Analysis of Volterra integrodifferential equations with nonlocal and boundary conditions via Picard operator

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Abstract
This article investigates the existence and uniqueness of solutions to the second-order Volterra integrodifferential equations with nonlocal and boundary conditions through its equivalent integral equations and fixed point of Banach. Furthermore, utilizing the Picard operator theory, we obtain the dependence of solutions on the initial nonlocal data and on functions involved on the right-hand side of the equations.

Keywords Nonlocal conditions · Integrodifferential equations · Fixed point theorem · Picard operator · Data dependency

Mathematics Subject Classification 45J05 · 34G20 · 47H10 · 34B15 · 65L10

1 Introduction

Over the years, the study of solutions of differential equations has been the subject of research, and continues for a number of reasons, from theoretical comfort, which is about existence, uniqueness, controllability, among others, and in the practical sense, involving the stability and continuous dependence on data (Byszewski 1991, 1999; Bednarz and Byszewski 2015; Byszewski 2017; Balachandran and Park 2003; Bednarz and Byszewski 2018). Therefore, there are several attractive points to investigate the solutions of the various types of differential equations. On the other hand, we can also highlight the integrodifferential equations that have gained prominence in the scientific community (Balachandran and Park 2003; Lin and Liu...
The nonlocal condition is a generalization of the classical initial condition. Studies with the nonlocal conditions are driven by theoretical premium, yet additionally, there are several events that happened in engineering, physics and life sciences that can be described by means of differential equations subject to nonlocal conditions (Bates 2006; Delgado et al. 2019). Therefore, differential equations with nonlocal condition have turned into an active zone of research.

In 1991, Byszewski (1991) introduced the nonlocal Cauchy problem in abstract spaces. In the following years, Byszewski (1999), carried out the work pertaining to the existence and uniqueness of classical solutions of nonlocal Cauchy type problem in a Banach space. In the literature, many researchers have been commented on nonlocal conditions and investigated the various class of differential and integrodifferential equations for existence, uniqueness and dependency of solutions (Byszewski and Akca 1998; Balachandran and Chandrasekaran 1997; Balachandran 1998; Byszewski and Teresa 2018).

Wang et al. (2012), utilizing the Picard and weakly Picard operator theory combined with Bielecki norms analyzed the nonlocal Cauchy problem in Banach spaces of the form:

\[ w'(t) = f(t, w(t)), \quad t \in [0, b], \quad w(0) = w_0 + g(w), \]

for existence, uniqueness, and dependency of solutions. On the other hand, Otrocol and Ilea [Otrocol and Ilea (2014), 27], using the technique of Picard weak operators, examined the existence and qualitative properties of solutions for differential and integrodifferential equations with abstract Volterra operators.

There are many other important and interesting works related to this theme that will contribute to the growth of the theory and make possible new research, so we suggest some works for a more in-depth reading [Byszewski 2017; Muresan 2007; Byszewski and Teresa 2018; Byszewski 2019; Rus and Egri 2006; Kucche and Shikhare 2020].

Motivated by Byszewski (1999), Wang et al. (2012), Otrocol and Ilea (2014), Byszewski (2019), the main objective of this paper is to discuss some basic problems, such as existence and uniqueness and dependency of solutions of the following second-order Volterra integrodifferential equations with nonlocal and boundary conditions (VIDNBC)

\[
\begin{align*}
    w''(t) &= \mathcal{D} \left( t, w(t), w'(t), \int_0^t \mathcal{G}(t, s, w(s), w'(s)) \, ds \right), \quad t \in J = [0, T], \quad T > 0, \quad (1.1) \\
    w(0) + \sum_{k=1}^{p} c_k w(t_k) &= w_0, \quad (1.2) \\
    w'(T) &= \beta w'(0), \quad 1 < \beta < +\infty, \quad (1.3)
\end{align*}
\]

by utilizing the fixed point theorem and Picard operator theory, where \(0 < t_1 < t_2 < \cdots < t_p \leq T\), \(\mathcal{D} : J \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\), \(\mathcal{G} : J \times J \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\) are given functions satisfying some assumptions that will be specified later and \(c_1, c_2, \ldots, c_p, (p \in \mathbb{N})\) are the constants such that \(\sum_{k=1}^{p} c_k \neq -1\).

The growth and branching of the field of differential and integrodifferential equations with nonlocal and boundary conditions provide the scientific and the academic community the main reasons for the elaboration of the present paper. A systematic and detailed study undertaken on the existence, uniqueness, and dependencies of the data, provides a wider range of tools and work that may prove to be useful and important for future research.
This paper is organized as follows. Find out preliminarily facts in Sects. 2. In Sect. 3, we will investigate the first main result of the article, that is, the existence and uniqueness of solutions of second order Volterra integrodifferential equations with nonlocal and boundary conditions, through Banach’s contraction principle. Sect. 4, identified with the dependency of solutions through Picard operators theory. In Sect. 5, we illustrate an example to verify the results. Concluding remakes closed the paper.

2 Preliminaries

Definition 2.1 (Wang et al. 2012; Rus 2007, 2003). Let \((X, d)\) be a metric space. An operator \(A : X \to X\) is a Picard operator (PO), if there exists \(w^* \in X\) satisfying the following conditions:

(a) \(F_A = \{w^*\}\), where \(F_A := \{w \in X : A(w) = w\}\).

(b) the sequence \((A^n(w_0))_{n \in \mathbb{N}}\) converges to \(w^*\) for all \(w_0 \in X\).

Theorem 2.1 (Wang et al. 2012; Rus 2007, 2003). Let \((Y, d)\) be a complete metric space and \(A, B : Y \to Y\) two operators. We suppose the following:

(a) \(A\) is a contraction with contraction constant \(\alpha\) and \(F_A = \{w^*_A\}\);

(b) \(B\) has fixed point and \(w^*_B \in F_B\);

(c) there exists \(\rho > 0\) such that \(d(A(w), B(w)) \leq \rho\) for all \(w \in Y\).

Then

\[d\left(w^*_A, w^*_B\right) \leq \frac{\rho}{1 - \alpha}.\]

3 Existence and uniqueness

Definition 3.1 The function \(w \in C^2(J, \mathbb{R})\) is said to be solution of the problem (1.1)–(1.3) if \(w\) satisfied (1.1) and nonlocal and boundary conditions (1.2) and (1.3), respectively.

Lemma 3.1 A function \(w \in C^2(J, \mathbb{R})\) is a solution of VIDNBC (1.1)–(1.3) if and only if \(w \in C^1(J, \mathbb{R})\) is a solution of the following integral equations:

\[
\begin{align*}
  w(t) &= \left( w_0 - \sum_{k=1}^{p} c_k \left[ \frac{t_k}{\beta - 1} \int_0^T \mathcal{F} \left( s, w(s), w'(s), \int_0^s \mathcal{G} \left( s, \sigma, w(\sigma), w'(\sigma) \right) d\sigma \right) ds \right] \\
  &\quad + \int_0^{t_k} (t_k - s) \mathcal{F} \left( s, w(s), w'(s), \int_0^s \mathcal{G} \left( s, \sigma, w(\sigma), w'(\sigma) \right) d\sigma \right) ds \right) \right] \right) + \left[ \sum_{k=1}^{p} c_k \right] \\
  &\quad + \frac{t}{\beta - 1} \int_0^T \mathcal{F} \left( s, w(s), w'(s), \int_0^s \mathcal{G} \left( s, \sigma, w(\sigma), w'(\sigma) \right) d\sigma \right) ds \\
  &\quad + \int_0^t (t - s) \mathcal{F} \left( s, w(s), w'(s), \int_0^s \mathcal{G} \left( s, \sigma, w(\sigma), w'(\sigma) \right) d\sigma \right) ds.
\end{align*}
\]
Proof Let \( w \in C^2(J, \mathbb{R}) \) is a solution of problem (1.1)–(1.3). Integrating (1.1) from 0 to \( t \), we get
\[
w'(t) = w'(0) + \int_0^t \mathcal{F} \left( s, w(s), w'(s), \int_0^s \mathcal{G} \left( s, \sigma, w(\sigma), w'(\sigma) \right) d\sigma \right) ds. \quad (3.2)
\]
Again integrate above equation from 0 to \( t \), we have
\[
w(t) = w(0) + w'(0)t + \int_0^t (t-s) \mathcal{F} \left( s, w(s), w'(s), \int_0^s \mathcal{G} \left( s, \sigma, w(\sigma), w'(\sigma) \right) d\sigma \right) ds. \quad (3.3)
\]
From (3.2), we have
\[
w'(T) = w'(0) + \int_0^T \mathcal{F} \left( s, w(s), w'(s), \int_0^s \mathcal{G} \left( s, \sigma, w(\sigma), w'(\sigma) \right) d\sigma \right) ds. \quad (3.4)
\]
From (3.4) and (1.3), we have
\[
\beta w'(0) = w'(0) + \int_0^T \mathcal{F} \left( s, w(s), w'(s), \int_0^s \mathcal{G} \left( s, \sigma, w(\sigma), w'(\sigma) \right) d\sigma \right) ds.
\]
This gives
\[
w'(0) = \frac{1}{\beta - 1} \int_0^T \mathcal{F} \left( s, w(s), w'(s), \int_0^s \mathcal{G} \left( s, \sigma, w(\sigma), w'(\sigma) \right) d\sigma \right) ds. \quad (3.5)
\]
Putting the value of \( w'(0) \) from (3.5) in equation (3.3), we obtain
\[
w(t) = w(0) + \frac{t}{\beta - 1} \int_0^T \mathcal{F} \left( s, w(s), w'(s), \int_0^s \mathcal{G} \left( s, \sigma, w(\sigma), w'(\sigma) \right) d\sigma \right) ds
+ \int_0^t (t-s) \mathcal{F} \left( s, w(s), w'(s), \int_0^s \mathcal{G} \left( s, \sigma, w(\sigma), w'(\sigma) \right) d\sigma \right) ds. \quad (3.6)
\]
Therefore, for any \( k (k = 1, 2, \ldots, p) \)
\[
w(t_k) = w(0) + \frac{t_k}{\beta - 1} \int_0^T \mathcal{F} \left( s, w(s), w'(s), \int_0^s \mathcal{G} \left( s, \sigma, w(\sigma), w'(\sigma) \right) d\sigma \right) ds
+ \int_0^{t_k} (t_k-s) \mathcal{F} \left( s, w(s), w'(s), \int_0^s \mathcal{G} \left( s, \sigma, w(\sigma), w'(\sigma) \right) d\sigma \right) ds. \quad (3.7)
\]
Using (3.7) in the nonlocal conditions (1.2), we obtain
\[
w(0) = \left( w_0 - \sum_{k=1}^{p} c_k \left[ \frac{t_k}{\beta - 1} \int_0^T \mathcal{F} \left( s, w(s), w'(s), \int_0^s \mathcal{G} \left( s, \sigma, w(\sigma), w'(\sigma) \right) d\sigma \right) ds + \int_0^{t_k} (t_k-s) \mathcal{F} \left( s, w(s), w'(s), \int_0^s \mathcal{G} \left( s, \sigma, w(\sigma), w'(\sigma) \right) d\sigma \right) ds \right] \right) / \left( 1 + \sum_{k=1}^{p} c_k \right).
\]
Putting this value of \( w(0) \) in (3.6), we get
\[
w(t) = \left( w_0 - \sum_{k=1}^{p} c_k \left[ \frac{t_k}{\beta - 1} \int_0^T \mathcal{F} \left( s, w(s), w'(s), \int_0^s \mathcal{G} \left( s, \sigma, w(\sigma), w'(\sigma) \right) d\sigma \right) ds \right] \right) / \left( 1 + \sum_{k=1}^{p} c_k \right).
\]
Which is Eq. (3.1). Conversely let \( w \in C^1(J, \mathbb{R}) \) is solution of (3.1) we prove that \( w \) satisfied (1.1)–(1.3). Differentiating (3.1) with respect to \( t \) we get

\[
\frac{d}{dt} w(t) = \frac{1}{\beta - 1} \int_0^T \mathcal{F} \left( s, w(s), w'(s), \int_0^s \mathcal{I} \left( s, \sigma, w(\sigma), w'(\sigma) \right) d\sigma \right) ds
\]

and

\[
\frac{d}{dt} w(T) = \frac{1}{\beta - 1} \int_0^T \mathcal{F} \left( s, w(s), w'(s), \int_0^s \mathcal{I} \left( s, \sigma, w(\sigma), w'(\sigma) \right) d\sigma \right) ds
\]

Furthermore, from (3.8), we have

\[
w(0) = \left( w_0 - \sum_{k=1}^p c_k \left[ \frac{t_k}{\beta - 1} \int_0^T \mathcal{F} \left( s, w(s), w'(s), \int_0^s \mathcal{I} \left( s, \sigma, w(\sigma), w'(\sigma) \right) d\sigma \right) ds \right] \right) / \left( 1 + \sum_{k=1}^p c_k \right)
\]

and for each \( i (i = 1, 2, \ldots, p) \),

\[
w(t_i) = \left( w_0 - \sum_{k=1}^p c_k \left[ \frac{t_k}{\beta - 1} \int_0^T \mathcal{F} \left( s, w(s), w'(s), \int_0^s \mathcal{I} \left( s, \sigma, w(\sigma), w'(\sigma) \right) d\sigma \right) ds \right] \right)
\]
\[
+ \int_{0}^{t_k} (t_k - s) \mathcal{F} \left( s, w(s), w'(s), \int_{0}^{s} \mathcal{G} \left( s, \sigma, w(\sigma), w'(\sigma) \right) d\sigma \right) ds \right) \bigg) \bigg) \\
\left( 1 + \sum_{k=1}^{p} c_k \right) \\
+ \frac{t_i}{\beta - 1} \int_{0}^{T} \mathcal{F} \left( s, w(s), w'(s), \int_{0}^{s} \mathcal{G} \left( s, \sigma, w(\sigma), w'(\sigma) \right) d\sigma \right) ds \\
+ \int_{0}^{t_i} (t_i - s) \mathcal{F} \left( s, w(s), w'(s), \int_{0}^{s} \mathcal{G} \left( s, \sigma, w(\sigma), w'(\sigma) \right) d\sigma \right) ds.
\]

Therefore,
\[
\begin{aligned}
\sum_{i=1}^{p} c_i w(t_i) \\
&= \left( w_0 - \sum_{k=1}^{p} c_k \left[ \frac{t_k}{\beta - 1} \int_{0}^{T} \mathcal{F} \left( s, w(s), w'(s), \int_{0}^{s} \mathcal{G} \left( s, \sigma, w(\sigma), w'(\sigma) \right) d\sigma \right) ds \\
\left( 1 + \sum_{k=1}^{p} c_k \right) \\
+ \int_{0}^{t_k} (t_k - s) \mathcal{F} \left( s, w(s), w'(s), \int_{0}^{s} \mathcal{G} \left( s, \sigma, w(\sigma), w'(\sigma) \right) d\sigma \right) ds \\
\left( 1 + \sum_{k=1}^{p} c_k \right) \\
+ \frac{t_i}{\beta - 1} \int_{0}^{T} \mathcal{F} \left( s, w(s), w'(s), \int_{0}^{s} \mathcal{G} \left( s, \sigma, w(\sigma), w'(\sigma) \right) d\sigma \right) ds \\
+ \int_{0}^{t_i} (t_i - s) \mathcal{F} \left( s, w(s), w'(s), \int_{0}^{s} \mathcal{G} \left( s, \sigma, w(\sigma), w'(\sigma) \right) d\sigma \right) ds \right) \\
= w_0.
\end{aligned}
\]

Which is conditions (1.3). This complete the proof. \(\square\)

**Theorem 3.2** Assume that:

(H1) Let \( \mathcal{F} \in (J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}) \), \( \mathcal{G} \in (J \times J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}) \) and there exist constants \( L_{\mathcal{F}}, L_{\mathcal{G}} > 0 \) such that

\[
|\mathcal{F}(t, w_1, w_2, w_3) - \mathcal{F}(t, v_1, v_2, v_3)| \leq L_{\mathcal{F}} \left( \sum_{j=1}^{3} |w_j - v_j| \right)
\]

\(\square\) Springer
There exist a constant 

\[ \gamma > 0, \text{ for all } t, s \in J \text{ and } w_j, v_j \in \mathbb{R} (j = 1, 2, 3). \]

(H2) There exist a constant \( \gamma > 0 \) such that

\[ q = \frac{L_\mathcal{F}}{\gamma} \left( 1 + \frac{L_\mathcal{F}}{\gamma} \right) \left( 1 + \left\{ T\beta + T\beta \left| \frac{\sum_{k=1}^{p} c_k}{1 + \sum_{k=1}^{p} c_k} \right| \right\} \frac{e^{-\beta t}}{\beta - 1} \right) < 1. \]

Then, the VIDNBC (1.1)–(1.3) has a unique solution in \( C^2(J, \mathbb{R}) \).

**Proof** Consider the space \( C^1(J, \mathbb{R}) \) with the norm

\[ \| w \|_1 = \max_{t \in J} \left\{ \left| w(t) \right| + \left| w'(t) \right| \right\}, \gamma > 0, w \in C^1(J, \mathbb{R}). \]

Then \( (C^1(J, \mathbb{R}), \| \cdot \|_1) \) is a Banach space. Define the operator \( \mathcal{P} : (C^1(J, \mathbb{R}), \| \cdot \|_1) \to (C^1(J, \mathbb{R}), \| \cdot \|_1) \)

\[
\mathcal{P}(w)(t) = \left( w_0 - \sum_{k=1}^{p} c_k \left[ \frac{t_k}{\beta - 1} \int_{0}^{T} \mathcal{F}(s, w(s), w'(s), \int_{0}^{s} \mathcal{F}(s, \sigma, w(\sigma), w'(\sigma)) d\sigma) \right] ds \right) \\
+ \int_{0}^{t} (t_k - s) \mathcal{F} \left( s, w(s), w'(s), \int_{0}^{s} \mathcal{F}(s, \sigma, w(\sigma), w'(\sigma)) d\sigma \right) ds) \right) / \left( 1 + \sum_{k=1}^{p} c_k \right) \\
+ \left( T - s \right) \mathcal{F} \left( s, w(s), w'(s), \int_{0}^{s} \mathcal{F}(s, \sigma, w(\sigma), w'(\sigma)) d\sigma \right) ds.
\]

Then the fixed point of \( w = \mathcal{P}w, \ w \in (C^1(J, \mathbb{R}), \| \cdot \|_1) \) is the solution of (1.1)–(1.3). Let any \( w, v \in C^1(J, \mathbb{R}) \) and \( t \in J \). Then

\[
| (\mathcal{P}w)(t) - (\mathcal{P}v)(t) |
\leq \left| \frac{\sum_{k=1}^{p} c_k}{1 + \sum_{k=1}^{p} c_k} \right| \left| \frac{t_k}{\beta - 1} \int_{0}^{T} \left\{ \mathcal{F} \left( s, w(s), w'(s), \int_{0}^{s} \mathcal{F}(s, \sigma, w(\sigma), w'(\sigma)) d\sigma \right) \right\} ds \right|
\]
\[-\mathcal{F}\left(s, v(s), v'(s), \int_{0}^{s} \mathcal{A}(s, \sigma, v(\sigma), v'(\sigma)) \, d\sigma\right) \, ds \]

\[+ \left|\frac{t}{\beta - 1}\right| \int_{0}^{T} \left\{ \mathcal{F}\left(s, w(s), w'(s), \int_{0}^{s} \mathcal{A}(s, \sigma, w(\sigma), w'(\sigma)) \, d\sigma\right) \right\} \, ds \]

\[-\mathcal{F}\left(s, v(s), v'(s), \int_{0}^{s} \mathcal{A}(s, \sigma, v(\sigma), v'(\sigma)) \, d\sigma\right) \, ds \]

\[+ \int_{0}^{T} \left\{ \mathcal{F}\left(s, w(s), w'(s), \int_{0}^{s} \mathcal{A}(s, \sigma, w(\sigma), w'(\sigma)) \, d\sigma\right) \right\} \, ds \]

\[-\mathcal{F}\left(s, v(s), v'(s), \int_{0}^{s} \mathcal{A}(s, \sigma, v(\sigma), v'(\sigma)) \, d\sigma\right) \, ds \]

\[+ \int_{0}^{T} \left\{ \mathcal{F}\left(s, w(s), w'(s), \int_{0}^{s} \mathcal{A}(s, \sigma, w(\sigma), w'(\sigma)) \, d\sigma\right) \right\} \, ds \]

\[\leq \left(1 + \left|\frac{\sum_{k=1}^{p} c_k}{1 + \sum_{k=1}^{p} c_k}\right|\right) \left|\frac{T}{\beta - 1}\right| \int_{0}^{T} \left\{ \mathcal{F}\left(s, w(s), w'(s), \int_{0}^{s} \mathcal{A}(s, \sigma, w(\sigma), w'(\sigma)) \, d\sigma\right) \right\} \, ds \]

\[-\mathcal{F}\left(s, v(s), v'(s), \int_{0}^{s} \mathcal{A}(s, \sigma, v(\sigma), v'(\sigma)) \, d\sigma\right) \, ds \]

\[+ \int_{0}^{T} \left\{ \mathcal{F}\left(s, w(s), w'(s), \int_{0}^{s} \mathcal{A}(s, \sigma, w(\sigma), w'(\sigma)) \, d\sigma\right) \right\} \, ds \]

\[-\mathcal{F}\left(s, v(s), v'(s), \int_{0}^{s} \mathcal{A}(s, \sigma, v(\sigma), v'(\sigma)) \, d\sigma\right) \, ds \]

\[+ \int_{0}^{T} \left\{ \mathcal{F}\left(s, w(s), w'(s), \int_{0}^{s} \mathcal{A}(s, \sigma, w(\sigma), w'(\sigma)) \, d\sigma\right) \right\} \, ds \]
\[
\begin{align*}
&\leq \left(1 + \frac{\sum_{k=1}^{p} c_k}{1 + \sum_{k=1}^{p} c_k}\right) \left\{ \frac{T}{\beta - 1} \int_{0}^{T} L_{\phi} e^{\gamma t} e^{-\gamma s} \left[ |w(s) - v(s)| + |w'(s) - v'(s)| \right] ds \right. \\
&\quad + \frac{T}{\beta - 1} \int_{0}^{T} \int_{0}^{s} L_{\phi} L_{\sigma} e^{\gamma \sigma} e^{-\gamma \sigma} \left[ |w(\sigma) - v(\sigma)| + |w'(\sigma) - v'(\sigma)| \right] d\sigma ds \\
&\quad + T \int_{0}^{T} L_{\phi} e^{\gamma t} e^{-\gamma s} \left[ |w(s) - v(s)| + |w'(s) - v'(s)| \right] ds \\
&\quad + T \int_{0}^{T} \int_{0}^{s} L_{\phi} L_{\sigma} e^{\gamma \sigma} e^{-\gamma \sigma} \left[ |w(\sigma) - v(\sigma)| + |w'(\sigma) - v'(\sigma)| \right] d\sigma ds \right) \\
&\leq \left(1 + \frac{\sum_{k=1}^{p} c_k}{1 + \sum_{k=1}^{p} c_k}\right) \left\{ \frac{T}{\beta - 1} L_{\phi} \left( \frac{e^{\gamma T}}{\gamma} - \frac{1}{\gamma} \right) \|w - v\|_1 \\
&\quad + \frac{T}{\beta - 1} L_{\phi} L_{\sigma} \left( \frac{e^{\gamma T}}{\gamma^2} - \frac{1}{\gamma^2} - \frac{T}{\gamma} \right) \|w - v\|_1 \\
&\quad + T L_{\phi} \left( \frac{e^{\gamma T}}{\gamma} - \frac{1}{\gamma} \right) \|w - v\|_1 + T L_{\phi} L_{\sigma} \left( \frac{e^{\gamma T}}{\gamma^2} - \frac{1}{\gamma^2} - \frac{T}{\gamma} \right) \|w - v\|_1 \right) \\
&\leq \left(1 + \frac{\sum_{k=1}^{p} c_k}{1 + \sum_{k=1}^{p} c_k}\right) \left\{ \frac{T}{\beta - 1} L_{\phi} \frac{e^{\gamma T}}{\gamma} + \frac{T}{\beta - 1} L_{\phi} L_{\sigma} \frac{e^{\gamma T}}{\gamma^2} \right. \\
&\quad + T L_{\phi} \frac{e^{\gamma T}}{\gamma} + T L_{\phi} L_{\sigma} \frac{e^{\gamma T}}{\gamma^2} \right\} \|w - v\|_1 \\
&= L_{\phi} \left(1 + L_{\sigma}\right) \left\{ T\beta + T\beta \left| \sum_{k=1}^{p} \frac{c_k}{1 + \sum_{k=1}^{p} c_k} \right| \right\} \frac{e^{\gamma T}}{\beta - 1} \|w - v\|_1 .
\end{align*}
\]

Similarly, we have

\[
\begin{align*}
\left| (Pw)'(t) - (Pv)'(t) \right| &\leq \left| \frac{1}{\beta - 1} \int_{0}^{T} \left[ F \left( s, w(s), w'(s), \int_{0}^{s} G \left( s, \sigma, w(\sigma), w'(\sigma) \right) d\sigma \right) \\
&\quad - F \left( s, v(s), v'(s), \int_{0}^{s} G \left( s, \sigma, v(\sigma), v'(\sigma) \right) d\sigma \right) \right] ds \right| \\
&\quad + \left| \int_{0}^{T} \left[ F \left( s, w(s), w'(s), \int_{0}^{s} G \left( s, \sigma, w(\sigma), w'(\sigma) \right) d\sigma \right) \\
&\quad - F \left( s, v(s), v'(s), \int_{0}^{s} G \left( s, \sigma, v(\sigma), v'(\sigma) \right) d\sigma \right) \right] ds \right| \\
&\leq \frac{L_{\phi}}{\gamma} \left(1 + L_{\sigma}\right) \left( \frac{e^{\gamma T}}{\beta - 1} + e^{\gamma T} \right) \|w - v\|_1 .
\end{align*}
\]

(3.10)
Thus, from (3.9) and (3.10), we have

\[
| (Pw)(t) - (Pv)(t) | + | (Pw)'(t) - (Pv)'(t) | 
\leq \frac{L_F}{\gamma} \left( 1 + \frac{L_G}{\gamma} \right) \left\{ T \beta + T \beta \left( \frac{\sum_{k=1}^{p} c_k}{1 + \sum_{k=1}^{p} c_k} \right) \right\} e^{\frac{\gamma T}{\beta - 1}} + e^{\frac{\gamma T}{\beta - 1} + e^{\gamma t}}
\]

\[\| w - v\|_1, \ t \in J. \text{ (3.11)}\]

Therefore,

\[
\| Pw - Pv \|_1 = \max_{t \in J} \frac{1}{e^{\gamma T}} \left\{ | (Pw)(t) - (Pv)(t) | + | (Pw)'(t) - (Pv)'(t) | \right\}
\leq \frac{L_F}{\gamma} \left( 1 + \frac{L_G}{\gamma} \right) \left\{ 1 + \left[ T \beta + T \beta \left( \frac{\sum_{k=1}^{p} c_k}{1 + \sum_{k=1}^{p} c_k} \right) \right] \right\} e^{\frac{\gamma T}{\beta - 1}}
\]

\[\| w - v\|_1.\]

Choose \( \gamma > 0 \) such that

\[
\frac{L_F}{\gamma} \left( 1 + \frac{L_G}{\gamma} \right) \left\{ 1 + \left[ T \beta + T \beta \left( \frac{\sum_{k=1}^{p} c_k}{1 + \sum_{k=1}^{p} c_k} \right) \right] \right\} e^{\frac{\gamma T}{\beta - 1}} < 1.
\]

Hence, \( P \) is contraction and by Banach contraction principle \( P \) has a fixed point \( \tilde{w} \) in \( (C^1(J, \mathbb{R}), \| \cdot \|_1) \) which is solution of VIDNBC (1.1)–(1.3).

\[\Box\]

4 Dependency of solutions via PO

In this section, we give the dependence of solution of problem (1.1)–(1.3) on the functions involved in the right hand side of equations (1.1)–(1.2) and on the initial nonlocal data via Picard operator theory. Note that Wang et al. (2012) every contraction operator is a Picard operator.

\[
w''(t) = \tilde{F} \left( t, w(t), w'(t), \int_0^t \tilde{F} \left( t, s, w(s), w'(s) \right) ds \right), \ t \in J = [0, T], \ T > 0, \text{ (4.1)}
\]

\[
w(0) + \sum_{k=1}^{p} c_k w(t_k) = \tilde{w}_0, \text{ (4.2)}
\]

\[
w'(T) = \beta w'(0), \ 1 < \beta < +\infty. \text{ (4.3)}
\]

Then, its equivalent Volterra integral equation is

\[
w(t) = \left( \tilde{w}_0 - \sum_{k=1}^{p} c_k \left[ \frac{t_k}{\beta - 1} - 1 \right] \int_0^T \tilde{F} \left( s, w(s), w'(s), \int_0^s \tilde{F} \left( s, \sigma, w(\sigma), w'(\sigma) \right) d\sigma \right) ds \right)
\]

\[\Box\] Springer
Theorem 4.1 Suppose the following:

(H1) All the conditions in Theorem 3.2 are satisfied and \( w^* \in C^1(J, \mathbb{R}) \) is the unique solution of the integral equation (3.1).

(H2) There exists \( L_{\tilde{\mathcal{F}}}, L_{\tilde{\mathcal{G}}} > 0 \) such that

\[
\left| \tilde{\mathcal{F}}(t, w_1, w_2, w_3) - \tilde{\mathcal{F}}(t, v_1, v_2, v_3) \right| \leq L_{\tilde{\mathcal{F}}} \left( \sum_{j=1}^{3} |w_j - v_j| \right)
\]

and

\[
\left| \tilde{\mathcal{G}}(t, s, w_1, w_2) - \tilde{\mathcal{G}}(t, s, v_1, v_2) \right| \leq L_{\tilde{\mathcal{G}}} \left( \sum_{j=1}^{2} |w_j - v_j| \right)
\]

for all \( t, s \in J \) and \( w_j, v_j \in \mathbb{R} (j = 1, 2, 3) \).

(H3) There exists a function \( \mu(\cdot) \in L^1(J, \mathbb{R}_+) \cap C(J, \mathbb{R}_+) \) such that

\[
\left| \tilde{\mathcal{F}}(t, u, v, w) - \tilde{\mathcal{F}}(t, u, v, \tilde{w}) \right| \leq \mu(t)
\]

for all \( t \in J \) and \( u, v, w, \tilde{w} \in \mathbb{R} \).

Then if \( v^* \) is the solution of integral equations (4.4) then

\[
\| w^* - v^* \|_1 \leq \frac{\left| w_0 - \tilde{w}_0 \right| + \frac{\beta L_{\mu}}{\beta - 1} \left[ 1 + \left( 1 + \frac{\sum_{k=1}^{p} c_k}{1 + \sum_{k=1}^{p} c_k} \right) T \right]}{1 - q},
\]

where \( L_{\mu} = \int_0^T \mu(s) \, ds \).

Proof Consider the operators \( \mathcal{P}, \mathcal{S} : (C^1(J, \mathbb{R}), \|\cdot\|_1) \to (C^1(J, \mathbb{R}), \|\cdot\|_1) \) defined by

\[
\mathcal{P}(w)(t) = \left( w_0 - \sum_{k=1}^{p} c_k \left[ \frac{t_k}{\beta - 1} \int_0^T \tilde{\mathcal{F}}(s, w(s), w'(s), \int_0^s \tilde{\mathcal{G}}(s, \sigma, w(\sigma), w'(\sigma)) \, d\sigma) \, ds \right] \right) + \int_0^t (t_k - s) \tilde{\mathcal{F}}(s, w(s), w'(s), \int_0^s \tilde{\mathcal{G}}(s, \sigma, w(\sigma), w'(\sigma)) \, d\sigma) \, ds.
\]
\[
\left(1 + \sum_{k=1}^{p} c_k\right) \\
+ \frac{t}{\beta - 1} \int_{0}^{T} \mathcal{F} \left( s, w(s), w'(s), \int_{0}^{s} \mathcal{G} (s, \sigma, w(\sigma), w'(\sigma)) \, d\sigma \right) \, ds \\
+ \int_{0}^{t} (t - s) \mathcal{F} \left( s, w(s), w'(s), \int_{0}^{s} \mathcal{G} (s, \sigma, w(\sigma), w'(\sigma)) \, d\sigma \right) \, ds.
\]

and
\[
S(w)(t) = \left( \tilde{w}_0 - \sum_{k=1}^{p} c_k \, \left[ \frac{t_k}{\beta - 1} \int_{0}^{T} \tilde{\mathcal{F}} \left( s, w(s), w'(s), \int_{0}^{s} \tilde{\mathcal{G}} (s, \sigma, w(\sigma), w'(\sigma)) \, d\sigma \right) \, ds \right. \right. \\
\left. \left. + \int_{0}^{t_k} (t_k - s) \tilde{\mathcal{F}} \left( s, w(s), w'(s), \int_{0}^{s} \tilde{\mathcal{G}} (s, \sigma, w(\sigma), w'(\sigma)) \, d\sigma \right) \, ds \right]\right) \\
\left(1 + \sum_{k=1}^{p} c_k\right) \\
+ \frac{t}{\beta - 1} \int_{0}^{T} \tilde{\mathcal{F}} \left( s, w(s), w'(s), \int_{0}^{s} \tilde{\mathcal{G}} (s, \sigma, w(\sigma), w'(\sigma)) \, d\sigma \right) \, ds \\
+ \int_{0}^{t} (t - s) \tilde{\mathcal{F}} \left( s, w(s), w'(s), \int_{0}^{s} \tilde{\mathcal{G}} (s, \sigma, w(\sigma), w'(\sigma)) \, d\sigma \right) \, ds.
\]

By condition \( (H1) \) \( \mathcal{P} \) is a contraction with contraction constant \( q \). Let \( \mathbf{F}_\mathcal{P} = \{w^*\} \). Following same steps in the proof of the Theorem 3.2, the operator that \( S \) is contraction with contraction constant
\[
\tilde{q} = \frac{L_{\tilde{\mathcal{F}}}}{\gamma} \left(1 + \frac{L_{\tilde{\mathcal{G}}}}{\gamma}\right) \left(1 + \left[ 1 + \left\{ T\beta + T\beta \left[ \frac{\sum_{k=1}^{p} c_k}{1 + \sum_{k=1}^{p} c_k} \right] \right\} \right\rangle e^{\gamma T} \right) < 1.
\]

Hence, the VIDNDBC (4.1)–(4.3) has a unique solution. Let \( \mathbf{F}_S = \{v^*\} \). For any \( w \in C^1(J, \mathbb{R}), \|\cdot\|_1 \). Then, any \( t \in J \), we have
\[
|\left(\mathcal{P}w\right)(t) - (Sw)(t)|
\leq\frac{1}{1 + \sum_{k=1}^{p} c_k} \left| w_0 - \tilde{w}_0 \right| + \left(1 + \left[ \frac{\sum_{k=1}^{p} c_k}{1 + \sum_{k=1}^{p} c_k} \right] \right) \frac{T}{\beta - 1} \\
\int_{0}^{T} \left\{ \mathcal{F} \left( s, w(s), w'(s), \int_{0}^{s} \mathcal{G} (s, \sigma, w(\sigma), w'(\sigma)) \, d\sigma \right) \\
- \tilde{\mathcal{F}} \left( s, w(s), w'(s), \int_{0}^{s} \tilde{\mathcal{G}} (s, \sigma, w(\sigma), w'(\sigma)) \, d\sigma \right) \right\} \, ds \\
+ \int_{0}^{T} T \left\{ \mathcal{F} \left( s, w(s), w'(s), \int_{0}^{s} \mathcal{G} (s, \sigma, w(\sigma), w'(\sigma)) \, d\sigma \right) \right\} \, ds.
\]
\[ -\mathcal{F}\left(s, w(s), w'(s), \int_0^s \mathcal{G}(s, \sigma, w(\sigma), w'(\sigma)) d\sigma\right) \right\} ds \]

\[ \leq \left| \frac{1}{1 + \sum_{k=1}^p c_k} |w_0 - \tilde{w}_0| + \left( 1 + \frac{\sum_{k=1}^p c_k}{1 + \sum_{k=1}^p c_k} \right) \frac{T}{\beta - 1} \int_0^T \mu(s) ds + T \int_0^T \mu(s) ds \right| \]

\[ \leq \frac{1}{1 + \sum_{k=1}^p c_k} |w_0 - \tilde{w}_0| + \left( 1 + \frac{\sum_{k=1}^p c_k}{1 + \sum_{k=1}^p c_k} \right) \frac{\beta TL_\mu}{\beta - 1}. \] \hspace{1cm} (4.5)

Similarly, we have

\[ |(Pw)'(t) - (Sw)'(t)| \leq \left| \frac{1}{\beta - 1} \int_0^T \left[ \mathcal{F}\left(s, w(s), w'(s), \int_0^s \mathcal{G}(s, \sigma, w(\sigma), w'(\sigma)) d\sigma\right) \right] ds \right| \]

\[ + \left| \int_0^T \left[ \mathcal{F}\left(s, w(s), w'(s), \int_0^s \mathcal{G}(s, \sigma, w(\sigma), w'(\sigma)) d\sigma\right) \right] ds \right| \]

\[ \leq \frac{1}{\beta - 1} \int_0^T \mu(s) ds + \int_0^T \mu(s) ds \]

\[ \leq \frac{\beta L_\mu}{\beta - 1}. \] \hspace{1cm} (4.6)

From (4.5) and (4.6)

\[ \|Pw - Sw\|_1 = \max \left\{ \frac{1}{e^{\nu \gamma}} \right\} \left\{ |(Pw)(t) - (Sw)(t)| + |(Pw)'(t) - (Sw)'(t)| \right\} \]

\[ \leq \left| \frac{1}{1 + \sum_{k=1}^p c_k} |w_0 - \tilde{w}_0| + \frac{\beta L_\mu}{\beta - 1} \left[ 1 + \left( 1 + \frac{\sum_{k=1}^p c_k}{1 + \sum_{k=1}^p c_k} \right) T \right] \right\}. \]

Using the Theorem 2.1, we get following inequality

\[ \|w^* - v^*\|_1 \leq \frac{\frac{1}{1 + \sum_{k=1}^p c_k} |w_0 - \tilde{w}_0| + \frac{\beta L_\mu}{\beta - 1} \left[ 1 + \left( 1 + \frac{\sum_{k=1}^p c_k}{1 + \sum_{k=1}^p c_k} \right) T \right]}{1 - q}. \] \hspace{1cm} (4.7)

\[ \square \]

**Remark 4.2**

(i) From inequality (4.7) it follows that the solutions of the VIDNBC (1.1)–(1.3) depends on the initial nonlocal data and functions involved on the right hand side of equation.

(ii) If \( \mu(t) = 0 \) then \( L_\mu = 0 \). In this case inequality (4.7) gives dependency of solutions on initial nonlocal data.
(iii) If $w_0 = \tilde{w}_0$ then inequality (4.7) gives dependency of solutions on functions involved on the right hand side of equation.
(iv) If $\mu(t) = 0$ and $w_0 = \tilde{w}_0$ then $L_\mu = 0$, in this case inequality (4.7) gives uniqueness of solution.

5 Examples

**Example 5.1** Consider the second-order Volterra integrodifferential equations with nonlocal and boundary conditions:

$$w''(t) = 0.010540 + \frac{1}{10} \sin\left(\frac{1}{10}\right) - \frac{\cos(w(t))}{1000} - \frac{\sin(w'(t))}{100}$$

$$+ \frac{1}{100} \int_0^t \frac{1}{10} \left[w(s) - \frac{e^{\frac{s}{10}}}{10} \sin(w(s)) + \frac{e^{\frac{s}{10}}}{10} \cos(w'(s))\right] ds, \ t \in J = [0, 1], \quad (5.1)$$

$$w(0) + w(t_1) + w(t_2) - w(t_2) + (0)w(t_4) + w(t_5) = 3.10, \quad (5.2)$$

$$w'(1) = e^{\frac{1}{10}} w'(0). \quad (5.3)$$

Comparing with the equation (1.1)–(1.3) we have $T = 1$, $\beta = e^{\frac{1}{10}}$, $t_1 = 0.2$, $t_2 = 0.4$, $t_3 = 0.6$, $t_4 = 0.8$, $t_5 = 1$ and $c_1 = c_2 = 1$, $c_3 = -1$, $c_4 = 0$, $c_5 = 1$ such that $\sum_{k=1}^{5} c_k = 2 \neq 1$.

(i) Define $\mathcal{G} : [0, 1] \times [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\mathcal{G}(t, s, w(s), w'(s)) = \frac{1}{10} \left[w(s) - \frac{e^{\frac{s}{10}}}{10} \sin(w(s)) + \frac{e^{\frac{s}{10}}}{10} \cos(w'(s))\right].$$

Then for any $t, s \in [0, 1]$ and $w_1, w_2, v_1, v_2 \in \mathbb{R}$, we have

$$|\mathcal{G}(t, s, w_1, w_2) - \mathcal{G}(t, s, v_1, v_2)| \leq \frac{1}{10} |w_1 - v_1| + \frac{e^{\frac{s}{10}}}{10} |\sin w_1 - \sin v_1|$$

$$+ \frac{e^{\frac{s}{10}}}{10} |\cos w_2 - \cos v_2|$$

$$\leq \frac{1}{10} |w_1 - v_1| + \frac{e^{\frac{s}{10}}}{10} |w_1 - v_1| + \frac{e^{\frac{s}{10}}}{10} |w_2 - v_2|$$

$$\leq \frac{1 + e^{\frac{s}{10}}}{10} \{|w_1 - v_1| + |w_2 - v_2|\}.$$

(ii) Define $\mathcal{F} : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\mathcal{F}\left(t, w(t), w'(t), \int_0^t \mathcal{G}(t, s, w(s), w'(s)) ds\right)$$

$$= 0.010540 + \frac{1}{10} \sin\left(\frac{1}{10}\right) - \frac{\cos(w(t))}{1000} - \frac{\sin(w'(t))}{100}$$

$$+ \frac{1}{100} \int_0^t \frac{1}{10} \left[w(s) - \frac{e^{\frac{s}{10}}}{10} \sin(w(s)) + \frac{e^{\frac{s}{10}}}{10} \cos(w'(s))\right] ds.$$
Then, for any $t \in [0, 1]$ and $w_1, w_2, w_3, v_1, v_2, v_3 \in \mathbb{R}$, we have

$$\left| \mathcal{F}(t, w_1, w_2, w_3) - \mathcal{F}(t, v_1, v_2, v_3) \right| \leq \frac{1}{1000} |\cos w_1 - \cos v_1| + \frac{1}{100} |\sin w_2 - \sin v_2| + \frac{1}{100} |w_3 - v_3| \leq \frac{1}{100} (|w_1 - v_1| + |w_2 - v_2| + |w_3 - v_3|).$$

We have proved that $L, G$ and $L_{\mathcal{G}}$ satisfied the conditions (H1) with $L = \frac{1}{100}$, and $L_{\mathcal{G}} = \frac{1+e^{\frac{t}{10}}}{10}$.

$$q = \frac{L_{\mathcal{G}}}{\gamma} \left( 1 + \frac{L_{\mathcal{G}}}{\gamma} \right) \left( 1 + \left| \frac{T \beta + T \beta}{\beta - 1} \left\{ \sum_{k=1}^{p} \frac{c_k}{1 + \sum_{k=1}^{p} c_k} \right\} \right| e^{\gamma t} \right) = \frac{1}{\gamma} \left( 1 + \frac{1+e^{\frac{t}{10}}}{10} \right) \left( 1 + \left| \frac{1 e^{\frac{t}{10}} + 1 e^{\frac{t}{10}}}{1 + 2} \right| \right) e^{\frac{\gamma t}{e^{\frac{t}{10}} - 1}}.$$ 

Note for $\gamma = 1$, we have

$$q = \frac{1}{100} \left( 1 + \frac{1+e^{\frac{t}{10}}}{10} \right) \left( 1 + \left| \frac{1 e^{\frac{t}{10}} + e^{\frac{t}{10}}}{3} \right| \right) e^{\frac{t}{e^{\frac{t}{10}} - 1}} = 0.901278 < 1.$$ 

Thus, all the assumptions of the Theorem 3.2 are satisfied. Applying the Theorem 3.2, the problem (5.1)–(5.3) has unique solution on $[0, 1]$. One can verify that $w(t) = e^{\frac{t}{10}}, t \in [0, 1]$ is the unique solution (5.1)–(5.3).

**Example 5.2** Consider the following second-order Volterra integrodifferential equations:

$$w''(t) = \frac{2}{10} - \frac{t^2}{1000} - \frac{(9 - t)}{1000} + \frac{\cos(w(t))}{100} - \frac{w'(t)}{100} + \frac{1}{100} \int_0^t \left( \frac{1 + 2s}{10} \right) \sin(w(s)) + w'(s) \, ds, \; t \in J = [0, 2], \quad (5.4)$$

subject to

$$w(0) + w(t_1) + w(t_2) + w(t_3) + w(t_4) = 1.35, \quad (5.5)$$

$$w'(2) = 5 w'(0) \quad (5.6)$$

Comparing with the equation (1.1)–(1.3) we have $T = 2, \beta = 5, t_1 = 0.5, t_2 = 1, t_3 = 1.5, t_4 = 2$ and $c_1 = c_2 = c_3 = c_4 = 1$ such that $\sum_{k=1}^{4} c_k = 4 \neq -1$.

Define $\mathcal{G} : [0, 2] \times [0, 2] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\mathcal{G}(t, s, w(s), w'(s)) = \left[ \left( \frac{1 + 2s}{10} \right) \sin(w(s)) + w'(s) \right].$$
Then
\[ |G(t, s, w_1, w_2) - G(t, s, v_1, v_2)| \leq (|w_1 - v_1| + |w_2 - v_2|) \quad t, s \in [0, 2] \text{ and } w_1, w_2, v_1, v_2 \in \mathbb{R}. \]

Furthermore, define \( \mathcal{F} : [0, 2] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) by
\[
\mathcal{F}(t, w(t), w'(t), \int_0^t G(t, s, w(s), w'(s)) \, ds) = \frac{2}{10} - \frac{t^2}{1000} - \frac{(9 - t)}{1000} + \frac{\cos(w(t))}{100} - \frac{w'(t)}{100} + \frac{1}{100} \int_0^t G(t, s, w(s), w'(s)) \, ds.
\]

Then,
\[
|\mathcal{F}(t, w_1, w_2, w_3) - \mathcal{F}(t, v_1, v_2, v_3)| \leq \frac{1}{100} (|w_1 - v_1| + |w_2 - v_2| + |w_3 - v_3|), \quad t \in [0, 2] \text{ and } w_1, w_2, w_3, v_1, v_2, v_3 \in \mathbb{R}.
\]

All the conditions of the Theorem 3.2 are satisfied with \( L_{\mathcal{F}} = \frac{1}{100} \) and \( L_G = 1 \). Take \( \gamma = 2 \) then we have
\[
q = \frac{L_{\mathcal{F}}}{\gamma} \left( 1 + \frac{L_G}{\gamma} \right) \left[ 1 + \left\{ T \beta + T \beta \left\{ \sum_{k=1}^{p} c_k \right\} \right\} e^{\gamma t} \right] = 0.8395 < 1.
\]

Thus, the Theorem 3.2 guarantee uniqueness solutions of (5.4)–(5.6). By actual substitution, one can verify that \( w(t) = \frac{(t + t^2)}{10}, \quad t \in [0, 2] \) is the unique solution (5.4)–(5.6).

### 6 Concluding remarks

We conclude the paper with the objectives achieved: the existence, uniqueness and dependency of solution for a second order Volterra integrodifferential equations with nonlocal and boundary conditions through the Picard operators theory. Note that the fractional calculus is the branch of mathematical analysis, which has been on the rise over the decade, for its well-established theory and its important results that have resulted to the point it has attained today (da C. Sousa and de Olveira 2018; Hscsc et al. 2019). Although many papers have been published and the theory has been expanded, there are many avenues to be covered. In this sense, an interesting idea is to adapt some results obtained here for the field of fractional calculus, especially fractional differential equations, to put other results, and to present the improvements that can be obtained, in particular, making some comparisons through examples, to elucidate the results obtained and consequently, the differences between the whole case and the fractional one (da C. Sousa et al. 2019; Kucche et al. 2019).

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