Abstract

With the development of fast routing algorithms for public transit the optimization of criteria other than arrival time has moved into the spotlight. Due to the often intricate nature of fare regulations, the price of a journey, while certainly being one of the most interesting criteria, is seldom considered. In this work, we present a novel framework to mathematically describe the fare systems of local public transit companies. The model allows us to solve the price-sensitive earliest arrival problem (PSEAP) even if prices depend on a number of parameters and non-linear conditions. We base our approach on conditional fare networks (CFN). A CFN consists of a ticket graph modeling tickets and the relations between them and a ticket transition function defined over a partially ordered monoid and a set of fare events. Typically, shortest path algorithms rely on the subpath optimality property which is usually lost when dealing with intricate fare systems. We show how subpath optimality can be restored by relaxing domination rules for tickets depending on the structure of the ticket graph. An exemplary model for the fare system of Mitteldeutsche Verkehrsbetriebe (MDV) is provided. PSEAP is NP-hard, even when considering a single-criterion version that optimizes for price only. Instances on finite monoids are, however, polynomial time solvable when the monoid is considered part of the encoding length. By integrating our framework in the multi-criteria RAPTOR algorithm, we provide an algorithm for PSEAP and assess its performance on data obtained from MDV. We introduce two simple speed-up techniques that in combination with the recently introduced Tight-BMRAPTOR pruning scheme allow us to solve PSEAP in mere milliseconds on our data set.

Index terms — shortest path, public transportation, optimization, raptor, fare

1 Introduction

Recent progress in the field of routing algorithms for public transportation has led to several very fast algorithms ([Bast et al. 2016]). Usually, these algorithms determine the best itinerary with respect to travel time in mere milliseconds. This led to the desire to optimize additional criteria such as the number of transfers or the reliability of the connection. For many users of public transportation systems the price of a journey is one of the most important criteria for assessing its quality. Unfortunately, public transportation fare systems are notoriously complex and therefore algorithmically hard to deal with. The ticket price can depend on a variety of variables such as the set of fare zones, the distance traveled, the number of stops visited, surcharges for night buses or ferries and so forth. To reduce this complexity, previous research has usually focused only on specific aspects such as zone- or distance-based prices and/or dealt with them heuristically. In this study, we present the conditional fare network (CFN), a novel and flexible framework to model the price structures of (regional) public transportation companies, that is able to take all of the aforementioned criteria into account. We use CFNs to formalize the price-sensitive earliest arrival problem (PSEAP).
When comparing journeys by price, the subpath optimality principle is usually lost. Consider the following example: A traveller takes a detour that passes through an additional fare zone to avoid paying the surcharge of a special connection (e.g. a ferry). It is not unlikely that, at a later point in the journey, the surcharge has to be paid regardless (e.g. because the target stop can only be reached via a ferry). In that case, taking the detour was a suboptimal decision and using the corresponding label to prune other partial journeys breaks the optimality guarantee of shortest path algorithms. In CFNs this problem is avoided by basing dominance relations between labels not on the price, but on paths in a directed ticket graph modeling the relations between tickets. Transitions between different tickets are modeled as directed arcs and usually depend on a number of criteria such as fare zones or the distance traveled. We model these criteria as positive partially ordered monoids. Using the modified dominance rules PSEAP can be solved by using typical multi-criteria shortest path algorithms such as the multi-criteria variants of RAPTOR (McRAP) or Dijkstra’s algorithm.

1.1 Related Literature

For an exhaustive overview of shortest path algorithms in road and public transportation networks please refer to Bast et al. (2016). It has previously been observed that shortest path problems can be generalized to ordered monoids (Zimmermann 1981) and semirings (Mohri 2002). In the study of public transit routing, prices are taken into account to varying degrees. Müller-Hannemann and Schnee (2005) study fare systems that entail distance- and relation-based prices, i.e., fare systems that at least in Germany are usually associated with long-distance public transportation. They approximate fares by assigning a fixed price to every edge. Their approach, however, does not take into account fares based on fare zones and short-distance city tickets. Both of these are usually more prominent in local public transportation. Schöbel and Urban (2021) gave polynomial algorithms for various simple fare systems. Delling, Pajor, and Werneck (2015) used their well-known RAPTOR (RAP) algorithm to compute journeys that touch the smallest number of fare zones. A variant of zone-based fare systems in the context of overflight costs was studied by Blanco et al. (2016) and Blanco et al. (2017). Recently, Gündling (2020) considered price-optimized routing in intermodal transportation. They, however, only considered mileage-based and flat fares. All these works consider quite specify fare systems and are usually not suited to model the complexities found in real world systems of public transportation. Recent work introduces a version of RAP, Tight-BMRAP, for computing restricted Pareto-sets (Delling, Dibbelt, and Pajor 2019). The idea of restoring subpath optimality by relaxing rules for label domination is discussed in a different context by Berger and Müller-Hannemann (2009). Our ticket graph approach bears similarity to the finite automata used to find language-constrained shortest paths (Barrett, Jacob, and Marathe 2000). It is different, however, in that it serves to evaluate paths instead of restricting the set of feasible paths. Furthermore, our approach also covers fares based on numerical attributes that are not expressed as part of a formal language. For an overview of automata theory, we refer to (Hopcroft and Ullman 1979). Schöbel and Urban (2021) showed that finding a cheapest path with prices depending on the number of fare zones visited is NP-hard. The proof relies on a reduction to the problem of finding a path with the minimum number of colored edges. The same proof was given earlier in an unpublished paper by Blanco et al. (2016). Since our approach generalizes this problem, the proof of NP-hardness extends to PSEAP. Nonetheless, we provide an additional proof of NP-completeness via a reduction of the shortest path with forbidden pairs problem (Gabow, Maheshwari, and Osterweil 1976) to PSEAP in Chapter 5.

This paper is an extended and improved version of work presented at the ATMOS’19 conference (Euler and Borndörfer 2019). We first presented the idea of using a ticket graph in (Borndörfer et al. 2018 German language).
1.2 Our Contribution

This paper makes the following contributions. Conditional fare networks (CFN) and the price-sensitive earliest arrival problem (PSEAP) over CFNs are introduced. The aspects of fare systems that can be modeled include (but are not limited to): zone-based fares, distance-based fares, surcharges for special vehicles or night liners, and discounted short-distance tickets that do not allow transfers. Our second contribution is a formal definition of label domination rules for tickets while retaining the subpath optimality property. We prove that using these rules in a multi-criteria shortest path algorithm does in fact yield lowest-price journeys. Furthermore, we both establish NP-hardness of PSEAP and show it to be solvable in polynomial time over finite monoids when considering the monoid part of the input length. Finally, we show that using our approach as a building block in a Tight-BMRAP algorithm allows us to solve PSEAP in less than ten milliseconds even over an intricate fare system for a mid sized public transit provider.

1.3 Overview

In Chapter 2, we introduce the fare system of MVD, a German public transit provider. This fare system will serve as a running example for the rest of the paper. We present conditional fare networks in detail in Chapter 3 and show how they can be used to model various aspects of fare systems. The algorithmic treatment of fares and domination rules is laid out in Chapter 4. In Chapter 5, we conduct a complexity analysis of PSEAP and explore links to automata theory. Chapter 6 discusses how the multi-criteria RAP and Tight-BMRAP algorithms can be modified to use CFNs for price-sensitive search. An evaluation of the framework’s performance is conducted in Chapter 7 using the network and fare system of MDV. Chapter 8 concludes the paper with some closing remarks.

2 Running Example: MDV

Fare systems in public transit often are designed as a compromise between the companies interest to generate revenue, political factors and also the function of pricing as a tool for demand steering. This can lead to quite intricate fare systems that pose significant challenges for the development of routing algorithms. We introduce the reader to some of those intricacies using the example of the fare system (as of 2019) of “Mitteldeutscher Verkehrsverbund” (MDV), a German public transit company. Throughout this paper, we will use MDV to illustrate our modeling concepts.

The features of the MDV fare system are exemplified in Figure 1.

Example 2.1 (The Fare System of MDV). MDV’s area of operations covers large rural areas in eastern Germany as well as the conurbation of Halle and Leipzig. This area is divided into a set of disjoint fare zones which we denote by \( Z \). As of 2019, there were 56 such fare zones. In most cases, the price depends on the number of visited fare zones: there is a new price level for one to seven fare zones. We denote the respective tickets by \( Z_i \) with \( i \in [7] \). For example, traveling from station A to station L in Figure 1 requires the ticket \( Z_4 \). For all paths covering more zones, a ticket for MDV’s whole area of operations has to be purchased, which we denote by \( M \). The two larger cities Halle and Leipzig each form a single fare zone. Travelling in these zones requires special tickets more expensive than \( Z_1 \). These we denote by \( H \) and \( L \), respectively.

For all paths that pass through multiple fare zones, they, however, count as normal zones, i.e., one of the tickets \( Z_2, \ldots, Z_7, M \) is applied. Hence, the paths \( A - C \) and \( H - L \) incur tickets \( H \) and \( L \), respectively, while the path \( A - G \) incurs \( Z_2 \). Several smaller towns are part of larger fare zones, but allow for discounted fares (town fares) when traveling only in that town. For each such town \( c \), we denote the ticket by \( C_c \). The path \( E - F \) in Merseburg (\( m \)), hence, requires the
Figure 1: Schematic depiction of a MDV-like fare plan with six fare zones and two lines. Two of them, colored in light gray, are the city zones of Halle and Leipzig. Vertically hatched hexagons represent overlap areas that can be counted as either of the neighboring zones. The horizontally hatched circle represents the small town Merseburg in which a special discounted fare is applicable. Small black nodes represent public transit stops. Footpaths are indicated by dotted lines.

ticket \( C_m \). When prolonging the path to stop \( G \), the ticket \( Z_1 \) becomes applicable. As of 2019, there were 17 towns with town fares and two price levels (which we denote by \( C_1 \) and \( C_2 \)). For paths starting in Halle and Leipzig, there are discounted tickets for short trips (\( D_H \) and \( D_L \)), which can be used for a maximum number of four stops without transfers. Hence, paths \( A - B \) and \( I - L \) are applicable for discounted tickets \( D_H \) and \( D_L \), respectively, while paths \( A - C \) and \( H - L \) are not. Discounted tickets exist also for other zones (\( D \)). These are a little cheaper and depend on the length of the journey (4 km maximum) instead of the number of visited stops. Sometimes it is possible to choose between town fares and length-based discounts. In this case, the town fare is applied because it is cheaper. To not unduly burden people living at the borders of fare zones, MDV uses overlap areas. These can be counted as part of either of their adjacent fare zones whichever is most benevolent to the traveller. For example, when traveling from \( J \) to \( M \), all stops are counted as part of fare zone 162 and thus ticket \( Z_1 \) would be applicable. When traveling from \( E \) to \( D \), \( D \) counts as part of the fare zone 233 but in the path \( A - D \) it counts as part of Halle. Hence, tickets \( Z_1 \) and \( H \) are applicable, respectively.

Note that we covered the most important features of the fare system but not its entirety. We do ignore some edge cases and explicit exceptions in the fare system. These are among other things: Slightly different discount rules for specify trains, counting stations that are passed without a stop for discounted tickets and exceptions for a specify tunnel. This is done in part because they are not properly reflected in our data set and in part to simplify presentation.

3 A Formal Framework for Fare Systems

We are given a (directed) routing graph \( G = (V, A) \), in which arcs represent either public transport connections, footpaths or transfers between lines and/or modes of transportation.
Every path $p$ in $G$ is associated with a ticket $t \in T$ from a ticket set $T$ that has to be bought to travel along $p$ and each ticket has a corresponding price $\pi(t) \in \mathbb{Q}^+$. In the following, we might also write $\pi(p)$ instead of $\pi(t)$ if $t$ is the ticket associated with $p$.

We aim at solving the price-sensitive earliest arrival problem (PSEAP).

**Definition 3.1** (Price-Sensitive Earliest Arrival Problem). Let a public transportation network be given as a graph $G = (V, A)$. Let for all $a \in A$ a time-dependent FIFO travel time function $c(a) : I \rightarrow I$ be given, where $I$ is the set of time points. Finally, let $P_{s,t}$ be the set of all $s,t$-paths in $G$ for some $s,t \in V$. Then, the price-sensitive earliest arrival problem (PSEAP) is defined as finding a Pareto-set of $s,t$-paths $P_{s,t}^0 \subseteq P_{s,t}$ in $G$, such that

$$\forall p^* \in P_{s,t}^0 \exists p \in P_{s,t} : \pi(p) \leq \pi(p^*) \land c(p) \leq c(p^*) \land (\pi(p) < \pi(p^*) \lor c(p) < c(p^*)).$$

We refer to the single criterion version of PSEAP that only optimizes for price as PSEAP$_0$.

Ideally, we want to solve this problem taking advantage of the vast literature on shortest-path algorithms. Ticket prices, however, are usually not as easy to deal with as travel times and cannot be modeled via arc weights. In order to efficiently compute cheapest paths using standard shortest path algorithms, we need to design labels for paths such that a) they can be updated quickly when a new arc is relaxed and b) dominance relationships between labels can be established that respect the subpath optimality property.

**Definition 3.2** (Subpath Optimality). Let $G = (V, A)$ be a directed graph and let all paths $p$ in $G$ be labeled with labels $l(p) \in L$ from a partially ordered label set $L$. We say that a $s,t$-path $p^* = (s = v_0, \ldots, v_k = t)$ is optimal or nondominated if there is no $p \in P_{s,t}$ such that $l(p^*) > l(p)$. The path $p^*$ is said to be subpath optimal if for all $i \in [k]$ there is no $p \in P_{s,v_i}$ such that $l((v_0, \ldots, v_i)) > l(p)$. We say that $(G, L)$ has the subpath optimality property if every optimal path is also subpath optimal.

All label-setting shortest path algorithms prune locally dominated labels to keep the search trees small. Here, the subpath optimality property ensures that no nondominated path is pruned.

In the remainder of this chapter, we detail how path labels for PSEAP$_0$ can be constructed and updated. We show how they can be made to fulfill a weaker but still sufficient form of subpath optimality in Chapter 4.

### 3.1 Modeling with Monoids

Note that in Example 2.1 the price of a path depends on several countable factors: the number of visited stations, the total distance traveled, the set of visited fare zones and on indicators reporting whether transfers where made or whether the path crossed a towns borders.

Observe that all these factors share several key properties: First, there is a natural partial order on them indicating which configuration requires a more expensive ticket. For example, for fare zones $A, B, C$ we have $\{A, B\} \subset \{A, B, C\}$ and one transfer can be naturally seen as being as most as price-driving as two. The factors can be summed up along a path using an appropriate notion of addition. For distances, this is the normal addition of natural numbers; for fare zones, it is the union of sets; we can use the logical OR ($\lor$) on the set $\{0, 1\}$ for indicators. Finally, we can assume the existence of a neutral element for every factor.

The above properties suggest that the structure of a partially ordered positive monoid is an appropriate model for a large number of fare-relevant factors.

**Definition 3.3** (Partially ordered monoid). A monoid $(H, +, e)$ is a set $H$ together with an associative operation $+$ (called addition) and a neutral element $e$, i.e., $h + e = h \forall h \in H$. When the definition of $e$ is obvious from context we just write $(H, +)$. We call $(H, +, \leq)$ a...
partially ordered monoid if \( \leq \) is a partial order on \( H \) that is translation-invariant with respect to the monoid operation +, i.e., \( h_1 \leq h_2 \Rightarrow h_1 + x \leq h_2 + x \forall h_1, h_2, x \in H \). If additionally \( e \leq h \forall h \in H \), we call \((H,+,\leq)\) a partially ordered positive monoid.

Note that the we can define the cross-product of two partially-ordered monoids \((H_1,+,\leq_1)\) and \((H_2,+,\leq_2)\) by \((H_1 \times H_2,+,\leq_{1,2})\) where \((h_1,h_2) +_{1,2} (i_1,i_2) := (h_1 +_1 i_1, h_2 +_2 i_2)\) and \((h_1,h_2) \leq_{1,2} (i_1,i_2)\) if and only if \( h_1 \leq_1 i_1 \) and \( h_2 \leq_2 i_2 \) for all \( h_1, i_1 \in H_1 \) and \( h_2, i_2 \in H_2 \). This allows us to represent all of the fare-relevant factors above as a single partially ordered positive monoid \((H,+,\leq)\).

Price-optimal paths can then be found in the following way: We label each arc \( a \in A \) with a weight from \( H \) living in \( H \) as well and can be obtained by summing up the weights of the arcs of \( p \). The ticket applicable for \( p \) can then be derived from \( h_p \) using the rules of the fare system. Finding a price-optimal \( s,t \)-path with \( s, t \in V \) can now be achieved by finding the Pareto-set of \( s,t \)-paths with regard to the partial order of \((H,+)\). Here, we can apply any multi-criteria shortest path algorithm by using elements of \( H \) as labels and the partial order of \((H,+)\) to establish dominance between labels. The subpath optimality property is fulfilled due to the positivity of \((H,+)\) and the translation-invariance of \( \leq \). Note, however, that this set will likely still contain many dominated paths with respect to price. This necessitates the filtering out of superfluous paths in a postprocessing step.

Example 3.1 (MDV). For the MDV we can construct a monoid in the following way: for each town zone \( c \in C \) we define the monoid \((H_c := \{0,1\}, \wedge, \leq)\) as an indicator whether our path started in \( c \) and then left the town. Hence, all arcs representing a connection leaving the town carry the weight \( 1 \in H_c \). Furthermore, we represent the distance traveled by the monoid \((H_d := \mathbb{N}, +, \leq)\), the number of visited stations by \((H_s := \mathbb{N}, +, \leq)\), the set of fare zone by \((H_z := \mathbb{Z}^2, \cup, \subseteq)\) and finally the transfers by \((H_t := \{0,1\}, \wedge, \leq)\). Price-optimal paths can then be computed by finding the Pareto-set over the monoid \((H_d \times H_s \times H_t \times H_z \times \prod_{c\in C} H_c, +, \leq)\) and filtering out dominated paths in a postprocessing step. Here, + and \( \leq \) are induced from the component monoids.

While the above modeling approach covers a reasonable set of real live cases, it is not expressive enough to be of much use for complex fare systems. First, note that the labels of a label-setting shortest path algorithm live in \( H \) and might be quite large. In most cases, however, it is not necessary to carry the whole label along. Consider again the MDV case. If a path has left a city \( c \), all town fares become unavailable and labels for all monoids \( H_c, c \in C \) need no longer be considered. This information, however, remains unavailable to a shortest path algorithm using the monoid based model.

Second, shortest path search over the monoid is agnostic to all pricing information and therefore tends to consider unnecessarily large search trees. In the MDV case, if we already know an \( s,t \)-path for which, e.g., the ticket \( D_L \) is applicable, all paths starting in \( s \) with a more expensive ticket can be pruned. To do this, however, we must find a way to quickly obtain the corresponding ticket to labels from \( H \).

Third, the calculated Pareto-set can become exponential in size. In contrast, \(|P^*_{s,d}|\) is bounded by a constant, namely the number of tickets \(|T|\).

Computational results in Section 7 reveal that monoid-based modeling quickly becomes intractable even when only considering the fare zone monoid \((H_z, \cup, \subseteq)\).

3.2 Ticket Graphs and Fare Events

In order to overcome the challenges laid out in the previous section, we aim to extend our modeling in two directions: First, we want to take a path’s ticket and hence its price into
account. To do so, we need to develop a model to represent tickets and their relationships. Second, we want to reduce the dimensionality of the monoid.

In order to address the first requirement, we notice that there is a natural progression of the applicable ticket along a path. When traveling a short distance a short-distance ticket might suffice. When we prolong the path by adding further stops to it, this ticket might no longer be applicable and now, e.g., a zone ticket might apply. When traveling even further, a ticket covering two zones might be needed. Hence, we can relate tickets to each other via their ability to transition into one another along paths in the routing graph. We formalize this observation by introducing a ticket graph \( T = (T,E) \) in which an arc \( e = (t_1,t_2) \in E \) is introduced if ticket \( t_1 \) can transition into ticket \( t_2 \). Additionally, each ticket \( t_1 \) carries a Boolean function determining the conditions under which the transition is performed.

**Example 3.2** (Running Example: Ticket Graph for the MDV). Consider the ticket graph in Figure 2. All possible tickets introduced in Example 2.1 are represented as nodes. Whenever, a ticket can transition into another one, we introduce an arc. Note that, e.g., a discounted ticket for Halle \( D_H \) can never transition into a Leipzig ticket \( L \).

To reduce the dimension of the monoid, note that the price of a path depends both on its fare state, e.g., its length or travel time, and some fare events, e.g., a transfer or the boarding of a train that requires a surcharge. Up to now, events were included in the monoid. However, this is not strictly necessary: A transfer arc could cause a ticket transition in the ticket graph. After that, transfers might be ineffectual and hence it is unnecessary to record them in the monoid. To do so, we need to annotate arcs in the routing graph not only with elements of a fare monoid but also with events. Note, that the distinction between states and events introduces some flexibility to the modeling as several factors of a fare system can be modeled as both. However, we naturally aim to keep the monoid as low-dimensional as possible.

**Example 3.3** (Running Example: Fare Events for MDV). Recall that the fare monoid in Example 3.1 was introduced as \( (H_d \times H_s \times H_t \times H_z \times \prod_{c \in C} H_c,+,\leq) \) However, we can see any transfer as an event \( t \) occurring on a transfer arc of the routing graph. In the same way, leaving any town \( \bar{c} \in C \) can be seen as an event \( c \) occurring on arcs crossing the town’s borders. Hence, we can shrink the fare monoid to \( (H_d \times H_s \times H_z,+,\leq) \). We also introduce events \( h \) and \( l \) for public transit arcs ending in the special fare zones of Halle and Leipzig, respectively. This is not strictly necessary as this information can also be read from the state of \( H_z \). However,
we want to store \( H_z \) in memory with one bit per fare zone. Hence, we minimize the need to read individual bits by adding events. We obtain a set of events \( S = \{c, t, h, l, s_0\} \) where \( s_0 \) is a dummy event with no effect.

### 3.3 Conditional Fare Networks

We now combine the three core ideas of a ticket graph, fare events and modeling with monoids to a formal model of public transit fare systems.

Again, let \( G = (V, A) \) be a routing graph. Additionally, let \((H, +, \leq)\) be positive, partially ordered fare monoid, \( T = (T, E) \) a ticket graph and \( S \) a set of fare events.

Now, let \( a \in A \) be an arc in the routing graph. We label \( a \) with a fare attribute, i.e., an element of the fare monoid \( H \) as well as an element from the set of fare events.

**Definition 3.4 (Fare Attribute Space).** We call the product space \( W = H \times S \) the fare attribute space. For every \( a \in A \), we denote the corresponding fare attribute by \( w(a) \in W \).

We denote the components of \( w(a) \in W \) by \( w(a) = (w^h(a), w^s(a)) \) and refer to \( w^h(a) \) as the monoid attribute and to \( w^s(a) \) as the event attribute of \( a \).

By collecting the fare attributes along a path \( p \) in \( G \), we build its fare state.

**Definition 3.5 (Fare State Space).** We call the product space \( F := T \times H \) the fare state space. An element \( f \in F \) is called a fare state. We use \( f^t \) and \( f^h \) to denote its components. We refer to \( f^h \) as the monoid state.

We denote the fare state of \( p \) by \( f(p) \). Every vertex \( v \in V \) is labeled with an initial fare state \( \mu(v) \in F \) giving an initial ticket and monoid state. Fare states will serve as path labels for shortest-path algorithms. They contain all information necessary for ticket updates as well as all information necessary to decide domination between paths. Note that, in contrast to common shortest-path applications, the arc labels \( W \) live not in the same space as the path labels \( F \).

We now want to enable the tracking of fare states along paths in \( G \). To do so, we formalize the notion of the fare transition function on tickets \( t \in T \) in the ticklet graph \( T = (T, E) \). A fare transition function returns the ticket \( t_2 \in T \) a ticket \( t_1 \in T \) transitions into given the current monoid state \( h \in H \) and a fare event \( s \in S \). Possible candidates are the neighborhood of \( t_1 \) in \( T \) as well as as \( t_1 \) itself.

**Definition 3.6 (Fare Transition Function).** We call a function \( \Gamma_1 : H \times S \rightarrow N^+_G(t) \) a fare transition function for the ticket \( t \). Here, \( N^+_G(t) = \delta^+(t) \cup \{t\} \) is the closed out-neighborhood of \( t \). The fare transition function \( \Gamma : T \times H \times S \rightarrow T \) of \( T \) is given by \( \Gamma(t, h, s) := \Gamma_1(h, s) \).

The definition is intentionally kept as general as possible to capture a large number of possible transition conditions.

We use the notion of fare transition functions to define the update of a fare state when relaxing an arc of the routing graph.

**Definition 3.7 (Ticket Update Function).** Let \( f \in F \) and \( a \in A \). Then, we define the the fare state update function \( \text{Up} : F \times A \rightarrow F \) by \( \hat{f} := \text{Up}(f, a) \) with

\[
\hat{f}^h := f^h + w^h(a) \\
\hat{f}^t := \Gamma(f^t, \hat{f}^h, w^s(a)).
\]

We now have a tool at our disposal to track the development of fare states along a path in \( G \) in the ticket graph \( T \). Every path in \( G \) can be associated with a sequence of fare states in \( F \) in the following way.

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Definition 3.8 (Path-Induced Fare Sequence). We call a sequence of fare states \((f_1,\ldots,f_n)\) path-induced if there is a path \(p = (v_1,\ldots,v_n)\) with the following properties:

1. \(f_1 = \mu(v_1)\)
2. \(f_i = \text{Up}(f_{i-1},(v_{i-1},v_i)) \forall i = 2,\ldots,n\).

We call the fare state \(f(p) := f_n\) the fare state of the path \(p\).

Combining all the above definitions, we arrive at the notion of conditional fare networks which can precisely describe a fare system.

Definition 3.9 (Conditional Fare Network). Let \(G = (V,A)\) be a routing graph and let the following be given:

1. a cycle-free ticket graph \(T = (T,E)\) with transition function \(\Gamma\),
2. a partially ordered, positive monoid \((H,+,\leq)\),
3. a set of fare events \(S\),
4. arc attributes \(w : A \to W = H \times S\),
5. initial fare states \(\mu : V \to F = T \times H\) and
6. a price function \(\pi : T \to \mathbb{Q}_+\) that is monotonously non-decreasing along directed paths in \(T\), i.e., if there is a directed \(t_1 - t_2\)-path in \(T\) for \(t_1, t_2 \in T\), then \(\pi(t_1) \leq \pi(t_2)\). We write \(\pi(p)\) instead of \(\pi((f(p)) t)\) for a path \(p \in P\).

We call the seven-tuple \((T,\Gamma,H,S,w,\mu,\pi)\) a conditional fare network \(N\) of \(G\).

Note that cycle-freeness in \(T\) and the monotonicity condition on \(\pi\) as well as the positivity of \(H\) ensure that no price-decreasing cycles exist in \(G\). We consider those assumptions natural enough that any reasonable fare systems should satisfy them. In Chapter 4, we will see that dominance relations between paths need to be based on their fare states instead of their price. Thus, we can drop the monotonicity condition on \(\pi\) while still retaining optimality. In this case, however, price-based target pruning (see Section 6.4), which proved essential in ensuring acceptable performance for our cheapest path search, cannot be applied.

Example 3.4 (Running Example: Conditional Fare Network for MDV). We can now give the complete conditional fare network \(N\) for the fare system of MDV. We have already introduced the ticket graph (Example 3.2). As in Example 3.3, we define the fare monoid \((H,+,\leq)\) as \((H_d \times H_s \times H_z,+,\leq)\). The components of \((H,+,\leq)\) are summarized in Table 1.

| Name \(H_d\) | Represents | Ground set \(\mathbb{N}\) | Operator | Partial Order | Neutral Element |
|-------------|------------|----------------|----------|--------------|----------------|
| \(H_d\)   | Distance   | \(\mathbb{N}\) | +        | \(\leq\)     | 0              |
| \(H_s\)   | Stops      | \(\mathbb{N}\) | +        | \(\leq\)     | 0              |
| \(H_z\)   | Fare Zones | \(2^\mathbb{Z}\) | \(\cup\) | \(\subseteq\)| \(\emptyset\)  |

Table 1: MDV Fare Monoid

The set of fare events is \(S = \{c,t,h,l,s_0\}\). The meaning of the events is summarized in Table 2. From \(T,(H,+,\leq)\) and \(S\), we directly obtain the space of fare attributes \(W\) and the space of fare states \(F\). Vertices \(v \in V\) in the routing graph can now be annotated with an initial fare state. For a station \(v\), in, e.g., in fare zone 156, we have \(\mu(v) = (D,\{156\},0,0)\), i.e. we start with the short-distance ticket, the current fare zone, a number of visited stops of 0 and 0m of distance. Arcs \(a \in A\) are annotated with fare attributes \(w(a) \in W\). For example, if a
represents a connection of length 231 m between two stations in the fare zone 233, it would be annotated with $w(a) = ((\{233\}, 1, 231), s_0)$. If a were to represent a footpath, it would be annotated with $w(a) = ((\emptyset, 0, 0), t)$.

Hence, to construct a conditional fare network for MDV, we now only need to give the transition functions for $T$. Let $s \in S$ and $h = (h_d, h_s, h_z) \in H$. Then, the fare transition function $\Gamma$ of $T$ is defined by

$$\Gamma(M, h, s) = M$$

$$\Gamma(Z_i, h, s) = \begin{cases} 
Z_{i+1} & |h_z| = i + 1 \\
Z_i & \text{otherwise}
\end{cases}$$

$$\Gamma(D, h, s) = \begin{cases} 
Z_1 & |h_z| = 1 \land (h_d > 4 \lor s = t) \\
Z_2 & |h_z| = 2 \land (h_d > 4 \lor s = t) \\
Z_3 & |h_z| = 3 \land (h_d > 4 \lor s = t) \\
D & \text{otherwise}
\end{cases}$$

$$\Gamma(L, h, s) = \begin{cases} 
Z_2 & s \neq l \land s \neq t \\
L & \text{otherwise}
\end{cases}$$

$$\Gamma(D_L, h, s) = \begin{cases} 
Z_2 & s \neq l \land h_s > 4 \\
L & s = t \lor (s = l \land h_s > 4) \\
D_L & \text{otherwise}
\end{cases}$$

$$\Gamma(H, h, s) = \begin{cases} 
Z_2 & s \neq h \land s \neq t \\
H & \text{otherwise}
\end{cases}$$

$$\Gamma(D_H, h, s) = \begin{cases} 
Z_2 & s \neq h \land h_s > 4 \\
H & s = t \lor (s = h \land h_s > 4) \\
D_H & \text{otherwise}
\end{cases}$$

$$\Gamma(C_1, h, s) = \begin{cases} 
Z_2 & s = c \lor h_d < 4 \\
C_1 & \text{otherwise}
\end{cases}$$

$$\Gamma(C_2, h, s) = \begin{cases} 
Z_1 & s = c \land h_d > 4 \\
D & s = c \land h_d \leq 4
\end{cases}$$

$$\Gamma(C_3, h, s) = \begin{cases} 
Z_3 & s = c \land h_d > 4 \\
D & s = c \land h_d \leq 4
\end{cases}$$

Note that we did not yet cover MDV’s overlap areas. We discuss in Section 3.4 why it’s best to address these in a preprocessing step.

Having introduced conditional fare networks, we can now finally formalize the price-sensitive earliest arrival problem.

**Definition 3.10** (Price-Sensitive Earliest Arrival Problem (Revisited)). Let a public transportation network be given as a graph $G = (V, A)$ together with a conditional fare network $(T, \Gamma, H, S, w, \mu, \pi)$ and a time-dependent FIFO travel time function $c(a) : I \rightarrow I \forall a \in A$. Then, the price-sensitive earliest arrival problem (PSEAP) is defined as finding a Pareto-set of $s,t$-paths $P^*_{s,t} \subseteq P_{s,t}$ in $G$, such that

$$\forall p^* \in P^*_{s,t} \exists p \in P_{s,t} : \pi(p) \leq \pi(p^*) \land c(p) \leq c(p^*) \land (\pi(p) < \pi(p^*) \lor c(p) < c(p^*)). \quad (2)$$

In Chapter [3], we will ignore the arrival time aspect of PSEAP and focus only on the fare framework. The correctness results carry over to the full version of PSEAP.
3.4 Some Hints on Modeling with CFNs

In the following, we elaborate on some common features of fare systems and how they can be modeled using conditional fare networks.

Transfer Penalties, Footpaths and Surcharges

Footpaths are modeled as arcs with the arc attribute \((e, s_0)\), where \(s_0 \in S\) is an event that cannot activate a ticket transition and \(e \in H\) is the neutral element of the monoid. Hence, a footpath does not change the current fare state. The transition from a footpath to a public transportation vehicle requires some care. Assume we walk from stop \(v_0\) to \(v_1\) along arc \(a_0 = (v_0, v_1)\) to take a vehicle along \(a_1 = (v_1, v_2)\) to reach \(v_2\). Some fare systems use the number of stops a path touches to calculate prices. Here, this number would be two. Counting a stop when relaxing \(a_0\) is a mistake if the optimal path would be to continue on foot. Counting both \(v_1\) and \(v_2\) when relaxing \(a_1\) is also wrong since this would overcount the number of stops for every journey that reaches \(v_1\) via a vehicle. Hence, the graph model needs to be extended by splitting up stops into vertices for every route and a vertex that is connected to footpaths. These vertices are then connected via transfer arcs and boarding arcs. We can also have arc attributes different from \((e, s_0)\) on transfer arcs. This allows us to make the applicable ticket dependent on the number of transfers. Arc attributes on arcs representing boarding can be used to model surcharges for the route boarded. For more details on how to build these expanded graphs, we refer to (Disser, Müller-Hannemann, and Schnee 2008).

Overlap areas

Some fare systems that are based on fare zones contain overlap areas. Stations in an overlap area can be counted as part of either of its neighboring zones, whichever is cheapest for the customer. This is meant to mitigate sharp price increases for short journeys at fare zone borders. MDV uses them as well as several other German railway companies (e.g. Verkehrsverbund Bremen/Niedersachsen GmbH).

At a first glance, one might be tempted to represent overlap area as tickets in the ticket graph. A label propagated along a path starting in an overlap area then keeps this ticket until a regular fare zone is picked up along the path and transitions in the zone ticket for this fare zone. This approach, however, becomes cumbersome when several overlap areas border each other. In this case a ticket for each combination of overlap areas needs to be introduced.

Alternatively, overlap areas can be incorporated by label duplication: Assume an overlap area neighbors \(n\) fare zones. We associate each arc \(a\) whose head\((a)\) represents a stop in the overlap area with \(n\) fare attributes, one for each fare zone it could possibly be part of. When settling the vertex in a shortest path search, the current fare state is updated once for each fare attribute thereby creating \(n\) new labels. This, however, leads to an increased need for dynamic memory allocation for labels which should be avoided.

Hence, we choose the simplest conceivable approach for the remainder of this work, namely route duplication. Whenever a route of the timetable contains a stop \(p\) in an overlap area neighboring \(n\) fare zones, we simply introduce \(n\) routes each with a single fare zone at \(p\). To avoid creating unnecessary duplicates, this is done block-wise, i.e., only for each consecutive sequence of stops along a route that are in the same overlap area. Hence, overlap areas are taken care of in a preprocessing step and are not represented in the conditional fare network.

4 The Fare Framework in Routing Algorithms

Classical shortest path algorithms rely on dynamic programming and the subpath optimality condition (Berger and Müller-Hannemann 2009). That is, every subpath of an optimal \(s, t\)-path is in itself an optimal path. When comparing paths in \(G\) naively by means of the price function \(\pi\), the subpath optimality condition is usually violated. Think about taking a local detour to
avoid a fare zone: Later on, travelers may be forced to cross the zone due to the infrastructure, turning the locally dominant detour into a suboptimal choice. On the other hand, a locally dominated subpath might still lead to an optimal \( s,t \)-path. This type of problem persists in our framework: the transition between tickets depends on the fare attributes already collected, but also on the structure of the reachable ticket graph and the transition functions of reachable fare arcs. Example 4.1 highlights that problems can already arise even with simple examples of ticket graphs.

![Routing Graph](image)

![Ticket Graph](image)

Figure 3: Example of a routing graph (a) with two possible conditional fare networks (b) and (c). For both networks, the underlying partially ordered monoid is \((\mathbb{R}, +, \leq)\), the fare events are \(S = \{s_0, s_1, s_2, s_3\}\) and the initial fare state for all vertices \(v_i\) with \(i = 1, \ldots, 5\) is \(\mu(v_i) = (A, 0)\). We set prices for the tickets as \(\pi(A) = 0\), \(\pi(B) = 2\), \(\pi(C) = 3\), \(\pi(D) = 1\) and \(\pi(E) = 5\). Transition functions are displayed as indicator functions on fare arcs. Using the ticket graph (b), the upper \(v_1,v_5\)-path yields ticket \(C\), while the lower path yields ticket \(E\). Using ticket graph (c), the upper path yields ticket \(B\), the lower path yields ticket \(C\).

**Example 4.1** (Label Dominance in Figure 3). Consider the routing graph (a) together with the conditional fare network (b). Examining the paths \(p_1 = (v_1,v_2,v_4)\) and \(p_2 = (v_1,v_3,v_4)\), we find their respective fare states are \(f(p_1) = (B,1)\) and \(f(p_2) = (D,2)\). Extending them by \(v_5\) to \(p_1'\) and \(p_2'\) yields \(f(p_1') = (C,3)\) and \(f(p_2') = (E,4)\). Comparing fare states by price would indicate that \(p_1\) could be pruned at \(v_4\) since \(\pi(B) > \pi(D)\). This is a suboptimal choice as \(p_1'\) dominates \(p_2'\) since \(\pi(C) < \pi(E)\). Hence, price cannot be used as dominance criterion for fare states. A natural alternative would be to use the partial order defined by paths in the ticket graph, instead. A ticket \(t_1\) then dominates a ticket \(t_2\) if there is a \(t_1,t_2\)-path. This would render the tickets \(B\) and \(D\) and the tickets \(C\) and \(E\) mutually incomparable. The idea, however, comes with problems of its own. To see this, consider now the conditional fare network (c). At \(v_4\), we have \(f(p_1) = (A,1)\) and \(f(p_2) = (A,2)\) and hence both paths are equivalent and it would be sensible to keep only one of them based on the relation between \(f^B(p_1)\) and \(f^B(p_2)\). By relaxing \((v_4,v_5)\), we obtain \(f(p_1') = (B,3)\) and \(f(p_2') = (C,4)\), which are incomparable, i.e., the fare states of \(p_1'\) and \(p_2'\) diverged from comparable to incomparable. Consequently, any dominance rule pruning either \(p_1\) or \(p_2\) would be defective.

To mitigate these and similar problems, we might assume a general incomparability of fare states. This comes down to enumerating all \(s,t\)-paths and simply sorting them by price.
However, in a sensibly designed fare system it is usually clear which ticket is better and taking a cheaper subpath should usually not turn out more expensive overall. In the remainder of this chapter, we propose a more tailored approach. It bases domination rules on path relationships but adds exceptions to cover cases in which it is not safe to do so.

### 4.1 Dominance for Fare States

We want to define a comparison operator for fare states that restores subpath optimality while not relaxing dominance too strongly.

To do so, we partition the ticket set $T$ into three disjoint comparability groups: $C_F$ (full comparability), $C_P$ (partial comparability), $C_N$ (no comparability). Based on the partition $C = (C_F, C_P, C_N)$, we define a comparison operator for fare states.

**Definition 4.1 (Comparability of Fare States).** Let $f_1 = (t_1, h_1)$, $f_2 = (t_2, h_2)$ be fare states. We say $f_1 \leq_C f_2$ if and only if $t_1 \notin C_N$, $h_1 \leq h_2$ and

$$ t_1 = t_2 \quad \text{if } t_1 \in C_P $$

$$ \exists t_1, t_2 \text{-path in } T \quad \text{if } t_1 \in C_F. $$

If and only if $f_1 \leq_C f_2$ and either $h_1 < h_2$ or $t_1 \neq t_2$, we say that $f_1$ is strictly lesser than $f_2$, i.e., $f_1 <_C f_2$.

We denote by $P^I_{s,t}$ the set of all paths Pareto-optimal with respect to $\leq_C$, i.e.,

$$ p^* \in P^I_{s,t} \Rightarrow \exists s, t \text{-path } p : f(p) <_C f(p^*). $$

Note that $P^I_{s,t}$ is not equal to $P^*_{s,t}$. Proposition 4.3 shows that it is in fact a superset of the set of all price-optimal paths $P^*_{s,t}$. We call paths in $P^I_{s,t}$ state-optimal and paths in $P^*_{s,t}$ price-optimal.

Shortest path algorithms on graphs with weights from partially ordered monoids require the monoid operation to be translation-invariant with respect to the partial order. Since the fare states and arc attributes do not belong to the same space, the notion of translation invariance needs to be generalized to a monotonicity formulation. Hence, the partition $C$ has to respect monotonicity of the update function along all arcs $a \in A$, i.e.,

$$ \forall f_1, f_2 \in F : f_1 \leq_C f_2 \implies \forall a \in A : \text{Up}(f_1, a) \leq_C \text{Up}(f_2, a). $$

This condition is enough to ensure that a weaker form of subpath optimality holds.

**Proposition 4.1 (Weak Subpath Optimality).** Let $G = (V, A)$ be a routing network and $\mathcal{N} = (T, \Gamma, H, S, w, \mu, \pi)$ be its conditional fare network. Let $p^* \in P^I_{s,t}$ be a state-optimal $s, t$-path in $G$ for some $s, t \in V$. Then, there is a path $p' = (s = v_0, v_1, \ldots, v_{n-1}, v_n = t) \in P^I_{s,t}$ with $f_{p'} = f_{p^*}$, such that every subpath $p'_l = (v_0, \ldots, v_l)$, $l < n$ of $p'$ is a state-optimal $v_0, v_l$-path.

**Proof.** Let $p^* = (s = v_0, v_1, \ldots, v_{n-1}, v_n = t) \in P^I_{s,t}$ be a state-optimal $s, t$-path and let $(f_0, \ldots, f_n)$ be the fare sequence associated with it. Assume there is another $s, t$-path $\bar{p} = (s = u_0, u_1, \ldots, u_{l-1} = v_{n-1}, u_l = t)$ with fare states $(f_0, \bar{f}_1, \ldots, \bar{f}_l)$. Let $k$ be the largest integer such that $v_{n-k} = u_{l-k}$, i.e., the paths $(v_{n-k}, \ldots, v_n)$ and $(u_{l-k}, \ldots, u_l)$ are equal. Now assume $\bar{f}_{l-k} <_C f_{n-k-1}$. By definition, $f_{n-k} = \text{Up}(f_{n-k-1}, (v_{n-k-1}, v_{n-k}))$ and $\bar{f}_{l-k} = \text{Up}(\bar{f}_{l-k-1}, (u_{l-k-1}, u_{l-k}))$. We apply Equation 6 to obtain $\bar{f}_{l-k} \leq_C f_{n-k}$. By repeating the process for $i \in \{k - 1, \ldots, 0\}$, we find $\bar{f}_l \leq_C f_n$. Since $p'$ was state-optimal, it follows that $\bar{f}_l =_C f_n$ and consequently $p$ is also state-optimal. Since the number of paths in $G$ is finite, we can repeat this procedure to find the path $p'$. \qed
4.2 Comparability Partitions

In choosing $C_F$, $C_P$ and $C_N$, there is some degree of freedom. We want $C_F$ to be as big and $C_N$ as small as possible while still fulfilling Equation $[8]$. It is clear that the choice does not only depend on the ticket graph $T$ and the transition function $\Gamma$ but also on the structure of $G$ and its arc attributes $A$. Choosing the partition depending on $G$ and $A$ would require some preprocessing of $G$. We propose a solution that depends only on $T$ and $\Gamma$ and needs no recomputation when changes in the routing network occur.

First, we introduce some notation.

**Definition 4.2** (Reach). Let $T = (T, E)$ be a directed graph. We define the reach $R(t)$ of a vertex $t \in T$ as the subgraph induced by all vertices reachable from $t$, i.e.,

$$R(t) := T[\{k \in T : \exists t, k \text{-path in } T\}] .$$

Furthermore, we introduce operators that represent path relations between tickets. If there is a directed path in $T$ between $t_1, t_2 \in T$, $t_1 \neq t_2$, we write $t_1 \rightarrow t_2$. We write $t_1 \leadsto t_2$ if either $t_1 \rightarrow t_2$ or $t_1 = t_2$.

**Definition 4.3** (No-overtaking Property). Let $t \in T$ be a ticket. We say its reach $R(t)$ has the no-overtaking property if for all tickets $k, l \in R(t)$ with $k \leadsto l$ and $(h, s) \in W$ it holds that

$$\forall \bar{h} \in H : \bar{h} \geq h \implies \Gamma_k(h, s) \leadsto \Gamma_l(\bar{h}, s).$$

The no-overtaking property bears some resemblance to the FIFO (first-in, first-out) property: A worse fare state, i.e., either a worse monoid state or a worse ticket, cannot give rise to a better fare state when relaxing the same arc in the routing graph. Note that the no-overtaking property has to be fulfilled not only for the neighborhood of a ticket $t$ but for the reach $R(t)$.

Subgraphs with the no-overtaking property allow for the strictest domination rules. We use them as comparability group $C_F$.

**Definition 4.4** (Comparability Partition). Let $G = (V, A)$ be a routing network and $N = (T, \Gamma, H, S, w, \mu, \pi)$ be its conditional fare network. We define

$$C_F := \{t \in T : R(t) \text{ traceable and has the no-overtaking property}\}$$

$$C_P := \{t \in T \setminus C_F : \forall k \in R(t) \forall s \in S \forall h_1, h_2 \in H : \Gamma_k(h_1, s) = \Gamma_k(h_2, s)\}$$

$$C_N := \{t \in T \setminus (C_F \cup C_P)\} .$$

It is not enough to fulfill Equation $[8]$ for a $t$ to be in the set $C_F$. Its reach $R(t)$ has also to be traceable, i.e., contain a Hamiltonian path. This condition is needed to avoid the divergence seen in Example $[3]$. If a ticket has non-traceable reach or does not have the no-overtaking property, it is placed in $C_P$. For tickets $t$ in $C_P$, the transition functions of tickets $k \in R(t)$ must be independent of $(H, +, \leq)$. In both cases, not only the ticket that we want to compare has to be considered but its reach. This, again, is necessary to ensure that comparable fare states do not diverge in an incomparable state after an update, i.e. all tickets that can be reached from a ticket in $C_F$ themselves need to be in $C_F$. All remaining tickets are added to $C_N$. Fare states containing tickets from $C_N$ can never be dominated.
Example 4.2 (Dominance Rules for MDV Fares). In the graph in Figure 3 all nodes have traceable reach and it is easy to verify that the no-overtaking property does indeed hold for all tickets. Hence, we can set the comparability partition as $C_F = T$, $C_F = C_N = \emptyset$.

A comparison operator $\leq_C$ defined with Definition 4.4 fulfills Equation 6.

Proposition 4.2 (Monotonicity of the Comparability Partition). The partial order defined by Definitions 4.1 and 4.4 fulfills the monotonicity condition

$$\forall f_1, f_2 \in F : f_1 \leq_C f_2 \implies \forall a \in A : \text{Up}(f_1, a) \leq_C \text{Up}(f_2, a).$$

Proof. Recall that we denote the components of fare states $f \in F$ by $f = (f_t, f^h)$ and those of fare attributes $w \in W$ by $w = (w^h, w^s)$.

Now, let $a \in A$ and $f_1, f_2 \in F$ such that $f_1 \leq_C f_2$. We write $\tilde{f}_1 := \text{Up}(f_1, a)$ and $\tilde{f}_2 := \text{Up}(f_2, a)$, i.e., $\tilde{f}_i^h = f_i^h + w^h(a)$ and $\tilde{f}_i^s = \Gamma(f_i^h, f^h_i, w^s(a))$ for $i \in \{1, 2\}$. By positivity of the monoid $(H, +, \leq)$, $\tilde{f}_i^h \leq \tilde{f}_2^h$ directly implies $\tilde{f}_1^h \leq \tilde{f}_2^h$. It remains to show that $\tilde{f}_1^s \equiv \tilde{f}_2^s$. To do so, we need to distinguish the cases $\tilde{f}_1^s \in C_F$ and $\tilde{f}_2^s \in C_F$.

First, assume that $\tilde{f}_1^s \in C_F$ and hence $\tilde{f}_1^s = \tilde{f}_2^s$. By applying the definition of $C_F$, we obtain

$$\tilde{f}_1^s = \Gamma(f_1^s, \tilde{f}_1^h, w^s(a)) = \Gamma(f_2^s, \tilde{f}_2^h, w^s(a)) = \tilde{f}_2^s.$$ 

Thus, $\tilde{f}_1^s = \tilde{f}_2^s$. Note that the definitions of $C_F$ and $C_F$ imply that $\tilde{f}_1^s \subset C_F \cup C_F$ since $\tilde{f}_1^s \in R(\tilde{f}_1^s)$ and hence $f_1 \leq_C f_2$.

Now, assume $\tilde{f}_1^s \in C_F$. Note that $R(\tilde{f}_1^s) \subset C_F$. This allows us to apply Equation 8 to obtain

$$\tilde{f}_1^s = \Gamma(f_1^s, \tilde{f}_1^h, w^s(a)) \equiv \Gamma(f_2^s, \tilde{f}_2^h, w^s(a)) = \tilde{f}_2^s$$

which concludes the proof.

Propositions 4.1 and 4.2 allow us to apply dynamic programming shortest path algorithms to the PSEAP using the comparability partition from Definition 4.4. However, we obtain only the set of state-optimal paths. It remains to show that this set contains the cheapest path.

Proposition 4.3 (Correctness). Let $\pi^* := \min_{P_{s,t}} \pi(p)$. Then, there is at least one $s, t$-path $p^*$ with $\pi^* = \pi(p^*)$ and $p^* \in P_{s,t}^f = \{ \hat{p} : s, t$-path : $\hat{p} : f(p) \prec f(\hat{p}) \}$. 

Proof. Consider $p \in P_{s,t}^f = \{ \hat{p} : s, t$-path : $\pi(f(\hat{p})) = \pi^* \} \neq \emptyset$. If there is a path $p' \in P_{s,t}^f$ with $f_t(p') = f_t(p)$, we are done. If not, there is a path $p' \in P_{s,t}^f$ with $f_t(p') \rightarrow f_t(p)$. This implies $\pi(f_t(p')) \leq \pi(f_t(p))$ and hence a path of the same price as $p$ is present in $P_{s,t}^f$.

5 Computational Complexity and Links to Automata Theory

In this section, we study the computational complexity of PSEAP. The intractability of general multi-objective shortest path problems is well-established (Hansen 1980). The standard argument is here, that the output might be exponential in size. Note that in PSEAP the number of tickets $T$ is finite. Therefore, it is always possible to find a valid set of Pareto-optimal solutions with size $\leq |T|$. This begs the question whether PSEAP can be solved in polynomial time. The answer depends on whether the fare monoid $(H, +, \leq)$ is considered as part of the encoding length.

If $(H, +, \leq)$ is not considered part of the input then even the single-criterion version of PSEAP with travel time functions $c \equiv 0$ (denoted by PSEAP0) is NP-hard. (Blanco et al. 2016) proved finding a shortest path with respect to weights from the monoid $(2^Z, \subseteq, \cup)$. with
fare zones $Z$ to be NP-hard. The proof was obtained using a reduction from the minimum-color single-path problem [Broersma et al. 2005]. As this is a special case of PSEAP, the NP-hardness of PSEAP follows immediately.

In the following, we provide an alternative reduction of the path with forbidden pairs problem to PSEAP. Its NP-completeness was established by [Gabow, Maheshwari, and Osterweil 1976].

**Definition 5.1** (Path with Forbidden Pairs Problem). Let $G = (V, A)$ be a directed graph and $(a_i, b_i), i \in I$ a list of forbidden pairs. Let $s, t \in V$. The shortest path with forbidden pairs problem asks whether there is a $s, t$-path $p$ in $G$ such that $\forall i \in I : a_i \not\equiv p \lor b_i \not\equiv p$. That is, $p$ contains at most one vertex of each pair $(a_i, b_i)$.

**Theorem 5.1** (NP-hardness of PSEAP). The single-criterion problem PSEAP is NP-hard. Its canonical decision problem PSEAP$_{DEC}$ is NP-complete if the evaluation time of $\Gamma$ is polynomially bounded.

**Proof.** The canonical decision problem PSEAP$_{DEC}$ of PSEAP asks whether there is a path $p$ in $G$ with $\pi(p) \leq k$ for some $k \in \mathbb{Q}^+$. We show NP-hardness by reducing the path with forbidden pairs problem to PSEAP. Let $G = (V, A)$ be a directed graph, $s, t \in V$ and $(a_i, b_i), i \in I$ a list of forbidden pairs. We construct a CFN $T = (T, \Gamma, H, S, w, \mu, \pi)$ as follows. Let $T = (T,E)$ be the ticket graph with tickets $T = \{t_1, t_2\}$ and $E = \{(t_1, t_2)\}$. Furthermore, we define $(H, +, \leq)$ as follows. Let $H = \{0, 1, 2\}^I$. The sum of $x, y \in H$, is defined as $x + y := (\min(2, x_i + y_i))_{i \in I}$. We have $x \leq y$ if and only if $x_i \leq y_i$ for all $i \in I$. We set the ticket prices to $\pi(t_1) = 0$ and $\pi(t_2) = 1$. Now, we let

$$w(v_1, v_2) = \begin{cases} e_i & v_1 = a_i \text{ or } v_1 = b_i \text{ for some } i \in I \\ 0 & \text{otherwise} \end{cases} \quad \forall (v_1, v_2) \in A \quad (13)$$

and

$$\Gamma(t_1, h, s) = \begin{cases} t_2 & \exists i \in I : h_i = 2 \\ t_1 & \text{otherwise} \end{cases} \quad (14)$$

where $e_i \in \{0, 1\}^I$ is the standard unit vector with $e_{ii} = 1$. Let $p$ be a simple path. W.l.o.g we assume that the last vertex of $p$ is not in $\bigcup_{i \in I}\{a_i, b_i\}$. Now, assume that $p$ has ticket $f^t(p) = t_2$. Hence, there must be an $i \in I$ s.t. $f^t_i = 2$ and therefore both $a_i \in p$ and $b_i \in p$. Conversely, if both $a_i \in p$ and $b_i \in p$, we must have $f^t_i(p) = 2$ and therefore $f^t(p) = t_2$. The CFN $\mathcal{N}$ can be built in polynomial time as $w$ can be built in $O(|A||I|)$. Hence, the shortest path with forbidden pairs problem can be polynomially reduced to PSEAP. Therefore, $\text{PSEAP} \leq \text{DEC}$.

It remains to show that PSEAP$_{DEC}$ is in NP. When evaluating the fare state of $p$ as many calls to $\Gamma$ have to be performed as there are arcs in $p$. Every simple $s, t$-path has clearly $|A| \leq |A|$ edges. Hence, the overall evaluation time of $\Gamma$ for $p$ is polynomially bounded. Finally, finding $\pi(p)$ from $f(p)$ requires a simple table-lookup. Hence, checking $\pi(p) \leq k$ can be done in polynomial time. \hfill $\Box$

Now, let us assume that the underlying monoid $(H, +, \leq)$ is of finite size and let’s consider $|H|$ a part of the encoding length. In this case, PSEAP can be solved in polynomial time using techniques already used by [Barrett, Jacob, and Marathe 2000] for the regular language constrained shortest path problem.

We begin with a short recapitulation of crucial results from automata theory. The following definition is taken from [Hopcroft and Ullman 1979]. For further background, we also refer to this book.

**Definition 5.2** (Deterministic finite automaton). A deterministic finite automaton (DFA) is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where $Q$ is a finite set of states, $\Sigma$ is a finite input alphabet, $q_0 \in Q$ is the initial state, $F \subseteq Q$ is the set of final states and $\delta : Q \times \Sigma \to Q$ is the transition function.
The words accepted by a DFA are exactly the words of a regular language. Hence, we can use DFAs to define the regular language constrained shortest path problem (REG-ShP) [Barrett, Jacob, and Marathe 2000]. In this problem, each arc \( a \in A \) of a directed graph \( G = (V, A) \) is associated with a letter \( \sigma(a) \in \Sigma \). The state of a path \( p = (v_0, \ldots, v_n) \) is then recursively defined via

\[
q((v_0, \ldots, v_i)) := \delta(q((v_0, \ldots, v_{i-1})), \sigma(v_{i-1}, v_i))
\]

\[
q(v_0) := q_0.
\]

**Definition 5.3** (Regular-language constrained shortest path problem (REG-ShP)). Let a directed graph \( G = (V, A) \), weights \( c : A \rightarrow \mathbb{Q}^+ \), a DFA \( (Q, \Sigma, \delta, q_0, F) \), letters \( \sigma(a), a \in A \), a source \( s \in V \) and a destination \( t \in V \) be given. Find a shortest \( s, t \)-path \( p \) such that \( q(p) \in F \).

Conditional fare networks \( (T, \Gamma, H, S, w, \mu, \pi) \) can be recast as DFAs for a fixed starting stop \( v \in V \) if \( H \) is finite. Thus, PSEAP can be interpreted as a version of a formal-language constrained shortest path problem where we additionally optimize over the set of accepting states.

Assume the monoid \((H, +, \leq)\) is trivial, i.e. \( H = \{x\} \) with \( x + x = x \) for some element \( x \). Then, the transition functions \( \Gamma(\cdot, \cdot, \cdot) \) can be seen as independent of \((H, +, \leq)\) and we can construct a DFA \( D(s) = (Q, \Sigma, \delta, q_0, F) \) in a straightforward manner by setting \( F := Q := T, \Sigma := S, q_0 := \tau \) where \((\tau, x) = \mu(s)\) is the initial fare state of \( s \). The transition function \( \delta \) is defined as

\[
\delta(q, \sigma) := \Gamma(q, x, \sigma) \forall q \in Q, \sigma \in \Sigma.
\]

As there is no direct equivalent for nontrivial \( H \) in a DFA, we incorporate it into the state set \( Q \). We set \( Q := F := T \times H, \Sigma := W = H \times S \) and \( q_0 := \mu(v) \). The transition function \( \delta \) is then defined by

\[
\delta(q, \sigma) := (\Gamma(q^h, q^h + \sigma^h, x^h), q^h + \sigma^h) \forall q = (q^h, q^h) \in Q, \sigma = (\sigma^h, \sigma^x) \in \Sigma.
\]

An example of this transformation can be seen in Figure [4].

Note that we can restrict \( \Sigma \) to those fare attributes that do really appear on arcs in \( A \). Hence, we can assume \( O(|\Sigma|) = O(|A|) \). Then, the transformation can be done in \( O(|Q| + |\Sigma| |Q| B) = O(|A| |T| H |B|) \) assuming the evaluation time of \( \Gamma \) to be bounded by a polynomial \( B \). We can use this insight to directly give a polynomial time algorithm for PSEAP over finite monoids. This is done by restricting the set of accepting states to a single element of \( Q \) and then solving an instance of REG-ShP. In this way, a superset \( M \) of the Pareto-set of PSEAP can be found in \( |Q| = |T| H |B| \) iterations of REG-ShP. REG-ShP can be solved in \( O(|V| H |B|) + |A| H |B| \) using the algorithm given by Barrett, Jacob, and Marathe (2000). As \( M \) has at most \(|T| H |B|\) entries it takes \( O(|T| H |B|) + |A| H |B| \) time to extract the actual Pareto-set [Kung, Luccio, and Preparata 1975]. Thus, we obtain an overall running time of \( O(|V| H |B|) + |A| H |B| \). A slight modification of the proof for Reg-ShP in [Barrett, Jacob, and Marathe 2000] allows us to obtain a tighter bound.

**Lemma 5.1** (PSEAP with constant travel time over finite monoids). Consider PSEAP over \( N = (T, \Gamma, H, S, w, \mu, \pi) \) with \( H \) finite, i.e., \(|H| < \infty\). Assume all travel time functions are constants and that the evaluation time of \( \Gamma \) can be bounded by a polynomial \( B \). Then, PSEAP can be solved in \( O(|V| H |B|) + |A| H |B| \).

**Proof.** Let \( D(s) = (Q, \Sigma, \delta, q_0, F) \) be the DFA constructed as described above. To keep consistency with automata terminology, we write \( \sigma(v_1, v_2) := w(v_1, v_2) \) for \( v_1, v_2 \in V \). We construct a product network \( G^\times = (V^\times, A^\times) \) of \( G \) and \( D(s) \) with

\[
V(G^\times) = V \times Q
\]

\[
E(G^\times) = \{(v_1, q_1), (v_2, q_2) \mid (v_1, v_2) \in E, q_2 = \delta(q_1, \sigma(v_1, v_2))\}.
\]
Lemma 5.1 solves PSEAP with FIFO travel time functions in \(O\) travel time functions is bounded by a constant. Thus, using the modified Dijkstra-variant in Proof. It is well-established that time-dependent shortest path problems can be solved using a evaluated in constant time, and that the evaluation time of \(\Gamma\) over the conditional fare network \(N\) is \(\mathcal{O}(1)\) \((Delling, Pajor, and Werneck 2015)\). Since RAP implicitly optimizes the number of trips, we obtain an multi-criteria algorithm that optimizes for travel time, number of trips and 

\[
\text{Figure 4: Transformation from CNF to DFA. The CFN } \mathcal{N} = (T, \Gamma, H, S, w, \mu, \pi) \text{ is given by the ticket graph } T \text{ and transition functions depicted in (a), the monoid } (H, +, \leq) \text{ with } H := \{0, 1, 2\} \text{ and } a + b := \min\{a + b, 2\} \text{ and } 0 \leq 1 \leq 2 \text{ and fare events } S = \{s_1, s_2\}. \text{ The definitions of } w \text{ and } \mu \text{ depend on the routing graph and are omitted in this example. Our transformation results in the DFA in (b). There is a state for each element from } T \times H. \text{ Each arc depicts a possible state transformation. For each arc, there is at least one letter from } \Sigma = \{a, b\} \text{ that the travel time functions } c(a) \text{ are sensitive version of the multi-criteria RAPTOR algorithm (abbr. RAP and McRAP, respectively) (Delling, Pajor, and Werneck 2015). Since RAP implicitly optimizes the number of trips, we obtain an multi-criteria algorithm that optimizes for travel time, number of trips and}
\]

\[
\begin{align*}
\text{Note that for every } a \in A \text{ there are at most } |T||H| \text{ edges in } E^\times \text{ and hence } |E^\times| \leq |A||T||H|. \text{ Thus, } G^\times \text{ can be constructed in } \mathcal{O}(|V||T||H| + |A||T||H|). \text{ Using Dijkstra’s algorithm we compute a shortest path tree rooted at } (s, q_0) \text{ in } G^\times. \text{ In particular, we obtain a shortest } (s, q_0), (t, q)-\text{path for all } q \in Q = T \times H. \text{ Using a Fibonacci heap, Dijkstra’s algorithm has a running time of } \mathcal{O}(|V^\times| \log(|V^\times|) + |E^\times|). \text{ We can again extract the Pareto-set in } \mathcal{O}(|T||H| \log(|T||H|)) \text{ time, giving an overall runtime of } \mathcal{O}(|V||T||H| \log(|V||T||H|) + |A||T||H||B).}
\end{align*}
\]

This result extends naturally to FIFO-travel time functions.

**Theorem 5.2** (PSEAP over finite monoids is polynomial time solvable in \(|H|\)). Consider PSEAP over the conditional fare network \(\mathcal{N} = (T, \Gamma, H, S, w, \mu, \pi)\) under the assumption that \(|H| < \infty\), that the travel time functions \(c(a) : I \rightarrow I, a \in A\) have the FIFO-property and can be evaluated in constant time, and that the evaluation time of \(\Gamma\) can be bounded by a polynomial \(B\). Then, PSEAP can be solved in \(\mathcal{O}(|V||T||H| \log(|V||T||H|) + |A||T||H||B)\).

**Proof.** It is well-established that time-dependent shortest path problems can be solved using a modified version of Dijkstra’s algorithm for FIFO-networks (Orda and Rom 1990). The modified algorithm exhibits the same running time as the standard algorithm if the evaluation time of travel time functions is bounded by a constant. Thus, using the modified Dijkstra-variant in Lemma 5.1 solves PSEAP with FIFO travel time functions in \(\mathcal{O}(|V||T||H| \log(|V||T||H|) + |A||T||H||B)\).

\[
\begin{align*}
\text{6 Price-Sensitive RAPTOR}
\end{align*}
\]

In this section, we will discuss how to use conditional fare networks to implement a price-sensitive version of the multi-criteria RAPTOR algorithm (abbr. RAP and McRAP, respectively) (Delling, Pajor, and Werneck 2015). Since RAP implicitly optimizes the number of trips, we obtain an multi-criteria algorithm that optimizes for travel time, number of trips and
price. So far, we presented our framework in a graph-based context. RAP, however, does not use a graph model but works directly on the timetable. The adaption for RAP is, however, straightforward.

This chapter is structured as follows. A review of the RAP-algorithm is provided in Section 6.1. Then, in Section 6.2, we lay out how McRAP can be modified to use conditional fare networks for price optimization. In Section 6.3, we shortly recap how improvements in running times can be achieved by only calculating a restricted Pareto-set using the recently introduced Bounded-RACTOR-algorithm (BMRAP) (Delling, Dibbelt, and Pajor 2019). The algorithm excludes all journeys that need significantly more transfers or take significantly more time than the journeys found with a price-insensitive RAP query. Finally in Section 6.4 we introduce two speed-up techniques that are tailored to our application.

Sections 6.1 and 6.3 give succinct summaries of the RAP and BMRAP algorithms. For a thorough presentation, see the original research in (Delling, Pajor, and Werneck 2015) and (Delling, Dibbelt, and Pajor 2019), respectively.

6.1 Multi-Criteria Search with McRAP

We largely follow the notation introduced in (Delling, Pajor, and Werneck 2015). In contrast to the original authors, we also consider change times at stops. The necessary modifications, however, are minuscule. The RAP algorithm does not use a graph model but works directly on the timetable. A timetable is a tuple \( (\Pi, \mathcal{P}, \mathcal{R}, \mathcal{D}, \mathcal{F}) \) consisting of a period of operation \( \Pi \), a set of stops \( \mathcal{P} \), a set of routes \( \mathcal{R} \), a set of trips \( \mathcal{D} \) and a set of footpaths \( \mathcal{F} \). A stop \( p \in \mathcal{P} \) is a location where a vehicle can be boarded or exited. Each stop \( p \) has a (possibly zero) transfer time \( \tau_{ch}(p) \in \mathbb{N} \) that is applied whenever a vehicle is boarded at \( p \). A trip is a sequence of stops together with arrival and departure times \( \tau_{arr}(t, p), \tau_{dep}(t, p) \in \Pi \). A route is a set of trips, where all trips have the same sequence of stops and no trip overtake another one. We denote the set of stops of a route \( r \) by \( \mathcal{P}(r) \). Finally, a footpath \((p_1, p_2, l)\) is a pair of stops \((p_1, p_2)\) combined with a walking time \( l \).

RAP operates in rounds \( k = 1, \ldots, K \) on \( \Pi \). It maintains arrival time labels \( \tau_{arr}(k, p) \) for every \( p \in \mathcal{P} \) and every round \( k = 1, \ldots, K \). Every entry of \( \tau_{arr} \) is initialized to \( \infty \). Each round begins with a set of marked stops. All routes touching these stops are collected and then processed in an arbitrary order. When processing a route \( r \in \mathcal{R} \), RAP begins with the first marked stop and from there on iterates the stops in order of travel. For each stop \( p \in \mathcal{P}(r) \), RAP finds the earliest trip \( t \) that can be taken at \( p \) after \( \tau_{arr}(k - 1, p) + \tau_{ch}(p) \). In the same sweep, the arrival times of \( t \) are used to update \( \tau_{arr}(k, p) \). All stops whose arrival times improved over \( \tau_{arr}(k - 1, p) \) are marked.

In a second step, a footpath search is performed. All footpaths \((p_1, p_2, l)\) starting in a marked stop are processed in an arbitrary order and labels in \( \tau_{arr}(\cdot, p_2) \) are updated accordingly. Again, each stop \( p \) with an improved arrival time \( \tau_{arr}(k, p) \) is marked for the next round. Note that this requires the footpath set to be transitively closed. The algorithm terminates after \( K \) rounds or when no more stops can be marked.

RAP can be modified slightly to allow for multi-criteria search. Instead of only the arrival times \( \tau_{arr}(k, p) \), McRAP now maintains a label bag \( B(k, p) \) for every stop \( p \) and round \( k \). Each label \( L \in B(k, p) \) contains an entry for every optimization criterion. When processing a route \( r \in \mathcal{R} \) at a starting stop \( p_s \), a route bag \( B_r \) is created and all labels from \( B(k - 1, p_s) \) are updated with \( \tau_{ch}(p) \) and copied into \( B_r \). Each label in \( B_r \) is associated with a trip \( t \in r \). At each stop \( p \in \mathcal{P}(r) \) after \( p_s \), McRAP updates all labels in \( B_r \), merges \( B_r \) into \( B(k, p) \) and finally merges all labels from \( B(k - 1, p) \) into \( B_r \) and assigns a trip to them. In each step, dominated labels are removed.
6.2 Using Conditional Fare Networks in McRAP

Adapting McRAP to incorporate fares is now fairly straightforward but requires several modifications to $T$.

For every trip $t \in D$, we additionally store two fare attributes for each stop $p \in t$. The first $w_1(t, p) \in W$ is considered when reaching the stop $p$ while iterating along $t$. The second $w_2(t, p) \in W$ is picked up when boarding $t$ at the stop $p$. This distinction is necessary since there is no direct equivalent in $T$ to the transfer arcs used in the graph-based setting to model, e.g., surcharges. Furthermore, the definition of a route needs a slight adjustment: A route $r \in R$ is a set of trips, where all trips have the same sequence of stops and the same fare attributes, i.e., $w_2(t_1, p) = w_2(t_2, p)$ and $w_1(t_1, p) = w_1(t_2, p)$ for all $t_1, t_2 \in r$ and $p \in P(r)$ and no trip overtakes another one.

Now, a label $L = (\tau, f)$ in a label bag $B(k, p)$ consists of an arrival time $\tau$ and a fare state $f$. Labels in route bags additionally hold the current trip $t$. When updating a label $L = (\tau, f, \tau)$ from route bag $B_r$ at a stop $p$, the arrival time $\tau$ is updated to $\tau_{arr}(t, p)$ and the fare state is updated to $Up(f, w_1(t, p))$. Update steps that are associated with transfers are performed whenever labels are merged into $B_r$. Here, $w_2(t, p)$ is used for updating. Dominance of labels is checked according to the theory developed in Chapter 4 while also taking arrival times into account. Since walking is usually free of charge, fare states do not need to be updated in the footpath stage. Hence, footpaths are also not enriched with fare attributes.

Using McRAP, we obtain the Pareto-set $J^f$, optimizing for arrival time, number of trips and fare state. The smaller set $J^* \subseteq J^f$, optimizing for price instead of fare state, can be calculated in a post-processing step.

Depending on the data set, it might be possible to save money by walking a long distance in the middle of a journey. This kind of journey, too, can be excluded during postprocessing.

6.3 Restricted Pareto-Sets

By design of fare systems the cheapest path is often among the fastest as detours are penalized by increases in price, arrival time and transfers. At other times a negligible reduction in price might be achievable at the expense of a significant increase in travel time. Such journeys are unlikely to be chosen by a traveller. Hence, it appears beneficial for a price-sensitive search to prune all labels that are worse by a certain margin (regarding both arrival time and transfers) than the results of a normal RAP query. This is achieved by the following pruning schemes first introduced in [Delling, Dibbelt, and Pajor 2019] for general multi-criteria search with RAP.

Let $J_A$ be the Pareto-set of all anchor journeys found with a RAP query, i.e. optimizing only for arrival time and number of transfers. We denote the arrival time of a journey $J$ by $\tau_{arr}(J)$ and its number of trips by $\text{tr}(J)$. We aim to calculate a restricted Pareto-set $J_R$ with $J_A \subseteq J_R \subseteq J^f$ of journeys that do not have a significantly higher arrival time or number of trips than some journey from $J_A$. Let $\sigma_{arr} \in \mathbb{R}^+$ and $\sigma_{tr} \in \mathbb{N}^+$ be the maximal acceptable slacks for arrival time and number of transfers, respectively. Then, we define

$$J_R = \{J \in J^f \mid \exists J_A \in J_A \text{ such that } \tau_{arr}(J) \leq \tau_{arr}(J_A) + \sigma_{arr} \text{ and } \text{tr}(J) \leq \text{tr}(J_A) + \sigma_{tr}\}.$$

To obtain a two-stage pruning scheme, we can first run a normal RAP query. The labels obtained in this first stage can then be used to prune the multi-criteria search. Let $\tau_k$ be the optimal arrival time at the target stop $p_t$ in round $k$ of the first stage (computed with RAP). During round $k$ of McRAP, we prune every label that has an arrival time $\tau$ with $\tau > \tau_k + \sigma_{arr}$. This pruning scheme is called Target-BMRAP. Note that Target-BMRAP works on the assumption that $\sigma_{tr} = \infty$. Also, the bound is not tight for all stops other than $p_t$.

The set $J_R$ can be computed with the more involved Tight-BMRAP. Here, three rounds are performed. As for Target-BMRAP, we first perform a normal RAP search, obtaining the Pareto-set $J_A$. Then, multiple reverse RAP queries are performed to build bounds at all stops.
Figure 5: Illustration of solution space of a Tight-BMRAP search with $\sigma_{arr} = 30$ min and $\sigma_{tr} = 1$. The circle marks represent the anchor journeys from $J_A$. Tight-BMRAP prunes all journeys to the right of the dotted line spanned by those journeys. The area to the left of the dashed line contains no Pareto-optimal journeys. The journeys from $J_f$ that fall into the area enclosed by the dashed and dotted lines form $J_R$. The light gray area forms $\bar{J_R}$. Note that the journey marked by the square mark is in $J_R$ but not in $\bar{J_R}$. The journey represented by the triangle mark is in $\bar{J_R}$ even though it uses more trips and has a later arrival time.

$p \in \mathcal{P}$. The third stage is the actual McRAP round using the previously computed bounds for pruning.

In the following, we described the second and third stage in more detail. Let $m := K + \sigma_{tr}$, where $K$ is the maximum number of trips in any journey $J_A \in J_A$. A backward RAP search with starting time $\tau_{arr}(J_A) + \sigma_{arr}$ and $n_{J_A} := \text{tr}(J_A) + \sigma_{tr}$ rounds is performed for every journey $J_A \in J_A$. The backward search works on departure times instead of arrival times and transfer times are not applied when boarding a vehicle but instead when disembarking.

Each backward search computes latest departure times $\tau_{dep}(J_A, k, p)$ such that $p_t$ can still be reached earlier than $\tau_{arr}(J_A) + \sigma_{arr}$ while using at most $k$ more trips onward from $p$. It is possible that $\tau_{dep}(J_A, k, p)$ remains at its initialization value of $-\infty$. Hence, using $\tau_{dep}(J_A, n_{J_A} - k, p)$ for pruning labels in round $k$ of a forward McRAP search computes $\{ J \in \mathcal{J}_f \mid \tau_{arr}(J) \leq \tau_{arr}(J_A) + \sigma_{arr} \text{ and } \text{tr}(J) \leq \text{tr}(J_A) + \sigma_{tr} \}$.

Carefully overlapping the $\tau_{dep}(J_A, k, p)$ labels results in a set of labels $\tau_{dep}(k, p)$ with $k = 1, \ldots, m$, i.e.,

$$\tau_{dep}(k, p) := \max_{J_A \in J_A: \ k \geq m - n_{J_A}} \{ \tau_{dep}(J_A, k - m + n_{J_A}, p) \}.$$

The third stage is now a normal McRAP search enriched with two pruning rules. Let $k$ be the current round.

- Labels $L$ are not merged into to $B(k, p)$ at stop $p$ if $\tau_{arr}(L) > \tau_{dep}(m - k, p)$,
- Labels $L$ can be removed from $B_r$ at stop $p$ if $\tau_{arr}(L) > \tau_{dep}(m - k + 1, p) + \tau_{ch}(p)$.

The summand $\tau_{ch}(p)$ in the second rule is required since it was already factored into $\tau_{dep}(m - k + 1, p)$ and no transfer is performed. This third stage computes exactly $\mathcal{J}_R$. Again, a post-processing step is necessary to remove all journeys from $\mathcal{J}_R$ that are not Pareto-optimal with regard to price.

Note that the definition we gave for $\mathcal{J}_f$ differs from the one given in [Delling, Dibbelt, and Pajor 2019]. The original authors assign each journey $J \in \mathcal{J}_f$ its anchor journey $J_A \in J_A$, that
has the maximum number of trips smaller or equal to \( \text{tr}(J) \). They then define the restricted Pareto-set \( \bar{J}_R \) to be
\[
\bar{J}_R := \{ J \in J^f \mid \text{its anch. journ. } J_A \in J_A \text{ has } \tau_{\text{arr}}(J) \leq \tau_{\text{arr}}(J_A) + \sigma_{\text{arr}} \text{ and } \text{tr}(J) \leq \text{tr}(J_A) + \sigma_{\text{tr}} \}
\]
\[
\subseteq \{ J \in J^f \mid \exists J_A \in J_A \text{ such that } \tau_{\text{arr}}(J) \leq \tau_{\text{arr}}(J_A) + \sigma_{\text{arr}} \text{ and } \text{tr}(J_A) + \sigma_{\text{tr}} \}
\]
\[
\subseteq J_R.
\]

Note that Tight-BMRAP does in fact calculate \( J_R \) and not \( \bar{J}_R \) and that \( J_R \) is a more interesting set to compute as it does not seem beneficial to cut off journeys that use less trips than their respective anchor journeys. The difference between \( J_R \) and not \( \bar{J}_R \) is visualized in Figure 5. The first fact can be seen easily by considering the case where \( \sigma_{\text{arr}} = \infty \). Then, the pruning scheme does not prune anything but all journeys with more than \( m \) trips. Specifically, no lower bound on the number of trips is applied.

6.4 Speed-Up Techniques

**Price-Based Target Pruning** In RAP as well as Dijkstra’s algorithm, it is possible to use target pruning (Delling, Pajor, and Werneck 2015) to delete labels that are worse than the labels that have already been found at the target stop. Naturally, the same speed-up technique is also possible for our algorithm. Moreover, we need not use \( \leq_C \) to compare fare states. Since the labels at the target stop are never updated and the price function \( \pi \) is non-decreasing, a partial journey already more expensive than the incumbent cheapest journey cannot be price-optimal. Hence, in round \( k \) of McRAP, we can prune all labels with a fare state \( f \) with \( \pi(f^t) \geq \pi^* \), with \( \pi^* \) being the best price at the target stop with at most \( k \) trips. We refer to this technique as **Price-Based Target Pruning (PTP)**.

**Fare Specific Speed-ups** Certain dimensions in \((H, +, \leq)\) might only be relevant for some tickets in \( T \). For example, many short-distance tickets depend on the number of stops visited while this number is irrelevant for all other tickets that can be reached from that ticket. We can therefore alter the comparison operator \( \leq_C \) for those tickets to ignore the number of stops. Hence, more labels become comparable which results in a smaller Pareto-set \( J^{f_{\text{ss}}} \) with \( J^* \subset J^{f_{\text{ss}}} \subset J^f \). When using Tight-BMRAP, this results in a set \( J^{f_{\text{ss}}}_R \) with \( J^*_R \subset J^{f_{\text{ss}}}_R \subset J^f_R \). We refer to this technique as **fare specific speed-up (FSS)**.

7 Computational Results

We implemented the McRAP algorithm in C++17 compiled with gcc 9.3.0 and \(-03\) optimization. All tests were conducted on Dell Poweredge M620 machines with 64 GB of RAM. While the general structure of the MDV price system is captured in our model, our computations deviate from the prices charged by MDV in the following two cases: A list of relations that is, contrary to the general rules, not eligible for the short-distance discount is not taken into account. Also, stops and fare zones that a route passes through without stopping are not represented in the available data and therefore cannot be taken into account.

Our dataset was built from two sources: first, the publicly available timetable data of MDV (available under [https://www.mdv.de/informationen/downloads/](https://www.mdv.de/informationen/downloads/)) and licensed under [CC BY 4.0](https://creativecommons.org/licenses/by/4.0/). Fare data is not part of the feed and was obtained separately via InfraDialog GmbH. The structure of the fare network was extracted from the publicly available fare regulations of MDV.

From the GTFS feed, we extracted a timetable spanning two days from 01.07.2019 to 02.07.2019. The resulting timetable contains 4371 stops, 36670 trips, 5576 routes and 845 footpaths. This original footpath set was not transitively closed. Since RAP requires a transitively closed footpath set (Delling, Pajor, and Werneck 2015), we computed its transitive closure.
## Table 3: Computational Results. Evaluation of different RAP-variants on the MDV dataset.

All experiments were conducted with a maximum of seven rounds. For each algorithm, the table reports the optimization criteria, the speed-up techniques employed and the arrival and trip slacks if applicable. We report the number of scanned routes (#Scan), the average running time (Time), the number of rounds performed (#Rounds), the number of journeys found (#Jn.) and the number of journeys that are dominated w.r.t. to the price (#PJn.). For each result, both the average and standard deviation are reported. The algorithms in rows 2-17 run at least one RAP and exactly one McRAP query. In this case, #Scan and Time are only given for the McRAP run, while the running time is summed up overall RAP and McRAP invocations.

| Criteria | Speed-Up | Slack | #Scan | Time[ms] | #Rounds | #Jn. | #PJn. |
|----------|----------|-------|-------|----------|---------|------|-------|
|          | trips    | time  | zones | fare     | PTP     | art  | trip  |
|          | Avg.     | Sd.   | Avg.  | Sd.      | Avg.    | Sd.  | Avg.  |
| RAP      |          |       |       |          |         |      |       |
| McRAP    | ● ● o o o o ● - | 17295 5883 3.27 1.31 6.64 0.79 1.53 0.66 - - |
| McRAP    | ● ● o o o o ● - | 29881 6118 4675 6172 6.92 0.48 47.00 45.95 - - |
| McRAP    | ● o o o o o ● - | 29891 5025 957.54 260.73 6.97 0.26 10.05 13.27 2.62 1.43 |
| McRAP    | ● o o o o o ● - | 23460 8263 243.09 283.08 6.86 0.55 3.00 1.80 2.62 1.43 |
| Target-BMRAP | ● o o o o o ● - | 20132 8502 71.16 87.10 6.72 0.72 1.72 0.82 1.66 0.74 |
| Target-BMRAP | ● o o o o o ● - | 20637 8412 74.86 88.75 6.75 0.70 1.80 0.89 1.72 0.80 |
| Target-BMRAP | ● o o o o o ● - | 29891 5025 957.54 260.73 6.97 0.26 10.05 13.27 2.62 1.43 |
| Target-BMRAP | ● o o o o o ● - | 23460 8263 243.09 283.08 6.86 0.55 3.00 1.80 2.62 1.43 |
| Target-BMRAP | ● o o o o o ● - | 20132 8502 71.16 87.10 6.72 0.72 1.72 0.82 1.66 0.74 |
| Target-BMRAP | ● o o o o o ● - | 20637 8412 74.86 88.75 6.75 0.70 1.80 0.89 1.72 0.80 |
| Target-BMRAP | ● o o o o o ● - | 29891 5025 957.54 260.73 6.97 0.26 10.05 13.27 2.62 1.43 |
| Target-BMRAP | ● o o o o o ● - | 23460 8263 243.09 283.08 6.86 0.55 3.00 1.80 2.62 1.43 |
| Tight-BMRAP | ● o o o o o ● - | 15 1 3192 1963 8.44 14.46 5.17 1.08 7.52 1.63 0.72 |
| Tight-BMRAP | ● o o o o o ● - | 30 1 3551 2071 9.40 16.47 5.17 1.08 9.06 12.29 1.68 0.78 |
| Tight-BMRAP | ● o o o o o ● - | 60 1 4316 2384 11.97 18.17 5.17 1.08 13.29 16.02 1.78 0.86 |
| Tight-BMRAP | ● o o o o o ● - | 15 2 4606 3167 20.79 260.73 5.84 0.70 11.90 21.05 1.66 0.74 |
| Tight-BMRAP | ● o o o o o ● - | 30 2 5147 3154 22.75 283.08 5.90 0.70 14.05 23.87 1.71 0.80 |
| Tight-BMRAP | ● o o o o o ● - | 60 2 6487 3359 29.47 107.53 5.90 0.70 20.98 29.70 1.83 0.88 |
| Tight-BMRAP | ● o o o o o ● - | 15 1 2954 1828 6.22 3.14 5.17 1.08 1.68 0.79 1.63 0.72 |
| Tight-BMRAP | ● o o o o o ● - | 30 1 3274 1919 6.69 3.29 5.17 1.08 1.75 0.85 1.69 0.78 |
| Tight-BMRAP | ● o o o o o ● - | 60 1 4316 2384 11.97 18.17 5.17 1.08 13.29 16.02 1.78 0.86 |
| Tight-BMRAP | ● o o o o o ● - | 15 2 4606 3167 20.79 260.73 5.84 0.70 11.90 21.05 1.66 0.74 |
| Tight-BMRAP | ● o o o o o ● - | 30 2 5147 3154 22.75 283.08 5.90 0.70 14.05 23.87 1.71 0.80 |
| Tight-BMRAP | ● o o o o o ● - | 60 2 6487 3359 29.47 107.53 5.90 0.70 20.98 29.70 1.83 0.88 |

and obtained 1,029 footpaths. We then chose a test set of 5,000 origin-destination pairs (OD-pairs) uniformly at random from the set of stops. After removing all OD-pairs that were not connected in the time interval starting at 08:00 a.m on the 01.07.2019, a total of 4964 OD-pairs remained.

As of 2019, the fare system of MDV contained 56 fare zones, 17 town zones and 30 overlap areas containing 191 stops. Overlap areas were implemented by route duplication as lined out in Section 3.4. After route duplication, the timetable contained 49072 trips in 7835 routes. All queries were performed with a starting time of 08:00 am.
Table 4: Computational Results. Evaluation of different RAP-variants on the MDV dataset.

All experiments were conducted with a maximum of twenty-five rounds. The maximum number of rounds performed across all algorithms was twenty-one. Hence, no query was cancelled prematurely. All algorithms other than RAP optimized for arrival time, number of trips and fare state. For all of those both FFS and PTP were activated. The same performance indicators as in Table 3 are reported. For Target-BMRAP and Tight-BMRAP, #Scan and #Rounds report only on the last McRAP call whereas Time reports the overall run time.

transfers a traveller is willing to undertake. Results of the experiment are reported in Table 3. A standard RAP run takes on average 3.27 ms with a standard deviation of 1.31 ms. The computed Pareto-set contains 1.53 journeys on average. If only fare zones are considered as additional operation criterion, the average runtime increases to 4.67 s with a standard deviation of 6.17 s. This indicates that run times of up to around 10 s are not out of the ordinary. Since McRAP optimizing for fare state would calculate even larger Pareto-sets, we did not include it in the experiments. While FSS in McRAP alone does not suffice to obtain acceptable run times, the combination of FSS and PTP produces an average run time of 243.09 ms. Although this performance is no longer prohibitive for practical application, the standard deviation remains high at 283 ms.

The Pareto-set computed with McRAP contains 3 journeys on average of which 2.62 are also price-optimal. Turning off PTP increases the number of computed journeys to 10.5; the Pareto-set of the zone-based McRAP contains 47 journeys on average. Hence, even though all tickets of MDV are in the full-comparability set $C_F$, a high number of superfluous journeys will be generated when no additional techniques are employed. This effect is mainly caused by the fare zones as they form an only partially ordered set. The 2.62 price-optimal journeys mark an increase of 71 % over the 1.53 journeys found with RAP. It might, however, contain journeys with an undesirable trade-off between arrival time and number of trips and price. To obtain restricted Pareto-sets with a reasonable trade-off, we ran Target-BMRAP and Tight-BMRAP with arrival time slacks of 15 min, 30 min and 60 min and in the case of Tight-BMRAP with trip slacks of 1 or 2. Using Target-BMRAP reduces run times to between 71.16 ms and 81.45 ms while restricting the size of the Pareto-set to between 1.66 and 1.85. This corresponds to between 8.5 % and 21 % more journeys compared to RAP. We ran Tight-BMRAP both with and without PTP and FSS. Tight-BMRAP performs reasonably well even without PTP and FSS on average with run times of up to 29.47 ms. However, a comparatively high standard deviation of up to 107.53 ms hints at high performance variability. When using both PTP and FSS run times decrease to at most 10.68 ms. Even more pronounced is the decrease in the standard deviation to at most 6.73 ms. Compared to RAP there are between 6.5 % and 19.6%.

| Slack | #Scan | Time[ms] | #Rounds | #Jn. | #PJn. |
|-------|-------|----------|---------|------|-------|
|       | Avg.  | Sd.      | Avg.    | Sd.  | Max   | Avg.  | Sd.  |
| RAP   | –     | 17496    | 6111    | 3.28 | 1331.9| 7.31  | 1.35 |
| McRAP | –     | 32757    | 16429   | 389.52| 710.75| 11.10 | 3.24 |
| Target-BMRAP 15 | – | 23007 | 12106 | 82.47 | 125.89 | 8.57 | 2.34 |
| Target-BMRAP 30 | – | 23724 | 12093 | 86.38 | 128.22 | 8.73 | 2.33 |
| Target-BMRAP 60 | – | 24980 | 12186 | 94.53 | 131.83 | 9.00 | 2.35 |
| Tight-BMRAP 15 1 | 2985 | 1871 | 6.68 | 3.28 | 5.20 | 1.14 | 9 | 1.69 | 0.81 | 1.63 | 0.73 |
| Tight-BMRAP 30 1 | 3309 | 1973 | 7.19 | 3.44 | 5.20 | 1.14 | 9 | 1.75 | 0.87 | 1.69 | 0.79 |
| Tight-BMRAP 60 1 | 4039 | 2307 | 8.36 | 3.89 | 5.20 | 1.14 | 9 | 1.89 | 0.96 | 1.79 | 0.89 |
| Tight-BMRAP 15 2 | 4360 | 3145 | 9.02 | 7.27 | 5.52 | 1.16 | 10 | 1.72 | 0.83 | 1.66 | 0.76 |
| Tight-BMRAP 30 2 | 4833 | 3151 | 9.68 | 7.43 | 5.58 | 1.15 | 10 | 1.79 | 0.90 | 1.72 | 0.81 |
| Tight-BMRAP 60 2 | 6071 | 3378 | 11.87 | 7.97 | 5.67 | 1.16 | 10 | 1.94 | 1.02 | 1.84 | 0.90 |
% more journeys. Consequently, conditional fare networks used within Tight-BMRAP appear well-suited to provide the user with price-optimized alternative routes while increasing run times only insignificantly.

A second experiment was conducted without an upper bound on the number of rounds. The table $\tau_{arr}$ was implemented as a fixed-size array with space for 25 rounds. Since the maximum number of rounds performed was 21, no journeys were cut off due to early termination. We excluded all algorithms that already performed poorly in the first experiment. Namely, these are the Tight-BMRAP-variants without additional speed-up techniques, McRAP for fare zones and McRAP for fare states without price-based target-pruning.

For McRAP, we see a significant increase of the run time of about 60 % and a even more pronounced increase of its standard deviation of 151 % compared to the variant with only seven rounds. All variants of Target-BMRAP exhibit similar behaviour, albeit to a lesser degree. Here, the average run time increased by about 16 % and the standard deviation by about 44 %. Note that for RAP the run time increased by only a marginal 0.01 ms and that when compared to RAP McRAP needs to perform significantly more rounds.

For all settings of slack variables for Tight-BMRAP, the increase of the run time remains minimal at around a millisecond. Hence, Tight-BMRAP remains highly competitive whereas the simple McRAP implementation suffers from considerably degraded performance. It is furthermore noteworthy that in all slack settings the multi-criteria part of Tight-BMRAP needs to both scan significantly less routes and perform fewer rounds than even the standard RAP. This clearly speaks to the strength of the pruning scheme used in Tight-BMRAP.

8 Conclusion

We presented a novel framework for modeling fare systems of public transportation companies. It is independent of the shortest path algorithm used and can be used to solve price-sensitive earliest arrival queries in real-world networks. The often complicated structure of fare regulations requires the optimization of several criteria. In our test case MDV, these were the actual ticket, fare zones, the number of stops visited and the number of kilometers driven. Naively optimizing for all these criteria results in slow run times with a high variance and the computation of many journeys that are not price-optimal. Using the speed-up techniques FSS and PTP mitigate these effects to some degree. Their effectiveness, however, is most likely highly dependent on the specific problem instance. Even though, run times of up to half a second are nothing out of the ordinary.

However, the combination of the speed-ups with the Tight-BMRAP-algorithm was clearly a success. Run times are reduced to at most 12 ms with low variance while still computing a reasonably-sized restricted Pareto-set when choosing appropriate arrival and trip slacks.

Fare regulations can differ quite significantly between public transportation companies. Hence, a systematic evaluation of the framework on other public transit networks is certainly worthwhile. As MDV operates in a largely rural area with two only medium-sized urban centers a study of larger urban centers such as Berlin or Madrid seems especially interesting. However, while timetables are widely available, fare data is not. Machine-readable mappings from stations to fare zones are in general not publicly available. Even when they are, the data is often incomplete and requires a significant manual polishing effort. This places such an effort outside the scope of this paper and leaves it as a direction for future research.

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