Snyder Like Modified Gravity in Newton’s Spacetime

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This work is focused on searching a geodesic interpretation of the dynamics of a particle under the effects of a Snyder like deformation in the background of the Kepler problem. In order to accomplish that task, a newtonian spacetime is used. Newtonian spacetime is not a metric manifold, but allows to introduce a torsion free connection in order to interpret the dynamic equations of the deformed Kepler problem as geodesics in a curved spacetime. These geodesics and the curvature terms of the Riemann and Ricci tensors show a mass and a fundamental length dependence as expected, but are velocity independent. In this sense, the effect of introducing a deformed algebra is examinated and the corresponding curvature terms calculated, as well as the modifications of the integrals of motion.

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I. INTRODUCTION

Nowadays noncommutative geometry is developing faster and faster as a candidate to be the arena of Physics at very high energies, where it is supposed quantum gravity should play a main role. Effectively, it is thought that the spacetime description itself must be modified at that scales. In that sense, main stream research on very high energy physics and on possible candidates for a suitable quantum gravity theory has lead to the idea of the existence of a fundamental length. Loop Quantum Gravity, String Theory and all their modifications and variations, propose the existence of a minimal fundamental length that is usually identified with the minimal size of the elements of these theories. The very existence

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of such fundamental length, the same for all observers, introduces a new observer invariant quantity that defies the Lorentz invariance based only on the invariance of speed of light. Hence, there are a number of effective theories like Double Special Relativity (DSR) \[1\] and Gravity’s Rainbow \[2\], where velocities or momenta enter in the connections and curvature terms.

A fundamental length, besides a fundamental speed, is usually introduced through modification of the Symplectic structure of the phase space, postulating a deformed algebra between momenta and position operators, as in the \( \kappa \)-deformed algebras.

There are different ways to introduce a minimal fundamental length, and many of them are incompatible with Lorentz symmetry, of course this is a very undesirable consequence.\[3\],[4],[5].

There is, however, a way to skip the Lorentz symmetry breaking that was proposed by H. Snyder, \[6\], who postulated a Lorentz invariant modification of the Heisenberg algebra that implies discrete spectra of the spacetime operators.

Snyder proposal researchers, being this an effective approach more than a fundamental theory, focus their efforts on finding ways to experimentally test the discreteness of spacetime, introducing the modified relations of the Snyder algebra in the quantum and in the classical realms. In the classical version of Snyder modifications there have been some efforts using planetary data and different approaches from Newtonian Mechanic and General Relativity dynamics, for instance Mignemi \[7\] for the last case and Romero \[8\] and Leiva \[9\] for the former.

The classic version of Snyder algebra is based on the noncanonical Poisson brackets:

\[
\{ \bar{x}_i, \bar{x}_j \} = i^2 L_{ij},
\]

\[
\{ \bar{x}_i, p_j \} = \delta_{ij} + l^2 p_ip_j,
\]

\[
\{ p_i, p_j \} = 0,
\]

where \( i, j = 1, 2, 3 \), and \( l \) is a parameter that measures the deformation introduced in the canonical Poisson brackets with dimensions of inverse of momentum, and \( L_{ij} \) is the angular momentum matrix.

Snyder space can be realized rather successfully on constant curvature momentum spaces (see e.g. \[10, 11\]), but it’s very difficult to find spacetime versions of the Snyder space due
the non trivial dependence of the dynamic on the momentum of the particle itself. Moreover, it's very debatable if it is possible to implement the Snyder relations in the realm of General Relativity, because the obtained dynamics depend in a non linear way on the mass, so the equivalence principle is broken and no frame it is possible to choose in order to cancel gravity effects, not even locally. Finally, among the difficulties that the models present it is remarkable the so called Soccer-Ball problem, consisting in the not clear treatment of multi-particles states that consists on the fact that the modified Lorentz transformations act on momenta in non linear way. (see e.g. \[12\]).

Taken all these difficulties into account, it is possible however, to think on implementing Snyder relations in Newton and in Minkowski spaces, the key is to find a good path to, at least, represent the dynamics of a particle in a right way. It is important to mention that even General Relativity seems to be the real and fundamental theory, the majority of phenomena in the universe can be adequately described by Newtonian gravity like the solar system, normal stars and galaxies. It would be then very interesting to find any clue of noncommutative Poisson brackets relations between variables in the dynamic of the (lets say), Newtonian Universe.

In this paper it is used a realization of the noncommutative Snyder relations between the space an momentum variables that allows to describe the dynamics of a particle in a Kepler central field and describe it as geodesics in a spacetime. It is a fact that newtonian spacetime is not a metric space, but it can be constructed as a manifold with a connection and consisting of copies of a spatial manifold, foliated by a parameter lineally depending on a universal time. In that way, it is possible to find geometrical quantities that represent correctly the dynamics of a "Snyder particle".

II. NEWTON'S SPACETIME

One of the main problem with Snyder and other similar proposals is that while it is common to find an interpretation of the phenomena as a curvature on the momenta space, it has not been possible to find a model of spacetime whit curvature that can explain the models. In order to attempt that, we are going to recall that it is possible to construct a Newtonian spacetime in order to include gravity as curvature. In this section the idea of a Newtonian Spacetime is constructed following the ideas presented in \[13\].
One can’t construct a metric in order to interpret the Newton gravity as curvature (see e.g. [14]). It is possible, however, to construct a quintuple \( \{ \mathcal{M}, \vartheta, A, \nabla, t \} \), where \( \{ \mathcal{M}, \vartheta, A \} \) that is a smooth manifold with \( \vartheta \) a standard topology and \( A \) an atlas. We can choose also the function \( t : \mathcal{M} \to \mathbb{R} \), and \( \nabla \) satisfying:

1. There is an absolute space, with \( dt \big|_p \neq 0 \), everywhere for all \( p \in \mathcal{M} \). The definition of this absolute space at time \( \tau \) is the set of points

\[
S_\tau = \{ p \in M/t(p) = \tau \}.
\]

In such way that \( \mathcal{M} = \dot{\bigcup} S_\tau \). Where \( \dot{\bigcup} \) is the disjoint union.

2. Absolute time flows uniformly, \( \nabla dt = 0 \), everywhere.

3. The connection \( \nabla \) is torsion free.

So, the smooth manifold \( \mathcal{M} \) is constructed from infinite copies of the \( S \) spatial set, each one labeled by the absolute time \( t \).

This is not a metric space, but it’s a manifold equipped with a connection \( \nabla \), usually called a Galilean Manifold. In such space it is totally possible to define a covariant derivative and Christoffel symbols, furthermore, it is possible to define the Riemann and the Ricci curvature tensors. It is not possible, however to define the Ricci scalar because one has not a metric to calculate a metric trace \( g^{\mu\nu} R_{\mu\nu} \).

Let’s write the motion equation for a probe particle in a gravity field due to another particle of mass \( M \), considering, as usual, that the movement is on a plane orbit with spatial coordinates \( \rho, \theta \) and temporal coordinate \( \tau \) that is a linear function of the time \( t \) that label the different copies of the spatial subspace and \( \dot{\tau} = 1 \).

\[
\ddot{\rho} - \rho \dot{\theta}^2 + \frac{GM}{\rho^2} \dot{\tau}^2 = 0, \quad (2)
\]

\[
\ddot{\theta} + \frac{2 \rho \dot{\theta}}{\rho} = 0, \quad (3)
\]

\[
\dot{\tau} = 0. \quad (4)
\]

Considering that these equations correspond to geodesics in the spacetime \( \mathcal{M} \), that are totally possible to define just with the connection, one can read the non zero Christoffel terms:
\[ \Gamma^\rho_{\theta\theta} = -\varphi; \quad \Gamma^\rho_{\theta\theta} = \frac{GM}{\rho^2}; \quad \Gamma^\rho_{\rho\theta} = \frac{1}{\rho}. \]

The non zero Riemann and Ricci curvature tensor components are:

\[ R^\rho_{\tau\rho\tau} = \frac{-2GM}{\rho^3}; \quad R^\theta_{\tau\theta\tau} = \frac{GM}{\rho^3}; \]

\[ R_{\tau\tau} = \frac{-GM}{\rho^3}. \]

It’s worth to note that the Ricci component \( R_{\tau\tau} \) coincides with the expression of the \(-\nabla^2 \phi\), where \( \phi \) is the Newtonian potential. So, the Poincaré equation for Newton gravitation law can be written as:

\[ R_{\tau\tau} = 4\pi G \sigma, \]

with \( \sigma \) the mass density generating the gravitational potential.

III. KEPLER POTENTIAL IN TERMS OF NONCOMMUTATIVE VARIABLES

After the proposal of Battisti and Meljanac [15], it is possible to find a a covariant realization of Snyder geometry [2]. Considering the noncommutative variables

\[ \tilde{x}_\mu = x_\mu \varphi_1(A) + l^2(xp)p_\mu \varphi_2(A), \]

(5)

where \( \mu, \nu = 0, 1, 2, 3 \), \( \varphi_1 \) and \( \varphi_2 \) are functions of the dimensionless quantity \( A = l^2 p^2 \). The function \( \varphi_2 \) depends on \( \varphi_1 \) by:

\[ \varphi_2 = \frac{1 + 2\dot{\varphi}_1 \varphi}{\varphi_1 - 2A \dot{\varphi}_1}. \]

(6)

It is possible to set two realizations for the noncommutative Snyder geometry. One from Snyder himself if one set \( \varphi_1 = 1 \), that implies that \( \varphi_2 = 1 \), and the second one from Maggiore [16, 17], where \( \varphi_1 = \sqrt{1 - sp^2} \) and \( \varphi_2 = 0 \).

The 3-D Kepler potential can be written in terms of spatial noncommutative vectorial variable \( \tilde{x} \):
\[ V = -\frac{MG}{\sqrt{x^2}} \]  \hspace{1cm} (7)

This can be implemented then, choosing either Snyder or Maggiore realization in terms of the commutative space variables \( x, p \) and at order \( l^2 \). In both cases one has:

\[ V(x) = -\frac{\kappa}{\sqrt{x^2 + 2l^2(xp)^2}}, \]  \hspace{1cm} (8)

Now, for calculations we need to choose coordinates. We are dealing with a central force even in this deformed version of the Kepler problem, so we can use a \( 3 - D \) spatial spherical coordinates \( (\rho, \theta, \phi) \) version. Doing so and identifying at this moment the momentum as \( m\dot{x} \), we obtain:

\[ x = \rho \dot{\rho}, \]  \hspace{1cm} (9)

\[ p = m(\dot{\rho} \dot{\rho} + \rho \dot{\theta} + \rho \dot{\phi} \sin(\theta) \dot{\phi}, \]  \hspace{1cm} (10)

and assuming \( l^2 m^2 \dot{\rho}^2 \ll 1 \), the potential for a probe particle can be written as

\[ V(\rho) = \frac{-k}{\rho} (1 - l^2 m^2 \dot{\rho}^2). \]  \hspace{1cm} (11)

With this potential we can construct, as usual, a reduced Lagrangian choosing \( \theta = \pi/2 \) that contains the modification due a Snyder like perturbation:

\[ L = \frac{1}{2} m[1 - \frac{2l^2 km}{\rho}] \dot{\rho}^2 + \frac{1}{2} m\rho^2 \dot{\phi}^2 + \frac{k}{\rho}. \]  \hspace{1cm} (12)

The Euler-Lagrange equations are:

\[ (m - \frac{2m^2 l^2}{\rho^2})\ddot{\rho} - m\rho \dot{\phi}^2 \dot{\phi}^2 + \frac{k}{\rho^2} - \rho^2 \rho^2 = 0, \]  \hspace{1cm} (13)

\[ \frac{d(m \rho^2 \dot{\phi})}{dt} = 0. \]  \hspace{1cm} (14)

One can discard the term proportional to \( l^2 \) in the acceleration in front of the mass. The presence of the mass of the particle in the equations is a characteristic feature of these models, furthermore the combination \( m^2 l^2 \) is usually present in Snyder models. On the other hand, in order to make a better comparison with section I, one can focus on a Newton’s
gravity problem and identify $k = GMm$, where $M$ is the mass of the particle that creates the gravitational attraction and $G$, the Newtonian constant of gravitation. Hence the equations are:

$$\ddot{\rho} - \rho \dot{\varphi}^2 + \frac{GM}{\rho^2} + \frac{GMm^2l^2}{\rho^2} \dot{\rho}^2 = 0,$$

(15)

$$\ddot{\varphi} + \frac{2\ddot{\rho} \dot{\varphi}}{\rho} = 0.$$

(16)

Clearly, after eq. 15 the perturbation breaks the equivalence principle, feature that is common to this kind of proposals. We cannot do General Relativity strictly speaking, but it is possible to attempt to find a spacetime that gives account of the dynamic and the Newtonian spacetime seems to be a suitable one.

IV. INTRODUCING SNYDER PERTURBATION IN THE NEWTON’S SPACETIME

Considering equations 15 and 16 and $\ddot{\tau} = 0$, one could interpret them as geodesics in the sense of Section II. This can be do because they effectively represent the free falling of the particle in a gravitational field (the perturbation depends on $GM$ and if $G = 0$ the perturbation disappears). Because of that, they can be read as the result of the equation $\nabla_v v = 0$, being $v$ the tangent vector to the worldline of the particle. One can then identify the non zero Christoffel connections:

$$\Gamma^\rho_{\tau\tau} = \frac{GM}{\rho^2}; \quad \Gamma^\rho_{\rho\rho} = \frac{GMm^2l^2}{\rho^2}; \quad \Gamma^\rho_{\varphi\varphi} = -\rho; \quad \Gamma^\rho_{\rho\varphi} = \frac{1}{\rho}.$$

The non zero Riemann curvature tensor components are:

$$R^\rho_{\tau\rho\tau} = \frac{-2GM}{\rho^3} + \frac{GMm^2l^2}{\rho^3}; \quad R^\rho_{\rho\varphi\varphi} = \frac{-GMm^2l^2}{\rho}.$$

$$R^\rho_{\rho\rho\rho} = \frac{GM}{\rho^3}; \quad R^\varphi_{\rho\varphi\rho} = \frac{GMm^2l^2}{\rho^3}.$$

And finally the non zero Ricci tensor components:

$$R_{\tau\tau} = \frac{-GM}{\rho^3} \left[1 - \frac{m^2l^2}{\rho}\right]; \quad R_{\varphi\varphi} = -\frac{GMm^2l^2}{\rho}; \quad R_{\rho\rho} = \frac{GMm^2l^2}{\rho^3}.$$
It’s possible to see now that there is curvature induced on the spatial part of the manifold $\mathcal{M}$ and, of course, one obtain the Newtonian gravity curvature interpretation in the limit $l = 0$. It is also important to see that the geodesics depend on the mass of the particle, this agrees totally with the idea that free falling is not longer mass independent in this kind of models. On the other hand, it is interesting that the metric is velocity independent, it is an interesting feature because this interpretation doesn’t inherit the momenta dependence from the momenta space geometry models (see e.g. [18]). However, the mass dependence seems to indicate that each particle sees a different spacetime and that is against the idea of a spacetime independent of a probe particle characteristics. But this is the characteristic of this kind of proposals, some kind of back reaction that affects the ambient spacetime of the particles in a strong nonlinear way. One can also recall the electron models in condensed matter, where there is a strong interaction between the particle characteristics and the surroundings.

The interpretation of $R_{\tau\tau}$ is now rather difficult; it contains the non perturbed part from the Newtonian gravity, but it isn’t sure that just this term gives account of a density interpretation as it was before due the existence of $R_{\rho\rho}$ and $R_{\varphi\varphi}$. Furthermore, one can think that $R_{\rho\rho}$ and $R_{\varphi\varphi}$ are related to some kind of density fluxes, so the interaction between the particle with a gravitating mass and the Snyder effect can be interpreted as the effect of a density-momentum quantity on this curved spacetime and construct a new quantity $T_{\mu\nu}$ such that:

$$R_{\mu\nu} = 4\pi GT_{\mu\nu}.$$ (17)

The properties of $T_{\mu\nu}$ should be examined in more detail elsewhere, but it should be done in a relativistic version of the model otherwise it has no sense. The curvature of the spatial part of $\mathcal{M}$ is given by $R_{\rho\rho}$ and $R_{\varphi\varphi}$. Anyway it has been found a curvature interpretation of the perturbative term of Newton’s gravity due to Snyder geometry in an suitable manifold that describes the particle dynamics.

V. INTEGRATION OF EQUATIONS

The dynamical equations are very non linear and are very difficult to solve, but one can find integrals of motion. First of all, due to the Lagrangian [12] is not time dependent the
energy is a constant and identifying the Hamiltonian as the energy operator, one has:

$$H = \frac{\Pi^2}{2m(1 - \frac{2km\ell^2}{\rho})} + \frac{\Pi^2}{2m\rho^2} = \text{Cte.}$$  \hspace{1cm} (18)

It’s worth to mention that one can have the exact Hamiltonian using the version of the potential in [8]. The second movement integral arises noting that because the Snyder perturbation is still a pure central force, the angular momentum is conserved as can be seen from [16]

$$m\rho^2\dot{\phi} = \Pi_{\phi} = \text{Cte.}$$  \hspace{1cm} (19)

There is another integral of motion that is very important in central force problems, the Laplace-Runge-Lenz $\mathbf{A}$ vector that fixes the major axis of the ellipse on the orbiting plane. For a force $f(\rho)$ acting in the radial direction it is possible to write:

$$\frac{d}{dt}(\mathbf{P} \times \mathbf{L}) = -mf(\rho)\rho^2 \frac{d\rho}{dt},$$  \hspace{1cm} (20)

where $\mathbf{P}$ is the linear momentum and $\mathbf{L}$ the angular momentum of the particle. Considering a general Kepler like force perturbed in the way proposed in this work, $f(\rho) = \frac{-k}{\rho^2} + \frac{m^2\ell^2k}{\rho^5}\rho^2$ it is possible to write:

$$\frac{d\mathbf{A}}{dt} = -km^2\ell^2 \frac{d\rho}{\rho^2} \dot{\phi} \hat{\phi}.$$  \hspace{1cm} (21)

For the integration of this equation it is possible to use the non perturbed solutions of the Kepler problem for $\rho$, $\dot{\rho}$ and $\dot{\phi}$, depending on the characteristics of the system, for example gravitation in the case of planet orbits or electrostatic in the case of semiclassical simple atomic problems and replacing $k$ in a suitable way. This could be the focus of a next research, specially in the atomic realm because, as have been stated (e. g. [7]), the Snyder effect is plausible to have consequences at a microscopic scale only.

### VI. FINAL REMARKS

The focus of this work was to find an interpretation of Snyder perturbation to the Kepler problem and specifically of Newton gravity in terms of curvature of a spacetime while the phase space has been examined in many other works. In this sense, a version of geodesics
interpretation and space curvature has been found, giving the possibility to extend the research to relativistic versions. The very existence of Riemann and Ricci tensors allows to visualize how the geometry of the space and the spacetime is affected due to the introduction of a independent length scale. In fact, Newtonian space isn’t a metric space itself, but it’s a starting point to research and the logical next step is to study the relativistic version and the conditions to connect with gravity. As was said before, it’s not clear how to do general relativity with the DSR theories because they brake the equivalence principle and this could be the aim of future researches. It’s important to state that the Snyder deformation is very interesting and even there are many works focused on the experimental effects, the properties of such modified spaces are very worthy to be examined in order to better understand it. The effects of this kind of perturbation are in fact compatible with dynamics just near the Planck scale, so it seems better to concentrate efforts in the geometrical properties that could shed some light on the basis of a new interpretation of the manifold that could be appropriate to represent spacetimes having a minimal fundamental length scale.

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