Nonlinear strain waves in a metamaterial defined a mass-to-mass chain

Vladimir Erofeev, Daniil Kolesov* and Alexey Malkhanov
Mechanical Engineering Research Institute of RAS, Nizhny Novgorod, Russia

* alandess@yandex.ru

Abstract. A mathematical model is proposed that is a chain of oscillators consisting of nonlinear elastic elements and masses, each of which contains an internal nonlinear oscillator. Such a model describes a mass-to-mass acoustic metamaterial class. It is shown that the obtained system of equations can be reduced to the nonlinear evolutionary equation of Benjamin-Bona-Mahoni, showing that spatially localized nonlinear deformation waves (solitons) can form in the metamaterial under dynamic action on it.

1. Introduction
Acoustic (or mechanical) metamaterials, being, in fact, not materials, but cellular periodic structures, in the long-wave range behave like continuous materials. The study of the dispersion, dissipation, and demonstration of the nonlinearity of acoustic waves in metamaterials is of great interest. This interest is due, both to the cognitive aspect of the issue, and to the prospects for practical applications of metamaterials [1-4], among which the possibility of creating superabsorbents of sound on their basis [5-8].

Guided by the mathematical analogy between acoustic and electromagnetic waves, many researchers have tried to construct continual models of mechanical metamaterials. However, great success on this path was not achieved, since the mechanical analogues of really existing materials with negative permittivity are deformable solids having a negative mass, density, or negative modulus of elasticity [9 - 13]. And there is no such material in nature. It was possible to avoid this shortcoming in the way of structural modeling of metamaterials.

In [14], was considered a one-dimensional chain containing identical masses $m_1$, connected by elastic elements (springs), having the same rigidity $k_1$, at the same time each mass inside itself contains another mass $m_2$ and one more elastic element - a spring with rigidity $k_2$ (Figure 1). Such a model, called the mass-in-mass chain, does not give the mentioned absurd results.

2. Mathematical Model
We generalize the model [14] by taking into account the quadratic nonlinearity of the external and internal elastic elements.
The potential energy of the unit cell of the mass-in-mass chain is written as:

\[ W^{(j)} = \frac{1}{2} \left[ k_1 \left( u_1^{(j+1)} - u_1^{(j)} \right)^2 + k_2 \left( u_2^{(j)} - u_1^{(j)} \right)^2 + h_1 \left( u_1^{(j+1)} - u_1^{(j)} \right)^3 + h_2 \left( u_2^{(j)} - u_1^{(j)} \right)^3 \right], \]  

(1)

and its kinetic energy in the form:

\[ T^{(j)} = \frac{1}{2} \left[ m_1 \left( \dot{u}_1^{(j)} \right)^2 + m_2 \left( \dot{u}_2^{(j)} \right)^2 \right]. \]  

(2)

Let us suppose that \( u_1(x) \) and \( u_2(x) \) are continuous functions, which describe the displacements of all masses \( m_1 \) and \( m_2 \), respectively. Taking into account the expansion of displacements in a Taylor series up to the second term, we obtain

\[ u_1^{(j+1)} = u_1(x + L) = u_1(x) + \frac{\partial u_1}{\partial x} L = u_1^{(j)} + \frac{\partial u_1}{\partial x} L. \]  

(3)

The technique of expansion displacements in (3) was effectively applied by Kunin I.A. [15] in the transformation of multi-mass discrete systems into a quasi-continuum.

The densities of the potential and kinetic energies for the equivalent continuum, obtained from (5) and (6), can be written in the form:

\[ W = \frac{1}{2L} \left[ k_1 \left( \frac{\partial u_1}{\partial x} L \right)^2 + k_2(u_2 - u_1)^2 + h_1 \left( \frac{\partial u_1}{\partial x} L \right)^3 + h_2(u_2 - u_1)^3 \right], \]  

(4)

\[ T = \frac{1}{2} \left[ m_1(\dot{u}_1)^2 + m_2(\dot{u}_2)^2 \right]. \]  

(5)

Let us construct from (4) and (5) the Lagrange function \( L = T - W = L(\dot{u}_1, \dot{u}_2, \dot{u}_{1x}, u_1, u_2) \) and take into account equations well known from analytical mechanics

\[
\left\{ \begin{array}{l}
\frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{u}_1} \right) + \frac{\partial}{\partial x} \left( \frac{\partial L}{\partial \dot{u}_{1x}} \right) - \frac{\partial L}{\partial u_1} = 0 \\
\frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{u}_2} \right) - \frac{\partial L}{\partial u_2} = 0
\end{array} \right.
\]

to get the system of equations in displacements:

\[
\frac{m_1}{L} \ddot{u}_1 - k_1 L \frac{\partial^2 u_1}{\partial x^2} - 3h_1 L^3 \frac{\partial u_1}{\partial x} \frac{\partial^2 u_1}{\partial x^2} - \frac{k_2}{L} (u_2 - u_1) - \frac{3h_2}{2L} (u_2 - u_1)^2 = 0,
\]

\[
\frac{m_2}{L} \ddot{u}_2 - \frac{k_2}{L} (u_2 - u_1) - \frac{3h_2}{2L} (u_2 - u_1)^2 = 0.
\]

(6)

Let us proceed in the system (6) to a moving system of coordinates \( \xi = x - ct, \tau = ct \), where \( c \) – wave velocity (previously unknown), \( \varepsilon \) – small parameter which characterized the maximal amplitude to wavelength ratio. The choice of variables is due to the fact that perturbations, propagating at a speed \( c \) along the \( x \) axis, slowly evolving over time due to nonlinearity and dispersion.

Figure 1. In finite mass-to-mass lattice structure.
Let us represent the displacements in the form of expansions into series in powers of a small parameter

\[ u_1(\xi, \tau) = u_1^{(0)}(\xi, \tau) + \varepsilon u_1^{(1)}(\xi, \tau) + \cdots, \]
\[ u_2(\xi, \tau) = u_2^{(0)}(\xi, \tau) + \varepsilon u_2^{(1)}(\xi, \tau) + \cdots. \]  

(7)

After substitution (6) into (7) we will get the system of equations of different order of smallness with respect to powers of parameter \( \varepsilon \). The zero-th approximation allows us to identify the value of wave velocity \( c = \sqrt{\frac{k}{m_1} L} \) and gives us the displacements relation

\[ u_1^{(0)} = u_2^{(0)} = m_2 k_1 L^2 \frac{\partial^2 u_2^{(0)}}{\partial \xi^2}. \]  

(8)

The first approximation gives us the evolutionary equation

\[-2\sqrt{m_1 k_1} \varepsilon \frac{\partial U}{\partial \tau} - \frac{m_2 k_1 L^2 \partial U}{m_1} \frac{\partial}{\partial \xi} + 2m_2 k_1 L \frac{1}{k_1} \frac{\partial^3 U}{\partial \xi^2 \partial \tau} - 3h_1 L^3 \frac{\partial U}{\partial \xi} + \frac{3h_1 L^5 m_2 k_1}{m_1 k_2} \frac{\partial U}{\partial \xi} \frac{\partial^2 U}{\partial \xi^2} - 3h_2 L^3 \frac{\partial^2 U}{\partial \xi^2 \partial \tau} - \frac{3h_2}{2L} \left( \frac{m_2 k_1 L^2}{m_1 k_2} \right)^2 \frac{\partial^3 U}{\partial \xi^3}, \]  

(9)

where \( U = \frac{\partial^2 u_2^{(0)}}{\partial \xi^2} \).

Let us introduce dimensionless variables \( \xi / \xi_0 = X, \tau / \tau_0 = T \), where spatial and time scales are chosen in the way they satisfy the relation \( \xi_0 / \tau_0 = \frac{3h_1 L^2}{2\sqrt{m_1 k_1} \varepsilon} \).

In the new variables the equation (9) takes the form

\[ \frac{\partial U}{\partial T} + d_1 \frac{\partial U}{\partial X} - d_2 \frac{\partial^3 U}{\partial X^2 \partial T} + U \frac{\partial U}{\partial X} + g_1 \left( \frac{\partial^2 U}{\partial X^2} \right) + g_2 \frac{\partial^2 U}{\partial X^2 \partial X} + g_3 \left( \frac{\partial U}{\partial X} \right)^2 = 0, \]  

(10)

where \( d_1 = \frac{m_2 k_1}{3m_1 h_1 L^2} \), \( d_2 = \frac{\varepsilon 10^2 m_2 k_1}{m_1 k_2} \), \( g_1 = -\frac{10^4 m_2 k_1}{m_1 k_2} \), \( g_2 = \frac{10^{-4} (m_2 k_1)}{(m_1 k_2)^2} \), \( g_3 = \frac{h_2 (m_2 k_1)^2}{20h_1 L^3 (m_1 k_2)^2} \).

For long wave processes (\( \xi_0 \sim 10L \)), \( g_1, g_2, g_3 \ll 1 \) and the equation (10) takes the form of well known in nonlinear wave dynamics equation by Benjamin-Bona-Mahoney:

\[ \frac{\partial U}{\partial T} + d_1 \frac{\partial U}{\partial X} - d_2 \frac{\partial^3 U}{\partial X^2 \partial T} + U \frac{\partial U}{\partial X} = 0. \]  

(11)

3. Solitary Stationary Wave

This equation has an analytical solution in the form of solitary stationary wave (soliton)

\[ U = 3(V - d_1) \text{sech}^2 \left[ \frac{\sqrt{V - d_1}}{2Vd_2} (X - VT) \right], \]  

(12)

where \( d_2 > 0, V > d_1 \).

From the (12) it follows that for the metamaterial under consideration the amplitude of the soliton of deformations is identified with the help of the relation

\[ A = 3 \left( V - \frac{m_2 k_1}{3m_1 h_1 L^2} \right), \]  

(13)

and its width is identified with the help of relation

\[ \Delta = \frac{2V m_2 k_1}{\varepsilon 10^2 m_2 k_1 L} \sqrt{V - \frac{m_2 k_1}{3m_1 h_1 L^2}}. \]  

(14)
4. Conclusions
As a result of analysis of the long-wavelength approximation of the obtained system, it is shown that spatially localized nonlinear deformation waves (solitons) can be formed in a metamaterial, under dynamic influence on it. The dependencies connecting the parameters of a localized wave are determined: amplitude, velocity and width with inertial and elastic characteristics of the metamaterial.

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