Restricted Lie algebras of polycyclic groups, II.

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Abstract

This is the second paper in the series of three. We study restricted Lie algebras of polycyclic groups and obtain conditions for existence of \( p \)-series with associated restricted Lie algebra abelian or free abelian with rank equal to the Hirsch number of the group. We develop methods for constructing such series, these methods are based on construction of filtrations and valuation functions in the group rings.

This paper continues author’s work [9] and [10].
1.1. Let $H$ be a group. A series of normal subgroups

$$H = H_1 \supseteq H_2 \supseteq \cdots$$  \hspace{1cm} (1.1)

is a $p$-series in $H$ if $[H_i, H_j] \subseteq H_{i+j}$ and $H_i^p \subseteq H_i$. We will denote by $L_p(H, H_i)$ the restricted Lie algebra associated to $p$-series (1.1) and by $U_p(L_p(H, H_i))$ its universal $p$-envelope. The algebra

$$L_p(H, H_i) = \sum_{i=1}^{\infty} H_i / H_{i+1}$$  \hspace{1cm} (1.2)

is obtained by the classical construction of Lazard [7]; we give in section 2 a brief description of this construction and of the main properties of it. The properties of the algebra $L_p(H, H_i)$ and the properties of the algebra $U_p(L_p(H, H_i))$ depend not only on the group $H$ but also on choice of the $p$-series in this group.

We studied in Lichtman [10] the restricted Lie algebras of polycyclic groups and obtained necessary and sufficient conditions for existence in a poly-infinite cyclic group with Hirsch number $r$ a series (1.1) with associated Lie algebra $L_p(H, H_i)$ free abelian of rank $r$. We obtained also in [10] some necessary conditions for existence of such series in an arbitrary polycyclic group.

The main results of this paper are Theorems I-XII. We begin by formulating Theorem V which will be proven in section 5.

**Theorem V.** Let $H$ be an infinite polycyclic-by-finite group with Hirsh number $r$. Assume that there exists a $p$-series (1.1) with unit intersection such that the corresponding restricted Lie algebra $L_p(H, H_i)$ is finitely generated. Then there exists a torsion free normal subgroup $F$ with index a power of $p$ such that the ideal $L_p(F, F_i)$ associated to the $p$-series $F_i = F \cap H_i$ ($i = 1, 2, \cdots$) is a restricted free abelian subalgebra of the center of $L_p(H, H_i)$ with index a power of $p$ and rank $1 \leq r_1 \leq r$.

Hence the center $Z$ of $L_p(H, H_i)$ has a finite index which is a power of $p$ and $L_p(H, H_i)$ is a nilpotent Lie algebra.
It is known that finitely generated restricted abelian Lie algebra $F$ has a representation

$$F = T + T_0$$

where $T$ is either a free abelian subalgebra or $T = 0$ and $T_0$ is finite; this follows also from classical theorems about modules over a polynomial ring. We will call throughout the paper the rank of the free abelian subalgebra $T$ the rank of $F$.

We recall also that an element $x$ of a restricted Lie algebra is nilpotent if there exists $p^n$ such that $x^{[p^n]} = 0$, and that a finitely generated restricted Lie algebra without nilpotent elements is free; the last fact follows also from the representation (1.3).

Before formulating Theorems VI-XII we recall first the elementary fact that if (1.1) is a $p$-series in an arbitrary group $H$, and $H_0 = \bigcap_{i=1}^{\infty} H_i$ then the Lie algebra $L_p(H, H_i)$ is isomorphic to the restricted Lie algebra $L_p(\bar{H}, \bar{H}_i)$ of the group $\bar{H} = H/H_0$ associated to the series $\bar{H}_i = H_i/H_0 (i = 1, 2, \cdots)$ (see section 2). The group $\bar{H}$ is a residually finite $p$-group, and the series $\bar{H}_i (i = 1, 2, \cdots)$ has unit intersection; this shows that in the study of the algebra $L_p(H, H_i)$ we can assume that series (1.1) has a unit intersection and, we can consider only the case when $H$ is a residually finite $p$-group. We will use throughout the paper the notation $H \in \text{res} \mathcal{N}_p$ for the fact that $H$ is a residually finite $p$-group.

Further, we will consider in this case the topology defined by series (1.1). If

$$M_n(Z_p H) \ (n = 1, 2, \cdots)$$

is the series of dimension subgroups in characteristic $p$ (see section 2) then the topology defined by this series will be called throughout the paper by $p$-topology. We will prove that if $H$ is a polycyclic group with Hirsch number $r$, (1.1) is a $p$-series with unit intersection and the algebra $L_p(H, H_i)$ is abelian of rank $r$ then the topology defined by this series is equivalent to the $p$-topology. This means that for every dimension subgroup $M_n(Z_p H)$ a number $i(n)$ can be found such that $H_{i(n)} \subseteq M_n(Z_p H)$; it is worth remarking that the inclusion $H_i \supseteq M_i(Z_p H) (i = 1, 2, \cdots)$ holds in an arbitrary group.

The necessary and sufficient conditions for existence in a polycyclic group of a $p$-series with associated restricted Lie algebra abelian of rank $r$ are
obtained in the following Theorems VII and VI.

**Theorem VII.** Let $H$ be a polycyclic group with Hirsch number $r$. There exists in $H$ a $p$-series (1.1) with unit intersection and associated restricted Lie algebra $L_p(H, H_i)$ abelian of rank $r$ if and only if $H$ contains normal subgroups $Q \supset N$ such that the quotient group $H/Q$ is a $p$-group, $Q/N$ is a free abelian group, $N$ is torsion free nilpotent, and for every element $h \in H$ the following condition holds

$$[h^p, N] \subseteq N'N^p$$

or, equivalently, if $R = gp(h, N)$ then the quotient group $\bar{R} = R/N'N^p \in resN_p$.

The necessity of the conditions of Theorem VII follow from statement vi) of Theorem VI which we will now formulate; we will show in subsection 1.3. that the sufficiency of these conditions follows from Theorem XII and Corollary 6.4.

**Theorem VI.** Let $H$ be a polycyclic group with Hirsch number $r$. Assume that there exists a $p$-series $H_i (i = 1, 2, \cdots)$ with unit intersection such that $L_p(H, H_i)$ is abelian of rank $r$.

i) Let $U$ be an arbitrary subgroup of $H$ with Hirsch number $k$, $U_i = U \cap H_i (i = 1, 2, \cdots)$. Then $L_p(U, U_i)$ is an abelian algebra of rank $k$.

ii) Let $U$ be a normal subgroup of $H$ with Hirsch number $k$, $\bar{H}_i$ be the image of the subgroup $H_i$ in $H/U$. Then the subgroup $\bigcap_{i=1}^{\infty} \bar{H}_i$ is finite and the algebra $L_p(\bar{H}, \bar{H}_i)$ is abelian of rank $r - k$. In particular, if $\bar{H} = H/U$ contains no finite normal subgroups then it is a residually $\{\text{finite } p\text{-group}\}$.

iii) Let $U$ be a normal subgroup of $H$. If $\bar{H} = H/U$ is a residually $\{\text{finite } p\text{-group}\}$ then $\bigcap_{i=1}^{\infty} \bar{H}_i = 1$.

iv) Let $W$ be the unique maximal normal nilpotent-by-finite subgroup of $H$. Then $W$ is an extension of a torsion free nilpotent group by a finite $p$-group. The quotient group $H/W$ is an extension of a free abelian group by a finite $p$-group.

v) The topology defined in $H$ by the $p$-series $H_i (i = 1, 2, \cdots)$ is equivalent to the $p$-topology.

The topologies defined in an arbitrary subgroup $U$ by the series $U_i = U \cap H_i (i = 1, 2, \cdots)$ and $U \cap M_n(H) (n = 1, 2, \cdots)$ are equivalent to the $p$-topology in $U$. 
vi) There exists an index $i_0$ such that if $i \geq i_0$ then the subgroup $Q = H_i$ contains a torsion free nilpotent subgroup $N$ which is invariant in $H$, $Q/N$ is free abelian, the algebra $L_p(Q, Q_i)$ is free abelian of rank $r$ and

$$H/N'N^p \in \text{res} \mathcal{N}_p$$

Clearly, $H/Q$ is a finite $p$-group.

vii) Let $F \supseteq S$ be two normal subgroups in $H$ such that $H/F$ and $F/S$ are residually \{finite $p$-groups\}. Then $H/S$ is a residually \{finite $p$-group\}.

viii) Let $H = H_1^* \supseteq H_2^* \supseteq \cdots$ (1.7)

be a series in $H$ with unit intersection and finitely generated associated graded Lie algebra $L_p(H, H_i^*)$. If the topology defined by series (1.7) is equivalent to the $p$-topology then the center of $L_p(H, H_i^*)$ has rank $r$. Moreover, there exists a number $k$ such that if $U = H_i (i \geq k)$ then the subalgebra $L_p(U, U_i^*) \cong \sum_{i \geq k} H_i/H_{i+1}$ associated to the $p$-series $U_i^* = U \cap H_i^* (i = 1, 2, \cdots)$ is a central free abelian subalgebra of rank $r$.

Condition (1.5) in Theorem VII holds in an arbitrary group $H$ which contains a normal subgroup $N$ such that the quotient group $H/N'N^p$ is a residually \{finite $p$-group\}. On the other hand, if $H$ is a polycyclic group with Hirsch number $r$ which contains a $p$-series (1.1) with associated graded algebra $L_p(H, H_i)$ abelian of rank $r$ and $Q$ and $N$ are the normal subgroups obtained in Theorems VII and VI we obtain from statement vi) of Theorem VI that $H/N'N^p$ is a residually \{finite $p$-group\}.

Theorem VI will be proven in section 6. We will show in subsection 1.3 that Theorem VII follows from Theorem VI and from Theorem XII which will be formulated in subsection 1.2.

We prove in section 6 Theorem 6.1. which gives an additional information about the normal subgroups $Q$ and $N$ which are obtained in Theorems VI and VII.

Theorem V implies that if the algebra $L_p(H, H_i)$ is finitely generated then the rank of the center of it is less than or equal to the Hirsch number of $H$. We prove in section 7 Theorems XI which gives the necessary and sufficient conditions for the center $Z$ to have rank equal to the Hirsch number of $H$.

1.2. Theorems VI and VII are related to an existence of a $p$-series $H_i (i = 1, 2, \cdots)$ with associated graded Lie algebra abelian of rank $r$, where $r$ is the
Hirsch number of $H$. We consider the question when the algebra $L_p(H, H_i)$ is free abelian of rank $r$ in Theorem XII. We have already pointed out that we should consider only the series with unit intersection; further, it is easy to see (Lemma 2.1.) that if the algebra $L_p(H, H_i)$ is free abelian then the group $H$ must be torsion free.

Theorem XII will be proven in section 8. This theorem provides a sufficient condition for existence in a torsion free polycyclic group $H$ with Hirsch number $r$ of a $p$ series (1.1) with restricted Lie algebra $L_p(H, H_i)$ free abelian of rank $r$. Theorem XII generalizes author’s results [10]; these results were obtained for the poly-{$\{\text{infinite cyclic}\}$ groups.

**Theorem XII.** Let $H$ be a torsion free polycyclic group with Hirsch number $r$, $U$ be a normal subgroup with Hirsch number $k$ and torsion free quotient group $\bar{H} = H/U$.

Assume that the following 3 conditions hold.

1) There exists in $U$ a $p$-series

$$U = U_1 \supset U_2 \supset \cdots$$

(1.8)

with associated restricted Lie algebra $L_p(U, U_i)$ free abelian of rank $k$.

2) There exists in the group $\bar{H} = H/U$ a $p$-series

$$\bar{H} = \bar{H}_1 \supset \bar{H}_2 \supset \cdots$$

(1.9)

with associated restricted Lie algebra $L_p(\bar{H}, \bar{H}_i)$ free abelian of rank $r - k$.

3) For every subgroup $R = gp(h, U)$ generated by $U$ and an element $h \in H$ the quotient group $\bar{R} = R/U'U^p \in res \mathcal{N}_p$ or, equivalently, $[h^k, U] \subseteq U'U^p$.

Then there exists a $p$-series (1.1) with unit intersection and associated restricted Lie algebra $L_p(H, H_i)$ free abelian of rank $r$ such that

$$\bar{H}_i = (H_iU)/U \ (i = 1, 2, \cdots)$$

(1.10)

The last statement of Theorem XII together with the classical results of Lazard implies immediately the following corollary.

**Corollary 8.3.** The natural homomorphism $\phi: H \longrightarrow \bar{H}$ defines a homomorphism of graded algebras

$$\bar{\phi}: L_p(H, H_i) \longrightarrow L_p(\bar{H}, \bar{H}_i)$$

(1.11)
Below are a few remarks on the conditions of Theorem XII.

First, the series $H_i, U_i, \bar{H}_i (i = 1, 2, \cdots)$ in Theorem XII have unit intersection. This follows from Proposition 3.5 in Lichtman [10]; it follows also from statement ii) of Theorem VI.

Second, everyone of conditions 1) and 3) is necessary. The necessity of condition 1) follows from the fact that the subalgebra of a free abelian algebra is free abelian, and the rank of $L_p(U, U_i)$ must be $k$ via statement i) of Theorem VI. The necessity of condition 3) follows from statement vii) of Theorem VI.

Third, it is easy to show that condition 3) does not follow from conditions 1 and 2 even if $H \in \text{res} N_p$ (see, for instance, [10], section 10.3.) but it holds if the group $H$ is an extension of a torsion free nilpotent group by a finite $p$-group. We have for this class of groups the following immediate corollary of Theorem XII.

**Corollary 1.1.** Let $H$ be a torsion free group which contains a nilpotent normal subgroup whose Hirsch number is $r$ and index is a power of $p$. Let $U$ be a normal subgroup of $H$ which satisfies conditions 1) and 2) of Theorem XII. Then there exists in $H$ a $p$-series which satisfies all the conclusions of Theorem XII.

We have one more corollary of Theorem XII.

**Corollary 1.2.** Let $H$ be a torsion free polycyclic group with Hirsch number $r$ which contains a nilpotent normal subgroup $U$ with Hirsch number $k$ such that the quotient group $H/U^pU'$ is a residually finite $p$-group. Assume that the quotient group $\bar{H} = H/U$ is torsion free and contains a $p$-series (1.9) such that the algebra $L_p(\bar{H}, \bar{H}_i)$ is free abelian of rank $r - k$. Then the group $H$ contains a $p$-series (1.1) with unit intersection such that the algebra $L_p(H, H_i)$ is free abelian of rank $r$.

**Proof.** The group $U$ contains a series of normal subgroups of length $k$ with infinite cyclic factors. Theorem XII together with a straightforward induction argument yields that $U$ contains a $p$-series (1.8) with associated restricted Lie algebra free abelian of rank $k$. We apply now Theorem XII to the group $H$ and its normal subgroup $U$ and the assertion follows.

We make in the proof of Theorem XII an essential use of Theorems VI, VII, IX and X.
Theorem X. Let $H$ be a torsion free polycyclic group with Hirsch number $r$ which contains a $p$-series (1.1) with unit intersection. Assume that the Lie algebra $L_p(H, H_i)$ is free abelian of rank $r$. Let $\Phi$ be a group of automorphisms of $H$ such that the order of every automorphism $\phi \in \Phi$ on the quotient group $H/H'^p$ is a power of $p$.

Then there exists a $p$-series

$$H = H^*_1 \supseteq H^*_2 \supseteq \cdots$$

with unit intersection such that all the subgroups $H^*_i$ ($i = 1, 2, \cdots$) are $\Phi$-invariant, $\Phi$ centralizes all the factors $H_i/H_i+1$ ($i = 1, 2, \cdots$) and the algebra $L_p(H, H^*_i)$ is free abelian of rank $r$.

Theorem IX. Let $H$ be a torsion free polycyclic group with Hirsch number $r$. Assume that there exists a $p$-series (1.1) with associated restricted Lie algebra $L_p(H, H_i)$ free abelian (abelian) of rank $r$. Then there exists a $p$-series of characteristic subgroups with associated restricted Lie algebra free abelian (abelian) of rank $r$.

Theorem IX is obtained as a corollary of the following more general result.

Theorem VIII. Let $H$ be a finitely generated torsion free group which has a $p$-series $H_i$ ($i = 1, 2, \cdots$) with unit intersection and with the associated restricted Lie algebra $L_p(H, H_i)$ free abelian (abelian) of finite rank. Assume that the topology defined by this $p$-series is equivalent to the $p$-topology. Then there exists a $p$-series

$$H = U_1 \supseteq U_2 \supseteq \cdots$$

(1.12)

whose terms $U_i$ ($i = 1, 2, \cdots$) are characteristic subgroups and the Lie algebra $L_p(H, U_i)$ is free abelian (abelian) of finite rank.

1.3. Derivation of Theorem VII from Theorem VI and Proposition 7.5. We have already observed that the necessity of the conditions of Theorem VII is in fact statement vi) of Theorem VI.

To prove the sufficiency we consider the normal subgroup $Q$ which was obtained in statement vi) of Theorem VI. Corollary 1.2. implies that $Q$ contains a $p$-series with unit intersection

$$Q = Q_1 \supseteq Q_2 \supseteq \cdots$$

(1.13)
and with associated restricted Lie algebra $L_p(Q, Q_i)$ free abelian of rank $r$. We see now that sufficiency of the conditions of Theorem VII will follow from Theorem XII and the following fact which will be proven in section 7.

**Proposition 7.5.** Let $H$ be a polycyclic group with Hirsch number $r$, $U$ be a normal subgroup of finite index $(H : U) = p^n$. Assume that there exists $p$-series

$$U = U_1 \supseteq U_2 \supseteq \cdots$$

(1.14)

with unit intersection such that the algebra $L_p(U, U_i)$ is abelian of rank $r$. Then there exists a $p$-series (1.1) such that the algebra $L_p(H, H_i)$ is abelian of rank $r$.

1.4. Polycentral systems in rings. Our methods are based on the results of section 3 on polycentral system in rings. Before formulating these results we define first the following concept which will be used throughout the whole paper.

Let $R$ be a ring, $t_1, t_2, \cdots, t_n$ be a system of elements in $R$. Assume that the element $t_1$ is central, the ideal $(t_1)$ generated by $t_1$ is residually nilpotent and $t_1$ is regular in $R$ or, equivalently, the graded ring associated to the filtration $(t_1)^i (i = 1, 2, \cdots)$ is isomorphic to the polynomial ring $(R/A)[t_1]$ (see Corollary 3.1.); further, for every $1 \leq i \leq n-1$ the element $t_{i+1}$ is central and regular modulo the ideal $A_i = \langle t_1, t_2, \cdots, t_i \rangle$ generated by $t_1, t_2, \cdots, t_i$ and the ideal $(t_{i+1})$ is residually nilpotent in $R/A_i$. If these conditions hold we will say that this system is polycentral independent; if all the elements $t_i (i = 1, 2, \cdots, n)$ are central in $R$ we will say that the system is central independent.

We will also consider this situation in a more detailed way taking into account the number of central elements in the system $T$ and its subsystems. Let $T_1$ be a central independent system in $R$, $A_1$ be the ideal generated by $T_1$; for every $1 \leq i \leq n - 1$ let $T_{i+1}$ be a system of elements which is central and independent modulo the ideal $A_i$ generated by the system $T_1 \cup T_2 \cup \cdots \cup T_i$. Clearly the system $T = \langle T_1, T_2, \cdots, T_m \rangle$ is polycentral independent in $R$. We will use this notation in order to make clear that $T$ is a polycentral independent system which is composed from the independent systems $T_1, T_2, \cdots, T_m$. It is worth remarking that we do not assume that the subsystems $T_i (i = 2, 3, \cdots, m)$ are central in $R$; we assume that
every subsystem $T_i$ must be central (and independent) in the quotient rings $R/A_i$ ($i = 2, 3, \cdots, m$).

We order the elements of every $T_i$ ($i = 1, 2, \cdots, m$) in an arbitrary way and then extend these orders to an order in $T$ by assuming that the elements of $T_i$ preceed the elements of $T_{i+1}$. The standard monomials on $T$ are defined in the usual way; it is convenient to assume that 1 is a standard monomial of degree zero. We will construct independent polycentral systems in some classes of group rings of torsion free polycyclic groups, and in other classes of rings but at this point we will only mention the following two cases where these systems are obtained easily.

1) Let $L$ is a nilpotent Lie algebra with a central series

$$L = L_1 \supseteq L_2 \supseteq \cdots \supseteq L_{k-1} \supseteq L_k = 0$$  \hspace{1cm} (1.15)

We pick in every $L_{k-i}$ ($i = 1, 2, \cdots, k - 1$) a system of elements $T_i$ which forms a basis of the quotient space $L_{k-i}/L_{k-i+1}$; in particular, the system $T_1$ is a basis of $L_{k-1}$ and is central in $L$. The system of elements $T_1 \cup T_2 \cup \cdots \cup T_{k-1}$ is an independent polycentral system in the universal enveloping algebra $U(L)$.

2) Let $H$ be a finitely generated torsion free nilpotent group. Let

$$H = H_1 \supseteq H_2 \supseteq \cdots \supseteq H_{k-1} \supseteq H_k = 0$$  \hspace{1cm} (1.16)

be a central series in $H$ with torsion free factors $H_i/H_{i+1}$ ($i = 1, 2, \cdots, k - 1$); we recall that the factors of the upper central series of $H$ have this property. We pick in $H_{k-i}$ ($i = 1, 2, \cdots, k - 1$) a system of elements $E_i$ which forms a basis of the free abelian group $H_{k-i}/H_{k-i+1}$. Let $E_i - 1 = \{e - 1 | e \in E_i\}$ ($i = 1, 2, \cdots, k - 1$). The system $E_1 - 1, E_2 - 1, \cdots, E_{k-1} - 1$ is a polycentral independent system in the group ring $KH$ of $H$ over an arbitrary field. Further, we consider the group ring of $H$ over the ring of integers or the ring of $p$-adic integers $\Omega$; in this case the system

$$p, E_1 - 1, E_2 - 1, \cdots, E_{k-1} - 1$$  \hspace{1cm} (1.17)

is an independent polycentral system in $\Omega H$. The same is true in a group ring $CH$ over an arbitrary ring $C$ of characteristic zero such that the powers of the ideal $(p)$ define a $p$-adic valuation in $C$. 

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More generally, we consider a torsion free nilpotent group $H$ without elements of infinite $p$-height and construct filtrations and valuations in the group rings over a field of $K$ of characteristic $p$ or over the ring of the integers $C$. These results are obtained in Theorem IV and Corollary 4.1. We prove also Theorem IV' which is an analog of these results when the characteristic of $K$ is zero. Theorem IV' provides also a new proof of Hall-Hartley Theorem about the residual nilpotence of the augmentation ideal $\omega(KH)$.

Polycentral ideals were considered by J.Roseblade and by P. Smith in group rings of polycyclic groups and in Noetherian rings (see Passman [13], section 11), and by Passman in [14] in connection with the AR-property and the localization theory. We consider here a different situation when the rings are in general non-noetherian and our results are related to the valuations defined by the polycentral systems.

Our applications of the polycentral systems are based on the following Theorem I and II which will be proven in section 3.

**Theorem I.** Let $R$ be a ring, $T = \langle t_1, t_2, \ldots, t_n \rangle$ be an independent polycentral system in $R$.

The ideal $A$ generated by the system $T$ is residually nilpotent. If $\hat{R}$ is the completion of $R$ in the topology defined by this ideal and $\mathcal{X}$ is a system of coset representatives for the elements of the quotient ring $R/A$ then every element $x \in \hat{R}$ has a unique representation

$$x = \sum_{n=0}^{\infty} \lambda_n \pi_n$$

(1.18)

where $\lambda_n \in \mathcal{X}$ ($n = 0, 1, \ldots$), $\pi_n$ are standard monomials on $T$, and $\lim_{n \to \infty} v(\pi_n) = \infty$.

Let $C$ be the set of integers. Here and throughout the paper we mean that a function $\rho$ from a ring $R$ into the set $C \cup \infty$ is a pseudovaluation if for every $x, y \in R$ we have

$$\rho(x + y) \geq \min\{\rho(x), \rho(y)\}$$

(1.19)

$$\rho(xy) \geq \rho(x) + \rho(y)$$

(1.20)

and $\rho(0) = \infty$. A pseudovaluation is a valuation if relation (1.20) is an equation. The pseudovaluations and valuations which will be considered
throughout this paper are discrete. We will assume also that \( \rho(x) = \infty \) only if \( x = 0 \). We refer the reader to Cohn’s book [3], or Bourbaki [2], chapter VI, for the basic concepts and properties of valuations and filtrations, and the graded rings associated with them.

The following Theorem II will be applied for construction of valuations and pseudovaluations in rings \( R \) with independent polycentral systems, and in particular in group rings of polycyclic groups; we will use then these pseudovaluations for the study of \( p \)-series in groups.

**Theorem II.** Let \( R \) be a ring, \( T = \langle t_1, t_2, \ldots, t_n \rangle \) be a polycentral independent system in \( R \) which is composed from the central systems \( T_1, T_2, \ldots, T_k \), \( A \) be the ideal generated by the system \( T \). Let \( f \) be a function on \( T \) whose values are natural numbers and \( f(t_1) > 2f(t_2) \) for \( t_1 \in T_i, t_2 \in T_{i+1} \) \( (i = 1, 2, \ldots, k-1) \).

Then there exists a pseudovaluation \( v \) of \( R \) such that

\[
v(t) = f(t) \quad \text{if} \quad (t \in T)
\]

and the graded ring \( \text{gr}_v(R) \) is isomorphic to the polynomial ring \( (R/A)[\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n] \) over the zero degree component \( R/A \), the topology defined in \( R \) by this pseudovaluation is equivalent to the topology defined by the powers of the ideal \( A \). Furthermore, \( v \) is a unique pseudovaluation such that \( v(t) = f(t) \ (t \in T) \) and the graded ring associated to it is isomorphic to \( R[\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n] \).

We have the following immediate corollary of Theorem II.

**Corollary 1.3.** Let \( R \) be a ring, \( T = \langle t_1, t_2, \ldots, t_n \rangle \) be a polycentral independent system in \( R \) which is composed from the central systems \( T_1, T_2, \ldots, T_k \), \( A \) be the ideal generated by the system \( T \). Let \( M_i \ (i = 1, 2, \ldots, k) \) be a system of natural numbers such that \( M_i > 2M_{i+1} \) \( (i = 1, 2, \ldots, k-1) \).

Then there exists a unique pseudovaluation \( v \) of \( R \) such that

\[
v(t) = M_i \quad \text{if} \quad t \in T_i; \quad (i = 1, 2, \ldots, k)
\]

and the graded ring \( \text{gr}_v(R) \) is isomorphic to the polynomial ring \( (R/A)[\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n] \), the topology defined in \( R \) by this pseudovaluation is equivalent to the topology defined by the powers of the ideal \( A \).
Theorem II will be derived from Theorem III which is formulated and proven in section 3.

Theorem II implies in particular that the completions $\tilde{R}_\rho$ and $\tilde{R}_\upsilon$ of $R$ in the $\rho$-topology and in the $\upsilon$-topology are homeomorphic. Let $\mathcal{X}$ be a system of coset representatives in the quotient ring $R/A$ for the ideal $A$. Theorems I and II now yield the following corollary of Theorem III.

**Corollary 3.1.** An arbitrary element $x \in \tilde{R}_\rho \cong \tilde{R}_\upsilon$ has a unique representation (1.18) where $\lambda_n \ (n = 1, 2, \cdots)$ are elements from a system of coset representatives $\mathcal{X}$ of the quotient ring $R/A$ and $\pi_n \ (n = 1, 2, \cdots)$ are standard monomials on the system $T$. The length $l(\pi_n)$ and the value $v(\pi_n)$ run to infinity if $n \to \infty$.

The following Corollary 3.8 of Theorems I and II will be proven in section 3; this corollary provides a method for construction of pseudovaluations in a ring $R$ using polycentral systems in a graded ring $\text{gr}(R)$.

**Corollary 3.9.** Let $R$ be a ring with a discrete pseudovaluation $\rho$, $\text{gr}_\rho(R)$ be the associated graded ring. Assume that there exists in $\text{gr}(R)$ an independent polycentral system $T$, let $A$ be the ideal generated in $\text{gr}(R)$ by this system. Then there exists in $R$ a discrete pseudovaluation $\upsilon$ such that the graded ring $\text{gr}_\upsilon(R)$ is isomorphic to a subring of the Laurent polynomial ring in the system of variables $T, t, t^{-1}$ over the ring $(\text{gr}(R))/A$.

Theorems I-III can be applied for constructions of valuations in some classes of skew fields, mainly the universal field of fractions of the free ring $D_K < X >$ over a field $D$.

**1.4.** Let $H$ be a torsion free polycyclic group with Hirsch number $r$ which satisfies conditions of Theorem VI, and hence has a $p$-series with unit intersection and associated graded Lie algebra $L_p(H, H_i)$ abelian of rank $r$. Theorem XII provides sufficient conditions for the existence of a $p$-series with associated algebra $L_p(H, H_i)$ free abelian of rank $r$. We will study the necessary and sufficient conditions for the existence in a polycyclic group of a series with associated graded algebra $L_p(H, H_i)$ free abelian of rank $r$ in the paper [11]; the results of the current paper reduce the problem to the case when the quotient group $H/H'$ is a finite $p$-group. Here we will only prove Proposition 1.1. and will sketch an example which show that there are torsion free polycyclic group which have $p$-series with unit intersection and
associated graded algebra abelian of rank equal to the Hirsch rank \( r \) of the group, nevertheless these groups do not have series with associated graded algebra free abelian of rank \( r \).

**Proposition 1.1.** Let \( H \) be a group. Assume that the quotient group \( H/\gamma_p(H) \) of \( H \) by the \( p^{th} \) term of the low central series has exponent \( p \). Then \( H \) can not have a \( p \)-series (1.1) with unit intersection and associated graded algebra free abelian.

**Proof.** Assume that there exists a \( p \)-series (1.1) with algebra \( L_p(H, H) \) free abelian. Let \( h \) be one of the elements with minimal weight, say \( \omega(h) = k \).

We consider now an arbitrary commutator \( u = [h_1, h_2, \cdots, h_l] \) with \( l \geq p \). Since the algebra \( L_p(H, H) \) is abelian we obtain from Lemma 2.2. below that the weight \( w([h_1, h_2]) \) of the commutator \([h_1, h_2]\) is greater that \( w(h_1) + w(h_2) \), and then that

\[
 w([h_1, h_2, \cdots, h_l]) > w(h_1) + w(h_2) + \cdots + w(h_l) \geq lk \geq pk \quad (1.23)
\]

Since the algebra \( L_p(H, H) \) is free abelian the weight of \( h^p \) must be \( pk \) by Lemma 2.1. On the other hand, since \( h^p \) is a product of commutators of length greater than or equal \( p \) its weight must be greater than \( pk \) by (1.23). This contradiction shows that \( L_p(H, H) \) can not be free abelian, and the proof is complete.

Proposition 1.1. provides a method for construction groups which can not have \( p \)-series with unit intersection and associated graded algebra free abelian. We will briefly sketch here with a proof one example, the proof and the computations will be given in [11]; we will provide there more examples.

**Example.** Let \( F \) be a free group with generators \( f_1, f_2, f_3 \), \( N \) be a normal subgroup such that the quotient group \( F/N \) has exponent 2. Let \( \bar{F} = F/N' \). The group \( \bar{F} \) is generated by the elements \( \bar{f}_1, \bar{f}_2, \bar{f}_3 \), the images of \( f_1, f_2, f_3 \), and it is an extension of the free abelian group \( N = N/N' \) by the abelian group \( \bar{F}/N \cong F/N \) of exponent 2 and rank 3. The group \( \bar{F} \) does have a \( p \)-series

\[
 \bar{F} = \bar{F}_1 \supseteq \bar{F}_2 \supseteq \cdots \quad (1.24)
\]

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with unit intersection and associated graded algebra free abelian of rank equal to the Hirsch rank of $\bar{F}$ which is equal to 17, this series can be constructed by methods of Lichtman [10]. We consider then the quotient group $H$ of $\bar{F}$ by the normal subgroup $U$ generated by the elements

$$\bar{f}_1^2, \bar{f}_2^2, \bar{f}_3^2, \bar{f}_2^3$$

(1.25)

We will prove in [11] that the group $H$ is torsion free; it is easy to verify that $H$ is an extension of a free abelian group by a group of exponent 2, it is generated by the elements $a, b, c$ which are the images of the elements $\bar{f}_1, \bar{f}_2, \bar{f}_3$, subject to the relations

$$a^2 = [b, c], b^2 = [c, a], c^2 = [a, b]$$

(1.26)

The group $H$ is a residually \{finite 2-group\}; relations (1.26) imply that the quotient group $H/H'$ has exponent 2, we conclude now from Proposition 1.1. that the group $H$ can not contain a 2-series with unit intersection and the algebra $L_2(H, H_i)$ free abelian.

To show that such series does not exist if $p \neq 2$ we observe that the group $H$ is not a residually-{finite $p$-group} for $p \neq 2$ so an arbitrary $p$-series will have in this case a non-unit intersection.

1.5. Our results on restricted Lie algebras of polycyclic groups are related to the cases when the algebra $L_p(H, H_i)$ is finitely generated and we deal almost entirely with the case when the rank of $L_p(H, H_i)$ coincides with the Hirsch number of $H$. We will construct in section 9 examples of $p$-series in a free abelian group of rank 2 with associated graded algebra free abelian of rank 1; we will construct also examples of series with associated graded algebra infinitely generated.

§2. Preliminaries.

2.1. We give now a brief account of the main concepts and results about the restricted Lie algebras of groups; these results are obtained in Lazard’s article [7].

Let $H$ be a group,
be a \( p \)-series. The restricted Lie algebra associated to series (2.1) is a Lie algebra over the prime field \( \mathbb{Z}_p \) whose vector space is \( \sum_{i=1}^{\infty} H_i / H_{i+1} \); an element \( \tilde{h} = hH_{i+1} \) from \( H_i / H_{i+1} \) is called homogeneous of degree \( i \) and the Lie operation for homogeneous elements is defined in the following way. If \( x_1 \in H_{i_1} / H_{i_1+1}, \ x_2 \in H_{i_2} / H_{i_2+1} \) and \( x_\alpha^i \) is the coset representative of \( x_\alpha (\alpha = 1, 2) \) then \( [x_1, x_2] \) is the element of \( H_{i_1+i_2} / H_{i_1+i_2+1} \) which contains the commutator \( [x_1^i, x_2^i] \). This operation extends by distributivity on arbitrary elements from \( \sum_{i=1}^{\infty} H_i / H_{i+1} \) and it defines the structure of graded Lie algebra on the set \( \sum_{i=1}^{\infty} H_i / H_{i+1} \). If \( x \in H_i / H_{i+1} \) and \( x^i \) is its representative in \( H \) then \( x^{[p]} \) is defined as the homogeneous component of the element \( (x^i)^p \) and we obtain the structure of a restricted Lie algebra in \( L_p(H, H_i) \). We will denote by \( \tilde{h} \) the homogeneous component of an element \( h \in H \).

Let \( U = \bigcap_{i=1}^{\infty} H_i \), \( \bar{H} = H/U \), \( \bar{H}_i = H_i/U \ (i = 1, 2, \cdots ,) \). The definition of the algebra \( L_p(H, H_i) \) implies that there exists a natural isomorphism between the graded algebras \( L_p(H, H_i) \) and \( L_p(\bar{H}, \bar{H}_i) \); this fact reduces the study of the algebra \( L_p(H, H_i) \) to the case when \( \bigcap_{i=1}^{\infty} H_i = 1 \).

If \( h \) is an element of \( H \) and \( h \in H_i \backslash H_{i+1} \) then the weight \( w(h) \) of \( h \) is \( i \); the weight of \( 1 \) is \( \infty \).

Let \( F \) be a subgroup of \( H \). Then the series (2.1) induces in \( F \) a \( p \)-series \( F_i = F \bigcap H_i \ (i = 1, 2, \cdots ,) \). We denote the associated Lie algebra of \( F \) by \( L_p(F, F_i) \) and we will use this notation throughout the paper. There is a natural imbedding of the graded Lie algebra \( L_p(F, F_i) \) in \( L_p(H, H_i) \); if \( F \) is normal in \( H \) then \( L_p(F, F_i) \) is an ideal in \( L_p(H, H_i) \). Let \( \phi : H \to H/F = G \) be an epimorphism of groups, and let \( G_i = \phi(H_i) = (H_iF)/F \ (i = 1, 2, \cdots ,) \). We will make a use of the following result which is Theorem 2.4. in Lazard’s article [7].

**Proposition 2.1.** Let \( F \) be a normal subgroup of \( H \), \( G = H/F \). The epimorphism \( \phi : H \to G \) defines in a natural way an epimorphism \( \tilde{\phi} : L_p(H, H_i) \to L_p(G, G_i) \) which preserves the degrees of the homogeneous elements. The kernel of \( \tilde{\phi} \) is the ideal \( L_p(F, F_i) \).

**Corollary 2.1.** Assume that the normal subgroup \( F \) in Proposition 2.1. has finite index. Let \( Q \) be the subgroup formed by all the elements of \( H \) whose images in \( G \) belong to \( \bigcap_{i=1}^{\infty} G_i \). Then \( Q \) is a normal subgroup which contains \( F \), the index \( (H : Q) \) is a power of \( p \) and \( L_p(Q, Q_i) = L_p(F, F_i) \).
Proof. Let $G_0 = \bigcap_{i=1}^{\infty} G_i$. Then $Q$ is the inverse image of $G_0$ in $H$ and $Q \supseteq F$. The quotient algebras $L_p(H, H_i)/L_p(Q, Q_i)$ is isomorphic to $L_p(G, G_i)$, so $L_p(F, F_i) = L_p(Q, Q_i)$.

The quotient group $\bar{H} = H/Q$ contains a $p$-series $\bar{H}_i = (H_iQ)/Q$ ($i = 1, 2, \cdots$) with unit intersection. This series has a finite length because the group $H/Q$ is finite. This implies that $H/Q$ is a finite $p$-group, and the proof is complete.

The following result is the first half of Theorem 2.2 in Lazard’s article [7].

Proposition 2.2. Let $\phi$ be a homomorphism from a group $H$ into a group $G$. Let $H_i$ and $G_i$ be $p$-series in $G$ and $H$ respectively and assume that $\phi(H_i) \subseteq G_i$ ($i = 1, 2, \cdots$). Then $\phi$ defines in the following way a homomorphism $\tilde{\phi}: L_p(H, H_i) \rightarrow L_p(G, G_i)$ such that if $h \in H$ is an element of weight $k$ then

$$
\tilde{\phi}(\tilde{h}) = \phi(h) + G_{k+1}
$$

The following two facts follow from this theorem immediately.

Corollary 2.2. Assume that the conditions of Proposition 2.2. hold. Let $h$ be an element of $H$ and $\tilde{h}$ be its homogeneous component. The element $\tilde{h}$ belongs to the kernel of $\tilde{\phi}$ iff the weight of $\phi(h)$ is greater than the weight of $h$. If $\phi(h) \neq 0$ then $\tilde{\phi}(\tilde{h}) = \phi(h)$.

Corollary 2.3. Let $\Phi$ be a group of automorphisms of $H$ such that $\phi(H_i) = H_i$ ($i = 1, 2, \cdots$). Then the group $\Phi$ defines in a natural way a group of automorphisms $\tilde{\Phi}$ of the restricted Lie algebra $L_p(H, H_i)$; the homogeneous components $H_i/H_{i+1}$ ($i = 1, 2, \cdots$) are $\tilde{\Phi}$-invariant.

Lemma 2.1. Let $h$ be a non-unit element of $H$, $\tilde{x}$ be its homogeneous component. Assume that $w(h) = k$. If $w(h^{p^n}) = kp^n$ then $\tilde{h}^{p^n} = \tilde{h}[p^n]$; if $w(h)p^n > p^n w(h)$ then $\tilde{h}[p^n] = 0$.

The proof is straightforward.

Lemma 2.2. i) Let $[h_1, h_2, \cdots, h_l]$ be a right normed commutator in $H$. If its weight is equal to $w(h_1) + w(h_2) + \cdots + w(h_l)$ then the homogeneous
component of this commutator is $[\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_l]$; if its weight is greater than $w(h_1) + w(h_2) + \cdots + w(h_l)$ then $[\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_l] = 0$.

ii) Let $h_1, h_2, \cdots, h_l$ be elements of $H$ with the same weight $q$. Assume that the weight of the element $h_1 h_2 \cdots h_l$ is also $q$. Then the homogeneous component of $h_1 h_2 \cdots h_l$ is

$$\tilde{h}_1 + \tilde{h}_2 + \cdots + \tilde{h}_l \quad (2.2)$$

If the weight of $h_1 h_2 \cdots h_l$ is greater than $q$ then $\tilde{h}_1 + \tilde{h}_2 + \cdots + \tilde{h}_l = 0$.

**Proof.** All the statements are known facts whose proofs are straightforward. For instance, the proof of statement ii) is obtained in the following way. If the condition of the assertion hold then the image of $h_1 h_2 \cdots h_l$ in the quotient group $H_q/H_{q+1}$ is the sum of the images of the elements $h_i$ ($i = 1, 2, \cdots, l$), hence its homogeneous component is (2.2). The rest of the statements are proven by similar arguments.

**Corollary 2.4.** Let $H$ be a nilpotent group. Then the restricted Lie algebra $L_p(H, H_i)$ is a nilpotent Lie algebra.

**Proof.** Follows immediately from statement i) of Lemma 2.2.

**2.2.** We need an analog of Proposition 2.1. and Lemmas 2.1. and 2.2. for rings. Let $R$ be a ring with a non-negative pseudovaluation $v$, $R_i$ is the filtration defined by $v$, that is

$$R_i = \{r \in R | v(r) \geq i\} \quad (i = 0, 1, \cdots)$$

It is clear that the pseudovaluation function $v$ is completely defined if the filtration $R_i$ is given. Let $A$ be an ideal in $R$, we have in $A$ an induced filtration $A_i = A \cap R_i$ ($i = 0, 1, \cdots$), let $gr(R)$ and $gr(A)$ be the graded ring of $R$ and $A$ associated to the filtrations $R_i$, $A_i$ ($i = 0, 1, \cdots$) respectively. Further if $\bar{X}$ denotes the image of a subset $X \in R$ under the natural homomorphism $\phi: R \to R/A$ we obtained in $\bar{R}$ a filtration $\bar{R}_i$ ($i = 0, 1, \cdots$) and a pseudovaluation $\bar{v}$ defined by this filtration. It is natural to say that pseudovaluation $\bar{v}$ is obtained from $v$ by the homomorphism $\phi$. The following fact is a modified version of Proposition 2 in section 3.4. of Bourbaki [2]; its proof can be read off from [2] or obtained by a straightforward argument.
Proposition 2.3. The homomorphism \( \phi: R \rightarrow R/A \) defines in \( \bar{R} \) a filtration \( \bar{R}_i \) \((i = 0, 1, \cdots)\), a pseudovaluation \( \bar{v} \) defined by this filtration and a homomorphism of graded rings \( gr(\phi): gr(R) \rightarrow gr(\bar{R}) \) with kernel \( gr(A) \).

Proposition 2.4. Let \( R \) be a ring, \( v \) and \( w \) be two non-negative pseudovaluations which define equivalent topologies in \( R \). Assume that there exists a system of elements \( T \subseteq R \) such that \( v(t) = w(t) \) \((t \in T)\) and the graded rings \( gr_v(R) \) and \( gr_w(R) \) are isomorphic to the polynomial rings \((R/A)[\tilde{T}_v]\) and \((R/A)[\tilde{T}_w]\) respectively, where \( A \) is the ideal generated by \( T \) and \( \tilde{T} \) is the set of the homogeneous components of elements \( t \in T \). Then \( v(r) = w(r) \) for every \( r \in R \).

Proof. Let \( \mathcal{X} \) be a system of coset representatives for the quotient ring \( R/A \), \( \pi_\alpha \) \((\alpha = 1, 2, \cdots, n)\) be distinct standard monomials with \( v \)-value \( k \). Then

\[
v(\sum_{\alpha=1}^{n} \lambda_\alpha \pi_\alpha) = k
\]  

(2.3)

and

\[
w(\sum_{\alpha=1}^{n} \lambda_\alpha \pi_\alpha) = k
\]  

(2.4)

because of the condition \( gr_v(R) \cong gr_w(R) \cong (R/A)[\tilde{T}] \); we obtain in particular that \( w(\pi) = v(\pi) \) for an arbitrary standard monomial.

Now assume that there exists an element \( r \in R \) such that \( v(r) = k \) and \( w(r) = l > k \). Let \( A_i, B_j \) \((i = 0, 1, \cdots; j = 0, 1, \cdots)\) be the filtration defined in \( R \) by the pseudovaluations \( v \) and \( w \) respectively. Since the topologies defined by \( v \) and \( W \) are equivalent we can find \( j > k \) such that \( A_j \subseteq B_{l+1} \). Let \( \bar{R} = R/A_j \). We have in \( \bar{R} \) pseudovaluations \( \bar{v} \) and \( \bar{w} \); since the \( v \)-values and \( w \)-values of elements from \( A_j \) are greater then \( l \) we obtain that

\[
\bar{v}(\bar{r}) = k, \bar{w}(\bar{r}) = l
\]  

(2.5)

The ring \( \bar{R} \) has now a pseudovaluation \( \bar{v} \) defined by the filtration \( \bar{A}_i = A_i/A_j \) \((i = 1, 2, \cdots; j, \) the graded ring \( gr_\bar{v}(\bar{R}) \) is an isomorphic image of the polynomial ring \((R/A)[\tilde{T}_v]\), it is generated over the subring \( R/A \) by all the standard monomials whose \( v \)-value less than or equal to \( j - 1 \) and the standard monomials with \( v \)-value \( i \) form a basis of the homogeneous component.
$\bar{A}_i/\bar{A}_{i+1}$ $(i = 1, 2, \cdots, j - 1)$. A routine argument shows that every element $\bar{x} \in \bar{R}$ has a unique representation

$$\bar{x} = \sum_{\alpha=0}^{m} \mu_{\alpha} \pi_{\alpha}$$

(2.6)

where $\mu_{\alpha} \in \mathcal{X}$ ($\alpha = 1, 2, \cdots, m$) and $v(\pi_{\alpha}) \leq j - 1$ ($\alpha = 1, 2, \cdots, m$).

Since $v(\bar{r}) = k$ we conclude that $\bar{r}$ has a unique representation

$$\bar{r} = \sum_{\beta} \mu_{\beta} \pi_{\beta}$$

(2.7)

where all the standard monomials $\pi_{\beta}$ have $v$-value greater than or equal to $k$.

We consider now the following element $r_1 \in R$

$$r_1 = \sum_{\beta} \mu_{\beta} \pi_{\beta}$$

(2.8)

The image of $r_1$ in $\bar{R}$ is $\bar{r}$; on the other hand representation (2.8) implies that $v(r_1) = w(r_1) = k$. Since all the elements of $A_j$ have $v$-values and $w$-values greater than $k$ we obtain that $v(\bar{r}) = w(\bar{r}) = k$. We obtained a contradiction with relation (2.5) and the assertion follows.

The proof of following analog of Lemma 2.2. for rings is obtained by the same straightforward argument.

**Lemma 2.3.** Let $R$ be a ring with a pseudovaluation $\rho$ and associated graded ring $gr(R)$, $r_1, r_2, \cdots, r_l$ be elements of $R$ with the same weight $q$.

i) If the weight of the element $r_1 + r_2 + \cdots + r_l$ is $q$ then its homogeneous component in $gr(R)$ is equal to $\bar{r}_1 + \bar{r}_2 + \cdots \bar{r}_l$; if the weight of element $r_1 + r_2 + \cdots + r_l$ is greater than $q$ then $\bar{r}_1 + \bar{r}_2 + \cdots + \bar{r}_l = 0$.

ii) If the weight of the element $r_1 r_2 \cdots r_l$ is $lq$ then its homogeneous component is $\bar{r}_1 \bar{r}_2 \cdots \bar{r}_l$; if the weight of this element is greater than $lq$ then $\bar{r}_1 \bar{r}_2 \cdots \bar{r}_l = 0$.

**2.3. Lemma 2.4.** Assume that the algebra $L_{\rho}(H, H_i)$ is generated by the first $l$ homogeneous components. Let $h \in H$ be an element of weight $r$. Then the homogeneous component $\bar{h}$ is a sum of Lie monomials

$$[\bar{h}_{\alpha_1}, \bar{h}_{\alpha_2}, \cdots, \bar{h}_{\alpha_s}]^{[p]^{\alpha}}$$

(2.9)
where the homogeneous elements $\tilde{h}_{\alpha_1}, \tilde{h}_{\alpha_2}, \cdots, \tilde{h}_{\alpha_s}$ are taken from the first $l$ factors $H_i/H_{i+1}$, the weight of every monomial $(2.9)$ is $r$, and

$$w([\tilde{h}_{\alpha_1}, \tilde{h}_{\alpha_2}, \cdots, \tilde{h}_{\alpha_s}]) = w(\tilde{h}_{\alpha_1}) + w(\tilde{h}_{\alpha_2}) + \cdots + w(\tilde{h}_{\alpha_s}) = r_1 \quad (2.10)$$

where $r = p^{n_\alpha} r_1$.

Further, if $h_{\alpha_i}$ is the coset representative of $\tilde{h}_{\alpha_i}$ ($i = 1, 2, \cdots, s$) then the element

$$[h_{\alpha_1}, h_{\alpha_2} \cdots, h_{\alpha_s}]^{p^{n_\alpha}} \quad (2.11)$$

is the representative in $H$ of the homogeneous component $(2.9)$.

**Proof.** Since the algebra $\mathcal{L}_p(H, H_i)$ is graded the element $\tilde{h}$ must be a sum of homogeneous Lie monomials $(2.9)$ of degree $r$. Lemma 2.2. implies that if a Lie monomial $(2.9)$ is non-zero then condition $(2.10)$ holds, and that the element $(2.11)$ is a coset representative of the element $(2.9)$ in $H_r/H_{r+1}$, and the proof is complete.

We will make an essential use of the following fact which is proven in Lichtman [10]. (See [10], Lemmas 2.6. and 3.1.)

**Proposition 2.5.** Let $H$ be a polycyclic group with Hirsch number $r$. Assume that there exists a $p$-series (1.1) such that the Lie algebra $\mathcal{L}_p(H, H_i)$ is abelian. Then the rank of the algebra $\mathcal{L}_p(H, H_i)$ does not exceed $r$; if this rank is equal $r$ then the subgroup $\bigcap_{i=1}^{\infty} H_i$ is finite.

We will need also the following result.

**Proposition 2.6.** Let $H$ be a polycyclic group with Hirsch number $r$, $F$ be an normal subgroup with a Hirsch number $r_1$. Assume that there exists a $p$-series with associated graded algebra $\mathcal{L}_p(H, H_i)$ abelian of rank $r$.

Then the Lie algebra $\mathcal{L}_p(F, F_i)$ associated to the $p$-series $F_i = F \cap H_i$ ($i = 1, 2, \cdots$) is abelian of rank $r_1$. The intersection of the $p$-series $\tilde{H}_i = (H_iF)/H_i$ ($i = 1, 2, \cdots$) is a finite subgroup and the Lie algebra $\mathcal{L}_p(\tilde{H}, \tilde{H}_i)$ of the group $\tilde{H} = H/F$ associated to the $p$-series $\tilde{H}_i$ ($i = 1, 2, \cdots$) is abelian of rank $r - r_1$.

**Proof.** We can assume that $H$ is infinite. Let $U$ be a torsion free normal subgroup of finite index, $V = U \cap F$. The subagebra $\mathcal{L}_p(U, U_i)$ associated to
the $p$-series $U_i = H_i \cap U$ ($i = 1, 2, \cdots$) has a finite index in $L_p(H, H_i)$ so its rank is equal to $r$; similarly the rank of the subalgebra $L_p(V, V_i)$ is equal to the rank of $L_p(F, F_i)$. We see that we can assume that the group $U$ is torsion free. The first statement follows now from Proposition 3.5 in Lichtman [10]; the second statement follows from Proposition 3.2 in [10].

2.4. The study of $p$-series in groups and Lie algebras associated to them is connected to filtrations and valuations in the group rings of these groups. We have already observed that in the study of the properties of the algebra $L_p(H, H_i)$ we can assume that $\bigcap_{i=1}^{\infty} H_i = 1$ and $p$-series (1.1) defines in a natural way a weight function in the group $H$ and this weight function defines a filtration in the group ring $KH$ (see Passman, [13]):

\[ A_0 = KH \supseteq A_1 \supseteq A_2 \supseteq \cdots \]  

(2.12)

where $A_n$ ($n \geq 1$) is the $K$-linear span of the set of all the products

\[ \{(x_{\alpha_1} - 1)(x_{\alpha_2} - 1) \cdots (x_{\alpha_s} - 1) \mid \sum_{i=1}^{s} w(x_{\alpha_i}) \geq n\} \]  

(2.13)

We recall that if $H_i = M_i(KH)$ ($i = 1, 2, \cdots$) is the Lazard-Zassenhaus-Jennings $p$-series (see Passman [13], section 11) then the filtration defined by it is $\omega^n(KH)$ ($n = 1, 2, \cdots$) where $\omega(KH)$ is the augmentation ideal of $KH$. Series (2.1) defines the $p$-topology in the group $H$ and filtration (2.12) defines a topology in the group ring $KH$ which we will call $p$-topology in $KH$. The following fact follows easily from the classical theorems on group rings, we give a sketch of the proof.

**Lemma 2.6.** If the topology defined by $p$-series (2.1) in the group $H$ is equivalent to the $p$-topology then the topology defined by the filtration (2.12) in the group ring $KH$ is equivalent to the $p$-topology of $KH$.

**Proof.** Let $\omega^k(KH)$ be a given power of the augmentation ideal. We have to find $i = i(k)$ such that $A_i \subseteq \omega^n(KH)$. Since $(h-1) \in \omega^n(KH)$ ($h \in M_n(H)$) the proof is easily reduced to the case of the group $H = H/M_n(H)$ and the $p$-series $\tilde{H}_i = (H_iM_n(H))/M_n(H)$ ($i = 1, 2, \cdots$); the series $\tilde{H}_i$ ($i = 1, 2, \cdots$) has a finite length because there exists a number $j$ such that $H_j \subseteq M_n(H)$. We see that we can assume that the original series $H_i$ ($i = 1, 2, \cdots$) has a finite length.
We obtain now that the weights of the factors \( x_\alpha - 1 \) in the left side of (2.13) are bounded; this implies that if the number \( n \) in (2.13) runs to infinity then \( s \to \infty \). We obtain from this that if \( n \) is sufficiently large then \( s \geq k \) and \( A_i \subseteq \omega^k(KH) \) if \( i \geq s \).

If \( \bigcap_{n=1}^\infty A_n = 0 \) the filtration (2.12) defines in a natural way a pseudovaluation \( \rho \) in \( KH \): if \( x \neq 0 \) then \( \rho(x) \) is equal to the maximal \( n \) such that \( x \in A_n \), and \( \rho(0) = \infty \). We will say that this pseudovaluation is defined by \( p \)-series (2.1). We recall that this pseudovaluation is a valuation if the graded ring \( gr(KH) \) is a domain; in the case of the group ring \( KH \) this condition can be replaced by the equivalent condition that \( U_p((L_p(H, H_i))) \) is a domain (see Proposition 2.7. below). Further it is known that if filtration (2.12) defines a valuation in \( KH \) the topological completion of \( KH \) must be a domain.

We will need the following fact (see Lichtman [10], Lemma 3.2.)

**Lemma 2.7.** Let \( H \) be a group which contains a \( p \)-series (2.1) with unit intersection, \( N \) be a normal subgroup of \( H \). Assume that the topology defined by the series \( N_i = N \cap H_i (i = 1, 2, \ldots) \) in \( N \) is equivalent to the \( p \)-topology in \( N \), and the topology defined by the series \( \hat{H}_i = (H_iN)/N \ (i = 1, 2, \ldots) \) in the quotient group \( \hat{H} = H/N \) is equivalent to the \( p \)-topology in \( \hat{H} \). Then the topology defined in \( H \) by series (2.1) is equivalent to the \( p \)-topology in \( H \).

**Lemma 2.8.** Let \( H \) be a free abelian group of rank \( r \geq 1 \) which contains a \( p \)-series (2.1.) with unit intersection and the associated graded algebra \( L_p(H, H_i) \) finitely generated.

i) If the rank of \( H \) is 1 then so is the rank of \( L_p(H, H_i) \).

ii) If the rank of \( L_p(H, H_i) \) is equal to \( r \) then the topology defined by series (2.1) is equivalent to the \( p \)-topology.

**Proof.** We will prove first both statements for the case when \( H \) has rank 1. Since \( L_p(H, H_i) \) is finitely generated we can find a number \( i_0 \) such that the subalgebra

\[
\sum_{i \geq i_0} H_i/H_{i+1}
\]  

(2.14)

contains no nilpotent elements. Hence this subalgebra is free abelian of rank 1.
On the other hand the subalgebra (2.14) is the restricted Lie algebra of the subgroup \( V = H_{i_0} \) which corresponds to the \( p \)-series

\[
V = H_{i_0} \supseteq H_{i_0+1} \supseteq \cdots \tag{2.15}
\]

Let \( v \) be the generator of \( V \), \( \tilde{v} \) be the homogeneous component of \( v \). The weight of \( v \) is \( i_0 \), since \( \tilde{v} \) is not nilpotent we obtain from Lemma 2.2 that the homogeneous component of the element \( v^p^n \) is equal to \( \tilde{v}^{[p]^n} \) and the weight of \( v^p^n \) is \( p^n i_0 \). We see that the topology defined in \( V \) by series (2.14) is equivalent to the \( p \)-topology. Since the quotient group \( H/H_{i_0} \) is a finite cyclic \( p \)-group we conclude from Lemma 2.7 that the topology defined in \( H \) by series (2.1) is equal to the \( p \)-topology. This completes the proof for the special case when the rank of \( H \) is 1.

We consider now the general case. Let \( u = h_1, h_2, \cdots, h_r \) be a free system of generators for \( H \), \( U \) be the subgroup generated by the element \( u \). Proposition 2.6 implies that the algebra \( L_p(U, U_i) \) associated to the \( p \)-series \( U_i = H \cap U_i \) is finitely generated abelian of rank 1, that the \( p \)-series \( \bar{H}_i = (H/U)/U \) in the group \( \bar{H} = H/U \) has a unit intersection and the algebra \( L_p(\bar{H}, \bar{H}_i) \) is finitely generated abelian of rank \( r - 1 \). We have already proven that the topology defined in \( U \) by the series \( U_i \) is equivalent to the \( p \)-topology, and the assumption of the induction implies that the topology defined in \( H \) by the series \( \bar{H}_i \) is equivalent to the \( p \)-topology. The assertion now follows from Lemma 2.7.

2.5. Let \( H \) be a group, \( N \) be its normal \( p \)-subgroup and \( V \supseteq N'N^p \) be an \( H \)-invariant subgroup of \( H \), \( G = H/N \). The conjugation in \( H \) defines in a natural way a structure of a \( \mathbb{Z}_pH \) module in the quotient group \( \bar{N} = N/V \); this module is in fact a \( \mathbb{Z}_pG \)-module.

**Lemma 2.9.** Assume that the module \( \bar{N} \) has dimension less than or equal to \( r \). Then the following conditions are equivalent

i) For every \( h \in H \) there exists a number \( n(h) \) such that \([h^{p^n(h)}, N] \subseteq V \)

ii) \([h^p, N] \subseteq V \)

iii) \( \bigcap_{i=1}^{\infty} \omega^i(Z_pG)^i \cdot \bar{N} = 0 \)

iv) \( \omega^r(Z_pG) \cdot \bar{N} = 0 \)

**Proof.** See [10], Lemma 2.15.
Corollary 2.5. Assume that $\bar{H} = H/N'N^p$ is a residually \{finite $p$-group\} and the rank of the vector space $\bar{N} = N/N'N^p$ does not exceed $r$. Then

$$\omega^r(Z_pG) \bullet \bar{N} = 0$$

and

$$[h^{p^r}, N] \subseteq N'N^p$$

Proof. If $\bar{H}$ is a residually \{finite $p$-group\} then condition iii) of Lemma 2.9 holds and the assertion now follows from Lemma 2.8.

The following lemma is a known fact, its proof is obtained by a routine argument.

Lemma 2.10. Let $H$ be a group which contains a finite normal $p$-subgroup $U$ such that that quotient group $H/U$ is an extension of a finitely generated torsion free nilpotent group by a finite $p$-group. Assume also that there exists a number $l$ such that every $h^{p^l}$ ($h \in H$) centralizes the factor $U/U'^{p^l}$. Then $H$ contains a finitely generated torsion free nilpotent normal subgroup $V$ whose index is a power of $p$.

2.6. We will need the following fact which follows immediately from Theorem 2.4 in Lichtman [12]; this fact is a generalization of Quillen’s theorem [15].

Proposition 2.7. Let $H$ be a group, $\{2.1\}$ be an arbitrary $p$-series in $H$, $K$ be a commutative domain of characteristic $p$ and $\{2.13\}$ be the filtration defined in $KH$ by this $p$-series. Let $gr(KH)$ be the graded ring associated to this filtration. Let $h \in H$ be an element of weight $i$. Then the correspondence

$$hH_{i+1} \longrightarrow (h - 1) + A_{i+1}$$

defines an isomorphism between the graded algebras $gr(KH)$ and $K \otimes U_p(L_p(H, H_i))$; in particular the restricted Lie subalgebra generated in $gr(KH)$ by all the elements $h - 1 + A_{i+1}(KH)$ is isomorphic to the restricted Lie algebra $L_p(H, H_i)$. If $U$ is a normal subgroup of $H$ then the elements $u - 1 + A_{i+1}$ ($u \in U$) generate a restricted Lie subalgebra isomorphic to the subalgebra $L_p(U, U_i)$. Further, the subring $gr(KU)$ is isomorphic to the
universal $p$-envelope $U_p(L_p(U_i))$ and the algebra $\text{gr}(KH)$ is isomorphic to a suitable smashed product of $\text{gr}(KU)$ with the restricted Lie algebra of the group $H/U$ associated to the $p$-series $(H_i U)/U$ ($i = 1, 2, \cdots$).

Let $R$ be an algebra over a field $K$, $v$ be a pseudovaluation in $R$, $R_i$ ($i \in \mathbb{Z}$) be the filtration defined by $v$, i.e. $R_i = \{r \in R|v(r) \geq i\}$, $\text{gr}(R)$ be the associated graded ring. We extend $v$ in a natural way to the Laurent polynomial ring $R[t, t^{-1}]$ assuming that $v(t) = 1$. Let $V$ be the valuation ring of $R[t, t^{-1}]$, i.e. $V = \{x \in R[t, t^{-1}]|v(x) \geq 0\}$. The following fact is Lemma 4.3. and Corollary 4.1. in Lichtman [9].

Proposition 2.8. There exists an isomorphism $\psi$ between the rings $\text{gr}(R)$ and $V/(t)$. If $e_j$ ($j \in J$) is a system of elements of $R_i$ which gives a basis of $R_i/R_{i+1}$ then the images of the elements $e_j t^{-i}$ ($j \in J$) in $V/(t)$ form a basis of the subspace $\psi(R_i/R_{i+1})$ of $V/(t)$.

We consider once again now a group $H$ with a $p$-series (2.1), let $v$ be the pseudovaluation defined in $KH$ by this series, we have also the corresponding filtration (2.12). Let $h_j$ be an arbitrary element of $H$ with weight $n_j$ so $v(h_j - 1) = n_j$, let $\tilde{h}_j$ be the homogeneous component of $h_j$ in the algebra $L_p(H, H_i)$. All the elements of $KH$ have non-negative values and we see that the quotient ring $V/(t)$ is generated over $K$ by the images $(h - 1)t^{-n_j}$ ($j \in J$) of the elements $(h - 1)t^{-n_j}$ ($j \in J$). We apply now Propositions 2.7. and 2.8. and obtain the following representation for the algebra $L_p(H, H_i)$.

Proposition 2.9. There exists an isomorphism $\theta: V/(t) \cong U_p(L_p(H, H_i))$ defined by the map $(h_j - 1)t^{-n_j} \mapsto \tilde{h}_j$ ($j \in J$). The elements $(h_j - 1)t^{-n_j}$ ($j \in J$) in $V/(t)$ generate with respect to the Lie operations in $V/(t)$ a subalgebra isomorphic to $L_p(H, H_i)$.

Now let $v$ be a discrete pseudovaluation in the group ring $KH$ where $K$ is a field of characteristic $p$ such that $v(h - 1) \geq 1$ for every $h \in H$. This pseudovaluation defines a filtration in $KH$

$$A_n(KH) = \{ x \in KH | v(x) \geq n \} \ (n = 0, 1, \cdots) \quad (2.19)$$

and also a $p$-series in $H$

$$H_i = \{ h \in H | v(h - 1) \geq i \} \ (i = 1, 2, \cdots) \quad (2.20)$$
Let \( \tilde{(h-1)} \) be the homogeneous component of the element \( h-1 \) in the graded ring associated to the filtration (2.19). The following fact is a special case of Theorem 6.6. in Lazard [7].

**Proposition 2.10.** The elements \( \tilde{(h-1)} \) \( (h \in H) \) generate in \( gr_v(KH) \) a graded Lie subalgebra isomorphic to the Lie algebra \( L_p(H,H_i) \).

**Corollary 2.6.** If the pseudovaluation \( v \) in Proposition 2.10. is a valuation then the algebra \( L_p(H,H_i) \) contains no nilpotent elements.

**Proof.** Follows from the fact that an element \( \tilde{(h-1)} \) \( (h \in H) \) is not nilpotent.

Now pick a natural number \( n \) and consider a discrete pseudovaluation \( v_1 \) in \( KH \) which is defined as \( v_1(x) = nv(x) \) \( (x \in KH) \); clearly, \( v_1 \) is equivalent to \( v \), the filtrations defined by \( v \) and \( v_1 \) coincide and we obtain the same graded ring \( gr(KH) \) for both pseudovaluations. We obtain from this the following fact which will be used in the proof of Theorem XII.

**Corollary 2.7.** The pseudovaluations \( v \) and \( v_1 \) in \( KH \) define in \( H \) the same \( p \)-series \( H_i \) \( (i = 1, 2, \ldots) \) and define the same algebra \( L_p(H,H_i) \).

2.7. We will need a few facts about free abelian graded Lie algebras. We assume that the algebras are graded by a set \( I \), which is either the set of natural numbers or the set of integers modulo some number \( m \).

**Lemma 2.11.** Let \( F \) be a restricted graded abelian Lie algebra without nilpotent elements over a field \( K \). Assume that

\[
F = \bigoplus_{i \in I} F_i
\]

is a grading in \( F \). Then the algebra \( F \) is free abelian. Further, there exists a subset \( I_1 \subseteq I \) such that

\[
F^{[p]} = \bigoplus_{i \in I_1} F_i
\]

and \( F \) has a direct sum representation

\[
F = M + F^{[p]}
\]
where $M$ is graded subspace of $F$ and

\[ M = \bigoplus_{i \in I_2} F_i \tag{2.24} \]

where $I_2$ is the complement of $I_1$ in $I$. Let $E_i$ be system of homogeneous elements of $F$ which forms a basis of the vector subspace $F_i (i \in I_2)$ and let $E = \bigcup_{i \in I_2} E_i$. Then $E$ is a free system of generators for $F$, i.e. the subset

\[ \bigcup_{n=1}^{\infty} E^{[p]^n} \tag{2.25} \]

is a basis of $F$ over $K$. The universal $p$-envelope of $F$ is isomorphic to the symmetric algebra $K[M]$ which is the polynomial algebra over $K$ in the system of variables $E$.

**Proof.** The map $x \mapsto x^{[p]} (x \in F)$ defines an epimorphism of graded algebras $F \rightarrow F^{[p]}$. Since $F$ contains no nilpotent elements we obtain immediately that the images of distinct homogeneous components $F_{i_1}$ and $F_{i_2}$ are distinct homogeneous components $F^{[p]}_{i_1}$ and $F^{[p]}_{i_2}$ in $F^{[p]}$ and relations (2.23) – (2.25) follow easily.

Since $E$ is a basis for $M$ we obtain that it is a free system of generators for $F$. This completes the proof.

The subspace $M \cong \bar{F}$ is in fact a graded subspace of $F$ and it is natural to say that the free restricted algebra $F$ is freely generated by the vector subspace $M$, and the universal $p$-envelope of $F$ is isomorphic to the symmetric algebra $K[M]$. Every subspace $F_i (i \in I_2)$ generate an ideal $K[F_i]$.

We will need a refinement of Lemma 2.11. for the case when $F$ is a restricted Lie algebra of a group $H$.

**Corollary 2.8.** Let $H$ be a group which contains a $p$-series (2.1) with unit intersection such that the algebra $L_p(H, H)$ free abelian. Let $E$ be a system of elements obtained in Lemma 2.11. Then every element of $E$ is a homogeneous component of an element $h \in H$ that is $e = \tilde{h}$ for a suitable $h \in H$. Moreover, every non-zero element of the subspace $F_i (i \in I_2)$ is a homogeneous component in the algebra $L_p(H, H)$.

**Proof.** The first statement follows from the definition of $E$.
We prove the second statement. Let \( i \in I_2 \) and \( 0 \neq h \in F_i \). Then

\[
x = \sum_{j=1}^{n} \lambda_j e_j
\]  

(2.26)

where \( e_j \in E_i \) (\( j = 1, 2, \cdots, n \)) are homogeneous elements of weight \( i \), \( e_j = \hat{h}_j \) (\( j = 1, 2, \cdots, n \)) and \( \lambda_j \in \mathbb{Z}_p \) (\( j = 1, 2, \cdots, n \)). Let \( 1 \leq n_j \leq p-1 \) be a natural number which is a coset representative of \( \lambda_j \) (\( j = 1, 2, \cdots, n \)). We obtain now from Lemma 2.3. ii) that the weight of the element \( h = \prod_{j=1}^{n} h_{j}^{n_j} \) is \( i \) and it is a coset representative for the the element \( x \), that is \( x = \hat{h} \), and the assertion follows.

2.8. Let \( R \) be a ring, \( \phi \) be an automorphism of \( R \). A pseudovaluation \( \rho \) is \( \phi \)-invariant if for every \( r \in R \)

\[
\rho(\phi(r)) = \rho(r)
\]

(2.27)

If the pseudovaluation \( \rho \) is \( \phi \)-invariant the automorphism \( \phi \) defines in a natural way an automorphism \( \tilde{\phi} \) of the ring \( gr(R) \). If \( \tilde{r} \) is a homogeneous element of degree \( k \) in \( gr(R) \) and \( r \) is an arbitrary element of \( R \) with \( \rho \)-value \( k \) then \( \tilde{\phi}(\tilde{r}) = 0 \) if \( \rho(\phi(r)) > k \) and

\[
\tilde{\phi}(\tilde{r}) = \tilde{\phi}(\tilde{r})
\]

(2.28)

if \( \rho(\phi(r)) = k \). We have also

\[
\tilde{\phi}\left(\sum_{i=1}^{n} \tilde{r}_i\right) = \sum_{i=1}^{n} \tilde{\phi}(\tilde{r}_i)
\]

(2.29)

for homogeneous elements \( \tilde{r}_i \) (\( i = 1, 2, \cdots, n \)).

We will say that an automorphism \( \phi \) of \( R \) centralizes the ring \( gr(R) \) if it acts trivially on \( gr(R) \). This is equivalent to the condition

\[
\tilde{\phi}(\tilde{r}) = \tilde{r}
\]

(2.30)

for all the homogeneous elements \( \tilde{r} \in gr(R) \) and it means that for every \( r \neq 0 \) there exists \( u \in R \) such that

\[
\phi(r) = r + u \text{ where } \rho(u) > \rho(r)
\]

(2.31)

or that
\[ \rho(\phi(r) - r) > \rho(r) \quad (r \neq 0) \quad (2.32) \]

We will need the version of these relations for group rings.

**Lemma 2.12.** Let \( H \) be a group, \( \phi \) be an automorphism of \( H \) and assume that a filtration and a pseudovaluation \( \rho \) of \( KH \) are defined by a \( \phi \)-invariant \( p \)-series (2.1). The automorphism \( \phi \) centralizes the graded ring \( \text{gr}_\rho(KH) \) iff it centralizes all the factors \( H_i/H_{i+1} \) \((i = 1, 2, \cdots)\) of the series i.e. if \( h \in H_i \setminus H_{i+1} \) then there exists \( u \in H_{i+1} \) such that \( \phi(h) = hu \).

**Proof.** If \( h \) and \( u \) are as in the condition of the lemma then relation
\[ hu - 1 = (h - 1) + (u - 1) + (h - 1)(u - 1) \quad (2.33) \]
implies that the homogeneous components of the elements \( h - 1 \) and \( hu - 1 \) coincide, hence the homogeneous components of the elements \( (h - 1) \) and \( \phi(h - 1) \) are equal iff the condition \( \phi(h) = hu \) holds, and the assertion follows.

**Corollary 2.9.** The automorphism \( \phi \) centralizes the graded ring \( \text{gr}_\rho(KH) \) iff it centralizes the algebra \( L_p(H, H_i) \).

**§3. The Lifting of the Valuations.**

3.1. Let \( R \) be a ring, \( A \) be a residually nilpotent ideal
\[ \bigcap_{i=1}^{\infty} A^i = 0 \quad (3.1) \]
and \( gr(R) \) be the graded ring associated to the filtration
\[ R \supseteq A \supseteq A^2 \supseteq \cdots \quad (3.2) \]

Let \( \rho \) be the pseudovaluation defined by filtration (3.2), \( \tilde{R} \) be the completion of \( R \) in the topology defined by \( \rho \) and \( \mathcal{X} \) be a system of coset representatives for the quotient ring \( R/A \). Now assume that \( A \) is generated by a central element \( t \) then an arbitrary element \( r \in \tilde{R} \) a representation
\[ r = \sum_{j=0}^{\infty} \lambda_j t^j \] \quad (\lambda_j \in \mathcal{X}; \ (j = 0, 1, \cdots) \quad (3.3)

In this notation we have the following simple fact.

**Lemma 3.1.** Let \( A \) be a residually nilpotent ideal of \( R \) generated by a central element \( t \). The following three conditions are equivalent:

1) The graded ring \( \text{gr}(R) \) associated to the pseudovaluation \( \rho \) is isomorphic to the polynomial ring \((R/A)[t]\).

2) Representation \((3.3)\) is unique.

3) The element \( t \) is regular.

**Proof.** If condition 1) hold then every element \( x \) in the factor \( A^n/A^{n+1} \) has a unique representation \( x = \lambda t^n \) (\( \lambda \in \mathcal{X} \)). We conclude from this that if we take an arbitrary natural number \( n \) and consider the quotient ring \( R/A^n \) then every element of \( R/A^n \) has a unique representation

\[ x = \sum_{j=0}^{m} \lambda_j t^j \] \quad (3.4)

where \( \lambda_j \in \mathcal{X} \) (\( j = 1, 2, \ldots, m \)). Since the ring \( \tilde{R} \) is the inverse limit of the system of rings \( R/A^n \) (\( i = 1, 2, \cdots \)) we obtain that every element of \( \tilde{R} \) has a unique representation \((3.3)\).

Assume that condition 2) holds. We derive from this that every element of \( \tilde{R} \) modulo \( A^n \) has a unique representation as a polynomial of degree less than or equal \( n \) with coefficients from \( \mathcal{X} \) and hence every element of \( A^n/A^{n+1} \) has a unique representation \( x = \lambda t^n \) (\( \lambda \in \mathcal{X} \)).

We pick now elements \( t^k \) and \( t^l \). The element \( t^k t^l \) has a unique representation as a monomial \( t^{k+l} \). Hence the associated graded ring \( \text{gr}(\tilde{R}) \) is isomorphic to the polynomial ring \((R/A)[t]\) and the same is true for the ring \( \text{gr}(R) \cong \text{gr}(\tilde{R}) \). This proves that \( 2) \rightarrow 1) \).

We prove now the equivalence of 2) and 3). If 2) holds then the element \( t \) is regular in \( R \) because its homogeneous component is regular in \( \text{gr}(R) \). Conversely, assume that \( t \) is regular and let \( x \in A^n \). There exists \( \lambda \in R \) such that \( x = \lambda t^n \). If now \( x \not\in A^{n+1} \) then \( \lambda \not\in A \); further we can assume that \( \lambda \in \mathcal{X} \). We see that for every element \( A^n/A^{n+1} \) there exists a representation \( x = \lambda t^n \) (\( \lambda \in \mathcal{X} \)). We will now verify that if \( \lambda_1 \neq \lambda_2 \) then \( \lambda_1 t^n \neq \lambda_2 t^n \) (mod \( A^{n+1} \)).
This will prove that the factor $A^n/A^{n+1}$ is isomorphic to the vector space $(R/A)t^n$ and hence $gr(R) \cong (R/A)[t]$.

In fact if $(\lambda_1 - \lambda_2)t^n \in A^{n+1}$ then there exists $y \in R$ such that $(\lambda_1 - \lambda_2)t^n = t^{n+1}y$ which yields $t^n(ty - \lambda_1 + \lambda_2) = 0$. This contradicts the assumption that $t$ is regular and the proof is complete.

3.2. We consider in this subsection an ideal $A$ of $R$ which is generated by a polycentral independent system of elements $< t_1, t_2 >$. We recall that this means that $t_1$ is a central element in $R$ such that $\bigcap_{i=1}^{\infty} (t_1)^i = 0$ and the graded ring $gr(R)$ associated to the filtration defined by the powers of the ideal $(t_1)$ is isomorphic to the polynomial ring $R_1[t]$, where $R_1 = R/(t_1)$; the element $t_2$ is central modulo the ideal $(t)$, the ideal $A_1 = A/(t_1)$ generated in the ring $R_1$ by the element $t_2$ is residually nilpotent and the graded ring associated to the filtration defined by the powers of $A_1$ is isomorphic to the polynomial ring $(R_1/A_1)[t_2] \cong (R/A)[t_2]$.

Pick an arbitrary $m$ and let $\bar{X}$ be the image of a subset $X$ under the natural homomorphism $R \rightarrow R/(t_1)^m$. Clearly we have a natural homomorphism $\bar{R} \rightarrow \bar{R}/(\bar{A}) \cong \bar{R}/A$ and the system of elements $X$ can be considered also as a system of coset representatives for quotient ring $\bar{R}/\bar{A} \cong R/A$. We order the system of elements $< t_1, t_2 >$ assuming that $t_1 < t_2$; since there is one to one correspondence between the systems $< t_1, t_2 >$ and $< \bar{t}_1, \bar{t}_2 >$ the system $< \bar{t}_1, \bar{t}_2 >$ is also well ordered. The standard monomials on $< t_1, t_2 >$ or on $< \bar{t}_1, \bar{t}_2 >$ are defined in the usual way.

**Lemma 3.2.** Let

$$x = r_1\bar{t}_2 \bar{r}_2 \bar{t}_2 \cdots r_n\bar{t}_2 \bar{r}_{n+1}$$  \hspace{1cm} (3.5)

where $r_\alpha \in \bar{R}$ ($\alpha = 1, 2, \cdots, n + 1$). Then there exists a representation

$$x = r\bar{t}_2^m + \bar{t}_1 x_1$$  \hspace{1cm} (3.6)

where $s \in \bar{R}$ and $x_1 \in \bar{A}^{n-1}$.

**Proof.** Since the element $\bar{t}_1$ is central modulo the ideal $(t_1)$ we obtain that for an arbitrary $\bar{r} \in \bar{R}$ there exists $\bar{s} \in \bar{R}$ such that

$$\bar{t}_2 \bar{r} = \bar{r} \bar{t}_2 + \bar{t}_1 \bar{s}$$  \hspace{1cm} (3.7)

We apply this identity to the factors $\bar{t}_2$ and $\bar{r}_{n+1}$ in (3.5) and obtain
\[ x = \bar{r}_1 \bar{t}_2 \bar{r}_2 \cdots \bar{r}_n \bar{r}_{n+1} \bar{t}_2 + \bar{t}_1 \bar{y} \]  
(3.8)

where \( \bar{y} \in \bar{A}^{n-1} \). We repeat this procedure, and after \( n \) steps obtain (3.6) where \( r = \bar{r}_1 \bar{r}_2 \cdots \bar{r}_{n+1} \).

**Corollary 3.1.** For an arbitrary natural \( n \)

\[ \bar{A}^n \cap (\bar{t}_1) \subseteq \bar{t}_1 \bar{A}^{n-1} \]  
(3.9)

**Proof.** Let \( x \in \bar{A}^n \cap (\bar{t}_1) \). Since \( x \in \bar{t}_1 \) we obtain from representation (3.6) that \( \bar{r}_1 \bar{t}_2 \bar{r}_2 \cdots \bar{r}_n \bar{r}_{n+1} \bar{t}_2 + \bar{t}_1 \bar{y} \). We obtain from the last inclusion that \( \bar{r} \in (\bar{t}_1) \) because the element \( t_2 \) is regular in the quotient ring \( \bar{R}/(\bar{t}_1) \cong R/(t_1) = R_1 \). We conclude from this and Lemma 3.2 that \( x \in \bar{r}_1 \bar{A}^{n-1} \) and the proof is complete.

**Lemma 3.3.**

\[ \bigcap_{n=1}^{\infty} \bar{t}_1^{m-1} \bar{A}^n = 0 \]  
(3.10)

**Proof.** We consider the regular representation \( \rho \) of \( \bar{R} \) in the ideal \( (\bar{t}_1)^{m-1} \) generated by the central element \( \bar{t}_1 \). We prove first of all that the kernel of this representation is the ideal \( (\bar{t}_1) \). Clearly, the kernel contains the ideal \( (\bar{t}_1) \).

On the other hand assume that \( \bar{r} \notin (\bar{t}_1) \), \( \bar{r} \bar{t}_1^{m-1} = 0 \) and let \( r \) is an element of \( R \) which is mapped in \( \bar{r} \) under the homomorphism \( R \longrightarrow R/(t_1)^m = \bar{R} \); then \( rt_1^{m-1} = t_1^m s \) for some element \( s \in R \). Hence \( t_1^{m-1}(r - t_1 s) = 0 \) which is impossible because \( t_1 \) is a regular element. This proves that the the kernel of \( \rho \) is the ideal \( (\bar{t}_1) \).

We obtain now that the ideal \( (\bar{t}_1)^{m-1} = \bar{t}_1^{m-1} \bar{R} \) is a one dimensional free module with generator \( \bar{t}_1^{m-1} \) over the ring \( \bar{R}/\bar{t} \cong R/(t_1) \cong R_1 \) and the \( R_1 \)-module \( \bar{t}_1^{m-1} A_1^n \) is isomorphic to \( A_1^n \). We have now

\[ \bar{t}_1^{m-1} \bar{R} = \bar{t}_1^{m-1} R_1 \]  
(3.11)

and we obtain from this that \( \bar{t}_1^{m-1} \bar{A}^n = \bar{t}_1^{m-1} A_1^n \) for every natural \( n \). Relation (3.10) now follows from the relation \( \bigcap_{n=1}^{\infty} A_1^n = 0 \). This completes the proof.

**Proposition 3.1.** Let \( \bar{X} \) denote the image of a subset \( X \subseteq R \) under the natural homomorphism the \( R \longrightarrow R/(t_1)^m \) and assume that \( n \geq (m - 1) \). Then

\[ \bar{A}^n \cap (\bar{t}_1)^{m-1} \subseteq (\bar{t}_1)^{m-1} \bar{A}^{n-(m-1)} \]  
(3.12)
**Proof.** We apply induction by the number $m$; the initial step of the induction when $m = 1$ is obvious and we can assume that the assertion has already been proven for all the quotient rings $R/(t_1)^k$ ($k = 1, 2, \ldots, m - 1$). This assumption implies that

$$\bar{A}^n \cap (\bar{t}_1)^{m-2} \subseteq \bar{t}_1^{m-2} \bar{A}^{n-(m-2)}$$

and hence

$$\bar{A}^n \cap (\bar{t}_1^{m-1}) \subseteq \bar{t}_1^{m-2} \bar{A}^{n-(m-2)}$$

and it is enough to prove that if $n \geq (m - 1)$ then

$$\bar{t}_1^{m-2} \bar{A}^{n-(m-2)} \cap (\bar{t}_1)^{m-1} \subseteq \bar{t}_1^{m-1} \bar{A}^{n-(m-1)}$$

if $n \geq (m - 1)$.

Let $\bar{y}$ be an element from the left side of (3.15). We obtain from Lemma 3.2 that $\bar{y}$ is a sum of elements which either have type

$$\bar{t}_1^{m-2} r \bar{t}_2^k \ (k \geq n - m + 2)$$

or they have type

$$\bar{t}_1^{m-2} \bar{t}_1 \bar{y}_1 = \bar{t}_1^{m-1} \bar{y}_1$$

with $\bar{y}_1 \in \bar{A}^{n-(m-2)-1}$.

Since element (3.17) belongs to $\bar{t}_1^{m-1} \bar{A}^{n-(m-1)}$ we can consider only the case when $\bar{y}$ is a sum of elements of type (3.16), that is

$$\bar{y} = \bar{t}_1^{m-2} \sum \bar{r}_a \bar{t}_2^{k_\alpha}$$

where $k_\alpha \geq n - m + 1$.

Since we assumed that $\bar{y} \in (\bar{t}_1)^{m-1}$ we conclude from (3.18) that

$$(\bar{t}_1^{m-2} \sum \bar{r}_a \bar{t}_2^{k_\alpha}) \in (\bar{t})^{m-1}$$

Let $u$ be an element of $R$ which is mapped in the element $\sum \bar{r}_a \bar{t}_2^{k_\alpha}$ under the homomorphism $R \rightarrow \bar{R} = R/(t_1)^m$. Relation (3.19) implies that there exist $a, b \in R$ such that $ut_1^{m-2} = at_1^{m-1} + bt_1^m$. Since $t_1$ is a regular element we can cancel the last relation by $t_1^{m-2}$ and obtain that $u = t_1 a + t_1^2 b$. Hence
\[ \sum_{\alpha=1}^{\kappa}(\bar{r}_\alpha \bar{t}^\kappa) \in (\bar{t}_1) \text{ and the relation } k_\alpha \geq (n-m+2) \text{ together with Corollary 3.1. imply that} \]
\[ \left( \sum_{\alpha=1}^{\kappa} \bar{r}_\alpha \bar{t}^\kappa \right) \in \bar{t}_1 \bar{A}^{n-m+1} \tag{3.20} \]

We obtain from this and (3.18) that \( \bar{y} \in (\bar{t}_1)^{m-1} \bar{A}^{(n-m+1)} \) and the proof is complete.

**Corollary 3.2.** Assume that the conditions of Proposition 3.1. hold. Then the ideal \( \bar{A} \) of \( \bar{R} \) is residually nilpotent.

**Proof.** We recall also that the definition of the independent polycentral system imply that the ideal \( A \) is residually nilpotent modulo \( (t)_1 \) and we can assume that \( A \) is residually nilpotent modulo \( (\bar{t}_1)^{m-1} \), i.e.
\[ \bigcap_{n=1}^{\infty} A^n \subseteq (\bar{t}_1)^{m-1} \tag{3.21} \]
We have now
\[ \bigcap_{n=1}^{\infty} \bar{A}^n = \left( \bigcap_{n=1}^{\infty} \bar{A}^n \right) \cap (\bar{t}_1)^{m-1} \subseteq \left( \bigcap_{n=1}^{\infty} \bar{A}^n \cap (\bar{t}_1)^{m-1} \right) \subseteq \]
\[ \bigcap_{n=2(m-1)}^{\infty} (\bar{A}^n \cap (\bar{t}_1)^{m-1}) \subseteq \bigcap_{n=2(m-1)}^{\infty} (\bar{t}_1)^{m-1} \bar{A}^{(n-m+1)} \tag{3.22} \]
Lemma 3.3. implies that the the last term in (3.22) is zero. This completes the proof.

**Theorem 3.1.** Let \( R \) be a ring, \( t_1, t_2 > \) be a polycentral independent system in \( R \). Let \( A = < t_1, t_2 > \) be the ideal generated by the elements \( t_1, t_2 \). Then the ideal \( A \) is residually nilpotent.

**Proof.** We pick an arbitrary natural \( m \) and consider the quotient ring \( \bar{R} = R/(t_1)^m \); let \( \bar{X} \) denote the image of a subset \( X \subseteq R \) under the natural homomorphism \( R \rightarrow R/(t_1)^m \). Since the ideal \( \bar{A} \subseteq \bar{R} \) is residually nilpotent by Corollary 3.2. the assertion follows now from the condition \( \bigcap_{m=1}^{\infty} (t_1)^m = 0. \)
3.3. We assume throughout this subsection that the conditions of Theorem 3.1. hold and hence $\bigcap_{n=1}^{\infty} A^n = 0$. We see that the powers of the ideal $A$ define a topology in $R$; let $\widetilde{R}$ be the completion of $R$. Similarly we pick an arbitrary $m$ and consider the ring $R_m = R/(t_1)^m$; we denote by $\widetilde{X}$ the image of a subset $X \subseteq R$ under the homomorphism $R \to R_m$. Corollary 3.2. yields that the ideal $\bar{A}$ is residually nilpotent in $R_m$. We denote by $\tilde{\mathcal{R}}_m$ the completion of $\mathcal{R}$ in the topology defined by the powers of $\bar{A}$.

Let $\mathcal{X}$ be a system of coset representatives for the quotient ring $R_m/(\bar{t}) \cong R/(t)$. and define the standard monomials on the set $\bar{T}$ in the usual way.

**Proposition 3.2.** Every element $r \in \tilde{R}_m$ has a unique representation

$$x = \lambda_0 + \lambda_1 \bar{t}_1 + \cdots + \lambda_{n-1} \bar{t}_1^{m-1}$$

(3.23)

where $\lambda_\alpha (\alpha = 0, 1, \ldots, m - 1)$ is a power series

$$\sum_{i=1}^{\infty} u_i \bar{t}_i$$

(3.24)

with $u_i \in \mathcal{X} \ (i = 1, 2, \cdots)$.

**Proof.** We consider the completion of the ideal $(\bar{t}_1)^{m-1} \subseteq R_m$. Proposition 3.1. implies that the topology induced in the subring $\widetilde{t}_1^{m-1} \widetilde{R}_m$ is equivalent to the topology defined by the system of ideals $(\bar{t}_1)^{m-1} A^n$; hence the completion of the ideal $(\bar{t}_1)^{m-1}$ is isomorphic to the ideal $\bar{t}_1^{(m-1)} \widetilde{R}_m$ of $\widetilde{R}_m$ and hence every element of this completion has a unique representation

$$u = \bar{t}_1^{m-1} \lambda$$

(3.25)

where $\lambda$ is a power series of type (3.24). Further the ideal $\bar{t}_1^{(m-1)} \widetilde{R}_m$ is the kernel of the natural homomorphism $\widetilde{R}_m \to \widetilde{R}_{m-1}$. We can assume that every element $y \in \widetilde{R}_{m-1}$ has a unique representation

$$y = \mu_0 + \mu_1 \bar{t}_1 + \mu_2 \bar{t}_1^2 + \cdots + \mu_{m-2} \bar{t}_1^{m-2}$$

(3.26)

where the coefficients $\mu_0, \mu_1, \ldots, \mu_{m-2}$ are power series of type (3.24) and this representation is unique.

The vector space $\tilde{R}_m$ is a direct sum $\tilde{R}_m = \tilde{R}_{m-1} + \bar{t}_1^{m-1} \tilde{R}_m$. We obtain from this and from the representations (3.25) and (3.26) that $x$ has representation (3.23); the uniqueness of this representation follows easily from the
uniqueness of the representations of (3.25) and (3.26). This completes the proof.

**Proposition 3.3.** Let \( \tilde{R} \) be the completion of \( R \) in the topology defined by the powers of the ideal \( A \). Then every element of \( r \in \tilde{R} \) has a unique representation

\[
r = \sum_{j=1}^{\infty} \lambda_j t_1^j \tag{3.27}
\]

where \( \lambda_j = \sum_{i=1}^{\infty} u_{ji} t_2^i \) (j = 1, 2, \cdots) with \( u_{ji} \in X \) (i = 1, 2, \cdots; j = 1, 2, \cdots).

**Proof.** The ring \( \tilde{R} \) is an inverse limit of the system of rings \( R/(t)^m \) (m = 1, 2, \cdots) and the existence and the uniqueness of representation (3.27) follows from Proposition 3.2.

**Corollary 3.3.** Every element \( r \in \tilde{R} \) has a unique representation

\[
r = \sum_{i=1}^{\infty} u_i \tau_i \tag{3.28}
\]

where \( u_i \in X \) (i = 1, 2, \cdots), \( \tau_i \) (i = 1, 2, \cdots) are standard monomials on the set \(<t_1, t_2>\) with \( \lim_{i \to \infty} l(\tau_i) = \infty \).

3.4. We set up now the notation for Theorem III and Lemmas 3.4. − 3.10. Let \( R \) be a ring, \( T = <t_1, t_2, \cdots, t_n> \) be an independent polycentral system which is composed from the central independent systems \( T_1, T_2, \cdots, T_k \), \( A \) be the ideal generated by the system \( T \), \( f \) be a function on \( T \) whose values form a bounded set of natural numbers and \( f(t_1) > 2f(t_2) \) for an arbitrary pair of elements \( t_1 \in T_i, t_2 \in T_{i+1} \) (1 ≤ i ≤ k − 1). The definition of the polycentral independent system implies that for an arbitrary (1 ≤ m ≤ k) the subsystem \( t_1, t_2, \cdots, t_m \) is an independent polycentral system in \( R \); let \( A_m \) be the ideal generated \(<t_1, t_2, \cdots, t_m>\). Further, the system \( t_{m+1}, t_{m+2}, \cdots, t_n \) is an independent polycentral system in the quotient ring \( R/A_m \).

Every element \( r \in A \) is sum

\[
\sum \mu_1 t_{\beta_1} \mu_2 t_{\beta_2} \cdots \mu_k t_{\beta_k} \mu_{k+1} \tag{3.29}
\]

where \( t_{\beta_1}, t_{\beta_2}, \cdots, t_{\beta_k} \) are elements from \( T \), \( \mu_1, \mu_2, \cdots, \mu_k \in R \). Let \( M \) be the maximum of the values of \( f(t) \) on \( T \). We define now a system of subsets
$B_j$ ($j = 0, 1, \cdots$) in $R$ as follows. Define $B_0 = R$ and then for $j \geq 1$ define that an element $x$ belongs to $B_j$ if there exists for it a representation (3.29) such that every summand satisfies the condition

$$\sum_{a=1}^{k} f(t_{\beta_a}) \geq j$$

(3.30)

It is clear that $B_1 = A$ and that $0 \in B_j$ ($j = 0, 1, \cdots$). Let $\mathcal{X}$ be a system of coset representatives for the ideal $A$.

**Theorem III.** i) The ideal $A$ is residually nilpotent and $\bigcap_{j=0}^{\infty} B_j = 0$. The system of subsets $B_j$ ($j = 0, 1, \cdots$) is a filtration in $R$.

ii) This filtration defines a pseudovaluation $v$ such that $v(t) = f(t)$ for an arbitrary $t \in T$; the homogeneous components $t_i$ ($i = 1, 2, \cdots, n$) are central in $gr_v(R)$ and the ring $gr_v(R)$ is isomorphic to the polynomial ring $(R/A)[\tilde{t}_1, \tilde{t}_2, \cdots, \tilde{t}_n]$.

The pseudovaluation $v$ is defined uniquely by the conditions $v(t) = f(t)$ ($t \in T$), and $gr_v(R) \cong (R/A)[\tilde{t}_1, \tilde{t}_2, \cdots, \tilde{t}_n]$; the topology defined by this pseudovaluation is equivalent to the topology defined by the powers of $A$.

**3.5.** We will prove in this subsection a few auxiliary facts.

We will use throughout this subsection the notation of subsection 3.4. and we will assume that the ideal $A$ generated by the system $T$ is residually nilpotent. We will prove under this condition Lemmas 3.4. – 3.10. We point out that Theorem 3.1. shows that the ideal $A$ is residually nilpotent if $T$ is the polycentral independent system $<t_1, t_2>$. This fact will make possible to apply Lemmas 3.4. – 3.10. in the proofs of Theorems 3.2. and 3.3.

**Lemma 3.4.** Every $B_j$ is an ideal in $R$,

$$R = B_0 \supseteq B_1 \supseteq B_1 \supseteq \cdots$$

(3.31)

and

$$A^j \subseteq B_j$$

$(j = 1, 2, \cdots)$

(3.32)

**Further**

$$B_{jM} \subseteq A^j$$

$(j = 1, 2, \cdots)$

(3.33)
Proof. The proof of the first statement is straightforward and relation (3.29) follows immediately as well as (3.32).

Now assume that \( r \in B_{jM} \). Then there exists for \( r \) representation where all the summands (3.31) satisfy relation

\[
\sum_{a=1}^{k} f(t_{\beta_a}) \geq jM \quad (3.34)
\]

We pick an arbitrary of these summands. Since \( f(t_{\beta_a}) \leq M (\alpha = 1, 2, \cdots, k) \) we must have \( k \geq j \) and hence this summand belongs to \( A^j \). This proves relation (3.33). The proof of the lemma is complete.

**Corollary 3.4.** The system of ideals \( B_j \ (j = 0, 1, \cdots) \) defines a pseudovaluation \( v \) in \( R \) as follows

\[
v(0) = \infty; \text{ if } x \neq 0 \text{ then } v(x) = \max \{ j | x \in B_j \} \quad (3.35)
\]

The topology defined by this pseudovaluation is equivalent to the topology defined by the powers of the ideal \( A \).

An element \( r \in R \) has \( v \)-value greater than zero iff it belongs to the ideal \( A \).

If \( t \in T \) then \( v(t) \geq f(t) \).

**Proof.** We obtain from (3.32) and the assumption \( \bigcap_{j=1}^{\infty} A^j = 0 \) that

\[
\bigcap_{j=1}^{\infty} B_j = 0 \quad (3.36)
\]

Further the definition of the system of ideals \( B_j \ (j = 0, 1, \cdots) \) shows that \( B_{j_1}B_{j_2} \subseteq B_{j_1 + j_2} \). We see that the system of ideals \( B_j \ (j = 0, 1, \cdots) \) forms a filtration in \( R \) so \( v \) is a pseudovaluation in \( R \).

The definition of the pseudovaluation \( v \) shows that \( v(x) > 0 \) iff \( x \) has a representation (3.29) with every summand containing elements of \( T \) which means that \( x \in A \).

The definition of \( B_j \) implies that if \( f(t) = j \) then \( t \in B_j \), so \( v(t) \geq j \) which proves the last statement.

**Definition 3.1.** The pseudovaluation \( v \) and the filtration \( B_j \ (j = 0, 1, \cdots) \) will be called the pseudovaluation and the filtration defined in \( R \) by the polycentral system \( T \) and the function \( f \).
Lemma 3.5. The graded ring associated to the pseudovaluation $v$ is generated by the zero component $R/A$ and the set of homogeneous components $T = \{ \tilde{t}_i \mid (t_i \in T, i \in I) \}$.

Proof. Let $r$ be an element of $R$ with $v(r) = j$. Then $r \in B_j$ and we will consider the image of this element in the factor $B_j/B_{j+1}$. Since we consider the image of $r$ modulo $B_{j+1}$ we can assume that it has representation (3.29) where every summand $r_1 = \mu_1 t_{\beta_1} \mu_2 t_{\beta_2} \cdots \mu_k t_{\beta_k} \mu_{k+1}$ satisfies condition

$$\sum_{\alpha=1}^{k} f(t_{\beta_\alpha}) = j$$

and none of these summands belongs to $B_{j+1}$. We conclude that $v(r_1) = j$ otherwise we would have $r_1 \in B_{j+1}$. The relation $v(r_1) = j$ implies that $v(\mu_\alpha) = 0$ ($\alpha = 1, 2, \cdots, k + 1$). In fact, if we assume $v(\mu_\alpha) > 0$ for some $\mu_\alpha$ then we obtain from this assumption together with condition (3.37) that $v(r_1) > j$. We conclude from this and from Lemma 2.3. that the homogeneous component $\tilde{r}_1$ of the summand $r_1$ has representation

$$\tilde{r}_1 = \tilde{\mu}_1 \tilde{t}_{\beta_1} \tilde{\mu}_2 \tilde{t}_{\beta_2} \cdots \tilde{\mu}_k \tilde{t}_{\beta_k} \tilde{\mu}_{k+1}$$

and that

$$\tilde{r} = \sum \tilde{\mu}_1 \tilde{t}_{\beta_1} \tilde{\mu}_2 \tilde{t}_{\beta_2} \cdots \tilde{\mu}_k \tilde{t}_{\beta_k} \tilde{\mu}_{k+1}$$

and the assertion follows.

The following fact was established in the proof of Lemma 3.5.

Corollary 3.5. Let $r$ be an element of $R$ such that $v(r) = j$. Then the homogeneous component $\tilde{r}$ is a sum of monomials (3.38) where $v(\tilde{u}_\alpha) = 0$ ($\alpha = 1, 2, \cdots, k + 1$) and $\sum_{\alpha=1}^{k} v(\tilde{t}_{\beta_\alpha}) = j$.

Lemma 3.6. Assume that the homogeneous components $\tilde{t}_i$ ($t_i \in T$) are central in $gr(R)$. Let $x \in B_j \setminus B_{j+1}$. Then

i) there exists a representation

$$x = x_1 + y$$

where
\[ x_1 = \sum_{i=1}^{n} \lambda_i \pi_i \]  

(3.41)

\[ \lambda_i \in \mathcal{X} \ (i = 1, 2, \ldots, n); \ \pi_i \ (i = 1, 2, \ldots, n) \] are standard monomials on \( T \) with \( v \)-value equal \( j \) and \( y \in B_{j+1} \).

ii) Let \( \tilde{R}_v \) be the completion of \( R \) in the topology defined by \( v \). Then the element \( y \) in the right side of (3.40) has a representation

\[ y = \sum_{i=n+1}^{\infty} \lambda_i \pi_i \]  

(3.42)

where \( \lambda_i \in \mathcal{X}, \ v(\pi_i) \geq j + 1 \) (\( i = 1, 2, \ldots \)), and \( \lim_{i \to \infty} v(\pi_i) = \infty \).

iii) Representations (3.40) - (3.42) yield a power series representation

\[ x = \sum_{i=1}^{\infty} \lambda_i \pi_i \]  

(3.43)

**Proof.**  

i) We have for \( x \) a representation (3.29). We consider now the homogeneous component (3.38) of one of the summands in this representation. Since the homogeneous component \( \tilde{t}_i \ (i \in I) \) are central in \( gr(R) \) we obtain that the element (3.38) is equal to the element \( \tilde{\mu}_1 \tilde{\mu}_2 \cdots \tilde{\mu}_{k+1} \pi(t_{\beta_1}, t_{\beta_2}, \cdots, t_{\beta_k}) \) where \( v(\tilde{\mu}_1 \tilde{\mu}_2 \cdots \tilde{\mu}_{k+1}) = 0 \) and \( \pi \) is a suitable standard monomial of value \( j \) on the set of elements \( t_{\beta_1}, t_{\beta_2}, \cdots, t_{\beta_k} \).

We obtain from this

\[ \mu_1 t_{\beta_1} \mu_2 t_{\beta_2} \cdots \mu_k t_{\beta_k} \mu_{k+1} = \mu_1 \mu_2 \cdots \mu_k \mu_{k+1} \pi + x_0 \]  

(3.44)

where \( \pi \) is a standard monomial with value \( j \) on the of elements \( t_{\beta_1}, t_{\beta_2}, \cdots, t_{\beta_k} \) obtained from \( \pi(t_{\beta_1}, t_{\beta_2}, \cdots, t_{\beta_k}) \) by substitution \( t_{\beta_\alpha} \rightarrow t_{\beta_\alpha} \) (\( \alpha = 1, 2, \cdots, k \)), \( x_0 \in B_{j+1} \) and \( \mu_1 \mu_2 \cdots \mu_k \mu_{k+1} = r \) is an element of \( R \). We obtain from this that \( x \) has a representation

\[ x = \sum_{i=1}^{m} r_i \pi_i + r_0 \]  

(3.45)

where \( \pi_i \ (i = 1, 2, \cdots, m) \) are standard monomials on \( T \) which belong to \( B_j \); \( r_i \in R \ (i = 1, 2, \cdots, m) \), \( r_0 \in B_{j+1} \). By adding, if necessary, the coefficients of the same monomial, we can assume that that all the monomials \( \pi_i \ (i = 1, 2, \cdots, m) \) are lexicographically distinct; further if a coefficient \( r_i \)
belongs to $A$ then the product $r_i\pi_i$ belongs to $B_{j+1}$ so we can assume that $r_i \in R \setminus A$ for $i = 1, 2, \cdots, m$. Hence, $r_i = \lambda_i + u_i, 0 \neq \lambda_i \in \mathcal{X}, u_i \in A = B_1 \subseteq B_1 (i = 1, 2, \cdots, m)$. We substitute these expressions in (3.45) and obtain

$$x = \sum_{i=1}^{n} \lambda_i \pi_i + \sum_{i=1}^{m} u_i \pi_i + r_0$$

(3.46)

The summands $u_i \pi_i (i = 1, 2, \cdots, m)$ belong to $B_{j+1}$, we denote now $y = \sum_{i=1}^{m} u_i \pi_i + r_0$ and obtain relations (3.40)- (3.42).

This proves statement i).

ii) Let $x \in R$ with $v(x) = j$. We obtain from statement i) representation (3.40). It is important that we can assume that all the coefficients $\lambda_i$ in the element $x_1 = \sum_{i=1}^{n} \lambda_i \pi_i$ are non-zero otherwise we would get $x \in B_{j+1}$. We consider now the element $y$ which has $v$-value $j_1 > j$. Once again, statement i) yields that $y = x_2 + z$, where $x_2$ is a linear combination with coefficients from $\mathcal{X}$ of standard monomials with $v$-value $j_1$ and $z \in B_{j_1+1}$.

We obtain by this argument that there exists a representation

$$x = x_1 + x_2 + x_3 + \cdots$$

(3.47)

where $x_1$ is a linear combination of monomials with value $j$, and $x_2, x_3, \cdots$ are linear combinations of monomials with values $j_1, j_2, \cdots$ greater than $j$. This series converge in the $\rho$-topology and in the topology defined by the pseudovaluation $v$. This completes the proof of statement ii).

Statement iii) follows from i) and ii).

**Corollary 3.6.** Assume that the conditions of Lemmas 3.6. and 3.7. hold. Then the homogeneous component $B_j/B_{j+1}$ is a left module over ring $R/A$, the standard monomials $\pi_i$ with value $j$ on the set $T$ form a system of generators for it.

**Proof.** Since $AB_j \subseteq B_{j+1}$ we obtain that $B_j/B_{j+1}$ is a left module over $R/B_1 \cong R/A$ and the assertion now follows from Lemma 3.6.

**Lemma 3.7.** Assume that conditions of Lemmas 3.6. hold. Then the following statements are equivalent:

1) Representation (3.42) is unique for every element $x \in B_j (j = 0, 1, \cdots)$. 

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2) The associated graded ring \( \text{gr}_v(R) \) is isomorphic to the polynomial ring \( (R/A)[\tilde{t}_1, \tilde{t}_2, \cdots, \tilde{t}_n] \).

**Proof.** We prove first that 1) \( \rightarrow \) 2). Assume that representation (3.42) is unique. We recall that the zero component of \( \text{gr}(R) \) is \( R/A \) and Corollary 3.6 implies that \( B_j/B_{j+1} \) is a module over the quotient ring \( R/A \).

We show now that standard monomials with value \( j \) must be linearly independent in \( B_j/B_{j+1} \) over \( R/A \). In fact if this is not true then there exists an element \( y \in B_{j+1} \) such that

\[
y = \sum_{i=1}^{m} \lambda_i \pi_i \quad (3.48)
\]

where

\[
v(\pi_i) = j, 0 \neq \lambda_i \in X (i = 1, 2, \cdots, n) \quad (3.49)
\]

On the other hand, since \( y \in B_{j+1} \) we obtain from Lemma 3.6. a power series representation for \( y \) where the minimal value of the monomials is greater than or equal \( j + 1 \). We obtained two representations for the element \( y \). This contradiction shows that the standard monomials of weight \( j \) must be linearly independent. Further, if \( \tilde{B}_j \) is the completion of \( B_j \) in the \( v \)-topology then \( \tilde{B}_j/\tilde{B}_{j+1} \cong B_j/B_{j+1} \). But \( \tilde{B}_j/\tilde{B}_{j+1} \) is generated over \( R/A \) by the standard monomials with value \( j \), we obtain from that the standard monomials on \( T \) with value \( j \) form a basis of \( B_j/B_{j+1} \), the homogeneous components of the elements of \( T \) are central in \( \text{gr}(R) \) so we conclude that the ring \( \text{gr}(R) \) is isomorphic to \( (R/A)[\tilde{t}_1, \tilde{t}_2, \cdots, \tilde{t}_n] \). This proves that 1 \( \rightarrow \) 2).

Now prove that 2) \( \rightarrow \) 1). Assume that \( \text{gr}(R) \cong (R/A)[T] \). Let \( x \in B_j \setminus B_{j+1} \). We have for \( x \) representation (3.40) where \( x_1 \) is given by (3.41) with the coefficients \( \lambda_1, \lambda_2, \cdots, \lambda_n \) uniquely defined. We repeat this procedure with the element \( y \) as in the proof of statement ii) of Lemma 3.6. and obtain a power series representation for \( x \) with uniquely defined coefficients \( \lambda_i \in X (i = 0, 1, \cdots) \). This completes the proof.

3.6. We use throughout this section the same notation as in subsections 3.4. – 3.5. In particular, \( T = < T_1, T_2, \cdots, T_k > \) is an independent polycentral system, \( A \) is the ideal generated by this system and we assume that this ideal is residually nilpotent. We consider now the central independent subsystem \( T_1 = < t_1, t_2, \cdots, t_{n_1} > \) and the ideal \( (t_1) \) of \( R \). Let \( \bar{X} \) denote the image of
a subseteq X \subseteq R under the natural homomorphism R \rightarrow \bar{R} = R/(t_1).

If n_1 > 1 then the definition of the independent polycentral system implies that the system \bar{T} = \langle t_2, t_3, \cdots, t_{n_1-1}; T_2, T_3, \cdots, T_k \rangle is an independent polycentral system in \bar{R}; if n_1 = 1 we obtain in \bar{R} an independent polycentral system \bar{T} = \langle T_2, T_3, \cdots, T_k \rangle. In both cases the system \bar{T} \subseteq \bar{R} has smaller length then the system \bar{T}, we keep for the restriction of the function f on \bar{T} the same notation \bar{f}.

Lemmas 3.8. and 3.9 that the conclusions of Theorem III hold in the ring \bar{R} = R/(t_1) for the polycentral independent system \bar{T} and the function f. This means that:

1) the ideal \bar{A} is residually nilpotent;
2) there exists a pseudovaluation \bar{v}_1 such that \bar{v}_1(\bar{t}) = f(\bar{t});
3) the homogeneous components of the elements \bar{t} \in \bar{T} are central in the ring gr_{\bar{v}_1}\bar{R};
4) these homogeneous components are algebraically independent in the graded ring gr_{\bar{v}_1}(\bar{R}) over the zero degree homogeneous component (\bar{R}/\bar{A}) \cong (R/A) and that they generate the ring gr_{\bar{v}_1}(\bar{R}) over \bar{R}/\bar{A} \cong R/A.

Lemmas 3.8. and 3.9. will be used in the induction proof of Theorem III in subsection 3.9. The initial step of the induction is the case when the system \bar{T} consists from one central element t such that \int_{i=1}^{\infty}(t)^i = 0 and the graded ring gr(R) asociated to the pseudovaluation \rho defined by the powers of t is isomorphic to the polynomial ring in the variable t over R/(t).

We consider now the homomorphism R \rightarrow \bar{R}. Proposition 2.3. implies that this homomorphism defines the filtration \bar{B}_j (j = 0, 1, \cdots) and a pseudovaluation \bar{v} defined by this filtration. We need the following fact.

**Lemma 3.8.** Assume that the conclusions of Theorem III hold in the ring \bar{R}. Then the pseudovaluation \bar{v} coincides with \bar{v}_1:

\[ \bar{v}(\bar{r}) = v_1(\bar{r}) \]  

for every \bar{r} \in \bar{R}.

**Proof.** We consider the filtration \bar{B}_j^* (j = 0, 1, \cdots) defined by the pseudovaluation \bar{v}_1 and will prove that \bar{B}_j = B_j^* (j = 0, 1, \cdots).

Assume that 0 \neq \bar{x} \in \bar{B} and let x \in R be an element whose image in \bar{R} is \bar{x}. Then there exists for x a representation as a sum of monomials (3.29)
which satisfy condition (3.30). If a summand (3.29) in the representation of
\( x \) contains the factors from \( t_1 \) its image in \( \bar{R} \) is zero. Otherwise its image is
the element

\[
\bar{\mu}_1 t_{\beta_1} \bar{\mu}_2 t_{\beta_2} \cdots \bar{\mu}_k t_{\beta_k} \bar{\mu}_{k+1}
\] (3.51)

We obtain from this that \( \bar{x} \in B_j^* \) which implies that \( B_j \subseteq B_j^* \) \((j = 0, 1, \ldots)\). The proof of the reverse inclusion is immediate, and the assertion
follows.

**Lemma 3.9.** Assume that the conclusions of Theorem III hold in the
quotient ring \( \bar{R} = R/(t_1) \). Then \( v(t) = f(t) \) if \( t \neq t_1 \) and all the homogeneous
components \( \bar{t} \) \((t \in T)\) are central in the graded ring \( gr_v(R) \).

**Proof.** The definition of \( v \) shows that \( v(t) \geq f(t) \) if \( t \in T \). Since
\( \bar{v}(\bar{r}) \geq v(r) \) for an arbitrary \( r \in R \) and \( v_1(t)) = \bar{v}(t) = f(t) \) for \( t \neq t_1 \) we
obtain the first statement of the lemma.

We prove now the second statement. Lemma 3.5. implies that the graded
ring \( gr_v(R) \) is generated by the zero homogeneous component \( R/A \) and the
homogeneous components of the elements \( t \in T \) so we have to prove only that

\[
v([t_{i_1}, t_{i_2}]) > v(t_{i_1}) + v(t_{i_2})
\] (3.52)

and for every \( t \in T \) and \( r \not\in A \)

\[
v([r, t]) > v(r) + v(t)
\] (3.53)

If any of the elements \( t_{i_1} \) or \( t_{i_2} \) belongs to \( T_1 \) then it is central and
relations (3.52) and (3.53) are obvious, so we can assume that \( t_{i_1}, t_{i_2} \in T_2 \cap T_3 \cap \cdots \cap T_k \). We have already proven that \( v(t_{i_1}) = f(t_{i_1}), v(t_{i_2}) = f(t_{i_2}) \) and the condition of the assertion imply that the homogeneous com-
ponents of \( t_{i_1} \) and \( t_{i_2} \) commute in the graded ring \( gr_v(R) \). We have therefore
in \( \bar{R} \)

\[
[t_{i_1}, t_{i_2}] = \bar{a}
\] (3.54)

where

\[
v_1(\bar{a}) = \bar{v}(\bar{a}) > (\bar{v}(t_{i_1}) + \bar{v}(t_{i_2}))
\] (3.55)
We find now an element \( a \in R \) whose image in \( \bar{R} \) is \( \bar{a} \) and \( v(a) = \bar{v}(\bar{a}) \) and obtain from (3.54)

\[
[t_{i_1}, t_{i_2}] = a + b \quad (3.56)
\]

where \( b \in (t_1) \) and \( a \) is an element with value \( v(a) > f(t_{i_1}) + f(t_{i_2}) \). Since \( b \) is a multiple of \( t_1 \) is a multiple of \( t_1 \) the \( v \)-value of \( b \) is greater than \( f(t_{i_1}) + f(t_{i_2}) \) because \( t_1 \in T_1, \ i_1, t_{i_2} \in (T_2 \cap T_3 \cap \cdots \cap T_k) \) and \( f(t_1) > 2f(t) \) for \( t \notin T_1 \). We conclude therefore that \( v(a + b) > f(t_{i_1}) + f(t_{i_2}) \), and this proves (3.52)

We will now prove (3.53). The image \( \bar{r} \) of \( r \) does not belong to \( A \), so \( v_1(\bar{r}) = \bar{v}(\bar{r}) = 0 \) and we have now in \( \bar{R} \)

\[
\bar{v}(\bar{r}, t) > \bar{v}(\bar{r}) + \bar{v}(t) = \bar{v}(t)
\]

and the proof of (3.53) can be completed in the same way as of (3.52).

We can now prove Theorem 3.2, which is an important step in the proof of Theorems I-III.

**Theorem 3.2.** Let \( R \) be a ring, \( < t_1, t_2 > \) be an independent polycentral system, \( A \) be the ideal generated by these elements, \( M_1, M_2 \) be two natural number such that \( M_1 > 2M_2 \). Then there exists a pseudovaluation \( v \) such that \( v(t_i) = M_i (i = 1, 2, \cdots) \), the graded ring associated to the pseudovaluation \( v \) is isomorphic to the polynomial ring \((R/A)[\tilde{t}_1, \tilde{t}_2] \), the topology defined by this valuation is equivalent to the \( \rho \)-topology defined by the powers of the ideal \( A \). The pseudovaluation \( v \) is defined uniquely by these properties.

**Proof.** Theorem 3.1. states that the ideal \( A \) formed by the system \( T = < t_1, t_2 > \) is residually nilpotent. We consider the filtration \( B_j (j = 0,1, \cdots) \) and the pseudovaluation \( v \) defined by the system \( T \) and the function \( f(t_i) = M_i (i = 1, 2) \) (see Definition 3.1.) Since the element \( t_1 \) is central in \( R \) its homogeneous component is central in \( gr(R) \). Lemma 3.9. implies that the homogeneous component \( t_2 \) is central in \( gr(R) \). Since the topology defined by the pseudovaluation \( v \) is equivalent to the \( \rho \)-topology defined by the powers of the ideal \( A \) we obtain from Corollary 3.3. that every element of the completion \( \bar{R} \) of \( R \) has a unique representation (3.43). We obtain now from Lemma 3.7. that \( gr_v(R) \cong (R/A)[\tilde{t}_1, \tilde{t}_2] \).

This completes the proof.
3.7. We will prove in this subsection the following Theorem 3.3, which is the main step in the proof of Theorems I-III.

**Theorem 3.3.** Let $R$ be a ring, $T = \langle t_1, t_2, \cdots, t_n \rangle$ be an independent polycentral system in $R$, $k \geq n$ be a natural numbers. Then the weight function defined by the function $f_k$ on $T$ as

$$f_k(t_i) = 3^{k-i+1}(i = 1, 2, \cdots, n) \quad (3.57)$$

extends to $t$-adic pseudovaluation $v_0$ of $R$ with associated graded ring isomorphic to the polynomial ring $(R/A)[\bar{t}_1, \bar{t}_2, \cdots, \bar{t}_n]$. The topologies defined by the pseudovaluation $v_0$ and by the powers of the ideal $A$ are equivalent.

Let $R$ be a ring, $t_i (i \in I)$ be a system of elements in $R$, $A$ be the ideal generated by them. Let $v$ be a pseudovaluation in $R$ such that $v(t_i) = mm_i (i \in I)$ where $m, m_i (i \in I)$ is a system of natural numbers and let $\bar{t}_i$ be the homogeneous component of $t_i$ in the associated graded ring $gr(R)$. Assume that the associated graded ring $gr(R)$ is isomorphic to the polynomial ring over $(R/A)$ in the system of variables $\bar{t}_i (i \in I)$. Let $R[t, t^{-1}]$ be the Laurent polynomial ring over $R$, we extend the pseudovaluation $v$ to this ring by defining $v(t) = m$. Let $V = \{ x \in R[t, t^{-1}] | v(x) \geq 0 \}$ be the ring of integers of $R[t, t^{-1}]$ and $\bar{X}$ be the image of a subset $X \subseteq V(R[t, t^{-1}])$ under the homomorphism $V \longrightarrow V/(t)$.

**Lemma 3.10.** The quotient ring $\bar{V} = V/(t)$ is isomorphic to the polynomial ring over $R/A$ in the system of variables $u_i = t_i t^{-m_i} (i \in I)$.

**Proof.** The pseudovaluation $v$ is equivalent to the pseudovaluation $v_1$ defined as $v_1(t_i) = m_i (i \in I); v_1(t) = m$ and the assertion now follows from Proposition 2.8.

We have the immediate corollary of Lemma 3.10.

**Corollary 3.8.** The map $\bar{t}_i \longrightarrow \bar{t}_i t^{-m_i}$ extends to an isomorphism between the rings $gr(R)$ and $V/(t)$.

**Proof of Theorem 3.3.** We recall that the definition of the independent polycentral system $\langle t_1, t_2, \cdots, t_n \rangle$ implies that for every $m < n$ the subsystem $\langle t_1, t_2, \cdots, t_m \rangle$ is an independent polycentral system in $R$ and the system $\langle t_{m+1}, t_{m+2}, \cdots, t_n \rangle$ is a polycentral independent system in the
quotient ring $R_m = R/A_m$ where $A_m$ is the ideal generated by $t_1, t_2, \ldots, t_m$. We make an assumption of an induction that the assertion is proven for an arbitrary ring $S$ with a polycentral independent system $x_1, x_2, \ldots, x_{n_1}$ of length $n_1 < n$ and a weight function $f_k$ ($k \geq n_1$) defined on this system by the function

$$f_k(x_1) = 3^{k_1}, f_k(x_2) = 3^{k_1-1}, \ldots, f_k(x_{n_1-1}) = 3^{k_1-n_1+1} \tag{3.58}$$

We have already pointed out in subsection 3.6. that the initial step of the induction when $T$ contains only one element is true. Further, assumption of the induction implies in particular that for every $k \geq n - 1$ there exists a pseudovaluation $v_{k-1}$ of $R$ such that $v_{k-1}(t_i) = 3^{k-i+1}$ ($i = 1, 2, \ldots, n - 1$) and the associated graded ring $gr_{v_{k-1}}(R_1)$ is isomorphic to the polynomial ring $R_{n-1}[\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_{n-1}]$; the definition of the independent polycentral system implies that the element $t_n$ is central in $R_{n-1}$, the ideal $(t_n)R_{n-1}$ is residually nilpotent and the graded ring associated to the filtration $(t_n)^iR_n$ ($i = 1, 2, \ldots$) is isomorphic to the polynomial ring generated over $R_{n-1}/(t_nR_{n-1}) \cong R/A$ by the homogeneous component of $t_n$.

We consider now the group ring $R[t, t^{-1}]$ and extend $v_{k-1}$ to this group ring defining $v_{k-1}(t) = 3^{k-n+2}$; let $V = \{ x \in R[t, t^{-1}]| v_{k-1}(x) \geq 0 \}$ be the subring of $v_{k-1}$-integers of $V$. We obtain from Lemma 3.10. that the quotient ring $V/(t)$ is isomorphic to the polynomial ring $R_{n-1}[\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{n-1}]$ where $u_1 = t_1t^{-3^{n-2}}, u_2 = t_2t^{-3^{n-3}}, \ldots, u_{n-1} = t_{n-1}t^{-1}$ and $v_{k-1}(u_i) = 0$ ($i = 1, 2, \ldots, n - 1$). We have in $R_{n-1}$ a central element $t_n$ which is also central in the polynomial ring $R_{n-1}[\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{n-1}]$ and we obtain now in $V$ an independent polycentral system $< t, t_n >$. We obtain now from Theorem 3.2. that there exists a pseudovaluation $v_0$ in $V$ such that $v_0(t) = 3^{k-n+2}, v_0(t_n) = 3^{k-n+1}$ with the associated graded ring a polynomial ring in the system of variables $< \bar{t}, \bar{t}_n >$ over the ring

$$(R_{n-1})[\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{n-1})/(t_n) \cong (R/A)[\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{n-1}]$$

The restriction of this pseudovaluation to $R$ defines a pseudovaluation in $R$; we will use for it the same notation $v_0$. We will now verify that this pseudovaluation satisfies all the conclusions of the theorem.

We have

$$t_1 = u_1t^{3^{n-2}}, t_2 = u_2t^{3^{n-3}}, t_{n-1} = u_{n-1}t \tag{3.59}$$

which yields

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We consider now the natural homomorphism \( \phi \) that the zero component of \( \text{gr} \) have in \( R \). This proves the first conclusion of Theorem 3.

We have a natural imbedding \( \text{gr}_{\text{v}_0}(R) \subseteq \text{gr}_{\text{v}_0}(V) \). We will prove first that the zero component of \( \text{gr}_{\text{v}_0}(R) \) is \( R/A \). This is equivalent to the fact that if \( x \in R \) then \( v_0(x) = 0 \) iff \( x \notin A \) which we will now verify. Clearly, if \( x \in A \) then \( v_0(x) > 0 \) because \( A \) is generated by the system of elements \( \langle t_1, t_2, \cdots, t_n \rangle \) and the elements of this system have values greater than zero.

Assume now that \( x \notin A \). Then \( x \notin A_{n-1} \) and the assumption of the induction implies that \( v_1(x) = 0 \) and the homogeneous component of \( x \) in the ring \( \text{gr}_{\text{v}_1}(R) \) belongs to the subring \( R/A_{n-1} \) of \( \text{gr}_{\text{v}_1}(R_{n-1}) \) and it coincides with the image \( \tilde{x} \) of \( x \) under the homomorphism \( R \rightarrow R_{n-1} \). Since \( x \notin A \) we obtain that image \( \tilde{x} \) is not contained in the ideal \( (A/A_{n-1}) = (t_n)R_{n-1} \). We recall now that \( V/(t) \cong \text{gr}_{\text{v}_1}(R_{n-1}) \) and obtain that the element \( x \) does not belong to the ideal \( (t, t_n) \). We obtain from Theorem 3.2 that \( v_0(x) = 0 \) and our claim is proven.

The system of elements \( \tilde{u}_1, \tilde{u}_2, \cdots, \tilde{u}_{n-1}, \tilde{t}_n, \tilde{t} \) is algebraically independent in the ring \( \text{gr}_{\text{v}_0}(V) \) over subring \( R/A \). We obtain now from (3.60) that \( \tilde{t}_i = \tilde{u}_i + 3^{n-i-1}t \) \( (i = 1, 2, \cdots, n-1) \) and conclude from this that the system of homogeneous components \( \tilde{t}_i, \tilde{t}_n \) \( (i = 1, 2, \cdots, n) \) is also algebraically independent and generates \( \text{gr}(V) \). The homogeneous components \( \tilde{t}_i \) \( (i = 1, 2, \cdots, n) \) belong to the subring \( \text{gr}_{\text{v}_0}(R) \subseteq \text{gr}_{\text{v}_0}(V) \) as well as the subring \( R/A \), we conclude from this that these homogeneous components are algebraically independent in the ring \( \text{gr}_{\text{v}_0}(R) \) over subring \( R/A \).

It remains to prove that \( \text{gr}_{\text{v}_0}(R) \) is generated by the elements \( \tilde{t}_1, \tilde{t}_2, \cdots, \tilde{t}_n \).

We consider now the natural imhomomorphism \( \phi: R \rightarrow R_1 = R/(t_1) \). We have in \( R_1 \) the polycentral independent system \( t_2, t_3, \cdots, t_n \) and the weight function obtained by restriction of \( f_k \) on the subsystem \( t_2, t_3, \cdots, t_n \)

\[
f_k(t_2) = 3^{k-1}, \quad f_k(t_3) = 3^{k-3}, \cdots, \quad f_k(t_n) = 3^{k-n+1} \quad (3.61)
\]

The assumption of the induction yields that this weight function extends to a \( t \)-adic pseudovaluation \( v_1 \) in the ring \( R_1 = R/(t_1) \) with associated graded
ring isomorphic to the polynomial ring in the system of homogeneous components of the elements $t_2, t_3, \ldots, t_n$ over the ring $R_1/(t_2, t_3, \ldots, t_n) \cong R/A$; for every $2 \leq i \leq n$ we denote by $t_i^*$ the homogeneous component of the element $t_i$ in the ring $gr_{\nu_i}(R_1)$.

Lemma 3.8. shows that the pseudovaluation $v_1$ coincides with the pseudovaluation $\bar{v}$ defined by the natural homomorphism $\phi: R \rightarrow R/(t_1) = R_1$ via Proposition 2.3. We obtain also from Proposition 2.3. that the homomorphism $\phi$ defines the homomorphism of graded rings $\bar{\phi}: gr_{\nu_0}(R) \rightarrow gr_{v_1}(R_1)$; the kernel of this homomorphism is the ideal $gr_{\nu_0}(A_1)$ where $A_1 = (t_1)$. The homogeneous component of degree zero in $gr(R)$ and in $gr(R_1)$ is $R/A$; since $ker(\bar{\phi}) \subseteq A$ we obtain that the restriction of $\bar{\phi}$ on $R/A$ defines an isomorphism between the homogeneous components of degree zero of $gr_{\nu_0}(R)$ and $gr_{v_1}(R_1)$.

For every $2 \leq i \leq n$ we have $\phi(t_i) = t_i$ and $v_0(t_i) = v_1(t_i) = 3^{k-i+1} (i = 2, 3, \ldots, n)$. Since the weights of the elements $t_i (i = 2, 3, \ldots, n)$ in $R$ and $R_1$ coincide we obtain that for every $2 \leq i \leq n$ $\phi$ maps the homogeneous component $t_i^* \in gr_{\nu_0}(R)$ on the homogeneous component $t_i^* \in gr_{v_1}(R_1)$. We see that the homomorphism $\bar{\phi}$ is an epimorphism of the ring $gr(R)$ on the polynomial ring $(R/A)[t_2^*, t_3^*, \ldots, t_n^*]$ which is isomorphic to the polynomial subring $(R/A)[\bar{t}_2, \bar{t}_3, \ldots, \bar{t}_n] \subseteq gr(R)$. We obtain therefore that $gr(R)$ is a direct sum of the subring $(R/A)[\bar{t}_2, \bar{t}_3, \ldots, \bar{t}_n]$ and the ideal $gr(A_1)$:

$$gr(R) \cong (R/A)[\bar{t}_2, \bar{t}_3, \ldots, \bar{t}_n] + gr(A_1) \quad (3.62)$$

We will now show that the ideal $gr(A_1)$ of $gr(R)$ is generated by the homogeneous component $\bar{t}_1$.

Let $x = rt_1$ be an element of $A_1$. If $v_0(r) + v_0(t_1) > v_0(x)$ then we would have in $gr(R)$ $\bar{r}\bar{t}_1 = 0$ for the homogeneous components of $r$ and $t_1$. This is impossible because $gr(V) \cong (R/A)[\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_n]$ so the homogeneous component of $t_1$ is regular in $gr(V)$ and because of this it is regular in the subring $gr(R) \subseteq gr(V)$. We see that $v_0(x) = v_0(r) + v_0(t)$ and we obtain $\bar{x} = \bar{r}\bar{t}_1$. This proves that

$$gr(A_1) \cong (\bar{t}_1)gr(R) \quad (3.63)$$

Let $y$ be a homogeneous element of degree $l$ in $gr(R)$. We obtain from (3.62) and (3.63) that there exist a homogeneous element $y_1 \in (R/A)[\bar{t}_2, \bar{t}_3, \ldots, \bar{t}_n]$ of degree $l$ such that $y = y_1 + \bar{t}_1y_2$ where $y_2 \in gr(R)$ is a homogeneous element of degree $l_1 < l$ because the degree of $t_1y_2$ is $l$. We can assume that $y_2$
has a representation as a polynomial in $\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n$ over $R/A$ and we obtain from this that $y$ is also a polynomial.

This completes the proof of Theorem 3.3.

3.8. **Theorem I.** Let $R$ be a ring, $T = \langle t_1, t_2, \ldots, t_n \rangle$ be an independent polycentral system in $R$.

The ideal $A$ generated by the system $t$ is residually nilpotent. If $\tilde{R}$ is the completion of $R$ in the topology defined by this ideal and $X$ is a system of coset representatives for the elements of the quotient ring $R/A$ then every element $x \in \tilde{R}$ has a unique representation

$$x = \sum_{n=0}^{\infty} \lambda_n \pi_n$$  \hspace{1cm} (3.64)

where $\lambda_n \in X$ ($n = 0, 1, \ldots$) $\pi_n$ are standard monomials on $T$, and $\lim_{n \to \infty} v(\pi_n) = \infty$.

**Proof of Theorem I.** Let $x \in \bigcap_{i=1}^{\infty} A^i$; we obtain from Theorem 3.3. that $v_0(x) = \infty$, hence $x = 0$. This proves the residual nilpotence of $A$.

We will now prove the second statement. Theorem 3.3. yields that $gr_{v_0} \cong (R/A)[\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n]$. We obtain from this and Lemma 3.7. that every element of the completion of $R$ in the $v_0$-topology has a unique representation (3.43). Since the $v_0$ and $\rho$ topology are equivalent we obtain that every element of $\tilde{R}$ has a unique representation (3.64).

The proof of Theorem I is complete.

3.9. **Proof of Theorem III.** We assume that the assertion is proven for the ring $\tilde{R} = R/(t_1)$ and the polycentral system $\tilde{T} = \langle \tilde{t}_2, \tilde{t}_3, \ldots, \tilde{t}_n \rangle$; we have already pointed out in subsection 3.6. that the initial step of the induction is true. Lemma 3.9. now implies that the homogeneous components $\tilde{t} \ (t \in T)$ are central in $gr_v(R)$. Lemma 3.6. implies that for every element $r$ there exists in $\tilde{R}_v$ a representation (3.43). Since the $v$-topology is equivalent to the $\rho$-topology we obtain now from Theorem 3.3. that it is equivalent also to the $v_0$-topology, and we conclude from this that representation (3.43) is unique in $\tilde{R}_v$. Finally, we obtain now from Lemma 3.7. that $gr_v(R) \cong (R/A)[\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n]$.

The uniqueness of $v$ follows from Proposition 2.4.

This completes the proof of Theorem III.
Corollary 3.8. Let \( v_m \) be the pseudovaluation defined in the quotient ring \( R_m = R/A_m \) by the independent polycentral system \( t_{m+1}, t_{m+2}, \ldots, t_n \) and the restriction of the function \( f \) on this system. Then the epimorphism \( \psi: R \rightarrow R_m \) defines in a natural way an epimorphism \( \tilde{\psi}: gr\nu(R) \rightarrow grv_m(R_m) \).

**Proof.** Lemma 3.6. implies that this statement is true for the homomorphism \( \psi_1: R \rightarrow R_1 = R/(t_1) \). We can assume that it is true for the epimorphism \( \tau: R_1 \rightarrow R_m \), and obtain that it is true for the product of homomorphisms \( \psi_1 \tau: R \rightarrow R_m \). This completes the proof.

Theorem II. Let \( R \) be a ring, \( T = \langle t_1, t_2, \ldots, t_n \rangle \) be a polycentral independent system in \( R \) which is composed from the central systems \( T_1, T_2, \ldots, T_k \), \( A \) be the ideal generated by the system \( T \). Let \( f \) be a function on \( T \) whose values are natural numbers and \( f(t_1) > 2f(t_2) \) for \( t_1 \in T_i, t_2 \in T_{i+1} \) (\( i = 1, 2, \ldots, k - 1 \)).

Then there exists a pseudovaluation \( v \) of \( R \) such that

\[
v(t) = f(t) \quad \text{if} \quad (t \in T)
\]

and the graded ring \( gr\nu(R) \) is isomorphic to the polynomial ring \( (R/A)[\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n] \), the topology defined in \( R \) by this pseudovaluation is equivalent to the topology defined by the powers of the ideal \( A \). Furthermore, \( v \) is the unique pseudovaluation such that \( v(t) = f(t) \) \( (t \in T) \) and the graded ring associated to it is isomorphic to \( (R/A)[\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n] \).

**Proof.** Follows immediately from Theorem III.

We finish this section with the following corollary of the results of this section.

Corollary 3.9. Let \( R \) be a ring with a discrete pseudovaluation \( \rho \), \( gr\rho(R) \) be the associated graded ring. Assume that there exists in \( gr(R) \) an independent polycentral system \( T \), let \( A \) be the ideal generated in \( gr(R) \) by this system. Then there exists in \( R \) a discrete pseudovaluation \( v \) such that the graded ring \( gr\nu(R) \) is isomorphic to a subring of the Laurent polynomial ring in the system of variables \( T, t, t^{-1} \) over the subring \( (gr(R))/A \).

**Proof.** We extend the pseudovaluation \( \rho \) to the ring \( R[t, t^{-1}] \) and apply Proposition 2.8. We obtain that \( V/(t) \cong gr(R) \), so the system of elements \( t, T \)
is an independent polycentral system in \( V \). Theorem II implies that there exists in \( V \) a discrete pseudovaluation \( v \) such that \( gr_v(V) \cong (gr(R)/A)[T,t] \).

Since the ring \( R[t, t^{-1}] \) is isomorphic to the ring of fractions of \( V \) with respect to the subsemigroup \( S \) generated by the element \( t \) the pseudovaluation \( v \) extends in a natural way to the ring to \( R[t, t^{-1}] \) and the graded ring of \( R[t, t^{-1}] \) associated to \( v \) is isomorphic to the ring of fractions \( (gr(R)/A)[T, t, t^{-1}] \) of \( (gr(R)/A)[T, t] \). Since \( R \subseteq R[t, t^{-1}] \) we obtain that \( gr_v(R) \) is the subring of this ring of fractions, and the proof is complete.

§4.

4.1. We will prove in this section Theorem IV which is an application of Theorems I-III to the group ring of torsion free nilpotent group. Its proof is based on the fact that if \( H \) is a finitely generated torsion free nilpotent group then every central series with torsion free factors provides in a natural way an independent polycentral system (see Proposition 4.1., statement ii). If \( H \) is not finitely generated we will reduce the proof to the finitely generated case by direct limit arguments.

**Lemma 4.1.** i) Let \( H \) be a torsion free nilpotent group which contains no elements of infinite \( p \)-height. Then
   i) all the factors of the upper central series contain no elements of infinite \( p \)-height.
   ii) Let \( H \) be a torsion free nilpotent group which has a central series
   \[
   H = U_1 \supseteq U_2 \supseteq \cdots \supseteq U_{k-1} \supseteq U_k = 1
   \] (4.1)
   with unit intersection and all the factors \( U_i/U_{i+1} \) are torsion free abelian group without elements of infinite \( p \)-height. Then \( H \) contains no elements of infinite \( p \)-height.

**Proof.** i) Let \( H \) be a torsion free nilpotent group without elements of infinite \( p \)-height, \( Z \) be its center. We will prove that the quotient group \( H/Z \) contains no elements of infinite \( p \)-height. Statement i) will follow from this by an induction on the nilpotency class of \( H \).
Assume that there exists a non-central element \( h \in H \) which has an infinite \( p \)-height in \( H/Z \). This means that for every given \( n \) there exist \( u \in H \) and \( z_n \in Z \) such that \( h = u^p z_n \). We pick an arbitrary element \( g_0 \in H \) which does not commute with \( h \) and obtain \( 1 \neq [g, h] = [g, u^p] \).

Now assume that the nilpotency class of \( H \) is \( c \), and pick \( n = mc \) where \( k \) is an arbitrary number. Since the element \( h = [g, u^p] \) is a product of two elements \( g^{-1}u^{-pmc}g \) and \( u^{pmc} \) we obtain from Malcev’s Lemma (see, for instance, Hartley [2], Lemma 2.4.2.) that \( h = v^{pm} \) for a suitable \( v \in H \).

Since \( m \) was an arbitrary integer we conclude that the element \( h \) has an infinite \( p \)-height. We obtained a contradiction and the proof of statement i) is complete.

ii) We can assume by an induction argument that the quotient group \( H/U_{k-1} \) contains no elements of infinite \( p \)-height. Assume that \( H \) has an element \( h \) of infinite \( p \)-height; if the image \( \bar{h} \) of \( h \) in \( H/U_{k-1} \) is non-unit it must have infinite \( p \)-height in \( H \). We conclude therefore that \( h \in U_{k-1} \). Once again, as in the proof of statement i), if there exist elements \( h_1, h_2, \ldots, h_n \in H \) such that \( h = h_1^{pm_1} h_2^{pm_2} \cdots h_n^{pm_n} \) then there exists \( u \in H \) such that \( h = u^{pm} \).

Since the quotient group \( H/U_{k-1} \) is torsion free we conclude that \( u \in U_{k-1} \). Since \( h \) has infinite \( p \)-height this contradicts the assumption that all the factors contain no elements of infinite \( p \)-height and the proof is complete.

4.2. We need a few elementary facts about Lie algebras of groups without elements of infinite \( p \)-height. Let \( H \) be a torsion free abelian group without elements of infinite \( p \)-height, \( E \) be a system of elements which forms a basis for the vector space \( H/H^p \). The system of elements \( E \) forms a free system of generators for every quotient group \( H/H^{pm} \) \((m = 1, 2, \ldots)\) and for the inverse limit of this system of groups. This inverse limit is isomorphic to the vector space \( C_p \otimes H \) where \( C_p \) is the subring of \( p \)-integers of the field of rationals, and the system \( E \) is also the basis of this vector space. If \( H_0 \) denote the subgroup generated by \( E \) then the quotient group \( H/H_0 \) is a periodic group without elements of order \( p \). Let \( E_j \ (j \in J) \) be the direct system of all the finite subsystems of \( E \), and for every \( j \in J \) let \( H_{0j} \) be the subgroup generated by \( E_j \). Then \( H \) is isomorphic to the direct limit of the system of group \( H_{0j} \ (j \in J) \).

**Lemma 4.2.** Let \( H \) be a torsion free abelian group without elements of infinite \( p \)-height, \( E \) be a system of elements which forms a basis of the vector space \( H/H^p \), \( K \) be a field of characteristic \( p \). Then
i) The restricted Lie algebra $L_p(H, H_i)$ associated to the $p$-series

$$H \supseteq H^p \supseteq \cdots$$

(4.2)

is free abelian Lie algebra with system of generators $\tilde{E}$ formed by homogeneous components of elements $e \in E$.

ii) The graded ring of $KH$ associated to the filtration $\omega^i(KH)$ $(i = 1, 2, \cdots)$ is isomorphic to the symmetric algebra $K[V]$ of the vector space $V = H/H^p$.

iii) Algebra $L_p(H)$ is the direct limit of the system of free abelian algebras $L_p(\tilde{E}_i)$ $(i \in I)$ and the graded ring $gr(KH)$ is the direct limit of the symmetric algebras $K[V_i]$ $(i \in I)$.

Proof. i) The group $H$ is an inverse limit of the system of groups $H^{p^n}$ $(n = 1, 2, \cdots)$. For every given $n$ the Lie algebra of the group $H/H^{p^n}$ is an abelian algebra of exponent $p^n$ and it is freely generated by the first factor $H/H^p$. Since the system $E$ forms a basis of the first factor we obtain from this statement i).

ii) The group ring $KH$ is an inverse limit of the system of rings $KH/\omega^i(KH)$ $(i = 1, 2, \cdots)$ where every of these rings is generated by the system $E$, and is isomorphic to the quotient ring of the polynomial ring $K[E]$ by the ideal $(E)^i$ where $(E)$ is the ideal generated by the system of elements $E$ and the proof can be completed easily.

Now let $f$ be an arbitrary function defined on the basis $E$ of the vector space $V = H/H^p$ with values in the set of the natural numbers, $K$ be a field of characteristic $p$. This function defines in a natural way a valuation of the ring $K[V]$ and of its completion $\widehat{K[V]}$. Since the group ring $KH$ imbeds into $K[V]$ we obtain a valuation $v$ of $KH$ whose values on the elements of $V$ coincide with the values of the function $f$. We see that the valuation $v$ can be defined by its values on an arbitrary basis of the vector space $H/H^p$. Since $H/H^p$ is the homogeneous component of degree 1 in the ring $K[V]$ we obtain also that the weight function $f$ defines a weight function on the vector space $H/H^p$.

4.3. Let $H$ be torsion free nilpotent group without elements of infinite $p$-height, (4.1) be a central series in $H$ whose factors $U_i/U_{i+1}$ $(i = 1, 2, \cdots, k - 1)$ contain no elements of infinite $p$-height. For every $1 \leq i \leq k - 1$ let $E_i$ be a system of elements in $U_i$ which is a basis of the vector space $U_i/U_i^pU_i'$ and let $E_i - 1$ denote the system of elements $e - 1$ $(e \in E_i)$, and

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\[ E - 1 = < E_1 - 1, E_2 - 1, \cdots, E_{k-1} - 1 > \]  \hspace{1cm} (4.3)

**Proposition 4.1.** Let \( H \) be a finitely generated torsion free nilpotent group, \( R \) be a ring. Then

i) The system of elements \( E_1 \) is an independent polycentral system in the group ring \( RH \).

ii) Let \( \pi \) be a central regular element in \( R \) such that \( \bigcap_{n=1}^{\infty} (\pi)^n = 0 \) and the quotient ring \( K = R/(t) \) is a field of finite characteristic \( p \). Then the system of elements \( \pi, E - 1 \) is an independent polycentral system in the group ring \( RH \).

**Proof.** We will prove statement ii); the proof of i) is obtained by an obvious simplification of the argument.

The system of elements \( E_{k-1} - 1 \) is a central independent system in \( KU_{k-1} \) by statement ii) of Lemma 4.2, and it is central and independent in \( KH \). The ideal generated by this system coincides with the ideal \( \omega(KU_{k-1})KH \) and the quotient ring \( KH/(\omega(KU_{k-1})KH) \) is isomorphic to the group ring of the group \( H/U_{k-1} \). The induction argument now implies that the system \( E - 1 \) is polycentral and independent in \( KH \). We obtain from this that the system \( < \pi, E - 1 > \) is polycentral and independent in \( RH \).

**Theorem IV.** Let \( H \) be a torsion free nilpotent group without elements of infinite \( p \)-height, \( R \) be a ring which contains a central regular element \( \pi \) such that \( \bigcap_{n=1}^{\infty} (\pi)^n = 0 \) and the quotient ring \( K = R/(t) \) is a field of finite characteristic \( p \).

Let \( f \) be a weight function on the system \( < \pi, E - 1 > \) whose values are natural numbers and

\[
\begin{align*}
  f(\pi) &> 2f(e - 1) \hspace{0.5cm} (e \in E_{k-1}); \\
  f(e_1 - 1) &> 2f(e_2 - 1) \hspace{0.5cm} (e_1 \in E_{i+1}, e_2 \in E_i) \hspace{0.5cm} (1 \leq i \leq k - 2) \hspace{0.5cm} (4.4)
\end{align*}
\]

This weight function has a unique extension to a \( t \)-adic pseudovaluation of \( RH \) with associated graded ring isomorphic to the polynomial ring \( K[\tilde{\pi}, (\tilde{E} - 1)] \) generated by the homogeneous components \( \tilde{\pi}, (\tilde{e} - 1) \ (e \in E) \).

The epimorphism \( : R \longrightarrow R/(\pi) \) defines an epimorphism of graded rings \( \tilde{\phi}: K[\tilde{\pi}, (\tilde{E} - 1)] \longrightarrow K[\tilde{(E - 1)}] \).

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Proof. The system of elements $E_i$ is a basis of the vector space $U_i/U_i^pU_{i+1}$, we have seen in the end of subsection 4.2. that the weight function on the system $E_i$ extends in a natural way to a weight function on the vector space $U_i/U_i^pU_{i+1}$; conversely a weight function on the vector space $U_i/U_i^pU_{i+1}$ defines a weight function on $E_i$. Because of this it is possible to assume that for every $i$ the weight function $f$ is defined on $U_i/U_i^pU_{i+1}$. We pick now an arbitrary finitely generated subgroup $V \subseteq H$ such that $V = \sqrt{V}$ i.e. if $h^n \in V$ for an element $h \in H$ then $h \in V$. We denote now $V_i = V \cap U_i$ ($i = 1, 2, \cdots, k$) and obtain now in $V$ a central series

$$V = V_1 \supseteq V_2 \supseteq \cdots \supseteq V_{k-1} \supseteq V_k = 1 \quad (4.5)$$

The condition $V = \sqrt{V}$ implies that for every $i$ the torsion free abelian subgroup $V_i/V_{i+1}$ is pure in the group $U_i/U_{i+1}$ and that the vector space $M_i' = V_i/V_i^pV_{i+1}$ naturally imbeds in the vector space $M_i = U_i/U_i^pU_{i+1}$. We obtain therefore a restriction of the weight function $f$ on the subspaces $V_i/V_i^pV_{i+1}$ ($i = 1, 2, \cdots, k-1$); we denote this restriction by $f'$. We consider the group ring $KV$ and obtain from Proposition 4.1. and Theorem II that there exists a valuation $v$ in $KV$ which extends the weight function $f'$ such that the graded ring $gr_v(KV)$ is isomorphic to the symmetric algebra $K[M']$ where $M' = \bigcup_{i=1}^{k-1} M_i'$. Further, Theorem II yields that there exists a valuation $\rho$ on the ring $RV$ such that $\rho(\pi) = f(\pi)$, $\rho(V_i) = f(V_i)$ ($i = 1, 2, \cdots, k-1$), the graded ring $gr_\rho(RV)$ is isomorphic to the polynomial ring $K[\bar{\pi}, T]$ where $T$ is an arbitrary basis of $V$ over $K$, and the epimorphism $RH \rightarrow KH$ defines the epimorphism of graded rings $K[\bar{\pi}, T] \rightarrow KT$.

We recall now that $V$ was an arbitrary finitely generated subgroup of $H$ such that $V = \sqrt{V}$. We construct for every $V_j$ ($j \in J$) a valuation $v_j$ and the proof is completed now by a standard direct limit argument.

We have the following immediate corollary of Theorem IV.

**Corollary 4.1.** Assume that the ring $R$ in Theorem IV is either the ring of integers $C$ or the ring of $p$-adic integers $\Omega$. Then the graded ring associated to the valuation $v$ is isomorphic to the polynomial ring $Z_p[t, T]$ generated by the homogeneous components $\tilde{p} = t$, and all the homogeneous components $e - 1$ ($e \in E$).

If $H$ is finitely generated the system of elements $p, E_1 - 1, E_2 - 1, \cdots, E_{k-1} - 1$ is an independent polycentral system in the group ring $CH$ or $\Omega H$. 

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4.4. Let $K$ be a field of characteristic zero, $H$ be a torsion free nilpotent group. The same type of an argument as in Theorem IV, with some simplifications, proves the following result.

**Theorem IV’.** Let $H$ be a torsion free nilpotent group, $K$ be a field of characteristic zero. Let $(4.1)$ be a central series with torsion free factors, and let $E_i$ be a system elements in $U_i$ which is a maximal linearly independent system modulo $U_{i+1}$ and let $E = \bigcup E_i (i = 1, 2, \cdots, k-1)$. Let $f$ be a weight function on on the system $E_1$ such that

$$f(e_1 - 1) > 2f(e_2 - 1) (e_1 \in E_{i+1}, e_2 \in E_i) (1 \leq i \leq k-2) \quad (4.6)$$

This weight function defines filtration $A_j (j = 0, 1, \cdots)$ in $KH$ with zero intersection and a valuation with graded ring isomorphic to $K[E]$.

If $H$ is finitely generated the system $E$ is a polycentral independent system in $H$.

We point out that Theorem IV’ gives a new proof Hall-Hartley Theorem about the residual nilpotence of the augmentation ideal $\omega(KH)$ (see Passman [13]. In fact the Hall-Hartley Theorem follows from Theorem IV’ because $\omega^j(KH) \subseteq A_j(KH)$ ($j = 1, 2, \cdots$).

The original proof of Hall-Hartley Theorem was based on extensive use of commutator calculus.

§5

**Lemma 5.1.** Let $L$ be a finitely generated restricted Lie algebra. Assume that every element of $L$ is nilpotent, i.e. for every $x \in L$ there exists a number $p^n$ such that $x^{p^n} = 0$ and that the Lie algebra $L$ is nilpotent, i.e. $\gamma_c(L) = 0$ for some number $c$. Then the algebra $L$ is finite, and its order is a power of $p$.

**Proof.** We will use induction by the nilpotency class of $L$. The assertion is obvious if $L$ is abelian, and we assume that it is true for the quotient algebra of $L$ by its center $Z$. Since $L/Z$ is finite and $L$ is finitely generated we obtain (see [1], 2.7.5.) that the subalgebra $Z$ is finitely generated. We obtain
therefore that $L$ is an extension of a finite algebra $Z$ by a finite algebra $L/Z$ hence it is finite. The same argument shows that the order of $L$ is a power of $p$.

**Lemma 5.2.** Let $L$ be a finitely generated restricted Lie algebra which contains an ideal $V$ which is nilpotent as a Lie algebra. Then there exists $n$ such that the ideal $V[p^n]$ generated by all the elements $v[p^n]$ $(v \in V)$ is central in $L$. If the quotient algebra $L/V$ is finite then there exists $m \geq n$ such that $V[p^m]$ is either zero or a central free abelian subalgebra of finite rank and the quotient algebra $L/V[p^n]$ is a restricted finite nilpotent algebra.

**Proof.** Let $v$ be an arbitrary element of $V$. Then $[u,v] \in V$ for an arbitrary element $u \in U$. Since $V$ is nilpotent we can find a number $p^n$ such that $[u,v,\ldots,v] = 0$ for every $u \in L$, and hence $[u, v[p^n]] = 0$. The last equation means that every element $v[p^n]$ is central.

If $L/V$ is finite then $V$ must be finitely generated. So $V[p^n]$ is a finitely generated central subalgebra of $L$. Since all the nilpotent elements in $V[p^n]$ form a finite subalgebra we can take number $m$ which is a suitable multiple of $n$ and to obtain from Lemma 2.4. that $V[p^m]$ is either zero or free abelian, and the quotient algebra $V/V[p^m]$ must be finite by Lemma 5.1.; hence $L/V[p^m]$ is finite and the proof is complete.

We need the following fact.

**Lemma 5.3.** Let $H$ be a polycyclic-by-finite group. Then $H$ contains a normal subgroup $V$ of finite index such that the nilpotent radical $N$ of $V$ is torsion free and the quotient group $V/N$ is free abelian. Moreover $V$ has an additional property that every element of it centralizes $N$ modulo the dimension subgroup $N'N^p$.

**Proof.** The first statement follows from the Kolchin-Malcev Theorem (see Kargapolov and Merzlyakov [6], Theorem VII. 3.3., or Segal [17]), the second one is obtained by a routine argument.

**Theorem V.** Let $H$ be an infinite polycyclic-by-finite group with Hirsh number $r$. Assume that there exists a $p$-series (1.1) with unit intersection such that the corresponding restricted Lie algebra $L_p(H, H_i)$ is finitely
generated. Then there exists a torsion free normal subgroup $F$ with index a power of $p$ such that the ideal $L_p(F, F_i)$ associated to the $p$-series $F_i = F \cap H_i$ ($i = 1, 2, \cdots$) is a restricted free abelian subalgebra of the center of $L_p(H, H_i)$ with index a power of $p$ and rank $1 \leq r_1 \leq r$.

Hence the center $Z$ of $L_p(H, H_i)$ has a finite index which is a power of $p$ and $L_p(H, H_i)$ is a nilpotent Lie algebra.

**Proof of Theorem V.** Let $V$ be a normal subgroup obtained in Lemma 5.3., $N$ be its nilpotent radical. Since the quotient group $H/V$ is finite the ideal $L_p(V, V_i)$ has a finite index in $L_p(H, H_i)$ and we obtain that the algebra $L_p(V, V_i)$ is finitely generated. Further the nilpotence of $N$ implies via Corollary 2.4. that the restricted Lie algebra $L_p(N, N_i)$ is nilpotent as a Lie algebra. Lemma 5.2. implies that there exists a number $p^n$ such that the subalgebra $L_p(N, N_i)[p^n]$ is a central subalgebra of $L_p(H, H_i)$. Let $c$ be the nilpotency class of $N$. Malcev’s Lemma (see Hartley [4]) implies that every element of the subgroup $N_p^{nc}$ has a form $x = y^{p^n}$ for a suitable $y \in N$. We consider now the normal subgroup $W = N_p^{nc}$ generated by all the elements $h^{p^n}$ ($h \in N$); every element of $L_p(W, W_i)$ is a homogeneous component of some element $y^{p^n}$ so either its homogeneous component is $\tilde{y}^{[p]n}$ or $\tilde{y}^{[p]n} = 0$. We obtain from this that the subalgebra $L_p(W, W_i)$ is central in $L_p(V, V_i)$. The quotient group $\tilde{V} = V/W$ is an extension of the finite $p$-group $N/N_p^{nc}$ by the free abelian group $\tilde{V}/\tilde{N} \cong V/N$, and $V$ acts trivially on the factor $\tilde{N}/\tilde{N}^p \cong N/N^p$, hence the group $\tilde{V}$ is nilpotent. Corollary 2.4. implies that the restricted Lie algebra $L_p(V, V_i)$ is a nilpotent Lie algebra. We see that the restricted Lie algebra $L_p(V, V_i)$ is an extension of a central ideal $L_p(W, W_i)$ by the algebra $L_p(\tilde{V}, \tilde{V}_i)$ which is a nilpotent Lie algebra, hence $L_p(V, V_i)$ is a nilpotent Lie algebra. Lemma 5.2. now implies that there exists a number $l$ such that the subalgebra $L_p(V, V_i)^{[p]^l}$ is central in $L_p(H, H_i)$. The index of $L_p(V, V_i)$ is finite because the subgroup $V$ has a finite index in $H$; since $L_p(V, V_i)^{[p]^l}$ has finite index in $L_p(V, V_i)$ we conclude that the index of $L_p(V, V_i)^{[p]^l}$ in $L_p(H, H_i)$ is finite.

We take now the subgroup $Q = V^{\cdot\cdot\cdot}$ and obtain by the same argument as above that the subalgebra $L_p(Q, Q_i)$ is central in $L_p(H, H_i)$ and has a finite index. Hence $L_p(Q, Q_i)$ is finitely generated once again by [1], 2.7.5. Since $L_p(Q, Q_i)$ contains a finite number of nilpotent elements we can find a number $i_0$ such that the central subalgebra
\[ \sum_{i \geq i_0} Q_i/Q_{i+1} \]  

is either the zero subalgebra or is free abelian. Subalgebra (5.1) is in fact the restricted Lie algebra \( L_p(R, R_i) \) of the subgroup \( R = Q_{i_0} \) associated to the \( p \)-series \( R_i = Q_{i_0} \cap H_i \); it has a finite index in \( L_p(H, H_i) \) because \( Q \) has a finite index in \( H \). If \( L_p(R, R_i) = 0 \) then \( \bigcap_{i=1}^{\infty} R_i = R \) which together with the relation \( \bigcap_{i=1}^{\infty} R_i = 1 \) implies \( R = 1 \). Since \( H \) is infinite and \( R \) has a finite index in \( H \), we obtain a contradiction. This means that \( L_p(R, R_i) \) is a central free abelian subalgebra of rank \( r_1 \geq 1 \). Proposition 2.5 implies that \( r_1 \leq r \).

We apply now Corollary 2.1 and obtain a normal subgroup \( F \supseteq R \) with index a power of \( p \) such that \( L_p(F, F_i) = L_p(R, R_i) \). The subgroup \( F \) is torsion free because the subalgebra \( L_p(F, F_i) \) is free abelian. We see that the subgroup \( F \) satisfies all the conclusion of the theorem.

We prove now the last statement of the theorem. Since \( Z \supseteq L_p(F, F_i) \) we obtain that the index of \( Z \) is a power of \( p \). Further, since the the quotient algebra \( L_p(H, H_i)/L_p(F, F_i) \) is a nilpotent Lie algebra and \( L_p(F, F_i) \) is a central subalgebra we obtain that the Lie algebra \( L_p(H, H_i) \) is nilpotent.

The proof of Theorem V is complete.

§6. Proof of Theorem VI.

6.1. Proposition 6.1. Let \( H \) be a group, \( U \) be its normal subgroup which satisfies a law \( \omega(x_1, x_2, \ldots, x_n) = 1 \). Assume that \( H \in \text{res} N_p \) and let \( V \supseteq U \) be the normal subgroup formed by all the elements which have an infinite \( p \)-height in the quotient group \( H/U \). Then \( V \) satisfies law \( \omega \).

Proof. Let \( \phi_i: H \rightarrow H_i \) be a homomorphism of \( H \) on a finite \( p \)-group \( H_i \), \( X_i \) be the image of a subset \( X \subseteq H \) under this homomorphism. The group \( V_i \) is an extension of the normal subgroup \( U_i \) by the group \( V_i/U_i \); since the elements of \( V \) have an infinite \( p \)-height in the quotient group \( V/U \), all the non-unit elements of \( V_i/U_i \) have an infinite \( p \)-height. But the group \( V_i/U_i \) is a subgroup of \( H_i \) which is a finite \( p \)-group. Hence \( V_i/U_i = 1 \), i.e. \( V_i = U_i \). We obtain therefore that \( \omega(V_i) = 1 \), which implies that \( \omega(V) = 1 \) and the assertion follows.
Corollary 6.1. Let $H$ be a finitely generated nilpotent-by-finite group, $U$ be the unique maximal nilpotent normal subgroup of $H$. If $H$ is a residually-\{finite \ p\text{-group}\} then the index of $U$ is a power of $p$.

Proof. Let $V$ be as in Proposition 6.1. Then $V$ is nilpotent, so we obtain that $V = U$. The definition of $V$ implies that the order of an arbitrary element of $H/V = H/U$ must be a power of $p$, so $H/U$ is a $p$-group.

Proposition 6.2. Let $H$ be a finitely generated abelian-by-finite group with Hirsh number $r$ and without finite normal subgroups. Assume that there exists a $p$-series

$$H = H_1 \supseteq H_2 \supseteq \cdots$$

(6.1)

with unit intersection such that the Lie algebra $L_p(H, H_i)$ is abelian of rank $r$. Then the topology defined by $p$-series (6.1) is equivalent to the $p$-topology, there exists a number $i_0$ such that $H_{i_0} \subseteq U$ where $U$ is the unique maximal free abelian characteristic subgroup of $H$, the index of $U$ is a power of $p$.

Proof. The nilpotent radical $\rho(H) = U$ is a torsion free nilpotent group. Since it is abelian-by-finite group it must be abelian. If $C(U)$ is the centralizer of $U$ then the commutator subgroup of $C(U)$ is finite by Schur’s Theorem, so $C(U)$ must be abelian because $H$ contains no finite normal subgroups. Since $U = \rho(H)$ we obtained that $C(U) = U$.

Let $\bar{H} = H/U$, $\bar{H}_i$ ($i = 1, 2, \cdots$) be the image of the $p$-series $H_i$ in $\bar{H}$. We will prove now that

$$\bigcap_{i=1}^{\infty} H_i = 1$$

(6.2)

In fact if $1 \neq \bar{h} \in \bigcap_{i=1}^{\infty} \bar{H}_i$ then there exists $h \in H$ such that $h \notin U$ but for every $i$ there exists $u_i \in U$ such that $hu_i \in H_i$. This implies that $[hu_i, u] = [h, u] \in H_i$ ($i = 1, 2, \cdots$) for every $u \in U$ and hence $[h, u] \in \bigcap_{i=1}^{\infty} H_i = 1$. But the centralizer of $U$ coincides with $U$, hence $h \in U$ and we obtained a contradiction, and relation (6.2) is proven.

Since the group $\bar{H}$ is finite we obtain from (6.2) that there exists $i_0$ such that $\bar{H}_{i_0} = 1$ that is $H_{i_0} \subseteq U$; this proves also that the index of $U$ is a power of $p$ because the index of $H_{i_0}$ is a power of $p$.

It remains to prove that the topology defined by series (6.1) is equivalent to the $p$-topology. The group $U$ is free abelian and the Lie algebra $L_p(U, U_i)$
associated to the $p$-series $U_i = U \cap H_i$ is abelian of rank $r$. Lemma 2.8 implies that the topology defined by this series in $U$ is equivalent to the topology defined by the lower $p$-series $U^{p^j}$ ($j = 1, 2, \ldots$). We can find therefore for every $j$ a number $i = i(j)$ such that $H_i \subseteq U^{p^j} \subseteq M_j(H)$ which proves that the series (6.1) defines the same topology as the series $M_j(H)$ ($j = 1, 2, \ldots$). This completes the proof.

**Proposition 6.3.** Let $H$ be a finitely generated abelian-by-finite group with Hirsch number $r$ which contains a $p$-series (6.1) with unit intersection such that the algebra $L_p(H, H_i)$ is free abelian of rank $r$. Then the topology defined by this $p$-series is equivalent to the $p$-topology and there exists a number $i_0$ such that the subgroup $H_{i_0}$ is torsion free abelian.

**Proof.** Let $tor(H)$ be the unique maximal normal finite subgroup of $H$. Since the intersection of the terms of series (6.1) is trivial we can finite a subgroup $F = H_{i_0}$ such that $F \cap tor(H) = 1$. The normal subgroup $F$ contains no finite normal subgroups and contains a series $F_i = F \cap H_i$ with unit intersection and if $i \geq i_0$ we have $F_i = H_i$. The algebra $L_p(F, F_i)$ is naturally imbedded in $L_p(H, H_i)$ so it must be abelian. The quotient algebra $L_p(H, H_i)/L_p(U, U_i)$ is finite because the index of $F$ in $H$ is finite. We conclude from this that the rank of $L_p(F, F_i)$ is equal $r$, and it coincides with the Hirsch number of $F$. We apply now Proposition 6.2. to the group $F$ and series $F_i (i = 1, 2, \ldots)$ and obtain that there exists a number $i_1 \geq i_0$ such that the subgroup $F_i = H_i$ is free abelian if $i \geq i_1$.

Proposition 6.2. implies that the topology defined in $F$ by the series $F_i (i = 1, 2, \ldots)$ is equivalent to the $p$-topology of this group. We obtain from this that for every subgroup $M_n(F)$ there exists $i(n)$ such that $F_i \subseteq M_n(F) \subseteq M_n(H)$ if $i \geq i(n)$. This together with the fact that $F_i = H_i (i \geq i_1)$ implies that $H_i \subseteq M_n(H)$ if $i \geq \max\{i(n), i_1\}$ and the proof is complete.

**Proposition 6.4.** Let $H$ be a finitely generated nilpotent-by-finite group with Hirsch rank $r$, $U$ be the nilpotent radical of $H$. Assume that there exists a $p$-series $H_i$ with unit intersection such that the Lie algebra $L_p(H, H_i)$ is abelian of rank $r$. Then the topology defined by this series is equivalent to the $p$-topology. Further, there exists an index $i_0$ such that the subgroup $H_{i_0}$ is torsion free nilpotent and the index of the nilpotent radical $U$ is a power of $p$.

**Proof.** We consider first the special case when $H$ contains no finite nor-
mal subgroups. Let in this case $V$ be a maximal abelian $H$-invariant subgroup of $U$ and $W$ be the inverse image in $H$ of the maximal finite normal subgroup of $H/V$. Clearly $V \subseteq W$ and $W$ is an abelian-by-finite normal subgroup of $H$ which contains no finite normal subgroups. Let $\rho(W)$ be the nilpotent radical of $W$. Then $\rho(W) \supseteq V$ and $\rho(W)$ is the unique maximal abelian normal subgroup of $W$; we obtain from this that $\rho(W) \subseteq U$. Since $V$ is a maximal abelian $H$-invariant subgroup of $U$ and $\rho(W) \supseteq V$ we conclude that $\rho(W) = V$. The group $W$ contains a $p$-series $W_i = H \cap W (i = 1, 2, \cdots)$, the Lie algebra $L_p(W, W_i)$ is abelian of rank $r_1$ equal to the Hirsh number of $W$ by Proposition 2.6. Proposition 6.2 now implies that the index $(W : V)$ is a power of $p$ and the topology defined by the series $W_i$ in $W$ is equivalent to the $p$-topology in $W$.

Let $\bar{H} = H/W$. The definition of $W$ implies that $\bar{H}$ contains no finite normal subgroups; we obtain from Proposition 2.6. that the image of the $p$-series $H_i (i = 1, 2, \cdots)$ is a $p$-series $\bar{H}_i$ with the associated Lie algebra $L_p(\bar{H}, \bar{H}_i)$ free abelian of rank $r - r_1$ which is equal to the Hirsh number of $\bar{H}$. We can assume by induction on the Hirsh number that the topology defined by the $p$-series $\bar{H}_i$ in $\bar{H}$ is equivalent to the $p$-topology. This, together with the fact that the topology defined by $W_i$ in $W$ is equivalent to the $p$-topology in $W$ implies via Lemma 2.7. that the topology defined by the $p$-series $H_i (i = 1, 2, \cdots)$ is equivalent to the $p$-topology.

Since $H$ is a residually finite $p$-group and $U$ is the nilpotent radical of it we can apply now Corollary 6.1 and to conclude that $H/U$ is a finite $p$-group. Hence there exists $n$ such that $M_n(H) \subseteq U$. Since the series $H_i (i = 1, 2, \cdots)$ and $M_i(H)$ define the same topology we can find $i_0$ such that $H_{i_0} \subseteq M_n(H) \subseteq U$.

This completes the proof for the special case.

Now consider the general case. Let $tor(H)$ be the unique maximal finite normal subgroup of $H$, $F = H_{i_0}$ be a term of series (6.1) that have a unit intersection with $tor(H)$. Then $F$ contains no finite normal subgroups and the assertion now follows from the proven special case by the same argument that was used in Proposition 6.3.

**Proposition 6.5.** Let $H$ be a finitely generated nilpotent-by-finite group with Hirsh number $r$. Assume that there exists a $p$-series $H_i$ such that the Lie algebra $L_p(H, H_i)$ is abelian of rank $r$. Let $F$ be a subgroup of $H$ with Hirsh rank $k$, $F_i = H \cap H_i (i = 1, 2, \cdots)$. Then the subalgebra $L_p(F, F_i)$ of $L_p(H, H_i)$ has rank $k$.
Proof. Let $H_0 = Q$ be the torsion free nilpotent normal subgroup of $H$, obtained in Proposition 6.4., $N = F \cap Q$, $Q_i = Q \cap H_i, N_i = N \cap H_i (i = 1, 2, \cdots)$. The algebra $L_p(Q, Q_i)$ is abelian of rank $r$ by Proposition 2.6.; further, $N_i = Q_i \cap N (i = 1, 2, \cdots)$. Since $N$ has a finite index in $F$ the subalgebra $L_p(N, N_i)$ has a finite index in $L_p(F, F_i)$, so it is enough to prove that $L_p(N, N_i)$ has rank $k$. We see that we can assume from the very beginning that the group $H$ is torsion free nilpotent, and $F$ is a subgroup of it.

Let $V$ be an infinite cyclic central subgroup of $H$ such that the quotient group $H/V$ is torsion free, $T = V \cap F, T_i = T \cap H_i (i = 1, 2, \cdots)$. We consider now the natural homomorphism $H \longrightarrow H/V = \overline{H}$ which defines the homomorphism $L_p(H, H_i) \longrightarrow L_p(\overline{H}, H_i)$ where $\overline{H_i} = (H_iV)/V (i = 1, 2, \cdots)$. The restriction of this homomorphism on $F$ defines a homomorphism $F \longrightarrow F/T$ and a homomorphism of Lie algebras $L_p(F, F_i) \longrightarrow L_p(\overline{F}, \overline{F_i})$ where $\overline{F} = F/T, \overline{F_i} = (F_iT)/T (i = 1, 2, \cdots)$; the kernel of the last homomorphism is a Lie subalgebra $L_p(T, T_i)$ of $L_p(F, F_i)$.

We have two possible cases: $T$ is the unit subgroup, or $T$ is an infinite cyclic subgroup of finite index in $V$. In the first case we obtain that $F \cong F$ and it is naturally imbedded in the quotient group $\overline{H} = H/V$, and $L_p(F, \overline{F_i})$ is the Lie algebra of the subgroup $\overline{F} = (FV)/V$ associated to the $p$-series $F_i = F \cap H_i (i = 1, 2, \cdots)$. The Hirsch number of $\overline{H}$ is $r - 1$ and we can assume by induction that the assertion holds for this case, that is the rank of the algebra $L_p(\overline{F}, \overline{F_i}) \cong L_p(F, F_i)$ is equal to its Hirsh number, so it is equal to $k$.

We consider now the second case. In this case the algebra $L_p(F, F_i)$ is an extension of the algebra $L_p(T, T_i)$ by the Lie algebra $L_p(\overline{F}, \overline{F_i})$. The algebra $L_p(T, T_i)$ is an abelian subalgebra of $L_p(V, V_i)$, its index is finite because $T$ has a finite index in $V$. Hence $L_p(T, T_i)$ is an abelian algebra of rank 1. The rank of $L_p(\overline{F}, \overline{F_i})$ is $k - 1$ by the induction argument. We obtain from this that the rank of $L_p(F, F_i)$ is $k$ and the proof is complete.

6.2. Theorem VI. Let $H$ be a polycyclic group with Hirsch number $r$. Assume that there exists a $p$-series $H_i (i = 1, 2, \cdots)$ with unit intersection such that $L_p(H, H_i)$ is abelian of rank $r$.

i) Let $U$ be an arbitrary subgroup of $H$ with Hirsch number $k$, $U_i = U \cap H_i (i = 1, 2, \cdots)$. Then the algebra $L_p(U, U_i)$ is abelian of rank $k$.

ii) Let $U$ be a normal subgroup of $H$ with Hirsch number $k$, $\overline{H_i}$ be the image of the subgroup $H_i$ in $H/U$. The subgroup $\bigcap_{i=1}^{\infty} \overline{H_i}$ is finite and the algebra $L_p(\overline{H}, \overline{H_i})$ is abelian of rank $r - k$. In particular, if $H = H/U$ contains
no finite normal subgroups then \( \bigcap_{i=1}^{\infty} \bar{H}_i = 1 \) and \( \bar{H} \) is a residually \{finite p-group\}.

iii) Let \( U \) be a normal subgroup of \( H \). If \( \bar{H} = H/U \) is a residually \{finite p-group\} then \( \bigcap_{i=1}^{\infty} \bar{H}_i = 1 \).

iv) Let \( W \) be the unique maximal normal nilpotent-by-finite subgroup of \( H \). Then \( W \) is an extension of a torsion free nilpotent group by a finite p-group. The quotient group \( H/W \) is an extention of a free abelian group by a finite p-group.

v) The topology defined in \( H \) by the p-series \( H_i \) \((i = 1, 2, \cdots)\) is equivalent to the p-topology. The topology defined in an arbitrary subgroup \( U \) by the series \( U_i = H_i \bigcap U \) \((i = 1, 2, \cdots)\) and \( M_n(H) \bigcap U \) \((n = 1, 2, \cdots)\) are equivalent to the p-topology in \( U \).

vi) There exists an index \( i_0 \) such that if \( i \geq i_0 \) then the subgroup \( Q = H_i \), contains a torsion free nilpotent subgroup \( N \) which is invariant in \( H \), \( Q/N \) is free abelian,

\[
\frac{H}{N'N} \in \text{res } \mathcal{N}_p
\]  \hspace{1cm} (6.3)

Clearly, \( H/Q \) is a finite p-group.

vii) Let \( F \supseteq S \) be two normal subgroups in \( H \) such that \( H/F \) and \( F/S \) are residually \{finite p-groups\}. Then \( H/S \) is a residually \{finite p-group\}.

viii) Let

\[
H = H_1^* \supseteq H_2^* \supseteq \cdots
\]  \hspace{1cm} (6.4)

be a series in \( H \) with finitely generated associated graded Lie algebra \( L_p(H, H_i^*) \). If the topology defined by series (6.4) is equivalent to the p-topology then the center of \( L_p(H, H_i^*) \) has rank \( r \). Moreover, there exists a number \( k \) such that if \( U = H_i \) \((i \geq k)\) then the subalgebra \( L_p(U, U_i^*) \cong \sum_{i \geq k} H_i/H_{i+1} \) associated to the p-series \( U_i^* = U \bigcap H_i^* \) \((i = 1, 2, \cdots)\) is a central free abelian subalgebra of rank \( r \).

**Proof of Theorem VI.** Statement ii) follows from Propositions 2.6. and 2.5.

We prove now statement i). Let \( r_1 \) be the Hirsch number of the unique maximal nilpotent-by-finite subgroup \( W \). Proposition 2.6. implies that the rank of the algebra \( L_p(W, W_i) \) is \( r_1 \). We consider now the subgroup of \( F = U \bigcap W \) and obtain from Proposition 6.5. that the rank of the algebra \( L_p(F, F_i) \) is equal to the Hirsch number of \( F \), say \( l \). Now consider the
natural homomorphism $H \rightarrow H/W$ and its restriction $U \rightarrow \bar{U} = U/F$. Statement ii) implies once again that the rank of the algebra $L_p(\bar{H}, \bar{H}_i)$ is equal to the Hirsch number $r - l$ of $\bar{H}$; since the group $\bar{H}$ is abelian-by-finite we obtain from Proposition 6.5 that the rank of $L_p(U, \bar{U}_i)$ is equal to the Hirsch number $k - l$. We see that the restricted abelian Lie algebra $L_p(U, U_i)$ is an extension of an algebra $L_p(F, F_i)$ with rank $l$ by the algebra $L_p(U, U_i)$ with rank $k - l$. This implies that the rank of $L_p(U, U_i)$ is $k$ which proves ii).

The proof of statements iv and v). The quotient group $\bar{H} = H/W$ is abelian-by-finite and it does not contain finite normal subgroups. Statement ii) implies that $L_p(\bar{H}, \bar{H}_i)$ is abelian with rank equal to the Hirsch number of $\bar{H}$ which is rank $r - r_1$, where $r_1$ is the Hirsch number of $W$ and $\bigcap_{i=1}^{\infty} H_i = 1$; we obtain from Proposition 6.2 that the group $\bar{H}$ is an extension of a free abelian group by a finite $p$-group. This completes the proof of statement iv).

Corollary 6.1 implies that $W$ is an extension of a torsion free nilpotent group by a finite $p$-group, and the topology defined by the series $W_i = H_i \cap W (i = 1, 2, \cdots)$ is equivalent to the $p$-topology of $W$. Proposition 6.3 implies that the topology defined in $H$ by series $H_i (i = 1, 2, \cdots)$ is equivalent to the $p$-topology in $\bar{H}$. We obtain now from Lemma 2.7 that the topology defined in $H$ by series $H_i (i = 1, 2, \cdots)$ is equivalent to the $p$-topology. We can apply this result to the subgroup $U$ because $U$ contains a $p$-series $U_i = U \cap H_i (i = 1, 2, \cdots)$ with unit intersection and with associated graded algebra $L_p(U, U_i)$ abelian of rank equal to the Hirsch number of $U$; the last fact follows from statement i). We obtain that the topology defined in $U$ by the series $U_i (i = 1, 2, \cdots)$ is equivalent to the $p$-topology.

We consider now the series $M_n(H) \cap U (n = 1, 2, \cdots)$. We have already proven that there exists a number $i(n)$ such that $U_{i(n)} \subseteq M_n(U)$. Hence

$$(M_{i(n)} \cap U) \subseteq (H_{i(n)} \cap U) \subseteq U_{i(n)} \subseteq M_n(U) \tag{6.5}$$

which proves that the topology defined by the series $M_n(H) \cap U (n = 1, 2, \cdots)$ is equivalent to the $p$-topology in $U$.

This completes the proofs of statement v).

To prove statements vi) and vii) we need the following lemma.

Lemma 6.1. Let $H$ be a group with Hirsch number $r$. Assume that there exists a $p$-series (6.1) with unit intersection and the restricted Lie algebra
$L_p(H, H_i)$ abelian of rank $r$. Let $R$ be a normal subgroup such that the quotient group $H/R$ is an extension of a free abelian group by a finite $p$-group. Then $H/M_n(R) \in \text{res} \mathcal{N}_p$ for every $n$.

**Proof.** Let $H_0 \supseteq R$ be a normal subgroup of index $p^k$ such that the quotient group $H_0/R$ is free abelian. The group $H_0$ has Hirsch number $r$ and it contains a $p$-series $H_{0i} = H_i \cap H_{0i}$ $(i = 1, 2, \cdots)$ with unit intersection such that the algebra $L_p(H_0, H_{0i})$ is abelian of rank $r$. Since $H/H_0$ is a finite $p$-group it is enough to prove that $H_0/M_n(R) \in \text{res} \mathcal{N}_p$. We see that we can assume from the very beginning that $H = H_0$, i.e. $H/R$ is free abelian.

Statement i) of Theorem VI implies that the group $R$ has a $p$-series $R_i = H_i \cap R$ ($i = 1, 2, \cdots$) such that the algebra $L_p(R, R_i)$ has rank equal to the Hirsch number of $R$; statement v) implies that we can find $m$ such $H_m \cap R = R_m \subseteq M_n(R)$. Let $\bar{H} = H/H_m$, $\bar{R} = R/R_m$. Since $H' \subseteq R$ we obtain

$$[R, H, H, \cdots, H] \subseteq (R \cap H_m) = R_m$$

and then

$$[R, \bar{H}, \bar{H}, \cdots, \bar{H}] = 1$$

Since $\bar{H}' \subseteq R$ we obtain from (6.7) that the group $\bar{H}$ is nilpotent. The group $H/M_n(R)$ is a homomorphic image of $\bar{H}$, hence it is also nilpotent. On the other hand, the group $H/M_n(R)$ is an extension of a finite $p$-group $R/M_n(R)$ by a free abelian group $H/R$; this together with the nilpotency of this group implies that $H/M_n(R) \in \text{res} \mathcal{N}_p$. This completes the proof of the lemma.

**Proof of statement vii) of Theorem VI.** Let once again $W$ be the nilpotent-by-finite radical of $H$. Since the rank of the Lie algebra $L_p(W, W_i)$ is equal to the Hirsch number of $W$ by statement i) Proposition 6.4. implies that there exists in $H$ a normal subgroup $W_{i0} = H_{i0} \cap W$ such that $W_{i0}$ is torsion free nilpotent and hence the group $W$ is an extension of a torsion free nilpotent group by a finite $p$-group. Hence we can find $l$ such that $M_l(W)$ is a torsion free nilpotent group. On the other hand we obtain from Lemma 6.1. that the group $\bar{H} = H/M_l(W) \in \text{res} \mathcal{N}_p$ so $\bar{H}$ is an extension of a finite $p$-group $\bar{W} = W/M_l(W)$ by the group $\bar{H}/\bar{W} \cong H/W$, which contains no
finite normal subgroups, and it is also an extension of a free abelian group by a finite $p$-group; hence $W$ is the unique maximal finite normal subgroup of $H$. Since $H \in \text{res} \mathcal{N}_p$ we can now find a normal subgroup $S$ with index $p^n$ in $H$ which does not intersect $W$. Hence $S$ is an extension of a free abelian group by a finite $p$-group; let $\bar{V}$ be a free abelian normal subgroup of index $p^m$ in $S$. Its inverse image $V$ has index $p^{n+m}$ in $H$, and $V$ is an extension of the torsion free nilpotent group $M_l(W)$ by the free abelian group $\bar{V}$. Since $\bar{V}$ is free abelian because it is a subgroup of the free abelian group $V/M_l(W)$. Lemma 6.1. implies that $(Q/N') \in \text{res} \mathcal{N}_p$. Since $H/Q$ is a finite $p$-group we obtain that $(H/N') \in \text{res} \mathcal{N}_p$.

This completes the proof of statement vi).

**Proof of statement vii).** Since the group $H/F$ is polycyclic it has a unique maximal finite normal subgroup $F_0/F$. Since $H/F \in \text{res} \mathcal{N}_p$ we obtain that $F_0/F$ is a $p$-group, and hence $F_0/S$ is an extension of a residually-finite $p$-group by a finite $p$-group $F_0/F$. Hence $(F_0/S) \in \text{res} \mathcal{N}_p$. Since $H/F_0$ contains no non-unit finite subgroups we obtain from statement ii) that $H/F_0 \in \text{res} \mathcal{N}_p$. We see that it is enough to prove the statement for the normal subgroups $F_0$ and $S$; we can assume from the very beginning that $F = F_0$ and hence $H/F$ contains no non-trivial finite normal subgroups.

Once again, we obtain from statement ii) that the quotient group $H/F$ contains a $p$-series with a unit intersection with associated restricted Lie algebra of rank equal to the Hirsch rank of $H/F$. Statement vi) now implies that $H/F$ contains a poly-$\{\text{infinite cyclic}\}$ normal subgroup $H_0/F$ with index $(H : H_0)$ a power of $p$. Since the index of $H_0$ is a power of $p$ it is enough to prove that $H_0 \in \text{res} \mathcal{N}_p$. We can assume therefore that $H = H_0$ i.e. the quotient group $H/F$ is poly-$\{\text{infinite cyclic}\}$.

We will now use induction on the Hirsch number of $H/F$. Let $R \supseteq F$ be a normal subgroup of $H$ such that $H/R$ is infinite cyclic group generated by an element $h \in H$. Statement i) implies that the algebra $L_p(R, R_i)$ is abelian with rank $r-1$ which is the Hirsch number of $R$. Since the Hirsch number of $R$ is $r-1$ we can assume that the assertion is proven for the group $R$ and its normal subgroups $F \supseteq S$, so $R/S \in \text{res} \mathcal{N}_p$.

Let $\bar{X}$ be the image of a subset $X \subseteq H$ under the natural homomorphism
Let $H \to H/S$ and $\bar{h} \neq 1$ be an element of $\bar{H}$. Since $R/S \in res \mathcal{N}_p$ we can find a dimension subgroup $M_n(\bar{R})$ which does not contain $\bar{h}$ and hence the image of $\bar{h}$ in the quotient group $H/M_n(\bar{R})$ is an element $h^* \neq 1$. But $H/M_n(\bar{R}) \in res \mathcal{N}_p$ by Lemma 6.1., hence there exists a homomorphism of this group on a finite $p$-group $G$ which maps $h^*$ in a non-unit element. We see that the composition of homomorphisms $H \to H/M_n(\bar{R})$ and $H/M_n(\bar{R}) \to G$ maps $\bar{h}$ on a non-unit element in $G$. This proves that $H \in res \mathcal{N}_p$ and completes the proof of statement vii).

**Proof of statement iii).** Assume that there exists an element

$$1 \neq \bar{x} \in \bigcap_{i=1}^{\infty} H_i \quad (6.8)$$

Since $H \in res \mathcal{N}_p$ we can find $M_n(H)$ such that $\bar{x} \not\in M_n(H)$. If $x$ is an element which is mapped on $\bar{x}$ under the homomorphism $H \to \bar{H}$ then $x \not\in M_n(H)U$. Since the topology defined by the series $H_i (i = 1, 2, \cdots)$ is equivalent to the $p$-topology we can find $i(n)$ such that $H_{i(n)} \subseteq M_n(H)$ and hence $x \not\in H_{i(n)}U$. This implies that $\bar{x} \not\in \bar{H}_{i(n)}$ which contradicts (6.8)

**Proof of statement viii).** Theorem V implies that there exists a torsion free normal subgroup $F$ with index $(H : F) = p^n$ such that the subalgebra $L_p(F, F_i)$ corresponding to the $p$-series $F_i = F \cap H_i^*$ $(i = 1, 2, \cdots)$ is finitely generated free abelian and central in $L_p(H, H_i^*)$ and its rank is $r_1 \leq r$. Since the topology defined by the series $H_i (i = 1, 2, \cdots)$ is equivalent to the $p$-topology by statement v) we can find a number $i_0$ such that $H_i^* \subseteq F$ if $i \geq i_0$.

On the other hand, the topology defined in $H$ by series $H_i (i = 1, 2, \cdots)$ is equivalent to the $p$-topology by statement v) so there exists $i_1$ such that $H_i \subseteq F$ if $i \geq i_1$. Since the algebra $L_p(H, H_i)$ is abelian of finite rank there exists $i_2$ such that the subalgebra $\sum_{i \geq i_2} H_i/H_{i+1}$ is free abelian of rank $r$. We pick now an arbitrary $i$ greater than or equal $k = \max\{i_0, i_1, i_2\}$ and obtain a normal subgroup $U = H_i$ in $H$ with the following properties:

1) $U \subseteq F$; 2) The index of $U = H_i$ is a power of $p$; 3) The algebra $L_p(U, U_i)$ associated to the $p$-series $U_i = U \cap H_i (i = 1, 2, \cdots)$ coincides with the subalgebra $\sum_{i \geq i_2} H_i/H_{i+1}$ so it is free abelian of rank $r$; 4) The algebra $L_p(U, U_i^*)$ associated to the $p$-series $U_i^* = U \cap F_i^* = U \cap H_i^* (i = 1, 2, \cdots)$ has a finite index in $L_p(H, H_i)$ because $U$ has a finite index in $H$, so $L_p(U, U_i^*)$
is finitely generated; it is free abelian with rank $r_1 \leq r$ because it is a subalgebra of finite index in $L_p(F, F^*)$.

Further since the index of $U$ in $H$ is a power of $p$ a straightforward argument show also that the topologies defined in $U$ by the series $U_i (i = 1, 2, \cdots)$ and $U^*_i (i = 1, 2, \cdots)$ are equivalent to the $p$-topology of $U$.

We see that the relation $r_1 = r$ will follow from the following lemma.

**Lemma 6.2.** Let $U$ be an arbitrary group,

$$U = U_1 \supseteq U_2 \cdots$$

(6.9)

$$U = U_1^* \supseteq U_2^* \supseteq \cdots$$

(6.10)

be two $p$-series with unit intersection. Assume that the topologies defined by these series are equivalent and that the algebras $L_p(U, U_i)$ and $L_p(U, U^*_i)$ are free abelian; then they are isomorphic.

**Proof.** The algebra $gr_\rho(Z_p(U, U_i))$ is isomorphic to the polynomial ring $Z_pT$ where $T$ is a free system of generators for $L_p(U, U_i)$. We consider now the completion $\hat{Z}_pU$ in the $\rho$-topology. A routine argument (see, for instance, the proof of statement ii) of Lemma 3.6. or the end of the proof of Lemma 3.7.) shows that every element $x \in \hat{Z}_pU$ has a unique representation

$$x = \sum_{i=0}^{\infty} \lambda_i \pi_i$$

where $\lambda_i \in Z_p (i = 0, 1, \cdots), \pi_i (i = 0, 1, \cdots)$ are standard monomials on the set of variables $T$. The algebra $\hat{Z}_pU$ has an augmentation ideal $\omega(\hat{Z}_pU)$ which is topologically generated by the of elements $T$. We consider the filtration defined by the powers of this ideal and obtain that the graded ring of $\hat{Z}_pU$ associated to this filtration is isomorphic to the polynomial ring $Z_p[T]$.

Since the completion of $Z_pU$ with respect to the topology defined by the valuation $\rho^*$ is isomorphic to $\hat{Z}_pU$ (see Lemma 2.6.) we obtain that $gr_{\rho^*}(Z_pU) \cong Z_p[T]$. On the other hand, $gr_{\rho^*}(Z_pU) \cong Z_p[T_1]$ where $T_1$ is a free system of generators $L_p(U, U^*_i)$, so we obtain that the cardinalities of $T$ and $T_1$ are equal and $L_p(U, U_i) \cong L_p(U, U^*_i)$.

This completes the proof of the lemma and the proof of Theorem VI.
6.3. We will need in the proof of theorem XII the following theorem which is a refined version of statement vi) of Theorem VI.

**Theorem 6.1.** Let $H$ be a polycyclic group with Hirsch number $r$. Assume that there exists a $p$-series (6.1) with unit intersection such that $L_p(H, H_i)$ is abelian of rank $r$. Let $i_0$ be the index defined in statement vi) of Theorem VI. Then there exists an index $i_1 \geq i_0$ such that if $i \geq i_1$ then the subgroups $Q = H_i$ and $N$ obtained in Theorem VI have the following series

$$Q = Q^{(1)} \supseteq Q^{(2)} \supseteq \cdots \supseteq Q^{(k-1)} \supseteq Q^{(k)} =$$

$$= N \supseteq Q^{(k+1)} \supseteq \cdots \supseteq Q^{(r-1)} \supseteq Q^{(r)} \supseteq Q^{(r+1)} = 1 \quad (6.11)$$

with following properties:

i) Every factor $Q^{(j)}/Q^{(j+1)}$ ($j = 1, 2, \cdots, r-1$) is an infinite cyclic group.

ii) Let $L_p(Q^{(j)}, Q^{(j)}_i)$ be the restricted Lie algebra of $Q^{(j)}$ associated to the $p$-series $Q^{(j)}_i \equiv Q^{(j)} \cap H_i$ ($i = 1, 2, \cdots$). Then the algebra

$$L_p(Q^{(j)}, Q^{(j)}_i)/L_p(Q^{(j+1)}, Q^{(j+1)}_i)$$

is free abelian of rank 1. If $q_j$ is an element of $Q^{(j)}$ which generates the quotient group $Q^{(j)}/Q^{(j+1)}$ then its homogeneous component $\tilde{q}_j$ in $L_p(Q^{(j)}, Q^{(j)}_i) \subseteq L_p(H, H_i)$ generate the quotient algebra $L_p(Q^{(j)}_i)/L_p(Q^{(j+1)}_i)$. Hence the system of elements $q_1, q_2, \cdots, q_r$ generate the subgroup $Q$ and the system of homogeneous components $\tilde{q}_1, \tilde{q}_2, \cdots, \tilde{q}_r$ freely generate the free abelian algebra $L_p(Q, Q_i)$.

We need first two lemmas.

**Lemma 6.3.** Let $H$ be a group with a $p$-series (6.1). Assume that there exists a normal subgroup $Q$ with $(H: Q) = p^n$ and an index $i_0$ such that $H_i \subseteq Q$ if $i \geq i_0$ and let

$$H = S_1 \supseteq S_2 \supseteq \cdots \supseteq S_n \supseteq S_{n+1} = Q \quad (6.12)$$

be an invariant series with factors $S_j/S_{j+1}$ ($j = 1, 2, \cdots, n$) cyclic groups of order $p$; let $s_j$ be an element of $S_j$ which generates $S_j/S_{j+1}$ ($j = 1, 2, \cdots, n$). Let $S_{j,i} = H_i \cap S_j$ ($i = 1, 2, \cdots$) for every given $1 \leq j \leq n$. Then every quotient algebra $L_p(S_j, S_{j,i})/L_p(S_{j+1}, S_{j+1,i})$ has dimension 1 and is generated by the homogeneous component $\tilde{s}_j$ ($j = 1, 2, \cdots, r-1$); the system of these homogeneous components together with the ideal $L_p(Q, Q_i)$ generates the algebra $L_p(H, H_i)$. 

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Proof. We denote \( U = S_2 \) and consider the series

\[
U = S_2 \supseteq S_3 \supseteq \cdots \supseteq S_n \supseteq S_{n+1} = Q
\]  

(6.13)

Let \( \bar{H} = H/U \) and \( \bar{H}_i = (H_i U)/U \) \( (i = 1, 2, \cdots) \). Since \( \bar{H}_{i_0} = 1 \) we see that \( \bigcap_{i=1}^{\infty} \bar{H}_i = 1 \) so the algebra \( L_p(\bar{H}, \bar{H}_i) \) is non-zero. On the other hand, \( \bar{H} \) is a cyclic group of order \( p \) so the dimension of \( L_p(\bar{H}, \bar{H}_i) \) can not be greater than 1. We obtain therefore that \( \dim(L_p(\bar{H}, \bar{H}_i)) = 1, \) and that \( L_p(\bar{H}, \bar{H}_i) \) is generated by the homogeneous component \( \tilde{s}_1 \). We see that the proof is now reduced to the subgroup \( U = S_2 \) and series (6.14). Since \( (U: Q) = p^{n-1} \) we obtain after \( n \) steps the system of elements \( \tilde{s}_1, \tilde{s}_2, \cdots, \tilde{s}_n \) which together with the subalgebra \( L_p(Q, Q_i) \) generates \( L_p(H, H_i) \).

**Corollary 6.2.** The dimension of the algebra \( L_p(H, H_i)/L_p(Q, Q_i) \) is \( n \).

**Corollary 6.3.** The system of elements \( \tilde{s}_j \) together with the ideal \( L_p(Q, Q_i) \) generates the algebra \( L_p(H, H_i) \).

We will need in the proof of Theorem XII one more corollary of Lemma 6.3.

**Corollary 6.4.** Let \( V \) be a polycyclic group with Hirsch number \( r \), \( W \) be a finite subgroup of index \( p \), \( s \) be an element of \( V \backslash W \), and let \( s^p = a \in W \). Assume that there exists in \( V \) a \( p \)-series \( V_i \) \( (i = 1, 2, \cdots) \) with unit intersection such that the algebra \( L_p(V, V_i) \) is free abelian of rank \( r \). Let \( p \) be the valuation in \( KV \) defined by this series, \( (s - 1), (a - 1) \) be the homogeneous components of the element \( s - 1, a - 1 \) in the rings \( gr(KV) \) and \( gr(KW) \) respectively.

Then the polynomial ring \( gr(KV) \) is a simple algebraic extension of the polynomial ring \( KW \)

\[
gr(KV) \cong gr(KW)[(s - 1)]
\]

(6.14)

where the minimal polynomial of \( (s - 1) \) is \( t^p - (a - 1) \).

**Proof.** Lemma 6.3. implies that the free abelian algebra \( L_p(V, V_i) \) is an extension of the free abelian algebra \( L_p(W, W_i) \) by the one dimensional algebra generated by the element \( \tilde{s} \), which is the homogeneous component of \( s \).
and $\tilde{s}^{|p|} = \tilde{a}$. We obtain from this that the ring $U_p(L_p(V, V_i))$ is a simple algebraic extension $U_p(L_p(V, V_i)) \cong U_p(L_p(W, W_i))[s]$. The assertion now follows from the isomorphism $U_p(L_p(V, V_i)) \cong gr(KV)$ obtained in Proposition 2.7.

**Lemma 6.4.** Let $H$ be a free abelian group of rank $r$. Assume that there exists a $p$-series

$$H = H_1 \supseteq H_2 \supseteq \cdots$$

with unit intersection such that the algebra $L_p(H, H_i)$ is free abelian of rank $r$. Then there exists a free system of generators $h_1, h_2, \cdots, h_r$ such that the homogeneous components $\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_r$ form a free system of generators for the free abelian algebra $L_p(H, H_i)$.

**Proof.** Let $u_1, u_2, \cdots, u_r$ be a free system of generators for $L_p(H, H_i)$. Since the algebra $L_p(H, H_i)$ is graded we can assume that the elements $u_i (i = 1, 2, \cdots, r)$ are homogeneous, so every element $u_i$ is the homogeneous component of an element $v_i \in H$. We pick now a system of elements $v_i \in H$ such that $\tilde{v}_i = u_i (i = 1, 2, \cdots, n)$; let $V$ be the subgroup generated by these elements.

The subgroup $V$ contains a $p$-series $V_i = V \cap H_i (i = 1, 2, \cdots)$ such that the Lie algebra $L_p(V, V_i)$ coincides with the algebra $L_p(H, H_i)$. We claim that this implies that the index $(H : V)$ is prime to $p$. In fact, if we assume that $p|(H : V)$ then we can find a subgroup $U \supseteq V$ with index $(H : U) = p$ and $L_p(U, U_i) = L_p(H, H_i)$, but Corollary 6.2. implies that the dimension of the algebra $L_p(H, H_i)/L_p(U, U_i)$ is 1. We see that $p$ is not a divisor of $(H : U) = p$.

We obtain therefore that there exists in $H$ a free system of generators $h_i (i = 1, 2, \cdots)$ such that $v_i = h_i^{m_i} (i = 1, 2, \cdots, r)$ where $m_1, m_2, \cdots, m_r$ are integers prime to $p$. The system of elements $h_1, h_2, \cdots, h_r$ generates $H$ and the homogeneous components of these elements generate $L_p(H, H_i)$. This system of generators satisfies the conclusion of the assertion.

**Proof of Theorem 6.1.** Let $i_0$ be as in statement vi) of Theorem VI, $H_{i_0} = Q_0$, $N_0 \subseteq Q_0$ be the torsion free nilpotent normal subgroup with $Q_0/N_0$ free abelian. Let $Z_0$ be the center of $N_0$, $l$ be its rank, and $\bar{H} = H/Z_0$. Statement ii) of Theorem VI implies and the algebra $L_p(\bar{H}, H_i)$ is abelian of rank $r - l$. Since $\bar{H}_{i_0} = H_{i_0}/Z_0$ is torsion free we obtain once again from statement ii) that $\bigcap_{i=1}^{\infty} \bar{H}_i = 1$. We can now apply induction by the Hirsch
number of $H$ and to assume that there exists an index $i_1 \geq i_0$ such that if $i \geq i_1$ is given and $Z = H_i \cap Z_0$ then the subgroup $\overline{Q} = \overline{H_i} = H_i/Z$ of $H$ contains the following series with infinite cyclic factors which satisfies the conclusions of the assertion

$$\overline{Q} = \overline{Q}^{(1)} \supseteq \overline{Q}^{(2)} \supseteq \cdots \supseteq \overline{Q}^{(k-1)} \supseteq \overline{Q}^{(k)} = \overline{N} \supseteq \overline{Q}^{(k+1)} \supseteq \cdots \supseteq \overline{Q}^{(n-l)} \supseteq \overline{Q}^{(n-l+1)} = 1 \quad (6.16)$$

We obtain from this the following series in $H$

$$Q = Q^{(1)} \supseteq Q^{(2)} \supseteq \cdots \supseteq Q^{(k-1)} \supseteq Q^{(k)} = N \supseteq Q^{(k+1)} \supseteq \cdots \supseteq Q^{(n-l)} \supseteq Q^{(n-l+1)} = Z \quad (6.17)$$

where every factor $Q^{(j)}/Q^{(j+1)}$ is an infinite cyclic groups generated by an element $q_j$, and every quotient algebra $L_p(Q^{(j)})/L_p(Q^{(j+1)})$ is free abelian of rank 1 generated by the element $\tilde{q}_j$. We apply now Lemma 6.4. to obtain a free system of generators whose homogeneous components freely generate the subalgebra $L_p(Z, Z_i)$ and the assertion follows.

Let $H$ be a polycyclic group with Hirsch number $r$. Assume that there exists a $p$-series (6.1) with unit intersection such that $L_p(H, H_i)$ is abelian of rank $r$. We will need in the proof of Theorem XII some special system of generators for the algebra $L_p(H, H_i)$ which is obtained in the following way. We obtain from Theorem VI and Theorem 6.1. that there exist a poly-

\{(infinite cyclic) normal subgroup $Q$ with quotient group $G = H/Q$ of index $p^n$, and a system of generators $q_1, q_2, \cdots, q_r$ such that the free abelian algebra $L_p(Q, Q_i)$ is freely generated by the system of elements $\tilde{q}_1, \tilde{q}_2, \cdots, \tilde{q}_r$. Lemma 6.3. implies that there exists a system of elements $s_1, s_2, \cdots, s_n$ which generates the quotient group $H/Q$ and the images of the elements $\tilde{s}_1, \tilde{s}_2, \cdots, \tilde{s}_n$ in $L_p(H, H_i)$ generate the quotient algebra $L_p(H, H_i)/L_p(Q, Q_i) \cong L_p(G, G_i)$. We have the following corollary of the results of this subsection.

**Corollary 6.5. and Definition 6.1.** Assume that the conditions of Theorem 6.1. hold. Then the system of elements

$$q_1, q_2, \cdots, q_r; s_1, s_2, \cdots, s_n$$

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generates the group \( H \); the system of their homogeneous components
\[ \tilde{q}_1, \tilde{q}_2, \cdots, \tilde{q}_r; \tilde{s}_1, \tilde{s}_2, \cdots, \tilde{s}_n \]
generates the algebra \( L_p(H, H_i) \). This system of generators for \( H \) is called a special system of generators.

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7.1. Lemma 7.1. Let \( H_j (j \in J) \) be a system of groups. Assume that every group \( H_j \) contains invariant subgroups \( S_j \supseteq U_j \), and \( U_j \) contains a \( p \)-series \( U_i \) \((i = 1, 2, \cdots) \) with unit intersection whose terms are invariant in \( S_j \). Let \( H = \prod_{j \in J} H_j, S = \prod_{j \in J} S_j, U = \prod_{j \in J} U_j \).

There exists in \( U \) a \( p \)-series \( U_i \) \((i = 1, 2, \cdots) \) such that all the subgroups \( U_i \) are \( S \)-invariant, \( U_i \cap U_j = U_i \) \((i = 1, 2, \cdots) \) and the associated graded algebra \( L_p(U, U_i) \) is isomorphic to the direct sum of the Lie algebras \( L_p(U_j, U_i) \)

\[ L_p(U, U_i) \cong \bigoplus_{j \in J} L_p(U_j, U_i) \] (7.1)

Proof. The \( p \)-series in \( U_j \) defines a weight function \( f_j \) in \( U_j \), and hence a weight function \( f \) on the set \( \bigcup_{j \in J} U_j \). We define now a weight function \( f(x) \) on the group \( U \). Let

\[ x = x_{j_1}x_{j_2}\cdots x_{j_n} \] (7.2)

be an element of \( U \). Then its weight in \( U \) is defined as

\[ f(x) = \min\{f(x_{j_1}), f(x_{j_2}), \cdots, f(x_{j_n})\} \] (7.3)

A straightforward verification shows that \( f([x, y]) \geq f(x) + f(y), f(x)^p \geq p(f(x)), f(xy^{-1}) \geq f(x) - f(y) \); hence the obtained weight function \( f \) on \( U \) defines a \( p \)-series \( U_i \) \((i = 1, 2, \cdots) \) in \( U \) and we obtain a Lie algebra \( L_p(U, U_i) \) corresponding to this \( p \)-series. The definition of \( f(x) \) implies that its restriction on every \( U_j \) coincides with \( f_j \), hence we obtain a natural imbedding

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of the Lie algebra $L_p(U^{(j)}, U^{(j)}_i)$ into $L_p(U, U_i)$. It is clear that for every $s \in S$ $f(s^{-1}us) = f(u)$ so every subgroup $U_i$ is $S$-invariant.

We consider now the homogeneous component of the element (7.2). Assume that $f(x) = k$. We can assume that $f(x_j^i) = f(x_j^i) = \cdots = f(x_j^i) = k$ and the weights of the remaining components is greater than $k$. Hence, the homogeneous component $\tilde{x}$ of $x$ coincides with the homogeneous component of the element $x_1x_2 \cdots x_l$. We apply Lemma 2.2. and obtain that the homogeneous component of the element $x$ is equal

$$\tilde{x} = \tilde{x}_{j_1} + \tilde{x}_{j_2} + \cdots + \tilde{x}_{j_l}$$

(7.4)

where $\tilde{x}_{j_\alpha} \in L_p(U^{(j_\alpha)}, U^{(j_\alpha)}_i) (\alpha = 1, 2, \cdots, l)$.

Hence the algebra $L_p(U, U_i)$ is generated by the system of subalgebras $L_p(U^{(j)}, U^{(j)}_i) (j \in J)$; we see that in order to prove relation (7.1) it is enough to verify that representation (7.4) is unique.

Assume that there exists another representation. A routine argument shows that we can assume now that the set $J$ is finite and that after a suitable numeration of it this second representation has a form

$$\tilde{x} = \tilde{x}_1 + \tilde{x}_2 + \cdots + \tilde{x}_m$$

(7.5)

with $\tilde{x}_j \in L_p(U^{(j)}, U^{(j)}_i) (j = 1, 2, \cdots, m)$. Since all the elements in the right side of (7.5) are homogeneous we obtain that their degrees and the degrees must be equal to $k$. We obtain now from (7.4) and (7.5) the following two representation for $x$

$$x = x_{j_1}x_{j_2}\cdots x_{j_l}y_1$$

(7.6)

and

$$x = x_1x_2\cdots x_my_2$$

(7.7)

where $f(y_\beta) > k (\beta = 1, 2)$ or, equivalently, $y_\beta \in \prod_{j \in J} U^{(j)}_{k+1}$ ($\beta = 1, 2$). We compare now representations (7.6) and (7.7) for $x$ and obtain that they coincide modulo the subgroup $\prod_{j \in J} U^{(j)}_{k+1}$; hence $l = m$ and after a renumeration we obtain that $x_{j_\alpha} \in (x_jU^{(j)}_{k+1}) (j = 1, 2, \cdots, l)$. Hence the homogeneous components of these elements in $L_p(U, U_i)$ coincide which proves the uniqueness of representation (7.4). This completes the proof.
Lemma 7.2. Let $W = S \wr G$ be the discrete wreath product of a group $S$ by a group $G$, $S^*$ be the base group of $W$. Assume that $S$ contains a normal subgroup $U$ with an $S$-invariant $p$-series
\[ U = U_1 \supseteq U_2 \supseteq \cdots \tag{7.8} \]
with unit intersection and let $U_i^* = \prod_{g \in G} g^{-1} U_i g$. Then there exists in $U^*$ a $W$-invariant $p$-series with unit intersection
\[ U^* = U_1^* \supseteq U_2^* \supseteq \cdots \tag{7.9} \]
with Lie algebra $L_p(U^*, U_i^*)$ isomorphic to the direct sum of copies of the algebra $L_p(U, U_i)$
\[ L_p(U^*, U_i^*) \cong \bigoplus_{g \in G} L_p(U, U_i) \tag{7.10} \]

Proof. We apply Lemma 7.1. and obtain a weight function $f$ on $U^*$ and a $p$-series in $U^*$; the terms of this $p$-series are invariant subgroups in $S^*$. We will prove now that they are $G$-invariant and hence $W$-invariant.

Indeed, let $u$ be a non-unit element of $U^*$, $u = u_{j_1} u_{j_2} \cdots u_{j_k}$ with $u_{j_\alpha} \in U_{j_\alpha} = g^{-1}_\alpha U g_\alpha$ ($\alpha = 1, 2, \cdots, k$). We can assume that $f(u) = f(u_{j_1}) = f(u_{j_2}) = \cdots f(u_{j_l}) = l$ and the rest of the factors, $u_{j_{l+1}}, u_{j_{l+2}}, \cdots, u_{j_k}$ have weights greater than $l$.

Let $g$ be an arbitrary element of $G$. Then $g^{-1} u g = (g^{-1} u_{j_1} g)(g^{-1} u_{j_2} g) \cdots (g^{-1} u_{j_k} g)$ where $g^{-1} u_{j_\alpha} g \in g^{-1}(g^{-1}_\alpha U g_\alpha) g$ ($\alpha = 1, 2, \cdots, k$) and we obtain from this that $f(g^{-1} u g) = f(u)$; we see that the weight function in $U^*$ is $G$-invariant and so is the $p$-series defined by it.

The rest of the statements follow immediately.

We apply now this lemma to the case when $U = S$ and the group $G$ is finite. Lemma 7.1. yields a $W$-invariant $p$-series $S_i^*$ ($i = 1, 2, \cdots$) in the base group $S^*$. We will need the following fact about this $p$-series.

**Corollary 7.1.** Assume that $S$ contains a $p$-series with unit intersection and that the topology defined by this series is equivalent to the $p$-topology, and that the group $G$ is finite. Then the topology defined by the series $S_i^*$ ($i = 1, 2, \cdots$) is equivalent to the $p$-topology in $S^*$.

**Proof.** For a given $n$ we can find a number $i(n)$ such that $S_{i(n)} \subseteq M_n(S)$ which implies that $S_{i(n)}^* \subseteq M_n(S^*)$ and the assertion follows.
**Proposition 7.1.** Let $H$ be a group, $U$ be a normal subgroup which contains a $p$-series (7.8) with unit intersection. Assume that there exists a normal subgroup $S \supseteq U$ of finite index $n$ in $H$ such that $h^{-1}U_i h = U_i (i = 1, 2, \cdots)$ for all $h \in S$.

Then there exists an $H$-invariant $p$-series

$$U = V_1 \supseteq V_2 \supseteq \cdots$$

(7.11)

with unit intersection such that the algebra $L_p(U, V_i)$ is isomorphic to a subalgebra of the direct sum of $n$ copies of the algebra $L_p(U, U_i)$.

**Proof.** Let $G = H/S$. We consider the usual imbedding of $H$ into the wreath product $W = S \text{wr} G$ where $S$ is isomorphically imbedded into the base group $S^*$ of $W$; this base group is the direct product of $(G : 1)$ copies of $S$.

We apply now Lemma 7.2. to the subgroup $U^* = \prod_{g \in G} g^{-1} U g$ and obtain that there exists a $W$-invariant series (7.9).

The series $V_i = U_i^* \cap U (i = 1, 2, \cdots)$ has unit intersection and the subalgebra $L_p(U, V_i)$ associated to this series is isomorphic to a subagebra of $L_p(U^*, U_i^*)$ which in its turn is isomorphic to the sum of $n$ copies of $L_p(U, U_i)$. This completes the proof.

**Corollary 7.2.** i) If the algebra $L_p(U, U_i)$ is free abelian, (abelian), (abelian of finite rank) then so is $L_p(U, V_i)$.

ii) If $H$ is a polycyclic group with Hirsch number $r$ and $L_p(H, H_i)$ is free abelian, (abelian of rank $r$) then so is $L_p(U^*, U_i^*)$.

**Proof.** We will prove statement ii); the proof of statement i) is obtained by obvious simplification of the argument. The group $U$ is a subgroup of the group $U^* = \prod_{i=1}^{n} U \times U \cdots \times U$. The algebra $L_p(U^*, U_i^*)$ has rank $rn$ because it is a direct sum of $n$ copies of $L_p(U, U_i)$. Since the rank of $L_p(U^*, U_i^*)$ coincides with the Hirsch number of $U^*$ we obtain from statement i) of Theorem VI that the rank of the algebra $L_p(U, V_i)$ must be equal to the Hirsch number of $U$ which is equal $r$.

Now assume that $L_p(U, U_i)$ is free abelian. Then the direct sum of $n$ copies of it is free abelian. Since $L_p(U, V_i)$ is a subalgebra of this direct sum it must be free abelian.

This completes the proof.
7.2. Proposition 7.2. Let $H$ be a group which contains a $p$-series

$$H = H_1 \supseteq H_2 \supseteq \cdots$$  \hspace{1cm} (7.12)

with unit intersection. Assume that the topology defined by this series is equivalent to the $p$-topology in $H$. Let $\Phi$ be a group of automorphisms of $H$.

i) Assume that the restricted Lie algebra $L_p(H, H_i)$ is generated by the first $l$ factors $H_i/H_{i+1}$ ($i = 1, 2, \cdots, l$) where the subgroups $H_i$ ($i = 1, 2, \cdots, l$) are $\Phi$-invariant. Then all the subgroups $H_i$ ($i = 1, 2, \cdots$) are $\Phi$-invariant.

ii) If

$$[\Phi, H_i] \subseteq H_{i+1} \quad (i = 1, 2, \cdots, l)$$  \hspace{1cm} (7.13)

i.e. $\Phi$ centralizes the first $l$ factors $H_i/H_{i+1}$ then it centralizes all the factors $H_i/H_{i+1}$.

**Proof.** We will prove first statement i) for the case when series (7.12) has a finite length, say $k$. Hence $H_k = 1$ but $H_{k-1} \neq 1$. We begin by proving that $H_{k-1}$ is $\Phi$-invariant. Let $h \in H_{k-1}$. Lemma 2.4, implies that $\tilde{h}$ can be expressed in $L_p(H, H_i)$ as a sum of the Lie monomials

$$[\tilde{h}_{\alpha_1}, \tilde{h}_{\alpha_2}, \cdots, \tilde{h}_{\alpha_s}] [p]^{n\alpha}$$  \hspace{1cm} (7.14)

where the homogeneous elements $\tilde{h}_{\alpha_1}, \tilde{h}_{\alpha_2}, \cdots, \tilde{h}_{\alpha_s}$ are taken from the first $l$ factors $H_i/H_{i+1}$ and the weight of every such monomial is $k - 1$. Since $H_k = 1$ we see that the element

$$[h_{\alpha_1}, h_{\alpha_2}, \cdots, h_{\alpha_s}] [p]^{n\alpha}$$  \hspace{1cm} (7.15)

of the group $H$ is a coset representative for the Lie monomial (7.15), and hence $h$ is a product of the elements (7.14) with weight $k - 1$.

Now let $\phi$ be an arbitrary automorphism from $\phi$. We obtain that the image of the element (7.15) is

$$\phi([h_{\alpha_1}, h_{\alpha_2}, \cdots, h_{\alpha_s}] [p]^{n\alpha}) = [\phi(h_{\alpha_1}), \phi(h_{\alpha_2}), \cdots, \phi(h_{\alpha_s})] [p]^{n\alpha}$$  \hspace{1cm} (7.16)

Let $w(x)$ denote the weight of an element $x \in H$. Since element (7.15) is the representative of the monomial (7.14) we obtain from Lemma 2.4. that its weight is

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\[ k - 1 = p^{\alpha_n} \sum_{i=1}^{s} w(h_{\alpha_i}) \quad (7.17) \]

Since \( \phi(H_i) = H_i \) \((i = 1, 2, \cdots, l)\) we obtain that \( w(h_{\alpha_i}) = w(\phi(h_{\alpha_i})) \) \((i = 1, 2, \cdots, s)\) and then

\[ w([\phi(h_{\alpha_1}), \phi(h_{\alpha_2}), \cdots, \phi(h_{\alpha_s})]^{p^{\alpha_n}}) \geq p^{\alpha_n} \sum_{i=1}^{s} w(h_{\alpha_i}) = k - 1 \]

This together with (7.16) implies that \( w(\phi([h_{\alpha_1}, h_{\alpha_2}, \cdots, h_{\alpha_s}]^{p^{\alpha_n}})) \geq k - 1 \), which means that \( \phi([h_{\alpha_1}, h_{\alpha_2}, \cdots, h_{\alpha_s}]^{p^{\alpha_n}}) \in H_{k-1} \). Hence \( \phi(h) \) is a product of elements from \( H_{k-1} \) and we obtain that \( \phi(h) \in H_{k-1} \). Since \( h \) was an arbitrary element of \( H_{k-1} \) we obtain that \( H_{k-1} \) is \( \Phi \)-invariant.

The group \( \bar{H} = H/H_{k-1} \) has a \( p \)-series \( H_i = H_i/H_{k-1} \) \((i = 1, 2, \cdots, k-1)\) whose length is shorter than the length of the series \( H_i \) \((i = 1, 2, \cdots, k)\) and the Lie algebra \( L_p(\bar{H}, H_i) \) is generated by the first \( l \) factors \( H_i/\bar{H}_{i+1} \) \((i = 1, 2, \cdots, l)\) because \( L_p(\bar{H}, H_i) \) is a homomorphic image of \( L_p(H, H_i) \). The action of \( \Phi \) on \( H \) defines in a natural way the action of \( \Phi \) on \( \bar{H} \) and we assume by induction that all the subgroups \( \bar{H}_i \) \((i = 1, 2, \cdots, k-1)\) are \( \Phi \)-invariant. This implies that their inverse images \( H_i \) \((i = 1, 2, \cdots, k)\) are \( \Phi \)-invariant; this completes the proof of statement i) for the special case when the series (7.12) has a finite length.

We consider now the general case. To prove that the term \( H_i \) \((i > l)\) of series (7.12) is \( \Phi \)-invariant we pick \( n > i \) and consider the quotient group \( G = H/M_n(H) \); let \( G_i \) be the image of \( H_i \) under the natural homomorphism \( H \rightarrow G \). Proposition 2.1 implies that the epimorphism \( H \rightarrow G \) defines an epimorphism \( L_p(H, H_i) \rightarrow L_p(G, G_i) \). Since the algebra \( L_p(H, H_i) \) is generated by its first \( l \) factors we conclude that the algebra \( L_p(G, G_i) \) is generated by the first \( l \) factors \( G_i/G_{i+1} \) \((i = 1, 2, \cdots, l)\). There exists \( m \geq i \) such that \( H_m \subseteq M_n(H) \) and hence \( G_m = 1 \). We see that the series \( G_i \) \((i = 1, 2, \cdots)\) has finite length in \( G \) and the algebra \( L_p(G, G_i) \) is generated by the factors \( G_i/G_{i+1} \cong (H_i/M_n(H))/(H_{i+1}/M_n(H)) \cong H_i/H_{i+1} \) \((i = 1, 2, \cdots, l)\). The group of automorphisms \( \Phi \) acts in a natural way on \( G \), we denote this group by \( \Psi \), and obtain from the proven special case that an arbitrary subgroup \( G_i \) is \( \Psi \)-invariant; hence its inverse image \( H_i \) is \( \Phi \)-invariant.

The proof of statement i) is complete.

ii) We prove now the second statement. Since all the subgroups \( H_i \) \((i = 1, 2, \cdots)\) are \( \Phi \)-invariant the action of the group \( \Phi \) on \( H \) defines in a natural...
way its action on the graded Lie algebra $L_p(H, H_i)$ by Corollary 2.3.; since $\Phi$ centralizes the factors $H_i/H_{i+1}$ $(i = 1, 2, \ldots, l)$ which generate the algebra $L_p(H, H_i)$ it centralizes all the factors and the proof is complete.

**Proposition 7.3.** Let $H$ be a finitely generated group, $\Phi$ be a group of automorphisms of $H$. Assume that $H$ contains a $p$-series (7.12) with unit intersection such that the topology defined by this $p$-series is equivalent to the $p$-topology and the Lie algebra $L_p(H, H_i)$ is finitely generated, or equivalently, is generated by a finite number of factors $H_i/H_{i+1}$ $(i = 1, 2, \ldots, l)$. Then

i) There exists a normal subgroup $\Phi_1$ of finite index in $\Phi$ such that all the subgroups $H_i$ $(i = 1, 2, \cdots) \Phi_1$-invariant and the factors $H_i/H_{i+1}$ $(i = 1, 2, \cdots)$ are centralized by $\Phi_1$.

ii) If the order of every automorphism $\phi \in \Phi$ in the quotient group $H/H'\Phi p$ is a power of $p$ then the index of $\Phi_1$ is also a power of $p$.

**Proof.** We pick an arbitrary $m > l$ and consider the quotient group $\bar{H} = H/M_m(H)$. The action of the group $\Phi$ on $H$ defines in a natural way a group $\bar{\Phi}$ of automorphisms of $\bar{H}$ and an epimorphism $\Phi \twoheadrightarrow \bar{\Phi}$.

Since $\bar{\Phi}$ is finite we obtain a normal subgroup $\Phi_1 \subseteq \Phi$ which acts on $\bar{H}$ trivially, hence $\Phi_1$ acts trivially on every subgroup $\bar{H}_i = H_i/M_m(H) \ (i = 1, 2, \cdots, m)$.

Since the subgroups $\bar{H}_i$ $(i = 1, 2, \cdots, m)$ are $\Phi_1$-invariant subgroups of $\bar{H}$ their inverse images $H_i$ $(i = 1, 2, \cdots, m)$ are $\Phi_1$-invariant in $H$. Proposition 7.2 now implies that all the subgroups $H_i$ $(i = 1, 2, \cdots)$ are $\Phi_1$-invariant. Finally, $\Phi_1$ acts trivially on every factor $\bar{H}_i/\bar{H}_{i+1} \cong H_i/H_{i+1}$ $(i = 1, 2, \cdots, l)$ so statement ii) of Proposition 7.2 implies that $\Phi_1$ centralizes all the factors $H_i/H_{i+1}$. This completes the proof of statement i).

We prove now statement ii). The subgroup $\Phi_1$ is the kernel of the map $\Phi \twoheadrightarrow \bar{\Phi}$; if the order of every $\phi \in \Phi$ in the quotient group $\bar{H}/\bar{H}'\Phi \cong H/H'\Phi p$ is a power of $p$ then $\bar{\Phi}$ is a $p$-group by Burnside’s Theorem, so the index of $\Phi_1$ is a power of $p$ if the conditions of statement ii) hold, and statement ii) follows.

**Proposition 7.4.** Let $H$ be a finitely generated group. Assume that it contains a $p$-series (7.12) with unit intersection such that the topology defined by this $p$-series is equivalent to the $p$-topology and the Lie algebra $L_p(H, H_i)$ is finitely generated.

There exists a $p$-series

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of characteristic subgroups with unit intersection whose associated graded algebra $L_p(V, V_i)$ is isomorphically imbeded into the direct sum of a finite number of isomorphic copies of the algebra $L_p(H, H_i)$.

**Proof.** Let $\Phi$ be the automorphism group of $H$. Proposition 7.3. implies that there exists a normal subgroup $\Phi_1$ of finite index in $\Phi$ such that $\Phi_1(H_i) = H_i$ ($i = 1, 2, \cdots$). We consider now the holomorph of $H$, that is the group $Hol(H)$ which is a split extension of the group $H$ by its automorphism group $\Phi$. Let $Q$ be its subgroup generated by $H$ and $\Phi_1$. Then $Q$ is a split extension of $H$ by $\Phi_1$, and it is a normal subgroup of finite index in $Hol(H)$; in fact the index of $Q$ in $Hol(H)$ is equal to the index $(\Phi: \Phi_1)$. We apply now Proposition 7.1. and obtain that there exists in $H$ a $p$-series (7.18) whose terms are $\Phi$-invariant and the algebra $L_p(V, V_i)$ isomorphically imbeded into the direct sum of a finite number of isomorphic copies of the algebra $L_p(H, H_i)$. This completes the proof.

**Corollary 7.3.** The algebra $L_p(V, V_i)$ has the following additional properties.

i) If $L_p(H, H_i)$ is abelian (free abelian) then so is $L_p(V, V_i)$; if $L_p(H, H_i)$ is abelian of finite rank then so is $L_p(V, V_i)$.

ii) Assume that the group $H$ in Proposition 7.4. is polycyclic with Hirsch number $r$ and the algebra $L_p(H, H_i)$ is abelian (free abelian) of rank $r$. Then the algebra $L_p(V, V_i)$ is abelian (free abelian) of rank $r$.

The first statement follows from Proposition 7.4. The second statement is obtained by the same argument as Corollary 7.2.

**Theorem VIII.** Let $H$ be a finitely generated group which has a $p$-series $H_i$ ($i = 1, 2, \cdots$) with unit intersection and with the associated restricted Lie algebra $L_p(H, H_i)$ abelian (free abelian) of finite rank. Assume that the topology defined by this $p$-series is equivalent to the $p$-topology. Then there exists a $p$-series

$$H = V_1 \supseteq V_2 \supseteq \cdots$$

(7.18) whose terms $U_i$ ($i = 1, 2, \cdots$) are characteristic subgroups and the Lie algebra $L_p(H, U_i)$ is abelian (free abelian) of finite rank.
Proof. The assertion follows from Proposition 7.4. and Corollary 7.3.

**Theorem IX.** Let \( H \) be a torsion free polycyclic group with Hirsch number \( r \). Assume that there exists a \( p \)-series (7.12) with unit intersection and associated graded Lie algebra \( L_p(H, H_i) \) free abelian (abelian) of rank \( r \). Then there exists a \( p \)-series

\[
H = U_1 \supset U_2 \cdots
\]

(7.20)

whose terms \( U_i \) \((i = 1, 2, \cdots )\) are characteristic subgroups and the Lie algebra \( L_p(H, U_i) \) is free abelian (abelian) of rank \( r \).

Proof. Theorem VI implies that the topology defined by series (7.12) is equivalent to the \( p \)-topology. The assertion now follows from Theorem VIII and Corollary 7.3.

**7.3. Theorem X.** Let \( H \) be a torsion free polycyclic group with Hirsch number \( r \) which contains a \( p \)-series (7.12) with unit intersection. Assume that the Lie algebra \( L_p(H, H_i) \) is free abelian of rank \( r \). Let \( \Phi \) be a group of automorphisms of \( H \) such that the order of every automorphism \( \phi \in \Phi \) on the quotient group \( H/H'_p \) is a power of \( p \).

Then there exists a \( p \)-series

\[
H = H_1^* \supset H_2^* \supset \cdots
\]

(7.21)

with unit intersection such that all the subgroups \( H_i^* \) \((i = 1, 2, \cdots )\) are \( \Phi \)-invariant, \( \Phi \) centralizes all the factors \( H_i/H_{i+1} \) \((i = 1, 2, \cdots )\) and the algebra \( L_p(H, H_i^*) \) is free abelian of rank \( r \).

We recall that series (7.21) defines a weight function \( f \) in \( H \), this weight function defines a valuation \( v \) in the group ring \( KH \); conversely, the function \( f \) completely defines series (7.21). The construction of the function \( f \) and valuation \( v \) in the proof of Theorem X will yield the following fact about the function \( f \).

**Corollary 7.4.** Let \( r \) be the Hirsch number of \( H \). Then \( f(h) \geq 2r \) \((h \in H)\).

Corollary 7.4. will not be used in the proofs of other results.

**Proof of Theorem X and Corollary 7.4.** The proof will be given in 4 steps.
Step 1. Theorem IX implies that we can assume that all the subgroups \( H_i \) \((i = 1, 2, \cdots)\) are characteristic. Series (7.12) defines a valuation \( \rho \) in the group ring \( \mathbb{Z}_p H \) with associated graded ring \( \text{gr}(\mathbb{Z}_p H) \). We extend now the valuation \( \rho \) to the Laurent polynomial ring \( \mathbb{Z}_p H[t, t^{-1}] \) assuming that \( \rho(t) = 1 \); let \( V \) be the valuation ring of \( \mathbb{Z}_p H[t, t^{-1}] \) and \( X \) be the image of a subset \( X \subset V \) under the homomorphism \( V \to V/(t) \). For an arbitrary element \( h_j \in H \) we denote \( \rho(h_j - 1) = n_j \) \((j \in J)\); Proposition 2.9 implies that the \( \mathbb{Z}_p \)-linear combinations of all the elements \( (h_j - 1)t^{-n_j} \) \((h_j \in H, j \in J)\) form a subalgebra isomorphic to \( L_p(H, H_i) \) and all these linear combinations form the set of homogeneous elements of \( L_p(H, H_i) \), and that \( V/(t) \cong U_p(L_p(H)) \). We will construct at this step some special free system of generators for this algebra and a weight function \( f \) on this system of generators.

Proposition 7.3. implies that there exists a normal subgroup \( \Phi_1 \) in \( \Phi \) which centralizes all the factors \( H_i/H_{i+1} \) and the index \( (\Phi: \Phi_1) \) is power of \( p \). Since all the subgroups \( H_i \) \((i = 1, 2, \cdots)\) are characteristic the group \( \Phi \) acts as a group of automorphisms of the algebra \( L_p(H, H_i) \) and every homogeneous component \( H_i/H_{i+1} \) is \( \Phi \)-invariant; since the subgroup \( \Phi_1 \) acts trivially we obtain that the finite \( p \)-group \( G = \Phi/\Phi_1 \) acts as a group of automorphisms of the algebra \( L_p(H, H_i) \) and of the vector spaces \( H_i/H_{i+1} \) so these vector spaces become \( G \)-modules.

Lemma 2.11 implies that the free abelian algebra \( L_p(H, H_i) \) is generated by the vector space \( Q \cong L_p(H, H_i)/L_p^0(H, H_i) \) which is in fact a subspace of \( L_p(H, H_i) \); this vector space has dimension \( r \) because the rank of \( L_p(H, H_i) \) is \( r \) and the group \( G \) acts in a natural way on this vector space. The grading in \( L_p(H, H_i) \) defines a grading in the vector subspace \( Q \). Since the homogeneous components of \( L_p(H, H_i) \) are \( G \)-invariant we obtain that the homogeneous components of \( Q \) are also \( G \)-invariant. Since \( Q \) has finite dimension \( r \) there will be a finite number \( k \leq r \) of non-zero homogeneous components \( Q_i \) \((i = 1, 2, \cdots, k)\)

\[
Q = \bigoplus_{i=1}^{k} Q_i \quad (7.22)
\]

It is worth remarking that every element \( x \in Q_i \) \((i = 1, 2, \cdots, k)\) is a homogeneous element in \( L_p(H, H_i) \).

We pick now an arbitrary homogeneous component \( Q_i \) and consider the
series
\[ Q_i \supseteq \omega(KG) \cdot Q_i \supseteq \omega^2(KG) \cdot Q_i \supseteq \ldots \supseteq \omega^{s-1}(KG) \cdot Q_i \supseteq \omega^s(KG) \cdot Q_i = 0 \quad (7.23) \]

The inclusions in this series are strict because if two consecutive terms had coincided, say \( \omega^n(KH) \cdot Q_i = \omega^{n+1}(KH) \cdot Q_i \neq 0 \), then we would have gotten that \( \omega^j(KH) \cdot Q_i = \omega^{j+1}(KH) \cdot Q_i \) for all \( j \geq n \) which is impossible because the ideal \( \omega(KG) \) is nilpotent. Since the inclusions are strict and the dimension of \( Q = r \) we obtain that in (7.23) \( s \leq r \).

We pick now in \( \omega^n(Z_pG) \cdot Q_i \) a system of elements \( \bar{T}_{i,n} \) which give a basis of the quotient module \( (\omega^n(Z_pG) \cdot Q_i) / (\omega^{n+1}(Z_pG) \cdot Q_i) \) \( (n = 0, 1, \ldots, s-1) \). It is also important that all the elements in the system \( \bar{T}_{i,n} \) are homogeneous elements of \( L_p(H, H_i) \). The system of elements \( \bar{T}_i = \bigcup_{n=1}^{s-1} \bar{T}_{i,n} \) forms a basis for \( Q_i \). We obtained a basis \( \bar{T}_i \) for every vector subspace \( Q_i \) \( (i = 1, 2, \ldots, k) \) and obtain then a basis \( \bar{T} \) for \( Q \) taking \( \bar{T} = \bigcup_{i=1}^{k} \bar{T}_i \), where all the elements of \( \bar{T} \) are homogeneous and \( \bar{T} \) freely generate the algebra \( L_p(H, H_i) \).

Let \( m \) be an arbitrary integer greater than \( 2r \). We define a weight function \( f \) on \( \bar{T}_{i,n} \) as follows
\[ f(x) = m + 2n + 1 \text{ if } x \in \bar{T}_{i,n} \quad (n = 0, 1, \ldots, s-1) \quad (7.24) \]

Since \( \bar{T} \) is a disjoint union of the systems \( \bar{T}_{i,n} \) we obtain a weight function \( f \) on \( \bar{T} \). It follows immediately that
\[ m + 1 \leq f(x) \leq m + 2r + 1 \quad (x \in \bar{T}) \quad (7.25) \]

We complete this step by reminding that it was pointed out in the beginning of the proof that \( V/\langle t \rangle \cong Z_p[\bar{T}] \) and that every element \( \bar{t}_j \in \bar{T} \) has in fact a form \( \bar{t}_j = (h_j - 1)t^{-n_j} \) \( (j = 1, 2, \ldots, r) \) where \( n_j = \rho(h_j - 1) \).

Step 2. We extend now the weight function \( f \) which was defined on the system of generators \( \bar{T} \) to a valuation \( \bar{v} \) of the algebra \( Z_p[\bar{T}] \cong \bar{V} \); this valuation will be used on Step 3 for a construction of a valuation \( v \) in \( V \) and in \( Z_pH \). We will show now that the group \( G \) centralizes the valuation \( \bar{v} \).

Let \( x \in \bar{T} \). The definition of \( \bar{T} \) implies that there exists a unique pair \( i, n \) such that \( x \in \bar{T}_{i,n} \) so \( \bar{v}(x) = m + 2n + 1 \). We see now from (7.23) that if \( x \in \bar{T}_{i,n} \) the the image of the element \( (g - 1) \cdot x \) in \( Q \) belongs to \( \omega^{n+1} \cdot Q \). We obtain from this that either there exists \( 0 \neq u \) which is a linear combination of elements from \( \bigcup_{k=n+1}^{s-1} \bar{T}_{i,k} \) and \( y \in Q^p \) such that

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\[ g \cdot x - x = (g - 1) \cdot x = u + y \]  

or

\[ g \cdot x - x = (g - 1) \cdot x = y \]  

We will now show that \( \bar{v}(u) \) and \( \bar{v}(y) \) are greater than \( m + 2n + 2 \). This will imply that the element of \( G \) centralize the elements of \( T \) with respect to the valuation \( \bar{v} \).

In fact, we obtain from (7.24) that the \( \bar{v} \)-values of the elements from \( \bar{T}_{i,l} \) are greater than or equal to \( m + 2n + 3 \) if \( l \geq (n + 1) \). Further, we see from relation (7.25) that the values of the function \( f(x) \) on the system \( \bar{T} \) are greater than or equal to \( m + 1 \); this together with the definition of the function \( \bar{v} \) implies that the \( \bar{v} \)-values of all the elements of \( Q \) are greater than or equal to \( m + 1 \). Since the element \( y \) belongs to the subalgebra \( Q^{p} \) the value of the element \( y \) is greater than or equal to \( p(m + 1) \geq 2(m + 1) > m + 2r + 1 \). We obtain from this that in both cases, when relation (7.26) or (7.27) hold,

\[ \bar{v}(g \cdot x - x) > \bar{v}(x) + 1 \]  

and our claim is proven.

**Step 3.** Let \( \bar{t}_j = (h - 1)t^{-n_j} \) be an arbitrary element of \( \bar{T} \). We denote now \( (h_j - 1)t^{-n_j} \) \( (j = 1, 2, \cdots, r) \) and obtain a system of elements

\[ < t_1, t_2, \cdots, t_r > = T \in \mathbb{Z}_p \mathbb{H} \] such that its image in \( V/(t) \) is \( \bar{T} \).

Consider now the system of elements \( < t, T > \) in \( V \). Since the algebra \( V/(t) \) is isomorphic to the polynomial algebra \( \mathbb{Z}_p[T] \) the system \( < t, T > \) is an independent polycentral system in \( V \). We extend now the weight function \( f(x) \) to the system \( < t, T > \) by defining \( f(t) = M \) where \( M \) is an arbitrary natural number greater than \( 2(m + 2r + 1) \) and \( f(t_j) = f(\bar{t}_j) \). Since the values of \( f \) on the subsystem \( T \) are less than or equal to \( m + 2r + 1 \) we obtain that \( f(t) > 2f(x) \) for every \( x \in T \) and we can apply Theorem II and obtain that this weight function extends to a valuation \( v \) in \( V \) with graded ring isomorphic to the polynomial ring \( \mathbb{Z}_p[t, T] \).

We will prove now that the group \( G \) centralizes the graded ring \( gr_v(V) \). Since \( \phi(t) = t \) \( (t \in T) \) by definition, we have to prove only that \( G \) centralizes the elements from \( T \). Let \( t_j \) be an element from \( T \). We have once again \( \bar{t}_j \in \bar{T}_{i,n} \) for some pair \( i, n \). If relation (7.26) holds in the ring \( V/(t) \cong \mathbb{Z}_p[T] \) then we obtain in \( V \).
\[ g \bullet t_j - t_j = (g - 1) \bullet t_j = u + y + u_1 \]  \hspace{1cm} (7.29)

where \( u \) and \( y \) are the same as in (7.26) and \( u_1 \) is an element from the ideal \((t)\). Since all the elements from the ideal \((t)\) have values greater than or equal to \( 2(m + 2r + 1) \) we obtain from (7.26) and (7.29) that \( v((g - 1) \bullet t_j) > v(t_j) + 1 \); the same relation is obtained in the case when (7.27) holds. This shows that \( G \) centralizes the system of elements \( t, T \); hence it centralizes the ring \( gr_v(V) \).

**Step 4.** We restrict now the valuation function \( v \) to the subring \( \mathbb{Z}_pH \subseteq V \). The graded ring \( gr_v(Z_pH) \) is a subring of \( gr_v(V) \), so \( gr_v(Z_pH) \) is centralized by \( \Phi \). Further, we denote

\[ H^*_i = \{ h \in H \mid v(h - 1) \geq i \} \quad (i = 1, 2, \cdots) \]  \hspace{1cm} (7.30)

and obtain \( o \) in \( H \) a new \( p \)-series with unit intersection

\[ H = H^*_1 \supseteq H^*_2 \supseteq \cdots \]  \hspace{1cm} (7.31)

Proposition 2.10. implies that the homogeneous components \( (h - 1) \) \((h \in H)\) generate in \( gr_v(Z_p(H)) \subseteq gr_v(V) \) a subalgebra isomorphic to the algebra \( L_p(H^*, H^*_i) \). Since \( gr(V) \) is centralized by \( \Phi \) the elements of this subalgebra are centralized by the group \( \Phi \). Since \( gr_v(V) \cong Z_p[t, T] \) we obtain that the algebra \( L_p(H, H^*_i) \) is abelian and it contains no nilpotent elements by Corollary 2.6., so it is free abelian.

We prove now that the rank of \( L_p(H, H^*_i) \) is \( r \). The system of homogeneous components \( t \) and \( \tilde{t}_j = (h_j - 1)t^{-n_j} \) \((j = 1, 2, \cdots, r)\) generate a ring isomorphic to the polynomial ring \( Z_p[t, T] \). This implies that the system of elements \( \tilde{t}_jt^{-n_j} = (h_j - 1) \) \((j = 1, 2, \cdots, r)\) is algebraically independent in \( gr(Z_pH) \) over \( Z_p \). Since \( U_p(L_p(H, H^*_i)) \cong gr(Z_pH) \) we obtain that \( U_p(L_p(H, H^*_i)) \) is a polynomial ring with \( r_1 \geq r \) variables. On the other hand, the rank of the algebra \( L_p(H, H^*_i) \) does not exceed \( r \) by Proposition 2.5. so \( r_1 \) can not be greater than \( r \). We obtain therefore that this rank is \( r \).

This completes the proof of Theorem X.

**Lemma 7.3.** Let \( G \) be a finite \( p \)-group, \( (G: 1) = p^n \). There exists a \( p \)-series

\[ G = G_1 \supseteq G_2 \supseteq \cdots \supseteq G_{n-1} \supseteq G_n = 1 \]  \hspace{1cm} (7.32)
such that the restricted Lie algebra $L_p(G,G_1)$ is abelian of dimension $n$ and exponent $p$.

**Proof.** Let $U$ be a central subgroup of order $p$ in $G$ generated by an element $u$. We can assume that there exists a $p$-series in the quotient group $\tilde{G} = G/U$

$$\tilde{G} = \tilde{G}_1 \supseteq \tilde{G}_2 \supseteq \cdots \supseteq \tilde{G}_{n-2} \supseteq \tilde{G}_{n-1} = 1 \quad (7.33)$$

such that the algebra $L_p(\tilde{G},\tilde{G}_i)$ is free abelian of dimension $n-1$ and of exponent $p$. Let $m$ be the maximum of the weights of non-unit elements of $\tilde{G}$. We pick now an integer $M > pm$ and for $u^n \in U$ $(1 \leq n \leq p-1)$ define the weight of $u^n$ in $G$ as $\omega(u) = nM, \omega(1) = \infty$. We define then for an element $g \not\in U$ its weight $\omega(g)$ to be equal to the weight of the coset $\bar{g} = gU$ in $\tilde{G}$. The weight function $\omega$ is now defined for all the elements $g \in G$ and $\omega(gu^k) = \omega(g)$ for an arbitrary $g \in G$ and a natural $k$.

Let $g_{i_1}, g_{i_2}$ be two elements of $G$. If the commutator $[g_{i_1}, g_{i_2}]$ does not belong to $U$ then

$$\omega([g_{i_1}, g_{i_2}]) > \omega(g_{i_1}) + \omega(g_{i_2}) \quad (7.34)$$

because this relation holds in $\tilde{G}$ for the elements $\bar{g}_{i_1}, \bar{g}_{i_2}$. On the other hand, if $[g_{i_1}, g_{i_2}] \in U$ then relation (7.34) holds because the elements of $U$ have weights greater than $pm \geq \omega(g_{i_1}) + \omega(g_{i_2})$.

The same argument shows that

$$\omega(g^p) > p\omega(g) \quad (7.35)$$

Relations (7.34) and (7.35) show that the weight function $\omega$ defines a $p$-series with associated graded algebra abelian of exponent $p$. Since 1 is the only element with infinite weight the intersection of all the terms of this series is 1 and the dimension of the associated Lie algebra is $n$.

This completes the proof.

**Proposition 7.5.** Let $H$ be a polycyclic with Hirsch number $r$, $U$ be a normal subgroup such that the quotient group $G = H/U$ is a finite $p$-group, $(G:U) = p^n$. Assume that there exists a $p$-series

$$U = U_1 \supseteq U_2 \supseteq \cdots \quad (7.36)$$
with unit intersection with the associated graded algebra \( L_p(U, U_i) \) is free abelian (abelian) of rank \( r \).

Then there exists a \( p \)-series

\[
H = H_1 \supseteq H_2 \supseteq \cdots
\]

with unit intersection such that the algebra \( L_p(H_i, H_i) \) is abelian of rank \( r \) and the algebra \( L_p(U, U_i^*) \) associated to the \( p \)-series \( U_i^* = U \cap H_i \) \( (i = 1, 2, \cdots) \) is free abelian of rank \( r \) (abelian of rank \( r \)).

**Proof.** *Step 1.* We consider first the case when the algebra \( L_p(U, U_i) \) is free abelian of rank \( r \). We pick in the group \( G \) a weight function \( \omega(G) \) obtained in Lemma 7.3. with the associated restricted Lie algebra abelian of dimension \( n \). Consider the wreath product \( W = U \wr G \) and obtain from Lemma 7.2. that there exists in the base group \( U^* \) a \( W \)-invariant \( p \)-series (7.9) such that the Lie algebra \( L_p(U^*, U_i^*) \) is isomorphic to the direct sum of \( p^n \) copies of \( L_p(U, U_i) \); so it is free abelian of rank \( p^n r \). Theorem X implies that we can get in \( U^* \) a \( G \)-invariant \( p \)-series \( V_i \) \( (i = 1, 2, \cdots) \) such that the Lie algebra \( L_p(U_i, V_i) \) is abelian of rank \( p^n r \). Let \( \rho \) be the weight function defined by this series and let \( m \) be the maximum of the weights \( \omega(g) \) \( (g \in G) \). We pick an arbitrary integer \( M \) greater than \( pm \).

We can assume (see Corollary 2.7.) that the values of the weight function \( \rho \) are multiples of \( M \).

We define now a weight function \( \Omega \) on \( W \) as follow.

\[
\Omega(u) = \rho(u) \text{ if } u \in U^*; \quad \Omega(gu) = \omega(g) \text{ for } 1 \neq g \in G; \quad u \in U^*
\]

The restrictions of \( \Omega \) on the groups \( G \) and \( U^* \) define algebras \( L_p(G, G_i) \) and \( L_p(U^*, V_i) \) respectively. This fact together with the definition of \( \Omega \) implies that for every two elements \( u_1, u_2 \) we have

\[
\Omega[u_1, u_2] > \Omega(u_1) + \Omega(u_2)
\]

in the following 2 cases:

1) If \( u_1, u_2 \in U^* \); 2) If \( u_1, u_2 \in G \).

We consider now the third case when when \( u_1 = g_1 v_1, u_2 = g_2 v_2 \) where \( g_1, g_2 \) are non-unit elements of \( G \), \( v_1, v_2 \) are non-unit elements of \( U \). In this case \( \Omega(u_i) = g_i \) \( (i = 1, 2) \) and this case is reduced to case 2.
It remains to prove relation (7.39) in the case when \( u_1 \in G \) and \( u_2 \in U^* \).

Since \( G \) centralizes the algebra \( L_p(U^*, V_i) \) we have in this case \( \rho(u_1^{-1}u_2^{-1}u_1u_2) > \rho(u_2) + 1 \). Since the value of \( \rho \) are multiples of \( M > m \) we obtain that

\[
\rho(u_1^{-1}u_2^{-1}u_1u_2) \geq \rho(u_2) + M > \rho(u_1) + \rho(u_2)
\]

(7.40)

which proves (7.39).

The same argument proves the relation

\[
\Omega(x^p) \geq p\Omega(x) \text{ for } (x \in W)
\]

(7.41)

We see that the weight function \( \Omega \) defines a \( p \)-series

\[
W = W_1 \supseteq W_2 \supseteq \cdots
\]

(7.42)

with unit intersection and \( W_i \cap U^* = V_i; W_i \cap G = G_i \) \((i = 1, 2, \ldots)\) and the algebra \( L_p(W, W_i) \) is a split extension of the free abelian algebra \( L_p(U^*, V_i) \) of rank \( p^n r \) by a finite abelian algebra \( L_p(G, G_i) \). We obtain from this together with relation (7.39) that the algebra \( L_p(W, W_i) \) is a direct sum of the algebra \( L_p(U^*, V_i) \) and the algebra \( L_p(G, G_i) \), so it is abelian of rank \( p^n r \) which is equal to the Hirsch number of \( W \). Since \( H \) is a subgroup of \( W \) statement i) of Theorem VI implies that the rank of the algebra \( L_p(H, H_i) \) must coincide with its Hirsch number of \( H \). Since the restriction of the function \( \Omega \) on the group \( U^* \) defines the free abelian algebra \( L_p(U^*, U_i^*) \) of rank \( p^n r \) equal to the Hirsch rank of \( U^* \) its restriction on \( U \) defines a free abelian algebra \( L_p(U, V_i) \), its rank must be equal \( r \) by statement i) of Theorem VI.

This completes the proof for the case when the algebra \( L_p(U, U_i) \) is free abelian of rank \( r \).

Step 2. Now assume that \( L_p(U, U_i) \) is abelian of rank \( r \). Apply Theorem 6.1 and find \( U_i = Q \) such that the algebra \( L_p(Q, Q_i) \) associated to the \( p \)-series \( Q_i = U_i \cap Q \) \((i = 1, 2, \ldots)\) is free abelian of rank \( r \). We find then in \( U \) a characteristic subgroup \( R \subseteq Q \) with index \( (U: R) = p^n \). The algebra \( L_p(R, R_i) \) associated to the \( p \)-series \( R_i = R \cap Q_i = R \cap U_i \) \((i = 1, 2, \ldots)\) is free abelian of rank \( r \) because it is a subalgebra of \( L_p(Q, Q_i) \). Since the index \( (H: R) \) is a power of \( p \) the assertion follows from the case which was considered at step 1.

**Theorem XI.** Let \( H \) be a polycyclic group with Hirsch number \( r \). Assume that there exists a \( p \)-series (1.1) with unit intercection such that the algebra
$L_p(H, H)$ is finitely generated. Then the center $Z$ of $L_p(H, H)$ has rank $r$ iff the following two conditions hold

i) $H$ contains normal subgroups $Q \supseteq N$ which satisfy the conditions of Theorem VII.

ii) The topology defined by series (1.1) is equivalent to the $p$-topology.

**Proof.** If conditions i) and ii) hold then the assertion follows from Theorem VII and statement viii) of Theorem VI.

We prove now the necessity of these conditions. Assume that the rank of $Z$ is $r$. Theorem V implies that there exists a normal subgroup $U$ of index $p^n$ such that the algebra $L_p(U, U_i)$ is free abelian subalgebra of $Z$ of finite index in $L_p(H, H)$. The index of $L_p(U, U_i)$ in $Z$ is also finite so the rank of $L_p(U, U_i)$ is $r$. Proposition 7.5. implies that there exists in $H$ a $p$-series with unit intesection and associated graded Lie algebra abelian of rank $r$. The necessity of the conditions i) and ii) now follows from statements v) and vi) of Theorem VI.

§8. Proof of Theorem XII.

8.1. Proposition 8.1. Let $R$ be an algebra over a field $K$ with a non-negative discrete pseudovaluation $\rho$, and associated graded ring $gr(R)$, $R \ast H$ be a suitable skew group ring of $R$ with infinite cyclic group $H$. Assume that there exists a natural number $k$ such that for every $x \in R$

$$\rho(hxh^{-1} - x) > \rho(x) + k \quad (8.1)$$

Then the pseudovaluation $\rho$ extends to a pseudovaluation $\rho_1$ of $R \ast H$ such that $\rho_1(h - 1) = k$. The graded ring $gr_{\rho_1}(R \ast H)$ is isomorphic to the polynomial ring $gr(R)_\rho[t]$ where $t = (h - 1)$ is the homogeneous component of the element $h - 1$. The extension $\rho_1$ is the only extension of $\rho$ with value of $h - 1$ equal $k$ and associated graded ring isomorphic to $gr_\rho(R)[t]$.

**Proof.** Let $h$ be the generator of $H$. We extend first $\rho$ to the skew polynomial subring $R[h]$. Every element $x \in R[h]$ has a unique representation...
\[ x = \sum_{i=0}^{n} \lambda_i (h-1)^i \quad (\alpha_i \in R; \ i = 0, 1, \ldots, n) \quad (8.2) \]

We define now the value of element (8.2) by

\[ \rho_1(x) = \min_i \{\rho(\lambda_i + ki)\} \quad (8.3) \]

and \( \rho_1(0) = \infty \).

We will now prove that \( \rho_1 \) is a pseudovaluation on in \( R[h] \). Assume that \( \rho_1(x) = \rho(\alpha_i) + i_0 k \), and let \( 0 \neq y = \sum_{j=1}^{m} \beta_j (h-1)^j \) be an element of \( R*H \) with \( \rho_1(y) = \rho(b_{j_0}) + j_0 k \). We see immediately that

\[ \rho_1(x + y) \geq \min \{\rho_1(x), \rho_1(y)\} \quad (8.4) \]

We prove now the relation

\[ \rho_1(xy) \geq \rho_1(x) + \rho_1(y) \quad (8.5) \]

We have \( x = x_1 + x_2, y = y_1 + y_2 \) where all the summands in the representations of \( x_1, y_1 \) have values \( \rho_1(x_1) \) and \( \rho_1(y_1) \) respectively and \( \rho_1(x_2) > \rho_1(x), \rho_1(y_2) > \rho_1(y) \). Relation (8.5) will follow if we prove that

\[ \rho_1(x_1 y_1) \geq \rho_1(x_1) + \rho_1(y_1) \quad (8.6) \]

Let \( \alpha(h-1)^i, \beta(h-1)^j \) be two arbitrary terms in the representations of \( x_1, y_1 \). We have \( (h-1)^i \beta - \beta(h-1) = h\beta - \beta h = (h\beta h^{-1} - \beta) \) and we obtain from this and (8.1) that

\[ (h-1)^i \beta = \beta(h-1) + u \quad (8.7) \]

where \( \rho(u) > \rho(\beta) + k \). We will use now an induction argument to show that for every natural \( i \)

\[ (h-1)^i \beta = \beta(h-1)^i + u_i \quad (8.8) \]

where \( u_i \) is an element of \( R*H \) with \( \rho_1(u_i) > \rho(\beta) + ki \)

In fact, assume that it has already been proven that

\[ (h-1)^{i-1} \beta = \beta(h-1)^{i-1} + u_{i-1} \quad (8.9) \]

where \( u_{i-1} \) is an element of \( R*H \) such that \( \rho_1(u_{i-1}) > \rho(\beta) + k(i-1) \).
We obtain from this

\[(h - 1)^i \beta = (h - 1)\beta (h - 1)^{i-1} + (h - 1)u_{i-1} = \]
\[= (\beta (h - 1) + u)(h - 1)^{i-1} + (h - 1)u_{i-1} = \]
\[= \beta (h - 1)^i + u(h - 1)^{i-1} + (h - 1)u_{i-1} \]  \hspace{1cm} (8.10)

We denote now \(u_i = u(h - 1)^{i-1} + (h - 1)u_{i-1}\) and obtain (8.8).

We have now from (8.8)

\[\alpha (h - 1)^i \beta (h - 1)^j = \alpha (\beta (h - 1)^{i+j} + u_i(h - 1)^j) = \]
\[= \alpha \beta (h - 1)^{i+j} + \alpha u_i(h - 1)^j = \alpha \beta (h - 1)^{i+j} + v \]  \hspace{1cm} (8.11)

where \(v = \alpha u_i(h - 1)^j\) and

\[\rho_1(v) > \rho(\alpha) + \rho(\beta) + k(i + j) \]  \hspace{1cm} (8.12)

Definition (8.3) implies that the summand \(\alpha \beta (h - 1)^{i+j}\) has value \(\rho(\alpha) + \rho(\beta) + k(i + j)\). This and (8.12) imply that the right side of (8.11) has value greater than or equal \(\rho(\alpha) + \rho(\beta) + k(i + j)\) and we obtain from this and (8.11)

\[\rho_1(\alpha (h - 1)^i \beta (h - 1)^j) \geq \rho_1(\alpha (h - 1)^i) + \rho_1(\beta (h - 1)^j) \]  \hspace{1cm} (8.13)

Since this relation holds for arbitrary summands in the representations of \(x_1, y_1\) we obtain that \(\rho_1(x_1y_1) \geq \rho(x_1) + \rho_1(y_1)\); this together with (8.4) proves that \(\rho_1\) is a pseudovaluation in \(R[h]\) which extends the valuation \(\rho\) of \(R\).

The element \(h - 1\) has value \(k\) and relation (8.7) implies that the homogeneous component \(t = (h - 1)\) of \(h - 1\) commutes with all the homogeneous components \(\beta\) of elements \((\beta \in R)\) which implies that \(t\) commutes with the subring \(gr_\rho(R) \cong gr_{\rho_1}(R)\). Further, if \(\rho_1(x) = l\) then \(x = u + x_2\) where \(\rho_1(x_2) > l\).

\[u = \sum_{i=0}^{n} \alpha_i(h - 1)^i \]  \hspace{1cm} (8.14)
and $\rho(\alpha_i) + ki = l$; we see that the homogeneous component of $x$ is equal to the homogeneous component of $x_1$. We will show now that this homogeneous component has a unique representation

$$\tilde{u} = \sum_{i=0}^{n} \tilde{\alpha}_it^i$$

(8.15)

This will imply that $gr_{\rho_1}(R[h])$ is isomorphic to the polynomial ring $gr_{\rho}(R)[t]$.

We observe first that the definition of $\rho_1$ implies that the weight of every summand $\alpha_i(h-1)^i$ in (8.14) is equal to $\rho_1(\alpha) + ki = l$ and Lemma 2.3 implies that the homogeneous component of this summand is $\tilde{\alpha}_it^i$. Since the weight of $x_1$ is $l$ we conclude now once again from Lemma 2.3 that $\tilde{x}_1$ does have representation (8.15), which proves that $gr_{\rho_1}(R[h]) \cong gr_{\rho}(R)[t]$.

To extend $\rho$ to the ring $R \ast H$ we use the fact that for every element $y \in R \ast H$ we can find $x \in R[h]$ and an integer $m$ such that $y = xh^m$. We define now $\rho_1(y) = \rho_1(x)$ and a straightforward argument shows that that we obtain a pseudovaluation in $R \ast H$ and that the graded ring $gr_{\rho_1}(R \ast H)$ is isomorphic to $gr_{\rho_1}(R[h])$.

To prove that $gr_{\rho_1}(R \ast H) \cong gr_{\rho}(R)[t]$ we have to prove the uniqueness of representation (8.15); once again, it is enough to do this for the subring $R[h]$.

If representation (8.15) is not unique then a standard argument yields that there exist non-zero elements $\lambda_j \in R$ ($j = 1, 2, \cdots, n$) such that

$$\sum_{j=0}^{m} \lambda_jt^j = 0$$

(8.16)

where

$$\rho(\lambda_j) + kj = l \quad (j = 1, 2, \cdots, m)$$

(8.17)

Equation (8.16) means that

$$\rho_1(\sum_{j=0}^{m} \lambda_j(h-1)^j) \geq (l + 1)$$

(8.18)

This contradicts the definition $\rho_1$ and we proved the uniqueness of representation (8.15).
It remains to prove that \( \rho_1 \) is the only extension of \( \rho \) such that \( \rho(h-1) = k \) and \( \text{gr}_{\rho_1}(R*H) \cong \text{gr}_{\rho}(R)[t] \). The proof of this fact is obtained by the same argument, with obvious simplifications, which will be used in step 2 of the proof of a similar statement in Proposition 8.2, and we omit it.

**Proposition 8.2.** Let \( R_1 \) be an algebra of characteristic \( p \), \( R \) be a subalgebra of \( R_1 \). Assume that there exists an invertible element \( g \in R_1 \) such that \( g^{-1}Rg = R, g^{p^m} = r_0 \in R \) and the elements

\[
1, g, g^2, \ldots, g^{p^m} - 1
\]

form a basis of the left \( R \)-module \( R_1 \). Assume also that there exists a non-negative discrete pseudovaluation \( \rho \), with commutative associated graded ring \( \text{gr}(R) \) such that \( \rho(r_0 - 1) = kp^m \) and

\[
\rho(g^{-1}rg - r) > \rho(r) + k
\]

for every \( r \in R \).

Then there exists an extension of \( \rho \) to a pseudovaluation \( \rho_1 \) of the ring \( R_1 \) such that \( \rho(g - 1) = k \), the graded ring \( \text{gr}_{\rho_1}(R_1) \) is isomorphic to the algebraic extension \( \text{gr}_{\rho}(R)[\theta] \) where \( \theta = (g - 1) \) is the homogeneous component of \( g - 1 \), the minimal polynomial of \( \theta \) is \( tp^m - (r_0 - 1) \).

The pseudovaluation \( \rho_1 \) is the unique extension such that the value of \( r_0 - 1 \) is \( k \) and the associated graded ring is isomorphic to the quotient ring \( \text{gr}_{\rho}(R)[t]/(tp^m - (r_0 - 1)) \).

**Proof.** Step 1. Let \( \phi \) be the inner automorphism \( x \rightarrow g^{-1}xg \) of \( R_1 \). Since the subring \( R \) is \( \phi \)-invariant we can consider the skew group ring \( R*H \) of \( R \) with infinite group \( H \) where \( h^{-1}rh = \phi(r) \) for the generator \( h \) of \( H \) and an arbitrary \( r \in R \). We apply Proposition 8.1 to obtain a pseudovaluation \( \tau \) of \( R*H \) which coincides with \( \rho \) on \( R \), \( \tau(h - 1) = k \), and \( \text{gr}_\tau(R*H) \cong R[t] \).

We consider now the homomorphism \( \psi: R*H \rightarrow R_1 \) whose kernel is the principal ideal \( A \) generated by the element \( (h - 1)^{p^m} - (r_0 - 1) \). Proposition 2.2. implies that we obtain a pseudovaluation \( \rho_1 \) of the ring \( R_1 \). We will show that this pseudovaluation satisfies all the conclusions of the assertion.

We will prove first the relation

\[
\text{gr}_{\rho_1}(R_1) \cong R/(tp^m - (r_0 - 1))
\]
The homogeneous component of the element $(h-1)^{p^m}-(r_0-1)$ in $gr_\tau(R*H)$ is $t^{p^m}-(r_0-1)$. So the element $t^{p^m}-(r_0-1)$ belongs to the kernel of the homomorphism $gr_\tau(R*H) \rightarrow gr_{\rho_1}(R_1)$. To prove relation (8.21) it is enough to verify that if $x \in ker(\phi) = (h^{p^m} - r_0)$ then $\tilde{x} \in (t^{p^m} - (r_0-1))$. Assume that $\tau(x) = \ell$ and let

$$x = \sum_{i,j} x_{ij}(h^{p^m} - r_0)x_j$$  \hspace{0.5cm} (8.22)$$

Since the graded ring $gr_\tau(R*H) \cong R[t]$ is commutative we obtain from (8.22) that

$$x = \sum_{i,j} (h^{p^m} - r_0)x_{ij}x_j + y$$  \hspace{0.5cm} (8.23)$$

where $\tau(y) > \ell$. We see that the homogeneous component of $x$ coincides with the homogeneous component of the element $(h^{p^m} - r_0)a$ where $a = \sum_{i,j} x_{ij}x_j$; we can assume in fact that $x = (h^{p^m} - r_0)a$. We have already observed that the homogeneous component of $(h^{p^m} - r_0)$ is equal to

$$(h^{p^m} - r_0) = ((h^{p^m} - 1) - (r_0 - 1)) = (t^{p^m} - (r_0-1)).$$

Since this element can not be a zero divisor in the ring $gr_\rho(R)[t]$ we obtain that the homogeneous component of $x$ is a product of homogeneous components of the element $h^{p^m} - r_0$ and of the element $a$:

$$\tilde{x} = (t^{p^m} - (r_0-1))\tilde{a}$$  \hspace{0.5cm} (8.24)$$

which means that $\tilde{x}$ belongs to the ideal generated by $(t^{p^m} - (r_0-1))$. This proves (8.21).

Step 2. We will prove now the uniqueness of the extension. If $\rho_2$ is a second extension of the pseudovaluation $\rho$ with associated graded ring isomorphic to $(gr_\rho(R)[t])/(t^{p^m} - (r_0-1))$ then $\rho_2(\rho - 1)$ must be less than or equal to $k$ because $(g-1)t^{p^m} = r_0-1$. If $\rho_2(\rho - 1) < k$ then the homogeneous component $(g-1)$ is nilpotent and the minimal polynomial of this component can not be $t^{p^m} - (r_0-1)$. We see that $\rho_2(\rho - 1) = k$. Further for every $\lambda \in R$ and $(g-1)^i$ ($1 \leq i \leq p^m - 1$) we have $\rho_2(\lambda(g-1)^i) \geq \rho(\lambda) + ki$; if we had in the last equation a strict inequality we would have $\lambda(g-1)^i = 0$ which is impossible because we assumed that the minimal polynomial of $g-1$ is $t^{p^m} - (r_0-1)$. We obtain therefore that $\rho_2(\lambda(g-1)^i) = \rho(\lambda) + ki$. 

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Let
\[
 u = \sum_{i=1}^{n} \lambda_i (g - 1)^i \quad (i \leq p^m - 1) \quad (8.25)
\]
be an arbitrary element of \( R_1 \), and let \( \min_{1 \leq i \leq n} \{ \rho(\lambda_i) + ik \} = l \); we can assume that \( \rho(\lambda_i) + ik = l \) if \( 1 \leq i \leq n_1 \) and \( \rho(\lambda_i) + ki > l \) if \( n_1 < i \leq n \). Let \( u_1 = \sum_{i=1}^{n_1} \lambda_i (g - 1)^i \). Clearly \( \rho_1(u) = \rho_1(u_1) \), \( \rho_2(u) = \rho_2(u_1) \) and \( \rho_1(\lambda_i (g - 1)^i) = \rho_2(\lambda_i (g - 1)^i) = l \) \((1 \leq i \leq n_1)\). The element \( \sum_{i=1}^{n_1} \lambda_i (g - 1)^i \) of \( gr_{\rho_1}(R_1) \) is nonzero and we conclude now from Lemma 2.3 that the \( \rho_1 \)-value of \( u \) must be \( l \). The same argument shows that \( \rho_2(u) = l \). We proved that the pseudovaluation \( \rho_2 \) must coincide with \( \rho_1 \). This completes the proof.

We will need in the proof of Theorem XII a corollary of Propositions 8.1, and 8.2.

**Corollary 8.1.** Assume that the conditions of Proposition 8.2. hold. Let \( A \) be an ideal of \( R \) such that \( g^{-1}Ag = A \) and \( A_1 = AR_1 \) be the ideal in \( R_1 \) generated by \( A \). Let \( x \) be an arbitrary element of \( A_1 \). Then the homogeneous component \( \tilde{x} \) of \( x \) in \( gr_{\rho_1}(A_1) \) has a unique representation
\[
 \tilde{x} = \sum_{i=1}^{m} \tilde{\mu}_i (\tilde{g} - 1)^{n_i} \quad (8.26)
\]
where \( \tilde{\mu}_i \) is the homogeneous component of an element \( \mu_i \in A_i(g - 1) \) is the homogeneous component of \( g - 1 \), \( 0 \leq n_i \leq p^m - 1 \). So the ideal \( gr_{\rho_1}(A_1) \) of the ring \( gr(R_1) \) is isomorphic to the ring \( A[\theta] \cong A[t]/(tp^m - (r_0 - 1)) \).

**Proof.** The element \( x \) has a representation
\[
x = \mu_0 + \mu_1 (g - 1) + \cdots + \mu_{p^m-1} (g - 1)^{p^m-1} \quad (\mu_i \in A; \; i = 1, 2, \cdots, p^m-1) \quad (8.27)
\]
Assume that \( \rho_1(x) = n \). Then \( x = x_1 + x_2 \) where \( \rho_1(x_2) > n \), \( \rho_1(x_1) = n \) and
\[
x_1 = \sum_{i=1}^{m} \mu_i (\tilde{g} - 1)^{n_i} \quad (8.28)
\]
which implies in particular that \( \tilde{\mu}_i(g-1)^{n_i} \) is \( \tilde{\mu}_i(g-1)^{i} \). Lemma 2.3. now implies that either the homogeneous component of \( x_1 \) is equal to the right side of (8.28), and in this case it is the homogeneous component of \( x \), or that

\[
\sum_{i=1}^{m} \tilde{\mu}_i(g-1)^{n_i} = 0 \tag{8.29}
\]

Relation (8.29) is impossible because the minimal polynomial of \( g-1 \) is \( t^n - (r_0 - 1) \) whereas all the coefficients \( \tilde{\mu}_i \) \( (i = 1, 2, \ldots, m) \) in (8.29) belong to \( gr(A) \). We obtain from this that the homogeneous component of \( x \) is given by (8.26). The uniqueness of this representation follows from Proposition (8.2).

This completes the proof.

We will need one more corollary of Proposition 8.2.

The ring \( \bar{R}_1 \) in Proposition 8.2. and Corollary 8.1. is isomorphic to a cross product \( R \ast G \) where \( G \) is the cyclic group of order \( p^n \). Assume that conditions of Corollary 8.1. hold and let \( \phi \) be the homomorphism \( R \longrightarrow R/A \) and \( \phi_1 \) be the homomorphism \( R \longrightarrow \bar{R}_1 = R/A_1 \) where \( A_1 = AR, \) and \( X \) be the image of a subset \( X \in R_1 \) under this homomorphism. Then the restriction of the homomorphism \( \phi_1 \) on \( R \) is \( \phi \), and \( \phi(R) \cong \bar{R} \), the ring \( \bar{R}_1 \) is isomorphic to a suitable cross product \( \bar{R} \ast \bar{G} \) where the group \( \bar{G} \) is isomorphic to \( G \). This implies that the ring \( \bar{R}_1 \) is a free left module with basis \( 1, (g-1), \ldots, (g-1)^{p^n-1} \) over the subring \( \bar{R} = R/A \) and \( (g-1)^{p^n} = (r_0 - 1) \).

We obtain now from Proposition 2.3. that the epimorphism \( \phi_1 \) together with the pseudovaluation \( \rho_1 \) in \( R_1 \) defines a pseudovaluation \( \bar{\rho}_1 \) in \( \bar{R}_1 \) and we have an epimorphism \( \tilde{\phi}_1: gr_{\rho_1}(R_1) \longrightarrow gr_{\bar{\rho}_1}(\bar{R}_1) \) with kernel \( gr_{\rho_1}(A_1) \). Similarly, the homomorphism \( \phi: R \longrightarrow \bar{R} \) and pseudovaluation \( \rho \) in \( R \) define a pseudovaluation \( \bar{\rho} \) in \( \bar{R} \) and the restriction of \( \tilde{\phi}_1 \) on \( gr(R) \) defines an epimorphism \( \tilde{\phi}: gr_{\rho}(R) \longrightarrow gr_{\bar{\rho}}(\bar{R}) \). The ring \( gr_{\rho_1}(\bar{R}_1) \) is generated by the subring \( \phi_1(gr_{\rho}(R)) \cong gr_{\bar{\rho}}(\bar{R}) \) and the element \( \phi(\theta) \) where \( \theta \) is the homogeneous component of the element \( g-1 \) in \( gr_{\rho_1}(R_1) \). We obtain now from Corollary 8.1. that \( gr_{\rho_1}(\bar{R}_1) \) is isomorphic to the quotient ring \( R[\theta]/A[\theta] \cong (R/A)[\theta] \cong \bar{R}[\theta] \) which implies in particular that \( \phi(\theta) \neq 0 \). We obtain from this and from
Proposition 2.3. that the homomorphism $\tilde{\phi}$ maps the homogeneous component $\theta = (g - 1)$ in $gr_{\rho_1}(R_1)$ on the homogeneous component of the element $g - 1$ in $gr_{\bar{\rho}_1}(\bar{R}_1)$; we denote this homogeneous component by $\bar{\theta}$.

We proved in these notations the following fact.

**Corollary 8.2.** Let $\theta$ and $\bar{\theta}$ be the homogeneous component of the elements $g - 1$ in $gr_{\rho_1}(R_1)$ and in $gr_{\bar{\rho}_1}(\bar{R}_1)$ respectively. Then

$$\tilde{\phi}(gr_{\rho_1}(R_1)) \cong gr_{\bar{\rho}_1}(\bar{R}_1) \cong \bar{R}[\bar{\theta}] \quad (8.30)$$

where $\tilde{\phi}(\theta)$ is the homogeneous component of the element $\tilde{\phi}(g - 1)$ in $\bar{R}_1$ and the minimal polynomial of $\tilde{\phi}(\theta)$ is $t^{r_\rho} - (r_0 - 1)$.

**8.2.** Let $R$ be a ring, $\phi$ be an automorphism of $R$, $A$ be an $\phi$-invariant ideal in $R$. Assume that there exists an $\phi$-invariant non-negative pseudovaluation $\rho$ of $R$, and let $\bar{\rho}$ be the pseudovaluation of the ring $\bar{R} = R/A$ obtained from the homomorphism $R \rightarrow \bar{R}$ (see Proposition 2.3.). Let $a_i (i \in I)$ be a system of elements in $A$ whose homogeneous components $\tilde{a}_i (i \in I)$ generate the ideal $gr(A) \subseteq gr(R)$. Proposition 2.3. implies that the homomorphism $R \rightarrow \bar{R}$ defines in a natural way a homomorphism $gr(R) \rightarrow gr(\bar{R})$. Let $b_j (j \in J)$ be a system of element in $R$ whose homogeneous components $\tilde{b}_j (j \in J)$ generate $gr(R)$ modulo the ideal $gr(A)$. We can assume that the images of $\tilde{b}_j (j \in J)$ in $gr(R)$ are non-zero. This implies that for every $j \in J$ the value $\bar{\rho}(b_j)$ is equal to the weight $\rho(b_j)$.

**Proposition 8.3.** Let $R$ be a ring, $\phi$ be an automorphism of $R$, $\rho$ be a non-negative $\phi$-invariant valuation in $R$, $A$ be a $\phi$-invariant ideal of $R$. Assume that there exists systems of elements $a_i (i \in I)$ whose homogeneous components $\tilde{a}_i (i \in I)$ generate the ideal $gr(A)$ of $gr(R)$, and a system of elements $b_j (j \in J)$ whose homogeneous components $\tilde{b}_j (j \in J)$ generate the ring $gr(R)$ modulo the ideal $gr(A)$, and natural number $k$ such that

$$\rho(\phi(a_i) - a_i) > k + \rho(a_i) \quad (i \in I) \quad (8.31)$$

$$\bar{\rho}(\phi(b_j) - \tilde{b}_j) > k + \bar{\rho}(\tilde{b}_j) \quad (j \in J) \quad (8.32)$$

Assume also that for every $i \in I, j \in J$

$$\rho(a_i) > k + \rho(b_j) \quad (8.33)$$

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Then for every $r \in R$

$$\rho(\phi(r) - r) > k + \rho(r)$$

(8.34)

We need first the following lemma.

**Lemma 8.3.** Let $R$ be a ring with a non-negative valuation $\rho$, $x_i (i \in I)$ be a system of elements whose homogeneous components $\tilde{x}_i (i \in I)$ generate the ring $gr(R)$. Let $r \in R$ be an element with $\rho(r) = n$ and $m > n$ be a natural number.

Then

$$r = \sum_{j=1}^{s} \pi_j + y$$

(8.35)

where $\rho(y) \geq m$ and every $\pi_j (j = 1, 2, \cdots, s)$ is a monomial with value $n \leq \rho(\pi_j) \leq m - 1$ on the set of elements $x_i (i \in I)$.

**Proof.** We find monomials $\pi_j (j = 1, 2, \cdots, s_1)$ with values $\rho(\pi_j) = n (j = 1, 2, \cdots, s_1)$ such that

$$r = \sum_{j=1}^{s_1} \pi_j + r_1$$

(8.36)

where $\rho(r_1) > n$. We can obtain a similar representation for the element $r_1$ and representation (8.35) is obtained in a finite number of steps.

**Proof of Proposition 8.3.** We see that the homogeneous components of elements $a_i, b_j (i \in I, j \in J)$ generate the ring $gr(R)$. Let $\rho(r) = n$. We pick the number $m = n + k$ and obtain from Lemma 8.3. that it is enough to prove the assertion for an arbitrary monomial $\pi = \pi_j$ in the representation (8.35) of $r$ where $n \leq \rho(\pi) = n_1 \leq m - 1$. We assume that

$$\pi = x_1x_2\cdots x_l$$

(8.37)

where the elements $x_\alpha (\alpha = 1, 2, \cdots, l)$ are taken from the set of generators $a_i, b_j (i \in I; j \in J)$ and $n \leq \rho(\pi) = n_1 \leq (m - 1)$. The conditions of the assertion yield that it is true when the element $\pi$ is equal to one of the generators, $a_i$ or $b_j$. We can assume that it is true when $\pi$ is a product of $l - 1$ generators. We apply now the identity
\[
\phi(uv) - w = \phi(u) (\phi(v) - v) + (\phi(u) - u) v \tag{8.38}
\]
to the elements \( u = x_1 x_2 \cdots x_{l-1} \) and \( y = x_l \) and obtain now for the first summand in the right side of (8.38)

\[
\rho(\phi(u) (\phi(v) - v)) \geq \rho(\phi(u)) + \rho(\phi(v) - v) = \\
= \rho(u) + \rho(\phi(v) - v) > \rho(u) + \rho(v) + k = \\
= \rho(uv) + k = \rho(\pi) + k = n_1 + k \tag{8.39}
\]

We obtain in the same way

\[
\rho((\phi(u) - u) v) > n_1 + k \tag{8.40}
\]

and hence

\[
\rho(\phi(\pi) - \pi) > n_1 + k \tag{8.41}
\]

Since \( \pi \) was an arbitrary summand in the representation (8.35) of \( r \) the assertion follows.

8.3. Theorem XII. Let \( H \) be a torsion free polycyclic group with Hirsch number \( r \), \( U \) be a normal subgroup with Hirsch number \( k \) and torsion free quotient group \( \bar{H} = H/U \).

Assume that the following 3 conditions hold.

1) There exists in \( U \) a \( p \)-series

\[
U = U_1 \supseteq U_2 \supseteq \cdots \tag{8.42}
\]

with associated restricted graded Lie algebra \( L_p(U, U_i) \) free abelian of rank \( k \).

2) There exists in the group \( \bar{H} = H/U \) a \( p \)-series

\[
\bar{H} = \bar{H}_1 \supseteq \bar{H}_2 \supseteq \cdots \tag{8.43}
\]

with associated restricted Lie algebra \( L_p(\bar{H}, \bar{H}_i) \) free abelian of rank \( r - k \).

3) For for every subgroup \( R = gp(h, U) \) generated by \( U \) and an element \( h \in H \) the quotient group \( \bar{R} = H/U'U^p \) is a residually \{finite \( p \)-group\} or equivalently \([U, h^{p^k}] \subseteq U'U^p\).

Then there exists a \( p \)-series
\[ H = H_1 \supseteq H_2 \supseteq \cdots \]  \hspace{1cm} (8.44)

with unit intersection and associated restricted Lie algebra \( L_p(H, H_i) \) free abelian of rank \( r \) such that \( \overline{H_i} = (H_iU)/U \) \((i = 1, 2, \cdots)\).

The last statement of Theorem XII together with Proposition 2.1. implies immediately the following corollary.

**Corollary 8.3.** The natural homomorphism \( \phi: H \longrightarrow \overline{H} \) defines a homomorphism \( \overline{\phi}: L_p(H, H_i) \longrightarrow L_p(\overline{H}, \overline{H}_i) \) of graded algebras.

**Proof.** Step 1. Apply Theorem 6.1. and Corollary 6.2. to obtain in the group \( \overline{H} \) a poly\{infinite cyclic\} normal subgroup \( \overline{Q} \) with the the quotient group \( G = \overline{H}/\overline{Q} \) a finite \( p \)-group of order \( p^n \), and a special system of generators

\[ \bar{s}_{\alpha}, \bar{q}_j \ (\alpha = 1, 2, \cdots, n; j = 1, 2, \cdots, r) \]  \hspace{1cm} (8.45)

where the free abelian algebra \( L_p(\overline{Q}, \overline{Q}_i) \) is freely generated by the system of elements \( \bar{q}_j \ (j = 1, 2, \cdots, r - k) \). Let \( m \) be the maximum of the weights of the elements from this system of generators. We pick an arbitrary natural \( M > pm \) and apply Theorem X and Corollary 7.4. to get an in \( U \) an \( H \)-invariant \( p \)-series \( U_i \) \((i = 1, 2, \cdots, k)\) with unit intersection such that the weights of all the elements from \( U \) are multiples of \( M \) and the algebra \( L_p(U, U_i) \) is centralized by the group \( H \).

Let \( Q \) be the inverse image of \( \overline{Q} \) in \( H \) and

\[ s_{\alpha}, q_j \ (\alpha = 1, 2, \cdots, n; j = 1, 2, \cdots, r) \]  \hspace{1cm} (8.46)

be a system of coset representatives for the elements of system (8.45). The group \( H \) is obtained from \( U \) by a chain of \( r - k \) infinite cyclic extensions, and then by a chain of \( n \) cyclic extensions of order \( p \); every infinite cyclic extension is generated by some element \( q_j \), every cyclic extension of order \( p \) is generated by an element \( s_{\alpha} \). Let \( V \supseteq W \) be two subgroups of this series with the quotient group \( V/W \) infinite cyclic or cyclic of order \( p \), \( \overline{V} = V/U, \overline{W} = W/U \). We define by \( \phi \) the natural homomorphism \( \overline{H} = H \longrightarrow H/U \) and use the same notation for the restrictions of this homomorphism on the subgroups \( V \) and \( W \).
We consider now the group ring $K\bar{H}$ where $K$ is an arbitrary field of characteristic $p$. Series (8.43) defines a filtration and a valuation $\bar{\rho}$ in the group ring $K\bar{H}$; we keep the same notation $\bar{\rho}$ for the restrictions of this valuation on $K\bar{V}$ and $KW$.

Assume that it has already been proven that the subgroup $W$ contains a $p$-series with unit intersection

$$W = W_1 \supseteq W_2 \supseteq \cdots \quad (8.47)$$

and with associated algebra $L_p(W, W_i)$ free abelian with rank equal to the Hirsch number of $W$, and that every subgroup $\bar{W}_i = \bar{H}_i \cap \bar{W}$ is isomorphic to $(W_iU)/U$. This assumption implies that the homomorphism $\phi: W \to \bar{W}$ defines the related homomorphism of algebras $L_p(W, W_i) \to L_p(\bar{W}, \bar{W}_i)$, and a homomorphism $U_p(L_p(W, W_i)) \to U_p(L_p(\bar{W}, \bar{W}_i))$; we will use for the last two homomorphisms the notation $\bar{\phi}$. Further, Proposition 2.7 imply that $gr_{\rho}(KW) \cong U_p(L_p(W, W_i))$, and $gr_{\rho}(KW) \cong U_p(L_p(\bar{W}, \bar{W}_i))$ so we obtain also a homomorphism $gr_{\rho}(KW) \to gr_{\bar{\rho}}(K\bar{W})$. Once again, we will use for this homomorphism the same notation $\bar{\phi}$.

We will prove that there exists a $p$-series

$$V = V_1 \supseteq V_2 \supseteq \cdots \quad (8.48)$$

such that $V_i \cap W = W_i \quad (i = 1, 2, \cdots)$, the algebra $L_p(V, V_i)$ is free abelian with rank equal to the Hirsch number of $V$ and

$$(V_iU)/U \cong \bar{V}_i \quad (i = 1, 2, \cdots) \quad (8.49)$$

This will imply that after a finite number of steps we will get a required $p$-series (8.44) in $H$. We will give the proof for for the existence of series (8.48) in the case when the quotient group $V/W$ is cyclic of order $p$; the case when this quotient group is infinite cyclic is obtained by the same argument with obvious simplifications.

**Step 2.** We will extend at this step the valuation $\rho$ in $KW$ to a valuation $\rho_1$ in $KV$.

Let $s \in V$ be the an element from system (8.46) which generates the quotient group $V/W$; its image $\bar{s}$ in $H/U$ belongs to $\bar{V}$ and generates the quotient group $\bar{V}/\bar{W} \cong V/W$, so $s^p = a \in W$ and $\bar{s}^p = \bar{a} \in \bar{W}$. Let $\bar{\rho}(\bar{s} - 1) = \delta$.

Let $\bar{\tau}$ be homogeneous component of the element $s - 1 = \bar{s} - 1$ in $gr_{\rho}(K\bar{V})$. Then we obtain from Proposition 8.2.
\[ \text{gr}_\rho(\bar{KV}) \cong \text{gr}_\rho(\bar{KW})[\bar{\tau}] \quad (8.50) \]

where the minimal polynomial of $\bar{\tau}$ is $t^p - (\bar{a} - 1)$; here $(\bar{a} - 1)$ denotes the homogeneous component of the element $\bar{a} - 1$ in $\text{gr}(\bar{KV})$.

Since the algebra $L_p(\bar{H}, \bar{H}_i)$ is free abelian with rank equal to the Hirsch number of $\bar{H}$ we obtain from statement i) of Theorem VI that the $p$-series $\bar{V}_i = \bar{H}_i \cap \bar{V}$ ($i = 1, 2, \ldots$) has unit intersection and the algebra $L_p(\bar{V}, \bar{V}_i)$ is free abelian of rank equal to the Hirsch number of $\bar{V}$; we have also $\bar{V}_i \cap \bar{W} = \bar{H}_i \cap \bar{W} = \bar{W}_i$ ($i = 1, 2, \ldots$).

Let $u$ be an arbitrary element whose homogeneous component belongs to the system of generators of the algebra $L_p(U, U_i)$. Since $s$ centralizes the series $U_i$ ($i = 1, 2, \ldots$) we obtain that the weight of the element $s^{-1}us - u$ is greater than the weight of $u$ (see subsection 2.8); we obtain from this

\[ \rho(s^{-1}(u - 1)s - (u - 1)) > \rho(u - 1) \quad (8.51) \]

It is important that the values of the elements of $L_p(U, U_i)$ are multiples of $M$; we recall also that $M > \delta = \rho(\bar{s} - 1)$ and obtain from (8.51)

\[ \rho(s^{-1}us - u) \geq \rho(u - 1) + M > \rho(u - 1) + \delta \quad (8.52) \]

We pick now in $\bar{W}$ an element $\bar{w}$ whose homogeneous component belongs to the system of generators of the algebra $L_p(\bar{W}, \bar{W}_i)$. Since the algebra $L_p(\bar{V}, \bar{V}_i)$ is commutative the homogeneous components of $\bar{s}$ and $\bar{w}$ commute and we obtain in the same way as in (8.51) and (8.52)

\[ \rho(\bar{s}^{-1}\bar{w}s - \bar{w}) > \delta + \rho(\bar{w} - 1) \quad (8.53) \]

Since the elements of $U$ have weights greater than $M$ all the elements of $\omega(\bar{KU})$ and all the elements of the ideal $\omega(\bar{KU})KW$ have $\rho$-values greater than $M$. Since $\omega(\bar{KU})KW$ is the kernel of the homomorphism $KW \to KW$ we obtain now from Proposition 8.3 that for every element $x \in KW$

\[ \rho(s^{-1}xs - x) > \rho(x - 1) + \delta \quad (8.54) \]

We recall now that $s^p = a$, $\bar{s}^p = \bar{a}$ and the weight of the element $\bar{a} = \bar{s}^p \in \bar{W}$ is equal to $p\delta$ because the algebra $L_p(\bar{V}, \bar{V}_i)$ is free abelian. This implies immediately that the element $a \in W$ has weight less than or equal to $p\delta$. We will now prove that this weight is equal to $p\delta$. 

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In fact, if the weight of \( a \) were less than \( p\delta \), say \( \delta_1 < p\delta \), then its image \( \bar{a} \) would also have weight \( \delta_1 \) because all the elements of the kernel \( \omega(KU)KW \) have weights greater than \( p\delta > \delta_1 \). This contradiction shows that the weight of \( a \) is \( p\delta \); we obtain from this that \( \rho(a - 1) = p\delta \).

We can apply now Proposition 8.2 to the group \( V \) and its normal subgroup \( W \). We obtain that there exists a unique extension of the valuation \( \rho \) to a pseudovaluation \( \rho_1 \) of the algebra \( KV \), such that \( \rho(s - 1) = \delta \), the graded ring \( gr(KV) \) is commutative and we have for the homogeneous components \( s - 1 \) and \( a - 1 \) of the elements \( s - 1 \) and \( a - 1 \).

\[
(s - 1)^p = (a - 1) \quad (8.55)
\]

We consider now once again the homomorphism \( \phi: V \rightarrow \bar{V} = V/U \) and the related homomorphism \( KV \rightarrow \bar{KV} \) of group rings. We obtain a pseudovaluation \( \bar{\rho}_1 \) of \( \bar{KV} \); if \( A_i \) is a filtration defined by \( \rho_1 \) in \( KV \) then \( \phi(A_i) = \bar{A}_i \) \( (i = 1, 2, \cdots) \) is the filtration in \( KV \) defined by \( \bar{\rho}_1 \). The restriction of \( \bar{\rho}_1 \) on \( KW \) coincides with the valuation \( \bar{\rho} \) of \( KW \).

The \( p \)-series \( \bar{V}'_i \) \( (i = 1, 2, \cdots) \) in \( \bar{V} \) defined by the pseudovaluation \( \bar{\rho}_1(\bar{V}) \) is in fact obtained as \( \bar{V}'_i = \phi(V_i) \) \( (i = 1, 2, \cdots) \); this means that

\[
\bar{V}'_i = (V_iU/U) \quad (i = 1, 2, \cdots) \quad (8.56)
\]

Corollary 8.2. implies that \( gr_{\bar{\rho}_1}(\bar{KV}) \cong gr_{\bar{\rho}}(KW)[\bar{\theta}] \) where \( \bar{\theta} \) is the homogeneous component of \( \bar{s} - 1 \) in \( gr_{\bar{\rho}_1}(\bar{KV}) \) and it has the same minimal polynomial \( t^p - \phi(a - 1) \) as the element \( \bar{t} \) in (8.47). We see that \( gr_{\bar{\rho}_1}(\bar{KV}) \cong gr_{\bar{\rho}}(\bar{KV}) \).

We obtained two extensions, \( \bar{\rho} \) and \( \bar{\rho}_1 \), of the valuation \( \bar{\rho}(KW) \) on \( KV \) with the same associated graded ring. Proposition 8.2. implies that \( \bar{\rho}_1 = \bar{\rho} \). We obtain from this that \( \bar{V}'_i = \bar{V}_i \) \( (i = 1, 2, \cdots) \) for the \( p \)-series \( \bar{V}_i \) \( (i = 1, 2, \cdots) \) defined by \( \rho \). This together with (8.56) implies that relation (8.49) holds and the proof is complete.

\section*{§9. Examples.}

We construct now two following examples.
Example 1. Let $H$ be an infinite cyclic group. We will construct a $p$-series $H_i$ ($i = 1, 2, \cdots$) such that the algebra $L_p(H, H_i)$ is not finitely generated. We define a weight function $w(h)$ on $H$ as follows. The elements of $H/H^p$ have weight 1, the elements of $H^p/H^{p^2}$ have weight $p + 1$, the elements of $H^{p^2}/H^{p^3}$ have weight $p(p + 1) + 1$; if the weight of the elements from $H^{p^n}/H^{p^{n+1}}$ is $w_n$ then the weight of the elements from $H^{p^{n+1}}/H^{p^{n+2}}$ is $pw_n + 1$.

This weight function defines a $p$-series $H_i$ ($i = 1, 2, \cdots$) in $H$ and we obtain that if $h \in H$ then $h^p = 0$, and $L_p(H, H_i)$ is a restricted infinite dimensional abelian Lie algebra of exponent $p$ over $Z_p$.

Example 2. We will construct now an example of a free abelian group of rank 2 which contains a $p$ series $H_i$ ($i = 1, 2, \cdots$) such that the algebra $L_p(H, H_i)$ is free abelian of rank 1.

We consider the ring of polynomials $Z_p[t]$ over the field $Z_p$. Let $R$ be the ring of fractions of $Z_p[t]$ with respect to the complement of the ideal $(t)$. Then $R$ has an ideal $(t)$. We define a $t$-adic valuation $\rho$ in $R$. The graded ring $gr_\rho(R)$ is isomorphic to the polynomial ring $Z_p[t]$. We pick the polynomials $p_1[t] = 1 + t, p_2[t] = 1 + t + t^p$. Every polynomial in $Z_p[t]$ has a unique representation as a product of irreducible polynomials and $p_2[t]$ is not divisible by $1 + t$ we obtain that these polynomials freely generate a free abelian semigroup in the ring $Z_p[t]$. Every element $p_i[t] (i = 1, 2)$ is invertible in the ring $R$, and we conclude easily, once again from the uniqueness of factorization in $Z_p[t]$, that the elements $u = p_1[t]$ and $v = p_2[t]$ freely generate a free abelian subgroup $H$ of the group of units of $R$.

Proposition 2.10. implies that the valuation $\rho$ defines a $p$-series $H_i$ ($i = 1, 2, \cdots$) in $H$, and the homogeneous components $(h - 1)$ ($h \in H$) generate in the algebra $gr_\rho(R) \cong Z_p[t]$ a subalgebra isomorphic to the algebra $L_p(H, H_i)$. This algebra is abelian; we obtain from Corollary 2.6. that it contains no nilpotent elements because the graded ring $gr_\rho(R)$ is a domain, so it is free abelian.

We show now that that the rank of $L_p(H, H_i)$ is 1.

Since the homogeneous components of $p_1[t] - 1$ and $p_2[t] - 1$ in $gr_\rho(R)$ are equal to $t$ we obtain that $\tilde{u} = \tilde{v} = t$. Let $h$ be a non-unit element of $H$, $h = u^n v^m$. The homogeneous components of the elements $u^n - 1$ and $v^m - 1$ are $nt$ and $mt$ respectively. Since $u^n v^m - 1 = (u^n - 1) + (v^m - 1) + (u^n - 1)(v^m - 1)$ we obtain that if $n \neq -m$ then the homogeneous component of the element $h - 1$ is $(n + m)t$. 

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Consider the case when \( n = -m \). We have to calculate in \( R \) the homogeneous component of the element

\[
u^n v^{-n} - 1 = (1 + t)^n (1 + t + t^p)^{-n} - 1 = ((1 + t)^n - (1 + t + t^p)^n)(1 + t + t^p)^{-n} \quad (9.1)
\]

The element \((1 + t + t^p)\) has \( \rho \)-value zero so the homogeneous component of \((1 + t + t^p)^{-n}\) is zero, because of this the homogeneous component of element (9.1) coincides with the homogeneous component of \((1 + t)^n - (1 + t + t^p)^n\). We consider first the subcase when \((n, p) = 1\) and replace the term \((1 + t + t^p)^n\) by its binomial expansion and obtain that the homogeneous component of the element \((1 + t)^n - (1 + t + t^p)^n\) coincides with the homogeneous component of \(-n(1 + t)^{n-1}t^p\) which is equal to \(-nt^p\).

In the general case, we have \( n = p^k n_1 \) where \((n_1, p) = 1\). Since the homogeneous component of the element \(u^{n_1} v^{n_1} - 1\) is \(-n_1 t^p\) the homogeneous component of the element \(u^n v^{-n} - 1\) is \((-n_1 t^p)^{p^k} = -n_1 t^{p^{k+1}}\).

We obtained that the homogeneous components \( \tilde{h} (h \in H) \) generate in \( gr(R) \) the free abelian restricted Lie subalgebra with the generator \( t \). This proves that the algebra \( L_p(H,H_i) \) is free abelian of rank 1.
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