Abstract

We point out (and then apply to a general situation) an unusual relationship among a variety of null geodesic congruences: (a) the generators of ordinary light-cones and (b) certain (related) shear-free but twisting congruences in Minkowski Space-time as well as (c) asymptotically shear-free null geodesic congruences that exist in the neighborhood of Penrose’s $I^+$ in Einstein or Einstein-Maxwell asymptotically flat-space-times. We refer to these geodesic congruences respectively as: Light-Cones (LCs), as “Almost-Complex” Light-Cones (ACLCs), though they are real they resemble complex light-cones in complex Minkowski space and finally to a family of congruences in asymptotically flat-spaces as ‘Almost Light-Cones’, (ALC). The two essential points of resemblance among the three families are: (1) they are all either shear-free or asymptotically shear-free and (2) in each family the individual members of the family can be labeled by the points in a real or complex four-dimensional manifold. As an example, the Minkowski space LCs are labeled by the (real) coordinate value of their apex. In the case of (ACLCs) (complex coordinate values), the congruences will have non-vanishing twist whose magnitude is determined by the imaginary part of the complex coordinate values.

In studies of gravitational radiation, Bondi-type of null surfaces and their associated Bondi coordinates have been almost exclusively used for calculations. It turns out that some surprising relations arise if, instead of the Bondi coordinates, one uses ALCs and their associated coordinate systems in the analysis of the Einstein-Maxwell equations in the neighborhood of $I^+$. More explicitly and surprisingly, the asymptotic Bianchi Identities (arising directly from the Einstein equations), expressed in the coordinates of the ALCs, turn directly into many of the standard definitions and equations and relations of classical mechanics coupled with Maxwell’s equations. These results extend and generalize the beautiful results of Bondi and Sachs with their expressions for, and loss of, mass and linear momentum.

1 Introduction

It has been known for many years that shear-free null geodesic congruences, in flat, Einstein and Einstein-Maxwell spaces have fascinating and useful properties. The Robinson-Trautman metrics and the Goldberg-Sachs theorem are among the most prominent examples.

It is one of our purposes to explore some new aspects of these congruences. We first note that the generators (the null geodesics) of ordinary light-cones, LCs, in Minkowski space-time are among the most familiar examples of shear-free congruences. Each of these can be identified by the four Minkowski coordi-
nate values at their apex. We will show, in Sec.II, that another set of SFGCs, can be described and identified by four complex valued coordinates that can be thought of as coordinates in complex Minkowski space-time. Although these congruences are real congruences in real Minkowski space, we will refer to them as Almost-Complex Light-Cones, (ACLCs), because of their identification labels - and their close relation to complex LCs cones. The real LCs are special cases of the ACLCs. The two sets, LCs and ACLCs, constitute the entire class of flat-space shear-free congruences that are diverging with isolated caustics.

A second type of related null geodesic congruence, described in Sec.III, occur in asymptotically flat-space-times - where, though they are not shear-free, are instead asymptotically shear-free. The individual members, as in the flat-space-time, are also labeled by four complex numbers and are referred to as Almost-Light-Cones, ALCs. The ALCs contain, as special cases, all the other cases. The four-complex numbers (in each of the cases) define a complex manifold, referred to as $\mathcal{H}$-space, containing an interesting variety of properties, e.g., a complex metric that satisfies the complex vacuum Einstein equations with a self-dual Weyl tensor.

In the cases when the labels are real the congruences are surface forming, - when they are complex, the imaginary parts are a measure of the twist of the congruence.

**Note** We have taken the liberty to slightly generalize the meaning of 'congruence'. Usually it refers to a three parameter family of curves filling a space-time region. We will use it to mean a two parameter family of curves, as, for example, the geodesics on one single light cone. A one-parameter family of such light-cones (based on a time-like world line) would yield the standard example of a congruence.

Since the 1950s most studies of the far-field gravitational properties, including gravitational radiation, have used, as technical tools in the analysis of the Einstein or Einstein-Maxwell equations, certain null surfaces (Bondi surfaces) and the associated coordinate systems referred to as Bondi coordinates. This led to Bondi’s and Sach’s, beautiful theorems on mass and linear momentum loss\(^2\)\(^3\) and eventually to the development of LIGO with its technology. Finally, after over 50 years of developments involving theory, numerical analysis and observational work, this led to the observation and analytic understanding of the collision and merger of the pair of black holes that produced the gravitational wave signal, GW105,\(^1\), that was seen by LIGO early in 2016.

It has seemed for many years as if the Bondi system was almost sacrosanct - the best and virtually only way to study, in general, the asymptotic behavior of the Einstein-Maxwell equations. Our contention is that this is not necessarily so - there appear to be very good reasons to consider the use of families of ALCs as our choice of asymptotic coordinate systems. First of all they very closely resemble the standard Minkowski space null coordinate systems in the neighborhood of null infinity; they are labeled by four coordinates and are asymptotically-shear-free - the Bondi surfaces are not. Second, when the asymptotic Bianchi Identities are studied in families of ALCs they turn out to explicitly be - in the low order, \((l = 0, 1)\) spherical harmonic decomposition -
many of the standard definitions and dynamic relations and equations of classical mechanics coupled with the Maxwell field. They are a large extension of the Bondi-Sachs results. Though this itself is surprising, the astonishing and so-far inexplicable fact is that these equations of classical mechanics take place in the $\mathcal{H}$-space rather than in physical space-time. Sec. IV will contain a description of these results.

1.1 Flat-Space Shear-Free Congruences

Using $x^a$ as standard Minkowski-space coordinates, an arbitrary null geodesic congruence can be described by

$$
x^a = \sqrt{2}ut^a - Lm^a - \overline{T}n^a + (r - r_0)l^a,
\sqrt{2}t^a = l^a + n^a
u_r = \sqrt{2}u,
$$

where the ‘parameters’ $(u, \zeta, \overline{\zeta})$ label the individual members of the geodesic congruence and $r$ is the affine parameter along each null geodesic, $u_r = t - r$, is the retarded time. The $L(u, \zeta, \overline{\zeta})$ (which is a null angle field\textsuperscript{[4]} and the primary source of information about the congruence) is an arbitrary regular complex function of the parameters, while $r_0(u, \zeta, \overline{\zeta})$ is a real function that determines the arbitrarily origin of the affine parameter along each geodesic. The null tetrad vectors, $l^a, m^a, \overline{m}^a, n^a$ are given by

$$
\begin{align*}
  l^a &= \frac{\sqrt{2}}{2P}(1 + \overline{\zeta}^2, \zeta + \overline{\zeta}, -i(\zeta - \overline{\zeta}), -1 + \zeta \overline{\zeta}); \\
  m^a &= \partial l^a = \frac{\sqrt{2}}{2P}(0, 1 - \zeta^2, -i(1 + \zeta^2), 2\zeta), \\
  \overline{m}^a &= \overline{\partial} l^a = \frac{\sqrt{2}}{2P}(0, 1 - \zeta^2, i(1 + \zeta^2), 2\zeta), \\
  n^a &= \frac{\sqrt{2}}{2P}(1 + \zeta \overline{\zeta}, -(\zeta + \overline{\zeta}), i(\zeta - \overline{\zeta}), 1 - \zeta \overline{\zeta}), \\
  P &= 1 + \zeta \overline{\zeta}. 
\end{align*}
$$

\textbf{Aside:} We note that Eq.(1) has the alternative interpretation as a coordinate transformation between the $x^a$ and the $(u, \zeta, \overline{\zeta}, r)$.

The optical parameters\textsuperscript{[5,6]} associated with the congruence, i.e., the complex divergence $\rho$, the complex shear $\sigma$ and the twist $\Sigma$, are given, after a rather lengthy calculation, by

$$
\begin{align*}
  \rho &= \frac{i\Sigma - (r - r_0^*)}{r^2 + \Sigma^2 - \overline{\sigma}_0 \sigma_0}, \\
  \sigma &= \frac{\overline{\sigma}_0}{(r - r_0^*)^2 + \Sigma^2 - \overline{\sigma}_0 \sigma_0} \\
\end{align*}
$$
with
\[ \sigma_0 = \partial L + LL_{\mu} \]
\[ 2i\Sigma = \partial L + L\mathcal{T}_{\mu} - \mathcal{T}L - L\mathcal{T}_{\mu} . \] (6) (7)

The arbitrary function \( r_0 \) has first been chosen as
\[ r_0 = -\frac{1}{2}(\partial L + LL_{\mu} + L\mathcal{T}_{\mu} ) \]
and then a new arbitrary \( r_0^* (u, \zeta, \overline{\zeta}) \) chosen again as the origin for the affine parameter \( r \).

We thus see that the optical parameters are determined by the choice of \( L(u, \zeta, \overline{\zeta}) \).

Our main interest lies in the class of regular null geodesic congruences with a vanishing shear. This condition is achieved by imposing the shear-free condition on \( L \), i.e., that \( L \) must satisfy the differential condition.\[9\]
\[ \partial L + LL_{\mu} = 0 \] (8)

and with regular solutions. This procedure has been well documented\[8,9\] in the literature and we only give the solution.

Changing the independent variable \( u \) to \( \tau \) via the function
\[ u = G^*(\tau, \zeta, \overline{\zeta}), \] (9)
with inverse,
\[ \tau = T(u, \zeta, \overline{\zeta}), \] (10)
we have the solution of Eq.(8), given parametrically, as
\[ L(u, \zeta, \overline{\zeta}) = \partial(\tau)G^*(\tau, \zeta, \overline{\zeta})|_{\tau=T(u, \zeta, \overline{\zeta})} \] (11)
\[ u = G^*(\tau, \zeta, \overline{\zeta}) = \xi^a(\tau)l_a(\zeta, \overline{\zeta}). \] (12)

The operator \( \partial(\tau) \) means \( \partial \) holding \( \tau \) constant. The \( z^a = \xi^a(\tau) \) determines a complex world-line, (parametrized by the complex \( \tau \)), in a complex four-dimensional space that can be identified with complex Minkowski space - a special case of \( H \)-space.

When the complex world-line is chosen as a real world-line in real Minkowski space, i.e., \( z^a = x^a = \xi^a_R(t), \) with \( t \) chosen so that the velocity vector, \( v^a = \xi^a_R \), is time-like with norm \( |v^a| = 1 \), we have the family of null geodesics given by the generators of the light-cones with apex on the world-line \( x^a = \xi^a_R(t) \). Explicitly, Eq.(11) becomes (after appropriately adjusting the \( r_0^* \)), our LCs,
\[ x^a = \xi^a(u) + r l^a(\zeta, \overline{\zeta}). \]

If we chose \( \xi^a(\tau) \) as a complex curve, (again with the velocity normalization, \( |v^a| = 1 \)) the construction of the congruence is a bit more complicated. In the
equations, (11) and (12), $u$, $\tau$, $G^*$ are complex while in principle we need $u$ to be real. The problem is handled as follows: $L$ and $\bar{L}$ are first constructed by

\begin{align*}
L &= \partial_{(\tau)} G(\tau, \zeta, \bar{\zeta}) = \xi^a(\tau) m_a(\zeta, \bar{\zeta}), \\
\bar{L} &= \bar{\xi}^a(\bar{\tau}) \bar{m}_a(\zeta, \bar{\zeta})
\end{align*}

(13)

In Eq. (9) we replace $\tau$ by $t + i\Lambda$ and decompose the complex $G$ into its real and imaginary parts,

$$G^*(t + i\Lambda, \zeta, \bar{\zeta}) = G_R^*(t, \Lambda, \zeta, \bar{\zeta}) + iG_I^*(t, \Lambda, \zeta, \bar{\zeta}).$$

By setting $G_I^*(t, \Lambda, \zeta, \bar{\zeta}) = 0$ we determine $\Lambda = \Lambda(t, \zeta, \bar{\zeta})$ and simultaneously make $u$ real, $u_R = G_R^*(t, \Lambda, \zeta, \bar{\zeta})$. Its inverse

$$t = T_R(u_R, \zeta, \bar{\zeta})$$

with $\Lambda(t, \zeta, \bar{\zeta})$ allow us to express $L$, $\bar{L}$, Eq. (13), as functions of $(u_R, \zeta, \bar{\zeta})$. We drop use of the subscript $(R)$ in the $u$.

The null geodesic congruence, Eq. (1), constructed with these $L$ and $\bar{L}$ are the ACLCs.

The twist, Eq. (7), of these congruences is determined by the imaginary part of the world-line, $\xi^a_I(\tau)$, $[\xi^a = \xi^a_R + i\xi^a_I]$ via

$$\Sigma = \xi^a_I(\tau)(n_a - l_a).$$

(14)

Note: It is important that we first construct the $L$ and $\bar{L}$ by taking the $\partial_{(\tau)}$ and $\overline{\partial}_{(\tau)}$ derivatives before choosing $u$ to be real.

Aside: If instead of $\bar{L}$ we had used $\tilde{L} = \xi^a(\tau) \bar{m}_a(\zeta, \bar{\zeta})$ with the complex $u$ from Eq. (11) the congruence would have been complex Minkowski space light-cones - i.e., the reason for referring those constructed with $\bar{L}$ as almost complex cones.

Remark: We mention, for later use, that the complex world-line $z^a = \xi^a(\tau)$ will be uniquely chosen, by definition, as both the complex center of mass, (center of mass $+$i angular momentum) and complex center of charge, (electric dipole $+$i magnetic dipole) world-line on which both vanish. That both 'centers' vanish on the same world-line is a special case of the more general situation.

1.1.1 An Alternative Means of Construction

For the insight that it gives and for use in the following section in the determination of asymptotically shear-free congruences, we describe an alternative method of construction of these congruences.

Starting with the family of null geodesics from the null cones with apex on the spatial origin

$$x^a = ut^a + rl^a(\zeta, \bar{\zeta}),$$

(15)
(augmented by $l^a, m^a, m^a, n^a$ from Eq.(3)) we can, very roughly or intuitively, define Penrose’s Null Infinity, $\mathcal{I}^+$, as all the points, $(u, \zeta, \bar{\zeta})$ obtained by taking the limit, $r => \infty$. $\mathcal{I}^+$ becomes the null surface at null infinity - with the structure of $\mathbb{R} \times S^2$. It is obviously coordinatized by the $(u, \zeta, \bar{\zeta})$. (The tetrad, with these coordinates, $(u, \zeta, \bar{\zeta})$, are a special case of a Bondi system.)

It turns out\cite{11} that the forward light-cone from any interior space-time point $x^a$ intersects $\mathcal{I}^+$ on the cut or slice, $S^2$, of $\mathcal{I}^+$ given by

$$u = x^a l_a (\zeta, \bar{\zeta})$$

so that the light-cones of a space-time world-line, $x^a = \xi^a_R (\tau)$, yields a one-parameter family of slicings of $\mathcal{I}^+$,

$$u = G^a (\tau, \zeta, \bar{\zeta}) = \xi^a_R (\tau) l_a (\zeta, \bar{\zeta}) .$$  \hspace{0.5cm} (17)

The null vectors, $l^a$, tangent to the geodesic congruence coming from the interior, that are normal to the slicings are given by

$$l^a = l^a + L m^a + \bar{L} n^a$$

with $L = \delta G$. The condition for the congruence to be shear-free is again, Eq.(8), $\delta L + L L_{,u} = 0$ which is satisfied by Eq.(17). We are thus back to the previous discussion and getting close to the discussion of the next section.

If the world-line is real then we can replace the $\tau$ by $t$, if it is complex, again we construct $L = \delta G^a$ as done earlier and evaluate it for real $u$.

## 2 Asymptotically Flat Space-Times

The study of solutions and properties of the asymptotically flat Einstein-Maxwell equations is a large subject with a great deal of literature. We will need only a small fraction of this material. Rather than rederiving what we do need, we will largely take from this literature - mainly from Newman-Penrose, (in Scholarpedia) and Adamo-Newman (in Living Reviews) - often making use of the NP formalism\cite{6}\cite{5}\cite{7}

A basic tool in these studies was the introduction, by Bondi, of null surfaces to be used as part of the asymptotic coordinate system. A one-parameter family of null surfaces, $\mathcal{B}_u$ labeled by $u$, was introduced. A two-parameter family of null geodesics, the generators or geodesics of each surface, are each labeled by sphere coordinates $(\theta, \phi)$ or equivalently (used here) by complex stereographic coordinates $(\zeta, \bar{\zeta})$, where $\zeta = e^{i \phi} \cot (\frac{\theta}{2})$. The 'length' along the geodesics is given by the affine parameter, $r$. Again (as in the previous section), roughly or intuitively, the future null boundary of space-time, i.e., Penrose’s Null Infinity, $\mathcal{I}^+$, is defined by points $(u, \zeta, \bar{\zeta})$ taken in the limit, $r => \infty$. The 'boundary', $\mathcal{I}^+$, \textit{(which can be mathematically more formally defined)}, is a null surface, $S^2 \times \mathbb{R}$, coordinatized by $(u, \zeta, \bar{\zeta})$ with $u$ as the intersection points of $\mathcal{I}^+$ with the Bondi null surfaces $\mathcal{B}_u$. The generators of $\mathcal{I}^+$, \textit{(the $S^2$ part)} are labeled by the
stereographic coordinates $(\zeta, \bar{\zeta})$ - and have the same labels as the generators of $\mathfrak{B}_u$ that they intersect.

**Aside:** The full set of coordinates, $(u, \zeta, \bar{\zeta}, r)$, called Bondi coordinates are not unique - there being an entire group, the BMS group of transformations\cite{9}, connecting the different members. This lack of uniqueness does not now play an important role for us - though that is likely to change in the future.

In addition to Bondi coordinates, a Bondi system, at and near $\mathcal{I}^+$ also contains a null tetrad, $(l^a_B, m^a_B, n^a_B, n^a_B)$. The $l^a_B$ are tangent vectors to the geodesics of $\mathfrak{B}_u$, the $n^a_B$, are tangent vectors of the generators of $\mathcal{I}^+$, $(m^a_B, \bar{m}^a_B)$ are tangent vectors to the $u = \text{const}$ slices of $\mathcal{I}^+$. The $m^a_B$ are parallel propagated down the generators of $\mathfrak{B}_u$ to the interior.

The points of $\mathcal{I}^+$ with constant value of $u$ are referred to as Bondi slices or Bondi cuts; any arbitrary cross-section or family of cross-section of $\mathcal{I}^+$, i.e., $u = K(\zeta, \bar{\zeta})$ or $u = F(s, \zeta, \bar{\zeta})$ are called slices or cuts. Much of our effort will be devoted to finding, studying and giving applications to certain preferred slicings (asymptotically shear-free) - that are very different from a Bondi slicing. An important fact is that the family of null geodesics of the surfaces $\mathfrak{B}_u$, in general, are not (asymptotically) shear-free. Their shear is given by

$$\sigma = \sigma^0(u, \zeta, \bar{\zeta}) \frac{r^2}{r^2} + O(r^{-4}),$$

with $\sigma^0(u, \zeta, \bar{\zeta})$ referred to as the asymptotic shear. It plays the role of arbitrary radiation data. ($\sigma^0(u)$ is referred to as the Bondi news function.) Our task is to find slicings of $\mathcal{I}^+$ so that the normal null congruences (normal to the slicing) are asymptotically shear-free.

Since this problem has been described and solved in the literature\cite{9}, we give the solution with just a brief explanation.

The Sachs theorem, which describes how $\sigma^0(u, \zeta, \bar{\zeta})$ transforms under a BMS super-translations, i.e., under $u' = u - \alpha(\zeta, \bar{\zeta})$, states that

$$\sigma^0 \lbrack(u', \zeta, \bar{\zeta}) = \sigma^0(u, \zeta, \bar{\zeta}) - \Box^2 \alpha.$$

Setting the new shear to zero, $\sigma^0 \lbrack = 0$ when $u' = 0$, (i.e., at $u = \alpha(\zeta, \bar{\zeta})$), leads, with $\alpha$ replaced by $G(\zeta, \bar{\zeta})$, to

$$\Box^2 G = \sigma^0(G(\zeta, \bar{\zeta})), \quad (19)$$

the so-called 'good-cut Equation'.

Solutions to Eq.\,(19) have been shown\cite{9,10} to depend on four arbitrary complex numbers, $z^a$, which in turn define a four-complex dimensional space, referred to as $\mathcal{H}$--space. Imposing coordinate conditions on the choices of these coordinates, the solution can always be written as

$$u = G(z^a, \zeta, \bar{\zeta}) = z^a l^a(\zeta, \bar{\zeta}) + H_{l \geq 2}(z^a, \zeta, \bar{\zeta}) \quad (20)$$

with $H_{l \geq 2}(z^a, \zeta, \bar{\zeta})$ expandable in spherical harmonics $l \geq 2$. 

7
Aside: We mention without further discussion that $\mathcal{H}$—space has a variety of interesting properties\[10]\[9]: it possesses a complex holomorphic metric, it is Ricci flat and is anti-self dual.

By choosing an arbitrary "world-line", $z^a = \xi^a(\tau)$, we have a one-complex parameter family of cuts of $\mathcal{J}^+$, (complexified in general).

\[
u = G(\xi^a(\tau), \zeta, \bar{\zeta}) \equiv G^*(\tau, \zeta, \bar{\zeta}) = \xi^a(\tau)l_a(\zeta, \bar{\zeta}) + H_{l\geq 2}(\xi^a(\tau), \zeta, \bar{\zeta}). \tag{21}\]

Using the freedom of reparametrization, $\tau^* = F(\tau)$, we make $\dot{\xi}^a \approx 1$ via a slow motion approximation, so that $\xi^0 \approx \tau$, $\xi^i \approx 0$. This leads to

\[
u = G = \frac{\tau}{\sqrt{2}} - \frac{1}{2} \xi^i(\tau)Y^0_{ij}(\zeta, \bar{\zeta}) + \xi^i(\tau)Y^2_{ij}(\zeta, \bar{\zeta}) + .. \tag{22}\]

In the following section $\xi^a(\tau)$ will be chosen in two separate ways: first $\xi^a(\tau)$ is taken - by definition - as the unique complex center of mass world-line,

\[
u = \xi^a_{\text{C}_\text{of} M}(\tau), \tag{23}\]

still to be determined.

Remark: As mentioned earlier, we specialize by assuming that the complex center of charge world-line coincides with the complex center of mass. This is not necessary but is a restriction on the class of solutions.

The second choice for $\xi^a$, again - by definition - is

\[
u = \tau^* \delta^a_0, \tag{24}\]

\[
u = G^*(\tau^*, \zeta, \bar{\zeta}) = \frac{\tau^*}{\sqrt{2}} + \xi^i(\tau^*)Y^2_{ij}(\zeta, \bar{\zeta}) + .. \tag{25}\]

yielding the "static-frame". Note that the these "static" slicing differ from Bondi slicing by $l \geq 2$ harmonics and are very close to Lorentzian-looking slicings.

The associated asymptotically shear-free congruences that are normal to the slicings are determined by

\[\tilde{l}^a = l^a_B + L m^a_B + \bar{L} m^a_B + \bar{L} \bar{m}^a_B\]

with

\[
\begin{align*}
  L &= \bar{\partial}(\tau)G^*, \\
  \bar{L} &= \bar{\partial}(\tau)\bar{G}^*.
\end{align*} \tag{26}\]

The $L$ automatically, from its construction, satisfies, parametrically, the generalization of Eq.\[8]\, namely

\[
\begin{align*}
  \partial L + LL_{\nu} &= \sigma^0(u, \zeta, \bar{\zeta}) \\
  u &= G^*(\tau, \zeta, \bar{\zeta}).
\end{align*}
\]

The asymptotic twist $\Sigma$ is almost the same as in Eq.\[14]\.
\[ \Sigma = \zeta_j^i(\tau)(n_a - l_a) + \text{higher harmonics} \]  
(27)

If the world-line and \( G^* \) are real then we can replace the \( \tau \) by \( t \), if complex, again we must construct \( L = \partial(\tau)G^* \) and then evaluate it for real \( u \), as in the previous section.

We are back to virtually the same results and discussion as that of the previous section in the flat-space 'Alternative Means of Construction'.

Our congruences are then the geodesics of the ALCs, i.e., they are asymptotically shear-free and they are labeled by points in a four complex dimensional space - \( \mathcal{H} \)-space. In the special case of passing to the limit of flat space, the congruences do then becomes those of LCs with the labeling remaining.

3 Application

Much of the material of this section - with detailed lengthy derivations - have appeared earlier[5][7][9] Here, in the context of our ALCs, we will simply describe these results, with some explanations but little in the way of derivation.

In this section 'prime' will denote the \( u \)-derivative.

We start with an asymptotically flat Einstein-Maxwell solution described in the neighborhood of null infinity, in a Bondi coordinate system with a Bondi tetrad. Using NP[5] notation, the Weyl and Maxwell tensors have the asymptotic (peeling) behavior,

\[
\begin{align*}
\Psi_0 &= \Psi_0^0 r^{-5} + O(r^{-6}), \\
\Psi_1 &= \Psi_0^0 r^{-4} + O(r^{-5}), \\
\Psi_2 &= \Psi_0^0 r^{-3} + O(r^{-4}), \\
\Psi_3 &= \Psi_0^0 r^{-2} + O(r^{-3}), \\
\Psi_4 &= \Psi_0^0 r^{-1} + O(r^{-2}).
\end{align*}
\]  
(28)

\[
\begin{align*}
\phi_0 &= \phi_0^0 r^{-3} + O(r^{-4}), \\
\phi_1 &= \phi_0^0 r^{-2} + O(r^{-3}), \\
\phi_2 &= \phi_0^0 r^{-1} + O(r^{-2}).
\end{align*}
\]  
(29)

The \( \Psi_n^0 \) and \( \phi_n^0 \), live on \( \mathcal{I}^+ \), i.e., are functions of \((u, \zeta, \bar{\zeta})\). They satisfy the asymptotic Bianchi Identities and asymptotic Maxwell equations,

\[
\begin{align*}
\Psi_2' &= -\delta \Psi_0^3 + \sigma \Psi_0^3 + k \phi_2 \phi_2, \\
\Psi_4' &= -\delta \Psi_2^3 + 2\sigma \Psi_2^3 + 2k \phi_4 \phi_2, \\
\Psi_0' &= -\delta \Psi_1^3 + 3\sigma \Psi_2^3 + 3k \phi_0 \phi_2, \\
k &= 2Gc^{-4},
\end{align*}
\]  
(30-33)
\begin{align}
\phi_0' &= -\sigma_0\phi_2, \\
\phi_0'' &= -\phi_1' + \sigma_0\phi_2.
\end{align}

The prime denoting the \(u\)-derivative. \(\sigma_0(u, \zeta, \vec{r})\) is the asymptotic shear, the free data. From the field equation we have that

\begin{align}
\Psi_3^0 &= \delta(\vec{\sigma})', \\
\Psi_4^0 &= -\delta(\vec{\sigma}).
\end{align}

It is very convenient to introduce, instead of the \(\Psi_2^0\), the mass aspect \(\Psi\), (which is real from the field equations) by

\begin{align}
\Psi = \Psi_2^0 + \phi_0\phi_2 + \sigma_0(\phi_0)' - \sigma_0(\sigma_0)',
\end{align}

Bondi defines the asymptotic mass, \(M_B\), and (R.Sachs) the 3-momentum, \(P_B^i\), as the \(l = 0 \& l = 1\) harmonic coefficients of \(\Psi\). Specifically,

**Definition 1 Identification of Physical Quantities:**

\begin{align}
\Psi &= \Psi_0 + \Psi_1 Y_1^0, \\
\Psi_0 &= -\frac{2\sqrt{2}G}{c^2} M_B, \\
\Psi_1^i &= -\frac{6G}{c^2} P_B^i.
\end{align}

By rewriting Eq.(30), replacing the \(\Psi_2^0\) by \(\Psi\) via Eq.(37), we have

\begin{align}
\Psi' &= (\phi_0)'(\sigma_0)' + k\phi_0'\phi_2' + \sigma_0(\sigma_0)',
\end{align}

and one immediately has the Bondi mass/energy loss theorem - which we return to later:

\begin{align}
M_B' = -\frac{c^2}{2\sqrt{2}G} \int \left((\phi_0)'(\sigma_0)' + k\phi_0'\phi_2' + \sigma_0(\sigma_0)\right) d^2 S \leq 0.
\end{align}

In addition to the Bondi/Sachs energy-momentum we define the complex center of mass by the \(l = 1\) (complex) spherical harmonic component of \(\Psi_1^1\). This definition, which came originally from linear theory, is now justified by the results that it leads to:

**Definition 2 Complex Center of Mass**

\begin{align}
\Psi_1^1 = -6\sqrt{2}Gc^{-2}(D_{(mass)}^i + ie^{-1}J^i)Y_1^1 + ....
\end{align}

with \(D_{(mass)}^i\) the mass dipole and \(J^i\), the total angular momentum, as seen at null infinity.

Our physical identification (standard) for the complex E&M dipole, (electric and magnetic dipoles) as the \(l = 1\) harmonic component of \(\phi_0^0\) and electric charge \(q\) are:
Definition 3  Complex E&M Dipole and Charge with the $Q$'s representing known quadrupole terms and $q$ the Coulomb charge.

\begin{align*}
(D_{\text{Elec}}^l + iD_{\text{Mag}}^l) &= q\xi^i \\
\phi_0^r &= 2q\xi^i Y_{1i}^1 + Q_0 \ldots \\
\phi_1^g &= q + \sqrt{2}q\xi^i Y_{1i}^0 + Q_1 + \ldots \\
\phi_2^q &= -2q\xi^i Y_{1i}^{-1} + Q_2 + \ldots,
\end{align*}

We have made, as mentioned earlier, a simplifying assumption here, namely that the complex center of charge coincides with the complex center of mass. This is not necessary but is a chosen special case.

Our main interests lie in the components $\Psi^0_0$ and $\Psi$, with their physical identifications and their evolution equations, (30) and (31).

Our modus operandi is now to consider both the tetrad and coordinate transformations from the Bondi tetrad and coordinates to the coordinates and associated tetrad of an ALC with (for the moment) an arbitrary complex world-line, $z^a = \xi^a(\tau)$. The transform of the tetrad and Weyl tensor components are

\begin{align*}
l^a &= l_B^a + bm_B^a + bm_B^0 + 0(r^{-2}), \\
m^a &= m_B^a + bn_B^a, \\
n^a &= n_B^a, \\
b &= -\frac{L}{r} + 0(r^{-2}).
\end{align*}

and

\begin{align*}
\Psi^*_{00} &= \Psi_0^0 - 4L\Psi_1^0 + 6L^2\Psi_2^0 - 4L^3\Psi_3^0 + L^4\Psi_4^0, \\
\Psi^*_{01} &= \Psi_0^0 - 3L\Psi_2^0 + 3L^2\Psi_3^0 - L^3\Psi_4^0, \\
\Psi^*_{02} &= \Psi_0^0 - 2L\Psi_3^0 + L^2\Psi_4^0, \\
\Psi^*_{03} &= \Psi_0^0 - L\Psi_4^0, \\
\Psi^*_{04} &= \Psi_0^0.
\end{align*}

The $L$ and its complex conjugate, $\overline{L}$, are determined by Eq.(20) with the coordinate transformation from $u$ to $\tau$ given by, Eq.(22),

\begin{equation}
u = \frac{\tau}{\sqrt{2}} - \frac{1}{2}\xi^i(\tau)Y_{1i}^0(\zeta, \overline{\zeta}) + \xi^{ij}(\xi^a(\tau))Y_{ij}^0(\zeta, \overline{\zeta}) + ..
\end{equation}

The world-line $\xi_{CoFM}^a = (\tau, \xi^i(\tau))$ is now determined by setting to zero the three components of the $l = 1$ coefficients of $\Psi_1^0$ in Eq.(10). Actually rather than doing that we reverse the process, using $\Psi_1^0 = 0$, and express the original Bondi $\Psi_1^0$ in terms of the $\xi_{CoFM}^a(\tau)$. Finally, after considerable effort, with Taylor and Clebsch-Gordon products and expansions, we have the $l = 1$ harmonic coefficient of the Bondi $\Psi_1^0$.
\[
\Psi_1^{0i} = -\frac{6\sqrt{2}G}{c^2} M_B \xi_{CoF_M}^i + i \frac{6\sqrt{2}G}{c^3} P_B^k \xi_{CoF_M}^j \epsilon_{kji} - \frac{576G}{5c^3} P_B^k \xi^k + i \frac{6912\sqrt{2}}{5} \xi^j \xi^k \epsilon_{jki} - i \frac{2\sqrt{2}G}{e^2} q \xi_{CoF_M}^j \xi^{ij} \epsilon_{kji} - \frac{48G}{5c^6} q^2 \xi^j \xi^{ij} - \frac{4G}{5c^7} q^2 \xi^j \xi_{CoF_M}^j \bar{Q}_C^{ij} - \frac{16\sqrt{2}G}{5c^7} q \xi^j \bar{Q}_C^{kij} \epsilon_{jki}.
\]

The Bondi-Sachs mass-momentum, **Definition 1**, has already been used. Prime indicates a derivative. Assuming that the quadrupole interactions (E&M and gravitational) and the high time derivatives are small, we are left with

\[
\Psi_1^{0i} = -\frac{6\sqrt{2}G}{c^2} M_B \xi_{CoF_M}^i + i \frac{6\sqrt{2}G}{c^3} P_B^k \xi_{CoF_M}^j \epsilon_{kji}.
\]

Finally using

\[
\xi_{CoF_M}^i = \xi^i_R + i \xi^i_I,
\]

and comparing Eq.(51) with our **Definition 2**, \( \Psi_1^{0i} = -6\sqrt{2}Gc^{-2}(D^{(mass)}_i + ic^{-1}J^i) \), we obtain our

**Result:1 - Dipole and Angular momentum**

\[
D^{(mass)}_i = M_B \xi^i_R - c^{-1} P_B^k \xi^j \epsilon_{kji} + \ldots \quad (53)
\]

\[
J^i = cM_B \xi^i_I + P_B^k \xi^j \epsilon_{kji} + \ldots \quad (54)
\]

or

\[
\bar{D}^{(mass)} = M_B \bar{P}^i + c^{-2} M_B^{-1} \bar{P}_B \times \bar{S} \quad (55)
\]

\[
\bar{P}^i = \xi_R^i = (\xi^1_R, \xi^2_R, \xi^3_R) \quad (56)
\]

\[
\bar{S} = cM_B \xi^i_I = cM_B (\xi^1_I, \xi^2_I, \xi^3_I) \quad (57)
\]

\[
\bar{J} = \bar{S} + \bar{P} \times \bar{P} \quad (58)
\]

The mass dipole is the usual term plus a 2nd term that is part of the standard relativistic angular momentum tensor[13]. We find for the angular momentum a spin term \( \bar{S} \) and the standard \( \bar{P} \times \bar{P} \) orbital angular momentum term.

**REMARK** Notice that once we have the **definition** of the complex center of mass and the complex center of mass world-line, these results for \( \bar{D}^{(mass)} \) and \( \bar{J} \) follow without any further calculations.

Next, replacing \( \Psi_1^{0i} \), from Eq.(51), in the evolutionary Bianchi Identity, Eq.(81),

\[
\Psi_1^{0'i} = -\partial \Psi_2^0 + 2 \sigma^0 \Psi_3^0 + 2k \phi^0_1 \phi_2^0 \quad (59)
\]

with **definition** 1 and Eq.(13), we find at **linear order**, directly from the real part, the linear momentum, \( P_B^i \). Looking at the lowest harmonic order, the \( l = 1 \), we see that \( P_B^i \) appears in the \( \Psi_2^0 \), the \( M_B \xi_R^i \) appears in \( \Psi_1^0 \) and \( \frac{2q^2}{5c^6} \xi_R^i \) is in the last term. This leads immediately - just by observation - to:
Result: 2 - Kinematic Linear Momentum

\[ P_B^i = M_B \xi_R^i - \frac{2q^2}{3c^3} \xi_R^i + H.O. \]  
\[ H.O. = \text{quadrupole and higher order terms.} \]

We have the Abraham-Lorentz-Dirac radiation reaction term appearing with virtually no derivation, no assumptions, no mass renormalization - just the starting definitions.

From the imaginary part of the same Bianchi Identity we have the angular momentum loss equation:

Result: 3 - Angular momentum Conservation

\[ J^i = \frac{2q^2}{3c^3} \xi_I^i + \frac{2q^2}{3c^3} (\xi_R^i \xi_R^i + \xi_I^i \xi_I^i) \epsilon_{kji} + H.O. \]  
\[ \text{Note} \] The first term on the right side can be moved to the left, which simply changes the definition of \( J^i \),

\[ J^{*i} = (J^i + \frac{2q^2}{3c^3} \xi_I^i)^* = \frac{2q^2}{3c^3} (\xi_R^i \xi_R^i + \xi_I^i \xi_I^i) \epsilon_{kji}, \]

i.e., it adds a spin dependent term.

\[ \text{Note} \] We have the exact Landau \& Lifschitz\[12\] expression for angular momentum loss in the special case of Eq.(62) when the derivatives of the spin terms \( \xi_I^i \) are considered to be zero.

Finally substituting the Bondi-Sachs terms and those of Eq. (36) into the first evolutionary Bianchi Identity, Eq.(30),

\[ \Psi_2^0 = -\sigma_3^0 \Psi_3^0 + \sigma_4^0 \Psi_4^0 + k \phi_2^0 \sigma_2^0, \]

we have for the \( l = 0 \) harmonic coefficient, the (Bondi) mass loss expression but now including the well known (classical) electromagnetic energy losses, i.e.,

Result: 4 - Energy loss

\[ M_B^i = -\frac{G}{5c^7} (Q_{Mass}^{kmn} Q_{Mass}^{kmn} + Q_{Spin}^{kmn} Q_{Spin}^{kmn}) - \frac{4q^2}{3c^5} (\xi_R^i \xi_R^i + \xi_I^i \xi_I^i) \]  
\[ - \frac{4}{45c^7} (Q_E^{kmn} Q_E^{kmn} + Q_M^{kmn} Q_M^{kmn}). \]

The first term is the standard Bondi quadrupole mass loss (now including the spin-quadrupole contribution to the loss - maybe new), the second and third terms are the classical E&M dipole and quadrupole energy loss - including the correct numerical factors. Note again that these results are just sitting in the Bianchi Identities - with no derivation - arising simply from the Ricci tensor expressed via the Maxwell stress tensor.

The \( l = 1 \) terms, the momentum loss expression, leads to

Result: 5 - Newton’s 2nd Law
\[ P_B^i = F_{\text{recoil}}^i \]  \hspace{1cm} (66)

where \( F_{\text{recoil}}^i \) is composed of many non-linear radiation terms involving the time derivatives of the gravitational quadrupole and the E&M dipole and quadrupole moments. These terms are known and given\(^9\) but not relevant to us now. Instead we substitute Eq.(60) into Eq.(66) leading to Newton’s second law:

\[ M_B \xi_R^{ii} = F^i \equiv M'_B \xi_R^{ii} + \frac{2g^2}{3c^3} \xi_R^{iii} + F_{\text{recoil}}^i. \]  \hspace{1cm} (67)

**Result: 6 - Rocket Force and Radiation Reaction Force**

We find this surprising - to have exactly the standard rocket mass loss expression, i.e., \( M'v' \), and the exact Abraham-Lorentz-Dirac radiation reaction force term - no mass renormalization needed.

### 3.1 The Last Step

For our last step, which turns out to be very easy, we must transform our results from the Bondi system to the 'static frame' of our asymptotic shear-free system, i.e., Eqs.(24) and (25). The coordinate transformation which takes us from the Bondi slicings to the 'static frame'\(^a\)

\[ u = \frac{\tau^*}{\sqrt{2}} + \xi^{ij}(\tau^*)Y^2_{ij}(\zeta, \bar{\zeta}) + .. \]

can within our approximations can be considered simply as

\[ u = \frac{\tau^*}{\sqrt{2}} \]

with the \( L \) and \( \bar{L} \) of the tetrad transformation, Eq.(44) considered as vanishing. We can thus treat the transformation as the identity.

All our six results then hold in the 'static frame' using the complex \( \tau^* \) instead of the \( u \). By forcing \( u \) to be real, as in the construction of the previous section, our approximations allow us to treat \( \Lambda = 0 \), and hence \( \tau^* = t \) as real. The 'prime' derivatives can then be thought of as simply \( t \) derivatives. The slicing of real \( \Im \) are given by

\[ u = t + [\xi^{ij}(t)Y^2_{ij}(\zeta, \bar{\zeta})]_R + .., \]

namely the Bondi slicing, \( (u = t) \), but with small higher, \( l \gg 2 \), harmonic corrections.

Our results are then the (real) standard relations of classical mechanics.

### 4 Discussion

The results of the previous section raise a variety of issues; some - so far - have been very difficult to resolve, others raise interesting questions that remain to be answered or even studied.
4.1 Meaning

Our prime problem is the following. We have found sitting in just the asymptotic Einstein-Maxwell equations, with no additional physical assumptions, (aside from a few definitions), a large number of the fundamental relations from classical mechanics coupled with the Maxwell field. These relations, (e.g., radiation reaction or angular momentum loss), that often involve considerable effort to obtain by standard procedures, are simply sitting in the Bianchi identities needing only the few definitions. The simplicity in finding them - to us - is rather surprising. But even more surprising is the fact that they seem to be basically unintelligible - they are "equations of motion" that appear to have nothing what-so-ever to do with space-time points. The 'motion' takes place in the rather unphysical complex $\mathcal{H}$-space. The imaginary values of the coordinates with their dynamics describe spin angular-momentum behavior. The real parts of the coordinates mimic real space-time and seem to describe, in $\mathcal{H}$-space, the motion of the center of mass. What does this mean.

A question that can be answered is: do any of these $\mathcal{H}$-space relations (e.g., the coordinates) appear in any aspects of real space-time and its $I^+$? For each real ALC cut constructed from the complex cut, Eq.(22),

$$u = \frac{\tau}{\sqrt{2}} - \frac{1}{2} \xi (\tau) Y^a_{l1}(\zeta, \bar{\zeta}) + \xi (\tau) Y^a_{l2}(\zeta, \bar{\zeta}) + ..$$

the coefficients of the $l = 0, 1$ harmonics are the real parts of the $\mathcal{H}$-space coordinates while the twist of the congruence determines the imaginary parts. They have the information to form a virtual image of a point. This is analogous but opposite to the case in flat-space time where the rays from the cut, Eq.(10),

$$u = x^a l_a(\zeta, \bar{\zeta}),$$

do focus back to the space-time point $x^a$. Unfortunately this does not seem to help clarify the issue of why these virtual points mimic the behavior of real space-time points.

4.2 The BMS Group

The role of the BMS group - the group of coordinate transformations between different Bondi coordinate systems - appears to be changed by the use of ALC coordinates. The members of the set of ALCs are geometric constructs and, as geometric objects, are not subject to intrinsic changes due to an arbitrary BMS transformation. On the other hand the ALCs are described in terms of any - but a specific choice of Bondi coordinates - and do undergo changes when the specific choice is changed. In other words under a BMS transformation the description of the ALCs will be changed. The details of these changes have not yet been worked out.
4.3 Queries

1. Are our results concerning the classical mechanical relationships at all significant - or are they just a curious coincidence of little consequence? We feel from the clarity and ease by which they sit in the Bianchi Identities with their associated Lorentzian-like structures (the ALCs), that they very likely are significant. But what is that significance? We also remember that almost every mother loves her own child - so we remain skeptical.

2. In either case, can the results or predictions of the angular momentum loss and spin contributions to gravitational radiation be considered as meaningful and correct? Or conceivably measured? These results, though small, appear to be new.

3. We know that the $\mathcal{H}$-space (and conjugate $\overline{\mathcal{H}}$-space) both contain complex-holomorphic Ricci-flat metrics with self-dual (and anti-self-dual) Weyl tensors. What happens to these structures when we go to the real $u$ and associated real $\mathcal{H}$-space coordinates? Do they remain - and if so with what structures?

4. The easy appearance of the radiation reaction force is both pleasing and disturbing. No heavy breathing nor hard work, no mass renormalization or further assumptions. It is just there. But then what can we say about the familiar instability - the runaway behavior of the associated motion due to radiation reaction? Is there a mechanism, that we do not see, that damps the motion. Or are the solutions to the Einstein-Maxwell equations unstable? We have no answer.

5. How do these results fold in with the attempts to construct a Quantum Theory of Gravity. If they do not fold in - then why not? They are part of GR. If they do fold in, what is their role? Do we get a Schrodinger-like Equation for the center of mass motion or a Dirac Equation for the spin? It appears highly unlikely.

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