The integrable harmonic map problem versus Ricci flow

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Abstract

We construct a zero-curvature representation for a four-parameter family of non-linear sigma models with a Kalb-Ramond term. The one-loop renormalization is performed that gives rise to a new set of ancient and eternal solutions to the Ricci flow with torsion. Our analysis provides an explicit illustration of the role of the dilaton field for the renormalization of the non-linear sigma model.
1 Introduction

Let $\mathcal{M}_D$ be a $D$-manifold (the target space) equipped with a Riemannian metric $G$ and an affine connection. Consider the system of PDE which describes a map of the two-dimensional worldsheet $\Sigma = (x^0, x^1)$ into the affine-metric manifold

$$\partial_+ \partial_- X^\mu + \Gamma^\mu_{\nu\sigma} \partial_+ X^\nu \partial_- X^\sigma = 0.$$ (1.1)

Here it is assumed that $\Sigma$ is equipped with a Minkowski metric, $\partial_\pm = \frac{1}{2} (\partial_0 \pm \partial_1)$ and $\Gamma^\mu_{\nu\sigma}$ stands for Christoffel symbol of the connection. For a general target space background, Eqs.(1.1) cannot be derived from the variational principal. However, as it was observed in Ref. [12], if the connection is compatible with the metric and the covariant torsion tensor

$$H_{\mu\nu\sigma} = G_{\mu\rho} \left( \Gamma^\rho_{\nu\sigma} - \Gamma^\rho_{\sigma\nu} \right)$$ (1.2)

is a closed three-form:

$$H_{\mu\nu\sigma} = \partial_\mu B_{\nu\sigma} + \partial_\nu B_{\sigma\mu} + \partial_\sigma B_{\mu\nu} ,$$ (1.3)

then (1.1) follows from the Polyakov action with the Kalb-Ramond term

$$\mathcal{A} = 2 \int_{\Sigma} d^2x \left( G_{\mu\nu} \partial_+ X^\mu \partial_- X^\nu - \frac{1}{2} B_{\mu\nu} \left( \partial_+ X^\mu \partial_- X^\nu - \partial_- X^\mu \partial_+ X^\nu \right) \right).$$ (1.4)

The 2-form $B_{\mu\nu}$ provides an anti-symmetric component to the affine connection and is sometimes known as the torsion potential. Field theories of the type (1.4) are important in many aspects of physics, from QCD to condensed matter and are known as Non-Linear Sigma Models (NLSM). The corresponding Euler-Lagrange equations are usually referred to as the (generalized) harmonic map problem [1]. In general, the NLSM is a complicated structure. To this order, any simplified example, that softens the sever mathematical problems can be considered useful and worth studying.

Starting at the end of the seventies, an approach was developed for a solution of the generalized harmonic map problem for certain classes of the integrable target space backgrounds [2–4]. More specifically, the term integrable here is used to imply that Eqs.(1.1) constitute a flatness condition

$$[D_+(\lambda), D_-(\lambda)] = 0$$ (1.5)

for some matrix valued worldsheet connection

$$D_\pm(\lambda) = \partial_\pm + A_\pm , \quad A_\pm = \alpha^{(\pm)}(\lambda) \partial_\pm X^\mu ,$$ (1.6)

which depends on an arbitrary complex parameter $\lambda$. The approach has been proven to be effective, especially concerning the models with homogeneous target manifolds $G/H$ and a Zero-Curvature Representation (ZCR) became a central issue in this class of harmonic map problems [5]. At the same time the integrability of NLSM with non-homogeneous target spaces
have been given considerably less attention. In particular, there are important examples of NLSM which are expected to be classical integrable systems despite having their ZCR remain unknown. Among them are the *sausage models* which were introduced in Refs. [6, 7] by taking advantage of a perturbative renormalizability of a general NLSM.

The quantum theory governed by the action (1.4) is perturbatively renormalizable and the scale dependence of its couplings can be computed order by order in perturbation theory [8–14]. This, in effect, induces deformations of the metric and the torsion potential with respect to the Renormalization Group (RG) time $t$, given by (up to the overall factor $\frac{1}{2\pi}$) the logarithm of worldsheet length scale, which can be formulated and studied systematically in all generality. The renormalization of the metric and the torsion potential, viewed as generalized couplings, takes the following form to one-loop [10,12]:

$$\frac{\partial}{\partial t} \left( G_{\mu\nu} + B_{\mu\nu} \right) = - \left( R_{\mu\nu} + 2 D_{\mu} V_{\nu} \right), \quad (1.7)$$

where $R_{\mu\nu}$ is the Ricci tensor built from the affine connection $D_{\mu}$. The RG equation (1.7) is no other but the *Ricci flow* which arose independently in mathematics (in the case of the Levi-Civita connection) as a tool to address a variety of non-linear problems in differential geometry and, in particular, the uniformization of compact Riemannian manifolds [15,16]. The one-loop RG equations (1.7) are highly non-linear and lead to solutions which typically develop singularities. However, the equations possess the *ancient* solutions which exist at $t \to -\infty$ and evolve forward in time until the formation of singularities. The NLSM underlining the ancient solutions have a good chance to be defined non-perturbatively as a local integrable quantum field theory. In the works [6] and [7] there were discovered remarkable ancient solutions which describe torsion-free deformations of the two- and three-spheres, respectively (see also Refs. [17,18] for comprehensive analysis of these solutions). The authors conjectured that the solutions describe the one-loop renormalization of certain quantum field theories and performed highly convincing non-perturbative analysis in favor of their quantum integrability. Note that their arguments were based on the $S$-matrix bootstrap and did not employ any classical integrable structures.

In this article we attempt to reverse the logic of Refs. [6,7] and apply the machinery of classical integrability to produce ancient solutions of the Ricci flow. We find, as the main result, the ZCR for a new four-parameter family of NLSM with three-dimensional target space background. Note that the proposed ansatz for the ZCR can be naturally understood in a context of the averaging procedure applied in the construction of elliptic and trigonometric solutions of the Yang-Baxter equation from the rational one [19,20]. A similar approach was used in Ref. [21] to explore reductions of the Lax representation (see also Part II, Chapter IV.2 in the book [22]). It turns out that modulo reparameterizations encoded by the second term in the r.h.s. of (1.7), the effect of one-loop renormalization within the obtained family of classical integrable NLSM, are reduced to the renormalization of the parameters and the *string tension* (the overall normalization of the action). Therefore, the family provides an interesting example of multi-parameter solution of the Ricci flow driven by the connection with torsion. In the case of a torsion-free background the solution is reduced to Fateev’s three-dimensional sausage [7].

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1 If a solution is defined for $-\infty < t < +\infty$, it is called an *eternal* solution.
Since the Fateev-Onofri-Zamolodchikov sausage [6] can be obtained from the three-dimensional one through a certain limiting procedure, the result of this work yields the ZCR for both models.

2 ZCR for the harmonic map problem

Before focusing on the case with \( D = 3 \) it is useful to rewrite Eqs.(1.1) in the matrix form for an arbitrary dimension \( D \). Suppose \( \mathcal{M}_D \) is an oriented manifold and its metric can be transformed from the coordinate basis to the vielbein one, \( G_{\mu \nu} = e^a_\mu e^a_\nu \). Introduce the conventional one-forms acting in a spinor representation of \( SO(N) \), namely the Levi-Civita spin connection \( \omega_\mu \), and \( \gamma_\mu = \gamma^a e^a_\mu \), where \( \gamma \)-matrixes obey the standard Dirac algebra, \( \{ \gamma^a, \gamma^b \} = 2 \delta^{ab} \). With the use of these notations the generalized harmonic map problem (1.1) can be brought to the form

\[
\partial_+ + \omega_+^\pm \partial_+ X^\mu, \gamma_\nu \partial_- X^\nu = 0, \quad \gamma_\nu \partial_+ X^\nu, \partial_- + \omega_-^\pm \partial_- X^\mu = 0 ,
\]

(2.1)

where \( \omega_\pm^\mu = \omega_\mu \mp \frac{i}{4} H_{\mu \sigma} \gamma^\sigma \) and as usual, \( \gamma^\mu = G^{\mu \nu} \gamma_\nu \).

For \( D = 3 \), \( \gamma \)-matrixes can be identified with the conventional Pauli matrixes \( \gamma^a = \sigma^a \), \( a = 1, 2, 3 \), whereas \( H_{\mu \sigma} \) must be proportional to the volume form,

\[
H_{\mu \sigma} = \sqrt{G} \frac{1}{3!} \epsilon_{\mu \sigma \rho} .
\]

(2.2)

In this case, the matrix valued one-forms \( \omega_\pm^\mu \) in (2.1) is simplified to

\[
\omega_\pm^\mu = \omega_\mu \mp \frac{i}{4} H \gamma_\mu .
\]

(2.3)

Without making any symmetry assumptions, the classification of NLSM possessing the zero-curvature representation seems to be a hopeless task even for lower dimensional target manifolds. We are therefore forced to impose some symmetry conditions on the metric and the torsion potential. Let us assume that the target space background possesses two commuting Killing vector fields. More specifically, there exist a local coordinate frame \( X^\mu = (u, v, w) \) with respect to which the metric takes the form

\[
G_{\mu \nu} \, dX^\mu dX^\nu = G_{uu} \, (du)^2 + G_{vv} \, (dv)^2 + G_{ww} \, (dw)^2 + 2 G_{vw} \, dv \, dw ,
\]

(2.4)

and the components of the metric tensor, as well as the torsion strength \( H \) in (2.2), do not depend on the coordinates \( v \) and \( w \). Without further loss of generality we set \( \sqrt{G_{uu}} \) to be a positive constant

\[
\sqrt{G_{uu}} = const > 0 .
\]

(2.5)

Given the metric \( G_{\mu \nu} \), the introduction of the tangent vectors \( e^a_\mu \) involves arbitrary choices at each point of \( \mathcal{M}_3 \). We are free to make local \( SO(3) \) rotation on the index \( a \), or equivalently adjoint \( SU(2) \) transformation on \( \gamma^\mu \). The transformation law of the 1-form \( \omega_\pm^\mu \) includes an inhomogeneous piece typical of gauge fields,

\[
\begin{align*}
\gamma_\mu & \rightarrow U^{-1} \gamma_\mu U, \\
\omega_\pm^\mu & \rightarrow U^{-1} \omega_\pm^\mu U + U^{-1} \partial_\mu U .
\end{align*}
\]

(2.6)
For the metric of the form (2.3) the gauge freedom can be used to set \( e_u^1 = e_u^2 = e_u^3 = 0 \). Then the non-vanishing components of the vielbein are defined by the formulas

\[
G_{uu} = (e_u^3)^2, \quad G_{uv} = e_u^+ e_v^-, \quad G_{ww} = e_v^+ e_w^-, \quad G_{vw} = \frac{1}{2} (e_v^+ e_w^- + e_w^+ e_v^-)
\]  

(2.7)

modulo \( U(1) \) rotations \( e_\mu^\pm \to e^{\pm \phi} e_\mu^\pm \), where

\[
e_\mu^\pm = e_\mu^1 \pm i e_\mu^2
\]

(2.8)

and \( \phi = \phi(u) \) is an arbitrary local phase. Below we use the fact that the residual freedom in the choice the vielbein implies the local symmetry (2.6), where \( U \) is substituted by the diagonal matrix

\[
U_\phi = \exp \left( \frac{i}{2} \phi(u) \sigma_3 \right).
\]  

(2.9)

Note that the sign of \( e_u^3 = \pm \sqrt{G_{uu}} \) is actually unambiguous for the chosen orientation of the vielbein (i.e., for the chosen sign of \( \sqrt{G} := \det(e^\mu_\mu) \)).

We turn now to the construction of the ZCR. Let \( \zeta_\mu(\lambda) \) be a matrix valued 1-form which depends on the spectral parameter

\[
\zeta_\mu(\lambda) = \sum_{a=\pm,3} f_a(\lambda) e_\mu^a \sigma_a.
\]  

(2.10)

Here \( \sigma_{\pm} = \frac{1}{2} (\sigma_1 \mp i \sigma_2) \) and \( f_a(\lambda) \) read explicitly as follows

\[
f_+(\lambda) = -f_-(\lambda) = \frac{1}{\sqrt{G_{uu}}} \frac{\vartheta_1(u - \frac{\lambda}{2}, q)}{2i \vartheta_1(\frac{u}{2}, q)} \vartheta_1(u, q)
\]

\[
f_3(\lambda) = \frac{1}{\sqrt{G_{uu}}} \frac{\vartheta_1'(\frac{\lambda}{2}, q)}{2i \vartheta_1(\frac{u}{2}, q)}.
\]  

(2.11)

In the l.h.s. of the above equations, we only indicate dependence on the spectral parameter, \( \vartheta_1 \) stands for the conventional theta function of the nome \( q = e^{i\pi \tau} \) \((0 < q < 1)\) and \( \vartheta_1'(u, q) := \partial_u \vartheta_1(u, q) \). To get a more informal feel for \( \zeta_\mu \), let us note that it can be alternatively defined through the principal value summation of the formal double series

\[
\zeta_\mu(\lambda) = \frac{1}{\sqrt{G_{uu}}} \text{V.P.} \sum_{n,m=-\infty}^{\infty} \frac{e^{i n u \sigma_3} \gamma_\mu e^{-i n u \sigma_3}}{1 + 2i \lambda + 2i \tau (m + n \tau)}.
\]  

(2.12)

Using \( \zeta_\mu \) as a building block, we define the worldsheet connection of the form (1.6) with

\[
\alpha_\mu^{(+)}(\lambda) = \alpha_\mu(\lambda \mid \eta_+, \phi_+), \quad \alpha_\mu^{(-)}(\lambda) = \alpha_\mu(\lambda - \pi \mid \eta_-, \phi_-),
\]  

(2.13)

and

\[
\alpha_\mu(\lambda \mid \eta, \phi) = \frac{1}{2i} \left( U_\phi^{-1} \zeta_\mu(i \eta + \lambda) U_\phi + \sigma_2 \right) U_\phi^{-1} \zeta_\mu(i \eta - \lambda) U_\phi \sigma_2.
\]  

(2.14)

Here \( \eta_+ \) and \( \eta_- \) stand for arbitrary parameters whereas \( \phi_\pm = \phi_{\pm}(u) \) are arbitrary local phases showing up in the matrices of the form (2.9). The local twists are included in (2.14) because of the residual freedom in the choice the vielbein.

Under these definitions, it is straightforward to establish the following properties:
• **Quasiperiodicity.**

\[ D_\pm(\lambda) = e^{inu_3} D_\pm(\lambda + 2\pi (m + n\tau)) e^{-inu_3} \quad (m, n \in \mathbb{Z}). \quad (2.15) \]

• **\(\lambda\)-parity.**

\[ D_\pm(-\lambda) = \sigma_2 D_\pm(\lambda) \sigma_2. \quad (2.16) \]

• **Singularities.** Let \(|\Im m(\eta_\pm)| < \pi\), \(|\Re e(\eta_\pm)| < \Im m(\pi\tau)\). In the fundamental parallelogram \((-\pi, \pi) \otimes (-\pi\tau, \pi\tau), D_+(\lambda)\) has two simple poles with the residues

\[
D_+(\lambda) = \frac{1}{\sqrt{G_{uu}}} \frac{1}{2(\lambda + i\eta_+)} U_+^{-1} \gamma_\mu U_+ \partial_+ X^\mu + O(1)
\]

\[
= \frac{1}{\sqrt{G_{uu}}} \frac{1}{2(\lambda - i\eta_-)} \sigma_2 U_+^{-1} \gamma_\mu U_- \partial_\tau X^\mu + O(1).
\]

Similarly, the singularities of \(D_-(\lambda)\) in the parallelogram \((0, 2\pi) \otimes (-\tau, \tau)\) are given by

\[
D_-(\lambda) = -\frac{1}{\sqrt{G_{uu}}} \frac{1}{2(\lambda + \pi + i\eta_-)} U_-^{-1} \gamma_\mu U_- \partial_- X^\mu + O(1)
\]

\[
= \frac{1}{\sqrt{G_{uu}}} \frac{1}{2(\lambda - \pi - i\eta_+)} \sigma_2 U_-^{-1} \gamma_\mu U_+ \partial_\tau X^\mu + O(1).
\]

Here we use the shortcut notations \(U_\pm = U_{\phi_\pm}\).

• **Hermiticity.** Let \(0 < q < 1\), \(|\Im m(\eta_\pm)| = 0\), \(|\eta_\pm| < \Im m(\pi\tau)\), then

\[ D_\pm^\dagger(\lambda) = -D_\pm(-\lambda^*) . \quad (2.19) \]

In particular, \(D_\pm\) are (formally) anti-Hermitian differential operators as \(\Re e(\lambda) = \pi n\) \((n = 0, \pm 1 \ldots)\).

The first two properties of the worldsheet connection imply that the field strength \(F(\lambda) = [D_+(\lambda), D_-(\lambda)]\) satisfies the conditions:

\[
F(\lambda) = e^{inu_3} F(\lambda + 2\pi (m + n\tau)) e^{-inu_3} \quad (m, n \in \mathbb{Z})
\]

\[
F(\lambda) = \sigma_2 F(-\lambda) \sigma_2 . \quad (2.20)
\]

We may try to adjust the target space background to make \(2 \times 2\) matrix \(F(\lambda)\) nonsingular in the whole complex plane of \(\lambda\). Using the matrix form \((2.11)\) of the harmonic map equations, it is easy to see that the cancellation of the poles of \(F(\lambda)\) yields the relations

\[
\omega_\mu^+ = U_+ \alpha_\mu(\pi - i\eta_+ | \eta_+, \phi_-) U_+^{-1} + U_+ \partial_\mu U_+^{-1}
\]

\[
\omega_\mu^- = U_- \alpha_\mu(\pi - i\eta_- | \eta_+, \phi_+) U_-^{-1} + U_- \partial_\mu U_-^{-1},
\]

\[
\omega_\mu^+ = U_+ \alpha_\mu(\pi - i\eta_+ | \eta_+, \phi_-) U_+^{-1} + U_+ \partial_\mu U_+^{-1}
\]

\[
\omega_\mu^- = U_- \alpha_\mu(\pi - i\eta_- | \eta_+, \phi_+) U_-^{-1} + U_- \partial_\mu U_-^{-1}.
\]
or, equivalently,

\[ \omega_\mu^\pm = \frac{1}{2i} \left[ \sigma_2 \left( U_+ U_- \right)^{-1} \zeta_\mu \left( \pi + 2i \eta \right) U_+ U_- \sigma_2 \right. \]

\[ \left. + U_\pm U_\mp^{-1} \zeta_\mu \left( \pi \mp 2i \nu \right) U_\pm U_\mp^{-1} + 2i U_\pm \partial_\mu U_\pm^{-1} \right], \]

(2.22)

where \( \eta \) and \( \nu \) stand for

\[ \eta = \frac{\eta_+ + \eta_-}{2}, \quad \nu = \frac{\eta_+ - \eta_-}{2}. \]

(2.23)

If we proceed further and impose an extra condition

\[ F(0) = 0, \]

(2.24)

then \( \text{Tr}[F^2(\lambda)] \) becomes an entire, doubly periodic function of \( \lambda \), vanishing at \( \lambda = 0 \). Hence it must be identically zero. Combining this fact with the hermiticity we find that the field strength vanishes for any pure imaginary \( \lambda \): \( F(\lambda) = 0, \Re(e(\lambda)) = 0 \). Of course, this implies that the conditions (2.21) and (2.24) guarantee the flatness of the worldsheet connection.

Let us take a closer look at the condition (2.24). Because of the \( \lambda \)-parity relation (2.16), the connection reduces at \( \lambda = 0 \) to the form

\[ D_\pm(0) = \partial_\pm \mp i I_\pm \sigma_2. \]

(2.25)

Therefore, the flatness condition implies a continuity equation \( \partial_+ I_- + \partial_- I_+ = 0 \). For the target space background with the two Killing vector fields \( \frac{\partial}{\partial v} \) and \( \frac{\partial}{\partial w} \) the NLSM possesses two Noether currents, \( V_A \) and \( W_A \). (Here we label the worldsheet components by the subscript \( A = \pm \).) Thus, we may conclude that the flatness condition at \( \lambda = 0 \) (2.24) is equivalent to the relation

\[ I_A = c_v V_A + c_w W_A, \]

(2.26)

where \( c_v \) and \( c_w \) are some real constants.

The conditions (2.22) and (2.24) can be treated as a set of equations for the determination of the non-vanishing vielbein components, the torsion strength \( H \) and the unknown phases \( \phi_\pm \). It is easy to see without actually doing any computation that the solution, if it exists, is not unique. Indeed, under the diffeomorphism \( X_\mu \to \tilde{X}_\mu \) the vielbein transforms as \( e_\mu = \tilde{e}_\nu \frac{\partial \tilde{X}_\nu}{\partial X_\mu} \), therefore the \( SL(2, R) \) coordinate transformations with a constant Jacobian matrix

\[ \frac{\partial \tilde{X}_\mu}{\partial X_\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & S_2^3 & S_3^3 \\ 0 & S_3^3 & S_2^3 \end{pmatrix}, \quad S_2^3 S_3^3 - S_2^3 S_3^3 = 1, \]

(2.27)

preserve the torsion strength and the general form of the metric (2.3). These transformations can be applied to generate a three parameter family from any given solution of Eqs. (2.22) and (2.24). It is also clear that the equations impose restrictions on the phase difference \( \phi_+ - \phi_- \) only, i.e., one out of two phases can be chosen at will. In Appendix we found the most general
solution of Eqs. (2.22). It turns out that using the \( SL(2,R) \) transformations (2.27) and the orientation-preserving transformation \((u,v,w) \rightarrow (-u,w,v)\), the solution can be brought to the form

\[
e^3_u = g^{-1} l \, \varepsilon(u)
\]

\[
e^\pm_v = g^{-1} e^{\pm i(\phi_+ - \frac{\pi}{2})} \rho(\pm u) \frac{\partial_4(i\eta \pm u, q^2)}{\partial_4(i\eta, q^2)}
\]  

\[
e^\pm_w = -g^{-1} e^{\pm i(\phi_+ - \frac{\pi}{2})} \rho(\pm u) \frac{\partial_1(i\eta \pm u, q^2)}{\partial_1(i\eta, q^2)}
\]  

and

\[
H = \frac{g}{l} \left[ i \frac{\partial_2(iv, q)}{\partial_2(iv, q)} - i \partial_u \log \left( \frac{\rho(u)}{\rho(-u)} \right) \right]
\]

\[
e^{i(\phi_+ - \phi_-)} = \frac{\rho(u)}{\rho(-u)}.
\]  

Here we use the notation

\[
\rho(u) = \frac{\partial_3(\frac{i\sigma - iv}{2}, q) \partial_4(\frac{i\sigma - iv}{2}, q)}{\partial_3(\frac{u + i\sigma - iv}{2}, q) \partial_4(\frac{u + i\sigma - iv}{2}, q)},
\]  

and \( \varepsilon(u) \) is a periodic step function

\[
\varepsilon(u) = \begin{cases} 
+1, & u/\pi \in (2n, 2n + 1) \\
-1, & u/\pi \in (2n + 1, 2n + 2) 
\end{cases} \quad (n \in \mathbb{Z}).
\]  

The above formulas call for a number of comments. The constant \( g > 0 \) (the string tension) just sets the overall normalization for the NLSM action (1.4), which does not affect the Euler-Lagrange equations. It can be set to be one without loss of any generality. We however reserve this parameter and make use of it later for the purpose of quantization. The parameter \( l \) in fact just replaces \( \sqrt{G}_{uv} \). It can be absorbed into \( g \) and the overall normalization of the Killing coordinates \( v \) and \( w \). It is convenient to choose it as

\[
l = i \frac{\partial_2(i\eta, q) \partial_3'(0, q)}{\partial_1(i\eta, q) \partial_2'(0, q)}.
\]  

Then, as it follows from (2.28)

\[
\sqrt{G} = \frac{l^2}{g^3} \rho(u)\rho(-u) \frac{\partial_1(u, q)}{\partial_1'(0, q)} \varepsilon(u).
\]  

Since under the complex conjugation, \( \rho^*(u) = \rho(-u) \), the periodic step function \( \varepsilon(u) \) provides the positivity of \( \sqrt{G} = \det(e^\mu_\nu) \) for an arbitrary real \( u \) except \( u = 0, \pm \pi, \pm 2\pi \ldots \). Thus nontrivial parameters of the solution are \( q, \eta, \nu \), which have already appeared in the ansatz for the worldsheet connection, and \( \sigma \) from the definition (2.30).

It turns out that the condition (2.24) does not impose any restriction on the above solution. More specifically, through proper choice of the constants \( c_v \) and \( c_w \), Eq. (2.26) is identically satisfied.
3 The integrable target space background

In this section we would like to discuss the integrable target space background, i.e., the metric and the torsion potential in the NLSM action (1.4). We start with the $q \to 0$ limit, assuming all other parameters are held constant. It is easy to see that

$$G_{\mu \nu} dX^\mu dX^\nu = \frac{1}{g^2} \left[ C^2 (du)^2 + (dv)^2 + (1 + (C^2 - 1) \sin^2(u)) (dw)^2 - 2 \cos(u) dv dw \right]$$

$$\frac{1}{2!} B_{\mu \nu} dX^\mu \wedge dX^\nu = \frac{C \tanh(\nu)}{g^2} \cos(u) \ dv \wedge \ dw \quad (q = 0),$$

(3.1)

where $C = \coth(\eta)$. Note that the parameter $\sigma$ does not appear in this limiting case. We may now send $\eta \to +\infty$:

$$G_{\mu \nu} dX^\mu dX^\nu = \frac{1}{g^2} \left[ (du)^2 + (dv)^2 + (dw)^2 - 2 \cos(u) dv dw \right]$$

$$\frac{1}{2!} B_{\mu \nu} dX^\mu \wedge dX^\nu = \frac{\tanh(\nu)}{g^2} \cos(u) \ dv \wedge \ dw \quad (q = 0, \ \eta \to +\infty).$$

(3.2)

Let $\vec{n}$ be a unit vector in $\mathbb{R}^4$ whose components are defined by the relations

$$n^1 \pm i n^2 = e^{\pm i(u_w - w/u)} \sin \left(\frac{\theta}{2}\right), \quad n^3 \pm i n^4 = e^{\pm i(u_w + w/u)} \cos \left(\frac{\theta}{2}\right),$$

(3.3)

then the metric (3.2) takes the form $\frac{1}{g^2} d\vec{n} \cdot d\vec{n}$, i.e. it coincides with the round metric on the three-sphere of radius $\frac{3}{2}$. The variables $\theta = \frac{u}{2}$, $\chi_1 = \frac{v - w}{2}$ and $\chi_2 = \frac{v + w}{2}$ are usually referred as to the Hopf coordinates and used in the description of the three-sphere as the Hopf bundle. For any value of $\theta$ between 0 and $\frac{\pi}{2}$, the pair $(\chi_1, \chi_2)$ parameterizes a two-dimensional torus $(\chi_a \sim \chi_a + 2\pi, \ a = 1, 2)$. The metric (3.2) degenerates (i.e. $\sqrt{G} = 0$) at two sub-manifolds of codimension two (circles) which correspond to $u = 0$ and $u = \pi$. However these are coordinate singularities that may be removed by introducing suitable coordinates. The same happens for the one-parameter family of metrics (3.1) which is sometimes referred as to a metric on a squashed three-sphere.

In order to give an explicit description of the general target space background it is convenient to replace the coordinate $u$ with another variable. As it follows from Eq.(2.33) the metric degenerates at $u = 0, \pm \pi, \pm 2\pi \ldots$. Suppose the coordinate $u$ runs over the segment

$$0 < u < \pi,$$

(3.4)

then the doubly periodic function $z = z(u, q)$,

$$z(u, q) = \frac{\varphi_3(u, q^2)\varphi_3(0, q^2)}{\varphi_3(u, q^2)\varphi_2(0, q^2)},$$

(3.5)

Note the slightly non-standard choice for the coupling $g$.

The NLSM with the target space background (3.1) is called the anisotropic $SU(2)$ Wess-Zumino-Witten-Novikov model. The ZCR for this model is known for a while (see [3], [23], Chapter II.1.5 in the monograph [22] and references therein).
varies monotonically along the segment $(-1, 1)$ and therefore $u$ can be replaced by $z$. Introduce
\[ p = -i \frac{\partial_1(i\eta, q^2)}{\partial_4(i\eta, q^2)}, \quad h = -i \sqrt{\kappa} \frac{\partial_1(i\nu - i\sigma, q^2)}{\partial_4(i\nu - i\sigma, q^2)}, \quad \bar{h} = -i \sqrt{\kappa} \frac{\partial_1(i\nu + i\sigma, q^2)}{\partial_4(i\nu + i\sigma, q^2)} \] (3.6)
and
\[ \sqrt{\kappa} = \frac{\partial_2(0, q^2)}{\partial_3(0, q^2)}, \] (3.7)
which can be viewed as a new set of parameters replacing $(\eta, \nu, \sigma, q)$. To make formulas more readable, we will also use the following combinations of the new parameters:
\[ c = +\sqrt{\frac{1 + h^2}{\kappa^2 + h^2}}, \quad \bar{c} = +\sqrt{\frac{1 + \bar{h}^2}{\kappa^2 + \bar{h}^2}} \] (3.8)
\[ m = +\sqrt{1 + \kappa^2 + \kappa p^2 + \kappa p^{-2}}. \]

With the new coordinate frame $(z, v, w)$ and the new set of parameters $(p, h, \bar{h}, \kappa)$, the metric defined by Eqs.(2.4), (2.7), (2.28), (2.32) can be brought to the form:
\[ G_{\mu\nu} \, dX^\mu \, dX^\nu = \frac{m^2}{g^2} \left( \frac{(dz)^2}{(1 - z^2)(1 - \kappa^2 z^2)} + \frac{(c + 1)(\bar{c} - 1)}{(1 - \kappa^2)(c + z)(\bar{c} - z)} \times \right. \]
\[ \left. \left[ (1 + \kappa p^2 - z^2 \kappa (\kappa + p^2)) \right. \right. \]
\[ \left. \left. (dv)^2 + (1 + \kappa p^{-2} - z^2 \kappa (\kappa + p^{-2})) \right) (dw)^2 \right. \]
\[ -2 (1 - \kappa^2) z \, dv \, dw \right], \] (3.9)

whereas the torsion potential $B$ ($\frac{1}{2} B_{\mu\nu} \, dX^\mu \wedge dX^\nu = B \, dv \wedge dw$) and torsion strength $H$ ($\partial_\kappa B = \sqrt{G} \, H$) are given by
\[ B = -\frac{m}{g^2} \frac{(c + 1)(\bar{c} - 1)}{(1 - \kappa^2)(c + z)} (1 - z) \left[ h \ \frac{c - 1}{c + z} + \bar{h} \ \frac{\bar{c} + 1}{\bar{c} - z} \right] \] (3.10)
\[ H = \frac{g}{m} \frac{h \ (c^2 - 1)(\bar{c} - z)^2 + \bar{h} \ (\bar{c}^2 - 1)(c + z)^2}{(c + z)(\bar{c} - z)}. \]

A few comments are in order here. Although the parameters $h$ and $\bar{h}$ appear in the metric through the combinations $c$ and $\bar{c}$ only, there are two reasons to choose $(h, \bar{h})$ as independent parameters. First, $h$ and $\bar{h}$ are fully unrestricted real numbers, i.e.,
\[ -\infty < h, \ \bar{h} < +\infty, \] (3.11)
whereas $1 < c \leq \kappa^{-1}$ and $1 < c \leq \kappa^{-1}$. Second, Eqs.(3.8) allow one to express $(h, \bar{h})$ through $(c, \bar{c})$ modulo sign factors requiring special care since the torsion potential substantially depends on the relative sign of $h$ and $\bar{h}$. Note the neither metric nor the torsion potential depends on a sign of $p$. Moreover the form of the target space background are invariant with respect
to the interchange $v \leftrightarrow w$ accompanied by the transformation of the parameters $(p, h, \bar{h}) \rightarrow (p^{-1}, -h, -\bar{h})$. Therefore, the actual parameter is $P^2$,

$$P = \frac{1}{2} (p - p^{-1}) ,$$  

(3.12)

rather than $p$ itself. We keep the notation $p$ to make the equations easier to visualize. Finally, the parameter $\kappa$ in (3.7) is no other but the elliptic modulus associated with the elliptic nome $q^2$, therefore

$$0 \leq \kappa < 1 .$$  

(3.13)

It is instructive to look at the metric (3.9) at the degeneration points $z = z_a$ ($a = 1, 2$) which correspond to $u = 0, \pi$. We introduce two different local coordinate frames $(\rho_1, \psi_1, \chi_1)$ and $(\rho_2, \psi_2, \chi_2)$ in the vicinity of $z_1 = +1$ and $z_2 = -1$, respectively. Let $\rho_a^2 = 2|z - z_a| + O(|z - z_a|^2)$,

$$\psi_1 = \frac{1}{2} ( (1 + \Delta) v + (1 - \Delta) w ) , \quad \psi_2 = \frac{1}{2R} ( (1 + \Delta) v - (1 - \Delta) w )$$  

(3.14)

and

$$\chi_1 = \frac{1}{2R} (v - w) , \quad \chi_2 = \frac{1}{2} (v + w) ,$$  

(3.15)

where $\Delta = \frac{\kappa}{m^2} (p^2 - p^{-2})$,

$$R = \sqrt{\frac{(c - 1)(\bar{c} + 1)}{(c + 1)(\bar{c} - 1)}} .$$  

(3.16)

Evaluating the metric relative to the local coordinate systems $(\rho_1, \psi_1, \chi_1)$ and $(\rho_2, \psi_2, \chi_2)$, one finds

$$G_{\mu\nu} \, dX^\mu dX^\nu = C_a^{(1)} \left( (d\rho_a)^2 + \rho_a^2 (d\psi_a)^2 + O(\rho_a^4) \right) + C_a^{(2)} (d\chi_a)^2 \left( 1 + O(\rho_a^4) \right) ,$$  

(3.17)

where $\rho_a \rightarrow 0$ ($a = 1, 2$) and $C_a^{(1,2)}$ stand for some positive constants. This general form implies that to avoid the formation of the conical singularities at $\rho_1 = 0$ and $\rho_2 = 0$, the local coordinates $\psi_1$ and $\psi_2$ have to be the angular type variables such that $\psi_a \sim \psi_a + 2\pi$.

In fact, we did not make any assumptions about global properties of the Killing coordinates $v$ and $w$ in the derivation of the ZCR. Therefore, it remains valid for any compactification of these variables. In what follows we will assume that

$$\chi_1 \sim \chi_1 + 2\pi , \quad \chi_2 \sim \chi_2 + 2\pi ,$$  

(3.18)

where $\chi_a$ are given by Eq. (3.15). In this case the “global” chart

$$\left( z, \chi_1, \chi_2 \mid -1 < z < 1, 0 \leq \chi_a < 2\pi \right)$$  

(3.19)
covers the whole target space $\mathcal{M}_3$ except two sub-manifolds of codimension two. The sub-manifolds are circles parameterized by the angular variables $\chi_1$ as $z = 1$ and $\chi_2$ as $z = -1$. Let us consider the neighborhoods of the circle at $z = 1$. We need at least two local charts with $\chi_1 \in (a, b)$ and $0 < b - a < 2\pi$ to cover the circle completely. As it follows from Eqs. (3.14), (3.15)

$$
\psi_1 = \chi_2 + R \Delta \chi_1 , \quad \psi_2 = \chi_1 + R^{-1} \Delta \chi_2 ,
$$

and hence, at a local chart with the decompactified coordinate $\chi_1$, the variable $\psi_1$ is of the angular type provided the compactification condition (3.18) is imposed. A similar analysis can be applied to the neighborhoods of the circle at $z = -1$.

To summarize, the formula (3.9) supplemented by the global conditions (3.15), (3.18), defines a nonsingular metric on a topological three-sphere $\mathcal{M}_3$.

An important integral characteristic of the target space background is a $H$-flux, i.e., a total flux of the closed three-form $H = H_{\mu\nu\sigma} \, dX^\mu \wedge dX^\nu \wedge dX^\sigma$ through the target manifold. Below we will use

$$
N = \frac{1}{16\pi^2} \int_{\mathcal{M}_3} H .
$$

(3.21)

With Eq. (3.10) and the compactification conditions (3.18) one finds

$$
N = \frac{m}{g^2} \frac{\sqrt{(c^2 - 1)(c^2 - 1)}}{(1 - \kappa^2)(c + \bar{c})} (h + \bar{h}), \quad -\infty < N < \infty .
$$

(3.22)

The target space background without torsion deserves a special mention. In this case $h = \bar{h} = 0$, the metric (3.9) reduces to the one from Ref. [7]. V.A. Fateev used the coordinates $z$ and $(\chi_1, \chi_2)$ related to $(u, v)$ through the formula (3.15) with $R = 1$ and compactified as in (3.18)\(^4\) The ZCR for the Fateev model was not known before. It is merely a specialization of the general case for $\nu = \sigma = 0$.

The three-dimensional target space background (3.9) can be used to build an integrable NLSM with $D = 2$. Let us set $h = \bar{h} = 0$, $g^2 = \kappa \tilde{g}^2 \, p^2$ and consider the limit $p \to \infty$ with $(\tilde{g}, \kappa)$ held constant. This formal procedure yields

$$
G^{(2)}_{\mu\nu} \, dX^\mu \wedge dX^\nu = \frac{1}{\tilde{g}^2} \left[ \frac{(dz)^2}{(1 - z^2)(1 - \kappa^2 z^2)} + \frac{(1 - z^2)(dv)^2}{1 - \kappa^2 z^2} \right] .
$$

(3.23)

The coordinate $w$ does not appear at this limit, consequently Eq. (3.23) can be interpreted as a metric for some NLSM with two-dimensional target space. In fact, this metric is equivalent

\(^4\) The five parameters $(u, a, b, c, d)$ from Ref. [7], subject to the constraints $(u + d)^2 = a^2 + c^2$, $d^2 = b^2 + c^2$, are related to the set $(g, p, \kappa)$ through the formulas

$$
\frac{a}{u} = \frac{1}{m} , \quad \frac{b}{u} = \frac{\kappa}{m} , \quad \frac{c}{u} = \frac{\kappa}{2m^2} (p^2 - p^{-2}) , \quad \frac{d}{u} = -\frac{\kappa}{2m^2} (2\kappa + p^2 + p^{-2}) , \quad u = \frac{g^2}{4} ,
$$

where $m = +\sqrt{1 + \kappa^2 + \kappa p^2 + \kappa p^{-2}}$. 

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to the sausage metric from Ref. [6], provided \( v \sim \nu + 2\pi \). One can check that the equations of motion for this NLSM admit the ZCR which follow from the general ZCR as \( \eta \to -i\pi \tau \) (see Eq. (3.6)). Taking the limit, Eqs. (2.28)-(2.32) with \( 0 < \nu < \pi \) and \( \nu = \sigma = 0 \) yield

\[
e_v = g^{-1} \partial_3^2(0, q^2), \quad e_v^\pm = g^{-1} e^{\pm i(\phi_+ - \nu)} \frac{\partial_1(u, q^2)\partial_3(0, q^2)}{\partial_1(u, q^2)\partial_2(0, q^2)}, \quad e_w^\pm = 0, \quad (3.24)
\]

whereas \( \phi_+ = \phi_- \). Then Eqs. (2.10)-(2.14) with \( \eta = -i\pi \tau \) can be applied literally.

4 Ricci flow with torsion

4.1 One-loop renormalization

We turn now to a discussion of the renormalization effects in the NLSM under consideration. The RG flow equations for a general target space background were computed up to two loops in Refs. [8–14]. At leading order, the equations can be written in somewhat symbolic form (1.7) [12]. For practical purposes, it is useful to rewrite them in terms of the symmetric Ricci tensor \( R_{\mu\nu} \) associated with the Levi-Civita connection \( \nabla_\mu \):

\[
\dot{G}_{\mu\nu} = - \left( R_{\mu\nu} - \frac{1}{4} H^\rho_{\mu \sigma \rho \nu} H_{\sigma \rho \nu} + \nabla_\mu V_\nu + \nabla_\nu V_\mu \right)
\]

\[
\dot{B}_{\mu\nu} = - \left( \frac{1}{2} \nabla_\sigma H^\sigma_{\mu \nu} - V_\sigma H^\sigma_{\mu \nu} \right). \quad (4.1)
\]

Here the dot stands for the (partial) derivative with respect to the RG “time” which is proportional to the logarithm of the RG energy scale \( E \):

\[
t = - \frac{1}{2\pi} \log \left( \frac{E}{E_*} \right), \quad (4.2)
\]

where \( E_* \) (the integration constant of the RG flow equations) sets the “physical scale” for the NLSM. Some clarification is needed for the terms depending on an arbitrary one-form \( V_\mu \). The general form of an infinitesimal RG transformation should admit the possibility of various coordinate transformations [10]. Under an arbitrary infinitesimal reparameterization \( \delta G_{\mu\nu} = -(\nabla_\mu V_\nu + \nabla_\nu V_\mu) \delta t \), whereas \( \delta B_{\mu\nu} = V_\sigma H^\sigma_{\mu \nu} \delta t + \delta \tilde{B}_{\mu\nu} \) with \( \delta \tilde{B}_{\mu\nu} = -(\partial_\mu V_\nu - \partial_\nu V_\mu) \delta t \).

The variation \( \delta \tilde{B}_{\mu\nu} \) is a pure gauge transformation which does not affect the torsion strength and therefore can be neglected. Thus the terms with \( V_\mu \) incorporate the effects of all possible diffeomorphisms and can be chosen arbitrarily (to a certain extent, see subsection 4.4 below) in order to simplify the equations. In what follows we assume that there exists a diffeomorphism generating function \( \Psi \) such that

\[
V_\mu = \partial_\mu \Psi. \quad (4.3)
\]

To establish the one-loop renormalizability of the finite-parameter family of NLSM, it is sufficient to demonstrate that, for some choice of the diffeomorphism generating function, the RG flow equations can be satisfied by allowing the parameters \( (p, h, \bar{h}, \kappa) \) and the string tension \( g \) to be \( t \)-dependent. With a brief look at (4.1) we conclude that \( \Psi \) is transformed as a scalar.
under $t$-independent coordinate transformations only. Therefore it essentially depends on a choice of the coordinate system or, widely speaking, on the RG scheme. It turns out that the $(u,v,w)$-coordinate frame is very useful for adjusting the diffeomorphism generating function. Using these coordinates one can show that Eqs.\((4.1), (4.3)\) are indeed satisfied, provided

$$e^{2\Psi} = e^{2\Psi_0} \rho(u)\rho(-u),$$  \hspace{1cm} (4.4)$$

where $\rho(u)$ is given by \((2.30)\) and $\Psi_0$ is an arbitrary coordinate-independent constant. More precisely, with this choice of $\Psi$ and for $G_{\mu\nu}, B_{\mu\nu}$ defined by Eqs.\((2.28)-(2.33)\), the one-loop RG flow equations are reduced to the following closed system of ODE:

$$\dot{\kappa} = -\frac{g^2}{m^2} \kappa (1 - \kappa^2)$$

$$\dot{g} = \frac{g^3}{4m^4} (1 - \kappa^2)^2 \left( 1 - N^2 g^4 \right)$$

$$\dot{p} = 0$$

$$\dot{c} = \frac{g^2}{m^2} \frac{(c^2 - 1)(\kappa^2 c\bar{c} + 1)}{(c + \bar{c})}$$

$$\dot{\bar{c}} = \frac{g^2}{m^2} \frac{(\bar{c}^2 - 1)(\kappa^2 c\bar{c} + 1)}{(c + \bar{c})},$$  \hspace{1cm} (4.5)$$

where $(m, c, \bar{c}, N)$ are expressed in terms of the independent set $(p, h, \bar{h}, \kappa)$ as in Eqs.\((3.8), (3.22)\).

The solution of the system (4.5) is a rather straightforward exercise. First of all, it is evident that $P$ (3.12) and $R$ (3.16) are the first integrals. Then we should recall that the path-integral quantization procedure requires that the $H$-flux, $\frac{1}{\pi} \int_{M_3} H$, must be an integer. Thus, $N$ given by Eq.\((3.22)\) must be the first integral as well as $P$ and $R$. This, of course, can be easily tested. Note that the flux essentially depends on a choice of compactification of the Killing coordinates, and therefore the RG invariance of (3.22) provides an additional support for the assumptions (3.18). A further analysis of the first two equations in (4.5) yields one more first integral which can be chosen in the form

$$M^2 = \frac{m^2}{4\kappa g^4} - \frac{(1 + \kappa)^2}{4\kappa} N^2.$$  \hspace{1cm} (4.6)$$

To summarize, for $N \neq 0$ the system of ODE (4.5) possesses the following complete set of the first integrals:

$$N \neq 0 : \hspace{0.5cm} (P, R, M, N).$$  \hspace{1cm} (4.7)$$

The condition $N = 0$ implies $R^2 = 1$. In this case the complete set of the first integrals can be chosen as follows

$$N = 0 : \hspace{0.5cm} (P, M, L) \hspace{0.5cm} \text{with} \hspace{0.5cm} L = \frac{h}{\sqrt{\kappa}} = -\frac{\bar{h}}{\sqrt{\kappa}}.$$  \hspace{1cm} (4.8)$$
The first integral $L^2$ can be also naturally introduced for nonvanishing $N$. Indeed, as it follows from the first and the last two equations in (4.5)

$$c \overline{c} = \frac{1 + L^2 \kappa}{\kappa (L^2 + \kappa)} , \quad c - \overline{c} = \frac{R^2 - 1}{R^2 + 1} \frac{1 - \kappa^2}{\kappa (L^2 + \kappa)} ,$$

(4.9)

where $L^2$ stands for the RG invariant which can be alternatively defined by Eq.(4.8) in the case $N = 0$. The first integrals $L^2$, $R$ and the ratio $\frac{N}{M}$ are functionally dependent (see Eqs.(4.19), (4.20) below).

The existence of a complete set of first integrals makes it possible to integrate the ODE system (4.5) explicitly. One just needs to substitute $\kappa$, which is an elliptic modulus associated with the elliptic nome $q^2$, for an elliptic modulus $k$ related to the elliptic nome $q$, or equivalently, to perform Landen’s transformation $\kappa \to k = \frac{2 \sqrt{\kappa}}{1 + \kappa}$. Then a simple calculation yields the result

$$e^t = \left| \frac{g^2 - g_0^2}{g^2 + g_0^2} \right|^{\frac{1}{2}} \left| \frac{1 + N g^2}{1 - N g^2} \right|^N \left| \frac{M g^2 + P}{M g^2 - P} \right|^{MP} ,$$

(4.10)

and

$$\kappa = \frac{1 - \sqrt{1 - k^2}}{1 + \sqrt{1 - k^2}} , \quad k^2 = \frac{(1 + N g^2)(1 - N g^2)}{(M g^2 + P)(M g^2 - P)} ,$$

(4.11)

where

$$b^2 = + \frac{1}{\sqrt{(N^2 + M^2)(1 + P^2)}} , \quad g_0 = + \sqrt{\frac{1 + P^2}{N^2 + M^2}} .$$

(4.12)

To evaluate the running coupling constant $g = g(t)$ as a function of the RG scale $E$ (4.2) requires an inversion of the relation (4.10). Note that, as follows from (4.11), $|\frac{M g^2 + P}{M g^2 - P}| \neq 0, \infty$ as $0 \leq \kappa \leq 1$. By following the logical structure of quantum field theory, the renormalized parameters of the target space background should be expressed in terms of the running coupling constant and the RG invariants. Eqs.(4.11) allow one to do this for $\kappa$. The relations (4.9) define $c, \overline{c} > 1$ unambiguously through the solution of a quadratic equation. Then, using Eqs.(3.8) one can determine $(h^2, \overline{h}^2)$. The signs of $h$ and $\overline{h}$ can recovered from (3.22). Note that $\text{sgn}(h + \overline{h}) = \text{sgn}(N)$. Finally, the parameter $p$ is a RG invariant itself.

Finishing with the solution of Eq.(4.1), let us note that the constant $\Psi_0$ in Eq.(4.4) does not contribute to the RG flow equations. However, if we set it to be

$$\exp \left(2 \Psi_0(t)\right) = \sqrt{\frac{g \kappa}{(1 - N^2 g^4)(1 - \kappa^2)}} ,$$

(4.13)

then the following relation is satisfied

$$\partial_t \left(2 \Psi - \log \sqrt{G}\right) = -\frac{1}{4} \left( -G^{\mu\nu} R_{\mu\nu} + \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + 4 \nabla_\mu \Psi \nabla^\mu \Psi - 4 \nabla_\mu \nabla^\mu \Psi \right) ,$$

(4.14)
and the constructed set of fields \( (G_{\mu\nu}, B_{\mu\nu}, \Psi) \) provides a solution to the coupled system of PDE (4.1), (4.3), (4.14). Note that, in practical calculations, it is usually desirable to rewrite (4.4) in terms of the doubly periodic function \( z = z(u, q) \) defined in (3.5):

\[
\exp \left( 2\Psi(u, t) \right) = \exp \left( 2\Psi_0(t) \right) \frac{\partial^2_{t} \left( 0, q^2 \right)}{\partial^2_{t} \left( u, q^2 \right)} \frac{(c + 1)(\bar{c} - 1)(1 - \kappa^2 z^2)}{(c + z)(\bar{c} - z)(1 - \kappa^2)}.
\]

(4.15)

Here the elliptic parameter \( q \) should be treated as a function of the RG time and, as it follows from the first equation in (4.5),

\[
\frac{\dot{q}}{q} = - \frac{g^2}{\partial^2_{t} \left( 0, q^2 \right)} m^2 = - \frac{1}{G_{uu}}.
\]

(4.16)

4.2 Ultraviolet behavior

As \( \kappa = 1 \) the ODE system (4.5) possesses constant solutions \( (p, h, \bar{h}, g) = (p_0, h_0, \bar{h}_0, g_0) \) which can be specified by the set of numbers

\[
\left( \alpha, \beta, \delta, b \mid -\frac{\pi}{2} < \alpha, \beta < \frac{\pi}{2}, \quad 0 < \delta < \pi, \quad b > 0 \right)
\]

through the relations

\[
h_0 = \tan(\alpha), \quad \bar{h}_0 = \tan(\beta), \quad p_0 = \cot \left( \frac{\delta}{2} \right), \quad g_0^2 = \frac{b^2}{\sin^2(\delta)}.
\]

(4.17)

Let \( S_{\delta | b}^{(\alpha, \beta)} \) be the RG trajectory which asymptotically approaches the constant solution characterized by a given set (4.17) as \( t \to -\infty \). The values of the functionally independent RG invariants for \( S_{\delta | b}^{(\alpha, \beta)} \) are determined through the formulas

\[
P = \cot(\delta), \quad R = \frac{\cos(\alpha)}{\cos(\beta)}, \quad N = \frac{1}{b^2} \sin(\alpha + \beta) \sin(\delta), \quad M = \frac{1}{b^2} \cos(\alpha + \beta) \sin(\delta),
\]

(4.18)

whereas the RG invariant \( L^2 \) from Eq. (4.9) is given by

\[
L^2 = \frac{\sin^2(\alpha) + \sin^2(\beta)}{\cos^2(\alpha) + \cos^2(\beta)}.
\]

(4.19)

(4.20)

It is interesting to look at the asymptotic form of the target space background corresponding to the RG trajectory \( S_{\delta | b}^{(\alpha, \beta)} \) at large negative \( t \). For this purpose, let us cut the chart defined by (3.19) into the three pieces \( U^{(0)}, U^{(1)} \) and \( U^{(2)} \) depending on the value of the coordinate \( z: -1 + \epsilon \leq z \leq 1 - \epsilon, \quad 0 < 1 - z \leq \epsilon \) and \( 0 < 1 + z \leq \epsilon \), respectively. Here \( \epsilon \) stands for some small number which is, in the case \( 1 - \kappa \ll 1 \), can be chosen to satisfy both conditions \( \epsilon \ll 1 \) and \( \epsilon \gg 1 - \kappa \) simultaneously. Then on the chart \( U^{(0)} \) covering the central region of \( M_3 \), we replace \( z \) by \( \rho \):

\[
\rho = \frac{2K}{\pi} \left( u - \frac{\pi}{2} \right),
\]

(4.21)
where $u$ stands for our original variable \((3.5)\) and $K$ is the elliptic quarter-period associated with the nome $q^2$, i.e., $K = \frac{\pi}{2} \vartheta_3'(0, q^2) \approx \frac{1}{2} \log(\frac{\alpha}{\gamma})$. It is straightforward to see that the metric on the chart $U^{(0)}$ is approximated by the form

\[
G^{(UV)}_{\mu\nu} dX^\mu dX^\nu|_{U^{(0)}} \approx \frac{4}{b^2} \left[ (d\rho)^2 + (d\chi_1^{(\alpha)})^2 + (d\chi_2^{(\beta)})^2 + 2 \cos(\delta) d\chi_1^{(\alpha)} d\chi_2^{(\beta)} \right],
\]

whereas the torsion strength $H \approx 0$. Here we use $\chi_1^{(\alpha)} = \cos(\alpha) \chi_1$, $\chi_2^{(\beta)} = \cos(\beta) \chi_2$, and as it follows from Eqs.\((3.18)\),

\[
\chi_1^{(\alpha)} \sim \chi_1^{(\alpha)} + 2\pi \cos(\alpha) , \quad \chi_2^{(\beta)} \sim \chi_2^{(\beta)} + 2\pi \cos(\beta).
\]

Thus, in the central region, the metric is almost flat and the target manifold $\mathcal{M}_3$ is well approximated by the Cartesian product of the two-torus and the line segment of total length $\ell \approx 2b^{-1} \log(\frac{1}{2})$.

Similarly to \((4.21)\), at the charts $U^{(1)}$ and $U^{(2)}$ we replace the coordinate $z$ by

\[
\rho_1 = \frac{2K}{\pi} u , \quad \rho_2 = \frac{2K}{\pi} (u - \pi) ,
\]

respectively. The target space background in the region covered by the chart $U^{(1)}$ is approximated as follows:

\[
G^{(UV)}_{\mu\nu} dX^\mu dX^\nu|_{U^{(1)}} \approx \frac{4}{b^2} \left[ (d\rho_1)^2 + \frac{\cos^2(\beta) \sinh^2(\rho_1)}{\cosh(\rho_1 - i\beta) \cosh(\rho_1 + i\beta)} (d\psi_1)^2 + \frac{\cosh^2(\rho_1)}{\cosh(\rho_1 - i\beta) \cosh(\rho_1 + i\beta)} (d\chi_1^{(\alpha,\delta)})^2 \right]
\]

\[
\frac{1}{2} B^{(UV)}_{\mu\nu} dX^\mu \wedge dX^\nu|_{U^{(1)}} \approx -\frac{4}{b^2} \frac{\sin(\beta) \sinh^2(\rho_1)}{\cosh(\rho_1 - i\beta) \cosh(\rho_1 + i\beta)} d\chi_1^{(\alpha,\delta)} \wedge d\psi_1 .
\]

Here we use the notations

\[
\psi_1 = \chi_2 + \frac{\cos(\alpha) \cos(\delta)}{\cos(\beta)} \chi_1, \quad \chi_1^{(\alpha,\delta)} = \sin(\delta) \cos(\alpha) \chi_1 .
\]

There is no need to present similar formulas for the region covered by the chart $U^{(2)}$. They are obtained by substituting $\chi_1 \leftrightarrow \chi_2$, $\psi_1 \leftrightarrow \psi_2$ and $\alpha \leftrightarrow \beta$ in the above expressions. Since $\rho = \rho_1 - K$, the metrics \((4.22)\) and \((4.25)\) are smoothly sewed together as $(-\rho) \sim \frac{1}{2} \log(\frac{1}{2}) \gg 1$ and $\rho_1 \sim \frac{1}{2} \log(\frac{\alpha}{\gamma}) \gg 1$. In this domain the torsion strength corresponding to the torsion potential \((4.26)\) becomes of the order $\frac{1}{\epsilon} \ll 1$.

Of course, the metric in the r.h.s. of \((4.22)\) combined with $H = 0$, provides a stationary solution of the RG flow equations. The background \((4.25)\) can be made into an RG fixed point in the precise sense of the world by an appropriate definition of the RG transformation or, in stringy speak, to introducing the dilaton field

\[
e^{2\Phi} = \frac{\cos^2(\beta)}{\cosh(\rho_1 - i\beta) \cosh(\rho_1 + i\beta)}.\]
Then, the limiting form of \((G_{\mu\nu}^{UV}, B_{\mu\nu}^{(UV)})_{U(1)}\) as \(t \to -\infty\), together with \(\Phi\) satisfy the so-called string equations \([13, 24]\) (the conditions for Weyl invariance to hold in the NLSM in the lowest nontrivial approximation)\(^5\):

\[
R_{\mu\nu} - \frac{1}{4} H_\sigma^{\sigma\rho} H_{\rho\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi = 0
\]

\[
\frac{1}{2} \nabla_\sigma H^{\sigma\mu\nu} - \nabla_\sigma \Phi H^{\sigma\mu\nu} = 0
\]

\[
-G^{\mu\nu} R_{\mu\nu} + \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + 4\nabla_\mu \Phi \nabla_\mu \Phi - 4\nabla_\mu \nabla_\mu \Phi = \text{const}
\]

Note that the dilaton \((4.27)\) does not vanish as \(\rho_1 \gg 1\) but approaches to \(\text{const} - \rho_1\). Therefore, the constant in the r.h.s. of the last equation in \((4.28)\) is equal to \(b^2\). It occurs because the region of \(M_3\) in the vicinity \(|z - 1| = \epsilon\) remains fixed with respect to the coordinate frame at \(U^{(0)}\). However, since \(\rho = \rho_1 - K(t)\), it “flows” uniformly without changing its shape with respect to the chosen coordinate frame at \(U^{(1)}\).

Of course, the string equations is a stationary version of \((4.1), (4.3), (4.14)\) and

\[
\Phi = \lim_{t \to -\infty} \frac{\Psi - \Psi_0}{\rho_1 - \text{fixed}}.
\]

It should be emphasized that, contrary to the diffeomorphism generating function \(\Psi\), the dilaton scalar field is a RG scheme-independent (universal), characteristic of the \textit{critical} target space background \([13]\).

**4.3 Infrared behavior**

Let us consider first the case with \(N \neq 0\) or, equivalently, \(\alpha + \beta \neq 0\). Then the RG trajectory \(S^{(\alpha, \beta)}_{(\delta b)}\) can be extended to a complete \textit{eternal} solution, i.e., it is defined for \(-\infty < t < +\infty\). As it follows from Eqs.\((4.10), (4.11)\) the parameter \(\kappa\) becomes zero at the limit \(t \to +\infty\) whereas

\[
\lim_{t \to +\infty} g^2 = \frac{1}{|N|}.
\]

It is also straightforward to see that there exist the limits

\[
h_* = \lim_{t \to +\infty} h, \quad \bar{h}_* = \lim_{t \to +\infty} \bar{h},
\]

and their values depend on the RG invariant \(R\) and on the sign of \(N\) only:

\[
\bar{h}_* = 0, \quad h_* = \text{sgn}(N) \frac{1}{2} (R^{-1} - R) \quad (0 < R \leq 1)
\]

\[
h_* = 0, \quad \bar{h}_* = \text{sgn}(N) \frac{1}{2} (R - R^{-1}) \quad (R \geq 1).
\]

---

\(^5\) This solution of the string equations is well known. Without regard to compactification conditions \((3.13), (4.26)\), it coincides with the marginal deformation of the Euclidean version of \(SL(2, \mathbb{R})\) (i.e., \(\mathbb{H}^+\)) WZWN background (see e.g. \([27]\) and references therein). The symmetric \(\mathbb{H}^+\)-background occurs in the properly taken limit \(i\beta, i\alpha \to \infty\). A compact version of the background (see Eqs.\((4.33), (4.34)\) below) were originally introduced in Refs. \([25, 26]\).
To describe the target space backgrounds corresponding to the infrared fixed point of the RG flow, it is convenient to use the RG invariants $R$, $N$ and the Hopf coordinates $(\theta, \chi_1, \chi_2)$ with $z = \cos(2\theta)$ and $\chi_a$ defined by \( (3.15) \). Then for any $R > 0$ one founds

$$G^{(IR)}_{\mu\nu} dX^\mu dX^\nu = 4 |N| \left[ (d\theta)^2 + \frac{R^2 \cos^2(\theta) (d\chi_1)^2}{\cos^2(\theta) + R^2 \sin^2(\theta)} + \frac{\sin^2(\theta) (d\chi_2)^2}{\cos^2(\theta) + R^2 \sin^2(\theta)} \right]$$

$$= \frac{1}{2!} B^{(IR)}_{\mu\nu} dX^\mu \wedge dX^\nu = -4 N R \frac{\sin^2(\theta)}{\cos^2(\theta) + R^2 \sin^2(\theta)} d\chi_1 \wedge d\chi_2 , \quad (4.33)$$

and for the dilaton field

$$\exp \left( 2\Phi^{(IR)} \right) = \frac{1}{\cos^2(\theta) + R^2 \sin^2(\theta)} . \quad (4.34)$$

As it was already mentioned, the RG invariant $N$ must satisfy the quantization condition

$$|N| = \frac{n}{16\pi} \quad \text{with} \quad n = 1, 2, \ldots . \quad (4.35)$$

The set of fields $(G^{(IR)}_{\mu\nu}, B^{(IR)}_{\mu\nu}, \Phi^{(IR)})$ obeys Eqs.\( (4.28) \). This solution of the string equations was introduced in Ref. \[25, 26\] and it is usually referred as to marginally deformed WZWN model.

Let us turn now to the case $N = 0$. The RG trajectory $S^{(a, -\alpha)}_{(d, b)}$ corresponds to the ancient solution terminating at $t = 0$ when the running coupling constant becomes infinite (see Eq.\( (4.10) \)). As $t \rightarrow -0$, the torsion strength vanishes whereas the metric asymptotically approaches to the round sphere metric \( (3.2) \) whose radius $\frac{2}{g(t)}$ shrinks to zero at $t = 0$ \[9\]:

$$\frac{4}{g^2} \sim -2 t = \frac{1}{\pi} \log \left( \frac{E}{E^*} \right) . \quad (4.36)$$

### 4.4 Comment on the diffeomorphism generating function

We finally discuss the relevance of the diffeomorphism dependent terms ($V$-terms below) in the Ricci flow equations. Since the diffeomorphism generating function depends on a choice of the coordinates, we can use it to simplify the general form of the Ricci flow equations somewhat. Namely, it seems natural to exclude the $V$-terms from \( (4.11) \) by a proper choice of the coordinate system $Z^\mu$, “moving” with respect to the frame $X^\mu = (u, v, w)$. The desirable coordinate frame is defined by the equation

$$\frac{dZ^\mu}{dt} := \left( \frac{\partial Z^\mu}{\partial t} \right)_X + G^{\mu\nu} \partial_\nu \Psi = 0 . \quad (4.37)$$

Let us chose the new coordinates in the form $Z^\mu = (Z, v, w)$ with $Z = Z(u, q)$ and $q = q(t)$. Then, Eq.\( (4.37) \) combined with \( (4.16) \), yields

$$q \partial_q Z - \partial_u \Psi \partial_u Z = 0 . \quad (4.38)$$

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This linear PDE should be supplemented by the initial condition \(Z|_{q=q_0} = Z_0(u)\). Of course, the desirable coordinate system is defined up to \(t\)-independent diffeomorphisms, so that \(Z_0(u)\) is a rather arbitrary monotonic function of \(u \in (0, \pi)\).

To be more specific at this point, let us consider the RG trajectory \(S^{(\alpha, -\alpha)}_{(\phi/b)}\). Then, as it follows from (4.15),

\[
\partial_u \Psi = -\frac{1}{2} \partial_u \log \left( \partial_1^2(u, q^2) + L^2 \partial_1^2(u, q^2) \right) ,
\]

(4.39)

where \(L = \tan(\alpha)\) stands for the RG invariant (4.8). The initial condition can be taken at \(q = 0\) with \(Z_0(u) = \cos(u)\). Given this initial setup, the solution of (4.37) is constructed as a power series in \(q\):

\[
Z = \cos(u) + 2L^2 \sin(u) \sin(2u) q + 2 \sin(u) \sin(2u) (1 - 4L^4 \cos^2(u)) \ q^2 + O(q^3) .
\]

(4.40)

Note that the \(n\)-th term of this series is a polynomial in \(L^2\) of order \(n\). The partial summation of the series yields

\[
Z = z + \frac{L^2}{2} (1 - z^2) \log \left( \frac{1 + \kappa z}{1 - \kappa z} \right) + O(L^4) ,
\]

(4.41)

where \(z = z(u, q)\) and \(\kappa = \kappa(q)\) are given by (3.5) and (3.7), respectively. Eq.(4.41) implies that, in the torsion-free case, the metric (3.9) with \(h = \bar{h} = 0\) satisfies \(\dot{G}_{\mu \nu} = -R_{\mu \nu}\). In fact, it was discovered by V.A. Fateev as a brute-force solution to this Ricci flow equation. The expansion (4.41) also suggests to consider \((z, \kappa)\) as an independent set of variables replacing the variables \((u, q)\). It is then straightforward to check that

\[
Z = Z[z, \kappa] : \quad \partial_\kappa Z - \frac{L^2 z (1 - z^2)}{1 + \kappa L^2 - \kappa (\kappa + L^2) z^2} \partial_z Z = 0 , \quad Z[z, 0] = z .
\]

(4.42)

The solution of this Cauchy problem can be obtained by the method of characteristic:

\[
Z = \frac{(1 + z)(1 + \kappa z)^{L^2} - (1 - z)(1 - \kappa z)^{L^2}}{(1 + z)(1 + \kappa z)^{L^2} + (1 - z)(1 - \kappa z)^{L^2}}.
\]

(4.43)

It is a nonsingular monotonic function of \(z \in [-1, 1]\) for any \(0 \leq \kappa < 1\). However, as \(\kappa \to 1\), the branch points at \(z = \pm \kappa^{-1}\) approach the ends of the segment. For this reason the regions \((z, \kappa | |1 - \kappa| \ll 1, \ |z \pm 1| \ll 1\) need a special attention. As it has been discussed in subsection 4.2, the target space backgrounds in these domains are asymptotically approaching the solutions of the string equations (4.28), and the each critical background required the dilaton field which cannot be absorbed by a nonsingular coordinate transformation.

In the case \(N \neq 0\) we can still chose \((z, \kappa)\) as an independent set of variables. Then, using Eqs.(4.9), (4.15), the linear PDE (4.38) can be brought to the form

\[
\partial_\kappa Z + F(z, \kappa) \partial_z Z = 0 ,
\]

(4.44)
where

\[ F(z, \kappa) = -\frac{\left(2\kappa L^2 z + A(1 + \kappa^2 z^2)\right)(1 - z^2)}{2\kappa \left(1 + \kappa L^2 - A(1 - \kappa^2) z - \kappa(\kappa + L^2) z^2\right)} \]  

(4.45)

and \( A \) stands for the RG invariant

\[ A = \frac{R^2 - 1}{R^2 + 1}. \]  

(4.46)

For \( A \neq 0 \), the solution of the characteristic curve equation

\[ \frac{dz}{d\kappa} = F(z, \kappa), \]  

(4.47)

is not available in a closed form; however, its small-\( \kappa \) asymptotic can be easily found. A simple calculation shows that the function \( Z \) can be chosen in the form

\[ Z \approx (1 + z) \left(\frac{1 - z}{\sqrt{\kappa}}\right)^A - (1 - z) \left(\frac{1 + z}{\sqrt{\kappa}}\right)^{-A} \]

\[ (1 + z) \left(\frac{1 - z}{\sqrt{\kappa}}\right)^A + (1 - z) \left(\frac{1 + z}{\sqrt{\kappa}}\right)^{-A} (\kappa \ll 1). \]  

(4.48)

It makes explicit non-analytic properties of the coordinate transformation \( z \rightarrow Z \) at the limit \( \kappa \rightarrow 0 \). As \( \kappa = 0 \) the target space background arrives at the infrared fixed-point which requires the introduction of the dilaton field.

Returning to the general one-loop RG equations, our analysis illustrates the role of the \( V \)-terms in Eq.(4.1). Namely, it suggests that, by means of a nonsingular reparameterization of the target manifold, these terms can be excluded everywhere except the RG fixed-point regime.

5 Conclusion

In this paper we have found the zero-curvature representation for a four-parameter family of the classical NLSM. In the context of a hierarchy of classical integrable systems the new family can be viewed as a three-parameter deformation of the \( SU(2) \) WZWN model. Also it contains, as a two-parameter subfamily, the Fateev sausage model. Thus the work resolves the long-standing question of classical integrability of that model.

We have discussed some aspects of the perturbative quantization. It was demonstrated the renormalizability of the integrable family of NLSM at the lowest perturbative order. The RG equations at the one-loop order describe a Ricci flow with torsion. Therefore, among the results of this paper is an interesting set of ancient and eternal solutions of the Ricci flow. In all likelihood these solutions correspond to the multi-parameter family of integrable quantum fields theories. Currently a non-perturbative description is available for the case of the Fateev model only.

An analysis of this paper is explicitly concerned with a relation between a particular classical integrable NLSM and explicit solutions of the Ricci flow. It seems extremely desirable to get a more general understanding about this remarkable relation, which may provide new analytical insights in searching for physically interesting string backgrounds.
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6 Appendix

Here we describe a general solution to Eqs. (2.22).

First of all one needs explicit formulas for the Levi-Civita spin connection. In the case at hand non-vanishing components of the connection read as follows:

$$\omega_\mu = \sum_{a=3, \pm} \omega^a_\mu \sigma_a : \quad \omega^a_\mu = \begin{cases} -\frac{1}{2} \left( \Omega^{+-} + \Omega^{-+} \right) e^3_\mu & a = 3 \text{ and } \mu = u, \\ \pm \frac{1}{2} \left( \Omega^{+-} - \Omega^{-+} \right) e^\pm_\mu + \Omega^{\mp} e^\mp_\mu & a = \pm \text{ and } \mu = v, w \end{cases} \quad (6.1)$$

where $\sigma_\pm = \frac{1}{2} (\sigma_1 \mp i \sigma_2)$ and $\Omega^{\alpha\beta} (\alpha, \beta = \pm)$ stands for

$$\Omega^{\alpha\beta} = \frac{1}{2 \sqrt{G_{uu}}} \left( e^\alpha_u \partial_v e^\beta_v - e^\alpha_v \partial_u e^\beta_u \right). \quad (6.2)$$

These formulas combined with the definitions (2.10), (2.11) allow one to rewrite Eqs. (2.22) in explicit form. Namely, the equations for $\omega_\mu = \frac{1}{2} (\omega^+_\mu + \omega^-_\mu)$ are given by

$$\Omega^{++} = -\frac{i}{2} f_+(\pi - 2i \eta) \exp^{i(\phi^+ + \phi^-)}$$

$$\Omega^{--} = \frac{i}{2} f_+(\pi + 2i \eta) \exp^{-i(\phi^+ + \phi^-)} \quad (6.3)$$

$$\Omega^{+-} + \Omega^{-+} = -i f_3(\pi + 2i \eta) + \frac{i}{2 \sqrt{G_{uu}}} \partial_u (\phi^+ + \phi^-)$$

$$\Omega^{+-} - \Omega^{-+} = \frac{1}{2i} \left( f_+(\pi - 2i \nu) \exp^{i(\phi^+ - \phi^-)} + f_+(\pi + 2i \nu) \exp^{-i(\phi^+ - \phi^-)} \right),$$

whereas the corresponding equations for the antisymmetric part $\omega^+_\mu - \omega^-_\mu = -\frac{i}{2} H \gamma_\mu$ read as

$$H = f_+(\pi - 2i \nu) \exp^{-i(\phi^+ - \phi^-)} - f_+(\pi + 2i \nu) \exp^{i(\phi^+ - \phi^-)}$$

$$H = 2 f_3(\pi - 2i \nu) + \frac{1}{\sqrt{G_{uu}}} \partial_u (\phi^+ - \phi^-). \quad (6.4)$$

Here we use the notations (2.23) and $\phi_\pm$ stand for $u$-dependent phases from the matrixes $U_\pm = \exp \left( \frac{i}{2} \phi_\pm \sigma_3 \right)$. Eqs. (6.4) can be immediately integrated and their general (one-parameter family) solution is given by

$$e^{i(\phi^+ - \phi^-)} = \frac{\vartheta_3 \left( \frac{u - i\varphi + i\nu}{2}, q \right) \vartheta_4 \left( \frac{u + i\varphi - i\nu}{2}, q \right)}{\vartheta_3 \left( \frac{u + i\varphi + i\nu}{2}, q \right) \vartheta_4 \left( \frac{u - i\varphi - i\nu}{2}, q \right)}, \quad (6.5)$$

$$H = \frac{1}{\sqrt{G_{uu}}} \frac{\vartheta_1'(0, q)}{2i \vartheta_2(i \nu, q) \vartheta_1(u, q)} \left[ \vartheta_2(u - i\nu, q) e^{i(\phi^+ - \phi^-)} - \vartheta_2(u + i\nu, q) e^{-i(\phi^+ - \phi^-)} \right].$$
Note that \( \Omega^+ - \Omega^- = \frac{1}{2\sqrt{\Omega}} \partial_u \log(S) \), where
\[
S = \frac{1}{2\xi} (e_v e_w^+ - e_v^+ e_w^+).
\] (6.6)

Therefore the last equation in (6.3) and (6.5) yield
\[
\partial_u \log(S) = \frac{\vartheta'_{0}(0, q)}{2\vartheta_{2}(u - iv, q) \vartheta_{1}(u, q)} \left[ \vartheta_{2}(u - iv, q) e^{i(\phi^+ - \phi^-)} + \vartheta_{2}(u + iv, q) e^{-i(\phi^+ - \phi^-)} \right],
\] (6.7)
or, equivalently,
\[
S = \frac{1}{g^2} \rho(-u) \rho(u) \frac{i \vartheta_{2}(u, q) \vartheta_{1}(u, q)}{\vartheta_{1}(u, q) \vartheta_{2}(0, q)}. \] (6.8)

Here we use the function \( \rho(u) \) defined in (2.30) and \( g \) is some \( u \)-independent real constant. Let us introduce \( E_{\mu}^\pm \) such that
\[
e_{\mu}^+ = g^{-1} e^{-i\phi^+} \rho(u) E_{\mu}^+, \quad e_{\mu}^- = g^{-1} e^{i\phi^-} \rho(-u) E_{\mu}^-.
\] (6.9)

They satisfy the conditions
\[
E_{v}^- E_{w}^+ - E_{v}^+ E_{w}^- = -2 \frac{\vartheta_{2}(u, q) \vartheta_{1}(u, q)}{\vartheta_{1}(u, q) \vartheta_{2}(0, q)}, \quad E_{\mu}^- = (E_{\mu}^+)\rangle,
\] (6.10)

and also solve a system of differential equations
\[
W[E_{w}^+, E_{v}^+] = \mp \frac{\vartheta_{2}(u \pm i\eta, q) \vartheta'_{0}(0, q)}{\vartheta_{1}(u, q) \vartheta_{2}(0, q)}
\]
\[
W[E_{w}^+, E_{w}^-] + W[E_{w}^-, E_{v}^+] = 2 \frac{\vartheta'_{0}(u, q) \vartheta_{1}(u, q)}{\vartheta_{1}(u, q) \vartheta_{2}(0, q)},
\] (6.11)

where \( W \) stands for the Wronskian, \( W[F, G] := F \partial_u G - G \partial_u F \). The system of Eqs. (6.10), (6.11) can be integrated explicitly, yielding the following expressions for \( e_{\mu}^\pm \):
\[
e_{v}^+ = g^{-1} e^{\pm i(\phi^+ - \phi^-)} \rho(\pm u) \left( a \frac{\vartheta_{4}(i\eta \pm u, q^2)}{\vartheta'_{4}(i\eta, q^2)} - b \frac{\vartheta_{1}(i\eta \pm u, q^2)}{\vartheta_{1}(i\eta, q^2)} \right),
\]
\[
e_{w}^+ = g^{-1} e^{\pm i(\phi^+ - \phi^-)} \rho(\pm u) \left( c \frac{\vartheta_{4}(i\eta \pm u, q^2)}{\vartheta'_{4}(i\eta, q^2)} - d \frac{\vartheta_{1}(i\eta \pm u, q^2)}{\vartheta_{1}(i\eta, q^2)} \right),
\] (6.12)

where real integration constant \( a, b, c \) and \( d \) obey a single constraint \( ad - bc = 1 \). Using \( SL(2, R) \) coordinate transformations (2.27), we can bring the solution to the forms with either \( a = d = 1, b = c = 0 \), or \( a = d = 0, b = -c = 1 \). Since this two cases related by the coordinate transformation \((u, v, w) \leftrightarrow (-u, w, v)\), we accept the form (2.28) without loss of generality. Note that in Eq. (2.28) we use the constant \( l \) which substitute the metric coefficient \( G_{uu} \) (2.5): \( \sqrt{G_{uu}} = \frac{l}{g} > 0 \). The ambiguity in sign of \( e_{u}^3 \) can be resolved by means of the condition \( \sqrt{G} := \det(e_{\mu}^\pm) \geq 0 \) that picks up an orientation for the vielbein.
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