t−ASPECT SUBCONVEXITY FOR $GL(2)$—$L$ FUNCTIONS

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ABSTRACT. Let $f$ be a holomorphic cusp form for $SL_2(\mathbb{Z})$ of weight $k > 1$. In these notes, we follow Munshi \[8\] to prove the Burgess bound
\[L(1/2 + it, f) \ll_{f, \varepsilon} (1 + |t|)^{1/2 - 1/8 + \varepsilon}.

1. Introduction

Let $f$ be a holomorphic cusp form for $SL_2(\mathbb{Z})$ of weight $k > 1$. The $L$-series is given by,
\[L(s, f) = \sum_{n \geq 1} \lambda_f(n)n^{-s} \quad \text{for } \text{Re}(s) > 1.\]

This extends to an entire function on the whole complex plane $\mathbb{C}$. The convexity principle gives the bound $L(1/2 + it, f) \ll_f (1 + |t|)^{1/2}$, known as the convexity bound. The purpose of this paper is to prove the following bound.

Theorem 1.1. Let $f$ be a holomorphic cusp form for $SL_2(\mathbb{Z})$. Then we have,
\[L(1/2 + it, f) \ll_{f, \varepsilon} (1 + |t|)^{1/2 - 1/8 + \varepsilon}.

The first such bound was obtained by Good \[2\]. The result was extended to Maass cusp forms by Jutila \[4\]. $t$-aspect subconvexity for higher $GL(n)$ is largely unknown. Subconvex bounds for $GL(1)$ and $GL(2)$, uniformly in all aspects is known by the works of Michel-Venkatesh \[7\]. $t$-aspect subconvexity for self dual Hecke-Maass forms for $GL(3)$ was first established by Li \[6\]. Munshi \[8\] used a different method (that we follow and execute) to extend the result to all Hecke-Maass cusp forms. Recently, Singh \[10\] did similar calculations for $t$-aspect subconvexity for $GL(2)$ $L$-functions of holomorphic and Hecke-Maass cusp forms and claims to get the Weyl bound.

We have followed the ideas of Munshi \[8\] and use a modification of the circle method. In the present situation, Kloosterman’s version of the circle method works best. Let,
\[\delta(n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases} \]

Then for any real number $Q > 0$, we have,
\[(1.1) \quad \delta(n) = 2 \text{Re} \int_0^1 \sum_{1 \leq q < Q} \sum_{\substack{a \leq Q \mod q \neq 0} \text{gcd}(a, q) = 1} \frac{1}{aq} e \left( \frac{na}{q} - \frac{nx}{aq} \right) dx \]
for $n \in \mathbb{Z}$. Here $e(\cdot) = e^{2\pi i \cdot}$ and the * on the inner sum means that $(a, q) = 1$. $\overline{a}$ is the multiplicative inverse of $a \mod q$. There are well understood drawbacks of this circle method. It will turn out that this circle method in itself will not be sufficient, and we will have to apply a ‘conductor lowering trick’ as used by Munshi in his various works \[8, 9\].
Suppose \( t > 2 \). The approximate functional equation gives
\[
L(1/2 + it, f) \ll t^\varepsilon \sup_{N \leq t^{1+\varepsilon}} \frac{|S(N)|}{N^{1/2}} + t^{-2015}
\]
where
\[
S(N) := \sum_{n \geq 1} \lambda(n) n^{-it} V\left(\frac{n}{N}\right).
\]

Let \( V \) be a smooth function supported on \([1, 2]\) satisfying \( V^{(j)} \ll j \). We further normalize \( V \) so that \( \int_{\mathbb{R}} V(x) dx = 1 \). We will apply (1.1) directly to \( S(N) \) with a conductor lowering integral to separate the oscillations of \( \lambda(n) \) and \( n^{-it} \).

(1.2)
\[
S(N) = \frac{1}{K} \int_{\mathbb{R}} V\left(\frac{v}{K}\right) \sum_{n \geq 1} \lambda(n) m^{-it} \left(\frac{n}{m}\right)^i V\left(\frac{n}{N}\right) U\left(\frac{m}{N}\right) \delta(n-m) dv.
\]
where \( t^\varepsilon < K < t \) is a parameter that will be optimized later. \( U \) is a smooth function which is supported on \([1/2, 5/2] \), with \( U(x) = 1 \) on \([1, 2] \) and satisfies \( U^{(j)} \ll j \). The extra integral introduced is
\[
\frac{1}{K} \int_{\mathbb{R}} \left(\frac{n}{m}\right)^i V\left(\frac{v}{K}\right) dv.
\]
For \( n, m \in \{N, 2N\} \), integration by parts shows that the above integral is small if \( |n-m| \gg Nt^\varepsilon / K \). This is the crucial ‘trick’ in the paper. As Munshi points out in the \( SL_3(\mathbb{Z}) \) case [8], introduction of this parameter \( K \) will seem to hurt us until the very last step, which we will justify in the proof sketch.

We can therefore write \( S(N) = S^+(N) + S^-(N) \) where
\[
S^\pm(N) = \frac{1}{K} \int_{\mathbb{R}} V\left(\frac{v}{K}\right) \sum_{1 \leq q \leq Q < a \leq Q} \frac{1}{aq} \sum_{n,m \geq 1} \lambda(n) n^\pm \left(\frac{n}{m}\right) m^{-i(t+v)}
\]
(1.3)
\[
e^{-\left(\frac{(n-m)a}{q} \mp \frac{(n-m)x}{aq}\right)} V\left(\frac{n}{N}\right) U\left(\frac{m}{N}\right) dv dx.
\]
The analysis and bounds for \( S^+(N) \) and \( S^-(N) \) are similar. We therefore analyze only \( S^+(N) \). We will justify later in Remark 3.2 that the natural choice for \( Q \) is \( Q = (N/K)^{1/2} \) (and thus the lowering of conductor by \( K^{1/2} \)).

We will take
\[
t^{3/4} \ll N < t^{1+\varepsilon} \text{ and } N^{1/2} \leq K \ll N^{1-\varepsilon}
\]
In this range, we will establish the following bound.

**Proposition 1.2.** For \( t^{3/4} \ll N < t^{1+\varepsilon} \), we have
\[
\frac{S^+(N)}{N^{1/2}} \ll t^{1/2+\varepsilon} \left(\frac{K^{1/2}}{N^{1/2}} + \frac{1}{K^{1/4}}\right).
\]

Same bound holds for \( S^-(N) \), and consequently for \( S(N) \). The optimal choice of \( K \) is therefore \( K = N^{2/3} \). With this choice of \( K \), \( S(N)/N^{1/2} \ll t^{1/2}/N^{1/6} \). For \( N \ll t^{3/4} \), the trivial bound \( S(N) \ll Nt^\varepsilon \) is sufficient. This follows by applying Cauchy’s inequality to the \( n \)-sum followed by Lemma 2.2 (Ramanujan bound on average). Theorem 1.1 then follows from Lemma 2.2 and Proposition 1.2.
1.1. **Proof Sketch.** We briefly explain the steps of the proof and provide heuristics in this subsection. Temporarily assume Ramanujan conjecture $\lambda(n) \ll n^\varepsilon$. This is not a serious assumption, since at any step we can apply Cauchy inequality and use Lemma 2.2. The circle method is used to separate the sums on $n$ and $m$, and we arrive at (1.3). Trivial estimate gives $S(N) \ll N^{2+\varepsilon}$. For simplicity, let $N = t$ and $q = Q$. So we are required to save $N$ and a little more in a sum of the form

$$
\int_{K}^{2K} \sum_{q=Q} \sum_{Q < a \leq Q + q} \sum_{n=N} \lambda(n)n^iv e\left(\frac{na}{q} - \frac{nx}{aq}\right) \sum_{m=N} m^{-i(t+v)} e\left(\frac{-ma}{q} + \frac{mx}{aq}\right) dv.
$$

The sum over $m$ has ‘conductor’ $Qt \approx N^{1/2}/t/K^{1/2}$. Roughly speaking, the conductor takes into account the arithmetic modulus $q$, with the size $= (t + v)$ of oscillation of the analytic weight. If we assume $K \ll t^{1-\varepsilon}$, then the size of the oscillation is $t$, so the extra oscillation of $m^{-iv}$ does not hurt us here. Poisson summation changes the length of summation to $Qt/N \approx N$ and contributes a factor of $N$ along with a congruence condition mod $q$ and an oscillatory integral. The oscillatory integral saves us $t^{1/2}$. In all, we will save $N/t^{1/2}$ in this step. So far the saving is independent of $K$. Next step is to apply Voronoi summation to the $n$-sum.

We need to save $t^{1/2}$ in a sum of the form

$$
\int_{K}^{2K} \sum_{q=Q} \sum_{(m,q)=1 \atop |m| < Qt/N} \left(\frac{(t + v)aq}{(x-ma)}\right)^{-i(t+v)} \sum_{n=N} \lambda(n) e\left(\frac{nm}{q} \right) n^iv e\left(\frac{-nx}{aq}\right) dv,
$$

where $a$ is the unique multiplicative inverse of $m \bmod q$ in the range $(Q, q + Q]$. Since the $n$-sum involves $GL(2)$ Fourier coefficients, the ‘conductor’ for the $n$-sum would be $(QK)^2$. The new length of sum would be $(QK)^2/N \approx K$. Voronoi summation would contribute a factor of $N/q$, a dual additive twist and an oscillatory weight function. The oscillation in the weight function would save us $K^{1/2}$. In all, we will save $Q/K^{1/2} = N^{1/2}/K$. If $K$ is large, we are actually making it worse. We are therefore left to save $t^{1/2}K/N^{1/2}$ in $S(N)$. Using stationary phase analysis, we will be able to save $K^{1/2}$ in the integral over $v$. At this point, $K$ seems to be hurting more than helping. The final step is to get rid of the $GL(2)$ oscillations using Cauchy inequality and then change the structure using Poisson summation formula. After Cauchy, the sum roughly looks like,

$$
\left[ \sum_{n \leq K} \sum_{q=Q \atop |m| < Qt/N} e\left(-\frac{nm}{q}\right) \int_{-K}^{K} n^{-i\tau} g(q, m, \tau) d\tau \right]^{1/2}.
$$

where $g(q, m, \tau)$ is an oscillatory weight function of size $O(1)$. The next steps would be to open the absolute value squared and, apply Poisson to the $n$-sum and analyze the $\tau$-integral. The $\tau$-integral gives us a saving of $K^{1/2}$. After Cauchy and Poisson summation, we will save $N^{1/2}/K^{1/2}$ in the diagonal term and $K^{1/4}$ in the off-diagonal term. Saving over convexity bound in the diagonal terms is $N^{1/2}/K^{1/2}$. Saving over convexity from the off-diagonal terms is $K^{1/4}$. We will therefore get maximum saving when $N^{1/2}/K^{1/2} = K^{1/4}$, that is $K = N^{2/3}$. That gives us a saving of $N^{1/6}$ over the convexity bound of $t^{1/2+\varepsilon}$. Matching this with the trivial bound $N^{1/2}$ for $N \ll t^{3/4}$ gives us the Burgess bound.

2. **$GL(2)$ Voronoi formula and stationary phase method**

2.1. **Voronoi summation formula for $SL_2(\mathbb{Z})$.** Suppose $f$ is a holomorphic cusp form for $SL_2(\mathbb{Z})$ which is an eigenfunction for all Hecke operators with $n^{th}$ Fourier coefficient $\lambda(n)$, normalized so that $\lambda(1) = 1$. In
this subsection, we will mention two important results— a summation formula for Fourier coefficients twisted by an additive character, and a bound on the average size of these Fourier coefficients, both of which will play a crucial role in our analysis.

Let $F$ be a smooth function compactly supported on $(0, \infty)$, and let $\tilde{F}(s) = \int_0^\infty g(x)x^{s-1}dx$ be its Mellin transform. An application of the functional equation of $L(s, f)$, followed by unwinding the integral and shifting the contour gives the Voronoi summation formula $J(2.2)$.

Lemma 2.1.

(2.1) $\sum_{n \geq 1} \lambda(n)e\left(\frac{n}{q}\right) F(n) = \frac{1}{q} \sum_{n \geq 1} \lambda(n)e\left(-\frac{n}{q}\right) \int_0^\infty F(x) \left[2\pi q J_{k-1}\left(\frac{4\pi \sqrt{nx}}{q}\right)\right] dx.$

For our calculations, we take a step back and use the following representation of $J_{k-1}$ as an inverse Mellin transform,

(2.2) $J_{k-1}(x) = \frac{1}{2\pi i} \int_{(\sigma)} \left(\frac{x}{2}\right)^{-s} \frac{\Gamma(s/2 + (k-1)/2)}{\Gamma(1-s/2 + (k-1)/2)}$ for $0 < \sigma < 1$.

We would need to study the oscillation of the gamma factors more closely. Recall the Stirling’s formula,

$\Gamma(\sigma + i\tau) = \sqrt{2\pi(i\tau)^{\sigma-1/2}e^{-\pi|\tau|/2}} \left(\frac{|\tau|}{e}\right)^{i\tau} \{1 + O\left(\frac{1}{|\tau|}\right)\}$

as $|\tau| \to \infty$. Letting $\gamma(s) = \frac{\Gamma(s/2 + (k-1)/2)}{\Gamma(1-s/2 + (k-1)/2)}$, we get

(2.3) $\gamma(1 + i\tau) = \left(\frac{|\tau|}{4e\pi}\right)^{i\tau} \Phi(\tau)$, where $\Phi'(\tau) \ll \frac{1}{|\tau|}.

We would also need the following bound, which gives Ramanujan conjecture on average. It follows from standard properties of Rankin-Selberg $L$-functions and is well known.

Lemma 2.2. We have,

$\sum_{n \leq x} |\lambda(n)|^2 \ll_{f,e} x^{1+\varepsilon}.$

2.2. Stationary phase method. We will need to estimate integrals of the type

(2.4) $I = \int_a^b g(x)e(f(x))dx.$

Let supp$(g) \subset [a, b]$ and $g^{(j)}(x) \ll_{j,a,b} 1$. Further suppose there is a $B > 0$ such that for $x \in [a, b]$, $|f'(x)| > B$ and $f^{(j)}(x) \ll B^{1+\varepsilon}$ when $j > 1$. Integration by parts $j$-times gives $I \ll_{j,a,b,e} B^{-j-\varepsilon}$.

In case $f'(x) = 0$ at a unique point $x = x_0 \in [a, b]$, there is an asymptotic expansion of the integral around $x_0$. $x_0$ is called the stationary phase. A sharp version useful for us can be found in [1, 3].

Lemma 2.3. Suppose $f$ and $g$ are smooth real valued functions satisfying

(2.5) $f^{(i)}(x) \ll \Theta_f/\Omega_f^i, \quad g^{(j)}(x) \ll 1/\Omega_g^j$

for $i = 2, 3$ and $j = 0, 1, 2$. Suppose $g(a) = g(b) = 0$. Define

$\bar{I} = \int_a^b g(x)e(f(x))dx.$
(a) Suppose \( f' \) and \( f'' \) do not vanish in \([a, b]\). Let \( \Lambda = \min_{[a, b]} |f'(x)| \). Then we have

\[
I \ll \frac{\Theta_f}{\Omega_f^2 \Lambda^3} \left( 1 + \frac{\Omega_f}{\Omega_g} + \frac{\Omega_f^2}{\Omega_g^2} \frac{\Lambda}{\Theta_f/\Omega_f} \right).
\]

(b) Suppose \( f' \) changes sign from negative to positive at the unique point \( x_0 \in (a, b) \). Let \( \kappa = \min\{b - x_0, x_0 - a\} \). Further suppose that \((2.5)\) holds for \( i = 4 \) and

\[
f^{(2)}(x) \gg \Theta_f/\Omega_f^2
\]

holds. Then

\[
I = \frac{g(x_0)e(f(x_0) + 1/8)}{\sqrt{f''(x_0)}} + O \left( \frac{\Omega_f^4}{\Theta_f^3 \kappa^3} + \frac{\Omega_f}{\Theta_f^{3/2} \Omega_g} + \frac{\Omega_f^3}{\Theta_f^{3/2} \Omega_g^2} \right).
\]

We will also need a second derivative bound for integrals in two variables. Let

\[
I(2) = \int_a^b \int_c^d g(x, y)e(f(x, y))dxdy.
\]

with \( f \) and \( g \) smooth real valued functions. Let \( \text{supp}(g) \subset (a, b) \times (c, d) \). Let \( r_1, r_2 \) be such that inside the support of the integral,

\[
f^{(2,0)}(x, y) \gg r_1^2, \quad f^{(0,2)}(x, y) \gg r_2^2, \quad f^{(2,0)}(x, y)f^{(0,2)}(x, y) - \left[f^{(1,1)}(x, y)\right]^2 \gg r_1^2 r_2^2,
\]

where \( f^{(i,j)}(x, y) = \frac{\partial^{i+j}}{\partial x^i \partial y^j} f(x, y) \). Then we have (see [11]),

\[
I(2) \ll \frac{1}{r_1 r_2}.
\]

Define the total variance of \( g \) by

\[
\text{var}(g) := \int_a^b \int_c^d \left| \frac{\partial^2}{\partial x \partial y} g(x, y) \right| dydx.
\]

Integration by parts along with the above bound gives us the following.

**Lemma 2.4.** Suppose \( f, g, r_1, r_2 \) are as above and satisfy condition \((2.10)\). Then we have

\[
I(2) \ll \frac{\text{var}(g)}{r_1 r_2}.
\]

2.3. **An integral of interest.** Following Munshi [8], let \( W \) be a smooth real valued function with \( \text{supp}(W) \subset [a, b] \subset (0, \infty) \) and \( W^{(j)}(x) \ll a, b, j \). Define

\[
W^\dagger(r, s) \int_0^\infty W(x)e(-rx)x^{s-1}dx
\]

where \( r \in \mathbb{R} \) and \( s = \sigma + i\beta \in \mathbb{C} \). This integral is of the form \((2.4)\) with

\[
g(x) = W(x)x^{\sigma-1} \quad \text{and} \quad f(x) = -rx + \frac{1}{2\pi}\beta \log x.
\]

Then,

\[
f'(x) = -r + \frac{1}{2\pi} \beta \quad \text{and} \quad f^{(j)}(x) = (-1)^j (j - 1)! \frac{1}{2\pi} \beta
\]
for \( j \geq 2 \). The unique stationary phase occurs at \( x_0 = \beta/2\pi r \). Note that we can write

\[
(2.12) \quad f'(x) = \frac{\beta}{2\pi} \left( \frac{1}{x} - \frac{1}{x_0} \right) = r \left( \frac{x_0}{x} - 1 \right).
\]

Applying Lemma 2.3 appropriately to \( W^\dagger(r, s) \), we get the following.

**Lemma 2.5.** Let \( W \) be a smooth real valued function with \( \text{supp}(W) \subset [a, b] \subset (0, \infty) \) and \( W^{(j)}(x) \ll_{a,b,j} 1 \). Let \( r \in \mathbb{R} \) and \( s = \sigma + i\beta \in \mathbb{C} \). We have

\[
(2.13) \quad W^\dagger(r, s) = \frac{\sqrt{2\pi}e(1/8)}{\sqrt{-\beta}} W \left( \frac{\beta}{2\pi r} \right) \left( \frac{\beta}{2\pi r} \right)^\sigma \left( \frac{\beta}{2\pi er} \right)^i\beta + O_{a,b,\sigma} \left( \min\{|\beta|^{-3/2}, |r|^{-3/2}\} \right).
\]

We also have

\[
(2.14) \quad W^\dagger(r, s) = O_{a,b,j,\sigma} \left( \min \left\{ \left( \frac{1 + |\beta|}{|r|} \right)^j, \left( \frac{1 + |r|}{|\beta|} \right)^j \right\} \right).
\]

3. **Application of dual summation formulas**

3.1. **Poisson summation to the \( m \)-sum.** The \( m \)-sum is given by

\[
\sum_{m \geq 1} m^{-i(t+v)} e \left( -\frac{ma}{q} + \frac{mx}{aq} \right) U \left( \frac{m}{N} \right) dvdx.
\]

Breaking the \( m \)-sum into congruence classes modulo \( q \), we get

\[
\sum_{\alpha \pmod{q}} e \left( -\frac{\alpha a}{q} \right) \sum_{m \in \mathbb{Z}} (\alpha + mq)^{-i(t+v)} e \left( \frac{(\alpha + mq)x}{aq} \right) U \left( \frac{\alpha + mq}{N} \right).
\]

Poisson to the \( m \)-sum gets us

\[
\sum_{\alpha \pmod{q}} e \left( -\frac{\alpha a}{q} \right) \sum_{m \in \mathbb{Z}} \int_{\mathbb{R}} (\alpha + yq)^{-i(t+v)} e \left( \frac{(\alpha + yq)x}{aq} \right) U \left( \frac{\alpha + yq}{N} \right) e(-my)dy.
\]

Making the change of variables \( (\alpha + yq) \mapsto u \) and executing the complete character sum mod \( q \), we arrive at

\[
N^{1-i(t+v)} \sum_{m \in \mathbb{Z}} \int_{\mathbb{R}} U(u)u^{-i(t+v)} e \left( \frac{N(x - ma)}{aq}u \right) du.
\]

The above integral equals

\[
(3.2) \quad U^\dagger \left( \frac{N(ma - x)}{aq}, 1 - i(t + v) \right).
\]

Everything together,

\[
S^+(N) = \frac{1}{K} \int_0^1 \int_{\mathbb{R}} V \left( \frac{v}{K} \right) \sum_{1 \leq q < Q < a} \sum_{Q \equiv \alpha \pmod{q}} \lambda(n)n^{iv} V \left( \frac{n}{N} \right) e \left( \frac{na - nx}{aq} \right) N^{1-i(t+v)} \sum_{m \equiv \alpha \pmod{q}} U^\dagger \left( \frac{N(ma - x)}{aq}, 1 - i(t + v) \right) dvdx.
\]

We can have \( m = 0 \) only when \( q = 1 \), in which case, \( N(ma - x)/aq \ll N/Qq \), so its contribution to the sum will be negligible (as soon as \( Q \) has size).
For $m \neq 0$, we have $N(ma - x)/aq \approx N|m|/q$. Bounds on $U^\dagger$ give

$$
U^\dagger \left( \frac{N(ma - x)}{aq}, 1 - i(t + v) \right) \ll j \left( \frac{1 + |t + v|}{N|m|q^{-1}} \right)^j
$$

(3.4)

Thus we get arbitrary saving for $|m| > (1 + |t + v|)q/N$. If we make sure $v < t$, that is $K < t$, we’ll have arbitrary saving for $|m| \gg qt^{1+\varepsilon}/N$. Noting the condition $m \equiv \dot{a} \mod q$ and rearranging the sums in $S^+(N)$,

$$
S^+(N) = \frac{N}{K} \int_0^1 \int_{\mathbb{R}} N^{-i(t+v)} V \left( \frac{v}{K} \right) \sum_{1 \leq q \leq Q} \sum_{1 \leq |m| \leq \frac{|t|+\varepsilon}{q}} \frac{1}{aq} U^\dagger \left( \frac{N(ma - x)}{aq}, 1 - i(t + v) \right)
$$

(3.5)

$$
\sum_{n \geq 1} \lambda(n)n^\nu V \left( \frac{n}{N} \right) e \left( \frac{nm}{q} \frac{nx}{aq} \right) \, dv dx
$$

where $a \in (Q, q + Q]$ is the unique multiplicative inverse of $m \mod q$.

**Remark 3.1.** Trivial bound here gives $S^+(N) \ll Nt^{1+\varepsilon}$. We need to save $t$ and a bit more.

We next split the $q-$sum into dyadic segments $(C, 2C)$

$$
S^+(N) = \frac{N}{K} \sum_{1 \leq C \leq Q} S(N, C)
$$

where

$$
S(N, C) = \int_0^1 \int_{\mathbb{R}} N^{-i(t+v)} V \left( \frac{v}{K} \right) \sum_{1 \leq q \leq 2C} \sum_{1 \leq |m| \leq \frac{|t|+\varepsilon}{q}} \frac{1}{aq} U^\dagger \left( \frac{N(ma - x)}{aq}, 1 - i(t + v) \right)
$$

(3.6)

$$
\sum_{n \geq 1} \lambda(n)n^\nu V \left( \frac{n}{N} \right) e \left( \frac{nm}{q} \frac{nx}{aq} \right) \, dv dx.
$$

3.2. **Voronoi summation to the $n$-sum.** Applying Lemma 2.1 to the $n$-sum gets us

$$
\sum_{n \geq 1} \lambda_f(n)e \left( \frac{m n}{q} \right) F(n) = \frac{\pi k}{q} \sum_{n \geq 1} \lambda_f(n)e \left( -na \frac{q}{q} \right) \int_0^\infty y^n V \left( \frac{y}{N} \right) e \left( -\frac{xy}{aq} \right) \, ds dy
$$

(3.7)

$$
\times \frac{1}{2\pi i} \int_{(\sigma)} \left( \frac{2\pi \sqrt{ny}}{q} \right)^{-s} \frac{\Gamma(s/2 + (k - 1)/2)}{\Gamma(1 - s/2 + (k - 1)/2)} ds dy
$$

where $F(y) = y^n V \left( \frac{y}{N} \right)e(-\frac{xy}{aq})$. We want to be able to interchange integrals. For this, we use the complex Stirling approximation

$$
|\Gamma(z)| = \sqrt{2\pi}e^{-\sigma}|z|^{|\sigma-1/2|e^{-\tau \arg(z)}} \left( 1 + O \left( \frac{1}{|z|} \right) \right)
$$

for $\arg(z) < \pi$ and $|z| \to \infty$. For

$$
\gamma(s) = \frac{(2\pi)^{-s} \Gamma(s/2 + (k - 1)/2)}{\Gamma(1 - s/2 + (k - 1)/2)}
$$

we have

$$
|\gamma(s)| \sim (2\pi)^{-s} |\tau|^{|\sigma-1|} e^{-\sigma|\tau|} \text{ as } |\tau| \to \infty
$$
Looking at the pole free regions of the $\Gamma$–factors in the definition of $\gamma(s)$, we get

$$|\gamma(s)| \ll 1 + |\tau|^{\sigma - 1} \quad \text{for} \quad \sigma > 1 - k$$

We cannot apply Fubini theorem to interchange integrals right away since the integral is not absolutely convergent for $0 < \sigma < 1$. But if we assume that $k > 1$, we can shift the integral to the line $\sigma = -1/2$ without picking any residues and the integral would be absolutely convergent, allowing us to apply Fubini and interchange integrals.

$$\sum_{n \geq 1} \lambda_f(n)e\left(\frac{m}{nq}\right) F(n) = \frac{\pi^k}{q} \sum_{n \geq 1} \lambda_f(n)e\left(-\frac{n}{q}\right) \frac{1}{2\pi i} \int_{(-1/2)} \left(\frac{\sqrt{n}}{q}\right)^{-s} \gamma(s)$$

$$\times \int_0^{\infty} y^{-s/2 + iv} V\left(\frac{y}{n}\right) e\left(-\frac{xy}{aq}\right) dy ds$$

$$= \frac{\pi^k N^{1 + iv}}{q} \sum_{n \geq 1} \lambda_f(n)e\left(-\frac{n}{q}\right) \frac{1}{2\pi i} \int_{(-1/2)} \left(\frac{\sqrt{nN}}{q}\right)^{-s} \gamma(s)$$

$$\times \int_0^{\infty} y^{-s/2 + iv} V(y)e\left(-\frac{xN}{aq} y\right) dy ds$$

$$= \frac{\pi^k N^{1 + iv}}{q} \sum_{n \geq 1} \lambda_f(n)e\left(-\frac{n}{q}\right) \frac{1}{2\pi i} \int_{(-1/2)} \left(\frac{\sqrt{nN}}{q}\right)^{-s} \gamma(s)$$

$$\times V^\dagger\left(\frac{xN}{aq}, 1 - s/2 + iv\right) ds$$

The bound on $V^\dagger$ gives

$$V^\dagger\left(\frac{xN}{aq}, 1 - s/2 + iv\right) \ll j \min\left\{1, \left(\frac{1 + |N x/aq|}{|v - \tau/2|}\right)^j\right\}$$

We can therefore shift the integral from $\sigma = -1/2$ to $\sigma = M$ for any large $M$ by choosing $j = M + 1$ (which kills the growth of $\gamma(s)$). We’ll thus get saving for large $n$.

**Remark 3.2.** Using the above bound on $V^\dagger$, we get

$$\left(\frac{\sqrt{nN}}{q}\right)^{-s} \gamma(s) V^\dagger\left(\frac{N x}{aq}, 1 - s/2 + iv\right) \ll j \left(\frac{\sqrt{nN}}{q}\right)^{-M} (1 + |\tau|^{M - 1}) \min\left\{1, \left(\frac{1 + |N x/aq|}{|v - \tau/2|}\right)^j\right\}$$

Since $v \asymp K$, the better bound on $V^\dagger$ would be $O(1)$ when $|\tau| \leq 8K$. In that case,

$$\int_{|\tau| \leq 8K} \left(\frac{\sqrt{nN}}{q}\right)^{-s} \gamma(s) V^\dagger\left(\frac{N x}{aq}, 1 - s/2 + iv\right) \ll \int_{|\tau| \leq 8K} \left(\frac{\sqrt{nN}}{q}\right)^{-M} |\tau|^{M - 1} d\tau$$

$$\ll \left(\frac{\sqrt{nN}}{qK}\right)^{-M}$$
We’ll thus get arbitrary saving for \( n \gg Q^2 K^2 \tau^\varepsilon / N \). On the other hand, when \( |\tau| > 8K \), we have the bound \( V \ll (N/aq|\tau|)^j \). Taking \( j = M + 1 \),

\[
\int_{|\tau| > 8K} \left( \frac{\sqrt{N}}{q} \right)^s \gamma(s) V^\dagger \left( \frac{N}{aq} \frac{1 - \frac{s}{2} + iv}{2} \right) \ll \int_{|\tau| > 8K} \left( \frac{\sqrt{N}}{q} \right)^{-M} |\tau|^{M-1} \left( \frac{N}{aq|\tau|} \right)^{M+1} d\tau
\]

\[
= \left( \frac{an^{1/2}}{N^{1/2}} \right)^{-M} \left( \frac{N}{aq} \right)^2
\]

We’ll thus get arbitrary saving for \( n \gg N\tau^\varepsilon / Q^2 \). It makes sense to choose \( Q \) so that the two bounds on \( n \) are equal. Therefore set \( Q = (N/K)^{1/2} \). We’ll get arbitrary saving for \( n \gg K\tau^\varepsilon \).

For smaller values of \( n \), we take \( \sigma = 1 \). Note that the \( \gamma \) factor will then be bounded.

\[
\sum_{n \geq 1} \lambda_j(n) e \left( \frac{m}{n} \right) F(n) = \pi^2 N^{1/2+iv} \sum_{n \in Q^2 K^2 / N} \lambda_j(n) \frac{1}{n^{1/2}} e \left( -\frac{n}{a} q \right) \frac{1}{2\pi} \int_{\mathbb{R}} \left( \frac{\sqrt{N}}{q} \right)^{-ir} \gamma(1 + i\tau)V^\dagger \left( \frac{xN}{aq}, 1/2 - i\tau/2 + iv \right) d\tau
\]

(3.10)

Assuming \( K \ll \tau^{1-\varepsilon} \), we get arbitrary saving for \( |\tau| > N\tau^\varepsilon / QC \) due to bounds on \( V^\dagger \). Thus we can restrict the integral to \( \tau \in [-N\tau^\varepsilon / QC, N\tau^\varepsilon / QC] \) by defining a smooth partition of unity on this set. Let \( W_j \) for \( J \in \mathcal{J} \) be smooth bump functions satisfying \( x^l W_j^{(l)} \ll 1 \) for all \( l \geq 0 \). For \( J = 0 \), let the support of \( W_0 \) be in \([-1, 1]\) and for \( J > 0 \) (resp. \( J < 0 \)), let the support of \( W_J \) be in \([J, 4J/3]\) (resp \([4J/3, J]\)). Finally, we require that

\[
\sum_{J \in \mathcal{J}} W_J(x) = 1 \quad \text{for} \quad x \in [-N\tau^\varepsilon / QC, N\tau^\varepsilon / QC]
\]

The precise definition of the functions \( W_J \) will not be needed. We note that we need only \( O(\log(t)) \) such \( J \in \mathcal{J} \). We can write the integral appearing in Voronoi summation as

\[
\int_{\mathbb{R}} \left( \frac{\sqrt{N}}{q} \right)^{-ir} \gamma(1 + i\tau)V^\dagger \left( \frac{xN}{aq}, 1/2 - i\tau/2 + iv \right) d\tau =
\]

\[
\sum_{J \in \mathcal{J}} \int_{\mathbb{R}} \left( \frac{\sqrt{N}}{q} \right)^{-ir} \gamma(1 + i\tau)V^\dagger \left( \frac{xN}{aq}, 1/2 - i\tau/2 + iv \right) W_J(\tau) d\tau + O(t^{-2015})
\]

Combining everything, we write \( S(N, C) \) as

\[
S(N, C) = \frac{\pi^2 N^{1/2-ir} K}{2} \sum_{J \in \mathcal{J}} \sum_{n \leq Q^2 K^2 / N} \lambda_j(n) \frac{1}{n^{1/2}} \sum_{C < q \leq 2C} e \left( -\frac{na}{q} \right) \frac{1}{aq} \int_{\mathbb{R}} \left( \frac{\sqrt{N}}{q} \right)^{-ir} \gamma(1 + i\tau) W_J(\tau) \tau^{*s}(q, m, \tau) d\tau + O(t^{-2015})
\]

(3.11)

where
\[
I^{**}(q, m, \tau) = \int_0^1 \int_{\mathbb{R}} V(v) U^1 \left( \frac{N(ma - x)}{aq}, 1 - i(t + K\nu) \right) V^1 \left( \frac{Nx}{aq}, \frac{1}{2} - \frac{i\tau}{2} + iK\nu \right) \, dvdx
\]

**Remark 3.3.** We can trivially bound \(I^{**}(q, m, \tau)\) by \(O(1)\), and the \(\tau\)-integral is over the interval \([-Nt^\varepsilon/QC, Nt^\varepsilon/QC]\). Trivial bound on \(S(N, C)\) will imply \(S(N, C) \ll K^{5/2}t^{1+\varepsilon}/N^{1/2}\). So \(S(N) \ll N^{1/2}K^{3/2}t^{1+\varepsilon}\). We need to save \(N^{1/2}K^{3/2}\) and a bit more.

4. **Analysis of the integrals**

We next analyze the integral \(I^{**}(q, m, \tau)\). Application of Lemma 2.5 to \(U^1\) gives us

\[
U^1 = \frac{e^{in/4}(t + K\nu)^{1/2}aq}{(2\pi)^{1/2}N(x - ma)} U \left( \frac{(t + K\nu)aq}{2\pi N(x - ma)} \right) \frac{(t + K\nu)^{1/2}}{2\pi eN(x - ma)} \frac{1}{2} - \frac{i(t + K\nu)}{2} + O(t^{-3/2}).
\]

Therefore,

\[
I^{**}(\tau) = \frac{c_1 aq}{N} \int_0^1 \int_{\mathbb{R}} V(v) V^1 \left( \frac{Nx}{aq}, \frac{1}{2} - \frac{i\tau}{2} + iK\nu \right) (t + K\nu)^{1/2} \frac{(t + K\nu)}{2\pi N(x - ma)} \frac{(t + K\nu)}{2\pi eN(x - ma)} \frac{1}{2} - \frac{i(t + K\nu)}{2} + O(t^{-3/2+\varepsilon})
\]

where \(c_1 = e^{in/4}/\sqrt{2\pi}\). We next apply Lemma 2.5 to \(V^1\).

\[
V^1 = 2\sqrt{\pi}e^{-in/4}(aq)^{1/2} \left( \frac{aq}{Nx} \right)^{1/2} V \left( \frac{2K\nu - \tau)aq}{4\pi Nx} \frac{(2K\nu - \tau)aq}{4\pi eNx} \right) (t + K\nu)^{1/2} \frac{(t + K\nu)}{2\pi N(x - ma)} \frac{(t + K\nu)}{2\pi eN(x - ma)} \frac{1}{2} - \frac{i(t + K\nu)}{2} + O \left( \min \left\{ \left( \frac{aq}{Nx} \right)^{3/2}, \frac{1}{\tau/2 - K\nu^{3/2}} \right\} \right).
\]

The integral then becomes

\[
(4.1)
I^{**}(q, m, \tau) = c_2 \left( \frac{aq}{N} \right)^{3/2} \int_0^1 \int_{\mathbb{R}} V(v) \left( \frac{1}{x} \right)^{1/2} V \left( \frac{2K\nu - \tau)aq}{4\pi Nx} \frac{(2K\nu - \tau)aq}{4\pi eNx} \right) (t + K\nu)^{1/2} \frac{(t + K\nu)}{2\pi N(x - ma)} \frac{(t + K\nu)}{2\pi eN(x - ma)} \frac{1}{2} - \frac{i(t + K\nu)}{2} + O(E^{**} + t^{-3/2+\varepsilon})
\]

with \(c_2 = 1/(2\pi)^{1/2}\) and since \(uU(u) \ll 1\),

\[
E^{**} = \frac{1}{t^{1/2}} \int_1^2 \min \left\{ \left( \frac{aq}{Nx} \right)^{3/2}, \frac{1}{\tau/2 - K\nu^{3/2}} \right\} \, dvdx
\]

(We note that more generally \(uU(u) \ll \delta 1\), but using this does not improve the error term.)
4.1. **Analysis of the error term** $E^{**}$. The first term is smaller than the second if and only if

$$\frac{\tau}{2K} - \frac{Nx}{aqK} < v < \frac{\tau}{2K} + \frac{Nx}{aqK}.$$ 

If $|\tau| \geq 10K$, this interval does not intersect $[1, 2]$ unless $Nx/aq \approx |\tau|$. For this, we use the trivial bound $O(1)$ for the inner integral over $v$. And if $|\tau| < 10K$, the inner integral is bounded by the length of the interval, which is $2Nx/aqK$. Hence the contribution where the first term is smaller than the second is of the order

$$\frac{1}{t^{1/2}} \int_0^1 \left( \frac{aq}{N} \right)^{1/2} \frac{1}{|\tau| < 10K} dx + \frac{1}{t^{1/2}} \int_0^1 \left( \frac{aq}{N} \right)^{1/2} \frac{1}{|\tau| > 10K} dx.$$ 

This is bounded by

$$O \left( \frac{Q}{t^{1/2} N^{1/2}} \min \left\{ 1, \frac{10K}{|\tau|} \right\} t^\epsilon \right).$$ 

Next we estimate the contribution to $E^{**}$ when the second term is smaller. This would be

$$\frac{1}{t^{1/2}} \int_0^1 \left( \frac{aq}{N} \right)^{1/2} \frac{1}{|\tau| - Kv| > Nax/aq} \int_0^2 \frac{1}{|\tau|/2 - Kv|^{1-\epsilon}} dx$$

$$\ll t^{\epsilon} \frac{Q}{t^{1/2} N^{1/2}} \min \left\{ 1, \frac{10K}{|\tau|} \right\}.$$ 

The total error term therefore is

$$E^{**} + t^{-3/2+\epsilon} \ll t^{\epsilon} \frac{Q}{t^{1/2} N^{1/2}} \min \left\{ 1, \frac{10K}{|\tau|} \right\} + t^{-3/2+\epsilon},$$

and we can write

$$I^{**}(q, m, \tau) = c_2 \left( \frac{aq}{N} \right)^{3/2} \int_0^1 V(v) \left( \frac{1}{x} \right)^{1/2} \frac{V \left( \frac{(2Kv - \tau)aq}{4\piNx} \right) \left( \frac{(2Kv - \tau)aq}{4\pi eNx} \right)^{i(Kv-\tau/2)}}{(x-ma)} U \left( \frac{(t + Kv)aq}{2\pi N(x-ma)} \right) \left( \frac{(t + Kv)aq}{2\pi eN(x-ma)} \right)^{-i(t+Kv)} dx

+ O \left( \frac{t^\epsilon}{t^{1/2} K^{3/2}} \min \left\{ 1, \frac{10K}{|\tau|} \right\} + t^{-3/2+\epsilon} \right).$$

**Remark 4.1.** The error term in the above estimate for $I^{**}$ saves a further $t^{1/2} K^{3/2}$. The main term saves $K^{1/2} t^{1/2}$. So we need to save $K$ and a bit more. Note that at this point $K$ seems to be hurting us rather than helping us. Moreover, if $K$ had no size, we would get the bound $S(N) \ll N^{1+\epsilon}$, which would get us the convexity bound.

4.2. **Analysis of integral over $v$.** The integral is given by

$$I_1 = c_2 \left( \frac{aq}{N} \right)^{3/2} \int_0^1 V(v) \left( \frac{1}{x} \right)^{1/2} \frac{V \left( \frac{(2Kv - \tau)aq}{4\pi Nx} \right) \left( \frac{(2Kv - \tau)aq}{4\pi eNx} \right)^{i(Kv-\tau/2)}}{(x-ma)} U \left( \frac{(t + Kv)aq}{2\pi N(x-ma)} \right) \left( \frac{(t + Kv)aq}{2\pi eN(x-ma)} \right)^{-i(t+Kv)} dx$$
Due to the argument of $U$, the integral vanishes if $m > 0$. Trivial estimate gives

$$I_1 \ll \left(\frac{aq}{N}\right)^{3/2} \int_0^1 \int_{\mathbb{R}} \left(\frac{t + Kv}{x^{1/2}(x - ma)}\right)^{1/2} V(v)V\left(\frac{(2Kv - \tau)aq}{4\pi N x}\right) U\left(\frac{(t + Kv)aq}{2\pi N(x - ma)}\right) dvdx$$

The length of the integral over $v$ is restricted due to the weight functions, respectively given by $1, -Nm/Kq$ and $N/x/aqK$. $N/x/aqK < -Nm/Kq$, so we can restrict the length of integral over $v$ to $N/x/aqK$. We restrict the integral over $x$ to $[0, 1/K]$ and estimate the resulting integral trivially.

$$\ll \left(\frac{aq}{N}\right)^{1/2} \frac{1}{t^{1/2}} \int_0^{1/2} \frac{1}{x^{1/2}/aq} dx$$

$$\ll \frac{1}{t^{1/2}K^{3/2+1}} \left(\frac{N}{aq}\right)^{1/2} = E$$

We write $I_1(\tau) = I_2(\tau) + O(E)$, where $I_2(\tau)$ is

$$I_2 = c_2 \frac{1}{t^{1/2}} \left(\frac{aq}{N}\right)^{3/2} \int_0^1 \int_{1/K}^{1/2} \left(\frac{(t + Kv)}{(x - ma)x^{1/2}}\right)^{1/2} V(v)V\left(\frac{(2Kv - \tau)aq}{4\pi N x}\right) \left(\frac{(2Kv - \tau)aq}{4\pi eN x}\right)^{i(Kv - \tau/2)}$$

$$\times U\left(\frac{(t + Kv)aq}{2\pi N(x - ma)}\right) U\left(\frac{(t + Kv)aq}{2\pi e N(x - ma)}\right) dvdx$$

where an extra $t^{1/2}$ is multiplied to balance the size of the function. Set

$$f(v) = -\frac{t + Kv}{2\pi} \log\left(\frac{(t + Kv)aq}{2\pi eN(x - ma)}\right) + \frac{2Kv - \tau}{4\pi} \log\left(\frac{(2Kv - \tau)aq}{4\pi eN x}\right)$$

and

$$g(v) = \frac{t^{1/2}(t + Kv)aq}{N(x - ma)} V(v)V\left(\frac{(2Kv - \tau)aq}{4\pi N x}\right) U\left(\frac{(t + Kv)aq}{2\pi N(x - ma)}\right)$$

So that

$$I_2 = c_2 \frac{1}{t^{1/2}} \left(\frac{aq}{N}\right)^{1/2} \int_0^1 \int_{1/K}^{1/2} g(v)e(f(v))dvdx$$

Then

$$f'(v) = -\frac{K}{2\pi} \log\left(\frac{2(t + Kv)x}{(2Kv - \tau)(x - ma)}\right), \quad f^{(j)}(v) = -\frac{(j - 1)!(Kj)}{2\pi(t + Kv)^{j-1}} + \frac{(j - 1)!(2Kj)}{4\pi(2Kv - \tau)^{j-1}}$$

The stationary phase is given by

$$v_0 = -\frac{(2t + \tau)x - \tau ma}{2Kma}$$

In support of the integral, we have

$$f^{(j)}(v) \approx \frac{N x}{aq} \left(\frac{Kaq}{N x}\right)^j$$
for $j \geq 2$, and for $j \geq 0$

$$g^{(j)}(v) \ll \left(1 + \frac{Kaq}{N\xi}\right)^j$$

We shall apply the sharp version of stationary phase method due to Huxley[3] (as given in Lemma 3 of Munshi[8]):

We can write

$$f'(v) = \frac{K}{2\pi} \log \left(1 + \frac{K(v_0 - v)}{(t + Kv)}\right) - \frac{K}{2\pi} \log \left(1 + \frac{2K(v_0 - v)}{(2Kv - \tau)}\right)$$

In the support of the integral, we have $0 \leq 2Kv - \tau < N/aq < t^{1+\epsilon}/Q$ (since $N/t^{1+\epsilon} < q$ and $a \approx Q$). Therefore

$$f''(v) = -\frac{K^2}{2\pi(t + Kv)} + \frac{K^2}{2\pi(Kv - \tau/2)}$$

is positive on the support of the integral for large enough $t$. So $f'$ changes sign from negative to positive at $v_0$. Support of the integral is contained in $[1, 2]$ due to the weight function $V(v)$. If $v_0 \notin [0.5, 2.5]$, then $v_0$ is not in the support of the integral and $|v_0 - v| > 0.5$. In the support of the integral, we will have

$$|f'(v)| \gg K^{1-\epsilon} \min \left\{1, \frac{Kaq}{N\xi}\right\}$$

Applying the first statement of Lemma (2.3) with

$$\Theta_f = \frac{N\xi}{aq}, \quad \Omega_f = \frac{N\xi}{Kaq}, \quad \Omega_g = \min \left\{1, \frac{N\xi}{Kaq}\right\}, \quad \Lambda = K^{1-\epsilon} \min \left\{1, \frac{Kaq}{N\xi}\right\}$$

we obtain the bound

$$\int_{\mathbb{R}} g(x)e(f(x))dx \ll \Theta_f \frac{\Omega_f}{\Omega_f^2} \frac{\Omega_f}{\Omega_g} \Lambda \left(1 + \frac{\Omega_f^2}{\Omega_g} + \frac{\Omega_f^3}{\Omega_f^2 \Theta_f} \frac{\Lambda}{\Omega_f}\right)^{1/\epsilon}$$

(4.4)

On the other hand, if $v_0 \in [0.5, 2.5]$, then treating the integral as one over the finite range $[0.1, 4]$ (so that $\kappa > 0.4$) and applying the second part of Lemma (2.3), we get

$$I = \frac{g(x_0)e(f(x_0) + 1/8)}{\sqrt{f''(x_0)}} + O \left(\frac{\Omega_f^4}{\Theta_f^2} + \frac{\Omega_f^3}{\Theta_f^2} \frac{\Omega_f^3}{\Omega_f^2 \Theta_f} \Lambda\right)$$

(4.5)

For the range $x \in [1/K, 1]$, we use the bound in lemma (2.3). In the case there is no stationary phase, we will use the first statement of lemma (2.3). We have,

$$\Theta_f = \frac{N\xi}{aq}, \quad \Omega_f = \frac{N\xi}{aqK}, \quad \Lambda = K^{1-\epsilon} \min \left\{1, \frac{Kaq}{N\xi}\right\}, \quad \Omega_g = \min \left\{1, \frac{N\xi}{aqK}\right\}.$$  

Next is the contribution of $x \in [1/K, 1]$ when there is no stationary phase. When $x < aqu/K$, $\Lambda = K$ and $\Omega_g = \Omega_f$. In that case, the contribution is

$$\left(\frac{2\pi aqu}{Nt}\right)^{1/2} \int_{1/K}^{\max(1, Kt^{1/2})} \frac{1}{x^{1/2}NK} dx \ll \frac{1}{t^{1/2}K^2}.$$  

This is always smaller than the contribution of the bound $E$. When $x > aqu/K$, $\Lambda = K^2 aqu/Nx$ and $\Omega_g = 1$. In that case, the contribution is $1/K^3 t^{1/2}$, which is better than above. We next calculate the contribution of the error term when there is a stationary phase. For that we have $\kappa > 0.4$. One can calculate that for both $x < aqu/K$ and $x > aqu/K$, the contribution is $1/K^2 t^{1/2}$.
With all of this, we summarize the analysis in the following Lemma. Let

\[(4.7)\]

\[B(C, \tau) = \frac{t^e}{t^{1/2}K^{3/2}} \min \left\{ 1, \frac{10K}{|\tau|} \right\} + \frac{1}{t^{1/2}K^{5/2}} \left( \frac{N}{QC} \right)^{1/2}.\]

Note that,

\[(4.8)\]

\[\int_{-Nt/\sqrt{QC}}^{Nt/\sqrt{QC}} B(C, \tau) d\tau \ll \frac{K}{t^{1/2}K^{3/2}} + \frac{1}{t^{1/2}K^{5/2}} \left( \frac{N}{QC} \right)^{3/2}.\]

Putting everything together, we have

**Lemma 4.2.** Suppose \(C < q \leq 2C\), with \(1 \ll C \leq (N/K)^{1/2}\) and \(K\) satisfies \(1 \leq K \ll t^{1-\epsilon}\). Suppose \(t > 2\) and \(|\tau| \ll N^{1/2}K^{1/2}t^{\epsilon}\). We have

\[I^{**}(q, m, \tau) = I_1(q, m, \tau) + I_2(q, m, \tau)\]

where

\[I_1(q, m, \tau) = \frac{c_4}{(t + \tau/2)^{1/2}K} \left( \frac{-(t + \tau/2)q}{2\pi NeM} \right)^{3/2-i(t+\tau/2)} V \left( \frac{(t + \tau/2)q}{2\pi NeM} \right) \int_0^1 V \left( \frac{\tau}{2K} - \frac{(t + \tau/2)x}{Kma} \right) dx\]

for some absolute constant \(c_4\) and

\[I_2(q, m, \tau) := I^{**}(q, m, \tau) - I_1(q, m, \tau) = O(B(C, \tau)t^{\epsilon})\]

with \(B(C, \tau)\) as defined in \((4.7)\).

Consequently, we have the following decomposition of \(S(N, C)\).

**Lemma 4.3.**

\[S(N, C) = \sum_{J \in \mathcal{J}} \{S_{1,J}(N, C) + S_{2,J}(N, C)\} + O(t^{-2015})\]

where

\[S_{1,J}(N, C) = \frac{i^JN^{1/2-iJ}K}{2} \sum_{n \ll Q^2K^2/N} \frac{\lambda_J(n)}{n^{1/2}} \sum_{C \ll q \leq 2C} \sum_{(m,q) = 1} \frac{e \left( -na \right)}{q} \frac{1}{aq} I_{1,J}(q, m, n)\]

and

\[I_{1,J}(q, m, n) = \int_{\mathbb{R}} \left( \frac{\sqrt{nN}}{q} \right)^{-i\tau} \gamma(1 + i\tau) W_J(\tau) I_1(q, m, \tau) d\tau\]

with \(I_1(q, m, \tau)\) as defined in the previous lemma.

**Remark 4.4.** The saving due to \(I_1(q, m, \tau)\) is still \(t^{1/2}K^{1/2}\), same as the main term before this analysis. The saving due to \(I_2(q, m, \tau)\) is \(t^{1/2}K^{9/4}/N^{1/4}\). In all, we need to save \(\max \{K, t^{1/4}/K^{3/4}\}\) and a bit more.
5. Application of Cauchy and Poisson summation - I

In this section, we will estimate

\[ S_2(N, C) := \sum_{J \in \mathcal{F}} S_{2,J}(N, C) \]

Here, we'll not apply any cancellation over the \( \tau \)-integral. Dividing the \( n \)-sum into dyadic segments and using the bound \( \gamma(1 + it) \ll 1 \), we get

\[
S_2(N, C) \ll \frac{t^2N^{1/2}K}{\zeta(t)^{1/2}} \sum_{1 \leq L \ll K^{1/2}} \sum_{n \text{ dyadic}} \left| \frac{\lambda_f(n)}{n^{1/2}} U \left( \frac{n}{L} \right) \right| \sum_{c < q \ll 2C (m,q) = 1} \sum_{1 \leq |m| \ll \frac{q^{1+\epsilon}}{K}} e \left( -\frac{n\alpha}{q} \right) \frac{1}{aq^{1-it}} I_2(q, m, \tau) dt.
\]

Applying Cauchy to the \( n \)-sum and using the Ramanujan bound on average (Lemma 2.2), we get

\[
S_2(N, C) \ll \frac{t^2N^{1/2}K}{\zeta(t)^{1/2}} \sum_{1 \leq L \ll K^{1/2}} L^{1/2} \left[ S_2(N, C, L, \tau) \right]^{1/2} dt.
\]

where

\[
S_2(N, C, L, \tau) = \sum_{n} \frac{1}{n} U \left( \frac{n}{L} \right) \sum_{c < q \ll 2C (m,q) = 1} \sum_{1 \leq |m| \ll \frac{q^{1+\epsilon}}{K}} e \left( -\frac{n\alpha}{q} \right) \frac{1}{aq^{1-it}} I_2(q, m, \tau)
\]

\[
\times \sum_{C < q' \ll 2C (m',q') = 1} \sum_{1 \leq |m'| \ll q'^{1+\epsilon}} e \left( \frac{n\alpha'}{q'} \right) \frac{1}{a'q'^{1+it}} I_2(q', m', \tau)
\]

\[
= \sum_{C < q \ll 2C (m,q) = 1} \sum_{1 \leq |m| \ll \frac{q^{1+\epsilon}}{K}} \sum_{C < q' \ll 2C (m',q') = 1} \sum_{1 \leq |m'| \ll q'^{1+\epsilon}} \frac{1}{aq^{1-it}} \frac{1}{a'q'^{1+it}} I_2(q, m, \tau) I_2(q', m', \tau) T
\]

where we set

\[
T = \sum_{n} \frac{1}{n} U \left( \frac{n}{L} \right) e \left( -\frac{n\alpha}{q} \right) e \left( \frac{n\alpha'}{q'} \right)
\]

We break the \( n \)-sum modulo \( qq' \) to get

\[
T = \sum_{\beta \mod qq'} e \left( \frac{\beta d - aq'}{qq'} \right) \sum_{\ell \in \mathbb{Z}} \frac{1}{\beta + lqq'} U \left( \frac{\beta + lqq'}{L} \right)
\]

Applying Poisson summation formula to \( l \)-sum,

\[
T = \sum_{\beta \mod qq'} e \left( \frac{\beta d - aq'}{qq'} \right) \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \frac{1}{\beta + yqq'} U \left( \frac{\beta + yqq'}{L} \right) e(-ny) dy
\]
Change variables \( w = (\beta + yqq')/L \) to get
\[
T = \frac{1}{qq'} \sum_{\beta \mod qq'} e \left( \frac{\beta(a'q - aq)}{qq'} \right) \sum_{n \in \mathbb{Z}} \frac{n\beta}{qq'} \int_{\mathbb{R}} \frac{1}{w} U(w) e \left( \frac{-nLw}{qq'} \right) dw
\]
Integration by parts will give arbitrary saving for \( n \gg C^2t^2/L \). Thus,
\[
T = \frac{1}{qq'} \sum_{n < \frac{C^2t^2}{L}} \left[ \sum_{\beta \mod qq'} e \left( \frac{\beta(a'q - aq)}{qq'} \right) e \left( \frac{n\beta}{qq'} \right) \right] \int_{\mathbb{R}} \frac{1}{w} U(w) e \left( \frac{-nLw}{qq'} \right) dw + O(t^{-2015})
\]
Plugging this in the expression for \( S_2(N, C, L, \tau) \), we get
\[
S_2(N, C, L, \tau) \ll \frac{K}{NC^2} B(C, \tau)^2 \sum_{C < q < 2C} \sum_{(m, q) = 1} \sum_{C < q' < 2C} \sum_{(m', q') = 1} \frac{\delta(n = aq' - a'q \mod qq')}{|C| + O(t^{-2015})}
\]
where
\[
C = \sum_{\beta \mod qq'} e \left( \frac{\beta(a'q - aq)}{qq'} \right) e \left( \frac{n\beta}{qq'} \right)
\]
Note that \( C = qq' \delta(n = aq' - a'q \mod qq') \). Plugging that into the above expression and rearranging the sums, we get

**Lemma 5.1.**
\[
S_2(N, C, L, \tau) \ll \frac{K}{NC^2} B(C, \tau)^2 \sum_{n < \frac{C^2t^2}{L}} \sum_{C < q < 2C} \sum_{(m, q) = 1} \sum_{C < q' < 2C} \sum_{(m', q') = 1} \delta(n = aq' - a'q \mod qq') + O(t^{-2015})
\]
We have to analyze the cases \( n = 0 \) and \( n \neq 0 \) separately. When \( n = 0 \), the congruence condition above gives \( q = q' \) and \( a = a' \). For a given \( m \), this fixes \( m' \) up to a factor of \( t^{1+\epsilon}/N \). Moreover, in the case \( Q^2 < K \), that is, \( K > N^{1/2} \), we’ll have only \( n = 0 \) for \( L > C^2 \). Therefore for \( n \neq 0 \), we will let \( L \) go up to \( \min\{C^2, K\} \).

We note that the congruence condition implies \( q|(n - aq') \) and \( q'| (n + a'q) \). Since \( a \) and \( a' \) lie in an interval of length \( q \), fixing \( n, q \) and \( q' \) fixes both \( a \) and \( a' \). That saves \( q, q' \) in the \( m, m' \)-sums respectively.

**Remark 5.2.** We haven’t used the conditions \((a, q) = 1\) and \((a', q') = 1\). But we can show that these conditions give us a saving of at most a power of \( \log t \).

Using \( I_2(q, m, \tau) \ll B(C, \tau) \), we get
\[
S_2(N, C, L, \tau) \ll t^4 \frac{K^2 B(C, \tau)^2}{N^3} \left[ \frac{1}{n=0} + \sum_{n \neq 0} \frac{C^2}{L} \right]
\]
so that
\[
S_2(N, C, L, \tau)^{1/2} \ll t^{1/2} \frac{K^{1/2} t B(C, \tau)}{N^{3/2}} \left[ 1 + \frac{C}{L^{1/2}} \right]
\]
Therefore,
\[
S_2(N, C) \ll t^e N^{1/2} K^{3/2} \int_{\mathbb{R}} e^{ \left[ \sum_{1 \leq L \leq K^{e \log \log K}} \frac{1}{L^{1/2}} \right] \sum_{1 \leq L \leq \min\{C^2, K\} t^e} \frac{K^{1/2} t B(C, \tau)}{N^{3/2}} } d\tau
\]
If $K \geq N^{1/2}$, then the contribution of the second term is smaller than that of the first. So we neglect the second term. Summing over $L$, using (4.8) (and noting $N \ll t^{1+\delta}$), we get

$$
S_2(N, C) \ll t^{K^2 t} N \left( \frac{1}{t^{1/2} K^{1/2} 2^{1/2}} + \frac{1}{t^{1/2} K^{3/2}} \left( \frac{N}{QC} \right)^{3/2} \right)
$$

Multiplying by $N^{1/2}/K$ and summing over $C$ dyadically,

$$
\frac{S_2(N)}{N^{1/2}} \ll t^{1/2+\delta} \left( \frac{K^{1/2}}{N^{1/2}} + \frac{N^{1/4}}{K^{3/4}} \right)
$$

where $K \geq N^{1/2}$.

6. Application of Cauchy and Poisson summation- II

$$
I_1(q, m, \tau) = \frac{c_4}{(t + \tau/2)^{1/2}} K \left( \frac{t + \tau/2}{2\pi N m} \right)^{3/2} \int_0^1 V \left( \frac{t + \tau/2}{2K} \right) dx
$$

$$
S_{1,j}(N, C) = \frac{2^{N^{1/2} - i t} K}{2} \sum_{1 \leq L \leq K} \left( \sum_{n \text{dyadic}} \left| \lambda_f(n) \right|^2 U \left( \frac{n}{L} \right) \right) \sum_{C \leq q \leq (2C, m, q) = 1} \sum_{1 \leq |m| < q^{1+\delta}/N} e \left( \frac{-na}{q} \right) \frac{1}{aq} \gamma(1 + i\tau) W_j(\tau) I_1(q, m, \tau) d\tau
$$

Using the two, rearranging $q, m-$sums and integral, taking absolute values and using Cauchy, we get

$$
|S_{1,j}(N, C)| \leq N^{1/2} K \sum_{1 \leq L \leq K} \left( \sum_{n \text{dyadic}} \left| \lambda_f(n) \right|^2 U \left( \frac{n}{L} \right) \right)^{1/2} |S_{1,j}(N, C, L)|^{1/2}
$$

where

$$
S_{1,j}(N, C, L) = \sum_n U \left( \frac{n}{L} \right) | \int_\mathbb{R} (\sqrt{N})^{-ir} \gamma(1 + i\tau) \sum_{|m| < q^{1+\delta}/N} e \left( \frac{-na}{q} \right) \frac{1}{aq^{1-ir}} W_j(\tau) I_1(q, m, \tau) d\tau |
$$

Opening $|...|^2$ and rearranging sums and integrals

$$
S_{1,j}(N, C, L) = \int_\mathbb{R} \int_\mathbb{R} (\sqrt{N})^{-ir'} \gamma(1 + i\tau)r' W_j(\tau) W_j(\tau')
$$

$$
\times \sum_{C \leq q \leq (2C, m, q) = 1} \sum_{1 \leq |m| < q^{1+\delta}/N} \sum_{1 \leq |m'| < q^{1+\delta}/N} \frac{1}{aq^{1-ir}} \frac{1}{aq^{1+ir}} I_1(q, m, \tau) I_1(q', m', \tau') T d\tau d\tau'
$$

where

$$
T = \sum_n n^{-1 + \frac{i(r'+r)}{2}} U \left( \frac{n}{L} \right) e \left( \frac{n(d'a - aq')}{qq'} \right)
$$
Analyzing \( T \): Breaking the sum modulo \( qq' \),

\[
T = \sum_{\beta(qq')} e \left( \frac{\beta(a'q - aq')}{qq'} \right) \sum_{l \in \mathbb{Z}} (\beta + qq'l)^{-1 + \frac{-ir + it'}{2}} U \left( \frac{\beta + qq'l}{L} \right)
\]

applying Poisson summation to the \( l \)-sum,

\[
T = \sum_{\beta(qq')} e \left( \frac{\beta(a'q - aq')}{qq'} \right) \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} (\beta + qq'y)^{-1 + \frac{-ir + it'}{2}} U \left( \frac{\beta + qq'y}{L} \right) e(-ny) dy
\]

and changing variables \( w = (\beta + qq'y)/L \),

\[
T = \frac{1}{qq'} \sum_{\beta(qq')} e \left( \frac{\beta(a'q - aq')}{qq'} \right) \sum_{n \in \mathbb{Z}} e \left( \frac{n\beta}{qq'} \right) L^{-(-ir + it')/2} U^\dagger \left( \frac{nnL}{qq'}, \frac{-ir}{2} + \frac{it'}{2} \right)
\]

with \( \mathbb{C} \) as before. Since \( |\tau - \tau'| \ll (NK)^{1/2}r^e/C \), the bound on \( U^\dagger \) gives arbitrary saving for \( |n| \gg C(NK)^{1/2}r^e/L \). We therefore get

**Lemma 6.1.**

\[
(6.3) \quad S_{1,J}(N, C, L) \ll \frac{K^{4}}{NC^4} \sum_{1 \leq |m| \leq C} \sum_{1 \leq |m'| \leq C} \sum_{1 \leq |q| \leq C} \sum_{1 \leq |q'| \leq C} |\mathbb{C}| |\mathcal{R}| + O(t^{-2015})
\]

where

\[
(6.4) \quad \mathcal{R} = \int_{\mathbb{R}^2} (NL)^{-ir/2 + it'/2} \gamma(1 + i\tau)\gamma(1 + it) \frac{1}{q - ir q' \tau} W_J(\tau) W_J(\tau') I_1(q, m, \tau) I_1(q', m', \tau') U^\dagger \left( \frac{nnL}{qq'}, \frac{-ir}{2} + \frac{it'}{2} \right) d\tau d\tau'
\]

Using the expression for \( I_1(q, m, \tau) \) as given in lemma (4.2), we get the expression

\[
(6.5) \quad \mathcal{R} = \frac{|\mathbb{C}|^2}{K^2} \int_{\mathbb{R}^2} \gamma(1 + i\tau)\gamma(1 + it\tau) W_J(q, m, \tau) W_J(q', m', \tau') (LN)^{-ir/2 - it'/2} \left( \frac{2\pi q q'}{2\pi q q'} \right)^{-i(t + \tau)/2} \frac{(2\pi q q')^{-it'/2}}{2\pi q q'} U^\dagger \left( \frac{nnL}{qq'}, \frac{-ir}{2} + \frac{it'}{2} \right) d\tau d\tau'
\]

where

\[
W_J(q, m, \tau) = \frac{1}{(t + \tau/2)^{1/2}} W_J(\tau) \left( -\frac{(t + \tau/2)q}{2\pi q q'} \right)^{3/2} V \left( -\frac{(t + \tau/2)q}{2\pi q q'} \right) \int_{0}^{1} V \left( \frac{\tau}{2K} - \frac{(t + \tau/2)q}{2\pi q q'} \right) dx
\]

Since \( u^{-3/2} V(u) \ll 1 \) and \( \tau \ll t^{1-\epsilon} \), it follows that

\[
(6.6) \quad \frac{\partial}{\partial \tau} W_J(q, m, \tau) \ll \frac{1}{t^{1/2} |\tau|}
\]

We also note that the \( x \)-integral inside the expression of \( W_J(q, m, \tau) \) contributes a factor of the size of its length, which is \( \ll Kma/(t + \tau) \). Since \( m \ll Ct^{1+\epsilon}/N \) and \( \tau \ll t \), the contribution is \( \ll KCQ t^e/N \). Therefore \( W_J(q, m, \tau) \ll K^{1/2}C/t^{1/2}N^{1/2} \).
We analyze the integral \( \mathcal{R} \) in two cases, when \( n = 0 \) and when \( n \neq 0 \). For \( n = 0 \), the expression for \( C \) gives \( q = q' \), and the bound on \( U^\dagger \) gives us arbitrary saving for \( |\tau - \tau'| > t^\varepsilon \). In this case,

\[
\mathcal{R} \ll \frac{|c_4|^2}{K^2} \int_{|\tau| < \frac{n}{NK}^{1/2}/C} \gamma(1 + i\tau)^2 W_J(q, m, \tau) \int_{|\tau'| < t^{\varepsilon/2}} W_J(q, m', \tau') d\tau' d\tau < \frac{t^\varepsilon C}{K^{1/2} N^{1/2} t^\varepsilon} =: B^*(C, 0)
\]

When \( n \neq 0 \),

\[
U^\dagger \left( \frac{nL}{qq'}, \frac{1 - i\tau/2 + it'}{2} \right) = \frac{c_5}{(\tau - \tau')^{1/2}} U \left( \frac{(\tau - \tau')qq'}{4\pi nL} \right) \left( \frac{(\tau - \tau')qq'}{4\pi nL} \right)^{-ir/2 + it'/2} + O \left( \min \left\{ \frac{1}{|\tau - \tau'|^{3/2}}, \frac{C^3}{(|nL|)^{3/2}} \right\} \right)
\]

for some absolute constant \( c_5 \).

Contribution of the error term towards \( \mathcal{R} \) is of the order of

\[
\frac{t^\varepsilon}{K^2} \int_{[J\mathcal{A}/3]^2} \int_{|\tau - \tau'| < |nL/C|^2} \frac{1}{t} \min \left\{ \frac{1}{|\tau - \tau'|^{3/2}}, \frac{C^3}{(|nL|)^{3/2}} \right\} d\tau' d\tau
\]

When the second term is smaller,

\[
\frac{t^\varepsilon}{K^2} \int_{[J\mathcal{A}/3]^2} \int_{|\tau - \tau'| < |nL/C|^2} \frac{1}{t} \frac{C^3}{(|nL|)^{3/2}} d\tau' d\tau' < \frac{1}{K^{3/2} t (|nL|)^{1/2}} t^\varepsilon
\]

When the first term is smaller,

\[
\frac{t^\varepsilon}{K^2} \int_{[J\mathcal{A}/3]^2} \int_{|\tau - \tau'| > |nL/C|^2} \frac{1}{t} \frac{1}{|\tau - \tau'|^{3/2}} d\tau' d\tau' < \frac{t^\varepsilon}{K^{3/2} t (|nL|)^{1/2}} \int_{[J\mathcal{A}/3]^2} \int_{|\tau - \tau'| > |nL/C|^2} \frac{1}{t} \frac{C}{(|nL|)^{1/2}} d\tau' d\tau' < \frac{1}{K^{3/2} t (|nL|)^{1/2}} t^\varepsilon
\]

The error contribution (for \( n \neq 0 \)) is

\[
B^*(C, n) = \frac{1}{K^{3/2} t (|nL|)^{1/2}} t^\varepsilon
\]

We finally analyze the main term. Striling’s formula is

\[
\Gamma(\sigma + i\tau) = \sqrt{2\pi} (i\tau)^{\sigma - 1/2} e^{-\pi|\tau|^2/2} \left( \frac{|\tau|}{e} \right)^i \left( 1 + O \left( \frac{1}{|\tau|} \right) \right)
\]

as \( |\tau| \to \infty \). That gives

\[
\gamma(1 + i\tau) = \left( \frac{|\tau|}{4\pi e} \right)^i \Phi(\tau), \quad \text{where} \quad \Phi'(\tau) \ll \frac{1}{|\tau|}
\]

By Fourier inversion, we write

\[
\left( \frac{4\pi nL}{(\tau - \tau')qq'} \right)^{1/2} U \left( \frac{(\tau - \tau')qq'}{4\pi nL} \right) = \int_{\mathbb{R}} U^\dagger(r, 1/2) e \left( \frac{(\tau - \tau')qq'}{4\pi nL} r \right) dr
\]
We conclude that for some constant $c_6$ (depending on the sign of $n$)

\[
R = \frac{c_6}{K^2} \left( \frac{q \pi}{|n|L} \right)^{1/2} \int_{\mathbb{R}^2} U^1(t, 1/2) \int g(\tau, \tau') \hat{e}(f(\tau, \tau')) d\tau d\tau' + O(B^*(C, n))
\]

where

\[
2\pi f(\tau, \tau') = -\tau \log \left( \frac{\tau}{4\pi e} \right) - \tau' \log \left( \frac{\tau'}{4\pi e} \right) - \frac{(\tau - \tau')}{2} \log(\tau) + \frac{\tau}{2} \log q - \frac{\tau'}{2} \log q' - (t + \tau/2) \log \left( \frac{\tau + \tau/2}{2\pi e N m} \right) + (t + \tau'/2) \log \left( \frac{(t + \tau'/2)q}{2\pi e N m'} \right)
\]

and

\[
g(\tau, \tau') = \Phi(\tau) \Phi(\tau') W_f(q, m, \tau) W_f(q', m', \tau')
\]

We intend to use the second derivative bound as given in Lemma 2.4. For that, we need the following

\[
2\pi \frac{\partial^2}{\partial \tau^2} f(\tau, \tau') = \frac{1}{4} \left( -\frac{4}{\tau} - \frac{1}{(t + \tau/2)} + \frac{2}{(\tau' - \tau)} \right), \quad 2\pi \frac{\partial^2}{\partial \tau'^2} f(\tau, \tau') = \frac{1}{4} \left( -\frac{4}{\tau'} + \frac{1}{(t + \tau'/2)} + \frac{2}{(\tau - \tau')} \right)
\]

and

\[
2\pi \frac{\partial^2}{\partial \tau \partial \tau'} f(\tau, \tau') = -\frac{1}{4} \left( \frac{2}{\tau' - \tau} \right)
\]

Also, by explicit computation,

\[
4\pi^2 \left[ \frac{\partial^2}{\partial \tau^2} f(\tau, \tau') \frac{\partial^2}{\partial \tau'^2} f(\tau, \tau') - \left( \frac{\partial^2}{\partial \tau \partial \tau'} f(\tau, \tau') \right)^2 \right] = -\frac{1}{2\tau'} + O \left( \frac{1}{t \tau} \right)
\]

for $\tau, \tau'$ such that $g(\tau, \tau') \neq 0$. So the conditions of lemma 4 of Munshi [8] hold with $r_1 = r_2 = 1/J^{1/2}$. To calculate the total variation of $g(\tau, \tau')$, recall that $\Phi'(\tau) \ll |\tau|^{-1}$ and $W_f(q, m, \tau) \ll t^{-1/2} \tau^{-1}$, so $\text{var}(g) \ll t^{-1+\epsilon}$. So the double integral in (6.11) over $\tau, \tau'$ is bounded by $O(t^{-1+\epsilon})$. Integrating trivially over $r$ using the rapid decay of the Fourier transform, we get that total contribution of the leading term in (6.11) towards $R$ is bounded by

\[
O \left( \frac{C}{K^2 (|n|L)^{1/2}} \left( \frac{NK}{C} \right)^{1/2} t^{-1+\epsilon} \right) = O(B^*(C, n))
\]

Putting everything together, we get the final bound

\[
S_{1,J}(N, C, L) \ll \frac{t^2 K}{NC^2} \left[ \sum_{C \leq q \leq 2C} \sum_{(m, q) = 1} \frac{(t / N)^2 B^*(C, 0)}{1 \leq m < q^{1+\epsilon / 2} / N} \sum_{n=0}^{C(NK)^{1/2}} \sum_{C < q < 2C} \frac{(t / N)^2 B^*(C, n)}{C^{(NK)^{1/4}}} \right]
\]

\[
= \frac{t^2 K}{NC^2} \left[ \frac{C^4}{N^{5/2} K^{1/2}} + \frac{C^{1/2} (NK)^{1/4}}{L} \frac{C^{2} t}{N^{3/2} K^{3/2}} \right]
\]
That gives
\[
S_{1,J}(N, C) \leq t^\epsilon N^{1/2}K \sum_{1 \leq L \leq K^\epsilon \text{dyadic}} L^{1/2} N^{1/2} C \left[ \frac{C^{3/2} t^{1/2}}{N^{5/4} K^{1/4}} + \frac{C^{1/4} (NK)^{1/8}}{L^{1/2} N^{3/4} K^{3/4}} \right] 
\]
\[
< t^\epsilon K^{3/2} \left( \frac{K^{1/4} C^{1/2} t^{1/2}}{N^{5/4}} + \frac{C^{1/4} t^{1/2}}{(NK)^{5/8}} \right)
\]

Multiplying by $N^{1/2}/K$ and summing over the dyadic range $C \ll Q$, we get

\[
(6.12) \quad \frac{S_1(N)}{N^{1/2}} \ll t^{1/2+\epsilon} \left( \frac{K^{1/2}}{N^{1/2}} + \frac{1}{K^{1/4}} \right)
\]

Finally, from equations (5.3) and (6.12), it follows that for $N \ll t^{1+\epsilon}$ and $K \gg N^{1/2}$,

\[
\frac{S(N)}{N^{1/2}} \ll t^{1/2+\epsilon} \left( \frac{K^{1/4}}{N^{1/2}} + \frac{N^{1/4}}{K^{3/4}} + \frac{K^{1/2}}{N^{1/2}} + \frac{1}{K^{1/4}} \right).
\]

The optimal choice for $K$ occurs at $K = N^{2/3}$ and we get Proposition 1.2.

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