is a restriction on the reachability, but the graph-lattice is infinite graphs, scan-graph it will also be the infinite graph with an even more complex structure and receiving such way the formulas for finding the quantity of \(2 - 2\) ways and formulas for the probability to go from the vertex \(O\) to the another vertex along \(2 - 2\) ways is a matter of doubt. The case of \(2 - 3\) ways is considered in [14].

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