Algebraic and Graph-Theoretic Conditions for the Herdability of Linear Time-Invariant Systems

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Abstract—In this paper we investigate a relaxed concept of controllability, known in the literature as herdability, namely the capability of a system to be driven towards the (interior of the) positive orthant. Specifically, we investigate herdability for linear time-invariant systems, both from an algebraic perspective and based on the graph representing the systems interactions. In addition, we focus on linear state-space models corresponding to matrix pairs \((A, B)\) in which the matrix \(B\) is a selection matrix that determines the leaders in the network, and we show that the weights that followers give to the leaders do not affect the herdability of the system. We then focus on the herdability problem for systems with a single leader in which interactions are symmetric and the network topology is acyclic, in which case an algorithm for the leader selection is provided. In this context, under some additional conditions on the mutual distances, necessary and sufficient conditions for the herdability of the overall system are given.

I. INTRODUCTION

There are many applications in control systems theory in which requiring that the system is controllable, namely that the system state can be driven towards any point in the state space, is unnecessary, due to the nature of the involved application. This is what happens, in particular, when dealing with networked systems [1], [6]. This kind of systems, in fact, comes into play in many applications related to biology [5], chemistry [2], sociology [13], neuroscience [4], etc. In all these contexts, asking whether the state can be brought to an arbitrary point of the state space may lead to unnecessarily restrictive conditions on the system model. For example, in the study of a chemical reaction it makes no sense to impose that variables representing solvent concentrations may reach negative values. In these cases it becomes of interest to study a weaker concept with respect to the one of controllability, known in the literature as herdability [11], [12]. Herdability indicates the capability of a system to be driven towards the positive orthant. In mathematical terms, a continuous-time linear time-invariant system

\[
\dot{x}(t) = Ax(t) + Bu(t),
\]

with \(A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}\), is herdable if, for every initial condition \(x(0)\), there exists a control input that drives all the state variables over a positive threshold.

In this work we investigate the herdability property of linear time-invariant (LTI) systems both from an algebraic and a topological perspective. In particular, we consider systems whose associated network has cooperative and competitive relationships and we try to understand when herdability can be deduced by simply looking at the type of relationships (namely at their signs) rather than at their values.

Controllability of multi-agent systems is a well-established research field. Extensive results have been obtained for sign controllable and structurally controllable networked systems [8], [9], [10], [16]. In [8] the controllability of LTI systems defined on a signed graph is discussed, by focusing on the controllability analysis of positive and negative eigenvalues of systems whose matrices share the same sign pattern. In [9], [16] complete controllability conditions for matrix pairs \((A, B)\) are deduced based on some graph properties. Finally, in [10] the controllability of multi-agent systems is studied by assuming that a subset of agents represent the leaders, while the remaining ones execute local protocols. It is also shown how the symmetry of the network structure affects the controllability property of the overall system.

Herdability of networked systems, on the other hand, is a much more recent line of research, as witnessed by the recent works [7], [11], [12], [14], [15]. In [14] the herdability property of dynamic leader-follower signed networks is studied from a topological point of view, under the assumption that the leaders are endowed with exogenous control inputs, thus developing sufficient conditions for herdability based on 1-walks and 2-walks on the graph. In [12] the topology and sign distribution of the underlying graph of an LTI system in related to its herdability property, while in [11] herdability of subsets of nodes in a graph is investigated, with a special focus on the herdability of directed out-branching rooted graphs with a single input. In the paper [15], the controllable subspace of a system is characterized based on graph partitions, and sufficient conditions for the system herdability are deduced. The concept of quotient graph is exploited in order to deduce the herdability of the original graph.

Inspired by the methods exploited in the study of the controllability of networked systems and motivated by the practical need to relax the controllability property, especially in the context of networked systems with cooperative and competitive interactions, we introduce here some sufficient conditions for the herdability of LTI systems. We focus, in particular, on systems with leader-follower networks and undirected acyclic graphs. We also introduce some algorithms to check some conditions ensuring the system herd-
ability with a special focus on tree topologies. Finally, we propose a leader selection strategy aimed at guaranteeing the system herdability.

In detail, Section II provides some sufficient conditions for the herdability of a generic matrix pair \((A, B)\), based on the algebraic structure of the system. In Section III herdability of leader-follower networks is studied, while Section IV focuses on systems whose associated graph has a single leader and tree topology. Section V concludes the paper.

**Notation.** Given \(k, n \in \mathbb{Z}\), with \(k < n\), the symbol \([k, n]\) denotes the integer set \(\{k, k + 1, \ldots, n\}\). The \((i, j)\)-th entry of a matrix \(A\) is denoted by \([A]_{ij}\), while the \(i\)-th entry of a vector \(v\) by \([v]_i\). The notation \(M = \text{diag}\{M_1, M_2, \ldots, M_n\}\) indicates a block diagonal matrix with diagonal blocks \(M_1, M_2, \ldots, M_n\). We let \(e_i\) denote the \(i\)-th vector of the canonical basis of \(\mathbb{R}^n\), where the dimension \(n\) will be clear from the context. Accordingly, \(Me_j\) denotes the \(j\)-th column of \(M\), \(e_i^t M\) the \(i\)-th row of \(M\), and \(e_i^t Me_j\) the \((i, j)\)-th entry of \(M\). Any nonzero multiple of a canonical vector is called monomial vector. The vectors \(1_n\) and \(0_n\) denote the \(n\)-dimensional vectors whose entries are all 1 or 0, respectively. Similarly, the symbol \(0_{p \times m}\) denotes the \(p \times m\) matrix with all zero entries.

Given a vector \(v \in \mathbb{R}^n\), the set \(\text{ZP}(v) = \{i \in [1, n] : [v]_i \neq 0\}\) denotes the non-zero pattern of \(v\) [17]. Similarly, we can define the non-zero pattern of a matrix \(A\). A nonzero vector \(v\) is said to be unsigned [12] if all its nonzero entries have the same sign. If \(v\) is a unsigned vector, then by \(\text{sign}(v)\) we mean the common sign of its nonzero entries. In other words, \(\text{sign}(v) = 1\) if the nonzero entries of \(v\) are positive, while \(\text{sign}(v) = -1\) if the nonzero entries of \(v\) are negative.

Given a matrix \(A \in \mathbb{R}^{n \times m}\), the notation \(\text{Im}(A)\) denotes the image of the matrix \(A\). A matrix (in particular, a vector) \(A\) is nonnegative (denoted by \(A \geq 0\)) [3] if all its entries are nonnegative. \(A\) is strictly positive (denoted by \(A \gg 0\)) if all its entries are positive. A matrix \(P \in \mathbb{R}^{n \times n}\) is a permutation matrix if its columns are a permuted version of the columns of the identity matrix \(I_n\).

Given a set \(S\), the cardinality of \(S\) is denoted by \(|S|\).

To any matrix \(A \in \mathbb{R}^{n \times n}\), we associate the signed and weighted directed graph \(\mathcal{G}(A) = (V, E, A)\), where \(V = [1, n]\) is the set of nodes. The set \(E \subseteq V \times V\) is the set of arcs (edges) connecting the nodes, while the matrix \(A \in \mathbb{R}^{n \times n}\) is the adjacency matrix of the graph. There is an arc \((j, i) \in E\) from \(j\) to \(i\), if and only if \([A]_{ij} \neq 0\). When so, \([A]_{ij}\) is the weight of the arc.

A sequence of \(k\) consecutive arcs \((j_1, j_2), (j_2, j_3), \ldots, (j_k, i) \in E\) is a walk of length \(k\) from \(j\) to \(i\). A walk from \(j\) to \(i\) is said to be positive (negative) if the product of the weights of the edges that compose the walk is positive (negative). A directed graph \(\mathcal{G}(A)\) is strongly connected if for every pair of vertices \(i, j \in V\) there exists a walk from \(i\) to \(j\). A minimum walk from \(j\) to \(i\) is a walk of minimum length connecting the two nodes. We define the distance \(d(j, i)\) from the node \(j\) to the node \(i\) as the length of the minimum walk from \(j\) to \(i\). The distance \(d(j, I)\) from the node \(j\) to the set of nodes \(I\) is the minimum among all the distances \(d(j, i), i \in I\). Similarly, the distance \(d(I, j)\) from the set of nodes \(I\) to the vertex \(j\) is the minimum among all the distances \(d(i, j), i \in I\).

Given a node \(i \in V\), we define the out-neighborhood of node \(i\) as the set of nodes \(j\) such that \(d(i, j) = 1\), namely \(\text{Out}(i) = \{j \in V : (i, j) \in E\}\). We define the positive out-neighborhood of node \(i\) as the set of nodes \(j\) such that \((i, j)\) is an arc of \(\mathcal{G}(A)\) of positive weight, namely \(\text{Out}_+(i) = \{j \in V : [A]_{ij} > 0\}\). The definition of negative out-neighborhood of a node is analogous. The out-neighborhood can be also defined for a set of nodes \(I \subseteq V\) as \(\text{Out}(I) = \{j \in V \setminus I : (i, j) \in E, \exists i \in I\}\). The definitions of \(\text{Out}_+(I)\) and \(\text{Out}_-(I)\) are analogous.

If \(A\) is a symmetric matrix, namely \(A = A^T\), the graph \(\mathcal{G}(A)\) is (signed, weighted and) undirected, and all previous concepts (in particular, the concepts of walk and distance) become symmetric.

A graph \(\mathcal{G}(A)\) is said to be structurally balanced if all its nodes can be partitioned into two disjoint subsets \(V_1\) and \(V_2\) in a way such that \(\forall i, j \in V_p, p \in \{1, 2\}, [A]_{ij} \geq 0\) and \(\forall i \in V_p, \forall j \in V_{q} = V_p, p, q \in \{1, 2\}, p \neq q\), it holds that \([A]_{ij} \leq 0\). Note that if \(V_1 = [1, m]\), while \(V_2 = [m + 1, n]\), the matrix \(A\) can be block partitioned as

\[
A = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix},
\]

where \(A_{11} \in \mathbb{R}^{m \times m}\) and \(A_{22} \in \mathbb{R}^{(n-m) \times (n-m)}\) are nonnegative matrices, while \(A_{12}\) and \(A_{21}\) are nonpositive matrices (i.e., the opposite of nonnegative matrices).

**II. SUFFICIENT CONDITIONS FOR HERDABILITY OF GENERAL PAIRS \((A, B)\)**

The concept of herdability of linear and time-invariant state space models described by a matrix pair \((A, B)\), with \(A \in \mathbb{R}^{n \times n}\) and \(B \in \mathbb{R}^{n \times m}\), has been defined in various ways [11], [12], [14]. In this paper we are interested in the behavior of all state variables, rather than in the behavior of a subset of them. Consequently, we assume the following definition (which is equivalent to Definition 3 in [12]).

**Definition 1:** Given a (continuous-time or discrete-time) (linear and time-invariant) state space model of dimension \(n\) with \(m\) inputs, described by a pair \((A, B)\), \(A \in \mathbb{R}^{n \times n}\) and \(B \in \mathbb{R}^{n \times m}\), the system (the pair) is said to be (completely) herdable if for every \(x(0)\) and every \(h > 0\), there exists a time \(t_f > 0\) and an input \(u(t), t \in [0, t_f]\), that drives the state of the system from \(x(0)\) to \(x(t_f) \geq h1_n\).

Both in the continuous-time case and in the discrete-time case, herdability reduces to a condition on the controllability matrix associated with the pair \((A, B)\).

**Proposition 1 (Corollary 1, [12]):** A pair \((A, B)\), \(A \in \mathbb{R}^{n \times n}\) and \(B \in \mathbb{R}^{n \times m}\), is herdable if and only if there exists a strictly positive vector belonging to \(\text{Im}(\mathcal{R}(A, B))\), where

\[
\mathcal{R}(A, B) := [B \ AB A^2B \ldots A^{n-1}B]
\]
is the controllability matrix of the pair \((A, B)\).

Clearly, every reachable pair \((A, B)\) is herdable, but the converse is not true. Also, if \(R(A, B)\) has zero rows then the problem is clearly not solvable. So, in the following we will investigate herdability by assuming that \(R(A, B)\) is devoid of zero rows and \(\text{Im}(R(A, B))\) is a proper subset of \(\mathbb{R}^n\).

In this section we present some sufficient conditions for the herdability of a generic matrix pair \((A, B)\). We will later focus on pairs \((A, B)\) that are endowed with specific structural properties.

**Lemma 2:** Given a pair \((A, B)\), let \(A \in \mathbb{R}^{n \times n}\) and \(B \in \mathbb{R}^{n \times m}\), assume that \(R := R(A, B) \in \mathbb{R}^{n \times nm}\) satisfies the following conditions:

i) \(R\) has no zero rows;

ii) the set \(J := \{ j \in [1, nm] : R e_j \text{ is unisigned} \}\) is such that \(|\cup_{j \in J} ZP(R e_j)| \geq n - 1\).

Then the pair \((A, B)\) is herdable.

**Proof:** Let us first suppose that \(|\cup_{j \in J} ZP(R e_j)| = n\), which means that \(\forall i \in [1, n]\), there exists \(j \in J\) such that the \(i\)-th entry of the unisigned vector \(R e_j\) is nonzero. By choosing the vector \(u\) with entries

\[
[u]_j = \begin{cases} 0, & \text{if } j \notin J; \\ \text{sign}(R e_j), & \text{if } j \in J; \end{cases}
\]

it is immediate to see that \(Ru \succ 0\), and hence the pair \((A, B)\) is herdable.

Let us assume now that \(|\cup_{j \in J} ZP(R e_j)| = n - 1\), and set \(J = \{ j_1, j_2, \ldots, j_k \}\). This implies that there exists a unique index \(i \in [1, n]\) such that \(e_i^T R [e_{j_1}, e_{j_2}, \ldots, e_{j_k}] = 0_k^T\). On the other hand, by hypothesis i), there exists \(h \in [1, nm], h \notin J\), such that \(e_i^T R e_h \neq 0\). Therefore, by choosing the vector \(u\) with entries

\[
[u]_j = \begin{cases} \text{sign}(e_i^T R e_h), & \text{if } j = h; \\ 0, & \text{if } j \notin J \cup \{ h \}; \\ k \cdot \text{sign}(R e_j), & \text{if } j \in J; \end{cases}
\]

there always exists \(k \in \mathbb{R}, k > 0\), sufficiently large such that \(Ru \succ 0\). □

We now introduce a technical lemma, whose proof is elementary and hence omitted.

**Lemma 3:** Given a matrix \(\Phi \in \mathbb{R}^{n \times k}\), assume that there exist two permutation matrices \(P_1 \in \mathbb{R}^{n \times n}\) and \(P_2 \in \mathbb{R}^{k \times k}\) such that

\[
P_1 \Phi P_2 = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ 0 & \Phi_{22} \end{bmatrix},
\]

and that both \(\text{Im}(\Phi_{11})\) and \(\text{Im}(\Phi_{22})\) include a strictly positive vector. Then there exists \(u \in \mathbb{R}^k\) such that \(\Phi u \succ 0\).

Based on Lemma 3, we can derive the following sufficient condition for herdability that bears some similarities with Lemma 3 in [12].

**Lemma 4:** Given a pair \((A, B)\), \(A \in \mathbb{R}^{n \times n}\) and \(B \in \mathbb{R}^{n \times m}\), assume that \(R := R(A, B) \in \mathbb{R}^{n \times nm}\) has no zero rows. Define the sets

\[
J := \{ j \in [1, nm] : R e_j \text{ is unisigned} \}
\]

\[
\mathcal{H} := \cup_{j \in J} ZP(R e_j),
\]

and suppose that \(\forall h \in [1, n] \setminus \mathcal{H}\) there exists \(j \in [1, nm] \setminus J\) such that

i) \(\{ \mathcal{H} \}_h = e_h^T R e_j \neq 0\), and

ii) \(\forall k \in [1, n] \setminus \mathcal{H}\), condition \(\{ \mathcal{H} \}_k = e_k^T R e_j \neq 0\) implies \(\text{sign}(\mathcal{H}_k) = \text{sign}(\mathcal{H}_j)\),

namely for every index \(h\) that does not belong to \(\mathcal{H}\) there exists a column of \(R\), say \(R e_j\), where the \(h\)-th entry and all the nonzero entries corresponding to indices that do not belong to \(\mathcal{H}\) are of the same sign. Then the pair \((A, B)\) is herdable.

**Proof:** Under the lemma assumptions there exists a set of indices \(T \subseteq [1, nm] \setminus J\) such that

a) \((\cup_{j \in J} ZP(R e_j)) \cup (\cup_{j \in T} ZP(R e_j)) = [1, n]\

b) if we denote by \(S \in \mathbb{R}^{(n-\mathcal{H}) \times n}\) the (selection) matrix whose rows are the \(n\)-dimensional canonical vectors indexed in \([1, n] \setminus \mathcal{H}\), then \(S R e_j\) is unisigned for every \(j \in T\).

This implies that there exist two permutation matrices \(P_1 \in \mathbb{R}^{n \times n}\) and \(P_2 \in \mathbb{R}^{nm \times nm}\) such that

\[
P_1 R P_2 = \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix},
\]

where \(R_{11}\) has all unisigned columns, while \(R_{22}\) has a subset of its columns that are unisigned and therefore \(\text{Im}(R_{22})\) includes a strictly positive vector. So, the result follows from Lemma 3. □

The idea behind Lemma 3 and Lemma 4 can be recursively iterated, thus leading to an algorithm that checks a sufficient condition for herdability. The algorithm receives as input the controllability matrix and returns, if the sufficient condition is verified, a confirmation that the pair \((A, B)\) is herdable. In detail, it proceeds as follows: at each step the algorithm detects a column vector that is unisigned, then sets to zero all the rows of \(R\) that correspond to the nonzero entries (the non-zero pattern) of such a column vector. Subsequently, the algorithm repeats the same step on the modified matrix \(R\), until either \(R\) becomes the zero matrix or the matrix \(R\) has no unisigned columns, thus iteratively applying the same strategy as in Lemma 3. In the former case the pair \((A, B)\) is herdable, in the second case the algorithm stops.

Algorithm 1, below, makes use of the following notation. Given a matrix \(R \in \mathbb{R}^{n \times nm}\) and a set \(I \subseteq [1, n]\), we denote by \(R_I\) the matrix obtained from \(R\) by (leaving unchanged all rows indexed in \(I\) and) replacing every row indexed in \([1, n] \setminus I\) with the zero row.

**III. SUFFICIENT CONDITIONS FOR HERDABILITY OF PAIRS \((A, B)\) CORRESPONDING TO A DIRECTED GRAPH \(G(A)\) WITH \(m\) LEADERS**

We now consider the case when the columns of the matrix \(B\) are \(m\) linearly independent canonical vectors. It entails no loss of generality assuming that

\[
B = \begin{bmatrix} I_m \\ 0 \end{bmatrix} \in \mathbb{R}^{n \times m},
\]
Algorithm 1 Greedy algorithm to check herdability

$$\mathcal{R} := [B|AB| \ldots |A^{n-1}B] \quad \triangleright \text{Initialization}$$
$$\mathcal{I} := [1,n]$$
$$\mathcal{J} := [1, nm]$$
while $\mathcal{I} \neq \emptyset$ do
  for $j \in \mathcal{J}$ do
    if $\mathcal{R}\{j\}$ is unsigned then
      $\mathcal{J} = \mathcal{J} \setminus \{j\}$
      $\mathcal{I} = \mathcal{I} \setminus \mathcal{Z}(\mathcal{R}\{j\})$
      $\mathcal{R} = \mathcal{R}_{\mathcal{J}}$
    end if
  end for
  if $\mathcal{I} = \emptyset$ then
    $(A,B)$ is herdable
  end if
if there are no unsigned column vectors in $\mathcal{R}$ then stop

since we can always permute the entries of the state vector so that this is the case. Accordingly, we can block-partition the matrix $A \in \mathbb{R}^{n \times n}$ as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

(8)

where $A_{11} \in \mathbb{R}^{m \times m}$. We want to investigate the herdability of the pairs $(A,B)$, where $A$ and $B$ are described as in (8) and (7), respectively.

One of the advantages of this set-up, that has already been considered in [14], [15], [7], is that it allows to investigate the herdability of the pair $(A,B)$ by resorting to the signed and weighted directed graph $\mathcal{G}(A)$ whose nodes are partitioned into leaders and followers, depending on whether the state variable associated to the node is endowed with an external and independent control input (leader) or not (follower). Specifically we introduce the following:

Assumption 1: We assume that in the signed and weighted directed graph $\mathcal{G}(A)$ the first $m$ vertices, associated with the $m$ canonical vectors in $B$, represent the set $\mathcal{L} = [1,m]$ of leaders and the remaining vertices are the set of followers, i.e., $\mathcal{F} = [m+1,n]$. We let $\mathcal{F}_k$ be the set of followers whose distance from the leaders is $k$, $k \in [1,K]$, by this meaning

$$\mathcal{F}_k := \{ j \in \mathcal{F} : d(\mathcal{L},j) = k \}.$$

We assume $\mathcal{F}_K \neq \emptyset$, $\mathcal{F}_k = \emptyset$, $k > K$. This means that $K$ is the maximum distance from the set of leaders to a follower. It entails no loss of generality assuming that $\mathcal{F}_1 = [m+1,m+m_1], \ldots, \mathcal{F}_k = [m+m_1+\cdots+m_{k-1}+1,m+m_1+\cdots+m_k]$, so that $|\mathcal{L}| = m$ and $|\mathcal{F}_k| = m_k$ and $m + m_1 + \cdots + m_K = n$.

Under Assumption 1, the controllability matrix in $K+1$ steps $\mathcal{R}_{K+1} = [B|AB|A^2B| \ldots |A^K B]$ takes the following structure

$$\mathcal{R}_{K+1} = \begin{bmatrix} I_m & \Phi_1 & \Phi_2 & \cdots & \Phi_K \\ 0 & \Phi_1 & \Phi_2 & \cdots & \Phi_K \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Phi_K \end{bmatrix}$$

(9)

and all the matrices $\Phi_k \in \mathbb{R}^{m_k \times m}$ have no zero rows.

In the following some sufficient conditions for the herdability of a pair $(A,B)$ satisfying Assumption 1 are provided.

Proposition 5: Consider a pair $(A,B)$, where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are described as in (8) and (7), respectively. Suppose that Assumption 1 holds and hence the controllability matrix in $K+1$ steps of the pair is described as in (9) and all the matrices $\Phi_k \in \mathbb{R}^{m_k \times m}$ have no zero rows. For every $k \in [1,K]$, introduce the sets

$$J_k := \{ j \in [1,m] : \Phi_k e_j \text{ is unsigned} \}$$

(10)

and

$$H_k := \cup_{j \in J_k} \mathcal{Z}(\Phi_k e_j).$$

(11)

If $\forall k \in [1,K]$, one has $|H_k| \geq m_k - 1$, then the pair $(A,B)$ is herdable.

Proof: The result follows from Lemma 2 and Lemma 3. Indeed, by making use of Lemma 2, we can claim that $\text{Im}(\Phi_k)$ includes a strictly positive vector, for every $k \in [1,K]$. But then, by recursively using Lemma 3, one can find a strictly positive vector in $\text{Im}(\mathcal{R}(K+1)).$ $\blacksquare$

Proposition 6: Consider a pair $(A,B)$, where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are described as in (8) and (7), respectively. Suppose that Assumption 1 holds and hence the controllability matrix in $K+1$ steps of the pair is described as in (9) and all the matrices $\Phi_k \in \mathbb{R}^{m_k \times m}$ have no zero rows. For every $k \in [1,K]$, consider the sets $J_k$ and $H_k$ as in (10) and (11).

If $\forall h \in [1,m_k] \setminus H_k$ there exists $i \in [1,m] \setminus J_k$ such that

i) $[\Phi_k]_{hi} = m_h^{-1} \Phi_k e_i \neq 0$, and

ii) $\forall \ell \in [1,m_k] \setminus H_k$, condition $[\Phi_k]_{\ell i} = e_{i}^{\ell} \Phi_k e_i \neq 0$ implies $\text{sign}([\Phi_k]_{\ell i}) = \text{sign}([\Phi_k]_{hi})$,

then the pair $(A,B)$ is herdable.

Proof: The proof is obtained by repeatedly applying Lemma 3 and Lemma 4. $\blacksquare$

Proposition 7: Consider a pair $(A,B)$, where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are described as in (8) and (7), respectively. If we denote by $\mathcal{R}(A,B)$ the controllability matrix of $(A,B)$ and by $\mathcal{R}(A_{22},A_{21})$ the controllability matrix of $(A_{22},A_{21})$, then for every choice of $v_1 \in \mathbb{R}^m$

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \text{Im}(\mathcal{R}(A,B)) \iff v_2 \in \text{Im}(\mathcal{R}(A_{22},A_{21})).$$

Therefore the pair $(A,B)$ is herdable if and only if the pair $(A_{22},A_{21})$ is herdable.
Proof: Since
\[ \mathcal{R}(A, B) = \begin{bmatrix} I_m & \Phi_{12} \\ 0 & \Phi_{22} \end{bmatrix} \]
where
\[ \Phi_{22} := \begin{bmatrix} A_{21} & A_{21}A_{11} + A_{22}A_{21} \\ A_{21}(A_{11} + A_{12}A_{21}) + A_{22}(A_{21}A_{11} + A_{22}A_{21}) \ldots \end{bmatrix} \]
and hence (by Cayley-Hamilton’s theorem)
\[ \text{Im}([0 \ I_{n-m} \ AB \ A^2B \ \ldots \ A^{n-1}B]) = \text{Im}([A_{21} \ A_{22}A_{21} \ \ldots \ A_{22}^{n-2}A_{21}]) = \text{Im}(\mathcal{R}(A_{22}, A_{21})). \]

Consequently, the pair \((A, B)\) is herdable if and only if the pair \((A_{22}, A_{21})\) is herdable. □

Proposition 7 allows to easily obtain two results that are already available in the literature. As we will see in the next section, however, the consequences of Proposition 7 can be further exploited.

Corollary 8 (Proposition 1 [7]): Consider a pair \((A, B)\), where \(A \in \mathbb{R}^{n \times n}\) and \(B \in \mathbb{R}^{n \times m}\) are described as in (8) and (7), respectively. If the directed graph \(\mathcal{G}(A)\) is strongly connected and structurally balanced, and the classes in which the agents split are \(V_1 = [1, m]\) and \(V_2 = [m + 1, n]\), then the pair \((A, B)\) is herdable.

Proof: We first note that as \(\mathcal{G}(A)\) is strongly connected then \(\mathcal{R}(A, B)\) cannot have zero rows, therefore (see Proposition 7) also \(\mathcal{R}(A_{22}, A_{21})\) has no zero rows. If \(V_1 = [1, m]\), then \(A_{21}\) is a nonpositive matrix, while \(A_{22}\) is a nonnegative matrix (see the end of the Notation part), therefore the controllability matrix of the pair \((A_{22}, A_{21})\) has all negative columns and no zero rows. This ensures that \((A_{22}, A_{21})\) is herdable. □

Remark 9: It is easily seen that the result of Corollary 8 would still be true if the set of leaders would include \(V_1\) rather than coincide with it.

Corollary 10 (Theorem 1 in [14]): Consider a pair \((A, B)\), where \(A \in \mathbb{R}^{n \times n}\) and \(B \in \mathbb{R}^{n \times m}\) are described as in (8) and (7), respectively. If every follower is reached by at least one of the leaders in a single step, namely through a walk of length 1, and for each leader the walks of length 1 to its followers have the same sign, then the pair \((A, B)\) is herdable.

Proof: By the corollary assumptions the matrix \(A_{21}\) is devoid of zero rows and all its columns are either zero vectors or unsigned vectors, therefore \(\text{Im}(A_{21})\) includes a strictly positive vector and, since \(\text{Im}(A_{21}) \subseteq \text{Im}(\mathcal{R}(A_{22}, A_{21}))\), also \(\text{Im}(\mathcal{R}(A_{22}, A_{21}))\) does. On the other hand, by Proposition 7, the pair \((A, B)\) is herdable if and only if the pair \((A_{22}, A_{21})\) is herdable, and this completes the proof. □

IV. HERDABILITY OF PAIRS \((A, B)\) WITH \(\mathcal{G}(A)\) AN UNDIRECTED TREE WITH A SINGLE LEADER

Let us now consider the case when \(B\) is a canonical vector and the matrix \(A\) is a symmetric real matrix whose associated undirected graph \(\mathcal{G}(A)\) is acyclic, namely \(\mathcal{G}(A)\) is a tree. This corresponds to the case of a tree with a single leader and \(n-1\) followers. This case has been investigated in [14], where a sufficient condition for the herdability of the pair \((A, B)\) has been provided. In this section we provide a sufficient condition for herdability that is less restrictive, and in the
case of trees whose followers have distance at most 2 from
the leader we provide necessary and sufficient conditions.

In order to investigate the problem we adopt the following

Assumption 2: The graph $G(A)$ is a signed, weighted,
connected and acyclic undirected graph, namely a tree. The
leader is $L = \{1\}$ (and hence $B = e_1$), while the followers
split into classes, based on their distance from the leader.
The followers at distance 1 from the leader are $F_1 =
[2, m_1 + 1]$, the followers at distance 2 from the leader are
$F_2 = [m_1 + 2, m_1 + m_2 + 1]$, and so on till the last class
$F_K = [m_1 + \ldots + m_{K-1} + 2, n]$, where $K$ is the maximum
distance between the leader and one of its followers.

Proposition 11: Consider a pair $(A, B)$, with $A \in \mathbb{R}^{n \times n}$
and $B \in \mathbb{R}^n$ satisfying the previous Assumption 2.
If, for every $k \in [0, K - 1]$, all the edges from the vertices
in $F_k$ to the vertices in $F_{k+1}$ have the same sign, then the
pair $(A, B)$ is herdable.

Proof: Under the previous assumption, it is easy to see
that every vertex in $F_k$ is reached for the first time by the
leader in $k$ steps, $k \in [0, K]$, and subsequently it is reached
after $k + 2h$ steps for every $h \in \{1, 2, 3, \ldots\}$ (since each
undirected edge of the graph can be crossed back and forth,
and hence condition $[A^k B]_{ij} \neq 0$ implies $[A^{k+2} B]_{ij} \neq 0$).
Therefore the controllability matrix of the pair $(A, B)$ takes the form

$$R = \begin{bmatrix}
1 & 0 & * & 0 & * & \ldots \\
0 & v_1 & 0 & * & 0 & \ldots \\
0 & 0 & v_2 & 0 & * & \ldots \\
0 & 0 & 0 & v_3 & \vdotso & \ldots \\
0 & 0 & \vdotso & \vdotso & \vdotso & \ldots \\
0 & 0 & \vdotso & \vdotso & \vdotso & \ldots \\
\end{bmatrix}, \quad (13)
$$

where $v_k \in \mathbb{R}^{m_k}$, $k \in [1, K]$, are, by assumption, unsigned,
while * denotes (nonzero) vectors/entries whose values are
not relevant. So, by making use of Proposition 5, we immedi-
ately deduce that there exists a strictly positive vector in
the image of $R$, and hence $(A, B)$ is herdable. □

Remark 12: Theorem 3 in [14] follows as a corollary of the
previous proposition, since it imposes that all paths from the
leader to the followers in $V_0 := \bigcup_{h \in \mathbb{Z}_+} F_1 + 2h$
have the same sign and, at the same time, all paths from the
leader to the followers in $V_k := \bigcup_{h \in \mathbb{Z}_+} F_{2h} + 2h$
have the same sign. This means that not only all the edges from vertices in $F_k$
to vertices in $F_{k+1}, k \in [0, K - 1], \ (where F_0 := L)$
have the same signs, but such signs are uniquely determined for
$k \geq 1$ once we choose the signs of the edges from $F_1$ to
$F_2$.

Example 1: Consider a pair $(A, B)$, with $A = A^T \in
\mathbb{R}^{9 \times 9}$ and $B = e_1$, and assume that the undirected
graph $G(A)$ associated with the matrix $A$ is a tree whose structure
and edge signs are described in Figure 1. The nodes $i = 2$
and $j = 9$ both belong to $V_n$, since both of them are reached
from the leader (node 1 in Fig. 1) in an odd number of steps
$(1 + 2h$ and $3 + 2h$, $h \in \{0, 1, 2, \ldots\}$, respectively). The

![Fig. 1: Tree structure of the herdable system of Example 1.](image)

node $i$ is reached by the leader with positive walks, while $j$
with negative ones, so the hypotheses of Theorem 3 in [14]
are violated. However, the controllability matrix of the pair
takes the structure in (13) for $K = 3$, with unsigned vectors
$v_1, v_2$ and $v_3$, the first one with a positive entry, while
the other two with negative entries, thus the pair is herdable
by Proposition 11.

Given a matrix $A$ and hence a graph $G(A)$ with a tree
structure, we propose now Algorithm 2 for the selection of
a (unique) leader $i$ in order to ensure, if possible, that the
pair $(A, e_i)$ is herdable. The algorithm searches for a single
node, if it exists, for which the sufficient condition given in
Proposition 11 is satisfied. For the meaning of the symbols
$\text{Out}_+(F), \text{Out}_-(F)$ etc., we refer the reader to the Notation
part, at the beginning of the paper.

Algorithm 2 Algorithm for the selection of a single leader
to ensure herdability of a pair $(A, B)$ when $G(A)$ is a tree

for $i \in V$ do
if $\text{Out}_+(i) = \text{Out}(i) \neq \emptyset$ or $\text{Out}_-(i) = \text{Out}(i) \neq \emptyset$ then
$L := \{i\}$
$F := \{j : (i, j) \in E\}$
$H := L \cup F$
if $|H| = n$ then
$(A, B)$ is herdable
else
while $\text{Out}_+(F) = \text{Out}(F) \neq \emptyset$ or
$\text{Out}_-(F) = \text{Out}(F) \neq \emptyset$ do
$F := F \cup \text{Out}(F)$
$H := H \cup F$
if $|H| = n$ then
$(A, B)$ is herdable

end

end
end

Propositions 13 and 14, below, provide complete character-
izations of herdability for trees in which followers have all
distance 1 from the leader or distance at most 2, respectively.

Proposition 13: Consider a pair $(A, B)$, with $A \in \mathbb{R}^{n \times n}$
and $B \in \mathbb{R}^n$ satisfying Assumption 2, and suppose that all
the followers have distance one from the leader.
Then the pair $(A, B)$ is herdable if and only if all the edges
have the same sign.

Proof: If all the followers have distance 1 from the
leader, namely $K = 1$, then

$$A = \begin{bmatrix}
0 & A_{12} \\
A_{21} & 0 & \ldots & \ldots & 0
\end{bmatrix}$$

$$g_0 = \left\{ \begin{array}{ll}
0 & \text{if } g_1 \\
g_1 & \text{otherwise}
\end{array} \right.$$
where \( A_{12} = A_{21}^T \in \mathbb{R}^{1 \times (n-1)} \) is devoid of zero entries. By Proposition 7, \((A, B)\) is herdable if and only if the pair \((0_{1 \times (n-1)}, A_{21})\) is herdable, and this is the case if and only if \( A_{21} \) is unsigned. \( \square \)

**Proposition 14**: Consider a pair \((A, B)\), with \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^n \) satisfying Assumption 2, and suppose that all the followers have distance at most 2 from the leader, and hence

\[
A = \begin{bmatrix}
0 & A_{12} & 0_{1 \times m_2} \\
A_{21} & 0_{m_1 \times m_2} & A_{23} \\
0_{m_2 \times 1} & A_{32} & 0_{m_2 \times m_2}
\end{bmatrix},
\]

where \( A_{21} = A_{12}^T \in \mathbb{R}^{m_1 \times 1} \) and \( A_{32} = A_{23}^T \in \mathbb{R}^{m_2 \times m_2} \). Then the pair \((A, B)\) is herdable if and only if for every \( i, j \in \mathcal{F}_1 = [2, m_1 + 1] \) (including \( i = j \))\(^1\) such that

\[
[A_{23}A_{32}]_{ij} = \sum_{k=1}^{m_2} ([A_{32}]_{ki})^2 = \sum_{k=1}^{m_2} ([A_{32}]_{kj})^2 = [A_{23}A_{32}]_{jj},
\]

(namely for every pair \((i, j)\) \( \in \mathcal{F}_1 \times \mathcal{F}_1 \) such that the sum of the squares of all edges \((i, k), k \in \mathcal{F}_2\), coincides with the sum of the squares of all edges \((j, k), k \in \mathcal{F}_2\) we have:

i) \([A_{21}]_i \cdot [A_{21}]_j > 0\) (namely the two edges from the leader \(\mathcal{L}\) to \(i\) and \(j\) have the same sign);

ii) \(A_{32}(e_i + e_j)\) is either zero or unsigned (namely all edges from \(i\) and \(j\) to their followers in \(\mathcal{F}_2\) have the same sign).

**Proof**: First of all, we highlight that, by Assumption 2, \(\Gamma := A_{21}\) is devoid of zero entries, and for every \( i \in [1, m_2] \) the row vector \( e_i^T A_{32} \) is a monomial vector (namely it has a single nonzero entry). Consequently, \(\Lambda := A_{23}A_{32}\) is a diagonal matrix (with nonnegative diagonal entries).

By Proposition 7, \((A, B)\) is herdable if and only if the pair

\[
\begin{bmatrix}
0_{m_1 \times m_2} & A_{23} \\
A_{32} & 0_{m_2 \times m_2}
\end{bmatrix},
\]

is herdable, and this is the case if and only if the image of the controllability matrix \(\mathcal{R}_1\) of the previous pair, given in (15) includes a strictly positive vector. This is the case if and only if the following two conditions simultaneously hold:

a) the image of the controllability matrix \(\mathcal{R}_1 := [A_{21} (A_{23}A_{32})A_{21} (A_{23}A_{32})^2A_{21} \ldots ]\) includes a strictly positive vector, namely the pair \((A_{23}A_{32}, A_{21})\) is herdable;

b) the image of the matrix \( A_{32}\mathcal{R}_1 \) includes a strictly positive vector.

As the matrix \(\Lambda = A_{23}A_{32}\) is diagonal, while the column vector \(\Gamma = A_{21}\) has no zero entries, by Lemma 15, the pair \((\Lambda, \Gamma)\) is herdable if and only if condition (14) implies \([A_{21}]_i \cdot [A_{21}]_j > 0\). This means that a) is equivalent to condition i).

So, we are now remained with proving that if i) (equivalently, a)) holds, then b) and ii) are equivalent. If i) holds, by referring to the proof of Lemma 15, we can assume without loss of generality that \(\Lambda\) and \(\Gamma\) take the form given in (16) and claim that

\[
\text{Im} (A_{32}\mathcal{R}_1) = \text{Im} (A_{32} \cdot \text{diag}\{\gamma_1, \ldots, \gamma_p, \gamma_{p+1}, \ldots, \gamma_s\}),
\]

where \(\gamma_i \in \mathbb{R}^{n_i}\) is strictly positive if \( i \in [1, p]\) and strictly negative if \( i \in [p+1, s]\).

Set \( W = [w_1 | \ldots | w_p | w_{p+1} | \ldots | w_s] := A_{32} \cdot \text{diag}\{\gamma_1, \ldots, \gamma_p, \gamma_{p+1}, \ldots, \gamma_s\}, \) where each vector \(w_i\) is obtained by combining with the coefficients of the vector \(\gamma_i\) (having all the same sign) the columns of \(A_{32}\) of indices \([h_1 + 1, h_i + n_i], \) where by definition \(h_1 := 0, \) while \(h_i := n_1 + n_2 + \cdots + n_{i-1}\) for \( i \in [2, s] \).

We observe that all columns of \(A_{32}\) are either zero (if a vertex in \(\mathcal{F}_1 = [2, m_1+1]\) has no followers) or have disjoint nonzero patterns, meaning that for every \(\ell, m \in [h_1 + 1, h_i + n_i], \ell \neq m, \text{ZP}(A_{32}e_\ell) \cap \text{ZP}(A_{32}e_m) = \emptyset.\) As a result also the columns \(w_i\) of \(W\) are either zero or have disjoint nonzero patterns.

We can now conclude that condition b) holds if and only if \(\text{Im} (A_{32}\mathcal{R}_1) = \text{Im}(W)\) contains a strictly positive vector, but this is possible if and only if all vectors \(w_i\) are unsigned. By the way the vectors \(w_i\) have been obtained, this is possible if and only if condition ii) holds. \(\square\)

**V. CONCLUSIONS**

In this paper herdability of linear time-invariant systems has been investigated. Special attention has been given to pairs \((A, B)\) corresponding to leader-follower networks \(\mathcal{G}(\mathcal{A}),\) and to networks with tree topologies and a single leader. For this latter case, an algorithm for leader selection is provided, and when the distance from the leader to the followers is at most 2, necessary and sufficient conditions for herdability are stated. Future research will focus on the study of herdability for networked systems interacting through more general topological structures.

**TECHNICAL LEMMA**

**Lemma 15**: Given a matrix pair \((A, \Gamma)\), with \(\Lambda = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \in \mathbb{R}^{n \times n}\) a diagonal matrix, and \(\Gamma \in \mathbb{R}^{n}\) devoid of zero entries, the pair is herdable if and only if \(\lambda_i = \lambda_j\) implies \([\Gamma_i, \cdot][\Gamma_j, \cdot] > 0\), namely the \(i\)-th and the \(j\)-th entries of \(\Gamma\) have the same sign.

**Proof**: We first prove that if the pair \((\Lambda, \Gamma)\) is herdable, then \(\lambda_i = \lambda_j\) implies \([\Gamma_i, \cdot][\Gamma_j, \cdot] > 0\). Suppose, by contradiction, that \(\lambda_i = \lambda_j =: \lambda\) and \([\Gamma_i, \cdot][\Gamma_j, \cdot] < 0\). Then it is easy to see that \(i \neq j\) and the vector \(w^T := [\Gamma_i, e_j^T - \Gamma_j, e_j^T]\) satisfies \(w^T \Lambda = \lambda w^T,\) namely \(w\) is a (left) eigenvector of \(\Lambda\) corresponding to \(\lambda,\) and \(w^T \Gamma = 0.\) Conversely, it is immediate to prove that \(w\) is orthogonal to \(\text{Im}(\mathcal{R}(\Lambda, \Gamma)),\) i.e. \(w^T \mathcal{R}(\Lambda, \Gamma) = 0_{n_1}.\) Since \(w^T\) is unsigned (since \([\Gamma_i, \cdot]\) and \((-\Gamma_j, \cdot)\) have the same sign), it is impossible that there exists a strictly positive vector \(v \in \text{Im}(\mathcal{R}(\Lambda, \Gamma)),\) since this would imply \(w^T v \neq 0.\) Therefore the pair \((\Lambda, \Gamma)\) cannot be herdable.

We now prove that if \(\lambda_i = \lambda_j\) implies \([\Gamma_i, \cdot][\Gamma_j, \cdot] > 0,\) then the pair \((\Lambda, \Gamma)\) is herdable.

\(^1\)Note that for \(i = j\) condition i) becomes trivial, while condition ii) becomes “\(A_{32}e_i\) is unsigned”.

If all entries of $\Gamma$ have the same sign, the pair $(\Lambda, \Gamma)$ is trivially herdable. So, suppose this is not the case. It entails no loss of generality to first permute the entries of $\Gamma$ (and accordingly the rows and columns of $\Lambda$) so that the first ones are positive and the last ones are negative. Then we can permute the entries in such a way that the identical diagonal entries of $\Lambda$ are consecutive. This implies that, under the previous assumptions, $\Lambda$ and $\Gamma$ take the following form

$$
\Lambda = \begin{bmatrix}
\lambda_1 & & & \\
& \ddots & & \\
& & \lambda_p & \\
& & & \lambda_{p+1}
\end{bmatrix},
\Gamma = \begin{bmatrix}
\gamma_1 \\
\vdots \\
\gamma_p \\
\gamma_{p+1}
\end{bmatrix}
$$

(16)

where each $\lambda_i$ is a scalar matrix of size say $n_i$, namely $\lambda_i = \lambda_i I_{n_i}$ with $\lambda_i \in \{\lambda_1, \ldots, \lambda_n\}$, while $\gamma_i \in \mathbb{R}^{n_i}$ is a strictly positive vector for every $i \in [1, p]$ and a strictly negative vector for every $i \in [p+1, n]$. Moreover, by the assumption that $\lambda_i = \lambda_j$ implies $[\Gamma]_i, [\Gamma]_j > 0$ we can claim that $\lambda_i \neq \lambda_k$ for $i \neq k$. It is immediate to see that the controllability matrix of the pair $(\Lambda, \Gamma)$ factorizes as in (17).

$$
\mathcal{R}(\Lambda, \Gamma) = \begin{bmatrix}
\gamma_1 \\
\vdots \\
\gamma_p \\
\gamma_{p+1}
\end{bmatrix}.
$$

Since $\lambda_1, \ldots, \lambda_n$ are all distinct, the Vandermonde matrix on the right of (17) is of full row rank. This ensures that

$$
\text{Im}(\mathcal{R}(\Lambda, \Gamma)) = \text{Im} \begin{bmatrix}
\gamma_1 \\
\vdots \\
\gamma_p \\
\gamma_{p+1}
\end{bmatrix}
$$

and since all columns of this latter matrix are unsigned, it is immediate to see that there exists a strictly positive vector in its image, and hence the pair $(\Lambda, \Gamma)$ is herdable. □

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