FOURIER EXPANSIONS OF VECTOR-VALUED AUTOMORPHIC
FUNCTIONS WITH NON-UNITARY TWISTS

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Abstract. We provide Fourier expansions of vector-valued eigenfunctions
of the hyperbolic Laplacian that are twist-periodic in a horocycle direction.
The twist may be given by any endomorphism of a finite-dimensional vector
space; no assumptions on invertibility or unitarity are made. Examples of such
eigenfunctions include vector-valued twisted automorphic forms of Fuchsian
groups. We further provide a detailed description of the Fourier coefficients
and explicitly identify each of their constituents, which intimately depend on
the eigenvalues of the twisting endomorphism and the size of its Jordan blocks.
In addition, we determine the growth properties of the Fourier coefficients.

Contents

1. Introduction 1
2. Essential objects and the refined statement of Theorem 1.2 14
3. Complexity reduction 28
4. Periodization 29
5. Differential equations for the Fourier coefficient functions 32
6. Solution of the system of differential equations in Proposition 5.4 if \( n = 0 \)
   and \( \alpha_0 = 0 \) 36
7. Asymptotic behavior of the Fourier coefficient functions 43
8. Solution of the system of differential equations in Proposition 5.4 if \( n \neq 0 \)
   or \( \alpha_0 \neq 0 \) 46
9. References 51

1. Introduction

For complex-valued functions on the hyperbolic plane that are simultaneously
eigenfunctions of the hyperbolic Laplacian and periodic in a horocycle direction\(^1\) it is
not only known that they admit a Fourier expansion in the direction of periodicity but
also rather precisely how the Fourier coefficients behave in the orthogonal direction.
This deep knowledge about the Fourier coefficients is indispensable in many studies
of the properties of automorphic functions and forms and their applications within

\(^1\)We refer to Section 2.2 for precise definitions. Using the upper half plane model,\(^1\) we may
restrict to Laplace eigenfunctions \( f: \mathbb{H} \to \mathbb{C} \) that are periodic in the real direction with period 1,
thus \( f(z) = f(z + 1) \) for all \( z \in \mathbb{H} \).
mathematics and physics. However, not only periodic Laplace eigenfunctions but also Laplace eigenfunctions with certain controlled non-periodicities occur naturally in many areas, most notably in spectral theory, number theory and mathematical physics. Further below, in Section 1.2, we provide a few instances of such occurrences.

These and further applications demand detailed information about the fine structures of Laplace eigenfunctions with the type of controlled non-periodicities (for short: twists) as considered in the present article. These demands and, in addition, the idea—as promoted already by Selberg [58]—to investigate twisted automorphic functions alongside their untwisted (periodic) relatives and to exploit their interactions in order to understand their properties inspires the quest for Fourier(-type) expansions of vector-valued eigenfunctions of the Laplacian that are not necessarily periodic but rather twist-periodic with a twist given by a, potentially non-unitary, endomorphism of a vector space. With this paper we develop such Fourier expansions for all twist-periodic vector-valued Laplace eigenfunctions and all finite-dimensional complex vector spaces. We give a detailed description of their Fourier coefficients, explicitly identifying each of their constituents and precisely determining their relevant growth properties. In what follows we provide a more detailed account of our main results, our motivation as well as an indication of the structural differences from the classical, untwisted case and the unitarily twisted case, including a brief discussion of the relation of our results to those in these more classical situations.

1.1. Main results. We use the upper half plane model

\begin{equation}
\mathbb{H} := \{ z \in \mathbb{C} : \text{Im } z > 0 \}, \quad ds_z^2 := \frac{dz \, d\bar{z}}{(\text{Im } z)^2}
\end{equation}

of the hyperbolic plane, and we fix a finite-dimensional complex vector space \( V \) as well as an endomorphism \( A \in \text{End}(V) \) of \( V \). We emphasize that \( A \) is not required to be unitary. We let

\begin{equation}
\Delta := -y^2 \left( \partial_x^2 + \partial_y^2 \right), \quad (z = x + iy \in \mathbb{H}, x \in \mathbb{R}, y > 0),
\end{equation}

denote the hyperbolic Laplacian in the upper half plane model, and we consider \( \Delta \) as an operator on smooth \( V \)-valued functions on \( \mathbb{H} \). We are interested in the Laplace eigenfunctions \( f : \mathbb{H} \to V \), thus

\begin{equation}
\Delta f = s(1 - s)f,
\end{equation}

that satisfy the twist-periodicity condition

\begin{equation}
f(z + 1) = Af(z)
\end{equation}

for all \( z \in \mathbb{H} \). We note that we could define the Laplacian \( \Delta \) on a much larger domain, e.g., \( V \)-valued eigendistributions. The elliptic regularity of \( \Delta \), however, immediately implies that all its eigendistributions are smooth (even real-analytic) eigenfunctions [26, Theorem 6.33]. For that reason we will refrain from introducing this additional freedom which will not add any additional generality to our results, and we will discuss smooth functions only. We emphasize that we do not assume any growth properties of these functions, and hence the spectral parameter \( s \) of the Laplace eigenfunction \( f \) in \( \{2\} \) can a priori take any value in \( \mathbb{C} \).
We fix a basis of the complex vector space $V$ with respect to which $A$ is represented by a matrix $J$ in (complex) Jordan-like normal form:

$$J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_p \end{pmatrix}.$$  

(4)

Here, each block $J_j$ is a square matrix whose only non-zero entries are on the diagonal and the superdiagonal and it is either of the form

$$\begin{pmatrix} 0 & 1 \\ & \ddots & \ddots \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$$  

(5)

or of the form

$$\begin{pmatrix} \lambda_j & \lambda_j & & \\ & \ddots & \ddots & \\ & & \ddots & \lambda_j \\ & & & \lambda_j \end{pmatrix}$$  

(6)

for some $\lambda_j \in \mathbb{C}$, $\lambda_j \neq 0$. We remark that such a basis of $V$ always exists as follows immediately from the existence of the standard Jordan normal form and elementary linear algebra. By splitting $V$ into a direct sum of subspaces

$$V = V_1 \oplus \ldots \oplus V_p$$  

(7)

corresponding to the generalized eigenspaces of the Jordan-like blocks in (4), and also splitting the Laplace eigenfunction $f$ analogously into a direct sum of functions

$$f = f_1 \oplus \ldots \oplus f_p$$  

(8)

with $f_j : \mathbb{H} \to V_j$, $j \in \{1, \ldots, p\}$, we may study the Fourier expansion of $f$ separately for each Jordan-like block, or equivalently, separately for each $f_j$.

We will first observe that for the Jordan-like blocks with vanishing eigenvalue (thus, if we consider a block of the form (5)) the corresponding components of $f$ vanish identically.

**Proposition 1.1** (Vanishing eigenvalue). For each $j \in \{1, \ldots, p\}$ for which the eigenvalue of $J_j$ is 0, the component function $f_j$ is the zero function.

For $j \in \{1, \ldots, p\}$ for which the Jordan-like block $J_j$ is of the form (5), the main constituents of the Fourier expansion of $f_j$ are extensions and adaptations of those from the untwisted case (i.e., $A$ is the identity), which we briefly survey in Section 1.2. First of all, to be able to allow any nonzero value for the eigenvalue $\lambda_j$ of the Jordan-like block $J_j$, we may need to choose a nonstandard branch cut for the holomorphic logarithm, say $\omega_j \in \mathbb{R}_{\geq 0}$, and adapt the modified Bessel functions $I$ and $K$ accordingly. For the sake of readability of this introduction, we invite all readers to ignore this subtlety for the moment and refer to Sections 2.3 and 2.5.1 for all details. However, we will reflect this issue in the notation of the modified Bessel functions of the first and second kind, by writing $I_\nu(\cdot; \omega_j)$ and $K_\nu(\cdot; \omega_j)$ instead of $I_\nu$ and $K_\nu$, respectively.

After this adaptation, all constituents can be built up from either the modified Bessel functions $I_\nu(\cdot; \omega_j)$ and $K_\nu(\cdot; \omega_j)$, or as linear combinations of $y^a$ and $y^a \log y$
for suitable values of $a \in \pm s + \frac{1}{2}\mathbb{Z}$, depending on $s$. More precisely, for any $\alpha \in \mathbb{C} \setminus \omega_j \mathbb{R}_{\geq 0}$, $m \in \mathbb{N}_0$ and $y > 0$ we set

$$I_j(m, \alpha, y, s) := \frac{y^{\frac{1}{4}}}{im} \partial_{\alpha}^m \left( I_{s-\frac{j}{2}}(\alpha y; \omega_j) \right)$$

and

$$K_j(m, \alpha, y, s) := \frac{y^{\frac{1}{4}}}{im} \partial_{\alpha}^m \left( K_{s-\frac{j}{2}}(\alpha y; \omega_j) \right).$$

As we will see, these maps are the constituents for almost all Fourier coefficients. We emphasize that in (9) and (10), the differential operator $\partial_{\alpha}^m$ is not the plain $m$-th derivative of $\Psi_{s-\frac{j}{2}}(\cdot; \omega_j)$ for $\Psi \in \{I, K\}$, but acts on the map

$$\alpha \mapsto \Psi_{s-\frac{j}{2}}(\alpha y; \omega_j),$$

as shall be indicated by the additional brackets around $\Psi_{s-\frac{j}{2}}(\alpha y; \omega_j)$. Further, we remark that it is always possible to choose the branch cut, and hence the $\omega_j$, uniformly for all $j$ so that the definitions of $I_j$ and $K_j$ in (9) and (10) only depend on the eigenvalues of $A$ but not on the sizes of the Jordan-like blocks.

The zeroth Fourier coefficient, however, may correspond to the case that $\alpha = 0$ and, as in the untwisted case, is then formed in a different way. In the untwisted case the value $\frac{1}{2}$ for the spectral parameter $s$ is special because of a resonance situation in a certain differential equation (see (17) below). In our twisted case, analogous resonance situations may occur, and a higher size of the Jordan-like block $J_j$ causes a larger set of special values for the spectral parameter, namely

$$E_j := \left\{ \frac{1}{2} \pm k : k = 0, \ldots, \left\lfloor \frac{d_j - 1}{2} \right\rfloor \right\},$$

where $d_j$ is the (complex) dimension of the vector space $V_j$. For any $m \in \mathbb{N}_0$ and $y > 0$ we define

$$I_j(m, 0, y, s)$$

to be a certain precise linear combination of

$$y^{s+2} \left\lfloor \frac{m}{2} \right\rfloor \text{ and } y^{s+2} \left\lfloor \frac{m}{2} \right\rfloor \log y$$

if

$$s \in E_j \cap \left[ \frac{1}{2} - \frac{m}{2}, \frac{1}{2} \right],$$

and to be a certain precise scalar multiple of

$$y^{s+2} \left\lfloor \frac{m}{2} \right\rfloor$$

for all other values of $s$. We refer to Section 2.5.3 for the precise expressions, and remark here only that the definition of $I_j(m, 0, y, s)$ depends on the size of the Jordan-like block $J_j$ and hence cannot be made uniform for all Jordan-like blocks of $A$, in contrast to the definition in [9]. Analogously we define

$$K_j(m, 0, y, s)$$

to be a certain precise linear combination of

$$y^{1-s+2} \left\lfloor \frac{m}{2} \right\rfloor \text{ and } y^{1-s+2} \left\lfloor \frac{m}{2} \right\rfloor \log y$$

if

$$s \in E_j \cap \left[ \frac{3}{2}, \frac{1}{2} \right].$$
and to be a certain scalar multiple of
\[ y^{1-s+2\left\lfloor \frac{m}{\pi} \right\rfloor} \]
for all other values of \( s \). As for \( f_j(\cdot, 0, \cdot, \cdot) \), the definition of \( K_j(\cdot, 0, \cdot, \cdot) \) cannot be made independent of \( j \). For each \( n \in \mathbb{Z} \) we set
\[
\alpha_{j,n} := 2\pi n - i \log(\lambda_j; \omega_j),
\]
where \( \log(\cdot; \omega_j) \) is the complex logarithm with branch cut \( \omega_j \mathbb{R}_{>0} \) (see Section 2.3 for details). Further we pick \( \varepsilon_{j,n} \in \{0, 1\} \) such that
\[
(-1)^{\varepsilon_{j,n}} \alpha_{j,n} \not\in \omega_j \mathbb{R}_{>0},
\]
and set
\[
\hat{\alpha}_{j,n} := (-1)^{\varepsilon_{j,n}} \alpha_{j,n}.
\]

**Theorem 1.2** (Single Jordan block; coarse statement). Let \( j \in \{1, \ldots, p\} \) be such that the eigenvalue \( \lambda_j \) of \( J_j \) is not 0. Then the Fourier expansion of \( f_j \) is
\[
f_j(z) = \sum_{n \in \mathbb{Z}} J_f \hat{f}_{j,n}(y, s)e^{2\pi inx} \quad (z = x + iy \in \mathbb{H}),
\]
where the Fourier coefficient function \( \hat{f}_{j,n}(\cdot, s) \) for \( n \in \mathbb{Z} \) is of the form
\[
\hat{f}_{j,n}(y, s) = \hat{C}_n \begin{pmatrix} I_j(d_j - 1, \hat{\alpha}_{j,n}, y, s) \\ I_j(1, \hat{\alpha}_{j,n}, y, s) \\ I_j(0, \hat{\alpha}_{j,n}, y, s) \end{pmatrix} + \hat{D}_n \begin{pmatrix} K_j(d_j - 1, \hat{\alpha}_{j,n}, y, s) \\ K_j(1, \hat{\alpha}_{j,n}, y, s) \\ K_j(0, \hat{\alpha}_{j,n}, y, s) \end{pmatrix}
\]
with \( \hat{C}_n, \hat{D}_n \) being appropriate matrices in \( \mathbb{C}^{d_j \times d_j} \).

We refer to Sections 2.6 and 2.7 for precise statements and the refined statement of Theorem 1.2 in particular to Theorem 2.5. Moreover, we will see that the matrices \( \hat{C}_n \) and \( \hat{D}_n \) in Theorem 1.2 belong to a certain precise subspace of \( \mathbb{C}^{d_j \times d_j} \) of dimension \( 2d_j \) if \( \lambda_j = 1, n = 0 \) (i.e., if \( \alpha_{j,n} = 0 \)) and \( d_j > 1 \), and of dimension \( d_j \) in all other cases. See Section 2.6 and Theorem 2.5.

We emphasize that in the case that the endomorphism \( A \) is the identity, we have \( \lambda_j = 1 \) and hence \( \alpha_{j,n} = 2\pi n \). We then may pick \( \omega_j = -1 \) so that \( \hat{\alpha}_{j,n} = 2\pi |n| \) and hence Theorem 1.2 reduces to the classical result stated in (19). Our desire to be able to recover the classical result from Theorem 1.2 is indeed the main motivation to introduce the \( \varepsilon_{j,n} \)-family in (13). See Section 1.2 for more details. In particular, in this case, the eigenvalue \( \lambda_j = 1 \) makes no visible appearance (as the relevant term \( \log(\lambda_j; \omega_j) \) vanishes). However, if \( A \) is not the identity and hence the eigenvalue \( \lambda_j \) is not 1 (for some \( j \in \{1, \ldots, p\} \)), then the modified Bessel functions have to account for a non-periodicity or a twist-periodicity in the \( y \)-direction of the function \( f_j \), which results in the dependence of the Bessel functions on \( \lambda_j \) (and hence \( \hat{\alpha}_{j,n} \)) as stated in (9)–(10).

The Fourier expansion of \( f \) can now be deduced by combining Proposition 1.1 and Theorem 1.2.

**Theorem 1.3** (Several Jordan blocks). The Fourier expansion of \( f \) is
\[
f(z) = \sum_{n \in \mathbb{Z}} J_f \hat{f}_n(y, s)e^{2\pi inx} \quad (z = x + iy \in \mathbb{H}),
\]
where the $n$-th Fourier coefficient is

\[ \hat{f}_n(y, s) = \begin{pmatrix} \hat{f}_{1,n}(y, s) \\ \vdots \\ \hat{f}_{p,n}(y, s) \end{pmatrix} \]

with

\[ \hat{f}_{j,n}(y, s) = \begin{cases} 0 & \text{if } \lambda_j = 0, \\ \text{as in Theorem 1.2} & \text{if } \lambda_j \neq 0 \end{cases} \]

for all $j \in \{1, \ldots, p\}$.

As in the untwisted case, we can distinguish the functions $I_j$ and $K_j$ by their growth properties. This result is certainly of independent interest and will also be used in the proof of Theorem 1.2. Due to the sometimes necessary non-standard choice of a branch cut for the holomorphic logarithm and the related adaptations of the modified Bessel functions (depending on the endomorphism $A$), the growth behavior of the functions $I_j$ and $K_j$ may be qualitatively different depending on the different domains for $\alpha$ in (9) and (10). In the “principal sector” (roughly, the sector in which the adapted and the classical modified Bessel functions are qualitatively equal) their growth behavior is as in the classical untwisted case, however outside of this sector it may change. We refer to Section 7 for the precise and detailed description of the situation and state here only a partial result.

**Theorem 1.4** (Growth behavior, partial statement). Let $s \in \mathbb{C}$, $m \in \mathbb{N}_0$, $j \in \{1, \ldots, p\}$, and suppose that $\alpha$ is an element of $\mathbb{C} \setminus \omega_j \mathbb{R}_{\geq 0}$ in the “principal sector.” Then:

(i) The absolute value of $I_j(m, \alpha, y, s)$ increases exponentially as $y \to \infty$.

(ii) The absolute value of $K_j(m, \alpha, y, s)$ decreases exponentially as $y \to \infty$.

In what follows we will first discuss, in Section 1.2, the mathematical context and the motivation of our investigations. After that, in Section 1.3 we will provide a survey of the proofs of the main theorems. Of course, the following two sections can also be read in converse order.

1.2. Motivation and mathematical context.

1.2.1. Brief review of the classical, untwisted case. If the endomorphism $A$ is the identity on $V$, then the system (2)-(3) asks for the Laplace eigenfunctions $f: \mathbb{H} \to V$ with spectral parameter $s \in \mathbb{C}$ that are periodic with period 1 in the $x$-direction. Thus, (2)-(3) becomes

\[ \Delta f = s(1-s)f \quad \text{and} \quad f(z+1) = f(z) \quad \text{for all } z \in \mathbb{H}. \]

In this classical, untwisted case the existence and structure of the Fourier expansion of $f$ is well-known (see, e.g., [3, Sections 1.2 and 8.1]). The periodicity of $f$ causes the Fourier expansion to take the form

\[ f(z) = f(x + iy) = \sum_{n \in \mathbb{Z}} \hat{f}_n(y, s)e^{2\pi inx} \quad (z \in \mathbb{H}). \]

As $f$ is a Laplace eigenfunction, the $n$-th Fourier coefficient function $\hat{f}_n(\cdot, s)$ necessarily satisfies the modified Bessel differential equation

\[ \left(y^2 \partial_y^2 + s(1-s) - (2\pi ny)^2\right) \hat{f}_n(y, s) = 0, \]
which depends on \( n \) and \( s \). If \( \dim V = 1 \) (i.e., the scalar case), and \( n \neq 0 \), then a fundamental set of solutions of the differential equation (17) is given by the two functions

\[
y \mapsto y^\frac{s}{2} I_{s-\frac{1}{2}}(2\pi n |y|) \quad \text{and} \quad y \mapsto y^\frac{s}{2} K_{s-\frac{1}{2}}(2\pi n |y|),
\]

which are both defined on the interval \((0, \infty)\). Here \( I_{s-\frac{1}{2}} \) and \( K_{s-\frac{1}{2}} \) are two specific linearly independent solutions of the modified Bessel differential equation, commonly known as the modified Bessel functions of the first and second kind with index \( s - \frac{1}{2} \), respectively. Among their important features are their growth behaviors as \( y \to \infty \): the function \( I_{s-\frac{1}{2}} \) is exponentially increasing, and \( K_{s-\frac{1}{2}} \) is exponentially decreasing. These growth properties are passed on to the functions in (18). Theorem 1.4 and its refined variants in Section 7 state analogous properties in the twisted case. For \( \dim V = 1 \) and \( n = 0 \), two independent solutions of (17) are \( y^s \) and \( y^{1-s} \) if \( s \neq \frac{1}{2} \), and \( y^\frac{s}{2} \) and \( y^\frac{s}{2} \log y \) if \( s = \frac{1}{2} \). For a vector space \( V \) of arbitrary finite dimension, the Fourier coefficient functions in (16) are therefore of the form

\[
\hat{f}_n(y, s) = \begin{cases} 
    c_n y^\frac{s}{2} I_{s-\frac{1}{2}}(2\pi n |y|) + d_n y^\frac{s}{2} K_{s-\frac{1}{2}}(2\pi n |y|), & n \neq 0, \\
    c_0 y^s + d_0 y^{1-s}, & n = 0, s \neq \frac{1}{2}, \\
    c_0 y^\frac{s}{2} + d_0 y^\frac{s}{2} \log y, & n = 0, s = \frac{1}{2}
\end{cases}
\]

for any \( n \in \mathbb{Z} \), where the coefficients \( c_n, d_n \) are suitable elements of \( V \), depending on \( f \).

### 1.2.2. Compatibility of Theorem 1.3 with the classical, untwisted result.

Theorem 1.3 and its full version, Theorem 2.5, are compatible with the classical result (19) in the untwisted setting, as we now indicate. Since the endomorphism \( A \) is the identity and hence all its eigenvalues equal 1, we may choose \( \omega = \omega_1 = -1 \) (see right after Proposition 1.1), which retrieves the principal logarithm. Then all adapted modified Bessel functions are identical to the classical modified Bessel functions and \( \alpha_n = 2\pi n \) for all \( n \in \mathbb{Z} \) (see (12)). For \( n > 0 \) we necessarily pick \( \varepsilon_n = 0 \) in (13), and for \( n < 0 \) we necessarily pick \( \varepsilon_n = 1 \). This results in

\[
\tilde{\alpha}_n = (-1)^{\varepsilon_n} \alpha_n = 2\pi |n|
\]

for all \( n \in \mathbb{Z} \). See (14). From this it immediately follows that Theorem 2.5 provides the classical result.

At this point we can explain more precisely our main motivation for working with the \( \varepsilon_n \)-family instead of using right away the values \( \alpha_n \). (See also the comment right after Theorem 1.2.) In the introduction of Section 8 we discuss an alternative method for handling negative \( \alpha_n \). If we would implement this method, then we would need to pick a value for \( \omega \) that is not in \( \mathbb{R}_{<0} \), but then we would be unable to reproduce the classical result.

### 1.2.3. Motivation I, untwisted case.

To illustrate the importance of such precise knowledge of the structure of the Fourier expansion and the Fourier coefficients in the untwisted case as in (19), we now briefly describe one of its many consequences: the growth thresholds of automorphic Laplace eigenfunctions at cusps. To that end we first note that (19) implies that the function \( f \) in (15) and (16) decomposes as

\[
f = f_I + f_0 + f_K
\]
with
\begin{equation}
  f_I(z) := \sum_{n \in \mathbb{Z}} c_n y^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi |n| y)e^{2\pi inx}, \quad z = x + iy \in \mathbb{H},
\end{equation}
growing exponentially in absolute value as \( y \to \infty \) as soon as not all coefficients \( c_n \) vanish,
\begin{equation}
  f_K(z) := \sum_{n \in \mathbb{Z}} d_n y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi |n| y)e^{2\pi inx}, \quad z = x + iy \in \mathbb{H},
\end{equation}
decaying exponentially as \( y \to \infty \) as soon as not all coefficients \( d_n \) vanish, and
\begin{equation}
  f_0(z) := \begin{cases} 
    c_0 y^s + d_0 y^{1-s} & \text{for } s \neq \frac{1}{2}, \\
    c_0 y^{\frac{1}{2}} + d_0 y^{\frac{1}{2}} \log y & \text{for } s = \frac{1}{2} 
  \end{cases}
\end{equation}
with a growth rate depending on \( s \) but being at most polynomial as \( y \to \infty \).

We now let \( \Gamma \) be a discrete subgroup of \( \text{SL}_2(\mathbb{R}) \) (or \( \text{PSL}_2(\mathbb{R}) \), if we want to consider Fuchsian groups); examples include \( \Gamma = \text{SL}_2(\mathbb{Z}) \) (or \( \Gamma = \text{PSL}_2(\mathbb{Z}) \)). Then \( \Gamma \) acts isometrically on the upper half plane \( \mathbb{H} \) by fractional linear transformations,
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}
\]
for \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \ z \in \mathbb{H}. \) We suppose further that the hyperbolic surface \( \Gamma \backslash \mathbb{H} \) that is the orbit space of the action of \( \Gamma \) on \( \mathbb{H} \) with the canonical Riemannian metric, has at least one infinite end of finite area, a so-called \textit{cusp}. In rough terms (and fully sufficient for our purposes), a \textit{cusp} can be described as follows: we take a (very large) compact subset \( K \) of \( \Gamma \backslash \mathbb{H} \) such that for all compact subsets \( \tilde{K} \) with \( K \subseteq \tilde{K} \) the spaces \( (\Gamma \backslash \mathbb{H}) \setminus K \) and \( (\Gamma \backslash \mathbb{H}) \setminus \tilde{K} \) have the same number of connected components. These connected components are the \textit{ends} of \( \Gamma \backslash \mathbb{H} \), and those of finite area are the \textit{cusps}. (It is irrelevant for our purposes that these notions of end and cusp depend on the choice of \( K \) and should be more correctly called \textit{end area} and \textit{cusp area}, respectively.) For example, the \textit{modular surface} \( \text{SL}_2(\mathbb{Z}) \backslash \mathbb{H} \) has one cusp and no further ends. In many applications in harmonic analysis, number theory, and physics, one needs to understand the growth behavior at cusps of \( \Gamma \)-\textit{automorphic functions}, that is of the functions \( u: \mathbb{H} \to V \) satisfying
\begin{equation}
  u(gz) = u(z) \quad \text{for all } g \in \Gamma, \ z \in \mathbb{H},
\end{equation}
that are simultaneously Laplace eigenfunctions, thus
\begin{equation}
  \Delta u = s(1-s)u.
\end{equation}
In particular, Maass wave forms for \( \Gamma \), real-analytic Eisenstein series, and \( \Gamma \)-automorphic forms of weight \( 0 \) are examples of such automorphic Laplace eigenfunctions. We remark that such functions satisfying (24) and (25) are sometimes called \( \Gamma \)-\textit{automorphic forms}. However, since we do not require any growth properties or regularity properties at cusps or other ends of the hyperbolic surfaces, we will refer to these objects as \textit{automorphic functions} despite the standing hypothesis of them being Laplace eigenfunctions.

\footnote{For simplicity, we refer to all of the orbit spaces \( \Gamma \backslash \mathbb{H} \) as hyperbolic surfaces, even though some of them are genuine orbifolds and not smooth manifolds. In this case, the Riemannian metric is to be understood as a Riemannian metric on an orbifold.}
In the case that \( \Gamma \) is generated by the element \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \text{SL}_2(\mathbb{R}) \), the system (24)–(25) reduces to (15). Also for any other choice of \( \Gamma \), the setting (15) is vital to understanding the solutions to (24)–(25): locally at each cusp of \( \Gamma \backslash \mathbb{H} \), after an appropriate change of coordinates (which we will suppress in what follows), the invariance (24) descends to 1-periodicity in the \( x \)-direction. Studying the local Fourier expansion of \( u \) in this cusp is therefore the same as studying the Fourier expansion of \( f \) as in (15). In particular, in this cusp, the function \( u \) admits a decomposition as \( f \) in (20). Therefore, if we know that \( u \) has at most polynomial growth in the cusp, then the term corresponding to \( f_I \) in the Fourier expansion of \( u \) in (20) vanishes and the expansion becomes

\[
u = u_0 + u_K
\]

with \( u_0 \) and \( u_K \) as in (20) (with \( u \) in place of \( f \)). This in turn provides us with the additional knowledge on the polynomial growth rate of \( u \). If \( u \) is bounded and \( s \not\in \{0, 1\} \), then in addition the term \( u_0 \) vanishes, and \( u = u_K \) is exponentially decreasing as \( y \to \infty \). Results of this type are crucial for several important applications, including obtaining dimension bounds for subspaces of solutions to (24) and (25) as in [45], as well as characterizing period functions for different types of automorphic functions as in [45, 6].

1.2.4. Motivation II, twisted case. In the untwisted case, the endomorphism \( A \) is the identity on \( V \), and studying the Fourier expansions of \( \Gamma \)-automorphic Laplace eigenfunctions in cusps is equivalent to studying the Fourier expansions of solutions of (15). In the twisted case, i.e., when the endomorphism \( A \) in (3) is not the identity on \( V \), the following similar relation exists: we consider functions \( u : \mathbb{H} \to V \) which satisfy the twisted invariance

\[
u(g.z) = \chi(g)u(z) \quad \text{for all } g \in \Gamma, \ z \in \mathbb{H},
\]

where \( \chi : \Gamma \to \text{GL}(V) \) is a representation of \( \Gamma \) on \( V \). Functions satisfying (26) are often called \( (\Gamma, \chi) \)-automorphic functions or vector-valued automorphic functions with multiplier \( \chi \). If we suppose for simplicity (to avoid any changes of coordinates) that

\[
T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma,
\]

then (26) applied for \( g := T \) becomes \( u(T.z) = \chi(T)u(z) \), or equivalently,

\[
u(z + 1) = Au(z) \quad \text{for all } z \in \mathbb{H}
\]

with \( A := \chi(T) \), which is precisely (3) for \( f = u \). An even stronger relation between (3) and (26) is valid if we suppose in addition that \( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \notin \Gamma \) for \( x \in (0, 1) \) or, in other words, if \( T \) is a generator of the stabilizer group of \( \infty \) in \( \Gamma \). Then \( \Gamma \backslash \mathbb{H} \) has a cusp of the form

\[
\langle T \rangle \backslash \{x + iy \in \mathbb{H} : y > y_0 \}
\]

for some \( y_0 > 0 \). Restricted to this cusp, the only twist-invariance of \( u \) that is preserved from (26) is the one by \( T \) (and its iterates). Thus, locally in cusps, twisted automorphic functions are exactly as in (3), and hence their local properties can be studied via investigating functions satisfying (3).

Those twisted automorphic functions that satisfy in addition the eigenfunction property (25), thus, the functions \( u : \mathbb{H} \to V \) that satisfy

\[
\Delta u = s(1-s)u \quad \text{and} \quad u(g.z) = \chi(g)u(z) \quad (g \in \Gamma, \ z \in \mathbb{H}),
\]
are the $\Gamma, \chi$-automorphic Laplace eigenfunctions. Prominent subclasses include various kinds of $V$-valued automorphic forms of weight $0$ with multiplier $\chi$, which are distinguished by regularity conditions or growth properties at the ends of $\Gamma \setminus \mathbb{H}$.

Twisted automorphic Laplace eigenfunctions have received considerable attention for several decades already as they arise naturally in mathematics and physics. For example, they help us to understand the fine structure of the untwisted automorphic Laplace eigenfunctions [58], and they play a crucial role in some areas of modern physics.

More precisely, several physical systems can only be adequately addressed with vector-valued functions rather than scalar functions, due to the necessity to describe several characteristics simultaneously, like a field of arbitrary spin. For example, vector-valued eigenfunctions of Laplace operators appear in the study of Casimir interaction between two conducting obstacles [46]. Vector-valued operators of Laplace-type help to investigate the behavior of a charged particle under the influence of a magnetic field [21]. They also play an important role in attacking tunneling problems [48].

In mathematical physics, both the thermal AdS$_3$ space-times and the BTZ black hole are modeled by a geometric setting closely related to that considered in the present article. Specifically, they are modeled by $\Gamma \setminus \mathbb{H}^3$ with $\Gamma \subseteq \text{SL}_2(\mathbb{C})$ being a discrete infinitely cyclic subgroup [11, p. 4]. For the BTZ black hole, the group $\Gamma$ is explicitly given by

$$\Gamma = \left\langle \begin{pmatrix} e^{a+ib} & 0 \\ 0 & e^{-a-ib} \end{pmatrix} \right\rangle$$

with $a, b \in \mathbb{R}$ depending on the black hole’s mass and angular momentum. See [7, (6-7)].

In the geometric setting of $\mathbb{H}^3$, fields of arbitrary spin-$s$ correspond to sections of homogeneous vector bundles [11, p. 2]. If the field should be defined not only on $\mathbb{H}^3$, but also on $\Gamma \setminus \mathbb{H}^3$, as used to model the aforementioned space times and black holes, then an additional $\Gamma$-invariance is required. Consequently, a homogeneous vector bundle with such a $\Gamma$-invariance becomes a locally homogeneous vector bundle.

To be more precise, let $G$ be a semisimple Lie group and $K$ its maximal compact subgroup. For BTZ black holes, $G = \text{SL}_2(\mathbb{C})$ and $K = \text{SU}(2)$. Let $\rho: G \to \text{GL}(V)$ be a finite-dimensional representation of $G$, and let $\rho_K$ and $\rho_\Gamma$ be the restrictions of $\rho$ to $K$ and $\Gamma$, respectively. We let $E_{\rho_K}$ be a flat bundle associated to the restriction of such a representation to $\Gamma$, and we let $E_{\rho_K}$ be the locally homogeneous vector bundle associated to $\rho_K$. In order to define the latter, we first define a homogeneous vector bundle

$$\bar{E}_{\rho_K} := (G \times V)/K \to G/K,$$

where $K$ acts on the right as

$$(g, v)k := (gk, \rho_K(k^{-1})v), \quad \text{for } g \in G, \ k \in K, \ v \in V$$

and let $E_{\rho_K} := \Gamma \setminus \bar{E}_{\rho_K}$. In [49, Proposition 3.1] it is stated that

$$E_{\rho_K} \cong E_{\rho_K}.$$

If we consider a spin-$s$ representation of $\text{SU}(2)$ for an integer $s$, then it has a lift to a representation of $\text{SL}_2(\mathbb{C})$. In this situation, we can construct a flat vector bundle associated to the restriction of such a representation to $\Gamma$, which will be in one-to-one correspondence with the original locally homogeneous vector bundle. Thus, fields of spin-$s$ for integer $s$ are in this way isomorphic to sections of a
flat vector bundle, analogous to the vector-valued functions in our present study. Moreover, non-unitary twists are essential: at least for $\text{SL}_2(\mathbb{R})$, restrictions of representations of $\text{SL}_2(\mathbb{R})$ to a subgroup of $\text{SL}_2(\mathbb{R})$, are in general not unitary. One may compare the explicit expression for such representations in the proof of [23 Proposition 5.1]. In the 3-dimensional case, these correspond to $\rho_\Gamma$ defined above which are in general non-unitary as well. Thus, spin-$s$ fields are isomorphic to sections of certain flat vector bundles corresponding to non-unitary representations. If these sections are Laplace eigenfunctions, then they are expected to correspond to zeros of the Selberg zeta function. In fact, for finite volume surfaces and unitary representations this is a direct consequence of the classical Selberg trace formula. These zeros encode the normal frequencies of spin-$s$ fields on thermal AdS$_3$ space-times. Consequently, detailed information on vector-valued Laplace eigenfunctions twisted by non-necessarily unitary representations should lead to a deeper understanding of Vasiliev theories [61]. Of course these applications also show that it would be of great interest to study Fourier expansions of twisted automorphic Laplace eigenfunctions for much more general spaces. However, such a generalization of our results is beyond the scope of this article.

In the applications we just discussed, the twist is provided by a genuine representation and hence the twisting endomorphism is invertible. However, with our investigations in this article, we can easily cover non-invertible endomorphisms as well. In view of understanding the twist as a perturbation, we are certain that also non-invertible twists will have far-reaching important applications and the transition from non-invertible twists to invertible ones will lead to interesting insights.

Due to the great significance of twisted automorphic Laplace eigenfunctions, much effort has already been spent on their study and several important results have been obtained, for various types of such functions and in various generalities in regard to the twist. A non-exhaustive list regarding non-unitary twists includes [20, 10, 50, 51, 13, 14, 12, 24, 23, 57] and [36, 37, 38, 39, 40, 42, 41, 44, 3, 9, 29, 17]. Analogous questions can be asked for higher-dimensional spaces as well; the first results in this direction for non-unitary twists were recently established in [51, 59, 22, 13, 12, 60].

Further examples of functions closely related to this area of research are vector-valued modular forms. Typically they are not Laplace eigenfunctions, but they obey the same type of controlled non-periodicities as supposed in the present article. Vector-valued modular forms play a crucial role in the generalized moonshine conjecture [16, 28, 18] being ultimately related to generating series for characters of rational vertex operator algebras [64, 58]. Vector-valued modular forms are also important objects in 2-dimensional conformal field theory [52, 27]. Deformations of vector-valued modular forms arise in the study of gravitational waves [4]. Additionally, vector-valued modular forms are used in the Standard Model of particle physics to construct example models for lepton masses [47]. Vector-valued mock modular forms appear in string theory while studying D3-instantons [2]. Interestingly, some components of vector-valued automorphic functions satisfy differential equations similar to those used to model the behavior of the 4-loop supergraviton [35, 31]. Considering the importance of vector-valued modular forms, it seems reasonable to expect that also a kind of non-holomorphic vector-valued “modular forms” that are Laplace eigenfunctions will have similar physical applications, for which the results of our present article are applicable.
1.2.5. Previous results on Fourier expansions in the presence of twists, and their relation to Theorem 1.3. Results regarding Fourier expansions of twisted automorphic Laplace eigenfunctions are—despite their importance—rather sparse. To date Fourier expansions have been established mostly for twists by unitary representations; for example, in [52, 53], they appear in the study of the behavior of counting functions of cusp forms. In particular, the authors employ the space of $L^2$-functions with zero Fourier coefficients in the cusp at $\infty$ vanishing close enough to the mentioned cusp. Other references include [33] and [62].

In order to show that our results are compatible with the mentioned results, we compare the structure of the Eisenstein series in [62] with the Fourier expansion we have obtained. We denote by $E_\chi(z, s) := E_{\Gamma, \chi}(z, s)$ the Eisenstein series twisted by a unitary representation $\chi$; without loss of generality we let $\dim C_\chi = 1$. If we assume that the eigenvalue of $\chi$ is equal to one as in [62, Theorem 3.1.2], then the problem reduces to the classical case that has been considered above. In the case when the eigenvalue, $e^{2\pi i \theta}$, is not equal to one, [62, Theorem 3.1.3] reads

$$E_\chi(z, s) = \sum_{j \in \mathbb{Z}} a_{n, \theta, s} e^{2\pi i (j + \theta) x} K_{s-1/2}(2\pi |j + \theta| y)$$

for some $a_{n, \theta, s}$ depending only on $n$, $s$, $\Gamma$ and $\theta$. The result in [14] is structurally similar; however, the coefficients, $a_{n, \theta, s}$, are different.

We note that the series above are, strictly speaking, not Fourier series, because $E_\chi(z, s)$ is not periodic in the $x$-direction. However, $e^{-2\pi i \theta x} E_\chi(z, s)$ is periodic. We will see later in Section 4 that a similar procedure of periodization, albeit more technically involved, plays a similar role for the Fourier expansions for non-unitary representations.

For non-unitary twists, Fourier expansions are known only for Eisenstein series, however with increasing complexity of twists, namely twists that are unitary in cusps [14] and twists with non-expanding cusp monodromy [23]. For the sake of completeness we remark that related results exist for (weighted) modular forms and cusp forms that are twisted with one-dimensional representations that are unitary in cusps, however asking for Fourier expansions in $z$ not in $x$, which are structurally easier. See, e.g., [58, 54, 55].

1.3. Survey of proofs and organization of this article. This article is rather technical and notation-heavy due to the very nature of the main results. For the convenience of the reader we provide now an informal survey of the proofs. Simultaneously, we present the organization of this article.

We collect the necessary background material and introduce the major part of the notation in Section 2. We start the investigations by reducing their complexity in the manner to which we have alluded above. Namely, we pick a basis of the vector space $V$ with respect to which the endomorphism $A$ is represented by a matrix in Jordan-like normal form. See (4)–(6). The associated decompositions of $V$ and $f$ as in (7) and (8) are preserved under the action of $A$ due to its block structure. Thus they are also compatible with Fourier expansions, by linearity. We may therefore reduce the investigations to the case that $A$ is represented by a single Jordan-like block, and we can easily deduce Theorem 1.3 from Proposition 1.1 and Theorem 1.2. This is discussed in further detail in Section 3.

We then suppose that $A$ is represented by a single Jordan-like block and split the investigations into the two cases whether $A$ has eigenvalue 0 or not. In the first
case, which is rather straightforward, $A$ is represented by a Jordan-like block of the form and hence nilpotent. An iteration of $f(z + 1) = Af(z)$ yields

$$f(z + d) = A^d f(z) = 0$$

for $d = \dim V$. From this, Proposition 1.1 follows immediately. We dispatch this case in Section 3.

The other case, in which the eigenvalue of $A$ is not 0, constitutes the main bulk of this article. To develop a Fourier expansion in this twisted case, we proceed as analogously to the classical approach for the untwisted case as possible. However, the presence of the twist not only causes some major differences to which we need to adapt but also makes these investigations much more involved than in the classical case.

The first main difference from the classical, untwisted case is that the map $f$ is not periodic in the $x$-direction. For periodic maps we can separate the $x$- and the $y$-component of the argument $z = x + iy$ of $f$ by representing $f$ with respect to the Fourier basis $\{e^{2\pi inx}\}_{n \in \mathbb{Z}}$. The representation coefficients in this basis depend then on $y$ only. In our situation these coefficients are necessarily dependent on both $x$ and $y$. To overcome this issue we use the following periodization of $f$, an approach which is also known from the study of monodromy of ordinary differential equations, vector bundles and modular forms. We represent the endomorphism $A$ by the Jordan-like block $J$ with eigenvalue $\lambda \neq 0$ of the form (6). The map $F: \mathbb{H} \to V, \ z = x + iy \mapsto J^{-x} f(z)$, is 1-periodic in the $x$-direction. By abstract Fourier theory we therefore know that the map $F$ has a Fourier expansion of the form

$$F(z) = \sum_{n \in \mathbb{Z}} \hat{f}_n(y) e^{2\pi inx},$$

where the coefficients $\hat{f}_n(y) \in V$ depend on $y$ only. Therefore, the original map $f$ admits the expansion

$$f(z) = \sum_{n \in \mathbb{Z}} J^x \hat{f}_n(y) e^{2\pi inx}.$$  

We observe that although the coefficients in this representation depend on both $x$ and $y$, the contributions of $x$ and $y$ are clearly separated. The details of this step are contained in Section 4.

We then turn to investigating the coefficient functions $\hat{f}_n: \mathbb{R}_{>0} \to V$. Exploring the property that $f$ is a Laplace eigenfunction, we find that the coefficient functions satisfy the ($n$-dependent) differential equation

$$(29) \quad \left( y^2 \partial_y^2 + s(1 - s) + y^2 \partial_x^2 \big|_{x=0} \left( e^{2\pi inx} J^x \right) \right) \hat{f}_n(y) = 0, \quad y \in \mathbb{R}_{>0},$$

which is reminiscent of the modified Bessel differential equation (17). At this point we encounter the second main difference to the untwisted case. If the size of the Jordan-like block $J$ is larger than 1, then the differential equation (29) is vector-valued or, in other words, a system of differential equations. Since $J$ is not diagonalizable, this system does not decompose into several independent differential equations; instead it is heavily interrelated. The deduction of (29) is presented in Section 5.

After a base change the system (29) becomes a cascade of differential equations that we can solve iteratively. As in the classical case, we may encounter resonance
situations for the zeroth coefficient function $\hat{f}_0$, depending on the value of the spectral parameter $s$ and the eigenvalue $\lambda$ of $A$. In Section 5 we present this base change, and in Sections 6–8 we provide a detailed discussion of the solution of (29).

The asymptotic behavior of the functions $I$ and $K$ as stated in Theorem 1.4 is essentially a consequence of the growth properties of the modified Bessel functions. Section 7 is devoted to the proof of the extended version of Theorem 1.4.

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2. Essential objects and the refined statement of Theorem 1.2

In this section we introduce the majority of the objects used throughout this article. In particular, we provide the precise definitions of all constituents of the Fourier coefficient functions. In addition, with all definitions in place, we will present the refined statement of Theorem 1.2, namely as Theorem 2.5 below.

2.1. General notation. We denote by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the set of the positive natural numbers, the non-negative integers, all integers, the real numbers and the complex numbers, respectively. We use

$$\mathbb{R}_{\geq x_0} := \{ x \in \mathbb{R} \mid x \geq x_0 \}$$

with $x_0 \in \mathbb{R}$ to denote the closed right half line, starting at $x_0$. We assign analogous meanings to $\mathbb{R}_{>x_0}$, $\mathbb{R}_{\leq x_0}$ and $\mathbb{R}_{<x_0}$. We write $i = \sqrt{-1}$ for the imaginary unit in $\mathbb{C}$, in contrast to $i$, which we will use as an index. For a complex number, $z \in \mathbb{C}$, we denote its real and imaginary part by $\text{Re} \, z$ and $\text{Im} \, z$, respectively. We will often write $x$ for $\text{Re} \, z$ and $y$ for $\text{Re} \, z$, with implicit understanding that $x$ and $y$ depend on $z$. Further, for any complex number $z \in \mathbb{C}$ we write $\bar{z}$ for its complex conjugate. We denote the set $\mathbb{C} \setminus \{0\}$ of invertible complex numbers by $\mathbb{C}^\times$. For any $n \in \mathbb{N}$, we use $\mathbb{C}^{n \times n}$ and $\text{Mat}(n \times n; \mathbb{C})$ interchangeably for the set of $(n \times n)$-matrices with complex entries. Further, for real numbers $r \in \mathbb{R}$ we let $\text{sgn}(r)$ denote the sign of $r$. We will use the floor function, $\lfloor \cdot \rfloor : \mathbb{R} \rightarrow \mathbb{Z}$,

$$\lfloor x \rfloor := \max \{ m \in \mathbb{Z} : m \leq x \}.$$

Further, from now on, $\Gamma$ always denotes the Gamma functions (contrary to the previous sections, where $\Gamma$ was used for a discrete subgroup of $\text{SL}_2(\mathbb{C})$ or $\text{SL}_2(\mathbb{R})$ or $\text{PSL}_2(\mathbb{R})$, which will not make any further appearances).
2.2. Setting. We use the upper half plane model for the hyperbolic plane, with the standard hyperbolic Riemannian metric:
\[
\mathbb{H} = \{ z = x + iy \in \mathbb{C} : \text{Im} z = y > 0 \}, \quad ds_z^2 = \frac{dx^2 + dy^2}{y^2}.
\]
In this model, the Laplace operator is
\[
\Delta = -y^2 \left( \partial_x^2 + \partial_y^2 \right).
\]
We fix a finite-dimensional complex vector space, \( V \), and consider \( \Delta \) as an operator on the smooth functions on \( \mathbb{H} \) with values in \( V \). We further fix an endomorphism \( A \) of \( V \). This endomorphism may well be non-unitary or non-invertible. We will develop Fourier expansions for the smooth functions \( f : \mathbb{H} \to V \) that are twist-periodic with respect to \( A \),
\[
(30) \quad f(z + 1) = Af(z) \quad \text{for all } z \in \mathbb{H},
\]
and eigenfunctions of the Laplacian
\[
(31) \quad \Delta f = s(1 - s)f
\]
for some \( s \in \mathbb{C} \). We remark that each eigenvalue of \( \Delta \) can be represented in the form \( s(1 - s) \) with some \( s \in \mathbb{C} \). Unless \( s(1 - s) = 1/4 \), in which case \( s = 1/2 \), there are always two choices of values for \( s \). We will see that it is more convenient to associate to the function \( f \) a spectral parameter \( s \) instead of its eigenvalue \( s(1 - s) \).

We further remark that this setting is more general than it may seem at first glance. To elaborate on this, we recall that any horocycle on \( \mathbb{H} \) is isomorphic to a maximal unipotent subgroup of orientation-preserving Riemannian isometries of \( \mathbb{H} \). Vice versa, any such subgroup generates a family of concentric horocycles \([19, 34]\). For this reason we call such a subgroup a horocycle direction. Each of these subgroups is conjugate within the orientation-preserving Riemannian isometries to the one-parameter group
\[
\{ \mathbb{H} \to \mathbb{H}, \ z \mapsto z + r \}_{r \in \mathbb{R}},
\]
and the associated family of horocycles in \( \mathbb{H} \) consists of the “horizontal” lines
\[
\{ iy + \mathbb{R} \}_{y > 0}.
\]
If a map is (twist-)periodic on such a horocycle, then we may always assume that the (twist-)period is 1 after suitable rescaling (which is indeed another conjugation within the orientation-preserving Riemannian isometries). Therefore, any map that is twist-periodic in any horocycle direction is conjugate to a map satisfying (30).

In the remainder of this section we will introduce several objects most of which have to be defined separately for each Jordan block of \( A \) and which depend intimately on the size of a Jordan block and its eigenvalue.

For that reason we suppose in what follows that \( A \) acts irreducibly on \( V \) and is invertible.

In other words, we suppose that \( A \) has a single Jordan block, and that its eigenvalue, \( \lambda \), is not equal to 0. Further we let \( d := \dim V \) denote the dimension of \( V \) (as a complex vector space). All definitions that we will present in what follows easily generalize to the generic setting by applying them separately to each Jordan block. We will use this approach in Section 3.
Several of the objects that we will define further below will depend on the value of the eigenvalue $\lambda$ of $A$. However, to simplify notation, we will usually suppress this dependence.

2.3. **Complex logarithm and branch cuts.** We denote the real logarithm by $\log$, thus

$$\log : \mathbb{R}^+ \to \mathbb{R}.$$ 

For any complex number $z \in \mathbb{C}^\times$, we choose the standard branch for its argument $\arg z$, thus $\arg z \in (-\pi, \pi]$. Further, we denote the standard choice of the complex logarithm, the *principal logarithm*, by $\text{Log}$. Thus,

$$\text{Log} z = \log |z| + i \arg z$$

for all $z \in \mathbb{C}^\times$. As is well-known, the domain of holomorphy of $\text{Log}$ is $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

For several constructions below we will require a complex logarithm that is holomorphic at several points related to the eigenvalue $\lambda$ of $A$. Because we allow here any (nonzero) complex number as the value of $\lambda$, we will also choose, if necessary or convenient, a non-standard complex logarithm. However, we will restrict the admissible choices to those complex logarithms that extend the real logarithm $\log$.

We pick $\omega \in \mathbb{C}$, $\omega \notin [0, \infty)$, such that for each $n \in \mathbb{Z}$, at least one of

$$-i \text{Log} \lambda + 2\pi n$$

is not in $\omega \mathbb{R}^+_0$. We note that $\pm (i \text{Log} \lambda - 2\pi n) = 0$ if and only if $\lambda = 1$ and $n = 0$. In all other cases, at least one of the values in (33) is not in $\omega \mathbb{R}^+_0$. Clearly, the possible values for $\omega$ depend on $\lambda$ (which we do not reflect in the notation in favor of simplification), but many choices are possible. The definitions and constructions that we will present in what follows intimately depend on the choice of $\omega$. However, a different choice will result only in phase changes that do not qualitatively affect the results of Theorems 1.2, 1.3 and 1.4 and their refinements and extensions further below.

We use the set $\omega \mathbb{R}^+_0$ as a branch cut for a complex logarithm. In other words, we will use throughout the realization of the complex logarithm that is holomorphic at several points related to the eigenvalue $\lambda$ of $A$. Because we allow here any (nonzero) complex number as the value of $\lambda$, we will also choose, if necessary or convenient, a non-standard complex logarithm. However, we will restrict the admissible choices to those complex logarithms that extend the real logarithm $\log$.

We pick $\omega \in \mathbb{C}$, $\omega \notin [0, \infty)$, such that for each $n \in \mathbb{Z}$, at least one of

$$-i \text{Log} \lambda + 2\pi n$$

is not in $\omega \mathbb{R}^+_0$. We note that $\pm (i \text{Log} \lambda - 2\pi n) = 0$ if and only if $\lambda = 1$ and $n = 0$. In all other cases, at least one of the values in (33) is not in $\omega \mathbb{R}^+_0$. Clearly, the possible values for $\omega$ depend on $\lambda$ (which we do not reflect in the notation in favor of simplification), but many choices are possible. The definitions and constructions that we will present in what follows intimately depend on the choice of $\omega$. However, a different choice will result only in phase changes that do not qualitatively affect the results of Theorems 1.2, 1.3 and 1.4 and their refinements and extensions further below.

We use the set $\omega \mathbb{R}^+_0$ as a branch cut for a complex logarithm. In other words, we will use throughout the realization of the complex logarithm that is holomorphic on $\mathbb{C} \setminus \omega \mathbb{R}^+_0$ and coincides with the real logarithm on $\mathbb{R}^+$. We denote this logarithm by $\log(\cdot; \omega)$. To be more specific, we first note that choosing $\omega = -1$ leads to the standard choice of the complex logarithm, thus

$$\text{Log} = \log(\cdot; -1) : \mathbb{C} \to \mathbb{C},$$

with $\text{Log}$ being defined in (32). For generic $\omega \in \mathbb{C} \setminus [0, \infty)$ we set

$$\Omega_\omega := \begin{cases} 
(-\pi, \arg \omega) & \text{for } \arg \omega > 0, \\
(\arg \omega, \pi] & \text{for } \arg \omega < 0,
\end{cases}$$

and define the logarithm map

$$\log(\cdot; \omega) : \mathbb{C}^\times \to \mathbb{C}$$

by

$$\log(z; \omega) := \begin{cases} 
\log(z; -1) & \text{for } \arg z \in \Omega_\omega, \\
\log(z; -1) - 2\pi i \text{sgn}(\arg z) & \text{otherwise}
\end{cases}$$

for any $z \in \mathbb{C}^\times$. The set $\Omega_\omega$ is chosen such that $\log(\cdot; \omega)$ coincides with $\log$ on $\mathbb{R}^+$. The logarithm map $\log(\cdot; \omega)$ is holomorphic on $\mathbb{C} \setminus \Omega_\omega$. The relation between $\log(\cdot; \omega)$ and $\text{Log} = \log(\cdot; -1)$ is shown in Figure 1.
2.4. **Jordan-like blocks.** For any \( \mu \in \mathbb{C}, \mu \neq 0 \), we let \( J(\mu) \) be the \((d \times d)\)-matrix all of whose entries on the diagonal and superdiagonal equal \( \mu \) and all of whose other entries are 0:

\[
J(\mu) = \begin{pmatrix}
\mu & \mu & \mu & \cdots \\
\mu & \mu & \cdots & \mu \\
& \ddots & \ddots & \ddots \\
& & \mu & \mu \\
& & & \mu
\end{pmatrix}
\]

We call \( J(\mu) \) the **Jordan-like matrix** with eigenvalue \( \mu \). Using the Kronecker delta function

\[
\delta_0: \mathbb{C} \to \{0, 1\}, \quad \delta_0(z) := \begin{cases} 
1 & \text{if } z = 0 \\
0 & \text{otherwise}
\end{cases}
\]

we may describe this Jordan-like matrix by

\[
J(\mu) = \left( (j_{mn})_{m,n=1}^d \right) = \left( \mu \delta_0(m - n) + \mu \delta_0(m + 1 - n) \right)_{m,n=1}^d .
\]

As is well-known, the map

\[
\mathbb{Z} \to \text{Mat}(d \times d; \mathbb{C}), \quad k \mapsto J(\mu)^k,
\]

is a group homomorphism. To understand the effect of the twisting endomorphism \( A \) on the components of the Fourier coefficients in the real direction (i.e., the \( x \)-direction) we will need an extension of (38) to a continuous homomorphism from \( \mathbb{R} \) to \( \text{Mat}(d \times d; \mathbb{C}) \), applied to \( \mu \) being the eigenvalue \( \lambda \) of \( A \). See Theorems 1.2 and 2.5. The existence of such a continuous extension of (38) and its degree of (non-)uniqueness is well-known. However, for the convenience of the reader we now provide details. This also allows us—as a by-product—to explicitly track down the influence of the choice of the logarithm in this extension, for which reason we are here sticking to a Jordan-like matrix representation of the considered endomorphism.

The extension we will use throughout is provided in the following lemma, for which we recall that \( \lambda \) denotes the eigenvalue of \( A \), and that \( \log(\cdot; \omega) \) is the complex logarithm.

**Figure 1.** The left figure indicates the definition of \( \log(\cdot; \omega) \) for \( \arg \omega > 0 \) in relation to \( \log(\cdot; -1) \). The right figure shows the relation between \( \log(\cdot; \omega) \) and \( \log(\cdot; -1) \) if \( \arg \omega < 0 \).
logarithm chosen in Section 2.3. For any \( x \in \mathbb{R} \), we set
\[
\lambda^x := e^{x \log(\lambda; \omega)}.
\]
We use the standard notation \( \binom{x}{p} \) for the generalized binomial coefficients of \( x \in \mathbb{R} \) and \( p \in \mathbb{N}_0 \).

**Lemma 2.1.** Let \( J := J(\lambda) \). Then the map
\[
\mathbb{R} \to \text{Mat}(d \times d; \mathbb{C}), \quad x \mapsto J^x,
\]
with
\[
J^x := \lambda^x \begin{pmatrix}
\binom{x}{0} & \binom{x}{1} & \binom{x}{2} & \ldots & \binom{x}{d-1} \\
\binom{x}{0} & \binom{x}{1} & \binom{x}{2} & \ldots & \binom{x}{d-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\binom{x}{0} & \binom{x}{1} & \binom{x}{2} & \ldots & \binom{x}{d-1} \\
\binom{x}{0} & \binom{x}{1} & \binom{x}{2} & \ldots & \binom{x}{d-2}
\end{pmatrix},
\]
is a continuous homomorphism that extends the homomorphism in (38) for \( \mu = \lambda \).

**Remark 2.2.** We emphasize two facts about Lemma 2.1.

(i) Formula (41) shows that if the considered endomorphism is represented as a Jordan-like matrix, then the influence of the choice of the logarithm is separated from the inner structure of the matrix.

(ii) The map in Lemma 2.1 and its properties can be derived from the matrix logarithm and the matrix exponential map as follows. Even though this fact is well-known from, e.g., Lie theory, we provide here some details for the convenience of the reader. The matrix \( J \) decomposes as
\[
J = \lambda(I + N),
\]
where \( I \) is the \((d \times d)\)-identity matrix and \( N \) is the nilpotent matrix with ones (1’s) on the superdiagonal and zeros (0’s) elsewhere. Then
\[
\log J = \log(\lambda; \omega) I + \log(I + N),
\]
where
\[
\log(I + N) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} N^k = \sum_{k=1}^{d-1} \frac{(-1)^{k+1}}{k} N^k
\]
\[
= \begin{pmatrix}
0 & 1 & -\frac{1}{2} & \ldots & \ldots & \frac{(-1)^d}{d-1} & \frac{(-1)^{d-1}}{d-2} \\
0 & 1 & -\frac{1}{2} & \ldots & \ldots & \frac{(-1)^{d-2}}{d-2} & \frac{(-1)^{d-3}}{d-3} \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 1 & -\frac{1}{2} & \ldots & \ldots & 1 & 0
\end{pmatrix}.
\]

Here, we suppressed in the notation that the matrix logarithm \( \log J \) depends on the choice of \( \omega \). The matrix logarithm \( \log(I + N) \), however, is independent
Fourier expansions for non-unitary twists

Further we have

\[J^x = \exp(x \log J) = \exp(x \log(\lambda; \omega) I) \exp(x \log(I + N))\]

\[= \exp(x \log(\lambda; \omega)) \cdot \prod_{k=1}^{d-1} \exp((-1)^{k+1} \frac{x}{k} N^k)\]

\[= \exp(x \log(\lambda; \omega)) \cdot \exp\left(\begin{array}{cccc}
\left(\frac{x}{0}\right) & \left(\frac{x}{1}\right) & \cdots & \left(\frac{x}{d-1}\right) \\
\left(\frac{x}{0}\right) & \left(\frac{x}{1}\right) & \cdots & \left(\frac{x}{d-2}\right) \\
\vdots & \vdots & \ddots & \vdots \\
\left(\frac{x}{0}\right) & \left(\frac{x}{1}\right) & \cdots & \left(\frac{x}{1}\right) \\
\left(\frac{x}{0}\right) & \left(\frac{x}{1}\right) & \cdots & \left(\frac{x}{0}\right)
\end{array}\right)\]

\[= \lambda^x \cdot \exp\left(\begin{array}{cccc}
\left(\frac{x}{0}\right) & \left(\frac{x}{1}\right) & \cdots & \left(\frac{x}{d-1}\right) \\
\left(\frac{x}{0}\right) & \left(\frac{x}{1}\right) & \cdots & \left(\frac{x}{d-2}\right) \\
\vdots & \vdots & \ddots & \vdots \\
\left(\frac{x}{0}\right) & \left(\frac{x}{1}\right) & \cdots & \left(\frac{x}{1}\right) \\
\left(\frac{x}{0}\right) & \left(\frac{x}{1}\right) & \cdots & \left(\frac{x}{0}\right)
\end{array}\right)\]

Since the complex logarithm, understood in its initial form as the inverse relation to the exponential function, is not univalent, neither is the matrix logarithm. In turn, there are several real analytic extensions of the homomorphism in (38). The map in Lemma 2.1 is the unique real analytic extension that stays within real matrices for all real values of \(\lambda\) if \(\omega \in \mathbb{R}_{<0}\). We remark that for Lemma 2.1 it is not important that the complex logarithm is holomorphic at \(\lambda\) (i.e., in some neighborhood of \(\lambda\)). Below we will provide an alternative, rather hands-on proof of Lemma 2.1.

Picking another branch of the complex logarithm and hence another real analytic extension of the homomorphism in (38) would cause a re-indexing of the Fourier coefficients in Theorems 1.2, 2.5 and 1.3, but not a qualitative change in the results.

Proof of Lemma 2.1. We first show that the map in (40) is a group homomorphism. To simplify notation we set

\[\mathcal{J}(x) := J^x\]

for all \(x \in \mathbb{R}\), and recall that the matrix entries of \(\mathcal{J}(x)\) are

\[(42) \quad \mathcal{J}(x)_{ij} = \begin{cases} 
\lambda^x \left(\frac{x}{j-i}\right) & \text{if } i \leq j, \\
0 & \text{otherwise},
\end{cases}\]

for \(i, j \in \{1, \ldots, d\}\). Let \(x_1, x_2 \in \mathbb{R}\) and set \(B := \mathcal{J}(x_1)\mathcal{J}(x_2)\). In order to show that \(B = \mathcal{J}(x_1 + x_2)\) we first note that the matrix \(B\) is upper triangular as a product of such matrices. For the matrix entries of \(B\) in the upper triangle we proceed by direct calculation, taking advantage of (42). For the matrix entry \(B_{ij}\) with
\(i, j \in \{1, \ldots, d\}, i \leq j\), we find
\[
B_{ij} = \sum_{k=1}^{d} J(x_1)_{ik} J(x_2)_{kj} = \sum_{k=i}^{j} J(x_1)_{ik} J(x_2)_{kj}
\]
\[
= \lambda^{x_1+x_2} \sum_{k=i}^{j} \left( \begin{array}{c} x_1 \\ k-i \end{array} \right) \left( \begin{array}{c} x_2 \\ j-k \end{array} \right)
\]
\[
= \lambda^{x_1+x_2} \left( \begin{array}{c} x_1 + x_2 \\ j-i \end{array} \right),
\]
where we used the Chu–Vandermonde identity for the last equality. This shows that \(B = J(x_1 + x_2)\) and hence the map in (40), called \(J\) above, is a group homomorphism.

It remains to show that the map \(J\) coincides with the homomorphism in (38). Since \(J\) is already known to be a group homomorphism, it suffices to show the equality of both maps on \(N\). A straightforward proof by induction confirms that for all \(m \in \mathbb{N}\), the \(m\)-fold product of the matrix \(J\) equals \(J(m)\), and hence the claimed equality of maps. \(\square\)

2.5. The \(y\)-dependent Fourier coefficient functions. The main constituents of the \(y\)-dependent part of the Fourier coefficient functions in the Fourier expansion of an \(A\)-twisted Laplace eigenfunction are the modified Bessel functions and their derivatives, or certain specific linear combinations of \(y^a\) and \(y^a \log y\) with \(a \in \pm s + \frac{1}{2} \mathbb{Z}\), all depending on the spectral parameter \(s\).

The definition of the modified Bessel functions admits some flexibility as to their domain of holomorphy. Classically, they are defined as functions that are analytic on \(\mathbb{C} \setminus (-\infty, 0]\). See [63, p. 45]. Because we allow here any complex numbers as eigenvalues of the twisting endomorphism \(A\) in (30), we need to slightly adapt this classical definition in order to match our needs (more precisely, to allow for more compact statements) and change the domain of holomorphy of the modified Bessel functions accordingly. The choice of the complex logarithm was already discussed in Section 2.3. We will now discuss the construction of the modified Bessel functions.

We recall that the endomorphism \(A\) is invertible and has a single Jordan block by hypothesis. We further recall that the eigenvalue of \(A\) is denoted \(\lambda\).

2.5.1. Modified Bessel functions with suitable domain of holomorphy. The modified Bessel functions are classically defined as functions that are holomorphic on the cut plane \(\mathbb{C} \setminus (-\infty, 0]\). More precisely, the modified Bessel function of the first kind with index \(\eta \in \mathbb{C}\) is the map
\[
I_\eta : \mathbb{C}^\times \rightarrow \mathbb{C}
\]
given by
\[
I_\eta(z) := e^{\eta \log(z) - 1} \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(\eta + m + 1)} \left( \frac{z^2}{2} \right)^{2m}.
\]
See [63, p. 77]. Since the principal logarithm \(\log(\cdot) - 1\) is holomorphic on the cut plane \(\mathbb{C} \setminus (-\infty, 0]\), so is \(I_\eta\). The modified Bessel function of the second kind with index \(\eta \in \mathbb{C}\) is the map
\[
K_\eta : \mathbb{C}^\times \rightarrow \mathbb{C},
\]
which, for \( \eta \notin \mathbb{Z} \), is defined as
\[
K_\eta(z) := \frac{\pi I_{-\eta}(z) - I_\eta(z)}{\sin(\eta \pi)}.
\]
For \( \eta \in \mathbb{Z} \), it is
\[
K_\eta(z) := \lim_{\xi \to \eta} K_\xi(z).
\]
See [63, p. 78]. For any value of \( \eta \), the map \( K_\eta \) is holomorphic on \( \mathbb{C} \setminus (-\infty, 0] \).

For our applications we need a realization of each modified Bessel function that is holomorphic on \( \mathbb{C} \setminus \omega \mathbb{R}_{\geq 0} \), where \( \omega \mathbb{R}_{\geq 0} \) is the branch cut chosen in Section 2.3.

For that reason we will slightly redefine the maps in (43), (44) and (45) using the complex logarithm map \( \log(\cdot; \omega) \) from Section 2.3 as follows.

We define our variant of the modified Bessel function of the first kind with index \( \eta \in \mathbb{C} \) to be the map
\[
I_\eta(\cdot; \omega) : \mathbb{C}^\times \to \mathbb{C}
\]
given by
\[
I_\eta(z; \omega) := e^{\eta \log(z/2\omega)} \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(\eta + m + 1)} \left( \frac{z}{2} \right)^{2m}.
\]
For our variant of the modified Bessel function of the second kind with index \( \eta \in \mathbb{C} \) we use the map
\[
K_\eta(\cdot; \omega) : \mathbb{C}^\times \to \mathbb{C},
\]
which for \( \eta \notin \mathbb{Z} \) is given by
\[
K_\eta(z; \omega) := \frac{\pi I_{-\eta}(z; \omega) - I_\eta(z; \omega)}{\sin(\eta \pi)}.
\]
For \( \eta \in \mathbb{Z} \) we set
\[
K_\eta(z; \omega) := \lim_{\xi \to \eta} K_\xi(z; \omega).
\]
For any value of \( \eta \), the maps \( I_\eta(\cdot; \omega) \) and \( K_\eta(\cdot; \omega) \) are holomorphic on \( \mathbb{C} \setminus \omega \mathbb{R}_{\geq 0} \).

They are closely related to the classical modified Bessel functions. For the modified Bessel function of the first kind we find
\[
I_\eta(z; \omega) = \begin{cases} 
I_\eta(z) & \text{for } \arg z \in \Omega_\omega, \\
I_\eta(z) \cdot e^{-2\pi i \eta \text{sgn}(\arg \omega)} & \text{otherwise}, 
\end{cases}
\]
where \( \Omega_\omega \) is the set defined in (34). See Figure 2. For the modified Bessel function of the second kind with a non-integer index, say \( \eta \in \mathbb{C} \setminus \mathbb{Z} \), we have
\[
K_\eta(z; \omega) = \frac{\pi I_{-\eta}(z) - I_\eta(z)}{\sin(\eta \pi)} = K_\eta(z)
\]
if \( \arg z \in \Omega_\omega \), and
\[
K_\eta(z; \omega) = \frac{\pi e^{2\pi i \eta \text{sgn}(\arg \omega)} I_{-\eta}(z) - e^{-2\pi i \eta \text{sgn}(\arg \omega)} I_\eta(z)}{\sin(\eta \pi)} = \frac{\pi}{2} I_{-\eta}(z) \cdot e^{2\pi i \eta \text{sgn}(\arg \omega)} - e^{-2\pi i \eta \text{sgn}(\arg \omega)} + e^{-2\pi i \eta \text{sgn}(\arg \omega)} K_\eta(z)
\]
if \( \arg z \notin \Omega_\omega \), and
\[
K_\eta(z; \omega) = 2\pi i \text{sgn}(\arg \omega) \cdot \cos(\pi \eta) \cdot I_{-\eta}(z) + e^{-2\pi i \eta \text{sgn}(\arg \omega)} K_\eta(z)
\]
Figure 2. The left figure indicates the relation between $I_\eta(\cdot;\omega)$ and $I_\eta$ for $\arg\omega > 0$; the right figure for $\arg\omega < 0$.

Figure 3. The left figure shows the relation between $K_\eta(\cdot;\omega)$ and $K_\eta$ for $\arg\omega > 0$; the right figure for $\arg\omega < 0$.

if $\arg z \notin \Omega_\omega$. See Figure 3. For any integer index $\eta \in \mathbb{Z}$, the relation simplifies to

\[
K_\eta(z;\omega) = \begin{cases} 
K_\eta(z) & \text{for } \arg z \in \Omega_\omega, \\
K_\eta(z) + 2\pi i s(\omega) \cdot (-1)^\eta \cdot I_{-\eta}(z) & \text{otherwise},
\end{cases}
\]

where $s(\omega) := \text{sgn}(\arg\omega)$. We note that for any value of $\eta$, the maps $I_\eta(\cdot;\omega)$ and $K_\eta(\cdot;\omega)$ coincide with $I_\eta$ and $K_\eta$, respectively, on $\mathbb{R} > 0$.

2.5.2. The functions $I$ and $K$ for $\alpha \neq 0$. With the complex logarithm and the modified Bessel function chosen adapted to the value of the eigenvalue $\lambda$ of $A$ we can now define the principal constituents for the $y$-dependent Fourier coefficient functions, namely the two functions

$I, K: \mathbb{N}_0 \times (\mathbb{C} \setminus \omega \mathbb{R} \geq 0) \times \mathbb{R} > 0 \times \mathbb{C} \to \mathbb{C},$

which provide the necessary generalization of the Fourier coefficient functions from the untwisted case.

If the second argument is not zero, then these functions are essentially given by derivatives of the modified Bessel functions. More precisely, for any $m \in \mathbb{N}_0$, $\alpha \in \mathbb{C} \setminus \omega \mathbb{R} \geq 0$, $y \in \mathbb{R} > 0$ and $s \in \mathbb{C}$ we set

\[
I(m,\alpha, y, s) := \frac{y^2}{i^m} \partial_\alpha^m \left( I_{s - \frac{1}{4}}(\alpha y; \omega) \right),
\]
We caution that the differential operator $\partial^m_\alpha$ is not the plain $m$-th derivative of $\Psi_{s-\frac{1}{2}}(\cdot; \omega)$ for $\Psi \in \{I, K\}$ but acts on the map

$$\alpha \mapsto \Psi_{s-\frac{1}{2}}(\alpha y; \omega),$$

as shall be indicated by the additional brackets around $\Psi_{s-\frac{1}{2}}(\alpha y; \omega)$. We further remark that the functions $I$ and $K$ depend on the choice of $\omega$. To keep the notational complexity to a minimum, we do not reflect the dependency in the symbols for these functions. However, it is visible from their domains.

If the second argument is zero, then the definition of the functions $I$ and $K$ is qualitatively different. We will provide it in the next section.

As we can see already in the statement of Theorem 1.2, the Fourier coefficients with nonzero indices are linear combinations of the two functions $I$ and $K$ for appropriate values of $\alpha \neq 0$. The zeroth Fourier coefficient, however, may also require the values of $I$ and $K$ for $\alpha = 0$.

### 2.5.3. The functions $I$ and $K$ for $\alpha = 0$

If the eigenvalue $\lambda$ of $A$ is 1 (and only in this case), then we will need in addition the values of the functions $I$ and $K$ with vanishing second argument for the Fourier expansion of the considered $A$-twisted Laplace eigenfunction, namely for its zeroth Fourier coefficient. As in the untwisted case, the zeroth Fourier coefficient is differently structured than all others, and attention needs to be paid to exceptional spectral parameters.

In the untwisted case, $s = 1/2$ is the only exceptional spectral parameter for the zeroth term of the Fourier expansion. In our situation, with the dimension $d$ of the vector space $V$ allowed to be any finite natural number, the set of exceptional values of the spectral parameter becomes

$$E(d) := \left\{ \frac{1}{2} \pm j : j = 0, 1, \ldots, \left\lfloor \frac{d-1}{2} \right\rfloor \right\}$$

$$= \left\{ \frac{1}{2} - \left\lfloor \frac{d-1}{2} \right\rfloor, \frac{1}{2} - \left( \left\lfloor \frac{d-1}{2} \right\rfloor + 1 \right), \ldots, \frac{1}{2} + \left\lfloor \frac{d-1}{2} \right\rfloor \right\}.$$

The definitions of $I$ and $K$ for vanishing second argument (thus, $\alpha = 0$) require three series of combinatorial coefficients, which we now discuss. Each series consists of a function that is defined on a certain subset of $\mathbb{N}_0 \times \mathbb{C}$.

The first series is given by a function, $p$, defined on the subset of $\mathbb{N}_0 \times \mathbb{C}$ that consists of the pairs $(k, s)$ that satisfy

$$s \notin \left\{ \frac{1}{2} - j : j = 1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor \right\}.$$

We set

$$p(0, s) := p(1, s) := 1.$$
and, for \( k \geq 2 \),
\[
(58) \quad p(k, s) := (-1)^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{k!}{2^{\left\lfloor \frac{k}{2} \right\rfloor} \prod_{j=1}^{\left\lfloor \frac{k}{2} \right\rfloor} (2j + 2s - 1) }.
\]

Using the convention that the empty product equals 1, the definitions in (57) may be subsumed into (58). Moreover, by taking advantage of the Gamma function, we can express the definitions in (57) and (58) as
\[
(59) \quad p(k, s) = \left( -\frac{1}{4} \right)^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{\Gamma(k + 1) \Gamma\left(s + \frac{1}{2}\right)}{\Gamma\left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) \Gamma\left(s + \left\lfloor \frac{k}{2} \right\rfloor + \frac{1}{2}\right)}
\]
for all pairs \((k, s) \in \mathbb{N}_0 \times \mathbb{C}\) obeying (56).

The second series is given by a function, \( q \), defined on the set \( \mathbb{N}_0 \times \left( E(d) \cap (-\infty, \frac{1}{2}] \right) \).

We set
\[
(60) \quad q(0, 1/2) := q(1, 1/2) := 1.
\]

For \( s \in E(d) \cap (-\infty, -\frac{1}{2}] \) we set
\[
(61) \quad q(0, s) := (2s)p(-2s - 1, s), \quad q(1, s) := (2s - 2)p(-2s, s).
\]

For \( k \geq 2 \) and any \( s \in E(d) \cap (-\infty, \frac{1}{2}] \), we define
\[
(62) \quad q(k, s) := -\frac{1}{4} \frac{(1 - 2s + k)(-2s + k)}{\frac{k}{2} \choose s} q(k - 2, s)
\]

Using (59), we may write (62) also as
\[
(63) \quad q(k, s) = \begin{cases} 
\left( -\frac{1}{4} \right)^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{k!}{\prod_{j=1}^{\left\lfloor \frac{k}{2} \right\rfloor} (1/2 - s + j)} q(0, s) & \text{for } k \text{ even}, \\
\left( -\frac{1}{4} \right)^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{k!}{\prod_{j=1}^{\left\lfloor \frac{k}{2} \right\rfloor} (1/2 - s + j)} q(1, s) & \text{for } k \text{ odd}.
\end{cases}
\]

**Remark 2.3.** In (61), we use the function \( p \) at the value \((k, s) = (-2s - 1, s)\) with \( s \in E(d) \cap (-\infty, -\frac{1}{2}] \). We now check that \( p \) is indeed defined at this point. From \( s \leq -\frac{1}{2} \) it follows that \( k \geq 0 \). Further, since \( s \in E(d) \), we have \( k \in \mathbb{N}_0 \) and
\[
\left\lfloor \frac{k}{2} \right\rfloor = -s - \frac{1}{2},
\]
and hence
\[
s < \frac{1}{2} - \left\lfloor \frac{k}{2} \right\rfloor = s + 1.
\]

Therefore, \((k, s)\) is an admissible pair for which \(p\) can be applied (see also (56)). Analogously, one checks that \(p(-2s, s)\) is well-defined.

We further remark that for \(s = \frac{1}{2}\), neither the point \((-2s - 1, s) = (-2, s)\) nor the point \((-2s, s) = (-1, s)\) are in the domain of \(p\). For this reason, (60) cannot be subsumed into (61).

The third series is given by a function, \(r\), defined on
\[
\mathbb{N}_0 \times \left( E(d) \cap \left( -\infty, \frac{1}{2} \right] \right).
\]
We set
\[
(64) \quad r(0, s) := r(1, s) := 0
\]
and, for \(k \in \{2, 3\},
\[
(65) \quad r(2j, s) := \frac{1}{2} - s + 2\left\lfloor \frac{k}{2} \right\rfloor \quad q(k, s).
\]
For \(k \geq 4\) we set
\[
(66) \quad r(k, s) := -\frac{1}{2} - s + 2\left\lfloor \frac{k}{2} \right\rfloor \quad q(k, s)
\]
\[
\text{with } j \geq 1 \quad (-1)^{\frac{k}{2} + j + 1} \quad \frac{1}{2j \left( \frac{1}{2} - s + j \right)} \quad q(2j + \delta_{\text{odd}}(k), s),
\]
where
\[
(67) \quad \delta_{\text{odd}}(k) := \begin{cases} 1 & \text{for } k \text{ odd}, \\ 0 & \text{for } k \text{ even}. \end{cases}
\]
Using the convention that empty sums equal 0, we may subsume (65) in (66).

With these preparations we can now define the functions \(I\) and \(K\) for arguments with vanishing second component. We recall that the function \(p\) is not defined on all of \(\mathbb{N}_0 \times \mathbb{C}\). See (56). To simplify the statements of the functions \(I\) and \(K\) in Definition 2.4, we extend the function \(p\)—somewhat arbitrarily—to the pairs \((k, s) \in \mathbb{N}_0 \times \mathbb{C}\) with \(s\) being contained in the “forbidden” set in (56) by setting
\[
p(k, s) := 1.
\]
In our applications, we will never use the functions \(I\) and \(K\) at those arguments for which the definition uses the function \(p\) at these special values.

**Definition 2.4.** Let \(m \in \mathbb{N}_0\), \(y \in \mathbb{R}_{>0}\) and \(s \in \mathbb{C}\).
(i) For
\[ s \in E(d) \cap \left[ \frac{1}{2} - m, \frac{1}{2} \right] \]
we set
\[
I(m, 0, y, s) := q(m - 1 + 2s, s)y^{s+2\left\lfloor \frac{m}{2} \right\rfloor} \log y
+ r(m - 1 + 2s, s)y^{s+2\left\lfloor \frac{m}{2} \right\rfloor}.
\]
For all other values of \( s \) we set
\[
I(m, 0, y, s) := p(m, s)y^{s+2\left\lfloor \frac{m}{2} \right\rfloor}.
\]

(ii) For
\[ s \in E(d) \cap \left[ \frac{3}{2}, \frac{1}{2} + m \right] \]
we set
\[
K(m, 0, y, s) := q(m + 1 - 2s, 1 - s)y^{1-s+2\left\lfloor \frac{m}{2} \right\rfloor} \log y
+ r(m + 1 - 2s, 1 - s)y^{1-s+2\left\lfloor \frac{m}{2} \right\rfloor}.
\]
For all other values of \( s \) we set
\[
K(m, 0, y, s) := p(m, 1 - s)y^{1-s+2\left\lfloor \frac{m}{2} \right\rfloor}.
\]

2.6. Matrices of scalar coefficients. As a last component for the statement of the refined version of Theorem 1.2 we now define two sets of coefficient matrices.

For \( n, k \in \mathbb{N}_0 \) we denote by \( s(n, k) \) the \textit{signed Stirling number of the first kind}, which is the number of permutations of \( n \) elements with \( k \) disjoint cycles. The Stirling numbers are also characterized as the coefficients in the polynomial
\[
\prod_{\ell=0}^{n-1} (x - \ell) = \sum_{k=0}^{n} s(n, k)x^k, \quad x \in \mathbb{R},
\]
where we use again the convention that empty products equal 1, and empty sums equal 0. In particular, we have
\[ s(0, 0) = 1, \quad \text{and} \quad s(n, k) = 0 \quad \text{for} \ n, k \in \mathbb{N}, \ n < k. \]

Further we use \( U = (u_{ij}) \) to denote the \((d \times d)\)-Stirling matrix. We will use it as a transformation to untangle a system of differential equations, hence the notation \( U \) for this matrix. Its entries are
\[
u_{ij} = \frac{s(d - i, d - j)}{(d - i)!}
\]
for \( i, j \in \{1, \ldots, d\} \).

We denote by \( \text{Coeff}_{1,d} \) the subset of matrices \((c_{ij}) \in \text{Mat}(d \times d; \mathbb{C})\) whose entries satisfy
\[ c_{ij} = 0 \quad \text{for} \ i > j, \]
and for \( i \in \{1, \ldots, d\} \) and \( p \in \{0, \ldots, d - i\} \)
\[
c_{i, i+p} = c_{1, i+p} \frac{\prod_{k=1}^{i-1} (d - p - k)}{\prod_{k=1}^{i-1} (d - k)}.
\]
Finally, we let \( \text{Coeff}_{0,d} \) denote the subset of matrices \((c_{ij}) \in \text{Mat}(d \times d; \mathbb{C})\) whose entries satisfy

\[
c_{i+2k,j+k} = c_{ij} \frac{2k-1}{\prod_{p=0}^{2k-1} (d - (i + p))} \prod_{p=0}^{2k-1} (d - (j + p))
\]

for all possible combinations of \( i, j, k \in \{1, \ldots, d\} \), and

\[
c_{ij} = 0 \quad \text{for } i \in \{3, \ldots, d\}, j \in \left\{1, \ldots, 2 \left\lfloor \frac{i-1}{2} \right\rfloor \right\}.
\]

We note that \( \text{Coeff}_{1,d} \) is a \(d\)-dimensional complex subspace of upper triangular matrices, the entries of which are completely determined by the entries of the first row. For \( d = 1 \), the set \( \text{Coeff}_{0,d} \) is \(1\)-dimensional, coinciding with \( \mathbb{C} \). For \( d \geq 2 \), the set \( \text{Coeff}_{0,d} \) is a \(2d\)-dimensional subspace of \( \text{Mat}(d \times d; \mathbb{C}) \), the entries of which are determined by the first two rows.

### 2.7. Refined statement of Theorem 1.2

With all definitions in place, we can now present one of our main results, the refined statement of Theorem 1.2. To that end, for each \( n \in \mathbb{Z} \), we set

\[
\alpha_n := 2\pi n - i \log(\lambda; \omega),
\]

pick \( \varepsilon_n \in \{0, 1\} \) such that

\[
(-1)^{\varepsilon_n} \alpha_n \notin \omega \mathbb{R}_{>0},
\]

and set

\[
\tilde{\alpha}_n := (-1)^{\varepsilon_n} \alpha_n.
\]

Further we let \( S \in \text{Mat}(d \times d; \mathbb{C}) \) be the “alternating” diagonal matrix

\[
S := \begin{pmatrix}
(-1)^{d-1} & & & \\
& (-1)^{d-2} & & \\
& & \ddots & \\
& & & -1
\end{pmatrix},
\]

the sign matrix.

**Theorem 2.5** (Single Jordan block; refined statement). Let \( V \) be a \(d\)-dimensional complex vector space, identified with \( \mathbb{C}^d \). Let \( f : \mathbb{H} \to V \) be a smooth map such that

\[
\Delta f = s(1 - s)f
\]

for some \( s \in \mathbb{C} \), and

\[
f(z + 1) = Jf(z) \quad \text{for all } z \in \mathbb{H},
\]

where \( J = J(\lambda) \) is a Jordan-like matrix with \( \lambda \neq 0 \). Then the Fourier expansion of \( f \) is

\[
f(x + iy) = \sum_{n \in \mathbb{Z}} J^x \hat{f}_n(y, s)e^{2\pi i nx}
\]
for all $z = x + iy \in \mathbb{H}$, where, for each $n \in \mathbb{Z}$, the Fourier coefficient function is

$$
\hat{f}_n(y, s) = UC_nS_n
\begin{pmatrix}
I(d - 1, \tilde{\alpha}_n, y, s) \\
\vdots \\
I(1, \tilde{\alpha}_n, y, s) \\
I(0, \tilde{\alpha}_n, y, s)
\end{pmatrix} + UD_nS_n
\begin{pmatrix}
K(d - 1, \tilde{\alpha}_n, y, s) \\
\vdots \\
K(1, \tilde{\alpha}_n, y, s) \\
K(0, \tilde{\alpha}_n, y, s)
\end{pmatrix}
$$

with

$$C_n, D_n \in \begin{cases}
\text{Coeff}_0 & \text{if } \tilde{\alpha}_n = \alpha_n = 0, \\
\text{Coeff}_1 & \text{otherwise.}
\end{cases}$$

The proof of Theorem 2.5 will be discussed in Sections 4–8.

### 3. Complexity Reduction

In this section we show how Theorem 1.3 follows immediately from Proposition 1.1 and Theorem 2.5 (or Theorem 1.2). In addition we dispatch Proposition 1.1, which is the “trivial case” in which the Fourier expansion vanishes identically.

Throughout this section let $V$ be a finite-dimensional complex vector space and let $A$ be any endomorphism of $V$. We do not assume that $A$ acts irreducibly or is invertible.

**Proof of Theorem 1.3 assuming Proposition 1.1 and Theorem 2.5.** We consider a smooth map $f : \mathbb{H} \to V$ that satisfies

$$\Delta f = s(1-s)f \quad \text{and} \quad f(z+1) = Af(z) \quad \text{for all } z \in \mathbb{H}.$$  

We fix a basis of $V$ with respect to which the endomorphism $A$ is represented by a matrix $J$ in Jordan-like normal form, say

$$J = 
\begin{pmatrix}
J_1 \\
\vdots \\
J_p
\end{pmatrix}
$$

such that for each $j \in \{1, \ldots, p\}$, the submatrix $J_j$ is a Jordan-like block matrix of size $(d_j \times d_j)$ with eigenvalue $\lambda_j \in \mathbb{C}$. See (4)–(6). Let

$$V = V_1 \oplus \ldots \oplus V_p \quad \text{and} \quad f = f_1 \oplus \ldots \oplus f_p$$

be the associated decompositions of $V$ and $f$. The linearity of $A$ and the property that $f$ is $A$-twisted imply that for each $j \in \{1, \ldots, p\}$, the map $f_j$ is $J_j$-twisted:

$$f_j(z + 1) = J_jf_j(z) \quad \text{for all } z \in \mathbb{H}.$$  

For $j \in \{1, \ldots, p\}$ with $\lambda_j = 0$, Proposition 1.1 now implies that $f_j = 0$. Moreover, the linearity of $\Delta$ and the property that $\hat{f}$ is a $\Delta$-eigenfunction with spectral parameter $s$ imply that $f_j$ is so for each $j \in \{1, \ldots, p\}$. Further, if we fix for all $j \in \{1, \ldots, p\}$ with $\lambda_j = 0$ any (possibly non-continuous) extension of the map

$$N_0 \to \text{Mat}(d_j \times d_j; \mathbb{C}), \quad n \mapsto J^n_j,$$

to a map (not necessarily a homomorphism)

$$\mathbb{R} \to \text{Mat}(d_j \times d_j; \mathbb{C}), \quad x \mapsto J^x_j,$$
and use the extensions discussed in Lemma \ref{lemma:extension} for those \( j \in \{1, \ldots, p \} \) with \( \lambda_j \neq 0 \), then
\[
J^x = \begin{pmatrix}
J^x_1 \\
\vdots \\
J^x_p
\end{pmatrix}
\]
for all \( x \in \mathbb{R} \). Therefore, linearity yields that the Fourier expansion of \( f \) is the direct sum of the separate Fourier expansions of the functions \( f_j, j \in \{1, \ldots, p\} \). Now taking advantage of the Fourier expansion for the components \( f_j \) for \( j \in \{1, \ldots, p\} \) with \( \lambda_j \neq 0 \) as developed in Theorem \ref{thm:fourier_expansion} completes the proof of Theorem \ref{thm:main}.

As a first step to establish the conditions on which the previous proof relies we now prove the following statement on vanishing eigenfunctions for nilpotent endomorphisms, of which Proposition \ref{prop:nilpotent_eigenfunctions} is a special case.

**Proposition 3.1.** Suppose that the map \( f: \mathbb{H} \to V \) satisfies
\[
f(z + 1) = Af(z)
\]
for all \( z \in \mathbb{H} \). If \( A \) is nilpotent, then \( f = 0 \).

**Proof.** Let \( d := \dim V \). For all \( z \in \mathbb{H} \) we have \( f(z + d) = A^d f(z) \). Now \( A^d = 0 \) yields \( f = 0 \).

\[
\square
\]

4. Periodization

In the course of this section and the following ones we will provide a proof of Theorem \ref{thm:fourier_expansion}. Throughout these sections we suppose that \( V \) is a complex vector space of dimension \( d \in \mathbb{N} \) and that \( A \) is an invertible endomorphism of \( V \) that acts irreducibly on \( V \). We fix a basis of \( V \) with respect to which \( A \) is represented by the Jordan-like matrix
\[
J = J(\lambda) = \begin{pmatrix}
\lambda & 0 & \cdots & 0 \\
1 & \lambda & \cdots & 0 \\
0 & 1 & \ddots & \ddots \\
0 & 0 & \ddots & \lambda \\
0 & 0 & \cdots & 1
\end{pmatrix}
\]
for some \( \lambda \in \mathbb{C}^\times \), which is then the eigenvalue of \( A \). Using this basis, we identify the vector spaces \( V \) and \( \mathbb{C}^d \). Further, we fix a complex logarithm via a choice of \( \omega \in \mathbb{C} \) satisfying the properties in Section \ref{sec:complex_logarithm} and we let \( f: \mathbb{H} \to V \) be a smooth map that is \( J \)-twist periodic (or \( A \)-twist periodic) with period 1:
\[
f(z + 1) = Jf(z) \quad \text{for all } z \in \mathbb{H}.
\]
Starting with the next section, we will suppose in addition that \( f \) is a Laplace eigenfunction. For the results of the present section, this additional hypothesis is not needed.

At the end of Section \ref{sec:preliminaries} we briefly outlined the strategy of the proof of Theorem \ref{thm:fourier_expansion}. As explained there, the first major obstacle towards a Fourier expansion of \( f \) is its non-periodicity. In this section we will show how to overcome it by means of a periodization, which uses in a crucial way that the map
\[
\mathbb{R} \to \text{Mat}(d \times d; \mathbb{C}), \quad x \mapsto J^x,
\]
is a group homomorphism (see Lemma \ref{lemma:extension}). We note that this approach is also known from the study of monodromy of ordinary differential equations, vector bundles and modular forms. In addition we will provide a vanishing statement
Lemma 4.2] and a differentiability statement (Lemma 4.3) that will be important for investigating the fine structure of the Fourier coefficient functions.

**Proposition 4.1.** The map $F : \mathbb{H} \rightarrow V$,

$$F(z) := J^{-z} f(z) \quad (z = x + iy \in \mathbb{H}),$$

is smooth and periodic with period 1.

**Proof.** To establish the periodicity of $F$, we recall from Lemma 2.1 that the map

$$(78) \quad \mathbb{R} \rightarrow \text{Mat}(d \times d; \mathbb{C}), \quad x \mapsto J^{x},$$

is a group homomorphism. For any $z = x + iy \in \mathbb{H}$ we find

$$F(z + 1) = J^{-(x+1)} f(z + 1) = J^{-x-1} J f(z) = J^{-z} f(z) = F(z).$$

The smoothness of $F$ follows immediately from the smoothness of $f$ and the real-analyticity of the map in (78) (see Lemma 2.1).

Since the map $F$ defined in Proposition 4.1 is smooth and periodic with period 1, we may expand it in a Fourier series:

$$(79) \quad F(z) = J^{-z} f(z) = \sum_{n \in \mathbb{Z}} \hat{f}_{n}(y) e^{2\pi inx}$$

for all $z = x + iy \in \mathbb{H}$, with suitable Fourier coefficient functions $\hat{f}_{n}$, $n \in \mathbb{Z}$. Re-arranging, we obtain a Fourier expansion for $f$,

$$(80) \quad f(z) = \sum_{n \in \mathbb{Z}} J^{x} \hat{f}_{n}(y) e^{2\pi inx} \quad (z = x + iy \in \mathbb{H}).$$

In order to find a more explicit description of the Fourier coefficient functions $\hat{f}_{n}$, $n \in \mathbb{Z}$, we will differentiate the Fourier series in (80) term by term. See Section 5. This will result in summands involving the functions

$$\mathbb{R} \rightarrow \mathbb{C}, \quad x \mapsto x^{k} e^{2\pi inx},$$

for $k \in \mathbb{N}_{0} \cap [0, d]$ and $n \in \mathbb{Z}$. The polynomial terms are caused by the presence of $J^{x}$. Whereas the functions in $\{ e^{2\pi inx} \}_{n \in \mathbb{Z}}$ are pairwise orthogonal in $L^{2}([0, 1])$, the functions in $\{ x^{k} e^{2\pi inx} \}_{n \in \mathbb{Z}, k \in \mathbb{N}_{0} \cap [0, d]}$ lose this property. We therefore require the following lemma in order to push the study of the coefficient functions further.

**Lemma 4.2.** Let $(P_{n})_{n \in \mathbb{Z}}$ be a sequence of complex polynomials with uniformly bounded degrees, i.e., sup_{n \in \mathbb{Z}} \deg P_{n} < \infty. Suppose that for all $x \in \mathbb{R},$

$$(81) \quad \sum_{n \in \mathbb{Z}} P_{n}(x) e^{2\pi inx} = 0.$$ 

Then $P_{n} = 0$ for all $n \in \mathbb{Z}$ and, in particular, $(e^{2\pi inx} P_{n}(x))_{|x=0} = 0$.

**Proof.** The second statement clearly follows from the first. We will prove the first statement by induction on $\max_{n \in \mathbb{Z}} \deg P_{n} := D$. We use the convention that the degree of the zero polynomial is $-1$. Let $D \leq 0$. Then each polynomial $P_{n}$, $n \in \mathbb{N}$, is constant (also allowing the constant 0). Say, $P_{n} = p_{n} \in \mathbb{C}$ for $n \in \mathbb{Z}$. By (81) we have

$$\sum_{n \in \mathbb{Z}} p_{n} e^{2\pi inx} = 0 \quad \text{for all } x \in \mathbb{R}.$$

Elementary Fourier analysis implies $p_{n} = 0$ for all $n \in \mathbb{Z}$. Now we proceed to the induction step by assuming that the statement holds true for all polynomials
degree at most $\mathcal{D} - 1$ with $\mathcal{D} > 0$. Let $(P_n)_{n \in \mathbb{Z}}$ be a sequence of polynomials of degree at most $\mathcal{D}$ which satisfies (81). For all $x \in \mathbb{R}$ we have therefore
\[
\sum_{n \in \mathbb{Z}} P_n(x) e^{2\pi i nx} = 0 \quad \text{and} \quad \sum_{n \in \mathbb{Z}} P_n(x+1) e^{2\pi i n(x+1)} = 0,
\]
and hence (note that $e^{2\pi i n} = 1$)
\[
\sum_{n \in \mathbb{Z}} e^{2\pi i nx} (P_n(x+1) - P_n(x)) = 0.
\]
Since $\mathcal{D} > 0$, $P_n(x+1) - P_n(x)$ is a polynomial of degree at most $\mathcal{D} - 1$ for all $n \in \mathbb{Z}$.

Thus, the inductive hypothesis implies that $P_n(x+1) - P_n(x) = 0$ for all $x \in \mathbb{R}$ and all $n \in \mathbb{Z}$. Consequently, each polynomial $P_n$, $n \in \mathbb{Z}$, is constant. By the base case of induction, $P_n = 0$ for all $n \in \mathbb{Z}$.

We end this section by showing a differentiability property of the series in (80).

**Lemma 4.3.** The series in (80) may be differentiated infinitely often term by term both in $x$ and in $y$.

**Proof.** We recall the map $F$ from Proposition 4.1,
\[
F(z) = J^{-x} f(z) \quad (z = x + iy \in \mathbb{H}).
\]
To prove that the series in (80) has the claimed differentiability properties, it suffices to establish them for the Fourier series of $F$, given in (79). For every fixed $y > 0$, the map
\[
\mathbb{R} \to V, \quad x \to F(x + iy),
\]
is smooth by Proposition 4.1, and the series in (79) is its Fourier series (with the functions $\hat{f}_n$ evaluated at the fixed value of $y$). Therefore the derivative of this Fourier series (in $x$) is given by differentiating it termwise in $x$. See [25, Theorem 2.3]. Iteration of this argument proves the claimed differentiability properties in $x$.

For differentiation in $y$ we use the Leibniz integral rule. For $k, m \in \mathbb{N}_0$, $n \in \mathbb{Z}$ and $y > 0$ we have
\[
\hat{f}_n(y) = \int_0^1 F(x + iy) e^{-2\pi i nx} dx
\]
and
\[
(2\pi n)^{4m} \partial_y^k \hat{f}_n(y) = \int_0^1 \partial_y^k F(x + iy) \cdot (2\pi n)^{4m} e^{-2\pi i nx} dx.
\]
The exchange of differentiation and integration in the second equation is justified by smoothness of the integrand. Integrating by parts together with the fact that $F$ is 1-periodic shows that the boundary terms vanish. Therefore we obtain
\[
(2\pi n)^{4m} \partial_y^k \hat{f}_n(y) = \int_0^1 \partial_x^{4m} \partial_y^k F(x + iy) \cdot e^{-2\pi i nx} dx.
\]
Hence for any compact set $K$ in $\mathbb{H}$, there exists $C_{k,K,m} > 0$ such that for any $z = x + iy \in K$,
\[
\left| \partial_y^k \hat{f}_n(y) \right| \leq C_{k,K,m} n^{-4m}.
\]
Consequently, for any \( z = x + iy \in K \),
\[
|e^{2\pi i nx} J^{-x} \partial_y^k \hat{f}_n(y)| \leq C \cdot |x^d \partial_y^k \hat{f}_n(y)| \leq \tilde{C}_{k,K,m}|x^d|n^{-4m}
\]
for appropriate positive constants \( C \) and \( \tilde{C}_{k,K,m} > 0 \), the latter potentially depending on \( k, K, m \). We recall that \( F(z) = J^{-z}f(z) \). Hence, for sufficiently large \( m \), the \( k \)-th termwise derivative of \( (79) \) in \( y \) converges absolutely and uniformly on compact subsets of \( \mathbb{H} \). Applying the Leibniz integral rule completes the proof. \( \square \)

5. Differential equations for the Fourier coefficient functions

Starting with this section we add to the hypotheses from Section 4 that \( f \) be a Laplace eigenfunction. Thus, in total we suppose that \( V \) is a complex vector space of dimension \( d \in \mathbb{N} \) which we identify throughout with \( \mathbb{C}^d \), and \( A \) is an endomorphism of \( V \) that is represented by the Jordan-like block matrix \( J = J(\lambda) \) with \( \lambda \in \mathbb{C}, \lambda \neq 0 \). Further \( f : \mathbb{H} \to V \) is \( J \)-twist periodic (or, equivalently, \( A \)-twist periodic), that is
\[
f(z+1) = Jf(z) \quad \text{for all } z \in \mathbb{H},
\]
and moreover, \( f \) is a Laplace eigenfunction
\[
\Delta f = s(1-s)f.
\]
Unless the Laplace eigenvalue \( s(1-s) \) of \( f \) is \( 1/4 \), there are two choices for the spectral parameter \( s \) of \( f \). In this section, all considerations will be independent of this choice as all results only involve the eigenvalue \( s(1-s) \), not the spectral parameter. Nevertheless, we will express the eigenvalue using the spectral parameter to keep the necessary notation to a minimum. We fix a complex logarithm by choosing \( \omega \in \mathbb{C} \) with the properties as in Section 2.3 and recall from (80) that \( f \) admits a Fourier expansion of the form
\[
f(z) = \sum_{n \in \mathbb{Z}} J^x \hat{f}_n(y) e^{2\pi i nx} \quad (z = x + iy \in \mathbb{H}).
\]
In this section we will show that just as in the scalar case, each of the Fourier coefficient functions, \( \hat{f}_n \), satisfies a certain differential equation, depending on \( n \in \mathbb{Z} \). In contrast to the scalar case, the differential equation is vector-valued, so indeed a system of differential equations.

Lemma 5.1. The Fourier coefficient function \( \hat{f}_n, n \in \mathbb{Z}, \) satisfies the system of second order differential equations (on \( \mathbb{R}_{>0} \))
\[
( y^2 \hat{\partial}_y^2 + s(1-s) + y^2 \partial_x^2 |_{x=0}(e^{2\pi i nx} J^x)) \hat{f}_n(y) = 0 .
\]

Proof. To evaluate the action of the Laplacian \( \Delta \) on the function \( f \) we take advantage of the Fourier expansion (83) of \( f \) and may pull the Laplace operator \( \Delta \) onto each summand by Lemma 4.3. From this and the property of \( f \) to be a Laplace eigenfunction with spectral parameter \( s \) (see (82)) we obtain
\[
0 = (s(1-s) - \Delta) f(z)
\]
\[
= \sum_{n \in \mathbb{Z}} (s(1-s) e^{2\pi i nx} J^x + y^2 \partial_x^2 (e^{2\pi i nx} J^x) + y^2 e^{2\pi i nx} J^x \partial_y^2) \hat{f}_n(y)
\]
for all \( z \in \mathbb{H} \). We consider (85) as a system of differential equations in the variable \( y \) on \( \mathbb{R}_{>0} \) with parameter \( x \in \mathbb{R} \), to which \( \hat{f}_n \) is an \( (x\)-independent) joint solution for
all values of $x$. To evaluate this system of differential equations further, we recall from Sections 2.3 and 2.4 that

\[ J^x = J(\lambda)^x = \lambda^x J(1)^x \]

for all $x \in \mathbb{R}$, and that $\lambda^x = \exp(x \log(\lambda; \omega))$. By direct calculation we find

\[
\partial_x^2(e^{2\pi i n x} J^x) = e^{2\pi i n x} \left((2\pi i n)^2 J^x + 4\pi i n \partial_x J^x + \partial_x^2 J^x\right) = \lambda^x e^{2\pi i n x} \left((2\pi i n)^2 J(1)^x + 4\pi i n (\log(\lambda; \omega)) J(1)^x + \partial_x J(1)^x\right) + \log(\lambda; \omega)^2 J(1)^x + \partial_x^2 J(1)^x + 2 \log(\lambda; \omega) \partial_x J(1)^x \right].
\]

The system of differential equations \((85)\) is therefore

\[
0 = \lambda^x \sum_{n \in \mathbb{Z}} e^{2\pi i n x} (s(1-s) J(1)^x + y^2 \left((2\pi i n)^2 J(1)^x + 4\pi i n (\log(\lambda; \omega)) J(1)^x + \partial_x J(1)^x\right)) f_n(y).
\]

(86)

We note that $\lambda$, $s$ and $\omega$ are fixed, and in particular are independent of $x$. Further, we recall from Lemma 2.4 that each of the entries of $J(1)^x$ is polynomial in $x$. Therefore for each fixed $y \in \mathbb{R}_{>0}$, the right hand side of \((86)\) is of the form

\[
\lambda^x \sum_{n \in \mathbb{Z}} e^{2\pi i n x} P_n(x) = 0,
\]

where, for each $n \in \mathbb{Z}$, the function $P_n$ is a polynomial in $x$, and its degree is bounded by the dimension, $d$, of the vector space $V$. Since $\lambda^x$ does not vanish, we obtain for each $n \in \mathbb{Z}$ that

(87) \[ 0 = \left. \left( e^{2\pi i n x} P_n(x) \right) \right|_{x=0} \]

by Lemma 4.2. Comparing \((86)\) to \((85)\) and using \((87)\) shows that for all $n \in \mathbb{Z}$ and $y \in \mathbb{R}_{>0}$ we have

\[
\left. \left[ s(1-s) e^{2\pi i n x} J^x \right] \right|_{x=0} + y^2 \left( \left. \partial_x^2 \right|_{x=0} (e^{2\pi i n x} J^x) + \left. e^{2\pi i n x} J^x \right|_{x=0} \partial_y \right) f_n(y) = 0.
\]

Now $e^{2\pi i n x} J^x \big|_{x=0} = \text{id}$, the identity matrix, which completes the proof. \hfill \Box

The system of the differential equations for the Fourier coefficients in Lemma 5.1 involves an upper triangular matrix. Consequently, this system is quite large. In the remainder of this section we will deduce an equivalent system of differential equations by means of a base change such that the upper triangular matrix is transformed into a three-band matrix. To obtain this simpler system we require the following two technical lemmas regarding the Stirling matrix $U$, which was defined in \((73)\).

**Lemma 5.2.** Let $U$ be the Stirling matrix from \((73)\). For each $x \in \mathbb{R}$ we have

\[
U \begin{pmatrix} x^{d-1} \\ x^{d-2} \\ \vdots \\ x \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{x}{(d-1)} \\ \frac{x}{(d-2)} \\ \vdots \\ \frac{x}{1} \end{pmatrix}.
\]
Proof. Using the very definition of the Stirling matrix $U = (u_{ij})$, we see that the entry in the $i$th row of the product on the left hand side of the claimed formula is

$$
\sum_{j=1}^{d} u_{ij} x^{d-j} = \sum_{j=i}^{d} \frac{s(d-i, d-j)}{(d-i)!} x^{d-j} = \sum_{k=0}^{d-i} \frac{s(d-i, k) x^{k}}{(d-i)!}.
$$

By the characterization of the Stirling numbers in (72) and the very definition of the generalized binomial coefficients we have

$$
\sum_{k=0}^{d-i} \frac{s(d-i, k) x^{k}}{(d-i)!} = \frac{1}{(d-i)!} \prod_{\ell=0}^{d-i-1} (x-\ell) = \binom{x}{d-\ell}.
$$

This completes the proof.

A simplification of the system of differential equations from Lemma 5.1 will be obtained by conjugating the matrix $J^x$ with the Stirling matrix $U$. We provide an explicit expression of the matrix $U^{-1} J^x U$ in the following lemma. For each $x \in \mathbb{R}$ we let $C(x) = (c_{ij}(x))$ be the $(d \times d)$-matrix with entries

$$
c_{ij}(x) := \begin{cases} 
(d-i) x^{j-i} & \text{if } i \leq j, \\
0 & \text{if } i > j.
\end{cases}
$$

Lemma 5.3. For each $x \in \mathbb{R}$, we have $U^{-1} J^x U = \lambda^x C(x)$.

Proof. Since $J^x = \lambda^x J(1)^x$, it suffices to show that $U^{-1} J(1)^x U = C(x)$ for all $x \in \mathbb{R}$. To that end we let $x \in \mathbb{R}$ and set

$$
M := M(x) := J(1)^x U \quad \text{and} \quad N := N(x) := UC(x).
$$

As usual we denote the entries of all matrices by corresponding small letters, and we omit their dependence on $x$ to simplify notation. Thus, $M = (m_{ij})$, $N = (n_{ij})$ and $C = C(x) = (c_{ij})$. Since $U$, $J(1)$ and $C$ are upper triangular matrices, so are $M$ and $N$. To show that $M$ and $N$ coincide on the upper triangle, we let $i, j \in \{1, \ldots, d\}$ with $i \leq j$. Then

$$
n_{ij} = \sum_{k=1}^{d} u_{ik} c_{kj} = \sum_{k=i}^{j} u_{ik} c_{kj}
$$

$$
= \sum_{k=i}^{j} \frac{s(d-i, d-k)}{(d-i)!} \binom{d-k}{j-k} x^{j-k}
$$

$$
= \sum_{q=0}^{j-i} x^q \frac{s(d-i, d-j+q)}{(d-i)!} \binom{d-j+q}{q},
$$
As we will see in Proposition 5.4, the required three-band matrix is a linear combination of the identity matrix with \(H\) for \(i,j\) where \(i,j\) are \(\alpha\)-valued. This completes the proof.

\[ (90) \]

\[ h_{ij} = \begin{cases} x \frac{s(d - k, d - j)}{(d - k)!} & \text{if } k = i \\ \frac{1}{(k - i)!} \sum_{\ell=0}^{k-i} s(k - i, \ell)x^\ell & \text{if } k \neq i \end{cases} \]

where we took advantage of Lemma 5.2 for the step (*). It remains to show that for all \(i, j \in \{1, \ldots, d\}\) with \(i \leq j\) and all \(q \in \{0, \ldots, j - i\}\) we have

\[ (89) \]

\[ \frac{s(d - i, d - j + q)}{(d - i)!} \left(\frac{d - j + q}{q}\right) = \sum_{\ell=d-j}^{d-i-q} \frac{s(\ell, d - j)s(d - i - \ell, q)}{\ell!(d - i - \ell)!}. \]

If we apply the substitutions \(a = d - i\) and \(b = d - j\), then (89) becomes

\[ \frac{s(a, b + q)}{a!} \left(\frac{b + q}{q}\right) = \sum_{\ell=b}^{a-q} \frac{s(\ell, b)s(a - \ell, q)}{\ell!(a - \ell)!}, \]

which is valid by \[ \text{II. 24.1.3 II.A}. \] This completes the proof.

With these preparations we can now show that the system of differential equations from Lemma 5.1 can be reformulated as a system of differential equations in which the interactions of the single differential equations are determined by a certain three-band matrix and hence are simpler than in the system from Lemma 5.1. For \(\alpha \in \mathbb{C}\) we let \(H(\alpha) = (h_{ij}(\alpha))\) denote the \((d \times d)\)-matrix with entries

\[ (90) \]

\[ h_{ij}(\alpha) := -2\alpha \left(\frac{d - i}{1}\right) \delta_0(j - i - 1) - 2 \left(\frac{d - i}{2}\right) \delta_0(j - i - 2) \]

for \(i, j \in \{1, \ldots, d\}\), where \(\delta_0\) denotes the Kronecker delta function [37]. The matrix \(H(\alpha)\) roughly looks like

\[
H(\alpha) = (h_{ij}(\alpha))_{i,j=1}^d = \begin{pmatrix}
0 & -2\alpha(d - 1) & -(d - 1)(d - 2) & 0 & 0 & \ldots & 0 \\
0 & 0 & -2\alpha(d - 2) & -(d - 2)(d - 3) & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & -2\alpha & -2 & 0 \\
0 & 0 & 0 & \ldots & 0 & -2\alpha \\
0 & 0 & 0 & \ldots & 0 & & 0
\end{pmatrix}.
\]

As we will see in Proposition 5.4, the required three-band matrix is a linear combination of the identity matrix with \(H(\alpha)\) for a specific value of \(\alpha\).

We recall that \(U\) denotes the Stirling matrix, defined in (73), and that \(\log(\cdot; \omega)\) is our choice from Section 2.3 for a complex logarithm that is holomorphic in the
eigenvalue $\lambda$ of the twisting endomorphism $A$ (which is here represented by the Jordan-like matrix $J$). Further we recall from \cite{74} that
\[ \alpha_n = 2\pi n - i \log(\lambda; \omega) \]
for all $n \in \mathbb{Z}$.

**Proposition 5.4.** For each $n \in \mathbb{Z}$, the Fourier coefficient function $\hat{f}_n$ satisfies the system of second order differential equations
\[ (y^2 \partial^2_y + s(1 - s))U^{-1} \hat{f}_n(y) = y^2 (\alpha_n^2 + H(\alpha_n))U^{-1} \hat{f}_n(y) \]
on $\mathbb{R}_{>0}$. Even stronger, this system is equivalent to the system \cite{84} in Lemma 5.1.

*Proof.* Multiplying the system of differential equations \cite{84} with $U^{-1}$ from the left, we get
\[ (y^2 \partial^2_y + s(1 - s))U^{-1} \hat{f}_n(y) = -y^2 U^{-1} \left( \partial^2_x \big|_{x=0} (e^{2\pi i n x} J^x) \right) \hat{f}_n(y) \]
\[ = -y^2 \left( \partial^2_x \big|_{x=0} (e^{2\pi i n x} U^{-1} J^x U) \right) U^{-1} \hat{f}_n(y) . \]

With the matrix $C(x)$ from \cite{88} and the identity $\lambda^x C(x) = U^{-1} J^x U$ shown in Lemma 5.3 we obtain
\[ \partial^2_x \big|_{x=0} (e^{2\pi i n x} U^{-1} J^x U) = \partial^2_x \big|_{x=0} (e^{2\pi i n x} \lambda^x C(x)) \]
\[ = C(x) + 2 \partial_x e^{i n x} \cdot \partial_x C(x) + e^{i n x} \partial^2_x C(x) \big|_{x=0} \]
\[ = -\alpha_n^2 \delta_0(i - j) + 2i\alpha_n \left( \begin{array}{c} d - i \\ 1 \end{array} \right) \delta_0(j - i - 1) \]
\[ + 2 \left( \begin{array}{c} d - i \\ 2 \end{array} \right) \delta_0(j - i - 2) \bigg|_{i,j=1} \]
\[ = -\alpha_n^2 \text{id} - H(\alpha_n) . \]

Thus, the system \cite{84} is equivalent to the system \cite{91}. \hfill \Box

6. Solution of the system of differential equations in Proposition 5.4

If $n = 0$ and $\alpha_0 = 0$

We continue to suppose that $V$ is a complex vector space of dimension $d \in \mathbb{N}$, and that $A$ is a linear isomorphism of $V$ that acts irreducibly. We fix a basis with respect to which $A$ is represented by the Jordan-like block matrix $J = J(\lambda)$ with $\lambda \in \mathbb{C}$, $\lambda \neq 0$, being the eigenvalue of $A$, and we identify $V$ with $\mathbb{C}^d$. We fix a complex logarithm $\log(\cdot; \omega)$ by choosing $\omega \in \mathbb{C}$ with the properties as in Section 2.3.

We let $f : \mathbb{H} \to V \cong \mathbb{C}^d$ be a $J$-twist periodic Laplace eigenfunction with spectral parameter $s$ and consider its Fourier expansion
\[ f(z) = \sum_{n \in \mathbb{Z}} J^x \hat{f}_n(y) e^{2\pi i n x} \quad (z = x + iy \in \mathbb{H}) . \]

In Proposition 5.4 we have shown that the Fourier coefficient function $\hat{f}_n$, for any $n \in \mathbb{Z}$, satisfies a certain system of second order differential equations. We will see that the solutions of this system of differential equations are based on the functions $I$ and $K$, defined in Sections 2.5.2 and 2.5.3. The second variable, $\alpha$, of these functions encodes a certain combination of the considered $n$ and the considered eigenvalue $\lambda$ of $J$. As $I$ and $K$ are qualitatively different depending on whether $\alpha$ equals 0 or
not, the discussion of the solution space of the differential equations naturally splits into these cases.

In this section, we will provide (see Proposition 6.5 below) the space of solutions of the system of differential equations for the case \( n = 0 \) and \( \lambda = 1 \), i.e., \( \alpha_n = 0 \) in the notation of Proposition 5.4. We recall the Stirling matrix \( U \) from (73), the two-band matrix \( H(0) \) from (90) and set

\[
w(y) := U^{-1} \hat{f}_0(y).
\]

Then the system of differential equations becomes

\[
(y^2 \partial_y^2 + s(1 - s)) w(y) = y^2 H(0) w(y), \quad y \in \mathbb{R}_{>0}.
\]

We will first provide two particular solutions of (92) and then construct from these all solutions. We note that this latter step is more involved than the standard argumentation with linear combinations as the space of solutions of (92) is \( 2d \)-dimensional, but we start with only two particular solutions. Proposition 6.5 immediately establishes Theorem 2.5 in the case \( n = 0 \) and \( \lambda = 1 \). We consider \( s \in \mathbb{C} \) to be fixed throughout and recall from (90) that

\[
H(0) = \left( -2 \left( \begin{array}{c} d - i \\ 2 \end{array} \right) \delta_0(j - i - 2) \right)_{i,j=1}^d
\]

\[
= \left( \begin{array}{ccccccc}
0 & 0 & -(d - 1)(d - 2) & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & -(d - 2)(d - 3) & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & -2 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{array} \right).
\]

6.1. Particular solutions. In this section we provide two solutions of the differential equation (92).

Proposition 6.1. Let \( \Psi \in \{I, K\} \). Then the map \( w: \mathbb{R}_{>0} \to V, \)

\[
w := \left( \begin{array}{c}
\Psi(d - 1, 0, \cdot, s) \\
\vdots \\
\Psi(1, 0, \cdot, s) \\
\Psi(0, 0, \cdot, s)
\end{array} \right)
\]

is a solution of the second order differential equation

\[
(y^2 \partial_y^2 + s(1 - s)) w(y) = y^2 H(0) w(y), \quad y \in \mathbb{R}_{>0}.
\]

Preparatory for the proof of Proposition 6.1 we present a few recursive identities between the combinatorial coefficients of the functions \( I \) and \( K \). We recall the set

\[
E(d) = \left\{ \frac{1}{2} \pm j : j = 0, 1, \ldots, \left\lfloor \frac{d - 1}{2} \right\rfloor \right\}
\]

of exceptional spectral parameters from (55) and the coefficient function \( p \) from (58). We will require the following recursive identity for \( p \).
Lemma 6.2. Let $m \in \mathbb{N}$, $m \geq 2$, and $s \in \mathbb{C}$, $s \notin E(d) \cap \left[\frac{1}{2} - \frac{m}{2}, -\frac{1}{2}\right]$. Then we have

$$2 \left\lfloor \frac{m}{2} \right\rfloor \left(2s - 1 + 2 \left\lfloor \frac{m}{2} \right\rfloor\right) p(m, s) = m(1 - m)p(m - 2, s).$$

Proof. Starting at the definition of $p$ in (58), we find

$$p(m, s) = (-1)^{\left\lfloor \frac{m}{2} \right\rfloor} \frac{m!}{2^{\left\lfloor \frac{m}{2} \right\rfloor} \left\lfloor \frac{m}{2} \right\rfloor! \prod_{j=1}^{\left\lfloor \frac{m}{2} \right\rfloor} (2j + 2s - 1)}$$

$$= (-1)^{\left\lfloor \frac{m-2}{2} \right\rfloor} \frac{(m-2)!}{2^{\left\lfloor \frac{m-2}{2} \right\rfloor} \left\lfloor \frac{m-2}{2} \right\rfloor! \prod_{j=1}^{\left\lfloor \frac{m-2}{2} \right\rfloor} (2j + 2s - 1)}$$

$$\times (-1)^{\left\lfloor \frac{m}{2} \right\rfloor} \frac{m(m-1)}{2^{\left\lfloor \frac{m}{2} \right\rfloor} \left(2 \left\lfloor \frac{m}{2} \right\rfloor + 2s - 1\right)}$$

$$= \frac{m(1-m)}{2^{\left\lfloor \frac{m}{2} \right\rfloor} \left(2 \left\lfloor \frac{m}{2} \right\rfloor + 2s - 1\right)} p(m - 2, s).$$

This completes the proof. \(\square\)

For the next set of recursive identities we recall the definitions of $q$ and $r$ from (60–62) and (64–66).

Lemma 6.3. Let $m \in \mathbb{N}$, $m \geq 2$, and $s \in E(d) \cap \left[\frac{3}{2} - \frac{m}{2}, \frac{1}{2}\right]$. Then we have

$$m(1 - m) q(m - 3 + 2s, s) = 2 \left\lfloor \frac{m}{2} \right\rfloor \left(2 \left\lfloor \frac{m}{2} \right\rfloor + 2s - 1\right) q(m - 1 + 2s, s),$$

and

$$m(1 - m) r(m - 3 + 2s, 2)$$

$$= \left(4 \left\lfloor \frac{m}{2} \right\rfloor + 2s - 1\right) q(m - 1 + 2s, s)$$

$$+ 2 \left\lfloor \frac{m}{2} \right\rfloor \left(2 \left\lfloor \frac{m}{2} \right\rfloor + 2s - 1\right) r(m - 1 + 2s, s).$$

Proof. We note that

$$2 \leq m - 1 + 2s \leq m.$$

Thus, using (62), we find

$$q(m - 1 + 2s, s) = -\frac{1}{4} \frac{m(m-1)}{\left[\frac{m-1+2s}{2}\right] \left(\frac{1}{2} - s + \left[\frac{m-1+2s}{2}\right]\right)} q(m - 3 + 2s, s)$$

$$= -\frac{1}{4} \frac{m(m-1)}{\left(\left[\frac{m}{2}\right] + s - \frac{1}{2}\right) \left[\frac{m}{2}\right]} q(m - 3 + 2s, s)$$

$$= \frac{m(1-m)}{2 \left[\frac{m}{2}\right] \left(2 \left[\frac{m}{2}\right] + 2s - 1\right)} q(m - 3 + 2s, s).$$
This proves the first of the claimed identities. For the second identity we start at the definition \((66)\) of \(r\) and obtain

\[
\begin{align*}
\tau(m - 1 + 2s, s) &= -\frac{\frac{1}{2} - s + 2 \left\lfloor \frac{m - 1 + 2s}{2} \right\rfloor \left( \frac{1}{2} - s + \left\lfloor \frac{m - 1 + 2s}{2} \right\rfloor \right)}{2} q(m - 1 + 2s, 2) \\
&\quad + \sum_{j=1}^{\left\lfloor \frac{m - 3 + 2s}{2} \right\rfloor} (-1)^{\left\lfloor \frac{m - 3 + 2s}{2} \right\rfloor + j + 1} \left( \frac{1}{2} - s + \left\lfloor \frac{m - 1 + 2s}{2} \right\rfloor \right) \frac{1}{2j (\frac{1}{2} - s + j)} \frac{1}{4^{\left\lfloor \frac{m - 1 + 2s}{2} \right\rfloor - j}} \\
&\quad \times \frac{m - 1 + 2s}{\prod_{\ell=j+1}^{\left\lfloor \frac{m - 1 + 2s}{2} \right\rfloor} \ell (\frac{1}{2} - s + \ell)} \\
&\quad \times \frac{\prod_{\ell=2j+1+\delta_{\text{odd}}(m - 1 + 2s)}^{m - 3 + 2s} (1 - 2s + \ell)}{\prod_{\ell=j+1}^{\left\lfloor \frac{m - 3 + 2s}{2} \right\rfloor} \ell (\frac{1}{2} - s + \ell)} \left(1 - 2s + \ell\right)
\end{align*}
\]

with \(\delta_{\text{odd}}\) as defined in \((67)\). We note that

\[
\delta_{\text{odd}}(m - 1 + 2s) = \delta_{\text{odd}}(m - 3 + 2s),
\]

and that, for any \(j \in \mathbb{N}\) with \(j \leq \left\lfloor \frac{m - 3 + 2s}{2} \right\rfloor\), we have

\[
\begin{align*}
\prod_{\ell=2j+1+\delta_{\text{odd}}(m - 1 + 2s)}^{m - 1 + 2s} (1 - 2s + \ell) &= \prod_{\ell=j+1}^{\left\lfloor \frac{m - 3 + 2s}{2} \right\rfloor} \ell (\frac{1}{2} - s + \ell) \\
&\quad \times \prod_{\ell=2j+1+\delta_{\text{odd}}(m - 3 + 2s)}^{m - 3 + 2s} (1 - 2s + \ell) \\
&\quad \times \prod_{\ell=j+1}^{\left\lfloor \frac{m - 3 + 2s}{2} \right\rfloor} \ell (\frac{1}{2} - s + \ell)
\end{align*}
\]

Using these identities in the equality above, we obtain

\[
\begin{align*}
\tau(m - 1 + 2s, s) &= -\frac{4 \left\lfloor \frac{m}{2} \right\rfloor + 2s - 1}{2 \left\lfloor \frac{m}{2} \right\rfloor (2 \left\lfloor \frac{m}{2} \right\rfloor + 2s - 1)} q(m - 1 + 2s, 2) \\
&\quad + \frac{m(1 - m)}{2 \left\lfloor \frac{m}{2} \right\rfloor (2 \left\lfloor \frac{m}{2} \right\rfloor + 2s - 1)} \sum_{j=1}^{\left\lfloor \frac{m - 3 + 2s}{2} \right\rfloor} (-1)^{\left\lfloor \frac{m - 3 + 2s}{2} \right\rfloor + j + 1} \\
&\quad \times \frac{\frac{1}{2} - s + 2j}{2j (\frac{1}{2} - s + j)} \frac{1}{4^{\left\lfloor \frac{m - 1 + 2s}{2} \right\rfloor - j}} \\
&\quad \times \prod_{\ell=2j+1+\delta_{\text{odd}}(m - 3 + 2s)}^{m - 3 + 2s} (1 - 2s + \ell) \\
&\quad \times \prod_{\ell=j+1}^{\left\lfloor \frac{m - 3 + 2s}{2} \right\rfloor} \ell (\frac{1}{2} - s + \ell) \left(1 - 2s + \ell\right)
\end{align*}
\]
The summand for \( j = \lfloor \frac{m-3+2s}{2} \rfloor \) in the “large” sum is

\[
(-1) \frac{1}{2} - s + 2 \left\lfloor \frac{m-3+2s}{2} \right\rfloor q(m - 3 + 2s, s),
\]

since, as one easily checks,

\[
2 \left\lfloor \frac{m-3+2s}{2} \right\rfloor + 1 + \delta_{\text{odd}}(m - 3 + 2s) = m - 2 + 2s.
\]

It now follows that

\[
\tau(m - 1 + 2s, s) = -\frac{4}{2} \left\lfloor \frac{m}{2} \right\rfloor \frac{2}{2} \left( 2 \left\lfloor \frac{m}{2} \right\rfloor + 2s - 1 \right) q(m - 1 + 2s, 2)
\]

\[
+ \frac{m(1-m)}{2} \left( 2 \left\lfloor \frac{m}{2} \right\rfloor + 2s - 1 \right),
\]

which establishes the second of the claimed identities. \(\square\)

With these preparations we can now present a proof of Proposition 6.1.

**Proof of Proposition 6.1.** Using Definition 2.4, we note that for all \( s \in \mathbb{C} \), \( s \neq \frac{1}{2} \), and all \( m \in \mathbb{N}_0 \), \( y \in \mathbb{R}_{>0} \), we have

\[
I(m, 0, y, s) = K(m, 0, y, 1-s).
\]

Therefore it suffices to prove the claimed statement for the two cases that \( \Psi = I \) and \( s \in \mathbb{C} \) arbitrary as well as that \( \Psi = K \) and \( s = \frac{1}{2} \). Equivalently, it suffices to establish the statement for the cases

(a) \( s \in \mathbb{C}, s \notin E(d) \cap \left[ \frac{1}{2} - \frac{d-1}{2}, -\frac{1}{2} \right] \) and for all \( m \in \{0, \ldots, d-1\} \),

\[
\Psi(m, 0, y, s) = p(m, s) y^{s+2\left\lfloor \frac{m}{2} \right\rfloor},
\]

(b) \( s = \frac{1}{2} - \frac{k}{2} \) for a (unique) \( k \in \{0, \ldots, d-1\} \cap 2\mathbb{N}_0 \) (thus, \( s \in E(d) \)) and

\[
\Psi(m, 0, y, s) = \begin{cases} 
q(m - 1 + 2s, s) y^{s+2\left\lfloor \frac{m}{2} \right\rfloor} \log y & \text{if } k \leq m \leq d - 1, \\
+\tau(m - 1 + 2s, s) y^{s+2\left\lfloor \frac{m}{2} \right\rfloor} & \text{if } 0 \leq m < k.
\end{cases}
\]

In both cases, establishing the claimed statement is equivalent to proving that

\[
\left( y^2 \partial_y^2 + s(1-s) \right) \Psi(m, 0, y, s) = m(1-m) y^2 \Psi(m - 2, 0, y, s)
\]

for all \( m \in \{0, \ldots, d-1\} \).

We consider the case (a). For each \( m \in \{0, \ldots, d-1\} \), a straightforward calculation shows

\[
\left( y^2 \partial_y^2 + s(1-s) \right) p(m, s) y^{s+2\left\lfloor \frac{m}{2} \right\rfloor}
\]

\[
= 2 \left\lfloor \frac{m}{2} \right\rfloor \left( 2s - 1 + 2 \left\lfloor \frac{m}{2} \right\rfloor \right) p(m, s) y^{s+2\left\lfloor \frac{m}{2} \right\rfloor}.
\]

For \( m \in \{0,1\} \), the right hand side of this equation obviously vanishes. For \( m \geq 2 \) it equals

\[
m(1-m)p(m - 2, s) y^{s+2\left\lfloor \frac{m}{2} \right\rfloor},
\]

by Lemma 6.2. This shows (93) for case (a).
We now discuss the case \([b]\). For \(m \in \{0, \ldots, k - 1\}\), the definition of \(\Psi\) is as in \([a]\), and also the sought-for equality in \([93]\) reads as in \([a]\) and hence follows as above. For \(m = k + p\) with \(p \in \{0, 1\}\), we have
\[
m - 1 + 2s = p
\]
and
\[
2s - 1 - \left\lfloor \frac{m}{2} \right\rfloor = 0.
\]
It follows that
\[
(y^2 \partial_y^2 + s(1-s)) \Psi(m, 0, y, s)
= (y^2 \partial_y^2 + s(1-s)) q(p, s) y^{s+2\lfloor \frac{m}{2} \rfloor} \log y
= q(p, s) \left( 2 \left\lfloor \frac{m}{2} \right\rfloor (2s - 1 - 2 \left\lfloor \frac{m}{2} \right\rfloor) y^{s+2\lfloor \frac{m}{2} \rfloor} \log y
+ (2s - 1 + 4 \left\lfloor \frac{m}{2} \right\rfloor) y^{s+2\lfloor \frac{m}{2} \rfloor} \right)
= q(p, s) \left( 2s - 1 + 4 \left\lfloor \frac{m}{2} \right\rfloor \right) y^{s+2\lfloor \frac{m}{2} \rfloor}
= m(1-m)p(m-2, s) y^{s+2\lfloor \frac{m}{2} \rfloor}
= m(1-m) y^2 \Psi(m-2, 0, y, s),
\]
having used \([61]\) in the penultimate equality. Finally, for \(m \geq k + 2\), we obtain
\[
(y^2 \partial_y^2 + s(1-s)) \Psi(m, 0, y, s)
= q(m-1+2s, s) 2 \left\lfloor \frac{m}{2} \right\rfloor (2s - 1 + 2 \left\lfloor \frac{m}{2} \right\rfloor) y^{s+2\lfloor \frac{m}{2} \rfloor} \log y
+ q(m-1+2s, s) \left( 2s - 1 + 4 \left\lfloor \frac{m}{2} \right\rfloor \right) y^{s+2\lfloor \frac{m}{2} \rfloor}
+ r(m-1+2s, s) 2 \left\lfloor \frac{m}{2} \right\rfloor (2s - 1 + 2 \left\lfloor \frac{m}{2} \right\rfloor) y^{s+2\lfloor \frac{m}{2} \rfloor}.
\]
An application of Lemma \([6.3]\) finishes the proof. \(\square\)

6.2. **Space of all solutions.** In this section we construct from the particular solutions in the previous section the 2d-dimensional space of all solutions of the differential equation \((92)\). See Proposition \([6.7]\). This establishes Theorem \([2.5]\) for the case \(n = 0\) and \(\lambda = 1\) or, equivalently, \(\alpha_n = 0\). We start by showing an independence result (Lemma \([6.4]\)) which will be crucial for the proof of Proposition \([6.5]\).

**Lemma 6.4.** For all \(m \in \mathbb{N}_0\) and all \(s \in \mathbb{C}\), the two functions \(I(m, 0, \cdot, s)\) and \(K(m, 0, \cdot, s)\) are linearly independent (over \(\mathbb{C}\)).

**Proof.** We need to distinguish three cases, depending on the value of \(s\). If
\[
s \in E(d) \cap \left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right],
\]
then \(I(m, 0, y, s)\) is a certain linear combination with nonzero coefficients of the functions \(y^{s+2\lfloor \frac{m}{2} \rfloor} \log y\) and \(y^{s+2\lfloor \frac{m}{2} \rfloor}\), and \(K(m, 0, y, s)\) is a nonzero scaling of \(y^{1-s+2\lfloor \frac{m}{2} \rfloor}\). See Section \([2.5.3]\). Therefore, in this case, the functions \(I(m, 0, \cdot, s)\) and \(K(m, 0, \cdot, s)\) are linearly independent as can be concluded from their different growth behaviors. If
\[
s \in E(d) \cap \left[ \frac{3}{2}, \frac{1}{2} + \frac{1}{2} \right],
\]
Then we have an analogous situation (with the roles of $I$ and $K$ interchanged). Thus, also in this case, $I(m, 0, \cdot, s)$ and $K(m, 0, \cdot, s)$ are linearly independent. For all other values of $s$, the function $I(m, 0, \cdot, s)$ is a nonzero scaling of $y^{s+2}[\frac{w}{x}]$ and $K(m, 0, \cdot, s)$ is a nonzero scaling of $y^{1-s+2}[\frac{w}{x}]$. Comparing the growth behaviors of these functions, we deduce that they are linearly dependent if and only if the two exponents are equal, thus, if $s = 1/2$. However, $1/2$ is not among the values $s$ may assume. Thus, also for this range of values for $s$, the functions $I(m, 0, \cdot, s)$ and $K(m, 0, \cdot, s)$ are linearly independent.

**Proposition 6.5.** Let $s \in \mathbb{C}$ and define the maps $P_I, P_K : \mathbb{R}_{>0} \to V$ by

$$P_I := \begin{pmatrix} I(d-1, 0, \cdot, s) \\ \vdots \\ I(1, 0, \cdot, s) \\ I(0, 0, \cdot, s) \end{pmatrix} \quad \text{and} \quad P_K := \begin{pmatrix} K(d-1, 0, \cdot, s) \\ \vdots \\ K(1, 0, \cdot, s) \\ K(0, 0, \cdot, s) \end{pmatrix}.$$

Then the $2d$-dimensional space of solutions of the second order differential equation

$$(y^2 \partial_y^2 + s(1 - s))w(y) = y^2 H(0)w(y), \quad y \in \mathbb{R}_{>0},$$

is

$$L_0 := \{CP_I + DP_K : C, D \in \text{Coeff}_{0,d} \},$$

with Coeff$_{0,d}$ as defined in Section 2.6.

**Proof.** The maps $P_I$ and $P_K$ are solutions of (94) by Proposition 6.1. Further, each matrix in Coeff$_{0,d}$ commutes with $H(0)$ (and indeed, Coeff$_{0,d}$ is the full centralizer of $H(0)$) as can be shown by a straightforward, but tedious calculation. Therefore, each element in $L_0$ is indeed a solution of (94). In what follows we will show that these are all solutions. As $L_0$ is a space of solutions of a $d$-dimensional system of second order linear differential equations, the (complex) vector space dimension of $L_0$ is at most $2d$. We will now find $2d$ linearly independent elements in $L_0$, which then immediately implies that $L_0$ is $2d$-dimensional and hence the full space of solutions of (94). To that end we consider the $d$ elements $C_1, \ldots, C_d \in \text{Coeff}_{0,d}$, where $C_j = (\Omega_j^{(0)}_{\min})$ for $j \in \{1, \ldots, d\}$ is the matrix with

$$c_{j}^{(j)} = 1$$

and as many other entries as possible equal to 0. The definition of Coeff$_{0,d}$ in Section 2.6 shows that in particular

$$(95) \quad c_{md}^{(j)} = 0 \quad \text{for} \ m > j$$

as well as

$$(96) \quad c_{jn}^{(j)} = 0 \quad \text{for} \ n \neq d. \quad$$

We now show that the $2d$ elements

$$(97) \quad C_1 P_I, \ldots, C_d P_I, C_1 P_K, \ldots, C_d P_K$$

of $L_0$ are linearly independent (over $\mathbb{C}$) by showing that any vanishing linear combination

$$(98) \quad \sum_{j=1}^{d} \mu_j C_j P_I + \nu_j C_j P_K = 0$$
with \( \mu_j, \nu_j \in \mathbb{C} \) is necessarily trivial (i.e., \( \mu_j = 0 = \nu_j \) for all \( j \in \{1, \ldots, d\} \)). The bottom vector entry of the linear combination (98) is

\[
\mu_d I(0, 0, \cdot, s) + \nu_d K(0, 0, \cdot, s) = 0.
\]

None of the other summands contribute to the bottom entry as follows immediately from (95) and (96). Since \( I(0, 0, \cdot, s) \) and \( K(0, 0, \cdot, s) \) are linearly independent by Lemma 6.4, we obtain

\[
\mu_d = 0 = \nu_d.
\]

Proceeding inductively implies the triviality of (98) and completes the proof. \( \square \)

7. Asymptotic behavior of the Fourier coefficient functions

In Section 6 we have observed that the growth properties of the functions \( I \) and \( K \) for \( \alpha = 0 \) are important for showing linear independence and ultimately for establishing Theorem 2.5 in the special case of \( \alpha = \alpha_n = 0 \) (in the notation of Theorem 2.5). In Section 8 we will see that also for \( \alpha \neq 0 \) the growth properties of \( I \) and \( K \) are crucial for an efficient proof of Theorem 2.5 this time in the case that \( \alpha = \alpha_n \neq 0 \).

In this section we prove growth asymptotics for the functions \( I \) and \( K \) for non-vanishing second variable \( \alpha \), thereby providing a complete and refined version of Theorem 1.4, split into Propositions 7.1 and 7.2. The “principal sector” in the statement of Theorem 1.4 refers to the case that \( \arg \alpha \in \Omega_\omega \). Depending on the sectorial location of \( \alpha \) we have the following asymptotic behavior of \( I(m, \alpha, y, s) \) as \( y \to \infty \):

\[
I(m, \alpha, y, s) \sim \begin{cases}
\frac{y^m}{i^m v_2 \sqrt{2\pi \alpha}} e_+(m, \alpha, y, s), & \text{arg } \alpha \in (-\frac{\pi}{2}, \pi] \cap \Omega_\omega, \\
\frac{y^m}{i^m v_2 \sqrt{2\pi \alpha}} e_-(m, \alpha, y, s), & \text{arg } \alpha \in (-\pi, -\frac{\pi}{2}] \cap \Omega_\omega, \\
\frac{y^m}{i^m v_2 \sqrt{2\pi \alpha}} e^{-2\pi i(s - \frac{1}{2})} e_+(m, \alpha, y, s), & \text{arg } \alpha \in (-\frac{\pi}{2}, \pi) \setminus \Omega_\omega, \\
\frac{y^m}{i^m v_2 \sqrt{2\pi \alpha}} e^{-2\pi i(s - \frac{1}{2})} e_-(m, \alpha, y, s), & \text{arg } \alpha \in (-\pi, -\frac{\pi}{2}) \setminus \Omega_\omega,
\end{cases}
\]

where

\[
e_+(m, \alpha, y, s) := e^{\alpha y} + (-1)^m e^{i\pi s} e^{-\alpha y},
\]

\[
e_-(m, \alpha, y, s) := e^{\alpha y} + (-1)^m e^{-i\pi s} e^{-\alpha y}
\]

and

\[
s(\omega) := \text{sgn}(\arg \omega).
\]

Proof. We recall from (53) that

\[
I(m, \alpha, y, s) = \exp(\frac{1}{2} \log(y; -1)) \frac{1}{i^m} \partial_x^m I_{\frac{1}{2}}(\alpha y; \omega)
\]

and note that

\[
\partial_x^m I_{\frac{1}{2}}(\alpha y; \omega) = \frac{\partial_x^m}{\partial z} I_{\frac{1}{2}}(z; \omega) \bigg|_{z=\alpha y}.
\]
We remark that here $y$ is not the imaginary part of $z$.) We further have the recurrence relation
\begin{equation}
\partial_z^m I_\eta(z; \omega) = \frac{1}{2m} \sum_{\ell=0}^m \binom{m}{\ell} I_{\eta-\ell+m}(z; \omega)
\end{equation}
for any $\eta \in \mathbb{C}$, as can be shown by a straightforward induction argument, starting with
\[\partial_z I_\eta(z; -1) = \frac{1}{2} (I_{\eta-1}(z; -1) + I_{\eta+1}(z; -1))\]
(see \cite{30} 8.486.2 or \cite{63} §3.71 (2), p. 79) and using the relation between the functions $I_\eta(z; -1)$ and $I_\eta(z; \omega)$ from \cite{49}. We therefore obtain
\begin{equation}
I(m, \alpha, y, s) = \frac{y^m \exp\left(\frac{1}{2} \log(y; -1)\right)}{i^m} \sum_{\ell=0}^m \binom{m}{\ell} I_{s - \frac{1}{2} - m + 2\ell}(\alpha y; \omega).
\end{equation}
Combining the classical results on the asymptotic behavior of the modified Bessel function $I$ of the first kind (see, e.g., \cite{30} 8.451.5 and \cite{15} 10.30.4, 10.30.5) with \cite{49} provides us with the following asymptotics for $I_\eta(\alpha y; \omega)$ as $y \to \infty$:
\[I_\eta(\alpha y; \omega) \sim \begin{cases}
\frac{1}{\sqrt{2\pi \alpha y}} e_+ (0, \alpha, y, \eta + \frac{1}{2}), & \text{arg } \alpha \in (-\frac{\pi}{2}, \pi) \cap \Omega_\omega, \\
\frac{1}{\sqrt{2\pi \alpha y}} e_- (0, \alpha, y, \eta + \frac{1}{2}), & \text{arg } \alpha \in (-\pi, -\frac{\pi}{2}) \cap \Omega_\omega, \\
\frac{1}{2\pi \alpha} e^{\pm 2\pi i y s}(\omega) e_+ (0, \alpha, y, \eta + \frac{1}{2}), & \text{arg } \alpha \in (-\frac{\pi}{2}, \pi) \setminus \Omega_\omega, \\
\frac{1}{2\pi \alpha} e^{-2\pi i y s}(\omega) e_- (0, \alpha, y, \eta + \frac{1}{2}), & \text{arg } \alpha \in (-\pi, -\frac{\pi}{2}) \setminus \Omega_\omega,
\end{cases}\]
where $\sqrt{2\pi \alpha y}$ is evaluated with respect to the principal logarithm. Using these asymptotics in \cite{100} completes the proof.}

**Proposition 7.2.** Let $s \in \mathbb{C}$, $\alpha \in \mathbb{C} \setminus \omega \mathbb{R}_{\geq 0}$, and $m \in \mathbb{N}_0$. Depending on the sectorial location of $\alpha$ we have the following asymptotic behavior of $K(m, \alpha, y, s)$ as $y \to \infty$:

(i) If $\text{arg } \alpha \in \Omega_\omega$, then
\[K(m, \alpha, y, s) \sim \sqrt{\frac{\pi}{2\alpha}} i^m y^m e^{-\alpha y}.
\]

(ii) If $\text{arg } \alpha \in (-\frac{\pi}{2}, \pi) \setminus \Omega_\omega$, then
\[K(m, \alpha, y, s) \sim -e^{-2\pi i y s}(\omega) \sqrt{\frac{\pi}{2\alpha}} i^m y^m e^{-\alpha y}
\]
\[+ \frac{\sqrt{2\pi}}{i^{m-1}\sqrt{\alpha}} s(\omega) \sin(\pi s) y^m (e^{\alpha y} - (-1)^m e^{-i \pi s} e^{-\alpha y}).
\]

(iii) If $\text{arg } \alpha \in (-\pi, -\frac{\pi}{2}) \setminus \Omega_\omega$, then
\[K(m, \alpha, y, s) \sim -e^{-2\pi i y s}(\omega) \sqrt{\frac{\pi}{2\alpha}} i^m y^m e^{-\alpha y}
\]
\[+ \frac{\sqrt{2\pi}}{i^{m-1}\sqrt{\alpha}} s(\omega) \sin(\pi s) y^m (e^{\alpha y} - (-1)^m e^{i \pi s} e^{-\alpha y}).
\]

As in Proposition 7.1 we set $s(\omega) = \text{sgn}(\text{arg } \omega)$.
Proof. Analogously as in the proof of Proposition 7.1 we recall from (54) that
\[ K(m, \alpha, y, s) = \exp\left(\frac{1}{2} \log(y; -1)\right) \partial^m_{\alpha} K_{s - \frac{1}{2}}(\alpha y; \omega) \]
and note that
\[
\partial^m_{\alpha} K_{s - \frac{1}{2}}(\alpha y; \omega) = y^m \partial^m_{\alpha} K_{s - \frac{1}{2}}(z; \omega) \bigg|_{z = \alpha y}.
\]
(We caution the reader that here \( y \) is not the imaginary part of \( z \).) We further have the recurrence relation
\[ \partial^m_{\alpha} z K_{\eta}(z; \omega) = (-1)^m \sum_{\ell=0}^{m} \left( \begin{array}{c} m \\ \ell \end{array} \right) K_{\eta - 2\ell}(z; \omega) \quad \text{for all } \eta \in \mathbb{C}. \]
This follows from a straightforward induction argument using the relation between \( K_{\eta}(\cdot; -1) \) and \( K_{\eta}(\cdot; \omega) \) from (50)–(52), the recurrence relation
\[ \partial_{z} K_{\eta}(z; -1) = -\frac{1}{2} (K_{\eta-1}(z; -1) + K_{\eta+1}(z; -1)) \]
as given in [30, 8.486.11] (see also [63, 3.71(2)]) for \( \arg z \in \Omega_{\omega} \), together with the recurrence relation
\[ \partial_{z} I_{\eta}(z; -1) = \frac{1}{2} (I_{\eta-1}(z; -1) + I_{\eta+1}(z; -1)) \]
(see [30, 8.486.2] or [63, §3.71 (2), p. 79]) if \( \arg z \not\in \Omega_{\omega} \). We hence obtain
\[ K(m, \alpha, y, s) = \frac{i^m}{2^m} y^m \sqrt{y} \sum_{\ell=0}^{m} \left( \begin{array}{c} m \\ \ell \end{array} \right) K_{s - \frac{1}{2} - 2\ell}(\alpha y; \omega), \]
where (here) \( \sqrt{y} = \exp\left(\frac{1}{2} \log(y; -1)\right) \). Combining the classical results regarding the asymptotic behavior of the modified Bessel function \( K \) of the second type (see, e.g., [30, 8.451.6] and also [63, p. 202, §7.23(1)] and [15, 10.25.3]) with (50)–(52) provides us with the following asymptotics for \( K_{\eta}(\alpha y; \omega) \) as \( y \to \infty \):
\[ K_{\eta}(\alpha y; \omega) \sim \sqrt{\frac{\pi}{2}} e^{-\alpha y} \quad \text{if } \arg \alpha \in \Omega_{\omega}, \]
and, if \( \arg \alpha \in (-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \Omega_{\omega}, \)
\[ K_{\eta}(\alpha y; \omega) \sim \sqrt{\frac{\pi}{2}} e^{-2\pi i \eta s(\omega)} e^{-\alpha y} \frac{1}{\sqrt{2\pi \alpha y}} \left( e^{\alpha y} + e^{-\eta + \frac{1}{2} i\pi} e^{-\alpha y} \right), \]
and, if \( \arg \alpha \in (-\pi, -\frac{\pi}{2}) \setminus \Omega_{\omega}, \)
\[ K_{\eta}(\alpha y; \omega) \sim \sqrt{\frac{\pi}{2}} e^{-2\pi i \eta s(\omega)} e^{-\alpha y} \frac{1}{\sqrt{2\pi \alpha y}} \left( e^{\alpha y} + e^{-\eta + \frac{1}{2} i\pi} e^{-\alpha y} \right). \]
In each of these asymptotics, \( \sqrt{y} \) is evaluated with respect to the principal logarithm. Combing these asymptotics with (102) completes the proof. \( \square \)
8. Solution of the system of differential equations in Proposition 5.4

If \( n \neq 0 \) or \( \alpha_0 \neq 0 \)

In this section we return to the situation that \( V \) is a complex vector space of dimension \( d \in \mathbb{N} \) and that \( A \) is a linear isomorphism of \( V \) that acts irreducibly. As before, we fix a basis with respect to which \( A \) is represented by the Jordan-like block matrix \( J = J(\lambda) \) with \( \lambda \in \mathbb{C} \), \( \lambda \neq 0 \), being the eigenvalue of \( A \), and we identify \( V \) with \( \mathbb{C}^d \). We again fix a complex logarithm \( \log(\cdot ; \omega) \) by choosing \( \omega \in \mathbb{C} \) with the properties as in Section 2.3. We let \( f : \mathbb{H} \to V \cong \mathbb{C}^d \) be a \( J \)-twist periodic Laplace eigenfunction with spectral parameter \( s \) and consider its Fourier expansion

\[
  f(z) = \sum_{n \in \mathbb{Z}} J^x \hat{f}_n(y) e^{2\pi i n x} \quad (z = x + iy \in \mathbb{H}).
\]

In Proposition 5.4 we have shown that the Fourier coefficient function \( \hat{f}_n \), \( n \in \mathbb{Z} \), is a solution of the differential equation

\[
  (y^2 \partial_y^2 + s(1 - s) - \alpha_n^2 y^2) U^{-1} \hat{f}_n(y) = y^2 H(i\alpha_n) U^{-1} \hat{f}_n(y),
\]

where \( y \in \mathbb{R}_{>0} \),

\[
  \alpha_n = 2\pi n - i \log(\lambda; \omega),
\]

the matrix \( U \) is the Stirling matrix from (73), and \( H(i\alpha_n) \) is the two-band matrix defined in (90). In Section 6 we provided the space of solutions of (103) in the case that \( \alpha_n = 0 \). In this section we will solve the differential equation (103) for \( \alpha_n \neq 0 \).

As in Section 6 we will see that the solutions of (103) are based on the functions \( I \) and \( K \), defined in Section 2.5.2. The appearance of \( \alpha_n^2 \) on the left hand side of (103) will \textit{a priori} lead to the wish to evaluate \( I \) and \( K \) with \( \alpha_n \) in the second argument. However, \( \alpha_n \) might not be in \( \mathbb{C} \setminus \omega \mathbb{R}_{>0} \), which is the domain for the second argument of \( I \) and \( K \). One way to overcome this issue is to reconsider the choice of \( \omega \). For \( \lambda = 1 \), however, choosing for \( \omega \) a value not in \( \mathbb{R}_{<0} \) would render Theorem 1.2 (Theorem 2.3) incompatible with the classical result. See (16) and (19). We refer to Section 6.2 for more discussion. Another way to overcome this issue is to take \(-\alpha_n\) as the second argument of \( I \) and \( K \). Such a sign change might seem to be rather innocent at first glance, but it influences the solution.

We will use here this second option and will first consider a more conceptual variant of the differential equation in (103). Indeed, we will provide solutions for the (vector-valued) differential equation

\[
  (y^2 \partial_y^2 + s(1 - s) - \alpha_n^2 y^2) w(y) = y^2 H((-1)^\varepsilon i\alpha) w(y), \quad y \in \mathbb{R}_{>0},
\]

for any \( \alpha \in \mathbb{C} \setminus \omega \mathbb{R}_{>0} \), \( \varepsilon \in \{0, 1\} \) and \( s \in \mathbb{C} \). As before, we will ultimately use \( w(y) = U^{-1} \hat{f}_n(y) \).

After establishing the full space of solutions of (104), we will show how this result and the one from Section 5 combine to a proof of Theorem 2.3 and briefly indicate how we may retrieve the classical, untwisted result from this theorem.

8.1. Particular solutions. In this section we will present two solutions of the differential equation (104) for any \( \alpha \in \mathbb{C} \setminus \omega \mathbb{R}_{>0} \), \( \alpha \neq 0 \) and any \( \varepsilon \in \{0, 1\} \). We consider \( s \in \mathbb{C} \) to be fixed throughout.

For the solution of (104) we will take advantage of the fact that it is a system of differential equations that is only sparsely entangled as a result of our base change from Section 5. Indeed, the differential operator on the left hand side of (104) acts coordinate-wise, and the matrix \( H((-1)^\varepsilon i\alpha) \) on the right hand side of (104)
is non-vanishing only on the first two upper diagonal bands. Therefore the last row of (104) depends only on the last row, \( w_d \), of \( w = (w_i)_{i=1}^d \), and can be solved independently. The penultimate row of (104) depends only on the last two entries of \( w \), hence on \( w_{d-1} \) and \( w_d \), and can be solved after having found the solution of the differential equation in the last row. In total, we may solve (104) in this "bottom-up" procedure. The final result is stated as Proposition 8.4.

We start with a series of lemmas in which we discuss some crucial relations between the functions \( I \) and \( K \) for different values of their first argument. In hindsight these results will be understood as the building blocks of the intermediate solutions of the separated rows of (104). We recall from (53) and (54) that

\[
I(0, \alpha, y, s) = y^{\frac{1}{2}}I_{s-\frac{1}{2}}(\alpha y; \omega), \quad K(0, \alpha, y, s) = y^{\frac{1}{2}}K_{s-\frac{1}{2}}(\alpha y; \omega)
\]

for all \( \alpha \in \mathbb{C} \setminus \omega \mathbb{R}_{\geq 0} \) and \( y \in \mathbb{R}_{>0} \).

**Lemma 8.1.** Let \( \alpha \in \mathbb{C} \setminus \omega \mathbb{R}_{\geq 0} \). Then the two functions \( I(0, \alpha, \cdot, s) \) and \( K(0, \alpha, \cdot, s) \) form a fundamental system of solutions of the second order differential equation

(105) \[ (y^2 \partial_y^2 + s(1-s) - y^2 \alpha^2)u(y) = 0, \quad y \in \mathbb{R}_{>0}. \]

**Proof.** As it is well-known, the two functions \( \mathbb{R}_{>0} \to \mathbb{C} \),

\[
y \mapsto y^{\frac{1}{2}}I_{s-\frac{1}{2}}(y) \quad \text{and} \quad y \mapsto y^{\frac{1}{2}}K_{s-\frac{1}{2}}(y)
\]

form a fundamental system of solutions of the differential equation

(106) \[ (y^2 \partial_y^2 + s(1-s) - y^2)u(y) = 0 \]

on \( \mathbb{R}_{>0} \). (A proof can be deduced, e.g., from [63, Section 3.7].) Using the relations between the modified Bessel functions \( I_{s-\frac{1}{2}} \) and \( K_{s-\frac{1}{2}} \) (using the standard complex logarithm) and their variants \( I_{s-\frac{1}{2}}(\cdot; \omega) \) and \( K_{s-\frac{1}{2}}(\cdot; \omega) \) (using the adapted choice of a complex logarithm) as discussed in Section 2.5.1 and the fact that also \( y^{\frac{1}{2}}I_{s-\frac{1}{2}}(y) \) solves (106), one easily shows that the functions 

\[
y^{\frac{1}{2}}I_{s-\frac{1}{2}}(\alpha y; \omega) \quad \text{and} \quad y^{\frac{1}{2}}K_{s-\frac{1}{2}}(\alpha y; \omega)
\]

are linearly independent solutions of (105). This completes the proof. \( \square \)

For the proofs of the following two lemmas we note that

(107) \[ i^mI(m, \alpha, y, s) = \partial_y^mI(0, \alpha, y, s), \quad i^mK(m, \alpha, y, s) = \partial_y^mK(0, \alpha, y, s) \]

for all \( \alpha \in \mathbb{C} \setminus \omega \mathbb{R}_{\geq 0} \) and \( m \in \mathbb{N}_0 \), as can be seen directly from the definitions in (53) and (54). We further recall that for all \( m \in \mathbb{N}_0 \) and \( s \in \mathbb{C} \) the maps \( I(m, \cdot, \cdot, s) \) and \( K(m, \cdot, \cdot, s) \) are smooth as functions of two variables.

**Lemma 8.2.** Let \( \alpha \in \mathbb{C} \setminus \omega \mathbb{R}_{\geq 0} \) and \( \Psi \in \{I, K\} \). Then \( \Psi(1, \alpha, \cdot, s) \) satisfies

\[
(y^2 \partial_y^2 + s(1-s) - y^2 \alpha^2)\Psi(1, \alpha, y, s) = -2iy^2\Psi(0, \alpha, y, s), \quad y \in \mathbb{R}_{>0}.
\]

**Proof.** Taking advantage of Lemma 8.1 and of (107), we find

\[
0 = \partial_\alpha \left[ (y^2 \partial_y^2 + s(1-s) - y^2 \alpha^2)\Psi(0, \alpha, y, s) \right]
\]

\[
= (y^2 \partial_y^2 + s(1-s) - y^2 \alpha^2)(\partial_\alpha \Psi(0, \alpha, y, s)) - 2\alpha y^2\Psi(0, \alpha, y, s)
\]

\[
= (y^2 \partial_y^2 + s(1-s) - y^2 \alpha^2)i\Psi(1, \alpha, y, s) - 2\alpha y^2\Psi(0, \alpha, y, s)
\]

for all \( y \in \mathbb{R}_{>0} \). Re-arranging this equality completes the proof. \( \square \)
While Lemma 8.2 discusses a relation between the functions $\Psi(0, \alpha, \cdot, s)$ and $\Psi(1, \alpha, \cdot, s)$ for $\Psi \in \{I, K\}$, the next lemma considers the situation for the first argument of $\Psi$ being at least 2.

**Lemma 8.3.** Let $\alpha \in \mathbb{C} \setminus \omega \mathbb{R}_{\geq 0}$, $m \in \mathbb{N}$ with $m \geq 2$ and $\Psi \in \{I, K\}$. Then the function $\Psi(m, \alpha, \cdot, s)$ satisfies the identity

$$y^2 \partial_y^2 + s(1 - s) - \alpha^2 y^2 \Psi(m, \alpha, y, s) = -y^2 \left(2mi\alpha \Psi(m - 1, \alpha, y, s) + m(m - 1)\Psi(m - 2, \alpha, y, s)\right)$$

for all $y \in \mathbb{R}_{>0}$.

**Proof.** Let $y \in \mathbb{R}_{>0}$. Using the Leibniz rule and (107) we find

$$\partial^m_{\alpha} \left(\alpha^2 \Psi(0, \alpha, y, s)\right)$$

$$= \sum_{k=0}^{m} \binom{m}{k} \left(\partial^k_{\alpha} \alpha^2\right) \left(\partial^{m-k}_{\alpha} \Psi(0, \alpha, y, s)\right)$$

$$= \alpha^2 \partial^m_{\alpha} \Psi(0, \alpha, y, s) + 2m\alpha \partial^{m-1}_{\alpha} \Psi(0, \alpha, y, s) + m(m - 1)\partial^{m-2}_{\alpha} \Psi(0, \alpha, y, s)$$

$$= i^m \alpha^2 \Psi(m, \alpha, y, s) + i^{m-1}2m\alpha \Psi(m - 1, \alpha, y, s) - i^m m(m - 1)\Psi(m - 2, \alpha, y, s).$$

Combining this equality and Lemma 8.1 we obtain

$$0 = \partial^m_{\alpha} \left[(y^2 \partial_y^2 + s(1 - s) - y^2 \alpha^2)\Psi(0, \alpha, y, s)\right]$$

$$= (y^2 \partial_y^2 + s(1 - s) - \alpha^2 y^2)i^m \Psi(m, \alpha, y, s) - y^2 \left(2m\alpha i^{m-1} \Psi(m - 1, \alpha, y, s) - m(m - 1)i^m \Psi(m - 2, \alpha, y, s)\right).$$

Re-arranging (and dividing by $i^m$) yields the claimed identity. \qed

We recall the sign matrix $S \in \text{Mat}(d \times d; \mathbb{C})$ from (77):

$$S := \begin{pmatrix}
(-1)^{d-1} & \cdots & 1 \\
(-1)^{d-2} & \cdots & -1 \\
& \cdots & \ddots \\
& & & 1 \\
& & & -1 \\
& & & & 1
\end{pmatrix}.$$

**Proposition 8.4.** Let $\alpha \in \mathbb{C} \setminus \omega \mathbb{R}_{\geq 0}$, $\varepsilon \in \{0, 1\}$ and $\Psi \in \{I, K\}$. Then the map $w: \mathbb{R}_{>0} \to V$

$$w := S^\varepsilon \begin{pmatrix}
\Psi(d - 1, \alpha, \cdot, s) \\
\vdots \\
\Psi(1, \alpha, \cdot, s) \\
\Psi(0, \alpha, \cdot, s)
\end{pmatrix}$$

is a solution of the second order differential equation

$$(y^2 \partial_y^2 + s(1 - s) - \alpha^2 y^2)w(y) = y^2 H((-1)^\varepsilon i\alpha)w(y), \quad y \in \mathbb{R}_{>0}.$$
Remark 8.5. Using the notation of Proposition 8.4 we note that

\[
S = \begin{pmatrix}
\Psi(d-1, \alpha, \cdot, s) \\
\vdots \\
\Psi(3, \alpha, \cdot, s) \\
\Psi(2, \alpha, \cdot, s) \\
\Psi(1, \alpha, \cdot, s) \\
\Psi(0, \alpha, \cdot, s)
\end{pmatrix} = \begin{pmatrix}
(-1)^{d-1} \Psi(d-1, \alpha, \cdot, s) \\
\vdots \\
-\Psi(3, \alpha, \cdot, s) \\
-\Psi(2, \alpha, \cdot, s) \\
-\Psi(1, \alpha, \cdot, s) \\
\Psi(0, \alpha, \cdot, s)
\end{pmatrix}.
\]

Thus, Proposition 8.4 states that

\[
\begin{pmatrix}
\Psi(d-1, \alpha, \cdot, s) \\
\vdots \\
\Psi(3, \alpha, \cdot, s) \\
\Psi(2, \alpha, \cdot, s) \\
\Psi(1, \alpha, \cdot, s) \\
\Psi(0, \alpha, \cdot, s)
\end{pmatrix}
\]

is a solution of

\[
(y^2 \partial_y^2 + s(1-s) - \alpha^2 y^2) w(y) = y^2 H(i\alpha) w(y), \quad y \in \mathbb{R}_{>0},
\]

and

\[
(-1)^{d-1} \begin{pmatrix}
\Psi(d-1, \alpha, \cdot, s) \\
\vdots \\
-\Psi(3, \alpha, \cdot, s) \\
-\Psi(2, \alpha, \cdot, s) \\
-\Psi(1, \alpha, \cdot, s) \\
\Psi(0, \alpha, \cdot, s)
\end{pmatrix}
\]

is a solution of

\[
(y^2 \partial_y^2 + s(1-s) - \alpha^2 y^2) w(y) = y^2 H(-i\alpha) w(y), \quad y \in \mathbb{R}_{>0}.
\]

Proof of Proposition 8.4. Let \( w = (w_j)_{j=1}^d \). Then

\[
w_{d-m} : \mathbb{R}_{>0} \to \mathbb{C}, \quad w_{d-m}(y) = (-1)^m \Psi(m, \alpha, y, s)
\]

for each \( m \in \{0, \ldots, d-1\} \). To show that \( w \) is a solution of the system of differential equations (108), we start to check this for the last row (i.e., for \( w_d \) or, equivalently, for \( m = 0 \)) and then proceed iteratively up to the first row (i.e., to \( w_1 \) or, equivalently, to \( m = d - 1 \)).

The last row of (108), thus the equation for \( w_d \), reads

\[
y^2 \partial_y^2 + s(1-s) - \alpha^2 y^2 \right) w_d(y) = 0
\]

for all \( y \in \mathbb{R}_{>0} \). Since \( w_d = \Psi(0, \alpha, \cdot, s) \), Lemma 8.1 shows that the equality (110) is satisfied.

The penultimate row of (108), thus the equation for \( w_{d-1} \), is

\[
(y^2 \partial_y^2 + s(1-s) - \alpha^2 y^2) w_{d-1}(y) = 2(-1)^{d+1} i\alpha y^2 w_d(y)
\]

for all \( y \in \mathbb{R}_{>0} \). Since \( w_{d-1} = (-1)^{d} \Psi(1, \alpha, \cdot, s) \) and \( w_d = \Psi(0, \alpha, \cdot, s) \), Lemma 8.2 yields that (111) is valid.
Then the Coeff is B. We recall from Section 2.6 that indeed the full matrix with vector space Coeff prescription. We now show that the 2d-dimensional system of second order linear differential equations, 

\[(y^2 \partial_y^2 + s(1-s) - \alpha^2 y^2)w_{d-m}(y) = 2(-1)^{m+1}i\alpha y^2 m\mathbf{w}(m-1, \alpha, y, s) - (-1)^m m(m-1)y^2 \mathbf{w}(m-2, \alpha, y, s) = 2(-1)^{m+1}i\alpha y^2 m(-1)^{m-1}\mathbf{w}(m-1, \alpha, y, s) - m(m-1)y^2 (-1)^{m-2}\mathbf{w}(m-2, \alpha, y, s) \]

for all \(y \in \mathbb{R}_{>0}\). Using the formulas in \(109\) for \(w\) completes the proof that \(112\) is satisfied.

\[\square\]

8.2. Space of all solutions. With these preparations and the asymptotics for \(\mathbf{K}\) and \(\mathbf{K}\) from Section 7, we are now able to present the full space of solutions of \(104\).

**Proposition 8.6.** Let \(\alpha \in \mathbb{C} \setminus \omega \mathbb{R}_{>0}, \varepsilon \in \{0, 1\}, s \in \mathbb{C}\) and define the maps \(\mathbf{P}_I, \mathbf{P}_K: \mathbb{R}_{>0} \to V\) by

\[\mathbf{P}_I := S^\varepsilon \begin{pmatrix} \mathbf{I}(d-1, \alpha, s) \\ \mathbf{I}(1, \alpha, s) \\ \mathbf{I}(0, \alpha, s) \end{pmatrix} \quad \text{and} \quad \mathbf{P}_K := S^\varepsilon \begin{pmatrix} \mathbf{K}(d-1, \alpha, s) \\ \mathbf{K}(1, \alpha, s) \\ \mathbf{K}(0, \alpha, s) \end{pmatrix}.\]

Then the 2d-dimensional space of solutions of the second order differential equation 

\[(y^2 \partial_y^2 + s(1-s) - \alpha^2 y^2)w(y) = y^2 H((-1)^{i\alpha}) w(y), \quad y \in \mathbb{R}_{>0}, \]

is 

\[L_1 := \{C\mathbf{P}_I + D\mathbf{P}_K : C, D \in \text{Coeff}_{1,d}\},\]

with Coeff_{1,d} defined as in Section 2.6.

**Proof.** By Proposition 8.3, the maps \(\mathbf{P}_I\) and \(\mathbf{P}_K\) are solutions of \(113\). Further, a straightforward calculation shows that each matrix in Coeff_{1,d} commutes with \(H((-1)^{i\alpha})\). Therefore, each element in \(L_1\) is indeed a solution of \(113\). (The vector space Coeff_{1,d} is the full centralizer of \(H((-1)^{i\alpha})\), a fact that we will not need here.) It remains to show that these are all solutions. As \(L_1\) is a space of solutions of a \(d\)-dimensional system of second order linear differential equations, the (complex) vector space dimension of \(L_1\) is at most \(2d\). In what follows we will detect \(2d\) linearly independent elements in \(L_1\), which then immediately yields that \(L_1\) is \(2d\)-dimensional and hence the full space of solutions of \(113\). To that end we consider the \(d\) elements \(B_1, \ldots, B_d \in \text{Coeff}_{1,d}\), where \(B_j = (b_{1n}^{(j)})\) for \(j \in \{1, \ldots, d\}\) is determined by

\[b_{1n}^{(j)} := \begin{cases} 1 & \text{for } n = j, \\ 0 & \text{for } n \neq j. \end{cases}\]

We recall from Section 2.6 that indeed the full matrix \(B_j\) is determined by this prescription. We now show that the \(2d\) elements

\[B_1\mathbf{P}_I, \ldots, B_d\mathbf{P}_I, B_1\mathbf{P}_K, \ldots, B_d\mathbf{P}_K\]
of $L_1$ are linearly independent (over $\mathbb{C}$). The topmost entry of the linear equation
\[ \sum_{j=1}^{d} \mu_j B_j P_I + \nu_j B_j P_K = 0, \]
where $\mu_j, \nu_j \in \mathbb{C}$, reads
\[ \sum_{j=1}^{d} \mu_j I(d - j, \alpha, \cdot, s) + \nu_j K(d - j, \alpha, \cdot, s) = 0. \]
The asymptotics for $I$ and $K$ from Propositions 7.1 and 7.2 now imply that $\mu_j = 0 = \nu_j$ for all $j \in \{1, \ldots, d\}$. This completes the proof. □

8.3. **Endgame.** Even though we have already indicated the proof of Theorem 2.5 in the previous sections, we provide here the brief, explicit discussion for the convenience of the readers. We further explain why Theorem 2.5 recovers the classical result in the untwisted setting (see Section 1.2).

**Proof of Theorem 2.5.** Proposition 5.4 shows that, for each $n \in \mathbb{Z}$, the Fourier coefficient function $\hat{f}_n$ satisfies the differential equation
\[ \left(y^2 \partial_y^2 + s(1 - s) - y^2 \alpha_n^2 \right) T^{-1} \hat{f}_n(y) = y^2 H(i\alpha_n) T^{-1} \hat{f}_n(y), \quad y \in \mathbb{R}_{>0}, \]
with
\[ \alpha_n = 2\pi n - i \log(\lambda; \omega). \]
If $\alpha_n = 0$, then Proposition 6.5 states that $\hat{f}_n$ is of the form claimed in Theorem 2.5. For $\alpha_n \neq 0$, we rewrite the differential equation as
\[ \left(y^2 \partial_y^2 + s(1 - s) - y^2 \tilde{\alpha}_n^2 \right) T^{-1} \hat{f}_n(y) = y^2 H((-1)^{\varepsilon_n} \tilde{\alpha}_n) T^{-1} \hat{f}_n(y), \quad y \in \mathbb{R}_{>0}, \]
with (see (76))
\[ \tilde{\alpha}_n = (-1)^{\varepsilon_n} \alpha_n. \]
This has the advantage that $\tilde{\alpha}_n \in \mathbb{C} \setminus \omega \mathbb{R}_{>0}$ by the choice of $\varepsilon_n$ (see (75)). Then Proposition 8.6 shows that $\hat{f}_n$ is of the claimed form. □

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