Self-Interaction Correction to Black Hole Radiance

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Abstract We consider the modification of the formulas for black hole radiation, due to the self-gravitation of the radiation. This is done by truncating the coupled particle-hole system to a small set of modes, that are plausibly the most significant ones, and quantizing the reduced system. In this way we find that the particles no longer move along geodesics, nor is the action along the rays zero for a massless particle. The radiation is no longer thermal, but is corrected in a definite way that we calculate. Our methods can be extended in a straightforward manner to discuss correlations in the radiation, or between incoming particles and the radiation.
1. Introduction

Black hole radiance [1] was originally derived in an approximation where the background geometry was given, by calculating the response of quantum fields to this (collapse) geometry. In this approximation the radiation is thermal, and much has been made both of the supposed depth of this result and of the paradoxes that ensue if it is taken literally. For if the radiation is accurately thermal there is no connection between what went into the hole and what comes out, a possibility which is difficult to reconcile with unitary evolution in quantum theory – or, more simply, with the idea that there are equations uniquely connecting the past with the future. To address such questions convincingly, one must go beyond the approximation of treating the geometry as given, and treat it too as a quantum variable. This is not easy, and as far as we know no concrete correction to the original result has previously been derived in spite of much effort over more than twenty years. Here we shall calculate what is plausibly the leading correction to the emission rate of single particles in the limit of large Schwarzschild holes, by a method that can be generalized in several directions, as we shall outline.

There is a semi-trivial fact about the classic results for black hole radiation, that clearly prevents the radiation from being accurately thermal. This is the effect that the temperature of the hole depends upon its mass, so that in calculating the “thermal” emission rate one must know what mass of the black hole to use – but the mass is different, before and after the radiation! (Note that a rigorous identification of the temperature of a hot body from its radiation, can only be made for sufficiently high frequencies, such that the gray-body factors approach unity. But it is just in this limit that the ambiguity mentioned above is most serious.) As has been emphasized elsewhere [2], this problem is particularly quantitatively acute for near-extremal holes — it is a general problem for bodies with finite heat capacity, and in the near-extremal limit the heat capacity of the black hole vanishes.

To resolve the above-mentioned ambiguity, one clearly must allow the geometry to fluctuate, namely to support black holes of different mass. Another point of view is that one must take into account the self-gravitational interaction of the radiation.

2. Model and Strategy of Calculation

To obtain a complete description of a self-gravitating particle it would be necessary to compute the action for an arbitrary motion of the particle and gravitational field. While writing down a formal expression for such an object is straightforward, it is of little use
in solving a concrete problem due to the large number of degrees of freedom present.
To arrive at a more workable description of the particle-hole system, we will keep only
those degrees of freedom which are most relevant to the problem of particle emission from
regions of low curvature. The first important restriction is made by considering only
spherically symmetric field configurations, and treating the particle as a spherical shell.
This is an interesting case since black hole radiation into a scalar field occurs primarily
in the s-wave, and virtual transitions to higher partial wave configurations are formally
suppressed by powers of $\hbar^2$. *

Before launching into the detailed calculation, which becomes rather intricate, it seems
appropriate briefly to describe its underlying logic. After the truncation to $s$-wave, the
remaining dynamics describes a shell of matter interacting with a black hole of fixed mass
and with itself. (The mass as seen from infinity is the total mass, including that from
the shell variable, and is allowed to vary. One could equally well have chosen the total
mass constant, and allowed the hole mass to vary.) There is effectively one degree of
freedom, corresponding to the position of the shell, but to isolate it one must choose
appropriate variables and solve constraints, since the original action superficially appears
to contain much more than this. Having done that, one obtains an effective action for the
true degree of freedom. This effective action is nonlocal, and its full quantization would
require one to resolve factor-ordering ambiguities, which appears very difficult. Hence
we quantize it semi-classically, essentially by using the WKB approximation. After doing
this one arrives at a non-linear first order partial differential equation for the phase of the
wave function. This differential equation may be solved by the method of characteristics.
According to this method, one solves for the characteristics, specifies the function to be
determined along a generic initial surface (intersecting the characteristics transversally),
and evolves the function away from the initial surface, by integrating the action along the
characteristics. (For a nice brief account of this, see [3].)

When the background geometry is regarded as fixed the characteristics for particle
motion are simply the geodesics in that geometry, and they are essentially independent
of the particle’s mass or energy — principle of equivalence — except that null geodesics
are used for massless particles, and timelike geodesics for massive particles. Here we
find that the characteristics depend on the mass and energy in a highly non-trivial way.
Also the action along the characteristics, which would be zero for a massless particle
and proportional to the length for a massive particle, is now a much more complicated
expression. Nevertheless we can solve the equations, to obtain the proper modes for our
problem.

Having obtained the modes, the final step is to identify the state of the quantum field

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*Since we do not address the ultraviolet problems of quantum gravity these corrections are actually
infinite, but one might anticipate that in gravity theory with satisfactory ultraviolet behavior the virtual
transitions will supply additive corrections of order $\omega^2/\Lambda^2$, where $\Lambda$ is the effective cutoff, but will not
alter the exponential factors we compute.
— that is, the occupation of the modes — appropriate to the physical conditions we wish to describe. We do this by demanding that a freely falling observer passing through the horizon see no singular behavior, and that positive frequency modes are unoccupied in the distant past. This, it has been argued, is plausibly the appropriate prescription for the state of the quantum field excited by collapse of matter into a black hole, at least in so far as it leads to late-time radiation. Using it, we obtain a mixture of positive- and negative-frequency modes at late times, which can be interpreted as a state of radiation from the hole. For massless scalar particles, we carry the explicit calculation far enough to identify the leading correction to the exponential dependence of the radiation intensity on frequency.

3. Effective Action

We now derive the Hamiltonian effective action for a self-gravitating particle in the s-wave. The Hamiltonian formulation of spherically symmetric gravity is known as the BCMN model [4]; our treatment of this model follows that of [5]. First, we would like to explain why the Hamiltonian form of the action is particularly well suited to our problem. As explained above, our physical problem really contains just one degree of freedom, but the original action appears to contain several. The reason of course is that Einstein gravity is a theory with constraints and one should only include a subset of the spherically symmetric configurations in the physical description, namely those satisfying the constraints. In general, in eliminating constraints Hamiltonian methods are more flexible than Lagrangian methods. This appears to be very much the case for our problem, as we now discuss.

In terms of the variables appearing in the Lagrangian description, the constraints have the form

$$C_L [\dot{r}, \ddot{r}; g_{\mu\nu}, \dot{g}_{\mu\nu}] = 0,$$

where $\dot{r}$ is the shell radius, and $\dot{}$ represents $\frac{d}{dt}$. When applied to the spherically symmetric, source free, solutions, one obtains the content of Birkhoff’s theorem – the unique solution is the Schwarzschild geometry with some mass, $M$. Since this must hold for the regions interior and exterior to the shell (with a different mass $M$ for each), and since $M$ must be time independent, we see that only those shell trajectories which are “energy conserving” are compatible with the constraints. This feature makes the transition to the quantum theory rather difficult, as one desires an expression for the action valid for an arbitrary shell trajectory. This defect is remedied in the Hamiltonian formulation, where the constraints are expressed in terms of momenta rather than time derivatives,

$$C_H [\dot{r}, p; g_{ij}, \pi_{ij}] = 0.$$
At each time, the unique solution is again some slice of the Schwarzschild geometry, but the constraints no longer prevent $M$ from being time dependent. Thus, an arbitrary shell trajectory $\hat{r}(t), p(t)$, is perfectly consistent with the Hamiltonian form of the constraints, making quantization much more convenient.

The starting point for the Hamiltonian formulation of gravity is to write the metric in ADM form [6]:

$$ds^2 = -N^t(t,r)^2dt^2 + L(t,r)^2[dr + N^r(t,r)dt]^2 + R(t,r)^2[\theta^2 + \sin^2 \theta d\phi^2]$$  \hspace{1cm} (3.1)

In considering the above form, we have restricted ourselves to spherically symmetric geometries at the outset. With this choice of variables, the action for the shell is written

$$S^s = -m \int \sqrt{-\hat{g}_{\mu\nu}d\hat{x}^\mu d\hat{x}^\nu} = -m \int dt \sqrt{\hat{N}t^2 - \hat{L}^2 (\dot{\hat{r}} + \hat{N}^r)^2} ,$$  \hspace{1cm} (3.2)

$m$ representing the rest mass of the shell, and the carets instructing one to evaluate quantities at the shell ($\hat{g}_{\mu\nu} = g_{\mu\nu}(\hat{t}, \hat{r})$).

The action for the gravity-shell system is then

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g}R - m \int dt \sqrt{(\hat{N}t)^2 - \hat{L}^2 (\dot{\hat{r}} + \hat{N}^r)^2} + \text{boundary terms}$$  \hspace{1cm} (3.3)

and can be written in canonical form as

$$S = \int dt p \dot{r} + \int dt dr \left[ \pi_R \dot{R} + \pi_L \dot{L} - N^t(\mathcal{H}_t^s + \mathcal{H}_t^G) - N^r(\mathcal{H}_r^s + \mathcal{H}_r^G) \right] - \int dt M_{\text{ADM}}$$  \hspace{1cm} (3.4)

with

$$\mathcal{H}_t^s = \sqrt{(p/\hat{L})^2 + m^2} \delta(r - \hat{r}) \hspace{1cm} \mathcal{H}_t^s = -p \delta(r - \hat{r})$$  \hspace{1cm} (3.5)

$$\mathcal{H}_r^G = \frac{L\pi_L^2}{2R^2} - \frac{\pi_L \pi_R}{R} + \left( \frac{R' \pi_L}{L} \right)' - \frac{R^2}{2L} - \frac{L}{2} \hspace{1cm} \mathcal{H}_r^G = R' \pi_R - L \pi_L'$$  \hspace{1cm} (3.6)

where $'$ represents $\frac{d}{dr}$. $M_{\text{ADM}}$ is the ADM mass of the system. The inclusion of this last term deserves some comment. Because the Einstein-Hilbert action contains second derivatives, a general variation of the metric variables gives a nonzero result even when the equations of motion are satisfied and the variation of the metric is zero on the boundary of the space. If we restrict the class of metrics to those which are asymptotically flat, with $N^t \rightarrow 1$ and $N^r \rightarrow 0$ as $r \rightarrow \infty$, then the variation of the term $\int dt M_{\text{ADM}}$ precisely cancels the unwanted terms, and so gives a well defined variational principle [7]. It is important to note that $M_{\text{ADM}}$ is a function of the metric variables (whose explicit form will be displayed in due course) and is numerically equal to the total mass of the combined gravity-shell system.
We now wish to eliminate the gravitational degrees of freedom in order to obtain an effective action which depends only on the shell variables. To accomplish this, we first identify the constraints which are obtained by varying with respect to $N^t$ and $N^r$:

$$H_t = H_t^e + H_t^G = 0; \quad H_r = H_r^e + H_r^G = 0. \quad (3.7)$$

By solving these constraints, and inserting the solutions back into (3.4) we can eliminate the dependence on $\pi_R$ and $\pi_L$. We first consider the linear combination of constraints

$$0 = \frac{R'}{L}H_t + \frac{\pi_L}{RL}H_r = -\mathcal{M}' + \frac{\hat{R}'}{L}H_t^e + \frac{\hat{\pi}_L}{RL}H_r^e \quad (3.8)$$

where

$$\mathcal{M} = \frac{\pi_L^2}{2R} + \frac{R}{2} - \frac{RR'^2}{2L^2}. \quad (3.9)$$

Away from the shell the solution of this constraint is simply $\mathcal{M} = \text{constant}$. By considering a static slice ($\pi_L = \pi_R = 0$), we see that the solution is a static slice of the Schwarzschild geometry with $\mathcal{M}$ the corresponding mass parameter. The presence of the shell causes $\mathcal{M}$ to be discontinuous at $\hat{r}$, so we write

$$\mathcal{M} = M \quad r < \hat{r}$$

$$\mathcal{M} = M_+ \quad r > \hat{r}. \quad (3.10)$$

As there is no matter outside the shell we also have $M_{\text{ADM}} = M_+$. Then, using (3.8) and (3.9) we can solve the constraints to find $\pi_L$ and $\pi_R$:

$$\pi_L = R\sqrt{(R'/L)^2 - 1 + 2M/R} \quad ; \quad \pi_R = \frac{L}{R'}\pi_L \quad r < \hat{r}$$

$$\pi_L = R\sqrt{(R'/L)^2 - 1 + 2M_+/R} \quad ; \quad \pi_R = \frac{L}{R'}\pi_L \quad r > \hat{r}. \quad (3.11)$$

The relation between $M_+$ and $M$ is found by solving the constraints at the position of the shell. This is done most easily by choosing coordinates such that $L$ and $R$ are continuous as one crosses the shell, and $\pi_{R,L}$ are free of singularities there. Then, integration of the constraints across the shell yields

$$\pi_L(\hat{r} + \epsilon) - \pi_L(\hat{r} - \epsilon) = -p/\hat{L}$$

$$R'(\hat{r} + \epsilon) - R'(\hat{r} - \epsilon) = -\frac{1}{R}\sqrt{p^2 + m^2\hat{L}^2} \quad (3.12)$$

Now, when the constraints are satisfied a variation of the action takes the form

$$dS = p\,d\hat{r} + \int dr(\pi_R\delta R + \pi_L\delta L) - M_+ \, dt \quad (3.13)$$
where \( \pi_{R,L} \) are now understood to be given by (3.11), and \( M_+ \) is determined by solving (3.12). We wish to integrate the expression (3.13) to find the action for an arbitrary shell trajectory. As discussed above, the geometry inside the shell is taken to be fixed (namely, \( M \) is held constant) while the geometry outside the shell will vary in order to satisfy the constraints. It is easiest to integrate the action by initially varying the geometry away from the shell. We first consider starting from an arbitrary geometry and varying \( L \) until \( \pi_R = \pi_L = 0 \), while holding \( \hat{r}, p, R, \hat{L} \) fixed:

\[
\int dS = \int_{r_{\text{min}}}^{\infty} dr \int_0^{L} \delta L \pi_L
\]

\[
= \int_{r_{\text{min}}}^{\hat{r}+\epsilon} dr \int_0^{L} \delta L R \sqrt{(R'/L)^2 - 1 + 2M/R} + \int_{\hat{r}-\epsilon}^{\infty} dr \int_0^{L} \delta L R \sqrt{(R'/L)^2 - 1 + 2M/R} \]

\[
= \int_{r_{\text{min}}}^{\hat{r}-\epsilon} dr \left[ RL \sqrt{(R'/L)^2 - 1 + 2M/R} + RR' \log \left| \frac{R'/L - \sqrt{(R'/L)^2 - 1 + 2M/R}}{\sqrt{1 - 2M/R}} \right| \right]
\]

\[
+ \int_{\hat{r}+\epsilon}^{\infty} dr \left[ RL \sqrt{(R'/L)^2 - 1 + 2M/R} + RR' \log \left| \frac{R'/L - \sqrt{(R'/L)^2 - 1 + 2M/R}}{\sqrt{1 - 2M/R}} \right| \right]
\]  

(3.14)

where the lower limit of integration, \( r_{\text{min}} \), properly extends to the collapsing matter forming the black hole; its precise value will not be important. We have discarded the constant arising from the lower limit of the \( L \) integration. In the next stage we can vary \( L \) and \( R \), while keeping \( \pi_{R,L} = 0 \), to some set geometry. Since the momenta vanish, there is no contribution to the action from this variation.

It remains to consider nonzero variations at the shell. If an arbitrary variation of \( L \) and \( R \) is inserted into the final expression of (3.14) one finds

\[
dS = \int_{r_{\text{min}}}^{\infty} dr \left[ \pi_R \delta R + \pi_L \delta L \right] - \left[ \frac{\partial S}{\partial \hat{R}} (\hat{r} + \epsilon) - \frac{\partial S}{\partial \hat{R}} (\hat{r} - \epsilon) \right] d\hat{R} + \frac{\partial S}{\partial M_+} dM_+. \tag{3.15}
\]

Since \( R' \) is discontinuous at the shell,

\[
\frac{\partial S}{\partial \hat{R}} (\hat{r} + \epsilon) - \frac{\partial S}{\partial \hat{R}} (\hat{r} - \epsilon)
\]

is nonvanishing and needs to be subtracted in order that the relations

\[
\frac{\delta S}{\delta R} = \pi_R \quad ; \quad \frac{\delta S}{\delta L} = \pi_L
\]

will hold. From (3.14), the term to be subtracted is
Similarly, arbitrary variations of $L$ and $R$ induce a variation of $M_+$ causing the appearance of the final term in (3.13). Thus we need to subtract

\[
\frac{\partial S}{\partial M_+} dM_+ = - \int_{\hat{r}+\epsilon}^{\hat{r}} dr L \frac{\sqrt{(R'/L)^2 - 1 + 2M_+/R}}{1 - 2M_+/R} dM_+. \tag{3.17}
\]

Finally, we consider variations in $p, \hat{r},$ and $t$. $t$ variations simply give $dS = -M_+ dt$. We do not need to separately consider variations of $p$ and $\hat{r}$, since when the constraints are satisfied their variations are already accounted for in our expression for $S$, as will be shown.

Collecting all of these terms, our final expression for the action reads

\[
S = \int_{r_{\text{min}}}^{\hat{r}-\epsilon} dr \left[ RL \sqrt{(R'/L)^2 - 1 + 2M/R} + \log \frac{R'/L - \sqrt{(R'/L)^2 - 1 + 2M/R}}{\sqrt{1 - 2M/R}} \right] \\
+ \int_{\hat{r}+\epsilon}^{\infty} dr \left[ RL \sqrt{(R'/L)^2 - 1 + 2M_+/R} + RR' \log \frac{R'/L - \sqrt{(R'/L)^2 - 1 + 2M_+/R}}{\sqrt{1 - 2M_+/R}} \right] \\
- \int dt \hat{R} \left[ \log \frac{R'(\hat{r} - \epsilon)/L - \sqrt{(R'(\hat{r} - \epsilon)/L)^2 - 1 + 2M/R}}{\sqrt{1 - 2M/R}} \right] \\
+ \log \frac{R'(\hat{r} + \epsilon)/L - \sqrt{(R'(\hat{r} + \epsilon)/L)^2 - 1 + 2M/R}}{\sqrt{1 - 2M_+/R}} \right] \\
+ \int dt \int_{\hat{r}+\epsilon}^{\infty} dr \frac{L \sqrt{(R'/L)^2 - 1 + 2M_+/R}}{1 - 2M_+/R} \dot{M}_+ - \int dt \dot{M}_+. \tag{3.18}
\]

To show that this is the correct expression we can differentiate it; then it can be seen explicitly that when the constraints are satisfied (3.13) holds.
We now wish to write the action in a more conventional form as the time integral of a Lagrangian. As it stands, the action in (3.18) is given for an arbitrary choice of \( L \) and \( R \) consistent with the constraints. There is, of course, an enormous amount of redundant information contained in this description, since many \( L \)'s and \( R \)'s are equivalent to each other through a change of coordinates. To obtain an action which only depends on the truly physical variables \( p, \hat{r} \) we make a specific choice for \( L \) and \( R \), ie. choose a gauge. In so doing, we must respect the condition

\[
R'(\hat{r} + \epsilon) - R'(\hat{r} - \epsilon) = -\frac{1}{\hat{R}}\sqrt{p^2 + m^2\hat{L}^2}
\]

which constrains the form of \( R' \) arbitrarily near the shell. Suppose we choose \( R \) for all \( r > \hat{r} \); then \( R'(\hat{r} - \epsilon) \) is fixed by the constraint, but we can still choose \( R \) for \( r < \hat{r} - \epsilon \), in other words, away from the shell. We will let \( R'^{\prime}_< \) denote the value of \( R' \) close to the shell but far enough away such that \( R \) is still freely specifiable. We employ the analogous definition for \( R'^{\prime}_> \), except in this case we are free to choose \( R'^{\prime}_> = R'(\hat{r} + \epsilon) \).

In terms of this notation the time derivative of \( S \) is

\[
L = \frac{dS}{dt} = \hat{r}\hat{R}\hat{L} \left[ \sqrt{(R'^{\prime}_</\hat{L})^2 - 1 + 2M/\hat{R}} - \sqrt{(R'^{\prime}_>/\hat{L})^2 - 1 + 2M+/\hat{R}} \right]
\]

\[
-\hat{R}\hat{R}\log \left| \frac{R'^{\prime}_-/\hat{L} - \sqrt{(R'^{\prime}_-/\hat{L})^2 - 1 + 2M/\hat{R}}}{R'^{\prime}_>/\hat{L} - \sqrt{(R'^{\prime}_>/\hat{L})^2 - 1 + 2M+/\hat{R}}} \right|
\]

\[
+ \int_{r_{\text{min}}}^{\hat{r} - \epsilon} dr [\pi_R \hat{R} + \pi_L \hat{L}] + \int_{\hat{r} + \epsilon}^{\infty} dr [\pi_R \hat{R} + \pi_L \hat{L}] - M_+.
\]

At this point we will, for simplicity, specialize to a massless particle \((m = 0)\) and define \( \eta = \pm = \text{sgn}(p) \). Then the constraints (3.12) read

\[
R'^{\prime}_>(\hat{r} - \epsilon) = R'^{\prime}_>(\hat{r} + \epsilon) + \frac{\eta p}{\hat{R}}
\]

\[
\sqrt{(R'^{\prime}_>(\hat{r} - \epsilon)/\hat{L})^2 - 1 + 2M+/\hat{R}} = \sqrt{(R'^{\prime}_>(\hat{r} + \epsilon)/\hat{L})^2 - 1 + 2M+/\hat{R}} + \frac{p}{L\hat{R}}.
\]

These relations can be inserted into (3.19) to yield

\[
L = \hat{r}\hat{R}\hat{L} \left[ \sqrt{(R'^{\prime}_</\hat{L})^2 - 1 + 2M/\hat{R}} - \sqrt{(R'^{\prime}_>/\hat{L})^2 - 1 + 2M+/\hat{R}} \right]
\]

\[
-\eta\hat{R}\hat{R}\log \left| \frac{R'^{\prime}_>/\hat{L} - \eta\sqrt{(R'^{\prime}_>/\hat{L})^2 - 1 + 2M+/\hat{R}}}{R'^{\prime}_>/\hat{L} - \eta\sqrt{(R'^{\prime}_>/\hat{L})^2 - 1 + 2M+/\hat{R}}} \right|
\]

\[
+ \int_{r_{\text{min}}}^{\hat{r} - \epsilon} dr [\pi_R \hat{R} + \pi_L \hat{L}] + \int_{\hat{r} + \epsilon}^{\infty} dr [\pi_R \hat{R} + \pi_L \hat{L}] - M_+.
\]
Now we can use the freedom to choose a gauge to make (3.21) appear as simple as possible. It is clearly advantageous to choose $L$ and $R$ to be time independent, so $\pi_R \dot{R} + \pi_L \dot{L} = 0$. Also, having $R' = L$ simplifies the expressions further. Finally, it is crucial that the metric be free of coordinate singularities. A gauge which conveniently accommodates these features is

$$L = 1 \quad ; \quad R = r$$

The Schwarzschild geometry in this gauge is reviewed in the Appendix. It is considered in more depth in [8].

The $L = 1, R = r$ gauge reduces the Lagrangian to

$$L = \dot{\hat{r}}\left[\sqrt{2M\hat{r}} - \sqrt{2M_+\hat{r}}\right] - \eta \dot{\hat{r}} \log \left|\frac{\sqrt{\hat{r}} - \eta \sqrt{M_+}}{\sqrt{\hat{r}} - \eta \sqrt{2M}}\right| - M_+ \quad (3.22)$$

where $M_+$ is now found from the constraints (3.20) to be related to $p$ by

$$p = \frac{M_+ - M}{\eta - \sqrt{2M_+/r}} \quad (3.23)$$

The canonical momentum conjugate to $\dot{\hat{r}}$ obtained from (3.22) is

$$p_c = \frac{\partial L}{\partial \dot{\hat{r}}} = \sqrt{2M\hat{r}} - \sqrt{2M_+\hat{r}} - \eta \dot{\hat{r}} \log \left|\frac{\sqrt{\hat{r}} - \eta \sqrt{2M_+}}{\sqrt{\hat{r}} - \eta \sqrt{2M}}\right| \quad (3.24)$$

in terms of which we write the action in canonical form as

$$S = \int dt[p_c \dot{\hat{r}} - M_+] \quad (3.25)$$

which identifies $M_+$ as the Hamiltonian. We should point out that $M_+$ is the Hamiltonian only for a restricted set of gauges. If we look back at (3.21) we see that the terms $\pi_R \dot{R} + \pi_L \dot{L}$ will in general contribute to the Hamiltonian.

4. Quantization

In this section we discuss the quantization of the effective action (3.25). First, it is convenient to rewrite the action in a form which explicitly separates out the contribution from the particle. We write

$$M_+ = M - p_t$$
so

\[ S = \int dt [p_c \dot{r} + p_t] \quad (4.1) \]

and the same substitution is understood to be made in (3.24). We have omitted a term, \( \int dt M \), which simply contributes an overall constant to our formulas. In order to place our results in perspective, it is useful to step back and consider the analogous expressions in flat space. Our results are an extension of

\[ p = \pm \sqrt{p_t^2 - m^2} \quad (4.2) \]

\[ S = \int dt [p \dot{r} + p_t] \quad (4.3) \]

Indeed, the \( G \to 0 \) limit of (3.24), (3.25) yields precisely these expressions (with \( m = 0 \)). To quantize, one is tempted to insert the substitutions \( p \to -i \frac{\partial}{\partial r} \), \( p_t \to -i \frac{\partial}{\partial t} \) into (4.2), so as to satisfy the canonical commutation relations. This results in a rather unwieldy, nonlocal differential equation. In this trivial case we know, of course, that the correct description of the particle is obtained by demanding locality and squaring both sides of (4.2) before substituting \( p \) and \( p_t \). So for this example it is straightforward to move from the point particle description to the field theory description, i.e. the Klein-Gordon equation. Now, returning to (3.24) we are again met with the question of how to implement the substitutions \( p \to -i \frac{\partial}{\partial r} \). In this case the difficulty is more severe; we no longer have locality as a guiding criterion instructing us how to manipulate (3.24) before turning the \( p \)'s into differential operators. This is because we expect the effective action (4.1) to be nonlocal on physical grounds, as it was obtained by including the gravitational field of the shell.

There is, however, a class of solutions to the field equations for which this ambiguity is irrelevant to leading order, and which is sufficient to determine the late-time radiation from a black hole. These are the short-wavelength solutions, which are accurately described by the geometrical optics, or WKB, approximation. Writing these solutions as

\[ \phi(t, r) = e^{iS(t,r)} \]

the condition determining the validity of the WKB approximation is that

\[ |\partial S| \gg |\partial^2 S|^{1/2}, |\partial^3 S|^{1/3} \ldots \]

and that the geometry is slowly varying compared to \( S \). In this regime, derivatives acting on \( \phi(t, r) \) simply bring down powers of \( \partial S \), so we can make the replacements

\[ p_c \to \frac{\partial S}{\partial r}; \quad p_t \to \frac{\partial S}{\partial t} \]
and obtain a Hamilton-Jacobi equation for $S$. Furthermore, it is well known that the solution of the Hamilton-Jacobi equation is just the classical action. So, if $\hat{r}(t)$ is a solution of the equations of motion found by extremizing (4.1), then

$$S(t, \hat{r}(t)) = S(0, \hat{r}(0)) + \int_0^t dt \left[ p_c(\hat{r}(t)) \dot{\hat{r}}(t) + p_t \right]$$  \hspace{1cm} (4.4)

where

$$p_c(0, \hat{r}) = \frac{\partial S}{\partial \hat{r}}(0, \hat{r}).$$  \hspace{1cm} (4.5)

Since the Lagrangian in (4.1) has no explicit time dependence, the Hamiltonian $p_t$ is conserved. Using this fact, it is easy to verify that the trajectories, $\hat{r}(t)$, which extremize (4.1) are simply the null geodesics of the metric

$$ds^2 = -dt^2 + \left(dr + \sqrt{\frac{2M_+}{r}} dt\right)^2.$$  \hspace{1cm} (4.6)

From (4.4) the geodesics are:

ingoing: $t + \hat{r}(0) + 2\sqrt{2M_+ \hat{r}(0)} + 4M_+ \log \left[\sqrt{\hat{r}(0) + \sqrt{2M_+}}\right]$

$$= \hat{r}(0) + 2\sqrt{2M_+ \hat{r}(0)} + 4M_+ \log \left[\sqrt{\hat{r}(0) + \sqrt{2M_+}}\right]$$

outgoing: $t - \hat{r}(0) - 2\sqrt{2M_+ \hat{r}(0)} - 4M_+ \log \left[\sqrt{\hat{r}(0) - \sqrt{2M_+}}\right]$

$$= -\hat{r}(0) - 2\sqrt{2M_+ \hat{r}(0)} - 4M_+ \log \left[\sqrt{\hat{r}(0) - \sqrt{2M_+}}\right].$$  \hspace{1cm} (4.7)

$M_+$, in turn, is determined by the initial condition $S(0, \hat{r})$ according to (3.24) and (4.5):

ingoing: $$\frac{\partial S}{\partial \hat{r}}(0, \hat{r}(0)) = \sqrt{2M\hat{r}(0)} - \sqrt{2M_+ \hat{r}(0)} + \hat{r}(0) \log \frac{\sqrt{\hat{r}(0) + \sqrt{2M_+}}}{\sqrt{\hat{r}(0) + \sqrt{2M}}}$$

outgoing: $$\frac{\partial S}{\partial \hat{r}}(0, \hat{r}(0)) = \sqrt{2M\hat{r}(0)} - \sqrt{2M_+ \hat{r}(0)} - \hat{r}(0) \log \frac{\sqrt{\hat{r}(0) - \sqrt{2M_+}}}{\sqrt{\hat{r}(0) - \sqrt{2M}}}.$$  \hspace{1cm} (4.8)

Finally, we can use this value of $M_+$ to determine $p_c(t)$:

ingoing: $$p_c(t) = \sqrt{2M\hat{r}(t)} - \sqrt{2M_+ \hat{r}(t)} + \hat{r}(t) \log \frac{\sqrt{\hat{r}(t) + \sqrt{2M_+}}}{\sqrt{\hat{r}(t) + \sqrt{2M}}}$$

outgoing: $$p_c(t) = \sqrt{2M\hat{r}(t)} - \sqrt{2M_+ \hat{r}(t)} - \hat{r}(t) \log \frac{\sqrt{\hat{r}(t) - \sqrt{2M_+}}}{\sqrt{\hat{r}(t) - \sqrt{2M}}}.$$  \hspace{1cm} (4.9)
These formulas are sufficient to compute \( S(t, r) \) given \( S(0, r) \).

As will be discussed in the next section, the relevant solutions needed to describe the state of the field following black hole formation are those with the initial condition

\[
S(0, r) = kr \quad k > 0
\]  

(4.10)
near the horizon. Here, \( k \) must be large (\( \gg 1/M \)) if the solution is to be accurately described by the WKB approximation. In fact, the relevant \( k \)'s needed to calculate the radiation from the hole at late times become arbitrarily large, due to the ever increasing redshift experienced by the emitted quanta as they escape to infinity. We also show in the next section that to compute the emission probability of a quantum of frequency \( \omega \), we are required to find the solution for all times in the region between \( r = 2M \) and \( r = 2(M + \omega) \). That said, we turn to the calculation of \( S(t, r) \) in this region, and with the initial condition (4.10). The solutions are determined from (4.4), (4.7)-(4.9). Because of the large redshift, we only need to keep those terms in these relations which become singular near the horizon. We then have for the outgoing solutions:

\[
S(t, r) = k \hat{r}(0) - \int_{\hat{r}(0)}^{r} \hat{r} \log \left[ \frac{\sqrt{\hat{r}} - \sqrt{2M_+}}{\sqrt{\hat{r}} - \sqrt{2M}} \right] - (M_+ - M)t \quad \text{(4.11)}
\]

\[
t - 4M_+ \log [\sqrt{\hat{r}} - \sqrt{2M_+}] = -4M_+ \log [\sqrt{\hat{r}(0)} - \sqrt{2M_+}] \quad \text{(4.12)}
\]

\[
k = -\hat{r}(0) \log \left[ \frac{\sqrt{\hat{r}(0)} - \sqrt{2M_+}}{\sqrt{\hat{r}(0)} - \sqrt{2M}} \right] \quad \text{(4.13)}
\]

To complete the calculation, we need to invert (4.12) and (4.13) to find \( M_+ \) and \( \hat{r}(0) \) in terms of \( t \) and \( r \), and then insert these expressions into (4.11). One finds that to next to leading order,

\[
\sqrt{2M_+} = \sqrt{2M} + (\sqrt{r} - \sqrt{2M}) \frac{(e^{k/2M'} - 1)}{1 + (e^{k/2M'} - 1)e^{-t/4M'}} \quad \text{(4.14)}
\]

\[
\sqrt{\hat{r}(0)} = \sqrt{2M} + (\sqrt{r} - \sqrt{2M}) \frac{e^{(k/2M' - t/4M')}}{1 + (e^{k/2M'} - 1)e^{-t/4M'}}
\]

where

\[
M' = M + \sqrt{2M} (\sqrt{r} - \sqrt{2M}) \frac{e^{(k/2M' - t/4M')}}{1 + e^{(k/2M' - t/4M')}} \quad \text{(4.15)}
\]

Plugging these relations into (4.11) and keeping only those terms which contribute to the late-time radiation, one finds after some tedious algebra,

\[
S(t, r) = -(2M^2 - r^2/2) \log \left[ 1 + e^{(k/2M' - t/4M')} \right]. \quad \text{(4.16)}
\]
5. Results

We will now discuss the application of these results to the problem of black hole radiance. We begin by recalling some general features of the quantization of a scalar field in the presence of a black hole [9]. The quantization proceeds by expanding the field operator in a complete set of solutions to the wave equation,

\[ \hat{\phi}(t, r) = \int dk \left[ \hat{a}_k f_k(t, r) + \hat{a}^+_k f^*_k(t, r) \right]. \quad (5.1) \]

There are, however, two inequivalent sets of modes which need to be considered: those which are natural from the standpoint of an observer making measurements far from the black hole, and those which are natural from the standpoint of an observer freely falling through the horizon subsequent to the collapse of the infalling matter. The appropriate modes for the observer at infinity are those which are positive frequency with respect to the Killing time, \( t \). Writing these modes as

\[ u_k(r) e^{-i\omega_k t}, \]

\[ \hat{\phi}(t, r) \] reads

\[ \hat{\phi}(t, r) = \int dk \left[ \hat{a}_k u_k(r) e^{-i\omega_k t} + \hat{a}^+_k u^*_k(r) e^{i\omega_k t} \right]. \quad (5.2) \]

These modes are singular at the horizon,

\[ \frac{du_k}{dr} \to \infty \quad \text{as} \quad r \to 2M. \]

Symptomatic of this is that the freely falling observer would measure an infinite energy-momentum density in the corresponding vacuum state,

\[ \langle 0| T_{\mu\nu}| 0 \rangle \to \infty \quad \text{as} \quad r \to 2M \]

where \( \hat{a}_k|0_t\rangle = 0 \). However, we do not expect this to be the state resulting from collapse, since the freely falling observer is not expected to encounter any pathologies in crossing the horizon, where the local geometry is entirely nonsingular for a large black hole. To describe the state resulting from collapse, it is more appropriate to use modes which extend smoothly through the horizon, and which are positive frequency with respect to the freely falling observer. Denoting a complete set of such modes by \( v_k(t, r) \), we write

\[ \hat{\phi}(t, r) \int dk \left[ \hat{b}_k v_k(t, r) + \hat{b}^+_k v^*_k(t, r) \right]. \quad (5.3) \]

Then, the state determined by

\[ \hat{b}_k|0_v\rangle = 0 \]
results in a non-singular energy-momentum density at the horizon, and so is a viable candidate. The operators $\hat{a}_k$ and $\hat{b}_k$ are related by the Bogoliubov coefficients,

$$\hat{a}_k = \int dk' \left[ \alpha_{kk'} \hat{b}_{k'} + \beta_{kk'} \hat{b}_{k'}^* \right]$$

(5.4)

where

$$\alpha_{kk'} = \frac{1}{2\pi u_k(r)} \int_{-\infty}^{\infty} dt e^{i\omega_k t} v_{kk'}(t, r)$$

$$\beta_{kk'} = \frac{1}{2\pi u_k(r)} \int_{-\infty}^{\infty} dt e^{i\omega_k t} v_{kk'}^*(t, r).$$

(5.5)

The average number of particles in the mode $u_k(r)e^{-i\omega_k t}$ is

$$N_k = \langle 0_v | \hat{a}_{k}^\dagger \hat{a}_k | 0_v \rangle = \int dk' |\beta_{kk'}|^2.$$  

(5.6)

If we treat the black hole as radiating for an infinite amount of time, then $N_k$ will be infinite. To obtain the rate of emission we can place the hole in a large box and use the density of states $d\omega/2\pi$ for outgoing particles. Further, if $|\alpha_{kk'}/\beta_{kk'}|$ is independent of $k'$, as will be shown to be the case, we can use the completeness relation

$$\int dk' \left[ |\alpha_{kk'}|^2 - |\beta_{kk'}|^2 \right] = 1$$

(5.7)

to obtain for the flux of outgoing particles with frequencies between $\omega_k$ and $\omega_k + d\omega_k$,

$$F(\omega_k) = \frac{d\omega_k}{2\pi} \frac{1}{|\alpha_{kk'}/\beta_{kk'}|^2 - 1}.$$  

(5.8)

This gives the flux of outgoing particles near the horizon. As the particles travel outwards, some fraction, $1 - \Gamma(\omega)$, of them will be reflected back into the hole by the spacetime curvature. Thus, for the flux seen at infinity we write

$$F_\infty(\omega_k) = \frac{d\omega_k}{2\pi} \frac{\Gamma(\omega_k)}{|\alpha_{kk'}/\beta_{kk'}|^2 - 1}.$$  

(5.9)

Next, we consider the issue of determining the modes $v_k(t, r)$. As stated above, we require these modes to be nonsingular at the horizon. Since the metric near the horizon is a smooth function of $t$ and $r$, a set of such modes can be defined by taking their behaviour on a constant time surface, say $t = 0$, to be

$$v_k(0, r) \approx e^{ikr} \quad \text{as} \quad r \to 2M.$$  

This is, of course, the initial condition given in (4.10). Now, the integrals in (5.5) determining the Bogoliubov coefficients depend on the values of $v_k(t, r)$ at constant $r$. Since
$v_k$ is evaluated in the WKB approximation, the highest accuracy will be obtained when $r$ is as close to the horizon as possible, since that is where $v_k$’s wavelength is short. On the other hand, in calculating the emission of a particle of energy $\omega_k$, we cannot take $r$ to be less than $2(M + \omega_k)$, since the solution $u_k(r)e^{-i\omega_k t}$ cannot be extended past that point. Therefore, we calculate the integrals with $r = 2(M + \omega_k)$.

The results of the previous section give us an explicit expression for $v_k$. From (4.16),

$$v_k(t, 2(M + \omega_k)) = e^{iS(t, 2(M + \omega_k))} = e^{i(4M\omega_k + 2\omega^2) \log[1 + e^{(k/2M' - t/4M')}]}$$

(5.10)

where $M'$ is

$$M' = M + \sqrt{2M(\sqrt{2(M + \omega_k)} - \sqrt{2M})} \frac{e^{(k/2M - t/4M')}}{1 + e^{(k/2M - t/4M')}} \approx M + \omega_k \frac{e^{(k/2M - t/4M')}}{1 + e^{(k/2M - t/4M')}}.$$ 

(5.11)

Then, the integrals are,

$$\int_{-\infty}^{\infty} dt e^{i\omega_k t} e^{\pm i(4M\omega_k + 2\omega^2) \log[1 + e^{(k/2M' - t/4M')}]} ,$$

(5.12)

the upper sign corresponding to $\alpha_{kk'}$, and the lower to $\beta_{kk'}$. We can compute the integrals using the saddle point approximation. It is readily seen that for the upper sign, the saddle point is reached when

$$e^{(k/2M' - t/4M')} \to \infty,$$

so $t$ is on the real axis. For the lower sign, the saddle point is

$$e^{(k/2M' - t/4M')} \approx -1/2,$$

which, to zeroth order in $\omega_k$, gives

$$t = 4i\pi M + \text{real}$$

and to first order in $\omega_k$, gives

$$t = 4i\pi(M - \omega_k) + \text{real}.$$ 

Inserting these values of the saddle point into the integrands gives for the Bogoliubov coefficients,

$$\left| \frac{\alpha_{kk'}}{\beta_{kk'}} \right| = e^{4\pi(M - \omega_k)\omega_k}.$$ 

(5.13)

The flux of radiation from the black hole is given by (5.9),

$$F_\infty(\omega_k) = \frac{d\omega_k}{2\pi} \frac{\Gamma(\omega_k)}{e^{8\pi(M - \omega_k)\omega_k} - 1}.$$ 

(5.14)
There is an alternative way of viewing the saddle point calculation, which provides additional insight into the physical origin of the radiation. Let us rewrite the integral (5.12) as
\[ \int_{-\infty}^{\infty} dt e^{i\omega_k t \pm iS(t,2(M+\omega_k))}. \] (5.15)
The saddle point is given by that value of \( t \) for which the derivative of the expression in the exponent vanishes:
\[ \omega_k \pm \frac{\partial S}{\partial t}(t,2(M+\omega_k)) = 0. \]
But \( \partial S/\partial t \) is just the negative of the Hamiltonian,
\[ \frac{\partial S}{\partial t} = p_t = M - M_+ \]
so the saddle point equation becomes
\[ M_+ = M \pm \omega_k. \]
To find the corresponding values of \( t \), we insert this relation into (4.12) and (4.13):
\[ t = 4(M \pm \omega_k) \log \left[ \frac{\sqrt{2(M + \omega_k) + \epsilon} - \sqrt{2(M + \omega_k)}}{\sqrt{\hat{r}(0)} - \sqrt{2(M \pm \omega_k)}} \right] \] (5.16)
\[ k = -\hat{r}(0) \log \left[ \frac{\sqrt{\hat{r}(0)} - \sqrt{2(M \pm \omega_k)}}{\sqrt{\hat{r}(0)} - \sqrt{2M}} \right], \] (5.17)
where we have written \( \hat{r} = 2(M + \omega_k) + \epsilon \) to make explicit that \( \hat{r} \) must lie outside the point where the solutions \( u_k(r) \) break down. We desire to solve for \( t \) as \( k \to \infty \). For the upper choice of sign, we find from (5.17) that
\[ \sqrt{\hat{r}(0)} = \sqrt{2(M + \omega_k)} + O(e^{-k/2M}), \]
which, from (5.16), then shows that the corresponding value of \( t \) is purely real.
For the lower choice of sign we have,
\[ \sqrt{\hat{r}(0)} = \sqrt{2(M + \omega_k)} - O(e^{-k/2M}). \]
Continuing \( t \) into the upper half plane, we find from (5.16) that
\[ t = 4i\pi(M - \omega_k) + \text{real}. \]
These results of course agree with our previous findings.
The preceding derivation invites us to interpret the radiation as being due to negative energy particles propagating in imaginary time. The particles originate from just inside the horizon, and cross to the outside in an imaginary time interval $4\pi(M - \omega_k)$. This, perhaps, helps clarify the analogy between black hole radiance and pair production in an electric field, which, in an instanton approach $[10]$, is also calculated by considering particle trajectories in imaginary time.

6. Comments

1. Let us return to the question of thermality. One might have guessed that the correct exponential suppression factor could be the Boltzmann factor for nominal temperature corresponding to the mass of the hole before the radiation, after the radiation, or somewhere in between. Thus one might have guessed that the exponential suppression of the radiance could take the form $e^{-\omega/T_\text{before}}$, $e^{-\omega/T_\text{after}}$, or something in between. Our result, to lowest order, corresponds to the nominal temperature for emission being equal to $T_\text{after}$.

2. Our methods clearly generalize in a completely straightforward manner to other forms of black holes (e.g., Reissner-Nordstrom or dilaton holes) and to charged or massive scalar fields. We have undertaken extensive calculations along these lines, whose results will be reported elsewhere. It should also be possible to consider emission into higher partial waves, though this involves some new issues in treating the collective modes for rotations of a non-symmetric hole, that we have not yet investigated seriously. Similarly, it would be interesting to consider emission with recoil.

3. One can also consider the case of two mutually gravitating particles. This involves geometries with two shell discontinuities. By analyzing this problem, we expect to be able to address the question whether there are non-trivial correlations in the radiation, or between incoming particles and subsequent radiation. There are however claims $[11, 12]$ that semiclassical methods become internally inconsistent, or at least highly singular, in the latter case.
Appendix: Schwarzschild Geometry with $L = 1, R = r$.

In this gauge the line element for the Schwarzschild geometry reads
\[
ds^2 = -dt^2 + (dr + \sqrt{\frac{2M}{r}} dt)^2 + r^2 d\theta^2 + \sin^2 \theta d\phi^2. \tag{A.1}\]

These coordinates are related to Schwarzschild coordinates,
\[
ds^2 = -(1 - \frac{2M}{r}) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \tag{A.2}\]

by a change of time slicing,
\[
t_s = t - 2\sqrt{2Mr} - 2M \log \left[\frac{\sqrt{r} - \sqrt{2M}}{\sqrt{r} + \sqrt{2M}}\right]. \tag{A.3}\]

In contrast to the surfaces of constant $t_s$, the constant $t$ surfaces pass smoothly through the horizon and extend to the future singularity free of coordinate singularities.

In terms of $r$ and $t$, the radially ingoing and outgoing null geodesics are given by
\[
\text{ingoing:} \quad t + r - 2\sqrt{2Mr} + 4M \log[\sqrt{r} + \sqrt{2M}] = v = \text{constant} \\
\text{outgoing:} \quad t - r - 2\sqrt{2Mr} - 4M \log[\sqrt{r} - \sqrt{2M}] = u = \text{constant}. \tag{A.4}\]
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