Permutability graph of cyclic subgroups

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Abstract

Let $G$ be a group. The permutability graph of cyclic subgroups of $G$, denoted by $\Gamma_c(G)$, is a graph with all the proper cyclic subgroups of $G$ as its vertices and two distinct vertices in $\Gamma_c(G)$ are adjacent if and only if the corresponding subgroups permute in $G$. In this paper, we classify the finite groups whose permutability graph of cyclic subgroups belongs to one of the following: bipartite, tree, star graph, triangle-free, complete bipartite, $P_n$, $C_n$, $K_4$, $K_1,3$-free, unicyclic. We classify abelian groups whose permutability graph of cyclic subgroups are planar. Also we investigate the connectedness, diameter, girth, totally disconnectedness, completeness and regularity of these graphs.

Keywords: Permutability graph, cyclic subgroup, bipartite graph, planar.

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1 Introduction

The properties of a group can be studied by assigning a suitable graph to it and by analyzing the properties of the associated graphs using the tools of graph theory. The Cayley graph is a well known example of a graph associated to a group, which have been studied extensively in the literature (see, for example, [9, 14]). In the past twenty five years many authors have assigned various graphs to study some specific properties of groups. For instance, see [1, 8, 12, 16].

Recall that two subgroups $H$ and $K$ of a group $G$ are said to permute if $HK = KH$; equivalently $HK$ is a subgroup of $G$. In [2], Aschbacher defined a graph corresponding to a group $G$ and for a fixed prime $p$, having all the subgroups of order $p$ as its vertices and two vertices are adjacent if they permute. To study the transitivity of permutability of subgroups, Bianchi, Gillio and Verardi in [3], defined a graph corresponding to a group $G$, called the permutability graph of non-normal subgroups of $G$, having all the proper non-normal subgroups of $G$ as its vertices and two vertices are adjacent if they permute (see, also in [4, 10]). In [19], the authors considered the generalized case of this graph, called the permutability graph of subgroups of $G$, denoted by $\Gamma(G)$, having the vertex set consisting of all proper subgroups of $G$ and two vertices are adjacent if they permute.

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In [5, p.14], Ballester-Bolinches et al. introduced a graph corresponding to a group \( G \), having all the cyclic subgroups of \( G \) as it vertices and two vertices are adjacent if they permute. In this paper, as a particular case, we consider a graph, denoted by \( \Gamma_c(G) \) with vertex set consists of all proper cyclic subgroups of \( G \) and two vertices are adjacent if they permute. We will call this graph as the \textit{permutability graph of cyclic subgroups of} \( G \). By investigating the properties of this graph, we study the permutability of cyclic subgroups of the corresponding group. Especially, Theorems 3.1, 3.3 and 4.6, Corollaries 3.1 and 3.2 in this paper are some of the main applications for group theory.

Now we introduce some notion from graph theory that we will use in this article. Let \( G \) be a simple graph with vertex set \( V(G) \) and edge set \( E(G) \). \( G \) is said to be \textit{complete} if any two of its vertices are adjacent. A complete graph with \( n \) vertices is denoted by \( K_n \). \( G \) is \textit{bipartite} if \( V(G) \) is the union of two disjoint sets \( X \) and \( Y \) such that no two vertices in the same subset are adjacent. Here \( X \) and \( Y \) are called a \textit{bipartition} of \( G \). A bipartite graph \( G \) with bipartition \( X \) and \( Y \) is called \textit{complete bipartite} if every vertex in \( X \) is adjacent with every vertex in \( Y \). If \( |X| = m \) and \( |Y| = n \), then the corresponding graph is denoted by \( K_{m,n} \). In particular, \( K_{1,3} \) is called the \textit{star graph} and \( K_{1,3} \) is called the \textit{claw graph}. A graph is \textit{planar} if it can be drawn in a plane so that no two edges intersect except possibly at vertices. The \textit{degree} of the vertex \( v \) in \( G \) is the number of edges incident with \( v \) and is denoted by \( \deg_G(v) \). A graph is said to be \textit{regular} if degrees of all the vertices are same. A \textit{path} joining two vertices \( u \) and \( v \) in \( G \) is a finite sequence \((u = )v_0, v_1, \ldots, v_n(= v)\) of distinct vertices, except, possibly, \( u \) and \( v \) such that \( u_i \) is adjacent with \( u_{i+1} \), for all \( i = 0, 1, \ldots, n - 1 \). A path joining \( u \) and \( v \) is a cycle if \( u = v \).

The length of a path or cycle is the number of edges in it. A path or cycle of length \( n \) is denoted by \( P_n \) or \( C_n \) respectively. A graph with exactly one cycle is said to be \textit{unicyclic}. A graph is a \textit{tree} if it has no cycles. The \textit{girth} of a graph \( G \) is the length of the smallest cycle in it and is denoted by \( girth(G) \).

A graph is said to be \textit{connected} if every pair of distinct vertices can be joined by a path. The \textit{distance} between two vertices \( u \) and \( v \) in \( G \), denoted by \( d(u, v) \), is the length of the shortest path between them, and \( d(u, v) = 0 \) if \( u = v \). If there exists no path between them, then we define \( d(u, v) = \infty \). The \textit{diameter} of \( G \), denoted by \( \text{diam}(G) \), is the maximum distance between any two vertices in the graph. An \textit{isomorphism} of graphs \( G_1 \) and \( G_2 \) is an edge-preserving bijection between the vertex sets of \( G_1 \) and \( G_2 \). \( G \) is said to be \textit{H-free} if \( G \) has no induced subgraph isomorphic to \( H \). Let \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) be two simple graphs. Their \textit{union} \( G_1 \cup G_2 \) is a graph with vertex set \( V_1 \cup V_2 \) and edge set \( E_1 \cup E_2 \). Their \textit{join} \( G_1 + G_2 \) is a graph consist of \( G_1 \cup G_2 \) together with all the lines joining points of \( V_1 \) to points of \( V_2 \). For any connected graph \( G \), we write \( nG \) for the graph with \( n \) components each isomorphic to \( G \). For basic graph theory terminology, we refer to [11].

The dihedral group of order \( 2n \), \( n \geq 3 \) is defined by \( D_{2n} = \langle a, b \mid a^n = b^2 = 1, ab = ba^{-1} \rangle \). For any integer \( n \geq 2 \), the generalized Quaternion group of order \( 4n \) is given by \( Q_{4n} = \langle a, b \mid a^{2n} = b^4 = 1, a^n = b^2 = 1, bab^{-1} = a^{-1} \rangle \). The modular group of order \( p^n, \alpha \geq 3 \) is given by \( M_{p^n} = \langle a, b \mid a^{p^{n-1}} = b^p = 1, bab^{-1} = a^{p^{n-2}+1} \rangle \). For an integer \( n \geq 1 \), \( S_n \) and \( A_n \) denotes the symmetric group and alternating group of degree \( n \) acting on \( \{1, 2, \ldots, n\} \) respectively. If \( n \) is a any positive integer, then \( \tau(n) \) denotes the number of positive divisors of \( n \). We denote the order of an element \( a \in \mathbb{Z}_n \) by \( \text{ord}_n(a) \). The number of Sylow \( p \)-subgroups of a group \( G \) is denoted by \( n_p(G) \); or simply by \( n_p \) if there is no ambiguity.

The rest of the paper is arranged as follows: In Section 2, we study some basic
properties of permutability graph of cyclic subgroups of groups.

Section 3 gives the classification of finite groups whose permutability graphs of cyclic subgroups are one of the following: bipartite, tree, star graph, triangle-free, complete bipartite, $P_n$, $C_n$, $K_4$, $K_{1,3}$-free, unicyclic. We estimate the girth of the permutability graphs of cyclic subgroups of finite groups. We also characterize the groups having totally disconnected permutability graphs of cyclic subgroups.

In Section 4, we investigate connectedness, diameter, regularity, completeness of the permutability graph of cyclic subgroups of a given group. Also we classify abelian groups whose permutability graph of cyclic subgroups are planar. We characterize the groups $Q_8$, $S_3$ and $A_4$ by using their permutability graph of cyclic subgroups. Moreover, we pose some open problems in this section.

We recall the following theorem, which we will use in the subsequent sections.

**Theorem 1.1.** ([19, Corollary 5.1]) Let $G$ be a finite group and $p$, $q$ be distinct primes. Then

(i) $\Gamma(G)$ is $C_n$ if and only if $n = 3$ and $G$ is either $\mathbb{Z}_{p^3}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$;

(ii) $\Gamma(G)$ is $P_n$ if and only if $n = 1$ and $G$ is either $\mathbb{Z}_{p^3}$ or $\mathbb{Z}_{pq}$;

(iii) $\Gamma(G)$ is claw-free if and only if $G$ is either $\mathbb{Z}_{p^\alpha}$ ($\alpha = 2, 3, 4$) or $\mathbb{Z}_{pq}$.

### 2 Some basic results

Note that the only groups having no proper cyclic subgroups are the trivial group, and the groups of prime order, so it follows that, we can define $\Gamma_c(G)$ only when the group $G$ is not isomorphic to either of these groups.

In this section, we study some basic properties about of permutability graph of cyclic subgroups of a given group. We start with the following result whose proof is immediate.

**Lemma 2.1.** Let $G$ be a group. If $G$ has $r$ proper cyclic subgroups, which are permutes with each other, then $\Gamma_c(G)$ has $K_r$ as a subgraph.

**Theorem 2.1.** Let $G_1$ and $G_2$ be two groups. If $G_1 \cong G_2$, then $\Gamma_c(G_1) \cong \Gamma_c(G_2)$.

**Proof.** Let $f : G_1 \to G_2$ be a group isomorphism. Define a map $\psi : V(\Gamma_c(G_1)) \to V(\Gamma_c(G_2))$ by $\psi(H) = f(H)$, for every $H \in V(\Gamma_c(G_1))$. Then it is easy to see that $\psi$ is a graph isomorphism.

**Remark 2.1.** The converse of Theorem 2.1 is not true. For example, consider the non-isomorphic groups $G_1 = \mathbb{Z}_{p^5}$, where $p$ is a prime and $G_2 = \mathbb{Z}_3 \times \mathbb{Z}_3$. Here $G_1$ has subgroups $\mathbb{Z}_{p^i}, i = 1, 2, 3, 4$ and $G_2$ has proper cyclic subgroups \langle (1, 0) \rangle, \langle (x, 1) \rangle, x = 0, 1, 2. It follows that $\Gamma_c(G_1) \cong K_4 \cong \Gamma_c(G_2)$.

**Theorem 2.2.** If $G$ is a group and $N$ is a subgroup of $G$, then $\Gamma_c(N)$ is a subgraph of $\Gamma_c(G)$.

### 3 Some classification related results for $\Gamma_c(G)$

The aim of this section is to classify the solvable groups whose permutability graphs of cyclic subgroups are one of the following: bipartite, complete bipartite, tree, star graph, $C_3$-free, $C_n$, $K_4$, $P_n$, $K_{1,3}$-free, unicyclic. First we consider the finite groups and then we deal with the infinite groups.
3.1 Finite abelian groups

Proposition 3.1. Let $G$ be a finite abelian group and $p$, $q$ be distinct primes. Then

(i) $\Gamma_c(G)$ is $C_3$-free if and only if $G$ is either $\mathbb{Z}_{p^\alpha}$ ($\alpha = 2, 3$) or $\mathbb{Z}_{pq}$;

(ii) $\Gamma_c(G)$ is bipartite if and only if it is $C_3$-free;

(iii) $\Gamma_c(G)$ is $C_n$ if and only if $n = 3$ and $G$ is either $\mathbb{Z}_{p^3}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$;

(iv) $\Gamma_c(G)$ is $P_n$ if and only if $n = 1$ and $G$ is either $\mathbb{Z}_{p^3}$ or $\mathbb{Z}_{pq}$;

(v) $\Gamma_c(G)$ is $K_4$ if and only if $G$ is one of $\mathbb{Z}_{p^3}$, $\mathbb{Z}_{p^2 q}$, $\mathbb{Z}_3 \times \mathbb{Z}_3$;

(vi) $\Gamma_c(G)$ is claw-free if and only if $G$ is one of $\mathbb{Z}_{p^\alpha}$ ($\alpha = 2, 3, 4$), $\mathbb{Z}_{pq}$, $\mathbb{Z}_2 \times \mathbb{Z}_2$;

(vii) $\Gamma_c(G)$ is unicyclic if and only if $G$ is either $\mathbb{Z}_{p^3}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. Let $|G| = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $p_i$’s are distinct primes and $a_i \geq 1$ for every $i = 1, 2, \ldots, k$. We divide the proof into two cases.

Case 1: If $G$ is cyclic, then $\Gamma_c(G) \cong \Gamma(G)$. So in view of this fact and by the proof of [19, Theorem 3.1], we have

$$\Gamma_c(G) \cong K_r,$$

where $r$ is the number of proper subgroups of $G$, which is given by $r = (\alpha_1 + 1)(\alpha_2 + 1)\cdots(\alpha_k + 1) - 2$. It follows that $\Gamma_c(G) \cong K_4$ if and only if $G$ is one of $\mathbb{Z}_{p^3}$ or $\mathbb{Z}_{p^2 q}$. Furthermore, $\Gamma_c(G)$ is bipartite or $C_3$-free if and only if $G$ is either $\mathbb{Z}_{p^\alpha}$ ($\alpha = 2, 3$) or $\mathbb{Z}_{pq}$. Note that the bipartiteness and $C_3$-freeness of permutability graphs of finite cyclic groups were proved in [20, Proposition 3.1 and corollary 3.1]. We repeated them here for the sake of completeness. Also by Theorem 1.1, we have

(i) $\Gamma_c(G)$ is $C_n$ if and only if $n = 3$ and $G \cong \mathbb{Z}_{p^3}$.

(ii) $\Gamma_c(G)$ is $P_n$ if and only if $n = 1$ and $G$ is either $\mathbb{Z}_{p^3}$ or $\mathbb{Z}_{pq}$.

(iii) $\Gamma_c(G)$ is claw-free if and only if $G$ is one of $\mathbb{Z}_{p^\alpha}$ ($\alpha = 2, 3, 4$), $\mathbb{Z}_{pq}$.

Case 2: If $G$ is non-cyclic, then we have the following cases to consider:

Subcase 2a: $k = 1$. If $\alpha_1 > 2$, then $G$ has a subgroup isomorphic to either $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ or $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$, for some prime $p$. It is easy to see that these groups have at least five proper cyclic subgroups, so they form $K_5$ as a subgraph of $\Gamma_c(G)$. If $\alpha_1 = 2$, then $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$, for some prime $p$. But the number of nontrivial subgroups of $\mathbb{Z}_p \times \mathbb{Z}_p$ is $p + 1$; they are $\langle (1, 0) \rangle$, $\langle (a, 1) \rangle$, for each $a \in \{0, 1, 2, \ldots, p - 1\}$. Thus, by Lemma 2.1,

$$\Gamma_c(G) \cong K_{p+1}.$$  \hfill (3.2)

Therefore, $\Gamma(G_1)$ contains $C_3$ as a subgraph; it is $C_3$ if and only if $p = 2$; it is $K_4$ if and only if $p = 3$; it is claw-free if and only if $p = 2$.

Subcase 2b: $k > 1$. If $\alpha_i > 1$ for some $i$, then $G$ has a subgroup $H$ isomorphic to $\mathbb{Z}_{pq} \times \mathbb{Z}_p$, for some distinct primes $p$ and $q$. It is easy to see that $H$ has at least five proper cyclic subgroups, so they form $K_5$ as a subgraph of $\Gamma_c(G)$.

The proof follows by combining these cases. \hfill $\Box$
3.2 Finite non-abelian groups

Proposition 3.2. Let \( G \) be a non-abelian of order \( p^\alpha \), where \( p \) is a prime and \( \alpha \geq 3 \). Then \( \Gamma_c(G) \) contains \( C_3 \) and \( K_{1,3} \) as proper subgraphs; \( \Gamma_c(G) \cong K_4 \) if and only if \( G \cong Q_8 \).

Proof. We first prove this result when \( \alpha = 3 \). According to the Burnside [7], up to isomorphism there are only four non-abelian groups of order \( p^3 \), where \( p \) is a prime, namely \( Q_8, M_8, M_{p^3} \) and \( (Z_p \times Z_p) \rtimes Z_p \), \( p > 2 \). If \( G \cong Q_8 \), then by [19, Theorem 4.3], we have

\[
\Gamma_c(G) \cong K_4. \tag{3.3}
\]

If \( G \cong M_8 \), then \( H_1 := \langle a \rangle, H_2 := \langle a^2 \rangle, H_3 := \langle b \rangle, H_4 := \langle ab \rangle, H_5 := \langle a^2b \rangle \) are proper cyclic subgroups of \( G \), so \( |V(\Gamma_c(G))| \geq 5 \). Since \( H_1, H_2 \) are normal in \( G \), they permutes with all the subgroups of \( G \). Thus, \( \Gamma_c(G) \) has \( C_3 \) as a subgraph induced by the vertices \( H_1, H_2, H_3 \); but it is not \( K_4 \) as it has five vertices. Also \( K_{1,3} \) is a subgraph of \( \Gamma_c(G) \) with bipartition \( X := \{H_2\} \) and \( Y := \{H_2, H_3, H_4\} \). If \( G \cong M_{p^3} \), where \( p \) is a prime and \( p > 2 \), then \( H_1 := \langle a \rangle, H_2 := \langle ab \rangle, H_3 := \langle ab^2 \rangle, H_4 := \langle b \rangle, H_5 := \langle a^p \rangle \) are proper cyclic subgroups of \( G \), so \( |V(\Gamma_c(G))| \geq 5 \). Here any two subgroups of \( G \) permutes, so \( K_5 \) is a subgraph of \( \Gamma_c(G) \). If \( G \cong (Z_p \times Z_p) \rtimes Z_p \), then \( Z_p \times Z_p \) is a subgroup of \( G \) and since \( p > 2 \), so by (3.2), \( \Gamma_c(G) \) contains \( K_4 \) as a proper subgraph. Clearly \( |V(\Gamma_c(G))| \geq 5 \).

Now we prove this result when \( \alpha \geq 4 \). We need to consider the following two cases:

Case 1: \( G \cong Q_{2^\alpha} \). Then \( G \) has two subgroups each isomorphic to \( Q_8 \), so in the view of (3.3), \( \Gamma_c(G) \) contains \( C_3 \) and \( K_{1,3} \) as proper subgraphs. Also \( G \) has at least five proper cyclic subgroups, so \( |V(\Gamma_c(G))| \geq 5 \).

Case 2: \( G \not\cong Q_{2^\alpha} \). By [22, Proposition 1.3], the number of subgroups of order \( p \) of \( G \) is not unique and so by [7, Theorem IV, p.129], \( G \) has at least three subgroups, say \( H_1, H_2, H_3 \); also it has a subgroup, say \( H \) of order \( p^3 \). Suppose \( \Gamma_c(H) \) contains \( C_3 \) and \( K_{1,3} \); also \( |V(\Gamma_c(H))| \geq 5 \), then \( \Gamma_c(G) \) also has the same. So by Propositions 3.1 and 3.2, the only cases remains to check are \( H \cong Z_{p^3} \) or \( Q_8 \). If \( H \cong Z_{p^3} \), then by (3.2), \( \Gamma_c(H) \cong K_2 \), so \( H \) together with its subgroups forms \( C_3 \) as a subgraph of \( \Gamma_c(G) \). The cyclic subgroups of \( H \) together with the subgroups \( H_1 \)'s make \( |V(\Gamma_c(G))| \geq 5 \). By [7, Corollary of Theorem IV, p.129], \( G \) has a normal subgroup of order \( p \), without loss of generality, say \( H_1 \). Then \( K_{1,3} \) is a subgraph of \( \Gamma_c(G) \) with bipartition \( X := \{H_1\} \) and \( Y := \{H, H_2, H_3\} \). If \( H \cong Q_8 \), then by (3.3), \( \Gamma_c(H) \cong K_4 \). Also the cyclic subgroups of \( H \) together with \( H_1 \)'s also make \( |V(\Gamma_c(G))| \geq 5 \).

The proof follows by combining all the above.

Proof. We have \( G \cong Z_q \rtimes Z_p \). Here every subgroup of \( G \) is cyclic, so \( \Gamma_c(G) \cong \Gamma(G) \). By the proof of Theorem 4.4 in [19], we have

\[
\Gamma_c(G) \cong K_{1, q}. \tag{3.4}
\]

This completes the proof.

Consider the semi-direct product \( Z_q \rtimes Z_{p^\alpha} \equiv \langle a, b | a^q = b^{p^\alpha} = 1, bab^{-1} = a^{-t}, ord_q(t) = p^t \rangle \), where \( p \) and \( q \) are distinct primes with \( p^t \mid (q - 1), t \geq 0 \). Then every semi-direct product \( Z_q \rtimes Z_{p^\alpha} \) is one of these types [6, Lemma 2.12]. In the future, when \( t = 1 \) we will suppress the subscript.
Proposition 3.4. Let $G$ be a non-abelian group of order $p^2q$, where $p, q$ are distinct primes. Then $\Gamma_c(G)$ contains $C_3$ as a proper subgraph; it is $K_{1,3}$-free if and only if $G \cong A_4$; it has at least five vertices.

Proof. Here we use the classification of groups of order $p^2q$ given in [7, p. 76-80]. We have the following cases to consider:

Case 1: $p < q$:

Case 1a: $p \nmid (q - 1)$. By Sylow's Theorem, it is easy to see that there is no non-abelian group in this case.

Case 1b: $p \mid (q - 1)$, but $p^2 \nmid (q - 1)$. In this case, there are two non-abelian groups.

The first group is $G_1 := \mathbb{Z}_q \rtimes \mathbb{Z}_{p^2} = \langle a, b \mid a^q = b^{p^2} = 1, bab^{-1} = a^i, \text{ord}_a(i) = p \rangle$. It has $H_1 := \langle a \rangle$, $H_2 := \langle ab \rangle$, $H_3 := \langle b \rangle$, $H_4 := \langle b^p \rangle$, $H_5 := \langle ab \rangle$ as its proper cyclic subgroups, so $|V(\Gamma_c(G_1))| \geq 5$. Here $H_1$ and $H_2$ are normal in $G$, so they permute with all the subgroups of $G$; $H_4$ is a subgroup of $H_3$ and $H_5$. So $K_4$ is a subgraph of $\Gamma_c(G_1)$ induced by $H_i$, $i = 1, 2, 3, 4$.

The second group in this case is $G_2 := \langle a, b, c \mid a^q = b^q = c^q = 1, bab^{-1} = a^i, ca = ac, cb = bc, \text{ord}_a(i) = p \rangle$. It has $H_1 := \langle a \rangle$, $H_2 := \langle b \rangle$, $H_3 := \langle c \rangle$, $H_4 := \langle bc \rangle$, $H_5 := \langle ab \rangle$ as its proper cyclic subgroups, so $|V(\Gamma_c(G_2))| \geq 5$. Here $H_3$ permutes with all the subgroups of $G_2$; $H_2$, $H_3$, $H_4$ permutes with each other. So $C_3$ is a subgraph of $\Gamma_c(G_2)$ induced by the vertices $H_1$, $H_2$, and $H_3$; and $K_{1,3}$ is a subgraph of $\Gamma_c(G_2)$ with bipartition $X := \{H_1\}$ and $Y := \{H_2, H_3, H_4\}$.

Case 1c: $p^2 \mid (q - 1)$. In this case, we have both groups $G_1$ and $G_2$ from Case 1b together with the group $G_3 := \mathbb{Z}_q \rtimes \mathbb{Z}_{p^2}$ as $\langle a, b \mid a^q = b^q = 1, bab^{-1} = a^i, \text{ord}_a(i) = p \rangle$. But in Case 1b, we already dealt with $G_1$ and $G_2$. Now we consider $G_3$. It has $H_1 := \langle a \rangle$, $H_2 := \langle b \rangle$, $H_3 := \langle b^p \rangle$, $H_4 := \langle ab \rangle$, $H_5 := \langle a^2b \rangle$ as its proper cyclic subgroups, so $|V(\Gamma_c(G_3))| \geq 5$. Since $H_1$ is normal in $G_3$, it permutes with all the subgroups of $G_3$; $H_3$ is a subgroup of $H_2$. So $C_3$ is a subgraph of $\Gamma_c(G)$ induced by $H_1$, $H_2$, $H_3$ and $K_{1,3}$ is a subgraph of $\Gamma_c(G_3)$ with bipartition $X := \{H_1\}$ and $Y := \{H_2, H_3, H_4\}$.

Case 2: $p > q$:

Case 2a: $q \nmid (p^2 - 1)$. In this case there is no non-abelian group.

Case 2b: $q \mid (p^2 - 1)$. In this case there are two groups. The first one is $G_4 := \langle a, b \mid a^{p^2} = b^q = 1, bab^{-1} = a^i, \text{ord}_a(i) = q \rangle$. It has $H_1 := \langle a \rangle$, $H_2 := \langle b \rangle$, $H_3 := \langle ab \rangle$, $H_4 := \langle b^2 \rangle$, $H_5 := \langle ab \rangle$ as its proper cyclic subgroups, so $|V(\Gamma_c(G_4))| \geq 5$. Since $H_1$ is a normal subgroup of $G_4$, so it permutes with all the subgroup of $G_4$; $H_2H_3H_5 = \langle ab^2, b \rangle = H_2H_4$; $H_2H_5 = \langle ab, ab \rangle$. So $C_3$ is a subgraph of $\Gamma_c(G_4)$ induced by $H_1$, $H_2$, $H_4$; $K_{1,3}$ is a subgraph of $\Gamma_c(G_4)$ with bipartition $X := \{H_1\}$ and $Y := \{H_2, H_3, H_4\}$.

Next, we have the family of groups $\langle a, b, c \mid a^p = b^q = c^t = 1, cac^{-1} = a^i, bcb^{-1} = b^i, ab = ba, \text{ord}_a(i) = q \rangle$. There are $(q + 3)/2$ isomorphism types in this family (one for $t = 0$ and one for each pair $\{x, x^{-1}\}$ in $\mathbb{F}_p^\times$. We will refer to all of these groups as $G_{5(1)}$ or $G_{5(1)}$ of order $p^2q$. They have a subgroup $H$ isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Since $p > 2$, by (3.2), $\Gamma_c(G_{5(1)})$ contains $K_4$ as a subgraph. In addition to these four vertices, $\Gamma_c(G_{5(1)})$ have $\langle c \rangle$ as their vertex, so $|V(\Gamma_c(G_{5(1)}))| \geq 5$.

Case 2c: $q \mid (p + 1)$. In this case we have only one group of order $p^2q$, given by $G_6 := \langle a, b, c \mid a^p = b^q = c^t = 1, ab = ba, cac^{-1} = a^ib, bcb^{-1} = a^kb, \rangle$, where $(i, j, k)$ has order $q$ in $GL_2(p)$. It has a subgroup $H$ isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Since $p > 2$, so by (3.2), $\Gamma_c(G_6)$ contains $K_4$ as a subgraph. In addition to these four vertices, $\Gamma_c(G_6)$ has $\langle c \rangle$ as its vertex, so $|V(\Gamma_c(G_6))| \geq 5$.

Note that if $(p, q) = (2, 3)$, the Cases 1 and 2 are not mutually exclusive. Up to isomorphism, there are three non-abelian groups of order 12: $\mathbb{Z}_3 \times \mathbb{Z}_4$, $D_{12}$, and $A_4$. In
Case 1b we already dealt with \( \mathbb{Z}_3 \times \mathbb{Z}_4 \) (the group \( G_1 \)), and \( D_{12} \) (the group \( G_2 \)). But for the case of \( A_4 \) (the group \( G_6 \)), we can not use the argument as in Case 2c, since \( p = 2 \). So we now separately deal with this case. Note that \( A_4 \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{Z}_3 \). Here \( H_i := \mathbb{Z}_2 \times \mathbb{Z}_2 \) is a subgroup of \( A_4 \) of order 4, and it has three nontrivial subgroups, say \( H_i, i = 2, 3, 4 \) each of order 2. Also \( A_4 \) has four subgroups of order 3, let them be \( H_j, j = 5, 6, 7, 8 \). These eight subgroups are the only proper subgroups of \( A_4 \), so \( |V(\Gamma_c(G))| \geq 5 \). Further, \( H_2, H_3 \) and \( H_4 \) permutes with each other, but no two subgroups \( H_5, H_6, H_7, H_8 \) permutes; for if they permutes, then \( G \) has a subgroup of order 9, which is not possible. Also, no \( H_i (i = 2, 3, 4) \) permutes with \( H_j (j = 5, 6, 7, 8) \); if they permutes, then \( G \) has a subgroup of order 6, which is not possible. Thus,

\[
\Gamma_c(G_6) \cong K_3 \cup K_4.
\]  

(3.5)

The proof follows by combining all the cases. \( \square \)

**Proposition 3.5.** If \( G \) is a non-abelian group of order \( p^\alpha q \), where \( p, q \) are two distinct primes with \( \alpha \geq 3 \), then \( \Gamma_c(G) \) has \( C_3 \) and \( K_{1,3} \) as proper subgraphs; it has at least five vertices.

**Proof.** Let \( P \) denote a Sylow \( p \)-subgroup of \( G \). We first prove this result for \( \alpha = 3 \). If \( p > q \), then \( n_p = 1 \), by Sylow’s Theorem and our group \( G \cong P \rtimes \mathbb{Z}_q \). Suppose \( \Gamma_c(P) \) contains \( C_3 \) and \( K_{1,3}; |V(\Gamma_c(G))| \geq 5 \), then \( \Gamma_c(G) \) also has the same. So by Propositions 3.1 and 3.2, the only possibilities are \( P \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q \) or \( \mathbb{Q}_8 \). If \( P \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q \), then \( G \cong \mathbb{Z}_{p^3} \rtimes \mathbb{Z}_q = \langle a, b \mid a^p = q = 1, bab^{-1} = a', ord_p(i) = q \rangle \) and it has \( H_1 := \langle a \rangle \), \( H_2 := \langle a^p \rangle \), \( H_3 := \langle a^{p^2} \rangle \), \( H_4 := \langle b \rangle \), \( H_5 := \langle ab \rangle \) as its proper cyclic subgroups, so \( |V(\Gamma_c(G))| \geq 5 \). Here \( H_1, H_2, H_3 \) are normal in \( G \), so they permutes with all the subgroups of \( G \). It follows that \( \Gamma_c(G) \) contains \( K_4 \) as a proper subgraph. If \( P \cong \mathbb{Q}_8 \), then by (3.3), \( \Gamma_c(P) \cong K_4 \). But this \( K_4 \) is a proper subgraph of \( \Gamma_c(G) \), since \( G \) has a cyclic subgroup isomorphic to \( \mathbb{Z}_q \), in addition and so \( |V(\Gamma_c(G))| \geq 5 \).

Now, let us consider the case \( p < q \) and \( (p, q) \neq (2, 3) \). Here \( n_q = p \) is not possible. If \( n_q = p^2 \), then \( q \mid (p + 1)(p - 1) \) which implies that \( q \mid (p + 1) \) or \( q \mid (p - 1) \). But this is impossible, since \( q > p > 2 \). If \( n_q = q^3 \), then there are \( p^3(q - 1) \) elements of order \( p \). But this only leaves \( p^3q - p^3(q - 1) = p^3 \) elements, and the Sylow \( p \)-subgroup must be normal, a case we already considered. Therefore, the only remaining possibility is that \( G \cong \mathbb{Z}_q \rtimes P \). Suppose \( \Gamma_c(P) \) contains \( C_3 \) and \( K_{1,3}; |V(\Gamma_c(P))| \geq 5 \), then \( \Gamma_c(G) \) also has the same. So by Propositions 3.1 and 3.2, we have the only possibilities \( P \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q \) or \( \mathbb{Q}_8 \). If \( P \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q \), then \( G \cong \mathbb{Z}_q \rtimes \mathbb{Z}_p = \langle a, b \mid a^p = b^q = 1, bab^{-1} = a', ord_p(i) = p^3 \rangle \) and it has \( H_1 := \langle a \rangle \), \( H_2 := \langle b \rangle \), \( H_3 := \langle b^p \rangle \), \( H_4 := \langle b^{p^2} \rangle \), \( H_5 := \langle ab \rangle \) as its proper cyclic subgroups, so \( |V(\Gamma_c(G))| \geq 5 \). Here \( H_1, H_2, H_3 \) are normal in \( G \), so they permutes with all the subgroups of \( G; H_4 \) is a subgroup of \( H_2 \). So \( K_4 \) is a subgraph of \( \Gamma_c(G) \) induced by \( H_1, H_2, H_3, H_4 \). The case \( P \cong \mathbb{Q}_8 \) is similar to the earlier case.

If \( (p, q) = (2, 3) \), then \( G \cong S_3 \) and it has a subgroup \( H \) isomorphic to \( D_8 \). Therefore, by Theorem 3.2, \( \Gamma_c(H) \) contains \( C_3 \) and \( K_{1,3} \) as proper subgraphs. Also \( H \) has more than four cyclic subgroups, so \( \Gamma_c(G) \) also has the same properties.

If \( \alpha \geq 4 \), then \( G \) has a subgroup, say \( H \) of order \( p^4 \). Suppose \( \Gamma_c(H) \) contains \( C_3 \) and \( K_{1,3} \); also \( |V(\Gamma_c(H))| \geq 5 \), then \( \Gamma_c(G) \) also has the same properties. So by Propositions 3.1 and 3.2, we need to check when \( H \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q \). If \( H \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q \), then by (3.2), \( \Gamma_c(H) \cong K_3 \), so \( H \) together with its subgroups forms \( K_4 \) as a subgraph of \( \Gamma_c(G) \). Also \( |V(\Gamma_c(G))| \geq 5 \), since \( G \) has a subgroup of order \( q \) in addition. \( \square \)
Proposition 3.6. If $G$ is a non-abelian group of order $p^2q^2$, where $p, q$ are two distinct primes, then $\Gamma_c(G)$ contains $C_3$ and $K_{1,3}$ as proper subgraphs; it has at least five vertices.

Proof. We use the classification of groups of order $p^2q^2$ given in [15]. Let $P$ and $Q$ denote a Sylow $p, q$-subgroups of $G$ respectively. Without loss of generality, we assume that $p > q$. By Sylow’s Theorem, $n_p = 1, q, q^2$. But $n_p = q$ is not possible, since $p > q$. If $n_p = q^2$, then $p \mid (q + 1)(q - 1)$, this implies that $p \mid (q + 1)$, which is true only when $(p, q) = (3, 2)$.

When $(p, q) \neq (3, 2)$, then $G \cong P \times Q$. Now we have the following possibilities.

If $G \cong \mathbb{Z}_p \times \mathbb{Z}_q^2 = \langle a, b \mid a^{p^2} = b^q = 1, bab^{-1} = a^i, i^q \equiv 1 \mod (p^2) \rangle$, then $H_1 := \langle a \rangle, H_2 := \langle a^p \rangle, H_3 := \langle b \rangle, H_4 := \langle b^q \rangle, H_5 := \langle ab \rangle$ are proper cyclic subgroups of $G$, so $|V(\Gamma_c(G))| \geq 3$. Here $H_1, H_2$ are normal in $G$; $H_3, H_4$ permutes with each other. So $K_4$ is a proper subgraph of $\Gamma_c(G)$ induced by $H_1, H_2, H_3, H_4$.

If $G \cong \mathbb{Z}_p \times (\mathbb{Z}_q \times \mathbb{Z}_q)$, then $H_1 := \langle a \rangle, H_2 := \langle a^p \rangle, H_3 := \langle b \rangle, H_4 := \langle c \rangle, H_5 := \langle bc \rangle$ are proper cyclic subgroups of $G$, so $|V(\Gamma_c(G))| \geq 5$. Here $H_1$ is a normal subgroup of $G$; $H_3, H_5$ permutes with each other. So $K_4$ is a proper subgraph of $\Gamma_c(G)$ induced by $H_i, i = 1, 3, 4, 5$.

If $G \cong (\mathbb{Z}_p \times \mathbb{Z}_p) \times \mathbb{Z}_q$ or $(\mathbb{Z}_p \times \mathbb{Z}_q) \times (\mathbb{Z}_q \times \mathbb{Z}_q)$, then $\mathbb{Z}_p \times \mathbb{Z}_p$ is a subgroup of $G$.

Next, we consider the case when $(p, q) = (3, 2)$ and $n_p = 1$. Consider the Sylow 3-subgroup $P$ and a Sylow 2-subgroup $Q$ of $G$. Let $H$ be a subgroup of $Q$ of order 2. Since $|G|$ does not divide $[G : P]$, so $P$ contains a subgroup, say $K$ of order 3, which is normal in $G$; $H_1 := P$ is a subgroup of order 12. Suppose $\Gamma_c(H)$ contains $C_3$ and $K_{1,3}$; also $|V(\Gamma_c(H))| \geq 5$, then $\Gamma_c(G)$ also has the same. So by Propositions 3.1 and 3.4, the only cases remains to check is when $H \cong \mathbb{Z}_{p^2q}$ or $A_4$. If $H_1 \cong \mathbb{Z}_{p^2q}$, then by (3.2), $\Gamma_c(H_1) \cong K_4$, so $H$ together with its subgroups forms $K_5$ as a proper subgraph of $\Gamma_c(G)$ and so $|V(\Gamma_c(G))| \geq 5$. If $H_1 \cong A_4$, then by (3.5), $\Gamma_c(H_1) \cong K_3 \cup K_4$, so $|V(\Gamma_c(G))| \geq 5$.

Also $K_{1,3}$ is a subgraph of $\Gamma_c(G)$ with bipartition $X := \{K\}$ and $Y := \{K_1, K_2, K_3\}$, where $K_i$’s are the vertices of $K_3$ in $\Gamma_c(H_1)$. $\Box$

Proposition 3.7. If $G$ is a non-abelian group of order $p^\alpha q^\beta$, where $p, q$ are distinct primes, and $\alpha, \beta \geq 2$, then $\Gamma_c(G)$ has $C_3$ and $K_{1,3}$ as proper subgraphs; it has at least five vertices.

Proof. We prove the result by induction on $\alpha + \beta$. If $\alpha + \beta = 4$, then by Propositions 3.1 and 3.6, the result is true in the case. Assume that the result is true for all non-abelian groups of order $p^m q^n$ with $m, n \geq 2$, and $m + n < \alpha + \beta$. We prove the result when $\alpha + \beta > 4$. Since $G$ is solvable, $G$ has a subgroup $H$ of prime index, with out loss of generality, say $q$. So $|H| = p^\alpha q^{\beta-1}$. If $H$ is abelian, then by Proposition 3.1, the result is true. If $H$ is non-abelian, then we have the following cases to consider:

Case 1: If $\beta = 2$, then $\alpha > 2$. So by Proposition 3.5, the result is true for $\Gamma_c(H)$.

Case 2: $\beta > 2$, then by induction hypothesis, the result is true for $\Gamma_c(H)$.

Case 3: $\alpha = 2$, then $\beta > 2$. So by Case 2, the result is true for $\Gamma_c(H)$.

Case 4: $\alpha > 2$, then by induction hypothesis, the result is true for $\Gamma_c(H)$.

Then by Theorem 2.2, result is true for $\Gamma_c(G)$ also. $\Box$

Proposition 3.8. Let $G$ be a finite group of order $p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$, $k \geq 3$, where $p_i$’s are distinct primes and $\alpha_i \geq 1$. Then $\Gamma_c(G)$ contains $C_3$, $K_{1,3}$ and it has more than four vertices.
Proof. If \( \alpha_i = 1 \), for every \( i \), then \( G \) is solvable. We consider the following cases:

**Case 1:** \( k = 3 \). If \( \alpha_1 = \alpha_2 = \alpha_3 = 1 \), then without loss of generality, we assume that \( p_1 < p_2 < p_3 \). Since \( G \) is solvable, it has a Sylow basis \( \{ P_1, P_2, P_3 \} \), where \( P_i \) is the Sylow \( p_i \)-subgroup of \( G \) for every \( i = 1, 2, 3 \). Also \( H_1 := \langle ab \rangle \) and \( H_2 := \langle bc \rangle \) are proper cyclic subgroups of \( G \), where \( a, b, c \) are generators of \( P_1, P_2, P_3 \) respectively, so we have \( |V(\Gamma_c(G))| \geq 5 \). Moreover, \( P_1, P_2, P_3 \) permutes with each other, so \( \Gamma_c(G) \) contains \( C_3 \) as a proper subgraph. Further, \( G \) has a normal subgroup, say \( N \) of order \( p_3 \), so it follows that \( \Gamma_c(G) \) contains \( K_{1,3} \) as a subgraph with bipartition \( X := \{ N \} \) and \( Y := \{ P_1, P_2, H_1 \} \).

**Case 2:** \( k > 3 \). Since \( G \) is solvable, it has a Sylow basis containing \( P_1, P_2, P_3 \), where \( P_i \) is the Sylow \( p_i \)-subgroup of \( G \) for every \( i = 1, 2, 3 \). Then \( H := P_1P_2P_3 \) is a subgroup of \( G \). So by Proposition 3.1 and by Case 1 of this proof, \( \Gamma_c(H) \) contains \( C_3 \) and \( K_{1,3} \) as subgraphs; also \( |V(\Gamma_c(H))| \geq 5 \). It follows that \( \Gamma_c(G) \) also has the same properties.

If \( \alpha_i > 1 \), for some \( i \), then without loss of generality, we assume that \( \alpha_1 > 1 \). By Sylow’s theorem, \( G \) has a Sylow \( p_1 \)-subgroup, say \( P \) and \( G \) has an element, say \( b \) of order \( p_2 \). If \( P \) is non-abelian, then by Proposition 3.2, \( \Gamma_c(P) \) contains \( C_3 \), \( K_{1,3} \) as a subgraph. By Theorem 3.2, taking the cyclic subgroups of \( P \) together with \( \langle b \rangle \), we have \( |V(\Gamma_c(G))| \geq 5 \).

If \( P \) is abelian, then we consider the following cases:

**Case 3:** \( P \) is cyclic. Let \( P := \langle a \rangle \). Now consider the subgroup \( \langle a, b \rangle \) of \( G \). Then by Propositions 3.1, 3.3, 3.4, 3.5, 3.6, and 3.7, we have \( \Gamma_c(\langle a, b \rangle) \) contains \( C_3 \), \( K_{1,3} \). Also by Propositions 3.3, 3.4, 3.5, 3.6 and 3.7, taking cyclic subgroups of \( \langle a, b \rangle \) together with \( \langle a, b \rangle \), we have \( |V(\Gamma_c(G))| \geq 5 \).

**Case 4:** \( P \) is non-cyclic. If \( \alpha_1 = 2 \), then \( P \cong \mathbb{Z}_p \times \mathbb{Z}_p := \langle a_1, a_2 \rangle \).

**Subcase 4a:** \( \langle a_1, a_2, b \rangle \not\cong A_4 \), then by Propositions 3.1, 3.3, 3.4, 3.5, 3.6 and 3.7, \( \Gamma_c(\langle a_1, a_2, b \rangle) \) contains \( C_3 \), \( K_{1,3} \). Also \( \langle a_1, a_2, b \rangle \) as a subgraph. Let \( c \) be an element of \( G \) of order \( p_3 \). If \( \langle c \rangle \) permutes with a cyclic subgroups of \( \langle a_1, a_2 \rangle \), then \( \Gamma_c(\langle a_1, a_2 \rangle) \cong C_3 \). So \( \langle c \rangle \) together with cyclic subgroups of \( \langle a_1, a_2 \rangle \) forms \( K_{1,3} \). If \( \langle c \rangle \) does not permutes with a cyclic subgroups of \( \langle a_1, a_2 \rangle \), then by Propositions 3.1, 3.3, 3.4, 3.5, 3.6 and 3.7, \( \Gamma_c(\langle a_1, c \rangle) \) contains \( K_{1,3} \) as a subgraph. Also \( |V(\langle a_1, a_2, b \rangle)| \geq 5 \), since by (3.5).

The proof follows by combining all these cases.

\[ \square \]

### 3.3 Main results for finite groups

Combining all the results obtained so-far in this section, we have the following main results which are applications for group theory.

**Theorem 3.1.** Let \( G \) be a finite group and \( p, q \) be distinct primes. Then

1. \( \Gamma_c(G) \) is \( C_3 \)-free if and only if \( G \) is one of \( \mathbb{Z}_{p\alpha} \) \( (\alpha = 2, 3) \), \( \mathbb{Z}_{pq} \), \( \mathbb{Z}_q \times \mathbb{Z}_p \);
2. \( \Gamma_c(G) \) is \( C_n \) if and only if \( n = 3 \) and \( G \) is either \( \mathbb{Z}_{p^4} \) or \( \mathbb{Z}_2 \times \mathbb{Z}_2 \);
3. \( \Gamma_c(G) \) is \( P_n \) if and only if \( n = 1 \) and \( G \) is either \( \mathbb{Z}_{p^3} \) or \( \mathbb{Z}_{pq} \);
4. \( \Gamma_c(G) \) is \( K_4 \) if and only if \( G \) is one of \( \mathbb{Z}_{p^5} \), \( \mathbb{Z}_{pq} \), \( \mathbb{Z}_3 \times \mathbb{Z}_3 \), \( Q_8 \);
5. \( \Gamma_c(G) \) is claw-free if and only if \( G \) is one of \( \mathbb{Z}_{p\alpha} \) \( (\alpha = 2, 3, 4) \), \( \mathbb{Z}_{pq} \), \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), \( A_4 \).
Corollary 3.1. Let $G$ be a finite group and $p, q$ are distinct primes.

(i) The following are equivalent:
(a) $\Gamma_c(G)$ is $C_3$-free;
(b) $\Gamma_c(G)$ is bipartite;
(c) $\Gamma_c(G)$ is complete bipartite;
(d) $\Gamma_c(G)$ is tree;
(e) $\Gamma_c(G)$ is star graph.

(ii) $\Gamma_c(G)$ is $P_2$-free if and only if $G$ is either $\mathbb{Z}_{p^\alpha}$ ($\alpha = 2, 3$) or $\mathbb{Z}_{pq}$.

(iii) girth$(\Gamma_c(G))$ is infinity if $G$ is one of $\mathbb{Z}_{p^\alpha}$ ($\alpha = 2, 3$), $\mathbb{Z}_{pq}$ or $\mathbb{Z}_q \rtimes \mathbb{Z}_p$; otherwise girth$(\Gamma_c(G)) = 3$.

Proof. To classify the groups whose permutability graph is either bipartite or complete bipartite, it is enough to consider the groups whose permutability graph of cyclic subgroups are $C_3$-free. By Theorem 3.1(i) and (3.1), (3.4), we have $(a) \iff (b) \iff (c)$. Now, to classify the groups whose permutability graphs of cyclic subgroups is one of tree, star graph or $P_2$-free, it is enough to consider the groups whose permutability graphs of cyclic subgroups are bipartite. So by the above argument and by Theorem 3.1(i), (3.1), (3.4), we have $(b) \iff (d) \iff (e)$ and $\Gamma_c(G)$ is $P_2$-free if and only if $\mathbb{Z}_{p^\alpha}$ ($\alpha = 2, 3$) or $\mathbb{Z}_{pq}$. This completes the proof of (i) and (ii). The proof of (iii) follows by the part (i) of this corollary and by Theorem 3.1(i).

Corollary 3.2. Let $G$ be a finite group. Then $\Gamma_c(G)$ is totally disconnected if and only if $G \cong \mathbb{Z}_{p^2}$.

Proof. Let $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, where $p_i$’s are distinct primes, $k \geq 1$ and $\alpha_i \geq 1$. If $\alpha_i = 1$, for every $i$, then $G$ is solvable. Suppose $k = 1$, then $G$ does not contains a proper subgroup. It follows that $k \geq 2$ and so any two subgroups in Sylow basis of $G$ permutes with each other. Therefore, $\Gamma_c(G)$ is not totally disconnected. If $\alpha_i > 1$, for some $i$, then without loss of generality we assume that $\alpha_1 > 1$ and so by Sylow’s Theorem, $G$ has a Sylow $p_1$ subgroup, say $P$. Suppose $P \not\cong \mathbb{Z}_{p^2}$, then by Propositions 3.1 and 3.2, $\Gamma_c(G)$ is not totally disconnected. If $P \cong \mathbb{Z}_{p^2}$, then $P$ and its subgroup of order $p$ permutes with each other. Thus $\Gamma_c(G)$ is not totally disconnected.

Remark 3.1. Not every graph is a permutability graph of cyclic subgroups of some group. For example, by Theorem 3.1 (3), the graph $C_n$, $n \geq 4$ is not a permutability graph of cyclic subgroups of any group.

3.4 Infinite groups

We now investigate the of permutability graph of cyclic subgroups of infinite groups. It is well known that any infinite group has infinite number of subgroups. Let $G$ be an infinite abelian group. If $G$ is finitely generated, then by fundamental theorem of finitely generated abelian groups, $\mathbb{Z}$ is a subgroup of $G$. Since $\mathbb{Z}$ is cyclic, it follows that $\Gamma_c(\mathbb{Z})$ contains $K_r$ as a proper subgraph for every positive integer $r$. Therefore, by Theorem 2.2, $\Gamma_c(G)$ also has the same property. If $G$ is not finitely generated, then we can take the cyclic groups generated by each generating element and so $\Gamma_c(G)$ contains $K_r$ as a proper subgraph, for every positive integer $r$. Thus we have the following result.
Theorem 3.2. The permutability graph of cyclic subgroups of any infinite abelian group contains $K_r$ as a subgraph, for every positive integer $r$.

Next, we consider the infinite non-abelian groups. Recall that an infinite non-abelian group $G$ in which every proper subgroups of $G$ have order a fixed prime number $p$ is called a Tarski monster group. Existence of such groups was given by Ol'shanskii in [17]. In general, the existence of infinite non-abelian groups in which the order of all proper subgroups are of prime order (primes not necessarily distinct) were also given by him in [18, Theorem 35.1]. Also M. Shahryari in [21, Theorem 5.2] give existence of countable non-abelian simple groups with the property that their all non-trivial finite subgroups are cyclic of order a fixed prime $p$ (of course this existence can also be deduced from the results of [18]). It is easy to see that the permutability graph of cyclic subgroups of the above mentioned first two class of non-abelian groups are totally disconnected and for the third class of non-abelian groups it is totally disconnected if that group does not have $Z$ as a subgroup. In the next result, we characterize the infinite non-abelian groups whose permutability graph of cyclic subgroups is totally disconnected.

Theorem 3.3. Let $G$ be an infinite group. Then $\Gamma_c(G)$ is totally disconnected if and only if every non-trivial finite subgroup of $G$ is of prime order (primes not necessarily distinct) and $Z$ is not a subgroup of $G$.

Proof. It is easy to see that if every proper subgroup of $G$ is of prime order (primes not necessarily distinct) and $Z$ is not a subgroup of $G$ then $\Gamma_c(G)$ is totally disconnected. Conversely, suppose that $\Gamma_c(G)$ is totally disconnected. Then by Theorem 3.2, $G$ must be non-abelian. Suppose not every proper subgroup of $G$ is of prime order, then we have the following possibilities.

(i) $G$ may have a subgroup whose order is a composite number; or
(ii) all the subgroups of $G$ may have infinite order.

If $G$ is of type (i), then let $H$ be a subgroup of $G$ of composite order. If $H \not\cong Z_{p^2}$, then by Corollary 3.2, $\Gamma_c(H)$ is not totally disconnected. If $H \cong Z_{p^2}$, then $H$ and its subgroup of order $p$ permutes with each other. So it follows that $\Gamma_c(G)$ is not totally disconnected.

If $G$ is of type (ii), then it must have $Z$ as a subgroup and so by Theorem 3.2, $\Gamma_c(G)$ is not totally disconnected. Hence the proof.

4 Further results on $\Gamma_c(G)$

Recall that a subgroup $H$ of a group $G$ is said to be permutable if it permutes with all the subgroups of $G$. In [13], Iwasawa characterized the groups whose subgroups are permutable.

Theorem 4.1. ([13]) A group whose subgroups are permutable is a nilpotent group in which for every Sylow $p$-subgroup $P$, either $P$ is a direct product of a quaternion group and an elementary abelian 2-group, or $P$ contains an abelian normal subgroup $A$ and an element $b \in P$ such that $P = A\langle b \rangle$ and there exists a natural number $s$, with $s \geq 2$ if $p = 2$, such that $a^b = a^{1+p^s}$ for every $a \in A$.

The next theorem classifies the groups whose permutability graph of cyclic subgroups are complete (see, also in [5, p.14]).

Theorem 4.2. Let $G$ be a group. Then $\Gamma_c(G)$ is complete if and only if $G$ is one of the groups given in Theorem 4.1.
Theorem 4.3. Let $G$ be a group with a permutable proper cyclic subgroup. Then $\Gamma_c(G)$ is regular if and only if $\Gamma_c(G)$ is complete.

Proof. Let $N$ be a permutable cyclic subgroup of $G$. Assume that $\Gamma_c(G)$ is regular. Since $N$ permutes with all the cyclic subgroup of $G$, so from the regularity of $\Gamma_c(G)$, it follows that any two vertices in $\Gamma_c(G)$ are adjacent and hence $\Gamma_c(G)$ is complete. Converse of the result is obvious. \(\square\)

Theorem 4.4. Let $G$ be a group with a permutable proper cyclic subgroup. Then $\Gamma_c(G)$ is connected and $\text{diam}(\Gamma_c(G)) \leq 2$.

Proof. If every cyclic subgroups of $G$ are permutable, then obviously $\Gamma_c(G)$ is connected and $\text{diam}(\Gamma_c(G)) = 1$. Let $N$ be a permutable proper cyclic subgroup of $G$. Suppose $H$ and $K$ are two proper cyclic subgroups of $G$ such that $HK \neq KH$. Then we have a path $H - N - K$ in $\Gamma_c(G)$ and so $\Gamma_c(G)$ is connected and $\text{diam}(\Gamma_c(G)) = 2$. \(\square\)

Problem 4.1. Which groups have connected permutability graph of cyclic subgroups and estimate their diameter.

In the next result, we classify the abelian groups whose permutability graph of cyclic subgroups are planar.

Theorem 4.5. Let $G$ be an abelian group and $p$, $q$ be distinct primes. Then $\Gamma_c(G)$ is planar if and only if $G$ is isomorphic to one of the following: $\mathbb{Z}_{p^\alpha}$ ($\alpha = 2, 3, 4, 5$), $\mathbb{Z}_{pq}$, $\mathbb{Z}_{p^\alpha q}$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Proof. If $G$ is infinite abelian, then by Theorem 3.2, $\Gamma_c(G)$ is non-planar. So in the rest of the proof, we assume that $G$ is finite.

Suppose $G$ is cyclic, then with the notations used in the proof of Proposition 3.1 and by (3.1), we have $\Gamma_c(G) \cong K_r$. So $\Gamma_c(G)$ is planar if and only if $r \leq 4$. This is true only when one of the following holds:

(i) $k = 1$ with $\alpha_1 < 6$;

(ii) $k = 2$ with $\alpha_1 = 1, \alpha_2 = 1$;

(iii) $k = 2$ with $\alpha_1 = 2, \alpha_2 = 1$.

If $G$ is non-cyclic, then we need to consider the following cases:

Case 1: $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Then the number of proper subgroups of $G$ is $p + 1$; they are $\langle (1, 0) \rangle$, and $\langle (x, 1) \rangle$, $x \in \{0, 1, \ldots, p - 1\}$. By (3.2), $\Gamma_c(G)$ is planar only when $p = 2, 3$.

Case 2: $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$. Then $\langle (1, 0) \rangle$, $\langle (1, 1) \rangle$, $\langle (p, 0) \rangle$, $\langle (0, 1) \rangle$, $\langle (p, 1) \rangle$ are proper subgroups of $G$, so $\Gamma_c(G)$ contains $K_5$ as a subgraph.

Case 3: $G \cong \mathbb{Z}_{pq} \times \mathbb{Z}_p$. Then $\mathbb{Z}_{pq}$, $\mathbb{Z}_p \times \mathbb{Z}_p$, $\mathbb{Z}_q \times \mathbb{Z}_p$ are proper subgroups of $G$. Here $\mathbb{Z}_p \times \mathbb{Z}_p$ has at least three proper subgroups of order $p$, so these three subgroups together with $\mathbb{Z}_{pq}$, $\mathbb{Z}_q$, forms $K_5$ as a subgraph of $\Gamma_c(G)$.

Case 4: $G \cong \mathbb{Z}_{p^l} \times \mathbb{Z}_p$, $k, l \geq 2$. Then $\mathbb{Z}_{p^l} \times \mathbb{Z}_p$ is a proper subgroup of $G$, so by Case 2 and by Theorem 2.2, $\Gamma_c(G)$ contains $K_5$ as a subgraph.

Case 5: $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, then $G$ has two subgroups each isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. It follows that $G$ has at least five subgroups of order $p$ and so they form $K_5$ as a subgraph of $\Gamma_c(G)$. 
Case 6: $G \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \ldots \times \mathbb{Z}_{p_k^{\alpha_k}}$, where $p_i$'s are primes and $\alpha_i \geq 1$. If $k = 2$ or $3$, then $\alpha_i > 1$, for some $i$ and if $k \geq 4$, then $\alpha_i \geq 1$. In either case, one of $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$, $\mathbb{Z}_{pq} \times \mathbb{Z}_p$, $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ is a proper subgroup of $G$, so by Cases 2, 3 and 5, $\Gamma_c(G)$ contains $K_5$ as a subgraph.

The result follows by combining all the above cases. \hfill \Box

Proposition 3.3 shows the existence of a finite non-abelian group whose permutability graph of cyclic subgroups is planar. Further, the Torski monster group is an example of an infinite non-abelian group whose permutability graph of cyclic subgroups is planar. Now we pose the following

**Problem 4.2.** Classify all non-abelian groups whose permutability graph of cyclic subgroups are planar.

The next result characterize some non-abelian groups by using their permutability graph of cyclic subgroups.

**Theorem 4.6.** Let $G$ be a finite group.

(i) If $G$ is non-abelian and $\Gamma_c(G) \cong \Gamma_c(Q_8)$, then $G \cong Q_8$.

(ii) If $\Gamma_c(G) \cong \Gamma_c(S_3)$, then $G \cong S_3$.

(iii) If $\Gamma_c(G) \cong \Gamma_c(A_4)$, then $G \cong A_4$.

**Proof.** (i): By Theorem 3.1(5), $Q_8$ is the only non-abelian group such that $\Gamma_c(Q_8) = K_4$, so the result follows.

(ii): By Theorem 3.1(1) and (3.1), (3.4), $S_3$ is the only group such that $\Gamma_c(S_3) = K_{1,3}$, so the result follows.

(iii): By Theorem 3.1(6) and (3.1), (3.2), (3.5), $A_4$ is the only group such that $\Gamma_c(A_4) = K_3 \cup \overline{K}_4$, so the result follows. \hfill \Box

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