Gaussian fluctuation for spatial average of super-Brownian motion

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ABSTRACT
Let \( f(t, x) \) be the density of one-dimensional super-Brownian motion starting from Lebesgue measure. Using the Laplace functional of super-Brownian motion, we prove that as \( N \to \infty \), the normalized spatial integral \( \frac{N}{C_0} \int_0^t f(u(t, x)) \, dx \) converges jointly in \((t, x)\) to Brownian sheet in distribution.

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1. Introduction and main result

Let \( X_t(dx) \) be the one-dimensional super-Brownian motion starting from Lebesgue measure. It has a density with respect to Lebesgue measure, that is, \( X_t(dx) = u(t, x)dx \) where almost surely \((t, x) \mapsto u(t, x)\) is jointly continuous on \((0, \infty) \times \mathbb{R}\) and satisfies the following stochastic partial differential equation (see Konno and Shiga [1])

\[
\partial_t u(t, x) = \frac{1}{2} \partial_x^2 u(t, x) + \sqrt{u(t, x)} \, \hat{W}(t, x), \quad t > 0, x \in \mathbb{R}
\]

where \( \hat{W} \) denotes the space-time white noise. The solution to the above stochastic heat equation is understood in the weak sense, that is, for any \( t > 0 \) and \( f \in C_c^\infty(\mathbb{R}) \) (collection of smooth functions with compact support), almost surely, we have

\[
\int_{\mathbb{R}} f(x) u(t, x) \, dx = \int_{\mathbb{R}} f(x) \, dx + \int_0^t \int_{\mathbb{R}} f(x) \sqrt{u(s, x)} \, W(ds, dx) + \int_0^t ds \int_{\mathbb{R}} \frac{1}{2} f''(x) u(s, x) \, dx,
\]

where \( W(ds, dx) \) denotes the stochastic integral with respect to space-time white noise (see Walsh [2]).

The goal of this paper is to establish the following central limit theorem for the solution to (1.1).
**Theorem 1.1.** Let \( \{u(t,x)\}_{(t,x) \in \mathbb{R}_+ \times \mathbb{R}} \) be the solution to (1.1) with \( u(0,x) \equiv 1 \). Then as \( N \to \infty \),

\[
\left\{ \frac{1}{\sqrt{N}} \int_0^{\tau N} [u(t,z) - 1] \, dz \right\} \xrightarrow{C(0,1)} \left\{ W(t,x) : (t,x) \in [0,1]^2 \right\},
\]

where \( W \) denotes the Brownian sheet and \( "C(0,1)" \) denotes the convergence in distribution in the space of continuous functions \( C([0,1]^2) \).

**Theorem 1.1** is motivated by the recent progress on the central limit theorem for stochastic partial differential equations. For example, let \( \{U(t,x)\}_{(t,x) \in \mathbb{R}_+ \times \mathbb{R}} \) be the solution to parabolic Anderson model subject to \( U(0) \equiv 1 \), driven by space-time white noise. Then according to Huang et al. [3, Theorem 1.2], for fixed \( x \in [0,1] \), as \( N \to \infty \),

\[
\left\{ \frac{1}{\sqrt{N}} \int_0^{\tau N} [U(t,z) - 1] \, dz \right\} \xrightarrow{C[0,1]} \left\{ \sqrt{\frac{x}{t}} \int_0^t \sqrt{E[U(s,0)^2]} \, dB_s \right\}_{t \in [0,1]},
\]

where \( B \) denotes the standard Brownian motion and \( "C[0,1]" \) denotes the convergence in law in the space of continuous functions \( C([0,1]) \). On the other hand, Chen et al [4, Theorem 2.3] have proved that for fixed \( t \in [0,1] \), as \( N \to \infty \),

\[
\left\{ \frac{1}{\sqrt{N}} \int_0^{\tau N} [U(t,z) - 1] \, dz \right\} \xrightarrow{C[tB_x : x \in [0,1]]} \{C_tB_x : x \in [0,1]\},
\]

where \( C_t^2 = \int_0^t E[U(s,0)^2] \, ds \). One might expect that as a process in \((t,x)\), the above normalized integral converges jointly to a two-parameter Gaussian process in distribution in the space \( C([0,1]^2) \) as \( N \to \infty \); see [4, Remark 2.5(2)]. Our **Theorem 1.1** provides such a result for the solution to stochastic heat equation (1.1).

Recently, there has been a lot of study on the Gaussian fluctuation for spatial average of SPDEs, initiated in [3] for stochastic heat equation driven by space-time white noise in dimension one, and later on developed in [4–6] with different driving Gaussian noises in higher dimensions. Notice that the same topic has been considered for parabolic Anderson model with rough initial data and/or rough noise in [7–9]. Moreover, the same type of problem has been investigated for stochastic wave equation in [10–14].

The SPDEs considered in the proceeding have Lipschitz continuous diffusion coefficient, where we can perform Malliavin calculus with the solution and combine with Stein’s method to obtain the CLT. For the stochastic heat equation (1.1) associated to super-Brownian motion, since the diffusion coefficient is not Lipschitz continuous, it is not clear if we can use Malliavin-Stein approach as in [3] to study the CLT for the solution. In the context of super-Brownian motion, one can make use of the Laplace functional to study the asymptotic behaviors of the process \( X_t \) as \( t \to \infty \) (see for example [15,16]). In fact, we will also appeal to the Laplace functional of super-Brownian to prove the central limit theorem for the spatial average of the solution to (1.1).

More details on the Laplace functional of super-Brownian motion will be presented in Section 2. In Sections 3 and 4, we will establish the convergence of finite dimensional distributions and the tightness respectively. Finally, we prove **Theorem 1.1** in Section 5.
2. Preliminaries

We write \( \langle \nu, f \rangle := \int_{\mathbb{R}} f(x) \nu(dx) \) and \( \langle g, f \rangle := \int_{\mathbb{R}} g(x)f(x)dx \) for a measure \( \nu \) and functions \( f \) and \( g \). Denote by \( \mathcal{M}_b(\mathbb{R})^+ \) the collection of nonnegative, bounded and measurable functions. Let \( \mathcal{M}(\mathbb{R}) \) be the collection of Radon measures on \( \mathbb{R} \). Set \( \mathcal{M}_2(\mathbb{R}) = \{ \mu \in \mathcal{M}(\mathbb{R}) : \int_{\mathbb{R}} (1 + x^2)^{-1} \mu(dx) < \infty \} \). Recall that the one dimensional super-Brownian motion \( X_t(x) \) is a measure-valued branching Markov process taking values in \( \mathcal{M}_2(\mathbb{R}) \) such that

\[
E_\mu[e^{-(X_t,f)}] = e^{-(\mu, V_t(f))}, \quad \text{for } f \in \mathcal{M}_b(\mathbb{R})^+.
\] (2.1)

where the notation \( E_\mu \) denotes the super-Brownian motion starting from a finite measure \( \mu \), and the function \((t,x) \mapsto V_t(f)(x)\) is the unique locally bounded and nonnegative solution to the following nonlinear partial differential equation

\[
\begin{align*}
\partial_t V_t(f)(x) &= \frac{1}{2} \partial_x^2 V_t(f)(x) - \frac{1}{2} [V_t(f)(x)]^2, \\
V_0(f) &= f.
\end{align*}
\] (2.2)

The solution to the above nonlinear PDE satisfies the following integral equation

\[
V_t(f)(x) = P_t f(x) - \frac{1}{2} \int_0^t P_{t-s} [V_s(f)]^2(x)ds,
\] (2.3)

where \( P_t f(x) := \int_{\mathbb{R}} p_t(x-y)f(y)dy \) with \( p_t(x) := (2\pi t)^{-1/2} e^{-x^2/(2t)} \) for \( t > 0 \) and \( x \in \mathbb{R} \).

Denote by \( \mathcal{F}_t \) the \( \sigma \)-algebra generated by super-Brownian motion up to time \( t \). The Markov property of super-Brownian motion states that for \( t \geq s \) and \( f \in \mathcal{M}_b(\mathbb{R})^+ \),

\[
E_\mu[e^{-(X_t,f)} | \mathcal{F}_s] = E_{X_s}[e^{-(X_t,f)}] = e^{-(X_t,V_t-s)}.
\] (2.4)

where the second equality holds by (2.1). Furthermore, according to [17, p.54] (see also [18, Lemma 2.1]), we have the following formulas for the moments of super-Brownian motion: for \( f \in \mathcal{M}_b(\mathbb{R})^+ \),

\[
E_\mu[\langle X_t, f \rangle] = \langle \mu, P_t f \rangle,
\] (2.5)

\[
E_\mu[\langle X_t, f \rangle^2] = \langle \mu, P_t f \rangle^2 + \int_0^t \langle \mu, P_{t-s}(P_s f)^2 \rangle ds.
\] (2.6)

We refer to [17–21] for more information on super-Brownian motion.

The above facts on the Laplace functional and moments are also true for super-Brownian motion starting from Lebesgue measure (denoted by \( \lambda \)); see, for example, [15, Theorem 1.1] and [1, Theorem 1.4]. In fact, we can decompose the Lebesgue measure as \( \lambda = \sum_{i=1}^{\infty} \lambda_i \), where \( \lambda_i \)'s are finite measures on \( \mathbb{R} \). By the branching property of super-Brownian motion, the process \( X_t \) starting from \( \lambda \) is the sum of independent copies of the process starting from \( \lambda_i \). Since each of the summand process satisfies the identities (2.1), (2.5) and (2.6), by independence, it implies that \( X_t \) starting from \( \lambda \) also satisfies theses properties. Denote \( L^b_b(\mathbb{R})^+ = L^1(\mathbb{R}) \cap \mathcal{M}_b(\mathbb{R})^+ \). Then, we have for \( f \in L^b_b(\mathbb{R})^+ \),

\[
E_\lambda[\langle X_t, f \rangle] = \langle \lambda, f \rangle,
\] (2.7)
The following technical lemma will be used later on.

**Lemma 2.1.** For all $t > 0$, $r_1, r_2 \geq 0$ and $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$,

$$
\lim_{N \to \infty} \int_0^t ds \int_{\mathbb{R}} dy \ P_{s+r_1} f^{(N)}(y) P_{s+r_2} g^{(N)}(y) = t \cdot \langle f, g \rangle,
$$

and

$$
\sup_{N > 0} \int_0^t ds \int_{\mathbb{R}} dy \left[ P_{s+r_1} f^{(N)}(y) \right]^2 \leq t \cdot \langle f, f \rangle.
$$

**Proof.** By the semigroup property of heat kernel, we write

$$
\int_{\mathbb{R}} dy \ P_{s+r_1} f^{(N)}(y) P_{s+r_2} g^{(N)}(y) \\
= \int_{\mathbb{R}} dy \int_{\mathbb{R}} p_{s+r_1}(y - z_1) f^{(N)}(z_1) dz_1 \int_{\mathbb{R}} p_{s+r_2}(y - z_2) g^{(N)}(z_2) dz_2 \\
= \int_{\mathbb{R}^2} p_{2s+r_1+r_2}(z_1 - z_2) f^{(N)}(z_1) g^{(N)}(z_2) dz_1 dz_2 \\
= \langle p_{2s+r_1+r_2} \ast f^{(N)}, g^{(N)} \rangle_{L^2(\mathbb{R})}.
$$

Appealing to the Plancherel's identity,
\begin{equation}
\langle p_{2x+r_1+r_2} f^{(N)} \rangle_{L^2(\mathbb{R})} = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\chi^2 / 2} \tilde{f}(N)(z) \tilde{g}(N)(z) \, dz
= \frac{N}{2\pi} \int_{\mathbb{R}} e^{-\chi^2 / 2} \tilde{f}(N) \tilde{g}(N) \, dz.
\end{equation}

Therefore, we obtain that
\begin{equation}
\int_{0}^{t} \int_{\mathbb{R}} dy \, P_{x+r_1} f^{(N)} (y) P_{x+r_1} g^{(N)} (y) = \frac{1}{2\pi} \int_{0}^{t} \int_{\mathbb{R}} e^{-\chi^2 / 2} \tilde{f}(N) \tilde{g}(N) \, dz ds,
\end{equation}
which implies (2.12) by dominated convergence theorem. Letting \( f = g \) and \( r_1 = r_2 \) in the proceeding display, we obtain (2.13). \( \square \)

Let us close this section with a brief description of some other notation of this paper. Throughout we write \( g_1(x) \leq g_2(x) \) for all \( x \in \mathbb{R} \) when there exists a real number \( L \) such that \( g_1(x) \leq L g_2(x) \) for all \( x \in \mathbb{R} \). Alternatively, we might write \( g_2(x) \geq g_1(x) \) for all \( x \in \mathbb{R} \). By \( g_1(x) \leq g_2(x) \) for all \( x \in \mathbb{R} \) we mean that \( g_1(x) \leq g_2(x) \) for all \( x \in \mathbb{R} \) and \( g_2(x) \leq g_1(x) \) for all \( x \in \mathbb{R} \). We denote \( \|g\|_{\infty} := \sup_{x \in \mathbb{R}} |g(x)| \) for a bounded function \( g \). Finally, we write \( \|X\|_k = (\mathbb{E}[|X|^k])^{1/k} \) for a random variable \( X \) and \( k \geq 1 \).

### 3. Convergence of finite-dimensional distributions

Denote \( \{B_t(f) : t \geq 0, f \in L^2(\mathbb{R})\} \) as the cylindrical Brownian motion, which is a centered Gaussian process such that
\[ \mathbb{E}[B_t(f)B_s(g)] = (t \wedge s) \cdot \langle f, g \rangle, \quad \text{for } t, s \geq 0 \text{ and } f, g \in L^2(\mathbb{R}). \]

We have the following result on the convergence of Laplace transform of super-Brownian motion.

**Proposition 3.1.** For \( 0 < t_m < \ldots < t_1 \) and \( f_1, \ldots, f_m \in L^1_b(\mathbb{R}), \)
\[ \lim_{N \to \infty} \mathbb{E}_x \left[ e^{-\sum_{k=1}^{m} (X_{t_k} f_k^{(N)})} - \mathbb{E}\left[ \langle X_{t_m} f_m^{(N)} \rangle \right] \right] = \mathbb{E}_x \left[ e^{-\sum_{k=1}^{m} B_k(f_k)} \right], \]
where the functions \( f_k^{(N)}, k = 1, \ldots, m, \) are defined as in (2.10).

**Proof.** By the Markov property (2.4), we write
\[ \mathbb{E}_x \left[ e^{-\sum_{k=1}^{m} (X_{t_k} f_k^{(N)})} \right] = \mathbb{E}_x \left[ e^{-\sum_{k=1}^{m} (X_{t_k} f_k^{(N)})} | \mathscr{F}_{t_1} \right] \]
\[ = \mathbb{E}_x \left[ e^{-\sum_{k=1}^{m} (X_{t_k} f_k^{(N)})} \mathbb{E}_x [e^{-\sum_{k=1}^{m} (X_{t_k} f_k^{(N)})} | \mathscr{F}_{t_1} ] \right] \]
\[ = \mathbb{E}_x \left[ e^{-\sum_{k=1}^{m} (X_{t_k} f_k^{(N)})} e^{-\langle X_{t_1} \rangle} \right] \]
\[ = \mathbb{E}_x \left[ e^{-\sum_{k=1}^{m} (X_{t_k} f_k^{(N)})} e^{-\langle X_{t_1} \rangle} \right] . \]

We can repeat the above procedure to obtain that
\[ \mathbb{E}_x \left[ e^{-\sum_{k=1}^{m} (X_{t_k} f_k^{(N)})} \right] = e^{-\langle \lambda, V_{t_m}(F_m) \rangle}, \]
where the function \( F_m \) is defined as...
\[
\begin{align*}
F_1 &= f_1^{(N)}, \\
F_k &= f_k^{(N)} + V_{t_k-t_{k-1}}(F_{k-1}), \quad \text{for } k = 2, \ldots, m.
\end{align*}
\] (3.3)

Since \( V_t(f)(x) \leq P_t f(x) \) for all \( x \in \mathbb{R} \) and nonnegative and bounded function \( f \),

\[
\|F_k\|_\infty \leq \|f_k^{(N)}\|_\infty + \|P_{t_k-t_{k-1}}F_{k-1}\|_\infty \\
\leq \|f_k^{(N)}\|_\infty + \|F_{k-1}\|_\infty,
\]

which implies that

\[
\|F_k\|_\infty \leq \sum_{j=1}^k \|f_j^{(N)}\|_\infty \leq \frac{1}{\sqrt{N}} \sum_{j=1}^k \|f_j\|_\infty,
\] (3.4)

Moreover,

\[
\langle \lambda, F_k \rangle \leq \langle \lambda, f_k^{(N)} \rangle + \langle \lambda, P_{t_k-t_{k-1}}F_{k-1} \rangle \\
= \langle \lambda, f_k^{(N)} \rangle + \langle \lambda, F_{k-1} \rangle,
\]

which implies that

\[
\langle \lambda, F_k \rangle \leq \sum_{j=1}^k \langle \lambda, f_j^{(N)} \rangle = \sqrt{N} \sum_{j=1}^k \langle \lambda, f_j \rangle.
\] (3.5)

Set \( t_{m+1} = 0 \). Using (2.9) and (3.3),

\[
\langle \lambda, V_{t_{m+1}}(F_m) \rangle = \langle \lambda, F_m \rangle - \frac{1}{2} \int_0^{t_{m+1}} \int_\mathbb{R} \left[ V_s(F_m)(y) \right]^2 dy ds \\
= \langle \lambda, f_m^{(N)} \rangle + \langle \lambda, V_{t_{m+1}}(F_{m-1}) \rangle - \frac{1}{2} \int_0^{t_{m+1}} \int_\mathbb{R} \left[ V_s(F_m)(y) \right]^2 dy ds \\
\vdots \\
= \sum_{k=1}^m \langle \lambda, f_k^{(N)} \rangle - \sum_{k=1}^m \frac{1}{2} \int_0^{t_{k-1}} \int_\mathbb{R} \left[ V_s(F_k)(y) \right]^2 dy ds.
\] (3.6)

Denote

\[
I_k^{(N)} = \int_0^{t_{k-1}} \int_\mathbb{R} \left[ V_s(F_k)(y) \right]^2 dy ds.
\] (3.7)

Hence, we see from (3.2), (3.6) and (2.7) that

\[
E_h \left[ e^{-\sum_{k=1}^m \langle X_k, f_k^{(N)} \rangle - E \langle X_k, f_k^{(N)} \rangle} \right] = \exp \left\{ \frac{1}{2} \sum_{k=1}^m I_k^{(N)} \right\}.
\] (3.8)

We next estimate \( I_k^{(N)} \). We apply the formula (2.3) to write

\[
I_k^{(N)} = \int_0^{t_{k-1}} ds \int_\mathbb{R} dy \left[ P_s F_k(y) - \frac{1}{2} \int_0^{t_s} P_{s-r}[V_s F_k]^2(y) dr \right]^2 \\
= I_{k,1}^{(N)} + I_{k,2}^{(N)} + I_{k,3}^{(N)},
\] (3.9)

where
By (3.4) and (3.5),
\[
\left| I_{k,2}^{(N)} \right| \leq \int_0^{t_k - t_{k+1}} ds \int_\mathbb{R} dy \ P_s F_k(y) \left\| V_r F_k \right\|_\infty^2 dr \\
\leq \int_0^{t_k - t_{k+1}} ds \int_\mathbb{R} dy \ P_s F_k(y) \left\| F_k \right\|_\infty^{\frac{3}{2}} \left( \sum_{j=1}^{k} \langle \lambda, f_j \rangle \left( \sum_{j=1}^{k} \left\| f_j \right\|_\infty \right)^2 \right)^{\frac{1}{2}} \\
\leq \frac{1}{\sqrt{N}} \frac{(t_k - t_{k+1})^2}{2} \sum_{j=1}^{k} \langle \lambda, f_j \rangle \left( \sum_{j=1}^{k} \left\| f_j \right\|_\infty \right)^2.
\]
Similarly,
\[
I_{k,3}^{(N)} \leq \frac{t_k - t_{k+1}}{4} \left\| F_k \right\|_\infty^{\frac{3}{2}} \int_0^{t_k - t_{k+1}} ds \int_\mathbb{R} dy \ P_{s-r} V_r F_k(y) dr \\
\leq \frac{t_k - t_{k+1}}{4} \left\| F_k \right\|_\infty^{\frac{3}{2}} \int_0^{t_k - t_{k+1}} ds \int_\mathbb{R} dy \ P_{s-r} F_k(y) dr \\
= \left\| F_k \right\|_\infty^{\frac{3}{2}} \frac{(t_k - t_{k+1})^3}{8} \langle \lambda, F_k \rangle \\
\leq \frac{1}{N} \frac{(t_k - t_{k+1})^3}{8} \sum_{j=1}^{k} \langle \lambda, f_j \rangle \left( \sum_{j=1}^{k} \left\| f_j \right\|_\infty \right)^3.
\]
The estimates in (3.10) and (3.11) yield that as \( N \to \infty \),
\[
I_k^{(N)} = I_{k,1}^{(N)} + o(1).
\]
Furthermore, appealing to the identity (3.3),
\[
I_{k,1}^{(N)} = \int_0^{t_k - t_{k+1}} ds \int_\mathbb{R} dy \ \left[ P_s y_{k}^{(N)}(y) + P_s V_{t_k-t_k} (F_k-1)(y) \right]^2 \\
= I_{k,1,1}^{(N)} + I_{k,1,2}^{(N)} + I_{k,1,3}^{(N)},
\]
where
\[
I_{k,1,1}^{(N)} = \int_0^{t_k - t_{k+1}} ds \int_\mathbb{R} dy \ \left[ P_s y_{k}^{(N)}(y) \right]^2, \\
I_{k,1,2}^{(N)} = 2 \int_0^{t_k - t_{k+1}} ds \int_\mathbb{R} dy \ P_s y_{k}^{(N)}(y) P_s V_{t_k-t_k} (F_k-1)(y), \\
I_{k,1,3}^{(N)} = \int_0^{t_k - t_{k+1}} ds \int_\mathbb{R} dy \ \left[ P_s V_{t_k-t_k} (F_k-1)(y) \right]^2.
\]
By Lemma 2.1, as \( N \to \infty \),
\[
I_{k,1,1}^{(N)} = (t_k - t_{k+1}) \cdot \langle f_k, f_k \rangle + o(1). \tag{3.14}
\]
Moreover, we use (2.3) to write
\[
I_{k,1,2}^{(N)} = 2 \int_0^{t_k - t_{k+1}} ds \int_\mathbb{R} dy \, P_{f_k}^{(N)}(y) P_{t_k - t_{k+1}(F_{k-1})}(y) \]
\[
- \int_0^{t_k - t_{k+1}} ds \int_\mathbb{R} dy \, P_{f_k}^{(N)}(y) \int_0^{t_k - t_k} P_{s, t_k - t_{k+1} - r} [V_r(F_{k-1})]^2(y) dr \]
\[
= 2 \int_0^{t_k - t_{k+1}} ds \int_\mathbb{R} dy \, P_{f_k}^{(N)}(y) P_{t_k - t_{k+1}(F_{k-1})}(y) + o(1), \quad \text{as } N \to \infty,
\]
where the second equality holds by (3.4) and (3.5). Using the formula (3.3), we have as \( N \to \infty \),
\[
I_{k,1,2}^{(N)} = 2 \int_0^{t_k - t_{k+1}} ds \int_\mathbb{R} dy \, P_{f_k}^{(N)}(y) P_{t_k - t_{k+1}(f_{k-1})}(y) \]
\[
+ 2 \int_0^{t_k - t_{k+1}} ds \int_\mathbb{R} dy \, P_{f_k}^{(N)}(y) P_{t_k - t_{k+1}(V_{t_k - t_{k+1}}(F_{k-2}))}(y) + o(1) \]
\[
= 2(t_k - t_{k+1}) \cdot \langle f_k, f_{k-1} \rangle + 2 \int_0^{t_k - t_{k+1}} ds \int_\mathbb{R} dy \, P_{f_k}^{(N)}(y) P_{t_k - t_{k+1}(V_{t_k - t_{k+1}}(F_{k-2}))}(y) \]
\[
+ o(1),
\]
thanks to Lemma 2.1. Since the estimates in (3.4) and (3.5) ensure that the dominated term of \( V_{t_k - t_{k+1}}(F_{k-2}) \) is \( P_{t_k - t_{k+1}}(F_{k-2}) \), we can repeat the proceeding argument and apply Lemma 2.1 to conclude that as \( N \to \infty \),
\[
I_{k,1,2}^{(N)} = 2(t_k - t_{k+1}) \sum_{j=1}^{k-1} \langle f_k, f_j \rangle + o(1). \tag{3.15}
\]
As for \( I_{k,1,3}^{(N)} \), we can use the same argument in (3.10) and (3.11) to see that as \( N \to \infty \),
\[
I_{k,1,3}^{(N)} = \int_0^{t_k - t_{k+1}} ds \int_\mathbb{R} dy \, [P_{t_k - t_{k+1}} F_{k-1}(y)]^2 + o(1). \tag{3.16}
\]
Therefore, we combine (3.12)-(3.15) to obtain that as \( N \to \infty \),
\[
I_k^{(N)} = (t_k - t_{k+1}) \left( \langle f_k, f_k \rangle + 2 \sum_{i=1}^{k-1} \langle f_k, f_i \rangle \right) + o(1) \tag{3.17}
\]
\[
+ \int_0^{t_k - t_{k+1}} ds \int_\mathbb{R} dy \, [P_{t_k - t_{k+1}} F_{k-1}(y)]^2.
\]
The estimate of the above double integral is similar to that of \( I_{k,1}^{(N)} \), with \( F_k \) replaced by \( F_{k-1} \). We proceed as follows. Using again the formula (3.3), we obtain
Therefore, as $N \to \infty$ by Lemma 2.1 and (3.4), (3.5). Similar to the estimate of $I_{k,1,2}^{(N)}$ in (3.15), we have as $N \to \infty$,

$$I_{k,2}^{(N)} = 2(t_k - t_{k+1}) \sum_{i=1}^{k-2} \langle f_{k-1} , f_i \rangle + o(1).$$

Now we see from (3.17)-(3.21) that as $N \to \infty$

$$I_k^{(N)} = (t_k - t_{k+1}) \sum_{j=1}^{k} \left( \langle f_j , f_j \rangle + 2 \sum_{i=1}^{j-1} \langle f_j , f_i \rangle \right) + o(1)
+ \int_0^{t_k-t_{k+1}} ds \int \ dy \left[ P_{t_k-t_s} F_{k-2}(y) \right]^2.

We can repeat the above argument to conclude that as $N \to \infty$,

$$I_k^{(N)} = (t_k - t_{k+1}) \sum_{j=1}^{k} \left( \langle f_j , f_j \rangle + 2 \sum_{i=1}^{j-1} \langle f_j , f_i \rangle \right) + o(1).$$

Therefore, as $N \to \infty$,
\[
\sum_{k=1}^{m} t_k^{(N)} = \sum_{k=1}^{m} (t_k - t_{k+1}) \sum_{j=1}^{k} \left( \langle f_j, f_j \rangle + 2 \sum_{i=1}^{j-1} \langle f_j, f_i \rangle \right) + o(1)
\]
\[
= \sum_{j=1}^{m} \left( \langle f_j, f_j \rangle + 2 \sum_{i=1}^{j-1} \langle f_j, f_i \rangle \right) \sum_{k=j}^{m} (t_k - t_{k+1}) + o(1)
\]
\[
= \sum_{j=1}^{m} t_j \left( \langle f_j, f_j \rangle + 2 \sum_{i=1}^{j-1} \langle f_j, f_i \rangle \right) + o(1),
\]
\[(3.22)\]

which together with (3.8) implies (3.1). The proof is complete.

**Remark 3.2.** The result in Proposition 3.1 also holds for \(d\)-dimensional super-Brownian motion starting from Lebesgue measure on \(\mathbb{R}^d\), with the scaled function in (2.10) defined as
\[
\bar{f}(x) = \frac{N^d}{C_0} f\left(\frac{x}{N}\right),
\]
\(x \in \mathbb{R}^d\) \((f \in L^1_0(\mathbb{R}^d))\) and the limit process replaced by the cylindrical Brownian motion \(B_t(f) : t \geq 0, f \in L^2(\mathbb{R}^d)\) on \(\mathbb{R}^d\). The proof follows along the same lines as in Proposition 3.1 using an analogue of Lemma 2.1.

### 4. Tightness

We first recall from [1, Lemma 2.5] that for \(t > 0\) and \(\phi \in C_c^\infty(\mathbb{R})\), almost surely,
\[
\langle X_t, \phi \rangle - \langle \lambda, P_t \phi \rangle = \int_0^t \int_\mathbb{R} \sqrt{u(r, z)} P_{t-r} \phi(z) W(dr \, dz).
\]
\[(4.1)\]

Using stochastic Fubini’s theorem, it implies that for all \(t > 0\) and \(x \in \mathbb{R}\), almost surely,
\[
u(t, x) = 1 + \int_0^t \int_\mathbb{R} P_{t-s}(x-y) \sqrt{u(s, y)} W(ds \, dy).
\]
\[(4.2)\]

**Lemma 4.1.** Let \(\nu(0, \cdot) \equiv 1\). For all \(T \geq 0\) and \(k \geq 2\), there exists a constant \(C > 0\) such that for all \((t, x), (s, y) \in [0, T] \times \mathbb{R}\)
\[
E \left[ |\nu(t, x) - \nu(s, y)|^k \right] \leq C \left( |t-s|^{k/4} + |x-y|^{k/2} \right).
\]
\[(4.3)\]

As a consequence, the solution to (1.1) subject to \(\nu(0, \cdot) \equiv 1\) admits a continuous version on \(\mathbb{R}_+ \times \mathbb{R}\).

**Proof.** The estimate of (4.3) is standard using the mild form (4.2) and the properties of heat kernel (see [1]). We present the details here for the particular choice of initial data \(\nu(0, \cdot) \equiv 1\). First, we use (4.2) and Burkholder’s inequality to conclude by induction that for all \(k \geq 2\),
\[
\sup_{t \in [0, T] \times \mathbb{R}} E \left[ \nu(t, x)^k \right] < \infty.
\]
\[(4.4)\]

Without loss of generality, we assume \(t \geq s \geq 0\). Then by Burkholder’s inequality,
\[ E \left[ |u(t, x) - u(s, y)|^k \right] \leq E \left[ \int_s^t \left( \int_{\mathbb{R}} p_{t-r}(x - z) \sqrt{u(r, z)} \ W(dr \ dz) \right)^k \right] \]
\[ + E \left[ \int_0^s \left( \int_{\mathbb{R}} (p_{t-r}(x - z) - p_{s-r}(y - z)) \sqrt{u(r, z)} \ W(dr \ dz) \right)^k \right] \]
\[ \leq E \left[ \int_s^t \left( p_{t-r}^2(x - z) u(r, z) \ dz \ dr \right)^{k/2} \right] \]
\[ + E \left[ \int_0^s \left( \int_{\mathbb{R}} (p_{t-r}(x - z) - p_{s-r}(y - z))^2 u(r, z) \ dz \ dr \right)^{k/2} \right] \]
\[ = \left( \int_s^t \int_{\mathbb{R}} p_{t-r}^2(x - z) u(r, z) \ dz \ dr \right)^{k/2} \]
\[ \leq (t - s)^{k/4}, \]

Using Minkowski’s inequality and (4.4),
\[ E \left[ \int_s^t \int_{\mathbb{R}} p_{t-r}^2(x - z) u(r, z) \ dz \ dr \right] \leq \left( \int_s^t \int_{\mathbb{R}} p_{t-r}^2(x - z) u(r, z) \ dz \ dr \right)^{k/2} \]
\[ \leq \left( \int_s^t \int_{\mathbb{R}} p_{t-r}^2(x - z) u(r, z) \ dz \ dr \right)^{k/2} \]
\[ \leq (t - s)^{k/4}, \]

thanks to the semigroup property of heat kernel. Similarly,
\[ E \left[ \int_0^s \int_{\mathbb{R}} (p_{t-r}(x - z) - p_{s-r}(y - z))^2 u(r, z) \ dz \ dr \right] \]
\[ \leq \left( \int_0^s \int_{\mathbb{R}} (p_{t-r}(x - z) - p_{s-r}(y - z))^2 u(r, z) \ dz \ dr \right)^{k/2} \]
\[ \leq \left( \int_0^s \int_{\mathbb{R}} (p_{t-r}(x - z) - p_{s-r}(y - z))^2 \ dz \ dr \right)^{k/2} \]
\[ \leq |t - s|^{k/4} + |x - y|^{k/2}, \]

where the last inequality follows from [22, Proposition 5.2]. Therefore, we combine (4.6) and (4.7) to obtain (4.3). Finally, by Kolmogorov continuity theorem, \( \{u(t, x) : (t, x) \in \mathbb{R}_+ \times \mathbb{R}\} \) has a continuous version on \( \mathbb{R}_+ \times \mathbb{R} \). \( \square \)

Recall the process \( V_N \) defined in (2.11). It is clear from Lemma 4.1 that for each \( N \geq 1 \), the process \( \{V_N(t, x) : (t, x) \in [0, 1]^2\} \) is continuous on \( [0, 1]^2 \). Moreover, we have the following moment estimates on the space-time difference of \( V_N \).

**Proposition 4.2.** There exists a constant \( C > 0 \) such that for all \( N \geq 1 \) and \( (t, x), (s, y) \in [0, 1]^2 \),
\[ E_k \left[ |V_N(t, x) + V_N(s, y) - V_N(t, y) - V_N(s, x)|^4 \right] \leq C(|t - s| \times |x - y|)^{5/4}. \] \( (4.8) \)
**Proof.** Assume $t \geq s$ and $x \geq y$ without loss of generality. According to (2.11), we write

$$V_N(t, x) + V_N(s, y) - V_N(t, y) - V_N(s, x)$$

$$= \int_s^t \int_{\mathbb{R}} \sqrt{u(r, z)} P_{t-r} 1_{[y, x]}(z) W(dr \, dz)$$

$$+ \int_0^s \int_{\mathbb{R}} \sqrt{u(r, z)} \left( P_{t-r} 1_{[y, x]} - P_{s-r} 1_{[y, x]} \right)(z) W(dr \, dz).$$

Hence,

$$E_x \left[ |V_N(t, x) + V_N(s, y) - V_N(t, y) - V_N(s, x)|^4 \right] \leq 8(A + B), \quad (4.9)$$

where

$$A = E_x \left[ \int_s^t \left( \frac{\sqrt{u(r, z)} P_{t-r} 1_{[y, x]}(z) W(dr \, dz)}{C_{20}} \right)^4 \right],$$

$$B = E_x \left[ \int_0^s \left( \frac{\sqrt{u(r, z)} \left( P_{t-r} 1_{[y, x]} - P_{s-r} 1_{[y, x]} \right)(z) W(dr \, dz)}{C_{21}} \right)^4 \right].$$

Applying the Burkholder’s inequality and the Cauchy-Schwarz inequality, we have

$$A \leq 4E_x \left[ \int_s^t \langle X_r, \left( P_{t-r} 1_{[y, x]} \right)^2 \rangle dr \right]^2$$

$$\leq 4(t-s) \int_s^t \left( \langle X_r, \left( P_{t-r} 1_{[y, x]} \right)^2 \rangle \right)^2 dr \quad (4.10)$$

$$= 4(t-s) \int_s^t \left( \langle \lambda, \left( P_{t-r} 1_{[y, x]} \right)^2 \rangle + \int_0^r \langle \lambda, \left( P_{r-t} 1_{[y, x]} \right)^2 \rangle \right) d\theta, \quad (4.11)$$

where the equality follows from (2.8).

Plancherel’s identity ensures that for all $F \in L^2(\mathbb{R})$ and $\theta > 0$,

$$\langle \lambda, (P_\theta F)^2 \rangle \leq \langle \lambda, F^2 \rangle. \quad (4.12)$$

Hence,

$$\langle \lambda, \left( P_{t-r} 1_{[y, x]} \right)^2 \rangle \leq \langle \lambda, \left( 1_{[y, x]} \right)^2 \rangle = |x - y|, \quad (4.13)$$

which implies that

$$(t-s) \int_s^t \langle \lambda, \left( P_{t-r} 1_{[y, x]} \right)^2 \rangle \leq |t-s|^2 |x - y|^2. \quad (4.14)$$

Denote $F = \left( P_{t-r} 1_{[y, x]} \right)^2$. By the Plancherel’s identity (see the calculation in (2.14)),

$$\langle \lambda, \left( P_\theta \left( P_{t-r} 1_{[y, x]} \right)^2 \right)^2 \rangle = \langle \lambda, (P_\theta F)^2 \rangle = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\theta z^2} |\tilde{F}(z)|^2 dz.$$

Note that for all $z \in \mathbb{R}$

$$|\tilde{F}(z)| \leq \langle \lambda, F \rangle = \langle \lambda, \left( P_{t-r} 1_{[y, x]} \right)^2 \rangle \leq |x - y|,$$
where the last inequality holds by (4.12). The proceeding yields that

\[
(t - s) \int_s^t \int_0^r \left( P_0 \left[ P_{t-r} 1_{[y,x]}^{(N)} \right]^2 \right) d\theta d\tau \leq \frac{(t - s)|x - y|^2}{2\pi} \int_s^t \int_0^r e^{-\theta \tau^2} d\tau d\theta \]

(4.14)

\[
\simeq (t - s)|x - y|^2 \int_s^t \sqrt{\tau} d\tau \leq |t - s|^2 |x - y|^2.
\]

Hence, we conclude from (4.10), (4.13) and (4.14) that

\[
\mathcal{A} \leq |t - s|^2 |x - y|^2.
\]

We proceed to estimate \( B \). Denote

\[
G = \left( P_{t-s+r} 1_{[y,x]}^{(N)} - P_{r} 1_{[y,x]}^{(N)} \right)^2.
\]

Again, applying the Burkholder’s inequality and the Cauchy-Schwarz inequality,

\[
B \leq 4E_{\lambda} \left[ \int_0^r \left( \int_0^r \langle X_\tau, G \rangle d\tau \right)^2 \right] \leq 4s \int_0^r E_{\lambda} \left[ \langle X_\tau, G \rangle^2 \right] d\tau
\]

\[
= 4s \int_0^r d\tau \left( \langle \lambda, G \rangle^2 + \int_0^r \langle \lambda, (P_0 G)^2 \rangle d\theta \right)
\]

(4.17)

\[
= B_1 + B_2,
\]

where the first equality holds by (2.8) and

\[
B_1 = 4s \int_0^r \langle \lambda, G \rangle^2 d\tau
\]

\[
B_2 = 4s \int_0^r d\tau \int_0^r \langle \lambda, (P_0 G)^2 \rangle d\theta.
\]

We first estimate \( B_1 \). According to (2.14),

\[
\langle \lambda, G \rangle = \langle \lambda, \left( P_{t-s+r} 1_{[y,x]}^{(N)} \right)^2 \rangle + \langle \lambda, \left( P_r 1_{[y,x]}^{(N)} \right)^2 \rangle - 2 \langle \lambda, P_{t-s+r} 1_{[y,x]}^{(N)} P_r 1_{[y,x]}^{(N)} \rangle
\]

\[
= \frac{N}{2\pi} \int_\mathbb{R} \left[ e^{-(t-s+r)\tau^2} + e^{-r\tau^2} - 2e^{-((t-s)/2+r)z^2} \right] \left\| 1_{[y,x]}^{(N)}(Nz) \right\|^2 d\tau
\]

(4.18)

\[
= \frac{N}{2\pi} \int_\mathbb{R} e^{-r\tau^2} \left( 1 - e^{-(t-s)\tau^2/2} \right)^2 \left\| 1_{[y,x]}^{(N)}(Nz) \right\|^2 d\tau,
\]

which implies that

\[
\langle \lambda, G \rangle \leq \frac{N}{2\pi} \int_\mathbb{R} \left\| 1_{[y,x]}^{(N)}(Nz) \right\|^2 d\tau = \left\| 1_{[y,x]}^{(N)} \right\|^2_{L^2(\mathbb{R})} = |x - y|.
\]

(4.19)

**Case 1:** \(|x - y| \leq |t - s|^{1/2}\). In this case, we use (4.19), (4.18) and the inequality \( 1 - e^{-a} \leq a \) for all \( a \geq 0 \) to see that
Hence, Case 2:

\[
B_1 \leq 4s|x-y| \int_0^t \langle \lambda, G \rangle dr
\]

\[
\leq 4s|x-y|N \int_0^t \frac{e^{-r^2}(t-s)z^2}{4\pi} |1_{[y,x]}(Nz)|^2 dz
\]

\[
\leq \frac{4|t-s||x-y|N}{4\pi} \int_0^t |1_{[y,x]}(Nz)|^2 dz = 2|t-s||x-y|^2 \leq 2|t-s|^{3/2}|x-y|^{3/2}.
\]  

(4.20)

**Case 2:** $|t-s|^{1/2} < |x-y|$. We observe that

\[
|1_{[y,x]}(a)|^2 = \frac{2(1 - \cos((x-y)a))}{a^2}, \quad \text{for all } a \in \mathbb{R},
\]

(4.21)

We appeal to (4.19), (4.18), (4.21) and the inequality $1 - e^{-a} \leq 1\wedge a$ for all $a \geq 0$ to see that

\[
B_1 \leq 4s|x-y| \int_0^t \langle \lambda, G \rangle dr
\]

\[
\leq 4\frac{|x-y|N}{2\pi} \int_0^t \left(1 - e^{-s^2z^2}\right) \left(1 \wedge |t-s|^2z^4\right) \frac{2(1 - \cos((x-y)Nz))}{(Nz)^2} dz
\]

\[
\leq 4\frac{|x-y|}{z^4} \int_0^t \frac{1\wedge |t-s|^2z^4}{z^4} dz = \frac{32}{3} |t-s|^{3/2}|x-y| \leq \frac{32}{3} |t-s|^{5/4}|x-y|^{5/4}.
\]  

We conclude from (4.20) and (4.22) that in both cases we have

\[
B_1 \leq |t-s|^{5/4}|x-y|^{5/4}.
\]  

(4.23)

We next estimate $B_2$. Recall the function $G$ defined in (4.16). We apply the Plancherel’s identity to see that

\[
\langle \lambda, (P_0G)^2 \rangle = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\theta z^2} |\hat{G}(z)|^2 dz
\]

\[
\leq \frac{|\hat{G}(0)|^2}{2\pi} \int_{\mathbb{R}} e^{-\theta z^2} dz \approx 0^{-1/2} \langle \lambda, G \rangle^2.
\]

Hence,

\[
B_2 \leq 4s \int_0^t \sqrt{r} \langle \lambda, G \rangle^2 dr \leq 4s \int_0^t \langle \lambda, G \rangle^2 dr
\]

\[
= B_1 \leq |t-s|^{5/4}|x-y|^{5/4},
\]  

(4.24)

where the last inequality follows from (4.23).

Therefore, the estimate (4.3) follows from (4.9), (4.15), (4.17), (4.23) and (4.24).

**Remark 4.3.** The exponent $\frac{5}{4}$ (greater than 1) in (4.3) will be sufficient for tightness. It might not be optimal and one may expect this exponent can be replaced by 2, as in (4.15).

**Proposition 4.4.** For any $k \geq 2$, there exists $C > 0$ such that for all $(t,s,x,y) \in [0,1]^4$ and $N \geq 1$
\[
E \left[ |V_N(t, x) - V_N(t, y)|^k \right] \leq C|x - y|^{k/2}, \quad (4.25)
\]
\[
E \left[ |V_N(t, x) - V_N(s, x)|^k \right] \leq C|t - s|^{k/2}. \quad (4.26)
\]

**Proof.** We first prove (4.25) and assume without loss generality \( y \leq x \). According to (2.11) and Burkholder’s inequality,

\[
E \left[ |V_N(t, x) - V_N(t, y)|^k \right] = E \left[ \left| \int_0^t \int \sqrt{u(r \cdot z)} P_{t-r} 1_{[y, x]}(z) W(\text{d}r \text{ d}z) \right|^k \right] \\
\leq E \left[ \left| \int_0^t \int u(r \cdot z) P^2_{t-r} 1_{[y, x]}(z) \text{d}z \text{d}r \right|^k \right] \\
\leq \left| \int_0^t \int \|u(r \cdot z)\|_{k/2} P^2_{t-r} 1_{[y, x]}(z) \text{d}z \text{d}r \right|^k \quad (4.27)
\]

where the second inequality holds by Minkowski’s inequality and the third by (4.4). Now we apply (2.13) to see that for all \( N \geq 1 \) and \((t, x, y) \in [0,1]^3\)

\[
\int_0^t \int P^2_{t-r} 1_{[y, x]}(z) \text{d}z \text{d}r \leq t \|1_{[y, x]}\|^2_{L^1(\mathbb{R})} \leq |x - y|,
\]

which together with (4.27) yields (4.25).

We proceed to prove (4.26) and assume without loss generality \( s \leq t \). We appeal to Burkholder’s inequality and Minkowski’s inequality again to see that

\[
E \left[ |V_N(t, x) - V_N(s, x)|^k \right] \leq E \left[ \left| \int_0^t \int \sqrt{u(r \cdot z)} P_{t-r} 1_{[0, x]}(z) W(\text{d}r \text{ d}z) \right|^k \right] \\
+ E \left[ \left| \int_0^s \int \sqrt{u(r \cdot z)} \left( P_{t-r} 1_{[0, x]}(z) - P_{s-r} 1_{[0, x]}(z) \right) W(\text{d}r \text{ d}z) \right|^k \right] \\
\leq \left| \int_0^t \int \|u(r \cdot z)\|_{k/2} P^2_{t-r} 1_{[0, x]}(z) \text{d}z \text{d}r \right|^k \\
+ \left| \int_0^s \int \|u(r \cdot z)\|_{k/2} \left( P_{t-r} 1_{[0, x]}(z) - P_{s-r} 1_{[0, x]}(z) \right)^2 \text{d}z \text{d}r \right|^k \\
\leq \left| \int_0^t \int P^2_{t-r} 1_{[0, x]}(z) \text{d}z \text{d}r \right|^k \\
+ \left| \int_0^s \int \left( P_{t-r} 1_{[0, x]}(z) - P_{s-r} 1_{[0, x]}(z) \right)^2 \text{d}z \text{d}r \right|^k,
\]

thanks to (4.4). The estimate (2.13) ensures that for all \( N \geq 1 \) and \((t, s, x) \in [0,1]^3\)

\[
\int_0^t \int P^2_{t-r} 1_{[0, x]}(z) \text{d}z \text{d}r \leq (t-s)\|1_{[0, x]}\|^2_{L^2(\mathbb{R})} \leq (t-s). \quad (4.29)
\]
Moreover, from the calculation in (4.18) (with \( y = 0 \)), we have for all \((t, s, x) \in [0, 1]^3\)
\[
\int_0^t \int_\mathbb{R} \left( P_{t-s+r} I_{(0,x)}^{(N)} (z) - P_{t} I_{(0,x)}^{(N)} (z) \right)^2 \, dz \, dr
\]
\[
= \frac{N}{2\pi} \int_\mathbb{R} \frac{1 - e^{-sz^2}}{z^2} \left( 1 - e^{-(t-s)z^2/2} \right)^2 \left| I_{(0,x)}^{(N)} (Nz) \right|^2 \, dz
\]
\[
\leq \frac{(t-s)N}{4\pi} \int_\mathbb{R} \left| I_{(0,x)}^{(N)} (Nz) \right|^2 \, dz = \frac{(t-s)x}{2} \leq (t-s),
\tag{4.30}
\]
where in the first inequality, we have used that \(1 - e^{-a} \leq 1 \wedge a\) for all \(a \geq 0\). Therefore, the estimate (4.26) follows from (4.28), (4.29) and (4.30). The proof is complete. \( \square \)

5. Proof of Theorem 1.1

Proof of Theorem 1.1. Choose \(0 < t_0 < ... < t_1 \leq 1\) and \(x_0, ..., x_m \in [0, 1]\). Recall the random variables \(V_N(t_k, x_k), k = 1, ..., m\) as defined in (2.11). By (2.8) and (2.13),
\[
\sup_{1 \leq k \leq m} \sup_{N > 0} E_{1} \left[ V_N(t_k, x_k)^2 \right] < \infty,
\]
which implies that the family of random vectors \((V_N(t_1, x_1), ..., V_N(t_m, x_m))_{N > 0}\) are tight on \(\mathbb{R}^m\). Hence, for any sequence \((n_j)_{j=1}^{\infty} (n_j \to \infty\) as \(j \to \infty\)), there exists a subsequence \((n'_j)_{j=1}^{\infty}\) and a random vector \(Y = (Y_1, ..., Y_m)\) such that as \(j \to \infty\),
\[
(V_{n'_j}(t_1, x_1), ..., V_{n'_j}(t_m, x_m)) \to (Y_1, ..., Y_m)\quad \text{in distribution.}
\]
Proposition 3.1 ensures that for all \(\theta_1 \geq 0, ..., \theta_m \geq 0\),
\[
\lim_{j \to \infty} E \left[ e^{-(\theta_1 V_{n'_j}(t_1, x_1) + ... + \theta_m V_{n'_j}(t_m, x_m))} \right] = E \left[ e^{-(\theta_1 W(t_1, x_1) + ... + \theta_m W(t_m, x_m))} \right],
\]
where \(W\) denotes the Brownian sheet. We can apply the same arguments as in [15, p.106–107] to see that for all \(\theta_1 \geq 0, ..., \theta_m \geq 0\)
\[
E \left[ e^{-(\theta_1 Y_1 + ... + \theta_m Y_m)} \right] = E \left[ e^{-(\theta_1 W(t_1, x_1) + ... + \theta_m W(t_m, x_m))} \right].
\]
By the analytic continuation theorem, the random vector \((Y_1, ..., Y_m)\) has the same distribution as \((W(t_1, x_1), ..., W(t_m, x_m))\). Therefore, we conclude that as \(N \to \infty\),
\[
(V_N(t_1, x_1), ..., V_N(t_m, x_m)) \to (W(t_1, x_1), ..., W(t_m, x_m))\quad \text{in distribution.}
\]
which implies that the finite-dimensional distributions of \(\left\{ N^{-1/2} \int_0^N [u(t, .) - 1] \, dz \right\}_{(t, x) \in [0, 1]^2}\) converge to those of Brownian sheet. The tightness follows from Propositions 4.2 and 4.4 (see [23, Theorem 3], [24, Theorem 1] or [25, Exercise 1.4.19]). The proof is complete. \( \square \)

Remark 5.1. Our approach might be applied to other SPDEs related to super-Brownian with more general branching mechanism; see for instance [1, (1.5)]. In this case, one can still try to apply the Laplace functional to obtain the convergence of finite-dimensional distributions and use the mild form of the SPDE to obtain the tightness.
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