Absorption in quantum electrodynamics cavities in terms of a quantum jump operator

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Abstract

We describe the absorption by the walls of a quantum electrodynamics cavity as a process during which the elementary excitations (photons) of an internal mode of the cavity exit by tunneling through the cavity walls. We estimate by classical methods the survival time of a photon inside the cavity and the quality factor of its mirrors.

Keywords: QED cavity, tunneling, exponential decay.

1 Introduction

Absorption of a photon by the walls and/or escape of the photons outside from a cavity is an intrinsically irreversible in time and non-unitary process. Its deep understanding would require in principle an accurate description of the properties of the environment (geometry and quality of the mirrors in the cavity, possible escape towards the empty space outside the cavity and so on). However this description is too complicated to perform in the practice and, commonly, one derives the effective master equation that incorporates the losses inside the cavity thanks to an “educated guess”. In this ad hoc description only one free parameter, the survival time of a single photon inside the cavity, characterizes the losses. The value of this parameter is fixed in last resort by experiments [1].

The aim of our work is to refine the usual description of a lossy cavity by

(1) describing the absorption and/or escape by the boundaries of a QED cavity as a tunneling process during which elementary excitations (photons) of an internal mode of the cavity exit the cavity by tunneling and

(2) estimating by classical methods the survival time of a photon inside the cavity.

The paper is organized as follows.

Firstly (section 2), we modelize the disappearance of the photons of the cavity as the tunnel process of a particle trapped between two square potential barriers (which represent in our view the boundaries of the cavity). By doing so, we show that the classical picture according to which photons bounce back and forth inside the cavity is misleading. Instead, the quantum state of light inside the cavity can be seen as a quasi-stationary state. This brings us to the next section (section 3), where we refine this elementary model by introducing a Quantum Jump Operator aimed at simulating the escape of the photons leaving the cavity. We show in the framework of this more sophisticated model that the survival time of a photon inside the cavity is in last resort defined by a single free parameter, which measures the coupling between a photon inside the cavity and a photon of same frequency leaving the cavity by tunneling through the barriers. This result is derived, in the weak coupling regime, along the lines of the so-called Friedrichs model which is very close in mind to the Wigner-Weisskopf treatment of exponential decay.

An interesting by-product of our approach (section 4) is
that it allows us to show in a straightforward manner that coherent states couple to the environment without getting entangled to it. Usually this property is derived by solving the master equation that describes the losses in the cavity. In our case this property is seen as a direct expression of the (bosonic) indistinguishability of the photons in and outside the cavity.

As, on the other side, coherent states are quasi-classical in the sense that they minimize Heisenberg-type uncertainty relations, it is consistent to neglect the quantum fluctuations in order to obtain a first estimation of the single free parameter that is not addressed by our quantum jump model, essentially the absorption rate of the cavity. In this coarse grained approach where the mode inside the cavity is treated as a classical oscillator, it is justified to resort to classical, Maxwell equations in order to estimate the losses. By doing so we find (section 5) a good estimation of the lifetime of a photon inside the cavity, and obtain an indirect estimation of the quality factor of the mirrors used in the QED cavity [17] developed by the group of Serge Haroche at the École Normale Supérieure de Paris [1, 16, 17, 18].

2 Tunneling of a particle trapped between two potential barriers

A first possible model for cavity quantum electrodynamics is that of a quantum particle initially confined between potential barriers. Cavity losses are then modeled by the tunneling of the particle’s wave function through the barriers. Initially we shall consider massive particles but later this approach will be generalized to particles of arbitrary mass, including zero mass particles, having in mind the application of our model to QED cavities.

2.1 The Gamow model for nuclear decay

The Gamow model [13][14][15] describes nuclear disintegrations, namely, α-type disintegrations

\[ ^{A}ZX \rightarrow ^{A-4}Z-2Y + \frac{4}{2}\alpha \]

Mother nucleus \hspace{5em} Daughter nucleus \hspace{5em} Helium nucleus

within a quantum mechanical framework. To this end, Gamow proposed a simple picture: within the mother nucleus, the α particle “sees” the potential created by the other part of the mother nucleus, namely the daughter nucleus. Both strong (nuclear) and electromagnetic (Coulomb) interactions were taken into account by Gamow, and the energy \( E_\alpha \) of the α particle was assumed in his model to be lower than the barrier height, so that it is in a “quasi-bound” state, but at the same time Gamow assumed that this energy was high enough so that the α particle can tunnel through the Coulomb barrier towards the outside world. The situation is schematically represented in Fig. 1. The Gamow model is often [11][12] presented as follows: one takes for granted that the survival probability decreases exponentially, and is simply interested in computing the decay constant \( \Gamma_G \). To this end, one writes it as a product of three factors [12]:

\[ \Gamma_G = p(\alpha) \cdot f \cdot T \]  \hspace{1em} (1)

where \( p(\alpha) \) is the probability that an α particle exists within the daughter nucleus but is independent from it (it is often taken to be equal to 1 since there is no easy way to evaluate it), \( f \) is the classical collision frequency between the α particle and the potential barrier created by the daughter nucleus\(^1\) and \( T \) is the tunneling probability through the potential barrier. This can be summed up informally in the following way: the α particle oscillates back and forth inside the potential well, every now and then it hits the barrier and a small part of the wave function is transmitted through the barrier. This approach allows to derive [12] the empirical Geiger-Nuttal law

\[ \text{log } \Gamma_G = \text{cst} - \frac{Z_{\text{daughter}}}{\sqrt{E_\alpha}} \]  \hspace{1em} (2)

where \( E_\alpha \) is the energy of the α particle and \( Z_{\text{daughter}} \) is the atomic number of the daughter nucleus. However, the

\[^1\text{If one denotes by } R_0 \text{ the daughter nucleus’s radius and } v_\alpha \text{ the particle’s speed, then we can write } f = \frac{v_\alpha}{2R_0}.\]
classical picture of a particle bouncing back and forth is misleading as we shall see and ought to be replaced by a purely quantum description.

2.2 Results for a double potential barrier

Inspired by the Gamow model, we study the so-called double potential barrier, sketched in Fig. 2. It features two potential barriers of height $V_0$ situated between $-l-w$ and $-l$, and $l$ and $l+w$ respectively. We are interested in the decay of a wave packet initially confined between the barriers.

To solve the Schrödinger equation numerically, we followed [10] and defined the imaginary part and the real part of the wave function to exist at half-step time intervals from each other, in order to use the central difference method to compute the discretized derivatives. Remarks about the stability of the numerical algorithm can be found in [10]. Let us note that we had to add absorbing layers to avoid reflections. Following the perfectly matched layer (PML) method [32, 33], the Laplacian operator was thus analytically continued in the complex plane [8].

In a first time we considered that the particle was prepared initially in a Gaussian wave packet (of energy denoted $E_{\text{free}}$). We let run different simulations for different values of the initial energy keeping constant $V_0$, the barrier height. Through numerical simulations we monitored the time evolution of the nonescape probability

$$P_{\text{nonesc}}(t) \equiv \int_{-l}^{l} \, dx \, |\psi(x,t)|^2$$

(3)

for different input energies [8]. Even though short-time behaviors are very different, namely, higher energy packets decay much faster at short times, it turns out that at long times, the decay is identical (and exponential) for all initial states. This means that the decay constant is independent of the initial energy, and solely depends on the potential profile which the particle evolves in. At first sight this seems to be a hard blow dealt at the Gamow model, since we expect the collision frequency and the tunneling probability to depend on the “input” energy. On the contrary, it appears that the Gamow model remains accurate if we focus on the specifications of the state at long times. The key step is to forget about the specifications of the initial state, and especially about the input energy, and to focus on the final, exponentially decaying state (which is the state the characteristics of which are accessible to the experimentalist who verifies the Geiger-Nuttal law). Explicit numerical examples are given in [8].

Our numerical results are comforted by the theory of Gamow functions and complex poles, in particular by the leaky eigenmode approach [20] approach which consists in finding the system’s normalized generalized eigenfunctions, which have poles in the complex $k$-plane. These poles correspond to the system’s so-called Gamow functions, i.e. solutions to the time-independent Schrödinger equation for complex energy values. These complex energy eigenvalues, the imaginary part of which is simply the corresponding Gamow functions’ lifetime, are, up to dimensional factors, the square of the $k$-values which are the normalized generalized eigenfunctions’ poles. Using the tetrachotomy algorithm (see [7] (sect. 3.4) or [8]) to find poles in the complex $k$-plane, we were able to find several Gamow states and thus to observe departure from the seemingly “universal” behavior initially witnessed. In Fig. 4 we plot the time evolution of the two longest-lived Gamow states.
we can write the wave function as
\[ \psi(x,t) = \varphi(x) e^{-\frac{\Gamma}{2} t}, \] (4)

where, in first approximation, the longest lived state is, up to a global normalisation factor \( N \), proportional to the fundamental mode of the potential well: \( \varphi(x) \approx N \cos(kx) \).

We find that, accordingly to what is inferred in [20], their space-dependence inside and between the barriers is time-invariant and that they simply “melt” with time. These Gamow states can thus be seen as leaky eigenmodes of the system for which the time-dependence can be factored out of a time-independent spatial envelope (see (1)). Let us note that they resemble the modes of an infinite potential well (the so-called harmonics), a fact that we shall exploit in the following section.

Since Gamow’s picture of a particle that bounces back and forth between the barriers proved rather irrelevant -the “final state” tends to be an (almost) standing wave which leaks through the barriers-, one might wonder why we find such a good agreement between the semi-classical model and the leaky modes approach. For instance, the estimation of the uranium decay constant obtained from (1) is \( \Gamma_G = 3.130 \times 10^{-13} \text{ s}^{-1} \) [8], to compare with the value \( \Gamma_{\text{pole}} = 2.6722 \times 10^{-13} \text{ s}^{-1} \) obtained from the complex poles treatment [20] and the experimentally observed, empirical, value \( \Gamma_{\text{exp}} = 2.5872 \times 10^{-13} \text{ s}^{-1} \). Other examples of this kind are given in ref. [8]. In order to explain why it is so, we propose the following argument: at sufficiently long times, and in the central region (\( x \in [-l-w, l+w] \)), we can write the wave function as

Making use of the probability conservation equation yields
\[ \Gamma = \frac{j(l+w,t) - j(-l-w,t)}{\int_{-l-w}^{l+w} dx \, |\psi(x,t)|^2} \]
where the probability current is given by
\[ j(x,t) = \frac{\hbar}{2m} \left[ e^{ikx} e^{-\Gamma t} \right. \]
(5)

The wave function is purely outgoing at the interface between the potential barrier and the outside world, that is, at \( x = \pm (l+w) \), which allows us to write
\[ \psi(x,t) = A_- e^{-ikx} e^{-\Gamma t} \quad \text{at} \ x = -l - w, \] (6a)
\[ \psi(x,t) = A_+ e^{ikx} e^{-\Gamma t} \quad \text{at} \ x = l + w \] (6b)

and finally yields
\[ \Gamma = \frac{\hbar k}{2m (l+w)} \frac{|A_+|^2 + |A_-|^2}{\langle \varphi|^2 \rangle_{\text{within}}} \] (7)

where \( \langle \cdot \rangle_{\text{within}} \) is the average taken over the central region (see Fig. 2) and \( k \) is equal to \( 2\pi/\lambda_0 \), where \( \lambda_0 \) is the wavelength of the quasi-stationary mode inside the cavity, due to energy conservation. The first factor in this last equation is of particular interest, since it strongly resembles the bouncing frequency of the “semiclassical” Gamow model (see (1)). As for the second factor, it is reminiscent of Gamow’s tunneling coefficient, since the latter links the intensity of the wave function at the outer end of the barrier to the intensity of the wave function at its inner end.

3 Tunneling processes in terms of a quantum jump operator

3.1 A quantum jump operator

As we noted before, the Gamow states resemble the infinite potential well eigenstates, which would motivate a perturbative treatment to the problem. However, the fact that the unperturbed system would feature infinite potential regions is a major obstacle since one cannot easily “perturbate oneself out of infinity”. Accordingly, we turn to a different treatment where we couple the inside and outside regions through an interaction Hamiltonian. This is a first step towards a second-quantization approach of the
problem which is a much more comfortable framework for photons.
We make the approximation that the only relevant mode between the barriers is the one which resembles the longest-lived Gamow state. Between the barriers (|x| < l) it is equal to \( \psi_{in}(x,t) = e^{-i\omega_0 t} \cos(k_0 x) \) where \( k_0 = \pi/(2l) \) and \( \hbar\omega_0 \) is the (real part of the) energy of the longest-lived Gamow state. It is taken to be equal to zero inside the potential barriers and in the outside region (that is, for all \( |x| > l \)). For “escaping” wave functions outside the barriers we make the natural choice of the basis which consists of plane waves \( \psi_{out}^{\omega,k}(x,t) = e^{i(kx-\omega t)} \), with \( k > 0 \) when \( x > w + l \), and \( k < 0 \) when \( x < -w - l \). These functions are taken to be zero between and inside the barriers (\( -w - l < x < w + l \)). This choice is motivated by our interest in the decay of initially confined states. The Hamiltonian operator for the system finally reads
\[
\hat{H} = \hat{H}_0^{in} + \hat{H}_0^{out} + \hat{V}^{jump} \tag{8}
\]
where \( \hat{H}_0^{in/out} \) is a standard free-space quantum Hamiltonian multiplied by the indicator function for the adequate region. The quantum jump operator is defined through
\[
\hat{V}^{jump} \equiv \int_0^{+\infty} d\omega(k) \left[ \lambda(\omega(k)) \left| \psi_{in}^{\omega,k} \right\rangle \langle \psi_{in}^{\omega,k} \right| \right. \\
\left. + \lambda^*(\omega(k)) \left| \psi_{in} \right\rangle \langle \psi_{out}^{\omega,k} \right] \tag{9}
\]
At this level we do not make any specifications on the coupling function \( \lambda \), but it will be useful in the following to ask that it be sufficiently regular in the vicinity of \( \omega_0 \), and also that \( |\lambda(\omega_0)|^2/h^2 \) is much smaller than \( \omega_0 \). As we shall see, the latter condition is typical of the Fermi golden rule regime, which is valid when the decay constant \( \Gamma \) is much smaller than the internal frequency \( \omega_0 \) of the state. Our model belongs to the class of the so-called Friedrichs models \[22\] in which a finite number \( (N) \) of discrete states is coupled to a continuum of decay products (here \( N = 1 \)).

3.2 Direct sum and tensor product

The natural structure of the system’s Hilbert space \( \mathcal{H} \) is given by \( \mathcal{H} = L^2(\mathbb{R}) = \mathcal{H}_{in} \oplus \mathcal{H}_{out} \), but here we turn to the different structure of the tensor product (this switch procedure from a direct sum structure to a tensor product structure is detailed in \[22\]):
\[
\mathcal{H} = \mathcal{H}_{in} \otimes \mathcal{H}_{out} \tag{10}
\]
We can further validate the procedure given in \[22\] with the following argument: for the direct sum structure one writes a state \( |\psi^{(i)}\rangle \) as
\[
\langle x | \psi^{(i)} \rangle = \langle x | \psi_{in}^{(i)} \rangle \mathbb{1}_{in}(x) + \langle x | \psi_{out}^{(i)} \rangle \mathbb{1}_{out}(x) \tag{11}
\]
where \( \mathbb{1}_{in} \) and \( \mathbb{1}_{out} \) are indicator functions for the in and out regions respectively. Thus the scalar product is given by
\[
\langle \psi^{(1)} | \psi^{(2)} \rangle = \langle \psi_{in}^{(1)} | \psi_{in}^{(2)} \rangle \mathbb{1}_{in} \mathbb{1}_{in} + \langle \psi_{out}^{(1)} | \psi_{out}^{(2)} \rangle \mathbb{1}_{out} \mathbb{1}_{out} \tag{12}
\]
Switching to the tensor product structure, we write
\[
| \psi^{(i)} \rangle = | \psi_{in}^{(i)} \rangle \otimes | 0_{out} \rangle + | 0_{in} \rangle \otimes | \psi_{out}^{(i)} \rangle \tag{13}
\]
where the vacuum states \( |0_{in/out}\rangle \) are normalized states. It is then readily found that one gets the same expression as \( (12) \) for the scalar product. We thus adopt this tensor product structure for the Hilbert space and write the state as follows:
\[
| \psi(t) \rangle = \alpha(t) | 1_{in} \rangle \otimes | 0_{out} \rangle \\
+ \int d\omega(k) \beta(k,t) | 0_{in} \rangle \otimes | \omega(k)_{out} \rangle, \tag{14}
\]
where \( \alpha \) and \( \beta \) are complex amplitudes. Writing in this tensor product structure the Hamiltonian for the system which is equivalent to the one defined by \( (8) \) and \( (9) \) we get
\[
\hat{H} = \hbar\omega_0 | 1_{in} \rangle \langle 1_{in} | \otimes \hat{\mathbb{1}}_{out} \\
+ \hat{\mathbb{1}}_{in} \otimes \int d\omega(k) \ h | \omega(k)_{out} \rangle \langle \omega(k)_{out} | \\
+ \int d\omega(k) \left[ \lambda(\omega(k)) | 0_{in} \rangle \langle 1_{in} | \otimes | \omega(k)_{out} \rangle \langle 0_{out} | \\
+ \lambda^*(\omega(k)) | 1_{in} \rangle \langle 0_{in} | \otimes | 0_{out} \rangle \langle \omega(k)_{out} | \right]. \tag{15}
\]

3.3 Time-dependent perturbation theory in the weak coupling regime

We shall now solve Schrödinger’s equation for the Hamiltonian defined by \[15\]. Our treatment is very close in mind to the Wigner-Weisskopf perturbative approach. Since the state is initially confined inside the cavity, we choose
\[
\left\{ \begin{array}{ll}
\alpha(t = 0) = 1, \\
\forall k \beta(k, t = 0) = 0.
\end{array} \right. \tag{16}
\]
These initial conditions \([16]\) yield the following equations:

\[
\beta (k,t) = -\frac{i}{\hbar} \lambda (\omega (k)) \int_0^t dt' e^{-i\omega (k)(t-t')} \alpha (t'),
\]

\[
\frac{d}{dt} (e^{i\omega_0 t} \alpha (t)) = -\int d\omega (k) \left| \frac{\lambda (\omega (k))}{\hbar} \right|^2 \int_0^t dt' e^{i(\omega_0 - \omega (k))(t-t')} \left( e^{i\omega_0 t'} \alpha (t') \right).
\]

(17a)

\[
(17b)
\]

Let us note that up to this point we assume a bijective map from \(k\) to \(\omega\). This means that we only consider outgoing modes to the right. We shall reestablish the contribution from the left-outgoing modes in the end. Now, since the “free” (non perturbed) Hamiltonian of the inside region reads \(\hat{H}_{in} = \hbar \omega_0 \mid 1_{in} \rangle \langle 1_{in} \mid\), we can reasonably assume, in the weak coupling regime, for which \(\lambda\) is a small parameter and the decay rate is slow, that \(\alpha\) is the product of an oscillating exponential of frequency \(\omega_0\) with a function \(f\) the variations of which are negligible on a time-scale of \(\omega_0^{-1}\):

\[
\alpha (t) = e^{-i\omega_0 t} f (t).
\]

(18)

The first consequence of this assumption is that energy is conserved in first approximation, as can be seen by integrating \([17a]\) considering that \(\alpha (t) \simeq e^{-i\omega_0 t} f (t = 0)\) which yields

\[
|\beta (k,t)|^2 \approx |\lambda f (t = 0) (\omega (k)) \sin ((\omega - \omega_0) t/2) / (\hbar (\omega - \omega_0))|^2
\]

which can be considered to go to 0 when \(t\) is large enough whenever \((\omega - \omega_0) > 2\pi / t\) which means that only the modes that vibrate at “nearly” the same frequency as the internal mode get coupled to it by the Hamiltonian defined at \([15]\). One can then reasonably assume that \(\lambda (\omega (k))\) is constant and takes the values \(\lambda (\omega_0)\) because anyhow the evolution will not couple the internal mode to external modes which would differ too much in energy. Coming back to this property at the end of the treatment it is easy to check that it is expressed by the weak coupling condition \(\omega_0 \gg \Gamma\). This allows us to write, developing the constraint \([17b]\) (see, e.g., \([23]\) for a more rigorous treatment)

\[
\frac{df}{dt} (t) = -\int d\omega (k) \left| \frac{\lambda (\omega (k))}{\hbar} \right|^2 \int_0^t dt' e^{i(\omega_0 - \omega (k))(t-t')} f (t')
\]

\[
\simeq -\left| \frac{\lambda (\omega_0)}{\hbar} \right|^2 \int_0^t dt' \int d\omega (k) e^{i(\omega_0 - \omega (k))(t-t')} f (t')
\]

\[
= -\left| \frac{\lambda (\omega_0)}{\hbar} \right|^2 \int_0^t dt' 2\pi \delta (t - t') f (t')
\]

\[
= -\pi \left| \frac{\lambda (\omega_0)}{\hbar} \right|^2 f (t)
\]

(19)

where the second step is approximately valid on energy conservation grounds. Under certain assumptions, \(f\) appears as the solution of the first-order differential equation \([19]\). Together with \([18]\) this motivates the following Wigner-Weisskopf exponential ansatz (with \(\Lambda \equiv \Gamma + 2i\omega_{LS}\) a complex-valued number):

\[
\alpha (t) = e^{-i\omega_0 t} e^{-\frac{1}{2} \Lambda t}.
\]

(20)

This leads to (following a computation very similar to the one found in ref. \([2]\), sect. 2.5):

\[
\frac{1}{2} \Lambda = i \int d\omega (k) \left( \frac{|\lambda (\omega (k))|^2}{\hbar^2} \right) \frac{1 - e^{-i(\omega (k) - \omega_0)t} e^{i\frac{1}{2} \Lambda t}}{\omega_0 - \omega (k) - \frac{1}{2} \Lambda}.
\]

Collecting contributions from left- and right-outgoing modes, and in the \(\Lambda \rightarrow 0^+\) limit the Sochcki-Plemelj theorem \([3]\) (sect. 8.1) allows to write, defining the transition rate

\[
\Gamma \equiv 4\pi \left| \frac{\lambda (\omega_0)}{\hbar^2} \right|^2
\]

(21)

as well as the Lamb shift

\[
\omega_{LS} \equiv - \frac{2}{\hbar^2} \text{vp} \int d\omega (k) \frac{|\lambda (\omega (k))|^2}{\omega (k) - \omega_0},
\]

(22)

where \(\text{vp}\) denotes the Cauchy principal value of the subsequent integral, the following expressions \([2]\) (sect. 3.5) for \(\alpha\) (compare with \([19]\)) and \(\beta (k, \cdot)\):

\[
\alpha (t) = e^{-i(\omega_0 + \omega_{LS})t} e^{-\frac{1}{2} \Gamma t},
\]

(23a)

\[
\beta (k,t) = e^{-i\omega_0 t} \frac{\lambda (\omega (k))}{\hbar} \left( 1 - e^{-i(\omega_0 + \omega_{LS} - \omega (k))t} e^{-\frac{1}{2} \Gamma t} \right)
\]

(23b)
### 3.4 Wavefront propagation

It is straightforward to treat “relativistic” massless particles with our model (in the following we focus on bosonic particles) which allows us to derive in a simple fashion an expression for the spacetime dependence of outgoing wave packets in the Wigner-Weisskopf regime\(^2\). For massless particles these wave packets obey a linear dispersion relation \(\omega(k) = ck\). Let us now derive the expression for the “photonic wave function” in the outside region. It admits a natural interpretation in terms of the photodetection probability function established by Glauber, as is shown in ref. [4], section 6.3, in the case of spontaneous emission.

The calculation (for details see [3]) yields
\[
\psi_{\text{right/left}}(x,t) = -i\Pi_{\text{right/left}}(x)\Theta(ct-x)\lambda(\bar{\omega}_0) \sqrt{\frac{2\pi}{\hbar}} e^{\frac{i}{\hbar} \left(\bar{\omega}_0 - \frac{i\pi}{2c} |\lambda(\omega)|^2\right)}(x-ct) .
\]

Thus we see that the wave function is nonvanishing inside the light cone only. At fixed time, it increases exponentially with increasing \(x\) or \(-x\), depending on whether we look at what happens on the right or on the left of the inside region. The exponential growth is strongly reminiscent of what is obtained in the framework of Gamow functions (for massive particles) [20], as well as in the case of a non-quantum treatment of an open electromagnetic system (see [31] for more details). Although in the Friedrichs formalism, as far as we know, it is not common to give an explicit spacetime dependence for the decay products, this has been done in quantum optics [31] and also in the leaky mode treatment of Maxwell’s equations [31].

### 3.5 A Bohmian description of the trajectories

Let us reconsider the right and left outgoing wave packets (27) and interpret them as the outgoing photonic wave functions. Interestingly, a simple Bohmian picture holds for these outgoing waves. Indeed the outgoing right (resp. left) wave function defined by (24) obeys\(^3\) \(\partial \Psi / \partial t = \pm c \partial \Psi / \partial x\), so that we can associate to the “wave function” \(\Psi\) a conserved density \(\rho(x,t) = |\psi(x,t)|^2\) and a density current \(j(x,t) = \pm c \rho(x,t)\). The Bohmian velocity reads then \(v_{\text{Bohm}}(x,t) \equiv j(x,t) / \rho(x,t) = \pm c\), and we recover the old Einsteins picture of a photon moving at the speed of light in the outside region. Between the barriers, however, the particles are nearly at rest in the Bohmian picture, regardless of their mass, because the wave function is nearly stationary in that region. This analysis confirms our conclusion of sect. 2.2 in the bohmian picture, particles are at rest inside the barriers (cavity) and only begin to move once they leave it.

### 4 On the time evolution of coherent states

In cavity QED experiments, photons are kept in a “box” (superconducting cavity) during “macroscopic” times (130...
milliseconds in the case of [13], after which they are absorbed by the cavity walls. We shall now use the model defined in sect. 3 to study the behavior of coherent states in QED cavities, in particular their status of “classical pointer states”.

4.1 Decoherence and pointer states

The decoherence program is aimed, among others, at explaining why the macroscopic world does not (directly) obey quantum mechanical laws but instead behave classically [25]. It is based on the insight that quantum systems are never isolated [27], but that their evolution is at least partly monitored by their environment with which they continuously interact (as is put in [11] (sect. 4), “the environment is watching”). The decoherence program’s goal is to build on this insight to explain how the “classical” properties of the macroscopic world can be inferred from a purely quantum mechanical paradigm at the microscopic level. In this approach, one can show [24, 25] how the interactions of the macroscopic world can be inferred from a purely quantum mechanical paradigm at the microscopic level. They are singled out by their robustness to the interactions between the system and its environment. Since decoherence is, as is well-known, the corollary of entanglement [21, 28], pointer states of the system can be defined as the states which become minimally entangled with the environment in the course of their evolution [26]. In our model, the inside region plays the role of the system while the outside region mimics the environment.

4.2 Decoherence-free behavior as a consequence of bosonic undistinguishability

Let us write symbolically

$$|1_{\text{out}}\rangle \equiv \int d\omega (k) \beta (k, t)|\omega (k)_{\text{out}}\rangle.$$  (28)

We know from sect. 3.3 that an initially confined state $|1_{\text{in}}\rangle \otimes |0_{\text{out}}\rangle$ evolves into a superposition of the type

$$\alpha (t) |1_{\text{in}}\rangle \otimes |0_{\text{out}}\rangle + \beta (t) |0_{\text{in}}\rangle \otimes |1_{\text{out}}\rangle.$$  (31)

Thus, we can write the following time evolution for $n$ (undistinguishable) bosons which we label with the index $k$:

$$\bigotimes_{k=1}^{n} (|1_{\text{in}}\rangle_k \otimes |0_{\text{out}}\rangle_k)$$

$$\rightarrow \bigotimes_{k=1}^{n} (\alpha (t) |1_{\text{in}}\rangle_k \otimes |0_{\text{out}}\rangle_k + \beta (t) |0_{\text{in}}\rangle_k \otimes |1_{\text{out}}\rangle_k)$$

$$= \sum_{j=0}^{n} \alpha^j (t) \beta^{n-j} (t) \sqrt{\frac{n!}{j!(n-j)!}} |j_{\text{in}}\rangle \otimes |(n-j)_{\text{out}}\rangle,$$  (29)

where the normalized states $|j_{\text{in}}\rangle \otimes |(n-j)_{\text{out}}\rangle$ are defined by symmetrization over all possible states having $j$ particles in and $(n-j)$ particles out. The corresponding nonnormalized states read

$$[|j_{\text{in}}\rangle \otimes |(n-j)_{\text{out}}\rangle]_{\text{non}}$$

$$\equiv (|1_{\text{in}}\rangle \otimes |0_{\text{out}}\rangle) \otimes (|1_{\text{in}}\rangle \otimes |0_{\text{out}}\rangle) \otimes \cdots \otimes (|1_{\text{in}}\rangle \otimes |0_{\text{out}}\rangle)$$

$$+ \text{ all other combinations with } j \text{ in and } (n-j) \text{ out.}$$

There are $(n!/j!(n-j)!)$ such combinations, which means that

$$|j_{\text{in}}\rangle \otimes |(n-j)_{\text{out}}\rangle = \sqrt{\frac{j!(n-j)!}{n!}} [|j_{\text{in}}\rangle \otimes |(n-j)_{\text{out}}\rangle]_{\text{non}},$$  (30)

It is straightforward to see that

$$\bigotimes_{k=1}^{n} (\alpha (t) |1_{\text{in}}\rangle_k \otimes |0_{\text{out}}\rangle_k + \beta (t) |0_{\text{in}}\rangle_k \otimes |1_{\text{out}}\rangle_k)$$

$$= \sum_{j=0}^{n} \alpha^j (t) \beta^{n-j} (t) [|j_{\text{in}}\rangle \otimes |(n-j)_{\text{out}}\rangle]_{\text{non}}$$

which along with (30) proves (29), which could be labeled a “quantum binomial” law. In particular, if at time $t=0$ the cavity is prepared in a coherent state and the exterior is in its vacuum state, then the evolution generates a tensor product of an interior and an exterior coherent states:

$$|\psi (t=0)\rangle = e^{-\frac{\xi^2}{2}} \sum_{n=0}^{+\infty} \frac{\xi^n}{\sqrt{n!}} |n_{\text{in}}\rangle \otimes |0_{\text{out}}\rangle$$

$$\rightarrow |\psi (t)\rangle = e^{-\frac{\xi^2}{2} |\alpha(t)|^2} \sum_{n=0}^{+\infty} \sum_{m=0}^{n} \frac{(\xi \alpha (t))^m}{\sqrt{m!}} |m_{\text{in}}\rangle$$

$$\otimes e^{-\frac{\xi^2}{2} |\beta(t)|^2} \frac{(\xi \beta (t))^{n-m}}{\sqrt{(n-m)!}} |(n-m)_{\text{out}}\rangle$$  (31)
where we used the identity \(|\alpha(t)|^2 + |\beta(t)|^2 = 1\). The state at time \(t\) can be rewritten as

\[
|\psi(t)\rangle = e^{-\frac{|\xi|^2}{2}|\alpha(t)|^2} \sum_{k=0}^{+\infty} \frac{(\xi \alpha(t))^k}{\sqrt{k!}} |k_{\text{in}}\rangle \\
\otimes e^{-\frac{|\xi|^2}{2}|\beta(t)|^2} \sum_{m=0}^{+\infty} \frac{(\xi \beta(t))^m}{\sqrt{m!}} |m_{\text{out}}\rangle. \tag{32}
\]

This establishes that coherent states of the cavity interact with the exterior without getting entangled with it. They can thus be considered as “classical pointers” according to the criterion for classicality derived by Zurek in \([26]\). A common argument for the classicality of coherent states is based on the Lindblad master equation, which is a widely used dissipative (i.e., nonunitary) equation which models the coherence loss which a quantum system undergoes when it interacts with its environment. It governs the studied system’s reduced density matrix, and, in the case of a QED cavity (at zero temperature), it reads \([1]\) (sect. 7.5.1)

\[
\frac{d}{dt} \hat{\rho}_{\text{in}}(t) = \frac{1}{i\hbar} \left[ \hat{H}_{\text{in}}, \hat{\rho}_{\text{in}}(t) \right] \\
+ \frac{\Gamma}{2} \left[ 2\hat{a}\hat{\rho}_{\text{in}}(t) \hat{a}^\dagger - \hat{a}^\dagger \hat{a} \hat{\rho}_{\text{in}}(t) - \hat{\rho}_{\text{in}}(t) \hat{a}^\dagger \hat{a} \right]. \tag{33}
\]

In \([21]\) we proposed an alternate derivation of the Lindblad master equation. Our proof is based on the fact that any damped coherent state of the form

\[
|\xi_{\text{in}}(t)\rangle = e^{-\frac{|\xi|^2}{2}|\alpha(t)|^2} \sum_{m=0}^{+\infty} \frac{(|\xi \alpha(t)|^m)}{\sqrt{m!}} |m_{\text{in}}\rangle, \tag{34}
\]

with \(\alpha\) given by \([23]\) and \(\hat{H} = \hbar \omega \hat{a}^\dagger \hat{a}\), obeys the Lindblad equation \([33]\). Since coherent states constitute a basis of the Hilbert space (rigorously, they form an overcomplete basis), we find that, by virtue of the linearity of the master equation, any state is a solution of the Lindblad master equation.

### 4.3 Damped coherent states and the classical limit

It is sometimes argued that the Ehrenfest theorem establishes that quantum average values behave like classical quantities. This is wrong because, even if one can derive in a variety of situations an equation that looks classical at first sight, of the type \(m \frac{d^2 \langle x \rangle}{dt^2} = \langle F \rangle\), this equation is not equivalent to its classical counterpart in general. For instance in the case where the force derives from a potential, \(F = -\nabla V(r,t)\) and \(\langle F \rangle = -\langle \nabla V(r,t) \rangle\), but, generically, \(\langle \nabla V(r,t) \rangle \neq \nabla V(\langle r \rangle, t)\) and it is only in situations where we can neglect quantum fluctuations that classical mechanics can be inferred from quantum mechanics, which is in a sense a tautology. Damped coherent states however constitute an exception in the sense that whichever the state of a damped oscillator could be, we have \(m \frac{d^2 \langle x \rangle}{dt^2} = F(\langle x \rangle, t), \frac{d\langle x \rangle}{dt}\). This is easily established by estimating thanks to the master equation \([33]\) the temporal evolution of \(\langle \hat{a} \rangle\). Elementary computations establish that

\[
\frac{d \langle \hat{a} \rangle}{dt} = \left(-i\omega - \frac{\Gamma}{2}\right) \langle \hat{a} \rangle(t). \tag{35}
\]

After integration over time we get

\[
\langle \hat{a} \rangle(t) = \langle \hat{a} \rangle(t = 0) e^{-\frac{i\omega-\frac{\Gamma}{2}}{t}}.
\]

Making use of the fact that \(\langle x \rangle(t) = \langle \hat{a} \rangle(t) + \langle \hat{a}^\dagger \rangle(t)\), we get finally that \(\langle x \rangle(t) = A \cos(\omega t + \phi_0) \exp\left((-\Gamma/2) t\right)\), which is the most general solution of the dynamical equation of a classical-one dimensional-damped oscillator (that is, \(d^2x/dt^2 + \Gamma dx/dt + \omega^2 x = 0\), with \(\omega^2 \equiv \omega^2 + \Gamma^2/4\)). From this point of view, it is justified to describe the average behavior of a damped oscillator trapped in a QED cavity as a classical oscillator, and thus to resort to the Maxwell equations in order to study its properties, as we shall do in the next section. It is by the way particularly relevant to approximate the behavior of the quantum oscillator by its average behavior in the case of coherent states because they minimize Heisenberg uncertainties, which emphasizes their status of “quasi-classical states”.

### 5 Quasimodal analysis of a QED cavity

In this section we study the open QED cavity described in \([17]\) by searching for its eigenmodes and complex eigenfrequency, using a Finite Element Method (FEM). This classical electrodynamic approach allow us to derive a number of features observed experimentally by Haroche and coworkers \([18]\).

#### 5.1 Searching resonances of open resonators using perfectly matched layers

The major difficulty in the treatment of open problems in a numerical scheme based on a finite computational window
is to deal with infinity issues. Since their introduction by Bérenger in [35] for the time dependent Maxwell’s equations, Perfectly Matched Layers (PMLs) have become a widely used technique in computational physics. The idea is to enclose the area of interest by surrounded layers which are absorbing and perfectly reflectionless. These absorbing boundary conditions can be understood in the global framework of transformation optics ([38]). The principle of the technique is to perform a geometrical transformation (here a complex stretch of coordinates), leading to equivalent material properties ([5, 36, 37]).

The spectral problem we are dealing with consists in finding the solutions of source free Maxwell’s equations, i.e. finding complex eigenvalues $\Lambda_n = (\omega_n/c)^2$ and non zero eigenvectors $\vec{E}_n$ such that:

$$\mathcal{M}(\vec{E}_n) := \vec{\nabla} \times \left( \mu^{-1} \vec{\nabla} \times \vec{E}_n \right) = \Lambda_n \vec{\varepsilon} \cdot \vec{E}_n.$$  \hfill (36)

where $\varepsilon$ and $\mu$ are the relative dielectric permittivity and magnetic permeability tensors describing the electromagnetic properties of the system (cavity + external world).

For Hermitian open problems, the generalized spectrum of Maxwell’s operator $\mathcal{M}$ is real and composed of two parts:
- the discrete spectrum with trapped modes exponentially decreasing at infinity, and the continuous spectrum with radiation modes oscillating at infinity. In addition, another type of solution is present and very useful to characterize the spectral properties of unbounded structures: the so-called leaky modes (also termed quasimodes, quasi normal modes) of quasi bounded structures ([5, 39]) or quasi guided modes ([10]) in the literature). These eigenmodes with complex associated frequency are an intrinsic feature of open waveguides.

PMLs have proven to be a very convenient tool to compute leaky modes in different configurations ([41, 42]). Indeed they efficiently mimic the infinite space provided a suitable choice of their parameters. The introduction of infinite PMLs rotates the continuous spectrum in the complex plane (since the operator involved in the problem is now a non self-adjoint extension of the original self-adjoint operator). The effect is not only to turn the continuous spectrum into complex values but it also unveils the leaky modes in the region swept by the rotation of this essential spectrum ([9]). It is important to note that leaky modes do not depend on the choice of a particular complex stretching: adding the PMLs is only a way to discover them. Finally, in order to apply the FEM, the PMLs have to be truncated at finite distance which results in an operator having only point spectrum with approximate radiation modes (also termed as PML modes or Bérenger modes) due to the discretization of the continuous spectrum by finite PMLs ([43]). Actually, the PML technique is analog to the theory of analytic dilation ([41, 45]) developed in the 1970s for Schrödinger Hamiltonians.

In the sequel, eigenvalues are denoted $\omega_n = \omega_n' + i\omega_n''$. The real part is the resonant angular frequency $\omega_n' = 2\pi f_n$ and the imaginary part is the damping coefficient, which is related to the lifetime $\tau_n$ of the photon in the cavity by $\omega_n'' = 2\pi/\tau_n$. The quality factor associated to a resonance is defined by $Q_n = \omega_n''/(2\omega_n')$.

### 5.2 Numerical results

#### 5.2.1 Numerical set up

A detailed description of the cavity can be found in [17]. It is composed of two mirrors of diameter $D = 50\text{mm}$ facing each other. The distance between their apexes is $L = 27.57\text{mm}$, and their surface is toroidal with radii of curvature $r = 39.4\text{mm}$ in the $Oxz$ plane and $R = 40.6\text{mm}$ in the $Oyz$ plane. The mirrors are coated with a thick layer of superconducting niobium. We take advantage of the symmetries of the problem and thus model only one eighth of the cavity to save memory and computation time. By setting a well chosen combination of Neumann and Dirichlet boundary conditions on the cutting planes, we can select the modes with desired symmetries. The eigenproblem defined by (36) is then solved by the FEM, using cartesian PMLs terminated by homogeneous Neumann boundary conditions to truncate the infinite space. The computational cell is meshed using second order edge elements, with a maximum size of an element set to $L^2/((9\pi\varepsilon N)$, where $L = 3.68\text{cm}$ is the approximate resonant wavelength of the cavity, and $N$ is an integer ($N = 7$ for the domain inside of the cavity, $N = 7$ for the domain outside the cavity, $N = 5$ for the PMLs and $N = N_m$ for the mirror surfaces). The final algebraic system is solved using a direct solver (PARDISO).

In a first approximation, both mirrors are assumed to be lossless, thus we set Perfect Electric Conductor boundary conditions (PEC, i.e. Dirichlet boundary conditions). In a second step, in order to account for losses, absorption is considered through a Surface Impedance Boundary Condition (SIBC) on the boundaries of both mirrors.
5.2.2 A first approach: Perfect mirrors

Let us consider first a lossless model, i.e., perfect mirrors. Among many others, we find only two eigenfrequencies with exceptionally low imaginary parts (or high associated lifetimes). Their real parts correspond to frequencies $f_1$ and $f_2$ located around 51.085 GHz (to be compared to 51.099 GHz found experimentally in [17]), with $\tau = f_1 - f_2 \approx 1.2$ MHz (same as in [17]). According to the well known Fabry-Perot interferometry principle, a modification of the distance between the mirrors would allow to retrieve the experimental frequency, as shown in the following section. The small discrepancy between the two complex eigenfrequencies is due to the removed degeneracy induced by the slight cylindrical symmetry breaking of cavity ($r \neq R$). The mode labelled 1 with higher frequency is $x$-polarized while mode 2 with lowest frequency is $y$-polarized, and they both have nine antinodes along $Oz$ (modes TEM$_{900}$, see the field maps on Fig. 5).

We study the convergence of the eigenfrequencies as a function of the mesh refinement. A satisfying convergence is obtained with increasing $N_m$ (Fig. 6a) for the real part of the frequencies, while all that can be said about lifetimes (Fig. 6b) is that they have an order of magnitude of a few seconds. This non-convergence is due to the fact that the resonances features are in that case dominated by the geometry of the mirrors surface, which is approximated by tetrahedral elements, thus not exactly reproducing the smooth curvature of the toroidal surfaces. To be more specific, Haroche et al. fabricated two cavities $M_1$ and $M_2$. They measured a fabricated maximum peak to valley deviation from the ideal toroidal mirror shape of only 300 nm. We can easily estimate a numerical maximum peak to valley deviation of our virtual tessellated mirrors. The 3D mesh is made of tetrahedra, so the surfaces of interest of each mirror are discretized into a set of triangles of side lengths set to a maximal value of $\lambda/|\text{Re}\varepsilon| N_M$. Let us consider an equilateral triangle $T$ of side length $\lambda/|\text{Re}\varepsilon| N_M$ whose vertices belong to a sphere of radius $r$. When approximating the corresponding piece of sphere by this triangle $T$, the maximum deviation from the original shape is the minimum distance between the barycenter of $T$ and the sphere. For $N_M = 22$, this characteristic length is approximately 300 nm, which corresponds to the experimental value. Our numerical mirrors are globally closer from the exact toroidal shape than the experimental ones. However, numerically taking into account the remaining local surface roughness of typically 10 nm is not only nu-
merically unthinkable but also irrelevant (details of $\lambda_r$ over half a million). Finally, the lossless PEC model represents a cavity so resonant that even a tiny change in a fine (setting $N_M$ to 60) leads to a maximum deviation from the exact shape of only 40 nm, i.e. $\lambda_r/1500000$! Discretization leads to a significantly different lifetime. In spite of the non-convergence of this first model, the order of magnitude of the lifetime is far greater than lifetimes found experimentally (seconds instead of 100 ms in [17]). A first conclusion can be drawn: Assuming a perfect geometry (perfect mirrors with ideal shape and alignment), losses are not dominated by radiation loss. We now try to give an estimate of the effective resistive losses. From a numerical point of view, including some friction mechanism is expected to benefit to the model convergence.

### 5.2.3 A more realistic approach: Lossy mirrors

To that extent, we now apply Surface Impedance Boundary Conditions (SIBC) $Z_s = X_s + iY_s$ on the mirrors faces. London penetration depth for niobium $L_L$ (independent of frequency) is set to a typical value of $0.1 \mu m$ [6]. In the framework of a two fluids model, at low temperature, the imaginary (inductive) part of the impedance can be approximated by $Y_s = \omega \mu_0 L_L = 6.4 \mu \Omega$. As for the real (resistive) part of the impedance, it is extremely difficult to measure and greatly depends on numerous experimental conditions. Therefore, our only option is to estimate this parameter to obtain lifetimes of the same order of magnitude as measured in [17]. Finally, we adjusted the length of the cavity to find resonant frequencies closer to those measured in [17].

With the updated value of $L = 27.562 \text{mm}$ (instead of $27.57 \text{mm}$ in [17]), the system exhibits two resonant frequencies $f_1 = 51.0984 \text{GHz}$ and $f_2 = 51.0997 \text{GHz}$ ($\tau = f_1 - f_2 \approx 1.29 \text{MHz}$). As in the PEC model, the convergence as function of mesh refinement is reached (see Fig. 7a) for $N_m = 40$.

Moreover, the discrepancy between lifetimes when refining the mesh size on the mirror is greatly reduced compared to the PEC case (see Fig. 7b). With $X_s = 1 \mu \Omega$, we obtain lifetimes of $\sim 100 \text{ms}$ for both modes, which corresponds to the average lifetime found in [17] for both cavities: $112 \pm 4 \text{ms}$ (LF mode) and $87 \pm 10 \text{ms}$ (HF mode) for $M_1$, $74 \pm 6 \text{ms}$ (LF mode) and $130 \pm 4 \text{ms}$ (HF mode) for $M_2$. This value of $X_s$ should be seen as an upper bound of the resistive phenomenon. It includes all loss processes beyond radiation loss: roughness of the mirrors, superconductor imperfections...

Finally, it is indeed much easier to “align the cavity numerically” than experimentally! The fact that the discrepancy between lifetimes corresponding to the two cavity modes is greater experimentally than numerically tends to indicate a residual tilt (3 Euler angles) between the two mirrors.

### 6 Conclusions and open questions.

Many ideas exploited by us in the present paper are already present in classical quantum optics textbooks. For instance it is well known that, when an oscillator is linearly coupled to an arbitrary number of oscillators, coherent states factor
out during the temporal evolution. In other words, coherent states are the quasi-classical pointer states of the problem. However our approach is slightly different because, among others, we treat the problem in a first quantization approach. This has as a consequence that it could be generalised to slow, massive particles, cold bosonic atoms for instance. It also sheds a new light onto certain “weird” features of quantum optics like the so-called Hong-Ou-Mandel two-photon interference effect [29], which is usually presented in terms of annihilation and creation operators. Roughly speaking, the effect is the following: whenever two indistinguishable photons are sent along two arms of a balanced (50-50) beamsplitter (schematized by the following diagram: \( \frac{A^{\text{out}}}{B^{\text{out}}} \times \frac{A^{\text{in}}}{B^{\text{in}}} \)), they interfere constructively in the outgoing arms so that both photons belong to the same outgoing arm. Schematically, the beamsplitter acts as follows: it sends an ingoing state \( |1_{A}^{\text{in}}, 1_{B}^{\text{in}}\rangle \) onto the state \( \left( \frac{|2_{A}^{\text{out}}, 0_{B}^{\text{out}}\rangle + |0_{A}^{\text{out}}, 2_{B}^{\text{out}}\rangle}{\sqrt{2}} \right) \), where the states \( |i_{A}\rangle (|i_{B}\rangle) \) \((i = 1, 2)\) are generated by letting act the creation operators \( a_{A}^{+} (a_{B}^{+})\) \(i\) times for the \( A\) (\( B\)) mode respectively.

In our approach, the initial state is the properly symmetrized two boson state

\[
\frac{1}{\sqrt{2}} \left( |1_{A}^{\text{in}}, 0_{B}^{\text{in}}\rangle_{A} |0_{A}^{\text{in}}, 1_{B}^{\text{in}}\rangle_{B} + |0_{A}^{\text{in}}, 1_{B}^{\text{in}}\rangle_{A} |1_{A}^{\text{in}}, 0_{B}^{\text{in}}\rangle_{B} \right).
\]

As described in [1], the passage through a beamsplitter can be modeled by the coherent exchange of quanta between the two modes \( A\) and \( B\). In the present case, its effect resorts, in the case of a balanced beamsplitter, for which the interaction time is the fourth of the Rabi period, to the transformation law [21]

\[
|1_{A}^{\text{in}}, 0_{B}^{\text{in}}\rangle \rightarrow \left( |1_{A}^{\text{out}}, 0_{B}^{\text{out}}\rangle + |0_{A}^{\text{out}}, 1_{B}^{\text{out}}\rangle \right) / \sqrt{2},
\]

\[
|0_{A}^{\text{in}}, 1_{B}^{\text{in}}\rangle \rightarrow \left( |1_{A}^{\text{out}}, 0_{B}^{\text{out}}\rangle - |0_{A}^{\text{out}}, 1_{B}^{\text{out}}\rangle \right) / \sqrt{2}.
\]

The passage through the beamsplitter is thus expressed, in this first quantization scheme, by the transformation law

\[
\frac{1}{2 \sqrt{2}} \left[ \left( |1_{A}^{\text{out}}, 0_{B}^{\text{out}}\rangle + |0_{A}^{\text{out}}, 1_{B}^{\text{out}}\rangle \right)^{\alpha} \left( |1_{A}^{\text{out}}, 0_{B}^{\text{out}}\rangle - |0_{A}^{\text{out}}, 1_{B}^{\text{out}}\rangle \right)^{\beta} \right.
\]

\[
+ \left( |1_{A}^{\text{out}}, 0_{B}^{\text{out}}\rangle - |0_{A}^{\text{out}}, 1_{B}^{\text{out}}\rangle \right)^{\alpha} \left( |1_{A}^{\text{out}}, 0_{B}^{\text{out}}\rangle + |0_{A}^{\text{out}}, 1_{B}^{\text{out}}\rangle \right)^{\beta} \right]
\]

\[
= \frac{1}{\sqrt{2}} \left( |1_{A}^{\text{out}}, 0_{B}^{\text{out}}\rangle_{A} |1_{A}^{\text{out}}, 0_{B}^{\text{out}}\rangle_{B} - |0_{A}^{\text{out}}, 1_{B}^{\text{out}}\rangle_{A} |0_{A}^{\text{out}}, 1_{B}^{\text{out}}\rangle_{B} \right).
\]

The effect is obviously exactly the same as the Hong-Ou-Mandel effect, but no increase of entanglement appears during the beamsplitting process: the initial and final states are both maximally entangled Bell states in this

---

**Figure 7:** Convergence of the eigenfrequencies with mesh parameter \( N_{m} \) for modes 1 and 2: (a) resonant frequency \( f \); (b) lifetime \( \tau \).
picture, contrarily to what happens in the second quantization formulation where the initial state is a factorizable state. Although this program lies out of the scope of the present work, it would be highly interesting to figure out to which extent our alternative, first quantized, formulation of quantum optics could be pursued.

Another non-standard feature of our treatment is that we never introduce the concept of quantum jump. This contrasts for instance with a popular approach for deriving the master equation of a lossy QED cavity (33), the so-called Monte-Carlo approach [1, 30] according to which (here we consider the temperature of the “environment”, that is to say of the modes outside the cavity, to be equal to zero), after an infinitesimally small time increase \( dt \), either the mode inside the cavity loses one photon (with probability \( \Gamma d t \)), or it remains unaffected by the absorption (with probability \( 1 - \Gamma d t \)). This random process can be simulated as a Monte-Carlo process and it leads to the derivation of the master equation (33) after averaging on the two possible outcomes [21] (either a detector ‘clicks’ and reveals the presence of an outgoing photon or it does not click). Certain features of this model are somewhat puzzling. For instance, in the case where the initial state of the quasi-stationary mode inside the cavity is a coherent state, it disentangles from the outside world throughout its temporal evolution, and it is difficult to figure out how the measurement of a photon outside the cavity would affect the state of the mode inside the cavity, even at the mere informational level, because the corresponding modes are totally decorrelated.

In contrast, our approach makes it possible to provide a coherent, unitary, description of the evolution of the full system (modes inside AND outside the cavity). In our view, decoherence (which manifests itself through the Lindblad equation (33)) is simply seen to be a corollary of entanglement and during the whole process, the state of the full system (cavity + external world outside the boundaries) remains pure and evolves unitarily. Coming back to the famous Schrödinger cat problem, where a cat trapped in a box is prepared into a superposition of a living and of a dead cat, our approach makes it possible to give a complete description of the dead-living cat AND of its direct environment, inside the box. In this view, it is only when we open the box that the superposition breaks down.

Besides these foundational issues, the classical techniques that we used for estimating the survival time of a photon inside the cavity (sect. 5) confirm that classical electromagnetism is sufficient to obtain the main features of the losses inside the cavity. For instance our numerical simulations also predict that the two longest-lived modes inside the cavity are characterized by orthogonal polarizations and possess 9 nodes, and they provide a quantitatively correct estimation of their frequencies as well as of their frequency separation.

As we have explained in sect. 5.2 our estimations are more qualitative than quantitative for what concerns the life times of these modes but at least we derive a correct order of magnitude in that case. Even then, our numerical treatment delivers some information relatively to the losses of the cavity, information that would be difficult to gather by other methods. For instance, our analysis suggests that the main source of losses is not the escape of the photons to the free space which surrounds the cavity but is rather due to the absorption by the mirrors. Similarly, it reveals that the roughness is not a weakness of the cavity regarding losses but that the alignment of the mirrors is a crucial parameter.

Finally, our results outlined in sections sects. 2, 3.4 and 3.5 suggest to abandon the naive classical picture of the photon that is represented as a pointlike particle which oscillates back and forth between the mirrors and to replace it by the more appropriate picture where the photon inside the cavity is associated to a quasi-stationary standing wave trapped between the two mirrors.

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