UNIVERSAL GYSIN FORMULAS FOR THE UNIVERSAL HALL-LITTLEWOOD FUNCTIONS

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Abstract. It is known that the usual Schur $S$- and $P$-polynomials can be described via the Gysin homomorphisms for flag bundles in the ordinary cohomology theory. Recently, P. Pragacz generalized these Gysin formulas to the Hall-Littlewood polynomials. In this paper, we introduce a universal analogue of the Hall-Littlewood polynomials, which we call the universal Hall-Littlewood functions, and give Gysin formulas for various flag bundles in the complex cobordism theory. Furthermore, we give two kinds of the universal analogue of the schur polynomials, and some Gysin formulas for these functions are established.

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2010 Mathematics Subject Classification. 05E05, 14M15, 14N15, 55N20, 55N22, 57R77.

Key words and phrases. Schur $S$, $P$, and $Q$-functions, Hall-Littlewood function, Complex-oriented generalized cohomology theory, Gysin map.

The first author is partially supported by the Grant-in-Aid for Scientific Research (C) 15K04876, Japan Society for the Promotion of Science.

The second author is partially supported by the Grant-in-Aid for Scientific Research (B) 16H03921, Japan Society for the Promotion of Science.
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References

1. Introduction

1.1. Gysin formulas. Any continuous map \( f : X \to Y \) between topological spaces defines pull-back homomorphism \( f^* : H^i(Y) \to H^i(X) \) in cohomology, and push-forward homomorphism \( f_* : H_*(X) \to H_*(Y) \) in homology for all \( i \in \mathbb{Z} \). If \( X \) is a compact oriented smooth manifold of dimension \( m \), the \( m \)-th homology group \( H_m(X) \) is isomorphic to \( \mathbb{Z} \), with a generator \([X]\) the fundamental class of \( X \), and the following Poincaré duality map

\[ P_X : H^i(X) \to H_{m-i}(X), \quad \alpha \mapsto \alpha \cap [X] \]

is an isomorphism. If \( f : X \to Y \) is a smooth map, with \( X \) and \( Y \) compact oriented smooth manifolds of dimensions \( m \) and \( n \) respectively, then we get a push-forward homomorphism in cohomology:

\[ f_* : H^i(X) \xrightarrow{f_*} H_{m-i}(X) \xrightarrow{f_*} H_{m-i}(Y) \xleftarrow{f_*} H^{i-(m-n)}(Y). \]

This map is called a Gysin map (homomorphism), push-forward, or Umkehr map. Intuitively the intersection of cycles in homology is turned into the product of cohomology classes by means of Poincaré duality. This conversion, together with the computation of the Gysin maps, enables us to make some geometric problems into algebraic computations. For example, the computation of the Gysin map for various flag bundles is used to determine the cohomology class corresponding to a Schubert variety (see e.g., Akyildiz [2], Damon [17 Theorem 3 (Chern's formula)]). Furthermore, the similar computation is applied to determining the cohomology class of degeneracy locus (see e.g., Damon [17 Corollary 3], Fulton [23 Chapter 14], Porteous [57 p.298]). Thus the computation of various Gysin maps has a lot of applications in geometry, and there are many formulas describing Gysin maps. These formulas are called Gysin formulas or push-forward formulas in general. Although we do not intend to survey these formulas thoroughly here, we shall quote some results related to our work:

- Gysin formulas for flag bundles, Grassmann bundles, or projective bundles are described in Borel-Hirzebruch [7 §8], [8 §20], Buch [14 §7], Damon [16, 17], Darondeau-Pragacz [18], Fel’dman [22 §4], Fulton [23 §14.2], Fulton-Pragacz [25 Chapter IV, Appendices E, F], Harris-Tu [27 §2], Ilori [35], Kajimoto-Sugawara [40], Pragacz [58 §2], [60 §4], [61], Quillen [62], Sugawara [65], Tu [68], Vishik [70 §5.7].

- For a connected complex (semi-simple) Lie group \( G_\mathbb{C} \) with a Borel subgroup \( B \) and a parabolic subgroup \( P \) containing \( B \), Gysin formulas for the natural projection \( G_\mathbb{C}/B \to G_\mathbb{C}/P \) are described in Akyildiz [2], Akyildiz-Carrell [3], [4], Fulton-Pragacz [25 Appendix E], Brion [12], Kajimoto [39].

As we mentioned above, most of these formulas are formulated in the ordinary cohomology (or Chow) theory, and many different approaches such as the residue symbol (Damon [16]), the zeros of holomorphic vector fields on flag varieties (Akyildiz-Carrell [3], [4]), representation theory (Brion [12]), and the equivariant localization formula of Atiyah-Bott-Berline-Vergne for a torus action (Tu [68], [69]) have been used to prove them.
On the other hand, it is known that the ordinary cohomology theory is a special case of complex-oriented generalized cohomology theories which corresponds to an additive formal group law (see Example 2.2 (1)). Therefore it is natural to ask if the above Gysin formulas in the ordinary cohomology theory can be generalized to any complex-oriented generalized cohomology theory. For example, Buch [14, universal cobordism theory which is proved a additive complex-oriented case of back to the full flag bundle $F$. As shown by Quillen [62], the complex cobordism theory $MU^*$ is universal among complex-oriented generalized cohomology theories, and therefore it is desirable to formulate these Gysin formulas in the complex cobordism theory. The first main purpose of this paper is to establish the Gysin formulas for general flag bundles in the complex cobordism theory. The first main result of this paper seems to be valid in the algebraic cobordism theory. The Becker-Gottlieb transfer [6, Theorem 4.3] and the Brumfiel-Madsen theorem due to Levine-Morel [47].

1.2. Gysin formulas for Schur functions. In the previous subsection 3.1.1 we collected various Gysin formulas related to our work. Some formulas involve symmetric functions such as Schur S- and P-functions (see e.g., Fulton-Pragacz [25, Chapter IV], Pragacz [58 §2], [60 §4]). In order to clarify what we are considering, we shall give one typical example: Let $E \rightarrow X$ be a complex vector bundle of rank $n$ over a variety $X$. Let $\pi : G^1(E) \rightarrow X$ be the associated Grassmann bundle of hyperplanes in $E$. On $G^1(E)$, we have the tautological exact sequence of vector bundles:

$$0 \rightarrow S \hookrightarrow \pi^*(E) \rightarrow Q \rightarrow 0.$$ 

Let $\xi := c_1(Q) \in H^2(G^1(E))$ be the first Chern class of the line bundle $Q$, and $x_1 := \xi, x_2, \ldots, x_n$ be the Chern roots of $E$. Then as for the Gysin homomorphism $\pi_* : H^*(G^1(E)) \rightarrow H^*(X)$, it is well-known that

$$\pi_*(\xi^k) = s_{k-n+1}(E) \quad (k \geq 0),$$

where $s_i(E)$ is the $i$-th Segre class of $E$ (see e.g., Fulton-Pragacz [25, §4.1]). Since the Chern classes $c_i(E)$ can be identified with the $i$-th elementary symmetric polynomial $e_i(x_n)$ in $x_n = (x_1, \ldots, x_n)$, the Segre class $s_j(E)$ can be identified with the $j$-th homogeneous complete symmetric polynomial $h_j(x_n)$, which is nothing but the Schur S-polynomial $s_{(j)}(x_n)$ corresponding to the “one-row” $(j)$. Therefore the formula (1.1) can be interpreted as

$$\pi_*(x_1^k) = s_{(k-n+1)}(x_n).$$

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1 Universal Gysin formulas in the title of this paper signifies Gysin formulas in the complex cobordism theory which is universal among all complex-oriented generalized cohomology theories.

2 One can naturally identify $G^1(E)$ with the associated projective bundle $P(E^\vee)$ of lines in the dual bundle $E^\vee$.

3 By the splitting principle, the vector bundle $E$ splits into the sum of line bundles when pulled-back to the full flag bundle $F(l)$ via the projection $\tau : F(l) \rightarrow X$. The Chern roots of $E$ are the first Chern classes of these line bundles on $F(l)$.

4 By means of $\tau^* : H^*(X) \rightarrow H^*(F(l))$, which is known to be injective.

5 Strictly speaking, this formula should be considered in $H^*(F(l))$. 


The formula (1.2) can be generalized to the full flag bundle $\tau : F\ell(E) \to X$ as follows: Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ ($\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$) be a partition of length $\leq n$. Then Fulton-Pragacz \cite[(4.1)]{25} (see also Pragacz \cite[Lemma 2.3]{58}, \cite[Proposition 4.4]{60}, \cite[Example 8]{61}) showed the following formula \cite[(1.3)]{81}:

\begin{equation}
\tau_*(x_1^{\lambda_1+n-1}x_2^{\lambda_2+n-2} \cdots x_n^{\lambda_n}) = s_\lambda(x_n).
\end{equation}

The formula (1.3) gives the usual Schur $S$-polynomial as the push-forward image of the Gysin map from the full flag bundle, and is called the Jacobo-Trudi identity in Fulton-Pragacz \cite[p.42]{25}. The essentially identical formula has been obtained by many authors (Damon \cite[Corollary 2]{17}, Harris-Tu \cite[Proposition 2.3]{27}, Manivel \cite[Example 8]{50}, Pragacz \cite[Proposition 4.4]{60}, \cite[Example 8]{61}, Sugawara \cite[Theorem 6.2]{65}). Recently, Pragacz \cite[Example 8]{61} succeeded in generalizing the above Gysin formulas for Schur $S$- and $P$-polynomials to the Hall-Littlewood polynomials, which interpolate between Schur $S$- and $P$-polynomials (see Macdonald \cite[Chapter III, §2]{48}).

On the other hand, it is well-known that the usual Schur $S$-polynomials (resp. $P$-polynomials) represent the Schubert classes of the complex Grassmannian (resp. Lagrangian Grassmannian) in the ordinary cohomology (see e.g., Fulton \cite[§9.4]{24}, Pragacz \cite[§6]{61}). In order to generalize the above facts to other cohomology theories such as $K$-theory, complex cobordism theory (or algebraic cobordism theory) and their torus equivariant versions, various generalizations, analogues, and deformations of Schur functions have been introduced by many authors. We shall quote some of them for convenience of the readers (in the following, $\lambda$ (resp. $\nu$) is understood to be a partition (resp. strict partition) of length $\leq n$, and $x_n = (x_1, \ldots, x_n)$ is a sequence of $n$ independent variables, and $b = (b_1, b_2, \ldots)$ is a sequence of “deformation parameters”):

- The factorial Schur polynomials $s_\lambda(x_n|b)$ (see Ikeda-Naruse \cite[§5.1]{32}), Macdonald \cite[Chapter 1, Examples 20]{48}, \cite[6th variation]{49}, Molev-Sagan \cite{51}) represent the Schubert classes of the torus equivariant cohomology of the complex Grassmannian (see Knutson-Tao \cite[§6]{42}, Ikeda-Naruse \cite[Theorem 5.4]{32}).

- The factorial Schur $P$- and $Q$-polynomials $P_\nu(x_n|b)$, $Q_\nu(x_n|b)$ introduced by Ivanov \cite[Definitions 2.10 and 2.13]{36} (see also Ikeda-Mihalceaa-Naruse \cite[§4.2]{33}) represent the Schubert classes of the torus equivariant cohomology of the orthogonal and Lagrangian Grassmannians (see Ikeda \cite[Theorem 6.2]{31}, Ikeda-Naruse \cite[Theorem 8.7]{32}).

- The factorial Grothendieck polynomials $G_\lambda(x_n|b)$ introduced by McNamara \cite[Definition 4.1]{51} (see also Ikeda-Naruse \cite[2.13, 2.14]{34}) represent the Schubert classes of the torus equivariant $K$-theory of the complex Grassmannian (see Ikeda-Naruse \cite[34]).

- The $K$-theoretic factorial $P$- and $Q$-polynomials $GP_\nu(x_n|b)$, $GQ_\nu(x_n|b)$ introduced by Ikeda-Naruse \cite[Definition 2.1]{34} represent the Schubert classes of the torus equivariant $K$-theory of the orthogonal and Lagrangian Grassmannians (see Ikeda-Naruse \cite[Theorem 8.3]{34}).

- The universal factorial Schur $(S)$-functions $s_\lambda^u(x_n|b)$, $P$- and $Q$-functions $P_\nu^u(x_n|b)$, $Q_\nu^u(x_n|b)$ were introduced by the authors (see \cite[Definitions 4.1]{55}).

\[ \text{This formula is also considered in } H^*(F\ell(E)). \]

\[ \text{Notice that } s_\lambda(x_n|b) \text{ and } G_\lambda(x_n|b) \text{ are polynomials in } x_n, \text{ whereas } s_\lambda^u(x_n|b) \text{ is a formal power series in } x_n. \]
and 4.10] or (3.1) and (3.4) in this paper). These functions are universal analogue of the above polynomials. Thus our second main purpose of this paper is the introduction of the universal analogue of the Hall-Littlewood polynomials, and to establish the Gysin formulas for various Schur functions in the complex cobordism theory. The universal analogue of the Hall-Littlewood polynomials denoted by \( H^L_\lambda(x_n; t) \) are defined in Definition 3.2, and we call them the universal Hall-Littlewood functions. Then our main result in this direction is Corollary 4.11 which gives the universal Hall-Littlewood function as the push-forward image of the Gysin map from a partial flag bundle, thus generalizing Pragacz’s result (see Corollary 4.4). The Jacobi-Trudi identity (1.3) can also be formulated in the complex cobordism theory (see Corollary 4.8). With regard to the universal Schur functions \( s^L_\lambda(x_n) \) and the universal Hall-Littlewood functions \( H^L_\lambda(x_n; t) \), a comment is in order: The usual Hall-Littlewood polynomial denoted by \( P_\lambda(x_1, \ldots, x_n; t) \) in Macdonald’s book [48, Chapter III, §2] reduces to the usual Schur polynomial \( s_\lambda(x_1, \ldots, x_n) \) under the specialization \( t = 0 \). However, we found that our \( H^L_\lambda(x_n; t) \) does not necessarily reduce to the universal Schur function \( s^L_\lambda(x_n) \) when \( t = 0 \). Thus the specialization \( H^L_\lambda(x_n; 0) \) gives another universal analogue of Schur functions, which we call the new universal Schur functions\(^8\). We shall discuss these new functions denoted \( S^L_\lambda(x_n) \) (and their factorial version denoted \( S^L_\lambda(x_n|b) \)) separately in the final section §5. Among our results in this section are Theorem 5.6 which reveals the difference between the “old” and the “new” universal factorial Schur functions. As an application of the Gysin formulas for the new universal factorial Schur functions, we formulate the Thom-Porteous formula in the complex cobordism theory (see Theorem 5.11). Moreover, we shall show that the new universal factorial Schur functions represent the Schubert classes of the complex Grassmannian in the complex cobordism theory (see Theorem 5.12).

1.3. Organization of the paper. The paper is organized as follows: In Section 2, we shall give topological preliminaries needed to develop our work. The key concepts are complex-oriented generalized cohomology theory, Gysin maps, Becker-Gottlieb transfer, Brunfiehl-Madsen formula, Bressler-Evens formula. Various types of Gysin formulas are reviewed at the end of this section. Section 3 is devoted to the introduction of the universal analogue of the usual Schur \( S-, P-, Q-, \) and Hall-Littlewood polynomials. Especially, the universal Hall-Littlewood functions, which are the central theme of this paper, are introduced. In Section 4, after reviewing the Gysin formulas for various Schur functions, we shall give the universal analogues of these formulas. In order to establish these formulas, the Bressler-Evens formula plays the crucial role. In Section 5, we introduce the new universal factorial Schur functions. If we set all the deformation parameters to be 0, the new universal Schur functions can be obtained, and these functions coincide with the universal Hall-Littlewood functions under the specialization \( t = 0 \). We also give some Gysin formulas for these new universal Schur functions.

Acknowledgments. We would like to thank Takeshi Ikeda, Thomas Hudson, Tomoo Matsumura for helpful comments and valuable conversations. Especially, Tomoo Matsumura kindly explained his recent work (Hudson-Matsumura [29]) to us, that is closely related to our current work.

2. Topological preliminaries

\(^8\) It is desirable to give these functions more specific name which characterize them (a proposal will be given in §5.5).
2.1. Complex-oriented generalized cohomology theory. A generalized cohomology theory $h^*(-) = \bigoplus_{n \in \mathbb{Z}} h^n(-)$ is a contravariant functor from the category CW of CW complexes to the category of graded abelian groups which satisfies all the axioms of Eilenberg-Steenrod [21, Chapter I, 3c] except the dimension axiom. Thus $h^0(pt)$ is not necessarily zero even if $n \neq 0$. Here pt means a space consisting of a single point. The cohomology $h^*: := h^*(pt)$ of a point is called the coefficient group. In what follows, we assume that the theory $h^*(-)$ is multiplicative, that is, for CW pairs $(X, A)$ and $(Y, B)$, there exists an external product

$$h^k(X, A) \otimes h^l(Y, B) \to h^{k+l}((X, A) \times (Y, B)), \quad k, l \in \mathbb{Z},$$

that satisfies certain axioms (see e.g., Dold [20, §4]). Under this assumption, $h^*$ becomes a graded-commutative ring, and for a space $X$, the cohomology ring $h^*(X)$ has an $h^*$-module structure. Furthermore, if we consider the infinite CW complexes, we need suitable axioms about limits such as the additivity axiom or wedge axiom due to Milnor [52]. In what follows, when we refer to a generalized cohomology theory, it means a multiplicative generalized cohomology theory defined on CW satisfying the additivity axiom unless otherwise stated. Let $h^*(-)$ denote the corresponding reduced cohomology theory, 9 and let $j: \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^\infty$ be the canonical inclusion of $\mathbb{C}P^1 \approx S^2$ into the infinite complex projective space.

Definition 2.1 (Adams [1], Part II, p.37; Switzer [66], §16.27). A generalized cohomology theory $h^*(-)$ is called complex-orientable if there exists an element $x^h \in \tilde{h}^2(\mathbb{C}P^\infty)$ such that $j^*(x^h)$ is a generator of $\tilde{h}^2(\mathbb{C}P^1) \cong h^2(S^2) \cong h^0(S^0) \cong h^0(pt) = h^0$.

If this element $x^h$ is specified, then $h^*(-)$ is said to be complex-oriented, and $x^h$ is called the orientation class. Then it is known that the cohomology ring of the infinite projective space $\mathbb{C}P^\infty$ is $h^*(\mathbb{C}P^\infty) = h^*[[x^h]]$, a formal power series ring with the given generator $x^h \in \tilde{h}^2(\mathbb{C}P^\infty)$ (see Adams [1], Part II, Lemma 2.5).

Complex-orientability implies a lot of useful properties. We recall here some of them.

2.1.1. Chern classes. For a complex vector bundle $E \to X$, one can define the $h^*$-theory Chern classes $c_i^h(E) \in h^i(X)$ ($i = 0, 1, 2, \ldots, n = \text{rank } E$) by the usual Grothendieck’s method (see Conner-Floyd [15, Theorem 7.6], Grothendieck [26, §3], Switzer [66, Theorem 16.2]). The total Chern class of $E$ is given by $c_i^h(E) := \sum_{i=0}^n c_i^h(E)$. Then the usual Whitney product formula is given by $c_i^h(E \oplus F) = c_i^h(E) \cdot c_i^h(F)$ for two complex vector bundles $E$, $F$.

2.1.2. Formal group law. If $L$, $M$ are complex line bundles over $X$, then

$$c_i^h(L \otimes M) = F_h(c_i(L), c_i^h(M)),$$

where

$$F_h(X, Y) = X + Y + \sum_{i, j \geq 1} a_{i,j}^h X^i Y^j \in h^*[[X, Y]] \quad (a_{i,j}^h \in h^{2(1-i-j)})$$

is a (one dimensional commutative) formal group law over the graded ring $h^*$ associated with the cohomology theory $h^*(-)$. Then the formal power series $F_h(X, Z)$ satisfies the conditions

(i) $F_h(X, 0) = X, \quad F_h(0, Y) = Y,$
(ii) $F_h(X, Y) = F_h(Y, X),$

9 For a CW-complex $X$ with base point $x_0$, the reduced cohomology $\tilde{h}^*(X)$ is defined to be $h^*(X, x_0)$. 

6
(iii) \( F_h(X, F_h(Y, Z)) = F_h(F_h(X, Y), Z) \).

We shall use this formal group law to define the formal sum, formal inverse, and formal subtraction. For two indeterminates \( X, Y \), the formal sum \( X +_F Y \) is defined as

\[ X +_F Y := F_h(X, Y) = X + Y + \sum_{i,j \geq 1} a^h_{i,j} X^i Y^j \in h^*[[X, Y]]. \]

Denote by

\[ [-1]_F(X) = t_F(X) = \overline{X} = -X + \sum_{j \geq 2} c^j_i X^j \in h^*[[X]] \]

the formal inverse series. To be precise, \([-1]_F(X)\) is the unique formal power series satisfying the condition \( F_h(X, [-1]_F(X)) = 0 \), or equivalently \( X +_F [-1]_F(X) = 0 \). This formal inverse allows us to define the formal subtraction:

\[ X -_F Y := X +_F [-1]_F(Y) = X + F Y. \]

Finally, we define \([0]_F(X) := 0\), and inductively,

\[ [n]_F(X) := [n - 1]_F(X) +_F X = F_h([n - 1]_F(X), X) = X +_F X +_F \cdots +_F X \]

for \( n \geq 1 \). We also define \([-n]_F(X) := [n]_F([-1]_F(X)) = [-1]_F([n]_F(X))\) for \( n \geq 1 \). We call \([n]_F(X)\) the \( n \)-series in the following.

Example 2.2.

1. For the ordinary cohomology theory (with integer coefficients) \( h = H \), the coefficient ring is \( H^* = H^*(pt) = Z \) \((H^0 = Z, H^k = 0 \ (k \neq 0))\). We choose the standard orientation, namely the class of a hyperplane \( x^H \in H^2(\mathbb{C}P^\infty) \). Then the associated formal group law is the additive formal group law \( F_H(X, Y) = F_h(X, Y) = X + Y \), and the formal inverse is given by \([-1]_H(X) = -X\).

2. For the (topological) \( K \)-theory \( h = K \), the coefficient ring is \( K^* = K^*(pt) = \mathbb{Z}[\beta, \beta^{-1}] \), with \( \beta := 1 - \eta_1 \in K^{-2}(pt) \cong K(S^2) \), where \( \eta_1 \) stands for the tautological (or Hopf) line bundle over \( \mathbb{C}P^1 \cong S^2 \), and \( \eta_1 \) its dual. We choose the standard orientation \( x^K := \beta^{-1}(1 - \eta_\infty) \in \tilde{K}^2(\mathbb{C}P^\infty) \), where \( \eta_\infty \) stands for the tautological line bundle over \( \mathbb{C}P^\infty \). Then the associated formal group law is the multiplicative formal group law \( F_K(X, Y) = F_m(X, Y) = X + Y - \beta XY \), and the formal inverse is given by

\[ [-1]_K(X) = - \frac{X}{1 - \beta X} = -X - \beta X^2 - \beta^2 X^3 - \beta^3 X^4 - \cdots. \]

3. For the complex cobordism theory \( h = MU \), the coefficient ring \( MU^* = MU^*(pt) \) is a polynomial algebra over \( \mathbb{Z} \) on generators of degrees \(-2, -4, \ldots\) (see e.g., Adams \([11] \), Part II, Theorem 8.1]). As in Adams \([11] \), Part II, Examples (2.4)], Ravenel \([64] \), Example 4.1.3], we take the orientation class \( x^{MU} \in MU^2(\mathbb{C}P^\infty) \) to be the (stable) homotopy class of the map \( \mathbb{C}P^\infty \cong BU(1) \rightarrow MU(1) \), where \( MU(1) \) denotes the Thom space of the universal line bundle over \( BU(1) \). Then the associated formal group law

\[ F_{MU}(X, Y) = X + Y + \sum_{i,j \geq 1} a^i_j X^i Y^j, \quad a^i_j \in MU^2(1 - i - j), \]

\[ 10 \] We adopt the convention due to Bott \([9] \), Theorem 7.1], Levine-Morel \([47] \), Example 1.1.5] so that the \( K \)-theory first Chern class of a line bundle \( L \) (over a space \( X \)) is given by \( c_i^K(L) = \beta^{-1}(1 - L^\vee) \), where \( L^\vee \) denotes the dual bundle of \( L \). In this convention, the orientation class \( x^K \) is equal to the \( K \)-theory first Chern class of the bundle \( \eta_\infty \), namely \( c_1^K(\eta_\infty) = \beta^{-1}(1 - \eta_\infty) \).
is a universal formal group law first shown by Quillen [62] Theorem 2. To be precise, for any formal group law $F$ over a commutative ring $R$ with unit, there exists a unique ring homomorphism $\theta : MU^* \to R$ such that $F(X,Y) = (\theta,F_{MU})(X,Y) := X + Y + \sum_{i,j \geq 1} \theta(a_{ij}^{MU})X^iY^j$. Quillen also showed that the coefficient ring $MU^*$ is isomorphic to the Lazard ring $\mathbb{L}$ (see §5.1 in this paper).

### 2.2. Gysin maps

For a certain kind of map $f : X \to Y$ between spaces, the so-called Gysin map, push-forward, or Umkehr map, usually denoted by $f_* : h^*(X) \to h^*(Y)$ can be defined. Here are some examples of Gysin maps:

- (Classical Gysin map in the ordinary cohomology theory): For a smooth map $f : M \to N$ between compact oriented smooth manifolds, the Gysin map
  
  $$ f_* : H^q(M) \to H^{q-(\dim M - \dim N)}(N) $$

  can be defined by $f_* := \mathcal{P}_N^{-1} \circ f_* \circ \mathcal{P}_M$, where $\mathcal{P}_M$ (resp. $\mathcal{P}_N$) denotes the Poincaré duality isomorphism from cohomology to homology.

- (Integration along (over) the fiber): For a fibration $F \hookrightarrow E \xrightarrow{\pi} B$ with the base $B$ simply-connected and the fiber $F$ a compact connected manifold, Borel-Hirzebruch [21] §8 defined a push-forward map called the integration along the fiber:
  
  $$ \pi_* : H^q(E) \to H^{q-\dim F}(B). $$

- (Gysin map in the $K$-theory $K(-)$): For a proper morphism $f : X \to Y$ of non-singular varieties, Grothendieck constructed an additive map $f_1 : K_0(X) \to K_0(Y)$ defined by
  
  $$ f_1([\mathcal{F}]) := \sum_{i \geq 0} (-1)^i [R^if_*(\mathcal{F})], $$

  for $\mathcal{F}$ a coherent sheaf. Here $R^if_*(\mathcal{F})$ is Grothendieck's higher direct image sheaf (see e.g., Fulton [23] §15.1).

- (Gysin map in the complex cobordism theory $MU^*(-)$): In [63], Quillen gave a geometric interpretation of the complex cobordism theory $MU^*(X)$, where $X$ is assumed to be a manifold. In his interpretation, an element of $MU^*(X)$ is given by a cobordism class of a proper and complex-oriented map $f : Z \to X$. A proper complex-oriented map $g : X \to Y$ of dimension $d$ induces a map
  
  $$ g_* : MU^q(X) \to MU^{q-d}(Y) $$

  which sends the cobordism class of $f : Z \to X$ into the cobordism class of $g \circ f : Z \to Y$.

All these Gysin maps have the common properties which can be axiomatized as follows (see Bressler-Evens [10] p.801, (E)], Levine-Morel [47] Definition 1.1.2], Quillen [63] §1): Let $h^*(-)$ be a complex-oriented generalized cohomology theory defined on a suitable category of spaces. For a morphism $f : X \to Y$, one has a Gysin map $f_* : h^*(X) \to h^*(Y)$ having the following basic properties:

1. (Naturality): For a composite $g \circ f : X \to Y \to Z$, one has $(g \circ f)_* = g_* \circ f_*$.  
2. (Projection formula): For $x \in h^*(X)$ and $y \in h^*(Y)$, one has $f_* (f^*(y) \cdot x) = y \cdot f_*(x)$. 


Furthermore the Gysin map and the Becker-Gottlieb transfer are related as follows: the following formula holds:

\[ \tau \circ g_* = g_\tau \circ f^*. \]

Here \( X \times_Z Y \) denotes the fiber product of \( X \) and \( Y \) over \( Z \), namely \( X \times_Z Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\} \).

### 2.3. Becker-Gottlieb transfer

Let \( F \hookrightarrow E \xrightarrow{\pi} B \) be a fiber bundle whose fiber \( F \) is a compact smooth manifold, whose structure group \( G \) is a compact Lie group acting smoothly on \( F \), and whose base space \( B \) is a finite complex. In this setting, Becker-Gottlieb \([6, \S 3]\) constructed a \textit{stable map} \[ \tau(\pi) : B^+ \longrightarrow E^+, \]

called the \textit{Becker-Gottlieb transfer}. Here \( B^+ \) means the union of \( B \) with a point. For any generalized cohomology theory \( h^*(-) \), the map \( \tau(\pi) : B^+ \longrightarrow E^+ \) induces a “wrong-way” degree-preserving homomorphism

\[ \tau(\pi)^* : h^*(E) \longrightarrow h^*(B), \]

which is also called the Becker-Gottlieb transfer. \( \tau(\pi)^* \) is \textit{not} a ring homomorphism, but is an \( h^*(B) \)-module homomorphism (Becker-Gottlieb \([6, (5.3)]\)), that is, the following formula holds:

\[ \tau(\pi)^*(\pi^*(x) \cdot y) = x \cdot \tau(\pi)^*(y), \quad x \in h^*(B), \quad y \in h^*(E). \]

Furthermore the Gysin map and the Becker-Gottlieb transfer are related as follows (see Becker-Gottlieb \([6\, \text{Theorem 4.3}]\)):

\[ (2.1) \quad \tau(\pi)^*(x) = \pi_* \left( \chi^h(T_\pi) \cdot x \right) \quad (x \in h^*(E)). \]

Here \( T_\pi \) is the \textit{(tangent) bundle along the fibers of \( \pi \)} (see e.g., Borel-Hirzebruch \([7\, \S 7.4]\)) and \( \chi^h(T_\pi) \) denotes the \( h^* \)-theory Euler class of \( T_\pi \). Here \( T_\pi \) is regarded as a real vector bundle, and assumed to be \textit{\( h^* \)-oriented} (For the notion of \( h^* \)-orientability, see Atiyah-Bott-Shapiro \([3\, \S 12]\), Dold \([20\, \S 4]\)). The \( h^* \)-theory Euler class \( \chi^h(T_\pi) \) is defined with respect to this orientation. In practice (see Bressler-Evens \([10\, \S 1]\) and \([2,5]\)), we require that the fiber \( F \) is smooth and \textit{almost complex}, and the structure group of \( F \) preserves the almost complex structure. Hence the tangent bundle \( T(F) \) has a complex vector bundle structure, and so does the bundle along the fibers \( T_\pi \). If the cohomology theory \( h^*(-) \) is complex-oriented, then the \( h^* \)-theory Euler class \( \chi^h(T_\pi) \) is nothing but the \( h^* \)-theory \textit{top} Chern class \( c^h_j(T_\pi) \), where \( 2f \) is the \textit{real} dimension of \( F \).

---

11 For a suitable positive integer \( m \), we have a map

\[ \tau(\pi) : S^m(B^+) \longrightarrow S^m(E^+), \]

where \( S \) denotes the reduced suspension.

12 If we assume that \( E, B \) are also smooth manifolds, \( T_\pi \) is a subbundle of the tangent bundle \( T(E) \) of \( E \). The fiber of \( T_\pi \) over a point \( y \in E \) consists of all tangent vectors at the point \( y \) which are tangent to the fiber \( \cong F \) through \( y \). If we denote by \( \iota : F \longrightarrow E \) the fiber inclusion, then \( \iota^*(T_\pi) \cong T(F) \), the tangent bundle of \( F \).
2.4. **Brumfiel-Madsen formula.** Let $G$ be a compact connected (semi-simple) Lie group (of rank $\ell$) with a maximal torus $T \cong (U(1))^\ell$. Let $H$ be a closed connected subgroup of $G$ of maximal rank, i.e., $T \subset H$. Denote by $W_G$ and $W_H$ the Weyl group of $G$ and $H$ respectively. We then have a natural inclusion $W_H \subset W_G$. Suppose that $G \hookrightarrow P \twoheadrightarrow B$ is a principal $G$-bundle, and consider the following associated bundles:

\[
G/T \hookrightarrow E_1 := P \times_G (G/T) \xrightarrow{\pi_1} B,
\]

\[
G/H \hookrightarrow E_2 := P \times_G (G/H) \xrightarrow{\pi_2} B.
\]

Then there is a fiber bundle $H/T \hookrightarrow E_1 \xrightarrow{\pi_1} E_2$, where the projection $\pi_1$ is induced from the natural projection (also denoted by the same symbol) $\pi : G/T \twoheadrightarrow G/H$, and we have the following commutative diagram:

\[
\begin{array}{ccc}
E_1 = P \times_G (G/T) & \xrightarrow{\pi_1} & E_2 = P \times_G (G/H) \\
\downarrow & & \downarrow \\
B & \xrightarrow{=} & B.
\end{array}
\]

The usual right action of the Weyl group $W_G$ on $G/T$ induces a right action on $E_1 = P \times_G (G/T)$ over $B$, i.e., a bundle map over $B$. As a subgroup of $W_G$, the Weyl group $W_H$ of $H$ also acts on $E_1$, which is a bundle map over $E_2$. Therefore the coset $\overline{w} = wW_H \in W_G/W_H$ defines a well-defined map $\pi \circ w : E_1 \xrightarrow{w} E_1 \xrightarrow{\pi} E_2$, which induces a homomorphism in cohomology: $w \circ \pi^* : h^*(E_2) \xrightarrow{\pi^*} h^*(E_1) \xrightarrow{w} h^*(E_1)$. Then Brumfiel-Madsen established the following useful formula:

**Theorem 2.3** (Brumfiel-Madsen [13], Theorem 3.5; Bressler-Evens [10], Theorem 1.3). In the above setting, we have

\[
\pi_1^* \circ \tau(\pi_2)^* = \sum_{\overline{w} \in W_G/W_H} w \circ \pi^*.
\]

As a special case where $H = T$, we have

**Corollary 2.4** (Bressler-Evens [10], Corollary 1.4).

\[
\pi_1^* \circ \tau(\pi_1)^* = \sum_{w \in W_G} w.
\]

We apply Corollary 2.4 to the case where $P = EG$, the universal space\(^\text{\textsuperscript{13}}\) for $G$, so that $B = BG$, the classifying space of $G$, and $E_1 = EG \times_G (G/T) \simeq BT$. In this case, the fibration $G/T \hookrightarrow E_1 \xrightarrow{\pi_1} B$ becomes the following classical Borel fibration:

\[
G/T \hookrightarrow BT \xrightarrow{\rho = \rho(T,G)} BG.
\]

Thus we have the following:

\[
\rho^* \circ \tau(\rho)^* = \sum_{w \in W_G} w.
\]

Combining (2.1) and (2.3), we have

\[
\rho^* \circ \rho^* (\chi^h(T_\rho) \cdot f) = \sum_{w \in W_G} w \cdot f \quad \text{for } f \in h^*(BT). \quad (2.4)
\]

From this formula, Bressler-Evens [10] derived a useful formula which is explained briefly in the next subsection.

\(^{13}\) $EG$ is a contractible space on which $G$ acts freely.
2.5. Bressler-Evens formula. Before stating their result, we shall recall some facts from Lie theory. Let $T \subset G$ be as above and $G/T \hookrightarrow BT \xrightarrow{\rho} BG$ the Borel fibration. Let $G_C$ and $T_C$ be the complexification of $G$ and $T$ respectively. Thus $G_C$ is a connected complex (semi-simple) Lie group with maximal compact subgroup $G$, and $T_C \cong (\mathbb{C}^*)^l$. Denote by $B$ a Borel subgroup of $G_C$ containing $T_C$. Then the natural inclusion $G \hookrightarrow G_C$ induces a diffeomorphism $G/T \cong G_C/B$. By this identification, the full flag manifold $G/T$ is equipped with a complex structure. Let $\mathfrak{g}$ and $\mathfrak{t}$ be the Lie algebras of $G$ and $T$, and $\mathfrak{g}_C = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathfrak{t}_C = \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}$ their complexification. Then $\mathfrak{g}_C$ and $\mathfrak{t}_C$ are the Lie algebras of $G_C$ and $T_C$ respectively. Then we have the root space decomposition

$$\mathfrak{g}_C = \mathfrak{t}_C \oplus \bigoplus_{\alpha \in \Delta^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}),$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g}_C \mid [h, x] = \alpha(h)x \ (\forall h \in \mathfrak{t}_C)\}$, and the system of positive roots $\Delta^+ \subset \text{Hom}_{\mathbb{C}}(\mathfrak{t}_C, \mathbb{C})$ corresponds to the Lie algebra $\mathfrak{b}$ of $B$. Thus $\mathfrak{b} = \mathfrak{t}_C \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$. We set $\Delta^- := -\Delta^+$, the system of negative roots, and $\Delta := \Delta^+ \sqcup \Delta^-$, the system of roots. It is well-known that the tangent bundle $T(G/T)$ to $G/T$ is isomorphic to the vector bundle $G \times_T (\mathfrak{g}/\mathfrak{t})$ associated with the principal $T$-bundle $T \hookrightarrow G \twoheadrightarrow G/T$ and $T$-module $\mathfrak{g}/\mathfrak{t}$. Since we have the natural identification $\mathfrak{g}/\mathfrak{t} \cong \mathfrak{g}_C/\mathfrak{b}$, we have the following isomorphism as complex vector bundles:

$$T(G/T) \cong G \times_T (\mathfrak{g}_C/\mathfrak{b}).$$

Hence the tangent bundle along the fibers $T_\rho$ is isomorphic to the complex vector bundle $ET \times_T (\mathfrak{g}_C/\mathfrak{b})$ associated with the universal $T$-bundle $T \hookrightarrow ET \twoheadrightarrow BT$. Thus

$$T_\rho \cong ET \times_T (\mathfrak{g}_C/\mathfrak{b}).$$

For each character $\chi \in \text{Hom}(T(U(1))) \cong \text{Hom}(T_C, \mathbb{C}^*) = \hat{T}_C$, we have the associated complex line bundle $L_\chi$ over $BT$ defined by $L_\chi := ET \times_T \mathbb{C} = (ET \times \mathbb{C})/(y, v) \sim (y \cdot t, \chi(t)^{-1}v)$. Each root $\alpha \in \Delta \subset \text{Hom}_{\mathbb{C}}(\mathfrak{t}_C, \mathbb{C})$ defines a character $\chi_\alpha \in \text{Hom}(T_C, \mathbb{C}^*)$, and we have the associated complex line bundle $L_{\chi_\alpha}$, which is also denoted by $L_\alpha$ for simplicity. By (2.5) and (2.6), we have

$$T_\rho \cong ET \times_T \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha} \cong \bigoplus_{\alpha \in \Delta^+} L_{-\alpha}.$$ 

Therefore the $h^*$-theory top Chern class (Euler class) of $T_\rho$ is given by

$$c^h_{\top}(T_\rho) = c^h_{\top}\left(\bigoplus_{\alpha \in \Delta^+} L_{-\alpha}\right) = \prod_{\alpha \in \Delta^+} c^h_1(L_{-\alpha}).$$

From (2.7) and the formula (2.4), Bressler-Evens deduced the following\footnote{For the ordinary cohomology theory $h = H$, this formula was already proved by Borel-Hirzebruch \cite[Theorem 20.3]{8} (see also Tu \cite[§11.1]{69}).}

**Theorem 2.5** (Bressler-Evens \cite[Theorem 1.8]{10}). Let $h^*(-) \text{ be a complex-oriented generalized cohomology theory. We assume that the coefficient ring } h^* \text{ is torsion-free. Then for } f \in h^*(BT), \text{ we have}$$

$$\rho^* \circ \rho_*(f) = \sum_{w \in W_G} w \cdot \left[ \prod_{\alpha \in \Delta^+} f_{\alpha}(L_{-\alpha}) \right].$$

\footnote{We made this assumption for simplicity. See Bressler-Evens \cite[Remark 1.10]{11} for less restrictive assumptions.}
One can easily extend Theorem 2.3 to the case of partial flag manifolds. Thus let Θ ⊂ Π := \{simple roots\} ⊂ Δ⁺ be a subset of the set of simple roots, and P = P_Θ be the corresponding parabolic subgroup, and put H = H_Θ := G ∩ P. Then it is known that H is the centralizer of the toral subgroup defined by “α = 0” for ∀α ∈ Θ, and hence a closed connected subgroup of G of maximal rank, i.e., T ⊂ H. The homogeneous manifold G/H has a complex structure (see e.g., Borel-Hirzebruch [7, §13.5]), and we have the following Borel fibration

\[ G/H \xrightarrow{\pi} BH \xrightarrow{\sigma=\rho(H,G)} BG. \]

There exists a natural identification G/H ∼= G_C/P, and this homogeneous manifold is called a partial flag manifold (see e.g., Borel-Hirzebruch [7, §14.3]). Denote by Δ_H = Δ_H^+ ∪ Δ_H^- the system of roots of H with respect to T. The Lie algebra p = p_Θ of P is given by

(2.8) \[ p = b \oplus \bigoplus_{\alpha \in \Delta_H^+} g_{-\alpha} \oplus t_C \oplus \bigoplus_{\alpha \in \Delta^+} g_\alpha \oplus \bigoplus_{\alpha \in \Delta_H^-} g_{-\alpha}. \]

The tangent bundle of G/H ∼= G_C/P is given by

\[ T(G/H) \cong G \times_H (g_C/p). \]

Hence the tangent bundle along the fibers T_σ is isomorphic to the complex vector bundle EH ×_H (g_C/p) associated with the universal H-bundle H → EH → BH. Thus

(2.9) \[ T_\sigma \cong EH \times_H (g_C/p). \]

We apply Theorem 2.3 to the case where P = EG and H = H_Θ, so that B = BG, and E2 = EG × G (G/H) ∼= BH. Then the fibration \( G/H \xrightarrow{\pi} E_2 \xrightarrow{\pi_2} B \) becomes the Borel fibration \( G/H \xrightarrow{\pi} BH \xrightarrow{\sigma} BG \), and the commutative diagram (2.2) yields the following commutative diagram:

(2.10) \[ \begin{array}{ccc} BT & \xrightarrow{\pi=\rho(T,H)} & BH \\ \rho=\rho(T,G) \downarrow & & \downarrow \sigma=\rho(H,G) \\ BG & \xrightarrow{=} & BG. \end{array} \]

Then by Theorem 2.3 we have

(2.11) \[ \rho^* \circ \tau(\sigma)^* = \sum_{w \in W_G/W_H} w \circ \pi^*. \]

Combining (2.1) and (2.11), we have

(2.12) \[ \rho^* \circ \sigma_*(c_{\text{top}}(T_\sigma) \cdot f) = \sum_{w \in W_G/W_H} w \cdot \pi^*(f) \quad \text{for} \quad f \in h^*(BH). \]

On the other hand, pulling back the tangent bundle along the fibers T_σ to BT via the map \( \pi = \rho(T, H) \), we have from (2.9), (2.5), and (2.8),

\[ \pi^*(T_\sigma) \cong ET \times_T (g_C/p) \cong ET \times_T \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_H^-} g_{-\alpha} \cong \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_H^-} L_{-\alpha}. \]

Therefore the \( h^* \)-theory top Chern class (Euler class) of \( \pi^*(T_\sigma) \) is given by

(2.13) \[ \pi^*c_{\text{top}}^h(T_\sigma) = c_{\text{top}}^h \left( \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_H^+} L_{-\alpha} \right) = \prod_{\alpha \in \Delta^+ \setminus \Delta_H^+} c_1^h(L_{-\alpha}). \]
Then by the analogous argument to that of Bressler-Evens \cite{Bressler-Evens}, Theorem 1.8, and the formula (2.12), one obtains the following (Note that by the commutativity of the diagram (2.10), one has $\rho^* \circ \sigma_* = \pi^* \circ \sigma^* \circ \sigma_*$):

**Corollary 2.6.** For $f \in H^*(BH)$, we have

$$\pi^* \circ \sigma^* \circ \sigma_*(f) = \sum_{w \in W_G/W_H} w \cdot \left[ \frac{\pi^*(f)}{\prod_{\alpha \in \Delta^+ \setminus \Delta^+_H} c_1^H(L-\alpha)} \right].$$

### 2.6. Various Gysin formulas.

As mentioned in the introduction, various types of Gysin formulas related to the Gysin maps are known (see e.g., Akyildiz \cite{Akyildiz}, Akyildiz-Carrell \cite{Akyildiz-Carrell}, Buch \cite{Buch}, Damon \cite{Damon}, \cite{Damon2}, Darondeau-Pragacz \cite{Darondeau-Pragacz}, Fel’dman \cite{Fel’dman}, Fulton \cite{Fulton}, Fulton-Pragacz \cite{Fulton-Pragacz}, Harris-Tu \cite{Harris-Tu}, Ilori \cite{Ilori}, Jozefiak-Lascoux-Pragacz \cite{Jozefiak-Lascoux-Pragacz}, Kamimoto \cite{Kamimoto}, Quillen \cite{Quillen}, Pragacz \cite{Pragacz}, \cite{Pragacz2}, Sugawara \cite{Sugawara}, Tu \cite{Tu}, \cite{Tu2}). In this subsection, we shall take up typical examples of these formulas.

#### 2.6.1. Gysin formulas of type $G_C/B \longrightarrow G_C/P$.

First recall the result due to Akyildiz-Carrell \cite{Akyildiz-Carrell}. In order to state their result, we shall use the same notation as in (2.5) with a slightly minor change. So let $G_C \supset B \supset T_C$ be as in (2.5). Consider the parabolic subgroup $P = P_\Theta$ corresponding to a subset $\Theta \subset \Pi = \{\text{simple roots}\} \subset \Delta^+$. Thus the homogeneous variety $G_C/P$ is a partial flag variety. Denote by $W_\Theta$ (resp. $\Delta_\Theta$) the Weyl group (resp. root system) corresponding to $P_\Theta$. Let $\chi \in \hat{T}_C = \text{Hom}(T_C, \mathbb{C}^*)$ be a character. By composing the natural projection $B \longrightarrow T_C$ with $\chi : T_C \longrightarrow \mathbb{C}^*$, we have a character $\chi_B = \chi : B \longrightarrow \mathbb{C}^*$. Then one can define a complex line bundle $M_\chi$ over $G_C/B$ in the usual manner. By assigning each character $\chi \in \hat{T}_C$ the first Chern class $c_1(M_\chi) \in H^2(G_C/B; \mathbb{C})$, the characteristic homomorphism\footnote{Recall that $B$ is the semi-direct product of $T_C$ and its unipotent part.} $c : R := \text{Sym}(\hat{T}_C) \longrightarrow H^*(G_C/B; \mathbb{C})$

is defined. Here $\text{Sym}(\hat{T}_C)$ means the symmetric algebra of $\hat{T}_C$ over $\mathbb{C}$. Let $\pi : G_C/B \longrightarrow G_C/P$ be the natural projection. Then Akyildiz-Carrell showed the following formula (see also Brion \cite{Brion}):

**Theorem 2.7** (Akyildiz-Carrell \cite{Akyildiz-Carrell}, Theorem 1; Brion \cite{Brion}, Proposition 1.1). The Gysin homomorphism $\pi_* : H^*(G_C/B; \mathbb{C}) \longrightarrow H^*(G_C/P; \mathbb{C})$ is given by

$$\pi^* \circ \pi_* (c(f)) = c \left( \sum_{w \in W_\Theta} \det(w) w \cdot \frac{f}{\prod_{\alpha \in \Delta^+ \setminus \Delta^+_H} \alpha} \right) \quad \text{for } f \in R.$$ \hfill (2.15)

Here $\det(w)$ means $(-1)^{\ell(w)}$, where $\ell(w)$ denotes the length of the Weyl group element $w$. Since $w \cdot \prod_{\alpha \in \Delta^+ \setminus \Delta^+_H} \alpha = (-1)^{\ell(w)} \prod_{\alpha \in \Delta^+_H} \alpha$ for any $w \in W_\Theta$, the above formula (2.15) can also be written as follows:

$$\pi^* \circ \pi_* (c(f)) = c \left( \sum_{w \in W_\Theta} \frac{w \cdot f}{\prod_{\alpha \in \Delta^+_H} \alpha} \right) \quad \text{for } f \in R.$$
Akyildiz-Carrell proved this formula by the method based on the zeros of holomorphic vector fields on relevant flag varieties. Brion proved this formula by the Weyl character formula and Grothendieck-Riemann-Roch theorem.

2.6.2. Gysin formulas of type \( \mathcal{F}(E) \to X \). Next recall the result due to Fulton-Pragacz [25, Chapter IV]. Let \( E \xrightarrow{\rho} X \) be a complex vector bundle of rank \( n \) over a variety. Denote by \( \tau = \tau_E : \mathcal{F}(E) \to X \) the associated flag bundle parametrizing successive flags of quotients of \( E \) of ranks \( n-1, \ldots, 2, 1 \). Thus we have the tautological sequence of flag of quotient bundles

\[
\tau^* E = Q^n \to Q^{n-1} \to \cdots \to Q^2 \to Q^1 \to Q^0 = 0,
\]

where \( \text{rank} (Q^i) = i \) \((i = 0, 1, 2, \ldots, n)\). Define the line bundles \( L^i := \ker (Q^i \to Q^{i-1}) \) \((i = 1, 2, \ldots, n)\) over \( \mathcal{F}(E) \). Put \( x_i := c_1(L^i) \in H^2(\mathcal{F}(E)) \) \((i = 1, 2, \ldots, n)\) (the Chern roots of \( E \)). Then Fulton-Pragacz showed the following formula:

**Theorem 2.8** (Pragacz [58], Lemma 2.4; [60], Proposition 4.3 (ii); Fulton-Pragacz [25, p.41]). For a polynomial \( f(X_1, \ldots, X_n) \in H^*(X)[X_1, \ldots, X_n], \) we have

\[
\tau^* \circ \tau_* (f(x_1, \ldots, x_n)) = \sum_{w \in S_n} w \cdot \frac{f(x_1, \ldots, x_n)}{\prod_{1 \leq i < j \leq n} (x_i - x_j)}.
\]

Thus the Gysin map \( \tau_* \) is given by a certain symmetrizing operator called the Jacobi symmetrizer in Fulton-Pragacz [25, §4.1]. As is well-known, the flag bundle \( \mathcal{F}(E) \) can be constructed as a sequence of projective bundles, and Fulton-Pragacz proved this formula by the induction on the rank of \( E \).

2.6.3. Application of the Bressler-Evens formula. Most of these Gysin formulas are formulated in the ordinary cohomology rings or Chow rings, and proved by many different ways. We remark that the Bressler-Evens formulas (Theorem 2.5 and Corollary 2.6) enable us to show these Gysin formulas by a unified manner.

For Theorem 2.7, one can argue as follows: Put \( H = G \cap P \) as in [25]. Then by the classification theorem of principal bundles, we have a classifying map \( h : G/H \to BH \) and its lift \( \tilde{h} : G/T \to BT \), and the following diagram is commutative:

\[
\begin{array}{ccc}
G_C/B & \cong & G/T \\
\pi \downarrow & & \downarrow \rho = \rho(T,H) \\
G_C/P & \cong & G/H \\
& \cong \tilde{h} &BH.
\end{array}
\]

Note that the above classifying map \( h \) (resp. \( \tilde{h} \)) coincides with the fiber inclusion \( \iota_H : G/H \hookrightarrow BH \) (resp. \( \iota : G/T \hookrightarrow BT \)) up to homotopy. Then by Theorem 2.5

\[\text{rank} (S_i) = i \quad (i = 0, 1, 2, \ldots, n). \]

These two tautological sequences are related by \( Q^i = \tau^* E/S_{n-1} \). Therefore if we define the line bundles \( L_i := S_i/S_{i-1} \) \((i = 1, 2, \ldots, n)\), then we have \( L_i = L_{n+1-i} \) \((i = 1, 2, \ldots, n)\).
and the base-change property of Gysin maps, we compute
\[ \pi^* \circ \pi_*(c(f)) = \pi^* \circ \pi_* \circ \iota^*(f) = \pi^* \circ \iota^*_H \circ \rho_*(f) = \iota^* \circ \rho^* \circ \rho_*(f) \]
\[ = \iota^* \left( \sum_{w \in W_\phi} w \cdot \frac{f}{\prod_{\alpha \in \Delta^+} c_1(L_{-\alpha})} \right) \]
\[ = c \left( \sum_{w \in W_\phi} w \cdot \frac{f}{\prod_{\alpha \in \Delta^+} \alpha} \right), \]
as required. Here we used the convention that \( c_1(L_n) = -\alpha \) for a root \( \alpha \in \Delta \).

For Theorem 2.8, one can argue as follows: By the classification theorem of complex vector bundles, we have the classifying map \( h : X \to BU(n) \), and its lift \( \tilde{h} : F\ell(E) \to BT^n \), and the following diagram is commutative:
\[
\begin{array}{ccc}
F\ell(E) & \xrightarrow{\tilde{h}} & BT^n \\
\tau \downarrow & & \downarrow \rho \\
X & \xrightarrow{h} & BU(n)
\end{array}
\]

Let \( \chi_i : T^n \to U(1) \) be the character which takes an element \( t = \text{diag}(t_1, \ldots, t_n) \in T^n \) to the \( i \)-th entry \( t_i \in U(1) \). The line bundles \( L_{\chi_i} \) over \( BT^n \) can be constructed as in §2.5. Put \( y_i := -c_1(L_{\chi_i}) = c_1((L_{\chi_i})^\vee) \in H^2(BT^n) \) \( (i = 1, 2, \ldots, n) \) (notice our convention). Then the positive root system for \( G = U(n) \) is given by \( \Delta^+ = \{ y_i - y_j \mid 1 \leq i < j \leq n \} \) as a subset of \( H^2(BT^n) \). The Weyl group \( W_U(n) \) of \( U(n) \) can be identified with the symmetric group \( S_n \) by the usual manner. Let \( \gamma^n \to BU(n) \) be the universal or canonical vector bundle over \( BU(n) \) (see Milnor-Stasheff [53, §14, p.161]). Then the associated flag bundle \( F\ell(\gamma^n) \to BU(n) \) can be identified with the Borel fibration \( BT^n \to BU(n) \). As noted in [2.6.2], there is the tautological sequence of flag of subbundles \( S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_{n-1} \subset S_n = \rho^*(\gamma^n) \) over \( BT^n \). The usual line bundles \( L_i \) \( (i = 1, 2, \ldots, n) \) over \( BT^n \) are defined by \( L_i := S_i/S_{i-1} \) \( (i = 1, 2, \ldots, n) \). Then it is easily verified that the line bundle \( L_i \) can be identified with the line bundle \( L_{\chi_i} \). Therefore as for the Chern roots of \( E \), we have
\[ x_i = c_1(L^i) = c_1(L_{n+1-i}) = c_1(\tilde{h}^*(L_{n+1-i})) = \tilde{h}^*(c_1(L_{n+1-i})) = \tilde{h}^*(-y_{n+1-i}). \]

\begin{enumerate}
\item Then by Theorem 2.5 and the base-change property of Gysin maps, we compute
\[
\tau^* \circ \tau_*(f(x_1, \ldots, x_n)) = \tau^* \circ \tau_*(\tilde{h}^*(f(-y_n, \ldots, -y_1)))
\]
\[ = \tilde{h}^* \circ \rho^* \circ \rho_*(f(-y_n, \ldots, -y_1)) \]
\[ = \tilde{h}^* \left( \sum_{w \in W_{U(n)}} w \cdot \frac{f(-y_n, \ldots, -y_1)}{\prod_{\alpha \in \Delta^+} c_1(L_{-\alpha})} \right) \]
\[ = \tilde{h}^* \left( \sum_{w \in S_n} w \cdot \frac{f(-y_1, \ldots, -y_n)}{\prod_{1 \leq i < j \leq n} (y_i - y_j)} \right) \]
\[ = \sum_{w \in S_n} w \cdot \frac{f(x_1, \ldots, x_n)}{\prod_{1 \leq i < j \leq n} (x_i - x_j)}, \]
as required.
\end{enumerate}

\[ \text{We also used the well-known fact that the map } \tilde{h}^* : H^*(BT^n) \to H^*(F\ell(E)) \text{ is surjective.} \]
2.6.4. Thom-Porteous formula. Finally, we briefly review the Thom-Porteous formula (see Porteous [57, p.298, Proposition 1.3]) as an application of Gysin formulas. Here we adopt the formulation as in Fulton [23, §14.4], Fulton-Pragacz [25, §2.1], Pragacz [58], [60] which is slightly different from Porteous’ original one. Let $E \overset{p_E}{\to} X$ and $F \overset{p_F}{\to} X$ be complex vector bundles of ranks $e$ and $f$ on a variety $X$. Let $\phi : E \to F$ be a vector bundle homomorphism. For each point $x$, denote by $\phi_x : E_x = p_E^{-1}(x) \to F_x = p_F^{-1}(x)$ the linear map on the fiber. Then we set

$$D_r(\phi) := \{ x \in X \mid \text{rank } \phi_x \leq r \} \subset X,$$

which is called the $r$th degeneracy locus of $\phi$ ($r = 0, 1, \ldots, \min(e, f)$). It is known that if the map $\phi$ is sufficiently generic, the subvariety $D_r(\phi)$ has codimension $(e-r)(f-r)$, and defines a cohomology class $[D_r(\phi)] \in H^{2(e-r)(f-r)}(X)$. Then Thom [67] observed that there must be a polynomial in the Chern classes of $E$ and $F$ which is equal to $[D_r(\phi)]$. Thom posed a problem to find such a polynomial, and later Porteous gave the answer (see Fulton-Pragacz [25, Chapter II, (2.1), (2.5)], Pragacz [58, p.414]):

$$[D_r(\phi)] = \det (e_{f-r-i+j}(F - E))_{1 \leq i, j \leq e-r}. \quad (2.16)$$

Here $c(F - E)$ is defined to be $c(F)/c(E)$. Notice that the right-hand side of (2.16) is equal to the relative version of the Schur polynomial $s_{(e-r)(f-r)}(F - E)$, where $((e-r)(f-r))$ is a rectangular partition with $(f-r)$ rows and $(e-r)$ columns (for the notation, see Fulton-Pragacz [25, §3.2]), and the above formula becomes as follows:

$$[D_r(\phi)] = s_{(e-r)(f-r)}(F - E). \quad (2.17)$$

We shall give an outline of the proof of the above formula for reader’s convenience: Let $\pi_F : G^{f-r}(F) \to X$ be the Grassmann bundle parametrizing rank $(f-r)$ quotient bundles of $F$. On $G^{f-r}(F)$, we have the tautological exact sequence of vector bundles:

$$0 \to S_F \hookrightarrow \pi_F^*(F) \to Q_F \to 0.$$

Then the vector bundle homomorphism $\pi_F^*(E) \overset{\pi_F^*}{\to} \pi_F^*(F) \to Q_F$ over $G^{f-r}(F)$ gives a cross-section $s_\varphi \in \Gamma(\text{Hom}(\pi_F^*(E), Q_F)) \cong \Gamma(\pi_F^*(E)^\vee \otimes Q_F)$. Denote by $Z(s_\varphi) \subset G^{f-r}(F)$ the zero locus of $\varphi$. Then for an element $W \in G^{f-r}(F)$ with $\pi_F(W) = x \in X$, one sees immediately that $W \in Z(s_\varphi)$ implies $\text{Im } \varphi_x \subset W$, and hence rank $\varphi_x \leq \dim W = r$. Thus we have $x \in D_r(\varphi)$. From this, the set $Z(s_\varphi)$ maps onto $D_r(\varphi)$. Then under appropriate conditions, the class $[Z(s_\varphi)]$ is given by the top Chern class $e_{c(f-r)}(\pi_F^*(E)^\vee \otimes Q_F)$. Therefore we have the following formula:

$$\pi_{F*}(e_{c(f-r)}(\pi_F^*(E)^\vee \otimes Q_F)) = [D_r(\varphi)],$$

and we have to compute the left-hand side of the above equation. This can be done by making use of Gysin formulas. Let $x_1, \ldots, x_f$ (resp. $a_1, \ldots, a_e$) be the Chern roots of $F$ (resp. $E$) as in [2.6.2]. The Chern roots of $Q_F$ are $x_1, \ldots, x_{f-r}$. By the splitting principle, the top Chern class $e_{c(f-r)}(\pi_F^*(E)^\vee \otimes Q_F)$ is given by the product $\prod_{i=1}^{f-r} \prod_{j=1}^{e} (x_i - a_j)$. On the other hand, by a similar argument as in the previous subsection [2.6.2], the Gysin map $\pi_{F*} : H^*(G^{f-r}(F)) \to H^*(X)$ is described by the following symmetrizing operator (see also Pragacz [58, Lemma 2.5], [60, Proposition 4.2]):

$$\pi_{F*}(g(x_1, \ldots, x_f)) = \sum_{w \in S_f/S_{f-r} \times S_e} w \cdot \left[ \frac{g(x_1, \ldots, x_f)}{\prod_{1 \leq i \leq f-r} \prod_{f-r+1 \leq j \leq f} (x_i - x_j)} \right].$$

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for a polynomial $g(X_1, \ldots, X_f) \in H^*(X)[X_1, \ldots, X_f]^{S_f - r \times S_r}$. From this description, one can compute $\pi_{F_*}(\prod_{i=1}^{f-r} \prod_{j=1}^{r} (x_i - a_j))$, and obtain the formula \[ (2.17) \] We remark that the $K$-theoretic analogue of this formula is also given by Buch [14, Theorem 2.3]. In \[ 5.4 \] we shall generalize the Thom-Porteous formula for cohomology to the complex cobordism theory.

### 3. Universal Hall-Littlewood functions

As mentioned in the introduction (see also Example 2.2), Quillen [62] showed that the complex cobordism theory $MU^*(\text{--})$ (with the associated formal group law $F_{MU}$) has the following universal property: for any complex-oriented cohomology theory $h^*(\text{--})$ (with the associated formal group law $F_h$), there exists a homomorphism of rings $\theta : MU^* \longrightarrow h^*$ such that $F_h(X, Y) = (\theta_*F_{MU})(X, Y) = X + Y + \sum a_{i,j}^{M_k} X^i Y^j$. Thus it will be sufficient to consider the case when $h = MU$, for general case follows immediately from the universal one by the specialization $a_{i,j}^{M_k} \mapsto \theta(a_{i,j}^{M_k}) (i, j \geq 1)$. Recall that, by Quillen again, the coefficient ring $MU_* = MU^{-*}$ is isomorphic to the Lazard ring $\mathbb{L}$. In our previous paper [55], we introduced the universal Schur $(S)$-functions $s^{\lambda}_X(x_n)$ for $\lambda$ partitions, and the universal Schur $P$- and $Q$-functions $P^{\lambda}_X(x_n), Q^{\lambda}_X(x_n)$ for $\nu$ strict partitions. In this section, we introduce the universal Hall-Littlewood functions $H^{\lambda}_X(x_n; t)$ which will be expected to interpolate the universal Schur $S$-functions and the universal Schur $P$-functions. Since these functions will be of independent interest in terms of, e.g., algebraic combinatorics, so apart from geometry, we shall deal with these functions purely algebraically, and slightly changes the notation concerning the formal group law in this section.

#### 3.1. Lazard ring $\mathbb{L}$ and the universal formal group law.

We begin with collecting the basic facts about the Lazard ring. We use the convention as in Levine-Morel’s book [17]. In [14], Lazard considered a universal commutative formal group law of rank one $(\mathbb{L}, F_{\mathbb{L}})$, where the ring $\mathbb{L}$, called the Lazard ring, is isomorphic to the polynomial ring in countably infinite number of variables with integer coefficients, and $F_{\mathbb{L}} = F_{\mathbb{L}}(u, v)$ is the universal formal group law (for a construction and basic properties of $\mathbb{L}$, see Levine-Morel [17, §1.1]):

$$F_{\mathbb{L}}(u, v) = u + v + \sum_{i,j \geq 1} a_{i,j}^{\mathbb{L}} u^i v^j \in \mathbb{L}[[u, v]].$$

This is a formal power series in $u, v$ with coefficients $a_{i,j}^{\mathbb{L}}$ of formal variables which satisfies the axiom of the formal group law (see (2.1)). For the universal formal group

\[21\] As in Ikeda-Naruse [62] [§5.1], Molev-Sagan [51, §2], let us introduce the following notation (cf. 432): Set

$$(t|a)^k := \prod_{i=1}^{k} (t - a_i) = (t - a_1)(t - a_2) \cdots (t - a_k)$$

for any integer $k \geq 0$ (Here $a = (a_1, \ldots, a_e) = (a_1, \ldots, a_e, 0, 0, \ldots)$ is the Chern roots of $E$). Then one can rewrite

$$\prod_{i=1}^{f-r} e \prod_{j=1}^{e} (x_i - a_j) = \prod_{i=1}^{f-r} (x_i|a)^e.$$  

Then one computes $\pi_{F_*}(\prod_{i=1}^{f-r} (x_i|a)^e)$ by the above symmetrizing operator description of $\pi_{F_*}$, and obtains the factorial Schur polynomial $s_{(\ell-r)\cdot)(\ell-r)}(F - E)$.  

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law, we shall use the notation (see Levine-Morel [47 §2.3.2])
\[ u +_L v = F_L(u, v) \quad \text{(formal sum)}, \]
\[ \overline{u} = [-1]_{L}(u) = \chi_{L}(u) \quad \text{(formal inverse of } u). \]

Note that $\overline{u} \in \mathbb{L}[[u]]$ is a formal power series in $u$ with initial term $-u$, and first few terms appear in Levine-Morel [47, p.41]. The $n$-series $[n]_{F_L}(u)$ introduced in [2.1.2] shall be denoted simply by $[n]_{L}(u)$ in the sequel. In what follows, we regard $\mathbb{L}$ as a graded algebra over $\mathbb{Z}$, and the grading of $\mathbb{L}$ is given by $\deg (a_{i,j}^{L}) = 1 - i - j$ ($i, j \geq 1$) (see Levine-Morel [47, p.5]). Be aware that in topology, it is customary to give $a_{i,j}^{L}$ the cohomological degree $2(1 - i - j)$.

### 3.2. Universal factorial Schur $S$-, $P$-, and $Q$-functions

In this subsection, we recall the definitions of the universal factorial Schur $S$-, $P$-, and $Q$-functions following Nakagawa-Naruse [55 §4]. Besides the variables $x = (x_1, x_2, \ldots)$, we prepare another set of variables $b = (b_1, b_2, \ldots)$. We provide the variables $x = (x_1, x_2, \ldots)$ and $b = (b_1, b_2, \ldots)$ with degree $\deg (x_i) = \deg (b_i) = 1$ for $i = 1, 2, \ldots$. In what follows, when considering polynomials or formal power series $f(x_1, x_2, \ldots)$ with coefficients in $\mathbb{L}$ (or $\mathbb{L}[[b]]$), we shall call the degree with respect to $x_1, x_2, \ldots, b_1, b_2, \ldots$, and $a_{i,j}^{L}$ the total degree of $f(x_1, x_2, \ldots)$.

For an integer $k \geq 1$, we define a generalization of the ordinary $k$-th power $t^k$ by
\[ [t|b]^k := \prod_{i=1}^{k} (t +_L b_i) = (t +_L b_1)(t +_L b_2) \cdots (t +_L b_k) \]
and its variant by
\[ [[[t|b]]]^k := (t +_L t)[[t|b]]^{k-1} = (t +_L t)(t +_L b_1)(t +_L b_2) \cdots (t +_L b_{k-1}), \]
where we set $[[t|b]]^0 := [t|b]^0 := 1$. For a partition\(^{22}\) i.e., a non-increasing sequence of non-negative integers $\lambda = (\lambda_1, \ldots, \lambda_r)$ ($\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0$)\(^{23}\) we set
\[ [x|b]^\lambda := \prod_{i=1}^{r} [x_i|b]^\lambda_i \quad \text{and} \quad [[[x|b]]]^\lambda := \prod_{i=1}^{r} [[[x_i|b]]]^\lambda_i. \]

Let $\mathcal{P}_n$ denote the set of all partitions of length $\leq n$. For a positive integer $n$, we set $\rho_n = (n, n-1, \ldots, 2, 1)$. For partitions $\lambda, \mu \in \mathcal{P}_n$, $\lambda + \mu$ is a partition of length $\leq n$ defined by $(\lambda + \mu)_i := \lambda_i + \mu_i$ ($1 \leq i \leq n$). With this notation, the universal factorial Schur ($S$-) function $s_{X}^{L}(x_1, \ldots, x_n|b) = s_{X}^{L}(x_1, \ldots, x_n|b)$ corresponding to a partition $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathcal{P}_n$ is defined to be
\[ s_{X}^{L}(x_n|b) = s_{X}^{L}(x_1, \ldots, x_n|b) := \sum_{w \in S_n} w \cdot \left[ [x|b]^{\lambda + \rho_n - 1} / \prod_{1 \leq i < j \leq n} (x_i +_L x_j) \right]. \]

We also define
\[ s_{X}^{L}(x_n) = s_{X}^{L}(x_1, \ldots, x_n) := s_{X}^{L}(x_1, \ldots, x_n|0). \]

The function $s_{X}^{L}(x_n)$ will be called just the universal Schur function.

\(^{22}\) For notation and terminology on partitions which will be used throughout this paper, we mainly follow those in Macdonald’s book [48 Chapter I].

\(^{23}\) It is customary not to distinguish two partitions which differ only by a string of zeros at the end.
Remark 3.1. The non-equivariant version $s^L_\lambda(x_n)$ is already defined by Fel’dman [22, Definition 4.2]. These are called the generalized Schur polynomials there. In that paper, the author also established a Gysin formula for these generalized Schur polynomials (see [22, Theorem 4.5]) [24].

Since

$$x_i +_L x_j = (x_i - x_j)(1 + \text{higher degree terms in } x_i \text{ and } x_j \text{ with coefficients in } \mathbb{L}),$$

the function $s^L_\lambda(x_n|b)$ is a formal power series with coefficients in $\mathbb{L}$ in the variables $x_1, \ldots, x_n$ and $b_1, b_2, \ldots, b_{\lambda_1 + n - 1}$. It is also a homogeneous formal power series of total degree $|\lambda| = \sum_{i=1}^n \lambda_i$, the size of $\lambda$. In (3.1), if we put $a^L_{i,j} = 0$ for all $i, j \geq 1$ and $b_i = -a_i$ ($i = 1, 2, \ldots$), where $a = (a_1, a_2, \ldots, \ldots)$ is another sequence of parameters, the functions $s^L_\lambda(x_n|b)$ reduce to the factorial Schur polynomials usually denoted by $s_\lambda(x_n|a)$ (for its definition, see Ikeda-Naruse [32] §5.1, Macdonald [18] I, §3, Examples 20], Molev-Sagan [54, p.4431]). If we put $a^L_{i,j} = \beta$ and $a^L_{i,j} = 0$ for all $(i, j) \neq (1, 1) [25]$ then $s^L_\lambda(x_n|b)$ reduce to the factorial Grothendieck polynomials $G_\lambda(x_n|b)$ (for its definition, see Ikeda-Naruse [31] (2.12), (2.13)], McNamara [51] Definition 4.1]). Thus our functions $s^L_\lambda(x_n|b)$ are generalizations of these polynomials and hence universal in this sense. Note that unlike the usual factorial Schur and Grothendieck polynomials, the function $s^L_\lambda(x_n|b)$ corresponding to the empty partition $\emptyset = (0^n)$ is not equal to 1. For instance, we have

$$s^L_\emptyset(x_2|b) = \frac{x_1 +_L b_1}{x_1 +_L x_2} + \frac{x_2 +_L b_1}{x_2 +_L x_1} = 1 + a^L_{1,2} x_1 x_2 + a^L_{1,1} a^L_{1,2} b_1 x_1 x_2 + \cdots \neq 1.$$

As mentioned in Remark 3.1 in order to formulate Fel’dman’s Gysin formula for the universal Schur functions, we need to extend the above definition to arbitrary sequences of non-negative integers. Thus, for a sequence $I = (I_1, I_2, \ldots, I_n)$ of non-negative integers, we define $s^L_I(x_n)$ to be

$$s^L_I(x_n) := \sum_{w \in S_n} w \cdot \left[ \frac{x_{I+\rho_n}}{\prod_{1 \leq i < j \leq n} (x_i +_L x_j)} \right].$$

Next let us recall the definition of the universal factorial Schur $P$- and $Q$-functions. Denote by $SP_n$ the set of all strict partitions of length $\leq n$, i.e., a sequence of positive integers $\nu = (\nu_1, \ldots, \nu_k)$ with $k \leq n$ such that $\nu_1 > \cdots > \nu_k > 0$. Then, for a strict partition $\nu = (\nu_1, \ldots, \nu_k) \in SP_n$, the universal factorial Schur $P$- and $Q$-functions are defined to be

$$P^L_\nu(x_n|b) = P^L_\nu(x_1, \ldots, x_n|b) := \frac{1}{(n-k)!} \sum_{w \in S_n} w \cdot \left[ [x|b]^{\nu} \prod_{i=1 \atop j=i+1}^{n} \frac{x_i +_L x_j}{x_i +_L x_j} \right],$$

$$Q^L_\nu(x_n|b) = Q^L_\nu(x_1, \ldots, x_n|b) := \frac{1}{(n-k)!} \sum_{w \in S_n} w \cdot \left[ [x|b]^{\nu} \prod_{i=1 \atop j=i+1}^{n} \frac{x_i +_L x_j}{x_i +_L x_j} \right],$$

where the symmetric group $S_n$ acts only on the $x$-variables $x_1, \ldots, x_n$ by permutations.

[24] Fel’dman’s formula is a generalization of Fulton-Pragacz’ formula [23] (4.2) (see also [12]) in this paper) to the complex cobordism theory as well as to general partial flag bundles. However, we think that his formula should be modified correctly. In the later section [44,2.2], we shall establish a Gysin formula for the universal Schur functions, thus correcting Fel’dman’s formula.

[25] Notice that the sign convention of $\beta$ is opposite from the one given in Example 22. In the rest of this paper, we shall use this sign convention that fits in with the listed references here.
We also define

\[
P^\nu_{\ell}(x_n) = P^\nu_{\ell}(x_1, \ldots, x_n) := P^\nu_{\ell}(x_1, \ldots, x_n|0),
\]

\[
Q^\nu_{\ell}(x_n) = Q^\nu_{\ell}(x_1, \ldots, x_n) := Q^\nu_{\ell}(x_1, \ldots, x_n|0).
\]

The functions \(P^\nu_{\ell}(x_n|b)\) and \(Q^\nu_{\ell}(x_n|b)\) are formal power series with coefficients in \(\mathbb{L}\) in the variables \(x_1, \ldots, x_n\) and \(b_1, b_2, \ldots, b_n\) for \(P^\nu_{\ell}\) (resp. \(b_1, b_2, \ldots, b_{n-1}\) for \(Q^\nu_{\ell}\)). These are homogeneous formal power series of total degree \(|\nu|\). In (3.3), if we put \(a^\nu_{i,j} = 0\) for all \(i, j \geq 1\) and \(a_i = -a_i\) \((i = 1, 2, \ldots)\), the functions \(P^\nu_{\ell}(x_n|b), Q^\nu_{\ell}(x_n|b)\) reduce to the factorial Schur P- and Q-polynomials \(P_\nu(x_n|a), Q_\nu(x_n|a)\) (for their definitions, see Ikeda-Mihalcea-Naruse [33, §4.2]), Ikeda-Naruse [32] Definition 8.1, Ivanov [36] Definitions 2.10 and 2.13). If we put \(a^\nu_{1,1} = \beta\) and \(a^\nu_{i,j} = 0\) for all \((i, j) \neq (1, 1)\), then \(P^\nu_{\ell}(x_n|b), Q^\nu_{\ell}(x_n|b)\) reduce to the \(K\)-theoretic factorial Schur P- and Q-polynomials \(GP_\nu(x_n|b), GQ_\nu(x_n|b)\) due to Ikeda-Naruse [34] Definition 2.1. Thus our functions \(P^\nu_{\ell}(x_n|b), Q^\nu_{\ell}(x_n|b)\) are generalizations of these polynomials and hence universal in this sense.

### 3.3. Universal Hall-Littlewood functions

In this subsection, we introduce the universal Hall-Littlewood functions which interpolate the \(S\)-functions and the \(P\)-functions.

We use the notation as in Pragacz [61]. Let \(\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathcal{P}_n\) be a partition of length \(\ell(\lambda) \leq n\). Consider the maximal “intervals” \(I_1, I_2, \ldots, I_d\) in \([n] := \{1, 2, \ldots, n\}\), where the sequence \(\lambda\) is “constant”. Thus we have a decomposition

\[
[n] = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_d \quad \text{(disjoint union)}
\]

Here and in what follows, we keep the following convention: when we refer to such a decomposition, we always arrange “intervals” \(I_1, I_2, \ldots\) in increasing order, that is, \(\max I_r < \min I_{r+1}\) for each \(r\). Let \(m_r\) be the “length” of the interval \(I_r\) for \(r = 1, 2, \ldots, d\), namely the cardinality of \(I_r\), so that \(\sum_{r=1}^{d} m_r = n\). We write \(n(i)\) the number of the interval containing \(i\) for \(i \in [n]\), namely if \(i\) is in \(I_r\), then \(n(i) = r\). Notice that, since \(\lambda = (\lambda_1, \ldots, \lambda_n)\) is a partition, \(n(i) < n(j)\) is equivalent to \(\lambda_i > \lambda_j\) for \(i, j \in \mathbb{N}\). We define a subgroup \(S^\lambda_n\) of \(S_n\) as the stabilizer of \(\lambda\). Thus

\[
S_n^\lambda = \prod_{i=1}^{d} S_{m_i} = S_{m_1} \times S_{m_2} \times \cdots \times S_{m_d}.
\]

Denote by \(\ell_L(x) \in \mathbb{L}[x]\) the logarithm\(^{26}\) of \(F_L\), i.e., a unique formal power series with leading term \(x\) such that

\[
\ell_L(a + Lb) = \ell_L(a) + \ell_L(b).
\]

Using the logarithm \(\ell_L(x)\), one can rewrite the \(n\)-series \([n]_L(x)\) for a non-negative integer \(n\), aforementioned in [33,1\] as

\[
\ell_L([n]_L(x)) = \ell_L(x + L + \cdots + L) = \ell_L(x) + \cdots + \ell_L(x) = n \cdot \ell_L(x),
\]

\(^{26}\) See e.g., Kono-Tamaki [13] Lemma 6.27, Levine-Morel [47] Lemma 4.1.29, Quillen [62], Ravenel [64] Appendix A2]. If we put \(a^\nu_{i,j} = 0\) for all \(i, j \geq 1\), then \(F_\ell(u, v)\) reduces to the additive formal group law \(F_\ell(u, v) = u + v\), and both \(\ell_L(x)\) and \(\ell_L^{-1}(x)\) reduce to \(x\). If we put \(a^\nu_{1,1} = \beta\) and \(a^\nu_{i,j} = 0\) for all \((i, j) \neq (1, 1)\), then \(F_\ell(u, v)\) reduces to the multiplicative formal group law \(F_\ell(u, v) = e^{uv}\). In this case, \(\ell_L(x)\) reduces to \(\beta^{-1} \log (1 + \beta x) = \sum_{i=0}^{\infty} \frac{(-\beta)^i}{i+1} x^{i+1}\), and \(\ell_L^{-1}(x)\) reduces to \(\beta^{-1} (e^{\beta x} - 1) = \sum_{i=0}^{\infty} \frac{\beta^i}{(i + 1)!} x^{i+1}\).
in other words, \([n]_\ell(x) = \ell_{\ell}(n \cdot \ell_{\ell}(x))\). This formula allows us to define

\[
[t]_\ell(x) := \ell_{\ell}(t \cdot \ell_{\ell}(x))
\]

for an indeterminate \(t\). This is a natural extension of \(t \cdot x\) as well as the \(n\)-series \([n]_\ell(x)\)\(^{27}\).

**Definition 3.2** (Universal Hall-Littlewood function). With the above notation, for a partition \(\lambda \in \mathcal{P}_n\), we define

\[
(3.5) \quad H^\lambda_\nu(x_n; t) := \sum_{w \in S_n/\mathcal{S}_\lambda} \mathcal{w} \cdot \left[ x^\lambda \prod_{1 \leq i < j \leq n, n(i) < n(j)} \frac{x_i + t t_j}{x_i + t x_j} \right].
\]

If we put \(a_{i,j}^\ell = 0\) for all \(i,j \geq 1\), the functions \(H^\lambda_\nu(x_n; t)\) reduce to the usual Hall-Littlewood polynomials denoted by \(P_\lambda(x_1, \ldots, x_n; t)\) in Macdonald’s book [48, Chapter III, §2, (2.2)]. For the usual Hall-Littlewood polynomial \(P_\lambda(x_1, \ldots, x_n; t)\), it is known that

\[
P_\lambda(x_1, \ldots, x_n; 0) = s_\lambda(x_1, \ldots, x_n)
\]

under the specialization \(t = 0\) (see Macdonald [48, Chapter III, §2, (2.3)]), and

\[
P_\lambda(x_1, \ldots, x_n; 1) = m_\lambda(x_1, \ldots, x_n),
\]

under the specialization \(t = 1\) (see Macdonald [48, Chapter III, §2, (2.4)]). Here \(m_\lambda\) denotes the monomial symmetric polynomial corresponding to \(\lambda\). Moreover, for a strict partition \(\nu\) of length \(\ell(\nu) \leq n\), one obtains that

\[
(3.6) \quad P_\nu(x_1, \ldots, x_n; -1) = P_\nu(x_1, \ldots, x_n)
\]

under the specialization \(t = -1\) (see Macdonald [48, Chapter III, §8, Examples 1.]).

For the universal Hall-Littlewood functions \(H^\lambda_\nu(x_n; t)\), it follows immediately from (3.5) that \(H^\lambda_\nu(x_n; 1) = m_\lambda(x_n)\) under the specialization \(t = 1\). Let us next consider the specialization \(t = -1\). Let \(\nu = (\nu_1, \ldots, \nu_k) \in \mathcal{S}_n\) be a strict partition with length \(\ell(\nu) = k \leq n\). Then we have a decomposition \([n] = I_1 \sqcup \cdots \sqcup I_k \sqcup I_{k+1}\), where \(I_r = \{r\} (r = 1, \ldots, k)\) and \(I_{k+1} = \{k+1, \ldots, n\}\). Therefore we have \(m_{\nu_r} = 1 (r = 1, \ldots, k)\), \(m_{\nu_{k+1}} = n-k\) and \(n(i) = i (i = 1, \ldots, k)\), \(n(i) = k+1 (i = k+1, \ldots, n)\). The stabilizer of \(\nu\) is given by \(S^\nu_n = (S_1)^k \times S_{n-k}\). Therefore it follows from Definition 3.2 and (3.4), we have

\[
H^\nu_\nu(x_n; -1) = \sum_{w \in S_n/((S_1)^k \times S_{n-k})} \mathcal{w} \cdot \left[ x^\nu \prod_{1 \leq i < j \leq n, 1 \leq i < k} \frac{x_i + t x_j}{x_i + t x_j} \right]
= \frac{1}{(n-k)!} \sum_{w \in S_n} \mathcal{w} \cdot \left[ x^\nu \prod_{i=1}^k \prod_{i+1 < j \leq n} \frac{x_i + t x_j}{x_i + t x_j} \right] = P_\nu^L(x_n).
\]

We now consider the specialization \(t = 0\). Let \(\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathcal{P}_n\) be a partition with length \(\ell(\lambda) \leq n\). Then we have a decomposition \([n] = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_d\) as above. Letting \(m_r\) be the cardinality of \(I_r\) for \(r = 1, \ldots, d\), one can rewrite \(\lambda =
(n_1^{m_1} n_2^{m_2} \ldots n_d^{m_d}) \ (n_1 > n_2 > \cdots > n_d \geq 0). \ We \ put \ \nu(r) := m_1 + \cdots + m_r \ for \ \ r = 1, 2, \ldots, d \ and \ \nu(0) := 0. \ Then \ the \ specialization \ t = 0 \ gives \\
(3.7) \\
H^r_X(\chi_n; 0) = \sum_{\pi \in S_n/S^2_n} w \cdot \left[ x^\lambda \prod_{1 \leq i < j \leq n, \ n(i) < n(j)} \frac{x_i}{x_i + L \bar{x}_j} \right] \\
= \sum_{\pi \in S_n/S^2_n} w \cdot \left[ \prod_{i=1}^{n} x_i^{\lambda_i} \prod_{1 \leq i < j \leq n, \ n(i) < n(j)} \frac{x_i}{x_i + L \bar{x}_j} \right] \\
= \sum_{\pi \in S_n/S^2_n} w \cdot \left[ \prod_{r=1}^{d} \left( \prod_{m_1 + \cdots + m_{r-1} < i \leq m_1 + \cdots + m_r} x_i^{n_r + n - (m_1 + \cdots + m_r)} \right) \right] \left[ \prod_{1 \leq i < j \leq n, \ n(i) < n(j)} (x_i + L \bar{x}_j) \right] \\
= \sum_{\pi \in S_n/S^2_n} w \cdot \left[ \prod_{r=1}^{d} \left( \prod_{\nu(r-1) < i \leq \nu(r)} x_i^{n_r + n - \nu(r)} \right) \right] \left[ \prod_{1 \leq i < j \leq n, \ n(i) < n(j)} (x_i + L \bar{x}_j) \right]. \\

We \ shall \ consider \ this \ function \ in \ later \ section \ 4.3. 

4. Applications of Gysin Formulas to the Schur Functions

In \[2.6\] we reviewed various Gysin formulas in the ordinary cohomology (or Chow) theory, and in the previous section \[3\] we introduced universal analogues of the ordinary Schur \( S \)-, \( P \)-, \( Q \)-, and Hall-Littlewood functions. In this section, we pursue the Gysin formulas in the complex cobordism theory which relate Gysin maps for flag bundles of quotient bundles of \( E \) to these universal Schur functions. The following \[4.1\] is devoted to the recollection of various known Gysin formulas in the ordinary cohomology theory, and their generalization to the complex cobordism theory will be treated in \[4.2\]. Our main tool for establishing Gysin formulas in the complex cobordism theory is the Bressler-Evens formula reviewed in \[2.5\].

4.1. Gysin formulas for various Schur functions. We use the same notation as in \[2.6,2\]. Let \( E \xrightarrow{p} X \) be a complex vector bundle of rank \( n \) (over a variety). Denote by \( \tau = \tau_E : \mathcal{F}(E) \to X \) the associated flag bundle parametrizing successive flags of quotient bundles of \( E \) of rank \( n-1, \ldots, 2, 1 \). The usual Schur polynomial \( s_\lambda(X_1, \ldots, X_n) \) corresponding to a partition \( \lambda \in \mathcal{P}_n \) is a symmetric polynomial in the \( n \)-variables \( X_1, \ldots, X_n \), and therefore it can be written as a polynomial in the elementary symmetric polynomials \( e_i(X_1, \ldots, X_n) \)'s. Let \( x_1, \ldots, x_n \) be the Chern roots of \( E \) as in \[2.6,2\]. Then the Chern classes \( c_i(E) \)'s can be identified with \( e_i(x_1, \ldots, x_n) \)'s as usual, and hence \( s_\lambda(x_1, \ldots, x_n) \) can be expressed as a polynomial in \( c_i(E) \)'s. Let us define a cohomology class \( s_\lambda(E) \in H^{2|\lambda|}(\mathcal{F}(E)) \) to be \( \tau^*(s_\lambda(X)) := s_\lambda(x_1, \ldots, x_n) \in H^{2|\lambda|}(\mathcal{F}(E)) \). Then the following formula is known:

**Proposition 4.1** (Pragacz \[58\], Lemma 2.3; Fulton-Pragacz \[23\], (4.1)). The image of the monomial \( x^{\lambda + \rho_{n-1}} = x_1^{\lambda_1 + n-1} x_2^{\lambda_2 + n-2} \cdots x_n^{\lambda_n} \) under the Gysin homomorphism \( \tau_* : H^*(\mathcal{F}(E)) \to H^*(X) \) is given by

\[
(4.1) \quad \tau_*(x^{\lambda + \rho_{n-1}}) = s_\lambda(E).
\]

\[28\] It is well-known that the induced homomorphism \( \tau^* : H^*(X) \to H^*(\mathcal{F}(E)) \) is injective, and hence the cohomology class \( s_\lambda(E) \) is well-defined.
The formula (4.1) is called the Jacobi-Trudi identity, from which Fulton-Pragacz \[25\] derived some useful formulas for Grassmann bundles, which we recall a bit later.

Furthermore, the analogous Gysin formulas which relate the Hall-Littlewood polynomials and more general flag bundles are considered in Pragacz \[61\]. Let us recall these formulas. We use the same notation as in \[3.3\] (see also Pragacz \[61\]). Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathcal{P}_n \) be a partition of length \( \leq n \). Then we have a decomposition

\[ [n] = \{1, 2, \ldots, n\} = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_d, \]

where \( \lambda \) is “constant” on each \( I_r \) (\( r = 1, 2, \ldots, d \)). Denote by \( m_r \) the length of the interval \( I_r \) for \( r = 1, 2, \ldots, d \) so that \( \sum_{r=1}^{d} m_r = n \), and \( n(i) \) the number of the interval containing \( i \) for \( i \in [n] \). We put \( \nu(p) = \sum_{r=1}^{p} m_r \) for \( p = 1, \ldots, d \) and \( \nu(0) = 0 \). The stabilizer of \( \lambda \) is denoted by \( S^n_\lambda \). Associated to a complex vector bundle \( E \to X \), one can define a “\((d - 1)\)-step flag bundle” with steps of lengths \( m_r \)

\[ \eta_\lambda : \mathcal{F}^\lambda(E) \to X, \]

parametrizing flags of quotient bundles of \( E \) of ranks

\[ n - m_d = \nu(d - 1), n - m_d - m_{d-1} = \nu(d - 2), \ldots, n - m_d - m_{d-1} - \cdots - m_2 = \nu(1). \]

If \( \lambda = \emptyset \), the empty partition, then \( \mathcal{F}^\emptyset(E) \) is understood to be the base space \( X \). Here, for later discussion, we shall fix the notation about partial flag bundles associated to a complex vector bundle \( E \overset{p}{\to} X \) of rank \( n \) (see Fulton’s book \[24\] §9.1, 10.6): For a sequence of integers \( 0 < r_1 < r_2 < \cdots < r_k < n \), let us denote by \( \mathcal{F}^{r_1, r_2, \ldots, r_k}(E) \) a partial flag bundle consisting of flags of subbundles \( 0 \subsetneq S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_k \subsetneq E \) with rank \( S_i = r_i \) (\( i = 1, \ldots, k \)). Since giving a flag of subbundles of \( E \) as above is equivalent to giving a flag of quotient bundles \( E \to Q^1 \to Q^2 \to \cdots \to Q^1 \to 0 \) with rank \( Q^i = n - r_{k+i-1} \) (\( i = 1, \ldots, k \)), this partial flag bundle is also denoted by \( \mathcal{F}^{n-r_1, n-r_2, \ldots, n-r_k}(E) \). Moreover, dualizing the sequence of vector bundles

\[ 0 \leftrightarrow S_1 \leftrightarrow S_2 \leftrightarrow \cdots \leftrightarrow S_k \leftrightarrow E \to Q^k \to Q^{k-1} \to \cdots \to Q^1 \to 0 \]

gives the sequence of vector bundles

\[ 0 \leftrightarrow (Q^1)^\vee \leftrightarrow (Q^2)^\vee \leftrightarrow \cdots \leftrightarrow (Q^k)^\vee \leftrightarrow E^\vee \to (S_k)^\vee \to (S_{k-1})^\vee \to \cdots \to (S_1)^\vee \to 0. \]

Therefore we have the canonical isomorphism

\[ \mathcal{F}^{r_1, r_2, \ldots, r_k}(E) = \mathcal{F}^{n-r_1, n-r_2, \ldots, n-r_k}(E) \cong \mathcal{F}^{\nu', \nu, \nu, \nu}(E^\vee) = \mathcal{F}^{\nu', \nu, \nu, \nu}(E^\vee). \]

Using this notation, we can write

\[ \mathcal{F}^\lambda(E) = \mathcal{F}^{\nu(d-1), \nu(d-2), \ldots, \nu(1)}(E) = \mathcal{F}^{\nu(d-1), \nu(d-2), \ldots, \nu(1)}(E) = \mathcal{F}^{\nu', \nu, \nu, \nu}(E^\vee) = \mathcal{F}^{\nu', \nu, \nu, \nu}(E^\vee). \]

**Example 4.2** (See Pragacz \[61\], Example 2).

1. Let \( \nu = (\nu_1, \ldots, \nu_k) \in \mathcal{S}_P \) be a strict partition with length \( \ell(\nu) = k \leq n \). Then as we saw at the end of \[3.3\] we have \( d = k + 1 \), and

\[ (m_1, \ldots, m_k, m_{k+1}) = (1, 1, \ldots, 1, n - k), \quad S_\nu = (S_1)^k \times S_{n-k}. \]

Then the corresponding flag bundle \( \eta_\nu : \mathcal{F}^\nu(E) \to X \) is often denoted by \( \tau^k_{E} : \mathcal{F}^{k-1, \ldots, 2, 1}(E) \to X \), and parametrizes flags of successive quotient bundles of \( E \) of ranks \( k, k-1, \ldots, 2, 1 \). As a special case of this example, the
flag bundle corresponding to the partition \( \rho_{n-1} = (n-1, n-2, \ldots, 2, 1, 0) \) is the full flag bundle \( F^{\ell - 1, n - 2, \ldots, 1} = F^{\ell}(E) \to X \).

(2) Let \( \lambda = (a, \ldots, a, b, \ldots, b) = (a^q b^{n-q}) \in P_n \) be a partition of two rows with \( a > b \geq 0 \). Then we have a decomposition \([n] = I_1 \sqcup I_2\), where \( I_1 = [1, q] = \{1, \ldots, q\} \) and \( I_2 = [q+1, n] = \{q+1, \ldots, n\} \). Therefore we have

\[
(m_1, m_2) = (q, n-q), \quad \text{and} \quad S^\lambda_n = S_q \times S_{n-q}.
\]

Thus the corresponding flag bundle is the Grassmann bundle \( \pi : G^q(E) \to X \) parametrizing rank \( q \) quotient bundles of \( E \).

Now we recall the useful formula for Grassmann bundle derived from the Jacobi-Trudi identity mentioned above. Let \( \pi : G^q(E) \to X \) be the Grassmann bundle parametrizing rank \( q \) quotient bundles of \( E \) as in Example 4.2 (2). On \( G^q(E) \), we have the tautological exact sequence of vector bundles:

\[
0 \to S \xrightarrow{\pi^*(E)} Q \to 0,
\]

where \( \text{rank } S = n - q \) and \( \text{rank } Q = q \). Set \( r := n - q \). Then by the repeated applications of the Jacobi-Trudi identity (4.1), Fulton-Pragacz [25 (4.2)] showed the following formula:[29]

\[
\pi^*(s_{\lambda}(Q) \cdot s_{\mu}(S)) = s_{\lambda_1 - r, \ldots, \lambda_r - r, \mu_1, \ldots, \mu_r}(E).
\]

In [4.2.2] below, we shall give a generalization of the above formula (4.2) as a special case of the corrected version of Fel’dman’s Gysin formula.

Let us recall Gysin formulas for more general flag bundles. The following proposition is a generalization of Theorem 2.8 to the partial flag bundle \( \eta_\lambda : F^\lambda(E) \to X \), which was proved by Pragacz [61] as a particular case of Brion [12, Proposition 2.1]:

**Proposition 4.3** (Pragacz [61], Proposition 5). For an \( S^\lambda_n \)-invariant polynomial \( f(x_1, \ldots, x_n) \in H^*(X)[x_1, \ldots, x_n]^{S^\lambda_n} \), we have

\[
(\eta_\lambda)_*(f(x_1, \ldots, x_n)) = \sum_{w \in S_n/S^\lambda_n} w \left[ \frac{f(x_1, \ldots, x_n)}{\prod_{1 \leq i < j \leq n, n(i) < n(j)} (x_i - x_j)} \right].
\]

Here the element \( f(x_1, \ldots, x_n) \) is regarded as an element in \( H^*(F^\lambda(E)) \to H^*(F^\lambda(E)) \)[30].

From this, Pragacz showed (implicitly) the following formula:

**Corollary 4.4.** For the Gysin homomorphism \( (\eta_\lambda)_* : H^*(F^\lambda(E)) \to H^*(X) \), the following formula holds:

\[
(\eta_\lambda)_* \left( x^\lambda \prod_{1 \leq i < j \leq n, n(i) < n(j)} (x_i - tx_j) \right) = P_\lambda(E; t),
\]

where the cohomology class \( P_\lambda(E; t) \) is defined from the Hall-Littlewood polynomial \( P_\lambda(x_1, \ldots, x_n, t) \) in the same way as \( s_\lambda(E) \) at the beginning of this subsection.

By Corollary 4.4 one can deduce Gysin formula for Schur \( P \)-polynomials (see Pragacz [61] Examples 2 and 11, Corollary 6):

---

[29] Note that in case the sequence \( (\lambda_1 - r, \ldots, \lambda_r - r, \mu_1, \ldots, \mu_r) \) is not a partition, the right-hand side is either 0 or \( \pm s_\mu(E) \) for some partition (see Fulton-Pragacz [25, p.42, Footnote]).

[30] Strictly speaking, this formula should be considered in \( H^*(F^\lambda(E)) \) via the pull-back \( H^*(X) \xrightarrow{\eta^* \lambda} H^*(F^\lambda(E)) \) (cf. Corollary 2.6). In what follows, we often use such abbreviation to simplify the presentation.
Corollary 4.5 (Pragacz [61], Corollary 6). In the setting as in Example 4.2 (1), the following formula holds:

\[(\tau^k)_* \left( x^\nu \prod_{1 \leq i \leq k, 1 \leq i < j \leq n} (x_i - tx_j) \right) = P_\nu(E; t). \]

Since we know that \( P_\nu(x_n; -1) = P_\nu(x_n) \) (see (3.6)), we obtain the following:

Corollary 4.6. With the above notation, the following formula holds:

\[(\tau^k)_* \left( x^\nu \prod_{1 \leq i \leq k, 1 \leq i < j \leq n} (x_i + x_j) \right) = P_\nu(E). \]

4.2. Universal Gysin formulas for the universal Schur functions.

4.2.1. Application of the Bressler-Evens formula. Since the Bressler-Evens formula (Theorem 2.8) is formulated in complex-oriented generalized cohomology theories, it can be applied especially to the complex cobordism theory \( MU^*(-) \). We use the same notation as in (2.6.2). For a complex vector bundle \( E \xrightarrow{p} X \) of rank \( n \), one can associate the \( MU^* \)-theory Chern classes \( c^\mu(E) \in MU^{2i}(X) \) (\( i = 1, 2, \ldots, n \)) and \( c_0^\mu(E) := 1 \). Put \( x_i = x_i^\mu := c_1^{\mu}(L_i) \in MU^2(F\ell(E)) \) (\( i = 1, 2, \ldots, n \)) (the \( MU^* \)-theory Chern roots of \( E \)). Then the \( MU^* \)-cohomology of \( F\ell(E) \) is given as follows (see e.g., Hornbostel-Kiritchenko [28, Theorem 2.6]):

\[ MU^*(F\ell(E)) = MU^*(X)[x_1, \ldots, x_n]/(1 + x_i) = c^{\mu}(E), \]

where \( c^{\mu}(E) = \sum_{i=0}^n c_i^{\mu}(E) \) is the total Chern class of \( E \). Then we obtain the following result, which is a universal analogue of Theorem 2.8

Theorem 4.7. With the same notation as in Theorem 2.8 we have

\[ \tau^* \circ \tau_*(f(x_1, \ldots, x_n)) = \sum_{w \in S_n} w \cdot \left[ \frac{f(x_1, \ldots, x_n)}{\prod_{1 \leq i < j \leq n} (x_i + tx_j)} \right] \]

for a polynomial \( f(X_1, \ldots, X_n) \in MU^*(X)[X_1, \ldots, X_n] \).

Thus the Gysin map \( \tau_* \) in the complex cobordism theory is also given by a certain symmetrizing operator which is a universal analogue of the Jacobi symmetrizer (see Theorem 2.8).

By Theorem 4.7 and (3.1), we obtain the following corollary, which is a universal analogue of Proposition 4.1

Corollary 4.8 (Characterization of the universal Schur functions). The image of the monomial \( x^{\lambda+\rho_n-1} = x_1^{\lambda_1+n-1}x_2^{\lambda_2+n-2}\ldots x_n^{\lambda_n} \) under the Gysin homomorphism \( \tau_* : MU^*(F\ell(E)) \to MU^*(X) \) is given by

\[ \tau_*(x^{\lambda+\rho_n-1}) = s_{\lambda}(E). \]

Here the characteristic class \( s_{\lambda}(E) \in MU^{2|\lambda|}(X) \) is defined by the same manner as \( s_{\lambda}(E) \in H^{2|\lambda|}(X) \).

31 For the historical reason, they are also called Conner-Floyd Chern classes (see Adams [1] Part I, §4], Conner-Floyd [15, Corollary 8.3]).
Remark 4.9. Since $\tau^*(s^\lambda_X(E)) = s^\lambda_X(x_n) \in MU^*(F\ell(E))$, and $\tau^*$ is injective, the above formula can be written as

$$\tau^*(x^{\lambda + p_{n-1}}) = s^\lambda_X(x_n).$$

More generally, the symmetrizing operator description of $\tau^*$ (Theorem 4.7) yields formally the following formula:

$$(4.3) \quad \tau^*(|x|^\lambda + p_{n-1}) = s^\lambda_X(x_n)|b).$$

Here $b = (b_1, b_2, \ldots)$ is a sequence of certain elements in $MU^*(X)$, which behaves as scalars with respect to $\tau^*$.

The universal analogue of Proposition 4.3 can also be obtained. Under the same setting as in Proposition 4.3, we have a classifying map $h : X \to BU(n)$ and its lift $\tilde{h} : F\ell^\lambda(E) \to B(U(m_1) \times U(m_2) \times \cdots \times U(m_d))$, and the following diagram is commutative:

$$\begin{array}{ccc}
F\ell^\lambda(E) & \xrightarrow{\tilde{h}} & B(U(m_1) \times U(m_2) \times \cdots \times U(m_d)) \\
\eta_\lambda & & \downarrow \sigma \\
X & \xrightarrow{h} & BU(n).
\end{array}$$

By Corollary 4.6 and the base-change property of Gysin maps, we obtain immediately the following generalization of Proposition 4.3.

Theorem 4.10. For an $S^\lambda_n$-invariant polynomial $f(X_1, \ldots, X_n) \in MU^*(X)[X_1, \ldots, X_n]^S^\lambda_n$, we have

$$(\eta_\lambda)_*(f(x_1, \ldots, x_n)) = \sum_{w \in S_n/S^\lambda_n} w \cdot \left[ \frac{f(x_1, \ldots, x_n)}{\prod_{1 \leq i < j \leq n, n(i) < n(j)} (x_i + \ell \tau_j)} \right].$$

From this, we have the following corollary which is a universal analogue of Corollary 4.6.

Corollary 4.11 (Characterization of the universal Hall-Littlewood functions). For the Gysin homomorphism $(\eta_\lambda)_* : MU^*(F\ell^\lambda(E)) \to MU^*(X)$, the following formula holds:

$$(\eta_\lambda)_* \left( x^\lambda \prod_{1 \leq i < j \leq n, n(i) < n(j)} (x_i + \ell \tau_j) \right) = H^\lambda_X(E; t).$$

For a strict partition $\nu \in \mathcal{SP}_n$, we saw that $H^\nu_X(x_n; -1) = P^\nu_X(x_n)$ at the end of [3.3]. From this and Corollary 4.11, we obtain the following corollary, which is a universal analogue of Corollary 4.6.

Corollary 4.12. With the same notation as in Corollary 4.6, we have

$$(\tau^k)_* \left( x^\nu \prod_{1 \leq i < j, 1 \leq i < j \leq n} (x_i + \ell \tau_j) \right) = P^\nu_X(E).$$

4.2.2. Fel’dman’s Gysin formula. As promised in the footnote after Remark 3.1 and we shall reformulate Fel’dman’s Gysin formula [22, Theorem 4.5] in the complex cobordism theory, and prove it. To this end, we use the universal Schur function $s^I_F(x_n)$ for a sequence of non-negative integers $I$ defined in (3.2). For two sequences $I = \ldots$
concerned with the induced Gysin map in the complex cobordism theory:

\[(I_1, I_2, \ldots, I_q), J = (J_1, J_2, \ldots, J_r)\] of non-negative integers, denote by \(IJ\) their juxtaposition, i.e.,

\[IJ := (I_1, I_2, \ldots, I_q, J_1, J_2, \ldots, J_r).\]

Let \(\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathcal{P}_n\) be a partition of length \(\leq n\), and rewrite it as \(\lambda = (n_1^{m_1}, n_2^{m_2}, \ldots, n_d^{m_d})\), \(n_1 > n_2 > \cdots > n_d \geq 0\), according to the decomposition \([n] = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_d\). Then consider the \((d-1)\)-step partial flag bundle

\[\eta_\lambda : \mathcal{F}^\lambda(E) \rightarrow X\]

associated with a complex vector bundle \(E \rightarrow X\) (for the notation, see \(\text{(1.1)}\)). We are concerned with the induced Gysin map in the complex cobordism theory:

\[(\eta_\lambda)_* : MU^*(\mathcal{F}^\lambda(E)) \rightarrow MU^*(X).\]

On \(\mathcal{F}^\lambda(E)\), we have the tautological sequence of flag of sub and quotient bundles:

\[S_0 = 0 \hookrightarrow S_1 \hookrightarrow \cdots \hookrightarrow S_r \hookrightarrow \cdots \hookrightarrow S_{d-1} \hookrightarrow S_d = \eta_\lambda^*(E),\]

\[\eta_\lambda^*(E) = Q^d \rightarrow Q^{d-1} \rightarrow \cdots \rightarrow Q^r \rightarrow \cdots \rightarrow Q^1 \rightarrow Q^0 = 0,\]

where, \(Q^r := \eta_\lambda^*(E)/S_{d-r}\) \((r = 1, 2, \ldots, d)\). Define the vector bundles over \(\mathcal{F}^\lambda(E)\) by

\[E_r := \text{Ker} (Q^r \rightarrow Q^{r-1}) \quad (r = 1, 2, \ldots, d).\]

Then we have \(\eta_\lambda^*(E) \cong \bigoplus_{r=1}^d E_r\), and the \(MU^*\)-theory Chern roots of \(E_r\) are given by \(x_{\nu(r)+1}, \ldots, x_{\nu(r)}\) \((r = 1, 2, \ldots, d)\). Here \(x_1, \ldots, x_n\) are the \(MU^*\)-theory Chern roots of \(E\) (see the beginning of \(\text{(4.2.1)}\)), and \(\nu(r) := \sum_{i=1}^r m_i\) for \(r = 1, 2, \ldots, d\) and \(\nu(0) := 0\) (see the end of \(\text{(3.3)}\)). With this notation, our version of Fel’dman’s Gysin formula is stated as follows:

**Theorem 4.13** (cf. Fel’dman \([22]\), Theorem 4.5). For sequences of non-negative integers \(I^{(r)} = (I_1^{(r)}, I_2^{(r)}, \ldots, I_m^{(r)})\) \((r = 1, 2, \ldots, d)\), we have

\[
(\eta_\lambda)_* \left( \prod_{r=1}^d s^L_{I^{(r)}+(n-\nu(r))m_r}(E_r) \right) = s^L_{\{I^{(1)}(1), \ldots, I^{(d)}(d)}(E).\]

Here the partition \(((n-\nu(r))m_r)\) means \((n-\nu(1), n-\nu(2), \ldots, n-\nu(d))\).

**Proof.** As in Fel’dman’s proof, it suffices to prove the assertion for the case of a Grassmann bundle. Therefore we may assume that the partition \(\lambda\) is of the form \(\lambda = (a^b b^{n-q})\) with \(a > b \geq 0\), and consider the associated Grassmann bundle \(\pi : G^a(E) \rightarrow X\) (see Example \(\text{(4.2)}\) (2)). On \(G^a(E)\), we have the tautological exact sequence of vector bundles:

\[0 \rightarrow S \hookrightarrow \pi^*(E) \rightarrow Q \rightarrow 0.\]

Note that \(E_1 = Q\), and \(E_2 = \text{Ker} (\pi^*(E) \rightarrow Q) \cong S\). For given two sequences of non-negative integers \(I = (I_1, I_2, \ldots, I_q), J = (J_1, J_2, \ldots, J_{n-q})\), consider the universal Schur function corresponding to the juxtaposition \(IJ = (I_1, \ldots, I_q, J_1, \ldots, J_{n-q})\):

\[s_{IJ}^n(x_1, \ldots, x_q, x_{q+1}, \ldots, x_n) = \sum_{w \in S_n} w \cdot \frac{x^{I+J+\rho_{n-1}}}{\prod_{1 \leq i < j \leq n}(x_i + x_j)}.\]
We decompose the denominator and the numerator inside the bracket in the right-hand side as
\[
\prod_{1 \leq i < j \leq n} (x_i + \under{x_j}) = \prod_{1 \leq i < j \leq q} (x_i + \under{x_j}) \times \prod_{q+1 \leq i < j \leq n} (x_i + \under{x_j}) \times \prod_{1 \leq i \leq q, q+1 \leq j \leq n} (x_i + \under{x_j}),
\]
\[
x^{IJ+\rho_n-1} = \prod_{i=1}^{q} x_i^{I_i+n-i} \times \prod_{i=q+1}^{n} x_i^{J_i+q-n-i} = (x_1 \cdots x_q)^{n-q} \times \prod_{i=1}^{q} x_i^{I_i+q-i} \times \prod_{i=q+1}^{n} x_i^{J_i+q-n-i}.
\]

Since \( \prod_{1 \leq i \leq q, q+1 \leq j \leq n} (x_i + \under{x_j}) \) and \((x_1 \cdots x_q)^{n-q}\) are both \( S_q \times S_{n-q}\)-invariant, we compute
\[
s^I_j(x_1, \ldots, x_q, x_{q+1}, \ldots, x_n) = \sum_{w \in S_n} w \cdot \left[ \frac{x^{IJ+\rho_n-1}}{\prod_{1 \leq i \leq q, q+1 \leq j \leq n} (x_i + \under{x_j})} \right]
\times \sum_{(u, v) \in S_q \times S_{n-q}} u \cdot \left[ \prod_{1 \leq i \leq q} x_i^{I_i+q-i} \right] \times v \cdot \left[ \prod_{q+1 \leq i \leq n} x_i^{J_i+q-n-i} \right]
\]
\[
= \sum_{w \in S_n} w \cdot \left[ \frac{(x_1 \cdots x_q)^{n-q} \cdot s^I_j(x_1, \ldots, x_q) \cdot s^I_j(x_{q+1}, \ldots, x_n)}{\prod_{1 \leq i \leq q, q+1 \leq j \leq n} (x_i + \under{x_j})} \right]
\]
\[
= \pi_\ast(s^{I+(n-q)v}_j(x_1, \ldots, x_q) s^I_j(x_{q+1}, \ldots, x_n))
\]
\[
= \pi_\ast(s^{I+(n-q)v}_j(x_1, \ldots, x_q) s^I_j(x_{q+1}, \ldots, x_n)).
\]

In the final step, we used the formula \(\pi_\ast((x_1 \cdots x_n)^c) \times s^I_j(x_n) = s^{I+(c\eta)}_j(x_n)\) for any non-negative integer \(c\), which follows immediately from the definition of \(s^I_j(x_n)\). Thus we have
\[
\pi_\ast(s^{I+(n-q)v}_j(x_1, \ldots, x_q) s^I_j(x_{q+1}, \ldots, x_n)) = s^I_j(x_1, \ldots, x_n).
\]

Rewriting this expression in terms of vector bundles, we have the desired formula:
\[
\pi_\ast(s^{I+(n-q)v}_j(E_1) s^I_j(E_2)) = s^I_j(E).
\]

\[
\square
\]

5. New universal factorial Schur functions

5.1. Definition of the new universal factorial Schur functions. In [3.3] we defined the universal Hall-Littlewood function \(H^I_\lambda(x_n; t)\) for a partition \(\lambda \in P_n\). We saw that for a strict partition \(\nu \in SP_n\), this function reduces to the universal Schur \(P\)-function \(P^I_\nu(x_n)\) under the specialization \(t = -1\). As announced in [3.3] we shall consider the specialization \(t = 0\) in this subsection. In fact, unlike the usual Hall-Littlewood polynomial \(P_\lambda(x_1, \ldots, x_n; t)\), the universal Hall-Littlewood function \(H^I_\lambda(x_n; t)\) need not coincide with the universal Schur function \(s^I_\lambda(x_n)\) under the specialization \(t = 0\). Thus one obtains another \(universal\) analogue of the Schur polynomial, which we call the “new universal (factorial) Schur function”. We shall use the same notation as in [3.3] Let \(\lambda = (\lambda_1, \ldots, \lambda_n) \in P_n\) be a partition of length \(\leq n\). Then we rewrite \(\lambda = (n_1^{m_1} \ n_2^{m_2} \ \cdots \ n_d^{m_d})\), \(n_1 > n_2 > \cdots > n_d \geq 0\) as before. Put
\( \nu(r) = \sum_{i=1}^{r} m_i \) for \( r = 1, 2, \ldots, d \) and \( \nu(0) = 0 \). Define

\[
(5.1) \quad (x|b)^{[\lambda]} := \prod_{r=1}^{d} \left( \prod_{\nu(r-1) < i \leq \nu(r)} [x_i|b]^{n_r + n - \nu(r)} \right).
\]

When the parameters \( b = 0 \), then we write simply \( x^{[\lambda]} \) for \( (x|0)^{[\lambda]} \). With the above notation, we make the following definition:

**Definition 5.1** (New universal factorial Schur functions). For a partition \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathcal{P}_n \), we define

\[
S_{\lambda}^L(x_n|b) = S_{\lambda}^L(x_1, \ldots, x_n|b) := \sum_{w \in S_n/S_n^L} w \cdot \left[ \frac{(x|b)^{[\lambda]}}{\prod_{1 \leq i < j \leq n, n(i) < n(j)} (x_i + L x_j)} \right].
\]

We also define

\[
S_{\lambda}^L(x_n) = S_{\lambda}^L(x_1, \ldots, x_n) := S_{\lambda}^L(x_1, \ldots, x_n|0).
\]

It follows immediately from Definition 5.1 and (3.7) that when \( t = 0 \), the universal Hall-Littlewood function \( H_{\lambda}^L(x_n; t) \) reduces to the new universal Schur function \( S_{\lambda}^L(x_n) \). Moreover, one sees directly from the definition that if we specialize the universal formal group law \( F_{\lambda}(u, v) \) to the additive one \( F_{n}(u, v) = u + v \) (and \( b_i = -a_i \) (\( i = 1, 2, \ldots, n \))), then the new universal factorial Schur function \( S_{\lambda}^L(x_n|b) \) reduces to the factorial Schur function \( s_{\lambda}(x_n|a) \) (see the proof of Macdonald [48, Chapter III, §1, (1.5)]). On the other hand, it is not obvious from Definition 5.1 that \( S_{\lambda}^L(x_n|b) \) reduces to the factorial Grothendieck polynomial \( G_{\lambda}(x_n|b) \) under the specialization from \( F_{\lambda}(u, v) \) to the multiplicative one \( F_{n}(u, v) = u + v + \beta uv \). We shall show that this is the case after establishing some Gysin formulas for the new universal factorial Schur functions (see §3.2). Thus the new universal factorial Schur functions are also universal analogues of the usual Schur functions. Note that, by definition, \( S_{\lambda}^L(x_n|b) = 1 \) for the empty partition \( \emptyset \), whereas \( s_{\lambda}^L(x_n|b) \neq 1 \) (see §3.2).

**Example 5.2.**

(1) Let \( \lambda \in \mathcal{P}_n \) be a partition such that \( \lambda_1 > \lambda_2 > \cdots > \lambda_{n-1} > \lambda_n \geq 0 \). Then we have a decomposition \([n] = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_n\), where \( I_1 = \{1\} \). Therefore we have

\[
(m_1, m_2, \ldots, m_n) = (1, 1, \ldots, 1) \quad \text{and} \quad S_{\lambda}^L = S_{1} \times \cdots \times S_{1} = \{1\}.
\]

In this case, one sees immediately from (5.1) that \( (x|b)^{[\lambda]} = [x|b]^{\lambda+n-1} \), and hence

\[
s_{\lambda}^L(x_n|b) = S_{\lambda}^L(x_n|b).
\]

Thus for such a distinct partition \( \lambda \), the function \( s_{\lambda}^L(x_n|b) \) coincides with the function \( S_{\lambda}^L(x_n|b) \). However, for a partition with equal parts, the difference does occur (see the end of §3.2).

(2) Let \( k \geq 1 \) be a positive integer, and consider the case where \( \lambda \) is “one-row”, namely \( \lambda = (k) = (k \ 0^{n-1}) \). Then we have a decomposition \([n] = I_1 \sqcup I_2\), where \( I_1 = \{1\}, I_2 = [2, n] = \{2, \ldots, n\} \). Thus we have

\[
(m_1, m_2) = (1, n-1) \quad \text{and} \quad S_{n}^L = S_{1} \times S_{n-1},
\]
and one sees immediately that $x^{[k]} = x_1^{k+1}$. Therefore the function $S_k^L(x_n) := S_k^L(x_n)$ corresponding to the one-row $(k)$ is given by

\[(5.2) \quad S_k^L(x_n) = \sum_{\varpi \in S_k \times S_{n-1}^{(k)}} w \cdot \left[ \frac{x_1^{k+n-1}}{\prod_{j=2}^n (x_1 + L \varpi_j)} \right] = \sum_{i=1}^n \frac{x_i^{k+n-1}}{\prod_{j \neq i} (x_i + L \varpi_j)}.
\]

Here we assumed that $k \geq 1$ is a positive integer. However, the right-hand side of (5.2) makes sense for $k \geq 1 - n$, which could be non-positive, or actually negative if $n \geq 2$. Therefore one can formally define $S_k^L(x_n)$ for each integer $k \geq 1 - n$ by the right-hand side of the above expression. For instance, one has

$$S_0^L(x_n) = \sum_{i=1}^n \frac{x_i^{n-1}}{\prod_{j \neq i} (x_i + L \varpi_j)}.$$  

which differs from $1^3$. Moreover, as we will see in the next subsection 5.3, one can define $S_k^L(x_n)$ for any integer $k \in \mathbb{Z}$.

**Remark 5.3.** In our previous paper [55 §4.5], we investigated various properties of the universal factorial Schur functions $s_k^L(x_n|b)$ such as “vanishing property”, “basis theorem”. The new universal factorial Schur functions $S_k^L(x_n|b)$ also have the similar properties. We shall discuss this problem elsewhere.

5.2. **Gysin formulas for the new universal Schur functions.** In this subsection, we shall establish Gysin formulas for the new universal Schur functions. Using these formulas, we are able to compare the new universal factorial Schur functions (“new” functions for short) $S_k^L(x_n|b)$ with the universal factorial Schur functions (“old” functions for short) $s_k^L(x_n|b)$ introduced in (3.2).

5.2.1. **Gysin formulas for the new universal Schur functions.** We begin with the following theorem: From Theorem 4.10, we have immediately

**Theorem 5.4 (Characterization of the new universal Schur functions).** For the Gysin homomorphism $(\eta_\lambda)_*: MU^*(\mathcal{F}_L^\lambda(E)) \rightarrow MU^*(X)$, the following formula holds:

$$(\eta_\lambda)_*(x^{[\lambda]}) = S^L_\lambda(x_n),$$

where $x_1, \ldots, x_n$ are the $MU^*$-theory Chern roots of $E$ as before. More generally, the following factorial version holds for a sequence $b = (b_1, b_2, \ldots)$ of certain elements in $MU^*(X)$:

$$(\eta_\lambda)_*((x|b)^{[\lambda]}) = S^L_\lambda(x_n|b).$$

**Example 5.5.**

1. Let us consider the “one-row” case $\lambda = (k)$ as in Example 5.2 (2). Thus the corresponding flag bundle is the projective bundle $\pi_1 : G^1(E) \rightarrow X$, and Theorem 5.4 implies the following formula:

$$\pi_1^*(x_1^{k+n-1}) = S^L_k(x_n).$$

2. Let $\lambda = (a, \ldots, a, b, \ldots, b) = (a^q b^{n-q}) \in \mathcal{P}_n$ be a partition of two rows with $a > b \geq 0$ as in Example 4.2 (2). Then the corresponding flag bundle is the Grassmann bundle $\pi : G^q(E) \rightarrow X$, and one sees directly that $x^{[\lambda]} = x_1^{a+n-q} \cdots x_q^{a+n-q} x_{q+1}^b \cdots x_n^b$, and Theorem 5.4 implies

$$\pi_*^*(x_1^{a+n-q} \cdots x_q^{a+n-q} x_{q+1}^b \cdots x_n^b) = S^L_{(a^q b^{n-q})}(x_n).$$

\[32\] Note that $S^L_0(x_n) = 1$ by definition.
(3) Let $\lambda \in \mathcal{P}_n$ be a distinct partition such that $\lambda_1 > \lambda_2 > \cdots > \lambda_{n-1} > \lambda_n \geq 0$ as in Example 5.2 (1). Then the corresponding partial flag bundle is the full flag bundle $\tau : \mathcal{F}(E) \to X$. In this case, one obtains that $x^{[\lambda]} = x^{\lambda+\rho_{n-1}}$, and Corollary 5.3 and Theorem 5.4 imply

$$s_\lambda^x(x_n) = \tau_n(x^{\lambda+\rho_{n-1}}) = S_\lambda^x(x_n),$$

which was already seen in Example 5.2 (1) (with $b = 0$).

5.2.2. Comparison of “new” functions $S_\lambda^x(x_n|b)$ with “old” functions $s_\lambda^x(x_n|b)$. More generally, the above Gysin formulas (Theorem 5.4) can be formulated as those between two partial flag bundles of the form “$\pi^\mu_\lambda : \mathcal{F}(\ell(E)) \to \mathcal{F}(\ell^\mu(E))$”, where the partition $\mu$ is a “refinement” of $\lambda$. Let us explain what the word “refinement” means: We use the notation in the beginning of this subsection (see also 3.3 and 3.4). For two positive integers $a, b$ with $a < b$, denote by $[a, b]$ the set of integers $i$ such that $a \leq i \leq b$. Let $\lambda, \mu \in \mathcal{P}_n$ be two partitions of length $\leq n$. Then as explained in 3.3, one obtains two decompositions

$$[1, n] = \bigcup_{\ell=1}^k I_\ell, \quad [1, n] = \bigcup_{\ell=1}^e J_\ell$$

of the interval $[1, n]$ corresponding to $\lambda, \mu$ respectively. Suppose that the decomposition $[1, n] = \bigcup_{\ell=1}^k I_\ell$ is a refinement of the decomposition $[1, n] = \bigcup_{\ell=1}^e J_\ell$ in the usual sense, i.e., $I_1 = J_1 \cup \cdots \cup J_{k_1}$, $I_2 = J_{k_1+1} \cup \cdots \cup J_{k_2}$, and so on, for some positive integers $1 \leq k_1 < k_2 < \cdots \leq e$. Then we say that $\mu$ is a “refinement” of $\lambda.$ By the construction of the associated partial flag bundle $\mathcal{F}(\ell(E))$ (see 3.4), we have a natural projection from $\mathcal{F}(\ell(E))$ to $\mathcal{F}(\ell^\mu(E))$, denoted by $\pi^\mu_\lambda$. Furthermore, for three partitions $\lambda, \mu, \nu \in \mathcal{P}_n$, if $\nu$ is a refinement of $\mu$, and $\mu$ is a refinement of $\lambda$, then $\nu$ is also a refinement of $\lambda$, and the relation $\pi^\mu_\lambda \circ \pi^\nu_\mu = \pi^\nu_\lambda$ holds. For example, the partition $\rho_{n-1} = (n-1, n-2, \ldots, 1, 0) \in \mathcal{P}_n$ is a refinement of any partition $\lambda \in \mathcal{P}_n$, and any partition $\lambda \in \mathcal{P}_n$ is a refinement of the empty partition $\emptyset = (0^n)$.

In particular, we are concerned with the following three projections and the induced Gysin homomorphisms in complex cobordism:

$$\pi^\rho_{n-1} : \mathcal{F}(\ell(E)) \to \mathcal{F}(\ell^\rho(E)) = X,$$

(5.4)

$$\pi^\lambda : \mathcal{F}(\ell^\mu(E)) \to \mathcal{F}(\ell^\lambda(E)) = X,$$

$$\pi^{\rho_{n-1}} : \mathcal{F}(\ell(E)) \to \mathcal{F}(\ell^\lambda(E)).$$

The theorem below (Theorem 5.6) will be useful in describing the difference between the “new” function $S_\lambda^x(x_n|b)$ and the “old” one $s_\lambda^x(x_n|b)$. Note that $\tau$ is written as a composite $\eta_\lambda \circ \pi^\rho_{n-1}$, and hence $\tau_* = (\eta_\lambda)_* \circ (\pi^\rho_{n-1})_*$. Theorem 5.6.

1. For the Gysin map $(\pi^\rho_{n-1})_* = \tau_* : MU^*(\mathcal{F}(\ell(E))) \to MU^*(X)$, the following formula holds:

$$((\pi^\rho_{n-1})_*)_*([x|b]^{\lambda+\rho_{n-1}}) = s_\lambda^x(x_n|b).$$

2. For the Gysin map $(\pi^\lambda)_* = (\eta_\lambda)_* : MU^*(\mathcal{F}(\ell^\lambda(E))) \to MU^*(X)$, the following formula holds:

$$((\pi^\lambda)_*)_*([x|b]^{[\lambda]}) = S_\lambda^x(x_n|b).$$

Note that even though $\mu$ is a “refinement” of $\lambda$, this does not necessarily mean $\mu \subset \lambda$. 31
(3) For the Gysin map \((\pi^p_{n-1})_* : MU^*(F\ell(E)) \to MU^*(F\ell^\lambda(E))\), the following formula holds:

\[
(\pi^p_{n-1})_*([x|b]^{\lambda+p_{n-1}}) = ([x|b]^{[\lambda]}, \prod_{r=1}^{d} s^{\nu}_q(x_{\nu(r-1)+1}, \ldots, x_{\nu(r)}|b|[+n_r+n-\nu(r)])).
\]

Here \(b = (b_1, b_2, \ldots)\) is a sequence of elements in \(MU^*(X)\), and \(b[m] := (b_{m+1}, b_{m+2}, \ldots)\) for a positive integer \(m\).

**Proof.** (1) and (2): The Gysin maps \((\pi^p_{n-1})_* = \tau_*\) and \((\pi^q_{\lambda})_* = (\eta)_*\) were already considered in (5.3) and (5.3) respectively.

(3) We shall show the assertion when the partition \(\lambda \in \mathcal{P}_n\) is of the form \(\lambda = (a^q b^{n-q})\) (see Example 5.6 (2)). In this case, one sees

\[
[x|b]^{\lambda+p_{n-1}} = \prod_{i=1}^{q} [x_i|b]^{a+n-i} \times \prod_{i=q+1}^{n} [x_i|b]^{b+n-i}
\]

\[
= \prod_{i=1}^{q} [x_i|b]^{a+n-q} \times \prod_{i=q+1}^{n} [x_i|b]^{b} \times \prod_{i=1}^{q} [x_i|b]^{+(a+n-q)}]^{q-i} \times \prod_{i=q+1}^{n} [x_i|b]^{[+]b}]^{n-i}
\]

Since \([x|b]^{[\lambda]}\) is \(S_{q} \times S_{n-q}\)-invariant, we have by the Bressler-Evens formula (Theorem 2.5)

\[
(\pi^p_{n-1})_*([x|b]^{\lambda+p_{n-1}}) = \sum_{w \in S_q \times S_{n-q}} w \cdot \left[ \prod_{1 \leq i < j \leq q} \frac{[x|b]^{\lambda+p_{n-1}}}{(x_i + \nu_{\lambda} x_j)} \times \prod_{q+1 \leq i < j \leq n} (x_i + \nu_{\lambda} x_j) \right]
\]

\[
= ([x|b]^{[\lambda]} \times \sum_{w \in S_q \times S_{n-q}} w \cdot \left[ \prod_{1 \leq i < j \leq q} \frac{[x|b]^{+(a+n-q)}]^{q-i} \times \prod_{i=q+1}^{n} [x_i|b]^{[+]b}]^{n-i} \right])
\]

\[
= ([x|b]^{[\lambda]} \times s^{\nu}_q(x_1, \ldots, x_q|b) \times s^{\nu}_q(x_{q+1}, \ldots, x_n|b) \times \prod_{1 \leq i < j \leq q} \frac{[x|b]^{[+]b}]^{n-i}}{\prod_{q+1 \leq i < j \leq n} (x_i + \nu_{\lambda} x_j)}
\]

and the theorem holds.

For an arbitrary partition \(\lambda \in \mathcal{P}_n\), write \(\lambda\) as the following form: \(\lambda = (n_1^{m_1} n_2^{m_2} \ldots n_d^{m_d})\), \(n_1 > n_2 > \ldots > n_d \geq 0\). Here \(m_i \geq 0\) for \(i = 1, 2, \ldots, d\), and \(\sum_{i=1}^{d} m_i = n\). Put \(\nu(r) = \sum_{i=1}^{d} m_i\) for \(r = 1, 2, \ldots, d\) and \(\nu(0) = 0\). Then analogous computation to the above case leads to the result of \((\pi^p_{n-1})_*([x|b]^{\lambda+p_{n-1}})\). Note that in the case where \(\lambda = (a^q b^{n-q})\), the “parameter shift” is given by \(a + n - q = n_1 + n - \nu(1)\) and \(b = n_2 + n - \nu(2)\).

With the aide of Theorem 5.6 we observe that

\[
S^L_{\lambda}(x_n|b) = \tau_*([x|b]^{\lambda+p_{n-1}}) = (\eta)_* \circ (\pi^p_{n-1})_*([x|b]^{\lambda+p_{n-1}})
\]

\[
= (\eta)_* \left( ([x|b]^{[\lambda]} \times \prod_{r=1}^{d} s^{\nu}_q(x_{\nu(r-1)+1}, \ldots, x_{\nu(r)}|b)\times [+
\]

Thus the difference between the “new” function \(S^L_{\lambda}(x_n|b)\) and the “old” one \(S^L_{\lambda}(x_n|b)\) is given by the product of “old” functions corresponding to the empty partition
∅. From this, “new” function $S_k^E(x_n|b)$ reduce to $G_\ell(x_n|b)$ under the specialization from $F_\ell(u,v)$ to $F_m(u,v)$ because of the fact $G_\ell(x_n|b) = 1$ (see e.g., Ikeda-Naruse [34 §2.4]).

5.3. Generating function for $S_k^E(x_n)$. As we mentioned in Example 5.2 (2), the new universal Schur functions $S_k^E(x_n) := S_{(k)}^E(x_n)$ for $k \geq 1$ can be extended to nonpositive integers. Thus one has the functions $S_k^E(x_n)$ for all integers $k \in \mathbb{Z}$. In this subsection, we shall give the generating function for these. For this purpose, we make use of Quillen’s result. Recall from Quillen [62] that the normalized invariant differential form\footnote{If we put $a_{i,j}^k = 0$ for all $i,j \geq 1$, then $\omega_\ell(t)$ reduces to $dt$. If we put $a_{1,1}^k = \beta$ and $a_{i,j}^k = 0$ for all $(i,j) \neq (1,1)$, then $\omega_\ell(t)$ reduces to $\frac{dt}{1+\beta t}$.} $\omega_\ell(t)$ associated with the universal formal group law $F_\ell(u,v) = u + v + \sum_{i,j \geq 1} a_{i,j}^\ell u^i v^j$ is defined by

$$\omega_\ell(t) := \frac{dt}{\partial F_\ell(t,0)} = \frac{dt}{1 + \sum_{i \geq 1} a_{i,1}^\ell t^i}.$$ 

The logarithm $\ell_\ell(x)$ of $F_\ell$ is then determined by the equations

$$\ell_\ell(x) dt = \omega_\ell(t), \quad \ell_\ell(0) = 0.$$ 

Then Quillen gave the following formula (see also Damon [16 p.650, Proposition]):

**Theorem 5.7** (Quillen [62], Theorem 1). Let $E \longrightarrow X$ be a complex vector bundle of rank $n$, let $\pi_1 : G^1(E) \cong P(E^\vee) \longrightarrow X$ be the associated projective bundle of lines in the dual $E^\vee$ of $E$, and let $Q \cong \mathcal{O}(1)$ be the canonical quotient line bundle on $G^1(E) \cong P(E^\vee)$. Then the Gysin homomorphism $\pi_1^* : MU^*(G^1(E)) \longrightarrow MU^*(X)$ is given by the residue formula

\begin{equation}
\pi_1^*(f(\xi)) = \text{Res}_{t=0} \frac{f(t) \omega_\ell(t)}{\prod_{i=1}^n (t + \xi_i)} = \text{Res}_{t=0} \frac{f(t) dt}{(1 + \sum_{i \geq 1} a_{i,1}^\ell t^i) \prod_{i=1}^n (t + \xi_i)}.
\end{equation}

Thus $\pi_1^*(f(\xi))$ is given by the coefficient of $t^{-1}$ in the Laurent series

$$\frac{f(t)}{(1 + \sum_{i \geq 1} a_{i,1}^\ell t^i) \prod_{i=1}^n (t + \xi_i)}.$$ 

Here $f(t) \in MU^*(X)[t]$, $\xi := c_1^{MU}(Q) \in MU^2(G^1(E))$, and $x_1 = \xi, x_2, \ldots, x_n$ are the $MU^*$-theory Chern roots of $E$.

On the other hand, in the same setting as in Examples 5.2 and 5.3 together with Theorem 4.10, we obtain

\begin{equation}
\pi_1^*(f(\xi)) = \sum_{i=1}^n \frac{f(x_i)}{\prod_{j \neq i} (x_i + \xi_j)}.
\end{equation}

The equivalence of (5.6) and (5.7) is shown by Vishik [70] Lemma 5.36]. Therefore if we take $f(t) = t^{k+n-1}$ $(k \geq 1)$, we have by Example 5.2 (2),

\begin{equation}
S_k^E(x_n) = \sum_{i=1}^n \frac{x_i^{k+n-1}}{\prod_{j \neq i} (x_i + \xi_j)} = \pi_1^*(\xi^{k+n-1})
\end{equation}

$$= \text{Res}_{t=0} \frac{t^{k+n-1} dt}{(1 + \sum_{i \geq 1} a_{i,1}^\ell t^i) \prod_{i=1}^n (t + \xi_i)}.$$
Thus the coefficient of \( t^{-1} \) in the formal Laurent series
\[
\frac{t^{k+n-1}}{(1 + \sum_{i \geq 1} a_{i,1}^L t^i) \prod_{i=1}^n (t + x_i)}
\]
is equal to \( S^L_k(x_n) \).

**Remark 5.8.** It should be remarked that the above residue formula is closely related to the recent work of Darondeau-Pragacz \cite{18}. The following comment might be useful when one wants to relate the residue formula to their work. We borrow the notation from \cite{18} §0. For a Laurent series \( f(t) \) in one indeterminate \( t \), we shall denote by \([t^k](f(t))\) the coefficient of \( t^k \) in \( f(t) \). The above residue formula means
\[
S^L_k(x_n) = [t^{-1}] \left( \frac{t^{k+n-1}}{(1 + \sum_{i \geq 1} a_{i,1}^L t^i) \prod_{i=1}^n (t + x_i)} \right)
\]

If one wants to consider the ordinary cohomology theory, we put \( a_{i,j}^L = 0 \) for all \( i, j \geq 1 \). Then the above formula leads to
\[
s_k(x_n) = [t^{n-1}] \left( \frac{t^{k+n-1}}{\prod_{i=1}^n (t - x_i)} \right).
\]

Here the left-hand side is the usual Schur polynomial corresponding to the one row \( (k) \), namely, the \( k \)-th homogeneous complete symmetric polynomial. We see easily that the rational function \( \prod_{i=1}^n (t - x_i) \) in the right-hand side is the “reversed Segre polynomial” \( s_{1/t}(E) \) of \( E \) if we think the variables \( x_1, \ldots, x_n \) the Chern roots of a complex vector bundle \( E \). This is the fundamental formula given in \cite{18} §0, p.2 \( ^{35} \)

From the above interpretation, we can obtain the generating function for \( S^L_k(x_n) \) \((k \in \mathbb{Z})\). We argue as follows: Set
\[
F_n(t) := \frac{t^n}{(1 + \sum_{i \geq 1} a_{i,1}^L t^i) \prod_{i=1}^n (t + x_i)}.
\]

Then it follows from the above interpretation that the coefficient of \( t^{-k} \) of \( F_n(t) \) is equal to \( S^L_k(x_n) \) for each positive integer \( k \geq 1 \). By defining \( S^L_k(x_n) \) for \( k \leq 0 \) by the same procedure, one obtains
\[
F_n(t) = \sum_{k \in \mathbb{Z}} S^L_k(x_n) t^{-k}.
\]

Thus we have the following:

**Theorem 5.9.** The generating function for \( S^L_k(x_n) \) \((k \in \mathbb{Z})\) is given by
\[
(5.9) \quad \sum_{k \in \mathbb{Z}} S^L_k(x_n) u^k = F_n(t)|_{t = u^{-1}} = \frac{u^{-n}}{(1 + \sum_{i \geq 1} a_{i,1}^L u^{-i}) \prod_{i=1}^n (u^{-1} + x_i)}.
\]

**Remark 5.10.**
\(^{35}\) Since Darondeau-Pragacz uses the projective bundle of lines \( G_1(E) = P(E) \rightarrow X \), not \( G^1(E) \cong P(E^*) \), in their formulation, one has to change \( E \) in their result to its dual \( E^* \).
(1) In Hudson-Matsumura \([21]\) Definition 3.1, the Segre class in the algebraic cobordism theory \(\mathcal{S}_m(E)\) of a complex vector bundle \(E\) is defined by using the push-forward image from the projective bundle \(G^1(E) \cong P(E^\vee)\)\(^{36}\). By definition, these classes coincide with our \(\mathcal{S}_m(E)\). Furthermore, they obtained the generating function \(\mathcal{S}(E; u) := \sum_{m \in \mathbb{Z}} \mathcal{S}_m(E) u^m\) of the Segre classes \(\mathcal{S}_m\) \([29]\, \text{Theorem 3.6}\) by a different method from ours. One can check directly that their result coincides with our Theorem 5.9 as for their arguments, readers are recommended to consult Hudson-Ikeda-Matsumura-Naruse \([30]\, \text{§3.1}\). In that paper, the K-theoretic Segre classes are introduced by the same way as above, and the generating function of the stable Grothendieck polynomials is given \([30]\, \text{Theorem 3.2, Appendix 8}\). Their result can also be obtained from Theorem 5.9 by the specialization \(a_{1,1}^L = \beta\) and \(a_{1,j}^L = 0\) for all \((i, j) \neq (1, 1)\).

(2) Since our formula \((5.9)\) is universal, one can obtain the generating function of the \(h^*-\text{theory Segre classes}\) of vector bundle by sepecializing the universal formal group law \(F_l(u, v)\) to the formal group law \(F_h(u, v)\) corresponding to a given complex-oriented cohomology theory \(h^*(-)\). For instance, we recently obtained a concrete expression of the “Elliptic Schur functions” corresponding to a cohomology theory, denoted \(SE^*(-)\), whose formal group law is that of a singular cubic curve in Weierstrass form, called hyperbolic (see Lenart-Zainoulline \([15]\, \text{[46]}\)).

5.4. Application of the Gysin formulas for the new universal Schur functions -Thom-Porteous formula for the complex cobordism theory-. Using the new universal Schur functions, one can formulate the Thom-Porteous formula (2.17) in the universal setting. We use the same notation as in \([2.6.4]\) As explained in that subsection, in order to obtain the class determined by \(D_r(\varphi)\), we have to compute the image \(\pi_F^*(c_{e(f-r)}(\pi_F^*(E)^\vee \otimes Q_F))\). Let \(x_1, \ldots, x_f\) (resp. \(b_1, \ldots, b_e\)) be the \(MU^*\)-theory Chern roots of \(F\) (resp. \(E\)). The Chern roots of \(Q_F\) are \(x_1, \ldots, x_{f-r}\). Let \(\lambda = ((e-r)(f-r))\) be the rectangular partition with \((f-r)\) rows and \((e-r)\) columns as in \([2.6.4]\). Then by the splitting principle, the top Chern class is given by

\[
c_{e(f-r)}(\pi_F^*(E)^\vee \otimes Q_F) = \prod_{i=1}^{f-r} \prod_{j=1}^{e} (x_i + \ell \bar{b}_j) = \prod_{i=1}^{f-r} [x_i | \bar{b}_e]^\ell = (x | \bar{b}_e)^{[\lambda]}.
\]

Here \(\bar{b}_e = (\bar{b}_1, \ldots, \bar{b}_e, 0, 0, \ldots)\). Therefore by \((5.3)\) (see also Theorem 5.4 and Example 5.5 (2)), one obtains\(^{37}\)

\[
\pi_F^*(c_{e(f-r)}(\pi_F^*(E)^\vee \otimes Q_F)) = \pi_F^*(x | \bar{b}_e)^{[\lambda]} = S^L_{\lambda}(x_f | \bar{b}_e).
\]

**Theorem 5.11** (Thom-Porteous formula for the complex cobordism theory). If the codimension of \(D_r(\varphi)\) is \((e-r)(f-r)\), then the class determined by \(D_r(\varphi)\) is given by the new universal factorial Schur function \(S^L_{(e-r)(f-r)}(x_f | \bar{b}_e)\).

5.5. Application of the Gysin formulas for the new universal Schur functions -Class of Damon’s resolution-. As we mentioned in the introduction \([11.2]\) it is well-known that the usual Schur polynomials \(s_\lambda(x_d)\), with \(\lambda\) contained in the

\(^{36}\) For the \(K\)-theoretic analogue of the Segre classes of a vector bundle, see Buch \([14]\, \text{Lemma 7.1}\).

\(^{37}\) Since we have \(S^L_{(e-r)}(x_f | \bar{b}_e) = \prod_{i=1}^{f-r} [x_i | \bar{b}_e]^\ell\), this formula can be written as

\[
\pi_F^*(S^L_{(e-r)}(x_f | \bar{b}_e)) = S^L_{(e-r)(f-r)}(x_f | \bar{b}_e).
\]
rectangular partition \( ((n - d)^d) \), represent the Schubert classes in the ordinary cohomology ring \( H^*(G_d(\mathbb{C}^n)) \) of the complex Grassmannian \( G_d(\mathbb{C}^n) \) of \( d \)-dimensional linear subspaces in \( \mathbb{C}^n \) (see, e.g., Fulton [24, §9.4]). In this subsection, by making use of the Damon’s resolution of the Schubert varieties (Damon [17, §3, p.258]), together with our Gysin formula (5.3), we shall show that the new universal Schur functions here classes” in the complex cobordism ring \( MU^*(G_d(\mathbb{C}^n)) \), give the “correct” representatives of the “Schubert classes” in the complex cobordism ring \( MU^*(G_d(\mathbb{C}^n)) \).

5.5.1. Bundle of Schubert varieties. In fact, we can formulate our assertion in more general situation: Let \( E \longrightarrow X \) be a complex vector bundle of rank \( n \) over a variety \( X \). For a positive integer \( 1 \leq d \leq n - 1 \), consider the associated Grassmann bundle \( \pi : G_d(E) \longrightarrow X \) of \( d \)-dimensional subspaces in the fibers of \( E \). On \( G_d(E) \), one has the tautological exact sequence of vector bundles

\[
0 \longrightarrow S \longrightarrow \pi^*(E) \longrightarrow Q \longrightarrow 0,
\]

where rank \( S = d \) and rank \( Q = n - d \). Suppose that we are given a complete flag of subbundles \( F_* : 0 = F_0 \subset F_1 \subset \cdots \subset F_{d-1} \subset F_d = E \), where rank \( F_i = i \) \((i = 0, 1, 2, \ldots, n)\). For any partition \( \lambda = (\lambda_1, \ldots, \lambda_d) \subset ((n - d)^d) \), the bundle of Schubert varieties \( \Omega_\lambda(F_*) \subset G_d(E) \) is defined by the so-called “Schubert conditions”, namely \( \Omega_\lambda(F_*) := \{ W \in G_d(E) \mid W \subset E_x \ (x \in X), \ \text{dim} \ (W \cap (F_{n-i+1} \cdot \lambda_1)_x) \geq i \ (i = 1, 2, \ldots, d) \}. \)

5.5.2. Damon’s resolution of singularities of \( \Omega_\lambda(F_*) \). Following Damon [17, §3, p.258], we will construct a resolution of singularities (or desingularization) of \( \Omega_\lambda(F_*) \) (with a slight modification). For a partition \( \lambda = (\lambda_1, \ldots, \lambda_d) \subset ((n - d)^d) \), we have a decomposition \( [d] = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_k \) as in [38]. Putting \( m_i := \sharp I_i \) \((i = 1, 2, \ldots, k)\), we can rewrite

\[
\lambda = (n_1^{m_1}, n_2^{m_2}, \ldots, n_k^{m_k}) \quad (n - d \geq n_1 > n_2 > \cdots > n_k \geq 0).
\]

We put \( \nu(p) := \sum_{i=p}^n m_i \) \((p = 1, 2, \ldots, k)\) and \( \nu(0) := 0 \). Since the variety \( \Omega_\lambda(F_*) \) is determined by the Schubert conditions corresponding to “outside corners” of the Young diagram of \( \lambda \) (see Fulton [24, §9.4, Exercise 18]), one can also define \( \Omega_\lambda(F_*) \) by

\[
\Omega_\lambda(F_*) = \{ W \in G_d(E) \mid W \subset E_x \ (x \in X), \ \text{dim} \ (W \cap (F_{n-d+p-\nu(p)-p})_x) \geq \nu(p) \ (p = 1, 2, \ldots, k) \}.
\]

Now consider the partial flag bundle \( \varpi : F_\ell(\nu_1(1), \nu_2(2), \ldots, \nu_{k-1}(k-1))(S) \longrightarrow G_d(E) \) associated with the tautological subbundle \( S \) over \( G_d(E) \). The fiber over a point \( W \in G_d(E) \) consists of partial flags in a \( d \)-dimensional linear subspace \( W \subset E_x \) for some point \( x \in X \), namely a nested sequence of linear subspaces of the form

\[
W_1 \subset W_2 \subset \cdots \subset W_{k-1} \subset W_k = W,
\]

where \( \text{dim} \ W_p = \nu(p) \) \((p = 1, \ldots, k)\). On \( F_\ell(\nu_1(1), \nu_2(2), \ldots, \nu_{k-1}(k-1))(S) \), one has the tautological sequence of flag of sub and quotient bundles

\[
D_1 \hookrightarrow D_2 \hookrightarrow \cdots \hookrightarrow D_p \hookrightarrow \cdots \hookrightarrow D_{k-1} \hookrightarrow D_k = \varpi^*(S),
\]

\[
\varpi^*(S) \twoheadrightarrow Q_1 \twoheadrightarrow Q_2 \twoheadrightarrow \cdots \twoheadrightarrow Q_p \twoheadrightarrow \cdots \twoheadrightarrow Q_{k-1} \twoheadrightarrow Q_k = 0,
\]

[38] cf. Damon [17 §1, p.251], Hudson-Ikeda-Matsumura-Naruse [38 §4.1], Daroneau-Pragacz [19 §1.1].
where rank $D_p = \nu(p)$, and $Q_p$ is defined by $Q_p := \varpi^*(S)/D_p (p = 1, 2, \ldots, k)$. Then the partial flag bundle $\mathcal{F}_{\nu(1), \nu(2), \ldots, \nu(k-1)}(S)$ is constructed as a tower of Grassmann bundles (here we omit the pull-back notation of vector bundles as is customary):

$$
\mathcal{F}_{\nu(1), \nu(2), \ldots, \nu(k-1)}(S) = G_{m_k}(Q_{k-1}) \xrightarrow{\varpi_k} G_{m_{k-1}}(Q_{k-2}) \xrightarrow{\varpi_{k-1}} \cdots \xrightarrow{\varpi_1} G_{m_1}(Q_1) \xrightarrow{\varpi_0} G_d(S) \xrightarrow{\varpi} G_d(E).
$$

The natural projection $\varpi = \varpi_1 \circ \varpi_2 \circ \cdots \circ \varpi_k : \mathcal{F}_{\nu(1), \nu(2), \ldots, \nu(k-1)}(S) \to G_d(E)$ sends a point $(0 \subset W_1 \subset \cdots \subset W_{k-1} \subset W_k = W) \in \mathcal{F}_{\nu(1), \nu(2), \ldots, \nu(k-1)}(S)$ to the point $W \in G_d(E)$. Note that the tautological exact sequence of vector bundles over the Grassman bundle $G_{m_p}(Q_{p-1})$ is regarded as

$$
0 \to D_p/D_{p-1} \hookrightarrow Q_{p-1} \to Q_p \to 0.
$$

We then define a subvariety $X_k$ of $\mathcal{F}_{\nu(1), \nu(2), \ldots, \nu(k-1)}(S)$ by

$$
X_k := \{(0 \subset W_1 \subset \cdots \subset W_{k-1} \subset W_k = W) \in \mathcal{F}_{\nu(1), \nu(2), \ldots, \nu(k-1)}(S) \mid W \subset E_x (x \in X), W_p \subset (F_{n-d+\nu(p)-n_p})_x (p = 1, 2, \ldots, k) \}. \tag{5.10}
$$

By definition, a point $(0 \subset W_1 \subset \cdots \subset W_{k-1} \subset W_k = W) \in X_k$ satisfies the conditions $W_p \subset (F_{n-d+\nu(p)-n_p})_x$, and hence $W \cap (F_{n-d+\nu(p)-n_p})_x \supset W_p$ for $p = 1, 2, \ldots, k$. Therefore its image $W \in G_d(E)$ under the projection $\varpi$ satisfies the conditions $\dim (W \cap (F_{n-d+\nu(p)-n_p})_x) \geq 1 \nu(p)$ ($p = 1, 2, \ldots, k$). Thus $W$ is in $\Omega_\lambda(F_s)$ by the definition of $\Omega_\lambda(F_s)$. The map $\varpi | X_k : X_k \to \Omega_\lambda(F_s)$ is a resolution of singularities of $\Omega_\lambda(F_s)$ constructed by Damon.

### 5.5.3. Class of Damon’s resolution

Having constructed the resolution $X_k$ of $\Omega_\lambda(F_s)$, we then define the **Damon class** $\delta_\lambda \in MU^*(G_d(E))$ associated to a partition $\lambda \subset ((n-d)^d)$ as the push-forward image $\varpi_*([X_k])$ of the class $[X_k] \in MU^*(\mathcal{F}_{\nu(1), \nu(2), \ldots, \nu(k)}(S))$. One can compute the class $[X_k]$ explicitly by the standard fact about the top Chern class of a vector bundle (see e.g., Quillen [63], Levine-Morel [47], Lemma 6.6.7). The vector bundle homomorphism $D_p/D_{p-1} \to E \to E/F_{n-d+\nu(p)-n_p}$ over $\mathcal{F}_{\nu(1), \nu(2), \ldots, \nu(k-1)}(S)$ defines a section $s_p$ of the vector bundle Hom $(D_p/D_{p-1}, E/F_{n-d+\nu(p)-n_p}) \cong (D_p/D_{p-1})^\nu \otimes E/F_{n-d+\nu(p)-n_p}$ for $p = 1, 2, \ldots, k$, and the conditions $W_p \subset (F_{n-d+\nu(p)-n_p})_x$ for $p = 1, 2, \ldots, k$ means that this section $s_p$ vanishes. Thus the variety $X_k \subset \mathcal{F}_{\nu(1), \nu(2), \ldots, \nu(k)}(S)$ is given by the zero locus of the section $s = \bigoplus_{p=1}^k s_p$ of the vector bundle $\bigoplus_{p=1}^k (D_p/D_{p-1})^\nu \otimes E/F_{n-d+\nu(p)-n_p}$. Therefore the class $[X_k]$ is given by the top Chern class of this vector bundle, that is,

$$
[X_k] = \prod_{p=1}^k c_{m_p+n_p+d-\nu(p)}(x) \cdot (D_p/D_{p-1})^\nu \otimes E/F_{n-d+\nu(p)-n_p}).
$$

In order to proceed with the computation, we consider the full flag bundle $\mathcal{F}_{l, 1, 2, \ldots, d-1}(S) = \mathcal{F}(S) \to G_d(E)$. On $\mathcal{F}(S)$, one has the tautological sequence of flag of subbundles

$$
S_1 \hookrightarrow S_2 \hookrightarrow \cdots \hookrightarrow S_i \hookrightarrow \cdots \hookrightarrow S_{d-1} \hookrightarrow S_d = S,
$$

where rank $S_i = i$ ($i = 1, 2, \ldots, d$). By the natural projection $\mathcal{F}(S) \to \mathcal{F}_{\nu(1), \nu(2), \ldots, \nu(k)}(S)$, the bundle $D_p$ over $\mathcal{F}_{\nu(1), \nu(2), \ldots, \nu(k)}(S)$ is pulled-back to $\bigoplus_{i=1}^{\nu(p)} S_i$. Now we set

$$
y_i := c_1^{MU}(S_i/S_{i-1})^\nu \quad (i = 1, 2, \ldots, d).
$$

---

\[\text{Damon’s resolution is one of the \textit{small resolution} of singularities of a Schubert variety constructed by Zelevinsky [71].}\]
These are the $MU^*$-theory Chern roots of the dual bundle $S^\vee$. Also we set
\[ b_i := c_i^{MU}(F_{a_{i+1}}/F_{a_{i}}) \in MU^*(X) \ (i = 1, 2, \ldots, n). \]
These are the $MU^*$-theory Chern roots of $E$. Then by the splitting principle, we have
\[ [X_k] = \prod_{p=1}^{k} c_{m_p+n_p+d-\nu(p)}^{MU}((D_p/D_{p-1})^\vee \otimes E/F_{n-d+\nu(p)-n_p}) = \prod_{p=1}^{k} \prod_{\nu(p)-1 \leq \nu(p)(1 \leq p \leq n_p+d-\nu(p))} (y_i + b_j) \]
\[ = \prod_{p=1}^{k} \prod_{\nu(p)-1 \leq \nu(p)} [y_i | b_n]^{m_p+n_p+d-\nu(p)} = (y | b_n)^{[\lambda]}. \]
Here $b_n = (b_1, \ldots, b_n, 0, 0, \ldots)$. Thus we have
\[ \delta_\lambda = \varpi_* ((y | b_n)^{[\lambda]}). \]

Now, for a partition
\[ \lambda = (n_1 \geq n_2 \geq \cdots \geq n_k \geq 0), \quad n - d \geq n_1 > n_2 > \cdots > n_k \geq 0, \]
consider the partial flag bundle $\eta_\lambda : \mathcal{F}^\ell(S^\vee) \rightarrow G_d(E)$ associated with the vector bundle $S^\vee \rightarrow G_d(E)$. Since we have
\[ \mathcal{F}^\ell(S^\vee) = \mathcal{F}^{\ell - m_k - \cdots - m_2}(S^\vee) = \mathcal{F}^{\ell - m_k - \cdots - m_2}(S^\vee) \]
\[ \cong \mathcal{F}^{\ell - m_1 - \cdots - m_2}(S) = \mathcal{F}^{\ell - \nu(1), \nu(2), \ldots}(S), \]
the partial flag bundle $\varpi : \mathcal{F}^{\ell - \nu(1), \nu(2), \ldots}(S) \rightarrow G_d(E)$ is identified with the partial flag bundle $\eta_\lambda : \mathcal{F}^\ell(S^\vee) \rightarrow G_d(E)$. Therefore computing $\varpi_* ((y | b_n)^{[\lambda]})$ is equivalent to computing $(\eta_\lambda)_* ((y | b_n)^{[\lambda]})$. Our Gysin formula (5.3) then yields the following:

**Theorem 5.12.** The Damon class $\delta_\lambda$ is represented by the new universal factorial Schur function. More precisely, we have
\[ \delta_\lambda = S^\ell_\lambda (y_d | b_n) \in MU^*(G_d(E)). \]

By Theorem 5.12, one may call the new universal factorial Schur function $S^\ell_\lambda (x_n | b)$ the universal factorial Schur function of Damon type.

5.6. **Concluding remarks and related work.** So far, we have considered two types of universal analogues of usual Schur polynomials, namely, the universal factorial Schur functions $S^\ell_\lambda (x_n | b)$ and the new universal factorial Schur functions or the universal factorial Schur functions of Damon type $S^\ell_\lambda (x_n | b)$. In closing this paper, we shall introduce another type of universal analogue of Schur polynomials, which was essentially introduced by Hudson-Matsumura [29, Definition 4.1], and state briefly our new results. Details will be discussed in our forthcoming paper [50]. Hudson-Matsumura introduced the class $\kappa_\lambda$ (in the algebraic cobordism ring $\Omega^*(G_d(E))$) using the Kempf-Laksov resolution (see op. cit. [29, §4.2], Kempf-Laksov [11]) of the Schubert varieties. They call the class $\kappa_\lambda$ the Kempf-Laksov class. Using the Gysin formula (5.3) and Theorem 4.10 we are able to show that the Kempf-Laksov class $\kappa_\lambda$ is given explicitly by (here $r$ is the length of $\lambda$)
\[ \kappa_\lambda = (\eta_{\nu_r})_* ([y | b_n]^{\lambda + \rho_{r-1} + (d-r)^r}) \]
\[ = \sum_{\sigma \in S_d(S_{\lambda})} \omega \cdot \left[ \left[ \frac{y | b_n}{\prod_{i \leq r} \prod_{i < j \leq d} (y_i + \lambda_j)} \right] \right]. \]

Motivated by this formula, we make the following definition:
Definition 5.13 (Universal factorial Schur functions of Kempf-Laksov type\textsuperscript{[40]}. For a partition $\lambda \in \mathcal{P}_n$ with length $\ell(\lambda) = r \leq n$, we define

$$s^{KL}_{\lambda}(x_1, \ldots, x_r) := \sum_{w \in S_n/(S_1)^r \times S_{n-r}} w \cdot \left[ \frac{[x|b]^{\lambda + \rho_{r-1} + (n-r)r}}{\prod_{1 \leq i \leq r} \prod_{i < j \leq n} (x_i + \ell \cdot x_j)} \right]$$

$$= \sum_{w \in S_n/(S_1)^r \times S_{n-r}} w \cdot \left[ \frac{\prod_{i=1}^r [x|b]^{\lambda_i + n - i + n}}{\prod_{1 \leq i \leq r} \prod_{i < j \leq n} (x_i + \ell \cdot x_j)} \right].$$

We also define

$$s^{KL}_{\lambda}(x_n) := s^{KL}_{\lambda}(x_1, \ldots, x_n|0).$$

From Theorem 4.10, we obtain the following theorem:

Theorem 5.14 (Characterization of the universal Schur functions of K-L type). For the Gysin homomorphism $(\tau^*)_* : MU^*(\mathcal{F}(r-1, \ldots, 2, 1)(E)) \longrightarrow MU^*(X)$, the following formula holds:

$$(\tau^*)_*(x^{\lambda + \rho_{r-1} + (n-r)r}) = s^{KL}_{\lambda}(E).$$

Here $x_1, \ldots, x_n$ are the $MU^*$-theory Chern roots of $E$ as in \textsuperscript{[41,2,1]}

On the other hand, following Hudson-Matsumura \textsuperscript{[29] Definition 3.1}, we define the $MU^*$-theory $k$-th Segre class $\mathcal{S}_k(E)$ of a complex vector bundle $E$ to be

$$(5.12) \quad \mathcal{S}_k(E) := \pi_{1*}(x_1^{k+n-1})$$

in the same setting as in Theorem 5.1. Let

$$\mathcal{S}_t(E) = \mathcal{S}_t(E; t) = \sum_{k \in \mathbb{Z}} \mathcal{S}_k(E)t^k$$

be the Segre series of $E$ in the complex cobordism theory. Then by Example 5.5 (1) and (5.12), we see immediately that $S_k(E) = \mathcal{S}_k(E)$, and therefore the Segre series $\mathcal{S}_t(E)$ is given by Theorem 5.9. Using the Segre series $\mathcal{S}_t(E)$, the universal push-forward formula for full flag bundle of type $A$ established by Darondeau-Pragacz \textsuperscript{[18]} pp.4–5\textsuperscript{[41]} can be generalized to the complex cobordism theory.

Theorem 5.15 (Darondeau-Pragacz formula in complex cobordism). In the same setting as in Theorem 5.14 one has

$$(\tau^*)_*(f(x_1, \ldots, x_r)) = \left[ t_1^{n-1} \cdots t_r^{n-1} \right] \left( f(t_1, \ldots, t_r) \prod_{1 \leq i < j \leq r} (t_j + \ell t_i) \prod_{1 \leq i \leq r} \mathcal{S}_{1/t_i}(E) \right)$$

for a polynomial $f(X_1, \ldots, X_r) \in MU^*(X)[X_1, \ldots, X_r]$.\textsuperscript{40,41}

This formula, together with Theorem 5.14 enables us to obtain the generating function of the universal Schur functions of K-L type (see Nakagawa-Naruse \textsuperscript{[56]}).

\textsuperscript{40} “K-L type” for short.

\textsuperscript{41} Note that this type of Gysin formula is also obtained by Ilori \textsuperscript{[35]} p.623, Theorem, Kaji-Terasoma \textsuperscript{[33]} Theorem 0.4 (Push-Forward Formula)].


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