On the Orlicz space generated from a random normed module

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Abstract

In this paper, we first introduce the notion of the Orlicz space generated from a random normed module $E$. Then we give a representation theorem which identify the dual of the Orlicz heart of $E$ with the Orlicz space generated from the random conjugate space of $E$. Finally, we establish relations between the strict convexity and uniform convexity of this Orlicz space and the random strict convexity and random uniform convexity of the underlying random normed module, respectively.

Keywords. Random normed module, Orlicz space, Dual representation theorem, Strict convexity, Uniform convexity

1 Introduction

Random normed modules are a generalization of ordinary normed spaces. Since random normed modules do not always admit a nontrivial continuous linear functional, the theory of traditional conjugate spaces can no longer apply universally to the study of random normed modules, thus the theory of random conjugate spaces for random normed modules has been developed in order to overcome the above difficulty. In the last 20 years, the theory of random conjugate spaces has played an essential role in both the development of the theory of random normed modules and their applications to various topics, see \cite{14,16,18} and the references therein for details.

In the development of the theory of random normed modules and its random conjugate spaces, one of the most powerful tools is the precise connection between the random conjugate space $E^*$ of a random normed module $E$ and the classical conjugate space of the abstract normed space $L^p(E)$ generated from
$E$, namely

$$(L^p(E), \| \cdot \|_p)' \cong (L^q(E^*), \| \cdot \|_q), \quad (1 \leq p < \infty, \frac{1}{p} + \frac{1}{q} = 1)$$

under the canonical embedding mapping. This connection was established in Guo [8, 10]. It unifies all the dual representation theorems of Lebesgue-Bochner function spaces (see [10]). Making use of this connection, Guo and Li [11] proved the James theorem in complete random normed modules; Guo, Xiao and Chen [13] established a basic strict separation theorem in random locally convex modules; Zhang and Guo [21] established a mean ergodic theorem on random reflexive random normed modules, and Wu [20] proved the Bishop-Phelps theorem in complete random normed modules endowed with the $(\varepsilon, \lambda)$-topology.

Orlicz spaces are a generalization of $L^p$ spaces. They share many useful properties with $L^p$ spaces, among which the most important is that, they are Banach spaces and admit nice duality. Recently, there is an increasing interest in Orlicz spaces theory in some topics in mathematical finance. Cheridito and Li [2, 3] studied convex risk measures defined on Orlicz spaces. Biagini and Frittelli [1] indicated that Orlicz spaces can be used as the unified framework for utility maximization problems. Kupper and Vogelpoth [19] and Eisele and Taieb [6] studied Orlicz type modules in the hope to use them in the study of conditional risk measures.

In this paper, we first introduce the notion of the Orlicz space generated from a random normed module $E$. Then, we give a representation theorem which identify the dual of the Orlicz heart of $E$ with the Orlicz space generated from the random conjugate space of $E$. Finally, we study strict convexity and uniform convexity of this Orlicz space, and establish some basic connections between these properties and random strict convexity and random uniform convexity of the underlying random normed module. Our results generalize some useful results concerning the connection between random normed modules and classical normed spaces, respectively. Perhaps most importantly, our results show that there are possibilities of systematically utilizing the Orlicz space theory in the study of random normed modules.

2 Terminology and notation

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $K$ the scalar field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$, and $L^0(\mathcal{F}, K)$ ($\tilde{L}^0(\mathcal{F}, \mathbb{R})$) the algebra of all equivalence classes of $\mathcal{F}$-measurable $K$-valued (accordingly, extended real-valued) random variables on $\Omega$. We write $L^0$ and $\tilde{L}^0$ for $L^0(\mathcal{F}, \mathbb{R})$ and $\tilde{L}^0(\mathcal{F}, \mathbb{R})$, respectively.

As usual, $\tilde{L}^0$ is partially ordered by $\xi \leq \eta$ iff $\xi^0(\omega) \leq \eta^0(\omega)$ for $P$-almost all $\omega \in \Omega$, where $\xi^0$ and $\eta^0$ are arbitrarily chosen representatives of $\xi$ and $\eta$, respectively. According to [4], $(\tilde{L}^0, \leq)$ is a complete lattice, and $(L^0, \leq)$ is a conditionally complete lattice. For a subset $H$ of $\tilde{L}^0$, $\vee H$ stands for
the supremum of $H$, and if $H$ is upward directed, namely there exists $c \in H$ for any $a, b \in H$ such that $a \leq c$ and $b \leq c$, then there exists a sequence $\{a_n, n \in \mathbb{N}\}$ in $H$ such that $\{a_n, n \in \mathbb{N}\}$ converges to $\forall A$ in a nondecreasing way. 

$I_A$ always denotes the equivalence class of $I_A$, where $A \in \mathcal{F}$ and $I_A$ is the characteristic function of $A$. For any $\xi \in L^0$, $|\xi|$ denotes the equivalence class of $|\xi^0| : \Omega \to [0, \infty)$ defined by $|\xi^0|(\omega) = |\xi^0(\omega)|$, where $\xi^0$ is an arbitrarily chosen representative of $\xi$.

Denote $L^0_+ = \{\xi \in L^0 | \xi \geq 0\}$.

Let us first recall the notion of random normed modules, which was first formulated in [9] with the following form.

**Definition 2.1** An ordered pair $(E, \| \cdot \|)$ is called a random normed module (briefly, an RN module) over $K$ with base $(\Omega, \mathcal{F}, P)$ if $E$ is a left module over $L^0(\mathcal{F}, K)$, and the mapping $\| \cdot \| : E \to L^0_+$ satisfies:

1. $\|x\| = 0$ if and only if $x = \theta$ (the null element of $E$);
2. $\|\xi x\| = |\xi| \|x\|$ for all $\xi \in L^0(\mathcal{F}, K)$ and $x \in E$;
3. $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$ for all $x_1, x_2 \in E$.

In this paper, given an RN module $(E, \| \cdot \|)$, $E$ is always endowed with the $(\varepsilon, \lambda)$-topology. It suffices to say that the $(\varepsilon, \lambda)$-topology is a metrizable linear topology, a sequence $\{x_n, n \in \mathbb{N}\}$ in $E$ converges in the $(\varepsilon, \lambda)$-topology to $x$ iff the sequence $\{|x_n - x|, n \in \mathbb{N}\}$ in $L^0_+$ converges in probability to 0.

Specially, $(L^0(\mathcal{F}, K), | \cdot |)$ is an RN module, and the $(\varepsilon, \lambda)$-topology is exactly the topology of convergence in probability.

Let $(E, \| \cdot \|)$ be an RN module over $K$ with base $(\Omega, \mathcal{F}, P)$. $E^*$ denotes the $L^0(\mathcal{F}, K)$-module of all continuous module homomorphisms $f$ from $(E, \| \cdot \|)$ to $(L^0(\mathcal{F}, K), | \cdot |)$. According to [7, Theorem 3.1], given a linear mapping $f : E \to L^0(\mathcal{F}, K)$, then $f \in E^*$ iff $f$ is a.s bounded, which means that for some $\xi \in L^0_+$, $|f(x)| \leq \xi \|x\|$ holds for all $x \in E$. Define $\|f\|^* = \vee\{|f(x)| : x \in E, \|x\| \leq 1\}$ for each $f \in E^*$, then $(E^*, \| \cdot \|^*)$ is an RN module, called the random conjugate space of $(E, \| \cdot \|)$.

### 3 The Orlicz space generated from a random normed module

#### 3.1 Some basic facts of Orlicz space theory

We shortly review some basic facts of Orlicz space theory. A function $\Phi : [0, \infty) \to [0, \infty]$ is called a Young function if it is convex, left-continuous, $\lim_{t \to 0} \Phi(t) = \Phi(0) = 0$, and $\lim_{t \to \infty} \Phi(t) = \infty$. It is easy to see that $\Phi$ is increasing, and continuous except possibly at a single point, where it jumps to $\infty$.

The conjugate of $\Phi$ defined by

$$
\Psi(s) := \sup_{t \geq 0} \{ts - \Phi(t)\}, \quad s \geq 0,
$$

is called a Young function, and if $H$ is upward directed, namely there exists $c \in H$ for any $a, b \in H$ such that $a \leq c$ and $b \leq c$, then there exists a sequence $\{a_n, n \in \mathbb{N}\}$ in $H$ such that $\{a_n, n \in \mathbb{N}\}$ converges to $\forall A$ in a nondecreasing way.
is also a Young function. We can check that the conjugate of \( \Psi \) is \( \Phi \).

In the sequel, we use \( E[\xi] \) to denote \( \xi \)'s expectation with respect to the probability \( P \).

The Orlicz space corresponding to \( \Phi \) is given by:

\[
L^\Phi = \{ \xi \in L^0 : E[\Phi(\xi)] < \infty \text{ for some } c > 0 \},
\]

and the Orlicz heart is given by:

\[
M^\Phi = \{ \xi \in L^0 : E[\Phi(\xi)] < \infty \text{ for all } c > 0 \}.
\]

The Luxemburg norm

\[
|\xi|_{\Phi L} = \inf \{ \lambda > 0 : E[\Phi(\xi/\lambda)] \leq 1 \},
\]

and the Orlicz norm

\[
|\xi|_{\Phi O} = \sup \{ |E[\xi \eta]| : \eta \in L^\Psi, |\eta|_{\Phi L} \leq 1 \}
\]

are equivalent norms on \( L^\Phi \) under which and with the usual partial order \( L^\Phi \) is a Banach lattice.

If \( \Phi(t) = \infty \) for some \( t \in (0, \infty) \), then \( M^\Phi \) is the trivial space \{0\}. Thus in the sequel, \( \Phi \) is assumed to be real valued, equivalently, \( \Phi \) is continuous. According to [5, Theorem 2.1.14], we always have that \( S \subset L^\infty \subset M^\Phi \), where \( S \) stands for the set of all simple measurable functions. Moreover, \( S \) is dense in \( (M^\Phi, |\cdot|_{\Phi L}) \), which implies that \( M^\Phi \) is the \( |\cdot|_{\Phi L} \)-closure of \( L^\infty \) in \( L^\Phi \). Furthermore, according to [5, Theorem 2.2.11], the norm dual of \( (M^\Phi, |\cdot|_{\Phi L}) \) \( (M^\Phi, |\cdot|_{\Phi O}) \) is given by \( (L^\Psi, |\cdot|_{\Psi O}) \) (accordingly, \( (L^\Psi, |\cdot|_{\Psi L}) \)), where \( \eta \in L^\Psi \) is identified with the bounded linear functional \( f_\eta : M^\Phi \to \mathbb{R} \) defined by

\[ f_\eta(\xi) = E[\xi \eta], \forall \xi \in M^\Phi. \]

In the following, we give some simple examples.

**Example 3.1**

1. If \( \Phi(t) = t \), then \( \Psi(s) = 0 \) for \( s \leq 1 \), and \( \infty \) otherwise. We have:

\[
M^\Phi = L^\Phi = L^1, |\cdot|_{\Phi L} = |\cdot|_{\Phi O} = |\cdot|_1,
\]

\[
L^\Psi = L^\infty, M^\Psi = \{0\}, |\cdot|_{\Psi O} = |\cdot|_{\Psi L} = |\cdot|_\infty.
\]

2. If \( \Phi(x) = x^p \) for \( p \in (1, \infty) \), then \( \Psi(s) = s^{1-1/q} s^{q-1} \). We have:

\[
M^\Phi = L^\Phi = L^p, |\cdot|_{\Phi L} = |\cdot|_p, |\cdot|_{\Phi O} = p^{1-q} q^{q-1} |\cdot|_p,
\]

\[
M^\Psi = L^\Psi = L^q, |\cdot|_{\Psi O} = |\cdot|_q, |\cdot|_{\Psi L} = p^{-1/q} q^{-1} |\cdot|_q.
\]

### 3.2 The Orlicz space generated from a random normed module

Let \( (E, \|\cdot\|) \) be an RN module and \( \Phi \) a Young function. We introduce the Orlicz space corresponding to \( \Phi \) generated from \( E \) as:

\[
L^\Phi(E) = \{ x \in E : \|x\| \in L^\Phi \},
\]

...
and the Orlicz heart of $E$ as:

$$\mathcal{M}^\Phi(E) = \{x \in E : \|x\| \in \mathcal{M}^\Phi\}.$$  

Induced by the norm on $L^\Phi$, the Orlicz norm $\| \cdot \|_\Phi$ and the Luxemburg norm $\| \cdot \|_\Psi$ on $L^\Phi(E)$ are given by

$$\|x\|_\Phi = \|\|x\||\|_\Phi,$$

and

$$\|x\|_\Psi = \|\|x\||\|_\Psi,$$

for each $x \in L^\Phi(E)$.

**Example 3.2** 1. When $(E, \| \cdot \|) = (L^0(F, K), | \cdot |)$, $L^\Phi(E)$ and $M^\Phi(E)$ are exactly $L^\Phi$ and $M^\Phi$, respectively.

2. If $\Phi(t) = t^p$ for $p \in [1, \infty)$, then we have $M^\Phi(E) = L^\Phi(E) = L^p(E)$ and $\| \cdot \|_\Psi = \| \cdot \|_p$, where $L^p(E) = \{x \in E : (E[|x|^p]) < \infty\}$ and $\|x\|_p = (E[|x|^p])^{\frac{1}{p}}$. Thus $(M^\Phi(E), \| \cdot \|_\Psi)$ is exactly the abstract $L^p$-space generated from $E$.

3. Let $\Psi(y) = 0$ for $y \leq 1$, and $\infty$ otherwise. Then $M^\Phi(E) = \{0\}$, $L^\Phi(E) = L^\infty(E) := \{x \in E : \|x\| \in L^\infty\}$ and $\|x\|_\Psi = \|x\|_\infty$.

**Proposition 3.3** Let $(E, \| \cdot \|)$ be a complete RN module and $\Phi$ a continuous Young function. Then both $(L^\Phi(E), \| \cdot \|_\Phi)$ and $(M^\Phi(E), \| \cdot \|_\Phi)$ are Banach spaces.

**proof.** We only need to prove the completeness. To show the completeness of $(L^\Phi(E), \| \cdot \|_\Phi)$, let $\{x_n, n \in \mathbb{N}\}$ be an arbitrary Cauchy sequence in $(L^\Phi(E), \| \cdot \|_\Phi)$. Then $\{x_n, n \in \mathbb{N}\}$ must also be a Cauchy sequence in $(E, \| \cdot \|)$, otherwise, there exist $\varepsilon > 0$ and $\lambda \in (0, 1)$ such that for any given $N_0 \in \mathbb{N}$, we can always find some $m, n \geq N_0$ such that $P\{\omega \in \Omega : \|x_m - x_n\|_\Phi \geq \varepsilon\} \geq \lambda$, then

$$1 \geq E[\Phi(\frac{\|x_m - x_n\|}{c})] \geq \lambda \Phi(\frac{\varepsilon}{c})$$

yields that $c \geq c_0 = \varepsilon(\Phi^{-1}(\frac{1}{\lambda}))^{-1}$, which means that $\|x_m - x_n\|_\Phi \geq c_0 > 0$ for these $m$'s and $n$'s, contradicting to the assumption that $\{x_n, n \in \mathbb{N}\}$ is a Cauchy sequence in $(L^\Phi(E), \| \cdot \|_\Phi)$. Then using the fact that $(E, \| \cdot \|)$ is complete, $\{x_n, n \in \mathbb{N}\}$ converges to some $x \in E$, namely, $\|x_n - x\| \to 0$ in probability as $n \to \infty$. Using Fatou’s lemma, we get

$$\lim_{n \to \infty} E[\Phi(\frac{\|x_n - x\|}{c})] \leq \lim_{n \to \infty} \lim_{m \to \infty} E[\Phi(\frac{\|x_n - x_m\|}{c})] = 0, \forall c > 0,$$

which implies that $\lim_{n \to \infty} \|x_n - x\|_\Phi = 0$.

$(M^\Phi(E), \| \cdot \|_\Phi)$ is a Banach space follows the next proposition. \hfill \square
Lemma 4.2 For any fixed proof. for each T 4.3. Lemma 4.2 shows that where the isometric isomorphism T conjugate space, and Φ is an arbitrarily chosen representative of ∥ · ∥, thus the ∥ · ∥-closure of MΦ(E) contains MΦ(E).

Fixed x ∈ MΦ(E), for each n ∈ N, let A_n = {ω ∈ Ω : ||x||₀(ω) ≤ n} and take x_n = 1A_n x, where ∥x∥₀ is an arbitrarily chosen representative of ∥x∥, then x_n ∈ L∞(E) and ∥x_n − x∥ΦL = |1A_n||x||ΦL → 0 as n → ∞. Thus the ∥ · ∥ΦL-closure of L∞(E) contains MΦ(E).

□

Remark 3.5 Let (E, || · ||) be a complete RN module and Φ a continuous Young function. Since the Orlicz norm || · ||ΦO and the Luxemburg norm || · ||ΦL are equivalent norms on LΦ(E), both (LΦ(E), || · ||ΦO) and (MΦ(E), || · ||ΦO) are also Banach spaces.

4 Dual representation of the conjugate space of MΦ(E)

We first state the main result as follows.

Theorem 4.1 Let (E, || · ||) be an RN module over K with base (Ω, F, P), (E∗, || · ||∗) its random conjugate space, and Φ a continuous Young functions with conjugate Ψ. Then

(MΦ(E), || · ||ΦL)' ∼= (LΨ(E∗), || · ||ΨO),

where the isometric isomorphism T : (LΨ(E∗), || · ||ΨO) → (MΦ(E), || · ||ΦL)' is given by

[Tf](x) = E[f(x)], ∀x ∈ MΦ(E),

for each f ∈ LΨ(E∗).

For the sake of clearness, the proof of Theorem 4.1 is divided into the following two Lemmas 4.2 and 4.3. Lemma 4.2 shows that T is well defined, and isometric and Lemma 4.3 shows that T is surjective.

In the following, U(E) := {x ∈ E : ||x|| ≤ 1} denotes the random closed unit ball of E.

Lemma 4.2 T is well defined and isometric.

proof. For any fixed f ∈ LΨ(E∗), we will prove Tf ∈ (MΦ(E), || · ||ΦL)' and ||Tf|| = ||f||ΨO.

For any x ∈ MΦ(E), according to “Hölder inequality”, we have:

||Tf|| = |E[f(x)]| ≤ E[||f||∗]|x|| ≤ ||f||∗||ΨO||x||ΦL = ||f||ΨO||x||ΦL.
This shows that $Tf \in (M^\Phi(E), \| \cdot \|_{\Phi L})'$ and $\|Tf\| \leq \|f\|_{\Phi O}$, namely, $T$ is well-defined. We remain to show that $\|Tf\| = \|f\|_{\Phi O}$, or equivalently, $\|Tf\| \geq \|f\|_{\Phi O}$.

Note that

$$\|f\|_{\Phi O} = \|f\|^{*}_{\Phi O} = \sup\{|E[f^{*}\xi]| : \xi \in M^\Phi, |\xi|_{\Phi L} \leq 1\}$$

$$= \sup\{|E[f^{*}\xi]| : \xi \in M^\Phi, \xi \geq 0, |\xi|_{\Phi L} \leq 1\},$$

thus for any fixed $\xi \in M^\Phi$ with $\xi \geq 0, |\xi|_{\Phi L} \leq 1$, it suffices to show $\|Tf\| \geq E[\|f^{*}\xi\|]$. It is easy to verify that the family $\{|f(x)| : x \in U(E)\}$ is upward directed, thus there exists a sequence $\{x_n, n \in \mathbb{N}\}$ in $U(E)$ such that $\{|f(x_n)|, n \in \mathbb{N}\}$ converges to $\vee\{|f(x)| : x \in U(E)\} = \|f\|^{*}$ in a nondecreasing way. Further, we can assume that $f(x_n) = |f(x_n)|$ for every $n$, otherwise we can replace each $x_n$ with $(sgnf(x_n))x_n$ (here $sgn(z)$ for an element $z \in L^0(F,K)$ means the equivalence class of $sgn(z^0)$ defined by $sgn(z^0)(\omega) = \frac{z^0(\omega)}{|z^0(\omega)|}$ if $z^0(\omega) \neq 0$, and 0 otherwise, where $z^0$ is an arbitrarily chosen representative of $z$). Thus, $\lim_{n \to \infty} f(\xi x_n) = \lim_{n \to \infty} \xi f(x_n) = \xi \|f\|^{*}$. For each $n$, since $|\xi x_n| = \xi |x_n| \leq \xi$, we have $\|\xi x_n\|_{\Phi L} \leq |\xi|_{\Phi L} \leq 1$, thus $E[f(\xi x_n)] = |Tf|(\xi x_n) \leq \|Tf\|$. Then according to Levi’s monotone convergence theorem, we finally get

$$E[\|f\|^{*}\xi] = \lim_{n \to \infty} E[f(\xi x_n)] \leq \|Tf\|,$$

which completes the proof. \hfill $\Box$

**Lemma 4.3** $T$ is surjective.

**Proof.** Let $F$ be an arbitrary element in $(M^\Phi(E), \| \cdot \|_{\Phi L})'$, we want to prove that there exists an $f \in L^\Phi(E^*)$ such that $F = Tf$.

For any fixed $x \in M^\Phi(E)$, define $\mu_x : \mathcal{F} \to K$ by $\mu_x(A) = F(\tilde{I}_A x), \forall A \in \mathcal{F}$, then $\mu_x$ is a countably additive $K$-valued measure, which is absolutely continuous with respect to the probability measure $P$. Thus according to Radon-Nikodým theorem, there exists an unique $\xi_x \in L^1$ such that $\mu_x(A) = F(\tilde{I}_A x) = E[\tilde{I}_A \xi_x], \forall A \in \mathcal{F}$. Moreover, $|\mu_x|(\Omega) = E[|\xi_x|] \leq \|F\|\|x\|_{\Phi L}$, since $|\mu_x(A)| = |F(\tilde{I}_A x)| \leq \|F\|\|x\|_{\Phi L}, \forall A \in \mathcal{F}$. Define $g : (M^\Phi(E), \| \cdot \|_{\Phi L}) \to (L^1, | \cdot |_1)$ by $g(x) = \xi_x, \forall x \in M^\Phi(E)$, then $g$ is a bounded linear operator, and $g(\tilde{I}_A x) = \tilde{I}_A g(x), \forall A \in \mathcal{F}, x \in M^\Phi(E)$. Immediately, we have that for each $x \in M^\Phi(E)$, $g(\xi x) = \xi g(x)$ holds for every simple function $\xi \in L^0(\mathcal{F}, K)$. Further we verify that $g(\xi x) = \xi g(x)$ holds for every $x \in U(E)$ and $\xi \in M^\Phi$. In fact, fix $x \in U(E)$ and $\xi \in M^\Phi$, according to [5] Theorem 2.1.14, there exists a sequence $\{\xi_n, n \in \mathbb{N}\}$ consisting of simple measurable functions such that $|\xi_n - \xi|_{\Phi L} \to 0$ which implies that $\xi_n \to \xi$ in probability and $\|\xi_n x - \xi x\|_{\Phi L} = \|(\xi_n - \xi)x\|_{\Phi L} \leq |\xi_n - \xi|_{\Phi L} \to 0$ as $n \to \infty$, thus $g(\xi x) = L^1 - \lim_{n \to \infty} g(\xi_n x) = L^1 - \lim_{n \to \infty} \xi_n g(x) = \xi g(x)$.
Consider the subset \( \{ |g(x)| : x \in U(E) \} \) of \( L^1 \subset L^0 \), since \( |g(I_A x + I_{A'} y)| = I_A |g(x)| + I_{A'} |g(y)| \) holds for every \( x, y \in U(E) \) and \( A \in \mathcal{F} \), we see that \( \{ |g(x)| : x \in U(E) \} \) is upward directed. As in the proof of Lemma 4.1, there exists a sequence \( \{ x_n, n \in \mathbb{N} \} \) in \( U(E) \) such that \( |g(x_n)| = g(x_n) \) converges to \( X_g := \vee \{ |g(x)| : x \in U(E) \} \in \bar{L}^0 \) in a nondecreasing way as \( n \to \infty \). Then, for any \( \xi \in M^\Psi \) with \( \xi \geq 0 \), by Levi’s monotone convergence theorem,

\[
E[\xi X_g] = \lim_{n \to \infty} E[\xi g(x_n)] = \lim_{n \to \infty} E[|g(x_n)|] \leq \lim_{n \to \infty} \|\xi x_n\|_{\Phi L} \leq \|F\|\|\xi\|_{\Phi L} < \infty,
\]

which implies that \( X_g \in L^\Psi \) with \( |X_g|_{\Psi O} \leq \|F\| \).

Define \( f : E \to L^0(\mathcal{F}, K) \) by \( f(x) = \lim_{n \to \infty} g(I_{A_n} x) \) for each \( x \in E \), where \( A_n \) is taken as in the proof of Proposition 3.3 and the limit on the right side is taken with respect to the convergent in probability. Since \( |g(I_{A_n} x) - g(I_{A_m} x)| = |g((I_{A_m} - I_{A_n}) x)| \leq X_g |I_{A_m} - I_{A_n}| |x| \to 0 \) in probability as \( m, n \to \infty \), we see that \( f \) is well defined. Moreover, it is easily seen that \( f \) is linear, \( f(x) = g(x) \) holds for every \( x \in L^\infty(E) \) and \( |f(x)| \leq X_g \|x\|, \forall x \in E \), it follows from \cite{14} Lemma 2.12 that \( f \) is \( L^0(\mathcal{F}, K) \)-linear, thus \( f \in E^* \) and

\[
\|f\| = \vee \{ |f(x)| : x \in U(E) \} = \vee \{ |g(x)| : x \in U(E) \} = X_g,
\]

which implies that \( f \in L^\Psi(E^*) \).

We remain to show \( F = Tf \). Note that

\[
[Tf](x) = E[f(x)] = E[g(x)] = F(x), \forall x \in L^\infty(E),
\]

since \( L^\infty(E) \) is a dense subset of \( (M^\Phi(E), \| \cdot \|_{\Phi L}) \) by Proposition 3.3 the two continuous functionals \( F \) and \( Tf \) must equal to each other on the whole space \( (M^\Phi(E), \| \cdot \|_{\Phi L}) \).

Obviously, if the Luxemburg norm \( \| \cdot \|_{\Phi L} \) on \( M^\Phi(E) \) is replaced by Orlicz norm \( \| \cdot \|_{\Psi O} \), then the operator norm on the dual space \( L^\Psi(E) \) changes accordingly from Orlicz norm \( \| \cdot \|_{\Psi O} \) to Luxemburg norm \( \| \cdot \|_{\Psi L} \). Precisely, we have:

**Proposition 4.4** Let \( (E, \| \cdot \|) \) be an RN module over \( K \) with base \( (\Omega, \mathcal{F}, P), (E^*, \| \cdot \|_{E^*}) \) its random conjugate space, and \( \Phi \) a continuous Young functions with conjugate \( \Psi \). Then

\[
(M^\Phi(E), \| \cdot \|_{\Phi L})' \cong (L^\Psi(E^*), \| \cdot \|_{\Psi L}),
\]

where the isometric isomorphism is the same as Theorem 4.1.
Furthermore, we can show the following:

**Proposition 4.5** Let \((E, \| \cdot \|)\) be an RN module over \(K\) with base \((\Omega, \mathcal{F}, P)\), \(\Phi\) and \(\Psi\) a pair of Young functions both of which satisfy the \((\Delta_2)\) condition. Then \((E, \| \cdot \|)\) is random reflexive if and only if \((L^\Phi(E), \| \cdot \|_{\Phi, L})\) is reflexive.

**Remark 4.6** In Theorem 4.4, if we choose \(\Phi(t) = t^p\) for \(p \in [1, \infty)\), then we get

\[(L^p(E), \| \cdot \|_p)' \cong (L^q(E^*), \| \cdot \|_q).\]

In Proposition 4.5, if we choose \(\Phi(t) = t^p\) for \(p \in (1, \infty)\), then we have that \((E, \| \cdot \|)\) is random reflexive if and only if \((L^p(E), \| \cdot \|_p)\) is reflexive. Thus the results in this section generalize some known results.

## 5 Strict convexity and uniform convexity of \(L^\Phi(E)\)

Recall that a normed space \((X, \| \cdot \|)\) is said to be:

- Strictly convex: if for any two different elements \(x, y \in X\) with \(\|x\| = \|y\| = 1\), we have \(\|\frac{x+y}{2}\| < 1\);
- Uniformly convex: if for every \(\epsilon \in (0, 2)\), there exists \(\delta(\epsilon) > 0\) such that for any \(x, y \in X\) with \(\|x\| = \|y\| = 1\) and \(\|x - y\| \geq \epsilon\), we have \(\|\frac{x+y}{2}\| \leq 1 - \delta(\epsilon)\).

It is well known that strictly convex and uniformly convex Banach spaces have played key roles in many important topics in nonlinear functional analysis and geometry of Banach spaces. In this section, we study the strict convexity and uniform convexity of \(L^\Phi(E)\). Naturally, these properties have close connection with the random strict convexity and random uniform convexity of \(E\). For convenience, we also recall the notions of random strict convexity and random uniform convexity of an RN module which were introduced by Guo and Zeng [15].

Let \(A \in \mathcal{F}\), then the equivalence class of \(A\), denoted by \(\tilde{A}\), is defined by \(\tilde{A} = \{B \in \mathcal{F} : P(A \triangle B) = 0\}\), where \(A \triangle B = (A \setminus B) \cup (B \setminus A)\) is the symmetric difference of \(A\) and \(B\), and \(P(\tilde{A})\) and \(I_{\tilde{A}}\) are defined to be \(P(A)\) and \(I_A\), respectively. For two \(\mathcal{F}\)-measurable sets \(G\) and \(D\), \(G \subset D\) a.s. means \(P(G \setminus D) = 0\), in which case we also say \(\tilde{G} \subset \tilde{D}\); \(\tilde{G} \cap \tilde{D}\) denotes the equivalence class determined by \(G \cap D\). Other similar symbols are easily understood in an analogous manner. For any \(\xi, \eta \in L^0\) and \(A \in \mathcal{F}\), \(\xi > \eta\) on \(\tilde{A}\) means that \(\xi^0(\omega) > \eta^0(\omega)\) for almost all \(\omega \in A\), where \(\xi^0\) and \(\eta^0\) are arbitrarily chosen representatives of \(\xi\) and \(\eta\), respectively.

Let \((E, \| \cdot \|)\) be an RN module. For any \(x, y \in E\), denote the equivalence class of \(\mathcal{F}\)-measurable set \(\{\omega \in \Omega : \|x\|^0(\omega) \neq 0\}\) by \(A_x\), called the support of \(x\), where \(\|x\|^0\) is an arbitrarily chosen representative of \(\|x\|\). Let \(B_{xy} = A_x \cap A_y \cap A_{x-y}\). The random unite sphere of \(E\) refers to

\[S(E) = \{x \in E : x \neq \theta, \|I_{A_x}x\| = I_{A_x}\}.\]
Lemma 5.3 If \( L^\# = \{ \xi \in L^0_\#: \exists \lambda \in \mathbb{R}, \lambda > 0 \text{ such that } \xi \geq \lambda \} \) is the set of elements in \( L^0_\# \) which is bounded away from 0. Furthermore, \( L^\#[0, 1] = \{ \xi \in L^\# : \xi \leq 1 \} \) and \( L^\#[0, 2] = \{ \xi \in L^\# : \xi \leq 2 \} \).

Definition 5.1 Let \((E, \| \cdot \|)\) be an RN module, \(E\) is said to be random strictly convex, if for any \(x, y \in S(E)\) with \(P(B_{xy}) > 0\), we have \(\|\frac{x+y}{2}\| < 1\) on \(B_{xy}\); and \(E\) is said to be random uniformly convex if for any \(\epsilon \in L^\#_\#[0, 2]\) there exists a \(\delta \in L^\#_\#[0, 1]\) such that the following condition holds: \(I_D\|x - y\| \geq \epsilon I_D\) implies that \(I_D\|\frac{x+y}{2}\| \leq (1 - \delta)I_D\) for any \(x, y \in S(E)\) and \(D \in \mathcal{F}\) with \(D \subset B_{xy}\) and \(P(D) > 0\).

Let \((E, \| \cdot \|)\) be an RN module, it is easy to check that \(\{\|x\| : x \in E, \|x\| \leq 1\}\) is directed, then there is a sequence of elements \(x_n \in E\) with \(\|x_n\| \leq 1\) such that \(\{\|x_n\|, n \in \mathbb{N}\}\) converges to \(I_{H(E)} = \vee\{\|x\| : x \in E, \|x\| \leq 1\}\) in a nondecreasing way. When \(I_{H_E} = 1\), \(E\) is called having full support. We restate [12, Lemma 3.1] as follows.

Proposition 5.2 Assume that \((E, \| \cdot \|)\) is a complete RN module, then there exists an \(x_0 \in E\) such that \(\|x_0\| = I_{H(E)}\), specially, when \((E, \| \cdot \|)\) has full support, then there exists an \(x_0 \in E\) such that \(\|x_0\| = 1\).

In this Section, an RN module is assumed to have full support.

In the sequel, the norm on \(L^\Phi\) is denoted by \(| \cdot |_\Phi\), which can be either the Luxemburg norm \(| \cdot |_{\Phi_L}\) or the Orlicz norm \(| \cdot |_{\Phi_O}\), accordingly, the norm on \(L^\Phi(E)\) induced by \(| \cdot |_\Phi\) is denoted by \(| \cdot |\).

Lemma 5.3 If \((L^\Phi, | \cdot |_\Phi)\) is strictly convex, specially when \((L^\Phi, | \cdot |_{\Phi})\) is uniformly convex, then for any different \(\xi, \eta \in L^\Phi\) with \(\xi \geq \eta \geq 0\), it holds that \(|\xi|_\Phi \geq |\eta|_\Phi\).

\textbf{proof.} The assumption \(\xi \geq \eta \geq 0\) yields that \(|\xi|_\Phi \geq |\eta|_\Phi\). We prove the conclusion by contradiction. Suppose that \(|\xi|_\Phi = |\eta|_\Phi = \lambda\). Since \(x, y\) are different, \(\lambda\) must be a positive number. Let \(\xi_0 = \frac{\xi}{\lambda}\) and \(\eta_0 = \frac{\eta}{\lambda}\), then \(|\xi_0|_\Phi = |\eta_0|_\Phi = 1\), clearly \(\xi_0 \geq \frac{\xi + \eta}{2} \geq \eta_0 \geq 0\), implying that \(|\frac{\xi_0 + \eta_0}{2}|_\Phi = 1\), but this contradicts to the assumption that \((L^\Phi, | \cdot |_\Phi)\) is strictly convex. \(\square\)

5.1 Strict convexity of \(L^\Phi(E)\)

Theorem 5.4 Let \((E, \| \cdot \|)\) be a complete RN module, then \((L^\Phi(E), | \cdot |_\Phi)\) is strictly convex if and only if \((L^\Phi, | \cdot |_\Phi)\) is strictly convex and \((E, \| \cdot \|)\) is random strictly convex.

\textbf{proof.} \((\leftarrow)\) For any pair of different \(x, y \in L^\Phi(E)\) with \(|x|_\Phi = |y|_\Phi = 1\), it is divided into two cases to show that \(|\frac{x+y}{2}|_\Phi < 1\).
Case 1: When $\|x\| \neq \|y\|$, due to the assumption that $(L^\Phi, | \cdot |_\Phi)$ is strictly convex and $\|x|_\Phi = \|y|_\Phi = 1$, we have $\frac{\|x + y\|}{2} < 1$, then $\|x + y\|_\Phi < 1$ follows immediately from that $\|x + y\| \leq \frac{\|x\| + \|y\|}{2}$.

Case 2: When $\|x\| = \|y\|$, then $\|x + y\| = \frac{\|x\| + \|y\|}{2}$, and due to the assumption that $(E, \| \cdot \|)$ is random strictly convex we have $\frac{\|x + y\|}{2} \neq \|x\|$. According to Lemma 5.3, $\frac{\|x + y\|}{2} < \|x\|_\Phi = 1$.

($\Rightarrow$) For any pair of different $\xi, \eta \in L^\Phi$ with $|\xi|_\Phi = |\eta|_\Phi = 1$, according to Proposition 5.2, there exists an $x_0 \in E$ such that $\|x_0\| = 1$. Choose $x = \xi x_0$ and $y = \eta x_0$, then $x, y \in L^\Phi(E)$ and $\|x\|_\Phi = \|y\|_\Phi = 1$. Note that $x \neq y$, according to the assumption that $(L^\Phi(E), \| \cdot \|_\Phi)$ is strictly convex, we have $\frac{\|x + y\|}{2} = \|x\|_\Phi < 1$, which means that $(L^\Phi, | \cdot |_\Phi)$ is strictly convex.

We show the random strict convexity of $(E, \| \cdot \|)$ by contradiction. Suppose that there exist $x, y \in S(E)$ with $P(B_{xy}) > 0$, such that for some $D \in \tilde{F}$, $D \subset B_{xy}$ with $P(D) > 0$, it holds that $I_D \frac{\xi + \eta}{2} = I_D$. Clearly, $\|I_D x\| = \|I_D y\| = I_D = \|I_D y\|_\Phi = \lambda$. Let $x' = \frac{\xi + \eta}{2}$ and $y' = \frac{\xi - \eta}{2}$, then we have $x', y' \in L^\Phi(E)$ and $\|x'\|_\Phi = \|y'\|_\Phi = 1$. Note that $x' \neq y'$, the assumption $(L^\Phi(E), \| \cdot \|_\Phi)$ is strictly convex then yields that $\frac{\|x' + y'\|}{2} < \lambda$, equivalently $\|I_D \frac{\xi + \eta}{2}\|_\Phi < \lambda$, however $I_D \frac{\xi + \eta}{2} = I_D$ implies that $\|I_D \frac{\xi + \eta}{2}\|_\Phi = |I_D|_\Phi = \lambda$.

\[\Box\]

Notice that we use the completeness of $(E, \| \cdot \|)$ only in the process to show that “$(L^\Phi(E), \| \cdot \|_\Phi)$ is strictly convex” implies that “$(L^\Phi, | \cdot |_\Phi)$ is strictly convex”, thus we have the following:

**Proposition 5.5** Let $(E, \| \cdot \|)$ be an RN module and assume that $(L^\Phi, | \cdot |_\Phi)$ is strictly convex, then $(E, \| \cdot \|)$ is random strictly convex if and only if $(L^\Phi(E), \| \cdot \|_\Phi)$ is strictly convex.

Since for every $p \in (1, \infty)$, $(L^p, | \cdot |_p)$ is strictly convex, thus we have the following:

**Corollary 5.6** Let $(E, \| \cdot \|)$ be an RN module and $1 < p < \infty$, then $(E, \| \cdot \|)$ is random strictly convex if and only if $(L^p(E), \| \cdot \|_p)$ is strictly convex.

This corollary is exactly Theorem 3.3 in Guo and Zeng [13].

### 5.2 Uniform convexity of $L^\Phi(E)$

**Theorem 5.7** Let $(E, \| \cdot \|)$ be a complete RN module. If $(L^\Phi(E), \| \cdot \|_\Phi)$ is uniformly convex, then $(L^\Phi, | \cdot |_\Phi)$ is uniformly convex and $(E, \| \cdot \|)$ is random uniformly convex.

**proof.** (1) We first show that $(L^\Phi, | \cdot |_\Phi)$ is uniformly convex. Given $\epsilon \in (0, 2]$. For any two elements $\xi, \eta \in L^\Phi$ with $|\xi|_\Phi = |\eta|_\Phi = 1$ and $|\xi - \eta|_\Phi \geq \epsilon$, according to Proposition 5.2, there exists an $x_0 \in E$ such that $\|x_0\| = 1$, and if we take $x = \xi x_0$ and $y = \eta x_0$, then we have $\|x\|_\Phi = |\xi|_\Phi = 1$, $\|y\|_\Phi = |\eta|_\Phi = 1$ and $\|x + y\|_\Phi < 1$.

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and \( \|x - y\|_\Phi = |\xi - \eta|_\Phi \geq \epsilon \). Since \( (L^\Phi(E), \|\cdot\|_\Phi) \) is uniformly convex, there exists a \( \delta \in (0, 1] \) such that 
\[
\frac{\xi + \eta}{2} |\Phi = \frac{\xi + \eta}{2} |\Phi \leq (1 - \delta).
\]
This \( \delta \) is decided by \( (L^\Phi(E), \|\cdot\|_\Phi) \), not depending on \( x, y \) and therefore neither on \( \xi, \eta \). This means that \( (L^\Phi, |\cdot|_\Phi) \) is uniformly convex.

(2) We show that \( (E, \|\cdot\|) \) is random uniformly convex by contradiction. Suppose that \( E \) is not random uniformly convex. Then we can find an \( \epsilon \in L_\#[0, 2] \) such that for any \( \delta \in L_\#[0, 1] \), there exist \( x_\delta, y_\delta \in S(E) \) and \( D_\delta \in \mathcal{F} \) satisfying \( D_\delta \subset B_{x_\delta y_\delta}, \ P(D_\delta) > 0 \) and \( I_{D_\delta} \|x_\delta - y_\delta\| \geq \epsilon I_{D_\delta} \) and 
\[
I_{D_\delta} \frac{\|x_\delta + y_\delta\|}{2} > (1 - \delta)I_{D_\delta} \text{ on } D_\delta.
\]
Since \( \epsilon \in L_\#[0, 2] \), there exists some positive number \( \lambda \) such that \( \lambda < \epsilon \leq 2 \) on \( \Omega \). For this \( \lambda \), by the uniform convexity of \( L^\Phi(E) \), there exists a number \( \delta_1 \in (0, 1] \) such that for any \( u, v \in L^\Phi(E) \) with \( \|u\|_\Phi = \|v\|_\Phi = 1, \ \|u + v\|_\Phi > 2(1 - \delta_1) \) implies \( \|u - v\|_\Phi < \lambda \).

Since \( \delta_1 \) can be regarded as an element in \( L_\#[0, 1] \), there exist \( x_{\delta_1}, y_{\delta_1} \in S(E) \) and \( D_{\delta_1} \in \mathcal{F} \) such that \( D_{\delta_1} \subset B_{x_{\delta_1} y_{\delta_1}}, \ P(D_{\delta_1}) > 0 \) and \( I_{D_{\delta_1}} \|x_{\delta_1} - y_{\delta_1}\| \geq \epsilon I_{D_{\delta_1}} \). Let \( c = I_{D_{\delta_1}} |\Phi > 0 \), since \( I_{D_{\delta_1}} x_{\delta_1} \| = I_{D_{\delta_1}} y_{\delta_1} \| = I_{D_{\delta_1}} \), we have \( I_{D_{\delta_1}} x_{\delta_1} \|_\Phi = I_{D_{\delta_1}} y_{\delta_1} \|_\Phi = c \). Choose \( x = \frac{I_{D_{\delta_1}} x_{\delta_1}}{c} \) and \( y = \frac{I_{D_{\delta_1}} y_{\delta_1}}{c} \), then \( \|x\|_\Phi = \|y\|_\Phi = 1 \). It follows from \( I_{D_{\delta_1}} \|x_{\delta_1} - y_{\delta_1}\| \geq \epsilon I_{D_{\delta_1}} \geq \lambda I_{D_{\delta_1}} \), we have \( c\|x - y\|_\Phi \geq \lambda \|x_{\delta_1}\|_\Phi = \lambda c \), and from \( I_{D_{\delta_1}} \|\frac{x_{\delta_1} + y_{\delta_1}}{2}\| > (1 - \delta_1)I_{D_{\delta_1}} \) on \( D_{\delta_1} \) and Lemma 5.3, we have \( c\|\frac{x + y}{2}\|_\Phi > (1 - \delta_1)I_{D_{\delta_1}} |\Phi = (1 - \delta_1)c \). That is to say, we have both \( \|x - y\|_\Phi \geq \lambda \) and \( \|\frac{x + y}{2}\|_\Phi > (1 - \delta_1) \), which is impossible since \( x, y \in L^\Phi(E) \) and \( \|x\|_\Phi = \|y\|_\Phi = 1 \).

\[ \square \]

**Remark 5.8** Let \( (E, \|\cdot\|) \) be a complete RN module, Guo and Zeng [12, 17] proved that, for any fixed \( p \in (1, \infty) \), \( (L^p(E), \|\cdot\|_p) \) is uniformly convex if and only if \( (E, \|\cdot\|) \) is random uniformly convex. Clearly, Theorem 5.7 generalizes this result in one direction. For the reverse direction, we guess that \( \text{"}(E, \|\cdot\|) \) is random uniformly convex and \( (L^\Phi, |\cdot|_\Phi) \) is uniformly convex\" also implies that \( \text{"}(L^\Phi(E), \|\cdot\|) \) is uniformly convex\". However, we are not able to prove it up to now. We leave it as a problem to be settled down in the future.

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