On the densities of rational multiples

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Abstract

For two subsets of natural numbers $A, B \subset \mathbb{N}$ define the set of rational numbers $M(A, B)$ with the elements represented by $m/n$, where $m, n$ are coprime, $m$ is divisible by some $a \in A$ and $n$ by some $b \in B$, respectively. Let $I$ be some interval of positive real numbers and $F^I_x$ denotes the set of rational numbers $m/n$, such that $m, n$ are coprime and $n \leq x$. The analogue to the Erdős-Davenport theorem about multiples is proved: under some constraints on $I$ the limits $\sum \{1/mn : m/n \in F^I_x \cap M(A, B)\} / \sum \{1/mn : m/n \in F^I_x\}$ exist for all subsets $A, B \subset \mathbb{N}$.

1 INTRODUCTION

For a subset $A$ of natural numbers $\mathbb{N}$ and $x > 1$ denote

$$\nu^0_x(A) = \frac{1}{x} \sum_{n \in A \cap [1; x]} 1, \quad \nu^1_x(A) = \frac{1}{\log x} \sum_{n \in A \cap [1; x]} \frac{1}{n}.$$ 

The lower and upper limits as $x \to \infty$ will be denoted by $\nu^r(A)$, $\nu^r_1(A)$ ($r = 0, 1$); the value of the limit if it exists by $\nu^r(A)$, respectively.

It follows from the chain of inequalities

$$\nu^0(A) \leq \nu^1(A) \leq \nu^0_1(A) \leq \nu^1_1(A)$$

that the existence of $\nu^0(A)$ implies the existence of $\nu^1(A)$. If $\nu^0(A)$ exists, we say that $A$ possesses asymptotic density, and if $\nu^1(A)$ exists, $A$ possesses logarithmic density. Even the subsets $A$ of apparently simple structure may not possess asymptotic density.

Let $A \subset \mathbb{N}$. The set of natural numbers divisible by some $a \in A$ will be denoted by $M(A)$, i.e. $M(A)$ is the set of multiples of $A$.

A.S. Besicovitch gave an example of $A$ such that $M(A)$ does not possess asymptotic density, see [1]. In 1937 H. Davenport and P. Erdős proved that every set of multiples have logarithmic density. Their original proof in [2] is based on Tauberian theorems, see also [6], Theorem 02. The direct and elementary proof of this theorem was provided by the authors in [3], it can be found also in the monograph of H. Halberstam and K.F. Roth, [5]. We formulate the Erdős-Davenport theorem in the form, which results from the arguments in [5].

**Theorem 1.** Let $A \subset \mathbb{N}$ and $A_N = A \cap [1; N]$ for $N \in \mathbb{N}$. Then $\nu^1(M(A_N)), \nu^1(M(A))$ exist, and

$$\nu^1(M(A)) = \lim_{N \to \infty} \nu^1(M(A_N)).$$

The main aim of this paper is to investigate the density questions related to the sets of multiples of rational numbers.
Let $\mathbb{Q}^+$ be the set of positive rational numbers. For the natural numbers $m, n$ we denote as usually by $(m, n)$ their greatest common divisor. If $(m, n) = 1$, i.e. the numbers are coprime, we write $m \perp n$ (suggestion of R.L. Graham, D.E. Knuth and O. Potashnik, see [11], p.115). For the rational numbers $r \in \mathbb{Q}^+$ we shall always use the unique representation $r = m/n$, $m, n \in \mathbb{N}, m \perp n$.

For two subsets $A, B \subset \mathbb{N}$ and $q \in \mathbb{N}$ we define the set of multiples in $\mathbb{Q}^+$ by

$$\mathcal{M}(A, B|q) = \left\{ \frac{m}{n} : m \in \mathcal{M}(A), n \in \mathcal{M}(B), mn \perp q \right\}.$$ 

If $q = 1$ we write $\mathcal{M}(A, B)$ instead of $\mathcal{M}(A, B|1)$.

Let $I_x = (\lambda_1(x), \lambda_2(x))$ be some system of intervals, $I_x \subset (0; +\infty), x \geq 1$. We shall write in the following briefly $I = (\lambda_1, \lambda_2)$ and introduce the sets of rational numbers

$$F^I_x = \left\{ \frac{m}{n} : m \in \mathbb{Q}^+, n \leq x \right\} \cap I.$$ 

Let $R \subset \mathbb{Q}^+$ and $r_1, r_2 \in \{0, 1\}$. Then if $F^I_x \neq \emptyset$, we denote

$$S_{x,I}^{r_1 r_2}(R) = \sum_{m/n \in F^I_x \cap R} m^{-r_1} n^{-r_2}, \quad \nu_{x,I}^{r_1 r_2}(R) = \frac{S_{x,I}^{r_1 r_2}(R)}{S_{x,I}^{r_1 r_2}(F^I_x)}.$$ 

If the limit of $\nu_{x,I}^{r_1 r_2}(R)$ exists for $R \subset \mathbb{Q}^+$ as $x \to \infty$, it will be denoted by $\nu^{r_1 r_2}(R)$, and the lower and upper limits by $\nu_{\leq}^{r_1 r_2}(R), \nu_{\geq}^{r_1 r_2}(R)$, respectively.

We investigate the limit behaviour of $\nu^{r_1 r_2}(\mathcal{M}(A, B|q))$ as $x \to \infty$ under some conditions imposed on $\lambda_i$. In the case of unit interval $I = (0, 1)$ related problems were considered in authors paper [9].

2 OVERVIEW OF RESULTS

If interval $I = (\lambda_1, \lambda_2)$ does not depend on $x$, the inequalities of type (1) can be proved.

**Theorem 2.** Let the interval $I = (\lambda_1, \lambda_2)$ be fixed. Then for an arbitrary $A \subset \mathbb{N}$

$$\nu_{i,0}^{00}(A) \leq \nu_{i,0}^{01}(A) \leq \nu_{i,0}^{11}(A),$$

$$\nu_{i,0}^{10}(A) \leq \nu_{i,1}^{11}(A) \leq \nu_{i,1}^{10}(A).$$

If $A, B$ are finite subsets of $\mathbb{N}$ the following statement holds.

**Theorem 3.** Let $\lambda_1 < \lambda_2$ satisfy the following conditions:

if $\lambda_1 = 0$, then $\lambda_2 > x^{-c}$ for some $0 < c < 1$;

if $\lambda_1 > 0$, then $\lambda_1 \log(\lambda_2/\lambda_1) \log x \to \infty$ as $x \to \infty$.

Then for finite sets $A, B \subset \mathbb{N}$ and $q \in \mathbb{N}$ all densities $\nu^{r_1 r_2}(\mathcal{M}(A, B|q))$ exist and are equal.

Note that if $\lambda_1 > 0$ and $(\lambda_2 - \lambda_1)/\lambda_1$ remains bounded, the constraints on $\lambda_i$ are equivalent to requirement $(\lambda_2 - \lambda_1) \cdot \log x \to \infty$ as $x \to \infty$.

It is possible to prove under appropriate conditions on $\lambda_i$ this statement for the sets satisfying

$$\sum_{d \in A \cup B} \frac{1}{d^2} < \infty,$$

but we shall not pursue this question.

The inequality for densities in the following theorem should be compared to Heilbronn-Rohrbach inequality proved in [7], [8]; see also [9].
Theorem 4. Let the sets $A, B \subset \mathbb{N}$ be finite and satisfy the following conditions: $a \perp b$ for all $a \in A, b \in B$ if $a_1, a_2 \in A, b_1, b_2 \in B$, then $a_1 \perp a_2/(a_1, a_2), b_1 \perp b_2/(b_1, b_2)$. Let $\nu(M(A, B|q))$ denote the common value of densities from Theorem 3. Then the following inequality holds:

$$1 - \nu(M(A, B|q)) \geq \prod_{p|q} \left(1 - \frac{2}{p+1}\right) \cdot \prod_{c \in A \cup B} \left(1 - \frac{1}{c} \prod_{p|c} \left(1 - \frac{1}{p+1}\right)\right).$$

The sets satisfying conditions of Theorem 4 can be constructed as follows. Let $a_j = \prod_{k \in I_j} r_k$, where $I_j$ is some finite subset of naturals then, obviously, $a_i \perp a_j/(a_j, a_i)$ for all pairs $i, j$.

The main result of the paper is an analogue or Erdős-Davenport theorem for the sets of rational multiples.

Theorem 5. Let for the intervals $I = (\lambda_1, \lambda_2)$ the following conditions be satisfied:

- if $\lambda_1 = 0$ then $\lambda_2 x \to \infty$ and $\log x / \log(\lambda_2 x) < c_1$ as $x \to \infty$ with some $c_1 > 0$;
- if $\lambda_1 > 0$ then with some positive constants $c_2, c_3$

$$\frac{c_2}{\log(\lambda_2 + 2)} < \lambda_1 < \lambda_2 < x^{c_3}, \quad \frac{1}{\log(\lambda_2 + 2)} \cdot \log \left(\frac{\lambda_2}{\lambda_1}\right) \cdot \log x \to \infty, \quad x \to \infty.$$

Then for arbitrary $A, B \subset \mathbb{N}$ and $q \in \mathbb{N}$ the limit

$$\nu^{11}(M(A, B|q)) = \lim_{x \to \infty} \nu_x^{11}(M(A, B|q))$$

exists.

Let $\lambda_1 > c(c > 0)$ and $\lambda_2$ be bounded. Then the conditions of Theorem 5 for $\lambda_i$ can be reduced to requirement

$$x \to \infty.$$

3 PROOFS

Let $q_0, q_1, q_2$ be some coprime natural numbers and

$$\mathbb{Q}_{q_0, q_1, q_2} = \left\{ \frac{m}{n} \in \mathbb{Q}^+, mn \perp q_0, mq_1 \perp q_2 \right\}.$$ (2)

We investigate the asymptotical behaviour of the sums $S_{x, I}^{q_0, q_1}(\mathbb{Q}_{q_0, q_1, q_2})$ as $x \to \infty$. Methods being used are elementary, the remainder terms in the asymptotics depend on $q_i$.

Lemma. Let for the coprime integers $q_0, q_1, q_2$

$$\Pi(q_0, q_1, q_2) = \prod_{p|q_0} \left(1 - \frac{2}{p+1}\right) \prod_{p|q_1q_2} \left(1 - \frac{1}{p+1}\right).$$

Then the following asymptotics hold

$$\frac{S_{x, I}^{q_0, q_1}(\mathbb{Q}_{q_0, q_1, q_2})}{\Pi(q_0, q_1, q_2)} = \frac{3}{\pi^2}(\lambda_2 - \lambda_1)x^2 \left\{1 + O\left(\frac{\log x}{x} + \frac{\log x}{(\lambda_2 - \lambda_1)x}\right)\right\},$$

$$\frac{S_{x, I}^{q_1, q_2}(\mathbb{Q}_{q_0, q_1, q_2})}{\Pi(q_0, q_1, q_2)} = \frac{6}{\pi^2}(\lambda_2 - \lambda_1)x \left\{1 + O\left(\frac{\log x}{x} + \frac{\log^2 x}{(\lambda_2 - \lambda_1)x}\right)\right\}. $$

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If \( \lambda_1 > 0 \) then

\[
\frac{S^1_{x,I}(q_0,q_1,q_2)}{\Pi(q_0,q_1,q_2)} = \frac{6}{\pi^2} \log \left( \frac{\lambda_2}{\lambda_1} \right) x \left\{ 1 + O \left( \frac{\log x}{x} + \frac{\log^2 x}{\lambda_1 \log \left( \frac{\lambda_2}{\lambda_1} \right) x} \right) \right\},
\]

\[
\frac{S^1_{x,I}(q_0,q_1,q_2)}{\Pi(q_0,q_1,q_2)} = \frac{6}{\pi^2} \log \left( \frac{\lambda_2}{\lambda_1} \right) x \left\{ 1 + O \left( \frac{1}{\log x} + \frac{1}{\lambda_1 \log \left( \frac{\lambda_2}{\lambda_1} \right) \log x} \right) \right\}.
\]

In the case \( \lambda_1 = 0 \) we have

\[
\frac{S^1_{x,I}(q_0,q_1,q_2)}{\Pi(q_0,q_1,q_2)} = \frac{6}{\pi^2} x \log(\lambda_2 x) \left\{ 1 + O \left( \frac{1}{\log(\lambda_2 x)} + \frac{x}{\log x} \right) \right\},
\]

\[
\frac{S^1_{x,I}(q_0,q_1,q_2)}{\Pi(q_0,q_1,q_2)} = \left\{ \begin{array}{ll}
\frac{3}{\pi^2} \log^2(\lambda_2 x) \left\{ 1 + O \left( \frac{\log x}{\log(\lambda_2 x)} \right) \right\}, & \text{if } \frac{1}{2} < \lambda_2 \leq 1,
\frac{3}{\pi^2} \log x \cdot \log(\lambda_2^2 x) \left\{ 1 + O \left( \frac{\log(\lambda_2 x)}{\log x \log(\lambda_2^2 x)} \right) \right\}, & \text{if } \lambda_2 > 1.
\end{array} \right.
\]

The functions in O-signs of the Lemma are different. It is easily seen, that if \( \lambda_1 = 0 \), then the condition \( x^{-c} < \lambda_2 \) with some \( 0 < c < 1 \) is sufficient for all functions in O-signs related to the case \( \lambda_1 = 0 \) to be vanishing.

Consider now the case \( \lambda_1 > 0 \). The function

\[
f(u) = u - c \log \left( 1 + \frac{u}{c} \right), \quad u \geq 0, \quad c > 0,
\]

is not decreasing, hence

\[
\lambda_1 \log \left( \frac{\lambda_2}{\lambda_1} \right) = \lambda_2 \log \left( 1 + \frac{\lambda_2 - \lambda_1}{\lambda_1} \right) \leq \lambda_2 - \lambda_1.
\]

It follows from this, that under condition

\[
\lambda_1 \log \left( \frac{\lambda_2}{\lambda_1} \right) x \log x \rightarrow \infty, \quad x \rightarrow \infty,
\]

all functions in O-signs of \( S^{r_1r_2}_{x,I}(q_0,q_1,q_2) \), with \( r_1 + r_2 < 2 \), are vanishing. We include \( S^{11}_{x,I} \) if we use the stronger requirement

\[
\lambda_1 \log \left( \frac{\lambda_2}{\lambda_1} \right) \log x \rightarrow \infty, \quad x \rightarrow \infty.
\]

If \( q_0 = q_1 = q_2 = 1 \), then

\[
S^{r_1r_2}_{x,I}(Q_1,q_1,q_2) = S^{r_1r_2}_{x,I}(Q^+) = \sum \left\{ m^{-r_1}n^{-r_2} : \frac{m}{n} \in F^I \right\}.
\]

The following Corollary follows easily from the Lemma.

**Corollary.** Let \( \lambda_i \) fulfill the following conditions

- if \( \lambda_1 = 0 \) then \( x^{-c} < \lambda_2 \) with some \( 0 < c < 1 \);
- if \( \lambda_1 > 0 \) then \( \lambda_1 \log \left( \frac{\lambda_2}{\lambda_1} \right) x \log x \rightarrow \infty, \quad x \rightarrow \infty.
\]

Then for all \( r_1, r_2 \) and fixed coprime numbers \( q_0, q_1, q_2 \)

\[
\frac{S^{r_1r_2}_{x,I}(q_0,q_1,q_2)}{S^{r_1r_2}_{x,I}(Q^+)} \rightarrow \Pi(q_0,q_1,q_2) \quad \text{as} \quad x \rightarrow \infty.
\]
Proof. We abbreviate the notation as \( S^{r_1 r_2} = S^{r_1 r_2}_n(q_0, q_1, q_2) \) and start with the expression

\[
S^{r_1 r_2} = \sum_{n \leq x} n^{-r_2} \sum_{\lambda_1 n < \lambda_2 n} m^{-r_1}.
\]

With the Möbius function \( \mu(n) \) we proceed as follows

\[
S^{r_1 r_2} = \sum_{\lambda_1 n < \lambda_2 n} \sum_{d \mid (m, n q_0 q_1)} \mu(d) \sum_{\lambda_1 n < \lambda_2 n} m^{-r_1} = \sum_{\lambda_1 n < \lambda_2 n} \sum_{d \mid (m, n q_0 q_1)} \mu(d) m^{-r_1} = \sum_{\lambda_1 n < \lambda_2 n} \sum_{d \mid (m, n q_0 q_1)} m^{-r_1}. \tag{3}
\]

For the last sum over \( m \) we shall use the following equalities

\[
\sum_{\lambda_1 n < \lambda_2 n} m^{-r_1} = \begin{cases} 
(\lambda_2 - \lambda_1) \frac{\theta}{x} + \theta_{n,d}, & \text{if } r_1 = 0, \\
\log \left( \frac{\lambda_2}{\lambda_1} \right) + \theta_{n,d} \frac{d}{X^{r_1}}, & \text{if } \lambda_1 > 0, r_1 = 1, \\
\log \left( \frac{\lambda_2}{\lambda_1} \right) + \theta_{n,d}, & \text{if } \lambda_1 = 0, r_1 = 1, \text{ and } \frac{\lambda_2}{\lambda_1} > 1,
\end{cases}
\]

where \( \theta_{n,d} \) are bounded by some absolute constant.

Consider the case \( r_1 = r_2 = 0 \) first. Then

\[
S^{00} = (\lambda_2 - \lambda_1) \sum_{d \mid (d, q_0 q_1)} \frac{\mu(d)}{d} \sum_{n \leq x} n + O\left( \sum_{d \mid x q_0 q_1} \mu^2(d) \sum_{n \leq x} 1 \right). \tag{4}
\]

Let \( S^{00}_1 \) stands for the main term in \( S^{00} \). Using the divisibility property \( d \mid (d, q_0 q_2) \) and the asymptotics

\[
\sum_{n \leq u} n = \frac{1}{2} u^2 \prod_{p \mid q} \left( 1 - \frac{1}{p} \right) + O(u),
\]

we rewrite the main term of \( S^{00} \) as

\[
S^{00}_1 = \frac{1}{2} (\lambda_2 - \lambda_1)x^2 \prod_{p \mid q_1} \left( 1 - \frac{1}{p} \right) \sum_{d \mid (d, q_0 q_2)} \frac{\mu(d)}{d^2} + O((\lambda_2 - \lambda_1)x \log x). \tag{5}
\]

Note that \( d \mid (d, q_0 q_2) \perp q_0 q_1 \) is equivalent to \( d \perp q_1 \), hence

\[
\sum_{d \mid (d, q_0 q_2)} \frac{\mu(d)}{d^2} = \sum_{d \mid q_1} \frac{\mu(d)}{d^2} + O\left( q_0 q_2 \sum_{d > x q_0 q_1} \frac{1}{d^2} \right)
= \prod_{p} \left( 1 - \frac{1}{p^2} \right) \prod_{p \mid q_1} \left( 1 - \frac{1}{p} \right)^{-1} \prod_{p \mid q_0 q_2} \left( 1 + \frac{1}{p} \right)^{-1} + O\left( \frac{1}{x} \right).
\]

Setting this in \( S^{00}_1 \) one gets

\[
S^{00}_1 = \frac{3}{\pi^2} (\lambda_2 - \lambda_1)x^2 \Pi(q_0, q_1, q_2) + O((\lambda_2 - \lambda_1)x \log x).
\]
For the remainder term in (6) we use the bound
\[
\sum_{d \leq xq_0q_1} \mu^2(d) \sum_{n \leq xq_0q_1 \atop n \equiv r \pmod{d/\phi(d,q_0q_2)}} 1 \leq \sum_{d \leq xq_0q_1} \mu^2(d) \sum_{n \leq x(d,q_0q_2)/d} 1 = O\left(x \log x\right).
\]

Hence putting all together we obtain
\[
S^{00} = \frac{3}{\pi^2} (\lambda_2 - \lambda_1) x^2 \Pi(q_0,q_1,q_2) \left\{ 1 + O\left(\frac{\log x}{x} + \frac{\log x}{(\lambda_2 - \lambda_1)x}\right) \right\}.
\]

Consider now the case \(r_1 = 0, r_2 = 1\). Then instead of (4) we have
\[
S^{01} = (\lambda_2 - \lambda_1) x \prod_{p \nmid q_0q_1} \left( 1 - \frac{1}{p} \right) \sum_{d \leq xq_0q_1 \atop d/\phi(d,q_0q_2) = 1} (d,q_0q_2) \frac{\mu(d)}{d^2} + O\left((\lambda_2 - \lambda_1) \log x\right)
\]
\[
= \frac{6}{\pi^2} (\lambda_2 - \lambda_1)x \Pi(q_0,q_1,q_2) + O\left((\lambda_2 - \lambda_1) \log x\right).
\]

For the remainder term in (6) use the obvious bound
\[
\sum_{d \leq xq_0q_2} \mu^2(d) \sum_{n \leq xq_0q_1 \atop n \equiv r \pmod{d/\phi(d,q_0q_2)}} 1 \leq \sum_{d \leq xq_0q_1} \mu^2(d) \sum_{n \leq xq_0q_1} \frac{1}{n} = O\left(\log^2 x\right).
\]

Hence the asymptotics
\[
S^{01} = \frac{6}{\pi^2} (\lambda_2 - \lambda_1)x \Pi(q_0,q_1,q_2) \left\{ 1 + O\left(\frac{\log x}{x} + \frac{\log^2 x}{(\lambda_2 - \lambda_1)x}\right) \right\}.
\]
is established.

Suppose now that \(r_1 = 1, r_2 = 0\). From (3) one gets
\[
S^{10} = \sum_{d \leq xq_0q_1 \atop d/\phi(d,q_0q_2) = 1} \frac{\mu(d)}{d} \sum_{n \leq xq_0q_1 \atop n \equiv r \pmod{d/\phi(d,q_0q_2)}} \frac{1}{m}.
\]

Let first \(\lambda_1 > 0\). Then
\[
S^{10} = \log \left(\frac{\lambda_2}{\lambda_1}\right) \sum_{d \leq xq_0q_1 \atop d/\phi(d,q_0q_2) = 1} \frac{\mu(d)}{d} \sum_{n \leq xq_0q_1 \atop n \equiv r \pmod{d/\phi(d,q_0q_2)}} \frac{1}{m} + O\left(\frac{1}{\lambda_1} \sum_{d \leq xq_0q_1} \mu^2(d) \sum_{n \leq xq_0q_1} \frac{1}{n}\right).
\]
Expression for $S^{10}$ differs from that one in (6) in term involving $\lambda_i$ only. Hence, in the same way as above we get

$$S^{10} = \frac{6}{\pi^2} \log \left( \frac{\lambda_2}{\lambda_1} \right) x \Pi(q_0, q_1, q_2) \left\{ 1 + O \left( \frac{\log x}{x} + \frac{\log^2 x}{\lambda_1 \log \left( \frac{\pi}{\lambda_1} x \right)} \right) \right\}.$$ 

Let now $\lambda_1 = 0$. Then

$$S^{10} = \sum_{d \mid (d, q_0q_1)^2} \frac{\mu(d)}{d} \sum_{n \leq x \mid (d, q_0q_2)} \log \left( \frac{\lambda_2 n}{d} \right) + O \left( \sum_{d \mid (d, q_0q_1)^2} \frac{\mu^2(d)}{d} \sum_{n \leq x \mid (d, q_0q_2)} 1 \right).$$

The remainder term does not exceed

$$x \sum_{d \mid (d, q_0q_2)} \frac{\mu^2(d)}{d^2} = O(x).$$

Using the divisibility condition $d \mid (d, q_0q_2)$, we proceed as follows

$$S^{10} = \sum_{d \mid (d, q_0q_1)^2} \frac{\mu(d)}{d} \sum_{n \leq x \mid (d, q_0q_2)} \log \left( \frac{\lambda_2 n}{d} \right) + O(x) = \sum_{d \mid (d, q_0q_1)^2} \frac{\mu(d)}{d} \sum_{n \leq x \mid (d, q_0q_2)} \log(\lambda_2 n)$$

$$- \sum_{d \mid (d, q_0q_1)^2} \frac{\mu(d)}{d} \log(d, q_0q_2) \sum_{n \leq x \mid (d, q_0q_2)} 1 + O(x).$$

The second minus term is $O(x)$, hence

$$S^{10} = \sum_{d \mid (d, q_0q_1)^2} \frac{\mu(d)}{d} \sum_{n \leq x \mid (d, q_0q_2)} \log(\lambda_2 n) + O(x). \quad (7)$$

Using

$$\sum_{n \leq x \mid (d, q_0q_2)} 1 = u \prod_{p \mid q} \left( 1 - \frac{1}{p} \right) + O(1)$$

and integrating by parts one derives for $c > 0$ easily

$$\sum_{n \leq x \mid (d, q_0q_2)} \log(cn) = u \log(cu) \prod_{p \mid q} \left( 1 - \frac{1}{p} \right) + O(u + |\log(cu)|), \text{ as } u \to \infty.$$ 

Using this in (7) we get

$$S^{10} = \prod_{p \mid q_0q_2} \left( 1 - \frac{1}{p} \right) x \sum_{d \mid (d, q_0q_1)^2} \frac{\mu(d)}{d^2} (d, q_0q_2) \log \left( \frac{\lambda_2 x (d, q_0q_2)}{d} \right)$$

$$+ O \left( x + x \sum_{d \mid (d, q_0q_2)^2} \frac{\mu^2(d)}{d^2} (d, q_0q_2) + \sum_{d \mid (d, q_0q_1)^2} \frac{\mu^2(d)}{d^2} \log \left( \frac{\lambda_2 x (d, q_0q_2)}{d} \right) \right).$$

It is easily seen that the remainder term can be reduced to $O(x + \log x \cdot \log(\lambda_2 x) + \log^2 x) = O(x + \log x \log(\lambda_2 x^2))$. Using additivity property for the logarithm in the first sum we split the
Using the asymptotics

\[ S^{10} = \prod_{p | q_0 q_2} \left( 1 - \frac{1}{p} \right) x \log(\lambda_2 x) \sum_{d / (d, q_0 q_2) \perp q_0 q_1} \frac{\mu(d)}{d^2} (d, q_0 q_2) + O(x + \log x \cdot \log(\lambda_2 x^2)). \]

The remaining sum was calculated above, then simplifying the remainder terms one gets

\[ S^{10} = \frac{6}{\pi^2} \Pi(q_0, q_1, q_2) x \log(\lambda_2 x) \left\{ 1 + O\left( \frac{1}{\log(\lambda_2 x)} + \frac{\log x}{x} \right) \right\}. \]

With \( r_1 = r_2 = 1 \) we have

\[ S^{11} = \sum_{d \leq \epsilon q_0 q_1 \atop d / (d, q_0 q_2) \perp q_0 q_1} \frac{\mu(d)}{d} \sum_{n \leq x \atop n \perp (d, q_0 q_2) / n} \frac{1}{n} + \frac{1}{n} \log \left( \frac{\lambda_2}{\lambda_1} \right) \sum_{d \leq \epsilon q_0 q_1 \atop d / (d, q_0 q_2) \perp q_0 q_1} \frac{\mu(d)}{d^2} (d, q_0 q_1) \sum_{n \leq x \atop n \perp (d, q_0 q_2) / n} \frac{1}{n} + O(\lambda^{-1}). \]

The sum over \( n \) in the remainder term is \( O(d^{-2}) \), hence

\[ S^{11} = \log \left( \frac{\lambda_2}{\lambda_1} \right) \prod_{p | q_0 q_1} \left( 1 - \frac{1}{p} \right) \log x + O(1), \]

using the asymptotics

\[ \sum_{n \leq x} \frac{1}{n} = \prod_{p | q} \left( 1 - \frac{1}{p} \right) \log u + O(1), \]

we derive

\[ S^{11} = \log \left( \frac{\lambda_2}{\lambda_1} \right) \prod_{p | q_0 q_1} \left( 1 - \frac{1}{p} \right) \sum_{d \leq \epsilon q_0 q_1 \atop d / (d, q_0 q_2) \perp q_0 q_1} \frac{\mu(d)}{d^2} (d, q_0 q_2) \log \left( \frac{d, q_0 q_2}{d} \right) + O\left( \frac{1}{\lambda_1} + \log \left( \frac{\lambda_2}{\lambda_1} \right) \right) \]

\[ = \log \left( \frac{\lambda_2}{\lambda_1} \right) \prod_{p | q_0 q_1} \left( 1 - \frac{1}{p} \right) \log x \sum_{d \leq \epsilon q_0 q_1 \atop d / (d, q_0 q_2) \perp q_0 q_1} \frac{\mu(d)}{d^2} (d, q_0 q_2) + O\left( \frac{1}{\lambda_1} + \log \left( \frac{\lambda_2}{\lambda_1} \right) \right). \]

Simplifying the sum over \( d \) as above we arrive finally to

\[ S^{11} = \frac{6}{\pi^2} \log \left( \frac{\lambda_2}{\lambda_1} \right) \Pi(q_0, q_1, q_2) \log x \left\{ 1 + O\left( \frac{1}{\log x} + \frac{1}{\lambda_1} \log \left( \frac{\lambda_2}{\lambda_1} \right) \log x \right) \right\}. \]

Consider now the case \( \lambda_1 = 0 \):

\[ S^{11} = \sum_{d \leq \epsilon q_0 q_1 \atop d / (d, q_0 q_2) \perp q_0 q_1} \frac{\mu(d)}{d} \sum_{n \leq \lambda_2 \perp n \atop n / (d, q_0 q_2) \perp n} \frac{1}{n} \sum_{d / \lambda_2 < n} \frac{1}{n} \log \left( \frac{\lambda_2 n}{d} \right) + O\left( \sum_{d \leq \epsilon q_0 q_1 \atop d / (d, q_0 q_2) \perp q_0 q_1} \frac{\mu(d)}{d^2} \sum_{n \leq \lambda_2 \perp n} \frac{1}{n} \right). \]

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Using the divisibility condition \( d/(d, q_0 q_2)|n \) we reduce the term in O-sign to \( O(\log x) \) and simplify the expression as follows

\[
S^{11} = \sum_{d \leq x \in q_0 q_1 \atop d/(d, q_0 q_2) \perp q_0 q_1} \frac{\mu(d)}{d^2} (d, q_0 q_2) \sum_{\lambda_2 n \in x/(d, q_0 q_2)/d \atop \lambda_2 \in \mathbb{Q} \setminus (d, q_0 q_2)/d} \frac{1}{n} \log \left( \frac{\lambda_2 n}{(d, q_0 q_2)} \right) + O(\log x)
\]

\[
= \sum_{d \leq x \in q_0 q_1 \atop d/(d, q_0 q_2) \perp q_0 q_1} \frac{\mu(d)}{d^2} (d, q_0 q_2) \sum_{\lambda_2 n \in x/(d, q_0 q_2)/d \atop \lambda_2 \in \mathbb{Q} \setminus (d, q_0 q_2)/d} \frac{\log(\lambda_2 n)}{n} + O(\log x).
\]

Extending the sum over \( n \) to the range \( 1/\lambda_2 < n \leq x \) we introduce the error term \( O(\log x + \log(\lambda_2 x)) \). Hence

\[
S^{11} = \sum_{d \leq x \in q_0 q_1 \atop d/(d, q_0 q_2) \perp q_0 q_1} \frac{\mu(d)}{d^2} (d, q_0 q_2) \sum_{\lambda_2 n \in x/(d, q_0 q_2)/d \atop \lambda_2 \in \mathbb{Q} \setminus (d, q_0 q_2)/d} \frac{\log(\lambda_2 n)}{n} + O(\log x + \log(\lambda_2 x)).
\]

The main term is expressed as the product of two sums, the first one equals to

\[
\frac{6}{\pi^2} \Pi(q_0, q_1, q_2) \prod_{p \neq q_0, q_1} \left( 1 - \frac{1}{p} \right)^{-1} + O(x^{-1}).
\]

The second sum of the main term can be calculated by partial integration, the final result would be

\[
\sum_{1/\lambda_2 < n \leq x \atop \lambda_2 \neq q_0 q_1} \frac{\log(\lambda_2 n)}{n} = \begin{cases} \frac{1}{2} \prod_{p \neq q_0, q_1} \left( 1 - \frac{1}{p} \right) (\log x + \log \lambda_2)^2 + O(1), & \text{as } \lambda_2 < 1, \\ \frac{1}{2} \prod_{p \neq q_0, q_1} \left( 1 - \frac{1}{p} \right) (\log^2 x + 2 \log \lambda_2 \log x) + O(\log(\lambda_2 + 1)), & \text{as } \lambda_2 \geq 1. \end{cases}
\]

If we write \( \log x + \log \lambda_2)^2 = \log^2(\lambda_2 x) \) and \( \log^2 x + 2 \log \lambda_2 \log x = \log x \cdot \log(\lambda_2 x) \), then after manipulating with the remainder terms we arrive to the following expressions

\[
S^{11} = \begin{cases} \frac{3}{2} \log^2(\lambda_2 x) \left( 1 + O \left( \frac{\log x}{\log(\lambda_2 x)} \right) \right), & \text{if } \frac{1}{2} < \lambda_2 \leq 1, \\ \frac{3}{2} \log x \cdot \log(\lambda_2 x) \left( 1 + O \left( \frac{\log x}{\log(\lambda_2 x)} \right) \right), & \text{if } \lambda_2 > 1. \end{cases}
\]

Note that the remainder term for \( \lambda_2 > x^{-c} \), where \( 0 < c < 1 \), is \( O(\log^{-1} x) \). The Lemma is proved.

**Proof of Theorem 2.** Let us start with the first chain of inequalities. Because of the interval \( I \) is fixed

\[
S_{x, t}^{01}(Q^+) \sim \frac{3}{\pi^2} \beta I \rightarrow \infty, \quad S_{x, t}^{01}(Q^+) \sim \frac{6}{\pi^2} \beta I, \quad \beta x \rightarrow \infty,
\]

where \( |I| = \lambda_2 - \lambda_1 \). For an arbitrary subset \( A \subset \mathbb{Q}^+ \) we have

\[
S_{x, t}^{01}(A) = \int_{t-1}^{t+1} \frac{1}{t^2} dS_{x, t}^{01}(A) = \frac{S_{x, t}^{01}(A)}{x} + \int_{t-1}^{t+1} \frac{S_{x, t}^{01}(A)}{t^2} dt.
\] (8)

For an arbitrary fixed \( \epsilon > 0 \) we shall have

\[
S_{x, t}^{01}(A) \leq (\pi^0(A) + \epsilon) \frac{3}{\pi^2} |I| t^2
\]

as \( t \geq t_0 \). From this observation and \( \square \) we derive

\[
S_{x, t}^{01}(A) \leq (\pi^0(A) + \epsilon) \frac{6}{\pi^2} I x + C,
\]

\[9\]
with some $C > 0$. Then, consequently, $\mathcal{M}^{11}(A) \leq \mathcal{M}^{10}(A)$. The inequality for lower limits follows from the inequality for complement set $\mathcal{M}^{11}(A^c) \leq \mathcal{M}^{10}(A^c)$.

The second chain of inequalities can be derived in an analogous manner from the equality $$S^{11}_{x,t}(A) = \int_{x}^{\infty} \frac{1}{t} d\mathcal{S}^{10}_{x,t}(A).$$

Theorem 2 is proved.

**Proof of Theorem 3.** Consider now the sets of multiples $\mathcal{M}(A,B|q)$. If $A = \{a\}, B = \{b\}$ we shall write $\mathcal{M}(A,B|q) = \mathcal{M}(a,b|q)$. For natural numbers $a, b$ with $(a, b) > 1$ or $(ab, q) > 1$ we have $\mathcal{M}(a,b|q) = \emptyset$. Let $a \perp b$ and $ab \perp q$. Then using the notation $\mathcal{S}^{r,s}_{x,t}(\mathcal{M}(a,b|q)) = a^{-r} b^{-s} s_{b-a^{-r}}^{s}(\mathbb{Q},a,b)$.

After examining the asymptotics of Lemma we conclude that under conditions of Theorem 3 for $\lambda_1$

$$S^{r,s}_{x,t}(\mathcal{M}(a,b|q)) \sim a^{-r_1} b^{-s_2} s_{b-a^{-r_1}}^{s_2}(\mathbb{Q},a,b), \quad x \to \infty.$$ 

From the Corollary we obtain

$$\nu^{r,s}_{x}(\mathcal{M}(a,b|q)) \to \frac{1}{ab} \prod_{p|q} \left(1 - \frac{2}{p+1}\right) \prod_{p|ab} \left(1 - \frac{1}{p+1}\right), \quad x \to \infty.$$ (9)

Let now $A, B$ be two finite sets. By the sieve arguments we have

$$\nu^{r,s}_{x}(\mathcal{M}(A,B|q)) = \sum_{C \subseteq A \times B} (-1)^{|C|+1} \nu^{r,s}_{x}(\bigcap_{(a,b) \in C} \mathcal{M}(a,b|q)).$$ (10)

For $C = \{(a_1,b_1), (a_2,b_2), \ldots, (a_{|C|},b_{|C|})\} \subset A \times B$ let us introduce the notations

$$[C]_A = [a_1,a_2,\ldots,a_{|C|}], \quad [C]_B = [b_1,b_2,\ldots,b_{|C|}],$$

here $[\cdot]$ stands for the least common multiples of numbers in the brackets. Then clearly

$$\bigcap_{(a,b) \in C} \mathcal{M}(a,b|q) = \mathcal{M}([C]_A,[C]_B|q).$$

Due to (9) all the summands in (10) tend to their limits as $x \to \infty$. Hence the statement of Theorem 3 follows.

**Proof of Theorem 4.** The inequality follows by induction over the number of elements $|A| + |B|$. If $A = \{a\}, B = \{b\}$, then either $\nu(\mathcal{M}(a,b|q)) = 0$ or

$$\nu(\mathcal{M}(a,b|q)) = \prod_{p|q} \left(1 - \frac{2}{p+1}\right) \prod_{p|ab} \left(1 - \frac{1}{p+1}\right).$$ (11)

In the first case the inequality is trivial, and in the second one we have

$$1 - \nu(\mathcal{M}(a,b|q)) = 1 - \prod_{p|q} \left(1 - \frac{2}{p+1}\right) \prod_{p|ab} \left(1 - \frac{1}{p+1}\right) \geq \prod_{p|q} \left(1 - \frac{2}{p+1}\right) \prod_{p|ab} \left(1 - \frac{1}{p+1}\right).$$

Let the inequality holds for some finite sets $A, B$ and we add a new number $a^*$ to $A$. We shall show that the inequality will be satisfied for $\mathcal{M}(A^*,B|q)$ with $A^* = A \cup \{a^*\}$, too. Let us introduce the
following notations: \( [a^*, A] = \{ [a^*, a] : a \in A \}, [a^*, A] = \{ a/(a, a^*) : a \in A \} \), where \([a^*, a]\) denotes the least common multiple of numbers in brackets; if \( C \) is some finite set of numbers, then \([C]\) stands for the least common multiple of all elements of \( C \). We start with

\[
\mathcal{M}(A^*, B|q) = \mathcal{M}(A, B|q) \cup (\mathcal{M}(A^*, B|q) \setminus \mathcal{M}(A, B|q)).
\]

Denote briefly \( \mathcal{M}(A, B|q) = \mathcal{M}(A^*, B|q) \setminus \mathcal{M}(A, B|q) \). Then

\[
\nu(\mathcal{M}(A^*, B|q)) = \nu(\mathcal{M}(A, B|q)) + \nu(\mathcal{M}(A, B|q) \setminus \mathcal{M}(A, B|q)),
\]

\[
\nu(\mathcal{M}(A, B|q)) = \nu(\mathcal{M}(A^*, B|q)) - \nu(\mathcal{M}(A^*, A, B|q)).
\]

Using the sieve arguments and the properties of \( A \) one derives

\[
\nu(\mathcal{M}(a^*, B|q)) = \sum_{a < p \leq b} (-1)^{1+|C|} \nu(\mathcal{M}(a^*, [C]|q)) = \frac{1}{a^*} \prod_{p|a^*} \left( 1 - \frac{1}{p+1} \right) \nu(\mathcal{M}(1, B|q)),
\]

\[
\nu(\mathcal{M}([a^*, A], B|q)) = \sum_{c < a < a^*} (-1)^{1+|C|} \nu(\mathcal{M}([C], A|, [C]|B|q)) = \frac{1}{a^*} \prod_{p|a^*} \left( 1 - \frac{1}{p+1} \right) \nu(\mathcal{M}(a^*, A, B|q)).
\]

It follows now from this that

\[
\nu(\mathcal{M}(A^*, B|q)) = \nu(\mathcal{M}(A, B|q)) + \frac{1}{a^*} \prod_{p|a^*} \left( 1 - \frac{1}{p+1} \right) \left( \nu(\mathcal{M}(1, B|q)) - \nu(\mathcal{M}(a^*, A, B|q)) \right).
\]

Because of \( \nu(\mathcal{M}(1, B|q)) \leq 1 \) and \( \nu(\mathcal{M}(a^*, A, B|q)) \geq \nu(\mathcal{M}(A, B|q)) \) we obtain

\[
1 - \nu(\mathcal{M}(A^*, B|q)) \geq 1 - \nu(\mathcal{M}(A, B|q)) - \frac{1}{a^*} \prod_{p|a^*} \left( 1 - \frac{1}{p+1} \right) (1 - \nu(\mathcal{M}(A, B|q))),
\]

and the inequality for the sets \( A^*, B \) follows. If instead of \( A \) we add a new element to \( B \), the arguments proving the inequality would be essentially the same. The Theorem is proved.

**Proof of Theorem 5.** Recall that for \( A \subset \mathbb{N} \) we denote by \( \mathcal{M}(A) \) the set of multiples of elements \( a \in A \). If \( N > 1 \) let \( A_N = A \cap [1; N] \).

We start with the equality

\[
\nu_x^{\epsilon, \tau}(\mathcal{M}(A, B|q)) = \nu_x^{\epsilon, \tau}(\mathcal{M}(A_N, B_N|q)) + \nu_x^{\epsilon, \tau}(\mathcal{M}(A, B|q) \setminus \mathcal{M}(A_N, B_N|q)). \tag{12}
\]

It suffices to show that for any \( \epsilon > 0 \) the upper limit of the second term in (12) is less than \( \epsilon \) as \( x \to \infty \), supposed that \( N \) is large enough. Define two subsets of rational numbers

\[
\mathcal{M}_N^1 = \left\{ \frac{m}{n} : m \in \mathcal{M}(A) \setminus \mathcal{M}(A_N) \right\}, \quad \mathcal{M}_N^2 = \left\{ \frac{m}{n} : n \in \mathcal{M}(B) \setminus \mathcal{M}(B_N) \right\}.
\]

Then

\[
\mathcal{M}(A, B|q) \setminus \mathcal{M}(A_N, B_N|q) \subset \mathcal{M}_N^1 \cup \mathcal{M}_N^2.
\]

We are going to prove that for fixed \( \delta > 0 \) and \( N \) sufficiently large we shall have \( S^{i, \epsilon}_x(\mathcal{M}_N^i) \leq \delta \) for \( i = 1, 2 \). Denote for the sake of brevity \( \mathcal{M}(A)_N = \mathcal{M}(A) \setminus \mathcal{M}(A_N), \mathcal{M}(B)_N = \mathcal{M}(B) \setminus \mathcal{M}(B_N) \). Then

\[
S^{i, \epsilon}_x(\mathcal{M}_N^i) \leq \sum_{\lambda_1 < \lambda_2 \leq x} \frac{1}{\lambda_2} \sum_{n_i \in \mathcal{M}(\lambda_i)} \frac{1}{\lambda_2}.
\]

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Let $\lambda_1 = 0$ first. With some constant $c > 0$ we have

$$S_{x,I}^1(\mathcal{M}_N^2) \leq (\log(\lambda_2 x) + c) \sum_{n \in \mathcal{M}(B)_{N}} \frac{1}{n}.$$ 

The Erdős-Davenport statement as formulated in the Theorem 1 implies that there exists some vanishing sequence $\delta_N$ such that $\varphi^1(\mathcal{M}(B)_N) < \delta_N$. It follows then that for $x$ a sufficiently large we shall have

$$S_{x,I}^1(\mathcal{M}_N^2) \leq \delta_N \log x (\log(\lambda_2 x) + c).$$

Compare now the functions on the right-side of this inequality to that ones in the asymptotics of $S_{x,I}^1(\mathbb{Q}^+)$ (see Lemma):

$$S_{x,I}^1(\mathcal{M}_N^2) \leq \left\{ \log^2(\lambda_2 x) \left\{ \frac{\delta_N}{\log(\lambda_2 x)} + \frac{\delta_N}{\log(\lambda_2 x)} \right\}, \log x \cdot \log(\lambda_2 x) \left\{ \frac{\delta_N}{\log(\lambda_2 x)} + \frac{\delta_N}{\log(\lambda_2 x)} \right\} \right\}.$$

Having in mind the conditions on $\lambda_i$ we conclude that $\varphi^1(\mathcal{M}_N^2) \leq \delta$ for $N$ large enough.

We shall show now that $\varphi^1(\mathcal{M}_N^1) \leq \delta$ as well. If $m/n < \lambda_2$ and $n \leq x$, then $m \leq \lambda_2 x$ and $n > m/\lambda_2$. We start with

$$S_{x,I}^1(\mathcal{M}_N^1) \leq \sum_{m \in \mathcal{M}(A)_N} \frac{1}{m} \sum_{m/\lambda_2 < n \leq x} \frac{1}{n} + \sum_{m \in \mathcal{M}(A)_N} \frac{1}{m} \sum_{m/\lambda_2 < n \leq x} \frac{1}{n}.$$

Consider the first summand. Using the Erdős-Davenport theorem as above we obtain that for $x$ large enough

$$\sum_{m \in \mathcal{M}(A)_N} \frac{1}{m} \sum_{m/\lambda_2 < n \leq x} \frac{1}{n} \leq (\log x + c) \sum_{m \in \mathcal{M}(A)_N} \frac{1}{m} \leq \delta_N (\log x + c) \log \lambda_2,$$

where $\delta_N \to 0$ as $N \to \infty$. Using similar arguments for the second sum we get

$$S_{x,I}^1(\mathcal{M}_N^1) \leq \delta_N (\log x + c) \log \lambda_2 + \sum_{m \in \mathcal{M}(A)_N} \frac{1}{m} \log \left( \frac{\lambda_2 x}{m} \right) + \frac{\lambda_2}{m} \leq \delta_N (\log x + c) \log(\lambda_2 x) + c_1, \quad c_1 > 0,$$

and $\varphi(\mathcal{M}_N^1) \leq \delta$. This completes the proof in the case $\lambda_1 = 0$.

Let now $\lambda_1 > 0$. Then using the Erdős-Davenport theorem again we have

$$S_{x,I}^1(\mathcal{M}_N^2) \leq \sum_{m \in \mathcal{M}(A)_N} \frac{1}{m} \sum_{m/\lambda_1 n < m/\lambda_2 n} \frac{1}{n} \leq \sum_{m \in \mathcal{M}(A)_N} \frac{1}{m} \left\{ \log \left( \frac{\lambda_2}{\lambda_1} \right) + \frac{1}{\lambda_1 n} \right\} \leq \log \left( \frac{\lambda_2}{\lambda_1} \right) \log x \left( \delta_N + \frac{1}{\lambda_1 \log(\lambda_2/\lambda_1)} \log x \right).$$

Note that under conditions on $\lambda_1, \lambda_2$

$$\lambda_1 \log(\lambda_2/\lambda_1) \log x = \lambda_1 \log(\lambda_2 + 2) \left( \log \left( \frac{\lambda_2}{\lambda_1} \right) \log x \right) \to \infty,$$
as }x \to \infty{. Consequently }S_{x,I}^{11}(\mathcal{M}_N^1) \leq \delta.

For }S_{x,I}^{11}(\mathcal{M}_N^1)\text{ we proceed as follows:

\[
S_{x,I}^{11}(\mathcal{M}_N^1) \leq \sum_{m \in \mathcal{M}(A)} \frac{1}{m} \sum_{n \in \mathcal{M}(A)} \frac{1}{n} + \sum_{\lambda_2 \leq \lambda_1} \sum_{m \leq \lambda_2 \leq n} \frac{1}{n}. \tag{13}
\]

If }\lambda_2\text{ remains bounded, then the first sum in (13) is zero for }N\text{ sufficiently large. Otherwise we have

\[
\sum_{m \in \mathcal{M}(A)} \frac{1}{m} \sum_{n \in \mathcal{M}(A)} \frac{1}{n} \leq \sum_{m \in \mathcal{M}(A)} \frac{1}{m} \left( \log \left( \frac{\lambda_2}{\lambda_1} \right) + c \right) \leq \log \left( \frac{\lambda_2}{\lambda_1} \right) \log x \left\{ \delta_N \frac{\log \lambda_2}{\log x} + \delta_N \frac{c \log \lambda_2}{\log \left( \lambda_2/\lambda_1 \right) \log x} \right\}.
\]

For the second sum in (13) we obtain

\[
\sum_{\lambda_2 \leq \lambda_1 \leq n} \frac{1}{n} \leq \sum_{\lambda_2 \leq \lambda_1 \leq n} \frac{1}{m} \left( \log \left( \frac{\lambda_2}{\lambda_1} \right) + \frac{c \lambda_2}{m} \right) \leq \delta_N \log \left( \frac{\lambda_2}{\lambda_1} \right) \log \lambda_2 + c
\]

\[
\leq \log \left( \frac{\lambda_2}{\lambda_1} \right) \log x \left\{ \delta_N \frac{\log \lambda_2}{\log x} + \delta_N \frac{c \log \lambda_2}{\log \left( \lambda_2/\lambda_1 \right) \log x} \right\}.
\]

It follows from both estimates that for given }\delta > 0\text{ under conditions on }\lambda_i\text{ we shall have }S_{x,I}^{11}(\mathcal{M}_N^1) \leq \delta S_{x,I}^{11}(\mathcal{Q}^+), \text{ supposed }x,N\text{ are large enough. Hence }\mathfrak{v}^{11}(\mathcal{M}_N^1) \leq \delta, \text{ and the proof of theorem is completed.}

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