Chiral Susceptibility in (2+1)-flavour QCD

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We calculate chiral susceptibilities in (2+1)-flavour QCD for different masses of the light quarks using the functional renormalisation group (fRG) approach to first-principles QCD. We follow the evolution of the chiral susceptibilities with decreasing masses as obtained from both the light-quark and the reduced quark condensate. The latter compares very well with recent results from the HotQCD collaboration for pion masses $m_{\pi} \gtrsim 100$ MeV. For smaller pion masses, the fRG and lattice results are still consistent. In particular, the estimates for the chiral critical temperature are in very good agreement. We close by discussing different extrapolations to the chiral limit.

I. INTRODUCTION

The phase structure of QCD probed with heavy-ion collisions is well described by (2+1)-flavour QCD. While the charm, bottom and top quarks are too heavy to significantly add to the dynamics of the system, the dynamics of the strange and, most importantly, of the light up and down quarks determine the rich phase structure in particular at large densities. Recently, functional methods for first-principles QCD have made significant progress in the description of this regime, see Refs. [1, 2] for functional renormalisation group studies (fRG) and, e.g., Refs. [3–5] for Dyson-Schwinger studies. Still, in the high-density regime the systematic error of the current computations grows large. This asks for both, systematically improved computations and a better error control. In turn, lattice simulations are obstructed by the sign problem at vanishing chemical potential [6–10] or on analytic continuations from imaginary to real quark chemical potential [6–10, 11]. In summary, this suggests a two-tier strategy to tackle the high-density regime by direct systematically improved functional computations and a quantitative access to the zero-density limit.

Interestingly, the mass dependence of the phase structure at vanishing density can potentially constrain the phase structure at large density. For instance, low-energy effective theory computations indicate that the chiral phase transition temperature in the limit of massless up and down quarks is a possible upper bound for the transition temperature at the critical endpoint, see Ref. [16] for a recent review. Accordingly, the nature of the chiral transition in QCD with three quark flavours is very actively researched. For sufficiently small masses of the three quarks, one expects a finite mass range with a first-order chiral transition [19]. Interestingly, this first-order regime may even extend to the limit of infinitely heavy strange quarks, see, e.g., Refs. [17, 20, 21]. This intricate question regarding the existence and range of such a regime is tightly connected to the fate of the axial $U_A(1)$ anomaly at finite temperature: depending on the strength of the associated $U_A(1)$ breaking, the phase transition may indeed be of first order [19, 22, 23].

For physical masses of the three quarks, the transition in (2+1)-flavour QCD from a low-temperature hadronic phase to a high-temperature quark-gluon plasma phase has been found in lattice and functional QCD studies to be a crossover, see, e.g., Refs. [18, 24–29] for lattice studies and Refs. [11, 15] for functional studies.

In the chiral limit of the light quarks, the critical behaviour is controlled by the three-dimensional (3d) $O(4)$ universality class, if the anomalous breaking of the $U_A(1)$ symmetry is sufficiently strong. In turn, if the $U_A(1)$ symmetry is effectively restored sufficiently close to the chiral transition, the critical behaviour may no longer be controlled by the 3d $O(4)$ universality class [30–33]. Within a very recent lattice QCD study investigating pion masses in the range of $50 \text{MeV} \lesssim m_{\pi} \lesssim 160 \text{MeV}$ with a physical strange quark mass, the scaling properties of the chiral susceptibility are now found to be compatible with the 3d $O(4)$ universality class [29]. An extrapolation to the chiral limit of the light quarks leads to $T_c = 132^{+3}_{-6} \text{MeV}$ for the chiral critical temperature [29].

The reconstruction of the chiral critical temperature from an extrapolation to the chiral limit is in general a non-trivial task as it is affected by non-universal aspects, such as the order of the transition and the dependence of the pseudocritical temperature on the pion mass. Moreover, the definition of a pseudocritical temperature is not unique. Indeed, the strength of the pion-mass dependence of the pseudocritical temperature is different for different definitions. Assuming that the chiral phase transition is of second order in the limit of massless up and down quarks, it follows from universal scaling arguments [34] that the pion-mass scaling of the pseudocritical temperature defined as the position of the peak of the chiral susceptibility is controlled by the critical exponents of the underlying universality class. Unfortunately, the size of the scaling regime is also a non-universal quantity. These statements also hold for other definitions of the pseudocritical temperature and it is also reasonable to expect that the size of the scaling regime is of the same order for different definitions, provided that they rely on
In low-energy effective theories of QCD it has been found that the pseudocritical temperature defined as the position of the peak of the chiral susceptibility scales roughly linearly over a wide range of pion masses which appears compatible with scaling arguments at first glance \cite{35,37}. Even more, it was found that the results for the suitably rescaled chiral order parameter fall almost on one line \cite{37} for \( m_\pi \gtrsim 75 \text{ MeV} \), seemingly suggesting scaling behavior. However, a comparison of these results with the corresponding scaling function extracted within the model studies exhibits clear deviations from scaling. A detailed analysis then revealed that the size of the actual scaling regime is very small, i.e., scaling behavior of the chiral susceptibility and the chiral order parameter is only observed for very small pion masses, \( m_\pi \lesssim 1 \text{ MeV} \), see Ref. \cite{37}.

Our present first-principles fRG study corroborates these findings in low-energy effective theories: the actual scaling regime in QCD is indeed small, with a conservatively estimated upper bound of \( m_\pi \approx 30 \text{ MeV} \). In addition, our present work shows that the glue dynamics softens the strong dependence of the pseudocritical temperature on the pion mass observed in low-energy effective theories of QCD. In fact, it is found to be in very good agreement with recent lattice QCD results \cite{29}. Moreover, we shall discuss different extrapolations to the chiral limit, leading us consistently to \( T_c \approx 142 \text{ MeV} \) for the critical temperature, see also Figs. \ref{fig:fig1} and \ref{fig:fig2} below.

This work is organized as follows: In Sect. \ref{sect:sect2} we briefly discuss the methodological framework of our present study. Our results for the chiral susceptibility as obtained from the light-quark condensate are presented in Sect. \ref{sect:sect3}. There, we also show a comparison of these results with those for the susceptibility extracted from the reduced condensate, also used in lattice computations. The results for the reduced condensate are then compared to lattice QCD data \cite{29}, including a discussion of the dependence of the pseudocritical temperature on the pion mass. Our conclusions can be found in Sect. \ref{sect:sect4}.

II. CONDENSATES & FUNCTIONAL QCD

In this section we discuss different chiral condensates and the associated susceptibilities which we compute to access the mass dependence and scaling of the pseudocritical temperature. We also briefly introduce the fRG approach to QCD used for this computation. The basis for our present study is discussed in detail in Ref. \cite{11}.

A. Chiral condensates

In order to obtain the (chiral) susceptibility in \((2+1)\)-flavour QCD for various (current) quark masses \( m_{q_i} \), we have to compute the chiral condensates \( \Delta_{q_i} \), associated with the three quark flavours \( q_i \). Here, \( q_i = u, d, s \) refers to the up, down, and strange quark, respectively. The \( \Delta_{q_i} \) can be obtained from the logarithmic derivative of the thermodynamic grand potential \( \Omega \) with respect to the corresponding current quark mass,

\[
\Delta_{q_i} = m_{q_i} \frac{\partial \Omega(m_{q_i}, T, \mu_u, \mu_d, \mu_s)}{\partial m_{q_i}} = m_{q_i}^0 \frac{T}{V} \int_0^\tau \int_V d^3 x \langle \bar{q}_i(r, \vec{x}) q_i(r, \vec{x}) \rangle . \tag{1}
\]

Here, \( T \) is the temperature and \( V \) is the spatial volume. The logarithmic derivative with respect to the current quark mass is taken since \( \Delta_{q_i} \) then carries the same scaling properties as the grand potential \( \Omega \). Consequently, it is not sensitive to details of the setup that typically change the precise value of the current quark mass, in particular the renormalisation scheme, see Ref. \cite{11} for a discussion. In the present work, we only consider the zero-density limit and therefore we set the quark chemical potentials to zero, \( \mu_u = \mu_d = \mu_s = 0 \). Thus, the quark condensates are only functions of the temperature and the current quark masses from here on. Moreover, we use identical current masses for the two light quarks, \( m_{q_u}^0 = m_{q_d}^0 = m_{q_s}^0 \). This allows to define the light quark condensate \( \Delta_l = \Delta_u = \Delta_d \).

The computation of the quark condensates via the expectation value in the last line of Eq. \ref{eq:1} requires renormalisation and hence the result depends on the renormalisation procedure. Both, the necessity for renormalising the operator, and the renormalisation scheme dependence is removed when considering finite difference of chiral condensates. Two possible choices are the renormalised and the reduced condensate. Both are commonly used in lattice QCD studies and have also been studied in functional approaches to QCD. The fRG approach naturally provides a renormalised finite expression for the condensate \( \Delta_{q_i} \), as it is based on a free energy and hence the current mass derivative in Eq. \ref{eq:1} is finite, for more details see Sect. \ref{sect:sect3b} and Ref. \cite{11}. For our study of the chiral phase transition, the condensate \( \Delta_l \) is therefore the key observable in the present work since it has the smallest systematic error within the truncation used for the computation.

The renormalised condensate associated with the light-quark flavours can be defined as

\[
\Delta_{l,R}(T) = \frac{1}{\mathcal{N}_R} (\Delta_l(T) - \Delta_l(0)) . \tag{2}
\]

The normalisation constant \( \mathcal{N}_R \) is at our disposal: in the following, it is chosen independent of the current masses and is typically used to render \( \Delta_{l,R}(T) \) dimensionless. Note that, by including a \( m_{q_i}^0 \)-dependence of the form \( \mathcal{N}_R \sim m_{q_i}^0/m_{q_i}^0 \), we recover the definition of the renormalised condensate conventionally employed in lattice QCD studies, see, e.g., Ref. \cite{35}.

The reduced condensate \( \Delta_{l,s} \) is a combination of the light-quark condensate \( \Delta_l \) and the strange quark con-
densate $\Delta_s$,

$$
\Delta_{i,s} = \frac{1}{N_{i,s}} \left( \Delta_i(T) - \left( \frac{m_i^0}{m_i^s} \right)^2 \Delta_s(T) \right),
$$

(3)

where the definition of the $m_i^0$-independent normalisation constant $N_{i,s}$ is again irrelevant for our discussion of the susceptibilities below. Instead, if we choose $N_{i,s}$ to be $m_i^0$-dependent, $N_{i,s} = (\Delta_i(0) - (m_i^0/m_i^s)^2 \Delta_s(0))^{-1}$, we arrive at the standard lattice definition of the reduced condensate, see, e.g., Ref. [29]. Alternatively, we could simply choose $N_{i,s} \sim m_i^0/m_i^s$ which yields the observable defined in Ref. [29] (up to numerical factors) to compute susceptibilities.

The subtraction in Eq. (3) renders the reduced condensate finite as in the case for the renormalised condensate. However, the systematic error of results of such a light-strange quark mixture is a combination of that in the strange and in the light quark sector and requires a quantitative treatment of both. Consequently, it is affected by larger systematic errors in our present fRG study than the light-quark condensate $\Delta_l$.

The corresponding susceptibilities are readily obtained from all three condensates. We define them as follows,

$$
\chi^{(i)}_M(T) = -\frac{\partial}{\partial m_i^0} \left( \frac{\Delta_i(T)}{m_i^0} \right),
$$

(4)

where $(i) = (l), (l,R), (l,s)$. We shall refer to these susceptibilities as light-quark susceptibility, renormalised susceptibility, and reduced susceptibility, respectively. Leaving an overall normalisation aside, it follows from the definition of the light-quark condensate $\Delta_l$ and the renormalised condensate $\Delta_{i,R}$ that the associated susceptibilities only differ by a temperature-independent shift. From our discussion above, it moreover follows that our definition of the reduced susceptibility matches the one used in lattice studies [29].

By multiplying the magnetic susceptibilities [4] with $(m_i^0)^2$, they also carry the scaling properties of the grand potential and there is no dependence on the renormalisation procedure left. This is in one-to-one correspondence to the lack of renormalisation scheme dependence of $\Delta_l$ defined by the logarithmic $m_i^0$-derivative of the grand potential, and to that of the renormalised and reduced condensates. However, for the sake of a straightforward comparison with the lattice results from Ref. [29], we have not included these factors in Eq. (4). Moreover, for a comparison of the susceptibilities for different pion masses (or current quark masses) as well as for a comparison with results from other methods, it is convenient to normalise the magnetic susceptibilities $\chi^{(i)}_M(T)$ with the respective peak value for the physical pion mass $m_\pi \approx 140$ MeV,

$$
\chi^{(i)}_M = \max_T \left. \chi^{(i)}_M(T) \right|_{m_\pi=140\text{ MeV}},
$$

(5)

where again $(i) = (l), (l,R), (l,s)$. Thus, we have $\chi^{(i)}_M(T)/\chi^{(i)}_M = 1$ at the peak position in case of the physical pion mass. The size and evolution of the increasing peak towards the chiral limit gives a rough estimate for the “distance to criticality.”

B. Functional renormalisation group approach

In this work, we use the fRG approach for the computation of the light-quark and reduced susceptibilities from first principles. In this approach, quark, gluon and hadron correlation functions of QCD are computed from functional relations that are derived from the flow equation for the finite effective action $\Gamma$. Accordingly, the finite effective action is easily accessible in this approach, and is self-consistent. In particular, it automatically encodes the same RG scheme as the correlation functions.

The thermodynamic grand potential $\Omega$ is then given by the effective action evaluated on the equations of motion (EoM), i.e., the ground state: $\Omega = \langle T/V \rangle |_{\text{EoM}}$. Hence, the fRG approach provides a finite thermodynamic grand potential $\Omega$. This leads to a finite quark condensate $\Delta_q$, within the RG scheme used for the computation of the correlation functions, see Eq. (1).

The computation of the susceptibilities in our fRG study requires the computation of the light-quark and reduced quark condensates for various temperatures and quark masses. To this end, we have to follow the RG flow from the classical QCD action in the ultraviolet to the long-range (infrared) limit where the dynamics is effectively described by hadronic degrees of freedom rather than quarks and gluons. To facilitate the description of the transition between the degrees of freedom in the ultraviolet and infrared limit, we employ dynamical hadronisation techniques [1, 39–42], see Refs. [1, 40, 43–47] for their application to QCD. The chiral susceptibilities can then be obtained with two different – formally equivalent procedures – from Eq. (1), both of which are detailed below around Eq. (6) as their comparison provides an important self-consistency and reliability check for our present truncation.

Our present study has been done within the fQCD collaboration [18], and is a follow-up of a recent work [1] within this collaboration. It also builds on previous advances made within this collaboration, see, e.g., Refs. [2, 44–47]. Therefore, we refrain from showing the flow equations required to compute the light-quark and reduced quark condensate because of the size of this set of equations. All these equations are derived, documented and discussed in detail in Ref. [1].

We only would like to discuss two aspects of our computation which are particularly relevant for the estimate of the systematic error of our results:

First, we only take into account the sigma–pion channel in the computation of the order-parameter potential, and do not allow for an effective $U_A(1)$ restoration at, e.g., high temperatures. Thus, we tacitly assume that the chiral transition falls into the 3d $O(4)$ universality class. This assumption is based on the fact that Fierz-complete finite-temperature studies of the
Within the second method, the light-quark susceptibility $\chi^{(l)}_M$ is simply obtained by computing the light-quark condensate as a function of the temperature and the light quark mass $m^0_l$, and then taking a numerical derivative of this data set with respect to $m^0_l$. Note that we also compute the reduced condensate in this way.

Evidently, both methods should give the same results for the light-quark susceptibility, provided that no approximations are involved. Therefore, deviations give access to the reliability of the underlying approximations. A comparison of results from the two methods is depicted in Fig. 1. From this, we conclude that our present approximation is trustworthy for pion masses $m_\pi \gtrsim 30$ MeV. In particular, the peak positions obtained from the two methods are in very good agreement with each other in this pion-mass regime. In fact, they only differ by about 1 MeV. Based on this comparison, we consider it sufficient to show in the following only those results for the light-quark susceptibility which have been obtained by taking a derivative of our numerical data set for the light-quark condensate with respect to $m^0_l$. As the light-quark condensate is part of the reduced condensate, we may assume that the uncertainty for the pseudocritical temperatures extracted from the reduced susceptibilities is the same as in the case of the light-quark susceptibilities. However, the systematic error is likely to be bigger in this case, as already indicated in Sect. II A.

Regarding smaller pion masses, we add that an extension in this direction (including the chiral limit) is possible but requires either an expansion of the effective action about a scale-independent point [49] or the use of, e.g., recently developed techniques to access the full order-parameter potential [51].

### III. RESULTS

Let us now discuss the susceptibilities associated with the light-quark and reduced condensate for various temperatures and pion masses. To this end, we shall keep the strange quark mass fixed at its physical value and only vary the light quark mass. As already discussed in the previous section, the renormalised condensate only differs from the light-quark condensate by a temperature-independent shift. This implies that the peak positions of these two susceptibilities are the same for a given pion mass. Therefore, we shall not further discuss the renormalised susceptibility in this section.

Susceptibilities as introduced in the previous section are of great interest as their maxima can be used to define pseudocritical temperatures. From general scaling arguments [54], it then follows that the pseudocritical temperatures extracted from the light-quark and reduced susceptibility scale as

$$T_{pc}^{(i)}(m_s) \approx T_c + c(i) m^0_s,$$

at least within the scaling regime. Here, $(i) = (l), (l,s)$. The chiral critical temperature is given by $T_c$ in Eq. [7].
The quantity $c_{(i)}$ is a non-universal constant depending on the susceptibility under consideration whereas the exponent $p$ can be related to the universal critical exponents $\beta$ and $\delta$, $p = 2/(\beta\delta)$. The relation (7) follows from the fact that the position of the peak of the susceptibility as a function of the scaling variable $z = t/h^{1/(\beta\delta)}$ is constant in the scaling regime. Here, $t = (T - T_c)/T_0$ is the reduced temperature with $T_0$ being a suitably chosen normalisation factor and $h = H/H_0$ is the symmetry breaking field normalised by a suitably chosen normalisation $H_0$. In the present case, $H$ can be identified with the current mass of the light quarks $m_L^0$ which, in turn, is directly related to the pion mass via $m_L^0 \simeq m_\pi/\chi_{(0)}^l$.

For example, employing the critical exponents of the 3d $O(4)$ universality class [55–57], we have $p \approx 1.08$. Based on previous fRG studies of critical exponents [58–61], however, we expect $p$ to be slightly smaller in our present study. In any case, this suggests an almost linear dependence of the peak positions of the susceptibilities on the pion mass, at least within the scaling regime. Note that the size of the latter is not universal but depends on the details of the theory under consideration.

In Fig. 2 (left panel), we show our results for the light-quark susceptibilities $\chi_{(i)}^l(T)$ as a function of the temperature $T$, where we have normalised the susceptibilities with $\chi_{(i)}^l$, i.e., the value of $\chi_{(i)}^l(T)$ for $m_\pi = 140$ MeV evaluated at its maximum, see Eq. (6). As expected, we find that the susceptibility increases when the pion mass is decreased. Indeed, by decreasing the pion mass, we approach the chiral limit associated with a diverging susceptibility at the chiral phase transition temperature $T_c$.

Our results for the pseudocritical temperature indeed appear to depend almost linearly on the pion mass, see Fig. 2 (left panel). Fitting the scaling relation (7) to our numerical results for $T_{pc}^{(l)}(m_\pi)$ for $m_\pi = 30, 35, 40, \ldots, 140$ MeV, we obtain $T_c \approx 141.4^{+0.5}_{-0.4}$ MeV, $c_{(l)} \approx 0.19^{+0.05}_{-0.05} \text{MeV}^{1-p}$, and $p \approx 0.88^{+0.05}_{-0.05}$. At this point, we would like remind the reader that the renormalised susceptibility obeys the same temperature dependence as the light-quark susceptibility. Therefore, the pseudocritical temperatures extracted from these two susceptibilities are identical.

The deviation of our estimate for the exponent $p$ from the value associated with the 3d $O(4)$ universality class [55–57] already suggests that QCD is not within the scaling regime, not even for the smallest pion masses considered in the present work. As already mentioned in Sect. 4 this is in line with studies of low-energy effective theories of QCD. There, it has been found that deviations from scaling are still sizeable, even if the pseudocritical temperature scales approximately linearly for large pion masses and also the suitably rescaled chiral susceptibilities for pion masses $m_\pi > 75$ MeV appear to fall approximately on one line [37]. Even worse from a practical standpoint, actual scaling behavior of the chiral susceptibility and the chiral order parameter has only been observed for very small pion masses, $m_\pi \lesssim 1$ MeV [37]. Given these results from low-energy QCD model studies and the fact that the exponent $p$ is close to one for the 3d $O(4)$ universality class anyhow, a linear fit for the pseudocritical temperature may be considered reasonable. Performing such a linear fit, we obtain $T_c = 142.4^{+0.3}_{-0.4}$ MeV from the extrapolation to $m_\pi = 0$, in very good agreement with our estimate for $T_c$ presented above.

Let us now consider the ratio

$$D_{(l)}(m_\pi) = \frac{T_{pc}^{(l)}(m_\pi) - T_c}{T_c},$$

which is an estimate for the relative dependence of the

Figure 2. Left panel: Light-quark susceptibility $\chi_{(i)}^{l}(T)$ as a function of the temperature. The inset shows the peak positions of the depicted susceptibilities as a function of the pion mass. Right panel: Comparison of the light-quark susceptibility $\chi_{(i)}^{l}(T)$ and the reduced susceptibility $\chi_{(i)}^{l}(T, m_\pi)/\chi_{(l)}^{l}$ as a function of the temperature. The normalisations are the maxima of the respective susceptibilities at the physical pion mass, see Eq. (5).
pseudocritical temperature on the pion mass. For the physical pion mass, $m_\pi = 140$ MeV, this ratio in our present first-principles fRG study is about a factor of three smaller than typical values for $D(l)$ found in low-energy QCD model studies [35, 36]. For example,

$$D_{(l)}^{QM}(m_\pi = 140 \text{ MeV}) \approx 0.28$$  \hspace{1cm} (9)

was reported in Ref. [36] for the quark-meson (QM) model. In our present QCD study, we instead find

$$D_{(l)}^{QCD}(m_\pi = 140 \text{ MeV}) \approx 0.10,$$  \hspace{1cm} (10)

where we have employed the value for $T_c$ obtained from an extrapolation of the pseudocritical temperature $T_{pc}^{(l)}$ to the limit $m_\pi = 0$.

Next, we turn to the reduced susceptibility $\chi_{M}^{(l,s)}$ as defined in Eq. (4). In Fig. 2 (right panel), we show a comparison of the light-quark susceptibility and the reduced susceptibility for three pion masses. As expected, the qualitative behaviour of the reduced susceptibility is the same as the one found for the light-quark susceptibility. More specifically, the susceptibilities increase for decreasing pion mass, indicating the approach to a singularity in the chiral limit. Fitting the relation (7) to our numerical results for $T_{pc}^{(l,s)}(m_\pi)$ for $m_\pi = 30, 35, 40, \ldots, 140$ MeV, we obtain $T_c \approx 141.6^{+0.3}_{-0.1}$ MeV, $c_l^{(l,s)} \approx 0.17^{+0.03}_{-0.02}$ MeV$^{1-p}$, and $p \approx 0.91^{+0.03}_{-0.02}$. Thus, the critical temperature $T_c$ is in excellent agreement with the one extracted from our analysis of the light-quark susceptibilities, as it should be. With respect to the exponent $p$, we note that it also deviates clearly from the expected $O(4)$ value. However, we observe that it is consistent within fit errors with the value for $p$ which we obtained from our analysis of the light-quark susceptibility. Overall, we therefore cautiously conclude that QCD is not within the scaling regime for the range of pion masses considered here, providing us with $m_\pi \approx 30$ MeV as a conservative estimate for the upper bound of this regime. An actual determination of the size of the scaling regime is beyond the scope of present work as it requires to study very small pion masses.

In analogy to the definition (8), we can also define the relative dependence $D_{(l,s)}(m_\pi)$ of the pseudocritical temperature on the pion mass in case of the reduced susceptibility. For $m_\pi = 140$ MeV, we then find that this quantity is only slightly smaller than the corresponding quantity associated with the light-quark susceptibility.

In Fig. 3 (right panel), we finally compare our fRG results for the reduced susceptibility to very recent results from the HotQCD collaboration [29]. We observe excellent agreement between the results from the two approaches for pion masses $m_\pi \geq 100$ MeV. The deviations of the results from the two approaches for smaller pion masses may at least partially be attributed to cutoff artefacts in the lattice data. Note that cutoff effects are expected to shift the maxima to smaller temperatures. We refer to Ref. [18] for a respective discussion.

It is also worthwhile to compare the peak positions of the reduced susceptibilities extracted from the lattice QCD data with those from our fRG study, see Tab. I and Fig. 3 (left panel). As discussed above, the peak position can be used to define a pseudocritical temperature. For the presently available pion masses on
In this work, we have studied the magnetic susceptibility in \((2 + 1)\)-flavour QCD within a first-principles fRG calculation. Specifically, we have presented results for the susceptibilities associated with the light-quark condensate and the reduced condensate.

The chiral pseudocritical temperatures have been determined from the peak positions of the susceptibilities. Interestingly, we found that its dependence on the pion mass in the present QCD study is milder than in low-energy QCD model studies. From an extrapolation to the chiral limit, we obtained \(T_c \approx 142\,\text{MeV}\) for the chiral phase transition temperature.

Our results for the susceptibilities and the scaling of the corresponding pseudocritical temperature indicate that QCD is not within the scaling regime for the considered pion masses \(m_\pi \gtrsim 30\,\text{MeV}\). As discussed in detail, this conclusion is at least in accordance with low-energy QCD model studies \cite{27}. There, a qualitatively similar behaviour of the susceptibilities and the pseudocritical temperature has been observed for the same pion mass range as considered here. However, the actual size of the scaling regime turned out to be significantly smaller. A detailed analysis of this issue within our present first-principles fRG approach is deferred to future work as it requires to study (very) small pion masses.

We have also compared our results for the reduced susceptibility with very recent results from the HotQCD collaboration \cite{29} and found that the results from both approaches are in very good agreement for pion masses \(m_\pi \gtrsim 100\,\text{MeV}\). For smaller pion masses, the lattice and fRG results are still consistent with each other. Following the analysis of the HotQCD collaboration, we have also estimated the phase transition temperature in the chiral limit based on scaling properties of the susceptibility in the temperature regime below the temperature defined by the peak of the susceptibility. As it should be, this provides us with the same value for the chiral phase transition temperature as in the case of an extrapolation of the peak positions of the light-quark and reduced susceptibility.

As a next step, it will be important to further extend our present study. For example, we plan to improve the stability of our numerical calculations for very small pion masses to eventually reach the chiral limit by employing recent developments for the solution of fRG equations \cite{54}. Moreover, an analysis of the effect of the breaking of the \(U_A(1)\) symmetry and effective \(U_A(1)\) restoration at high temperatures is in order. A detailed analysis of the latter issue requires the study of Fierz-
complete truncations of the effective action. First steps into this direction have been taken [63], suggesting that effective \( U_A(1) \) restoration already sets in closely above the chiral phase transition temperature, in accordance with recent lattice QCD studies [63]. An understanding of the effect of this almost coincidence of chiral and \( U_A(1) \) restoration on the critical behaviour is indeed an intriguing and not yet fully resolved problem in QCD.

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