User Selection and Minimum Probability of Error Multiuser Transmit Beamforming

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Abstract

Motivated by the challenges to the existing multiuser transmission methods in a low signal to noise ratio (SNR) regime, in this paper we show that it is beneficial in this regime to incorporate knowledge of modulation type in the design of the multiuser transmit beamformer. We propose a transmit beamforming method by utilizing the concept of minimum probability of error (MPE). In a multiuser scenario, an objective function is designed based on the weighted sum of error probabilities of each user. We take a convex optimization approach to transform and solve this ill-behaved nonconvex optimization problem. It is shown that by minimizing this objective function, the performance of the system could be dramatically improved. We then develop a user selection algorithm compatible with MPE beamforming that selects the maximum number of users with their error probabilities approaching zero. Based on line packing principles in Grassmannian manifolds, it is shown that the number of selected users could potentially be more than the number of transmit antennas.

Index Terms

Broadcast channels, convex optimization, Grassmannian manifolds, line packing, minimum probability of error, scheduling, transmit beamforming.

I. INTRODUCTION

Wireless multiple input multiple output (MIMO) channels have been attracting a great deal of interest in the last decades [1]–[4]. MIMO technologies are already at the core of several wireless standards, since they provide improvements in terms of spectral efficiency and reliability compared to single input single output (SISO) channels. Hence, the design of MIMO systems has been usually posed under two different perspectives: either the increase of the data transmission rate through spatial multiplexing or the improvement of the system reliability through the increased antenna diversity. Spatial multiplexing is a simple MIMO transmit technique in that
it does not require channel state information (CSI) at the transmitter and allows a high spectral efficiency by splitting the incoming data into multiple independent substreams and transmitting each substream on a different antenna as in V-BLAST [1]. When CSI is available at the transmitter, channel-dependent linear precoding (transmit beamforming) of the data substreams can further improve the performance by adapting the transmitted signal to the instantaneous channel state. In this case, employing multiuser MIMO techniques allows for gain in sum capacity obtained by channel reuse [2]–[5].

Although channel reuse for multiple users is advantageous in terms of throughput, when multiple uncoordinated links share a common communication medium, e.g., in a broadcast system, cochannel interference caused by the transmission of multiple users’ data on the same carrier frequency could limit channel reuse [6]. Most wireless systems deal with interference by orthogonalizing the communication links in time or frequency, so that they do not interfere with one another. It is clear that this approach could be suboptimal since it entails a priori loss of degrees of freedom in both links independent of the amount of interference. Power control, beamforming, and scheduling techniques, with capability of reducing interference, are conventional solutions to the cochannel interference problem [4]. From a practical point of view, using multiple antennas to communicate with many users simultaneously is especially appealing in wireless local area network (WLAN) environments, WiMAX, and other time-division duplex (TDD) systems where channel conditions can readily be learned by all parties [3].

Classically, a beamformer controls the pattern of the antenna array by setting the antenna array weights to satisfy a certain optimization criterion. Many beamforming methods aim at maximizing throughput. However, the problem with this criterion is that it implicitly presumes that an unrealizable ideal continuous Gaussian code is used instead of a signal constellation [7]. On the other hand, whenever the transmitter sends a modulated signal the throughput is determined largely by the order of modulation. Signal to noise ratio (SNR), signal to interference plus noise ratio (SINR), mean square error (MSE) between the desired signal and the array output, and signal to leakage ratio (SLR) are other common criteria in formulating the beamforming optimization problem [2], [3], [5]. However, for a communication system, probability of error or the achievable bit error rate (BER) is a more direct system performance metric [8]–[12]. Therefore, the performance of a system could be improved if the beamformer is designed in a way to directly minimize the error probability.
The error probability of each user in a multiuser downlink system depends on the modulation type. Therefore, to design a beamformer that minimizes the error probability, one ought to account for the modulation type which is not considered in classical beamforming methods such as minimum mean square error (MMSE), maximum signal to leakage plus noise ratio (MSLNR), and block diagonalization (BD) [13]. In this paper, it is established that by considering the modulation type in the design of beamforming weights the performance of the system could be significantly improved.

As mentioned earlier, selecting a subset of users for transmission in MIMO broadcast channel, is another conventional solution to to the cochannel interference problem for increasing the system throughput and reliability. The gain in throughput and reliability is obtained by multiuser diversity presented by user selection when the number of users is large. Although the optimal user subset can be found by brute-force search over all possible combinations of user subsets, its computational complexity is prohibitive. Hence, low complexity scheduling algorithms are desired in practice [4], [14]–[17]. For example in [4], [14], low complexity algorithms based on semi-orthogonal user selection are presented are based on zero forcing beamforming and achieve the same asymptotic (high SNR) sum rate as dirty paper coding for broadcast channels. In [15], a greedy based user selection is proposed based on BD beamforming and increases the total throughput of the users. In this paper, we propose a user selection algorithm for MPE beamforming that semi-greedily selects the set of users by a geometric approach such that the number of selected users is made as large as possible. It is shown that for a one-dimensional modulation such as binary phase-shift keying (BPSK), it is possible to utilise the extra dimensions provided by the complex channel and transmit information at the same time and frequency to more users than the number of antennas of the transmitter.

The rest of the paper is organized as follows: in Section II, the system model is introduced. In Section III, the error probability of a user in the downlink of a multiuser system is calculated assuming the transmitter is using BPSK modulation and linear beamforming for transmission. Section IV presents the minimum probability of error (MPE) transmit beamforming considering two different scenarios. First, it is assumed that the receiver uses a single-user maximum likelihood detector. Second, it is assumed that the transmit beamforming weights and the receive filter coefficients are calculated jointly at the transmitter. In both scenarios we try to transform the beamforming optimization problem to convex optimization subproblems and present algorithms.
for finding the beamforming vectors of users. In Section V by taking a geometric approach, a user selection algorithm compatible with MPE beamforming is presented. Numerical results are demonstrated in Section VI. Finally, conclusions are drawn in Section VII.

II. SYSTEM MODEL

We consider a broadcast system with one transmitter and $K$ receivers (users) which are preselected out of $K_T$ total users as in Fig. 1. It is assumed that the transmitter consists of an array of antennas with $M$ elements and each receiver $j$, $1 \leq j \leq K$, has one antenna in its array. It is also assumed that the transmitter has one symbol encoded in $s_j$ for each receiver $j$, $1 \leq j \leq K$ to be transmitted in the same time and frequency slots. The transmitter uses an $M \times 1$ precoding vector $u_j$ to encode the transmitted symbols intended for receiver $j$. Ignoring the noise at the transmitter’s output, and assuming that the transmitter has information symbols for all of the $K$ preselected receivers we can model the $M \times 1$ transmitted signal vector as

$$x = \sum_{l=1}^{K} u_l s_l = Us,$$

where $U = [u_1, \ldots, u_K]$ and $s = [s_1, \ldots, s_K]^T$. The channel matrix between $M$ antennas of the transmitter and the single antenna of receiver $j$ can be represented by the $1 \times M$ vector $h_j$ with entries following an independent identically distributed (i.i.d.) circularly complex Gaussian distribution with zero mean and variance 1. This channel model is valid for narrow-band (frequency non-selective) systems if the transmit and receive antennas are in non line-of-sight rich-scattering environments with sufficient antenna spacing [14], [18]. At receiver $j$ the received

Fig. 1. System model.
signal can be modeled as
\[ r_j = h_j x + z_j = \sum_{l=1}^{K} h_j u_l s_l + z_j = \bar{r}_j + z_j, \quad 1 \leq j \leq K, \]
where \( z_j \) is modeled as circularly complex Gaussian noise with zero mean and variance \( \sigma_z^2 \).

At receiver \( j \) the received signal is processed by a filter. Therefore the output at the receiver \( j \) can be written as
\[ y_j = y_j(w_j, U) = w_j r_j = \sum_{l=1}^{K} w_j h_j u_l s_l + w_j z_j = \bar{y}_j + z'_j, \]
where \( w_j \) is the complex-valued filter coefficient at receiver \( j \) and \( z'_j = w_j z_j \) is circularly Gaussian noise with variance \( \sigma_z^2 w_j w_j^H \). We also express \( y_j \) as a function of \( w_j \) and \( U \) to emphasize its dependence on these parameters.

### III. ERROR PROBABILITY

To calculate the error probability of each user in a broadcast channel one needs to know the estimation technique that the receiver is using in addition to the modulation type. Motivated by the facts that a low dimensional modulation like BPSK is employed in wireless systems such as IEEE 802.11a,n,ac, and it is often selected in systems with adaptive modulation when SNR is low, we assume BPSK modulation for all users [19], [20]. It is also assumed that the system operates with the following decision rule for estimating the transmitted symbols of user \( j \) when the output noise is additive white Gaussian:
\[ \hat{s}_j = \text{sign}(y_j^R), \]
where the superscript \( R \) denotes taking the real part operation, i.e., \( x^R = \text{Re}\{x\} \).

Here, the error probability of each user \( j \), \( 1 \leq j \leq K \), is calculated as a function of its receive filter coefficient and its transmit beamforming weight vector as well as the transmit beamforming weight vectors of all other users \( (P_{e_j}(w_j; U), 1 \leq j \leq K) \). Later in Section IV we use this probability of error to calculate beamforming weights for two different scenarios, namely, a scenario in which each receiver estimates the transmitted signal according to a classical single-user maximum likelihood scheme without relying on knowledge of other channels, and
a joint transmit-receive (Tx-Rx) beamforming scenario in which both the receive and transmit beamforming weights are calculated at the transmitter and then the calculated receive filter coefficient of each user is provided to its receiver via the forward channel.

The error probability at the output of receiver $j$ is expressed as

$$P_{e_j} = P_{e_j}(y_j; w_j, U) = P(\hat{s}_j \neq s_j) = P_0 P(\hat{s}_j = 1 | s_j = -1) + P_1 P(\hat{s}_j = -1 | s_j = 1) = P(\hat{s}_j = -1 | s_j = 1) = P(y_j^R < 0 | s_j = 1),$$

where $P_0 = P(s_j = -1)$, $P_1 = P(s_j = +1)$, and $P(.)$ is the probability of an event. It is assumed that $P_0 = P_1 = 1/2$, i.e., the transmitted BPSK modulated signal $s_j$ takes its elements from the set $\{\pm 1\}$ with equal probability, which could be the result of source compression and hard decision. It should be remarked that, using uniform rather than Gaussian distribution over signal sets causes an asymptotic loss in throughput which could be compensated to some extent by using constellation shaping techniques [20]. The symmetry in the estimation of $+1$ and $-1$ for user $j$ is also considered, which is the result of the symmetry between $p(y_j^R | s_j = -1)$ and $p(y_j^R | s_j = +1)$, where $p$ is the probability density function (pdf), and consequently results in $P(\hat{s}_j = -1 | s_j = +1) = P(\hat{s}_j = +1 | s_j = -1)$. More generally, let us denote the number of constellation points in the modulation by $L$, e.g., $L = 2$ for BPSK. Therefore, $N_b = L^K$ is the number of possible symbol sequences for all $K$ users in one transmission, i.e., there could be $L^K$ different possible sets of $K$-tuple symbols $s_b$, $1 \leq b \leq N_b$, for $K$ users. Assuming BPSK transmission, we denote $N_{pb} = 2^{K-1}$ as the number of possible symbol sequences for transmission if the transmitted symbol of user $j$ is already known to be either $+1$ or $-1$.

Using the total probability theorem, equal probability for transmission of BPSK constellation points, and Gaussian output noise $\text{Re}\{z_j^R\}$, we have

$$p(y_j^R | s_j = +1) = \frac{1}{N_{pb}} \sum_{\substack{t \in \{1, \ldots, K\} \\ t \neq j \\ s_t \in \{\pm 1\}}} p(y_j^R | s_j = +1, \{s_t\})$$

$$= \frac{1}{N_{pb}} \sum_{b=1}^{N_{pb}} \sum_{s_j \neq -1} \frac{1}{\sqrt{\pi\sigma_z^2 w_j^H w_j}} \exp \left( -\frac{(y_j^R - \bar{y}_{j,b}^R(w_j, U))^2}{\sigma_z^2 |w_j|^2} \right),$$
where $\bar{y}_{j,b}^R(w_j, U) = \text{Re}\{\tilde{y}_j\}$ when $s_b$ is transmitted, i.e.,

$$
\bar{y}_{j,b}^R(w_j, U) = \text{Re}\{w_j h_j U s_b\}, \quad 1 \leq b \leq N_b.
$$

Therefore, the probability of error can be calculated as

$$
P_{e_j} = \int_{-\infty}^{0} p(y_j^R | s_j = +1) \, dy_j^R = \frac{1}{N_{pb}} \sum_{b=1}^{N_{pb}} Q \left( \frac{\bar{y}_{j,b}^R}{\sqrt{\frac{\sigma_z^2 |w_j|^2}{2}}} \right)
$$

$$
= \frac{1}{N_b} \sum_{b=1}^{N_b} Q \left( \frac{s_{b,j} \text{Re}\{w_j h_j \sum_{l=1}^{K} u_l s_{b,l}\}}{\sigma^2 \sqrt{2} |w_j|} \right),
$$

where $s_{b,j}$ is the $j$th symbol of $s_b$ and $Q(x)$ is defined as

$$
Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-\frac{u^2}{2}} \, du.
$$

It should be mentioned that the above system model and error probability calculation could be easily extended to the multiple receive antenna scenario.

### IV. Minimum Probability of Error Beamforming

In this section, our objective is to minimize the weighted sum of error probabilities for two different scenarios:

1) First scenario- the receiver only needs its own channel information to calculate its filter coefficient and there is no need for extra feedback. It is assumed that the receivers use ML detection. We try to minimize the weighted sum of error probabilities only over transmit beamforming weights since each receive filter coefficient could be expressed as a function of transmit beamforming weights. This method, while yielding suboptimal performance as compared with joint transmit-receive beamforming is nevertheless easier to practically realize.

2) Second scenario- we assume joint transmit-receive beamforming in which we minimize the error sum over both transmit-receive beamforming weights. This approach could be considered as the optimal method for communication in our broadcast system when considering the error probability as the measure of quality.

If one wants to minimize error probabilities of all users, it means that several objective functions have to be minimized which are all interdependent by the common transmit beamforming matrix. A standard approach to this multiple-objective optimization problem is to combine the individual objective functions into a single composite function \cite{21}, \cite{22}. Similar to \cite{7}, we use the weighted
average error probability of users $P_e = \frac{1}{\sum_{j=1}^{K} \alpha_j} \sum_{j=1}^{K} \alpha_j P_{ej}$. If $\alpha_j = 1$ for all users, i.e., all users have the same priority, the weighted average turns to a simple average of the error probability of users in (1) as

$$P_e = \frac{1}{K} \sum_{j=1}^{K} P_{ej},$$

which is considered as the system performance criterion hereafter. The parameters $w = [w_1, \ldots, w_K]$ and $U$ have a bilinear dependence in each $Q$-function argument of (2). Now, we state the following constraints that are required in both forthcoming scenarios.

**Lemma 1:** For the $j$th user not to have an error floor (a lower bound on error probability with a rough value of $1/N_b$), it is necessary for $w_j$ to comply with the following constraints:

$$s_{b,j} \text{Re} \left\{ w_j \sum_{l=1}^{K} h_j u_l s_{b,l} \right\} \geq 0, \quad 1 \leq b \leq N_b. \quad (3)$$

**Proof:** Assume that there exists a $b = b_1$ such that, for a given $w_j$ and $U$, $s_{b_1,j} \text{Re} \{ w_j \sum_{l=1}^{K} h_j u_l s_{b_1,l} \} < 0$. Therefore,

$$s_{b_1,j} \text{Re} \left\{ w_j \sum_{l=1}^{K} h_j u_l s_{b_1,l} \right\} < 0.$$

Hence,

$$Q \left( \frac{s_{b_1,j} \text{Re} \{ w_j \sum_{l=1}^{K} h_j u_l s_{b_1,l} \}}{\sigma \sqrt{2} |w_j|} \right) > \frac{1}{2}. \quad (4)$$

Moreover, if there exists a $b_1$ such that (4) is true, then there also exists a $b_2$ where each bit in $s_{b_1}$ is inverted. For $b = b_2$ we also have $Q \left( \frac{s_{b_2,j} \text{Re} \{ w_j \sum_{l=1}^{K} h_j u_l s_{b_2,l} \}}{\sigma \sqrt{2} |w_j|} \right) > \frac{1}{2}$, and therefore,

$$P_{ej} > \frac{1}{N_b} + \frac{1}{N_b} \sum_{b=1}^{N_b} \sum_{b \neq b_1, b_2} Q \left( \frac{s_{b,j} \text{Re} \{ w_j \sum_{l=1}^{K} h_j u_l s_{b,l} \}}{\sigma \sqrt{2} |w_j|} \right).$$

In other words, there always exists an error floor of $1/N_b$. 

\[\blacksquare\]
A. Transmit beamforming with ML receiver

For the first scenario, it is assumed that the receivers use single-user maximum likelihood detection similar to [5]. Additionally, for the transmit beamforming design the receive beamforming weights are assumed to be available at the transmitter, because they could be calculated in closedform at the transmitter. For ML detection the receive filter coefficient of a user can be expressed as a function of its transmit weight and channel as [5]

\[ w_j = \frac{u_j^H h_j^H}{|h_j u_j|^2}, \quad 1 \leq j \leq K. \]

Consequently, the error probability of a user in (1) will be only a function of \( U \) and so will be the average error probability of users in (2):

\[ P_{ML}^e(U) = \frac{1}{K N_b} \sum_{j=1}^{K} \sum_{b=1}^{N_b} Q\left( \frac{\sum_{l=1}^{K} \text{Re}\left\{ u_j^H h_j^H h_j u_j s_{b,l} s_{b,j} \right\}}{\sigma_z \sqrt{2} |h_j u_j|} \right). \] (5)

Now, we try to minimize the average error probability \( P_{ML}^e(U) \) by constraining the total transmit power. Based on Lemma 1 without loss of generality the arguments of \( Q \)-functions are constrained to be nonnegative. Thus, the transmit beamforming optimization problem is stated as

\[
\min_U P_{ML}^e \\
\text{subject to} \quad \text{Tr}(UU^H) \leq P_{tot}, \\
\sum_{l=1}^{K} \text{Re}\left\{ u_j^H h_j^H h_j u_j s_{b,l} s_{b,j} \right\} \geq 0, \quad 1 \leq j \leq K, \quad 1 \leq b \leq N_b.
\]

Now, let us restate the transmit beamforming weight \( u_j \) of each user as

\[ u_j = a_j \bar{u}_j, \quad 1 \leq j \leq K, \]

where \( a_j = \frac{u_j}{||u_j||_2} \) is the amplitude of \( u_j \) and \( ||\bar{u}_j||_2 = 1 \). By using the Cauchy-Schwarz inequality the following upper bound on (5) is obtained:

\[ P_{ML}^{e,U} = \frac{1}{K N_b} \sum_{j=1}^{K} \sum_{b=1}^{N_b} Q\left( \frac{\sum_{l=1}^{K} \text{Re}\left\{ \bar{u}_j^H h_j^H h_j \bar{u}_j s_{b,l} s_{b,j} a_l \right\}}{\sigma_z \sqrt{2} ||h_j||} \right). \] (6)
Now, we minimize (6) as follows:

\[
\min_{\bar{U}, a} P_{e}^{\text{ML-U}}
\]

subject to

\[
\sum_{j=1}^{K} a_j^2 \leq P_{\text{tot}},
\]

\[
|h_j \bar{u}_j|^2 a_j + \sum_{l=1}^{K} \text{Re}\{\bar{u}_j^H h_j^H h_j \bar{u}_l s_b l s_{b,j} a_l\} \geq 0, \quad 1 \leq j \leq K,
\]

\[
1 \leq l \neq j \leq N_b,
\]

\[
\|\bar{u}_j\| = 1, \quad 1 \leq j \leq K,
\]

where \(\bar{U} = [\bar{u}_1, \cdots, \bar{u}_K]\) and \(a = [a_1, \cdots, a_K]\).

We propose the alternating minimization algorithm of Table I which minimizes \(P_{e}^{\text{ML-U}}\) over \(\bar{U}\) and \(a\) in alternation to solve (7). In Minimization 1 of the proposed algorithm, when (7a) is minimized over \(\bar{U}\), constraint (7b) does not depend on the minimization parameter and can be removed. Similarly, in Minimization 2, when (7a) is minimize over \(a\), constraints in (7d) are not included since they are independent of the optimization parameters. Now, we observe that the alternating minimization algorithm has the following properties:

**Property 1:** Minimization 2 in Table I is a convex optimization problem.

**Proof:** Assuming \(\alpha \in \mathbb{R}\), we define \(a_0\) as

\[
a_0 = \alpha a_1 + (1 - \alpha) a_2.
\]

We also define the function \(g(a)\) to denote the argument of the \(Q\)-function in (6). Therefore,

\[
g(a_0) = \frac{\|h_j\| a_{0j} + \sum_{l \neq j}^{K} \text{Re}\{\bar{u}_j^H h_j^H h_j \bar{u}_l s_b l s_{b,j} (a_{0l})\}}{\frac{\sigma_j}{\sqrt{2}} \|h_j\|}
\]

\[
= \alpha g(a_1) + (1 - \alpha) g(a_2),
\]

which means that the argument of each \(Q\)-function in (6) is affine with respect to \(a\). Now, considering the fact that \(Q\)-function is convex over positive arguments and that affine mapping does not change the convexity it becomes clear that \(Q(g(a))\) is a convex function \([23]\). Taking into account that the sum of two convex functions is a convex function it could be inferred that (6), i.e., the objective function of (7) is convex with respect to \(a\).
TABLE I
ALTERNATING MINIMIZATION ALGORITHM FOR SOLVING (7)

\begin{tabular}{l}
\textbf{Initialization:} \\
\( \bar{U}^0 \leftarrow \text{random complex } M \times K \text{ matrix such that each column is}
\text{normalized.} \\
\( a^0 \leftarrow \text{random real positive } 1 \times K \text{ vector such that } \sum_{j=1}^{K} a_{j}^2 \leq \) \\
\( P_{\text{tot}}. \) \\
\end{tabular}

Initialize \( P_{1e}^1 \) and \( P_{2e}^2 < P_{1e}^1 \) with proper values to start the while loop, e.g., \( P_{1e}^1 = 1 \) and \( P_{2e}^2 = 0.5. \)

\begin{tabular}{l}
\textbf{while} \( P_{1e}^1 - P_{2e}^2 > P_{\text{threshold}} \textbf{do} \) \\
\( P_{2e}^2 \leftarrow P_{1e}^2 \). \\
\end{tabular}

\begin{tabular}{l}
\textbf{Minimization 1:} \\
Minimize (7) over \( \bar{U} \) assuming that \( a = a^0 \), and using the \text{initial value of } \( \bar{U}^0 \text{ for } \bar{U}. \) \\
\( \bar{U}^0 \leftarrow \bar{U}^{\text{opt}}. \) \\
\end{tabular}

\begin{tabular}{l}
\textbf{Minimization 2:} \\
Minimize (7) over \( a \) assuming \( \bar{U} = \bar{U}^0 \), by using a numerical \text{convex optimization method and the initial value of } a^{0} \text{ for } a. \\
a^{0} \leftarrow a^{\text{opt}}. \\
P_{2e}^2 \leftarrow P_{e}^{\text{ML-U}}(\bar{U}^{0}, a^{0}). \\
\end{tabular}

\begin{tabular}{l}
\textbf{end while} \\
\textbf{end while} \\
\( U = \bar{U}^{0} \text{diag}(a) \text{ is the transmit beamforming matrix.} \) \\
\end{tabular}

It is obvious that the constraint (7b) is a convex set with respect to \( a \) since it represents the volume inside of a \( K \)-dimensional sphere. By using the definition of a convex set [23], it could be shown that the constraints in (7c) are also convex sets with respect to \( a \). Also as mentioned earlier, the constraints defined in (7d) are not included in Minimization 2. Therefore, Minimization 2 of Table I is a convex optimization problem with a convex objective function and closed convex constraints.

\textbf{Theorem 1:} The algorithm presented in Table I converges to a local minimum of (7).

\textbf{Proof:} The objective function \( P_{e}^{\text{ML-U}}(\bar{U}, a) \) is minimized in Minimization 1 of Table I over \( \bar{U} \) while keeping \( a \) fixed at the values obtained in Minimization 2 of the previous iteration. Therefore, at each iteration of the algorithm the value of the objective function after Minimization 1 is non-increasing compared to the value of the objective function after Minimization 2 of the
previous iteration. In Minimization 2 of Table I, $P_{e, \text{ML}}^{\bar{U}}(\bar{U}, \mathbf{a})$ is minimized over $\mathbf{a}$ while keeping $\bar{U}$ at the values obtained in Minimization 1. Therefore, the value of $P_{e, \text{ML}}^{\bar{U}}$ does not increase compared to the result of Minimization 1 of the same iteration. Therefore, each iteration of the algorithm causes $P_{e, \text{ML}}^{\bar{U}}$ to be non-increasing. Moreover, considering the fact that $P_{e, \text{ML}}^{\bar{U}}$ is always nonnegative, $0 \leq P_{e, \text{ML}}^{\bar{U}} \leq 1$, guarantees that the algorithm converges to a stationary (minimum) point.

**B. Joint transmit-receive beamforming**

For the second scenario, it is assumed that the receive filter coefficients are calculated by the transmitter in conjunction with the transmit beamforming weights, and each receiver is provided with its receive filter coefficient. In this scenario, the transmitter finds the transmit beamforming weights and receive filter coefficients that minimize the average error probability of users. Therefore, we may formulate the problem as

$$
\min_{\mathbf{w}, \mathbf{U}} \bar{P}_e
$$

subject to

$$
\text{Tr}(\mathbf{U} \mathbf{U}^H) \leq P_{\text{tot}},
$$

(8)

where $P_e$ is the average error probability of (2). It should be noted that the error probability of user $j$, $P_{e_j}$ in (1), depends on its receive filter coefficient $w_j$, and the transmit beamforming matrix of all users $\mathbf{U}$, but it does not depend on the receive filter coefficients of other users. Based on this observation we strive to develop an alternating minimization algorithm to solve (8). Before designing the optimization algorithm, we determine key properties of (1), (2), and (8) that will be utilized.

**Property 2:** The error probability in (1) and therefore the average error probability (2) are invariant to the scaling of $w_j$ by a positive constant. Based on this property, one can set $|w_j| = 1$ in the error probability of (1) and (2) and add $|w_j| = 1$, $1 \leq j \leq K$, as additional constraints to (8).

**Property 3:** For a fixed $\mathbf{U}$ that satisfies $\text{Tr}(\mathbf{U} \mathbf{U}^H) \leq P_{\text{tot}}$ we have $\arg \min_w P_e = [\arg \min_{w_1} P_{e_1}, \ldots, \arg \min_{w_K} P_{e_K}]$. In other words, $\arg \min_w P_e = \arg \min_{w_j} P_{e_j}$. This is obvious since each $P_{e_j}$ depends only on $w_j$ but not on $w_l$, $l \neq j$. Therefore, minimizing (2) over all $w_j$s could be partitioned into $K$ decoupled minimizations of $P_{e_j}$ over $w_j$s, $1 \leq j \leq K$. 
If one wants to solve (8) over \( w \) for a given \( U \), it is clear from Properties 2 and 3 and Lemma 1 that when there exists no error floor, without loss of generality, the constraints \( |w_j| = 1 \) and (3), for \( 1 \leq j \leq K \), could be added to the optimization problem. Therefore, for a given \( U \), the \( K \) optimization problems arising from (8) could be rewritten as follows:

\[
\min_{w_j} \frac{1}{N_b} \sum_{b=1}^{N_b} Q \left( \frac{\sqrt{2}}{\sigma_z} s_{b,j} \text{Re} \left\{ w_j \sum_{l=1}^{K} h_j u_{l,s_{b,l}} \right\} \right)
\]

subject to \( |w_j| = 1 \),
\[
s_{b,j} \text{Re} \left\{ w_j \sum_{l=1}^{K} h_j u_{l,s_{b,l}} \right\} \geq 0, \quad 1 \leq b \leq N_b,
\]

for \( 1 \leq j \leq K \). Now, we have the following proposition:

**Proposition 1:** If the constraints in the minimization problem (9) are satisfied, any local minimizer of error probability function \( P_{e_j} \), the objective function of the optimization problem (9), is also a global minimizer. Moreover, the global minimizer is unique.

**Proof:** See Appendix A. ■

**Property 4:** \( \min_{w_j} P_{e_j} \), where \( P_{e_j} \) is calculated as in (1), could be transformed to a convex optimization problem with a unique global minimizer.

**Proof:** Properties 2 and 3, Lemma 1, and Proposition 1 show that without loss of generality \( \min_{w_j} P_{e_j} \) could be stated as (9) which does not have any local minimizer and has only one global minimizer. Problem (9) could be equivalently rewritten in the form of convex optimization problem as

\[
\min_{w_j} \frac{1}{N_b} \sum_{b=1}^{N_b} Q \left( \frac{\sqrt{2}}{\sigma_z} s_{b,j} \text{Re} \left\{ w_j \sum_{l=1}^{K} h_j u_{l,s_{b,l}} \right\} \right)
\]

subject to \( |w_j| \leq 1 \),
\[
s_{b,j} \text{Re} \left\{ w_j \sum_{l=1}^{K} h_j u_{l,s_{b,l}} \right\} \geq 0, \quad 1 \leq b \leq N_b,
\]

since \( |w_j| \leq 1 \) is an active constraint. ■

Hence, based on Properties 2-4, (8) could be written as

\[
\min_{w,U} \frac{1}{KN_b} \sum_{j=1}^{K} \sum_{b=1}^{N_b} Q \left( \frac{\sqrt{2}}{\sigma_z} s_{b,j} \text{Re} \left\{ w_j \sum_{l=1}^{K} h_j u_{l,s_{b,l}} \right\} \right)
\]

subject to \( \text{Tr}(UU^H) \leq P_{tot} \),

(11a)

subject to \( \text{Tr}(UU^H) \leq P_{tot} \),

(11b)
\[ |w_j| \leq 1, \quad 1 \leq j \leq K, \tag{11c} \]

\[ s_{b,j} \text{Re}\{w_j \sum_{l=1}^{K} h_{j} u_{l} s_{b,l}\} \geq 0, \quad 1 \leq j \leq K, \quad 1 \leq b \leq N_b. \tag{11d} \]

**Property 5:** Optimization problem (II) is a convex optimization problem with respect to U.

**Proof:** See Appendix B.

Using Properties 4 and 5, we propose the alternating minimization algorithm of Table II for problem (8). It should be noted that in this algorithm in Minimization 1 when (IIa) is minimized over U, constraints in (11c) do not depend on the optimization parameter and therefore are not included. It is easily shown that the algorithm in Table II converges to a local minimum.

**Theorem 2:** The algorithm in Table II converges to a local minimum of (8).

**Proof:** Since the objective function \( P_e(U, w) \) is minimized at both Minimization 1 and 2, each iteration causes \( P_e \) to be non-increasing. Moreover, considering the fact that \( P_e \) is always nonnegative, \( 0 \leq P_e \leq 1 \), it guarantees that the algorithm converges to a stationary (minimum) point.

V. USER SELECTION

In this section, we develop a user selection algorithm by taking into account that MPE transmit beamforming is utilized at the transmitter. It is assumed that in total \( K_T \) users are in the system such that \( K_T \gg M \) and the set of all users is given by \( A = \{1, \cdots, K_T\} \). The transmitter selects a subset, \( S \), of \( K \) users out of \( K_T \) users, \( S \subseteq A \), for transmission. The users should be selected in such a way that certain criteria are met. We are interested in maximizing the number of selected users and minimizing the error probabilities at the same time. Due to MPE beamforming, all users receive BPSK signals, so maximizing the number of users may increase the throughput. Hence, ideally we would like to solve the following multiobjective optimization problem which only has an extra objective function compared with (II):

\[
\min_{w, U, S \subseteq A} \frac{1}{K N_b} \sum_{j \in A} \sum_{b=1}^{N_b} Q\left( \sqrt{\frac{2}{\sigma^2}} s_{b,j} \text{Re}\{w_j \sum_{l \in A} h_{j} u_{l} s_{b,l}\} \right) \\
\max_{w, U, S \subseteq A} K \\
\text{subject to} \quad \text{Tr}(UU^H) \leq P_{tot}, \\
|w_j| \leq 1, \quad j \in S,
\]
TABLE II
ALTERNATING MINIMIZATION ALGORITHM FOR SOLVING (8)

| Initialization: |
|----------------|
| $U_0 \leftarrow \text{random complex } M \times K \text{ matrix.}$ |
| $U_0 \leftarrow \frac{U_0}{\|U_0\|_F} P_{\text{tot}}.$ |
| $w_0 \leftarrow \text{normalized random complex number.}$ |
| $w_0 \leftarrow [w_0]^1 \times K.$ |
| Initialize $P_1^e = [P_1^e, \cdots, P_K^e]$ and $P_2^e \prec P_1^e$ with proper values to start the while loop, e.g. $P_1^e = [1]_1 \times K$ and $P_2^e = [0.5]_1 \times K.$ |
| while $\sum_{k=1}^{K} P_1^e_k > P_{\text{threshold}}$ do |
| $P_1^e \leftarrow P_2^e.$ |
| Minimization 1: |
| Minimize (11) over $U$ assuming that $w = w_0$ by using a numerical convex optimization method and the initial value of $U_0$ for $U.$ |
| $U_0 \leftarrow U_{\text{opt}}.$ |
| Minimization 2: |
| for $j = 1 : K$ do |
| Minimize (10) over $w_j$ assuming $U = U_0$ by using a numerical convex optimization method and the initial value of $w_{0j}$ for $w_j.$ |
| $w_{0j} \leftarrow w_{0j}^\text{opt}.$ |
| $P_2^e_j \leftarrow P_{s_j}(U_0, w_{0j}).$ |
| end for |
| end while |

$U_0$ and $w_0$ are the transmit beamforming weights and receive filter coefficients, respectively.

$$s_{b,j} \text{Re}\{w_j \sum_{l \in S} h_{j,l} u_l s_{b,l}\} \geq 0, \ j \in S, \ 1 \leq b \leq N_b,$$  

(12)

where $K$ is the cardinality of the set of selected users, $|S|$. In the majority of existing user selection algorithms, e.g. [4], [14], the number of selected users $K$ should be less than or equal to the degrees of freedom (DoF) of the system, which in this case is equal to $M$. However, the number of selected users could possibly be more than the DoF of the system, in the user selection method that is proposed in this section as will be confirmed in the numerical results. This is because of the fact that in the proposed user selection method the constraints of the optimization...
(12) originate from the MPE beamforming problem which in turn considers a one-dimensional discrete modulation rather than a continuous Gaussian input distribution.

From the viewpoint of system design, it is more practical to solve the user selection and beamforming problems separately. Although suboptimal, (12) is separated into a minimization over \( U \) and \( w \) and a maximization over \( S \). Since the beamforming problem has been addressed in Section IV, user selection will be the focus in this section. Therefore, instead of (12) we solve

\[
\begin{align*}
\text{max}_{S \subseteq A} |S| \\
\text{subject to} \quad \text{Tr}(UU^H) &= \sum_{j \in S} ||u_j||^2 \leq P_{tot}, \\
|w_j| &\leq 1, \quad j \in S, \\
s_{b,j}\text{Re}\{w_j \sum_{l \in S} h_j u_l s_{b,l}\} &\geq 0, \quad j \in S, \quad 1 \leq b \leq N_b.
\end{align*}
\]

In other words, (13) maximizes the cardinality of the set of selected users such that none of the selected users experience error floors (Lemma 1) and the transmit power constraint is met, i.e., the number of selected users is maximized subject to the constraint set of the MPE beamforming problem. This means that we try to find the maximum number of users such that the feasible region of the MPE beamforming problem is not an empty set. This combinatorial optimization problem has very high computational complexity. Therefore, we try to find a suboptimal algorithm for finding a good set of selected users.

For a tractable solution, we are interested in user selection as a separate entity from beamforming. Therefore, we try to remove the dependence of the optimization problem (13) from the transmit and receive beamforming weights. But before doing this, we first simplify the problem by finding a lower bound for the left side of the inequality of the constraint (13d) as

\[
s_{b,j}\text{Re}\{w_j \sum_{l \in S} h_j u_l s_{b,l}\} \geq \text{Re}\{w_j h_j u_j\} - \sum_{l \in S \setminus \{l \neq j\}} |\text{Re}\{w_j h_j u_l\}|.
\]

This lower bound is obtained by using the fact \( s_{b,j}s_{bl} = \pm 1 \) and \( x \geq -|x| \). Since this lower bound is independent of \( b \), all \( N_b \) constraints for a fixed \( j \) in (13d) are replaced with the single constraint:

\[
\text{Re}\{w_j h_j u_j\} - \sum_{l \in S \setminus \{l \neq j\}} |\text{Re}\{w_j h_j u_l\}| \geq 0, \quad j \in S.
\]
Replacing (13d) with (14) reduces the size of the feasible region in the optimization problem (13).

Now, if we further assume that the receivers use ML detection with normalized filter coefficients we have \( w_j = \frac{u_j^H h_j^H}{|u_j^H h_j^H|} \) and therefore (13c) could be removed from (13). The modified constraint of (13d), i.e. (14) is then substituted with the following bound

\[
\sum_{l \in \mathcal{S}, \ l \neq j} |\text{Re}\{u_j^H h_j^H h_l u_l\}| \leq |h_j u_j|^2.
\]

Now, to remove the dependence of the user selection problem on transmit beamforming vectors \( u_j \)'s, normalized maximum ratio transmission (conjugate beamforming) is assumed for transmission such that \( u_j = h_j^H \sqrt{\frac{P_r}{\sum_{i \in \mathcal{S}} |h_i|^2}} \) [24], [25]. Consequently, this removes (13b) and simplifies (13d) further to

\[
\sum_{l \in \mathcal{S}, \ l \neq j} |\text{Re}\{h_j h_l^H\}| \leq |h_j|^2.
\]

Therefore, the user selection optimization problem has been simplified to

\[
\begin{align*}
\max_{\mathcal{S} \subseteq \mathcal{A}} & \quad |\mathcal{S}| \\
\text{subject to} & \quad \sum_{l \in \mathcal{S}, \ l \neq j} |\text{Re}\{h_j h_l^H\}| \leq |h_j|^2, \ \forall j \in \mathcal{S}. \quad (15a) \quad (15b)
\end{align*}
\]

This problem could be interpreted as follows: There is an \( M \) dimensional vector space on the complex field \( \mathbb{C} \), i.e., \( \mathbb{C}^M \), in which there are \( K_T \gg M \) given elements. We want to choose the maximum number of elements with the following properties: \( \sum_{l \in \mathcal{S}} |\text{Re}\{h_j h_l^H\}| \leq |h_j|^2, \ \forall j \in \mathcal{S}. \)

Again we try to simplify the problem by reducing the feasible region. A sufficient condition for (15b) to hold is that

\[
|\text{Re}\{h_j h_l^H\}| \leq \frac{\min(|h_j|^2, |h_l|^2)}{|\mathcal{S}| - 1}, \ \forall j, l \in \mathcal{S}.
\]

Substituting (15b) with the above sufficient condition gives us

\[
\begin{align*}
\max_{\mathcal{S} \subseteq \mathcal{A}} & \quad |\mathcal{S}| \\
\text{subject to} & \quad \frac{|\text{Re}\{h_j h_l^H\}|}{\min(|h_j|^2, |h_l|^2)} \leq \frac{1}{|\mathcal{S}| - 1}, \ \forall j, l \in \mathcal{S}. \quad (16)
\end{align*}
\]

From a geometric point of view, problem (16) is similar to packing lines in a Grassmannian manifold of dimension \( M \), i.e., \( G(1, M) \) [26]–[29]. Namely, problem (16) is to pack the \( M \)-complex space with the maximum number of lines passing through the origin such that \( \text{real} \)
correlation distance between any two lines is less than some value, where we define the real correlation distance between two lines as

\[ d_{RC}(\mathbf{h}_i, \mathbf{h}_j) \triangleq \frac{|\text{Re}\{\mathbf{h}_j \mathbf{h}_i^H\}|}{\min(|\mathbf{h}_j|^2, |\mathbf{h}_i|^2)}. \]

It should be remarked that \( d_{RC} \) is not actually a distance or metric but only possess some properties of a metric. To solve (16) we propose the geometric user selection (GUS) algorithm of Table III.

This suboptimum low-complexity algorithm semi-greedily selects users based on both their channel strength and their real correlation distance. In the Main Body of Algorithm in Table III.
first the user with the strongest channel in the set of candidate users, \( C \), is opted for the set of selected users \( S \). Then, any user from the set of available users, \( A \), with real correlation distance of more than \( \frac{1}{K-1} \) from the previously selected user, \( \pi_i \), is removed for the next set of available users. To make the packing tighter, the algorithm selects users from a set of candidate users rather than from all available users. A user is considered for the next set of candidate users if its distance from the last selected user, \( \pi_i \), is close to the upper bound \( \frac{1}{K-1} \). The parameter \( \alpha \) determines how close the distance of the next candidate users from the last selected user should be to the upper bound. The Main Body of Algorithm iterates until either \( K \) users are selected or the set of candidate users is empty. It should be remarked that the cardinality of the selected set in this algorithm is upper bounded by the initial guess for the number of users \( K \).

In Decision for Next Iteration in Table III it is decided whether the set of selected users could be improved either in the sense of size or the distance among users. If \( \text{iter} = 1 \), then depending on whether the cardinality of the selected set is equal to \( K_{\text{iter}} \) or less than \( K_{\text{iter}} \), \( K \) is incremented or decremented respectively, i.e., \( K_{\text{iter}+1} = K_{\text{iter}} + 1 \) or \( K_{\text{iter}+1} = K_{\text{iter}} - 1 \).

As a result the number of iterations is always greater than one. If \( \text{iter} \neq 1 \) then one of four different scenarios may occur:

1) If \( K_{\text{iter}} > K_{\text{iter}-1} \) and the size of the current selected set is larger than or equal to the size of the previous selected set ( \( |S_{\text{iter}}| \geq |S_{\text{iter}-1}| \) ), then the algorithm gets greedy and prepares to check if the size of the selected set could be further improved by incrementing \( K \) for the next iteration, i.e., \( K_{\text{iter}+1} = K_{\text{iter}} + 1 \).

2) If \( K_{\text{iter}} > K_{\text{iter}-1} \) and \( |S_{\text{iter}}| < |S_{\text{iter}-1}| \) then the previous selected set is considered as the selected set (\( S = S_{\text{iter}-1} \)) and the algorithm breaks from the loop.

3) If \( K_{\text{iter}} < K_{\text{iter}-1} \) and \( |S_{\text{iter}}| \geq K_{\text{iter}} - 1 \) then the current selected set is chosen as the selected set (\( S = S_{\text{iter}} \)) and the algorithm breaks from the loop.

4) Otherwise, \( K_{\text{iter}+1} = K_{\text{iter}} - 1 \).

Remarks on initialization: Now, we justify the choice of \( K = M + 1 \) as the initial value for \( K \) in GUS algorithm. In the context of packing, geodesic distance and chordal distance are more common distance metrics with existing bounds for packing problems. Therefore, to find an approximate value for the maximum number of users \( K \), the size of the feasible region in
is reduced, again by replacing the constraint of (16) with a sufficient condition as

\[
\max_{S \subseteq A} |S| 
\]  
(17a)

subject to

\[
\frac{|h_j h_l^H|}{\min(|h_j|^2, |h_l|^2)} \leq \frac{1}{|S| - 1}, \quad \forall j, l \in S,
\]  
(17b)

allowing us to utilize existing bounds from differential geometry.

Next, we find an approximation for the maximum number of lines that could fill the \(M\) space such that (17b) is satisfied. In \(G(1, M)\), the principal angle between lines \(h_j\) and \(h_l\) is defined as \([30]\)

\[
\theta_{j,l} = \arccos \frac{|\langle h_j, h_l \rangle|}{|h_j| |h_l|}
\]

and the chordal distance between two lines is defined as \([26]\), \([27]\)

\[
d_c(h_j, h_l) = |\sin(\theta_{j,l})| = \sqrt{1 - \frac{|h_j h_l^H|^2}{|h_j|^2 |h_l|^2}}.
\]  
(18)

Similar to the Rankin bound for spherical codes, there exists the following upper bound on the chordal distance between any two lines in the Grassmannian manifolds for packing \(K\) lines \([26]\), \([27]\):

\[
d_c^2 \leq \left\{ \begin{array}{ll} \frac{(M-1)K}{M(K-1)} & \text{if } K \leq \left( \frac{M+1}{2} \right) \\ \frac{(M-1)}{M} & \text{if } K > \left( \frac{M+1}{2} \right). \end{array} \right.
\]  
(19)

Using (18) and (19) we have

\[
\frac{|h_j h_l^H|^2}{\min(|h_j|^2, |h_l|^2)} \geq \frac{|h_j h_l^H|^2}{|h_j|^2 |h_l|^2} \geq 1 - \frac{(M-1)K}{M(K-1)} \quad \text{if } K \leq \left( \frac{M+1}{2} \right)
\]

and by taking (17b) into account, we have

\[
\left\{ \begin{array}{ll} 1 - \frac{(M-1)K}{M(K-1)} \leq \frac{1}{(K-1)^2} & \text{if } K \leq \left( \frac{M+1}{2} \right) \\ 1 - \frac{(M-1)}{M} \leq \frac{1}{(K-1)^2} & \text{if } K > \left( \frac{M+1}{2} \right). \end{array} \right.
\]

This is equivalent to the following sufficient (but not necessary) condition:

\[
K \leq M + 1,
\]

which serves as an approximation to the number of selected users in the GUS algorithm.
VI. Numerical Results

A. MPE beamforming methods

In this section, we first consider a broadcast channel with a 3-antenna transmitter sending information to 3 users each equipped with 1 antenna. It is assume that the transmitter has one information symbol for each receiver in the same time and frequency slots. The channel gains are assumed to be quasi static and follow a Rayleigh distribution. Since our focus is on various beamforming methods rather than on the effect of channel estimation, we assume that perfect CSI of all channels is available at the transmitter and each receiver only has perfect knowledge of its own channels [3], [5]. At the receivers, white Gaussian noise is added to the received signal.

Fig. 2 compares the bit error rates of MSLNR, ZF, MMSE, MPE-ML (from Section IV-A), and joint Tx-Rx MPE (from Section IV-B) beamforming methods. For all of these methods, maximum likelihood is used at the receiver except for joint Tx-Rx MPE beamforming in which each receive filter coefficient is calculated by the transmitter jointly with the transmit beamforming weights and is provided to the corresponding receiver. As can be seen, MPE transmit beamforming methods substantially improve the performance of all users. For example, at BER of $10^{-2}$ both MPE beamforming methods show a gain of about 6.5dB compared with MMSE beamforming and much more gain compared with ZF and MSLNR. It should be mentioned that in Fig. 2, theoretical BER curves of MPE-ML and Tx-Rx MPE beamforming methods are obtained by substituting the calculated beamforming weights for each channel realization into the error probability expressions of (6) and (11a) respectively. All other curves in Fig. 2 are the result of Monte Carlo simulations. As can be seen in Fig. 2, the simulations confirm the theoretical results.

It is interesting to observe that the performance of MPE-ML beamforming closely follows that of the joint Tx-Rx MPE beamforming up to a certain SNR. At higher SNRs, although still outperforming classical beamforming methods, MPE-ML beamforming shows a large gap in performance compared with Tx-Rx MPE beamforming. The reason that MPE-ML beamforming cannot follow the performance of Tx-Rx MPE beamforming is that in Minimization 1 of Table 1, the optimization problem is not convex. Therefore, the solver of the optimization might get stuck in a local minimizer instead of the global minimizer of (7).

In Fig. 3 we show the average number of iterations needed for the convergence of the
algorithms in Table I and II. In both algorithms the average number of iterations is less than 20. It is very interesting to observe that as SNR increases the number of iterations needed for the convergence of the algorithm in Table II decreases. This is because $P_{\text{threshold}}$ is set fixed at $10^{-8}$ for all SNRs while the error probability decreases from about $10^{-1}$ to $10^{-5}$. If it is assumed that the error probability is known in advance, it would be better to change $P_{\text{threshold}}$ to a small fraction of the error probability to have a faster convergence with the same reliability.

Although in principle MPE beamforming may be extended to higher-order modulations, its main advantage lies in lower-order modulations. If a higher-order modulation is used the optimizations contain more terms and become more computationally complex.

**B. User selection**

First, the proposed GUS algorithm is compared with semi-orthogonal user selection (SUS) algorithm of [4]. Fig. 4 shows the average number of selected users over 1000 different channel realizations versus the total number of available users for three different cases: when the number of transmit antennas is 2, 4, and 6. We observe that for SUS it is possible to have at most as many users as the number of transmit antennas, as it is obvious by its algorithm in [4], while
Fig. 3. Average number of iterations for convergence of algorithms of Table I and II when $M = 3$ antennas and $K = 3$ users without user selection.

GUS may select more users. For example, when there are $K_T = 10000$ users available, GUS can select 6 users with only 4 antennas compared to the 4 selected users of the SUS algorithm. When the transmitter has 2 antennas and there are only 10 users available, GUS can select 2.48 users on average which is still more than the number of transmit antennas) while SUS selects 1.92 users on average.

As expected, it can also be seen that until saturation, the number of selected users for both GUS and SUS increases as the total number of available users increases. Each user corresponds to a line passing through the origin of the complex $M$ dimensional hypersphere. Hence, when the number of users increases, the density of possible lines to choose from increases. This makes it more likely to pack the space with more lines obeying the specific distance.

Now, we study the effect of user selection in conjunction with beamforming. Consider a system with a 2-antenna transmitter that first selects a set of users out of $K_T = 50$ users by using either GUS or SUS, and then sends information to the selected users using various beamforming methods. Fig. 5 shows the error probabilities of different combinations of user selection and beamforming methods. As can be seen, the lowest error probabilities are achieved

\[1\text{Of course it is not practical to service 10000 users with one transmitter. This large number of users is just for illustration purposes to gain insight into the system.} \]
by SUS rather than GUS, and by a small margin, MPE beamforming outperforms all other beamforming methods used for SUS. However, it could not be simply concluded that SUS algorithm outperforms GUS since the average number of selected users over all SNRs is 2 for SUS while it is 3.19 for GUS. In other words, although GUS selects more users, they achieve higher error probabilities in comparison with the selected users of SUS, as expected. It could also be noticed that for the proposed geometric user selection algorithm, ZF and MMSE beamforming are absent in Fig. 5. Owing to the fact that the number of selected users by GUS is more than the number of transmit antennas, GUS is not suitable for ZF and MMSE beamforming.

As observed in Fig. 5 since the error probability alone is not a good indicator of the performance when there are different numbers of users in the system, Figs. 6 and 7 are provided to give more insight into the GUS and SUS performances. In Fig. 6 the throughput is shown for different combinations of user selection and beamforming. Similar to [19], we use the notion of expected throughput for frame-based transmission as $E[\text{Thr}] = (1 - P_e)\ell K_{\text{avg}}$ bits per channel use, where $\ell$ is the frame size and $K_{\text{avg}}$ is the average number of selected users. We consider two different frame sizes: 100 and 500 bits. No channel coding is assumed, and a transmission is considered to be successful if the entire frame is decoded error-free. It can be seen in Fig. 6 that as SNR increases the achievable expected throughput approaches limits determined by the average numbers of selected users by SUS and GUS. For this example, at higher SNRs
Fig. 5. Average bit error rate of users when $M = 2$ antennas and $K_T = 50$ users. In this scenario, the average number of selected users over all SNRs is 2 for SUS while it is 3.19 for GUS.

Fig. 6. Average frame-wise rate when $M = 2$, $K_T = 50$, and frame size equals 100 and 500.

the achievable throughput by GUS and MPE beamforming is about 160% of the achievable throughput by SUS and any of the other beamforming methods. Moreover, Fig. 5 shows that as the frame size increases the throughput decreases. However, for all frame sizes the throughput eventually approaches the upper limit of the throughput dictated by the average number of selected users.
In Fig. 7 the error probabilities of different combinations of user selection and beamforming methods are shown when the transmitter has 4 antennas. We modified our GUS algorithm to only select at most $M$ users so that the number of selected users would be the same as that of SUS algorithm. Since GUS design is specific to MPE beamforming, it is only for MPE beamforming that it performs comparably with SUS. Nevertheless, for the other beamforming methods GUS shows reasonably good results but SUS outperforms as expected. In short, the major advantages of GUS to SUS lie in its ability to select more users than the number of transmit antennas for MPE beamforming, resulting in higher throughput for MPE beamforming, in addition to its lower complexity.

VII. CONCLUSION

In this paper, it has been demonstrated that by exploiting the type of modulation, transmit beamforming can be designed by directly minimizing the average error probability of multiple users. Two algorithms were developed based on the concepts of alternating minimization and convex optimization for two different cases: 1- when the receivers use single-user maximum likelihood detection, 2- when the transmitter optimizes the transmit beamforming weights and the receive filter coefficients jointly. It has been shown that minimum probability of error beamforming dramatically outperforms conventional transmit beamforming methods such as ZF,
MMSE, and MSLNR. Moreover, a geometric user selection (GUS) algorithm was developed that selects as many users as possible, assuming that the transmitter employs MPE beamforming. It has been shown that the number of selected users by the proposed GUS algorithm could potentially be greater than the number of transmit antennas.

**APPENDIX A**

**PROOF OF PROPOSITION [1]**

The minimization problem is considered over the following set:

\[ \mathcal{W}_j = \{ w_j : |w_j| = 1, s_{b,j} \text{Re} \{ w_j \sum_{l=1}^{K} h_j u_l s_{b,l} \} \geq 0, 1 \leq b \leq N_b \}. \]  

(20)

Assume that \( w_{j_1} \in \mathcal{W}_j \) is a global minimizer of the optimization problem (9), and \( w_{j_2} \in \mathcal{W}_j \) is a local minimizer of the problem such that

\[ P_{e,j}(w_{j_1}) < P_{e,j}(w_{j_2}). \]  

(21)

Assuming \( 0 < \alpha < 1 \), we define \( w_{j_0} \) as

\[ w_{j_0} = \frac{\alpha w_{j_1} + (1 - \alpha) w_{j_2}}{\| \alpha w_{j_1} + (1 - \alpha) w_{j_2} \|}. \]

Therefore, we have \( \| w_{j_0} \| = 1 \), and for \( 1 \leq b \leq N_b \), we have \( s_{b,j} \text{Re} \{ w_{j_0} \sum_{l=1}^{K} h_j u_l s_{b,l} \} \geq 0 \). Hence, it can be inferred that \( w_{j_0} \in \mathcal{W}_j \). It is also obvious that

\[ \| \alpha w_{j_1} + (1 - \alpha) w_{j_2} \| \leq \alpha \| w_{j_1} \| + (1 - \alpha) \| w_{j_2} \| = 1. \]

Consequently,

\[ s_{b,j} \text{Re} \left\{ w_{j_0} \sum_{l=1}^{K} h_j u_l s_{b,l} \right\} \geq \alpha s_{b,j} \text{Re} \left\{ w_{j_1} \sum_{l=1}^{K} h_j u_l s_{b,l} \right\} + (1 - \alpha) s_{b,j} \text{Re} \left\{ w_{j_2} \sum_{l=1}^{K} h_j u_l s_{b,l} \right\}, \]  

(22)
for \(1 \leq b \leq N_b\). Therefore,

\[
Q\left(\frac{\sqrt{2}}{\sigma_z} s_{b,j} \text{Re}\{w_{j_0} \sum_{l=1}^{K} h_{j,u} s_{b,l}\}\right) \leq \\
\alpha s_{b,j} \text{Re}\{w_{j_1} \sum_{l=1}^{K} h_{j,u} s_{b,l}\} + (1 - \alpha) s_{b,j} \text{Re}\{w_{j_2} \sum_{l=1}^{K} h_{j,u} s_{b,l}\}
\]

\[
\leq \alpha Q\left(\frac{\sqrt{2}}{\sigma_z} s_{b,j} \text{Re}\{w_{j_1} \sum_{l=1}^{K} h_{j,u} s_{b,l}\}\right) + \\
(1 - \alpha) Q\left(\frac{\sqrt{2}}{\sigma_z} s_{b,j} \text{Re}\{w_{j_2} \sum_{l=1}^{K} h_{j,u} s_{b,l}\}\right)
\]

(23)

where the first inequality is the result of (22) and due to the fact that \(Q(x)\) is a decreasing function for \(x \geq 0\), and the second inequality stands because \(Q(x)\) is a convex function for \(x \geq 0\).

From (1) and (23), it can be inferred that

\[
P_{e_j}(w_{j_0}) = \frac{1}{N_b} \sum_{b=1}^{N_b} Q\left(\frac{\sqrt{2}}{\sigma_z|w_{j_0}|} s_{b,j} \text{Re}\{w_{j_0} \sum_{l=1}^{K} h_{j,u} s_{b,l}\}\right) \leq \\
\alpha \frac{1}{N_b} \sum_{b=1}^{N_b} Q\left(\frac{\sqrt{2}}{\sigma_z|w_{j_1}|} s_{b,j} \text{Re}\{w_{j_1} \sum_{l=1}^{K} h_{j,u} s_{b,l}\}\right) + \\
1 - \alpha \frac{1}{N_b} \sum_{b=1}^{N_b} Q\left(\frac{\sqrt{2}}{\sigma_z|w_{j_2}|} s_{b,j} \text{Re}\{w_{j_2} \sum_{l=1}^{K} h_{j,u} s_{b,l}\}\right) = \\
\alpha P_{e_j}(w_{j_1}) + (1 - \alpha) P_{e_j}(w_{j_2}) < P_{e_j}(w_{j_2}), \quad \forall \alpha \in (0,1),
\]

where the last inequality is due to the fact that \(w_{j_1}\) is the global minimizer of \(P_{e_j}(w_j)\). Now, let \(\alpha \to 0\), \(w_{j_0} \to w_{j_2}\). Hence, in a small neighborhood of \(w_{j_2}\), there always exists a \(w_{j_0}\), so that \(P_{e_j}(w_{j_0}) < P_{e_j}(w_{j_2})\), i.e., \(w_{j_2}\) is not a local minimizer. In other words, there does not exist any local minimizer such that (21) holds. Therefore, it can be concluded that either there is no local minimizer at all which proves the proposition, or there exists a local minimizer such that \(P_{e_j}(w_{j_1}) \geq P_{e_j}(w_{j_2})\). However, since \(w_{j_1}\) is a global minimizer of \(P_{e_j}(w_j)\), we have \(P_{e_j}(w_{j_1}) \leq P_{e_j}(w_{j_2})\). Therefore, it can be concluded that \(P_{e_j}(w_{j_1}) = P_{e_j}(w_{j_2})\), i.e., the local minimizer (if exists) is also a global minimizer.
To show the uniqueness of the global minimizer, first the following set is considered:

$$\mathcal{W}_j^0 = \{ w_j : |w_j| = 1, \text{Re} \left\{ w_j \sum_{l=1}^{K} h_j u_l \right\} = 0, 1 \leq b \leq N_b \}.$$ 

It is obvious that each point in this set is a global maximizer of error probability function in (1) constrained by the set defined in (20), because of the fact that the arguments in all $Q$-functions in error probability will be zero. Therefore, to solve the minimization problem it is sufficient to solve the problem over the set $$\mathcal{W}_j^1 = \mathcal{W}_j - \mathcal{W}_j^0.$$ 

$P_{e_j}(w_j)$ is strictly convex on $\mathcal{W}_1$, because $Q(x)$ is strictly convex for $x > 0$. Assume that $w_{j1} \neq w_{j2}$ are two global minimizers of the optimization problem (9). We define $w_{j0}$ as follows:

$$w_{j0} = \frac{\alpha w_{j1} + (1 - \alpha) w_{j2}}{\| \alpha w_{j1} + (1 - \alpha) w_{j2} \|}, \quad \forall \alpha \in (0, 1).$$

Since $w_{j1}$ is a global minimizer, it is obvious that

$$P_{e_j}(w_{j0}) \geq P_{e_j}(w_{j1}). \quad (24)$$

On the other hand, we have

$$P_{e_j}(w_{j0}) < \alpha P_{e_j}(w_{j1}) + (1 - \alpha) P_{e_j}(w_{j2}) = P_{e_j}(w_{j1}), \quad (25)$$

because $P_{e_j}(w_j)$ is strictly convex on $\mathcal{W}_1$. Since (25) contradicts (24), it can be inferred that the global minimizer is unique.

**APPENDIX B**

**PROOF OF PROPERTY 5**

Before stating the proof it should be noted that the Re\{\} operator is not a linear function. Assuming $\alpha \in \mathbb{R}$, we define $U_0$ as

$$U_0 = \alpha U_1 + (1 - \alpha) U_2.$$

We also define the function $g(U)$ to denote the argument of the $Q$-function in (11). Therefore,

$$g(U_0) = \frac{\sqrt{2}}{\sigma_z} s_{b,j} \text{Re} \{ w_j \sum_{l=1}^{K} h_j (\alpha u_{1l} + (1 - \alpha) u_{2l}) s_{b,l} \} = \alpha g(U_1) + (1 - \alpha) g(U_2),$$

which means that the argument of each $Q$-function in (11a) is affine with respect to $U$. Now, considering the fact that the $Q$-function is convex for positive arguments and that affine mapping
does not change the convexity, \( Q(g(U)) \) is a convex function \(^23\). Since the sum of two convex functions is a convex function it becomes clear that the objective function in (11), \( P_e(U) \), is convex with respect to \( U \).

It is obvious that the constraint (11b) is a convex set with respect to \( U \) since \( \text{Tr}(UU^H) = \|U\|_F^2 \) and every norm is a convex function. The second constraint is not defined over \( U \) and the third constraint could be shown to be a convex set by using the definition. Therefore, problem (11) is a convex optimization problem over \( U \) with a convex objective function and closed convex constraints.

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