Abstract

We introduce the notion of mild supersolution for an obstacle problem in an infinite dimensional Hilbert space. The minimal supersolution of this problem is given in terms of a reflected BSDEs in an infinite dimensional Markovian framework. The results are applied to an optimal control and stopping problem.

1 Introduction

The connection between backward stochastic differential equations in $\mathbb{R}^n$ and semilinear parabolic PDEs is known since the seminal paper of Pardoux and Peng [18]. This result was extended to the case of reflected BSDEs and correspondingly of obstacle problem for PDEs in [6]. Moreover it is also well known that the above equations are related to optimal stochastic control problems (in the first case) and optimal stopping or optimal control/stopping problems in the second see [19]. We notice that in the finite dimensional framework the above mentioned partial differential equations are intended either in classical sense (see [18]) or, more frequently, in viscosity sense.

On the other hand the relation between backward stochastic differential equations in infinite dimensional spaces, optimal control of Hilbert valued stochastic evolution equations and parabolic equation on infinite dimensional spaces was investigated in [8] and in several successive papers. In the above mentioned literature it appears that the concept of solution of the PDE has to be modified in the infinite dimensional case. Namely classical solutions require too much regularity while the theory of viscosity solutions can be applied only in special cases with trace class noise and very regular value function (see [14]). The type of definition that was seen to fit the infinite dimensional framework and the BSDE approach is the classical notion of mild solution. Namely if we consider a semilinear parabolic PDE such as

$$\begin{cases}
\frac{\partial u}{\partial t}(t,x) = \mathcal{L}_t u(t,x) + \psi(t,x,u(t,x),\nabla u(t,x)) \\
u(T,x) = \phi(x),
\end{cases} \quad t \in [0,T], \; x \in H$$

and $(P_{s,t})_{0 \leq s \leq t \leq T}$ is the transition semigroup related to the second order differential operators $(\mathcal{L}_t)_{t \in [0,T]}$ then a function $u : [0, T] \times H \to \mathbb{R}$ is called a mild solution of the above PDE whenever
setting a gradient (in a suitable sense) and it holds:
\[ u(s, x) = P_{s,t}[u(t, \cdot)](x) + \int_s^t P_{s,\tau} \left[ \psi(\tau, \cdot, u(\tau, \cdot), \nabla u(\tau, \cdot)) \right](x) \, d\tau. \]

Large amount of literature has then extended the BSDE approach to control problems to several different situations both in the finite and in the infinite framework but, at our best knowledge, the problem of relating reflected BSDEs in infinite dimensional spaces and obstacle problems for PDEs with infinitely many variables was never investigated. The point is that it is not obvious how one should include the reflection term (which is not absolutely continuous with respect to Lebesgue measure on \([0, T]\)) into the definition of mild solution.

In this paper, inspired by the work of A. Bensoussan see [2], to overcome such a difficulty, we propose the notion of \textit{mild-supersolution} (see Definition 1.2). To be more specific, our main result will be to prove that if \((X^{s,x}, Y^{s,x}, Z^{s,x}, K^{s,x})\) is the solution of the following forward backward system with reflected BSDE:
\[
\begin{cases}
    dX^{s,x}_t = AX^{s,x}_t + F(t, X^{s,x}_t) \, dt + G(t, X^{s,x}_t) \, dW_t & t \in [s, T] \\
    X^{s,x}_s = x, \\
    -dY^{s,x}_t = \psi(t, X^{s,x}_t, Y^{s,x}_t, Z^{s,x}_t) \, dt + dK^{s,x}_t - Z^{s,x}_t \, dW_t, & t \in [0, T], \\
    Y^{s,x}_T = \phi(X^{s,x}_T), \\
    Y^{s,x}_t \geq h(X^{s,x}_t), \\
    \int_0^T (Y^{s,x}_t - h(X^{s,x}_t)) \, dK^{s,x}_t = 0.
\end{cases}
\]

setting \(u(t, x) := Y^{t,x}_t\) then \(u\) is the minimal mild supersolution of the obstacle problem
\[
\begin{cases}
    \min \left( u(t, x) - h(x), -\frac{\partial}{\partial t}(t, x) - L_t u(t, x) - \psi(t, x, u(t, x), \nabla u(t, x) G(t, x)) \right) \geq 0 & t \in [0, T], \ x \in H \\
    u(T, x) = \phi(x),
\end{cases}
\tag{1.1}
\]

Another issue that is considered in this paper is that we do not assume any nondegeneracy on the coefficient \(G\) (and consequently any strong ellipticity on the second order differential operator in the PDE). Therefore we can not expect to have regular solutions of the obstacle problem. Thus we have to precise how the directional gradient \(\nabla uG\) has to be intended. We choose here to employ the definition of generalized gradient (in probabilistic sense) introduced in [11]. It was proved in [11] that such generalized gradient exists for all locally Lipschitz functions. In Theorem 2.9 we prove that our candidate solution \(u(t, x) := Y^{t,x}_t\) is indeed locally Lipschitz). Moreover we notice that we work under general growth assumptions with respect to \(x\) on the nonlinear term \(\psi\) and on the final datum \(\phi\). This forces us to obtain \(L^p\) estimates on the solution on the reflected BSDE that extend the ones proved in [6].

The structure of the paper is the following. In section 2 we study reflected BSDEs obtaining the desired \(L^p\) estimates and the local lipschitzianity with respect to the initial datum in the markovian framework. In section 3 we introduce the notion of minimal mild supersolution of the obstacle problem in the sense of the generalized gradient and we show how it is related to the reflected BSDEs. Finally in section 4 we apply the above results to an optimal control and stopping problem.

## 2 Reflected BSDEs

In a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) we consider a cylindrical Wiener process \(\{W_\tau, \tau \geq 0\}\) in a Hilbert space \(\Xi\) and \((\mathcal{F}_\tau)_{\tau \geq 0}\) is its natural filtration, augmented in the usual way. We consider
the following reflected backward stochastic differential equation (RBSDE in the following):

\[
\begin{cases}
  dY_t = -f(t, Y_t, Z_t) \, dt - dK_t + Z_t \, dW_t, & t \in [0,T], \\
  Y_T = \xi, \\
  Y_t \geq S_t, \\
  \int_0^T (Y_t - S_t) \, dK_t = 0
\end{cases}
\tag{2.1}
\]

for the unknown adapted processes \(Y, Z\) and \(K\). \(Y\) and \(K\) are real processes, and \(Z\) is a \(\Xi^*\)-valued process. \(Y\) and \(Z\) are square integrable processes, \(Y\) admits a continuous modification and \(K\) is a continuous non-decreasing process with \(K_0 = 0\). The equation is understood in the usual integral way, namely:

\[
Y_t + \int_t^T Z_r \, dW_r = \xi + \int_t^T f(r, Y_r, Z_r) \, dr + K_T - K_t, \quad t \in [0,T], \, \mathbb{P} \text{-a.s.}
\tag{2.2}
\]

We also consider equation (2.1) with \(f\) not depending on \((y, z)\):

\[
\begin{cases}
  dY_t = -f(t) \, dt - dK_t + Z_t \, dW_t, & t \in [0,T], \\
  Y_T = \xi, \\
  Y_t \geq S_t, \\
  \int_0^T (Y_t - S_t) \, dK_t = 0
\end{cases}
\tag{2.3}
\]

In the following, if \(E\) is a separable Hilbert space, \(0 < a < b\) and \(p \geq 1\) we denote by \(L^p_p(\Omega \times [a,b], E)\) the space of \(E\)-valued \(\mathcal{F}_t\)-predictable processes \(\ell\) s.t.

\[\mathbb{E} \int_a^b |\ell(t)|^p \, dt < \infty.\]

If \(E = \mathbb{R}\) we write \(L^p_p(\Omega \times [a,b])\) instead of \(L^p_p(\Omega \times [a,b], \mathbb{R})\).

Moreover by \(L^p_p(\Omega, C([a,b], E))\) we denote the subspace of \(L^p_p(\Omega \times [a,b], E)\) given by processes admitting a continuous version and verifying

\[\mathbb{E} \sup_{t \in [a,b]} |\ell(t)|^p < \infty.\]

An analogous definition is given to \(L^p_p(\Omega, C([a,b]))\) It proved in [3, proposition 5.1], that if \(f \in L^2_p(\Omega \times [0,T]), \xi \in L^2(\Omega)\) and \(\sup_{t \in [0,T]} S_t^+ \in L^2(\Omega)\) then equation (2.3) admits a unique solution \((Y, Z, K)\) with \((Y, Z) \in L^2_p(\Omega \times [0,T]) \times L^2_p(\Omega \times [0,T], H)\), moreover \(Y\) admits a continuous version and \(\mathbb{E} \sup_{t \in [0,T]} |Y|^2 < \infty;\) finally \(K_T \in L^2(\Omega)\).

In the following we need to prove regular dependence of the solution to the above equation with respect to parameters, namely the initial data of a related (forward) stochastic differential equation. Due to the assumptions that we choose on the nonlinearity \(\psi\) we will need \(L^p\) estimates (both on the solution and on its approximations corresponding to suitable penalized approximating equations).

We make the following assumptions on the generator, on the final datum and on the obstacle of the RBSDE (2.1):

**Hypothesis 2.1** \(f : (\Omega \times [0,T]) \times \mathbb{R} \times \Xi \rightarrow \mathbb{R}\) is measurable with respect to \(\mathcal{P} \times \mathcal{B}(\mathbb{R} \times \Xi^*)\) (where by \(\mathcal{P}\) we mean the predictable \(\sigma\)-algebra on \(\Omega \times [0,T]\), and by \(\mathcal{B}(\Lambda)\) the Borel \(\sigma\)-algebra on any topological space \(\Lambda\)).
Moreover \( f \) is Lipschitz with respect to \( y \) and \( z \) uniformly in \( t \) and \( \omega \), and for some \( p \geq 2 \)

\[
\mathbb{E} \int_0^T |f(t,0,0)|^p < \infty
\]

The final data \( \xi \) is \( \mathcal{F}_T \) measurable and \( p \)-integrable.
Finally the obstacle \( S \) is a continuous, \( \mathcal{P} \)-measurable, real valued process satisfying

\[
\mathbb{E} \sup_{t \in [0,T]} |S_t|^{2p-2} < \infty.
\]

We notice that the integrability requests are not optimal (for instance we assume \( p \)-integrability jointly in \( \Omega \times [0,T] \)) for the generator \( f \) and \( 2(p-1) \) integrability for the obstacle \( S \). Nevertheless such assumptions are verified in the Markovian framework (see Section 2.1) and will allow us to treat general obstacle problems under general assumptions (see Section 3).

By a penalization procedure, we can prove the following:

**Theorem 2.2** If hypothesis 2.1 holds true, equation (2.1) admits a unique adapted solution \((Y,Z,K)\) such that \( Y \) admits a continuous version and \( K \) is non decreasing with \( K_0 = 0 \). Moreover \((Y,Z,K)\) satisfy

\[
\mathbb{E} \sup_{t \in [0,T]} |Y_t|^p + \mathbb{E} \left( \int_0^T |Z_t|^2 \, dt \right)^{p/2} + \mathbb{E} |K_T|^p 
\leq C \mathbb{E} |\xi|^p + C \mathbb{E} \int_0^T |f(t,0,0)|^p \, dt + C \left( \mathbb{E} \sup_{t \in [0,T]} |S_t|^{2p-2} \right)^{p/(2p-2)}.
\]

Where \( C \) only depends on \( T \) and on the Lipschitz constant of \( f \).

We first need an analogous result on the corresponding penalized equation, that we now introduce. Let us consider the following BSDE

\[
\begin{align*}
- dY^n_t &= f(t,Y^n_t, Z^n_t) \, dt + n(Y^n_t - S_t)^- \, dt - Z^n_{t \cdot} \, dW_t, & t \in [0,T], \\
Y^n_0 &= \xi.
\end{align*}
\]

It is shown in [3] that the penalized BSDE (2.5) admits a unique solution \((Y^n, Z^n)\) in \( L^p_\mathbb{P}(\Omega,C([0,T])) \times L^p_\mathbb{P}(\Omega \times [0,T], \mathcal{F}) \), whose norm (in the above spaces) is uniformly bounded with respect to \( n \). Moreover such a solution \((Y^n, Z^n)\) converges in \( L^p_\mathbb{P}(\Omega,C([0,T])) \times L^p_\mathbb{P}(\Omega \times [0,T], \mathcal{F}) \) to \( Y^{s,x}, Z^{s,x} \), solution of the RBSDE. Next we want to prove an \( L^p \)-estimate, uniform with respect to \( n \).

**Proposition 2.3** If hypothesis 2.1 holds true then equation (2.1) admits a unique adapted solution \((Y^n, Z^n)\) such that \( Y^n \) admits a continuous version. Moreover \((Y,Z,K)\) satisfy

\[
\mathbb{E} \sup_{t \in [0,T]} |Y^n_t|^p + \mathbb{E} \left( \int_0^T |Z^n_t|^2 \, dt \right)^{p/2} 
\leq C \mathbb{E} |\xi|^p + C \mathbb{E} \int_0^T |f(t,0,0)|^p \, dt + C \left( \mathbb{E} \sup_{t \in [0,T]} |S_t|^{2p-2} \right)^{p/(2p-2)}.
\]

Finally if \( K^n_t = n \int_0^t (Y^n_s - S_s)^- \, ds \) then \( K^n \) is an adapted, continuous, non-decreasing process satisfying

\[
\mathbb{E}|K^n_T|^p \leq C \mathbb{E}|\xi|^p + C \mathbb{E} \int_0^T |f(t,0,0)|^p \, dt + C \left( \mathbb{E} \sup_{t \in [0,T]} |S_t|^{2p-2} \right)^{p/(2p-2)}
\]

where \( C \) only depends on \( p, T \) and on the Lipschitz constant of \( f \).
Proof. First of all we notice that we can always reduce ourselves to the case in which
\[
\frac{y}{|y|} f(t, y, z) \leq |f(t, 0, 0)| + \mu |y| + \lambda |z| \quad \text{with } \mu + \lambda^2 \leq 0. \tag{2.8}
\]
Indeed, setting \(\tilde{Y}_t^n = e^{at}Y_t^n\), \(\tilde{Z}_t^n = e^{at}Z_t^n\), we get that \((\tilde{Y}_t^n, \tilde{Z}_t^n)\) satisfies
\[
\begin{cases}
-d\tilde{Y}_t^n = e^{at}f(t, e^{-at}\tilde{Y}_t^n, e^{-at}\tilde{Z}_t^n) dt - a\tilde{Y}_t^n dt + n(\tilde{Y}_t^n - \tilde{S}_t)) dt \\
-\tilde{Z}_t^n dW_t, \\
Y_T^n = \xi.
\end{cases}
\]
So the generator is given by
\[
\tilde{f}(t, y, z) := e^{at} f(t, e^{-at}y, e^{-at}z) - ay,
\]
so by choosing \(a\) sufficiently large (depending only on the Lipschitz constant of \(f\)) we can assume \(\mu + \lambda^2 \leq -1\). From now on we assume that \((2.8)\) holds true and for simplicity we omit the superscript \(\sim\) where necessary.

Moreover by \(c\) we shall denote a constant that depends only on the Lipschitz constant of \(f\), \(T\) and \(p\) and by \(c(\delta)\) a constant that depends, beside the above parameters, on an auxiliary constant \(\delta > 0\). Their value can change from line to line. We apply Itô formula to \(|Y_t^n|^p\) for \(s \leq t \leq T\) and we get,
\[
-d|Y_t^n|^p = p|Y_t^n|^{p-1}\dot{Y}_t^n f(t, Y_t^n, Z_t^n) dt + pm|Y_t^n|^{p-1}\dot{Y}_t^n (Y_t^n - S_t)^- dt \\
- p|Y_t^n|^{p-1}\dot{Y}_t^n Z_t^n dW_t - \frac{p(p-1)}{2}|Y_t^n|^p |Z_t^n|^2 dt.
\]
where \(\dot{Y}_t^n := \frac{Y_t^n}{|Y_t^n|}\). Integrating between \(s\) and \(T\), \(0 \leq s \leq t \leq T\), we get
\[
|Y_s^n|^p + \frac{p(p-1)}{2} \int_s^T |Y_t^n|^{p-2} |Z_t^n|^2 dt \\
= |\xi|^p + p \int_s^T |Y_t^n|^{p-1}\dot{Y}_t^n f(t, Y_t^n, Z_t^n) dt + np \int_s^T |Y_t^n|^{p-1}\dot{Y}_t^n (Y_t^n - S_t)^- dt \\
- p \int_s^T |Y_t^n|^{p-1}\dot{Y}_t^n Z_t^n dW_t \\
\leq |\xi|^p + p \int_s^T |Y_t^n|^{p-1}|f(t, 0, 0)| dt + p\mu \int_s^T |Y_t^n|^p dt + p\lambda \int_s^T |Y_t^n|^{p-1} |Z_t^n| dt \\
+ np \int_s^T |S_t|^{p-1} (Y_t^n - S_t)^- dt - p \int_s^T |Y_t^n|^{p-1}\dot{Y}_t^n Z_t^n dW_t \\
\leq |\xi|^p + c \int_s^T |f(t, 0, 0)|^p dt + p \int_s^T |Y_t^n|^p + p\mu \int_s^T |Y_t^n|^p dt + \frac{p\lambda^2}{(p-1)} \int_s^T |Y_t^n|^p dt \\
+ \frac{p(p-1)}{4} \int_s^T |Y_t^n|^{p-2} |Z_t^n|^2 dt + \sup_{t \in [s, T]} |S_t|^{p-1} n \int_s^T (Y_t^n - S_t)^- dt \\
- p \int_s^T |Y_t^n|^{p-1}\dot{Y}_t^n Z_t^n dW_t,
\]
where we have applied Young inequality. So recalling that by (2.8) $\mu + \lambda^2 \leq 0$ and also since $p \geq 2$, we get

$$
|Y^n_s|^p + \frac{p(p-1)}{4} \int_s^T |Y^n_t|^p - 2|Z^n_t|^2 \, dt
\leq |\xi|^p + c \int_t^T |f(t,0,0)|^p \, dt
+ \sup_{t \in [s,T]} |S_t|^{p-1} \int_s^T (Y^n_t - S_t)^- \, dt - c \int_s^T |Y^n_t|^p - 1 \tilde{Y}_t^n Z^n_t \, dW_t
$$

By the penalized BSDE (2.5) in integral form we deduce that

$$
\int_s^T n(Y^n_t - S_t)^- \, dt = -\xi + Y^n_n - \int_s^T f(t,Y^n_t, Z^n_t) \, dt + \int_s^T Z^n_t \, dW_t,
$$

and so

$$
|Y^n_s|^p + \frac{p(p-1)}{4} \int_s^T |Y^n_t|^p - 2|Z^n_t|^2 \, dt
\leq |\xi|^p + c \int_s^T |f(t,0,0)|^p \, dt - p \int_s^T |Y^n_t|^p - 1 \tilde{Y}_t^n Z^n_t \, dW_t
+ \sup_{t \in [s,T]} |S_t|^{p-1} \left( -\xi + Y^n_s - \int_s^T f(t,Y^n_t, Z^n_t) \, dt + |\int_s^T Z^n_t \, dW_t| \right)
\leq |\xi|^p + c \int_s^T |f(t,0,0)|^p \, dt - p \int_s^T |Y^n_t|^p - 1 \tilde{Y}_t^n Z^n_t \, dW_t + c \sup_{t \in [s,T]} |S_t|^p + \frac{1}{2} |Y^n_s|^p
+ \sup_{t \in [s,T]} |S_t|^{p-1} \left( |\xi| + |\int_s^T f(t,Y^n_t, Z^n_t) \, dt| + |\int_s^T Z^n_t \, dW_t| \right)
$$

Now we recall that, by the $L^p$-estimates on BSDEs, see e.g. [8], $(Y^n, Z^n) \in L_P^p(\Omega, C([0,T])) \times L_P^p(\Omega, L^2([0,T], \mathbb{X}))$, and so the Itô integral $\int_s^T |Y^n_t|^p - 1 \tilde{Y}_t^n Z^n_t \, dW_t$ has null expectation. Computing expectation in the above inequality

$$
\frac{1}{2} \mathbb{E}|Y^n_s|^p + \frac{p(p-1)}{4} \mathbb{E} \int_s^T |Y^n_t|^p - 2|Z^n_t|^2 \, dt \leq \mathbb{E}|\xi|^p + c\mathbb{E} \int_s^T |f(t,0,0)|^p \, dt
+ \mathbb{E} \sup_{t \in [s,T]} |S_t|^{p-1} \left( |\xi| + |\int_s^T f(t,Y^n_t, Z^n_t) \, dt| + |\int_s^T Z^n_t \, dW_t| \right)
\leq C\mathbb{E}|\xi|^p + C\mathbb{E} \int_s^T |f(t,0,0)|^p \, dt + \left( \mathbb{E} \sup_{t \in [s,T]} |S_t|^{2(p-1)} \right)^{1/2}
$$

As already mentioned, it is well known that the penalized BSDE admits a unique solution whose norm is uniformly bounded in $L_P^2(\Omega, C([0,T])) \times L_P^2(\Omega \times [0,T], \mathbb{X})$. Namely estimates in section 6 of [6] reed:

$$
\mathbb{E} \sup_{s \in [0,T]} |Y^n_s|^2 + \mathbb{E} \int_0^T |Z^n_t|^2 \, dt \leq c\mathbb{E}|\xi|^2 + c\mathbb{E} \int_0^T |f(t,0,0)|^2 \, dt,
So plugging the above in (2.11) we get, also by the BDG inequality,

\[ \frac{1}{2} \mathbb{E}|Y_s^n|^p + \frac{p(p-1)}{4} \mathbb{E} \int_s^T |Y_t^n|^{p-2} |Z_t^n|^2 \, dt \leq \mathbb{E}|\xi|^p + c \mathbb{E} \int_0^T |f(t,0,0)|^p \, dt \]

\[ + c \left( \mathbb{E} \sup_{t \in [s,T]} |S_t|^{2(p-1)} \right)^{\frac{1}{2}} \mathbb{E} \left[ |\xi|^2 + \int_0^T |f(t,0,0)|^2 \, dt + \int_s^T |Y_t^n|^2 + \int_s^T |Z_t^n|^2 \, dt \right]^{\frac{1}{2}} \]

\[ \leq \mathbb{E}|\xi|^p + c \mathbb{E} \int_0^T |f(t,0,0)|^p \, dt + c \left( \mathbb{E} \sup_{t \in [0,T]} |S_t|^{2(p-1)} \right)^{\frac{1}{2}} \left( \mathbb{E}|\xi|^2 + C \mathbb{E} \int_0^T |f(t,0,0)|^2 \, dt \right)^{\frac{1}{2}} \]

So we can deduce that

\[ \frac{p(p-1)}{4} \mathbb{E} \int_s^T |Y_t^n|^{p-2} |Z_t^n|^2 \, dt \]

\[ \leq C \mathbb{E}|\xi|^p + C \mathbb{E} \int_0^T |f(t,0,0)|^p \, dt + C \left( \mathbb{E} \sup_{t \in [0,T]} |S_t|^{2(p-1)} \right)^{\frac{1}{2}} \left( \mathbb{E}|\xi|^2 + C \mathbb{E} \int_0^T |f(t,0,0)|^2 \, dt \right)^{\frac{1}{2}} \]

By (2.10), with \( r \) in the place of \( s \), such that \( 0 \leq s \leq r \leq T \) we get

\[ |Y_r^n|^p \leq 2|\xi|^p + c \int_0^T |f(t,0,0)|^p \, dt \]

\[ - 2p \int_r^T |Y_r^n|^{p-1} \dot{Y}_t^n Z_t^n \, dW_t + c \sup_{r \in [s,T]} |S_t|^p + \]

\[ + 2 \sup_{r \in [s,T]} |S_t|^{p-1} \left[ |\xi| + \int_r^T |f(t,Y_t^n, Z_t^n)| \, dt \right] + \int_r^T Z_t^n \, dW_t \].

By taking the supremum over the time \( r \), by taking expectation and with calculations in part
similar to the ones we have performed in (2.12) we arrive at

\[
\begin{align*}
\mathbb{E} \sup_{r \in [s, T]} |Y_r^n|^p & \leq 2\mathbb{E} |\xi|^p + c \mathbb{E} \int_s^T |f(t, 0, 0)|^p dt \\
& + c \mathbb{E} \sup_{r \in [s, T]} \left| \int_r^T |Y_t^n|^p - \hat{Y}_t^n Z_t^n dW_t \right| + c \mathbb{E} \sup_{t \in [s, T]} |S_t|^p \\
& + 2 \left( \mathbb{E} \sup_{t \in [s, T]} |S_t|^{2(p-1)} \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ |\xi|^2 + \int_s^T |f(t, 0, 0)|^2 dt \right] \right)^{\frac{1}{2}} \\
& \leq c \mathbb{E} |\xi|^p + c \mathbb{E} \int_s^T |f(t, 0, 0)|^p dt + c \mathbb{E} \left( \sup_{t \in [s, T]} |Y_t^n|^p \int_s^T |Y_t^n|^{p-2} |Z_t^n|^2 dt \right)^{\frac{1}{2}} \\
& + c \left( \mathbb{E} \sup_{t \in [s, T]} |S_t|^{2(p-1)} \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ |\xi|^2 + \int_s^T |f(t, 0, 0)|^2 dt \right] \right)^{\frac{1}{2}} \\
& \leq c \mathbb{E} |\xi|^p + c \mathbb{E} \int_s^T |f(t, 0, 0)|^p dt + c \mathbb{E} \left( \sup_{t \in [s, T]} |Y_t^n|^p \int_s^T |Y_t^n|^{p-2} |Z_t^n|^2 dt \right)^{\frac{1}{2}} \\
& + c \left( \mathbb{E} \sup_{t \in [s, T]} |S_t|^{2(p-1)} \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ |\xi|^2 + \int_s^T |f(t, 0, 0)|^2 dt \right] \right)^{\frac{1}{2}} \\
& \leq c \mathbb{E} |\xi|^p + c \mathbb{E} \int_s^T |f(t, 0, 0)|^p dt + \frac{1}{2} \mathbb{E} \sup_{r \in [s, T]} |Y_r^n|^p + c \mathbb{E} \int_s^T |Y_t^n|^{p-2} |Z_t^n|^2 dt \\
& + c \left( \mathbb{E} \sup_{t \in [s, T]} |S_t|^{2(p-1)} \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ |\xi|^2 + \int_s^T |f(t, 0, 0)|^2 dt \right] \right)^{\frac{1}{2}} \\
\end{align*}
\]

So we get, also by applying estimate (2.13)

\[
\begin{align*}
\mathbb{E} \sup_{r \in [s, T]} |Y_r^n|^p & \leq c \mathbb{E} |\xi|^p + c \mathbb{E} \int_s^T |f(t, 0, 0)|^p dt \\
& + \left( \mathbb{E} \sup_{t \in [s, T]} |S_t|^{2(p-1)} \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ |\xi|^2 + \int_s^T |f(t, 0, 0)|^2 dt \right] \right)^{\frac{1}{2}} \\
\end{align*}
\]

Next we estimate \( \mathbb{E} \left( \int_s^T |Z_t^n|^2 dt \right)^{\frac{1}{2}} \); we apply Itô formula to \( |Y_t^n|^2 \), \( s \leq t \leq T \) obtaining

\[
d|Y_t^n|^2 = -2Y_t^n f(t, Y_t^n, Z_t^n) dt - nY_t^n (Y_t^n - S_t)^- dt + 2Y_t^n Z_t^n dW_t + |Z_t^n|^2 dt.
\]
We integrate on \([s, T]\) and we raise to the power \(\frac{p}{2}\):

\[
|Y^n_s|^p + \left( \int_s^T |Z^n_t|^2 \, dt \right)^{\frac{p}{2}} \leq |\xi|^p + \left( 2 \int_s^T Y^n_t f(t, Y^n_t, Z^n_t) \, dt \right)^{\frac{p}{2}} + \left( \int_s^T Y^n_t (Y^n_t - S_t)^{+} \, dt \right)^{\frac{p}{2}} + \left( \int_s^T Y^n_t Z^n_t \, dW_t \right)^{\frac{p}{2}}
\]

Using the expression (2.34) for \(n \int_s^T (Y^n_t - S_t)^{+} \, dt\) that comes from the penalized BSDE (2.5), we get

\[
\left( \int_s^T |Z^n_t|^2 \, dt \right)^{\frac{p}{2}} \leq |\xi|^p + 2 \left| \int_s^T Y^n_t f(t, Y^n_t, Z^n_t) \, dt \right|^{\frac{p}{2}} + \left| \int_s^T Y^n_t Z^n_t \, dW_t \right|^{\frac{p}{2}} + \sup_{t \in [s, T]} |S_t|^\frac{p}{2} \left( \sup_{t \in [s, T]} \left| \xi + Y^n_t - \int_s^T f(t, Y^n_t, Z^n_t) \, dt + \int_s^T Z^n_t \, dW_t \right| \right)^{\frac{p}{2}}
\]

Computing expectation, by BDG and Young inequalities, and by using estimate (2.15), we get

\[
E \left( \int_s^T |Z^n_t|^2 \, dt \right)^{\frac{p}{2}} \leq E|\xi|^p + c E \sup_{r \in [s, T]} |Y^n_r|^p + c E \left( \int_s^T |f(t, 0, 0)|^2 \, dt \right)^{\frac{p}{2}} + \frac{1}{4} \left( E \int_s^T |Z^n_t|^2 \, dt \right)^{\frac{p}{2}} + E \left( \int_s^T |Y^n_t Z^n_t|^2 \, dt \right)^{\frac{p}{2}} + c E \sup_{t \in [s, T]} |S_t|^p
\]

\[
\leq c E|\xi|^p + c E \int_0^T |f(t, 0, 0)|^p \, dt + \frac{1}{2} E \left( \int_s^T |Z^n_t|^2 \, dt \right)^{\frac{p}{2}} + c E \sup_{r \in [s, T]} |Y^n_r|^p
\]
Concluding by estimate (2.15), we obtain:
\[
\mathbb{E} \left( \int_s^T |Z^n_t|^2 \, dt \right)^{\frac{p}{2}} \leq c\mathbb{E}|\xi|^p + c\mathbb{E} \int_0^T |f(t, 0, 0)|^p \, dt + c \left( \mathbb{E} \sup_{t \in [s, T]} |S_t|^{2(p-1)} \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ |\xi|^2 + \int_s^T |f(t, 0, 0)|^2 \, dt \right] \right)^{\frac{1}{2}}
\]
and this concludes the estimate of \( \mathbb{E} \left( \int_s^T |Z^n_t|^2 \, dt \right)^{\frac{p}{2}} \).

The estimate of \( \mathbb{E}|K^n_T|^p \) is then easy consequence of the previous ones and of relation (2.9). \( \square \)

We are now ready to prove Theorem 2.2.

**Proof of Theorem 2.2.** By [6], section 6, we know that \( Y^n_t \uparrow Y_t \) and
\[
\mathbb{E}(\sup_{t \in [0, T]} (Y^n_t - Y^n_t))^2 \to 0.
\]
Thus choosing a suitable subsequence we can assume the \( \mathbb{P} \)-a.s. convergence of \( \sup_{t \in [0, T]} (Y^n_t - Y^n_t) \) towards 0. Consequently by Fatou Lemma and (2.6) we get
\[
\mathbb{E} \sup_{t \in [0, T]} |Y^n_t|^p \leq C\mathbb{E}|\xi|^p + C\mathbb{E} \int_0^T |f(t, 0, 0)|^p \, dt + C \left( \mathbb{E} \sup_{t \in [0, T]} |S_t|^{2(p-2)} \right)^{p/(2p-2)}.
\]
For what concerns the convergence of \( Z^n \), again by [6], section 6, we already know that \( Z^n \to Z \) in \( L^p_P(\Omega \times [0, T]) \), and by proposition 2.2 we know that \( Z^n \) is bounded in \( L^p_P(\Omega \times [0, T]) \), so, extracting, if needed, a subsequence, we can assume that such that \( (Z^n) \) converges weakly in \( L^p_P(\Omega \times [0, T]) \) and consequently also weakly in \( L^2_P(\Omega \times [0, T]) \). Therefore the weak limit of \( (Z^n) \) in \( L^2_P(\Omega \times [0, T]) \) must coincide with the strong limit \( Z \) in \( L^2_P(\Omega \times [0, T]) \) topology. Consequently again by (2.6) we have that \( Z \) satisfies
\[
\mathbb{E} \left( \int_0^T |Z_t|^2 \, dt \right)^{p/2} \leq C\mathbb{E}|\xi|^p + C\mathbb{E} \int_0^T |f(t, 0, 0)|^p \, dt + C \left( \mathbb{E} \sup_{t \in [0, T]} |S_t|^{2(p-2)} \right)^{p/(2p-2)}.
\]
For what concerns \( K \), by [6] we already know (see again [6], section 6) that \( \mathbb{E}|K^n_T - K_T|^2 \to 0 \).

The claim follows as before by Fatou lemma by extracting a subsequence that converges \( \mathbb{P} \)-a.s. and exploiting estimate (2.9). \( \square \)

### 2.1 Reflected BSDEs in a Markovian framework

Now we consider a RBSDE depending on a forward equation with values in another real and separable Hilbert space \( H \). Namely, we consider the forward backward system
\[
\begin{cases}
  dX_t^{s,x} = AX_t^{s,x} + F(t, X_t^{s,x})dt + G(t, X_t^{s,x})dW_t & t \in [s, T], \\
  X_s^{s,x} = x, \\
  -dY_t^{s,x} = \psi(t, X_t^{s,x}, Y_t^{s,x}, Z_t^{s,x}) \, dt + dK_t^{s,x} - Z_t^{s,x} \, dW_t, & t \in [0, T], \\
  Y_T^{s,x} = \phi(X_T^{s,x}), \\
  Y_t^{s,x} \geq h(X_t^{s,x}), \\
  \int_0^T (Y_t^{s,x} - h(X_t^{s,x}))dK_t^{s,x} = 0.
\end{cases}
\]
We denote the solution of the RBSDE in the above equation by \( (Y^{s,x}, Z^{s,x}, K^{s,x}) \), to stress the dependence on the initial conditions, or by \( (Y, Z, K) \) if no confusion is possible.

On the coefficients of the forward equation we make the following assumptions:
Hypothesis 2.4  1. A is the generator of a strongly continuous semigroup of linear operators 
\((e^{tA})_{t \geq 0}\);

2. The mapping \(F : [0, T] \times H \to H\) is measurable and satisfies, for some constant \(C > 0\) and \(0 \leq \gamma < 1\),
\[ |e^{sA}F(\tau, x)| \leq Cs^{-\gamma}(1 + |x|), \quad t \in [0, T], \]
\[ |e^{sA}F(\tau, x) - e^{sA}F(\tau, y)| \leq Cs^{-\gamma}|x - y|, \quad s > 0, \quad t \in [0, T], \quad x, y \in H. \]  
(2.17)

3. \(G\) is a mapping \([0, T] \times H \to L(\Xi, H)\) such that for every \(v \in \Xi\), the map \(Gv : [0, T] \times H \to H\) is measurable and for every \(s > 0, \tau \in [0, T]\) and \(x \in H\) we have \(e^{sA}G(\tau, x) \in L_2(\Xi, H)\). Moreover there exists \(0 < \theta < \frac{1}{2}\) such that
\[ |e^{sA}G(\tau, x)|_{L_2(\Xi, H)} \leq Ls^{-\theta}(1 + |x|), \]
\[ |e^{sA}G(\tau, x) - e^{sA}G(\tau, y)|_{L_2(\Xi, H)} \leq Ls^{-\theta}|x - y|, \quad s > 0, \tau \in [0, T], \quad x, y \in H. \]  
(2.18)

The next existence and uniqueness Proposition is proved in [8].

Proposition 2.5  Under hypothesis 2.4, the forward equation in (2.16) admits a unique continuous mild solution. Moreover \(E\sup_{t \in [s, T]} |X_t^{s,x}|^p \leq C_p (1 + |x|)^p\), for every \(p \in (0, \infty)\), and some constant \(C_p > 0\).

We will work under the following assumptions on \(\psi\):

Hypothesis 2.6  The function \(\psi : [0, T] \times H \times \mathbb{R} \times \Xi \to \mathbb{R}\) is Borel measurable and satisfies the following:

1. there exists a constant \(L > 0\) such that
\[ |\psi(t, x, y_1, z_1) - \psi(t, x, y_2, z_2)| \leq L(|y_1 - y_2| + |z_1 - z_2|_{\Xi}), \]  
for every \(t \in [0, T]\), \(x \in H\), \(y_1, y_2 \in \mathbb{R}\), \(z_1, z_2 \in \Xi\);

2. for every \(t \in [0, T]\), \(\psi(t, \cdot, \cdot, \cdot)\) is continuous \(H \times \mathbb{R} \times \Xi^* \to \mathbb{R}\);

3. there exists \(L' > 0\) and \(m \geq 0\) such that
\[ |\psi(t, x_1, y, z) - \psi(t, x_2, y, z)| \leq L'(1 + |x_1|^m + |x_2|^m + |y|^m)(1 + |z|_{\Xi}), \]  
for every \(t \in [0, T]\), \(x_1, x_2 \in H\), \(y \in \mathbb{R}\), \(z \in \Xi\).

4. as far as the final datum \(\phi\) and the obstacle \(h\) are concerned there exists \(L > 0\) such that:
\[ |\phi(x_1) - \phi(x_2)| \leq L|x_1 - x_2|(1 + |x_1|^m + |x_2|^m), \]
\[ |h(x_1) - h(x_2)| \leq L|x_1 - x_2|(1 + |x_1|^m + |x_2|^m), \]  
for all \(x_1, x_2 \in H\).

We notice that hypothesis 2.6 implies that, for all \(p > 0\)
\[ |\psi(t, x, y, z)| \leq L(1 + |x|^{m+1} + |y| + |z|_{\Xi}), \quad |\phi(x)| \leq L(1 + |x|^{m+1}), \quad |h(x)| \leq L(1 + |x|^{m+1}), \]  
(2.19)

for all \(t \in [0, T], x \in H, y \in \mathbb{R}, z \in \Xi\), and for all \(p \geq 2\).
Let hypotheses 2.4 and 2.6 hold true and fix $s \in [0, T]$, $x \in H$. Then the RBSDE in (2.16) admits a unique adapted solution $(Y^{s,x}, Z^{s,x}, K^{s,x})$. Moreover, $Y^{s,x}$ is continuous and non-decreasing ($K^{s,x}_t = 0$) and, for all $p \geq 2$ there exists $C_p > 0$ such that

$$
E \sup_{t \in [0, T]} |Y^{s,x}_t|^p + \mathbb{E} \left( \int_0^T |Z^{s,x}_t|^2 \, dt \right)^{p/2} + \mathbb{E}|K^{s,x}_T|^p < C(1 + |x|^{p(m+1)}).
$$

(2.20)

We consider also the penalized version of the RBSDE in (2.16):

$$
\begin{cases}
-dY^{n,s,x}_t = \psi(t, X^{s,x}_t, Y^{n,s,x}_t, Z^{n,s,x}_t) \, dt + n(Y^{n,s,x}_t - h(X^{s,x}_t)) - dt - Z^{n,s,x}_t \, dW_t, & t \in [0, T], \\
Y^{n,s,x}_T = \phi(X^{s,x}_T).
\end{cases}
$$

(2.21)

The same holds for the penalized equation with constant $C$ independent on $n$.

**Proof.** It suffices to notice that by setting

$$f(t, y, z) := \psi(t, X^{s,x}_t, y, z), \quad S_t := h(X^{s,x}_t), \quad \xi := \phi(X^{s,x}_T)$$

for all $t \in [0, T]$, $y \in \mathbb{R}$, $z \in \Xi$ and with $X^{s,x}$ solution to the forward equation in the FBSDE (2.16), by (2.19) $f$, $h$, $S$ satisfy hypothesis 2.1 and in particular:

$$
\mathbb{E} \int_0^T |f(t, 0, 0)|^p \, dt = \mathbb{E} |\psi(t, X^{s,x}_t, 0, 0)|^p \leq c \left( 1 + |x|^{p(m+1)} \right)
$$

(2.22)

$$
\mathbb{E} \sup_{t \in [0, T]} |S_t|^{2(p-1)} = \mathbb{E} \sup_{t \in [0, T]} |h(X_t)|^{2(p-1)} \leq c \left( 1 + |x|^{2(p-1)(m+1)} \right)
$$

(2.23)

$$
\mathbb{E} |\xi|^p = \mathbb{E} |\phi(X_T)|^p \leq (1 + |x|^{p(m+1)}).
$$

So we can apply Proposition 2.7 and Theorem 2.2 to obtain the claim.

**Remark 2.8** Notice that $(Y^{s,x}_t, Z^{s,x}_t)$ is independent on $\mathcal{F}_s$ so, fixed $0 \leq s \leq T$ the compositions

$$Y^{s,x}_{t \wedge T}, \quad Z^{s,x}_{t \wedge T}, \quad t \in [s, T]$$

are well defined. Moreover by uniqueness of the solution to the forward equation in (2.16) we have $X^{r,x}_t = X^{s,x}_{t \wedge r}$ and consequently

$$Y^{s,x}_{t \wedge T} = Y^{r,x}_t \mathbb{P} - \text{a.s.}, \quad \forall t \in [s, T]$$

$$Z^{s,x}_{t \wedge T} = Z^{r,x}_t \mathbb{P} - \text{a.s.} \quad \text{for a.e.} \quad t \in [s, T]$$

The next theorem is devoted to the local Lipschitz continuity of $Y^{s,x}$ with respect to $x$.

**Theorem 2.9** Let hypotheses 2.4 and 2.6 hold true and let $(Y^{s,x}, Z^{s,x}, K^{s,x})$ be the unique solution of the RBSDE in (2.16). Then there exists a constant $L > 0$ such that, $\forall x_1, x_2 \in H$,

$$|Y^{s,x}_1 - Y^{s,x}_2| \leq L \left( 1 + |x_1|^{m(m+1)} + |x_2|^{m(m+1)} \right) |x_1 - x_2|.
$$

(2.24)
is a solution of the BSDE (to be intended in mild form):

\[
\frac{d}{dt}Y_t^{n,s,x} = \psi(t, X_t^{n,s,x}, Y_t^{n,s,x}, Z_t^{n,s,x}) + n\gamma(y) \frac{d}{dt}Z_t^{n,s,x} \quad \text{for } t \in [0,T],
\]

\[
Y_T^{n,s,x} = \phi(X_T^{s,x}),
\]

and we notice that estimates obtained in proposition 2.7 are still true for the pair of processes \((Y^{n,s,x}, Z^{n,s,x})\) solution of equation (2.27).

Notice that it is still true that \(|y|^{p-1} \gamma(y - s) \leq |s|^{p-1} \gamma(y - s)\) for all \(y, s \in \mathbb{R}\).

By [8] we know that we can differentiate \((Y^{n,s,x}, Z^{n,s,x})\) with respect to \(x\), and that \((\nabla_x Y^{n,s,x}, \nabla_x Z^{n,s,x})\) is the solution of the BSDE (to be intended in mild form):

\[
\begin{cases}
-d\nabla_x Y_t^{n,s,x} = \nabla_x \psi(t, X_t^{n,s,x}, Y_t^{n,s,x}, Z_t^{n,s,x}) \nabla_x X_t^{s,x} dt + n\gamma(y) \nabla_y \psi(t, X_t^{n,s,x}, Y_t^{n,s,x}, Z_t^{n,s,x}) \nabla_x Y_t^{n,s,x} dt \\
\quad + n\gamma(y) \nabla_y (Y_t^{n,s,x} - h(X_t^{s,x})) \nabla_x Y_t^{n,s,x} dt - \nabla_x Z_t^{n,s,x} dW_t,
\end{cases}
\]

where (see again [8]) \(\nabla_x X_t^{s,x}\) is the mild solution to the following forward equation

\[
\begin{cases}
\nabla_x Y_t^{n,s,x} = A \nabla_x X_t^{s,x} dt + \nabla_x F(t, X_t^{s,x}) \nabla_x X_t^{s,x} dt + \nabla_x G(t, X_t^{s,x}) \nabla_x X_t^{s,x} dW_t,
\end{cases}
\]

\(I : H \to H\) being the identity operator in \(H\).

We set \(\hat{\mathbb{P}} := \mathcal{E}_T \mathbb{P}\), with

\[
\mathcal{E}_T = \exp \left( -\int_s^T \nabla_x \psi(t, X_t^{s,x}, Y_t^{n,s,x}, Z_t^{n,s,x}) dW_t - \frac{1}{2} \int_s^T |\nabla_x \psi(t, X_t^{s,x}, Y_t^{n,s,x}, Z_t^{n,s,x})|^2 dt \right).
\]

By the Girsanov theorem \(\hat{\mathbb{P}}\) is a probability measure equivalent to the original one \(\mathbb{P}\) (recall that by hypothesis 2.6 \(\nabla_x\) is bounded and)

\[
\hat{W}_t = -\int_s^t \nabla_x \psi(t, X_t^{s,x}, Y_t^{n,s,x}, Z_t^{n,s,x}) dt + W_t, \quad s \leq t \leq T
\]

is a \(\hat{\mathbb{P}}\)-cylindrical Wiener process.
In $(\Omega, \mathcal{F}, \mathbb{P})$ the pair $(\nabla_x Y^{n,s,x}, \nabla_x Z^{n,s,x})$ solve the following BSDE for $t \in [s, T]$:

\[
\begin{cases}
-d\nabla_x dY^{n,s,x}_t = \nabla_x \psi(t, X^{s,x}_t, Y^{n,s,x}_t, Z^{n,s,x}_t) \nabla_x X^{s,x}_t dt \\
+ \nabla_y \psi(t, X^{s,x}_t, Y^{n,s,x}_t, Z^{n,s,x}_t) \nabla_x Y^{n,s,x}_t dt \\
+ n^\gamma(Y^{n,s,x}_t - h(X^{s,x}_t)) (\nabla_x Y^{n,s,x}_t - \nabla h(X^{s,x}_t)) \nabla_x X^{s,x}_t dt - \nabla_x Z^{n,s,x}_t d\bar{W}_t,
\end{cases}
\]

\[
\nabla_x Y^{n,s,x}_t = \nabla \phi(X^{s,x}_T) \nabla_x X^{s,x}_T, 
\]

(2.27)

Multiplying $\nabla_x Y^{n,s,x}_t$ by $\exp \{ \int_s^t (\nabla_y \psi(t, X^{s,x}_\sigma, Y^{n,s,x}_\sigma, Z^{n,s,x}_\sigma) + n^\gamma(Y^{n,s,x}_\sigma - h(X^{s,x}_\sigma))) d\sigma \}$ and writing the obtained equation in $t = s$ we get:

\[
\nabla_x Y^{n,s,x}_s = \mathbb{E}_T \int_s^T \exp \left\{ \int_s^T \nabla_y \psi(t, X^{s,x}_\tau, Y^{n,s,x}_\tau, Z^{n,s,x}_\tau) + n^\gamma(Y^{n,s,x}_\tau - h(X^{s,x}_\tau))) d\tau \right\} 
\]

\[
\left( \nabla_x \psi(t, X^{s,x}_\tau, Y^{n,s,x}_\tau, Z^{n,s,x}_\tau) \nabla_x X^{s,x}_t - n^\gamma(Y^{n,s,x}_\tau - h(X^{s,x}_\tau))) \nabla_x X^{s,x}_\tau \right) d\tau \]

\[
+ \mathbb{E}_T \exp \left\{ \int_s^T \nabla_y \psi(t, X^{s,x}_\tau, Y^{n,s,x}_\tau, Z^{n,s,x}_\tau) + n^\gamma(Y^{n,s,x}_\tau - h(X^{s,x}_\tau)) d\sigma \right\} \nabla \phi(X^{s,x}_T) \nabla_x X^{s,x}_T,
\]

so that, since $\hat{\gamma} \leq 0$ and $\nabla_y \psi$ is bounded by hypothesis [2.6] point 1,

\[
|\nabla_x Y^{n,s,x}_s| 
\]

\[
\leq c \mathbb{E}_T \left| \nabla \phi(X^{s,x}_T) \nabla_x X^{s,x}_T + \int_s^T \nabla_x \psi(t, X^{s,x}_\tau, Y^{n,s,x}_\tau, Z^{n,s,x}_\tau) \nabla_x X^{s,x}_\tau d\tau \right|
\]

\[
+ c \mathbb{E}_T \left| \int_s^T \exp \left\{ \int_s^\tau n^\gamma(Y^{n,s,x}_\sigma - h(X^{s,x}_\sigma))) d\sigma \right\} d\tau \right| 
\]

\[
\left( -n^\gamma(Y^{n,s,x}_\tau - h(X^{s,x}_\tau))) \nabla h(X^{s,x}_\tau) \nabla_x X^{s,x}_\tau \right) d\tau \]

\[
= I + II,
\]

We start by estimating $I$. Here and in the following we again denote by $c$ a constant whose value can vary from line to line and that may depend on $T$, on the coefficients $A, F, G, \psi, , h, \phi$, on $p$ but not on $n$ and $x$.

\[
I \leq c \mathbb{E}_T \left| \nabla \phi(X^{s,x}_T) \nabla_x X^{s,x}_T \right| + c \mathbb{E}_T \left| \int_s^T |\nabla_x \psi(t, X^{s,x}_\tau, Y^{n,s,x}_\tau, Z^{n,s,x}_\tau) \nabla_x X^{s,x}_\tau | d\tau \right|.
\]

Taking into account that $\mathbb{E}^{\mathbb{P}} \leq c$, by Holder inequality, with $p, q, r$ conjugate exponents $p > 1$, $1 < q < 2$, $qm > 2$, (where $m$ is the same as in hypothesis [2.6]) we get:

\[
\mathbb{E}_T \left| \nabla \phi(X^{s,x}_T) \nabla_x X^{s,x}_T \right| \leq c \left( \mathbb{E} \left| \nabla \phi(X^{s,x}_T) \right|^p \right)^{1/p} \left( \mathbb{E} \left| \nabla_x X^{s,x}_T \right|^q \right)^{1/q} \leq c(1 + |x|^m),
\]

where we have used the estimate on $\nabla_x X^{s,x}_T$ stated in [3], proposition 3.3.
In a similar way we can estimate (for $q > 2$)

$$
\mathbb{E}\left[ \mathcal{E}_T \int_s^T |\nabla_x \psi(\tau, X^{s,x}_\tau, Y^{n,s,x}_\tau, Z^{n,s,x}_\tau) \nabla_x X^{s,x}_\tau| \, d\tau \right]
\leq c \mathbb{E}\left[ \mathcal{E}_T \int_s^T (1 + |X^{s,x}_\tau|^m + |Y^{n,s,x}_\tau|^m) |Z^{n,s,x}_\tau| (1 + |\nabla_x X^{s,x}_\tau|) \, d\tau \right]
\leq c \left( \mathbb{E}\left[ \left( 1 + \sup_{\tau \in [s,T]} |X^{s,x}_\tau|^m \right) \sup_{\tau \in [s,T]} |\nabla_x X^{s,x}_\tau| \left( \int_s^T |Z^{n,s,x}_\tau| \, d\tau \right)^q \right)^{1/q}
\leq c \left( \mathbb{E}\left[ \left( 1 + \sup_{\tau \in [s,T]} |X^{s,x}_\tau|^{2mq} \right) \sup_{\tau \in [s,T]} |Y^{n,s,x}_\tau|^{2mq} \left( \int_s^T |Z^{n,s,x}_\tau| \, d\tau \right)^{2q} \right)^{1/2q}
\leq c \left( 1 + |x|^{m(m+1)} \right).
$$

where we have used estimates $\text{(2.20)}$ and Proposition $\text{2.5}$.

For what concerns $\text{II}$, let $p, q$ and $\bar{p}, \bar{q}$ be two pairs of conjugate exponents, and let

$$
l(\tau) := -n\bar{q}(Y^{n,s,x}_\tau - h(X^{s,x}_\tau)) \geq 0, \ \tau \in [s,T]
$$

Then

$$
\mathbb{E}\left[ \mathcal{E}_T \left| \int_s^T \exp\left( - \int_s^\tau l_\sigma \, d\sigma \right) \nabla h(X^{s,x}_\tau) \nabla_x X^{s,x}_\tau \, d\tau \right| \right]
\leq c \left( \mathbb{E}\left[ \left( 1 + \sup_{\tau \in [s,T]} |X^{s,x}_\tau|^m \right) \sup_{\tau \in [s,T]} |\nabla_x X^{s,x}_\tau| \int_s^T \exp\left( - \int_s^\tau l_\sigma \, d\sigma \right) l_\tau \, d\tau \right)^{1/q}
\leq c \left( \mathbb{E}\left[ \left( 1 + \sup_{\tau \in [s,T]} |X^{s,x}_\tau|^m \right) \sup_{\tau \in [s,T]} |\nabla_x X^{s,x}_\tau| \left( \int_s^T \exp\left( - \int_s^\tau l_\sigma \, d\sigma \right) l_\tau \, d\tau \right)^{\bar{q}} \right)^{1/(\bar{q}q)}
\leq c (1 + |x|^m) \left( \mathbb{E}\left[ 1 - \exp\left( - \int_s^\tau l_\sigma \, d\sigma \right) \right]^{\bar{q}} \right)^{1/(\bar{q}q)} \leq c (1 + |x|^m)
$$

where in the last passage we have used that

$$
\int_s^T \exp\left( - \int_s^\tau l(\sigma) \, d\sigma \right) l(\tau) \, d\tau = 1 - \exp\left( - \int_s^T l(\sigma) \, d\sigma \right)
$$

So

$$
|\nabla_x Y^{n,s,x}_s| \leq c \left( 1 + |x|^{m(m+1)} \right), \quad (2.29)
$$

where $c$ may depend on $T$, on the coefficients $A, F, G, \psi, h, \phi$, but not on $n$. By $\text{(2.29)}$ we get that $\forall x, y \in H$

$$
|Y^{n,s,x}_s - Y^{n,s,y}_s| \leq c|x - y|(1 + |x|^{m(m+1)} + |y|^{m(m+1)}).
$$

By letting $n \to \infty$, arguing as in section 6 in $\text{[6]}$ finally get the desired Lipschitz continuity of $\text{Y}_s^{s,x}$:

$$
|Y^{s,x}_s - Y^{s,y}_s| \leq c|x - y|(1 + |x|^{m(m+1)} + |y|^{m(m+1)}), \quad \forall x, y \in H. \quad (2.30)
$$

Finally we have to remove the assumption of differentiability on the coefficient $\psi, h, \phi$ in the reflected BSDE. Since $\psi$ is Lipschitz continuous with respect to $y$ and $z$, and $\forall t, y, z \in [0, T] \times \mathbb{R} \times \mathbb{E}, \psi(t, \cdot, y, z)$, $h, \phi$ are locally Lipschitz continuous with respect to $x$, then by taking their inf-sup convolution ($\psi_k, \phi_k, h_k)_{k \geq 1}$ we obtain differentiable functions where the derivative is
bounded by the Lipschitz constant in the Lipschitz case, and the derivative has the polynomial
growth imposed by the locally Lipschitz growth, see e.g. [4] for the notion of inf-sup convolution,
and [15] and [16] for the use of inf-sup convolutions in the Lipschitz and locally Lipschitz case.
So in particular the growth of the derivatives the inf-sup convolutions is uniform with respect
to k: it follows that Lipschitz and locally Lipschitz constants are uniform with respect to k,
and this allows to pass to the limit as \( k \to \infty \) and to preserve Lipschitz and locally Lipschitz
properties. Coming into more details, we denote by \((Y^{n,k,s,x}, Z^{n,k,s,x}, K^{n,k,s,x})\) the solution
of the penalized RBSDEs with regularized coefficients:

\[
\begin{aligned}
-dY^{n,k,s,x}_t &= \psi_k(t, X^{s,x}_t, Y^{n,k,s,x}_t, Z^{n,k,s,x}_t) \, dt + n \gamma (Y^{n,k,s,x}_t - h_k(X^{s,x}_t)) \, dt - Z^{n,k,s,x}_t \, dW_t, \\
Y^{n,k,s,x}_T &= \phi_k(X^{s,x}_T), 
\end{aligned}
\]

By the previous calculations we get that \( \forall x, y \in H \)

\[ |Y^{n,k,s,x}_s - Y^{n,k,s,x}_y| \leq c|x - y|(1 + |x|^{m(m+1)} + |y|^{m(m+1)}), \]

where \( c \) does not depend on \( n \) nor on \( k \). By standard results on BSDEs (see [2]) we know that

\((Y^{n,k,s,x}, Z^{n,k,s,x}) \to (Y^{n,s,x}, Z^{n,s,x})\) in \( L^p_p(\Omega, C([0,T])) \times L^p_p(\Omega \times [0,T], \Xi)\),

where \((Y^{n,s,x}, Z^{n,s,x})\) is solution to the smooth penalized BSDE (2.32). In particular \( Y^{n,k,s,x} \to Y^{n,s,x}\). Finally proceeding as in [6] if we let \( n \to \infty \), we have already recalled that \( Y^{n,s,x} \uparrow Y^{s,x}\),
since for this monotone convergence what matters is the monotonicity of the penalization term
corresponding to \( K \), and we get the desired Lipschitz continuity of \( Y^{s,x}\):

\[ |Y^{s,x}_s - Y^{s,x}_y| \leq c|x - y|(1 + |x|^{m(m+1)} + |y|^{m(m+1)}), \quad \forall x, y \in H. \]  

Remark 2.10 Notice that if \( h \) and \( \phi \) are bounded and lipschitz continuous functions, if for every
\( s \in [0,T] \), \( \sup_{x \in H} |\psi(s, x, 0, 0)| < \infty \) and as a function of \( x \), \( \psi \) is lipschitz continuous uniformly
with respect to the other variables, that is hypothesis [2,6] point 3 holds true with \( m = 0 \), then by
repeating the same argument in proposition [2,7] we can prove that the processes \( Y^{s,x}, Z^{s,x}\) are bounded processes with respect to \( x \), that is namely

\[ \mathbb{E} \sup_{t \in [0,T]} |Y^{s,x}_t|^p + \mathbb{E} \left( \int_0^T |Z^{s,x}_t|^2 \, dt \right)^{p/2} < C. \]  

3 Obstacle problem for a semilinear parabolic PDE: solution via RBSDEs

In this section we consider an obstacle problem for a semilinear PDE in an infinite dimensional
Hilbert space \( H \) and we solve it in a suitable sense by means of reflected BSDEs. An informal
description is as follows: we study an obstacle problem of the following form

\[
\begin{aligned}
\min \left( u(t, x) - h(x), -\frac{\partial u}{\partial t}(t, x) - \mathcal{L} u(t, x) - \psi(t, x, u(t, x), \nabla u(t, x) G(t, x)) \right) &= 0, \\
u(T, x) &= \phi(x), 
\end{aligned}
\]

\( t \in [0, T], \ x \in H \)  

(3.1)
where $G : [0, T] \times H \to L(\Xi, H)$, and $\nabla u(t, x) G(t, x)$ is the directional generalized gradient of $u$ with respect to $x$, see [11], section 3, and the following for the definition of generalized gradient. For a function $f : H \to \mathbb{R}$, the operator $L_t$ is formally defined by
\[
L_t f(x) = \frac{1}{2} \text{Trace} \left( G(t, x) G^*(t, x) \nabla^2 f(x) \right) + \langle Ax, \nabla f(x) \rangle_H + \langle F(t, x), \nabla f(x) \rangle_H,
\]
and it arises as the generator of an appropriate Markov process $X$ in $H$. More precisely if $X$ is the mild solution to the stochastic differential equation in $H$
\[
\begin{cases}
    dX_t^{s, x} = [AX_t^{s, x} + F(t, X_t^{s, x})] \, dt + G(t, X_t^{s, x}) \, dW_t, \quad t \in [s, T] \\
    X_s^{s, x} = x,
\end{cases}
\]
where $T > 0$ is fixed. For $t \in [s, T]$ we denote by $P_{s, t}$ the transition semigroup
\[
P_{s, t}[\phi](x) = \mathbb{E}\phi(X_{s, x}^t).
\]
where $\phi : H \to \mathbb{R}$ is bounded and measurable.

Note that $L_t$ is formally the generator of the transition semigroup $(P_{s, t})_{t \in [s, T]}$. This leads us to consider solutions of the obstacle problem (3.1) in mild sense, as we are going to state.

### 3.1 The generalized directional gradient

We observe that, under our assumptions, it is reasonable to expect that function $u$ is locally Lipschitz but not that it is differentiable. To this aim, we briefly show an example where the value function of a deterministic optimal stopping problem is not differentiable. Let us consider, as state equation without control,
\[
\begin{cases}
    dX_t^{s, x} = 0 \\
    X_s^{s, x} = x \in \mathbb{R}
\end{cases}
\]
We consider the following cost functional:
\[
J(s, x, \tau) = \phi \left( X_T^{u, s, x} \right) \chi_{\{\tau = T\}} + h(\tau, X_T^{u, s, x}) \chi_{\{\tau < T\}},
\]
So the value function is given by
\[
u(x) = \sup_{\tau} \left( \phi \left( X_T^{u, s, x} \right) \chi_{\{\tau = T\}} + h(\tau, X_T^{u, s, x}) \chi_{\{\tau < T\}} \right) = \sup_{x \in \mathbb{R}}(\phi(x), h(x)),
\]
and it is evident that, even if the data are differentiable, the value function may fail to be differentiable.

Notice that in the above example and statement we take into account that we allow degeneracy of the noise. The issue of differentiability of $u$ when noise is non degenerate is very interesting but falls out of the scope of the present work.

To take into account the lack of regularity of $u$ the derivative $\nabla u$ must not appear in the precise formulation of the problem. Indeed it will be substituted by the notion of generalized gradient, whose definition is given in the next subsection.

We start by giving the definition of generalized gradient

**Theorem 3.1** Assume that Hypothesis [2.4] holds and that $u : [0, T] \times H \to \mathbb{R}$ is a Borel measurable function satisfying, for some $r > 0$
\[
|u(t, x) - u(t, x')| \leq c(1 + |x| + |x'|)^r|x - x'|.
\]
Then there exists a Borel measurable function $\zeta : [0, T] \times H \to \Xi^*$ with the following properties.
(i) For every \( s \in [0, T] \), \( x \in H \) and \( p \in [2, \infty) \),

\[
E \int_s^T |\zeta(\tau, X_t^{s,x})|^p \, d\tau < +\infty. \tag{3.4}
\]

(ii) For \( \xi \in \Xi, x \in H \) and \( 0 \leq s \leq T' < T \) the processes \( \{u(t, X_t^{s,x}), t \in [s, T]\} \) and \( W^\xi \) admit a joint quadratic variation on the interval \([s, T']\) and

\[
\langle u(\cdot, X^{s,x}), W^\xi \rangle_{[s, T']} = \int_s^{T'} \zeta(t, X_t^{s,x}) \xi \, dt, \quad \mathbb{P} - a.s.
\]

(iii) Moreover there exists a Borel measurable function \( \rho : [0, T] \times H \rightarrow H^* \) such that for all \( t \in [s, T] \) and all \( x \in H \)

\[
\zeta(t, X_t^{s,x}) = \rho(t, X_t^{s,x}) G(t, X_t^{s,x}) \quad \mathbb{P} - a.s. \text{ for a.a. } t \in [s, T]
\]

Proof. The proof is given in [11], section 4. In that paper it is also noticed, see remark 3.1, that uniqueness can be stated in the following sense: if \( \hat{\zeta} \) is another function with the stated properties then for \( 0 \leq s \leq t \leq T \) and \( x \in H \) we have \( \zeta(t, X_t^{s,x}) = \hat{\zeta}(t, X_t^{s,x}), \mathbb{P} - a.s. \) for a.a. \( t \in [s, T] \). \( \square \)

Definition 3.1 Let \( u : [0, T] \times H \rightarrow \mathbb{R} \) be a Borel measurable function satisfying (3.3). The family of all measurable functions \( \zeta : [0, T] \times H \rightarrow \Xi^* \) satisfying properties (i) and (ii) in Theorem 3.1 will be called the generalized directional gradient of \( u \) and denoted by \( \tilde{\nabla} G u \).

3.2 Mild solutions of the obstacle problem in the sense of the generalized directional gradient

Having defined the generalized directional gradient, we are in the position to give the precise definition of supersolution for the problem (3.1).

Definition 3.2 We say that a Borel measurable function \( \bar{u} : [0, T] \times H \rightarrow \mathbb{R} \) is a mild supersolution of the obstacle problem (3.1) in the sense of the generalized directional gradient if the following holds:

1. for some \( C > 0 \) and for every \( s \in [0, T], x, y \in H \)

\[
|\bar{u}(s, x) - \bar{u}(s, y)| \leq C|x - y|(1 + |x| + |y|)^r, \quad |\bar{u}(s, 0)| \leq C;
\]

2. for every \( s \in [0, T], x \in H \)

\[
\bar{u}(s, x) \geq h(x);
\]

3. for all \( 0 \leq s \leq t \leq T \) and \( x \in H \)

\[
\bar{u}(s, x) \geq P_{s,t}[u(t, \cdot)](x) + \int_s^t P_{s,\tau} \left[ \psi(\tau, \cdot, \bar{u}(\tau, \cdot), \zeta(\tau, \cdot)) \right](x) \, d\tau, \tag{3.5}
\]

where \( \zeta \) is an arbitrary element of the generalized gradient \( \tilde{\nabla} G \bar{u} \);

4. \( \bar{u}(T, \cdot) = \phi \).

We are now ready to state the main result of this paper.
Theorem 3.2 Assume that hypotheses 2.4 and 2.6 hold true. Let us define

\[ u(s, x) = Y^s_{\cdot, x}, \]

where \((Y^{s,x}, Z^{s,x})\) is solution to the reflected BSDE in (2.16). Then \(u\) is a mild supersolution in the sense of the generalized directional gradient for the obstacle problem (3.1).

Moreover \(u\) is minimal in the sense that given any \(\tilde{u}\), supersolution of (3.1) in the sense of definition 3.2, it holds \(u(s, x) \leq \tilde{u}(s, x)\), and \(s \in [0, T], \ x \in H\).

Finally, if in addition \(\sup_{s \in [0, T], x \in H} |\psi(s, x, 0, 0)| < \infty\) and \(\phi\) and \(h\) are bounded then \(u\) is also bounded.

Proof. By theorem 2.9 by defining \(u(s, x) := Y^s_{\cdot, x}\), \(u\) has the regularity required in definition 3.2, point 1, and moreover points 2 and 4 immediately follow since \(Y\) is solution to the RBSDE in (2.16).

For what concerns point 3 of definition 3.2, since \(Y\) is solution to the reflected BSDE, we get

\[ u(s, x) = Y^s_{\cdot, x} + \int_s^t \psi(\tau, X^{s,x}_\tau, Y^{s,x}_\tau, Z^{s,x}_\tau) \ d\tau + K^s_{\cdot, x} - K^s_{s, x} - \int_s^t Z^{s,x}_\tau \ dW_\tau, \]  

(3.7)

Fixed \(\xi \in \Xi\), let us consider the joint quadratic variation of both side of (3.7) with \(W^\xi\). Proposition 2.1 in [11] and Theorem 3.1 yield that \(\tilde{\nabla}G^u\) exists and letting \(\zeta \in \tilde{\nabla}G^u\), we have

\[ \langle u(\cdot, X^{s,x}), W^\xi \rangle_{[s,t]} = \int_s^t \zeta(\sigma, X^{s,x}_\sigma) \xi \ d\sigma, \]

where

\[ W^\xi_t := \int_s^t \langle \xi, dW_\sigma \rangle, \quad 0 \leq s \leq t \leq T. \]

On the other hand by the Markov property stated in Remark 2.8

\[ u(t, X^{s,x}_t) = Y^{t,x}_{t, X^{s,x}_t} = Y^{s,x}_{t}, \]

and since \(Y\) is solution to the RBSDE in (2.16) we deduce:

\[ \langle Y^{s,x}, W^\xi \rangle_{[s,t]} = \int_s^t Z^{s,x}_\sigma \xi \ d\sigma. \]

So, by these two expression of the joint quadratic variation of \(u(\cdot, X^{s,x})\) and \(W^\xi\) we get

\[ \int_s^t \zeta(\sigma, X^{s,x}_\sigma) \xi \ d\sigma = \int_s^t Z^{s,x}_\sigma \xi \ d\sigma, \]  

(3.8)

\(\mathbb{P}\) a.s. Since both sides of (3.8) are continuous with respect to \(t\), it follows that, \(\mathbb{P}\) a.s., they coincide for all \(t \in [s, T]\). This implies that \(\zeta(t, X^{s,x}_t) = Z^{s,x}_t, \mathbb{P}\) a.s. for a.a. \(t \in [s, T]\). Therefore equation (3.7) can be rewritten as

\[ u(s, x) = Y^{s,x}_t + \int_s^t \psi(\tau, X^{s,x}_\tau, u(\tau, X^{s,x}_\tau), \zeta(\tau, X^{s,x}_\tau)) \ d\tau + K_{s, x}^{s,x} - K^{s,x}_s - \int_s^t Z^{s,x}_\tau \ dW_\tau, \]  

(3.9)

By taking the conditional expectation \(\mathbb{E}^{\mathcal{F}_t}\) and since \(K\) is a nondecreasing process, we get

\[ u(s, x) \geq Y^{s,x}_t + \int_s^t P_{s, \tau} \left[ \psi(\tau, \cdot, u(\tau, \cdot), \zeta(\tau, \cdot)) \right](x) \ d\tau, \]  

(3.10)
and we have proved that \( u \) is a mild supersolution along the Definition 3.2.

We have to prove that \( u \) is the minimal supersolution. Let \( \bar{u} \) be any supersolution and let us define \( \bar{Y}^{s,x}_t = \bar{u}(t, X^{s,x}_t) \). Then for every \( \sigma \in [s, t] \), with \( 0 \leq s \leq t \), by point 3 of definition 3.2 having replaced \( x \) with \( X^{s,x}_\sigma \) which is \( \mathcal{F}_\sigma \)-measurable,

\[
\bar{u}(\sigma, X^{s,x}_\sigma) \geq \mathbb{E}^{\mathcal{F}_s} \bar{u}(t, X^{s,x}_t) + \mathbb{E}^{\mathcal{F}_s} \int_s^t \psi(\tau, X^{s,x}_\tau, \bar{Y}^{s,x}_\tau, \bar{\zeta}(\tau, X^{s,x}_\tau)) \, d\tau. \tag{3.11}
\]

So it turns out that

\[
(L^{s,x}_\sigma)_{\sigma \in [s,T]} := \left( \bar{u}(\sigma, X^{s,x}_\sigma) - \int_s^\sigma \psi(\tau, X^{s,x}_\tau, \bar{Y}^{s,x}_\tau, \bar{\zeta}(\tau, X^{s,x}_\tau)) \, d\tau \right)_{\sigma \in [s,T]}.
\]

is a submartingale. By hypothesis 2.0 on \( \psi \), by the growth property of \( u \) as required in definition 3.2, point 1, by relation 3.4 and finally by Proposition 2.5 we get that \( L^{s,x} \) is a uniformly integrable martingale, so it is of class (D) and the Doob-Meyer decomposition can be applied, see e.g. Definition 4.8 and Theorem 4.10 in Chapter 1 of [13]. So \( L^{s,x} \) can be decomposed into:

\[
L^{s,x}_\sigma = \bar{M}^{s,x}_\sigma + \bar{K}^{s,x}_\sigma,
\]

where \( \bar{K}^{s,x} \) is an integrable nondecreasing process such that \( \bar{K}^{s,x}_\sigma = 0 \), and \( \bar{M}^{s,x} \) is a uniformly integrable martingale. Moreover, see [5], Chapter VII relation (15.1), since

\[
\mathbb{E} \sup_{\sigma \in [s,T]} |L^{s,x}_\sigma|^2 < \infty
\]

we have

\[
\bar{K}^{s,x}_T \in L^2(\Omega).
\]

Notice that we are working in a complete probability space filtered with the filtration generated by the Wiener process, so by the martingale representation theorem, see again [13] and [3] for its infinite dimensional version, there exists a process \( \bar{Z} \in L^2(\Omega \times [s,T]; L^2(\mathbb{R})) \) such that

\[
\bar{M}^{s,x}_\sigma = -\left[u(s, x) + \int_s^{\sigma} \bar{Z}^{s,x}_\tau \, dW_\tau \right].
\]

We finally get \( \forall \sigma \in [s, T] \)

\[
u(s, x) = \bar{u}(\sigma, X^{s,x}_\sigma) + \int_s^{\sigma} \psi(\tau, X^{s,x}_\tau, \bar{Y}^{s,x}_\tau, \bar{\zeta}(\tau, X^{s,x}_\tau)) \, d\tau + \bar{K}^{s,x}_\sigma - \bar{K}^{s,x}_s - \int_s^{T} \bar{Z}^{s,x}_\tau \, dW_\tau, \tag{3.12}
\]

that is, \( \forall 0 \leq s \leq t \leq T \)

\[
\bar{Y}^{s,x}_t = \bar{Y}^{s,x}_s + \int_s^{T} \psi(\tau, X^{s,x}_\tau, \bar{Y}^{s,x}_\tau, \bar{\zeta}(\tau, X^{s,x}_\tau)) \, d\tau + \bar{K}^{s,x}_t - \bar{K}^{s,x}_s - \int_s^{T} \bar{Z}^{s,x}_\tau \, dW_\tau. \tag{3.13}
\]

Finally we have to identify \( \tau \zeta(\tau, X^{s,x}_\tau) \) with \( \bar{Z}^{s,x}_\tau \), \( \mathbb{P} \)-a.s. for a.a. \( \tau \in [s, T] \). To this aim, for \( \xi \in \Xi \), let us consider the joint quadratic variation of both sides of (3.12) with \( W^\xi \). Notice that the finite variation term \( K \) does not give any contribution to the joint quadratic variation with \( W^\xi \); so Proposition 2.1 in [11] and Theorem 3.1 yield, for \( s \leq \sigma < T \) and \( \zeta \in \nabla^G u \),

\[
\int_s^{\sigma} \zeta(\tau, X^{s,x}_\tau) \, d\tau = \int_s^{\sigma} \bar{Z}^{s,x}_\tau \xi \, d\tau, \tag{3.14}
\]

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\( \mathbb{P} \)-a.s.. Since both sides of (3.14) are continuous with respect to \( \sigma \), it follows that, \( \mathbb{P} \)-a.s., they coincide for all \( \sigma \in [s, T] \). This implies that \( \zeta(\sigma, X_{\sigma}^{s,x}) = \tilde{Z}_{\sigma}^{s,x}, \mathbb{P} \)-a.s. for a.a. \( \sigma \in [s, T] \). So we get that, by defining, \( Y_{s,x}^s := \tilde{u}(s, x) \) and \( \tilde{Y}_{s,x}^s := \tilde{u}(-, X_{s,x}^s) \), the couple of processes \((Y_{s,x}^s, \tilde{Z}_{s,x}^s)\) solves the following problem

\[
\begin{align*}
\frac{dY_{s,x}^s}{dt} &= \psi(t, X_{s,x}^s, \tilde{Y}_{s,x}^s, \tilde{Z}_{s,x}^s) dt + d\tilde{K}_{s,x}^s - \tilde{Z}_{s,x}^s dW_t, \\
Y_T &= \phi(X_T^s), \\
Y_{s,x}^s &\geq h(X_{s,x}^s),
\end{align*}
\]

(3.15)

which is “almost” a reflected BSDE, what is lacking is the requirement that \( \tilde{K}_{s,x}^s \) is the minimal increasing process, namely it is not required the condition

\[
\int_s^T (\tilde{Y}_{s,x}^s - h(X_{s,x}^s)) d\tilde{K}_{s,x}^s = 0
\]

Now we have to compare \( \tilde{Y}_{s,x}^s \) with \( Y_{s,x}^s \). To this aim, extending a procedure used in [2], we compare \( \tilde{Y}_{s,x}^s \) with the penalized solution \( Y_{s,x}^{n,s,x} \) of equation (3.21) that we rewrite in integral form, for \( t \in [s, T] \)

\[
Y_{s,x}^{n,s,x} = \phi(X_T^s) + \int_t^T \psi(t, X_{s,x}^s, Y_{s,x}^{n,s,x}, Z_{s,x}^{n,s,x}) d\tau + \int_t^T n(Y_{s,x}^{n,s,x} - h(X_{s,x}^s)) - dt - \int_t^T Z_{s,x}^{n,s,x} dW_t.
\]

(3.16)

Applying Itô formula to the process \( e^{n(T-t)}Y_{s,x}^{n,s,x} \) we get

\[
\begin{align*}
\frac{d}{dt}e^{n(T-t)}Y_{s,x}^{n,s,x} &= ne^{n(T-t)}\psi(t, X_{s,x}^s, Y_{s,x}^{n,s,x}, Z_{s,x}^{n,s,x}) dt + ne^{n(T-t)}Y_{s,x}^{n,s,x} \nabla h(X_{s,x}^s) dW_t, \\
Y_T &= \phi(X_T^s),
\end{align*}
\]

(3.17)

Applying Itô formula to the process \( e^{n(T-t)}\tilde{Y}_{s,x}^s \) we get

\[
\begin{align*}
\frac{d}{dt}e^{n(T-t)}\tilde{Y}_{s,x}^s &= ne^{n(T-t)}\tilde{Y}_{s,x}^s + ne^{n(T-t)}\psi(t, X_{s,x}^s, \tilde{Y}_{s,x}^s, \tilde{Z}_{s,x}^s) dt + e^{n(T-t)} d\tilde{K}_{s,x}^s, \\
\tilde{Y}_T &= \phi(X_T^s)
\end{align*}
\]

(3.18)

Notice that in (3.18) we can replace \( \tilde{Y}_{s,x}^s \) by \( \tilde{Y}_{s,x}^s \) \( h(X_{s,x}^s) \) (recall that since \( \tilde{u} \) is a supersolution to the obstacle problem (3.11) it holds \( \tilde{u} \geq h \)). Assume for a moment the following lemma.

**Lemma 3.3** Let \( f^i : \Omega \times [0, T] \times \mathbb{R} \times \Xi \rightarrow \mathbb{R}, i = 1, 2 \) satisfy hypothesis (2.1) with \( p = 2 \), fix \( \xi \in L^2_{\mathbb{P},f}(\Omega) \) and let \( K \) be a progressively measurable nondecreasing processes with \( \mathbb{E} K_T^2 < \infty \). If \( (Y^1, Z^1) \) and \( (Y^2, Z^2) \) with \( Y^i \in L^2_{\mathbb{P},f}(\Omega, C([0, T])) \) and \( Z^i \in L^2_{\mathbb{P}}(\Omega \times [0, T], \Xi), i = 1, 2 \), are the solutions to the following equations of backward type:

\[
\begin{align*}
\frac{dY^1_t}{dt} &= f^1(t, Y^1_t, Z^1_t) dt + dK_t - Z^1_t dW_t, \\
Y^1_0 &= \xi,
\end{align*}
\]

(3.19)

and

\[
\begin{align*}
\frac{dY^2_t}{dt} &= f^2(t, Y^2_t, Z^2_t) dt - Z^2_t dW_t, \\
Y^2_0 &= \xi,
\end{align*}
\]

(3.20)

then we have that \( Y^1_t \geq Y^2_t \) \( \mathbb{P} \)-almost surely for any \( t \in [0, T] \).

(3.21)

\( \delta f_t := f^1(t, Y^1_t, Z^1_t) - f^2(t, Y^2_t, Z^2_t) \geq 0, d\mathbb{P} \times dt \) a.s.

\( 21 \)
By applying lemma 3.3 to the BSDEs 3.17 and 3.18 we get a comparison for the processes \((e^{n(T-t)}Y_t^{n,s,x})_{t \in [s,T]}\) and \((e^{n(T-t)}\tilde{Y}_t^{n,s,x})_{t \in [s,T]}\), namely we get

\[ e^{n(T-t)}\tilde{Y}_t^{n,s,x} \geq e^{n(T-t)}Y_t^{n,s,x} \tag{3.22} \]

almost surely and for any time \(t\), and consequently

\[ Y_t^{s,x} \geq Y_t^{n,s,x} \tag{3.23} \]

Now we let \(n \to \infty\) by [6], section 6, \(Y_t^{n,s,x} \uparrow Y_t^{s,x}\) for any \(s \leq t \leq T\) and \(\mathbb{P}\)-a.s.. So taking \(s = t\) in (3.23) we finally get

\[ \tilde{u}(s, x) \geq u(s, x), \tag{3.24} \]

for any \(\tilde{u}\) supersolution for the obstacle problem 3.1. So the minimality of \(u\) is proved: the unique solution to the obstacle problem 3.1 is given by formula (3.6) and the other properties follows by estimates (2.20), which passes to the limit as \(n \to \infty\) as stated in proposition 2.7 on the solution of the RBSDE in terms of the growth of the \(\psi, h\) and \(\phi\).

\[ \square \]

**In order to complete the proof of theorem 3.2, we have to prove lemma 3.3.**

**Proof of Lemma 3.3.** We adequite the proof of the classical comparison theorem for BSDEs given in [7], Theorem 2.2, to the equations 3.19 and 3.20. By denoting

\[ \Delta_y f_t^1 = \frac{f^1(t, Y_t^1, Z_t^1) - f^1(t, Y_t^2, Z_t^1)}{Y_t^1 - Y_t^2} \quad \text{if } Y_t^1 - Y_t^2 \neq 0, \quad \Delta_y f_t^1 = 0 \text{ otherwise}, \]

\[ \Delta_z f_t^1 = \frac{f^1(t, Y_t^2, Z_t^1) - f^1(t, Y_t^2, Z_t^2)}{|Z_t^1 - Z_t^2|^2} (Z_t^1 - Z_t^2) \quad \text{if } Z_t^1 - Z_t^2 \neq 0, \quad \Delta_z f_t^1 = 0 \text{ otherwise}, \]

\[ \delta_2 \text{ as defined in (3.21), } \delta Y_t = Y_t^1 - Y_t^2 \text{ and } \delta Z_t = Z_t^1 - Z_t^2 \text{ we get} \]

\[ \left\{ \begin{array}{l}
-\delta_t Y_t = \Delta_y f_t^1 \delta_t Y_t dt + (\Delta_z f_t^1)^* \delta_t Z_t dt + \delta_2 f_t dt + dK_t - \delta_t Z_t dW_t, \\
\delta Y_T = 0
\end{array} \right. \quad t \in [0, T], \tag{3.25} \]

We notice that \(\Delta_y f_t^1\) and \(\Delta_z f_t^1\) are bounded and that \(\delta_2 f \in L_T^2(\Omega \times [0, T], \mathbb{R})\).

Multiplying \(\delta Y_t\) by \(\exp(\int_0^T \Delta_y f_t^1 d\tau)\) and then applying Girsanov theorem we obtain:

\[ \delta Y_t = \mathbb{E} \left( \rho_{t,T} \left[ \int_t^T \exp \left( \int_t^s \Delta_y f_t^1 d\tau \right) dK_s + \int_t^T \exp \left( \int_t^s \Delta_y f_t^1 d\tau \right) \delta_2 f_s ds \right] \right) \tag{3.26} \]

where \(\rho_{t,T}\) is the Girsanov density:

\[ \rho_{t,T} = \exp \left( \int_t^T (\Delta_z f_t^1)^* dW_s - \frac{1}{2} \int_t^T |\Delta_z f_t^1|^2 ds \right). \]

The claim obviously follows from (3.26) being \((K)\) non decreasing and \(\delta_2 Y\) non negative. \[ \square \]
4 The Optimal Control-Stopping problem

An Admissible Control System is a set

\[ S = (\Omega^S, \mathcal{F}^S, (\mathcal{F}^S_t)_{t \geq 0}, \mathbb{P}^S, (W^S_t)_{t \geq 0}) \]

where \((\Omega^S, \mathcal{F}^S, (\mathcal{F}^S_t)_{t \geq 0}, \mathbb{P}^S)\) is a complete probability space endowed with a filtration satisfying the usual assumptions and \((W^S_t)_{t \geq 0}\) is a cylindrical Wiener process in \(\mathfrak{X}\). Fixed a closed subset \(U\) of a normed space \(U_0\) an admissible control in the setting \(S\) is any \((\mathcal{F}^S_t)\)-predictable process \(\alpha : [0, T] \to U\). The set of all admissible controls will be denoted by \(U^S\).

We fix a function \(R: H \times U \to \mathbb{R}\) bounded, continuous such that:

\[ |R(\alpha, x) - R(\alpha, x')| \leq |x - x'| \quad \forall u \in U, x, x' \in H \quad (4.1) \]

Moreover, given an admissible setting \(S\) and an admissible control \(\alpha \in U^S\) and fixed \(x \in H, s \in [0, T]\) by \(X^\alpha_{s,x}\) we will denote the solution to the following stochastic differential equation in a Hilbert space \(H\)

\[
\begin{aligned}
&\begin{cases}
  dX^\alpha_{s,x} = AX^\alpha_{s,x} dt + F(t, X^\alpha_{s,x})dt + G(t, X^\alpha_{s,x})(R(X^\alpha_{s,x},\alpha_t)dt + dW^S_t),
  &t \in [s, T] \\
  X^\alpha_{s,x} = x \in H.
\end{cases}
\end{aligned}
\]

Moreover given \(l : [0, T] \times H \times U_0 \to \mathbb{R}\) we introduce the cost functional:

\[
J(s, x, \tau, \alpha) = \mathbb{E} \int_s^\tau l(r, X^\alpha_{r,x}, \alpha_r) \, dr + \mathbb{E}[\phi(X^\alpha_{T,x}) \chi_{\tau=T}] + \mathbb{E}[h(\tau, X^\alpha_{\tau,x}) \chi_{\tau<T}],
\]

that we wish to maximize over all admissible control \(\alpha \in U^S\) and over all \((\mathcal{F}^S_t)\)-stopping times \(\tau\) satisfying \(t \leq \tau \leq T\).

For \(s \in [0, T], x \in H, z \in \mathfrak{X}\) we define the hamiltonian function in the usual way as

\[ \psi(s, x, z) = \sup_{\alpha \in \mathcal{U}} \{ zR(x, \alpha) + l(s, x, \alpha) \}. \quad (4.4) \]

We notice that since \(R\) is bounded \(\psi\) is Lipschitz with respect to \(z\). We will assume throughout this section that \(A, F \) and \(G\) verify Hypothesis (2.4) and that \(\phi, \psi\) and \(h\) verify Hypothesis (2.6). Moreover we assume that \(|l(s, x, \alpha)| \leq c(1 + |x|^r)\) for some \(c, r > 0\).

We notice that under the above assumptions, fixed \(s \in [0, T]\) and \(x \in H\) then for all \(\alpha \in U^S\) there exists a unique mild solution \(X^\alpha_{s,x}\) to equation (4.2). Moreover \(X^\alpha_{s,x} \in L_p^p(\Omega, C([s, T], H))\) for all \(p \geq 1\), see [3]. Consequently \(J(s, x, \tau, \alpha)\) is a well defined real number for all \(\alpha \in U^S\) and all \((\mathcal{F}^S_t)\)-stopping time \(\tau \leq T\). We also notice that \(X^\alpha_{s,x}\) is adapted to the filtration generated by \((W^S_t)\).

By the Girsanov theorem, there exists a probability measure \(\mathbb{P}^{S,\alpha}\) such that the process

\[ W^S_{t,\alpha} := W^S_t + \int_s^t R(X^\alpha_{r,x}, \alpha_r) \, dr \quad t \geq s \]

is a cylindrical \(\mathbb{P}^{S,\alpha}\)-Wiener process in \(\mathfrak{X}\). We denote by \((\mathcal{F}^S_{t,\alpha})_{t \geq s}\) its natural filtration, augmented in the usual way. \(X^\alpha_{s,x}\) satisfies the following equation:

\[
\begin{aligned}
&\begin{cases}
  dX^\alpha_{t,x} = AX^\alpha_{t,x} dt + F(t, X^\alpha_{t,x})dt + G(t, X^\alpha_{t,x})dW^S_{t,\alpha},
  &t \in [s, T] \\
  X^\alpha_{s,x} = x.
\end{cases}
\end{aligned}
\]

Consequently (notice that the above equation enjoys strong existence, in probabilistic sense, and pathwise uniqueness) \(X^\alpha_{t,x}\) turns out to be adapted to \((\mathcal{F}^S_{t,\alpha})_{t \geq s}\).
In \( \left( \Omega^S, \mathcal{F}^S, (\mathcal{F}^S_t)_{t \geq 0}, \mathbb{P}^S, \alpha \right) \) we consider the solution \((\tilde{Y}^{s,x}, \tilde{Z}^{s,x}, \tilde{K}^{s,x})\) of the following reflected backward stochastic differential equation:

\[
- \frac{d\tilde{Y}^{s,x}}{dt} = \psi(s, X_t^{\alpha,s,x}, \tilde{Z}^{s,x}) \ dt + d\tilde{K}^{s,x}_t - \tilde{Z}^{s,x}_tdW^S_t, \quad t \in [0, T],
\]

\[
\tilde{Y}^{s,x}_T = \phi(X_T^{\alpha,s,x}),
\]

\[
\tilde{Y}^{s,x}_t \geq h(t, X_t^{\alpha,s,x}),
\]

\[
\int_0^T (\tilde{Y}^{s,x}_t - h(t, X_t^{\alpha,s,x}))d\tilde{K}^{s,x}_t = 0,
\]

We omit to indicate the dependence on the admissible setting \( S \) and on the admissible control \( \alpha \) since the law of \( \tilde{Y}^{s,x}, \tilde{Z}^{s,x} \) and \( \tilde{K}^{s,x} \) is uniquely determined by \( A, F, G, x, \psi \) and \( \phi \), and does not depend on the probability space and on the Wiener process, and in particular \( \tilde{Y}^{s,x}_s \) is a real number that does not depend on \( S \) and on \( \alpha \). We argue as in [6], proposition 2.3. Rewriting (4.6) in terms of the original noise \((W^S)\) and integrating it between \( s \) and any \((\mathcal{F}^S)\)-stopping time \( \tau \), we get that \( \mathbb{P}^S\)-a.s., and consequently \( \mathbb{P}^S, \alpha \)-a.s.,

\[
\tilde{Y}^{s,x}_s = \tilde{Y}^{s,x}_\tau + \int_s^\tau \psi(r, X_r^{\alpha,s,x}, \tilde{Z}^{s,x}_r) \ dr + K^{s,x}_\tau - K^{s,x}_s - \int_s^\tau \tilde{Z}^{s,x}_r dW^S - \int_s^\tau \tilde{Z}^{s,x}_r R(X_r^{\alpha,s,x}, \alpha_r) \ dr
\]

Noticing that \( \left( \int_0^T \tilde{Z}^{s,x}_r dW^S \right)_{t \geq 0} \) is a \( \mathbb{P}^S \)-martingale and that \( \tilde{Y}^{\alpha,s,x}_r \geq h(r, X_r^{\alpha,s,x}) \) by computing expectation with respect to \( \mathbb{P}^S \) we get:

\[
\tilde{Y}^{s,x}_s \geq \mathbb{E}\left[ \int_s^\tau \psi(r, X_r^{\alpha,s,x}, \tilde{Z}^{s,x}_r) \ dr - \mathbb{E}\left[ \int_s^\tau \tilde{Z}^{s,x}_r \alpha_r \ dr + \mathbb{E}[K^{s,x}_\tau - K^{s,x}_s] + \mathbb{E}[h(\tau, X_\tau^{\alpha,s,x})\chi_{\{\tau < T\}}] + \mathbb{E}[\phi(X_\tau^{\alpha,s,x})\chi_{\{\tau = T\}}]\right]\right],
\]

Finally adding and subtracting the current cost we have:

\[
\tilde{Y}^{s,x}_s \geq J(s, x, \tau, \alpha) + \mathbb{E}\left[ \int_s^\tau \left[ \psi(r, X_r^{\alpha,s,x}, \tilde{Z}^{s,x}_r) - l(r, X_r^{\alpha,s,x}, \alpha_r) - Z^{s,x}_r \alpha_r \right] \ dr + \mathbb{E}[K^{s,x}_\tau - K^{s,x}_s] \right].
\]

We have therefore proved the following result

**Theorem 4.1** For every admissible setting \( S \) and every admissible control \( u \in \mathcal{U}^S \) we have:

\[
J(s, x, \tau, \alpha) \leq \tilde{Y}^{s,x}_s
\]

moreover the equality holds if and only if

\[
\psi(r, X_r^{\alpha,s,x}, \tilde{Z}^{s,x}_r) - l(r, X_r^{\alpha,s,x}, \alpha_r) - Z^{s,x}_r \alpha_r = 0, \quad \mathbb{P} - \text{a.s. for a.e.r} \in [s, \tau]
\]

\[
K^{s,x}_\tau - K^{s,x}_s = 0, \quad \mathbb{P} - \text{a.s.}
\]

\[
\tilde{Y}^{s,x}_s \chi_{\{\tau < T\}} = h(\tau, X_\tau^{\alpha,s,x})I_{\{\tau < T\}}, \quad \mathbb{P} - \text{a.s.}
\]

**Remark 4.2** Fixed an admissible setting \( S \) and admissible control \( \alpha \in \mathcal{U}^S \) let \( \bar{\tau} \) be defined as

\[
\bar{\tau} = \inf\{t \leq \tau \leq T : \tilde{Y}^{s,x}_t = h(r, X_t^{\alpha,s,x})\} \wedge T.
\]

The condition \( \int_0^T (\tilde{Y}^{s,x}_t - h(t, X_t^{\alpha,s,x}))dK^{s,x}_t = 0 \) together with continuity and monotonicity of \( \tilde{K} \) imply that

\[
\tilde{K}^{s,x}_s - \tilde{K}^{s,x}_s = 0.
\]

Moreover (4.10) follows by definition. Consequently we have:

\[
\tilde{Y}^{s,x}_s = J(s, x, \bar{\tau}, \alpha) + \mathbb{E}\left[ \int_s^\bar{\tau} \left[ \psi(r, X_r^{\alpha,s,x}, \tilde{Z}^{s,x}_r) - l(r, X_r^{\alpha,s,x}, \alpha_r) - Z^{s,x}_r \alpha_r \right] \ dr \right].
\]
Taking into account equations (4.5), (4.6) and Proposition 5.2, the above results can be reformulated as follows.

**Corollary 4.3** Let \( u \) be the minimal mild supersolution to the obstacle problem and let \( \zeta \) be any element of its generalized gradient. Given any admissible setting \( S \) and any admissible control \( \alpha \in U^S \), we have:

\[
J(s, x, \tau, \alpha) \leq u(s, x)
\]

moreover the equality holds if and only if

\[
\psi(r, X_r^{\alpha, s, x}, \zeta(r, X_r^{\alpha, s, x})) - l(r, X_r^{\alpha, s, x}, \alpha_r) - \zeta(r, X_r^{\alpha, s, x})\alpha_r = 0, \quad \mathbb{P} \text{ a.s. for } a.e. r \in [s, \tau]
\]

\[
K_r^{s, x} - \tilde{K}_r^{s, x} = 0, \quad \mathbb{P} \text{ a.s.,}
\]

\[
u(\tau, X_r^{\alpha, s, x})I_{\{\tau < T\}} = h(\tau, X_r^{\alpha, s, x})I_{\{\tau < T\}}, \quad \mathbb{P} \text{ a.s.}
\]

Finally if

\[
\tau = \inf\{t \leq r \leq T : u(r, X_r^{\alpha, s, x}) = h(r, X_r^{\alpha, s, x})\} \wedge T.
\]

then the equality holds if and only if (4.13) holds.

We come now to the existence of optimal controls. We shall exploit the weak formulation of the control problem and select a suitable admissible setting \( \tilde{S} \). We assume the following.

**Hypothesis 4.4** The minimum in the definition (4.4) is attained for all \( t \in [s, T] \), \( x \in H \) and \( z \in \Xi^* \) e.g. if we define

\[
\Gamma(s, x, z) = \{ \alpha \in U : zR(x, \alpha) + l(s, x, \alpha) = \psi(s, x, z)\}
\]

then \( \Gamma(s, x, z) \neq \emptyset \) for every \( s \in [0, T] \), every \( x \in H \) and every \( z \in \Xi^* \).

**Remark 4.5** By [1], see Theorems 8.2.10 and 8.2.11, under the above assumption \( \Gamma \) always admits a measurable selection, i.e. there exists a measurable function \( \gamma : [0, T] \times H \times \Xi^* \rightarrow U \) with \( \gamma(s, x, z) \in \Gamma(s, x, z) \) for every \( s \in [0, T] \), every \( x \in H \) and every \( z \in \Xi^* \).

Moreover we notice that if \( U \) is compact then Hypothesis 4.4 always hold.

**Theorem 4.6** Assume Hypothesis 4.4 and fix a measurable selection \( \gamma \) of \( \Gamma \), \( s \in [0, T] \), \( x \in H \) and an element \( \zeta \) of the generalized gradient of the minimal supersolution \( u \) of the obstacle problem (3.1); then there exists at least an admissible setting \( \tilde{S} \) in which the closed loop equation

\[
\begin{cases}
\frac{d\tilde{X}_t}{dt} = A\tilde{X}_t + F(t, \tilde{X}_t)dt + G(t, \tilde{X}_t)[R(t, \gamma(t, \zeta(t, \tilde{X}_t))) + dW^S_t] \\
\tilde{X}_s = x.
\end{cases}
\]

admits a mild solution.

**Proof.** We fix any admissible setting

\[
S = (\Omega^S, \mathcal{F}^S, (\mathcal{F}^S_t)_{t \geq 0}, \mathbb{P}^S, (W^S_t)_{t \geq 0})
\]

and consider the uncontrolled forward SDE

\[
\begin{cases}
\frac{dX_t}{dt} = AX_t + F(t, X_t)dt + G(t, X_t)dW^S_t, \\
X_s = x.
\end{cases}
\]

By the Girsanov theorem, there exists a probability measure \( \hat{\mathbb{P}} \) such that the process

\[
\hat{W}_t := W^S_t - \int_s^t R(X_r^{s,x}, \zeta(s, X_r^{s,x}))dr,
\]

\( t \geq s \)

25
is a cylindrical $\hat{\mathbb{P}}$-Wiener process in $\Xi$. We denote by $(\hat{\mathcal{F}}_t)_{t \geq s}$ its natural filtration, augmented in the usual way. Clearly $X$ solves

$$
\begin{align*}
\begin{cases}
    dX_t = AX_t dt + F(t, X_t) dt + G(t, X_t) [R(X_t, \gamma(t, X_t) dt + d\hat{W}_t], & t \in [s, T] \\
    \hat{X}_s = x.
\end{cases}
\end{align*}
$$

(4.17)

and $(\Omega^S, \mathcal{F}^S, (\hat{\mathcal{F}}_t)_{t \geq 0}, \hat{\mathbb{P}}, (\hat{W}_t)_{t \geq 0})$ is the desired admissible system. $\square$

We finally get the following

**Theorem 4.7** Assume Hypothesis [4.4] and fix a measurable selection $\gamma$ of $\Gamma$, $s \in [0, T]$, $x \in H$ and an element $\zeta$ of the generalized gradient of the minimal supersolution $u$ of the obstacle problem [4.14]. Moreover let $\mathcal{S}$ be an admissible setting in which the closed loop equation [4.15] admits a mild solution then there exists $\bar{\alpha} \in \mathcal{U}^S$ and an $(\mathcal{F}^S)$ stopping time $\bar{\tau}$ for which

$$
\bar{Y}^s_{s,x} = u(s,x) = J(s,x, \bar{\tau}, \bar{\alpha}).
$$

*Proof:* Just let $\bar{X}$ be the mild solution of equation [4.15] and define $\bar{\alpha} = \gamma(t, \zeta(t, \bar{X}_t))$ clearly $\bar{X}_t = X^{\bar{\alpha}, s, x}$ and relation [4.13] holds. Thus by Corollary [4.3] it is enough to choose

$$
\bar{\tau} = \inf\{t \leq r \leq T : u(r, \bar{X}_r) = h(r, \bar{X}_r)\} \wedge T.
$$

$\square$

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