Finite Temperature Expectation Values of Local Fields in the sinh-Gordon model

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Abstract

Sklyanin’s method of separation of variables is employed in a calculation of finite temperature expectation values. An essential element of the approach is Baxter’s $Q$-function. We propose its explicit form corresponding to the ground state of the sinh-Gordon theory. With the method of separation of variables we calculate the finite temperature expectation values of the exponential fields to one-loop order of the semi-classical expansion.

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1. Introduction

One-point functions have numerous applications in Statistical Mechanics and Condensed Matter Physics [1,2]. They determine “generalized susceptibilities” i.e. the linear response of a system to external fields. In a path integral formulation the one-point function of a local field $\mathcal{O}$ is represented by a Euclidean path integral of the form

$$\langle \mathcal{O} \rangle = Z^{-1} \int \mathcal{D}[\varphi] \mathcal{O} e^{-A}. \quad (1.1)$$

Recently some progress has been achieved in the calculation of one-point functions in integrable Quantum Field Theory (QFT) defined on a two dimensional Euclidean plane [3,4]. In this case, the integral (1.1) can also be viewed as a Vacuum Expectation Value (VEV) in 1 + 1-dimensional QFT associated with the action $A$. For many applications, especially in Condensed Matter Physics, it is important to generalize the results of Refs.[3,4] to the case of Euclidean path integrals defined on an infinite cylinder. In the Matsubara imaginary time formalism such path integrals are interpreted as thermal averages

$$\langle \mathcal{O} \rangle_R = \frac{\text{Tr}[e^{-R\mathcal{H}} \mathcal{O}]}{\text{Tr}[e^{-R\mathcal{H}}]}, \quad (1.2)$$

where $\mathcal{H}$ is the Hamiltonian of the corresponding QFT and the temperature coincides with the inverse circumference of the cylinder.

The path integral defined on a cylinder also allows another interpretation. It is an expectation value for the ground state $|\text{vac}\rangle_R$ of the 1 + 1-dimensional theory in the finite geometry where the spatial coordinate is compactified on a circle. Hence, the VEVs contain important information about Renormalization Group flow controlled by the parameter $R$.

An exact calculation of the finite volume (finite $R$) VEVs is a challenge even in integrable QFT. Recent progress made in papers [5,6] should be mentioned here. In [5] A. Leclair and G. Mussardo proposed an integral representation which makes it possible to generate a low-temperature ($R \to \infty$) expansion for the VEVs in terms of infinite volume form-factors of local fields and some thermodynamical data. Their conjecture works for theories with trivial $S$-matrices such as Ising and Free Dirac Fermion models [6], but its validity remains questionable for models with non-trivial scattering amplitudes (see e.g. [7]). Another line of research was proposed in the work [8]. F. Smirnov applied the method of separation of variables [8,10] to the semi-classical study of finite volume matrix elements in the quantum KdV equation. The model does not constitute a relativistic field
theory. Nevertheless, it is of prime importance for the sinh-Gordon QFT since both of the equations are in the same integrable hierarchy.

In this paper we will implement the method of separation of variables in the case of the quantum sinh-Gordon theory. The problem is defined by a Euclidean action,

$$A_{shG} = \int_{-\infty}^{\infty} dx_1 \int_0^R dx_2 \left\{ \frac{1}{16\pi} \left( \partial_\sigma \varphi \right)^2 + 2\mu \cosh(b\varphi) \right\},$$

(1.3)

where $\varphi$ is a scalar field with periodic boundary condition along the $x_2$-coordinate. We are focusing on the VEVs of the exponential fields,

$$\mathcal{O} = e^{a\varphi}.$$

(1.4)

For our purposes it will be useful to rewrite (1.2) in the form,

$$\langle e^{a\varphi} \rangle_R = Z^{-1} \int D[\chi] \Psi_0^{2}[\chi] e^{a\chi}.$$

(1.5)

Here $\Psi_0[\chi]$ is an integral taken over field configurations on the half-cylinder, $x_1 < 0$, satisfying the boundary condition,

$$\varphi(x_1, x_2)|_{x_1=0} = \chi(x_2),$$

i.e. it is a wave functional corresponding to the ground state $|\text{vac}\rangle_R$. The method of separation of variables [9,10,5] allows one to introduce a change of integration variables in (1.5),

$$\chi(x_2) \to \{\gamma_k\}_{k=-\infty}^{\infty},$$

from the function $\chi(x_2)$ to the infinite discrete set of $\gamma_k$. A notable advantage of the new variables is that the wave functional in the “$\gamma$-representation” has a factorizable form,

$$\Psi_0[\{\gamma_k\}] \sim \prod_k Q[\gamma_k].$$

Notice that the integration measure $D[\{\gamma_k\}]$ does not factorize in the variables $\gamma_k$. At this moment, we are not able to elaborate on all steps of the changing of variables on a rigorous basis. Therefore, we suggest the deduced integral representation for $\langle e^{a\varphi} \rangle_R$ as a conjecture rather than a well established result. To test the validity of this integral representation, we carry out a semi-classical expansion of the VEV.
More explicitly, the parameter \( b^2 \) in the action (1.3) can readily be identified with the Planck constant. Then, for finite \( \alpha = ab \) and \( b^2 \to 0 \), the functional integral (1.1) is dominated by a non-trivial saddle-point configuration and admits the semi-classical expansion,

\[
\langle e^{a\varphi} \rangle_R = e^{-\frac{S}{b^2}} D \left( 1 + O(b^2) \right).
\]

(1.6)

Here \( S \) is a Euclidean action on the cylinder evaluated in the saddle-point configuration and the pre-exponential factor \( D \) is the result of evaluating the functional integral (1.1) in the Gaussian approximation around the classical solution. With the proposed integral representation we calculate the functions \( S \) and \( D \) and find complete agreement with the expected high- and low-temperature behavior of the VEVs. In particular, our result matches well with the Leclair-Mussardo conjecture.

2. Integral representation for VEVs

2.1. Flaschka-McLaughlin variables

In the paper [11] H. Flaschka and D. McLaughlin found remarkable canonically conjugate variables in the phase spaces of the classical Toda chain and KdV equations. Their approach can be straightforwardly adapted to the classical sinh-Gordon equation. Here we give a brief review of the Flaschka-McLaughlin variables for this model. For more information and proofs, the reader is referred to Refs.[11][12].

The sinh-Gordon equation admits a zero curvature formulation: There exists a \( sl(2,\mathbb{R}) \)-valued connection 1-form, depending on an auxiliary parameter \( \lambda \), such that the condition of vanishing curvature is equivalent to the equation of motion. Dealing with the theory on cylinder, one can integrate this 1-form along some cycle, say,

\[
x_1 = 0, \quad 0 \leq x_2 < R,
\]

(2.1)

and obtain the so-called monodromy matrix,

\[
M(\lambda) = \begin{pmatrix}
A(\lambda) & \lambda B(\lambda) \\
\lambda^{-1} C(\lambda) & D(\lambda)
\end{pmatrix} \in SL(2,\mathbb{R}) \quad (\Im m \lambda = 0).
\]

(2.2)

This matrix satisfies important analytical conditions which are readily obtained from an explicit form of the connection. In particular, the matrix elements in (2.2) are real analytical functions of the variable \( \lambda^2 \) with two essential singularities at the points \( \lambda^2 = 0, \infty \). Zeroes of \( B(\lambda) \),

\[
\lambda_k^2 : \quad B(\lambda_k) = 0,
\]

(2.3)
are of prime importance in the construction. It is possible to show that all zeroes \((2.3)\) are simple, real, positive and accumulate towards the essential singularities. Thus we can order them,

\[
0 \leftarrow \ldots \lambda_{-N}^2 < \lambda_{-N+1}^2 \ldots < \lambda_0^2 < \ldots \lambda_{N-1}^2 < \lambda_N^2 \ldots \rightarrow +\infty
\]

and define two infinite sets:

\[
\begin{align*}
\{\gamma_k\}_{k=-\infty}^{\infty} : \gamma_k &= \log \lambda_k^2, \\
\{\pi_k\}_{k=-\infty}^{\infty} : \pi_k &= 4 \log |A(\lambda_k)|.
\end{align*}
\]

\[(2.4)\]

The mapping of the canonical Poisson data, \(\varphi\) and \(\partial_x \varphi\), to the variables \((2.4)\) is found to be a canonical transformation, i.e.

\[
\{\pi_k, \gamma_m\} = \delta_{km}, \quad \{\pi_k, \pi_m\} = \{\gamma_k, \gamma_m\} = 0.
\]

\[(2.5)\]

Hence \((2.4)\) are canonically conjugate variables in the phase space of the sinh-Gordon model. At the same time, we can treat \(\{\gamma_k\}_{k=-\infty}^{\infty}\) as coordinates in the corresponding configuration space.

2.2. \(\gamma\)-representation

The Flaschka-McLaughlin variables proved to be useful in quantum theory as was demonstrated in the seminal work \([9]\) on the example of the Toda chain equation (see also \([13,14]\)). We refer to a quantization in these variables as a quantization in \(\gamma\)-representation. Recently the \(\gamma\)-representation was employed to quantize “real” KdV theory \([5]\). In fact, Smirnov presented heuristic, but convincing, model-independent arguments which can also be applied to the sinh-Gordon equation. Following these arguments we introduce the integral,

\[
\mathcal{I}_N(R, a) = \prod_{k=-N}^{N} \int_{-\infty}^{+\infty} \frac{d\gamma_k}{b} \prod_{N \geq k > m \geq -N} \sinh(\gamma_k - \gamma_m) \prod_{k=-N}^{N} Q^2[\gamma_k] \ e^{\frac{2\pi c}{b^2} (ab + k)}.
\]

\[(2.6)\]

\(^1\) The monodromy matrix in the “real” KdV model has the form \((2.2)\), whereas the monodromy matrix of the “imaginary” equation belongs to the group \(SU(2)\) (\(\Im \lambda = 0\)). The imaginary KdV model is related to the sine-Gordon theory and (perturbed) CFT with the central charge \(c < 1\) (see e.g. \([11]\)). A sensible \(\gamma\)-representation for the imaginary equation has, to our knowledge, not been found.
We shall also use slightly different form of $\mathcal{I}_N$: With the identity,

$$\prod_{N \geq k > m \geq -N} 2 \sinh \left( \frac{\gamma_k - \gamma_m}{b^2} \right) = \text{Det} \left| \exp \left( \frac{2k\gamma_j}{b^2} \right) \right|_{-N \leq j, k \leq N},$$

the integral (2.6) can be rewritten in the form

$$\mathcal{I}_N(R, a) = \frac{1}{(2N + 1)!} \prod_{k=-N}^{N} \int_{-\infty}^{+\infty} \frac{d\gamma_k}{b} \times \prod_{N \geq k > m \geq -N} 2 \sinh(\gamma_k - \gamma_m) \sinh \left( \frac{\gamma_k - \gamma_m}{b^2} \right) \prod_{k=-N}^{N} Q[\gamma_k] e^{\frac{2\gamma_k a}{b}}.$$  \hspace{1cm} (2.7)

The function $Q[\gamma]$ appearing in Eqs.(2.6), (2.7) is the so-called Baxter’s $Q$-function. It is a non-singular function for real $\gamma$ with leading asymptotic behavior

$$Q[\gamma] \sim e^{-2C_0 \cosh(bq\gamma)} \quad \text{as} \quad \gamma \to \infty.$$  \hspace{1cm} (2.8)

Here and below we use the notation,

$$q = b + b^{-1}.$$  

The positive constant $C_0$ in (2.8) reads explicitly,

$$C_0 = \frac{mR}{4 \sin \left( \frac{\pi b}{q} \right)},$$

and $m$ is a mass of the sinh-Gordon particle. Therefore, $\mathcal{I}_N$ (2.7) is a convergent integral for any finite $N$. Notice that the configuration space of the model under consideration is an infinite-dimensional space. In writing the $2N + 1$-fold integral, we truncate it to $2N + 1$-dimensional space with coordinates $\{\gamma_k\}_{k=-N}^{N}$. As well as in the KdV theory [3], we can treat

$$\Psi_N[\{\gamma_k\}] = \prod_{k=-N}^{N} Q[\gamma_k]$$

as a wave functional in the $\gamma$-representation. For $N \to \infty$, it corresponds to the ground state $|\text{vac}\rangle_R$ of the sinh-Gordon theory with periodic boundary conditions. Furthermore, the double product in (2.7) is an integration measure and we shall denote it $\mathcal{D}_N[\{\gamma\}]$. It was calculated in the semi-classical approximation in [3]. The semi-classical analysis suggests also that the product

$$\mathcal{O}_N = \prod_{k=-N}^{N} e^{\frac{2\gamma_k a}{b}}$$

is...
represents (in the limit $N \to \infty$) the exponential field $e^{a\varphi}$ located at the point $(R/2, R/2)$ on the cylinder. Therefore, the integral (2.7) has the form of a quantum mechanical diagonal matrix element,

$$I_N = \int D_N[\{\gamma\}] \, \Phi_N \, \sigma_N \, \Phi_N \, (2.9)$$

We shall consider the Vacuum Expectation Values only. In this case, $Q$ is an eigenvalue of the Baxter $Q$-operator corresponding to the ground state.

2.3. *Baxter’s Q-function in sinh-Gordon theory*

We now turn to the most delicate point of our construction: an explicit form of the function $Q$. To the best of our knowledge a rigorous derivation of $Q$ is not currently available. Here we formulate a conjecture for the sinh-Gordon $Q$-function based on the following heuristic arguments. First, we note that the substitution $b \to i\beta$ transforms $A_{shG}$ (1.3) to the action of the sine-Gordon model. Naively we could try to obtain $Q$ by means of analytical continuation from $-1 < b^2 < 0$. In this coupling constant domain, the Baxter $Q$-operator is relatively well studied \[15\]. Unfortunately, the sine-Gordon $Q$-function has an essential singularity at $b^2 = 0$ the analytical structure of which is unknown. This makes the analytical continuation to the domain of the sinh-Gordon model a highly questionable procedure. One can guess an explicit form for $Q$ by examining its asymptotic behavior. The $Q$-function in the sine-Gordon model admits the following asymptotic expansion for $\gamma \to +\infty$ [15]:

$$\log Q_{sinG} \simeq -C_0 \, e^\theta + \sum_{n=1}^{\infty} C_n \, \mathbb{I}_{2n-1} \, e^{-(2n-1)\theta} + \sum_{n=1}^{\infty} \tilde{G}_n \, e^{2n\theta(b^{-2}+1)} \quad (-1 < b^2 < 0) \, .$$

Here we introduce a new notation,

$$\theta \equiv \gamma \, bq \, . \quad (2.10)$$

The leading term of this expansion has already appeared in our consideration (see (2.8)) while $\mathbb{I}_{2n-1}$ and $\tilde{G}_n$ are vacuum eigenvalues of the so-called local and dual unlocal Integrals of Motion (IM) respectively. The constants $C_n$ depend on the normalization of the local IM.

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2 The position of the insertion is determined by choosing of the integration contour for the monodromy matrix. Here we assume that the contour is given by (2.1).

3 In this work we use a convention for the $Q$-function which differs from the one of \[15\] by an overall shift of the argument.
In Appendix A (see Eq.(A.2)) we present their form for the normalization adopted in [13]. A similar asymptotic form holds for \( \gamma \to -\infty \). The eigenvalues \( \mathbb{I}_{2n-1} \) are regular functions of \( b^2 \), and can be continued to the domain of the sinh-Gordon model without problems. Contrary to \( \mathbb{I}_{2n-1} \), the eigenvalues of the dual unlocal IM, \( \tilde{G}_n \), are highly singular functions at \( b^2 = 0 \). One can expect that the appearance of these IM is a consequence of the existence of the soliton sector of the sine-Gordon QFT. This sector is absent for \( b^2 > 0 \). All these observations suggest the following large \( \gamma \) asymptotic behavior in the sinh-Gordon model,

\[
\log Q \simeq -C_0 e^{\theta} - \sum_{n=1}^{\infty} C_n \mathbb{I}_{2n-1} e^{-(2n-1)\theta} \quad (b^2 > 0).
\]

In Appendix A we give numerical evidence that the values of local IM can be expressed in terms of a solution of the Thermodynamic Bethe Ansatz (TBA) equation:

\[
C_n \mathbb{I}_{2n-1} = C_0 \delta_{n1} + (-1)^n \int_{-\infty}^{\infty} \frac{d\theta}{\pi} e^{(2n-1)\theta} \log \left( 1 + e^{-\epsilon(\theta)} \right).
\]

Here the function \( \epsilon(\theta) \) solves the TBA equation [16,17,18]

\[
\epsilon(\theta) - mR \cosh(\theta) + \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \Phi(\theta - \theta') \log \left( 1 + e^{-\epsilon(\theta')} \right) = 0,
\]

with the kernel

\[
\Phi(\theta) = \frac{4 \sin \left( \frac{\pi b}{q} \right) \cosh(\theta)}{\cosh(2\theta) - \cos \left( \frac{2\pi b}{q} \right)}.
\]

The series (2.11) is an asymptotic expansion. In fact, it is a divergent geometrical series which can easily be summed up and one can guess an explicit form of \( Q \):

\[
\log Q(\theta) = -\frac{mR}{2 \sin \left( \frac{\pi b}{q} \right)} \cosh(\theta) + \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{\log \left( 1 + e^{-\epsilon(\theta')} \right)}{\cosh(\theta - \theta')}.
\]

We leave an examination of the properties of this function for future publications.

One more aspect of the \( \gamma \)-representation deserves a comment. The sinh-Gordon model manifests an important non-perturbative symmetry. The couplings \( b \) and \( b^{-1} \) correspond to physically indistinguishable theories. \( Q \) in (2.15) is a self-dual function in a sense that it is invariant under the substitution \( b \to b^{-1} \). Furthermore, it is easy to see that

\[
\mathcal{I}_N|_b = \mathcal{I}_N|_{b^{-1}}.
\]

This supports our choice of the measure in (2.7). Strictly speaking, the measure was obtained in [5] in the semi-classical approximation. The exact invariance of the semi-classical measure suggests its applicability for an arbitrary value of the coupling constant \( b^2 \).
2.4. Large $N$ limit

As was noted above, in writing the $2N + 1$-fold integral (2.7), we truncate the configuration space of the theory to $2N + 1$-dimensional one. In fact, the truncation amounts to an ultraviolet regularization with a momentum cutoff given by

$$\Lambda_N = \frac{2\pi N}{R}.$$ 

Now we let $N \to \infty$. From (2.9), it is clear that the ratio

$$\bar{T}_N(R,a) = \frac{T_N(R,a)}{T_N(R,0)} \quad (2.16)$$

represents the VEV $\langle e^{a\varphi} \rangle_R$ in the large $N$ limit. More explicitly, dimensional analysis suggests that

$$\bar{T}_N(R,a) \propto \left(\frac{\Lambda_N}{m}\right)^{2a^2},$$

thus the correct relation has the form

$$\langle e^{a\varphi} \rangle_R = \kappa_a \lim_{N \to \infty} \left(\frac{4\pi N}{mR}\right)^{-2a^2} \bar{T}_N(R,a). \quad (2.17)$$

Here $\kappa_a$ is an arbitrary $R$-independent constant. To eliminate this ambiguity one has to impose some normalization condition on the fields. For example, the so-called conformal normalization stipulates that the exponential fields with sufficiently small $|a|$ are normalized in accordance with the short distance behavior of the two-point function

$$\langle e^{a\varphi}(x)e^{-a\varphi}(y) \rangle \to |x-y|^{4a^2} \quad \text{as} \quad |x-y| \to 0.$$ 

The key result of Refs.[3,4] is a calculation of the limit,

$$\lim_{R \to \infty} \langle e^{a\varphi} \rangle_R = G_a \quad (2.18)$$

with this normalization. Explicitly,

$$G_a = \left[ \frac{m \Gamma \left( \frac{1}{2q} \right) \Gamma \left( 1 + \frac{b}{2a} \right)}{4\sqrt{\pi}} \right]^{-2a^2} \times \exp \left\{ \int_0^\infty \frac{dt}{t} \left[ - \frac{\sinh^2(2abt)}{2\sinh(b^2t) \sinh(t) \cosh(qbt)} + 2a^2 e^{-2t} \right] \right\}. \quad (2.19)$$
Once we adopt the conformal normalization for the exponential fields, the constant $\kappa_a$ is uniquely determined by the condition $(2.18)$. In particular, the semi-classical consideration (see below) leads to the relation,

$$\kappa_a = G_a \left( 1 + O(b^2) \right).$$

3. Semi-classical expansion

In this section we will study the VEVs in the semi-classical approximation. The VEV $(2.17)$ can be represented by the Euclidean path integral $(1.1)$ on a cylinder with $A$ and $O$ given by $(1.3)$, $(1.4)$. For fixed $\alpha$,$$
\alpha = a b
$$
and $b^2 \to 0$, the path integral is dominated by a saddle-point configuration $\phi = b \varphi$ and the VEV has the form $(1.6)$, where $S$ coincides with the regularized Euclidean classical action on the cylinder:

$$S = \lim_{\varepsilon \to 0} \left[ \int_{|x-y| > \varepsilon} \frac{d^2x}{8\pi} \left\{ \frac{(\partial_\sigma \phi)^2}{2} + m^2 \sinh^2 \left( \frac{\phi}{2} \right) \right\} + \alpha \oint_{|x-y| = \varepsilon} ds \frac{2\pi}{\phi - 2\alpha^2 \log \varepsilon} \right].$$

Here the function $\phi$ is a solution of the classical equation of motion,

$$\partial^2_\sigma \phi = m^2 \sinh(\phi),$$

such that,

$$\phi \to 4\alpha \log |x-y| + O(1) \quad \text{as} \quad |x-y| \to 0,$$

and

$$\phi \to 0 \quad \text{as} \quad |x-y| \to \infty,$$

$$\phi(x_1, x_2 + R) = \phi(x_1, x_2).$$

The field configuration $\phi$ develops a singularity at the point $y$ where the exponential field is inserted. Therefore, in the definition $(3.2)$ we cut the small disc of radius $\varepsilon$ around this point and add the boundary term to the action to ensure $(3.4)$. We also add a field independent term such that the action is finite at $\varepsilon \to 0$. The pre-exponential factor $D$ $(1.4)$ is a result of evaluating the path integral $(1.1)$ in the Gaussian approximation around the classical solution defined above,

$$D = \left( \frac{m \varepsilon}{2} e^{\gamma_E} \right)^{-2\alpha^2} \left[ \text{Det} \left( \frac{-\partial^2_\sigma + m^2 \cosh(\phi)}{-\partial^2_\sigma + m^2} \right) \right]^{-\frac{1}{2}},$$

where $\gamma_E = 0.577216\ldots$ is the Euler constant. The first factor in $(3.6)$ appears as a result of the mass renormalization.
3.1. Main semi-classical order

We now proceed to the semi-classical calculation of the VEV $\langle e^{a\phi} \rangle$ using the representation (2.17). In order to apply the saddle-point machinery, it is convenient to begin with the form (2.6) for the integral $I_N$. To the lowest semi-classical order,

$$Q[\gamma] \sim e^{-\frac{r}{2\pi b^2} \cosh(\gamma)}.$$  

(3.7)

Here and below the notation

$$r = mR$$  

(3.8)

is used. The corresponding saddle-point equations have the form,

$$r \sinh(\rho_k) = 2\pi (k + \alpha), \quad k = 0, \pm 1, \ldots, \pm N.$$  

(3.9)

In writing (3.9) we have assumed that $b^2 \to 0$ and $\alpha = ab$ is fixed. Thus we find,

$$\log I_N = -\frac{1}{b^2} \sum_{k=-N}^{N} \left\{ \frac{r}{\pi} \cosh(\rho_k) - 2 \rho_k (k + \alpha) \right\} + O(1).$$  

(3.10)

The sum can be evaluated with the result,

$$-\sum_{k=-N}^{N} \left\{ \frac{r}{\pi} \cosh(\rho_k) - 2 \rho_k (k + \alpha) \right\} = T - \left( \frac{r}{2\pi} \right)^2 \log \left( \frac{4\pi Ne}{r} \right) + 2 \alpha^2 \log N +$$

$$\left( 2N(N + 1) + \frac{1}{3} + 2\alpha^2 \right) \log \left( \frac{4\pi}{r} \right) + 4 \sum_{k=1}^{N} k \log(k/e) - 4 \log A_G + O\left( \frac{1}{N} \right),$$  

(3.11)

where $A_G = 1.282427\ldots$ is the Glaisher constant and the function $T = T(r, \alpha)$ reads explicitly,

$$T = r \int_{-\infty}^{\infty} \frac{d\tau}{\pi^2} \tau \sinh(\tau) \Re e \left[ \log \left( 1 - e^{-r \cosh(\tau) + 2\pi i\alpha} \right) \right].$$  

(3.12)

Now, combining Eqs.(2.17), (3.10), (3.11) one obtains

$$\langle e^{a\phi} \rangle_R \sim e^{-\frac{S}{2\pi}},$$

with

$$S = S_0(\alpha) - T(r, \alpha) + T(r, 0),$$  

(3.13)

and

$$S_0 = 2\alpha^2 \log \left( \frac{m}{4} \right) + \int_{0}^{\infty} \frac{dt}{t} \left\{ \frac{\sinh^2(2\alpha t)}{t \sinh(2t)} - 2\alpha^2 e^{-2t} \right\}.$$  

(3.14)

Thus the function (3.13) coincides with the regularized Euclidean action (3.2)\textsuperscript{4}. This result was obtained by a different method in [7].
3.2. Semi-classical expansion of Q-function

To compute the VEV to one-loop order, we have to find the next term in the semi-classical expansion of $Q$. It can be obtained by means of the TBA equation (2.13). The kernel $\Phi$ (2.14) allows an expansion in $b^2$,

$$\Phi(\theta) = 2\pi \delta(\theta) + 2\pi b^2 \ P.V. \frac{\cosh(\theta)}{\sinh^2(\theta)} + O(b^4), \quad (3.15)$$

thus to lowest order the solution of the TBA equation has the form,

$$e^\epsilon = e^{r \cosh(\theta)} - 1 + O(b^2).$$

With this equation and the definition (2.15) we calculate

$$\log Q[\gamma] = -\frac{r \cosh(\gamma)}{2\pi b^2} - \frac{r \cosh(\gamma)}{2\pi} + \frac{r \gamma \sinh(\gamma)}{2\pi} - \int_{-\infty}^\infty \frac{dt}{2\pi} \log \left(1 - e^{-r \cosh(\tau)} \right) \cosh(\gamma - \tau) + O(b^2). \quad (3.16)$$

In order to see an analytical structure of this function it is instructive to represent $Q$ in the form of an infinite product,

$$\frac{1}{Q[\gamma]} = e^{\frac{r \cosh(\gamma)}{2\pi b^2}} \left(\frac{r e^{\gamma E}}{4\pi} \right)^{\frac{r \cosh(\gamma)}{2\pi}} \sqrt{2r} \times \cosh\left(\frac{\gamma}{2}\right) \prod_{n=1}^\infty \left\{ \sqrt{1 + \left(\frac{r}{2\pi n}\right)^2 + \frac{r \cosh(\gamma)}{2\pi n}} \right\} e^{-\frac{r \cosh(\gamma)}{2\pi n}} \left(1 + O(b^2)\right). \quad (3.17)$$

Here $\gamma_E$ is the Euler constant.

3.3. One-loop order

The saddle-point approximation allows one to find the one-loop order in the semi-classical expansion. To this order,

$$I_N = W \prod_{N \geq k > m \geq -N} \sinh(\rho_k - \rho_m) \prod_{k=-N}^N Q^2[\rho_k] \ e^{2\rho_k(\alpha+k)} \left(1 + O(b^2)\right). \quad (3.18)$$

Here the function $W$ is a result of the Gaussian integrations in (2.6) around the saddle points $\gamma_k = \rho_k$ (3.9),

$$W = (2\pi^2)^{N+\frac{1}{2}} \prod_{k=-N}^N \frac{1}{\sqrt{r \cosh(\rho_k)}}.$$
The main steps in the calculation of (3.18) are given in Appendix B. Our final result has the form (1.6) with the function $S$ given by (3.13) and
\[
\log D = \log D_0 + T(r, \alpha) - T(r, 0) - \alpha \partial_\alpha T(r, \alpha) - \int_r^\infty \frac{dr}{8} r \left\{ \left( \partial_r \partial_\alpha T \right)^2 - \frac{1}{2\pi^2} \left( \partial_\alpha^2 T |_{\alpha=0} \partial_\alpha^2 T - \left( \partial_\alpha^2 T \right)^2 |_{\alpha=0} \right) \right\}. 
\]
(3.19)

Here
\[
\log D_0 = -2\alpha^2 \log 2 + \frac{1}{2} \int_0^\infty dt \frac{\sinh^2(2\alpha t)}{\cosh^2(t)}. 
\]

Recall that (3.19) should coincide with the functional determinant (3.6).

4. High-temperature behavior

Now that we have computed (1.6), let us check the result for some limiting cases. Here we argue for $R \to 0$ behavior of the VEVs.

Due to the scaling properties of the interaction operator in (1.3) one can rescale the problem to a circle of circumference $2\pi$. Thus, the Hamiltonian of the model under consideration takes the form
\[
H_{shG} = \frac{2\pi}{R} \int_0^{2\pi} d\tau \left\{ 4\pi \Pi^2 + \frac{1}{16\pi} (\partial_\tau \chi)^2 + \mu \left( \frac{R}{2\pi} \right)^{2bq} \left( e^{b\chi} + e^{-b\chi} \right) \right\}, 
\]
(4.1)
where $\Pi = \frac{1}{i} \frac{\delta}{\delta \chi}$ is the momentum conjugate to $\chi = \varphi|_{x_1=0}$. The mass of the sinh-Gordon particle is related to the parameter $\mu$ by [19],
\[
\mu = -\frac{\Gamma(-b^2)}{\pi \Gamma(1 + b^2)} \left[ \frac{m \Gamma(\frac{1}{2bq}) \Gamma(1 + \frac{b}{2q})}{4\sqrt{\pi}} \right]^{2bq}.
\]
(4.2)

For $r \to 0$ and $a > 0$ the main contribution in the path integral (1.4) comes from a region of the configuration space corresponding to
\[
\chi \sim -2q \log(r) \gg 1. 
\]
(4.3)

In this region we can neglect the term $e^{-b\chi}$ in the Hamiltonian (4.1), and approximate the ground state wave functional $\Psi_0[\chi]$ by a proper wave functional from the Liouville Conformal Field Theory (CFT).
More explicitly, the Hilbert space of the Liouville CFT contains a continuous set of primary states \( |p \rangle \) parameterized by \( p > 0 \) with the conformal dimension \[ \Delta_p = p^2 + \frac{q^2}{4} . \] (4.4)

We will assume that these states are canonically normalized,
\[
\langle p' | p \rangle = 2\pi \delta(p - p') .
\]

Let \( \Psi_p[\chi] \) be a normalized wave functional corresponding to the state \( |p \rangle \). As was discussed in \[17,18\], the following relation holds
\[
\Psi_0[\chi] \approx L_p \Psi_p[\chi] \quad (4.5)
\]
in the region \((4.3)\). Here \( p = p(r) \) solves the equation,
\[
p(r) : 2pq \log \left[ \frac{r\Gamma(\frac{1}{2bq})\Gamma(1 + \frac{1}{2bq})}{8\pi^2 (b^2)^{\frac{1}{2}}} \right] = -\frac{\pi}{2} + 3m \left[ \log \left\{ \Gamma(1 + 2ip/b) \Gamma(1 + 2ipb) \right\} \right]. \quad (4.6)
\]

We emphasize that \( L_p \) is a unique coefficient provided a normalization of \( \Psi_0 \) is chosen. Therefore, one can expect the following relation for \( r \ll 1 \):
\[
\langle e^{a\varphi} \rangle_R \approx L_p L_{-p} \langle p | e^{a\varphi} | p \rangle_{\text{Liouv}} \quad R \langle \text{vac} | \text{vac} \rangle_R . \quad (4.7)
\]

The matrix element \( \langle p | e^{a\varphi} | p \rangle_{\text{Liouv}} \) was found in \[21,18\]. It reads explicitly,
\[
\langle p | e^{a\varphi} | p \rangle_{\text{Liouv}} = \left( \frac{R}{2\pi} \right)^{2a(q-a)} \left[ \frac{\pi \mu \Gamma(b^2) b^2 - 2b^2}{\Gamma(1 - b^2)} \right]^{-a/b} \frac{\Upsilon_0 \Upsilon(a) \Upsilon(2ip) \Upsilon(-2ip)}{\Upsilon^2(a) \Upsilon(a + ip) \Upsilon(a - ip)} . \quad (4.8)
\]

Here we use the notations \[18\]
\[
\log \Upsilon(a) = -\left( \frac{q}{2} - a \right)^2 \log(2b) + \int_0^\infty \frac{dt}{t} \left[ \left( \frac{q}{2} - a \right)^2 e^{-2t} - \frac{\sinh \left( (qb - 2ab)t \right)}{\sinh(2t) \sinh(2tb^2)} \right] ,
\]
and
\[
\Upsilon_0 = \partial_a \Upsilon(a) \bigg|_{a=0} .
\]

It is easy to see that the function \( p = p(r) \) \((4.6)\) satisfies the condition,
\[
\lim_{r \to 0} p(r) = 0 . \quad (4.9)
\]
Using Eqs. (4.7)-(4.9), we can derive

\[ \langle e^{a\varphi} \rangle_R \approx N^2 \left[ \frac{m \Gamma\left(\frac{1}{2}aq\right) \Gamma(1 + \frac{b}{2}q)}{4\sqrt{\pi}} \right]^{-2aq} b^{2aq} \left( \frac{R}{2\pi} \right)^{2a(a-q)} \frac{\Upsilon(2a) \Upsilon^3_3}{\Upsilon^4(a)}, \quad (4.10) \]

where

\[ N^2 = \lim_{p \to 0} \left\{ \frac{4p^2 L_p L_{-p}}{R \langle \text{vac} | \text{vac} \rangle_R} \right\} \quad (4.11) \]

does not depend on \( a \). In writing (4.10) we also used the relation (4.2). Unfortunately, the function \( N = N(r, b) \) is not known in closed form for an arbitrary value of the coupling constant. One can obtain its limiting value as \( b^2 \to 0 \). For \( b^2 \ll r \ll 1 \), it is sufficient to consider the dynamics of the zero-mode \( X \) [17,18]:

\[ X = \int_0^{2\pi} \frac{d\tau}{2\pi} \chi(\tau). \quad (4.12) \]

In this approximation, known as the mini-superspace approach [22], the Hamiltonian (4.1) is substituted by,

\[ H_{ms} = \frac{2\pi}{R} \left\{ -2 \partial_X^2 + \left( \frac{r}{4\pi b} \right)^2 \cosh(bX) \right\}. \]

The Schrödinger equation

\[ H_{ms} \Psi_0(X) = E_{ms} \Psi_0(X), \]

coincides with the modified Mathieu equation and the wave functional \( \Psi_0 \) is represented by its lowest eigenfunction. We will use the common normalization condition

\[ \int_{-\infty}^{\infty} dX \Psi_0^2(X) = 1. \quad (4.13) \]

The Liouville wave functionals \( \Psi_p \) in the mini-superspace approximation have the form,

\[ \Psi_p(X) = \frac{2}{\Gamma(2ip/b)} \left( \frac{r}{8\pi b^2} \right)^{2ip/b} K_{2ip} \left( \frac{r}{4\pi b^2} e^{bX/2} \right). \]

Here \( K_{\nu}(z) \) is the MacDonald function. Now it is clear that the mini-superspace approximation for \( N \) (4.11) can be obtained from the large \( X \) behavior of the normalized Mathieu function \( \Psi_0(X) \) (4.13),

\[ \Psi_0(X) \to \sqrt{\frac{2}{r}} \frac{\pi}{\cosh(bX/4)} \exp \left\{ -\frac{r}{2\pi b^2} \cosh(bX/2) \right\} \quad \text{as} \quad X \to \pm \infty. \quad (4.14) \]
Of concern to us is the behavior of $N_{m_n}$ in the domain $b^2 \ll r \ll 1$. In this case, we replace $\Psi_0(X)$ by its WKB asymptotic (4.14) and readily obtain,

$$\lim_{b^2 \to 0} N^2 = \frac{\sqrt{2} r^{\frac{3}{2}}}{(2\pi)^3}, \quad r \ll 1.$$ (4.15)

Having arrived at Eq. (4.15), we can straightforwardly expand (4.10) in a power series of $b^2$,

$$\langle e^{a\phi} \rangle_R \approx F \left( 1 + O(b^2) \right) \quad (b^2 \ll r \ll 1),$$ (4.16)

with

$$F = \left( \frac{R}{2\pi} \right)^{2\alpha(\alpha - 1)} 2^{-\frac{2\alpha}{\beta}} \exp \left\{ \frac{1}{2b^2} \left( 4S_0(1/2 - \alpha) - S_0(1/2 - 2\alpha) \right) \right\} \times$$

$$2^{2 + \frac{1}{2\alpha}} m^{-\frac{3}{4\alpha}} A_G^{-\frac{\alpha^2}{\beta}} \left( \frac{r}{4\pi} \right)^{\frac{3}{4} - 2\alpha} \sqrt{\frac{\Gamma(1 - 2\alpha)}{\Gamma(2\alpha)}} \frac{\Gamma^2(\alpha)}{\Gamma^2(1 - \alpha)},$$

where $S_0$ is given by Eq. (3.14) and $A_G$ is the Glaisher constant. It is possible to show that the high-temperature expansion of (1.6) exactly matches (4.16) (see Appendix B for some details).

5. Low-temperature expansion

We have mentioned in the Introduction that the finite volume VEVs can be understood as thermal averages (1.2). Hence, $\langle e^{a\phi} \rangle_R$ admits the low-temperature ($R \to \infty$) expansion in the form,

$$\log \left( \langle e^{a\phi} \rangle_R / \mathcal{G}_a \right) = 1 + \sum_{k=1}^{\infty} G_k(r).$$ (5.1)

Here $G_k$ represents $k$-particle contributions in the infinite-volume channel and

$$G_k(r) \sim e^{-kr}.$$ 

Recently A. Leclair and G. Mussardo [6] proposed an integral representation which is sufficient to generate $G_k(r)$ systematically in terms of form-factors of the field $e^{a\phi}$ at $R = \infty$ and the solution of the TBA equation (2.13). Taking into account contributions of one- and two-particle states to the trace (1.2), they obtained

$$\log \left( \langle e^{a\phi} \rangle_R / \mathcal{G}_a \right) = 4[a] \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} f_-(-\theta) +$$

$$[2a] \int_{-\infty}^{\infty} \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \frac{\Phi(\theta_1 - \theta_2)}{\cosh(\theta_1 - \theta_2)} f_-(\theta_1)f_-(\theta_2) + \ldots,$$ (5.2)
where the notations

\[ f_- (\theta) = \frac{1}{1 + e^{\epsilon(\theta)}} \]

and

\[ [a] = \frac{\sin^2 \left( \frac{\pi a}{q} \right)}{\sin \left( \frac{\pi b}{q} \right)} \]

are used. The function \( \Phi \) is the kernel in the TBA equation (2.13). With (5.2) and the TBA equation one can calculate the first two terms in the low-temperature expansion (5.1):

\[
G_1 = \frac{4[a]}{\pi} K_0(r) ,
\]

\[
G_2 = 4[a] \int_{-\infty}^{\infty} \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \left( \Phi(\theta_1 - \theta_2) - 2\pi \delta(\theta_1 - \theta_2) \right) e^{-r \cosh \theta_1 + r \cosh \theta_2} + \]

\[ [2a] \int_{-\infty}^{\infty} \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \frac{\Phi(\theta_1 - \theta_2)}{\cosh(\theta_1 - \theta_2)} e^{-r \cosh \theta_1 + r \cosh \theta_2} , \]

where \( K_n(r) \) is the MacDonald function. We now expand (5.3) as a power series in \( b^2 \),

\[
G_1 = 4 \left\{ \frac{s^2(\alpha)}{b^2} + s^2(\alpha) - \alpha s(2\alpha) + O(b^2) \right\} K_0(r) ,
\]

where

\[ s(\alpha) = \frac{\sin(\pi \alpha)}{\pi} . \]

To expand \( G_2 \) one needs to use Eq.(3.15),

\[
G_2 = \left\{ \frac{s^2(2\alpha)}{b^2} + s^2(2\alpha) - 2\alpha s(4\alpha) \right\} K_0(2r) -
\]

\[ 4 s^2(\alpha) r^2 \left( K_1^2(r) - K_0^2(r) \right) - s^2(2\alpha) r^2 \left( K_2(r) K_0(r) - K_1^2(r) \right) + O(b^2) . \]

It is quite straightforward to verify that the low-temperature expansion of (1.6) exactly reproduces (5.4) and (5.5).

6. Conclusion

The proposed \( \gamma \)-representation (2.17) is the main result of this paper. Its rigorous derivation has not yet been achieved. Although (2.17) are conjectures, the evidence presented in this paper appears to make it reasonable to take them as the starting point for further investigation.
One can expect that similar representations exist for non-minimal CFT, say, the Liouville theory and $SL(2,\mathbb{R})/U(1)$ non-compact $\sigma$-model. It may cast new light on many unsolved problems of 2D Quantum Gravity. In this connection an intriguing similarity between the integrals appeared in the $\gamma$-representation and Matrix Models of 2D Quantum Gravity [23,24,25] can be mentioned.

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Note added

After finishing this paper it was drawn to my attention that Al.B. Zamolodchikov independently introduced and studied the function (2.15) in Ref.[26]. I am grateful to him for the communication of that paper, and sharing insights.

Appendix A.

The QFT defined by (1.3) possesses infinitely many local IM $\hat{I}_{2n-1}$ whose vacuum eigenvalues have appeared in the equation (2.11). They can be represented in the form,

$$\hat{I}_{2n-1} = \int_0^R \frac{dx_2}{2\pi} \left( T_{2n}(x_2 + ix_1, x_2 - ix_1) + \Theta_{2n-2}(x_2 + ix_1, x_2 - ix_1) \right),$$

where the local fields $T_{2n}$ and $\Theta_{2n-2}$ satisfy the continuity equations,

$$\partial_z T_{2n}(z, \bar{z}) = \partial_{\bar{z}} \Theta_{2n-2}(z, \bar{z}).$$

Although a general expression for the densities $T_{2n}$, $\Theta_{2n-2}$ is not known, they are determined up to normalization by the commutativity conditions,

$$[ \hat{I}_{2n-1}, \hat{I}_{2m-1} ] = 0.$$

In Refs.[15,27] the following normalization was adopted,

$$T_{2n} = 2^{-2n} (\partial_z \varphi)^{2n} + \ldots ,$$

(A.1)
where omitted terms contain higher derivatives of $\varphi$ and exponential fields. Notice that the condition (A.1) does not depend on the regularization scheme defining the composite field $(\partial_\varphi)^{2n}$. With normalization (A.1) the constants $C_n$ (2.11) were found in Refs. [28,15],

$$C_n = \frac{\Gamma\left(\frac{2n-1}{2q}\right) \Gamma\left(\frac{2n-1}{2bq}\right)}{2 \sqrt{\pi} n! q} \left[ \frac{m \Gamma\left(\frac{b}{2q}\right) \Gamma\left(\frac{1}{2q}\right)}{8 q \sqrt{\pi}} \right]^{1-2n}. \quad (A.2)$$

The local IM $\hat{I}_{2n-1}$ are certain deformations of the local IM of the Liouville CFT. Let $I_{2n-1}(p)$ be an eigenvalue of the Liouville local IM corresponding to the state $|p\rangle$ (4.4), while $\hat{I}_{2n-1}$ is the sinh-Gordon ground state eigenvalues of $\hat{I}_{2n-1}$. Eq. (4.5) suggests the following relation for $r \ll 1$:

$$\hat{I}_{2n-1} = I_{2n-1}(p(r)) + O(r^{4q}b, r^{4q/b}) \quad (A.3)$$

where $p(r)$ solves (4.6). The power corrections in $r$ (A.3) can be obtained by means of Conformal Perturbation Theory. Explicit forms of the functions $I_{2n-1}(p)$ (for $n = 1, \ldots, 8$) are given in Appendix B of Ref. [27]. Here we present only the first two of them,

$$I_1(p) = \frac{2\pi}{R} \left( p^2 - \frac{1}{24} \right),$$
$$I_3(p) = \left( \frac{2\pi}{R} \right)^3 \left( p^4 - \frac{p^2}{4} + \frac{4b^4 + 17b^2 + 4}{960b^2} \right).$$

In Tables 1-4 we list numerical values of the local IM $\hat{I}_{2n-1}$ for some $0.01 \leq r \leq 1$ and $b^2 = 0.81$ which were obtained by means of numerical solution of the TBA equation (2.13) with use of Eqs. (2.12) and (A.2). These data are compared against values of the Liouville local IM $I_{2n-1}(p(r))$. We consider the content of Tables 1-4 to be an impressive evidence in support of the relation (2.12).

---

5 For $n = 1$ this relation was discussed in Ref. [18].
| $r$  | $\mathbb{I}_1$     | $I_1((p(r))$ |
|------|-----------------|--------------|
| 1.0  | -0.0059897196942029 | -0.0059891933248581 |
| 0.8  | -0.0123637695005731 | -0.0123636397906571 |
| 0.6  | -0.018395019094376 | -0.0183954816200409 |
| 0.4  | -0.0242191017573388 | -0.0242191003738075 |
| 0.2  | -0.0301582132435655 | -0.0301582132312211 |
| 0.1  | -0.0335250692109914 | -0.0335250692108919 |
| 0.01 | -0.0381898149656469 | -0.0381898149656469 |

| $r$  | $\mathbb{I}_3$ | $I_3((p(r))$ |
|------|----------------|--------------|
| 1.0  | 0.01857760504756 | 0.01858088002375 |
| 0.8  | 0.01975951264163 | 0.01976027692024 |
| 0.6  | 0.0209510471765  | 0.0209511804696 |
| 0.4  | 0.02216988492643 | 0.0221698925147 |
| 0.2  | 0.02348269733500 | 0.02348269739676 |
| 0.1  | 0.02425825249985 | 0.02425825250033 |

| $r$  | $\mathbb{I}_5$ | $I_5((p(r))$ | $\mathbb{I}_7$ | $I_7((p(r))$ |
|------|----------------|--------------|--------------|--------------|
| 1.0  | -0.02830178173 | -0.02830149724 | 0.0731035360 | 0.0731032652 |
| 0.8  | -0.02936366562 | -0.02936359281 | 0.0750097817 | 0.0750097140 |
| 0.6  | -0.03040878256 | -0.03040877074 | 0.0768644292 | 0.0768644184 |
| 0.4  | -0.03145618592 | -0.03145618508 | 0.0787032289 | 0.0787032281 |
| 0.2  | -0.03256438579 | -0.03256438578 | 0.0806285556 | 0.0806285556 |

| $r$  | $\mathbb{I}_9$ | $I_9((p(r))$ | $\mathbb{I}_{11}$ | $I_{11}((p(r))$ |
|------|----------------|--------------|-------------|-------------|
| 1.0  | -0.29532264 | -0.29532199 | 1.73235 | 1.73235 |
| 0.8  | -0.30116225 | -0.30116209 | 1.75983 | 1.75983 |
| 0.6  | -0.30680503 | -0.30680500 | 1.78627 | 1.78627 |
| 0.4  | -0.31236318 | -0.31236317 | 1.81220 | 1.81220 |
| 0.2  | -0.31814537 | -0.31814537 | 1.83907 | 1.83907 |

Tables 1-4. Comparison of the LHS and RHS of equation (A.3) ($b^2 = 0.81$).
Appendix B.

Here we proceed with calculation of the products in (3.18) and give some technical hints on the study of their high-temperature behavior. First, let us consider the product,

\[ e^{M_1} = W \prod_{N \geq k > j \geq -N} \sinh(\rho_k - \rho_m) \, . \]

Using the relation

\[ \partial_r \rho_k = -\frac{1}{r} \tanh(\rho_k) \, , \]

which follows from the saddle-point equations (3.9), one obtains

\[ \partial_r M_1 = -\frac{1}{2r} \sum_{k, m = -N}^{N} \frac{\cosh(\rho_k - \rho_m)}{\cosh(\rho_k) \cosh(\rho_m)} \, . \]

This sum can be rewritten in the form,

\[ \partial_r M_1 = -\frac{(2N + 1)^2}{2r} + \frac{1}{2r} \left( \sum_{k = -N}^{N} \tanh(\rho_k) \right)^2 \, . \] (B.1)

Notice that the sum in (B.1) converges for \( N \to \infty \),

\[ \lim_{N \to \infty} \sum_{k = -N}^{N} \tanh(\rho_k) = -\frac{r}{2} \partial_r \partial_\alpha T + 2\alpha \, . \]

To derive this formula we used (3.11) and the saddle-point equations (3.9). Thus we obtain,

\[ M_1 = M_N - \frac{(2N + 1)^2}{2} \log \left( \frac{r}{4\pi} \right) - 2\alpha^2 \log \left( \frac{4\pi N}{r} \right) - \alpha \partial_\alpha T(\alpha) - \int_r^\infty \frac{dr}{8} \left( \partial_r \partial_\alpha T \right)^2 + O(N^{-1}) \, . \] (B.2)

The constant \( M_N \) here does not depend on \( r \). To find how it depends on \( \alpha \), let us consider \( \partial_\alpha M_1 \). A similar calculation leads to the equation,

\[ \partial_\alpha M_1 = \frac{1}{4} \left( r \partial_r \partial_\alpha T - 4\alpha \right) \left( \partial_\alpha^2 T + 4 \log(4\pi N/r) \right) + \frac{2\pi}{r} \sum_{m = -N}^{N} \frac{\tanh(\rho_m)}{\cosh(\rho_0) + \cosh(\rho_m)} + \frac{2\pi}{r} \sum_{m = -N}^{N} \sum_{k = 1}^{N} \left[ \frac{\tanh(\rho_k(\alpha))}{\cosh(\rho_k(\alpha)) + \cosh(\rho_m)} - \frac{\tanh(\rho_k(-\alpha))}{\cosh(\rho_k(-\alpha)) + \cosh(\rho_m)} \right] \, . \] (B.3)
It follows immediately from the last equation that
\[ \partial_{\alpha} M_1 \bigg|_{r \to \infty} = -4 \alpha \log \left( \frac{4\pi N}{r} \right). \]

Therefore, we conclude that the constant \( M_N \) in (B.3) does not depend on \( \alpha \). Notice that Eq. (B.3) is very convenient for studying the high-temperature limit \( r \to 0 \). It is straightforward to show that for \( \alpha > 0 \),
\[ e^{M_1} \bigg|_{r \to 0} \rightarrow \left( \frac{4\pi}{r} \right)^{2N(N+1)} N^{-2\alpha^2} e^{M_N} \times \]
\[ \frac{2^{-2\alpha}}{\Gamma \left( \frac{1}{2} + \alpha \right)} \sqrt{2\pi} \Gamma(\alpha) e^{-2\alpha^2 \gamma} \prod_{k=1}^{\infty} \frac{\Gamma^2 \left( \frac{k+1}{2} \right) e^{2\alpha^2}}{\Gamma \left( \frac{k+1}{2} - \alpha \right) \Gamma \left( \frac{k+1}{2} + \alpha \right)}. \]

To finish the calculation of (3.18) one needs to evaluate the product,
\[ e^{M_2} = \prod_{k=-N}^{N} Q^2[\rho_k] e^{2\rho_k(\alpha+k)}. \quad (B.4) \]

The first two terms of the semi-classical expansion for \( Q \) are given by (3.16). With the saddle-point equation (3.9) \( M_2 \) in (B.4) can be written as,
\[ M_2 = M'_2 + M''_2, \quad (B.5) \]
where
\[ M'_2 = - \left( b^{-2} + 1 \right) \sum_{k=-N}^{N} \left\{ \frac{r}{\pi} \cosh(\rho_k) - 2 \rho_k (k + \alpha) \right\} \quad (B.6) \]
and
\[ M''_2 = -2 \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} \log \left( 1 - e^{-r \cosh(\tau)} \right) \sum_{k=-N}^{N} \frac{1}{\cosh(\rho_k - \tau)}. \quad (B.7) \]
The sum \( M'_2 \) is evaluated by means of Eq. (3.11). To calculate \( M''_2 \) we note that
\[ \partial_r M''_2 = -2 \sum_{k=-N}^{N} \frac{1}{\cosh(\theta_k)} \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} m e^{r \cosh(\tau) - 1}. \]

With the relations
\[ \sum_{k=-N}^{N} \frac{1}{\cosh(\theta_k)} = \frac{r}{2\pi} \sum_{k=-N}^{N} \partial_\alpha \theta_k = \frac{r}{4\pi} \left\{ \partial^2_\alpha T + 4 \log \left( \frac{4\pi N}{r} \right) \right\} + O \left( \frac{1}{N} \right). \]
\[ \int_{-\infty}^{\infty} \frac{d\tau}{e^{r \cosh(\tau)} - 1} = \frac{1}{4} \partial_{\alpha}^2 T|_{\alpha=0}, \]
onumber

one obtains,

\[ M''_2 = \int_{r}^{\infty} \frac{dr}{16\pi^2} r \partial_{\alpha}^2 T|_{\alpha=0} \partial_{\alpha}^2 T + \int_{r}^{\infty} \frac{dr}{4\pi^2} r \partial_{\alpha}^2 T|_{\alpha=0} \log \left( \frac{4\pi N}{r} \right). \quad (B.8) \]

We specify the integration constant here using the condition,

\[ M''_2|_{r \to \infty} \to 0, \]

which follows from the definition (B.7). The function \( T \) (3.12) satisfies Laplace’s equation,

\[ r^{-1} \partial_r \left( r \partial_r T \right) + \frac{1}{4\pi^2} \partial_{\alpha}^2 T = 0. \]

This allows one to calculate the second integral in (B.8). Thus we find,

\[ M''_2 = \int_{r}^{\infty} \frac{dr}{16\pi^2} r \partial_{\alpha}^2 T|_{\alpha=0} \partial_{\alpha}^2 T + T|_{\alpha=0} + r \partial_r T|_{\alpha=0} \log \left( \frac{4\pi N}{r} \right). \]

Finally we note that the most efficient way to study the \( r \to 0 \) limit of \( M_2 \) (B.4) is based on the representation (3.17). It shows that for \( r \to 0 \) with \( r \cosh(\gamma) \) fixed,

\[ Q[\gamma]|_{r \to 0} \to \frac{\Gamma(1 + \frac{r \cosh(\gamma)}{2\pi})}{\sqrt{2r \cosh(\gamma/2)}} \left( \frac{r}{4\pi} \right)^{-\frac{r \cosh(\gamma)}{2\pi}} \left( 1 + O(b^2) \right). \]

Hence, examination of the product (B.4) at this limit creates no difficulties at all.
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24