Relation between quantum $\kappa$-Poincaré framework and Doubly Special Relativity

Jerzy Lukierski
Institute of Theoretical Physics, Wrocław University, 50-204 Wrocław, pl. M. Borna 9, Poland

Abstract

We describe firstly the basic features of quantum $\kappa$-Poincaré symmetries with their Hopf algebra structure. The quantum $\kappa$-Poincaré framework in any basis relates rigidly the quantum $\kappa$-Poincaré algebra with quantum $\kappa$-Poincaré group, noncommutative space-time and $\kappa$-deformed phase space. Further we present the approach of Doubly Special Relativity (DSR) theories, which introduce (in the version DSR1) kinematically the frame-independent fundamental mass parameter as described by maximal three-momentum $|\vec{p}| = \kappa c$. We argue why the DSR theories in one-particle sector can be treated as the part of quantum $\kappa$-Poincaré framework. The DSR formulation has been extended to multiparticle states either in a way leading to nonlinear description of classical relativistic symmetries, or providing the identification of DSR approach with full quantum $\kappa$-Poincaré framework.

1 Introduction

The aim of this paper is to present the relation between the formalism of quantum $\kappa$-Poincaré symmetries ($\kappa$-Poincaré quantum algebra and $\kappa$-deformed energy-momentum dispersion relation [1–8], $\kappa$-Poincaré quantum group and $\kappa$-Minkowski space [9,4–6], $\kappa$-deformed phase space [4–6,10]) and recently developed Doubly Special Relativity (DSR) theories [11–19].

Quantum $\kappa$-deformed Poincaré symmetries are described by quite rigid scheme of Hopf algebras which generalize the notion of classical symmetries.
to the case of noncommuting transformation parameters. Such a scheme describes simultaneously the deformation of classical Poincaré algebra $P^{3,1}$

\begin{align}
[M_{\mu\nu}^{(0)}, M_{\rho\tau}^{(0)}] &= i(g_{\mu\rho}M_{\nu\tau}^{(0)} + g_{\nu\tau}M_{\mu\rho}^{(0)} - g_{\mu\tau}M_{\nu\rho}^{(0)} - g_{\nu\rho}M_{\mu\tau}^{(0)}) \\
[M_{\mu\nu}^{(0)}, P_{\rho}^{(0)}] &= i(g_{\mu\rho}P_{\nu}^{(0)} - g_{\nu\rho}P_{\mu}^{(0)}) \\
[P_{\mu}^{(0)}, P_{\nu}^{(0)}] &= 0
\end{align}

as well as the deformation of classical Poincaré group $P^{3,1} = O(3,1) \times T_4$ to quantum one, with noncommuting translation and Lorentz rotation parameters [9,6]. Using the duality of Hopf algebras (see e.g. [20]) one can show that the deformation of quantum Poincaré group (e.g. $\kappa$-Minkowski space-time described by the translation sector $T_4$) follows uniquely from the Hopf-algebraic structure of quantum $\kappa$-Poincaré algebra.

Doubly Special Relativity (DSR) theories guided by some theoretical challenges in astrophysics and quantum gravity (see e.g. [21–24]) derive their framework from the explicit form of $\kappa$-deformed mass-shell condition originally postulated in DSR1 theories as follows (see e.g. [15])

\begin{equation}
C_2^\kappa(\vec{P}, 2, p_0) = \left(2\kappa \sin \frac{P_0}{2\kappa}\right)^2 - \vec{P}^2 e^{\frac{P_0}{\kappa}} = M^2,
\end{equation}

with the deformation parameter $\kappa$ identified with the Planck mass $M_{\mu\nu}$. It appears that [2] is identical with particular form of $\kappa$-deformed mass Casimir for quantum $\kappa$-Poincaré algebra in bicrossproduct basis [4,5,25].

We shall present firstly in Sect. 2 the basic features of the Hopf-algebraic framework of quantum $\kappa$-Poincaré symmetries, in particular we will describe the Hopf-algebraic approach the $\kappa$-deformed Lorentz transformations of four-momenta and will point out the arbitrariness of the choice of basic generators defining quantum $\kappa$-Poincaré algebra. Further the interpretation (see [8]) of the classical Lie algebra basis for quantum $\kappa$-deformed Poincaré algebra will be given. In Sect. 3 we shall consider the $\kappa$-deformed Lorentz transformation in DSR1 theories and show that they are a part of the framework of quantum $\kappa$-Poincaré algebra. We shall point out the absence of definite coproduct rules in DSR framework, and recall two alternative choices proposed in [17]. Further we recall that one can introduce three classes of DSR theories [26],

\footnote{For quantum $\kappa$-Poincaré algebra in DSR papers there is used the new name of $\kappa$-Poincaré-Hopf algebra.}
with bounded $|\vec{p}|$ (DSR1), bounded $E$ (DSR3) and both variables $|\vec{p}|$ and $E$ bounded (DSR2) with the first model in DSR2 class provided by Magueijo and Smolin [14]. In Sect. 4 we present final remarks.

2 Hopf-algebraic structure of quantum $\kappa$-Poincaré symmetries

The quantum $\kappa$-Poincaré algebra $U_\kappa P^{3,1}_\kappa$, introduced in 1991 [1–3] has been rewritten in 1994 [4–6] in so-called bicrossproduct basis as follows:

\begin{enumerate}
\item[a)] algebraic sector ($M_{\mu\nu} = (M_i, N_i)$; $P_\mu = (P_i, P_0 = \frac{E}{c}$)

\begin{equation}
\begin{aligned}
[M_i, M_j] &= i \epsilon_{ijk} M_k, \\
[M_i, N_j] &= i \epsilon_{ijk} N_k, \\
[N_i, N_j] &= -i \epsilon_{ijk} M_k, \\
[M_i, P_j] &= \epsilon_{ijk} P_k, \\
[N_i, P_j] &= \frac{i}{2} \delta_{ij} \left[ \kappa c \left( 1 - e^{-\frac{E}{\kappa c^2}} \right) + \frac{1}{\kappa c} \vec{P}^2 \right] - \frac{i}{\kappa c} P_i P_j, \\
[N_i, P_0] &= i P_i, \\
[P_\mu, P_\nu] &= 0,
\end{aligned}
\end{equation}

\item[b)] coalgebra sector

\begin{equation}
\begin{aligned}
\Delta(E) &= E \otimes 1 + 1 \otimes E \\
\Delta(\vec{P}) &= \vec{P} \otimes 1 + e^{-\frac{E}{\kappa c^2}} \otimes \vec{P} \\
\Delta(\vec{M}) &= \vec{M} \otimes 1 + 1 \otimes \vec{M} \\
\Delta(N_i) &= N_i \otimes 1 + e^{-\frac{E}{\kappa c^2}} \otimes N_i + \frac{1}{\kappa c} \epsilon_{ijk} P_j \otimes M_k
\end{aligned}
\end{equation}

\item[c)] antipodes (quantum coinverses)

\begin{equation}
\begin{aligned}
S(E) &= -E \\
S(\vec{P}) &= -\vec{P} e^{\frac{E}{\kappa c^2}} \\
S(\vec{M}) &= -\vec{M} \\
S(N_i) &= -e^{\frac{E}{\kappa c^2}} N_i + \frac{1}{\kappa c} \epsilon_{ijk} e^{\frac{E}{\kappa c^2}} P_j M_k
\end{aligned}
\end{equation}

It appears that in the basis given by formulae (3) the deformed dispersion relation (2) is the deformed mass Casimir $C^\kappa_2$ for quantum $\kappa$-Poincaré algebra.
If we introduce the Hopf-algebraic duality relations (see e.g. [20]) between the enveloping algebra $U_\kappa(P^{3,1})$ and functions $f(P_\kappa)$ on deformed Poincaré group $P_\kappa$ one obtains the full description of deformed Poincaré group. We obtain one-to-one correspondence

$$
\text{algebraic sector of } U_\kappa(P^{3,1}) \leftrightarrow \text{duality} \rightarrow \text{coalgebraic sector of } f(P_\kappa)
$$

(6)

$$
\text{coalgebraic sector of } U_\kappa(P^{3,1}) \leftrightarrow \text{duality} \rightarrow \text{algebraic sector of } f(P_\kappa)
$$

(7)

In particular because the fourmomenta $P_\mu$ describe Hopf subalgebra, we obtain

$$
\text{\(\kappa\)-deformed coproducts for } P_\mu \leftrightarrow \text{duality} \rightarrow \text{\(\kappa\)-deformed Minkowski space}
$$

(8)

where we identified the space-time with the the translations sector of the Poincaré group. From (4) and (8) one gets explicitly the algebra of $\kappa$-Minkowski space-time, firstly obtained in such a way in 1994 [9,4,5]

$$
[\hat{x}_i, \hat{x}_j] = 0, \quad [\hat{x}_0, \hat{x}_i] = \frac{i}{\kappa} \hat{x}_i.
$$

(9)

If we add to the relations (9) the commuting momenta (see (3)), we obtain the generators $(\hat{x}_\mu, \hat{p}_\mu \equiv P_\mu)$ of noncommutative $\kappa$-deformed relativistic phase space. The cross commutators between the quantum coordinates and momenta $\hat{p}_\mu$ are derived in a unique way from the quantum $\kappa$-deformed Poincaré algebra via the double cross-product construction, called Heisenberg double [20,10]. One obtains [4–6]

$$
[\hat{x}_0, \hat{p}_i] = \frac{i}{\kappa} \hat{p}_i, \quad [\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij},
$$
\[ [\hat{x}_0, \hat{p}_i] = 0, \quad [\hat{x}_0, \hat{p}_0] = -i\hbar \delta_{ij}. \] (10)

One can also introduce generalized $\kappa$-deformed phase space, which is described by the Heisenberg double of complete $\kappa$-deformed Poincaré algebra (generators $P_\mu \equiv \hat{P}_\mu, M_{\mu\nu}$) and $\kappa$-deformed Poincaré group (generators $\hat{x}_\mu, \Lambda_{\mu\nu}$). The relations (10) have been extended in 1997 [10] by the following cross relations:

\[ [M_{\mu\nu}, \hat{x}_\rho] = i\hbar (\eta_{\nu\rho}\hat{x}_\mu - \eta_{\mu\rho}\hat{x}_\nu) + \frac{i}{\kappa} (\eta_{\rho0}M_{\mu\nu} - \eta_{\mu0}M_{\nu\rho}) \]
\[ [M_{\mu\nu}, \Lambda_{\rho\tau}] = i\hbar (\eta_{\nu\rho}\Lambda_{\mu\tau} - \eta_{\mu\rho}\Lambda_{\nu\tau}) \]
\[ [\hat{p}_\mu, \Lambda_{\rho\tau}] = 0 \] (11)

In such a way we obtain a generalized $\kappa$-deformed phase space with 20 generators ($\hat{x}_\mu, \Lambda_{\rho\tau}, \hat{p}_\nu, M_{\lambda\rho}$) as a $\kappa$-deformed counterpart of the classical generalized phase space, which includes also relativistic spin degrees of freedom.

Summarizing, the complete information on quantum $\kappa$-deformed Poincaré symmetries is given by the Hopf-algebraic form of $\kappa$-Poincaré algebra or, equivalently, the quantum $\kappa$-deformed Poincaré group. Such a structure extends simultaneously the notion of classical symmetries in their infinitesimal (Lie-algebraic) as well as global (Lie group) form. Further, having $\kappa$-deformed Poincaré algebra one can uniquely define via Heisenberg double construction the $\kappa$-deformed relativistic phase space, in its standard (8-dimensional) or extended (20-dimensional) form.

The formulation of $\kappa$-deformed Poincaré algebra can be given in different bases, related by nonlinear transformations of the generators. In particular the first formulation of $\kappa$-Poincaré algebra has been given in so-called standard basis [1–3], with deformed boost sector of the Lorentz subalgebra. The bicrossproduct basis (3–5) with classical Lorentz sector can be rewritten in any nonlinearly related coordinate frame

\[ p'_i = p_i f_\kappa(\vec{p}^2, p_0), \quad p'_0 = g_\kappa(\vec{p}^2, p_0), \] (12)

where $f_\infty(\vec{p}, p_0) = 1$, $g_\infty(\vec{p}^2, p_0) = p_0$ and we have chosen the dependence on the length $|\vec{p}|$ of the three-momentum in order to preserve in all bases the classical $O(3)$-covariance. In particular choosing [7]

\[ f_\kappa(\vec{p}^2, p_0) = A_{\kappa} e^{\frac{p_0}{2\kappa}}, \quad g_\kappa(\vec{p}^2, p_0) = A \left( e^{\frac{p_0}{2\kappa}} - 1 - \frac{C_{\kappa}}{2\kappa^2} \frac{\vec{p}^2}{p_0} \right) \] (13)
where $C_2^\kappa$ is given in by (2) we obtain $P_{\mu}' = P_{\mu}^{(0)}$, (see (1)), i.e. we arrive at the classical Poincaré algebra basis. The standard choice of the formula (12) is obtained if $A = \kappa - \frac{M^2}{2\kappa}$ and $C = \kappa + \frac{M^2}{2\kappa}$. One gets the following inverse deformation map: [25,7]

\[ P_{\mu}^{(0)} = e^{\frac{P_{\mu}}{\kappa}} P_{\mu}, \quad P_0^{(0)} = \kappa \sinh \frac{P_0}{\kappa} + \frac{1}{2\kappa} e^{\frac{P_0}{\kappa}} P^2 \] (14)

where

\[ \left( P_0^{(0)} \right)^2 - \left( P_i^{(0)} \right)^2 = M_0^2 = M^2 \left( 1 + \frac{M^2}{4\kappa^2} \right), \] (15)

and the formula (15) describes the relation between the $\kappa$-deformed rest mass $M$ and its classical counterpart $M_0$ in classical Poincaré basis. Using (14) we obtain the coproducts of $P_{\mu}^{(0)}$ given by (see [8])

\[ \Delta \left( P_0^{(0)} \right) = P_0^{(0)} \otimes K^2 + K^{-2} \otimes P_0^{(0)} + \frac{1}{\kappa} K^{-2} P_i^{(0)} \otimes P_i^{(0)} \] (16a)

\[ \Delta \left( P_i^{(0)} \right) = P_i^{(0)} \otimes K + 1 \otimes P_i^{(0)} \] (16b)

where

\[ K = \kappa^{-\frac{1}{2}} \left[ P_0^{(0)} + \left( \left( P_0^{(0)} \right)^2 - \left( P_i^{(0)} \right)^2 + \kappa^2 \right) \right]^{\frac{1}{2}} \] (17)

The coproduct $\Delta(E)$ of energy $E = P_0^{(0)}$ (we put here $c = 1$) describes the Hamiltonian of the system composed out of two constituents [27,10]. The nonprimitive nature of the coproduct (16a) describes the geometric interaction, which tells us that the system invariant under the $\kappa$-Poincaré symmetry describes geometrically interacting 2-particle system. We obtain from (16a) that [8]

\[ E_{1+2} = E_1 + E_2 + \frac{1}{\kappa} \vec{P}_1 \cdot \vec{P}_2 + \frac{1}{2\kappa^2} [E_2(E_2^2 - \vec{P}_2^2)] + (E_1^2 + \vec{P}_1^2) E_2 - 2E_1(\vec{P}_1 \cdot \vec{P}_2)] \] + O(\frac{1}{\kappa^3}), \] (18a)

i.e. the terms of order $\frac{1}{\kappa^k}$ for $k \geq 2$ become nonsymmetric. Assuming that $E_1 = \frac{P_1^2}{2\kappa}$, $E_2 = \frac{P_2^2}{2\kappa}$ the formula (18a) can be rewritten as follows:

\[ E_{1+2} = \frac{(\vec{P}_1 + \vec{P}_2)^2}{2\kappa} + O\left( \frac{1}{\kappa^2} \right), \] (18b)
It would be very interesting to find a physical interpretation of the nonlinear terms on rhs of (18a) as due to some algebraic approximation to universal quantum gravity effects (in such a case one should put $\kappa = M_{pl}$).

It should be added that the transformation of the $\kappa$-Poincaré algebra generators from standard to classical basis was studied already in 1993 by Maślanka [28].

In order to introduce the $\kappa$-deformed boost transformations of the four-momentum variables one should use the formula (see [6], formula (2.37))

$$P_{\mu}(\alpha) = ad_{\exp i\alpha_i N_i}(P_{\mu}) = \sum_{k=0}^{\infty} \frac{\alpha_{l_1} \ldots \alpha_{l_k}}{k!} ad_{N_{l_1}} \left( ad_{N_{l_2}} \ldots (ad_{N_{l_k}}(P_{\mu})) \ldots \right)$$ \hfill (19)

where $ad_{a}b \equiv a_1 b S(a_2)$ denotes quantum adjoint operation [20], $\Delta(a) = a_1 \otimes a_2$ (Swedler notation) and $\alpha_i$ denote three boost parameters. The $\kappa$-deformed Poincaré algebra is covariant under the $\kappa$-deformed Lorentz transformations (19), and the $\kappa$-deformed Casimirs remains invariant. We see therefore that in the $\kappa$-deformed framework we obtain the equivalence of $\kappa$-deformed frames, contrary to the framework with modification of mass-shell condition due to broken Poincaré invariance.

In bicrossproduct basis (3–5), with classical Lorentz subalgebra one can derive the following relation [6]

$$ad_{N_i} P_{\mu} = [N_i, P_{\mu}]$$ \hfill (20)

and the formula (19) takes the form

$$P_{\mu}(\alpha_i) = \exp(i \alpha_i N_i) P_{\mu} \exp(-i \alpha_i N_i) ,$$ \hfill (21)

as in the case of classical Poincaré symmetry. It should be pointed out that the formula (21) is also the base for the derivation of $\kappa$-deformed boost transformations in DSR1 theory [12]. In quantum $\kappa$-deformed framework the formula (21) is consistent as well with the addition law of the momenta described by the coproduct $\Delta(P_{\mu})$, (see (4)) i.e. one can show that [17]

$$[\Delta(P_{\mu})](\alpha_i) = \exp \{ i \alpha_i \Delta(N_i) \} \Delta(P_{\mu}) \exp \{ -i \alpha_i \Delta(N_i) \} . \hfill (22)$$

The relation (22) shows how to extend the equivalence of $\kappa$-deformed Poincaré frames to multiparticle states.
3 Doubly Special Relativity (DSR) Theories

The main idea of DSR theories is based on the physical interpretation of the results of quantum κ-deformed framework. It was observed [11–13] that in the case of κ-deformed mass-shell condition (2) one can define operationally the deformation parameter κ as the limiting three-momentum. Indeed from (2) if \( M = 0 \) it follows that

\[
\bar{p}^2 = c^2 \kappa^2 \left( 1 - e^{-\frac{E}{\kappa c}} \right)^2 \xrightarrow{E \to \infty} c^2 \kappa^2
\]

i.e. one obtains that the maximal value of three-momentum \( p \equiv |\bar{p}| = \kappa c \) determines the parameter κ. It appears therefore that the κ-deformed Lorentz transformations (21) which were explicitly calculated in [12] (see [17] for arbitrary boost three-vector) leave invariant two parameters: the observer-independent velocity \( c \) and maximal momentum \( |\bar{p}| = \kappa c \), defining the masslike geometric parameter κ. Identifying \( \kappa = M_p \) one gets the generalized relativity theory with two fundamental constants \( c \) and \( M_p \), called doubly special relativity theory.

Unquestionable merit of the DSR research is stressing the presence in the κ-deformed framework of modified Lorentz transformation laws between inertial observers, and the interpretation of \( \kappa = M_p \) as the observer-independent limit. It should be pointed out however that the equivalence between inertial observers in κ-deformed framework is the basic feature of the whole quantum group approach, and it is a build-in property of quantum κ-Poincaré framework also in general basis (see [25]). The DSR approach borrows however only the algebraic part of the quantum κ-Poincaré framework, and then exposes the property that the deformed energy-momentum dispersion relation (2) is valid in all κ-deformed frames.

At present DSR theories determine only the properties of the one-particle sector, described by single irreducible representations of κ-deformed Poincaré algebra. In DSR approach the attitude toward the choice of coproduct is ambiguous. Usually (see e.g. [15], where the postulates of DSR theories are listed and discussed) there is presented the symmetric addition law of fourmomenta. In such a case the DSR theory is bound to be a classical relativistic theory with classical linear coproduct, rewritten in nonlinear basis via the formulae \( P_\mu = P_\mu(P^{(0)}) \) inverse to the relations (14) [7]. Such an interpretation of DSR1 theories was firstly given in [17], where also the nonlinear symmetric addition law for energy and momentum was derived [17,18].
Other coalgebra extension of DSR1 theory by supplementing the addition law which is described by the coproduct of quantum $\kappa$-Poincaré algebra (see (4)) is advocated by Kowalski-Glikman et al (see e.g. [16]). This version of DSR1 theory is identical with the framework of quantum $\kappa$-deformed framework. In particular within this approach there was investigated the subalgebra of $(\hat{x}_\mu, M_{\mu\nu})$ of $\kappa$-deformed generalized phase space (10–11), which is the $D = 4$ de-Sitter algebra of DSR approach [29].

It should be stressed that from the mathematical point of view the choice of basis in the algebraic sector and corresponding choice of $\kappa$-deformed mass shell (e.g. given by (2)) is a matter of convention. In particular for quantum $\kappa$-Poincaré algebra only physical arguments can select a given fourmomentum basis. A good example of such considerations which provide the definite form of $\kappa$-deformed algebra is the recent paper by Amelino-Camelia, Smolin and Starodubtcev [30]. In [30] it was shown that for $D = 3$ quantum gravity with cosmological term the space-time symmetry is a $q$-deformed Drinfeld-Jimbo $D = 3$ de-Sitter algebra $SO_q(3, 1)$ where $q$ is proportional to the inverse $dS$ radius. In such a way it was shown twelve years later that the first method of obtaining in 1991 the quantum $\kappa$-Poincaré algebra by the contraction of Drinfeld-Jimbo deformation of $(D = 4$ anti-de-Sitter algebra) $SO(3, 2)$ (see [1]) is based on interesting physical ground.

Finally we would like to recall that besides the DSR1 theories, with the range of energies extending to infinity and limit value $|\vec{p}| = \kappa c$ of three-momenta (see (23)) we can have other two types of DSR theories [26].

i) with both energy $E$ and three-momentum $|\vec{p}|$ bounded by $\kappa$: $|\vec{p}| \leq \kappa c$ and $D \leq \kappa c^2$. Such theories can be called DSR2 theories [15]. The first example of DSR2 theory, with symmetric coproduct, i.e. equivalent to nonlinear description of classical relativistic theories, was presented in [14].

ii) Remaining class of DSR3 theories is provided by energy bounded ($E \leq \kappa c^2$) and three-momenta unbounded. The example of such basis was presented in [26].

It should be added that one can have obviously the deformed relativistic theories with three-momentum and energy unbounded, without the interpretation of the parameter $\kappa$ as frame-independent limiting value of momentum and/or energy.
4 Final Remarks

We have an impression that in the framework of DSR theories there is some confusion what should be included in the DSR scheme. The general attitude of DSR approach is to take only the postulates which are physically motivated, and that selects which parts of the quantum $\kappa$-Poincaré framework are incorporated. It is however not always so easy: because of the arguments originating in quantum gravity the DSR theories do not renounce the noncommutative space-time framework, but simultaneously the nonsymmetric coproducts of quantum group approach usually are not accepted. Unfortunately you can not have both - or we postulate symmetric coproduct and classical space-time, or nonsymmetric coproduct and noncommutative Minkowski space. The relation of DSR approach to quantum $\kappa$-Poincaré framework is therefore somewhat schizophrenic. In order to illustrate such a statement we recall that in some papers (see e.g. [16]) it can be explicitly seen that DSR theories are described by quantum $\kappa$-Poincaré framework, but in other one (see [31]) we find the subtitle about "an illustrative example of $\kappa$-Poincaré algebra which is not admissible in DSR".

In conclusion whatever are the disputes between these two approaches one should say that DSR and quantum $\kappa$-Poincaré points of view have the same common aim - to find a physically plausible formalism for modification of classical relativistic symmetries. At present, however, this goal still has not been fully achieved.

Acknowledgments

The discussion with G. Amelino-Camelia, J. Kowalski-Glikman, N. Mavromatos and A. Nowicki are gratefully acknowledged.

References

[1] J. Lukierski, A. Nowicki, H. Ruegg and V.N. Tolstoy, Phys. Lett. B264, 331 (1991).

[2] S. Giller, P. Kosiński, M. Majewski P. Maślanka and J. Kunz, Phys. Lett. B286, 57 (1992).

[3] J. Lukierski, A. Nowicki and H. Ruegg, Phys. Lett. B293, 344 (1992).
[4] S. Majid, H. Ruegg, Phys. Lett. **B334**, 348 (1994); [hep-th/9405107](http://arxiv.org/abs/hep-th/9405107).

[5] J. Lukierski, H. Ruegg, W. Zakrzewski, Ann. Phys. **243**, 90 (1995); [hep-th/9312153](http://arxiv.org/abs/hep-th/9312153).

[6] J. Lukierski, H. Ruegg and V.N. Tolstoy, in "Quantum Groups: Formalism and Applications", Proc. XXX-th Karpacz Winter School, Feb. 1994, ed. by J. Lukierski, Z. Popowicz and J. Sobczyk, Polish Scientific Publishers (1995), p. 359.

[7] P. Kosinski, J. Lukierski, P. Maslanka, and J. Sobczyk, Mod. Phys. Lett. **10A**, 2599 (1995); [hep-th/9412114](http://arxiv.org/abs/hep-th/9412114).

[8] J. Lukierski, in “Quantum Group Symposium at Group 21”, ed. H.D. Doebner and V.K. Dobrev, Heron Press, Sofia 1997, p. 173.

[9] S. Zakrzewski, J. Phys. **A27**, 2075 (1994).

[10] J. Lukierski and A. Nowicki, in “Quantum Group Symposium at Group 21”, ed. H.D. Doebner and V.K. Dobrev, Heron Press, Sofia 1997, p. 186; [q-alg/9702003](http://arxiv.org/abs/q-alg/9702003).

[11] G. Amelino-Camelia, Phys. Lett. **B510**, 255 (2001); [hep-th/0012238](http://arxiv.org/abs/hep-th/0012238).

[12] R. Bruno, G. Amelino-Camelia and J. Kowalski-Glikman, Phys. Lett. **B522**, 133 (2001); [hep-th/0107039](http://arxiv.org/abs/hep-th/0107039).

[13] G. Amelino-Camelia, Int. J. Mod. Phys. **D11**, 35 (2002); [hep-th/0012051](http://arxiv.org/abs/hep-th/0012051).

[14] J. Magueijo and L. Smolin, Phys. Rev. Lett. **88**, (2002)190403; [hep-th/0112090](http://arxiv.org/abs/hep-th/0112090).

[15] G. Amelino-Camelia, D. Benedetti, F. D’Andrea and A. Bocaccini, Class. Quant. Grav. **20**, 5353 (2003); [hep-th/0201245](http://arxiv.org/abs/hep-th/0201245).

[16] J. Kowalski-Glikman and S. Nowak, Phys. Lett. **B539**, 126 (2002); [hep-th/0203040](http://arxiv.org/abs/hep-th/0203040).

[17] J. Lukierski and A. Nowicki, Int. Journ. Mod. Phys. **A18**, 7 (2003); [hep-th/0203065](http://arxiv.org/abs/hep-th/0203065).
[18] S. Judes and M. Visser, Phys. Rev. D68, (2003) 045001; gr-qc/0205067
[19] J. Magueijo and L. Smolin, Phys. Rev. D67, (2003) 044017; hep-th/0207085
[20] S. Majid “Fundations of Quantum Group Theory”, Cambridge University Press, 1995.
[21] G. Amelino-Camelia, J. Ellis, N.E. Mavromatos and D.V. Nanopoulos, Int. J. Mod. Phys. A12, 607 (1997); hep-th/9605211
[22] G. Amelino-Camelia, J. Ellis, N.E. Mavromatos, D.V. Nanopoulos and S. Sarkar Nature 393, 763 (1998); astro-ph/9712103
[23] R.J. Protheroe and H. Meyer, Phys. Lett. B493, 1(2000).
[24] R. Aloisio, P. Blasi, P.L. Ghia and A.F. Grillo, Phys. Rev. D62 (2000) 053010.
[25] H. Ruegg and V.N. Tolstoy, Lett. Math. Phys. 32, 85 (1994); hep-th/9406146
[26] J. Lukierski and A. Nowicki, Proceedings of International Colloquium “Group 24”, Paris, July 2002, ed. J.P. Gazeau et al. IOP Publishing House, Bristol, 2003, p. 287; hep-th/0210111
[27] J. Lukierski and P. Stichel, Czech. Journ. Phys. 47, 55 (1997); hep-th/9606170
[28] P. Maślanka, J. Math. Phys. 34, 6025 (1993).
[29] J. Kowalski-Glikman, Phys. Lett. B547, 291 (2002); hep-th/0207279
[30] G. Amelino-Camelia, L.Smolin and Starodubtsev, hep-th/0306134
[31] G. Amelino-Camelia, J. Kowalski-Glikman, G. Mandacini and A. Pro- caccini, “Phenomenology of Doubly Special Relativity”, gr-qc/0312124