One-loop Partition Functions in Hyperbolic Space $\mathbb{H}^n(\mathbb{R})$

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Abstract

We first study the problem of the one-loop partition function for a free massive quantum field theory living on a fixed background hyperbolic space on the field of real numbers, $\mathbb{H}^n(\mathbb{R})$, $n \geq 2$. Earlier attempts were limited to $n = 3$ dimensions due to the computational complexity. We have developed a new method to determine the fundamental solution of the heat equation and techniques to specify its asymptotics in the small time limit. These enable us to determine the regular part and the ultra violet divergences of the one-loop effective action in the scalar case. The contribution of the Abelian gauge excitations to the one-loop partition function were treated separately using Fourier analysis and bi-tensor techniques on $\mathbb{H}^n(\mathbb{R})$. Finally, by employing the DeWitt’s method, we confirmed the correctness and extended our results to any dimension regarding the first three heat kernel coefficients.

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1 Introduction

In this paper our goal is to determine explicitly the Euclidean partition function
\begin{equation}
Z = \int_{\mathbb{H}^n} D\phi e^{-\frac{1}{2}S_E(\phi)}
\end{equation}
of a free quantum scalar field $\phi$ and of a $U(1)$ gauge quantum field. The factor $g^{-2}$ included in front of the Euclidean action is proportional to $1/\hbar$. This problem along with the graviton excitations were solved in three dimensions by [13]. We pursuit the more general $n$-dimensional case for mathematical completeness.

If we partial integrate the corresponding actions of the fields, by imposing suitable boundary conditions, we arrive at
\begin{equation}
S_E(\phi) = \frac{1}{2} \left( \langle d\phi, d\phi \rangle + m^2 \langle \phi, \phi \rangle \right) = \frac{1}{2} \int_{\mathbb{H}^n} \phi \hat{D}_\phi \phi \sqrt{g} d^n x, \quad \hat{D}_\phi = -\nabla^\mu \partial_\mu + m^2
\end{equation}
\begin{equation}
S_E(A, c, b) = \frac{1}{2} \left( \langle F, F \rangle + \langle \ast d \ast A, \ast d \ast A \rangle + m^2 \langle A, A \rangle \right) + \langle dc, db \rangle
\end{equation}
\begin{equation}
\hat{D}_{F,\mu\nu} = g_{\mu\nu}(-\Delta + m^2) - R_{\mu\nu}, \quad \hat{D}_{gh} = -\nabla^\mu \partial_\mu
\end{equation}
where $\langle \cdot, \cdot \rangle$ is the Hodge inner product (see [A1] for the definition), $\ast$ is the Hodge star operation and $d$ the exterior derivative [I], [7]. In [3] we adopted the Feynman gauge, to simplify the form of the operator $\hat{D}_{F,\mu\nu}$, and the Fadeev-Popov ghosts $c, b$ have been introduced by the gauged fixed Yang-Mills action.
Since we are dealing with determinants of operators, it is instructive to provide some information about their structure. The operator $\hat{D}_\phi$ is semi-bounded from below by $\lambda_0 = (n - 1)^2/4 + m^2$ and symmetric on its domain $\text{dom}(C^\infty_0(\mathbb{H}^n))$. If $\hat{D}_\phi$ has a self-adjoint closure, denoted by $\overline{\hat{D}_\phi}$, then it is said to be an essentially self-adjoint operator [19]. Since the hyperbolic manifold is geodesically complete, such self-adjoint extension is uniquely determined [12], [20], [21] and the problem of defining the Feynman propagator (i.e., the Green’s operator $\hat{D}_\phi^{-1}$) can naturally be solved. In the remainder of the paper we will consider $\overline{\hat{D}_\phi}$ and from now on the bar will be omitted from the extension of $\hat{D}_\phi$.

Negative powers of $\hat{D}_\phi$ can be expressed in terms of the heat operator $e^{-t\hat{D}_\phi}$, $t > 0$ by the following integral

$$\hat{D}_\phi^{-l} = g(l) \frac{1}{\Gamma(l)}, \quad \text{where} \quad g(l) = \int_0^\infty e^{-t\hat{D}_\phi} t^{l-1} dt$$

and $\ln(\hat{D}_\phi)$ by

$$\ln(\hat{D}_\phi) = -\lim_{\epsilon \to 0^+} \left( \int_\epsilon^\infty \frac{e^{-t\hat{D}_\phi}}{t} dt + (\gamma + \ln \epsilon) \hat{I} \right)$$

where $\hat{I}$ denotes the identity operator and $\gamma$ is Euler’s or Mascheroni’s constant

$$\gamma = -\int_0^\infty e^{-s} \ln s \, ds.$$ (6)

The Hilbert space $\mathcal{H} = L^2(\mathbb{H}^n, \langle \cdot, \cdot \rangle)$ with inner product defined by

$$\langle u, v \rangle = \int_{\mathbb{H}^n} u(x)v(x) \sqrt{g} d^n x$$

for real one-component scalar fields. Adopting Dirac’s notation, consider the abstract basis $|x'\rangle \in \mathcal{H}$ and the dual basis $\langle x| \in \mathcal{H}^*$, then the orthonormal condition reads $\langle x|x' \rangle = \delta^{(n)}(x, x')$. The heat kernel $e^{-t\hat{D}_\phi}$ has elements

$$K_n(x, x'; \tau) = \langle x | e^{-t\hat{D}_\phi} | x' \rangle.$$ (8)

The contribution of quantum effects to the one-loop partition function in the scalar field case is given by, [23] (and references therein), the functional

$$W^{(1\text{-loop})} = -\frac{1}{2} \ln \det(\hat{D}_\phi) = -\frac{1}{2} \text{Tr} \ln(\hat{D}_\phi).$$ (9)

If the manifold $\mathcal{M}$ were compact then $\hat{D}_\phi$ would have a discrete spectrum and [9] with the help of [5] would become

$$W^{(1\text{-loop})} = \frac{1}{2} \lim_{\epsilon \to 0^+} \int_\epsilon^\infty \frac{dt}{t} \text{Tr} \left( e^{-t\hat{D}_\phi} \right) = \frac{1}{2} \lim_{\epsilon \to 0^+} \int_\epsilon^\infty \frac{dt}{t} \int_{\mathcal{M}} K_n(x, x; t) \sqrt{g} d^n x.$$ (10)

The hyperbolic space is not compact and $\hat{D}_\phi$ would also have a continuous spectrum giving a divergent contribution to (10) proportional to the volume of $\mathcal{M}$.

Relation (10) can be calculated by evaluating the heat kernel at the coincident limit $x' \to x$. Also, due to the presence of the mass term, one is not confronted with infra-red (IR) divergences caused by zero or negative eigenvalues of $\hat{D}_\phi$. Only ultra-violet (UV) divergences are present. These can be
found by introducing a lower cut off $\Lambda^{-2}$ and performing the $\Lambda \to \infty$ limit. For $\mathbb{H}^{2k+1}$ the fundamental solution of the heat equation is expressed in terms of elementary functions and this allows us to obtain an explicit formula for the one-loop effective action. In even dimensions the analysis is more involved but the recurrence relation (63) enables us to solve this problem.

Regarding the Abelian gauge excitations, the operator $\hat{D}_{F,\mu\nu}$ is semi-bounded from below by $\lambda_0 = (n-3)^2/4 + m^2 - (n-1)$ and symmetric on its domain $\text{dom}(C_0^\infty(\mathbb{H}^n, u(1)))$. One can repeat operator theory arguments, as the ones developed in the scalar case, to establish essential self-adjointness of $\hat{D}_{F,\mu\nu}$. The one-loop partition function now becomes

\[
W^{(1\text{-loop})}_{U(1)} = -\frac{1}{2} \left( \ln \det(\hat{D}_{F,\mu\nu}) - 2 \ln \det(\hat{D}_{gh}) \right) = \frac{\text{Vol}(\mathbb{H}^n)}{2} \lim_{\epsilon \to 0^+} \int_{\epsilon}^{\infty} dt \left( \text{Tr}(K_{U(1)}^n)(t) - K_{gh}^n(t) \right),
\]

where the factor $-2$ is due to the presence of $b, c$ fields. The vector nature of the $U(1)$ Abelian gauge field implies a delicate treatment for the one-loop effective action. Adopting a special ansatz for the heat kernel, first proposed by [9], and using Fourier analysis as well as bi-tensor identities on $\mathbb{H}^n$, we reach our goal. Motivated by the DeWitt’s powerful method, after some modifications, we performed an independent calculation of the heat kernel coefficients. This method produced identical results and extended the first ones.

The paper is organized as follows:

In Section 2 we briefly review the definition of the real $n$–dimensional hyperbolic space $\mathbb{H}^n$ and list some fundamental properties.

In Section 3, depending on the dimension of the manifold $\mathbb{H}^n$ (even or odd), we derive a recurrent relation for the fundamental solution of the heat equation in terms of the probability density functions (17) and (18) respectively. The method is new and elegant and deserves a special mention. In sections five and six these recurrent relations will be used repeatedly.

In Section 4 we present the Fourier transform on $\mathbb{H}^n$ dictated by a version of Helgason’s treatment who involves group theoretic machinery. We write the unique $SO(n)$ invariant spherical eigenfunction $\Phi_\lambda(x)$ of the Laplace-Beltrami operator subjected to the condition $\Phi_\lambda(0) = 1$. This help us to identify the Harish-Chandra $c$–function (Proposition 4.2) which plays a key role in harmonic analysis on $\mathbb{H}^n$. We state the basic theorem without proof for the Fourier transformation on symmetric spaces of non-compact type and concentrate on rotationally symmetric functions.

In Section 5 we determine the one-loop partition function of a massive and real scalar quantum field theory. The regular and divergent parts are calculated in any odd, two and four dimensions respectively. A different approach based on the W.K.B. approximation of the heat kernel is also performed to secure our results. The latter provides results independent of the dimension of the hyperbolic space. We also derive a weighted Poincaré inequality for radial functions.

In Section 6 we solve the same problem as previously but in the presence of a massive Abelian $U(1)$ gauge field. Adopting a convenient ansatz for the $U(1)$ heat kernel and using Fourier analysis we specify the trace of the corresponding kernel. Again an independent W.K.B. heat kernel approximation is performed to check the agreement of the first three heat kernel coefficients with those predicted by (11).

In Section 7 we provide additional mathematical background to clarify technical issues.

\footnote{u(1) is the unitary Lie Algebra.}
2 Background

2.1 Definition of $\mathbb{H}^n(\mathbb{R})$ and properties

**Definition 2.1** The hyperbolic space $\mathbb{H}^n$, $n \geq 2$ geometrically can be realised as the upper sheet of the hyperboloid embedded into the Minkowski space, $\mathcal{M}^{(1,n)}$, namely

$$\mathbb{H}^n = \{ X \in \mathcal{M}^{(1,n)} : [X, X] = 1, X_0 > 0 \}$$

where $[\cdot, \cdot]$ is the bilinear form defined by

$$[X, Y] = X_0 Y_0 - \sum_{i=1}^{n} X_i Y_i = X^t J_{1,n} Y$$

with $J_{1,n}$ be the $(n+1) \times (n+1)$ diagonal matrix with signature $(+1, -1, \ldots, -1)$.

The hyperbolic space $\mathbb{H}^n$ is the maximally symmetric, simply connected, $n$-dimensional Riemannian manifold with constant negative sectional curvature $K = -1$. The group of all orientation preserving isometries, $\text{Iso}^+(\mathbb{H}^n)$, consists of the proper, orthochronous Lorentz group

$$G = \text{Iso}^+(\mathbb{H}^n) = SO^+(1, n) = SO(1, n) \cap O^+(1, n)$$

$$= \{ \text{GL}(n + 1, \mathbb{R}) : \ h^t J_{1,n} h = J_{1,n}, \ \det h = 1, \ h_{00} > 0 \}$$

where $\text{GL}(n + 1, \mathbb{R})$ is the group of all nonsingular real $(n + 1) \times (n + 1)$ matrices $h$. This group is isomorphic to the Möbius group of $\mathbb{R}^{n-1} = \mathbb{R}^{n-1} \cup \{ \infty \}$. An element of $h \in G$ acts on $x \in \mathbb{H}^n$ as follows

$$x_i' = \frac{(h_{i0} + h_{in})x_i^2 + 2 \sum_{k=1}^{n-1} h_{ik} x_k + h_{i0} - h_{in}}{A_h x_i^2 + 2 \sum_{k=1}^{n-1} (h_{0k} - h_{nk}) x_k + B_h} \quad i = 1, \ldots, n - 1$$

$$x_n' = \frac{2x_n}{A_h x_i^2 + 2 \sum_{k=1}^{n-1} (h_{0k} - h_{nk}) x_k + B_h}$$

where $|x|^2 = \sum_{k=1}^{n} x_k^2$, $A_h = h_{00} + h_{nn} - h_{0n} - h_{nn}$ and $B_h = h_{00} + h_{nn} - h_{00} - h_{0n}$. A metric on $\mathbb{H}^n$ can be constructed by

$$g_{\mathcal{M}(1,n)} = \sum_{i=1}^{n} (dX_i)^2 - (dX_0)^2.$$  

2.2 Heat kernel recurrence relations between different dimensions

**Definition 2.2** A fundamental solution $p_n$ for the heat operator $\partial/\partial t - \Delta_{\mathbb{H}^n, x}$ is a function

$$p_n : \mathbb{H}^n \times \mathbb{H}^n \times (0, \infty) \rightarrow \mathbb{R}$$

with the following properties

i) $p_n \in C(\mathbb{H}^n \times \mathbb{H}^n \times (0, \infty))$, $C^2$ in the first variable and $C^1$ in the second variable,

ii) $(\partial_t - \Delta_{\mathbb{H}^n, x}) p_n(x, y; t) = 0$, $\forall t > 0$,

iii) $\lim_{t \to 0^+} p_n(\cdot, y; t) = \delta_y(\cdot)$, $\forall y \in \mathbb{H}^n$ where $\delta_y$ is the Dirac distribution centered at $y$ and the limit is considered in the distributional sense,

$$\lim_{t \to 0^+} \int_{\mathbb{H}^n} p_n(x, y; t) f(y) d^n y = f(x), \ \forall f \in C_0^\infty(\mathbb{H}^n), \ \forall y \in \mathbb{H}^n$$

where $C_0^\infty(\mathbb{H}^n)$ denotes the set of $C^\infty$—smooth functions with compact support and $d^n y = \sqrt{g} dy^1 \wedge \cdots \wedge dy^n$ is the volume form.
Theorem 2.3 Let

\[ p_1(r, t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{r^2}{4t}} \] (17)

\[ p_2(r, t) = \frac{\sqrt{2}}{(4\pi t)^{\frac{3}{2}}} e^{-\frac{t}{4\pi t}} \int_{r}^{\infty} \frac{se^{-\frac{s^2}{4t}}}{\sqrt{cosh s - cosh r}} ds \] (18)

be the fundamental solutions for the heat operator on the real line and the hyperbolic space \( \mathbb{H}^2 \) respectively. Then the following recurrence relations hold

\[ p_{2k+1}(r, t) = \frac{(-1)^k}{(2\pi)^k} e^{-k^2 t} \left( \frac{1}{\sinh r} \frac{\partial}{\partial r} \right)^k p_1(r, t) \] (19)

\[ p_{2(k+1)}(r, t) = \frac{(-1)^k}{(2\pi)^k} e^{-k(k+1)t} \left( \frac{1}{\sinh r} \frac{\partial}{\partial r} \right)^k p_2(r, t) \] (20)

depending on whether \( \text{dim} \mathbb{H}^n = n \) is an odd or even natural number.

The formulas (19) and (20) are not new [15], [8] but the proposed method is inspired by a different viewpoint.

**Proof**

The Laplacian commutes with the action of the isometry group \( G = SO^+(1, n) \) on \( \mathbb{H}^n \) space and as a result \( p_n \) is a point-pair invariant; that is, it satisfies

\[ p_n(x, y; t) = p_n(hx, hy; t), \quad \forall x, y \in \mathbb{H}^n, \forall h \in G, t > 0. \] (21)

Therefore

\[ p_n(x, y; t) = p_n(d_{\mathbb{H}^n}(x, y); t) = p_n(r, t) \] (22)

where \( r(x, y) = d_{\mathbb{H}^n}(x, y) \) is the geodesic distance between the points \( x, y \in \mathbb{H}^n \). A straightforward calculation shows that (see Appendix (A.13) for the generalization)

\[ \Delta_x d_{\mathbb{H}^n}(x, y) = (n - 1) \coth(d_{\mathbb{H}^n}(x, y)) \] (23)

and the diffusion equation turns out to be

\[ \partial_t p_n(r, t) = \partial_r^2 p_n(r, t) + (n - 1) \coth r \partial_r p_n(r, t). \] (24)

We want to establish a recurrence relation between \( p_m \) and \( p_n \), \( m > n \) in different dimensions of hyperbolic spaces. Suppose there exists a differential operator \( \hat{B}(r, t) \) such that

\[ p_m(r, t) = \hat{B}(r, t)p_n(r, t) = f(r, t)\partial_r p_n(r, t). \] (25)

Then, by demanding the heat operator to commute with the \( \hat{B} \) operator,

\[ \left[ \partial/\partial t - \Delta_{\mathbb{H}^m}, \hat{B}(r, t) \right] = 0 \] (26)

we obtain the following system of partial differential equations

\[ \partial_t f - \partial_r^2 f - (m - 1) \coth r \partial_r f - \frac{(n - 1)}{\sinh^2 r} f = 0 \] (27)

\[ 2\partial_r f + (m - n) \coth rf = 0. \] (28)
Eliminating the partial derivatives with respect to \( r \), from equation (27) using (28), we end up with a constraint which relates the dimensions of the hyperbolic spaces and a first order, w.r.t. time \( t \), partial differential equation

\[
\dim \mathbb{H}^m = \dim \mathbb{H}^n + 2 \\
\partial_t f + nf = 0.
\]  

The general solution of (30) is \( f(r, t) = \exp(-nt)g(r) \). Substituting this back into (28) we determine \( g(r) \). The operator \( \hat{B} \) is found to be given, up to a constant, by

\[
\hat{B}(r, t) \propto e^{-nt} \frac{1}{\sinh r} \partial_r.
\]  

Applying successively \( \hat{B} \) we recover (19) and (20) with the help of the finite sum formulas \( \sum_{k=0}^{k-1} 2(l+1) = k(k+1) \).

### 2.3 Fourier transform on \( \mathbb{H}^n(\mathbb{R}) \)

Consider the boundary cone of \( \mathbb{H}^n \)

\[
A = \{ X \in \mathbb{R}^n : [X, X] = 0, X_{n+1} > 0 \}
\]  

and the mapping

\[
g : S^{n-1} \to B
\]  

with \( g(\omega) = (1, \omega) \) and

\[
B = \{ X \in A : X_{n+1} = 1 \}.
\]

**Proposition 2.4** The functions

\[
h_{\lambda, \omega}(x) = [x, g(\omega)]^{i\lambda - \rho}, \quad \rho = \frac{n-1}{2}
\]

are eigenfunctions of the Laplace-Beltrami operator \( \Delta_{\mathbb{H}^n} \) with eigenvalues \( -(\lambda^2 + \rho^2) \).

**Proof**

Using the spherical polar representation, the bilinear form is written as

\[
[x, g(\omega)]^s = [(\cosh r, \phi), (1, \omega)]^s, \quad \phi, \omega \in S^{n-1} \\
= (\cosh r - \sinh r \langle \phi, \omega \rangle)^s \\
= (\cosh r - \sinh r \cos \theta)^s.
\]

The Laplace-Beltrami operator when expressed in the geodesic polar representation can be applied on \( h \) to give

\[
\Delta_r h(r, \theta) = s(s - 1 + n)h - \frac{1}{\sinh^2 r} \Delta_{S^{n-1}} h.
\]

Setting \( s = i\lambda - \rho \), in (37), we recover the desired result. Note that for \( s = (n-1)^2/4 \) the \( L^2 \) spectrum of \( \hat{D}_\phi \) is the whole of the interval

\[
\text{Spec}(\hat{D}_\phi) = \left[ \frac{(n-1)^2}{4}, \infty \right).
\]
For an alternative approach the reader may refer to [16].

The spherical function

\[ \Phi_\lambda(x) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} [x, g(\omega)]^{i\lambda \rho} d\omega, \quad \omega_{n-1} = |S^{n-1}| = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \]

is the unique \( SO(n) \) invariant eigenfunction of \( \Delta_{S^n} \) satisfying the condition \( \Phi_\lambda(0) = 1 \). It is an even function w.r.t. \( \lambda \) and \( r \). For calculational purposes relation (3.1) is more convenient. Motivated by [17] we are led to the following proposition.

**Proposition 2.5** Let \( \text{Re}(i\lambda) > 0 \) then

\[
\lim_{r \to \infty} e^{-i(\lambda - \rho)r} \Phi_\lambda(r) = c(\lambda), \quad \text{where} \quad c(\lambda) = \frac{2^{2\rho - 1} \Gamma\left(\rho + \frac{1}{2}\right) \Gamma(i\lambda)}{\sqrt{\pi} \Gamma(\rho + i\lambda)}
\]

is the Harish-Chandra \( c \)-function.

**Proof**

Using the transformation \( u = \tan(\theta/2) \) the integral (39) becomes

\[
\Phi_\lambda(r) = e^{(i\lambda - \rho)r} \frac{2^{2\rho} \Gamma\left(\rho + \frac{1}{2}\right)}{\sqrt{\pi} \Gamma(\rho)} \int_0^{\infty} \left(1 + e^{-2ru^2}\right)^{i\lambda - \rho} \frac{u^{2\rho - 1}}{(1 + u^2)^{i\lambda + \rho}} du.
\]

Assuming \( \lambda = a + ib \) and \( \text{Re}(i\lambda) = -b > 0 \) the integrand is absolutely integrable since for \( \rho \leq |b| \) we have

\[
|f(r, u)| = \left|(1 + e^{-2ru^2})^{ia}(1 + u^2)^ia \right| \leq (1 + e^{-2ru^2})^{-\rho}(1 + u^2)^{-\rho} \leq (1 + u^2)^{-2\rho} u^{2\rho - 1}.
\]

Note that by definition \( \rho \geq 1/2 \). If \( \rho > |b| \) then

\[
|f(r, u)| < (1 + u^2)^{-|b| - \rho} u^{2\rho - 1}
\]

which is again integrable. Therefore by applying the dominated convergence theorem and changing once more variable, by setting \( t = (1 + u^2)^{-1} \), we arrive at the following pointwise estimate

\[
\lim_{r \to \infty} e^{-i(\lambda - \rho)r} \Phi_\lambda(r) = \frac{2^{2\rho - 1} \Gamma\left(\rho + \frac{1}{2}\right)}{\sqrt{\pi} \Gamma(\rho)} \int_0^{1} (1 - t)^{\rho - 1} t^{i\lambda - 1} dt = c(\lambda).
\]

In (44) we used the definition of the Beta function \( B(\rho, i\lambda) = \Gamma(\rho)\Gamma(i\lambda)/\Gamma(\rho + i\lambda) \).

Let now the Fourier transform of \( f \in C_0^\infty(\mathbb{H}^n) \) be defined by

\[
\tilde{f}(\lambda, \omega) = \int_{\mathbb{H}^n} f(x) h_{\lambda, \omega}(x) dx.
\]

On symmetric spaces of non-compact type, Helgason [17] has given the following Theorem of Fourier transform.

**Theorem 2.6** Let \( f \in C_0^\infty(\mathbb{H}^n) \). Then

(i) The inverse Fourier transform is given by

\[
f(x) = \frac{2^{2\rho}}{2\pi \omega_{n-1}} \int_{\mathbb{R}^+} (P_\lambda f)(x) \frac{d\lambda}{|c(\lambda)|^2} \quad \text{with} \quad (P_\lambda f)(x) = \int_{S^{n-1}} \tilde{f}(\lambda, \omega) h_{\lambda, \omega}(x) d\omega.
\]
Plancherel

\[ \| f \|_2^2 = \frac{2^\rho}{2\pi^{n-1}} \int_{\mathbb{R}^+} \left( \frac{P_\lambda f(x)}{c(\lambda)} \right)^2 d\lambda \]

and the Fourier transform extends as an isometry of \( L^2(\mathbb{H}^n) \) functions onto

\[ L^2(\mathbb{R}^+ \times S^{n-1}, \frac{2^\rho}{\sqrt{2\pi^n} \omega_{n-1}} |c(\lambda)|^{-2} d\omega d\lambda). \]

For rotationally symmetric functions the Fourier transform extends as an isometry of radial \( L^2(H_n) \) functions onto \( L^2(\mathbb{R}^+ \times S^{n-1}, \frac{2^\rho}{2\pi^{n-1}} |c(\lambda)|^{-2} d\omega d\lambda) \). Note that the Harish-Chandra measure \( \frac{2^\rho}{2\pi^{n-1}} |c(\lambda)|^{-2} \) corresponds to the density of zero angular momentum radial functions \cite{4}. Relation (45) becomes

\[ \tilde{f}(\lambda) = \omega_{n-1} \int_0^\infty f(r) \Phi_\lambda(r)(\sinh r)^{2\rho} dr \]

and the inverse transform is then given by

\[ f(r) = \frac{2^\rho}{2\pi^{n-1}} \int_{\mathbb{R}^+} \tilde{f}(\lambda) \Phi_\lambda(r) \frac{d\lambda}{|c(\lambda)|^2}. \]

3 One-loop determinants in \( \mathbb{H}^n(\mathbb{R}) \)

3.1 Scalar fields

We consider a real scalar field \( \phi \) of mass \( m \) on \( \mathbb{H}^n \). Its action is described by (2) where Green’s formula has been applied on the condition that either \( \text{supp} \phi \) or \( \text{supp} \nabla \phi \) is compact. The scalar heat kernel on \( \mathbb{H}^n \) has been determined by Theorem 3.2, apart from a multiplicative constant \( \exp(-m^2 t) \) due now to the presence of the mass term.

The quantum effects generated by the background field \( \phi \) in the one-loop approximation of quantum field theory are given by the functional (10) and taking into account that \( K_n(x, x; t) = p_n(0, t) \) it can be rewritten as

\[ W^{(1\text{-loop})} = \frac{1}{2} \text{Vol}(\mathbb{H}^n) \lim_{\epsilon \to 0^+} \int_\epsilon^\infty dt \frac{dt}{t} K_n(0, t) \]

where the volume of \( \mathbb{H}^n \) is set to be equal to the volume of the open ball \( B(\alpha, r) \)

\[ \text{Vol}(B)(r) = |S^{n-1}| \int_0^r (\sinh w)^{n-1} dw. \]

The asymptotic behaviour of the heat kernel for \( n = 2k + 1 \), in the \( x' \to x \) coincident limit, or equivalently, using the spherical polar representation, in the \( r \equiv d_{\mathbb{H}^n}(x, x') \to 0 \) limit, can be written as the finite \( t \)-series

\[ K_{2k+1}(0, t) = \frac{1}{(4\pi t)^{k+\frac{3}{2}}} e^{-(k^2+m^2)t} \sum_{l=0}^{k-1} t^l a_{k,l}. \]

\footnote{Actually, in order to remove this volume divergence one may replace \( \hat{D}_\alpha \) in \cite{9} by \( \hat{D}_\phi / \hat{D}_{0,\phi} \) where \( \hat{D}_{0,\phi} \equiv -\partial^2 \rightleftharpoons + m^2 \) is a reference operator describing the propagation of a massive particle in Euclidean space \cite{23}. Alternatively, the regularisation of this divergence can be performed by introducing, under the trace in \cite{9}, a smooth smearing function \( f \) defined on \( \mathbb{H}^n \).}
The truncation of the series at the finite positive integer \( k - 1 \) is due to the fact that the coefficients \( a_{k,l} \) vanish for \( l \geq k \). This can be proved by taking the \( r \to 0^+ \) limit of the multiple derivative (B.6) (see Appendix B for some analytic expressions of \( p_{2k+1} \)). Substituting (52) into (50) we obtain

\[
W_{2k+1,\text{reg.}}^{(1-\text{loop})} = \frac{(-1)^{k+1}}{2^{2(k+1)\pi + 1}} Vol(\mathbb{H}^{2k+1})(k^2 + m^2)^{k+1} \sum_{l=0}^{k-1} \frac{(-1)^l}{\Gamma(k-l+(2k+1))(k^2+m^2)^l} a_{k,l}
\]

where the t-integral was performed by analytic continuation of the Gamma function to negative arguments. In general the integral in (50) can fail to provide a finite result due to IR \((t \to \infty)\) and UV \((t \to 0^+)\) divergences. The first class of divergences cannot occur since the operator \( \hat{D} \) is semi-bounded from below by \( \lambda_0 = (n-1)^2/4 + m^2 \) and the Green’s function of \( \hat{\Delta} \) is exponentially decreasing at infinity. As a consequence, a negative constant sectional curvature provides a natural infrared cut-off. The ultra-violet divergencies of \( W_{2k+1,\text{div.}}^{(1-\text{loop})}(\Lambda) \) can be found by introducing a lower cut off at \( \epsilon = \Lambda^{-2} \), performing the partial integration and then taking the limit \( \Lambda \to \infty \). In particular, for \( \mathbb{H}^3, k = 1 \), and we would have

\[
W_{\mathbb{H}^3}^{(1-\text{loop})}(\Lambda) = \frac{Vol(\mathbb{H}^3)}{2(4\pi)^{\frac{3}{2}}} b^3 a_{1,0} \int_{\frac{\Lambda}{b}}^{\infty} u^{-\frac{3}{2}} e^{-u} du
\]

where

\[
W_{\mathbb{H}^3,\text{div.}}^{(1-\text{loop})}(\Lambda) = \frac{Vol(\mathbb{H}^3)}{3(4\pi)^{\frac{3}{2}}} (m^2 + 1)^{\frac{3}{2}} a_{1,0} \left(-\frac{2\Lambda}{b} + \left(\frac{\Lambda}{b}\right)^3\right) e^{-\left(\frac{\Lambda}{b}\right)^2} \quad \text{and}
\]

\[
W_{\mathbb{H}^3,\text{reg.}}^{(1-\text{loop})}(\Lambda) = \frac{Vol(\mathbb{H}^3)}{12\pi} (m^2 + 1)^{\frac{3}{2}}
\]

and Erf\(x\) is the error function. Taking the \( \Lambda \to 0^+ \) limit of (55),

\[
\lim_{\Lambda \to 0^+} W_{\mathbb{H}^3,\text{reg.}}^{(1-\text{loop})}(\Lambda) = \frac{Vol(\mathbb{H}^3)}{12\pi} (m^2 + 1)^{\frac{3}{2}}
\]

we recover the result of [13]. In the general case we find

\[
W_{\mathbb{H}^{2k+1}}^{(1-\text{loop})}(\Lambda) = \frac{Vol(\mathbb{H}^{2k+1})}{2(4\pi)^{k+1/2}} b^{2k+1} \sum_{l=0}^{k-1} \frac{a_{k,l}}{b^{2k+1}} \int_{\frac{\Lambda}{b}}^{\infty} u^{-k-\frac{3}{2}} e^{-u} du
\]

where the ultra-violet divergency and the regular part of the integral are both computed using the formula

\[
\int_{\frac{\Lambda}{b}}^{\infty} u^{-\left(s+\frac{3}{2}\right)} e^{-u} du = \frac{e^{-\left(\frac{\Lambda}{b}\right)^2}}{(2s-1)!!} \sum_{r=1}^{s-1} 2^r (-1)^{r-1} (2s-2r-1)!! \left(\frac{\Lambda}{b}\right)^{2(s-r)+1}
\]

\[
+ \frac{(-1)^{s-1} 2^s}{(2s-1)!!} \left[\left(\frac{\Lambda}{b}\right) e^{-\left(\frac{\Lambda}{b}\right)^2} - \int_{\frac{\Lambda}{b}}^{\infty} u^{-\frac{3}{2}} e^{-u} du\right]
\]

where \( s = k - l + 1 \) is a positive integer.

In the even dimensional case, \( n = 2(k+1) \), the asymptotic behaviour of the heat kernel at coincident points is more involved. For \( k = 0 \) we obtain (see Appendix C for details)

\[
K_2(0, t) = \sqrt{\frac{4t}{\pi}} \frac{1}{4\pi t} e^{-\left(m^2 + \frac{1}{4}\right)t} \int_{0}^{\infty} \frac{ue^{-u^2}}{\sinh(\sqrt{t}u)} du
\]

\[
= \frac{1}{4\pi t} e^{-\left(m^2 + \frac{1}{4}\right)t} \left(1 + \sum_{l=1}^{\infty} (21-2l-1) B_{2l} t^l \right).
\]

(60)
In [60] $B_{2l}$ are the Bernoulli numbers and this expression is identical to the one derived in [6]. The ultra violet divergency in the one-loop partition function is given by

$$W_{H^2, \text{div.}}^{(1\text{-loop})}(\Lambda) = \frac{\text{Vol}(H^2)}{4\pi} \left[ \left( \Lambda^2 - 2b^2 \ln \left( \frac{\Lambda}{b} \right) \right) e^{-\left(\frac{\Lambda}{b}\right)^2} + (m^2 + \frac{1}{4})\gamma \right], \quad b^2 = m^2 + \frac{1}{4} \tag{61}$$

while the finite part is found to be

$$W_{H^2, \text{reg.}}^{(1\text{-loop})} = \frac{\text{Vol}(H^2)}{4\pi} (m^2 + \frac{1}{4}) \sum_{l=1}^{\infty} (2^{1-2l} - 1)B_{2l} \frac{1}{l!(m^2 + \frac{1}{4})^l} \Gamma(l-1) \tag{62}$$

where $\gamma$ is the Euler’s constant [6]. In (62) the series converges provided the mass for big $l$ behaves like $cl^2$ with the constant $c$ taking values greater than $1/\pi e$ as one may check using the asymptotic behaviour of Bernoulli’s numbers $|B_{2l}| \approx 4\sqrt{\pi l}/(l/2\pi)^{2l}$.

In the $H^4$ case we use the recurrence relation [8]

$$K_{n+1}(r, t) = \sqrt{2} e^{\frac{r}{4}(2n+1)} \int_r^{\infty} \frac{K_{n+2}(r, t)}{\cosh s - \cosh r} \sinh s \, ds \tag{63}$$

which connects the heat kernels in even and odd dimensions. Setting $n = 3$, in (63), the integrand involves $K_5$ and its Maclaurin expansion around $t = 0$ provides the asymptotic expansion

$$K_4(0, t) = \frac{1}{(4\pi t)^2} e^{-\left(\frac{4}{m^2} + m^2\right)t} \left( 1 + \frac{t}{4} + \sum_{l=2}^{\infty} t^l \left[ (l - 1)(2^{1-2l} - 1)B_{2l} + \frac{l}{4}(2^{3-2l} - 1)B_{2l-l} \right] \right). \tag{64}$$

Substituting (64) back into (50) we derive the one-loop UV divergency.

The simplest prescription for defining a finite one-loop partition function is to subtract from $K_n(x, x; t)$ the divergences proportional to $a_{k,t}$. It is easily checked that the coefficient $b_{1,2k+1} = -k^2 + a_{k,1}$ (or $\tilde{b}_{1,2k+1} = -k(k + 1) - 1/4 + a_{2k+1,1}$) of the linear term in $t$ equals to $-R/6$ where $R$ is the Ricci scalar curvature. On the other hand the coefficient $b_{2,2k+1/2} = k^4/4! - k^2a_{k,1} + a_{k,2}$ (or $\tilde{b}_{2,2k+1/2} = (k(k + 1) + 1/4)^2/2! - (k(k + 1) + 1/4)a_{2k+1,1} + a_{2k+1,2}$) of $t^2$ depends on a linear combination of $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$, $R_{\mu\nu}^2$ and $R^2$ (see (67)). The following proposition provides an analytic result for the first three heat kernel coefficients in the scalar field case utilizing De Witt’s iterative procedure [10].

**Proposition 3.1** The coincidence limits of the heat kernel coefficients

$$\tilde{b}_{l,n}(0) \equiv \lim_{r \to 0^+} b_{0,n}(r) \equiv [b_{0,n}(r)], \quad l = 0, 1, 2, \quad n = \text{dim}(H^n)$$

are given by

$$\tilde{b}_{0,n} = 1 \tag{65}$$

$$\tilde{b}_{1,n} = -\frac{R}{6} \tag{66}$$

$$\tilde{b}_{2,n} = \frac{n^4}{36} - \frac{n^3}{15} + \frac{13n^2}{180} - \frac{n}{30}$$

$$= \frac{1}{90} (R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - R_{\mu\nu}R^{\mu\nu}) + \frac{R^2}{36}. \tag{67}$$

**Proof**

For the study of the asymptotic behaviour of $K_n$ in the $r \to 0^+$ limit a W.K.B. expansion suffices,

$$K_n(x, x', t) = \frac{1}{(4\pi t)^{n/2}} e^{-m^2 t} e^{-\frac{x(x')}{4t}} \mathcal{P}(x, x') D^\frac{1}{2}(x, x') \Omega(x, x', t) \tag{69}$$
where $\sigma(x, x') = d_{\Pi, n}^2/2$ is the Synge’s world function, $\mathcal{P}(x, x')$ is identically one in the neighbourhood of $x'$ but vanishes outside a normal neighbourhood of $x'$.

$$D(x, x') = g^{-\frac{1}{2}}(x)\text{Det}(\frac{\partial^2 \sigma(x, x')}{\partial \mu \partial \nu'})g^{-\frac{1}{2}}(x')$$

is the Van Vleck-Morette determinant which represents the density of geodesics. $\Omega$ is a function which satisfies the initial condition $\Omega(x, x', 0) = 1$ and is going to be determined. Substituting (69) into the heat equation and using the identities (see Appendix C for their proof)

$$\sigma = \frac{1}{2} \sigma^\mu \sigma_\mu = \frac{1}{2} \sigma^\mu \sigma'_\mu, \quad \sigma_\mu = \partial_\mu \sigma \quad (70)$$

$$n = D^{-1} \nabla_\mu (D \sigma^\mu) \quad (71)$$

$$D(r) = \left(\frac{r}{\sinh r}\right)^{n-1} \quad (72)$$

we end up with the equation

$$\left(\partial_t + \frac{1}{t} \sigma_\mu \partial^\mu - \hat{\mathcal{N}}\right) \Omega(x, x', t) = 0, \quad \text{where} \quad \hat{\mathcal{N}} = D^{-\frac{1}{2}} \Box D^{\frac{1}{2}}. \quad (73)$$

The solution that vanishes as $t$ goes to zero may be expressed as a power series

$$\Omega(x, x', t) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} b_{l,n}(x, x') t^l \quad (74)$$

and the coefficients are determined by the differential recursion relation

$$\left(1 + \frac{1}{t} \sigma_\mu \partial^\mu\right) b_{l,n}(x, x') = \hat{\mathcal{N}} b_{l-1,n}(x, x') \quad (75)$$

or

$$\left(1 + \frac{1}{t} r \partial_r\right) b_{l,n}(r) = \hat{\mathcal{N}} b_{l-1,n}(r) \quad (76)$$

by using the geodesic distance $r$. The solution of (76) is

$$b_{l,n}(r) = l r^{l-1} \int_0^r \tilde{r}^{l-1} \tilde{\mathcal{N}} b_{l-1,n}(\tilde{r}) d\tilde{r}. \quad (77)$$

For $l = 0$, (65) is obvious, bearing in mind the initial condition $\Omega(r, t = 0) = 1$. For $l = 1$, (77) gives

$$b_{1,n}(r) = \frac{(n-1)}{4} \left[ (n-3) \left( \frac{1}{r^2} - \frac{\coth(r)}{r} \right) + n - 1 \right] \quad (78)$$

which reproduces (66) in the coincidence limit $r \to 0^+$. The third heat kernel coefficient is recovered in the same way. The explicit expression of $b_{2,n}(r)$ is omitted due to its lengthy appearance. This method becomes very cumbersome beyond $b_{4,n}$ and a refined nonrecursive procedure was used by [2], [3] to calculate $b_{8,n}$. For a vector field $\vec{\phi}$, having $N_0$ real components, relations (65)-(67) can be generalised by multiplying them by $N_0$.

It is worth noting that the function

$$\Phi(r) = -D^{-1/2}(r) \Delta D^{1/2}(r) = -\frac{(n-1)}{4r^2} \left[ (n-3) \left( 1 - r^2 \coth^2 r \right) - 2r^2 \right], \quad (79)$$

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is strictly positive, right-continuous, decreasing for \( n = 2 \) and increasing for \( n > 3 \). In addition

\[
\sup_{r \in (0, \infty)} \Phi(r) = \frac{1}{3}, \quad \inf_{r \in (0, \infty)} \Phi(r) = \frac{1}{4}, \quad \text{for } n = 2 \quad (80)
\]

\[
\sup_{r \in (0, \infty)} \Phi(r) = \inf_{r \in (0, \infty)} \Phi(r) = 1, \quad \text{for } n = 3 \quad (81)
\]

\[
\sup_{r \in (0, \infty)} \Phi(r) = \frac{(n-1)^2}{4}, \quad \inf_{r \in (0, \infty)} \Phi(r) = \frac{n(n-1)}{6}, \quad \text{for } n > 3 \quad (82)
\]

and satisfies the weighted Poincaré inequality

\[
\int_{\mathbb{R}^n} |\nabla f|^2 d\mu \geq \int_{\mathbb{R}^n} \Phi f^2 d\mu \geq \inf_{r \in D \subset \mathbb{R}^n} \Phi(r) \int_{\mathbb{H}^n} f^2 d\mu \quad (83)
\]

w.r.t. the measure \( d\mu(r) = (\sinh r)^{n-1} dr \) and for all compactly supported, smooth and radial functions \( f \in C_0^\infty(\mathbb{R}^n) \). In \( n = 2 \) dimensions the \( \inf \Phi(r) \geq 1/4 \), as \( (80) \) requires, but not in \( n > 3 \). Recall that \((n-1)^2/4\) is the greatest lower bound of the spectrum of the Laplacian acting on \( \mathcal{L}^2 \) functions. This inequality is generalized to every complete Riemannian manifold under certain conditions \( [18] \).

### 3.2 \( U(1) \) vector field

Our starting point is a massive \( U(1) \) gauge field \( A_\mu \) on \( \mathbb{R}^n \), with a \( \xi \) dependent gauge fixed action,

\[
S_E(A) = \int_{\mathbb{R}^n} \left( \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{\xi}{2} (\nabla^\mu A_\mu)^2 + \frac{1}{2} m^2 A_\mu A^\mu \right) \sqrt{g} d^n x
\]

\[
= \frac{1}{2} \int_{\mathbb{H}^n} A_\nu \left( -\nabla^\rho \nabla_\rho A^\nu + \nabla^\rho \nabla^\sigma A_\rho - \xi \nabla^\rho \nabla^\sigma A_\rho + m^2 A^\nu \right) \sqrt{g} d^n x
\]

\[
= \frac{1}{2} \int_{\mathbb{H}^n} A^\mu \left( -g_{\mu\nu} \nabla_\rho - R_{\mu\nu} + (1 - \xi) \nabla_\mu \nabla_\nu + g_{\mu\nu} m^2 \right) A^\nu \sqrt{g} d^n x
\]

\[
= \frac{1}{2} \int_{\mathbb{H}^n} A^\mu \hat{D}_{F,\mu\nu} A^\nu \sqrt{g} d^n x, \quad \hat{D}_{F,\mu\nu} = g_{\mu\nu}(-\Delta + m^2) - R_{\mu\nu} + (1 - \xi) \nabla_\mu \nabla_\nu \quad (84)
\]

where \( F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \) is the field strength. We use the following conventions: \( \nabla^\mu A_\mu = g^{\mu\nu}(\partial_\nu A_\mu - \Gamma^\mu_{\nu\rho} A_\rho) \) and the components of the Ricci tensor are given by \( R^\lambda_{\mu\lambda\nu} = \partial_\nu \Gamma^\lambda_{\rho\mu} + \Gamma^\nu_{\rho\sigma} \Gamma^\lambda_{\sigma\mu} \).

The last definition leads to the identity \( [\nabla_\mu, \nabla_\nu] A^\nu = R_{\mu\nu} A^\nu \) with \( R_{\mu\nu} = (n-1)g_{\mu\nu} \) for the hyperbolic space. The Green’s function satisfies the equation

\[
(-\delta^\mu_\nu \nabla^\rho - R^\rho_{\mu\nu} + (1 - \xi) \nabla_\mu \nabla_\nu + \delta^\nu_\nu m^2) G_{\nu\nu}(x, x') = \frac{g_{\mu\nu}(x)}{\sqrt{g}} \delta^{(n)}(x - x'). \quad (85)
\]

In the path integral approach of Quantum Field Theory one has to add the contribution of the Fadeev-Popov ghosts which in the Feynman gauge (\( \xi = 1 \)) is described by the action

\[
S_{ghost} = \int_{\mathbb{R}^n} \partial^\mu c \partial_\mu b \sqrt{g} d^n x = \int_{\mathbb{H}^n} c \hat{D}_{gh} b \sqrt{g} d^n x, \quad \hat{D}_{gh} = -\nabla^\mu \partial_\mu. \quad (86)
\]

with \( b \) and \( c \) be real anti-commuting scalar fields. In the previous derivation of the actions \( (84) \) and \( (86) \) we again utilized Green’s formula considering that \( \text{supp} A \) is compact and either \( \text{supp} c \) or \( \text{supp} \nabla b \) is compact.

The total massless action, in the Feynman gauge, is invariant under the B.R.S.T. symmetry

\[
\delta g A_\mu = \theta \partial_\mu b, \quad \delta g c = -\theta \nabla_\mu A^\mu \quad \delta gb = 0, \quad \theta^2 = 0 \quad (87)
\]

In this notation the components of the Ricci tensor for the hyperbolic space are positive.
which is parametrized by the infinitesimal anticommuting constant \( \theta \) \( \{c, \theta\} = \{b, \theta\} = 0 \).

The following bi-tensor identities for the derivatives of \( u \) (see Appendix \( A.11 \)) are very crucial for solving the \( U(1) \) heat equation

\[
\Box (\partial_\nu \partial_\nu u) = \partial_\nu \partial_\nu u, \quad \text{where} \quad \Box = g^{\mu \lambda} \nabla_\mu \nabla_\lambda
\]

\[
(\nabla_\mu u) \nabla^\mu (\partial_\nu u \partial_\nu u) = 4(1 + u) \partial_\nu u \partial_\nu u
\]

\[
(\Box n) \nabla (\partial_\nu u \partial_\nu u) = (n + 1) \partial_\nu u \partial_\nu u + 2(u + 1) \partial_\nu \partial_\nu u
\]

\[
(\Box + n - 1) \partial_\nu \partial_\nu Q(t, u) = \partial_\nu \partial_\nu (\Box Q(t, u)).
\]

(88)

The \( U(1) \) kernel \( K_{\mu \nu} \) is a \((1, 1)\) bi-tensor \([1]\) which can be written in the form (see Appendix \( C.6 \))

\[
K_{\mu \nu}(x, x'; t) = F(t, u) \partial_\mu \partial_\nu u + \partial_\mu \partial_\nu Q(t, u)
\]

(89)

and solves the initial-boundary valued problem

\[
\partial_t K_{\mu \nu}(x, x'; t) = -\hat{D}_{\nu \mu} K_{\nu \mu}(x, x'; t)
\]

\[
K_{\mu \nu}(x, x'; 0) = g_{\mu \nu}(x) \delta^{(n)}(x, x').
\]

(90)

Using (88) and (89) one finds that

\[
-\hat{D}^\nu_{\nu \mu} K_{\nu \mu}(x, x'; t) = (\partial_\mu \partial_\nu u) \Box n + 2 - (n - 2) F + 2 \left(F \partial_\mu \partial_\nu u + F' \partial_\nu u \partial_\nu u\right)
\]

+ \partial_\mu \partial_\nu (\Box Q) - m^2 Q \partial_\mu u \partial_\nu u.

(91)

Substituting (91) back into (90) we obtain the following partial differential equations

\[
\partial_t F(t, u) = (\Box_{H^n} + (n - 2) - m^2) F(t, u)
\]

(92)

\[
F(0, u) = -\delta^{(n)}(x, x')
\]

(93)

\[
\partial_t Q(t, u) = \Box_{H^n} Q(t, u) - m^2 Q(t, u) - 2 \int_u^\infty F(t, v)dv
\]

(94)

\[
\partial_a Q(0, u) = \partial^2_a Q(0, u) = 0
\]

(95)

where \( \Box_{H^n} = u(2 + \partial_u^2) + n(u + 1) \partial_u \). The solution of (92) subjected to condition (93) is

\[
F_{2k+1}(t, r) = \frac{(-1)^{k+1} e^{-m^2 t}}{(2\pi)^k \sqrt{4\pi t}} \left( \frac{1}{\sinh r} \partial_r \right)^k e^{-\frac{r^2}{4t}}.
\]

(96)

This is justified by noticing that (27) receives now an extra contribution \(-nf\) and, with the help of (28) which remains unchanged, it leads to a time independent solution \( f \), in the massless case. Therefore \( B(r, t) \propto \frac{1}{\sinh r} \partial_r \). Fourier transforming equation (95) we obtain, for \( n = 2k + 1 \), the kernel equation

\[
\partial_t \tilde{Q}(t, \lambda) = -(\lambda^2 + \rho^2 + m^2) \tilde{Q}(t, \lambda) + \frac{(-1)^{\rho-1}}{\pi (2\pi)^{\rho-1} \sqrt{4\pi t}} \tilde{F}_{2\rho-1}(t, \lambda).
\]

(97)

The solution of (97) turns out to be

\[
\tilde{Q}(t, \lambda) = \frac{1}{\lambda^2 + \rho^2 + m^2} e^{-A t} - \frac{2(-1)^{\rho-1}}{\lambda[(1 + \rho^2 + m^2)^2 + 4\lambda^2]} \left[(1 + \rho^2 + m^2) \sin(2t \lambda) - 2\lambda \cos(2t \lambda)\right] e^{-A t}
\]

\[
+ \frac{4(-1)^{\rho}}{[(1 + \rho^2 + m^2)^2 + 4\lambda^2]} e^{-A t} \left[\frac{1}{\lambda^2 + \rho^2 + m^2} \int_0^1 e^{(1 + \rho^2 + m^2)\xi} \sin(2t \lambda \xi) d\xi\right]
\]

(98)

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and the inverse Fourier transform is therefore given by

\[ Q(t, r) = \frac{1}{2\rho^{\rho+1}\Gamma(\rho)(\sinh r)^{2\rho-1}} \sum_{l_1=0}^{\rho-1} \left( \frac{\rho - 1}{l_1} \right) (-1)^{l_1} (\cosh r)^{l_1} \]

\[ \times \int_0^\infty \tilde{Q}(t, \lambda) \left( \int_0^r \cos(s\lambda)(\cosh s)^{\rho-1-l_1} ds \right) \prod_{l_2=1}^\rho [(\rho - l_2)^2 + \lambda^2] d\lambda. \]  

(99)

The trace of the heat kernel for a \( U(1) \) gauge field is

\[ Tr_{U(1)}^{2k+1} K = \lim_{x' \to x} \left( g^{\mu\nu}(x, x') K_{\mu\nu}(t, x, x') \right) \]

\[ = -(2k + 1) \lim_{r \to 0^+} \left( F(t, r) + \frac{1}{\sinh r} \partial_r G(t, r) \right) \]  

(100)

since

\[ [g^{\mu\nu}(x, x')] = g^{\mu\nu} \]  

(101)

\[ [g^{\mu\nu} \partial_\mu \partial_\nu u] = -\dim(\mathbb{H}^n) = -n \]  

and

\[ [g^{\mu\nu} \partial_\mu u \partial_\nu u] = 0 \]  

(103)

as one may prove by a straightforward calculation.

We investigate the massive-massless cases in \( n = 3 \) dimensions. Using (96) we have

\[ \lim_{r \to 0^+} F_3(t, r) = -\frac{e^{-m^2 t}}{(4\pi t)^\frac{3}{2}} \lim_{r \to 0^+} \left( \frac{r}{\sinh r} e^{-r^2/4t} \right) = -\frac{e^{-m^2 t}}{(4\pi t)^\frac{3}{2}}. \]  

(104)

On the other hand from (99) we obtain

\[ Q(t, r) = \frac{e^{-(1+m^2)t}}{2|S^2| \sinh r} \left[ 2 \left( e^{(1+m^2)t} \sinh(r(1 + m^2/2)) - e^{(1+m^2)t} \sinh(r\sqrt{1+m^2}) \right) \right. \]

\[ + e^{(1+m^2)t} e^{-r(1+m^2)} \text{Erf}\left( \sqrt{t(1+m^2)} - \frac{r}{2\sqrt{t}} \right) - e^{(1+m^2)t} e^{-r\sqrt{1+m^2}} \text{Erf}\left( \sqrt{t(1+m^2)} - \frac{r}{2\sqrt{t}} \right) \]

\[ + e^{(1+m^2)t} e^{r\sqrt{1+m^2}} \text{Erf}\left( \sqrt{t(1+m^2)} + \frac{r}{2\sqrt{t}} \right) - e^{(1+m^2)t} e^{r(1+m^2)} \text{Erf}\left( \sqrt{t(1+m^2)} + \frac{r}{2\sqrt{t}} \right) \]

\[ + \sqrt{\frac{t}{\pi}} \frac{2e^{-\frac{r^2}{4t}}}{|S^2| \sinh r} \int_0^1 e^{-tm^2(1-\xi)} e^{-t(1-\xi)^2} \sinh(r\xi) d\xi. \]  

(105)

The partial derivative of (105) and the integral of the last term can be performed exactly. We only present now the massless case for simplicity, which gives

\[ \lim_{m \to 0} \lim_{r \to 0} \left( \frac{1}{\sinh r} \partial_r Q(t, r) \right) = -\frac{(4t + e^{-t} - 1)}{3(4\pi t)^\frac{3}{2}}. \]  

(106)

Finally the trace of the heat kernel for a massless vector field in \( \mathbb{H}^3 \) is

\[ Tr_{U(1)}^{\mathbb{H}^3} K = \frac{(2 + 4t + e^{-t})}{(4\pi t)^\frac{3}{2}}. \]  

(107)
Proposition 3.2 The coincidence limits of the first three heat kernel coefficients, in the presence of a U(1) gauge field, are given by

\begin{align*}
\tilde{b}_{0,\mu
u'} &= g_{\mu
u'}(x, x) \\
\tilde{b}_{1,\mu
u'} &= \frac{1}{n} \left(1 - \frac{n}{6}\right) R g_{\mu
u'} \\
\tilde{b}_{2,\mu
u'} &= \left(\frac{n^4}{72} - \frac{n^3}{5} + \frac{313n^2}{360} - \frac{27n}{20} + \frac{2}{3}\right) g_{\mu
u'}
\end{align*}

(108) (109) (110)

where \(g_{\mu
u'}(x, x')\) is the parallel displacement operator of the field along the geodesic from the point \(x'\) to the point \(x\). The \(U(1)\) trace of \(\tilde{b}_{2,\mu
u'}(x, x)\) is then given by

\[
\text{Tr}_{U(1)} \tilde{b}_2 = \frac{1}{360} \left(2n - 30\right) R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + (180 - 2n) R_{\mu\nu} R^{\mu\nu} + (5n - 60) R^2
\]

(111)

Proof

Substituting the W.K.B. expansion of the \(U(1)\) heat kernel

\[
K_{\mu
u'}(x, x', t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-m^2 t} e^{-\frac{g_{\mu\nu'}^a(x,x')}{2t}} D^\frac{1}{2}(x, x') \Omega_{\mu
u'}(x, x', t) \text{ with}
\]

\[
\lim_{t \to 0^+} K_{\mu
u'}(x, x', t) = g_{\mu
u'}(x, x) \delta^{(n)}(x - x')
\]

(112)

back into (90) and using the power series expansion

\[
\Omega_{\mu
u'}(x, x', t) = \sum_{l=0}^{\infty} b_{l,\mu
u'}(x, x') t^l
\]

(113)

we obtain the following recurrent differential equation

\[
\left(1 + \frac{1}{l} \sigma_{\lambda} \nabla^\lambda\right) b_{l,\mu
u'}(x, x') = \frac{1}{l} \left(\hat{N} + n - 1\right) b_{l-1,\mu
u'}(x, x'), \quad l \geq 1 \quad \text{and}
\]

\[
\sigma_{\lambda} b_{0,\mu
u'}^{';}^\lambda(x, x') = 0
\]

(114) (115)

where the semicolon denotes covariant derivative in (115). The solution of (113) is \(b_{0,\mu\nu'}(x, x') = g_{\mu\nu'}(x, x')\) which in the \(x' \to x\) limit becomes the unit matrix. The recurrent partial differential equation (114) at the coincidence limit is rewritten as

\[
\tilde{b}_{2,\mu
u'} = \frac{1}{2} \left[\hat{N} b_{1,\mu
u'}(x, x') + \frac{(n-1)}{2} \tilde{b}_{1,\mu
u'}\right].
\]

(116)

Using the identity

\[
\Box b_{1,\mu\nu'}(x, x') = \Box \left(\hat{N} b_{0,\mu\nu'}(x, x') + (n - 1)b_{0,\mu\nu'}(x, x') - d\eta^\lambda b_{1,\mu\nu',\lambda}(x, x')\right)
\]

(117)

and the coincidence limits

\[
\left[\Box \hat{N}\right] = -\frac{n}{30} (n - 3)(n - 1)
\]

(118)

\[
\left[\hat{N} (\Box g_{\mu\nu'}(x, x'))\right] = -(n - 1) g_{\mu\nu'}
\]

(119)

\[
\left[D^{-\frac{1}{2}}(\Box D^{-\frac{1}{2}}) b_{1,\mu\nu'}(x, x')\right] = -\frac{n(n - 1)^2}{6} \left(1 - \frac{n}{6}\right) g_{\mu\nu'}
\]

(120)

we find that

\[
\left[\hat{N} b_{1,\mu\nu'}(x, x')\right] = \left(\frac{n^4}{36} - \frac{7n^3}{30} - \frac{73n^2}{180} + \frac{8n}{15} - \frac{1}{3}\right) g_{\mu\nu'}
\]

(121)
which combined with (116) the second heat kernel coefficient (110) is recovered. The \( U(1) \) trace of the heat kernel, in the massless case, is identical to the one predicted by the Fourier method in \( n = 3 \) dimensions.

The general solution of the recurrent equation (115), for arbitrary value of \( l \geq 1 \), can be written formally as

\[
b_{l,\mu\nu}(x, x') = \frac{1}{l!} \left( 1 + \frac{1}{\hat{A}} \right)^{-1} \hat{F} \left( 1 + \frac{1}{l-1} \hat{A} \right)^{-1} \hat{F} \cdots \left( 1 + \hat{A} \right)^{-1} b_{0,\mu\nu}(x, x')
\]

(122)

where \( \hat{A} = \sigma \lambda \nabla^{\lambda} \) and \( \hat{F} = \hat{N} + n - 1 \). Adding the contribution of the ghosts fields the \( U(1) \) traces in the massless case become

\[
\begin{align*}
\text{Tr}_{U(1)}^{H_n} b_{0,\text{tot.}} &= n - 2 \\
\text{Tr}_{U(1)}^{H_n} b_{1,\text{tot.}} &= \frac{1}{6} (8 - n) R \\
\text{Tr}_{U(1)}^{H_n} b_{3,\text{tot.}} &= \frac{1}{180} \left( (n - 17) R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + (92 - n) R_{\mu\nu} R^{\mu\nu} \right) + \frac{1}{72} (n - 14) R^2.
\end{align*}
\]

(123) (124) (125) (126)

4 Conclusions

In this paper we have calculated the one-loop partition function for a free, massive, real quantum scalar field and \( U(1) \) gauge field, living on the hyperbolic space \( \mathbb{H}^n(\mathbb{R}) \). In the scalar case we have provided closed expressions for the regular and UV divergent parts of the partition function. The \( U(1) \) vector case was solved by adapting a suitable ansatz and applying Fourier analysis on the corresponding heat equation. The regular parts of both partition functions were justified by an independent W.K.B. approximation in any dimension. We have also proposed an alternative approach to derive recurrent relations of the fundamental solutions of the heat equation in different dimensions.

Appendices

Appendix A

The Hodge inner product of a \( p \)-form \( (p \leq n - 1) \), taking values on a real vector bundle \( E \), is defined generally by

\[
\langle a_p, a_p \rangle = (-1)^s \int_{\mathcal{M}_n} a_p \wedge *a_p = \frac{(-1)^s}{p!} \int_{\mathcal{M}_n} a_{i_1 \ldots i_p} a^{i_1 \ldots i_p} \sqrt{g} d^n x
\]

(A.1)

where the index \( s \) denotes the dimension of the maximal subspace on which \( g \) is negative definite. For the real hyperbolic space \( s = 0 \).

There exist two distinct coordinate representations of the hyperbolic space that will be utilized throughout the present work.

The *geodetic spherical polar representation*. The equation of hyperboloid is satisfied by the
transformations

\[
X^1 = \sinh r \cos \theta_1, \quad r \in (0, \infty), \quad \theta_1 \in [0, 2\pi) \\
X^2 = \sinh r \sin \theta_1 \cos \theta_2 \\
\vdots \\
X^{n-1} = \sinh r \sin \theta_1 \sin \theta_2 \cdots \cos \theta_{n-1} \\
X^n = \sinh r \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-1}, \quad \theta_k \in [0, \pi], \quad k = 2, \cdots, n - 1 \\
X^0 = \cosh r
\] (A.2)

and the space has the topology of \( \mathbb{R}^+ \times S^{n-1} \). A direct calculation leads to the metric tensor and the volume element in the form

\[
d s_{\mathbb{H}^n}^2 = dr^2 + \sinh^2 r \gamma_{ij} d\theta^i d\theta^j \\
d V = (\sinh r)^{n-1} dr d\Omega_{n-1}
\] (A.3, A.4)

where \( \gamma_{ij} \) is the metric on \( S_{n-1} \) and \( d\Omega_{n-1} \) its volume element. Note that the hyperbolic spherical polar coordinates are the same as the Euclidean ones with the substitution \( \sinh r \) for \( r \). The Laplace-Beltrami operator is given by

\[
\Delta_{\mathbb{H}^n} = \partial_r^2 + (n - 1) \coth r \partial_r + \frac{1}{\sinh^2 r} \Delta_{S^{n-1}}.
\] (A.5)

The \textit{Poincaré half-space representation}. The second class of transformations that preserves the bilinear form is

\[
X^i = \frac{x_i}{x_n}, \quad i = 1, \cdots, n - 1 \\
X^n + X^0 = \frac{1}{x_n} \\
X^n - X^0 = \frac{1}{x_n} \sum_{i=1}^{n} x_i^2
\] (A.6)

The metric tensor

\[
ds_{\mathbb{H}^n}^2 = \frac{1}{x_n^2} \sum_{i=1}^{n} (dx_i)^2
\] (A.7)

is manifestly conformally invariant and superior in some calculations as compared to (A.2). The Laplace-Beltrami operator in this model is

\[
\Delta_{\mathbb{H}^n} = x_n^2 \Delta_{S^n} - (n - 2)x_n \partial_n
\] (A.8)

where \( \Delta_{S^n} \) is the Euclidean Laplacian.

Proposition 1 The geodesics of \( \mathbb{H}^n \) are straight lines perpendicular to the hyperplane \( x_n = 0 \) (the boundary \( \partial \mathbb{H}^n \)) and circles whose planes are perpendicular to \( x_n = 0 \) with centers on the hyperplane.

The proof is based on finding the solutions to the geodesics equation

\[
\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0
\] (A.9)
where $\tau$ is an affine parameter and
\[
\Gamma_{\nu\lambda}^\mu = \frac{1}{2}g^{\mu\sigma} (\partial_\nu g_{\lambda\sigma} + \partial_\lambda g_{\nu\sigma} - \partial_\sigma g_{\nu\lambda}), \quad g_{\mu\nu} = \frac{1}{x_n^2} \delta_{\mu\nu}
\]
are the components of the Levi-Civita connection (the well-known Christoffel symbols of the second kind). The Levi-Civita connection is an affine connection which is metric compatible and torsion free. Considering two points on $\mathbb{H}^n$ their geodesic distance is given by
\[
d_{\mathbb{H}^n}(x, x') = \inf \left\{ \int_0^1 \sqrt{g_{\mu\nu}(x(\tau)) \dot{x}^\mu \dot{x}^\nu d\tau}, x(\tau) \in C^1([0, 1]), x(0) = x, x(1) = x' \right\}
\]
which is found to be
\[
cosh d_{\mathbb{H}^n}(x, x') = 1 + \frac{1}{2x_n x'_n} d_{\mathbb{H}^n}^2(x, x') = 1 + u(x, x') \tag{A.10}
\]
and $u(x, x')$ is the chordal distance. Using the half-space representation one can have the alternative expression for $u(x, x')$
\[
u(x, x') = \frac{1}{2} \eta_{AB} (X - X')^A (X - X')^B = \frac{\delta_{ij}(x - x')_i (x - x')_j}{2x_n x'_n},
\]
\[A, B = 0, \cdots, n, \quad i, j = 1, \cdots, n \tag{A.11}
\]
where the Minkowski metric $\eta_{AB}$ has signature $(-, +, \cdots, +)$ and Einstein’s summation convention for each repeated pair of indices is adopted.

In general a function of the hyperbolic distance satisfies the equation
\[
\Delta f(d_{\mathbb{H}^n}) = (n - 1) f'(d_{\mathbb{H}^n}) \coth(d_{\mathbb{H}^n}) + f''(d_{\mathbb{H}^n}). \tag{A.12}
\]
This can be proved by first showing the following relation
\[
x_n^2 \left( \Delta_{\mathbb{H}^{n-1}} + \partial_{n}^2 \right) d_{\mathbb{H}^n} = 1. \tag{A.14}
\]

Appendix B

In the Fourier transform we are going to use a form of $\Phi_\lambda$ which is derived by changing $r$ into the variable $s = -\ln(cosh r - \cos \theta \sinh r)$ and thus obtaining
\[
\Phi_\lambda(r) = \frac{(-1)^{\rho+1}2^{\rho-1}\Gamma(\rho + \frac{1}{2})}{(\sinh r)^{2\rho-1}\sqrt{\pi}\Gamma(\rho)} \int_{-r}^{r} e^{is\lambda}(\cosh r - \cosh s)^{\rho-1} ds
\]
\[
= \frac{2^{\rho}\Gamma(\rho + \frac{1}{2})}{(\sinh r)^{2\rho-1}\sqrt{\pi}\Gamma(\rho)} \sum_{l=0}^{\rho-1} \left( \frac{\rho - 1}{l} \right) (-1)^l (\cosh r)^l \int_0^r \cos(s\lambda)(\cosh s)^{\rho-1-l} ds. \tag{B.1}
\]
Also the modulus of the Harish-Chandra $c$-function is given by
\[
|c(\lambda)|^2 = \frac{4^{2\rho-1}\Gamma^2(\rho + \frac{1}{2})}{\pi} \left| \frac{\Gamma(i\lambda)}{\Gamma(\rho + i\lambda)} \right|^2 \frac{1}{\prod_{l_2=1}^\rho [(\rho - l_2)^2 + \lambda^2]}. \tag{B.2}
\]
where the gamma function identities, for $z \in \mathbb{C}$,
\begin{align}
\Gamma(z) &= \Gamma(z) \\
\Gamma(1 + z) &= z\Gamma(z) \\
\Gamma(1 - z)\Gamma(z) &= \frac{\pi}{\sin(\pi z)}.
\end{align}

have been used.

The derivation of the one-loop effective actions (54), (61), (62), on $\mathbb{H}^3, \mathbb{H}^2$, is based on $p_3, p_5$ which are obtained by applying the recurrence relation
\begin{equation}
p_{2k+1}(r, t) = -\frac{1}{2\pi} e^{-(2k+1)\frac{t}{4}} \frac{1}{\sinh r} \frac{\partial}{\partial r} p_{2k-1}(r, t).
\end{equation}

We thus have
\begin{align}
p_3(r, t) &= \frac{1}{(4\pi t)^{\frac{3}{2}}} \frac{r}{\sinh r} e^{-\left(\frac{r^2}{4} + t\right)} \\
p_5(r, t) &= \frac{1}{(4\pi t)^{\frac{3}{2}}} \left[ \left(\frac{2r}{\sinh r}\right)^2 + g(r, t) \right] e^{-\left(\frac{r^2}{4} + t\right)} , \quad g(r, t) = 2t \frac{r \cosh r - \sinh r}{\sinh^3 r} \\
p_7(r, t) &= \frac{1}{(4\pi t)^{\frac{3}{2}}} \left[ \left(\frac{3r}{\sinh r}\right)^3 + \frac{3r}{\sinh r} g(r, t) - \frac{2t}{\sinh r} \frac{\partial}{\partial r} g(r, t) \right] e^{-\left(\frac{r^2}{4} + 9t\right)}
\end{align}

where \( \frac{\partial}{\partial r} g(r, t) = \frac{2t}{\sinh^2 r} (r + 3 \coth r - 3r \coth^2 r) \).

It is interesting to observe that all the heat kernels in $\mathbb{H}^{2k+1}$ are given by elementary functions while in even dimensions by integrals.

For $\mathbb{H}^3$ we have $a_{1,0} = 1$, $\mathbb{H}^5 (a_{2,0}, a_{2,1}) = (1, 2/3)$ and $\mathbb{H}^7 (a_{3,0}, a_{3,1}, a_{3,2}) = (1, 2, 16/15)$. Note that
\begin{equation}
a_{k+1,0} = \lim_{r \to 0^+} \left(\frac{r}{\sinh r}\right)^k = 1, \quad \forall k \in \mathbb{N}.
\end{equation}

In the derivation of (60) we used the trigonometric identity
\begin{equation}
\sinh\left(\frac{x}{2}\right) = \sqrt{\cosh x - 1} \text{sign}(x),
\end{equation}

the Taylor expansion of $1/ \sinh x$ around $x = 0$, namely
\begin{equation}
\frac{1}{\sinh x} = \frac{1}{x} \left(1 - \sum_{l=1}^{\infty} \frac{2(2l-1) - 1}{2l} \frac{x^{2l}}{(2l)!}\right)
\end{equation}

and
\begin{equation}
\frac{2^l (2l-1)!!}{(2l)!} = \frac{1}{l!}.
\end{equation}

Another useful Maclaurin expansion, used in (64), is
\begin{equation}
\frac{1}{\cosh x} = \sum_{k=0}^{\infty} \frac{(-1)^k 2(1 - 2^{k+1}) B_k + 1}{(k + 1)(2k)!} x^{2k}.
\end{equation}
Appendix C

The proof of (72) is based on the following relation

\[ D(x, x') = \left( \frac{r}{\sinh r} \right)^n \text{Det} A_{\mu\nu' \prime}, \quad \text{where} \]

\[ A_{\mu\nu' \prime} = \frac{1}{\sinh r} \left[ \left( \text{coth } r - \frac{1}{r} \right) \partial_{\mu} u \partial_{\nu'} u + \partial_{\mu} \partial_{\nu'} u \right] \]

\[ = \frac{1}{\sinh r} \left[ \left( \text{coth } r - \frac{1}{r} \right) \left( \frac{(x-x')_\mu}{x'_n} - u \delta_{\mu n} \right) \left( \frac{(x-x')_{\nu'}}{x_n} - u \delta_{\nu' n} \right) \right] + \left( \delta_{\mu\nu' \prime} + \frac{(x-x')_\mu}{x'_n} \delta_{\nu' n} + \frac{(x-x')_{\nu'}}{x_n} \delta_{\mu n} - u \delta_{\nu' n} \delta_{\mu n} \right) \]

\[ \text{Det} A_{\mu\nu' \prime} = \frac{\sinh r}{r}. \quad \text{(C.2)} \]

The ansatz (89) we used for the \( U(1) \) heat kernel is justified by the observation that any \((1,1)\) maximally symmetric bi-tensor can be expressed as the following combination

\[ K_{\mu\nu'}(x, x'; t) = f_1(t, u)g_{\mu\nu'}(x, x') + f_2(t, u)\eta_\mu(x, x')\eta_{\nu'}(x, x') \quad \text{(C.3)} \]

where \( \eta_\mu(x, x') = \partial_\mu d_{\xi\eta}(x, x') \) and \( \eta_{\nu'}(x, x') = \partial_{\nu'} d_{\xi\eta}(x, x') \) are the unit tangents to the chordal distance at \( x \) and \( x' \) respectively. Taking into account the relation

\[ \nabla_\mu \eta_{\nu'} = -\frac{1}{\sinh d} (g_{\mu\nu'} + \eta_\mu \eta_{\nu'}) \quad \text{(C.4)} \]

equation (C.3) can be rewritten as

\[ K_{\mu\nu'}(t, u) = A(t, u) \partial_\mu \partial_{\nu'} u + B(t, u) \partial_\mu u \partial_{\nu'} u \quad \text{where} \]

\[ A(t, u) = \frac{1}{u + 2} \left( f_1(t, u) + \frac{f_2(t, u)}{u} \right) \quad \text{and} \quad B(t, u) = -f_1(t, u). \]

The derivation of (105) requires the following integrals

\[ \int_0^1 e^{(1+\rho^2+m^2)t} \sin(2t \lambda \xi) d\xi = \frac{2\lambda + e^{(1+\rho^2+m^2)t} \left[ (1 + \rho^2 + m^2) \sin(2t \lambda) - 2 \lambda \cos(2t \lambda) \right]}{t' \left[ 1 + (1 + \rho^2 + m^2)^2 + 4\lambda^2 \right]} \quad \text{(C.6)} \]

and

\[ \int_0^\infty e^{-\beta x^2} \sin(\alpha x) \frac{x}{(\gamma^2 + x^2)} dx = -\frac{\pi}{4} e^{\beta \gamma^2} \left[ 2 \sinh(\alpha \gamma) + e^{-\alpha \gamma} \text{Erf} \left( \gamma \sqrt{\beta} - \frac{\alpha}{2\sqrt{\beta}} \right) \right] \]

\[ - e^{\alpha \gamma} \text{Erf} \left( \gamma \sqrt{\beta} + \frac{\alpha}{2\sqrt{\beta}} \right). \quad \text{(C.7)} \]

**Useful identities and proofs involving coincident limits**

Using the definition of the Syng's function \( \sigma(x, x') = d_{\eta \eta}(x, x')/2 \) \[22\] we can prove the following identities \((d \equiv d_{\xi \eta})\):

\[ \eta_{\mu'} = -g_{\mu''}^\nu \eta_\nu, \quad \eta_\mu \eta^\mu = \eta_{\mu'} \eta^{\mu'} = 1 \quad \text{(C.8)} \]

\[ \eta_{\mu', \nu} = A(d)(g_{\mu\nu} - \eta_\mu \eta_\nu), \quad \text{where} \quad A(d) = \coth d \quad \text{(C.9)} \]

\[ \eta_{\mu' \nu'} = C(d)(g_{\mu' \nu'} + \eta_{\mu'} \eta_\nu), \quad \text{where} \quad C(d) = -\frac{1}{\sinh d} \quad \text{(C.10)} \]

\[ \eta^\nu \eta_{\mu' \nu'} = 0 \quad \text{(C.11)} \]

\[ \Box \eta_\mu = -(n-1)A^2 \eta_\mu, \quad \Box \eta_{\mu'} = -(n-1)C^2 \eta_{\mu'} \quad \text{(C.12)} \]
\[ g_{\mu'\nu;\lambda} = -(A + C)(g_{\nu\lambda}\eta_{\mu'} + g_{\mu'\lambda}\eta_{\nu}) \] (C.13)

\[ \Box g_{\mu'\nu} = -(A + C)^2 (g_{\mu'\nu} - (n - 2) \eta_{\mu'} \eta_{\nu}) \] (C.14)

\[ \eta^\lambda g_{\mu'\nu;\lambda} = 0 \] (C.15)

\[ \sigma_{\mu;\nu} = dA g_{\mu\nu} + (1 - dA)\eta_{\mu} \eta_{\nu}, \quad \sigma_{\mu} = d\eta_{\mu} \] (C.16)

\[ \sigma_{\mu';\nu} = dC g_{\mu'\nu} + (1 + dC)\eta_{\mu'} \eta_{\nu} \] (C.17)

\[ \sigma_{\mu;\nu;\lambda} = \sigma_{\mu;\lambda;\nu} = R^{\rho}_{\mu
u\lambda} \sigma_{\rho} \] (C.18)

Next we list some identities related to coincidence limits

\[ [\eta_{\mu}] = [\eta_{\mu'}] = [\sigma_{\mu}] = [\sigma_{\mu'}] = [D_{\mu}] = 0 \quad \text{and} \quad [D] = 1 \] (C.19)

\[ [\sigma_{\mu;\nu}] = -[\sigma_{\mu';\nu}] = -[g_{\mu';\nu}] = -g_{\mu'\nu} \] (C.20)

\[ \left[ D^\frac{1}{2}_{\mu';\nu} \right] = -\frac{1}{6} R_{\mu'\nu}, \quad \left[ \Box D^\frac{1}{2} \right] = -\frac{1}{6} R \] (C.21)

\[ [\Box \Box g_{\mu'\nu}] = -(n - 1) g_{\mu'\nu} \] (C.22)

We now establish the proof of relation (71). Observe that

\[ D^{-1} \nabla_{\mu} (D \partial \sigma_{\mu}) = \Box \sigma + dD^{-1} D', \quad \text{where} \quad D' \equiv \frac{dD(r)}{dr} \] (C.23)

\[ = n \] (C.24)

since \( \Box \sigma = d(n - 1) \coth(d) + 1 \) by (A.13).

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