Regression-based variance reduction approach for strong approximation schemes

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Abstract In this paper we present a novel approach towards variance reduction for discretised diffusion processes. The proposed approach involves specially constructed control variates and allows for a significant reduction in the variance for the terminal functionals. In this way the complexity order of the standard Monte Carlo algorithm ($\varepsilon^{-3}$) can be reduced down to $\varepsilon^{-2}\sqrt{|\log(\varepsilon)|}$ in case of the Euler scheme with $\varepsilon$ being the precision to be achieved. These theoretical results are illustrated by several numerical examples.

1 Introduction

Let $T > 0$ be a fixed time horizon. Consider a $d$-dimensional diffusion process $(X_t)_{t \in [0,T]}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$ by the Itô stochastic differential equation

$$dX_t = \mu(X_t) \, dt + \sigma(X_t) \, dW_t, \quad X_0 = x_0 \in \mathbb{R}^d,$$

for Lipschitz continuous functions $\mu: \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma: \mathbb{R}^d \to \mathbb{R}^{d \times m}$, where $(W_t)_{t \in [0,T]}$ is a standard $m$-dimensional $(\mathcal{F}_t)$-Brownian motion. Suppose we want to find a continuous function

$$u = u(t,x): [0,T] \times \mathbb{R}^d \to \mathbb{R},$$
which has a continuous first derivative with respect to \( t \) and continuous first and second derivatives with respect to the components of \( x \) on \([0, T) \times \mathbb{R}^d\), such that it solves the partial differential equation
\[
\frac{\partial u}{\partial t} + \mathcal{L} u = 0 \quad \text{on} \ [0, T) \times \mathbb{R}^d, \tag{2}
\]
\[
u(T, x) = f(x) \quad \text{for} \ x \in \mathbb{R}^d, \tag{3}
\]
where \( f \) is a given Borel function on \( \mathbb{R}^d \). Here, \( \mathcal{L} \) is the differential operator associated with the equation (1):
\[
(\mathcal{L} u)(t, x) := d \sum_{k=1}^{d} \mu_k(x) \frac{\partial u}{\partial x_k}(t, x) + \frac{1}{2} \sum_{k, l=1}^{d} (\sigma \sigma^\top)_{kl}(x) \frac{\partial^2 u}{\partial x_k \partial x_l}(t, x),
\]
where \( \sigma^\top \) denotes the transpose of \( \sigma \). Under appropriate conditions on \( \mu, \sigma \) and \( f \), there is a solution of the Cauchy problem (2)–(3), which is unique in the class of solutions satisfying certain growth conditions, and it has the following Feynman-Kac stochastic representation
\[
u(t, x) = \mathbb{E}[f(X_t^{0,x})] \tag{4}
\]
(see Section 5.7 in [5]), where \( X_t^{x,x} \) denotes the solution started at time \( t \) in point \( x \). Moreover it holds
\[
\mathbb{E}[f(X_T^{0,x})] = u(t, X_t^{0,x}), \quad \text{a.s.}
\]
for \( t \in [0, T] \) and
\[
f(X_T^{0,x}) = \mathbb{E}[f(X_T^{0,x})] + M_T^x, \quad \text{a.s.} \tag{5}
\]
with
\[
M_T^x := \int_0^T \nabla_x u(t, X_t^{0,x}) \sigma(X_t^{0,x}) \, dW_t = \int_0^T \sum_{k=1}^{d} \frac{\partial u}{\partial x_k}(t, X_t^{0,x}) \sum_{i=1}^{m} \sigma_i(X_t^{0,x}) \, dW_t. \tag{6}
\]

The standard Monte Carlo (SMC) approach for computing \( u(0, x) \) at a fixed point \( x \in \mathbb{R}^d \) basically consists of three steps. First, an approximation \( \overline{X}_T \) for \( X_T^{0,x} \) is constructed via a time discretisation in equation (1) (we refer to [6] for a nice overview of various discretisation schemes). In this paper we focus on the Euler-Maruyama approximation to the exact solution (the Euler scheme). Next, \( N_0 \) independent copies of the approximation \( \overline{X}_T \) are generated, and, finally, a Monte Carlo estimate \( V_{N_0} \) is defined as the average of the values of \( f \) at simulated points:
\[
V_{N_0} := \frac{1}{N_0} \sum_{n=1}^{N_0} f(\overline{X}_T^{n}). \tag{7}
\]
In the computation of $u(0, x) = \mathbb{E}[f(X_T^{0, x})]$ by the SMC approach there are two types of error inherent: the discretisation error $\mathbb{E}[f(X_T^{0, x})] - \mathbb{E}[f(X_T)]$ and the Monte Carlo (statistical) error, which results from the substitution of $\mathbb{E}[f(X_T)]$ with the sample average $V_N$. The aim of variance reduction methods is to reduce the statistical error. For example, in the so-called control variate variance reduction approach one looks for a random variable $\zeta$ with $\mathbb{E}\zeta = 0$, which can be simulated, such that the variance of the difference $f(X_T) - \zeta$ is minimised, that is,

$$
\text{Var}[f(X_T) - \zeta] \to \min \text{ under } \mathbb{E}\zeta = 0.
$$

The use of control variates for solving (1) via Monte Carlo path simulation approach was initiated by Newton [10] and further developed in Milstein and Tretyakov [8]. In fact, the construction of the appropriate control variates in the above two papers essentially relies on identities (5) and (6) implying that the zero-mean random variable $M_T$ can be viewed as an optimal control variate, since

$$
\text{Var}[f(X_T^{0, x}) - M_T] = \text{Var}[\mathbb{E}[f(X_T^{0, x})]] = 0.
$$

Let us note that it would be desirable to have a control variate reducing the variance of $f(X_T)$ rather than the one of $f(X_T^{0, x})$ because we simulate from the distribution of $f(X_T)$ and not from the one of $f(X_T^{0, x})$. Moreover, the control variate $M_T$ cannot be directly computed, since the function $u(t, x)$ is unknown. This is why Milstein and Tretyakov [8] proposed to use regression for getting a preliminary approximation for $u(t, x)$ in a first step.

The contribution of our work is as follows. We propose an approach for the construction of control variates that reduce the variance of $f(X_T)$ rather than the one of $f(X_T^{0, x})$ because we simulate from the distribution of $f(X_T)$ and not from the one of $f(X_T^{0, x})$. Moreover, the control variate $M_T$ cannot be directly computed, since the function $u(t, x)$ is unknown. This is why Milstein and Tretyakov [8] proposed to use regression for getting a preliminary approximation for $u(t, x)$ in a first step.

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Summing up, we propose a new regression-type approach for the construction of control variates in case of the Euler scheme. It takes advantage of the smoothness in $\mu, \sigma$ and $f$ (which is needed for nice convergence properties of regression methods) in order to significantly reduce the variance of the random variable $f(X_T)$.

This work is organised as follows. In Section 2 we describe the construction of control variates for strong approximation schemes. Section 3 describes the use of regression algorithms for the construction of control variates and analyses their
convergence. A complexity analysis of the variance reduced Monte Carlo algorithm is conducted in Section 4. Section 5 is devoted to a simulation study. Finally, all proofs are collected in Section 6.

Notational convention. Throughout, elements of \( \mathbb{R}^d \) (resp. \( \mathbb{R}^{1 \times d} \)) are understood as column-vectors (resp. row-vectors). Generally, most vectors in what follows are column-vectors. However, gradients of functions and some vectors defined via them are row-vectors. Finally, we record our standing assumption that we do not repeat explicitly in the sequel.

Standing assumption. The coefficients \( \mu \) and \( \sigma \) in (1) are globally Lipschitz functions.

2 Control variates for strong approximation schemes

To begin with, we introduce some notations, which will be frequently used in the sequel. Throughout this paper, \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \) denotes the set of nonnegative integers, \( J \in \mathbb{N} \) denotes the time discretisation parameter, we set \( \Delta := T/J \) and consider discretisation schemes defined on the grid \( \{t_j = j \Delta : j = 0, \ldots, J\} \). We set \( \Delta_j W := W_j - W_{(j-1)\Delta} \), and by \( W^i \) we denote the \( i \)-th component of the vector \( W \). Further, for \( k \in \mathbb{N}_0 \), \( H_k: \mathbb{R} \to \mathbb{R} \) stands for the (normalised) \( k \)-th Hermite polynomial, i.e.

\[
H_k(x) := \frac{(-1)^k}{\sqrt{k!}} e^{\frac{x^2}{2}} \frac{d^k}{dx^k} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}.
\]

Notice that \( H_0 \equiv 1 \), \( H_1(x) = x \), \( H_2(x) = \frac{1}{\sqrt{2}} (x^2 - 1) \).

2.1 Series representation

Let us consider a scheme, where \( d \)-dimensional approximations \( X_{\Delta,j \Delta} \), \( j = 0, \ldots, J \), satisfy \( X_{\Delta,0} = x_0 \) and

\[
X_{\Delta,j \Delta} = \Phi_{\Delta} \left(X_{\Delta,(j-1) \Delta}, \Delta_j W \right), \tag{8}
\]

where \( \Delta_j W := W_j - W_{(j-1)\Delta} \), for some Borel measurable functions \( \Phi_{\Delta}: \mathbb{R}^{d \times m} \to \mathbb{R}^d \) (clearly, the Euler scheme is a special case of this setting).

Theorem 1. Let \( f: \mathbb{R}^d \to \mathbb{R} \) be a Borel measurable function such that it holds \( \mathbb{E}|f(X_{\Delta,T})|^2 < \infty \). Then we have the representation (cf. Theorem 2.1 in [2])
\begin{align*}
    f(X_{\Delta,T}) &= \mathbb{E}[f(X_{\Delta,T})] + \sum_{j=1}^{J} \sum_{k \in \mathbb{N}_0^d \setminus \{0_m\}} a_{j,k}(X_{\Delta,(j-1)\Delta}) \prod_{r=1}^{m} H_{k_r} \left( \frac{\Delta_r W_r}{\sqrt{\Delta}} \right),
    \tag{9}
\end{align*}

where \( k = (k_1, \ldots, k_m) \) and \( 0_m := (0, \ldots, 0) \in \mathbb{R}^m \) (in the second summation), and the coefficients \( a_{j,k} : \mathbb{R}^d \rightarrow \mathbb{R} \) are given by the formula

\begin{align*}
    a_{j,k}(x) &= \mathbb{E}\left[ f(X_{\Delta,T}) \prod_{r=1}^{m} H_{k_r} \left( \frac{\Delta_r W_r}{\sqrt{\Delta}} \right) \Big| X_{\Delta,(j-1)\Delta} = x \right],
    \tag{10}
\end{align*}

for all \( j \in \{1, \ldots, J\} \) and \( k \in \mathbb{N}_0^d \setminus \{0_m\} \).

**Remark 1.** Representation \( (9) \) shows that we have a perfect control variate, namely

\begin{align*}
    M_{\Delta,T} := \sum_{j=1}^{J} \sum_{k \in \mathbb{N}_0^d \setminus \{0_m\}} a_{j,k}(X_{\Delta,(j-1)\Delta}) \prod_{r=1}^{m} H_{k_r} \left( \frac{\Delta_r W_r}{\sqrt{\Delta}} \right),
    \tag{11}
\end{align*}

for the functional \( f(X_{\Delta,T}) \), i.e. \( \text{Var}[f(X_{\Delta,T}) - M_{\Delta,T}] = 0 \).

The control variate \( M_{\Delta,T} \) is not implementable because of the infinite summation in \( (11) \) and because the coefficients \( a_{j,k} \) are unknown. In the later sections we estimate the unknown coefficients in this and other (related) representations via regression and present bounds for the estimation error.

Now we introduce the following “truncated” control variate

\begin{align*}
    M_{\Delta,T}^{\text{ser},1} := \sum_{j=1}^{J} \sum_{i=1}^{m} a_{j,e_i}(X_{\Delta,(j-1)\Delta}) \Delta_i W_i \frac{\Delta_i W_i}{\sqrt{\Delta}},
    \tag{12}
\end{align*}

where \( e_i \) denotes the \( i \)-th unit vector in \( \mathbb{R}^m \). The superscript “ser” comes from “series”. In the next subsection, performing a quite different argumentation, we derive another control variate, which will turn out to be theoretically equivalent to \( M_{\Delta,T}^{\text{ser},1} \).

### 2.2 Integral representation

**Integral representation for the exact solution.** We first motivate what we call “integral representation for the discretisation”, which will be presented below in this subsection, in that we recall in more detail the main idea of constructing control variates in Milstein and Tretyakov [8]. As was already mentioned in the introduction, the control variate in \( [8] \) is an approximation of \( M_T^* \) of \( (6) \), where the function \( u \) is given in \( (4) \) and is therefore unknown, which rises the question about a possible practical implementation of \( (6) \).

To this end, let us define the “derivative processes” \( \delta^i X_{k,i}^{k}(t) := \frac{\partial X_{k,i}^{k}(t)}{\partial x_i} \) for \( i, k \in \{1, \ldots, d\} \), where \( X_{k,i}^{k}(t) \) means the \( k \)-th component of the solution of \( (1) \) started...
at time \( s \) in \( x \) evaluated at time \( t \geq s \), and simply write \( \delta^i X^k_t \) rather than \( \delta^i X^k_{t,0}(t) \) below. Further, we define the matrix \( \delta X_i := \begin{pmatrix} \delta^1 X^i_1 & \cdots & \delta^d X^i_1 \\ \vdots & \ddots & \vdots \\ \delta^1 X^i_d & \cdots & \delta^d X^i_d \end{pmatrix} \in \mathbb{R}^{d \times d} \) as well as the vectors \( \delta^i X_i := (\delta^i X^1_i \cdots \delta^i X^d_i)^\top \in \mathbb{R}^d \). Assuming \( \mu, \sigma \in C^1 \), we notice that \( \delta^i X_i \) satisfies the following SDE

\[
\begin{align*}
\delta^i X_i &= \sum_{k=1}^d \delta^i X^k_i \left[ \frac{\partial \mu(X_i)}{\partial x_k} dt + \frac{\partial \sigma(X_i)}{\partial x_k} dW_t \right], \\
\delta^i X^k_0 &= \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases}.
\end{align*}
\]

(13)

Milstein and Tretyakov \cite{8} exploit (13) to prove that, provided \( f, \mu, \sigma \in C^1 \), the integral in (8) can be expressed by means of \( \delta X_i \) as follows

\[
M^*_T := \int_0^T \nabla u(t, X_t) \sigma(X_t) dW_t = \int_0^T \mathbb{E}[\nabla f(X_T) \delta X_T | X_t] \delta X^k_0 \sigma(X_t) dW_t,
\]

(14)

where \( \nabla u(t, x) \in \mathbb{R}^{1 \times d} \) denotes the gradient of \( u \) w.r.t. \( x \). The second integral here can be used for a practical construction of an approximation of \( M^*_T \) because the conditional expectation can be approximated via regression.

The preceding description lacks assumptions under which the procedure works (the mentioned ones are not enough). We refer to \cite{8} for more detail.

**Integral representation for the discretisation.** As was mentioned in the introduction, we are going to reduce not the variance in \( f(X_T) \) but rather the one in \( f(X_{\Delta,T}) \), that is, we aim at constructing control variates directly for the discretised process. The fine details of the construction must of course depend on the discretisation scheme. For the rest of the paper, we focus on the Euler scheme, that is, we have

\[
\Phi_{\Delta}(x,y) = x + \mu(x) \Delta + \sigma(x)y.
\]

(15)

We define the “discretised derivative process” \( \delta^i X^k_{i,j}(\Delta,t_l) := \frac{\partial X^k_{i,j}(\Delta,t_l)}{\partial x_i} \), for \( l \geq j \) and \( i,k \in \{1, \ldots, d\} \), where \( X^k_{i,j}(\Delta,t_l) \) means the \( k \)-th component of the (Euler) discretisation for (1) started at time \( t_j \) in \( x \) and evaluated at time \( t_l \geq t_j \), and use \( \delta^i X^k_{\Delta,t_l} \) as an abbreviation of \( \delta^i X^k_{i,j}(\Delta,t_l) \). Assuming \( \mu, \sigma \in C^1 \), we get that the process \( (\delta^i X_{\Delta,i})_{j=1,\ldots,j} \) has the dynamics

\[
\delta^i X_{\Delta,i} = \delta^i X_{\Delta,(j-1)i} + \sum_{k=1}^d \delta^i X^k_{\Delta,(j-1)i} \left[ \frac{\partial \mu(X_{\Delta,(j-1)i})}{\partial x_k} \Delta + \frac{\partial \sigma(X_{\Delta,(j-1)i})}{\partial x_k} \Delta_j W \right],
\]

(16)
Regression-based variance reduction approach for strong approximation schemes 7

(cf. (13)), where $\delta X_{\Delta,0} = I_d$, and in what follows $I_d$ denotes the identity matrix of size $d$.

Given a Borel function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfying $\mathbb{E}|f(X_{\Delta,T})| < \infty$, it can be verified by a direct calculation that, for $t \in [t_{j-1}, t_j)$,

$$\mathbb{E}[f(X_{\Delta,T})|\mathcal{F}_t] = u_{\Delta}(t, X_{\Delta,t_{j-1}}, W_t - W_{t_{j-1}}),$$

where the function $u_{\Delta} : [0, T] \times \mathbb{R}^{d+m} \rightarrow \mathbb{R}$ is constructed via the backward recursion as follows

$$u_{\Delta}(t, x, y) = \mathbb{E}[u_{\Delta}(t_j, \Phi_{\Delta}(x, y + \sqrt{t_j-t}, 0)], \quad t \in [t_{j-1}, t_j),$$  

$$u_{\Delta}(T, x, 0) = f(x),$$

where $t_j := \frac{T}{j}, \; j \in \{0, \ldots, J\}$, and $z_1, \ldots, z_J \sim \mathcal{N}(0_n, I_n)$.

We now introduce the following assumptions: for any $j \in \{1, \ldots, J\}$ and $x \in \mathbb{R}^d$, it holds

(Ass1) $f(X_{t_{j-1},x}(\Delta, T)) \in L^1$,  

(Ass2)$_n$ $|\Delta_j|^{\pi n} \mathbb{E}[f(X_{t_{j-1},x}(\Delta, T))|\mathcal{F}_{t_j}] \in L^1$.

(Ass1) is just a minimal assumption that allows to have (17) with the function $u_{\Delta}$ constructed via (18)–(19). (Ass2)$_n$ is a technical assumption, which depends on $n$, allowing to replace integration and differentiation in several cases of interest (see below). In most places we need the variant (Ass2)$_1$, i.e. with $n = 1$, but at a couple of instances we will need stronger variants (Ass2)$_n$ with $n \geq 1$. That is why we have the parameter $n$ in the formulation of that assumption.

An attractive feature of such an approach via the discretised process (in contrast to the one via the exact solution) is that, under (Ass1) and (Ass2)$_1$, due to the smoothness of the Gaussian density, the function $u_{\Delta}$ is continuously differentiable in $y$ regardless of whether $f$ is smooth, and, moreover, $u_{\Delta}$ is continuously differentiable in $x$, provided $f, \mu, \sigma$ are continuously differentiable. More precisely, we obtain the above statements because, for $t \in [t_{j-1}, t_j)$, we can write (for simplicity, in the one-dimensional case)

$$u_{\Delta}(t, x, y) = \int_{\mathbb{R}} u_{\Delta}(t_j, \Phi_{\Delta}(x, w), 0) \frac{1}{\sqrt{2\pi(t_j-t)}} e^{-\frac{(w-y)^2}{2(t_j-t)}} \, dw,$$

and differentiation under the integral applies due to (Ass2)$_1$ together with the dominated convergence theorem (notice that the expression $|f(X_{t_{j-1},x}(\Delta, T))| \in L^1$ in (Ass2)$_n$ is nothing else than $u_{\Delta}(t_j, \Phi_{\Delta}(x, \Delta_j W), 0)$).

Theorem 2. Suppose (Ass1) and (Ass2)$_1$.

(i) It holds
\[ f(X_{\Delta, T}) = \mathbb{E}[f(X_{\Delta, T})] + \sum_{j=1}^{J} \int_{t_j}^{t_{j+1}} \nabla_x u_\Delta(t, X_{\Delta, t_j-1}, W_t - W_{t_j-1}) \, dW_t, \]

where \( \nabla_x u_\Delta(t, x, y) \in \mathbb{R}^{1 \times m} \) denotes the gradient of \( u_\Delta \) w.r.t. \( y \).

(ii) Assume additionally that \( f, \mu, \sigma \in C^1 \). Then we also have the alternative representation

\[ f(X_{\Delta, T}) = \mathbb{E}[f(X_{\Delta, T})] + \sum_{j=1}^{J} \int_{t_j}^{t_{j+1}} \mathbb{E} \left[ \nabla f(X_{\Delta, T}) \delta X_{\Delta, T} \delta X_{\Delta, t_j}^{-1} \bigg| \mathcal{F}_t \right] \sigma(X_{\Delta, t_j-1}) \, dW_t. \]

Let us define the function \( g_j : \mathbb{R}^d \to \mathbb{R}^{1 \times d}, j \in \{1, \ldots, J\} \), through

\[ g_j(x) = (g_{j,1}(x), \ldots, g_{j,d}(x)) := \mathbb{E} \left[ \nabla f(X_{\Delta, T}) \delta X_{\Delta, T} \delta X_{\Delta, t_j}^{-1} \bigg| X_{\Delta, t_j-1} = x \right]. \quad (20) \]

Note that it holds (see the proof of Theorem 2)

\[ g_j(x) = \mathbb{E} \left[ \nabla_x u_\Delta(t, X_{\Delta, t_j-1}, 0) \bigg| X_{\Delta, t_j-1} = x \right], \quad (21) \]

\[ \nabla_x u_\Delta(t_{j-1}, x, 0) = g_j(x) \sigma(x), \quad (22) \]

where \( \nabla_x u_\Delta(t, x, y) \) denotes the gradient of \( u_\Delta \) w.r.t. \( x \), and we conditioned on \( X_{\Delta, t_j-1} \) instead of \( \mathcal{F}_{t_j-1} \) because \((X_{\Delta, t_j})_{j=0, \ldots, J}\) is a Markov chain (one can do that for grid points only). Theorem 2 inspires to introduce the control variate

\[ M^{int,1}_{\Delta, T} := \sum_{j=1}^{J} \sum_{i=1}^{m} \frac{\partial u_\Delta(t_{j-1}, X_{\Delta, t_j-1}, 0)}{\partial y_i} \Delta_j W_i \]

\[ = \sum_{j=1}^{J} \sum_{k=1}^{d} g_{j,k}(X_{\Delta, t_j-1}) \sum_{i=1}^{m} \sigma_{ik}(X_{\Delta, t_j-1}) \Delta_j W_i. \quad (23) \]

It will turn out that \( M^{int,1}_{\Delta, T} = M^{ser,1}_{\Delta, T} \). To this end, we derive a connection between the series and integral representations.

**Theorem 3.** Under (Ass1) and (Ass2), for all \( n \in \mathbb{N} \), provided that it holds

\[ \left| D^\alpha \left( \frac{\partial}{\partial y_i} u_\Delta(t, x, y) \right) \right| := \left| \frac{\partial^K \left( \frac{\partial}{\partial y_i} u_\Delta(t, x, y) \right)}{\partial t^{a_1} \partial y_{i_1}^{a_2} \cdots \partial y_{m+1}^{a_m}} \right| \leq C^K \quad (24) \]

for all \( K \in \mathbb{N}, r \in \{1, \ldots, m\}, |\alpha| = K, t \in [t_{j-1}, t_j), x \in \mathbb{R}^d, y \in \mathbb{R}^m \), with some constant \( C > 0 \), we have for the Euler scheme.
Regression-based variance reduction approach for strong approximation schemes

\begin{equation}
    f(X_{\Delta,T}) = \mathbb{E}[f(X_{\Delta,T})] + \sum_{j=1}^{J} \sum_{l=1}^{\infty} \Delta^{l/2} \sum_{k \in \mathbb{N}^m} \prod_{r=1}^{m} \frac{\partial^l u_\Delta(t_{j-1}, X_{\Delta,t_{j-1}}, 0)}{\partial y_1^{k_1} \cdots \partial y_m^{k_m}} \prod_{r=1}^{m} \frac{H_r(\Delta^{1/2})}{\sqrt{k_r!}}
\end{equation}

whenever \( 0 < \Delta < \frac{1}{\mathbb{E}[f]} \). (The series converge in \( L^2 \).) Consequently, we obtain for \( l = \sum_{r=1}^{m} k_r \in \mathbb{N} \)

\begin{equation}
    \frac{\Delta^{l/2}}{\sqrt{k_1! \cdots k_m!}} \cdot \frac{\partial^l u_\Delta(t_{j-1}, X_{\Delta,t_{j-1}}, 0)}{\partial y_1^{k_1} \cdots \partial y_m^{k_m}} = a_{j,l}(X_{\Delta,t_{j-1}}).
\end{equation}

**Remark 2.** In the one-dimensional case \( (d = m = 1) \), a representation of a similar type as (25) appears in [1] in a somewhat different form. Our form is aimed at constructing control variates via regression methods.

In particular, we see from Theorem 3 that \( M_{\Delta,T}^{\text{int},1} = M_{\Delta,T}^{\text{ser},1} \) provided that (24) holds. However, we can prove the equality of the aforementioned control variates without assuming (24).

**Theorem 4.** Under (Ass1) and (Ass2), we have for \( i \in \{1, \ldots, m\} \)

\[ a_{j,i}(x) = \sqrt{\Delta} \frac{\partial}{\partial y_j} u_\Delta(t_{j-1}, x, 0), \]

and consequently,

\[ M_{\Delta,T}^{\text{int},1} = M_{\Delta,T}^{\text{ser},1}. \]

It is interesting to remark that, although we assumed \( f(X_{\Delta,T}) \in L^2 \) when speaking about the series representation, the coefficients \( a_{j,i} \) are well-defined already under (Ass1) and (Ass2).

We can now investigate the order of the truncation error, which arises when we replace the control variate \( M_{\Delta,T} \) of (11) with the control variate \( M_{\Delta,T}^{\text{ser},1} \) of (12).

**Theorem 5.** Suppose (Ass1) and (Ass2). Provided that the function \( u_\Delta(t,x,y) \) has bounded partial derivatives in \( y \) of orders 2 and 3, it holds

\[ \text{Var} \left[ f(X_{\Delta,T}) - M_{\Delta,T}^{\text{int},1} \right] = \text{Var} \left[ f(X_{\Delta,T}) - M_{\Delta,T}^{\text{ser},1} \right] \lesssim \Delta. \]

**Remark 3.** (i) Below we will present sufficient conditions in terms of the functions \( f, \mu, \sigma \) that ensure the assumption on \( a_\Delta \) in Theorem 5 (see Theorem 6 in Section 3).

(ii) The control variate \( M_{\Delta,T}^{\text{int},1} \) differs from the one suggested in [8] only in an index concerning the inverted matrix, i.e. we have \( \delta X_{\Delta,t_{j-1}}^{-1} \) inside of \( g_j(X_{\Delta,t_{j-1}}) \) rather than the \( \mathcal{F}_{t_{j-1}} \)-measurable random variable \( \delta X_{\Delta,t_{j-1}}^{-1} \) which arises in case of the exact solution \( f(X_T) \) from a simple discretisation of the stochastic integral in (14).
Regarding the weak convergence order of the Euler scheme, we have the following result (cf. Theorem 2.1 in [7]).

**Proposition 1.** Assume that \( \mu \) and \( \sigma \) in (1) are Lipschitz continuous with components \( \mu_k, \sigma_{ki} : \mathbb{R}^d \to \mathbb{R}, k = 1, \ldots, d, i = 1, \ldots, m \), being 4 times continuously differentiable with their partial derivatives of orders up to 4 having polynomial growth. Let \( f : \mathbb{R}^d \to \mathbb{R} \) be 4 times continuously differentiable with partial derivatives of orders up to 4 having polynomial growth. Then, for the Euler scheme (15), we have

\[
| \mathbb{E} f(X_T) - \mathbb{E} f(X_{\Delta T}) | \leq c \Delta,
\]

where the constant \( c \) does not depend on \( \Delta \).

We remark that the assumption that, for sufficiently large \( n \in \mathbb{N} \), the expectations \( \mathbb{E}|X_{\Delta,j\Delta}|^{2n} \) are uniformly bounded in \( J \) and \( j = 0, \ldots, J \) (cf. Theorem 2.1 in [7]) is automatically satisfied for the Euler scheme because \( \mu \) and \( \sigma \), being globally Lipschitz, have at most linear growth.

### 3 Regression analysis

In the previous sections we have given several representations for the control variates. Now we discuss how to compute the coefficients in these representations via regression. For the sake of clarity, we will focus on the control variate given by (23), that is, we will estimate the functions \( g_j, k \) in (20) via linear regression. Let us start with a general description of the global Monte Carlo regression algorithm.

#### 3.1 Global Monte Carlo regression algorithm

Fix a \( q \)-dimensional vector of real-valued functions \( \psi = (\psi^1, \ldots, \psi^q) \) on \( \mathbb{R}^d \). Simulate a set of \( N \) “training paths” of the Markov chains \( X_{\Delta,j\Delta} \) and \( \delta X_{\Delta,j\Delta} \), \( j = 0, \ldots, J \). We should choose \( N > q \). In what follows these \( N \) training paths are denoted by \( D_N^{tr} \):

\[
D_N^{tr} := \left\{ (X_{\Delta,j\Delta}^{tr,(n)}, \delta X_{\Delta,j\Delta}^{tr,(n)}) : j = 0, \ldots, J, n = 1, \ldots, N \right\}.
\]

Let \( \alpha_{j,k} = (\alpha^1_{j,k}, \ldots, \alpha^q_{j,k}) \), where \( j \in \{1, \ldots, J\}, k \in \{1, \ldots, d\} \), be a solution of the following least squares optimisation problem:

\[
\arg\min_{\alpha \in \mathbb{R}^q} \sum_{n=1}^N \left[ e_{j,k}^{tr,(n)} - \alpha^1 \psi^1(X_{\Delta,(j-1)\Delta}^{tr,(n)}) - \cdots - \alpha^q \psi^q(X_{\Delta,(j-1)\Delta}^{tr,(n)}) \right]^2
\]

with
The number of basis functions at each regression is of order $O(n_q^2)$, since each $\hat{g}_{j,k}$ is of the form $\hat{g}_{j,k} = B^{-1}b$ with

$$B_{l,o} := \frac{1}{N} \sum_{n=1}^{N} \psi^l(X_{\Delta,(j-1)\Delta}^{(n)}) \psi^{o}(X_{\Delta,(j-1)\Delta}^{(n)})$$

and

$$b_{j,k} := \frac{1}{N} \sum_{n=1}^{N} \psi^l(X_{\Delta,(j-1)\Delta}^{(n)}) \hat{g}_{j,k}^{(n)},$$

$l,o \in \{1,\ldots,q\}$. The cost of approximating the family of the coefficient functions $g_{j,k}$, $j \in \{1,\ldots,J\}$, $k \in \{1,\ldots,d\}$, is of order $O(JdNq^2)$.

### 3.2 Piecewise polynomial regression

There are different ways to choose the basis functions $\psi = (\psi^1, \ldots, \psi^q)$. In this section we describe piecewise polynomial partitioning estimates and present $L^2$-upper bounds for the estimation error.

From now on, we fix some $p \in \mathbb{N}_0$, which will denote the maximal degree of polynomials involved in our basis functions. The piecewise polynomial partitioning estimate of $g_{j,k}$ works as follows: consider some $R > 0$ and an equidistant partition of $[-R,R]^d$ in $Q_d$ cubes $K_1,\ldots,K_{Q_d}$. Further, consider the basis functions $\psi^{l,1},\ldots,\psi^{l,q}$ with $l \in \{1,\ldots,Q_d\}$ and $q = (p+d)$ such that $\psi^{l,1}(x),\ldots,\psi^{l,q}(x)$ are polynomials with degree less than or equal to $p$ for $x \in K_l$ and $\psi^{l,1}(x) = \cdots = \psi^{l,q}(x) = 0$ for $x \notin K_l$. Then we obtain the least squares regression estimate $\hat{g}_{j,k}(x)$ for $x \in \mathbb{R}^d$ as described in Section 3.1 based on $Q_d^q = O(Q_d^dp^d)$ basis functions. In particular, we have $\hat{g}_{j,k}(x) = 0$ for any $x \notin [-R,R]^d$. We note that the cost of computing $\hat{g}_{j,k}$ for all $j,k$ is of order $O(JdNQ_d^dp^{2d})$ rather than $O(JdNQ_d^dp^{2d})$ due to a block diagonal matrix structure of $B$ in (29). An equivalent approach, which leads to the same estimator $\hat{g}_{j,k}(x)$, is to perform separate regressions for each cube $K_1,\ldots,K_{Q_d}$. Here, the number of basis functions at each regression is of order $O(p^d)$ so that the overall cost is of order $O(JdNQ_d^dp^{2d})$, too. For $x = (x_1,\ldots,x_d) \in \mathbb{R}^d$ and $h \in [1,\infty)$, we will use the notations
|x|_h := \left( \sum_{k=1}^{d} |x_k|^h \right)^{1/h}, \quad |x|_\infty := \max_{k=1,\ldots,d} |x_k|.

For $s \in \mathbb{N}_0$, $C > 0$ and $h \in [1, \infty]$, we say that a function $F: \mathbb{R}^d \to \mathbb{R}$ is $(s+1,C)$-smooth w.r.t. the norm $|\cdot|_h$ whenever, for all $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ with $\sum_{k=1}^d \alpha_k = s$, we have

$$|D^\alpha F(x) - D^\alpha F(y)| \leq C|x - y|_h, \quad x, y \in \mathbb{R}^d,$$

i.e. the function $D^\alpha F$ is globally Lipschitz with the Lipschitz constant $C$ with respect to the norm $|\cdot|_h$ on $\mathbb{R}^d$ (cf. Definition 3.3 in [4]). In what follows, we use the notation $\mathbb{P}_{\Delta,j-1}$ for the distribution of $X_{\Delta,(j-1)\Delta}$. In particular, we will work with the corresponding $L^2$-norm:

$$\|F\|^2_{L^2(\mathbb{P}_{\Delta,j-1})} := \int_{\mathbb{R}^d} F^2(x) \mathbb{P}_{\Delta,j-1}(dx) = \mathbb{E} \left[ F^2 \left(X_{\Delta,(j-1)\Delta}\right) \right].$$

We now define $\zeta_{j,k}$ as the $k$-th component of the vector $\zeta_j = (\zeta_{j,1}, \ldots, \zeta_{j,d}) := \nabla f(X_{\Delta,t}) \delta X_{\Delta,t} \delta X_{\Delta,t}^{j-1,\Delta}$ and remark that $g_{j,k}(x) = \mathbb{E} [\zeta_{j,k} | X_{\Delta,(j-1)\Delta} = x]$. In what follows, we consider the following assumptions: there exist $h \in [1, \infty]$ and positive constants $\Sigma, A, C_h, \nu, B_V$ such that, for all $J \in \mathbb{N}$, $j \in \{1, \ldots, J\}$ and $k \in \{1, \ldots, d\}$, it holds

\begin{align*}
(A1) & \sup_{x \in \mathbb{R}^d} \text{Var}[\zeta_{j,k} | X_{\Delta,(j-1)\Delta} = x] \leq \Sigma < \infty, \\
(A2) & \sup_{x \in \mathbb{R}^d} |g_{j,k}(x)| \leq A < \infty, \\
(A3) & g_{j,k} \text{ is } (p+1, C_h)\text{-smooth w.r.t. the norm } |\cdot|_h, \\
(A4) & \mathbb{P}(|X_{\Delta,(j-1)\Delta}|_\infty > R) \leq B_V R^{-\nu} \text{ for all } R > 0.
\end{align*}

Remark 4. Let us notice that it is only a matter of convenience which $\delta$ to choose in (A3) because all norms $|\cdot|_\delta$ are equivalent. Furthermore, since $\mu$ and $\sigma$ are assumed to be globally Lipschitz, hence have linear growth, then, given any $\nu > 0$, (A4) is satisfied with a sufficiently large $B_V > 0$. In other words, (A4) is needed only to introduce the constant $B_V$, which appears in the formulations below.

In the next theorem we, in particular, present sufficient conditions in terms of the functions $\mu$, $\sigma$ and $f$ that imply the preceding assumptions.

Theorem 6. (i) Under (Ass1) and (Ass2), let all functions $f, \mu_k, \sigma_{i,k}$, $k \in \{1, \ldots, d\}$, $i \in \{1, \ldots, m\}$, be continuously differentiable with bounded partial derivatives. Then (A1) and (A2) hold with appropriate constants $\Sigma$ and $A$.

(ii) If, moreover, (Ass1) and (Ass2) are satisfied, all functions $\sigma_{i,k}$ are bounded and all functions $f, \mu_k, \sigma_{i,k}$ are $3$ times continuously differentiable with bounded partial derivatives up to order $3$, then the function $u_{\Delta}(t,x,y)$ has bounded partial derivatives in $y$ up to order $3$. In particular, (27) holds true.

Remark 5. As a generalisation of Theorem 6, it is natural to expect that (A3) is satisfied with a sufficiently large constant $C_h > 0$ if, under (Ass1) and (Ass2)$_{p+2}$, all
Lemma 1. Under (A1)–(A4), we have
\[
\mathbb{E} \| \tilde{g}_{j,k} - g_{j,k} \|_{L^2(\mathbb{P}_{\Delta,j-1})}^2 \leq \tilde{c} \left( \Sigma + A^2 (\log N + 1) \right) \frac{(p+d) Q^d}{N} + \frac{8 C_h^2}{(p+1)^{2}d^{2-2/h}} \left( \frac{Rd}{Q} \right)^{2p+2} + 8 A^2 B_c R^{-v},
\]
where \( \tilde{c} \) is a universal constant.

Theorem 7. Let us assume \( \sup_{x \in \mathbb{R}^d} | \sigma_k(x) | \leq \sigma_{\text{max}} < \infty \) for all \( k \in \{1, \ldots, d\} \) and \( i \in \{1, \ldots, m\} \). Then we have under (A1)–(A4)
\[
\Var[f(X_{\Delta,T}) - \tilde{M}^{\text{int},1}_{\Delta,T}] \lesssim \frac{1}{J} + d^2 T m \sigma_{\text{max}}^2 \left\{ \tilde{c} \left( \Sigma + A^2 (\log N + 1) \right) \frac{(p+d) Q^d}{N} + \frac{8 C_h^2}{(p+1)^{2}d^{2-2/h}} \left( \frac{Rd}{Q} \right)^{2p+2} + 8 A^2 B_c R^{-v} \right\}.
\]
We finally stress that $\tilde{M}^{int,1}_{\Delta,T}$ is a valid control variate in that it does not introduce bias, i.e. $\mathbb{E}[\tilde{M}^{int,1}_{\Delta,T}|D_N^f] = 0$, which follows from the martingale transform structure in (32).

3.3 Summary of the algorithm

The algorithm of the “integral approach” consists of two phases: training phase and testing phase. In the training phase, we simulate $N$ independent training paths $D_N^f$ and construct regression estimates $\hat{g}_{j,k}(\cdot, D_N^f)$ for the coefficients $g_{j,k}(\cdot)$, $k \in \{1, \ldots, d\}$. In the testing phase, independently from $D_N^f$ we simulate $N_0$ independent testing paths $(X^{(n)}_{\Delta,T})_{j=0, \ldots, J, n=1, \ldots, N_0}$, and build the Monte Carlo estimator for $\mathbb{E}f(X_T)$ as
\[
\frac{1}{N_0} \sum_{n=1}^{N_0} \left( f(X^{(n)}_{\Delta,T}) - \tilde{M}^{int,1}_{\Delta,T}(n) \right). \tag{34}
\]
The expectation of this estimator equals $\mathbb{E}f(X_{\Delta,T})$, and the upper bound for the variance is $\frac{1}{N_0}$ times the expression in (33).

4 Complexity analysis

The results presented in previous sections provide us with “building blocks” to perform the complexity analysis.

**Standing assumption for Complexity Analysis** consists in

(Ass1), (Ass2), (27) and (28).

Combining Theorem 5, Theorem 6 and Proposition 1, we recall that this standing assumption is satisfied whenever we have (Ass1), (Ass2), $\sigma$ is bounded, $f, \mu, \sigma \in C^4$, the partial derivatives of $f, \mu$ and $\sigma$ up to order 3 are bounded and of order 4 have polynomial growth. However, we prefer to formulate the standing assumption for complexity analysis as above because one might imagine other sufficient conditions for it.

4.1 Integral approach

Below we present a complexity analysis which explains how we can approach the complexity order $\varepsilon^{-2} \sqrt{|\log(\varepsilon)|}$ with $\varepsilon$ being the precision to be achieved.

For the integral approach we perform $d$ regressions in the training phase and $d$ evaluations of $\hat{g}_{j,k}$ in the testing phase (using the regression coefficients from the
Performing the full complexity analysis via Lagrange multipliers one can see that these parameter values are not optimal if $2(p + 1) < d$ or $\nu < \frac{2d(p+1)}{\sum(p+1)-d}$. Thus, we have for the complexity

$$JQ^2d^c_{p,d} \max \{c_{p,d}N,N_0\},$$

(35)

where $c_{p,d} := \binom{p+d}{p}$. Under (A1)–(A4) and boundedness of $\sigma$ (cf. Theorem 7), we have the following constraints

$$\max \left\{ \frac{1}{J^2}, \frac{Q^2d^c_{p,d} \log(N)}{MN_0} , \frac{d^2m_{c,d} \log(N)}{MN_0} \left( \frac{Rd}{Q} \right)^{2(p+1)} , \frac{d^2m_{B,v}}{N_0R^v} \right\} \lesssim \epsilon^2,$$

(36)

to ensure a mean squared error (MSE) of order $\epsilon^2$. Note that the first term in (36) comes from the squared bias of the estimator (due to (28)) and $\mathbb{E}[\mathcal{M}_T^{\Delta T}] = 0$ and the remaining four ones come from the variance of the estimator (see (33) and (34)).

**Theorem 8.** Under (A1)–(A4) and boundedness of $\sigma$, we obtain the following solution for the integral approach:

$$J \asymp \epsilon^{-1}, \quad Q \asymp \left[ \frac{B_v^{(p+1)}d^{2(p+1)}(v+1)m^{v+2(p+1)}}{\epsilon^{2(p+1)v+4(p+1)}c_{p,d}^{(p+1)}} \right],$$

(37)

$$N \asymp \left[ \frac{B_v^{(p+1)}d^{2d(p+1)}(v+1)m^{d(p+1)+2v+2(p+1)}}{\epsilon^{2d(p+1)v+4(p+1)}c_{p,d}^{(p+1)}} \right],$$

(38)

$$N_0 \asymp Nc_{p,d}$$

(39)

$$R \asymp \left[ \frac{B_v^{(p+1)}d^{2d(p+1)}m^{2(p+1)}}{\epsilon^{4(p+1)c_{p,d}d^{2(p+1)}(p+1)}} \right],$$

(40)

provided that $2(p + 1) > d$ and $\nu > \frac{2d(p+1)}{\sum(p+1)-d}$. Thus, we have for the complexity

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1 Performing the full complexity analysis via Lagrange multipliers one can see that these parameter values are not optimal if $2(p + 1) < d$ or $\nu < \frac{2d(p+1)}{\sum(p+1)-d}$ (a Lagrange multiplier corresponding to a “$\leq 0$” constraint is negative, cf. proof of Theorem 6). Therefore, the recommendation is to choose
\[ \mathcal{C}_{int} \asymp J Q^d d c_{p,d} N \asymp J Q^d d c_{p,d} N_0 \]
\[ \asymp \left[ \left( B_v^{(d)(p+1)} e_{p,d} + \int_0^{(d)(p+1)} \int_0^{(d)(p+1)} \int_0^{(d)(p+1)} \int_0^{(d)(p+1)} \int_0^{(d)(p+1)} m_{3d+6(p+1)(d+v)} \right)^{\frac{1}{(d+2v)}} e^{5d^2+2(p+1)(5d^2+4v)} \right]^{\frac{1}{(d+2v)}} \]
\[ \cdot \log \left( \epsilon \right) \]  
\[ \text{(41)} \]

**Remark 6.** (i) For the sake of comparison with the SMC and MLMC approaches, we recall at this point that their complexities are
\[ \mathcal{C}_{SMC} \asymp \epsilon^{-3} \text{ and } \mathcal{C}_{MLMC} \asymp \epsilon^{-2} \]
at best\(^2\). Complexity estimate (41) shows that one can approach the complexity order \( \epsilon^{-2} \sqrt{\log(\epsilon)} \), when \( p, \nu \to \infty \), i.e. if the coefficients \( g_{j,k} \) are smooth enough and the solution \( X \) of SDE (1) lives in a compact set.

(ii) Note that we would have obtained the same complexity even when the variance in (37) were of order \( \Delta^2 \) with \( K > 1 \). This is due to the fact that the second constraint in (36) is the only inactive one and this would still hold if the condition were \( \frac{1}{\nu \Delta^2 N_0} \lesssim \epsilon^2 \). Hence, it is not useful to derive a control variate with a higher variance order for the Euler scheme.

### 4.2 Series approach

Below we present a complexity analysis for the series representation, defined in Section 2.1. Again we focus on the Euler scheme (15). Then we compare the resulting complexity with the one in (41).

Similarly to Section 3.2, we define \( \zeta_{j,i} \) as the \( i \)-th component of the vector \( \zeta_j = (\zeta_j,1, \ldots,\zeta_j,m)^\top \) := \( f(X_{\Delta_j}) \Delta_j^W \) and remark that \( a_{j,\iota}(x) = \mathbb{E}[\zeta_{j,\iota}X_{\Delta_j,j-1}\Delta = x] \) (compare with (10)). We will work under the following assumptions: there exist \( h \in [1,\infty) \) and positive constants \( \Sigma, A, C_h \) such that, for all \( J \in \mathbb{N}, j \in \{1, \ldots, J\} \) and \( i \in \{1, \ldots, m\} \), it holds:

(B1) \( \sup_{x \in \mathbb{R}^d} \mathbb{V}[\zeta_{j,i}X_{\Delta_j,j-1}\Delta = x] \leq \Sigma < \infty \),

(B2) \( \sup_{x \in \mathbb{R}^d} \mathbb{V}[a_{j,\iota}(x)] \leq A \sqrt{\Delta} < \infty \),

(B3) \( a_{j,\iota} \) is \((p+1,C_h)\)-smooth w.r.t. the norm \( \| \cdot \|_h \).

---

the power \( p \) for our basis functions according to \( p > \frac{d+2}{2} \). The opposite choice is allowed as well (the method converges), but theoretical complexity of the method would be then worse than that of the SMC, namely, \( \epsilon^{-3} \).

\(^2\)For the Euler scheme, there is an additional logarithmic factor in the complexity of the MLMC algorithm (see [3]).
Note the difference between (B2) and (A2) of Section 3.2, while (B1) has the same form as (A1). This is due to (26), hence the additional factor \( \sqrt{\Delta} \) in (B2).

In what follows the \( N \) training paths are denoted by

\[
D_N^\ell := \left\{ (X_{\Delta,j}\Delta)_{j=0,\ldots,J} : n = 1,\ldots,N \right\},
\]

that is, we do not need to simulate paths for the derivative processes \( \delta X_{\Delta,j}\Delta \). Let \( \hat{a}_{j,e,i}(x) \) be the piecewise polynomial partitioning estimate of \( a_{j,e,i} \), described in Section 3.2.

By \( \hat{a}_{j,e,n}(x) \) we denote the truncated estimate, which is defined as follows:

\[
\hat{a}_{j,e,n}(x) := T_{A\sqrt{\Delta}} \hat{a}_{j,e,i}(x) := \begin{cases} 
\hat{a}_{j,e,i}(x) & \text{if } |\hat{a}_{j,e,i}(x)| \leq A\sqrt{\Delta}, \\
A\sqrt{\Delta} \text{sgn} \hat{a}_{j,e,i}(x) & \text{otherwise}.
\end{cases}
\]

Lemma 2. Under (B1)–(B3) and (A4), we have

\[
\mathbb{E} \| \hat{a}_{j,e,i} - a_{j,e,i} \|^2 \leq \tilde{c} \left( \Sigma + A^2 \Delta (\log N + 1) \right) \frac{c_p d\sigma_d^l}{N} + \frac{8C_h^2}{(p+1)^2d^{-2}} \left( \frac{R}{Q} \right)^{2p+2} + 8A^2 \Delta B \nu R^{-\nu},
\]

where \( \tilde{c} \) is a universal constant.

Let us now estimate the variance of the random variable \( f(X_{\Delta,T}) - \tilde{M}_{\Delta,T}^{\text{err},1} \), where

\[
\tilde{M}_{\Delta,T}^{\text{err},1} := \sum_{j=1}^J \sum_{i=1}^m \hat{a}_{j,e,i}(X_{\Delta,(j-1)\Delta},D_N^\ell) \frac{\Delta_j W_i}{\sqrt{\Delta}}.
\]

Theorem 9. Under (B1)–(B3) and (A4), we have

\[
\text{Var}[f(X_{\Delta,T}) - \tilde{M}_{\Delta,T}^{\text{err},1}] \leq \frac{1}{J} + Jm \left\{ \tilde{c} \left( \Sigma + A^2 \Delta (\log N + 1) \right) \frac{c_p d\sigma_d^l}{N} + \frac{8C_h^2}{(p+1)^2d^{-2}} \left( \frac{R}{Q} \right)^{2p+2} + 8A^2 \Delta B \nu R^{-\nu} \right\}.
\]

Let us study the complexity of the following “series approach”: In the training phase, we simulate \( N \) independent training paths \( D_N^\ell \) and construct regression estimates \( \hat{a}_{j,e,i}(\cdot,D_N^\ell) \) for the coefficients \( a_{j,e,i}(\cdot) \), \( i \in \{1,\ldots,m\} \). In the testing phase, independently from \( D_N^\ell \) we simulate \( N_0 \) independent testing paths \( (X_{\Delta,j}\Delta)_{j=0,\ldots,J} \), \( n = 1,\ldots,N_0 \), and build the Monte Carlo estimator for \( \mathbb{E} f(X_{\Delta,T}) \) as

\[
\frac{1}{N_0} \sum_{n=1}^{N_0} \left( f(X_{\Delta,T}) - \tilde{M}_{\Delta,T}^{\text{err},1,(n)} \right).
\]
Therefore, the overall cost is of order
\[ JQ^d mc_{p,d} \max \{ c_{p,d} N, N_0 \} \quad (46) \]
The expectation of the estimator in \((45)\) equals \(E f(X_{\Delta,T})\), and the upper bound for the variance is \(1/N_0\) times the expression in \((44)\). Hence, we have the following constraints
\[
\max \left\{ \frac{1}{J_1^2}, \frac{1}{J_1^2 N_0}, \frac{Jm}{N_0}, \frac{Jm}{(p+1)^2 N_0} \right\} \frac{Rd}{\mathcal{Q}}^{2(p+1)} \frac{mB_v}{N_0 R^v} \lesssim \varepsilon^2, \quad (47)
\]
to ensure a MSE of order \(\varepsilon^2\) (due to \(E[M^\varepsilon,1\mathcal{Q}] = 0\) as well as \(44\) and \(45\)). Note that there is no longer a log-term in \((47)\). This is due to the factor \(\Delta\) in \((44)\) such that \(\Sigma\) is of a higher order, compared to \(\Delta \log N + 1\).

**Theorem 10.** Under \((B1)\)–\((B3)\) and \((A4)\), we obtain the following solution for the series approach:
\[ J \asymp \varepsilon^{-1}, \quad Q \asymp \frac{B_v^{(p+1)} d^{4v(p+1)} m^{(p+1)\nu+2}}{\mathcal{E}^{5v+2(p+1)} d^{2v(p+1)} m^{(p+1)\nu+2} (p+1)!^4v}, \quad (48) \]
\[ N \asymp \frac{B_v^{2d(p+1)} d^{2d(p+1)} m^{(p+1)\nu+4} 2(p+1)(d+v)}{\mathcal{E}^{2d(p+1)+3v} d^{2d(p+1)} m^{(p+1)\nu+4} (p+1)!^{2d(p+)}} d^{2v(p+1)} m^{(p+1)\nu+2} (p+1)!(d+v), \quad (49) \]
\[ N_0 \asymp N_{c_{p,d}} \asymp \frac{B_v^{4d(p+1)} d^{4d(p+1)} m^{(p+1)\nu+4} 2d(p+1)\nu+2(2p+1)(d+v)}{\mathcal{E}^{4d(p+1)+5v} d^{4d(p+1)} m^{(p+1)\nu+4} (p+1)!^{2d(p+)}} d^{2v(p+1)+3v} m^{(p+1)\nu+4} (p+1)!^{2d(p+)}, \quad (50) \]
\[ R \asymp \frac{B_v^{d+4(p+1)} (p+1)!^{2d(p+1)} d^{2d(p+1)}}{\mathcal{E}^{2d(p+1)+4d(p+1)} d^{2d(p+1)}} \quad (51) \]
provided that \(2(p+1) > d\) and \(v > \frac{2(p+1)}{2(p+1) - d}\). Thus, we have for the complexity
\[ \mathcal{E}_{ser} \asymp JQ^d mc_{p,d}^2 \asymp JQ^d mc_{p,d} N \asymp JQ^d mc_{p,d} N_0 \]
\[ \asymp \frac{B_v^{4d(p+1)} d^{2d(p+1)} (4v-d) \nu 6d(p+1)!^{3d\nu+6} (p+1)!^{5d\nu}}{\mathcal{E}^{6d(p+1)+4v+5d(p+1)!^{6d\nu}} (p+1)!^{3d\nu}}, \quad (52) \]
\[ \footnote{\text{Compare with footnote [1] on page 15}} \]
Remark 7. (i) Complexity estimate (52) shows that one cannot go beyond the complexity order $\epsilon^{-2.5}$ in this case, no matter how large $p, \nu$ are. This is mainly due to the factor $J$ within the third constraint in (47) which does not arise in (36).

(ii) Similarly to Section 4.1, we would have obtained the same complexity even when we used a control variate with a higher variance order $\Delta^K$ for some $K > 1$.

(iii) When comparing (52) with (41), one clearly sees that (41) always achieves a better complexity for $\nu > \frac{2(p+1)}{2(p+1-d)}$ (in terms of $\epsilon$).

(iv) Furthermore, also from the pure computational point of view it is preferable to consider the integral approach rather than the series approach, even though the control variates $M_{\Delta, T}^{\text{ser,1}}$ and $M_{\Delta, T}^{\text{int,1}}$ are theoretically equivalent (recall Theorem 4). This is mainly due to the factor $\Delta_j W_i$ in $a_{j,i}$ (see (10)), which is independent of $X_{\Delta,(j-1)\Delta}$ and has zero expectation and thus may lead to poor regression results (cf. “RCV approach” in [2]). Regarding the integral approach, such a destabilising factor is not present in $g_{j,k}$ (see [20]).

5 Numerical results

In this section, we consider the Euler scheme and compare the numerical performance of the SMC, MLMC, series and integral approaches. For simplicity we implemented a global regression (i.e. the one without truncation and partitioning). Regarding the choice of basis functions, we use in both series and integral approaches the same polynomials $\psi(x) = \prod_{k=1}^{d} x_k^{l_k}$, where $l_1, \ldots, l_d \in \{0, 1, \ldots, p\}$ and $\sum_{k=1}^{d} l_k \leq p$. In addition to the polynomials, we consider the function $f$ as a basis function. Hence, we have overall $(p^d + 1)$ basis functions in each regression. As for the MLMC approach, we use the same simulation results as in [2].

The following results are based on program codes written and vectorised in MATLAB and running on a Linux 64-bit operating system.

5.1 One-dimensional example

Here $d = m = 1$. We consider the following SDE (cf. [2])

$$dX_t = -\frac{1}{2} \tanh(X_t) \sech^2(X_t) dt + \sech(X_t) dW_t, \quad X_0 = 0,$$

(53)

for $t \in [0, 1]$, where $\sech(x) := \frac{1}{\cosh(x)}$. This SDE has an exact solution $X_t = \text{arsinh}(W_t)$. Furthermore, we consider the functional $f(x) = \sech(x) + 15 \arctan(x)$, that is, we have
\[ \mathbb{E}[f(X_1)] = \mathbb{E}[\text{sech}(\text{arsinh}(W_1))] = \mathbb{E}\left[ \frac{1}{\sqrt{1 + W_1^2}} \right] \approx 0.789640. \quad (54) \]

We choose \( p = 3 \) (that is, 5 basis functions) and, for each \( \varepsilon = 2^{-i}, i \in \{2, 3, 4, 5, 6\} \), we set the parameters \( J, N \) and \( N_0 \) as follows (compare with the formulas in Section 4 for \( \nu \to \infty \), \( \lim_{\nu \to \infty} B_\nu = 1 \) and ignore the log-terms for the integral approach):

\[
J = \left\lfloor \varepsilon^{-1} \right\rfloor, \quad N = 256 \cdot \left\{ \begin{array}{ll}
0.6342 \cdot \varepsilon^{-1.0588} & \text{integral approach}, \\
0.6342 \cdot \varepsilon^{-1.5882} & \text{series approach},
\end{array} \right.
\]

\[
N_0 = 256 \cdot \left\{ \begin{array}{ll}
2.5367 \cdot \varepsilon^{-1.0588} & \text{integral approach}, \\
2.5367 \cdot \varepsilon^{-1.5882} & \text{series approach}.
\end{array} \right.
\]

Regarding the SMC approach, the number of paths is set \( N_0 = 256 \cdot \varepsilon^{-2} \). The factor 256 is here for stability purposes. As for the MLMC approach, we set the initial number of paths in the first level \( (l = 0) \) equal to \( 10^3 \) as well as the “discretisation parameter” \( M = 4 \), which leads to time steps of the length \( \frac{1}{M} \) at level \( l \) (the notation here is as in [3]). Next we compute the numerical RMSE (the exact value is known, see (54)) by means of 100 independent repetitions of the algorithm. As can be seen from left-hand side in Figure 1, the estimated numerical complexity is about \( \mathrm{RMSE}^{-1.82} \) for the integral approach, \( \mathrm{RMSE}^{-2.43} \) for the series approach, \( \mathrm{RMSE}^{-1.99} \) for the MLMC approach and \( \mathrm{RMSE}^{-3.02} \) for the SMC approach, which we get by regressing the log-time (logarithmic computing time of the whole algorithm in seconds) vs. log-RMSE. Thus, the complexity reduction works best with the integral approach.

![Fig. 1](image-url) Numerical complexities of the integral, series, SMC and MLMC approaches in the one- and five-dimensional cases.
5.2 Five-dimensional example

Here \( d = m = 5 \). We consider the SDE (cf. [2])

\[
\begin{align*}
\frac{dX_i}{dt} &= - \sin(X_i) \cos^3(X_i) \, dt + \cos^2(X_i) \, dW_i, & X_i^0 = 0, & i \in \{1, 2, 3, 4\}, \\
\frac{dX_5}{dt} &= \sum_{i=1}^{4} \left[ -\frac{1}{2} \sin(X_i) \cos^2(X_i) \, dt + \cos(X_i) \, dW_i \right] + dW_5, & X_5^0 = 0. \quad (55)
\end{align*}
\]

The solution of (55) is given by

\[
\begin{align*}
X_i &= \arctan(W_i), & i \in \{1, 2, 3, 4\}, \\
X_5 &= \sum_{i=1}^{4} \text{arsinh}(W_i) + W_5.
\end{align*}
\]

for \( t \in [0, 1] \). Further, we consider the functional

\[
E[f(X_1)] = (\mathbb{E} \left[ \cos(\arctan(W_1)) + \text{arsinh}(W_1) \right])^4 \mathbb{E} \left[ \cos(W_1^5) \right] \approx 0.002069.
\]

We again choose \( p = 3 \) (this now results in 57 basis functions), consider the same values of \( \varepsilon \) as above (and, in addition, consider the values \( \varepsilon = 2^{-7} \) and \( \varepsilon = 2^{-8} \) for the SMC approach to obtain similar computing times as for the series and integral approaches). Moreover, we set (compare with the formulas in Section 4 for \( \nu \to \infty \), \( \lim_{\nu \to \infty} B_\nu = 1 \) and ignore the log-terms for the integral approach):

\[
J = \left[ \varepsilon^{-1} \right], \quad N = \left\{ \begin{array}{ll}
35.9733 \cdot e^{-1.2381} & \text{integral approach}, \\
4 \cdot 4.9044 \cdot e^{-1.8571} & \text{series approach},
\end{array} \right. \\
N_0 = \left\{ \begin{array}{ll}
2014.5030 \cdot e^{-1.2381} & \text{integral approach}, \\
4 \cdot 274.6480 \cdot e^{-1.8571} & \text{series approach}.
\end{array} \right.
\]

The number of paths for the SMC approach is again set \( N_0 = 256 \cdot \varepsilon^{-2} \). Regarding the MLMC approach, we again choose \( M = 4 \), but the initial number of paths in the first level is increased to \( 10^4 \). As in the one-dimensional case, we compute the numerical RMSE by means of 100 independent repetitions of the algorithm. Our empirical findings are illustrated on the right-hand side in Figure 1. We observe the numerical complexity \( \text{RMSE}^{-1.95} \) for the integral approach, \( \text{RMSE}^{-2.05} \) for the series approach, \( \text{RMSE}^{-2.01} \) for the MLMC approach and \( \text{RMSE}^{-3.03} \) for the SMC approach. Even though here the complexity order of the series approach is better than that of the SMC approach and close to that of MLMC approach, the series approach is practically outperformed by the other approaches (see Figure 1).
plicative constant influencing the computing time is obviously very big). However, the integral approach remains numerically the best one also in this five-dimensional example.

6 Proofs

Proof of Theorem 1

Cf. the proof of Theorem 2.1 in [2].

Proof of Theorem 2

First of all, we derive
\[
\lim_{t \to t_j} u_{\Delta}(t, X_{\Delta,t_{j-1}}, W_t - W_{t_{j-1}}) = \lim_{t \to t_j} E \left[ u_{\Delta}(t_j, \Phi_{\Delta}(x, y + z_j \sqrt{t_j - t}), 0) \right]_{x = X_{\Delta,t_{j-1}}, y = W_t - W_{t_{j-1}}}
\]

By means of Itô’s lemma and the fact that \( u_{\Delta} \) satisfies the heat equation
\[
\frac{\partial u_{\Delta}}{\partial t} + \frac{1}{2} \sum_{i=1}^{m} \frac{\partial^2 u_{\Delta}}{\partial y_i^2} = 0
\]
due to its relation to the normal distribution, we then obtain
\[
\begin{align*}
    f(X_{\Delta,T}) - \mathbb{E}[f(X_{\Delta,T})] & = u_{\Delta}(T,X_{\Delta,T},0) - u_{\Delta}(0,x_0,0) \\
    & = \sum_{j=1}^{J} \left( u_{\Delta}(t_j, X_{\Delta,t_j}, 0) - u_{\Delta}(t_{j-1}, X_{\Delta,t_{j-1}}, 0) \right) \\
    & = \sum_{j=1}^{J} \lim_{t_j \to t_{j-1}} \left( u_{\Delta}(t, X_{\Delta,t_{j-1}}, W_t - W_{t_{j-1}}) - u_{\Delta}(t_{j-1}, X_{\Delta,t_{j-1}}, 0) \right) \\
    & = \sum_{j=1}^{J} \sum_{i=1}^{m} \lim_{t_j \to t_{j-1}} \int_{t_{j-1}}^{t_j} \frac{\partial u_{\Delta}}{\partial y_{i}}(x, X_{\Delta,t_{j-1}}, W_s - W_{t_{j-1}}) \, dW_s \\
    & = \int_{t_{j-1}}^{t_j} \nabla_y u_{\Delta}(x, X_{\Delta,t_{j-1}}, W_s - W_{t_{j-1}}) \, dW_s.
\end{align*}
\]

Next, let us derive a relation between \( \nabla_y u_{\Delta} \) and \( \nabla_x u_{\Delta} \). We have for \( t \in [t_{j-1}, t_j) \)
\[
\nabla_x u_{\Delta}(t,x,y) = \nabla_x \mathbb{E}[u_{\Delta}(t_j, \Phi_{\Delta}(x, y + z_j \sqrt{t_j - t}), 0)] = \nabla_x \mathbb{E}[u_{\Delta}(t_j, \Phi_{\Delta}(x, y + z_j \sqrt{t_j - t}), 0)] \sigma(x).
\]

Thus, the term \( \nabla_y u_{\Delta}(s, X_{\Delta,t_{j-1}}, W_s - W_{t_{j-1}}) \) in (58) takes the form
\[
\nabla_y u_{\Delta}(s, X_{\Delta,t_{j-1}}, W_s - W_{t_{j-1}}) = \mathbb{E}[\nabla_x u_{\Delta}(t_j, X_{\Delta,t_j}, 0) \mid \mathcal{F}_s] \sigma(X_{\Delta,t_{j-1}}).
\]

Note that it holds
\[
u_{\Delta}(t_j, x, 0) = \mathbb{E}[f(X_{\Delta,T})],
\]
where we recall that \( X_{\Delta,T}(\Delta, t_l) \), for \( l \geq j \), denotes the Euler discretisation starting at time \( t_j \) in \( x \) (analogous to \( X_{\Delta,T}(t) \) for the exact solution). Hence, we have for \( \nabla_y u_{\Delta} \)
\[
\nabla_y u_{\Delta}(t_j, x, 0) = \mathbb{E}[\nabla f(X_{\Delta,T}(\Delta, T)) \delta X_{\Delta,T}(\Delta, T)]
\]
or, in another form,
\[
\nabla_y u_{\Delta}(t_j, X_{\Delta,t_j}, 0) = \mathbb{E}[\nabla f(X_{\Delta,T}(\Delta, T)) \delta X_{\Delta,T}(\Delta, T) \mid \mathcal{F}_{t_j}],
\]
where \( \delta X_{\Delta,T}(\Delta, t_l) := \frac{\partial X_{\Delta,T}(\Delta, t_l)}{\partial x_k} \) with \( l \geq j \) and \( i, k \in \{1, \ldots, d\} \). We also notice at this point that \( X_{\Delta,T} = X_{0,x_0}(\Delta, t_l) \) and \( \delta X_{\Delta,t_l} = \delta X_{0,x_0}(\Delta, t_l) \).

Let us define \( \sigma_k(x) := (\sigma_{k,1}(x), \ldots, \sigma_{k,d}(x))^\top \) for \( k \in \{1, \ldots, d\} \). Further, we denote with \( f_{\mu} \in \mathbb{R}^{d \times d} \), \( f_{\sigma_k} \in \mathbb{R}^{d \times d} \) the Jacobi matrices of the functions \( \mu \), \( \sigma_k \). Regarding the discretisation \( \delta X_{\Delta,t} \) of \( \delta X \) we can use, alternatively to (16), the matrix form
\[\text{(59)}\]
\[ \delta X_{\Delta,j} = A_j \delta X_{\Delta,(j-1)\Delta} = A_j A_{j-1} \cdots A_1, \quad (60) \]

where

\[ A_k := I_d + \mathcal{J}_\mu(X_{\Delta,(k-1)\Delta}) \Delta + \frac{(\Delta t^\top \mathcal{J}_\sigma_1(X_{\Delta,(k-1)\Delta}))}{1!} + \cdots + \frac{(\Delta t^\top \mathcal{J}_\sigma(\Delta t))}{(k-1)!}. \]

This gives us

\[ X_{j,X_{\Delta,j}}(\Delta,t_l) = \Phi_\Delta(\cdots(\Phi_\Delta(X_{\Delta,j}, \Delta_{j+1} W), \cdots, \Delta_{1} W)) \]

\[ = \Phi_\Delta(\cdots(\Phi_\Delta(X_{\Delta,0}, \Delta_{1} W), \cdots, \Delta_{1} W)) = X_{\Delta,t_l}, \]

\[ \delta X_{j,X_{\Delta,j}}(\Delta,t_l) = A_j A_{j-1} \cdots A_{j+1} = A_j A_{j-1} \cdots A_1 (A_j A_{j-1} \cdots A_1)^{-1} \]

\[ = \delta X_{\Delta,t_l} \delta X_{\Delta,t_j}^{-1}, \]

where \( \Phi_\Delta \) is defined through (15). Finally, we obtain for \( s \in [t_{j-1}, t_j) \)

\[ \nabla_s u_\Delta(s,X_{\Delta,t_{j-1}}, W_s - W_{t_{j-1}}) = \mathbb{E} \left[ \mathbb{E} \left[ \nabla f(X_{\Delta,T}) \delta X_{\Delta,T} \delta X_{\Delta,t_{j-1}}^{-1} \bigg| \mathcal{F}_s \right] \bigg| \mathcal{F}_s \right] \sigma(X_{\Delta,t_{j-1}}) \]

\[ = \mathbb{E} \left[ \nabla f(X_{\Delta,T}) \delta X_{\Delta,T} \delta X_{\Delta,t_{j-1}}^{-1} \bigg| \mathcal{F}_s \right] \sigma(X_{\Delta,t_{j-1}}). \]

**Proof of Theorem 3**

Below we simply write \( u_{\Delta,t_{j-1}} \) rather than \( u_\Delta(t_{j-1},X_{\Delta,t_{j-1}}, 0) \). Let us consider the Taylor expansion for \( \frac{\partial}{\partial y_{r}} u_\Delta(t,X_{\Delta,t_{j-1}}, W_s - W_{t_{j-1}}) \) of order \( K \in \mathbb{N}_0 \) around \( (t_{j-1},X_{\Delta,t_{j-1}}, 0) \), with \( r \in \{1, \ldots, m\} \), that is, for \( t \in [t_{j-1}, t_j) \), we set

\[ T_{j,r}^K(t) := \sum_{|\alpha| \leq K} \frac{D^\alpha \left( \frac{\partial}{\partial y_{r}} u_\Delta(t_{j-1}) \right)}{\alpha_1! \cdots \alpha_{m+1}!} (t - t_{j-1})^\alpha (W_{r}^{1} - W_{t_{j-1}}^{1})^{\alpha_1} \cdots (W_{r}^{m} - W_{t_{j-1}}^{m})^{\alpha_{m+1}}, \quad (61) \]

where \( \alpha \in \mathbb{N}_0^{m+1} \) and \( D^\alpha \left( \frac{\partial}{\partial y_{r}} u_\Delta(t_{j-1}) \right) = \frac{\partial^{\alpha}}{\partial y_{r_1} \partial y_{r_2} \cdots \partial y_{r_{m+1}}}. \) Via Taylor’s theorem we obtain
\[
\frac{\partial}{\partial y_r} u_\Delta(t, X_{t,j-1}, W_t - W_{t,j-1}) - T_{j,r}^K(t)
\]

\[
= \sum_{|\alpha| = K+1} \frac{(K+1)!}{\alpha_1! \cdots \alpha_{m+1}!} \int_0^1 (1-z)^K D^\alpha \left( \frac{\partial}{\partial y_r} u_\Delta(t_{j-1} + z(t-t_{j-1}), X_{t,j-1}, z(W_t - W_{t,j-1})) \right) dz
\]

\[
\cdot (t-t_{j-1})^{\alpha_1} (W_t^1 - W_{t,j-1}^1)^{\alpha_2} \cdots (W_t^m - W_{t,j-1}^m)^{\alpha_{m+1}}
\]

Provided that (24) holds, we get

\[
\text{Var} \left[ \sum_{j=1}^J \sum_{r=1}^m \int_{t_{j-1}}^{t_j} \left( \frac{\partial}{\partial y_r} u_\Delta(t, X_{t,j-1}, W_t - W_{t,j-1}) - T_{j,r}^K(t) \right) dW_r^r \right]
\]

\[
= \sum_{j=1}^J \sum_{r=1}^m \int_{t_{j-1}}^{t_j} \mathbb{E} \left[ \left( \frac{\partial}{\partial y_r} u_\Delta(t, X_{t,j-1}, W_t - W_{t,j-1}) - T_{j,r}^K(t) \right)^2 \right] dt
\]

\[
\lesssim C^{2(K+1)} \sum_{j=1}^J \sum_{\alpha=1}^{K+1} \int_{t_{j-1}}^{t_j} \mathbb{E} \left[ (t-t_{j-1})^{2\alpha_1} (W_t^1 - W_{t,j-1}^1)^{2\alpha_2} \cdots (W_t^m - W_{t,j-1}^m)^{2\alpha_{m+1}} \right] dt
\]

\[
\lesssim (C^2 \Delta)^{K+1} \rightarrow \infty \rightarrow 0,
\]

and thus \( T_{j,r}^K \) converges for \( K \rightarrow \infty \) in \( L^2(\Omega \times [0,T]) \) to \( \frac{\partial u_\Delta}{\partial y_r} (t, X_{t,j-1}, W_t - W_{t,j-1}) \).

Moreover, due to (57), the limit of \( T_{j,r}^K \) simplifies to (cf. (61))
This gives us finally

\[ \frac{\partial u_{t,j-1}}{\partial y_{t}} + \sum_{i=1}^{m} \frac{\partial^{2} u_{t,j-1}}{\partial y_{t} \partial y_{t}^{i}} (W_{t}^{i} - W_{t-1}^{i}) \]

\[ + \frac{1}{2} \sum_{i=1}^{m} \frac{\partial^{3} u_{t,j-1}}{\partial y_{t} \partial y_{t}^{i} \partial y_{t}^{j}} ((W_{t}^{i} - W_{t-1}^{i})^2 - (t - t_{j-1})) + \sum_{i_{1},i_{2}=1}^{m} \frac{\partial^{2} u_{t,j-1}}{\partial y_{t} \partial y_{t}^{i_{1}} \partial y_{t}^{i_{2}}} (W_{t}^{i_{1}} - W_{t-1}^{i_{1}}) (W_{t}^{i_{2}} - W_{t-1}^{i_{2}}) \]

\[ + \left[ \frac{1}{6} \sum_{i_{1},i_{2} \leq 2}^{m} \frac{\partial^{4} u_{t,j-1}}{\partial y_{t} \partial y_{t}^{i_{1}} \partial y_{t}^{i_{2}}} ((W_{t}^{i_{1}} - W_{t-1}^{i_{1}})^3 - 3(W_{t}^{i_{1}} - W_{t-1}^{i_{1}})(W_{t-1}^{i_{1}} - W_{t-1}^{i_{2}})(W_{t-1}^{i_{2}} - W_{t-1}^{i_{1}})) \right] \]

+ ... \]

\[ = \sum_{i=1}^{m} (t - t_{j-1})^{\frac{l}{2} - 1} \sum_{k_{1},...k_{l-1} \in \mathbb{N}_{0}}^{\sum_{i=1}^{m} k_{i} = l-1} \frac{\partial^{l} u_{t,j-1}}{\partial y_{t} \partial y_{t}^{k_{1}} \ldots \partial y_{t}^{k_{l-1}}} \prod_{i=1}^{m} H_{k_{i}} \left( \frac{W_{t}^{i} - W_{t-1}^{i}}{\sqrt{t-t_{j-1}}} \right). \]

To compute the stochastic integral

\[ \int_{t_{j-1}}^{t_{j}} \nabla_{y} u_{t} \left( t, X_{t,j-1}, W_{t} - W_{t-1} \right) dW_{t} \]

\[ = \sum_{i=1}^{m} \sum_{r=1}^{t_{j}} \int_{t_{j-1}}^{t_{j}} (t - t_{j-1})^{\frac{l}{2} - 1} \sum_{k_{1},...k_{l-1} \in \mathbb{N}_{0}}^{\sum_{i=1}^{m} k_{i} = l-1} \frac{\partial^{l} u_{t,j-1}}{\partial y_{t} \partial y_{t}^{k_{1}} \ldots \partial y_{t}^{k_{l-1}}} \prod_{i=1}^{m} \frac{H_{k_{i}} \left( \frac{W_{t}^{i} - W_{t-1}^{i}}{\sqrt{t-t_{j-1}}} \right)}{\sqrt{k_{i}!}} dW_{t}^{r}, \]

we apply Itô’s lemma w.r.t. the functions \( F_{k}(t, y_{1}, \ldots, y_{m}) := t^{l/2} \prod_{i=1}^{m} \frac{H_{k_{i}} \left( \frac{y_{i}}{\sqrt{k_{i}}} \right)}{\sqrt{k_{i}!}}, \) where \( \sum_{i=1}^{m} k_{i} = l. \) Thus, we obtain

\[ dF_{k}(t - t_{j-1}, W_{t-1}^{1} - W_{t-1}^{1}, \ldots, W_{t-1}^{m} - W_{t-1}^{m}) \]

\[ = (t - t_{j-1})^{\frac{l}{2} - 1} \sum_{r=1}^{m} H_{k_{i} - 1} \left( \frac{W_{t}^{i} - W_{t-1}^{i}}{\sqrt{t-t_{j-1}}} \right) \prod_{i=p}^{m} \frac{H_{k_{i}} \left( \frac{W_{t}^{i} - W_{t-1}^{i}}{\sqrt{t-t_{j-1}}} \right)}{\sqrt{k_{i}!}} dW_{t}^{r}. \]

This gives us finally
Regression-based variance reduction approach for strong approximation schemes

\[ \int_{t_{j-1}}^{t_j} \nabla_x u_\Delta(t, X_{\Delta,j-1}, W_t - W_{t_{j-1}}) \, dW_t = \sum_{l=1}^{\infty} \Delta^{l/2} \sum_{k_i \in \mathbb{N}^m} \frac{\partial^l u_\Delta(t_{j-1}, X_{\Delta,j-1}, 0)}{\partial y_1^{k_1} \cdots \partial y_m^{k_m}} \prod_{i=1}^m \frac{H_{k_i} \left( \frac{\Delta_j y_i}{\sqrt{\Delta}} \right)}{\sqrt{k_i^2}}. \]

**Proof of Theorem 4**

We define the (random) function \( G_{l,j}(x) \) for \( J \geq l \geq j \geq 0, \ x \in \mathbb{R}^d \), as follows

\[
G_{l,j}(x) = \Phi_{\Delta,l} \circ \Phi_{\Delta,j-1} \circ \cdots \circ \Phi_{\Delta,1}(x), \quad l > j, \quad G_{l,j}(x) = x, \quad l = j,
\]

where \( \Phi_{\Delta,l}(x) := \Phi_{\Delta}(x, \Delta_j W) \) for \( l = 1, \ldots, J \). Note that it holds

\[
u_\Delta(t_j, x, 0) = E[f(G_{j,j}(x))].
\]

Similar to \( G \) we define the function \( \tilde{G}_{j,j}(x,z) \), \( 0 \leq j < J, \ x \in \mathbb{R}^d, \ z := (z_1, \ldots, z_{J-j}) \in \mathbb{R}^{m \times (J-j)}, \ z_l := (z_l^1, \ldots, z_l^m) \in \mathbb{R}^m \) for \( l = 1, \ldots, J - j \), as follows

\[ \tilde{G}_{j,j}(x,z) := \Phi_{\Delta,z_{j-1}} \circ \cdots \circ \Phi_{\Delta,z_1}(x), \]

where \( \Phi_{\Delta,z_l}(x) := \Phi_{\Delta}(x, z_l \sqrt{\Delta}) \). Note that \( G \) and \( \tilde{G} \) are in the following relation

\[
G_{j,j}(x) = \tilde{G}_{j,j} \left( x, \frac{1}{\sqrt{\Delta}} (\Delta_{j+1} W, \Delta_{j+2} W, \ldots, \Delta_j W) \right), \quad j < J.
\]

Let us represent \( \sqrt{\Delta} \frac{\partial}{\partial y_i} u_\Delta(t_{j-1}, x, 0) \), where \( j \in \{1, \ldots, J\} \) and \( i \in \{1, \ldots, m\} \), as a \((J - j + 1) m\)-dimensional integral, that is (cf. (65))

\[
\sqrt{\Delta} \frac{\partial}{\partial y_i} u_\Delta(t_{j-1}, x, 0) = \sqrt{\Delta} \frac{\partial}{\partial y_i} E[f(G_{j,j}(\Phi_\Delta(x,\Delta_j W + y))) | y = 0_m] = \int_{\mathbb{R}^{(J - j + 1)m}} \sqrt{\Delta} \frac{\partial}{\partial y_i} f \left( \tilde{G}_{j-1} \left( x, \left( z_1 + \frac{y}{\sqrt{\Delta}}, z_2, \ldots, z_{j-1} \right) \right) \right) \varphi_{(J-j+1)m}(z) \, dz \big|_{y=0_m},
\]

where \( \varphi_{(J-j+1)m} \) denotes the \((J - j + 1)m\)-dimensional standard normal density function. Since it holds
\[
\sqrt{\Delta} \frac{\partial}{\partial y_i} \left[ f \left( \tilde{G}_{j-1} \left( x, \left( z_1 + \frac{y}{\sqrt{\Delta}}, z_2, \ldots, z_{j-1} \right) \right) \right) \right] \\
= \frac{\partial}{\partial z_i^1} \left[ f \left( \tilde{G}_{j-1} \left( x, \left( z_1 + \frac{y}{\sqrt{\Delta}}, z_2, \ldots, z_{j-1} \right) \right) \right) \right].
\]

we obtain via integration by parts
\[
\sqrt{\Delta} \frac{\partial}{\partial y_i} u_{\Delta}(t_{j-1}, x) = \int_{\mathbb{R}^{(j-j+1)m}} \frac{\partial}{\partial z_i^1} \left[ f \left( \tilde{G}_{j-1} \left( x, z \right) \right) \right] \phi_{j-j+1}(z) dz
\]
\[
= \int_{\mathbb{R}^{(j-j+1)m}} f \left( \tilde{G}_{j-1} \left( x, z \right) \right) \frac{\partial}{\partial z_i^1} \phi_{j-j+1}(z) dz
\]
\[
= \int_{\mathbb{R}^{(j-j+1)m}} f \left( \tilde{G}_{j-1} \left( x, z \right) \right) \phi_{j-j+1}(z) dz
\]
\[
= \mathbb{E} \left[ f(\tilde{G}_{j-1}(x)) \frac{\Delta W^i}{\sqrt{\Delta}} \right] = \mathbb{E} \left[ f(X_\Delta, t) \frac{\Delta W^i}{\sqrt{\Delta}} | X_\Delta, (j-1) = x \right] = a_{j,e}(x).
\]

We finally remark that we have only the integral term in the integration by parts above because the function \( z_1 \mapsto f(\tilde{G}_{j-1}(x,z))\phi_{j-j+1}(z) \) is integrable over \( \mathbb{R} \) w.r.t. the Lebesgue measure.

**Proof of Theorem**

Via Taylor’s theorem we get
\[
\frac{\partial u_{\Delta}(t, X_\Delta, t_{j-1}, W_t - W_{t_{j-1}})}{\partial y_i} = \frac{\partial u_{\Delta}(t_{j-1}, X_\Delta, t_{j-1}, 0)}{\partial y_i} + (t - t_{j-1}) \int_0^t \frac{\partial^2 u_{\Delta}(t_{j-1} + z(t - t_{j-1}), X_\Delta, t_{j-1}, z(W_t - W_{t_{j-1}}))}{\partial y_i \partial y} dz
\]
\[
+ \sum_{r=1}^m (W^r_t - W^r_{t_{j-1}}) \int_0^t \frac{\partial^2 u_{\Delta}(t_{j-1} + z(t - t_{j-1}), X_\Delta, t_{j-1}, z(W_t - W_{t_{j-1}}))}{\partial y_i \partial y_r} dz. \tag{66}
\]

Due to (57), (66) simplifies to
Regression-based variance reduction approach for strong approximation schemes

where \( \Phi \) is the regression-based variance reduction approach for strong approximation schemes. Provided that the second and third derivatives of \( u \) are bounded, we have

\[
\var\left[ \int_{t_{j-1}}^{t_j} \frac{\partial u(\tau, X_{\Delta, t_{j-1}}, W_t - W_{t_{j-1}})}{\partial y_i} \, dW^i_t - \frac{\partial u(\tau, X_{\Delta, t_{j-1}}, 0)}{\partial y_i} \, \Delta_j \right] \leq \sum_{r=1}^m \int_{t_{j-1}}^{t_j} \mathbb{E} \left[ (W^r_t - W^r_{t_{j-1}})^2 + (t - t_{j-1})^2 \right] dt \leq \Delta^2.
\]

Thus, we finally obtain

\[
\var \left[ f \left( X_{\Delta, t} \right) - M_{\Delta, t}^{m,1} \right] \leq \Delta.
\]

**Proof of Theorem 6**

We start the calculations, which will lead to the proof of part (ii). At some point we will get the proof of part (i) as a by-product.

In this proof we will use the shorthand notation \( \xi_k := \Delta_k W, k \in \{1, \ldots, J\} \). For \( j \in \{0, \ldots, J-1\} \), we have

\[
u \Delta(\tau, x, y) = \mathbb{E} \left[ f \left( \Phi_{\Delta, J} \circ \Phi_{\Delta, J-1} \circ \cdots \circ \Phi_{\Delta, j+2} \circ \Phi_{\Delta, j+1} \left( x, y + \xi_{j+1} \right) \right) \right],
\]

where \( \Phi_{\Delta, k}(x) := \Phi_{\Delta}(x, \xi_k) \). Denote, for \( k > j \),

\[
G_{k, j}(x, y) := \Phi_{\Delta, k} \circ \Phi_{\Delta, k-1} \circ \cdots \circ \Phi_{\Delta, j+2} \circ \Phi_{\Delta}(x, y + \xi_{j+1}.
\]
Assume that for any \( n \in \mathbb{N}, l \in \{1, \ldots, d\}, \alpha \in \mathbb{N}_0^d, \)

\[
\left| \mathbb{E} \left[ \left( D^\alpha \Phi_{\Delta,k+1}^l(G_{k,j}(x,y)) \right)^n \right| \mathcal{F}_k \right| \leq \begin{cases} 
(1 + A_{n,j} \Delta), & \beta = \alpha = 1 \\
B_{n,l,\alpha} \Delta, & (\beta > 1) \vee (\alpha_j \neq 1) 
\end{cases} \tag{67}
\]

with probability one for \( \beta = |\alpha| \in \mathbb{N} \) and some constants \( A_{n,j} > 0, B_{n,l,\alpha} > 0. \) We recall the notation \( D^\alpha f(x) = \frac{\partial^{(|\alpha|)}}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}} f, \) which was used here. Clearly, for the Euler scheme (15), condition (67) is satisfied if all the derivatives of order \( \beta \) for \( \mu_k, \sigma_{ki}, \)

\( k \in \{1, \ldots, d\}, i \in \{1, \ldots, m\}, \) are bounded. Moreover, suppose that for any \( n_1, n_2 \in \mathbb{N}, l \in \{1, \ldots, d\}, \alpha_1, \alpha_2 \in \mathbb{N}_0^d, \) with \( \beta_1 = |\alpha_1| > 0, \beta_2 = |\alpha_2| > 0, (\beta_1 > 1) \vee (\beta_2 > 1) \vee ((\alpha_1)_l \neq 1) \vee ((\alpha_2)_l \neq 1), \)

\[
\left| \mathbb{E} \left[ \left( D^{\alpha_1} \Phi_{\Delta,k+1}^l(G_{k,j}(x,y)) \right)^{n_1} \left( D^{\alpha_2} \Phi_{\Delta,k+1}^l(G_{k,j}(x,y)) \right)^{n_2} \right| \mathcal{F}_k \right| \leq C_{n_1,n_2,l,\alpha_1,\alpha_2} \Delta \tag{68}
\]

for some constant \( C_{n_1,n_2,l,\alpha_1,\alpha_2} > 0. \) Again, for the Euler scheme (15), condition (68) is satisfied if all the derivatives of orders \( \beta_1 \) and \( \beta_2 \) for \( \mu_k, \sigma_{ki} \) are bounded.

We have for some \( i \in \{1, \ldots, m\} \) and \( l \in \{1, \ldots, d\}, \)

\[
\frac{\partial}{\partial y_i} G_{k+1,j}(x,y) = \sum_{s=1}^{d} \frac{\partial}{\partial x_s} \Phi_{\Delta,k+1}^l(G_{k,j}(x,y)) \frac{\partial}{\partial y_i} G_{k,j}^s(x,y)
\]

and \( \frac{\partial}{\partial y_i} G_{j+1,i}(x,y) = \frac{\partial}{\partial y_i} \Phi_{\Delta}^s(x,y + \xi_{j+1}). \) Hence

\[
\mathbb{E} \left[ \left( \frac{\partial}{\partial y_i} G_{k+1,j}(x,y) \right)^2 \right] \leq \mathbb{E} \left[ \left( \frac{\partial}{\partial y_i} \Phi_{\Delta,k+1}^l(G_{k,j}(x,y)) \frac{\partial}{\partial y_i} G_{k,j}^s(x,y) \right)^2 \right]
\]

\[
+ \sum_{s \neq l} \left\{ 2 \frac{\partial}{\partial x_l} \Phi_{\Delta,k+1}^l(G_{k,j}(x,y)) \frac{\partial}{\partial x_s} \Phi_{\Delta,k+1}^l(G_{k,j}(x,y)) \frac{\partial}{\partial y_i} G_{k,j}^s(x,y) \frac{\partial}{\partial y_l} G_{k,j}^l(x,y) \right. \\
\left. + (d-1) \left( \frac{\partial}{\partial x_s} \Phi_{\Delta,k+1}^l(G_{k,j}(x,y)) \frac{\partial}{\partial y_l} G_{k,j}^l(x,y) \right)^2 \right\}
\]

Denote

\[
\rho_{k+1,n}^{i,s} \leq \mathbb{E} \left[ \left( \frac{\partial}{\partial y_i} G_{k+1,j}^s(x,y) \right)^n \right], \tag{69}
\]

then, due to \( 2ab \leq a^2 + b^2, \) we get for \( k = j + 1, \ldots, J - 1, \)

\[
\rho_{k+1,2}^{i,l} \leq (1 + A_{2,j} \Delta) \rho_{k+1,2}^{i,l} + \sum_{s \neq l} \left\{ C_{1,1,l,s} \Delta (\rho_{k+1,2}^{i,l} + \rho_{k+1,2}^{i,s}) + (d-1)B_{2,l,s} \Delta \rho_{k+1,2}^{i,s} \right\}
\]
Further, denote
\[ \rho_{i_k+1,n} = \sum_{l=1}^{d} \rho_{i_k+1,l,n}, \]
then we get for \( k = j+1, \ldots, J-1, \)
\[ \rho_{i_k+1,2} \leq (1 + A_2 \Delta) \rho_{i_k+2} + 2(d-1)C_{1,1} \Delta \rho_{i_k+2} + (d-1)^2 B_2 \Delta \rho_{i_k+2}, \]
where \( A_2 := \max_{l=1, \ldots, d} A_{2,l}, \ B_2 := \max_{l,s=1, \ldots, d} B_{2,l,s,l,s}, \) \( C_{1,1} := \max_{l,s=1, \ldots, d} C_{1,1,l,s,l,s}, \) This gives us
\[ \rho_{i_k+1,2} \leq (1 + \kappa_1 \Delta) \rho_{i_k+2}, \quad k = j+1, \ldots, J-1, \]
for some constant \( \kappa_1 > 0, \) leading to
\[ \rho_{i_k+2} \leq (1 + \kappa_1 \Delta)^{k-j-1} \rho_{i_j+2}, \quad k = j+1, \ldots, J-1, \]
where
\[ \rho_{i_j+2} = \sum_{s=1}^{d} E \left[ \left( \frac{\partial}{\partial y_i} \Phi^s_{\Delta}(x,y+\xi_{j+1}) \right)^2 \right] = \sum_{s=1}^{d} \sigma_{s,i}^2(x). \]
Thus, we obtain the boundedness of
\[ \frac{\partial}{\partial y_i} u_{\Delta}(t_j,x,y) = \sum_{s=1}^{d} E \left[ \frac{\partial}{\partial x_s} f(G_{j,s}(x,y)) \frac{\partial}{\partial y_i} G_{j,s}^s(x,y) \right], \]
provided that \( \sigma_{i} \) and all the derivatives of order 1 of \( f, \mu_k, \sigma_{i} \) are bounded.

Similar calculations show that the boundedness of \( \sigma_{i} \) is not necessary to assume in order to get that \( \frac{\partial}{\partial y_i} u_{\Delta}(t_j,x,y) \) and consequently \( g_{j,l}(x) \) for \( l \in \{ 1, \ldots, d \} \) are bounded (recall (21)). This yields (A2) under the assumptions in part (i) of Theorem 6 (that is, the boundedness of \( \sigma_{i} \) is not needed).

Furthermore, we have, due to \( (\sum_{k=1}^{d} a_k)^n \leq d^{n-1} \sum_{k=1}^{d} a_k^n, \)
where
\[
\rho\leq s_i = \frac{\rho}{\partial x_i} + \sum_{s \neq i} \rho^l_{\Delta k+1}(G_{k, j}(x, y)) \frac{\partial}{\partial y_i} G_{k, j}(x, y) + \frac{\partial}{\partial x_i} \rho^l_{\Delta k+1}(G_{k, j}(x, y)) \frac{\partial}{\partial y_j} G_{k, j}(x, y)
\]

and thus, due to \(4a^3b \leq 3a^4 + b^4\) and \(2a^2b^2 \leq a^4 + b^4\),
\[
\rho^{i l}_{k+1, 4} \leq (1 + A_4 \Delta) \rho^{i l}_{k, 4} + \sum_{s \neq l} \left\{ C_{3, 1, l, \xi, \epsilon_1} \Delta(3 \rho^{i l}_{k, 4} + \rho^{i s}_{k, 4}) + (d - 1)C_{2, 2, l, \xi, \epsilon_1} \Delta(\rho^{i l}_{k, 4} + \rho^{i s}_{k, 4}) \right\}
\]

This gives us
\[
\rho^{i l}_{k+1, 4} \leq (1 + A_4 \Delta) \rho^{i l}_{k, 4} + 4(d - 1)C_{3, 1} \Delta \rho^{i l}_{k, 4} + 6(d - 1)^2 C_{2, 2} \Delta \rho^{i l}_{k, 4}
\]

where \(A_4 := \max_{l = 1, \ldots, d} A_{4, l}, B_4 := \max_{l, s = 1, \ldots, d} B_{4, l, \xi, \epsilon}, C_{3, 1} := \max_{l, s = 1, \ldots, d} C_{3, 1, l, \xi, \epsilon}, C_{2, 2} := \max_{l, s = 1, \ldots, d} C_{2, 2, l, \xi, \epsilon}, C_{1, 3} := \max_{l, s = 1, \ldots, d} C_{1, 3, l, \xi, \epsilon}\). Hence, we obtain
\[
\rho^{i l}_{k+1, 4} \leq (1 + \kappa_2 \Delta) \rho^{i l}_{k, 4}, \quad k = j + 1, \ldots, J - 1
\]

for some constant \(\kappa_2 > 0\), leading to
\[
\rho^{i l}_{k, 4} \leq (1 + \kappa_2 \Delta)^{k-j-1} \rho^{i j_{j+1}, 4}, \quad k = j + 1, \ldots, J - 1
\]

where
\[
\rho^{i j_{j+1}, 4} = \sum_{s=1}^d \mathbb{E} \left[ \left( \frac{\partial}{\partial y_i} \Phi^l_{\Delta}(x, y + \xi_{j+1}) \right)^4 \right] = \sum_{s=1}^d \sigma^l_{st}(x).
\]
Thus, we obtain boundedness of $\rho_{i,k}^j$ uniformly in $x, y, j, k \in \{j + 1, \ldots, J\}$ and $J$, for all $i \in \{1, \ldots, m\}$, provided $\sigma_{ki}$ and all derivatives of order 1 of $f, \mu_k, \sigma_k$ are bounded.

Now we set $\tilde{G}_{j,j}(x) := G_{j,j}(x, 0)$ and observe that similar calculations involving derivatives w.r.t. $x_k$ show that the quantities
\[
E \left[ \left( \frac{\partial}{\partial x_k} \tilde{G}_{j,j}(x) \right)^4 \right]
\]
(cf. with (69)) are all bounded uniformly in $x, J$ and $j \in \{0, \ldots, J - 1\}$, provided all derivatives of order 1 of $f, \mu_k, \sigma_k$ are bounded (that is, boundedness of $\sigma_{ki}$ is not needed at this point). Using the identity $\tilde{G}_{j,j}(X_{\Delta,t_j}) = X_{\Delta,T}$ one can check that
\[
\mathcal{J}_{G_{j,j}}(X_{\Delta,t_j}) = \delta X_{\Delta,T} \delta X_{\Delta,t_j}^{-1},
\]
where $\mathcal{J}_{G_{j,j}}$ denotes the Jacobi matrix of the function $G_{j,j}$. Recalling the definition
\[
\zeta_j = (\zeta_j, \ldots, \zeta_j) := \nabla f(X_{\Delta,T}) \delta X_{\Delta,T} \delta X_{\Delta,t_j}^{-1}
\]
of the vector $\zeta_j$, we get from (70) that
\[
\zeta_{j,k} = \sum_{s=1}^d \frac{\partial}{\partial x_s} \tilde{G}_{j,j}(X_{\Delta,t_j}) \frac{\partial}{\partial x_k} \tilde{G}_{j,j}(X_{\Delta,t_j}).
\]

Then we obtain for $k \in \{1, \ldots, d\}$ and $j \in \{1, \ldots, J\}$
\[
\begin{align*}
\text{Var} \left[ \zeta_{j,k} | X_{\Delta,t_{j-1}} = x \right] & \leq E \left[ \zeta_{j,k}^2 | X_{\Delta,t_{j-1}} = x \right] \\
& = E \left[ \left( \sum_{s=1}^d \frac{\partial}{\partial x_s} \tilde{G}_{j,j}(X_{\Delta,t_j}) \frac{\partial}{\partial x_k} \tilde{G}_{j,j}(X_{\Delta,t_j}) \right)^2 | X_{\Delta,t_{j-1}} = x \right] \\
& \leq d \sum_{s=1}^d E \left[ \left( \frac{\partial}{\partial x_s} \tilde{G}_{j,j-1}(x) \frac{\partial}{\partial x_k} \tilde{G}_{j,j}(\Phi_{\Delta,j}(x)) \right)^2 \right] \\
& \leq d \sum_{s=1}^d \left[ E \left( \frac{\partial}{\partial x_s} \tilde{G}_{j,j-1}(x) \right)^4 \right] E \left[ \left( \frac{\partial}{\partial x_k} \tilde{G}_{j,j}(\Phi_{\Delta,j}(x)) \right)^4 \right].
\end{align*}
\]

Due to the discussion above, the latter expression is bounded in $x$, provided all derivatives of order 1 of $f, \mu_k, \sigma_k$ are bounded. That is, we get (A1), and the proof of part (i) is completed.

Proceeding with part (ii), we have
\[4\] Notice that thus defined $\tilde{G}_{j,j}$ is the same as $G_{j,j}$ of (63) (in the proof of Theorem 4).
\[
E \left[ \left( \frac{\partial}{\partial y_j} G_{k+1,j}(x,y) \right)^6 \right] \\
\leq E \left[ \left( \frac{\partial}{\partial x_k} \Phi_{\Delta,k+1}(G_{k,j}(x,y)) \frac{\partial}{\partial y_j} G_{k,j}(x,y) \right)^6 \right] \\
+ \sum_{s \neq l} \left\{ 6 \left( \frac{\partial}{\partial x_l} \Phi_{\Delta,k+1}(G_{k,j}(x,y)) \frac{\partial}{\partial y_j} G_{k,j}(x,y) \right)^5 \frac{\partial}{\partial x_s} \Phi_{\Delta,k+1}(G_{k,j}(x,y)) \frac{\partial}{\partial y_j} G_{k,j}(x,y) \\
+ 15(d-1) \left( \frac{\partial}{\partial x_l} \Phi_{\Delta,k+1}(G_{k,j}(x,y)) \frac{\partial}{\partial y_j} G_{k,j}(x,y) \right)^4 \left( \frac{\partial}{\partial x_s} \Phi_{\Delta,k+1}(G_{k,j}(x,y)) \frac{\partial}{\partial y_j} G_{k,j}(x,y) \right)^2 \\
+ 20(d-1)^2 \left( \frac{\partial}{\partial x_l} \Phi_{\Delta,k+1}(G_{k,j}(x,y)) \frac{\partial}{\partial y_j} G_{k,j}(x,y) \right)^3 \left( \frac{\partial}{\partial x_s} \Phi_{\Delta,k+1}(G_{k,j}(x,y)) \frac{\partial}{\partial y_j} G_{k,j}(x,y) \right)^3 \\
+ 15(d-1)^3 \left( \frac{\partial}{\partial x_l} \Phi_{\Delta,k+1}(G_{k,j}(x,y)) \frac{\partial}{\partial y_j} G_{k,j}(x,y) \right)^2 \left( \frac{\partial}{\partial x_s} \Phi_{\Delta,k+1}(G_{k,j}(x,y)) \frac{\partial}{\partial y_j} G_{k,j}(x,y) \right)^4 \\
+ 6(d-1)^4 \frac{\partial}{\partial x_l} \Phi_{\Delta,k+1}(G_{k,j}(x,y)) \frac{\partial}{\partial y_j} G_{k,j}(x,y) \left( \frac{\partial}{\partial x_s} \Phi_{\Delta,k+1}(G_{k,j}(x,y)) \frac{\partial}{\partial y_j} G_{k,j}(x,y) \right)^5 \\
+ (d-1)^5 \frac{\partial}{\partial x_l} \Phi_{\Delta,k+1}(G_{k,j}(x,y)) \frac{\partial}{\partial y_j} G_{k,j}(x,y) \left( \frac{\partial}{\partial x_s} \Phi_{\Delta,k+1}(G_{k,j}(x,y)) \frac{\partial}{\partial y_j} G_{k,j}(x,y) \right)^6 \right\} \\
\]

and thus, due to \(6a^5b \leq 5a^6 + b^6, 3a^4b^2 \leq 2a^6 + b^6\) and \(2a^3b^3 \leq a^6 + b^6\),

\[
\rho_{k+1,6}^l \leq (1 + A_6 \Delta) \rho_{k,6}^l + \sum_{s \neq l} \left\{ C_{5,1,l,e_1,\varepsilon} \Delta(5p_{k,6}^{l,s} + \rho_{k,6}^{l,s}) + 5(d-1)C_{4,2,1,\varepsilon,e_1,\varepsilon} \Delta(2p_{k,6}^{l,s} + \rho_{k,6}^{l,s}) \\
+ 10(d-1)^2C_{3,3,l,e_1,\varepsilon} \Delta(\rho_{k,6}^{l,s} + \rho_{k,6}^{l,s}) + 5(d-1)^3C_{2,4,1,\varepsilon,e_1,\varepsilon} \Delta(\rho_{k,6}^{l,s} + 2\rho_{k,6}^{l,s}) \\
+ (d-1)^4C_{1,5,l,e_1,\varepsilon} \Delta(\rho_{k,6}^{l,s} + 5\rho_{k,6}^{l,s}) + (d-1)^5 B_{6,l,e_1,\varepsilon,\varepsilon} \Delta \rho_{k,6}^{l,s} \right\} .
\]

This gives us

\[
\rho_{k+1,6}^l \leq (1 + A_6 \Delta) \rho_{k,6}^l + 6(d-1)C_{5,1} \Delta \rho_{k,6}^l + 15(d-1)^2C_{4,2} \Delta \rho_{k,6}^l + 20(d-1)^3C_{3,3} \Delta \rho_{k,6}^l \\
+ 15(d-1)^4C_{2,4} \Delta \rho_{k,6}^l + 6(d-1)^5C_{1,5} \Delta \rho_{k,6}^l + (d-1)^6 B_{6} \rho_{k,6}^l,
\]

where \(A_6 := \max_{l=1,\ldots,d} A_6, B_6 := \max_{l,s=1,\ldots,d} B_{6,l,e_1,\varepsilon}, C_{5,1} := \max_{l,s=1,\ldots,d} C_{5,1,l,e_1,\varepsilon,\varepsilon}, C_{4,2} := \max_{l,s=1,\ldots,d} C_{4,2,l,e_1,\varepsilon,\varepsilon} \), \(C_{3,3} := \max_{l,s=1,\ldots,d} C_{3,3,l,e_1,\varepsilon,\varepsilon}, C_{2,4} := \max_{l,s=1,\ldots,d} C_{2,4,l,e_1,\varepsilon,\varepsilon} \) and \(C_{1,5} := \max_{l,s=1,\ldots,d} C_{1,5,l,e_1,\varepsilon,\varepsilon} \). Hence, we obtain

\[
\rho_{k+1,6}^l \leq (1 + \kappa_3 \Delta) \rho_{k,6}^l, \quad k = j + 1, \ldots, J - 1
\]
for some constant $\kappa_3 > 0$, leading to

$$\rho_{k,6}^j \leq (1 + \kappa_1 \Delta)^{k-j-1} \rho_{j+1,6}^{j+1}, k = j+1, \ldots, J-1,$$

where

$$\rho_{j+1,6}^j = \sum_{s=1}^d \mathbb{E} \left[ \left( \frac{\partial}{\partial y_i} \Phi_{\Delta,k+1}^i G_{k,j}(x,y) + \xi_{j+1,6}^{i,j} \right)^6 \right] = \sum_{s=1}^d \sigma_{0,s}^i(x).$$

Moreover, we have

$$\mathbb{E} \left[ \left( \frac{\partial}{\partial y_i} G_{k+1,j}^i(x,y) \right)^8 \right]$$

$$\leq \mathbb{E} \left[ \left( \frac{\partial}{\partial x_j} \Phi_{\Delta,k+1}^j (G_{k,j}(x,y)) \frac{\partial}{\partial y_i} G_{k,j}^i(x,y) \right)^8 \right]$$

$$+ \sum_{s \neq i} \left\{ 8 \left( \frac{\partial}{\partial x_j} \Phi_{\Delta,k+1}^j (G_{k,j}(x,y)) \frac{\partial}{\partial y_i} G_{k,j}^i(x,y) \right)^7 \left( \frac{\partial}{\partial x_j} \Phi_{\Delta,k+1}^j (G_{k,j}(x,y)) \frac{\partial}{\partial y_i} G_{k,j}^i(x,y) \right)^6 \right\}$$

$$+ 28(d-1) \left( \frac{\partial}{\partial x_j} \Phi_{\Delta,k+1}^j (G_{k,j}(x,y)) \frac{\partial}{\partial y_i} G_{k,j}^i(x,y) \right)^6 \left( \frac{\partial}{\partial x_j} \Phi_{\Delta,k+1}^j (G_{k,j}(x,y)) \frac{\partial}{\partial y_i} G_{k,j}^i(x,y) \right)^2$$

$$+ 56(d-1)^2 \left( \frac{\partial}{\partial x_j} \Phi_{\Delta,k+1}^j (G_{k,j}(x,y)) \frac{\partial}{\partial y_i} G_{k,j}^i(x,y) \right)^5 \left( \frac{\partial}{\partial x_j} \Phi_{\Delta,k+1}^j (G_{k,j}(x,y)) \frac{\partial}{\partial y_i} G_{k,j}^i(x,y) \right)^3$$

$$+ 70(d-1)^3 \left( \frac{\partial}{\partial x_j} \Phi_{\Delta,k+1}^j (G_{k,j}(x,y)) \frac{\partial}{\partial y_i} G_{k,j}^i(x,y) \right)^4 \left( \frac{\partial}{\partial x_j} \Phi_{\Delta,k+1}^j (G_{k,j}(x,y)) \frac{\partial}{\partial y_i} G_{k,j}^i(x,y) \right)^4$$

$$+ 56(d-1)^4 \left( \frac{\partial}{\partial x_j} \Phi_{\Delta,k+1}^j (G_{k,j}(x,y)) \frac{\partial}{\partial y_i} G_{k,j}^i(x,y) \right)^5 \left( \frac{\partial}{\partial x_j} \Phi_{\Delta,k+1}^j (G_{k,j}(x,y)) \frac{\partial}{\partial y_i} G_{k,j}^i(x,y) \right)^5$$

$$+ 28(d-1)^5 \left( \frac{\partial}{\partial x_j} \Phi_{\Delta,k+1}^j (G_{k,j}(x,y)) \frac{\partial}{\partial y_i} G_{k,j}^i(x,y) \right)^6 \left( \frac{\partial}{\partial x_j} \Phi_{\Delta,k+1}^j (G_{k,j}(x,y)) \frac{\partial}{\partial y_i} G_{k,j}^i(x,y) \right)^6$$

$$+ 8(d-1)^6 \frac{\partial}{\partial x_j} \Phi_{\Delta,k+1}^j (G_{k,j}(x,y)) \frac{\partial}{\partial y_i} G_{k,j}^i(x,y) \left( \frac{\partial}{\partial x_j} \Phi_{\Delta,k+1}^j (G_{k,j}(x,y)) \frac{\partial}{\partial y_i} G_{k,j}^i(x,y) \right)^7$$

$$+ (d-1)^7 \left( \frac{\partial}{\partial x_j} \Phi_{\Delta,k+1}^j (G_{k,j}(x,y)) \frac{\partial}{\partial y_i} G_{k,j}^i(x,y) \right)^8 \right\}$$

and thus, due to $8a^7 b \leq 7a^8 + b^8$, $4a^6 b^2 \leq 3a^8 + b^8$, $8a^5 b^3 \leq 5a^8 + 3b^8$ and $2a^4 b^4 \leq a^8 + b^8$, ...
\[
\rho_{k+1,8}^{j,l} \leq (1 + A_8 \Delta) \rho_{k,8}^{j,l} + \sum_{s=1}^{3} \left\{ C_{7,1,j,e_i,e_s} \Delta (7 \rho_{k,8}^{j,l} + \rho_{k,8}^{i,s}) + 7(d - 1) C_{6,2,j,e_i,e_s} \Delta (3 \rho_{k,8}^{j,l} + \rho_{k,8}^{i,s}) \right. \\
+ 7(d - 1)^2 C_{3,3,j,e_i,e_s} \Delta (5 \rho_{k,8}^{j,l} + 3 \rho_{k,8}^{i,s}) + 35(d - 1)^3 C_{4,4,j,e_i,e_s} \Delta (\rho_{k,8}^{j,l} + \rho_{k,8}^{i,s}) \\
+ 7(d - 1)^4 C_{3,5,j,e_i,e_s} \Delta (3 \rho_{k,8}^{j,l} + 5 \rho_{k,8}^{i,s}) + 7(d - 1)^5 C_{2,6,j,e_i,e_s} \Delta (\rho_{k,8}^{j,l} + 3 \rho_{k,8}^{i,s}) \\
+ (d - 1)^6 C_{1,7,j,e_i,e_s} \Delta (\rho_{k,8}^{j,l} + 7 \rho_{k,8}^{i,s}) + (d - 1)^7 B_{8,j,e_i,e_s} \Delta \rho_{k,8}^{i,s} \right\}.
\]

This gives us
\[
\rho_{k+1,8}^{j,l} \leq (1 + A_8 \Delta) \rho_{k,8}^{j,l} + 8(d - 1) C_{7,1,j,e_i,e_s} \Delta \rho_{k,8}^{j,l} + 28(d - 1)^2 C_{6,2,j,e_i,e_s} \Delta \rho_{k,8}^{j,l} + 56(d - 1)^3 C_{3,3,j,e_i,e_s} \Delta \rho_{k,8}^{j,l} \\
+ 70(d - 1)^4 C_{4,4,j,e_i,e_s} \Delta \rho_{k,8}^{j,l} + 56(d - 1)^5 C_{3,5,j,e_i,e_s} \Delta \rho_{k,8}^{j,l} + 28(d - 1)^6 C_{2,6,j,e_i,e_s} \Delta \rho_{k,8}^{j,l} \\
+ 8(d - 1)^7 C_{1,7,j,e_i,e_s} \Delta \rho_{k,8}^{j,l} + (d - 1)^8 B_{8,j,e_i,e_s} \Delta \rho_{k,8}^{i,s},
\]

where
\[
A_8 := \max_{l=1,\ldots,d} A_{8,l}, \quad B_{k,j} := \max_{l=1,\ldots,d} B_{k,8,l}, \quad C_{7,1} := \max_{l=1,\ldots,d} C_{7,1,l,e_i,e_s}, \quad C_{6,2} := \max_{l=1,\ldots,d} C_{6,2,l,e_i,e_s}, \quad C_{3,3} := \max_{l=1,\ldots,d} C_{3,3,l,e_i,e_s}, \quad C_{4,4} := \max_{l=1,\ldots,d} C_{4,4,l,e_i,e_s}, \quad C_{3,5} := \max_{l=1,\ldots,d} C_{3,5,l,e_i,e_s}, \quad C_{2,6} := \max_{l=1,\ldots,d} C_{2,6,l,e_i,e_s}, \quad C_{1,7} := \max_{l=1,\ldots,d} C_{1,7,l,e_i,e_s}.
\]

Hence, we obtain
\[
\rho_{k+1,8}^{j,l} \leq (1 + \kappa_8 \Delta) \rho_{k,8}^{j,l}, \quad k = j + 1,\ldots,J - 1
\]

for some constant \(\kappa_8 > 0\), leading to
\[
\rho_{k,8}^{j,l} \leq (1 + \kappa_8 \Delta)^{k-j-1} \rho_{j+1,8}^{j,l}, \quad k = j + 1,\ldots,J - 1,
\]

where
\[
\rho_{j+1,8}^{j,l} = \sum_{i=1}^{d} \mathbb{E} \left[ \left( \frac{\partial}{\partial y_i} \Phi_{\Delta}^s (x,y + \xi_{j+1}) \right)^8 \right] = \sum_{i=1}^{d} \sigma_{i}^8 (x).
\]

Next, we have for some \(i,o \in \{1,\ldots,m\}\) and \(l \in \{1,\ldots,d\}\)
\[
\frac{\partial^2}{\partial y_i \partial y_o} G_{k+1,j}^l (x,y) = \sum_{x=1}^{d} \frac{\partial}{\partial x_i} \Phi_{\Delta,k+1} (G_{k,j}(x,y)) \frac{\partial^2}{\partial y_i \partial y_o} G_{k,j}^l (x,y) \\
+ \sum_{x=1}^{d} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_o} \Phi_{\Delta,k+1} (G_{k,j}(x,y)) \frac{\partial}{\partial y_i} G_{k,j}^l (x,y) \frac{\partial}{\partial y_o} G_{k,j}^l (x,y)
\]

and
\[
\frac{\partial^2}{\partial y_i \partial y_o} G_{j+1,j}^l (x,y) = \frac{\partial^2}{\partial y_i \partial y_o} \Phi_{\Delta}^s (x,y + \xi_{j+1}).
\]

Hence
\[ E \left[ \left( \frac{\partial^2}{\partial y_j \partial y_0} G^l_{k+1,j}(x,y) \right)^2 \right] \]

\[ \leq E \left[ \left( \frac{\partial}{\partial x_l} \Phi_{\Delta,k+1}^l(G_{k,j}(x,y)) \frac{\partial^2}{\partial y_j \partial y_0} G^l_{k,j}(x,y) \right)^2 \right] \]

\[ + \sum_{s \neq l} \left\{ 2 \frac{\partial}{\partial x_l} \Phi_{\Delta,k+1}^l(G_{k,j}(x,y)) \frac{\partial}{\partial x_s} \Phi_{\Delta,k+1}^l(G_{k,j}(x,y)) \frac{\partial^2}{\partial y_j \partial y_0} G^l_{k,j}(x,y) \frac{\partial^2}{\partial y_i \partial y_0} G^l_{k,j}(x,y) \right\} \]

\[ + (d-1) \left( \frac{\partial}{\partial x_s} \Phi_{\Delta,k+1}^l(G_{k,j}(x,y)) \frac{\partial^2}{\partial y_j \partial y_0} G^s_{k,j}(x,y) \right)^2 \]

\[ + 2 \sum_{s,m=1}^d \frac{\partial}{\partial x_s} \Phi_{\Delta,k+1}^l(G_{k,j}(x,y)) \frac{\partial}{\partial x_m} \Phi_{\Delta,k+1}^l(G_{k,j}(x,y)) \frac{\partial^2}{\partial y_j \partial y_0} G^s_{k,j}(x,y) \frac{\partial}{\partial y_0} G^s_{k,j}(x,y) \]

\[ + d^2 \sum_{s,m=1}^d \left( \frac{\partial^2}{\partial x_s \partial x_m} \Phi_{\Delta,k+1}^l(G_{k,j}(x,y)) \frac{\partial}{\partial y_j} G^s_{k,j}(x,y) \frac{\partial}{\partial y_0} G^s_{k,j}(x,y) \right)^2 \]

Denote

\[ \psi_{k+1,\alpha}^{l,\alpha} = E \left[ \left( \frac{\partial^2}{\partial y_j \partial y_0} G^l_{k+1,j}(x,y) \right)^\alpha \right] \]

and \( e_{\alpha,\alpha} := e_\alpha + e_\alpha \), then we get, due to

\[ 2E[XYZ] \leq 2 \sqrt{E[X^2]} \sqrt{E[Y^4]} \sqrt{E[Z^4]} \leq E[X^2] + \sqrt{E[Y^4]} \sqrt{E[Z^4]} \leq E[X^2] + \frac{1}{2} (E[Y^4] + E[Z^4]), \]

for \( k = j + 1, \ldots, J - 1, \)

\[ \psi_{k+1,2}^{l,\alpha} \leq (1 + A_{2,j} \Delta) \psi_{k,2}^{l,\alpha} + \sum_{s \neq l} \left\{ C_{1,1,l,e_s,e_s} \Delta (\psi_{k,2}^{l,\alpha} + \psi_{k,2}^{l,s \alpha}) + (d-1)B_{2,l,e_s} \Delta \psi_{k,2}^{l,s \alpha} \right\} \]

\[ + \sum_{s,m=1}^d C_{1,1,l,e_s,e_s} \Delta \left( \psi_{k,2}^{l,v} + \frac{1}{2} (\rho_{k,4}^{l,s} + \rho_{k,4}^{l,o}) \right) + d^2 \sum_{s,m=1}^d B_{2,l,e_s} \Delta \frac{1}{2} (\rho_{k,4}^{l,s} + \rho_{k,4}^{l,o}). \]

Further, denote

\[ \psi_{k+1,\alpha}^{l,\alpha} = \sum_{l=1}^d \psi_{k+1,\alpha}^{l,\alpha}, \]

then we get for \( k = j + 1, \ldots, J - 1, \)

\[ \psi_{k+1,2}^{l,\alpha} \leq (1 + A_{2,j} \Delta) \psi_{k,2}^{l,\alpha} + 2(d-1)C_{1,1,l,e_s,e_s} \Delta \psi_{k,2}^{l,\alpha} + (d-1)^2 B_2 \Delta \psi_{k,2}^{l,\alpha} \]

\[ + d^3 \tilde{C}_{1,1} \Delta \left( \psi_{k,2}^{l,\alpha} + \frac{1}{2} (\rho_{k,4}^{l,s} + \rho_{k,4}^{l,o}) \right) + d^4 \tilde{B}_2 \Delta \frac{1}{2} (\rho_{k,4}^{l,s} + \rho_{k,4}^{l,o}). \]

where \( \tilde{C}_{1,1} := \max_{l,s,a,v=1,\ldots,d} C_{1,1,l,e_s,e_s} \) and \( \tilde{B}_2 := \max_{l,s,a=1,\ldots,d} B_{2,l,e_s} \). This gives us
\[ \psi_{k+1,2}^{j,\alpha} \leq (1 + \kappa_5 \Delta) \psi_{k,2}^{j,\alpha} + \psi_{k,1} \] 
for some constants \( \kappa_5, \kappa_6 > 0 \), leading to 
\[ \psi_{k,2}^{j,\alpha} \leq (1 + \kappa_5 \Delta)^{k-j} \psi_{j+1,2}^{j,\alpha} + \kappa_7 \psi_{j,1}, k = j + 1, \ldots, J - 1, \]
where \( \kappa_7 > 0 \) and 
\[ \psi_{j+1,2}^{j,\alpha} = \sum_{s=1}^{d} \mathbb{E} \left[ \left( \frac{\partial^2}{\partial y_i \partial y_o} \Phi_{\Delta} (x, y + \xi_{j+1}) \right)^2 \right] = 0. \]
Thus, we obtain the boundedness of 
\[ \frac{\partial^2}{\partial y_i \partial y_o} \Phi_{\Delta} (x, y) = \mathbb{E} \left[ \sum_{s=1}^{d} \frac{\partial}{\partial x_s} f(G_{J, j}(x, y)) \frac{\partial^2}{\partial y_i \partial y_o} G_{J, j}^s (x, y) \right. \]

\[ + \left. \sum_{s', u=1}^{d} \frac{\partial^2}{\partial x_s \partial x_{u'}} f(G_{J, j}(x, y)) \frac{\partial}{\partial y_i} G_{J, j}^s (x, y) \frac{\partial}{\partial y_o} G_{J, j}^u (x, y) \right], \]
provided that \( \sigma_{ki} \) and all the derivatives of order 1 and 2 for \( f, \mu_k, \sigma_{ki} \) are bounded.
Moreover, we have
Regression-based variance reduction approach for strong approximation schemes

\[ E \left[ \left( \frac{\partial^2}{\partial y_1 \partial y_0} G_{k+1,j}^l(x,y) \right)^4 \right] \]
\[ \leq E \left[ \left( \frac{\partial}{\partial x_j} \Phi_{\Delta,k+1}(G_{k,j}(x,y)) \frac{\partial^2}{\partial y_1 \partial y_0} G_{k,j}^l(x,y) \right)^4 \right] \]
\[ + \sum_{s \neq l} \left\{ 4 \left( \frac{\partial}{\partial x_j} \Phi_{\Delta,k+1}(G_{k,j}(x,y)) \frac{\partial^2}{\partial y_1 \partial y_0} G_{k,j}^l(x,y) \right)^3 \frac{\partial}{\partial x_s} \Phi_{\Delta,k+1}(G_{k,j}(x,y)) \frac{\partial^2}{\partial y_1 \partial y_0} G_{k,j}^l(x,y) \right. \]
\[ \left. + 6(d-1) \left( \frac{\partial}{\partial x_j} \Phi_{\Delta,k+1}(G_{k,j}(x,y)) \frac{\partial^2}{\partial y_1 \partial y_0} G_{k,j}^l(x,y) \frac{\partial}{\partial x_s} \Phi_{\Delta,k+1}(G_{k,j}(x,y)) \frac{\partial^2}{\partial y_1 \partial y_0} G_{k,j}^l(x,y) \right)^2 \right\} \]
\[ + 4(d-1)^2 \left( \frac{\partial}{\partial x_j} \Phi_{\Delta,k+1}(G_{k,j}(x,y)) \frac{\partial^2}{\partial y_1 \partial y_0} G_{k,j}^l(x,y) \left( \frac{\partial}{\partial x_s} \Phi_{\Delta,k+1}(G_{k,j}(x,y)) \frac{\partial^2}{\partial y_1 \partial y_0} G_{k,j}^l(x,y) \right)^3 \right. \]
\[ \left. + (d-1)^3 \left( \frac{\partial}{\partial x_j} \Phi_{\Delta,k+1}(G_{k,j}(x,y)) \frac{\partial^2}{\partial y_1 \partial y_0} G_{k,j}^l(x,y) \right)^4 \right\} \]
\[ + \sum_{s,t,u,v=1}^d \left\{ 4d^2 \left( \frac{\partial}{\partial x_s} \Phi_{\Delta,k+1}(G_{k,j}(x,y)) \frac{\partial^2}{\partial y_1 \partial y_0} G_{k,j}^l(x,y) \right)^3 \frac{\partial^2}{\partial x_t \partial x_u} \Phi_{\Delta,k+1}(G_{k,j}(x,y)) \right. \]
\[ \cdot \frac{\partial}{\partial y_v} G_{k,j}^l(x,y) \frac{\partial}{\partial y_v} G_{k,j}^l(x,y) \]
\[ + 6d^3 \left( \frac{\partial}{\partial x_s} \Phi_{\Delta,k+1}(G_{k,j}(x,y)) \frac{\partial^2}{\partial y_1 \partial y_0} G_{k,j}^l(x,y) \frac{\partial^2}{\partial x_t \partial x_u} \Phi_{\Delta,k+1}(G_{k,j}(x,y)) \right)^2 \]
\[ \cdot \left( \frac{\partial}{\partial y_t} G_{k,j}^l(x,y) \frac{\partial}{\partial y_v} G_{k,j}^l(x,y) \right)^2 \]
\[ + 4d^4 \frac{\partial}{\partial x_s} \Phi_{\Delta,k+1}(G_{k,j}(x,y)) \frac{\partial^2}{\partial y_1 \partial y_0} G_{k,j}^l(x,y) \]
\[ \cdot \left( \frac{\partial^2}{\partial x_t \partial x_u} \Phi_{\Delta,k+1}(G_{k,j}(x,y)) \frac{\partial}{\partial y_t} G_{k,j}^l(x,y) \frac{\partial}{\partial y_v} G_{k,j}^l(x,y) \right)^3 \]
\[ + d^6 \sum_{s,t,u,v=1}^d \left( \frac{\partial^2}{\partial x_s \partial x_t \partial x_u} \Phi_{\Delta,k+1}(G_{k,j}(x,y)) \frac{\partial}{\partial y_v} G_{k,j}^l(x,y) \frac{\partial}{\partial y_v} G_{k,j}^l(x,y) \right)^4 \]

and thus, due to $4a^3bc \leq 3a^4 + \frac{1}{2}(b^8 + c^8)$, $2a^2b^2c^2 \leq a^4 + \frac{1}{2}(b^8 + c^8)$ and $4ab^3c^3 \leq a^4 + \frac{3}{2}(b^8 + c^8)$,
\[
\psi^i_{k+1,4} \leq (1 + A_4 \Delta) \psi^i_{k,4} + \sum_{s \neq i} \left\{ C_{3,1,l,e_{i,1}} \Delta (3 \psi^i_{k,4} + \psi^{i,o,s}_{k,4}) + 3(d - 1) C_{2,2,l,e_{i,1}} \Delta (\psi^i_{k,4} + \psi^{i,o,s}_{k,4}) \right. \\
+ (d - 1)^2 C_{1,3,l,e_{i,1}} \Delta (\psi^i_{k,4} + 3 \psi^{i,o,s}_{k,4}) + (d - 1)^3 B_{4,l,e_{i,1}} \Delta \psi^{i,o,s}_{k,4} \right\} \\
+ \sum_{s,u,v=1}^d \left\{ d^2 C_{3,1,l,e_{i,1}} \Delta \left(3 \psi^i_{k,4} + \frac{1}{2} (\rho^{i,s}_{k,8} + \rho^{o,u}_{k,8})\right) + 3d^3 C_{2,2,l,e_{i,1},u} \Delta \left(\psi^{i,o,v}_{k,4} + \frac{1}{2} (\rho^{i,s}_{k,8} + \rho^{o,u}_{k,8})\right) \right. \\
+ d^4 C_{1,3,l,e_{i,1},u} \Delta \left(\psi^i_{k,4} + \frac{3}{2} (\rho^{i,s}_{k,8} + \rho^{o,u}_{k,8})\right) \left. + d^6 \sum_{s,u,v=1}^d B_{4,l,e_{i,1},u} \Delta \left(\psi^{i,o,v}_{k,4} + \frac{1}{2} (\rho^{i,s}_{k,8} + \rho^{o,u}_{k,8})\right) \right\}.
\]

This gives us
\[
\psi^i_{k+1,4} \leq (1 + A_4 \Delta) \psi^i_{k,4} + 4(d - 1) C_{3,1} \Delta \psi^i_{k,4} + 6(d - 1)^2 C_{2,2} \Delta \psi^i_{k,4} \\
+ 4(d - 1)^3 C_{1,3} \Delta \rho^{i}_{k,8} + (d - 1)^4 B_4 \Delta \psi^i_{k,4} + d^2 \tilde{C}_{3,1} \Delta \left(3 \psi^i_{k,4} + \frac{1}{2} (\rho^{i,s}_{k,8} + \rho^{o,u}_{k,8})\right) \]
\[
+ 3d^3 \tilde{C}_{2,2} \Delta \left(\psi^i_{k,4} + \frac{1}{2} (\rho^{i,s}_{k,8} + \rho^{o,u}_{k,8})\right) + d^4 \tilde{C}_{1,3} \Delta \left(\psi^i_{k,4} + \frac{3}{2} (\rho^{i,s}_{k,8} + \rho^{o,u}_{k,8})\right) + d^6 B_4 \Delta \left(\rho^{i,s}_{k,8} + \rho^{o,u}_{k,8}\right),
\]
where \( \tilde{C}_{3,1} := \max_{l,s,u,v=1,...,d} C_{3,1,l,e_{i,1},u}, \tilde{C}_{2,2} := \max_{l,s,u,v=1,...,d} C_{2,2,l,e_{i,1},u}, \tilde{C}_{1,3} := \max_{l,s,u,v=1,...,d} C_{1,3,l,e_{i,1},u} \) and \( B_4 := \max_{l,s,u,v=1,...,d} B_{4,l,e_{i,1},u}. \) Hence, we obtain
\[
\psi^i_{k+1,4} \leq (1 + k_3 \Delta) \psi^i_{k,4} + k_0, \quad k = j + 1, \ldots, J - 1
\]
for some constants \( k_3, k_0 > 0, \) leading to
\[
\psi^i_{k,4} \leq (1 + k_3 \Delta)^{k-j-1} \psi^i_{j+1,4} + k_{10} = k_{10}, k = j + 1, \ldots, J - 1,
\]
where \( k_{10} > 0 \) and
\[
\psi^i_{j+1,4} = \sum_{s=1}^d \mathbb{E} \left[ \left( \frac{\partial^2}{\partial y_1 \partial y_2} \Phi^i_\Delta(x, y + \xi_{j+1}) \right)^4 \right] = 0.
\]
Next, we have for some \( i, o, r \in \{1, \ldots, m\} \) and \( l \in \{1, \ldots, d\} \)
\[
\frac{\partial^3}{\partial y_i \partial y_o \partial y_r} G_{k+1,j}^i(x,y) \\
= \sum_{s=1}^d \frac{\partial}{\partial x_s} \Phi_{\Delta,k+1}^i(G_{k,j}(x,y)) \frac{\partial^3}{\partial y_i \partial y_o \partial y_r} G_{k,j}^i(x,y) \\
+ \sum_{s,u=1}^d \frac{\partial^2}{\partial x_s \partial x_u} \Phi_{\Delta,k+1}^i(G_{k,j}(x,y)) \left( \frac{\partial^2}{\partial y_i \partial y_o} G_{k,j}^i(x,y) \frac{\partial}{\partial y_r} G_{k,j}^i(x,y) + \frac{\partial^2}{\partial y_i \partial y_r} G_{k,j}^i(x,y) \frac{\partial}{\partial y_o} G_{k,j}^i(x,y) \right) \\
+ \sum_{s,u,v=1}^d \frac{\partial^3}{\partial x_s \partial x_u \partial x_v} \Phi_{\Delta,k+1}^i(G_{k,j}(x,y)) \frac{\partial}{\partial y_i} G_{k,j}^i(x,y) \frac{\partial}{\partial y_o} G_{k,j}^i(x,y) \frac{\partial}{\partial y_r} G_{k,j}^i(x,y) \\
\text{and } \frac{\partial^3}{\partial y_i \partial y_o \partial y_r} G_{j+1,j}^s(x,y) = \frac{\partial^3}{\partial y_i \partial y_o \partial y_r} \Phi_{\Delta}^s(x,y+\xi_{j+1}). \text{ Hence}
\]
\[ \begin{align*}
\mathbb{E} \left[ \left( \frac{\partial^3}{\partial y_l \partial y_o \partial y_r} G'_{k+1,j} (x,y) \right)^2 \right] \\
\leq \mathbb{E} \left[ \left( \frac{\partial}{\partial x_l} \Phi'_{\Delta,k+1} (G_{k,j}(x,y)) \frac{\partial^3}{\partial y_l \partial y_o \partial y_r} G'_{k,j} (x,y) \right)^2 \\
+ \sum_{s \neq l} \left\{ 2 \frac{\partial}{\partial x_l} \Phi'_{\Delta,k+1} (G_{k,j}(x,y)) \frac{\partial}{\partial x_s} \Phi'_{\Delta,k+1} (G_{k,j}(x,y)) \frac{\partial^3}{\partial y_l \partial y_o \partial y_r} G'_{k,j} (x,y) \right. \\
+ \left. (d-1) \left( \frac{\partial}{\partial x_s} \Phi'_{\Delta,k+1} (G_{k,j}(x,y)) \frac{\partial^3}{\partial y_l \partial y_o \partial y_r} G'_{k,j} (x,y) \right)^2 \right\} \\
+ 2 \sum_{s,u,j=1}^d \frac{\partial^3}{\partial x_s \partial x_u \partial x_j} \Phi'_{\Delta,k+1} (G_{k,j}(x,y)) \frac{\partial}{\partial x_j} \Phi'_{\Delta,k+1} (G_{k,j}(x,y)) \frac{\partial^3}{\partial y_s \partial y_o \partial y_r} G''_{k,j} (x,y) \\
\cdot \left( \frac{\partial^2}{\partial y_j \partial y_o} G''_{k,j} (x,y) \frac{\partial}{\partial y_r} G''_{k,j} (x,y) + \frac{\partial^2}{\partial y_j \partial y_r} G''_{k,j} (x,y) \frac{\partial}{\partial y_o} G''_{k,j} (x,y) \\
+ \frac{\partial}{\partial y_j} G''_{k,j} (x,y) \frac{\partial^2}{\partial y_o \partial y_r} G''_{k,j} (x,y) \right) \\
+ 6 d^2 \sum_{s,u=1}^d \left( \frac{\partial^2}{\partial x_s \partial x_u} \Phi'_{\Delta,k+1} (G_{k,j}(x,y)) \right)^2 \left( \left( \frac{\partial^2}{\partial y_s \partial y_o} G''_{k,j} (x,y) \frac{\partial}{\partial y_r} G''_{k,j} (x,y) \right)^2 \\
+ \left( \frac{\partial^2}{\partial y_s \partial y_r} G''_{k,j} (x,y) \frac{\partial}{\partial y_o} G''_{k,j} (x,y) \right)^2 \\
+ \left( \frac{\partial}{\partial y_s} G''_{k,j} (x,y) \frac{\partial^2}{\partial y_o \partial y_r} G''_{k,j} (x,y) \right)^2 \right) \\
+ 2 d^3 \sum_{s,u,j=1}^d \left( \frac{\partial^3}{\partial x_s \partial x_u \partial x_j} \Phi'_{\Delta,k+1} (G_{k,j}(x,y)) \frac{\partial}{\partial x_j} \Phi'_{\Delta,k+1} (G_{k,j}(x,y)) \frac{\partial}{\partial y_s} G''_{k,j} (x,y) \frac{\partial}{\partial y_o} G''_{k,j} (x,y) \frac{\partial}{\partial y_r} G''_{k,j} (x,y) \right)^2 \right] \\
\end{align*} \]

Denote
\[ \rho_{k+1}^{i,o,r,s} = \mathbb{E} \left[ \left( \frac{\partial^3}{\partial y_l \partial y_o \partial y_r} G'_{k+1,j} (x,y) \right)^2 \right] \]

and \( e_{s,u,j} := e_s + e_u + e_j \), then we get, due to \( 3a^2 b^2 c^2 \leq a^6 + b^6 + c^6 \) and
\[ 2\mathbb{E}[XYZU] \leq 2 \sqrt{\mathbb{E}[X^2] \sqrt{\mathbb{E}[Y^2] \sqrt{\mathbb{E}[Z^2] \sqrt{\mathbb{E}[U^2]}}} \leq \mathbb{E}[X^2] + \sqrt{\mathbb{E}[Y^2]} \sqrt{\mathbb{E}[Z^2]} \sqrt{\mathbb{E}[U^2]}, \]

for \( k = j + 1, \ldots, J - 1, \)

\[ r_{i,o,r}^{j+1} \leq (1 + A_j \Delta) r_{i,o,r}^{j+1} + \sum_{i=1}^{d} C_{1,1,l,e_i,\epsilon_i} \Delta (s_{k}^{j+1} + s_{k}^{j+1}) + (d-1)B_{2,l,e_i} \Delta s_{k}^{j+1}, \]

Further, denote

\[ r_{i,o,r}^{j+1} = \sum_{i=1}^{d} r_{i,o,r}^{j+1}, \]

then we get for \( k = j + 1, \ldots, J - 1, \)

\[ s_{k+1}^{j+2} \leq (1 + A_2 \Delta) s_{k+1}^{j+2} + 2(d-1)C_{1,1,l,e_i,\epsilon_i} \Delta s_{k+1}^{j+2} + (d-1)^2 B_{2,l,e_i} \Delta s_{k+1}^{j+2}, \]

where \( C_{1,1,l,e_i,\epsilon_i} := \max_{l,i=1,\ldots,d} C_{1,1,l,e_i,\epsilon_i} \) and \( B_2 := \max_{l,i=1,\ldots,d} B_{2,l,e_i}. \) This gives us

\[ r_{i,o,r}^{j+2} \leq (1 + \kappa_{11} \Delta) r_{i,o,r}^{j+2} + \kappa_{12}, \quad k = j + 1, \ldots, J - 1 \]

for some constants \( \kappa_{11}, \kappa_{12} > 0, \) leading to

\[ s_{k+1}^{j+2} \leq (1 + \kappa_{11} \Delta)^{k-j-1} r_{i,o,r}^{j+2} + \kappa_{13} = \kappa_{13}, \quad k = j + 1, \ldots, J - 1, \]

where \( \kappa_{13} > 0 \) and
\[ \zeta_{j+1,2}^{i,o,r} = \sum_{s=1}^{d} \mathbb{E} \left[ \left( \frac{\partial^3}{\partial y_i \partial y_o \partial y_r} \Phi_{\Delta_s}^s (x, y + \xi_{j+1}) \right)^2 \right] = 0. \]

Thus, we obtain the boundedness of

\[
\frac{\partial^3}{\partial y_i \partial y_o \partial y_r} u_{\Delta}(t_j, x, y)
= \mathbb{E} \left[ \sum_{s=1}^{d} \frac{\partial}{\partial x_s} f(G_{I_j}(x, y)) \frac{\partial^3}{\partial y_i \partial y_o \partial y_r} G_{I_j}^s(x, y) + \sum_{s,u=1}^{d} \frac{\partial^2}{\partial x_s \partial x_u} f(G_{I_j}(x, y)) \left( \frac{\partial^2}{\partial y_i \partial y_o} G_{I_j}^s(x, y) \frac{\partial}{\partial y_r} G_{I_j}^s(x, y) + \frac{\partial^2}{\partial y_i \partial y_r} G_{I_j}^s(x, y) \frac{\partial}{\partial y_o} G_{I_j}^s(x, y) \right) + \sum_{s,u,v=1}^{d} \frac{\partial^3}{\partial x_s \partial x_u \partial x_v} f(G_{I_j}(x, y)) \frac{\partial}{\partial y_i} G_{I_j}^s(x, y) \frac{\partial}{\partial y_o} G_{I_j}^s(x, y) \frac{\partial}{\partial y_r} G_{I_j}^s(x, y) \right],
\]

provided that \( \sigma_{ki} \) and all the derivatives of order 1, 2 and 3 for \( f, \mu_k, \sigma_{ki} \) are bounded.

**Proof of Lemma**

Cf. Theorem 5.2 in \cite{Reference}.

**Proof of Theorem**

We have, by the martingale property of \( (\tilde{M}_{\Delta, j}^{i})_{j=0,...,J} \), where \( \tilde{M}_{\Delta, j}^{i} \) is given by \cite{Reference} with \( J \) being replaced by \( j \), and by the orthogonality of the system \( \Delta_j W^i \),
Regression-based variance reduction approach for strong approximation schemes

\[ \text{Var}[f(X_{\Delta,T}) - \bar{M}_{\Delta,T}^{\text{int},1}] = \text{Var}[f(X_{\Delta,T}) - \bar{M}_{\Delta,T}^{\text{int},1}] + \text{Var}[\bar{M}_{\Delta,T}^{\text{int},1} - \bar{M}_{\Delta,T}^{\text{int},1}] \]

\[ \leq \frac{1}{J} + \Delta \sum_{j=1}^{J} \sum_{i=1}^{m} \mathbb{E} \left[ \sum_{k=1}^{d} (\bar{g}_{j,k} - \tilde{g}_{j,k}) \sigma_{kl} \right]_{L^2(\pi_{\Delta,j-1})} \]

\[ \leq \frac{1}{J} + d\Delta \sum_{j=1}^{J} \sum_{i=1}^{m} \sum_{k=1}^{d} \mathbb{E} \left[ \left( \bar{g}_{j,k} - \tilde{g}_{j,k} \right) \sigma_{kl} \right]_{L^2(\pi_{\Delta,j-1})} \]

\[ \leq \frac{1}{J} + d^2 T m \sigma_{\text{max}}^2 \left\{ \bar{c} \left( \Sigma + A^2 (\log N + 1) \right) \left( \frac{p+d}{d} \right)^{Q^d} N \right\} + \frac{8 C^2_h}{(p+1)^{1/2} d^{2-2/h}} \left( \frac{Rd}{Q} \right)^{2p+2} + 8 A^2 B \nu R^{-\nu} \].

**Proof of Theorem 8**

Let us, for simplicity, first ignore the log(N)-term in (36) and only consider the terms w.r.t. the variables \( J, N, N_0, Q, R \) which shall be optimised, since the constants \( d, m, c_{p,d}, (p+1)!, B_\nu \) do not affect the terms on \( \epsilon \). Further, we consider the log-cost and log-constraints rather than (35) and (36). Let us subdivide the optimisation problem into two cases:

1. \( N \leq N_0 \). This gives us the Lagrange function

\[ L_{\lambda_1, \ldots, \lambda_N}(J, N, N_0, Q, R) \]

\[ := \log(J) + \log(N_0) + d \log(Q) + \lambda_1 (-2 \log(J) - 2 \log(\epsilon)) \]

\[ + \lambda_2 (- \log(J) - \log(N_0) - 2 \log(\epsilon)) \]

\[ + \lambda_3 (d \log(Q) - \log(N) - \log(N_0) - 2 \log(\epsilon)) \]

\[ + \lambda_4 (2(p+1)(\log(R) - \log(Q)) - \log(N_0) - 2 \log(\epsilon)) \]

\[ + \lambda_5 (-v \log(R) - \log(N_0) - 2 \log(\epsilon)) + \lambda_6 (\log(N) - \log(N_0)). \]

where \( \lambda_1, \ldots, \lambda_6 \geq 0 \). Thus, considering of the conditions \( \frac{\partial L}{\partial J} = \frac{\partial L}{\partial N} = \frac{\partial L}{\partial Q} = \frac{\partial L}{\partial R} = 0 \) gives us the following relations

\[ \lambda_1 = \frac{1 - \lambda_2}{2}, \]

\[ \lambda_3 = \frac{2(p+1)(v(1 - \lambda_2) - d) - dv}{dv + 2(p+1)(d + 2v)} = \lambda_6, \]

\[ \lambda_4 = \frac{dv(3 - \lambda_2)}{dv + 2(p+1)(d + 2v)}, \]

\[ \lambda_5 = \frac{2d(p+1)(3 - \lambda_2)}{dv + 2(p+1)(d + 2v)}. \]
The case $\lambda_1, \ldots, \lambda_6 > 0$ is not feasible, since all constraints in (71) can not be active, that is they cannot become zero simultaneously because of six (linearly independent) equalities on five unknowns. Hence, we derive the solutions under $\lambda_i = 0$ for different $i$ and observe which one is actually optimal.

a. $\lambda_1 = 0 \Rightarrow \lambda_3 = \lambda_6 = -\frac{d(2(p+1)+v)}{dv+2(p+1)(d+2v)} < 0$. Due to negative $\lambda_3$, this case is not optimal.

b. $\lambda_2 = 0 \Rightarrow \lambda_4, \lambda_5 > 0, \lambda_3 = \lambda_6 = \frac{2(p+1)(v-d)-dv}{dv+2(p+1)(d+2v)}$. Again, we make a case distinction:

i. $\lambda_3 = \lambda_6 = 0 \Rightarrow v > \frac{2d(p+1)}{2(p+1)}$ for $2(p+1) > d$. This gives us, due to $\lambda_1, \lambda_4, \lambda_5 > 0$,

$$J \asymp \epsilon^{-1},$$

$$Q \asymp \left[\frac{1}{N_0 \epsilon^2}\right]^\frac{1}{2},$$

$$JQ^dN_0 \asymp \epsilon^{-3}.$$

This solution is no improvement compared to the SMC approach.

ii. $\lambda_3 = \lambda_6 > 0 \Rightarrow v > \frac{2d(p+1)}{2(p+1)}$ for $2(p+1) > d$. In this case, all constraints apart from the second one in (71), corresponding to $\lambda_2$, are active. Then we obtain

$$J \asymp \epsilon^{-1},$$

$$Q \asymp \epsilon^{-\frac{2d+4(p+1)}{2d+2(p+1)(d+2v)}},$$

$$N_0 \asymp \epsilon^{-\frac{2d+4(p+1)}{2d+2(p+1)(d+2v)}},$$

$$JQ^dN_0 \asymp \epsilon^{-\frac{5d+2(p+1)(5d+4v)}{2d+2(p+1)(d+2v)}},$$

which is a better solution than the previous one. Moreover, the remaining constraint $\frac{1}{N_0} \lesssim \epsilon^2$ is also satisfied under this solution.

c. $\lambda_3 = \lambda_6 = 0 \Rightarrow \lambda_4, \lambda_5 > 0, \lambda_2 = \frac{2(p+1)(v-d)-dv}{2(v+1)}$. The case $\lambda_2 = 0$ is the same as the last but one and thus gives us $JQ^dN_0 \asymp \epsilon^{-3}$. The case $\lambda_2 > 0$ leads to four active constraints in (71), namely the ones corresponding to $\lambda_1, \lambda_2, \lambda_4, \lambda_5$, such that

$$J \asymp \epsilon^{-1},$$

$$Q \asymp \epsilon^{-\frac{2d+4(p+1)}{2d+2(p+1)}},$$

$$N_0 \asymp \epsilon^{-1},$$

$$JQ^dN_0 \asymp \epsilon^{-\frac{5d+2(p+1)(5d+4v)}{2d+2(p+1)(d+2v)}}.$$
This solution seems to be nice at the first moment. However, it does not satisfy both constraints corresponding to \( \lambda_3, \lambda_6 \). On the one hand, we have for the third constraint \( N \gtrsim \varepsilon^{-1 - \frac{2d'(p+1)}{2\nu(p+1)}} \). On the other hand, we have for the sixth constraint \( N \lesssim \varepsilon^{-1} \). Hence, this is not an admissible solution.

d. \( \lambda_4 = 0 \Rightarrow \lambda_1 = -1 \). Since \( \lambda_1 \) is negative, this case is not optimal.

e. \( \lambda_5 = 0 \Rightarrow \lambda_1 = -1 \). As for the previous one, this case is not optimal.

2. \( N \gtrsim N_0 \). This gives us the Lagrange function

\[
\tilde{L}_{\lambda_1, \ldots, \lambda_6}(J, N, N_0, Q, R) := \log(J) + \log(N) + d\log(Q) + \lambda_1(-2\log(J) - 2\log(\varepsilon)) + \lambda_2(-\log(J) - \log(N_0) - 2\log(\varepsilon)) + \lambda_3(d\log(Q) - \log(N) - \log(N_0) - 2\log(\varepsilon)) + \lambda_4(2(p+1)(\log(R) - \log(Q)) - \log(N_0) - 2\log(\varepsilon)) + \lambda_5(-\nu\log(R) - \log(N_0) - 2\log(\varepsilon)) + \lambda_6(\log(N_0) - \log(N)).
\]

Analogously to the procedure above we get the same optimal solution, that is

\[
J \gtrsim \varepsilon^{-1}, \quad Q \gtrsim \varepsilon^{-\frac{2d'(p+1)}{2\nu(p+1)}}\left(d+\nu\right), \quad N \gtrsim \varepsilon^{-\frac{2d'(p+1)}{2\nu(p+1)}}\left(d+\nu\right),
\]

\[
\tilde{Q}^d\tilde{N} \gtrsim \varepsilon^{-\frac{5d'(p+1)}{2\nu(p+1)}}\left(d+\nu\right)\left(5d+4\nu\right).
\]

Now we consider also the remaining terms \( c_{p,d}, (p+1)! \), \( B_\nu \) and obtain (37)–(41) via equalising all constraints in (36) apart from the second one. Finally, we add the log-term concerning \( \varepsilon \) in the parameters \( N, N_0 \) to ensure that all constraints are really satisfied.

**Proof of Lemma** [2]

Cf. Theorem 5.2 in [2].

**Proof of Theorem** [9]

We have, by the martingale property of \( \tilde{M}_{\Delta_j}^{\nu, 1} \), where \( \tilde{M}_{\Delta_{j,1}}^{\nu, 1} \) is given by (43) with \( J \) being replaced by \( j \), and by the orthonormality of the system \( \Delta_j W_{\sqrt{\nu}} \).
\[ \text{Var}[f(X_{\Delta,T}) - \bar{M}^{\text{ser},1}] = \text{Var}[f(X_{\Delta,T}) - M_{\Delta,T}^{\text{ser},1}] + \text{Var}[M_{\Delta,T}^{\text{ser},1} - \bar{M}^{\text{ser},1}] \]
\[ \lesssim \frac{1}{J} + \sum_{j=1}^{m} \sum_{i=1}^{m} E|\tilde{a}_{j,e_{i}} - a_{j,e_{i}}|^{2}_{L^{2}(\mathbb{P}_{\Delta,j-1})} \]
\[ \lesssim \frac{1}{J} + Jm \left\{ \tilde{c} \left( \Sigma + A^{2} \Delta (\log N + 1) \right) \frac{c_{p,d} Q^{d}}{N} \right. \]
\[ \left. + \frac{8 C_{h}^{2}}{(p+1)!^{2}} \left( \frac{R}{Q} \right)^{2p+2} + 8 A^{2} \Delta B_{t} R^{-\nu} \right\} . \]

**Proof of Theorem 10**

The proof is similar to the one of Theorem 8.

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