Spin-statistics theorem and geometric quantisation

Charis Anastopoulos *

Spinoza Instituut, Leuvenlaan 4, 3584HE Utrecht, The Netherlands

March 31, 2022

Abstract

We study the relation of the spin-statistics theorem to the geometric structures on phase space, which are introduced in quantisation procedures (namely a $U(1)$ bundle and connection). The relation can be proved in both the relativistic and the non-relativistic domain (in fact for any symmetry group including internal symmetries) by requiring that the exchange can be implemented smoothly by a class of symmetry transformations that project in the phase space of the joint system system. We discuss the interpretation of this requirement, stressing the fact that any distinction of identical particles comes solely from the choice of coordinates - the exchange then arises from suitable change of coordinate system. We then examine our construction in the geometric and the coherent-state-path-integral quantisation schemes. In the appendix we apply our results to exotic systems exhibiting continuous “spin” and “fractional statistics”. This gives novel and unusual forms of the spin-statistics relation.

1 Introduction

The relation between spin and statistics is a theorem of relativistic quantum field theory. In the original proof of Pauli [1] the spin-statistics theorem arises as a consequence of: i) the existence of a representation of the Poincaré group, ii) positivity of energy, iii) the necessity that two fields at spacelike separation either commute or anticommute.

This particular proof was valid for free fields: for generic field theories one has to recourse to the axiomatic method: identify a number of postulates as fundamental for a relativistic quantum field theory and recover general properties

* anastop@phys.uu.nl
of the fields as consequence of these axioms. Then indeed, the spin-statistics connection is verified as a theorem [2].

These proofs and their variations assume either the postulate (iii) we gave earlier or a logically equivalent form for it (see [3] for an analysis). However familiar this postulate might be, it is not apparently intuitive in the context of a quantum theory in which fields are the sole fundamental objects. It is true that commutativity at spacelike separations is an indication of locality, in the sense that measurements of commuting quantities can be carried out simultaneously. However, this is not true for anticommutativity. Hence, the physical significance of postulate (iii) arises from the consideration of the relation between quantum fields and relativistic free particles with spin. This requires the introduction of the duality field-particle in our interpretation.

This is one reason, why there has been an effort to prove the relation between spin and statistics in the context of particle quantum mechanics, without making any reference to quantum fields. Another motivation for this effort comes from the fact that non-relativistic quantum mechanics (with the Galilei group as a group of covariance) is a logically complete theory. For this reason it would be desirable to prove the spin-statistics relation making reference solely to concepts of this theory.

Many proofs of the spin-statistics theorem have been found in this context. They typically employ a configuration space representation for the wave functions. The configuration space for a single particle is taken as a product of \( \mathbb{R}^3 \), for the translational degrees of freedom, times the two-sphere \( S^2 \) or the rotation group \( SO(3) \) to account for the spin degrees of freedom.

However, all such proofs use axioms or principles, which lie outside the scope of particle mechanics. Hence, Finkelstein and Rubinstein considered particles as extended solitonic objects and used topological arguments from rubber band twisting [4, 5]; Balachandran et al introduced antiparticles and symmetries reminiscent of the field theoretic CTP [6]; Berry and Robbins employed a particular construction for the transported basis of the spins (using the Schwinger representation of spins) [7, 8]. All constructions need essentially to satisfy a condition identified in [9].

In this paper we shall study the manifestation of the spin statistics relation on the classical phase space of the particles rather than the configuration space. The phase space itself as a symplectic manifold does not have enough structure to support a spin-statistics theorem; we need to add some additional structure to move towards quantum theory. In geometric quantisation [10, 12] this a \( U(1) \) bundle and a connection compatible with the symplectic form.

There are various reasons why we think it is of interest to see the spin-statistics relation in this perspective:

i) We are of the opinion that indistinguishability is a statistical rather than an intrinsic (or ontological) property of physical systems. By this we mean that if it is not possible to distinguish between two particles at all times by properties either intrinsic or extrinsic to them, then any statistical scheme we introduce in order to describe the combined system has to treat these particles as identical.
It makes no difference, whether the corresponding probability theory is quantum or classical. Hence, the study the spin-statistics relationship enables us to compare the quantum and classical notions of indistinguishability, and their consequences, with the aim to identify their geometric origins. The quantum to classical transition is also of interest in this context.

ii) There exists a theory of symplectic group actions (e.g. the Poincaré group) in close analogy and with the same degree of generality as the theory of group representations on a Hilbert space. The spin degrees of freedom arise naturally by the consideration of the symplectic actions of the spacetime symmetry group and do not have to be put in by hand. Furthermore, by stating the spin-statistics relation in a geometric language we may find such relations in more general systems, than ones that have so far been studied (such systems may not involve actual spin degrees of freedom).

iii) The geometric structure that is responsible for the non-trivial spin-statistics relation is known as prequantisation of a symplectic manifold. This is present in all quantisation algorithms either as an object that needs to be introduced \textit{a priori} (geometric quantisation, Klauder’s quantisation \cite{[13]}) or as a structure that arises \textit{a posteriori} after the quantum theory has been constructed (from the study of coherent states). It can, therefore, be argued that this is the minimal structure one needs to add to classical mechanics, before starting the construction of quantum theories. As such, we expect our results to be relevant to formulations of quantum theory, which try to sidestep the Hilbert space formalism (see \cite{[15],[16]} for our perspective).

In our effort to prove the spin-statistics connection, we find, like all previous works, that one needs to introduce an additional postulate. Indeed, we identify a postulate that is simple from a geometrical perspective and show its equivalence to the spin-statistics connection. However, it is equally \textit{ad hoc} as far as the relation with the standard formulation of quantum mechanics is concerned.

In fact, our study does not need to take into account the full quantisation algorithm: the spin-statistics connection can be phrased at the level of prequantisation, \textit{i.e.} before constructing the physical Hilbert space. This latter construction can be achieved in different ways through the introduction of additional structures: in standard geometric quantisation a \textit{polarisation}, in Klauder’s theory a homogeneous metric by which to support a Wiener process. Our condition is compatible with such additional structures. However, it remains equally \textit{ad hoc} in virtue of standard quantum theory. The reason is that the probabilistic/statistical concepts of quantum theory make only indirect (if at all) reference to the geometrical objects that were used in their construction.

In the next section we give a brief, but self-contained summary of the ba-

\footnote{Note, however, that in a deterministic theory particles are distinguishable by virtue of their initial conditions.}
sic ideas of geometric quantisation. We then explain how the combination of subsystems is effected. The study of the systems with $SO(3)$ symmetry is our guide, in order to identify a general group-theoretic postulate that is equivalent to the spin-statistics connection. It is then easy (but rather involved) to generalise for the Poincaré and Galilei groups, and also to systems with more exotic spin and statistics structure. This generalisation is found in the appendix.

Overall, our presentation relies on Souriau’s monograph [10], to which we refer for a detailed treatment of the existing material we have found necessary to include in our paper. We use a different notation, though.

2 Geometric quantisation

2.1 Prequantisation

The state space of classical mechanics is a symplectic manifold, i.e. a manifold $\Gamma$ equipped with a non-degenerate two-form $\Omega$, which is closed ($d\Omega = 0$). $\Omega$ is known as the symplectic form; its physical significance lies in the fact that it provides a map from observables $f$ (functions on $\Gamma$) to vector fields $X_f$ (that generate one-parameter groups of diffeomorphisms) through the assignment

\[ df = \iota_{X_f} \Omega \]  \hspace{1cm} (2.1)

(Here $\iota$ denotes the interior product). Vector fields that can be written as $X_f$ for some $f$ are called Hamiltonian. The Poisson bracket between two functions $f$ and $g$ is then defined as $X_{\{f, g\}} = -\{X_f, X_g\}$.

A group action on the manifold is called symplectic if its generating vector fields are all Hamiltonian.

Passing from classical to quantum mechanics necessitates the introduction of complex-valued objects. The most natural way to achieve this is through a gauge $U(1)$ symmetry. This allows us to implement the rule that Poisson bracket goes to operator commutator.

More precisely the prequantisation of a symplectic manifold $(\Gamma, \Omega)$ consists of a fiber bundle $(Y, \Gamma, \pi)$ with total space $Y$, base space $\Gamma$, fiber and structure group $U(1)$, with $\pi : Y \rightarrow M$ the projection map. In addition $Y$ is equipped with a connection, whose form $\omega$ satisfies $d\omega = \pi^* \Omega$.

An immediate consequence of this definition is that one cannot prequantise all symplectic manifolds: an integrability condition arises, which the symplectic form has to satisfy. This comes from the fact that a connection generates parallel transport along paths. If $A$ is a potential of a connection $\omega$, then the holonomy along a loop $\gamma$ $\exp(i \int_\gamma A)$ equals $\int_\Sigma dA = \int_\Sigma \Omega$, where $\Sigma$ is a two-surface spanning $\gamma$. Since the holonomy is independent of the choice of $\Sigma$, $\int_\Sigma \Omega$ is an integral multiple of $2\pi$ \[12\].

We need to point out two facts that we will use in what follows:

\[ \text{\footnote{One also employs the associated line bundle $(L, \Gamma, \pi')$, which has } \mathbb{C} \text{ as fiber.}} \]
i) The inequivalent prequantisations -if any exist- of a symplectic manifold are classified by the characters of its homotopy group. Hence a simply-connected manifold has a unique prequantisation.

ii) If there exists a symplectic potential (i.e. an one-form $\theta$ on $\Gamma$ such that $d\theta = \Omega$ globally) the prequantizing bundle is trivial.

2.2 Group actions

To any function on $\Gamma$ there corresponds a unique vector field $Y_f$ on $Y$, such that $\omega(Y_f) = \pi^* f$ and $\iota_{Y_f} d\omega = \pi^* df$. Vector fields of the type $Y_f$ generate diffeomorphisms on $Y$ that are known as quantomorphisms.

A group $G$ that acts on $Y$ by symplectomorphisms as $(g \in G, \xi \in Y) \rightarrow g \cdot \xi$ has also a symplectic action on $\Gamma$ as $(g \in G, x \in \Gamma) \rightarrow \pi(g \cdot \xi)$, for any $\xi$ such that $\pi(\xi) = x$. The action of $G$ on $Y$ is then called a lifting of the symplectic action of $G$ on $\Gamma$. Because of topological obstructions not all symplectic group actions can be isomorphically lifted. Often one needs introduce a larger group $G'$ acting on $Y$, which is a covering group of $G$.

We shall denote the defining $U(1)$ action on the bundle as $(z \in U(1), \xi \in Y) \rightarrow z \cdot \xi$.

2.3 The next step

One can construct a Hilbert space from the cross-sections of the line bundle associated with $Y$. From this, a natural assignment of functions on $\Gamma$ to operators on this Hilbert space follows. The Hilbert space is, however, too big compared with the ones of standard quantum mechanics. One needs to restrict in one of its subspaces.

Standard geometric quantisation proceeds by choosing a polarisation $P$, which amounts to choosing a maximal Lagrangian subspace $P_x$ of the complexified tangent space $T^C_x \Gamma$ at each point $x \in \Gamma$. The physical Hilbert space is constructed by all cross-sections of the bundle that are constant along the vector fields of the polarisation. For instance, in the position representation of the free particle the polarisation is generated by the vector fields $\frac{\partial}{\partial p_i}$. In general, there is substantial freedom in the choice of polarisation, but when the system has a symmetry, it is preferable to choose a polarisation that is preserved by the symplectic action of the corresponding group.

A different way to proceed is to consider the space of complex-valued functions on $\Gamma$ and then identify a projection operator onto the physical Hilbert space. In Klauder’s coherent state quantisation the projector is constructed by a path integral, in which the connection form plays dominant role. An homogeneous metric on $\Gamma$ is also necessary, in order to support a Wiener process for the path integral’s definition. The relation between these two types of quantisation is found in [14].

---

3I.e. a subspace in which the symplectic form vanishes.
There are other variations of these themes of quantisation schemes based on geometry. We only want to point out that the justification of the geometric structures introduced in the quantisation are viewed as intermediate steps towards the construction of the Hilbert space. Once arriving there, all physical interpretation takes place through the Hilbert space concepts. In particular, there does not exist an apparent relation between geometric objects and the statistical ones of standard quantum theory.

3 Combination of subsystems

In classical mechanics (or any classical statistical system) the combination of subsystems is effected through the Cartesian product. That is, if \((\Gamma_1, \Omega_1)\) and \((\Gamma_2, \Omega_2)\) are phase spaces associated with two physical systems, then the combined system is described by the Cartesian product \(\Gamma_1 \times \Gamma_2\) and the symplectic form \(\Omega = \Omega_1 \oplus \Omega_2\).

As we mentioned in the introduction, the notion of identical systems is meaningful also in a classical setting, as it is essentially a statistical one. Two systems are identical if they cannot be distinguished at all times by virtue of any internal or external characteristics.

Even though two particles can always be distinguished by virtue of their initial conditions in a deterministic theory, symplectic geometry is also an arena for statistical description of physical systems. Symmetries are generated by a Hamiltonian flow, but this is the case for dynamics only if time-translation is a symmetry. This is not the case in, for instance, open systems.

If we have then two identical systems, any function on \(\Gamma \times \Gamma\), has to be symmetric with respect to the exchange

\[(x_1, x_2) \rightarrow (x_2, x_1).\] (3.1)

The existence of this symmetry amounts to having a probabilistic description in terms of functions on a phase space \(\Gamma_S\). The latter is obtained the following way. We first define the diagonal set \(\Delta = \{(x, x), x \in \Gamma\}\). Then \(\Gamma_S\) is defined as the quotient of \(\Gamma \times \Gamma - \Delta\) with respect to the permutation (3.1). If \(p: \Gamma \times \Gamma - \Delta \rightarrow \Gamma_S\) is the corresponding projection map, there exists a unique symplectic form \(\Omega_S\) on \(\Gamma_S\), such that \(p_* \Omega_S = \Omega \oplus \Omega\).

This definition is easily extended for more than a pair of identical systems. For \(n\) systems we define the diagonal \(\Delta = \{(x_1, x_2, \ldots, x_n) \in \Gamma^n | \exists i, j, s.t. x_i = x_j\}\). The resulting space \(\Gamma_S\) is the quotient of \(\Gamma^n - \Delta\) with respect to the group of permutations.

When two systems \((Y_1, \Gamma_1, \pi_1; \omega_1)\) and \((Y_2, \Gamma_2, \pi_2; \omega_2)\) are combined at the prequantisation level, the total system is described by a fiber bundle with basis space \(\Gamma_1 \times \Gamma_2\) and a total space \(Y\), which is constructed as follows. We define the 1-form \(\tilde{\omega} = \omega_1 \oplus \omega_2\) on \(Y_1 \times Y_2\) and then identify the null direction of \(\tilde{\omega}\).

\[\text{If } X_1, Y_1 \text{ are vector fields on } \Gamma_1 \text{ and } X_2, Y_2 \text{ on } \Gamma_2, \text{ then } \Omega_1 \oplus \Omega_2 \text{ is defined by } \Omega_1 \oplus \Omega_2[(X_1, X_2), (Y_1, Y_2)] = \Omega_1(X_1, Y_1) + \Omega_2(X_2, Y_2). \text{ It is similarly defined for all tensor fields.}\]
i.e a vector field \( Z \) on \( Y_1 \times Y_2 \), such that \( \tilde{\omega}(Z) = 0 \). This defines a foliation on \( Y_1 \times Y_2 \). In fact, a leaf of this foliation is characterised by the group action
\[
(\xi_1, \xi_2) \in Y_1 \times Y_2 \rightarrow (z \cdot \xi_1, z^{-1} \cdot \xi_2), z \in U(1).
\]

Hence one can define \( Y \) as the quotient of \( Y_1 \times Y_2 \) by this group action. The one-form \( \tilde{\omega} \) naturally projects into an 1-form \( \omega \) on \( Y \). Also the projection map
\[
\pi : Y \rightarrow \Gamma_1 \times \Gamma_2
\]
is defined as \( \pi((\xi_1, \xi_2)) = (\pi_1(\xi_1), \pi_2(\xi_2)) \), where we denoted as \([\xi_1, \xi_2]\) the equivalence class of \((\xi_1, \xi_2)\) under the group action (3.2). The action of \( U(1) \) on \( Y \) along the fibers is then \( z \cdot [\xi_1, \xi_2] := [z \cdot \xi_1, z \cdot \xi_2] \).

Let us consider now the case of identical systems. From a pair of \( \Gamma \)'s one can construct uniquely the bundle \((Y, \Gamma \times \Gamma, \pi)\). There exists the action of the permutation group on \( \Gamma \times \Gamma - \Delta \), which can be lifted on \( Y \). If \( \Gamma \) is simply connected, so is \( \Gamma \times \Gamma - \Delta \) and there are two possible ways by which the permutation group may act \( \text{I7} \). Either
\[
[\xi_1, \xi_2] \rightarrow [\xi_2, \xi_1],
\]
or
\[
[\xi_1, \xi_2] \rightarrow [(-1) \cdot \xi_2, \xi_1].
\]

Now if \( i : \Gamma \times \Gamma - \Delta \rightarrow \Gamma \times \Gamma \) is the inclusion map, the actions above also pass into the pull-back bundle \( i^*Y \). From each of these actions we obtained two different quotient spaces from \( i^*Y \) and essentially to different bundles over \( \Gamma_S \) for the prequantisation of the combined system. They correspond respectively to Bose-Einstein and Fermi-Dirac statistics and their total spaces will be denoted as \( Y_B \) and \( Y_F \) respectively. It is easy to see that the connection 1-form and the projection maps pass down from \( Y \) to \( Y_S \) or \( Y_B \).

In fact, the same results are valid for combination of more than two systems. The inequivalent prequantisations of a connected manifold are classified by the characters of its homotopy group. If \( \Gamma \) is connected then the homotopy group of \( \Gamma_S \) is the permutation group, which has only two characters, namely \( \chi_+(P) = 1 \) and \( \chi_-(P) = \sigma(P) \); here \( P \) denotes a permutation and \( \sigma(P) \) its parity.

4 The spin-statistics relation

Symplectic geometry has the attractive feature of an intimate relation with Lie group theory. Given a Lie group, one can determine all symplectic manifolds, upon which it acts with symplectic transformations. They are essentially orbits of the coadjoint action of the group on the dual of its Lie algebra (for details see \[10, 11\]).

This fact provides one of the motivation for studying the spin-statistics relation in the present context, because spin degrees of freedom appear naturally from the representation theory of groups containing space rotations and they do not have to be postulated ad hoc.
For instance, in analogy with Wigner representation theory for the Poincaré group, one can get the symplectic manifolds corresponding to a free massive or massless relativistic particle with spin.

4.1 $SO(3)$ and spin

The easiest way to understand the appearance of spin classically is through the study of the symplectic actions of the group $SO(3)$ of rotations in 3-dimensional space (ignoring all translational degrees of freedom). The symplectic manifolds upon which $SO(3)$ acts transitively have the topology of a two sphere $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3; x_1^2 + x_2^2 + x_3^2 = 1\}$, with the symplectic form

$$\Omega = \frac{1}{2} s \varepsilon_{ijk} x^i dx^j \wedge dx^k$$

(4.1)

(It is a different symplectic manifold for each choice of $s$). The group $SO(3)$ acts as $x^i \rightarrow O_{ij} x^j$ in terms of its fundamental representation.

The prequantisation of $S^2$ is achieved with the use of spinors. First, one can show that a necessary and sufficient integrability condition for $S^2$ to be prequantizable is $s$ to equal $n/2$, with $n$ an integer, i.e. the usual quantum notion of a spin.

Let us consider a two-spinor

$$\xi = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

(4.2) which is normalised to unity $\bar{\xi} \xi = 1$. All such unit spinors span a 3-sphere $S^3$. There exists a natural projection map $\pi: S^3 \rightarrow S^2$ given by

$$[\pi(\xi)]^i = \bar{\xi} \sigma^i \xi,$$

(4.3)

and a natural connection form

$$\omega = -i \bar{\xi} d\xi,$$

(4.4)

and a $U(1)$ action along the fibers $e^{i\phi} \cdot \xi = e^{i\phi} \xi$. The ensuing bundle is known as the Hopf bundle.

For each choice of $n$ we have the action of the group of $n$-th roots of unity on $S^3$:

$$e^{2i\pi r/n} \cdot \xi, \quad r = 0, \ldots, n-1.$$  

(4.5)

The prequantisation of a system characterised by a given value of $s$ is a bundle with total space $Y_n$, which is the space of orbits $[\xi]_n$ of $S^3$ under the action (4.5). The projection $\pi_n: Y_n \rightarrow S^2$ is defined as $\pi_n([\xi]_n) = \pi(\xi)$, while the connection form on $Y_n$ has as pullback on $S^3$ the connection

$$\omega_n = -i n \bar{\xi} d\xi$$

(4.6)

It is easy to check that $d\omega_n = (\pi_n)_* \Omega$ for the value $s = n/2$. 

8
Again using the properties of spinors one obtains a lift of the $SO(3)$ action, or rather of its double cover $SU(2)$ on $Y_n$. If $\alpha \in SU(2)$ then to it there corresponds an $SO(3)$ matrix $O^{ij}(\alpha) = \frac{1}{2} Tr(\alpha^i \sigma^j \alpha \sigma^j)$ acting on $\mathbb{R}^3$ and hence on $S^2$. (As is well known, the map $\alpha \rightarrow O(\alpha)$ is two-to-one.) The action of $SU(2)$ on $S^3$, which lifts the $SO(3)$ symplectic action on $S^2$, is $\xi \rightarrow \alpha \xi$. Since this action commutes with the action (4.5), it passes through the equivalence classes into the bundle $Y_n$ that prequantises the spin system, with arbitrary value of $n$.

Due to the equivalence relation coming from (4.5), if $X$ is an element of the Lie-algebra of $SU(2)$, the corresponding $SU(2)$ group element for the action on $Y_n$ is

$$
\cos(s|X|)1 - i \frac{|X|}{|X|} \sin(s|X|) \sigma^i X_i
$$

(4.7)

This shows that a rotation of $|X| = 2\pi$ performs the transformation

$$
\xi \rightarrow (-1)^n \xi,
$$

(4.8)
i.e. for even values of $n$ the action of the $SU(2)$ matrix $-1$ is identified with the action of unity.

Consider now the combination of two spin systems with spin $s = n/2$. According to the general construction presented earlier the total space for the bundle characterising the system is an equivalence class of a pair of unit spinors $[\xi_1, \xi_2]$ modulo the equivalence relation

$$
(\xi_1, \xi_2) \sim (e^{i\phi} \xi_1, e^{-i\phi} \xi_2) \quad \text{and} \quad (\xi_1, \xi_2) \sim (e^{i2\pi r/n} \xi_1, e^{i2\pi r'/n} \xi_2), \quad r, r' = 0, \ldots, n-1.
$$

There is a natural projection to $S^2 \times S^2$ as well as the two possible actions of the permutation group

$$
[\xi_1, \xi_2] \rightarrow [\xi_2, (-1)^f \xi_1],
$$

(4.9)

We denoted $f = 0$ for the bosonic and $f = 1$ for the fermionic action, each value corresponding to the two possible prequantisations of the combined system.

Clearly, if $G$ is the symmetry group of a phase space $\Gamma$, then $G \times G$ is a symmetry on $\Gamma \times \Gamma$ and its action can be lifted on its prequantizing bundle.

The action of a symmetry group like $SO(3)$ or the Poincaré group can be thought of as corresponding to a change of coordinate system. In the case of identical systems, it is natural to assume that the exchange takes place through a continuous change of coordinates.

In the spin system the $SO(3)$ group action has as integral curves of its generators circles on $S^2$. If we restrict to transformations generated by one element $X$ of the Lie algebra of $SO(3)$, we notice that there are two possible routes from one point $x_1$ to another $x_2$, corresponding to a smooth path of $SO(3)$ actions. The reason is that two points can be connected by two segments of the circle that is defined by the group action, say $\gamma$ and $\gamma'$. We take the convention that $\gamma$ starts from $x_1$ and ends at $x_2$ and $\gamma'$ starts at $x_2$ and ends at $x_1$. 


Let us denote by $g_1$ and $g_2$ the elements of the group $SU(2)$ that satisfy:

$$(g_1, g_2) \cdot [\xi_1, \xi_2] := [g_1 \cdot \xi_1, g_2 \cdot \xi_2] = (-1)^f [\xi_2, \xi_1].$$

One has then that

$$(g_2 g_1) \cdot \xi_1 = (-1)^f \xi_1. \quad (4.10)$$

Since the systems are identical one can represent a state of the system, by two non-coinciding points on a single sphere $S^2$. Let us consider that the exchange $g_1 x_1 = x_2$ and $g_2 x_2 = x_1$ is effected by an one-parameter group of $SO(3)$ actions for each point, that have a common generator. That is, we restrict to an exchange that takes place within the orbit of a single generator, hence both subsystems have to move in the same circle. If the first derivative of the map $t \to g(t)x$ exists, then the transformation is along an integral curve of the vector fields generating the $SO(3)$ action on the sphere.

According to our earlier discussion, our restriction implies that there are two choices for the orbit of the transformation:

i. $x_1 \to x_2$ through $\gamma$ and $x_2 \to x_1$ through $\gamma^{-1}$ (or similarly for $\gamma'$).

ii. $x_1 \to x_2$ through $\gamma$ and $x_2 \to x_1$ through $\gamma'$.

In the first case the orbit of the transformation necessarily passes through the diagonal set and cannot be continuous on the phase space $\Gamma_S$ of the combined system. The second case implies that the action of $g_2 g_1$ on $x_1$ is a rotation of $2\pi$, hence $(g_2 g_1) \cdot \xi_1 = (-1)^n \xi_1$. Comparing with (4.10) we get

$$f = n \mod 2, \quad (4.11)$$

which amounts to the spin-statistics theorem. (The reader may easily verify that the conclusions remain unchanged, if we allow rotations of more than $2\pi$ along the circle: unless $g_2 g_1$ is a rotation of an odd number times $2\pi$ the diagonal set is crossed by the transformation.)

The generalisation to systems of $n$ particles is immediate as any permutation of $n$ objects can be written as a product of exchanges each involving two of them.

### 4.2 Generalisation

Note that $G \times G$ acts on $\Gamma \times \Gamma$, but its action does not descend on $\Gamma_S$, because the diagonal is not preserved. This action can be decomposed into one of the type $(x_1, x_2) \to (gx_1, gx_2)$ and one of the type $(x_1, x_2) \to (gx_1, g^{-1}x_2)$. The latter type does not preserve the diagonal, while the former generically does. It, therefore, descends into an action of $G$ on $\Gamma_S$, the diagonal action. It is clear from our previous discussion that the exchange takes place along the orbits of one-parameter subgroups, which correspond to the diagonal action of $G$ on $\Gamma_S$.

This allows us to identify a postulate, which leads to an analogue of the spin-statistics connection for any quantum mechanical systems, which is characterised by the transitive symplectic action of a group $G$ on the classical state space. In other words, this postulate refers to elementary systems associated to the group $G$. 

10
**Postulate 1:** In combination of two identical systems, each characterised by a symmetry group $G$, it should be possible to obtain the permutation $(3,3)$ by smooth transformations along the orbits of the diagonal action of $G$ on $\Gamma_S$.

This statement can be made explicit as follows:

We assume that the Lie group $G$ acts transitively by symplectomorphisms on $\Gamma$. Then we demand that there should exist two elements $Z_1$ and $Z_2$ of the Lie algebra of $G$, each a scalar multiple of the other, such that

i) one can define the paths $(t \in [0,1], (x_1, x_2)) \rightarrow (e^{Z_1 t} x_1, e^{Z_2 t} x_2)$, on $\Gamma \times \Gamma$ that do not cross the diagonal.

ii) if $g_1 = e^{Z_1}$, $g_2 = e^{Z_2}$, then $g_1 x_1 = x_2$ and $g_2 x_2 = x_1$.

iii) in the lift in the bundle $Y$ we should have $[g_1 \cdot \xi_1, g_2 \cdot \xi_2] = e^{i \theta} \cdot [\xi_2, \xi_1]$, where $e^{i \theta}$ is the phase associated with the exchange.

Then we expect that the spin-statistics relation arises as a consequence.

It is important to emphasise that our postulate singles one particular class of paths by which the exchange should be performed: these are the *orbits of one-parameter subgroups of $G$*, in its diagonal action on $\Gamma_S$.

**Zero spin**  
Our postulate suffices to establish the spin-statistics connection for single spins. Moreover, it is compatible with the bosonic character of the spin zero particles. In the relativistic case the phase space is $\mathbb{R}^6$. It is parametrised by a 4-vector $x$ for a fixed value of $x^0$ (say $x^0 = 0$) and a unit timelike vector $I$.

Then the symplectic form reads

\[
\Omega = m dx^\mu \wedge dI_\mu = dp^i \wedge dx_i,
\]

where $p^i = m I^i$ and the Poincaré groups acts as $x \rightarrow \Lambda x + C$, $I \rightarrow \Lambda I$. Here $\Lambda \in SO(3,1)$ and $C \in \mathbb{R}^4$.

The prequantisation proceeds by constructing a trivial bundle $\mathbb{R}^6 \times S^1$, with elements $(x, I, e^{i \phi})$. The connection form is $\omega = p^i dx_i + d\phi$. The action of the Poincaré group lifts then $(x, I, e^{i \phi}) \rightarrow (\Lambda x + C, \Lambda I, e^{i \phi})$, i.e. it is trivial on the fibers. This implies that the only possible choice of prequantisation for combined systems is the bosonic one, because there does not exist any symmetry transformation that could reproduce the fermionic action of the permutation group.

**Relativistic particles:** In appendix A we demonstrate that postulate 1 provides the correct spin statistic relation also for the case of relativistic and non-relativistic particles with spin. The proof involves no concepts other than the

---

\[5\] A more covariant way to construct it is by the unit timelike vector $I$ and the equivalence class of spacetime points, where $x \sim x'$ if $x - x'$ is parallel to $I$. 

---

11
ones we used in the case of a single spin, but is provided in some detail for reasons of completeness.

**Non-relativistic particles:** For non-relativistic systems one can obtain the description of spin by studying the symplectic actions of the Galilei group. The phase space is a product of the sphere and $\mathbb{R}^6$, which we have already studied. The symmetry group, however, does not factorise into a piece acting on $\mathbb{R}^6$ and one acting on $S^2$. However, it is easy for the reader to verify that our postulate 1 reproduces the spin-statistics connection, with reference now to the Galilei group, rather than the Poincaré.

**Non-trivial systems:** In effect, the spin-statistics theorem in the familiar setting of particles in three spatial dimensions is equivalent to the statement that *a rotation by $2\pi$ of a single particle is physically identical with an exchange* [9]. Postulate 1 reproduces this fact in a general group-theoretical language, thereby providing a generalisation that can be used in a wider class of systems. We demonstrate this in Appendix B. There we study the case of the relativistic particle in three dimensions, where rather surprisingly only bosonic statistics seem to be acceptable. Also we study a simple example for combination of systems with non-simply connected phase spaces. Such systems employ non-trivial prequantizing bundles, hence they have more alternatives than Bose-Fermi for statistics and provide novel versions of the spin-statistics theorem.

### 4.3 After prequantisation

We proceed to study the possible consequences of the spin-statistics relation, with respect to the remaining part of the geometric quantisation procedure.

#### 4.3.1 Wave functions

Standard geometric quantisation proceeds by specifying a complex polarisation $P$ on the phase space of the system. Let us denote by $\Xi_P(\Gamma)$ the space of vector fields, such that at each $x \in M$ the corresponding tangent vector lies in the polarisation. Let us also consider the line bundle $(B, \Gamma, \tilde{\pi})$, which is associated to the bundle $(Y, \Gamma, \pi)$ of the prequantisation of $\Gamma$. This bundle has total space $B = Y \times \mathbb{R}^+$, projection map $\pi(\xi, r) = \pi(\xi)$ and $U(1)$ action $e^{i\phi} \cdot (\xi, r) = (e^{i\phi} \cdot \xi, r)$, for $\xi \in Y$ and $r \in \mathbb{R}^+$.

The connection form $\omega$ induces a covariant derivative $\nabla$ on the cross-sections of $B$. A cross-section $\psi$ of $B$ corresponds to a quantum mechanical wave function if $\nabla_X \psi = 0$ for all vector fields $X \in \Xi_P(M)$, i.e. if the cross-section vanishes in the directions of the polarisation.

Clearly in a system of two identical particles on $\Gamma$, the wave functions are cross-sections $\psi(x_1, x_2)$ of a bundle over $\Gamma \times \Gamma$. Assuming that $|\psi|^2(x_1, x_2)$
remains invariant from the exchange \(^6\), and that both \(\Gamma\) carry the same polarisation, we obtain the two possible behaviors for \(\psi\) according to the choice of the action of the permutation (3.3)
\[
\psi(x_2, x_1) = (-1)^f \psi(x_1, x_2) \tag{4.13}
\]
A continuous cross-section \(\psi\) satisfying (4.13) for either choice of \(f\) can be viewed as as a cross-section of either of the bundles \(Y_B\) or \(Y_F\) defined earlier.

Given a group action on \(\Gamma\), one can often construct polarisations that are left invariant under the symplectomorphisms by which the group acts; this is true, for instance, for the Poincaré group. Therefore, if \(g(t)\) is an one-parameter group of transformations of such a group, then the transformation
\[
\psi \to g(t) \cdot \psi(x) = \psi(g^{-1}(t)x), \quad x \in \Gamma \tag{4.14}
\]
can be defined acting on the wave functions. A smooth cross-section will remain smooth, whenever the \(g(t)\) is smooth, as we have demanded.

One can then define the action of \(G \times G\) on the wave functions of a combined system (on \(\Gamma \times \Gamma\))
\[
\psi(x_1, x_2) \to \psi(g_1(t)x_1, g_2(t)x_2). \tag{4.15}
\]
This action descends into the action on wave functions on \(\Gamma_S\) (of either the bosonic or fermionic type) if it preserves the diagonal i.e. if for \(x_1 \neq x_2\), \(g_1(t)x_1 \neq g_2(t)x_2\) for all \(t \in [0,1]\). In that case one can write the law
\[
\psi(x_1, x_2) \to \psi(e^{Zt_1}x_1, e^{Zt_2}x_2), \tag{4.16}
\]
where \(Z\) is an element of the Lie algebra of \(G\) and \(t_1, t_2 \in \mathbb{R}\). In light of these remarks, postulate 1 can be rephrased as

**Postulate 1a:** One can perform the exchange (4.13) by means of a smooth transformation of the type (4.16) acting on wave functions defined on \(\Gamma_S\).

### 4.3.2 Path integrals

In Klauder’s coherent state quantisation, one considers the Hilbert space of complex valued functions on the phase space \(\Gamma\) and identifies a relevant physical subspace by means of a projection operator, which is defined by a positive hermitian kernel \(K(x_1|x_2), x_1, x_2 \in \Gamma\). A wave function \(\Psi\) on the physical Hilbert space is a function on \(\Gamma\) that can be written in the form
\[
\Psi_{\alpha_l, x'_l}(x) = \sum_l \alpha_l K(x|x'_l) \tag{4.17}
\]
Such functions are parametrised by a finite number of complex numbers \(\alpha_l\) and points \(x'_l \in \Gamma\). A group of transformations on phase space amounts to a
\[^6\text{The exchange is inherited from the principal bundle } Y \text{ so the value of } r = \sqrt{\bar{\psi}\psi} \text{ is not affected.}\]
transformation \( K(x|x') \rightarrow K(gx|gx') \), which can be immediately translated in terms of wave functions.

The kernel \( K \) is constructed by path integration as

\[
K(x_1|x_2) = \lim_{\nu \to \infty} \int Dx(x)e^{i\int x(A^{\nu} - \frac{1}{2}\int d\theta \theta^2)},
\]

(4. 18)

where \( x(\cdot) \) is a path on phase space in the time interval \([0, T]\), \( A \) is a \( U(1) \) potential one-form on \( \Gamma \) (its pullback by \( \pi \) on the bundle \( Y \) is \( \omega \)), \( h \) is a homogeneous Riemannian metric on \( \Gamma \) and the path integration refers to the Wiener measure as supported by the metric \( g \) and constrained by \( x(0) = x_1, x(T) = x_2 \). This quantisation scheme, then, introduces a homogeneous metric in order to arrive at the physical Hilbert space.

A manifold upon which a Lie group \( G \) acts transitively is a homogeneous space. Namely, there exists a metric that accepts \( G \) as a group of isometries and the integral curves of the group action correspond to geodesics. Hence, a transformation \( x_1 \rightarrow gx_1, x_2 \rightarrow gx_2 \) corresponds to diffeomorphisms, which leave the connection form and the metric invariant.

In the combination of two systems the metric goes to \( h_1 \oplus h_2 \). When the systems are identical, there exists a metric on \( h \) such that its pullback on \( \Gamma \times \Gamma \) equals \( h \oplus h \). The holonomy is \( \exp(i\int_{(x_1^2),x_2(1)} A_S) \), with \( A_S \) a potential corresponding to the \( U(1) \) connection over \( \Gamma_S \). Under the exchange the potentials transform according to the structure of the bundle over \( \Gamma_S \). Effectively, the holonomies transform as \( (3.3) \) and since the term with the metric remains invariant under the exchange, we obtain

\[
K(x_1, x_1'|x_2, x_2') = (-1)^f K(x_1', x_1|x_2', x_2)
\]

(4. 19)

This leads, through (4.17), to equation (4.13) for the wave function.

If \( G \) is a symmetry group of \( \Gamma \) then on the kernels for the quantum theory on \( \Gamma \times \Gamma \) there exists the action of the symmetry group \( G \times G \) as

\[
K(x_1, x_2|x_1', x_2') \rightarrow K(g_1 x_1, g_2 x_2|g_1 x_1', g_2 x_2')
\]

(4. 20)

Again, it is not always projected on kernels defined on \( \Gamma_S \) as it does not preserve the diagonal. However, the diagonal action is preserved. It is important to stress that the corresponding vector fields on \( \Gamma_S \) generate isometries of the metric on \( \Gamma_S \). The exchange would then not affect the Wiener measure for the process on \( \Gamma_S \).

Postulate 1 is equivalent to the following one

**Postulate 1b:** The exchange (4.19) can be performed by a transformation along the integral curves of the diagonal action of \( G \) on \( \Gamma_S \).

### 4.4 Internal degrees of freedom:

We would like to consider the spin-statistics connection for degrees of freedom that correspond to internal symmetries (isospin, flavour, ...). One possible procedure would be to introduce a symplectic manifold \( \Gamma_{int} \) for these degrees of freedom.
freedom. Typically $\Gamma_{\text{int}}$ would be a coadjoint orbit of the corresponding symmetry group (SO(3), SU(3), etc).

However, this procedure does not work, because the internal degrees of freedom are genuinely discrete: isospin in nuclear physics takes only the value “proton” or the value “neutron”: there is absolutely no physical meaning to a classical symplectic manifold underlying isospin or flavor. The internal symmetry label the type of particles: in a first-quantised version these symmetries can only act on the indices of the wave function; in a second quantised version on the indices of the fields. They do not enter the process of quantisation.

The correct way is to consider that the different values of the internal degrees of freedom correspond to different copies of the particle phase space. Hence, in the description of isospin we have an elementary phase space consisting of $\Gamma_0 = \Gamma^{(n)} \times \Gamma^{(p)}$. Now $\Gamma^{(n)}$ is isomorphic to $\Gamma^{(p)}$, however, the presence of the internal characterisation as neutron or proton does not render the particles identical: so the physical phase space that describes a neutron and a proton should not be quotiented out by the permutation group. The same would hold for the corresponding prequantising bundles.

An exchange of isospin corresponds to a map $Ex: \Gamma_0 \to \Gamma_0$, such that for $(x, y) \in \Gamma^{(n)} \times \Gamma^{(p)}$, we have $Ex[(x, y)] = (y, x)$. Clearly, such a transformation cannot be effected by the Cartesian product of Poincaré groups that is the symmetry group of $\Gamma_0$.

However, if the physical state of the system is invariant under $Ex$, the neutron and the proton are identified. Hence, the correct physical space is $\Gamma_S$ and the corresponding prequantising bundle $Y_B$ or $Y_F$. The sign $(-1)^f$ characterising the bundle is then interpreted in terms of isospin exchange. Hence, our result $f = n \mod 2$ would be interpreted as saying that the particles with half-integer spin produce a phase of $(-1)$ in the isospin exchange.

In other words, we can either say that we have two identical particles, or that we have two particles characterised by internal quantum numbers, the exchange of which is a symmetry of the physical description. In absence of interactions that break the internal symmetry, there is no way to distinguish between particles with internal quantum numbers (a neutron is distinguished from a proton by virtue of the electromagnetic interaction): hence, at the fundamental level the spin-statistics theorem for internal degrees of freedom is tautological with the spin-statistics theorem for identical particles.

5 Conclusions

Souriau in his monograph notes that “... geometry does not provide the relation between spin and the character $\chi$ (of the permutation) as suggested by experiments...”. We showed that, on the contrary, there is a simple geometric postulate that leads to this relation, but the question remains at the level of the relevance of the geometric description to the basic statistical principles of quantum theory.

There are two points one can make in relation to our result.
First, a transitive group acting on phase space may be viewed passively as corresponding to coordinate changes: this is definitely true for the Poincaré group. The demand then that the exchange should be implementable with a smooth group action might be said to correspond to a statement that the distinction between two particles done in basis of their coordinates is arbitrary and one should be able to exchange them in the statistical description by means of a change of reference frame. Even though our results at present cannot yet fully ascertain this statement, they definitely assert the relevance of the action of the symmetry group to the existence of a spin-statistics theorem.

The requirement that the coordinate transformation proceeds continuously (indeed smoothly) is also of interest. More so, because the description of identical systems on $\Gamma_S$ is specially relevant when we consider continuous wave functions. The Hilbert space formulation focuses on the measurability properties of the quantum state (over the spectrum of any self-adjoint operator) - similarly as in classical probability Our results might be taken as a hint that continuity (or smoothness) over the phase space is an important ingredient of quantum probability.

In any case, we have showed the spin-statistics connection by means of geometric structures over the phase space, while employing the statistical notion of indistinguishability. Nowhere, was there any need to employ Hilbert space concepts. Indeed, the whole analysis is consistent with formulations of quantum theory phrased solely in terms of geometrical objects - even classes of hidden variable theories. This latter point has indeed been an underlying motivation for this work. In light of our results, it is important to find a relation between our geometric description of the spin-statistics theorem and the standard one that relies on the positivity of the Hamiltonian of the corresponding field. This necessitates a geometric understanding of the procedure of “second quantisation”, which is a focus of our present research.

Aknowledgements

This research was supported through a European Community Marie Curie Fellowship with contract number HPMF-CT-2000-0817.

References

[1] W. Pauli, The Connection Between Spin and Statistics, Phys. Rev. 58, 716 (1942).
[2] R. F. Streater and A. S. Wightman, PCT, Spin, Statistics and All That, (Addison-Wesley, 1989).
[3] I. Duck and E. C. G. Sudarshan, Pauli and the Spin-Statistics Theorem, Am. J. Phys 66, 284 (1998).
[4] D. R. Finkelstein and J. Rubinstein, *Connection Between Spin, Statistics and Kinks*, J. Math. Phys. 9, 1762 (1968).

[5] R. Tscheuschner, *Towards a Topological Spin Statistics Theorem in Quantum Field Theory*, Int. J. Theor. Phys. 28, 1269, 1989; Erratum-ibid. 29, 1437-1438, 1990.

[6] A.P. Balachandran, A. Daughton, Z.C. Gu, G. Marmo, R.D. Sorkin and A.M. Srivastava, *A Topological Spin-Statistics Theorem or a Use of the Antiparticle* Mod. Phys. Lett. A5, 1575 (1990); *Spin Statistics Theorem without Relativity or Field Theory*, Int. J. Mod. Phys. A8, 2993 (1993).

[7] M. V. Berry and J. M. Robbins, *Indistinguishability for Quantum Particles: Spin, Statistics and the Geometric Phase*, Proc. Roy. Soc. Lond. A453, 1771 (1997); *Alternative Constructions of the Transformed Basis*, J. Phys. A: Math. Gen. 33, L207 (2000).

[8] J. Anandan, *Relativistic Spin-Statistics Connection and Kaluza-Klein Spacetime*, Phys. Lett. A248, 124 (1998).

[9] F. Guerra and R. Marra, *A Remark on a Possible Form of the Spin-Statistics Theorem in Nonrelativistic Quantum Mechanics*, Phys. Lett. B 141, 94 (1984).

[10] J. M. Souriau, *Structure of Dynamical Systems: a Symplectic View of Physics*, (Birkhäuser, Boston, 1997).

[11] V. Guillemin and S. Sternberg, *Symplectic Techniques in Physics*, (Cambridge University Press, Cambridge, 1990).

[12] N. M. J. Woodhouse, *Geometric Quantization*, (Clarendon Press, Oxford, 1992).

[13] J. R. Klauder, *Quantization is Geometry After All*, Annals Phys. 188, 120 (1988); *Is Quantization Geometry?*, quant-ph/9604032.

[14] J. R. Klauder and E. Onofri, *Landau Levels and Geometric Quantization*, Int. J. Mod. Phys. A4, 3939 (1989); J. R. Klauder, *Geometric Quantization from a Coherent State Viewpoint*, quant-ph/9510008.

[15] C. Anastopoulos, *Quantum Theory without Hilbert spaces*, Found. Phys. 31, 1545 (2001).

[16] C. Anastopoulos, *Quantum Processes on Phase Space*, Ann. Phys. 303, 275 (2003).

[17] F.J. Bloore, *Configuration Spaces for Identical particles*, in *Differential Geometrical Methods in Mathematical Physics*, edited by P. L. Garcia, A. Perez-Rendon and J. M. Souriau, Lecture Notes in Mathematics 836 (Springer-Verlag, Berlin, 1980).

[18] F. Wilczek, *Quantum Mechanics of Fractional Spin Particles*, Phys. Rev. Lett. 9, 957 (1982).
A Relativistic particles

The symplectic actions of the Poincaré group are classified in complete analogy with Wigner’s classification of Hilbert space representations. If $M^\mu{}^\nu$ and $P^\mu$ are the generators of the Lorentz and translation group respectively, we can define the Pauli-Lubanski four-vector

$$W^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}.$$  \hfill (A. 1)

Then the mass $m$ defined by $P^\mu P_\mu = m^2$, and the spin $s$ defined by $W^\mu W_\mu = -s^2m^2$ are invariants of the action.

The case $s = 0$ was explained in section 4.2.

A.1 $s \neq 0, m \neq 0$

The phase space is $\mathbb{R}^6 \times S^2$. It is parametrised by a 4-vector $x$ for a fixed value of $x^0$, by a unit timelike vector $I$ (corresponding as earlier to 4-momentum) and a unit spacelike vector $J$ (corresponding to a normalised Pauli-Lubanski vector), such that $I^\mu J_\mu = 0$. Note that for a fixed value of $I$, $J$ takes value in a two-sphere (one can readily check that for the case $I = (1, 0)$).

The symplectic form is

$$\Omega = m dx^\mu \wedge dI_\mu + \frac{s}{2} \epsilon_{\mu\nu\rho\sigma} I^\mu J^\nu (dI^\rho \wedge dI^\sigma - dJ^\rho \wedge dJ^\sigma),$$  \hfill (A. 2)

with the Poincaré group acting as $(x, I, J) \rightarrow (\Lambda x + C, \Lambda I, \Lambda J)$.

The prequantisation can be achieved with the use of the Dirac spinor $s$. We remind that a Dirac spinor consists of a pair of two-spinors as

$$\psi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix},$$  \hfill (A. 3)

while the $\gamma$ matrices are defined as

$$\gamma^\mu = \begin{pmatrix} 0 & \tilde{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix},$$  \hfill (A. 4)

where $\sigma^\mu = (1, \sigma^i)$ and $\tilde{\sigma}^\mu = (1, -\sigma^i)$.

Again, it can be shown that one has to restrict to the choice of $s = n/2$.

The explicit construction of the prequantisation is as follows:

Consider the space $Y$ which is the Cartesian product of $\mathbb{R}^3$ (in which the spatial variables $x^i$ live) times a manifold $S$, which consists of Dirac spinors $\psi$ which satisfy the equation

$$\tilde{\psi} \psi = 1 \quad \tilde{\psi} \gamma_5 \psi = 0.$$  \hfill (A. 5)

As usually we have denoted $\tilde{\psi} = \psi^\dagger \gamma^0$ and $\gamma_5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3$. 

18
There exists a projection map \( \pi(x, \psi) = (x^i, I^\mu = \bar{\psi} \gamma^\mu \psi, J^\mu = \bar{\psi} \gamma^\mu \gamma^5 \psi) \), while the \( U(1) \) action up the fibers is

\[
e^{i\phi} \cdot (x, \psi) = (x, e^{i\phi} \psi) \tag{A. 6}
\]

Thus a \( U(1) \) bundle \((Y, \mathbb{R}^6 \times S^2, \pi)\) is constructed.

As in the case of a single spin, we have the action of the group of \( n \)-th roots of the unity on the fibers

\[
e^{i\psi r/n} \cdot (x, \psi), \quad r = 0, \ldots, n - 1 \tag{A. 7}
\]

Taking the quotient of this action we can obtain a bundle \((Y_n, \mathbb{R}^6 \times S^2, \pi_n)\), which has as elements equivalence classes \([x, \psi]_n\) and projection map \(\pi_n([x, \psi]_n) = \pi(x, \psi)\). Upon this bundle one can a connection form, whose pullback on \(Y\) is

\[
\omega_n = -in\bar{\psi} d\psi - m' I_\mu dx^\mu, \tag{A. 8}
\]

where \(m' = nm\).

One can lift the action of the Lorentz group \(SO(3,1)\) to the one of its double cover \(SL(2,\mathbb{C})\). An element \(\alpha \in SL(2,\mathbb{C})\) acts on Dirac spinors by means of the matrix

\[
U(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^\dagger \end{pmatrix} \tag{A. 9}
\]

By virtue of the map \(\psi \to \bar{\psi} \gamma^\mu \psi\), we verify that for each \(\alpha\) there corresponds an element \(\Lambda(\alpha)\) of \(SO(3,1)\) such that

\[
\Lambda^{\mu\nu}(\alpha) = \frac{1}{4} Tr(\bar{U} \gamma^\mu U \gamma^\nu) = \frac{1}{2} Tr(\alpha^\dagger \sigma^\nu \alpha \sigma^\mu), \tag{A. 10}
\]

where \(\bar{U} = \gamma^0 U^\dagger \gamma^0\). The map is two-to-one as \(\pm\alpha\) go to the same \(SO(3,1)\) element.

The action then of \(SL(2,\mathbb{C}) \rtimes \mathbb{R}^4\) on \(Y_n\) is

\[
[x, \psi]_n \to [\Lambda(\alpha)x + C, \alpha \psi]_n. \tag{A. 11}
\]

Note that we have written as \([x, \psi]\) the equivalence class of elements of \(Y\) modulo the action (4.19). We should note again that for even values of \(n\) the action of \(-1 \in SL(2,\mathbb{C})\) is identified - due to (4.19) - with the one of unity. Hence for a rotation of \(2\pi\) we have

\[
(x, \psi) \to (x, (-1)^n \psi) \tag{A. 12}
\]

We are at the position now to consider the combination of identical relativistic particles say \((x_1, I_1, J_1)\) and \((x_2, I_2, J_2)\). Any two points \(x_1, x_2\) can be identified by a space translation; we can, therefore, focus on the \(I\) and \(J\) degrees of freedom and the corresponding action of the Lorentz group. We can choose a coordinate system such as \(I_1 = (1, 0)\), \(J_1 = (0, n)\), with \(n \in S^2\). Consider now the special case that \(I_2 = I_1\) and \(J_2 = (0, n_2)\), the group actions reducing to the ones of \(SO(3)\). The analysis of section 4.1 passes unchanged in this case.
In particular, we can write the path from \( n_1 \) to \( n_2 \) as \( e^{Z_1 t} n_1 \) and from \( n_2 \) to \( n_1 \) as \( e^{Z_2 t} n_2 \), \( Z_1 \) and \( Z_2 \) being elements of the Lie algebra of \( SO(3) \) and \( t \in [0, 1] \). According to our previous analysis, if motion takes place such that the diagonal set is not crossed \( e^{Z_1 t} e^{Z_2 t} \) corresponds to a rotation of \( 2\pi \).

Now we keep \( I_1, J_1 \) fixed in its previous value, but we consider a generic value of \( I_2, J_2 \) by the action of a Lorentz transformation \( \Lambda \) on \( I_2 = (1, 0), J_2 = (0, n_2) \) we considered earlier. The corresponding paths will transform under the adjoint group action \( \Lambda e^{Z_1 t} \Lambda^{-1} \) and \( \Lambda e^{Z_2 t} \Lambda^{-1} \). However, the whole analysis remains identical; if two paths intersect then this property is preserved by the adjoint action. In particular \( (\Lambda e^{Z_1} \Lambda^{-1})(\Lambda e^{Z_1} \Lambda^{-1}) \) still corresponds to a rotation of \( 2\pi \) and hence an action on the bundle as (4.8). The spin-statistics connection then follows.

**A.2 \( m = 0, s \neq 0 \)**

This case can be proven in a similar fashion to the previous one. The construction of the phase space is more intricate, though. For massless particles the Pauli-Lubanski vector is parallel to the momentum four-vector, which is null. The state of the system is, then, more conveniently specified by the use of a spacetime point \( x^\mu \), a null vector \( I \) corresponding to the momentum four-vector and another null vector \( J \), such that \( I^\mu J_\mu = -1 \). Let us denote the space consisting of the triple \( (I, J, x) \) as \( M \).

We then construct the closed two-form

\[
\Omega = -\chi s \epsilon_{\mu\nu\rho\sigma} I^\mu J^\nu dI^\rho \wedge dJ^\sigma + dX^\mu \wedge dI_\mu, \tag{A.13}
\]

where \( \chi = \pm 1 \) is the *helicity* of the particle. The Poincaré group acts as follows as \( (x, I, L) = (\Lambda x + C, \Lambda I, \Lambda J) \). Note that from the pair \( I, J \) one can define the vectors \( K_\pm = \frac{1}{\sqrt{2}} (I \mp J) \), which are unit timelike and spacelike respectively and satisfy \( K_- \cdot K_+ = 0 \).

We can repeat a similar procedure as in the massive case, in order to get a prequantisation on a bundle \( Y \) together with an action of the \( SL(2\mathbb{C}) \rtimes \mathbb{R}^4 \) on \( Y \), with the property that a rotation of \( 2\pi \) corresponds to \( (x, \psi) \rightarrow (x, (1)^n \psi) \) \[18\].

Concerning the combination of subsystems, one can repeat without a change the analysis of the last two paragraphs of A.1; only now it has to make reference to the vectors \( K_- \) and \( K_+ \), which we defined earlier. This establishes the spin-statistics connection for the massless relativistic particles. The only difference is that the phase space is topologically \( \mathbb{R}^4 \times S^2 \), because the symplectic form (A.13) has a null direction corresponding to the vector fields

\[
\alpha^\sigma \left( \chi s \frac{\partial}{\partial x^\sigma} + \epsilon^{\mu\nu\rho\sigma} I_\nu J_\rho \frac{\partial}{\partial J_\mu} \right), \tag{A.14}
\]

where \( \alpha^\mu \) is a four-vector that satisfies \( \alpha_\mu I^\mu = 0 \). These null directions have to be excised if we are to construct the physical phase space for the massless particles.
B  Topologically non-trivial systems

B.1 Three dimensions

An important feature of the phase space analysis is that it can be phrased with respect to any symmetry group: it is not just restricted to groups associated with change of reference frames in four-dimensional Minkowski spacetime. For instance, it is meaningful for the study of systems in three dimensional Minkowski spacetime. The symmetry group there is \( SO(2,1) \cong \mathbb{R}^3 \) with \( SO(2) \) being its subgroup that generates spatial rotations. The phase space of the relativistic particle consists of \( x^\mu \) at constant time \( x^0 \) and a unit timelike vector \( I^\mu \). The symplectic form depends on two parameters: the mass \( m \) and the spin \( s \). However, the topology is \( \mathbb{R}^4 \), hence there exists no sphere corresponding to the spin degrees of freedom. Rather “spin” arises out of a non-trivial symplectic two-form

\[
\Omega = m dx^\mu \wedge dI_\mu + s \epsilon_{\mu\nu\rho} I^\mu dI^\nu \wedge dI^\rho
\]  

(B. 1)

However in \( \mathbb{R}^4 \) all closed two-forms are exact, the phase space has a global symplectic potential and hence the prequantizing bundle is trivial. The spin then takes all real values in the quantum theory.

Also in complete correspondence with the system in 4.3.1 the \( SO(2,1) \) group acts trivially on the fibers. Our postulate then implies that in three dimensions the only possible statistics are the Bose statistics.

We should note that Bose statistics is the only possibility in all systems with a simply-connected phase space that have a global symplectic potential. This includes cotangent bundles over simply connected configuration spaces.

B.2 Fractional statistics

It is important to remark that there is no alternative to the Bose-Fermi statistics if the phase space of the single particle is simply connected. However, for a connected, non-simply connected state spaces \( \Gamma \), the possible prequantisations are classified by the characters \( \chi \) of the homotopy group \( \pi_1(\Gamma) \), that is all homomorphisms from \( \pi(\Gamma) \) to \( U(1) \).

This comes from the fact that if \( \tilde{\Gamma} \) is the universal cover of \( \Gamma \), there exists by definition an action of \( \pi(\Gamma) \) on \( \tilde{\Gamma} \), such that its space of orbits is \( \Gamma \). If \( \tilde{\Gamma} \) admits a prequantisation to a bundle \( (\tilde{Y}, \Gamma, \pi) \) and the action of \( \pi(\Gamma) \) can be lifted then one can quotient as in section 4.3.2 to obtain the prequantizing bundle for \( \Gamma \). If \( (g \in \pi_1(\Gamma), \xi \in \tilde{Y}) \rightarrow g \cdot \xi \) denotes an action of \( \pi_1(\Gamma) \) in the bundle, then also \( (g \in \pi_1(\Gamma), \xi \in \tilde{Y}) \rightarrow \chi(g)g \cdot \xi \) is an inequivalent action that can be used to construct an inequivalent prequantisation of \( \Gamma \).

Fractional statistics of particles were postulated for motion of particles in two spatial dimensions in the presence of a solenoid with flux \( \Phi \) [15]. We can crudely substitute for the effects of the solenoid by excising a point from the configuration space of the particle making the configuration space and hence the phase space non-simply connected. In fact, the resulting phase space has topology \( \mathbb{R}^2 \times (\mathbb{R}^2 - \{0\}) = \mathbb{R}^3 \times S^1 \). The homotopy group is clearly \( \mathbb{Z} \) and
all possible characters of the group are of the form $\chi_{\alpha}(n) = e^{i\alpha n}$, for arbitrary values of $\alpha \in \mathbb{R}$. Now, $\mathbb{R}^4$ is the universal cover of $\Gamma$ and this has a unique prequantisation with a trivial bundle. We shall denote it as $\tilde{\Gamma}$.

Let us denote the action of $\pi(\Gamma) = \mathbb{Z}$ on $\Gamma$ as $(n \cdot (x, I) \rightarrow (n \cdot x, I)$. Explicitly this equals

$$(x_1, x_2) \rightarrow (x_1 + n, x_2 + n)$$

These coordinates are not the homogeneous coordinates $(x, y)$ on $\mathbb{R}^2$ that correspond to the definition of the spatial translations. The latter read in terms of them as $x = e^{(x_1 + x_2)/2} \cos \pi(x_1 - x_2), y = e^{(x_1 + x_2)/2} \sin \pi(x_1 - x_2)$. Effectively the action $n \cdot x$ corresponds to a rotation of $2\pi n$ around the origin.

The possible actions of $\mathbb{Z}$ on the trivial bundle $\mathbb{R}^4 \times U(1)$ are then $n \cdot (x, I, e^{i\phi}) \rightarrow (n \cdot x, I, e^{i\alpha n} e^{i\phi})$. Each of these actions gives a different bundle $Y_\alpha$ as a quotient, this being a different prequantisation of $\Gamma$. This fact is often referred to as implying the need of multivalued wave-functions [18]. Clearly from the perspective of geometric quantisation each choice of $\alpha$ is an intrinsic property of the quantum system and defines a physically different Hilbert space, with wave functions corresponding to cross-sections of different bundles.

We should note that at least the spatial rotations (around the origin) are well defined in our resulting phase space. Its action is in terms of the coordinates $x$ and $y$: $(x, y) \rightarrow (x \cos \phi + y \sin \phi, -x \sin \phi + y \cos \phi)$. This action commutes with the action of $\mathbb{Z}$ and is therefore well defined on $\Gamma$. A rotation of $2\pi n$ is equivalent to a transformation $x \rightarrow n \cdot x$ and hence a phase change of $e^{i\alpha n}$.

Consider now the combination of two such systems. The phase space has as universal cover $\mathbb{R}^4 \times \mathbb{R}^4$ which has a unique trivial prequantisation. On this bundle we have the action of $\mathbb{Z} \times \mathbb{Z}$ as

$$(x_1, x_2, I_1, I_2, e^{i\phi}) = (n \cdot x_1, m \cdot x_2, I_1, I_2, e^{i\alpha_1 n + i\alpha_2 m} e^{i\phi})$$

What constitutes identity of particles is now an issue, because of the nature of the parameters $\alpha_1$ and $\alpha_2$ that have no classical analogue. From the standard quantum treatment of such systems $\alpha$ depends on the details of the experimental setup. If the system represents a solenoid with magnetic flux $\Phi$ through a plane, then $\alpha = q \Phi$, where $q$ is the charge of the particles [18]. Both $q$ and $\Phi$ are parameters characterising uniquely a given system (they are an intrinsic and an extrinsic property respectively), hence identity necessitates both particles to be characterised by the same value of $\alpha$. Now, the resulting phase space for the system is obtained by excising the diagonal, i.e. the set of all $x_1$ and $x_2$ such that $n \cdot x_1 = m \cdot x_2$ or otherwise $(n - m) \cdot x_1 = x_2$. Transformations of the form $(n, -n)$ do not preserve the diagonal and hence cannot be used to define $\Gamma \times \Gamma - \Delta$ as a quotient of $\tilde{\Gamma} \times \tilde{\Gamma} - \Delta$. This implies that $\pi_1(\Gamma \times \Gamma - \Delta) = \mathbb{Z}$, a fact that as can be also checked directly.

So on $\tilde{\Gamma} \times \tilde{\Gamma} - \Delta$ we have a pullback of the bundle on $\tilde{\Gamma} \times \tilde{\Gamma}$ and an action of $\mathbb{Z}$ of the form

$$(x_1, x_2, I_1, I_2, e^{i\phi}) = (n \cdot x_1, n \cdot x_2, I_1, I_2, e^{i\alpha n} e^{i\phi}).$$
Taking the quotient with respect to this action we obtain the bundle that prequantises the combined system, before the implementation of the exchange symmetry (this bundle is what we referred to as $i_*Y$ in section 3). The action of the permutation group is established by the demand that two repeated exchanges ought to give an identity. If the exchange induces a phase change $e^{i\theta}$ on the fibers one needs have $e^{2i\theta} = e^{2in\alpha}$ for some integer $n$. Then $e^{i\theta} = \pm e^{in\alpha}$. But equation (B.4) states that for even $n$ the action of $e^{in\alpha}$ is identical to that of unity. Hence there are only four distinct ways of implementing the phase change: the standard two of Bose and Fermi statistics with multiplication in the fibers by 1 and (-1) respectively, but also two more corresponding to multiplication by $+e^{i\alpha}$ and $-e^{i\alpha}$ respectively.

This means that the permutation group acts as

$$([x_1, x_2, I_1, I_2, e^{i\phi}]_{\alpha} \rightarrow [x_1, x_2, I_1, I_2, (-1)^f e^{i\alpha} e^{i\phi}])_{\alpha},$$ (B. 5)

where both $f$ and $l$ take values 0 and 1. The brackets $[\ldots]_{\alpha}$ denote equivalence classes with respect to the action (B.4).

For $\alpha$ an integer multiple of $\pi$ we have only Bose and Fermi statistics. In this case the charge of the particle is quantised $q = n\frac{\pi}{\Phi}$.

If we employ our principle in this phenomenological system, we will notice that the condition that the smooth paths implementing the rotation do not cross the diagonal again imply that the total rotation $g_1g_2$ (as explained in section 4.1) has to take place an odd number of times $2\pi$. This corresponds to a phase change of $e^{i\alpha}$. Hence, of all possible statistics the one characterised by $f = 0$ and $l = 1$ is selected. Note that for $\alpha = (2k + 1)\pi$ this is Fermi statistics, for $\alpha = 2k\pi$ it is Bose, in the general case it is neither.

To summarise, the system we described admits generically four different statistics for the combined systems and only one of them is allowed by our version of the spin-statistics theorem. This analysis is an illustration of how the framework of geometric quantization and our analysis can be employed to deal with combination of identical systems that admit action of a symmetry group and is not restricted to the standard case of particle in three spatial dimensions.