Irrational charge from topological order

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Topological or deconfined phases of matter exhibit emergent gauge fields and quasiparticles that carry a corresponding gauge charge. In systems with an intrinsic conserved $U(1)$ charge, such as all electronic systems where the Coulombic charge plays this role, these quasiparticles are also characterized by their intrinsic charge. We show that one can take advantage of the topological order fairly generally to produce periodic Hamiltonians which endow the quasiparticles with continuously variable, generically irrational, intrinsic charges. Examples include various topologically ordered lattice models, the three dimensional RVB liquid on bipartite lattices as well as water and spin ice. By contrast, the gauge charges of the quasiparticles retain their quantized values.

Introduction: The phenomenon of quantum number fractionalization [1] is among the most striking collective phenomena in condensed matter physics: the low-energy excitations of a many-body system can exhibit quantum numbers which cannot be obtained by simple addition or subtraction of those of the high-energy degrees of freedom. The best-known example is charge fractionalization: excitations can carry fractional charge, e.g. $e/3$ in the case of Laughlin quasiparticles, even though we are quite confident that the underlying degrees of freedom are electrons with quantized elementary charge $e$.

In one dimension fractionalization is associated with soliton formation, but in the known examples in higher dimensions it goes along with phases of matter that exhibit emergent gauge fields [2]. Such phases are variously called fractionalized, deconfined or topological; the last term is the one we shall use in this paper even though strictly it refers only to phases where the emergent gauge field is described by a purely topological action [3]. As a consequence, the low energy excitations are classified not only by their fractionalized intrinsic quantum numbers but also by their emergent gauge charge(s) and/or fluxes under the emergent gauge field(s).

In the case of the Laughlin quasiparticles, this distinction is not crucial as they carry a gauge flux and an intrinsic electric charge that are slaved to each other by a constraint [4]. However, as we show here, in general this distinction is important: While the emergent gauge quantum numbers of a state can effectively be considered to be fixed [5], it is possible to tune the intrinsic quantum numbers continuously by perturbing the Hamiltonian.

Specifically, we consider systems that are characterized by a microscopically conserved $U(1)$ charge which we call the intrinsic charge. When the Coulomb interaction is explicitly included, this charge is, as usual, the source of an electric field; in other cases it is simply proportional to a conserved particle number. We show that the intrinsic charge of the quasiparticles in our examples can take on continuous, generally irrational, values. This generalizes the one dimensional irrational charge found by Brazovskii [6] and by Rice and Mele [7] to higher dimensions and ex post facto reinterprets their result as establishing the independence of the intrinsic charge from the topological charge characterizing solitons in their problem. [We also note the stimulating work by Chamon and collaborators [8, 9] who have generalized the field theoretic treatment [10] of solitonic fractionalization and the Brazovskii–Rice–Mele effect to two dimensions. Interestingly, they find logarithmically confined excitations with continuously variable charge which re-rationalizes upon inclusion of a gauge field to deconfine them.]

In the balance of the paper we describe this phenomenon in more detail in three different contexts. First, we introduce a variant of the well studied quantum dimer model on hypercubic lattices which allows a controlled computation of the irrational charge. Next, we consider the short ranged RVB liquid phase on hypercubic lattices in $d \geq 3$ and show that its excitations can be made to carry irrational intrinsic charges in the presence of an (inevitable) electron-phonon coupling. Finally, we review the corresponding physics in water and spin ice, which exhibit a classical limit of topological order, and provide experimentally realized instances of this phenomenon.

Topologically ordered lattice models: We first present a simple lattice model for which an explicit calculation is possible. This model describes bosons with charge $q$, created by $b^\dag_{i\alpha}$, living on the links of a hypercubic lattice in $d \geq 3$. The links are labeled by the sites they join, the Roman/Greek letters referring to the A/B sublattices of these bipartite lattices, with Hamiltonian

$$H = H_U + H_{QDM} + H_s , \tag{1}$$

$$H_U = U \sum_{i} \sum_{\alpha,\beta \in \eta n(i)} (b^\dag_{i\alpha} b_{i\alpha})(b^\dag_{j\beta} b_{j\beta}) \tag{2}$$

$$H_{QDM} = - t \sum_p \sum_{\{i,j,\alpha,\beta\} \in p} b^\dag_{i\alpha} b_{j\beta} b_{j\alpha} b_{i\beta} + h.c. \tag{3}$$

$$+ v \sum_p \sum_{\{i,j,\alpha,\beta\} \in p} (b^\dag_{i\alpha} b_{i\alpha})(b^\dag_{j\beta} b_{j\beta}) + \{i \leftrightarrow j\}$$

$$H_s = -s \sum_i \sum_{\alpha,\beta \in \eta n(i)} b^\dag_{i\alpha} b_{i\beta} + h.c. \tag{4}$$
The sum on $i$ runs over all $A$ sublattice sites, $nn(i)$ refers to the $B$ sublattice neighbors of site $i$, and $p$ indexes the plaquettes of the lattice. The boson density per site is fixed to be $1/2$. Note that the boson number is conserved and defines the intrinsic charge in this problem.

We now consider the terms in the Hamiltonian in order of their strength, $U \gg v, t \gg s$, initially ignoring $H_s$.

In this limit, $H_U$ singles out a low energy manifold of boson configurations with exactly one boson occupying the bonds emanating from a given site—these are in precise correspondence with hard core dimer coverings of the lattice. Fig. 1. This manifold is separated from all excited states by a gap; in our ordering of energy scales the leading order dynamics takes place entirely within it. Indeed, the action of $H_{QDM}$ within this dimer manifold is exactly that of the quantum dimer model which is known to exhibit a deconfined $U(1)$ “Coulomb” phase for $v \lesssim t$ on the hypercubic lattice in $d \geq 3$ [11, 12].

The gauge field that is deconfined in this phase can be traced to the structure of the dimer states as follows. Let us assign an oriented lattice flux to each link [11]:

$$F_{i\alpha} = b^\dagger_{i\alpha} b_{i\alpha} - \frac{1}{2d} = -F_{i\alpha}. \quad (5)$$

Each dimer configuration then has a vanishing lattice divergence $\nabla \cdot F = 0$ at each site of the lattice. The solution of this sourceless “Gauss’s law” constraint, $F = \nabla \times A$ uncovers the gauge field at issue. The low energy physics of the dimer manifold is that of Maxwell theory with a gapless mode corresponding to the photon [11].

In addition, there are point-like excitations just above the gap of order $U$, violating Gauss’s law, which are thus charged under $A$. Specifically, these sites are associated with monomers and trimers, i.e. they have no or two dimers emanating from them. Crucially, there are two species of each carrying opposite gauge charge, one for each sublattice. Monomers on sublattice $A/B$ (and trimers on sublattice $B/A$) act as sinks/sources of the “electric” flux $F$. These particles interact much as their analogs in fundamental electrodynamics.

Finally, let us identify the the intrinsic charge of the monomers and trimers at this stage of the analysis. As a pair of monomers/trimers is created by removing/adding one boson, they are seen to carry intrinsic charges $\mp q/2$. While we have derived the gauge and intrinsic charges in a leading order analysis in the large $U$ limit, these are exact results at large $U$ for the combination $H_U + H_{QDM}$.

Now let us turn to the last term $H_s$, which breaks the sublattice symmetry by introducing boson hopping only between pairs of links meeting in the $A$ sublattice. A characteristic property of topologically ordered systems is that any small perturbation leaves the low energy structure qualitatively intact. Thus quite generally we expect a sufficiently small value of $s$ to lead to another Hamiltonian with a gapless photon and monomers/trimers with the gauge charges identified above [3]. For our model, the situation is particularly robust, as $H_s$ has no matrix elements within the dimer manifold. It is only at order $s^2/U$ that we get a renormalization of $t$, innocuous at small $s/U$ as it moves the system to a nearby point in the parameter space of the dimer model. Higher order corrections generate other terms but the general conclusion will remain.

![FIG. 1: (a) A dimer covering representing either bosons living on the bonds or singlets formed between spins on the adjacent sites. The arrows indicate the orientation of the induced dipole moments on the dimers. The modification of the state along the (fat green) oriented string of dipoles generates the dimer configuration with two monomers/spinons shown in (b). We have indicated the sublattices where appropriate. We are displaying a two-dimensional cartoon, even though many of our results hold in $d \geq 3$ only.]

However, there is another effect of turning on $H_s$—it leads to a redistribution of charge in the perturbatively modified dimer states. This modifies the intrinsic charge of the gauge charged excitations. The easiest way to compute the change in intrinsic charge is to observe that a given dimer in each configuration acquires a dipole moment of magnitude $P = \frac{1}{2} q_{ad}(s/U)^2$ ($a$ is the eventually irrelevant lattice constant). This moment is oriented from sublattice $A$ to $B$, and is computed with respect to its nominal position at the center of a bond. The creation of a pair of monomers/trimers then involves the insertion of a suitable string of such dipoles oriented head to tail (see Fig. 1 for an example). Such a string exhibits a pair of endpoint charges of magnitude $P/a$.

Adding this correction to the starting values we find that monomers/trimers now carry the intrinsic charges

$$Q_m/q = -\frac{1}{2} \pm \frac{d}{2} \left( \frac{s}{U} \right)^2; \quad Q_t/q = +\frac{1}{2} \pm \frac{d}{2} \left( \frac{s}{U} \right)^2, \quad (6)$$

up to corrections of $O(s/U)^4$ on sublattice $A/B$. The reader can check that a direct computation of the coarse grained intrinsic charge agrees with these values.

The ingredients needed above to decouple gauge and intrinsic charges – topological order and sublattice specific quasiparticles – are also present in various other models with topological phases to which our analysis can be applied mutatis mutandis. Examples of these are the bosonic models of [13, 14] which exhibit a conserved boson number in addition to emergent $U(1)$ gauge fields coupled to Higgs scalars of various charge. Specifically, our construction works readily for the $\mathbb{Z}_2 \times \mathbb{Z}_3$ Higgs phase on a $d = 2$ triangular lattice of [14] and for the
Coulomb phase in $d \geq 3$ for the charge 3 and higher generalizations of the models presented in [13]. Specifically, what is required in these cases is the introduction of a staggered potential that breaks the symmetry between sublattices. Finally our construction also generalizes to quantum Hall states in the Tao-Thouless limit [15].

This is also a good place to note that there are cases when our construction does not work. These include the charge 2 Higgs condensed phase of [13] and the de-confined phase in the triangular lattice quantum dimer model [10] both of which exhibit $Z_2$ topological order and an absence of sublattice specific quasiparticles. The coarse grained intrinsic charge in such cases cannot be altered by breaking sublattice symmetries.

**RVB on bipartite lattices:** The quantum dimer model was constructed to capture the essential physics of the short ranged resonating valence bond (SR-RVB) state proposed by Anderson [17] as a description of a disordered, non-magnetic ground state of a Mott insulator – the prototypical spin liquid. As such, our above analysis suggests that the phenomenon of charge irrationalization should extend to SR-RVB physics in its electronic setting. This is indeed true but the details work out somewhat differently, as we describe next.

The largest energy scale in the Mott insulator is the charge gap $\Delta_c$ stemming from the Coulomb repulsion. Below it the dynamical degrees of freedom in the singly occupied manifold are the spins of the electrons. In an SR-RVB insulator, there is a second scale $\Delta_{RVB} < \Delta_c$, which characterizes the formation of local singlets, with each spin forming a singlet with exactly one neighbor. At lower energies the basic degrees of freedom of the SR-RVB models are thus again hard core dimers on bonds, but now they encode the formation of a singlet bond between the two spins at the ends of the dimer, as opposed to the boson living on the bond in our model above. A concomitant key difference is that the conserved intrinsic charge is not the number of dimers on bonds but instead the number of electrons on sites.

The physics in the valence bond/dimer manifold is again well captured by the processes contained in $H_{QDM}$ [4] and the states and spectra are isomorphic [18]. Thus on hypercubic lattices in $d \geq 3$, the low energy physics of the SR-RVB state is also described by an emergent $U(1)$ gauge field with a Maxwell action.

The gauge charged excitations are spinons which are the analogs of the monomers above: spins which are not involved in a singlet bond. Again they come in two sublattice flavors with opposite gauge charges and *prima facie* they carry spin $S_s = 1/2$ and intrinsic charge $Q_s = 0$ as the distribution of electronic charge appears undisturbed by their formation.

However, that last statement ceases to be true when we add two ingredients to this discussion. First, we follow our prescription above and explicitly break the symmetry between the sublattices—this time via a weak staggered potential which favors electronic occupancy of the $A$ sublattice [14]. Secondly, note that electron hopping between neighboring sites is increased if they are connected by a spin singlet, because both of the Pauli principle forbidding hopping between aligned spins, and due to the electron-lattice coupling. These two ingredients combine so that each dimer now involves not only a spin singlet but also a nonzero excess probability $\delta$ of finding both electrons on its sublattice $A$ rather than $B$ end. Thus dimers carry an electric dipole moment $P = 2e\delta a$ oriented from sublattice $B$ to $A$. As before, the result is to endow spinons created by the insertions of strings of dipoles with charges $\pm Q_s = e\delta$, which, being continuously tunable by means of the sublattice potential and/or electron-lattice coupling, are in general again irrational. We note that upon acquiring charge, the spinons contribute to the optical conductivity in the Mott gap.

Finally, we would be remiss if we did not remind the reader that the SR-RVB liquid [10] on the non-bipartite triangular lattice, is free of irrationality following our comments at the end of the last section.

**Water and spin ice:** The above constructions establish a point of principle but they do not currently apply directly to an experiment. To remedy this, we turn to a pair of materials one of which should be familiar even to string theorists among our readers. We find that deconfined irrational electric charge exists even in the absence of any explicit sublattice symmetry breaking. This we trace to a classical limit of $U(1)$ topological order.

The starting point of our analysis once again consists of identifying a solenoidal vector field which can be done in either the cubic or hexagonal ice phases I$_h$ or I$_h$ [12]. In either phase, the structure of ice can be described as a tetrahedrally coordinated network of oxygen ions with single protons placed asymmetrically between the pair of oxygens they bond. The ice rules state that exactly two protons sit close to each oxygen (thereby forming an “H$_2$O molecule”). Assigning a flux $\pm 1$ to the links with a close (distant) proton, this defines the solenoidal field we require. Note that the emergent “electric” flux on a bond comes automatically with an actual electric dipole moment and thus we need not take recourse to the sublattice constructions used earlier.

Bjerrum defects are the most common violation of the ice rules carrying a gauge charge. They involve taking an “H$_2$O molecule” and rotating it so that one of its bonds has no protons sitting on it, and another two (one near and one far). A simple point-charge model gives an irrational charge of [22]

$$Q_{\text{Bjerrum}} = \pm \sqrt{3}\mu/a \approx \pm 0.38e,$$

where $\mu$ is the dipole moment of the “H$_2$O molecule”, $a$ the distance between neighbouring oxygens, and $e$ the electronic charge.

Closely related to this are the ionic defects, a pair of which carrying opposite gauge and electric charge can be
created by finding an “electric” flux tube/dipole string and reversing it. This creates two excitations located at the ends which carry gauge charge ±2 and electric charge

\[ Q_{\text{ion}} = \pm (|e| - |Q_{\text{Bjerrum}}|) \, . \]  

With \( \mu \) an accident of local chemistry, and as the oxygen separation is tunable by the application of pressure, it is clear that \( Q_{\text{Bjerrum,ion}} \) are continuously tunable as well.

In this discussion we appear to have dispensed with topological order altogether but this is misleading. We need our defects to be deconfined. This can be accomplished most robustly by picking a quantum Hamiltonian acting on the ice ground state manifold that is in a topological phase [20]. In actual ice, instead, there is (to good accuracy) a classical Hamiltonian – that leaves ice rule compliant configurations energetically degenerate – which also exhibits a purely static emergent gauge field that does not confine defects charged under it [21].

Thus ice also supports deconfined irrationally charged excitations on account of (classical) topological order.

While the full context discussed in this paper is new, the fact that defects in water ice carry irrational electric charge has been long recognized [22]. In chemistry, irrational charge is of course not an unusual occurrence, as ions can ‘share’ an electron in a chemical bond in a non-quantized way. However, turning this charge into a deconfined quasiparticle is not trivial. The fact that common water ice pulls off this magic trick is remarkable.

This story takes another fascinating twist in the case of spin ice, a magnetic analogue of water ice. The statistical mechanics of water ice is locally that of spin ice. The basic electric degrees of freedom, however, are replaced by magnetic ones: Ising magnetic reside on the midpoints of the links of a diamond lattice of bond length \( a \), whose local easy axes are the bond directions.

This analogy led to the recent realization [23] that there are excitations in spin ice that carry irrational magnetic charge \( Q_{\text{monopole}} = \pm 2\mu/a \approx \pm q_D/a \), where \( q_D \) is the size of Dirac’s monopole. These magnetic monopoles interact Coulombically and are sources of the magnetic field \( H \) (rather than the magnetic flux density \( B \)). In the absence of microscopic monopoles, magnetic charge arises from the fractionalization of the dipoles.

**In closing:** The two central messages of this paper are first, that gauge and intrinsic charges in topologically ordered systems are logically distinct and second, that this distinction permits the intrinsic charge to take on a continuous set of, generically, irrational values. We have presented examples in \( d = 2 \) and \( d = 3 \) which involve \( Z_3 \) and \( U(1) \) gauge fields in quantum and classical settings. We expect that such examples could be multiplied for other instances of topological order with gapped matter, including cases with non Abelian gauge fields, and also for quantum numbers other than the intrinsic charge.

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