Spectral solution of urn models for interacting particle systems

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Using generating function methods for diagonalizing the transition matrix in 2-Urn models, we provide a complete classification into solvable and unsolvable subclasses, with further division of the solvable models into the Martingale and non-Martingale subcategories, and prove that the stationary distribution is a Gaussian function in the latter. We also give a natural condition related to the symmetry of the random walk in which the non-Martingale Urn models lead to an increase in entropy from Gaussian states. Certain models of social opinion dynamics, treated as Urn models, do not increase in entropy, unlike isolated mechanical systems.

Physical applications of urn problems can be traced to the Ehrenfest model to describe the Second Law of Thermodynamics [1]. This model assumes that a closed container is subdivided into two equal sides. Particles are chosen randomly to change sides independently of other particles. The model is often posed as an urn problem in which balls are chosen randomly to move to the other urn. In a more general model, the urns can correspond to notions such as spins, chemical substrates, or opinions. However, systems such as the Ehrenfest model do not adequately describe the dynamics of interacting particle systems [2]. In the systems that we introduce, the particles change urns by interacting with each other. Naturally, these models have a wide range of physical applications for various interpretations of the urns themselves, such as well mixed kinetic reactions [3, 5] and thermodynamics [1, 6]. These models also have applications to social opinion dynamics, in which the voter model [7, 9] is the only case that is a martingale. When the system is generalized to three urns, one can pose Naming Game dynamics on the complete graph [10, 13] in a similar fashion. Further instances of interacting particle systems are the contact process, exclusion processes, and stochastic Ising models [2, 5].

In addition to the wider class of models that we introduce to describe interacting particle systems, we also provide their exact solutions. The method is an extension of the generating function solution of the Ehrenfest model formulated by Mark Kac in 1947 [6]. Despite the inherent differences between interacting particle systems and the Ehrenfest model, the method of solution we provide is powerful enough to solve both types of models. We utilize a generating function method for solving the spectral problem of the Markov transition matrix for each model, which allows us to easily find the m step propagator of the model. Since all future probability distributions of macro-states are found explicitly, we can compute several quantities depending on the application of the model. In sociophysics [13], the expected time for all people to have the same opinion is a quantity of great interest in the literature [10, 12, 14, 15]. For thermodynamic systems, one may wish to know how entropy changes as the system evolves with time. For a general non-martingale interacting particle system proposed here, we show that entropy is not guaranteed to increase or decrease monotonically. We will provide a condition in which entropy will decrease from Gaussian states.

![Classification tree for the 2-Urn problems that shows all relevant subclasses. Among the linear cases (2γ1 = 2γ2 − α1 + α2 + β1 − β2 = 0), the martingales (α1 = β1, α2 = β2 = γ1 = γ2 = 0) are equivalent to the voter model and the non-martingales constitute a much larger class of models. We show that an entropy increase in the non-martingale cases is equivalent to macroscopic symmetry of the model given by Eqn. 20.](image)

In the models that we consider, there are two urns (A and B) with N balls distributed between them. In a discrete time step, two balls are drawn randomly, one after the other. The balls are then redistributed between the urns stochastically. The redistribution probabilities depend on the urns from which the balls came and the order that they were drawn. We consider all interacting particle systems of this type.

The post-selection probability distributions are utilized to define rate parameters which specify the model in the macro-state. Let n_A(m) denote the number of balls in urn A at discrete time m. The model is completely
characterized through the specification of the rate parameters, which we denote as \{\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2\}. Given that the balls came from different urns, \alpha_1 and \beta_1 are the probabilities that \(n_A\) increases/decreases by one respectively. If they were drawn from the same urn, \(\alpha_2\) and \(\beta_2\) are the probabilities that \(n_A\) increases or decreases by one respectively. Similarly, \(\gamma_1\) and \(\gamma_2\) are the probabilities that \(n_A\) increases or decreases by two. These parameters correspond to probability distributions, they should be chosen such that the system is physically viable. For example, \(\alpha_2 + \gamma_1 \leq 1\) is one such constraint.

The parameters of the urn model have interpretations based on the influence of people in social settings. Values of \(\alpha_1\) and \(\beta_1\) correspond to the impact a person has on another with an opposite opinion. In this interaction, two opposing individuals enter a discussion and one of them changes their opinion as a result. The voter model on the complete graph has this interpretation and has parameters \(\alpha_1 = \beta_1 = 1\), with all other parameters set to zero. The other parameters, \(\alpha_2, \beta_2, \gamma_1,\) and \(\gamma_2\) can correspond to mutation and competition of individuals. Agents of the same type that interact with each other may compete with each other, with one or both changing as a result.

The parameters affect the transition probabilities of the urn model when \(n_A = i\), which are given to be

\[
\begin{align*}
\alpha_1 & = \frac{i(N-i)}{N(N-1)} + \frac{\alpha_1(N-i)(N-i-1)}{N(N-1)} + \alpha_2(1 - \gamma_1) (N-i) \\
\beta_1 & = \frac{(N-i)(N-i-1)}{N(N-1)} \\
\alpha_2 & = \frac{i(N-i)}{N(N-1)} + \frac{\beta_1 i(N-i-1)}{N(N-1)} \\
\beta_2 & = \frac{i(N-i)}{N(N-1)} + \frac{\gamma_1 i(N-i-1)}{N(N-1)} \\
\gamma_1 & = \frac{i(N-i)}{N(N-1)} + \frac{\gamma_2 i(N-i-1)}{N(N-1)}
\end{align*}
\]

Here, we define \(\alpha_1(k) = Pr(\Delta n_A = k | n_A = i)\) and \(\gamma_1(k) = Pr(\Delta n_A = -k | n_A = i)\). Notice that the parameter choice \(\{1, 1, 1, 1, 0, 0\}\) exactly simplifies to the Ehrenfest model. Let \(a_i^{(m)} = Pr(n_A(m) = i)\). We introduce the finite difference operator \(\Delta_i\) acting on a grid function \(\phi_i\) defined as \(\Delta_i [\phi_i] = \phi_{i+k} - \phi_i\). With this notation, we can form a single step difference equation that describes the probability distribution in macro-state:

\[
\begin{align*}
\Delta_{+m} a_i^{(m)} &= \Delta_{-1} [\alpha_1^{(m)}] + \Delta_{-2} [\alpha_2^{(m)}] \\
&+ \Delta_{+1} [\beta_1^{(m)}] + \Delta_{+2} [\beta_2^{(m)}].
\end{align*}
\]

Here, \(\Delta_{+m}\) is the forward difference operator over the time index \(m\). This constitutes a pentadiagonal Markov transition matrix for the macro-state probability distribution. We solve for all eigenvalues and eigenvectors of this model by extending the procedure described in detail in Ref. [10], which solved the voter model on various networks. The method is outlined in the context of these general interacting particle system as follows. For eigenvalue \(\lambda\) and eigenvector \(v\) with components \(c_i\), let \(G(x, y) = \sum_i c_i x^i y^{N-i}\). We rewrite the spectral problem for the single step propagator given in Eqn. [5] as an equivalent partial differential equation for \(G\). This is done using the differentiation and shift properties of \(G\) [16 18 20]. Doing so shows that the PDE for \(G\) is

\[
N(N-1)(\lambda - 1)G = \gamma_1(x^2 - y^2)G_{yy} + \alpha_1 x(x-y)G_{xy} + \alpha_2 y(x-y)G_{yy} - \gamma_1(x^2 - y^2)G_{xx} - \beta_1 y(x-y)G_{xy} - \beta_2 x(x-y)G_{xx}.
\]

To solve this equation, we make the change of variables \(u = x - y\) and \(H(u, y) = G(x, y)\). We show below that \(H\) to have the same structure as \(G\). That is, we can define \(H(u, y) = \sum_i b_i u^i y^{N-i}\). This change of variables will allow us to solve the system when \(2\gamma_1 - 2\gamma_2 - \alpha_1 + \alpha_2 + \beta_1 - \beta_2 = 0\). Under this parameter restriction, the change of variables will transform the pentadiagonal structure of the transition matrix into a lower triangular matrix. Since the transformed matrix is lower triangular, the difference equation for \(b_i\) is explicit, which allows us to find both \(\lambda\) and \(c_i\). Collecting coefficients in the transformed equation for \(H\) yields

\[
b_i = \frac{((-2\gamma_1 + \alpha_1)(i-1) + (2\gamma_1 + \alpha_2)(N-i))b_{i-1} + \gamma_1(N-i+2)b_{i-2}(N-i+1)}{N(N-1)(\lambda - 1) + (2\gamma_1 + 2\gamma_2 + \alpha_2 + \beta_2) i(N-i) + \frac{1}{2} (\alpha_1 + \alpha_2 + \beta_1 + \beta_2)(i-1)}.
\]

This allows us to find the exact expression for all eigenvalues. Since the coefficients of \(G\) are zero for \(i < 0\) and \(i > N\), we require \(b_i = 0\) for \(i < 0\) and \(i > N\) as well. Since Eqn. [7] is an explicit linear difference equation, we would expect that all \(b_i = 0\) unless the equation is singular for some \(i = k\). However, since this would correspond to the trivial solution to the eigenvalue problem, we discard these solutions. Thus, requiring a singularity at \(i = k\) implies that the denominator of Eq. [7] must be zero. Solving for \(\lambda\) shows that the eigenvalues are...
\[ \lambda_k = 1 - \frac{(2\gamma_1 + 2\gamma_2 + \alpha_2 + \beta_2)k(N-k)}{N(N-1)} - \frac{(\alpha_1 + \alpha_2 + \beta_1 + \beta_2)k(1-k)}{2N(N-1)} \] for \( k = 0 \ldots N \). This allows \( b_k \) to take any value. Values for \( b_i \) for any \( i > k \) can be found by repeated application of Eq. 7. The components of each eigenvector can be found in terms of \( b_i \). We do this by expressing \( H(u, y) \) in the original coordinates, giving

\[ G(x, y) = \sum_{i=0}^{N} \sum_{j=1}^{N} (-1)^{j-i} \binom{j}{i} b_j x^i y^{N-i} \]

as the generating function for the eigenvectors \( \text{Ref. [16]} \), which shows that \( H \) and \( G \) have the same form. Thus the spectral problem is solved for all urn models that satisfy \( 2\gamma_1 - 2\gamma_2 - \alpha_1 + \alpha_2 + \beta_1 - \beta_2 = 0 \). It happens that this parameter constraint holds if and only if the dynamical system for mean density \( \{ \rho_A(t), \rho_E(t) \} \) is linear. We define a linear urn model as any of the above models that satisfy this constraint. We conjecture that no other change of variables \( (x, y) \rightarrow (u, v) \) will solve the nonlinear cases in this fashion, although a rigorous proof of this claim is not given here.

The treatment of the spectral problem by generating functions is equivalent to solving it by a similarity transformation of the transition matrix. Let \( \mathbf{T} \) denote the transition matrix given by Eqn. 5 and let \( \mathbf{v} = \mathbf{P} \mathbf{w} \) for some transformation matrix \( \mathbf{P} \). Then, the spectral problem for \( \mathbf{w} \) is given by \( \mathbf{P}^{-1} \mathbf{T} \mathbf{P} \mathbf{w} = \lambda \mathbf{w} \). The generating function method prescribes the matrix \( \mathbf{P} \) so that the new matrix \( \mathbf{L} = \mathbf{P}^{-1} \mathbf{T} \mathbf{P} \) is lower triangular with a bandwidth of at most two. The components of the transformation matrices that do this are determined to be

\[
[\mathbf{P}]_{ij} = (-1)^{j-i} \binom{j}{i} \\
[\mathbf{P}^{-1}]_{ij} = \binom{j}{i}.
\]

We use the convention that \( \binom{j}{i} = 0 \) when \( i > j \), which suggest that both \( \mathbf{P} \) and \( \mathbf{P}^{-1} \) are upper triangular. Furthermore, these matrices would be difficult to construct analytically without this generating function theory.

The spectral decomposition of the transition matrix can be found by this similarity transformation. We do this by diagonalizing the matrix \( \mathbf{L} = \mathbf{W} \mathbf{A} \mathbf{W}^{-1} \). Here, \( \mathbf{A} = \text{diag}(\lambda_1, \ldots, \lambda_N) \) and \( \mathbf{W} \) are the eigenvectors of \( \mathbf{L} \). The components of these eigenvectors are \( b_i \) corresponding to eigenvalue \( \lambda_k \). Since \( b_i = 0 \) for \( i < j \), the matrix of eigenvectors \( \mathbf{W} \) is lower triangular. Therefore, \( \mathbf{W}^{-1} \) can be found explicitly via forward substitution. Diagonalization of \( \mathbf{L} \) allows us to explicitly diagonalize the transition matrix as

\[ \mathbf{T} = (\mathbf{P} \mathbf{W}) \mathbf{A} (\mathbf{P} \mathbf{W})^{-1}. \]

The immediate consequence of the explicit diagonalization of the transition matrix is the solution of the \( m \) step propagator. Beginning with the initial probability distribution of macro-states, \( \mathbf{a}^{(0)} \), the future distributions are given by \( \mathbf{a}^{(m)} = (\mathbf{P} \mathbf{W})^m (\mathbf{P} \mathbf{W})^{-1} \mathbf{a}^{(0)} \). With this solution, we can find several valuable quantities. Table I summarizes some of the exact solutions that directly follow from the diagonalization. The quantity \( d_k \) is the initial distribution expressed in the eigenbasis. That is, \( d_k \) are the components of \( \mathbf{d} = (\mathbf{P} \mathbf{W})^{-1} \mathbf{a}^{(0)} \).

| Macro-state probability | Discrete Solution |
|-------------------------|-------------------|
| \( a_i^{(m)} \) | \( \sum_{k=0}^{N} d_k \gamma_{vk}^i \lambda_k^m \) |

| Consensus time | \( E[\tau^p] \) | \( \sum_{k=1}^{N} \frac{d_k p!}{[N(1-\lambda_k)]^{p+1}} \times \) |
|---------------|----------------|--------------------------------------------------|
|                | \( \{ \alpha_i \gamma_{vk} | N-1 + \frac{2g_1}{N-1} \} \) |

| Local time | \( E[M] \) | \( \frac{1}{N} \sum_{k=0}^{N} \frac{d_k \gamma_{vk}}{1-\lambda_k} \) |

| Gibbs entropy | \( S(m) \) | \( -\sum_{i=0}^{N} a_i^{(m)} \log a_i^{(m)} / N \) |

The consensus time is the amount of scaled time \( (\tau = m/N) \) until all of the balls are in a single urn and the dynamics of the system halt. We assume that only one of the consensus points is a Stopped state and without loss in generality, we assume that it is when \( n_A = N \) instead of \( n_A = 0 \). When both consensus points are absorbing, the linear urn model we solved reduces to the voter model on the complete graph, which is well studied \( [12, 15, 16, 22] \). Not only can the expected time to consensus be found, but all moments thereof. The mathematical derivation of this formula for the voter model is explored in detail in Ref. \( [16] \).

The local time is the total amount of scaled time spent in each macro-state prior to the Stopped consensus. The sum of the local times is equivalent to the consensus time, which makes this a more detailed quantity, although the higher moments are not computed here. The expected local time \( M_i \) for macro-state \( n_A = i \) is known to be \( E[M_i] = \frac{1}{N} \sum_{m=0}^{\infty} a_i^{(m)} \), which we can compute exactly by the diagonalization \( [16] \).

Next, we consider entropy defined in the sense of Gibbs \( [23] \). We also make the common assumption in statistical mechanics that each micro-state is equally likely
for a given macro-state [24]. When the probability distribution is $\mathcal{N}(\mu, \sigma^2)$, the entropy can be shown to be $S \sim N \log 2 - 2\sigma^2 - 2(\mu - \frac{1}{2})^2 + \log(2\sqrt{\piN}) + \frac{1}{2}$. We use this when calculating the change in entropy between the stationary distribution and an initial Gaussian distribution, $\mathcal{N}(\mu_0, \sigma_0^2)$. Let $\pi_i$ to be the discrete stationary distribution and let $\pi(\rho; N) \approx \pi_i N$ be the probability density function that approximates $\pi_i$ for large $N$. The following results prove that the stationary distribution is asymptotically Gaussian whose mean and variance are also calculated exactly, which allows us to efficiently compute the change in entropy.

**Theorem:** The following holds for all non-martingale linear urn models $[22]$: (1) $\pi(\rho; N) = \mathcal{N}(\mu_f, \sigma_f^2) + O\left(\frac{1}{\sqrt{N}}\right)$ (13) (2) $E_x[n_A] = b_1$ (14) (3) $Var_x(n_A) = 2b_2 + b_1 - b_1^2$ (15)

**Proof:** The proof of statement (1) begins by finding the Fokker-Planck equation for the probability density as $N \to \infty$:

$$u_t = -\frac{\partial}{\partial \rho}[\nu(\rho)u] + \frac{1}{2N} \frac{\partial^2}{\partial \rho^2}[D(\rho)u].$$

Here, $u(\rho, t)$ is the probability density function, $\nu(\rho) = 2p^{(2)}(\rho) + p^{(1)}(\rho) - q^{(1)}(\rho) - 2q^{(2)}(\rho)$, and $D(\rho) = 4p^{(2)}(\rho) + p^{(1)}(\rho) + q^{(1)}(\rho) + 4q^{(2)}(\rho)$. The functions $p^{(k)}(\rho)$ and $q^{(k)}(\rho)$ are the continuous analogs of their discrete counterparts. At $t \to \infty$, the probability density will approach the stationary distribution $\pi(\rho; N)$, and $u_t(\rho, t) \to 0$. The ODE for the stationary distribution is

$$0 = \frac{d}{d\rho}[\nu(\rho)\pi(\rho; N)] + \frac{1}{2N} \frac{d^2}{d\rho^2}[D(\rho)\pi(\rho; N)].$$

(17)

Since we assume that the drift is linear, $\nu(\rho)$ is a linear function. Furthermore, there exists $\rho_0$ such that $\nu(\rho_0) = 0$. Therefore, we can write $\nu(\rho) = \nu_1(\rho - \rho_0)$. We make the linear change of variables $\rho \to \xi$ defined as $\rho = \rho_0 + \delta \xi$. We will choose $\delta = o(1)$ so that the ODE has non-trivial balance. Also, we let $\pi(\rho; N) = \pi_0(\rho; N) + \delta \pi_1(\rho; N) + \ldots$ be an asymptotic expansion of the stationary probability density. Making these substitutions into Eqn. (17) yields

$$-\nu_1 \frac{d}{d\xi}[\pi_0(\xi; N)] + \frac{D_0}{2N\delta^2} \frac{d^2}{d\xi^2}[\pi_0(\xi; N)] + O\left(\frac{1}{N\delta}\right) = 0.$$ (18)

The coefficients $D_0$ and $\nu_1$ depend only of the choice of parameters $\{\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2\}$. For the leading order terms to balance, choose $\delta = 1/\sqrt{N}$. As $N \to \infty$, the leading order terms give an ODE for $\pi_0(\xi; N)$. The solution of this ODE is a Gaussian function centered at $\xi = 0$. As a function of $\rho$, the Gaussian is centered at $\rho_0$ with mean $\mu$ and variance $\sigma^2$. To characterize the mean and variance, we prove statements (2) and (3).

The proof of (2) follows by considering the generating function $G(x, y)$ that corresponds to the eigenvalue $\lambda = 1$. This is defined by $G(x, y) = \sum \pi_i x^i y^{N-i}$. Note that $G(1,1) = 1$ for normalization and $E_x[n_A] = G_x(1,1)$. In terms of $H(u, y) = \sum_i b_i u^i y^{N-i}$, we have that $H(0,1) = 1$ and $G_x(1,1) = H_y(0,1)$. This implies that $b_0 = 1$ and $H_y(0,1) = b_1$. Therefore, $E_x[n_A] = b_1$. Since $b_0 = 0$ for $i < 0$, we can use Eqn. (17) to find $b_1$ exactly. The proof of (3) is demonstrated in a similar fashion.

The values of $b_1$ and $b_2$ depend only on the choice of parameters and $N$, which allow us to exactly characterize the stationary distribution in terms of the parameterization of the urn model. For the Ehrenfest model, Mark Kac proved that the stationary distribution of the Ehrenfest model is $\pi_1 = \left(\frac{N}{2}\right)^2 N^{-2}$ [6], which is consistent with this result due to the central limit theorem applied to Bernoulli trials [22].

When $b_2$ and $b_1$ are combined to give $Var_x(n_A)$, the result is $O(N)$. Thus, the variance for the density is $\sigma_f^2 = O(1/N)$. This is particularly relevant to the entropy of the system as $m \to \infty$. Since the stationary distribution is also Gaussian, the change in entropy is $\Delta S = \log(\sigma_f/\sigma_0) - 2N\sigma_f^2 + 2N\sigma_0^2 + (\mu_0 - 1/2)^2 - (\mu_f - 1/2)^2$. As $N \to \infty$, this implies that the entropy of the system will decrease when

$$\mu_f(1 - \mu_f) < \mu_0(1 - \mu_0) - \sigma_f^2.$$ (19)

A significant consequence of Eqn. (19) is that unless $\mu_f(1 - \mu_f)$ achieves its maximum value, there will always
exist an initial condition, $(\mu_0, \sigma_0^2)$, that will cause entropy to decrease with time for large $N$. Thus, for an entropy increase as $N \to \infty$, we require $\mu_f = b_1/N = 1/2$.

We show that the condition given in Eqn. (19) for an increase in entropy is equivalent to a form of symmetry in the model. Let $U_1, U_2$ denote the urns that the first and second balls are selected respectively and $U_c$ denotes the urn that is opposite to $U$. We define a macroscopically symmetric urn model when

$$E[\Delta n_A|U_1, U_2] = E[\Delta n_B|U_1^c, U_2^c] \quad (20)$$

for all permutations of $U_1, U_2$. The following result relates entropy to this form of symmetry.

**Theorem:** If the urn model is linear and non-martingale, then Eqn. (20) is necessary and sufficient for an increase in entropy from Gaussian states.

**Proof:** If the system is macroscopically symmetric, then $2\gamma_1 - 2\gamma_2 + \alpha_2 - \beta_2 = 0$ and $\alpha_1 = \beta_1$. These constraints together imply that the system has linear drift. Furthermore, using Eqn (10), we have that $\mu_f = 1/2$, which is sufficient to show that entropy increases from Gaussian initial states by Eqn. (19). To show the converse, an entropy increase requires $\mu_f = 1/2$, which implies $2\gamma_1 - 2\gamma_2 + \alpha_2 - \beta_2 = 0$. Using linearity, we have $\alpha_1 = \beta_1$, which is sufficient to show macroscopic symmetry. □

If the microscopic behavior of the system is invariant under an urn permutation, then $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \gamma_1 = \gamma_2$. If microscopic symmetry holds, then macroscopic symmetry holds, but the converse is not necessarily true. A counterexample to this is to consider $\gamma_1 = 1/2, \beta_2 = 1$, and all other parameters are zero.

In conclusion, the solutions that we provide here can be applied to a variety of physical applications. This method of analysis is extremely powerful for studying systems of interacting particles. Furthermore, we can derive several exact quantities of interest that could not be found otherwise. For many problems, it is assumed that $N$ must be large for the solutions to be valid. Since we have provided exact discrete solutions, the results given in this Letter are valid even for small $N$. In social systems, these correspond to the dynamics of crowds, which require this spectral solution to address. An extension of this framework for social models with multiple opinions and zealots can be studied in this way.

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