Identification Fractional Linear Dynamic Systems with fractional errors-in-variables

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Abstract. A new algorithm is proposed for linear dynamic systems of the identification single-input-single-output (SISO) discrete fractional order with fractional errors-in-variables. The estimates are proved to be convergent to the true values with a probability one. The results of a simulated example indicate that the proposed algorithm provides good estimates.

1. Introduction

The fractional calculus based on integro-differential operators of the generalized fractional order is a very old topic in mathematics [1]. At the end of the 19th century, Liouville and Riemann introduced the first definition of the fractional derivative.

As far as the applications of fractional calculus are concerned, there is a large volume of research on viscoelasticity/damping, [2,3] and chaos/fractals [4], dielectric materials [5], electrochemical processes and flexible robots [6], traffic in information networks [7].

Upon automatic control, control algorithms both in frequency [8, 9] and time [10] domains based on the concepts of fractional calculus have been proposed. More examples, physical interpretations and areas of applications of fractional calculus are to be found in [11-13].

Due to their long memory behavior, the identification of fractional order models is more difficult compared with those of the integer order models. Several algorithms, based on the frequency domain, were proposed to solve this problem [6]. In [14–16], time domain identification techniques of the discrete time models of the fractional order, based on the least nonrecursive squares, are presented. For an overview of different identification methods based on fractional models, let us refer to Malti et al. [17].

In [18], a generalization of the Kalman filter for linear and nonlinear fractional order discrete state-space systems is presented.

Identification of fractional systems is much more difficult when there is a color noise. Therefore, the choice of instrumental variables is difficult because of the long memory of the fractional systems. The paper proposes a method to identify fractional systems for one of the colored noise classes: a fractional noise.

A new algorithm generalizes the results [19] in case of fractional noises.

The paper is organized as follows. In the next section, we present the problem statement. In section 3, the criteria for identifying systems are defined. In section 4, the iterative algorithm is determined. The simulation results are presented in section 5. Finally section 6 concludes this paper.
2. Problem statement

Let us consider the linear dynamic system of the fractional order described by the following stochastic discrete time equations:

\[ i = \ldots -1,0,1 \ldots \]

\[ z_i = \sum_{m=1}^{\infty} b_m^{(m)} \Delta^{\alpha_m} z_{i-1} + \sum_{m=1}^{\infty} a_m^{(m)} \Delta^{\beta_m} x_i, \quad y_i = z_i + \Delta^\phi \xi_i, \quad w_i = x_i + \Delta^\psi \zeta_i, \quad (1) \]

where \( \Delta^{\alpha_m} z_i = \sum_{j=0}^{\infty} (-1)^j \left( \begin{array}{c} \alpha_m \\ j \end{array} \right) z_{i-j}, \quad \Delta^{\beta_m} x_i = \sum_{j=0}^{\infty} (-1)^j \left( \begin{array}{c} \beta_m \\ j \end{array} \right) x_{i-j}, \quad \Delta^\phi \xi_i = \sum_{j=0}^{\infty} (-1)^j \left( \begin{array}{c} \phi \\ j \end{array} \right) \xi_{i-j}, \]

\[ \Delta^\psi \zeta_i = \sum_{j=0}^{\infty} (-1)^j \left( \begin{array}{c} \psi \\ j \end{array} \right) \zeta_{i-j}, \]

\[ 0 < \alpha_1 \ldots < \alpha_r, \quad 0 < \beta_1 \ldots < \beta_n, \quad 0 < \phi, \quad 0 < \psi. \]

Euler’s function \( \Gamma \) is defined as \( \Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt. \)

Newton’s binomial is generalized to non-integer orders using the Euler’s function:

\[ \left( \begin{array}{c} \alpha_m \\ j \end{array} \right) = \frac{\Gamma(\alpha_m + 1)}{\Gamma(j+1)\Gamma(\alpha_m - j + 1)}, \quad \left( \begin{array}{c} \beta_m \\ j \end{array} \right) = \frac{\Gamma(\beta_m + 1)}{\Gamma(j+1)\Gamma(\beta_m - j + 1)}, \]

\[ \left( \begin{array}{c} \phi \\ j \end{array} \right) = \frac{\Gamma(\phi + 1)}{\Gamma(j+1)\Gamma(\phi - j + 1)}, \quad \left( \begin{array}{c} \psi \\ j \end{array} \right) = \frac{\Gamma(\psi + 1)}{\Gamma(j+1)\Gamma(\psi - j + 1)}, \]

where \( z_i, \ x_i \) are noise-free output and input;

\( y_i, w_i \) are the measured output and input;

\( \Delta^{\phi} \zeta_i, \Delta^{\psi} \zeta_i \) are the measurement noise in the output and input signals;

The following assumptions are introduced:

1. Dynamic system (1) is asymptotically stable.

2. Noises \( \{ \xi_i \} \) and \( \{ \zeta_i \} \) are statistically independent sequences with \( E(\xi_i) = 0, \quad E(\zeta_i) = 0, \)

\[ E(\xi_i^2) = \sigma^2_{\xi_i} < \infty, \quad E(\zeta_i^2) = \sigma^2_{\zeta_i} < \infty, \quad a.s., \]

where \( E \) is the expectation operator.

3. Sequences \( \{ \xi_i \} \) and \( \{ \zeta_i \} \) are mutually do not depend and do not depend with \( z_i, x_i \), respectively.

4. Sequence \( \{ x_i \} \) is random signals with \( E(x_i) = 0, \quad E(x_i^2) = \sigma^2_x < \infty \) and

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \left( \begin{array}{c} \Phi^{(i)}_{\xi} \\ \Phi^{(i)}_{x} \end{array} \right) \left( \begin{array}{c} \Phi^{(i)}_{\xi} \\ \Phi^{(i)}_{x} \end{array} \right)^{T} = H \quad a.s., \]

where \( H \) is restricted positive definite.

5. Ratio \( \gamma = \sigma^2_{\xi}/\sigma^2_{\zeta} \) is known a priori.

It is required to estimate unknown coefficients \( b_m^{(m)}, a_m^{(m)} \) of the dynamical system of the linear fractional order, described by equation (1) in observable sequences \( \{ y_i \}, \{ w_i \} \). System orders \( \alpha_m, \beta_m, \phi, \psi \) are known.
3. Criteria for identification

Equation (1) is represented in the vector form as a linear regression:

\[ y_i = \Phi_i^T \theta_0 + \varepsilon_i, \]  

(2)

where \( \Phi_i = \left( (\Phi_{yi})^T \quad (\Phi_{\omega i})^T \right)^T \), \( \Phi_{yi} = \left( \sum_{j=0}^{k} (-1)^j \left( \alpha_i \right)_{yi-j-1}, \ldots, \sum_{j=0}^{k} (-1)^j \left( \alpha_i \right)_{yi-j-1} \right)^T \), \( \Phi_{\omega i} = \left( \sum_{j=0}^{k} (-1)^j \left( \beta_i \right)_{\omega i-j}, \ldots, \sum_{j=0}^{k} (-1)^j \left( \beta_i \right)_{\omega i-j} \right)^T \), \( \theta_0 = \left( b_0, a^T_0 \right)^T = \left( b_0^{(0)}, \ldots, b_0^{(r)}, a_0^{(0)}, \ldots, a_0^{(r)} \right)^T \), \( \varepsilon_i = \Delta \varepsilon_i - b_0^T \Phi_{\varepsilon_i} - a_0^T \Phi_{\varepsilon_i} \),

\( \Phi_{\varepsilon_i} = \left( \sum_{j=0}^{k} (-1)^j \left( \alpha_i + \phi \right)_{\varepsilon i-j-1}, \ldots, \sum_{j=0}^{k} (-1)^j \left( \alpha_i + \phi \right)_{\varepsilon i-j-1} \right)^T \),

\( \Phi_{\varepsilon_i} = \left( \sum_{j=0}^{k} (-1)^j \left( \beta_i + \psi \right)_{\varepsilon i-j-1}, \ldots, \sum_{j=0}^{k} (-1)^j \left( \beta_i + \psi \right)_{\varepsilon i-j-1} \right)^T \).

From requirement 2, it follows that generalized error \( \varepsilon_i \) has zero mean. We obtain that variance of the generalized error equal to

\[ \sigma^2_\varepsilon = \sigma^2_\varepsilon^2 h_0^{(0)} + \sigma^2_\varepsilon^2 b_0^T H_{\alpha+\phi} b_0 + \sigma^2_\varepsilon^2 a_0^T H_{\beta+\psi} a_0 - 2 \sigma^2_\varepsilon h_0^{\omega} b_0 = \sigma^2_\varepsilon^2 (1 + b_0^T H_{\alpha+\phi} b_0 + a_0^T H_{\beta+\psi} a_0 - 2 h_0^{\omega} b_0) = \sigma^2_\varepsilon^2 \omega(b_0, a_0), \]

\[ h_0^{(0)} = \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-1} \left( \frac{\Phi_i(j)}{N-j} \right) = \frac{1}{N} \sum_{j=0}^{N-1} \left( \frac{\Phi_i(j)}{N-j} \right) \]

\[ H_{\alpha+\phi} = \left( H_{\alpha+\phi}^{(1)}, \ldots, H_{\alpha+\phi}^{(r)} \right), \quad H_{\beta+\psi} = \left( H_{\beta+\psi}^{(1)}, \ldots, H_{\beta+\psi}^{(r)} \right), \]

\[ h_0^{(m)} = \frac{1}{N} \sum_{j=0}^{N-1} \left( \frac{\Phi_i(j)}{N-j} \right) \]

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Let us define estimates \( \hat{\theta} \) of the unknown true value from the requirement of minimum considered square variance \( \sigma^2_\varepsilon \) weighted \( \omega(b, a) \):
\[
\min_{\alpha \in \mathbb{B}} \sum_{i=1}^{N} \frac{(y_i - \varphi_i^{T}\theta)^2}{1 + b^T H_{\alpha} b + \gamma a^T H_{\beta} a - 2h_{\alpha}^T b}
\]

**Theorem.** Suppose that assumptions1-5, then the estimates defined by equation (3) \( \hat{\theta}(N) \xrightarrow{N \to \infty} \theta_0 \)
a.s.

4. Iterative algorithm

4.1 Approximation algorithm

Define estimates \( \hat{\theta} \) of the unknown true value from the requirement of minimum considered square variance \( \sigma^2_{\theta} \). To define initial estimate \( \hat{\lambda}(0) \), which is necessary for further using, the exact algorithm or to define that there is not the root \( \hat{\lambda}'_1(N) \).

**Step 0.** \( \hat{\lambda}'(0) = 0 \);

**Step 1.** \( \hat{\lambda}'(i) = \frac{\lambda_{\min} + \hat{\lambda}'(i - 1)}{2} \), \( \lambda_{\min} \) - the smallest root of equation
\[
\det\left( \begin{pmatrix} \Phi_y^T \Phi_y & \Phi_y^T \Phi_w \\ \Phi_w^T \Phi_y & \Phi_w^T \Phi_w \end{pmatrix} - \lambda \begin{pmatrix} H_{\alpha}^T(N) & 0 \\ 0 & H_{\beta}^T(N) \end{pmatrix} \right) = 0
\]

where \( Y = (y_1, ..., y_N)^T \),
\[
\Phi_y = \begin{pmatrix} \varphi_1^T(0) \\ \vdots \\ \varphi_N^T(N-1) \end{pmatrix}, \quad \Phi_w = \begin{pmatrix} \varphi_1^T(0) \\ \vdots \\ \varphi_N^T(N-1) \end{pmatrix}
\]

**Step 2.** Calculate \( \hat{b}(N, \hat{\lambda}') \), \( \hat{\alpha}(N, \hat{\lambda}') \) from the following equation system:
\[
\begin{pmatrix} \hat{b}(N, \hat{\lambda}') \\ \hat{\alpha}(N, \hat{\lambda}') \end{pmatrix} = \begin{pmatrix} \Phi_y^T \Phi_y - \hat{\lambda}'(i)H_{\alpha}^T(N) \\ \Phi_w^T \Phi_y \end{pmatrix}^{-1} \begin{pmatrix} \Phi_y^T Y - \hat{\lambda}'(i)h_{\alpha}^T(N) \\ \Phi_w^T Y \end{pmatrix}.
\]

**Step 3.** Calculate
\[
V_N(\hat{\lambda}'(i)) = Y^T Y - \hat{\lambda}'(i) - \begin{pmatrix} \Phi_y^T Y - \hat{\lambda}'(i)h_{\alpha}^T(N) \\ \Phi_w^T Y \end{pmatrix} \begin{pmatrix} \hat{b}(N, \hat{\lambda}) \\ \hat{\alpha}(N, \hat{\lambda}) \end{pmatrix}.
\]

**Step 4.** Examine requirement \( V_N(\hat{\lambda}'(i)) \leq 0 \).

Then, if equation \( V_N(\hat{\lambda}'(i)) = 0 \) has root \( \hat{\lambda}'(N) \in [0, \lambda_{\min}(N)] \), then \( \hat{\lambda}'(0), \hat{\lambda}'(1), ..., \hat{\lambda}'(N) \) is finite, and
\( \lambda(0) \in \{ \hat{\lambda}'_1(N), \lambda_{\min}(N) \} \).

4.2 Exact algorithm

Suppose that there is \( \hat{\lambda}(0) \in \{ \hat{\lambda}'_1(N), \lambda_{\min}(N) \} \), then \( \lim_{i \to \infty} \hat{\lambda}'(i) = \hat{\lambda}(N), \lim_{i \to \infty} \hat{\alpha}(N, \hat{\lambda}) = \hat{\alpha}(N) \),

\( \lim_{i \to \infty} \hat{b}(i, \hat{\lambda}) = \hat{b}(N, \hat{\lambda}) \), where \( \hat{\lambda}(i), \hat{b}(i, \hat{\lambda}) \) and \( \hat{\alpha}(i, \hat{\lambda}) \), are defined together with the following algorithm:

**Step 1.** Calculate \( \hat{b}(N, \hat{\lambda}), \hat{\alpha}(N, \hat{\lambda}) \), from equation system (4);

**Step 2.** Calculate
\[
\hat{\lambda}(i+1) = \left(1 + \left[\tilde{b}(N, \hat{\lambda}) \right]^T H_{\alpha+\tau}(N) \tilde{b}(N, \hat{\lambda}) + \gamma \left[\tilde{a}(N, \hat{\lambda}) \right]^T H_{\beta+\psi}(N) \tilde{a}(N, \hat{\lambda}) - 2 h_{\alpha+\tau}(N) \tilde{b}(N, \hat{\lambda}) \right]^{-1} \times
\times \left[\hat{Y}^T + \hat{\lambda}(i) \left[\tilde{b}(N, \hat{\lambda}) \right]^T H_{\alpha+\tau}(N) \tilde{b}(N, \hat{\lambda}) + \gamma \left[\tilde{a}(N, \hat{\lambda}) \right]^T H_{\beta+\psi}(N) \tilde{a}(N, \hat{\lambda}) - 2 h_{\alpha+\tau}(N) \tilde{b}(N, \hat{\lambda}) \right]^{-1} \times

\left( \Phi_Y^T - \hat{\lambda}(i) h_{\alpha+\tau}(N) \right) \tilde{b}(N, \hat{\lambda}) \right).
\]

**Step 3.** Transition to step 1.

The calculus is over if the following requirement is met:

\[
\frac{\|V_N(\hat{\lambda}(i+1)) - V_N(\hat{\lambda}(i))\|}{\|V_N(\hat{\lambda}(i+1))\|} \leq \delta,
\]

where \(\delta\) - a priori given accuracy of the estimation method.

5. **Iterative algorithm**

The offered algorithm has been realized in Matlab and compared to the least square (LS) method. The dynamic system is defined by equations

\[
\begin{align*}
\zeta_i &= 0.5 \Delta^{0.5} \zeta_{i-1} - 0.2 \Delta^{1.7} \zeta_{i-1} + \Delta^{0.7} x_i - 0.2 \Delta^{1.2} x_{i-1} + \Delta^{0.3} \zeta_i, \\
y_i &= z_i + \Delta^{0.3} \zeta_i, \\
w_i &= x_i + \Delta^{0.5} \zeta_i.
\end{align*}
\]

The noise-free input is defined as

\[
x_i + 0.8 \cdot x_{i-1} + 0.6 \cdot x_{i-2} = \zeta_i + 1.7 \cdot \zeta_{i-1} + 0.5 \cdot \zeta_{i-2},
\]

where \(\zeta_i\) is a white noise.

The normalized root mean square error, defined as

\[
\delta \theta = \sqrt{\frac{\| \hat{\theta} - \theta \|}{\| \theta_0 \|}} \cdot 100\%,
\]

is given in Table 1.

| \(\sigma_{\hat{\theta}} / \sigma_\theta\) | \(\sigma_{\hat{\zeta}_i} / \sigma_{\zeta}\) | \(\delta \theta, \%\) | \(\delta \theta_{LS}, \%\) |
|---|---|---|---|
| 0.1 | 0.1 | 2.95±2.11 | 5.73±2.73 |
| 0.25 | 0.25 | 8.57±4.44 | 24.56±7.98 |
| 0.5 | 0.5 | 20.14±16.04 | 58.74±10.60 |

6. **Conclusion**

In this paper, the method of consistent estimation parameters of linear dynamic systems of the fractional order with fractional errors-in-variables has been studied. Simulation results indicate that the LS method gives biased results. An errors-in-variables model have received much attention because of its important applications in signal processing, communications and control systems.

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