Extension of a Borel subalgebra symmetry into the $sl_2$ loop algebra symmetry for the twisted XXZ spin chain at roots of unity and the Onsager algebra *

Tetsuo Deguchi

Department of Physics, Ochanomizu University
2-1-1 Ohtsuka, Bunkyo-ku, Tokyo 112-8610, Japan

Abstract

We discuss a conjecture that the twisted transfer matrix of the six-vertex model at roots of unity with some discrete twist angles should have the $sl(2)$ loop algebra symmetry. As an evidence of this conjecture, we show the following mathematical result on a subalgebra of the $sl(2)$ loop algebra, which we call a Borel subalgebra: any given finite-dimensional highest weight representation of the Borel subalgebra is extended into that of the $sl(2)$ loop algebra, if the parameters associated with it are nonzero. Thus, if operators commuting or anti-commuting with the twisted transfer matrix of the six-vertex model at roots of unity generate the Borel subalgebra, then they also generate the $sl(2)$ loop algebra. The result should be useful for studying the connection of the $sl(2)$ loop algebra symmetry to the Onsager algebra symmetry of the superintegrable chiral Potts model.

1 Introduction

Spectral properties of the XXZ spin chain under the twisted boundary conditions have attracted much attention in mathematical physics and condensed matter physics [1, 2, 3, 4, 5]. The XXZ Hamiltonian on a ring of $L$ sites is given by

$$
\mathcal{H}_{XXZ} = J \sum_{j=1}^{L} \left( \sigma_j^X \sigma_{j+1}^X + \sigma_j^Y \sigma_{j+1}^Y + \Delta \sigma_j^Z \sigma_{j+1}^Z \right).
$$

(1)

where $\sigma_j^\alpha$ ($\alpha = X, Y, Z$) are the Pauli matrices defined on the $j$th site, and they satisfy the twisted boundary conditions:

$$
\sigma_{L+1}^\pm = \exp(\pm i\phi)\sigma_1^\pm, \quad \sigma_{L+1}^Z = \sigma_1^Z.
$$

(2)

*Talk given at the workshop RAQIS, LAPTH, Annecy, France, September 11-14, 2007
We call the parameter $\phi$ the twist angle. When $\phi = 0$, conditions (2) reduces to the periodic boundary conditions. We define parameter $q$ by $\Delta = (q + q^{-1})/2$. We also introduce twist parameter $q$ by $q^2 = \exp(i\phi)$. (3)

It has been shown that when $q$ is a root of unity the XXZ spin chain under the periodic boundary conditions commutes with the $\mathfrak{sl}_2$ loop algebra, $U(L(\mathfrak{sl}_2))$ [6]. (See also, [7, 8, 9, 10, 11].) Through the similar derivation in terms of the Temperley-Lieb algebra as given in [6], it was shown that the twisted XXZ spin chain at roots of unity commutes with the $\mathfrak{sl}_2$ loop algebra for $\phi = \pi$, i.e. under the anti-periodic boundary conditions [12]. It was also shown that when $q$ is a root of unity such as $q^{2N} = 1$ and $\varphi$ is an integer, there exist some operators commuting or anti-commuting with the twisted transfer matrix of the six-vertex model [12]. Furthermore, it was pointed out by Korff that in some sectors such operators generate a subalgebra $U(\mathcal{B}_0)$ of the $\mathfrak{sl}_2$ loop algebra $U(L(\mathfrak{sl}_2))$, which we call a Borel subalgebra [13]. Let $x^\pm_{m}$ and $h_n$ for $m,n \in \mathbb{Z}$ be the generators of the $\mathfrak{sl}_2$ loop algebra $U(L(\mathfrak{sl}_2))$. Then, the Borel subalgebra $U(\mathcal{B}_0)$ is generated by the following operators: $x^+_k, h_k, x^-_k$ for $k = 0, 1, \ldots$, and $x^-_k$ for $k = 1, 2, \ldots$.

In the paper we show a mathematical result that every highest weight representation of the Borel subalgebra $U(\mathcal{B}_0)$ is extended into that of the $\mathfrak{sl}_2$ loop algebra if the parameters associated with the representation are nonzero. It follows from the result that if the twisted transfer matrix has the Borel subalgebra symmetry, then it has also the $\mathfrak{sl}_2$ loop algebra symmetry. We thus give a conjecture that the $\mathfrak{sl}_2$ loop algebra is generated by the operators constructed in [12] which commute or anti-commute with the twisted transfer matrix of the six-vertex model at roots of unity. Here we note that Benkart and Terwilliger have shown that the action of $U(\mathcal{B}_0)$ on a finite-dimensional irreducible $U(\mathcal{B}_0)$ module extends uniquely to an action of $U_q(L(\mathfrak{sl}_2))$ on it [14]. The mathematical result in the paper is new for reducible highest weight representations of $U(\mathcal{B}_0)$. We also discuss construction of generators of the Onsager algebra from a highest weight representation of the $\mathfrak{sl}_2$ loop algebra. The result should be useful for investigating the connection of the $\mathfrak{sl}_2$ loop algebra to the Onsager algebra symmetry of the super-integrable chiral Potts model [15]. Quite recently in an independent research [16], eigenvectors of the superintegrable model associated with the superintegrable chiral Potts model have been studied by making use of the $\mathfrak{sl}_2$ loop algebra symmetry of some XXZ spin chain. They should be closely related to Ref. [15], and some results of the present paper should also be relevant.

The content of the paper consists of the following: In section 2, we review the infinite-dimensional symmetries of the twisted transfer matrix of the six-vertex model at roots of unity. In particular, we review operators commuting or anti-commuting with the twisted transfer matrix at roots of unity. In some sectors they generate the Borel subalgebra. In section 3, we show that any given highest weight representation of the Borel subalgebra is extended to that of the $\mathfrak{sl}_2$ loop algebra if the associated parameters are nonzero. In section 4, we summarize some results on the infinite-dimensional symmetry of the twisted transfer matrix of the six-vertex model at roots of unity, and then suggest a conjecture that
the twisted transfer matrix of the six-vertex model at roots of unity should have the $sl_2$ loop algebra symmetry. In section 5, we give a method for constructing a representation of the Onsager algebra from a finite-dimensional highest weight representation of the $sl_2$ algebra.

2 Infinite dimensional symmetry of the twisted XXZ spin chain

2.1 Definition of the twisted transfer matrix

In order to formulate the twisted transfer matrix of the six-vertex model, we review some formulas of the algebraic Bethe ansatz. The $R$ matrix of the XXZ spin chain is defined by

$$R(z-w) = \begin{pmatrix} f(w-z) & 0 & 0 & 0 \\ 0 & g(w-z) & 1 & 0 \\ 0 & 1 & g(w-z) & 0 \\ 0 & 0 & 0 & f(w-z) \end{pmatrix}$$

where $f(z-w)$ and $g(z-w)$ are given by

$$f(z-w) = \frac{\sinh(z-w-2\eta)}{\sinh(z-w)}, \quad g(z-w) = \frac{\sinh(-2\eta)}{\sinh(z-w)}.$$  

We introduce $L$ operators for the XXZ spin chain

$$L_n(z) = \begin{pmatrix} L_n(z)_{11} & L_n(z)_{12} \\ L_n(z)_{21} & L_n(z)_{22} \end{pmatrix} = \begin{pmatrix} \sinh(z I_n + \eta \sigma_n^Z) & \sinh 2\eta \sigma_n^- \\ \sinh 2\eta \sigma_n^+ & \sinh(z I_n - \eta \sigma_n^Z) \end{pmatrix}$$

Here $I_n$ and $\sigma_n^a$ ($n = 1, \ldots, L$) are acting on the $n$th vector space $V_n$. We recall that $\sigma^\pm$ denote $E_{12}$ and $\sigma^- = E_{21}$, and $\sigma^X, \sigma^Y, \sigma^Z$ the Pauli matrices. In terms of the $R$ matrix and $L$ operators, the Yang-Baxter equation is expressed as

$$R(z-w)(L_n(z) \otimes L_n(w)) = (L_n(w) \otimes L_n(z)) R(z-w)$$

We define the monodromy matrix $T(z)$ by $T(z) = L_L(z) \cdots L_2(z)L_1(z)$. The monodromy matrix satisfies the Yang-Baxter equations

$$R(z-w) (T(z; \{\xi_n\}) \otimes T(w; \{\xi_n\})) = (T(w; \{\xi_n\}) \otimes T(z; \{\xi_n\})) R(z-w)$$

Let us denote the matrix elements of $T(z)$ as follows:

$$T(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$$

The twisted transfer matrix $\tau_{6V}(z; \varphi)$ is defined by

$$\tau_{6V}(z; \varphi) = \text{tr} \left( q^{\varphi \sigma_0^Z} T(z) \right) = q^\varphi A(z) + q^{-\varphi} D(z).$$
The twisted Hamiltonian is given by the following logarithmic derivative:
\[
\sinh 2\eta \times \frac{d}{dz} \log \tau(z; \varphi)|_{z=\eta} \\
= \sum_{j=1}^{L-1} \left( 2\sigma_{j+1}^{+} \sigma_{j}^{-} + 2\sigma_{j}^{-} \sigma_{j+1}^{+} + \cosh 2\eta \sigma_{j}^{Z} \sigma_{j+1}^{Z} \right) \\
+ q^{-2\varphi} 2\sigma_{0}^{+} \sigma_{1}^{-} + q^{2\varphi} 2\sigma_{0}^{-} \sigma_{1}^{+} + \cosh 2\eta \sigma_{0}^{Z} \sigma_{1}^{Z} + L \cosh 2\eta \\
= \mathcal{H}_{XXZ}(\phi)/J + L\Delta.
\]

2.2 Roots of unity conditions

Let us formulate roots of unity conditions explicitly as follows [12, 11].

**Definition 1** (Roots of unity conditions). We say that $q_0$ is a root of unity with $q_0^N = 1$, if one of the three conditions hold: (1) $N$ is odd and $q_0$ is a primitive $N$th root of unity, i.e. $q_0^N = 1$; (2) $N$ is odd and $q_0$ is a primitive $2N$th root of unity, i.e. $q_0^N = -1$; (3) $N$ is even and $q_0$ is a primitive $2N$th root of unity, i.e. $q_0^N = -1$.

Let us denote by $S^Z \pm \varphi$ either $S^Z + \varphi$ or $S^Z - \varphi$. We now consider the condition of $q_0^{2S^Z \pm 2\varphi} = 1$. The values of $S^Z$ and $\varphi$ are given by integers or half-integers under the twisted boundary conditions.

1. When $N$ is odd and $q_0^N = 1$, we have $q_0^{2S^Z \pm 2\varphi} = 1$ if and only if $S^Z \pm \varphi \equiv 0 \pmod{N}$ or $S^Z \pm \varphi \equiv N/2 \pmod{N}$. When $S^Z \pm \varphi \equiv 0 \pmod{N}$, $\varphi$ is given by an integer for even $L$, and a half-integer for odd $L$. When $S^Z \pm \varphi \equiv N/2 \pmod{N}$, $\varphi$ is given by a half-integer for even $L$, and an integer for odd $L$.

2. When $N$ is odd and $q_0^N = -1$, we have $q_0^{2S^Z \pm 2\varphi} = 1$ if and only if $S^Z \pm \varphi \equiv 0 \pmod{N}$. $\varphi$ is given by an integer for even $L$, and a half-integer for odd $L$.

3. When $N$ is even and $q_0^N = -1$, we have $q_0^{2S^Z \pm 2\varphi} = 1$ if and only if $S^Z \pm \varphi \equiv 0 \pmod{N}$. $\varphi$ is given by an integer for even $L$, and a half-integer for odd $L$.

Here we note that if the number of lattice sites $L$ is given by an even integer, then $S^Z$ takes integral values, while if $L$ is odd, $S^Z$ takes half-integral values.

2.3 Operators commuting with the twisted XXZ Hamiltonian

We now formulate operators commuting or anti-commuting with the twisted transfer matrix of the six-vertex model at roots of unity [12]. We introduce operators $S_{j}^{\pm}$ and $T_{j}^{\pm}$ by

\[
S_{j}^{\pm} = q^{\sigma_{j}^{Z}/2} \otimes \cdots \otimes q^{\sigma_{j}^{Z}/2} \otimes \sigma_{j}^{\pm} \otimes q^{-\sigma_{j}^{Z}/2} \otimes \cdots \otimes q^{-\sigma_{j}^{Z}/2}, \\
T_{j}^{\pm} = q^{-\sigma_{j}^{Z}/2} \otimes \cdots \otimes q^{-\sigma_{j}^{Z}/2} \otimes \sigma_{j}^{\pm} \otimes q^{\sigma_{j}^{Z}/2} \otimes \cdots \otimes q^{\sigma_{j}^{Z}/2} \quad (j = 1, 2, \ldots, L). \tag{11}
\]

We define $S^{\pm}$ and $T^{\pm}$ by

\[
S^{\pm} = \sum_{j=1}^{L} S_{j}^{\pm}, \quad T^{\pm} = \sum_{j=1}^{L} T_{j}^{\pm}. \tag{12}
\]
They are generators of the affine quantum group $\hat{U}_q(sl(2))$.

Let us introduce the $q$-integer $[n]$ and the $q$-factorial $[m]!$, respectively, by the following:

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [m]! = \prod_{k=1}^{m}[k]. \quad (13)$$

It is easy to show

$$(S^\pm)^m = q^{\pm m(m-1)/2}[m]! \sum_{1 \leq i_1 < \cdots < i_m \leq L} S_{i_1}^\pm \cdots S_{i_m}^\pm, \quad (14)$$

$$(T^\pm)^m = q^{\mp m(m-1)/2}[m]! \sum_{1 \leq i_1 < \cdots < i_m \leq L} T_{i_1}^\pm \cdots T_{i_m}^\pm. \quad (14)$$

The symbols $S^\pm(N)$ and $T^\pm(N)$ are defined in Ref. [6] by

$$S^\pm(N) = \lim_{q \to q_0} \frac{(S^\pm)^N}{[N]!}, \quad T^\pm(N) = \lim_{q \to q_0} \frac{(T^\pm)^N}{[N]!}. \quad (15)$$

Here we define $(S^\pm)^{(m)}$ and $(T^\pm)^{(m)}$ for all positive integers $m$ by

$$(S^\pm)^{(m)} = \lim_{q \to q_0} \frac{(S^\pm)^m}{[m]!}, \quad (T^\pm)^{(m)} = \lim_{q \to q_0} \frac{(T^\pm)^m}{[m]!}. \quad (16)$$

Explicitly, we have $(S^\pm)^{(m)}$ for any positive integer $m$ as follows.

$$\left( S^\pm \right)^{(m)} = \sum_{1 \leq j_1 < \cdots < j_m \leq L} q_0^{m/2} \sigma_{j_1}^z \otimes \cdots \otimes q_0^{m/2} \sigma_{j_m}^z \otimes q_0^{m/2} \sigma_{j_1}^\pm \otimes \cdots \otimes q_0^{m/2} \sigma_{j_m}^\pm \otimes \cdots \otimes q_0^{m/2} \sigma_{j_m}^z \otimes \cdots \otimes q_0^{m/2} \sigma_{j_1}^z. \quad (17)$$

Let $m$ and $n$ be integers such that $|m - n| = kN$ for some integer $k$. When $q_0$ is a root of unity with $q_0^{2N} = 1$, we have the following.

1. In the sectors of $S^\pm \equiv -\varphi + n \text{(mod} N\text{)},$ we have

$$\left( S^+ \right)^{(m)}(T^-)^{(m)} \tau(z; \varphi) = q_0^{m-n} \tau(z; \varphi)(S^+)^{(m)}(T^-)^{(m)}$$

2. In the sectors of $S^\pm \equiv -\varphi - n \text{(mod} N\text{)},$ we have

$$\left( T^- \right)^{(m)}(S^+)^{(m)} \tau(z; \varphi) = q_0^{m-n} \tau(z; \varphi)(T^-)^{(m)}(S^+)^{(m)}$$

3. In the sectors of $S^\pm \equiv \varphi - n \text{(mod} N\text{)},$ we have

$$\left( S^- \right)^{(m)}(T^+)^{(m)} \tau(z; \varphi) = q_0^{m-n} \tau(z; \varphi)(S^-)^{(m)}(T^+)^{(m)}$$

4. In the sectors of $S^\pm \equiv \varphi + n \text{(mod} N\text{)},$ we have

$$\left( T^+ \right)^{(m)}(S^-)^{(m)} \tau(z; \varphi) = q_0^{m-n} \tau(z; \varphi)(S^-)^{(m)}(T^+)^{(m)}$$

Here we note that $q_0^N = \pm 1$ when $q_0$ is a root of unity with $q_0^{2N} = 1$. Thus we have $q_0^{m-n} = (\pm 1)^k$. For simplicity, we have not considered the case when $N$ is odd with $q_0^N = 1$ and $S^\pm + \varphi \equiv N/2 \text{ (mod} N\text{)}$ or $S^\pm - \varphi \equiv N/2 \text{ (mod} N\text{)}$. 

5
2.4 Examples

For an illustration, we consider the case of a root of unity where $N = 3$ ($q_0^3 = 1$) and $L$ is even. Some of the operators commuting or anti-commuting with the twisted transfer matrix are given as follows.

(1a) $\varphi = 0$ and $S^Z \equiv 0 \pmod{N}$

$$(S^+)^{(3)}, \ (S^-)^{(3)}, \ (T^+)^{(3)}, \ (T^-)^{(3)}$$

They generate the $sl_2$ loop algebra [6].

(1b) $\varphi = 0$ and $S^Z \equiv 1 \pmod{N}$:

$$(S^+)^{(4)}(T^-)^{(1)}, \ (T^-)^{(5)}(S^+)^{(2)}, \ (S^-)^{(5)}(T^+)^{(2)}, \ (T^+)^{(4)}(S^-)^{(1)},$$

$$(S^+)^{(1)}(T^-)^{(4)}, \ (T^-)^{(2)}(S^+)^{(5)}, \ (S^-)^{(2)}(T^+)^{(5)}, \ (T^+)^{(1)}(S^-)^{(4)}, . . . .$$

It is conjectured that they should generate the $sl_2$ loop algebra [6].

(1c) $\varphi = 0$ and $S^Z \equiv 2 \pmod{N}$:

$$(S^+)^{(5)}(T^-)^{(2)}, \ (T^-)^{(4)}(S^+)^{(1)}, \ (S^-)^{(4)}(T^+)^{(1)}, \ (T^+)^{(5)}(S^-)^{(2)},$$

$$(S^+)^{(2)}(T^-)^{(5)}, \ (T^-)^{(1)}(S^+)^{(4)}, \ (S^-)^{(1)}(T^+)^{(4)}, \ (T^+)^{(2)}(S^-)^{(5)}, . . . .$$

It is conjectured that they should generate the $sl_2$ loop algebra [6].

(2a) $\varphi = 1$ and $S^Z \equiv 0 \pmod{N}$:

$$(S^+)^{(4)}(T^-)^{(1)}, \ (T^-)^{(5)}(S^+)^{(2)}, \ (S^-)^{(5)}(T^+)^{(2)}, \ (T^+)^{(4)}(S^-)^{(1)}, . . . .$$

$$(S^+)^{(1)}(T^-)^{(4)}, \ (T^-)^{(2)}(S^+)^{(5)}, \ (S^-)^{(2)}(T^+)^{(5)}, \ (T^+)^{(1)}(S^-)^{(4)}, . . . .$$

(2b) $\varphi = 1$ and $S^Z \equiv 1 \pmod{N}$:

$$(S^+)^{(5)}(T^-)^{(2)}, \ (T^-)^{(4)}(S^+)^{(1)}, \ (S^-)^{(3)}, \ (T^+)^{(3)}, . . . .$$

$$(S^+)^{(2)}(T^-)^{(5)}, \ (T^-)^{(1)}(S^+)^{(4)}, . . . .$$

$(S^-)^{(3)}$ and $(T^+)^{(3)}$ generate a Borel subalgebra [13].

(2c) $\varphi = 1$ and $S^Z \equiv 2 \pmod{N}$

$$(S^+)^{(3)}, \ (T^-)^{(3)}, \ (S^-)^{(5)}(T^+)^{(2)}, \ (T^+)^{(4)}(S^-)^{(1)}, . . . .$$

$$(S^-)^{(2)}(T^+)^{(5)}, \ (T^+)^{(1)}(S^-)^{(4)}, . . . . \quad (18)$$

$(S^+)^{(3)}$ and $(T^-)^{(3)}$ generate a Borel subalgebra [13].

(3a) $\varphi = 2$ and $S^Z \equiv 0 \pmod{N}$:

$$(S^+)^{(5)}(T^-)^{(2)}, \ (T^-)^{(4)}(S^+)^{(1)}, \ (S^-)^{(5)}(T^+)^{(2)}, \ (T^+)^{(4)}(S^-)^{(1)}, . . . .$$

$$(S^+)^{(2)}(T^-)^{(5)}, \ (T^-)^{(1)}(S^+)^{(4)}, \ (S^-)^{(2)}(T^+)^{(5)}, \ (T^+)^{(1)}(S^-)^{(4)}, . . . .$$
(3b) \( \varphi = 1 \) and \( S^{Z} \equiv 1 \) (mod \( N \)):

\[
(S^{+})^{(3)}, \quad (T^{-})^{(3)}, \quad (S^{-})^{(4)}(T^{+})^{(1)}, \quad (T^{+})^{(5)}(S^{-})^{(2)}, \quad (S^{-})^{(1)}(T^{+})^{(4)}, \quad (T^{+})^{(2)}(S^{-})^{(5)}, \ldots.
\]

\( (S^{+})^{(3)} \) and \( (T^{-})^{(3)} \) generate a Borel subalgebra \([13]\).

(3c) \( \varphi = 1 \) and \( S^{Z} \equiv 2 \) (mod \( N \)):

\[
(S^{+})^{(4)}(T^{-})^{(1)}, \quad (T^{-})^{(5)}(S^{+})^{(2)}, \quad (S^{-})^{(3)}, \quad (T^{+})^{(3)}, \ldots,
\]

\( (S^{+})^{(1)}(T^{-})^{(4)}, \quad (T^{-})^{(2)}(S^{+})^{(5)}, \ldots \).

\( (S^{-})^{(3)} \) and \( (T^{+})^{(3)} \) generate a Borel subalgebra \([13]\).

### 3 Extension of the Borel subalgebra symmetry

#### 3.1 Definition of the Borel subalgebra of \( U(L(sl_{2})) \)

We recall that the Borel subalgebra, \( U(B_0) \), is generated by the following operators:

\[
x^{+}_k, h_k \quad \text{for} \quad k = 0, 1, \ldots, \quad \text{and} \quad x^{-}_k \quad \text{for} \quad k = 1, 2, \ldots.
\]

They satisfy the defining relations given as follows:

\[
[h_j, x^{+}_k] = 2x^{+}_{j+k}, \quad \text{for} \quad j, k \geq 0,
\]

\[
[h_j, x^{-}_k] = (-2)x^{-}_{j+k}, \quad \text{for} \quad j \geq 0 \text{ and } k \geq 1,
\]

\[
[x^{+}_j, x^{-}_k] = \delta_{j,k}h_{j+k}, \quad \text{for} \quad j \geq 0 \text{ and } k \geq 1,
\]

\[
[h_j, h_k] = 0, \quad \text{for} \quad j, k \geq 0,
\]

\[
[x^{+}_j, h_k] = 0 \quad \text{for} \quad j \geq 0 \text{ and } k \geq 0,
\]

\[
[x^{-}_j, h_k] = 0 \quad \text{for} \quad j \geq 1 \text{ and } k \geq 1.
\]

#### 3.2 Highest weight vectors and highest weight parameters

Let us define highest weight vectors of the Borel subalgebra \( U(B_0) \).

**Definition 2.** In a representation of \( U(B_0) \), we call a vector \( \Psi \) a highest weight vector if it is annihilated by all \( x^{+}_k \)'s, i.e. \( x^{+}_k \Psi = 0 \) for \( k = 0, 1, \ldots \), and is a simultaneous eigenvector of all \( h_k \)'s, i.e. \( h_k \Psi = d_k \Psi \) for \( k = 0, 1, \ldots \). We call the set of eigenvalues \( d_k \) the highest weight of \( \Psi \). We call the representation generated by a highest weight vector \( \Psi \), the highest weight representation of \( \Psi \). We denote it by \( U(B_0)\Psi \).

**Definition 3.** Let \( \Psi \) be a highest weight vector of \( U(B_0) \). If \((x^{-}_1)^r \Psi = 0 \) and \((x^{-}_1)^{r+1} \Psi \neq 0 \) for an integer \( r \), we say that \( x^{-}_1 \) is nilpotent of degree \( r \) in the highest weight representation.
In a finite-dimensional representation of $U(B_0)$, $x_1^-$ is nilpotent, i.e. $(x_1^-)^s = 0$ for some integer $s$. For a highest weight vector in a finite-dimensional representation of $U(B_0)$, we can define the highest weight polynomial and highest weight parameters $a_j$ similarly as in the case of the $sl_2$ loop algebra [10] [17] [18].

**Definition 4.** Let $\Psi$ be a highest weight vector of $U(B_0)$. By applying the Poincaré-Birkhoff-Witt theorem to $U(B_0)$, it follows that the highest weight representation $U(B_0)\Psi$ is decomposed into the direct sum of subspaces with respect to eigenvalues of $h_0$, and that every vector $v$ in the subspace of weight $d_0 - 2n$ is written as follows:

$$v = \sum_{1 \leq k_1 \leq \cdots \leq k_n} C_{k_1, \ldots, k_n} x_{k_1}^- \cdots x_{k_n}^- \Psi.$$ 

We call the subspace of weight $d_0 - 2n$ the sector of degree $n$.

**Proposition 5.** Let $\Psi$ be a highest weight vector of $U(B_0)$. If $x_1^-$ is nilpotent of degree $r$ in the highest weight representation $U(B_0)\Psi$, then the sector of degree $2r$ in $U(B_0)\Psi$ is one-dimensional.

We can show proposition 3 through the following lemma [18] [10].

**Lemma 6.** Let $\Psi$ be a highest weight vector of $U(B_0)$. We assume that $x_1^-$ is nilpotent of degree $r$ in $U(B_0)\Psi$. Let us take a non-negative integer $n$ satisfying $n \leq r$. Then, for any set of positive integers, $k_1, \ldots, k_n$, we have

$$(x_1^-)^{r-n} x_{k_1}^- \cdots x_{k_n}^- \Psi = A_{k_1, \ldots, k_n} (x_1^-)^r \Psi. \quad (20)$$

Here, $A_{k_1, \ldots, k_n}$ is given by a complex number.

Let us denote by $(X)^{(n)}$ the $n$th power of operator $X$ divided by the $n$ factorial, i.e. $(X)^{(n)} = X/n!$.

**Lemma 7.** Let $\Psi$ be a highest weight vector of $U(B_0)$. If $x_1^-$ is nilpotent of degree $r$ in $U(B_0)\Psi$, then $\Psi$ is a simultaneous eigenvector of $(x_0^+)^{(n)}(x_1^-)^{(n)}$:

$$(x_0^+)^{(j)}(x_1^-)^{(j)} \Omega = \lambda_j \Omega, \quad \text{for} \quad j = 1, 2, \ldots, r. \quad (21)$$

Here $\lambda_j$ are eigenvalues.

**Proof.** From the Poincaré-Birkhoff-Witt theorem of $U(B_0)$, it follows that the sector of degree 0 in $U(B_0)\Psi$ is one-dimensional. Since $(x_0^+)^{(j)}(x_1^-)^{(j)} \Psi$ is in the sector of degree 0 in $U(B_0)\Psi$, it is proportional to the basis vector $\Psi$. 

Let $\Psi$ be a highest weight vector in a finite-dimensional representation of the Borel subalgebra $U(B_0)$. We now introduce parameters expressing the highest weight of $\Psi$. We denote by $\lambda = (\lambda_1, \ldots, \lambda_r)$ the sequence of eigenvalues $\lambda_k$ which are defined in eq. (21). Here we recall that in a finite-dimensional representation, $x_1^-$ is nilpotent of some degree. We define a polynomial $P_{\lambda}(u)$ by the following relation [10]:

$$P_{\lambda}(u) = \sum_{k=0}^{r} \lambda_k (-u)^k. \quad (22)$$
We call it the *highest weight polynomial* of $\Psi$.

Let us factorize polynomial $P_\lambda(u)$ as follows

$$P_\lambda(u) = \prod_{k=1}^{s} (1 - a_k u)^{m_k}, \quad (23)$$

where $a_1, a_2, \ldots, a_s$ are distinct, and their multiplicities are given by $m_1, m_2, \ldots, m_s$, respectively. We denote by $a$ the sequence of $s$ parameters $a_j$:

$$a = (a_1, a_2, \ldots, a_s). \quad (24)$$

Here we note that $r$ is equal to the sum of multiplicities $m_j$: $r = m_1 + \cdots + m_s$. We define parameters $\hat{a}_i$ for $i = 1, 2, \ldots, r$, as follows.

$$\hat{a}_i = a_k \quad \text{if} \quad m_1 + m_2 + \cdots + m_{k-1} < i \leq m_1 + \cdots + m_{k-1} + m_k. \quad (25)$$

Then, the set $\{\hat{a}_j | j = 1, 2, \ldots, r\}$ corresponds to the set of parameters $a_j$ with multiplicities $m_j$ for $j = 1, 2, \ldots, s$. We denote by $\hat{a}$ the sequence of $r$ parameters $\hat{a}_i$:

$$\hat{a} = (\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_r). \quad (26)$$

We call parameters $\hat{a}_i$ the *highest weight parameters* of $\Psi$. It follows from the definition of highest weight polynomial $P_\lambda(u)$ given by (23) and that of highest weight parameters (23) that we have

$$\lambda_n = \sum_{1 \leq j_1 < \cdots < j_n \leq r} \hat{a}_{j_1} \cdots \hat{a}_{j_n}. \quad (27)$$

If the highest weight parameters are nonzero, we define $\tilde{\lambda}_n$ for $n = 0, 1, \ldots, r$ by

$$\tilde{\lambda}_n = \sum_{1 \leq j_1 < \cdots < j_n \leq r} \hat{a}_{j_1}^{-1} \cdots \hat{a}_{j_n}^{-1}. \quad (28)$$

We remark that we may call the highest weight polynomial of $\Psi$ and the highest weight parameters of $\Psi$ the *loop-highest weight polynomial* of $\Psi$ and the *loop-highest weight parameters* of $\Psi$, respectively [19].

### 3.3 Borel subalgebra generators with parameters

Let $\alpha$ denote a finite sequence of complex parameters such as $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. We define generators with $n$ parameters, $x^\pm_m(\alpha)$ and $h_m(\alpha)$, as follows [10] [17] [18]:

$$x^\pm_m(\alpha) = \sum_{k=0}^{n} (-1)^k x^\pm_{m-k} \sum_{\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}} \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_k},$$

$$h_m(\alpha) = \sum_{k=0}^{n} (-1)^k h_{m-k} \sum_{\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}} \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_k}. \quad (29)$$

Here, in the case of the Borel subalgebra $U(B_0)$, we define $x^\pm_m(\alpha)$ and $h_m(\alpha)$ for $m \geq n \geq 0$, and $x_m(\alpha)$ for $m > n \geq 0$. 
Let $\alpha$ and $\beta$ be arbitrary sequences of $n$ and $p$ parameters, respectively. Here we have $n, p \geq 0$. In terms of generators with parameters we express the defining relations of the Borel subalgebra as follows:

$$[x_\ell^+(\alpha), x_m^-(\beta)] = h_{\ell+m}(\alpha\beta), \quad [h_\ell(\alpha), x_m^-(\beta)] = -2x_{\ell+m}^-(\alpha\beta), \quad (30)$$

for $\ell \geq n$ and $m > p$, and

$$[h_\ell(\alpha), x_m^+(\beta)] = 2x_{\ell+m}^+(\alpha\beta). \quad (31)$$

for $\ell \geq n$ and $m \geq p$, Here the symbol $\alpha\beta$ denotes the composite sequence of $\alpha$ and $\beta$:

$$\alpha\beta = (\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_p). \quad (32)$$

**Lemma 8.** Let $\Psi$ be a highest weight vector of $U(\mathcal{B}_0)$. If $x_{t}^- (\alpha) \Psi = 0$ for a positive integer $t$ and a sequence of parameters $\alpha = (\alpha_1, \ldots, \alpha_n)$ where $t > n$, we have $h_{t+m}(\alpha) = 0$ and $x_{t+m}^\pm (\alpha) = 0$ in the highest weight representation of $\Psi$ for $m \in \mathbb{Z}_{\geq 0}$: for any set of positive integers, $k_1, k_2, \ldots, k_n$, and for $m \in \mathbb{Z}_{\geq 0}$, we have the following:

$$x_{t+m}^- (\alpha) x_{k_1}^- x_{k_2}^- \cdots x_{k_n}^- \Psi = 0, \quad (33)$$

$$h_{t+m}(\alpha) x_{k_1}^- x_{k_2}^- \cdots x_{k_n}^- \Psi = 0, \quad (34)$$

$$x_{t+m}^+ (\alpha) x_{k_1}^- x_{k_2}^- \cdots x_{k_n}^- \Psi = 0. \quad (35)$$

**Proof.** Following the Poincaré-Birkhoff-Witt theorem, one can show that every vector in the sector of degree $n$ of $U(\mathcal{B}_0)$ is expressed as a linear combination of monomial vectors $x_{k_1}^- x_{k_2}^- \cdots x_{k_n}^- \Psi$. It is easy to show \((33)\). By induction on $n$, we can show \((34)\), and then \((35)\). \hfill \square

### 3.4 Recurrence relations

Let us denote by $\mathcal{B}_0^+$ such a subalgebra of $U(\mathcal{B}_0)$ that is generated by $x_k^+$ for $k \in \mathbb{Z}_{\geq 0}$. Similiarly as the case of the $sl_2$ loop algebra \([10]\), we can show the following:

**Lemma 9.** The following recurrence relations hold for $n \in \mathbb{Z}_{\geq 0}$:

$$(A_n) : \quad (x_0^+)^{(n-1)}(x_1^-)^{(n)} = \sum_{k=1}^{n} (-1)^{k-1} x_k^-(x_0^+)\left(x_k^-\right)^{(n-k)}\left(x_1^-\right)^{(n-k)} \bmod U(\mathcal{B}_0)\mathcal{B}_0^+, \quad (36)$$

$$(B_n) : \quad (x_0^+)^{(n)}(x_1^-)^{(n)} = \frac{1}{n} \sum_{k=1}^{n} (-1)^{k-1} h_k(x_0^+)\left(x_0^+\right)^{(n-k)}\left(x_1^-\right)^{(n-k)} \bmod U(\mathcal{B}_0)\mathcal{B}_0^+, \quad (37)$$

$$(C_n) : \quad [h_j(a), (x_0^+)^{(m)}(x_1^-)^{(m)}] = 0 \bmod U(\mathcal{B}_0)\mathcal{B}_0^+ \quad \text{for } m \leq n \text{ and } j \in \mathbb{Z}_{\geq 0}. \quad (38)$$
Proposition 10 (Reduction relations).

\[ x_{r+1+m}^- \Psi = \sum_{k=1}^{r} (-1)^{r-k} \lambda_{r+1-k} x_{k+m}^- \Psi, \quad \text{for } m \in \mathbb{Z}_{\geq 0}, \quad (36) \]

\[ d_{r+1+m}^+ = \sum_{k=1}^{r} (-1)^{r-k} \lambda_{r+1-k} d_{k+m}, \quad \text{for } m \in \mathbb{Z}_{\geq 0}. \quad (37) \]

Proof. Reduction relation (36) for \( m = 0 \) is derived from \((A_{r+1})\) of lemma 9 and lemma 7. Applying \( h_n \) for \( n \geq 0 \) to reduction relation (36) for \( m = 0 \), we have reduction relation (36) for \( m = n \). Applying \( x_0^+ \) to (36) from the left, we derive relations (37). \( \square \)

Corollary 11. Let \( \Psi \) be a highest weight vector of \( U(\mathcal{B}_0) \) and \( \hat{a} = (\hat{a}_1, \ldots, \hat{a}_r) \) the highest weight parameters. In the highest weight representation of \( \Psi \) we have \( h_{r+m}(\hat{a}) = 0 \), \( x_{r+m}^+(\hat{a}) = 0 \) and \( x_{r+1+m}^- (\hat{a}) = 0 \) for \( m \in \mathbb{Z}_{\geq 0} \).

Proof. It follows from lemma 8 and reduction relations (36). \( \square \)

3.5 A theorem on the Borel subalgebra

We first recall a simple fact in linear algebra. Let \( x_n \) for \( n = 0, 1, \ldots \), be an infinite sequence of numbers satisfying a linear recurrence relation:

\[ x_{n+r} = \sum_{k=1}^{r} \gamma_k x_{n+r-k} \quad (38) \]

We denote by \( x_n = (x_{n+1}, \ldots, x_{n+r}) \). Then, for any integer \( n \), there exists a matrix \( A^{[n]} \) such that

\[ x_n = A^{[n]} x_1. \quad (39) \]

Furthermore, we have for any \( m \) the following:

\[ x_{n+m} = A^{[n]} x_{m+1}. \quad (40) \]

Theorem 12. Let \( \Psi \) be a highest weight vector in a finite-dimensional representation of the Borel subalgebra \( U(\mathcal{B}_0) \). If all the highest weight parameters of \( \Psi \), i.e. \( \hat{a}_j \), are nonzero, then the action of \( \mathcal{B}_0 \) on \( \Psi \) can be extended to that of the \( sl_2 \) loop algebra: Suppose that \( x_1^- \) is nilpotent of degree \( r \) in the highest weight representation of \( \Psi \). We define \( \tilde{h}_0 \) by \( \tilde{h}_0 = h_0 - d_0 + r \), and \( x_0^- \) by

\[ x_0^- = \sum_{j=1}^{r} (-1)^{j-1} \tilde{\lambda}_j x_j^- \quad (41) \]

where \( \tilde{\lambda}_j \) are given by

\[ \tilde{\lambda}_j = \sum_{i_1 < \cdots < i_j} \hat{a}_{i_1}^{-1} \cdots \hat{a}_{i_j}^{-1}. \quad (42) \]

11
We also define $x^\pm_{-\ell}$ and $h_{-\ell}$ for $\ell \in \mathbb{Z}_{>0}$ by
\[
x^\pm_{-\ell} = \sum_{j=1}^{r} (-1)^{j-1} \lambda_{j-1} x^\pm_{j-\ell},
\]
\[
h_{-\ell} = \sum_{j=1}^{r} (-1)^{j-1} \lambda_{j-1} h_{j-\ell}.
\]

Then, they satisfy the defining relations of the $\mathfrak{sl}_2$ loop algebra in the highest weight representation $U(\mathcal{B}_0)\Psi$.

**Proof.** If a set of operators $x^+_j$, $h_k$ for $j = 0, 1, \ldots$, and $x^-_k$ for $k = 1, 2, \ldots$, satisfy the defining relations of the Borel subalgebra, then the set operators with $h_0$ being replaced by $\tilde{h}_0$ also satisfy the defining relations [19]. We can also show that $x^-_0$ satisfies the defining relations [19]. Making use of corollary [11] we can show $[x^+_m, x^-_{-\ell}] = h_{m-\ell}$, $[h_m, x^\pm_{-\ell}] = (\pm 2)x^\pm_{m-\ell}$, $[x^\pm_{-\ell}, x^-_m] = h_{m-\ell}$ and $[h_{-\ell}, x^\pm_m] = (\pm 2)x^\pm_m$ for $m, n \in \mathbb{Z}_{\geq 0}$. Here we express $h_{-\ell}$ and $x^\pm_m$ in terms of linear combinations of $h_j$ and $x^\pm_j$ for $j = 1, 2, \ldots, r$.

We calculate commutation relations among $h_j$ and $x^\pm_k$ for $j, k = 1, 2, \ldots, r$, and then show the defining relations through [10]. For an illustration, we show $[h_{-\ell}, x^\pm_m] = (\pm 2)x^\pm_{-\ell+m}$ as follows.

\[
[h_{-\ell}, x^\pm_m] = [(A^{[-\ell-1]}h_0)_{1}, x^\pm_m] = \sum_{j=1}^{r} A^{[-\ell-1]}_{1,j} [h_j, x^\pm_m] = (\pm 2) \sum_{j=1}^{r} A^{[-\ell-1]}_{1,j} x^\pm_{j+m} = (\pm 2) \left( A^{[-\ell-1]}x^\pm_m \right)_{1} = (\pm 2)x^\pm_{-\ell+m}.
\]

Similarly, we can show $[x^+_m, x^-_{-\ell}] = h_{-\ell+m}$. Furthermore, we can show $[x^+_m, x^-_n] = h_{m-n}$ and $[h_m, x^\pm_n] = (\pm 2)x^\pm_n$ for $m, n \in \mathbb{Z}_{\geq 0}$.

We note that in a finite-dimensional representation of the Borel subalgebra, the highest weight $d_0$ is not necessarily given by an integer.

We should note that Benkart and Terwilliger (2004) have shown that an irreducible finite-dimensional representation of the Borel subalgebra is extended uniquely to an irreducible representation of the $\mathfrak{sl}_2$ loop algebra [14]. Thus, if the highest weight representation is irreducible, then theorem [12] should be equivalent to the result [13]. Here we recall that an irreducibility criterion is known for a finite-dimensional highest weight representation of the Borel algebra with nonzero highest weight parameters as follows [18]:

**Proposition 13.** Let $\Psi$ be a highest weight vector in a finite-dimensional representation of $U(\mathcal{B}_0)$. We denote by $\hat{a}_j$ the highest weight parameters. It generates an irreducible
representation if and only we have
\[
\sum_{j=0}^{s} (-1)^{s-j} \mu_{s-j} x_{j+1}^{-} \Psi = 0, \tag{44}
\]

where \( \mu_k (k = 1, 2, \ldots, s) \) are given by
\[
\mu_k = \sum_{1 \leq i_1 < \cdots < i_k \leq s} a_{i_1} \cdots a_{i_k}.
\]

Thus, in a finite-dimensional highest weight representation of \( U(B_0) \), if the highest weight vector satisfies the condition (44), it is irreducible and we can also show by making use of the result of Ref. [14] that it is extended into a finite-dimensional highest weight representation of the \( sl_2 \) loop algebra.

4 Application to the twisted XXZ spin chain

4.1 Regular Bethe vectors under the twisted B.C. as highest weight of \( U(L(sl_2)) \)

We briefly discuss some important points of the infinite-dimensional symmetry of the twisted XXZ spin chain at roots of unity. Some details will be given elsewhere.

For the periodic XXZ spin chain at a root of unity \( q_0 \) with \( q_0^{2N} = 1 \), Fabricius and McCoy conjectured [9] that every Bethe state should be a highest weight vector of the \( sl_2 \) loop algebra. Then, it has been explicitly proved for some sectors of \( S^Z \mod N \) [11] that every regular Bethe state is a highest weight vector of the \( sl_2 \) loop algebra. For the twisted XXZ spin chain, we can also show in some sectors of \( S^Z \mod N \) that every regular Bethe state is highest weight with respect to the Borel subalgebra \( U(B_0) \).

Let us denote by \( t_1, t_2, \ldots, t_R \) solutions of the twisted BA equations
\[
\frac{a_{SV}(t_j)}{d_{SV}(t_j)} = q^{-2 \varphi} \prod_{k=1, k \neq j}^{R} \frac{f(t_k - t_j)}{f(t_j - t_k)}, \quad \text{for } j = 1, 2, \cdots, R. \tag{45}
\]
Here \( a_{SV}(z) \) and \( d_{SV}(z) \) are given by
\[
a_{SV}(z) = \sinh^{L}(z + \eta), \quad d_{SV}(z) = \sinh^{S}(z - \eta). \tag{46}
\]

We recall that function \( f(z - w) \) is given by
\[
f(z - w) = \frac{\sinh(z - w - 2\eta)}{\sinh(z - w)},
\]
and \( q = \exp(2\eta) \).

**Definition 14.** Let \( t_1, t_2, \ldots, t_R \) satisfy the twisted Bethe ansatz equations (45). We call them Bethe roots. We call a set of Bethe roots, \( t_1, t_2, \ldots, t_R \), regular, if they are finite
and distinct. In terms of a set of regular Bethe roots, $t_1, t_2, \ldots, t_R$, we define the regular Bethe state $|R\rangle$ by

$$|R\rangle = B(t_1)B(t_2)\cdots B(t_R)|0\rangle$$  \hspace{1cm} (47)

Here $|0\rangle$ denotes the vacuum state in which all spins are up.

We recall the following conjecture [11].

**Conjecture 15.** For the twisted Bethe ansatz equations (45) at a root of unity $q_0$, every set of regular Bethe roots $\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_R$ gives an isolated solution of eqs. (45).

We can show conjecture [15] for some particular cases. For instance, it is trivial for such Bethe states with one down-spin.

Assuming conjecture [15] we can show the following theorem.

**Theorem 16.** Let $q_0$ be a root of unity with $q_0^{2N} = 1$ for some integer $N$, and $\varphi$ an integer or a half-integer such that we have $q_0^{2N\varphi} = 1$. Every regular Bethe state $|R\rangle$ gives a highest weight vector of the Borel subalgebra $U(B_0)$ if it is in sector $A$: $S^Z \equiv +\varphi \pmod{N}$ or $S^Z \equiv -\varphi \pmod{N}$ where $q_0^{2N} = 1$, or in sector $B$: $S^Z \equiv N/2 + \varphi \pmod{N}$ or $S^Z \equiv N/2 - \varphi \pmod{N}$ where $q_0^N = 1$ with $N$ odd.

The proof of theorem [16] will be given in a different paper.

### 4.2 Derivation of the degree of nilpotency for $x_1^-$

Let us assume that a regular Bethe state $|R\rangle$ of the twisted XXZ spin chain is in such a sector of $S^Z$ where theorem [16] holds, i.e. $|R\rangle$ is a highest weight vector of $U(B_0)$. For a highest weight representation of $U(B_0)$ generated by $|R\rangle$, it follows from the finite dimensionality that $x_1^-$ is nilpotent.

Let $\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_R$ be such a set of regular Bethe roots at a root of unity $q_0$ with $q_0^{2N} = 1$ that leads to the regular Bethe state $|R\rangle$. Let us define $\eta_0$ by $q = \exp 2\eta_0$. We now introduce the following function:

$$Y(v; \varphi) = \sum_{\ell=0}^{N-1} q_0^{-\varphi(2\ell+1)} \prod_{j=1}^{L} \sinh(v - (2\ell + 1)\eta_0) \prod_{j=1}^{R} \sinh(v - \tilde{t}_j - 2(\ell + 1)\eta_0)$$  \hspace{1cm} (48)

It follows from the twisted Bethe ansatz equations (45) that $Y(v; \varphi)$ is a Laurent polynomial of $\exp 2Nv$. Here we recall that when $\varphi = 0$ it is nothing but the polynomial introduced by Fabricius and McCoy for the XXZ spin chain at roots of unity under the periodic boundary conditions [9, 11].

Making use of the Laurent polynomial $Y(v; \varphi)$, we can show the following.

**Proposition 17.** Let $|R\rangle$ be a regular Bethe state of the twisted XXZ spin chain at $q_0$ a root of unity with $q_0^{2N} = 1$ and in a sector of $S^Z$ and with twist parameter $\varphi$ such that the conditions of theorem [16] hold. Then, it is a highest weight vector of $U(B_0)$, and all the highest weight parameters of $|R\rangle$ are nonzero.
Let us consider a regular Bethe state \( |R\rangle \) in such a sector of \( S^Z \) where we have \( S^Z \pm \varphi \equiv 0 \) (mod \( N \)). It follows from theorem 12 and proposition 17 that the highest weight representation generated by any given regular Bethe state in a sector of \( S^Z \pm \varphi \equiv 0 \) (mod \( N \)) extends to a highest weight representation of the \( sl_2 \) loop algebra. Thus, the Borel subalgebra symmetry of the twisted XXZ spin chain is extended to the \( sl_2 \) loop algebra symmetry.

4.3 Conjecture of the \( sl_2 \) loop algebra symmetry of the twisted transfer matrix

We now present the following conjecture:

**Conjecture 18.** The twisted XXZ spin chain at roots of unity with twist parameter \( \varphi \) being integers should have the \( sl_2 \) loop algebra symmetry in every sector of \( S^Z \) mod \( N \).

5 Representations of the Onsager algebra derived from highest weight representations of \( U(L(sl_2)) \)

The Onsager algebra is generated by operators \( A_m \) and \( G_\ell \) \((\ell, m = 0, \pm 1, \pm 2, \ldots)\) satisfying the following defining relations \([20, 21, 22, 23, 24, 25]\):

\[
\begin{align*}
[A_\ell, A_m] &= 4G_{\ell - m}, \\
[G_\ell, A_m] &= 2A_{m+\ell} - 2A_{m-\ell}, \\
[G_\ell, G_m] &= 0.
\end{align*}
\] (49)

We remark that Davies has shown that if generators \( A_n \) satisfy a linear recurrence relation

\[
\sum_{k=-n}^{n} \gamma_k A_{k-n} = 0,
\] (50)

then they are expressed in terms of the generators of \( sl_2 \) as follows \([22]\):

\[
\begin{align*}
A_m &= 2 \sum_{j=1}^{n} \left( e_+^j \otimes z_j^m + e_-^j \otimes z_j^{-m} \right), \\
G_k &= \sum_{j=1}^{n} \left( h_j \otimes z_j^k - h_j \otimes z_j^{-k} \right)
\end{align*}
\] (51)

Here \( e_\pm \) and \( h_k \) satisfy

\[
[h_j, e_\pm^j] = \pm 2 e_\pm^j \delta_{jk}, \quad [e_+^j, e_-^j] = h_j \delta_{j,k}
\] (52)

Let \( \alpha \) denote a finite sequence of complex parameters such as \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \).
Similarly as (29), we define generators with \( \ell \) parameters, \( A_m(\alpha) \) and \( G_m(\alpha) \), as follows:

\[
A_m(\alpha) = \sum_{k=0}^{\ell} (-1)^k A_{m-k} \sum_{\{i_1, \ldots, i_k\} \subset \{1, \ldots, \ell\}} \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_k}, \\
G_m(\alpha) = \sum_{k=0}^{\ell} (-1)^k G_{m-k} \sum_{\{i_1, \ldots, i_k\} \subset \{1, \ldots, \ell\}} \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_k}.
\]

(53)

Let \( \Omega \) be a highest weight vector in a finite-dimensional representation of the \( sl_2 \) loop algebra, \( U(L(sl_2)) \). We define operators \( A_m \) and \( G_k \) in terms of generators \( x_m^\pm \) and \( h_k \) of \( U(L(sl_2)) \) by

\[
A_m = x_m^+ + x_m^-, \quad G_k = h_k - h_{-k}.
\]

(54)

Then, operators \( A_m \) and \( G_k \) satisfy the defining relations of the Onsager algebra. Furthermore, we can show recurrence relations of \( A_m \)'s.

**Proposition 19.** Let \( \Omega \) be a highest weight vector in a finite-dimensional representation of the \( sl_2 \) loop algebra, \( U(L(sl_2)) \). If generators \( x_m^- \) of \( U(L(sl_2)) \) satisfy a recurrence relation, \( x_m^- (\beta) \Omega = 0 \), for a sequence of nonzero parameters \( \beta = (\beta_1, \beta_2, \ldots, \beta_n) \), then we have

\[
A_n(\beta; \bar{\beta}) = 0.
\]

(55)

Here \( \bar{\beta} \) denotes \( \bar{\beta} = (\beta_1^{-1}, \beta_2^{-1}, \ldots, \beta_n^{-1}) \).

In Ref. [15], the \( sl_2 \) loop algebra symmetry is derived for a spin-\( N/2 \) fusion model of the six-vertex model at \( q_0 \) being an \( N \)th root of unity, which is associated with the superintegrable chiral Potts model. From the representations of the \( sl_2 \) loop algebra derived from the fusion model, we can thus construct representations of the Onsager algebra. Then, through proposition 19 we derive recurrence relations for generators of the Onsager algebra. It should thus be an interesting problem to discuss connections to the Onsager algebra symmetry of the \( Z_N \) symmetric Hamiltonian given by von Gehlen and Rittenberg. We shall discuss them elsewhere.

**Acknowledgement**

The author would like to thank the organizers of the workshop RAQIS, Sep. 11-14, 2007, LAPTH, Annecy France, for giving him the opportunity to participate it. He would also like to thank Prof. M. Jimbo, Prof. B.M. McCoy and Dr. A. Nishino for helpful comments. He is quite grateful to Prof. P. Baseilhac for useful discussion on the Onsager algebra during the stay in Tours in September 2006. This work is partially supported by Grant-in-Aid for Scientific Research (C) No. 17540351.
References

[1] F.C. Alcaraz, M. Barber and M. Batchelor, Conformal invariance, the XXZ chain and the operator content of two-dimensional critical systems Ann. Phys., NY 182 (1988) 280-343.

[2] F.C. Alcaraz, U. Grimm and V. Rittenberg, The XXZ Heisenberg chain, conformal invariance and the operator content of $c < 1$ systems, Nucl. Phys. B 316 (1989) 735-768.

[3] B.S. Shastry and B. Sutherland, Twisted boundary conditions and effective mass in Heisenberg-Ising and Hubbard rings Phys. Rev. Lett. 65, 243 (1990).

[4] V.E. Korepin and A.C.T. Wu, Int. J. Mod. Phys. B5, 497 (1991).

[5] N. Yu and M. Fowler, Twisted boundary conditions and the adiabatic ground state for the attractive XXZ Luttinger liquid, Phys. Rev. B 46, 14583 (1992).

[6] T. Deguchi, K. Fabricius and B. M. McCoy, The $sl_2$ loop algebra symmetry of the six-vertex model at roots of unity, J. Stat. Phys. 102 (2001) 701-736.

[7] K. Fabricius and B. M. McCoy, Bethe’s Equation Is Incomplete for the XXZ Model at Roots of Unity, J. Stat. Phys. 103(2001) 647-678.

[8] K. Fabricius and B. M. McCoy, Completing Bethe’s Equations at Roots of Unity, J. Stat. Phys. 104(2001) 573-587.

[9] K. Fabricius and B. M. McCoy, Evaluation Parameters and Bethe Roots for the Six-Vertex Model at Roots of Unity, Progress in Math. Phys. 23 (MathPhys Odyssey 2001), edited by M. Kashiwara and T. Miwa, (Birkhäuser, Boston, 2002) 119-144.

[10] T. Deguchi, Irreducibility criterion for a finite-dimensional highest weight representation of the $sl_2$ loop algebra and the dimensions of reducible representations, J. Stat. Mech. (2007) P05007.

[11] T. Deguchi, XXZ Bethe states as highest weight vectors of the $sl_2$ loop algebra at roots of unity, J. Phys. A 40 (2007) 7473-7508.

[12] T. Deguchi, The $sl_2$ loop algebra symmetry of the twisted transfer matrix of the six-vertex model at roots of unity, J. Phys. A: Math. Gen. 37 (2004) 347-358.

[13] C. Korff, The twisted XXZ chain at roots of unity revisited, J. Phys. A 37 (2004) 1681-1689.

[14] G. Benkart and P. Terwilliger, Irreducible modules for the quantum affine algebra $U_q(sl_2)$ and its Borel subalgebra, J. Algebra 282 (2004) 172-194.

[15] A. Nishino and T. Deguchi, The $L(sl_2)$ symmetry of the Bazhanov-Stroganov model associated with the superintegrable chiral Potts model, Phys. Let. A 356 (2006) pp. 366-370.

[16] H. Au-Yang and J.H.H. Perk, Eigenvectors in the superintegrable Model, arXiv:0710.5257 [math-ph].
[17] T. Deguchi, The Six-Vertex Model at Roots of Unity and some Highest Weight Representations of the $sl_2$ Loop Algebra, Ann. Henri Poincaré 7 (2006), 1531-1540.

[18] T. Deguchi, Generalized Drinfeld polynomials for highest weight vectors of the Borel subalgebra of the $sl_2$ loop algebra, “Differential Geometry and Physics”, the Proc. of the 23rd International Conference of Differential Geometric Methods in Theoretical Physics, Tianjin, China, 20-26 August 2005, eds. M.-L. Ge and W. Zhang (Chern Institute of Mathematics, Tianjin, China, 2006) pp. 169-178.

[19] It was pointed out by Prof. M. Jimbo during the workshop RAQIS07 that technical terms such as loop-highest weight modules are employed by H. Nakajima (see for instance, arXiv:math/0103008v2 [math.QA]).

[20] L. Onsager, Crystal Statistics. I. A Two-Dimensional Model with an Order-Disorder Transition, Phys. Rev. 65 (1944) 117-149.

[21] L. Dolan and M. Grady, Conserved charges from self-duality, Phys. Rev. D 25 (1982) 1587-1604.

[22] B. Davies, Onsager’s algebra and superintegrability, J. Phys. A: Math. Gen. 23 (1990) 2245-2261.

[23] S. Roan, Onsager’s algebra, loop algebra and chiral Potts model, Bonn preprint 1991.

[24] D.B. Uglov and I.T. Ivanov, $sl(N)$ Onsager’s algebra and Integrability, J. Stat. Phys. 112 (1996) 87-113.

[25] E. Date and S. Roan, The structure of the Onsager algebra by closed ideals, J. Phys. A: Math. Gen. 33 (2000) 3275-3296.