ITERATING THE COFINALITY-\(\omega\) CONSTRUCTIBLE MODEL

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Abstract. We investigate iterating the construction of \(C^*\), the \(L\)-like inner model constructed using first order logic augmented with the “cofinality \(\omega\)” quantifier. We first show that \((C^*)^* = C^* \neq L\) is equiconsistent with ZFC, as well as having finite strictly decreasing sequences of iterated \(C^*\)’s. We then show that in models of the form \(L[U]\) we get infinite decreasing sequences of length \(\omega\), and that an inner model with a measurable cardinal is required for that.

§1. Introduction. The model \(C^*\), introduced by Kennedy, Magidor, and Väänänen in [5], is the model of sets constructible using the logic \(L(Q_{\text{cf}}\kappa)\)—first order logic augmented with the “cofinality \(\omega\)” quantifier. To be precise, for a regular cardinal \(\kappa\) we define a logic \(L(Q_{\text{cf}}\kappa)\) where we add to the syntax of first order logic a quantifier \(Q_{\text{cf}}\kappa\) whose semantics are given by

\[
\mathcal{M} \models Q_{\kappa}^{\text{cf}} xy\varphi(x, y, \vec{b}) \iff \{(c, d) \in M^2 \mid \mathcal{M} \models \varphi(c, d, \vec{b})\}
\]

is a linear order of cofinality \(\kappa\).

This logic, first introduced by Shelah [10], is a proper extension of first order logic, which is fully compact. Using this logic we construct inner models of set theory, akin to Gödel’s constructible universe \(L\):

Definition 1.1. \(C^*_\kappa\), the class of \(L(Q_{\text{cf}}\kappa)\)-constructible sets, is defined by induction:

\[
L'_0 = \varnothing,
\]

\[
L_{\alpha+1} = \text{Def}_{L(Q_{\text{cf}}\kappa)}(L'_\alpha),
\]

\[
L'_\beta = \bigcup_{\alpha < \beta} L'_\alpha \quad \text{for limit } \beta.
\]

\[
C^*_\kappa = \bigcup_{\alpha \in \text{Ord}} L'_\alpha,
\]

where for any logic \(L\) extending first-order logic and a set \(M\),

\[
\text{Def}_L(M) := \left\{ a \in M \mid (M, \in) \models_L \varphi(a, \vec{b}) \mid \varphi \in L; \vec{b} \in M^{<\omega} \right\}.
\]

We focus on the case \(\kappa = \omega\) and denote \(C^* := C^*_\omega\). As shown in [5], \(C^*\) is a model of ZFC, and in fact it is the same as \(L[\text{Ord}_\omega]\) where \(\text{Ord}_\omega = \{ \alpha \in \text{Ord} \mid \text{cf}(\alpha) = \omega \}\).
Like in the case of $L$, one can phrase the formula \( V = C^* \), i.e., $\forall x \exists \alpha (x \in L^*_\alpha)$ where $L^*_\alpha$ is the $\alpha$th level in the construction of $C^*$. Unlike $L$, however, it is not always true that $C^* \models V = C^*$, which is equivalent to the question whether $(C^*)^{C^*} = C^*$. This is clearly the case if $V = L$, so the interesting question is whether this can hold with $C^* \neq L$. In Section 2 we show that this is consistent relative to the consistency of ZFC. Next we investigate the possibilities of $C^* \not\models V = C^*$. In such a case, it makes sense to define recursively the iterated $C^*$s:

\[
\begin{align*}
C^*^0 &= V, \\
C^*^{\alpha+1} &= (C^*)^{C^*^\alpha} \text{ for any } \alpha, \\
C^*^\alpha &= \bigcap_{\beta < \alpha} C^*^\beta \text{ for limit } \alpha.
\end{align*}
\]

This type of construction was first investigated by McAloon [8] regarding HOD, the class of hereditarily ordinal definable sets, where he showed that it is equiconsistent with ZFC that there is a strictly decreasing sequence of iterated HODs of length $\omega$, and the intersection of the sequence can be either a model of ZFC or of ZF + $\neg$AC. Harrington also showed (in unpublished notes; cf. [14]) that the intersection might not even be a model of ZF. Jech [4] used forcing with Suslin trees to show that it is possible to have a strictly decreasing sequence of iterated HODs of any arbitrary ordinal length, and later Zadrożny [13] improved this to an Ord length sequence. In [14] Zadrożny generalized McAloon’s method and gave a more flexible framework for coding sets by forcing, which he used to give another proof of this result. HOD can be described also as the model constructed using second order logic (as shown by Myhill and Scott [9]), so it is natural to ask which of the results for HOD apply to other such models, specifically to $C^*$. In Section 3 we show that unlike the case of HOD, without large cardinals we can only have finite decreasing sequences of iterated $C^*$, and that the existence of an inner model with a measurable cardinal is equivalent to the existence of an inner model with a strictly decreasing $C^*$ sequence of length $\omega$.

§2. Relative consistency of \( V = C^* \neq L \). In this section we follow the method of Zadrożny [14] to obtain the following result:

**Theorem 2.1.** If ZFC is consistent then so is ZFC + $V = C^* \neq L + 2^{\aleph_0} = \aleph_1$.

The idea of Zadrożny’s proofs, which are based on results of McAloon’s [7, 8], is to add a generic object (to make $V \neq L$), code it using some other generic object, then code the coding, and so on, iterating until we catch our tail. Our coding tool will be the modified Namba forcing of [5, Section 6], which adds a countable cofinal sequence to any element of some countable sequence of regular cardinals $> \aleph_1$ (and only to them).

**Definition 2.2.** Suppose $S = \langle \lambda_n \mid n < \omega \rangle$ is a sequence of regular cardinals $> \omega_1$ such that every $\lambda_n$ occurs infinitely many times in the sequence. Let $\langle B_n \mid n < \omega \rangle$
be a partition of \( \omega \) into infinite sets. The modified Namba forcing with respect to \( S \) is defined as follows. Conditions are trees \( T \) with \( \omega \) levels, consisting of finite sequences of ordinals, such that if \( (\alpha_0, \ldots, \alpha_i) \in T \) and \( i \in B_n \) then:

1. \( \alpha_i < \lambda_n \).
2. \(|\{ \beta \mid (\alpha_0, \ldots, \alpha_{i-1}, \beta) \in T \}| \in \{1, \lambda_n\} \).
3. For every \( m \) there are \( \alpha_{i+1}, \ldots, \alpha_{k-1} \) such that \( k \in B_m \) and \(|\{ \beta \mid (\alpha_0, \ldots, \alpha_{k-1}, \beta) \in T \}| = \lambda_m \).

A condition \( T' \) extends another condition \( T, T' \leq T \), if \( T' \subseteq T \).

If \( \langle T_n \mid n < \omega \rangle \) is a generic sequence of conditions, then the stems of the trees \( T_n \) form a sequence \( \langle \alpha_n \mid n < \omega \rangle \) such that \( \langle \alpha_i \mid i \in B_n \rangle \) is cofinal in \( \lambda_n \). Thus, in the generic extension of \( \langle \lambda_n \rangle = \omega \) for all \( n < \omega \), and KMV show that these are the only regular cardinals that change their cofinality to \( \omega \). Furthermore, it is shown that revised countable support iterations of this forcing preserves \( \omega_1 \).

In [5, Theorem 6.7], these tools are used to produce a model of ZFC + \( V = C^* \neq L + 2^{\aleph_0} = \aleph_2 \), but this requires an inaccessible cardinal (as proven there as well). Here we show that we can get a model of ZFC + \( V = C^* \neq L \) from ZFC alone, and this model will satisfy \( 2^{\aleph_0} = \aleph_1 \). These two results covers all possibilities, since in [5, Corollary to Theorem 5.20], it is shown that the statement \( V = C^* \) implies that \( 2^{\aleph_0} \in \{ \aleph_1, \aleph_2 \} \), and for any \( \kappa > \aleph_0 \) \( 2^\kappa = \kappa^+ \).

We say that a set \( X \) is nice if it is a countable set of ordinals which does not contain any of its limit points, and such that \( L[X] \) agrees with \( L \) on \( \omega_1 \) and on regular cardinals starting from \( \aleph_{\sup X + 2} \). Note that if \( S \in L \) is a countable sequence of regular \( L \) cardinals \( \geq \aleph_1 \), and \( X \) is generic for the modified Namba forcing corresponding to \( S \), then \( X \) is nice as the size of the forcing is \( \leq \aleph_{\sup X + 1} \) (we assume GCH, at least starting from this point). Assume \( V = L[A_0] \) where \( A_0 \) is nice. Set \( P_0 = \{1\} \), and inductively we assume that \( P_n \) forces the existence of a nice set of ordinals \( A_n \), denote \( \xi_n = \sup A_n \) and we set \( P_{n+1} = P_n \ast \dot{Q}_{\alpha+1} \) where \( \dot{Q}_{\alpha+1} \) is the modified Namba forcing to add a Namba sequence \( E_\alpha \) to each \( \aleph_\alpha^{L[\mathbb{P}_{\alpha+1}]} \) such that \( \alpha \in A_n \) (note that by niceness, \( \aleph_\alpha^{L[\mathbb{P}_{\alpha+1}]} \) is still a regular uncountable cardinal in \( V[\mathbb{P}_n] \)). We can require that \( E_\alpha \subseteq \bigcup_{\alpha \in A_n} \{ E_\alpha \mid \alpha \in A_n \} \) does not contain any of its limit points, and as we noted, it is nice. \( P_{\alpha+1} \) is the full support (which is in our case also the revised countable support) iteration. Let \( G \subseteq P_\omega \) generic, and denote \( A = \bigcup_{n \in \omega} A_n \). By the properties of the modified Namba forcing, for any \( \gamma, V[G] \vDash \text{cf} (\aleph_\gamma^{L[\mathbb{P}_n]}) = \omega \iff \gamma = \xi_n + \alpha \) for \( \alpha \in A_n \).

Remark 2.3.

1. \( \xi_{n+1} = \aleph_{\xi_n^{L[\mathbb{P}_n]}} \), so inductively depends only \( A_0 \) and not on the generics, \( \xi_0 := \text{otp} A \) satisfies \( \xi_0 = \aleph_{\xi_0}^{L[\mathbb{P}_0]} \) and is of cofinality \( \omega \).
2. From \( \text{otp}(A_0) \) and \( A \) we can inductively reconstruct each \( A_n - A_0 \) is the first \( \text{otp}(A_0) \) elements of \( A \), and if we know \( A_n \), then \( A_{n+1} \) are the first \( \text{otp}(A_n) \cdot \omega \) elements of \( A \) above \( \text{sup} A_n \).
3. Hence for each \( n \), \( \text{otp}(A_n) = \text{otp}(A_0) \cdot \omega^n \).
4. If \( \alpha \in A_n \), then \( E_\alpha = A \cap (\aleph_\alpha^{L[\mathbb{P}_{\alpha+1}]} \setminus \aleph_\alpha^{L[\mathbb{P}_{\alpha+1}]} + \omega) \).

Proposition 2.4. \((C^*)^{V[G]} = L[A] = V[G] \).
Proof. By the properties of the modified Namba forcing, at each stage of the iteration the only cardinals of $L[A_0]$ receiving cofinality $\omega$ are the ones in $A_0$. The whole iteration will also add new $\omega$ sequences to sup $A$, but this already had cofinality $\omega$ as we noted earlier. So $V[G] \models \text{cf} (\gamma) = \omega$ if and only if $V \models \text{cf} (\gamma) = \omega$ or there is $n$ s.t. $V \models \text{cf} (\gamma) = \aleph_{\kappa + n}^{L} + 2$ for $\alpha \in A_n$. And on the other hand, if $\alpha \in A$ then $V[G] \models \text{cf} (\aleph_{\kappa + n}^{L} + 2) = \omega$. So

$$A = \bigcup_{n<\omega} \{ \alpha \in [\xi_{n-1}, \xi_n) \mid V[G] \models \text{cf} (\aleph_{\kappa + n}^{L}) = \omega \} \in (C^*)^{V[G]}$$

(where $\xi_{-1} = 0$); hence $L[A] \subseteq (C^*)^{V[G]}$.

As we noted, for every $\alpha \in A$, $E_\alpha$ can be reconstructed from $A$ and $\alpha$, so $\langle E_\alpha \mid \alpha \in A \rangle$ is in $L[A]$. $G$ can be reconstructed from this sequence; hence $V[G] \subseteq L[A] \subseteq (C^*)^{V[G]}$, so the equality follows.

To finish the proof of Theorem 2.1, we start with a model of $V = L + \text{“there is no inaccessible cardinal,”}$ and take, e.g., $A_0 = \omega$. Then $L[A]$ will satisfy $V = C^* \neq L$, and $2^{\aleph_0} = \aleph_1$ will still hold since by [5, Theorem 6.7], violating CH in a model of $V = C^*$ requires an inaccessible cardinal.\footnote{It can in fact be proved that our forcing does not add new reals in general, but as we are looking for a consistency result this is not required.}

§3. Iterating $C^*$.

**Theorem 3.1.** If ZFC is consistent then so is the existence of a model with a decreasing $C^*$-sequence of any given finite length.

**Proof.** Going back to the proof of Theorem 2.1, we note that for any $n$,

$$(C^*)^{L[\bigcup_{k=0}^n A_k]} = L \left[ \bigcup_{k=0}^n A_k \right]$$

$A_n$ can be computed from $A_{n+1}$ using the cofinality-$\omega$ quantifier, which gives $\sup$, and on the other hand, from $\bigcup_{k=0}^n A_k$ we know exactly which ordinals will have cofinality $\omega$ in $L \left[ \bigcup_{k=0}^{n+1} A_k \right]$, which gives $\subseteq$. So by starting, e.g., from $A_0 = \omega$, $L \left[ \bigcup_{k=0}^n A_k \right]$ has the decreasing $C^*$ chain

$$L \left[ \bigcup_{k=0}^n A_k \right] = C^{*0} \supsetneq C^{*1} \supsetneq \cdots \supsetneq C^{*n} = L.$$\onlinebreak

We now show that without large cardinals this is best possible.

**Lemma 3.2.** Let $E = \{ \alpha < \omega_2^V \mid \text{cf} (\alpha) = \omega \}$.

1. If $0^d$ does not exist, then $C^* = L[E]$.
2. If there is no inner model with a measurable cardinal, then $C^* = K[E]$ where $K$ is the Dodd–Jensen core model.

**Proof.** 1. Clearly $E \subseteq C^*$ so $L[E] \subseteq C^*$. Let $\alpha \in \text{Ord}$. If $\text{cf} (\alpha) \geq \omega_2^V$ then also $L \models \text{cf} (\alpha) \geq \omega_2^V$ so in particular $L \models \text{cf} (\alpha) > \omega$. If $\text{cf} (\alpha) < \omega_2^V$, let $A \subseteq \alpha$.
be cofinal, so \(|A| \leq \aleph_1\). By the covering theorem, there is \(B \in L, A \subseteq B \subseteq \alpha\) s.t. \(|B| = \aleph_1 + |A| = \aleph_1\). Let \(\bar{\alpha} = \text{otp}(B)\), so \(\bar{\alpha} < \omega^V_2\), and \(\text{cf}(\alpha) = \text{cf}(\bar{\alpha})\), so \(\text{cf}(\alpha) = \omega\) iff \(\bar{\alpha} \in E\).

To summarize, we get that for every \(\alpha\), in \(L[E]\) we can determine whether \(\text{cf}(\alpha) = \omega\) or not, so \(C^* \subseteq L[E]\).

2. The proof is exactly the same, noting that \(K \subseteq C^*\) by [5, Theorem 5.5], and that our assumption implies the covering theorem holds for \(K\).

**Theorem 3.3.** If there is no inner model with a measurable cardinal, then there is \(k < \omega\) such that \(C^{*k} = C^{*(k+1)}\).

**Proof.** By applying Lemma 3.2 clause 2 inside each \(C^{*n}\), for every \(n\) we have \(C^{*(n+1)} = K[E_n]\) where

\[
E_n = \left\{ \alpha < \omega^2 \mid C^{*n} \models \text{cf}(\alpha) = \omega \right\}.
\]

The sequence \(\left\{ (\omega_1^{*n}, \omega_2^{*n}) \mid n < \omega \right\}\) is non-increasing in both coordinates; hence it stabilizes. Let \(k\) be such that \((\omega_1^{*k}, \omega_2^{*k}) = (\omega_1^{*k(k+1)}, \omega_2^{*k(k+1)})\), and we claim that \(C^{*(k+1)} = C^{*(k+2)}\). To simplify notation we assume w.l.o.g. \(k = 0\), i.e., \(\omega_1^{C^*} = \omega_1^V\) for \(i = 1, 2\) (so we can omit the superscript) and we want to show that \((C^*)^{C^*} = C^*\). We have

\[
C^* = K \left\{ \alpha < \omega_2 \mid V \models \text{cf}(\alpha) = \omega \right\} = K[E_0],
\]

\[
(C^*)^{C^*} = K \left\{ \alpha < \omega_2 \mid C^* \models \text{cf}(\alpha) = \omega \right\} = K[E_1].
\]

Clearly \(E_1 \subseteq E_0\). On the other hand, if \(\alpha \in \omega_2 \setminus E_1\), this means that \(C^* \models \text{cf}(\alpha) = \omega_1\), and since \(\omega_1^{C^*} = \omega_1\), we get that also \(V \models \text{cf}(\alpha) = \omega_1\), so \(\alpha \in \omega_2 \setminus E_0\), thus \(E_1 = E_0\), and our claim is proved.

Our next goal is to show that this is precisely the consistency strength of a decreasing sequence:

**Theorem 3.4.** If there is an inner model with a measurable cardinal, then there is an inner model in which the sequence \(\langle C^{*n} \mid n < \omega \rangle\) is strictly decreasing.

We work in \(L[U]\) where \(U\) is a normal ultrafilter on \(\kappa\), and denote by \(M_\alpha\) the \(\alpha\)th iterate of \(L[U]\) by \(U\), \(j_{\alpha, \beta} : M_\alpha \to M_\beta\) the elementary embedding, \(\kappa_\alpha = j_{0, \alpha}(\kappa)\) and \(U^{(\alpha)} = j_{0, \alpha}(U)\).

In [5, Theorem 5.16] the authors show that if \(V = L[U]\) then \(C^* = M_{\omega^2}[E]\) where \(E = \left\{ \kappa_{\omega^2, n} \mid n < \omega \right\}\). We improve this by showing that \(C^*\) is unchanged after adding a Prikry sequence to \(\kappa\), and then investigate the \(C^*\)-chain of \(L[U]\). First we prove two useful lemmas.

**Lemma 3.5.** \(E = \left\{ \kappa_{\omega^2, n} \mid n < \omega \right\}\) is generic over \(M_{\omega^2}\) for the Prikry forcing on \(\kappa_{\omega^2}\) defined from the ultrafilter \(U^{(\omega^2)}\) \(\in M_{\omega^2}\).

**Proof.** We use Mathias’s characterization of Prikry forcing:

**Fact 3.6** (Mathias, cf. [6]). Let \(M\) be a transitive model of ZFC, \(U\) a normal ultrafilter on \(\kappa\), then \(S \subseteq \kappa\) of order type \(\omega\) is generic over \(M\) for the Prikry forcing defined from \(U\) iff for any \(X \in U, S \setminus X\) is finite.
So we need to show that for any \( X \in U^{(\omega^2)}, E \setminus X \) is finite. The ultrafilter \( U^{(\omega^2)} \) is defined by \( X \in U^{(\omega^2)} \) iff \( \exists \alpha < \omega^2 \{ \kappa_\beta | \alpha \leq \beta < \omega^2 \} \subseteq X \). For \( X \in U^{(\omega^2)} \), choose some \( \alpha < \omega^2 \) such that \( \{ \kappa_\beta | \alpha \leq \beta < \omega^2 \} \subseteq X \), then \( E \models \alpha = \{ \kappa_{\omega \cdot n} | \omega \cdot n < \alpha \} \), and since \( \kappa_{\omega \cdot n} = \sup \{ \kappa_{\omega \cdot n} | n < \omega \} \), this set is finite. Hence \( E = \{ \kappa_{\omega \cdot n} | n < \omega \} \) satisfies the characterization.

**Lemma 3.7.** For any \( \beta < \kappa \) and any \( \alpha, M_\beta \models \text{cf}(\alpha) = \kappa \) iff \( V \models \text{cf}(\alpha) = \kappa \).

**Proof.** \( \kappa \) is regular in \( V \); thus it is regular in every \( M_\beta \) which is an inner model of \( V \). If \( M_\beta \models \text{cf}(\alpha) = \kappa \), then there is a cofinal \( \kappa \)-sequence in \( \alpha \) (in both \( M_\beta \) and \( V \)), and since \( \kappa \) is regular we get \( V \models \text{cf}(\alpha) = \kappa \). If \( V \models \text{cf}(\alpha) = \kappa \), then the same argument rules out \( M_\beta \models \text{cf}(\alpha) < \kappa \). So the only case left to rule out is \( V \models \text{cf}(\alpha) = \kappa \land M_\beta \models \text{cf}(\alpha) > \kappa \). If \( \beta = \gamma + 1 \), then \( M_\beta \) is contained in \( M_\gamma \) and closed under \( \kappa \)-sequences in it, so they agree on cofinality \( \kappa \). and by induction we get that they agree with \( V \) as well. So assume \( \beta \) is limit and let \( \langle \alpha_\eta \mid \eta < \kappa \rangle \) be a cofinal sequence in \( \alpha \). By definition of the limit ultrapower, each \( \alpha_\eta \) is of the form \( j_{\beta, \beta}(\tilde{\alpha}_\eta) \) for some \( \tilde{\beta} < \beta \). We can also assume that each such \( \tilde{\beta} \) is large enough so that \( \alpha \in \text{Range}(j_{\beta, \beta}) \). Since \( \beta < \kappa \), there is some \( \tilde{\beta} \) fitting \( \kappa \) many \( \alpha_\eta \)'s, so without loss of generality we can assume \( \tilde{\beta} \) fits all of them. We can assume \( \tilde{\beta} > 0 \) so \( \kappa \) is a fixed point of \( j_{\beta, \beta} \). If \( \tilde{\alpha} = \sup \{ \tilde{\alpha}_\eta \mid \eta < \kappa \} \), then, since \( \alpha = \sup \{ j_{\beta, \beta}(\tilde{\alpha}_\eta) \mid \eta < \kappa \} \) and \( \alpha \in \text{Range}(j_{\beta, \beta}) \), we must have that \( \alpha = j_{\beta, \beta}(\tilde{\alpha}) \). \( \tilde{\alpha} \) is of cofinality \( \kappa \) in \( V \), so by the induction hypothesis also in \( M_\beta \), hence by elementarity \( M_\beta \models \text{cf}(\alpha) = \kappa \).

**Proposition 3.8.** If \( V = L[U] \) where \( U \) is a measure on \( \kappa \), \( G \) is generic for Prikry forcing on \( \kappa \), then \( (C^*)^{V[G]} = (C^*)^V \).

**Proof.** After forcing with Prikry forcing, the only change of cofinalities is that \( \kappa \) becomes of cofinality \( \omega \). So \( V[G] \models \text{cf}(\alpha) = \omega \) iff \( V \models \text{cf}(\alpha) \in \{ \omega, \kappa \} \). We now follow the proof of [5, Theorem 5.16].

Consider \( M_{\omega^2} \), the \( \omega^2 \) iterate of \( V \), and let \( E = \{ \kappa_{\omega \cdot n} | n < \omega \} \). As we noted above, by [5, Theorem 5.16] we know that \( (C^*)^V = M_{\omega^2}[E] \), so it is enough to show that also \( (C^*)^{V[G]} = M_{\omega^2}[E] \).

Fix an ordinal \( \alpha \). As in the proof of [5, Theorem 5.16], \( V \models \text{cf}(\alpha) = \omega \) iff \( M_{\omega^2}[E] \models \text{cf}(\alpha) \in \{ \omega \} \cup E \cup \{ \sup E \} \). Regarding cofinality \( \kappa \)-by Lemma 3.7, \( V \models \text{cf}(\alpha) = \kappa \) iff \( M_{\omega^2} \models \text{cf}(\alpha) = \kappa \). As we noted, \( M_{\omega^2} = L[u'] \) where \( u' \) is a measure on \( \kappa_{\omega^2} \) and by Lemma 3.5, \( E \) is Prikry generic over it: hence, since cofinality \( \kappa \) is unaffected by Prikry forcing on \( \kappa_{\omega^2} \), we get \( V \models \text{cf}(\alpha) = \kappa \) iff \( M_{\omega^2}[E] \models \text{cf}(\alpha) = \kappa \).

Putting these facts together, in \( M_{\omega^2}[E] \) we can detect whether \( V \models \text{cf}(\alpha) \in \{ \omega, \kappa \} \), so we know whether \( V[G] \models \text{cf}(\alpha) = \omega \); hence we can construct \( (C^*)^{V[G]} \) inside \( M_{\omega^2}[E] \).

The other direction of the proof is almost the same as in [5, Theorem 5.16]: \( E \) is the set of ordinals in the interval \( (\kappa, \kappa_{\omega^2}) \) which have cofinality \( \omega \) in \( V[G] \) and are regular in the core model, which is contained in any \( C^* \), so \( E \in (C^*)^{V[G]} \), and from \( E \) one can define \( M_{\omega^2} \), so \( M_{\omega^2}[E] \subseteq (C^*)^{V[G]} \).

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1Note that we can’t assume this sequence is in \( M_{\tilde{\beta}} \).

4Note that here we had to avoid \( \kappa \) which also satisfies this.
Now we can analyze the \( C^* \)-chain of \( V = L[U] \). We stick to the notation \( C^{*\alpha} \), starting from \( C^{*0} = L[U] = M_0 \), with \( M_\alpha \) being the \( \alpha \)th iterate of \( L[U] \) and \( \kappa_\alpha \) the \( \alpha \)th image of the measurable cardinal. So by [5, Theorem 5.16] we have \( C^{*1} = M_{\omega^2} [E_1] \) for \( E_1 = \{ \kappa_{n} \mid n < \omega \} \). As we noted earlier \( M_{\omega^2} \) is also of the form \( L[U'] \) for the measurable \( \kappa_{\omega^2} \), and by Lemma 3.5, \( E_1 \) is Prikry generic over it, so by Proposition 3.8,

\[
C^{*2} = (C^*)^{M_{\omega^2} [E_1]} = (C^*)^{M_{\omega^2}},
\]

which is again the \( \omega^2 \)th iterate of \( M_{\omega^2} \), i.e., \( M_{\omega^2+\omega^2} \), plus the corresponding sequence–\( E_2 = \{ \kappa_{\omega^2+n} \mid n < \omega \} \). So \( C^{*2} = M_{\omega^2+2} [E_2] \). We can continue inductively, and get the following:

**Theorem 3.9.** If \( V = L[U] \) then for every \( 1 \leq m < \omega \) \( C^{*m} = M_{\omega^2-m} [E_m] \) where \( M_\alpha \) is the \( \alpha \)th iterate of \( V \), \( \kappa_\alpha \) the \( \alpha \)th image of the measurable cardinal, and \( E_m = \{ \kappa_{\omega^2-(m-1)+\omega^2} \mid n < \omega \} \).

To see that this gives us a strictly decreasing sequence of models, consider the following. For every \( m \), \( \kappa_{\omega^2-m} \) is the single measurable in \( M_{\omega^2-m} \), by which we iterate, so it remains regular in all subsequent models, in particular in \( M_{\omega^2-(m+1)} \). Since \( E_{m+1} \) is Prikry generic for \( \kappa_{\omega^2-(m+1)} \) which is larger than \( \kappa_{\omega^2-m} \), the latter remains regular in \( M_{\omega^2-(m+1)}[E_{m+1}] \), while in \( M_{\omega^2-m}[E_m] \) it is singular, so indeed

\[
C^{*m} = M_{\omega^2-m} [E_m] \neq M_{\omega^2-(m+1)} [E_{m+1}] = C^{*(m+1)}.
\]

This concludes the proof of Theorem 3.4.

To analyze \( (C^{*\omega})^{L[U]} \), we will use the following theorem, due to Bukovsky [1, 2] and Dehornoy [3]:

**Fact 3.10.** If \( \kappa \) is measurable, \( M_\alpha \) is the \( \alpha \)th iterate of \( V \) by a normal ultrafilter on \( \kappa \), and \( \kappa_\alpha \) is the \( \alpha \)th image of \( \kappa \), then for any limit ordinal \( \lambda \) exactly one of the following holds:

1. If \( \exists \alpha < \lambda \) s.t. \( M_\alpha \models \text{cf} (\lambda) > \omega \) then \( \bigcap_{\alpha < \lambda} M_\alpha = M_\lambda \).
2. If \( \lambda = \alpha + \omega \) for some \( \alpha \), then \( \langle \kappa_{\alpha+n} \mid n < \omega \rangle \) is Prikry generic over \( M_\alpha \) and \( \bigcap_{\alpha < \lambda} M_\alpha = M_\lambda \).
3. Otherwise, \( \bigcap_{\alpha < \lambda} M_\alpha \) is a quasi-generic extension of \( M_\lambda \), hence satisfies \( \text{ZF} \), but doesn’t satisfy \( \text{AC} \).

**Corollary 3.11.** If \( V = L[U] \) then \( C^{*\omega} = \bigcap_{\alpha < \omega^3} M_\alpha \) and it satisfies \( \text{ZF} \) but not \( \text{AC} \).

**Proof.** By definition and our previous calculation,

\[
C^{*\omega} = \bigcap_{m < \omega} C^{*m} = \bigcap_{1 \leq m < \omega} M_{\omega^2-m} [E_m]
\]

and for each \( m \geq 1 \), \( E_m \not\in M_{\omega^2-(m+1)}[E_{m+1}] \) so

\[
\bigcap_{1 \leq m < \omega} M_{\omega^2-m} [E_m] = \bigcap_{m < \omega} M_{\omega^2-m} = \bigcap_{\alpha < \omega^3} M_\alpha.
\]

Since \( \omega^3 \) is of cofinality \( \omega \) but not of the form \( \alpha + \omega \), the conclusion follows from Fact 3.10 clause 3.
§4. Conclusion and open questions. We summarize what is now known in terms of equiconsistency:
1. ZFC is equiconsistent with \( V = C^* \neq L + 2^{\aleph_0} = \aleph_1 \).
2. Existence of an inaccessible cardinal is equiconsistent with \( V = C^* + 2^{\aleph_0} = \aleph_2 \).
3. Existence of a measurable cardinal is equiconsistent with \( \forall n < \omega (C^{*n} \supset C^{*(n+1)}) \) and \( C^{*\omega} \models ZF + \neg AC \).

Compared to the results regarding HOD, the following questions remain open:

**Question 4.1.**

1. Is it possible, under any large cardinal hypothesis, that \( \forall n < \omega C^{*n} \supset C^{*(n+1)} \) and \( C^{*\omega} \models ZFC \)? More generally, for which ordinals \( \alpha \) can we get a decreasing \( C^* \) sequence of length \( \alpha \)?
2. Is it possible, under any large cardinal hypothesis, that \( \forall n < \omega C^{*n} \supset C^{*(n+1)} \) and \( C^{*\omega} \not\models ZF \)?

A natural first attempt towards answering the first question would be to try and work in a model with more measurable cardinals. However, it seems that it would require at least *measurably many* measurables: in a forthcoming paper [12], we generalize [5, Theorem 5.16] and our Proposition 3.8 and show the following:

**Theorem 4.2.** Assume \( V = L[U] \) where \( U = \langle U_\gamma \mid \gamma < \chi \rangle \) is a sequence of measures on the increasing measurables \( \langle \kappa^\gamma \mid \gamma < \chi \rangle \) where \( \chi < \kappa^0 \). Iterate \( V \) according to \( U \) where each measurable is iterated \( \omega^2 \cdot 2 \) many times, to obtain \( \langle M^\alpha_\omega \mid \gamma < \chi, \alpha \leq \omega^2 \rangle \), with iteration points \( \langle \kappa^\gamma_\alpha \mid \gamma < \chi, \alpha \leq \omega^2 \rangle \), and set \( M^\chi \) as the directed limit of this iteration. Let \( G \) be generic over \( V \) for the forcing adding a Prikry sequence to every \( \kappa^\gamma \). Set for every \( \gamma < \chi \) \( E^\gamma = \langle \kappa^\gamma_\alpha \mid 1 \leq \alpha < \omega \rangle \) and \( \tilde{E}^\gamma = \langle \kappa^\gamma_0 \mid 0 \leq \alpha < \omega \rangle \) then

\[
(C^*)^V = M^\gamma \langle (E^\gamma \mid \gamma < \chi) \rangle,
\]

\[
(C^*)^V[G] = M^\gamma \langle (\tilde{E}^\gamma \mid \gamma < \chi) \rangle.
\]

So, if \( V = L[U] \) as above, \( C^* \) is of the form \( L[U_\alpha][G_\alpha] \) for some sequence of measures and a sequence of Prikry sequences on its measures, and so \( C^{*2} \) is again of that form, where we iterated the measures in \( U_\omega \cdot 2 \) many times and add Prikry sequences. So again, as we’ve done here, we’ll get that \( C^{*\omega} \) is the intersection of the models \( M^\chi_\omega \), where we iterated each measure \( \omega^2 \cdot n \) times. This is due to the facts that changing the order of iteration between the measures doesn’t change the final result, and that the Prikry sequences “fall out” during the intersection. Now, we don’t have a complete analysis of intersections of iterations by more than one measure, but Dehornoy proves the following more general fact:

**Fact 4.3** [3, Section 5.3. Proposition 3]. For every \( \alpha \) let \( N_\alpha \) be the \( \alpha \)th iteration of \( V \) by some measure. Assume \( \lambda \) is such that for every \( \alpha < \lambda, N_\alpha \models \text{cf}(\lambda) = \omega \) but there is no \( \rho \) such that \( \lambda = \rho + \omega \). Then if \( M \) is a transitive inner model of ZFC containing \( \bigcap_{\alpha < \omega} N_\alpha \), then there is some \( \alpha \) such that \( N_\alpha \subseteq M \).

So, if we take \( N_\alpha \) to be the iteration of \( V \) by the first measure in \( U \), we get that \( C^{*\omega} \) contains \( \bigcap_{\alpha < \omega^3} N_\alpha \), but doesn’t contain any \( N_\alpha \) for \( \alpha < \omega^3 \), so \( C^{*\omega} \) cannot satisfy AC.

Furthermore, a recent result by Welch [11] shows that under the assumption that \( O^k \) (O-kukri) doesn’t exist, i.e., there is no elementary embedding of an inner
model with a proper class of measurables to itself (and in particular no measurable limit of measurables), the characterization $\left(C^\ast\right)^V = M^\chi \left[\left\{E^\gamma : \gamma < \chi\right\}\right]$ holds for any $\chi \leq \text{Ord}$, and it seems likely that the analysis of the iterated $C^\ast$ will be the same as well. Hence a different approach, or larger cardinals, would be required to answer this question.

A different line of inquiry stems from the following fact:

**Fact 4.4** [5]. If there is a proper class of Woodin cardinals then the theory of $C^\ast$ is unchanged by forcing.

So the question whether $C^\ast \models V = C^\ast$ cannot be changed under forcing in the presence of class many Woodin cardinals. If the sequence of $C^{\ast\alpha}$ is definable (perhaps up to some ordinal) then this will also be in the theory of $C^\ast$ (note that on the face of it even the sequence up to $\omega$ may not be definable).

**Question 4.5.** What can be deduced on the sequence of iterated $C^\ast$ from a proper class of Woodin cardinals?

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