New potentials for conformal mechanics

G Papadopoulos
Department of Mathematics, King’s College London, Strand, London WC2R 2LS, UK
E-mail: george.papadopoulos@kcl.ac.uk

Received 14 December 2012
Published 13 March 2013
Online at stacks.iop.org/CQG/30/075018

Abstract
We find under some mild assumptions that the most general potential of one-dimensional conformal systems with time-independent couplings is expressed as 

$$V = V_0 + V_1,$$

where $V_0$ is a homogeneous function with respect to a homothetic motion in configuration space and $V_1$ is determined from an equation with source a homothetic potential. Such systems admit at most an $SL(2, \mathbb{R})$ conformal symmetry which, depending on the couplings, is embedded in Diff$(\mathbb{R})$ in three different ways. In one case, $SL(2, \mathbb{R})$ is also embedded in Diff$(S^1)$. Examples of such models include those with potential $V = \alpha x^2 + \beta x^{-2}$ for arbitrary couplings $\alpha$ and $\beta$, the Calogero models with harmonic oscillator couplings and nonlinear models with suitable metrics and potentials. In addition, we give the conditions on the couplings for a class of gauge theories to admit a $SL(2, \mathbb{R})$ conformal symmetry. We present examples of such systems with general gauge groups and global symmetries that include the isometries of $AdS_2 \times S^3$ and $AdS_2 \times S^3 \times S^3$ which arise as backgrounds in $AdS_2/CFT_1$.

PACS numbers: 03.65.-w, 11.25.Hf

1. Introduction

It has been known for sometime that one-dimensional models with potential $V = \beta x^{-2}$ are conformally invariant [1, 2]. de Alfaro, Fubini and Furlan (DFF) explored the $SL(2, \mathbb{R})$ conformal symmetry of this theory and noted that the Hamiltonian operator does not have a ground state [2]. To overcome this problem, they suggested to choose the eigenstates of

$$\mathcal{O} = \frac{p^2}{2} + \alpha x^2 + \beta x^{-2},$$

(1.1)
as a basis in the Hilbert space. $\mathcal{O}$ is not the Hamiltonian operator, but a linear combination of conserved charges associated with the $SL(2, \mathbb{R})$ conformal symmetry of the theory. Choosing suitably the coupling constants $\alpha$, $\beta$, this operator exhibits a ground state and discrete energy spectrum. As a result the DFF formulation of the theory has been widely accepted in the literature. However, although a Hilbert space has been defined for the theory, the Hamiltonian...
operator is not diagonal in the chosen basis and so the energy levels of the theory cannot be identified. There have been many generalizations of the $V = \beta x^{-2}$ model, see e.g. [3–8], including the construction of nonlinear theories [9, 10], which exhibit similar properties, see also reviews [11, 12] and references within. The DFF treatment of the theory and its generalizations have found widespread applications in the description of near horizon black hole dynamics [13–16] and in the understanding of black hole moduli spaces [17–21].

Another application of conformal mechanics is in the context of $AdS_2/CFT_1$ correspondence [22], and for further exploration see e.g. [23, 24]. It is expected that string theory or M-theory on an $AdS_2 \times X$ background is dual to a conformal theory on the boundary. After analytic continuation the Lorentzian boundary of $AdS_2$, which is two copies of $\mathbb{R}$, is mapped to a circle, see e.g. [23]. In the Euclidean regime, the associated dual theory should be a conformal theory defined on the circle. As we shall demonstrate, there are such conformal theories but they are based on different potentials from a conformal theory defined on the circle. As we shall demonstrate, there are such conformal theories but they are based on different potentials from a conformal theory defined on the circle.

In this paper, we investigate the conformal properties of theories with Lagrangian

$$L = \frac{1}{2} \dot{q}_i \dot{q}^i - V,$$

where $g$ is a metric on the configuration space, $V$ is a potential and $\dot{q}$ is the time derivative of the position. The conditions required for such theories to be invariant under the conformal transformations (2.1) have been stated in (2.2). Assuming that the configuration space of these theories admits a homothetic vector field $Z$ associated with a homothetic potential $h$, the conditions for conformal invariance (2.2) can be solved. The potential of the theory can be written as

$$V = V_0 + V_1,$$

where $V_0$ is a homogeneous function with respect to the homothetic motion $Z$ and $V_1$ obeys the inhomogeneous equation (2.13) which has as a source the homothetic potential $h$. The dimension of the conformal group of these models is at most 3 and one of the generators is time translations. This is because the parameter of the transformation obeys a third-order equation (2.10). The maximal conformal group is $SL(2, \mathbb{R})$ and it is embedded in $Diff(\mathbb{R})$ in three different ways generating the vector fields

1. $\partial_t, \ t\partial_t, \ t^2 \partial_t$;
2. $\partial_t, \ \cosh(\omega t) \partial_t, \ \sinh(\omega t) \partial_t$;
3. $\partial_t, \ \cos(\omega t) \partial_t, \ \sin(\omega t) \partial_t$;

for some $\omega$ related to the couplings.

The first $SL(2, \mathbb{R})$ embedding (i) in (1.4) is realized for the models with $V_1 = 0$. These class of models has a homogeneous potential $V_0$ and includes the DFF model, and its linear and nonlinear generalizations [9, 10]. Furthermore, if $V_1 \neq 0$, the $SL(2, \mathbb{R})$ conformal group is embedded in $Diff(\mathbb{R})$ generating the vector fields (ii) or (iii). These are Newton–Hooke transformations and the two cases are distinguished by the sign of the inhomogeneous term in equation (2.13), which determines $V_1$. The models with conformal transformations (ii) and (iii) are related by a naive analytic continuation, and the $SL(2, \mathbb{R})$ group in the latter case can be embedded in $Diff(S^1)$.

The class of conformal models with conformal symmetries (ii) and (iii) in (1.4) includes those with potential [3]

$$V = \alpha x^2 + \beta x^{-2},$$

1 With the term ‘linear theories’ we mean those for which the configuration space is $\mathbb{R}^n$ equipped with the Euclidean metric but they may exhibit a non-trivial potential. ‘Nonlinear theories’ are those with curved configuration space.

2 The $SL(2, \mathbb{R})$ conformal symmetry of the $V = -\beta x^{-2}$ model acts with fractional linear time re-parameterizations and it cannot be embedded in $Diff(S^1)$. We allow for diffeomorphisms with some discontinuities.
where $V_0 = \beta x^{-2}$ and $V_1 = \alpha x^2$. For $\alpha < 0$ the conformal group generates the vector fields (ii) in (1.4), while for $\alpha > 0$ the conformal group generates the vector field (iii). There are also several multi-particle models which exhibit types (ii) and (iii) in (1.4) conformal symmetry. Such systems include the Calogero model with harmonic oscillator couplings of equal frequency [27], and the multi-particle linear models of [28] for which $V_0$ satisfies additional symmetries. We shall present some additional linear and nonlinear systems with (ii) and (iii) conformal symmetries. Observe that the theories with $\alpha, \beta > 0$ in (1.5) have a ground state and discrete energy spectrum, and so there is no need to choose another operator different from the Hamiltonian to give a basis in the Hilbert space of the theory. This also applies to several other models in this class.

One result which follows from the general analysis of this paper is that the most general linear conformal model admits a potential (1.3), where $V_0$ is a homogeneous function of the positions $q$ of degree $-2$ and $V_1 = \alpha |q|^2$. This rigidity result is based on the uniqueness of homothetic motions in flat space associated with a homothetic potential. The homothetic motion is the homogeneous scaling of all coordinates, $q' \to \ell q'$. These models admit an $SL(2, \mathbb{R})$ conformal symmetry generated by the vector fields (ii) and (iii) in (1.4) and depending on whether $\alpha < 0$ or $\alpha > 0$, respectively.

More recently, conformal models in one dimension have been investigated which apart from scalar fields also contain vectors [25]. So far such theories have been based on gauging models with homogeneous potentials. We shall demonstrate that such models can be generalized to include potentials of the type (1.5). In particular, we derive the conditions (4.9) for gauged nonlinear sigma models with Lagrangian (4.1) to admit a conformal symmetry, and determine the equations that restrict the potentials. We find that for a large class of such conformal theories the potential can be written as in (1.3), where both $V_0$ and $V_1$ must also be gauge invariant. In addition, we give some examples which include conformal models with a general gauged group and global symmetries. Some of these models exhibit the isometries of $AdS_2 \times S^3$ and $AdS_2 \times S^3 \times S^3$ backgrounds as global symmetries. A class of these models is solvable, and the Hamiltonian has a ground state and discrete spectrum. A similar investigation of $SL(2, \mathbb{R})$ symmetries in the context of matrix models has been done in [26] and the associated potentials have been identified.

This paper is organized as follows. In section 2, we derive and investigate the conditions for conformal invariance of nonlinear one-dimensional theories and derive the scalar potential (1.3). In section 3, we give several examples of such models. In section 4, we derive the conditions on the couplings gauged sigma models with a potential to admit conformal invariance, and give several examples. In section 5, we present our conclusions.

2. Conformal models

2.1. Lagrangian

Consider the Lagrangian (1.2) of a sigma model on a manifold $M$ with metric $g$ and with a potential $V$. This describes either the propagation of a non-relativistic particle in a curved manifold $M$ or a multi-particle system with a non-trivial configuration space $M$. One can assign mass dimensions such that $q$ is dimensionless $[q] = 0$ while $[\ell t] = -1$. Thus $[L] = 2$ provided one takes the coupling $V$ terms to have dimension 2. This is not the most general Lagrangian that one can consider, as a coupling with dimension 1 has not been included. This will be done elsewhere [29].
2.2. Conformal transformations

All time re-parameterizations \( t' = u(t) \) are conformal transformations of the Euclidean metric on \( \mathbb{R} \) as \( ds^2 = (dt')^2 = (\dot{u})^2 dt^2 \). Therefore, one can choose any of these transformations and demand that leave the action (1.2) invariant. Apart from time translations, such transformations will not leave the action invariant unless there is a compensating additional transformation on the positions generated by a vector field \( X \) on \( M \) [9]. As a result, one considers the infinitesimal transformations [10]

\[
\delta q^i = -\epsilon a(t)q^i + \epsilon X^i(t, q),
\]

where \( \epsilon \) is a small parameter. The first term in the transformation of \( q \) is induced by the infinitesimal transformation \( \delta t = \epsilon a(t) \), where \( a(t) \) is the vector field on \( \mathbb{R} \) which generates the time re-parameterizations, while the second term containing \( X \) is the compensating transformation which may explicitly depend on \( t \).

The conditions for the invariance of the action (1.2), up to surface terms, under the transformations (2.1) are [10]

\[
L_X g_{ij} = \partial_i X_j g_{ij}, \quad \partial_i \partial_j X^i g_{ij} = -\partial_i f, \quad \partial_i X^i \partial_j V = -\partial_j f,
\]

where \( f = f(t, q) \) is the contribution from the surface term, and \( \partial_i \) denotes differentiation of the explicit dependence of \( X \) and \( f \) on \( t \), i.e.

\[
\frac{d}{dt} f(q, t) = \partial_i f + \dot{q}^i \partial_i f.
\]

The conserved charges associated with the above symmetries are

\[
Q(a, X) = \frac{a}{2} g_{ij} \dot{q}^i \dot{q}^j - g_{ij} \dot{q}^i X^j + a V + f.
\]

It can be easily shown that \( Q(a, X) \) is conserved subject to field equations.

2.3. Solution of conformal conditions and new models

It is clear that the first condition in (2.2) implies that \( X \) generates a family of homothetic transformations on \( M \) which may depend on \( t \). Since all \( \text{Diff}(\mathbb{R}) \) are conformal transformations, the system can be invariant under any subgroup of \( \text{Diff}(\mathbb{R}) \). So, one should consider at most as many homothetic motions in \( M \) as the dimension of the subgroup of conformal transformations. However, in most examples of interest \( M \) admits one homothetic motion generated by a vector field \( Z \) which does not depend explicitly on \( t \),

\[
L_Z g_{ij} = \ell g_{ij},
\]

where \( \ell \) is a constant. Then, the first condition can be solved by setting

\[
X^i(t, q) = \ell^{-1} a(t) Z^i(q).
\]

Assuming that \( Z \) arises from a homothetic potential, i.e.

\[
Z^i g_{ij} = \partial_j h,
\]

where \( h = h(q) \), \( f \) can be chosen [4]

\[
f = \ell^{-1} \dot{a} h.
\]

---

3 We have chosen the couplings \( g \) and \( V \) not to depend explicitly on time. However, it is straightforward to carry out the analysis of this section for models with time-dependent couplings.

4 We assume that \( \dot{a} \neq 0 \). If \( \dot{a} = 0 \), \( Z \) does not have to be associated with a homothetic potential and \( V \) is a homogeneous function of the homothetic transformation. The models do not have an \( \text{SL}(2, \mathbb{R}) \) symmetry but rather are invariant under time translations and scale transformations generated by the vector fields \( \partial_i, \tau \dot{a}_i \).
The last equation in (2.2) can now be rewritten as
\[ \dot{a}(V + \ell^{-1}Z^l\partial_l V) = -\ell^{-1}\partial^j a^2 h. \] (2.9)

Since we are seeking to find potentials $V$ which solve the above equations and do not depend explicitly on $t$, we have to take
\[ \partial^3 t a = \lambda \dot{a}, \] (2.10)
where $\lambda$ is a constant. Of course, if $\dot{a} = 0$, there is no condition on $V$ as the only symmetry of the action is time translations. Thus, we take $\dot{a} \neq 0$ and as a result the equation which determines the potential is
\[ V + \ell^{-1}Z^l\partial_l V = -\ell^{-1}\lambda h. \] (2.11)

The general solution for the potential can be written as in (1.3), i.e. $V = V_0 + V_1$, where $V_0$ is the most general solution of the homogeneous equation
\[ V_0 + \ell^{-1}Z^l\partial_l V_0 = 0, \] (2.12)
and $V_1$ is a solution of
\[ V_1 + \ell^{-1}Z^l\partial_l V_1 = -\ell^{-1}\lambda h. \] (2.13)

Clearly, there are three cases to consider depending on whether $\lambda = 0$, or $\lambda > 0$ or $\lambda < 0$. In these three choices, the vector field $a$ is determined from (2.10) as follows. For $\lambda = 0$, one has
\[ a = a_0 + a_1 t + a_2 t^2, \] (2.14)
where $a_0$, $a_1$ and $a_2$ are integration constants. For $\lambda = \omega^2$, one has
\[ a = a_0 + be^{\omega t} + ce^{-\omega t}, \] (2.15)
and for $\lambda = -\omega^2$, one has
\[ a = a_0 + b \cos(\omega t) + c \sin(\omega t), \] (2.16)
where $a_0$, $b$, $c$ are integration constants. The new conformal models arise from the last two cases.

Before we proceed to investigate individual models, let us examine the algebra of these transformations. A basis in the space of vector fields of the infinitesimal transformations (2.14), (2.15) and (2.16) is given in (i), (ii) and (iii) of (1.4), respectively, with $|\lambda| = \omega^2$. The group of transformations generated by (2.14), (2.15) and (2.16) is $SL(2, \mathbb{R})$. However, $SL(2, \mathbb{R})$ is embedded into Diff$(\mathbb{R})$ in three different ways$^5$. The group of transformations generated by (2.16) is also embedded in the Diff$(S^1)$ as the associated vector fields are periodic in $t$. The two cases (2.15) and (2.16) are related to each other by analytic continuation.

Substituting the above expressions of $X$ into the conserved charges and using the properties of the homothetic motion on $M$, one finds that
\[ Q(a, Z) = \frac{a}{2} g_{ij} \dot{q}^i \dot{q}^j - \dot{a} \ell^{-1} \partial^i h q^i + a(V_0 + V_1) + \ell^{-1}\dot{a} h. \] (2.17)

These can be easily computed explicitly in the examples described below.

$^5$ In the (2.14) case, $SL(2, \mathbb{R})$ acts with fractional linear transformations on $\mathbb{R}$.
3. Examples

3.1. Conformal particle in flat space

The most illuminating model is that of a single particle propagating on the real line. Here we shall show that \( L = \frac{1}{2} \dot{x}^2 - V(x) \), which has been found previously in [3], is the only potential consistent with conformal invariance. For this we shall take the Lagrangian

\[
L = \frac{1}{2} \dot{x}^2 - V(x),
\]

and we shall determine \( V \) such that the action is conformally invariant. For this, consider the homothetic vector field

\[
Z = \frac{1}{2} x \partial_x,
\]

on the configuration space. For this choice of \( Z, \ell = 1 \). The homothetic potential in this case is

\[
h = \frac{1}{4} x^2.
\]

Then equation (2.12) can be solved for \( V_0 \) to yield

\[
V_0 = \beta x^{-2},
\]

for some constant \( \beta \), which is the potential of the DFF model. However, we have seen that the potential \( V \) also receives a contribution from \( V_1 \) which is determined in (2.13). The latter equation can be solved as

\[
V_1 = \alpha x^2, \quad \alpha = -\frac{\lambda}{8}.
\]

Thus the most general potential \( V = V_0 + V_1 \) of such conformal models is given in (1.5).

The Hamiltonian of this class of conformal models is given in (1.1). As it has already been mentioned, the associated Hamiltonian operator with \( \alpha > 0, \beta \geq 0 \) has a ground state and discrete spectrum.

3.2. Conformal multi-particle systems

Consider next the linear model of \( N \) particles propagating in \( \mathbb{R} \) and interacting with a potential \( V \). The Lagrangian of such a system is

\[
L = \frac{1}{2} \sum_i (\dot{x}_i)^2 - V(x_i).
\]

To find the potentials \( V \) consistent with conformal invariance, consider the homothetic motion

\[
Z = \frac{1}{2} \sum_{i=1}^N x_i \partial_{x_i},
\]

of the \( \mathbb{R}^N \) configuration space. The homothetic potential in this case is

\[
h = \frac{|x|^2}{4}, \quad |x|^2 = \delta_{ij} x_i x_j.
\]

As has been mentioned in the introduction, \( Z \) in (3.7) is the unique homothetic motion in \( \mathbb{R}^N \) associated with a homothetic potential\(^6\) up to an overall scale which does not affect the form of the potential. After solving conditions (2.12) and (2.13), one finds that the potential \( V \) is

\[
V = \alpha |x|^2 + V_0(x), \quad \alpha = -\frac{\lambda}{8}
\]

\(^6\) If the requirement of the homothetic potential is removed, the scaling transformation (3.7) can mix with other isometries, like \( SO(N) \) rotations, to give rise to new homothetic motions. These can be used to construct invariant theories under subgroups of \( SL(2, \mathbb{R}) \) involving at most two generators.
and $V_0$ is a homogeneous function of degree $-2$,
$$x^i \partial_i V_0 = -2V_0. \quad (3.10)$$
Equation (3.9) is the most general potential of linear models.

Of course, there are many choices for $V_0$. A minimal choice for $V_0$ is $V_0 = \beta |x|^{-2}$.
However, this is not unique. For example, one can also choose
$$V_0 = \sum_{i \neq j} \beta_{ij} (x^i - x^j)^2. \quad (3.11)$$
The models with potentials $V$ given in (3.9) and (3.11) are the Calogero models with harmonic
couplings of equal frequency. Our results demonstrate that these models are conformally
invariant. It is well known that such models with $\alpha > 0$ and $\beta \geq 0$ have a vacuum state and
discrete energy spectrum [27, 30]. Of course, there are many more potential functions $V_0$ which
satisfy the homogeneity condition (3.10) above than those appearing in the Calogero models.
The above models also include those presented in [28] where some additional symmetry
assumptions were made on the form of $V_0$ potential.

To summarize, we have shown that all the above models admit an $SL(2, \mathbb{R})$ conformal
symmetry which is embedded in $\text{Diff}(\mathbb{R})$ as in (i), (ii) or (iii) of (1.4) depending on whether
$\alpha = 0$, $\alpha < 0$ or $\alpha > 0$, respectively. The associated conserved charges can be computed by a
direct substitution in (2.17).

3.3. Particles propagating on cones

So far, we have presented linear models as examples. For a nonlinear example, consider
particles propagating on a cone and interacting with a potential $V$. The Lagrangian of such a
system is
$$\mathcal{L} = \frac{1}{2} \left( \dot{r}^2 + r^2 \gamma_{ij} \dot{x}^i \dot{x}^j \right) - V(r, x), \quad (3.12)$$
where $\gamma$ is the metric of the cone section which does not depend on the radial coordinate $r$ but
it may depend on the rest of the coordinates $x$. The cone metric
$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \gamma_{ij} \mathrm{d}x^i \mathrm{d}x^j, \quad (3.13)$$
adopts a homothetic motion generated by the vector field
$$Z = \frac{1}{2} r \partial_r, \quad (3.14)$$
with the homothetic potential
$$h = \frac{r^2}{4} + k(x), \quad (3.15)$$
where $k$ is an arbitrary function of $x$. It is straightforward to show that the most general potential
compatible with conformal symmetry is
$$V = \alpha r^2 + \beta(x) r^{-2} + 8\alpha k(x), \quad \alpha = -\lambda/8. \quad (3.16)$$
Again these models admit an $SL(2, \mathbb{R})$ conformal symmetry generating the vector fields (i),
(ii) or (iii) of (1.4) depending on whether $\alpha = 0$, $\alpha < 0$ or $\alpha > 0$, respectively.

4. Conformal gauge theories in one dimension

4.1. Action

Motivated by applications in $\text{AdS/CFT}$, which typically requires dual theories with a gauge
symmetry, and to enhance the class of one-dimensional conformal systems, we shall also
examine the conditions for a gauged sigma model to admit conformal invariance. For this, we assume that $M$ admits a group of isometries $G$, generating the vector fields $\xi$, which leave $V$ invariant. Gauging the isometries of (1.2), one finds the Lagrangian
\[ L = \frac{1}{2} g_{ij} \nabla_t q^i \nabla_t q^j - V, \] (4.1)
where
\[ \nabla_t q^i = \dot{q}^i - A^a \xi_a^i, \quad [\xi_a, \xi_b] = -f_{ab}^c \xi_c, \] (4.2)
$A$ is the gauge potential and $f$ are the structure constants of $G$. We assign mass dimension to $A$ as $[A] = 1$ so that $L$ has a mass dimension 2.

The equations of motion of the theory are
\[ g_{ij} D_t \nabla_t q^j + \partial_i V = 0, \quad \xi_a \nabla_t q^j = 0, \] (4.3)
where
\[ D_t \nabla_t q^j = \partial_t \nabla_t q^j - A^a \partial_j \xi_a^i \nabla_t q^j + \Gamma^j_{ik} \nabla_t q^i \nabla_t q^k. \] (4.4)
Under certain conditions the gauge connection $A$ can be eliminated from the equations of motion leading to a theory with dynamical variables just the $q$s. In particular note that the second equation of motion can be rewritten as
\[ \xi_{ab} A^b = \xi_{ab} q^i \] (4.5)
where $\ell_{ab} = g_{ij} \xi_a^i \xi_b^j$. If $\ell$ is invertible, then all $A$ can be eliminated. However, we shall not elaborate on this here. Instead, we shall proceed to find the conditions such that the action (4.1) is invariant under some conformal symmetries.

4.2. Conformal and gauge symmetries

The action (4.1) is invariant under the gauge transformations
\[ \delta q^i = \eta^a \xi_a^i, \quad \delta A^a = \nabla_i \eta^a, \] (4.6)
where $\eta$ is the gauge infinitesimal parameter.

Next as in the un-gauged case, one expects that the transformations on $q$ and $A$, which induce the conformal symmetries of the action (4.1), to contain two parts. One part is associated with time re-parameterizations and an additional term which generates compensating transformations on the configuration space. As a result, we postulate the conformal transformations
\[ \delta q^i = -\epsilon a(t) \partial_t q^i + \epsilon X^i(t, q, A), \]
\[ \delta A^a = -\epsilon a A^a - \epsilon a A^a + \epsilon W^a(t, q, A), \] (4.7)
where the first term in the variation of $q$ and the first two terms in the variation of $A$ are the transformations induced on $q$ and $A$ from the infinitesimal re-parameterization of $t$, $\delta t = \epsilon a(t)$, and the rest are the compensating transformations.

These transformations mix with the gauge transformations above. In particular, the coordinate transformation induced on $A$ by $a$ can be rewritten as a gauge transformation with parameter $-aA^a$. Since the action is invariant under gauge transformations, this can be used to simplify the conformal transformations as
\[ \delta q^i = -\epsilon a(t) \nabla_t q^i + \epsilon X^i, \]
\[ \delta A^a = \epsilon W^a. \] (4.8)

This is not the most general Lagrangian of dimension 2 as couplings of dimension 1 have not been included.
For the same reason $X$ and $Z$ are not uniquely defined. In particular, $X$ and $W$ are defined up to terms $\ell^a \xi_a$ and $\nabla_t \ell^a$, respectively, where $\ell = \ell(t, q, A)$.

Assuming that $X$ and $W$ do not depend on time derivatives of $q$, a straightforward computation reveals that the conditions required for the invariance of the action, up to surface terms, are

$$
\mathcal{L}_X g_{ij} = \dot{a} g_{ij},
$$

$$
g_{ij} \partial_t X^j + g_{ij} A^a [\xi_a, X]^j - g_{ij} \xi^j_i W^b = \partial_t f,
$$

$$
\dot{a} V + X^j \partial_k V = -\partial_t f,
$$

(4.9)

where $f = f(t, q)$ is the contribution from the surface term. $f$ is taken to be gauge invariant, $\xi^i_a \partial_i f = 0$. To find conformal models, one has to solve (4.9).

4.3. Solution of conformal conditions

Here, we shall not seek the most general solution to the conformal invariance conditions (4.9). Instead, we shall take $[\xi_a, X] = 0$, $W^a = 0$.

(4.10)

In this case, the above conditions (4.9) reduce to those of (2.2) but with the additional assumption that $f$ is gauge invariant.

To find solutions, we proceed as in section 2.3. The potential is given as $V = V_0 + V_1$, (1.3), with $V_0$ and $V_1$ determined by equations (2.12) and (2.13), respectively. There is an additional restriction here that the homothetic potential $h$ is gauge invariant, $\xi_i^a \partial_i h = 0$.

As in the systems without gauge symmetry, there are three cases to consider depending on whether $\lambda = 0$, $\lambda > 0$ or $\lambda < 0$. In all cases the conformal group is $SL(2, \mathbb{R})$ but it is embedded in three different ways into Diff($\mathbb{R}$). The $\lambda > 0$ and $\lambda < 0$ models are related by analytic continuation.

4.4. Examples

4.4.1. Gauged nonlinear models on a cone. Examples of nonlinear gauge theories exhibiting conformal symmetry are those that describe the propagation of particles on a cone. Assuming that the cone section metric $\gamma$ admits a group of isometries generating the vector fields $\xi$, the Lagrangian of the theory can be written as

$$
\mathcal{L} = \frac{1}{2} \left( \dot{r}^2 + r^2 \gamma_{ij} \nabla_i x^j \nabla_k x^l \right) - V(r, x),
$$

(4.11)

where

$$
\nabla_i x^j = \dot{x}^i - \xi^i_a A^a.
$$

(4.12)

The homothetic vector field is again given by $Z = \frac{1}{2} r \partial_r$, and commutes with the Killing vector fields $\xi_a$ satisfying the assumption (4.10).

The rest of the analysis proceeds as in the cone example in section 3.3 for the un-gauged model yielding a potential

$$
V = \alpha r^2 + \beta(x) r^{-2} + 8 \alpha k(x), \quad \alpha = -\lambda/8,
$$

(4.13)

where now $\beta(x)$ and $k(x)$ are gauge invariant functions of the cone section, $\xi^i_a \partial_i \beta = \xi^i_a \partial_i k = 0$.

The simplest explicit example is to consider the flat cone $\mathbb{R}^2$ and as the gauged symmetry the rotational symmetry. The potential of this model is given as in (4.24) with $\beta$ and $k$ constants.
4.4.2. Gauge theories. A large class of linear conformal models can be constructed beginning from some gauge group $G$ and some linear representation $D$ of its Lie algebra $\mathfrak{g}$ on a vector space $V$. Suppose that $D$ leaves invariant a (constant) metric $g$ on $V$. Then one can consider the Lagrangian

$$L = \frac{1}{2} g_{m n} \nabla_t x^m \nabla_t x^n - V(x),$$

where

$$\nabla_t x^m = \dot{x}^m - A^a (D_a)^m_{n} x^n.$$  

(4.15)

To determine $V$ such that this theory is conformal, observe that the metric admits a homothetic motion generated by the vector field

$$Z = \frac{1}{2} x^m \partial_m.$$  

(4.16)

Moreover, this commutes with the Killing vector fields

$$\xi_a = \frac{1}{2} (D_a)^m_{n} x^n \partial_m,$$

i.e. $[Z, \xi_a] = 0$. As a consequence (4.10) is satisfied. Furthermore, the homothetic potential of $Z$ is

$$h = \frac{1}{4} g_{m n} x^m x^n.$$  

(4.18)

Using this, the potential $V$ can be determined by solving (2.12) and (2.13) as

$$V = \alpha g_{m n} x^m + V_0, \quad \alpha = -\frac{\lambda}{8},$$

(4.19)

and $V_0$ is a function of $x$ of homogeneous degree $-2$,

$$x^m \partial_m V_0 = -2V_0,$$

(4.20)

which is also invariant under $G$. The minimal choice is

$$V_0 = \beta g_{m n} x^m x^n.$$  

(4.21)

However, such a choice is not unique for general gauge groups and representations $D$. A similar potential has been derived in the investigation of $SL(2, \mathbb{R})$ invariant matrix models in [26].

Amongst these models, one can take $D = \text{ad} \otimes I^2$, where $\text{ad}$ is the adjoint representation of a group $G$ and $I$ is the trivial representation. In such a case, the Lagrangian can be written as

$$L = \frac{1}{2} g_{a b} \kappa_{i j} \nabla_t x^a \nabla_t x^b - V(x),$$

(4.22)

where

$$\nabla_t x^a = \dot{x}^a - A^b f_{bc}^a x^c,$$

(4.23)

$g_{a b}$ is an invariant metric on the adjoint representation of $G$ and $\kappa$ a metric on the $k$-copies of the trivial representation. The potential in this case can be written as

$$V = \alpha g_{a b} \kappa_{i j} x^a x^b + V_0, \quad \alpha = -\frac{\lambda}{8},$$

(4.24)

and $V_0$ is a function of $x$ of homogeneous degree $-2$ which is also invariant under $G$. Now there are several options for $V_0$. For example, $V_0$ can be any homogeneous function of degree $-2$ expressed in terms of the gauge invariant functions like

$$m^{i j} = g_{a b} x^a x^b, \quad m^{i j k} = f_{a b c} x^a x^b x^c,$$

(4.25)

These can also be thought of as special cases of the cone models above.

---

8 These can also be thought of as special cases of the cone models above.
and many others which can be constructed from all the invariant tensors of $g$ under the action of the adjoint representation. One example is a gauged Calogero model for which the potential is given in (4.24) with

$$V_0 = \sum_{i\neq j} \frac{\beta_{ij}}{g_{ab}(x^a - x^b)}(x^b - x^a).$$

(4.26)

Further restrictions can be put on the form of the potential by requiring that the theory is invariant under the global symmetry $\times O(n)$ which leaves $\kappa$ invariant. The above construction can also be done by replacing $a\otimes j$ with another representation of the gauge group.

This class of conformal theories has all the bosonic symmetries required for the CFT duals of backgrounds like $AdS_2 \times S^3$ or $AdS_2 \times S^1 \times S^3$. In particular, one can easily construct models with rigid symmetry $SL(2,\mathbb{R}) \times SO(4)$, which is the isometry group of $AdS_2 \times S^3$, and any gauge symmetry including $U(N)$, and similarly there are models which exhibit the isometries of $AdS_2 \times S^3 \times S^3$ backgrounds as symmetries. It is also worth remarking that the analytic continuation of a $\lambda > 0$ theory which exhibits $SL(2,\mathbb{R})$ conformal symmetry is equivalent to taking $\lambda$ to $-\lambda$ and $V_0$ to $-V_0$ and leads to a model with $SL(2,\mathbb{R})$ conformal invariance but now embedded in $Diff(S^1)$ as expected in the context of $AdS_2/CFT_1$.

The quantum theory of the model with action (4.14) can be easily described in the case when $V_0 = 0$ and $\alpha > 0$. The Hilbert space of these theories can be constructed starting from the Hilbert space of $\text{dim} D$ harmonic oscillators. Then, gauge invariance requires that one has to consider only those states which are invariant under the gauge group. The Hamiltonian operator has a ground state and the spectrum is discrete. However, the details of the construction depend on the choice of the gauge group and representation $D$. If $V_0 \neq 0$, the quantum theory depends on the choice of $V_0$. It is likely that some of the properties of the $V_0 = 0$ models can be maintained in the presence of a large class of $V_0$ potentials as it happens for the Calogero models with harmonic oscillator couplings.

5. Concluding remarks

We have demonstrated that the potential $V$ of conformal mechanics models admitting a homothetic motion in configuration space can be expressed as a sum $V = V_0 + V_1$, where $V_0$ is a homogeneous function of the homothetic motion and $V_1$ is determined from an equation which has as a source the homothetic potential. Depending on the couplings, the maximal conformal group $SL(2, \mathbb{R})$ is embedded in $Diff(\mathbb{R})$ in three different ways. Furthermore, one of these can also be thought as an embedding of $SL(2, \mathbb{R})$ in $Diff(S^1)$. This is significant from the point of view of $AdS_2/CFT_1$ as the dual Euclidean theory must be defined on the boundary which is a circle.

Examples of conformal one-dimensional systems include models with potential $V = \alpha x^2 + \beta x^{-2}$ [3]. The $SL(2, \mathbb{R})$ conformal symmetry of this model is embedded in $Diff(\mathbb{R})$ in three different ways depending on whether $\alpha = 0$, $\alpha < 0$ or $\alpha > 0$, respectively. Moreover, if $\alpha > 0$, $SL(2, \mathbb{R})$ can also be embedded in $Diff(S^1)$.

We have described all one-dimensional linear conformal theories described by the Lagrangian (3.6). The potential of all such models is $V = \alpha |x|^2 + V_0$, where $V_0$ is a homogeneous of degree $-2$ function of the positions $x$. This rigidity result is based on the uniqueness of the homothetic motion in flat space associated with a homothetic potential and the analysis in section 2. Examples of such theories include the Calogero models with harmonic oscillator couplings of equal frequency as well as the models given in [28]. We have also presented examples of nonlinear models.
It is clear from the analysis of section 2 that if the configuration space of a system admits a single homothetic motion associated with a homothetic potential, then the vector field \( a(t) \partial_t \) which generates the time re-parameterizations obeys the third-order equation (2.10). Because of this, the conformal group can be at most three dimensional. Therefore, if there are theories with larger conformal groups than \( SL(2, \mathbb{R}) \), then necessarily must have additional fields, like vectors or spinors, and possibly must couple to gravity. As a consequence all linear models admit at most an \( SL(2, \mathbb{R}) \) conformal symmetry.

We have also investigated the conformal properties of one-dimensional systems with scalar and vector fields based on the Lagrangian (4.1). We have derived the conditions for such systems to admit a conformal symmetry (4.9) and present several examples. The potential of a class of such theories is again the sum of a homogeneous function, under the action the homothetic motion, and a term that depends on the homothetic potential. Examples of such conformal models can exhibit general gauge groups and global symmetries. In particular, we have constructed models with arbitrary gauge group which have the isometries of \( AdS_2 \times S^3 \) and \( AdS_2 \times S^3 \times S^3 \) backgrounds as global symmetries. Similar potentials have arisen in the investigation of matrix models with \( SL(2, \mathbb{R}) \) invariance in [26].

Gravitational backgrounds that have applications in \( AdS_2/CFT_1 \) typically preserve some of the spacetime supersymmetry and as a result the dual theories must be superconformal. The supersymmetric extension of some of the conformal models we have considered here has already been done; see e.g. [30] and [9, 10] for the supersymmetric extension of Calogero model with harmonic oscillator couplings and that of nonlinear conformal theories with homogeneous potentials, respectively (see also [32] for matrix models). Conformal linear models with extended supersymmetry and homogeneous potentials have been reviewed in [12], see also [33]. It is straightforward to construct superconformal models with potentials \( V = V_0 + V_1 \) especially those that exhibit a small number of supersymmetries. Such supersymmetric extensions can be based on the results of [31, 19] and they will be reported elsewhere.

Acknowledgments

I would like to specially thank Anton Galajinsky and Jeong-Hyuck Park for their comments as they have led to significant improvements in the paper. I would also like to thank Evgeny Ivanov and Roman Jackiw for correspondence. GP is partially supported by the STFC rolling grant ST/J002798/1.

References

[1] Jackiw R 1972 Introducing scale symmetry Phys. Today 25 23
[2] de Alfaro V, Fubini S and Furlan G 1976 Conformal invariance in quantum mechanics Nuovo Cimento A 34 569
[3] Jackiw R 1980 Dynamical symmetry of the magnetic monopole Ann. Phys. 129 183
[4] Akulov V P and Pashnev I A 1983 Quantum superconformal model in (2,1) space Theor. Math. Phys. 56 862
[5] Fubini S and Rabinovici E 1984 Superconformal quantum mechanics Nucl. Phys. B 245 17
[6] Ivanov E, Krivonos S and Leviant V 1989 Geometry of conformal mechanics J. Phys. A: Math. Gen. 22 345
[7] Jackiw R 1990 Dynamical symmetry of the magnetic vortex Ann. Phys. 201 83
[8] Wyllard N 2000 (Super)conformal many body quantum mechanics with extended supersymmetry J. Math. Phys. 41 2826 (arXiv:hep-th/9910160)
[9] Michelson J and Strominger A 2000 The geometry of (super)conformal quantum mechanics Commun. Math. Phys. 213 1 (arXiv:hep-th/9907191)
[10] Papadopoulos G 2000 Conformal and superconformal mechanics Class. Quantum Grav. 17 3715 (arXiv:hep-th/0002007)
[11] Britto-Pacumio R, Michelson J, Strominger A and Volovich A 1999 Lectures on superconformal quantum mechanics and multiblack hole moduli spaces arXiv:hep-th/9911066
[12] Fedoruk S, Ivanov E and Lechtenfeld O 2012 Superconformal mechanics J. Phys. A: Math. Theor. 45 173001 (arXiv:1112.1947[hep-th])
[13] Claus P, Derix M, Kallosh R, Kumar J, Townsend P and van Proeyen A 1998 Black holes and superconformal mechanics Phys. Rev. Lett. 81 4553 (arXiv:hep-th/9804177)
[14] de Azcarraga J A, Izquierdo J M, Buono J C Perez and Townsend P K 1999 Superconformal mechanics, black holes, and non-linear realizations Phys. Rev. D 59 084015 (arXiv:hep-th/9810230)
[15] Gibbons G W and Townsend P K 1999 Black holes and Calogero models Phys. Lett. B 454 187 (arXiv:hep-th/9812034)
[16] Galajinsky A 2008 Particle dynamics on AdS(2) × S^2 background with two-form flux Phys. Rev. D 78 044014 (arXiv:0806.1629[hep-th])
Galajinsky A 2010 Particle dynamics near extreme Kerr throat and supersymmetry J. High Energy Phys. JHEP11(2010)126 (arXiv:1009.2341[hep-th])
[17] Gibbons G W and Ruback P J 1986 The motion of extreme Reissner–Nordström black holes in the low velocity limit Phys. Rev. Lett. 57 1492
[18] Shiraishi K 1993 Moduli space metric for maximally-charged dilaton black holes Nucl. Phys. B 402 399
[19] Gibbons G W, Papadopoulos G and Stelle K S 1997 HKT and OKT geometries on soliton black hole moduli spaces Nucl. Phys. B 508 623 (arXiv:hep-th/9706207)
[20] Michelson J and Strominger A 1999 Superconformal multi-black hole quantum mechanics (HUPTP-99/A047) arXiv:hep-th/9908044
[21] Gutowski J and Papadopoulos G 2000 The dynamics of very special black holes Phys. Lett. B 472 45 (arXiv:hep-th/9910022)
Gutowski J and Papadopoulos G 2000 Moduli spaces for four-dimensional and five-dimensional black holes Phys. Rev. D 62 064023 (arXiv:hep-th/0002242)
[22] Maldacena J 1998 The large N limit of superconformal field theories and supergravity Adv. Theor. Math. Phys. 2 231 (arXiv:hep-th/9711200)
[23] Sen A 2011 State operator correspondence and entanglement in AdS_2/CFT_1 Entropy 13 1305 (arXiv:1101.4254 [hep-th])
[24] Chamoun C, Jackiw R, Pi S-Y and Santos L 2011 Conformal quantum mechanics as the CFT_1 dual to AdS_2 Phys. Lett. B 701 503 (arXiv:1106.0726[hep-th])
[25] Fedoruk S, Ivanov E and Lechtenfeld O 2009 Supersymmetric Calogero models by gauging Phys. Rev. D 79 105015 (arXiv:0904.4276 [hep-th])
[26] Erdmenger J, Park J-H and Sochichiu C 2006 Matrix models from D-particle dynamics and the string/black hole transition Class. Quantum Grav. 23 6873 (arXiv:hep-th/0603243)
[27] Calogero F 1971 Solution of the one-dimensional N body problems with quadratic and/or inversely quadratic pair potentials J. Math. Phys. 12 419
[28] Galajinsky A 2010 Conformal mechanics in Newton–Hooke spacetime Nucl. Phys. B 832 586 (arXiv:1002.2290 [hep-th])
[29] Papadopoulos G to appear
[30] Freedman D Z and Mende P 1990 An exactly solvable N particle system in supersymmetric quantum mechanics Nucl. Phys. B 344 317
[31] Coles R A and Papadopoulos G 1990 The geometry of the one-dimensional supersymmetric nonlinear sigma models Class. Quantum Grav. 7 427
[32] Copland N B, Ko S M and Park J-H 2012 Superconformal Yang–Mills quantum mechanics and Calogero model with OSP(N(2|2), R) symmetry J. High Energy Phys. JHEP07(2012)076 (arXiv:1205.3869 [hep-th])
[33] Kuznetsova Z and Toppan F 2012 D-module representations of N = 2, 4, 8 superconformal algebras and their superconformal mechanics J. Math. Phys. 53 043513 (arXiv:1112.0995[hep-th])
Khodaee S and Toppan F 2012 Critical scaling dimension of D-module representations of N = 4, 7, 8 superconformal algebras and constraints on superconformal mechanics arXiv:1208.3612 [hep-th]