Dynamical Renormalization Group Study of a Conserved Surface Growth with Anti-Diffusive and Nonlinear Currents

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Abstract

Based on dynamical renormalization group (RG) calculations to the one-loop order, the surface growth described by a nonlinear stochastic conserved growth equation,

$$\frac{\partial h}{\partial t} = \pm \nu_2 \nabla^2 h + \lambda \nabla \cdot (\nabla h)^3 + \eta \quad (\nu_2 > 0),$$

is studied analytically. The universality class of the growth described by the above equation with $+\nu_2$ (diffusion) is shown to be the same as that described by the Edwards-Wilkinson (EW) equation (i.e. $+\nu_2$ and $\lambda = 0$). In contrast our RG recursion relations manifest that the growth described by the above equation with $-\nu_2$ (anti-diffusion) is an unstable growth and do not reproduce the recent results from a numerical simulation by J. M. Kim [Phys. Rev. E 52, 6267 (1995)].

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Recently, there have been considerable interests in the various surface growth models \[1\] to understand roughenings in the growing surfaces theoretically. Since the surface structures of many growth processes are self-affine, most efforts have concentrated on the surface width \( W \), which is defined by the root mean square fluctuation of the surface height. In a finite system of lateral size \( L \), the width \( W \) starting from a flat substrate scales as \[2\]

\[
W(t) \sim L^\alpha f(t/L^z) \sim t^\beta, \quad t \ll L^z
\]

\[
\sim L^\alpha, \quad t \gg L^z
\]

where the scaling function \( f(x) \) is \( x^\beta \) for \( x \ll 1 \) and constant for \( x \gg 1 \). The exponents \( \beta \) and \( z \) are connected by the relation \( z = \alpha/\beta \). Among the growth models which satisfy Eq. (1), the class of models known as “conserved” models \[2-9\], which conserve the total number of particles after being deposited, has been extensively studied as a possible description for the real molecular beam epitaxy (MBE) growth \[10\]. In these conserved models the height \( h \) describing the local position of the surface of growing materials is known to obey \[3\]

\[
\frac{\partial h(x, t)}{\partial t} = -\nabla \cdot J(x, t) + \eta(x, t),
\]

where \( J \) is the surface current and \( \eta(x, t) \) is an uncorrelated random noise

\[
\langle \eta(x, t)\eta(x', t') \rangle = 2D\delta^d(x-x')\delta(t-t').
\]

There have been several studies on the surface growth described by the following current \[11-16\]

\[
J(x, t) = -\nu_2\nabla h + \nu_4\nabla^3 h - \lambda(\nabla h)^3
\]

and thus by the corresponding continuum equation

\[
\frac{\partial h(x, t)}{\partial t} = \nu_2\nabla^2 h - \nu_4\nabla^4 h + \lambda\nabla \cdot (\nabla h)^3 + \eta(x, t).
\]

For \( \nu_2 > 0 \), the critical property of the surface growth described by Eq. (5) is solely dependent on the first term \( \nu_2\nabla^2 h \) in the right hand side, which is the most relevant term
in a renormalization group (RG) sense and it belongs to the Edwards and Wilkinson (EW) universality class [17] with \( \alpha = (2 - d)/2 \) and \( z = 2 \) for any substrate dimension \( d \) [11,12]. When \( \nu_2 = \lambda = 0 \), Eq. (5) becomes the Mullins-Herring equation [18], which can be solved exactly to give \( \alpha = (4 - d)/2 \) and \( z = 4 \) [3]. When \( \nu_2 = 0, \nu_4 > 0 \) and \( \lambda > 0 \), the recent studies [11,12] have shown that the most relevant nonlinear \( \lambda \) term generates an effective \( \nu_2^{\text{eff}} \nabla^2 h \) term with \( \nu_2^{\text{eff}} > 0 \) and the corresponding growth belongs to the EW universality class, even though both the scaling argument and the dimensional analysis [4] suggest the exponent values \( \beta = (4 - d)/2 (4 + d) \) and \( \alpha = (4 - d)/4 \).

More recently the growth described by Eq. (5) with \( \nu_2 < 0, \nu_4 \geq 0 \) and \( \lambda > 0 \) has been studied numerically [13]. From this numerical results it has been argued that the critical property of such growth belongs to the EW universality class even for \( \nu_2 < 0 \), on the ground that the nonlinear \( \lambda \) term could supress any negative \( \nu_2 \), so that \( \nu_2 \) could become effectively positive, i.e. \( \nu_2^{\text{eff}} > 0 \) [13]. If this argument is right, then the \( \lambda \) term could make the unstable growth with \(-|\nu_2|\nabla^2 h\) term the stable EW growth with \( \nu_2^{\text{eff}} > 0 \). However if one analyzes Eq. (5) from a point of view of the scaling and dimensional analyses, the most relevant term is \( \nu_2 \nabla^2 h \) regardless of the sign of \( \nu_2 \). So it is somewhat strange that the less relevant \( \lambda \) term renormalizes the most relevant \( \nu_2 \) term so effectively as to make the change of sign of \( \nu_2 \). It is thus our motivation to check whether this strange renormalization is physically plausible or not by use of dynamical renormalization group calculations [19,20] and other analytical methods.

To achieve this goal, we first discuss the dynamical renomalization group study of Eq. (5) with \( \nu_2 \neq 0, \nu_4 = 0 \) and \( \lambda > 0 \). In the large distance and long-time hydrodynamic limits, Eq. (5) for \( \nu_4 = 0 \) in Fourier space can be written as

\[
h(k, \omega) = G(k, \omega)\eta(k, \omega) \\
= G_0(k, \omega)\eta(k, \omega) + \lambda G_0(k, \omega) \\
\times \int \int \int \int d\Omega_1 d^d q_1 d\Omega_2 d^d q_2 \\
\times (q_1 \cdot q_2)[k \cdot (k - q_1 - q_2)]h(q_1, \Omega_1)
\]
\begin{align}
\times h(q_2, \Omega_2) h(k - q_1 - q_2, \omega - \Omega_1 - \Omega_2) \tag{6}
\end{align}

where \( G_0(k, \omega) \) is the bare propagator defined by the expression
\begin{align}
G_0(k, \omega) = \frac{1}{\nu_2 k^2 - i\omega}. \tag{7}
\end{align}

Using \( \langle \eta(k, \omega)\eta(k', \omega') \rangle = 2D\delta^{d}(k + k')\delta(\omega + \omega') \) and performing the internal frequency integrals of Eq. (6) to the one-loop order, one obtains
\begin{align}
G(k, \omega) = G_0(k, \omega) - \frac{\lambda D}{\nu_2} K_{d} \frac{d + 2}{d} k^2 G_0^2(k, \omega) \Sigma(q), \tag{8}
\end{align}

in the limits \( \omega \to 0 \) and \( k \to 0 \). Here \( \Sigma(q) = \int \Lambda dqq_{-1} \), \( \Lambda \) is the momentum cutoff (\( \Lambda \equiv 1 \)), and \( K_{d} = S_{d}/(2\pi)^{d} \) with \( S_{d} = 2\pi^{d/2}/\Gamma(d/2) \). Since \( \Sigma(q) \) has no infrared divergence for any dimension, we can expect that the nonlinear \( \lambda \) term is irrelevant in RG sense and that the \( \lambda \) will not renormalize negative \( \nu_2 \) term to be positive.

For the confirmation’s sake, we want to calculate the dynamical RG recursion relations for \( \nu_2 \) and \( \lambda \). Integrating out the fast modes in the momentum shell \( k^{>} \in [\Lambda e^{-\ell}, \Lambda] \) and restoring the slow modes \( k^{<} \in [0, \Lambda e^{-\ell}] \) with \( e^{-\ell} = 1 - \delta \ell + \cdots \), we find an effective surface tension \( \nu_2^{<} \) for the long wavelength modes,
\begin{align}
\nu_2^{<} = \nu_2 \left[ 1 + \delta \ell K_{d} \frac{\lambda D d + 2}{d} \right]. \tag{9}
\end{align}

In an analogous way we get
\begin{align}
\lambda^{<} = \lambda \left[ 1 - \delta \ell K_{d} a(d) \frac{\lambda D}{\nu_2^2} \right], \tag{10}
\end{align}

where \( a(d) = 9 \) for \( d = 1 \), and \( a(d) = \frac{9}{2} \) for \( d = 2 \). Upon requiring that the equation stays invariant under the scale(\( e^{\ell} \)) transformations \( k \to e^{-\ell}k \), \( t \to e^{\varepsilon t}t \) and \( h \to e^{\alpha t} \), the parameters should transform as follows: \( \nu_2 \to b^{z-2} \nu_2 \), \( D \to b^{z-2\alpha - d} D \) and \( \lambda \to b^{z+2\alpha-4} \lambda \). Combining these scale transformations and Eqs. (9) and (10), we get the following RG recursion relations:
\begin{align}
\frac{d\nu_2}{d\ell} = \nu_2 \left[ z - 2 + gK_{d} \frac{d + 2}{d} \right] \tag{11}
\end{align}

\begin{align}
\frac{dD}{d\ell} = D \left[ z - 2\alpha - d \right] \tag{12}
\end{align}

\begin{align}
\frac{d\lambda}{d\ell} = \lambda \left[ z + 2\alpha - 4 - ga(d)K_{d} \right], \tag{13}
\end{align}

4
where the coupling constant \( g \) is defined as

\[
g \equiv \frac{\lambda D}{\nu_2^2} .
\]  

(14)

The RG recursion relation of the effective coupling constant \( g \) is thus

\[
\frac{dg}{d\ell} = g \left[ -d - gA(d)K_d \right] ,
\]  

(15)

where \( A(1) = 15 \) and \( A(2) = \frac{17}{2} \) for \( d = 1 \) and \( 2 \) and we have also confirmed \( A(d) > 0 \) for any \( d \). As one can expect from Fig. 1, the RG flow of \( g \) has only one stable fixed point which is a Gaussian fixed point, i.e. \( g^* = 0 \) and does not have any nontrivial fixed points. The RG flow of \( g \) for \( \nu_2 < 0 \) is the same as those for \( \nu_2 > 0 \). If the results from the numerical calculations in Ref. [13] would be explained by the dynamical RG, then the RG flow of \( g \) for \( \nu_2 < 0 \) should be different from those for \( \nu_2 > 0 \). As shown in Fig. 2, the RG flows starting at \( (\nu_2 > 0, \lambda > 0) \) will eventually arrive at \( (\nu_2^{\text{eff}} > 0, \lambda \to 0) \). This means that the growth described by Eq. (5) with \( \nu_1 = 0, \nu_2 > 0 \) and \( \lambda > 0 \) in the large distance scale belongs to the EW universality class as we have expected. The RG flows starting at \( (\nu_2 < 0, \lambda > 0) \), on the while, will eventually arrive at \( (\nu_2^{\text{eff}} < 0, \lambda \to 0) \). This means that the growth described by Eq. (5) with \( \nu_4 = 0, \nu_2 < 0 \) and \( \lambda > 0 \) in the large distance scale has nearly the same critical property as the unstable linear growth described by

\[
\frac{\partial h(x,t)}{\partial t} = -|\nu_2|\nabla^2 h + \eta .
\]  

(16)

Based on our dynamical RG calculations we conclude that the growth by Eq. (5) with \( \nu_2 < 0, \lambda > 0 \) and \( \nu_4 = 0 \) should be an unstable growth and we cannot reproduce the recent results by a numerical simulation [13].

The RG recursion relations (11), (12) and (13) are based on the one-loop order perturbative calculations and are intrinsically exact only for \( \lambda \ll 1 \) or \( g = \frac{\lambda D}{\nu_2^2} \ll 1 \). The main result of the dynamical RG calculations physically indicates that the growth described by Eq. (5) with \( \nu_2 < 0, \nu_4 = 0 \) and the small \( \lambda(> 0) \) does not belong to the same universality class as that of the EW growth. To understand physically the unstable growth by Eq. (5)
with \( \nu_2 < 0, \lambda > 0 \) and \( \nu_4 = 0 \) for the finite \( \lambda \) including the case for \( \lambda \ll 1 \), let’s discuss the growth from an Hamiltonian-based argument or from an equilibrium physics. If one believe that the dynamical equation (14) with \( \nu_4 = 0 \) can be derived by an equilibrium Hamiltonian \( \mathcal{H} \) via the Langevin equation \( \frac{\partial h}{\partial t} = -\delta_{\mathcal{H}} + \eta \) and the Langevin equation will reach a steady state or an equilibrium where the states are controlled by the Boltzman distribution \( P(\mathcal{H}) \propto \exp(-\beta \mathcal{H}) \), \( \mathcal{H} \) should be

\[
\mathcal{H} = \int d^d x \left[ \frac{\nu_2}{2} (\nabla h)^2 + \frac{\lambda}{4} (\nabla h)^4 \right].
\] (17)

If one puts \( \nabla h \) to be equal to a field \( \phi(x) \) (i.e., \( \nabla h \equiv \phi(x) \) ) and use a mean-field theoretic argument for Hamiltonian (17), the corresponding Landau-Ginzburg function \( \mathcal{L} \) becomes

\[
\mathcal{L} = \frac{\nu_2}{2} \phi^2 + \frac{\lambda}{4} \phi^4. \] (18)

When \( \nu_2 > 0 \) and \( \lambda > 0 \), the mean field theory with \( \frac{\partial \mathcal{L}}{\partial \phi} = 0 \) gives \( \langle \phi \rangle = \langle \nabla h \rangle = 0 \) as in a disordered phase of the ordinary magnetic phase transitions \([21]\). The fluctuations around the mean-field \( \langle \phi \rangle = \langle \nabla h \rangle = 0 \) should be a Gaussian-type and it seems quite plausible to believe that the surface roughenings with \( \nu_2 > 0 \) and \( \lambda > 0 \) belong to the EW universality class. However when \( \nu_2 < 0 \) and \( \lambda > 0 \), the mean field theory predicts

\[
\langle \phi^2 \rangle = \langle (\nabla h)^2 \rangle_{eq} = -\frac{\nu_2}{\lambda} > 0
\] (19)
as in an ordered phase of the ordinary magnetic phase transitions. In the surface growth phenomena, Eq. (19) means that the local slope satisfies \( \langle |\nabla h| \rangle_{eq} \simeq \sqrt{|\nu_2|/\lambda} \) in an equilibrium state, i.e. in a saturation state. Dynamically, in the growing process when \( \langle |\nabla h| \rangle < \sqrt{|\nu_2|/\lambda} \), the anti-diffusion term \(-\nu_2 \nabla h \) of the current (see Eq. (14)) dominates. Dynamical RG recursion relations (11), (12) and (13) which are intrinsically exact for \( \lambda \ll 1 \) or \( g = \frac{10}{\nu_2} \ll 1 \) correspond to this growing stage of the local slopes. For \( g \ll 1 \) where RG calculations are correct, it takes very long time for the local slopes to reach the equilibrium value \( \sqrt{|\nu_2|/\lambda} \) and this dynamical process should be described by Eq. (16), because \( \langle |\nabla h| \rangle_{eq} \simeq \sqrt{|\nu_2|/\lambda} \gg 1 \). We have confirmed this kind of growth by using several simulations for the growthes with different \( g \)’s \([22]\). In contrast for finite \( g \)’s the time interval in which the anti-diffusion dominates
become rather finite and sooner or later the growth reaches the saturation state or the equilibrium state where the effect of the anti-diffusion term is balanced to give 
\[ \langle |\nabla h| \rangle \sim \sqrt{\frac{\nu_2}{\lambda}}. \]
After that local slopes fluctuates around the value in Eq. (19). We have also confirmed that this kind of the unstable growth occurs when \( g \) is finite by simulations [22]. It is thus not physically sound that the growth with \( \nu_2 < 0 \) and \( \lambda > 0 \) belongs to the same universality class as that of the EW-like growth with \( \nu_2^{\text{eff}} \nabla^2 h \) with \( \nu_2^{\text{eff}} > 0 \) as in Ref. [13]. Instead we do expect an unstable growth.

We now want to discuss on the point why the numerical simulations in Ref. [13] couldn’t see such an unstable growth and did see only the stable growth which belongs to the EW universality class. To do a numerical simulation in Ref. [13], the local currents in Eq. (4) in \( d = 1 \) has been set to be equal to
\[ \hat{j}(k, i) = \nu_2[h(k) - h(i)] - \nu_4[h(k + 1) + h(k - 1)] - 2h(k) - h(i + 1) - h(i - 1) + 2h(i) + \lambda[h(k) - h(i)]^3, \] (20)
where \( k \) is either \( i + 1 \) and \( i - 1 \) and a particle is added on site \( k \) if \( \hat{j}(k) \) is negative. If both \( \hat{j}(i + 1, i) \) and \( \hat{j}(i - 1, i) \) are positive, a particle is added on site \( i \). In case both \( \hat{j}(i + 1, i) \) and \( \hat{j}(i - 1, i) \) are negative, a particle is added on either site randomly. When \( \nu_4 = 0 \), Eq. (21) can be written as
\[ \hat{j}(k, i) = \nu_2[h(k) - h(i)]\left\{ 1 + \frac{\lambda}{\nu_2}[h(k) - h(i)]^2 \right\}. \] (21)
In the initial growing process with \( \nu_2 < 0 \) when \( \langle [h(k) - h(i)]^2 \rangle \) is quite small and \( \{ \} \) term in Eq. (21) is positive, the anti-diffusion should dominate the growth. As the growth is developed, \( \langle [h(k) - h(i)]^2 \rangle \) increases to \( -\nu_2/\lambda \) rapidly and after that \( \{ \} \) term in Eq. (21) fluctuates around zero. We have confirmed that at this stage the local slope \( \langle |h(k) - h(i)| \rangle \) is nearly equal to \( \sqrt{-\nu_2/\lambda} [22]. \) In contrast the numerical simulations in Ref. [13] have been done only for the cases \( -\nu_2/\lambda = 1, 3 \) and the corresponding saturated local slopes \( |h(k) - h(i)| \) are only 1 and \( \sqrt{3} \). Since \( h(k) \) must be integer numbers 0,1,2,\ldots in a simulation,
\[ |h(k) - h(i)| = 0, 1 \text{ or } 2 \] could be easily generated even by the noise effect only, so the \{ \} term in Eq. (21) being negative for \(-\nu^2/\lambda = 1, 3\). This fact explains why the numerical simulation in Ref. [13] had seen only the EW-type growths instead of the unstable growth. But for the considerably large \(-\nu^2/\lambda\), the anti-diffusion term dominates from the beginning and this effect stays longer and longer until being cancelled out by the nonlinear \(\lambda\) term. Hence hardly realizes the EW-like behavior.

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FIGURES

FIG. 1. The RG flows of the coupling constant $g$. There is only one fixed point $g^* = 0$ which is attractive fixed point. The RG flows of $g$ for $\nu_2 < 0$ are the same as those for $\nu_2 > 0$. (See Eq. (15))

FIG. 2. Schematic RG flows in $(\lambda, \nu_2)$-plane. for $d = 1$ and 2, respectively. The RG flows starting at $(\nu_2 > 0, \lambda > 0)$ will eventually arrive at $(\nu_2^{\text{eff}} > 0, \lambda \to 0)$. In contrast the RG flows starting at $(\nu_2 < 0, \lambda > 0)$ will eventually arrive at $(\nu_2^{\text{eff}} < 0, \lambda \to 0)$. 

