Boundary Value Problem for
\[ r^2 \frac{d^2 f}{dr^2} + f = f^3 \] (III): Global Solution
and Asymptotics

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Abstract

Based on the results in the previous papers that the boundary value problem
\[ y'' - y' + y = y^3, \quad y(0) = 0, \quad y(\infty) = 1 \]
with the condition \( y(x) > 0 \) for \( 0 < x < \infty \) has a unique solution \( y^*(x) \), and \( a^* = y^*(0) \) satisfies
\( 0 < a^* < 1/4 \), in this paper we show that \( y'' - y' + y = y^3, \quad -\infty < x < 0 \),
with the initial conditions \( y(0) = 0, \quad y'(0) = a^* \) has a unique solution
by using functional analysis method. So we get a globally well defined bounded function \( y^*(x), -\infty < x < +\infty \). The asymptotics of \( y^*(x) \) as \( x \to -\infty \) and as \( x \to +\infty \) are obtained, and the connection formulas for the parameters in the asymptotics and the numerical simulations are also given. Then by the properties of \( y^*(x) \), the solution to the boundary value problem \( r^2 f'' + f = f^3, \quad f(0) = 0, \quad f(\infty) = 1 \) is well described by the asymptotics and the connection formulas.

1 Introduction

In this paper, we use the results obtained in \([2, 3]\) to study the boundary value problem
\[ r^2 f'' + f = f^3, \quad 0 < r < \infty, \quad (1.1) \]

\[ f(r) \to 0, \quad \text{as} \quad r \to 0, \quad (1.2) \]

\[ f(\infty) = 1. \quad (1.3) \]

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If we make a transformation $r = e^x$, $f(r) = y(x)$, the equation is changed to $y'' - y' + y = y^3$.

The results in [2] [3] can be briefly summarized as follows. The following boundary value problem
\[ y'' - y' + y = y^3, \quad 0 < x < \infty, \]
\[ (P^+) \quad y(0) = 0, y(\infty) = 1, \]
\[ y(x) > 0, \quad 0 < x < \infty, \]
has a unique solution $y^*(x)$ for $0 < x < \infty$. The formula of
\[ a^* = y^{*'}(0), \tag{1.4} \]
is obtained, and
\[ 0 < a^* < 1/4. \tag{1.5} \]

In this paper, we show that the initial value problem
\[ y'' - y' + y = y^3, \quad -\infty < x < 0, \]
\[ (P^-) \quad y(0) = 0, \quad y'(0) = a^*, \]
has a unique solution by discussing the corresponding integral equation on a Banach space. Then we get a global solution $y^*(x)$ of $y'' - y' + y = y^3$ for $-\infty < x < +\infty$ and the asymptotics as $x \to -\infty$ and as $x \to +\infty$. And any bounded solution $y(x)$ of this equation satisfying $y(-\infty) = 0, y(+\infty) = 1$ is expressed as $y(x) = y^*(x - \tau)$, where $\tau$ is the largest zero of $y(x)$. Therefore we have solved the boundary value problem (1.1) (1.2) (1.3).

This paper is organized as follows. In Sect. 2, $y^*(x)$ is extended to the negative axis by using contraction mapping theorem. In Sect. 3 we give numerical approximations for some important values including $a^*$, which are used to represent asymptotics of the solution. In Sect. 4, the asymptotic expressions and connection formulas are given.

\section{Global Solution}

In [2], we have proved problem $(P^+)$ has a unique solution $y^*(x) = y(x, a^*)$. In this section we want to extend the solution to the negative axis. Consider the following problem
\[ y'' - y' + y = y^3, \quad -\infty < x < 0, \tag{2.1} \]
\[ (P^-) \quad y(0) = 0, \tag{2.2} \]
\[ y'(0) = a^*. \tag{2.3} \]
Let 
\[ t = -x, \quad u(t) = e^{-x/2}y(x). \]
Then problem \((P^-)\) is equivalent to the following integral equation
\[ u(t) = -2\sqrt{3}a^* \sin \frac{\sqrt{3}}{2}t + 2\sqrt{3} \int_0^t e^{-s} \sin \frac{\sqrt{3}}{2}(t-s)u^3(s) \, ds, \quad (2.4) \]
where \( t \geq 0. \) Define
\[ X = \{ f(t) | f \text{ is continuous on } [0, \infty), |f| \leq \frac{1}{2}, t \in [0, \infty) \}, \quad (2.5) \]
\[ d(f, g) = \sup_{t \in [0, \infty)} |f(t) - g(t)|, \quad (2.6) \]
for \( f, g \in X. \) It is not difficult to prove the following lemma.

**Lemma 1** \((X,d)\) is a complete metric space.

Define
\[ T(u)(t) = -2\sqrt{3}a^* \sin \frac{\sqrt{3}}{2}t + 2\sqrt{3} \int_0^t e^{-s} \sin \frac{\sqrt{3}}{2}(t-s)u^3(s) \, ds, \quad (2.7) \]
for \( u \in X. \)

**Theorem 1** \( T \) has precisely one fixed point \( u^* \) in \( X. \)

**Proof.** Let us first prove \( T(u) \in X \) when \( u \in X. \) By \((2.7), T(u)(t) \) is continuous on \([0, \infty)\). By \((1.3),\)
\[ a^* < \frac{1}{4}, \]
which implies that if \(|u| \leq 1/2,\)
\[ |T(u)(t)| \leq 2\sqrt{3}a^* + 2\sqrt{3} \left( \frac{1}{2} \right)^3 < \frac{\sqrt{3}}{4} < \frac{1}{2}. \]
So \( T(u) \in X. \)

Next we show \( T \) is a contraction on \( X. \) In fact, if \( u_1, u_2 \in X, \) there is
\[ |T(u_1) - T(u_2)| = \left| \frac{2}{\sqrt{3}} \int_0^t e^{-s} \sin \frac{\sqrt{3}}{2}(t-s)(u_1^3 - u_2^3)(u_1^2 + u_1u_2 + u_2^2) \, ds \right| \]
\[ \leq \frac{3}{2\sqrt{3}} \sup_{t \in [0, \infty)} |u_1(t) - u_2(t)| \]
\[ \leq \frac{\sqrt{3}}{2} d(u_1, u_2), \]
which implies
\[ d(T(u_1), T(u_2)) \leq \frac{\sqrt{3}}{2} d(u_1, u_2). \]
So \( T \) is a contraction. By Banach fixed point theorem, this theorem is proved. \( \square \)

**Theorem 2** There is a unique solution \( y^* \) to the problem
\[
\begin{align*}
y'' - y' + y &= y^3, \quad -\infty < x < \infty, \\
(P) \quad y(0) &= 0, \quad y(\infty) = 1, \\
y(x) &= 0, \quad 0 < x < \infty.
\end{align*}
\] (2.8) (2.9) (2.10)
And \( y^* \) has infinitely many zeros \( x_n(n = 0, 1, 2, \ldots) \)
\[-\infty < \ldots < x_n < \ldots < x_1 < x_0 = 0,\]
also
\[ y(-\infty) = 0, \quad |y^*(x)| \leq \frac{1}{2}, \]
for \( x \in (-\infty, 0] \).

**Proof.** The Theorem 2 in [2] and Theorem 1 imply that problem (P) has a unique solution \( y^* \). By (2.8), we see that \( y^* \) has infinitely many zeros, and \( y(-\infty) = 0, |y^*(x)| = |u^*(x)e^x| \leq \frac{1}{2}, \) for \( x \in (-\infty, 0] \). \( \square \)

**Theorem 3** For any integer \( n \geq 0 \), there is a unique solution \( y^{(n)} \) to the following problem
\[
\begin{align*}
y'' - y' + y &= y^3, \quad 0 < x < \infty, \\
y(0) &= 0, y(\infty) = 1, \\
y \text{ has precisely } n \text{ zeros in } (0, \infty).
\end{align*}
\]

**Proof.** By Theorem 2, it is easy to check that
\[ y^{(n)}(x) = y^*(x + x_n) \]
is a solution to this problem, where \( x_n \) is given in Theorem 2.

Now suppose there is another solution \( \bar{y}^{(n)} \) to this problem. Let \( 0 = \bar{x}_0 < \bar{x}_1 < \bar{x}_2 < \ldots < \bar{x}_n \) be the zeros of \( \bar{y}^{(n)} \) in \( [0, \infty) \). Then \( y(s) = \bar{y}^{(n)}(s + \bar{x}_n) \), satisfies (2.8),(2.9),(2.10) with \( x = s \). By Theorem 2, we have \( y(s) = y^*(s), -\infty < s < \infty, \) or \( \bar{y}^{(n)}(x) = y^*(x - \bar{x}_n) \). Then \( \bar{x}_k - \bar{x}_n(k = 0, 1, \ldots, n) \) are the zeros of \( y^*, y^*(\bar{x}_k - \bar{x}_n) = \bar{y}^{(n)}(\bar{x}_k) = 0, \) satisfying \( -\bar{x}_n = \bar{x}_0 - \bar{x}_n < \bar{x}_1 - \bar{x}_n < \cdots < \bar{x}_k - \bar{x}_n < \cdots < \bar{x}_n - \bar{x}_n = 0 \). Because \( 0, \bar{x}_k(k = 1, \ldots, n) \) are the consecutive zeros of \( \bar{y}^{(n)}, \bar{x}_k - \bar{x}_n(k = 0, 1, \ldots, n) \) are the consecutive zeros of \( y^* \). So we see that \( x_n = -\bar{x}_n \). Thus \( \bar{y}^{(n)}(x) = y^*(x - \bar{x}_n) = y^*(x + x_n) = y^{(n)}(x) \). So the theorem is proved. \( \square \)
3 Numerical Results

In this section, we present numerical computation results for some important values related to $y^\ast$. First of all, as seen in the previous sections, the most important number is $a^\ast$, which defines $y^\ast$. In the next section, we investigate the asymptotics of $y^\ast(x)$ as $x$ approaches to $-\infty$ and as $x$ approaches to $\infty$.

The following values will be used to represent the asymptotics,

\begin{align*}
b^\ast &= \int_{-\infty}^{0} e^{-\frac{x}{2}} \cos \frac{\sqrt{3}}{2} s (y^\ast(s))^3 ds, \\
c^\ast &= \int_{-\infty}^{0} e^{-\frac{x}{2}} \sin \frac{\sqrt{3}}{2} s (y^\ast(s))^3 ds, \\
d^\ast &= \int_{0}^{\infty} e^{s} \left(1 - y^\ast(s)^2 - \frac{1}{3}(1 - y^\ast(s))^3\right) ds.
\end{align*}

(1) $a^\ast$

In Theorem 2 of [4], we obtained that

$$\left(\frac{1}{2} + 2 \sum_{n=1}^{N_1} \frac{b_n}{n+1}\right)^{\frac{1}{2}} < a\ast < -\sum_{n=1}^{N_2} b_n,$$

for any $N_1 \geq 2$, and $N_2 \geq 1$. If we choose large numbers $N_1 = 170$, $N_2 = 170$, the Mathematica shows the following result

$$0.16871221576 \cdots < a\ast < 0.16871221594 \cdots.$$  \hspace{1cm} (3.4)

(2) $y^\ast(x)$ and $P(z)$

In last section, we have proved that $y^\ast(x)$ is the unique solution to problem (P). And it was also shown that $y^\ast(x)$ is the solution of the following initial value problem

\begin{align*}
y'' - y' + y = y^3, \quad -\infty < x < \infty, \\
y(0) = 0, \quad y'(0) = a^\ast.
\end{align*}

So we can approach the function $y^\ast(x)$ by numerically solving the initial value problem with properly approximate value of $a^\ast$ given above. The Mathematica gives the graph of $y^\ast(x)$. See Figure 1.

In section 4 and 5, we discussed the function $P(z)$ which is the solution to equation $PP' - P = z(z-1)(z-2), z \in [0, 1]$ with $P(z) \sim -z$ as $z \to 0$. 

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Figure 1: Graph of $y^*(x)$ (solution of (P)). The value of $a^* = y'^*(0)$ is chosen $0.168712 \cdots$.

As mentioned in [3], $P(z)$ gives the relation between $y'^*(x)$ and $y^*(x)$ for $x > 0$, which is

$$y'^*(x) = -P(1 - y^*(x)).$$

The graph of $P(z)$ is computed also by Mathematica, and given in Figure 2. The numerical computation for $P(z)$ shows that $P(1) \approx -0.1687 \cdots$, which matches the result (3.4).

(3) $b^*$, $c^*$, and $d^*$

The values of $b^*$, $c^*$, $d^*$ will be used to represent the asymptotics of $y^*(x)$ as $x \to \pm\infty$, and the asymptotics of $f(r)$ as $r \to 0$ and $r \to \infty$ in the next section. We do not have further analytic properties for these three numbers since we have not found the analytic representation of $y^*$. Computer shows the approximate values of these three numbers based on
Figure 2: Graph of $P(z)$ as solution to the equation $PP' - P = z(z-1)(z-2), z \in [0, 1]$ with $P(z_0) = -z_0$, where $z_0 = 10^{-40}$.

The numerical computation for $y^*(x)$,

$$b^* \approx \int_{-30}^{0} e^{-\frac{s}{2}} \cos \frac{\sqrt{3}}{2}s (y^*(s))^3 \, ds \approx -0.0005497 \cdots, \quad (3.7)$$

$$c^* \approx \int_{-30}^{0} e^{-\frac{s}{2}} \sin \frac{\sqrt{3}}{2}s (y^*(s))^3 \, ds \approx 0.001939 \cdots, \quad (3.8)$$

$$d^* \approx \int_{0}^{15} e^{s} \left( (1 - y^*(s))^2 - \frac{1}{3}(1 - y^*(s))^3 \right) \, ds \approx 4.1728 \cdots. \quad (3.9)$$

4 Asymptotics and Connection Formulas

We have seen in Sect. 2 that the problem (P) has a unique solution $y^*(x)$. Now, let us consider the asymptotics of $y^*(x)$ as $x \to -\infty$, and $x \to \infty$. 

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Theorem 4  The function $y^*(x)$ has the following asymptotics

(i) As $x \to -\infty$,

$$y^*(x) = Ae^{\frac{x}{2}} \sin \left( \frac{\sqrt{3}}{2}x + \phi \right) + O(e^{\frac{3x}{2}}), \quad (4.1)$$

with the connection formula

$$A = \frac{2}{\sqrt{3}} \sqrt{(a^* - b^*)^2 + c^2} \approx 0.196, \quad (4.2)$$

$$\phi = \arctan \frac{c^*}{a^* - b^*} \approx 0.0115 \approx 0.00375 \pi. \quad (4.3)$$

(ii) As $x \to 0$, there is

$$y^*(x) \sim a^* x. \quad (4.4)$$

(iii) As $x \to \infty$,

$$y^*(x) = 1 - Be^{-x} + O(e^{-2x}), \quad (4.5)$$

with the connection formula

$$B = 2 + a^* \approx 4.90. \quad (4.6)$$

Proof. Let us first consider (4.1). Recall that if we let

$$t = -x, u^*(t) = e^{-\frac{x}{2}} y^*(x),$$

then $u^*$ satisfies for $t > 0$

$$u^{**} + \frac{3}{4} u^* = e^{-t} u^* (t),$$

and

$$u^*(t) = -\frac{2}{\sqrt{3}} a^* \sin \left( \frac{\sqrt{3}}{2} t + \frac{2}{\sqrt{3}} \int_0^t e^{-s} \sin \left( \frac{\sqrt{3}}{2} (t - s) u^* (s) \right) ds \right)$$

$$= -\frac{2}{\sqrt{3}} a^* \sin \left( \frac{\sqrt{3}}{2} t + \frac{2}{\sqrt{3}} \int_0^\infty e^{-s} \sin \left( \frac{\sqrt{3}}{2} (t - s) u^* (s) \right) ds + O(e^{-t}),$$

as $t \to \infty$. Direct calculation shows that

$$y^*(x) = \frac{2}{\sqrt{3}} (a^* - b^*) e^{\frac{x}{2}} \sin \left( \frac{\sqrt{3}}{2} x + \frac{2}{\sqrt{3}} e^{\frac{x}{2}} \cos \frac{\sqrt{3}}{2} x + O(e^{\frac{3x}{2}})$$

$$= Ae^{\frac{x}{2}} \sin \left( \frac{\sqrt{3}}{2} x + \phi \right) + O(e^{\frac{3x}{2}}),$$
where $A$ and $\phi$ are given by (4.2)–(4.3).

The asymptotics (4.4) is obvious. Finally let us prove (4.5). Let
\[ y^*(x) = 1 - z(x). \]

It’s easy to check that $z(x)$ satisfies the differential equation
\[ z'' - z' - 2z = -3z^2 + z^3, \] (4.7)
and integral equation
\[
z(x) = \frac{1 - a^*}{3} e^{2x} + \frac{2 + a^*}{3} e^{-x} + \frac{e^{2x}}{3} \int_0^x e^{-2s}(-3z^2(s) + z^3(s))\,ds - \frac{e^{-x}}{3} \int_0^x e^s(-3z^2(s) + z^3(s))\,ds,
\]
since $z(0) = 1, z'(0) = -a^*$. Using equation (4.7) and Lemma 3.1 (iii) in [4], we have
\[
\int_0^\infty e^{-2s}(-3z^2(s) + z^3(s))\,ds = \int_0^\infty e^{-2s}(z''(s) - z'(s) - 2z(s))\,ds = -z'(0) - z(0) + ((-2)^2 + (-2) - 2) \int_0^\infty e^{-2s}z(s)\,ds = a^* - 1.
\]

Thus
\[
z(x) = \frac{2 + a^*}{3} e^{-x} - \frac{e^{-x}}{3} \int_0^x e^s(-3z^2(s) + z^3(s))\,ds + O(e^{-2x}) = \left(\frac{2 + a^*}{3} + d^*\right) e^{-x} + O(e^{-2x}).
\]
So (4.3) holds. □

**Theorem 5** Any solution $y(x)$ to the problem
\[
y'' - y' + y = y^3, \quad -\infty < x < \infty, \] (4.8)
y(x) → 0, as $x \to -\infty$ \hspace{1cm} (4.9)
y(x) → 1, as $x \to \infty$ \hspace{1cm} (4.10)
has the representation
\[ y(x) = y^*(x - \tau), \] (4.11)
where $\tau$ is the largest zero point of $y(x)$. Hence the solution to (4.8), (4.9), (4.10) is unique up to a transition of $x$. 

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Proof. Let $y_1 = y, y_2 = y'$, and change equation (4.8) into the system
\[
\begin{align*}
y_1' &= y_2, \\
y_2' &= y_2 - y_1 + y_1^3.
\end{align*}
\]
It is not hard to see that the point $(0, 0)$ in the phase plane is a stable focus of the system (see [1]). If $y(x)$ is a solution to (1.8), (4.9) and (4.10), then $y(x)$ has infinitely many zeros. Because $y(\infty) = 1$, the zeros of $y(x)$ have upper bound, and the largest zero $\tau$ exists (finite). Let
\[
y_1(x) = y(x + \tau).
\]
Then $y_1(x)$ solves (P). By Theorem 2, $y_1(x) = y^*(x)$, and then (4.11) holds. \qed

The solution to problem (P) is clear now. Let us come back to the original equation (1.1).

**Theorem 6** Any solution to (1.1), (1.2) and (1.3) has the representation
\[
f(r) = y^*(\log \frac{r}{r_0}),
\]
where $r_0$ is the largest zero of $f(r)$. With $a^*, b^*, c^*$ and $d^*$ defined by (1.4), (3.1), (3.2) and (1.3) respectively, $f(r)$ has the following asymptotics.

(i) As $r \to 0$,
\[
f(r) = A \left( \frac{r}{r_0} \right)^{\frac{1}{2}} \sin \left( \frac{\sqrt{3}}{2} \log \frac{r}{r_0} + \phi \right) + O(r^{\frac{3}{2}}),
\]
where again
\[
A = \frac{2}{\sqrt{3}} \sqrt{(a^* - b^*)^2 + c^2} \approx 0.196,
\]
\[
\phi = \arctan \frac{c^*}{a^* - b^*} \approx 0.0115 \approx 0.00375 \pi.
\]

(ii) As $r \to r_0$,
\[
f(r) \sim a^* \log \frac{r}{r_0},
\]

(iii) As $r \to \infty$,
\[
f(r) = 1 - \frac{Br_0}{r} + O\left( \frac{1}{r^2} \right),
\]
where
\[
B = \frac{2 + a^*}{3} + d^* \approx 4.90.
\]

Proof. By Theorem 4 and Theorem 5, using $r = e^x, r_0 = e^{\tau}, \log \frac{r}{r_0} = x - \tau, f(r) = y(x)$, direct calculations show the results. \qed
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