Structure of $k$-closures of finite nilpotent permutation groups

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Abstract

Let $G$ be a permutation group on a set $\Omega$, and $k$ a positive integer. The $k$-closure $G^{(k)}$ of $G$ is the largest subgroup of $\text{Sym}(\Omega)$, with the same $G$ orbits of componentwise action on $\Omega^k$. We prove that the $k$-closure of a finite nilpotent permutation group is the direct product of $k$-closures of its Sylow subgroups.

1 Introduction

Let $G$ be a permutation group on a finite nonempty set $\Omega$, and $k$ a positive integer. Denote by $\text{Orb}_k(G)$ the set of the orbits of componentwise action of $G$ on the Cartesian power $\Omega^k$. Elements of the set $\text{Orb}_k(G)$ are called $k$-orbits of $G$, and they form a partition of the set $\Omega^k$. The $k$-closure $G^{(k)}$ of $G$ is the automorphism group of $k$-orbits of $G$ [1, Def. 5.3],

$$G^{(k)} = \{ g \in \text{Sym}(\Omega) \mid O^g = O \ \forall O \in \text{Orb}_k(G) \}.$$ 

Equivalently, $G^{(k)}$ is the largest subgroup of the symmetric group $\text{Sym}(\Omega)$ such that $\text{Orb}_k(G) = \text{Orb}_k(G^{(k)})$. The following inclusions for closures of $G$ are known [1, Thm. 5.8]:

$$G \leq G^{(k)} \leq G^{(k-1)}, k \geq 2.$$  (1)
The main result of this paper establishes a simple link between the $k$-closure of a finite nilpotent permutation group and $k$-closures of its Sylow subgroups.

**Theorem 1.1.** If $G$ is a finite nilpotent permutation group, and $k \geq 2$, then

$$G^{(k)} = \prod_{P \in \text{Syl}(G)} P^{(k)},$$

i.e. the $k$-closure of $G$ is the direct product of $k$-closures of its Sylow subgroups. In particular, $G^{(k)}$ is nilpotent.

A group $G$ is $k$-closed if $G^{(k)} = G$. Theorem 1.1 implies necessary and sufficient conditions for a nilpotent group to be $k$-closed.

**Corollary 1.2.** A finite nilpotent permutation group is $k$-closed if and only if every Sylow subgroup of $G$ is $k$-closed.

Theorem 1.1 generalizes recent results of [2,3], where similar theorems were proved for 2-closures of nilpotent groups [2] and for $k$-closures of abelian groups [3]. It is worth mentioning that nilpotency of $k$-closure of nilpotent group is already known and it easily follows from [2, Thm. 1.1] and the inclusions (1).

## 2 Preliminaries

We will mainly use standard notations of permutation group theory (see [4]). We especially note the following ones. If a group $G$ (an element $x \in G$) acts on a set $\Omega$, then $G^\Delta$ ($x^\Delta$ respectively) denotes the permutation group (the permutation respectively) induced by this action. If $G \leq \text{Sym}(\Omega)$, and $\Delta \subseteq \Omega$ is a $G$-invariant subset, then $G_{\{\Delta\}} = \{x \in G \mid \Delta^x = \Delta\}$ is the setwise stabilizer of $\Delta$ in $G$. Finally, $1_\Delta$ denotes the identity element of $\text{Sym}(\Delta)$.

Several times we will use the following criterion for a permutation to belong to a $k$-closure.

**Theorem 2.1.** [1, Thm. 5.6] If $G \leq \text{Sym}(\Omega)$, $k \geq 1$, and $x \in \text{Sym}(\Omega)$, then $x \in G^{(k)}$ if and only if for all $\alpha_1, \ldots, \alpha_k \in \Omega$ there exists $g \in G$ such that $\alpha_i^x = \alpha_i^g$, $i = 1, \ldots, k$.

Let $\pi$ be a set of prime numbers, and $n$ is a positive integer. Denote by $n_\pi$ the divisor of $n$ such that $(n_\pi, \frac{n}{n_\pi}) = 1$ and the set of prime divisors of $n_\pi$ is equal to $\pi$. 

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Lemma 2.2. [2, Lemma 3.1] If $G$ is a finite nilpotent permutation group of degree $n$, and $H$ is a Hall subgroup of $G$, then

1) the size of every orbit of $H$ is equal to $n_\pi$, where $\pi = \pi(H)$ is the set of prime divisors of $|H|$, 

2) $G$ acts on $\text{Orb}(H)$, and $H$ is the kernel of this action.

The following two lemmas are well known, but, apparently, they are not completely proved anywhere. We fill this gap. Some variations of these statements can be found in [5–7].

Lemma 2.3. If $G_i \leq \text{Sym}(\Omega_i)$, $i = 1, 2$, and the group $G_1 \times G_2$ acts on the disjoint union $\Omega_1 \cup \Omega_2$, then for all integers $k \geq 1$,

$$(G_1 \times G_2)^{(k)} = G_1^{(k)} \times G_2^{(k)}.$$ 

Proof. $\geq$ It suffices to prove that $G_1^{(k)} \times 1$ and $1 \times G_2^{(k)}$ are contained in $(G_1 \times G_2)^{(k)}$. Fix a set of elements $\alpha_1, \ldots, \alpha_k \in \Omega_1 \cup \Omega_2$ and its subset $\alpha_{j_1}, \ldots, \alpha_{j_l}$ consisting of elements from $\Omega_1$. If $(g, 1) \in G_1^{(k)} \times 1$, then the inclusions (1) imply that $g \in G_1^{(l)}$, and by Theorem 2.1 exists $h \in G_1$ such that $\alpha_{j_i} = \alpha_{i}^h$ for all $i = 1, \ldots, l$.

Now consider the element $(h, 1) \in G_1 \times G_2$. By construction for all $i = 1, \ldots, k$

$$\alpha_i^{(h, 1)} = \begin{cases} \alpha_i^h = \alpha_i^{g, 1} = \alpha_i^{g, 1} & \text{if } \alpha_i \in \Omega_1, \\ \alpha_i^h = \alpha_i^{g, 1} & \text{if } \alpha_i \in \Omega_2. \end{cases}$$

It follows from Theorem 2.1 that $(g, 1) \in (G_1 \times G_2)^{(k)}$. So that, $G_1^{(k)} \times 1 \leq (G_1 \times G_2)^{(k)}$. The inclusion $1 \times G_2^{(k)} \leq (G_1 \times G_2)^{(k)}$ can be proved analogously. $\leq$

Let $x \in (G_1 \times G_2)^{(k)}$, so $\Omega_i^x = \Omega_i$. For $i = 1, 2$ put $x = x^{\Omega_i}$. Then $x$ coincides with $(x_1, x_2) \in \text{Sym}(\Omega_1 \cup \Omega_2)$, which is acting on $\Omega_i$ as $x_i$, $i = 1, 2$.

We show that $x_i \in G_i^{(k)}$ for $i = 1, 2$. Let $\alpha_1, \ldots, \alpha_k \in \Omega_i$. By Theorem 2.1 there exist $(h_1, h_2) \in G_1 \times G_2$ such that $\alpha_{j}^x = \alpha_{j}^{(h_1, h_2)}$, $j = 1, \ldots, k$. In particular, $\alpha_{j}^x = \alpha_{j}^{h_i}$ in view of $\alpha_{j}^x = \alpha_{j}^{(h_1, h_2)} = \alpha_{j}^{h_i}$. Then Theorem 2.1 implies that $x_i \in G_i^{(k)}$, and $x \in G_1^{(k)} \times G_2^{(k)}$ then. $\square$

Lemma 2.4. If $G_i \leq \text{Sym}(\Omega_i)$, $i = 1, 2$, and the group $G_1 \times G_2$ acts on the direct product $\Omega_1 \times \Omega_2$, then for every integer $k \geq 1$,

$$(G_1 \times G_2)^{(k)} = G_1^{(k)} \times G_2^{(k)}.$$
Proof. \[ \geq \] It suffices to prove that \( G_1^{(k)} \times 1 \) and \( 1 \times G_2^{(k)} \) are contained in \((G_1 \times G_2)^{(k)}\). Let \((g, 1) \in G_1^{(k)} \times G_2^{(k)}\), and \((\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k) \in \Omega_1 \times \Omega_2\). Since \( g \in G_1^{(k)} \), by Theorem 2.1 there exists \( h \in G_1 \) such that \( \alpha_j^g = \alpha_j^h \), \( j = 1, \ldots, k \). This means that for the element \((h, 1) \in G_1 \times G_2\) the equality \((\alpha_j, \beta_j)^{g,1} = (\alpha_j^g, \beta_j^g) = (\alpha_j^h, \beta_j^h) = (\alpha_j, \beta_j)^{h,1}\) holds, and it follows from Theorem 2.1 that \((g, 1) \in (G_1 \times G_2)^{(k)}\). So that, \( G_1^{(k)} \times 1 \leq (G_1 \times G_2)^{(k)}\). Inclusion \( 1 \times G_2^{(k)} \leq (G_1 \times G_2)^{(k)}\) can be proved similarly.

\[ \leq \] First, we will study the structure of elements from \((G_1 \times G_2)^{(2)}\). Put \( \Sigma_1 = \{\{\alpha_1\} \times \Omega_2 \mid \alpha_1 \in \Omega_1\}, \Sigma_2 = \{\Omega_1 \times \{\alpha_2\} \mid \alpha_2 \in \Omega_2\}\). Obviously, the group \( G_1 \times G_2 \) acts on the sets \( \Sigma_1 \) and \( \Sigma_2 \). We will show that the group \((G_1 \times G_2)^{(2)}\) acts on these sets too, i.e. for all \( x \in (G_1 \times G_2)^{(2)}\) and all \( \Delta \in \Sigma_i, i = 1, 2\), either \( \Delta^x = \Delta \), or \( \Delta^x \cap \Delta = \emptyset \).

If \( \Delta \) is a singleton, then there is nothing to prove. Suppose that there exist two different elements \( u, v \in \Delta \) such that \( u^x \in \Delta \), and \( v^x \notin \Delta \). By Theorem 2.1, there exists \( h \in G_1 \times G_2 \) such that \( (u^x, v^x) = (u^h, v^h)\). But that would imply that \( u^h \in \Delta \), and \( v^h \notin \Delta \), which is impossible, because the group \( G_1 \times G_2 \) acts on the set \( \Sigma_i \).

So the group \((G_1 \times G_2)^{(2)}\) acts on sets \( \Sigma_1 \) and \( \Sigma_2 \), and hence on the set \( \Sigma_1 \times \Sigma_2 \). The bijection

\[
\rho : \Omega_1 \times \Omega_2 \to \Sigma_1 \times \Sigma_2, \quad (\alpha_1, \alpha_2) \mapsto (\{\alpha_1\} \times \Omega_2, \Omega_1 \times \{\alpha_2\}).
\]

establishes a permutation isomorphism between the group \((G_1 \times G_2)^{(2)}\) and a subgroup of the group \(\text{Sym}(\Sigma_1 \times \Sigma_2)\). For \( i = 1, 2 \) define the permutation \( x_i \) acting on \( \alpha \in \Omega_i \) in the following way:

\[
\alpha^{x_1} = \beta \iff (\{\alpha\} \times \Omega_2)^x = (\beta) \times \Omega_2,
\]
\[
\alpha^{x_2} = \beta \iff (\Omega_1 \times \{\alpha\})^x = \Omega_1 \times \{\beta\}.
\]

Then the image of the permutation \( x \) under the permutation isomorphism defined above coincides with the permutation \((x_1, x_2) \in \text{Sym}(\Omega_1 \times \Omega_2)\).

Let us return to the proof of the lemma. Let \( x \in (G_1 \times G_2)^{(k)}\). From the previous observations and the inclusions (1) follows that \( x = (x_1, x_2), x_i \in \text{Sym}(\Omega_i), i = 1, 2 \). We show that \( x_i \in G_i^{(k)} \). Let \( \alpha_1^i, \ldots, \alpha_k^i \in \Omega_i \). It follows from Theorem 2.1 that for \( x \in (G_1 \times G_2)^{(k)}\) and for \( (\alpha_1^1, \alpha_2^1), \ldots, (\alpha_1^k, \alpha_2^k) \in \Omega_1 \times \Omega_2 \) there exists \( (h_1, h_2) \in G_1 \times G_2 \) such that \( (\alpha_j^1, \alpha_j^2)^x = (\alpha_j^1, \alpha_j^2)^{(h_1, h_2)} \), \( j = 1, \ldots, k \). In particular, \( \alpha_j^{x_i} = \alpha_j^{x_{h_i}}, j = 1, \ldots, k \), so Theorem 2.1 implies that \( x_i \in G_i^{(k)} \), and \( x \in G_1^{(k)} \times G_2^{(k)} \) then.
The following technical lemma is of particular interest.

**Lemma 2.5.** Let $G$ be a finite nilpotent permutation group, $P \in \text{Syl}(G)$, $\Delta_1, \ldots, \Delta_k \in \text{Orb}(P)$, and $\Delta = \bigcup_{i=1}^{k} \Delta_i$. Then

$$\left( \bigcap_{i=1}^{k} G_{\{\Delta_i\}} \right)^{\Delta} \leq P^{\Delta}.$$ 

*Proof.* Let $g \in \left( \bigcap_{i=1}^{k} G_{\{\Delta_i\}} \right)^{\Delta}$. Since the group $G$ is nilpotent, $G = P \times H$, with $H$ being a Hall subgroup of $G$, and then $g = xy$ for some $x \in P$ and $y \in H$. The choice of $g$ and $x$ yields that $\Delta_i^g = \Delta_i^x = \Delta_i$ for all $i = 1, \ldots, k$, so the same holds true for $y$, because

$$\Delta_i = \Delta_i^g = \Delta_i^x = \Delta_i^{x^{-1}g} = \Delta_i^y.$$ 

We show that $y^{\Delta_i} = 1_{\Delta_i}$ for all $i = 1, \ldots, k$. Indeed, by the construction of $y$, the element $y^{\Delta_i}$ belongs to the centralizer $Z_i$ of the transitive group $P^{\Delta_i} \leq \text{Sym}(\Delta_i)$, which is semiregular by [4, Exer. 4.5']. Therefore $|Z_i|$ divides $|\Delta_i|$, which is a $p$-power. So that $Z_i$ is a $p$-group. In particular, the order of the element $y^{\Delta_i}$ is $p$-power, and therefore $y^{\Delta_i} \in P^{\Delta_i}$. Since $P^{\Delta_i} \cap H^{\Delta_i} = 1$, we have $y^{\Delta_i} = 1_{\Delta_i}$.

Thus, $y^{\Delta} = 1_{\Delta}$, which means that

$$g^{\Delta} = (xy)^{\Delta} = x^{\Delta}y^{\Delta} = x^{\Delta} \in P^{\Delta}.$$ 

\[\square\]
3 Proof of the theorem

First, consider the case where the group $G$ is transitive. We use induction on $|\pi(G)|$. If $|\pi(G)| = 1$, then $G$ is a $p$-group, and by virtue of [1, Exer. 5.28] the 2-closure $G^{(2)}$ is also a $p$-group, and due to the inclusions (1), $G^{(k)}$ is a $p$-group too for all $k \geq 2$.

Now let $G = P \times H$, where $P \in \text{Syl}_p(G)$, and $H$ is the Hall subgroup of $G$. Applying Lemma 2.2(1) for orbits $\Delta \in \text{Orb}(P)$ and $\Gamma \in \text{Orb}(H)$, we obtain that

$$|\Delta| = n_p \quad \text{and} \quad |\Gamma| = n_{p'},$$

where $p' = \pi(H)$. Since $P$ and $H$ are normal subgroups of $G$, it follows that $\Delta$ and $\Gamma$ are blocks of some imprimitivity system of the group $G$. Moreover, the set $\Delta \cap \Gamma$ is either empty, or it is also a block whose order divides both $|\Delta|$ and $|\Gamma|$, which means $|\Delta \cap \Gamma| \leq 1$.

Now each point $\alpha \in \Omega$ can be uniquely associated with the orbits (containing this point) $\Delta_\alpha$ and $\Gamma_\alpha$ of the groups $P$ and $H$, respectively. Previous arguments imply that $|\Delta_\alpha \cap \Gamma_\alpha| = 1$.

So the mapping

$$\rho : \Omega \rightarrow \text{Orb}(H) \times \text{Orb}(P), \quad \alpha \mapsto (\Delta_\alpha, \Gamma_\alpha)$$

is a bijection. Denote by $P'$ and $H'$ the permutation groups induced by the action of $G$ on $\text{Orb}(H)$ and $\text{Orb}(P)$, respectively. By Lemma 2.2(2) we obtain that

$$P^\rho = P' \times 1 \quad \text{and} \quad H^\rho = 1 \times H'.$$

Thus, $G$ is permutationally isomorphic to the group $P' \times H'$ acting on the set $\text{Orb}(H) \times \text{Orb}(P)$. It follows from the Lemma 2.4 that

$$(P' \times H')^{(k)} = (P')^{(k)} \times (H')^{(k)}.$$ 

The proof concludes by applying the induction assumption to the group $H$, which is possible by the equality $\pi(G) = \pi(P) \cup \pi(H)$.

Now let $G$ be intransitive. Each transitive constituent $H$ of $G$ is nilpotent (as a homomorphic image of a nilpotent group), and $\pi(H) \subseteq \pi(G)$. It follows from the previous arguments that the group $H^{(k)}$ is also nilpotent, and $\pi(H) = \pi(H^{(k)})$. Applying Lemma 2.3 we obtain that

$$G \leq G^{(k)} \leq \left( \prod\limits_{H} H^{(k)} \right) \leq \prod\limits_{H} H^{(k)}.$$
which implies the equality $\pi(G) = \pi(G^{(k)})$.

Now we consider the Sylow subgroups of $G$ and $G^{(k)}$.

**Lemma 3.1.** If $P \in \text{Syl}_p(G)$ and $Q \in \text{Syl}_p(G^{(k)})$, then $P^{(k)} \leq Q$ and $\text{Orb}(P) = \text{Orb}(Q)$.

**Proof.** The inclusion $P^{(k)} \leq Q$ follows from [1, Exer. 5.28] and the inclusions (1). Therefore, to prove the equality $\text{Orb}(P) = \text{Orb}(Q)$ it suffices to prove that every $P$-orbit is a $Q$-orbit.

Let $\Delta \in \text{Orb}(P)$ and $\Gamma \in \text{Orb}(G)$ such that $\Delta \subseteq \Gamma$. Since the $k$-closure preserves 1-orbits, $\Gamma$ is also a $G^{(k)}$-orbit. Denote by $\Delta'$ the orbit of the group $Q$ such that $\Delta \subseteq \Delta' \subseteq \Gamma$.

The groups $G^\Gamma$ and $(G^{(k)})^\Gamma$ are transitive and nilpotent, so the double application of Lemma 2.2 implies $|\Delta| = |\Gamma|_p = |\Delta'|$, and $\Delta = \Delta'$.

**Lemma 3.2.** In above notations, $P^{(k)} = Q$.

**Proof.** It was established in Lemma 3.1 that $P^{(k)} \leq Q$. To prove the inverse inclusion, let $(\alpha_1, \ldots, \alpha_k) \in \Omega^k$ and $g \in Q$. By Theorem 2.1, there exists $h \in G$ such that $(\alpha_1, \ldots, \alpha_k)^g = (\alpha_1, \ldots, \alpha_k)^h$.

Denote by $\Delta_i$ an orbit of $Q$ containing $\alpha_i$, $i = 1, \ldots, k$. By Lemma 3.1, it follows that every such $\Delta_i$ is also an orbit of $P$. The element $h$ fixes every $\Delta_i$ as a set, so $\alpha_i^h = \alpha_i^g \in \Delta_i$. In other words, $h \in \bigcap_{i=1}^k G(\Delta_i)$, and by Lemma 2.5 there exists $u \in P$ such that $h^\Delta = u^\Delta$ (here $\Delta = \bigcup_{i=1}^k \Delta_i$), and then $$(\alpha_1, \ldots, \alpha_k)^g = (\alpha_1, \ldots, \alpha_k)^h = (\alpha_1, \ldots, \alpha_k)^u.$$ It follows from Theorem 2.1 that $g \in P^{(k)}$ and $Q \leq P^{(k)}$.

The proof of the theorem is completed by the application of the Lemma 3.2 to the following equalities:

$$G^{(k)} = \prod_{\text{Syl}(G^{(k)})} Q = \prod_{\text{Syl}(G)} P^{(k)}.$$
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