Spin Network States in Gauge Theory

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Abstract

Given a real-analytic manifold $M$, a compact connected Lie group $G$ and a principal $G$-bundle $P \to M$, there is a canonical ‘generalized measure’ on the space $A/G$ of smooth connections on $P$ modulo gauge transformations. This allows one to define a Hilbert space $L^2(A/G)$. Here we construct a set of vectors spanning $L^2(A/G)$. These vectors are described in terms of ‘spin networks’: graphs $\phi$ embedded in $M$, with oriented edges labelled by irreducible unitary representations of $G$, and with vertices labelled by intertwining operators from the tensor product of representations labelling the incoming edges to the tensor product of representations labelling the outgoing edges. We also describe an orthonormal basis of spin networks associated to any fixed graph $\phi$. We conclude with a discussion of spin networks in the loop representation of quantum gravity, and give a category-theoretic interpretation of the spin network states.

1 Introduction

Penrose [14] introduced the notion of a spin network as an attempt to go beyond the concept of a manifold towards a more combinatorial approach to spacetime. In his definition, a spin network is a trivalent graph labelled by spins $j = 0, \frac{1}{2}, 1, \ldots$, satisfying the rule that if edges labelled by spins $j_1, j_2, j_3$ meet at a vertex, the Clebsch-Gordon condition holds:

$$|j_1 - j_2| \leq j_3 \leq j_1 + j_2.$$ 

In fact, the spins should be thought of as labelling finite-dimensional irreducible representations of $\text{SL}(2, \mathbb{C})$, and the Clebsch-Gordon condition is necessary and sufficient for there to be a nontrivial intertwining operator from $j_1 \otimes j_2$ to $j_3$. One can use the representation theory of $\text{SL}(2, \mathbb{C})$ to obtain numerical invariants of such labelled graphs.

Recently, spin networks and their generalizations have played an important role in topological quantum field theories such as Chern-Simons theory [13] and the Turaev-Viro model [12, 18] in dimension three, and the Crane-Yetter model [11] in dimension four. In these theories, the category of finite-dimensional representations of $\text{SL}(2, \mathbb{C})$ is replaced by a suitable category of representations of a quantum group. Again,
the key idea is a method for obtaining invariants of graphs whose edges are labelled by irreducible representations. Here, however, the graphs are regarded as embedded in $\mathbb{R}^3$, and are equipped with a framing. The edges are oriented, and reversing the orientation of an edge has the same effect as replacing the representation $\rho$ labelling it by the dual representation $\rho^*$. Moreover, the graphs need not be trivalent, but each vertex must be labelled with an intertwining operator from the tensor product of the representations labelling the incoming edges to the tensor product of the representations labelling the outgoing edges. The $\text{SL}_q(2)$ case is very similar to the situation studied by Penrose, and effectively reduces to it in the limit $q \to 1$. In this case every representation is self-dual, so edges do not need orientations, and the space of intertwining operators from $j_1 \otimes j_2$ to $j_3$ is at most 1-dimensional, so trivalent vertices do not need labels.

In parallel with these developments, mathematical work on the loop representation of quantum gravity \cite{1} has led to a theory of functional integration on spaces $\mathcal{A}/\mathcal{G}$ of connections modulo gauge transformations which allows one to define a rigorous version of the Hilbert space $L^2(\mathcal{A}/\mathcal{G})$. This space is not defined using the purely formal ‘Lebesgue measure’ on $\mathcal{A}/\mathcal{G}$, but instead using the canonical ‘generalized measure’ on $\mathcal{A}/\mathcal{G}$ coming from Haar measure on the (compact, connected) gauge group $\mathcal{G}$. Here we construct an explicit set of vectors spanning $L^2(\mathcal{A}/\mathcal{G})$ using spin networks. Each of these ‘spin network states’ $\Psi_{\phi,\rho,\iota}$ is labelled by a choice of: a) an oriented, unframed graph embedded in the base manifold $M$, b) a labelling of each edge $e$ of $\phi$ by an irreducible representation $\rho_e$ of $\mathcal{G}$, and c) a labelling of each vertex $v$ of $\phi$ by a vector $\iota_v$ in the space of intertwining operators from the tensor product of ‘incoming’ representations to the tensor product of the ‘outgoing’ representations.

In the language of canonical quantum gravity, vectors in $L^2(\mathcal{A}/\mathcal{G})$ represent states at the ‘kinematical’ level. Spin networks have also been used by the physicists Rovelli and Smolin to describe states at the ‘diffeomorphism-invariant’ level \cite{17}. Mathematically, states at the diffeomorphism-invariant level are thought to be given by diffeomorphism-invariant generalized measures on $\mathcal{A}/\mathcal{G}$. Such states have already been characterized, and examples constructed, using the language of graphs \cite{7}. To give a rigorous foundation to the work of Rovelli and Smolin, one would like to construct large classes of such states using spin networks. This is likely to involve a generalization of techniques due to Ashtekar et al \cite{2,5} for constructing diffeomorphism-invariant generalized measures from certain knot invariants. The present work is intended as a first step in this direction. For more remarks on the applications of spin network states to quantum gravity, as well as a category-theoretic interpretation of the spin network states, see Section 5.

2 Gauge Theory on a Graph

In this section we develop the basic concepts of gauge theory on a graph, which we apply in the next section to gauge theory on manifolds. Readers familiar with lattice
gauge theory may find it useful to think of what follows as a slight generalization of gauge theory on a finite lattice. In the case of a trivial bundle, a connection on a graph will assign a group element to each edge of the lattice, or ‘bond’, and a gauge transformation will assign a group element to each vertex, or ‘site’. In fact, we only consider trivializable bundles. However, to apply our results to gauge theory on manifolds, it is convenient not to assume the bundles are equipped with a fixed trivialization.

Let $G$ be a compact Lie group, and let $\phi$ be a (finite, directed) graph, by which we mean a finite set $E$ of edges, a finite set $V$ of vertices, and functions

$$s: E \to V, \quad t: E \to V.$$ 

We call the vertex $s(e)$ the source of the edge $e$, and the vertex $t(e)$ the target of $e$. Let $P$ be a principal $G$-bundle over $V$, regarding $V$ as a space with the discrete topology. Any such bundle is trivializable, but we do not assume $P$ is equipped with a fixed trivialization. Given any vertex $v$, we write $P_v$ for the fiber of $P$ over $v$.

Given any edge $e$, let $A_e$ denote the space of smooth maps $F: P_{s(e)} \to P_{t(e)}$ that are compatible with the right action of $G$ on $P$:

$$F(xg) = F(x)g.$$ 

We define $A$, the space of connections on $\phi$, by

$$A = \prod_{e \in E} A_e.$$ 

Given $A \in A$, we write $A_e$ for the value of $A$ at the edge $e \in E$. Note a trivialization of $P$ lets us to identify each space $A_e$ with a copy of $G$, with elements acting as maps from $P_{s(e)} \cong G$ to $P_{t(e)} \cong G$ by left multiplication. We may then identify $A$ with $G^E$. This equips $A$ with the structure of a smooth manifold, and also endows it with a probability measure $\mu$, namely the product of copies of normalized Haar measure on $G$. One can check that this smooth structure and measure are independent of the choice of trivialization. It follows that the space $C(A)$ of continuous functions on $A$ and the Hilbert space $L^2(A)$ of square-integrable functions on $A$ are well-defined in a trivialization-independent manner.

Similarly, we define the group of gauge transformations on $\phi$, written $G$, by

$$G = \prod_{v \in V} P_v \times_{\text{Ad}} G.$$ 

This is just the usual group of gauge transformations of the bundle $P$, so a trivialization of $P$ allows us to identify $G$ with $G^V$. Given $g \in G$, we write $g_v$ for the value of $g$ at $v$. We may regard $g_v$ as a map from $P_v$ to itself, so that the group $G$ acts on $A$ by

$$(gA)_e = g_{t(e)}A_e g_{s(e)}^{-1}.$$ 

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Note that $G$ acts on $\mathcal{A}$ in a smooth, measure-preserving manner.

We can push forward the measure on $\mathcal{A}$ to a probability measure on $\mathcal{A}/G$ using the quotient map $\mathcal{A} \to \mathcal{A}/G$. The space $L^2(\mathcal{A}/G)$ is then naturally isomorphic to the $G$-invariant subspace of $L^2(\mathcal{A})$. In what follows, we describe an orthonormal basis of $L^2(\mathcal{A}/G)$ using spin networks.

Let $\Lambda$ denote a set of irreducible unitary representations of $G$, one for each equivalence class. We assume the trivial representation of $G$ on $\mathbb{C}$ is a member of $\Lambda$. Given $\rho \in \Lambda$, we write $\rho^*$ for the representation in $\Lambda$ equivalent to the dual of $\rho$. Recall that $G \times G$ acts on $G$ by

$$(g_1, g_2)(g) = g_2 gg_1^{-1},$$

and that this makes $L^2(G)$ into a unitary representation of $G \times G$, which by the Peter-Weyl theorem is isomorphic to

$$\bigoplus_{\rho \in \Lambda} \rho \otimes \rho^*.$$ 

As an immediate consequence we have:

**Lemma 1.** Any trivialization of $P$ determines a unitary equivalence of the following representations of $G$:

$$L^2(\mathcal{A}) \cong \bigotimes_{e \in E} \bigoplus_{\rho \in \Lambda} \rho \otimes \rho^*,$$

where $g \in G$ acts on the latter space by

$$\bigotimes_{e \in E} \bigoplus_{\rho \in \Lambda} \rho(g_s(e)) \otimes \rho^*(g_t(e)).$$

However, to describe the spin network states, a slightly different description of $L^2(\mathcal{A})$ is preferable. An element $\rho \in \Lambda^E$ is a labelling of all edges $e \in E$ by irreducible representations $\rho_e \in \Lambda$. In these terms, when we fix a trivialization of $P$ we obtain a unitary equivalence

$$L^2(\mathcal{A}) \cong \bigoplus_{\rho \in \Lambda^E} \bigotimes_{e \in E} \rho_e \otimes \rho_e^*,$$

with $g \in G$ acting on the right hand side by

$$\bigoplus_{\rho \in \Lambda^E} \bigotimes_{e \in E} \rho_e(g_s(e)) \otimes \rho_e^*(g_t(e)).$$

Now, given a vertex $v \in V$, let $S(v)$ denote the set of all edges of \( \phi \) having $v$ as source, and $T(v)$ the set of all edges having $v$ as target. Then the above formula for $L^2(\mathcal{A})$ gives:
Lemma 2. Any trivialization of $P$ determines a unitary equivalence of the following representations of $G$:

$$L^2(A) \cong \bigoplus_{\rho \in \Lambda^E} \bigotimes_{v \in V} \left( \bigotimes_{e \in S(v)} \rho_e \otimes \bigotimes_{e \in T(v)} \rho_e^* \right),$$

where $g \in G$ acts on the latter space by

$$\bigoplus_{\rho \in \Lambda^E} \bigotimes_{v \in V} \left( \bigotimes_{e \in S(v)} \rho_e(g_v) \otimes \bigotimes_{e \in T(v)} \rho_e^*(g_v) \right).$$

This allows us to describe $L^2(A/G)$ in terms of spin networks as follows. For each vertex $v$ of the graph $\phi$, and for each choice $\rho \in \Lambda^E$ of labellings of the edges of $\phi$ by irreducible representations of $G$, let $\text{Inv}(v, \rho)$ denote the subspace of invariant elements of the following representation of $G$:

$$\bigotimes_{e \in S(v)} \rho_e \otimes \bigotimes_{e \in T(v)} \rho_e^*.$$

Note that elements $f \in \text{Inv}(v, \rho)$ may be thought of as intertwining operators

$$f: \bigotimes_{e \in S(v)} \rho_e \rightarrow \bigotimes_{e \in T(v)} \rho_e.$$

Lemma 2 implies the following:

Lemma 3. In a manner independent of the choice of trivialization of $P$, $L^2(A/G)$ is isomorphic as a Hilbert space to

$$\bigoplus_{\rho \in \Lambda^E} \text{Inv}(v, \rho).$$

As a consequence, $L^2(A/G)$ is spanned by spin network states

$$\Psi_{\rho, \iota} = \bigotimes_{v \in V} \iota_v,$$

where $\rho \in \Lambda^E$ is any labelling of the edges $e$ of $\phi$ by irreducible representations $\rho_e \in \Lambda$, and $\iota_v \in \text{Inv}(v, \rho)$ for each vertex $v$ of $\phi$. In particular, if we let $\rho$ range over $\Lambda^E$ and for each vertex $v$ let $\iota_v$ range over an orthonormal basis of $\text{Inv}(v, \rho)$, the spin network states $\Psi_{\rho, \iota}$ form an orthonormal basis of $L^2(A/G)$.

To deal with gauge theory on manifolds we also need to study the dependence of $L^2(A/G)$ on the graph $\phi$, and particularly the situation where a graph $\psi$ is included in the graph $\phi$. For this we need to write subscripts such as $\psi$ or $\phi$ on the symbols $E, V, P, A, G, \mu, S(v), T(v)$ and $\text{Inv}(v, \rho)$ to indicate the dependence on the graph.

Here is an example of what we have in mind by a graph $\psi$ being ‘included’ in a graph $\phi$. 

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Here we have labelled the vertices of $\psi$ and $\phi$ but not the edges. Note that every vertex of $\psi$ is a vertex of $\phi$, but edges of $\psi$ may be ‘products’ of edges in $\phi$ and their inverses.

To precisely define the notion of one graph being ‘included’ in another, we first define a path in a graph $\phi$ to be a sequence of vertices $v_1, \ldots, v_n \in V_\phi$, together with, for each $i$, $1 \leq i \leq n$, an edge $f_i \in E_\phi$ such that either:

$$s(f_i) = v_i, \quad t(f_i) = v_{i+1},$$  

or

$$t(f_i) = v_i, \quad s(f_i) = v_{i+1}.$$  

In this situation, we write the path as a product $f_1^{\pm 1} \cdots f_n^{\pm 1}$, where the exponents are either $+1$ or $-1$ depending on whether case (1) or case (2) holds, and we say that the edges $f_1, \ldots, f_n$ appear in the path. We say the path is *simple* if the vertices $v_1, \ldots, v_n$ are distinct, with the exception that we allow $v_1 = v_n$. Heuristically, a simple path in is one that never retraces or intersects itself, the only exception being that it may end where it began.

If we have graphs $\psi$ and $\phi$, an *inclusion* of $\psi$ in $\phi$, written $i: \psi \hookrightarrow \phi$, is a one-to-one map $i: V_\psi \to V_\phi$ together with an assignment to each edge $e \in E_\psi$ of a simple path $i(e)$ in $\phi$ from $i(s(e))$ to $i(t(e))$, such that each edge of $\phi$ appears in at most one path $i(e)$. In the rest of this section we assume we are given an inclusion $i: \psi \hookrightarrow \phi$ and principal $G$-bundles $P_\psi \to V_\psi$, $P_\phi \to V_\phi$ such that $i^*P_\phi = P_\psi$. To simplify the notation we assume, without loss of generality, that $V_\psi$ is a subset of $V_\phi$ and $i: V_\psi \to V_\phi$ is given by $i(v) = v$. In this situation the bundle $P_\psi$ is just the restriction of $P_\phi$ to $V_\psi$.

In this situation there is a map

$$i^*: A_\phi \to A_\psi$$

given as follows. Recall that a connection $A$ on $\phi$ assigns to each edge $f$ of $\phi$ a map $A_f: P_{s(f)} \to P_{t(f)}$ compatible with the right $G$-action on $P_\phi$. Similarly, the connection

Fig. 1. Graph $\psi$ included in graph $\phi$
\(i^* (A)\) on \(\psi\) must assign to each edge \(e\) of \(\psi\) a map \(i^* (A)_e : P_{s(e)} \to P_{t(e)}\) compatible with the right \(G\)-action. If the inclusion \(i\) assigns to \(e\) the simple path

\[i(e) = f_1^{\pm 1} \cdots f_n^{\pm 1},\]

we let

\[i^* (A)_e = A_{f_1^{\pm 1}} \cdots A_{f_n^{\pm 1}}.\]

One can check that \(i^* (A)\) is well-defined and indeed a connection on \(\psi\). The order-reversal here is due to the unfortunate fact that the convention for writing products of paths is the opposite of that for composites of maps.

By arguments already given in the more concrete cases treated earlier [3, 6, 8, 13], the map \(i^*\) is smooth and onto, and the measure \(\mu_\psi\) on \(A_\psi\) pushes forward by \(i^*\) to give the measure \(\mu_\phi\) on \(A_\phi\). This implies that \(i^*\) yields a one-to-one algebra homomorphism from \(C(A_\psi)\) to \(C(A_\phi)\), which we write simply as

\[i : C(A_\psi) \to C(A_\phi),\]

and also that \(i^*\) yields an isometry

\[i : L^2(A_\psi) \to L^2(A_\phi).\]

There is also a surjective homomorphism

\[i^* : G_\phi \to G_\psi\]

given by the natural projection

\[G_\phi = \prod_{v \in V_\phi} P_v \times \text{Ad } G \to \prod_{v \in V_\psi} P_v \times \text{Ad } G = G_\psi.\]

The action of \(G_\phi\) on \(A_\phi\) is related to that of \(G_\psi\) on \(A_\psi\) by

\[i^* (gA) = i^* (g)i^* (A)\]

for all \(g \in A_\phi\), \(A \in A_\phi\). It follows that \(i : C(A_\psi) \to C(A_\phi)\) restricts to a one-to-one algebra homomorphism

\[i : C(A_\psi/\mathcal{G}_\psi) \to C(A_\phi/\mathcal{G}_\phi),\]

and \(i : L^2(A_\psi) \to L^2(A_\phi)\) restricts to an isometry

\[i : L^2(A_\psi/\mathcal{G}_\psi) \to L^2(A_\phi/\mathcal{G}_\phi).\]

In short, when the graph \(\psi\) is included in \(\phi\), we can think of \(L^2(A_\psi/\mathcal{G}_\psi)\) as a subspace of \(L^2(A_\phi/\mathcal{G}_\phi)\). The spin network states for \(L^2(A_\psi/\mathcal{G}_\psi)\) are then automatically spin network states for \(L^2(A_\phi/\mathcal{G}_\phi)\):
Lemma 4. Suppose \( \psi \) and \( \phi \) are graphs, \( P_\psi \to V_\psi \) and \( P_\phi \to V_\phi \) are principal \( G \)-bundles, and \( i: \psi \hookrightarrow \phi \) is an inclusion such that \( i^* P_\phi = P_\psi \). Then the induced isometry

\[ i: L^2(A_\psi/G_\psi) \to L^2(A_\phi/G_\phi) \]

maps spin network states for the former space into spin network states for the latter space.

Proof - We assume without loss of generality that \( V_\psi \) is a subset of \( V_\phi \), \( i: V_\psi \to V_\phi \) is given by \( i(v) = v \), hence that bundle \( P_\psi \) is the restriction of \( P_\phi \) to \( V_\psi \). Note that there is a natural way to compose inclusions, and that any inclusion of the above sort can be written as a product of a finite sequence of inclusions, each of which is of one of the following four forms:

1. Adding a vertex:

\[ \implies \cdot \vspace{1cm} v \]

More precisely, \( V_\phi \) is the disjoint union of \( V_\psi \) and \( \{v\} \), and \( E_\phi = E_\psi \). The source and target functions are the same for \( \phi \) as for \( \psi \), and the inclusion \( i: \psi \to \phi \) sends each edge \( e \) of \( \psi \) to the path \( e \) in \( \phi \).

2. Adding an edge:

\[ v_1 \bullet v_2 \implies v_1 e v_2 \]

Here \( V_\phi = V_\psi \), for some \( e \not\in E_\psi \) we have \( E_\phi = E_\psi \cup \{e\} \), and the source and target functions for \( \phi \) agree with those of \( \psi \) on \( E_\psi \), while

\[ s(e) = v_1, \quad t(e) = v_2 \]

for some \( v_1, v_2 \in V_\phi \). The inclusion \( i: \psi \hookrightarrow \phi \) assigns to each edge \( f \) of \( \psi \) the path \( f \) in \( \phi \).

3. Subdividing an edge:

\[ s(e) e t(e) \implies s(e) e_1 v e_2 t(e) \]
For some \( v \notin V_\psi \) we have \( V_\phi = V_\psi \cup \{v\} \), for some \( e \in E_\psi \) and \( e_1, e_2 \notin E_\psi \) we have \( E_\phi = (E_\psi - \{e\}) \cup \{e_1, e_2\} \), and the source and target functions of \( \phi \) agree with those of \( \psi \) on \( E_\psi - \{e\} \), while

\[
\begin{align*}
  s(e_1) &= s(e), & t(e_1) &= v, \\
  s(e_2) &= v, & t(e_2) &= t(e).
\end{align*}
\]

The inclusion \( i: \psi \hookrightarrow \phi \) assigns to each edge \( f \neq e \) of \( \psi \) the path \( f \) in \( \phi \), and assigns to the edge \( e \) the path \( e_1 e_2 \).

4. Reversing the orientation of an edge:

Here \( V_\psi = V_\phi \), and for some \( e \in E_\psi \) and \( e' \notin E_\psi \) we have \( E_\phi = (E_\psi - \{e\}) \cup \{e'\} \).

The source and target functions for \( \phi \) agree with those of \( \psi \) on \( E_\psi - \{e\} \), while \( s(e) = v_1 \), \( t(e) = v_2 \) then \( s(e') = v_2 \), \( t(e') = v_1 \).

The inclusion \( i: \psi \hookrightarrow \phi \) assigns to each edge \( f \neq e \) of \( \psi \) the path \( f \) in \( \phi \), but assigns to \( e \) the path \( e^{-1} \).

Thus, to show that \( i \) maps each spin network state \( \Psi_{\rho, \iota} \in L^2(A_\psi/G_\psi) \) into a spin network state \( \Psi_{\rho', \iota'} \in L^2(A_\phi/G_\phi) \), it suffices to show this in each of the four cases above.

Calculations give the following results for each case:

1. \( i(\Psi_{\rho, \iota}) = \Psi_{\rho', \iota'} \) is given as follows: \( \rho' = \rho \), and \( \iota'_w = \iota_w \) for all vertices \( w \in V_\psi \), while \( \iota'_v = 1 \in \mathbf{C} \). (Note that the \( S(v) \) and \( T(v) \) are the empty set, and the empty tensor product of representations is defined to be the trivial representation \( \mathbf{C} \), so \( \text{Inv}_\phi(v, \rho') = \mathbf{C} \).)

2. Here \( \rho'_f = \rho_f \) for all edges \( f \in E_\psi \) except \( e \), while \( \rho_e = \mathbf{C} \). Moreover, \( \iota'_v = \iota_v \) for all \( v \in V_\psi \) except \( v_1 \) and \( v_2 \). Given

\[
\iota_{v_1} \in \text{Inv}_\psi(v_1, \rho) = \text{Inv} \left( \bigotimes_{f \in S_\psi(v)} \rho_f \otimes \bigotimes_{f \in T_\psi(v)} \rho_f^* \right),
\]

then \( \iota'_{v_1} \) is the vector in

\[
\text{Inv}_\phi(v_1, \rho') = \text{Inv} \left( \mathbf{C} \otimes \bigotimes_{f \in S_\psi(v)} \rho_f \otimes \bigotimes_{f \in T_\psi(v)} \rho_f^* \right)
\]
corresponding to \( v_1 \) under the natural isomorphism between these spaces. The case of \( v_2 \) is analogous.

3. Here \( \rho'_f = \rho_f \) for all edges \( f \in E_\psi \) except \( e_1 \) and \( e_2 \), while \( \rho'_{e_1} = \rho'_{e_2} = \rho_e \). Moreover, \( \iota'_w = \iota_w \) for all \( w \in V_\psi \) except \( v \), while \( \iota'_v \) is the vector in \( \text{Inv}(\rho_e \otimes \rho'_e) \) corresponding to the identity intertwining operator \( 1 : \rho_e \rightarrow \rho_e \).

4. Here \( \rho'_f = \rho_f \) for all edges \( f \in E_\psi \) except \( e'_1 \), while \( \rho'_{e'_1} = \rho'_e \). Moreover, \( \iota'_v = \iota_v \) for all \( v \in V_\psi \) except \( v_1 \) and \( v_2 \). Given

\[
\iota_{v_1} \in \text{Inv}_\psi(v_1, \rho) = \text{Inv}\left( \bigotimes_{f \in S_\psi(v_1)} \rho_f \otimes \bigotimes_{f \in T_\psi(v_1)} \rho'_f \right)
\]

\[
\cong \text{Inv}\left( \rho_e \otimes \bigotimes_{f \in S_\psi(v_1), f \neq e} \rho_f \otimes \bigotimes_{f \in T_\psi(v_1)} \rho'_f \right),
\]

then \( \iota'_{v_1} \) is the vector in

\[
\text{Inv}_\phi(v_1, \rho') = \text{Inv}\left( \bigotimes_{f \in S_\phi(v_1)} \rho_f \otimes \bigotimes_{f \in T_\phi(v_1)} \rho'_f \right)
\]

\[
\cong \text{Inv}\left( \rho_e \otimes \bigotimes_{f \in S_\phi(v_1)} \rho_f \otimes \bigotimes_{f \in T_\phi(v_1), f \neq e'} \rho'_f \right)
\]

corresponding to \( \iota_{v_1} \) under the natural isomorphism between these spaces. The case of \( v_2 \) is analogous.

The proof of Lemma 4 not only shows that \( i \) maps each spin network state \( \Psi_\iota,\rho \in L^2(A_\psi/G_\psi) \) into a spin network state \( \Psi_{\iota',\rho'} \in L^2(A_\phi/G_\phi) \); it also gives an algorithm for computing \( \iota', \rho' \) from \( \iota, \rho \). This is likely to be useful in applications. It also yields:

**Lemma 5.** Given the hypothesis of Lemma 4, suppose \( \Psi \in L^2(A_\psi/G_\psi) \) is such that \( i(\Psi) \) is a spin network state in \( L^2(A_\phi/G_\phi) \). Then \( \Psi \) is a spin network state.

**Proof -** Given any spin network state \( \Psi_{\rho,\iota} \in L^2(A_\psi/G_\psi) \), write

\[
i(\Psi_{\rho,\iota}) = \Psi_{\rho',\iota'}.
\]

By Lemma 2 we can write

\[
\Psi = \sum_{\rho} \sum_{\iota} c_{\rho,\iota} \Psi_{\rho,\iota}
\]

where \( \rho \) ranges over \( \Lambda^E \) and for each \( \rho, \iota \) ranges over an orthonormal basis of \( \text{Inv}_\psi(v, \rho) \).

Then

\[
i(\Psi) = \sum_{\rho} \sum_{\iota} c_{\rho,\iota} \Psi_{\rho',\iota'}.
\]
Note from the proof of Lemma 4 that $\rho'$ depends only on $\rho$, not $\iota$, and the function $\rho \mapsto \rho'$ is one-to-one. Furthermore, if $i(\Psi)$ is spin network state all the summands in the above equation must vanish except those involving a particular choice of $\rho'$. Thus for some particular choice of $\rho$, 

$$i(\Psi) = \sum c_{\rho, \iota} \Psi_{\rho', \iota'}.$$ 

Now the requirement that $i(\Psi)$ be a spin network state implies that this vector, which lies in 

$$\bigotimes_{v \in V_\phi} \text{Inv}_\phi(v, \rho'),$$ 

is a tensor product of vectors in the factors. It follows that 

$$\Psi = \sum c_{\rho, \iota} \Psi_{\rho, \iota}$$ 

and that this vector, which lies in 

$$\bigotimes_{v \in V_\phi} \text{Inv}_\phi(v, \rho'),$$ 

is a tensor product of vectors in the factors. Thus $\Psi$ is a spin network state. \qed 

## 3 The Loop Representation

To apply the result of the previous section to gauge theory on a manifold, we need to recall some facts about the loop representation. All the material in this section can be found in existing mathematically rigorous work on the loop representation [3, 4, 6, 7, 8, 9, 13].

Let $M$ be a real-analytic manifold and let $P$ be a smooth principal $G$-bundle over $M$, with $G$ a compact connected Lie group. Let $\mathcal{A}$ be the space of smooth connections on $P$ and $\mathcal{G}$ the group of smooth gauge transformations. By a path in $M$ we will always mean a piecewise analytic path. Given a path $\gamma$ in $M$, let $\mathcal{A}_\gamma$ denote the space of smooth maps $F: P_{\gamma(a)} \to P_{\gamma(b)}$ that are compatible with the right action of $G$ on $P$: 

$$F(xg) = F(x)g.$$ 

Note that for any connection $A \in \mathcal{A}$, the parallel transport map 

$$T\exp \int_\gamma A : P_{\gamma(a)} \to P_{\gamma(b)}$$ 

lies in $\mathcal{A}_\gamma$. Of course, if we fix a trivialization of $P$ at the endpoints of $\gamma$, we can identify $\mathcal{A}_\gamma$ with the group $G$. 

11
Let the algebra $\text{Fun}_0(\mathcal{A})$ of cylinder functions be the algebra of functions on $\mathcal{A}$ generated by those of the form

$$F(T \exp \int_{\gamma} A)$$

where $F$ is a continuous function on $\mathcal{A}$. Let $\text{Fun}(\mathcal{A})$ denote the completion of $\text{Fun}_0(\mathcal{A})$ in the sup norm:

$$\|f\|_\infty = \sup_{A \in \mathcal{A}} |f(A)|.$$ 

Equipped with this norm, $\text{Fun}(\mathcal{A})$ is a commutative C*-algebra. A generalized measure on $\mathcal{A}$ is defined to be a continuous linear functional $\nu: \text{Fun}(\mathcal{A}) \to \mathbb{C}$.

We say that the generalized measure $\nu$ on $\mathcal{A}$ is strictly positive if $\nu(f) > 0$ for all nonzero $f \geq 0$ in $\text{Fun}(\mathcal{A})$. The group $\mathcal{G}$ acts as gauge transformations on $\mathcal{A}$, and as automorphisms of $\text{Fun}(\mathcal{A})$ by

$$gf(A) = f(g^{-1}A),$$

where $A \in \mathcal{A}$. We say that $\nu$ is gauge-invariant if for all $g \in \mathcal{G}$ and $f \in \text{Fun}(\mathcal{A})$ we have $\nu(gf) = \nu(f)$. In the next section we focus on a particular gauge-invariant, strictly positive generalized measure on $\mathcal{A}$, the ‘uniform’ generalized measure. This serves as a kind of substitute for the purely formal ‘Lebesgue measure’ on $\mathcal{A}$, but it is constructed using Haar measure on $G$ rather the structure of $\mathcal{A}$ as an affine space.

In fact, any gauge-invariant, strictly positive generalized measure $\nu$ on $\mathcal{A}$ allows us to define analogues of the space of $L^2$ functions on $\mathcal{A}$ and $\mathcal{A}/\mathcal{G}$, as follows. First, we define the Hilbert space $L^2(\mathcal{A}, \nu)$ to be the completion of $\text{Fun}(\mathcal{A})$ in the norm

$$\|f\|_2 = \nu(|f|^2)^{1/2}.$$ 

Then, let $\text{Fun}(\mathcal{A}/\mathcal{G})$ to be the subalgebra of gauge-invariant functions in $\text{Fun}(\mathcal{A})$. These functions can also be regarded as continuous functions $\mathcal{A}/\mathcal{G}$ with its quotient topology. We define a generalized measure on $\mathcal{A}/\mathcal{G}$ to be a continuous linear functional from $\text{Fun}(\mathcal{A}/\mathcal{G})$ to $\mathbb{C}$. Any generalized measure $\nu$ on $\mathcal{A}$ restricts to a generalized measure on $\mathcal{A}/\mathcal{G}$. This restriction process defines a one-to-one correspondence between gauge-invariant generalized measures on $\mathcal{A}$ and generalized measures on $\mathcal{A}/\mathcal{G}$. For example, the uniform generalized measure on $\mathcal{A}$ corresponds to a measure on $\mathcal{A}/\mathcal{G}$ called the ‘Ashtekar-Lewandowski’ generalized measure \cite{Ashtekar-Lewandowski}.

Finally, given a gauge-invariant, strictly positive generalized measure $\nu$ on $\mathcal{A}$, define $L^2(\mathcal{A}/\mathcal{G}, \nu)$ to be the completion of $\text{Fun}(\mathcal{A}/\mathcal{G})$ in the above norm $\| \cdot \|_2$. It turns out that the representation of $\mathcal{G}$ on $\text{Fun}(\mathcal{A})$ extends uniquely to a unitary representation of $\mathcal{G}$ on $L^2(\mathcal{A}, \nu)$, and that $L^2(\mathcal{A}/\mathcal{G}, \nu)$ is naturally isomorphic as a Hilbert space to the subspace of $\mathcal{G}$-invariant elements of $L^2(\mathcal{A})$. Moreover, the algebra $\text{Fun}_0(\mathcal{A}/\mathcal{G})$ of gauge-invariant cylinder functions on $\mathcal{A}$ is dense in $\text{Fun}(\mathcal{A}/\mathcal{G})$, hence in $L^2(\mathcal{A}/\mathcal{G}, \nu)$. 

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4 Spin Network States

While none of the results on the loop representation in Section 3 explicitly mention graphs, the proofs of some, and the actual construction of interesting generalized measures on $\mathcal{A}$, turn out to be closely related to gauge theory on graphs [6, 7, 8, 9, 13]. In this section we recall this relationship and use it to prove the main result about spin network states.

There is an equivalence relation on paths $\gamma: [0, 1] \to M$ that are embeddings when restricted to $(0, 1)$, namely, $\gamma_1 \sim \gamma_2$ if $\gamma_1$ is obtained from $\gamma_2$ by an orientation-preserving continuous reparametrization with continuous inverse. We call an equivalence class $e = [\gamma]$ of such paths an embedded edge in $M$. We call the endpoints $\gamma(0)$ and $\gamma(1)$ are independent of a choice of representative $\gamma$ for $e$, as is the set $\gamma[0, 1] \subseteq M$. We write these as $e(0), e(1)$, respectively. We define an embedded graph $\phi$ to be a finite collection $e_i$ of embedded edges such that for all $i \neq j$, $e_i[0, 1]$ and $e_j[0, 1]$ intersect, if at all, only at their endpoints. We call $e_i$ edges of $\phi$, and call the points $e_i(0), e_i(1)$ the vertices of $\phi$. Somewhat redundantly, we write $E_\phi$ for the set of edges of $\phi$, and $V_\phi$ for the set of vertices.

Note that any graph $\phi$ embedded in $M$ determines a graph in the sense of Section 2 — which by abuse of notation we also call $\phi$ — having edges $E_\phi$, vertices $V_\phi$, and $s(e_i) = e_i(0), \quad t(e_i) = e_i(1)$.

(Sometimes we will call graphs in the sense of Section 2 abstract graphs, to distinguish them from embedded graphs.) Moreover, if we restrict the bundle $P$ to the vertices of $\phi$, we obtain a principal $G$-bundle $P_\phi$ over $V_\phi$. We may thus define the space $\mathcal{A}_\phi$ of connections on $\phi$, the group $G_\phi$ of gauge transformations, and so on, as in Section 2.

The uniform generalized measure $\mu$ on $\mathcal{A}$ can be efficiently described using embedded graphs, as follows. There is an onto map

$$p_\phi: \mathcal{A} \to \mathcal{A}_\phi$$

given by

$$(p_\phi(A))_e = T \exp \int_\gamma A,$$

where $\gamma$ is any representative of the embedded edge $e$. This map allows us to identify $C(\mathcal{A}_\phi)$ with a subalgebra of the algebra $\text{Fun}_0(\mathcal{A})$ of cylinder functions, and $\text{Fun}_0(\mathcal{A})$ is the union of the algebras $C(\mathcal{A}_\phi)$ as $\phi$ ranges over all graphs embedded in $M$. The uniform generalized measure $\mu$ on $\mathcal{A}$ is then uniquely characterized by the property that for any embedded graph $\phi$, and any $f \in C(\mathcal{A}_\phi)$,

$$\mu(f) = \int_{\mathcal{A}_\phi} f \mu_\phi,$$

where $\mu_\phi$ is the measure on $\mathcal{A}_\phi$ introduced in Section 2 (essentially a product of copies of normalized Haar measure on $G$).
As in the previous section we define $L^2(A)$ to be the completion of $\text{Fun}(A)$ in the norm
\[ \|f\|_2 = \mu(|f|^2)^{1/2}, \]
and $L^2(A/G)$ to be the completion of $\text{Fun}(A/G)$ in the same norm. By the defining property of $\mu$, the inclusion $C(A_\phi) \subset \text{Fun}(A_\phi)$ extends uniquely to an isometry $L^2(A_\phi) \rightarrow L^2(A)$ which in turn restricts to an isometry $L^2(A_\phi/G_\phi) \rightarrow L^2(A/G)$. Therefore, we can think of $L^2(A_\phi)$ as a closed subspace of $L^2(A)$, and $L^2(A_\phi/G_\phi)$ as a closed subspace of $L^2(A/G)$. By results recalled in Section 3, the union of the subspaces $L^2(A_\phi)$ is dense in $L^2(A)$, and similarly the union of the subspaces $L^2(A_\phi/G_\phi)$ is dense in $L^2(A/G)$.

Theorem 1. The set of all vectors of the form $\Psi_{\phi,\rho,\iota}$ spans $L^2(A/G)$.

Proof - The union of the subspaces $L^2(A_\phi/G_\phi)$ is dense in $L^2(A/G)$, and for each embedded graph $\phi$ the vectors $\Psi_{\rho,\iota}$ span $L^2(A_\phi/G_\phi)$, by Lemma 3. \hfill \Box

We call the vectors $\Psi_{\phi,\rho,\iota}$ spin network states. Note that this concept is unambiguous, in the sense that the question of whether a given vector in $L^2(A/G)$ is a spin network state can be answered irrespective of a choice of embedded graph:

Theorem 2. Given $\Psi_{\phi,\rho,\iota} \in L^2(A_\phi/G_\phi) \subset L^2(A/G)$

for some embedded graph $\phi$, if $\Psi_{\phi,\rho,\iota} \in L^2(A_{\phi'}/G_{\phi'}) \subset L^2(A/G)$

for some other embedded graph $\phi'$, then

$\Psi_{\phi,\rho,\iota} = \Psi_{\phi',\rho',\iota'}$

for some labellings $\rho'$ and $\iota'$.

Proof - First note that given two embedded graphs, there is always a third embedded graph including both, so that it suffices to consider the cases where $\phi \leftrightarrow \phi'$ or $\phi' \leftrightarrow \phi$. In the former case, we obtain $\Psi_{\phi,\rho,\iota} = \Psi_{\phi',\rho',\iota'}$ using Lemma 3. In the latter case, the result follows using Lemma 4. \hfill \Box
5 Conclusions

One aim of this work is to provide tools for work on the loop representation of quantum gravity and other diffeomorphism-invariant gauge theories. In the loop representation of quantum gravity [16], it is typical to proceed towards the description of physical states in three stages. The first two stages have been formalized in a mathematically rigorous way, but until the crucial third stage has been dealt with rigorously, the success of the whole program is an open question. In particular, it is quite possible that the work done so far will need refinement and revision to provide a sufficient platform for the third stage.

In the first stage, kinematical states are taken to be generalized measures on $\mathcal{A}/\mathcal{G}$, where $\mathcal{A}$ is the space of SU(2) connections on a bundle isomorphic to the spin bundle of the (real-analytic, oriented) 3-manifold $M$ representing ‘space’ in the theory. Such generalized measures can be characterized in terms of embedded graphs [7]. Briefly, they are in one-to-one correspondence with ‘consistent’ uniformly bounded families of measures on the spaces $\mathcal{A}_\phi/\mathcal{G}_\phi$ for all graphs $\phi$ embedded in $M$. Here consistency means that when the embedded graph $\psi$ is included in $\phi$, the measure on $\mathcal{A}_\phi/\mathcal{G}_\phi$ must push forward to the measure on $\mathcal{A}_\psi/\mathcal{G}_\psi$ under the induced map from $\mathcal{A}_\phi/\mathcal{G}_\phi$ to $\mathcal{A}_\psi/\mathcal{G}_\psi$.

In the second stage, diffeomorphism-invariant states are taken to be generalized measures on $\mathcal{A}/\mathcal{G}$ that are invariant under the action of Diff$_0(M)$, the identity component of the group of real-analytic diffeomorphisms of $M$. (It is worth noting that analyticity plays a technical role here and a purely $C^\infty$ version of the theory would be preferable in some ways.) Diffeomorphism-invariant states have also been characterized in terms of embedded graphs [7]. Unfortunately, while this characterization allows the explicit construction of many diffeomorphism-invariant states, of which the Ashtekar-Lewandowski generalized measure is the simplest, it does not give a concrete recipe for constructing ‘all’ diffeomorphism-invariant states, or even a dense set. In particular, the ‘loop states’, so important in the heuristic work of Rovelli and Smolin [16], remain mysterious from this viewpoint.

Finally, one hopes that physical states are diffeomorphism-invariant states that satisfy a certain constraint, the Hamiltonian constraint. Formulating this constraint rigorously is a key technical problem in the loop representation of quantum gravity, much studied [10] but still insufficiently understood. One key aspect, the interplay between SU(2) and SL(2, $\mathbb{C}$) connections which is so important in the theory, has recently been clarified using embedded graph techniques [3]. But one would also like to make precise various arguments such as Rovelli and Smolin’s argument that the ‘loop states’ satisfy the Hamiltonian constraint. In order to do this, it is important to understand the diffeomorphism-invariant states as explicitly as possible.

In this direction, recent work by Ashtekar and collaborators [2, 5] has given a rigorous construction of the Rovelli-Smolin ‘loop states’. The construction is applicable to any compact connected gauge group $G$, and it produces a diffeomorphism-invariant
generalized measure $\nu$ on $\mathcal{A}/\mathcal{G}$ from an isotopy equivalence class of knots $K$ and an irreducible representation $\rho$ of $G$, as follows. For the present purposes, we define a knot to be an equivalence class of analytically embedded circles in $M$, two embeddings being equivalent if they differ by an orientation-preserving continuous reparametrization with continuous inverse. Also, we define two knots to be isotopic if one can be obtained from the other by the action of Diff$_0(M)$. Now, given an isotopy equivalence class of knots $K$, choose for each knot $k \in K$ an analytically embedded circle $\gamma: S^1 \to M$ representing $k$. Note that $\gamma$, regarded as a map from $[0,1]$ to $M$ with $\gamma(0) = \gamma(1)$, defines an embedded edge $e$ in $M$. Associated to $e$ there is an embedded graph $\phi$ having $e$ as its only edge and $v = e(0) = e(1)$ as its only vertex. If we label the edge $e$ with the representation $\rho \in \Lambda$ and label the vertex $v$ with the identity operator (as an intertwining operator from $\rho$ to itself), we obtain a spin network state $\psi_k \in L^2(\mathcal{A}/\mathcal{G})$. Next, consider the formal sum over all knots $k$ in the isotopy class $K$,

$$\nu = \sum_{k \in K} \psi_k.$$  

This sum does not converge in $L^2(\mathcal{A}/\mathcal{G})$ — indeed, it is an uncountable sum — but one can show that for any function $f \in \text{Fun}(\mathcal{A}/\mathcal{G})$, the sum

$$\nu(f) = \sum_{k \in K} \langle \psi_k, f \rangle$$  

does converge, where the inner product is that of $L^2(\mathcal{A}/\mathcal{G})$. In fact, $\nu$ defines a generalized measure on $\mathcal{A}/\mathcal{G}$. By the nature of the construction it is clear that $\nu$ is Diff$_0(M)$-invariant. We call $\nu$ the loop state associated to the knot class $K$ and the representation $\rho$.

The framework of spin network states appears to allow an interesting generalization of the above construction. Namely, one should be able to construct diffeomorphism-invariant generalized measures on $\mathcal{A}/\mathcal{G}$ from isotopy classes of spin networks. The key idea is to treat a knot labelled by a group representation as a very special case of a spin network, in a manner that our exposition above should make clear. This idea, currently under investigation by Ashtekar and the author, might make possible the sort of explicit description of ‘all’ diffeomorphism-invariant states that one would like for rigorous work on the Hamiltonian constraint.

Another aim of this work is to clarify the relationship between the loop representation of quantum gravity and a body of recent work on topological quantum field theories \cite{11, 12, 13, 18}. As noted in Section 1, spin networks arise naturally in the study of the category of representations of a group or quantum group. More generally, we may define them for any category with the appropriate formal properties. It is a striking fact that the most efficient construction of many topological quantum field theories involves category theory and the use of spin networks. For example, Euclidean 3-dimensional quantum gravity with nonzero cosmological constant can be identified with the Turaev-Viro theory \cite{18}, and the latter is a topological quantum
field theory that is most easily constructed using $SU_q(2)$ spin networks. It is unclear whether 4-dimensional quantum gravity is a topological quantum field theory (or some generalization thereof), but the present work at least begins to make precise the role of spin networks in the loop representation of 4-dimensional quantum gravity. In what follows we briefly comment on the category-theoretic significance of our results.

Associated to any abstract graph $\phi$ in the sense of Section 2 there is a category $\mathcal{C}_\phi$, or more precisely, a groupoid (a category in which all the morphisms are invertible). This is the free groupoid on the objects $V_\phi$ and morphisms $E_\phi$. If we fix a trivial $G$-bundle $P$ over $V_\phi$, the connections $A \in \mathcal{A}_\phi$ are precisely the functors from $\mathcal{C}_\phi$ to $G$, where we regard the compact connected Lie group $G$ as a groupoid with one object. Similarly, the gauge transformations $g \in G_\phi$ are precisely the natural transformations between such functors. As we have seen, the set $\mathcal{A}_\phi/G_\phi$ of ‘functors modulo natural transformations’ inherits the structure of a measure space from $G$, and Lemma 3 gives an explicit description of $L^2(\mathcal{A}_\phi/G_\phi)$ in terms of the category of finite-dimensional unitary representations of $G$.

Similarly, given a real-analytic manifold $M$ and a smooth principal $G$-bundle $P$ over $M$, we may define the holonomy groupoid $\mathcal{C}$ to have as objects points of $M$ and as morphisms equivalence classes of piecewise analytic paths in $M$, where two paths $\gamma, \gamma'$ are regarded as equivalent if $T \exp \int_\gamma A = T \exp \int_{\gamma'} A$

for all connections $A$ on $P$. This has as a subgroupoid the ‘holonomy loop group’ of Ashtekar and Lewandowski [4]. If we fix a trivialization of $P_x$ for all $x \in M$, any connection on $P$ determines a functor from $\mathcal{C}$ to $G$, while conversely any such functor can be thought of as a ‘generalized connection’ [5, 6]. Similarly, any gauge transformation determines a natural transformation between such functors, and any natural transformation between such functors can be thought of as a ‘generalized gauge transformation’.

The relation between gauge theory on graphs and gauge theory on manifolds then turns upon the fact that for any graph $\phi$ embedded in $M$ we obtain a subcategory of $\mathcal{C}$ isomorphic to $\mathcal{C}_\phi$. Moreover, an inclusion $i: \psi \hookrightarrow \phi$ induces a functor $i_*: \mathcal{C}_\psi \hookrightarrow \mathcal{C}_\phi$, and the holonomy groupoid $\mathcal{C}$ is the colimit of the groupoids $\mathcal{C}_\phi$ as $\phi$ ranges over all embedded graphs in $M$. This explains the importance of ‘projective limit’ techniques in studying generalized measures on the space of connections and the space of connections modulo gauge transformations [3]. In particular, it is this that lets us obtain a spanning set of spin network states for $L^2(\mathcal{A}/\mathcal{G})$ from the spin network states for $L^2(\mathcal{A}_\phi/\mathcal{G}_\phi)$ as $\phi$ ranges over all graphs embedded in $M$.

Finally, it is interesting to note that the holonomy groupoid $\mathcal{C}$ has as a quotient the fundamental groupoid of $M$, in which morphisms are given by homotopy equivalence classes of paths. A functor from $\mathcal{C}$ to $G$ that factors through the fundamental groupoid
is just a flat connection on $P$. In certain cases there is a natural measure on the space of $A_0/G$ of flat connections modulo gauge transformations, and then the space $L^2(A_0/G)$ is the Hilbert space for a theory closely related to quantum gravity, namely $BF$ theory [9].

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