Second-Order Asymptotics in Covert Communication

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Abstract

We study the second-order asymptotics of covert communication over binary-input Discrete Memoryless Channels (DMCs) for three different metrics of covertness. When covertness is measured in terms of the relative entropy between the channel output distributions induced with and without communication, we characterize the exact second-order asymptotics of the number of bits that can be reliably transmitted with a probability of error less than $\epsilon$ and a relative entropy less than $\delta$. When covertness is measured in terms of the variational distance between the channel output distributions or in terms of the probability of missed detection for fixed probability of false alarm, we establish the exact first-order asymptotics and bounds on the second-order asymptotics. The main conceptual contribution of this paper is to clarify how the choice of a covertness metric impacts the information-theoretic limits of the number of covert and reliable bits. The main technical contribution of the underlying results is a detailed analysis of probability of existence of a code satisfying the reliability and covert criteria.

I. INTRODUCTION

While most information-theoretic security works to date have revolved around the issues of confidentiality and authentication \cite{5, 23}, the growing concern around mass communication surveillance programs has rekindled interest for investigating the covertness of communications, also known as Low Probability of Detection (LPD). In LPD problems, the objective is to hide the presence of communication and not necessarily to prevent information leakage about the messages transmitted. Following the analysis of LPD with space-time codes \cite{11}, recent works have established the information-theoretic limits of covert communications over noisy channel \cite{2}, \cite{15}. In particular, building upon concepts from steganography \cite{6}, \cite{15}, \cite{2} has proved the existence of a “square-root law” for covert communication, which essentially states that no more than $O(\sqrt{n})$ bits can be communicated covertly over $n$ channel uses of a memoryless channel.

The square-root law of covert communication has been refined in several follow-up works \cite{7}; in particular, the information-theoretic limits are now known for classical discrete and Gaussian memoryless channels \cite{5}, \cite{23}, classical-quantum channels \cite{22}, \cite{19}, and multiple access channels \cite{11}, when covertness is measured in terms of the relative entropy between the channel output distributions induced with and without communication. Note that the choice of relative entropy as a metric for covertness is guided in part by the natural connection between relative entropy and information-theoretic metrics, such as entropy and mutual information, which have been largely explored in the context of information-theoretic security \cite{12}.

The contribution of the present paper is twofold. First, as an attempt to develop operational characterizations of co covertness, we study the information-theoretic limits of covert communication for alternative metrics, including variational distance, and probability of missed detection. Second, motivated by the likely time-limited nature of covert communications, we make a first step towards a finite transmission length analysis and extend the first-order analysis of information-theoretic limits to second-order asymptotics. The specific results developed in the present paper focus on binary-input DMCs to simplify analytical expressions and are the following.

- We characterize the exact second-order asymptotics of the maximum number of reliable and covert bits that can be transmitted with average error probability $\epsilon \leq \frac{1}{2}$ and relative entropy $\delta$ between channel output distributions with and without communication (Theorem 1); this corrects an unfortunate error in the conference version \cite{20}, in which we claimed erroneous second-order asymptotics, and generalizes known results about first-order asymptotics \cite{5}, \cite{23}.
- We characterize the exact first-order asymptotics of the maximum number of reliable and covert bits that can be transmitted with average error probability $\epsilon \leq \frac{1}{2}$ and variational distance $\delta$ between channel output distributions with and without communication; we also develop bounds for the second-order asymptotics (Theorem 2).
- Finally, we characterize the exact first-order asymptotics of the maximum number of reliable and covert bits that can be transmitted with average error probability $\epsilon \leq \frac{1}{2}$ and probability of missed detection $\gamma = 1 - \alpha - \delta$, where $\alpha$ is the adversary’s probability of false alarm; we again develop bounds for the second-order asymptotics (Theorem 3).

The second-order asymptotics obtained with a relative entropy metric for co covertness are what could have been expected by extrapolating the results of channel coding in the finite length regime \cite{17} to the first-order asymptotics of covert communication \cite{5}, \cite{23}; however, the proof requires specific techniques beyond those used to study the first-order asymptotics. First, the achievability proof relies on Pulse Position Modulation (PPM) codes \cite{4}, which can be viewed as a highly structured form of constant composition codes, instead of independent and identically distributed (i.i.d.) random codes. Second, the optimal
coding scheme identified in [5] exploits a code with a bin structure, in which each bin forms a reliability code for the legitimate channel indexed by the secret key, while the overall code forms a resolvability code for the adversary’s channel. To guarantee the positive probability of existence of a code with the desired characteristics, we resort to concentration of measure inequalities such as McDiarmid’s Inequality and carefully analyze the probability of error. We point out that our current results only identify the second-order asymptotics for the number of transmitted message bits and do not characterize the second-order asymptotics for the number of key bits.

The remainder of the paper is organized as follows. In Section III we formally introduce the model of covert communication and state our main results. In Section III we develop a series of general results that form the basis of our analysis of covert communication. In Section IV-A, Section IV-B, and Section IV-C, we exploit the results of Section III to study a relative entropy metric, variational distance metric, and probability of missed-detection metric for covertness, respectively. In Section V we conclude the paper with a discussion of possible further extensions and improvements.

II. MODEL AND MAIN RESULTS

A. Notation

Throughout the paper, log and exp should be understood in base e. Moreover, random variables are denoted with upper case letters, e.g. X, while their realization are denoted with lower case letters, e.g., x. The distribution of a random variable such as X is denoted by P_X. Calligraphic letters are used for sets, e.g., X, and boldface fonts are used for vectors e.g., x. For two integers a and b, if a ≤ b, we define [a, b] ≜ {a, a + 1, · · · , b − 1, b}; otherwise [a, b] ≜ ∅. For any distribution P over X, P^n denotes the product distribution over X^n, i.e., P^n(x) ≜ \prod_{i=1}^n P(x_i). For two distributions P and Q over the same set X, we define the quantities

\[ \mathbb{D}(P||Q) \triangleq \sum_x P(x) \log \frac{P(x)}{Q(x)} \] (relative entropy)

\[ \mathbb{V}(P, Q) \triangleq \frac{1}{2} \sum_x |P(x) - Q(x)| \] (variational distance)

\[ \chi^2(P||Q) \triangleq \sum_x \frac{(P(x) - Q(x))^2}{Q(x)} \] (chi-squared distance)

\[ \beta_\alpha(P, Q) \triangleq \inf_{T \leq \alpha} \mathbb{E}_X : P(T) \leq \alpha Q(T) \] (optimal probability of missed detection).

The notation P ≪ Q means that P is absolutely continuous with respect to (w.r.t.) Q, i.e., if Q(x) = 0 for some x ∈ X, then P(x) = 0. A DMC (X, W_Y|X, Y) consists of a finite input alphabet X, a finite output alphabet Y, and a transition probability W_Y|X where W_Y|X(y|x) indicates the probability of obtaining y at the output given that x is transmitted at the input. For a DMC (X, W_Y|X, Y) and two distributions P_X and Q_Y on X and Y, we define

\[ P_Y(y) \triangleq \sum_x P_X(x)W_Y|X(y|x), \]

\[ I(P_X, W_Y|X) \triangleq \sum_{x,y} P_X(x)W_Y|X(y|x) \log \frac{W_Y|X(y|x)}{P_Y(y)}, \]

and for any γ > 0,

\[ F_{XY|Q_Y}(\gamma) \triangleq \mathbb{P}_{W_Y|X|P_X} \left( \log \frac{W_Y|X(Y|X)}{Q_Y(Y)} \leq \gamma \right) = \sum_{x,y} P_X(x)W_Y|X(y|x) \mathbb{I} \left\{ \log \frac{W_Y|X(y|x)}{Q_Y(y)} \leq \gamma \right\}, \]

\[ F_{XY}(\gamma) \triangleq F_{XY|Q_Y}(\gamma), \]

\[ F_X(\gamma) \triangleq 1 - F_{XY}(\gamma). \]

Moreover, given codewords \{x_i\}_{i=1}^M ∈ X^M and a uniform random variable W ∈ [1, M], \widehat{P}_{WXY} denotes the joint distribution induced on (W, X, Y), i.e.,

\[ \widehat{P}_{WXY}(w, x, y) \triangleq \frac{1}{M} \mathbb{I}\{x_w = x\} W_Y(x). \]

For any discrete random variable A, let \mu_A ≜ \min_{\mathbb{P}(A = a) > 0} \mathbb{P}(A = a). Finally, for x ∈ \{0, 1\}^n, wt(x) denotes the weight of x defined as |

| {i ∈ [1, n] : x_i = 1} |.
which induces the distribution \( \tilde{\eta} \).

**Definition 1.** A formal description of a code is as follows.

1. Communicate a uniformly distributed message \( W \) Bernoulli(\( \eta \)) over communication and other cases have been investigated in [5, Section V]. Moreover, for \( \epsilon \), an \( \epsilon \)-reliable code defined by

   \[
   D(\hat{W} || W^n) \leq \delta, \quad \epsilon = \frac{1}{\epsilon}, \quad \eta = \frac{1}{\eta}, \quad \alpha = \frac{1}{\alpha}, \quad \beta = \frac{1}{\beta}.
   \]

2. An \( \epsilon \)-reliable code, if the pair \((f, \phi)\) is such that

   \[
   P_e^* = \max_{s \in [1, K^2]} P(\phi(Y), s) | W | S = s \leq \epsilon.
   \]

Moreover, an \( \epsilon \)-reliable code defined by \((f, \phi)\) is

- an \((M, K, n, \epsilon, \delta)_D\) code, if it satisfies

  \[
  \mathbb{D}(\hat{P}_Z || Q_0^n) \leq \delta;
  \]

- an \((M, K, n, \epsilon, \delta)_V\) code, if it satisfies

  \[
  \forall (\hat{P}_Z, Q_0^n) \leq \delta;
  \]

- an \((M, K, n, \epsilon, \delta, \alpha)_\beta\) code, if it satisfies

  \[
  \beta_{\alpha}(Q_0^n, \hat{P}_Z) \geq 1 - \alpha - \delta.
  \]

The maximum number of messages that can be transmitted by an \((M, K, n, \epsilon, \delta)_D\), \((M, K, n, \epsilon, \delta)_V\), and \((M, K, n, \epsilon, \delta, \alpha)_\beta\) code is denoted by \(M_D(n, \epsilon, \delta), M_V(n, \epsilon, \delta),\) and \(M_\beta(n, \epsilon, \delta, \alpha)\), respectively.

We point out that the definition of the probability of error in (15) differs from previous works [23, 5]: we ask that the average probability of error be small for *any* choice of the key \( S \). This more stringent condition not only captures a perhaps more practical requirement, but also greatly simplifies our converse analysis. Also note that the three metrics for covertness in (16)-(18) have different operational meanings. When enforcing \( \beta_{\alpha}(Q_0^n, \hat{P}_Z) \geq 1 - \alpha - \delta \) for a fixed probability of false alarm...
For as defined above as function of the channel characteristics and the block length \( n \), one implicitly assumes that the adversary is known to optimize its detection at a specific point of its Receiver Operating Characteristic (ROC) curve. In contrast, when enforcing \( \mathbb{V}(\hat{P}_Z, Q_{\alpha}^{n}) \leq \delta \), since \( \alpha + \beta_{\alpha}(Q_{\alpha}^{n}, \hat{P}_Z) \geq 1 - \mathbb{V}(\hat{P}_Z, Q_{\alpha}^{n}) \) for any probability of false alarm \( \alpha \), one essentially wishes to enforce coarseness irrespective of the exact operating point on the adversary’s ROC curve. Finally, since \( \mathbb{V}(\hat{P}_Z, Q_{\alpha}^{n}) \leq \mathbb{V}(\hat{P}_Z || Q_{\alpha}^{n}) \) by Pinsker’s Inequality, the constraint \( \mathbb{D}(\hat{P}_Z || Q_{\alpha}^{n}) \leq \delta \) is more stringent than when using variational distance, but otherwise has the same significance. The relations between the metrics for coarseness immediately lead to the following ordering of the maximum number of covert bits.

\[
M_{D}^\ast(n, \epsilon, \delta) \leq M_{V}^\ast(n, \epsilon, \sqrt{\delta}) \leq \min_{\alpha \in [0, 1]} M_{\ast}^\ast(n, \epsilon, \sqrt{\delta}, \alpha). \tag{19}
\]

Our main results in Theorem 1, Theorem 2, and Theorem 3 characterize the maximum number of reliable and covert bits defined above as function of the channel characteristics and the block length \( n \). In all cases, the first and second terms behave as \( \Theta(n^{\frac{1}{2}}) \) and \( \Theta(n^{\frac{3}{4}}) \), respectively, as expected; nevertheless, the constant behind \( \Theta(.) \) is metric-specific.

**Theorem 1.** For \( \epsilon \in [0, \frac{1}{2}] \) and \( \delta > 0 \), we have

\[
\log M_{D}^\ast(n, \epsilon, \delta) = \sqrt{\frac{2\delta}{\chi_2(Q_1||Q_0)}} D_{P} n^{\frac{1}{2}} - \sqrt{\frac{2\delta}{\chi_2(Q_1||Q_0)}} V_{\epsilon} Q^{-1}(\epsilon) n^{\frac{1}{2}} + O(\log n), \tag{20}
\]

with

\[
D_{P} \triangleq \mathbb{D}(P_1||P_0), \quad V_{\epsilon} \triangleq \text{Var} \left( \log \frac{P_1(Y)}{P_0(Y)} \right| P_1). \tag{21}
\]

This optimal message length is obtained with a first-order optimal number of key bits

\[
\log K = (1 + \rho) \sqrt{\frac{2\delta}{\chi_2(Q_1||Q_0)}} [D_{Q} - D_{P}] n^{\frac{1}{2}}, \tag{22}
\]

where \( \rho > 0 \) can be arbitrarily small and \( D_{Q} = \mathbb{D}(Q_1||Q_0) \).

**Theorem 2.** For \( \epsilon \in [0, \frac{1}{2}] \) and \( \delta \in [0, 1] \), we have

\[
\log M_{V}^\ast(n, \epsilon, \delta) \leq \sqrt{\frac{2}{\chi_2(Q_1||Q_0)}} Q^{-1} \left( \frac{1 - \delta}{2} \right) D_{P} n^{\frac{1}{2}} - \sqrt{\frac{2}{\chi_2(Q_1||Q_0)}} V_{\epsilon} Q^{-1}(\epsilon) n^{\frac{1}{2}} + O(\log n), \tag{23}
\]

and

\[
\log M_{V}^\ast(n, \epsilon, \delta) \geq \sqrt{\frac{2}{\chi_2(Q_1||Q_0)}} Q^{-1} \left( \frac{1 - \delta}{2} \right) D_{P} n^{\frac{1}{2}} - \sqrt{\frac{2}{\chi_2(Q_1||Q_0)}} V_{\epsilon} Q^{-1}(\epsilon) n^{\frac{1}{2}} + O(\log n), \tag{24}
\]

where \( D_{P} \) and \( V_{\epsilon} \) are as in (21) and \( U_{\epsilon} \triangleq V_{\epsilon} + D_{P}^2 \). This message length is obtained with a number of key bits

\[
\log K = (1 + \rho) \frac{2}{\sqrt{\chi_2(Q_1||Q_0)}} Q^{-1} \left( \frac{1 - \delta}{2} \right) [D_{Q} - D_{P}] n^{\frac{1}{2}}, \tag{25}
\]

where \( \rho > 0 \) can be arbitrarily small.

**Theorem 3.** For \( \epsilon \in [0, \frac{1}{2}] \), \( \delta \in [0, 1 - \alpha] \), and \( \alpha \in [0, 1] \), we have

\[
\log M_{\ast}^\ast(n, \epsilon, \delta, \alpha) \leq \sqrt{\frac{2}{\chi_2(Q_1||Q_0)}} Q^{-1}(1 - \alpha - \delta) + Q^{-1}(\alpha) D_{P} n^{\frac{1}{2}} - \sqrt{\frac{2}{\chi_2(Q_1||Q_0)}} V_{\epsilon} Q^{-1}(\epsilon) n^{\frac{1}{2}} + O(\log n), \tag{26}
\]

and

\[
\log M_{\ast}^\ast(n, \epsilon, \delta, \alpha) \geq \sqrt{\frac{2}{\chi_2(Q_1||Q_0)}} Q^{-1}(1 - \alpha - \delta) + Q^{-1}(\alpha) D_{P} n^{\frac{1}{2}} - \sqrt{\frac{2}{\chi_2(Q_1||Q_0)}} V_{\epsilon} Q^{-1}(\epsilon) n^{\frac{1}{2}} + O(\log n). \tag{27}
\]

This message length is obtained with a number of key bits

\[
\log K = (1 + \rho) \frac{2}{\sqrt{\chi_2(Q_1||Q_0)}} Q^{-1}(1 - \alpha - \delta) + Q^{-1}(\alpha) [D_{Q} - D_{P}] n^{\frac{1}{2}}, \tag{28}
\]

\(^1\)See also the discussion in [5, Appendix A].
where $\rho > 0$ can be arbitrarily small.

We illustrate these different results with a simple numerical example. We consider the situation in which $(\mathcal{X}, W_{Y|X}, \mathcal{Y})$ and $(\mathcal{X}, W_{Z|X}, \mathcal{Z})$ are Binary Symmetric Channels (BSCs) with cross-over probability $p_m = 0.25$ and $p_w = 0.33$, respectively, and $\epsilon = 10^{-2}$, $\delta = 5 \cdot 10^{-2}$, and $\alpha = 0.2$. As shown in Fig. 2, the choice of the covertness metric results in different number of bits, which of course raises the question of which number to settle on. We argue that $M^*_{\beta}(n, \epsilon, \delta)$ is the number to focus on since total variation distance satisfies two desirable properties: it is directly connected to the operation of the adversary, through the inequality $\alpha + \beta \geq 1 - \mathbb{V}(\hat{P}_2, Q_0^{\alpha})$, and it does not presume any knowledge about the exact operating point on the adversary’s ROC curve.

**Remark 1.** It is not straightforward to find the optimal throughput for covert communication, as defined in [23], [5], from our results. The reason is that our second-order asymptotics include an $O(\log n)$ term that depends on $\epsilon$ and $\delta$, and for which we do not characterize the behavior as $\epsilon$ and $\delta$ tend to zero. However, for the covert metrics we consider, one can check that second and third order terms are negligible, and the optimal throughput is the pre-factor in the first-order term.

Fig. 2. Maximum number of covert and reliable bits as a function of blocklength. Both DMCs are BSCs with cross-over probability $p_m = 0.25$ and $p_w = 0.33$, respectively, and $\epsilon = 10^{-2}$, $\delta = 5 \cdot 10^{-2}$, and $\alpha = 0.2$.

Before we detail the achievability and converse proofs in the next sections, we provide here a high-level sketch of the proofs. Following [5], the coding scheme in the achievability proof consists of $MK$ randomly generated codewords $x_{sw}$ with $s \in [1, K]$ and $w \in [1, M]$. The code is designed such that the following properties hold:

(P1) the codebook $\{x_{sw}\}_{(s,w)\in[1,K]\times[1,M]}$ forms a resolvability code for the channel $(\mathcal{X}, W_{Z|X}, \mathcal{Z})$ approximating $Q_0^{\alpha n}$;

(P2) for every $s \in [1, K]$, the sub-codebook $\{x_{sw}\}_{w\in[1,M]}$ forms a reliability code for the channel $(\mathcal{X}, W_{Y|X}, \mathcal{Y})$ with an average probability of error $\epsilon$.

The analysis does not follow from standard arguments for two reasons. First, if the codewords are generated according to an i.i.d. distribution, we do not obtain the optimal dispersion; instead, we resort to PPM codes [4]. Second, one cannot ensure (P2) by merely expurgating some of the sub-codebooks, since this expurgation may change the output distribution on $\mathcal{Z}^n$ induced by the code; we address this by carefully analyzing the probability of error and using concentration of measure results such as McDiarmid's Inequality. The converse proof adapts arguments from [17] to the case of covert communication.

III. COVERT COMMUNICATIONS WITH GENERIC COVERTNESS QUASI-METRIC

In this section, we develop results for a generic covertness metric, which will be specialized to the three metrics highlighted in Definition 1 in Section IV-A, Section IV-B and Section IV-C. This organization allows us to handle separately the part of the analysis that depends solely on the code structure and not on the exact metric choice. Specifically, throughout this section, we consider an arbitrary quasi-metric $d$ measuring the distance between distributions. The quasi-metric $d$ does not necessarily possess the standard characteristics of a metric; however, we assume that it is non-negative and satisfies a modified triangle inequality, i.e.,

- $\forall P, Q : d(P, Q) \geq 0$
\[ \forall P, Q, R : \quad d(R, Q) \leq d(P, Q) + D(R\|P) + \sqrt{D(R\|P)} \max \left( 1, \log \frac{1}{\min \max Q(x)} \right). \]

This modified version of the triangle inequality is intimately linked to the relative entropy, and is used mainly for convenience; this will avoid repetitions when we study different metrics and should not be given much operational significance. In brief, we will use the modified triangle inequality as follows: we will pick an appropriate distribution \( P \) such that \( d(R, Q) \) is essentially on the order of \( d(P, Q) \), which depends on \( d \), while the remaining terms that depend on relative entropies will be made negligible.

We define an \((M, K, n, \epsilon, \delta)\) covert code and \( M_d^\gamma(n, \epsilon, \delta) \) for a covert communication channel \((\mathcal{X}, W_{Y|X}, W_{Z|X}, \mathcal{Y}, \mathcal{Z})\) as in Definition [1].

### A. One-shot achievability analysis

We now develop one-shot random coding results for codes simultaneously ensuring channel reliability and channel resolvability. Given a DMC \((\mathcal{X}, W_{Y|X}, \mathcal{Y})\) and a codebook with \( M \) codewords \( x_1, \ldots, x_M \in \mathcal{X} \), our analysis of reliability is based on a threshold decoder [17] operating as follows. Given \( \gamma > 0 \) and a distribution \( Q_Y \) on \( \mathcal{Y} \), and upon observing \( y \), the decoder forms the estimate \( \hat{W} = w \) of the transmitted message \( W \) if there exists a unique \( w \in \{1, M\} \) such that

\[
\log \frac{W_{Y|X}(y|x_w)}{Q_Y(y)} > \gamma.
\]

If there is no \( w \) satisfying (29), the decoder declares an error. It is known that the conditional probability of error when \( W = w \) is upper bounded by \( \epsilon_w^{(1)} + \epsilon_w^{(2)} \) where

\[
\epsilon_w^{(1)} \doteq \sum_y W_{Y|X}(y|x_w) \mathbb{I} \left\{ \log \frac{W_{Y|X}(y|x_w)}{Q_Y(y)} \leq \gamma \right\},
\]

\[
\epsilon_w^{(2)} \doteq \sum_y W_{Y|X}(y|x_w) \mathbb{I} \left\{ \exists w' \neq w : \log \frac{W_{Y|X}(y|x_{w'})}{Q_Y(y)} > \gamma \right\}.
\]

Under random coding, \( \epsilon_w^{(1)}, \ldots, \epsilon_w^{(1)} \) are independent, and we can therefore bound their average using well-known concentration inequalities. The following lemma upper bounds the expectation of \( \epsilon_w^{(1)} \) and \( \epsilon_w^{(2)} \).

**Lemma 1.** Let \((\mathcal{X}, W_{Y|X}, \mathcal{Y})\) be a DMC and \( P_X \) and \( Q_Y \) be two distributions on \( \mathcal{X} \) and \( \mathcal{Y} \). If we choose \( X_1, \ldots, X_M \) independently according to \( P_X \), for all \( \gamma \in \mathbb{R} \) and \( w \in \{1, M\} \), we have

\[
\mathbb{E} \left( \epsilon_w^{(1)} \right) = F_{XY|Q_Y}(\gamma),
\]

\[
\mathbb{E} \left( \epsilon_w^{(2)} \right) \leq \frac{M}{\exp(\gamma)} \mathbb{E}_{P_Y} \left( \frac{P_Y(Y)}{Q_Y(Y)} \right).
\]

**Proof:** The proof is very similar to the proof of [5, Lemma 3] or [17, Theorem 18], but for completeness we provide the proof. We know that

\[
\mathbb{E} \left( \epsilon_w^{(1)} \right) = \sum_{x_1, \ldots, x_M} \prod_{k=1}^M P_X(x_k) \sum_y W_{Y|X}(y|x_w) \mathbb{I} \left\{ \log \frac{W_{Y|X}(y|x_w)}{Q_Y(y)} \leq \gamma \right\}
\]

\[
= \sum_{x_1, \ldots, x_M} P_X(x_w) W_{Y|X}(y|x_w) \mathbb{I} \left\{ \log \frac{W_{Y|X}(y|x_w)}{Q_Y(y)} \leq \gamma \right\}
\]

\[
= \mathbb{P}_{W_{Y|X}} \left( \log \frac{W_{Y|X}(Y|X)}{Q_Y(Y)} \leq \gamma \right)
\]

\[
= F_{XY|Q_Y}(\gamma).
\]
Moreover, we have
\[ \mathbb{E} \left( e_w^{(2)} \right) \leq \sum_{x_1, \ldots, x_M} \prod_{k=1}^{M} P_X(x_k) \sum_y W_{Y|X}(y|x_w) \mathbb{1} \left\{ \exists w' \neq w : \log \frac{W_{Y|X}(y|x_w)}{Q_Y(y)} \geq \gamma \right\} \]
\[ \leq \sum_{x_1, \ldots, x_M} \prod_{k=1}^{M} P_X(x_k) \sum_y W_{Y|X}(y|x_w) \sum_{w' \neq w} \mathbb{1} \left\{ \log \frac{W_{Y|X}(y|x_{w'})}{Q_Y(y)} \geq \gamma \right\} \]
\[ = \sum_{w' \neq w} \sum_{x_1, \ldots, x_M} W_{Y|X}(y|x_w) \prod_{k=1}^{M} P_X(x_k) \mathbb{1} \left\{ \log \frac{W_{Y|X}(y|x_w)}{Q_Y(y)} \geq \gamma \right\} \]
\[ = \sum_{w' \neq w} \sum_{x_1, \ldots, x_M} Q_Y(y) \frac{P_Y(y)}{Q_Y(y)} P_X(x_{w'}) \mathbb{1} \left\{ \log \frac{W_{Y|X}(y|x_{w'})}{Q_Y(y)} \geq \gamma \right\} \]
\[ \leq \sum_{w' \neq w} \sum_{x_1, \ldots, x_M} \exp(-\gamma) W(y|x_{w'}) \frac{P_Y(y)}{Q_Y(y)} P_X(x_{w'}) \mathbb{1} \left\{ \log \frac{W_{Y|X}(y|x_{w'})}{P_Y(y)} \geq \gamma \right\} \]
\[ \leq \frac{M}{\exp(\gamma)} \mathbb{E} \left( P_Y(Y) \right) \quad \exp(-2M\lambda^2). \]

**Remark 2.** In a second-order analysis, \( \gamma \) is generally chosen such that \( \mathbb{E} \left( e_w^{(1)} \right) \) dominates \( \mathbb{E} \left( e_w^{(2)} \right) \); we follow this approach as well, which is convenient since we have some flexibility in the bounding of the expected value of \( e_w^{(1)} \).

Next, we develop one-shot channel resolvability results. In our proofs, we treat the distance between induced output distribution and the desired distribution as a function of independently generated codewords, which allows us to prove a super-exponential concentration inequality in Lemma 2 using Mc Diarmid’s Theorem. This may be viewed as an alternative approach to [9] and we recall this concentration inequality below for convenience.

**Theorem 4** (McDiarmid’s Theorem). Let \( X \triangleq (X_1, \ldots, X_n) \) be a sequence of independent random variables defined on \( \mathcal{X} \). Furthermore, suppose \( g : \mathcal{X}^n \rightarrow \mathbb{R} \) is a function satisfying
\[ \sup_{x_1, \ldots, x_i, x_i'} \left| g(x_1, \ldots, x_i, \ldots, x_n) - g(x_1, \ldots, x_i', \ldots, x_n) \right| \leq c_i, \quad \forall i \in [1, n]. \]
Then, for all \( \lambda > 0 \), we have
\[ \mathbb{P}(g(X) - \mathbb{E}(g(X)) \geq \lambda) \leq \exp \left( \frac{-2\lambda^2}{\sum c_i^2} \right). \]

**Proof:** See [18] Theorem 2.2.2. 

**Lemma 2.** Consider a DMC \( (\mathcal{X}, W_{Z|X}, \mathbb{P}) \) and a distribution \( P_X \) on \( \mathcal{X} \). If \( \{x_w\}_{w=1}^{M} \in \mathcal{X}^M \) are \( M \geq 2 \) codewords and \( \hat{P}_Z \) is the corresponding induced distribution on \( \mathcal{Z} \), we define the functions \( g_1, g_2 : \mathcal{X}^M \rightarrow \mathbb{R} \) as
\[ g_1(x_1, \ldots, x_M) \triangleq \mathbb{V} \left( \hat{P}_Z, P_Z \right), \]
\[ g_2(x_1, \ldots, x_M) \triangleq \mathbb{D} \left( \hat{P}_Z \| P_Z \right), \]
with \( P_Z(z) = \sum_x P_X(x)W_{Z|X}(z|x) \). Then, for all \( x_1, \ldots, x_M, x_i' \in \mathcal{X} \), we have
\[ \left| g_1(x_1, \ldots, x_i, \ldots, x_M) - g_1(x_1, \ldots, x_i', \ldots, x_M) \right| \leq \frac{1}{M}, \]
and
\[ \left| g_2(x_1, \ldots, x_i, \ldots, x_M) - g_2(x_1, \ldots, x_i', \ldots, x_M) \right| \leq \frac{1}{M} \log \left( \frac{M|\mathcal{Z}|}{\mu_Z^2} \right). \]
Moreover, if \( X_1, \ldots, X_M \) are i.i.d. with distribution \( P_X \), then
\[ \mathbb{P} \left( \mathbb{V} \left( \hat{P}_Z, P_Z \right) - \mathbb{E} \left( \mathbb{V} \left( \hat{P}_Z, P_Z \right) \right) \geq \lambda \right) \leq \exp \left( -2M\lambda^2 \right), \]
and
\[
P\left(D\left(\tilde{P}_Z\|P_Z\right) - E\left(D\left(\tilde{P}_Z\|P_Z\right)\right) \geq \lambda\right) \leq \exp\left(-\frac{2M\lambda^2}{\log^2\left(M|Z|/\mu_Z^2\right)}\right). \tag{52}
\]

Proof: We assume that \(\tilde{P}_1^1\) and \(\tilde{P}_2^2\) are the distributions induced by the codebooks \(C_1 \triangleq \{x_1, \ldots, x_i, \ldots, x_M\}\) and \(C_2 \triangleq \{x_1, \ldots, x'_i, \ldots, x_M\}\), respectively. First, note that
\[
g_1(x_1, \ldots, x_i, \ldots, x_M) - g_1(x_1, \ldots, x'_i, \ldots, x_M) = \mathbb{V}\left(\tilde{P}_1^1, P_Z\right) - \mathbb{V}\left(\tilde{P}_2^2, P_Z\right) \leq \mathbb{V}\left(\tilde{P}_1^1, \tilde{P}_2^2\right) \leq \frac{1}{2} \sum_z \left|\tilde{P}_1^1(z) - \tilde{P}_2^2(z)\right| \leq \frac{1}{2} \sum_z \left|\sum_{x \in C_1} W_{Z|X}(z|x) - \sum_{x \in C_2} W_{Z|X}(z|x)\right| = \frac{1}{2} \sum_{x \in C_1} W_{Z|X}(z|x) \leq \frac{1}{2M} \sum_z \left|W_{Z|X}(z|x) - W_{Z|X}(z|x_i)\right| \leq \frac{1}{M} \mathbb{V}\left(W_{Z|X}(.|x_i), W_{Z|X}(.|x'_i)\right) \leq \frac{1}{M}. \tag{59}
\]

Next, for any two distributions \(P\) and \(Q\), by \([8\text{ Lemma 2.7}]\), we have
\[
|H(P) - H(Q)| \leq \mathbb{V}(P, Q) \log \frac{\mathbb{V}(P, Q)}{|X|}. \tag{60}
\]
Therefore, we have
\[
|H(\tilde{P}_1^1) - H(\tilde{P}_2^2)| \leq \mathbb{V}\left(\tilde{P}_1^1, \tilde{P}_2^2\right) \log \frac{\mathbb{V}(\tilde{P}_1^1, \tilde{P}_2^2)}{|Z|} \leq \frac{1}{M} \log \frac{1}{|M|Z|} \equiv (a) \leq \frac{1}{M} \log (M|Z|), \tag{62}
\]
where \((a)\) follows from the fact that \(x \log(x/|Z|)\) is decreasing for \(0 < x < |Z|/2\). Accordingly, we obtain
\[
\left|D\left(\tilde{P}_1^1\|P_Z\right) - D\left(\tilde{P}_2^2\|P_Z\right)\right| = \left|\mathbb{H}(\tilde{P}_1^1) + \sum_z \tilde{P}_1^1(z) \log \frac{1}{\tilde{P}_1^1(z)} + \mathbb{H}(\tilde{P}_2^2) - \sum_z \tilde{P}_2^2(z) \log \frac{1}{\tilde{P}_2^2(z)}\right| \leq \frac{1}{M} \log (M|Z|) + \sum_z \left|\tilde{P}_1^1(z) - \tilde{P}_2^2(z)\right| \log \frac{1}{\tilde{P}_2^2(z)} \leq \frac{1}{M} \log (M|Z|) + \frac{2}{M} \log \frac{1}{\mu_Z} \leq \frac{1}{M} \log \left(\frac{M|Z|}{\mu_Z^2}\right). \tag{67}
\]

Additionally, using Theorem \([4]\) we obtain \((51)\) and \((52)\).

**Lemma 3.** Let \((X, W_{Z|X}, Z)\) be a DMC and \(P_X\) be a distribution on \(X\). If \(X_1, \ldots, X_M\) are i.i.d. with distribution \(P_X\), and \(\tilde{P}_Z(z)\) is the corresponding induced distribution on \(Z\), we have
\[
\mathbb{E}\left(D\left(\tilde{P}_Z\|P_Z\right)\right) \leq \log \left(1 - \frac{1}{\mu_Z} + 1\right) F_{XZ}(\gamma) + \frac{\exp(\gamma)}{M}, \tag{68}
\]
for all \(\gamma \in \mathbb{R}\) where \(P_Z(z) = \sum_x P_X(x)W_{Z|X}(z|x)\).

Proof: From \([13\text{ Equation 10}]\), we know that
\[
\mathbb{E}\left(D\left(\tilde{P}_Z\|P_Z\right)\right) \leq \mathbb{E}_{W_{Z|X}P_X}\left(\log \left(\frac{W_{Z|X}(Z|X)}{M \bar{P}_Z(Z)} + 1\right)\right). \tag{69}
\]
If we define the event $\mathcal{E} \triangleq \{(x, z) : \log \frac{W_{X|Z}(z|x)}{MP_Z(z)} \leq \gamma\}$, then we get

$$E\left(\log \left(\frac{W_{Z|X}(Z|X)}{MP_Z(Z)} + 1\right)\right) = E\left(\log \left(\frac{W_{Z|X}(Z|X)}{MP_Z(Z)} + 1\right) | \mathcal{E}\right)P(\mathcal{E}) + E\left(\log \left(\frac{W_{Z|X}(Z|X)}{MP_Z(Z)} + 1\right) | \mathcal{E}^c\right)P(\mathcal{E}^c)$$

(70)

$$\leq \log \left(\frac{\exp(\gamma)}{M} + 1\right) + F_{XZ}(\gamma) \log \left(\frac{1}{\mu_Z} + 1\right)$$

(71)

$$\leq \frac{\exp(\gamma)}{M} + F_{XZ}(\gamma) \log \left(\frac{1}{\mu_Z} + 1\right).$$

(72)

Consider now the DMC $(\mathcal{X}, \mathcal{Y}, \mathcal{W}, \mathcal{Z})$ and a randomly generated code $\{X_{sw} : s \in [1, K], w \in [1, M]\}$. The next lemma leverages Lemma 1 and Lemma 2 to guarantee that, with high probability, the entire code is a channel resolvability code for the channel $W_{Z|X}$, and simultaneously, the subcode corresponding to every $s \in [1, K]$ is a channel reliability code for the channel $W_{Y|X}$. Note that a code with these properties is not necessarily a covert code, but we show in the next sections how a careful choice of the random coding distribution leads to covertness.

**Lemma 4.** Consider a DMC $(\mathcal{X}, \mathcal{Y}, \mathcal{W}, \mathcal{Z})$ and two probability distributions $P_X$ and $Q_Y$ on $\mathcal{X}$ and $\mathcal{Y}$. We sample codewords $\{X_{sw} : s \in [1, K], w \in [1, M]\}$ independently according to $P_X$. Moreover, for every $\lambda_1$, $\lambda_2$, $\gamma_1$, and $\gamma_2 > 0$, we define events

$$\mathcal{E}_1 \triangleq \left\{ P_{err}^* \leq F_{XY|Q_Y}(\gamma_1) + \lambda_1 + \frac{M}{\exp(\gamma_1)}E_{P_Y}(P_Y(Y) | Q_Y(Y)) \right\}$$

(73)

and

$$\mathcal{E}_2 \triangleq \left\{ \mathbb{D}(\hat{P}_Z \| P_Z) \leq \log \left(\frac{1}{\mu_Z} + 1\right) F_{XZ}(\gamma_2) + \frac{\exp(\gamma_2)}{MK} + \lambda_3 \right\}.$$  

(74)

Then $P(\mathcal{E}_1 \cap \mathcal{E}_2) > 0$, if

$$1 > \exp \left( -\frac{2M\lambda_3^2}{\log^2 \left(\frac{MKZ}{\mu_Z^2}\right)} \right) - \exp \left( -2M\lambda_1^2 \right) - \frac{1}{\lambda_2}.$$  

(75)

**Proof:** We analyze the probability of each event separately. First, note that

$$P(\mathcal{E}_1) = P\left( P_{err}^* \leq F_{XY|Q_Y}(\gamma_1) + \lambda_1 + \frac{M}{\exp(\gamma_1)}E_{P_Y}(P_Y(Y) | Q_Y(Y)) \right)$$

(76)

$$\geq P\left( \max_{s \in [1, K]} \frac{1}{M} \sum_{w=1}^{M} \left( e_{(s)w}^{(1)} + e_{(s)w}^{(2)} \right) \leq F_{XY|Q_Y}(\gamma_1) + \lambda_1 + \frac{M}{\exp(\gamma_1)}E_{P_Y}(P_Y(Y) | Q_Y(Y)) \right)^K$$

(77)

$$= P\left( \frac{1}{M} \sum_{w=1}^{M} \left( e_{(s)w}^{(1)} + e_{(s)w}^{(2)} \right) \leq F_{XY|Q_Y}(\gamma_1) + \lambda_1 + \frac{M}{\exp(\gamma_1)}E_{P_Y}(P_Y(Y) | Q_Y(Y)) \right)^K$$

(78)

$$\geq \left( \frac{1}{M} \sum_{w=1}^{M} \epsilon_{1w} \leq F_{XY|Q_Y}(\gamma_1) + \lambda_1 + \frac{M}{\exp(\gamma_1)}E_{P_Y}(P_Y(Y) | Q_Y(Y)) \right)^K$$

(79)

$$\geq \left( 1 - \exp \left( -2M\lambda_2^2 \right) - \exp \left( \frac{1}{M} \sum_{w=1}^{M} \epsilon_{1w}^{(2)} > \frac{M}{\exp(\gamma_1)}E_{P_Y}(P_Y(Y) | Q_Y(Y)) \right) \right)^K$$

(80)

$$\geq \left( 1 - \exp \left( -2M\lambda_2^2 \right) - \frac{1}{\lambda_2} \right)^K,$$  

(81)

where (a) follows from Hoeffding’s Inequality and the fact that $\epsilon_{11}, \cdots, \epsilon_{1M}$ are independent random variables taking value in $[0, 1]$ with mean $F_{XY}(\gamma_1)$, and (b) follows from Markov’s Inequality. Furthermore, we have

$$P(\mathcal{E}_2) = P\left( \mathbb{D}(\hat{P}_Z \| P_Z) \leq \log \left(\frac{1}{\mu_Z} + 1\right) F_{XZ}(\gamma_2) + \frac{\exp(\gamma_2)}{MK} + \lambda_3 \right)$$

(82)

$$\geq P\left( \mathbb{D}(\hat{P}_Z \| P_Z) \leq \mathbb{E}\left( \mathbb{D}(\hat{P}_Z \| P_Z) \right) + \lambda_3 \right)$$

(83)

$$= 1 - \exp \left( \frac{2MK\lambda_3^2}{\log^2 \left(\frac{MKZ}{\mu_Z^2}\right)} \right),$$  

(84)
where (a) follows from Lemma 3 and (b) follows from Lemma 2. Combining these inequalities, we get

\[
\mathbb{P}(E_1) + \mathbb{P}(E_2) \geq \left(1 - \exp\left(-2M\lambda_1^2 - \frac{1}{\lambda_2}\right)\right)^K + 1 - \exp\left(-\frac{2MK\lambda_3^2}{\log^2\left(\frac{MK|Z|}{\mu_Z^2}\right)}\right).
\]  

(85)

Thus, we obtain

\[
\begin{align*}
\mathbb{P}(E_1 \cap E_2) &= 1 - \mathbb{P}(E_1' \cup E_2') \\
&\geq 1 - \mathbb{P}(E_1') - \mathbb{P}(E_2') \\
&= \mathbb{P}(E_1) + \mathbb{P}(E_2) - 1 \\
&\geq \left(1 - \exp\left(-2M\lambda_1^2 - \frac{1}{\lambda_2}\right)\right)^K - \exp\left(-\frac{2MK\lambda_3^2}{\log^2\left(\frac{MK|Z|}{\mu_Z^2}\right)}\right).
\end{align*}
\]

(86) 

(87) 

(88) 

(89)

where (a) follows from the (85). Finally, (75) completes the proof.

\[\square\]

B. Asymptotic results for covert communications

We now specialize the one-shot result of Lemma 4 to show the existence of covert codes for large blocklength \(n\).

**Theorem 5.** Consider a sequence of distributions \(\{P^n_X\}_{n \geq 1}\) where \(P^n_X\) is defined on \(X^n\) and two sequences of natural numbers \(\{M_n\}_{n \geq 1}\) and \(\{K_n\}_{n \geq 1}\). For a quasi-metric \(d\) and \(n\) large enough, if we have

\[
1 - \exp\left(-\frac{2M_n}{n^8 \log^2\left(\frac{M_nK_n|Z|}{\mu_Z^2}\right)}\right) - \exp\left(-\frac{2M_n}{n^2}\right) - \frac{1}{n} > 0,
\]

(90)

\[
F_{XY|P_0^n}(\log(n^2M_n)) + \frac{1 + \mathbb{E}_P\left(\frac{P(Y)}{P_0^n(Y)}\right)}{n} \leq \epsilon,
\]

(91)

\[
F_{XZ}\left(\log\frac{M_nK_n}{n^4}\right) \leq \frac{1}{n^6},
\]

(92)

and

\[
d(P_Z, Q_0^n) \leq \delta - \frac{1}{\sqrt{n}},
\]

(93)

then \(M^*(n, \epsilon, \delta) \geq M_n\).

Theorem 5 is useful in that it identifies sufficient conditions for the existence of codes. In particular, subsequent achievability proofs are reduced to choosing an appropriate sequence of random coding distributions \(\{P^n_X\}_{n \geq 1}\) and verifying inequalities (90)-(93). We use both PPM and i.i.d. distributions, which we introduce and analyze precisely in Section III-C.

**Proof:** For a fixed \(n\), we choose codewords \(X_{sw}\) for \(s \in [1, K_n]\) and \(w \in [1, M_n]\) independently according to \(P^n_X\). In Lemma 4, if we set

\[
\lambda_1 = \frac{1}{n}, \quad \lambda_2 = n, \quad \lambda_3 = \frac{1}{n^4}, \quad \gamma_1 = \log(n^2M_n), \quad \text{and} \quad \gamma_2 = \log\frac{M_nK_n}{n^4},
\]

(94)

because of (90), we have with positive probability for \(n\) large enough

\[
P^*_\text{err} \leq F_{XY|P_0^n}(\gamma_1) + \lambda_1 + \lambda_2 \frac{M_n}{\exp(\gamma_1)} \mathbb{E}_P\left(\frac{P(Y)}{P_0^n(Y)}\right)\]

(95)

\[
= F_{XY|P_0^n}(\log(n^2M_n)) + \frac{1 + \mathbb{E}_P\left(\frac{P(Y)}{P_0^n(Y)}\right)}{n} \leq \epsilon,
\]

(96)

and with \(\mu_Z \triangleq \min_Z P_Z(x)\)

\[
D\left(P_Z||P_X\right) \leq \log\left(1 + \frac{1}{\mu_Z}\right) F_{XZ}(\gamma_2) + \frac{\exp(\gamma_2)}{M_nK_n} + \lambda_3
\]

(97)

\[
\leq n \log\left(1 + \frac{1}{\mu_Z}\right) \frac{1}{n^6} + \frac{1}{n^4} + \frac{3}{n^4} \leq \frac{3}{n^4},
\]

(98)
where (a) follows from \((91)\) and (b) follows from \((92)\). Finally, to prove the covertness of the code, note that for \(n\) large enough

\[
d(\hat{P}_Z, Q_0^n) \overset{(a)}{\leq} d(P_n^\eta, Q_0^n) + \mathbb{D}(\hat{P}_Z \| P_Z^n) + \sqrt{\mathbb{D}(\hat{P}_Z \| P_Z^n) \max \left(1, \log \frac{1}{\min_z Q_0^n(z)} \right)}
\]

\[
\overset{(b)}{\leq} \delta - \frac{1}{\sqrt{n}} + \mathbb{D}(\hat{P}_Z \| P_Z^n) + \sqrt{\mathbb{D}(\hat{P}_Z \| P_Z^n) \max \left(1, \log \frac{1}{\min_z Q_0^n(z)} \right)}
\]

\[
\overset{(c)}{\leq} \delta - \frac{1}{\sqrt{n}} + \frac{3}{n^4} \max \left(1, \log \frac{1}{\min_z Q_0^n(z)} \right)
\]

\[
\leq \delta,
\]

where (a) follows because \(d\) is a quasi-metric, (b) follows from \((93)\), and (c) follows from \((98)\).

C. Covert distributions

In this section, we study specific “covert distributions,” which we subsequently use to generate random codes and combine with Theorem 5. We first define the PPM distribution which was used in [4] for covert communication.

**Definition 2** (i.i.d. covert distribution). For \(n \in \mathbb{N}\) and \(\eta > 0\), we define the distribution \(P_{X,\text{PPM}}^{n,\eta}\) as \(\Pi_{\eta}^n\) on \(\mathcal{X}^n = \{0,1\}^n\). The output distributions for DMCs \((\mathcal{X}, W_{Y|X}, \mathcal{Y})\) and \((\mathcal{X}, W_{Z|X}, \mathcal{Z})\) are denoted by \(P_{Y,\text{PPP}}^{n,\eta}\) and \(P_{Z,\text{PPP}}^{n,\eta}\), respectively.

**Definition 3** ((\(n, \ell\))-PPM covert distribution). Given \(\mathcal{X} = \{0,1\}\) and \(n \geq \ell \geq 1\), we will define distribution \(P_{X,\text{PPM}}^{n,\ell}\) on \(\mathcal{X}^n\). If \(n = n\ell + r\) for \(0 \leq r < \ell\), we partition the set \([1, n]\) into \(\ell\) sets of size \(m\) and one set of size \(r\). For simplicity, we consider the following partition: \(B_i = \{(i-1)m+1, \ldots, im\}\) for \(i \in [1, \ell]\) and \(B_{\ell+1} = \{(m+1, \ldots, n)\}\). Then, \(P_{X,\text{PPM}}^{n,\ell}\) is the uniform distribution on the sequences with exactly one “1” in all \(B_i\) for \(i \in [1, \ell]\) and no “1” in \(B_{\ell+1}\), i.e.,

\[
\left\{ x \in \mathcal{X}^n : \forall i \in [1, \ell] : \sum_{j \in B_i} x_j = 1 \quad \text{and} \quad \sum_{j \in B_{\ell+1}} x_j = 0 \right\}.
\]

We denote the output distribution of \(P_{X,\text{PPM}}^{n,\ell}\) for DMCs \((\mathcal{X}, W_{Y|X}, \mathcal{Y})\) and \((\mathcal{X}, W_{Z|X}, \mathcal{Z})\) by \(P_{Y,\text{PPP}}^{n,\ell}\) and \(P_{Z,\text{PPP}}^{n,\ell}\).

The i.i.d. covert distribution \(P_{X,\text{PPP}}^{n,\eta}\) is merely a Bernoulli product distribution, but we allow the parameter \(\eta\) to depend on \(n\); note that a randomly generated sequence according to a covert distribution contains \(\eta n\) 1-symbols, on average. The PPM covert distribution \(P_{X,\text{PPM}}^{n,\ell}\) may be viewed as a way to generate structured constant composition codewords; specifically, every codeword generated according to \(P_{X,\text{PPM}}^{n,\ell}\) contains exactly \(\ell\) 1-symbols, each located in a window of size \(m\), the quotient of the Euclidean division of \(n\) by \(\ell\). The structure of the PPM covert distribution plays a pivotal role in our analysis.

We devote the remainder of this section to the development of properties of covert distributions, which are geared towards the use of Theorem 5. We make repeated use of the Berry-Esseen Theorem, which we recall here for convenience.

**Theorem 6** (Berry-Esseen Theorem). Let \(X_1, \ldots, X_n\) be independent random variables s.t. for \(k \in [1, n]\) we have \(\mathbb{E}(X_k) = \mu_k, \sigma_k^2 = \text{Var}(X_k)\), and \(t_k = \mathbb{E}\left(|X_k - \mu_k|^3\right)\). If we define \(\sigma^2 = \sum_{i=1}^n \sigma_i^2\) and \(T = \sum_{k=1}^n t_i\), then we have

\[
\mathbb{P}\left(\sum_{k=1}^n (X_k - \mu_k) \geq \lambda \sigma\right) - Q(\lambda) \leq \frac{6T}{\sigma^3},
\]

**Proof:** See [17] Theorem 44.

**Lemma 5.** Let \((\mathcal{X}, W_{Y|X}, \mathcal{Y})\) be a binary-input DMC.

- If \((X, Y)\) is distributed according to \(P_{X,\text{PPM}}^{n,\ell} W_{Y|X}^{n,\ell}\), we have

\[
F_{X,\text{PPM}}(\gamma) \leq Q\left(\frac{\ell D_P - \gamma}{\sqrt{4\ell D_P}}\right) + \frac{6T_P}{\sqrt{4\ell D_P}}.
\]

with

\[
T_P := \mathbb{E}_{P_0}\left(\left|\log\left(\frac{P_1(Y)}{P_0(Y)}\right) - \mathbb{D}(P_1 \| P_0)\right|\right)^3.
\]
Lemma 6. If \((X, Y)\) is distributed according to \(P_{X\text{,IID}}^{n}\|W_{Y|X}^{\otimes n}\), we have

\[
F_{XY|^{\gamma}_{P_0^n}}(\gamma) \leq Q\left(\frac{n\eta D_P - \gamma}{\sqrt{n}(\eta U_P - \eta^2 D_P^2)}\right) + C\sqrt{\frac{1}{n\eta}},
\]

with \(C\) a constant that only depends on the channel.

**Proof:** When \((X, Y)\) is distributed according to \(P_{X\text{,PPM}}^{n,\ell}\|W_{Y|X}^{\otimes n}\), we know that

\[
F_{XY|^{\gamma}_{P_0^n}}(\gamma) = \sum_{x \in \mathcal{X}} P_{X\text{,PPM}}^{n,\ell}(x) F_{X|Y|^\gamma_{P_0^n}}(\gamma|X = x) \leq \sum_{x \in \mathcal{X}} P_{X\text{,PPM}}^{n,\ell}(x) \left(\log \frac{W_{Y|X}^{\otimes n}(Y|X)}{P_0^{\otimes n}(Y)} \leq \gamma|X = x\right)
\]

\[
= \sum_{x \in \mathcal{X}} P_{X\text{,PPM}}^{n,\ell}(x) \left(\log \frac{W_{Y|X}^{\otimes n}(Y|x_0)}{P_0^{\otimes n}(Y)} \leq \gamma|X = x_0\right)
\]

where \(x_0\) is an arbitrary element with \(P_{X\text{,PPM}}^{n,\ell}(x_0) > 0\). Since channel is memoryless, by relabeling, we can assume without loss of generality that the support of \(x_0\) is \([1, \ell]\). Hence, we have

\[
P_{X\text{,PPM}}^{n,\ell}(x) \leq \sum_{i=1}^{\ell} P_{x_i}(Y_i) \leq \gamma|X = x_0\)

\[
\leq \sum_{i=1}^{\ell} \left(\log \frac{P_1(Y_i)}{P_0(Y_i)}\right) \leq \gamma
\]

\[
\leq \left(\frac{\ell D_P - \gamma}{\sqrt{\ell V_P}}\right) + \frac{6T_P}{\sqrt{\ell V_P}},
\]

where \((a)\) follows from Theorem 6.

When \((X, Y)\) is distributed according to \(P_{X\text{,IID}}^{n,\eta}\|W_{Y|X}^{\otimes n}\), first notice that for all \(i \in [1, n]\), we have

\[
\mathbb{E}\left(\log \frac{W_{Y|X}^{\otimes n}(Y_i|X_i)}{P_0(Y_i)}\right) = \eta D_P,
\]

\[
\operatorname{Var}\left(\log \frac{W_{Y|X}^{\otimes n}(Y_i|X_i)}{P_0(Y_i)}\right) = \eta U_P - \eta^2 D_P^2,
\]

and

\[
\mathbb{E}\left(\left|\log \frac{W_{Y|X}^{\otimes n}(Y_i|X_i)}{P_0(Y_i)} - \eta D_P\right|^3\right) \leq \mathbb{E}\left(\left|\log \frac{W_{Y|X}^{\otimes n}(Y_i|X_i)}{P_0(Y_i)} \right| + |\eta D_P|\right)^3
\]

\[
= \eta^3 D_P^3 + 3\eta^3 D_P^2 \mathbb{E}_{P_1}\left(\log \frac{P_1(Y)}{P_0(Y)}\right) + 3\eta^2 D_P U_P + \eta \mathbb{E}_{P_1}\left(\log \frac{P_1(Y)}{P_0(Y)}\right)^3
\]

\[
\leq \eta C_1.
\]

Therefore, we have

\[
F_{XY|^{\gamma}_{P_0^n}}(\gamma) = \mathbb{P}\left(\log \frac{W_{Y|X}^{\otimes n}(Y|X)}{P_0^{\otimes n}(Y)} \leq \gamma\right)
\]

\[
= \mathbb{P}\left(\sum_{i=1}^{n} \log \frac{W_{Y|X}^{\otimes n}(Y_i|X_i)}{P_0(Y_i)} \leq \gamma\right)
\]

\[
\leq \left(\frac{n\eta D_P - \gamma}{\sqrt{n}(\eta U_P - \eta^2 D_P^2)}\right) + \frac{C_2}{\sqrt{n\eta}},
\]

where \((a)\) follows from Theorem 6.

**Lemma 6.** For a binary-input DMC \((\mathcal{X}, W_{Y|X}, Y)\), we have

\[
\mathbb{E}_{P_{Y|X}^{n,\ell}}\left(\frac{P_{Y|X}^{n,\ell}(Y)}{P_0^{n}(Y)}\right) = \left(1 + \frac{\chi_2(P_1||P_0)}{[n/\ell]}\right)^\ell
\]

\[
\leq \exp \left(\frac{\ell (\ell + 1)}{n} \chi_2(P_1||P_0)\right),
\]
Therefore, we have

\[ m \leq \exp \left( m n^2 \chi_2 (P_1 \| P_0) \right). \] (124)

**Proof:** To prove (122), we define \( m \triangleq \lfloor n / \ell \rfloor \). We first consider \( P_{X, \text{PPM}}^{n, \ell} \) for which we have

\[
\mathbb{E}_{P_{X, \text{PPM}}^{n, \ell}} \left( \frac{P_{Y, \text{PPM}}^{n, \ell}(Y)}{P_0^{n}(Y)} \right) = \sum_{y} P_{Y, \text{PPM}}^{n, \ell}(y) \frac{P_{Y, \text{PPM}}^{n, \ell}(y)}{P_0^{n}(y)}
\]

(125)

\[
= \sum_{y} P_0^{n}(y) \left( \frac{P_{Y, \text{PPM}}^{n, \ell}(y)}{P_0^{n}(y)} \right)^2
\]

(126)

\[
= \sum_{y} P_0^{n}(y) \left( \frac{1}{m} \sum_{i=1}^{m} P_1(y_i) \right)^2
\]

(127)

\[
= \frac{1}{m^2} (m(m-1) + m(\chi_2(P_1 \| P_0) + 1))
\]

(128)

\[
= 1 + \frac{\chi_2(P_1 \| P_0)}{m}.
\]

(129)

Since \( P_{Y, \text{PPM}}^{n, \ell} \) is the product of the distributions over different blocks, i.e. \( P_{Y, \text{PPM}}^{n, \ell} = \left( P_{Y, \text{PPM}}^{n, 1} \right)^{\ell} \otimes Q_0^{n \mod \ell} \), upon defining \( Y_i = (Y_{(i-1)m+1}, \ldots, Y_{im}) \) for \( i \in [1, \ell] \), we obtain

\[
\mathbb{E}_{P_{Y, \text{PPM}}^{n, \ell}} \left( \frac{P_{Y, \text{PPM}}^{n, \ell}(Y)}{P_0^{n}(Y)} \right) = \mathbb{E}_{P_{Y, \text{PPM}}^{n, 1}} \left( \prod_{i=1}^{\ell} \frac{P_{Y, \text{PPM}}^{n, 1}(Y_i)}{P_0^{n}(Y_i)} \right)
\]

(130)

\[
= \prod_{i=1}^{\ell} \mathbb{E}_{P_{Y, \text{PPM}}^{n, 1}} \left( \frac{P_{Y, \text{PPM}}^{n, 1}(Y_i)}{P_0^{n}(Y_i)} \right)
\]

(131)

\[
= \left( 1 + \frac{\chi_2(P_1 \| P_0)}{m} \right)^{\ell}
\]

(132)

\[
\leq \exp \left( \frac{(\ell + 1) \chi_2(P_1 \| P_0)}{n m} \right).
\]

(133)

Moreover, (124) follows from [5, Equation (147)].

**Lemma 7.** Given a binary-input DMC \((X, W_{Z|X}, Z)\),

- if \((X, Z)\) is distributed according to \( P_{X, \text{PPM}}^{n, \ell} W_{Z|X}^{n} \), then, for \( \gamma \geq D_Q \), we have
  \[
  \mathcal{F}_{XZ}(\gamma) \leq \exp \left( -\frac{(\gamma - \ell D_Q)^2}{2\ell \log^2 \mu_Z} \right); \] (134)

- if \((X, Z)\) is distributed according to \( P_{X, \text{IID}}^{n, \eta} W_{Z|X}^{n} \), then, for \( \gamma \geq n \eta D_Q \), we have
  \[
  \mathcal{F}_{XZ}(\gamma) \leq \exp \left( -\frac{\frac{1}{2}(\gamma - n \eta D_Q)^2}{n C_1 + C_2 \gamma} \right), \] (135)

for two positive constants \( C_1 \) and \( C_2 \) that depend on the channel.

**Proof:** When \((X, Z)\) is distributed according to \( P_{X, \text{PPM}}^{n, \ell} W_{Z|X}^{n} \), for the random vector \( Z \in \mathcal{Z}^n \), the first \( \ell \) blocks of length \( m \triangleq \lfloor n / \ell \rfloor \) are denoted by \( Z_1, \ldots, Z_\ell \) with \( Z_i = (Z_{(i-1)m+1}, \ldots, Z_{im}) \). Moreover, \( P_{Z_i} \) denotes the distribution of block \( Z_i \). Therefore, we have

\[
\mathcal{F}_{XZ}(\gamma) = P_{P_{X, \text{PPM}}^{n, \ell} W_{Z|X}^{n}} \left( \log \frac{W_{Z|X}^{n}(Z|X)}{P_{Z, \text{PPM}}^{n}(Z)} \geq \gamma \right)
\]

(136)

\[
= P_{P_{X, \text{PPM}}^{n, \ell} W_{Z|X}^{n}} \left( \sum_{i=1}^{\ell} \log \frac{W_{Z|X}^{n}(Z_i|X_i)}{P_{Z_i}(Z_i)} \geq \gamma \right)
\]

(137)

\[
\leq \exp \left( -\frac{2(\gamma - \ell I(P_{X, \text{PPM}}^{n, 1} W_{Z|X}^{n})^2}{\ell B^2} \right),
\] (138)
where \( (a) \) follows from Hoeffding’s Inequality with
\[
B \triangleq \sup_{x, z} \left| \log \frac{W_{Z|X}^m(z|x_i)}{P_{Z_i}(z_i)} \right| > 0
\]  
(139)
and \( I(P_{X,\text{PPM}}^m, W_{Z|X}^m) \) is defined as in [5]. By [4, Lemma 1.5], we know that \( I(P_{X,\text{PPM}}^m, W_{Z|X}^m) \leq D_Q \) and \( B \leq 2 \log \frac{1}{\mu_Z} \). Hence, we obtain
\[
\exp \left( - \frac{2 (\gamma - \ell I(P_{X,\text{PPM}}^m, W_{Z|X}^m))^2}{\ell B^2} \right) \leq \exp \left( - \frac{(\gamma - \ell D_Q)^2}{2 \ell \log^2 \mu_Z} \right).
\]  
(140)
Moreover, when \((X, Z)\) is distributed according to \( P_{X,\text{iid}}^n W_{Z|X}^m \), we define \( \mu \triangleq \mathbb{E} \left( \log \frac{W_{Z|X}(Z_1|X_1)}{Q_n(Z_1)} \right) \), \( \sigma^2 \triangleq \text{Var} \left( \log \frac{W_{Z|X}(Z_1|X_1)}{Q_n(Z_1)} \right) \), and \( L = \max_{x, z} \log \frac{W_{Z|X}(z|x)}{Q_n(z)} \). Therefore, by Bernstein’s Inequality, we obtain
\[
F_{XZ} = \mathbb{P} \left( \log \frac{W_{Z|X}(Z|X)}{P_{Z,\text{iid}}^n} > \gamma \right) 
\leq \exp \left( - \frac{1}{2} (\gamma - \nu \mu)^2 \right) \frac{n \sigma^2 + L (\gamma - \nu \mu)}{n \sigma^2 + L \gamma} \]  
(141)
(142)
(143)
Notice that
\[
\mu = \mathbb{E} \left( \log \frac{W_{Z|X}(Z_1|X_1)}{Q_n(Z_1)} \right) = I(X_1; Z_1)
\]  
(144)
\[
\overset{(a)}{=} \eta D_Q - \mathcal{D}(Q_1||Q_0)
\]  
(145)
\[
< \eta D_Q,
\]  
(146)
where \( (a) \) follows from [5, Equation (12)] and
\[
\sigma^2 = \text{Var} \left( \log \frac{W_{Z|X}(Z_1|X_1)}{Q_n(Z_1)} \right)
\]  
(147)
\[
\leq \mathbb{E} \left( \log^2 \frac{W_{Z|X}(Z_1|X_1)}{Q_n(Z_1)} \right)
\]  
(148)
\[
= \eta \mathbb{E}_{Q_1} \log^2 \frac{Q_1(Z)}{Q_0(Z)} + (1 - \eta) \mathbb{E}_{Q_0} \log^2 \frac{Q_0(Z)}{Q_0(Z)}
\]  
(149)
\[
\leq \eta \mathbb{E}_{Q_1} \log^2 \frac{Q_1(Z)}{Q_0(Z)} + \mathbb{E}_{Q_0} \log^2 \frac{Q_0(Z)}{Q_0(Z)} + \mathbb{E}_{Q_1} \log^2 \frac{Q_1(Z) - Q_0(Z)}{Q_0(Z)}
\]  
(150)
\[
= \eta U_P + \mathbb{E}_{Q_0} \log^2 \left( 1 + \eta \frac{Q_1(Z) - Q_0(Z)}{Q_0(Z)} \right)
\]  
(151)
\[
\overset{(a)}{=} \eta^2 \left( \frac{Q_1(Z) - Q_0(Z)}{Q_0(Z)} \right)^2 \]  
(152)
Using inequality \( \log^2 (1 + x) \leq x^2 (1 + 1/(1 + x)^2) \) for \( x > -1 \), we get
\[
\log^2 \left( 1 + \eta \frac{Q_1(Z) - Q_0(Z)}{Q_0(Z)} \right) \leq \eta^2 \left( \frac{Q_1(Z) - Q_0(Z)}{Q_0(Z)} \right)^2 \left( 1 + \frac{1}{1 + \eta \frac{Q_1(Z) - Q_0(Z)}{Q_0(Z)}} \right)^2
\]  
(153)
If we assume that for all \( z \in Z \), we have \( \left| \frac{Q_1(z) - Q_0(z)}{Q_0(z)} \right| \leq s \), then
\[
\eta^2 \left( \frac{Q_1(Z) - Q_0(Z)}{Q_0(Z)} \right)^2 \left( 1 + \frac{1}{1 + \eta \frac{Q_1(z) - Q_0(z)}{Q_0(z)}} \right)^2 \leq \eta^2 s^2 \left( 1 + \frac{1}{(1 - \eta)^2} \right)
\]  
(154)
which is less than \( 3 \eta^2 s^2 \) for \( \eta \) small enough. Therefore, we have \( \sigma^2 \leq \eta (U_P + 3 s^2) \). We finally get
\[
\exp \left( - \frac{1}{n \sigma^2 + L \gamma} \right) \leq \exp \left( - \frac{1}{n \eta(U_P + 3 s^2) + L \gamma} \right).
\]  
(155)
Lemma 8. For a DMC \((\mathcal{X}, W_{|X}, Z)\), there exists a constant \(C > 0\) such that for \(n/\ell\) large enough, we have
\[
\mathbb{D}(P^{n,\ell}_{Z,\text{PPM}} || Q_0^n) \leq \frac{\ell^2}{2n} \chi_2(Q_1 || Q_0) + \frac{\ell^3C}{n^2}.
\] (156)

Proof: If we define \(m = \lfloor \frac{n}{\ell} \rfloor\), by [4, Lemma 1], we have for some constant \(C_1, C_2,\) and \(C_3\)
\[
\mathbb{D}(P^{n,\ell}_{Z,\text{PPM}} || Q_0^n) \leq \frac{\ell}{2m} \chi_2(Q_1 || Q_0) + \frac{\ell C_1}{m^2} + \frac{\ell C_2}{m^3}.
\] (157)
\[
\leq \frac{\ell}{2m} \chi_2(Q_1 || Q_0) + \frac{\ell C_3}{m^2}.
\] (158)
\[
\leq \frac{\ell^2}{2(n/\ell - 1)} \chi_2(Q_1 || Q_0) + \frac{\ell C_3}{(n/\ell - 1)^2}.
\] (159)
\[
\leq \frac{\ell^2}{2n} \chi_2(Q_1 || Q_0) + \frac{\ell^3C_4}{n^2}.
\] (160)

Lemma 9. For a DMC \((\mathcal{X}, W_{|X}, Z)\) and \(\eta \in [0, 1]\), we have
\[
\mathbb{V}(P^{n,\eta}_{Z,\text{ID}}, Q_0^n) = 1 - 2Q\left(\frac{\sqrt{n}\eta \sqrt{\chi_2(Q_1 || Q_0)}}{2}\right) + O\left(\frac{1}{\sqrt{n}} + \eta^3 n\right),
\] (161)
and
\[
\beta_\alpha(Q_0^n, P^{n,\eta}_{Z,\text{ID}}) \geq Q\left(\sqrt{n}\eta \sqrt{\chi_2(Q_1 || Q_0)} - Q^{-1}(\alpha)\right) + O\left(\frac{1}{\sqrt{n}} + \eta^3 n\right).
\] (162)

Note that the constants behind \(O(.)\) are independent of \(\eta\) and just depend on the channel. Moreover, it refers to \(n \to \infty\) and \(\eta \to 0\).

Before proving this lemma, we state an auxiliary lemma.

Lemma 10. For a DMC \((\mathcal{X}, W_{|X}, Z)\) and \(\eta \in [0, 1]\), we have
\[
\mathbb{P}_{P^{n,\eta}_{Z,\text{ID}}}(\frac{P^{n,\eta}_{Z,\text{ID}}(Z)}{Q_0^n(Z)} \geq \exp \gamma) = Q\left(\frac{\gamma - n\eta^2 \chi_2(Q_1 || Q_0)/2}{\eta \sqrt{n} \chi_2(Q_1 || Q_0)}\right) + O\left(\frac{1}{\sqrt{n}} + \eta^3 n\right),
\] (163)
and
\[
\mathbb{P}_{Q_0^n}(\frac{P^{n,\eta}_{Z,\text{ID}}(Z)}{Q_0^n(Z)} \geq \exp \gamma) = Q\left(\frac{\gamma + n\eta^2 \chi_2(Q_1 || Q_0)/2}{\eta \sqrt{n} \chi_2(Q_1 || Q_0)}\right) + O\left(\frac{1}{\sqrt{n}} + \eta^3 n\right).
\] (164)

Proof: If we define \(A(z) \triangleq \frac{Q_1(z) - Q_0(z)}{Q_0(z)}\) and \(B(z) \triangleq \log (1 + \eta A(z))\), we have
\[
\mathbb{P}_{P^{n,\eta}_{Z,\text{ID}}}(\frac{P^{n,\eta}_{Z,\text{ID}}(Z)}{Q_0^n(Z)} \geq \exp \gamma) = \left\{ z : P^{n,\eta}_{Z,\text{ID}}(z) \geq \exp \gamma \right\} = \left\{ z : \prod_{i=1}^{n} (1 + \eta A(z_i)) \geq \exp \gamma \right\}
\] (165)
\[
= \left\{ z : \sum_{i=1}^{n} \log (1 + \eta A(z_i)) \geq \gamma \right\}
\] (166)
\[
= \left\{ z : \sum_{i=1}^{n} B(z_i) \geq \gamma \right\}.
\] (167)

Furthermore, we have
\[
\mu_0 \triangleq \mathbb{E}_{Q_0}(B(Z))
\] (168)
\[
= \sum_z Q_0(z) \log (1 + \eta A(z))
\] (169)
\[
= \sum_z Q_0(z) \left( \eta A(z) - \frac{\eta^2}{2} A^2(z) + O(\eta^3) \right)
\] (170)
\[
= -\frac{\eta^2}{2} \chi_2(Q_1 || Q_0) + O(\eta^3),
\] (171)
and
\[
\mu_1 \triangleq \mathbb{E}_{Q_1}(B(Z))
\]
\[
= \sum_z Q_1(z) \log (1 + \eta A(z))
\]
\[
= \sum_z Q_1(z) (\eta A(z) + O(\eta^2))
\]
\[
= \eta \chi_2(Q_1\|Q_0) + O(\eta^2).
\]

Therefore, we get
\[
\mu_\eta \triangleq \mathbb{E}_{Q_\eta}(B(Z))
\]
\[
= \eta \mu_1 + (1 - \eta) \mu_0
\]
\[
= \frac{\eta^2 \chi_2(Q_1\|Q_0)}{2} + O(\eta^3).
\]

Likewise, we obtain
\[
s_0 \triangleq \mathbb{E}_{Q_1}(B^2(Z))
\]
\[
= \sum_z Q_0(z) \log^2 (1 + \eta A(z))
\]
\[
= \sum_z Q_0(z) (\eta^2 A^2(z) + O(\eta^3))
\]
\[
= \eta^2 \chi_2(Q_1\|Q_0) + O(\eta^3),
\]
\[
\text{and}
\]
\[
s_1 \triangleq \mathbb{E}_{Q_1}(B^2(Z)) = O(\eta^2).
\]

Accordingly, we have
\[
\sigma_0^2 \triangleq \text{Var}_{Q_0}(B(Z))
\]
\[
= s_0 - \mu_0^2
\]
\[
= \eta^2 \chi_2(Q_1\|Q_0) + O(\eta^3),
\]
\[
\text{and}
\]
\[
\sigma_\eta^2 \triangleq \text{Var}_{Q_\eta}(B(Z))
\]
\[
= \eta s_1 + (1 - \eta) s_0 - \mu_\eta^2
\]
\[
= \eta^2 \chi_2(Q_1\|Q_0) + O(\eta^3).
\]

Finally, with similar calculations, we get
\[
T_0 \triangleq \mathbb{E}_{Q_0}(|B(Z) - \mu_0|^3) = O(\eta^3),
\]
\[
\text{and}
\]
\[
T_\eta \triangleq \mathbb{E}_{Q_\eta}(|B(Z) - \mu_\eta|^3) = O(\eta^3).
\]

Thus, Theorem 6 yields
\[
\mathbb{P}_{P^n_{Z,HD}}(P^n_{Z,HD}(Z) \geq \exp \gamma) = Q \left( \frac{\gamma - n \eta \mu_\eta}{\sqrt{n} \sigma_\eta^2} \right) + O \left( \frac{T_\eta}{\sqrt{n} \sigma_\eta^{3/2}} \right)
\]
\[
= Q \left( \frac{\gamma - n \eta^2 \chi_2(Q_1\|Q_0)}{\sqrt{n} \eta^2 \chi_2(Q_1\|Q_0)} + O(\eta^3 n) \right) + O \left( \frac{1}{\sqrt{n}} \right)
\]
\[
= Q \left( \frac{\gamma - n \eta^2 \chi_2(Q_1\|Q_0)/2}{\eta \sqrt{n} \eta \chi_2(Q_1\|Q_0)} + O(\eta^3 n) \right) + O \left( \frac{1}{\sqrt{n}} + \eta^3 n \right),
\]
and
\[
P_{Q_0^n} \left( \frac{P_{Z,\text{IID}}(Z)}{Q_0^n(Z)} \geq \exp \gamma \right) = Q \left( \frac{\gamma - n\mu_0}{\sqrt{n\sigma_0^2}} \right) + O \left( \frac{T_0}{\sqrt{n\sigma_0^2}} \right)
\]
(195)
\[
= Q \left( \frac{\gamma + \eta^2 \chi_2(Q_1 || Q_0)/2}{\eta \sqrt{n \chi_2(Q_1 || Q_0)}} \right) + O \left( \frac{1}{\sqrt{n}} + \eta^3 n \right)
\]
(196)
\[
= Q \left( \frac{\gamma + \eta^2 \chi_2(Q_1 || Q_0)/2}{\eta \sqrt{n \chi_2(Q_1 || Q_0)}} \right) + O \left( \frac{1}{\sqrt{n}} + \eta^3 n \right).
\]
(197)

**Proof of Lemma 9** By definition of total variation and Lemma 10, we have
\[
\mathbb{V} \left( P_{Z,\text{IID}}, Q_0^n \right) = \mathbb{P} P_{Z,\text{IID}}(Z) \geq Q_0^n(Z) - \mathbb{P} Q_{0^n}(P_{Z,\text{IID}}(Z) \geq Q_0^n(Z))
\]
(198)
\[
= \mathbb{P} P_{Z,\text{IID}}(Z) \geq 1 - \mathbb{P} Q_{0^n}(P_{Z,\text{IID}}(Z) \geq 1)
\]
(199)
\[
= Q \left( \frac{0 - \eta^2 \chi_2(Q_1 || Q_0)/2}{\eta \sqrt{n \chi_2(Q_1 || Q_0)}} \right) - Q \left( \frac{0 + \eta^2 \chi_2(Q_1 || Q_0)/2}{\eta \sqrt{n \chi_2(Q_1 || Q_0)}} \right) + O \left( \frac{1}{\sqrt{n}} + \eta^3 n \right)
\]
(200)
\[
= 1 - 2Q \left( \frac{\sqrt{n} \eta \sqrt{n \chi_2(Q_1 || Q_0)}}{2} \right) + O \left( \frac{1}{\sqrt{n}} + \eta^3 n \right).
\]
(201)

For \( \beta_\alpha \left( Q_0^n, P_{Z,\text{IID}} \right) \), by the Neyman-Pearson Lemma, if for some \( \gamma \) we have
\[
\alpha \leq \mathbb{P}_{Q_0^n} \left( \frac{P_{Z,\text{IID}}(Z)}{Q_0^n(Z)} \geq \exp \gamma \right)
\]
(202)
\[
= Q \left( \frac{\gamma + \eta^2 \chi_2(Q_1 || Q_0)/2}{\eta \sqrt{n \chi_2(Q_1 || Q_0)}} \right) + O \left( \frac{1}{\sqrt{n}} + \eta^3 n \right),
\]
(203)
then
\[
\beta_\alpha \left( Q_0^n, P_{Z,\text{IID}} \right) \geq \mathbb{P} P_{Z,\text{IID}}(Z) \leq \exp \gamma
\]
(204)
\[
= Q \left( \frac{\gamma - \eta^2 \chi_2(Q_1 || Q_0)/2}{\eta \sqrt{n \chi_2(Q_1 || Q_0)}} \right) + O \left( \frac{1}{\sqrt{n}} + \eta^3 n \right).
\]
(205)

Hence, by choosing
\[
\gamma = Q^{-1} \left( \alpha - O \left( \frac{1}{\sqrt{n}} + \eta^3 n \right) \right) \eta \sqrt{n \chi_2(Q_1 || Q_1)} - \eta^2 \chi_2(Q_1 || Q_0)/2,
\]
(206)
we obtain
\[
\beta_\alpha \left( Q_0^n, P_{Z,\text{IID}} \right)
\geq Q \left( -Q^{-1} \left( \alpha - O \left( \frac{1}{\sqrt{n}} + \eta^3 n \right) \right) \eta \sqrt{n \chi_2(Q_1 || Q_1)} - \eta^2 \chi_2(Q_1 || Q_0)/2 - \eta^2 \chi_2(Q_1 || Q_0)/2 \right) + O \left( \frac{1}{\sqrt{n}} + \eta^3 n \right)
\]
(207)
\[
= Q \left( -Q^{-1} \left( \alpha \right) + \eta \sqrt{n \chi_2(Q_1 || Q_0)} \right) + O \left( \frac{1}{\sqrt{n}} + \eta^3 n \right).
\]
D. Converse

We now develop a generic converse for a quasi-metric $d$. The following result states that the number of covert and reliable bits that one can transmit may be characterized by establishing an upper bound on the weight of codewords. Such upper bounds are metric-specific, and we develop them in subsequent sections.

**Theorem 7.** Let $d$ be a quasi-metric and $(X, W_{Y|X}, W_{Z|X}, Y, Z)$ be a covert communication channel. If there exists some $h > 0$ such that for every $(M, K, n, \epsilon, \delta)_d$ code $C$, there exists a subset of the codewords, $D$, with $|D| \geq MK/n^h$ and

$$\max_{x \in D} \text{wt}(x) \leq g(\delta) + \frac{C}{\sqrt{n}}. \quad (208)$$

then

$$\log M_d^*(n, \epsilon, \delta) \leq g(\delta)D_P^* n^{1/2} - \frac{\sqrt{g(\delta)V_P}Q^{-1}(\epsilon)}{\sqrt{n}} + (h + 1) \log n + \log(\sqrt{ng(\delta)} + C) + O(1). \quad (209)$$

**Proof:** Let $C$ be an $(M, K, n, \epsilon, \delta)_d$ code. For $s \in \llbracket 1, K \rrbracket$, we denote the sub-codebook of all codewords characterized by the key value $s$ by $C^s$. By our assumptions, we can choose $D \subset C$ of size at least $MK/n^h$ satisfying (208). By the pigeonhole principle, there should be at least one sub-codebook $C^s$ such that $|D \cap C^s| \geq M/n^h$. Let us define

$$D_i^* \triangleq \{x \in D \cap C^s : \text{wt}(x) = i\}. \quad (210)$$

Since $D_i^*$ is included in $C^s$, it is a reliability code for the channel $W_{Y|X}$ with probability of error less than or equal to $\epsilon$. Moreover, the type of the codewords in $D_i^*$ is fixed, and therefore for any $x \in D_i^*$, we can use [17, Theorem 28] to get

$$\log |D_i^*| \leq -\beta_{1-\epsilon} (W_{Y|X=x}, P_0^n). \quad (211)$$

Next, similar to [24], by [17, Equation 102], we obtain for any $\gamma > 0$

$$-\beta_{1-\epsilon} (W_{Y|X=x}, P_0^n) \leq \gamma - \log \left(1 - \epsilon - P_{W_{Y|X=x}} \left(\log \frac{W_{Y|X}(Y|x)}{P_0^n(Y)} \geq \gamma\right)\right). \quad (212)$$

However, we know that $W_{Y|X}(Y|x) = \prod_{k:x_k=1} P_1(Y_k) \prod_{j:x_j=0} P_0(Y_j)$ which yields that if $\tilde{Y}_1, \cdots, \tilde{Y}_s$ are i.i.d. according to $P_1$, then

$$P_{W_{Y|X=x}} \left(\log \frac{W_{Y|X}(Y|x)}{P_0^n(Y)} \geq \gamma\right) = P_{P_0^n} \left(\sum_{k=1}^i \log \frac{P_1(\tilde{Y}_k)}{P_0(Y_k)} \geq \gamma\right). \quad (213)$$

By Theorem 6 we have

$$P \left(\sum_{k=1}^i \log \frac{P_1(\tilde{Y}_k)}{P_0(Y_k)} \geq \gamma\right) \leq Q \left(\frac{\gamma - iD_P}{\sqrt{iV_P}}\right) + \frac{B}{\sqrt{i}}, \quad (214)$$

where $B$ just depends on the channel. Setting $\gamma = iD_P + \sqrt{iV_P}Q^{-1}(1 - \epsilon)$ and combining all above equations, we have

$$\log |D_i^*| \leq iD_P + \sqrt{iV_P}Q^{-1}(1 - \epsilon) - \log \left(\frac{B}{\sqrt{i}}\right). \quad (215)$$

Hence, we obtain

$$\log \frac{M}{n^h} \leq \log |D \cap C^s| \leq \log \left(\sum_{i=0}^{\gamma_0} |D_i^*|\right) \quad (217)$$

$$\leq \log \left(\sum_{i=0}^{\gamma_0} \exp(iD_P + \sqrt{iV_P}Q^{-1}(1 - \epsilon) - \log \left(\frac{B}{\sqrt{i}}\right)\right) \quad (218)$$

$$\leq \log \left((\sqrt{ng(\delta)} + C) \exp(g(\delta)D_Pn^{1/2} - \sqrt{V_P g(\delta)}Q^{-1}(\epsilon)n^{1/2} + O(1))\right). \quad (219)$$

Thus, we get

$$\log M \leq g(\delta)D_P n^{1/2} - \sqrt{g(\delta)V_P Q^{-1}(\epsilon)n^{1/2}} + (h + 1) \log n - \log(B) + \log(\sqrt{ng(\delta)} + C) + O(1). \quad (220)$$
IV. COVERT COMMUNICATION WITH SPECIFIC COVERTNESS METRICS

We now leverage the general results established in Section III and specialize them to study three covertness metrics: relative entropy (Subsection IV-A), variational distance (Subsection IV-B), and probability of missed detection (Subsection IV-C). As alluded to earlier, all that needs to be done is: (i) establish that the metric under consideration is a quasi-metric, as defined at the beginning of Section III (ii) verify that the conditions of Theorem 5 and Theorem 7 are satisfied.

A. Covertness in relative entropy

In this subsection, we prove Theorem 1

Lemma 11. Relative entropy is a quasi-metric, i.e.,

\[
\forall P, Q, R: \mathbb{D}(R\|Q) \leq \mathbb{D}(P\|Q) + \mathbb{D}(R\|P) + \sqrt{\mathbb{D}(R\|P)} \max \left(1, \log \frac{1}{\min_z Q(z)} \right). \tag{221}
\]

Proof: Note that

\[
\mathbb{D}(R\|Q) = \sum_z R(z) \log \frac{R(z)}{Q(z)} \leq \sum_z R(z) \log \frac{R(z)}{P(z)} + \sum_z R(z) \log \frac{P(z)}{Q(z)} = \mathbb{D}(R\|P) + \sum_z R(z) \log \frac{P(z)}{Q(z)} \tag{222}
\]

\[
= \mathbb{D}(R\|P) + \sum_z R(z) \log \frac{P(z)}{Q(z)} + \sum_z (R(z) - P(z)) \log \frac{P(z)}{Q(z)} \tag{223}
\]

\[
\leq \mathbb{D}(R\|P) + \mathbb{D}(P\|Q) + \sum_z (R(z) - P(z)) \log \frac{P(z)}{Q(z)} \tag{224}
\]

\[
\leq \mathbb{D}(R\|P) + \mathbb{D}(P\|Q) + \sqrt{\mathbb{D}(R\|P)} \max \left(1, \log \frac{1}{\min_z Q(z)} \right). \tag{225}
\]

Proof of Theorem 1 - Achievability: Fix \( \epsilon \in (0, 1) \) and \( \delta > 0 \), and define \( \omega \triangleq \sqrt{\frac{28}{\chi_2(Q_1\|Q_0)}} \) and \( \ell_n \triangleq \lfloor \omega \sqrt{n} - t \rfloor \) where the value of \( t \) will be determined later. To use Theorem 5, we choose \( P_X^n \triangleq P_{X^{\ell_n}} \), and we set

\[
\log M_n \triangleq \ell_n D_P - \sqrt{\ell_n V_P Q_n^{-1}} \left( \epsilon - 1 + 6 T_p / V_P^2 \right) - 2 \log n \tag{226}
\]

\[
= \omega D_P n^2 - \sqrt{\omega V_P Q_n^{-1}} (\epsilon) n^2 - 2 \log n + O(1), \tag{227}
\]

and

\[
\log M_n + \log K_n \triangleq \max (\log M_n, (1 + \rho) \ell_n D_Q) \tag{228}
\]

\[
= \max (\log M_n, (1 + \rho) \omega D_Q n^2 + O(1)) \tag{229}
\]

By Lemma 8, we know that

\[
\mathbb{D}(P_{Z_{\ell_n}\|Q_0^n}) \leq \frac{\ell_n^2}{2n} \chi_2(Q_1\|Q_0) + \frac{\ell_n^2 C}{n^2} \tag{230}
\]

\[
\leq \frac{(\omega \sqrt{n} - t)^2}{2n} \chi_2(Q_1\|Q_0) + \frac{(\omega \sqrt{n} - t)^3 C}{n^2} \tag{231}
\]

\[
\leq \frac{1}{2} \omega^2 \chi_2(Q_1\|Q_0) - \frac{1}{\sqrt{n}}, \tag{232}
\]

where (a) is true for \( t \) and \( n \) large enough. Furthermore, by Lemma 6, we have

\[
\mathbb{E}(P_{X^{\ell_n}} \| P_{Y^{\ell_n}}(Y) \) \) \leq \exp \left( \frac{\ell_n (\ell_n + 1)}{n} \chi_2(P_1\|P_0) \right) \) = O(1). \tag{233}
\]
Hence, for large enough $n$, we have

\[
F_{XY|P_0^n} \left( \log \left( n^2 M_n \right) \right) + \frac{1 + \mathbb{E}_{P_{Y^n}^{P_0^n}} \left( \frac{P_{Y^n|P_0^n}^{P_0^n}(Y)}{P_{Y^n|P_0^n}^{P_0^n}(Y)} \right)}{n} \\
= F_{XY|P_0^n} \left( \ell_n D_P - \sqrt{\ell_n V_P} Q^{-1} \left( \epsilon - \frac{1 + 6 T_P / V_P^2}{\sqrt{\ell_n}} \right) \right) + O \left( \frac{1}{n} \right) \\
\leq Q \left( \ell_n D_P - \ell_n D_P + \sqrt{\ell_n V_P} Q^{-1} \left( \epsilon - \frac{1 + 6 T_P / V_P^2}{\sqrt{\ell_n}} \right) \right) + \frac{6 T_P}{\sqrt{\ell_n V_P}} + O \left( \frac{1}{n} \right) \quad (238)
\]

where (a) follows from Lemma 5. By Lemma 7, we have

\[
F_{xz} \left( \log \frac{M_n K_n}{n^4} \right) \leq \exp \left( - \frac{(\log \frac{M_n K_n}{n^2} - \ell_n D_Q)^2}{2 \ell_n \log^2 \mu_Z} \right) \leq \exp \left( - \frac{(1 + \rho) \ell_n D_Q - 4 \log n - \ell_n D_Q)^2}{2 \ell_n \log^2 \mu_Z} \right) \leq \exp \left( - \frac{\ell_n (\rho D_Q - 4 \log n)^2}{2 \log^2 \mu_Z} \right) \leq \frac{1}{n^6}, \quad (241)
\]

where (a) is true for large enough $n$. Thus, all the conditions in Theorem 5 hold, and we obtain

\[
\log M_P(n, \epsilon, \delta) \geq \omega D_P n^{\frac{3}{4}} - \sqrt{\omega V_P} Q^{-1} (\epsilon) n^{\frac{3}{4}} - 2 \log n + O(1). \quad (243)
\]

Proof of Theorem 7 - Converse: By [5, Equations (11) and (96)], for every $(M, K, n, \epsilon, \delta)_D$ code, we have

\[
\sum_{s=1}^{K} \sum_{w=1}^{M} \text{wt}(x_{sw})/(MK) \leq (B + \sqrt{n}) \sqrt{\frac{2 \delta}{\chi_2(Q_1\|Q_0)}}, \quad (244)
\]

for some $B > 0$ depending on the channel. Thus, we can choose a subset of codewords of size $MK/n$ such as $D$ such that

\[
\frac{\max_{x \in D} \text{wt}(x)}{\sqrt{n}} \leq \frac{n}{n - 1} \left( \frac{B}{\sqrt{n}} + 1 \right) \sqrt{\frac{2 \delta}{\chi_2(Q_1\|Q_0)}}, \quad (245)
\]

Applying Theorem 7 completes the proof.

B. Covertness with variational distance

In this subsection, we prove Theorem 2.

Lemma 12. Total variation is a quasi-metric for distributions, i.e., for all distributions $P$, $Q$, and $R$ defined over same set, we have

\[
\mathbb{V}(R, Q) \leq \mathbb{V}(P, Q) + D(R\|Q) + \sqrt{D(R\|P)} \max \left( 1, \log \frac{1}{\min_x Q(x)} \right). \quad (246)
\]

Proof: Total variation is a metric, and therefore we have

\[
\mathbb{V}(R, Q) \leq \mathbb{V}(P, Q) + \mathbb{V}(R, P) \leq \mathbb{V}(P, Q) + \sqrt{D(R\|P)} \leq \mathbb{V}(P, Q) + D(R\|Q) + \sqrt{D(R\|P)} \max \left( 1, \log \frac{1}{\min_x Q(x)} \right), \quad (249)
\]

where (a) follows from Pinsker Inequality.
Proof of Theorem 2 - Achievability: Fix $\epsilon$ and $\delta$ in $[0, 1]$, and define $\omega \triangleq \frac{2Q^{-1}(1-\delta/2)}{\chi_2(Q_1\|Q_0)}$ and $\eta_n \triangleq \left\lfloor \frac{\omega}{\sqrt{n}} - \frac{\epsilon}{n} \right\rfloor$ where the value of $t$ will be determined later. To use Theorem 3, we choose $P^n_X \triangleq P^n_{X,\text{IID}}$, and we set

$$\log M_n \triangleq nD_p n - \left( (nD_p n - \eta_n D_p n) Q^{-1} \left( \epsilon - \frac{1 + C}{\sqrt{n}} \right) \right) - 2 \log n$$

(250)

and

$$\log M_n + \log K_n \triangleq \max(\log M_n, (1 + \rho) nD_p n)$$

(252)

$$= \max(\log M_n, (1 + \rho) \omega D_p n^{1/2} + O(1)).$$

(253)

By Lemma 9, we know that

$$\mathcal{V} \left( P^n_{X,\text{IID}}, Q_0^{\alpha n} \right) \leq 1 - 2Q \left( \frac{(\omega - t/\sqrt{n}) \sqrt{\chi_2(Q_1\|Q_0)}}{2} \right) + C \sqrt{\frac{1}{n}}$$

(a)

(254)

$$\leq 1 - 2Q \left( \frac{\omega \sqrt{\chi_2(Q_1\|Q_0)}}{2} \right) - \frac{1}{\sqrt{n}}$$

(255)

$$= \delta - \frac{1}{\sqrt{n}},$$

(256)

where (a) is true for $t$ large enough. Furthermore, by Lemma 6, we have

$$\mathbb{E}_{P^n_{X,\text{IID}}} \left( \frac{P^n_{X,\text{IID}}(Y)}{P^n_0(Y)} \right) \leq \exp \left( n\eta_n^2 \chi_2(P_1\|P_0) \right) = O(1).$$

(257)

Hence, for large enough $n$, we have

$$F_{X|P_0^n} \left( \log \left( n^2 M_n \right) \right) + 
\frac{1 + \mathbb{E}_{P^n_{X,\text{IID}}} \left( \frac{P^n_{X,\text{IID}}(Y)}{P^n_0(Y)} \right)}{n}
= F_{X|P_0^n} \left( nD_p n - \sqrt{nU_p n} Q^{-1} \left( \epsilon - \frac{1 + C}{\sqrt{n}} \right) \right) + O \left( \frac{1}{n} \right)
\leq Q \left( \frac{nD_p n - \eta_n D_p n - \sqrt{nU_p n - \eta_n^2 D_p n} Q^{-1} \left( \epsilon - \frac{1 + C}{\sqrt{n}} \right)}{nU_p n - \eta_n^2 D_p n} \right) + C \sqrt{\frac{1}{n}} + O \left( \frac{1}{n} \right)
\leq \epsilon,$$

where (a) follows from Lemma 5. By Lemma 7, for some $\zeta > 0$, we have

$$F_{XZ} \left( \log \frac{M_n K_n}{n^3} \right) \leq \exp \left( - \frac{\left( \log \frac{M_n K_n}{n^3} - \eta_n D_p n \right)^2}{\eta_n C_1 n + C_2 \log \frac{M_n K_n}{n^3}} \right)
\leq \exp \left( -n\eta_n \zeta \right)
\leq \frac{1}{n^6},$$

(258)

(259)

(260)

(261)

where (a) is true for large enough $n$. Thus, all conditions in Theorem 5 hold, and we obtain

$$\log M^n_v (n, \epsilon, \delta) \geq \omega D_p n^{1/2} - \sqrt{\omega U_p n} Q^{-1} \left( \epsilon \right) n^{1/2} - 2 \log n + O(1).$$

(262)

To develop the converse for variational distance, we start by relating the variational distance $\mathcal{V} \left( \hat{P}_Z, Q_0^{\alpha n} \right)$ to the minimum weight of the codewords.

Lemma 13. Consider a binary-input DMC $(\mathcal{X}, W_{Z|X}, Z)$ and $M$ codewords $x_1, \cdots, x_M \in \mathcal{X}^n$ with induced distribution $\hat{P}_Z$ on $\mathcal{Z}^n$. If $w_{\min} \triangleq \min_{m \in [1,M]} w(x_m)$, then we have

$$\mathcal{V} \left( \hat{P}_Z, Q_0^{\alpha n} \right) \geq 1 - 2Q \left( \frac{w_{\min} \sqrt{\chi_2(Q_1\|Q_0)}}{2\sqrt{n}} \right) - \frac{B}{\sqrt{n}} - \frac{w_{\min}^2 B}{n^2},$$

(263)

where $B$ is a constant that only depends on the channel.
Proof: To lower bound $\mathbb{V}(\hat{P}_z, Q_0^n)$, we introduce hypothesis testing problem with two hypotheses $H_0$ and $H_1$ corresponding to distributions $Q_0^n$ and $\hat{P}_z$, respectively. We know that for any test with probability of false alarm and missed detection $\alpha$ and $\beta$, respectively, we have

$$\mathbb{V}(\hat{P}_z, Q_0^n) \geq 1 - \alpha - \beta. \quad (264)$$

Hence, to link the variational distance to the weight of codewords, it suffices to introduce a test for which $\alpha$ and $\beta$ conveniently relate to the weight of codewords. We consider here the sub-optimal test

$$T(z) \triangleq 1 \left\{ \sum_{i=1}^{n} A(z_i) > \tau \right\}, \quad (265)$$

where $A(z) \triangleq \frac{Q_1(z) - Q_0(z)}{Q_0(z)}$ and $\tau$ is an arbitrary constant that is determined later. Intuitively, this test plays the same role for DMCs as the radiometer played for Gaussian channels \cite{2} in that it only depends on the codeword weight. To bound the probability of false alarm, we use Theorem 6 to obtain

$$\mathbb{P}_{H_0} \left( \sum_{i=1}^{n} A(Z_i) \geq \tau \right) \leq Q \left( \frac{\tau - n\mu_0}{\sqrt{n}\sigma_0} \right) + \frac{6t_0}{\sigma_0^{3/2}} \sqrt{n}, \quad (266)$$

with

$$\mu_0 \triangleq \mathbb{E}_{Q_0}(A(Z)), \quad \sigma_0^2 \triangleq \text{Var}_{Q_0}(A(Z)), \quad t_0 \triangleq \mathbb{E}_{Q_0}(|A(Z) - \mu_0|^3). \quad (267)$$

Note that all above quantities are finite. For the probability of missed detection, we condition on the codeword transmitted by the channel to obtain

$$\mathbb{P}_{H_1} \left( \sum_{i=1}^{n} A(Z_i) \leq \tau \right) = \sum_{m=1}^{M} \frac{1}{M} \mathbb{P} \left( \sum_{i=1}^{n} A(Z_i) \leq \tau | X = x_m \right) \quad (268)$$

\[ \leq \sum_{m=1}^{M} \frac{1}{M} \left( Q \left( \frac{-\tau + n\mu_0 + wt(x_m)(\mu_1 - \mu_0)}{\sqrt{n}\sigma_0^2 + wt(x_m)(\sigma_1^2 - \sigma_0^2)} \right) + \frac{6(t_0 + wt(x_m))}{\sigma_0^2 + wt(x_m)} \right) \quad (269) \]

\[ \leq \sum_{m=1}^{M} \frac{1}{M} \left( Q \left( \frac{-\tau + n\mu_0 + wt(x_m)(\mu_1 - \mu_0)}{\sqrt{n}\sigma_0^2 + wt(x_m)(\sigma_1^2 - \sigma_0^2)} \right) + \frac{B_1}{\sqrt{n}} \right) \quad (270) \]

with

$$\mu_1 \triangleq \mathbb{E}_{Q_1}(A(Z)), \quad \sigma_1^2 \triangleq \text{Var}_{Q_1}(A(Z)), \quad t_1 \triangleq \mathbb{E}_{Q_1}(|A(Z) - \mu_1|^3). \quad (271)$$

If we choose $\tau = n\mu_0 + \frac{w_{\min}}{2}(\mu_1 - \mu_0)$, we have

$$\mathbb{P}_{H_1} \left( \sum_{i=1}^{n} A(Z_i) \leq \tau \right) \leq \sum_{m=1}^{M} \frac{1}{M} \left( Q \left( \frac{wt(x_m)(\mu_1 - \mu_0)}{2\sqrt{n}\sigma_0^2 + wt(x_m)(\sigma_1^2 - \sigma_0^2)} \right) + \frac{B_1}{\sqrt{n}} \right). \quad (272)$$

Note that if we have $\sigma_1 < \sigma_0$, then we get

$$\beta \triangleq \mathbb{P}_{H_1} \left( \sum_{i=1}^{n} A(Z_i) \leq \tau \right) \leq Q \left( \frac{w_{\min}(\mu_1 - \mu_0)}{2\sqrt{n}\sigma_0^2} \right) \quad (273)$$

Otherwise, we split the summation in (272) into two parts: if $wt(x_m) > \frac{\sigma_1}{\sigma_0} w_{\min}$, then we have

$$Q \left( \frac{wt(x_m)(\mu_1 - \mu_0)}{2\sqrt{n}\sigma_0^2 + wt(x_m)(\sigma_1^2 - \sigma_0^2)} \right) \overset{(a)}{=} Q \left( \frac{\sigma_1}{\sigma_0} w_{\min}(\mu_1 - \mu_0) \right) \quad (274)$$

$$= Q \left( \frac{w_{\min}(\mu_1 - \mu_0)}{2\sqrt{n}\sigma_0^2} \right). \quad (275)$$
where (a) follows from $\sigma_1 > \sigma_0$. If $w_{\text{min}} \leq \text{wt}(x_m) \leq \frac{\sigma_1}{\sigma_0} w_{\text{min}}$, then we obtain

$$Q\left(\frac{\text{wt}(x_m)(\mu_1 - \mu_0)}{2\sqrt{n\sigma_0^2 + \text{wt}(x_m)(\sigma_1^2 - \sigma_0^2)}}\right) \leq Q\left(\frac{w_{\text{min}}(\mu_1 - \mu_0)}{2\sqrt{n\sigma_0^2}} \left(\frac{1}{1 + \frac{\sigma_1}{\sigma_0} w_{\text{min}}(\sigma_1^2 - \sigma_0^2)}\right)\right)$$

(276)

$$= Q\left(\frac{w_{\text{min}}(\mu_1 - \mu_0)}{2\sqrt{n\sigma_0^2}} \left(1 - \frac{w_{\text{min}}B_2}{n}\right)\right)$$

(277)

$$\leq Q\left(\frac{w_{\text{min}}(\mu_1 - \mu_0)}{2\sqrt{n\sigma_0^2}} + \frac{w_{\text{min}}^2 B_3}{n^2}\right)$$

(278)

for $B_2 = \frac{\sigma_1(\sigma_1^2 - \sigma_0^2)}{2\sigma_0^2} > 0$ and $B_3 = \frac{(\mu_1 - \mu_0)B_2}{2\sqrt{2\pi}\sigma_0} > 0$ where (a) follows from $\frac{1}{\sqrt{1 + x}} \geq -\frac{x}{2} + 1$ for all $x > 0$, and (b) follows from $Q(x - y) \leq Q(x) + \frac{y}{\sqrt{2\pi}}$ for all $0 < y < x$. Therefore, we always have

$$\beta \leq Q\left(\frac{w_{\text{min}}(\mu_1 - \mu_0)}{2\sqrt{n\sigma_0^2}}\right) + \frac{w_{\text{min}}^2 B_3}{n^2} + \frac{B_1}{\sqrt{n}}.$$  

(280)

Moreover, plugging in the value $\tau$ in (266), we obtain

$$\alpha \triangleq P_{\mathcal{H}_0}\left(\sum_{i=1}^{n} A(Z_i) \geq \tau\right) \leq Q\left(\frac{w_{\text{min}}(\mu_1 - \mu_0)}{2\sqrt{n\sigma_0^2}}\right) + \frac{B_0}{\sqrt{n}},$$

(281)

for some positive constant $B_0$. Using (264), (280), and (281), we obtain

$$\forall \left(P_Z, Q_0^n\right) \geq 1 - 2Q\left(\frac{w_{\text{min}}(\mu_1 - \mu_0)}{2\sqrt{n\sigma_0^2}}\right) - \frac{B_0 + B_1}{\sqrt{n}} - \frac{w_{\text{min}}^2 B_3}{n^2}.$$  

(282)

Finally, since $\mu_0 = 0$ and $\sigma_0^2 = \mu_1 = \chi_2(Q_1||Q_0)$, we have

$$\forall \left(P_Z, Q_0^n\right) \geq 1 - 2Q\left(\frac{w_{\text{min}}\chi_2(Q_1||Q_0)}{2\sqrt{n}}\right) - \frac{B_0 + B_1}{\sqrt{n}} - \frac{w_{\text{min}}^2 B_3}{n^2}. $$

(283)

Next, we relate the weight of the codewords in a covert code to the channel characteristics and the specific amount of covertness measured in variational distance.

**Lemma 14.** Let $\mathcal{C}$ be an $(M, K, n, \epsilon, \delta)_V$ code for a binary-input covert communication channel $(X, W_{Y|X}, W_{Z|X}, Y, Z)$. For all $\gamma \in [0, 1]$, there exists a subset of codewords $\mathcal{D}$ such that $|\mathcal{D}| \geq \gamma MK$ and

$$\frac{1}{\sqrt{n}} \max\{|\text{wt}(x)|: x \in \mathcal{D}\} \leq \frac{2}{\sqrt{\chi_2(Q_1||Q_0)}}Q^{-1}\left(\frac{1 - \delta}{2} - \frac{C}{\sqrt{n} - \gamma}\right),$$

(284)

where $C$ is a constant that depends only on the channel.

**Proof:** We define

$$A \triangleq \frac{2}{\sqrt{\chi_2(Q_1||Q_0)}}Q^{-1}\left(\frac{1 - \delta}{2} - \frac{C_2}{\sqrt{n}} - \gamma\right)$$

(285)

where $C_2 > 0$ is specified later and $\mathcal{D} \triangleq \{x \in \mathcal{C}: \text{wt}(x) \leq A\sqrt{n}\}$. Clearly, $\mathcal{D}$ satisfies (284), and we just need to check $|\mathcal{D}| \geq \gamma MK$. To this end, let $P_1$ and $P_2$ be the induced output distributions for codes $\mathcal{D}$ and $\mathcal{C} \setminus \mathcal{D}$, respectively. Then, we
have
\[ \delta \geq \mathcal{V}(\hat{P}_Z, Q_0) \] (286)
\[ \geq (1 - \frac{|D|}{MK}) \mathcal{V}(\hat{P}_2, Q_0') - \frac{|D|}{MK} \mathcal{V}(\hat{P}_1, Q_0') \] (287)
\[ \geq (1 - \frac{|D|}{MK}) \left( 1 - 2Q \left( \frac{A\sqrt{n}}{\chi_2(Q_1||Q_0)} \right) \right) - \frac{|D|}{MK} \] (288)
\[ \geq \left( 1 - 2Q \left( \frac{A\sqrt{\chi_2(Q_1||Q_0)}}{2} \right) \right) - \frac{B(1 + A^2)}{\sqrt{n}} - 2\frac{|D|}{MK} \] (289)
\[ \geq \left( 1 - 2 \left( \frac{1 - \delta}{2} - \frac{C_2}{\sqrt{n}} - \gamma \right) \right) - \frac{B(1 + A^2)}{\sqrt{n}} - 2\frac{|D|}{MK} \] (290)
\[ \geq \delta + 2\gamma - 2\frac{|D|}{MK}, \] (291)

where (a) follows from the definition of an \((M, K, n, \epsilon, \delta)_V\) code, (b) follows from the triangle inequality and \(\hat{P}_Z = \frac{|D|}{MK} \hat{P}_1 + (1 - \frac{|D|}{MK}) \hat{P}_2\), (c) follows from Lemma 13, (d) follows from the definition of \(A\), and (e) follows by choosing

\[ C_2 > \frac{1}{2} B \left( 1 + \left( \frac{2}{\sqrt{\chi_2(Q_1||Q_0)}} Q^{-1} \left( \frac{1 - \delta}{2} - \gamma \right) \right)^2 \right). \] (292)

Therefore, we get \(|D| \geq \gamma MK\).

Proof of Theorem 2 - Converse: In Lemma 14 if we set \(\gamma = \frac{1}{\sqrt{n}}\) for any \((M, K, n, \epsilon, \delta)_V\) code, we have a subset of codewords \(D\) with \(|D| \geq MK/\sqrt{n}\) and

\[ \max_{x \in D} \frac{\text{wt}(x)}{\sqrt{n}} \leq \frac{2}{\sqrt{\chi_2(Q_1||Q_0)}} Q^{-1} \left( \frac{1 - \delta}{2} - \frac{C}{\sqrt{n}} - \gamma \right) \] (293)
\[ = \frac{2}{\sqrt{\chi_2(Q_1||Q_0)}} Q^{-1} \left( \frac{1 - \delta}{2} \right) + O \left( \frac{1}{\sqrt{n}} \right). \] (294)

Therefore, by Theorem 7 we have

\[ \log M_V(n, \epsilon, \delta) \leq \frac{2}{\sqrt{\chi_2(Q_1||Q_0)}} Q^{-1} \left( \frac{1 - \delta}{2} \right) D_{\text{max}} - \sqrt{\frac{2}{\sqrt{\chi_2(Q_1||Q_0)}} Q^{-1} \left( \frac{1 - \delta}{2} \right)} V_P Q^{-1}(\epsilon)n^{\frac{1}{2}} + O(\log n). \] (295)

\[ \square \]

C. Covertness with probability of missed detection at fixed significance level

In this subsection, we prove Theorem 5

Lemma 15. For a fixed \(\alpha \in [0, 1]\) and two distributions \(P\) and \(Q\) over some set \(Z\), we define \(d(P, Q) \triangleq 1 - \alpha - \beta_\alpha(Q, P)\); then \(d\) is a quasi-metric, and we have

\[ d(R, Q) \leq d(P, Q) + \mathbb{D}(R||P) + \sqrt{\mathbb{D}(R||P)} \max \left( 1, \log \frac{1}{\min_z Q(z)} \right). \] (296)

Proof: By definition of \(\beta_\alpha(Q, R)\), there exists a set \(T\) with \(Q(T) \leq \alpha\) and \(R(T) = \beta_\alpha(Q, R)\); therefore we have

\[ \beta_\alpha(Q, P) \leq P(T) \] (297)
\[ = R(T) + (P(T) - R(T)) \] (298)
\[ \leq R(T) + \mathbb{V}(R, P) \] (299)
\[ = \beta_\alpha(Q, R) + \mathbb{V}(R, P). \] (300)

By Pinsker’s Inequality, we obtain

\[ \beta_\alpha(Q, R) + \mathbb{V}(R, P) \leq \beta_\alpha(Q, R) + \sqrt{\mathbb{D}(R||P)}. \] (301)

Finally, using the definition of \(d(P, Q)\), we get the result.

Proof of Theorem 3 - Achievability: If we choose \(\omega = \frac{Q^{-1}(1-\alpha-\delta)+Q^{-1}(\alpha)}{\sqrt{\chi_2(Q_1||Q_0)}}\), the remainder of proof is similar to the total variation case.
The proof of the converse of Theorem [5] requires the following steps similar to those of coventional with variational distance. We first relate the probability of missed detection \( \beta_\alpha \left( Q_0^n, \hat{P}_Z \right) \) at significance level \( \alpha \) to the minimum weight of codewords.

**Lemma 16.** Consider a binary-input DMC \((\mathcal{X}, W_{Z|X}, Z)\) and \( M \) codewords \( x_1, \ldots, x_M \in \mathcal{X}^n \) with induced distribution \( \hat{P}_Z \) on \( Z^n \). If \( \frac{n^{Q^{-1}(\alpha)}}{\sqrt{n}Q(1)} \leq w_{\min} \leq \min_{m \in [1,M]} \text{wt}(x_m) \), we have

\[
\beta_\alpha \left( Q_0^n, \hat{P}_Z \right) \leq Q \left( \frac{w_{\min} \sqrt{\chi_2(Q_1||Q_0)}}{\sqrt{n}} - Q^{-1}(\alpha) \right) + \frac{B}{\sqrt{n}} + \frac{w_{\min}^2 B}{n^2},
\]

(302)

where \( B \) is a constant that depends just on the channel.

**Proof:** For the hypothesis testing problem consisting of two hypotheses \( H_0 \) with distribution \( Q_0^n \) and \( H_1 \) with distribution \( \hat{P}_Z \), we introduce again the test

\[
T(z) \equiv 1 \left\{ \sum_{i=1}^n A(z_i) > \tau \right\},
\]

(303)

where \( A(z) \) is defined as \( A(z) = \frac{Q_1(z)-Q_0(z)}{Q_0(z)} \). Using Theorem [6] and the calculations in the proof of Lemma [13], we have

\[
\mathbb{P}_{H_0} \left( \sum_{i=1}^n A(Z_i) \geq \tau \right) = Q \left( \frac{\tau}{\sqrt{n}Q(1)} \right) + O \left( \frac{1}{\sqrt{n}} \right).
\]

(304)

Thus, if we choose \( \tau = \sqrt{n} \chi_2(Q_1||Q_0)Q^{-1} \left( \alpha + O \left( n^{-\frac{1}{2}} \right) \right) \), the false alarm probability would be less than or equal to \( \alpha \). Hence, by definition of \( \beta_\alpha \left( Q_0^n, \hat{P}_Z \right) \), we get

\[
\beta_\alpha \left( Q_0^n, \hat{P}_Z \right) \leq \mathbb{P}_{H_1} \left( \sum_{i=1}^n A(Z_i) \leq \tau \right)
\]

(305)

\[
= \sum_{i=1}^M \frac{1}{M} \mathbb{P} \left( \sum_{i=1}^n A(Z_i) \leq \tau | X = x_i \right)
\]

(306)

\[
= \sum_{i=1}^M \frac{1}{M} \left( Q \left( \frac{-\tau + \chi_2(Q_1||Q_0)\text{wt}(x_i)}{\sqrt{n}\chi_2(Q_1||Q_0) + \text{wt}(x_i)(\sigma_1^2 - \sigma_0^2)} \right) + O \left( n^{-\frac{1}{2}} \right) \right).
\]

(307)

with

\[
\sigma_0^2 \triangleq \text{Var}_{Q_0}(A(Z)) \quad \text{and} \quad \sigma_1^2 \triangleq \text{Var}_{Q_1}(A(Z)).
\]

(308)

Plugging in the value of \( \tau \), we obtain

\[
\beta_\alpha \left( Q_0^n, \hat{P}_Z \right) \leq \sum_{i=1}^M \frac{1}{M} \left( Q \left( \frac{-\tau + \chi_2(Q_1||Q_0)\text{wt}(x_i)}{\sqrt{n}\chi_2(Q_1||Q_0) + \text{wt}(x_i)(\sigma_1^2 - \sigma_0^2)} \right) + O \left( n^{-\frac{1}{2}} \right) \right)
\]

(309)

\[
= \sum_{i=1}^M \frac{1}{M} \left( Q \left( \frac{-Q^{-1}(\alpha)}{\sqrt{1 + \frac{\text{wt}(x_i)}{n}\chi_2(Q_1||Q_0)}} + \frac{\chi_2(Q_1||Q_0)\text{wt}(x_i)}{\sqrt{n}\chi_2(Q_1||Q_0) + \text{wt}(x_i)(\sigma_1^2 - \sigma_0^2)} \right) + O \left( n^{-\frac{1}{2}} \right) \right).
\]

(310)

Analogous to the proof of Lemma [13] we obtain

\[
\beta_\alpha \left( Q_0^n, \hat{P}_Z \right) \leq Q \left( \frac{w_{\min} \sqrt{\chi_2(Q_1||Q_0)}}{\sqrt{n}} - Q^{-1}(\alpha) \right) + O \left( n^{-\frac{1}{2}} \right) + O \left( \frac{w_{\min}^2}{n^2} \right).
\]

(311)

Next, we develop an upper bound on the weight of codewords as a function of channel characteristics and covertseness measured in terms of probability of missed detection.

**Lemma 17.** Let \((\mathcal{X}, W_{Y|X}, W_{Z|X}, Y, Z)\) be a binary-input covert communication channel and \( \mathcal{C} \) be an \((M, K, n, \epsilon, \delta, \alpha)_\beta\) code. For all \( \gamma \in [0, 1] \), we can choose a subset of codewords \( \mathcal{D} \) such that \( |\mathcal{D}| \geq \gamma MK \) and

\[
\frac{1}{\sqrt{n}} \max \{ \text{wt}(x) : x \in \mathcal{D} \} \leq \frac{1}{\sqrt{\chi_2(Q_1||Q_0)}} \left( Q^{-1} \left( 1 - \alpha - \delta - \frac{C}{\sqrt{n}} - \gamma \right) + Q^{-1}(\alpha) \right),
\]

(312)
where \( C \) is a constant that depends just on the channel.

Proof: If \( A \triangleq \frac{1}{\sqrt{\chi_2(Q_1||Q_0)}} \left( Q^{-1} \left( 1 - \alpha - \delta - \frac{C}{\sqrt{n}} - \gamma \right) + Q^{-1}(\alpha) \right) \) for \( C_2 > 0 \) specified later, we define

\[
\mathcal{D} \triangleq \{ x \in \mathcal{C} : \text{wt}(x) \leq A\sqrt{n} \}. 
\]

Obviously, \( \mathcal{D} \) satisfies (312), and we just need to check \( |\mathcal{D}| \geq \gamma MK \). To do so, let \( \hat{P}_1 \) and \( \hat{P}_2 \) be the induced output distributions for codes \( \mathcal{D} \) and \( \mathcal{C} \setminus \mathcal{D} \), respectively. Using Lemma 16 for \( \mathcal{C} \setminus \mathcal{D} \), we get

\[
\beta_\alpha \left( Q_0^{\text{in}}, \hat{P}_2 \right) \leq Q \left( A\sqrt{\chi_2(Q_1||Q_0)} - Q^{-1}(\alpha) \right) + \frac{B(1 + A^2)}{\sqrt{n}} \tag{314}
\]

\[
\leq 1 - \alpha - \delta - \gamma, \tag{315}
\]

where (a) follows from the definition of \( A \) and by choosing

\[
C_2 > \frac{1}{2} B \left( 1 + \left( \frac{1}{\sqrt{\chi_2(Q_1||Q_0)}} Q^{-1}(1 - \alpha - \delta - \gamma) \right)^2 \right). \tag{316}
\]

This means that there exists \( T \subset \mathbb{Z}^n \) such that \( Q_0^{\text{in}}(T) \leq \alpha \) and \( \hat{P}_2(T) \leq 1 - \alpha - \delta - \gamma \). Accordingly, we have

\[
1 - \alpha - \delta \leq \beta_\alpha \left( Q_0^{\text{in}}, \hat{P}_2 \right) \tag{317}
\]

\[
\leq \hat{P}_2(T) \tag{318}
\]

\[
= \frac{|\mathcal{D}|}{MK} \hat{P}_1(T) + (1 - \frac{|\mathcal{D}|}{MK}) \hat{P}_2(T) \tag{319}
\]

\[
\leq |\mathcal{D}| + 1 - \alpha - \delta - \gamma, \tag{320}
\]

where (a) follows from the definition of an \((M, K, n, \epsilon, \delta, \alpha)_\beta \) code, and (b) follows from \( \hat{P}_2 = \frac{|\mathcal{D}|}{MK} \hat{P}_1 + (1 - \frac{|\mathcal{D}|}{MK}) \hat{P}_2 \). Simplifying the above inequality, we get \( |\mathcal{D}| \geq \gamma MK \). ■

Proof of Theorem 3 - Converse: In Lemma 17, if we set \( \gamma = \frac{1}{\sqrt{n}} \), for any \((M, K, n, \epsilon, \delta, \alpha)_\beta \) code, there exists a subset of codewords \( \mathcal{D} \) with \( |\mathcal{D}| \geq MK/\sqrt{n} \) and

\[
\max_{x \in \mathcal{D}} \frac{\text{wt}(x)}{\sqrt{n}} \leq \frac{1}{\sqrt{\chi_2(Q_1||Q_0)}} \left( Q^{-1} \left( 1 - \alpha - \delta - \frac{C}{\sqrt{n}} - \gamma \right) + Q^{-1}(\alpha) \right) \tag{321}
\]

\[
= \frac{1}{\sqrt{\chi_2(Q_1||Q_0)}} \left( Q^{-1}(1 - \alpha - \delta) + Q^{-1}(\alpha) \right) + O \left( \frac{1}{\sqrt{n}} \right). \tag{322}
\]

Therefore, by Theorem 7 we have

\[
\log M_\beta(n, \epsilon, \delta, \alpha) \leq \frac{1}{\sqrt{\chi_2(Q_1||Q_0)}} \left( Q^{-1}(1 - \alpha - \delta) + Q^{-1}(\alpha) \right) D_P n^{\frac{1}{2}} - \frac{(Q^{-1}(1 - \alpha - \delta) + Q^{-1}(\alpha))}{\sqrt{\chi_2(Q_1||Q_0)}} V_P Q^{-1}(\epsilon) n^{\frac{1}{2}} + O(\log n). \tag{323}
\]

V. STRONG CONVERSE FOR PROBABILITY OF ERROR

In this section, we briefly discuss how the second-order results can be exploited to obtain a strong converse for the probability of error in covert communication.

Theorem 8. Let \( d \) be a quasi-metric and \((X, W_{Y|X}, W_{Z|X}, Y, Z)\) be a covert communication channel. We assume that for a function \( g : \mathbb{R} \to \mathbb{R} \) and \( \delta > 0 \), the conditions in Theorem 7 are satisfied. Then, for \( \delta > 0 \) and a sequence of codes \( \{C_n\}_{n \geq 1} \), if \( C_n \) is an \((M_n, K_n, n, \epsilon_n, \delta)_{D_P}\) code with

\[
\liminf_{n \to \infty} \frac{1}{\sqrt{n}} \log M_n > g(\delta) D_P, \tag{324}
\]

then \( \lim_{n \to \infty} \epsilon_n = 1 \). ■
Proof: If \( \lim_{n \to \infty} \epsilon_n \) does not exist or is less than 1, there exists a sub-sequence \( \{ C_{n_k} \}_{k=1}^{\infty} \) such that \( \sup_k \epsilon_{n_k} \leq \epsilon < 1 \). Each \( C_{n_k} \) is an \((M_{n_k}, K_{n_k}, n_k, \epsilon, \delta)_d\) code. Furthermore, (324) yields that
\[
\lim \inf_{k \to \infty} \frac{1}{\sqrt{n_k}} \log M_{n_k} > g(\delta)D_p.
\] (325)
By Theorem [7] we know that
\[
\log M^*_d(n, \epsilon, \delta) \leq g(\delta)D_p n^{\frac{1}{2}} + O\left(n^{\frac{1}{2}}\right),
\] (326)
which means that
\[
\lim \sup_{n \to \infty} \frac{1}{\sqrt{n}} \log M^*_d(n, \epsilon, \delta) \leq g(\delta)D_p,
\] (327)
which contradicts with (325).
\[\square\]

Note that the form of the first-order asymptotics developed earlier also shows that there is no strong converse for the covertness metric. If the amount \( \delta \) of covertness is relaxed, more covert bits can be sent through the channel. This is unlike other situations in information-theoretic security, for which a strong converse seems to hold fairly generally for both secrecy and reliability [21], [10].

VI. CONCLUSION

We have developed the tools to study covert communication when covertness is measured with several “quasi-metrics,” as defined in Section [3]. As discussed in Section [V], in the absence of strong converse for the covertness metric, it is legitimate to ask which metric would make most sense from an operational perspective. While relative entropy is amenable to a fairly extensive information-theoretic analysis, as illustrated by our complete characterization of second-order asymptotics in Theorem [1], variational distance and probability of missed detection are probably more adequate since they are used in the operation of an adversary attacking to detect communication. Since measuring covertness in terms of variational distance does not impose any constraint on where the adversary operates on its ROC curve, we believe that variational distance is perhaps the metric of choice for covertness.

Our results are presently limited to binary-input DMCs, but extensions to arbitrary finite input alphabets do not present major difficulties by following the approach of [23], however several research questions remain open. We have not characterized the exact second-order asymptotics of covert communication with a variational distance and probability of missed detection metrics. We have also not characterized the second-order asymptotics for the number of key bits required. A close inspection of our current proof technique shows that we explicitly rely on a law of large numbers to analyze the number of key bits, which one would have to circumvent.

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