Multiplicity for a strongly singular quasilinear problem via bifurcation theory

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Abstract

A $p$-Laplacian elliptic problem in the presence of both strongly singular and $(p-1)$-superlinear nonlinearities is considered. We employ bifurcation theory, approximation techniques and sub-supersolution method to establish the existence of an unbounded branch of positive solutions, which is bounded in positive $\lambda$-direction and bifurcates from infinity at $\lambda = 0$. As consequence of the bifurcation result, we determine intervals of existence, nonexistence and, in particular cases, global multiplicity.

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1 Introduction

In this paper, we investigate the existence of an unbounded connected branch of solutions for the following $\lambda$-parameter problem

\begin{equation*}
(P) \begin{cases}
-\Delta_p u = \lambda (u^{-\delta} + u^q) \text{ in } \Omega, \\
u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,
\end{cases}
\end{equation*}

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a smooth bounded domain, $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ is the $p-$Laplacian operator, $1 < p < \infty$, $q > p - 1$, $\delta > 0$ and $\lambda > 0$ is a real parameter.

As a consequence of the singular nature at zero of the source term considered in $(P)$, the solutions of $(P)$ are not smooth up to the boundary (see Theorem 2 in \cite{28}). In fact, for $\delta \geq (2p - 1)/(p - 1)$ the gradient blows up near the boundary in such a way that no solution belongs to $W^{1,p}_{0}(\Omega)$ (see Corollary 3.6 in \cite{33}), but $u^\gamma \in W^{1,p}_{0}(\Omega)$ for some $\gamma > \gamma(\delta) > 1$. The interested reader can consult \cite{10} and \cite{29} for more details about an optimal $\gamma(\delta)$. For this reason, we adopt the following definition.

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Definition 1.1 We say that $u$ is a solution for (P) if $u \in W^{1,p}_{\text{loc}}(\Omega) \cap C_0(\Omega)$, $u > 0$ in $\Omega$, $(u - \varepsilon)^+ \in W^{1,p}_{\text{loc}}(\Omega)$ for any $\varepsilon > 0$, and

$$
\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \varphi dx = \lambda \int_{\Omega} (u^{-\delta} + u^q) \varphi dx \quad \text{for all } \varphi \in C_c^\infty(\Omega).
$$

(1.1)

Singular problems appear in non-newtonian fluids models, turbulent flows in porous media, glaciology and many other contexts and have been widely investigated since the remarkable work of Crandall, Rabinowitz and Tartar [13] (see also reviews on the subject [23] and [26]). Concerning problems like

$$
\Delta_p u = \lambda u^{-\delta} + \mu u^q, \quad \text{in } \Omega,
$$

$$
u|_{\partial \Omega} = 0, \quad u > 0 \text{ in } \Omega,
$$

$$
\delta > 0, \quad q > p - 1,
$$

(1.2)

with singular terms combined with $(p - 1)$-superlinear ones, we can quote the pioneer work of Coclite and Palmeri [12], in which the authors considered (1.2) with $p = 2$, $\delta > 0$, $q \geq 1$, $\lambda = 1$ and proved existence of $\mu_\ast > 0$ such that the problem (1.2) has classical solution for $0 < \mu < \mu_\ast$ and has no solution for $\mu > \mu_\ast$. Mainly highlighting the studies in which strongly singular problems ($\delta > 1$) were considered, we can also mention Hirano et. al in [27] and Arcoya and Mérida [3]. In [27], the authors explored (1.2) with $\lambda = 1$, $p = 2$, $1 < q < 2^* - 1$ and using non-smooth analysis tools proved the existence of $\mu_\ast > 0$ such that (1.2) has at least two weak solutions for $0 < \mu < \mu_\ast$, at least one solution for $\mu = \mu_\ast$ and no solution for $\mu > \mu_\ast$. Whereas in [3] the problem (1.2) was studied with $p = 2$, $\mu = 1$, $1 < q < 2^* - 1$ and a local multiplicity of $W^{1,2}_{\text{loc}}$-solutions was established through penalization arguments, a priori estimates and continuation theorem of Leray-Schauder.

In the case $p \neq 2$, few are known about (1.2), especially when $\delta > 1$. In [24], the problem (1.2) was considered with $\mu = 1$, $0 < \delta < 1$, $p - 1 < q \leq 2^* - 1$ and global multiplicity with respect to the parameter $\lambda$ was established by combining Brezis-Nirenberg type result with sub-supersolution ones. Recently, Bal and Garain in [6] generalized the results of Arcoya and Mérida [3] by considering $(2N + 2)/(N + 2) < p < N$.

The main goal of this paper is to study (P) by combining bifurcation theory, comparison principle for sub-supersolutions in $W^{1,p}_{\text{loc}}(\Omega)$ with approximations arguments to prove the existence of a global unbounded connected of solutions for the problem (P). The main advantage of this approach is that, in addition to establish multiplicity of solutions, we obtain a global connected branch of solutions of (P). In the environment of singular problems, this kind of approach was considered in [1] and [8], where an analytic globally path connected branch of solutions was obtained for the cases $p = 2$ and fractional Laplace operator, respectively. In these works, analytic bifurcation theory is used and requires to deal with analytic operators. In the case of nonlinear diffusion operators as $-\Delta_p$, it is not clear if it is true.

Before presenting our main results, let us set some terminologies and notations. Denoting by $S = \{(\lambda, u) \in \mathbb{R} \times C_0(\overline{\Omega}) : (\lambda, u) \text{ solves (P) in the sense of Definition 1.1}\}$ and $\Sigma \subset S$ an unbounded connected set of $S$, we say that $\lambda \in \mathbb{R}$ is a bifurcation value of $\Sigma$ from infinity if there exists a sequence $\{(\lambda_n, u_n)\}_{n=1}^\infty \subset \Sigma$ such that $\lambda_n \to \lambda$ and $\|u_n\|_\infty \to \infty$ as $n \to \infty$.

The normalized positive eigenfunction associated to the first eigenvalue of $(-\Delta_p, W^{1,p}_0(\Omega))$ is denoted by $\phi_1 \in C^1_0(\overline{\Omega})$, that is,

$$
-\Delta_p \phi_1 = \lambda_1 \phi_1^{p-1} \quad \text{in } \Omega, \quad \phi_1|_{\partial \Omega} = 0.
$$

Now we can state our main results.

Theorem 1.1 Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\delta > 0$ and $q > p - 1$. Then, there exists an unbounded connected set $\Sigma \subset \mathbb{R} \times \left(W^{1,p}_{\text{loc}} \cap C_0(\overline{\Omega})\right)$ of positive solutions of (P) satisfying the following:

a) $(0, 0) \in \Sigma$,

b) $\Sigma$ contains the branch of minimal solutions of (P) and bifurcates from infinity at $\lambda = 0$.

Moreover, by letting $\overline{\text{Proj}_{\mathbb{R}^2} \Sigma} = [0, \Lambda^\ast]$, then:
involved nonlinearity in the non-singular case, we quote [9], [16], [17], [21] and [32] for instance.

Theorem 1.2

In the proof of Theorem 1.1, we performed two approximation arguments, one in the \((p - 1)\)-superlinear term \(u^q\) truncating it by \(\min\{n, u\}^{q-p+1}u^{p-1}\), and the other one in the singular nonlinearity \(u^{-\delta}\) by considering \((u + \epsilon)^{-\delta}\) for \(\epsilon > 0\). The existence of an unbounded connected set of solutions for \((P)\) will be obtained through a limit process of the continua \(\Sigma^n\) as \(n \to \infty\) and \(\epsilon \to 0^+\), in that order, where \(\Sigma^n\) are continua of positive solutions the \((\epsilon, n)\)-problems

\[
\begin{cases}
-\Delta_p u = \lambda [(u + \epsilon)^{-\delta} + \min\{n, u\}^{q-p+1}u^{p-1}] & \text{in } \Omega, \\
u > 0 \text{ in } \Omega, & u = 0 \text{ on } \partial\Omega.
\end{cases}
\]

The advantage of this approach is that several qualitative informations may be extracted from continua \(\Sigma^n\) due to the linear and non-singular nature of the nonlinearity in \((P^n)\). Moreover, no additional conditions on the \((p - 1)\)-superlinear power \(q\) are required.

The knowledge in literature (see [5] and [24]) and the behavior of the \(\Sigma^n\) suggest that \((P)\) should admit a solution on the extremal parameter \(\Lambda^*\), but we were not able to prove such existence in Theorem 1.1 mainly by the fact that, under the assumptions considered there, any \(\lambda \in (0, \Lambda^*)\) may be a bifurcation parameter of \(\Sigma\) from infinity. This type of behavior may be ruled out when it is possible to assure the existence of a priori estimates for the solutions of \((P)\) for each fixed \(\lambda > 0\).

Inspired by Azizieh and Clément [4], we proved a priori estimates by using the blow-up technique combined with Liouville type theorem for strictly convex and smooth domain. A crucial point in the proof presented by them is that the global maxima of the solutions are uniformly distance from the boundary \(\partial\Omega\). In our case, this fact will be established through the results of monotonicity of Damascelli and Sciunzi \[15\], but in this direction restrictions in the domain are essential.

**Theorem 1.2** Let \(\Omega\) be a bounded strictly-convex and smooth domain in \(\mathbb{R}^N\), \(N \geq 2\). In addition, assume \(p - 1 < q < p^* - 1\). Then \(\lambda = 0\) is the only bifurcation parameter of \(\Sigma\) from infinity, that is, for \(\{(\lambda_n, u_n)\}_{n=1}^{\infty} \subset \Sigma\) with

\[
\left\{
\begin{array}{l}
\lambda_n \to \lambda \\
\|u_n\|_\infty \to \infty
\end{array}
\right.
\]

one has \(\lambda = 0\). As a consequence, \((P)\) admits a solution for \(\lambda = \Lambda^*\), that is, global existence and local multiplicity hold.

About other bifurcation diagrams and existence of extremal solutions with respect to the behaviour of the involved nonlinearity in the non-singular case, we quote [9], [10], [17], [21] and [32] for instance.

The main novelties of this paper are the following:

(i) we propose here a different way to approach strongly-singular and \((p - 1)\)-superlinear problems, compared to variational aproaches used in most of papers in the current literature,

(ii) the existence of a branch of positive solutions for \((P)\) is established without any restriction on the singular and \((p - 1)\)-superlinear powers,

(iii) global multiplicity is proved for weak singularities, that is, \(\delta \in (0, 1)\). This complements the principal result in [27] for \(q > p^* - 1\),

(iv) local multiplicity is proved in the case of strong singularity \((\delta \in [1, \infty))\), which is completely new and complements the main results in [3] and [6].
admits a unique solution $\lambda, v$ which associates each pair $(\lambda, v)$ to the only weak solution of (1.5), is well-defined.

Let $\lambda, u \in \mathbb{R}^+ \times C(\overline{\Omega})$, the problem
\[
-\Delta_p u = \lambda \left[|u|^{p-\delta} + f_n(|u|)\right] \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial \Omega
\]
admits a unique solution $u \in C^{1,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$. Thus, the operator $T : \mathbb{R}^+ \times C(\overline{\Omega}) \to C(\overline{\Omega})$, which associates each pair $(\lambda, v) \in \mathbb{R}^+ \times C(\overline{\Omega})$ to the only weak solution of (1.5), is well-defined.

It is classical to show that $T$ is a compact operator by using Arzelà-Ascoli theorem. Hence, we are able to apply the bifurcation theorem by Rabinowitz (see Theorem 3.2 in [33]) to get an unbounded $\epsilon$-continuum $\Sigma^\epsilon_n \subset \mathbb{R}^+ \times C(\overline{\Omega})$ of solutions of
\[
T(\lambda, u) = u
\]
By the definition, $T(0, v) = 0$. Moreover, if $T(\lambda, 0) = 0$, then $\lambda = 0$. So we conclude that $\Sigma^\epsilon_n \setminus \{(0, 0)\}$ is formed by nontrivial solutions of (1.5).

Finally, using that $0 < (|v| + \epsilon)^{-\delta} + f_n(|v|) \in L^\infty(\Omega)$ for each given $v \in C(\overline{\Omega})$ and classical strong maximum principle from Vazquez [39], we obtain that $T((\mathbb{R}^+ \setminus \{0\}) \times C(\overline{\Omega})) \subset \{u \in C(\overline{\Omega}) : u > 0 \text{ in } \Omega\}$. Therefore, $\Sigma^\epsilon_n$ is a continuum of positive solutions of $(P^\epsilon_n)$, for any $\epsilon > 0$ and $n \in \mathbb{Z}^+$ given.

Lemma 2.2 Let $\epsilon > 0$ and $n \in \mathbb{N}$. Then:

a) there exists $\Lambda_{\epsilon,n} = \Lambda_{\epsilon,n}(\epsilon, n, \Omega, \lambda_1, p, q)$ such that $(P^\epsilon_n)$ has no solution for $\lambda > \Lambda_{\epsilon,n}$.

b) the value $\lambda = \lambda_1/n^{q-p+1}$ is the unique bifurcation point of $\Sigma^\epsilon_n$ from infinity.

Proof
a) Since \((t + \epsilon)^{-\delta} + f_n(t)\) tends to \(n^{q-p+1}\) as \(t \to +\infty\), we can find \(C_* > 0\) such that
\[
(t + \epsilon)^{-\delta} + f_n(t) \geq C_* t^{p-1}, \quad \text{for all } t \geq 0.
\] (1.7)

We claim that there is no solution of \((P_{\lambda, \epsilon}^n)\) for \(\lambda > \lambda_1 C_*^{-1}\). Indeed, if \(u_* > 0\) solves \((P_{\lambda, \epsilon}^n)\) for some \(\lambda > \lambda_1 C_*^{-1}\), then \(u_*\) is a supersolution of
\[
\begin{cases}
-\Delta_p u = (\lambda_1 + \epsilon) u^{p-1} & \text{in } \Omega, \\
u > 0, \quad u|_{\partial\Omega} = 0,
\end{cases}
\] (1.8)
for all \(\kappa \in (0, \lambda C_* - \lambda_1)\). On the other hand, \(s\phi\) is a supersolution of \((1.3)\) satisfying \(s\phi < u_*\) in \(\Omega\) for \(s \in (0, 1)\) small enough. So, the monotone iteration method provides a solution of \((1.8)\) for any \(\kappa \in (0, \lambda C_* - \lambda_1)\), which contradicts the fact that \(\lambda_1\) is an isolate point in the spectrum of \((-\Delta_p, W_0^{1,p}(\Omega))\) (see [2]).

b) Since \(\Sigma^n_\epsilon\) is bounded in the \(\lambda\)-direction, it has to become unbounded in the direction of the Banach space \(C_0(\Omega)\). Hence, there exists \(\lambda_* > 0\) and a sequence \(\{(\lambda_k, u_k)\}^{\infty}_{k=1} \subset \Sigma^n_\epsilon\) such that
\[
\begin{cases}
\|u_k\|_{\infty} \to +\infty \\
\lambda_k \to \lambda_*,
\end{cases}
\]
Then \(v_k := u_k/\|u_k\|_{\infty}\) satisfies
\[
\begin{cases}
-\Delta_p v_k = \lambda_k \left[n^{q-p+1} t^{p-1} + \frac{g(u_k)}{\|u_k\|_{\infty}^{p-1}} \right], & \text{in } \Omega, \\
v_k > 0, \quad \|v_k\|_{\infty} = 1, \quad v_k|_{\partial\Omega} = 0,
\end{cases}
\] (1.9)
where
\[
g(t) = (t + \epsilon)^{-\delta} + f_n(t) - n^{q-p+1} t^{p-1} = \begin{cases} (t + \epsilon)^{-\delta} + t^q - n^{q-p+1} t^{p-1}, & \text{if } 0 \leq t \leq n \\ (t + \epsilon)^{-\delta}, & \text{if } t > n
\end{cases}
\]
Notice that \(|g(t)| \leq \epsilon^{-\delta} + 2n^q\), for all \(t \in \mathbb{R}^+\), so that the right hand side of the equation in \((1.9)\) is uniformly bounded in \(L^\infty(\Omega)\). Hence, we can employ the regularity result due to Lieberman ([29], Theorem 1) to conclude that \(\{v_k\}^{\infty}_{k=1}\) is uniformly bounded in \(C^{1,\beta}(\overline{\Omega})\), for some \(\beta \in (0, 1)\). Therefore, it follows from Arzel\'a-Ascoli theorem that \(v \in C^1(\overline{\Omega})\) and a subsequence \(\{v_{k_j}\}^{\infty}_{j=1}\) such that \(v_{k_j} \to v\) in \(C^1(\overline{\Omega})\) as \(k_j \to \infty\). Moreover, \(v \geq 0\) in \(\Omega\), \(\|v\|_{\infty} = 1\) and \(v\) solves
\[-\Delta_p v = \lambda_* n^{q-p+1} t^{p-1} \quad \text{in } \Omega, \quad v|_{\partial\Omega} = 0.
\]
Since \(v\) does not change signs, it follows from (Theorem 7, [30]) that \(\lambda_* = \lambda_1/n^{q-p+1}\). This concludes the proof of Lemma [22].

The next two lemmas claim that the continuum \(\Sigma^n_\epsilon\) cross the line \(\lambda = \lambda_1/n^{q-p+1}\).

**Lemma 2.3** Let
\[
\varepsilon_0 = 2^{-\frac{1}{(p-1)(r-1)+1}}, \quad N_0 = \lambda^{\frac{1}{p(r-1)}} \left(\|\omega_{1,0}\|_{\infty}^{-1} \varepsilon_0^{-\frac{1}{p(r-1)-1}}\right)^{\frac{1}{p(r-1)-1}} + 1,
\]
where \(\omega_{1,0}\) is the only solution of \((1.4)\), with \(\lambda = 1\) and \(\epsilon = 0\). Then for each \(0 < \epsilon < \varepsilon_0\) and \(n > N_0\), there exists \(\varrho > 0\), independent of \(\epsilon > 0\) and \(n\), such that \((P_{\lambda, \epsilon}^n)\) admits a solution for all \(\lambda \in [\lambda_1/n^{q-p+1}, \lambda_1/n^{p(r-1)+\varrho}]\).

**Proof** Clearly the only solution \(\omega_{\lambda, \epsilon} \in W_0^{1,p}(\Omega) \cap C_0(\Omega)\) of the problem \((1.4)\) is a subsolution of \((P_{\lambda}^n)\). To construct a supersolution of \((P_{\lambda}^n)\), let us consider \(\varpi_{\lambda, \epsilon} = M \omega_{\lambda, \epsilon}\), where \(M > 1\) will be chosen later. In order to \(\varpi_{\lambda, \epsilon}\) be a supersolution of \((P_{\lambda}^n)\), we must have
\[-\Delta_p \varpi_{\lambda, \epsilon} = M^{p-1} \lambda (\omega_{\lambda, \epsilon} + \epsilon)^{-\delta} \geq \lambda \left[(M \omega_{\lambda, \epsilon} + \epsilon)^{-\delta} + f_n(M \omega_{\lambda, \epsilon})\right] \quad \text{in } \Omega.
\] (1.10)
For the validity of the inequality \( (1.10) \), it is sufficient that

\[
\begin{align*}
\frac{M^{p-1}}{2(\omega_{\lambda,\epsilon} + \epsilon) \delta} & \geq \frac{1}{(\omega_{\lambda,\epsilon} + \epsilon) \delta}, \\
\frac{M^{p-1}}{2(\omega_{\lambda,\epsilon} + \epsilon) \delta} & \geq M^q(\omega_{\lambda,\epsilon} + \epsilon) \delta
\end{align*}
\] (1.11)

hold.

Since \( \omega_{\lambda,\epsilon} \) is a subsolution of \( (1.4) \) and \( \omega_{\lambda,0} = \lambda^{\frac{p-1+\delta}{\delta}} \omega_{1,0} \) is the only solution of \( (1.4) \) with \( \epsilon = 0 \), we can apply the comparison principle of [10, 36] to conclude that

\[
\omega_{\lambda,\epsilon} \leq \lambda^{\frac{p-1+\delta}{\delta}} \omega_{1,0} \text{ in } \Omega.
\] (1.12)

On the other hand, by the choice of \( N_0 \) and \( \epsilon_0 \), we have

\[
\left( \frac{\lambda_1}{n^{q-p+1}} \right)^{\frac{1}{p-1+\delta}} \| \omega_{1,0} \|_\infty + \epsilon < 2^{\frac{1}{(q-p+1)(\delta+q)}}
\]

for all \( 0 < \epsilon < \epsilon_0 \) and \( n > N_0 \). Hence, for any \( 0 < \rho \left( \| \omega_{1,0} \|_\infty \right)^{p-1+\delta} - \frac{\lambda_1}{N_0^{q-p+1}} \) and for all \( \lambda \in [\lambda_1/n^{q-p+1}, \lambda_1/n^{q-p+1} + \rho] \) we obtain

\[
\lambda^{\frac{1}{p-1+\delta}} \| \omega_{1,0} \|_\infty + \epsilon < 2^{\frac{1}{(q-p+1)(\delta+q)}},
\] (1.13)

for all \( 0 < \epsilon < \epsilon_0 \) and \( n > N_0 \). Thus, combining (1.12) and (1.13) one gets

\[
\| \omega_{\epsilon,\lambda} \|_\infty + \epsilon < 2^{\frac{1}{(q-p+1)(\delta+q)}},
\] (1.14)

for all \( \lambda \in [\lambda_1/n^{q-p+1}, \lambda_1/n^{q-p+1} + \rho] \), \( \epsilon \in (0, \epsilon_0) \) and \( n > N_0 \). So, by setting \( M = 2^{\frac{1}{(p-1+\delta)\delta}} \), it follows from (1.14) that

\[
M < \left( \frac{1}{2(\| \omega_{\epsilon,\lambda} \|_\infty + \epsilon) \delta} \right)^{\frac{1}{p-1+\delta}}.
\] (1.15)

As a consequence of (1.15), both inequalities in (1.11) are fulfilled for such \( M \), whence \( \overline{\omega}_{\epsilon,\lambda} = M \omega_{\epsilon,\lambda} \) is a supersolution of \( (P^n_r) \).

Now, let us prove that for \( \lambda \in [\lambda_1/n^{q-p+1}, \lambda_1/n^{q-p+1} + \rho] \), \( 0 < \epsilon < \epsilon_0 \) and \( n > N_0 \), the problem \( (P^n_r) \) admits a solution in \( [\omega_{\lambda,\epsilon}, \overline{\omega}_{\lambda,\epsilon}] \). For this, consider

\[
g(x, t) = \begin{cases} 
(\omega_{\lambda,\epsilon} + \epsilon)^{-\delta} + f_n(\omega_{\lambda,\epsilon}) & \text{if } t < \omega_{\lambda,\epsilon}, \\
(t + \epsilon)^{-\delta} + f_n(t) & \text{if } \omega_{\lambda,\epsilon} \leq t \leq \overline{\omega}_{\lambda,\epsilon}, \\
(\overline{\omega}_{\lambda,\epsilon} + \epsilon)^{-\delta} + f_n(\overline{\omega}_{\lambda,\epsilon}) & \text{if } \overline{\omega}_{\lambda,\epsilon} < t
\end{cases}
\]

and the functional \( I : W^{1,p}_0(\Omega) \to \mathbb{R} \) given by

\[
I(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \lambda \int_{\Omega} G(x, u) dx,
\]

where \( G(x, t) = \int_0^t g(x, s) ds \). As usual, we can prove that \( I \in C^1(W^{1,p}_0, \mathbb{R}) \) is coercive and weakly lower semi-continuous. Hence, \( I \) achieves its global minimum at some \( u_0 \in W^{1,p}_0(\Omega) \), which satisfies \( -\Delta_p u_0 = \lambda g(x, u_0) \) in \( \Omega \).

Moreover,

\[
0 \leq \int_{\Omega} (|\nabla \omega_{\lambda,\epsilon} - | \nabla u_0|^p \nabla \omega_{\lambda,\epsilon} - | \nabla u_0|^p \nabla u_0) \nabla (\omega_{\lambda,\epsilon} - u_0)^+ dx \\ \leq \lambda \int_{[\omega_{\lambda,\epsilon}, u_0]} ((\omega_{\lambda,\epsilon} + \epsilon)^{-\delta} + f_n(\omega_{\lambda,\epsilon}) - g(x, u_0)) (\omega_{\lambda,\epsilon} - u_0)^+ dx = 0,
\]

whence \( u_0 \geq \omega_{\lambda,\epsilon} \) in \( \Omega \). In a similar way, we obtain \( u_0 \leq \overline{\omega}_{\lambda,\epsilon} \) in \( \Omega \). Therefore, \( u_0 \) is the solution claimed. ■
Lemma 2.4 For all $0 < \epsilon < 1$, if $g = \frac{p-1+\delta}{q-p+1}$, then there exists a solution $u$ such that $\|u\|_{\infty} \leq \frac{1}{2} \left( \frac{p-1+\delta}{q-p+1} \right)^{\frac{1}{g+1}}$.

Proof Define

$$g_{\epsilon}(t) = \frac{(t+\epsilon)^{-\delta} + f_{\epsilon}(t)}{t^{p-1}}, \quad \text{for } t > 0.$$ 

If $0 < t \leq n$, then $g_{\epsilon}$ becomes $g_{\epsilon}(t) = t^{1-p}(t+\epsilon)^{-\delta} + t^{q-p+1}$, whose derivative is

$$g_{\epsilon}'(t) = (1-p)t^{-p}(t+\epsilon)^{-\delta} - \delta t^{1-p}(t+\epsilon)^{-\delta-1} + (q-p+1)t^{q-p}$$

Thus, for all $0 < t \leq n$ satisfying

$$t + \epsilon \leq \left( \frac{p-1+\delta}{q-p+1} \right)^{\frac{1}{g}} \tag{1.16}$$

we have

$$(q-p+1)t^{q-1}(t+\epsilon)^{1+\delta} < (q-p+1)(t+\epsilon)^q < (p-1+\delta),$$

which implies in $g_{\epsilon}'(t) < 0$. In particular, if

$$0 \leq \epsilon < \frac{1}{2} \left( \frac{p-1+\delta}{q-p+1} \right)^{\frac{1}{g}}, \quad \text{and} \quad 0 < t < \frac{1}{2} \left( \frac{p-1+\delta}{q-p+1} \right)^{\frac{1}{g}},$$

then (1.16) holds and as a result, $g_{\epsilon}$ is decreasing. Hence, we can employ the uniqueness result of Díaz-Ortiz (see Theorem 1 in [20]) to conclude that for at least $\lambda \in \text{Proj}_{R}^{n}$, the problem $(P_{\epsilon}^{n})$ has a unique solution satisfying $\|u\|_{\infty} \leq \frac{1}{2} \left( \frac{p-1+\delta}{q-p+1} \right)^{g+1}$.

Combining the previous lemmas, we have the following.

Lemma 2.5 There exist $\epsilon_1 > 0$ (independent of $n > 0$) and $N_1 > 0$ (independent of $\epsilon > 0$) such that $(\text{Proj}_{\Sigma}^{n}) \cap (\lambda_1/n^{q-p+1}, +\infty)) \times C_{0}(R)_{+} \neq \emptyset$, for all $0 < \epsilon < \epsilon_1$ and $n > N_1$.

Proof Let $\epsilon_0$, $N_0$, $K$ and $\rho$ be the constants introduced in Lemma 2.3. In Lemma 2.3, we proved that $(P_{\epsilon}^{n})$ admits a positive solution for all $\lambda \in \left[ \frac{\lambda_1}{n^{q-p+1}} , \frac{\lambda_2}{n^{q-p+1}} + \rho \right]$, whenever $0 < \epsilon < \epsilon_0$ and $n > N_0$. Moreover, for $\lambda \in \left[ \frac{\lambda_1}{n^{q-p+1}} , \frac{\lambda_2}{n^{q-p+1}} + \rho \right]$ the solution obtained there, say $u$, satisfies

$$\|u\|_{\infty} \leq 2^{\frac{1}{p^*}} \frac{\lambda^{1+\delta}}{\rho^{1+\delta}} \|\omega_{1,0}\|_{\infty}. \tag{1.17}$$

Thus, by taking $\epsilon_1 = \min \left\{ \epsilon_0, \frac{1}{2} \left( \frac{p-1+\delta}{q-p+1} \right)^{\frac{1}{g}} \right\}$,

$$N_1 = \max \left\{ N_0, \frac{1}{2} \left( \frac{p-1+\delta}{q-p+1} \right)^{\frac{1}{g}}, \left( 2^{\frac{1}{p^*}} \frac{g-p+1}{p-1+\delta} \lambda_1^{1+\delta} \|\omega_{1,0}\|_{\infty} \right)^{\frac{p-1+\delta}{q-p+1}} \right\},$$

and reducing $\rho$, if it is necessary, we conclude from (1.17) that $\|u\|_{\infty} < \frac{1}{2} \left( \frac{p-1+\delta}{q-p+1} \right)^{\frac{1}{g}}$, whenever $0 < \epsilon < \epsilon_1$, $n > N_1$ and $\lambda \in \left[ \frac{\lambda_1}{n^{q-p+1}} , \frac{\lambda_2}{n^{q-p+1}} + \rho \right]$. Indeed, for $n > N_1$, $\epsilon \in (0, \epsilon_1)$ and for all $\lambda \in \left[ \frac{\lambda_1}{n^{q-p+1}} , \frac{\lambda_2}{n^{q-p+1}} + \rho \right]$ one has

$$\lambda^{1+\delta} \|\omega_{1,0}\|_{\infty} + \epsilon < 2^{\frac{1}{p^*}} \frac{\lambda^{1+\delta}}{\rho^{1+\delta}}.$$
\[
\|u\|_{\infty} + \epsilon \leq \left( \frac{p - 1 + \delta}{q - p + 1} \right)^{1/q}.
\]

After all these, the result follows by combining Lemmas 2.3 and 2.4.

The previous lemmas suggest that \( \Sigma_n^\epsilon \) may have one of the shapes given in the figure below.

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### 3 Asymptotic singular problem

The unbounded connected set of solutions of \((P)\) will be obtained through limit process of \( \Sigma_n^\epsilon \) as \( n \to +\infty \) and \( \epsilon \to 0^+ \).

**Definition 3.1** Let \( X \) be a Banach space and let \( \{\Sigma_n\}_{n=1}^\infty \) be a family of subsets of \( X \). The set of all points \( x \in X \) such that every neighborhood of \( x \) contains points of infinitely many sets \( \{\Sigma_n\}_{n=1}^\infty \) is called the limit superior of \( \{\Sigma_n\}_{n=1}^\infty \) and is written \( \lim_{n \to \infty} \sup \Sigma_n \). The set of all points \( y \) such that every neighborhood of \( y \) contains points of all but a finite number of the sets of \( \{\Sigma_n\}_{n=1}^\infty \) is called the limit inferior of \( \{\Sigma_n\}_{n=1}^\infty \) and is written \( \lim_{n \to \infty} \inf \Sigma_n \).

**Lemma 3.1** \((\text{[40]})\) Let \( X \) be a normal space and let \( \{\Sigma_n\}_{n=1}^\infty \) be a sequence of unbounded connected subsets of \( X \). Assume that:

i) there exists \( z^* \in \lim_{n \to \infty} \inf \Sigma_n \) with \( \|z^*\| < +\infty \),

ii) for every \( R > 0 \), \( \left( \bigcup_{n=1}^{+\infty} \Sigma_n \right) \cap \overline{B_R(0)} \) is a relatively compact set of \( X \), where \( \overline{B_R(0)} = \{ x \in X : \|x\| \leq R \} \).

Then \( C = \lim_{n \to \infty} \sup \Sigma_n \) is unbounded, closed and connected set.

**Lemma 3.2** \((\text{[40]})\) Let \( (X, \| \cdot \|) \) be a normed vector space and \( \{\Sigma_n\}_{n=1}^\infty \) a sequence of unbounded sets whose limit superior is \( C \) and satisfies the following conditions:
Moreover, taking thus once again invoking Lieberman regularity result \([29]\) and applying Arzelà-Ascoli theorem we obtain that \(\zeta\). So, denoting by

\[
\left\{ \left( \lambda_n, u_n \right) \right\}_{n = 1}^{\infty} \subset \bigcup_{n=N_1+1}^{+\infty} \Sigma_n \cap \overline{B_R(0,0)},
\]

it follows from the mapping properties of the inverse \(p\)-Laplacian (see \([29]\)) that \(\|u_j\|_{C^{1,\beta}(\Omega)}\) is uniformly bounded, for some \(\beta \in (0,1)\). Thus, Arzelà-Ascoli theorem assures us that

\[
\lambda_j \to \lambda \geq 0 \quad \text{and} \quad u_j \to u \quad \text{in} \quad C^1(\overline{\Omega}),
\]

up to a subsequence. Hence, we are able to apply Lemma \(3.1\) to conclude that \(\Sigma : = \limsup_{n \to \infty} \Sigma_n\) is unbounded, closed and connected set in \(\mathbb{R} \times C_0(\overline{\Omega})\).

We claim that \(\Sigma\) is formed by solutions of \((P)\). In fact, if \((\lambda, u) \in \Sigma\), then

\[
(\lambda_{n_j}, u_{n_j}) \to (\lambda, u) \quad \text{in} \quad \mathbb{R} \times C_0(\overline{\Omega}),
\]

for some subsequence \(\{\lambda_{n_j}, u_{n_j}\}\), where \((\lambda_{n_j}, u_{n_j}) \in \Sigma_{n_j}^{\infty}\). In particular, \(\{\|u_{n_j}\|_{\infty}\}_{j = 1}^{\infty}\) is uniformly bounded, thus once again invoking Lieberman regularity result \([29]\) and applying Arzelà-Ascoli theorem we obtain that

\[
(\lambda_{n_j}, u_{n_j}) \to (\lambda, u) \quad \text{in} \quad \mathbb{R} \times C_0^1(\Omega) \quad \text{and} \quad (\lambda, u) \quad \text{solves} \quad (P).
\]

To prove that \(\Sigma\) is bounded in the \(\lambda\)-direction, notice that the function

\[
g_{\epsilon}(t) = \frac{(t + \epsilon)^{-\delta} + t^q}{t^{p-1}}, \quad \text{for} \quad t > 0
\]

admits a global minimum at \(t_{\min} = h^{-1}(0)\), where \(h : \mathbb{R}^+ \to \mathbb{R}\) is an invertible function given by

\[
h(t) = -\delta(t + \epsilon)^{-\delta-1}t^{1-p} + (1-p)(t + \epsilon)^{-\delta}t^{1-p} + (q-p+1)t^{q-p}.
\]

Moreover,

\[
\left( \frac{p-1+\delta}{q-p+1} \right)^{\frac{1}{q-p+1}} - \epsilon < t_{\min} < \left( \frac{p-1+\delta}{q-p+1} \right)^{\frac{1}{q-p+1}}.
\]

(1.18)

So, denoting by \(\zeta = \left( \frac{p-1+\delta}{q-p+1} \right)^{\frac{1}{q-p+1}}\), we conclude from (1.18) that

\[
g_{\epsilon}(t) \geq g_{\epsilon}(t_{\min}) \geq (\zeta + 1)^{-\delta} \zeta^{1-p}
\]
for all \( t > 0 \) and \( 0 < \epsilon < \varepsilon_1 < 1 \).

Suppose there exists \((\lambda_*, u_*) \in \Sigma_\epsilon\) with \( \lambda_* > \lambda_1(\zeta + 1)^{\delta \zeta p^{-1}} \). Then, \( \lambda_* g_\epsilon(t) \geq \lambda_1 + \kappa \) for every \( \kappa > 0 \) small enough, that is,

\[
\lambda_* \left((t + \epsilon)^{-\delta} + t^\delta\right) \geq (\lambda_1 + \kappa) t^{p-1}, \quad \text{for all } t > 0.
\]

In particular, \( u_* \) is a supersolution of

\[
\begin{aligned}
-\Delta_p u &= (\lambda_1 + \kappa) u^{p-1} \quad \text{in } \Omega, \\
u > 0, \quad u|_{\partial \Omega} &= 0,
\end{aligned}
\]

for all \( \kappa > 0 \) small enough. Moreover, for \( s > 0 \) small \( s \phi_1 \) is a subsolution of \((1.19)\) and satisfies \( u_* \geq s \phi_1 \) in \( \Omega \).

Hence, by monotone interaction we obtain a solution of \((1.19)\) for any \( \kappa > 0 \) small, contradicting the fact that \( \lambda_1 \) is an isolated point in the spectrum of \((-\Delta_p, W_0^{1, p}(\Omega))\) (see [2]). Therefore, \( \text{Proj}_{\mathbb{R}^+} \Sigma_\epsilon \subset [0, \lambda_1(\zeta + 1)^{\delta \zeta p^{-1}}] \), for any \( 0 < \epsilon < \varepsilon_1 \).

Finally, let us prove that \( \Sigma_\epsilon \) joins \((0, 0)\) to \((0, +\infty)\). In this direction, we first observe that there exists a sequence \( \{(\lambda_k, u_k)\}_{k=1}^\infty \subset \Sigma_\epsilon \) such that \( \lambda_k \rightarrow 0^+ \) and \( u_k \neq u_{\lambda_k} \), where \( u_{\lambda_k} \) denotes the minimal solution of \((P_\lambda)\) for \( \lambda = \lambda_k \). Indeed, otherwise we could find some \( \lambda_* > 0 \) small enough such that \( \Sigma_\epsilon \cap [0, \lambda_*] \times C_0(\overline{\Omega}) \) contains only elements in the branch of minimal solutions of \((P_\lambda)\) (see Proposition 3.2 below), which is not possible by invoking Lemma 3.2. Therefore, consider a sequence \( \{(\lambda_k, u_k)\}_{k=1}^\infty \subset \Sigma_\epsilon \) satisfying \( \lambda_k \rightarrow 0^+ \) and \( u_k \neq u_{\lambda_k} \). In this case, we must have \( \|u_k\|_\infty \rightarrow \infty \), up to a subsequence. On the contrary, \( \|u_k\|_\infty \) would be uniformly bounded, which combined with the \( \lambda_k \rightarrow 0^+ \) and Arzelà-Ascoli theorem would lead us to \((\lambda_k, u_k) \rightarrow (0, 0)\) in \( \mathbb{R} \times C_0(\overline{\Omega}) \), but this is not possible by uniqueness of solution for small \( \lambda \) and small norm (note that \( t \rightarrow [(t + \epsilon)^{-\delta} + t^\delta]/t^{p-1} \) is decreasing for \( 0 < t < \eta, \eta \) small). Hence the continuum \( \Sigma_\epsilon \) joins \((0, 0)\) to \((0, +\infty)\).

\[\blacksquare\]

**Proposition 3.2** For each \( 0 < \epsilon < \varepsilon_1 \), let \( \text{Proj}_{\mathbb{R}^+} \Sigma_\epsilon = [0, \Lambda_\epsilon] \) be the closure of the projection of \( \Sigma_\epsilon \) onto the \( \lambda \)-axis. Then:

i) \( \Sigma_\epsilon \) contains the branch of minimal solutions of \((P_\epsilon)\),

ii) for \( \lambda > \Lambda_\epsilon \) there is no solution of \((P_\epsilon)\),

iii) for \( 0 < \lambda < \Lambda_\epsilon \) there are at least two solutions of \((P_\epsilon)\) on \( \Sigma_\epsilon \),

iv) the map \( \epsilon \mapsto \Lambda_\epsilon \) is non decreasing.

Besides this, \( 0 < \Lambda_\epsilon \leq \lambda_1(\zeta + 1)^{\delta \zeta p^{-1}} \) for all \( \epsilon > 0 \) sufficiently small, where \( \zeta = \left(\frac{p-1+\delta}{q-p+1}\right)^{1/p} \).

**Proof** Part i): Since \( \Sigma_\epsilon \subset \mathbb{R}^+ \times C_0(\overline{\Omega})_+ \), it follows from the theory of regularity for elliptic equations (see [20], Theorem 1) that \( \Sigma_\epsilon \subset \mathbb{R}^+ \times C^1_0(\overline{\Omega})_+ \). Let us denote by \((\Sigma_\epsilon, \mathbb{R} \times C)\) the set \( \Sigma_\epsilon \) with the topology induced by \( \mathbb{R} \times C_0(\overline{\Omega}) \) and represent by \((\Sigma_\epsilon, \mathbb{R} \times C^1)\) the set \( \Sigma_\epsilon \) with the topology induced by \( \mathbb{R} \times C^1_0(\overline{\Omega}) \). As we have proved, \((\Sigma_\epsilon, \mathbb{R} \times C)\) is connected.

Claim: \((\Sigma_\epsilon, \mathbb{R} \times C^1)\) is connected. Indeed, let \( Z \) be the set of integers with the topology induced by the usual topology on \( \mathbb{R} \) and \( h : (\Sigma_\epsilon, \mathbb{R} \times C^1) \rightarrow Z \) be a continuous function. Then \( h : (\Sigma_\epsilon, \mathbb{R} \times C) \rightarrow Z \) is also continuous. Since \((\Sigma_\epsilon, \mathbb{R} \times C)\) is connected, it follows that \( h : (\Sigma_\epsilon, \mathbb{R} \times C) \rightarrow Z \) is constant, hence \( h : (\Sigma_\epsilon, \mathbb{R} \times C^1) \rightarrow Z \) is constant as well, which proves that \((\Sigma_\epsilon, \mathbb{R} \times C^1)\) is connected.

Now we are able to prove that \( \Sigma_\epsilon \) contains the branch of minimal solutions of \((P_\epsilon)\), that is, if \( \lambda' \in (0, \Lambda_\epsilon) \) and \( u_{\lambda'} \) is a minimal solution of \((P_\epsilon)\) with \( \lambda' = \lambda' \), then \( (\lambda', u_{\lambda'}) \in \Sigma_\epsilon \). On the contrary, consider

\[
A = (0, \lambda') \times \left\{ u \in C^1_0(\overline{\Omega}) : 0 < u < u_{\lambda'}, \text{ in } \Omega, \ 0 > \frac{\partial u}{\partial \nu} > \frac{\partial u_{\lambda'}}{\partial \nu} \text{ on } \partial \Omega \right\}
\]
an open and bounded set in $C^1_0(\overline{\Omega})_+$, where $v$ is the outward unit normal to $\partial \Omega$. Notice that $A \cap \Sigma_e \neq \emptyset$ and, by our contradiction hypothesis, $\Sigma_e \cap (\{\lambda'\} \times \{0, u_{\lambda'}\}) = \emptyset$. Moreover, for $(\lambda, u) \in \Sigma_e \cap A$ with $\lambda \in [0, \lambda')$ we have

$$\begin{cases} -\Delta_p u - \lambda (u + \epsilon)^{-\delta} = \lambda u^q, \\
-\Delta_p u_{\lambda'} - \lambda (u_{\lambda'} + \epsilon)^{-\delta} = \lambda u_{\lambda'}^q + (\lambda' - \lambda)(u_{\lambda'} + \epsilon)^{-\delta}, 
\end{cases} \quad (1.20)$$

where $\lambda u^q < \lambda u_{\lambda'}^q + (\lambda' - \lambda)(u_{\lambda'} + \epsilon)^{-\delta}$ in $\Omega$ because $0 \leq u \leq u_{\lambda'}$ in $\Omega$. Thus, by taking advantage of the proof of the Theorem 2.3 in \cite{24}, we conclude from \eqref{1.20} that $u < u_{\lambda'}$ in $\Omega$. Therefore, $\Sigma_e \cap \partial A = \{(0, 0)\}$, which contradicts the unboundedness and $C^1$–connectedness of $\Sigma_e$. Hence, $\Sigma_e$ contains the branch of minimal solutions of $(P_e)$. Part ii): We argue by contradiction. Suppose there exists a pair $(\lambda_*, u_*)$ of solution of the problem $(P_e)$ with $\lambda_* > \Lambda_*$. Without loss of generality, we can assume that $u_*$ is a minimal solution of $(P_e)$ with $\lambda = \lambda_*$. Consider the open and bounded set in $C^1_0(\overline{\Omega})_+$ defined by

$$A = (0, \lambda_*) \times \left\{ u \in C^1_0(\overline{\Omega})_+ : 0 < u < u_* \text{ in } \Omega \text{ and } 0 > \frac{\partial u}{\partial \nu} > \frac{\partial u_*}{\partial \nu} \text{ on } \partial \Omega \right\}$$

and notice that $A \cap \Sigma_e \neq \emptyset$. Proceeding exactly as in Part i) one gets $\Sigma_e \cap \partial A = \{(0, 0)\}$, again contradicting the unboundedness and connectedness of $\Sigma_e$. Part iii): Let $\lambda' \in (0, \Lambda_*)$. In the following discussion, $u_{\lambda_\Lambda}$ denotes the minimal solution of $(P_e)$ with $\lambda = \Lambda_*$. If $(P_e)$ does not admit a solution for $\lambda = \Lambda_*$, then just replace $u_{\lambda_\Lambda}$ with $u_{\lambda}'$, where $\lambda'' \in (\lambda', \Lambda_*)$. Now, we argue by contradiction. Suppose that $u < u_{\lambda_\Lambda}$, whenever $(\lambda', u) \in \Sigma_e$. In this case, it follows from the strong comparison principle \cite{24} that $u < u_{\lambda_\Lambda}$ in $\Omega$ and $\partial u/\partial \nu > \partial u_{\lambda_\Lambda}/\partial \nu$ on $\partial \Omega$. Consider the open and bounded set

$$V = \left\{ u \in C^1_0(\overline{\Omega})_+ : u(x) \leq u_{\lambda_\Lambda}(x) \text{ in } \Omega \text{ and } \frac{\partial u}{\partial \nu}(x) > \frac{\partial u_{\lambda_\Lambda}}{\partial \nu}(x) \text{ on } \partial \Omega \right\}$$

and $B = [0, \lambda'] \times \overline{V^c}$. Clearly $\Sigma_e \cap B^c \neq \emptyset$ and $\Sigma_e \cap B \neq \emptyset$, because $\Sigma_e$ bifurcates from infinity at $\lambda = 0$ and emanates from $(0, 0)$. On the other hand, we have

$$\partial B = \left\{(0, \lambda') \times \overline{V^c} \right\} \cup \left\{(0, \lambda'] \times \partial V^c \right\},$$

where

$$\partial V^c = \overline{V} \cap \overline{V^c} \subseteq \left\{ u \in C^1_0(\overline{\Omega})_+ : u(x) \leq u_{\lambda_\Lambda}(x) \text{ in } \Omega \text{ and } \frac{\partial u}{\partial \nu}(x) \geq \frac{\partial u_{\lambda_\Lambda}}{\partial \nu}(x) \text{ on } \partial \Omega \right\}$$

$$\cap \left\{ u \in C^1_0(\overline{\Omega})_+ : u(x) \geq u_{\lambda_\Lambda}(x) \text{ for some } x \in \Omega \text{ or } \frac{\partial u}{\partial \nu}(x) \leq \frac{\partial u_{\lambda_\Lambda}}{\partial \nu}(x) \text{ for some } x \in \partial \Omega \right\}$$

$$\subseteq \left\{ u \in C^1_0(\overline{\Omega})_+ : u(x) \leq u_{\lambda_\Lambda}(x) \text{ in } \Omega \text{ and } u(x) = u_{\lambda_\Lambda}(x) \text{ for some } x \in \partial \Omega \text{ or } \frac{\partial u}{\partial \nu}(x) = \frac{\partial u_{\lambda_\Lambda}}{\partial \nu}(x) \text{ for some } x \in \partial \Omega \right\},$$

which implies again by Theorem 2.3 in \cite{24} that $\Sigma_e \cap [0, \lambda'] \times \partial V^c = \emptyset$. Since $\Sigma_e \cap \left(\{0, \lambda\} \times \overline{V^c}\right) = \emptyset$, we have $\Sigma_e \cap \partial B = \emptyset$, contradicting the $C^1$–connectedness of $\Sigma_e$. From this, the proof of item $–iiii$) is established. Part iv): Let $\kappa > 0$ small, $0 < \epsilon_1 < \epsilon_2$ and denote by $u_{\min}$ the minimal solution of the problem $(P_{\epsilon_i})$ with $\lambda = \Lambda_{\epsilon_i} - \kappa$, $i = 1, 2$. In this case, $u_{\min}$ is a supersolution of

$$\begin{cases} -\Delta_p u = (\Lambda_{\epsilon_i} - \kappa) ((u + \epsilon_2)^{-\delta} + u^q) \text{ in } \Omega \\
u = 0 \text{ on } \partial \Omega, \end{cases} \quad (1.21)$$

and $v = 0$ is subsolution. So, \eqref{1.21} admits a positive solution in $[0, u_{\min}]$ and by arbitrariness of $\kappa$ we conclude that $\Lambda_{\epsilon_1} \leq \Lambda_{\epsilon_2}$. Below, we present some of the possible bifurcation diagrams of $\Sigma_e$.\[\hfill\]
Lemma 3.3 Let \( g \in C(\overline{\Omega} \times \mathbb{R}) \) be a non-negative function and \( \{ \epsilon_k \}_{k=1}^{\infty} \subset (0,1) \) a sequence satisfying \( \epsilon_k \to 0^+ \). If \( u_k \in W^{1,p}_0(\Omega) \cap C(\overline{\Omega}) \) is a solution of
\[
\begin{cases}
-\Delta_p u = \lambda_k [(u + \epsilon_k)^{-\delta} + g(x,u)] & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega \tag{1.22}
\end{cases}
\]
such that \( 0 < \inf_k \lambda_k \leq \lambda_k \leq \lambda \) and \( 0 < u_k(x) \leq h(x) \), for some \( \lambda > 0 \) and \( h \in C_0(\overline{\Omega})_+ \), then there exists \( (\lambda_*, u_*) \in \mathbb{R}^+ \times \left( W^{1,p}_0(\Omega) \cap C_0(\overline{\Omega}) \right) \) such that
\[
\lambda_k \to \lambda_* \quad \text{and} \quad u_k \to u_* \quad \text{in } C(\overline{\Omega}) \quad \text{and} \quad W^{1,p}_0(\Omega),
\]
up to a subsequence. Moreover, \( u_* \) solves
\[
\begin{cases}
-\Delta_p u = \lambda_* [u^{-\delta} + g(x,u)] & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega \tag{1.23}
\end{cases}
\]
Therefore, combining (1.24) and (1.25) one has
\[
\omega_{\lambda_{i} - \tau, 1} \leq u_k \text{ in } \Omega, \quad \omega_{\lambda_{i} - \tau, 1} \in C_0^1(\Omega)
\]
the only solution of (1.23) with \( \lambda = \lambda_i - \tau \) and \( \epsilon = 1 \).

Consider a sequence \( (\Omega_i) \) of open sets in \( \Omega \) such that \( \Omega_i \subset \subset \Omega_{i+1} \), \( \bigcup_i \Omega_i = \Omega \) and define \( \gamma_i = \min \omega_{\lambda_i - \tau, 1} \), for each \( i \in \mathbb{N} \). Using that \( g(x, u_k) \leq \max_{\mathbb{R} \times [0, \|h\|_\infty]} g(x, t) \) in \( \Omega \) and testing the problem (1.22) against \( \varphi = (u_k - \gamma_i)^+ \), we obtain
\[
\int_{\{u_k \geq \gamma_i\}} |\nabla u_k|^p dx = \lambda_k \int_{\{u_k \geq \gamma_i\}} [(u_k + \epsilon_k)^- + g(x, u_k)] (u_k - \gamma_i)^+ dx \leq C_1,
\]
where \( C_1 > 0 \) is a real constant independent of \( k \). Hence, the sequence \( \{u_k\}_{k=1}^\infty \) is bounded in \( W^{1,p}(\Omega_i) \) and there exists \( u_{\Omega_i} \in W^{1,p}(\Omega_i) \) and a subsequence \( \{u_{k_j}\} \) of \( \{u_k\} \) such that
\[
\begin{cases}
  u_{k_j} \rightharpoonup u_{\Omega_i} \text{ weakly in } W^{1,p}(\Omega_i) \text{ and strongly in } L^s(\Omega_i) \text{ for } 1 \leq s < p^* \\
  u_{k_j} \to u_{\Omega_i} \text{ a.e. in } \Omega_i.
\end{cases}
\]

By construction, we have \( u_{\Omega_{i+1}} |_{\Omega_i} = u_{\Omega_i} \). Hence,
\[
u_* := \begin{cases}
  u_{\nu_{i+1}} & \text{in } \Omega_i, \\
  u_{\Omega_{i+1}} & \text{in } \Omega_{i+1}\setminus\Omega_i
\end{cases}
\]
belongs to \( W^{1,p}(\Omega) \) and satisfies \( \omega_{\lambda_{i} - \tau, 1} \leq u_* \leq h \) in \( \Omega \).

We claim that \( u_* \) is a solution of (1.23). Indeed, by taking \( \varphi \in C_c^\infty(\Omega) \) and using Theorem 2.1 of Boccardo and Murat [7], we obtain
\[
\int_\Omega |\nabla u_k|^p \nabla \varphi dx \to \int_\Omega |\nabla u|^p \nabla \varphi dx,
\]
up to a subsequence. On the other hand, it follows from the convergence \( u_k \to u_* \) a.e in \( \Omega \), continuity of \( g \), uniform boundedness of \( \{u_k\}_{k=1}^\infty \) and Lebesgue dominated convergence theorem that
\[
\lambda_k \int_\Omega [(u_k + \epsilon_k)^- + g(x, u_k)] \varphi dx \to \lambda_* \int_\Omega [u_*^- + g(x, u_*)] \varphi dx.
\]
Therefore, combining (1.23) and (1.25) one has
\[
\int_\Omega |\nabla u_*|^p \nabla \varphi dx = \lambda_* \int_\Omega [u_*^- + g(x, u_*)] \varphi dx,
\]
for all \( \varphi \in C_c^\infty(\Omega) \), which proves that \( u_* \) solves (1.23).

To conclude that \( (\lambda_k, u_k) \to (\lambda_*, u_*) \) in \( \mathbb{R} \times C(\Omega) \) as well, we just need to combine \( L^\infty \)-uniform bound of \( \{u_k\}_{k=1}^\infty \) and Arzelà-Ascoli theorem with Theorem 1.8 of [25].
Lemma 3.4 Let \( B_R(0,0) \subset \mathbb{R} \times C_0(\Omega) \) be the ball centered at \((0,0)\) with radius \( R \), \( \epsilon \in (0,1) \) and \((\lambda, u_*) \in \left((0,\infty) \times (W_0^{1,p}(\Omega)) \cap C_0(\Omega)\right) \cap \overline{B}_{R}(0,0)\) be a pair of solution of \( (1.28) \) and using that \( \text{Lemma 3.4} \)

\[
\left\{ \begin{array}{l}
-\Delta_p u = \lambda \left[ (u + \epsilon)^{-\delta} + u^q \right] \quad \text{in} \quad \Omega, \\
u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,
\end{array} \right.
\]

\( (1.26) \)

satisfying \( \| (\lambda, u_*) \|_\infty > \rho \), for some \( \rho \in (0, R) \). Then, there exist positive constants \( \mathcal{K}_1 = \mathcal{K}_1(R, \rho) \) and \( \mathcal{K}_2 = \mathcal{K}_2(r, R) \) such that

\[
\lambda \mathcal{K}_1 \mathcal{K}_2 (R, \rho) \phi_1 \leq u_* \leq r + \lambda \mathcal{K}_2 (r, R) e_p \text{ in } \Omega,
\]

\( (1.27) \)

for each \( r \in (0, R) \) fixed, where \( e_p \) is defined in \( (1.3) \).

Proof To prove the first inequality in \( (1.27) \), we set

\[
\mathcal{K}_2 (r, R) = \max \left\{ t^{-\delta} + t^q : r \leq t \leq R + 1, x \in \Omega \right\},
\]

where \( r \) is a fixed number on \((0, R)\), and \( \partial \Omega = \{ x \in \Omega : u_* > r \} \). Then, it follows from the definition of \( \mathcal{K}_2 \) that

\[
-\Delta_p \left( r + \lambda \mathcal{K}_2 (r, R) e_p \right) = \lambda \mathcal{K}_2 (r, R) \geq \lambda \left( u_* + \epsilon \right)^{-\delta} + u_*^q \geq -\Delta_p u_* \text{ in } \partial \Omega.
\]

Since \( r + \lambda \mathcal{K}_2 (r, R) e_p - u_* = \lambda \mathcal{K}_2 (r, R) e_p \) \( \geq 0 \) on \( \partial \partial \Omega \), the claim is valid in \( \partial \partial \Omega \) by classical comparison principles. In the complementary of \( \partial \partial \Omega \), the inequality is obvious.

To show the first inequality in \( (1.27) \), we start by proving that

\[
\lambda \geq C_* := \min \left\{ \frac{1}{\mathcal{K}_1 (\rho / 4, R)} \left( \frac{\rho}{4} \right) \right\},
\]

In fact, otherwise by taking \( r = \rho / 4 \) in the second inequality in \( (1.29) \) we would have \( (\lambda, u_*) \in B_{\rho / 4}(0,0) \subset \mathbb{R} \times C(\Omega) \), which contradicts the fact that \( \| (\lambda, u_*) \|_\infty > \rho \).

Now, let us define \( u_* = \lambda \mathcal{K}_2 (r, R) \phi_1 \), where \( \mathcal{K}_1 (R, \rho) \) will be chosen later. It follows from Picone’s inequality that

\[
0 \leq \int_\Omega \left[ |\nabla u_*|^{p-2} \nabla u_* \nabla \left( \frac{u_*^p}{u_*^{p-1}} \right) \right] + \int_\Omega \left[ \lambda \mathcal{K}_2 (r, R) e_p \right] \left( u_* + \epsilon \right)^{-\delta} + u_*^q \left( u_*^p - u_*^{p-1} \right) \]

\( (1.28) \)

Since \( ((t+1)^{-\delta} + t^q)/t^{p-1} \to +\infty \) as \( t \to 0^+ \), for \( \mathcal{K} > \max \{ R, \lambda_1 / C_* \} \) given we can find \( a \) such that \( (t+1)^{-\delta} + t^q \leq \mathcal{K} t^{p-1} \), for all \( 0 < t < a \). Hence, for \( \mathcal{K}_1 (R, \rho) = a / \left( 2 \mathcal{K} \mathcal{K}_1 (\rho / 4, R) \right) \) the first inequality in \( (1.27) \) holds. Indeed, if \( \| u_* \| > u_* \) then

\[
\lambda \geq C_* \left( \frac{\rho}{4} \right) \left( \frac{\rho}{4} \right) > \frac{a}{2} \text{ on } \{ u_* > u_* \}.
\]

Therefore, going back to \( (1.28) \) and using that \( \lambda_1 / \lambda_* \leq \lambda_1 / C_* \), we get

\[
0 \leq \lambda \int_\Omega \left[ \lambda \mathcal{K}_2 (r, R) e_p \right] \left( \frac{u_* + \epsilon}{u_*^{p-1}} \right) \left( u_*^p - u_*^{p-1} \right) \]

\( (1.28) \)

\[
\leq \lambda \int_\Omega \left[ \frac{\lambda_1}{C_*} - \mathcal{K}_1 (R, \rho) \right] \left( u_*^p - u_*^{p-1} \right) < 0,
\]
which is an absurd. Hence, $\lambda_0^\frac{1}{\delta} \mathcal{K}_2(R, \rho) \phi_1 \leq u_\ast$ in $\Omega$ and the inequality (1.27) is proved.

**Proof Theorem 1.3.** Our proof will be based again on the Lemma 3.1. Initially, notice that $(0, 0) \in \Sigma$, for all $0 < \epsilon < \epsilon_1$, whence the pair $(0, 0)$ fulfills the first condition of the mentioned lemma. To prove that the second condition in Lemma 3.1 is also satisfied, let $B_R(0, 0) \subset \mathbb{R} \times C_0(\Omega)$ be the ball centered at $(0, 0)$ with radius $R > 0$, $\{\epsilon_n\}_{n=1}^\infty \subset (0, \epsilon_1)$ a sequence such that $\epsilon_n \to 0^+$, and $\{((\lambda_k, u_k))\}_{k=1}^\infty$ a sequence in

$$\bigcup_{\lambda \in \Sigma} (\epsilon_n \times \Omega) \cap B_R(0, 0).$$

We have three cases to consider:

a) an infinite amount terms of the sequence $\{((\lambda_k, u_k))\}_{k=1}^\infty$ belongs to some $\Sigma_{\epsilon_n}$.

b) $(0, 0)$ is a limit point of $\{((\lambda_k, u_k))\}_{k=1}^\infty$.

c) $\{((\lambda_k, u_k))\}_{k=1}^\infty$ has terms on infinite amount of $\Sigma_{\epsilon_n}$ and $(0, 0)$ is not a limit point of this sequence.

If a) occurs, by using Arzelà-Ascoli theorem, we get a convergent subsequence in the $\mathbb{R} \times C_0(\Omega)$—topology. If condition b) holds, naturally we have a convergent subsequence as well. In the case of c) be true, we can assume without loss of generality that $(\lambda_k, u_k) \in \Sigma_{\epsilon_k}$ and $\rho \leq |(\lambda_k, u_k)|_{\infty} \leq R$, for some $\rho > 0$ and for all $k \in \mathbb{N}$. Thus we are able to use Lemma 3.4 to obtain positive constants $\mathcal{K}_1 = \mathcal{K}_1(R, \rho)$ and $\mathcal{K}_2 = \mathcal{K}_2(r, R)$ such that

$$\lambda_k \frac{1}{\delta} \mathcal{K}_1(R, \rho) \phi_1 \leq u_k \leq r + \lambda_k \frac{1}{\delta} \mathcal{K}_2(r, R) \phi_1$$

in $\Omega$, (1.29) for each $r \in (0, R]$ fixed.

Suppose that $\lambda_k \to \lambda \geq 0$. If $\lambda = 0$, then by (1.29) we have $(\lambda_k, u_k) \to (0, 0)$ in $\mathbb{R} \times C_0(\Omega)$, which contradicts the fact that $(0, 0)$ is not a limit point of the sequence $\{((\lambda_k, u_k))\}_{k=1}^\infty$. Therefore, $0 < \inf_k \lambda_k \leq \lambda_k \leq R$ for all $k$ sufficiently large. From this and (1.29), the existence of the subsequence convergent of $\{((\lambda_k, u_k))\}_{k=1}^\infty$ is a consequence of Lemma 3.3.

Therefore, from Lemma 3.1, Lemma 3.2 and Proposition 3.2 we obtain that

$$\Sigma' := \lim_{n \to \infty} \sup \Sigma_{\epsilon_n}$$

is unbounded, closed, connected and joins $(0, 0)$ to $(0, +\infty)$. Moreover, by Proposition 3.2 we also have\n
$$\text{Proj}_{\mathbb{R}} \Sigma' \subset [0, \Lambda^\ast] \subset [0, \lambda_1(\zeta + 1)^{-\delta} \phi^-]$$

and $\Lambda^\ast \leq \Lambda_\ast$ (see item iv in Proposition 3.2).

Let us prove that $\Sigma := \Sigma' \setminus \{(0, 0)\}$ has the properties stated in the theorem. It is a direct consequence of the Lemma 3.3 and the construction of $\Sigma$ that $\Sigma$ is formed by solutions of $(P)$.

Next, let us show that $\Sigma$ contains the branch of minimal solutions of $(P)$. In fact, assume $\lambda_\ast \in \text{Proj}_{\mathbb{R}} \Sigma$, let $(\lambda_\ast, u_\ast) \in \mathbb{R} \times C_0(\Omega)$ be a pair of solution of $(P)$ and consider the iterative process

$$\begin{cases}
-\Delta_p u_n - \lambda_\ast u_n^{-\delta} = \lambda_\ast u_{n-1}^\eta & \text{in } \Omega, \\
u_0 = 0, & u_n \in W_{\text{loc}}^{1,p}(\Omega) \cap C_0(\Omega)
\end{cases}$$

(1.30)

It is clear that $u_0 \leq u_1$ in $\Omega$. By induction, we assume $u_{n-1} \leq u_n$ in $\Omega$ and let us prove that $u_n \leq u_{n+1}$ in $\Omega$. Indeed,

$$-\Delta_p u_n = \lambda_\ast u_n^{-\delta} + \lambda_\ast u_{n-1}^\eta$$

and

$$-\Delta_p u_{n+1} = \lambda_\ast u_{n+1}^{-\delta} + \lambda_\ast u_n^\eta \geq \lambda_\ast u_{n+1}^{-\delta} + \lambda_\ast u_{n-1}^\eta,$$

that is, $u_n$ is a solution and $u_{n+1}$ is a supersolution of

$$-\Delta_p u = \lambda_\ast u^{-\delta} + \lambda_\ast u_n^\eta$$

in $\Omega$, $u|_{\partial \Omega} = 0$.

respectively. So, we can apply the comparison principle of [36] to conclude that $u_n \leq u_{n+1}$ in $\Omega$, as claimed.

Analogously, we can show that $0 < u_n \leq u_\ast$ in $\Omega$, for all $n \in \mathbb{N}$. Since $0 < u_1 \leq u_n \leq u_\ast$ for all $n \geq 1$, we are able to employ the same steps of proof of Lemma 3.3 to ensure the existence of a solution
\( u_\ast \in W^{1,p}_{\text{loc}}(\Omega) \cap C_0(\overline{\Omega}) \) of (P) such that \( u_n \to u_\ast \) in \( W^{1,p}_{\text{loc}}(\Omega) \) and in \( C_0(\overline{\Omega}) \), up to a subsequence. Furthermore, the construction of \( u_\ast \) assures us that this must be the minimal solution of (P) with \( \lambda = \lambda_\ast \).

Finally, we will show that \( (\lambda_\ast, u_\ast) \in \Sigma \). To this end, let us consider \( \epsilon_k \searrow 0^+ \) as \( k \to +\infty \) and denote by \( u_{\epsilon_k} \) the minimal solution of \( (P_{\epsilon_k}) \) with \( \lambda = \lambda_\ast \). Once again by monotonic iteration and the comparison principle in [36], we have

\[
\underline{u}_k \leq u_{\epsilon_k} \leq \overline{u}_k \tag{1.31}
\]

for all \( \epsilon_k \in (0,1] \). It follows from Lemma 3.3 and inequalities (1.31) that \( u_{\epsilon_k} \to u_\ast \) as \( k \to \infty \). Since \( (\lambda_\ast, u_\ast) \in \Sigma_{\epsilon_k} \), the construction of \( \Sigma \) provides \( (\lambda_\ast, u_\ast) \in \Sigma \).

Now let us verify that item \( -ii \) holds. On the contrary, we could find a pair \( (\lambda_\ast, u_\ast) \in \mathbb{R} \times \left(W^{1,p}_{\text{loc}}(\Omega) \cap C_0(\overline{\Omega})\right) \) of solution of the problem (P) with \( \lambda_\ast > \Lambda^* \).

Let \( \epsilon_k \searrow 0^+ \) as \( k \to +\infty \). Given \( \tau = (\lambda_\ast - \Lambda^*)/2 \), we can apply Lemma 3.2 to find some \( k_0 \in \mathbb{N} \) such that \( \Sigma_{\epsilon_0} \subset V_{\tau}(\Sigma' \setminus 0) \), for all \( k > k_0 \). In particular, \( \Lambda_{\epsilon_k} \leq \Lambda^* + \tau < \lambda_\ast \) for all \( k > k_0 \), where \( \Lambda_{\epsilon_k} \) is the threshold parameter for the existence of solutions of \( (P_{\epsilon_k}) \). Let us fix \( k > k_0 \), \( \lambda \in (0, \Lambda_{\epsilon_k}) \) and consider \( \tilde{u}_{\epsilon_k} \in W^{1,p}_{\text{loc}}(\Omega) \cap C_0(\overline{\Omega}) \) the minimal solution of \( (P_{\epsilon_k}) \) with \( \lambda = \lambda_\ast \). Since \( u_\ast \) is a supersolution of the problem \( (P_{\epsilon_k}) \) with \( \lambda = \lambda_\ast \), once again by monotonic iteration we can conclude that \( \tilde{u}_{\epsilon_k} \leq u_\ast \) in \( \Omega \). So, about the problem

\[
\begin{aligned}
-\Delta_p u &= \lambda_\ast ((u + \epsilon_k)^{-\delta + u^q}) \quad \text{in } \Omega, \\
u > 0 \text{ in } \Omega, \quad &u = 0 \text{ on } \partial \Omega,
\end{aligned}
\]

we can summarize the following facts:

- \( \tilde{u}_{\epsilon_k} \) is a subsolution of (1.32);
- \( u_\ast \) is a supersolution of (1.32);
- \( 0 < \tilde{u}_{\epsilon_k} \leq u_\ast \) in \( \Omega \).

Hence, we are able to apply Theorem 2.4 of [31] to get a \( W^{1,p}_0(\Omega) \)-solution of (1.32) in \([\tilde{u}_{\epsilon_k}, u_\ast]\), which contradicts the fact that (1.32) does not admits any solution since \( \lambda_\ast \notin \text{Proj}_{\Sigma_{\epsilon_k}} \). This proves item \( -ii \).

Regarding item \( -iii \), the multiplicity for \( \lambda > 0 \) small follows from the facts that \( \Sigma \) is connected and \( \lambda = 0 \) is a bifurcation value of \( \Sigma \) from the infinity and from the trivial solution. Indeed, let \( C_\ast > 0 \) be a positive constant such that (P) admits at most a positive solution satisfying \( \|u\|_\infty \leq C_\ast \) (such constant exists by Lemma 2.4). If for some \( \lambda > 0 \) small enough (P) does not admit two distinct solutions in \( \Sigma \), then we can define the open set \( U = (0, \lambda) \times V \), where \( V := \{ u \in C(\overline{\Omega}) : \|u\|_\infty > C_\ast \} \), and conclude that \( \partial U \cap \Sigma \neq \emptyset \) (since \((0,0) \in \Sigma\)) and \( U \cap \Sigma \neq \emptyset \) (because \( \lambda = 0 \) is a bifurcation value of \( \Sigma \) from the infinity). However \( \partial U \cap \Sigma \neq \emptyset \) because \( \{0, \lambda\} \times V = \emptyset \), due to our contradiction assumption, and \((0,\lambda) \times V = \emptyset \), because \( \|u_\ast\|_\infty \leq C_\ast \) for any \( \lambda \leq \lambda_\ast \) since we are supposing \( \lambda \) small enough. This contradiction leads us to conclude that there exists \( \Lambda_\ast > 0 \) such that (P) admits at least two solutions for \( \lambda \in (0, \Lambda_\ast) \).

In the particular case, when \( \delta \in (0,1) \), the proof of the existence of at least two solutions for \( \lambda \in (0, \Lambda^*) \) is obtained by redoing the proof of item \( -iii \) in Proposition 3.2 and noting that in this case any continuous solution of (P) belongs to \( C^1_0(\overline{\Omega}) \) (see Theorem B.1 in [24]).

Our goal from now on is to establish the proof of Theorem 1.2, which is essentially inspired by [4], see also [11] and [22]. For this, we need to introduce some definitions and preliminary results.

Lemma 3.5 ([4], Lemma 4.1) Let \( \Omega \subset \mathbb{R}^N \) be a convex and bounded domain with \( C^2 \) boundary. Then

\[
\{ x \in \mathbb{R}^N : x = y + tv(y), \ 0 < t < 2\rho \} \subset \Omega,
\]

for some \( \rho > 0 \), where \( v(y) \) denotes the inward unit normal to \( \partial \Omega \) at \( y \).

Let \( v \) be a direction in \( \mathbb{R}^N \) with \( |v| = 1 \). For \( \lambda \in \mathbb{R} \) we set
be denoted by \( L \). Since we are supposing that there are no global maximum points of \( u_\rho \) or equal to \( \rho \) at \( \tau \), by Lemma 3.7. So the claim is proved.

Proof of Theorem 1.2: We argue by contradiction, that is, let us assume that there exists \( \tilde{\lambda} \in (0, \lambda^*] \) being a bifurcation parameter of \( \Sigma \) from infinity. Then, by the construction of \( \Sigma \), there would exist a continuous function which is strictly positive and locally Lipschitz continuous in \((0, \infty)\) and \( u \in C^1(\Omega) \) a weak solution of

\[
\begin{align*}
-\Delta_p u &= f(u) \quad \text{in } \Omega \\
u > 0 & \quad \text{in } \Omega \\
u = 0 & \quad \text{on } \partial \Omega.
\end{align*}
\]

For any direction \( \nu \) and for \( \lambda \) in the interval \((a(\nu), \lambda_1(\nu)]\), we have

\[
u(x) \leq u(x), \quad \forall x \in \Omega_\lambda^\nu.
\]

Now we are able to proof Theorem 1.2 which follows similar strategy considered in [3], with minor changes. However, for the reader convenience, we include the details here.

**Proof of Theorem 1.2** We argue by contradiction, that is, let us assume that there exists \( \tilde{\lambda} \in (0, \lambda^*] \) being a bifurcation parameter of \( \Sigma \) from infinity. Then, by the construction of \( \Sigma \), there would exist a sequence \( \{\epsilon_n\}_{n \in \mathbb{N}} \) such that \( \epsilon_n \rightarrow 0 \), and pairs \( (\lambda_n, u_n) \in \Sigma_{\epsilon_n} \) satisfying

\[
\begin{align*}
\lambda_n &\rightarrow \tilde{\lambda}, \\
\|u_n\|_\infty &\rightarrow \infty.
\end{align*}
\]

Claim 1: For each \( n \in \mathbb{N}' \), there exists a global maximum point \( \tau_n \in \Omega \) of \( u_n \) (that is, \( u_n(\tau_n) = \|u_n\|_\infty \)) such that \( \text{dist}(\tau_n, \partial \Omega) \geq \rho \).

Proof of claim 1: Assume by contradiction that every global maximum point \( \tau \) of \( u_n \) satisfies \( \text{dist}(\tau, \partial \Omega) < \rho - \epsilon \), for some \( \epsilon \in (0, \rho) \). By fixing \( \tilde{\tau} \) a such maximum and considering \( \tilde{x} \in \partial \Omega \) the nearest point of \( \partial \Omega \) from \( \tilde{\tau} \), we have that \( \text{dist}(\tilde{\tau}, \tilde{x}) = \text{dist}(\tilde{\tau}, \partial \Omega) < \rho - \epsilon \). Moreover, \( \tilde{\tau} \) belongs to the normal line to \( \partial \Omega \) at \( \tilde{x} \), which will be denoted by \( L \). From Lemma 3.6, we are able to find \( y \in L \cap \Omega_{\lambda_n(\tilde{x})} \) with \( \text{dist}(y, \tilde{x}) = \text{dist}(y, \partial \Omega) = \rho - \epsilon \). Since we are supposing that there are no global maximum points of \( u_n \) at a distance of \( \partial \Omega \) greater than or equal to \( \rho - \epsilon \), we conclude that \( u(y) < u(\tilde{\tau}) \), but this fact contradicts the monotonicity established in Lemma 3.7. So the claim is proved.
In what follows, we employ a blow-up method to derive a contradiction with the existence of the positive bifurcation parameter $\bar{\lambda} \in (0, \Lambda^*)$. For this proposal, denote by

$$M_n = \|u_n\|_{\infty} = u_n(\tau_n),$$

where $\tau_n$ is a maximum point of $u_n$ given by Claim 1, and define

$$w_n(y) = \frac{u_n(M_n^{-k}y + \tau_n)}{M_n}, \quad y \in \Omega_n := M_n^k(\Omega - \tau_n),$$

where $k = (q - p + 1)/p > 0$. Then, from the fact that $(\lambda_n, u_n) \in \Sigma_{\epsilon_n}$ and using change of variable in the integral one obtains

$$\int_{\Omega_n} |\nabla w_n|^{p-2}\nabla w_n \nabla \varphi dy = \lambda_n \int_{\Omega_n} \left[ w_n^q + M_n^{-kp-1}(M_n w_n + \epsilon_n)^{-\beta} \right] \varphi dy, \quad \forall \varphi \in C^\infty_c(\Omega_n).$$

Given any $R > 0$, we obtain from $M_n \to \infty$ and $k > 0$ that $\overline{B_R(0,0)} \subset \Omega_n$ for $n$ large enough, where $B_R(0,0)$ is the ball in $\mathbb{R}^n$ centered at the origin with radius $R$. Fixing a such ball, notice that

$$(w_n(y)M_n + \epsilon_n)^{-\beta} \leq [w_n(M_n^{-k}y + x_n)]^{-\beta} \leq [\omega_{\lambda_n,1}(M_n^{-k}y + x_n)]^{-\beta}, \quad y \in \overline{B_R(0,0)},$$

(1.33)

where $\omega_{\lambda_n,1}$ is the solution of (1.3) with $\lambda = \lambda_n$ and $\epsilon = 1$. Noting that $\lambda_n \to \bar{\lambda} > 0$, we get from (1.33) that $(w_n(y)M_n + \epsilon_n)^{-\beta} \leq C_R$ in $B_R(0,0)$, for some $C_R$ depending on $R$ but not of $n$. In this way, we can apply once again the regularity results of Lieberman [29] to conclude that $w_n$ is $C^{1,\alpha}(\overline{B_R(0,0)})$ uniformly bounded. Hence, using Arzelà-Ascoli theorem and a diagonalization argument one obtains a subsequence which converges locally uniformly in $C^{1,\beta}(\mathbb{R}^n)$ to a $w \in C^1(\mathbb{R}^n)$ satisfying

$$\left\{ \begin{array}{l}
\int_{\mathbb{R}^n} |\nabla w|^{p-2}\nabla w \nabla \varphi dx = \bar{\lambda} \int_{\mathbb{R}^n} w^q \varphi dx, \quad \forall \varphi \in C^\infty_c(\mathbb{R}^n) \\
\|w\|_{\infty} = 1, \quad w > 0 \text{ in } \mathbb{R}^n,
\end{array} \right.$$ 

that contradicts the result of Serrin and Zou (see Theorem II in [37]). Therefore, $\lambda = 0$ is the only bifurcation value of $\Sigma$ from infinity. As a consequence, if $\lambda_n \not\to \Lambda^*$ and $u_{\lambda_n}$ is the minimal solution of $(P)$ with $\lambda = \lambda_n$, then $\|u_{\lambda_n}\|_{\infty}$ is uniformly bounded. So, from Lemma 3.3 we obtain the existence of $u \in W^{1,p}_0(\Omega) \cap C_0(\overline{\Omega})$ solution of $(P)$ with $\lambda = \Lambda^*$ such that $u_n \to u$ in $W^{1,p}_0(\Omega)$. \hfill $\blacksquare$

4 Conclusion

In this paper, we present a new approach to deal with elliptic quasilinear problems perturbed by strongly-singular terms combined with $(p - 1)$-superlinear nonlinearities on smooth bounded domains. With this approach, we were able to establish not only $\lambda$-ranging for existence and multiplicity but also qualitative information of the solutions depending on the parameter $\lambda > 0$. However, mainly due to the lack of a priori estimates and strong comparison principle for strongly-singular problems (that in general requires $C^1(\overline{\Omega})$-regularity of the solutions), we only provided a local multiplicity in the strong singular case ($\delta \geq 1$). An important challenge in this class of problems is to establish conditions to obtain global multiplicity.

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