Dynamics of condensation in growing complex networks

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A condensation transition was predicted for growing technological networks evolving by preferential attachment and competing quality of their nodes, as described by the fitness model. When this condensation occurs a node acquires a finite fraction of all the links of the network. Earlier studies based on steady state degree distribution and on the mapping to Bose-Einstein condensation, were able to identify the critical point. Here we characterize the dynamics of condensation and we present evidence that below the condensation temperature there is a slow down of the dynamics and that a single node (\textit{not} necessarily the best node in the network) emerges as the winner for very long times. The characteristic time $t^*$ at which this phenomenon occurs diverges both at the critical point and at $T \to 0$ when new links are attached deterministically to the highest quality node of the network.

Questions concerning the dynamics of condensation in within the fitness model, might also shed some light on the relation between the condensation phenomena occurring in this model an the other off-equilibrium condensation phase transitions occurring in complex systems (traffic jam, wealth distribution, urn and network models) \cite{15, 16, 17, 18, 19, 20, 21, 22, 23}. Off-equilibrium phenomena have attracted large interest in the last ten years. Mainly the attention has been addressed to the characterization of the steady state of these out-of-equilibrium systems, above the condensation transition \cite{2, 3, 4, 5, 7, 16, 17}. On the contrary the dynamics by which the condensate emerges \cite{17, 18, 19, 22} has been studied only in the case of urn models and models which can be reduced to the Zero Range Process \cite{15}. However the models where the condensation occurs as a structural phase transition of a network are not in general reducible to the Zero-Range-Process.

In this paper we study the dynamics of condensation in the fitness model. We apply the rank statistics for the description of the dynamics of condensation and we will show that below the condensation phase transition a condensate node emerges only after a characteristic time $t^*$. This node is then one of the nodes with highest fitness. The probability that later-comers high fitness nodes will overcome the condensate is decaying with time and there is a slow down of the dynamics. At the condensation transition the characteristic time $t^*$ diverges and the condensation is only marginal, i.e. we have a sequence of highly connected high-fitness nodes each one overcoming the other each of them grabbing a very small fraction of the total links of the network.

The fitness model and the condensation phase transition. The fitness model \cite{14} is a growing network model. We start from a finite connected network of $N_0$ nodes. At each time $t_i$ a new node $i$ and $m < N_0$ new links are added to the network. To the node $i$ it is assigned a quenched variable, $\varepsilon_i$ (‘energy’ of the node) drawn from a $g(\varepsilon)$ distribution. The variable $\varepsilon_i$ indicates

Condensation phenomena \cite{1, 2, 3, 4, 5, 6, 7} in complex networks \cite{8, 9, 10, 11} are structural phase transitions in which a node grabs a finite fraction of all the links of the network. This phenomenon is of particular interest in the case of technological networks. The maps of the Internet at the Autonomous System Level show that the fraction of nodes connected to the most connected node is increasing in time reaching a order the order of 10% in recent maps \cite{12}. Also in the World-Wide Web the share of webpages linked to Google webpages is of the order of 1% a large number if one takes into account the size of the World-Wide-Web. Are the Internet and the World-Wide-Web close to a condensation transition? Which are the dynamical signatures of a condensation? Which are the consequences of a condensation of technological networks? The problem might have relevant implications for the monitoring of technological networks and there might be important difference between the statistics of lead change below and above \cite{13} the condensation transition. In the following we are studying these problems focusing on the dynamics of the fitness model \cite{2, 14} in which nodes acquire links in proportion to their connectivity and in proportion of their fitness which indicates the quality of the node. This model, has been considered a good stylized model for the Internet \cite{11} and is a good stylized model describing the emergence of high quality search engines in the World-Wide-Web and show a condensation phase transition depending on the parameters of the model. The study of the dynamical properties of this model will be able give some estimates of the characteristic time scale at which a condensate node might emerge in a condensation scenario and will allow us to evaluate, once a condensate node is formed, the probability that a new node would overcome the condensate at later times.

PACS numbers: 89.75.Hc, 89.75.Da, 89.75.Fb
there can be a critical temperature condensation, (i.e for the condensation phase transition. A necessary condition for this to occur is that

$$\Pi_j(t) = \frac{\eta_j k_j(t)}{\sum_i \eta_i k_i(t)}.$$  

(1)

where we have introduced the fitness $\eta_j$ associated to node $j$ as

$$\eta_j = e^{-\beta \varepsilon_j}.$$  

(2)

The parameter $\beta = 1/T$ tunes the relevance of the quality of the nodes in the choice of the target node. In the limit case $T \to \infty$ the dynamics of the attachment of new links is only dependent on the connectivity of the nodes and we recover the Barabási-Albert model [24]. For $T \ll 1$ the intrinsic quality of the nodes highly affects the dynamics of the system.

For the $g(\varepsilon)$ distribution such that $g(\varepsilon) \to 0$ as $\varepsilon \to 0$ a phase transition can occur at a critical temperature $T_c$. Above this critical temperature every node has an infinitesimal fraction of all the links, below the critical temperature one node grabs a finite fraction of all the links. These results can be obtained by consideration of the characteristics of the steady state of the model above the phase transition. In fact the model can be solved in a mean-field approximation by making a self-consistent assumption. In particular in [14] it is assumed that $Z_t = \sum j \eta_j k_j(t) \to \langle Z_t \rangle = m(t/t_i)^{f(\varepsilon)}$ with $f(\varepsilon) = e^{-\beta(\varepsilon-\mu)}$ and with the constant $\mu$ determined by the self-consistent equation

$$1 = I(\beta, \mu) = \int d\varepsilon g(\varepsilon) \frac{1}{e^{\beta(\varepsilon-\mu)} - 1}.$$  

(3)

In [2] it was shown that the self-consistent equation is equivalent to the equation for the conservation of the total number of links present in the network

$$2mt = mt + mtI(\beta, \mu).$$  

(4)

If Eq. (3) has a solution then the degree distribution of nodes of fitness $\eta = e^{-\beta \varepsilon}$ is given by $P_\eta(k) \approx k^{-1} f(\varepsilon) - 1$ and $P(k) = \int d\varepsilon g(\varepsilon) P_\eta(k)$ is dominated by the term $P_\theta(\varepsilon) = k^{-\gamma}$ with $\gamma = e^{-\beta \mu} + 1$ but can have logarithmic corrections [14].

Nevertheless, if $I(\beta, 0) < 1$ the self-consistent equation cannot be solved. In this case the self-consistent approach fails and we have the condensation phase transition. A necessary condition for condensation, (i.e for $I(\beta, 0) < 1$) is that the distribution $g(\varepsilon) \to 0$ for $\varepsilon \to 0$. For this type of $g(\varepsilon)$ distributions there can be a critical temperature $T_c = 1/\beta_c$ such that $I(\beta, 0) < 1 \ \forall \ \beta > \beta_c$. This phase transition is formally equivalent to the Bose-Einstein condensation for non-interacting Bose gases in dimensions $d > 2$. Using the similarity to the Bose-Einstein condensation in Bose gases, in Ref. [2] it was assumed that below the condensation temperature the equation for conservation of the links can be written as

$$2mt = mt + mtI(\beta, 0) + n_0(\beta)mt$$  

(5)

where $n_0(\beta)$ indicates the number of links attached to the most connected node. Consistently with [5], simulation results reported in [2] showed that indeed the fitness model undergoes a condensation phase transition at $T = T_c$. An example of $g(\varepsilon)$ distributions for which there is a condensation is

$$g(\varepsilon) = (\theta + 1)\varepsilon^\theta$$  

(6)

and $\varepsilon \in [0, 1]$. Here and in the following we will always consider $g(\varepsilon)$ distributed according to [6]. Assuming [4] and a distribution of the type [8] the fraction of condensed links, similarly to the corresponding result for the Bose gas, will take the form

$$n_0(\beta) = 1 - \left(\frac{T}{T_c}\right)^{\theta+1}.$$  

(7)

Dynamics of the fitness model. The necessary condition for condensation $g(\varepsilon) \to 0$ as $\varepsilon \to 0$ is a clear indication that dynamical effects not captured by the stationary state approaches [2, 4, 7] will be very important for the emergence of a long lasting condensate node in the network. Moreover all considerations regarding the dynamical aspects of the condensation phenomena on the fitness network cannot take advantage to the similarity with the condensation phenomena in the non-interacting Bose gas. In fact, in a Bose gas, the equivalent of $n_0(\beta)$ is the occupation of the state at energy $\varepsilon = 0$. On the contrary, in the fitness networks, $n_0(\beta)$ cannot be the connectivity of the fittest node ($\varepsilon = 0$). Indeed, in the fitness model, the fittest node appears in the network only in the infinite time limit because the probability $g(\varepsilon)$ of having a node of quality $\varepsilon$ goes to zero as $\varepsilon \to 0$. Therefore the question is: is the condensate node the best node in the network? Once the condensate has appeared in the network is there a possibility for later-comers high fitness nodes to take over the condensate? To narrow down this general question we define a specific problem we want to address. First of all, let us introduce the notion of a record. A record $\varepsilon_R(t)$ at time $t$ is the node (or equivalently the value of its $\varepsilon$) with minimal $\varepsilon$ (maximal fitness) present in the network. In the fitness model below the condensation transition, all records are good candidates as condensate nodes. Suppose that a record $\varepsilon$, arrived in the network at time $t$, is a condensate with $k$ links at the time $t_r$ where $t_r$ is the $r$-th record occurs.
the subsequent record \( \epsilon \)
fitness \( \eta \).
Records-to answer this question we have to use some results re-
1 for a pictorial representation of our problem). In order
to answer this question we have to use some results related
related to record statistics.
Records- The typical value of the record at time \( t \) is given by
\[
\int_0^{\epsilon_R} g(x) dx = \frac{1}{t}, \tag{8}
\]
indicating that the average number of nodes with \( \varepsilon > \varepsilon_R(T) \) in a network of \( t \) nodes is less than one. Using (3) for the distribution (6) we obtain
\[
\varepsilon_R(t) \simeq t^{-\frac{1}{\theta+1}}. \tag{9}
\]
The statistics of records is a field on its own (25), here we cite only another result which will be used in the following: given the \( r \)-th record at time \( t_r \) the probability to have the following record at time \( t_{r+1} \) \( t \) in the approximation of a continuous time is given by (25)
\[
P_R(t_{r+1} > t, t_r) = \frac{t_r}{t} \tag{10}
\]
where \( t > t_r \).
The effective model- At low temperatures, below the con-
densation phase transition, the linking probability \( \Pi_j \)
(Eq. (11)) of the fitness model will be relevant only for
nodes of very high fitness (the series of records) or for the
condensate node. Therefore we consider the extremely simplified effective model of just two competing nodes (the condensate node and the record) disregarding all
the other nodes present in the network. This simplifica-
tion will provide us a scenario for condensation which we
will subsequently compare with simulation results on
the original fitness model. At time \( t_r \) there is a node (the
condensate) arrived in the network at time \( \bar{t} \) with fitness
\( \bar{\eta} = e^{-\beta \bar{\varepsilon}} \) and \( \bar{k} \) links, and a record node with fitness
\( \eta_r = e^{-\beta \varepsilon_r} \) which is just born and has degree 1. At each
time we distribute one link to one of the two nodes with
probability given by expression (11).
The probability \( p_t(\tau) \) that the node with fitness \( \eta_r \) has
\( \tau \) links at time \( t \) satisfies in the continuous time limit
\[
\frac{\partial p_t(\tau)}{\partial t} = \frac{\eta_r(\tau - 1)}{Z_{\tau-1}} p_t(\tau - 1) - \frac{\eta_r \tau}{Z_{\tau}} p_t(\tau), \tag{11}
\]
where the competition is included in the formula through
\( Z_{\tau} = \bar{\eta}(\bar{k} + t - t_r - \tau) + \eta \tau \). We are interested in
the dynamics of this effective model until the node with
fitness \( \eta_r \) takes over, i.e. until
\[
\tau < \tau_C = (\bar{k} + t - t_r)/2. \tag{12}
\]
Equation (11) is non-trivial to study and some approxi-
mations are necessary. Therefore we assume that
\( Z_{\tau-1} \simeq Z_{\tau-1} \), (i.e. \( \beta \bar{\varepsilon} \ll 1 \)) which, taking into account the scal-
ing of the records with the time of their appearance, Eq. (8)
reduces to the condition
\[
\bar{t} > T^{-((\theta+1)}. \tag{13}
\]
Furthermore we assume that \( C \simeq Z_{\tau} / (\bar{k} + t - t_r) \)
can be approximated by a constant for \( \tau < \tau_C \). In-
deed for \( \tau < \tau_C \) we have that \( C \) varies in a narrow
interval \( C \in ([\bar{\eta}, (\bar{\eta} + \eta_r)/2]) \). In this approxima-
tion we can rewrite equation (11) using the generating function
\( q(z, t) = \sum_{\tau=1}^{\infty} p(t, \tau) z^\tau \). The equation becomes
\[
C(\bar{k} + t - t_r) \frac{\partial q}{\partial t} = \eta_r z(1 - \frac{\partial q}{\partial z}) = \eta_r \frac{z}{z - 1} \tag{14}
\]
which, if we impose the initial condition \( q(z, t_r) = z \), has
the solution
\[
q(z, t) = \frac{z}{z - 1} \left( \frac{\bar{k} + t - t_r}{\bar{k}} \right)^{-\eta_r/C}, \tag{15}
\]
and the expression for the probability \( p_t(\tau) \) becomes
\[
p_t(\tau) = \left( \frac{\bar{k}}{\bar{k} + t - t_r} \right)^{\eta_r/C} \left( 1 - \left( \frac{\bar{k}}{\bar{k} + t - t_r} \right)^{\eta_r/C} \right)^{\tau - 1}. \tag{16}
\]
The probability that, at some time \( t \), \( \tau < \tau_C \), i.e. the
probability that the record has not become the most con-
Cected node of the network can be calculated:
\[
P_1(\tau < \tau_C) \simeq 1 - \exp\left[ -\frac{\bar{k}}{2} \left( \frac{\bar{k} \tau}{\bar{k} + t - t_r} \right)^{(\eta_r - C)/C} \right], \tag{17}
\]
where in the last step we have assumed that \( \tau_C - t_r \gg 1 \).
Expression (16) for \( P_1(\tau < \tau_C) \) allow us to evaluate the
characteristic time \( \tau_C \) at which the record \( r \) becomes the
most connected node, i.e.
\[
t_C - t_r \simeq \bar{k}^{1+C/(\eta_r - C)}. \tag{18}
\]
Therefore, by making use of (10) we can evaluate the probability $P_C^{\text{eff}}$ that the record $r$ has become the most connected node of the network before the appearance of the record $r+1$,

$$P_C^{\text{eff}} = P_R(t_{r+1} > t_C, t_r) \approx \frac{1}{n k^{C/(n_0 - C)}} = \frac{1}{n k^\alpha} \tag{16}$$

where $\tilde{n} = \tilde{k}/t_r$. Taking into account the interval of definition of $C$, the scaling exponent $\alpha = C/(n_0 - C) \in [1/\beta(\tilde{\varepsilon} - \varepsilon_r), 2/\beta(\tilde{\varepsilon} - \varepsilon_r)]$. Therefore if $\tilde{\varepsilon} \gg \varepsilon_r$ then

$$\alpha \approx \frac{1}{\beta \tilde{\varepsilon}} \approx 1T\tilde{\varepsilon}^{1/\alpha}. \tag{17}$$

In other words as long as Eq. (13) is satisfied, the exponent $\alpha$ is greater than one, (i.e. $\alpha > 1$) and it increases with larger $\tilde{t}$ times and for higher temperatures $T$.

If we fix the condition $P_C^{\text{eff}} < 1/B$ we find a characteristic time for the appearance of a long lasting condensate node in the effective model is given by

$$t^* = \max \left( T^{-(\theta+1)}, \left( \frac{B}{n} \right)^{1/\alpha} \frac{1}{\tilde{n}} \right) \tag{18}$$

where $\tilde{n} = \tilde{k}/t_r$.

**Dynamics of condensation.** We expect that the effective model, describing the competition of only two nodes, provides an upper bound for the probability $P_C(\tilde{k})$ that in the fitness model a new record overcomes the most connected node before the appearance of a subsequent record. We define $P_C(\tilde{k})$ the probability that a record, introduced in a network with a condensate of connectivity $\tilde{k}$, is able to overcome the connectivity of the condensate node before the subsequent record. To numerically evaluate $P_C(\tilde{k})$ we consider all the records that follow the emergence of a condensate and we evaluate the ratio between the number of positive events and the total number of records that appear in the network when the condensate has $\tilde{k}$ links. The data reported in Figure 2 are taken for $\theta = 1$ at different temperatures $T$ below the phase transition. The results shown are statistics taken over $10^4$ networks for a time lapse $T^{-(\theta+1)} < t < 10^5$.

![Figure 2](image2.png)

**Figure 2:** Probability $P_C$ that a record overcomes the most connected node of the network in the fitness model averaged over $10^4$ runs for a time lapse of $t_{\text{max}} = 10^5$.

This characteristic time is depicted in Figure 3 for different values of $\alpha$ and the results are consistent with a divergence of $t^*$ at $T \to T_c$ and at $T \to 0$. In order to evaluate the time life of a condensate for times $t > t^*$ we can use the scaling of the probability $P_C \sim n_0^{-1}\tilde{k}^{-\alpha}$ and we can assume that for the condensate node $\tilde{k} \approx n_0t$. Moreover if the relevant competition is only between the condensate node and the node with highest fitness, the probability to have the same condensate after $S$ records is given by

$$P_W^{\text{eff}}(S) \approx \prod_{s=\tau+1}^{\tau+s} \left( 1 - \frac{1}{n_0(n_0 t_\tau)^{\alpha}} \right).$$

where $t_\tau$ are the times of record occurrence after the condensate has become the most connected node in the network (i.e. for $s > \tau^*$). The average of this quantity over the distributions of the times of the records $\prod_s P_R(t_s, t_{s-1}) = \prod_s t_s^{-1/\tau^2}$ gives

$$\langle P_W^{\text{eff}}(S) \rangle \approx \exp \left\{ - \left[ 1 - \frac{1}{(1 + \alpha)S} \right] \frac{1 + \alpha}{\alpha n_0(n_0 t_\tau)^{\alpha}} \right\}. \tag{21}$$

![Figure 3](image3.png)

**Figure 3:** Characteristic time $t^*$ calculated with the estimation (20) ($B=10$) after which a long lasting condensate node emerges for the fitness model for $\theta = 1$ and $\alpha = 1, \infty$.
Therefore for long times, we expect $\langle P_{\text{eff}}^W(S) \rangle \rightarrow \text{const}$ as $S \rightarrow \infty$. Consequently we can conclude that for times $t > t^*$ the condensate node is typically not the best node of the network nevertheless it dominates the network for very long times. Only for $T \rightarrow T_c$ and $T \rightarrow 0$ the time needed to havelong lasting condensate diverges and in the network there is a fair competition and a succession of high fitness nodes on which condensation occurs.

Conclusions- In conclusion we have studied the dynamics of condensation in growing network models within the fitness model. We show that below the condensed phase transition there is a characteristic time $t^*$. For times $t < t^*$ the network dynamics is dominated by a succession of high fitness nodes with a finite fraction of the links, above the characteristic time $t^*$ a long-lasting winner takes over acquiring a finite fraction of all the links and slowing down the dynamics. The characteristic time $t^*$ is diverging at the critical point of the condensation phase transition and in the limit $T \rightarrow 0$. These results may have strong implications for the monitoring of technological networks in which we expect a fair competition lasting for very long times only if close to the condensation transition.

We acknowledge fruitful discussions with Claude Godrèche and Jean Marc Luck and Marcello Mamino.

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