Common idempotents in compact left topological left semirings

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A classical result of topological algebra states that any compact left topological semigroup has an idempotent. This result, in its final form due to Ellis, became crucial for numerous applications in number theory, algebra, topological dynamics, and ergodic theory.

In this talk, I consider more complex structures than semigroups: left semirings, the algebras with two associative operations one of which is also left distributive w.r.t. another one. By Ellis’ result, compact left topological left semirings have additive idempotents as well as multiplicative ones. I show that they have, moreover, common, i.e., additive and multiplicative simultaneously, idempotents.

As an application, I partially answer a question related to algebraic properties of the Stone–Čech compactification of natural numbers. Finally, I show that similar arguments establish the existence of common idempotents in far more general structures than left semirings.
One operation:

Ellis’ result and applications
Here I recall Ellis’ theorem on semigroups and discuss its importance for various applications.
**Terminology.** A groupoid is a set with a binary operation. The usual notation is \((X, \cdot)\) or its variants, like \((X, +)\), or simply \(X\). In the multiplicative notation, the symbol \(\cdot\) is often omitted. When the operation satisfies the associativity law
\[(xy)z = x(yz),\]
the groupoid is called a semigroup.

A groupoid is left topological iff for any fixed first argument \(a\) the left translation
\[x \mapsto ax\]
is continuous. Right topological groupoids are defined dually. A groupoid is semitopological iff it is left and right topological simultaneously, and topological iff its operation is continuous. It is not difficult to verify that this hierarchy is not degenerate, even for semigroups.

We shall be interested in left topological groupoids.
R. Ellis in his “Lectures on topological dynamics” (1969) published the following simple but remarkable result (see [1]):

**Theorem (Ellis).** Every compact Hausdorff left topological semigroup has an idempotent.

(Earlier A. Wallace and K. Numakura stated independently this result for topological semigroups. However just one sided continuity is important for applications, as we'll see later.)

In particular, *any minimal compact left topological semigroup consists of a unique element*. Note that this statement has a (trivial) purely algebraic counterpart: *Any minimal semigroup consists of a unique element*. Here a groupoid is *minimal* iff it includes no proper subgroupoids, and *minimal compact* iff it is compact and includes no proper compact subgroupoids.
This result, interesting in itself, became crucial for various applications (of Ramsey-theoretic character) in number theory, algebra, topological dynamics, and ergodic theory. These applications are based on using of *idempotent ultrafilters*. (For a general reference, see [2].) Perhaps, the simplest illustration of such applications is Hindman’s famous *Finite Sums Theorem* (1974):

**Theorem** (Hindman). Any finite partition of $\mathbb{N}$, the set of natural numbers, has a part that is “big” in the following sense: it contains an infinite sequence all finite sums of distinct elements of which belong to this part.

(Previously weaker results were given by various authors, including D. Hilbert and I. Schur).

The original purely combinatorial proof was unbelievably cumbersome. (Recently N. Hindman said: “I never understood the original complicated proof (no, I did not plagiarize it)...”.) Soon after this, however, F. Galvin and S. Glazer found a way to obtain this result in a few lines. To explain the idea, we need some preparations.
The space of ultrafilters. Let $X$ be a set. Recall that the set $\beta X$ of all ultrafilters over $X$ carries a natural topology with elementary (cl)open sets

$$O_A = \{u \in \beta X : A \in u\}$$

for each $A \subseteq X$. The following facts are standard in general topology: $\beta X$ is a compact Hausdorff extremally disconnected space. Moreover, it is the Stone–Čech compactification of the discrete space $X$.

Recall that the Stone–Čech compactification of a space $X$ is a compact space $Z$ such that $X$ is dense in $Z$, and for every compact space $Y$, any continuous mapping $f : X \to Y$ can be extended to a continuous mapping $\tilde{f} : Z \to Y$. Such a $Z$ is unique up to homeomorphism with fixed points of $X$. It is customary to identify elements of $X$ with principal ultrafilters, while non-principal ultrafilters form the remainder:

$$X^* = \beta X \setminus X.$$ 

Under this identification, any unary operation on a discrete space $X$ can be extended to an operation on the space of ultrafilters over $X$. 
The algebra of ultrafilters. Consider now extensions of binary operations. (In principle, the argument works for arbitrary operations.)

Let \((X, \cdot)\) be a groupoid. To extend \(\cdot\) on \(X\) to \(\cdot\) on \(\mathcal{B}X\) (which let me denote by the same symbol), we proceed as follows: First we extend \(\cdot\) by fixing each second argument, then by fixing each first argument.

We can define this extension in a straightforward way by putting

\[
uv = \{S \subseteq X : \{a \in X : \{b \in X : ab \in S\} \in u\} \in v\}
\]

for all \(u, v \in \mathcal{B}X\). This looks slightly complicated but has the same clear meaning.

As a result, the extended operation is continuous for any fixed first argument, i.e. the groupoid \((\mathcal{B}X, \cdot)\) is left topological. And although it is never right topological, except for trivial cases, the operation is continuous for any fixed second argument whenever it is in \(X\). Moreover, it is a unique operation with these properties.
Not many algebraic properties are stable under this extension.

E.g. consider the extension of \((\mathbb{N}, +, \cdot)\), the semiring of natural numbers with the usual addition and multiplication. In \((\mathcal{B}\mathbb{N}, +, \cdot)\), none of the extended operations is commutative. Also both distributivity laws fail (I'll return to this example below).

However, the associativity law is stable:

**Lemma.** If \(X\) is a semigroup, so is \(\mathcal{B}X\).

So any semigroup \(X\) extends to the compact left topological semigroup \(\mathcal{B}X\) of ultrafilters over \(X\). By Ellis’ theorem, the latter has an idempotent.

Of course, it is in \(X^*\) if \(X\) has no idempotents. Another case when one can get an idempotent in \(X^*\) is when \(X\) is (weakly) cancellative; in this case, \(X^*\) forms a subsemigroup of \(\mathcal{B}X\), and as \(X^*\) is closed in \(\mathcal{B}X\) and so compact, one can apply Ellis’ theorem to \(X^*\).
After this preparations, the Finite Sums Theorem is reduced to the following fact:

**Lemma.**  *Idempotents of $(\mathbb{N}^*, +)$ consist just of sets “big” in sense of that theorem.*

The proof of this lemma is not hard.

Actually, the lemma is true not only for $(\mathbb{N}, +)$ but for any groupoid $(X, \cdot)$ whenever $(X^*, \cdot)$ has *idempotents*. But to establish their existence we use associativity and apply Ellis’ theorem.

On the other hand, no specific properties of semigroups are used, thus the argument works for any semigroup. E.g. one can apply it to $(\mathbb{N}, \cdot)$ and prove the *Finite Products Theorem* (formulated analogously).
Hindman’s theorem, even in its general form, i.e. for every semigroups, is the simplest illustration here — since its proof uses arbitrary idempotent ultrafilters. By using idempotent ultrafilters of specific form, one can prove a lot of other significant results. Most popular examples include:

van der Waerden’s and Szemerédi’s Arithmetic Progressions Theorem,

Furstenberg’s Common Recurrence Theorem,

Hales–Jewett’s theorem,

etc. Some of these results can be proved elementarily but using of ultrafilters much simplifies this; for other results, no elementary proofs are known.
Two operations:

Left semirings
Here I establish the main result of this talk, the existence of common idempotents in compact left topological left semirings. Also I give some its application and consider its algebraic version.
Terminology. An algebra means here a universal algebra. I shall consider algebras with two binary operations. Let me denote such an algebra by \((X, +, \cdot)\), or simply \(X\) again, and refer to its operations as its addition and multiplication. (However, no their properties, like commutativity, associativity, etc., are assumed \textit{a priori}.)

Given an algebra \((X, +, \cdot)\), recall that the multiplication is \textit{left distributive w.r.t.}\ the addition iff the algebra satisfies the law

\[ x(y + z) = xy + xz. \]

An algebra \((X, +, \cdot)\) is a left semiring iff both its groupoids are semigroups and \(\cdot\) is left distributive w.r.t. \(+\). \textit{Right distributivity} and \textit{right semirings} have the dual definitions; \textit{semirings} are algebras which are left and right semirings simultaneously.

An algebra with binary operations is \textit{left topological} iff any of its groupoids is left topological.
Now we are ready to establish our main result:

**Theorem.** Any compact Hausdorff left topological left semiring has a common idempotent (and so consists of a unique element whenever is minimal compact).

Here a *common idempotent* means an element that is an idempotent for each operations, in other words, an element forming a subalgebra.

This theorem generalizes Ellis’ result on semigroups — since any left topological semigroup can be turned into a left topological semiring in a trivial way. E.g.:

If \((X, \cdot)\) is a left topological semigroup, let \(+\) be the projection onto the *first* argument, then \((X, +, \cdot)\) is a left topological semiring. Or else:

If \((X, +)\) is a left topological semigroup, let \(\cdot\) be the projection onto the *second* argument, then \((X, +, \cdot)\) is a left topological semiring, too.
Proof. Let \((X, +, \cdot)\) be a compact Hausdorff left topological left semiring.

1. First we show that *if X is minimal compact, then it consists of a unique element.*

Let \(a \in X\). The set \(aX = \{ax : x \in X\}\) is compact (as the image of \(X\) under \(x \mapsto ax\)). Moreover, \(aX\) forms a subalgebra. Therefore, \(aX = X\) (by minimality).

Furthermore, the set \(A = \{x \in X : ax = a\}\) is nonempty (since \(aX = X\)) and compact (as the preimage of \(\{a\}\) under \(x \mapsto ax\)). Does \(A\) form a subalgebra? In general, *no*: \((A, \cdot)\) is a semigroup but \((A, +)\) should not be a semigroup.

But assume that \(a\) is an additive idempotent; it exists by Ellis’ theorem applied to \((X, +)\). Then \((A, +, \cdot)\) is a subalgebra. Therefore, \(A = X\) (by minimality), and thus \(a\) is also a multiplicative idempotent. It follows \(X = \{a\}\) (by minimality again).
2. Now we consider the general case, when $X$ is not assumed minimal compact, and show that it has a common idempotent.

The intersection of any $\subseteq$-decreasing chain of compact subalgebras of $X$ is a compact subalgebra. By Zorn’s Lemma, there is a minimal compact subalgebra $A$. By the first part of the proof, it consists of a unique element, which is thus a common idempotent.

\[ \square \]

**Remark.** The proof shows a stronger fact: Any additive idempotent of every minimal compact left topological left semiring is also a multiplicative one. Simple examples show that multiplicative idempotents should not be additive, and additive idempotents of nonminimal compact left topological left semiring should not be multiplicative.
Algebraic variant. As we noted, Ellis’ result on minimal compact semigroups has an obvious purely algebraic counterpart: Every minimal semigroup consists of a unique element. One can ask whether the result on minimal compact semirings have the similar algebraic version:

**Question.** Can a minimal left semiring have more than one element?

Although the question looks not difficult, I was able to get the (expected) negative answer only in partial cases:

**Proposition.**
1. Any minimal finite left semiring consists of a unique element.
2. Any minimal semirings consists of a unique element.

Clause 1 follows from Theorem (consider the discrete topology), while the proof of Clause 2 uses different arguments. It remains to exclude a possibility of a minimal countable left semiring.
An application. We use the theorem to partially answer a long-standing question on \((\mathbb{N}^*, +, \cdot)\), the algebra of non-principal ultrafilters over \(\mathbb{N}\) with their (extended) addition and multiplication:

**Question.** Can non-principal ultrafilters over \(\mathbb{N}\) be instances of left or right distributivity? i.e. can some \(u, v, w \in \mathbb{N}^*\) satisfy \(u(v + w) = uv + uw\) or \((u + v)w = uw + vw\)?

E. van Douwen proved that such instances, even if exist, are topologically rare (see [3]):

**Theorem (van Douwen).** The sets

\[
\{u \in \mathbb{N}^* : \forall v, w \in \mathbb{N}^* \ u(v + w) = uv + uw\}
\]

and

\[
\{u \in \mathbb{N}^* : \forall v, w \in \mathbb{N}^* \ (u + v)w = uw + vw\}
\]

are nowhere dense in \(\mathbb{N}^*\).

We produce a negative result in another direction:
A well-known (and difficult) result says that the algebra \((\mathbb{N}^*, +, \cdot)\) has no common idempotents. It follows

**Corollary.** No closed subalgebras of \((\mathbb{N}^*, +, \cdot)\) satisfy the left distributivity law.

In this in mind, we may specify the question as follows:

**Question.** Can some non-closed subalgebra of \((\mathbb{N}^*, +, \cdot)\) be a left semiring?

**Remark.** In fact, \(\mathbb{N}^*\) does not have even \(u\) with \(u + u = uu\). And it is open if some \(u, v, w, x \in \mathbb{N}^*\) satisfy \(u + v = wx\) (or other linear equations). On the other hand, the closure of the set of additive idempotents forms a right ideal of \((\mathbb{N}^*, \cdot)\), so (by Ellis’ result) there must be multiplicative idempotents close to additive ones. This allows to refine Hindman’s theorem: *For any finite partition of \(\mathbb{N}\) there is a part containing all finite sums of an infinite sequence as well as all finite products of another one.* (This refinement was proved by Hindman and Bergelson, see [2].)
Generalizations:

Non-associative case
Here we note that the used arguments work in more general situations, when the operations are not assumed associative but satisfy certain other, much weaker conditions.

For simplicity, let me consider only the case with one operation. Results for algebras with two (and in fact, with any number of) operations can be established along the same line as this was done above for left semirings (see [4]).
**Terminology.** An occurrence of a variable $x$ into a term $t(x,\ldots)$ is *right-most* iff the occurrence is non-dummy and whenever

$$t_1(x,\ldots) \cdot t_2(x,\ldots)$$

is a subterm of $t$, then $x$ occurs non-dummy into $t_2(x,\ldots)$ but not $t_1(x,\ldots)$.

**Examples.** All the occurrences of the variable $x$ into the terms

$$x, \ vx, \ (vv)x, \ v(vx), \ (uv)(wx)$$

are right-most, while all its occurrences into the terms

$$v, \ uv, \ xv, \ xx, \ x(vx) \ (ux)(vx)$$

are not.

A *left-most* occurrence can be defined dually but is not used in what follows. Note that if the occurrence of $x$ into $t$ is right-most (or left-most), then $x$ occurs into $t$ just one time.
Lemma. Let $X$ be a left topological groupoid, and let $t(v,\ldots,x)$ be a term with the right-most occurrence of the last argument. Then for every $a,\ldots \in X$, the mapping

$$x \mapsto t(a,\ldots,x)$$

is continuous.

Proof. By induction on construction of $t$. \hfill \Box

Now we are ready to formulate our non-associative version of Ellis’ result:
**Theorem.** Let $X$ be a compact left topological groupoid, and $r$, $s$, and $t$ some one-, two-, and three-parameter terms respectively, where $s$ has the right-most occurrence of the last argument. Suppose that $X$ satisfies

\[ s(x, y) = s(x, z) = r(x) \rightarrow s(x, y \cdot z) = r(x) \]

and

\[ s(x, y) \cdot s(x, z) = s(x, t(x, y, z)). \]

Then $X$ has an idempotent.

The proof refines the argument used for semigroups. First one shows that if such a groupoid is *minimal compact*, then it consists of a unique element (the lemma above is necessary here). Then one applies Zorn’s Lemma to isolate a minimal compact subgroupoid (which satisfies the same universal formulas).

The theorem indeed generalizes Ellis’ one — since associativity implies the conditions of this theorem, with

\[ r(x) = x, \quad s(x, y) = xy, \quad t(x, y, z) = yxz. \]

But many other *identities* imply these conditions as well. Let me give several examples.
Examples. 1. Consider four following identities ("in Moufang style"):

\[(1) \quad (xy)(yz) = ((xy)y)z,\]
\[(2) \quad (xz)(yz) = ((xz)y)z,\]
\[(3) \quad (xy)(xz) = x(y(xz)),\]
\[(4) \quad (xx)(yz) = ((xx)y)z\]

Each of them is strictly weaker than associativity.

(1), as well as (2), implies the first condition of the theorem, while (3) implies the second one, all with \(r(x) = x\) and \(s(x, y) = xy\) again. (4) implies the first condition with \(r(x) = x\) and \(s(x, y) = (xx)y\). Therefore, every compact left topological groupoid satisfying any of (1), (2), (4), together with (3), has an idempotent.

2. The identity

\[(5) \quad x(yz) = (xz)y\]

is incomparable with associativity but, like it, implies both conditions of the theorem, again with \(r(x) = x\) and \(s(x, y) = xy\). Hence, every compact left topological groupoid satisfying (5) has an idempotent.
3. The left self-distributivity

\[(6) \quad x(yz) = (xy)(xz)\]

appeared in the algebra of elementary embeddings arising under (very) large cardinal hypotheses (R. Laver). Also, P. Dehornoy found its deep connection with infinite braid groups. (See [5].) It is not hard to check that the conjunction of (6) with the identity

\[(7) \quad x(xx) = (xx)x\]

implies both conditions of the theorem, with
\[r(x) = x(xx), \quad s(x, y) = xy, \quad t(x, y, z) = yz.\]

Hence, any compact left topological groupoid satisfying (6) and (7) has an idempotent.

(The identity (7) is necessary; left self-distributivity alone implies only finiteness of minimal groupoids.)
Our non-associative theorem can be interesting from the following point:

**Question.** Which formulas (or, at least, which identities) are stable under passing to the algebra of ultrafilters?

**Conjecture.** Identities that follow from associativity are stable under passing to the algebra of ultrafilters.

E.g. so are the identities (1)–(4) above.

**Observation.** If some identities, on the one hand, imply the conditions of the theorem, and, on the other hand, are stable under passing to ultrafilters, then the technique based on ultrafilters and developed for semigroups can be extended to groupoids satisfying these identities. E.g. one can prove for such groupoids (appropriate analogs of) Hindman’s theorem, van der Waerden’s theorem, etc.
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