CHENG-YAU OPERATOR AND GAUSS MAP OF SURFACES OF REVOLUTION

DONG-SOO KIM\textsuperscript{1}, JONG RYUL KIM AND YOUNG HO KIM\textsuperscript{3}

Abstract. We study the Gauss map $G$ of surfaces of revolution in the 3-dimensional Euclidean space $\mathbb{E}^3$ with respect to the so called Cheng-Yau operator $\Box$ acting on the functions defined on the surfaces. As a result, we establish the classification theorem that the only surfaces of revolution with Gauss map $G$ satisfying $\Box G = AG$ for some $3 \times 3$ matrix $A$ are the planes, right circular cones, circular cylinders and spheres.

1. Introduction

The theory of Gauss map of a surface in a Euclidean space and a pseudo-Euclidean space is always one of interesting topics and it has been investigated from the various viewpoints by many differential geometers (\cite{2, 3, 8, 9, 10, 6, 11, 13, 14, 15, 16, 18, 19, 20, 21}).

2010 Mathematics Subject Classification. 53A05, 53B25.

Key words and phrases. Gauss map, Cheng-Yau operator, surfaces of revolution.

\textsuperscript{1} supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2010-0022926).

\textsuperscript{3} supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2012R1A1A2042298) and supported by Kyungpook National University Research Fund, 2012.
Let $M$ be a surface of the Euclidean 3-space $\mathbb{E}^3$. The map $G : M \rightarrow S^2 \subset \mathbb{E}^3$ which sends each point of $M$ to the unit normal vector to $M$ at the point is called the Gauss map of the surface $M$, where $S^2$ is the unit sphere in $\mathbb{E}^3$ centered at the origin. It is well known that $M$ has constant mean curvature if and only if $\Delta G = ||dG||^2 G$, where $\Delta$ is the Laplace operator on $M$ corresponding to the induced metric on $M$ from $\mathbb{E}^3$ ([23]). Surfaces whose Gauss map is an eigenfunction of Laplacian, that is, $\Delta G = \lambda G$ for some constant $\lambda \in \mathbb{R}$, are the planes, circular cylinders and spheres ([6]).

Generalizing this equation, F. Dillen, J. Pas and L. Verstraelen ([11]) studied surfaces of revolution in a Euclidean 3-space $\mathbb{E}^3$ such that its Gauss map $G$ satisfies the condition

\[(1.1) \quad \Delta G = AG, \quad A \in R^{3 \times 3}.
\]

As a result, they proved ([11])

**Proposition 1.1.** Among the surfaces of revolution in $\mathbb{E}^3$, the only ones whose Gauss map satisfies (1.1) are the planes, the spheres and the circular cylinders.

Baikoussis and Blair also studied ruled surfaces in $\mathbb{E}^3$ and proved ([2])

**Proposition 1.2.** Among the ruled surfaces in $\mathbb{E}^3$, the only ones whose Gauss map satisfies (1.1) are the planes and the circular cylinders.

Generalized slant cylindrical surfaces (GSCS’s) are natural extended notion of surfaces of revolution ([17]). Surfaces of revolution, cylindrical surfaces and tubes along a plane curve are special cases of GSCS’s. In [19], the first author and B. Song proved that among the GSCS’s in $\mathbb{E}^3$, the only ones whose Gauss map satisfies (1.1) are the planes, the spheres and the circular cylinders.
The so-called Cheng-Yau operator $\Box$ (or, $L_1$) is a natural extension of the Laplace operator $\Delta$ (cf. [1], [7]). Hence, following the condition (1.1), it is natural to ask as follows.

**Question 1.3.** Among the surfaces of revolution in a Euclidean 3-space $\mathbb{E}^3$, which one satisfy the following condition?

\[(1.2) \quad \Box G = AG, \quad A \in R^{3 \times 3}.
\]

In this paper, we give a complete answer to the above question.

Throughout this paper, we assume that all objects are smooth and connected, unless otherwise mentioned.

**2. Cheng-Yau Operator and Lemmas**

Let $M$ be an oriented surface in $E^3$ with Gauss map $G$. We denote by $S$ the shape operator of $M$ with respect to the Gauss map $G$. For each $k = 0, 1$, we put $P_0 = I, P_1 = tr(S)I - S$, where $I$ is the identity operator acting on the tangent bundle of $M$. Let us define an operator $L_k : C^\infty(M) \to C^\infty(M)$ by $L_k(f) = -tr(P_k \circ \nabla^2 f)$, where $\nabla^2 f : \chi(M) \to \chi(M)$ denotes the self-adjoint linear operator metrically equivalent to the hessian of $f$. Then, up to signature, $L_k$ is the linearized operator of the first variation of the $(k + 1)$-th mean curvature arising from normal variations of the surface. Note that the operator $L_0$ is nothing but the Laplace operator acting on $M$, i.e., $L_0 = \Delta$ and $L_1 = \Box$ is called the Cheng-Yau operator introduced in [7].

Now, we state a useful lemma as follows ([1]).
Lemma 2.1. Let $M$ be an oriented surface in $E^3$ with Gaussian curvature $K$ and mean curvature $H$. Then, the Gauss map $G$ of $M$ satisfies

\[(2.1) \quad \Box G = \nabla K + 2HKG,\]

where $\nabla K$ denotes the gradient of $K$.

Now, using Lemma 2.1 we give some examples of surfaces with Gauss map satisfying (1.2).

Examples.

(1) Flat surfaces. In this case, we have $\Box G = 0$, and hence flat surfaces satisfy $\Box G = AG$ for some $3 \times 3$ matrix $A$. Note that the matrix $A$ must be singular.

(2) Spheres: $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$. In this case, we have $G = \frac{1}{r}(x - a, y - b, z - c)$ so the sphere satisfies $\Box G = AG$ with $A = -\frac{1}{r^2}I$, where $I$ denotes the identity matrix.

3. Gauss map of surfaces of revolution

We consider a unit speed plane curve $C : (x(s), 0, z(s))$ with $x(s) > 0$ in the $xz$ plane which is defined on an interval $I$. By rotating the curve $C$ around $z$-axis, we get a surface of revolution $M$, which is parametrized by

\[(3.1) \quad X(s, t) = (x(s) \cos t, x(s) \sin t, z(s)).\]
The adapted frame field \( \{e_1, e_2, G\} \) on the surface of revolution \( M \) are given by

\[
e_1 = X_s = (x'(s) \cos t, x'(s) \sin t, z'(s)),
\]

\[
e_2 = \frac{1}{x} X_t = (- \sin t, \cos t, 0),
\]

\[
G = e_1 \times e_2 = (-z' \cos t, -z' \sin t, x').
\]

The principal curvatures \( k_1, k_2 \) of \( M \) with respect to the Gauss map \( G \) are respectively \([12]\)

\[
k_1 = \langle S(e_1), e_1 \rangle = x' z'' - x'' z' = \kappa,
\]

\[
k_2 = \langle S(e_2), e_2 \rangle = \frac{z'}{x},
\]

where \( S \) and \( \kappa \) denote the shape operator of \( M \) and the plane curvature of the plane curve \( C \), respectively.

Since the parametrization \((x(s), 0, z(s))\) of the plane curve \( C \) is of unit speed, there exists a smooth function \( \theta = \theta(s) \) such that \( x' = \cos \theta \) and \( z' = \sin \theta \). Then, the Gaussian curvature \( K \) and the mean curvature \( H \) of \( M \) are, respectively, given by

\[
K = k_1 k_2 = \frac{\theta'(s) \sin \theta}{x},
\]

\[
2H = k_1 + k_2 = \theta'(s) + \frac{\sin \theta}{x}.
\]

Hence, the gradient \( \nabla K \) of the Gaussian curvature \( K \) of \( M \) is given by

\[
\nabla K = K'(s)e_1,
\]

where

\[
K'(s) = \frac{1}{x^2} \{x\theta''(s) \sin \theta + x\theta'(s)^2 \cos \theta - \theta'(s) \cos \theta \sin \theta \}.
\]
We now suppose that the Gauss map $G$ of the surface of revolution $M$ satisfies for a $3 \times 3$ matrix $A = (a_{ij})$

\[(3.7) \quad \Box G = AG.\]

Recall that the Gauss map $G$ is given by

\[(3.8) \quad G(s, t) = (-\sin \theta \cos t, -\sin \theta \sin t, \cos \theta).\]

Then, it follows from (2.1), (3.4) and (3.5) that

\[(3.9) \quad \left\{ K'(s) \cos \theta - 2KH \sin \theta \right\} \cos t \]
\[= -a_{11} \sin \theta \cos t - a_{12} \sin \theta \sin t + a_{13} \cos \theta,\]

\[(3.10) \quad \left\{ K'(s) \cos \theta - 2KH \sin \theta \right\} \sin t \]
\[= -a_{21} \sin \theta \cos t - a_{22} \sin \theta \sin t + a_{23} \cos \theta\]

and

\[(3.11) \quad K'(s) \sin \theta + 2KH \cos \theta = -a_{31} \sin \theta \cos t - a_{32} \sin \theta \sin t + a_{33} \cos \theta.\]

First, we suppose that the set $J = \{ s \in I | \theta'(s) \neq 0 \}$ is nonempty. Then $\theta(I)$ contains an interval, hence we get from (3.9)-(3.11) that $a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 0$ and $a_{11} = a_{22}$. Thus we obtain $A = \text{diag}(\lambda, \lambda, \mu)$,

\[(3.12) \quad K'(s) \cos \theta - 2KH \sin \theta = -\lambda \sin \theta,\]

and

\[(3.13) \quad K'(s) \sin \theta + 2KH \cos \theta = \mu \cos \theta.\]

Note that (3.12) and (3.13) are equivalent to the following:

\[(3.14) \quad K'(s) = a \cos \theta \sin \theta,\]
and

$$(3.15) \quad 2KH = -a \sin^2 \theta + \mu,$$

where we put $a = \mu - \lambda$.

We prove the following lemma, which plays a crucial role in the proof of our main theorem.

**Lemma 3.1.** Let $M$ be a surface of revolution given by (3.1) with nonempty set $J = \{s \in I|\theta'(s) \neq 0\}$. Suppose that the Gauss map $G$ of $M$ satisfies $\Box G = AG$ for some $3 \times 3$ matrix $A$. Then $A$ is of the form $\lambda I$, where $I$ is an identity matrix.

**Proof.** The above discussions show that $A$ is a diagonal matrix of the form $A = \text{diag}(\lambda, \lambda, \mu)$ for some constants $\lambda$ and $\mu$. We put $a = \mu - \lambda$. Then, it follows from (3.4), (3.6), (3.14) and (3.15) that

$$(3.16) \quad x\theta''(s) \sin \theta + x\theta'(s)^2 \cos \theta - \theta'(s) \cos \theta \sin \theta = ax^2 \cos \theta \sin \theta$$

and

$$(3.17) \quad x\theta'(s)^2 \sin \theta + \theta'(s) \sin^2 \theta = (-a \sin^2 \theta + \mu)x^2.$$ 

By differentiating the both sides of (3.17) with respect to $s$, we get

$$(3.18) \quad \theta''(s) \sin^2 \theta + 2x\theta'(s)\theta''(s) \sin \theta + x\theta'(s)^3 \cos \theta + 3\theta'(s)^2 \sin \theta \cos \theta$$

$$+ 2ax^2 \theta'(s) \sin \theta \cos \theta = 2x \cos \theta(-a \sin^2 \theta + \mu).$$

If we substitute $\theta''(s)$ in (3.16) into (3.18), then we have

$$(3.19) \quad - x^2 \theta'(s)^3 \cos \theta + 4x\theta'(s)^2 \cos \theta \sin \theta + \{\cos \theta \sin^2 \theta + 4ax^3 \cos \theta \sin \theta\}\theta'(s)$$

$$+ 3ax^2 \cos \theta \sin^2 \theta - 2\mu x^2 \cos \theta = 0.$$
Let us substitute $\theta'(s)^2$ in (3.17) into (3.19). Then we obtain

\begin{equation}
5x\theta'(s)^2 \cos \theta \sin \theta + \{\cos \theta \sin^2 \theta + 5ax^3 \cos \theta \sin \theta - \mu x^3 \cot \theta\} \theta'(s) \\
+ 3ax^2 \cos \theta \sin^2 \theta - 2\mu x^2 \cos \theta = 0.
\end{equation}

(3.20)

Once more, we substitute $\theta'(s)^2$ in (3.17) into (3.20). Then we get

\begin{equation}
\theta'(s) = \frac{\gamma x^2}{\alpha x^3 + \beta},
\end{equation}

where we put

\begin{equation}
\alpha(s) = -5a \sin^2 \theta(s) + \mu, \beta(s) = 4 \sin^3 \theta(s),
\end{equation}

(3.22)

\begin{equation}
\gamma(s) = -2a \sin^3 \theta(s) + 3\mu \sin \theta(s).
\end{equation}

(3.23)

Now, we replace $\theta'(s)$ in (3.17) with that in (3.21). Then we have

\begin{equation}
a_6 x^6 + a_3 x^3 + a_0 = 0,
\end{equation}

(3.23)

where we use the following notations:

\begin{equation}
a_6(\theta) = 25a^3 \cos^6 \theta + 5a^2(15\lambda - 8\mu) \cos^4 \theta \\
+ a(5a - \mu)(4\mu - 15\lambda) \cos^2 \theta + \lambda(5a - \mu)^2,
\end{equation}

(3.24)

\begin{equation}
a_3(\theta) = 26a^2 \sin^7 \theta - 19a\mu \sin^5 \theta - 4\mu^2 \sin^3 \theta
\end{equation}

(3.25)

and

\begin{equation}
a_0(\theta) = -8a \sin^8 \theta + 4\mu \sin^6 \theta.
\end{equation}

(3.26)

Let us differentiate (3.23) with respect to $s$. Here, we denote by $\dot{a}_i(\theta)$ the derivative of $a_i(\theta)$ with respect to $\theta, i = 0, 3, 6$. Using $x' = \cos \theta$ and $\theta'(s)$ given by (3.21), we get

\begin{equation}
b_6 x^6 + b_3 x^3 + b_0 = 0,
\end{equation}

(3.27)
where we denote

\[(3.28) \quad b_6(\theta) = 6\alpha \cos \theta a_6(\theta) + \gamma a_6(\theta),\]

\[(3.29) \quad b_3(\theta) = 3\alpha \cos \theta a_3(\theta) + 6\beta \cos \theta a_6(\theta) + \gamma a_3(\theta)\]

and

\[(3.30) \quad b_0(\theta) = 3\beta \cos \theta a_3(\theta) + \gamma a_0(\theta).\]

If we compute \(b_i(\theta)\) for \(i = 0, 3, 6\), then we have

\[(3.31) \quad b_6(\theta) = \{1050a^4 \sin^8 \theta + \sum_{i=0}^6 p_i(\lambda, \mu) \sin^i \theta\} \cos \theta,\]

\[(3.32) \quad b_3(\theta) = \{-214a^3 \sin^9 \theta + \sum_{i=0}^7 q_i(\lambda, \mu) \sin^i \theta\} \cos \theta\]

and

\[(3.33) \quad b_0(\theta) = \{440a^2 \sin^{10} \theta + \sum_{i=0}^8 r_i(\lambda, \mu) \sin^i \theta\} \cos \theta,\]

where \(p_i(\lambda, \mu), q_i(\lambda, \mu)\) and \(r_i(\lambda, \mu)\) are respectively some polynomials in \(\lambda\) and \(\mu\).

Eliminating \(x^6\), it follows from (3.23) and (3.27) that

\[(3.34) \quad c_3 x^3 + c_0 = 0,\]

where

\[(3.35) \quad c_3 = a_3 b_6 - b_3 a_6, c_0 = a_0 b_6 - b_0 a_6.\]

Due to (3.24)-(3.26) and (3.31)-(3.33), we may compute \(c_3\) and \(c_0\) as follows:

\[(3.36) \quad c_3 = \{32650a^6 \sin^{15} \theta + \sum_{j=0}^{13} p_{3j}(\lambda, \mu) \sin^j \theta\} \cos \theta,\]
and

\[(3.37) \quad c_0 = \{-19400a^5 \sin^{16} \theta + \sum_{j=0}^{14} p_{0j}(\lambda, \mu) \sin^j \theta \} \cos \theta,\]

where each \(p_{ij}(\lambda, \mu)(i = 0, 3)\) is a polynomial in \(\lambda\) and \(\mu\).

Let us replace \(x^3\) in (3.23) with \(x^3 = -c_0/c_3\) given in (3.34). Then we have

\[(3.38) \quad a_6c_0^2 - a_3c_0c_3 + a_0c_3^2 = 0.\]

Using (3.31)-(3.33), (3.36) and (3.37), we may compute the leading terms of those in (3.38) as follows:

\[(3.39) \quad a_6c_0^2 = -25(19400)^2a^{13}\sin^{40} \theta + \text{lower degree terms in } \sin \theta,\]

\[(3.39) \quad a_3c_0c_3 = 26(19400)(32650)a^{13}\sin^{40} \theta + \text{lower degree terms in } \sin \theta,\]

\[(3.39) \quad a_0c_3^2 = 8(32650)^2a^{13}\sin^{40} \theta + \text{lower degree terms in } \sin \theta.\]

Hence we obtain

\[(3.40) \quad a_6c_0^2 - a_3c_0c_3 + a_0c_3^2 = -17349480000a^{13}\sin^{40} \theta + \text{lower degree terms in } \sin \theta.\]

Since \(\theta(I)\) contains an interval, together with (3.38), (3.40) shows that \(a\) must be zero. Thus we have \(\mu = \lambda\) and hence \(A = \lambda I\). This completes the proof. \(\square\)

4. Main Theorems and Corollaries

Finally, we prove the main theorem as follows.

**Theorem 4.1.** Let \(M\) be a surface of revolution. Then the Gauss map \(G\) of \(M\) satisfies \(\Box G = AG\) for some \(3 \times 3\) matrix \(A\) if and only if \(M\) is an open part of the following surfaces:

1) a plane,
2) a right circular cone,
3) a circular cylinder,
4) a sphere.

**Proof.** We consider a surface of revolution $M$ obtained by rotating the unit speed plane curve $C : (x(s), 0, z(s))$ with $x(s) > 0$ around $z$-axis which is defined on an interval $I$.

Suppose that the Gauss map $G$ of $M$ satisfies $\square G = AG$ for some $3 \times 3$ matrix $A$. For a function $\theta = \theta(s)$ satisfying $(x'(s), z'(s)) = (\cos \theta(s), \sin \theta(s))$, let us put $J = \{ s \in I | \theta'(s) \neq 0 \}$.

We divide by two cases.

**Case 1.** Suppose that $J$ is nonempty. Then, as in the proof of Lemma 3.1 we have $A = diag(\lambda, \lambda, \mu)$ with $a = \lambda - \mu$. Furthermore, Lemma 3.1 shows that $a = 0$ that is, $\lambda = \mu$. Hence it follows from (3.14) and (3.15) that the Gaussian curvature $K$ is constant and the mean curvature $H$ satisfies $2KH = \lambda$.

If $\lambda \neq 0$, then both of $K$ and $H$ are nonzero constant. Hence it follows from a well-known theorem (cf. [22]) that $M$ is an open part of a sphere. Using (3.17) and (3.21) with $a = 0$, it can be directly shown that $\theta'(s)$ is constant and $x(s) = r \sin \theta$ for a positive constant $r$. This shows that the profile curve $C$ is an open part of a half circle centered on the rotation axis of $M$. Thus, $M$ is an open portion of a round sphere.

If $\lambda = 0$, then $K$ is constant with $2KH = 0$. Suppose that $K \neq 0$. Then we have $H = 0$. But catenoids are the only minimal nonflat surfaces of revolution, of which Gaussian curvature $K$ are nonconstant. This contradiction shows that $K = 0$. Thus $M$ is a flat surfaces of revolution. Therefore, $M$ is an open part of a plane, a right circular cone or a circular cylinder.
Case 2. Suppose that \( J \) is empty. Then the profile curve \( C \) of \( M \) is a straight line. Thus, \( M \) is an open part of a plane, a right circular cone or a circular cylinder.

The converse is obvious from (2.1). \( \square \)

Combining the results of [11, 19], the following characterization theorems can be obtained.

Corollary 4.2. Let \( M \) be a surface of revolution. Then the following are equivalent.
1) \( M \) is an open part of a round sphere.
2) The Gauss map \( G \) of \( M \) satisfies \( \Box G = AG \) for some nonsingular \( 3 \times 3 \) matrix \( A \).

Corollary 4.3. Let \( M \) be a surface of revolution. Then the following are equivalent.
1) \( M \) is an open part of a right circular cone.
2) The Gauss map \( G \) of \( M \) satisfies \( \Box G = AG \) for some \( 3 \times 3 \) matrix \( A \), but not satisfies \( \Delta G = AG \) for any \( 3 \times 3 \) matrix \( A \).

References

[1] Luis J. Alias and N. Gurbuz, An extension of Takahashi theorem for the linearized operators of the higher order mean curvatures, Geom. Dedicata 121 (2006), 113-127.
[2] C. Baikoussis and D. E. Blair, On the Gauss map of ruled surfaces, Glasgow Math. J. 34 (1992), no. 3, 355-359.
[3] C. Baikoussis and L. Verstraelen, On the Gauss map of helicoidal surfaces, Rend. Sem. Mat. Messina Ser. II 2(16) (1993), 31-42.
[4] B.-Y. Chen, Total mean curvature and submanifolds of finite type, World Scientific Publ., New Jersey (1984).
[5] B.-Y. Chen, *Finite type submanifolds and generalizations*, University of Rome (1985).

[6] B.-Y. Chen and P. Piccinni, *Submanifolds with finite type Gauss map*, Bull. Austral. Math. Soc. 35 (1987), no. 2, 161-186.

[7] S. Y. Cheng and S. T. Yau, *Hypersurfaces with constant scalar curvature*, Math. Ann. 225 (1977), no. 3, 195-204.

[8] S. M. Choi, *On the Gauss map of surfaces of revolution in a 3-dimensional Minkowski space*, Tsukuba J. Math. 19 (1995), no. 2, 351-367.

[9] S. M. Choi, *On the Gauss map of ruled surfaces in a 3-dimensional Minkowski space*, Tsukuba J. Math. 19 (1995), no. 2, 285-304.

[10] S. M. Choi, D.-S. Kim, Y. H. Kim and D. W. Yoon, *Circular cone and its Gauss map*, Colloq. Math. 129 (2012), no. 2, 203-210.

[11] F. Dillen, J. Pas and L. Verstraelen, *On the Gauss map of surfaces of revolution*, Bull. Inst. Math. Acad. Sinica 18 (1990), no. 3, 239-246.

[12] Manfredo P. do Carmo, *Differential geometry of curves and surfaces*. Translated from the Portuguese, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1976.

[13] U. Dursun, *Flat surfaces in the Euclidean space $E^3$ with pointwise 1-type Gauss map*, Bull. Malays. Math. Sci. Soc. (2) 33 (2010), no. 3, 469-478.

[14] U-H. Kí, D.-S. Kim, Y. H. Kim and Y.-M. Roh, *Surfaces of revolution with pointwise 1-type Gauss map in Minkowski 3-space*, Taiwanese J. Math. 13 (2009), no. 1, 317-338.

[15] D.-S. Kim, *On the Gauss map of quadric hypersurfaces*, J. Korean Math. Soc. 31 (1994), no. 3, 429-437.

[16] D.-S. Kim, *On the Gauss map of hypersurfaces in the space form*, J. Korean Math. Soc. 32 (1995), no. 3, 509-518.

[17] D.-S. Kim and Y. H. Kim, *Surfaces with planar lines of curvature*, Honam Math. J. 32 (2010), 777-790.

[18] D.-S. Kim, Y. H. Kim and D. W. Yoon, *Extended B-scrolls and their Gauss maps*, Indian J. Pure Appl. Math. 33 (2002), no. 7, 1031-1040.

[19] D.-S. Kim and B. Song, *On the Gauss map of generalized slant cylindrical surfaces*, J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math., 20(3), 149-158.
[20] Y. H. Kim and N. C. Turgay, *Surfaces in $E^3$ with $L_1$-pointwise 1-type Gauss map*, Bull. Korean Math. Soc. 50 (2013), no. 3, 935-949.

[21] Y. H. Kim and D. W. Yoon, *On the Gauss map of ruled surfaces in Minkowski space*, Rocky Mountain J. Math. 35 (2005), no. 5, 1555-1581.

[22] T. Levi-Civita, *Famiglie di superficie isoparametriche nell’ordinario spazio euclideo*, Rend. Acad. Lincei 26 (1937), 355-362.

[23] E. A. Ruh and J. Vilms, *The tension field of the Gauss map*, Trans. Amer. Math. Soc. 149 (1970), 569-573.

**Department of Mathematics, Chonnam National University, Gwangju 500-757, South Korea,**

**Department of Mathematics, Kunsan National University, Kunsan 573-701, South Korea,**

**Department of Mathematics, Kyungpook National University, Daegu 702-701, South Korea**

*E-mail address: dosokim@chonnam.ac.kr, kimjr0@yahoo.com and yhkim@knu.ac.kr*