CONNECTED HOPF ALGEBRAS OF DIMENSION $p^2$

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Abstract. Let $H$ be a finite-dimensional connected Hopf algebra over an algebraically closed field $\mathbb{k}$ of characteristic $p > 0$. We provide the algebra structure of the associated graded Hopf algebra $gr H$. Then, we study the case when $H$ is generated by a Hopf subalgebra $K$ and another element and the case when $H$ is cocommutative. When $H$ is a restricted universal enveloping algebra, we give a specific basis for the second term of the Hochschild cohomology of the coalgebra $H$ with coefficients in the trivial $H$-bicomodule $\mathbb{k}$. Finally, we classify all connected Hopf algebras of dimension $p^2$ over $\mathbb{k}$.

1. Introduction

Let $\mathbb{k}$ denote a base field, algebraically closed of characteristic $p > 0$. In [5], all graded cocommutative connected Hopf algebras of dimension less than or equal to $p^3$ are classified by using W.M. Singer’s theory of extensions of connected Hopf algebras [13]. In this paper, we classify all connected Hopf algebras of dimension $p^2$ over $\mathbb{k}$. We use the theories of restricted Lie algebras and Hochschild cohomology of coalgebras for restricted universal enveloping algebras.

Let $H$ denote a finite-dimensional connected Hopf algebra in the sense of [9, Def. 5.1.5] with primitive space $P(H)$, and $K$ be a Hopf subalgebra of $H$. In Section 2, basic definitions related to and properties of $H$ are briefly reviewed. In particular, we describe a few concepts concerning the inclusion $K \subseteq H$. We say that the $p$-index of $K$ in $H$ is $n - m$ if $\dim K = p^m$ and $\dim H = p^n$. The notion of the first order of the inclusion and a level-one inclusion are also given in Definition 2.3.

In Section 3, the algebra structure of a finite-dimensional connected coradically graded Hopf algebra is obtained (Theorem 3.1) based on a result for algebras representing finite connected
group schemes over $k$. It implies that the associated graded Hopf algebra $\text{gr} H$ is isomorphic to as algebras

$$k[x_1, x_2, \cdots, x_d] / (x_1^p, x_2^p, \cdots, x_d^p)$$

for some $d \geq 0$.

Section 4 concerns a simple case when $H$ is generated by $K$ and another element $x$. Suppose the $p$-index of $K$ in $H$ is $d$. Under an additional assumption, the basis of $H$ as a left $K$-module is given in terms of the powers of $x$ (Theorem 4.5). Moreover, if $K$ is normal in $H$ [9, Def. 3.4.1], then $x$ satisfies a polynomial equation as follows:

$$x^{pd} + \sum_{i=0}^{d-1} a_i x^{p^i} + b = 0$$

for some $a_i \in k$ and $b \in K$.

Section 5 deals with the special case when $H$ is cocommutative. It is proved in Proposition 5.2 that such Hopf algebra $H$ is equipped with a series of normal Hopf subalgebras $k = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_n = H$ satisfying certain properties. If we apply these properties to the case when $P(H)$ is one-dimensional, then we have $N_1$ is generated by $P(H)$ and each $N_i$ has $p$-index one in $N_{i+1}$ (Corollary 5.3). In Theorem 5.4 we give locality criterion for $H$ in terms of its primitive elements. This result, after dualization, is equivalent to a criteria for unipotency of finite connected group schemes over $k$, as shown in Remark 5.5.

In section 6, we take the Hopf subalgebra $K = u(g)$, the restricted universal enveloping algebra of some finite-dimensional restricted Lie algebra $g$. We consider the Hochschild cohomology of the coalgebra $K$ with coefficients in the trivial bicomodule $k$, namely $H^\bullet(k, K)$. Then the Hochschild cohomology can be computed as the homology of the cobar construction of $K$. In Proposition 6.2 we give a specific basis for $H^2(k, K)$. We further show, in Lemma 6.5 that $\bigoplus_{n \geq 0} H^n(k, K)$ is a graded restricted $g$-module via the adjoint map. When the inclusion $K \subseteq H$ has first order $n \geq 2$, the differential $d^1$ in the cobar construction of $H$ induces a restricted $g$-module map from $H_n$ into $H^2(k, K)$, whose kernel is $K_n$ (Theorem 6.6). Concluded in Theorem 6.7 if $K \neq H$, we can find some $x \in H \setminus K$ with the following comultiplication

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \omega \left( \sum_i \alpha_i x_i \right) + \sum_{j<k} \alpha_{jk} x_j \otimes x_k$$

where $\{x_i\}$ is a basis for $g$. 
Finally, the classification of connected Hopf algebras of dimension \( p^2 \) over \( k \) is accomplished in section 7. Assume \( \dim H = p^2 \). We apply results on \( H \) from previous sections, i.e., Corollary 5.3 and Theorem 6.7. The main result is stated in Theorem 7.4 and divided into two cases. When \( \dim P(H) = 2 \), based on the classification of two-dimensional Lie algebras with restricted maps (see Appendix A), there are five non-isomorphic classes

1. \( k[x, y] / (x^p, y^p), \)
2. \( k[x, y] / (x^p - x, y^p), \)
3. \( k[x, y] / (x^p - y, y^p), \)
4. \( k[x, y] / (x^p - x, y^p - y), \)
5. \( k[x, y] / ((x, y) - y, x^p - x, y^p), \)

where \( x, y \) are primitive. When \( \dim P(H) = 1 \), \( H \) must be commutative and there are three non-isomorphic classes

6. \( k[x, y] / (x^p, y^p), \)
7. \( k[x, y] / (x^p, y^p - x), \)
8. \( k[x, y] / (x^p - x, y^p - y), \)

where \( \Delta (x) = x \otimes 1 + 1 \otimes x \) and \( \Delta (y) = y \otimes 1 + 1 \otimes y + \omega(x) \). Moreover, all local Hopf algebras of dimension \( p^2 \) over \( k \) are classified by duality, see Corollary 7.6.

2. Preliminaries

Throughout this paper, \( k \) denotes a base field, algebraically closed of characteristic \( p > 0 \). All vector spaces, algebras, coalgebras, and tensor products are taken over \( k \) unless otherwise stated. Also, \( V^* \) denotes the vector space dual of any vector space \( V \).

For any coalgebra \( C \), the coradical \( C_0 \) is defined to be the sum of all simple subcoalgebras of \( C \). Following [13, 5.2.1], \( \{C_n\}_{n=0}^\infty \) is used to denote the coradical filtration of \( C \). If \( C_0 \) is one-dimensional, \( C \) is called connected. If every simple subcoalgebra of \( C \) is one-dimensional, \( C \) is called pointed. Let \( (C, \Delta, \varepsilon) \) be a pointed coalgebra, and \( (M, \rho_l, \rho_r) \) be a \( C \)-bicomodule via the structure maps \( \rho_l : M \to C \otimes M \) and \( \rho_r : M \to M \otimes C \). We denote the identity map of \( C^\otimes n \) by \( I_n \) and \( C^\otimes 0 = k \). The Hochschild cohomology \( H^* (M, C) \) of \( C \) with coefficients in \( M \) is defined by the homology of the complex \( (C^n(M, C), d^n) \), where \( C^n(M, C) = \text{Hom}_k (M, C^\otimes n) \) and

\[
d^n(f) = -(1 \otimes f)\rho_l - (\Delta \otimes I_{n-1})f + \cdots + (-1)^{n}(I_{n-1} \otimes \Delta)f + (-1)^{n+1}(f \otimes I)\rho_r.
\]
For any Hopf algebra $H$, we use $P(H)$ to indicate the subspace of primitive elements. Following the terminology in [2, Def. 1.13], we recall the definition of graded Hopf algebras.

**Definition 2.1.** Let $H$ be a Hopf algebra with antipode $S$. If

1. $H = \bigoplus_{n=0}^{\infty} H(n)$ is a graded algebra,
2. $H = \bigoplus_{n=0}^{\infty} H(n)$ is a graded coalgebra,
3. $S(H(n)) \subseteq H(n)$ for any $n \geq 0$,

then $H$ is called a graded Hopf algebra. If in addition,

4. $H = \bigoplus_{n=0}^{\infty} H(n)$ is a coradically graded coalgebra,

then $H$ is called a coradically graded Hopf algebra. Also, the associated graded Hopf algebra of $H$ is defined by $\text{gr}H = \bigoplus_{n \geq 0} H_n / H_{n-1}$ (where $H_{-1} = 0$) with respect to its coradical filtration.

There are some basic properties of finite-dimensional Hopf algebras, which we use frequently.

**Proposition 2.2.** Let $H$ be a finite-dimensional Hopf algebra.

1. $H$ is local if and only if $H^*$ is connected.
2. If $H$ is local, then any quotient or Hopf subalgebra of $H$ is local.

Furthermore assume that $H$ is connected. Denote by $u(P(H))$ the restricted universal enveloping algebra of $P(H)$.

3. Any quotient or Hopf subalgebra of $H$ is connected.
4. $\dim P(H) = \dim J / J^2$, where $J$ is the Jacobson radical of $H^*$.
5. $H$ is primitively generated if and only if $H \cong u(P(H))$.
6. $\dim u(P(H)) = p^{\dim P(H)}$.
7. $\dim H = p^n$ for some integer $n$.

**Proof.** (1) and (4) are derived from [9, Prop. 5.2.9].

For (3) assume $H$ is connected, $H/I$ is connected by [9, Cor. 5.3.5], where $I$ is any Hopf ideal of $H$. And for any Hopf subalgebra $K$ of $H$, by [9, Lemma 5.2.12], $K_0 = K \cap H_0$. Since $H_0$ is one-dimensional, so is $K_0$. Thus $K$ is connected.

(2) is the dual version of (3) by (1).

(5) is a standard result from [12, Prop. 13.2.3] and (6) comes from [9, P. 23].
(7) is true because the associated graded ring $\text{gr}_J(H^*)$ with respect to its $J$-adic filtration is connected and primitively generated. Hence $\dim H = \dim H^* = \dim \text{gr}_J(H^*) = p^n$, where $n = \dim \text{P}(\text{gr}_J(H^*))$ by (6).

\textbf{Definition 2.3.} Consider an inclusion of finite-dimensional connected Hopf algebras $K \subseteq H$.

1. If $\dim K = p^m$ and $\dim H = p^n$, then the \textit{p-index} of $K$ in $H$ is defined to be $n - m$.

2. The \textbf{first order} of the inclusion is defined to be the minimal integer $n$ such that $K^n \subseteq H_n$.

And we say it is infinity if $K = H$.

3. The inclusion is said to be \textbf{level-one} if $H$ is generated by $H_n$ as an algebra, where $n$ is the first order of the inclusion.

4. The inclusion is said to be \textbf{normal} if $K$ is a normal Hopf subalgebra of $H$.

\textbf{Remark 2.4.} By [9, Lemma 5.2.12], if $D$ is a subcoalgebra of $C$, we have $D_n = D \cap C_n \subseteq C_n$.

Also the coradical filtration is exhaustive for any coalgebra by [9, Thm. 5.2.2]. As a result of [9, Lemma 5.2.10], a connected bialgebra is automatically a connected Hopf algebra. Furthermore, it is well known that any sub-bialgebra of a connected Hopf algebra is a Hopf subalgebra. Let $H$ be a connected Hopf algebra. Then the algebra generated by each term of the coradical filtration $H_n$ is a connected Hopf subalgebra of $H$. Because each term of the coradical filtration $H_n$ is a subcoalgebra and the algebra generated by it is certainly a sub-bialgebra. Throughout the whole paper we will use the following convention:

\textbf{Convention 2.5.} Define the expression $\omega(x) = \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} x^i \otimes x^{p-i}$, where $\frac{(p-1)!}{i!(p-i)!} \in k$ for each $1 \leq i \leq p - 1$.

3. \textbf{Associated graded Hopf algebras for finite-dimensional connected Hopf algebras}

\textbf{Theorem 3.1.} Let $H = \bigoplus_{n=0}^{\infty} H(n)$ be a finite-dimensional connected coradically graded Hopf algebra. Then $H$ is isomorphic to $k[x_1, x_2, \ldots, x_d]/(x_1^p, x_2^p, \ldots, x_d^p)$ for some $d \geq 0$ as algebras.

\textit{Proof.} Denote by $K = \bigoplus_{n=0}^{\infty} H(n)^*$ the graded dual of $H$. It is a graded Hopf algebra and connected for $K_0 \subseteq K(0) = H(0)^* = k$ by [9] Lemma 5.3.4]. Moreover since $H$ is coradically graded, by [11, Lemma 5.5], $K$ is generated in degree one and hence cocommutative. Therefore by duality $H$ is commutative and local. Then according to [15, Thm. 14.4], $H$ is isomorphic to
\[ k[x_1, x_2, \cdots, x_d]/(x_1^{p_1}, x_2^{p_2}, \cdots, x_d^{p_d}) \] for some \( d \geq 0 \) as an algebra. Thus it suffices to prove inductively that for any homogeneous element \( x \in H(n) \), we have \( x^p = 0 \) for all \( n \geq 1 \). Since \( H \) is coradically graded, \( P(H) = H(1) \). Then for any \( x \in H(1) \), we have \( x^p \in (H(1))^p \cap H(1) \subseteq H(p) \cap H(1) = 0 \). Assume the assertion holds for \( n \leq m - 1 \). Let \( x \in H(m) \). By the definition of graded Hopf algebras we have:

\[
\Delta(x) = x \otimes 1 + 1 \otimes x + \sum_{i=1}^{m-1} y_i \otimes z_{m-i},
\]

where \( y_i, z_i \in H(i) \) for all \( 1 \leq i \leq m - 1 \). Therefore \( \Delta(x^p) = x^p \otimes 1 + 1 \otimes x^p + \sum_{i=1}^{m-1} y_i^p \otimes z_{m-i}^p = x^p \otimes 1 + 1 \otimes x^p \) by induction. Thus \( x^p \in (H(m))^p \cap H(1) \subseteq H(pm) \cap H(1) = 0 \). □

**Corollary 3.2.** The associated graded Hopf algebra of a finite-dimensional connected Hopf algebra is isomorphic to \( k[x_1, x_2, \cdots, x_d]/(x_1^{p_1}, x_2^{p_2}, \cdots, x_d^{p_d}) \) for some \( d \geq 0 \) as algebras.

**Proof.** The associated graded space \( \text{gr}H = \bigoplus_{n \geq 0} H_n/H_{n-1} \) is a graded Hopf algebra by [9, P. 62]. Also mentioned in [2, Def. 1.13], \( \text{gr}H \) is coradically graded. Therefore \( \text{gr}H \) is a coradically graded Hopf algebra, which is clearly connected because \( H \) is connected. Hence \( \text{gr}H \) satisfies all the conditions in Theorem 3.1 and the result follows. □

As a consequence of the commutativity of the associated graded Hopf algebra for any finite-dimensional connected Hopf algebra we conclude that:

**Corollary 3.3.** Let \( H \) be a finite-dimensional connected Hopf algebra. Then \( [H_n, H_m] \subseteq H_{n+m-1} \) for all integers \( n, m \).

### 4. Finite-dimensional connected Hopf algebras with Hopf subalgebras

In this section, we always assume \( K \subseteq H \) is an inclusion of finite-dimensional connected Hopf algebras.

**Lemma 4.1.** Suppose the inclusion \( K \subseteq H \) has first order \( n \). Then the \( p \)-index of \( K \) in \( H \) is greater or equal to \( \dim(H_n/K_n) \).

**Proof.** By Remark 2.4 the inclusion \( K \hookrightarrow H \) induces an injection \( K_i/K_{i-1} \hookrightarrow H_i/H_{i-1} \) for all \( i \geq 1 \). Thus \( \text{gr}K = \bigoplus_{i \geq 0} K(i) \hookrightarrow \text{gr}H = \bigoplus_{i \geq 0} H(i) \) and \( K(i) = H(i) \) for all \( 0 \leq i \leq n - 1 \) since \( n \) is the first order of the inclusion. Moreover by [2, Def. 1.13], \( (\text{gr}H)_m = \bigoplus_{0 \leq i \leq m} H(m) \) for all
m \geq 0 and the same is true for \text{gr}K. Therefore it is enough to prove the result in the associated graded Hopf algebras inclusion \text{gr}K \subseteq \text{gr}H.

For simplicity, we write \(K\) for \(\text{gr}K\), \(H\) for \(\text{gr}H\) and use \(d(H/K)\) to denote the \(p\)-index of \(K\) in \(H\). We will prove the result by induction on \(\dim(H_n/K_n)\). When \(\dim(H_n/K_n) = 1\), it is trivial. Now suppose that \(\dim(H_n/K_n) = n\) and choose any \(x \in H(n) \setminus K(n)\). Because \(H\) is a graded coalgebra,\[
\Delta(x) = x \otimes 1 + 1 \otimes x + \sum_{i=1}^{n-1} y_i \otimes z_{n-i},
\]
where \(y_i, z_i \in H(i) = K(i)\) for all \(1 \leq i \leq n-1\). Hence \(K\) and \(x\) generate a Hopf subalgebra of \(H\) by Remark 2.4, which we denote as \(L\). Now according to Theorem 3.1, we have \(x^p = 0\). Thus \(K \subseteq L\) has \(p\)-index one and first order \(n\). Because \(H\) is a graded algebra, it is clear that \(L_n\) is spanned by \(K_n\) and \(x\). Hence \(\dim(L_n/K_n) = 1\) and \(\dim(H_n/L_n) = \dim(H_n/K_n) - 1\). Therefore by induction we have \[
\dim(H_n/K_n) = \dim(H_n/L_n) + \dim(L_n/K_n) = \dim(H_n/L_n) + 1 \leq d(H/L) + 1 = d(H/L) + d(L/K) = d(H/K).
\]

\[
\square
\]

**Lemma 4.2.** Let \(K \subseteq H\) be a level-one inclusion with first order \(n\). Then \(K\) is normal in \(H\) if and only if \([K, H_n] \subseteq K\).

**Proof.** First suppose that \(K\) is normal in \(H\). By [9 Lemma 5.3.2] for any \(x \in H_n\), \(\Delta(x) = x \otimes 1 - 1 \otimes x \in H_{n-1} \otimes H_{n-1} = K_{n-1} \otimes K_{n-1} \subseteq K \otimes K\). Thus we can write \(\Delta(x) = x \otimes 1 + 1 \otimes x + \sum a_i \otimes b_i\) where \(a_i, b_i \in K\). Apply the antipode \(S\) to get \[
S(x) = \varepsilon(x) - x - \sum a_i S(b_i).
\]

By the definition of normal Hopf subalgebras [9 Def. 3.4.1], for any \(y \in K\)
\[
\sum x_1 y S(x_2) = xy + y S(x) + \sum a_i y S(b_i) = u \in K.
\]

Therefore \[
[y, x] = yx - xy = y \left(\varepsilon(x) - \sum a_i S(b_i)\right) + \sum a_i y S(b_i) - u \subseteq K,
\]
which shows that \([K, H_n] \subseteq K\). Conversely suppose that \([K, H_n] \subseteq K\). Then it is clear that 
\(K^+ H_n \subseteq H_n K^+ + K^+ \subseteq HK^+\) since \([K^+, H_n] \subseteq K\). We claim that 
\(K^+ (H_n)^i \subseteq HK^+\) for all \(i \geq 0\) by induction. Suppose the inclusion holds for \(i\) and then for \(i + 1\):
\[
K^+ (H_n)^{i+1} = K^+ (H_n)^i H_n \subseteq (HK^+) H_n \subseteq H (HK^+) \subseteq HK^+.
\]
Therefore \(K^+ H = \bigcup K^+ (H_n)^i \subseteq HK^+\) and by symmetry \(K^+ H = HK^+\). According to [9, Cor. 3.4.4], \(K\) is normal.

\[\]

**Lemma 4.3.** If \(x \in H\) satisfies \([K, x] \subseteq K\) and \(\Delta(x) - x \otimes 1 - 1 \otimes x \in K \otimes K\), then \(\Delta(x^n) - x^n \otimes 1 - 1 \otimes x^n \in K \otimes K\) for all \(n \geq 0\).

**Proof.** First, we prove \(\Delta(x^n) - x^n \otimes 1 - 1 \otimes x^n \in K \otimes K\) for \(n \geq 1\). Denote \(\Delta(x) = x \otimes 1 + 1 \otimes x + u\), where \(u \in K \otimes K\). By Lemma A.1 we have:
\[
\Delta(x^n) = \left( x \otimes 1 + 1 \otimes x + u \right)^n = x^n \otimes 1 + 1 \otimes x^n + u^n + \sum_{i=1}^{p-1} S_i,
\]
where \(iS_i\) is the coefficient of \(\lambda^{i-1}\) in \(u (\text{ad}(\lambda u + x \otimes 1 + 1 \otimes x))^{p-1}\). Hence it suffices to show inductively that
\[
u (\text{ad}(\lambda u + x \otimes 1 + 1 \otimes x))^n \in (K \otimes K)[\lambda]
\]
for all \(n \geq 0\). Notice that when \(n = 0\), it is just the assumption. Suppose it’s true for \(n - 1\) then for \(n\)
\[
u (\text{ad}(\lambda u + x \otimes 1 + 1 \otimes x))^n \in ([K \otimes K][\lambda], \lambda u + x \otimes 1 + 1 \otimes x]
\]
\[
\subseteq \{[K \otimes K, u] + [K, x] \otimes K + K \otimes [K, x]\} [\lambda]
\]
\[
\subseteq (K \otimes K)[\lambda].
\]
Now replace \(x\) with \(x^{n-1}\) and we have \([K, x^{n-1}] = K (\text{ad}(x))^{p-1} \subseteq K\) by Lemma A.1. Then the other cases can be proved in the similar way. 

\[\]

**Lemma 4.4.** If \(x \in H\) satisfies \(\Delta(x) - x \otimes 1 - 1 \otimes x \in K \otimes K\) and \([K, x] \subseteq \sum_{0 \leq i \leq 1} K x^i\). For each \(n \geq 0\), set \(L_n = \sum_{i \leq n} K x^i\). Then we have the following:

1. \([K, x^n] \subseteq L_n\) and \(L_n\) is a \(K\)-bimodule via the multiplication in \(H\).
2. \(\Delta(x^n) - x^n \otimes 1 - 1 \otimes x^n \in L_{n-1} \otimes L_{n-1}\).
3. \(L_n\) is a subcoalgebra of \(H\).
(4) If $H$ is generated by $K$ and $x$ as an algebra, then $H = \bigcup_{n \geq 0} L_n$.

Proof. (1) Since $xL_n \subseteq L_{n+1}$, we have $x^n L_1 \subseteq L_{n+1}$ for all $n \geq 0$. By assumption, it holds that $[K, x] \subseteq L_1$. Suppose $[K, x^{n-1}] \subseteq L_{n-1}$. For any $a \in K$, it follows that

$$x^n a \in x^{n-1} (ax + L_1) \subseteq (ax^{n-1} + L_{n-1}) x + x^{n-1} L_1 \subseteq ax^n + L_n.$$ 

Hence $[K, x^n] \subseteq L_n$ for each $n \geq 0$. Moreover, we have $L_n K \subseteq L_n$ for each $n \geq 0$, the left $K$-module $L_n$ now becomes $K$-bimodule.

(2) Denote $\Delta(x) = x \otimes 1 + 1 \otimes x + u$, where $u \in K \otimes K$. We still prove by induction. When $n = 1$, it is just the assumption. Suppose it’s true for $n-1$. Write $\Delta(x^{n-1}) = x^{n-1} \otimes 1 + 1 \otimes x^{n-1} + \sum a_i \otimes b_i$, where $a_i, b_i \in L_{n-2}$. Therefore

$$\Delta(x^n) - x^n \otimes 1 - 1 \otimes x^n$$

$$= (x \otimes 1 + 1 \otimes x + u) \left( x^{n-1} \otimes 1 + 1 \otimes x^{n-1} + \sum a_i \otimes b_i \right) - x^n \otimes 1 - 1 \otimes x^n$$

$$\in x \otimes x^{n-1} + x^{n-1} \otimes x + xL_{n-2} \otimes L_{n-2} = L_{n-1} \otimes L_{n-1}$$

(3) Now because of (1) and (2), it is enough to check that $L_n$ is a coalgebra by induction.

(4) Furthermore if $H$ is generated by $K$ and $x$ as an algebra, it is easy to see $H = \bigcup_{n \geq 0} L_n$. □

**Theorem 4.5.** Let $H$ be a finite-dimensional connected Hopf algebra with Hopf subalgebra $K$. Suppose the $p$-index of $K$ in $H$ is $d$ and $H$ is generated by $K$ and some $x \in H$ as an algebra. Also assume that $\Delta(x) = x \otimes 1 + 1 \otimes x + u$, where $u \in K \otimes K$ and $[K, x] \subseteq \sum_{0 \leq i \leq d} Kx^i$. Then $H$ is a free left $K$-module such that $H = \bigoplus_{i=0}^{d-1} Kx^i$. Furthermore if $K$ is normal in $H$, then $x$ satisfies a polynomial equation as follows:

$$x^p + \sum_{i=0}^{d-1} a_i x^p + b = 0$$

for some $a_i \in k$ and $b \in K$.

Proof. Denote $L_n = \sum_{0 \leq i \leq n} Kx^i$ for all $n \geq 0$. By the Lemma 4.4(3), $L_n$ is a subcoalgebra. Also $H$ is a left $K$-module with generators $\{x^i | i \geq 0\}$ for $H = \sum Kx^i$. Because $H$ is finite-dimensional, there exist some nontrivial relations between the generators such as

$$d_m x^m + d_{m-1} x^{m-1} + \cdots + d_1 x + d_0 = 0,$$
where \( d_i \in K \) and \( d_m \neq 0 \), among which we choose the lowest degree in terms of \( x \), say degree \( m \). Furthermore denote \( D = K, L = L_{m-1}, F = x^m \) and \( V = \{ a \in D|aF \in L \} \). As a result of Lemma \( \cite{14}, \text{Lemma 1.1} \), we know \( \Delta(F) = x^m \otimes 1 + 1 \otimes x^m \in L \otimes L \). Then \( D, L, F \) satisfy all the conditions listed in \( \cite{14}, \text{Lemma 1.1} \). Hence \( V = D \) for \( 0 \neq d_m \in V \). Thus \( x^m \in \bigoplus_{i \leq n} K x^i \) and consequently \( H \) is a free left \( K \)-module with the free basis \( \{ x^i | 0 \leq i \leq m-1 \} \). Since \( \dim H = m \dim K \), it is easy to see \( m = p^d \) by definition.

Now assume that \( K \) is normal. Follow the proof in Lemma \( \cite{12} \) we can show that \( [K, x] \subseteq K \).

From previous discussion there exists a general equation for \( x \):

\[
 x^{p^d} + \sum_{i=0}^{p^d-1} a_i x^i = 0,
\]

where all \( a_i \in K \). According to Lemma \( \cite{13} \) we can write \( \Delta \left( x^{p^n} \right) = x^{p^n} \otimes 1 + 1 \otimes x^{p^n} + u_n \), where \( u_n \in K \otimes K \) for all \( n \geq 0 \). Now apply the comultiplication \( \Delta \) to the above identity (1) to get

\[
 x^{p^d} \otimes 1 + 1 \otimes x^{p^d} + u_d + \sum_{i=0}^{p^d-1} \Delta(a_i)(x \otimes 1 + 1 \otimes x + u)^i = 0.
\]

Replacing \( x^{p^d} \) with \( \left( -\sum_{i=0}^{p^d-1} a_i x_i \right) \), the following equation is straightforward:

\[
\left( -\sum_{i=0}^{p^d-1} a_i x_i \right) \otimes 1 + 1 \otimes \left( -\sum_{i=0}^{p^d-1} a_i x_i \right) + \sum_{i=0}^{d-1} \Delta(a_{p^i}) \left( x^{p^i} \otimes 1 + 1 \otimes x^{p^i} + u_i \right) + \sum_{i \in S} \Delta(a_i)(x \otimes 1 + 1 \otimes x + u)^i + \Delta(a_0) + u_d = 0
\]

with the \( p \)-index set \( S = \{1, 2, \cdots, p^d\} \setminus \{1, p, p^2, \cdots, p^d\} \).

We first prove that \( a_i = 0 \) for all \( i \in S \) by contradiction. If not, suppose \( n \in S \) is the largest integer such that \( a_n \neq 0 \). The free \( K \)-module structure for \( H \) implies that the \( K \otimes K \)-module \( H \otimes H \) has a free basis \( \{ x^i \otimes x^j | 0 \leq i < j < p^d \} \). Thus the term \( K x^{n-i} \otimes K x^i \) would only come from \( \Delta(a_n)(x \otimes 1 + 1 \otimes x + u)^n \) for all \( 1 \leq i \leq n-1 \). Moreover it exactly comes from \( \Delta(a_n)(x \otimes 1 + 1 \otimes x)^n \) by the choice of \( n \). Therefore \( \binom{n}{i} \Delta(a_n)(x^{n-i} \otimes x^i) = 0 \) for all \( 1 \leq i \leq n-1 \).

Suppose \( n = p^a m \) where \( m > 1 \) and \( m \neq 0 \) (mod \( p \)). Choose \( i = p^a \). Hence by \( \cite{7}, \text{Lemma 5.1} \), \( \binom{n}{p^a} \equiv m \) (mod \( p \). Then \( \Delta(a_n) = 0 \), which implies that \( a_n = 0 \), a contradiction. Therefore from equation (2), we deduce that \( \Delta(a_{p^i}) (x^{p^i} \otimes 1) = a_{p^i} x^{p^i} \otimes 1 \) for all \( 0 \leq i \leq d-1 \). Thus \( \Delta(a_{p^i}) = a_{p^i} \otimes 1 \).

Then since \( H \) is counital, all of \( a_{p^i} \) are coefficients in the base field \( k \). \( \square \)
5. Finite-dimensional cocommutative connected Hopf algebras

Notice that the following lemma holds over any arbitrary base field. In the remaining of this section, we still assume $k$ to be algebraically closed of characteristic $p > 0$.

**Lemma 5.1.** Let $H$ be a finite-dimensional Hopf algebra with normal Hopf subalgebras $K \subseteq L \subseteq H$. Then there exists a natural isomorphism:

$$\left(\frac{H/K^+H}{(H/L^+H)^+}\right)\left(\frac{H/K^+H}{(H/L^+H)^+}\right) \cong \left(\frac{L/K^+L}{L^+L}\right).$$

**Proof.** By [9, Thm. 2.1.3], $L$ is Frobenius. Hence the injective left $L$-module map $L \hookrightarrow H$ splits since $L$ is self-injective. Therefore we can write $H = L \bigoplus M$ as a direct sum of two left $L$-modules. Because $K \subseteq L$, we have $L \cap K^+H = L \cap K^+ (L \bigoplus M) = L \cap (K^+L \bigoplus K^+M) = K^+L$. Then the inclusion map $L \hookrightarrow H$ induces an injective Hopf algebra map $L/K^+L \hookrightarrow H/K^+H$, since $K^+L$ and $K^+H$ are Hopf ideals of $L$ and $H$ by [9, Lemma 3.4.2].

It is clear that the composition map $L/K^+L \hookrightarrow H/K^+H \twoheadrightarrow H/L^+H$ factors through $k$ by the counit. Thus the dualized map restricted on $(H/L^+H)^+ = (H/L^+H)^* \cap \text{Ker } u^* \hookrightarrow (L/K^+L)^*$ is the zero map, where $u$ is the unit map in $H$.

Therefore the natural surjective map $(H/K^+H)^* \twoheadrightarrow (L/K^+L)^*$, which is induced by the inclusion $L/K^+L \hookrightarrow H/K^+H$, factors through $(H/K^+H)^* / (H/L^+H)^+ (H/K^+H)^*$. In order to show that it is an isomorphism, it is enough to prove that both sides have the same dimension. By [9, Theorem 3.3.1], we have

$$\dim \left(\frac{H/K^+H}{(H/L^+H)^+}\right)\left(\frac{H/K^+H}{(H/L^+H)^+}\right) = \frac{\dim H/\dim K}{\dim H/\dim L} = \frac{\dim L/\dim K}{\dim(L/K^+L)^*}.$$  

□
Let $H$ be any Hopf algebra over $k$, and $k \subseteq E$ be a field extension. In the proof of [9, Cor. 2.2.2], we know that $H \otimes E$ is also a Hopf $E$-algebra, via
\[
\Delta(h \otimes \alpha) := \Delta(h) \otimes \alpha \in H \otimes H \otimes E \cong (H \otimes E) \otimes_E (H \otimes E)
\]
\[
\varepsilon(h \otimes \alpha) := \varepsilon(h) \alpha \in E
\]
\[
S(h \otimes \alpha) := S(h) \otimes \alpha
\]
for all $h \in H, \alpha \in E$. Now consider any automorphism $\sigma$ of $k$. By taking $E = k$ and $\sigma$ to be the embedding in the discussion above, $H \otimes_\sigma k$ is also a Hopf $k$-algebra, which we will denote by $H_\sigma$. Note that in $H_\sigma$, we have $h \alpha \otimes 1 = h \otimes \sigma(\alpha)$ for all $h \in H, \alpha \in k$. Let $id_\sigma$ be the map $id \otimes 1$ from $H$ to $H_\sigma$. The following hold for all $h, l \in H$ and $\alpha \in k$
\[
id_\sigma(hl) = id_\sigma(h)id_\sigma(l), \quad \Delta id_\sigma(h) = (id_\sigma \otimes id_\sigma)\Delta h, \quad S(id_\sigma(h)) = id_\sigma(S(h)) \]
\[
\varepsilon id_\sigma(h) = \sigma(\varepsilon(h)), \quad id_\sigma(h \alpha) = id_\sigma(h)\sigma(\alpha).
\]
Generally, let $A$ be another Hopf algebra over $k$, and $\phi$ be a map from $A$ to $H$. We say that $\phi : A \mapsto H$ is a $\sigma$-linear Hopf algebra map if the composition $id_\sigma \circ \phi : A \mapsto H_\sigma$ is a $k$-linear Hopf algebra map. Suppose $H, A$ are both finite-dimensional. Note that $(H_\sigma)^* \cong (H^*)_\sigma$ since $\text{Hom}_E(H \otimes E, E) \cong \text{Hom}_k(H, k) \otimes E$ for any field extension $k \subseteq E$. Let $f$ be a $\sigma$-linear Hopf algebra map from $A$ to $H$. It is clear that the dual of $f$ is a $\sigma^{-1}$-linear Hopf algebra map from $H^*$ to $A^*$. Also quotients of $\sigma$-linear Hopf algebra maps are still $\sigma$-linear.

**Proposition 5.2.** Let $H$ be a finite-dimensional cocommutative connected Hopf algebra. Then $H$ has an increasing sequence of normal Hopf subalgebras: $k = N_0 \subset N_1 \subset \cdots \subset N_n = H$ satisfying the following properties:

1. Denote by $J$ the Jacobson radical of $H^*$. Then the length $n$ is the minimal integer such that $x^n = 0$ for all $x \in J$.
2. $N_1$ is the Hopf subalgebra of $H$ generated by all primitive elements.
3. There are $\sigma$-linear injective Hopf algebra maps:
   \[
   N_m/N_{m-1}^+N_m \hookrightarrow N_{m-1}/N_{m-2}^+N_{m-1}
   \]
   for all $2 \leq m \leq n$, where $\sigma$ is the Frobenius map of $k$.
4. $0 = \dim P(H/N_n^+H) \leq \dim P(H/N_{n-1}^+H) \leq \cdots \leq \dim P(H/N_0^+H) = \dim P(H)$. 

Proof. (1) By duality, $H^*$ is a finite-dimensional commutative local Hopf algebra. Therefore by [15] Thm. 14.4] we can write:

$$H^* = k[x_1, x_2, \ldots, x_d] \rightarrow \left( x_1^{p^m}, x_2^{p^m}, \ldots, x_d^{p^m} \right)$$

for some $d \geq 0$, in which we can define a decreasing sequence of normal Hopf ideals [9 Def. 3.4.5]

\[ (J_m = (x_1^{p^m}, x_2^{p^m}, \ldots, x_d^{p^m}))_{m \geq 0}. \]

By [9 P. 36], in the dual vector space $H$ we have an increasing sequence of normal Hopf subalgebras:

\[ k = N_0 \subset N_1 \subset \cdots \subset N_m \subset \cdots \subset H, \]

where $N_m = (H^*/J_m)^*$ for all $m \geq 0$. For the length of this sequence, notice that $N_m = H \iff J_m = 0 \iff x_1^{p^m} = 0$ for all $1 \leq i \leq d \iff x_i^{p^m} = 0$ for all $x \in J_0 = J$.

(2) Denote by $L$ the Hopf subalgebra of $H$ generated by $P(H)$. By [9 Prop. 5.2.9], $k \bigoplus P(H) = \{ h \in H | (J^2, h) = 0 \}$. Hence under the natural identification, $P(H) \subset (H^*/J^2)^* \subset (H^*/J_1)^* = N_1$. Because $L$ is generated by $P(H)$ as an algebra, we have $L \subseteq N_1$. Moreover we know $\dim L = p^{\dim P(H)} = p^{\dim J/J^2} = p^d$ by Proposition 2.2(4). On the other side, $\dim N_1 = \dim H^*/J_1 = p^d$, which implies that $L = N_1$.

(3) Define a decreasing sequence of normal Hopf subalgebras of $H^*$ by

$$A_m = \{ h^{p^m} | h \in H^* \} = k \left[ x_1^{p^m}, x_2^{p^m}, \ldots, x_d^{p^m} \right].$$

Notice that $A_m^+H^* = J_m$ for all $m \geq 0$. Moreover, by Lemma 5.1 we have

\[ (A_m/A_{m+1}^+A_m)^* \cong \left( H^*/A_{m+1}^+H^* \right)^*/ \left( H^*/A_m^+H^* \right)^*/ \left( H^*/A_{m+1}^+H^* \right)^* = N_{m+1}/N_m + N_{m+1}. \]

Let $\sigma$ be the Frobenius map of $k$ (i.e., the $p$-th power map). For any $2 \leq m \leq n$, we can take $(A_{m-2})_{\sigma^{-1}} = A_{m-2} \otimes_{\sigma^{-1}} k$ such that $ak \otimes 1 = a \otimes \sigma^{-1}(k)$ for any $a \in A_{m-2}$ and $k \in k$. Hence it is easy to see that there exists a series of $\sigma^{-1}$-linear surjective $p$-th power Hopf algebra maps $\phi_{m-2} : A_{m-2} \rightarrow A_{m-1}$ such that $\phi_{m-2}(x) = x^p$ for all $x \in A_{m-2}$. Therefore $\phi_{m-2}$ induces a series of $\sigma^{-1}$-linear surjective maps on their quotients $A_{m-2}/A_{m-1}^+A_{m-2} \rightarrow A_{m-1}/A_{m}^+A_{m-1}$. By dualizing all the maps and the above natural isomorphism [9], we have a series of $\sigma$-linear injective Hopf algebra maps:

$$N_m/N_{m-1}N_m \rightarrow N_{m-1}/N_{m-2}N_{m-1}$$

for all $2 \leq m \leq n$. 
(4) In Lemma 5.1 let $K = k$ and $L = A_m$. Then we have the special isomorphism:

$$A^*_m \cong H / N^+_m H.$$ 

Therefore, by Proposition 2.2(4),

$$\dim P(H / N^+_m H) = \dim J(A_m) / J(A_m)^2 = \# \left\{ \{ x_1^{p^m}, x_2^{p^m}, \cdots, x_d^{p^m} \} \setminus \{ 0 \} \right\},$$

which is the number of generators among $\{ x_1, x_2, \cdots, x_d \}$, whose $p^m$-th power does not vanish. Thus the inequalities follow.

**Corollary 5.3.** Let $H$ be a finite-dimensional connected Hopf algebra with $\dim P(H) = 1$. Then $H$ has an increasing sequence of normal Hopf subalgebras:

$$k = N_0 \subset N_1 \subset \cdots \subset N_n = H,$$

where $N_1$ is generated by $P(H)$ and each $N_i$ has $p$-index one in $N_{i+1}$.

**Proof.** Denote by $H^*$ the dual Hopf algebra of $H$. By duality, $H^*$ is local. Set $J = J(H^*)$, the Jacobson radical of $H^*$. Since $\dim P(H) = 1$, by Proposition 2.2(4), $\dim J / J^2 = 1$. Suppose that $\dim H = p^n$ by Proposition 2.2(7). It is clear that $H^* \cong k[x] / (x^{p^n})$ as algebras and $J = (x)$. Hence $H$ is cocommutative and it has an increasing sequence of normal Hopf subalgebras $k = N_0 \subset N_1 \subset \cdots \subset N_n = H$ such that $N_1$ is generated by $P(H)$ and $\dim N_m = p^m$ for all $0 \leq m \leq n$ by Proposition 5.2.

**Theorem 5.4.** Let $H$ be finite-dimensional cocommutative connected Hopf algebra. Denote by $K$ the Hopf subalgebra generated by $P(H)$. Then the following are equivalent:

1. $H$ is local.
2. $K$ is local.
3. All the primitive elements of $H$ are nilpotent.

**Proof.** (1) $\Rightarrow$ (2) is from Proposition 2.2(2) and (2) $\Rightarrow$ (3) is clear since $K$ contains $P(H)$ and its augmentation ideal is nilpotent.

In order to show that (3) $\Rightarrow$ (2), denote $g = P(H)$, which is a restricted Lie algebra. Then (3) is equivalent to the statement that $g^{p^n} = 0$ for sufficient larger $n$. Therefore $(\text{ad}x)^{p^n} = \text{ad}(x^{p^n}) = 0$ for all $x \in g$. By Engel’s Theorem [6 I §3.2], $g$ is nilpotent. Any representation of $K \cong u(g)$ is a restricted representation of $g$. Therefore any irreducible representation of $K$ is one-dimensional.
with trivial action of the augmentation ideal of \( K \). Hence the augmentation ideal of \( K \) is nilpotent and \( K \) is local.

Finally, we need to show (2) \( \Rightarrow \) (1). Suppose \( k = N_0 \subset N_1 \subset \cdots N_n = H \) is the sequence of normal Hopf subalgebras stated in Proposition 5.2 for \( H \). By Proposition 5.2(2), we know \( N_1 = K \) is local. We will show inductively that each \( N_m \) is local. Assume \( N_m \) to be local and denote \( \sigma \) as the Frobenius map of \( k \). We have the following injective Hopf algebra map according to Proposition 5.2(3) and the definition of \( \sigma \)-linear Hopf algebra maps:

\[
\frac{N_{m+1}/N_m N_{m+1}}{N_m/N_{m-1} N_m}_\sigma.
\]

Note that any finite-dimensional Hopf algebra \( A \) is local if and only if its augmented ideal \( A^+ \) is nilpotent. Since \( (A \otimes_\sigma k)^+ = (A^+) \otimes_\sigma k \), we see that \( A \) is local if and only if \( A_{\sigma} \) is local. Hence \( (N_m/N_{m+1} N_m)_\sigma \) is local. Moreover, by Proposition 5.2(2), \( N_{m+1}/N_m N_{m+1} \) is local. Therefore there exist integers \( l, d \) such that \( (N_{m+1}^+)^d \subseteq N_m^+ N_{m+1} \) and \( (N_m^+)^l = 0 \). Hence \( (N_{m+1}^+)^{ld} \subseteq (N_m^+)^d N_{m+1} = 0 \). Here we have used \( N_m^+ N_{m+1} = N_{m+1} N_m^+ \), which follows from [9, Cor. 3.4.4] and the fact that \( N_m \) is normal. This completes the proof. \( \square \)

Remark 5.5. Let \( G \) be a connected affine algebraic group scheme over \( k \), and \( G_1 \) be the first Frobenius kernel of \( G \). By [3, Prop. 4.3.1 Exp. XVII], we know that \( G \) is unipotent if and only if \( \text{Lie}(G) \) is unipotent, i.e., for any \( x \in \text{Lie}(G_1) \), there exists integer \( n > 0 \), such that \( x^{p^n} = 0 \). Moreover, \( \text{Lie}(G) = \text{Lie}(G_1) \). Hence \( G \) is unipotent if and only if \( G_1 \) is unipotent. Denote the coordinate ring \( A = k[G] \). Then \( k[G_1] = A/A^{+(p)}A \), where \( A^{+(p)} = \{ a^p \mid a \in A \} \). We can state the above assertion in another way: \( A \) is connected if and only if \( A/A^{+(p)}A \) is connected. If \( A \) is finite-dimensional, as shown in Proposition 5.2(2), \( (A/A^{+(p)}A)^* \) is the Hopf subalgebra of \( A^* \) generated by its primitive elements. This provides an alternative proof for Theorem 5.4 and shows that the locality criterion in Theorem 5.4 for finite-dimensional cocommutative connected Hopf algebras parallel the criteria for unipotency of finite connected group schemes over \( k \).

6. Hochschild cohomology of restricted universal enveloping algebras

Suppose \( H \) is a Hopf algebra. Denote by \( k \) the trivial \( H \)-bicomodule. The Hochschild cohomology \( H^*(k, H) \) of \( H \) with coefficients in \( k \) can be computed as the homology of the differential graded algebra \( \Omega H \) defined as follows [11, Lemma 1.1]:

- As a graded algebra, \( \Omega H \) is the tensor algebra \( T(H) \),
The differential in $\Omega H$ is given by $d^0 = 0$ and for $n \geq 1$

$$d^n = 1 \otimes I_n + \sum_{i=0}^{n-1} (-1)^{i+1} I_i \otimes \Delta \otimes I_{n-i-1} + (-1)^{n+1} I_n \otimes 1.$$ 

This DG algebra is usually called the cobar construction of $H$. See [4, §19] for the basic properties of cobar constructions. Throughout, we will use $H^*(k, H)$ to denote the homology of the DG algebra $(\Omega H, d)$.

**Lemma 6.1.** Let $H$ be a finite-dimensional Hopf algebra. Thus

$$H^n(k, H) \cong H^n(H^*, k) \cong \text{Ext}_H^n(k, k),$$ 

for all $n \geq 0$.

**Proof.** We still denote by $k$ the trivial $H$-bimodule. Then the first isomorphism comes from [11, Prop. 1.4]. Let $M$ be a $H$-bimodule with the trivial right structure. We define the right structure of $M^\text{ad}$ by $m.h = S(h)m$ using the antipode $S$ of $H$ for any $m \in M, h \in H$. Then it is easy to see $k^\text{ad} \cong k$ as trivial right $H$-modules. Hence the second isomorphism is derived from [11, Thm. 1.5].

Let $\mathfrak{g}$ be a restricted Lie algebra. We denote by $u(\mathfrak{g})$ the restricted universal enveloping algebra of $\mathfrak{g}$. analogue to ordinary Lie algebras, restricted $\mathfrak{g}$-modules are in one-to-one correspondence with $u(\mathfrak{g})$-modules, i.e., a vector space $M$ is a restricted $\mathfrak{g}$-module if there exists an algebra map $T: u(\mathfrak{g}) \to \text{End}_k(M)$.

**Proposition 6.2.** Let $\mathfrak{g}$ be a restricted Lie algebra with basis $\{x_1, x_2, \cdots, x_n\}$. Then the image of

$$\{\omega(x_i), x_j \otimes x_k \mid 1 \leq i \leq n, 1 \leq j < k \leq n\}$$

is a basis in $H^2(k, u(\mathfrak{g}))$.

**Proof.** Denote $K = u(\mathfrak{g})$ and let $C_p^n$ be the elementary abelian $p$-group of rank $n$. It is clear that $K^*$ is isomorphic to $k[C_p^n]$ as algebras. Then it follows from, e.g., [10, P. 558 (4.1)] that $\dim H^2(K^*, k) = \dim H^2(C_p^n, k) = n(n+1)/2$. Thus by Lemma 6.1 $\dim H^2(k, K) = n(n+1)/2$. First, it is direct to check that all $\omega(x_i)$ and $x_j \otimes x_k$ are cocycles in $\Omega K$. We only check for $x_j \otimes x_k$
here. Notice that $d^2 = 1 \otimes I \otimes I - \Delta \otimes I + I \otimes \Delta - I \otimes I \otimes 1$. Thus

$$d^2 (x_j \otimes x_k) = 1 \otimes x_j \otimes x_k - \Delta(x_j) \otimes x_k + x_j \otimes \Delta(x_k) - x_j \otimes x_k \otimes 1$$

$$= 1 \otimes x_j \otimes x_k - (x_j \otimes 1 + 1 \otimes x_j) \otimes x_k + x_j \otimes (x_k \otimes 1 + 1 \otimes x_k) - x_j \otimes x_k \otimes 1$$

$$= 0.$$

Secondly, we need to show they are linearly independent in $H^2(k, K) = \text{Ker} \ d^2/\text{Im} \ d^1$. We only deal with the case when $p \geq 3$. The remaining case of $p = 2$ is similar. By the PBW Theorem, $K$ has a basis formed by

$$\{x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \mid 0 \leq i_1, i_2, \ldots, i_n \leq p - 1\}.$$

Because the differential $d^1 = 1 \otimes I - \Delta + 1 \otimes 1$ in $\Omega K$ only uses the comultiplication, without loss of generality, we can assume $g$ to be abelian. Suppose each variable $x_i$ of $K$ has degree one. Assign the usual total degree to any monomial in $K$. Also the total degree of a tensor product $A \otimes B$ in $K \otimes K$ is the sum of the degrees of $A$ and $B$ in $K$. Therefore $d^1$ preserves the degree from $K$ to $K \otimes K$ for any monomial. Notice that $\omega(x_i)$ has degree $p$ and $x_j \otimes x_k$ has degree two. We can treat them separately. Suppose that $\sum_i \alpha_i \omega(x_i) \in \text{Im} d^1$. First, we consider the ideal $I = (x_2, \ldots, x_n)$ in $K$. By passing to the quotient $K/I$, we have $\alpha_1 \omega(x_1) \in \text{Im} d^1$, where $d^1 : K/I \to K/I \otimes K/I$. But every monomial in $K/I$, which is generated by $x_1$, has degree less than $p$. This forces that $\alpha_1 = 0$. The same argument works for all the coefficients. Now suppose $\sum_{j<k} \alpha_{jk} x_j \otimes x_k \in \text{Im} d^1$. Therefore there exists $\sum_{j<k} \lambda_{jk} x_j x_k \in K$ such that

$$\sum_{j<k} \alpha_{jk} x_j \otimes x_k = d^1 \left( \sum_{j<k} \lambda_{jk} x_j x_k \right)$$

$$= \sum_{j<k} \lambda_{jk} (1 \otimes x_j x_k - \Delta(x_j x_k) + x_j x_k \otimes 1)$$

$$= - \sum_{j<k} \lambda_{jk} (x_j \otimes x_k + x_k \otimes x_j).$$

By applying the PBW Theorem to $K \otimes K$, we have all the coefficients equal zero. This completes the proof. \qed

**Lemma 6.3.** Let $g$ be a restricted Lie algebra. Then the cocycle

$$\sum_{i=1}^n \alpha_i^p \omega(x_i) - \omega \left( \sum_{i=1}^n \alpha_i x_i \right)$$
is zero in $H^2(k, u(\mathfrak{g}))$, where $x_i \in \mathfrak{g}$ and $\alpha_i \in k$ for all $1 \leq i \leq n$.

Proof. Denote by $K$ the restricted universal enveloping algebra of $\mathfrak{g}$. First, it is direct to check that $\omega(x)$ is a cocycle in $(\Omega K, d)$ for any $x \in \mathfrak{g}$. Hence the expression in the statement is also a cocycle in $(\Omega K, d)$. We only need to show that it lies in the coboundary $\text{Im } d^1$. Without loss of generality, we can assume $\mathfrak{g}$ to be finite-dimensional. Because $k$ is algebraically closed in $F_p$, we can replace $k$ with some finite field $F_q$. By basic algebraic number theory, there exists some number field $L \supset \mathbb{Q}$, where $p$ remains prime in the ring of integers $O_L$ such that $O_L/(p) = F_q$.

Now by choosing representatives for $F_q$ in $O_L$, we can view $\mathfrak{g}$ as a free module over $O_L$ with a Lie bracket $\left[ , \right]$, representing all the relations between a chosen basis for $\mathfrak{g}$. Denote by $A = U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$ over $O_L$, which is a Hopf algebra as usual. There is a quotient map $\pi : A \to u(\mathfrak{g})$, which factors through $A/(p)$. Therefore it suffices to prove that for any $x, y \in \mathfrak{g}$, there exists some $\Theta \in A$ such that

$$\omega(x) + \omega(y) - \omega(x + y) = 1 \otimes \Theta - \Delta(\Theta) + \Theta \otimes 1.$$  

(4)

The general result will follow by applying the quotient map $\pi$ to (4), and the induction on the number of variables appearing in the expression. By Lemma A.1 in $A \otimes_{O_L} O_L/(p) = A \otimes_{O_L} F_q = A/(p)$, there exists some $z \in \mathfrak{g}$ such that

$$(x + y)^p = x^p + y^p + z.$$  

So back in $A$, we have some $\Theta \in A$ such that

$$(x + y)^p = x^p + y^p + z + p \Theta.$$  

Thus in $A$, we can calculate $\Delta(x + y)^p$ in two different ways:

(I) \quad $\Delta(x + y)^p = (\Delta(x + y))^p$

\hspace{1cm} $= ((x + y) \otimes 1 + 1 \otimes (x + y))^p$

\hspace{1cm} $= (x + y)^p \otimes 1 + 1 \otimes (x + y)^p + p \omega(x + y)$

\hspace{1cm} $= (x^p + y^p + z) \otimes 1 + 1 \otimes (x^p + y^p + z) + p \Theta \otimes 1 + 1 \otimes p \Theta + p \omega(x + y)$.  

On the other hand,

\[(\Delta(x + y)^p = \Delta(x^p + y^p + z + p \Theta)\]

\[= x^p \otimes 1 + 1 \otimes x^p + p \omega(x) + y^p \otimes 1 + 1 \otimes y^p + p \omega(y) + z \otimes 1 + 1 \otimes z + p \Delta(\Theta)\]

\[= (x^p + y^p + z) \otimes 1 + 1 \otimes (x^p + y^p + z) + p \omega(x) + p \omega(y) + p \Delta(\Theta).\]

Therefore we have the following identity in \(A \otimes A\).

\[p \{\omega(x) + \omega(y) - \omega(x + y)\} = p \{1 \otimes \Theta - \Delta(\Theta) + \Theta \otimes 1\}.\]

Since \(A\) is a domain, we can cancel \(p\) from both sides. This completes the proof. \(\square\)

**Definition 6.4.** Let \(H\) be a Hopf algebra. For any \(x \in H\), define the adjoint map \(T_x\) on \(\Omega H\) by

\[T_x^n = \sum_{i=0}^{n-1} I_i \otimes \text{ad}(x) \otimes I_{n-i-1},\]

where \(\text{ad}(x)(H) = [x, H]\).

**Lemma 6.5.** If \(H\) is any Hopf algebra, then \(T_x\) is a degree zero cochain map from \(\Omega H\) to itself for all \(x \in P(H)\). Moreover, \(P(H) = H^1(k, H)\) and \(\bigoplus_{n \geq 0} H^n (k, H)\) is a graded restricted \(P(H)\)-module via the adjoint map.

**Proof.** First, for simplicity write \(T = T_x\) for some \(x \in P(H)\). We prove \(d^n T^n = T^{n+1} d^n\) inductively for all \(n \geq 0\). It is easy to check that it holds for \(n = 0, 1\). Notice that

\[d^n = d^{n-1} \otimes I + (-1)^{n-1} I_{n-1} \otimes d^1,\]
Hence the induced map is injective, suppose by Lemma 6.5. □

Let \( g, d \) as a subcomplex of \( \Omega^p \) restricted Lie algebra via the Theorem 6.6. Let \( x, y \) \( g \) for all \( n \geq 2 \). Thus

\[
d^n T^n = (d^{n-1} \otimes I + (-1)^{n-1} I_{n-1} \otimes d^1) (T^{n-1} \otimes I + I_{n-1} \otimes T^1)
\]

\[
= d^{n-1} T^{n-1} \otimes I + d^{n-1} \otimes T^1 + (-1)^{n-1} T^{n-1} \otimes d^1 + (-1)^{n-1} I_{n-1} \otimes d^1 T^1
\]

\[
= T^n d^{n-1} \otimes I + d^{n-1} \otimes T^1 + (-1)^{n-1} T^{n-1} \otimes d^1 + (-1)^{n-1} I_{n-1} \otimes T^2 d^1
\]

\[
= T^n d^{n-1} \otimes I + d^{n-1} \otimes T^1 + (-1)^{n-1} (T^{n-1} \otimes I_2 + I_{n-1} \otimes T^1 \otimes I) (I_{n-1} \otimes d^1) + (-1)^{n-1} I_{n-1} \otimes (I \otimes T^1) d^1
\]

\[
= T^n d^{n-1} \otimes I + d^{n-1} \otimes T^1 + (-1)^{n-1} (T^n \otimes I) (I_{n-1} \otimes d^1) + (-1)^{n-1} I_{n-1} \otimes (I \otimes T^1) d^1
\]

\[
= (T^n \otimes I + I_n \otimes T^1) (d^{n-1} \otimes I + (-1)^{n-1} I_{n-1} \otimes d^1)
\]

\[
= T^{n+1} d^n
\]

Therefore \( T \) induces an action of \( P(H) \) on \( H^n(k, H) \) for each \( n \). Moreover, we know \( P(H) \) is a restricted Lie algebra via the \( p \)-th power map in \( H \). It is clear that \( [T_x, T_y] = T_{[x,y]} \) and \( T_x^p = T_{x^p} \) for any \( x, y \in P(H) \). Hence \( \bigoplus_{n \geq 0} H^n(k, H) \) becomes a graded restricted \( P(H) \)-module via \( T \). Finally, \( P(H) \cong H^1(k, H) \) by definition. □

**Theorem 6.6.** Let \( K \subseteq H \) be an inclusion of connected Hopf algebras with first order \( n \geq 2 \). Then the differential \( d^1 \) induces an injective restricted \( g \)-module map

\[
H_n/K_n \longrightarrow H^2(k, K),
\]

where \( g = P(H) \).

**Proof.** By Corollary 3.3 \( H_n \) becomes a restricted \( g \)-module via the adjoint action since \( [P(H), H_n] \subseteq [H_1, H_n] \subseteq H_n \). We know \( g = P(H) = P(K) \) for the inclusion has first order \( n \geq 2 \). Hence the \( g \)-action factors through \( H_n/K_n \). Choose any \( x \in H_n \). We know \( d^1(x) = 1 \otimes x - \Delta(x) + x \otimes 1 \in H_{n-1} \otimes H_{n-1} = K_{n-1} \otimes K_{n-1} \subseteq K \otimes K \) by [9] Lemma 5.3.2]. Furthermore, we can view \((\Omega K, d_K)\) as a subcomplex of \((\Omega H, d_H)\). Then \( d_K^2 d_K(x) = d_H^2 d_K(x) = 0 \). Hence \( d^1(x) \) is a cocycle in \( \Omega K \) and \( d^1 \) maps \( H_n \) into \( H^2(k, K) \). The map \( d^1 \) factors through \( H_n/K_n \) for \( d^2 d^1(K_n) = 0 \). To show the induced map is injective, suppose \( d^1(x) \in \text{Im } d_K^1 \). Then there exists some \( y \in K \) such that \( d^1(x) = d^1(y) \), which implies that \( d^1(x - y) = 0 \). By definition, we have \( x - y \in P(H) = P(K) \). Hence \( x \in K \cap H_n = K_n \) by Remark 2.4. Finally, \( d^1 \) is compatible with the \( g \)-action on \( H^2(k, K) \) by Lemma 6.5 □
**Theorem 6.7.** Let \( g \) be a restricted Lie algebra with basis \( \{ x_1, x_2, \ldots, x_n \} \). Suppose \( u(g) \subseteq H \) is an inclusion of connected Hopf algebras. Then there exists some \( x \in H \setminus u(g) \) such that

\[
\Delta(x) = x \otimes 1 + 1 \otimes x + \omega \left( \sum_{i} \alpha_i x_i \right) + \sum_{j<k} \alpha_{jk} x_j \otimes x_k
\]

with coefficients \( \alpha_i, \alpha_{jk} \in k \). Moreover, the first order for the inclusion can only be 1, 2 or \( p \).

**Proof.** Denote by \( d \) the first order for the inclusion. By definition, \( d = 1 \) implies that \( g \subseteq \mathfrak{p}(H) \). Then we can find some primitive element \( x \in \mathfrak{p}(H) \setminus g \subseteq H \setminus u(g) \) such that \( \Delta(x) = x \otimes 1 + 1 \otimes x \). In the following, we may assume \( d \geq 2 \). By Theorem 6.6 and Proposition 6.2 there exists \( x \in H_d \setminus u(g) \) such that

(I) \[
1 \otimes x - \Delta(x) + x \otimes 1 = d^1(x) = - \sum_i \alpha_i^p \omega(x_i) - \sum_{j<k} \alpha_{jk} x_j \otimes x_k.
\]

By the choice of \( x \), we know the coefficients are not all zero. By Lemma 6.3 there exists some \( y \in u(g) \) such that

(II) \[
1 \otimes y - \Delta(y) + y \otimes 1 = d^1(y) = \sum_i \alpha_i^p \omega(x_i) - \omega \left( \sum_i \alpha_i x_i \right).
\]

If we add (I) to (II), then we have

\[
(x + y) \otimes 1 - \Delta(x + y) + 1 \otimes (x + y) = - \omega \left( \sum_i \alpha_i x_i \right) - \sum_{j<k} \alpha_{jk} x_j \otimes x_k.
\]

This implies that

\[
\Delta(x + y) = (x + y) \otimes 1 + 1 \otimes (x + y) + \omega \left( \sum_i \alpha_i x_i \right) + \sum_{j<k} \alpha_{jk} x_j \otimes x_k.
\]

It is clear that \( x + y \in H \setminus u(g) \). Finally, because the associated graded Hopf algebra \( \text{gr}H \) is coradically graded as mentioned in [2] Def. 1.13, it is easy to see that if all \( \alpha_i = 0 \) then \( d = 2 \). Otherwise \( d = p \). Hence the first order \( d \) can only be 1, 2 or \( p \). This completes the proof. \( \square \)

**7. Connected Hopf Algebras of Dimension \( p^2 \)**

The starting point for classifying finite-dimensional connected Hopf algebras turns out to be when the dimension of the Hopf algebras is just \( p \). It is obvious that such Hopf algebras are primitively generated, i.e., by some primitive element \( x \). As a consequence of the characteristic of the base field, \( x^p \) is still primitive. This implies that \( x^p = \lambda x \) for some \( \lambda \in k \), since the dimension of the
primitive space is one. By rescaling of the variable, we can always assume the coefficient $\lambda$ to be zero or one. Thus we have the following result:

**Theorem 7.1.** All connected Hopf algebras of dimension $p$ are isomorphic to either $k[x]/(x^p)$ or $k[x]/(x^p - x)$, where $x$ is primitive.

**Corollary 7.2.** All local Hopf algebras of dimension $p$ are isomorphic to $k[x]/(x^p)$ with comultiplication either $\Delta(x) = x \otimes 1 + 1 \otimes x$ or $\Delta(x) = x \otimes 1 + 1 \otimes x + x \otimes x$.

**Proof.** By Proposition 2.2(1), $p$-dimensional local Hopf algebras are in one-to-one correspondence with $p$-dimensional connected Hopf algebras by vector space dual. Therefore by Theorem 7.1, there are two non-isomorphic classes of local Hopf algebras of dimension $p$. It is clear that $k[x]/(x^p)$ is a local algebra of dimension $p$. Regarding the coalgebra structure, when $\Delta(x) = x \otimes 1 + 1 \otimes x$, it is connected. When $\Delta(x) = x \otimes 1 + 1 \otimes x + x \otimes x$, $\Delta(x + 1) = (x + 1) \otimes (x + 1)$, which is a group-like element. Therefore it is cosemisimple. They are certainly non-isomorphic as coalgebras.

In the rest of the section, we concentrate on the classification of connected Hopf algebras of dimension $p^2$. We first consider the case when $\dim P(H) = 1$. By Corollary 5.3, we have $k \subset K \subset H$, where $K$ is generated by some $x \in P(H)$. By Proposition 2.2(5), we know $K$ is isomorphic to the restricted universal enveloping algebra of the one-dimensional restricted Lie algebra spanned by $x$. Therefore by Proposition 6.2, $H^2(k, K)$ is one-dimensional with the basis representing by the element

$$
\omega(x) = \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} x^i \otimes x^{p-i}.
$$

Furthermore, by Theorem 6.7 there exists some $y \in H \setminus K$ such that $\Delta(y) = y \otimes 1 + 1 \otimes y + \omega(x)$.

**Lemma 7.3.** Let $H$ be a connected Hopf algebra of dimension $p^2$ with $\dim P(H) = 1$. Then $H$ is isomorphic to one of the following

1. $k[x, y]/(x^p, y^p)$,
2. $k[x, y]/(x^p, y^p - x)$,
3. $k[x, y]/(x^p - x, y^p - y)$. 

where the coalgebra structure is given by

\[
\Delta(x) = x \otimes 1 + 1 \otimes x, \\
\Delta(y) = y \otimes 1 + 1 \otimes y + \omega(x).
\]

Proof. By the previous argument, we can find elements \(x, y \in H\) with the comultiplications given in (5). They generate a Hopf subalgebra of \(H\) by Remark 2.4. Since \(H\) has dimension \(p^2\), \(H\) is generated by \(x, y\). It is clear that \([x, y]\) is primitive since

\[
\Delta([x, y]) = [\Delta(x), \Delta(y)] \\
= [x \otimes 1 + 1 \otimes x, y \otimes 1 + 1 \otimes y + \omega(x)] \\
= [x, y] \otimes 1 + 1 \otimes [x, y].
\]

In other words, we can write \([x, y] = \lambda x\) for some \(\lambda \in k\), which implies that \([x^n, y] = n\lambda x^n\) for any \(n \geq 1\). Therefore we can show that

\[
[x^n, y] = \lambda x^n = \left(\sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-1)!} x^i \otimes x^{p-i}, y \otimes 1 + 1 \otimes y\right)
\]

\[
= \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-1)!} \left([x^i, y] \otimes x^{p-i} + x^i \otimes [x^{p-i}, y]\right)
\]

\[
= \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-1)!} \left(i\lambda x^i \otimes x^{p-i} + x^i \otimes (p-i)\lambda x^{p-i}\right)
\]

\[
= \sum_{i=1}^{p-1} \frac{p!}{i!(p-i)!} \lambda x^i \otimes x^{p-i}
\]

\[
= 0.
\]

Since \(\omega(x)^p = \omega(x^p)\), we have

\[
\Delta(y^p) = (y \otimes 1 + 1 \otimes y + \omega(x))^p = y^p \otimes 1 + 1 \otimes y^p + \omega(x^p).
\]

By Theorem 7.1, we can assume that \(x^p = 0\) or \(x^p = x\). When \(x^p = 0\), according to the above equation (7), \(y^p\) is primitive. Then we can write \(y^p = \mu x\) for some \(\mu \in k\). Thus \(\lambda^p x = x \text{ad}(y)^p = [x, y^p] = [x, \mu x] = 0\), which implies that \(\lambda = 0\). By further rescaling of the variables, we can assume \(\mu\) to be either one or zero, which yields the first two classes. On the other hand, when \(x^p = x\), by (7) again, \(y^p - y\) is primitive. Then we can write \(y^p = y + \mu x\) for some \(\mu \in k\). Moreover,
\[ [x, y] = [x^p, y] = \text{ad}(x^p)y = 0. \] After the linear translation \( y = y' + \sigma x \) satisfying \( \sigma^p = \sigma + \mu \), we have \( y^p = y' \) while \( \Delta(y') = y' \otimes 1 + 1 \otimes y' + \omega(x) \). This gives the third class. It remains to show those three Hopf algebras are non-isomorphic. The first two are local with different number of minimal generators and the third one is semisimple. Hence they are non-isomorphic as algebras. This completes the classification.

Finally, the classification for connected Hopf algebras of dimension \( p^2 \) follows:

**Theorem 7.4.** Let \( H \) be a connected Hopf algebra of dimension \( p^2 \). When \( \dim P(H) = 2 \), it is isomorphic to one of the following:

1. \( k[x, y] / (x^p, y^p) \),
2. \( k[x, y] / (x^p - x, y^p) \),
3. \( k[x, y] / (x^p - y, y^p) \),
4. \( k[x, y] / (x^p - x, y^p - y) \),
5. \( \langle x, y \rangle / ([x, y] - y, x^p - x, y^p) \),

where \( x, y \) are primitive. When \( \dim P(H) = 1 \), it is isomorphic to one of the following:

6. \( k[x, y] / (x^p, y^p) \),
7. \( k[x, y] / (x^p, y^p - x) \),
8. \( k[x, y] / (x^p - x, y^p - y) \),

where \( \Delta(x) = x \otimes 1 + 1 \otimes x \) and \( \Delta(y) = y \otimes 1 + 1 \otimes y + \omega(x) \).

**Proof.** By Proposition 2.2(6), we know \( \dim P(H) \leq 2 \). If \( \dim P(H) = 2 \), then \( H \) is primitively generated and \( H \cong u(\mathfrak{g}) \) for some two-dimensional restricted Lie algebra \( \mathfrak{g} \) by Proposition 2.2(5). Therefore Proposition A.3 provides the classification. When \( \dim P(H) = 1 \), it is directly from Lemma A.3. Finally, it is clear that the Hopf algebras given in (1)-(5) are non-isomorphic to the ones given in (6)-(8), since their primitive spaces have different dimension. The Hopf algebras in (1)-(5) are obviously non-isomorphic as algebras. Neither are the ones in (6)-(8). This completes the proof.

**Corollary 7.5.** Let \( H \) be a local Hopf algebra of dimension \( p^2 \). Then \( H \) is isomorphic to either \( k[\xi, \eta] / (\xi^p, \eta^p) \) or \( k[\xi] / (\xi^p) \) as algebras. When \( H \cong k[\xi, \eta] / (\xi^p, \eta^p) \), the coalgebra structure is given by one of the following:
(1) $\Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi,$
    $\Delta(\eta) = \eta \otimes 1 + 1 \otimes \eta,$
(2) $\Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi + \xi \otimes \xi,$
    $\Delta(\eta) = \eta \otimes 1 + 1 \otimes \eta,$
(3) $\Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi,$
    $\Delta(\eta) = \eta \otimes 1 + 1 \otimes \eta + \omega(\xi),$
(4) $\Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi + \xi \otimes \xi,$
    $\Delta(\eta) = \eta \otimes 1 + 1 \otimes \eta + \eta \otimes \eta,$
(5) $\Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi + \xi \otimes \xi,$
    $\Delta(\eta) = \eta \otimes 1 + 1 \otimes \eta + \xi \otimes \eta.$

When $H \cong k[\xi]/(\xi^p),$ the coalgebra structure is given by

(6) $\Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi,$
(7) $\Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi + \omega(\xi^p),$
(8) $\Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi + \xi \otimes \xi.$

Proof. Denote the dual Hopf algebra of $H$ by $H^*$. By Proposition 2.2(1), $H^*$ is a connected Hopf algebra of dimension $p^2$. When $\dim P(H^*) = 2$, as shown in Theorem 7.4, there are five non-isomorphic classes for $H^*$. By duality, there are also five non-isomorphic classes for $H$. Furthermore, from Proposition 2.2(4), $\dim J/J^2 = \dim P(H^*) = 2$, where $J$ is the Jacobson radical of $H$. Notice that $H^*$ is cocommutative. Then $H$ is commutative and we have $H \cong k[\xi, \eta]/(\xi^p, \eta^p)$ by [15, Thm. 14.4]. It is easy to check that the coalgebra structures given in (1)-(5) are non-isomorphic. The same argument applies to the other case. Theorem 7.4 shows that when $\dim P(H^*) = 1$, there are three non-isomorphic classes. Since $\dim J/J^2 = \dim P(H^*) = 1$, $H$ is isomorphic to $k[\xi]/(\xi^p)$ as algebras. Because those given in (6)-(8) are non-isomorphic as coalgebras. They complete the list. \qed

Remark 7.6. In fact, the Hopf algebras in Corollary 7.5 (1)-(8) are in one-to-one correspondence with those in Theorem 7.4 (1)-(8) via duality. Below, in each case, we describe the generator(s)
ξ, η as linear functional(s) on the basis \( \{x^i y^j \mid 0 \leq i, j \leq p - 1 \} \).

\[
\begin{align*}
\xi(x^i y^j) &= \begin{cases} 
1 & i = 1, j = 0 \\
0 & \text{otherwise}
\end{cases}, & \eta(x^i y^j) &= \begin{cases} 
1 & i = 0, j = 1 \\
0 & \text{otherwise}
\end{cases} \\
\end{align*}
\]

\[
\begin{align*}
\xi(x^i y^j) &= \begin{cases} 
1 & i \neq 0, j = 0 \\
0 & \text{otherwise}
\end{cases}, & \eta(x^i y^j) &= \begin{cases} 
1 & i = 0, j \neq 0 \\
0 & \text{otherwise}
\end{cases} \\
\end{align*}
\]

\[
\begin{align*}
\xi(x^i y^j) &= \begin{cases} 
1 & i = 1, j = 0 \\
0 & \text{otherwise}
\end{cases}, & \eta(x^i y^j) &= \begin{cases} 
-1 & i = 0, j = 1 \\
0 & \text{otherwise}
\end{cases} \\
\end{align*}
\]

\[
\begin{align*}
\xi(x^i y^j) &= \begin{cases} 
1 & i \neq 0, j = 0 \\
0 & \text{otherwise}
\end{cases}, & \eta(x^i y^j) &= \begin{cases} 
1 & j = 1 \\
0 & \text{otherwise}
\end{cases} \\
\end{align*}
\]

\[
\begin{align*}
\xi(x^i y^j) &= \begin{cases} 
1 & i = 1, j = 0 \\
0 & \text{otherwise}
\end{cases}.
\end{align*}
\]

**Theorem 7.7.** Let \( H \) be a finite-dimensional connected Hopf algebra with \( \dim P(H) = 1 \). Then the center of \( H \) contains \( P(H) \).

**Proof.** Suppose \( P(H) \) is spanned by \( x \). By Corollary 5.3, \( H \) has an increasing sequence of normal Hopf subalgebras:

\[
k = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_n = H
\]

satisfying \( N_1 \) is generated by \( x \) and \( N_{n-1} \subset H \) is normal with \( p \)-index one. We show by induction on \( n \) such that the center of \( H \) contains \( x \). It is trivial when \( n = 1 \). Assume that \( n \geq 2 \). Then by Theorem 6.6, we can find some \( y \in H \setminus N_{n-1} \) such that \( \Delta(y) = y \otimes 1 + 1 \otimes y + u \), where \( u \in N_{n-1} \otimes N_{n-1} \), which together with \( N_{n-1} \) generate \( H \). Apply Theorem 4.5 to \( N_{n-1} \subset H \), we have \( y^p + \lambda y + a = 0 \) for some \( \lambda \in k \) and \( a \in N_{n-1} \).

By induction, \( x \in Z(N_{n-1}) \). Then it suffices to show \([x, y] = 0\). It is easy to check that \([x, y]\) is primitive. Therefore we can write \([x, y] = \mu x\) for some \( \mu \in k \). By rescaling, we can further
assume either \(x^p = 0\) or \(x^p = x\). When \(x^p = 0\), by Theorem 5.4, \(H\) is local. Then its quotient \(H/N_{n-1}H\), which is generated by the image of \(y\), is local too. Hence the image of \(y\) in \(H/N_{n-1}H\) is nilpotent since it is primitive. Thus in the relation \(y^p + \lambda y + a = 0\), we must have \(\lambda = 0\) and \(y^p + a = 0\). A calculation therefore shows that \(\mu y^p + x \cdot y^p + a = 0\). When \(x^p = x\), we have \([x, y] = [x^p, y] = (\text{ad}x)^p y = 0\). This completes the proof.

\[\square\]

### Appendix A. Restricted Lie algebras

We state the following technical lemma which is the key to our classification of finite-dimensional connected Hopf algebras.

**Lemma A.1.** [8, P. 186-187] For any associative \(k\)-algebra \(A\), we have

\[(x + y)^p = x^p + y^p + \sum_{i=1}^{p-1} s_i(x, y)\]

where \(s_i(x, y)\) is the coefficient of \(\lambda^{i-1}\) in \(x(\text{ad}(\lambda x + y))^{p-1}\) and

\([x^p, y] = (\text{ad}x)^p (y)\]

for any \(x, y \in A\).

**Definition A.2.** [8, Chapter V Def. 4] A **restricted Lie algebra** \(g\) over \(k\) is a Lie algebra in which there is defined a map \(g \rightarrow g\), i.e., \(x \mapsto x^{[p]}\) such that

1. \((\alpha x)^{[p]} = \alpha^p x^{[p]}\),
2. \((x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)\), where \(s_i(x, y)\) is the coefficient of \(\lambda^{i-1}\) in \(x(\text{ad}(\lambda x + y))^{p-1}\),
3. \([x^{[p]}, y^{[p]}] = x (\text{ad} y)^p\),

for all \(x, y \in g\) and \(\alpha \in k\).

If \(g\) is restricted and \(U(g)\) is the usual universal enveloping algebra, let \(B\) be the ideal in \(U(g)\) generated by all \(x^p - x^{[p]}, x \in g\), and define \(u(g) = U(g)/B\). Then \(u(g)\) is called the restricted universal enveloping algebra of \(g\). A version of the PBW theorem holds for \(u(g)\): given a basis for \(g\), the ordered monomials in this basis, where the exponent of each basis element is bounded by \(p - 1\), form a basis for \(u(g)\). Consequently if \(\dim g = n\), then \(\dim u(g) = p^n\).

Let \(g\) be a two-dimensional Lie algebra with basis \(\{x, y\}\). There is, up to isomorphism, a unique two-dimensional non-abelian Lie algebra, and we can assume \([x, y] = y\) without loss of generality.
The following result about two-dimensional restricted Lie algebras probably is well-known, see, e.g., [8, Chapter V §8].

Proposition A.3. Let \( g \) be a two-dimensional restricted Lie algebra with basis \( \{x, y\} \). Then the restricted maps can be classified as follows: When \( g \) is abelian:

(1) \( x^{[p]} = 0, y^{[p]} = 0 \),
(2) \( x^{[p]} = x, y^{[p]} = 0 \),
(3) \( x^{[p]} = y, y^{[p]} = 0 \),
(4) \( x^{[p]} = x, y^{[p]} = y \).

When \( g \) is non-abelian such that \( [x, y] = y \):

(5) \( x^{[p]} = x, y^{[p]} = 0 \).

Proof. First suppose \( g \) is abelian. Then by [8, Ex. 19], \( g \) can be decomposed into a direct sum \( g = g_0 \oplus g_1 \), where \( g_0^{n} = 0 \) for sufficient large \( n \) and \( g_1^{p} = g_1 \). Define the non-commutative polynomial ring \( \Phi = \{\alpha_0 + \alpha_1 t + \cdots + \alpha_n t^n | \alpha_i \in k \} \), where \( t \) is an indeterminate such that \( t^\alpha = \alpha^p t \).

By comments [8, P. 192], \( g_0 \) can be viewed as a module over \( \Phi \) with \( t \) acts on \( g_0 \) by the restricted map. Hence \( g_0 \) is annihilated by \( t^n \) for \( n > 0 \). Notice that \( \Phi \) is a PID. Thus

\[ g_0 \cong \bigoplus_i \Phi / (t^{n_i}) \]

as \( \Phi \)-modules. Suppose \( \dim g_1 = 0 \). Then \( g_0 \) is either isomorphic to the cyclic module of dimension two over \( \Phi \), or isomorphic to the direct sum of two copies of the one-dimensional cyclic module over \( \Phi \). By applying [8, Chapter V §8 Thm. 13] to \( g_1 \), it is easy to see that the first one gives case (3) and the second one gives case (1). If \( \dim g_1 = 1 \), we have case (2). If \( \dim g_1 = 2 \), it is case (4). Moreover, they are all non-isomorphic because of the different decompositions and module structures over \( \Phi \). When \( g \) is non-abelian, by the condition (3) of Definition [A.2], we have \( [x, x^{[p]}] = [y, y^{[p]}] = [x, y^{[p]}] = 0 \) and \( [x^{[p]}, y] = y \). Since \( [x, y] = y \), we have \( x^{[p]} = x, y^{[p]} = 0 \). \( \square \)

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