A class formula for $L$-series in positive characteristic

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Abstract

We prove a formula for special $L$-values of Anderson modules, analogue in positive characteristic of the class number formula. We apply this result to two kinds of $L$-series.

1 Introduction

Let $\mathbb{F}_q$ be a finite field with $q$ elements and $\theta$ an indeterminate over $\mathbb{F}_q$. We denote by $A$ the polynomial ring $\mathbb{F}_q[\theta]$ and by $K$ the fraction field of $A$. For a finite $A$-module $M$, we denote by $[M]_A$ the monic generator of the Fitting ideal of $M$. The Carlitz zeta value at a positive integer $n$ is defined as

$$\zeta_A(n) := \sum_{a \in A_+} \frac{1}{\theta^a} \in K_\infty := \mathbb{F}_q((\theta^{-1})),$$

where $A_+$ is the set of monic polynomials of $A$.

The Carlitz module $C$ is the functor that associates to an $A$-algebra $B$ the $A$-module $C(B)$ whose underlying $\mathbb{F}_q$-vector space is $B$ and whose $A$-module structure is given by the homomorphism of $\mathbb{F}_q$-algebras

$$\varphi_C: \quad A \to \text{End}_{\mathbb{F}_q}(B) \quad \theta \mapsto \theta + \tau,$$

where $\tau$ is the Frobenius endomorphism $b \mapsto b^q$. For $P$ a prime of $A$ (i.e. a monic irreducible polynomial), one can show (see [8, theorem 3.6.3]) that $[C(A/PA)]_A = P - 1$. Thus

$$\zeta_A(1) = \prod_{P \text{ prime}} \left(1 - \frac{1}{P}\right)^{-1} = \prod_{P \text{ prime}} \frac{[\text{Lie}(C)(A/PA)]_A}{[C(A/PA)]_A}.$$  \hfill (1.1)

Recently, Taelman [10] associates, to a Drinfeld module $\phi$ over the ring of integers $R$ of a finite extension of $K$, a finite $A$-module called the class module $H(\phi/R)$ and an $L$-series value $L(\phi/R)$. In particular, if $\phi$ is the Carlitz module and $R$ is $A$, thanks to (1.1), we have

$$L(C/A) = \zeta_A(1).$$

These objects are related by a class formula: $L(\phi/R)$ is equal to the product of $[H(\phi/R)]_A$ times a regulator (see theorem 1 of loc. cit.).

This class formula was generalized by Fang [7], using the theory of shtukas and ideas of Vincent Lafforgue, to abelian $t$-modules over $A$, which are $n$-dimensional analogues of Drinfeld modules. In particular, for $C^\otimes n$, the $n^{th}$ tensor power of the Carlitz module, introduced by Anderson and Thakur [2], we have

$$L(C^\otimes n/A) = \zeta_A(n)$$
and this is related to a class module and a regulator as in the work of Taelman.

On the other hand, Pellarin [9] introduced a new class of \( L \)-series. Let \( t_1, \ldots, t_s \) be indeterminates over \( \mathbb{C}_\infty \), the completion of a fixed algebraic closure of \( K_\infty \). For each \( 1 \leq i \leq s \), let \( \chi_{t_i} : A \to \mathbb{F}_q[t_1, \ldots, t_s] \) be the \( \mathbb{F}_q \)-linear ring homomorphism defined by \( \chi_{t_i}(\theta) = t_i \). Then, the Pellarin’s \( L \)-value at a positive integer \( n \) is defined as

\[
L(\chi_{t_1} \cdots \chi_{t_s}, n) := \sum_{a \in A_+} \frac{\chi_{t_1}(a) \cdots \chi_{t_s}(a)}{a^n} \in \mathbb{F}_q[t_1, \ldots, t_s] \otimes_{\mathbb{F}_q} K_\infty.
\]

In this paper, inspired by ideas developed by Taelman in [10], we prove a class formula for abelian \( t \)-modules over \( \mathbb{F}_q(t_1, \ldots, t_s)[\theta] \). In particular, for \( s = 0 \), we recover theorem 1.10 of [7]. Then, we express Pellarin’s \( L \)-values as a product of quotients of Fitting ideals in the manner of (1.1). Thus, we obtain a class formula for these \( L \)-values (see section 4.2.3). This result was already used by Anglès, Pellarin and Tavares Ribeiro [4] in the 1-dimensional case, i.e. for Drinfeld modules.

Finally, let \( a \in A_+ \) be squarefree and \( L \) be the cyclotomic field associated with \( a \), i.e. the finite extension of \( K \) generated by the \( a \)-torsion of the Carlitz module. It is a Galois extension of group \( \Delta_a \simeq \langle a/A \rangle^\times \). Let \( \chi : (A/aA)^\times \to F^\times \) be a homomorphism where \( F \) is a finite extension of \( \mathbb{F}_q \). The special value at a positive integer \( n \) of Goss \( L \)-series associated to \( \chi \) is defined as

\[
L(n, \chi) := \sum_{b \in A_+} \frac{\chi(b)}{b^n} \in F \otimes_{\mathbb{F}_q} K_\infty,
\]

where \( \overline{b} \) is the image of \( b \) in \( (A/aA)^\times \). We can group all the \( L(n, \chi) \) together in one equivariant \( L \)-value \( L(n, \Delta_a) \). Then, we prove an equivariant class formula for these \( L \)-values (see theorem 4.15), generalizing that of Anglès and Taelman [5] in the case \( n = 1 \).

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2 Anderson modules and class formula

2.1 Lattices

Let \( \mathbb{F}_q \) be the finite field with \( q \) elements and \( \theta \) an indeterminate over \( \mathbb{F}_q \). We denote by \( A \) the polynomial ring \( \mathbb{F}_q[\theta] \) and by \( K \) the fraction field of \( A \). Let \( \infty \) be the unique place of \( K \) which is a pole of \( \theta \) and \( v_\infty \) the discrete valuation of \( K \) corresponding to this place with the normalization \( v_\infty(\theta) = -1 \). The completion of \( K \) at \( \infty \) is denoted by \( K_\infty \). We have \( K_\infty = \mathbb{F}_q((\theta^{-1})) \). We denote by \( \mathcal{C}_\infty \) a fixed completion of an algebraic closure of \( K_\infty \). The valuation on \( \mathcal{C}_\infty \) that extends \( v_\infty \) is still denoted by \( v_\infty \).

Let \( s \geq 0 \) be an integer and \( t_1, \ldots, t_s \) indeterminates over \( \mathbb{C}_\infty \). We set \( k_s := \mathbb{F}_q(t_1, \ldots, t_s) \), \( R_s := k_s[\theta] \), \( K_s := k_s(\theta) \) and \( K_{s, \infty} := k_s((\theta^{-1})) \). For \( f \in \mathbb{C}_\infty[t_1, \ldots, t_s] \) a polynomial expanded as a finite sum

\[
f = \sum_{i_1, \ldots, i_s \in \mathbb{N}} \alpha_{i_1, \ldots, i_s} t_1^{i_1} \cdots t_s^{i_s},
\]

with \( \alpha_{i_1, \ldots, i_s} \in \mathbb{C}_\infty \), we set

\[
v_\infty(f) := \inf \{ v_\infty(\alpha_{i_1, \ldots, i_s}) \mid i_1, \ldots, i_s \in \mathbb{N} \}.
\]
For $f \in \mathbb{C}_\infty(t_1, \ldots, t_s)$, there exists $g$ and $h$ in $\mathbb{C}_\infty[t_1, \ldots, t_s]$ such that $f = g/h$, then we define $v_\infty(f) := v_\infty(g) - v_\infty(h)$. We easily check that $v_\infty$ is a valuation, trivial on $k_s$, called the Gauss valuation. For $f \in \mathbb{C}_\infty[t_1, \ldots, t_s]$, we set $\|f\|_\infty := q^{-v_\infty(f)}$ if $f \neq 0$ and $\|0\|_\infty = 0$. The function $\|\cdot\|_\infty$ is called the Gauss norm. We denote by $\mathbb{C}_{s,\infty}$ the completion of $\mathbb{C}_\infty(t_1, \ldots, t_s)$ with respect to $v_\infty$.

Let $V$ be a finite dimensional $K_{s,\infty}$-vector space and $\| \cdot \|$ be a norm on $V$ compatible with $\| \cdot \|_\infty$ on $K_{s,\infty}$. For $r > 0$, we denote by $B(0, r) := \{ v \in V \mid \|v\| < r \}$ the open ball of radius $r$, which is a $K_s$-subspace of $V$.

**Definition.** A sub-$R_s$-module $M$ of $V$ is an $R_s$-lattice of $V$ if it is free of rank one and the $K_{s,\infty}$-vector space spanned by $M$ is $V$.

We can characterize these lattices.

**Lemma 2.1.** Let $V$ be a $K_{s,\infty}$-vector space of dimension $n \geq 1$ and $M$ be a sub-$R_s$-module of $V$. The following assertions are equivalent:

1. $M$ is an $R_s$-lattice of $V$;
2. $M$ is discrete in $V$ and every open subspace of the $k_s$-vector space $V/M$ is of finite co-dimension.

**Proof.** Let us suppose that $M$ is an $R_s$-lattice of $V$, i.e. there exists a family $(e_1, \ldots, e_n)$ of elements of $M$ such that

$$M = \bigoplus_{i=1}^n R_s e_i \quad \text{and} \quad V = \bigoplus_{i=1}^n K_{s,\infty} e_i.$$ 

Any element $v$ of $V$ can be uniquely written as $v = \sum_{i=1}^n v_i e_i$ with $v_i \in K_{s,\infty}$. Then, we set $\|v\| := \max \{ \|v_i\|_\infty \mid i = 1, \ldots, n \}$. Since $R_s$ is discrete in $K_{s,\infty}$, this implies that $M$ is discrete in $V$. Now, let $m \geq 0$ be an integer. We have

$$B(0, q^{-m}) = \bigoplus_{i=1}^n \theta^{-m-1} k_s[[\theta^{-1}]] e_i.$$ 

In particular, we have $V = M \oplus B(0, 1)$ and

$$\dim_{K_s} \frac{B(0, q^{-m})}{B(0, q^{-m-1})} = n.$$ 

This implies that every open $k_s$-subspace of $V/M$ is of finite co-dimension.

Reciprocally, let us suppose that $M$ is discrete in $V$ and every open subspace of the $k_s$-vector space $V/M$ is of finite co-dimension. Let $W$ be the $K_{s,\infty}$-subspace of $V$ generated by $M$ and $m$ be its dimension. There exist $e_1, \ldots, e_m$ in $M$ such that

$$W = \bigoplus_{i=1}^m K_{s,\infty} e_i.$$ 

Set

$$N = \bigoplus_{i=1}^m R_s e_i.$$ 

This is a sub-$R_s$-module of $M$ and an $R_s$-lattice of $W$. In particular, $M/N$ is discrete in $W/N$. Since any open $k_s$-subspace of $W/N$ is of finite co-dimension, we deduce that $M/N$ is a finite dimensional $k_s$-vector space. This implies that $M$ is a free $R_s$-module of rank $m$. Finally, observe that, if $m < n$, $V/M$ can not verify the co-dimensional property, thus $W = V$. \qed
2.2 Anderson modules and exponential map

Let $L$ be a finite extension of $K$, $L \subseteq C_\infty$. We define $R_{L,s}$ to be the subring of $L_s := L(t_1, \ldots, t_s)$ generated by $k_s$ and $O_L$, where $O_L$ is the integral closure of $A$ in $L$. We set $L_{s,\infty} := L \otimes_K K_{s,\infty}$. This is a finite dimensional $K_{s,\infty}$-vector space. We denote by $S_\infty(L)$ the set of places of $L$ above $\infty$. For a place $\nu \in S_\infty(L)$, we denote by $L_\nu$ the completion of $L$ with respect to $\nu$. Let $\pi_\nu$ be a uniformizer of $L_\nu$ and $F_\nu$ be the residue field of $L_\nu$. Then, we define $L_{s,\nu} := F_\nu(t_1, \ldots, t_s)((\pi_\nu))$ viewed as a subfield of $C_{s,\infty}$. We have an isomorphism of $K_{s,\infty}$-algebras

$$L_{s,\infty} \simeq \prod_{\nu \in S_\infty(L)} L_{s,\nu}.$$ 

Observe that $R_{L,s}$ is an $R_s$-lattice in the $K_{s,\infty}$-vector space $L_{s,\infty}$.

Let $\tau: C_{s,\infty} \to C_{s,\infty}$ be the morphism of $k_s$-algebras given by the $q$-power map on $C_{s,\infty}$.

**Lemma 2.2.** The elements of $C_{s,\infty}$ fixed by $\tau$ are those of $k_s$.

**Proof.** Obviously, $k_s \subseteq C_{s,\infty}^\tau$. Reciprocally, observe that $\mathbb{C}_{s,\infty}^\tau = \{f \in \mathbb{C}_{s,\infty} \mid v_\infty(f) = 0\}$. But we have the direct sum of $\mathbb{F}_q(\tau)$-modules

$$\{f \in \mathbb{C}_{s,\infty} \mid v_\infty(f) \geq 0\} = \mathbb{F}_q(t_1, \ldots, t_s) \oplus \{f \in \mathbb{C}_{s,\infty} \mid v_\infty(f) > 0\}.$$ 

Since $\mathbb{F}_q(t_1, \ldots, t_s)^{\tau = 1} = k_s$, we get the result. \qed

The action of $\tau$ on $L_{s,\infty} = L \otimes_K K_{s,\infty}$ is the diagonal one $\tau \otimes \tau$.

**Definition.** Let $r$ be a positive integer. An Anderson module $E$ over $R_{L,s}$ is a morphism of $k_s$-algebras

$$\phi_E: R_s \to M_n(R_{L,s})(\tau)$$

$$\theta \mapsto \sum_{j=0}^{r-1} A_j \theta^j$$

for some $A_0, \ldots, A_r \in M_n(R_{L,s})$ such that $(A_0 - \theta I_n)^n = 0$.

These objects are usually called abelian $t$-motives as in the terminology of [1] but, to avoid confusion between $t$ and the indeterminates $t_1, \ldots, t_s$, we prefer called them Anderson modules. Note also that Drinfeld modules are one-dimensional Anderson modules.

For a matrix $A = (a_{ij}) \in M_n(\mathbb{C}_{s,\infty})$, we set $v_\infty(A) \equiv \min_{1 \leq i \leq n} \{v_\infty(a_{ij})\}$ and $\tau(A) := (\tau(a_{ij})) \in M_n(\mathbb{C}_{s,\infty})$.

**Proposition 2.3.** There exists a unique skew power series $\exp_E := \sum_{j \geq 0} e_j \tau^j$ with coefficients in $M_n(L_s)$ such that

1. $e_0 = I_n$;
2. $\exp_E A_0 = \phi_E(\theta) \exp_E$ in $M_n(L_s)(\{\tau\})$;
3. $\lim_{j \to \infty} \frac{v_\infty(e_j)}{q^j} = +\infty$.

**Proof.** See proposition 2.1.4 of [1]. \qed
Observe that $\exp_E$ is locally isometric. Indeed, by the third point,

$$c := \sup_{j \geq 1} \left( -v_\infty(e_j) \right)$$

is finite. Then, for any $x \in L^n_{s,\infty}$ such that $v_\infty(x) > c$, we have

$$v_\infty \left( \sum_{j \geq 0} e_j \tau^j(x) - x \right) \geq \min_{j \geq 1} \left( v_\infty(e_j) + q^j v_\infty(x) \right) > v_\infty(x).$$

If $B$ is an $R_{L,s}$-algebra, we denote by $E(B)$ the $k_s$-vector space $B^n$ equipped with the structure of $R_s$-module induced by $\phi_E$. We can also consider the tangent space $\text{Lie}(E)(B)$ which is the $k_s$-vector space $B^n$ whose $R_s$-module structure is given by the morphism of $k_s$-algebras

$$\partial: R_s \rightarrow M_n(R_{L,s}) \quad \theta \mapsto A_0.$$ 

In particular, by the previous proposition, we get a continuous $R_s$-linear map

$$\exp_E: \text{Lie}(E)(L_{s,\infty}) \rightarrow E(L_{s,\infty}).$$

### 2.3 The class formula

In this section, we define a class module and two lattices in order to state the main result.

**Lemma 2.4.**

1. $A^n_0 = \theta^n I_n$ ;
2. $\inf_{j \in \mathbb{Z}} \left( v_\infty(A^n_j) + j \right)$ is finite.

**Proof.** See lemma 1.4 of [7].

By the second point, for any $a_j \in k_s$ and $m \in \mathbb{Z}$, the series $\sum_{j \geq m} a_j A_0^{-j}$ converges in $M_n(L_{s,\infty})$. Thus, $\partial$ can be uniquely extended to a morphism of $k_s$-algebras by

$$\partial: K_{s,\infty} \rightarrow M_n(L_{s,\infty}) \quad \sum_{j \geq m} a_j \frac{1}{\theta^j} \mapsto \sum_{j \geq m} a_j A_0^{-j},$$

where $a_j \in k_s$ and $m \in \mathbb{Z}$. Then, $\text{Lie}(E)(L_{s,\infty})$ inherits a $K_{s,\infty}$-vector space structure. Observe, by the first point of the lemma, that, for any $f \in k_s((\theta^{-q^n}))$, we have $\partial(f) = f I_n$, i.e. the action is the scalar multiplication for these elements. In particular, we get an isomorphism $\text{Lie}(E)(L_{s,\infty}) \cong L^n_{s,\infty}$ as $k_s((\theta^{-q^n}))$-modules. We deduce that $\text{Lie}(E)(L_{s,\infty})$ is a $k_s((\theta^{-q^n}))$-vector space of dimension $n q^n$, so of dimension $n$ over $K_{s,\infty}$.

**Proposition 2.5.** The $R_s$-module $\text{Lie}(E)(R_{L,s})$ is an $R_s$-lattice of $\text{Lie}(E)(L_{s,\infty})$. Furthermore, if $L = K$, the canonical basis is an $R_s$-base of $\text{Lie}(E)(R_s)$.
Proof. By the first point of the previous lemma, Lie($E$)($R_{L,s}$) and $R^m_{L,s}$ are isomorphic as $k_s[\theta^n]$-modules. Thus, Lie($E$)($R_{L,s}$) is a finitely generated $k_s[\theta^n]$-module. On the other hand, the action of an element $a \in R_s$ is the left multiplication by $a I_n + N$ where $N$ is a nilpotent matrix. Since $a I_n + N$ is an invertible matrix, Lie($E$)($R_{L,s}$) is a torsion-free $R_s$-module. Moreover, the $k_s((\theta^{-q^n}))$-vector space generated by Lie($E$)($R_{L,s}$) and $K_{s,\infty}$ is $L^n_{s,\infty} \cong$ Lie($E$)($L_{s,\infty}$). Therefore, Lie($E$)($R_{L,s}$) is a free $R_s$-module of finite rank. Looking at the dimension as $k_s$-vector space, the rank is necessarily $n$.

For the second assertion, denote by $e_i$ the $i^{th}$ vector of the canonical basis. Firstly, we show that this family spans Lie($E$)($R_s$). We proceed by induction on $\max_{1 \leq i \leq n} \deg_{\theta} x_i$ where $(x_1, \ldots, x_n)$ is in $R^n_s$. The case of degree 0 is trivial because the action of an element of $k_s$ is the scalar multiplication. Now let $m$ be a positive integer and $(x_1, \ldots, x_n)$ be a vector of $R^n_s$ such that $\max_{1 \leq i \leq n} \deg_{\theta} x_i = m$. We can write

$$
\begin{pmatrix}
    x_1 \\
    \vdots \\
    x_n 
\end{pmatrix} = \theta^m \begin{pmatrix}
    \zeta_1 \\
    \vdots \\
    \zeta_n 
\end{pmatrix},
$$

where $\zeta_1, \ldots, \zeta_n$ are elements of $k_s$. Since we have $\partial_{\theta^m} = \theta^m I_n \mod \theta^{m-1}$, we get

$$
\begin{pmatrix}
    x_1 \\
    \vdots \\
    x_n 
\end{pmatrix} = \partial_{\zeta_1 \theta^m e_1} + \cdots + \partial_{\zeta_n \theta^m e_n} \mod \theta^{m-1}.
$$

Thus we obtain the spanning property by induction.

Finally, suppose that there exists $(a_1, \ldots, a_n) \neq (0, \ldots, 0)$ in $R^n_s$ such that

$$
\sum_{i=1}^{n} \partial_{a_i} e_i = 0.
$$

Let $d := \max_{1 \leq i \leq n} \deg_{\theta} a_i$. Looking at the above equality modulo $\theta^d$, since $\partial_{\theta^d} = \theta^d I_n \mod \theta^{d-1}$, we obtain that $a_i = 0$ if $\deg_{\theta} a_i = d$, thus necessarily all the $a_i$ are zero, i.e. $e_1, \ldots, e_n$ are linearly independent in Lie($E$)($R_s$).

\begin{proposition}

1. Set

$$
H(E/R_{L_s}) := \frac{E(L_{s,\infty})}{\exp(E(L_{s,\infty})) + E(R_{L_s})}.
$$

This is a finite dimensional $k_s$-vector space, thus a finitely generated $R_s$-module and a torsion $R_s$-module, called the class module.

2. The $R_s$-module $\exp^1_E(E(R_{L,s}))$ is an $R_s$-lattice in Lie($E$)($R_{L,s}$).

\end{proposition}

\begin{proof}
Let $V$ be an open neighbourhood of 0 in $L^n_{s,\infty}$ on which $\exp_E$ acts as an isometry and such that $\exp_E(V) = V$. We have a natural surjection of $k_s$-vector spaces

$$
\frac{L^n_{s,\infty}}{R^n_{L,s} + V} \twoheadrightarrow H(E/R_{L,s}).
$$

By proposition 2.5, the left hand side is a finite dimensional $k_s$-vector space, hence a fortiori $H(E/R_{L,s})$ too.

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Now, let us prove that $\exp_E^1(E(R_{L,s}))$ is an $R_s$-lattice in $\Lie(E)(R_{L,s})$. Since the kernel of $\exp_E$ and $\Lie(E)(R_{L,s})$ are discrete in $\Lie(E)(R_{L,s})$, so is $\exp_E^1(E(R_{L,s}))$. Let $V$ be an open neighbourhood of 0 on which $\exp_E$ is isometric and such that $\exp_E(V) = V$. The exponential map induces a short exact sequence of $k_s$-vector spaces

$$0 \to \frac{\Lie(E)(R_{L,s})}{\exp_E^1(E(R_{L,s})) + V} \to \frac{E(L_{s,\infty})}{E(R_{L,s}) + V} \to H(E/R_{L,s}) \to 0.$$ 

Since the last two $k_s$-vector spaces are of finite dimension, the first one is of finite dimension too; thus $\exp_E^1(E(R_{L,s}))$ satisfies the co-dimensional property. 

An element $f \in K_{s,\infty}$ is monic if

$$f = \frac{1}{\theta^n} + \sum_{i > m} x_i \frac{1}{\theta^n},$$

where $m \in \mathbb{Z}$ and $x_i \in k_s$. For an $R_s$-module $M$ which is a finite dimensional $k_s$-vector space, we denote by $[M]_{R_s}$ the monic generator of the Fitting ideal of $M$.

Let $V$ be a finite dimensional $K_{s,\infty}$-vector space. Let $M_1$ and $M_2$ be two $R_s$-lattices in $V$. There exists $\sigma \in \GL(V)$ such that $\sigma(M_1) = M_2$. Then, we define $[M_1 : M_2]_{R_s}$ to be the unique monic representative of $k_s^\times \det \sigma$.

The aim of the next section is to prove a class formula à la Taelman for Anderson modules:

**Theorem 2.7.** Let $E$ be an Anderson module over $R_{L,s}$. The infinite product

$$L(E/R_{L,s}) := \prod_{m \text{ maximal ideal of } \mathcal{O}_L} \frac{[\Lie(E)(R_{L,s}/mR_{L,s})]_{R_s}}{[E(R_{L,s}/mR_{L,s})]_{R_s}}$$

converges in $K_{s,\infty}$. Furthermore, we have

$$L(E/R_{L,s}) = [\Lie(E)(R_{L,s}) : \exp_E^1(E(R_{L,s}))]_{R_s} [H(E/R_{L,s})]_{R_s}.$$

### 3 Proof of the class formula

The proof is very close to ideas developed by Taelman in [10] so we will only recall some statements and point out differences.

#### 3.1 Nuclear operators and determinants

Let $k$ be a field and $V$ a $k$-vector space equipped with a non-archimedean norm $\| \cdot \|$. Let $\varphi$ be a continuous endomorphism of $V$. We say that $\varphi$ is locally contracting if there exist an non empty open subspace $U \subseteq V$ and a real number $0 < c < 1$ such that $\|\varphi(u)\| \leq c\|u\|$ for all $u \in U$. Any such open subspace $U$ which moreover satisfies $\varphi(U) \subseteq U$ is called a nucleus for $\varphi$. Observe that any finite collection of locally contracting endomorphisms of $V$ has a common nucleus. Furthermore if $\varphi$ and $\phi$ are locally contracting, then so are the sum $\varphi + \psi$ and the composition $\varphi \psi$.

For every positive integer $N$, we denote by $V[[Z]]/Z^N$ the $k[[Z]]/Z^N$-module $V \otimes_k k[[Z]]/Z^N$ and by $V[[Z]]$ the $k[[Z]]$-module $V[[Z]] := \lim V[[Z]]/Z^N$ equipped with the limit topology. Observe that any continuous $k[[Z]]$-linear endomorphism $\Phi: V[[Z]] \to V[[Z]]$ is of the form

$$\Phi = \sum_{n \geq 0} \varphi_n Z^n,$$
where the $\varphi_n$ are continuous endomorphisms of $V$. Similarly, any continuous $k[[Z]]/Z^N$-linear endomorphism of $V[[Z]]/Z^N$ is of the form

$$\sum_{n=0}^{N-1} \varphi_n Z^n.$$

We say that the continuous $k[[Z]]$-linear endomorphism $\Phi$ of $V[[Z]]$ (resp. of $V[[Z]]/Z^N$) is nuclear if for all $n$ (resp. for all $n < N$), the endomorphism $\varphi_n$ of $V$ is locally contracting.

From now on, we assume that for any open subspace $U$ of $V$, the $k$-vector space $V/U$ is of finite dimension.

Let $\Phi$ be a nuclear endomorphism of $V[[Z]]/Z^N$. Let $U_1$ and $U_2$ be common nuclei for the $\varphi_n$, $n < N$. Since Proposition 8 in [10] is valid in our context,

$$\det_{k[[Z]]/Z^N}(1 + \Phi | V) \in k[[Z]]/Z^N$$

is independent of $i \in \{1, 2\}$. We denote this determinant by

$$\det_{k[[Z]]/Z^N}(1 + \Phi | V).$$

If $\Phi$ is a nuclear endomorphism of $V[[Z]]$, then we denote by $\det_{k[[Z]]}(1 + \Phi | V)$ the unique power series that reduces to $\det_{k[[Z]]/Z^N}(1 + \Phi | V)$ modulo $Z^N$ for every $N$.

Note that Proposition 9, Proposition 10, Theorem 2 and Corollary 1 of [10] are also valid in our context. We recall the statements for the convenience of the reader.

**Proposition 3.1.**

1. Let $\Phi$ be a nuclear endomorphism of $V[[Z]]$. Let $W \subseteq V$ be a closed subspace such that $\Phi(W[[Z]]) \subseteq W[[Z]]$. Then $\Phi$ is nuclear on $W[[Z]]$ and $(V/W)[[Z]]$, and

$$\det_{k[[Z]]}(1 + \Phi | V/W) = \det_{k[[Z]]}(1 + \Phi | V) \det_{k[[Z]]}(1 + \Phi | V/W).$$

2. Let $\Phi$ and $\Psi$ be nuclear endomorphisms of $V[[Z]]$. Then $(1 + \Phi)(1 + \Psi) - 1$ is nuclear, and

$$\det_{k[[Z]]}(1 + \Phi)(1 + \Psi) | V) = \det_{k[[Z]]}(1 + \Phi | V) \det_{k[[Z]]}(1 + \Psi | V).$$

**Theorem 3.2.**

1. Let $\varphi$ and $\psi$ be continuous $k$-linear endomorphisms of $V$ such that $\varphi$, $\varphi \psi$ and $\psi \varphi$ are locally contracting. Then

$$\det_{k[[Z]]}(1 + \varphi \psi | V) = \det_{k[[Z]]}(1 + \psi \varphi | V).$$

2. Let $N \geq 1$ be an integer. Let $\varphi$ and $\psi$ be continuous $k$-linear endomorphisms of $V$ such that all compositions $\varphi$, $\varphi \psi$, $\psi \varphi$, $\varphi^2$, etc. in $\varphi$ and $\psi$ containing at least one endomorphism $\varphi$ and at most $N - 1$ endomorphisms $\psi$ are locally contracting. Let $\Delta = \sum_{n=1}^{N-1} \gamma_n Z^n$ such that

$$1 + \Delta = \frac{1 - (1 + \varphi) \psi Z}{1 - \psi (1 + \varphi) Z} \mod Z^N.$$

Then $\Delta$ is a nuclear endomorphism of $V[[Z]]$ and

$$\det_{k[[Z]]}(1 + \Delta | V) = 1 \mod Z^N.$$
3.2 Taelman’s trace formula

Let $L$ be a finite extension of $K$ and $E$ be the Anderson module given by

$$\phi: R_s \rightarrow M_n(\mathcal{R}_{L,s})\{\tau\}$$

$$\theta \mapsto \sum_{j=0}^{r} A_j \tau^j$$

for some $A_0, \ldots, A_r \in M_n(\mathcal{R}_{L,s})$ such that $(A_0 - \theta I_n)^n = 0$. Let $M_n(\mathcal{R}_{L,s})\{\tau\}[Z]$ be the ring of formal power series in $Z$ with coefficients in $M_n(\mathcal{R}_{L,s})\{\tau\}$, the variable $Z$ being central.

We set

$$\Theta := \sum_{n \geq 1} (\theta - \Phi \theta) \theta_0^{n-1} Z^n \in M_n(\mathcal{R}_{L,s})[Z].$$

**Lemma 3.3.** Let $\mathfrak{m}$ be a maximal ideal of $\mathcal{O}_L$. In $K_{s,\infty}$, the following equality holds:

$$\frac{[\text{Lie}(E)](\mathcal{R}_{L,s}/\mathfrak{m}\mathcal{R}_{L,s})_{R_s}}{[E(\mathcal{R}_{L,s}/\mathfrak{m}\mathcal{R}_{L,s})]_{R_s}} = \det_{k_s(\mathcal{O}_L)}(1 + \Theta | (\mathcal{R}_{L,s}/\mathfrak{m}\mathcal{R}_{L,s})^n)^{1-n}|_{Z=\theta^{-1}}.$$

**Proof.** It is an easy computation using the definition of Fitting ideal and of $\Theta$. $\blacksquare$

Let $S$ be a finite set of places of $L$ containing $S_{\infty}(L)$. Denote by $\mathcal{O}_S$ the ring of regular functions outside $S$. In particular $\mathcal{O}_L \subseteq \mathcal{O}_S$. Let $R_{S,s}$ be the subring of $L_s$ generated by $\mathcal{O}_S$ and $k_s$. For example, if $S = S_{\infty}(L)$, we have $R_{S,s} = \mathcal{R}_{L,s}$.

Let $\mathfrak{p}$ be a maximal ideal of $\mathcal{O}_L$ which is not in $S$. The natural inclusion $\mathcal{O}_L \hookrightarrow \mathcal{O}_S$ induces an isomorphism $\mathcal{R}_{L,s}/\mathfrak{p}\mathcal{R}_{L,s} \cong R_{S,s}/\mathfrak{p}R_{S,s}$. By the previous lemma, we obtain

$$\frac{[\text{Lie}(E)](\mathcal{R}_{L,s}/\mathfrak{p}\mathcal{R}_{L,s}))_{R_s}}{[E(\mathcal{R}_{L,s}/\mathfrak{p}\mathcal{R}_{L,s})]_{R_s}} = \det_{k_s(\mathcal{O}_L)}(1 + \Theta | (\mathcal{R}_{L,s}/\mathfrak{p}\mathcal{R}_{L,s})^n)^{1-n}|_{Z=\theta^{-1}}. \quad (3.1)$$

Denote by $\mathcal{L}_{s,\mathfrak{p}}$ the $\mathfrak{p}$-adic completion of $L_s$, i.e. the completion of $L_s$ with respect to the valuation $v_\mathfrak{p}$ defined on $L[t_1, \ldots, t_s]$ by

$$v_\mathfrak{p}\left(\sum_{i_1, \ldots, i_s \in \mathbb{N}} \alpha_{i_1, \ldots, i_s} t_1^{i_1} \cdots t_s^{i_s}\right) := \inf_{i_1, \ldots, i_s \in \mathbb{N}} v_\mathfrak{p}(\alpha_{i_1, \ldots, i_s}),$$

where $v_\mathfrak{p}$ is the normalized $\mathfrak{p}$-adic valuation on $L$. Denote by $\mathcal{O}_{s,\mathfrak{p}}$ the valuation ring of $\mathcal{L}_{s,\mathfrak{p}}$. By the strong approximation theorem, for any $n > 0$, there exists $\pi_n \in L$ such that $v_\mathfrak{p}(\pi_n) = -n$ and $v(\pi_n) \geq 0$ for all $v \not\in S \cup \mathfrak{p}$. Thus, we have

$$\mathcal{L}_{s,\mathfrak{p}} = \mathcal{O}_{s,\mathfrak{p}} + R_{\mathcal{O}_{s,\mathfrak{p}},s} \quad \text{and} \quad R_{s,s} = \mathcal{O}_{s,\mathfrak{p}} + R_{\mathcal{O}_{s,\mathfrak{p}},s}. \quad (3.2)$$

Finally, denote by $\mathcal{L}_{s,S}$ the product of the completions of $L_s$ with respect to places of $S$. For example, if $S = S_{\infty}(L)$, we have $\mathcal{L}_{s,S} = \mathcal{L}_{s,\infty}$.

Recall that $R_{S,s}$ is a Dedekind domain, discrete in $\mathcal{L}_{s,S}$ and such that every open subspace of $L_{s,S}/R_{S,s}$ is of finite co-dimension. Observe also that any element of $M_n(\mathcal{R}_{S,s})\{\tau\}$ induces a continuous $k_s$-linear endomorphism of $(L_{s,S}/R_{S,s})^n$ which is locally contracting. In particular, the endomorphism $\Theta$ is a nuclear operator of $(L_{s,S}/R_{S,s})^n[[Z]].$

**Lemma 3.4.** Let $\mathfrak{p}$ be a maximal ideal of $\mathcal{O}_L$ which is not in $S$. Then

$$\det_{k_s(\mathcal{O}_L)}(1 + \Theta | (\mathcal{R}_{L,s}/\mathfrak{p}\mathcal{R}_{L,s})^n) = \frac{\det_{k_s(\mathcal{O}_L)}(1 + \Theta | (\mathcal{R}_{L,s} \times L_{s,\mathfrak{p}})^n)}{\det_{k_s(\mathcal{O}_L)}(1 + \Theta | (L_{s,\mathfrak{p}})^n)}.$$

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Proof. The proof is the same as that of lemma 1 of [10], using equalities (3.2).

Proposition 3.5. The following equality holds in $K_{s,\infty}$:

$$L(E/R_{L,s}) = \prod_m \det_{k_i[[z]]}(1 + \Theta \mid (R_{L,s}/mR_{L,s})^n)^{-1}|_{Z=\theta^{-1}}.$$

In particular, $L(E/R_{L,s})$ converges in $K_{s,\infty}$.

Proof. By lemma 3.3, we have

$$L(E/R_{L,s}) = \prod_m \det_{k_i[[z]]}(1 + \Theta \mid (R_{L,s}/mR_{L,s})^n)^{-1}|_{Z=\theta^{-1}},$$

where the product runs through maximal ideals of $O_L$. Fix $S \supseteq S_{\infty}(L)$ above (the case $S = S_{\infty}(L)$ suffices). By equality (3.1), we have

$$\prod_m \det_{k_i[[z]]}(1 + \Theta \mid (R_{L,s}/mR_{L,s})^n)^{-1} = \prod_m \det_{k_i[[z]]}(1 + \Theta \mid (R_{S,s}/mR_{S,s})^n)^{-1},$$

where the products run through maximal ideals of $O_L$ which are not in $S$.

Define $S_{D,N}$ as in [10]. It suffices to prove that for any $1 + F \in S_{D,N}$, the infinite product

$$\prod_{m \notin S \setminus S_{\infty}(L)} \det_{k_i[[z]]/Z^N}(1 + F \mid \left(\frac{R_{S,s}}{mR_{S,s}}\right)^n)$$

converges to

$$\det_{k_i[[z]]/Z^N}(1 + F \mid \left(\frac{L_{s,s}}{R_{S,s}}\right)^n)^{-1}.$$

Let $m_1, \ldots, m_r$ be the maximal ideals of $O_L$ which are not in $S$ and such that $m_i R_{S,s}$ is a maximal ideal of $R_{S,s}$ verifying $\dim_{k_i} R_{S,s}/m_i R_{S,s} < D$. Applying successively lemma 3.4 to $R_{S,s}$, $R_{S\cup\{m_1\},s}$, $R_{S\cup\{m_1,m_2\},s}$, etc., we obtain the following equality:

$$\frac{\det_{k_i[[z]]}(1 + F \mid \left(\frac{L_{s,s}}{R_{S,s}}\right)^n) \prod_m \det_{k_i[[z]]}(1 + F \mid \left(\frac{R_{S,s}}{mR_{S,s}}\right)^n)}{\prod_{m \notin \{m_1, \ldots, m_r\}} \det_{k_i[[z]]}(1 + F \mid \left(\frac{R_{S,s}}{mR_{S,s}}\right)^n)}.$$ 

This allows us, replacing $R_{S,s}$ by $R_{S\cup\{m_1, \ldots, m_r\},s}$, to suppose that $R_{S,s}$ has not maximal ideal of the form $mR_{S,s}$ with $m$ maximal ideal of $O_L$ which is not in $S$ such that $\dim_{k_i} R_{S,s}/mR_{S,s} < D$. Then, we can finish the proof as in [10].

3.3 Ratio of co-volumes

Let $V$ be a finite dimensional $K_{s,\infty}$-vector space and $\| \cdot \|$ be a norm on $V$ compatible with $\| \cdot \|_\infty$ on $K_{s,\infty}$. Let $M_1$ and $M_2$ be two $R$-lattices in $V$ and $N \in \mathbb{N}$. A continuous $k_i$-linear map $\gamma: V/M_1 \to V/M_2$ is $N$-tangent to the identity on $V$ if there exists an open $k_i$-subspace $U$ of $V$ such that

1. $U \cap M_1 = U \cap M_2 = \{0\}$;

2. $\gamma$ restricts to an isometry between the images of $U$;
3. for any \( u \in U \), we have \( \| \gamma(u) - u \| \leq q^{-N} \| u \| \).

The map \( \gamma \) is infinitely tangent to the identity on \( V \) if it is \( N \)-tangent for every positive integer \( N \).

**Proposition 3.6.** Let \( \gamma \in M_n(L_s) \{ \{ \tau \} \} \) be a power series convergent on \( L_s^{n,\infty} \) with constant term equal to 1 and such that \( \gamma(M_1) \subseteq M_2 \). Then \( \gamma \) is infinitely tangent to the identity on \( L_s^{n,\infty} \).

**Proof.** See proposition 12 of [10].

For example, by proposition 2.3, the map

\[
\exp_E : \frac{\operatorname{Lie}(E)(L_s,\infty)}{\exp_E^1(E(R_{L,s}))} \longrightarrow \frac{E(L_s,\infty)}{E(R_{L,s})}
\]

is infinitely tangent to the identity on \( L_{s,\infty} \).

Now, let \( H_1 \) and \( H_2 \) two finite dimensional \( k_s \)-vector spaces which are also \( R_s \)-modules and set \( N_i := \frac{1}{2^i} \times H_i \) for \( i = 1, 2 \). A \( k_s \)-linear map \( \gamma : N_1 \rightarrow N_2 \) is \( N \)-tangent (resp. infinitely tangent) to the identity on \( V \) if the composition

\[
\frac{V}{M_1} \rightarrow N_1 \xrightarrow{\gamma} N_2 \rightarrow \frac{V}{M_2}
\]

is so. For a \( k_s \)-linear isomorphism \( \gamma : N_1 \rightarrow N_2 \), we define an endomorphism

\[
\Delta_\gamma := \frac{1 - \gamma^{-1} \partial_0 \gamma Z}{1 - \partial_0 Z} - 1 = \sum_{i \geq 1} (\partial_0 - \gamma^{-1} \partial_0 \gamma) \partial_0^{-1} Z^n
\]

of \( N_1[[Z]] \).

**Proposition 3.7.** If \( \gamma \) is infinitely tangent to the identity on \( V \), then \( \Delta_\gamma \) is nuclear and

\[
\det_{k_s[[Z]]}(1 + \Delta_\gamma \mid N_1) \mid Z=\theta^{-1} = [M_1 : M_2] R_s \frac{[H_2]_{R_s}}{[H_1]_{R_s}}
\]

**Proof.** See theorem 4 of [10].

### 3.4 Proof of theorem 2.7

By theorem 3.5, \( L(E/R_{L,s}) \) converges in \( K_{s,\infty} \) and

\[
L(E/R_{L,s}) = \det_{k_s[[Z]]}(1 + \Theta \mid (L_{s,\infty}/R_{L,s})^n) \mid Z=\theta^{-1}.
\]

The exponential map \( \exp_E \) induces a short exact sequence of \( R_s \)-modules

\[
0 \longrightarrow \frac{\operatorname{Lie}(E)(L_s,\infty)}{\exp_E^1(E(R_{L,s}))} \longrightarrow \frac{E(L_s,\infty)}{E(R_{L,s})} \longrightarrow H(E/R_{L,s}) \longrightarrow 0.
\]

By proposition 2.6, the \( k_s \)-vector space \( H(E/R_{L,s}) \) is of finite dimension. Moreover, since the \( R_s \)-module on the left is divisible and \( R_s \) is principal, the sequence splits. The choice of a section gives rise to an isomorphism of \( R_s \)-modules

\[
\frac{\operatorname{Lie}(E)(L_s,\infty)}{\exp_E^1(E(R_{L,s}))} \times H(E/R_{L,s}) \simeq \frac{E(L_s,\infty)}{E(R_{L,s})}.
\]
This isomorphism can be restricted to an isomorphism of $k_s$-vector space

$$
\gamma: \text{Lie}(E)(L_{s,\infty}) \times H(E/R_{L,s}) \xrightarrow{\sim} \left(\frac{L_{s,\infty}}{R_{L,s}}\right)^n.
$$

Observe that $\gamma$ corresponds with the map induced by $\exp_E$. By proposition 3.6, $\gamma$ is infinitely tangent to the identity on $L_{s,\infty}$. By second point of proposition 2.3, we have $\exp_E \partial_\theta \exp_E^{-1} = \phi_\theta$, hence the equality of $k_s[[Z]]$-linear endomorphisms of $\left(\frac{L_{s,\infty}}{R_{L,s}}\right)^n$:

$$
1 + \Theta = \frac{1 - \gamma \partial_\theta \gamma^{-1} Z}{1 - \partial_\theta Z}.
$$

Thus, by theorem 3.7, we obtain

$$
\det_{k_s[[Z]]}(1 + \Theta | (L_{s,\infty}/R_{L,s})^n) |_{Z=\theta^{-1}} = [\text{Lie}(E)(R_{L,s}) : \exp_E^1(E(R_{L,s}))]_{R_s}[H(E/R_{L,s})]_{R_s}.
$$

This concludes the proof.

4 Applications

4.1 The $n$th tensor power of the Carlitz module

Let $\alpha$ be a non-zero element of $R_s$. Let $E_\alpha$ be the Anderson module defined by the morphism of $k_s$-algebras $\phi: R_s \to M_n(R_s)\{\tau\}$ given by

$$
\phi_\theta = \partial_\theta + N_\alpha \tau,
$$

where

$$
\partial_\theta = \begin{pmatrix}
\theta & 1 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 \\
0 & \cdots & 0 & \theta
\end{pmatrix}
$$

and

$$
N_\alpha = \begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 \\
\alpha & 0 & \cdots & 0
\end{pmatrix}.
$$

In other words, if $\tau(x_1, \ldots, x_n) \in C^n_{s,\infty}$, we have

$$
\phi_\theta \left( \begin{array}{c}
x_1 \\
\vdots \\
x_n
\end{array} \right) = \left( \begin{array}{c}
\theta x_1 + x_2 \\
\vdots \\
\theta x_{n-1} + x_n \\
\theta x_n + \alpha \tau(x_1)
\end{array} \right).
$$

The case $\alpha = 1$ is denoted by $C^n\otimes$, the $n$th tensor power of Carlitz module, introduced in [2]. In this section, we show that the exponential map associated to $C^n\otimes$ is surjective on $C^n_{s,\infty}$ and we recall its kernel.

4.1.1 Surjectivity and kernel of $\exp_{C^n\otimes}$

By proposition 2.3, there exists a unique exponential map $\exp_{C^n\otimes}$ associated with $C^n\otimes$ and by [2, section 2], there exists a unique formal power series

$$
\log_{C^n\otimes} = \sum_{i \geq 0} P_i \tau^i \in M_n(C_{s,\infty})\{\{\}\}.
$$
Thus τ
y

The following assertions are equivalent:

\begin{enumerate}
\item \(\exists x\) such that \(\exp_C(x) = y\).
\item \(\exists x\) such that \(\log_C(x) = y\).
\end{enumerate}

Proposition 4.1. The exponential map \(\exp_{\mathbb{C}^n}\) is surjective on \(\mathbb{C}^n_r\).

To prove this, we reduce to the one dimensional case.

Lemma 4.2. The following assertions are equivalent:

\begin{enumerate}
\item \(\exp_{\mathbb{C}^n}\) is surjective on \(\mathbb{C}^n_r\);
\item \(C^\otimes_n\) is surjective on \(\mathbb{C}^n_r\);
\item \(\tau - 1\) is surjective on \(\mathbb{C}^n_r\).
\end{enumerate}

Proof. It is easy to show that that (1) implies (2). Indeed, let \(y \in \mathbb{C}^n_r\). By hypothesis, there exists \(x \in \mathbb{C}^n_r\) such that \(\exp_{\mathbb{C}^n}(x) = y\). Hence we have

\[
C^\otimes_n \exp_{\mathbb{C}^n}(\theta_n^{-1} x) = \exp_{\mathbb{C}^n}(x) = y.
\]

Next we prove that (2) implies (3). Since \(C^\otimes_n\) is supposed to be surjective on \(\mathbb{C}^n_r\), for any \(y = (y_1, \ldots, y_n) \in \mathbb{C}^n_r\), there exists \(x = (x_1, \ldots, x_n) \in \mathbb{C}^n_r\) such that

\[
\begin{align*}
\theta x_1 + x_2 &= y_1 \\
\vdots \quad & \\
\theta x_{n-1} + x_n &= y_{n-1} \\
\theta x_n + \tau(x_1) &= y_n
\end{align*}
\]

In particular, we get

\[
\tau(x_1) - (-\theta)^n x_1 = \sum_{i=1}^n (-\theta)^{n-i} y_i.
\]

Thus \(\tau - (-\theta)^n\) is surjective on \(\mathbb{C}^n_r\). But we have

\[
\tau \left( (-\theta) \frac{-\theta}{\tau - \theta} \right) = (-\theta)^n (-\theta) \frac{-\theta}{\tau - \theta},
\]

hence \(\tau - 1\) is also surjective on \(\mathbb{C}^n_r\).

In fact, it is also easy to check that (3) implies (2). As in the previous case, the surjectivity of \(\tau - (-\theta)^n\) is deduced from the surjectivity of \(\tau - 1\). Hence, for a fixed \(y = (y_1, \ldots, y_n) \in \mathbb{C}^n_r\), there exists \(x_1 \in \mathbb{C}^n_r\) verifying equation (4.1). Then, by back-substitution, we find successively \(x_2, \ldots, x_n \in \mathbb{C}^n_r\) such that \(x = (x_1, \ldots, x_n)\) satisfies \(C^\otimes_n(x) = y\).

We finally prove that (2) implies (1). Since \(\log_{\mathbb{C}^n}\) converges on the polydisc \(D_n(n-i - \frac{m}{\tau - n}, i = 1, \ldots, n)\) and \(\exp_{\mathbb{C}^n} \log_{\mathbb{C}^n}\) is the identity map on it, this polydisc is included in the image of
the exponential. We will "grow" this polydisc to show that \( \exp_{C^n} \) is surjective. For \( i = 1, \ldots, n \), we define

\[
r_{0,i} := n - i - \frac{nq}{q-1} = -i - \frac{n}{q-1},
\]

and for \( k \geq 1 \),

\[
r_{k+1,i} = \begin{cases} r_{k,i+1} & \text{if } 1 \leq i \leq n-1, \\ qr_{k,1} & \text{if } i = n. \\ \end{cases}
\]

By induction, we prove that for any integer \( k \geq 0 \) and any \( 1 \leq i \leq n-1 \),

\[
r_{k,i+1} \leq r_{k,i} - 1.
\]

We also prove that for any integer \( k \geq 0 \) and \( i \in \{1, \ldots, n\} \), we have \( r_{k,i} \leq r_{0,i} - k \). In particular, for any \( 1 \leq i \leq n \), the sequence \( (r_{k,i}) \) tends to \(-\infty\), i.e. the polydiscs \( D_n(r_{k,i}, i = 1, \ldots, n) \) cover \( \mathbb{C}_s^{n,\infty} \). Thus, it suffices to show that \( D_n(r_{k,i}, i = 1, \ldots, n) \subseteq \Im \exp_{C^n} \) for any integer \( k \geq 0 \).

The case \( k = 0 \), corresponding to the convergence domain of \( \log_{C^n} \), is already known. Let us suppose that \( D_n(r_{k,i}, i = 1, \ldots, n) \) is included in the image of \( \exp_{C^n} \) for an integer \( k \geq 0 \). Let \( y \) be an element of \( D_n(r_{k+1,i}, i = 1, \ldots, n) \setminus D_n(r_{k,i}, i = 1, \ldots, n) \).

We claim that there exists \( x \in D_n(r_{k,i}, i = 1, \ldots, n) \) such that \( C^n_\theta(x) = y \).

Assume temporarily this. Since \( D_n(r_{k,i}, i = 1, \ldots, n) \subseteq \Im \exp_{C^n} \), there exists \( z \in \mathbb{C}_s^{n,\infty} \) such that \( \exp_{C^n}(z) = x \). Thus

\[
\exp_{C^n}(\partial z) = C^n_\theta \exp_{C^n}(z) = C^n_\theta(x) = y.
\]

In particular \( y \) is in the image of the exponential as expected.

It only remains to prove the claim. By hypothesis, there exists \( x = (x_1, \ldots, x_n) \in \mathbb{C}_s^{n,\infty} \) such that

\[
\left\{ \begin{array}{l} x_2 = y_1 - \theta x_1 \\ \vdots \\ x_n = y_{n-1} - \theta x_n \\ 
\tau(x_1) - (-\theta)^n x_1 = \sum_{i=1}^{n} (-\theta)^{n-i} y_i 
\end{array} \right.
\]

We need to show that \( x \) is in \( D_n(r_{k,i}, i = 1, \ldots, n) \). Let begin by showing \( v_\infty(x_1) > r_{k,1} \). If \( v_\infty(x_1) = \frac{n\theta}{q-1} \), then \( v_\infty(x_1) > r_{0,1} > r_{k,1} \). So we may suppose that \( v_\infty(x_1) \neq \frac{n\theta}{q-1} \). Then

\[
v_\infty(\tau(x_1) - (-\theta)^n x_1) = \min(qv_\infty(x_1) ; v_\infty(x_1) - n) = \min(qv_\infty(x_1) ; v_\infty(x_1) - n).
\]

In particular,

\[
qv_\infty(x_1) \geq v_\infty \left( \sum_{i=1}^{n} (-\theta)^{n-i} y_i \right) \geq \min_{1 \leq i \leq n} (v_\infty(y_i) - n + i) > \min_{1 \leq i \leq n} (r_{k+1,i} - n + i),
\]

where the last inequality comes from the fact that \( y \) is in \( D_n(r_{k+1,i}, i = 1, \ldots, n) \). But, by the inequality (4.2), we have

\[
r_{k+1,n} \leq r_{k+1,n-1} - 1 \leq \cdots \leq r_{k+1,1} - n + 1.
\]

Hence we get

\[
qv_\infty(x_1) > r_{k+1,n} = qr_{k,1},
\]

as desired.
Finally, we show that $v_\infty(x_i) > r_{k,i}$ for all $2 \leq i \leq n$. Since $y \in D_n(r_{k+1,i}, i = 1, \ldots, n)$, we have

$$v_\infty(x_2) \geq \min(v_\infty(y_1) ; v_\infty(x_1) - 1) > \min(r_{k+1,1} ; r_{k,1} - 1) = r_{k,2},$$

where the last equality comes from the definition of $r_{k+1}$ and from inequality (4.2). On the same way, we obtain the others needed inequalities.

**Lemma 4.3.** The application $\tau - 1 : C_{s,\infty} \to C_{s,\infty}$ is surjective.

**Proof.** Since $\sum_{i \geq 0} \tau^i(x)$ converges for $x \in C_{s,\infty}$ such that $v_\infty(x) > 0$, we have

$$\{x \in C_{s,\infty} | v_\infty(x) > 0\} \subseteq \operatorname{Im}(\tau - 1).$$

Thus, since $C_{s,\infty}(t_1, \ldots, t_s)$ is dense in $C_{s,\infty}$, it suffices to show that $C_{s,\infty}(t_1, \ldots, t_s) \subseteq (\tau - 1)(C_{s,\infty})$. Observe that $(\tau - 1)(C_{s,\infty}[t_1, \ldots, t_s]) = C_{s,\infty}[t_1, \ldots, t_s]$. Now let $f \in C_{s,\infty}(t_1, t_2)$. We can write

$$f = \frac{g}{h} \quad \text{with} \quad g, h \in C_{s,\infty}[t_1, \ldots, t_s] \quad \text{and} \quad v_\infty(h) = 0.$$ 

Now write $h = \delta - z$ with $\delta \in \mathbb{F}_q[t_1, \ldots, t_s] \setminus \{0\}$ and $z \in C_{s,\infty}[t_1, \ldots, t_s]$ such that $v_\infty(z) > 0$. Then, in $C_{s,\infty}$, we have

$$f = \frac{g}{h} = \sum_{k \geq 0} \frac{g^k}{\delta^{k+1}}.$$

On the one hand, since the series converges, there exists $k_0 \in \mathbb{N}$ such that

$$v_\infty\left(\sum_{k \geq k_0} \frac{g^k}{\delta^{k+1}}\right) > 0.$$ 

In particular, this sum is in the image of $\tau - 1$. On the other hand, we have

$$\sum_{k = 0}^{k_0 - 1} \frac{g^k}{\delta^{k+1}} \in \frac{1}{\delta^{k_0}}C_{s,\infty}[t_1, \ldots, t_s].$$

But we can write $\frac{1}{\delta} = \frac{\beta}{\gamma}$ with $\beta \in \mathbb{F}_q[t_1, \ldots, t_s]$ and $\gamma \in \mathbb{F}_q[t_1, \ldots, t_s] \setminus \{0\}$. Hence

$$\sum_{k = 0}^{k_0 - 1} \frac{g^k}{\delta^{k+1}} \in \frac{1}{\gamma}C_{s,\infty}[t_1, \ldots, t_s] \subseteq (\tau - 1)\left(\frac{1}{\gamma}C_{s,\infty}[t_1, \ldots, t_s]\right).$$

Thus, by linearity of $\tau - 1$, we get $f \in (\tau - 1)(C_{s,\infty})$. 

Denote by $\Lambda_n$ the kernel of the morphism of $R_s$-modules

$$\exp_{C_{s,\infty}} : \operatorname{Lie}(C_{s,\infty}) \to C_{s,\infty}.$$ 

**Proposition 4.4.** The $R_s$-module $\Lambda_n$ is free of rank 1 and is generated by a vector with $\tilde{\pi}^n$ as last coordinate.

**Proof.** See [2, section 2.5].
4.1.2 Characterization of Anderson modules isomorphic to $C^\otimes n$

We characterize Anderson modules which are isomorphic, in a sense described below, to the $n^{th}$ tensor power of the Carlitz module. We obtain an $n$-dimensional analogue of proposition 6.2 of [4].

**Definition.** Two Anderson modules $E$ and $E'$ are isomorphic if there exists a matrix $P \in \text{GL}_n(C_{s,\infty})$ such that $E \theta \overset{P}{=} P E' \theta$ in $M_n(C_{s,\infty})\{\tau\}$.

Let $\alpha \in R_s$. Denote by $E_\alpha$ the Anderson module defined at the beginning of section 4.1. Note that $E_\alpha$ and $C^\otimes n$ are isomorphic if and only if there exists a matrix $P \in \text{GL}_n(C_{s,\infty})$ such that
\[
\partial_\theta P = P \partial_\theta \quad \text{and} \quad N_1(\tau(P)) = P N_\alpha.
\]

(4.3)

Let us set $U_s := \{ \alpha \in C_{s,\infty}^* | \exists \beta \in C_{\infty}^*, \gamma \in \mathbb{F}_q(t_1, \ldots, t_s), v_\infty \left( \alpha - \beta \frac{\tau(\gamma)}{\gamma} \right) > v_\infty(\alpha) \}$.

**Lemma 4.5.** The map which associates to any element $x$ of $C_{s,\infty}^*$ the element $\frac{\tau(x)}{x}$ of $C_{s,\infty}^*$ induces a short exact sequence of multiplicative groups
\[
1 \rightarrow k_s^* \rightarrow C_{s,\infty}^* \rightarrow U_s \rightarrow 1.
\]

**Proof.** The kernel comes from lemma 2.2.

Let $\alpha \in C_{s,\infty}^*$ such that there exists $x \in C_{s,\infty}^*$ verifying $\tau(x) = \alpha x$. Since $C_{\infty}$ is an algebraically closed field, one can suppose that $v_\infty(\alpha) = 0$. We write $x = \gamma + m$ with $\gamma \in \mathbb{F}_q(t_1, \ldots, t_s)$ and $m \in C_{s,\infty}^*$ such that $v_\infty(m) > 0$. Then, we have $v_\infty(\tau(\gamma) - \alpha \gamma) > 0$, i.e. $\alpha \in U_s$.

Reciprocally, let $\alpha \in U_s$ and $\beta \in C_{\infty}^*, \gamma \in \mathbb{F}_q(t_1, \ldots, t_s)$ such that
\[
v_\infty \left( \alpha - \beta \frac{\tau(\gamma)}{\gamma} \right) > v_\infty(\alpha).
\]

We set $\delta := \beta \frac{\tau(\gamma)}{\gamma}$. Observe that $\prod_{i \geq 0} \frac{\tau^i(\delta)}{\tau^i(\alpha)}$ converges in $C_{s,\infty}^*$. Now, since $\tau$ is $k_s$-linear, there exists $\varepsilon \in C_{\infty}^* \mathbb{F}_q(t_1, \ldots, t_s)$ such that $\tau(\varepsilon) = \delta$. Then, we set
\[
\omega_\alpha := \varepsilon \prod_{i \geq 0} \frac{\tau^i(\delta)}{\tau^i(\alpha)} \in C_{s,\infty}^*.
\]

(4.4)

Thus, we have $\tau(\omega_\alpha) = \alpha \omega_\alpha$. Observe that $\omega_\alpha$ is defined up to a scalar factor in $\mathbb{F}_q^*$ whereas it depends a priori on the choices of $\beta$, $\gamma$ and $\varepsilon$.

We are now able to characterize Anderson modules which are isomorphic to $C^\otimes n$.

**Proposition 4.6.** The following assertion are equivalent:

1. $E_\alpha$ is isomorphic to $C^\otimes n$,
2. $\alpha \in U_s$,
3. $\exp_\alpha$ is surjective,
4. $\ker \exp_\alpha$ is a free $R_s$-module of rank 1,
where $\exp_\alpha$ is the exponential map associated with $E_\alpha$ by proposition 2.3.

**Proof.** Setting $P = \omega_n I_n$ where $\omega_n$ is defined by (4.4), we see that (2) implies (1).

We prove that (1) implies (3). Let $P \in \text{GL}_n(C_{s,\infty})$ such that $C_{\theta}^{\otimes n}P = PE_\theta$. Using equalities (4.3), we check that

$$P^{-1}\exp_{C_{\theta}^{\otimes n}} P \partial_\theta = E_\theta P^{-1}\exp_{C_{\theta}^{\otimes n}} P.$$

Thus, by unicity in proposition 2.3, we get $P^{-1}\exp_{C_{\theta}^{\otimes n}} P = \exp_\alpha$. In particular, by proposition 4.1, we deduce that $\exp_\alpha$ is surjective.

Next, we prove that (3) implies (2). We can assume that $v_\infty(\alpha) = 0$. By lemma 4.5, it suffices to show that $\ker(\alpha \tau - 1)$ is not trivial. Let us suppose the converse. As at the beginning of the proof of lemma 4.2, we easily show that the surjectivity of $\exp_\alpha$ on $C_{s,\infty}$ implies that of $\alpha \tau - 1$ on $C_{s,\infty}$. Thus, $\alpha \tau - 1$ is an automorphism of the $k_s$-vector space $C_{s,\infty}$. We verify that $v_\infty(f) = 0$ if and only if $v_\infty(\alpha \tau(f) - f) = 0$. Let $\overline{\eta} \in \mathbb{F}_q(t_1, \ldots, t_s)$ such that $v_\infty(\alpha - \overline{\eta}) > 0$. Then, $\overline{\eta} \tau - 1$ is an automorphism of the $k_s$-vector space $\mathbb{F}_q(t_1, \ldots, t_s)$, which is obviously false.

It is easy to show that (1) implies (4). Indeed, since $E_\alpha$ is isomorphic to $C_{\otimes n}$, we have

$$\ker \exp_\alpha = \frac{1}{\omega_n} \ker \exp_{C_{\otimes n}}.$$

Thus, by proposition 4.4, $\ker \exp_\alpha$ is a free $R_s$-module of rank 1 generated by a vector with $\frac{\overline{\eta}^n}{\omega_n}$ as last coordinate.

Finally, we prove that (4) implies (2). Let $f$ be a non zero element of $\ker \exp_\alpha$ such that $\partial_\theta^n f \notin \ker \exp_\alpha$. Thus, the vector $g := \exp_\alpha(\partial_\theta^n f) \in C_{s,\infty}^{\otimes n}$ is non zero and $E_\theta(g) = 0$. Denote by $g_1, \ldots, g_n$ its coordinates. We have

$$\begin{cases}
\theta g_1 + g_2 = 0 \\
\vdots \\
\theta g_{n-1} + g_n = 0 \\
\theta g_n + \alpha \tau(g_1) = 0
\end{cases}$$

As $g \neq 0$, we deduce that $g_i \neq 0$ for all $1 \leq i \leq n$. Summing, we obtain $\alpha \tau(g_1) - (-\theta)^n g_1 = 0$. Thus

$$\alpha \tau(-\theta)^n g_1 = (-\theta)^n g_1.$$

We conclude, by lemma 4.5, that $\alpha \in \mathcal{U}_s$. \hfill \Box

**Example.** Looking at the degree in $t_1$, we easily show that $t_1 \notin \mathcal{U}_s$. So $E_{t_1}$ is not isomorphic to $C_{\otimes n}$ and $\exp_{t_1}$ is not surjective.

### 4.2 Pellarin’s $L$-functions

Let $\alpha \in R_s \setminus \{0\}$ and $E_\alpha$ be the Anderson module defined at the beginning of section 4.1. By theorem 2.7, we have a class formula for

$$L(E_\alpha/R_s) := \prod_{P \in \text{prime}} \frac{[\text{Lie}(E_\alpha)(R_s/PR_s)]_{R_s}}{[E_\alpha(R_s/PR_s)]_{R_s}}.$$

We compute the $R_s$-module structure of $\text{Lie}(E_\alpha)(R_s/PR_s)$ and $E_\alpha(R_s/PR_s)$. Then, we show that we recover special values of Pellarin’s $L$-functions if we take $\alpha = (t_1 - \theta) \cdots (t_s - \theta)$.
4.2.1 Fitting ideal of \( \text{Lie}(E_\alpha)(R_\alpha/PR_\alpha) \)

Let us recall some facts about hyperdifferential operators. For more details, we refer the reader to [6].

Let \( j \geq 0 \) be an integer. The \( j^{\text{th}} \) hyperdifferential operator \( D_j \) is the \( k_\alpha \)-linear endomorphism of \( R_\alpha \) given by \( D_j(\partial^k) = \binom{j}{k} \theta^{k-j} \) for \( k \geq 0 \). For any \( f, g \in R_\alpha \), we have the Leibnitz rule

\[
D_j(fg) = \sum_{k=0}^j D_k(f)D_{j-k}(g).
\]

**Lemma 4.7.** For any \( a \in R_\alpha \), we have

\[
\theta(a) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} D_{n-1}(a) \\ \vdots \\ D_1(a) \\ a \end{pmatrix}.
\]

**Proof.** By linearity, it suffices to prove the equality for \( a = \theta^k \), \( k \in \mathbb{N} \). The action of \( \partial(\theta^k) \) is the left multiplication by

\[
\begin{pmatrix} \theta & 1 & \cdots & \cdots \\ \vdots & \ddots & \ddots & \vdots \\ \cdots & \cdots & 1 & \theta \\ 0 & \cdots & \cdots & 0 \end{pmatrix}^k = \theta I_n + \sum_{i=0}^k \binom{k}{i} \theta^{k-i} \begin{pmatrix} 0 & 1 & \cdots & \cdots \\ \vdots & \ddots & \ddots & \vdots \\ \cdots & \cdots & 1 & 0 \end{pmatrix}^i,
\]

hence the result comes from the definition of hyperdifferential operators.

**Lemma 4.8.** Let \( P \) be a prime of \( A \) and \( m \) a positive integer. Then \( \partial(P^m) \) is zero modulo \( P \) if and only if \( m \) is greater than or equal to \( n \).

**Proof.** By the previous lemma, it suffices to show that for any \( k \geq 0 \), the congruence \( D_k(P^m) = 0 \) mod \( P \) holds if and only if \( m \geq k + 1 \). The case \( k = 0 \) being obvious, let us suppose the result for an integer \( k \). By the Leibnitz rule, we have

\[
D_{k+1}(P^m) = \sum_{i+j=k+1} D_i(P^{m-1})D_j(P).
\]

which is zero modulo \( P \) if \( m \geq k + 2 \). Reciprocally, observe that

\[
D_{k+1}(P^{k+1}) = P D_{k+1}(P^k) + D_1(P)D_k(P^k) + D_2(P)D_{k-1}(P^k) + \cdots + D_{k+1}(P)P^k
\]

which is non zero modulo \( P \) by hypothesis.

Thanks to this lemma, we can compute the first Fitting ideal.

**Proposition 4.9.** Let \( P \) be a prime of \( A \). The \( R_\alpha \)-module \( \text{Lie}(E_\alpha)(R_\alpha/PR_\alpha) \) is isomorphic to \( R_\alpha/P\alpha R_\alpha \) and is generated by the residue class of \( \binom{1}{0}, \ldots, \binom{1}{0} \).

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Proof. By definition, \( \text{Lie}(E_\alpha)(R_s/PR_s) \) is the \( k_s \)-vector space \( (R_s/PR_s)^n \) equipped with the \( R_s \)-module structure given by \( \partial \). This \( R_s \)-module is finitely generated and, since \( \partial(P^n) = P^n I_n \) by lemma 2.4, the polynomial \( P^n \) annihilates it. Since \( R_s \) is principal, by the structure theorem, there exists integers \( e_1 \leq \cdots \leq e_m \) such that

\[
\text{Lie}(E_\alpha)(R_s/PR_s) \cong \frac{R_s}{P^{e_1}R_s} \times \cdots \times \frac{R_s}{P^{e_m}R_s}.
\]

Since \( \text{Lie}(E_\alpha)(R_s/PR_s) \) is a \( k_s \)-vector space of dimension \( n \deg P \), we have \( e_1 + \cdots + e_m = n \).

But, by the previous lemma, the residue class of \( f(0, \ldots, 0, 1) \) is not annihilated by \( P^{n-1} \), hence \( e_m \geq n \). Thus, \( \text{Lie}(E_\alpha)(R_s/PR_s) \) is cyclic and generated by the residue class of this vector. \( \square \)

### 4.2.2 Fitting ideal of \( E_\alpha(R_s/PR_s) \)

Let \( P \) be a prime of \( A \) and denote its degree by \( d \). We consider \( R := R_s/PR_s \) and \( E_\alpha(R) \) the \( R_s \)-module \( R^n \) where the action of \( R_s \) is given by \( \phi \), as defined at the beginning of section 4.1.

For \( i = 1, \ldots, n \), we denote by \( e_i : \mathbb{C}_s, \infty \to \mathbb{C}_s, \infty \) the projection on the \( i \)-th coordinate. By analogy with [2], we define the \( R_s \)-module

\[
W_n(R) := \{ w \in R((t^{-1}))/R[t] \mid \alpha \tau(w) = (t-\theta)^n w \mod R[t] \},
\]

where \( \tau(w) = \sum \tau(r_i)t^i \) if \( w = \sum r_it^i \in R((t^{-1})) \).

**Proposition 4.10.** The map

\[
\psi : \quad E_\alpha(R) \longrightarrow \frac{R((t^{-1}))/R[t]}{\theta} \quad c \quad \mapsto \quad -\sum_{i=1}^\infty e_i \phi_{\theta^{i-1}}(c) t^{-i}
\]

induces an isomorphism of \( R_s \)-modules between \( E_\alpha(R) \) and \( W_n(R) \).

**Proof.** See proposition 1.5.1 of [2]. \( \square \)

Observe that for any \( c \in E_\alpha(R) \), we have \( \psi(\phi_\theta(c)) = t\phi_\theta(c) \mod R[t] \). Moreover, since it is a \( k_s \)-vector space of dimension \( nd \), \( W_n(R) \) is a finitely generated and torsion \( k_s [t] \)-module.

For \( w \in W_n(R) \), applying \( d-1 \) times \( \alpha \tau \) to the relation \( \alpha \tau(w) = (t-\theta)^n w \), we get

\[
\alpha \tau(\alpha) \cdots \alpha \tau^{d-1}(\alpha) \tau^d(w) = \prod_{i=0}^{d-1} \left( t - \theta^i \right)^n w.
\]

But \( \tau^d(w) = w \in W_n(R) \) and \( \prod_{i=0}^{d-1} (t-\theta^i) = P(t) \mod R[t] \) where \( P(t) \) denotes the polynomial in \( t \) obtained substituting \( t \) form \( \theta \) in \( P \). Thus we obtain

\[
P^n(t) - \alpha \tau(\alpha) \cdots \alpha \tau^{d-1}(\alpha) = 0 \text{ in } W_n(R).
\]

Since we have the isomorphism

\[
\frac{R_s}{PR_s} \cong \frac{A}{PA} \otimes_{k_s} k_s,
\]

for any \( x \in R_s \), there exists a unique \( y \in k_s \) such that \( x \tau(x) \cdots \tau^{d-1}(x) = y \mod PR_s \). We denote by \( \rho_\alpha(P) \) the element of \( k_s \) such that \( \rho_\alpha(P) = \alpha \tau(\alpha) \cdots \alpha \tau^{d-1}(\alpha) \mod PR_s \). Note that, since \( P \) is prime, \( \rho_\alpha(P) = 0 \mod P \) if and only if \( P \) divides \( \alpha \) in \( R_s \). Then, by (4.5), we deduce that \( W_n(R) \) is annihilated by \( P^n(t) - \rho_\alpha(P) \), or equivalently

\[
E_\alpha(R) \subseteq \ker \phi_{P^n - \rho_\alpha(P)} = \{ x \in R^n \mid \phi_{P^n - \rho_\alpha(P)}(x) = 0 \}.
\]
Lemma 4.11. For any \(a \in k_s[t]\) prime to \(P(t) := P|_{t=1}\), the \(k_s\)-vector space \(W_n(R)[a]\) of \(a\)-torsion points of \(W_n(R)\) is of dimension at most \(\deg a\).

Proof. By definition, we have

\[
W_n(R)[a] = \left\{ w \in \frac{1}{a} R[t]/R[t] \mid \alpha \tau(w) = (t - \theta)^n w \mod R[t] \right\} \subseteq R((t^{-1}))/R[t].
\]

Let \(w \in W_n(R)[a]\). Since the \(t^i/a\) for \(i \in \{0, \ldots, \deg a - 1\}\) form an \(R\)-basis of \(\frac{1}{a} R[t]/R[t]\), we can write

\[
w = \sum_{i=0}^{\deg a - 1} \lambda_i \frac{t^i}{a},
\]

where the \(\lambda_i\) are in \(R\). Using the binomial formula and writing \(t^j/a\) for \(j \geq \deg a\) in the above basis, the functional equation verified by \(w\) becomes

\[
\sum_{i=0}^{\deg a - 1} \alpha \tau(\lambda_i) \frac{t^i}{a} = \sum_{i=0}^{\deg a - 1} \sum_{j=0}^{\deg a - 1} b_{i,j} \lambda_j \frac{t^i}{a},
\]

where the \(b_{i,j}\) are in \(R\). Identifying the two sides, we obtain \(\tau(\Lambda) = BA\) where \(\Lambda\) is the vector \(t(\lambda_0, \ldots, \lambda_{\deg a - 1})\) and \(B\) is the matrix of \(M_{\deg a}(R)\) with coefficients \(b_{i,j}/a\).

But the \(k_s\)-vector space \(V := \{ X \in R^{|\deg a|} \mid \tau(X) = BX \}\) is of dimension at most \(\deg a\). Indeed, observe that, if \(v_1, \ldots, v_m\) are vectors of \(R^{|\deg a|}\) such that \(\tau(v_i) = Bv_i\) for all \(i \in \{1, \ldots, m\}\), linearly independent over \(R\), there are also linearly independent over \(R^t = k_s\) (by induction on \(m\), see [12, lemma 1.7]). \(\square\)

Proposition 4.12. Let \(P\) be a prime of \(A\). We have the isomorphism of \(R_s\)-modules

\[
E_\alpha(R) \simeq \frac{R_s}{(P^n - \rho_\alpha(P))R_s}.
\]

Proof. Observe that if \(P\) divides \(\alpha\), we have \(\rho_\alpha(P) = 0\) and the isomorphism of \(R_s\)-modules \(\text{Lie}(E_\alpha)(R) \simeq E_\alpha(R)\). Then, the result is the same as in proposition 4.9.

Hence, let us suppose that \(\alpha\) and \(P\) are coprime. The \(k_s\)-vector space \(E_\alpha(R)\) is of dimension \(nd\). We deduce from lemma 4.11 that \(E_\alpha(R)\) is a cyclic \(R_s\)-module, i.e.

\[
E_\alpha(R) \simeq \frac{R_s}{fR_s},
\]

for some monic element \(f\) of \(R_s\) of degree \(nd\). On the other hand, by the inclusion (4.6), \(E_\alpha(R)\) is annihilated by \(P^n - \rho_\alpha(P)\) thus \(f\) divides \(P^n - \rho_\alpha(P)\). Since these two polynomials are monic and have the same degree, they are equal. \(\square\)

4.2.3 \(L\)-values

Let \(a\) be a monic polynomial of \(A\) and \(a = P_1^{e_1} \cdots P_r^{e_r}\) be its decomposition into a product of primes. Then, we define

\[
\rho_\alpha(a) := \prod_{i=1}^{r} \rho_\alpha(P_i)^{e_i}.
\]
By propositions 4.9 and 4.12, we get

\[
L(E_\alpha/R_s) = \prod_{P \in A \text{ prime}} \frac{[\text{Lie}(E_\alpha)(R_s/PR_s)]_{R_s}}{[E_\alpha(R_s/PR_s)]_{R_s}} = \prod_{P \in A \text{ prime}} \frac{P^n}{P^n - \rho_\alpha(P)} = \sum_{a \in A_+} \frac{\rho_\alpha(a)}{a^n} \in K_{s,\infty}.
\]

As in [4, section 4.1], observe that for any prime \(P\) of \(A\), \(\rho_\alpha(P)\) is the resultant of \(P\) and \(\alpha\) seen as polynomials in \(\theta\). In particular, if \(\alpha = (t_1 - \theta) \cdots (t_s - \theta)\), we obtain \(\rho_\alpha(P) = P(t_1) \cdots P(t_s)\).

Thus, by theorem 2.7, we get a class formula for \(L\)-values introduced in [9]:

\[
L(\chi_1, \cdots \chi_t, n) = \sum_{a \in A_+} \frac{\chi_t(a) \cdots \chi_t(a)}{a^n} = [\text{Lie}(E_\alpha)(R_s) : \exp_\theta^1(E_\alpha(R_s))]_{R_s} [H(E_\alpha/R_s)]_{R_s},
\]

where \(\chi_t : A \rightarrow F_q[t_1, \ldots, t_s]\) are the ring homomorphisms defined respectively by \(\chi_t(\theta) = t_i\).

### 4.3 Goss abelian \(L\)-series

This section is inspired by [5].

Let \(a \in A_+\) be squarefree and \(L\) be the cyclotomic field associated with \(a\), i.e. the finite extension of \(K\) generated by the \(a\)-torsion of the Carlitz module. We denote by \(\Delta_a\) the Galois group of this extension, it is isomorphic to \((A/aA)^\times\).

Note that \(A[\Delta_a] = \prod F_i[\theta]\) for some finite extensions \(F_i\) of \(F_q\). In particular, \(A[\Delta_a]\) is a principal ideal domain and Fitting ideals are defined as usual. If \(M\) is a finite \(A[\Delta_a]\)-module, we denote by \([M]_{A[\Delta_a]}\) the unique generator \(f\) of \(\text{Fitt}_{A[\Delta_a]} M\) such that each component \(f_i \in F_i[\theta]\) of \(f\) is monic.

We denote by \(\hat{\Delta}_a\) the group of characters of \(\Delta_a\), i.e. \(\hat{\Delta}_a = \text{Hom}(\Delta_a, \mathbb{F}_q^\times)\). For \(\chi \in \hat{\Delta}_a\), we denote by \(F_q(\chi)\) the finite extension of \(F_q\) generated by the values of \(\chi\) and we set \(e_\chi := \frac{1}{\# \Delta_a} \sum_{\sigma \in \Delta_a} \chi^{-1}(\sigma) \sigma \in F_q(\chi)[\Delta_a]\).

Then \(e_\chi\) is idempotent and \(\sigma e_\chi = \chi(\sigma) e_\chi\) for every \(\sigma \in \Delta_a\).

Let \(F\) be the finite extension of \(F_q\) generated by the values of all characters, i.e. \(F\) is the compositum of all \(F_q(\chi)\) for \(\chi \in \hat{\Delta}_a\). If \(M\) is an \(A[\Delta_a]\)-module, we have the decomposition into \(\chi\)-components

\[
F \otimes_{\mathbb{F}_q} M = \bigoplus_{\chi \in \hat{\Delta}_a} e_\chi \left( F \otimes_{\mathbb{F}_q} M \right).
\]

Let \(V\) be a free \(K_\infty[\Delta_a]\)-module of rank \(n\). A sub-\(A[\Delta_a]\)-module \(M\) of \(V\) is a lattice of \(V\) if \(M\) is free of rank one and \(K_\infty[\Delta_a] \cdot M = V\). Let \(M\) be a lattice of \(V\) and \(\chi \in \hat{\Delta}_a\). Then \(M(\chi) := e_\chi \left( F_q(\chi) \otimes_{\mathbb{F}_q} M \right)\) is a free \(A(\chi)\)-module of rank \(n\), discrete in \(V(\chi) := e_\chi \left( F_q(\chi) \otimes_{\mathbb{F}_q} V \right)\), where \(A(\chi) := F_q(\chi) \otimes_{\mathbb{F}_q} A\). Now let \(M_1\) and \(M_2\) be two lattices of \(V\). For each \(\chi \in \hat{\Delta}_a\), there exists \(\sigma_\chi \in \text{GL}(V(\chi))\) such that \(\sigma_\chi(M_1(\chi)) = M_2(\chi)\). Then, we define \([M_1(\chi) : M_2(\chi)]_{A(\chi)}\) to be the unique monic representative of \(\det \sigma_\chi\) in \(K_\infty(\chi) := F_q(\chi) \otimes_{\mathbb{F}_q} K_\infty\). Finally, we set

\[
[M_1 : M_2]_{A[\Delta_a]} := \sum_{\chi \in \hat{\Delta}_a} [M_1(\chi) : M_2(\chi)]_{A(\chi)} e_\chi \in K_\infty[\Delta_a]^\times.
\]
4.3.1 Gauss-Thakur sums

We review some basic facts on Gauss-Thakur sums, introduced in [11] and generalized in [3].
We begin with the case of only one prime. Let \( P \) be a prime of \( A \) of degree \( d \) and \( \zeta \in \mathbb{F}_q \) such that \( P(\zeta) = 0 \). We denote by \( \lambda_P \) the \( P \)-torsion of the Carlitz module and let \( \lambda_P \) be a non zero element of \( \Lambda_P \). We consider the cyclotomic extension \( K_P := K(\Lambda_P) = K(\lambda_P) \) and we denote its Galois group by \( \Delta_P \). We have \( \Delta_P \cong (A/PA)^\times \). More precisely, if \( b \in (A/PA)^\times \), the corresponding element \( \sigma_b \in \Delta_P \) is uniquely determined by \( \sigma_b(\lambda_P) = C_b(\lambda_P) \). We denote by \( \mathcal{O}_{K_P} \) the integral closure of \( A \) in \( K_P \). We have \( \mathcal{O}_{K_P} = A[\lambda_P] \).

We define the Teichmüller character

\[
\omega_P: \Delta_P \to \mathbb{F}_q^\times, \quad \sigma_b \mapsto b(\zeta_P),
\]

where \( \sigma_b \) is the unique element of \( \Delta_P \) such that \( \sigma_b(\lambda_P) = C_b(\lambda_P) \). Let \( \chi \in \hat{\Delta}_P \). Since the Teichmüller character generates \( \hat{\Delta}_P \), there exists \( j \in \{0, \ldots, q^d - 2\} \) such that \( \chi = \omega_P^j \). We expand \( j = j_0 + j_1 q + \cdots + j_{d-1} q^{d-1} \) in base \( q \) \((j_0, \ldots, j_{d-1} \in \{0, \ldots, q - 1\})\). Then, the Gauss-Thakur sum (see [11]) associated with \( \chi \) is defined as

\[
g(\chi) := \prod_{i=0}^{d-1} \left( -\sum_{\delta \in \Delta_P} \omega_P^{-\delta}(\delta(\lambda_P))^{j_i} \right) \in \mathbb{F}_q(\chi) \otimes_{\mathbb{F}_q} \mathcal{O}_{K_P}.
\]

We compute the action of \( \tau = 1 \otimes \tau \) on these Gauss-Thakur sums (see [11, proof of Theorem II]). Let \( 1 \leq j \leq d - 1 \). Since by the Carlitz action \( \sigma_b \sigma_\delta(\lambda_P) = \theta \sigma_b(\lambda_P) + \tau(\sigma_b(\lambda_P)) \), we have

\[
\tau \left( g(\omega_P^{j_i}) \right) = -\sum_{\sigma_b \in \Delta_P} \omega_P^{-\delta}(\sigma_b)(\sigma_b \sigma_\delta(\lambda_P) - \theta \sigma_b(\lambda_P))
\]

Then, by substitution, we get

\[
\tau \left( g(\omega_P^{j_i}) \right) = \left( \zeta_P^{-q^j} - \theta \right) g(\omega_P^{j_i}). \tag{4.7}
\]

Now, we return to the general case. Since \( a \) is squarefree, we can write \( a = P_1 \cdots P_r \) with \( P_1, \ldots, P_r \) distinct primes of respective degrees \( d_1, \ldots, d_r \). Since \( \Delta_a \cong \hat{\Delta}_{P_1} \times \cdots \times \hat{\Delta}_{P_r} \), for every character \( \chi \in \hat{\Delta}_a \), we have

\[
\chi = \omega_{P_1}^{N_{P_1}} \cdots \omega_{P_r}^{N_{P_r}}, \tag{4.8}
\]

for some integers \( 0 \leq N_i \leq q^{d_i} - 2 \) and where \( \omega_{P_i} \) is the Teichmüller character associated with \( P_i \). The product \( f_\chi := \prod_{N_i \neq 0} P_i \) is the conductor of \( \chi \). Then, the Gauss-Thakur sum (see [3, section 2.3]) associated with \( \chi \) is defined as

\[
g(\chi) := \prod_{i=1}^{r} \prod_{j=0}^{d_i-1} g(\omega_{P_i}^{j})^{N_{P_i}},
\]

or equivalently

\[
g(\chi) = \prod_{i=1}^{r} \prod_{j=0}^{d_i-1} g(\omega_{P_i}^{j})^{N_{P_i}},
\]

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where the $N_{i,j}$ are the $q$-adic digits of $N_i$. By equality (4.7), we obtain

$$
\tau(g(\chi)) = \prod_{i=1}^{r} \prod_{j=0}^{d_i-1} (\zeta_{\mathcal{P}_i}^{N_{i,j}} - \theta)^{N_{i,j}} g(\chi). 
$$

(4.9)

Lemma 4.13. The ring $\mathcal{O}_L$ is a free $A[\Delta_a]$-module of rank one generated by $\eta_0 := \sum_{\chi \in \Delta_a} g(\chi)$.

Proof. See lemma 16 of [3].

4.3.2 The Frobenius action on the $\chi$-components

Recall that $L$ is the extension of $K$ generated by the $a$-torsion of the Carlitz module. Let $L_\infty := L \otimes_K K_\infty$ on which $\tau$ acts diagonally and $\Delta_a$ acts on $L$. As in section 2.2, we have a morphism of $A[\Delta_a]$-modules

$$
\exp_{C^{\otimes n}} : \text{Lie}(C^{\otimes n})(L_\infty) \longrightarrow C^{\otimes n}(L_\infty).
$$

Let $\chi \in \widehat{\Delta}_a$. We get an induced map

$$
\exp_{C^{\otimes n}} : \epsilon(\text{Lie}(C^{\otimes n})(\mathbb{F}_q(\chi) \otimes_{\mathbb{F}_q} L_\infty)) \longrightarrow C^{\otimes n}(\epsilon(\mathbb{F}_q(\chi) \otimes_{\mathbb{F}_q} L_\infty)),
$$

where the action of $\tau$ on $\mathbb{F}_q(\chi) \otimes_{\mathbb{F}_q} L_\infty$ is on the second component. But, by lemma 4.13, we have

$$
\epsilon(\mathbb{F}_q(\chi) \otimes_{\mathbb{F}_q} L_\infty) = g(\chi)K_\infty(\chi),
$$

where $K_\infty(\chi) := \mathbb{F}_q(\chi) \otimes_{\mathbb{F}_q} K_\infty$.

We have the obvious isomorphism of modules over $A(\chi) := \mathbb{F}_q(\chi) \otimes_{\mathbb{F}_q} A$

$$
g(\chi)K_\infty(\chi) \sim \sim K_\infty(\chi),
$$

where the action on the right hand side is denoted by $\tilde{\tau}$ and given by $\tilde{\tau}(f) = \alpha(\chi)(1 \otimes \tau)(f)$ for any $f \in K_\infty(\chi)$, where $\alpha(\chi)$ is defined by equality 4.9. In particular, this isomorphism maps $C^{\otimes n}_q$ into $\partial \theta + N_i \tilde{\tau} = \partial \theta + N_i(\alpha(\chi))\tau$ with notation of section 4.1 and $\exp_{C^{\otimes n}}$ into $\exp_{\alpha(\chi)}$. Thus, by lemma 4.13, we have the isomorphism of $A(\chi)$-modules

$$
\epsilon(\mathbb{F}_q(\chi) \otimes_{\mathbb{F}_q} H(C^{\otimes n}/\mathcal{O}_L)) \cong \frac{E_{\alpha(\chi)}(K_\infty(\chi))}{\exp_{\alpha(\chi)}(\text{Lie}(E_{\alpha(\chi)}(K_\infty(\chi)))) + E_{\alpha(\chi)}(A(\chi))}.
$$

We denote the right hand side by $H(E_{\alpha(\chi)}/A(\chi))$. Note that we have also

$$
\epsilon(\mathbb{F}_q(\chi) \otimes_{\mathbb{F}_q} \exp_{C^{\otimes n}}(C^{\otimes n}(\mathcal{O}_L)))) = \exp_{\alpha(\chi)}(E_{\alpha(\chi)}(A(\chi))).
$$

4.3.3 $L$-values

Let $\chi \in \widehat{\Delta}_a$ and denote its conductor by $f_\chi$. Recall that the special value at $n \geq 1$ of Goss $L$-series (see [8, chapter 8]) associated with $\chi$ is defined by

$$
L(n, \chi) := \sum_{b \in A_+} \frac{\chi(b^n)}{b^n} \in K_\infty(\chi),
$$

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where the sum runs over the elements \( b \in A_+ \) relatively prime to \( f_\chi \). If \( b \in A_+ \) and \( f_\chi \) are not coprime, we set \( \chi(\sigma_b) = 0 \). Then, define the Goss abelian \( L \)-series

\[
L(n, \Delta_a) := \sum_{\chi \in \hat{\Delta}_a} L(n, \chi) e_\chi \in K_\infty[\Delta_a]^\times.
\]

**Lemma 4.14.** The infinite product

\[
\prod_{P \in A \text{ prime}} \frac{[\text{Lie}(C^{\otimes n})(\mathcal{O}_L/P\mathcal{O}_L)]_{A[\Delta_a]}}{[C^{\otimes n}(\mathcal{O}_L/P\mathcal{O}_L)]_{A[\Delta_a]}}
\]

converges in \( K_\infty[\Delta_a] \) to \( L(n, \Delta_a) \).

**Proof.** On the one hand, for all \( \chi \in \hat{\Delta}_a \), we have

\[
L(n, \chi) = \prod_{P \in A \text{ prime}} \left(1 - \frac{\chi(\sigma_P)}{P^n}\right)^{-1},
\]

where \( \chi(\sigma_P) = 0 \) if \( P \) divides \( f_\chi \). On the other hand, let \( \chi \in \hat{\Delta}_a \). We write \( \chi = \omega_1^{N_1} \cdots \omega_r^{N_r} \) as in equality (4.8) and denote by \( N_{i,j} \) the \( q \)-adic digits of \( N_i \). Then, as in section 4.2.2, we can prove that

\[
[E_\alpha(\chi)(A(\chi)/PA(\chi))]_{A(\chi)} = P^n - \prod_{i=1}^r \prod_{j=0}^{d_i-1} P \left( \zeta_{P_i}^{N_{i,j}} \right) = P^n - \prod_{i=1}^r P(\zeta_{P_i})^{N_i} = P^n - \chi(\sigma_P).
\]

Thus, we obtain

\[
L(n, \chi) = \prod_{P \in A \text{ prime}} \frac{[\text{Lie}(E_\alpha(\chi)(A(\chi)/PA(\chi))]_{A(\chi)}}{[E_\alpha(\chi)(A(\chi)/PA(\chi))]_{A(\chi)}}
\]

Hence, we get the result by the discussion of section 4.3.2 and definition of \( L(n, \Delta_a) \).

Finally, we obtain a generalization of theorem A of [5]:

**Theorem 4.15.** Let \( a \in A_+ \) be squarefree and denote by \( L \) the extension of \( K \) generated by the \( a \)-torsion of the Carlitz module. In \( K_\infty[\Delta_a] \), we have

\[
L(n, \Delta_a) = \left[\text{Lie}(C^{\otimes n})(\mathcal{O}_L) : \exp_{C^{\otimes n}}(C^{\otimes n}(\mathcal{O}_L))\right]_{A[\Delta_a]} \left[H(C^{\otimes n}/\mathcal{O}_L)\right]_{A[\Delta_a]}.
\]

**Proof.** By the previous lemma, \( L(n, \Delta_a) \) is expressed in terms of Anderson module and Fitting. Then, as in proposition 3.5, we express \( L(n, \Delta_a) \) as a determinant. The proof is similar but we deal with the \( \chi \)-components \( e_\chi(\mathcal{P}_q(\chi) \otimes \mathcal{O}_L) \) for all \( \chi \in \hat{\Delta}_a \). Then, since \( A[\Delta_a] \) is principal, we conclude as in section 3.4. We refer to [5, paragraph 6.4] for more details.
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