Agmon–Kolmogorov inequalities on $\ell^2(\mathbb{Z}^d)$

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Abstract

Landau–Kolmogorov inequalities have been extensively studied on both continuous and discrete domains for an entire century. However, the research is limited to the study of functions and sequences on $\mathbb{R}$ and $\mathbb{Z}$, with no equivalent inequalities in higher-dimensional spaces. The aim of this paper is to obtain a new class of discrete Landau–Kolmogorov type inequalities of arbitrary dimension:

$$\|\varphi\|_{\ell^\infty(\mathbb{Z}^d)} \leq \mu_{p,d} \|\nabla D\varphi\|_{\ell^2(\mathbb{Z}^d)}^{p/2d} \|\varphi\|_{\ell^2(\mathbb{Z}^d)}^{1-p/2d},$$

where the constant $\mu_{p,d}$ is explicitly specified. In fact, this also generalises the discrete Agmon inequality to higher dimension, which in the corresponding continuous case is not possible.

1 Introduction

In 1912, G. H. Hardy, J. E. Littlewood and G. Pólya (see [HLP52]) proved the following inequalities for a function $f \in L^2(\mathbb{R})$:

$$\|f'\|_{L^2(-\infty, \infty)} \leq \|f\|_{L^2(-\infty, \infty)}^{1/2} \|f''\|_{L^2(-\infty, \infty)}^{1/2},$$

$$\|f'\|_{L^2(0, \infty)} \leq \sqrt{2} \|f\|_{L^2(0, \infty)}^{1/2} \|f''\|_{L^2(0, \infty)}^{1/2},$$

with the constants $1$ and $\sqrt{2}$ being sharp. These results sparked interest in inequalities involving functions, their derivatives and integrals for a century to come. Specifically, in 1913, E. Landau (see [Lan13]) proved the following inequality: For $\Omega \subseteq \mathbb{R}$, and $f \in L^\infty(\Omega)$:

$$\|f'\|_{L^\infty(\Omega)} \leq \sqrt{2} \|f''\|_{L^\infty(\Omega)}^{1/2} \|f\|_{L^\infty(\Omega)}^{1/2},$$

with the constant $\sqrt{2}$ being sharp. This result in turn was motivation for A. Kolmogorov (see [Kol39]), where in 1939 he found sharp constants for the more general case, using a simple, but
very effective inductive argument to extend the case to higher order derivatives:

\[ \| f^{(k)} \|_{L^\infty(\Omega)} \leq C(k, n) \| f^{(n)} \|_{L^\infty(\Omega)}^{k/n} \| f \|_{L^\infty(\Omega)}^{1-k/n}, \]

where, for \( k, n \in \mathbb{N} \) with \( 1 \leq k < n \), he determined the best constants \( C(k, n) \in \mathbb{R} \) for \( \Omega = \mathbb{R} \). Since then, there has been a great deal of work on what are nowadays known as the Landau–Kolmogorov inequalities, which are in their most general form:

\[ \| f^{(k)} \|_{L^p} \leq K(k, n, p, q, r) \| f^{(n)} \|_{L^q}^\alpha \| f \|_{L^r}^\beta, \]

with the minimal constant \( K = K(k, n, p, q, r) \). The real numbers \( p, q, r \geq 1 \); \( k, n \in \mathbb{N} \) with \( (0 \leq k < n) \) and \( \alpha, \beta \in \mathbb{R} \) take on values for which the constant \( K \) is finite (see [Gab67]).

However, literature on discrete equivalents of those inequalities remained very limited for a long time. In 1979, E. T. Copson (see [Cop79]) was one of the first to find equivalent results for sequences, series and difference operators. Indeed, he found the discrete equivalent to (1) and (2). For a square summable sequence, \( \{a(n)\}_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) \) and a difference operator \((Da)(n) := a(n+1) - a(n)\), we have:

\[ \| Da \|_{\ell^2(-\infty, \infty)} \leq \| a \|_{\ell^2(-\infty, \infty)}^{1/2} \| D^2 a \|_{\ell^2(-\infty, \infty)}^{1/2}, \]

\[ \| Da \|_{\ell^2(0, \infty)} \leq \sqrt{2} \| a \|_{\ell^2(0, \infty)}^{1/2} \| D^2 a \|_{\ell^2(0, \infty)}^{1/2}, \]

with the constants 1 and \( \sqrt{2} \) yet again being sharp. Z. Ditzian (see [Dit83]) then extended those results to establish best constants for a variety of Banach spaces, adding equivalent results for continuous shift operators \( f(x + h) - f(x); \ x \in \mathbb{R}, f \in L^2(\mathbb{R}) \).

Comparing inequalities such as (1) and (2), with (3) and (4) respectively, it was suspected that sharp constants were identical for equivalent discrete and continuous Landau–Kolmogorov inequalities for \( 1 \leq p = q = r \leq \infty \). Indeed, in the cases \( p = 1, 2, \infty \), this was true for the whole and semi-axis. However, the general case has since been shown to be false, as for example demonstrated in [KKZ88] by M. K. Kwong and A. Zettl, where they prove that for many values of \( p \), the discrete constants are strictly greater than the continuous ones.

Another important special case of the Landau–Kolmogorov inequalities is the Agmon inequality, proven by S. Agmon (see [Agm10]). Viewed as an interpolation inequality between \( L^\infty(\mathbb{R}) \) and \( L^2(\mathbb{R}) \), he states the following:

\[ \| f \|_{L^\infty(\mathbb{R})} \leq \| f \|_{L^2(\mathbb{R})}^{1/2} \| f' \|_{L^2(\mathbb{R})}^{1/2}. \]

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Thus, throughout this paper we shall call, for a domain $\Omega$, a function $f \in L^2(\Omega)$, a sequence $\phi \in \ell^2(\Omega)$, $\alpha, \beta$ being $\mathbb{Q}$-valued functions of the integers $k, n$ with $k \leq n$ and constants $C(\Omega, k, n)$, $D(\Omega, k, n) \in \mathbb{R}$:

\[
\|f^{(k)}\|_{L^\infty(\Omega)} \leq C(\Omega, k, n) \|f\|_{L^2(\Omega)}^{(k,n)} \|f^{(n)}\|_{L^2(\Omega)}^{(k,n)},
\]

\[
\|D^k \phi\|_{\ell^\infty(\Omega)} \leq D(\Omega, k, n) \|\phi\|_{\ell^2(\Omega)}^{(k,n)} \|D^n \phi\|_{\ell^2(\Omega)}^{(k,n)},
\]

Agmon–Kolmogorov inequalities, where $\Omega := \mathbb{Z}^d$ will be the central concern of this paper. Specifically we only require the case where $k = 0$ and $n = 1$, whereas the other inequalities, i.e. those concerned with higher order, have been discussed in [Sah13].

\section{Agmon–Kolmogorov Inequalities over $\mathbb{Z}^d$}

We introduce our notation for the $d$-dimensional inner product space of square summable sequences. For a vector of integers $\zeta := (\zeta_1, \ldots, \zeta_d) \in \mathbb{Z}^d$, we say $\{\phi(\zeta)\}_{\zeta \in \mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$, if and only if the following norm is finite:

\[
\|\phi\|_{\ell^2(\mathbb{Z}^d)} := \left( \sum_{\zeta \in \mathbb{Z}^d} |\phi(\zeta)|^2 \right)^{1/2}.
\]

Then, for $\varphi, \phi \in \ell^2(\mathbb{Z}^d)$, we let $<\varphi, \phi>_d$ be the inner product on $\ell^2(\mathbb{Z}^d)$:

\[
<\varphi, \phi>_d := \sum_{\zeta \in \mathbb{Z}^d} \varphi(\zeta)\overline{\phi(\zeta)}.
\]

We then let $D_1, \ldots, D_d$ be the partial difference operators defined by:

\[
(D_i \varphi)(\zeta) := \varphi(\zeta_1, \ldots, \zeta_i + 1, \ldots, \zeta_d) - \varphi(\zeta_1, \ldots, \zeta_d),
\]

The discrete gradient $\nabla_D$ shall thus take the following form:

\[
\nabla_D \varphi(\zeta_1, \zeta_2, \ldots, \zeta_d) = (D_1 \varphi(\zeta), D_2 \varphi(\zeta), \ldots, D_d \varphi(\zeta)).
\]

Thus, combining this definition with that of our norm above, we obtain:

\[
\|\nabla_D \varphi\|_{\ell^2(\mathbb{Z}^d)}^2 = \|D_1 \varphi\|_{\ell^2(\mathbb{Z}^d)}^2 + \cdots + \|D_d \varphi\|_{\ell^2(\mathbb{Z}^d)}^2.
\]

Further, we require the following notation:

\section*{Definition 2.1.}

For a sequence $\varphi(\zeta) \in \ell^2(\mathbb{Z}^d)$ with $\zeta := (\zeta_1, \ldots, \zeta_d) \in \mathbb{Z}^d$, for $0 \leq k \leq d$ we define:

\[
[\varphi]_k := \left( \sum_{\zeta_1 \in \mathbb{Z}} \cdots \sum_{\zeta_k \in \mathbb{Z}} |\varphi(\zeta)|^2 \right)^{1/2}.
\]
Remark. We identify that \([\varphi]_0 := |\varphi(\zeta)|\) and if we apply this operator for \(k = d\), i.e. sum across all coordinates, we obtain the \(\ell^2(\mathbb{Z}^d)\)-norm:

\[
[\varphi]_d = \|\varphi\|_{\ell^2(\mathbb{Z}^d)}.
\]

We are interested in a higher-dimensional version of the discrete Agmon inequality (see [Sah10]), which estimates the sup-norm of a sequence \(\phi \in \ell^2(\mathbb{Z})\) as follows:

\[
\|\phi\|_{\ell^\infty(\mathbb{Z})} \leq \|\phi\|_{\ell^2(\mathbb{Z})} \|D\phi\|_{\ell^2(\mathbb{Z})}.
\]

Thus we commence by 'lifting' this estimate to encompass more variables:

**Lemma 2.2 (Agmon–Cauchy Inequality).** For the operator \(D_{k+1}\), acting on a sequence \(\varphi(\zeta) \in \ell^2(\mathbb{Z}^d)\), we have:

\[
\sup_{\zeta_{k+1} \in \mathbb{Z}} [\varphi]_k \leq [D_{k+1}\varphi]_{k+1}^{1/2} [\varphi]_{k+1}^{1/2}.
\]

**Proof.** Using the discrete Agmon inequality on the \((k+1)\)th coordinate, we find:

\[
|\varphi(\zeta_1, \ldots, \zeta_d)|^2 \leq \left( \sum_{l \in \mathbb{Z}} |D_{k+1}\varphi(\zeta_1, \ldots, \zeta_k, l, \zeta_{k+2}, \ldots, \zeta_d)|^2 \right)^{1/2} \left( \sum_{l \in \mathbb{Z}} |\varphi(\zeta_1, \ldots, \zeta_k, l, \zeta_{k+2}, \ldots, \zeta_d)|^2 \right)^{1/2}.
\]

Now we sum with respect to the other coordinates:

\[
\sum_{\zeta_1 \in \mathbb{Z}} \cdots \sum_{\zeta_k \in \mathbb{Z}} |\varphi(\zeta_1, \ldots, \zeta_d)|^2 \leq \sum_{\zeta_1 \in \mathbb{Z}} \cdots \sum_{\zeta_k \in \mathbb{Z}} \left[ \left( \sum_{l \in \mathbb{Z}} |D_{k+1}\varphi(\zeta_1, \ldots, \zeta_k, l, \zeta_{k+2}, \ldots, \zeta_d)|^2 \right)^{1/2} \left( \sum_{l \in \mathbb{Z}} |\varphi(\zeta_1, \ldots, \zeta_k, l, \zeta_{k+2}, \ldots, \zeta_d)|^2 \right)^{1/2} \right],
\]

and use the Cauchy–Schwarz inequality on the \(k\)th coordinate:

\[
\sum_{\zeta_1 \in \mathbb{Z}} \cdots \sum_{\zeta_k \in \mathbb{Z}} |\varphi(\zeta_1, \ldots, \zeta_d)|^2 \leq \sum_{\zeta_1 \in \mathbb{Z}} \cdots \sum_{\zeta_k \in \mathbb{Z}} \left[ \left( \sum_{l \in \mathbb{Z}} \sum_{\zeta_{k-1} \in \mathbb{Z}} |D_{k+1}\varphi(\zeta_1, \ldots, \zeta_k, l, \zeta_{k+2}, \ldots, \zeta_d)|^2 \right)^{1/2} \left( \sum_{l \in \mathbb{Z}} \sum_{\zeta_{k-1} \in \mathbb{Z}} |\varphi(\zeta_1, \ldots, \zeta_k, l, \zeta_{k+2}, \ldots, \zeta_d)|^2 \right)^{1/2} \right].
\]

We repeat this process to finally obtain:

\[
\sum_{\zeta_1 \in \mathbb{Z}} \cdots \sum_{\zeta_k \in \mathbb{Z}} |\varphi(\zeta_1, \ldots, \zeta_d)|^2 \leq \left( \sum_{\zeta_1 \in \mathbb{Z}} \cdots \sum_{\zeta_k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} |D_{k+1}\varphi(\zeta_1, \ldots, \zeta_k, l, \zeta_{k+2}, \ldots, \zeta_d)|^2 \right)^{1/2} \left( \sum_{\zeta_1 \in \mathbb{Z}} \cdots \sum_{\zeta_k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} |\varphi(\zeta_1, \ldots, \zeta_k, l, \zeta_{k+2}, \ldots, \zeta_d)|^2 \right)^{1/2}.
\]

\[\square\]

We estimate the \(\ell^2(\mathbb{Z}^d)\)-norm of a partial difference operator with the \(\ell^2(\mathbb{Z}^d)\)-norm of the sequence itself:
Lemma 2.3. For a sequence \( \varphi \in \ell^2(\mathbb{Z}^d) \) and for \( i \in \{1, \ldots, d\} \), we have:

\[
\|D_i \varphi\|_{\ell^2(\mathbb{Z}^d)} \leq 2\|\varphi\|_{\ell^2(\mathbb{Z}^d)}.
\]

Proof. We show the argument for \( D_1 \) and note that due to symmetry the other cases follow immediately.

\[
\|D_1 \varphi\|_{\ell^2(\mathbb{Z}^d)}^2 = \sum_{\zeta \in \mathbb{Z}^d} |\varphi(\zeta_1 + 1, \ldots, \zeta_d) - (\zeta_1, \ldots, \zeta_d)|^2
\]
\[
\leq 2 \left( \sum_{\zeta \in \mathbb{Z}^d} |\varphi(\zeta_1 + 1, \ldots, \zeta_d)|^2 + \sum_{\zeta \in \mathbb{Z}^d} |\varphi(\zeta_1, \ldots, \zeta_d)|^2 \right)
\]
\[
= 4 \sum_{\zeta \in \mathbb{Z}^d} |\varphi(\zeta_1, \ldots, \zeta_d)|^2
\]
\[
= 4\|\varphi\|_{\ell^2(\mathbb{Z}^d)}^2.
\]

This implies that we can obtain an estimate for any mixed difference operator as follows:

\[
\|D_1 \ldots D_k \varphi\|_{\ell^2(\mathbb{Z}^d)} \leq 2\|D_1 \ldots D_{l-1}D_{l+1} \ldots D_k \varphi\|_{\ell^2(\mathbb{Z}^d)}.
\]

Therefore, by eliminating \( l \) difference operators, our inequality will contain the constant \( 2^l \).

We arrive at our main result, the Agmon–Kolmogorov inequalities on \( \ell^2(\mathbb{Z}^d) \).

Theorem 2.4. For a sequence \( \varphi \in \ell^2(\mathbb{Z}^d) \), and \( p \in \{1, \ldots, 2^d-1\} \):

\[
\|\varphi\|_{\ell^\infty(\mathbb{Z}^d)} \leq \mu_{p,d} \|
abla D \varphi\|_{\ell^2(\mathbb{Z}^d)}^{p/2d} \|\varphi\|_{\ell^2(\mathbb{Z}^d)}^{1-p/2d},
\]

where

\[
\mu_{p,d} := \left( \frac{\kappa_{p,d}}{db^{p/2}} \right)^{1/2d},
\]

and \( \kappa_{p,d} \) is a constant to be determined in the following section.

Proof.

We use Lemma 2.2 and Lemma 2.3 repeatedly:

\[
\|\varphi\|_{\ell^\infty(\mathbb{Z}^d)} \leq \|D_1 \varphi\|_{\ell^2(\mathbb{Z}^d)}^{1/2} \|\varphi\|_{\ell^2(\mathbb{Z}^d)}^{1/2}
\]
\[
\leq \|D_2 D_1 \varphi\|_{\ell^2(\mathbb{Z}^d)}^{1/4} \|D_1 \varphi\|_{\ell^2(\mathbb{Z}^d)}^{1/4} \|D_2 \varphi\|_{\ell^2(\mathbb{Z}^d)}^{1/4} \|\varphi\|_{\ell^2(\mathbb{Z}^d)}^{1/4}
\]
\[
\vdots
\]
\[
\leq \|D_d \ldots D_1 \varphi\|_{\ell^2(\mathbb{Z}^d)}^{1/2d} \ldots \|\varphi\|_{\ell^2(\mathbb{Z}^d)}^{1/2d}
\]
\[
= \|D_d \ldots D_1 \varphi\|_{\ell^2(\mathbb{Z}^d)}^{1/2d} \ldots \|\varphi\|_{\ell^2(\mathbb{Z}^d)}^{1/2d}
\]
\[
\Rightarrow \|\varphi\|_{\ell^\infty(\mathbb{Z}^d)}^{2d} \leq \|D_d \ldots D_1 \varphi\|_{\ell^2(\mathbb{Z}^d)} \ldots \|\varphi\|_{\ell^2(\mathbb{Z}^d)}.
\]
We have generated an estimate by $2^d$ norms, with exactly $2^{d-1}$ norms originating from the term $[D_1 \varphi]^{1/2}$. All those will thus involve the operator $D_1$, or more formally: $|\Xi_1| = 2^{d-1}$, where we let

$$\Xi_1 := \{ \| D_{a_1} \ldots D_{a_k} D_1 \varphi \|_{\ell^2(\mathbb{Z}^d)} | a_i \neq a_j \forall i \neq j; \{a_1, \ldots, a_k\} \subset \{2, \ldots, d\} \}.$$ 

We note that we could also employ estimates by $\| D_1 \varphi \|_{\ell^2(\mathbb{Z}^d)}$ for any $i \in \{1, \ldots, 2^d\}$, but our inequality will not change due to our symmetrising argument. Similarly, we have $2^{d-1}$ norms originating from the term $[\varphi]^{1/2}$, whose estimates will not involve the operator $D_1$. Hence $|\Xi_2| = 2^{d-1}$, where we let

$$\Xi_2 := \{ \| D_{a_1} \ldots D_{a_k} \varphi \|_{\ell^2(\mathbb{Z}^d)} | a_i \neq a_j \forall i \neq j; \{a_1, \ldots, a_k\} \subset \{2, \ldots, d\} \}.$$ 

We will now apply Lemma 2.3 repeatedly, to reduce the order of the operator inside the norms to either 0 or 1. We recognise that we have to estimate all $1^\xi \in \Xi_1$ by $1^\xi_1 := \| D_1 \varphi \|_{\ell^2(\mathbb{Z}^d)}$ or alternatively by $\| \varphi \|_{\ell^2(\mathbb{Z}^d)}$.

Hence, we choose a $p \in \{0, \ldots, 2^{d-1}\}$ to estimate $p$ elements in $\Xi_1$ by $\| D_1 \varphi \|_{\ell^2(\mathbb{Z}^d)}$, leaving $2^{d-1} - p$ elements in $\Xi_1$ to be estimated by $\| \varphi \|_{\ell^2(\mathbb{Z}^d)}$. However, for all $2^{d-1}$ elements $2^\xi \in \Xi_2$, we have to provide an estimate by $2^\xi_1 := \| \varphi \|_{\ell^2(\mathbb{Z}^d)}$ only. This means we have $2^d - p$ elements in $\Xi := \Xi_1 \cup \Xi_2$ to be estimated by $\| \varphi \|_{\ell^2(\mathbb{Z}^d)}$:

$$\| \varphi \|_{\ell^2(\mathbb{Z}^d)}^{2^d} \leq \kappa_{p,d} \| D_1 \varphi \|_{\ell^2(\mathbb{Z}^d)}^p \| \varphi \|_{\ell^2(\mathbb{Z}^d)}^{2^d - p},$$

where $\kappa_{p,d}$ remains a constant of the form $2^z$ with $z \in \mathbb{Q}$, which we leave to be identified in the next section. We thus obtain the following estimate:

$$\| \varphi \|_{\ell^\infty(\mathbb{Z}^d)}^{2^{d+1}/p} \leq \kappa_{p,d}^{2/p} \| D_1 \varphi \|_{\ell^2(\mathbb{Z}^d)}^2 \| \varphi \|_{\ell^2(\mathbb{Z}^d)}^{(2^{d+1}-2p)/p},$$

We now exploit the symmetry of the argument:

$$d \| \varphi \|_{\ell^\infty(\mathbb{Z}^d)}^{2^{d+1}/p} \leq \kappa_{p,d}^{2/p} \left( \| D_1 \varphi \|_{\ell^2(\mathbb{Z}^d)}^2 + \ldots + \| D_d \varphi \|_{\ell^2(\mathbb{Z}^d)}^2 \right) \| \varphi \|_{\ell^2(\mathbb{Z}^d)}^{(2^{d+1}-2p)/p}$$

$$= \kappa_{p,d}^{2/p} \| \nabla D \varphi \|_{\ell^2(\mathbb{Z}^d)}^2 \| \varphi \|_{\ell^2(\mathbb{Z}^d)}^{(2^{d+1}-2p)/p},$$

and finally rearrange:

$$\| \varphi \|_{\ell^\infty(\mathbb{Z}^d)} \leq \left( \frac{\kappa_{p,d}}{d^{1/2}} \right)^{1/2^d} \| \nabla D \varphi \|_{\ell^2(\mathbb{Z}^d)}^{p/2^d} \| \varphi \|_{\ell^2(\mathbb{Z}^d)}^{1-p/2^d}.$$
3 The Constant $\kappa_{p,d}$

It remains to identify the constant $\kappa_{p,d}$, we thus give:

**Theorem 3.1.** We have, for arbitrary dimension $d$ and $p \in \{1, \ldots, 2^{d-1}\}$:

$$\kappa_{p,d} = 2^d \cdot 2^{d-1} - p.$$ 

We will break the proof down into several steps. The method for finding $\kappa_{p,d}$ will rely largely on the following observation:

Let $\tau(\xi)$ be the order of the operator contained in any given $\xi \in \Xi$. Then we let $\Omega_i := \{\xi \mid \tau(\xi) = i\}$, be the set of all terms in the estimate whose operator has a given order $i$. In $\Xi_1$ we have $1 \leq i \leq d$, and in $\Xi_2$, $0 \leq i \leq d - 1$.

**Lemma 3.2.** For the size of $\Omega_i$, we have for $d \geq 2$:

For $\Xi_1$:

$$|\Omega_i| = \binom{d-1}{i-1}, \quad 1 \leq i \leq d,$$

and $\Xi_2$:

$$|\Omega_i| = \binom{d-1}{i}, \quad 0 \leq i \leq d - 1.$$

**Proof.** We follow by induction and prove the case of $\Xi_2$, noting that the argument for $\Xi_1$ is symmetrically identical. We have already seen that the formula is correct for $d = 2$, and now we assume it is true for $d = l$, i.e. for $0 \leq i \leq l - 1$:

$$|\Omega_i| = \binom{l-1}{i},$$

and thus we have the following list:

$$\Xi_2 \frac{2\xi_{2^{l+1}}}{} \ldots \frac{2\xi_{2^2}}{2\xi_1} |\Omega_0| |\Omega_1| |\Omega_2| \ldots |\Omega_{l-1}|$$

$$\Xi^{l+1}_2: \frac{D_l \ldots D_1 \ldots}{2\xi_{2^d}} \ldots \frac{2\xi_{2^2}}{2\xi_1} |\Omega_0| |\Omega_1| |\Omega_2| \ldots |\Omega_{l-1}|$$

Now each term of a given order $\tau$ will, by the Agmon–Cauchy inequality (Lemma 2.2), generate a term of order $\tau$ and one of order $\tau + 1$. Thus we have:

$$\Xi_2 \frac{2\xi_{2^{l+1}}}{} \ldots \frac{2\xi_{2^2}}{2\xi_1} |\Omega_0| |\Omega_1| |\Omega_2| \ldots |\Omega_{l-1}|$$

$$\Xi^{l+1}_2: \frac{D_l \ldots D_1 \ldots}{2\xi_{2^d}} \ldots \frac{2\xi_{2^2}}{2\xi_1} |\Omega_0| |\Omega_1| |\Omega_2| \ldots |\Omega_{l-1}| \binom{l-1}{0} \binom{l-1}{1} \binom{l-1}{2} \ldots \binom{l-1}{l-1}$$
Now we apply the standard combinatorial identity $^aC_b + ^aC_b+1 = ^{a+1}C_{b+1}$ and consider $^aC_0 = ^aC_{a} = 1$, which immediately implies:

$$
\sum_{\xi_1}^2 2^{\xi_1} \cdots 2^{\xi_d} |\Omega_0| |\Omega_1| |\Omega_2| \cdots |\Omega_{l+1}|
$$

and hence for $d = l + 1$, we have:

$$|\Omega_i| = \binom{l}{i},$$

completing our inductive step.

As discussed previously, if we consider to estimate a given $\xi \in \Xi$ using Lemma 2.3, we will, for example, obtain $\|D_1 \cdots D_k \varphi\|_{\ell^2(\mathbb{Z}^d)} \leq 2\|D_1 \cdots D_{l-1} D_l \cdots D_k \varphi\|_{\ell^2(\mathbb{Z}^d)}$. We can see that we generate a factor of 2 for every partial difference operator we eliminate, and thus have, for $1\xi \in \Xi_1$ and $2\xi \in \Xi_2$ with order $\tau(1\xi)$ and $\tau(2\xi)$ respectively:

$$1\xi \leq 2^{\tau(1\xi)-1} \|D_1 \varphi\|_{\ell^2(\mathbb{Z}^d)}, \quad \text{and} \quad 2\xi \leq 2^{\tau(2\xi)} \|\varphi\|_{\ell^2(\mathbb{Z}^d)}.$$

We note here that $\kappa_{p,d}$ will not depend on which $\ell^2(\mathbb{Z}^d)$-norms in $\Xi_1$ are chosen to be estimated by $2\xi_1 := \|\varphi\|_{\ell^2(\mathbb{Z}^d)}$. The reason for this is transparent when considering that the sum of all the orders $\sum_{i=1}^{2d-1} \tau(1\xi_i)$ is a constant and needs to be reduced to the constant $p \cdot \tau(1\xi_1) = p$, generating a unique $\kappa_{p,d}$.

**Lemma 3.3.** The $\min_p \kappa_{p,d}$ will be attained at $p = 2^{d-1}$ and takes on the following explicit form:

$$
\kappa_{2^{d-1},d} = \prod_{i=0}^{d-1} 2^{(\tau(d^{-1}))}.
$$

**Proof.** Our minimum constant for $\Xi_1$ in fact occurs if we choose all $1\xi_1 \in \Xi_1$ to be estimated by $\|D_1 \varphi\|_{\ell^2(\mathbb{Z}^d)}$, i.e. choose $p = 2^{d-1}$, the maximum $p$ possible. Our minimum constant, denoted by $\rho^1_d$, for all terms in $\Xi_1$ will thus be:

$$\rho^1_d = \prod_{k=1}^{2^{d-1}} 2^{\tau(1\xi_k)-1}.$$

Instead of examining each individual element $1\xi$, we consider that all $1\xi$ of equal order $i$ generate the same constant, namely $2^{i-1}$. Thus we collect all $1\xi$ of the same order, and obtain:

$$
\rho^1_d = \prod_{i=1}^{d} 2^{(i-1)|\Omega_i|} = \prod_{i=1}^{d} 2^{(i-1)(\ell_{i-1})}.
$$
Then we need to estimate all $\xi \in \Xi_2$, and we proceed as for $\Xi_1$. All $\xi$ need to be estimated by $\|\varphi\|_{L^2(\mathbb{Z}^d)}$, each generating the constant $2^i$, forming the equivalent pattern as that of $\Xi_1$. We thus obtain, for the minimal constant $\rho^2_\ell$:

$$\rho^2_\ell = \prod_{i=0}^{d-1} 2^{||\Omega_i||} = \prod_{i=0}^{d-1} 2^{i(d-1)}.$$  

We now see that $\rho^2_\ell = \rho^1_\ell$, and:

$$\kappa_{2^{d-1},d} = \rho^2_\ell \rho^1_\ell = \prod_{i=0}^{d-1} 2^{2i(d-1)}.$$ 

We are now finally in a position to prove Theorem 3.1:

Proof of Theorem 3.1 We are left to analyse the constant’s dependence on our choice of $p$. First we note that in addition to the constant generated above, we will have chosen $2^{d-1} - p$ terms to be further reduced to $\|\varphi\|_{L^2(\mathbb{Z}^d)}$, each generating a power of 2. Hence we additionally need to multiply $\kappa_{2^{d-1},d}$ by $2^{2^{d-1} - p}$. Thus our final constant will be:

$$\kappa_{p,d} = 2^{2^{d-1} - p} \prod_{i=0}^{d-1} 2^{2i(d-1)} = 2^{2^{d-1} - p + 2 \sum_{i=0}^{d-1} i(d-1)}.$$ 

Then we can simplify this further by considering the binomial formula $(1 + X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k$. We differentiate with respect to $X$ and set $X = 1$:

$$n \cdot 2^{n-1} = \sum_{k=0}^{n} \binom{n}{k}.$$

Thus we arrive at:

$$\kappa_{p,d} = 2^d 2^{d-1} - p.$$ 

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