Use strong coupling strength to coherently preserve quantum entanglement

Guihua Tian†
School of Science, Beijing University of Posts And Telecommunications. Beijing 100876 China.

Shuquan Zhong‡
State Key Laboratory of Information Photonics and Optical Communications, Beijing University of Posts And Telecommunications. Beijing 100876 China.

The dynamics of two qubits ultra-strongly coupled with a quantum oscillator is investigated by the adiabatic approximation method. The evolution formula of the initial four Bell states are studied under the control mechanism of the coherent state of the quantum oscillator. The influential parameters for the preservation of the entanglement are the four parameters: the average number of the coherent state, the ultra-strong coupling strength, the ratio of two frequencies of qubit and oscillator, and the inter-interaction coupling of the two qubits. The novel results show that the appropriate choice of these parameters can enable this mechanism to be utilized to preserve the entanglement of the two qubits, which is initially in the state $|I_0\rangle$ of the four Bell states. We give two different schemes to choose the respective parameters to maintain the entangled state $|I_0\rangle$ almost unchanged. The results will be helpful for the quantum information process.

PACS numbers: 42.50.Pq, 42.50.Md, 03.65.Ud

I. INTRODUCTION

Entanglement is indispensable in the quantum information process. Quantum entanglement states have been applied in quantum key distribution and teleportation, entanglement purification, factorization of integers, random searches[1]-[5]. Generation and preservation of entanglement of qubits are crucial for all the quantum information process. It is still challenge to externally control entanglement. Recently, using quantum bus to coherently and controllably manipulate quantum entanglement is provided in Ref.[6], where the famous Jaynes-Cummings (JC) model is applied as the control mechanism.

Theoretically, the entanglement generation and maintaining of two qubits can be achieved by use of Jaynes-Cummings (JC) model Hamiltonian[7]. The entanglement reciprocation is also studied between the field variables and a pair of qubits in JC cavity[8]-[12]. Later, study shows that coherent-state control of a pair of non-local atom-atom entanglement between two spatially separated sites is possible[13]. There are some time-dependent entanglement death and rebirth effects in these investigations.

JC model is a main mechanism to be used to study how to control the manipulation of quantum qubits, whose validity rely on the assumption of the weak coupling of quantum oscillator, qubits and their near resonance condition. It is obtained from the Rabi model by discarding the count-rotating-wave terms[2]. The strong and ultra-strong coupling region of qubit and oscillator provide many new and anti-intuitive results for the Rabi model[14]-[21], which is treated in Ref.[9] with zero detuning. Further, ultra-strong coupling and large detuning is investigated, and novel results appear, like frequency modification, collapse and revival of Rabi oscillation for one qubit with the initial state of the single mode field being thermal or coherent[14]-[21]. In Ref.[15], Rabi model is extended to two qubits case, where the authors studied the death and revival phenomena for the two qubits’ entanglement for the initial coherent state of the oscillator. However, they only study very special case of the coupling parameter $\beta^2 \ll \Omega_{1N}$ (see Ref.[15] or the following for details), which will greatly simplify the eigenvectors and subsequent calculation. There are no investigation concerning the range where the coupling strength does not satisfy $\beta^2 \ll \Omega_{1N}$.

 Enlightened by these works, we investigate use the Rabi mechanism control. In our study, we will not restrict our study by the condition $\beta^2 \ll \Omega_{1N}$, and this in turn will make calculation complicated. Never the less, it also give one chance to obtain some unexpected phenomena, that is the new way to use coherent quantum mode to control one of the full entangled Bell state and novel results are obtained based on the complex calculation, which will helpful for the quantum information process.

In addition, the condition $\beta^2 \ll \Omega_{1N}$ in Ref.[15] is not easy to be satisfied due to the fact $\Omega_{1N}$ depends on the parameters $N$, $\beta$, $\Omega$ nonlinearily. So the investigation without the condition is crucial for their further application in quantum information process. Furthermore, the above mentioned complexity unexpectedly promotes our ability to preserve the entanglement of the two qubits in the state $|I_0\rangle$, one of the four Bell states. This unexpected result can be exploited in quantum information process involved the entanglement of two qubits.

The paper organized as follows: we first introduce the two-qubit system with inter-qubit coupling briefly and give a simple model for it. Then we give a view of the
method to to be used to study the Rabi model in ultralong coupling regime with larger detuning, the adiabatic approximation method (AA). The spectrum of the qubits coupled with quantum mode filed is given in subsequent section. Then the evolution of the system are investigated and followed by the section to study the preservation of the entanglement.

II. THE TWO QUBITS WITH INTER-QUBIT COUPLING

Because we will treat the two qubits system as three-level-energy system, which will be used both for circuit QED and cavity QED. Here we give a brief introduction about it. The Hamiltonian for the system of a two-qubit with inter-qubit coupling \cite{22, 23}

\[ H_q = \frac{1}{2} \hbar \omega_0 (\sigma_z^{(1)} + \sigma_z^{(2)}) + \kappa \hbar \omega_0 (\sigma_+^{(1)} \sigma_-^{(2)} + \sigma_-^{(1)} \sigma_+^{(2)}) \]

where \( \sigma_z^{(i)} \), \( i = 1, 2 \) are the ith qubit’s Pauli matrices and \( \kappa \) is the coupling strength between the qubits. \( \sigma_{\pm}^{(i)} \) are

\[ \sigma_+^{(i)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_-^{(i)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \]

for ith-qubit. The Hamiltonian could be a diagonal matrix under the collective states

\[ |3\rangle = |\uparrow\uparrow\rangle, \]

\[ |s\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \]

\[ |a\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle), \]

\[ |1\rangle = |\downarrow\downarrow\rangle \]

as basis \cite{22, 23}:

\[ H_q^\prime = \hbar \omega_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & -1 \end{pmatrix}. \]

There are two transition channels, the symmetric one \( |3\rangle \rightarrow |s\rangle \rightarrow |1\rangle \) and the asymmetric one \( |3\rangle \rightarrow |a\rangle \rightarrow |1\rangle \). The two channels are not correlated \cite{22, 23}, so we will divide the \( H_q \) as

\[ H_q = \hbar \omega_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & -1 \end{pmatrix} \]

for symmetric transition channel or

\[ H_q = \hbar \omega_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\kappa & 0 \\ 0 & 0 & -1 \end{pmatrix} \]

for asymmetric channel transition. We will unite the two cases as

\[ H_q = \hbar \omega_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -1 \end{pmatrix} \]

with a positive and negative for symmetric and asymmetric channels respectively. We also denote \( |2\rangle = |s\rangle \) for symmetric channel and \( |2\rangle = |a\rangle \) for asymmetric channel respectively.

The interacting two identical qubits will correspond to two three-level-energy systems with the same top and bottom eigenstates and different middle states. The top and bottom states’ energies are \( \hbar \omega_0, -\hbar \omega_0 \). The two middle states will have the same energy if the two qubits do not couple with each other. In this case, the two three-level-energy systems are the same concerning their energy distributions. Generally, they will be regarded as just one. However, the two middle states will have different energies, one positive and the other negative. The corresponding two three-level-energy systems are different even concerning with their energy distribution. Nevertheless, relating the transition, the two three-level-energy systems are not correlated with each other, so we could consider one.

III. THE RABI HAMILTONIAN

As the Hamiltonian of the two qubits can be represented a \( 3 \times 3 \) matrix, the Rabi model is extended as describing the dynamics of the two qubits interacting with a single quantum mode field by

\[ H = \hbar \omega_0 S_z + \hbar \omega a^\dagger a + \hbar \beta (a + a^\dagger) S_x, \]

which will be the Rabi model as \( S_x, S_z \) reduce to the usual Pauli matrices. The qubits is described by \( H_q = \hbar \omega_0 \) before stated. In the matrix form, \( S_x, S_z \) are

\[ S_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad S_z = \begin{pmatrix} 0 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]

The operator \( S_x \) is connected with the operators \( \sigma^+ \), \( \sigma^- \)

\[ S_x = \sigma^+ + \sigma^-, \]

where

\[ \sigma^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \]
The eigenstates \( |1\rangle, |2\rangle, |3\rangle \) change under the operators \( \sigma^+, \sigma^- \) as following
\[
\sigma^+ |1\rangle = |2\rangle, \quad \sigma^+ |2\rangle = |3\rangle, \quad \sigma^+ |3\rangle = 0,
\]
\[
\sigma^- |1\rangle = 0, \quad \sigma^- |2\rangle = |1\rangle, \quad \sigma^- |3\rangle = |2\rangle.
\]
As stated before, the Hamiltonian of the system is not analytically integrable. The RWA approximation supposes the resonate condition \( \omega_0 \approx \omega \) and the weak coupling \( \beta \ll 1 \) and becomes completely solvable by discarding the non-energy conserving terms \( a\sigma^-, a^\dagger \sigma^+ \).

IV. ADIABATIC APPROXIMATION IN ULTRA-STRONG COUPLING RANGE

Whenever the coupling is strong or the detuning is large, the counter-RWA terms \( a\sigma^-, a^\dagger \sigma^+ \) can not be omitted. This belongs to the regime of adiabatic approximation. In adiabatic approximation, \( \omega_0 \) is small relative to the other terms in the Hamiltonian, and one could first omit it to study the rest as the non-RWA Hamiltonian, then take it as perturbation in later. Physically, this focuses on the quantum oscillator influenced by the term \( h\beta(a + a^\dagger)S_x \). The Hamiltonian reads
\[
H^0 = \hbar \omega a^\dagger a + h\beta(a + a^\dagger)S_x.
\] (10)

If studied classically, the oscillator undergoes some forced motion by the qubits. The quantum oscillator system interacting with one qubit and two non-interacting qubits have been solved in the adiabatic approximation (see references [14]-[15]). We now employ the similar method to solve the non-equal-level qubits system. The eigenvectors \( |1, 1\rangle, |1, 0\rangle, |1, -1\rangle \) of the operator \( S_x \) are
\[
S_x |1, m\rangle = \sqrt{2m}|1, m\rangle, \quad m = 0, \pm 1,
\] (11)
which are written as
\[
\begin{pmatrix}
|1, 1\rangle \\
|1, 0\rangle \\
|1, -1\rangle
\end{pmatrix} = \begin{pmatrix}
1/2 & 1/\sqrt{2} & 1/2 \\
1/\sqrt{2} & 0 & -1/\sqrt{2} \\
1/2 & -1/\sqrt{2} & 1/2
\end{pmatrix} \begin{pmatrix}
|3\rangle \\
|2\rangle \\
|1\rangle
\end{pmatrix}
\] (12)

With the help of these vectors, the eigen-vectors \( |\Psi\rangle \) for the eigenstates of operator \( H^0 \) will be written as \([14], [15]\)
\[
|\Psi_{n,m}\rangle = |1, m\rangle |N_m\rangle = |1, m\rangle D(-\sqrt{2m}\beta)|N\rangle
\]
with the corresponding eigen-values \( E^0_{n,m} = \hbar \omega (N - 2\beta^2m^2) \). The displaced operator \( D(\alpha) \) of the quantum oscillator is defined as \( D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a) \) for the arbitrary complex number \( \alpha \). The interaction with the qubits makes potential well of the quantum oscillator displaced according to the states of the qubits. From the physical view, the interaction term \( h\beta(a + a^\dagger)S_x \) has influence to displace the equilibrium position of the oscillator to different points by the different states of the qubits \( |1, m\rangle, m = 0, \pm 1 \), which result in three displaced number vectors \( |N_m\rangle, m = 0, \pm 1 \). From the mathematical view, the eigenstates \( |1, m\rangle |N_m\rangle, m = 0, \pm 1, N = 0, 1, 2, \cdots \) constitute as a complete basis for the composite system and are useful for the later calculation, that is, any vector for the composite system of the qubits and oscillator could be decomposed in the basis \( |1, m\rangle |N_m\rangle, m = 0, \pm, N = 0, 1, 2, \cdots \). This basis is not orthogonal due to the fact
\[
\langle N_1| M_r \rangle \neq 0 \text{ for } r \neq l.
\] (13)

In order to obtain the spectrum for the Hamiltonian \( H \), we need to make calculation of the terms \( \langle 1, l| S_z |1, r \rangle \langle N_1| M_r \rangle, l, r = 0, \pm 1 \). Due to the fact \( \omega_0 \ll \omega \), the transition of the qubits generally contributes little in exciting the quantum oscillator, so the corresponding terms \( \langle 1, l| S_z |1, r \rangle \langle N_1| M_r \rangle, N \neq M \) could be omitted. This approximation is called adiabatic approximation (AA).

V. THE SPECTRUM OF THE HAMILTONIAN BY AA METHOD

Under this adiabatic approximation, the Hamiltonian \( H \) becomes block-diagonal with the nth diagonal block \( \tilde{H}_N \) as a \( 3 \times 3 \) matrices defined under the basis \( |1, m\rangle |N_m\rangle, m = 1, 0, -1 \) as
\[
\tilde{H}_N = \begin{pmatrix}
\tilde{N} & \Omega_{1N} & \Omega_{2N} \\
\Omega_{1N} & \Omega_{1N} & N \\
\Omega_{2N} & \Omega_{1N} & \Omega_{1N}
\end{pmatrix},
\] (14)
where
\[
\tilde{N} = N - 2\beta^2 + \frac{a \omega_0}{2 \omega},
\] (15)
\[
\Omega_{1N} = \frac{\omega_0}{\omega} \langle 1, 1| S_z |1, 0 \rangle |N_1| N_0 \rangle = \frac{1}{\sqrt{2}} \omega_0 \exp(-\beta^2) L_N(2\beta^2),
\] (16)
\[
\Omega_{2N} = \frac{\omega_0}{\omega} \langle 1, -1| S_z |1, 1 \rangle |N_{-1}| N_1 \rangle = -\frac{a \omega_0}{2} \omega \exp(-4\beta^2) L_N(8\beta^2).
\] (17)
The parameter \( a \) enters \( \tilde{H}_N \) contributing two diagonal terms in \( \tilde{N} \) and two off-diagonal ones in \( \Omega_{2N} \), which will give rise to the transition between \( |1, -1\rangle |N_{-1}\rangle \) and \( |1, 1\rangle |N_1\rangle \). This is a new transition due to the non-equal-level parameter \( a \neq 0 \) and is absent in the equal-level case. The solutions to the eigen-value problem of the
operator $\hat{H}_N$ are

$$\tilde{E}_{N,0}^0 = \hbar \omega \left( N - 2\beta^2 + \frac{a \omega_0}{2} - \Omega_{2N} \right)$$

$$= \hbar \omega \left( N + \tilde{T}_0 - 2\Omega_{2N} \right), \quad (18)$$

$$\tilde{E}_{N,\pm}^0 = \hbar \omega \left( N + \frac{\tilde{T}_0 \pm \sqrt{\tilde{T}_0^2 + 8\Omega_{1N}^2}}{2} \right), \quad (19)$$

and

$$|\tilde{E}_{N,0}^0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad |E_{N,\pm}^0\rangle = \frac{1}{L_{N,\pm}} \begin{pmatrix} 1 \\ \tilde{Y}_{N,\pm} \end{pmatrix}, \quad (21)$$

$$\tilde{Y}_{N,\pm} = \left( -\tilde{T}_0 \pm \sqrt{\tilde{T}_0^2 + 8\Omega_{1N}^2} \right) \frac{2}{2\Omega_{1N}}, \quad (22)$$

$$\tilde{L}_{N,\pm}^2 = \tilde{Y}_{N,\pm}^2 + 2. \quad (23)$$

In Ref.[15], the authors discussed the special case where $\Omega_{1N} \gg 2\beta^2$ with $a = 0$. The other extreme case is that $\Omega_{1N} \approx 0$: $a = 0$ means that the spectrum of $\hat{H}_0^0$ is the same as that of $\hat{H}$ in AA method, but they are different whenever $a \neq 0$.

All eigen-values $\tilde{E}_N^m$, $m = 0, \pm$ are influenced by the parameter $a$ through the quantity $\tilde{T} = -2\beta^2 + \frac{a \omega_0}{2} (1 - \exp (-4\beta^2)L_N(8\beta^2))$, so are the eigenvectors $|E_{N}^m\rangle$. Of course, the dynamics of the qubits will definitely be differently from that of the equal-level one.

VI. THE PHYSICAL IMPACT OF THE RESULTS

The dynamics of the qubits is important for the real system. The evolutionary behavior of the qubits depends crucially on the initial states, the initial states both for the qubits and the quantum oscillator. Here we employ the initial state for the qubits are $|1, m\rangle$, $m = 0, \pm 1$, which are applicable in the strong coupling (see reference[13] for detail.). $|1, m\rangle$, $m = 0, \pm 1$ are different from the states $|m\rangle$, $m = 1, 2, 3$. Similarly, the natural initial states for the quantum oscillator are the displaced number states or the displaced coherent states. In the report, we treat the simplest case of the initial states $|\Phi_{m}^N(0)\rangle = |1, m\rangle |N_m\rangle$, $m = 0, \pm 1$. We mainly focus on the two kinds of probabilities, that is, $P_N(t)$ for the system remains unchanged and $T_{m 	o l}(t)$ for it transits to new states $|1, l\rangle |N_l\rangle$, $l \neq m$. It is easy to obtain the following

$$P_N^1(t) = \frac{1}{4} + \frac{1}{L_{N,+}^4} \frac{1}{L_{N,-}^4} + \frac{1}{L_{N,+}^4} \cos \omega_{N,1} t + \frac{1}{L_{N,-}^4} \cos \omega_{N,0} t,$$

$$P_N^0(t) = \frac{\tilde{Y}_{N,+}^4}{L_{N,+}^4} + \frac{\tilde{Y}_{N,-}^4}{L_{N,-}^4} + \frac{2}{L_{N,+}^4 + L_{N,-}^4} \cos \omega_{N,0} t,$$

$$(24)$$

FIG. 1: Schematic diagram of $P_N^1(t)$ with the four parameters as $N = 2$, $\omega_0 = 0.25$, $\beta = 0.2$, $a = 0.2$, $0$, $-0.2$ from the top to bottom respectively. The apparent difference in these three figures strongly implies that the parameter $a$ influences the qubits dynamically.
The parameter $\omega = \omega_{N,1}$ oscillates with three frequencies $\omega_{N,1}, \omega_{N,2}, \omega_{N,0}$. The non-equal level parameter $a$ changes the three frequencies as well as the amplitude. For $a = 0$, the detailed dynamics of the qubits is given in Ref. [15]. The special case with $\beta^2 < 8\Omega_1^2$, $a = 0$ is studied in Ref. [14], where $\omega_{N,1} = -\omega_{N,2} = -\omega_{N,0} = \sqrt{2}\Omega_1^2$.

Figs. [11] show the general trait for the qubit remaining in its initial states $|1, \pm \rangle$ for different parameter $a = 0.2, 0, -0.2$. Obviously, $P_N^0(t)$ is influenced by four parameters $\beta, a, N, \omega_{\pm}$. In Ref. [14], it is shown that the coupling strength $\beta$ ranges from 0.01 to 1 for the application of adiabatic approximation (weak coupling will not be discussed here). From Eqs. (25)-(28), we see that the parameter $a$ will come to action apparently whenever $\beta \approx 0.01 - 0.6$. As stated before, the qubits is equivalent to a two qubits system and the non-equal-energy-level parameter $a$ represents the coupling strength between the two qubits. This shows the coupling of the two qubits changes their dynamics considerably in the range of $\beta \approx 0.1 - 0.6$ for the adiabatic approximation method to be applied, and this is our limit on the coupling parameter $\beta$.

\[ \dot{T}_{1 \rightarrow -1}^N(t) = \frac{1}{4} L_{N,+}^4 - \frac{1}{L_{N,-}^4} + \frac{1}{L_{N,+}^2 - L_{N,-}^2} \cos \omega_{N,1} t \]
\[ - \frac{1}{L_{N,-}^4} \cos \omega_{N,2} t + \frac{1}{L_{N,+}^4 + L_{N,-}^4} \cos \omega_{N,0} t, \]

(26)

and

\[ \dot{T}_{1 \rightarrow 0}^N(t) = \frac{Y_{N,+}^4 + Y_{N,-}^4}{L_{N,+}^4} + \frac{2Y_{N,+}Y_{N,-}}{L_{N,+}^4 + L_{N,-}^4} \cos \omega_{N,0} t, \]

(27)

where

\[ \omega_{N,1} = \omega \left( \frac{4\Omega_{2N} - \tilde{T}_0 + \sqrt{\tilde{T}_0^4 + 8\Omega_{1N}^2}}{2} \right), \]

(28)

\[ \omega_{N,2} = \omega \left( \frac{4\Omega_{2N} - \tilde{T}_0 - \sqrt{\tilde{T}_0^4 + 8\Omega_{1N}^2}}{2} \right), \]

(29)

\[ \omega_{N,0} = \omega \sqrt{\tilde{T}_0^2 + 8\Omega_{1N}^2}. \]

(30)

VII. DYNAMICS OF THE SYSTEM

Here we consider the dynamics of the composite system with different initial conditions. Furthermore, we could also calculate the probability for the two qubits stay in their other fully entangled states. There are four fully entangled Bell states

\[ \Psi_\pm = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle) \]

(31)

and

\[ \Phi_\pm = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle \pm |\downarrow\downarrow\rangle). \]

(32)

It is easy to see that $|1,0\rangle = \Phi_-$, and the others are related with the vectors $|1,1\rangle$, $|1,-1\rangle$. Due to the facts that

\[ |2\rangle = \frac{1}{\sqrt{2}}(|1,1\rangle - |1,-1\rangle) \]

and

\[ |2\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle) \]

for the symmetrical case $a = \kappa > 0$ and

\[ |2\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle) \]

for the symmetrical case $a = -\kappa < 0$. We could write them as

\[ \Phi_- = |1,0\rangle \]

(33)

\[ \Phi_+ = \frac{1}{\sqrt{2}}(|1,1\rangle + |1,-1\rangle) \]

(34)

\[ \Psi_\pm = |2\rangle = \frac{1}{\sqrt{2}}(|1,1\rangle - |1,-1\rangle). \]

(35)

Note that $\Psi_+, \Psi_- \pm$ are vectors correspond to the positive and negative sign of the parameter $a$ respectively. So the initially fully entangled Bell states of qubits can be united to be written as $|I_0\rangle = \frac{1}{\sqrt{2}}(|1,1\rangle + \delta|1,-1\rangle)$ ($\delta = \pm 1$) for $\Phi_+$, $\Psi_\pm$ and $|I_0\rangle = |1,0\rangle$ for $\Phi_-$. We will consider the dynamics of the system with the two qubits in $|I_{\pm 1}\rangle$ or $|I_0\rangle$ and quantum mode field in $|\alpha\rangle$.

A. The qubits are initially in states $|I_{\pm 1}\rangle$

In first case, the initial state for the two qubits is $|I_{\delta}\rangle$ with $\delta = \pm 1$ and quantum mode field in $|\alpha\rangle$. Suppose $\beta < 0.7$. Then in the adiabatic approximation, the qubits will evolve into the states $|I_{\delta}\rangle$ with $\delta = \pm 1$ with probability...
\[ P(\delta, \tilde{\delta}, \alpha, t) = \frac{1}{2} + \frac{\delta \tilde{\delta}}{N!} (\alpha^2 - \beta^2)^N e^{-(\alpha^2 + \beta^2)} \langle N-1 | N \rangle \]

\[ -\frac{1}{2} \sum_{N=0}^{\infty} \left( p(N, \alpha + \beta) + p(N, \alpha - \beta) + \frac{2\delta}{N!} \alpha^2 - \beta^2)^N e^{-(\alpha^2 + \beta^2)} (1 + \langle N-1 | N \rangle) \right) \frac{\Omega^2_N}{T^2 + 8\Omega^2_N} (1 - \cos(\omega_N,0,t)), \]

where

\[ p(N, \alpha) = \frac{e^{-\alpha \alpha^*} |\alpha|^{2N}}{N!} \quad (37) \]

is the probability of quantum field state \(|\alpha|\) in its number state \(|N\rangle\) and \(\delta = \pm 1\), \(\Delta \alpha = \pm 1\). From the above Eq.(36), we see that the initial entangled states \(|I_{\pm 1}\rangle\) of the two qubits will have large possibility \(p(\delta, -\delta, \alpha, t)\) being around \(\frac{1}{2}\) to evolve into the states \(|I_{\pm 1}\rangle\). So it is hard for the two qubits to remain its entangled states. In the following subsection, we will check that for the initial state \(|I_0\rangle\).

**B. The qubits are initially in states \(|I_0\rangle\)**

suppose the qubits is in the state \(|I_0\rangle\) initially, while the oscillator naturally stays in its coherent state \(|\alpha\rangle\). This is a very general state for a quantum processing system. The system will evolve accordingly and the probability for the qubits remain in its initial state is

\[ P_0(\alpha) = 1 - T(\alpha, t), \]

where \(T(\alpha, t)\) is the probability of the two qubits transiting to other non-entangled states and is given as

\[ T(\alpha, t) = \sum_{N=0}^{\infty} 2p(N, \alpha) \frac{\overline{Y}_2}{L^4_{\alpha, +}} (1 - \cos(\omega_N,0,t)). \quad (38) \]

where \(p(N)\) is the probability of \(N\) photons in the coherent state \(|\alpha\rangle\). In the large quantity \(|\alpha|^2 \gg 1\) approximation, the quantity \(P(t)\) could be simplified greatly due to the fact

\[ p(N, \alpha) = \frac{e^{-\langle N-\alpha \rangle^2/2\alpha^2}}{\sqrt{2\pi\alpha^2}}. \quad (39) \]

We denote the term \(2\frac{\overline{Y}_2}{L^4_{\alpha, +}}\) as \(B(N)\) in the above equation and will show the general properties of \(B(N)\) by some special parameters in Fig.**. Fig** also gives some examples concerning the \(p(N, \alpha)\) in case of \(|\alpha|^2 \gg 1\). Under these assumption of rapid falling to zero of \(p(N, \alpha)\) as \(N\) deviated from its average \(|\alpha|^2\), we could safely approximate \(B(N)\) in Eq.(38) as

\[ B(N) \approx b_0 + b_1 (N - \bar{n}) + b_2 (N - \bar{n})^2 \]

where \(\bar{n} = \lfloor |\alpha|^2 \rfloor\) is the integer part of \(|\alpha|^2\) and

\[ b_0 = 2\frac{\overline{Y}_2^2}{L^4_{\alpha, +}}, \quad b_1 = \left[ \frac{dB(N)}{dN} \right]_{N=\bar{n}}, \quad b_2 = \left[ \frac{d^2 B(N)}{dN^2} \right]_{N=\bar{n}}. \]

It is easy to calculate \(T(\alpha, t)\) as two parts

\[ T(\alpha, t) = T_1(\alpha) - T_2(\alpha, t) \quad (40) \]

\[ T_1(\alpha) = \sum_{N=0}^{\infty} B(N) \frac{e^{-\langle N-\alpha \rangle^2/2\alpha^2}}{\sqrt{2\pi\alpha^2}} = b_0 + b_2 \bar{n} \quad (41) \]

\[ T_2(\alpha, t) = \sum_{N=0}^{\infty} B(N) \frac{e^{-\langle N-\alpha \rangle^2/2\alpha^2}}{\sqrt{2\pi\alpha^2}} \cos(\omega_N,0,t). \quad (42) \]

In the assumption that \(|\alpha|^2 \gg 1\) and Gauss form of \(p(N, \alpha)\), it is reasonable to extend the summation in \(T_2(\alpha, t)\) from 0 to \(-\infty\). Then the use of Poisson summation formula gives

\[ T_2(\alpha, t) = \sum_{k=-\infty}^{+\infty} \tilde{g}(k, t) \quad (43) \]

\[ \tilde{g}(k, t) = \int_{-\infty}^{+\infty} B(N) \frac{e^{-\langle N-\alpha \rangle^2/2\alpha^2}}{\sqrt{2\pi\alpha^2}} \cos(\omega_N,0,t) e^{i2\pi k N} dN. \quad (44) \]

Generally, \(\omega_N,0,t\) could be simplified as

\[ \omega_N,0,t = \omega_{\bar{n},0,t} + c_1 (N - \bar{n}) + c_2 (N - \bar{n})^2 \]

with

\[ c_1 = \left[ \frac{d\omega_{\bar{n},0,t}}{dN} \right]_{N=\bar{n}} \quad c_2 = \left[ \frac{d^2 \omega_{\bar{n},0,t}}{dN^2} \right]_{N=\bar{n}}. \]
that when useful to delineate them in some special cases: one case that
phenomena with its $k$.

It is clear that $T_2(\alpha, t)$ exhibits the collapse and revival phenomena with its $k - \theta \text{th}$ term being $\tilde{g}(k, t)$. It is more useful to delineate them in some special cases: one case that when $c_1 \neq 0$ with $c_2 = 0$, and the other extreme case that $c_1 = 0$ and $c_2 \neq 0$. In the first case, we obtain that

$$A(k, t) = e^{-\tilde{n}(2\pi k + c_1 t)^2}$$

The revival time

$$t_{rev}(k) = 2\pi \frac{k}{c_1}$$

and the height for its amplitude is

$$A(k, t_{rev}) = b_0 + b_2 \tilde{n} = T_1,$$

which is constant in contrast to the decreasing height as time goes in Ref. [15].

In the second case, it is easy to see that

$$A(k, t) = \frac{e^{-\frac{4\pi^2 k^2}{(1 + 4\tilde{n}^2 c_2^2 t^2)}}}{(1 + 4\tilde{n}^2 c_2^2 t^2)\tilde{n}} \left( b_0 + \frac{\tilde{n} b_2}{(1 + 4\tilde{n}^2 c_2^2 t^2)\tilde{n}} \right) - \frac{4\pi^2 k^2 \tilde{n}^3 b_2}{(1 + 4\tilde{n}^2 c_2^2 t^2)},$$

(51)

where the fact

$$\tan \theta(t) = 2\tilde{n} c_2 t, \quad \cos \theta(t) = \frac{1}{(1 + 4\tilde{n}^2 c_2^2 t^2)^\frac{1}{2}}$$

are used. Obviously, there is no revival phenomena in $A(k, t)$ in this case. $A(k, t)$ also will generally decreases as time goes except for the initially irregular transiting change.

This two cases are not possible absolutely. Anyway, $c_2$ may be very small with $c_1 \gg c_2$, and this case is close to the first case, where collapse and revival appear in $T_2(\alpha, t)$. However, the small and non-equal-zero quantity $c_2$ contributes both the decreasing heights of the revival amplitude, as is shown in Eq. 15. and the broadening of revivals with the time growing. The broadening of revivals also makes the collapse interval shorter and shorter until its disappearing.

Similarly, $c_2 \gg c_1$ and $c_1 \neq 0$ means the revival time gap is greater than that in the first case. All these features all shown in Fig. 3.

![FIG. 3: The transition $2T(\alpha, t)$ with $|\alpha|^2 = 14$, $a = -0.48$, $\beta = 0.102$, $\tilde{\omega} = 0.21$.](image-url)
the quantity intricacy of the Rabi model could be utilized to make that of its RWA counterpart JC model. To initial en-

cination if the average number approximation must be added. As this is seldom, we just stop here.

VIII. THE PRESERVATION OF ENTANGLEMENT OF TWO QUBITS

From the previous section, we see that the dynamics of the Rabi model is much more complicated than that of its RWA counterpart JC model. To initial entangled state $|I_0\rangle$ of the two qubits with control field mode in its coherent state, its evolution involves on the various parameters in Eq.(38). It depends on the number $N$ in an extremely nonlinear and intricate way. The kind perplexity makes it hard to study the Rabi model, never the less, it also provide the opportunity to preserve the entanglement of the two qubits by careful choice of the appropriate parameters. Because the coherent state has the probability of Poisson distribution, which will be approximated by a Gauss distribution if the average number $|\alpha|^2$ is large enough. The intricacy of the Rabi model could be utilized to make the quantity $\frac{\gamma^2}{\Omega^2} = B(N)$ in Eq.(38) extremely small when $N$ is in the neighborhood of $|\alpha|^2$ by some selection of the appropriate parameters, which will guaranty the initial state of the qubits unchanging. This is shown in Fig.4. It can be easy to see that whenever we select the parameters appropriate, for example, the parameters as $|\alpha|^2 = 55, a = -0.6, \beta = 0.5599, \frac{\omega}{\omega_0} = 0.24$, the Bell state $|I_0\rangle = |1, 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle)$ have the probability about $1 - 0.005 = \frac{95}{98}$ to remain unchanged.

\[ \frac{\gamma^2}{\Omega^2} \] in Eq.(38) being small in the neighborhood of $N = |\alpha|^2$ is crucial for $T(\alpha, t) \approx 0$. So we will select the zeros ($N_1, N_2, \cdots$) of $\Omega_1N$ as possible parameters for $|\alpha|^2$ under definite quantities $\beta, a$. Then $|\bar{t}_0|^2 = |-2\beta^2 + \frac{2}{\omega_0^2} + \Omega_2N|^2$ larger around zeros of $\Omega_1N$ will advantage $T(\alpha, t) \approx 0$. As a result, a negative and $\Omega_2N$ negative around zeros of $N = |\alpha|^2$ of $\Omega_1N$ are keys to make $T(\alpha, t) \approx 0$, that is, to keep the entangled state $|I_0\rangle$ unchanging.

There is an alternative method for the realization of $T(\alpha, t) \approx 0$. One could first determine the average number $T(\alpha, t) \approx 0$ of the coherent state of the control field, then chooses $\beta$ and $a$ by similar requirement that $|\bar{t}_0|^2 = |-2\beta^2 + \frac{2}{\omega_0^2} + \Omega_2N|^2$ as larger as possible around $N = |\alpha|^2$.

The parameter $a$ is connected with the inter-qubit coupling strength $\kappa$ as $a = \pm \kappa$ in symmetric and asymmetric transition cases respectively. Study also shows that the parameter $a$ negative is favorable for the maintaining the initial entanglement, especial with asymmetrical transition case ($a < 0$).

All the others’ Bell states have not this nice property because there is a simple factor $\frac{1}{2}$ in quantities $P(\delta, -\delta, \alpha, t)$ in Eq.(39). Never the less, the preservation of the entangled Bell state $|I_0\rangle$ is still useful for its application in quantum information process. Also, the complex formula of $T(\alpha, t)$ make the appropriate choice of the parameters much easier and will beneficial to the experiment application.

In summary, coupled strongly with a quantum mode field, the two qubits’ dynamics is influenced by three parameters $\beta, \frac{\omega}{\omega_0}, a$ and the initial conditions in a very complicated form. We investigate the evolution of the four Bell entangled states with the control mode in its coherent state. Three out of the four Bell states will become the combination of the four Bell states and can not remain in their initial entangled states. Nevertheless, the above mentioned complexity unexpectedly promotes our ability to preserve the entanglement of the two qubits in one Bell state $|I_0\rangle = |1, 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle)$, that is, it could remain in its initial states by suitable choice of the controlled parameters. It is shown that the parameter $a$ negative is more favorable for the maintaining the state $|I_0\rangle = |1, 0\rangle$. These results will be useful for the information process.

Acknowledgments

The work was partly supported by the Major State Basic Research Development Program of China (973 Program: No.2010CB923202) and the National Natural Sci-
ence of China (No. 10875018).

[1] M.A. Nielsen and I.L. Chuang, Quantum Computation and Quantum Information (Cambridge: Cambridge University Press) (2000)
[2] L. Mandel and E. Wolf, Optical Coherence and Quantum Optics (Cambridge University Press, Cambridge, England), (1995)
[3] S.J. van Enk, J.I. Cirac and P. Zoller, Phys. Rev. Lett. 78, 4293 (1997)
[4] P.W. Shor, In Proceedings of the 35th Annual Symposium on the foundations of Computer Science, (IEEE Press, Los Alamitos, CA, USA), (1994)
[5] A.K. Ekert and R. Josza, Phys. Rev. Lett. 67, 661 (1991)
[6] M. Yüçac and J. H. Eberly, Phys. Rev. A 82, 022321 (2010)
[7] T. Yu and J. H. Eberly, Science, 323, 598 (2009)
[8] B. Kraus and J.I. Cirac, Phys. Rev. Lett. 92, 013602 (2004)
[9] A.T. Sornborger, A.N. Cleland and M.R. Geller, Phys. Rev. A. 70, 052315 (2004)
[10] M. Paternostro, W. Son, M.S. Kim, G. Falci and G.M. Palma, Phys. Rev. A 70, 022320 (2004)
[11] M. Paternostro, W. Son, M.S. Kim, Phys. Rev. Lett. 92, 197901 (2004)
[12] L. Zhou and G. Yang, J. Phys. B 39, 5143 (2006)
[13] I. Rabi, Phys. Rev. 49, 324 (1936)
[14] E.K. Irish, J. Gea-Banacloche, I. Martin, and K.C. Schwab, Phys. Rev. A 72, 195410 (2005)
[15] S. Agarwal, S.M. Hashemi Rafsanjani, and J.H. Eberly, Phys. Rev. B 85, 043815 (2012).
[16] E.K. Irish, Phys. Rev. Lett. 99, 173601 (2007).
[17] L. Di Carlo, J. M. Chow, J. M. Gambetta, Lev S. Bishop, B. R. Johnson, D. I. Schuster, J. Majer, A. Blais, L. Frunzio, S. M. Girvin, and R. J. Schoelkopf, Nature 460, 240-244 (2009)
[18] S. Filipp, P. Maurer, P. J. Leek, M. Baur, R. Bianchetti, J. M. Fink, M. G?pppl, L. Steffen, J. M. Gambetta, A. Blais and A. Wallraff, Phys. Rev. Lett 102, 200402 (2009)
[19] A. Crespi, S. Longhi and R. Osellame, Phys. Rev. Lett. 108, 163601 (2012).
[20] Qing Ai, Yong Li, Hang Zheng and C.P. Sun, Phys. Rev. A 81, 042116 (2010).
[21] S. Ashhab and F. Nori, Phys. Rev. A 81, 042116 (2010).
[22] J.Jing, Z. G. L¨u and Z. Ficek, Phys. Rev. A 79, 044305 (2009)
[23] Z. Ficek, J.Jing and Z. G. L¨u, Phys. Scr. T 140, 014005 (2010)
[24] R.H. Dicke, Phys. Rev. 93, 99 (1954)
[25] G. H. Tian, S.Q. Zhong, 2013, submitted.
