Reinforcement Learning of Markov Decision Processes with Peak Constraints

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Abstract
In this paper, we consider reinforcement learning of Markov Decision Processes (MDP) with peak constraints, where an agent chooses a policy to optimize an objective and at the same time satisfy additional peak constraints. The agent has to take actions based on the observed states, reward outputs, and constraint-outputs, without any knowledge about the dynamics, reward functions, and/or the knowledge of the constraint functions. We introduce a transformation of the original problem in order to apply reinforcement learning algorithms where the agent maximizes a bounded and unconstrained objective. We show that the policies obtained from the transformed problem are optimal whenever the original problem is feasible. Our solution is memory efficient and doesn’t require to store the values of the constraint functions. To the best of our knowledge, this is the first time learning algorithms guarantee convergence to optimal stationary policies for the MDP problem with peak constraints for discounted and expected average rewards, respectively.

Keywords: Markov Decision Process, Reinforcement Learning, Peak Constraints, Memory Efficient Learning

1. Introduction
1.1. Motivation

Reinforcement learning is concerned with optimizing an objective function that depends on a given agent’s action and the state of the process to be controlled. However, many applications in practice require that we take actions that are subject to additional constraints that need to be fulfilled. One example is wireless communication where the total transmission power of the connected wireless devices is to be minimized subject to constraints on the quality of service (QoS) such as maximum delay constraints Djonin and Krishnamurthy (2007). Another example is the use of reinforcement learning methods to select treatments for future patients. To achieve just the right effect of a drug for specific patients, one needs to make sure that other important patient values satisfy some peak constraints that should not be violated in the short or long run for the safety of the patient.

Informally, the problem of reinforcement learning for Markov decision processes with peak constraints is described as follows (note that bandit optimization with peak constraints becomes a special case). Given a stochastic process with state $s_k$ at time step $k$, reward function $r$, and a discount factor $0 < \gamma < 1$, the constrained reinforcement learning problem is that for the optimizing agent to find a stationary policy $\pi(s_k)$, taking values in some finite set $A$, that minimizes the
discounted reward
\[
\sum_{k=0}^{\infty} \gamma^k \mathbb{E} (r(s_k, \pi(s_k)))
\]
(1)
or the expected average reward
\[
\lim_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T-1} \mathbb{E} (r(s_k, \pi(s_k)))
\]
(2)
subject to the constraints
\[
r^j(s_k, a_k) \geq 0, \quad a_k \in A, \quad \text{for all } k, \quad j = 1, \ldots, J
\]
(3)
(a more formal definition of the problem is introduced in the next section).

The \textit{peak} constrained reinforcement learning problem of Markov decision processes is that of finding an optimal policy that satisfies a number of peak constraints of the form (3). The constraints (3) can be equivalently written as \(a_k \in A(s_k)\), where the feasibility set \(A(s_k)\) is state dependent and is given by the inequality constraints (3). If the functions \(r_1, \ldots, r_J\) where known, then the optimization problem is straightforward and standard \(Q\)-learning solves the problem in the case of unknown process and reward \(r\). However, if we don’t know the constraint functions \(r_1, \ldots, r_J\), then we don’t know the feasibility set \(A(s_k)\), and hence we need to learn it. It’s important to note that since the constraint functions are unknown, it’s inevitable that we violate the constraints in the learning process. The agent may, however, measure the samples \(r_1(s_k, a_k), \ldots, r_J(s_k, a_k)\). The goal of this paper is to provide an algorithm that asymptotically converges to a feasible and optimal solution. One way could be to wait until one passes through all pairs \((s, a)\) to observe \(r_1(s, a), \ldots, r_J(s, a)\), learn them, and then construct the feasibility sets \(A(s)\) and run \(Q\)-learning for optimizing. However, this imposes huge cost in terms of memory, as one must store \(r_1(s, a), \ldots, r_J(s, a)\) for all \((s, a)\), which mounts to \(S \times A \times J\) elements. Furthermore, not optimizing along the learning process could be costly.

The following example from wireless communication describes in more detail a model where we have a Markov decision process with hard (peak) constraints and where the agent doesn’t have knowledge of the process and reward functions, but only observations of the state \(s\) and reward samples \(r(s, a), r_1(s, a), \ldots, r_J(s, a)\).

\textbf{Example 1 (Wireless communication)} Consider the problem of wireless communication were the goal is to minimize the average of the transmitted power subject to a strict quality of service (QoS) constraint. Let \(s_k\) denote the channel state at time step \(k\) which belongs to a finite set and let the \(a_k\) be the bandwidth allocation action, also belonging to a finite set of actions. The power required to occupy a bandwidth \(a\) given the channel state is \(P(s, a)\), which is unknown to the agent. The power affects the channel state, and hence, the channel evolves according to a probability distribution given by \(p(s_{k+1} | s_k, a_k)\) which is also unknown. The QoS is given by a lower bound \(b\) on the bit error rate, given by \(q(s_k, a_k) \geq b\). The function \(q(s_k, a_k)\) is not known as it is affected by noise that is not accessible to the agent. By introducing \(r(s, a) = -P(s, a)\) and \(r^1(s, a) = q(s, a) - b\), the task is to solve the following optimization problem
\[
\sup_{a_k} \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathbb{E}(r(s_k, a_k))
\]
\[
\text{s. t. } r^1(s_k, a_k) \geq 0
\]
Example 2 (Search Engine) In a search engine, there is a number of documents that are related to a certain query. There are two values that are related to every document, the first being a (advertisement) value $u_i$ of document $i$ for the search engine and the second being a value $v_i$ for the user (could be a measure of how strongly related the document is to the user query). The task of the search engine is to display the documents in a row some order, where each row has an attention value, $A_j$ for row $j$. We assume that $u_i$ and $v_i$ are known to the search engine for all $i$, whereas the attention values $\{A_j\}$ are not known. The action of the search engine is to display document $i$ in position $j$, $a_i = j$. Thus, the expected average reward for the search engine is

$$R = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}(u_i A_{a(i)})$$

The search engine has multiple objectives here where it wants to maximize the rewards for the user and itself. One solution is to define a measure for the quality of service, $v_i A_{a(i)} \geq q$ and then maximize its reward subject to the quality of service constraint, that is

$$\sup_{a_i} \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}(u_i A_{a(i)})$$

s. t. $v_i A_{a(i)} - q \geq 0$

Although constrained Markov decision process problems are fundamental and have been studied extensively in the literature (see Altman (1999) and the references therein), the reinforcement learning counterpart of finding the optimal policies seem to be still open, and even less is known for the case of peak constraints considered in this paper. When an agent has to take actions based on the observed states, rewards outputs, and constraint-outputs solely (without any knowledge about the dynamics, reward functions, and/or the knowledge of the constraint functions), a general solution seem to be lacking to the best of the authors’ knowledge.

1.2. Previous Work

Most of the work on Markov decision processes and bandit optimization considers constraints in the form of discounted or expected average rewards Altman (1999). Constrained MDP problems are convex and hence one can convert the constrained MDP problem to an unconstrained zero-sum game where the objective is the Lagrangian of the optimization problem Altman (1999). However, when the dynamics and rewards are not known, it doesn’t become apparent how to do it as the Lagrangian will itself become unknown to the optimizing agent. Previous work regarding constrained MDP:s, when the dynamics of the stochastic process are not known, considers scalarization through weighted sums of the rewards, see Roijers et al. (2013) and the references therein. Another approach is to consider Pareto optimality when multiple objectives are present Moffaert and Nowé (2014). However, none of the aforementioned approaches guarantee to satisfy lower bounds for a given set of reward functions simultaneously. In Gabor et al. (1998), a multi-criteria problem is considered where the search is over deterministic policies. In general, however, deterministic policies are not optimal Altman (1999). Also, the multi-criteria approach in Gabor et al. (1998) may provide a deterministic solution to a multi-objective problem in the case of two objectives and it’s not clear how to generalize to a number of objectives larger than two. In Geibel (2006), the author considers a single constraint and allowing for randomized policies. However, no proofs of convergence are
provided for the proposed sub-optimal algorithms. Sub-optimal solutions with convergence guarantees are provided in Chow et al. (2017) for the single constraint problem, allowing for randomized polices. In Borkar (2005), an actor-critic sub-optimal algorithm is provided for one single constraints and it’s claimed that it can generalized to an arbitrary number of constraints. Sub-optimal solutions to constrained reinforcement learning problems with expected average rewards in a wireless communications context were considered in Djonin and Krishnamurthy (2007). Sub-optimal reinforcement learning algorithms were presented in Lizotte et al. (2010) for controlled trial analysis with multiple rewards, again by considering a scalarization approach. In Drugan and Nowé (2013), multi-objective bandit algorithms were studied by considering scalarization functions and Pareto partial orders, respectively, and present regret bounds. As previous results, the approach in Drugan and Nowé (2013) doesn’t guarantee to satisfy the constraints that correspond to the multiple objectives. In Achiam et al. (2017), constrained policy optimization is studied for the continuous MDP problem and some heuristics algorithms were suggested.

1.3. Contributions

We consider the problem of optimization and learning for Markov decision processes with peak constraints given by (3), for both discounted and expected average rewards, respectively. We reformulate the optimization problem with peak constraints to a zero-sum game where the opponent acts on a finite set of actions (and not on a continuous space of actions). This transformation is essential in order to achieve a tractable optimal algorithm. The reason is that using Lagrange duality without model knowledge requires infinite dimensional optimization since the Lagrange multipliers are continuous (compare to the intractability of a partially observable MDP, where the beliefs are continuous variables). Furthermore, the Lagrange multipliers imply that the reward function is unbounded. We introduce a reformulation of the problem that is proved to be equivalent, where we get bounded reward functions and thereafter apply reinforcement learning algorithms that converge to optimal policies for the Markov decision process problem with peak constraints. The algorithm works without imposing a cost in terms of memory, where there is no need to store the learned values of $r_1(s,a), ..., r_J(s,a)$ for all $(s,a)$, saving memory of size $S \times A \times J$. We give complete proofs for both cases of discounted and expected average rewards, respectively.
1.4. Notation

- $\mathbb{N}$: The set of nonnegative integers.
- $[J]$: The set of integers $\{1, ..., J\}$.
- $\mathbb{R}$: The set of real numbers.
- $\mathbb{E}$: The expectation operator.
- $\mathbb{P}$: $\Pr(x \mid y)$ denotes the probability of the stochastic variable $x$ given $y$.
- $\arg \max$: $\pi^* = \arg \max_{\pi \in \Pi} f_\pi$ denotes an element $\pi^* \in \Pi$ that maximizes the function $f_\pi$.
- $\geq$: For $\lambda = (\lambda^1, ..., \lambda^J)$, $\lambda \geq 0$ denotes that $\lambda^i \geq 0$ for $i = 1, ..., J$.
- $1_{\{X\}}(x)$: $1_{\{X\}}(x) = 1$ if $x \in X$ and $1_{\{X\}}(x) = 0$ if $x \notin X$.
- $1_n$: $1_n = (1,1, ..., 1) \in \mathbb{R}^n$.
- $N(t, s, a)$: $N(t, s, a) = \sum_{k=1}^t 1_{(s,a)}(s_k, a_k)$.
- $e$: $e : (s,a) \mapsto 1$.
- $|S|$: Denotes the number of elements in $S$.
- $s_+$: For a state $s = s_k$, we have $s_+ = s_{k+1}$.

2. Problem Formulation

Consider a Markov Decision Process (MDP) defined by the tuple $(S, A, P)$, where $S = \{S_1, S_2, ..., S_n\}$ is a finite set of states, $A = \{A_1, A_2, ..., A_m\}$ is a finite set of actions taken by the agent, and $P : S \times A \times S \rightarrow [0,1]$ is a transition function mapping each triple $(s, a, s_+)$ to a probability given by

$$P(s, a, s_+) = \Pr(s_+ \mid s, a)$$

and hence,

$$\sum_{s_+ \in S} P(s, a, s_+) = 1, \quad \forall (s,a) \in S \times A$$

Let $\Pi$ be the set of policies that map a state $s \in S$ to a probability distribution of the actions with a probability assigned to each action $a \in A$, that is $\pi(s) = a$ with probability $\Pr(a \mid s)$. The agent’s objective is to find a stationary policy $\pi \in \Pi$ that maximizes the expected value of the total the discounted reward

$$\sum_{k=0}^{\infty} \gamma^k \mathbb{E}(r(s_k, \pi(s_k)))$$

or the expected average reward

$$\lim_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T-1} \mathbb{E}(r(s_k, \pi(s_k)))$$

for $s_0 = s \in S$, where $r : S \times A \rightarrow \mathbb{R}$ is some unknown reward function. The parameter $\gamma \in (0,1)$ is a discount factor which models how much weight to put on future rewards. The expectation is taken with respect to the randomness introduced by the policy $\pi$ and the transition mapping $P$. 

The (hard or peak) constrained reinforcement learning problem is concerned with finding a policy that satisfies a set of peak constraints given \((3)\), where \(r^j : S \times A \rightarrow \mathbb{R}\) are bounded functions, for \(j = 1, \ldots, J\). The agent doesn’t have knowledge of the process and reward functions, but only measurements of the state \(s\) and reward samples \(r(s_k, a_k), r_1(s_k, a_k), \ldots, r_J(s_k, a_k)\). However, the agent knows that the reward functions are bounded by some constant \(c\).

**Assumption 1** The absolute values of the reward functions \(r\) and \(\{r^j\}_{j=1}^J\) are bounded by some constant \(c\) known to the agent.

**Assumption 2** \(r(s, a) > 0\) for all \((s, a) \in S \times A\).

Note that Assumption 2 is not restrictive since we can replace \(r\) with \(r + c + \epsilon\) for some positive real number \(\epsilon > 0\) and obtain an equivalent problem with positive reward \(r\) in the objective function and it will remain bounded by the constant \(2c + \epsilon\).

3. Reinforcement Learning for Markov Decision Processes

3.1. Discounted Rewards

Consider a Markov Decision Process where the agent is maximizing the total discounted reward given by

\[
V(s_0) = \sum_{k=0}^{\infty} \gamma^k \mathbb{E}(R(s_k, a_k))
\]

for some initial state \(s_0 \in S\). Let \(Q^*(s, a)\) be the expected reward of the agent taking action \(a \in A\) from state \(s \in S\), and continuing with an optimal policy thereafter.

Then for any stationary policy \(\pi\), we have that

\[
Q^*(s, a) = R(s, a) + \max_{\pi \in \Pi} \sum_{k=1}^{\infty} \gamma^k \mathbb{E}(R(s_k, \pi(s_k)))
\]

\[
= R(s, a) + \gamma \cdot \max_{\pi \in \Pi} \mathbb{E}(Q^*(s_+, \pi(s_+)))
\]

Equation (21) is known as the Bellman equation, and the solution to (21) with respect to \(Q^*\) that corresponds to the optimal policy \(\pi^*\) and optimal actions of the opponent is denoted \(Q^*\). If we have the function \(Q^*\), then we can obtain the optimal policy \(\pi^*\) according to the equation

\[
\pi^*(s) = \arg \max_{\pi \in \Pi} \mathbb{E}(Q^*(s, \pi(s)))
\]

which maximizes the total discounted reward

\[
\sum_{k=0}^{\infty} \gamma^k \mathbb{E}(R(s_k, \pi^*(s_k))) = \mathbb{E}(Q^*(s, \pi^*(s)))
\]

for \(s = s_0\). Note that the optimal policy may not be deterministic, as opposed to reinforcement learning for unconstrained Markov Decision Processes, where there is always an optimal policy that is deterministic.

In the case we don’t know the process \(P\) and the reward function \(R\), we will not be able to take advantage of the Bellman equation directly. The following results show that we will be able to design an algorithm that always converges to \(Q^*\).
**Definition 1 (Unichain MDP)** An MDP is called unichain, if for each policy $\pi$ the Markov chain induced by $\pi$ is ergodic, i.e. each state is reachable from any other state.

Unichain MDP:s are usually considered in reinforcement learning problems with discounted rewards, since they guarantee that we learn the process dynamics from the initial states. Thus, for the discounted reward case we will make the following assumption.

**Assumption 3 (Unichain MDP)** The MDP $(S, A, P)$ is assumed to be unichain.

**Proposition 1** Consider a Markov Decision Process given by the tuple $(S, A, P)$, where $(S, A, P)$ is unichain, and suppose that $R$ is bounded. Let $Q = Q^*$ and $\pi = \pi^*$ be solutions to the Bellman equation

$$Q(s, a) = R(s, a) + \gamma \max_{\pi \in \Pi} \mathbb{E}(Q(s_{+}, \pi(s)))$$

$$\pi(s) = \arg \max_{\pi \in \Pi} \mathbb{E}(Q(s, \pi(s))) \quad (9)$$

Let $\alpha_k(s, a) = \alpha_k \cdot 1_{\{s, a\}}(s_k, a_k)$ satisfy

$$0 \leq \alpha_k(s, a) < 1$$

$$\sum_{k=0}^{\infty} \alpha_k(s, a) = \infty$$

$$\sum_{k=0}^{\infty} \alpha_k^2(s, a) < \infty$$

$$\forall (s, a) \in S \times A \quad (10)$$

Then, the update rule

$$Q_{k+1}(s, a) = (1 - \alpha_k(s, a))Q_k(s, a) + \alpha_k(s, a)(R(s, a) + \gamma \max_{a \in A} Q_k(s_{+}, a)) \quad (11)$$

converges to $Q^*$ with probability 1. Furthermore, the optimal policy $\pi^* \in \Pi$ given by (8) maximizes (6).

**Proof** Consult Jaakkola et al. (1994).

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### 3.2. Expected Average Rewards

The agent’s objective here is to maximize the total reward given by

$$\lim_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T-1} \mathbb{E}(R(s_k, a_k)) \quad (12)$$

for some initial state $s_0 \in S$.

We will make a simple assumption regarding the existence of recurring state, a standard assumption in Markov decision process problems with expected average rewards to ensure that the expected value of the reward is independent of the initial state.
**Assumption 4** There exists a state $s^* \in S$ which is recurrent for every stationary policy $\pi$ played by the agent.

Assumption 4 implies that $E(r_j(s_k, a_k))$ is independent of the initial state at stationarity.

**Proposition 2** If Assumption 4 holds, then the value of the Markov Decision Process $(S, A, P)$, with finite state and action spaces, is independent of the initial state.

**Proof** Consult Bertsekas (2005).

**Proposition 3** Under Assumption 4, there exists a number $v$ and a vector $(h(S_1), \ldots, h(S_n)) \in \mathbb{R}^n$, such that for each $s \in S$, we have that

$$h(s) + v = T h(s) = \max_{\pi \in \Pi} E \left( R(s, \pi(s)) + \sum_{s_+ \in S} P(s_+ | s, \pi(s)) h(s_+) \right)$$

(13)

Furthermore, the value of $v$ is equal to $v$.

**Proof** Consult Bertsekas (2005).

Similar to $Q$-learning (but still different), our goal is to find a function $Q^*(s, a)$ that satisfies

$$Q^*(s, a) + v^* = R(s, a) + \sum_{s_+ \in S} P(s_+ | s, a) h^*(s_+)$$

(14)

for any solutions $h^*$ and $v^*$ to Equations (13)-(14). Note that we have

$$h^*(s) = \max_{\pi \in \Pi} E(Q^*(s, \pi(s)))$$

In the case we don’t know the process $P$ and the reward function $R$, we will not be able to take advantage of (14) directly. The next proposition shows that we will be able to design an algorithm that always converges to $Q^*$. It’s worth to note here that the operator $T$ in Equation (13) is not a contraction, so the standard $Q$-learning that is commonly used for reinforcement learning in Markov decision processes with discounted rewards can’t be applied here.

**Assumption 5 (Learning rate)** The sequence $\beta(k)$ satisfies:

1. For every $0 < x < 1$, $\sup_k \beta(\lfloor xk \rfloor) / \beta(k) < \infty$
2. $\sum_{k=1}^\infty \beta(k) = \infty$ and $\sum_{k=0}^\infty \beta(k)^2 < \infty$.
3. For every $0 < x < 1$, the fraction

$$\frac{\sum_{k=\lfloor yt \rfloor}^{\lfloor yt+1 \rfloor} \beta(k)}{\sum_{k=1}^t \beta(k)}$$

converges to 1 uniformly in $y \in [x, 1]$ as $t \to \infty$. 

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For example, \( \beta(k) = \frac{1}{k} \) and \( \beta(k) = \frac{1}{k \log k} \) (for \( k > 1 \)) satisfy Assumption 5.

Now define \( N(t, s, a) \) as the number of times that state \( s \) and actions \( a \) and \( b \) were played up to time \( t \), that is
\[
N(t, s, a) = \sum_{k=1}^{t} 1_{\{s, a\}}(s_k, a_k)
\]

The following assumption is needed to guarantee that all combinations of the triple \((s, a)\) are visited often.

**Assumption 6 (Often updates)** There exists a deterministic number \( d > 0 \) such that for every \( s \in S \), and \( a \in A \), we have that
\[
\liminf_{t \to \infty} \frac{N(t, s, a)}{t} \geq d
\]
with probability 1.

**Definition 2** We define the set \( \Phi \) as the set of all functions \( f : \mathbb{R}^{n \times m} \to \mathbb{R} \) such that

1. \( f \) is Lipschitz
2. For any \( c \in \mathbb{R} \), \( f(cQ) = cf(Q) \)
3. For any \( r \in \mathbb{R} \) and \( \hat{Q}(s, a) = Q(s, a) + r \) for all \( (s, a) \in \mathbb{R}^{n \times m} \), we have \( f(\hat{Q}) = f(Q) + r \)

**Proposition 4** Suppose that \( R \) is bounded and that Assumption 4, 5, and 6 hold. Introduce
\[
FQ(s, a) = \max_{\pi \in \Pi} E(Q(s, \pi(s))) \tag{15}
\]
and let \( f \in \Phi \) be given, where the set \( \Phi \) is defined as in Definition 2. Then, the asynchronous update algorithm given by
\[
Q_{t+1}(s, a) = Q_t(s, a) + 1_{\{s, a\}}(s_t, a_t) \times \\
\times \beta(N(t, s, a))(R(s_t, a_t) + FQ_t(s_{t+1}, a_t) - f(Q_t(s_t, a_t)) - Q_t(s, a)) \tag{16}
\]
converges to \( Q^* \) in (13)-(14) with probability 1.

**Proof** Consult Abounadi et al. (2001).  

4. **Reinforcement Learning for Constrained Markov Decision Processes**

4.1. **Discounted Rewards**

Consider the optimization problem of finding a stationary policy \( \pi \) subject to the initial state \( s_0 = s \) and the constraints (1), that is
\[
\begin{aligned}
&\sup_{a_k \in A} \sum_{k=0}^{\infty} \gamma^k E(r(s_k, a_k)) \\
&\text{s. t. } r^j(s_k, a_k) \geq 0, \; k \in \mathbb{N} \\
&\quad \; j = 1, \ldots, J \\
&\quad \; s_0 = s
\end{aligned} \tag{17}
\]
Since $\gamma > 0$, optimization problem (17) is equivalent to
\[
\sup_{a_k \in A} \sum_{k=0}^{\infty} \gamma^k E\left(r(s_k, a_k)\right) \\
\text{s. t. } \gamma^k r^j(s_k, a_k) \geq 0, \ k \in \mathbb{N} \\
\quad j = 1, \ldots, J \\
\quad s_0 = s
\] (18)

The Lagrange dual of the problem gives the following equivalent formulation
\[
\sup_{a_k \in A} \min_{\lambda_k \geq 0} \sum_{k=0}^{\infty} \gamma^k \left( E\left(r(s_k, a_k)\right) + \sum_{j=1}^{J} \lambda^j r^j(s_k, a_k) \right) \\
\text{s. t. } s_0 = s
\] (19)

Now introduce
\[
R(s, a, \lambda) = r(s, a) + \sum_{j=1}^{J} \lambda^j r^j(s, a)
\] (20)

Let $Q^*(s, a)$ be the expected reward of the agent taking action $a_0 = a \in A$ from state $s_0 = s$ satisfying the constraints in (17), and continuing with an optimal policy satisfying the constraints in (17) thereafter. Then, we have that
\[
Q^*(s, a) = \min_{\lambda \geq 0} R(s, a, \lambda) + \gamma \cdot \max_{\pi \in \Pi} E\left(Q^*(s_+, \pi(s_+))\right)
\] (21)

and the optimal policy $\pi^*$ is given by
\[
\pi^*(s) = \arg \max_{\pi \in \Pi} E\left(Q^*(s, \pi(s))\right)
\]

The next theorem states that if the optimization problem (17) is feasible, then it’s equivalent to an unconstrained optimization problem with bounded rewards, in the sense that an optimal strategy of the agent in unconstrained problem for (17), and the value of the unconstrained problem is equal to the value of (17).

**Theorem 1** Suppose that Assumption 1 and 2 hold and set $C = c \cdot \gamma/(1 - \gamma)$. Let $Q^*$ and $\pi^*$ be solutions to
\[
Q^*(s, a) = \max \left( -C, \min_{\lambda \geq 0} R(s, a, \lambda) \right) + \gamma \cdot \max_{\pi \in \Pi} E\left(Q^*(s_+, \pi(s_+))\right)
\]
\[
\pi^*(s) = \arg \max_{\pi \in \Pi} E\left(Q^*(s, \pi(s))\right)
\] (22)

If $Q^*(s, a) > 0$ for all $(s, a) \in S \times A$, then (19) is feasible and $\pi^*$ is an optimal solution. Otherwise, if $Q^*(s, a) \leq 0$ for some $(s, a) \in S \times A$, then (19) is not feasible.

**Proof** See the appendix.

The reason why we need Theorem 1 is that we want to transform the optimization problem (17) to an equivalent problem where the reward functions are bounded. Now that we are equipped with Theorem 1, we are ready to state and prove our first main result.
Theorem 2  Consider the constrained MDP problem (17) and suppose that it’s feasible and that Assumption 3, 1, and 2 hold. Introduce \( C = c \cdot \gamma / (1 - \gamma) \) and

\[
R(s, a) = \max \left( -C, \min_{\lambda \geq 0} R(s, a, \lambda) \right)
\]

Let \( Q_k \) be given by the recursion according to (11). Then, \( Q_k \to Q^* \) as \( k \to \infty \) where \( Q^* \) is the solution to (22). Furthermore, the policy

\[
\pi^*(s) = \arg \max_{\pi \in \Pi} \mathbb{E} \left( Q^*(s, \pi(s)) \right)
\]

is an optimal solution to (17).

Proof  Follows from Theorem 1, that \( R(s, a) \) is bounded for all \((s, a) \in S \times A\), and Proposition 1.

4.2. Expected Average Rewards

Consider the optimization problem of finding a stationary policy \( \pi \) subject to the constraints (2), that is

\[
\sup_{a_k \in A} \lim_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T-1} \mathbb{E} (r(s_k, a_k)) \quad \text{s. t. } r^j(s_k, a_k) \geq 0 \quad \text{for } j = 1, \ldots, J
\]

Under Assumption 4, the value of the objective function is independent of the initial state according to Proposition 2. The Lagrange dual of the problem gives the following equivalent formulation

\[
\sup_{a_k \in A} \min_{\lambda_k \geq 0} \lim_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T-1} \left( \mathbb{E} (r(s_k, a_k)) + \sum_{j=1}^{J} \lambda^j_k r^j(s_k, a_k) \right)
\]

Let \( Q^*(s, a) \) be the expected reward of the agent taking action \( a_0 = a \in A \) from state \( s_0 = s \) satisfying the constraints in (17), and continuing with an optimal policy satisfying the constraints in (17) thereafter. Also, let \( R(s, a, \lambda) \) be given by (20). Then, we have that

\[
\begin{align*}
 h^*(s) + v^* &= \max_{\pi \in \Pi} \left( \min_{\lambda \geq 0} R(s, a, \lambda) + \sum_{s_+ \in S} P(s_+ | s, \pi(s))h^*(s_+) \right) \\
 Q^*(s, a) + v^* &= \min_{\lambda \geq 0} R(s, a, \lambda) + \sum_{s_+ \in S} P(s_+ | s, a)h^*(s_+)
\end{align*}
\]

and the optimal policy \( \pi^* \) is given by

\[
\pi^*(s) = \arg \max_{\pi \in \Pi} \mathbb{E} \left( Q^*(s, \pi(s)) \right)
\]

The value of (23) and (24) is given by \( v^* \).
Theorem 3  Suppose that Assumption 1 and 2. Also, let $Q^*$, $h^*$, and $\pi^*$ be solutions to

$$
\begin{align*}
    &h^*(s) + v^* = \max_{\pi \in \Pi} \mathbb{E} \left( \max \left( -c, \min_{\lambda \geq 0} R(s, a, \lambda) \right) + \sum_{s_+ \in S} P(s_+ | s, \pi(s)) h^*(s_+) \right) \\
    &Q^*(s, a) + v^* = \max \left( -c, \min_{\lambda \geq 0} R(s, a, \lambda) \right) + \sum_{s_+ \in S} P(s_+ | s, a) h^*(s_+) \\
    &\pi^*(s) = \arg \max_{\pi \in \Pi} \mathbb{E} \left( Q^*(s, \pi(s)) \right)
\end{align*}
$$

If $Q^*(s, a) + v^* > 0$ for all $(s, a) \in S \times A$, then (24) is feasible and $\pi^*$ is an optimal solution. Otherwise, if $Q^*(s, a) + v^* \leq 0$ for some $(s, a) \in S \times A$, then (24) is not feasible.

Proof  See the appendix.

The reason why we need Theorem 3 is that we want to transform the optimization problem (24) to an equivalent problem where the reward functions are bounded.

Now that we are equipped with Theorem 3, we are ready to state and proof the second main result.

Theorem 4  Consider the constrained Markov Decision Process problem (23) and suppose that Assumption 4, 5, and 6 hold. Introduce

$$
R(s, a) = \max \left( -c, \min_{\lambda \geq 0} R(s, a, \lambda) \right)
$$

Let $Q_k$ be given by the recursion according to (16) and suppose that Assumptions 5 and 6 hold. Then, $Q_k \to Q^*$ as $k \to \infty$ where $Q^*$ is the solution to (26). Furthermore, the policy

$$
\pi^*(s) = \arg \max_{\pi \in \Pi} \mathbb{E} \left( Q^*(s, \pi(s)) \right)
$$

is a solution to (23).

Proof  Follows from Theorem 3, that $R(s, a)$ is bounded for all $(s, a) \in S \times A$, and Proposition 4.

5. Conclusions

We considered the problem of optimization and learning for peak constrained Markov decision processes, for both discounted and expected average rewards, respectively. We transformed the original problems in order to apply reinforcement learning to a bounded unconstrained problem. The algorithm works without imposing a cost in terms of memory, where there is no need to store the learned values of $r_1(s, a), ..., r_J(s, a)$ for all $(s, a)$, saving memory of size $S \times A \times J$.

It would be interesting to combine our algorithms with deep reinforcement learning and study the performance of the approach introduced in this paper when the value function $Q$ is modeled as a deep neural network.
References

J. Abounadi, D. Bertsekas, and V. S. Borkar. Learning algorithms for markov decision processes with average cost. *SIAM J. Control and Optimization*, 40:681–698, 11 2001.

J. Achiam, D. Held, A. Tamar, and P. Abbeel. Constrained policy optimization. In *Proceedings of the 34th International Conference on Machine Learning - Volume 70*, ICML’17, pages 22–31. JMLR.org, 2017. URL http://dl.acm.org/citation.cfm?id=3305381.3305384.

E. Altman. Constrained markov decision processes, 1999.

D. P. Bertsekas. *Dynamic Programming and Optimal Control*, volume 1. Athena Scientific, 2005. ISBN 1886529094.

V. S. Borkar. An actor-critic algorithm for constrained markov decision processes. *Systems & Control Letters*, 54(3):207 – 213, 2005. ISSN 0167-6911. doi: https://doi.org/10.1016/j.sysconle.2004.08.007. URL http://www.sciencedirect.com/science/article/pii/S0167691104001276.

Y. Chow, M. Ghavamzadeh, L. Janson, and M. Pavone. Risk-constrained reinforcement learning with percentile risk criteria. *Journal of Machine Learning Research*, 18:167:1–167:51, 2017.

D. V. Djonin and V. Krishnamurthy. Mimo transmission control in fading channels: A constrained markov decision process formulation with monotone randomized policies. *IEEE Transactions on Signal Processing*, 55(10):5069–5083, Oct 2007. ISSN 1053-587X. doi: 10.1109/TSP.2007.897859.

M. M. Drugan and A. Nowé. Designing multi-objective multi-armed bandits algorithms: A study. In *The 2013 International Joint Conference on Neural Networks (IJCNN)*, pages 1–8, Aug 2013. doi: 10.1109/IJCNN.2013.6707036.

Z. Gabor, Z. Kalmár, and C. Szepesvari. Multi-criteria reinforcement learning. In *ICML*, 1998.

P. Geibel. Reinforcement learning for MDPs with constraints. In Johannes Fürnkranz, Tobias Scheffer, and Myra Spiliopoulou, editors, *Machine Learning: ECML 2006*, pages 646–653, Berlin, Heidelberg, 2006. Springer Berlin Heidelberg. ISBN 978-3-540-46056-5.

T. Jaakkola, M. I. Jordan, and S. P. Singh. On the convergence of stochastic iterative dynamic programming algorithms. *Neural Comput.*, 6(6):1185–1201, November 1994. ISSN 0899-7667. doi: 10.1162/neco.1994.6.6.1185. URL http://dx.doi.org/10.1162/neco.1994.6.6.1185.

D. Lizotte, M. H. Bowling, and A. S. Murphy. Efficient reinforcement learning with multiple reward functions for randomized controlled trial analysis. In *Proceedings of the 27th International Conference on International Conference on Machine Learning*, ICML’10, pages 695–702, USA, 2010. Omnipress. ISBN 978-1-60558-907-7. URL http://dl.acm.org/citation.cfm?id=3104322.3104411.
K. V. Moffaert and A. Nowé. Multi-objective reinforcement learning using sets of pareto dominating policies. *Journal of Machine Learning Research*, 15:3663–3692, 2014. URL http://jmlr.org/papers/v15/vanmoffaert14a.html.

D. M. Roijers, P. Vamplew, S. Whiteson, and R. Dazeley. A survey of multi-objective sequential decision-making. *J. Artif. Int. Res.*, 48(1):67–113, October 2013. ISSN 1076-9757. URL http://dl.acm.org/citation.cfm?id=2591248.2591251.
Appendix

Proof of Theorem 1

First note that the value of optimization problem (17) is positive if it’s feasible since $r$ is positive according to Assumption 2. Thus, the value of (19) is either positive if the constraints in (17) are satisfied, or it goes to $-\infty$ if (17) is not feasible. Furthermore, the value of (19) (which is equal to the value of (17) is at most $c/(1 - \gamma)$ since $r$ is bounded by $c$ according to Assumption 1. Thus,

$$\gamma \max_{\pi \in \Pi} \mathbb{E}(Q^*(s, \pi(s))) \leq \gamma \cdot c/(1 - \gamma) = C$$

for all $s \in S$. Suppose that $a$ is such that $r^j(s, a) < 0$ for some $j$. Then,

$$\max \left( -C, \min_{\lambda \geq 0} R(s, a, \lambda) \right) = -C$$

and

$$Q^*(s, a) = \max \left( -C, \min_{\lambda \geq 0} R(s, a, \lambda) \right) + \gamma \cdot \max_{\pi \in \Pi} \mathbb{E}(Q^*(s_+, \pi(s_+))) \leq 0$$

On the other hand, if $a$ is such that $r^j(s, a) \geq 0$ for all $j \in [J]$, then we have that

$$\max \left( -C, \min_{\lambda \geq 0} R(s, a, \lambda) \right) = \min_{\lambda \geq 0} R(s, a, \lambda)$$

and

$$Q^*(s, a) = \min_{\lambda \geq 0} R(s, a, \lambda) + \gamma \cdot \max_{\pi \in \Pi} \mathbb{E}(Q^*(s_+, \pi(s_+)))$$

which is identical to (21). Thus, by choosing a policy that satisfies the constraints in (17), $r^j(s, a) \geq 0$, $Q^*(s, a)$ will be positive. We conclude that if the policy $\pi^*$ implies that $Q^*(s, a) > 0$, then it’s optimal for (19). Otherwise at least one of the constraints $r^j(s, a) \geq 0$ is not satisfied and the value of

$$\max_{\pi \in \Pi} \mathbb{E}(Q^*(s, \pi(s))) = \mathbb{E}(Q^*(s, \pi^*(s)))$$

is non positive which implies that the value of (19) goes to $-\infty$, and so (17) is not feasible.

Proof of Theorem 3

First note that the value of optimization problem (23) is positive if it’s feasible since $r$ is positive according to Assumption 2. Thus, the value of (24) is either positive if the constraints in (23) are satisfied, or it goes to $-\infty$ if (23) is not feasible. Now we have that $\min_{\lambda \geq 0} R(s, a, \lambda) \leq c$ according to Assumption 1. Thus,

$$h^*(s) = \max_{\pi \in \Pi} \mathbb{E}(Q^*(s, \pi(s))) \leq c$$

for all $s \in S$. Suppose that $a$ is such that $r^j(s, a) < 0$ for some $j$. Then,

$$\max \left( -c, \min_{\lambda \geq 0} R(s, a, \lambda) \right) = -c$$
Then
\[ Q^*(s, a) + v^* = \max \left( -c, \min_{\lambda \geq 0} R(s, a, \lambda) \right) + \sum_{s_+ \in S} P(s_+ | s, a) h^*(s_+) \]
\[ = -c + \sum_{s_+ \in S} P(s_+ | s, a) h^*(s_+) \]
\[ \leq -c + c \]
\[ = 0 \]
(28)

On the other hand, if \( a \) is such that \( r^j(s, a) \geq 0 \) for all \( j \in [J] \), then we have that
\[ \max \left( -c, \min_{\lambda \geq 0} R(s, a, \lambda) \right) = \min_{\lambda \geq 0} R(s, a, \lambda) \]

and
\[ Q^*(s, a) + v^* = \min_{\lambda \geq 0} R(s, a, \lambda) + \sum_{s_+ \in S} P(s_+ | s, a) h^*(s_+) \]

which is identical to (25). Thus, by choosing a policy that satisfies the constraints in (23), \( r^j(s, a) \geq 0 \), \( Q^*(s, a) + v^* \) will be positive. We conclude that if the policy \( \pi^* \) implies that \( Q^*(s, a) + v^* > 0 \), then it's optimal for (24). Otherwise at least one of the constraints \( r^j(s, a) \geq 0 \) is not satisfied and the value of
\[ \max_{\pi \in \Pi} \mathbb{E}(Q^*(s, \pi(s))) + v^* = \mathbb{E}(Q^*(s, \pi^*(s))) + v^* \]
is non positive which implies that the value of (24) goes to \(-\infty\), and so (23) is not feasible.