PIECEWISE-LINEAR APPROXIMATION FOR FEATURE SUBSET SELECTION IN A SEQUENTIAL LOGIT MODEL

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Abstract This paper concerns a method of selecting a subset of features for a sequential logit model. Tanaka and Nakagawa (2014) proposed a mixed integer quadratic optimization formulation for solving the problem based on a quadratic approximation of the logistic loss function. However, since there is a significant gap between the logistic loss function and its quadratic approximation, their formulation may fail to find a good subset of features. To overcome this drawback, we apply a piecewise-linear approximation to the logistic loss function. Accordingly, we frame the feature subset selection problem of minimizing an information criterion as a mixed integer linear optimization problem. The computational results demonstrate that our piecewise-linear approximation approach found a better subset of features than the quadratic approximation approach.

Keywords: Optimization, statistics, feature subset selection, sequential logit model, piecewise-linear approximation, information criterion

1. Introduction
The analysis of ordinal categorical data is required in various areas including finance, econometrics, and bioinformatics. For instance, credit ratings of financial instruments are typical ordinal categorical data, and thus, many previous studies have analyzed such data by means of ordinal classification models (see, e.g., [1, 12, 28]). A sequential logit model, also known as the continuation ratio model, is a commonly used ordinal classification model. It predicts an ordinal class label for each sample by successively applying separate logistic regression models. One can find various applications of sequential logit models: Kahn and Morimune used this model to explain the duration of unemployment of workers; Weiler investigated the choice behavior of potential attendees in higher education institutions; Fu and Wilmot estimated dynamic travel demand caused by hurricane evacuation.

In order to enhance the reliability of these data analyses, it is critical to carefully choose a set of relevant features for model construction. Such a feature subset selection problem is of essential importance in statistics, data mining, artificial intelligence, and machine learning (see, e.g., [11, 16, 20, 24]). The mixed integer optimization (MIO) approach to feature subset selection has recently received a lot of attention as a result of algorithmic advances and hardware improvements (see, e.g., [10, 21, 22, 25, 26]). In contrast to heuristic algorithms, e.g., stepwise regression, $L_1$-penalized regression, and metaheuristic strategies, the MIO approach has the potential of providing an optimality guarantee for the selected set of features under a given goodness-of-fit measure.

Tanaka and Nakagawa recently devised an MIO formulation for feature subset selection in a sequential logit model. It is hard to exactly solve the feature subset selection problem for a sequential logit model, because its objective contains a nonlinear function called the logistic loss function. To resolve this issue, they employed a quadratic approximation of the logistic loss function.
function. The resultant mixed integer quadratic optimization (MIQO) problem can be solved with standard mathematical optimization software; however, there is a significant gap between the logistic loss function and its quadratic approximation. As a result, the MIQO formulation may fail to find a good-quality solution to the original feature subset selection problem.

The purpose of this paper is to give a novel MIO formulation for feature subset selection in a sequential logit model. Sato et al. [29] used a piecewise-linear approximation in the feature subset selection problem for binary classification. In line with Sato et al. [29], we shall apply a piecewise-linear approximation to the sequential logit model for ordinal multi-class classification. Consequently, the problem is posed as a mixed integer linear optimization (MILO) problem. This approach is capable of approximating the logistic loss function more accurately than the quadratic approximation does. Moreover, our MILO formulation has the advantage of selecting a set of features with an optimality guarantee on the basis of an information criterion, such as the Akaike information criterion (AIC, [4]) or Bayesian information criterion (BIC, [31]).

The effectiveness of our MILO formulation is assessed through computational experiments on several datasets from the UCI Machine Learning Repository [7]. The computational results demonstrate that our piecewise-linear approximation approach found a better subset of features than the quadratic approximation approach did.

2. Sequential Logit Model

Let us suppose that we are given $n$ samples of pairs, $(x_i, y_i)$ for $i = 1, 2, \ldots, n$. Here, $x_i = (x_{i1}, x_{i2}, \ldots, x_{ip})^\top$ is a $p$-dimensional feature vector, and $y_i \in \{1, 2, \ldots, m+1\}$ is an ordinal class label to be predicted for each sample $i = 1, 2, \ldots, n$. In the sequential logit model for ordinal classification, we sequentially apply the following logistic regression models in order to predict a class label of each sample (see, e.g., [5, 17, 33]),

$$q_k(x) = \Pr(y = k \mid y \geq k, x) = \frac{1}{1 + \exp(-(w_k^\top x + b_k))} \quad (k = 1, 2, \ldots, m), \quad (2.1)$$

where the intercept, $b_k$, and the $p$-dimensional coefficient vector, $w_k = (w_{1k}, w_{2k}, \ldots, w_{pk})^\top$, are parameters to be estimated.

As shown in Figure 1, a feature vector $x$ is moved into class 1 with a probability $q_1(x)$. In the next step, it falls into class 2 with a probability $(1 - q_1(x))q_2(x)$. In the similar manner, it reaches class $k$ with a probability $(1 - q_1(x))(1 - q_2(x)) \cdots (1 - q_{k-1}(x))q_k(x)$.

Here, we define

$$\delta_{ik} = \begin{cases} 1 & \text{if } y_i = k, \\ 0 & \text{otherwise}, \end{cases} \quad (i = 1, 2, \ldots, n; k = 1, 2, \ldots, m). \quad (2.2)$$

It then follows that

$$1 - \sum_{j=1}^{k} \delta_{ij} = \begin{cases} 1 & \text{if } k < y_i, \\ 0 & \text{otherwise}, \end{cases} \quad (i = 1, 2, \ldots, n; k = 1, 2, \ldots, m). \quad (2.3)$$

Therefore, the occurrence probability of a sample $(x_i, y_i)$ is modeled as follows:

$$\prod_{k=1}^{m} (1 - q_k(x_i))^{1 - \sum_{j=1}^{k} \delta_{ij}} (q_k(x_i))^{\delta_{ik}} \quad (i = 1, 2, \ldots, n). \quad (2.4)$$
Convex optimization problem.

The function \( f \) makes binary classification in the reverse order from \( k = m \) to \( k = 1 \). From (2.2), (2.3) and (2.5), we obtain a compact formulation of the log likelihood function,

\[
L(b, W) = -\sum_{i=1}^{n} \left( \sum_{k=1}^{m} f(-w_k^T x_i + b_k) + \sum_{k=1}^{m} \delta_{ik} f(w_k^T x_i + b_k) \right) \tag{2.5}
\]

where

\[
f(v) = \log(1 + \exp(-v)). \tag{2.6}
\]

The function \( f(v) \) is called the logistic loss function. This function is convex because its second derivative always has a positive value. Hence, maximizing the log likelihood function \( L(b, W) \) is a convex optimization problem.

From (2.2), (2.3) and (2.5), we obtain a compact formulation of the log likelihood function,

\[
L(b, W) = -\sum_{i=1}^{n} \left( \sum_{k=1}^{m} f(-w_k^T x_i + b_k) + \sum_{k=1}^{m} \delta_{ik} f(w_k^T x_i + b_k) \right) = -\sum_{i=1}^{n} \sum_{k=1}^{m} \psi_{ik} f(\psi_{ik} (w_k^T x_i + b_k)),
\]

We refer to model (2.4) as a forward sequential logit model because binary classification models \( (2.1) \) are applied in order from \( k = 1 \) to \( k = m \). We can also consider the backward model that makes binary classification in the reverse order from \( k = m \) to \( k = 1 \). It is known that these two models do not produce the same results (see [32]).

The maximum likelihood estimation method estimates the parameters, \( b = (b_1, b_2, \ldots, b_m)^T \) and \( W = (w_1, w_2, \ldots, w_m) \), so that the log likelihood function, \( L(b, W) \), is maximized:

\[
\begin{align*}
L(b, W) &= \log \prod_{i=1}^{n} \prod_{k=1}^{m} (1 - q_k(x_i))^{1-\sum_{j=1}^{k} \delta_{ij}} (q_k(x_i))^{\delta_{ik}} \\
&= \sum_{i=1}^{n} \sum_{k=1}^{m} \left( 1 - \sum_{j=1}^{k} \delta_{ij} \right) \log(1 - q_k(x_i)) + \delta_{ik} \log(q_k(x_i)) \\
&= \sum_{i=1}^{n} \sum_{k=1}^{m} \left( 1 - \sum_{j=1}^{k} \delta_{ij} \right) \log \left( \frac{1}{1 + \exp(w_k^T x + b_k)} \right) + \delta_{ik} \log \left( \frac{1}{1 + \exp(-(w_k^T x + b_k))} \right) \\
&= -\sum_{i=1}^{n} \sum_{k=1}^{m} \left( 1 - \sum_{j=1}^{k} \delta_{ij} \right) f(-w_k^T x_i + b_k) + \sum_{k=1}^{m} \delta_{ik} f(w_k^T x_i + b_k) \tag{2.5}
\end{align*}
\]

where

\[
f(v) = \log(1 + \exp(-v)). \tag{2.6}
\]
where
\[
\psi_{ik} = \begin{cases} 
-1 & \text{if } k < y_i, \\
1 & \text{if } k = y_i, \\
0 & \text{otherwise}, 
\end{cases} 
\quad (i = 1, 2, \ldots, n; k = 1, 2, \ldots, m).
\]

3. Mixed Integer Optimization Formulations for Feature Subset Selection

This section presents mixed integer optimization (MIO) formulations for feature subset selection in the sequential logit model.

3.1. Mixed integer nonlinear optimization formulation

Similarly to the previous research [6, 18, 27, 34], we shall employ information criteria, e.g., the Akaike information criterion (AIC, [4]) and Bayesian information criterion (BIC, [31]), as a goodness-of-fit measure for the sequential logit model.

Let \( S \subseteq \{1, 2, \ldots, p\} \) be a set of selected features. Accordingly, by setting the coefficients of other candidate features to zero, most information criteria can be expressed as follows:
\[
-2 \max \{L(b, W) \mid w_{jk} = 0 \ (j \notin S; k = 1, 2, \ldots, m)\} + Fm(|S| + 1),
\]
where \( F \) is a penalty for the number of selected features. For instance, \( F = 2 \) and \( F = \log(n) \) correspond to the AIC and BIC, respectively.

Let \( z = (z_1, z_2, \ldots, z_p)^\top \) be a vector of 0-1 decision variables; \( z_j = 1 \) if \( j \in S; z_j = 0 \), otherwise. The feature subset selection problem for minimizing the information criterion (3.1) of the sequential logit model can be formulated as a mixed integer nonlinear optimization (MINLO) problem,
\[
\begin{align*}
\text{minimize} & \quad 2 \sum_{i=1}^n \sum_{k=1}^m |\psi_{ik}| f(\psi_{ik}(w_k^\top x_i + b_k)) + Fm \left( \sum_{j=1}^p z_j + 1 \right) \\
\text{subject to} & \quad z_{j} = 0 \Rightarrow w_{jk} = 0 \quad (j = 1, 2, \ldots, p; k = 1, 2, \ldots, m), \\
& \quad z_{j} \in \{0, 1\} \quad (j = 1, 2, \ldots, p).
\end{align*}
\]

The logical implications (3.3) can be represented by a special ordered set type one (SOS1) constraint [8, 9]. This constraint implies that not more than one element in the set can have a nonzero value, and it is supported by standard MIO software. Therefore, to incorporate the logical implications (3.3), it is only necessary to impose the SOS1 constraint on \( \{1 - z_j, w_{jk}\} \ (j = 1, 2, \ldots, p; k = 1, 2, \ldots, m) \). Indeed, if \( z_j = 0 \), then \( 1 - z_j \) has a nonzero value, and \( w_{jk} \) must be zero from the SOS1 constraints.

3.2. Quadratic approximation

The objective function (3.2) to be minimized is a convex but nonlinear function, which may cause numerical instabilities in the computation. Moreover, most MIO software cannot handle such a nonlinear objective function. In view of these facts, Tanaka and Nakagawa [32] used a quadratic approximation of the logistic loss function.

The second-order Maclaurin series of the logistic loss function (2.6) is written as follows:
\[
f(v) \approx \frac{v^2}{8} - \frac{v}{2} + \log 2.
\]
Figure 2: Logistic loss function and its quadratic approximation

This quadratic approximation of the logistic loss function reduces the MINLO problem (3.2)–(3.4) to a mixed integer quadratic optimization (MIQO) problem,

\[
\begin{align*}
\text{minimize} \quad & 2 \sum_{i=1}^{n} \sum_{k=1}^{m} |\psi_{ik}| \left( \frac{\psi_{ik}^2 (w_k^\top x_i + b_k)^2}{8} - \frac{\psi_{ik} (w_k^\top x_i + b_k)}{2} + \log 2 \right) \\
& + F_m \left( \sum_{j=1}^{p} z_j + 1 \right) \\
\text{subject to} \quad & z_j = 0 \Rightarrow w_{jk} = 0 \quad (j = 1, 2, \ldots, p; k = 1, 2, \ldots, m), \\
& z_j \in \{0, 1\} \quad (j = 1, 2, \ldots, p).
\end{align*}
\]

Figure 2 shows the graphs of the logistic loss function (2.6) (dashed curve) and its quadratic approximation (3.5) (solid curve). We can see that the approximation error sharply increases with distance from \( v = 0 \). More importantly, the quadratic approximation function increases on the right side, while the logistic loss function monotonically decreases so that it reduces penalties on correctly classified samples. This means that the quadratic approximation imposes large penalties on correctly classified samples. Consequently, the MIQO problem (3.6)–(3.8) may fail to find a good subset of features.

### 3.3. Piecewise-linear approximation

In order to approximate the logistic loss function more accurately, we propose the use of a piecewise-linear approximation instead of a quadratic approximation.

By following Sato et al. [29], we make a piecewise-linear approximation of the logistic loss function. Let \( V = \{v_1, v_2, \ldots, v_h\} \) be a set of \( h \) discrete points. Since the graph of a convex function lies above its tangent lines, the logistic loss function (2.6) can be approximated by the
Figure 3: Logistic loss function and its tangent lines

Pointwise maximum of a family of tangent lines, that is,

\[
f(v) \approx \max\left\{ f'(v_\ell)(v - v_\ell) + f(v_\ell) \mid \ell = 1, 2, \ldots, h \right\}
= \min\{t \mid t \geq f'(v_\ell)(v - v_\ell) + f(v_\ell) \quad (\ell = 1, 2, \ldots, h)\}.
\]

Figure 3 shows the graph of the logistic loss function \(2.6\) (dashed curve) together with the tangent lines (solid lines) at \(v_1 = -\infty, v_2 = -1, v_3 = 1,\) and \(v_4 = \infty\). Also note that

\[
f'(v_1)(v - v_1) + f(v_1) = -v,
\]
\[
f'(v_4)(v - v_4) + f(v_4) = 0.
\]

As shown in Figure 3, the pointwise maximum of the four tangent lines creates a piecewise-linear underestimator of the logistic loss function. It is clear that this approach approximates the logistic loss function more accurately than the quadratic approximation approach does.

By utilizing a piecewise-linear approximation of the logistic loss function, the feature subset selection problem for the sequential logit model can be posed as a mixed integer linear optimization (MILO) problem,

\[
\begin{align*}
\text{minimize} & \quad 2 \sum_{i=1}^{n} \sum_{k=1}^{m} |\psi_{ik}| t_{ik} + Fm \left( \sum_{j=1}^{p} z_j + 1 \right) \\
\text{subject to} & \quad t_{ik} \geq f'(v_\ell)(\psi_{ik}(w_k^\top x_i + b_k) - v_\ell) + f(v_\ell) \\
& \quad (i = 1, 2, \ldots, n; k = 1, 2, \ldots, m; \ell = 1, 2, \ldots, h), \\
& \quad z_j = 0 \Rightarrow w_{jk} = 0 \quad (j = 1, 2, \ldots, p; k = 1, 2, \ldots, m), \\
& \quad z_j \in \{0, 1\} \quad (j = 1, 2, \ldots, p),
\end{align*}
\]

where \(T = (t_{ik}; i = 1, 2, \ldots, n, k = 1, 2, \ldots, m)\) is a decision variable for calculating the value of piecewise-linear approximation function.
Table 1: List of instances

| abbreviation | n  | p  | #class | original dataset |
|--------------|----|----|--------|------------------|
| Wine-R       | 1599 | 11 | 6      | Wine Quality (red wine) |
| Wine-W       | 4898 | 11 | 7      | Wine Quality (white wine) |
| Skill        | 3338 | 18 | 7      | SkillCraft1 Master Table Dataset |
| Choice       | 1474 | 21 | 3      | Contraceptive Method Choice |
| Tnns-W       | 118  | 31 | 7      | Tennis Major Tournament (Wimbledon-women) |
| Tnns-M       | 113  | 33 | 7      | Tennis Major Tournament (Wimbledon-men) |
| Stdnt-M      | 395  | 40 | 18     | Student Performance (mathematics) |
| Stdnt-P      | 649  | 40 | 17     | Student Performance (Portuguese language) |

This MILO problem approaches the original MINLO problem \( (3.2) - (3.4) \) by increasing the number of tangent lines at appropriate points. Moreover, this MILO problem, as well as the MIQO problem \( (3.6) - (3.8) \), can be solved with standard mathematical optimization software.

4. Computational Results

This section compares the effectiveness of our piecewise-linear approximation approach with that of the quadratic approximation approach employed by Tanaka and Nakagawa \[32\].

We downloaded eight datasets for ordinal classification from the UCI Machine Learning Repository \[7\]. Table 1 lists these instances, where \( n \) and \( p \) are the number of samples and number of candidate features, respectively; and \#class is the number of ordinal class labels, i.e., \( m + 1 \).

For all the instances, each integer and real variable was standardized so that its mean was zero and its standard deviation was one. Each categorical variable was transformed into dummy variable(s). Variables having missing values for samples of over 10% were eliminated. After that, samples including missing values were all eliminated. In the Tnns-W and Tnns-M instances, the variables “Player 1” and “Player 2” were removed because they are not suitable for prediction purposes.

The computational experiments compared the performances of the following methods:

**Quad** MIQO formulation \( (3.6) - (3.8) \) based on quadratic approximation,

**PWL** MILO formulation \( (3.9) - (3.12) \) based on piecewise-linear approximation using the following set of points for tangent lines, similarly to Sato et al. \[29\]:

\[
V = \{ 0, \pm 0.44, \pm 0.89, \pm 1.37, \pm 1.90, \pm 2.63, \pm 3.55, \pm 5.16, \pm \infty \} \quad (|V| = 17).
\]

All computations were performed on a Linux computer with an Intel Core i7-4820 CPU (3.70 GHz) and 32 GB memory. Gurobi Optimizer 6.0.0 [http://www.gurobi.com](http://www.gurobi.com) was used to solve the MILO and MIQO problems. Here, the logical implications \( (3.7) \) and \( (3.11) \) were represented by the \texttt{SOS\_TYPE1} function in Gurobi Optimizer.

Tables 2–5 show the computational results of minimizing AIC/BIC in the forward/backward sequential logit models. The columns labeled “AIC” and “BIC” are the values of the corresponding information criteria calculated from the selected set of features. Note that the smaller of the AIC/BIC values between Quad and PWL are bold-faced for each instance. The column labeled “objval” is the value of the objective function, i.e., \( (3.6) \) and \( (3.9) \). The column labeled “|S|” is the number of selected features, and the column labeled “time (s)” is computation time in seconds. The computation for solving the MILO/MIQO problems was terminated if it did not finish by
Tables 2 and 3 show the results of AIC minimization in the forward and backward sequential logit models. These tables reveal that AIC values of PWL were smaller than those of Quad for all the instances. The computation time of PWL was longer than that of Quad because the problem size of PWL is dependent on the number of samples (see the constraints (3.10)). Nevertheless, we itself after 10000 seconds. In this case, the tables show the result of the best solution obtained within 10000 seconds.

Table 2: AIC minimization in forward sequential logit model

| instance  | n   | p  | #class | method | AIC   | objval | |S| | time (s) |
|-----------|-----|----|--------|--------|-------|--------|---|----|---------|
| Wine-R    | 1599| 11 | 6      | Quad   | 3057.5| 4204.6 | 4  | 0.03 |
|           |     |    |        | PWL    | 3028.4| 3013.2 | 10 | 428.05 |
| Wine-W    | 4898| 11 | 7      | Quad   | 10859.6| 14343.0| 8  | 0.07 |
|           |     |    |        | PWL    | 10726.7| 10671.2| 9  | 1899.54 |
| Skill     | 3338| 18 | 7      | Quad   | 9108.8| 11289.1| 9  | 5.13 |
|           |     |    |        | PWL    | 9080.2| 8939.0 | 15 | >10000 |
| Choice    | 1474| 21 | 3      | Quad   | 2816.2| 2850.2 | 9  | 7.07 |
|           |     |    |        | PWL    | 2813.1| 2804.4 | 12 | 1632.37 |
| Tnns-W    | 118 | 31 | 7      | Quad   | 331.1 | 331.1  | 0  | >10000 |
|           |     |    |        | PWL    | 316.2 | 315.4  | 4  | >10000 |
| Tnns-M    | 113 | 33 | 7      | Quad   | 278.5 | 296.1  | 2  | >10000 |
|           |     |    |        | PWL    | 278.3 | 277.6  | 4  | >10000 |
| Stdnt-M   | 395 | 40 | 18     | Quad   | 1052.7| 2251.2 | 1  | >10000 |
|           |     |    |        | PWL    | 946.2 | 941.9  | 6  | >10000 |
| Stdnt-P   | 649 | 40 | 17     | Quad   | 1709.9| 3929.7 | 1  | >10000 |
|           |     |    |        | PWL    | 1653.6| 1645.2 | 7  | >10000 |

Table 3: AIC minimization in backward sequential logit model

| instance  | n   | p  | #class | method | AIC   | objval | |S| | time (s) |
|-----------|-----|----|--------|--------|-------|--------|---|----|---------|
| Wine-R    | 1599| 11 | 6      | Quad   | 3062.5| 4073.5 | 5  | 0.04 |
|           |     |    |        | PWL    | 3050.1| 3034.6 | 10 | 357.76 |
| Wine-W    | 4898| 11 | 7      | Quad   | 10811.6| 14697.4| 7  | 0.05 |
|           |     |    |        | PWL    | 10786.9| 10734.7| 9  | 1785.95 |
| Skill     | 3338| 18 | 7      | Quad   | 9024.5| 11089.6| 8  | 22.65 |
|           |     |    |        | PWL    | 8961.2| 8925.9 | 10 | >10000 |
| Choice    | 1474| 21 | 3      | Quad   | 2829.4| 2940.8 | 9  | 11.51 |
|           |     |    |        | PWL    | 2826.1| 2816.7 | 11 | 3513.40 |
| Tnns-W    | 118 | 31 | 7      | Quad   | 331.1 | 447.4  | 0  | 253.13 |
|           |     |    |        | PWL    | 327.0 | 307.5  | 8  | >10000 |
| Tnns-M    | 113 | 33 | 7      | Quad   | 307.4 | 419.8  | 0  | 9.53 |
|           |     |    |        | PWL    | 280.5 | 279.3  | 4  | >10000 |
| Stdnt-M   | 395 | 40 | 18     | Quad   | 1065.5| 2584.6 | 2  | >10000 |
|           |     |    |        | PWL    | 1044.1| 1038.1 | 2  | >10000 |
| Stdnt-P   | 649 | 40 | 17     | Quad   | 1640.7| 3532.1 | 2  | >10000 |
|           |     |    |        | PWL    | 1620.7| 1611.6 | 3  | >10000 |
The BIC minimization in forward sequential logit model is shown in Table 4:

| instance    | n   | p   | #class | method | BIC   | objval | |S| | time (s) |
|-------------|-----|-----|--------|--------|-------|--------|---|---|----------|
| Wine-R      | 1599| 11  | 6      | Quad   | 3191.9| 4339.1 | 4  | 0.05|
|             |     |     |        | PWL    | 3191.9| 3175.3 | 4  | 331.58|
| Wine-W      | 4898| 11  | 7      | Quad   | 11127.4| 14543.6| 3  | 0.05|
|             |     |     |        | PWL    | 11077.0| 11019.4| 4  | 6901.04|
| Skill       | 3338| 18  | 7      | Quad   | 9434.7 | 11518.1| 3  | 1.41|
|             |     |     |        | PWL    | 9333.3 | 9291.3 | 6  | >10000|
| Choice      | 1474| 21  | 3      | Quad   | 2903.4| 2932.4 | 5  | 3.00|
|             |     |     |        | PWL    | 2903.4| 2894.5 | 5  | 1372.28|
| Tnns-W      | 118 | 31  | 7      | Quad   | 347.7  | 347.7  | 0  | 99.67|
|             |     |     |        | PWL    | 347.7  | 345.9  | 0  | 833.61|
| Tnns-M      | 113 | 33  | 7      | Quad   | 323.7  | 323.7  | 0  | 96.83|
|             |     |     |        | PWL    | 323.7  | 321.3  | 0  | 514.01|
| Stdnt-M     | 395 | 40  | 18     | Quad   | 1187.9 | 2386.4 | 1  | 220.77|
|             |     |     |        | PWL    | 1181.7 | 1175.9 | 2  | >10000|
| Stdnt-P     | 649 | 40  | 17     | Quad   | 1853.2 | 4073.0 | 1  | 725.73|
|             |     |     |        | PWL    | 1853.2 | 1835.3 | 1  | >10000|

The BIC minimization in backward sequential logit model is shown in Table 5:

| instance    | n   | p   | #class | method | BIC   | objval | |S| | time (s) |
|-------------|-----|-----|--------|--------|-------|--------|---|---|----------|
| Wine-R      | 1599| 11  | 6      | Quad   | 3257.3| 4197.6 | 2  | 0.02|
|             |     |     |        | PWL    | 3206.5| 3190.6 | 4  | 296.37|
| Wine-W      | 4898| 11  | 7      | Quad   | 11151.9| 14907.6| 3  | 0.03|
|             |     |     |        | PWL    | 11143.2| 11083.8| 4  | 8233.98|
| Skill       | 3338| 18  | 7      | Quad   | 9425.3 | 13155.8| 4  | 3.47|
|             |     |     |        | PWL    | 9275.4 | 9241.3 | 6  | >10000|
| Choice      | 1474| 21  | 3      | Quad   | 2917.3 | 3009.5 | 4  | 1.86|
|             |     |     |        | PWL    | 2917.0 | 2907.5 | 5  | 1601.25|
| Tnns-W      | 118 | 31  | 7      | Quad   | 347.7  | 464.0  | 0  | 0.10|
|             |     |     |        | PWL    | 347.7  | 344.1  | 0  | 3205.70|
| Tnns-M      | 113 | 33  | 7      | Quad   | 323.7  | 436.1  | 0  | 0.09|
|             |     |     |        | PWL    | 323.7  | 319.9  | 0  | 9767.70|
| Stdnt-M     | 395 | 40  | 18     | Quad   | 1207.0 | 2740.4 | 1  | 421.56|
|             |     |     |        | PWL    | 1207.0 | 1199.8 | 1  | >10000|
| Stdnt-P     | 649 | 40  | 17     | Quad   | 1822.4 | 3717.1 | 1  | >10000|
|             |     |     |        | PWL    | 1822.4 | 1809.3 | 1  | >10000|

We should notice that PWL provided better-quality solutions than Quad did in spite of the time limit of 10000 seconds. In addition, the number of features selected by PWL sometimes differed greatly from that selected by Quad; for instance, Quad and PWL respectively selected 4 and 10 features for Wine-R in Table 4.

We can see from Tables 2 and 3 that PWL approximated the logistic loss function very accurately, whereas Quad caused a major gap between AIC and objval. Additionally, since the
objective function of PWL is an underestimator relative to AIC, its optimal value serves as a lower bound of the smallest AIC. In the case of Wine-R in Table 2, AIC and objval of Quad were 3057.5 and 4204.6, whereas those of PWL were 3028.4 and 3013.2. This also implies that PWL found a subset of features such that the associated AIC value was 3028.4, and it collateral guaranteed that the smallest AIC value was greater than 3013.2. This optimality guarantee is the most notable characteristic of our piecewise-linear approximation approach, and it cannot be shared by the quadratic approximation approach.

Tables 4 and 5 show the results of BIC minimization in the forward and backward sequential logit models. We should recall that the penalty, \( F \), for the number of selected features in BIC (i.e., \( F = \log(n) \)) is larger than that in AIC (i.e., \( F = 2 \)). Hence, BIC minimization selected a small number of features, and accordingly, Quad and PWL often yielded the same subset of features for each instance in Tables 4 and 5. Meanwhile, when these two methods provided different subsets of features, PWL always found the better one.

5. Conclusions
This paper dealt with the feature subset selection problem for a sequential logit model. We formulated it as a mixed integer linear optimization (MILO) problem by applying a piecewise-linear approximation to the logistic loss functions. The computational results confirmed that our formulation has a clear advantage over the mixed integer quadratic optimization (MIQO) formulation proposed in the previous study [32].

In contrast to the MIQO formulation, the approximation accuracy of the logistic loss function can be controlled by the number of tangent lines in our MILO formulation. Furthermore, after the MILO problem is solved, it provides an optimality guarantee of the selected features on the basis of information criteria. To the best of our knowledge, this paper is the first to compute a subset of features with an optimality guarantee for a sequential logit model.

A future direction of study will be to extend our piecewise-linear approximation approach to other logit models. However, this will be a difficult task because it is imperative to approximate a multivariate objective function. Another direction of future research is to analyze actual data by means of our feature subset selection method. For instance, Sato et al. [30] investigated consumers’ store choice behavior by applying feature subset selection based on mixed integer optimization. Since proper feature subset selection is essential for data analysis, our approach has a clear advantage over heuristic algorithms.

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