On the effective size of a non-Weyl graph

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Abstract

We show how to find the coefficient of the leading term of the resonance asymptotics using the method of pseudo-orbit expansion for quantum graphs which do not obey Weyl asymptotics. For a non-Weyl graph we develop a method to reduce the number of edges of a corresponding directed graph. Through this method we prove bounds on the above coefficient depending on the structure of the graph, for graphs with the same lengths of internal edges. We explicitly find the positions of the resolvent resonances.

Keywords: quantum graphs, resonances, Weyl asymptotics

1. Introduction

Quantum graphs have been intensively studied, mainly in the last thirty years. There is an extensive literature to refer to, e.g. the book [BK13], the proceedings [AGA08], and references therein. One of the models studied is quantum graphs with attached semi-infinite leads. For this model, the notion of resonances can be defined; there are two main definitions: resolvent resonances (singularities of the resolvent) and scattering resonances (singularities of the scattering matrix). For the study of resonances in quantum graphs we refer the reader to, e.g., [ES94, Exn13, Exn97, KS04 BSS10, EL07, EL10].

The problem of finding resonance asymptotics in quantum graphs has been addressed in [DEL10, DP11, EL11]. A surprising observation by Davies and Pushnitski [DP11] shows that a graph has in some cases fewer resonances than one would expect from Weyl asymptotics. Criteria which can distinguish these non-Weyl graphs from the graphs with regular Weyl asymptotics have been presented in [DP11] (graphs with standard coupling) and in [DEL10] (graphs with general coupling). Although distinguishing between these two cases is quite easy (it depends on the vertex properties of the graph), finding the leading term of the asymptotics, which is closely related to the ‘effective size’ of the graph, is more difficult since it uses the structure of the whole graph.
In the paper we express the first term of non-Weyl asymptotics using the method of pseudo-orbits and find bounds on the effective size that depend on the structure of the graph. The paper is divided into the following sections: in section 2 we introduce the model itself, in section 3 we state known theorems on the asymptotics of the resonances, in section 4 we develop the method of pseudo-orbit expansion for the resonance condition, in section 5 we state how the effective size can be found for an equilateral graph (a graph with the same lengths of the internal edges), in section 6 we develop a method that allows us to delete some of the edges of a non-Weyl graph, in section 7 we state the main theorems on the bounds on the effective size and position of the resonances for equilateral graphs, and finally section 8 illustrates the results and the developed method using three examples.

2. Preliminaries

First, we describe the main notions. We consider a metric graph $\Gamma$ that consists of the set of vertices $V$, the set of $N$ finite internal edges $E_i$ of lengths $\ell_i$ (these edges can be parametrized by intervals $(0, \ell_i)$), and the set of $M$ semi-infinite external edges $E_e$ (parametrized by intervals $(0, \infty)$). The graph is equipped by the second order differential operator $H$ that acts as $-\frac{d^2}{dx^2}$ on internal and external edges. The domain of this operator consists of functions with edge components in Sobolev space $W^{2,2}$, and at the same time satisfying the coupling conditions at the vertices

$$(U_j - I)\psi_j + i(U_j + I)\psi'_j = 0,$$

where $U_j$ is a $d_j \times d_j$ unitary matrix ($d_j$ is the degree of the vertex), $I$ is a $d_i \times d_i$ identity matrix, $\psi_j$ is the vector of limits of functional values to the vertex from attached edges and, similarly, $\psi'_j$ is the vector of limits of outgoing derivatives. This general form of the coupling was independently introduced by Kostrykin and Schrader [KS00] and Harmer [Har00].

A trick is shown in [EL10, Kuc08] for how to describe the coupling on the whole graph by one large $(2N + M) \times (2N + M)$ unitary matrix $U$ which is similar to a block diagonal matrix with blocks $U_j$ (Square matrices $A$ and $B$ are similar if there exists a regular square matrix $P$ such that $A = PBP^{-1}$). This matrix encodes not only the coupling but also the topology of the graph.

$$(U - I)\psi + i(U + I)\psi' = 0.$$  

In the previous equation $I$ is a $(2N + M) \times (2N + M)$ identity matrix, and the vectors $\psi$ and $\psi'$ consist of entries $\psi_j$ and $\psi'_j$. This coupling condition corresponds to a graph where all the vertices are joined into one vertex.

One can describe the effective coupling on the compact part of the graph using energy-dependent $2N \times 2N$ coupling matrix $\tilde{U}(k)$, where $k$ is the square root of energy. There is a straightforward way [EL10] to construct this effective coupling matrix using a standard method named Schur complement (see e.g. [Zha05]),

$$\tilde{U}(k) = U_1 - (1 - k)U_2[(1 - k)U_4 - (1 + k)I]^{-1}U_3,$$

where the matrices $U_1, ..., U_4$ are blocks of the matrix $U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}$. $U_1$ corresponds to coupling between internal edges, $U_4$ corresponds to coupling between external edges, and $U_2$
and $U_3$ correspond to mixed coupling. The coupling condition has a similar form to (2) with $U$ replaced by $\tilde{U}(k)$ and $I$ now representing a $2N \times 2N$ identity matrix.

3. Asymptotics of the number of resonances

We are interested in the asymptotical behavior of the number of resolvent resonances of the system. The resolvent resonance is usually defined as the pole of the meromorphic continuation of the resolvent $(H - \lambda \text{id})^{-1}$ into the second sheet. The resolvent resonances can be obtained by the method of complex scaling (see [EL07] for further details). We use a simpler but equivalent definition.

**Definition 3.1.** We call $\lambda = k^2$ a resolvent resonance if there exists a generalized eigenfunction $f \in L^2(\Gamma)$, $f \not\equiv 0$ satisfying $-f''(x) = k^2 f(x)$ on all edges of the graph and fulfilling the coupling conditions (1) that on all external edges behaves as $c_\ell e^{ik\ell}$. This definition is equivalent to the previously mentioned one because these generalized eigenfunctions (which, of course, are not square integrable), become square integrable after complex scaling. It was proven in [EL07] that the set of resolvent resonances is equal to the union of the set of scattering resonances, and the eigenvalues with the eigenfunctions support only on the internal part of the graph.

We define the counting function $N(R)$ that counts the number of all resolvent resonances including their multiplicities contained in the circle of radius $R$ in the $k$-plane. One should note that using this method we count twice as many resonances than in the energy plane. This is clear for the case of a compact graph because $k$ and $-k$ correspond to the same eigenvalue.

The asymptotics of the resonances in most of the quantum graphs obeys Weyl’s law—equation (4) with $W$ equal to the sum of the lengths of the internal edges, i.e. $\text{vol } \Gamma$. However, there exist such quantum graphs for which the leading term of the asymptotics is smaller than expected; we call these graphs non-Weyl. The problem was studied for graphs with standard coupling (in the literature also called Kirchhoff, free, or Neumann coupling) in [DP11], and for graphs with general coupling in [DEL10]. We define the standard coupling and state main the results presented in these papers.

**Definition 3.2.** For the standard coupling functional values are continuous at each vertex and the sum of outgoing derivatives is equal to zero, i.e. for the vertex $v$ with degree $d$ we have

$$f_j(v) = f_j(v) \equiv f(v), \quad \forall i, j \in 1, ..., d,$$

$$\sum_{j=1}^d f_j'(v) = 0.$$  

**Theorem 3.3 (Davies and Pushnitski).** Let us assume a graph $\Gamma$ with standard coupling at all vertices and the sum of the lengths of all internal edges equal to $\text{vol } \Gamma$. Then the number of resolvent resonances has the following asymptotics

$$N(R) = 2 \frac{W}{\pi} R + O(1), \quad \text{as } R \to \infty,$$

where $0 \leq W \leq \text{vol } \Gamma$. One has $W < \text{vol } \Gamma$ iff there exists at least one balanced vertex, i.e. a vertex that connects the same number of internal and external edges.
Theorem 3.4 (Davies, Exner and Lipovský). Let us assume a graph \( \Gamma \) with general coupling (2) and with the sum of the internal edges equal to \( \text{vol} \Gamma \). The number of resolvent resonances has the asymptotics

\[
N(R) = \frac{2}{\pi} WR + \mathcal{O}(1), \quad \text{as } R \to \infty,
\]

where \( 0 \leq W \leq \text{vol} \Gamma \). One has \( W < \text{vol} \Gamma \) (the graph is non-Weyl) iff the effective coupling matrix \( \tilde{U}(k) \) has at least one eigenvalue equal to either \( \frac{1+k}{1-k} \) or \( \frac{1-k}{1+k} \) for all \( k \).

While it is quite easy to find whether a graph is Weyl or non-Weyl (it depends only on the vertex properties of the graph), determining the coefficient \( W \) (the effective size of the graph) is more complicated, because it depends on the properties of the whole graph. It is illustrated, e.g., in theorem 7.3 in [DEL10].

4. Pseudo-orbit expansion for the resonance condition

A theory has been developed that shows how to find the spectrum of a compact quantum graph using pseudo-orbit expansion (see e.g. [BHJ12, KS99]). In this section we adjust this method to find the resonance condition. The idea is to find the effective vertex-scattering matrix that acts only on the compact part of the graph. The vertex-scattering matrix maps the vector of the amplitudes of the incoming waves to the vertex, into the vector of the amplitudes of the outgoing waves. We show that in the case of standard coupling this matrix is not energy-dependent and has a nicely arranged form.

Definition 4.1. Let us assume a vertex \( v \) of the graph \( \Gamma \). Let there be \( n \) internal edges emanating from \( v \), all parametrized by \((0, \ell_j)\) with \( x = 0 \) corresponding to \( v \), and \( m \) halflines. Let the solution of the Schrödinger equation on these internal edges be

\[
\psi_i(x) = a_j^i e^{-ikx} + \tilde{a}_j^i e^{ikx},
\]

\( j = 1, \ldots, n \) and on the external edges \( g_s(x) = \beta_s e^{ikx}, s = 1, \ldots, m \). Then the \( n \times n \) effective vertex-scattering matrix \( \tilde{\sigma}^{(v)} \) is given by the relation \( \tilde{\sigma}^{(v)} \tilde{a}_v^i = \tilde{\sigma}^{(v)} \tilde{a}_v^o \), where \( \tilde{a}_v^i \) and \( \tilde{a}_v^o \) are vectors with entries \( a_j^i \) and \( \tilde{a}_j^o \), respectively.

For the next theorem we drop the superscript or subscript \( v \) denoting the vertex in the coupling and vertex-scattering matrices.

Theorem 4.2 (a general form of the effective vertex-scattering matrix). Let us consider a non-compact graph \( \Gamma \) with the coupling at the vertex \( v \) given by (1) and the matrix \( U \). Let the vertex \( v \) connect \( n \) internal edges and \( m \) external edges. Let the matrix \( I \) be an \((n + m) \times (n + m)\) identity matrix and by \( I_n \) we denote the \( n \times n \) identity matrix and by \( I_m \) the \( m \times m \) identity matrix. Then the effective vertex-scattering matrix is given by

\[
\tilde{\sigma}(k) = -[(1-k)\tilde{U}(k) - (1+k)I_n]^{-1}[(1+k)\tilde{U}(k) - (1-k)I_n].
\]

The inverse relation is \( \tilde{U}(k) = [(1+k)\tilde{\sigma}(k) + (1-k)I_n] [(1-k)\tilde{\sigma}(k) + (1+k)I_n]^{-1} \).

Proof. Let the wavefunction components on the internal edges emanating from the vertex \( v \) be \( f_j(x) = a_j^i e^{-ikx} + \tilde{a}_j^o e^{ikx} \) and on the external edges \( g_s(x) = \beta_s e^{ikx} \). The vector of the functional values is therefore \( \Psi = \left( \tilde{a}_v^i \tilde{a}_v^o \right) / \beta \) and the vector of outgoing derivatives is
\[ \Psi' = ik\left( -\bar{\alpha}^{\text{in}} + \bar{\alpha}^{\text{out}} \right) \]. The coupling condition (1) gives

\[ (U - I) \left( \bar{\alpha}^{\text{in}} + \bar{\alpha}^{\text{out}} \right) + i k (U + I) \left( -\bar{\alpha}^{\text{in}} + \bar{\alpha}^{\text{out}} \right) = 0. \]

Hence we have the set of equations

\[ [U_i - I_n - k(U_i + I_n)]\bar{\alpha}^{\text{out}} + [(U_i - L_n) + k(U_i + I_n)]\bar{\alpha}^{\text{in}} + (1 - k)U_i \bar{\beta} = 0, \]

\[ (1 - k)U_j \bar{\alpha}^{\text{out}} + (1 + k)U_j \bar{\alpha}^{\text{in}} + [(U_i - I_m) - k(U_i + I_m)]\bar{\beta} = 0. \]

Expressing \( \bar{\beta} \) from the second equation and substituting it in the first one has

\[ \{ (1 - k)U_i - (1 + k)I_n - (1 - k)U_j [(1 - k)U_i - (1 + k)I_m]^{-1} (1 - k)U_j \} \bar{\alpha}^{\text{out}} + \{ (1 + k)U_i - (1 - k)I_n - (1 - k)U_j [(1 - k)U_i - (1 + k)I_m] (1 + k)U_j \} \bar{\alpha}^{\text{in}} = 0, \]

from which using (3) the claim follows. Expressing the inverse relation is straightforward. \( \square \)

Using the previous theorem one can straightforwardly compute the effective vertex-scattering matrix for standard conditions.

**Corollary 4.3.** Let \( \nu \) be a vertex with \( n \) internal and \( m \) external edges with standard coupling conditions (i.e. \( U = \frac{2}{n + m} I_{n+m} - I_{n+m} \), where \( I_n \) denotes an \( n \times n \) matrix with all entries equal to one). Then the effective vertex-scattering matrix is 
\[ \bar{\sigma}(k) = \frac{2}{n + m} I_{n} - I_n, \]

in particular, for a balanced vertex we have 
\[ \bar{\sigma}(k) = \frac{1}{n} I_n. \]

**Proof.** There are two ways to prove this corollary. The first is to compute the effective vertex-scattering matrix from the definition in a similar way to the proof of theorem 4.2. The second, to some extent longer but more straightforward, uses the theorem 4.2 itself. Using the formula 
\[ (aI_n + bE_n)^{-1} = \frac{1}{b} \left( -\frac{a}{a + b} I_n + I_n \right) \]

we compute the effective coupling matrix and obtain 
\[ \bar{U}(k) = \frac{2}{m + n} I_n - I_n. \]

Substituting it into the formula in theorem 4.2 and using the above expression of the inverse matrix, and recalling the fact that \( I_n \cdot I_n = n I_n \), one obtains the result. \( \square \)

Having defined the effective vertex-scattering matrix, we can proceed to the lines of the method shown in [BHJ12]. The only difference is that the vertex-scattering matrix \( \sigma^{(v)} \) introduced in [BHJ12] is replaced by the effective vertex-scattering matrix \( \bar{\sigma}^{(v)} \). We replace the graph \( \Gamma \) by a compact oriented graph \( \Gamma_2 \), which is constructed from the compact part of \( \Gamma \) by the following rule: each internal edge \( e_j \) of the graph \( \Gamma \) is replaced by two oriented edges (bonds) \( \hat{b}_j, \hat{\hat{b}}_j \) of the same length as \( e_j \); the oriented edges \( \hat{b}_j \) and \( \hat{\hat{b}}_j \) have the opposite orientations. On these edges we use the ansatz

\[ f_{\hat{b}_j}(x) = \alpha^{\text{in}}_{\hat{b}_j} e^{-ikx_{\hat{b}_j}} + \alpha^{\text{out}}_{\hat{b}_j} e^{ikx_{\hat{b}_j}}, \]

\[ f_{\hat{\hat{b}}_j}(x) = \alpha^{\text{in}}_{\hat{\hat{b}}_j} e^{-ikx_{\hat{\hat{b}}_j}} + \alpha^{\text{out}}_{\hat{\hat{b}}_j} e^{ikx_{\hat{\hat{b}}_j}}, \]

where \( x_{\hat{b}_j} \) is the coordinate on the bond \( \hat{b}_j \) and \( x_{\hat{\hat{b}}_j} \) is the coordinate on the bond \( \hat{\hat{b}}_j \). Since there is a relation \( x_{\hat{b}} + x_{\hat{\hat{b}}} = \ell_j \), with \( \ell_j \) being the lengths of the bond \( \hat{b}_j \) or \( \hat{\hat{b}}_j \), and since the functional values for both directed bonds must correspond to each other
\[ f_h(x_h) = f_b(t_x - x_b), \] we obtain the following relations between coefficients
\[ \alpha_{b_j}^{in} = e^{i\ell_{b_j}} \alpha_{b_j}^{out}, \quad \alpha_{b_j}^{in} = e^{i\ell_{b_j}} \alpha_{b_j}^{out}. \] (5)

Now we define several matrices that we will use later.

**Definition 4.4.** The matrix \( \tilde{\Sigma}(k) \) (which is in general energy-dependent) is a matrix that is similar to a block diagonal matrix with blocks \( \tilde{\sigma}_r(k) \). The similarity transformation is transformation from the basis \( (\mathbf{a}^{in}, \ldots, \mathbf{a}^{in}_{b_1}, \mathbf{a}^{in}_{b_2}, \ldots) \) to the basis \( (\mathbf{a}^{in}_{b_1}, \ldots, \mathbf{a}^{in}_{b_1}, \mathbf{a}^{in}_{b_2}, \ldots) \),

where \( b_{rij} \) is the \( j \)th edge ending in the vertex \( v_1 \).

Furthermore, we define \( Q(k) = \begin{pmatrix} 0 & I_N \\ I_N & 0 \end{pmatrix} \), scattering matrix \( S(k) = Q \tilde{\Sigma}(k) \), and \( L = \text{diag}(\ell_1, \ldots, \ell_N, \ell_1, \ldots, \ell_N) \).

Using these matrices we can state the following theorem.

**Theorem 4.5.** The resonance condition is given by
\[ \det(e^{i\ell}Q \tilde{\Sigma}(k) - I_{2N}) = 0. \]

**Proof.** If we define \( \tilde{\alpha}_{b_j}^{in} = (\alpha_{b_j}^{in}, \ldots, \alpha_{b_j}^{in}) \) and similarly for the outgoing amplitudes and bonds in the opposite direction \( b_j \), we can subsequently obtain
\[ \begin{pmatrix} \tilde{\alpha}_{b_j}^{in} \\ \tilde{\alpha}_{b_j}^{out} \end{pmatrix} = e^{i\ell_{b_j}} \begin{pmatrix} \tilde{\alpha}_{b_j}^{out} \\ \tilde{\alpha}_{b_j}^{in} \end{pmatrix} = e^{i\ell_{b_j}} Q \tilde{\Sigma}(k) \begin{pmatrix} \tilde{\alpha}_{b_j}^{out} \\ \tilde{\alpha}_{b_j}^{in} \end{pmatrix}. \]

First we used relations (5) and then the definitions of matrices \( Q \) and \( \tilde{\Sigma} \). Since the vectors on the lhs and rhs are the same, the equation of the solvability of the system gives the resonance condition.

Now we define orbits on the graph using the same notation as in [BHJ12].

**Definition 4.6.** A periodic orbit \( \gamma \) on the graph \( \Gamma_2 \) is a closed path that begins and ends in the same vertex. It can be denoted by the directed edges that it subsequently contains, e.g. \( \gamma = (b_1, b_2, \ldots, b_n) \). Cyclic permutation of directed bonds does not change the periodic orbit. A pseudo-orbit is a collection of periodic orbits \( \tilde{\gamma} = \{ \gamma_1, \gamma_2, \ldots, \gamma_m \} \). An irreducible pseudo-orbit \( \tilde{\gamma} \) is a pseudo-orbit that contains no directed bond more than once. The metric length of a periodic orbit is defined as \( \ell_\gamma = \sum_{b \in \gamma} \ell_b \), the length of a pseudo-orbit is the sum of the lengths of all periodic orbits the pseudo-orbit is composed of. We denote the product of scattering amplitudes along the periodic orbit \( \gamma = (b_1, b_2, \ldots, b_n) \) as \( A_\gamma = S_{b_1} S_{b_2} \ldots S_{b_n} \).
here $S_{b_i b_j}$ denotes the entry of the matrix $S$ in the row corresponding to the bond $b_i$ and column corresponding to the bond $b_j$. For a pseudo-orbit we define $A_\gamma = \prod_{\ell \in \gamma} A_{\ell}$. By $m_\gamma$ we denote the number of periodic orbits in the pseudo-orbit $\tilde{\gamma}$. The set of irreducible pseudo-orbits also contains an irreducible pseudo-orbit on the zero edges with $m_\gamma = 0$, $\ell_\gamma = 0$ and $A_\gamma = 1$.

Without a separate proof, which immediately follows from theorem 1 in [BHJ12], we provide a theorem for finding the resonance condition using pseudo-orbits.

**Theorem 4.7.** The resonance condition is given by

$$\sum_\gamma (-1)^{m_\gamma} A_\gamma (k) e^{i\ell_\gamma} = 0,$$

where the sum goes over all irreducible pseudo-orbits $\tilde{\gamma}$.

As summary, we highlight the main steps of the pseudo-orbit expansion:

• construct effective vertex-scattering matrices for each vertex (see theorem 4.2) and find the scattering matrix $S(k) = Q^* \hat{S}(k)$;
• construct the oriented graph $\Gamma_2$: ‘cut off’ the halflines and replace each edge of the compact part of $\Gamma$ by two edges of opposite direction and the same length;
• for each irreducible pseudo-orbit find $m_\gamma$, $\ell_\gamma$, and $A_\gamma$ (this is found from the scattering matrix or directly from the vertex-scattering matrices);
• find the contribution of each irreducible pseudo-orbit on $\Gamma_2$ by theorem 4.7 and sum them up;
• the contribution of the irreducible pseudo-orbit on zero edges is 1.

For an illustration of the procedure we refer to the example in section 8.1 or to [Lip15].

5. Effective size of an equilateral graph

As we stated in section 3, the effective size of a graph is defined as a $\frac{n}{\pi}$-multiple of the coefficient of the leading term of the asymptotics. In this section we find the effective size of an equilateral graph from matrices $Q$ and $\Sigma$.

**Definition 5.1.** By an equilateral graph we mean a graph that has the same lengths for all internal edges.

First, we show a general criterion whether the graph is non-Weyl using the notions of theorem 4.5.

**Theorem 5.2.** A graph is non-Weyl iff $\det \hat{\Sigma}(k) = 0$ for all $k \in \mathbb{C}$. In other words, a graph is non-Weyl iff there exists a vertex for which $\det \hat{\sigma}_i(k) = 0$ for all $k \in \mathbb{C}$.

**Proof.** The term of the determinant in theorem 4.5, which has the highest multiple of $ik$ in the exponent, is $\det \{Q^* \Sigma(k)\} e^{i\ell_k \Sigma_{\ell_k}}$ and the term with the lowest multiple of $ik$ is 1. Theorem 3.1 in [DEL10] shows that the number of zeros of this determinant in the circle of radius $R$ is asymptotically equal to $\frac{2}{\pi} \text{vol } \Gamma$ iff the coefficient by the first term is nonzero. Since
multiplying by \( Q \) means only rearranging the rows, \( \det Q\tilde{\Sigma}(k) = 0 \) iff \( \tilde{\Sigma}(k) = 0 \). Since \( \Sigma(k) \) is similar to the matrix with blocks \( \delta_i \), the second part of the theorem follows.

From this theorem and corollary 4.3, theorem 1.2 from [DP11] follows, which states that a graph with standard coupling is non-Weyl iff there is a vertex connecting the same number of finite and infinite edges.

Now we state a theorem that gives the effective size of an equilateral graph.

**Theorem 5.3.** Let us assume an equilateral graph (all internal edges have lengths \( \ell \)). Then the effective size of this graph is \( \frac{\ell}{2}n_{\text{nonzero}} \) where \( n_{\text{nonzero}} \) is the number of nonzero eigenvalues of the matrix \( Q\tilde{\Sigma}(k) \).

**Proof.** We use theorem 4.5 again. First, we notice that the matrix \( Q\tilde{\Sigma}(k) \) can be in this theorem replaced by its Jordan form \( \tilde{\Sigma}(k) = S - D \) with \( V \) unitary. The matrix \( L \) (defined in section 4) is for an equilateral graph a multiple of the identity matrix, and therefore \( V \) commutes with \( e^{i\ell\tau} \). Unitary transformation does not change the determinant of a matrix, hence we have the resonance condition \( \det(e^{i\ell\tau}D(k) - I_N) = 0 \). Since the matrix under the determinant is upper triangular, the determinant is equal to multiplication of its diagonal elements. The term of the determinant, which contains the highest multiple of \( ik \) in the exponent, is \( e^{i\ell n_{\text{nonzero}}} \). The term with the lowest multiple of \( ik \) in the exponent is 1. The claim follows from theorem 3.1 in [DEL10].

Clearly, if there are \( n_{\text{bal}} \) balanced vertices with standard coupling, then there is at least \( n_{\text{bal}} \) zeros in the eigenvalues of the matrix \( Q\tilde{\Sigma} \), and hence the effective size is bounded by \( W \leq \text{vol } \Gamma - \frac{\ell}{2}n_{\text{bal}} \). The following corollary of the previous theorem gives a criterion saying when this bound can be improved.

**Corollary 5.4.** Let \( \Gamma \) be an equilateral graph with standard coupling and with \( n_{\text{bal}} \) balanced vertices. Then the effective size \( W < \text{vol } \Gamma - \frac{\ell}{2}n_{\text{bal}} \) if and only if \( \text{rank}(Q\tilde{\Sigma}) < \text{rank}(\tilde{\Sigma}) = 2N - n_{\text{bal}} \).

**Proof.** For the vertex of degree \( d \) the rank of \( \delta_i \) is either \( d - 1 \) for a balanced vertex or \( d \) otherwise. Hence \( \text{rank}(Q\tilde{\Sigma}) = \text{rank}(\tilde{\Sigma}) = 2N - n_{\text{bal}} \). The effective size is smaller than \( \text{vol } \Gamma - \frac{\ell}{2}n_{\text{bal}} \) if there is at least one 1 just above the diagonal in the block with zeros on the diagonal in the Jordan form of \( Q\tilde{\Sigma} \). Each Jordan block consisting of \( n \) zeros on the diagonal and \( n - 1 \) ones above the diagonal has rank \( n - 1 \); its square has rank \( n - 2 \). The blocks with eigenvalues other than zero have (as well as their squares) rank maximal. Hence the rank of \( Q\tilde{\Sigma}Q\tilde{\Sigma} \) is smaller than the rank of \( Q\tilde{\Sigma} \).

6. Deleting edges of the oriented graph

In this section we develop a method with the help of which the resonance condition for non-Weyl graphs can be constructed more easily. We restrict ourselves to equilateral graphs with standard coupling. For each balanced vertex we delete one directed edge of the graph \( \Gamma_2 \), which ends in this vertex, and replace it by one or several ‘ghost edges’. These edges allow the pseudo-orbits to hop from a vertex to a directed edge that is not connected with the vertex. The ‘ghost edges’ do not contribute to the resonance condition with the
Let us assume an equilateral graph $\Gamma$ (all internal edges of length $\ell$) with standard coupling. Let us assume that there is no edge that starts and ends in one vertex, and that no two vertices are connected by two or more edges. We use the following construction:

- Let the vertex $v_2$ be balanced in the graph $\Gamma$ and let the directed bonds $b_1, \ldots, b_d$ in the graph $\Gamma_2$ end in the vertex $v_2$ (part of the corresponding directed graph $\Gamma_2$ is shown in figure 1);
- delete the directed edge $b_1$ that starts at the vertex $v_1$ and ends at $v_2$;
- there are new directed ‘ghost edges’ $b_1', b_1'', \ldots, b_1^{d-1}$ introduced that start in the vertex $v_1$ and are connected to the edges $b_2, b_3, \ldots, b_d$, respectively (see figure 2);
- use irreducible pseudo-orbits to obtain the resonance condition from the new graph according to theorem 4.7 and obeying the following rules;
- if the ‘ghost edge’ $b_1'$ is contained in the irreducible pseudo-orbit $\gamma$, the length of this ‘ghost edge’ does not contribute to $\ell$;
- let, e.g., the ‘ghost edge’ $b_1'$ be included in $\gamma$; then the scattering amplitude from the bond $b$ ending in $v_1$ to the bond $b_2$ is the scattering amplitude in the former graph $\Gamma_2$ between $b$ and $b_1$ taken with the opposite sign;
- in the irreducible pseudo-orbit each ‘ghost edge’ can be used only once;
- the above procedure can be repeated; for each balanced vertex we can delete one directed edge that ends in this vertex and replace it by the ‘ghost edges’.

Figure 1. Part of the graph $\Gamma_2$. To vertex $v_1$ and to undenoted vertices which neighbor $v_2$ other bonds can be attached.
Theorem 6.1. The described construction under the assumptions above does not change the resonance condition.

Proof. Unitary transformation of the matrix $\tilde{S}Q$ does not change the determinant in theorem 4.5, since for an equilateral graph $e_kL_i$ is replaced by $I_e^k\ell N_i$ and $(\tilde{S})^{-1} = S^{-1} = S$:

$$\det e \det e \det e.$$ We want to delete edge $b_1$. Let $v_2$ be a balanced vertex and let the other edges ending in $v_2$ be $b_2, b_3, \ldots, b_d$. We choose as $V_1$ the $2N \times 2N$ matrix with the following entries

$$(V_1)_i^{j} = 1 \quad \text{for } i = 1, \ldots, 2N; \quad (V_1)_{b_1}^{j} = 1 \quad \text{for } i = 2, \ldots, d; \quad (V_1)_j = 0 \quad \text{otherwise},$$

where $(V_1)_{b_1}^{j}$ is the entry with the row corresponding to bond $b_1$ and the column corresponding to $b_1$. One can easily show that the entries of $V_1^{-1}$ are

$$(V_1^{-1})_i^{j} = 1 \quad \text{for } i = 1, \ldots, 2N; \quad (V_1^{-1})_{b_1} = -1 \quad \text{for } i = 2, \ldots, d; \quad (V_1^{-1})_j = 0 \quad \text{otherwise}.$$ The matrix $\tilde{\sigma}_v = \frac{1}{\sqrt{d}}J_d - I_d$ has linearly dependent columns, hence, if one multiplies it from the right by a matrix with all diagonal entries equal to 1, and all the entries in one of its columns equal 1, other entries being 0, one obtains a matrix which has one column with all zeros and the other columns the same as $\tilde{\sigma}_v$. The matrix $Q\tilde{\Sigma}$ has entries of $\tilde{\sigma}_v$ in the columns corresponding to bonds ending at $v_2$ and rows corresponding to bonds starting from $v_2$. By the same reasoning as for $\tilde{\sigma}_v$, the column of $Q\tilde{\Sigma}V_1$ corresponding to $b_1$ has all entries equal to 0 and other columns are the same as the corresponding columns of $Q\tilde{\Sigma}$.

Now it remains to show how multiplying from the left by $V_1^{-1}$ changes the matrix. Since nondiagonal entries are only in the rows of $V_1^{-1}$ corresponding to bonds $b_2, b_3, \ldots, b_d$, multiplying by $V_1^{-1}$ changes only these rows. Since nondiagonal entries are only in the column corresponding to $b_1$, the change can happen only in columns in which there is a nontrivial entry in the row corresponding to $b_1$. These columns correspond to bonds which end in the vertex $v_1$. We have to multiply these columns by rows of $V_1^{-1}$ which have 1 in the $b_j$th position, $j = 2, 3, \ldots, d$ and $-1$ in the $b_1$th position. Since no two vertices are connected.
by two or more edges, the only bond starting at $v_1$ and ending at $v_2$ is $b_1$ and the entries of $Q^S V_i$ in the column corresponding to the edges ending at $v_1$ and in the row corresponding to the edges $b_2$, $b_3$, ..., $b_j$ are zero (the edges in the column cannot be followed in an orbit by edges in the row). Hence, $1$ in the above row of $V_i^{-1}$ is multiplied by $0$ and $-1$ is multiplied by the scattering amplitude between the bonds ending in $v_1$ and $b_1$. Therefore, the only change is that there is this scattering amplitude taken with the opposite sign in the columns corresponding to the bonds which end in $v_1$, and in the row corresponding to bonds $b_2$, $b_3$, ..., $b_j$. These entries are represented by the ‘ghost edges’.

It is clear now that one has to take the entry of $I_{2N}$ in the column corresponding to $b_1$ in the determinant in theorem 4.5 with $Q^S$ replaced by $V_i^{-1} Q^S V_i$, therefore this edge effectively does not exist. The ‘ghost edge’ does not contribute to $\ell_g^+$, it only says which bonds are connected in the pseudo-orbit. Similar arguments can be used for other balanced vertices; for each of them we delete one edge which ends in it. Note that this method does not delete edges to which a ‘ghost edge’ leads.

\[\square\]

7. Main results

In this section we give two main theorems on the bounds on the effective size for equilateral graphs with standard coupling, and a theorem that gives the positions of the resonances.

**Theorem 7.1.** Let us assume an equilateral graph with $N$ internal edges of lengths $\ell$, with standard coupling, $n_{bal}$ balanced vertices, and $n_{nonneig}$ balanced vertices that do not neighbor any other balanced vertex. Then the effective size is bounded by $W \leq N\ell - \frac{\ell}{2} n_{bal} - \frac{\ell}{2} n_{nonneig}$.

**Proof.** Clearly, for each balanced vertex, we can delete one directed edge of the graph $G_2$, the size of the graph is reduced by $\frac{\ell}{2} n_{bal}$. In the balanced vertex of the degree $d$ that does not neighbor any other balanced vertex we have $d - 1$ incoming directed bonds and $d$ outgoing directed bonds. No outgoing bond is deleted and no ‘ghost edge’ ends in the outgoing edge, because there is no balanced vertex which neighbors the given vertex. Hence, we cannot use one of the outgoing directed edges in the irreducible pseudo-orbit (there is no way to obtain this vertex for the $d$th time). The longest irreducible pseudo-orbit does not include $n_{nonneig}$ bonds and the effective size of the graph must be reduced by $\frac{\ell}{2} n_{nonneig}$.

\[\square\]

**Theorem 7.2.** Let us assume an equilateral graph (N internal edges of the lengths $\ell$) with standard coupling. Let there be a square of balanced vertices $v_1$, $v_2$, $v_3$, and $v_4$ without diagonals, i.e. $v_1$ neighbors $v_2$, $v_2$ neighbors $v_3$, $v_3$ neighbors $v_4$, $v_4$ neighbors $v_1$, $v_1$ does not neighbor $v_3$, and $v_2$ does not neighbor $v_4$. Then the effective size is bounded by $W \leq (N - 3)\ell$.

**Proof.** Let us denote the bond from $v_1$ to $v_2$ by 1, the bond from $v_2$ to $v_3$ by 2, the bond from $v_3$ to $v_4$ by 3 and the bond from $v_4$ to $v_1$ by 4, the bonds in the opposite directions by $\hat{1}$, $\hat{2}$, $\hat{3}$, and $\hat{4}$ (see figure 3). Let $G_2'$ be the rest of the graph $G_2$; it can be connected with the square by bonds in both directions, we denote them in figure 3 by edges with arrows in both directions. Now we delete bonds $\hat{1}$, $\hat{2}$, $\hat{3}$, and $\hat{4}$ (see figure 4), there arise ‘ghost edges’ in the square (explicitly shown), and there may arise ‘ghost edges’ from vertices of the square to edges of the rest of the graph (represented by dashed edges between the square and $G_2'$). Since the pseudo-orbit can continue from bond 1 to bond 4 (with the scattering amplitude equal to the
scattering amplitude from 1 to $\hat{\Gamma}$ with the opposite sign), from bond 2 to bond 1, etc, one can effectively represent the oriented graph by figure 5.

It is clear that the effective size has been reduced by $\ell^2$, because four edges have been deleted. The term of the resonance condition with the highest multiple of $i$ in the exponent corresponds to the contribution of irreducible pseudo-orbits on all remaining ‘non-ghost’ bonds (the pseudo-orbits may or may not use the ‘ghost edges’). The contribution of the pseudo-orbits (1234) and (12)(34) cancels out, because both pseudo-orbits differ only in the number of orbits, hence there is a factor of $-1$. A similar argument holds also for the pair
of pseudo-orbits (1432) and (14)(32) and all irreducible pseudo-orbits that include these pseudo-orbits.

Now we show why also the second highest term is zero. It includes the contribution of all bonds but one. If the not-included bond is not 1, 2, 3, or 4, the contributions are canceled due to the previous argument. If, e.g., bond 4 is not included, it would mean that one must go from one of the lower vertices in figure 5 to the other through \( \Gamma_2' \). This is not possible because the irreducible pseudo-orbit has to include all the bonds but 4, none of the bonds in part \( \Gamma_2' \) is deleted and there is no ‘ghost edge’ ending in the bond 1, 2, 3 or 4. If there exists a path through \( \Gamma_2' \) from one vertex to another, then the path in the opposite direction cannot be covered by the irreducible pseudo-orbit.

Therefore, at least six directed edges of the former graph \( \Gamma_2 \) are not used and the effective size is reduced by 3\( \ell \).

Finally, we state what the positions of the resolvent resonances are.

**Theorem 7.3.** Let us assume an equilateral graph (lengths \( \ell \)) with standard coupling. Let the eigenvalues of \( \tilde{Q} \Sigma \) be \( c_j = r_j e^{i\phi} \). Then the resolvent resonances are \( \lambda = k^2 \) with \( k = \frac{1}{\ell} (\pi + 2n\pi + i \ln r_j), \ n \in \mathbb{Z} \). Moreover, \( |c_j| \leq 1 \) and for a graph with no edge starting and ending in one vertex also \( \sum_{j=1}^{2N} c_j = 0 \).

**Proof.** The resonance condition is

\[
\prod_{j=1}^{2N} (e^{ik\ell c_j} - 1) = 0,
\]

hence we have for \( k = k_R + ik_\ell \)

\[
r_j e^{-ik\ell e^{ik_\ell \phi}} = 1,
\]
from which the claim follows. For $r_j > 1$ we would have the positive imaginary part of $k$ which would contradict the fact that eigenvalues of the self-adjoint Hamiltonian are real (the corresponding generalized eigenfunction would be square integrable). If the graph does not have any edge starting and ending in one vertex, then there are zeros on the diagonal of $\tilde{S}_Q$, hence its trace (the sum of its eigenvalues) is zero.

8. Examples

In this section we show three specific examples that illustrate the general behavior. In the first example the methods of pseudo-orbit expansion and deleting edges are explained. In the second example we deal with deleting the directed bonds again. The third example shows that the symmetry of the graph is not sufficient in order to obtain an effective size smaller than it is expected from the bound in theorem 7.1.

8.1. Three abscissas and three halflines

In this example we show in detail how the method of pseudo-orbit expansion and the method of ‘deleting edges’ are used. Let us consider a tree graph with three internal edges of lengths $\ell$ and three halflines attached in the central vertex (see figure 6). There is standard coupling in the central vertex and Dirichlet coupling at the spare ends of the internal edges.

Figure 6. Graph $\Gamma$ in example 8.1.
Since there is Dirichlet coupling for the vertices $v_1$, $v_2$ and $v_3$, one has $U_0 = -1$ for $v = v_1$, $v_2$, $v_3$. From (3) we have $U_0 = -1$ and by theorem 4.2 $\sigma_i = -1$. Since there is standard coupling in the middle vertex, we have from corollary 4.3 $\sigma_i = s_i$. From (4) we have $\tilde{\sigma}_1 = \tilde{\sigma}_2 = \tilde{\sigma}_3 = -1$. Since there is standard coupling in the middle vertex, we have from corollary 4.3 $\tilde{\sigma}_i = -1$. Since there is standard coupling in the middle vertex, we have from corollary 4.3 $\tilde{\sigma}_i = -1$.

The graph $G_2$ is obtained by ‘cutting off’ the halflines and replacing each edge of the compact part by two directed bonds (see figure 7). One can easily see that there are no irreducible pseudo-orbits on the odd number of edges. There is one (trivial) pseudo-orbit on the zero edges. There are three irreducible pseudo-orbits on two edges $1 \times 1$ (trivial), $1 \times 2$ and $1 \times 3$. For each of these pseudo-orbits we have $A_i = (-1)\left(-\frac{2}{3}\right)$ (there is $-1$ for the scattering from $1$ to $1$), $m_i = 1$ and $\ell_i = 2$. We have six irreducible pseudo-orbits on four edges $(1 \times 1)(2 \times 2)$; $(1 \times 2)(3 \times 3)$; $(1 \times 2)(3 \times 3)$; $(1 \times 2)(3 \times 3)$; $(1 \times 2)(3 \times 3)$; $(1 \times 2)(3 \times 3)$; $(1 \times 2)(3 \times 3)$. And there are six irreducible pseudo-orbits on six edges $(1 \times 1)(2 \times 2)(3 \times 3)$; $(1 \times 2)(3 \times 3)$; $(1 \times 2)(3 \times 3)$; $(1 \times 2)(3 \times 3)$; $(1 \times 2)(3 \times 3)$; $(1 \times 2)(3 \times 3)$. And there are six irreducible pseudo-orbits on six edges $(1 \times 1)(2 \times 2)(3 \times 3)$; $(1 \times 2)(3 \times 3)$; $(1 \times 2)(3 \times 3)$; $(1 \times 2)(3 \times 3)$; $(1 \times 2)(3 \times 3)$; $(1 \times 2)(3 \times 3)$. The first one consists of three periodic orbits, the next three of two periodic orbits, and the last two of one periodic orbit. The coefficients by the exponentials are hence the following

\[
\begin{align*}
\exp 0: & 1, \\
\exp (2ik\ell): & (-1)\left(-\frac{2}{3}\right)(-1)^1 \cdot 3 = -2, \\
\exp (4ik\ell): & (-1)^2\left(-\frac{2}{3}\right)^2 (-1)^2 \cdot 3 + (-1)^2\left(\frac{1}{3}\right)^2 (-1)^1 \cdot 3 = 1, \\
\exp (6ik\ell): & (-1)^3\left(-\frac{2}{3}\right)^3 (-1)^3 + (-1)^3\left(\frac{1}{3}\right)^3 (-1)^2 \cdot 3 \\
& + (-1)^3\left(\frac{1}{3}\right)^3 (-1)^1 \cdot 2 = 0.
\end{align*}
\]
Since the vertex $v_4$ was balanced, we can delete one directed bond, e.g. bond 3. The graph $G_2$ after deleting the edge and introducing ‘ghost edges’ is shown in figure 8. One can easily see that again we do not have irreducible pseudo-orbits on an odd number of ‘non-ghost’ edges. We again have the trivial irreducible pseudo-orbit on the zero edges. We have the following four irreducible pseudo-orbits on two ‘non-ghost’ edges: $(1\tilde{1})$; (2$\tilde{2}$); (3$\tilde{3}$'); and (3$\tilde{3}$'2). The scattering amplitude for the orbit (3$\tilde{3}$') from 3 via 3' to 1 is $+1$, because the scattering amplitude in the former graph from 3 to 3 was $-1$. Finally, we have the following five irreducible pseudo-orbits on four ‘non-ghost’ edges: (1$\tilde{1}$)(2$\tilde{2}$); (1$\tilde{1}$)(3$\tilde{3}$'2); (2$\tilde{2}$)(3$\tilde{3}$'); (1$\tilde{1}$2$\tilde{2}$); (13$\tilde{3}$'21); and (23$\tilde{3}$'12). Clearly, there are no pseudo-orbits on six edges, since we have deleted one edge. The coefficient by the exponentials in the resonance condition are

\[
\exp 0 : 1, \quad \exp (2ik\ell) : (-1)^{\frac{2}{3}}(-1)^1 \cdot 2 + \frac{1}{3}(-1)^{\frac{1}{3}} \cdot 2 = -2, \\
\exp (4ik\ell) : (-1)^2\left(-\frac{2}{3}\right)^2(-1)^2 + (-1)\left(-\frac{2}{3}\right)\frac{1}{3}(-1)^3 \cdot 2 \\
+ (-1)^2\left(\frac{1}{3}\right)^2(-1)^1 + (-1)\left(\frac{1}{3}\right)^21(-1)^1 \cdot 2 = 1,
\]

In both cases the resonance condition is $1 - 2 e^{2ik\ell} + e^{4ik\ell} = 0$. There are resonances at $k^2$, $k = \frac{\pi d}{2\pi}$, $n \in \mathbb{Z}$ with multiplicity 2.

8.2. Square with the diagonal and all vertices balanced

We assume an equilateral graph with four internal edges in the square and the fifth edge as a diagonal of this square. All four vertices are balanced, i.e. to two of them two halflines are
attached and to the other two vertices three halflines are attached. There is standard coupling in all vertices. The oriented graph $G_2$ is shown in figure 9.

Let us denote the vertex from which the edge 1 starts by $v_1$ and the vertex where 1 ends by $v_2$. Then the effective vertex-scattering matrices are

$$\varphi_{v_1} = \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \varphi_{v_2} = \frac{1}{3} \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \end{pmatrix}$$

similarly for the other two vertices. Hence, the matrix $Q_\Sigma$ is equal to

$$
\begin{array}{ccccccccccc}
1 & 2 & 3 & 4 & 5 & \hat{1} & \hat{2} & \hat{3} & \hat{4} & \hat{5} \\
1 & 0 & 0 & 0 & 1/2 & 0 & -1/2 & 0 & 0 & 0 \\
2 & 1/3 & 0 & 0 & 0 & 1/3 & 0 & -2/3 & 0 & 0 \\
3 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 \\
4 & 0 & 0 & 1/3 & 0 & 0 & 0 & 0 & 0 & -2/3 & 1/3 \\
5 & 0 & 0 & 1/3 & 0 & 0 & 0 & 0 & 0 & 1/3 & -2/3 \\
\hat{1} & -2/3 & 0 & 0 & 0 & 1/3 & 0 & 1/3 & 0 & 0 & 0 \\
\hat{2} & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\
\hat{3} & 0 & 0 & -2/3 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 \\
\hat{4} & 0 & 0 & 0 & -1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\
\hat{5} & 1/3 & 0 & 0 & 0 & -2/3 & 0 & 1/3 & 0 & 0 & 0 \\
\end{array}
$$

Figure 9. The graph $G_2$ for the graph in example 8.2.
The edges, to which the rows and columns correspond, are denoted on the left and on the top of the matrix.

Now we delete the bond $\hat{1}$ (figure 10). Deleting this edge is equivalent to the unitary transformation $V^{-1}Q\Sigma V_1$, where $V_1$ has 1 on the diagonal, and 1 in the sixth column (corresponding to the bond $\hat{1}$) and the fourth row (corresponding to the bond 4); the other entries of this matrix are zero ($(V_1)_{ii} = 1, \forall i, (V_1)_{i4} = 1, (V_1)_{i6} = 0$ otherwise). Its inverse has 1 on the diagonal and $-1$ in the sixth column and fourth row. We obtain the matrix

\[
\begin{array}{cccccccc}
 1 & 2 & 3 & 4 & 5 & \hat{1} & \hat{2} & \hat{3} & \hat{4} & \hat{5} \\
1 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\
2 & 1/3 & 0 & 0 & 0 & 1/3 & 0 & -2/3 & 0 & 0 \\
3 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 \\
4 & 2/3 & 0 & 1/3 & 0 & -1/3 & 0 & -1/3 & 0 & -2/3 & 1/3 \\
5 & 0 & 0 & 1/3 & 0 & 0 & 0 & 0 & 1/3 & -2/3. \\
\hat{1} & -2/3 & 0 & 0 & 0 & 1/3 & 0 & 1/3 & 0 & 0 & 0 \\
\hat{2} & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\
\hat{3} & 0 & 0 & -2/3 & 0 & 0 & 0 & 0 & 1/3 & 1/3 \\
\hat{4} & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hat{5} & 1/3 & 0 & 0 & 0 & -2/3 & 0 & 1/3 & 0 & 0 & 0
\end{array}
\]

In this matrix the sixth column consists of all zeros and there are three other new entries in the fourth row that are printed in bold. Since the new entries are in the fourth row, there must be a
‘ghost edge’ $\hat{1}'$ from the vertex $v_2$ pointing to bond 4. We can use it in pseudo-orbits containing one of the bonds 1, 5, $\hat{2}$ continuing then to bond 4 with the scattering amplitudes in bold above.

Now we delete the bond $\hat{2}$ (figure 11). Since two other bonds end in the vertex $v_2$ (bonds 1 and 5), there will be two ‘ghost edges’. We use the unitary transformation $V_2^{-1}V_1^{-1}Q\Sigma V_1V_2$. $V_2$ has 1 on the diagonal and 1 in the seventh column (corresponding to the edge $\hat{2}$) and rows 1 and 5. Its inverse has nondiagonal terms with opposite signs. After the transformation we obtain

```
|   | 1 | 2  | 3  | 4  | 5  | ˆ1 | ˆ2 | ³ | 4  | 5  |
|---|---|----|----|----|----|----|----|---|----|----|
| 1 | 0 | 1/2| 0  | 1/2| 0  | 0  | 0  | 0 | -1/2| 0  | 0  |
| 2 | 1/3| 0  | 0  | 0  | 1/3| 0  | 0  | 0 | 0   | 0  | 0  |
| 3 | 0  | 1/2| 0  | 0  | 0  | 0  | 0  | -1/2| 0  | 0  | 0  |
| 4 | 2/3| 0  | 1/3| 0  | -1/3| 0  | 0  | -2/3| 1/3| 0  | 0  |
| 5 | 0  | 1/2| 1/3| 0  | 0  | 0  | 0  | -1/2| 1/3| -2/3| 0  |
| ³ | -2/3| 0  | 0  | 0  | 1/3| 0  | 0  | 0  | 0  | 0  | 0  |
| 4 | 0  | -1/2| 0  | 0  | 0  | 0  | 0  | 1/2| 0  | 0  | 0  |
| 5 | 0  | 0  | -2/3| 0  | 0  | 0  | 0  | 0  | 1/3| 1/3| 0  |
| ³ | 0  | 0  | 0  | -1/2| 0  | 0  | 0  | 0  | 0  | 0  | 0  |
```

The seventh column (corresponding to $\hat{2}$) has all entries equal to zero; other new entries are printed in bold. These entries correspond to the new ‘ghost edges’ $\hat{2}'$ and $\hat{2}''$. Similarly, we
delete edges $\hat{3}$ and $\hat{4}$ (figure 12); the matrix $\tilde{S}Q$ after these transformations is

$$
\begin{array}{c|cccccccc}
 & 1 & 2 & 3 & 4 & 5 & \hat{1} & \hat{2} & \hat{3} & \hat{4} & \hat{5} \\
1 & 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1/3 & 0 & 2/3 & 0 & 1/3 & 0 & 0 & 0 & 0 & -1/3 \\
3 & 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 2/3 & 0 & 1/3 & 0 & -1/3 & 0 & 0 & 0 & 0 & 1/3 \\
5 & 0 & 1/2 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 & -2/3 \\
\hat{1} & -2/3 & 0 & 0 & 0 & 0 & 1/3 & 0 & 0 & 0 & 0 \\
\hat{2} & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hat{3} & 0 & 0 & -2/3 & 0 & 0 & 0 & 0 & 0 & 0 & 1/3 \\
\hat{4} & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hat{5} & 1/3 & 0 & 0 & 1/2 & -2/3 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
$$

The eigenvalues of the matrix $\tilde{S}Q$ are $-2/3, -1/3, -1, 1$ with multiplicity 2, 0 with multiplicity 5. There is one Jordan block $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in the Jordan form of $\tilde{S}Q$. The resonance condition can be obtained from the eigenvalues by the equation at the beginning of the proof of theorem 7.3

$$1 - \frac{16}{9}e^{2\lambda t} - \frac{2}{9}e^{4\lambda t} + \frac{7}{9}e^{4\lambda t} + \frac{2}{9}e^{8\lambda t} = 0.$$ 

By theorem 7.3 we obtain that the positions of the resonances are $\lambda = k^2$ with $k = \frac{1}{7}[2n + 1] \pi - i \ln 3]$, $k = \frac{1}{7}[2n + 1] \pi - i \ln \frac{3}{7}$, $k = \frac{1}{7}[2n + 1] \pi$, and $k = \frac{1}{7}[2n \pi$ with multiplicity 2, $n \in \mathbb{Z}$. 

---

Figure 12. The graph $\Gamma_2$ after deleting bonds $\hat{1}, \hat{2}, \hat{3},$ and $\hat{4}$. 

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8.3. Fully connected graph on four vertices with all vertices balanced

In this subsection we assume a graph on four vertices, every two vertices are connected by one edge of length $\ell$ and there are three halflines attached at each vertex, hence each vertex is balanced. The directed graph $\Gamma_2$ is shown in figure 13. For each vertex we have the effective vertex-scattering matrix

$$\tilde{\sigma}_v = \frac{1}{3} \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}. $$

The matrix $\tilde{Q}_v$ is

\[
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & \hat{1} & \hat{2} & \hat{3} & \hat{4} & \hat{5} & \hat{6} \\
1 & 0 & 0 & 0 & 1/3 & 0 & 0 & -2/3 & 0 & 0 & 0 & 1/3 \\
2 & 1/3 & 0 & 0 & 0 & 1/3 & 0 & 0 & -2/3 & 0 & 0 & 0 \\
3 & 0 & 1/3 & 0 & 0 & 0 & 1/3 & 0 & 0 & -2/3 & 0 & 0 \\
4 & 0 & 0 & 1/3 & 0 & 0 & 0 & 0 & 0 & -2/3 & 1/3 & 0 \\
5 & 0 & 0 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & -2/3 \\
6 & 0 & 0 & 0 & 1/3 & 0 & 0 & 1/3 & 0 & 0 & 0 & -2/3 \\
\hat{1} & -2/3 & 0 & 0 & 0 & 1/3 & 0 & 0 & 1/3 & 0 & 0 & 0 \\
\hat{2} & 0 & -2/3 & 0 & 0 & 0 & 1/3 & 0 & 0 & 1/3 & 0 & 0 \\
\hat{3} & 0 & 0 & -2/3 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 0 \\
\hat{4} & 0 & 0 & 0 & -2/3 & 0 & 0 & 1/3 & 0 & 0 & 0 & 1/3 \\
\hat{5} & 1/3 & 0 & 0 & 0 & -2/3 & 0 & 0 & 1/3 & 0 & 0 & 0 \\
\hat{6} & 0 & 1/3 & 0 & 0 & 0 & -2/3 & 0 & 0 & 1/3 & 0 & 0 \\
\end{array}
\]
Its eigenvalues are $-1$ with multiplicity 2, $1$ with multiplicity 3, $-1/3$ with multiplicity 3, and $0$ with multiplicity 4. There is no Jordan block. The resonance condition is

$$1 - \frac{8}{3} e^{2i\ell t} - \frac{8}{27} e^{4i\ell t} + \frac{62}{27} e^{6i\ell t} + \frac{16}{27} e^{8i\ell t} - \frac{16}{27} e^{10i\ell t} - \frac{8}{27} e^{12i\ell t} - \frac{1}{27} e^{14i\ell t} = 0.$$ 

Similarly to the previous example we can find the positions of the resolvent resonances $\lambda = k^2$ with $k = \frac{1}{\ell} (2n + 1) \pi$ with multiplicity 2, $k = \frac{1}{\ell} 2n \pi$ with multiplicity 3, and $k = \frac{1}{\ell} [ (2n + 1) \pi - i \ln 3]$ with multiplicity 3, $n \in \mathbb{Z}$.

This example shows that the symmetry of the graph does not assure that it will have the effective size smaller than the bound in theorem 7.1. This graph has four zeros as eigenvalues of $\tilde{S}$, hence the effective size is $4\ell$, as we expect from four balanced vertices. Although this graph is very symmetric, we do not have smaller effective size in contrast to the example in theorem 7.3 in [DEL10].

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