MORAL HAZARD, DYNAMIC INCENTIVES, AND AMBIGUOUS PERCEPTIONS

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Abstract. This paper considers dynamic moral hazard settings, in which the consequences of the agent’s actions are not precisely understood. In a new continuous-time moral hazard model with drift ambiguity, the agent’s unobservable action translates to drift set that describe the evolution of output. The agent and the principal have imprecise information about the technology, and both seek robust performance from a contract in relation to their respective worst-case scenarios. We show that the optimal long-term contract aligns the parties’ pessimistic expectations and broadly features compressing of the high-powered incentives. Methodologically, we provide a tractable way to formulate and characterize optimal long-run contracts with drift ambiguity. Substantively, our results provide some insights into the formal link between robustness and simplicity of dynamic contracts, in particular high-powered incentives become less effective in the presence of ambiguity.

Key words and phrases: Dynamic moral hazard, ambiguity, robustness, continuous-time methods.

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1. Introduction

This paper studies dynamic agency problems, in which the worker’s action translates to outcomes in ways that are not precisely understood. These situations are common, especially for white collar employees in large organizations, constituting a large portion of jobs people work in. Lack of specific responsibilities on the part of these workers for either generating sales or the overall performance of a major organizational unit, such as a division or the entire firm, limits the availability of appropriate performance measures sufficient to support exclusive reliance on explicit financial incentives. Lack of apparent sensitivity of compensation to performance is a common occurrence. In such settings, it is plausible to imagine that the parties understand the technological possibilities only to some degree, but not precisely. What kind of a contract would the principal offer?

The main contribution of this paper is twofold. Formally, it gives a justification for incentive schemes that do not vary sensitively with performance in an environment where the parties’ ambiguous perception of consequences of the actions lead them to seek robustness as in worst-case guaranteed value. In the process, the paper also provides new insights into a more general question about incentive contracting, namely, what happens to “high-powered incentives” as the information available about technology becomes more imprecise in comparison to standard Bayesian models? Our results suggest that broadly such incentives lose their impact.

In the model, the risk-neutral principal and the risk-averse agent engage in an agency relationship. They both know the common set of actions available, but for any action, the consequences are ambiguous. Namely, consequences are perceived imprecisely as drift ambiguity, or sets of distributions over output. The special case where sets collapse to singletons reduce to Bayesian formulations of dynamic moral hazard problems with precise information about technology, for instance in Sannikov (2008)’s classic framework where each action’s consequences are precisely understood as a unique probability distribution over outcomes. The novel modeling approach in the current paper extends the dynamic moral hazard framework by allowing for imprecise information about technology, where each action induces a set of distributions, modeled with drift ambiguity a la Chen and Epstein (2002). The parties have imprecise understanding of technology, and perceive drift-ambiguity associated with each action.

The key challenge is as follows: for any given contract, due to imprecise information, the principal and the agent can differ in their perceptions of worst-case scenarios. This divergence in worst-case perceptions implies non-separability between the agency cost of implementing an action and its profitability, unlike the classic moral hazard problem (for instance in Grossman and Hart (1983)). It is difficult to analyze the optimal contract as a result (Mukerji and Tallon, 2004, pg.290). In particular, in the analysis of dynamic moral hazard problems recursive structures in the classic Bayesian mould as in Sannikov (2008) are not applicable. By systematically incorporating drift ambiguity a la Chen and Epstein (2002)

\footnote{See, for instance, Holmström (2017), especially Section II.C.}
in a dynamic moral hazard model the current paper presents a tractable approach to the problem of ambiguous perceptions in the agency relationship and characterizes the optimal contract in two steps.

In the first-step, for any arbitrary contract the agent’s value is represented recursively using his continuation value \( W \) as a state variable. Analogous to Sannikov (2008) it yields a quantity \( Y \) that represents the sensitivity of the agent’s future payments to performance uncertainty. As a novel element, concern for imprecision about technology, say with strength \( \kappa \) that parameterizes the size of the drift set, adds one additive term \( \kappa Y \) relative to the agent’s value representation in a Bayesian model with a unique prior. This added term \( \kappa Y \) discounts the agent’s expected value according to his worst-case scenario due to ambiguity aversion. Since the sensitivity \( Y \) reflects the expected continuation value, the presence of ambiguity induces a preference for the certainty of payments today over the expected payments in the uncertain future. Back-loading payments that are effective in providing incentives in the Bayesian model entails larger agency costs of implementation when there is drift-ambiguity.

In the second step, we provide a recursive representation for the principal’s contracting problem which facilitates the characterization of the optimal contract. Assuming stationary drift-sets, the principal’s problem subject to the agent’s incentive constraints reduces to an optimal stochastic control problem with a single variable, the agent’s continuation value \( W \). The approach in this paper heuristically sets up a recursive functional equation, Hamiltonian-Jacobian-Bellman-Isaacson (HJBI), which extends the familiar HJB formulation of the contracting problem and incorporates drift ambiguity. A novel result establishes that such a heuristic description formally represents the contracting problem via an extension of verification theorem appropriate for drift ambiguity.

With the formulation of non-probabilistic beliefs, and robustness concerns of the parties, the analysis shows that presence of imprecision compresses the incentives in the optimal long-run contract. The driving force behind our results is the observation that imprecision in technology reduces the expectations and also contributes to the agency costs of long-run incentives, and the principal’s optimal contract design features lower incentives as compared to the Bayesian formulation. Formalization of this results flows from a novel application of monotone comparative statics inspired by Quah and Strulovici (2013)’s analysis of optimal control decisions. Complementing the result on lack of incentive sensitivities, consumption profile is smooth under ambiguity as the contract is less sensitive to performance. Furthermore, effort profile and wage scheme are compressed.

The analysis of the HJBI shows that during the employment relationship the principal and the agent align their worst-cases as the lower envelopes of drift sets. Agreement on the worst-case is an endogenous property of the optimal contract, which takes into account the values outside the employment. Given that outside the employment there is no ambiguity, under the lower envelopes as the common worst-case scenarios, the optimal contract becomes observationally equivalent to the model with the lower-envelope as the specification of technology.
Related Literature. Methodologically, this paper offers a flexible framework to formulate and characterize optimal long-run contracts in dynamic moral hazard relationships in continuous-time by extending the classical Bayesian formulation as in [Sannikov (2008)] and incorporating drift ambiguity as in [Chen and Epstein (2002)]. Substantively, our results provide some insights into the formal link between robustness and dynamic contracts. These qualitative features of the optimal contract flow from extending and applying [Quah and Strulovici (2013)]’s analysis of the time-preferences on optimal control decisions in an appropriate manner for the contracting problem analyzed in the current paper. Specific relations to these contributions have been already noted.

The main theoretical development in this paper derives a recursive characterization of the optimal dynamic contract problem with drift ambiguity. Earlier contributions in dynamic agency problems fruitfully utilize formulations based on precise Bayesian information in various moral hazard settings.

A few papers have looked at what happens when the agent’s action is ambiguously perceived. The closest related paper is [Miao and Rivera (2013)] who also consider Sannikov-style model of moral hazard with ambiguity about the drift. In their model of one-sided ambiguity, the principal is ambiguity averse, while the agent is ambiguity neutral, that is, the agent evaluates the contract under the ‘true’ reference probability measure. In the current paper, both the principal and the agent are ambiguity averse and, potentially, evaluate the contract under different worst-case measures. Moreover, in the current setting the set of measures may depend on the actions that the agent takes, which is not possible in Miao and Rivera’s model. Therefore, in their model ambiguity does not affect incentive provision. In the current paper the main focus on the interaction of imprecise information about technology and the structure of incentives is best reflected in a model with two-sided ambiguity.

Work by [Szydlowski (2012)] introduced ambiguity into a dynamic contracting problem in continuous time. In that model, the principal is ambiguous about the agent’s effort cost. The formulation in the current paper instead models information imprecision in the decision-theoretic theme of ambiguity, in particular using drift ambiguity. [Szydlowski] shows that the agent receives excessively strong incentives in a dynamic contracting setting when the principal is ambiguous about the agent’s cost of effort. [Wu et al. (2017)] extend Holmstrom and Milgrom (1987)’s model by introducing belief distortions à la Sargent and Hansen. In such a model they find increase in pay-for-performance sensitivity and shed light on compensations schemes that reward luck. Complementing those contributions, the results here show that broadly high-powered incentives become less effective when there is drift-ambiguity.

Compressed wages arise in the literature in different model settings. In a relational contract setting, [MacLeod (2003)] relaxes the assumption of common knowledge of output between the parties. [MacLeod (2003)] shows that resulting disagreement is best resolved by flattening the wage profile in the optimal long term contract. [Fuchs (2007)] shows that such a compressing

[2] The prominent examples of this approach includes: Cvitanic et al. (2009), Williams (2009), Williams (2011), DeMarzo and Sannikov (2013), Kapicka (2013), Garrett and Pavan (2013) and Sannikov (2014).
of wages is robust to allowing the principal’s subjective evaluation to be her private information. Relative to these contributions, the current paper provides a complementary rationale for compressed wages in an environment where output is contractible but the parties have imprecise beliefs about technology.

Our paper fits more broadly in a small literature on moral hazard under ambiguity in a static model. A desire for robust contracts often leads to the use of simple contracts in this literature, including contributions [Harwicz and Shapiro (1978), and more recently, Lopomo et al. (2011), Chassang (2013), Antic (2014), Garrett (2014), Carroll (2015), and Dumav and Khan (2017, 2018)]. These papers provide foundations for contracts in simple forms, e.g. linear (Carroll (2015), Dumav and Khan (2017)) or step functions (Lopomo et al. (2011)). Robustness of high powered non-linear incentives have been analyzed in specific applications by Ghirardato (1994) and Mukerji (1998). The question under what conditions on imprecision about the actions rationalizes simple forms of contracts in dynamic setting is left to future research.

2. A MODEL OF DYNAMIC MORAL HAZARD WITH DRIFT AMBIGUITY

Consider an agency relation taking place in continuous time. The agent’s unobservable action translates to a flow of outputs in a stochastic manner according to drift ambiguity. In particular, for any action \( a_t \) its drift set \( \Theta(a_t) \) describes the set of possible drifts for this action, or expected increments in output diffusion. Given a sequence of actions \( (a_t)_{t \leq t} \) up to time \( t \) the total output \( X_t \) evolves according to a family of diffusion processes whose members are:

\[
dX_t = (a_t + \theta_t)dt + \sigma dB_t, \tag{2.1}
\]

where the expected increment \( \theta_t \) belongs to the drift set \( \Theta(a_t) \), and the noise process \( \{B_t\} \) is a Brownian motion adapted to the filtration \( \mathcal{F} \) on a standard filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}) = \{\mathcal{F}_t; 0 \leq t < \infty\}\). We model the drift sets as intervals centered at zero: for any \( a \in A \), the drift set of the action is \( \Theta(a) = [-\kappa(a), \kappa(a)] \) where \( \kappa(a) \) describes the size of the drift set of the action \( a \). For simplicity, we assume that \( 0 < \kappa(a) < a \) and interpret \( \kappa \) as strength of ambiguity. Indeed, taking \( \kappa(a) = 0 \) for all \( a \in A \) restricts the drifts sets to be singletons collapses into the precise information as in the classical model of dynamic moral hazard studied by Sannikov (2008). Unlike in Sannikov (2008)’s model, in the present setting, the agent’s unobservable actions controls the drift sets. The productivity \( a_t + \theta_t \) of the action \( a_t \) is therefore not precisely known; rather, it can be any one of the elements in the set \( a_t + \Theta(a_t) \). An interpretation of the specification of technology with drift-ambiguity is that the parties to the contract are aware of the possibility that they have erroneous beliefs about the true drift for actions and seek robust performance.

We assume that the drift sets \( \Theta(a_t) = [-\kappa(a_t), \kappa(a_t)] \) are stationary and independent (using terminology introduced by Chen and Epstein (2002)). The drift sets are stationary as they do not depend on time \( t \) beyond the action \( a_t \) at that time, and independent of history of output realizations up to that time.
In order to specify the contracting problem classically in terms of expected utilities, following Chen and Epstein (2002) we translate drift sets into sets of distributions over outputs. To illustrate this translation, for simplicity consider a given finite horizon $T$. For an action profile $a = \{a_t\}$, we consider, with a slight abuse of notation, a drift process $\theta := \{\theta_t\}$ such that $\theta_t \in \Theta(a_t)$. Using the drift process we define a process of Girsanov exponents as follows

$$z_t^{a+\theta} := \exp \left\{ -\frac{1}{2} \int_0^t (a_s + \theta_s)^2 ds - \int_0^t (a_s + \theta_s) dB_s \right\}, \quad 0 \leq t \leq T,$$

where $\theta_t \in \Theta(a_t) = [-\kappa(a_t), \kappa(a_t)]$. The Girsanov exponent $\{z_t^{a+\theta}\}$ generate a probability measure $P^{a+\theta}$ on the measurable space of sample paths $(\Omega, \mathcal{F})$ and the Radon-Nikodym derivative of this measure with respect to the reference measure $P$ is given by

$$\frac{dP^{a+\theta}}{dP} \bigg|_{\mathcal{F}_t} = z_t^{a+\theta}, \quad 0 \leq t \leq T. \quad (2.3)$$

For the action $a = \{a_t\}$ and the drift process $\{\theta_t\}$ such that $\theta_t \in \Theta(a_t)$ we have therefore constructed a measure $P^{a+\theta}$ equivalent to $P$ via a change of measure. Now, taking the collection of all such measures $P^{a+\theta}$ yields the set of distributions for the action $a = \{a_t\}$ and it is given by:

$$P^{\Theta(a)} := \{P^{a+\theta} : \theta = \{\theta_t \in \Theta(a_t), 0 \leq t \leq T\} \text{ and } P^{a+\theta} \text{ is defined by } (2.3)\} \quad (2.4)$$

This construction of the set of distributions follows the formulation in Chen and Epstein (2002) in a single-agent decision setting and allows the drift sets to depend on the actions (In Appendix, we provide the technical details on the construction of the drift sets and the sets of distributions). We have therefore represented drift sets as sets of probabilities over outputs, as in the classic theme of ambiguity.

We next study the benchmark model where the agent and the principal have the symmetric perception of imprecision: they both perceive $\Theta(a)$ to be the possible set of drifts associated to the action $a$. The assumption of common symmetrically drift-ambiguity perceptions allows for tractability and heterogenous perceptions are treated afterwards as an extension. Given ambiguity perceptions, we assume MaxMin criteria for ambiguity-sensitive preferences. Next we specify the contracting problem in this environment with drift-ambiguity.

2.1. Contracting problem with drift ambiguity. The principal’s contract to the agent specifies a stream of wage payments $\{c_t\}$ and an incentive-compatible recommendation of action $\{a_t\}$. The action $a := \{a_t \in A, 0 \leq t < \infty\}$ is a measurable function with respect to the filtration $\mathcal{F}$ generated by the standard Brownian motion and the set of possible effort levels $A$ is a compact subset of $R_+$ with zero as the smallest element. The agent’s disutility from effort level $a \in A$ is given by $h(a)$ measured in terms of utility of consumption, and the function $h$ is continuous, increasing and convex. Furthermore, we assume that $h(0) = 0$ as a normalization.
The principal’s contract specifies non-negative flow wage payments \( c := \{c_t; 0 \leq t < \infty\} \) measurable with respect to the filtration \( \mathcal{F} \). The principal can commit to any such long-term contract. Regarding the agent’s preferences, we assume that his utility function over monetary consequences \( u \) is bounded from below and satisfies a normalization \( u(0) = 0 \). Moreover, we assume that the utility function \( u : [0, \infty) \rightarrow [0, \infty) \) is an increasing, strictly concave, and \( C^2 \) function that satisfies \( u'(c) \rightarrow 0 \) as \( c \rightarrow \infty \). These assumptions on the primitives of the economic environment are adopted following the framework of Sannikov.

For a given contract \((c, a)\) the parties evaluate the contract according to ambiguity sensitive preferences. In particular, we assume that the parties’ ambiguity sensitive preferences are represented by MaxMin Expected Utility and hence the parties evaluate the contract according to their respective worst-case scenarios for drifts. For simplicity we assume that the principal and the agent use a common discount rate \( r > 0 \) on the flow of profit and utility. From the contract \((c, a)\) the principal’s ex-ante guaranteed value is therefore computed as follows

\[
\min_{\theta \in \Theta(a)} E^{a+\theta} \left[ r \int_0^\infty e^{-rt} (dX_t - c_t dt) \right] = \min_{\theta \in \Theta(a)} E^{a+\theta} \left[ r \int_0^\infty e^{-rt} (a_t + \theta_t - c_t) dt \right]
\]

and the agent obtains the ex-ante guaranteed value

\[
\min_{\theta \in \Theta(a)} E^{a+\theta} \left[ r \int_0^\infty e^{-rt} (u(c_t) - h(a_t)) dt \right]
\]

where the output process \( dX_t \) evolves according to (2.1). Here we abuse the notation slightly, and denote by \( \Theta(a) \) the drift-sets associated to the action \( a = \{a_t\} \), where a representative element of the set \( \Theta(a) \) is given by \( \theta = \{\theta_t\} \) such that \( \theta_t \in \Theta(a_t) \).

Following the classical definition of incentive-compatibility, in the current environment with drift-ambiguity we say that in the contract \((c, a)\) the action recommendation \( a \) is incentive-compatible if the action \( a \) maximizes the agent’s guaranteed value under the payment scheme \( c \). The payment scheme \( c \) is incentive-compatible for the action \( a \) as from any alternative action \( \hat{a} \neq a \) the agent obtains weakly smaller guaranteed value than the action \( a \)

\[
\min_{\theta \in \Theta(a)} E^{a+\theta} \left[ r \int_0^\infty e^{-rt} (u(c_t) - h(a_t)) dt \right] \geq \min_{\theta \in \Theta(\hat{a})} E^{\hat{a}+\theta} \left[ r \int_0^\infty e^{-rt} (u(c_t) - h(\hat{a}_t)) dt \right]
\]

(2.5)

Letting \( \mathcal{A}_t \) denote the set of incentive-compatible payment and recommended action pairs \((c, a)\), in this setting with drift ambiguity, the principal’s incentive-compatible contract offer maximizes his expected profit under his worst-case criterion

\[
\max_{(c, a) \in \mathcal{A}_t} \min_{\theta \in \Theta(a)} E^{a+\theta} \left[ r \int_0^\infty e^{-rt} (a_t + \theta_t - c_t) dt \right] \quad \text{(P)}
\]

subject to ensuring the agent ex-ante value of at least \( \hat{W} \geq 0 \)

\[
\min_{\theta \in \Theta(a)} E^{a+\theta} \left[ r \int_0^\infty e^{-rt} (u(c_t) - h(a_t)) dt \right] \geq \hat{W} \quad \text{(PK)}
\]
We assume and normalize to zero the value of the agent’s outside option. As the wages are non-negative and the agent can always choose zero effort, in the contracting problem \((P)\) the principal considers only the incentive-compatible contracts \((c, a)\) that ensure the agent’s participation: \((PK)\) holds for \(\hat{W} = 0\) in \((PK)\). Moreover, as the principal always has the option of not hiring the agent, without loss we focus on the contracting relationship that yields non-negative guaranteed payoff for the principal. We refer to the problem in \((P)\) as the sequential contracting problem.

In the contracting problem with drift ambiguity, even though the principal and the agent has symmetric perception of drift ambiguity, \(\Theta(a)\) for any action \(a\), their worst-case scenarios can differ as the parties have different objectives.

In order to analyze how drift ambiguity affects the contracting relationship, we introduce a parameterization on the size of drift sets to formalize different degrees of ambiguity. Intuitively, larger drift sets reflect higher degree of ambiguity. In particular, we introduce a parameter \(\phi\) and denote by \(\Theta(a; \phi)\) the drift set for the action under the parameter \(\phi\). The drift sets under the technology \(\phi\) is given by \(\Theta(a; \phi) = a + [-\kappa(a; \phi), \kappa(a; \phi)]\) for \(a \in \mathcal{A}\). As the function \(\kappa\) measures the size of the drift sets, we interpret it as a measure of ambiguity. In particular, we use this parameterization to rank drift ambiguity in different economic environments, and say that the degree of ambiguity is stronger if the drift sets are larger. Formally, we define this relation as follows.

**Definition 1.** Technology characterized by \(\hat{\phi}\) is more ambiguous than \(\phi\), written as \(\hat{\phi} \succ_{\kappa} \phi\) if for all actions \(a \in \mathcal{A}\), the drift sets satisfy \(\kappa(a; \phi) \leq \kappa(a; \hat{\phi})\).

This relation provides a partial order over the drift sets as the set containment in the definition may not hold uniformly over all actions. With this indexing we introduce a list of the key assumptions on ambiguity about technology that streamline comparative static analysis. These assumptions on the mapping \(\kappa(a; \phi)\) that determines strength of ambiguity are:

(A.1) Viability of ‘zero’ actions: \(\kappa(0; \phi) = 0\);
(A.2) Better pessimistic expectations from higher effort: \(\kappa_{a}(a; \phi) < 1\);
(A.3) Non-trivial role for drift ambiguity: the size of the drift set \(\kappa(a; \phi)\) is not too concave in effort, i.e., \(-\kappa_{aa} \frac{a}{1-\kappa_{a}} \leq 1\);
(A.4) Complementarity between effort and ambiguity: \(\kappa_{\phi a} > 0\).

**Discussion of Assumptions.** Assumption [A.1] ensures that the set of admissible actions \(\mathcal{A}_{R}\), include those actions that generate positive surplus according to the worst-case scenario, i.e., lower-envelope of the drift set. We call the set of such actions \(\mathcal{A}_{R}\) as viable.

Assumption [A.2] is a monotonicity condition describing that higher actions yield better pessimistic expectations, more precisely higher lower-envelopes of the drift sets. This assumption implies that increasing effort leads to the drift sets that have higher lower envelopes.
Assumption (A.3) ensures that the agent’s optimization problem (below) for any arbitrary viable contract is a concave problem and the first-order conditions are sufficient to characterize the agent’s decision rule for his action choice.

Single-crossing condition (A.4) in the current context says that as the strength of ambiguity increases in order to maintain the expected output effort must increase. This is analogous to the single-crossing property commonly used in models of contracting in private information environments. As in those model, “better” type of the agent, here corresponding to lower ambiguity technology, is also marginally better for all margins.

A minimalist example for drift-ambiguity consistent with the assumptions (A.1)-(A.4) is the linear model: \( \kappa(a; \phi) = \phi a \) with \( \phi \leq 1 \). This reflects technology where ambiguity increase in effort, arguably a worker exerting higher effort confronts more complex tasks that are harder to measure. This class of examples is examined in detail below as well as the complementary specification where higher actions reduce drift ambiguity.

**Admissible Contracts.** The description of a contract \((c_t, a_t)\) as a pair of wage and action recommendation allows for two kinds of retirement clauses. First, it can involve a retirement clause, also referred to as a ‘golden parachute,’ which specifies a stopping-time \( \tau \) such that the agent receives some constant payment \( c_t = \xi > 0 \) and does not exert effort \( a_t = 0 \) for \( t \geq \tau \) if this stopping-time arrives. For a second kind of retirement, the contract can specify a termination clause in which the principal effectively fires the agent. Such a termination can again be represented with a stopping time \( \tau \) such that \( a_t = 0 \) and \( c_t = 0 \) for \( t \geq \tau \). In either case of retirement with a lump-sum payment \( \xi \geq 0 \) at the stopping-time \( \tau \) measurable with respect to \( \mathcal{F} \) can be equivalently expressed as flow payment \( \xi \) after \( \tau \) using the observation that \( \xi = \int_\tau^\infty re^{-rt} \xi dt \).

To ensure that the parties’ payoffs in the contracting problem (P) are well-defined, we assume that the set of admissible contracts satisfy the integrability conditions. For simplicity, we also assume that the set of admissible contracts has a finite random horizon. As the action set \( \mathcal{A} \) is compact and for any action \( a \in \mathcal{A} \) the drift-set \( \Theta(a) \) is a compact set, it is without
loss of generality to assume that the set of admissible contracts \((c, a) = (c_t, a_t)\) satisfy the following square integrability conditions:

\[
\mathbb{E}\left( (e^{-r\tau} \xi)^2 + \int_0^\tau (e^{-rt} c_t)^2 \, dt \right) < \infty
\] (2.6)

and the random horizon \(\tau\) is finite

\[
\lim_{n \to \infty} P[\tau \geq n] = 0
\] (2.7)

We denote by \(\mathcal{Z}(W_0)\) the set of admissible contracts \((c, a)\) that satisfy the integrability conditions (2.6) and (2.7) and provide the agent with the ex-ante value \(W_0 \geq 0\).

We remark the assumption that in an admissible contract \((c_t, a_t)\) the payment scheme \(\{c_t\}\) and incentive compatible action \(\{a_t\}\) are measurable with respect to the filtration \(\mathcal{F}\) generated by the standard Brownian motion \(B\) does not entail loss of generality. Indeed, given that the payment scheme \(\{c_t\}\) is measurable with respect to the filtration \(\mathcal{F}\), the incentive-compatible action \(\{a_t\}\) determined by (2.5) is also a measurable function with respect to the filtration \(\mathcal{F}\). In this case, as noted by Cvitanic and Zhang (2013, Chapter 5) without loss of generality, for any finite horizon of contracting problem, an admissible contract can alternatively be specified as a function of history of output realizations, as in Samnikov (2008).

2.2. Incentive compatibility with drift-ambiguity. An action profile \(\{a_t\}\) is implementable if there is a contract with output contingent payments \(\{c_t\}\) so that the agent chooses this action profile. We use this usual definition of implementability while incorporating drift ambiguity.

For a given contract \((c, a)\), assume that the agent follows the recommended action \(a\). The agent’s continuation value \(W_t\) at time \(t\) is defined as

\[
W_t = \min_{\theta \in \Theta(a)} \mathbb{E}^{a+\theta}\left[ r \int_t^\tau e^{-rt} (u(c_t) - h(a_t)) \, dt \right]
\] (2.8)

where \(\tau\) is the random finite stopping-time associated to the contract \((c, a)\).

\[\text{As the action } \mathcal{A} \text{ is a compact subset of } \mathbb{R}_+ \text{ and for each action } a \in \mathcal{A} \text{ the drift-set } \Theta(a) \text{ is compact, the integrability conditions (2.6) and (2.7) are sufficient as they imply that the following integrability holds for any admissible contract } (c, a): \]

\[
\sup_{a \in \mathcal{A}} \sup_{\theta \in \Theta(a)} \mathbb{E}^{a+\theta}\left[ (e^{-r\tau} \xi)^2 + \int_0^\tau (e^{-rt} c_t)^2 \, dt \right] < \infty;
\]

and the random horizon \(\tau\) is finite

\[
\lim_{n \to \infty} \sup_{a \in \mathcal{A}} \sup_{\theta \in \Theta(a)} P^{a+\theta}[\tau \geq n] = 0
\]

Lin et al. (2020) uses these stronger conditions as in their more general formulation of dynamic moral hazard problem the agent’s action controls both the drift and the volatility of the output process. In the current model, the agent’s action controls on the drift of output process and hence the required integrability conditions are simpler.

\[\text{Many thanks to an anonymous referee for pointing out this clarification on the different forms of measurability requirements.}\]
To characterize implementability we start with representing as a diffusion process the agent’s value from any contract and action strategy. Following Chen and Epstein (2002)’s representation of recursive ambiguity sensitive preferences in continuous-time under the max-min criterion the agent’s payoff can be represented as a diffusion process:

**Proposition 1.** For any wage scheme \( \{c_t\} \) and any action strategy \( \{a_t\} \) with its associated set of drift terms \( a_t + \Theta(a_t; \phi) = [a_t - \kappa(a_t; \phi), a_t + \kappa(a_t; \phi)] \) there exists a progressively measurable process \( \{Y_t\} \) such that

\[
W_t = W_0 + \int_0^t r (W_s - u(c_s) + \min_{\theta_s \in \Theta(a_s; \phi)} \theta_s | Y_s |) ds + \int_0^t rY_s dB_s^a \tag{2.9}
\]

In this representation the action strategy \( \{a_t\} \) need not be optimal for the wage scheme \( \{c_t\} \) contingent on output history. In particular, it holds for the optimal action strategy.

The novel aspect of this representation due to drift-ambiguity is the effect that the agent’s aversion to ambiguity introduces one additive term and discounts his value according to his worst-case scenario in the drift set: \( \min_{\theta_t \in \Theta(a_t; \phi)} \theta_t Y_t \). In Miao and Rivera (2016)’s model of dynamic moral hazard with drift ambiguity, the agent is ambiguity neutral and hence in Miao and Rivera’s representation result ambiguity does not play a role. In our representation using MaxMin preferences to model ambiguity aversion, the effect of ambiguity disappears if the agent has precise information about the drift sets, i.e., \( \kappa = 0 \).

In any viable contracting relationship the provision of non-trivial incentives (that is needed to implement a positive action, \( a_t > 0 \)) requires a positive sensitivity process \( Y_t > 0 \). This in turn implies that the lower envelope is the worst-case scenario perceived by the agent. Since zero-contract that implements zero action at no cost to the principal is always feasible, non-trivial incentives in a viable contracting relationship requires a positive variation process. During employment the agent’s worst-case scenario for any non-trivial action \( a_t > 0 \) is then given by the drift term that minimizes the expected continuation value over the drift set: \( \arg\min_{\theta_t \in \Theta(a_t; \phi)} \theta_t Y_t = \min\{\theta \in \Theta(a_t; \phi)\} = a_t - \kappa(a_t; \phi) \) – the lower-envelope of the drift set. Denoting the agent’s worst-case scenario on drift sets by \( \theta^A(a_t; \phi) \) the following result summarizes this observation:

**Lemma 1.** For any non-trivial contract, the sensitivity \( \{Y_t\} \) process is non-negative and for any optimal action strategy \( \{a_t\} \) the agent’s worst-case scenario is the lower envelope of drift sets: \( \theta^A(a_t; \phi) = a_t - \kappa(a_t; \phi) \).

Using Lemma 1 in Proposition 1 from any viable contract \( \{c_t\} \) and optimal action strategy \( \{a_t\} \) the agent’s value as a diffusion takes the following form:

\[
W_t = W_0 + \int_0^t r (W_s - u(c_s) + h(a_s) - \kappa(a_s; \phi)Y_s) ds + \int_0^t rY_s dB_s^a \tag{2.10}
\]
An interpretation of this diffusion representation of the agent’s value up to the term involving \( \kappa \) due to ambiguity aversion is analogous to that in Sannikov (2008). As in the latter, after any history of output up to time \( t \) its drift represents the expected increment in the agent’s expected value. It involves accumulating interest coupon on the promised payments \( rW_t \), adding his effort cost \( h(a_t) \), and reducing it by utility of consumption from the payment received today \( u(c_t) \). The last term in the drift is a novel negative term added due to the drift ambiguity and aversion to ambiguity. This term reduces the expected continuation value according to the agent’s pessimistic expectations according to his worst-case scenario for the drift \( \kappa(a_t; \phi)Y_t \), which is proportional to the sensitivity of the agent’s continuation-value to output uncertainty or to the strength of ambiguity.

As Proposition 1 shows relative to the classic case (where \( \kappa = 0 \)) drift ambiguity reduces the agent’s value according to his worst-case scenario. The drift of the agent’s value is determined by the allocation of payments over time as in Sannikov (2008). All else equal, drift ambiguity \( \kappa > 0 \) and ambiguity aversion reduces the drift on the agent’s value diffusion and hence reflect stronger preference for front-loaded payments relative to the case without ambiguity. Therefore, the novel effect of ambiguity in contract design is to decrease the benefits of back-loaded payments, captured by \( \kappa(a_s; \phi)Y_s \). Since the backloaded payments are important to incentivize the agent’s actions in dynamic moral hazard problems, deferred payments may not be as effective when ambiguity is present.

To analyze the effect of ambiguity on the optimal contract we next derive incentive compatibility condition using the representation of the agent’s value (2.10) for any contract. The following result characterizes incentive compatibility in the current setting.

**Proposition 2.** For any action strategy \( a = \{a_t\} \) and wage scheme \( c = \{c_t\} \), let \( \{Y_t\} \) be the volatility process from Proposition 1. Then the action strategy \( a \) is incentive compatible if and only if

\[
\forall \tilde{a}_t \in \mathcal{A} \quad Y_t(a_t - \kappa(a_t; \phi)) - h(a_t) \geq Y_t(\tilde{a}_t - \kappa(\tilde{a}_t; \phi)) - h(\tilde{a}_t) \quad dt \otimes dP \ a.e. \quad (2.11)
\]

Notice that setting \( \kappa(a; \phi) = 0 \) in condition (2.11) for all \( a \in \mathcal{A} \) specializes it to Sannikov (2008)’s incentive compatibility condition in the classical case without drift ambiguity. Compared to the latter, drift-ambiguity introduces one additive term to the incentive compatibility comparison. These additional terms discounts the agent’s continuation value according to the worst-case scenario due to ambiguity aversion. With a tractable incentive compatibility condition as in (2.11) we next turn to analyze how ambiguity affects incentives needed to implement actions.

**On the incentive benefits of back-loaded payments under ambiguity.** The incentive compatibility condition (2.11) provides a tractable way to analyze how incentives needed to implement an action depend on ambiguity. It implies that given any non-trivial incentives \( Y > 0 \) the agent’s decision rule for his optimal action choice maximizes the agent’s expected continuation value net of effort cost (suppressing the dependence on employment history to
max(a - κ(a; ϕ)) Y - h(a)

In view of the assumptions (A.1)-(A.4), the objective function in this maximization programme is strictly concave. Therefore, the first-order condition is sufficient to characterize the optimal action and it is given by (1 - κ_a(a; ϕ)) Y - h'(a) = 0. This in turn implies that the minimum variation Y(a; ϕ) needed to implement a given action a > 0 solves the first-order condition:

\[ Y = \frac{h'(a)}{1 - κ(a; ϕ)}. \]  

(2.12)

For any non-trivial effort level a > 0 the incentives Y(a; ϕ) to implement it increases in the degree of ambiguity by the assumed complementarity (Assumption (A.4)) between effort and ambiguity, κ_aφ > 0. Moreover, for any degree of ambiguity ϕ, Assumption (A.3) and (2.12) together imply that implementing higher effort levels requires stronger incentives. These observations are summarized in the following lemma:

**Lemma 2.** Under the assumptions (A.1)-(A.4), for any non-trivial effort level a > 0, the required incentives to implement it increases in ambiguity: Y(a; ˆϕ) ≥ Y(a; ϕ) whenever ˆϕ ≻ κ ϕ. Moreover, for any given degree of ambiguity the incentives increase in effort level: Y( ˆa; ϕ) ≥ Y(a; ϕ) whenever ˆa ≥ a.

This result says that implementing an action requires stronger incentives as the strength of ambiguity increases. To illustrate in a special case, consider the κ-ignorance model where degree of ambiguity increases linearly in effort level: κ(a; ϕ) = φa with φ > 0. To implement any given action a notice from (2.12) that the principal needs to provide stronger incentives under ambiguity: Y(a; ϕ) = \(\frac{h'(a)}{1 - φ} > h'(a) = Y(a; 0)\). In other words, the incentive constraint become more difficult to satisfy under ambiguity. In particular, if the incentives Y become too high, the principal can be better off by retiring the agent by effectively setting Y = 0 and implementing zero-action a = 0.

As the incentive-compatibility condition describes the agent’s decision rule for his action choice, it depends only on the agent’s ambiguity but not on the principal’s ambiguity. We show later this observation in an extension of the model where the principal and the agent can perceive different ambiguity. In particular, if the principal does not perceive ambiguity, relative to the Sannikov (2008)’s model the agent’s ambiguity require stronger incentives to satisfy the incentive compatibility condition, because the agent’s worst-case is always the lower envelope of the drift set for any non-trivial contract. Now, if the principal perceives ambiguity, stronger incentives can make a high drift term her worst-case scenario, as a high drift term corresponds to a distribution under which a high payment is more likely. Since the principal’s worst-case scenario is endogenous, the contracting problem in the current setting
is not equivalent to a version of the Sannikov (2008)’s model by changing the drift of each action with the lower envelope of its drift set.

It is worth remarking that in Miao and Rivera (2016)’s model of dynamic moral hazard with drift ambiguity, as the agent is ambiguity neutral in their setting, the agent’s incentive compatibility condition does not depend on ambiguity. Therefore, in Miao and Rivera’s model ambiguity does not affect how difficult it is for the principal to incentivize high effort from the agent. In contrast, in the current setting the incentive compatibility condition depends on drift ambiguity perceived by the agent. As the incentive compatibility condition describes the agent’s decision rule for his action choice, it depends on ambiguity perceived by the agent but not on what ambiguity perception the principal can have.

Having obtained a tractable incentive compatibility condition for implementation we then turn to characterize the optimal contract and study how it depends on degree of ambiguity.

3. Basic Properties of Optimal Contracts

We next represent the (sequential) contracting problem recursively using the agent’s continuation value as a state variable and then characterize its properties.

For a given contract \((c, a)\), the agent’s value process is recursively represented by and its diffusion is given by

\[
dW_t = r \left( W_t - u(c_t) + h(a_t) + \theta_t^A Y_t \right) dt + r \sigma Y_t dB_t^a
\]  

Here \(\theta_t^A\) describes the agent’s worst-case scenario under the contract \((c, a)\) for the increments to output according to \(dX_t = (a_t + \theta_t^A)dt + \sigma dB_t^a\). In the contract \((c, a)\), the agent’s promised value \(W_t\) grows at the interest rate, it increases by the value of the agent’s effort \(h(a_t)\), and it is reduced by the value of the payment \(u(c_t)\). As noted, the sensitivity \(Y_t\) describes the change in the agent’s continuation value due to output uncertainty. As the principal commits to long-term contracts, and infers the agent’s worst-case scenario \(\theta_t^A\), she commits to provide the agent with the continuation value \(\{W_t\}\).

Due to aversion to ambiguity, the principal and the agent can have different perceptions about the evolution of output even if they have the same drift-sets. Letting \(\theta_t^P\) denote the principal’s worst-case scenario from the principal’s perspective the output process has the following distribution \(dX_t = (a_t + \theta_t^P)dt + \sigma dB_t^P\). Under the principal’s worst-case scenario, the agent’s promised value \(W_t\) therefore evolves according to the following diffusion:

\[
dW_t^P = r \left( W_t + h(a_t) - u(c_t) + \theta_t^P Y_t \right) dt + r \sigma Y_t dB_t^a
\]  

As \(dB_t^a\) is standard Brownian motion under the measure \(P^a\), (3.2) implies that the drift of the agent’s continuation value under the principal’s worst-case distribution, is \(r(W_t +

5 Many thanks to an anonymous referee for helping to point out this distinction.
6 Many thanks to anonymous referees for highlighting this novel effect in the current model.
h(a_t) - u(c_t) + \theta_t^2 Y_t) and its quadratic variation is \( r^2 Y^2 \sigma^2 \). By Ito’s lemma, the principal’s robust contracting problem \( P \) is represented recursively as an ordinary differential equation with the max-min criterion, which is of the Hamilton-Jacobian-Bellman-Isaac type (HJBI for short) and it is given as follows:

\[
F(W; \phi) = \max_{(c,a,Y) \in \Gamma} \min_{\theta \in \Theta(a;\phi)} \left\{ r(a + \theta - c) + rF'(W;\phi) (W - u(c) + h(a) + \theta Y) + \frac{F''(W;\phi)}{2} r^2 Y^2 \sigma^2 \right\}
\]

for any \( W > 0 \) subject to the boundary conditions:

\[
F(0; \phi) = 0, \quad \text{and} \quad F(W; \phi) \geq F_0(W).
\]

In this formulation, as in Sannikov, since the agent can always choose to exert zero effort, the representation of the agent’s continuation value implies that the agent’s participation holds with \( W > 0 \). Here the constraint set \( \Gamma \) includes non-negative wage payments \( c \geq 0 \), non-negative action recommendation \( a \geq 0 \), and non-negative sensitivity \( Y \geq 0 \) that satisfies incentive-compatibility for the action \( a \) according to the agent’s decision rule for his action choice as in Proposition 2.

In this representation, the first term of the objective function is the expected flow profits and it increases by expected output and decreases by payments to the agent. On the other hand, the second term reflects the contribution to the expected continuation profits through delaying the payments to the agent, and the last term captures the effect due to exposing the agent to payoff uncertainty. The principal’s aversion to ambiguity is reflected by pessimistic expectation formed by taking minimum over drift sets.

The function \( F_0(W) \) describes the principal’s guaranteed value after retiring the agent with the continuation value \( W \). The principal always has available the option to retire the agent with any value \( W \in [0, u(\infty)) \) where \( u(\infty) = \lim_{c \to \infty} u(c) \). Retiring the agent with value \( u(c) \), the principal pays the agent constant wage \( c \) and allows him to choose zero effort. The principal’s payoff from retiring the agent is given by \( F_0(u(c)) = -c \). As there is no ambiguity after retirement, the function \( F_0 \) does not depend on the strength of ambiguity \( \phi \). As a state variable the agent’s continuation value \( W \) evolves during the employment, the boundary condition on retirement ensures that the principal triggers retirement whenever the principal’s payoff from doing so exceeds the principal’s payoff from continuing to employ the agent.

The next result shows that the recursive equation describes a solution to the the principal’s robust contracting problem \( P \) in the sequential formulation. It provides a verification argument using dynamic programming approach and characterizes the principal’s contracting problem recursively using the agent’s continuation value as a state variable. In particular, following a standard verification argument in the analysis of continuous-time principal agent

\[\text{See Evans and Souganidis (1984) for more detailed material on the mathematical development of this class of problems.}\]
problems and adapting it to drift-ambiguity, it shows that HJB (3.3) characterizes a solution to the principal’s sequential contract design problem.

**Theorem 1.** Assume that the HJB equation (3.3) has a unique smooth solution \( F \in C^2(R_+) \) and that the stopping region takes the form \( S = \{0\} \cup [W_{gp}, \infty) \) for some finite \( W_{gp} < \infty \), then the solution \((c(W), a(W), Y(W))\) for \( W \in (0, W_{gp}) \) characterizes an optimal contract with a positive profit to the principal. Such a contract is based on the agent’s continuation value as a state variable, which starts at \( W_0 \) and evolves according to

\[
dW_t = r(W_t - u(c_t) + h(a_t) + \theta_t^A Y(a_t))dt + r\sigma Y(a_t)dB_t^a
\]

with the payments \( c_t = c(W_t) \) and the action \( a_t = a(W_t) \) with termination time \( \tau := \inf\{t \geq 0 : W_t \in S\} \).

This theorem says that the principal’s (sequential) problem \((P)\) can be alternatively specified in recursive form given in the associated HJB (3.3) using the agent’s continuation value as the summary of the payoff relevant history. Moreover, the optimal controls \((a)\) and \((c)\) to the HJB describe the optimal long-term contract.

To establish this result we use Possamaï and Touzi (2020)’s dynamic programming approach to study principal-agent problem in continuous time and adapt it drift ambiguity. In the formulation of Possamaï and Touzi (2020), the agent’s action controls the drift of the output diffusion, like in the current paper, which follows the general approach of Lin et al. (2020) allowing that the agent’s action can control both the drift and the volatility of output diffusion.

A version of Theorem 1 in a classical setting where there is no drift ambiguity is given by Sannikov (2008). In the analysis of the HJB equation (3.3), unlike Sannikov’s approach we do not require an upper-boundary constraint to determines retirement with a smooth-pasting condition \( F'(W_{gp}) = F'_0(W_{gp}) \). As illustrated by Possamaï and Touzi (2020), such a smooth-pasting constraint as in Sannikov’s approach can entail loss of generality. Instead, following Possamaï and Touzi (2020)’s dynamic programming approach and for simplicity, we model retirement with a finite random horizon for the contracting relationship. A possible generalization to infinite valued random horizon where with a positive probability the contracting relationship does not terminate with a retirement is beyond the scope of this paper. Arguably, in many economic applications, modeling with a finite random horizon does not entail loss of generality as contracting relationships typically come to an end in a finite time.

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8The formulation of Lin et al. (2020) generalize to random time-horizon the formulation of principal-agent problem in continuous-time with a finite time-horizon developed by Cvitanić et al. (2018). These contributions provide justification for reducing the principal’s contracting problem into a standard stochastic control problem as introduced by Sannikov (2008).

9Possamaï and Touzi (2020) discusses that such a generalization to infinite random horizon in a classical setting of Sannikov (2008) is possible as the dynamic contracting problem reduces to a standard stochastic control problem for which the classical tools of stochastic analysis applies (see their Remark 3.8.). By analogous reasoning, we conjecture that the formulation of dynamic contracting with drift ambiguity can be generalized to infinite random-horizon.
We also assume that the HJB equation (3.3) has a smooth solution $F \in C^2(R_+)$. In the classical setting without drift ambiguity, analogous HJB equation admits a smooth solution. We conjecture that by adopting Possamai and Touzi’s approach using classical tools of stochastic analysis, smooth solution obtains in our framework. However, such a generalization is beyond the scope of the current paper.

The analysis of HJB functional equation in (3.3) yields further basic properties of the optimal long-run contract: both the principal and the agent perceive the lower envelope as common worst-case scenario; and the principal’s value function $F$ displays monotonicity with respect to the strength of ambiguity. We next take up these properties of the optimal contract and then turn to perform monotone comparative statics analysis with respect to the strength of ambiguity.

3.1. Agreeing on the worst-case scenario. In the contracting relation, a priori the principal and the agent do not have to agree on the worst-case scenario in drift sets due to the differences in objectives. From the agent’s perspective the worst-case minimizes the expectation of payments received net of effort costs. We have already seen in Lemma 1 that non-trivial incentive provision implies that the agent’s worst-case perception is always the lower envelope of the drift set. The principal’s worst-case drift on the other hand minimizes the expectation of profits and need not be the same as the agent’s worst-case scenario.

The recursive representation (3.3) of the optimal contract provides a tractable characterization for the worst-case scenario perceived by the principal. For any incentive compatible action strategy $\{a_t\}$, the worst-case drift scenario perceived by the principal $\theta^P$ is the drift that minimizes her expected payoff over the drift sets:

$$
\theta^P(a; \phi) = \operatorname{argmin}_{\theta \in [-\kappa(a;\phi), \kappa(a;\phi)]]} \theta \{ 1 + F'(W; \phi)Y \}.
$$

Equivalently,

$$
\theta^P(a; \phi) = \begin{cases} 
-\kappa(a; \phi) & \text{if } 1 + F'(W; \phi)Y \geq 0; \\
\kappa(a; \phi) & \text{if } 1 + F'(W; \phi)Y < 0. 
\end{cases}
$$

(L)

Here the principal’s worst-case analysis involves two terms of HJB (3.3) due to drift ambiguity: the first term involving 1 reflects the contemporaneous effect on the expected output and the other involving $F'(W; \phi)Y$ captures effect on the expected continuation profits. \(^{10}\) The condition (L) says that the lower-envelope is the worst-case scenario for the principal, so long as for each euro/(dollar) wage paid today the principal expects to make positive profits from continuing to employ the agent. Otherwise, additional payments incur net losses and in this case the upper-envelope (which maximizes expected losses) is her perceived worst-case scenario. However, the latter does not arise during employment in any optimal contract since the principal always has the option of retiring the agent.

\(^{10}\) Notice also that there is no effect through the third and the last term of HJB (3.3) since there is no ambiguity about the volatility of the output diffusion (by assumption for tractability).
Early in the contracting relationship when the agent’s continuation value $W$ is sufficiently low, marginal profitability from delaying payments contingent on future performance is positive $F'(W; \phi) > 0$ as it incentivizes the agent to work. The principal’s perceived worst-case scenario for low promised-values is therefore the lower envelope as it minimizes the expectation of marginal profits. Since the profit function is concave, for high enough $W$, the marginal benefit of delaying payments becomes negative $F'(W; \phi) < 0$ as incentivizing the agent who has accumulated high continuation value becomes costlier due to the income effects. For high enough promised continuation value the principal’s payment made today exceeds its expected benefit in the continuation value, and hence her worst-case scenario is then the upper envelope which maximizes such expected losses. However, at such high continuation values the principal would rather retire the agent. Therefore, within the contracting relationship the principal’s worst-case perception is always the lower envelope of the set of drift terms. The following result summarizes this observation:

**Proposition 3.** In any optimal long-term contract, the principal and the agent both perceive the lower envelopes of drift sets as the common worst-case scenario:

$$\theta^A(a; \phi) = \theta^P(a; \phi) = \min \{ \theta \in \Theta(a; \phi) \} = a - \kappa(a; \phi) \text{ for each } a \in A.$$

It is worth remarking that the result in Proposition 3 shows that the principal and the agent both perceive the lower envelope to be the common worst-case scenario of drift sets during employment relationship. As this agreement on the worst-case scenario is an endogenous property in the optimal contract, the model is not equivalent to a formulation that obtains by taking the lower-envelope as the specification of technology in the model of Sannikov (2008). Despite this difference, however, Proposition 3 implies that the current model becomes observationally equivalent to the model in Sannikov (2008). In the current setting the principal can perceive the upper envelope as her worst-case for high enough continuation values where the cost of incentives exceed the benefits to the principal. In such a case, the principal optimally chooses to retire the agent. Therefore, the principal and the agent can use different worst-case scenarios outside of employment. As outside of employment there is no ambiguity, this does not bring new possibilities into the principal’s contract design.

**On concavity of the principal’s value function.** In a model of dynamic moral hazard where there is no drift ambiguity [Sannikov 2008] and Possamaï and Touzi (2020) show that the solution $F$ to the HJB equation is concave. This property extends to the current model with drift-ambiguity. Using the characterization of the principal’s worst-case scenario in (L) and rearranging (3.3) implies

$$F''(W) = \frac{F(W) - a(W) + c(W) - F'(W)(W - u(c(W)) + h(a(W)))}{r\sigma^2 Y(W)^2/2} < 0$$

$$-\kappa(a(W)) \frac{1 + F'(W)Y(W)}{r\sigma^2 Y(W)^2/2} < 0$$

$$< 0$$

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When there is no ambiguity, i.e., $\kappa = 0$, in the expression above the second term on the right-hand side disappears and the model reduces to the classical model of Sannikov (2008) and hence the first term is negative. In the presence of ambiguity, the value function is also concave and in particular more concave relative to the classical case. In Miao and Rivera (2013)’s framework where the agent is ambiguity neutral and the principal’s ambiguity sensitive preferences are modeled with variational preferences based on a representation of Maccheroni et al. (2006), the value function need not be concave. In particular, Miao and Rivera (2013) observe that the value function can be convex whenever the principal’s ambiguity aversion is high enough. This in turn implies that the principal can benefit from using a public randomization device in the optimal contract. In contrast, in the framework considered here both the principal and the agent have MaxMin preferences the principal’s value function coincides with its concave hull and hence such a public randomization cannot (strictly) improve upon her payoff.

### 3.2. Basic monotonicity properties

Using the common worst-case scenario from Proposition 3 basic properties on the monotonicity of the principal’s value with respect to the strength of drift ambiguity flows from the analysis of HJB (3.3).

**Proposition 4.** In the optimal long-term contract with drift ambiguity, the principal’s value has the following features with respect to a change in the strength of drift ambiguity:

- **(B1)** Higher ambiguity reduces the principal’s profits: $\hat{\phi} \succ_{\kappa} \phi$ implies $F(\cdot; \hat{\phi}) < F(\cdot; \phi)$;
- **(B2)** Higher ambiguity reduces the increments of the principal’s profits: $\hat{\phi} \succ_{\kappa} \phi$ implies $F'(\cdot; \hat{\phi}) < F'(\cdot; \phi)$;
- **(B3)** Higher ambiguity leads to more concave profit function: $\hat{\phi} \succ_{\kappa} \phi$ implies $F''(\cdot; \hat{\phi}) < F''(\cdot; \phi)$.

Here inequalities hold strictly except for the common initial value $W_0$. While the first two properties that higher degree of ambiguity reduces the principal’s guaranteed value and its increments, in parts the parts (B1) and (B2) of Proposition 4 respectively, are intuitively, the fact that increasing ambiguity uniformly makes the principal’s profit function more concave is probably less so. Intuitively, increasing drift ambiguity for each action uniformly lowers the lower envelope which in turn reduces expected profits monotonically, and also increases the agency cost of implementation by exposing the agent to larger uncertainty. These two effects reinforce each other and imply increase in the curvature of the principal’s value function. The parts (B2) and (B3) of Proposition 4 are broadly technical and used in the comparative static analysis.

If the mapping $\phi \mapsto \kappa(a; \phi)$ is differentiable, the results in Proposition 4 can be expressed in its differential form. In particular, for a differential change in $\phi$ that expands drift-set or increases $\kappa(a; \phi)$ for each action $a$, the resulting changes in the principal’s profit function

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11For the analogous argument in Bayesian formulation see in particular Sannikov (2008)’s Lemma 5 and his remark following Proposition 4.
corresponding to the parts of the proposition are denoted as $F_\phi(W; \phi) < 0$, $F_\phi'(W; \phi) < 0$, and $F_\phi''(W; \phi) < 0$, respectively. The differential form will make it easier to develop the comparative static analysis in the next section.

Before we turn to monotone comparative static analysis on the optimal contract with respect to the strength of ambiguity, we give an additional implication that continuation domain shrinks in the strength of ambiguity.

More specifically, define the *continuation domain* at time $t$ by

$$C(\phi) = \{W_t : F(W_t; \phi) \geq F_0(W_t)\}$$

where $F_0(W_t)$ is the principal’s outside option value from retiring the agent that yields the agent utility $W_t$. Since the retirement option is always available to the principal, $F_0(W_t)$ determines the lower bound on the principal’s value. The principal optimally retires the agent whenever the principal’s value from the relationship $F(W_t; \phi)$ hits this lower boundary. The set $C(\phi)$ therefore describes the states at which stopping and terminating the contract at time $t$ is strictly suboptimal. The part [B1] of Proposition 4 then implies that the continuation domain shrinks as the degree of ambiguity increases.

**Proposition 5.** In any optimal long-term contract, the continuation domain shrinks in the strength of ambiguity: $C(\hat{\phi}) \subset C(\phi)$ for all $t$ whenever $\hat{\phi} \succ \kappa \phi$.

### 4. High-Powered Incentives and Ambiguity

This section presents the main results of the paper on the nature of the optimal contract and show that broadly speaking ambiguity flattens the power of incentives. This result follows from extending and applying monotone comparative statics analysis in continuous time framework inspired by Quah and Strulovici (2013)’s approach. In the context of single-agent decision problem, Quah and Strulovici’s powerful analysis shows a monotonic relationship between optimal control variables and time-preference parameters. For this development their key idea is that HJB equation describing the agent’s decision problem can alternatively be viewed as an objective function for a monotone comparative static analysis. We start with adapting and extending this observation to the agency problem here and then analyze monotonicity between optimal compensation mix and strength of ambiguity.

The principal’s optimal contract problem using the lower-envelope as the common worst-case scenario and Ito’s formula on HJBI (3.3) is rewritten as

$$\max_{(a,Y,c) \geq 0} \left\{ r(a - \kappa(a; \phi) - c) + r F'(W; \phi)(W - u(c) + h(a) - \kappa(a; \phi)Y) + r^2 \sigma^2 F''(W; \phi)Y^2/2 - r F(W; \phi) \right\} = 0$$

subject to the same boundary conditions and the action recommendation $a$ and the required sensitivity $Y$ to implement it are related via the incentive compatibility (2.12). The latter implies the agent’s decision rule for action choice $a(Y; \phi)$ as a function of incentives $Y$ and the degree of ambiguity $\phi$. Using this relationship and letting $m(Y; \phi) =$


\[(W - u(c) + h(a(Y; \phi)) - \kappa(a(Y; \phi); \phi)Y)\] denote the drift of the agent’s continuation value (to simplify notation) we can express the contracting problem in terms of the compensation mix in the following form:

\[
\max_{(Y,c) \geq 0} \left\{ r(a(Y; \phi) - \kappa(a(Y; \phi); \phi) - c) + rF'(W; \phi)m(Y; \phi) + r^2 \sigma^2 F''(W; \phi) Y^2 / 2 - rF(W; \phi) \right\} = 0
\]

(4.1)

The interest is in analyzing qualitative monotonic between the strength of ambiguity and the optimal compensation mix. The model of drift ambiguity described so far makes assumptions in (A.1) through (A.4) about the strength of ambiguity or the length of drift sets \(\kappa(a; \phi)\) up to second-order derivatives. These assumptions enable tractable characterization of monotonic relationships between the strength of ambiguity and the incentives to the agent for implementation given in Lemma 2 on the one hand, and the principal’s value function in Proposition 4 on the other hand.

Monotone comparative statics analysis in its full generality, however, requires the knowledge of higher order derivatives beyond the second due to the endogenous relationship (2.12) connecting actions to incentives. Such a complication in principle is not difficult to solve and yet it does not yield transparent economic analysis. Instead, sharper characterization follows from focusing on linear relationships between actions and the degree of ambiguity. For this we assume a linear relationship in how actions translate to strength of ambiguity: \(\kappa(a; \phi) = \phi_0 + \phi_1 a\), where the vector \(\phi := (\phi_0, \phi_1)\) describes the parameterization of drift-ambiguity. In this linear specification, the constant \(\phi_0 \geq 0\) denotes base-level drift-ambiguity independent of action, and the slope \(\phi_1\) describes how the agent’s effort affects the strength of drift-ambiguity. Consistent with the assumptions (A.1) through (A.4), we consider \(\phi\) so that \(\kappa(a; \phi) \geq 0\) and \(a - \kappa(a; \phi) \geq 0\), or equivalently, \(\phi_0 \geq 0\) and \(\phi_1 \leq 1\). For the linear drift-ambiguity model, we have the following characterization of monotone comparative statics.

**Theorem 2.** Consider linear drift ambiguity model \(\kappa(a; \phi) = \phi_0 + \phi_1 a\) with \(\phi_0 \geq 0\). In the optimal long-term contract, as \(\phi_0\) or \(\phi_1\) increases, the optimal sensitivity \(Y(W; \phi)\) decreases, for low enough continuation values such that \(W \in [W_0, \bar{W}]\). On other hand, as it increases the optimal sensitivity \(Y\) may be non-decreasing, for large enough continuation value such that \(W \in [\bar{W}, W_{gp}]\). Here the principal’s profits obtain the maximum at the unique value \(\bar{W} \in (W_0, W_{gp})\).

This result says that the optimal sensitivity of the agent’s value to output realizations broadly becomes flatter as the strength of ambiguity increases except possibly if the agent’s continuation value is very high. As lower-envelope of drift-sets becomes smaller, or the expected productivity of each action in the worst-case scenarios decreases, the power of incentives become less responsive or less contingent to output realizations.

There is an intuitive explanation for why ambiguity flattens power of incentives. Drift ambiguity has an contemporaneous effect and ambiguity aversion reduces the expected benefits
from each action due to pessimistic expectation. Drift ambiguity also has a continuation effect and it reduces the principal's value because the principal has to provide stronger incentives \( Y \) to implement any viable action according to Lemma 2. The need for strong incentives contribute to the agency costs of implementing actions by exposing the agent to larger output uncertainty captured by the term \( F'(W; \phi)Y^2 \). All else being equal, both the contemporaneous and the continuation effects reduce the marginal profitability and by the monotone comparative statics argument the principal optimally chooses to provide lower powered incentives under higher drift ambiguity. However, not all else remain equal as there is an additional continuation effect of ambiguity interacting with marginal profitability of delayed payments captured by the term involving \( F'(W; \phi) \). This effect due to delayed payments depends on the level of promised value \( W \).

For low enough continuation values, at the beginning of the relationship, delaying payments are beneficial for the principal as \( F'(W; \phi) > 0 \). Since higher ambiguity makes the back-loaded payments less desirable for the agent due to ambiguity aversion (see Proposition 1 and discussion following it), this delayed-payment effect reinforces the two effects previously identified, and further reduces incentive benefits of the continuation value; the principal optimally chooses lower powered incentives. If, on the other hand, the continuation value is sufficiently high that the delaying payments become costly for the principal due to wealth effects, i.e., \( F'(W; \phi) < 0 \), increases in ambiguity has a positive effect on the principal’s profits. Therefore, depending on the relative strength of this effect from delaying payments in comparison to the contemporaneous and the continuation effects, the strength of incentives may increase in ambiguity for high enough continuation values. However, in typical employment relationships analyzed by Sannikov (2008), where the optimal incentives are first increasing and then decreasing in continuation value, the optimal incentives become flatter with increases in ambiguity.

Drift ambiguity has a similar flattening effect on the evolution of optimal effort stream and reduces the optimal effort profile. When drift ambiguity increases in action in the linear model \( \kappa(a; \phi) = \phi_1 a \), from the incentive compatibility condition (2.12) the optimal action is given by \( a = (1 - \phi_1)Y \) and higher ambiguity (corresponding to higher \( \phi \)) implies decrease in optimal action. If, on the other hand, the strength of ambiguity decreases in actions, that is the size of the drift set is of the form \( \kappa(a; \phi) = \phi_0 - \phi_1 a \), then the incentive compatibility condition (2.12) implies \( a = (1 + \phi_1)Y \), which again decreases in the strength of ambiguity as in this case smaller \( \phi_1 \) corresponds to larger drift sets.

Drift ambiguity has an ambiguous effect on the duration of the agency relationship, or the optimal stopping time. Notice that Theorem 2 says that expected worst-case increment in the agent’s continuation value (drift of the diffusion \( W \)) and its volatility \( Y \) both becomes smaller. While the first effect reduces the length of time it takes to trigger the end of the relationship via retiring or firing the agent, the second effect delays such an event. The effect of ambiguity on the duration of the contracting relationship is therefore ambiguous, even
if according to Proposition 5 the upper bound of the continuation domain is smaller with ambiguity.

The analysis so far has characterized the effect of ambiguity on optimal power of incentives and on the duration of the contracting relationship. Next we further characterize the optimal payment schemes and show that the presence of ambiguity depresses wages and makes the optimal contract more reliant on the long-term incentives.

**Proposition 6.** Higher ambiguity depresses wages: $c(W; \hat{\phi}) < c(W; \phi)$ for all $W$ whenever $\hat{\phi} \succ_{\kappa} \phi$. Moreover, the ratio of volatilities of the agent’s consumption and continuation values is greater in the contract associated with $\hat{\phi}$ whenever $c(W; \hat{\phi}) = c(W; \phi)$.

This result is reminiscent of Sannikov (2008)’s Theorem 4 that studies how the contractual environment affects the optimal long-term incentives. In particular, imprecise information about technology as compared to precise information compresses wage profile, and reduces the sensitivity of current wage with respect to the continuation value. The latter implies that the contract associated with precise information environment relies less on short-term incentives and more on long-term incentives.

Turning to the interpretation of the result in Proposition 6, notice that as in the classical case, the variation in the continuation value with output realizations is still present to incentivize the agent’s effort, see from the incentive compatibility condition (2.11). The familiar intuition is that to solve agency conflict the principal’s design of contract aligns interests by letting agent’s compensation to positively vary with her profits. This intuition is still applicable to interpret our result as during employment the parties endogenously agree on the worst-case. On the other hand, as we have seen the presence of ambiguity and aversion to it makes back-loaded incentives a less effective tool relative to the classical case because ambiguity-averse agent prefers certainty of payments today over uncertainty of future payments.

### 5. Extension to Heterogeneous Drift-Ambiguity

The agency problem posed with symmetric ambiguity and the approach to its analysis is flexible enough to allow for heterogeneity in ambiguity. In particular, we now allow for drift-sets perceived by the principal and the agent to be different by setting for each action $a \in A$, $\kappa(a; \phi^A)$ and $\kappa(a; \phi^P)$, respectively. We assume that the parties share common knowledge about the heterogeneity in ambiguity. The principal and the agent can evaluate a contract using different drift sets. To keep track of drift-ambiguity perceived by the parties, we use the notation $\phi := (\phi^A, \phi^P)$ by extending the notation used in the symmetric case.

For tractability, we make regularity assumptions analogous to those in the symmetric case and assume that Assumptions (A.1)-(A.4) hold for the mappings $\kappa(a; \phi^A)$ and $\kappa(a; \phi^P)$, respectively.

With this reformulation Proposition 1, which uses only the drift set of the agent, applies using the drift set $\Theta(a_t; \phi^A) = [a_t - \kappa(a_t; \phi^A), a_t + \kappa(a_t; \phi^A)]$ and yields a representation of
the agent’s value as a diffusion. Using this diffusion representation then, Lemma 1 applies analogously and the agent’s worst-case scenario for any non-zero contract is now given by the lower-envelope of his drift-set: \( \theta^A(a_t; \phi^A) = a_t - \kappa(a_t; \phi^A) \). Applying Proposition 2 incentive compatible sensitivity \( Y \) to implement a non-trivial action \( a > 0 \) analogous to (2.12) in with minimized cost is therefore characterized by

\[
Y = \frac{a}{1 - \kappa_a(a; \phi^A)}. \tag{5.1}
\]

This in turn extends Lemma 2 and implies that as the ambiguity perceived by the agent increases the required variation \( Y(a; \phi^A) \) to implement an action \( a \), which solves for \( Y \) the equation (5.1), increases and that for any \( \phi^A \) the incentives \( Y(a; \phi^A) \) increase in \( a \). Notice that for any contract the agent’s decision rule given by (5.1) depends only on the agent’s ambiguity \( \phi^A \) but not on the principal’s ambiguity \( \phi^P \).

Using the characterization of the agent’s decision rule for his action choice, HJBI formulation for the optimal contracting problem for heterogeneous ambiguity \( \phi = (\phi^A, \phi^P) \) then takes the following form

\[
rF(W; \phi) = \max_{(a,c) \geq 0} \min_{\theta \in \Theta(a; \phi^P)} \left\{ r(a + \theta - c) - rF'(W; \phi) (W - u(c) + h(a) + \theta Y(a; \phi^A)) \right. \\
\left. + \frac{F''(W; \phi)}{2} r^2 Y(a; \phi^A) \sigma^2 \right\} \tag{5.2}
\]

subject to the boundary conditions and the smooth pasting condition:

\[
F(0; \phi) = 0, \quad F(W_{gp}; \phi) = F_0(W_{gp}) \quad \text{and} \quad F'(W_{gp}; \phi) = F'_0(W_{gp}).
\]

Comparing the contracting problem (5.3) with symmetric ambiguity notice here the distinct roles played by \( \phi^A \) and \( \phi^P \). The principal’s value depends on her ambiguity \( \phi^P \) directly through his worst-case expectation over \( \Theta(a; \phi^P) \). Her value indirectly depends on the agent’s ambiguity \( \phi^A \) as it determines via (5.1) the agent’s incentive compatible decision rule. The principal’s optimal contract design takes into account both sources of ambiguity.

The argument for Theorem 1 extends and applies to heterogeneous case as the regularity conditions analogously hold. The recursive HJBI formulation (5.2) therefore provides an equivalent characterization of the optimal contracting problem. In turn, the recursive representation (5.2) provides a tractable characterization for the worst-case scenario perceived by the principal as in the symmetric case. For any incentive compatible action strategy \( \{a_t\} \) and the corresponding incentives \( (Y_t) \), the worst-case drift scenario \( \theta^P \) perceived by the principal is the drift term that minimizes her expected payoff over the drift set:

\[
\theta^P(a; \phi) = \arg\min_{\theta \in [-\kappa(a; \phi^P), \kappa(a; \phi^P)]} \theta \left[ 1 + F'(W; \phi)Y \right]
\]

Equivalently,

\[
\theta^P(a; \phi) = \begin{cases} 
-\kappa(a; \phi^P) & \text{if } 1 + F'(W; \phi)Y \geq 0; \\
\kappa(a; \phi^P) & \text{if } 1 + F'(W; \phi)Y < 0.
\end{cases} \tag{L'}
\]
The interpretation of this condition is analogous to that, in the symmetric ambiguity case: during employment in any optimal contract the lower-envelope of drift-set is the worst-case scenario for the principal. For otherwise, additional payments incur net losses and the principal always has the option of retiring the agent instead of making such losses.

5.1. Basic properties with heterogeneous ambiguity. Proposition 4 has established basic monotonicity properties of the principal’s value function with respect to symmetric drift ambiguity. Its analysis extends to heterogeneous ambiguous perceptions and shows that qualitatively the analogous results hold.

To represent different degrees of drift ambiguity in heterogeneous case, we define the relation $\hat{\phi} \succ_K \phi$ to mean contracting environment where at least one of the parties perceive larger drift ambiguity: formally, $\kappa(\cdot; \hat{\phi}^i) \geq \kappa(\cdot; \phi^i)$ for each $i = A, P$ and $\kappa(\cdot; \hat{\phi}^i) > \kappa(\cdot; \phi^i)$ and for at least one of $i = A, P$. This naturally extends to the heterogeneity in drift ambiguity the definition of comparative ambiguity for the symmetric case.

Using the lower envelopes as worst-case scenarios for each party, analogous to Proposition 4 the basic monotonicity properties for the principal’s guaranteed value with respect to the strength of ambiguity $\phi = (\phi^A, \phi^P)$ follow from the analysis of HJBI (5.2):

Proposition 7. In the optimal long-term contract with drift ambiguity, the principal’s guaranteed value has the following features,

(B1') Higher ambiguity reduces the principal’s profits: $\hat{\phi} \succ_K \phi$ implies $F(\cdot; \hat{\phi}) < F(\cdot; \phi)$;

(B2') Higher ambiguity reduces the increments of the principal’s profits: $\hat{\phi} \succ_K \phi$ implies $F'(\cdot; \hat{\phi}) < F'(\cdot; \phi)$;

(B3') Higher ambiguity leads to a more concave profit function: $\hat{\phi} \succ_K \phi$ implies $F''(\cdot; \hat{\phi}) < F''(\cdot; \phi)$.

Here inequalities hold strictly except for the common initial value $W_0$. In other words, higher ambiguity perceived by the principal or the agent reduces the principal’s value function, decreases its increments, and makes it more concave. These basic properties with heterogeneous drift ambiguity are qualitative similar to those with homogeneous drift ambiguity given in Proposition 4. Relative to the latter allowing for heterogeneous drift-ambiguity in the analysis of HJBI (5.2) highlights two distinct effects of the parties’ ambiguity perceptions. Firstly, increases in the agent’s ambiguous perceptions increase agency costs due to higher powered incentives required for implementation and hence reduce the principal’s value. Secondly, increases in the principal’s ambiguity reduces her guaranteed value by deteriorating expectations in her worst-case scenario.

5.2. Optimal incentives and heterogeneous drift-ambiguity. We now turn to analyze the monotone comparative static analysis on the strength of ambiguity and the power of incentives in the optimal compensation mix. For this, using the basic monotonicity properties of HJBI’s objective function from Proposition 7 we extend Theorem 2 to heterogenous drift
ambiguity. This analysis shows broadly that the qualitative feature of flattening of incentives at higher ambiguity levels holds allowing for heterogeneity in drift-ambiguity.

**Theorem 3.** Consider a linear drift ambiguity model \( \kappa(a; \phi^i) = \phi_0^i + \phi_1^i a \) with \( \phi_0^i \geq 0 \) for each \( i = A, P \). In the optimal long-term contract, the incentives with respect to ambiguity have the following characterization.

(a) As \( \phi_0^P, \phi_1^P \) or \( \phi_1^A \) increases, the optimal sensitivity \( Y(W; \phi) \) decreases for low enough continuation values \( W \in [W_0, \overline{W}] \). On other hand, as it increases the optimal sensitivity may be non-decreasing, for large enough continuation value such that. The principal’s profits obtain the maximum at the unique \( \overline{W} \in (W_0, W_{gp}) \).

(b) As \( \phi_1^A \) increases the optimal sensitivity \( Y(W; \phi) \) decreases for all continuation values during employment \( W \in [W_0, \overline{W}] \).

This characterization is qualitatively similar to that of the symmetric linear-drift-ambiguity model described in Theorem 2. Formulation with heterogeneous ambiguity perceptions reveals different possible effects of the strength of ambiguity perceived by the parties. Compared to the symmetric formulation, notice that the agent’s heterogeneous constant ambiguity perception has unambiguous effect and flattens the incentives. This is because an increase in the agent’s constant ambiguity \( \phi_0^A \) worsens the expected value for the agent from any viable contract and hence constrains the set of contracts the principal can implement.

On the other hand, an increase in the constant ambiguity \( \phi_0^P \) perceived by the principal has an effect of incentives that can depend on the level of the promised value to the agent \( W \). First, increase in \( \phi_0^P \) lowers the expectation in the flow benefit, which unambiguously call for lower incentives because its benefits are now lower. Second, the principal’s worsening of lower envelope also limits the increments to the agent’s continuation value from the principal’s perspective. This second effect’s sign depends on where the relationship stands during the contract. For low enough \( W \in [W_0, \overline{W}] \) backloading payments is beneficial for the principal and an increase in \( \phi_0^P \) limits this gain by reducing the agent’s continuation value from the principal’s perspective. This reinforces the first effect and lowers the optimal incentive provision. On the other hand, for high enough \( W \) backloading incentives is costly due to wealth effects and in this case reducing the increments to the continuation value as \( \phi_0^P \) increases can decrease the cost of incentive provision. This in turn counteracts the first effect and hence can induce an increase in incentives. This effect is also present for changes in \( \phi_1^P \) or \( \phi_1^A \) that increases drift-sets. In either scenario, incentive path becomes compressed, i.e., it takes values in a narrower range, as constant ambiguity increases since typically incentives are first increasing and then decreasing in \( W \) in a continuous manner (see Sannikov (2008)).

Applying similar arguments extends Proposition 6 to the heterogenous drift-ambiguity case. Therefore, the optimal wage scheme becomes more depressed as the strength of drift ambiguity perceived by the principal and/or the agent increases.
6. Concluding Remarks

This paper presents a dynamic model of moral hazard relationship whose central feature is imprecise information about the consequences of actions. Imprecise information about technology translating actions into outcomes are important for applications, yet known to lead to difficulties of analysis in the classic Bayesian formulation of dynamic moral hazard problems. Methodologically, the paper provides a model of dynamic moral hazard that formulates imprecise information using drift-ambiguity i) can sustain nontrivial dynamic incentives; ii) has a tractable solution, characterized by a one-dimensional differential equation; and iii) incorporates novel monotone comparative statics in a dynamic agency relationship.

Substantively, the paper sheds new lights on the effects of imprecise information and ambiguity aversion on dynamic contracts (compressing of the incentive effects of “high-powered” incentive payments), and we provide empirical predictions on the “high-powered” incentives (compressing of the incentive payments in imprecisely understood environments relative to Bayesian environments with precise understanding of connection between actions and their consequences). This resonates well with empirical observations that in workplace settings where output and/or quality are difficult to measure, for instance performance of white collared workers, the workers’ contracts are not sensitively fine-tuned to performance realizations (Lazear, 2018).

We want to make a remark about our use of the MaxMin criteria as a formalization of attitude towards imprecise information or non-probabilistic beliefs. One class of axiomatizations of MaxMin preferences come from Gilboa and Schmeidler’s (1989)’s approach towards ambiguity that takes the state space as a primitive. We do our analysis on the space of payoff-relevant outcomes, or more specifically over paths of output diffusions. In general, the state-space approach towards ambiguity is not conceptually amenable to agency problems, as the probabilities of outcomes in the latter are conditional on actions (see Karni (2006) and Karni (2009) for discussions on this issue). We therefore do not invoke such axiomatizations as a foundation for our formulation, instead we see this simply as a plausible way to capture the idea of robustness concerns when designing contracts. Our take is that such concerns are more appropriately modeled as evaluations of outcome prospects. We therefore follow the behaviorally justifiable approach of taking as a primitive the decision maker’s preferences over sets of distributions over the outcome space (see Ahn (2008), Olszewski (2007) and Dumav and Stinchcombe (2018) for similar approaches).

For tractability the current model uses, for ambiguity sensitive preferences an extreme form of pessimism represented by the formulation with MaxMin Expected Utility. In the dynamic framework considered here, MEU is the only form that allows for non-separability between time and ambiguity attitudes (Strzalecki, 2013). This provides therefore a transparent insight into effect of ambiguity on dynamic incentives. However, the framework is analytically flexible enough to accommodate other formulations of ambiguity sensitive preferences.
While interpreting our results, one has to be conscious of our formalization that the agent and the principal have stationary perceptions of drift ambiguity. Generally, our intuitions about the dynamics of the agent’s wages and incentives do not appear to depend on this assumption. However, the assumption that the parties have stationary ambiguity matters if one addresses the practically important issues that the contracting parties experiment about the technological possibilities. These highly interesting issues introduce forms of non-stationarity in the contracting problem and further analysis is left to future research.

APPENDICES

THE STRUCTURE OF THE SETS OF PRIORS: RECTANGULARITY AND REGULARITY

In this section, we provide a class of multi-priors that satisfy dynamic consistency and regularity conditions. These conditions in turn ensure that the contracting problem admits a solution and can be analyzed recursively.

The formalization of the drift-sets follows the approach in Chen and Epstein (2002) and adapts it to the contracting problem. Throughout we use as primitive a measure space \((\Omega, \mathcal{F}, P)\) and a Brownian motion \(B\) defined on this space.

We begin with observing that as the contraction problem has a finite random horizon \(P(\tau < \infty) = 1\), we fix any given finite time horizon \(T \leq \infty\) such that \(\tau(\omega) \leq T\) with probability one, and let time \(t\) vary over \([0, T]\). We also fix any given drift ambiguity scenario \(\phi\) and hence leave its dependence implicit to simplify notation.

For any action profile \(a = \{a_t\}\) in an admissible contract, we consider a drift process \(\theta = \{\theta_t\}\) adapted to the filtration \(\mathcal{F}\) such that \(\theta_t \in \Theta(a_t)\). Using the drift process we define a process of Girsanov exponents as follows

\[
z_{t+a}^{\theta} = \exp\left\{-\frac{1}{2} \int_0^t |a_s + \theta_s|^2 ds - \int_0^t (a_s + \theta_s) dB_s^a\right\}, \quad 0 \leq t \leq T, \tag{6.1}
\]

where \(\theta_t \in \Theta(a_t) = [-\kappa(a_t), \kappa(a_t)]\).

As the action set \(\mathcal{A}\) is a compact subset in \(R_+\) and the drift set \(\Theta(a)\) for any action \(a \in \mathcal{A}\) is a compact subset in \(R_+\), the process \(a + \theta := \{a_t + \theta_t\}\) is bounded and hence satisfies the following square-integrability condition

\[
\mathbb{E}\left[\exp\left(\frac{1}{2} \int_0^T |a_s + \theta_s|^2 ds\right)\right] < \infty.
\]

Using this property, and using Girsanov’s change of measure theorem ensure that the process \(\{z_{t+a}^{\theta}\}\) defines a \(P\)-density on \(\mathcal{F}_T\) (see, for instance, Duffie (2001, p. 337)). Therefore, the process \(a + \theta\) generates a probability measure \(P^{a+\theta}\) on the measurable space of sample paths \((\Omega, \mathcal{F})\) and the Radon-Nikodym derivative of this measure with respect to the reference measure \(P\) is given by

\[
\frac{dP^{a+\theta}}{dP}\bigg|_{\mathcal{F}_t} = z_{t}^{a+\theta}, \quad 0 \leq t \leq T. \tag{6.2}
\]
For the action \( a = \{a_t\} \) and the measurable drift process \( \{\theta_t\} \) such that \( \theta_t \in \Theta(a_t) \) we have therefore constructed a measure \( P^{a+\theta} \) equivalent to \( P \) via a change of measure. Now, as in Chen and Epstein (2002) we define the set of multi-priors for the given action \( a = \{a_t\} \) as the set of all such equivalent measures \( P^{a+\theta} \) that can be constructed using some measurable selection \( \{\theta_t\} \) from the process of correspondences \( \Theta(a_t) \). The multi-prior for the action is then given by:

\[
P^{\Theta(a)} := \{ P^{a+\theta} : \{\theta_t\} \text{ is a measurable selection from } \{\Theta(a_t)\} \text{ and } P^{a+\theta} \text{ is defined by } (6.2) \}
\]

(6.3)

For any admissible action \( a = \{a_t\} \), by analogous argument and terminology used by Chen and Epstein (2002) the set of priors \( P^{\Theta(a)} \) satisfies rectangularity.

We next turn to verify that the drift sets \( \{\Theta(a_t)\} \) and the set of priors \( P^{\Theta(a)} \) satisfy regularity conditions that facilitate the recursive analysis of the contracting problem.

As the action \( a = \{a_t\} \) is measurable with respect to the filtration \( \mathcal{F} \), and the function \( \kappa \) defined on \( \mathcal{A} \) is continuous, the process of compact and convex-valued correspondences \( \{\Theta(a_t)\} \) is measurable with respect to the same filtration.\(^{12}\) Moreover, a normalization property \( 0 \in \Theta(a_t) \) holds almost surely as \( 0 \in \Theta(a) = [-\kappa(a), \kappa(a)] \) for any \( a \in \mathcal{A} \). We summarize these regularity properties on the drift sets as follows

**Lemma 3.** The process of drift sets \( \{\Theta(a_t)\} \) satisfies:

(a) \( \{\Theta(a_t)\} \) is uniformly bounded.

(b) The correspondence \((t, \omega) \mapsto \Theta(a_t(\omega))\) is compact-valued and convex-valued, and is \( \mathcal{F}_t \)-measurable for any \( 0 < t \leq T \).

(c) Normalization \( 0 \in \Theta(a_t(\omega)) \) holds \( dt \otimes dP \) a.e.

We next turn to verify regularity properties for the set of priors \( P^{\Theta(a)} \). For this we use the weak topology on the space of countably additive probabilities \( \text{ca}(\Omega, \mathcal{F}_T) \). To characterize this topology we use the Prokhorov metric \( \rho \). For any two measures \( Q \) and \( Q' \) in the space \( \text{ca}(\Omega, \mathcal{F}_T) \) the Prokhorov distance \( \rho(Q, Q') \) is defined by

\[
\rho(Q, Q') := \inf\{ \varepsilon > 0 : \text{ for any closed set } F \in \mathcal{F}_T, \ Q(F) \leq Q'(F^\varepsilon) + \varepsilon \text{ and } Q'(F) \leq Q(F^\varepsilon) + \varepsilon \}
\]

(6.4)

where \( F^\varepsilon \) is the \( \varepsilon \)-ball around the set \( F \). The Prokhorov metric \( \rho \) metrizes the weak topology on \( \text{ca}(\Omega, \mathcal{F}_T) \), indeed \( \rho(Q_n, Q) \to 0 \) if and only if for all bounded and continuous function \( f \int_\Omega f dQ_n \to \int_\Omega f dQ \) (see, instance, Corbae et al. (2009, Theorem 9.3.2)). Moreover, on the class compact and convex subsets of \( \text{ca}(\Omega, \mathcal{F}_T) \) we use Hausdorff topology corresponding to the Prokhorov metric \( \rho \).

\(^{12}\)Indeed, using the definition of measurability in the class \( K \) of compact subsets of \( R \) (see for instance, Aliprantis and Border (2006, Chapter 14.12)) for each compact set \( K \in K \), it is the case that \( \{(t, \omega) \in [0, s] \times \Omega : \Theta(a_t(\omega)) \cap K \neq \emptyset \} \in \mathcal{B}([0, s]) \times \mathcal{F}_t \).
Under the regularity conditions on the process of drift sets \( \{ \Theta(a_t) \} \) in Lemma 3, for any admissible action \( a = \{ a_t \} \), the following set of properties on the prior sets \( P^{\Theta(a)} \) hold.

**Lemma 4.** The set of priors \( P^{\Theta(a)} \) satisfies:

(a) \( P \in P^{\Theta(a)} \) and \( P^{\Theta(a)} \) is uniformly absolutely continuous with respect to \( P \). Each measure in \( P^{\Theta(a)} \) is equivalent to \( P \).

(b) \( P^{\Theta(a)} \) is convex and compact in the weak topology.

(c) For every \( \xi \in L^2(\omega, F_T, P) \), there exists \( P^{a+\theta^*} \in P^{\Theta(a)} \) such that
\[
E^{a+\theta^*}[\xi|F_t] = \min_{\theta \in \Theta(a)} E^{a+\theta}[\xi|F_t], \quad 0 \leq t \leq T.
\]

**Proof.** The properties in (a), (b) and (c) follows from the arguments analogous to those made in the proof of Theorem 2.1 by Chen and Epstein (2002). \( \square \)

**Proofs of the Results for Symmetric Drift Ambiguity in Sections 2-4**

**Proof of Proposition 1.** As in the structure of the prior sets, we begin with observing that as the contraction problem has a finite random horizon \( P(\tau < \infty) = 1 \), we fix any given finite time horizon \( T \leq \infty \) such that \( \tau(\omega) \leq T \) with probability one, and let time \( t \) vary over \([0, T]\). This is without loss of generality. Indeed, notice that if \( t \geq \tau(\omega) \), then the agent stops working \( a_t(\omega) = 0 \), and receives no payment \( c_t(\omega) = 0 \) or a positive payment \( c_t(\omega) = \xi > 0 \), depending on whether at the stopping time \( \tau(\omega) \) the principal terminates the contract by firing the agent or by retiring him, respectively.

For a given pair of processes of wages and actions \( \{c_t\} \) and \( \{a_t\} \), respectively, define the agent’s valuation process \( \{V_t\} \) by
\[
V_t = r \int_0^t e^{-rs}(u(c_s) - h(a_s))ds + e^{-rt}W_t(c,a) \quad (6.5)
\]
where \( W_t(c,a) \) is the continuation value defined by
\[
W_t = \min_{Q \in P^{\Theta(a)}} E_Q \left[ \int_t^\infty e^{-r(s-t)}(u(c_s) - h(a_s)) ds |F_t \right].
\]

Since, by construction, \( P^{\Theta(a)} \) is a rectangular set of multi-priors over the paths of output realizations, using Chen and Epstein (2002)’s representation theorem, then we find a progressively measurable process \( \{Y_t\} \) such that
\[
-dV_t = -\theta_t^*e^{-rt}|Y_t|dt - \sigma e^{-rt}Y_t dB_t^a \quad (6.6)
\]
where \( B_t \) is a Brownian motion under the reference measure \( P; \theta_t^* := \min_{\theta \in \Theta(a)} \theta_t|Y_t| \) is the agent’s worst-case drift for the action process \( \{a_t\} \); and the factor \( re^{-rt}\sigma \) is a convenient rescaling factor. On the other hand, differentiating (6.5) with respect to \( t \) one finds that
\[
dV_t = re^{-rt}(u(c_t) - h(a_t))dt - re^{-rt}W_t dt + e^{-rt}dW_t. \quad (6.7)
\]
Now, together (6.6) and (6.7) imply that
\[
re^{-rt}(u(c_t) - h(a_t))dt - re^{-rt}W_tdt + e^{-rt}dW_t = \theta_t^*re^{-rt}\kappa|Y_t|dt + \sigma re^{-rt}Y_tdB_t^a
\]

\[\implies dW_t = r\theta_t^*|Y_t|dt + \sigma rY_tdB_t^a - r(u(c_t) - h(a_t))dt + rW_tdt.\]

The latter further implies
\[
W_t = W_0 + \int_0^t r\left(W_s - u(c_s) + h(a_s) + \min_{\theta_s \in \Theta(a_s)} \theta_s|Y_s|\right) ds + \int_0^t rY_sdB_s^a
\]
where \((B_t)\) with \(B_t = X_t - a_t\) is a Brownian motion under the action strategy \(\{a_t\}\). □

**Proof of Proposition 2.** This result follows as a special case of the incentive compatibility condition for a slightly more general specification \(\Theta(\mu(a_t))\) for the drift-set. The special case then arises by taking the drift set \(\Theta(a)\) to be in \(\kappa\)–ignorance form: \(\Theta(a_t) = [a_t - \kappa(a_t), a_t + \kappa(a_t)]\).

**Proposition 8.** For a given strategy \(a = \{a_t\}\), let \(\{Y_t\}\) be the volatility process from Proposition 1. Then the action strategy \(a\) is optimal if and only if
\[
\forall a'_t \in \mathcal{A}, \; Y_t \mu(a_t) - h(a_t) + \min_{\theta_t \in \Theta(a_t)} \theta_t|Y_t| \geq Y_t \mu(a'_t) - h(a'_t) + \min_{\theta_t \in \Theta(a'_t)} \theta_t|Y_t| \; dt \otimes dP \text{ a.e.}
\]
(6.8)

**Proof.** Consider an arbitrary alternative strategy \(a' = (a'_t)\) that follows possibly different actions \(a'_t\) up to \(t\) and afterwards continues with \(a_t\). The action strategy process \(a'\) induces a set of densities \(\Theta(a)\) satisfying the regularity conditions as specified earlier, and since the set is IID (independent and indistinguishable in the sense of Chen and Epstein see Section 6 for details) the corresponding set of multiple priors \(\mathcal{P}^{a'}\) is rectangular. The agent’s expected payoff from this action strategy is well-defined by
\[
V'_t = \min_{Q \in \mathcal{P}^{a'(t)}} V_t^Q,
\]
where \(V_t^Q\) is unique solution (ensured by Duffie and Epstein (1992)) to BSDE
\[
V_t^Q = \mathbb{E}^Q \left[ \int_t^\infty f(c_s, a'_s, V_s^Q) \right],
\]
where \(f = u(c) - h(a) - \beta V\) is the standard aggregator. By \(g\)–martingale representation theorem of Chen and Epstein (2002, Theorem 2.2), \(V'\) is recursively represented in a unique manner:
\[
dV'_t = (-f(c_t, a'_t, V'_t) + \max_{\theta_t \in \Theta(a'_t)} \theta_t Y'_t)dt + Y'_t dB_t^a
\]
for a unique volatility process \(Y'\).

More generally, \(V'\) and \(Y'\) uniquely solves a BSDE of the following form
\[
dV_t = g'(V_t, Z_t, \omega, t)dt + Z_t dB_t^{a'},
\]
(6.9)
with terminal condition \( \xi \) for any path \( \omega \) over the output realizations. Specializing this result to the setup of the current formulation we have
\[
g'(V, Z, \omega, t) = -f(c_t(\omega), a'_t(\omega), V) + \max_{\theta \in \Theta(a'_t)} \theta(\omega)Y
\]
and the terminal condition is the value at the retirement. It therefore follows that under the action strategy \( \{a_t\} \) the agent’s value process \( V \) and its volatility \( Y \) solve (6.9).

Suppose that the condition given in (6.8) holds. Since the terminal conditions correspond to the retirement and at the retirement there is no uncertainty, the terminal conditions are invariant under different action strategies \( a \) and \( a' \). By the Comparison Theorem in El Karoui et al. (1997, Theorem 2.2)
\[
g(V, Z, \omega, t) \leq g'(V, Z, \omega, t) \ dt \otimes dP \ a.e.
\]
or equivalent the condition (6.8) in our model holds
\[
\mu(a_t)Y_t - h(a_t) - \max_{\theta \in \Theta(a_t)} \theta_t Y_t \geq \mu(a'_t)Y_t - h(a'_t) - \max_{\theta \in \Theta(a'_t)} \theta Y_t \ dt \otimes dP \ a.e.
\]
which implies that \( V \geq V' \) for almost every \( t \).

Suppose now that the condition (6.8) fails on a set of positive measures, and choose an action strategy \( a' \) that maximizes \( \mu(\tilde{a})Y_t - h(\tilde{a}) - \max_{\theta \in \Theta(\tilde{a})} \theta Y_t \) over \( \tilde{a} \in A \) for all \( t \geq 0 \). Then
\[
g(V, Z, \omega, t) \leq g'(V, Z, \omega, t) \ dt \otimes dP \ a.e. \quad \text{Since } a' \text{ specifies the same action as } a \text{ after } t, \text{ an application of the comparison theorem implies } V' > V. \quad \text{Therefore, } a \text{ is suboptimal.} \quad \Box
\]

If the volatility process is written as \(-Y_t\) the minimum replaces the maximum in (6.11). Notice that Proposition 8 is formulated for any generating process \( \Theta(a) \). Taking \( \Theta(a) := \{\theta_t : \mu(a_t) + |\theta_t| \leq \kappa(a_t)\} \) in the setup of Proposition 1, Proposition 8 now specializes to the characterization of incentive compatibility \( \kappa \)-ignorance model:

**Lemma 5.** For any action strategy \( a \), let \( \{Y_t\} \) be the volatility process from Proposition 1. Then the strategy \( a \) is incentive compatible if and only if
\[
\forall \tilde{a}_t \in A \quad Y_t \mu(a_t) - \kappa(a_t)|Y_t| - h(a_t) \geq Y_t \mu(\tilde{a}_t) - \kappa(\tilde{a}_t)|Y_t| - h(\tilde{a}_t) \quad dt \otimes dP \ a.e.
\]
Finally, taking \( \mu(a) = a \) for all \( a \in A \) specializes the result to Proposition 2. \Box

**Proof of Theorem 1.** In this proof, we provide a verification argument as a standard justification that establishes validity of the dynamic programming approach to characterize the principal’s contracting problem recursively using the agent’s continuation value as a state variable.

We follow closely dynamic programming approach of Possamaï and Touzi (2020) to analyze dynamic principal agent problems in a classical setting of Sannikov (2008), we adopt it to allow for drift ambiguity. In particular, the characterization of the optimal contract as a solution

\[\text{Possamaï and Touzi (2020)}\] uses the general approach of Lin et al. (2020) who develop formulation of the model where the agent’s action can control both the drift and the volatility of output process.
to the HJB equation follows closely the proof strategy used in the proof of Theorem 3.6 and Proposition 7.2 in Possamaï and Touzi (2020) for the case where the principal and the agent have a common discount factor. Indeed, the main new mathematical arguments will be due to possible heterogeneity in the worst-case drift scenarios perceived by the principal and the agent.

We prove the verification argument under a ‘regularity’ assumption that for any given promised value to the agent $W \geq 0$ the solution $(c(W), a(W), Y(W))$ of the HJB equation (6.12) yields an admissible contract. In particular, this contract is defined by $(c_t, a_t) := (c(W_t), a(W_t))$ with the associated incentive-compatible sensitivity is given by $Y_t := Y(W_t)$. For this contract, the agent’s promised value process $\{W_t\}$ evolves according to the flow diffusion equation (3.1). To streamline exposition, we first prove the verification result under this regularity assumption. Specifically, we show that under the regularity assumption recursive optimal scheme $(c, a)$ is a solution to the principal’s sequential contracting problem. We then turn to verify this regularity property by showing that recursive optimal scheme is indeed an admissible contract. To simplify notation, we develop the verification argument for any given drift-ambiguity scenario $\phi$ and leave it implicit.

**Step 1.** Reproducing the contracting problem in sequential and recursive formulations

We re-state the sequential problem (P) and denote by $v$ its value function defined by

$$V(W) := \max_{(c, a) \in Z(W_0)} \min_{\theta \in \Theta(a)} \mathbb{E}^{a+\theta} \left[ r \int_0^\infty e^{-rt}(dX_t - c_t dt) \right]. \tag{P}$$

Here we recall that the set $Z(W_0)$ describes the set of admissible contracts that satisfy incentive compatibility in (2.5) and integrability in (2.6) and (2.7), and yields to the agent ex-ante value $W_0$.

We also reproduce the recursive formulation of the contracting problem and simplify the representation of its solution

$$F(W; \phi) = \max_{(c, a, Y) \in \Gamma} \min_{\theta \in \Theta(a; \phi)} \left\{ r(a + \theta - c) + rF'(W; \phi) (W - u(c) + h(a) + \theta Y) + \frac{F''(W; \phi)}{2} r^2 Y^2 \sigma^2 \right\}$$

subject to the boundary conditions:

$$F(0) = 0, \quad \text{and} \quad F \geq F_0.$$  

As in the retirement region the value function is equal to the value of retiring the agent, i.e., $S = \{F = F_0\}$, the recursive formulation (6.12) can be rewritten in a compact form:

$$F(0) = 0, \quad \text{and} \quad L(F) = 0 \text{ on } W \in (0, \infty) \tag{6.13}$$
where for any continuation value \( W > 0 \) the operator \( L \) is defined as follows:

\[
L F(W) := F - \max_{(c,a,Y) \in \Gamma} \min_{\theta \in \Theta(a)} \left\{ r(a + \theta - c) + r F'(W) (W - u(c) + h(a) + \theta Y) + \frac{F''(W)}{2} r^2 Y^2 \right\}
\]

**Step 2.** The recursive optimal scheme yields a larger value than the principal’s value function in the sequential problem: \( F \geq V \).

Consider any admissible contract \((c,a) \in Z(W_0)\) which yields ex-ante value \( W_0 \) to the agent. Let \( \{W_t\} \) denote the corresponding value process that describes the evolution of the agent’s value from the contract \((c,a)\), according to the flow diffusion equation \(\text{[6.13]}\). Let \( \tau_n := \tau \land \inf\{t \geq 0 : W_t \geq n\} \) a finite stopping time. By Ito’s formula, the principal’s value from the contract \((c,a)\) can be expressed as follows

\[
F(W_0) = e^{-\tau_n} F(W_{\tau_n}) - \int_{0}^{\tau_n} e^{-rt} \{ -r F + \partial_t v + r F'(W_t)(W_t + u(c_t) - h(a_t) + \theta_t^P Y_t) + \frac{1}{2} r^2 Y_t^2 F''(W_t) \} dt
\]

As the solution to \(\text{HJB} \) satisfies \( L(F) = 0 \) by \(\text{[6.13]}\) and \( F \geq F_0 \) on the continuation region \( S^c \) the last equation implies

\[
\geq e^{-\tau_n} F_0(W_{\tau_n}) + \int_{0}^{\tau_n} e^{-rt} ( LF(W_t) + a_t + \theta_t^P - c_t ) dt - \int_{0}^{\tau_n} e^{-rt} F'(W_t) r Y_t \sigma dB_t^{a+\theta} + \int_{0}^{\tau_n} e^{-rt} F''(W_t) r^2 Y_t^2 dt
\]

As \( F' \) is bounded on \([0,\tau_n]\), and the admissible contract \((c,a)\) satisfies by assumption the integrability conditions in \(\text{[2.13]}\) and \(\text{[2.14]}\), taking expectations in turn implies

\[
F(W_0) \geq \mathbb{E}^{a+\theta} \left[ e^{-\tau_n} F_0(W_{\tau_n}) + \int_{0}^{\tau_n} e^{-rt} ( a_t + \theta_t^P - c_t ) dt \right]
\]

\[
\xrightarrow{n \to \infty} \mathbb{E}^{a+\theta} \left[ e^{-\tau} F_0(W_{\tau}) + \int_{0}^{\tau} e^{-rt} ( a_t + \theta_t^P - c_t ) dt \right]
\]

\[
= \min_{\theta \in \Theta(a)} \mathbb{E}^{a+\theta} \left[ e^{-\tau} F_0(W_{\tau}) + \int_{0}^{\tau} e^{-rt} ( a_t + \theta_t - c_t ) dt \right].
\]

Here the convergence as \( n \to \infty \) follows from integrability of the contract \((c,a)\) and the implied continuation value process \(\{W_t\}\). Moreover, the equality holds as the term \(\theta_t^P\) is the principal’s worst-case scenario under the contract \((c,a)\). Finally, since the inequality holds for an arbitrary admissible contract \((c,a)\) that yields to the agent value \( W_0 \), it implies that \( F(W_0) \geq V(W_0) \).

**Step 3.** The principal’s value function in the sequential problem exceeds the value from the recursive optimal scheme: \( F \geq V \).

For this, we start with considering the solution \((\tau(W), a(W), Y(W))\) to the \(\text{HJB} \) for any admissible \( W \geq 0 \). Associated to this solution, define a recursive optimal scheme by \((c_t, a_t) := (c(W_t), a(W_t))\) together with sensitivity process \( Y_t := (Y(W_t)) \) given by \(\text{[2.12]}\) for the action...
where the corresponding promised value process \( \{W_t\} \) is defined by the flow diffusion equation (3.1) starting with \( W_0 \) for this recursive optimal scheme.

For simplicity we assume that the solution \((c(W), a(W), Y(W))\) to the recursive equation HJB (3.3) is unique.\(^{14}\) Analogous to the argument as in the previous step, letting \( \tau_n := \tau \wedge \inf\{t \geq 0 : W_t \geq n\} \) denote a finite stopping time and applying Ito’s formula to the principal’s value yields

\[
F(W_0) = \mathbb{E}^{a+\theta P} \left[ e^{-r\tau_n} F(W_{\tau_n}) + \int_0^{\tau_n} e^{-rt} \left( a_t + \theta_t^P - c_t \right) dt \right]
\]

\[\xrightarrow{n \to \infty} \mathbb{E}^{a+\theta P} \left[ e^{-r\tau} F(W_{\tau}) + \int_0^{\tau} e^{-rt} \left( a_t + \theta_t^P - c_t \right) dt \right] \leq V(W_0)
\]

where we use the fact that in the retirement region \( F = F_0 \). Here the inequality holds as the recursive optimal scheme \((c, a) \in \mathcal{Z}(W_0)\) is an admissible contract and it yields the promised value \( W_0 \) to the agent. Since the recursive optimal contract is an admissible contract in the sequential primal formulation, the principal’s optimization in the sequential formulation over the (weakly) larger set of admissible contracts yields him a (weakly) better payoff and hence \( F(W_0) \leq V(W_0) \).

**Step 4.** Verifying the regularity condition that the recursive optimal scheme yields an admissible contract.

We have shown that the solution to the HJB equation describes an optimal contract under the regularity assumption that the recursive optimal solution yields an admissible contract \((c, a) \in \mathcal{Z}(W_0)\). In the remainder of this proof, we verify that this regularity property holds.

For this we start with noticing that as the action space \( \mathcal{A} \) and the consumption space \( \mathcal{C} \) are compact subsets in \( \mathbb{R}_+ \), respectively, the mappings \( a(W) \) and \( c(W) \) are bounded on the continuation domain \( \mathcal{S}^c \). Similarly, the optimal sensitivity \( Y(W) \) implied by incentive compatibility condition (2.12) is bounded on the continuation domain \( \mathcal{S}^c \). Furthermore, as the value function \( F \) is continuous in \( W \), the objective function of HJB equation is continuous and hence by the theorem of Maximum as an optimizer the mapping \( Y(W) \) is continuous on \( \mathcal{S}^c \). This therefore shows that rewriting the agent’s continuation value process (3.1) as the following

\[
dW_t = r(W_t + h(a_t) - u(c_t))dt + r\sigma Y_t dB_t^{a+\theta A}, \quad P^{a+\theta A} - a.s.
\]

(6.14)

where \( dB_t^{a+\theta A} = (a_t + \theta_t^A)dt + dB_t \) is a Brownian motion under the measure \( P^{a+\theta A} \) for the agent’s worst-case scenario, the drift term is a bounded function of \( W_t \) and the volatility term is a continuous function of it. Applying Stroock and Varadhan (1997) (Corollary 6.4.4) now

---

\(^{14}\)The analogous arguments apply to general case of allowing for multi-valued correspondence and working with each possible value in the correspondence.
implies that there exists a unique (weak) solution for the agent’s continuation-value process \((W_t)\) starting from \(W_0\).

It remains to verify that the stopping-time \(\tau := \inf\{t \geq 0 : W_t \in \mathcal{S}\}\) associated with the contract \((c,a)\) has a finite random horizon. As the value process \(\{W_t\}\) defined by (6.14) is a one-dimensional Markov process for which the boundaries 0 and \(W_{gp}\) are absorbing, the stopping-time \(\tau\) is finite with probability 1 under the distribution \(P^{a+\theta^A}\) (see, for instance, Helland (1996)). Moreover, as the action \(\{a_t\}\) takes values in the compact set \(\mathcal{A}\) and the drift \(\theta^A(a_t)\) for any \(a_t\) belongs to a compact set \(\Theta(a_t)\), Radon-Nikodym derivative of the distribution \(P^{a+\theta^A}\) with respect to the reference measure \(dP^{a+\theta^A}/dP^0\) given by the Girsanov exponent in (??) has finite moments of any order. Therefore, the stopping-time \(\tau\) is also finite almost surely under the measure \(P^0\). Similarly, for any alternative action process \(\{\hat{a}_t\}\) to which the agent can deviate and the drift process \(\{\hat{\theta}_t\}\), taking values in \(\Theta(\hat{a}_t)\), the resulting process \(\hat{W}_t\) defined by (6.14) using \(dB_t^{\hat{\theta}}\) is a Brownian motion and hence a one-dimensional Markov process. The stopping-time \(\tau\) is therefore finite with probability 1 under any such resulting distribution \(P^{\hat{a}+\hat{\theta}}\).

**Proof of Proposition 3** Recall that non-trivial incentives implies a positive variation in the continuation payoff \(Y > 0\) within the contracting relationship. This in turn implies that the agent perceives the lower envelope to be the worst-case scenario for him: \(\theta^A = a - \kappa(a;\phi)\).

Using the characterization of the principal’s worst-case scenario in (1), we next show that whenever the principal’s worst-case becomes upper envelope, i.e., \(a + \kappa(a;\phi)\), she would rather retire the agent, and hence during the contracting relationship the lower-envelope is the principal’s worst case. In particular whenever upper-envelope is active for some \(W'\) then \(W' > W_{gp}\). Suppose to the contrary and hence for some \(\hat{W} < W'\), and close to \(W_{gp}\) by continuity of \(F\) implies that \(F'(\hat{W}) > F'_o(W_{gp}) = -\frac{1}{\gamma_o}\) where \(\gamma_o\) is the minimum of continuous function \(Y(a) = h'(a)/(1 - r'(a))\) over \(a\) in a compact set \(\mathcal{A}\). Moreover, since the continuation value is in the interior of the continuation region, \(\hat{W} \in (0, W_{gp})\), and \(F(\hat{W}) > F_0(\hat{W})\), taking linear approximation at \(\hat{W}\) yields

\[
F'(\hat{W})W > F_o(W_{gp}) + F'_o(W_{gp}) W_{gp} > C_{gp} + F'(\hat{W})u(C_{gp}).
\]

As \(F'(\hat{W}) > F'_o(W_{gp})\) and the upper envelope in (1) is the worst case, the latter implies that

\[
F''(\hat{W}) = \frac{F(\hat{W}) - a - \kappa(a) + c - F'(\hat{W})(W - u(c) + h(a) + \kappa(a)Y)}{r\sigma^2 Y^2/2} > \frac{-a + h(a)/\gamma_o}{r\sigma^2 Y^2/2} > 0,
\]

contradicting concavity of the principal’s value function.

**Proof of Proposition 4** The proof of (B1) of Proposition 4 follows from a ‘revealed preference’ argument. Specifically, fix any initial promised value \(W\) and pick an optimal action
strategy \((\hat{a}_t)\) under \(\hat{\phi}\). Let \(\hat{c}^a\) and \(\hat{c}^\hat{a}\) be the least costly wage schemes to implement these action strategies under \(\hat{\phi}\) and \(\phi\), respectively. Since \(\hat{\phi} > \phi\), the contract \((\hat{c}^a)\) together with action strategy \((\hat{a}_t)\) yields first-order stochastically better distribution over output paths under \(\phi\) than under \(\hat{\phi}\)

\[
F(W; \phi) = E_{\hat{a} - \kappa(\hat{a}; \hat{\phi})} \left[ \int_0^\tau F_t - \hat{c}^\hat{a}_t dt \right] \leq E_{\hat{a} - \kappa(\hat{a}; \hat{\phi})} \left[ \int_0^\tau F_t - \hat{c}^\hat{a}_t dt \right] \tag{K1}
\]

Moreover, since \(\hat{c}^\hat{a}\) is the least costly way of implementing \(a_\hat{\phi}\) under \(\phi\), we have that:

\[
E_{\hat{a} - \kappa(\hat{a}; \hat{\phi})} \left[ \int_0^\tau X_t - \hat{c}^\hat{a}_t dt \right] \leq E_{\hat{a} - \kappa(\hat{a}; \hat{\phi})} \left[ \int_0^\tau X_t - \hat{c}^\hat{a}_t dt \right] \tag{K2}
\]

Finally, under \(\phi\) the optimal action strategy \((a^*_\phi)\) does weakly better than \((\hat{a}_t)\):

\[
E_{\hat{a} - \kappa(\hat{a}; \hat{\phi})} \left[ \int_0^\tau X_t - \hat{c}^\hat{a}_t dt \right] \leq E_{a^*_\phi - \kappa(a^*_\phi; \phi)} \left[ \int_0^\tau X_t - c^*_{a^*_\phi} dt \right] = F(W; \phi) \tag{K3}
\]

Together with \([K1]+[K3]\) now gives \(F(W; \phi) > F(W; \hat{\phi})\). The latter in turn holds for all \(W\) by Markov property of the solutions to HJBI equation \((3.3)\).

Since the profit function has the same initial condition and the same boundary conditions, and the profit function under higher ambiguity is lower at all values of \(W\). The latter implies that the slope of the profit function must be lower under higher ambiguity at every continuation value \(W\), which shows the part \([B2]\) of Proposition \([4]\)

To show the part \([B3]\) of Proposition \([4]\) we reformulate HJBI in the following form:

\[
P''(W; \phi) = \min_{(c,a,Y) \in \Gamma} \frac{F(W) - (a - \kappa(a; \phi)) + c - F(W)(W - u(c) + h(a) + \kappa(a; \phi)Y)}{r \sigma^2 Y^2 / 2}
\]

By a generalized envelope theorem argument [Milgrom and Segal (2002)]:

\[
P''(W; \phi) := \frac{\partial F''(W; \phi)}{\partial \phi} = \frac{F_{\phi}(W; \phi)}{r \sigma^2 Y^2 / 2} + \frac{\kappa_{\phi}(a; \phi) - 1 - F'(W; \phi)Y}{r \sigma^2 Y^2 / 2} - \frac{F_{\phi}(W; \phi)(W - u(c) + h(a) + \kappa(a; \phi)Y)}{r \sigma^2 Y^2 / 2} < 0
\]

evaluated at the optimal controls \(a, c\) and \(Y\)\[15\]. The first term is negative because \(F_{\phi}(W; \phi) \leq 0\) by the part \([B1]\) of Proposition \([4]\); the second term is negative in view of \(-1 - F'(W; \phi)Y < 0\) - property \([I]\) that characterizes in Proposition \([3]\) the lower envelope as the common worst-case scenario, and by Definition \([I]\) higher strength of ambiguity implies a larger drift-set \(\kappa_{\phi}(a; \phi) > 0\)\[15\]. The third term is also negative because \(F_{\phi}(W; \phi) \leq 0\) by the part \([B2]\) of Proposition \([4]\) and the drift of the agent’s continuation value \((W - u(c) + h(a) + \kappa(a; \phi)Y)\) is non-negative as the zero-action is always feasible.

\[\square\]

\[15\] The parameter \(\phi\) does not affect the distribution \(P\) of the underlying process, which justifies the application of the theorem.
Proof of Proposition 5. Suppose that \( W_t \in C(\phi', t) \). This implies that \( F(W_t, \phi') > F_0(W_t) \). The part (B1) of Proposition 4 implies that \( F(W_t, \phi) > F(W_t, \phi') \), and hence \( W_t \in C(\phi) \).

Proof of Theorem 2. The proof strategy implements monotone comparative statics for three kinds of affine forms relating actions to the degrees of drift ambiguity: constant, linear increasing, and linear decreasing. We treat each in turn below.

**Constant Ambiguity.** Consider a drift ambiguity model with \( \kappa(a; \phi) = \phi_0 \) for all \( a \). With this worst-case scenario the functional equation HJBI (3.3) describing the principal’s contracting problem therefore takes the following form:

\[
\begin{align*}
\sigma \frac{d}{dt} F(W; \phi_0) &= \text{max} \left\{ r(a - \phi_0 - c) + r F'(W; \phi_0) \left( W - u(c) + h(a) - \phi Y \right) \right. \\
&\left. + F''(W; \phi_0) r^2 Y^2 \sigma^2 / 2 \right\}
\end{align*}
\]

subject to the boundary conditions in (3.3).

The incentive compatibility for the constant ambiguity model implies the following relationship between an action \( a \) and incentive compatible variation in the continuation value to the agent \( Y \) required to implement it: \( Y = a \). It does not depend on the constant ambiguity as the agent’s action choice in this model does not affect it. Using this relationship and letting \( m(Y; \phi_0) := (W - u(c) + h(Y) - \phi_0 Y) \) denote the drift of the agent’s continuation value (from the principal’s perspective) expresses the HJBI equation (6.15) in the following form:

\[
\begin{align*}
\sigma \frac{d}{dt} F(W; \phi_0) &= \text{max} \left\{ r \left( Y - \phi_0 - c \right) + r F'(W; \phi_0) m(Y; \phi_0) \right. \\
&\left. + F''(W; \phi_0) r^2 Y^2 \sigma^2 / 2 \right\}
\end{align*}
\]

subject to the same set of boundary conditions as in (6.15).

Interest is in the monotone comparative statics argument for the optimal incentive \( Y \) respect to constant (common) strength of ambiguity \( \phi_0 \). This is based on analyzing the nature of modularity for the objective function with respect to \( Y \) and \( \phi_0 \). For this, we focus on the terms of the objective function that involve interactions between the incentives \( Y \) and strength of ambiguity \( \phi_0 \). The partial derivative of the related terms with respect to \( \phi_0 \):

\[
\mathcal{H}_{\phi_0}(Y) = r F'_\phi(W; \phi_0) m(Y; \phi_0) - r F'(W; \phi_0) Y + F''_{\phi_0}(W; \phi_0) r^2 Y^2 \sigma^2 / 2
\]

We next analyze whether this partial derivative is decreasing/increasing in \( Y \) and determine whether the objective function is sub(super)-modular in \( Y \) and \( \phi_0 \). By envelope theorem applied to the agent’s optimization problem, the first term of the partial derivative \( \mathcal{H}_{\phi_0}(Y) \) does not vary with \( Y \). The last term is also monotone decreasing in \( Y \) as \( F''_{\phi_0}(W; \phi_0) < 0 \) by part (B3) of Proposition 4. The second term, on the other hand, is first decreasing in \( Y \) for low enough continuation values such that \( W \in (W_0, \overline{W}) \) since on this set \( F'(W; \phi_0) \geq 0 \), and then it is increasing in \( Y \) for high enough continuation values such that \( W \in (\overline{W}, W_{gb}) \). As
the sum of these terms the partial derivative $H_{\phi_0}(Y)$ is therefore monotone decreasing in $Y$ and hence the objective function is sub-modular for low enough continuation values such that $W \in (W_o, W)$. For high enough continuation values $W \in (W, W_{gp})$ on the other hand it may be super-modular if super-modularity due to the second term overcomes the sub-modularity due to the other terms.

Summarizing, the characterization of monotone comparative statics for the optimal incentive $Y$ with respect to the constant ambiguity $\phi_0$ is as follows. For low enough continuation values $W \in (W_o, W)$, the objective function in (6.15) is sub-modular in $(Y, \phi_0)$ and by the monotone comparative static argument, the optimal incentive sensitivity $Y$ decreases in $\phi_0$. For high enough continuation values $W \in (W, W_{gp})$, on the other hand, the objective function in (6.15) can be super-modular in $(Y, \phi_0)$ and by the monotone comparative static argument, hence the optimal sensitivity $Y$ can be non-decreasing in $\phi_0$. This yields the result in the part (a) of Theorem 2. We next turn to analyze how the optimal $Y$ changes as the slope term $\phi_1$ changes.

**Ambiguity linearly increasing in action.** Consider a linear drift ambiguity model with $\kappa(a; \phi_1) = \phi_1 a$. With this formulation of the drift ambiguity, using the lower envelope $a(1-\phi)$ as the (common) worst-case scenario the HJBI (3.3) therefore takes the form:

$$rF(W; \phi_1) = \max_{(a,Y) \geq 0} \left\{ r(a(1-\phi_1) - c) + rF'(W; \phi_1)(W - u(c) + h(a) - \phi_1 a Y) + F''(W; \phi_1) r^2 Y^2 \sigma^2 / 2 \right\}$$

(6.16)

subject to the boundary conditions (corresponding to individual rationality requirements), and incentive compatibility [2,12]. The incentive compatibility implies the following relationship between an action $a$ and incentive compatible variation in the continuation value to the agent $Y$ required to implement it: $Y = a / (1 - \phi_1)$. Using this relationship so that $a(Y; \phi_1) = (1 - \phi_1)Y$ and letting $m(Y; \phi_1) = (W - u(c) + h(a(Y; \phi)) - \phi_1 a(Y; \phi) Y)$ denote the drift of the agent’s continuation value (from the principal’s perspective) expresses the HJBI equation (6.16) in the following form:

$$rF(W; \phi_1) = \max_{(Y,c) \geq 0} \left\{ r(a(Y; \phi_1)(1 - \phi_1) - c) + rF'(W; \phi_1)m(Y; \phi_1) + F''(W; \phi_1) r^2 Y^2 \sigma^2 / 2 \right\}$$

subject to the same set of boundary conditions as in (6.16). For monotone comparative statics argument, we focus on the terms of the objective function that involve interactions between incentives $Y$ and the strength of ambiguity $\phi_1$

$$H(Y; \phi_1) := r(1 - \phi_1)^2 Y + rF'(W; \phi_1)m(Y; \phi_1) + F''(W; \phi_1) r^2 Y^2 \sigma^2 / 2$$

(6.17)

We next analyze modularity of this function in terms of $Y$ and $\phi_1$ which in turn determines the monotone comparative statics in the optimal contract for $Y$ in terms of $\phi$. The partial derivative of $H(Y; \phi)$ with respect to $\phi$ is

$$H_{\phi_1}(Y) := -r 2 (1-\phi_1) Y + F'_{\phi_1}(W; \phi)m(Y; \phi_1) + F'(W; \phi_1)m_{\phi}(Y; \phi_1) + F''_{\phi_1}(W; \phi_1) r Y^2 \sigma^2 / 2$$

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The first term of this partial derivative is decreasing in \( Y \) since \( \phi_1 \leq 1 \) for any viable contracting relationship for otherwise the incentive compatibility condition (2.12) implies that the optimal contract would implement zero-action corresponding to termination. The second term at the optimum contract does not vary in \( Y \) by the envelope theorem applied to the agent’s optimization problem. Turning to the third term of the partial derivative, since \( m_{\phi_1}(Y; \phi_1) \) is decreasing in \( Y \), it is decreasing in \( Y \) for low enough continuation value so that \( W \in (W_o, \overline{W}) \) as in this range \( F'(W; \phi_1) \geq 0 \). On the other hand, it is then increasing in \( Y \) for high enough continuation values \( W \in (\overline{W}, W_{gp}) \) for which \( F'(W; \phi_1) < 0 \). Finally, the last term of the partial derivative \( \mathcal{H}_{\phi_1}(Y) \) is monotone decreasing in \( Y \) as \( F''_{\phi_1} < 0 \) by the part (B3) of Proposition 4. Considering the sum of these terms then, except possibly for high enough continuation values such that \( W \in (\overline{W}, W_{gp}) \) the objective function is sub-modular in \( Y \) and \( \phi_1 \) in the linear model, and hence by monotone comparative statics arguments the optimal incentives \( Y \) decreases in the principal’s linear ambiguity \( \phi_1 \).

**Ambiguity linearly decreasing in effort:** In this formulation, the length of drift-set is given by \( \kappa(a; \phi) = \phi_0 - \phi_1 a \). With this linear decreasing formulation of the drift ambiguity, using the lower envelope \( a - \kappa(a; \phi) \) as the (common) worst-case scenario in drift-sets the functional equation HJBI (3.3) describing the principal’s contracting problem therefore takes the form:

\[
rf(W; \phi) = \max_{(a,Y,c) \geq 0} \left\{ r(a - \kappa(a; \phi) - c) + rF'(W; \phi) (W - u(c) + h(a) - \kappa(a; \phi)Y) + F''(W; \phi) r^2 Y^2 \sigma^2 / 2 \right\}
\]

subject to boundary conditions in (3.3), and incentive compatibility (2.12). The latter for the linear decreasing model of drift set implies the following relationship between an action \( a \) and sensitivity in the continuation value to the agent \( Y \) to implement it: \( Y = a/1 + \phi_1 \). Using this relationship \( a(Y; \phi) := (1+\phi_1)Y \) and letting \( m(Y; \phi) = (W - u(c) + h(a(Y; \phi)) - (\phi_0 - \phi_1 a(Y; \phi))Y) \) denote the drift of the agent’s continuation value (from the principal’s perspective) expresses the objective function in the following form:

\[
rf(W; \phi) = \max_{(Y,c) \geq 0} \left\{ r (a(Y; \phi)(1 + \phi_1) - \phi_0 - c) + rF'(W; \phi)m(Y; \phi) + F''(W; \phi) r^2 Y^2 \sigma^2 / 2 \right\}
\]

Notice that in the linear decreasing formulation constant ambiguity \( \phi_0 \) and slope ambiguity \( \phi_1 \) enters into the formulation linearly additivity. Due to this separability, the monotone comparative static analysis of \( Y \) with respect to the constant ambiguity \( \phi_0 \) follows by the arguments analogous to that made earlier in this proof for the constant ambiguity model.

The monotone comparative static analysis with respect to \( \phi_1 \) follows from analogous arguments made above for linear increasing model by a change of variable. This change of variable replaces \( \phi_1 \) in the linear increasing ambiguity model with \(-\phi_1\) to represent drift set becoming smaller as effort increases. In this specification, higher drift ambiguity arises from an increase in \( \phi_0 \) and a decrease in \( \phi_1 \), which expand drift sets. From this change of variables, the analogous characterization follows.
Proof of Proposition 6. The optimal wage is determined by solving the principal’s optimization problem (4.1), and it solves the first-order condition:

\[ 1 = F'(W; \phi)u'(c(W; \phi)) \]

Since \( F'_\phi < 0 \) by the part (B2) of Proposition 4 and \( u \) is concave, this equation implies that \( c(W; \hat{\phi}) < c(W; \phi) \) whenever \( \hat{\phi} \succ \kappa \phi \). Moreover, as the principal’s profits are lower at higher ambiguity levels \( F(W; \hat{\phi}) \leq F(W; \phi) \), by Proposition 4 whenever the two contracts specify the same wage levels for the associated continuation values \( \hat{W} \) and \( W \), respectively, by analogous arguments in Sannikov (2008, Theorem 4) the ratio of volatilities of the agent’s consumption and continuation values is greater in the contract associated with \( \hat{\phi} \). Therefore, the optimal contract associated with higher ambiguity \( \hat{\phi} \) relies less on short-term incentives.

□

Proofs of the Results for Heterogeneous Drift Ambiguity in Section 5

Proof of Proposition 7. The proof of this proposition extends and applies the arguments made in Proposition 4 which characterizes the basic properties of the principal’s value function for the symmetric drift ambiguity. The proof of the part (B1’ Proposition 7) for heterogeneous ambiguity follows from a ‘revealed preference’ argument analogous to that made for symmetric case in the part (B1) of Proposition 4.

Consider first a change in the ambiguous perception of the principal. Specially, suppose that starting from the ambiguity scenario characterized by \( \phi = (\phi^P, \phi^A) \) the ambiguity perceived by the principal increases from \( \phi^P \) to \( \hat{\phi}^P \) i.e., \( \kappa(a; \hat{\phi}^P) > \kappa(a; \phi^P) \) for each \( a \in \mathcal{A} \) while the agent’s ambiguous perception remains unchanged at \( \phi^A \). Denote this new scenario by \( \hat{\phi} := (\hat{\phi}^P, \phi^A) \).

For a ‘revealed preference’ argument, fix any promised value \( W \) and let \((\hat{c}, \hat{a})\) be the optimal contract under the ambiguity scenario \( \hat{\phi} \) that yields value \( W \) to the agent. As \( \kappa(a; \hat{\phi}^P) \geq \kappa(a; \phi^P) \), the lower envelope being the worst-case scenario for the principal, implies that the action choice \((\hat{a}_t)\) yields a better worst-case distribution in first-order stochastic order under \( \phi^P \) than under \( \hat{\phi}^P \):

\[
F(W; \hat{\phi}) := \mathbb{E}_P \left[ \int_0^{\hat{\tau}} e^{-r t} (\hat{a}_t - \kappa(\hat{a}_t; \phi^P) - \hat{c}_t) dt \right] \\
\leq \mathbb{E}_P \left[ \int_0^{\hat{\tau}} e^{-r t} (\hat{a}_t - \kappa(\hat{a}_t; \phi^P) - \hat{c}_t) dt \right]. \quad (K1')
\]

Here the inequality is strict if \( \kappa(a; \hat{\phi}^P) > \kappa(a; \phi^P) \) for each \( a \in \mathcal{A} \).

Now consider the effect of the agent’s ambiguity perceptions and that it increases from \( \phi^A \) to \( \hat{\phi}^A \) such that \( \kappa(a; \hat{\phi}^A) \geq \kappa(a; \phi^A) \) for each \( a \in \mathcal{A} \). It implies from (5.1) that the agent’s action choice increases as his perceived ambiguity decreases and it reduces the agency cost of implementation. In particular, denote by \( \{a_t\} \) the action profile that the wage scheme \((\hat{c})\), which is optimal under the scenario \( \hat{\phi} \), with the associated sensitivity process \((\hat{Y}_t)\) implements under \( \phi^A \). Notice from the incentive compatibility condition (5.1) this incentive scheme
implements a higher action profile under \( \phi^A \) than \( \hat{\phi}^A \), or more precisely: \( \hat{\phi}^A \geq \phi^A \) implies \( a_t \geq \hat{a}_t \), with strict inequality if \( \hat{\phi}^A > \phi^A \). This in turn implies for the principal’s guaranteed payoff

\[
\mathbb{E}_P \left[ \int_0^T e^{-rt} (\hat{a}_t - \kappa(\hat{a}_t; \phi^P) - \hat{c}_t) dt \right] \leq \mathbb{E}_P \left[ \int_0^T e^{-rt} (a_t - \kappa(a_t; \phi^P) - c_t) dt \right] \tag{K2'}
\]

with strict inequality if \( \kappa(a; \hat{\phi}^A) > \kappa(a; \phi^A) \) for each \( a \in \mathcal{A} \).

As the principal’s optimal contract chooses optimally amongst the incentive compatible allocations, under ambiguity scenario \( \phi \) the optimal action strategy \( (a^*_t) \) does weakly better for the principal than the action strategy \( (\hat{a}_t) \), which is optimal under \( \hat{\phi} \):

\[
\mathbb{E}_P \left[ \int_0^T e^{-rt} (\hat{a}_t - \kappa(\hat{a}_t; \phi^P) - \hat{c}_t) dt \right] \leq \mathbb{E}_P \left[ \int_0^T e^{-rt} (a^*_t - \kappa(a^*_t; \phi^P) - c^*_t) dt \right] := F(W; \hat{\phi}) \tag{K3'}
\]

Summarizing, together with \( \text{[K1']} \cdot \text{[K3']} \) implies that \( F(W; \phi) > F(W; \hat{\phi}) \) whenever \((\phi^P, \phi^A) \succ_A (\hat{\phi}^P, \hat{\phi}^A) \). The latter in turn holds for all \( W \) by Markov property of the solutions to the HJBI equation \( \text{[5.2]} \), which shows the part \( \text{[B1']} \) of Proposition 7.

To see that the part \( \text{[B2']} \) holds notice that the profit function has the same initial conditions and the same boundary conditions both under \( \phi \) and \( \hat{\phi} \). This together with the part \( \text{[B1']} \) implies that the slope of the profit function must be lower under higher ambiguity.

To show the part \( \text{[B3']} \) of Proposition 7 we start with reformulating HJBI \( \text{[5.2]} \)

\[
F''(W; \phi) = \min_{(a,Y) \in \Gamma} \frac{F(W; \phi) - (a - \kappa(a; \phi^P)) + c - F'(W; \phi)(W - u(c) + h(a) - \kappa(a; \phi^A)Y)}{r \sigma^2 Y^2 / 2}
\]

and then analyze the effects of changes in the principal’s ambiguity \( \phi^P \) and the agent’s ambiguity \( \phi^A \) on the concavity of the principal’s value function, in turn. By a generalized envelope theorem argument [Milgrom and Segal (2002)] its derivative with respect to \( \phi^P \):

\[
F''_{\phi^P}(W; \phi) := \frac{\partial F''(W; \phi)}{\partial \phi^P} = \frac{F_{\phi^P}(W; \phi)}{r \sigma^2 Y^2 / 2} - \kappa_{\phi^P}(a; \phi^A) \frac{1 + F'(W; \phi)Y}{r \sigma^2 Y^2 / 2}
\]

+ \( \frac{F_{\phi^P}(W; \phi)(W - u(c) + h(a) - \kappa(a; \phi^A)Y)}{r \sigma^2 Y^2 / 2} \) < 0

evaluated at the optimal controls \( a, c \) and \( Y \). The first term is negative as \( F_{\phi^P}(W; \phi) < 0 \) by \( \text{[B1']} \). The second term is zero since the agent’s action choice depends on her perceived ambiguity \( \phi^A \) but not on the principal’s \( \phi^P \), i.e., \( \kappa_{\phi^P}(a; \phi^A) = 0 \). The third and the last term is also negative because \( F''_{\phi^P}(W; \phi) \leq 0 \) by the part \( \text{[B2']} \) of Proposition 7 in the current proof and the drift of the agent’s continuation value, \((W - u(c) + h(a) - \kappa(a; \phi^P)Y)\), is non-negative during the optimal contract as the zero-action is always feasible to implement.

Again by the generalized envelope theorem its derivative this time with respect to \( \phi^A \):
The first term is negative as $F_{\phi^A}(W; \phi) < 0$ by the part \textbf{(B1')} of Proposition \ref{pro:7}. The second term is also negative in view of $1 + F'(W; \phi)Y > 0$ – condition \textbf{(L')} that characterizes the lower envelope as the principal’s worst-case scenario, and by Definition \ref{def:1} that higher strength of ambiguity implies a larger drift-set $\kappa_{\phi^A}(a; \phi^A) > 0$ for each $a \in \mathcal{A}$. The third and the last term is also negative because $F'_{\phi^A}(W; \phi) \leq 0$ by the part \textbf{(B2')} of Proposition \ref{pro:7} and the drift of the agent’s continuation value is non-negative for any non-trivial action.

\textbf{Monotone Comparative Statics for Heterogeneous Drift-Ambiguity in Section\textsuperscript{5.2}}

\textbf{Proof of Theorem} \ref{thm:3}. The proof strategy using monotone comparative statics for heterogeneous drift-ambiguity that in Theorem \ref{thm:2} We first analyze the case of constant ambiguity independent of action, and then turn to the case that specify linear relationship between actions and drift sets.

\textbf{Constant Ambiguity.} For the constant ambiguity model, notice that the incentive compatibility condition \textbf{5.1} does not depend on the strength of (constant) ambiguity, and implies that the required sensitivity to implement a non-trivial action $a$ satisfies $Y(a; \phi^0_{\phi}) = a$. Using this relationship, notice that the drift of the agent’s continuation value from the principal’s perspective is given by $m(Y; \phi^P_0) := (W - u(c) + h(a) - \phi^P_0 Y)$, depends only on the principal’s ambiguity perception $\phi^P_0$ but not on the agent’s $\phi^A_0$. The HJBI formulation (5.2) of the contracting problem therefore takes the form:

$$rF(W; \phi_0) = \max_{(Y,c) \geq 0} \left\{ r(Y - \phi^P_0 - c) + rF'(W; \phi_0) m(Y; \phi^P_0) + \frac{F''(W; \phi_0)}{2} r^2 Y^2 \sigma^2 \right\}$$

subject to the same boundary and smooth-pasting conditions. The monotone comparative statics argument, is based on analyzing the nature of modularity for the objective function with respect to the incentives $Y$ and strength of ambiguity $\phi_0$. For this, we focus on the terms of the objective function that involve interactions between $Y$ and $\phi_0$. We analyze the effects of $\phi^P_0$ and $\phi^A_0$ on the optimal incentive scheme in turn. The partial derivative of the related terms with respect to $\phi^P_0$:

$$\mathcal{H}_{\phi^P_0}(Y) = rF'_{\phi^P_0}(W; \phi)m(Y; \phi^P_0) - rF'(W; \phi_0)Y + F''_{\phi^P_0}(W; \phi_0)r^2 Y^2 \sigma^2 / 2.$$  

We next analyze whether this partial derivative is decreasing/increasing in $Y$ and hence determine whether the objective function is sub(super)-modular in $Y$ and $\phi_0$. The first term of the partial derivative $\mathcal{H}_{\phi^P_0}(Y)$ is monotone decreasing in $Y$. This is because during the contracting relationship $F'_{\phi^P_0}(W; \phi_0) < 0$ by the part \textbf{(B2')} of Proposition \ref{pro:7} and the drift of the agent’s continuation value $m(Y; \phi_0)$ is increasing in $Y$ for any viable action, i.e.,
\( a - \phi_0^P > 0 \), otherwise the principal would optimally implement zero action. The last term is also monotone decreasing in \( Y \) as \( F''_{\phi_0^P}(W; \phi_0) < 0 \) by the part \([B3']\) of Proposition 7. The second term, on the other hand, is first decreasing in \( Y \) for low enough continuation values such that \( W \in (W_o, \overline{W}) \) since on this set \( F'(W; \phi_0) \geq 0 \), and then it is increasing in \( Y \) for high enough continuation values such that \( W \in (\overline{W}, W_{gp}) \). The sum is therefore monotone decreasing in \( Y \) and hence the objective function is sub-modular for low enough continuation values \( W \in (W_o, \overline{W}) \). For high continuation values \( W \in (\overline{W}, W_{gp}) \) on the other hand it may be super-modular if super-modularity due to the second term overcomes the sub-modularity due to the other two terms.

Summarizing then gives the following characterization. For low enough continuation values \( W \in (W_o, \overline{W}) \), the objective function in (6.18) is sub-modular in \((Y, \phi_0^P)\) and the optimal incentive sensitivity \( Y \) decreases in \( \phi_0^P \). On the other hand, for high enough continuation values \( W \in (\overline{W}, W_{gp}) \), the objective function can be super-modular in \((Y, \phi_0^P)\) and hence the optimal sensitivity \( Y \) can be non-decreasing in \( \phi_0^P \). This yields the result stated in the part (a) of Theorem 3 for an increase in \( \phi_0^P \).

We next turn to the comparative statics of \( Y \) with respect to \( \phi_0^A \). For this, we focus on the terms of the objective function (6.18) involving interaction between them and consider its partial derivative with respect to \( \phi_0^A \)

\[
\mathcal{H}_{\phi_0^A}(Y) = rF'_a(W; \phi_0)m(Y; \phi_0^P) + F''_{\phi_0^P}(W; \phi_0)r^2Y^2\sigma^2/2
\]

This partial derivative is monotone decreasing in \( Y \). This is because \( F'_a(W; \phi_0) < 0 \) and \( F''_{\phi_0^P}(W; \phi_0) < 0 \) by the parts \([B2']\) and \([B3']\) of Proposition 7 respectively, and the drift of the continuation value \( m(Y; \phi_0) \) is increasing in \( Y \). The objective function of HJBI (6.18) is therefore sub-modular in \( Y \) and \( \phi_0^A \). By monotone comparative statics argument then, the optimal incentive sensitivity \( Y \) decreases with the agent’s constant ambiguity \( \phi_0^A \). This yields the result stated in the part (b) of Theorem 3 for an increase in \( \phi_0^A \).

**Ambiguity linearly increasing in action.** Consider a linear drift ambiguity model with \( \kappa(a; \phi_1^P) = \phi_1^P a \) and \( \kappa(a; \phi_1^A) = \phi_1^A a \) for all \( a \in A \). Denote this heterogenous ambiguity by \( \phi_1 = (\phi_1^P, \phi_1^A) \). With this formulation of the drift ambiguity, the HJBI formulation (5.2) of the contracting problem now takes the form:

\[
rF(W; \phi_1) = \max_{(a, Y; c) \geq 0} \left\{ r \left( a(1 - \phi_1^P) - c \right) + rF'(W; \phi_1)(W - u(c) + h(a) - \phi_1^P aY) + F''(W; \phi_1)r^2Y^2\sigma^2/2 \right\}
\]

subject to boundary conditions and incentive compatibility (5.1). The latter implies the following relationship between an action \( a \) and incentive compatible variation in the continuation value to the agent \( Y \) required to implement it: \( Y = \frac{a}{1 - \phi_1^A} \). Notice that this condition depends only on the agent’s perception \( \phi_1^A \) but not the principal’s \( \phi_1^P \) as the incentive compatibility condition describes the agent’s decision rule for his action choice for any given \( Y \). Using this relationship so that \( a(Y; \phi_1) = (1 - \phi_1^A)Y \) and letting \( m(Y; \phi) = (W - u(c) + h(a(Y; \phi)) - \phi_1^P a(Y; \phi)Y \) denote the drift of the agent’s continuation value...
(from the principal’s perspective) now expresses the HJBI equation in the following form:

$$rF(W; \phi_1) = \max_{(Y,c) \geq 0} \left\{ r \left( a(Y; \phi_1)(1 - \phi_1^P) - c \right) + rF'(W; \phi_1)m(Y; \phi_1) + F''(W; \phi_1)r^2Y^2\sigma^2/2 \right\}$$  \hfill (6.19)

subject to the same set of boundary conditions.

Analogous to monotone comparative statics analysis in the proof of Theorem 2, the focus is on
the terms of the objective function of HJBI functional equation (6.19) that involve interactions
between the strength of incentives $Y$ and the strength of ambiguity $\phi_1 = (\phi_1^A, \phi_1^P)$

$$\mathcal{H}(Y; \phi_1) := r(1 - \phi_1^A)(1 - \phi_1^P)Y + rF'(W; \phi_1)m(Y; \phi_1) + F''(W; \phi_1)r^2Y^2\sigma^2/2$$  \hfill (6.20)

We next analyze modularity of this function to characterize the monotone comparative statics
of the optimal incentive sensitivity $Y$ with respect to $\phi_1$. We do so first with respect to the
principal’s ambiguity $\phi_1^P$ and then turn to the agent’s linear ambiguity $\phi_1^A$. The partial
derivative of $\mathcal{H}(Y; \phi_1)$ with respect to $\phi_1^P$ (up to positive constant $r$) is

$$\mathcal{H}_{\phi_1^P}(Y; \phi_1) = -(1 - \phi_1^A)Y + F'_{\phi_1^P}(W; \phi_1)m(Y; \phi_1) + F''_{\phi_1^P}(W; \phi_1)m_{\phi_1^P}(Y; \phi_1) + F'''_{\phi_1^P}(W; \phi_1)rY^2\sigma^2/2$$

The first term is decreasing in $Y$ because in any viable contracting relationship $\phi_1^A \leq 1$,
for otherwise the incentive compatibility condition (5.1) implies that the optimal contract
would implement zero-action corresponding to termination. By envelope theorem applied to
the agent’s decision problem, the agent’s continuation value at the optimum does not vary
with differential change in $Y$. Turning to the third term, since $m_{\phi_1^P}(Y; \phi_1) = -(1 - \phi_1^A)Y^2$ is
decreasing in $Y$, it is decreasing in $Y$ for low enough continuation values so that $W \in (W_o, \bar{W})$
as in this range $F'(W; \phi_1) \geq 0$. On the other hand, it is then increasing in $Y$ for high
enough continuation values $W \in (\bar{W}, W_{gp})$ for which $F'(W; \phi_1) < 0$. Finally, the last term of
$\mathcal{H}_{\phi_1^P}(Y; \phi_1)$ is monotone decreasing in $Y$ as $F'''_{\phi_1^P} < 0$ by the part \[B3’\] of Proposition 7.

Considering the sum of these terms then, except possibly for high enough continuation values
such that $W \in (\bar{W}, W_{gp})$ the objective function is sub-modular in $Y$ and $\phi_1^P$ in the linear
model. Therefore, in this case by monotone comparative statics arguments the optimal
incentives $Y$ decreases in $\phi_1^P$. This yields the result stated in the part (a) of Theorem 3 for
an increase in $\phi_1^P$ in an affine increasing model of drift ambiguity.

Next we turn to analyze the monotone comparative statics with respect to the agent’s linear component of ambiguity $\phi_1^A$. Consider the partial derivative the objective function in (6.19) involving interaction between $Y$ and $\phi_1^A$

$$\mathcal{H}_{\phi_1^A}(Y) = -(1 - \phi_1^A)Y + F'_{\phi_1^A}(W; \phi_1)m(Y; \phi_1) + F''_{\phi_1^A}(W; \phi_1)m_{\phi_1^A}(Y; \phi_1) + F'''_{\phi_1^A}(W; \phi_1)rY^2\sigma^2/2$$

For monotone comparative statics argument, we characterize monotonicity of this partial
derivative in $Y$ term by term. The first term of $\mathcal{H}_{\phi_1^A}(Y)$ is decreasing in $Y$. Its second term
does not vary with $Y$ by envelope theorem applied to the agent’s decision problem. Turning to
the third term of $\mathcal{H}_{\phi_1^A}(Y)$, notice that its second component $m_{\phi_1^A}(Y; \phi_1) = -aY + \phi_1^P Y^2 =
-Y^2 (1 - \phi_1^A - \phi_1^P)$ is decreasing in $Y$ for any viable contracting relationship: $1 - \phi_1^A -
\( \phi_1^P \geq 0 \), for otherwise, the principal would optimally implement zero-action. For any viable contract, the third term is then monotone decreasing in \( Y \) for low enough continuation values such that \( W \in (W_o, W) \), as in this range \( F'(W; \phi_1) > 0 \), and it is decreasing in \( Y \) for high enough continuation values such that \( W \in (W, W_{gp}) \) for which \( F'(W; \phi_1) < 0 \). Finally, the last term of \( \mathcal{H}_{\phi_1^A}(Y) \) is monotone decreasing in \( Y \) as \( F''_{\phi_1^A} < 0 \) by the part (B3) of Proposition 7. Summarizing, the partial derivative \( \mathcal{H}_{\phi_1^A}(Y) \) is monotone decreasing in \( Y \) and hence by monotone comparative static argument the optimal incentive sensitivity \( Y \) is decreasing in \( \phi_1^A \) except possibly for high enough continuation values \( W \in (W, W_{gp}) \). This yields the result stated in the part (a) of Theorem 8 for an increase in \( \phi_1^A \) in affine increasing model.

**Ambiguity linearly decreasing in action.** Consider a linear drift ambiguity model with \( \kappa(a; \phi^P) = \phi_0^P - \phi_1^P a \) and \( \kappa(a; \phi^A) = \phi_0^A - \phi_1^A a \). Denote this heterogeneous ambiguity by \( \phi = (\phi^P, \phi^A) \) with \( \phi^i = (\phi_0^i, \phi_1^i) \) for \( i = A, P \).

With this formulation of the drift ambiguity, the HJBI formulation (5.2) of the contracting problem now takes the form:

\[
rf(W; \phi) = \max_{(a, Y, c) \geq 0} \left\{ \left( r(a - \kappa(a; \phi^P)) - c \right) + rf'(W; \phi_1) (W - u(c) + h(a) - \kappa(a; \phi^P)Y) + F''(W; \phi_1)r^2Y^2\sigma^2 / 2 \right\}
\]

subject to boundary conditions and incentive compatibility (5.1).

The latter implies the following relationship between an action \( a \) and incentive compatible variation in the continuation value to the agent \( Y \) required to implement it: \( Y = \frac{a}{1 - \phi_1^A} \). As the incentive compatibility condition describes the agent’s decision rule for his action choice for any given \( Y \), it depends only on linear component of the agent’s perception \( \phi^A \) but not on the principal’s perception \( \phi^P \). Using this relationship so that \( a(Y; \phi^A) = (1 - \phi_1^A)Y \) let \( m(Y; \phi) = (W - u(c) + h(a(Y; \phi)) - \kappa(a(Y; \phi^A); \phi^P)Y) \) denote the drift of the agent’s continuation value (from the principal’s perspective). The latter together with the functional form for the principal’s kappa ambiguity now now yields the HJBI equation in the following form:

\[
rf(W; \phi) = \max_{(Y, c) \geq 0} \left\{ r \left( (1 - \phi_1^A)(1 - \phi_1^P)Y - \phi_0^P - c \right) + rf'(W; \phi)m(Y; \phi) \right. \\
\left. + \frac{F''(W; \phi)r^2Y^2\sigma^2}{2} \right\}
\]

subject to the same set of boundary conditions.

Since the affine from of drift ambiguity are additively separable in constant and slope part, the monotone comparative static analysis made for the constant ambiguity model and for the linear increasing ambiguity model apply analogously. Specifically, notice that in the linear decreasing formulation constant ambiguity \( \phi_0^i \) and slope ambiguity \( \phi_1^i \) for each actor \( i = A \) and \( i = P \) enter into the formulation linearly additively. Due to this separability, the monotone comparative static analysis of \( Y \) with respect to the constant ambiguity \( \phi_0^i \) follow
from the analogous arguments made for constant ambiguity model in the earlier part of this proof.

Similarly the monotone comparative static analysis with respect to $\phi^i_1$ for each actor $i = A$ and $i = P$ follows from analogous arguments made for linear increasing model by a change of variable. As in the monotone comparative static analysis of the model with symmetric ambiguity, this change of variable replaces $\phi^i_1$ in the linear increasing ambiguity model with $-\phi^i_1$ to represent drift set shrinking as effort increases. In this specification, higher drift ambiguity arises from an increase in $\phi^0_0$ and a decrease in $\phi^1_1$, which expand drift sets. From this change of variables, the analogous characterization follows for an affine decreasing model of drift ambiguity.

References

Ahn, D. S. (2008). Ambiguity without a state space. *Rev. Econom. Stud.* 75(1), 3–28.

Aliprantis, C. D. and K. C. Border (2006). *Infinite Dimensional Analysis: A Hitchhiker’s Guide*. Springer Science & Business Media.

Antic, N. (2014). Contracting with unknown technologies. *Unpublished working paper, Princeton University*.

Carroll, G. (2015). Robustness and linear contracts. *American Economic Review* 105(2), 536–63.

Chassang, S. (2013). Calibrated incentive contracts. *Econometrica* 81(5), 1935–1971.

Chen, Z. and L. Epstein (2002). Ambiguity, risk, and asset returns in continuous time. *Econometrica* 70(4), 1403–1443.

Corbae, D., M. B. Stinchcombe, and J. Zeman (2009). *An introduction to mathematical analysis for economic theory and econometrics*. Princeton University Press.

Cvitanic, J., D. Possamaï, and N. Touzi (2018). Dynamic programming approach to principal–agent problems. *Finance and Stochastics* 22(1), 1–37.

Cvitanic, J., X. Wan, and J. Zhang (2009). Optimal compensation with hidden action and lump-sum payment in a continuous-time model. *Applied Mathematics and Optimization* 59(1), 99–146.

Cvitanic, J. and J. Zhang (2013). *Contract theory in continuous-time models*. Springer Finance. Springer, Heidelberg.

DeMarzo, P. and Y. Sannikov (2013). Learning in dynamic incentive contracts. *Working paper, Princeton University*.

Duffie, D. (2001). *Dynamic asset pricing theory* (3rd ed.). Princeton University Press.

Duffie, D. and L. G. Epstein (1992). Stochastic differential utility. *Econometrica: Journal of the Econometric Society*, 353–394.
Dumav, M. and U. Khan (2017). Moral hazard, uncertain technologies, and linear contracts. *Working paper, Uni. Carlos III de Madrid.*

Dumav, M. and U. Khan (2018). Moral hazard with non-additive uncertainty: When are actions implementable? *Economics Letters 171*, 110–114.

Dumav, M. and M. Stinchcombe (2018). Ambiguity aversion and the interpretation of analogies. *Working paper, Department of Economics, Carlos III de Madrid.*

El Karoui, N., S. Peng, and M. C. Quenez (1997). Backward stochastic differential equations in finance. *Mathematical finance 7*(1), 1–71.

Evans, L. C. and P. E. Souganidis (1984). Differential games and representation formulas for solutions of hamilton-jacobi-isaacs equations. *Indiana University mathematics journal 33*(5), 773–797.

Fuchs, W. (2007). Contracting with repeated moral hazard and private evaluations. *American Economic Review 97*(4), 1432–1448.

Garrett, D. F. (2014). Robustness of simple menus of contracts in cost-based procurement. *Games and Economic Behavior 87*, 631–641.

Garrett, D. F. and A. Pavan (2015). Dynamic managerial compensation: A variational approach. *Journal of Economic Theory 159*, 775–818.

Ghirardato, P. (1994). Agency theory with non-additive uncertainty. *Unpublished working paper.*

Gilboa, I. and D. Schmeidler (1989). Maxmin expected utility with non-unique prior. *Journal of mathematical economics 18*(2), 141–153.

Grossman, S. J. and O. D. Hart (1983). An analysis of the principal-agent problem. *Econometrica: Journal of the Econometric Society*, 7–45.

Helland, I. (1996). One-dimensional diffusion processes and their boundaries. *Statistical Research Report.*

Holmström, B. (2017). Pay for performance and beyond. *American Economic Review 107*(7), 1753–77.

Holmstrom, B. and P. Milgrom (1987). Aggregation and linearity in the provision of intertemporal incentives. *Econometrica: Journal of the Econometric Society*, 303–328.

Hurwicz, L. and L. Shapiro (1978). Incentive structures maximizing residual gain under incomplete information. *The Bell Journal of Economics*, 180–191.

Kapićka, M. (2013). Efficient allocations in dynamic private information economies with persistent shocks: A first-order approach. *Review of Economic Studies 80*(3), 1027–1054.

Karni, E. (2006). Subjective expected utility theory without states of the world. *Journal of Mathematical Economics 42*(3), 325–342.
Karni, E. (2009). A reformulation of the maxmin expected utility model with application to agency theory. *Journal of Mathematical Economics* 45(1), 97–112.

Lazear, E. P. (2018). Compensation and incentives in the workplace. *Journal of Economic Perspectives* 32(3), 195–214.

Lin, Y., Z. Ren, N. Touzi, and J. Yang (2020). Random horizon principal-agent problem. *arXiv preprint arXiv:2002.10982*.

Lopomo, G., L. Rigotti, and C. Shannon (2011). Knightian uncertainty and moral hazard. *Journal of Economic Theory* 146(3), 1148–1172.

Maccheroni, F., M. Marinacci, and A. Rustichini (2006). Ambiguity aversion, robustness, and the variational representation of preferences. *Econometrica* 74(6), 1447–1498.

MacLeod, W. B. (2003). Optimal contracting with subjective evaluation. *American Economic Review* 93(1), 216–240.

Miao, J. and A. Rivera (2013). Robust contracts in continuous time. *Boston University*.

Miao, J. and A. Rivera (2016). Robust contracts in continuous time. *Econometrica* 84(4), 1405–1440.

Milgrom, P. and I. Segal (2002). Envelope theorems for arbitrary choice sets. *Econometrica* 70(2), 583–601.

Mukerji, S. (1998). Ambiguity aversion and incompleteness of contractual form. *American Economic Review*, 1207–1231.

Mukerji, S. and J.-M. Tallon (2004). An overview of economic applications of david schmeidler’s models of decision making under uncertainty. In D. Schmeidler and I. Gilboa (Eds.), *Uncertainty in economic theory: Essays in honor of David Schmeidler’s 65th birthday*, pp. 283–301. Routledge.

Olszewski, W. (2007). Preferences over sets of lotteries. *Rev. Econom. Stud.* 74(2), 567–595.

Possamaï, D. and N. Touzi (2020). Is there a golden parachute in sannikov’s principal-agent problem? *arXiv preprint arXiv:2007.05529*.

Quah, J. K.-H. and B. Strulovici (2013). Discounting, values, and decisions. *Journal of Political Economy* 121(5), 896–939.

Sannikov, Y. (2008). A continuous-time version of the principal-agent problem. *The Review of Economic Studies* 75(3), 957–984.

Sannikov, Y. (2014). Moral hazard and long-run incentives. *Unpublished working paper, Princeton University*.

Stroock, D. W. and S. S. Varadhan (1997). *Multidimensional diffusion processes*, Volume 233 of *Grundlehren der mathematischen Wissenschaften*. Springer, Berlin, Heidelberg.
Strzalecki, T. (2013). Temporal resolution of uncertainty and recursive models of ambiguity aversion. *Econometrica* 81(3), 1039–1074.

Szydlowski, M. (2012). Ambiguity in dynamic contracts. *Working paper*.

Williams, N. (2009). On dynamic principal-agent problems in continuous time. *Working paper, University of Wisconsin, Madison*.

Williams, N. (2011). Persistent private information. *Econometrica* 79(4), 1233–1275.

Wu, Y., J. Yang, and Z. Zou (2017). Ambiguity sharing and the lack of relative performance evaluation. *Economic Theory*, 1–17.