Asymptotics for a nonlinear integral equation with a generalized heat kernel

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Abstract

This paper is concerned with a nonlinear integral equation

\[ (P) \quad u(x, t) = \int_{\mathbb{R}^N} G(x - y, t)\varphi(y)dy + \int_0^t \int_{\mathbb{R}^N} G(x - y, t - s)f(y, s : u)dyds, \]

where \( N \geq 1, \varphi \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, (1 + |x|^K)dx) \) for some \( K \geq 0 \). Here \( G = G(x, t) \) is a generalization of the heat kernel. We are interested in the asymptotic expansions of the solution of \((P)\) behaving like a multiple of the integral kernel \( G \) as \( t \to \infty \).
1 Introduction

Let $u$ be a solution of a nonlinear integral equation,

$$u(x,t) = \int_{\mathbb{R}^N} G(x-y,t)\varphi(y)dy + \int_0^t \int_{\mathbb{R}^N} G(x-y,t-s)f(y,s;u)dyds$$  \hspace{1cm} (1.1)

in $\mathbb{R}^N \times (0,\infty)$, where $N \geq 1$, $f$ is an inhomogeneous term possibly depending on the solution $u$ itself, and $G = G(x,t)$ is an integral kernel satisfying the following condition:

$$(G) \quad \text{(i) } G \in C^\gamma(\mathbb{R}^N \times (0,\infty)) \text{ for some } \gamma \in \mathbb{N};$$

$$(\text{ii}) \text{ There exist positive constants } d \text{ and } L \text{ such that } G(x,t) = t^{-\frac{N}{d}} G\left(\frac{x}{t^{1/d}},1\right), \quad x \in \mathbb{R}^N, \ t > 0,$$

$$\sup_{x \in \mathbb{R}^N} (1 + |x|)^{N+L+j} |\nabla^j_x G(x,1)| < \infty, \quad j = 0, \ldots, \gamma;$$  \hspace{1cm} (1.2)

$$(\text{iii}) \ G(x,t) = \int_{\mathbb{R}^N} G(x-y,t-s)G(y,s)dy \text{ for } x \in \mathbb{R}^N \text{ and } t > s > 0.$$

Condition $(G)$ holds for the fundamental solutions of the following linear diffusion equations,

$$\partial_t u + (-\Delta)^{\theta/2} u = 0 \text{ in } \mathbb{R}^N \times (0,\infty) \quad (0 < \theta < 2),$$

$$\partial_t u + (-\Delta)^m u = 0 \text{ in } \mathbb{R}^N \times (0,\infty) \quad (m = 1, 2, 3, \ldots),$$

and integral equation (1.1) appears in the study of various nonlinear diffusion equations. In this paper we give the asymptotic expansions of the solutions of (1.1) behaving like a multiple of the kernel $G$ as $t \to \infty$. Our arguments are applicable to the large class of nonlinear diffusion equations, including the following semilinear parabolic equations (see Section 6):

- (Fractional semilinear parabolic equation)

$$\partial_t u + (-\Delta)^{\theta/2} u = |u|^{p-1} u \text{ in } \mathbb{R}^N \times (0,\infty),$$  \hspace{1cm} (1.4)

where $N \geq 1$, $0 < \theta < 2$ and $p > 1 + \theta/N$ (see e.g. [1], [13], [14], [19] and [32]);

- (Higher order semilinear parabolic equation)

$$\partial_t u + (-\Delta)^m u = |u|^p \text{ in } \mathbb{R}^N \times (0,\infty),$$  \hspace{1cm} (1.5)

where $N \geq 1$, $m = 1, 2, \ldots$ and $p > 1 + 2m/N$ (see e.g. [5], [10], [11], [15] and [17]).

See also a forthcoming paper [26], where the asymptotic expansions of the solutions of convection-diffusion equations will be discussed.

Asymptotic behavior of solutions of nonlinear parabolic equations has been extensively studied in many papers by various methods. See e.g. [3]–[37] and references therein. Among others, Fujigaki and Miyakawa [15] studied the large time behavior of the solution $u$
of the Cauchy problem for the incompressible Navier-Stokes equation, and gave higher order asymptotic expansions of the solution \( u \) satisfying

\[
\sup_{0 \leq t \leq N+1} \sup_{(x,t) \in \mathbb{R}^N \times (0,\infty)} (1 + |x|)(1 + t)^{(N+1-t)/2} |u(x,t)| < \infty. \tag{1.6}
\]

Their arguments can be also applied to convection-diffusion equations (see e.g. [34]–[36]). On the other hand, in [23] the first and the second authors of this paper considered the Cauchy problem for the semilinear heat equation

\[
\partial_t u = \Delta u + \lambda |u|^{p-1} u \quad \text{in} \quad \mathbb{R}^N \times (0,\infty), \tag{1.7}
\]

where \( \lambda \in \mathbb{R} \) and \( p > 1 + 2/N \), and gave the precise description of the asymptotic behavior of the solution behaving like a multiple of the heat kernel (see also [21]). Furthermore, in [24] they extended the results in [23], and established the method of obtaining higher order asymptotic expansions of the solutions behaving like a multiple of the heat kernel as \( t \to \infty \) for general nonlinear heat equations. The arguments in [24] are applicable to various nonlinear heat equations systematically without pointwise decay estimates of the solutions as \( |x| \to \infty \), such as (1.6).

In this paper we improve and generalize the arguments in [21], [23] and [24], and establish the method of obtaining the higher order asymptotic expansions of the solutions of nonlinear integral equation (1.1) behaving like a multiple of the integral kernel \( G \) as \( t \to \infty \). Our arguments are applicable to general nonlinear parabolic equations including (1.4) and (1.5), and they can also give some new and sharp decay estimates of the solutions even if we focus on the semilinear heat equation (1.7) (see also Remark 1.1).

We introduce some notation. For any \( k \geq 0 \), let \( [k] \in \mathbb{N} \) be such that \( k - 1 < [k] \leq k \). For any multi-index \( \alpha \in \mathbb{N}^N \), put

\[
|\alpha| := \sum_{i=1}^N \alpha_i, \quad \alpha! := \prod_{i=1}^N \alpha_i!, \quad x^\alpha := \prod_{i=1}^N x_i^{\alpha_i}, \quad \partial_x^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}.
\]

Let \( \mathcal{M}_k := \{ \alpha \in \mathbb{N} : |\alpha| \leq k \} \) for \( k \geq 0 \). For any \( \alpha = (\alpha_1, \ldots, \alpha_N), \beta = (\beta_1, \ldots, \beta_N) \in \mathcal{M}, \) we say

\[
\alpha \leq \beta
\]

if \( \alpha_i \leq \beta_i \) for all \( i \in \{1, \ldots, N\} \). For any \( 1 \leq r \leq \infty \), let \( \| \cdot \|_r \) be the usual norm of \( L^r := L^r(\mathbb{R}^N) \). For any \( k \geq 0 \), we denote by \( \| \cdot \|_k \) the norm of \( L^1_k := L^1(\mathbb{R}^N, (1 + |x|^k)dx) \), that is,

\[
\| f \|_k := \int_{\mathbb{R}^N} |f(x)|(1 + |x|^k)dx, \quad f \in L^1_k.
\]

For any \( \varphi \in L^q(\mathbb{R}^N) \) (\( 1 \leq q \leq \infty \)), we put

\[
e^{tE} \varphi(x) := \int_{\mathbb{R}^N} G(x - y, t) \varphi(y)dy, \quad x \in \mathbb{R}^N, \quad t > 0. \tag{1.8}
\]

Then, under assumption (G), we have the following (see also Section 2):
• Let \( 1 \leq q \leq r \leq \infty \). Then there exists a constant \( C \) such that
\[
\| e^{t \mathcal{L}} \varphi \|_r \leq C t^{-\frac{N}{d} \left( \frac{1}{q} - \frac{1}{r} \right)} \| \varphi \|_q, \quad t > 0,
\]
for any \( \varphi \in L^q \);

• For any \( \varphi \in L^q \) with \( 1 \leq q \leq \infty \),
\[
e^{t \mathcal{L}} \varphi(x) = e^{(t-s) \mathcal{L}} [e^{s \mathcal{L}} \varphi](x)
\]
for all \( x \in \mathbb{R}^N \) and \( 0 < s < t \).

Let \( 0 \leq k < L \) with \( [k] \leq \gamma \) and \( f \in L^1_k \). Put
\[
g(x, t) := G(x, t + 1), \quad g_\alpha(x, t) := \frac{(-1)^{|\alpha|}}{\alpha!} \partial_\alpha^\alpha g(x, t) \quad (\alpha \in \mathbb{M}_\gamma).
\]
Then, for any \( t \geq 0 \), we denote by \( P_k(t) f \in L^1_k \) by
\[
[P_k(t) f](x) := f(x) - \sum_{|\alpha| \leq k} M_\alpha(f, t) g_\alpha(x, t), \quad (1.9)
\]
where \( M_\alpha(f, t) \) \((|\alpha| \leq k)\) are defined inductively \((\alpha)\) by
\[
\begin{cases}
M_0(f, t) := \int_{\mathbb{R}^N} f(x) dx & \text{if } \alpha = 0,
M_\alpha(f, t) := \int_{\mathbb{R}^N} x^\alpha f(x) dx - \sum_{\beta \leq \alpha, \beta \neq \alpha} M_\beta(f, t) \int_{\mathbb{R}^N} x^\beta g_\beta(x, t) dx & \text{if } \alpha \neq 0.
\end{cases} \quad (1.10)
\]
Then it follows that
\[
\int_{\mathbb{R}^N} x^\alpha [P_k(t) f](x) dx = 0, \quad t > 0,
\]
for any \( \alpha \in \mathbb{M}_k \) (see Lemma 2.1 (ii)). This is a crucial property of the operator \( P_k(t) \) \((\text{on } L^1_k)\) in our analysis.

Now we are ready to state the main results of this paper, which give asymptotic expansions of the functions
\[
e^{t \mathcal{L}} \varphi(x) = \int_{\mathbb{R}^N} G(x - y, t) \varphi(y) dy,
\]
\[
\int_0^t e^{(t-s) \mathcal{L}} f(s) ds = \int_0^t \int_{\mathbb{R}^N} G(x - y, t - s) f(y, s) dy ds,
\]
as \( t \to \infty \), under suitable integrability conditions on \( \varphi \) and \( f \).

**Theorem 1.1** Assume condition \((G)\) for some \( \gamma \in \mathbb{N} \), \( d > 0 \) and \( L > 0 \). Let \( 0 \leq K < L \) with \([K] + 1 \leq \gamma \). For any \( \varphi \in L^1_K \), put
\[
v(x, t) := e^{t \mathcal{L}} \varphi(x) - \sum_{|\alpha| \leq K} M_\alpha(\varphi, 0) g_\alpha(x, t). \quad (1.11)
\]
Then, for any \( j \in \{0, \ldots, \gamma \} \), \( q \in [1, \infty] \) and \( \ell \in [0, K] \), there exists a constant \( C \) independent of \( \varphi \in L^1_K \) such that

\[
t^\frac{N}{d} (1 - \frac{1}{q}) + \frac{\ell}{q} \|\nabla^j v(t)\|_q + t^\frac{1}{d} (1 + t)^{-\frac{1}{d}} \|\nabla^j v(t)\|_\ell \leq C (1 + t)^{-\frac{K}{d}} \|\varphi\|_K \tag{1.12}
\]

for all \( t > 0 \). Furthermore, for any \( \varphi \in L^1_K \),

\[
\lim_{t \to \infty} t^\frac{N}{d} \left[ t^\frac{1}{d} (1 - \frac{1}{q}) + \frac{\ell}{q} \|\nabla^j v(t)\|_q + t^\frac{1}{d} (1 + t)^{-\frac{1}{d}} \|\nabla^j v(t)\|_\ell \right] = 0. \tag{1.13}
\]

**Theorem 1.2** Assume condition (G) for some \( \gamma \in \mathbb{N} \), \( d > 0 \) and \( L > 0 \). Let \( 0 \leq K < L \) with \( [K] + 1 \leq \gamma \) and \( 1 \leq q \leq \infty \). Let \( f \) be a measurable function in \( \mathbb{R}^d \times (0, \infty) \) such that

\[
E_{K,q}[f](t) := (1 + t)^\frac{K}{d} \left[ t^\frac{1}{d} \|f(t)\|_q + \|f(t)\|_1 + \|f(t)\|_K \right] \in L^\infty(0, T) \tag{1.14}
\]

for any \( T > 0 \). Then the following holds:

(i) For any \( \alpha \in M_K \), there exists a constant \( C_1 \) such that

\[
|M_\alpha(f(t), t)| \leq C_1 (1 + t)^{-\frac{K - |\alpha|}{d}} E_{K,q}[f](t) \tag{1.15}
\]

for almost all \( t > 0 \);

(ii) Put

\[
R_K[f](t) := \int_0^t e^{(t-s)\frac{V}{d}} P_K(s) f(s) ds
= \int_0^t e^{(t-s)\frac{V}{d}} f(s) ds - \sum_{|\alpha| \leq K} \left[ \int_0^t M_\alpha(f(s), s) ds \right] g_\alpha(t).
\]

Let \( j \in \{0, \ldots, \gamma \} \) with \( j < d \) and \( T_0 > 0 \). Then there exists a constant \( C_2 \) such that, for any \( \epsilon > 0 \) and \( T \geq T_0 \),

\[
t^\frac{N}{d} (1 - \frac{1}{q}) \|\nabla^j R_K[f](t)\|_q + t^\frac{1}{d} \|\nabla^j R_K[f](t)\|_\ell \\
\leq \epsilon t^{-\frac{K + \frac{1}{d}}{d}} + C_2 t^{-\frac{K}{d}} \int_T^t (t-s)^{-\frac{1}{d}} E_{K,q}[f](s) ds \tag{1.16}
\]

for all sufficiently large \( t > 0 \). In particular, if

\[
\int_0^\infty E_{K,q}[f](s) ds < \infty,
\]

then

\[
\lim_{t \to \infty} t^\frac{N}{d} \left[ t^\frac{1}{d} \|R_K[f](t)\|_q + t^\frac{1}{d} \|R_K[f](t)\|_\ell \right] = 0. \tag{1.17}
\]
By Theorems 1.1 and 1.2 we can give decay estimates of the distance in $L^q$ and $L^1$ ($0 \leq \ell \leq K$) from the solution of (1.1) to its asymptotic expansion

$$\sum_{|\alpha| \leq K} \left[ M_\alpha(\varphi, 0) + \int_0^t M_\alpha(f(s), s)ds \right] g_\alpha(t). \tag{1.18}$$

The higher order asymptotic expansions of the solutions depend on the nonlinearity of $f$ and are discussed in Sections 4 and 5.

**Remark 1.1** Let $G$ be the heat kernel, that is,

$$G(x, t) := (4\pi t)^{-\frac{N}{2}} \exp \left(-\frac{|x|^2}{4t}\right).$$

Let $\varphi \in L^1_K$ for some $K \geq 0$, and define a function $v$ by (1.11). In [21] the authors proved that, for any $1 \leq q \leq \infty$ and $0 \leq \ell \leq K$,

$$t^{\frac{N}{2}(1 - \frac{1}{q})} \|v(t)\|_q = \begin{cases} O(t^{-K}) & \text{if } K > [K], \\ o(t^{-\frac{N}{2}}) & \text{if } K = [K], \end{cases} \quad t^{\frac{N}{2}k} \|v(t)\|_\ell = O(t^{\frac{N}{2}+\sigma}), \tag{1.19}$$

as $t \to \infty$, for any $\sigma > 0$. This is one of the main ingredients of the asymptotic analysis in [21], [23] and [24] for parabolic equations.

On the other hand, since the heat kernel satisfies condition (G) for any $\gamma \in \mathbb{N}$ and $L > 0$ with $d = 2$, Theorem 1.1 gives better decay estimates of $v$ than (1.19), and enables us to improve the asymptotic analysis in [21], [23] and [24]. See Sections 5 and 6.

The rest of this paper is organized as follows. Section 2 presents some preliminaries on $e^{t\mathcal{L}} \varphi$ and $M_\alpha(f, t)$. In Section 3 we improve the argument in [21], and study the asymptotic expansion of $e^{t\mathcal{L}} \varphi$. This enables us to prove Theorem 1.1 Section 4 is devoted to the proof of Theorem 1.2 by using the arguments in the previous sections. In Section 5 we study the asymptotic behavior of solutions of integral equations with power nonlinearity. In Section 6 we apply our arguments to semilinear parabolic equations (1.4) and (1.5), and show the validity of our arguments.

## 2 Preliminaries

In this section we prove some preliminary results on $e^{t\mathcal{L}} \varphi$ and $M_\alpha(f, t)$. In what follows, for any two nonnegative functions $f_1$ and $f_2$ in a subset $D$ of $[0, \infty)$, we say

$$f_1(t) \preceq f_2(t), \quad t \in D$$

if there exists a positive constant $C$ such that $f_1(t) \leq Cf_2(t)$ for all $t \in D$. In addition, we say

$$f_1(t) \asymp f_2(t), \quad t \in D$$

if $f_1(t) \preceq f_2(t)$ and $f_2(t) \preceq f_1(t)$ for all $t \in D$.

We first state some properties on the kernel $G$, which immediately follow from condition (G) (see also [23]):
(i) \( \int_{\mathbb{R}^N} G(x,t)dx = 1 \) for any \( t > 0 \);

(ii) For any \( \alpha \in M_{\gamma} \),
\[
|\partial_x^\alpha G(x,t)| \leq t^{-\frac{N}{d} - \frac{|\alpha|}{d}} \left( 1 + \frac{|x|}{t^{1/d}} \right)^{-(N+L+|\alpha|)} , \quad (x,t) \in \mathbb{R}^N \times (0, \infty);
\] \hspace{1cm} (2.1)

(iii) For any \( 1 \leq r \leq \infty \), \( \alpha \in M_{\gamma} \) and \( \ell \in [0, L + |\alpha|) \),
\[
\sup_{t>0} \left[ t^{\frac{N}{d}(1-\frac{1}{r}) + \frac{|\alpha|}{d}} \left\| \partial_x^\alpha G(t) \right\|_r + (1 + t)^{-\frac{\ell}{d}} t^{\frac{|\alpha|}{d}} \left\| \left| \partial_x^\alpha G(t) \right| \right\|_\ell \right] < \infty.
\] \hspace{1cm} (2.2)

Furthermore, applying the Young inequality to (1.8) with the aid of property (iii), for any \( 1 \leq r \leq q \leq \infty \) and \( j \in \{0, \ldots, \gamma\} \), we can find a constant \( C \) such that
\[
\left\| \nabla \!^j e^{tL} \varphi \right\|_q \lesssim \left\| \varphi \right\|_r , \quad t \geq 0,
\] \hspace{1cm} (2.3)

Next we state a lemma on \( M_\alpha(f,t) \) and the operator \( P_k(t) \).

**Lemma 2.1** Assume condition \((G)\) for some \( \gamma \in \mathbb{N} \), \( d > 0 \) and \( L > 0 \). For any \( f, g \in L^1_k \),
with \( 0 \leq k < L \) and \( |k| \leq \gamma \), we can find a constant \( C \) such that
\[
\left\| \nabla \!^j e^{tL} \varphi \right\|_q \lesssim t^{-\frac{N}{d}(1-\frac{1}{r}) - \frac{j}{d}} \left\| \varphi \right\|_r , \quad t \geq 0,
\] \hspace{1cm} (2.4)

(i) For any \( a, b \in \mathbb{R} \) and \( \alpha \in M_k \),
\[
M_\alpha(af + bg,t) = aM_\alpha(f,t) + bM_\alpha(g,t) , \quad t \geq 0;
\]

(ii) For any \( \alpha \in M_k \),
\[
\int_{\mathbb{R}^N} x^\alpha [P_k(t)f](x)dx = 0 , \quad t \geq 0;
\]

(iii) Assume that there exists constants \( \{c_\alpha\}_{\alpha \in M_k} \) such that
\[
\int_{\mathbb{R}^N} x^\beta \left( f - \sum_{|\alpha| \leq k} c_\alpha g_\alpha(x,t) \right)dx = 0 , \quad \beta \in M_k,
\] for some \( t \geq 0 \). Then
\[
c_\alpha = M_\alpha(f,t) , \quad \alpha \in M_k;
\]

(iv) For any \( t \geq 0 \),
\[
M_\alpha(e^{t\xi}f,t) = M_\alpha(f,0) , \quad \alpha \in M_k;
\]

(v) Let \( f \in L^1_k \) be such that
\[
\int_{\mathbb{R}^N} x^\beta f(x)dx = 0 , \quad \beta \in M_k.
\] \hspace{1cm} (2.5)

Then
\[
\int_{\mathbb{R}^N} x^\beta e^{t\xi} f(x)dx = 0 , \quad \beta \in M_k,
\] for all \( t \geq 0 \).
Proof. Assertion (i) immediately follows from (1.10). We prove assertion (ii). For any \( f \in L^1_k \) and \( \alpha \in M_k \), since

\[
\int_{\mathbb{R}^N} x^\alpha g_{\beta}(x,t)dx = 0 \quad \text{if not } \beta \leq \alpha,
\]

by (1.9) and (1.10) we have

\[
\int_{\mathbb{R}^N} x^\alpha [P_k(t)f](x)dx = \int_{\mathbb{R}^N} x^\alpha f(x)dx - \sum_{\beta \leq \alpha} M\beta(f,t) \int_{\mathbb{R}^N} x^\alpha g_{\beta}(x,t)dx
\]

\[
= \int_{\mathbb{R}^N} x^\alpha f(x)dx - M\alpha(f,t) - \sum_{\beta \leq \alpha, \alpha \neq \beta} M\beta(f,t) \int_{\mathbb{R}^N} x^\alpha g_{\beta}(x,t)dx = 0
\]

for \( t \geq 0 \). This implies assertion (ii). Similarly assertion (iii) follows inductively in \( \alpha \).

We prove assertion (iv). For any \( f \in L^1_k \), put

\[
w(x,t) := e^{t\mathcal{L}}f(x) - \sum_{|\alpha| \leq k} M\alpha(f,0)g_{\alpha}(x,t).
\]

(2.6)

Since

\[
[e^{t\mathcal{L}}g_{\alpha}(0)](x) = \frac{(-1)^{|\alpha|}}{\alpha!} \int_{\mathbb{R}^N} G(x-y,t)\partial_y^\alpha G(y,1)dy
\]

\[
= \frac{(-1)^{|\alpha|}}{\alpha!} \partial_x^\alpha \int_{\mathbb{R}^N} G(x-y,t)G(y,1)dy = \frac{(-1)^{|\alpha|}}{\alpha!} \partial_x^\alpha G(x,t+1) = g_{\alpha}(x,t),
\]

we have

\[
w(x,t) = [e^{t\mathcal{L}}w(0)](x).
\]

On the other hand, it follows from assertion (ii) that

\[
\int_{\mathbb{R}^N} x^\beta w(x,0)dx = \int_{\mathbb{R}^N} x^\beta [P_k(0)f](x)dx = 0, \quad \beta \in M_k.
\]

Therefore, by the Fubini theorem and the binomial theorem we have

\[
\int_{\mathbb{R}^N} x^\beta w(x,t)dx = \int_{\mathbb{R}^N} x^\beta \left( \int_{\mathbb{R}^N} G(x-y,t)w(y,0)dy \right) dx
\]

\[
= \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} (x+y)^\beta G(x,t)dx \right) w(y,0)dy
\]

\[
= \sum_{\alpha \leq \beta} C\alpha(t) \int_{\mathbb{R}^N} y^\alpha w(y,0)dy = 0
\]

for \( \beta \in M_k \), where \( \{C\alpha(t)\} \) are constants depending on \( t \). Then assertion (iv) follows from assertion (iii) and (2.6).

It remains to prove assertion (v). Let \( f \in L^1_k \), and assume (2.5). By (1.10) we obtain inductively

\[
M\alpha(f,0) = 0, \quad \alpha \in M_k.
\]

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This together with assertion (iv) implies that
\[ (P_k(t)e^{t\ell}f)(x) = e^{t\ell}f(x) - \sum_{|\alpha| \leq k} M_\alpha(e^{t\ell}f,t)g_\alpha(x,t) = e^{t\ell}f(x) - \sum_{|\alpha| \leq k} M_\alpha(f,0)g_\alpha(x,t) = e^{t\ell}f(x). \]

Then assertion (v) follows from assertion (ii), and the proof of Lemma 2.1 is complete. \(\square\)

At the end of this section, we prove a lemma on the functions \(P_K(t)f(t)\) and \(E_K[f](t)\).

**Lemma 2.2** Assume the same conditions as in Theorem 1.2. Let \(1 \leq r \leq q \leq \infty\) and \(0 \leq \ell \leq K\). Then

\[
\|f(t)\|_r \leq t^{\frac{N}{q}(1 - \frac{\ell}{D})(1 + t)^{-\frac{\ell}{D}} E_{K,q}[f](t),}
\]

\[
\|f(t)\|_\ell \leq (1 + t)^{-\frac{N}{r} + \frac{\ell}{r}} E_{K,q}[f](t),
\]

\[
|E_\alpha(f(t),t)| \leq (1 + t)^{-\frac{N}{r} + \frac{\ell}{r}} E_{K,q}[f](t), \quad \alpha \in M_K,
\]

for almost all \(t > 0\). Furthermore,

\[
 t^{\frac{N}{q}(1 - \frac{\ell}{D})} \|P_K(t)f(t)\|_q + (1 + t)^{-\frac{\ell}{D}} \|P_K(t)f(t)\|_\ell \leq (1 + t)^{-\frac{K}{D}} E_{K,q}[f](t)
\]

for almost all \(t > 0\).

**Proof.** For \(1 \leq r \leq q\), by the Hölder inequality and (1.14) we have

\[
\|f(t)\|_r \leq \|f(t)\|_1^{\frac{r}{2}} \|f(t)\|_q^{\frac{r}{2}} \leq t^{\frac{N}{q}(1 - \frac{\ell}{D})(1 + t)^{-\frac{\ell}{D}} E_{K,q}[f](t)}
\]

for almost all \(t > 0\), where \(\eta := (r - 1)/(1 - 1/q)\), and we obtain (2.7). For any \(0 \leq \ell \leq K\), since

\[
\left(\frac{1 + |x|}{1 + t}\right)^{\ell} \leq 1 + \left(\frac{1 + |x|}{1 + t}\right)^{K},
\]

we get

\[
\|f(t)\|_\ell \leq (1 + t)^{\frac{\ell}{r}} \int_{\mathbb{R}^N} \left(\frac{1 + |x|}{1 + t}\right)^{\ell} |f(x,t)|dx
\]

\[
\leq (1 + t)^{\frac{\ell}{r}} \int_{\mathbb{R}^N} \left[1 + \left(\frac{1 + |x|}{1 + t}\right)^{K}\right] |f(x,t)|dx
\]

\[
\leq (1 + t)^{\frac{\ell}{r}} \|f(t)\|_1 + (1 + t)^{-\frac{K}{D}} \|f(t)\|_K \leq (1 + t)^{-\frac{K}{D} + \frac{\ell}{r}} E_{K,q}[f](t)
\]

for almost all \(t > 0\). This implies (2.8).

The proof of (2.9) is by induction in \(\alpha \in M_K\). For \(\alpha = 0\), by (1.10) and (2.7) we have

\[
|M_\alpha(f(t),t)| = \int_{\mathbb{R}^N} f(x,t)dx \leq \|f(t)\|_1 \leq (1 + t)^{-\frac{K}{D}} E_{K,q}[f](t)
\]
for almost all $t > 0$, and inequality (2.9) holds for $\alpha = 0$. Assume that inequality (2.9) holds for $\alpha \in M_n$ for some $n \in \{0, \ldots, [K] - 1\}$. Then, for any $\alpha \in M_{n+1} \setminus M_n$, it follows from (1.10), (2.2) and (2.8) that

$$
|M_\alpha(f, t)| \leq \left| \int_{\mathbb{R}^N} x^\alpha f(x, t) dx \right| + \sum_{\beta \leq \alpha, \beta \neq \alpha} |M_\beta(f, t)| \left| \int_{\mathbb{R}^N} x^\alpha g_\beta(x, t) dx \right|
$$

$$
\leq \|f(t)\|_{\alpha} + \sum_{\beta \leq \alpha, \beta \neq \alpha} (1 + t)^{-\frac{K}{\alpha} + \frac{1}{q}} E_{K, q}[f](t) \cdot (1 + t)^{\frac{|\alpha| - |\beta|}{\alpha}}
$$

$$
\leq (1 + t)^{-\frac{K}{\alpha} + \frac{1}{q} \max_{\beta \leq \alpha, \beta \neq \alpha} |\beta|} E_{K, q}[f](t)
$$

for almost all $t > 0$, and (2.9) holds. Thus inequality (2.9) holds for all $\alpha \in M_K$. Furthermore, by (1.9), (2.2) and (2.9) we obtain

$$
t^{\frac{N}{\alpha}(1 - \frac{1}{q})} \|P_K(t) f(t)\|_q + (1 + t)^{-\frac{1}{q}} \|P_K(t) f(t)\|_\ell
$$

$$
\leq t^{\frac{N}{\alpha}(1 - \frac{1}{q})} \|f(t)\|_q + (1 + t)^{-\frac{1}{q}} \|f(t)\|_\ell
$$

$$
+ \sum_{|\alpha| \leq K} |M_\alpha(f, t)| \left[ t^{\frac{N}{\alpha}(1 - \frac{1}{q})} \|g_\alpha(t)\|_q + (1 + t)^{-\frac{1}{q}} \|g_\alpha(t)\|_\ell \right]
$$

$$
\leq (1 + t)^{-\frac{K}{\alpha} + \frac{1}{q} \max_{\beta \leq \alpha, \beta \neq \alpha} |\beta|} E_{K, q}[f](t)
$$

for all $t > 0$. This implies (2.10), and the proof of Lemma 2.2 is complete. □

3 Proof of Theorem 1.1

In this section we prove the following proposition on the decay estimates of $e^{t\mathcal{L}} \varphi$. Proposition 3.1 is one of the main ingredients of this paper and improves [21, Lemmas 2.2 and 2.5]. Theorem 1.1 follows from Lemma 2.4 and Proposition 3.1.

**Proposition 3.1** Assume condition (G) for some $\gamma \in \mathbb{N}$, $d > 0$ and $L > 0$. Let $0 \leq k < L$ with $[k] + 1 \leq \gamma$ and $j \in \{0, \ldots, \gamma\}$.

(i) For any $\ell \in [0, k]$, there exists a constant $C_1$ such that

$$
\int_{\mathbb{R}^N} |x|^{\ell} |\nabla e^{t\mathcal{L}} \varphi(x)| dx \leq C_1 t^{-\frac{k-\ell}{d}} \int_{\mathbb{R}^N} |\varphi(x)| dx + C_1 t^{-\frac{1}{d}} \int_{\mathbb{R}^N} |x|^{\ell} |\varphi(x)| dx
$$

(3.1)

for all $t > 0$ and $\varphi \in L^1_k$.

(ii) For any $\ell \in [0, k]$, there exists a constant $C_2$ such that

$$
\int_{\mathbb{R}^N} |x|^{\ell} |e^{t\mathcal{L}} \varphi(x)| dx \leq C_2 t^{-\frac{k-\ell}{d}} \int_{\mathbb{R}^N} |x|^k |\varphi(x)| dx, \quad t > 0,
$$

(3.2)

for all $\varphi \in L^1_k$ satisfying

$$
\int_{\mathbb{R}^N} x^\alpha \varphi(x) dx = 0, \quad \alpha \in M_k.
$$

(3.3)
(iii) For any $\ell \in [0,k],$

$$\lim_{t \to \infty} t^{-\frac{k-\ell}{d}} \int_{\mathbb{R}^N} |x|^k |e^{t\ell} \varphi(x)| dx = 0$$

(3.4)

for $\varphi \in L^1_k$ satisfying (3.3).

**Proof.** Let $\varphi \in L^1_k$ and $0 \leq \ell \leq k.$ By (2.2) we have

$$\int_{\mathbb{R}^N} |x|^{\ell} |\nabla^j G(x-y,t)| dx = \int_{\mathbb{R}^N} |x+y|^{\ell} |\nabla^j G(x,t)| dx$$

$$\leq \int_{\mathbb{R}^N} (|x|^{\ell} + |y|^{\ell}) |\nabla^j G(x,t)| dx \leq t^{-\frac{k-\ell}{d}} + t^{-\frac{k}{d}} |y|^{\ell}$$

(3.5)

for all $y \in \mathbb{R}^N$ and $t > 0.$ This implies

$$\int_{\mathbb{R}^N} |x|^{\ell} |\nabla^j e^{t\ell} \varphi(x)| dx \leq \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} |x|^{\ell} |\nabla^j G(x-y,t)| dx \right) |\varphi(y)| dy$$

$$\leq t^{-\frac{k-\ell}{d}} \|\varphi\|_1 + t^{-\frac{k}{d}} \int_{\mathbb{R}^N} |x|^{\ell} |\varphi(x)| dx$$

for all $t > 0,$ and assertion (i) follows.

In order to prove assertions (ii) and (iii), we assume (3.3). Let $R > 1,$ and put

$$\varphi_1(x) := \varphi(x) \chi_{|x|<R^{1/d}}(x), \quad \varphi_2(x) := \varphi(x) \chi_{|x|\geq R^{1/d}}(x),$$

$$\psi_R(x) := [P_k(R)\varphi_1](x) = \varphi_1(x) - \tilde{\psi}_R(x), \quad \tilde{\psi}_R(x) := \sum_{|\beta| \leq k} M_\beta(\varphi_1, R)g_\beta(x, R).$$

We prove that, for any $\beta \in M_k,$ there exists a constant $C_1$ such that

$$|M_\beta(\varphi_1, R)| \leq C_1 R^{-\frac{k-|\beta|}{d}} \int_{\{|x|\geq R^{1/d}\}} |x|^k |\varphi(x)| dx, \quad R > 1.$$  

(3.6)

For any $\beta \in M_k,$ by (3.3) we obtain

$$\left| \int_{\mathbb{R}^N} x^\beta \varphi_1(x) dx \right| = \left| \int_{\{|x|<R^{1/d}\}} x^\beta \varphi(x) dx \right| = \left| \int_{\{|x|\geq R^{1/d}\}} x^\beta \varphi(x) dx \right|$$

$$\leq \int_{\{|x|\geq R^{1/d}\}} |x|^{|\beta|} |\varphi(x)| dx \leq R^{-\frac{k-|\beta|}{d}} \int_{\{|x|\geq R^{1/d}\}} |x|^k |\varphi(x)| dx.$$  

(3.7)

This implies (3.6) for $\beta \in M_0,$ and (3.6) holds in the case $0 \leq k < 1.$ In the case $k \geq 1,$ we assume that inequality (3.6) holds for all $\beta \in M_n,$ where $n \in \{0, \ldots, [k] - 1\}.$ Then, for any $\beta \in M_k$ with $|\beta| = n + 1,$ by (1.10), (2.2) and (3.7) we have

$$|M_\beta(\varphi_1, R)| \leq \int_{\mathbb{R}^N} x^\beta \varphi_1(x) dx + \sum_{\alpha \leq \beta, \alpha \neq \beta} |M_\alpha(\varphi_1, R)| \left| \int_{\mathbb{R}^N} x^\beta g_\alpha(x, R) dx \right|$$

$$\leq \int_{\mathbb{R}^N} x^\beta \varphi_1(x) dx + C_2 \sum_{\alpha \leq \beta, \alpha \neq \beta} (1 + R)^{\frac{|\beta|-|\alpha|}{d}} |M_\alpha(\varphi_1, R)|$$

$$\leq C_3 R^{-\frac{k-|\beta|}{d}} \int_{\{|x|\geq R^{1/d}\}} |x|^k |\varphi(x)| dx, \quad R > 1,$$
for some constants $C_2$ and $C_3$. This implies (3.6) for any $\beta \in \mathbb{M}_k$ with $|\beta| = n + 1$. Thus it follows (3.6) by induction.

Let $0 \leq \ell \leq k$. We apply a similar argument to (3.7) with the aid of assertion (i), and obtain

$$
\int_{\mathbb{R}^N} |x|^\ell |e^{t\ell} \varphi_2(x)| dx \leq t^\ell \int_{\mathbb{R}^N} |\varphi_2(x)| dx + \int_{\mathbb{R}^N} |x|^\ell |\varphi_2(x)| dx
$$

$$
\leq (t^\ell R^{-\frac{\beta}{\alpha}} + R^{-\frac{k-\ell}{\alpha}}) \int_{\{|x| \geq R^{1/\alpha}\}} |x|^k |\varphi(x)| dx
$$

$$
\leq t^{-\frac{k-\ell}{\alpha}} \left[ (t^{-1} R)^{-\frac{k}{\alpha}} + (t^{-1} R)^{-\frac{k-\ell}{\alpha}} \right] \int_{\{|x| \geq R^{1/\alpha}\}} |x|^k |\varphi(x)| dx \quad (3.8)
$$

for all $t > 0$ and $R > 1$. On the other hand, by (2.2) and (3.6) we have

$$
\int_{\mathbb{R}^N} |x|^\ell |e^{t\ell} \tilde{\psi}_R(x)| dx \leq \sum_{|\beta| \leq k} |M_\beta(\varphi_1, R)| \int_{\mathbb{R}^N} |x|^\ell |g_\beta(x, t + R)| dx
$$

$$
\leq R^{-\frac{k}{\alpha}} \sum_{|\beta| \leq k} \frac{R^{|\beta|}}{(1 + t + R)^{|\beta|+1}} \int_{\{|x| \geq R^{1/\alpha}\}} |x|^k |\varphi(x)| dx
$$

$$
\leq R^{-\frac{k}{\alpha}} (t + R)^\ell \int_{\{|x| \geq R^{1/\alpha}\}} |x|^k |\varphi(x)| dx \quad (3.9)
$$

for all $t \geq 0$ and $R > 1$. In particular, by (3.9) we have

$$
\int_{\mathbb{R}^N} |x|^\ell |\tilde{\psi}_R(x)| dx \leq R^{-\frac{k-\ell}{\alpha}} \int_{\{|x| \geq R^{1/\alpha}\}} |x|^k |\varphi(x)| dx, \quad (3.10)
$$

$$
\int_{\mathbb{R}^N} |x|^k |\psi_R(x)| dx \leq \int_{\mathbb{R}^N} |x|^k ||\varphi_1(x)|| + |\tilde{\psi}_R(x)|| dx \leq \int_{\mathbb{R}^N} |x|^k |\varphi(x)| dx, \quad (3.11)
$$

for all $R > 1$.

Put

$$
G_k(x, y, t) := G(x - y, t) - \sum_{|\alpha| \leq k} \frac{(-1)^{|\alpha|}}{\alpha!} \partial^\alpha_x G(x, t) y^\alpha.
$$

It follows from Lemma 2.1 (ii) that

$$
\int_{\mathbb{R}^N} y^\alpha \psi_R(y) dy = 0, \quad \alpha \in \mathbb{M}_k.
$$

This implies that

$$
e^{t\ell} \psi_R(x) = \int_{\mathbb{R}^N} G_k(x, y, t) \psi_R(y) dy
$$

$$
= \int_{\{|y| < R^{1/\alpha}\}} G_k(x, y, t) \psi_R(y) dy - \int_{\{|y| \geq R^{1/\alpha}\}} G_k(x, y, t) \tilde{\psi}_R(y) dy
$$

$$
=: I_1(x, t) - I_2(x, t). \quad (3.12)
$$
By the mean value theorem, for any \(y \in \mathbb{R}^N\), we can find \(\tilde{y} \in \mathbb{R}^N\) with \(|\tilde{y}| \leq |y|\) such that
\[
|G_k(x, y, t)| \leq C_4 |\nabla_x^{[k]+1} G(x - \tilde{y}, t)||y|^{[k]+1},
\]
where \(C_4\) is a constant independent of \(y\). Then, by (2.2), (3.11) and (3.12) we have
\[
\begin{align*}
&\int_{\mathbb{R}^N} |x|^{\ell} |I_1(x, t)|\,dx \\
&\leq \int_{\{|y| < R^{1/d}\}} \left( \int_{\mathbb{R}^N} |x|^{\ell} |\nabla_x^{[k]+1} G(x - \tilde{y}, t)||y|^{[k]+1}\psi_R(y)\,dy \right) \\
&\leq \int_{\{|y| < R^{1/d}\}} \left( \int_{\mathbb{R}^N} (|x|^{\ell} + |y|^{\ell}) |\nabla_x^{[k]+1} G(x, t)||y|^{[k]+1}\psi_R(y)\,dy \right) \\
&\leq \int_{\{|y| < R^{1/d}\}} \left[ \left| t^{-\frac{[k]+1-\ell}{d}} + |y|^{\ell} t^{-\frac{[k]+1}{d}} \right| |y|^{[k]+1}\psi_R(y)\,dy \right] \\
&\leq \int_{\{|y| < R^{1/d}\}} \left[ t^{-\frac{[k]+1-\ell}{d}} R^{-\frac{[k]+1-k}{d}} + t^{1-\frac{[k]+1}{d}} R^{-\frac{[k]+1+\ell-k}{d}} \right] \\
&\int_{\mathbb{R}^N} |y|^k \psi_R(y)\,dy \]
&\leq t^{-\frac{k-\ell}{d}} \left( (t^{-1}R)^{\frac{[k]+1-k}{d}} + (t^{-1}R)^{\frac{[k]+1+\ell-k}{d}} \right) \\
&\int_{\{|y| \geq R^{1/d}\}} |y|^k \varphi(y)\,dy
\end{align*}
\]
(3.13)
for all \(t > 0\) and \(R > 1\). On the other hand, since
\[
I_2(x, t) = \int_{\{|y| \geq R^{1/d}\}} G(x - y, t)\tilde{\psi}_R(y)\,dy - \sum_{|\alpha| \leq k} \frac{(-1)^{|\alpha|}}{\alpha!} \partial^{\alpha}_x G(x, t) \int_{\{|y| \geq R^{1/d}\}} y^\alpha \tilde{\psi}_R(y)\,dy,
\]
by (2.2), (3.5) and (3.10) we have
\[
\begin{align*}
&\int_{\mathbb{R}^N} |x|^{\ell} |I_2(x, t)|\,dx \\
&\leq \int_{\{|y| \geq R^{1/d}\}} \left( \int_{\mathbb{R}^N} |x|^{\ell} |G(x - y, t)|\,dx \right) \tilde{\psi}_R(y)\,dy \\
&\quad + \sum_{|\alpha| \leq k} \frac{1}{\alpha!} \int_{\mathbb{R}^N} |x|^{\ell} |\partial^{\alpha}_x G(x, t)|\,dx \int_{\{|y| \geq R^{1/d}\}} |y|^{|\alpha|} \tilde{\psi}_R(y)\,dy \\
&\leq \int_{\{|y| \geq R^{1/d}\}} \left( t^{\frac{\ell}{d}} + |y|^{\ell} \right) \tilde{\psi}_R(y)\,dy \\
&\quad + \sum_{|\alpha| \leq k} \left| t^{1-\frac{|\alpha|}{d}} R^{-\frac{k-|\alpha|}{d}} \right| \int_{\{|y| \geq R^{1/d}\}} |y|^k \varphi(y)\,dy \\
&\leq t^{-\frac{k-\ell}{d}} \left[ (t^{-1}R)^{-\frac{k}{d}} + (t^{-1}R)^{-\frac{k-|\alpha|}{d}} \right] \\
&\quad \times \int_{\{|y| \geq R^{1/d}\}} |y|^k \varphi(y)\,dy
\end{align*}
\]
(3.14)
for all \(t > 0\) and \(R > 1\).
Put $R = t + 1$. Since
\[ e^{tC} \varphi(x) = I_1(x, t) - I_2(x, t) + e^{tC} \tilde{\psi}_R(x) + e^{tC} \varphi_2(x), \]
by (3.8), (3.9), (3.13) and (3.14) we have
\[ \int_{R^N} |x|^k e^{tC} \varphi(x)dx \leq t^{-\frac{k}{2}} \int_{R^N} |x|^k |\varphi(x)|dx \]
for all $t > 0$, and we obtain assertion (ii). Furthermore, putting $R = \epsilon t + 1$ with $\epsilon > 0$, by (3.8), (3.9), (3.13) and (3.14) we have
\[ \limsup_{t \to \infty} t^{\frac{k+\alpha}{d}} \int_{R^N} |x|^k |e^{tC} \varphi(x)|dx \leq \limsup_{t \to \infty} t^{\frac{k+\alpha}{d}} \int_{R^N} |x|^k |I_1(x, t)|dx \]
\[ \leq C_5 \left[ \epsilon \frac{1}{d} \right] \int_{R^N} |y|^k |\varphi(y)|dy \]
for some constant $C_5$. Since $\epsilon$ is arbitrary, we obtain (3.1), and assertion (iii) follows. Thus the proof of Proposition 3.1 is complete. □

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let $v$ be the function given in Theorem 1.1 and $0 \leq \ell \leq K$. By Lemma 2.2 we have
\[ |||v(0)|||_K \leq C|||\varphi|||_K \] (3.15)
for some constant $C$. Then, by Proposition 3.1 (i), (2.3) and (2.4) we have
\[ t^{\frac{K}{d}} (1 + t)^{-\frac{\ell}{d}} |||\nabla^j v(t)|||_q \leq ||v(t/2)||_1 \leq ||v(0)||_1 \leq |||\varphi|||_K, \] (3.16)
\[ t^{\frac{K}{d}} (1 + t)^{-\frac{\ell}{d}} |||\nabla^j v(t)|||_\ell \leq ||v(t/2)||_1 + t^{-\frac{\ell}{d}} \int_{R^N} |x|^\ell |v(t/2)|dx, \] (3.17)
\[ t^{\frac{K}{d}} (1 + t)^{-\frac{\ell}{d}} |||\nabla^j v(t)|||_\ell \leq ||v(0)||_K \leq |||\varphi|||_K, \] (3.18)
for all $t > 0$. On the other hand, it follows from Lemma 2.1 that
\[ \int_{R^N} x^\alpha v(x, 0)dx = 0, \quad \alpha \in M_K. \]
Therefore, applying Proposition 3.1 (ii) with the aid of (3.15), we see that
\[ ||v(t)||_1 + t^{-\frac{\ell}{d}} \int_{R^N} |x|^\ell |v(x, t)|dx \leq t^{-\frac{K}{d}} \int_{R^N} |x|^K |v(x, 0)|dx \leq t^{-\frac{K}{d}} |||\varphi|||_K \] (3.19)
for all $t > 0$. Similarly, by Proposition 3.1 (iii) we have
\[ \lim_{t \to \infty} t^{\frac{K}{d}} \left[ ||v(t)||_1 + t^{-\frac{\ell}{d}} \int_{R^N} |x|^\ell |v(x, t)|dx \right] = 0. \] (3.20)
Hence, by (3.16)–(3.20) we have
\[ t^{\frac{K}{d}} (1 + t)^{-\frac{\ell}{d}} |||\nabla^j v(t)|||_q \leq C_1 \min \{ |||\varphi|||_K, t^{-\frac{K}{d}} |||\varphi|||_K \} \leq C_2 (1 + t)^{-\frac{K}{d}} |||\varphi|||_K, \quad t > 0, \]
where $C_1$ and $C_2$ are constants independent of $\varphi \in L^1_K$, and

$$
\lim_{t \to \infty} t^{\frac{K}{d}} \left[ t^{\frac{N}{d(1-\frac{1}{q})} + \frac{d}{2}} \left\| \nabla^j v(t) \right\|_q + t^{\frac{d}{2}} (1 + t)^{-\frac{d}{4}} \left\| \nabla^j v(t) \right\|_\ell \right] = 0.
$$

Thus we obtain (1.12) and (1.13), and the proof of Theorem 1.1 is complete. □

4 Proof of Theorem 1.2

In this section we prove Theorem 1.2 by using Proposition 3.1 and the operator $P_k(t)$.

Proof of Theorem 1.2. Assertion (i) follows from (2.9). We prove assertion (ii). Let $1 \leq q \leq \infty$ and $0 \leq \ell \leq K$. For any $j \in \{0, \ldots, \gamma\}$ with $j < d$, put

$$
I_1(t) := \int_{t/2}^t \nabla^j e^{(t-s)\mathcal{L}} P_K(s) f(s) ds,
$$

$$
I_2(t) := \int_{t/2}^t \nabla^j e^{(t-s)\mathcal{L}} P_K(s) f(s) ds = \int_0^{t/2} \nabla^j e^{\frac{t-s}{2} \mathcal{L}} P_K(s) f(s) ds.
$$

By (2.3) and (2.10) we have

$$
t^{\frac{N}{d}(1-\frac{1}{q})} \| I_1(t) \|_q \leq t^{\frac{N}{d}(1-\frac{1}{q})} \int_{t/2}^t (t-s)^{-\frac{d}{2}} \| P_K(s) f(s) \|_q ds
\leq \int_{t/2}^t (t-s)^{-\frac{d}{2}} s^{\frac{d}{4}(1-\frac{1}{q})} \| P_K(s) f(s) \|_q ds \leq \int_{t/2}^t (t-s)^{-\frac{d}{2}} (1 + s)^{-\frac{K}{d}} E_{K,q}[f](s) ds
\leq (1 + t)^{-\frac{K}{d}} \int_{t/2}^t (t-s)^{-\frac{d}{2}} E_{K,q}[f](s) ds
$$

(4.1)

for all $t > 0$. Furthermore, applying Proposition 3.1 (i) with the aid of (2.10), we obtain

$$
(1 + t)^{-\frac{d}{2}} \| I_1(t) \|_\ell
\leq (1 + t)^{-\frac{d}{2}} \int_{t/2}^t \left[ (t-s)^{-\frac{d}{2}} \| P_K(s) f(s) \|_1 + (t-s)^{-\frac{d}{2}} \| P_K(s) f(s) \|_\ell \right] ds
\leq \int_{t/2}^t (t-s)^{-\frac{d}{2}} \left[ \| P_K(s) f(s) \|_1 + (1 + s)^{-\frac{K}{d}} \| P_K(s) f(s) \|_\ell \right] ds
\leq (1 + t)^{-\frac{K}{d}} \int_{t/2}^t (t-s)^{-\frac{d}{2}} E_{K,q}[f](s) ds
$$

(4.2)

for all $t > 0$.

On the other hand, applying Proposition 3.1 (ii) with the aid of Lemma 2.1 (ii), for any $\delta > 0$, we deduce from (2.10) that

$$
\left\| e^{\frac{t-s}{2} \mathcal{L}} P_K(s) f(s) \right\|_\ell \leq (t-s)^{-\frac{K}{d} + \delta} \left\| P_K(s) f(s) \right\|_K \leq (t-s)^{-\frac{K}{d} + \delta} E_{K,q}[f](s)
$$

(4.3)
for all $t \geq s + \delta > 0$. Similarly to (4.1) and (4.2), we have
\[
t^\frac{\alpha}{\alpha-1} t^\frac{1}{\alpha-1} \|I_2(t)\|_q + (1 + t)^{-\frac{\alpha}{\alpha-1}} \|I_2(t)\|_1 \\
\leq t^\frac{\alpha}{\alpha-1} t^\frac{1}{\alpha-1} \int_0^{t/2} (t-s)^{-\frac{\alpha}{\alpha-1}} \|e^{L2tP_K(s)}f(s)\|_1 ds \\
+ (1 + t)^{-\frac{\alpha}{\alpha-1}} \int_0^{t/2} (t-s)^{-\frac{\alpha}{\alpha-1}} \|e^{L2tP_K(s)}f(s)\|_1 ds \\
+ (1 + t)^{-\frac{\alpha}{\alpha-1}} \int_0^{t/2} (t-s)^{-\frac{\alpha}{\alpha-1}} \|e^{L2tP_K(s)}f(s)\|_1 ds
\]
for all $t > 0$. On the other hand, for any $T > 0$, it follows from Proposition 3.1 (iii) and Lemma 2.1 (ii) that
\[
\lim_{t \to \infty} (t-s)^{K/k} \|e^{L2tP_K(s)}f(s)\|_k = 0
\]
for any $0 \leq k \leq K$ and $s \in (0, T)$. Then, by the Lebesgue dominated convergence theorem and Proposition 3.1 (ii) we see that
\[
\limsup_{t \to \infty} t^\frac{\alpha}{\alpha-1} t^\frac{1}{\alpha-1} \int_0^{T} (t-s)^{-\frac{\alpha}{\alpha-1}} \left[\|e^{L2tP_K(s)}f(s)\|_1 + (t-s)^{-\frac{\alpha}{\alpha-1}} \|e^{L2tP_K(s)}f(s)\|_1\right] ds \\
\leq \limsup_{t \to \infty} \int_0^{T} (t-s)^{K/k} \left[\|e^{L2tP_K(s)}f(s)\|_1 + (t-s)^{-\frac{\alpha}{\alpha-1}} \|e^{L2tP_K(s)}f(s)\|_1\right] ds
\]
for all $t \geq 2T$ and $T \geq T_0$. Therefore, by (4.4)–(4.6), for any $\epsilon > 0$ and $T \geq T_0$, we have
\[
t^\frac{\alpha}{\alpha-1} t^\frac{1}{\alpha-1} \|I_2(t)\|_q + (1 + t)^{-\frac{\alpha}{\alpha-1}} \|I_2(t)\|_1 \\
\leq C_1 t^\frac{\alpha}{\alpha-1} \int_0^{T/2} (t-s)^{-\frac{\alpha}{\alpha-1}} E_{K,q}[f](s) ds + C_2 t^{-\frac{\alpha}{\alpha-1}} \int_0^{T/2} (t-s)^{-\frac{\alpha}{\alpha-1}} E_{K,q}[f](s) ds
\]
for all sufficiently large $t$, where $C_3$ is a constant independent of $T \in [T_0, \infty)$ and $\epsilon > 0$. Hence, by (4.1), (4.2) and (4.7), we have (1.16). In addition, (1.17) immediately follows from (1.16). Thus the proof of Theorem 1.2 is complete. □

5 Integral equation with power nonlinearity

Let $F = F(x, t, u)$ be a function in $\mathbb{R}^N \times (0, \infty) \times \mathbb{R}$ such that
\[
F(x, t, 0) = 0,
\]
\[
|F(x, t, u_1) - F(x, t, u_2)| \leq C_4(1 + t)^{A} \max\{|u_1|^{p-1}, |u_2|^{p-1}\}|u_1 - u_2|,
\]
where $A \geq 1$ is a constant.
for $x \in \mathbb{R}^N$, $t > 0$ and $u_1, u_2 \in \mathbb{R}$, where $C_0 > 0$, $A \in \mathbb{R}$ and $p \geq 1$. Consider the integral equation

$$u(x,t) = \int_{\mathbb{R}^N} G(x-y,t)\varphi(y)dy + \int_0^t \int_{\mathbb{R}^N} G(x-y,t-s)F(y,s,u(y,s))dyds,$$

(5.3)

where $\varphi \in L^1_{K^*}$ for some $K \geq 0$. Problem (5.3) is a generalization of problems (1.4) and (1.5). In this section, under condition (G) and (5.2), we study the asymptotic behavior of the solution $u$ of (5.3) satisfying

$$\sup_{t>0} (1 + t)^{N(1-\frac{1}{q})}\|u(t)\|_q + \sup_{t>0} (1 + t)^{-\frac{d}{2}}||_{\ell}\|u(t)||_{\ell} < \infty$$

(5.4)

for any $q \in [1, \infty]$ and $\ell \in [0, K]$, and prove the following theorem. (For the existence of the solutions of (5.3) satisfying (5.4), see [25].)

**Theorem 5.1** Assume condition (G) for some $\gamma \in \mathbb{N}$, $d > 0$ and $L > 0$. Let $0 \leq K < L$ with $[K] + 1 \leq \gamma$ and $\varphi \in L^\infty \cap L^1_{K^*}$. Assume (5.1), (5.2) and $A_p := -A + N(p-1)/d - 1 > 0$.

Let $u$ be a global-in-time solution of (5.3) satisfying (5.4).

(i) For any $\alpha \in \mathbb{M}_K$, put

$$c_\alpha(t) := M_\alpha(\varphi, 0) + \int_0^t M_\alpha(F(s), s)ds,$$

(5.5)

where $F(x,t) := F(x,t,u(x,t))$. If $A_p > |\alpha|/d$, then there exists a constant $c_\alpha$ such that

$$|c_\alpha(t) - c_\alpha| = O(t^{-A_p+|\alpha|/d})$$

(5.6)

as $t \to \infty$. If $A_p \leq |\alpha|/d$, then

$$c_\alpha(t) = \begin{cases} O(t^{-A_p+|\alpha|/d}) & \text{if } A_p < |\alpha|/d, \\ O(\log t) & \text{if } A_p = |\alpha|/d, \end{cases}$$

(5.7)

as $t \to \infty$.

(ii) Define the functions $U_n = U_n(x,t)$ $(n = 0, 1, 2, \ldots)$ inductively by

$$U_0(x,t) := \sum_{|\alpha| \leq K} c_\alpha(t)g_\alpha(x,t),$$

(5.8)

$$U_n(x,t) := U_0(x,t) + \int_0^t e^{(t-s)\xi}P_K(s)F_{n-1}(s)ds$$

$$= \sum_{|\alpha| \leq K} \left[ M_\alpha(\varphi, 0) + \int_0^t M_\alpha(F(s) - F_{n-1}(s), s)ds \right] g_\alpha(x,t) + \int_0^t e^{(t-s)\xi}F_{n-1}(s)ds,$$

(5.9)
where \( n = 1, 2, \ldots \) and \( F_{n-1}(x, t) := F(x, t, U_{n-1}(x, t)) \). Then, for any \( q \in [1, \infty] \) and \( \ell \in [0, K] \),
\[
\sup_{t > 0} t^{\frac{N}{q}(1 - \frac{1}{q})} \|U_n(t)\|_q + \sup_{t > 0} (1 + t)^{-\frac{\ell}{q}} \|\|U_n(t)\|_\ell < \infty \tag{5.10}
\]
and
\[
t^\frac{N}{q}(1 - \frac{1}{q}) \|u(t) - U_n(t)\|_q + t^{-\frac{\ell}{q}} \|u(t) - U_n(t)\|_\ell = \begin{cases} 
    o(t^{-\frac{K}{q}}) + O(t^{-(n+1)A_p}) & \text{if } (n+1)A_p \neq K/d, \\
    O(t^{-\frac{K}{q}} \log t) & \text{if } (n+1)A_p = K/d,
\end{cases} \tag{5.11}
\]
as \( t \to \infty \).

Here we remark:

- \( U_0(\cdot, t) \) is a linear combination of \( \{g_\alpha(\cdot, t)\}_{|\alpha| \leq K} \) and plays a role of the projection of \( u(\cdot, t) \) into the finite dimensional space spanned by \( \{g_\alpha(\cdot, t)\}_{|\alpha| \leq K} \);
- For \( n = 1, 2, \ldots, U_n \) is a nonlinear approximation to the solution \( u \) and is constructed by \( U_0 \) systematically.

**Proof of Theorem 5.1.** Let \( f(x, t) = F(x, t, u(x, t)) \). It follows from (5.1) and (5.2) that
\[
|f(x, t)| \leq C_s(1 + t)^A|u(x, t)|^p.
\]
This together with (1.14) and (5.4) implies
\[
E_{K,q}[f](t) \leq C_s(1 + t)^A\|u(t)\|_\infty^{p-1}E_{K,q}[u](t) \leq (1 + t)^{A - \frac{N}{q}(p-1) + \frac{K}{d}} = (1 + t)^{-A_p + 1 + \frac{K}{d}} \tag{5.12}
\]
for all \( t > 0 \). Then, by Lemma 2.2 and (5.5) we have
\[
|c_\alpha(t_2) - c_\alpha(t_1)| \leq \int_{t_1}^{t_2} |M_\alpha(f(s), s)| ds \leq \int_{t_1}^{t_2} (1 + s)^{-A_p + 1 + \frac{K}{d}} ds
\]
for \( t_2 \geq t_1 > 0 \). This implies (5.6) and (5.7), and assertion (i) follows.

We prove assertion (ii). The proof is by induction. Assertion (i) together with (2.2) yields (5.10) for \( n = 0 \). Let \( v = v(x, t) \) and \( R_K[f](x, t) \) be functions given in Theorems 1.1 and 1.2. Since
\[
u(x, t) - U_0(x, t) = e^{t\mathcal{L}}\phi(x) + \int_0^t e^{(t-s)\mathcal{L}} f(s) ds - \sum_{|\alpha| \leq K} \left[ M_\alpha(\phi, 0) + \int_0^t M_\alpha(f(s), s) ds \right] g_\alpha(x, t)
\]
\[
= e^{t\mathcal{L}} \left[ \phi - \sum_{|\alpha| \leq k} M_\alpha(\phi, 0)g_\alpha(0) \right] + \int_0^t e^{(t-s)\mathcal{L}} \left[ f(s) - \sum_{|\alpha| \leq k} M_\alpha(f(s), s)g_\alpha(s) \right] ds
\]
\[
= v(x, t) + R_K[f](x, t),
\]
Thus assertion (ii) holds for assertion (ii) with $n$ where $f$ as $t$ as $t$ by induction we see that assertion (ii) holds for all $n$. Then, by Theorem 1.2, for any $T > \infty$, we obtain

\[
M := \lim_{t \to \infty} \int_{\mathbb{R}^N} u(x,t) dx = \int_{\mathbb{R}^N} \varphi(x) dx + \int_0^\infty \int_{\mathbb{R}^N} F(x,t,u(x,t)) dx dt.
\]

We remark that $M$ coincides with $c_0$, which is given in Theorem 5.1 (i).
Corollary 5.1 Assume the same conditions as in Theorem 5.1
(i) For any $1 \leq q \leq \infty$ and $0 \leq \ell \leq K$,
\[
\frac{N}{(1 - \frac{1}{q})} \|u(t) - Mg(t)\|_q + t^{-\frac{\ell}{q}} \|u(t) - Mg(t)\|_\ell
\]
\[= \begin{cases} 
  o(t^{-\frac{K}{q}}) + O(t^{-A_\ell}) & \text{if } 0 \leq K < 1 \text{ and } A_\ell \neq K/d, \\
  O(t^{-\frac{K}{q}} \log t) & \text{if } 0 \leq K < 1 \text{ and } A_\ell = K/d, \\
  O(t^{-\frac{1}{q}}) + O(t^{-A_\ell}) & \text{if } K \geq 1 \text{ and } A_\ell \neq 1/d, \\
  O(t^{-\frac{1}{q}} \log t) & \text{if } K \geq 1 \text{ and } A_\ell = 1/d,
\end{cases}
\] (5.16)
as $t \to \infty$.

(ii) Let $K \geq 1$. Let $f_M(x, t) = F(x, t, Mg(x, t))$, and assume that
\[
\int_0^\infty |M_\alpha(f_M(t), t)| \, dt < \infty
\] (5.17)
for any $\alpha \in M$ with $|\alpha| = 1$. Then, for any $1 \leq q \leq \infty$ and $0 \leq \ell \leq K$,
\[
\frac{N}{(1 - \frac{1}{q})} \|u(t) - Mg(t)\|_q + t^{-\frac{\ell}{q}} \|u(t) - Mg(t)\|_\ell = O(t^{-\frac{1}{q}}) + O(t^{-A_\ell})
\] (5.18)
as $t \to \infty$.

Proof. It follows from (5.8) that
\[
U_0(x, t) - Mg(x, t) = (c_0(t) - M)g(x, t) + \sum_{1 \leq |\alpha| \leq K} c_\alpha(t)g_\alpha(x, t).
\]
This together with Theorem 5.1 (i) and (2.2) implies that
\[
\frac{N}{(1 - \frac{1}{q})} \|U_0(t) - Mg(t)\|_q + t^{-\frac{\ell}{q}} \|U_0(t) - Mg(t)\|_\ell
\]
\[\leq |c_0(t) - M| + \sum_{1 \leq |\alpha| \leq K} (1 + t)^{-\frac{|\alpha|}{q}} \|c_\alpha(t)\|
\]
\[= \begin{cases} 
  O(t^{-A_\ell}) & \text{if } 0 \leq K < 1, \\
  O(t^{-\frac{1}{q}}) + O(t^{-A_\ell}) & \text{if } K \geq 1 \text{ and } A_\ell \neq 1/d, \\
  O(t^{-\frac{1}{q}} \log t) & \text{if } K \geq 1 \text{ and } A_\ell = 1/d,
\end{cases}
\] (5.19)
as $t \to \infty$. Combining (5.19) with Theorem 5.1 (ii), we see that
\[
\frac{N}{(1 - \frac{1}{q})} \|u(t) - Mg(t)\|_q + t^{-\frac{\ell}{q}} \|u(t) - Mg(t)\|_\ell
\]
\[= \begin{cases} 
  o(t^{-\frac{K}{q}}) + O(t^{-A_\ell}) & \text{if } 0 \leq K < 1 \text{ and } A_\ell \neq K/d, \\
  O(t^{-\frac{K}{q}} \log t) & \text{if } 0 \leq K < 1 \text{ and } A_\ell = K/d, \\
  O(t^{-\frac{1}{q}}) + O(t^{-A_\ell}) & \text{if } K \geq 1 \text{ and } A_\ell \neq 1/d, \\
  O(t^{-\frac{1}{q}} \log t) & \text{if } K \geq 1 \text{ and } A_\ell = 1/d,
\end{cases}
\]
as $t \to \infty$, and assertion (i) follows.
We prove assertion (ii). It suffices to consider the case $A_p = 1/d$. Let $K \geq 1$ and $\alpha \in \mathbf{M}$ with $|\alpha| = 1$. By (5.5) and (5.17) we apply Lemmas 2.1 and 2.2 to obtain

$$
|c_\alpha(t_2) - c_\alpha(t_1)| \leq \int_{t_1}^{t_2} [M_\alpha(f(s), s) - M_\alpha(f_M(s), s)] ds + 1
$$

for all $0 < t_1 < t_2$, where $f(x, t) = F(x, t, u(x, t))$. Then, by a similar argument to (5.12) with the aid of (5.16) we see that

$$
|c_\alpha(t_2) - c_\alpha(t_1)| \leq \int_{t_1}^{t_2} s^{-\frac{K}{d} + 1} E_{K,q}[f - f_M](s) ds + 1
$$

for all sufficiently large $t_1$ and $t_2$ with $t_1 < t_2$. This implies that $|c_\alpha(t)| = O(1)$ as $t \to \infty$. Therefore, by the same argument as in the proof of assertion (i) we have (5.18). Thus assertion (ii) follows, and the proof of Corollary 5.1 is complete.

Next, applying Theorem 5.1 with $n = 1$, we give more precise description of the asymptotic behavior of the solution of (5.3) than in Corollary 5.1.

**Corollary 5.2** Assume the same conditions as in Theorem 5.1 and $0 \leq K < 1$. Let

$$
f(x, t) := F(x, t, u(x, t)), \quad f_M(x, t) := F(x, t, M g(x, t)),
$$

and put

$$
\tilde{u}(x, t) := M'g(x, t) + \int_0^t e^{(t-s)\mathcal{L}} f_M(s) ds,
$$

where

$$
M' := \int_{\mathbb{R}^N} \varphi(x) dx + \int_0^\infty \int_{\mathbb{R}^N} [f(x, t) - f_M(x, t)] dx dt.
$$

Then, for any $q \in [1, \infty]$ and $\ell \in [0, K]$,

$$
t^{\frac{K}{d}(1-\frac{1}{q})} \|u(t) - \tilde{u}(t)\|_q + t^{\frac{K}{d}+\frac{q}{d}} \|u(t) - \tilde{u}(t)\|_\ell = \begin{cases} o(t^{-\frac{K}{d}}) + O(t^{-2A_p}) & \text{if } 2A_p \neq K/d, \\ O(t^{-\frac{K}{d}\log t}) & \text{if } 2A_p = K/d, \end{cases}
$$

as $t \to \infty$.

**Proof.** Put

$$
f_1(x, t) := F(U_0(x, t)) - f_M(x, t), \quad f_2(x, t) := f(x, t) - f_M(x, t),
$$

$$
w(x, t) := \int_0^t e^{(t-s)\mathcal{L}} P_K(s) f_1(s) ds.
$$
Similarly to (5.14), by Corollary 5.1 and (5.19) we have
\[
E_{K,q}[f_1](t) + E_{K,q}[f_2](t) = \begin{cases} 
 o(t^{-A_p-1}) + O(t^{\frac{K}{d} - 2A_p - 1}) & \text{if } A_p \neq K/d, \\
 O(t^{-A_p-1} \log t) & \text{if } A_p = K/d,
\end{cases}
\] (5.21)
as \( t \to \infty \). Then, by Theorem 1.2 and (5.21) we see that
\[
N_d(1 - \frac{1}{q}) \| w(t) \|_q + t^{-\frac{K}{d}} ||| w(t) |||_\ell = \begin{cases} 
 o(t^{-\frac{K}{d}}) + O(t^{-2A_p}) & \text{if } 2A_p \neq K/d, \\
 O(t^{-\frac{K}{d} \log t}) & \text{if } 2A_p = K/d,
\end{cases}
\] (5.22)
as \( t \to \infty \). Furthermore, by Lemma 2.2 and (5.21) we get
\[
|M_0(f_2(t), t)| \leq (1 + t)^{-\frac{K}{d}} E_{K,q}[f_2](t) = \begin{cases} 
 o(t^{-\frac{K}{d}} - A_p - 1) + O(t^{-2A_p - 1}) & \text{if } A_p \neq K/d, \\
 O(t^{-2A_p - 1} \log t) & \text{if } A_p = K/d,
\end{cases}
\] (5.23)
as \( t \to \infty \). On the other hand, it follows from (5.9) that
\[
\tilde{u}(x, t) - U_1(x, t) = \int_t^\infty M_0(f_2(s), s) ds \cdot g(x, t) - w(x, t).
\]
Therefore, by (5.22) and (5.23) we have
\[
t^{\frac{N}{d}(1 - \frac{1}{q})} \| U_1(t) - \tilde{u}(t) \|_q + t^{-\frac{K}{d}} ||| U_1(t) - \tilde{u}(t) |||_\ell = \begin{cases} 
 o(t^{-\frac{K}{d}}) + O(t^{-2A_p}) & \text{if } 2A_p \neq K/d, \\
 O(t^{-\frac{K}{d} \log t}) & \text{if } 2A_p = K/d,
\end{cases}
\]
as \( t \to \infty \). This together with (5.11) implies (5.20), and Corollary 5.2 follows. \( \Box \)

6 Applications

We apply the results in the previous sections to some nonlinear parabolic equations, and show the validity of our arguments.

6.1 Semilinear parabolic equations

Let \( u \) be a solution of the Cauchy problem for a semilinear parabolic equation
\[
\begin{align*}
\partial_t u &= \Delta u + a(x, t)|u|^{p-1}u & \text{in } \mathbb{R}^N \times (0, \infty), \\
u(x, 0) &= \varphi(x) & \text{in } \mathbb{R}^N,
\end{align*}
\] (6.1)
where \( p \geq 1, a \in L^\infty(0, \infty : L^\infty(\mathbb{R}^N)) \) and \( \varphi \in L^\infty \). Asymptotic behavior of the solutions of (6.1) has been studied by many mathematicians (see e.g. \([8, 13, 20–24, 27, 28, 30, 33]\) and references therein). In particular, the asymptotic expansions of the solutions of (6.1) behaving like a multiple of the heat kernel were discussed in \([21–24]\).

On the other hand, the heat kernel satisfies condition (G) for any \( \gamma \in \mathbb{N} \) and \( L > 0 \) with \( d = 2 \). Therefore, as a corollary of the results in the previous section, we have:
Theorem 6.1 Let \( \varphi \in L^\infty \cap L^1_k \) for some \( K \geq 0 \) and \( p \geq 1 \). Assume
\[
\sup_{t > 0} (1 + t)^{-A} \|a(t)\|_\infty < \infty \tag{6.2}
\]
for some \( A \in \mathbb{R} \) and \( A_p := -A + N(p - 1)/2 - 1 > 0 \). Let \( u \) be a solution of (6.1) satisfying
\[
\sup_{t > 0} (1 + t)^{\frac{N}{p}} \|u(t)\|_\infty < \infty. \tag{6.3}
\]
Then the conclusions as Theorem 5.1 and Corollaries 5.1 and 5.2 hold for any \( \gamma \in \mathbb{N} \) and \( L > 0 \) with \( d = 2 \).

Proof. Under assumptions (6.2) and (6.3), by a similar argument as in [24, Theorem 3.1] we see that
\[
\sup_{t > 0} t^{\frac{N}{p}(1 - \frac{1}{q})} \|u(t)\|_q + \sup_{t > 0} (1 + t)^{-\frac{A}{p}} \|u(t)\|_{\ell} < \infty
\]
for any \( q \in [1, \infty] \) and \( \ell \in [0, K] \). Then, applying the arguments in Section 5, we obtain the desired conclusions. Thus Theorem 6.1 follows. \( \square \)

Remark 6.1 Let \( G = G(x,t) \) be the heat kernel and \( \alpha \in \mathbb{M} \) with \( |\alpha| = 1 \). Then, by the radially symmetry of \( G \), we have
\[
M_\alpha(f, t) = \int_{\mathbb{R}^N} x^\alpha f(x) dx, \quad t > 0,
\]
for all \( f \in L^1_1 \). Furthermore, if \( a = a(x,t) \) is radially symmetric with respect to the space variable \( x \), then
\[
M_\alpha(f_M(t), t) = \int_{\mathbb{R}^N} x^\alpha f_M(x, t) dx = |M|^{p-1} M \int_{\mathbb{R}^N} x^\alpha a(x, t) g(x, t)^p dx = 0,
\]
where \( f_M \) is the function defined in Corollary 5.2 and assumption (5.17) is satisfied.

Theorem 6.1 gives sharper decay estimates of \( L^q(\mathbb{R}^N) \)-distance from the solution \( u \) to its asymptotic profiles than in [8], [21], [22]–[24] and [33]. Furthermore, similarly to [24], we see that similar results to Theorem 6.1 hold for more general nonlinear heat equations
\[
\partial_t u = \Delta u + F(x, t, u, \nabla u) \quad \text{in} \quad \mathbb{R}^N \times (0, \infty),
\]
under suitable assumptions on \( F \) (see conditions \( (C_A) \) and \( (F_A) \) in [24]). The details are left to the reader.

6.2 Fractional semilinear parabolic equations

Consider the Cauchy problem for a fractional semilinear parabolic equation
\[
\left\{ \begin{array}{l}
\partial_t u = -(-\Delta)^{\theta/2} u + a(x, t)|u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N \times (0, \infty), \\
u(x, 0) = \varphi(x) \quad \text{in} \quad \mathbb{R}^N,
\end{array} \right. \tag{6.4}
\]
where $0 < \theta < 2$, $p \geq 1$, $a \in L^\infty(0, \infty : L^\infty(\mathbb{R}^N))$ and $\varphi \in L^\infty$. A continuous function $u$ in $\mathbb{R}^N \times (0, \infty)$ is said to be a solution of (6.4) if $u$ satisfies

$$u(x, t) = \int_{\mathbb{R}^N} G_\theta(x - y, t) \varphi(y) dy + \int_0^t \int_{\mathbb{R}^N} G_\theta(x - y, t - s) a(y, s) |u(y, s)|^{p-1} u(y, s) dy ds$$

for all $x \in \mathbb{R}^N \times (0, \infty)$, where $G_\theta = G_\theta(x, t)$ be the fundamental solution of

$$\partial_t u + (-\Delta)^{\theta/2} u = 0 \quad \text{in} \quad \mathbb{R}^N \times (0, \infty).$$

Problem (6.4) has been studied extensively by many mathematicians in view of various aspects, for example, nonlinear problems with anomalous diffusion and the Laplace equation with dynamical boundary conditions (see [1], [13], [14], [19], [32], [37] and references therein). Among others, in the case where $a(x, t)$ is a negative constant function in $\mathbb{R}^N \times (0, \infty)$, Fino and Karch [14] proved the following (see also [19]):

- Let $\varphi \in L^1(\mathbb{R}^N)$ and $p > 1 + \theta/N$. Then there exists a constant $M$ such that

$$\lim_{t \to \infty} \int_{\mathbb{R}^N} u(x, t) dx = M \quad \text{and} \quad \lim_{t \to \infty} t^\frac{\theta}{N} \|u(t) - MG_\theta(t)\|_q = 0 \quad (6.5)$$

for any $q \in [1, \infty]$. If $1 < p \leq 1 + \theta/N$, then

$$\lim_{t \to \infty} \int_{\mathbb{R}^N} u(x, t) dx = 0.$$

(For the case $1 < p \leq 1 + \theta/N$, see [19].) In the case where $a(x, t)$ is a positive constant function in $\mathbb{R}^N \times (0, \infty)$, the following holds:

- If $1 < p \leq 1 + N/\theta$, then problem (6.4) has no positive global in time solutions (see [32]);

- Let $\varphi \in L^\infty \cap L^1$, $\theta = 1$ and $p > 1 + N$. If $\|\varphi\|_1 \|\varphi\|_\infty^{N(p-1)-1}$ is sufficiently small, then there exists a global in time solution $u$ of (6.4) such that (6.5) holds with $\theta = 1$ (see [13]).

As far as we know, there are few results giving the precise description of the asymptotic behavior of the global in time solutions of (6.4).

On the other hand, $G_\theta$ satisfies condition (G) for $d = L = \theta$ with $\gamma = 1$ if $0 < \theta \leq 1$ and $\gamma = 2$ if $1 < \theta < 2$ (see [3] Lemma 7.3, [4] Lemma 5.3 and [37] Lemma 2.1). Then we can apply the results in Section 5 to problem (6.4), and obtain the following theorem.

**Theorem 6.2** Consider problem (6.4). Assume (6.3) with $A_p := -A + N(p-1)/\theta - 1 > 0$. Then the conclusions of Theorem 5.1 and Corollaries 5.1 and 5.2 hold for $d = L = \theta$ with $\gamma = 1$ if $0 < \theta \leq 1$ and $\gamma = 2$ if $1 < \theta < 2$.

This enables us to study the precise description of the asymptotic behavior of the solution of (6.4) behaving like a multiple of $G_\theta$ as $t \to \infty$ for the case $p > 1 + \theta/N$, and improves [13] and [14].
Remark 6.2 Yamamoto recently studied the asymptotic behavior of the solutions of
\[ \partial_t u = -(-\Delta)^{\theta/2}u + a(x,t)u \quad \text{in} \quad \mathbb{R}^N \times (0, \infty), \quad 1 < \theta < 2, \quad (6.6) \]
and obtained higher order asymptotic expansions of the solutions, which are similar to those given in Theorem 6.2 with \( p = 1 \) and \( 1 < \theta < 2 \). However, his results require a stronger assumption on \( a = a(x,t) \) than (6.2) and a pointwise decay condition of the solution as \(|x| \to \infty\), such as (1.6).

6.3 Higher-order semilinear parabolic equations

Let \( m = 1, 2, \ldots \) and
\[ Lu := \sum_{|\alpha|=2m} A_\alpha \partial_\alpha^2 u \]
be a 2\( m \)-th order differential operator such that
\[ \sum_{|\alpha|=2m} (i\xi)^\alpha A_\alpha \leq -c_1|\text{Re}\xi|^{2m} + c_2|\text{Im}\xi|^m, \quad \xi \in \mathbb{C}^N, \quad (6.7) \]
for some positive constants \( c_1 \) and \( c_2 \), where \( \{A_\alpha\} \subset \mathbb{R} \). In this section, under assumptions (5.2) and (6.7), we consider the Cauchy problem for the 2\( m \)-th order semilinear parabolic equation
\[ \begin{aligned}
\partial_t u &= Lu + a(x,t)|u|^p \quad \text{in} \quad \mathbb{R}^N \times (0, \infty), \\
u(x,0) &= \varphi(x) \quad \text{in} \quad \mathbb{R}^N, 
\end{aligned} \quad (6.8)\]
where \( p \geq 1 \), \( a \in L^\infty(0, \infty : L^\infty(\mathbb{R}^N)) \) and \( \varphi \in L^\infty \cap L^1 \). In the case where \( a \) is a positive constant function in \( \mathbb{R}^N \times (0, \infty) \), problem (6.8) has been studied in several papers (see [5], [10]–[12], [16], [17] and references therein), and the following holds:

- Let \( 1 < p \leq 1 + 2m/N \). If \( \varphi \not\equiv 0 \) in \( \mathbb{R}^N \) and \( \int_{\mathbb{R}^N} \varphi(x)dx \geq 0 \), then problem (6.8) has no global in time solutions (see [10]);

- Let \( p > 1 + 2m/N \). Assume that \( \varphi \not\equiv 0 \) in \( \mathbb{R}^N \) and \( \int_{\mathbb{R}^N} \varphi(x)dx \geq 0 \). Then there exists a positive constant \( C_1 \) such that, if
\[ |\varphi(x)| \leq C_1 e^{-\frac{1}{2}|x|^{2m/(2m-1)}} \quad \text{for almost all} \quad x \in \mathbb{R}^N, \]
then problem (6.8) has a global in time solution behaving like a multiple of \( G_m(x,t) \) as \( t \to \infty \), where \( G_m = G_m(x,t) \) is the fundamental solution of
\[ \partial_t u + (-\Delta)^m u = 0 \quad \text{in} \quad \mathbb{R}^N \times (0, \infty) \]
(see [16]);

- Let \( p > 1 + 2m/N \). Assume
\[ 0 \leq \varphi(x) \leq \frac{C_2}{1 + |x|^\beta} \]
for some \( \beta > 2m/(p-1) \) and \( C_2 > 0 \). If \( \|\varphi\|_\infty \) is sufficiently small, then problem (6.8) has a global in time solution (see [5]). For the case \( \beta = 2m/(p-1) \), see [17].
Similarly to problem (6.4), as far as we know, there are few results giving the precise description of the asymptotic behavior of the global in time solutions of (6.8).

On the other hand, under assumption (6.7), the fundamental solution of $\partial_t u = Lu$ in $\mathbb{R}^N \times (0, \infty)$ satisfies condition (G) for any $\gamma \in \mathbb{N}$ and $L > 0$ with $d = 2m$ (see e.g. [7]). Then we apply the results in Section 5 to problem (6.8), and obtain the following theorem.

**Theorem 6.3** Assume (6.7), and consider problem (6.8). Assume (6.3) with $A_p := -A + N(p-1)/2m - 1 > 0$. Then the conclusions of Theorem 5.1 and Corollaries 5.1 and 5.2 hold for any $L > 0$ and $\gamma > 0$ with $d = 2m$.

Theorem 6.3 enables us to study the precise description of the asymptotic behavior of the solutions behaving like a multiple of the kernel $G_m$.

**References**

[1] H. Amann and M. Fila, A Fujita-type theorem for the Laplace equation with a dynamical boundary condition, Acta Math. Univ. Comenianae 66 (1997), 321–328.

[2] S. Benachour, G. Karch and P. Laurençot, Asymptotic profiles of solutions to viscous Hamilton-Jacobi equations, J. Math. Pures Appl. 83 (2004), 1275–1308.

[3] P. Biler, T. Funaki and W. A. Woyczynski, Fractal Burgers equations, J. Differential Equations 148 (1998), 9–46.

[4] P. Biler and W. A. Woyczyński, Global and exploding solutions for nonlocal quadratic evolution problems, SIAM J. Appl. Math. 59 (1999), 845–869.

[5] G. Caristi and E. Mitidieri, Existence and nonexistence of global solutions of higher-order parabolic problems with slow decay initial data, J. Math. Anal. Appl. 279 (2003), 710–722.

[6] A. Carpio, Large time behaviour in convection-diffusion equations, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 23 (1996), 551–574.

[7] S. Cui, Local and global existence of solutions to semilinear parabolic initial value problems, Nonlinear Anal. 43 (2001), 293–323.

[8] J. Dolbeault and G. Karch, Large time behavior of solutions to nonhomogeneous diffusion equations, Banach Center Publ. 74 (2006), 113–147.

[9] G. Duro and E. Zuazua, Large time behavior for convection-diffusion equations in $\mathbb{R}^N$ with asymptotically constant diffusion, Comm. Partial Differential Equations 24 (1999), 1283–1340.

[10] Yu. V. Egorov, V. A. Galaktionov, V. A. Kondratiev and S I. Pohozaev, On the necessary conditions of global existence to a quasilinear inequality in the half-space, C. R. Math. Acad. Sci. Paris 330 (2000), 93–98.
[11] Yu. V. Egorov, V. A. Galaktionov, V. A. Kondratiev and S I. Pohozaev, On the asymptotics of global solutions of higher-order semilinear parabolic equations in the supercritical range, C. R. Math. Acad. Sci. Paris 335 (2002), 805–810.

[12] Yu. V. Egorov, V. A. Galaktionov, V. A. Kondratiev and S I. Pohozaev, Global solutions of higher-order semilinear parabolic equations in the supercritical range, Adv. Differential Equations 9 (2004), 1009–1038.

[13] M. Fila, K. Ishige and T. Kawakami, Convergence to the Poisson kernel for the Laplace equation with a nonlinear dynamical boundary condition, Commun. Pure Appl. Anal. 11 (2012), 1285–1301.

[14] A. Fino and G. Karch, Decay of mass for nonlinear equation with fractional Laplacian, Monatsh. Math. 160 (2010), 375–384.

[15] Y. Fujigaki and T. Miyakawa, Asymptotic profiles of nonstationary incompressible Navier-Stokes flows in the whole space, SIAM J. Math. Anal. 33 (2001), 523–544.

[16] V. A. Galaktionov and S. I. Pohozaev, Existence and blow-up for higher-order semilinear parabolic equations: majorizing order-preserving operators, Indiana Univ. Math. J. 51 (2002), 1321–1338.

[17] F. Gazzola and H.-C. Grunau, Global solutions for superlinear parabolic equations involving the biharmonic operator for initial data with optimal slow decay, Calc. Var. Partial Differential Equations 30 (2007), 389–415.

[18] A. Gmira and L. Véron, Large time behaviour of the solutions of a semilinear parabolic equation in $\mathbb{R}^N$, J. Differential Equations 53 (1984), 258–276.

[19] N. Hayashi, E. I. Kaikina and P. I. Naumkin, Asymptotics for fractional nonlinear heat equations, J. London Math. Soc. 72 (2005), 663–688.

[20] L. A. Herraiz, Asymptotic behaviour of solutions of some semilinear parabolic problems, Ann. Inst. H. Poincaré Anal. Non Linéaire 16 (1999), 49–105.

[21] K. Ishige, M. Ishiwata and T. Kawakami, The decay of the solutions for the heat equation with a potential, Indiana Univ. Math. J. 58 (2009), 2673–2708.

[22] K. Ishige and T. Kawakami, Asymptotic behavior of solutions for some semilinear heat equations in $\mathbb{R}^N$, Commun. Pure Appl. Anal. 8 (2009), 1351–1371.

[23] K. Ishige and T. Kawakami, Refined asymptotic profiles for a semilinear heat equation, Math. Ann. 353 (2012), 161–192.

[24] K. Ishige and T. Kawakami, Asymptotic expansions of solutions of the Cauchy problem for nonlinear parabolic equations, to appear in J. Anal. Math.

[25] K. Ishige, K. Kawakami and K. Kobayashi, Global solutions for a nonlinear integral equation with a generalized heat kernel, preprint.
[26] K. Ishige, K. Kawakami and K. Kobayashi, Asymptotics for a nonlinear integral equation with a generalized heat kernel. II, in preparation.

[27] K. Ishige and K. Kobayashi, Convection-diffusion equation with absorption and non-decaying initial data, J. Differential Equations 254 (2013), 1247–1268.

[28] S. Kamin and L. A. Peletier, Large time behaviour of solutions of the heat equation with absorption, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 12 (1985), 393–408.

[29] T. Ogawa and M. Yamamoto, Asymptotic behavior of solutions to drift-diffusion system with generalized dissipation, Math. Models Methods Appl. Sci. 19 (2009), 939–967.

[30] P. Quittner and P. Souplet, Superlinear parabolic problems: Blow-up, global existence and steady states, Birkhäuser Advanced Texts, Basel, 2007.

[31] A. Raczyński, Diffusion-dominated asymptotics of solution to chemotaxis model, J. Evol. Equ. 11 (2011), 509–529.

[32] S. Sugitani, On nonexistence of global solutions for some nonlinear integral equations, Osaka J. Math. 12 (1975), 45–51.

[33] J. Taskinen, Asymptotical behaviour of a class of semilinear diffusion equations, J. Evol. Equ. 7 (2007), 429–447.

[34] T. Yamada, Higher-order asymptotic expansions for a parabolic system modeling chemotaxis in the whole space, Hiroshima Math. J. 39 (2009), 363–420.

[35] T. Yamada, Moment estimates and higher-order asymptotic expansions of solutions to a parabolic system in the whole space, Funkcial. Ekvac. 54 (2011), 15–51.

[36] M. Yamamoto, Asymptotic expansion of solutions to the drift-diffusion equation with large initial data, J. Math. Anal. Appl. 369 (2010), 144–163.

[37] M. Yamamoto, Asymptotic expansion of solutions to the dissipative equation with fractional Laplacian, SIAM J. Math. Anal. 44 (2012), 3786–3805.