A Separation Principle for Discrete-Time Fractional-Order Dynamical Systems and its Implications to Closed-loop Neurotechnology

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Abstract—Closed-loop neurotechnology requires the capability to predict the state evolution and its regulation under (possibly) partial measurements. There is evidence that neurophysiological dynamics can be modeled by fractional-order dynamical systems. Therefore, we propose to establish a separation principle for discrete-time fractional-order dynamical systems, which are inherently nonlinear and are able to capture spatiotemporal relations that exhibit non-Markovian properties. The separation principle states that the problems of controller and state estimator design can be done independently of each other while ensuring proper estimation and control in closed-loop setups. Lastly, we illustrate, as proof-of-concept, the application of the separation principle when designing controllers and estimators for these classes of systems in the context of neurophysiological data. In particular, we rely on real data to derive the models used to assess and regulate the evolution of closed-loop neurotechnologies based on electroencephalographic data.

I. INTRODUCTION

There is an increasing trend of looking into leveraging closed-loop control and estimation strategies for the continued monitoring and interaction of subjects in the form of closed-loop neurotechnologies. Such technologies bring the promise of improving the quality-of-life of patients affected by neurological disorders such as epilepsy [1], Parkinson’s disease [2], Alzheimer’s disease [3], anxiety [4], and depression [5]. For neurophysiological signals, lingering interacting effects originating from long-term temporal dependence properties have illustrated the potential for clinical applications of fractional-order based modeling, design, and analysis of such neurotechnologies [6]–[9].

Due to the highly dynamic nature of the neurophysiological processes, it is imperative that we consider feedback mechanisms [10]. A particularly successful closed-loop controller design strategy that has achieved remarkable success in several engineering applications is the strategy of model predictive control [11]. Indeed, the main advent of model-based approaches is that we can understand how an external signal or stimulus would craft the dynamics of the process. In [12], the authors propose an electrical neurostimulation MPC-based strategy for the mitigation of epileptic seizures by modeling brain dynamics through fractional-order systems.

Recent work provides evidence that fractional-order dynamical systems (FODS) exhibit great success in accurately modeling dynamics which undergo nonexponential power-law decay, and have long-term memory or fractal properties [13]–[18]. Not only have FODS found applications in domains such as gas dynamics [19], viscoelasticity [20], chaotic systems [21], and biological swarming [22], just to mention a few, but also in cyber-physical systems to model the interlaced evolution of the spatial and temporal components of complex networks [23], [24]. Some of these relationships have also been explored in the domain of neurophysiological signals such as electroencephalogram (EEG) and electrocardiogram (ECG) [25].

The separation principle, one of the cornerstones of modern feedback systems theory, states that the problems of optimal control and state estimation can be decoupled in certain specific instances [26]. These ideas were advanced early on in [27], [28] and [29] and are connected to the idea of certainty equivalence [30] in stochastic control theory. Since then, the separation principle has been proposed in a wide variety of settings, including, but not limited to, stochastic control systems [31], [32], hybrid systems [33], distributed control systems [34], quantum control [35], linear systems with Markovian jumps [36], wireless fading channels subject to channel capacity constraints [37], and discrete-time networked control systems with random packet drops [38].

However, the separation principle does not hold for nonlinear systems in general. Therefore, in this paper, we state and prove a separation principle result that stems from the problem of closed-loop discrete-time FODS and demonstrate the implications of our results in the context of closed-loop neurotechnology using real-world electroencephalographic data. Specifically, we prove that if a closed-loop controller and an observer are designed for discrete-time FODS, then the aforementioned design can be carried out independently of each other. FODS are inherently nonlinear and they possess long-term memory in the sense that the evolution of a FODS aggregates the effects of all time as the evolution of the system progresses. As a consequence, the intrinsic non-Markovian nonlinearity of FODS does not immediately ensure the existence of a separation simple for the reasons mentioned above. Furthermore, FODS are finding increasing applications in the field of model predictive control (MPC), where the problems of estimator and controller design need the existence of a separation principle. Although separation principle results such as [39]–[41] have been derived for FODS in continuous time, and, to the best of our knowledge, no such result has been previously proposed and analyzed for discrete-time FODS.

The remainder of the paper is organized as follows. Sec-
III. SEPARATION PRINCIPLE FOR FRACTIONAL-ORDER SYSTEMS

In this section, we will present the main result of our paper, i.e., the separation principle for discrete-time FODS. We first introduce the theory of state evolution in discrete-time FODS, presenting the relevant equations for the evolution of the dynamics of the system states in Lemma 1. We will then sequentially consider the problems of observer design (in Section III-A), which entails the construction of an observer for the dynamical system (1), followed by the problem of stabilizability (in Section III-B), which requires us to design a controller to stabilize the system (1). With the above ingredients, and some mathematical preliminaries, we present the statement and proof of the main result of our paper, the separation principle for discrete-time FODS in Section III-C (see Theorem 1).

We begin by reviewing some essential theory for fractional-order systems, including closed-form expressions for the state dynamics. Using the expansion of the fractional-order derivative in (2), the evolution of the state vector can be written as follows

\[ x[k + 1] = Ax[k] - \sum_{j=1}^{k+1} D(\alpha, j)x[k + 1 - j] + Bu[k] \]

where \( D(\alpha, j) = \text{diag}(\psi(\alpha_1, j), \psi(\alpha_2, j), \ldots, \psi(\alpha_n, j)) \).

Alternatively, (4) can be written as

\[ x[k + 1] = \sum_{j=0}^{k} A_j x[k - j] + Bu[k] \]

where \( A_0 = A - D(\alpha, 1) \) and \( A_j = -D(\alpha, j + 1) \) for \( j \geq 1 \).

Defining matrices \( G_k \) as

\[ G_k = \begin{cases} I_n & k = 0, \\ \sum_{j=0}^{k-1} A_j G_{k-1-j} & k \geq 1, \end{cases} \]

we can state the following result.

Lemma 1 ([43]). The solution to the system described by (1) is given by

\[ x[k] = G_k x[0] + \sum_{j=0}^{k-1} G_{k-1-j} Bu[j]. \]
substituting (9) into (5), we have

\[ x[k + 1] = \sum_{j=0}^{k} A_j \hat{x}[k - j] + Bu[k] + L(y[k] - \hat{y}[k]), \]

\[ \hat{y}[k] = C\hat{x}[k], \]

where the matrix \( L \in \mathbb{R}^{n \times n} \) is a weighting matrix that weights the difference between the outputs of the plant and the observer. Note that the observer consists of two parts, the first part being a copy of the plant’s dynamics as applied to the observer, and an innovation term being a scaled version of the difference between the outputs of the plant and the observer.

### B. Stabilizability and Output Feedback

In this section, we consider the problem of stabilizing (5) in a classical state-feedback control setting. Assume that the control input \( u \in \mathbb{R}^p \) can be written a weighted linear combination of the states of the observer with memory, i.e.,

\[ u[k] = F_0 \hat{x}[k] + F_1 \hat{x}[k - 1] + \ldots + F_k \hat{x}[0] \]

\[ = \sum_{j=0}^{k} F_j \hat{x}[k - j], \quad (9) \]

where \( F_j \in \mathbb{R}^{p \times n} \) for \( j = 0, 1, \ldots, k \). Substituting this into (8) and using the fact that \( y[k] = Cx[k] \) and \( \hat{y}[k] = C\hat{x}[k] \), we have

\[ \hat{x}[k + 1] = \sum_{j=0}^{k} (A_j + BF_j) \hat{x}[k - j] + L(Cx[k] - C\hat{x}[k]) \]

\[ = \sum_{j=0}^{k} (A_j + BF_j) \hat{x}[k - j] + LCe[k], \]

\[ = \sum_{j=0}^{k} (A_j + BF_j) e[k - j] - LCe[k]. \]

(10)

where \( e[k] = x[k] - \hat{x}[k] \) is defined as the error between the states of the plant and the observer.

We now turn our attention towards the problem of output feedback. Going back to the dynamics of the plant and substituting (9) into (5), we have

\[ x[k + 1] = \sum_{j=0}^{k} A_j x[k - j] + Bu[k] \]

\[ = \sum_{j=0}^{k} A_j x[k - j] + \sum_{j=0}^{k} BF_j \hat{x}[k - j] \]

\[ = \sum_{j=0}^{k} A_j x[k - j] + \sum_{j=0}^{k} BF_j (x[k - j] - e[k - j]) \]

\[ = \sum_{j=0}^{k} (A_j + BF_j) x[k - j] - \sum_{j=0}^{k} BF_j e[k - j]. \]

(11)

Next, we consider the dynamics of the error signal. Indeed, we have

\[ e[k + 1] = x[k + 1] - \hat{x}[k + 1] \]

\[ = \sum_{j=0}^{k} (A_j + BF_j) x[k - j] - \sum_{j=0}^{k} BF_j e[k - j] \]

\[ - \left( \sum_{j=0}^{k} (A_j + BF_j) \hat{x}[k - j] + LCe[k] \right) \]

\[ = \sum_{j=0}^{k} (A_j + BF_j) e[k - j] - \sum_{j=0}^{k} BF_j e[k - j] \]

\[ - LCe[k] \]

\[ = \sum_{j=0}^{k} A_j e[k - j] - LCe[k]. \]

(12)

### C. Separation Principle for Discrete-Time FODS

Having derived the expressions for the dynamics of the plant state and the error signal, we are now ready to state and prove the separation principle for discrete-time FODS. We first state some mathematical preliminaries that will aid our proof.

**Definition 1.** A Hilbert space is a vector space \( \mathcal{H} \) over \( \mathbb{R} \) or \( \mathbb{C} \) together with an inner product \( \langle \cdot, \cdot \rangle \) such that relative to the metric \( d(x, y) = \|x - y\| \) induced by the norm \( \| \cdot \|_2 = \langle \cdot, \cdot \rangle \), \( \mathcal{H} \) is a complete metric space.

**Definition 2.** The sequence space \( \ell^2(\mathbb{N}) \) denotes the Hilbert space of all square-summable sequences. Such sequences are represented by vectors with infinitely many elements \( \mathcal{X} = \{x[0], x[1], x[2], \ldots \} \). For \( \mathcal{X}, \mathcal{Y} \in \ell^2(\mathbb{N}) \), the space is equipped with the inner product

\[ \langle \mathcal{X}, \mathcal{Y} \rangle = \sum_{k=0}^{\infty} x[k] y[k]^*, \]

where the \( * \) denotes the complex conjugate. In other words, a sequence \( \mathcal{X} \in \ell^2(\mathbb{N}) \) if \( \| \mathcal{X} \|_2^2 = \langle \mathcal{X}, \mathcal{X} \rangle = \sum_{k=0}^{\infty} |x[k]|^2 < \infty \).

**Definition 3.** For a causal sequence \( \mathcal{X} \), we define the **backward shift operator** \( S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}) \) by

\[ SX = S\{x[0], x[1], x[2], \ldots \} = \{x[1], x[2], x[3], \ldots \}. \]

**Definition 4.** The **spectrum** of a matrix \( M \), denoted by \( \text{spec}(M) \), is the set of eigenvalues of the matrix \( M \).

Lastly, we present the main result of this paper.

**Theorem 1** (Separation Principle for discrete-time FODS). Consider the discrete-time fractional-order dynamical system given in (1), and consider the problems of

1. Designing an unbiased estimator (of the form (8)) for the system (1) by following the procedure outlined in Section III-A, and,
2. Designing a controller (of the form (9)) that stabilizes the system (1) by following the procedure outlined in Section III-B.
Then, given knowledge of the input \( u \in \mathbb{R}^p \) and the output \( y \in \mathbb{R}^n \), the above designs can be done independently of each other towards achieving closed-loop stabilizability with partial measurements.

Proof. With respect to our problem, we define the infinite column sequences

\[
X = \begin{bmatrix} x[0] \\ x[1] \\ x[2] \end{bmatrix}, \quad E = \begin{bmatrix} e[0] \\ e[1] \\ e[2] \end{bmatrix}.
\]

Using \( X \) and \( E \), we can now compactly write equations (11) and (12) as follows

\[
\begin{bmatrix} S X \\ S E \end{bmatrix} = \begin{bmatrix} J_1 & J_2 & J_3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ E \end{bmatrix},
\]

where \( S \) is the backward shift operator and the matrices \( J_i \) \( (i = 1, 2, 3) \) are Toeplitz with the following structures

\[
J_1 = \begin{bmatrix}
A_0 + BF_0 & 0 & 0 & \cdots & 0 \\
A_1 + BF_1 & A_0 + BF_0 & 0 & \cdots & 0 \\
A_2 + BF_2 & A_1 + BF_1 & A_0 + BF_0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots
\end{bmatrix},
\]

\[
J_2 = \begin{bmatrix}
-BF_0 & 0 & 0 & \cdots & 0 \\
-BF_1 & -BF_0 & 0 & \cdots & 0 \\
-BF_2 & -BF_1 & -BF_0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots
\end{bmatrix},
\]

\[
J_3 = \begin{bmatrix}
A_0 - LC & 0 & 0 & \cdots & 0 \\
A_1 - LC & A_0 - LC & 0 & \cdots & 0 \\
A_2 - LC & A_1 - LC & A_0 - LC & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots
\end{bmatrix}.
\]

Note that \( J_1 \) only contains terms that pertain to the stabilizability, and \( J_3 \) only contains terms that pertain to the observer design. From the block structure of (14), it can be seen that

\[
\text{spec}(J) = \text{spec}(J_1) \cup \text{spec}(J_3),
\]

and the design of \( J_1 \) and \( J_3 \) can be carried out independently of each other.

The key difference between the observers in equations (8) and (16) are that in the former we have a single weighting matrix that weights the difference of the outputs of the plant and the observer, and in the latter, we use multiple weighting matrices to weight the differences of the outputs of the plant and the observer with memory. We then have the following theorem.

**Theorem 2.** Consider the discrete-time fractional-order dynamical system given in (1), and consider the problems of

1. Designing an unbiased estimator (of the form (16)) for the system (1) by following the procedure outlined above, and,
2. Designing a controller (of the form (15)) that stabilizes the system (1) by following the procedure outlined in Section III-B.

Then, given knowledge of the input \( u \in \mathbb{R}^p \) and the output \( y \in \mathbb{R}^n \), the above designs can be done independently of each other towards achieving closed-loop stabilizability with partial measurements.

Proof. By setting \( L_0 = L \), and \( L_1 = L_2 = \ldots = L_k = 0 \) for \( k = 1, 2, \ldots \), the problem reduces to the statement of Theorem 1 and the proof follows by a similar line of reasoning.

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**IV. CLOSED-LOOP NEUROTECHNOLOGY**

In this section, we illustrate our results by designing a model predictive controller (MPC) that simulates a simple implantable closed-loop electrical neurostimulator. The controller is implemented on a discrete-time fractional-order plant, representing normal brain activity, whereas the predictive model will be based on an autoregressive finite-history approximation. Naturally, the controller will be designed as if it had access to the actual state of the system, and similarly the state observer (whose estimates are fed into the designed controller) is designed without consideration of the control strategy adopted.

We start by identifying the spatial and temporal parameters \( A \) and \( \alpha \) in (1), from a 4-channel sample of length 1 second of normalized EEG recordings. We model the \( n = 4 \) components of the state vector as denoting the different recorded channels (i.e., readings obtained from microelectrodes). The data used for these experiments are from subject 11 from the CHB-MIT Scalp EEG database [45]. To achieve this identification, we leveraged the tools developed in [46],...
which led us to

\[
A = \begin{bmatrix}
0.0350 & 0.0526 & -0.0034 & -0.0391 \\
0.0296 & -0.0496 & 0.0646 & 0.0610 \\
-0.0103 & -0.0028 & -0.0091 & 0.0068 \\
-0.0291 & 0.0143 & -0.0008 & 0.0394
\end{bmatrix}
\]  

(17)

and

\[
\alpha = [0.5945 \ 0.7176 \ 0.9603 \ 0.6279]^T,
\]  

(18)
as the main parameters in the system. We are interested in modeling the impact of an electrical stimulation signal \(u[k]\) originating from an integrated arbitrary voltage generator circuit. We start by considering the scenario \(B = [1 \ 1 \ 1 \ 1]^T\) corresponding to a stimulus that perturbs all channels uniformly (e.g., if the four electrodes are placed considerably near each other). The measurements \(y[k]\) used to estimate the state (through a simple Kalman-like filter) will be assumed simply as those given directly by the first channel, i.e., \(C = [1 \ 0 \ 0 \ 0]^T\).

At each step \(k\), the MPC controller will minimize a quadratic cost function

\[
J(u[k], \ldots, u[k+P-1]) = \sum_{j=1}^{P} \| x[k] - x_{ref}[k+j] \|^2,
\]

(19)

with the predicted evolution \(x[k]\) evolving not by the original system \(\{\}\), but by instead by a multivariate autoregressive (MVAR) approximation

\[
x[k+1] = \sum_{j=0}^{p-1} A_j x[k-j] + B u[k],
\]

(20)

based on \(\{\}\), by clipping off the infinite-horizon memory dependence by instead only a \(p\)-horizon one. The prediction horizon \(P\) was set to \(P = 8\) (50 milliseconds), whereas the control horizon \(M\) upon which the solution is implemented was set to \(M = 4\) (25 milliseconds). The reference signal \(x_{ref}[k]\) denotes a simple rectangular pulse of frequency 8 Hz, within the usual range of alpha rhythms that characterize relaxed, but conscious brain activity [47].

V. CONCLUSIONS AND FUTURE WORK

In this paper, we proposed and proved a separation principle result for discrete-time FODS. As a consequence, we can decouple the problems of designing, respectively, a controller for the stabilization of the system states, and an observer for the estimation of the system states. The ability of discrete-time FODS to model complex spatiotemporal relationships in neurophysiological signals have led to the use of these models in closed-loop neurotechnologies.

Very rarely in practical settings, however, do we have deterministic fractional-order models. EEG signals, for instance, are particularly prone to disturbances arising from outside the brain, which are referred to as artifacts in the neuroscience literature [48]. Furthermore, stabilizing these models in the presence of disturbances becomes relevant in the treatment of disorders like epilepsy, Parkinson’s disease, or Alzheimer’s disease, since, in recent years, there have been increasing research efforts into finding possible palliative therapies for the aforementioned using neurofeedback [49]. Future work, therefore, will focus on developing controllers and observers for FODS with associated process and measurement noise, and investigating the possible existence of separation principle-like results akin to those already existing in the field of linear stochastic control theory.

REFERENCES

[1] F. Sun and M. J. Morrell, “Closed-loop neurostimulation: The clinical experience,” Neurotherapeutics, vol. 11, no. 3, pp. 553–563, 2014.
[2] A. L. Benabid, “Deep brain stimulation for Parkinson’s disease,” Current opinion in neurobiology, vol. 13, no. 6, pp. 696–706, 2003.
[3] R. Nardone, Y. Hölter, F. Tetzon, M. Christova, K. Schwenker, S. Gołąbowski, E. Trinka, and F. Brigo, “Neurostimulation in Alzheimer’s disease: from basic research to clinical applications,” Neurological Sciences, vol. 36, no. 5, pp. 689–700, 2015.
[4] V. Sturm, D. Lenartz, A. Kouloussakis, H. Treuer, K. Herholz, J. C. Klein, and J. Klosterkötter, “The nucleus accumbens: a target for deep-brain stimulation in obsessive-compulsive and anxiety disorders,” in Proceedings of the Medtronic Forum for Neuroscience and NeuroTechnology 2005. Springer, 2007, pp. 62–67.
[5] L. Marangell, M. Martinez, R. Jardi, and H. Zboyan, “Neurostimulation therapies in depression: a review of new modalities,” Acta Psicatrica Psicoveraca, vol. 116, no. 3, pp. 174–181, 2007.
[6] N. Brodu, F. Lotte, and A. Lécuyer, “Exploring two novel features for EEG-based brain–computer interfaces: Multifractal cumulants and predictive complexity,” Neurocomputing, vol. 79, pp. 87–94, 2012.
[7] P. Ciuciu, G. Varoquaux, P. Abry, S. Sadaghiani, and A. Kleinschmidt, “Scale-free and multifractal properties of fMRI signals during rest and task,” Frontiers in Physiology, vol. 3, p. 186, 2012.
[8] T. Zorick and M. A. Mandelkern, “Multifractal detrended fluctuation analysis of human EEG: preliminary investigation and comparison with the wavelet transform modulus maxima technique,” PloS one, vol. 8, no. 7, p. e68360, 2013.
[9] Y. Zhang, W. Zhou, and S. Yuan, “Multifractal analysis and relevance vector machine-based automatic seizure detection in intracranial EEG,” International journal of neural systems, vol. 25, no. 06, p. 1550020, 2015.
[10] A. Dzieliński and D. Sierociuk, “Adaptive feedback control of fractional order discrete state-space systems,” in CIMCA-IWAITC, 2005.
[11] E. F. Camacho and C. B. Alba, Model predictive control. Springer Science & Business Media, 2013.
[12] O. Romero and S. Pequito, “Fractional-Order Model Predictive Control for Neuropsychiological Cyber-Physical Systems: A Framework for Electrical Neurostimulation in Epilepsy.” 2019. [Online]. Available: https://www.dropbox.com/s/lm0d41r3dripbr8/Frontiers-Romero.pdf
[13] F. C. Moon, Chaotic and fractal dynamics: introduction for applied scientists and engineers. John Wiley & Sons, 2008.
