Notes on Some Geometric and Algebraic Problems Solved by Origami

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Abstract
Details for known solutions of some geometric and algebraic problems with the help of origami are presented: two theorems of Haga, the general cubic equation, especially the heptagon equation, doubling the cube as well as the trisection of angles $\alpha$, $\pi - \alpha$ and $\pi + \alpha$.

Introductory remarks
These notes give details on some geometric and algebraic problems related to cubic equations which are solved using origami (Japanese for folding paper). Seven axioms for origami can be found in [16]. These notes mostly start with a square of some given length, called $R$ in some length unit. Given any (transparent) sheet larger than this square one can fold an $R \times R$ square, provided one can determine the distance $R$ between two points $P$ and $Q$ on some line (crease). This assumes that one has some way to measure $R$, e.g., a marked ruler. Then one starts with some crease, call it $c_1$, and folds perpendicular to this crease through some point, defined as the first corner $A$, another crease called $c_2$. A perpendicular folding with respect to some line $c$ (crease) and a point $P$ (not necessarily on $c$) can be accomplished (sometimes called axiom 4 or IV), but here it is useful to have a transparent sheet in order to see when the two parts of $c$ fit together for this folding through $P$. Then one finds the next corner of the square, called $D$, at a given distance $R$ from $A$ on the crease $c_1$. Next, through $D$ a crease $c_3$ perpendicular to $c_1$ is formed. The next crease $c_4$ is obtained by folding crease $c_1$ onto crease $c_2$ (point $A$ will lie on both creases; guaranteed by axiom 3 or III). This will define the next corner $C$ as the intersection point of $c_4$ with $c_3$. Finally a crease $c_5$ perpendicular to $c_3$ through point $C$ is formed to find the last corner $B$ as the intersection of $c_5$ with $c_2$. Alternatively one can fold crease $c_3$ onto crease $c_3$, with $D$ on both creases, to find $B$ as the intersection with crease $c_2$. This completes then the square $A, B, C, D$ oriented in the positive sense (on one of the transparent paper’s sides). In the following we will use the notation $|BC|$ to denote the straight line connecting points $B$ and $C$, as well as the length of this line segment. The latter should be denoted by $|BC|$, but it should be clear what is meant in each case.

Problem 1: Haga’s second Theorem
In the book of Bellos [2] one finds on p. 115 an origami leading to the “second theorem” of Kazuo Haga. For this one folds two neighboring corners of a square sheet of paper (length of the side $R$ in some unit), say $B$ and $C$, in turn on some point $B’ = C’$ of the opposite side (bordered by the corners $A$ and $D$). This is done by two intersecting creases (called $f$ and $g$ in Figure 3). The intersection point, called $S$ in Figure 3, will always lie on the crease which arises if one folds $C$ onto $B$ (which gives one of the medians of the square). This happens independently of the position of the point on which the two corners have been folded. In addition, the three distances between the intersection point $S$ and the chosen point $C’$ and the two corners $B$ and $C$ coincide.
In order to analyse this consider first Figure 1 where $C$ is folded onto $C’$ having distance $xR$ from the left upper corner $A$.

\begin{quote}
\text{Notes on Some Geometric and Algebraic Problems Solved by Origami}
\end{quote}
Figure 1: Folding $C$ onto $C'$

Figure 2: Folding $B$ onto $B'$

$$A,B = R = A,D, \quad A,C' = xR, \quad D,F = yR, \quad V,B = V,W = vR, \quad W,C' = B,C = R, \quad B,E = W,E = zR, \quad P,C = P,C' = cR, \quad P,F' = pR, \quad \angle(F,V,C) = \angle(C',V,F) = \angle(C',C,D) = \angle(P,C',F) = \alpha, \quad \angle(V,P,C') = \frac{\pi}{2} = \angle(V,C',F)$$ (indicated by the two bullets), $\angle(V,C',A) = \gamma = 2\alpha$.

$P$ is the intersection point of two perpendicular straight lines, viz $g''$ connecting $C'$ and $C$ and $g$ (the crease bringing $C$ to $C'$) connecting $V$ and $F$.

The analytic data, depending on $R$ (usually taken as 1 length unit) and $x$, is:

$$y = \frac{x(x-2)}{2}$$

$$2c = \sqrt{1 + (1-x)^2}, \quad P,F = R\sqrt{(1-y)^2 - c^2}, \quad \tan\alpha = \frac{FC}{VC} = \frac{z}{v} = \frac{PF}{cR} = \frac{C',D}{R} = 1 - x, \quad \tan(2\alpha) = \frac{2(1-x)}{x(1-x)}, \quad \sin\alpha = \frac{1-x}{\sqrt{1 + (1-x)^2}} = \frac{c}{1 + v}, \quad \cos\alpha = \frac{1}{\sqrt{1 + (1-x)^2}} = \frac{p}{c}, \quad p = \frac{1}{2}, \quad v = \frac{x^2}{2(1-x)}, \quad F,C = F,C' = R(1 - y) = R \frac{2 - 2x - x^2}{2}, \quad z = v \tan\alpha = \frac{x^2}{2}, \quad V,E = R\sqrt{z^2 + v^2} = R \frac{x^2\sqrt{1 + (1-x)^2}}{2(1-x)}, \quad \frac{W,\ell}{R} = R \tan(2\alpha) = R \frac{x^2(1-x)}{\sqrt{x(2-x)}}, \quad \frac{L,A}{R} = R x \tan(2\alpha) = R \frac{2(1-x)}{2-x}, \quad \frac{E,L}{R} = R \frac{zR}{\cos(2\alpha)} = R \frac{x(1 + (1-x)^2)}{2 - x}, \quad \frac{L,C'}{R} = R \frac{x}{\sin(\pi/2 - 2\alpha)} = R \frac{1 + (1-x)^2}{2 - x}, \quad \frac{L,B}{R} = R v \tan(2\alpha) = R \frac{x}{2 - x}, \quad \frac{E,\ell}{R} = \frac{pR}{\sin\alpha} - \frac{v,\ell}{R} = R \frac{(1+x)\sqrt{1 + (1-x)^2}}{2}.$$

$$\frac{B,F'}{R} = R(1 - c \sin\alpha) = R \frac{1 + x}{2}, \quad \frac{P,C'}{R} = R \frac{1 - x}{2}.$$

With the origin $O = B$ in the $(\hat{x}, \hat{y})$-plane (no confusion with the above $x$ and $y$ should arise) the straight lines $g, g'$ and $g''$ are given by:

$$g: \quad \hat{y} = (1 - x)(\hat{x} + vR), \quad g': \quad \hat{y} = \frac{2(1-x)}{x(2-x)}(\hat{x} + vR), \quad g'': \quad \hat{y} = -\frac{1}{1-x}(\hat{x} - R).$$

**Observation 1:** If $x$ varies from 0 to $R$ then $P$ moves on the middle line $\hat{y} = \frac{R}{2}$ from $\hat{x} = \frac{R}{2}$ to $R$.

Next, the analysis is done for the case when the left lower corner $B$ is folded onto $B'$ on the side connecting the corners $A$ and $D$, with distance $xR$ from $A$ (not necessarily the same $x$ as in Figure 1).
The analytic data which depends on \( \angle C \) opposite side. The distance of bringing \( B = A, B \) to \( R \) is the intersection point of the perpendicular straight lines \( g' \), with points \( B' \) and \( B \), and \( g \) (the crease bringing \( B \) to \( B' \)) with points \( V \) and \( E \). Figure 2:

\[
A, B = R = \frac{A, D}{2}, A, B' = x R, A, E = y R, \ D, C = \frac{V, W}{2}, v R, E, B = \frac{E, B'}{2} = (1 - y) R, \ W, B' = \frac{C, B}{2} = R, C, F = \frac{F, W}{2} = z R, P, B = \frac{P, B'}{2} = b R, P, E' = p R, \angle(P, B, V) = \angle(P, B, V) = \angle(A, B', B) = \angle(B', E, F) = \angle(P, E, B) = \angle(P', P, V) = \beta, \angle(L, B', D) = \delta = \pi - 2 \beta, \angle(C, L, V) = \angle(B', L, D) = \angle(E, B', A) = 2 \beta - \frac{\pi}{2}.
\]

\( P \) is the intersection point of the perpendicular straight lines \( g' \), with points \( B' \) and \( B \), and \( g \) (the crease bringing \( B \) to \( B' \)) with points \( V \) and \( E \). The analytic data which depends on \( R \) (usually taken as 1 length unit) and \( x \) is:

\[
y = \frac{1 - x^2}{2}, \quad 2b = \sqrt{1 + x^2}, \quad P, E = R \sqrt{(1 - y)^2 - b^2}, \quad \tan\left(\frac{x}{2} - \beta\right) = \frac{1}{\tan \beta} = \frac{E, B}{B', V} = \frac{E, B'}{B, V} = \frac{b R}{b R} = \frac{E, P}{x} = \frac{A, B'}{A, B} = x, \quad \tan(2 \beta - \frac{x}{2}) = - \frac{1}{\tan(2 \beta)} = \frac{y}{x} = \frac{1 - x^2}{2x}, \cos \beta = \frac{x}{\sqrt{1 + x^2}} = \frac{b R}{b R} = \frac{E, P}{b R} = \frac{A, B'}{A, B} = x, \quad v = \frac{(1 - x)^2}{2x}, \sin \beta = \frac{1}{\sqrt{1 + x^2}} = \frac{p}{b}, \quad p = \frac{1}{2}, \quad E, B = E, B' = R(1 - y) = \frac{R \left(1 - x^2\right)}{2}, \quad z = v / \tan \beta = \frac{(1 - x)^2}{2}, \quad V, F = R \sqrt{v^2 + v^2} = R \left(1 - x^2\right) \sqrt{1 + x^2}, \quad W, L = R \left(1 - y\right) = \frac{R \left(1 - x\right)}{2} \left(1 + x\right), \quad \tan\left(\frac{x}{2} - \beta\right) = \frac{1 - x}{\tan(2 \beta)} = \frac{R \left(1 - x\right)}{2} \left(1 + x\right), \quad \frac{E, P}{b R} = \frac{E, P}{b R} = \frac{A, B'}{A, B} = x, \quad \sin\left(2 \beta - \frac{x}{2}\right) = \frac{1}{\tan(2 \beta)} = \frac{v}{x} = \frac{R \left(1 - x\right)}{1 + x}, \quad F, P = \frac{R \left(1 - x\right)}{2} \left(1 + x\right), \quad \frac{P, B}{P, E} = \frac{R \left(1 - x\right)}{2} \left(1 + x\right), \quad \frac{C, L, V}{C, L, V} = \angle(B', L, D) = \angle(E, B', A) = 2 \beta - \frac{\pi}{2}.
\]

With the origin \( O = B \) in the \((\hat{x}, \hat{y})\)-plane the straight lines \( g \) and \( g' \) are given by:

\[
g: \quad \hat{y} = -x (\hat{x} + R(1 + v)) = -x \hat{x} - R \frac{1 + x^2}{2}, \quad g': \quad \hat{y} = \frac{\hat{x}}{x}
\]

Observation 1': If \( x \) varies from 0 to \( R \) then \( P \) moves on the middle line \( \hat{y} = \frac{R}{2} \) from \( \hat{x} = 0 \) to \( \frac{R}{2} \).

Like depicted in Figure 3, one now folds the corners \( B \) and \( C \) onto the same point \( C' = B' \) on the opposite side. The distance of \( C' \) from corner \( A \) is \( x R \).

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Figure 3: Folding \( C \) and \( B \) onto \( C' = B' \)

Figure 4: Haga’s triple of Egyptian triangles
Problem 2: Haga’s Theorem on Egyptian Triangles

This is found in the book of Bellos [2] on p. 114, and has also been attributed to Kazuo Haga. Three Egyptian triangles (or scaled Pythagorean triangles) appear when folding a corner (vertex) of a square sheet of paper (here C) onto the midpoint of one of the non-adjacent sides (see Figure 4 point C'). The crease is the straight line g. The right triangles are \( T_1 = \triangle(F, D, C') \), \( T_2 = \triangle(C', A, L) \), and \( T_3 = \triangle(L, W, E) \). \( T_3' = \triangle(G, B, E) \) folds onto \( T_3 \). Each of these right triangles has rational side lengths, and they are scaled (3, 4, 5)–Pythagorean triangles.
\textbf{Theorem 2 (K. Haga)}

If the length of the square $R$ is taken as 1 length unit then the sides of the three right triangles of Figure 4 have side lengths:

$T_1 : \left( \frac{3}{8}, \frac{1}{2}, \frac{5}{8} \right)$, $T_2 : \left( \frac{1}{2}, \frac{2}{3}, \frac{5}{6} \right)$, $T_3 : \left( \frac{1}{8}, \frac{1}{6}, \frac{5}{24} \right)$.

Therefore, if the length of the square $R$ is chosen as 24 length units, these triangles become \textit{Pythagorean} triangles $\overrightarrow{T_1} : (9, 12, 15) = 3 \ast (3, 4, 5)$, $\overrightarrow{T_2} : (12, 16, 20) = 4 \ast (3, 4, 5)$ and $\overrightarrow{T_3} : (3, 4, 5)$.

\textbf{Proof:} The notation, with the length of the side of the square being $R$ length units, is: $C$ maps to $C'$, $B$ maps to $W$. $L$ is the intersection point of $\overrightarrow{V, C'}$ (the straight line $g'$ with segment $\overrightarrow{W, L}$) and $A, B$. $L$ maps to $G$. $P$ is the midpoint of $C', C$, $Q$ is the midpoint of $\overrightarrow{L, G}$ and $S$ is the midpoint of $\overrightarrow{W, B}$. The angle $\gamma = \angle(L, C', A)$ equals $2 \alpha$ because $\angle(E, L, W) = \frac{\pi}{4} - 2 \alpha = \angle(C', A, L)$

Similar right triangles with angle $\alpha$ are $\triangle(V, B, E)$, $\triangle(\overrightarrow{V, C}, F)$, $\triangle(F, P, C)$, $\triangle(B, S, V)$, $\triangle(B, E, S)$, $\triangle(L, E, Q)$, and their mirrors obtained by folding along $g$. The four shaded right triangles with angle $\gamma = 2 \alpha$ are also similar. $\tan \alpha = \frac{1}{2}$, $\sin \alpha = \frac{\tan \alpha}{\sqrt{1 + (\tan \alpha)^2}} = \frac{1}{5} \sqrt{5}$, $\cos \alpha = \frac{1}{\sqrt{1 + (\tan \alpha)^2}} = \frac{2}{5} \sqrt{5}$, $\tan(2 \alpha) = \frac{2 \tan \alpha}{1 - (\tan \alpha)^2} = \frac{4}{3}$, $\sin(2 \alpha) = \frac{4}{5}$, $\cos(2 \alpha) = \frac{3}{5}$.

The analytic data is: $A, C' = \frac{1}{2} R = \overrightarrow{C', D}$, $\overrightarrow{C', P} = a = \overrightarrow{P, C} = R \frac{\sqrt{5}}{4}$, $\overrightarrow{L, Q} = b = \overrightarrow{Q, G} = R \frac{\sqrt{5}}{12}$, $\overrightarrow{W, E} = \overrightarrow{E, B} = R \frac{1}{8}$, $\overrightarrow{W, L} = \overrightarrow{B, G} = R \frac{1}{6}$, $\overrightarrow{L, E} = \overrightarrow{E, G} = R \frac{5}{24}$, $\overrightarrow{V, B} = \overrightarrow{V, W} = R \frac{1}{4} \sqrt{5}$, $\overrightarrow{V, F} = R \frac{5}{8} \sqrt{5}$, $\overrightarrow{V, E} = R \frac{1}{8} \sqrt{5}$, $\overrightarrow{E, Q} = R \frac{1}{24} \sqrt{5}$, $\overrightarrow{Q, P} = R \frac{1}{3} \sqrt{5}$, $\overrightarrow{V, S} = R \frac{1}{10} \sqrt{5}$, $\overrightarrow{S, B} = R \frac{1}{20} \sqrt{5}$, $\overrightarrow{S, E} = R \frac{1}{40} \sqrt{5}$.

\hfill $\square$

\textbf{Problem 3: Solving Third Order Equations using Origami}

It is well known that cubic (and higher) order equations cannot be solved geometrically using only an (unmarked) ruler and a compass (see e.g., Wantzel\cite{15}, Adler \cite{1}, \S 36, pp. 188-195). It is also known that the general cubic equation can be solved geometrically with two right angles (at least one of them should have a scale in order to mark the absolute value of the coefficients) \cite{6}, p. 267, Abb. 150, adapted from \cite{1}, pp. 259-261, Fig. 156 (where we use the solving chain of straight lines $A$, $X$, $Y$, $E$ with $\angle(B, A, X) = \hat{\omega}$ (not the $\omega$ of the figure), $x = \tan \hat{\omega} = \frac{\overrightarrow{B, X}}{a_0} = \frac{\overrightarrow{Y, C}}{X, C} = \frac{\overrightarrow{D, E}}{Y, D}$ and $\overrightarrow{X, C} = |a_1| - \overrightarrow{B, X} = |a_1| - a_0 x$, $\overrightarrow{D, Y} = |a_2| + \overrightarrow{Y, C} = |a_2| + x \overrightarrow{X, C}$. With \overrightarrow{D, E} = a_3 = x \overrightarrow{D, Y} this leads to the cubic equation $x^3 - |a_1| x^2 - |a_2| x + a_3 = 0$ if one takes $a_0 = 1$. Note that $a_0$ and $a_1$ are supposed to have opposite signs because, coming from $\overrightarrow{A, B}$, one takes a $90^\circ$ left turn at $B$ to get to $C$. Similarly, $a_1$ and $a_2$ have like signs because, coming from $\overrightarrow{B, C}$, one takes a left turn at $C$ to get to $D$. Then $a_2$ and $a_3$ have again opposite signs because of the left turn at $D$. The length of the lines are always positive. These sign rules are taken from \cite{6}, and in \cite{1} a different solving path, namely $A$, $F$, $G$, $H$, has been chosen. This type of figure is also found in \cite{5}, p. 198, referring to Fig. 3 on p. 207 (where the top vertex is $A'$ which is connected to $B'$ on line $B'$. The distance between $A'$ and $I$ is $x$, and the distance between $A$ and $B'$ is $y$).
In order to solve the general cubic equation \( X^3 + a X^2 + b X - c = 0 \) with origami, following Huzita [5], p. 197 and Fig. 2, one first folds, like in Figure 5, a point \( A \) onto \( A' \) on the \( x \)-axis. The coordinates in the \([x, y]\)-plane with length unit \( R \) are \( A : [a_x R, a_y R] \) and \( A' : [\tilde{x} R, 0] \). The \( y' \) axis with angle \( \alpha \) has been added for later purposes, and is not relevant for this folding. \( g_A \) is the crease perpendicular to \( \overrightarrow{A'A} \) with intersection point \( P_A : [P_{A,x}, P_{A,y}] \). The crease hits the \( x \)-axis at point \( R_A \). The inputs are \( a_x, a_y \) and \( \tilde{x} = \frac{\alpha}{R} \). Instead of \( \tilde{x} \) we shall use \( X := a_x - \tilde{x} \). The right triangles \( \triangle(A', A_x, A) \) and \( \triangle(P_A, P_{A,x}, R_A) \) are similar. The following analytic expressions are found immediately:

\[
A, P_A/R =: a = \frac{P_A, \overrightarrow{A'}/R}{\frac{1}{2} \sin \beta} = \frac{1}{2} \frac{a_y}{\cos \beta}, \\
\frac{P_{A,x}, R_A}{R} = \frac{R}{2} \frac{a_y \tan \beta, O, R_A}{a_x - X + \frac{a_y}{2 \tan \beta} + \frac{a_y \tan \beta}{2} R} = \frac{1}{2} \left( \frac{a_y^2}{X} + 2 a_x - X \right) R.
\]

The straight line \( g_A \) (the crease) satisfies \( \frac{y}{R} = -\frac{X}{a_y} \left( \frac{\alpha}{R} + \frac{1}{2} (X - \frac{a_y^2}{X} - 2 a_x) \right) \).

Then a similar folding, shown in Figure 6, is done, in order to map a point \( B \) (different from \( A \)) onto a point \( B' \) on the \( y' \)-axis, forming some angle \( \alpha \) from \( \left(0, \frac{\pi}{2}\right)\) with the \( x\)-axis. The crease \( g_B \) is perpendicular to the line \( \overrightarrow{B'B} \) with intersection at the midpoint \( P_B \). It hits the \( x \)-axis at the point \( R_B \). The \((x,y)\) coordinates of \( B' \) are \([b'_x R, b'_y R]\). \( \angle(B', B', T) = \sigma \). The inputs are \( B : [b_x R, b_y R] \) and \( \tilde{y} = 0, B'/R \). The right triangles \( \triangle(B', T, B) \) and \( \triangle(P_B, P_{B,x}, R_B) \) are similar. One finds:

\[
b'_y = \tilde{y} \sin \alpha, \\
b'_x = \tilde{y} \cos \alpha, \\
\tan \sigma = \frac{b_y - b'_y}{b_x - b'_x}, \\
\frac{b'_y^2 + b_x^2 - \tilde{y}^2}{2 b_x - b'_x}.
\]

The straight line \( g_B \) (the crease) satisfies \( y = -\frac{1}{\tan \sigma} (x - \frac{O}{R_B}) = -\frac{b_x - \tilde{y} \cos \alpha}{b_y - \tilde{y} \sin \alpha} x + R \left( \frac{1}{2} \left( \frac{b_x^2 + b_y^2 - \tilde{y}^2}{b_y - \tilde{y} \sin \alpha} \right) \right) \).

In [5] the coordinate axes \( y' \) and \( x' \) are used. The transformation between coordinates of a point \( P : [p_x R, p_y R] \) and \( [p_{x'}, R', p_{y'} R'] \) is \( p_{x'} = p_x - \frac{p_y}{\tan \alpha} \) and \( p_{y'} = \frac{p_y}{\sin \alpha} = \sqrt{1 + 1/(\tan \alpha)^2 p_y} \).
The typical origami which brings at the same time one point $A$ onto $A^\prime$ on, say the $x$-axis, and another point $B$ onto $B^\prime$ on some other axis (here $y'$) is then shown to correspond to a cubic equation for a certain line segment (here $X = a_x - a_x^\prime$). See Figure 7 where $\alpha = 45^\circ$.

**Theorem 3 (Huzita [5]):** There exists a folding which brings $A : [a_x R, a_y R]$ onto $A^\prime : [a_x^\prime R, 0]$ and $B : [b_x R, b_y R]$ onto $B^\prime : [b_x^\prime R, b_y^\prime \tan \alpha R]$ on the $y'$ axis, which forms an angle $\alpha \in \left(0, \frac{\pi}{2}\right)$ with the $x$-axis. The solution for crease $g$ is $y = \frac{-X}{a_y} (x - r R)$, with $r = \frac{1}{2} \left(\frac{a_y^2}{X} + 2 a_x - X\right)$, and $X := a_x - a_x^\prime$ is the real solution of the cubic equation

$$X^3 + \left(b_x - 2 a_x + \frac{b_y - a_y}{\tan \alpha}\right) X^2 + a_y \left(2 b_y - a_y + 2 \frac{a_x - b_x}{\tan \alpha}\right) X - a_y^2 \left(b_x - \frac{a_y - b_y}{\tan \alpha}\right) = 0.$$ 

Before giving the proof a remark and an example are in order.

**Remark:** In [5] the components of $A$ and $B$ with respect to the axis $X$ and $Y$ (with $k = \cos \angle (X, Y)$) correspond to the above given $(a_x^\prime, a_y^\prime)$ and $(b_x^\prime, b_y^\prime)$, and $k = \cos \alpha$. Therefore $z$ of eq. (1) on p. 197 is given by $z = a_x^\prime - a_x = -\frac{a_y}{\tan \alpha} - a_x = X - \frac{a_y}{\tan \alpha}$ (with our $X = a_x - a_x^\prime$ and $a_y^\prime = 0$).

**Example 1:** In Figure 7 we have chosen $\alpha = \frac{\pi}{4}$ and $R = 1$ length unit. With $A : [.8, .2]$ and $B : [.5, .3]$ one has the real solution of $X^3 - X^2 + 0.2 X - 0.024 = 0$, which is $X \approx 0.7839279132$ (Maple 10 digits). This leads to $r = 0.4335485933$. $X$ has been indicated by the fat (magenta) line in Figure 7. $a_x^\prime = \bar{x} = 0.8 - X \approx 0.160720868$.

**Proof:** One combines the foldings of Figure 5 and Figure 6 with the constraint that the two creases, the straight lines $g_A$ and $g_B$, coincide. This leads to two equations: (I) $\overrightarrow{O,A} = \overrightarrow{O,B}$ and (II) $\beta = \sigma$. 

$$\left(\begin{array}{c}
\overrightarrow{O,A} - \overrightarrow{O,B} \\
\left|\overrightarrow{O,A} - \overrightarrow{O,B}\right|
\end{array}\right) / R = \frac{1}{2 X (b_x - b_x^\prime)} \left(\begin{array}{c}
X^2 \hat{y}^2 + \sqrt{X^2 - 2 a_x X - a_x^2} \hat{y} - b_x X^2 + (2 a_x b_x - b_x^2 - b_y^2) X + a_y^2 b_x
\end{array}\right) = 0.$$
The pre-factor does not vanish and it is not divergent \((b_x \neq b'_x)\), therefore the bracket term has to vanish. The other restriction is
\[
\tan \sigma - \tan \beta = \frac{b_y - \tilde{y} \sin \alpha}{b_x - \tilde{y} \cos \alpha} - \frac{a_y}{X} = 0.
\]
Solving for \(\tilde{y}\) as a function of \(X\) yields
\[
\tilde{y} = \sqrt{1 + (\tan \alpha)^2} \frac{a_y b_x - b_y X}{a_y - X \tan \alpha}.
\]
Inserting this \(\tilde{y}\) into the bracket term of \(\mathcal{O}_{RB} - \mathcal{O}_{RA}/R\) results in a factorized form given by
\[
\frac{(b_y - b_x \tan \alpha) X}{(-a_y + \tan \alpha X)^2} \left[ (\tan \alpha) X^3 + ((b_x - 2 a_x) \tan \alpha + (b_y - a_y)) X^2 + a_y (2 b_y - a_y) \tan \alpha + 2 (a_x - b_x) X + a_y^2 (a_y - b_y - b_x \tan \alpha) \right] = 0.
\]
Because \(B\) does not lie on the \(y'\) axis \(b_y \neq b_x \tan \alpha\), and \(\beta \neq \alpha\), hence \(a_y \neq X \tan \alpha\). Therefore, the new pre-factor does neither vanish nor diverge, and the bracket term has to vanish.

Now two cases have to be considered:

i) \(\tan \alpha \neq 0\) and \(\neq \infty\), \(i.e.\), \(\alpha \in \left(0, \frac{\pi}{2}\right)\) and

ii) \(\tan \alpha = \infty\), or \(\alpha = 90^\circ\).

The case \(\alpha = 0^\circ\) will later be treated separately.

i): This case leads to the cubic equation for \(X\) given in Theorem 3 after dividing \(\tan \alpha\) out. The equation for the crease \(g\) is just given by \(g_A\) from above (see Figure 5), with \(r = \mathcal{O}_{RA}\) and \(\beta = \sigma\) from condition (II).

\[\Box\]

Special case ii) \(\alpha = \frac{\pi}{2}\) (see Figures 8 and 9)

**Theorem 4**

With the notation of Theorem 3 and \(\alpha = \frac{\pi}{2}\) the equation for \(X = a_x - a'_x\) is
\[
X^3 + (b_x - 2 a_x) X^2 + a_y (2 b_y - a_y) X - a_y^2 b_x = 0.
\]

**Proof:** Extract \(\tan \alpha\) from the bracket term of \(\mathcal{O}_{RB} - \mathcal{O}_{RA}/R = 0\), factorized above, and observe that the original pre-factor \(\frac{1}{2 X (b_x - b'_x)}\) when multiplied with the factor \(\frac{(b_y - b_x \tan \alpha) X}{(-a_y + \tan \alpha X)^2}\) and after extraction of the \(\tan \alpha\) from the bracket term becomes, in the limit \(\tan \alpha \to \infty\), \(\frac{1}{2 X^2}\), provided \(b_x \neq 0\).

This new factor does neither vanish nor diverge, and from the bracket term the claimed cubic equation for \(X\) is obtained.

This result can also be reached in the limit \(\tan \alpha \to \infty\) from the above equation for \(X\) in Theorem 3, which, however, has been derived assuming \(\alpha \neq 0\). \[\Box\]

For the case \(\alpha = \frac{\pi}{2}\) see Figures 8 and 9. The other quantities are found by first folding \(B\) onto \(B'\) on the \(y'\)-axis. See Figure 8. The free parameter is \(\tilde{y} = O, B'\). \(\delta = \frac{\pi}{2} - \gamma\), \(b = B, PB = \frac{PB, B'}{2} = \frac{1}{2} \sqrt{\tilde{y}^2 - b_y^2} + b_y^2\), \(\tan \gamma = \frac{O, PB_x - O, RB}{b_y + b \sin \gamma}\), \(PB_x = [R(b_x - \cos \gamma b), 0]\), \(PB_y : [0, R(b_y + \sin \gamma b)]\), \(RB : [PB_x[1] - \tan \gamma PB_y[2], 0]\). The crease is \(g : y = \frac{1}{\tan \gamma} (x - RB[1])\).

Then also \(A\) is folded onto \(A'\) on the \(x\)-axis. This has been treated in connection with Figure 5 (were the \(y'\)-axis was not important). The two constraints are \(RA = RB\) and \(\tan \delta = \tan (\frac{\pi}{2} + \beta)\), \(i.e.,\)
\[\tan \gamma = -\tan \beta.\] One obtains a real solution of the cubic equation for \(X = a_x - a'_x\) indicated by the thick (magenta) line segment in Figure 9, and \(\tilde{y} = -b_x \frac{a_y}{X} + b_y.\) 

\[\begin{aligned}
\text{Case } \alpha &= \frac{\pi}{2} \\
\text{Figure 8: Folding } B \text{ onto } B' &\quad \text{Figure 9: Folding } A \text{ onto } A' \text{ and } B \text{ onto } B'
\end{aligned}\]

**Example 2:** \(R = 1\) length unit, \(A: [.2, .7], B: [.8, .3].\) \(X = a_x - a'_x \approx .646416, \beta \approx 47.279^\circ, \delta = (90 + \beta)^\circ \approx 137.279, \overline{O,R_B} \approx .2558, P_A: [\approx -.123, .35], P_B: [.4, \approx -.133].\)

**Special case \(\alpha = 0\)**

The special case \(\alpha = 0\) is obtained from folding \(A\) onto \(A'\), discussed above (see Figure 5), and folding \(B\) onto \(B'\) (also on the \(x\)-axis). One uses the formulae given above in connection with Figure 5 (with \(\beta\) changed into \(\beta_A\)) and replaces there \(A\) by \(B\) and \(A'\) by \(B'\) (with \(\beta = \beta_B\)). Then Figure 10 is obtained by setting \(R_A = R_B\) and \(\beta_A = \beta_B.\) 

\[\begin{aligned}
\text{Figure 10: Folding } A \text{ on } A' \text{ and } B \text{ on } B' \text{ when } \alpha &= 0
\end{aligned}\]
The result of identifying both creases $g_A$ and $g_B$, calling them $g$, leads to the following data:

$$X := a_x - a'_x, \quad Y := b_x - b'_x.$$ Setting $O, R_A - O, R_B = 0$ leads to the equation involving $X$ and $Y$.

$$-X + Y + a_y^2 \frac{1}{X} - b_y^2 \frac{1}{Y} + 2(a_x - b_x) = 0.$$

The slope of $g$ is given by $\tan \beta_A = \tan \beta_B$, which expresses $Y$ in terms of $X$:

$$Y = \frac{b_y}{a_y} X.$$

The mid points of $A A'$ and $B B'$ are $P_A : \left[ R \frac{2a_x - X}{2}, R \frac{a_y}{2} \right]$ and $P_B : \left[ R \frac{2b_x - Y}{2}, R \frac{b_y}{2} \right]$, respectively. The intercept is $r = \frac{O, R_A = O, R_B}{R} = \frac{R}{2} \left( \frac{a_y^2}{X} + 2a_x - X \right)$.

For the following one assumes that $0 \neq a_y \neq b_y$. Plugging $Y$ into the previous equation results in a quadratic equation for $X$:

$$X^2 - 2a_y \frac{1}{\tan \sigma} X - a_y^2 = 0,$$

with $\tan \sigma = \frac{a_y - b_y}{a_x - b_x}$.

This equation can be obtained directly from the above given analysis for non-vanishing $\alpha$ before $\tan \alpha$ has been divided out. Just let there $\tan \alpha \to 0$, which eliminates the $X^3$ term, and divide by $b_y - a_y$.

The relevant solution for $X$ is then

$$X = a_x - a'_x = -a_y \frac{1 - \cos \sigma}{\sin \sigma}.$$

**Example 3:** $R = 1$ length unit, $A : [7, 8], B : [1, 1]$. $\sigma \approx 49.399^\circ$, $-X \approx 0.368$, and $r \approx 0.014$.

If $a_y = b_y$, the equation for $X$ becomes linear, in fact $X = 0$.

**Degenerate case: Parallel lines with $A'$ and $B'$**

As mentioned in [5], p. 197, the case of parallel lines, is also of interest. See Figure 12.

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**Figure 11:** Folding $B$ onto $B'$

**Figure 12:** Folding $A$ onto $A'$ and $B$ onto $B'$

First one folds $B$ onto $B'$ which lies on the horizontal $y'$-axis with $y = dR$, hence $B'_y = dR$, with some length scale $R$. This is shown in Figure 11.
Figure 11:

Given \( d = b_y \), A and B, the free parameter is \( b'_x = \frac{B'_x}{R} \), the position of \( B' \) on the \( y' \)-axis. The straight line \( g \) is perpendicular to the line segment \( \overline{B,B'} \) and passes through its midpoint \( P_B \). This line \( g \) intersects the parallel \( x \)- and \( y' \)-axes at \( R_B \) and \( S_B \), respectively. The angle \( \delta \) equals \( \frac{\pi}{2} - \gamma \). Half the distance between \( B \) and \( B' \) is \( R_b \). The formulae are: \( b = \frac{b'_x - b_x}{2 \cos \gamma} \), \( \tan \gamma = \frac{b'_y - b_y}{b'_x - b_x} \), \( \cos \gamma = \frac{1}{\sqrt{1 + (\tan \gamma)^2}} \), \( \sin \gamma = \frac{\tan \gamma}{\sqrt{1 + (\tan \gamma)^2}} \).

\[ \quad PB : [R (b_x + b \cos \gamma), R (b_y + b \sin \gamma)], R_B : \left[ R \left( \frac{b'_x^2 - b_x^2 + b'_y^2 - b_y^2}{2 (b'_x - b_x)} \right), 0 \right], \]

\[ g : y = -\frac{1}{\tan \gamma} (x - R_{B,x}), S_B : [R_{B,x} - d \tan(\gamma) R, d R] \]. If \( B \) lies on the \( y' \)-axis then \( \gamma = 0 \) and \( \delta = \frac{\pi}{2} \), a simple special case.

Now \( A \) is folded onto \( A' \) on the \( x \)-axis at the same time as \( B \) is folded onto \( B' \). See Figure 12. From the above results in connection with Figure 5 one takes \( R_A \) and \( \tan \beta \) in terms of \( X := a_x - \bar{x} \), where \( O, A' = R \bar{x} \). The solution is obtained from \( R_A = R_B \) and \( \tan \beta = \tan \gamma \). The latter equation can be used to eliminate \( b'_x \) by (remember that \( d = b'_y \))

\[ b'_x = \frac{1}{a_y} ((d - b_y) X + b_x a_y), \]

provided \( a_y \neq 0 \), which we assume. If \( a_y = 0 \) then \( b_y = b'_y = d, \beta = \gamma = 0 \) and \( \delta = \frac{\pi}{2} \), a simple special case. From \( 2 \overline{O,R_A} = 2 \overline{O,R_B} \) one obtains

\[ X + \frac{a_y^2}{X} + 2 a_x - \frac{b'_y^2 - b_x^2 + d^2 - b_y^2}{b'_x - b_x} = 0. \]

If \( b'_x \) is inserted one finds a quadratic equation for \( X \), assuming that \( a_y X \) does neither vanish nor diverge. Because \( a_y \neq 0 \) has been assumed, if \( X = 0 \) then \( a_x = a'_x, b'_x = b_x \gamma = \beta = \frac{\pi}{2}, \delta = 0 \), another simple special case. With \( a_y \) also \( X \) will be finite.

\[ (a_y - b_y + d) X^2 - 2 a_y (a_x - b_x) X - a_y^2 (a_y - b_y - d) = 0. \]

Therefore, assuming \( (b_y \neq d + a_y) \) one finds the positive solution for \( X \)

\[ X = \frac{a_y}{d + a_y - b_y} \left( a_x - b_x + \sqrt{(a_x - b_x)^2 + (a_y - b_y)^2 - d^2} \right). \]

This shows, that a solution is only possible if \( d R \) does not exceed the distance between \( A \) and \( B \), which has been observed in [5]. It is clear that \( P_A \) and \( P_B \) have coordinates which are the arithmetic mean between the corresponding coordinates of \( A \) and \( A' \) and \( B \) and \( B' \), e.g., \( P_{B,x} = \frac{R b_x + b'_x}{2} \). Note that \( \overline{B,B'} = \frac{d - b_y}{a_y} A, A' \). Therefore the trapezoid \( A', A, B' \) and \( B \) becomes a rectangle precisely if \( d = a_y + b_y \).

**Example 4:** Put \( R = 1 \) length unit and take \( d = A, a_x = .8, a_y = .3, b_x = .1, b_y = .2 \), then \( X = a_x - a'_x \approx 0.77, b'_x \approx 0.61 \).
Application 1: The case of the heptagon equation

The minimal polynomial of the algebraic number \( \rho(7) := 2 \cos \left( \frac{\pi}{7} \right) \approx 1.801937736 \) (the length ratio of the larger diagonal and the side of a regular 7-gon) is \( C(7, x) = x^3 - 2 x^2 - 2 x + 1 \) (see e.g., [7], Table 2 and section 3). The three real zeros are known to be \( x(7; k) = 2 \cos \left( k \frac{\pi}{7} \right) \), for \( k = 1, 3, \) and 5. They are \( x(7; 1) = \rho(7) \), \( x(7; 3) \approx 0.4450418670 \) and \( x(7; 5) = -2 \cos \left( 2 \frac{\pi}{7} \right) \approx -1.246979604 \).

Here we show how these zeros are obtained by three different origamis. We also treat the standard geometric solution of this cubic equation using two right angular rulers, as explained in [6] based on [1] (see also von Sanden [14], ch. III, sect. 2, pp. 55-61, with Fig. 17 on p. 55). The corresponding Figures are 13, 14, and 15. The slope of the \( y' \) axis is chosen as \( \alpha = 90^\circ \), thus \( y' = y \). The monic cubic equation \( C(7, x) = 0 \) has sign pattern ++--++. This leads, in the standard geometrical construction, to the right angle pattern \( l, r, l \), with \( l \) and \( r \) for a 90° left and right turn, respectively. One starts with some (oriented) horizontal line segment \( \overline{BC} \) of length \( a_1 = 1 \) (for the monic case in some length unit \( R \)). A 90° left turn gives \( C, \overline{D} \) of length \( a_1 = 1 \), then a 90° right turn leads to \( \overline{DE} \) of length \( a_2 = 2 \), and finally the 90° left turn leads to \( \overline{EA} \) of length \( a_3 = 1 \). (The starting point has been chosen as \( B \) in order to comply with the later origami solution). This pattern (‘Streckenzug’ or line segment zig-zag) is dictated by the cubic equation and will be the same for each of the three solutions.

In the origami version one needs the two perpendicular axes \( y' = y \) and \( x \). As explained in [5], p. 198-199 and Fig. 4 on p. 208, the \( y \)-axis is chosen parallel to \( \overline{CD} \), at a perpendicular distance \( 2 a_0 = 2 \) from point \( B \). The \( x \)-axis is parallel to \( \overline{DE} \) at a perpendicular distance \( 2 a_3 = 2 \) from point \( A \). See the present Figure 13. In our case point \( B \) has coordinates \([-2, 0] \) and \( A : [1, 2] \) (if \( R = 1 \) ). In the standard geometrical construction of a solution to the cubic equation one has to find a point \( F \) on the axis with line segment \( \overline{CD} \), here \( F : [-1, x] \), such that a line perpendicular to \( B, F \) through \( F \) hits point \( G \) on the straight line with segment \( \overline{DE} \), and a perpendicular line to \( F, G \) through \( G \) hits point \( A \). In Figure 13 the solution has \( F = P_B = PB \) and \( G = P_A = PA \). For a general cubic equation there will always be at least one real solution, and depending on its discriminant one will find either one, two or three real solutions. In general the discriminant is \( Disc = p^3 + q^2 \), with \( q := \frac{1}{2} \left( 2 \frac{a_1^3}{27} - \frac{a_1 a_2}{3} + a_3 \right) \) and \( p := \frac{1}{9} (3 a_2^2 - a_1^2) \). In our case \( Disc = -\frac{7^2}{22 \cdot 3^3} < 0 \), telling that there are three (different) real solutions, in accordance with the explicitly known ones. Therefore, one expects three different constructions for the given right angle zig-zag \( B, C, D, E, A \). In the origami version we expect to find three (different) creases \( g_1, g_2 \) and \( g_3 \) each for folding simultaneously \( A \) onto some \( A' \) on the \( x \)-axis and \( B \) onto some \( B' \) on the \( y \)-axis. The three Figures 13, 14 and 15 show these solutions.

For all three figures the zeroes of \( C(7, x) \) are \( x = \frac{C+D}{R} = \frac{P_B y}{R} > 0 \). \( F = P_B \) and \( G = P_A \). The folding \( A \to A' \) works like earlier described in connection with Figure 5. For \( B \to B' \) with \( B : [-2 R, 0] \) (we use the length unit \( R \) here) and \( B' : [0, \tilde{y}] \) one has for the mid-point \( P_B : \left[-R, \frac{\tilde{y}}{2}\right] \).

With \( \gamma = \angle(D, P_B, P_A) \), \( \tan \gamma = \frac{R + \overline{O, R_B}}{P_B y} = \frac{P_B y}{R} \). Hence \( \overline{O, R_B} = \frac{P_B^2 y}{R} - R = \frac{\tilde{y}^2}{4R} - R \).

\( \tan \delta = \tan \left( \frac{\pi}{2} + \gamma \right) = -\frac{1}{\tan \gamma} \). The equation for the crease is \( g_B : y = -\frac{1}{\tan \gamma} (x - \overline{O, R_B}) \).

(Here \( x \) is a cartesian variable.) Putting then \( \tan \beta = \tan \gamma \) (with \( \beta = \angle(A, A', O) \)) yields \( \tilde{y} = 2 \frac{a_1 y}{X} = \frac{4R}{X} \) with \( \hat{X} := \frac{X}{R} \), where \( X = R - a'_2 \). Together with \( \overline{O, R_A} = 0, \overline{R_B} = =: rR \) one finds the cubic equation for \( \hat{X} \):

\[
\hat{X}^3 - 4 \hat{X}^2 - 4 \hat{X} + 8 = 0
\]

for each of the three figures.
Figure 13:
Here $\tilde{x} = a'_x/R < 0$, i.e., $X = R + |a'_x| > 0$. $x = \frac{C, P_B}{R} = \frac{\tilde{y}}{2R} = \frac{2}{X}$. The cubic heptagon equation for $x$, given above, is compatible with the cubic equation for $\hat{X}$. Because the three solutions for $x$ are known from the heptagon (see above), and since here $x > 1$ one has $x = \rho(7) = 2 \cos \left( \frac{\pi}{7} \right) \approx 1.801937736$, corresponding to $\hat{X} = 2x = \frac{1}{\cos \left( \frac{\pi}{4} \right)} = \sqrt{1 + \tan \left( \frac{\pi}{7} \right)^2} \approx 1.109916264$.

Figure 14:
$\tilde{x} = a'_x/R < 0$, i.e., $X = R + |a'_x| > 0$. $x = \frac{C, P_B}{R} = \frac{\tilde{y}}{2R} = \frac{2}{X}$. Because $0 < x < 1$, one has $x = x(7; 3) = 2 \cos \left( \frac{3\pi}{7} \right) \approx .4450418670$, corresponding to $\hat{X} = 2x = \frac{1}{\cos \left( \frac{3\pi}{7} \right)} = \sqrt{1 + \tan \left( \frac{3\pi}{7} \right)^2} \approx 4.493959217$.

Figure 15:
$\tilde{x} = a'_x/R > 0$, i.e., $X = R - a'_x < 0$. $\tilde{y} = b'_y < 0$. $0 > x = \frac{C, P_B}{R} = \frac{\tilde{y}}{2R} = \frac{2}{X}$. Hence $x = x(7; 5) = -2 \cos \left( \frac{5\pi}{7} \right) \approx -1.246979604$, corresponding to $0 > \hat{X} = \frac{2}{x} = \frac{1}{\cos \left( \frac{5\pi}{7} \right)} = -\sqrt{1 + \tan \left( \frac{5\pi}{7} \right)^2} \approx -1.603875472$.
Figure 15: $\alpha = \frac{\pi}{2}$ heptagon, third origami

**Application 2: Doubling the cube**

This classical problem cannot be solved by ruler and compass, but with Origami this can be accomplished because one has to solve the third order equation $x^3 - 2 = 0$. See [8] and Figure 16. (i) First one has to find a third of $A, B$. This is a standard origami problem solved by finding the intersection point $X$ of the two creases $A, C$ and $A, E$ where $E$ is found by halving the square, bringing $B \rightarrow A$ and $C \rightarrow D$. If the length of the side of the square $A, B, C, D$ is taken as 1 (in some length unit) then $B, H = \frac{1}{3} = C, J$.

![Figure 16: Doubling the cube; finding $C', B$](image1.png)  
![Figure 17: Doubling the cube; standard version](image2.png)

To see this one just has to find the intersection of the two straight lines $y = \frac{1}{2}x$ and $y = -x + 1$ finding $X : \begin{bmatrix} 2 \\ 1 \\ 3 \\ 3 \end{bmatrix}$. (ii) Folding $A, D$ onto $H, J$ will then generate the crease $F, G$ which completes the task to
divide the square into three equal parts. \( F : \left[ 0, \frac{2}{3} \right] \), \( G : \left[ 1, \frac{2}{3} \right] \). (iii) The crucial origami is then to fold at the same time \( C \rightarrow C' \) with \( C' \) on the line \( \overline{A,B} \) and \( J \rightarrow J' \) with \( J' \) on the line \( F,G \). The claim is that \( x := \frac{A,C'}{C',B} = 2^{1/3} \), or with \( s := \overline{C',B} \), i.e., \( x = \frac{1-s}{s} \), \( s^3 - s^2 + s - \frac{1}{3} = 0 \). The discriminant is \( \text{Disc} = +\frac{1}{3^4} \) showing that there is only one real solution, which is \( s = \frac{1}{3}(2^{2/3} - 2^{1/3} + 1) \approx 0.442493339 \). See [9] \( A2466\) for the decimal expansion of \( s \).

The analytic proof is obtained from looking at the right triangle \( \triangle(C', F, J') \) with angle \( \tau := \angle(F J', C') \). This angle can be computed by identifying the trapezoid angle \( \angle(C', J', J) \) with the one \( \angle(J', J C) = \beta + \frac{\pi}{2} \). Then \( \beta + \tau + (\beta + \frac{\pi}{2}) = \pi \), i.e., \( \tau = \frac{\pi}{2} - 2\beta \). Now \( \sin \tau = (1 - s - \frac{1}{3})/(\frac{1}{3}) = 2 - 3s \), which is also \( \sin \left( \frac{\pi}{2} - 2\beta \right) = 2\cos^2 \beta - 1 \). But \( \cos \beta = \sqrt{1 + s^2} \) from \( \tan \beta = s \), and thus \( \sin \tau = \frac{1-s^2}{1+s^2} \). Equating this with \( 2 - 3s \) leads to the claimed equation for \( s \), hence the one for the ratio \( x \).

We list more analytic data for Figure 16 with \( s := \overline{C',B}, b := \overline{C',M}, M := \overline{M,C} and a := \overline{J',N}, N := \overline{N,J} : \tan \alpha = \frac{1}{2}, \tan \beta = s/1 = s, \sin \beta = \frac{s}{2b} = \frac{1}{6} \), \( \tan \tau = \frac{1-s^2}{2s} \) (from the second formula for \( \sin \tau \) given above). \( J' : \left[ \frac{3s-1}{3s}, \frac{2}{3s} \right] \) from \( F, J' = 1 - \overline{J',G} = 1 - \frac{1}{3} \tan \beta = 1 - \frac{1}{3s} \approx 0.2466929834 \). \( M : \left[ \frac{1}{2}, \frac{s}{2} \right], N : \left[ \frac{6s-1}{6s}, \frac{1}{2} \right], \frac{1}{2} = 1 + s - s^2 \), \( s = \frac{1}{2}, P : \left[ \frac{s^2 - 1}{s^2 - 2}, \frac{1}{s^2 - 1} \right], Q : \left[ \frac{1 + 2s - s^2}{2}, \frac{1 + s^2}{2} \right], S : \left[ \frac{2}{3}, \frac{3}{1 + 2s}, \frac{1}{3}, \frac{1 + 3s}{3} \right] \), \( T : \left[ \frac{1}{3}, \frac{1 - 3s}{3}, \frac{1}{3}, \frac{1 - s}{3} \right] \), \( U : \left[ \frac{2}{3}, \frac{1 + 2s}{1 + s}, \frac{1}{3}, \frac{1 + 2s}{2} \right] \), \( Y : \left[ \frac{1- s^2}{2}, 0 \right], Z : \left[ \frac{1 - s^2}{2}, s, 1 \right] \). The equation for the crease \( Y,Z \) is \( y = \frac{1}{s} \left( x - \frac{1-s^2}{2} \right) \).

Standard version to find \( s \) with \( s^3 - s^2 + s - \frac{1}{3} = 0 \)

The cubic equation for \( s = \overline{C',B} \) can also be solved geometrically in the standard fashion, similar to finding \( x \) in the heptagon case treated above. Here the sign pattern is +,−,+−, which means that the 90° chain pattern is \( \alpha \), \( \ell \), \( l \). This leads to the line chain \( B, C, D, E, A \) shown in Figure 17 which is identical with \( B, C, D, A, F \) in Figure 16 (the scale of both figures is different). In this case the \( y \)-axis is parallel to \( D,C \) such that the \( x \)-coordinates of \( B \) becomes −2 (for \( R = 1 \) length unit), and the \( x \)-axis is parallel to \( E,D \) in Figure 16 this is \( \overline{A,D} \) such that the \( y \)-coordinate of \( B \) becomes \( -\frac{1}{3} - 1 = -\frac{4}{3} \), using \( E,A = \frac{1}{3} = E,A'' \). Here the \( F \) in the standard geometrical construction (not to be confused with \( F \) in Figure 16) with \( C,F = s \) is shown in Figure 17. That is the \( [x,y] \) coordinates of this \( F = PB \) from the origami construction are \( \left[ -1, -\left( \frac{4}{3} - s \right) \right] \). \( B \rightarrow B' \) on the \( y \)-axis with coordinates \( B' : [0, -\frac{4}{3} + 2s] \) (from the continuation of the line element \( \overline{B,PB} \) to the \( y \)-axis). \( A \rightarrow A' \) with coordinates \( A' : \left[ -2 \left( 1 - \frac{1}{3s} \right), 0 \right] \) because \( A,A' \) is parallel to \( B,B' \) with slope \( \tan \beta = s \). The two midpoints defining the crease are \( F = P_B \) and \( P_A : \left[ -2(1 - \frac{1}{s}), -\frac{4}{3} \right] \). The slope of the crease \( \overline{X,Y} \), shown in Figure 17, is \( \tan \alpha = -\frac{1}{s} \), \( \tan \beta = \frac{1}{s} \). The equation for the crease is
\[ y = \frac{1}{s} \left( x + \frac{s^2 + 6s - 1}{3s} \right). \]

The data for the trapezoid \( A, B, B', A' \) is: 

- \( a := \overline{AB} = \overline{PA, A'} = \frac{\sqrt{1 + s^2}}{3s} \approx 0.8237612353 \),
- \( b := \overline{PB} = \overline{PB, B'} = \sqrt{1 + s^2} \approx 1.093526566 \),
- \( \overline{AB} = \overline{A, A''} = \frac{2}{3} \).

**Completing the task of doubling a given cube**

Up to now we have only found the doubling of the cube with side length \( s \). In Figure 16 we had \( 2^{1/3} \overline{C', B} = \overline{C', A} \), i.e., \( 2s^3 = (1 - s)^3 \). In Figure 17 we had \( 2^{1/3} \overline{PB, C} = \overline{D, PB} \). For the decimal expansion of \( 2^{1/3} \) see [9] A002580. But the task is to double a cube with given side length \( L \). If one takes as length of the side of the square \( R = \frac{L}{s} \), then \( L = \overline{B, C} \) and \( M = \overline{C', A} = 2^{1/3} L \) is the side length for the doubled cube. However, we first have to find via origami \( 1/s \approx 2.259921051 \). But this can be achieved by considering the parallel to \( \overline{C, C'} \) through \( A \). This parallel will hit the continuation of \( \overline{B, C} \) on some point \( C'' \) with coordinates \( \left[ \frac{1}{s}, 0 \right] \) (origin at \( B \), \( x \)-axis along \( \overline{B, C} \) and \( y \)-axis along \( \overline{B, A} \)). See Figure 18. This means that if we take the length scale \( R = L \) the searched length \( M = 2^{1/3} L \) for the doubled cube is given by \( \overline{C, C''} \) which is \( \left( 1 - \frac{1}{s} \right) L \). It is easy to find the parallel \( g_2 \) in Figure 18 by origami. First find \( g_1 \), the crease perpendicular to \( g \) through the point \( A \) (this can be done, as explained in the introductory remarks; axiom 4 or IV). Then find \( g_2 \) as the crease perpendicular to \( g_1 \) through point \( A \). Finally the square \( C, C'', D'', A'' \) can be completed.

The coordinates of some points are: 
- \( C : [L, 0], C' : [0, s L], C'' : \left[ \frac{L}{s} = (1 + 2^{1/3}) L, 0 \right], \)
- \( V : [-s L, 0], \)
- \( W : \left[ -\frac{s (1 - s)}{1 + s^2}, L \frac{s (1 + s)}{1 + s^2} L \right], \)
- \( D'' : \left[ \frac{L}{s}, 2^{1/3} L \right]. \)

![Figure 18: Doubling the cube: finding \( M = 2^{1/3} L \)](image)

**Application 3: Trisection of an angle**

This is another classical problem not solvable with ruler and compass but with origami. See [3], [5], [8]. We first discuss the origami shown in [5] Fig. 1, pp. 204-5, and also in [8]. See the present Figure 19, where the angle is \( \alpha = \angle(P, B, C) \). Because the origami solution will be based on a cubic equation with three real roots, the question of the meaning of the other two roots arises. The answer can be found.
in [5]: the origami prescription for trisecting a given angle is not unique and the other two solutions correspond to trisecting the angle \( \pi - \alpha \) and \( \pi + \alpha \). This will be treated at the end of this section.

In the square \( (A, B, C, D) \) the point \( P \) on \( \overline{AD} \) defines the angle \( \alpha = \angle PBC \) to be trisected and the crease \( g_1 \). Then an arbitrary horizontal crease \( g_2 \) defining points \( E \) and \( F \) with distance \( 2h < 1 \) from the base line \( \overline{BC} \) is folded. The dashed crease \( g_3 \) bringing \( B \) onto \( E \) and \( C \) onto \( F \) has then a distance \( h \) from the base line. The crucial folding \( g \) is then to bring point \( E \) onto \( E' \) on crease \( g_1 \) and simultaneously point \( B \) onto \( B' \) on crease \( g_3 \). This will also bring point \( G \) to \( G' \). The continuation of \( \overline{BC} \) will intersect the line \( \overline{D' C} \) at a point \( V \), defining crease \( g_4 \). The intersection point of crease \( g \) with crease \( g_3 \) is called \( Y \). This defines the blue crease \( g_5 \) with line segment \( \overline{BY} \) crossing the continuation of the top line \( \overline{AD} \) at a point \( Q \) (depending on the choice of \( P \) and \( h \) this point \( Q \) could also lie on \( \overline{AD} \), e.g., for \( \alpha = 70^\circ \) and \( h = 0.2 \)). The claim is now that the blue crease \( g_5 \) and the crease \( g_3 \) trisect the angle \( \alpha \) with \( \sigma = \frac{\pi}{3} \).

One also shows that \( G' \) lies on the blue crease \( g_5 \).

The following analytic data is given for a coordinate system with origin at \( B \), the \( x \)-axis along \( \overline{BC} \) and the \( y \)-axis along \( \overline{BA} \). The input quantities are \( R \), the length of the square in some length unit, \( A, P = x = \frac{1}{\tan \alpha} \), with input \( \alpha \) in radians, and \( B, C = h R \):

\[
\beta = \frac{\pi}{3} - \sigma, \quad B, C = R, E : [0, 2h R], \quad G : [0, h R], \quad F : [R, 2h R], \quad H : [R, h R], \quad B' : \left[ \frac{h}{\tan \sigma} - R, h R \right],
\]

\[
E' : \left[ \frac{h \cos \sigma}{\sin \sigma} R, \frac{h \sin \sigma}{\sin \sigma} R \right],
\quad G' : \left[ 1 + \frac{\cos(2 \sigma)}{\tan(2 \sigma)} h R, (1 + \cos(2 \sigma)) h R \right],
\quad Y : \left( \frac{h}{\tan(2 \sigma)} R, h R \right),
\]

\[
X : \left[ \frac{h}{2 \tan \sigma} R, \frac{h}{2 \tan \sigma} \right],
\quad L : \left[ 0, \frac{h}{\sin(2 \sigma) \tan \sigma} \right],
\quad J : \left[ \frac{h}{\sin(2 \sigma)} R, 0 \right],
\]

\[
K : \left[ \frac{\sin(2 \sigma)}{2} (\frac{L_y}{R} - h) R, (h + \sin^2(2 \sigma)(L_y/R - h)) R \right],
\quad Z : \left[ \frac{\sin(2 \sigma)}{2} (\frac{L_y}{R} - 2h) R, (2h + \sin^2(\sigma) \frac{L_y/R - 2h}{R}) R \right],
\quad Q : \left[ \frac{1}{\tan(2 \sigma)} R, R \right],
\quad V : [R, R \tan \sigma].
\]

The equations for the creases are (\( x \) is here the abscissa): \( g_1 : y = \tan(\alpha) x \), \( g_2 : y = 2h \), \( g_3 : y = h \), \( g : y = \tan(\frac{\pi}{2} + \sigma) \left( x - \frac{h}{\sin(\sigma)} \right) \), \( g_4 : y = \tan(\sigma) x \), \( g_5 : y = \tan(2 \sigma) x \).

Some lengths in the trapezoid \( (B, B', E', E) \) are:

\[
E'G' = E, \quad G = h R = \overline{G' B'}, \quad \overline{B B'} = \overline{B G}, \quad b := \overline{B X} = \frac{h}{2 \sin \sigma} R, \quad e := \overline{E_1 Z} = \overline{Z E'} = \frac{\sin \alpha - 2 \sin \sigma}{2 \sin^2 \sigma} h R, \quad \overline{B Y} = \frac{h}{\sin(2 \sigma)} R.
\]

Now to the proof of the trisection of \( \alpha = \angle (P, B, C) \). Name the three angles, called \( \sigma \) in Figures 19 and 20, as follows. \( \tau := \angle (X, B, J) \), \( \alpha := \angle (Y, B, X) \), and \( \eta := \angle (E', B, G') = \alpha - (\tau + \sigma) \). We want to show that \( \tau = \sigma \) and \( \eta = \sigma \) which implies \( \alpha = 3 \sigma \).

Consider the angle \( \varepsilon := \angle (G', Y, B') \). (Note that at this stage it is not yet clear that \( \varepsilon = \angle (Y, B, J) \) which would immediately show that \( \sigma = \tau \). This is because it is not yet clear that the line \( \overline{Y G'} \) (obtained from folding along \( g \) where \( Y \) is the intersection of \( g \) with \( \overline{GC} \)) really continues to point \( B \). For this one has to prove \( \sigma = \tau \).) Because \( \angle (G, Y, L) = \angle (B, J, L) = : \beta = \frac{\pi}{2} - \tau \), and also \( \angle (K, Y, G') = \angle (K, Y, G) = \beta \) from the folding along \( g \), we have \( \pi = 2 \beta + \varepsilon \), or \( \varepsilon = \pi - 2 \beta = 2 \tau \). The proof that \( \sigma = \tau \) is done by starting with \( \sigma = \angle (X, B', Y) \) from the folding along \( g \). Also \( \angle (Y, B', G') = \frac{\pi}{2} - \varepsilon \) because of the folding along \( g \) the right angle \( \angle (Y, G, E) \) appears also as \( \angle (Y, G', E') \) and \( E, G' \) and \( B' \) are on a straight line, like \( E, G \) and \( B \). Because \( \angle (X, B', Y) = \frac{\pi}{2} - \tau \) we have from the right angle \( \angle (J, B', E') \) also \( \angle (J, B', G') = \frac{\pi}{2} = \tau + \sigma + (\frac{\pi}{2} - \varepsilon) \). This proves that \( \varepsilon = \pi - \sigma - 2 \tau \) or \( \sigma = \tau \). Finally, \( \eta = \sigma \) because the blue line \( \overline{B, G'} \) is the height, let its length be \( k \), in \( \triangle (B, B', E') \) and \( \tan \sigma = \frac{k}{h} = \tan \eta \) (because \( \overline{G, B'} = \overline{B, E'} = \overline{h} \) from folding along \( g \)). This implies that this triangle is isosceles, i.e., \( \overline{B, E'} = \overline{B, B'} \).
As mentioned above, in [5] the cubic equation relevant for this origami trisection of an angle $\alpha$ is identified and the question of the other two roots is answered. The crucial prescription to fold $B \to B'$ and, at the same time $E \to E'$, has three solutions, corresponding to the three real solutions of the cubic equation governing this folding. In Figure 21 the origami for the trisection of the angle $\alpha$ is repeated but now $X_1 := \overrightarrow{G, B'}$ has been shown as a fat line segment (in blue). If one uses the results from above for this case one will find that the dimensionless $\hat{X}_1 := \frac{X_1}{R} = \frac{\overrightarrow{G, B'}}{R} \frac{h}{\tan \frac{\alpha}{3}}$ satisfies the following cubic equation.

$$\hat{X}^3 - \frac{3h}{\tan \alpha} \hat{X}^2 - 3h^2 \hat{X} + \frac{h^3}{\tan \alpha} = 0$$

The discriminant of this cubic is $Disc = -\frac{h^6(1 + \tan(\alpha)^2)^2}{\tan(\alpha)^4}$, hence negative, therefore there are three different real solutions. The other two solutions $X_2$ and $X_3$ are shown in the Figures 22 and 23, respectively.
Figure 21: Solution X1: Trisecting an angle $\alpha$

$\alpha = 60^\circ$ and $h = .2$

Figure 22: X2, Trisecting an angle $\pi - \alpha$

$180^\circ - 60^\circ = 120^\circ$, $h = .3$

Trisecting an angle $\pi + \alpha$

$180^\circ + 60^\circ = 240^\circ$, $h = .2$

Finally, we list some analytic data for the three trisection Figures. The data for Figure 21 has been given already. The origin is taken at $B$ with the $x$–axis along $B,C$ and the $y$-axis along $B,A$. Note that in the origami for the general cubic equation treated above the $y'$ axis was along the crease $g1$ and the origin was there at $O : \left[ \frac{h R}{\tan \alpha}, h R \right]$, $\sigma = \frac{\alpha}{3}$, as used also above.

Figure 21: $\hat{X}1 = \frac{X1}{R} = \frac{G.B'}{R} \ (\approx 0.5495$ for $R = 1, \alpha = 60^\circ, h = .2)$

$0 < \frac{h}{\hat{X}1} = \tan \sigma = \frac{e'_y - 2h}{e'_x} = \frac{\sin \alpha - 2 \sin \sigma}{\cos \alpha}$.
In the first equation the mapping $B \rightarrow B'$ and in the second one $E \rightarrow E'$ has been considered. Thus
$$\hat{X}1 = \frac{h}{\tan \sigma},$$
which is in the Figure 21 shown for $R = 1, \alpha = 60^\circ$ and $h = .2$ which has the above given value.

Figure 22: $\hat{X}2 = \frac{X2}{R} = -\frac{G.B'}{R} (\approx -2.3835 \text{ for } R = 1, \alpha = 60^\circ, h = .3)$

$$\beta = \frac{\pi - \alpha}{3}, \quad B', G = h R = \frac{G.E}{R}, \quad \tan \beta = \tan \left(\frac{\pi - \alpha}{3}\right) = \frac{h}{\hat{X}2} = \frac{2 h R - e'}{e'_x},$$

$$B, B' =: 2 b R = \frac{h R}{\sin \beta}, \quad E, E' =: 2 e R = \frac{e'_x}{\cos \beta} B': \left[\frac{-h R}{\tan \beta}, h R\right],$$

$$E': \left[\frac{2 h R}{\cos \alpha \cos \beta} \frac{\sin (\alpha + \beta)}{\sin (\alpha - \beta)}, \quad 2 h R \frac{\sin \alpha \cos \beta}{\sin (\alpha + \beta)}\right], \quad J: \left[\frac{-h R}{\sin (2 \beta)}, 0\right].$$

The equations for the creases are: $g: \ y = \frac{1}{\tan \beta} (x + \frac{h R}{\sin (2 \beta)}), \quad g1: \ y = (\tan \alpha) x,$

$g4: \ y = (\tan \beta) x, \quad g5: \ y = -\tan (2 \beta) x.$

Figure 23: $\hat{X}3 = \frac{X3}{R} = -\frac{G.B'}{R} (\approx +0.0353 \text{ for } R = 1, \alpha = 60^\circ, h = .2)$

$$\beta = \frac{\pi + \alpha}{3}, \gamma = \frac{\pi}{2} - \beta, \quad B', G = h R = \frac{G.E}{R}, \quad \tan \beta = \tan \left(\frac{\pi + \alpha}{3}\right) = \frac{h}{\hat{X}3} = \frac{2 h R + |e'_y|}{|e'_x|},$$

$$B, B' =: 2 b R = \frac{h R}{\sin \beta}, \quad E, E' =: 2 e R = \frac{|e'_x|}{\cos \beta} B': \left[\frac{h R}{\tan \beta}, h R\right],$$

$$E': \left[\frac{2 h R}{\cos \alpha \cos \beta} \frac{\sin (\beta + \alpha)}{\sin (\beta - \alpha)}, \quad 2 h R \frac{\sin \alpha \cos \beta}{\sin (\beta + \alpha)}\right], \quad J: \left[\frac{h R}{\sin (2 \beta)}, 0\right].$$

The equations for the creases are: $g: \ y = -\frac{1}{\tan \beta} \left(x + \frac{h R}{\sin (2 \beta)}\right), \quad g1: \ y = (\tan \alpha) x,$

$g4: \ y = -(\tan(\beta) x, \quad g5: \ y = -\tan(\beta - \alpha) x.$

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