Extending finite group actions on surfaces over $S^3$

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Abstract

Let $OE_g$ (resp. $CE_g$ and $AE_g$) be the maximum order of finite (resp. cyclic and abelian) groups $G$ acting on the closed orientable surfaces $\Sigma_g$ which extend over $(S^3, \Sigma_g)$ among all embeddings $\Sigma_g \to S^3$ and resp. unknotted embeddings $\Sigma_g \to S^3$.

It is known that $OE_g \leq 12(g - 1)$, and we show that $12(g - 1)$ is reached for an unknotted embedding $\Sigma_g \to S^3$ if and only if $g = 2, 3, 4, 5, 6, 9, 11, 17, 25, 97, 121, 241, 601$. Moreover $AE_g$ is $2g + 2$; and $CE_g$ is $2g + 2$ for even $g$, and $2g - 2$ for odd $g$.

Efforts are made to see intuitively how these maximal symmetries are embedded into the symmetries of the 3-sphere$^1$.

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1 Introduction

In this article, we use $\Sigma_g$ to denote the orientable closed surface of genus $g > 1$, and use $V_g$ to denote the handlebody of genus $g > 1$. All group actions will be faithful and orientation-preserving (on both surfaces and 3-manifolds).

Let $O_g$ (resp. $C_g$ and $A_g$) be the maximum order of all finite (resp. cyclic and abelian) groups $G$ which can act on $\Sigma_g$. A classical result of Hurwitz...
states that \(O_g\) is at most \(84(g-1)\), proved by applying the Riemann-Hurwitz formula. However for each fixed \(g\), \(O_g\) is hard to determine in general, see [Ac], [Ma] for some partial results. On the other hand \(C_g\) is \(4g+2\), see [St], also [Ha], and [Wa] for more direct constructions; and \(A_g\) is \(4(g+1)\), see [Ma].

Let \(OH_g\) (resp. \(CH_g\) and \(AH_g\)) be the maximum order of all finite (resp. cyclic and abelian) groups \(G\) which can act on the handlebody \(V_g\). By definition we have that \(O_g \geq OH_g, C_g \geq CH_g\) and \(A_g \geq AH_g\). It is a result due to Zimmermann [Zi1] that \(OH_g \leq 12(g-1)\), see also [Zi2], [MMZ], [MZ]. A handlebody orbifold theory was derived in [MMZ], see also [Zi3] for a more geometric approach, and then proved that \(CH_g\) is \(2g+2\) when \(g\) is even, and \(2g\) when \(g\) is odd [MMZ]. Moreover \(OH_g\) is bounded below by \(4(g+1)\), and \(OH_g\) is either \(12(g-1)\) or \(8(g-1)\) if \(g\) is odd, and each of these is achieved by infinitely many odd \(g\) [MZ].

In the present article, we consider the following extension problems: Suppose a finite group \(G\) acts on the surface \(\Sigma_g\). If there is an embedding \(e : \Sigma_g \subset S^3\) such that \(G\) can act on the pair \((S^3, \Sigma_g)\) and the restriction to \(\Sigma_g\) is the given \(G\) action on \(\Sigma_g\), we call the action of \(G\) on \(\Sigma_g\) extendable (over \(S^3\) with respect to \(e\)).

Call an embedding \(e_o : \Sigma_g \to S^3\) unknotted if each component of \(S^3 \setminus \Sigma_g\) is a handlebody. So each extendable \(G\) action w.r.t. an unknotted embedding \(e_o\) provides a \(G\)-invariant Heegaard splitting of \(S^3\). Similarly we define an action of \(G\) on \(V_g\) to be extendable, and an embedding \(e_o : V_g \to S^3\) to be unknotted if the complement \(S^3 \setminus V_g\) is also a handlebody. For each \(g\), an unknotted embedding is unique up to isotopy of \(S^3\) and automorphisms on \(\Sigma_g\) (resp. \(V_g\)).

In such extension problems, we first study the maximum orders in the present paper. Let \(OE_g\) (resp. \(CE_g\) and \(AE_g\)) be the maximum order of all extendable finite (resp. cyclic and abelian) groups \(G\) acting on \(\Sigma_g\). It is not obvious that \(OH_g \geq OE_g, CH_g \geq CE_g\) and \(AH_g \geq AE_g\). So finer notions may be useful at the moment: Let \(OE^o_g\) (resp. \(CE^o_g\) and \(AE^o_g\)) be the maximum order of a finite (resp. cyclic and abelian) group \(G\) acting on \(\Sigma_g\) which extends over \(S^3\) w.r.t. an unknotted embedding. Then

1. \(OE_g \geq OE^o_g, CE_g \geq CE^o_g\) and \(AE_g \geq AE^o_g\).
2. \(OH_g \geq OE^o_g, CH_g \geq CE^o_g\) and \(AH_g \geq AE^o_g\).

Now we are going to describe the results and the organization of the paper.

Even to determine \(OE_g\) and \(OE^o_g\) are harder, some discussions are made in Section 2. It is clear \(OE^o_g \leq 12(g-1)\). We show that there are only finitely many \(g\) such that \(OE^o_g = 12(g-1)\), and indeed we list all such \(g\) (Theorem 2.1, see also the Examples in Section 4). It is derived that for each \(g\), \(4(g+1)\) is a lower bound for \(OE^o_g\) (Example 4.3), and for each \(g = n^2\) a lower bound of \(OE^o_g\) is \(4(n+1)^2\) which is larger than \(4(g+1)\) (Example 4.4).

In Section 3, we discuss the abelian case and the cyclic case which are easier. By applying the handlebody orbifold theory, we will first derive the needed information about orders of large abelian and cyclic group action
on $V_g$ (Theorem 3.1 and Theorem 3.2). Then we show that $AE_g$ is $2g+2$ (Theorem 3.3), and $CE_g$ is $2g+2$ for even $g$, and $2g-2$ for odd $g$ (Theorem 3.5). All these maximum order group actions are realized by unknotted embedding (Examples 4.1 and 4.2), hence $CE^o_g = CE_g$ and $AE^o_g = AE_g$.

**Question 1.** If an embedding $\Sigma_g \to S^3$ realizes $OE_g$, should the embedding be unknotted? Weakly does $OE_g = OE^o_g$ for each $g > 1$?

The existence of extendable group actions on surfaces with large symmetry presented in Sections 2 and 3 are mostly derived from the orbifold theory. On the other hand in the process of this work, most large symmetries in Sections 2 and 3 are first constructed in a more direct and intuitive way without using orbifold theory. Section 4 presents those constructions which show us how those symmetries on surfaces stay in the symmetries on 3-sphere. A reason of doing so is given in the next paragraph.

We end the introduction by the some motivations of our study: Surfaces (as well as handlebodies) are very familiar subjects to us, mostly because we can see them staying in our 3-space in various manners. The symmetries of the surfaces have been studied for a long time, and it will be natural to wonder when these symmetries can be embedded into the symmetries of our 3-space (3-sphere). Another inspiring fact is a related problem on extending surface automorphisms over 4-space which had been addressed 30 years ago [Mo], and for recent developments see [Hi], [DLWY] and [LNSW].

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## 2 Maximum order of extendable group action

From now on all groups in this paper will be finite.

A simple picture we should keep in mind is the following: Suppose that the action of $G$ on $\Sigma_g$ is extendable w.r.t. some embedding $\Sigma \subset S^3$. Let $\Gamma = \{x \in S^3 | \exists g \in G, s.t. gx = x\}$; then $\Gamma$ is a graph, possibly disconnected, and $S^3/G$ is a 3-orbifold whose singular set $\Gamma/G$ is also a graph. Each edge of $\Gamma/G$ can be labeled by an integer $> 0$ which corresponds to the singular index of it. Also, $\Sigma_g/G$ is a 2-orbifold with singular set $\Sigma_g/G \cap \Gamma/G$, which are isolated points.

We recall the handlebody orbifold theory from [MMZ], [Zi3].

Let $G$ be a finite group acting on a handlebody of genus $g$. Associated to this action there is a handlebody orbifold $V_g/G$, a finite graph of finite groups $(\Gamma, \mathcal{G})$ and a surjection $\phi : \pi_1(\Gamma, \mathcal{G}) \to G$ whose kernel is isomorphic to a free group of rank $g$; in particular, $\phi$ is injective on the finite vertex groups of $(\Gamma, \mathcal{G})$. Here $\pi_1(\Gamma, \mathcal{G})$ denotes the fundamental group of the graph of groups or of the corresponding handlebody orbifold: this is the iterated free product with amalgamation and HNN-extension over the vertex groups, amalgamated over the edge groups of a maximal tree, with
the HNN-generators corresponding to the edges in the complement of the chosen maximal tree. We denote by

$$\chi = \chi(\Gamma, G) = \sum 1/|G_v| - \sum 1/|G_e| \quad (2.1)$$

the Euler characteristic of the graph of groups $(\Gamma, G)$ (the sum is taken over all vertex groups $G_v$ resp. edge groups $G_e$ of $(\Gamma, G)$; then

$$g - 1 = |G|(-\chi) \quad (2.2).$$

The vertex groups $G_v$ belong to one of the following five classes which correspond to the five types of finite subgroups of the orthogonal group $SO(3)$ given in Figure 1. The edge groups $G_e$ are cyclic groups which are either trivial or maximally cyclic in the adjacent vertex groups. We can also assume that the edge group of an edge which is not a loop (not closed) does not coincide with one of the two vertex groups of the edge.

![Figure 1](cyclic_dihedral_tetrahedral_octahedral_icosahedral)

Conversely, to each such finite graph of finite groups associated to a handlebody orbifold and surjection $\phi : \pi_1(\Gamma, G) \to G$, there is a corresponding action of $G$ on a handlebody $V_g$ of genus $g$.

**Theorem 2.1.** Suppose the $G$-action on $\Sigma_g$ is extendable over $S^3$ w.r.t. the unknotted embedding $e_o$ and the order of $G$ is $12(g-1)$. Then $g$ is as follows:

$$g = 2, 3, 4, 5, 6, 9, 11, 17, 25, 97, 121, 241, 601.$$  

**Proof.** The values of $g$ can be obtained as follows.

![Figure 2](p=2, 3, 4, or 5)

$$p, q=2, 3, 4, \text{ or } 5$$

![Figure 2](sigma)

$$\sigma$$

Figure 2
The unknotted embedding $\Sigma_g \rightarrow S^3$ provides a Heegaard splitting $S^3 = H_1 \cup \Sigma_g H_2$ such that each handlebody $H_i$ is invariant under the extendable $G$-action on $(S^3, \Sigma_g)$. Since $|G| = 12(g-1)$, each handlebody orbifold $H_i/G$ has the underlying space $B^3$ and singular set as indicated in Figure 2 (a). Then the quotient-orbifold $S^3/G$ is $S^3$, the singular set is a 2-bridge link with the two standard unknotted tunnels of branching orders $p$ and $q$ added, with $p, q \leq 5$, as indicated in Figure 2(b); three of the strands of the 2-bridge link have branching order 2, the remaining one branching order 3, where $\sigma$ is a braid on 3 strands, and these are the orbifolds $O(\sigma; p, q)$, see [Zi3] for details.

All spherical 3-orbifolds $S^3/G$ with underlying space $|S^3/G| = S^3$ are listed in Tables 6, 7, 8 of Dunbar’s paper [Du1] (pages 89-93): Each singular set is a graph with vertex valency at most 3, and each edge is labeled by an integer indicating the singular index of the edge, with the convention that each unlabeled edge has index 2. Each small box encoded by an integer $k$ indicates two parallel arcs with $k$-half twists in the box, and each small box encoded by two integers $m, n$ indicates a rational tangle given by $(m, n)$ with a “strut” connecting the two arcs of the tangle and labeled by the largest common divisor of $m$ and $n$.

Now we use Dunbar’s list [Du1] to see which of the orbifolds in Figure 2 are spherical. There are two cases.

(1) If not both $p$ and $q$ are equal to 2 then there is a singular point which is not dihedral, so it is of type $A_4$ (tetrahedral), $S_4$ (octahedral) or $A_5$ (icosahedral). In Dunbar’s list, these are only the non-fibered orbifolds on page 93 of [Du1]. By further checking which graph can meet a 2-sphere $S^2$ with four singular points of indices $(2, 2, 2, 3)$ so that each side of $S^2$ is a handlebody orbifold, we have only nine graphs left which are listed in Figure 3. From [Du2], we also know the fundamental groups of these nine orbifolds, and we indicate the groups and their orders under each graph. Here $O$ denotes the orientation-preserving symmetry group of the octahedron, and $J$ the orientation-preserving symmetry group of the icosahedron; all the symbols are from [Du2].
From the relation $|G| = 12(g - 1)$, the cases one finds are $g = 3, 5, 11, 6, 17, 601, \text{again} 11, 97 \text{and} 241$.

(2) If both $p$ and $q$ are equal to 2, then an easy and more interesting way to find the cases is as follows. Take any 2-bridge link $L(\sigma)$ in $S^3$ and associate branching index 3 to each of its components, obtaining an orbifold $L_3(\sigma)$. It is shown in [MeZ] that such an orbifold $L_3(\sigma)$ has an orientation-preserving symmetry group $\mathbb{Z}_2 \times \mathbb{Z}_2$, and the quotient orbifold is exactly $O(\sigma; 2, 2)$, for the same 2-bridge knot defined by $\sigma$. So if $O(\sigma; 2, 2)$ is spherical, also $L_3(\sigma)$ is spherical and the singular set is just a 2-bridge link now. The spherical orbifolds whose singular set is just a link are listed on pages 89-92 of [Du1]. When we restrict to the two bridge links such that each component has index 3, then only five links are left which are listed in Figure 4, and all of them are very simple torus links.

Let $\tilde{L}_3(\sigma)$ be the 3-fold cyclic branched covering of $L_3(\sigma)$ so that $\tilde{L}_3(\sigma)$ has no singularity. Then we have the orbifold covering

$$\tilde{L}_3(\sigma) \rightarrow L_3(\sigma) \rightarrow O(\sigma; 2, 2)$$

of degree 12.

For all $L_3(\sigma)$ in Figure 4, $\tilde{L}_3(\sigma)$ are well-known spherical Seifert fiber spaces: The first is $S^3$, the second the lens space space $L(3, 1)$, the third is the quaternion manifold, the fourth is the 3-manifold with binary tetrahedral fundamental group, and the fifth is the Poincare homology 3-sphere (indeed these five 3-manifolds are exactly the $k$-fold cyclic branched covering over the trefoil knot, for $k = 1, 2, 3, 4 \text{and} 5$; see [Ro], pp 304-309 for a discussion of the cyclic branched coverings of the trefoil). The fundamental groups of these orbifolds $\tilde{L}_3(\sigma)$ have orders 1, 3, 8, 24 and 120, so for the corresponding orbifolds $O(\sigma; 2, 2)$ one has orders 12, 36, 96, 288 and 1440; applying again $|G| = 12(g - 1)$ we obtain the genera 2, 4, 9, 25 and 121.

We end this section by a lemma which should be useful to study Question 1, and has also some applications in Section 4.

**Lemma 2.2.** Suppose a group $G$ acts on $(M, F)$ where $M$ is a 3-manifold and $F \subset M$ a surface, so we have the diagrams:

\[
\begin{array}{ccc}
F & \xrightarrow{i} & M \\
\downarrow{p} & & \downarrow{p} \\
F/G & \xrightarrow{\hat{i}} & M/G \\
\end{array}
\quad
\quad
\begin{array}{ccc}
\pi_1(F) & \xrightarrow{i_*} & \pi_1(M) \\
\downarrow{p_*} & & \downarrow{p_*} \\
\pi_1(F/G) & \xrightarrow{\hat{i}_*} & \pi_1(M/G) \\
\end{array}
\]

Suppose $F/G$ is connected. Then $F$ is connected if

$$\hat{i}_*(\pi_1(F/G)) \cdot p_*(\pi_1(M)) = \pi_1(M/G).$$
Proof. Suppose $F$ is not connected. Let $F_1 \subseteq F$ be a component of $F$ and $G_1$ its stabilizer in $G$, that is $G_1 = \{ h \in G | h(F_1) = F_1 \}$. Then $F_1 / G_1 = F / G$. Now $|\pi_1(M/G) : p_*(\pi_1(M))| = |G|$, and

\[
|\hat{i}_*(\pi_1(F/G)) : p_*(\pi_1(M))| = |\hat{i}_*(\pi_1(F/G)) : p_*(\pi_1(M))| / p_*(\pi_1(M))| \\
= |\hat{i}_*(\pi_1(F/G)) : p_*(\pi_1(M))| / p_*(\pi_1(M))| \\
= |\hat{i}_*(\pi_1(F/G)) : p_*(\pi_1(M))| / p_*(\pi_1(M))| \\
\leq |\hat{i}_*(\pi_1(F/G)) : p_*(\pi_1(M))| \\
= |\pi_1(F/G) / \hat{i}_* : p_*(\pi_1(F_1)) / \hat{i}_*| \\
= |\pi_1(F_1 / G_1) : p_*(\pi_1(F_1)) / \hat{i}_*| \\
\leq |\pi_1(F_1 / G_1) : p_*(\pi_1(F_1))| \\
= |G_1| < |G|.
\]

Hence $\hat{i}_*(\pi_1(F/G)) : p_*(\pi_1(M)) \subseteq \pi_1(M/G)$. \qed

Remark In Lemma 2.2, if $M$ is $S^3$, then $F$ is connected if $\hat{i}_*$ is surjective.

3 Maximum orders of extendable abelian and cyclic groups

To get the maximum orders $AE_g$ and $CE_g$, we first need some information about actions of abelian groups and cyclic groups on handlebodies which are contained in the following Theorem 3.1 and Theorem 3.2. Some facts in Theorem 3.1 and Theorem 3.2 have been either explicitly or implicitly stated, with or without proofs, in [MMZ], [MeZ]. For our later applications, we reorganize them into our statement and provide proofs.

We define two actions of a finite group $G$ to be equivalent if the corresponding groups of homeomorphisms of $V_g$ are conjugate (i.e., allowing isomorphisms of $G$).

**Theorem 3.1.** The largest order of a finite abelian group $G$ acting on the handlebody $V_g$ of genus $g \geq 2$ is $2(g + 1)$ if $g \neq 5$, and 16 if $g = 5$. The groups $G$ which realize the maximum orders are $\mathbb{Z}_2 \times \mathbb{Z}_{g+1}$ if $g \neq 5$, $(\mathbb{Z}_2)^4$ if $g = 5$, and in addition $(\mathbb{Z}_2)^3$ if $g = 3$. Moreover,

(i) there is one equivalence class for each of the groups $\mathbb{Z}_2 \times \mathbb{Z}_{g+1}$ and $(\mathbb{Z}_2)^4$ whereas there are three equivalence classes for the group $(\mathbb{Z}_2)^3$ acting on $V_3$;

(ii) no abelian group of order larger than 12 acts on $V_5$ except $(\mathbb{Z}_2)^4$.

Proof. Suppose that $G$ is an abelian group as in Theorem 3.1. Then $g \geq 2$ implies $-\chi > 0$ by (2.2). Also, every vertex group of $(\Gamma, G)$ is either cyclic or isomorphic to the dihedral group $(\mathbb{Z}_2)^2$ of order 4, since these are the finite abelian subgroups of $SO(3)$; then the group of every edge which is not a loop is either trivial or $\mathbb{Z}_2$, and in the second case the two adjacent vertex group are $(\mathbb{Z}_2)^2$.

Note that, if $|G| \geq 2g - 1$, then $-\chi = (g - 1)/|G| \leq (g - 1)/(2g - 1) < 1/2$. We will assume that $-\chi < 1/2$ in the following and discuss all possibilities for $(\Gamma, G)$ and $G$. The discussion is divided into two cases:
(I) Suppose first that \((\Gamma, \mathcal{G})\) has no vertex group \((\mathbb{Z}_2)^2\); then all vertex groups are cyclic. Let

\[ E = \{ \text{edge } e \in \Gamma|G_e \text{ is non-trivial}\}; \]

then all edges in \(E\) must be loops. Let \(\Gamma_0 = \Gamma - E\); if we view \(\Gamma_0\) as a usual graph, we denote it by \(|\Gamma_0|\). It is easy to see \(-\frac{1}{2} < \chi(\Gamma) \leq \chi(\Gamma_0) \leq \chi(|\Gamma_0|)\). However \(\chi(|\Gamma_0|)\) is an integer, so \(\chi(|\Gamma_0|) = 1\) or 0. Suppose \(\Gamma_0\) has \(k\) non-trivial vertices; then \(\chi(\Gamma_0) \leq \chi(|\Gamma_0|) - \frac{k}{2}\). So if \(\chi(|\Gamma_0|) = 0\), it is easy to see \(\chi(\Gamma) \notin (-\frac{1}{2}, 0)\). So we must have \(\chi(|\Gamma_0|) = 1\), which means \(|\Gamma_0|\) must be a tree. Notice that every end (degree-one vertex) of \(\Gamma_0\) must be non-trivial, hence the same reason as above shows that \(\Gamma_0\) is equal to some \(\Gamma(\mathbb{Z}_{n_1}, 1, \mathbb{Z}_{n_2})\) (consisting of one edge with trivial edge group, and two vertices). Furthermore, \((n_1, n_2) \in \{(3, 5), (3, 4), (3, 3), (2, n)\}, n \geq 3, \) and \(\Gamma\) must equal to \(\Gamma_0\). So the only possibilities for \(\Gamma\) are the graphs of groups \(\Gamma(\mathbb{Z}_2, 1, \mathbb{Z}_n)\), with \(-\chi = (n - 2)/2n\), and \(\Gamma(\mathbb{Z}_3, 1, \mathbb{Z}_n)\), for \(n = 3, 4, 5\), and \(-\chi = 1/3, 5/12\) or \(7/15\). The only finite abelian groups onto which the free product \(\pi_1(\Gamma, \mathcal{G})\) surjects with torsionfree kernel are the groups \(\mathbb{Z}_2 \times \mathbb{Z}_2\) and, if \(n\) is even, \(\mathbb{Z}_n\). In the first case we have \(g = n - 1\) and \(|G| = 2n = 2(g + 1)\), so for all genera \(g\) the order \(2g + 1\) of the Theorem is achieved for the group \(\mathbb{Z}_2 \times \mathbb{Z}_{g+1}\). In the three other cases, the possibilities for \(G\) are the groups \(\mathbb{Z}_3\) and \((\mathbb{Z}_2)^2, \mathbb{Z}_{12}\) or \(\mathbb{Z}_{15}\), and in each of these cases one has \(|G| < 2(g + 1)\).

(II) Suppose now that \((\Gamma, \mathcal{G})\) has some vertex group \((\mathbb{Z}_2)^2\). Let

\[ E = \{ \text{edge } e \in \Gamma|G_e \text{ is non-trivial, and the ends of } e \text{ are both cyclic groups}\}; \]

also all edges in \(E\) must be loops. Setting \(\Gamma_0 = \Gamma - E\), every non-trivial edge in \(\Gamma_0\) must have both ends \((\mathbb{Z}_2)^2\). We now also have \(-\frac{1}{2} < \chi(\Gamma) \leq \chi(\Gamma_0) \leq \chi(|\Gamma_0|)\). We explain the last inequality: suppose \(\Gamma_0\) has \(l\) vertices of type \((\mathbb{Z}_2)^2\); then \(\Gamma_0\) has no more than \(\frac{3l}{2}\) non-trivial edges, so \(\chi(\Gamma_0) \leq \chi(|\Gamma_0|) - \frac{3l}{4} + \frac{1}{2} \cdot \frac{3l}{4} = \chi(|\Gamma_0|)\). We also have \(\chi(|\Gamma_0|) = 1\) or 0. Now for a cyclic end, \(\chi(\Gamma_0)\) must decrease at least by \(-\frac{1}{2}\) compared with \(\chi(|\Gamma_0|)\). For a \((\mathbb{Z}_2)^2\) end, \(\Gamma_0\) has no more than \(\frac{3l - 2}{2}\) non-trivial edges, and this time \(\chi(\Gamma_0) \leq \chi(|\Gamma_0|) - \frac{3l}{4} + \frac{1}{2} \cdot \frac{3l - 2}{4} = \chi(|\Gamma_0|) - \frac{1}{2}\), so it also decrease at least by \(-\frac{1}{2}\). This argument show that \(\Gamma_0\) has at most 2 ends.

(1) If \(\chi(|\Gamma_0|) = 0\), \(\Gamma_0\) can even have no ends at all, so it is a loop divided by some \((\mathbb{Z}_2)^2\) vertices. The only possibility for \((\Gamma, \mathcal{G})\) is a graph of groups with exactly one vertex and one edge (a loop), with \(\chi = -1/4\); however, since the HNN-generator of \(\pi_1(\Gamma, \mathcal{G})\) corresponding to the loop has to conjugate a subgroup \(\mathbb{Z}_2\) of the vertex group \((\mathbb{Z}_2)^2\) into a different subgroup \(\mathbb{Z}_2\) (see [MMZ] or [Zi3]), its fundamental group does not surject onto an abelian group and this case does not occur.

(2) If \(\chi(|\Gamma_0|) = 1\), \(\Gamma_0\) is a segment. The segment may have inner vertices, but every inner vertex must be \((\mathbb{Z}_2)^2\). For every such inner vertex, \(\Gamma_0\) has no more than \(\frac{3l - 1}{2}\) non-trivial edges, and this time \(\chi(\Gamma_0) \leq \chi(|\Gamma_0|) - \frac{3l}{4} + \frac{1}{2} \cdot \frac{3l - 1}{4} = \chi(|\Gamma_0|) - \frac{1}{4}\). So there is at most one inner vertex. If there is no inner vertex, it is easy to see that \((\Gamma, \mathcal{G})\) is equal to \(\Gamma((\mathbb{Z}_2)^2, 1, \mathbb{Z}_2)\), with \(\chi = 1/4\), or to \(\Gamma((\mathbb{Z}_2)^2, 1, \mathbb{Z}_3)\), with \(\chi = 5/12\); the possibilities for \(G\) are the groups \((\mathbb{Z}_2)^2\) and \((\mathbb{Z}_2)^3\) in the first case, and \(\mathbb{Z}_2 \times \mathbb{Z}_6\) in the second one, and only
for the group \((\mathbb{Z}_2)^3\), with \(g = 3\), the bound \(|G| = 2(g + 1)\) of the Theorem is obtained. The last case is the graph of groups \(\Gamma((\mathbb{Z}_2)^2, \mathbb{Z}_2, (\mathbb{Z}_2)^2, \mathbb{Z}_2, (\mathbb{Z}_2)^2)\), with two edges and three vertices and \(-\chi = 1/4\), whose fundamental group surjects onto \((\mathbb{Z}_2)^n\) for \(n = 2, 3\) and 4; the group \((\mathbb{Z}_2)^4\) realizes the maximum order 16 for \(g = 5\) of the Theorem, whereas the group \((\mathbb{Z}_2)^3\) realizes again the maximum order \(8 = 2(g + 1)\) for \(g = 3\).

There is only one finite-injective surjective map from \(\mathbb{Z}_2 \times \mathbb{Z}_{g+1}\) or from \(\Gamma((\mathbb{Z}_2)^2, \mathbb{Z}_2, (\mathbb{Z}_2)^2, \mathbb{Z}_2, (\mathbb{Z}_2)^2)\) to an abelian group. There are two finite-injective surjective map from \(\Gamma((\mathbb{Z}_2)^2, \mathbb{Z}_2, (\mathbb{Z}_2)^2, \mathbb{Z}_2, (\mathbb{Z}_2)^2)\) to \((\mathbb{Z}_2)^3\) (either all three vertex groups \((\mathbb{Z}_2)^2\) are mapped to different subgroups of \((\mathbb{Z}_2)^3\), or two vertex groups are mapped to the same subgroup), and one finite-injective surjective map from \(\Gamma((\mathbb{Z}_2)^2, 1, \mathbb{Z}_2)\) to \((\mathbb{Z}_2)^3\). These show the result in (i). For (ii), an abelian group of order 13, 14 or 15 must be a cyclic group, and the above orbifolds can not finite-injectively surject to such a cyclic group when \(g = 5\).

This completes the proof of Theorem 3.1.

For cyclic groups, the proof of Theorem 3.1 implies also the following:

**Theorem 3.2.** Let \(G\) be a finite cyclic group acting on a handlebody of genus \(g \geq 2\). If \(G\) has order at least \(2g - 2\) then \(G\) is one of the following groups:

1. \(\mathbb{Z}_{2g+2}\) if \(g\) is even, associated to a surjection \(\mathbb{Z}_2 \ast \mathbb{Z}_{g+1} \to \mathbb{Z}_{2(g+1)}\);
2. \(\mathbb{Z}_g\) for all \(g\), associated to a surjection \(\mathbb{Z}_2 \ast \mathbb{Z}_g \to \mathbb{Z}_g\) and, for \(g = 6\), also to \(\mathbb{Z}_3 \ast \mathbb{Z}_4 \to \mathbb{Z}_{12}\);
3. \(\mathbb{Z}_{2g-1}\) for \(g = 2\) and 8, associated to surjections \(\mathbb{Z}_3 \ast \mathbb{Z}_3 \to \mathbb{Z}_3\) and \(\mathbb{Z}_3 \ast \mathbb{Z}_5 \to \mathbb{Z}_{15}\);
4. \(\mathbb{Z}_{2g-2}\) for all \(g\); for each \(g\) the graphs of groups are \(\Gamma((\mathbb{Z}_2, 1, \mathbb{Z}_n))\) with an additional loop with edge group \(\mathbb{Z}_n\) attached to the vertex of type \(\mathbb{Z}_n\), for each \(n \geq 1\) which divides \(2g - 2\); in addition, for \(g = 3\) and 2 there are actions associated to \(\mathbb{Z}_4 \ast \mathbb{Z}_4 \to \mathbb{Z}_4\) and \(\mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2 \to \mathbb{Z}_2\).

**Proof.** The cases \(|G| \geq 2g - 1\) are covered by the proof of Theorem 3.1. For \(|G| = 2g - 2\) one has to consider in addition graphs of groups with \(-\chi = 1/2\). If \(\Gamma_0\) has at least three non-trivial vertices, it gives the case \(\mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2 \to \mathbb{Z}_2\). Or \(\Gamma_0 = \Gamma((\mathbb{Z}_{n_1}, 1, \mathbb{Z}_{n_2}))\) and there are two more cases: \((n_1, n_2) = (4, 4)\), or \(\Gamma\) is obtained from \(\Gamma_0\) by adding a non-trivial loop as stated in (4).

**Theorem 3.3.** \(AE_g = 2g + 2\).

**Proof.** Suppose an abelian group \(G\) acts on \(\Sigma_g\) which is extendable over \(S^3\) for some embedding \(e : \Sigma_g \hookrightarrow S^3\). Then the action of \(G\) extends to each 3-manifold of \(S^3 \setminus \Sigma_g\). According to [RZ, Theorem 2], for each abelian group \(G\), the \(G\)-action on \(\Sigma_g\) extends to a compact 3-manifold \(M\) with \(\partial M = \Sigma_g\) if and only if the \(G\)-action on \(\Sigma_g\) extends to a handlebody \(V_g\) with \(\partial V_g = \Sigma_g\). Therefore we have \(AE_g \leq AH_g\).

It is a general fact of Smith fixed point theory that the finite 2-group \((\mathbb{Z}_2)^{(n+1)}\) does not act orientation-preservingly on a mod 2 homology \(n\)-sphere, see [Sm]. So a \((\mathbb{Z}_2)^4\)-action on \(F_3\) does not extend to an action on \(S^3\). (Alternatively, by the confirmation of the geometrization conjecture one
can use the fact that every finite group acting orientation-preservingly on $S^3$
can be conjugated into $SO(4)$, and $SO(4)$ contains no subgroup isomorphic
to $(\mathbb{Z}_2)^4$.) Now applying Theorem 3.1, we have indeed $AE_g \leq 2(g + 1)$ for
each $g > 1$.

By Example 4.1, for every $g > 1$ there is an abelian group $G \cong \mathbb{Z}_2 \times \mathbb{Z}_{g+1}$
which acts on $\Sigma_g$, and this action extends to a $G$-action on $S^3$ for the
unknotted embedding of $\Sigma_g \subset S^3$. Hence $AE_g^o \geq 2(g + 1)$.

Then

$$2(g + 1) \leq AE_g^o \leq AE_g \leq 2(g + 1).$$

So we have $AE_g = AE_g^o = 2(g + 1)$. and Theorem 3.3 is proved.

The following fact proved without using Smith theory is of independent
interest.

**Lemma 3.4.** Some order 2 element of the $(\mathbb{Z}_2)^4$ action on $V_5$ is not extend-
able.

![Figure 5](image)

*Figure 5*

**Proof.** We first describe a geometric model of the action of $(\mathbb{Z}_2)^4$ on $V_5$.

As in Figure 5, we view $V_5$ as four 3-balls $\{A, B, C, D\}$ with 8 handles
$\{a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2\}$ attached; here, for simplicity, we draw each han-
dle as an arc, and the four attaching disks on each ball are at front, back,
top and bottom respectively.
We will describe the four generators \( \{\sigma_1, \sigma_2, \rho_1, \rho_2\} \) of the action of \( G \cong (\mathbb{Z}_2)^4 \); on each ball \( \sigma_1 \) is a \( \pi \)-rotation about the axis through the front and back points along the arrows shown as in Figure 6, which interchanges \( a_1 \) and \( a_2 \) (resp. \( c_1 \) and \( c_2 \)) but keeps each \( b_j \) and \( d_j \), \( j = 1, 2 \). Similarly \( \sigma_2 \in G \) is a \( \pi \)-rotation about the axis through the top and bottom points on each ball, which interchanges \( b_1 \) and \( b_2 \), (resp. \( d_1 \) and \( d_2 \)), but keeps each \( a_i \) and \( c_i \), \( j = 1, 2 \). The generator \( \rho_i \) is a \( \pi \)-rotation of the whole handlebody about the axis \( l_i \), \( i = 1, 2 \). One can check that \( \{\sigma_1, \sigma_2, \rho_1, \rho_2\} \) generates the abelian group \( G \). Note that \( \rho_1 \) and \( \rho_2 \) are extendable.

For any embedding \( V_5 \hookrightarrow S^3 \), we construct a link \( \{K_1, K_2\} \) in the handlebody \( V_5 \) as in Figure 6 where the image \( \sigma_1\{K_1, K_2\} \) is shown on the right hand side. The linking numbers locally differ by 1 in the upper-left ball. So \( \sigma_1 \) is not extendable.

**Theorem 3.5.** \( CE_g = 2g + 2 \) if \( g \) is even, and \( 2g - 2 \) if \( g \) is odd.

**Proof.** In Example 4.1 we shall describe a cyclic group action of order \( 2g + 2 \) on \( \Sigma_g \) which extends over \( S^3 \) for each even \( g > 1 \), and in Example 4.2 a cyclic group action of order \( 2g - 2 \) on \( \Sigma_g \) which extends over \( S^3 \) for each odd \( g > 1 \).

Hence \( CE_g \geq 2g + 2 \) for even \( g > 1 \), \( CE_g \geq 2g - 2 \) for odd \( g > 1 \). (3.3)

Suppose the \( G \) action on \( \Sigma_g \) is extendable. Still applying [RZ, Theorem 2], we have that the \( G \) action on \( \Sigma_g \) extends to \( (V_g, \partial V_g = \Sigma_g) \).

By Theorem 3.2 (1) for each even \( g \) we have \( AH_g = 2g + 2 \). By Theorem 3.2 (2) (3) for each odd \( g \) a cyclic group \( G \) of order \( |G| > 2g - 2 \) acting on \( V_g \) must be \( \mathbb{Z}_{2g} \), is associated to surjection \( \mathbb{Z}_2 * \mathbb{Z}_{2g} \rightarrow \mathbb{Z}_{2g} \).

**Claim:** The \( \mathbb{Z}_{2g} \) action on \( \partial V_g = \Sigma_g \) which is the restriction of the \( \mathbb{Z}_{2g} \) acts on \( V_g \) is not extendable.

With this Claim we have

\[
CH_g \leq 2g + 2 \quad \text{for even } g > 1, \quad CH_g \leq 2g - 2 \quad \text{for odd } g > 1. \tag{3.4}
\]

Combining (3.3) and (3.4), Theorem 3.5 is proved.

**Proof of the Claim.** We must have a close look on the \( \mathbb{Z}_{2g} \) action on \( V_g \). By the discussion made in [MMZ], [Zi3], the handlebody orbifold \( X = V_g / \mathbb{Z}_{2g} \) must consist of two 3-balls with singular arcs of indices 2 and \( 2g \), respectively, connected by a regular 1-handle as shown in the left hand side of Figure 7. Now the pre-image of the 3-ball with singular arc of index \( 2g \) is just an ordinary 3-ball \( B^3 \) in \( V_g \), the \( \mathbb{Z}_{2g} \)-action on it is a \( \frac{\pi}{2} \)-rotation, and the pre-image of the remaining part of the handlebody orbifold \( X \) in \( V_g \) consists just of \( g \) 1-handles attached to opposite \( \mathbb{Z}_{2g} \)-equivariant disks on \( B^3 \). The right hand side of Figure 7 is the case of \( V_3 \).

Hence the \( \mathbb{Z}_{2g} \) action on \( \Sigma_g = \partial V_g = S^2_+ \cup \{N_1, ..., N_g\} \) is obtained from the 2-sphere \( S^2_+ \) with \( 2g \) punctures by attaching \( g \) tubes \( N_1, ..., N_g \) along \( g \) pairs of opposite punctures.
Now we are going to give two different proofs that this $\mathbb{Z}_{2g}$ action on $\Sigma_g$ is not extendable.

**Proof 1.** The first one invokes Smith theory.

Suppose the $\mathbb{Z}_{2g}$ action on $\Sigma_g$ extends an action on $(S^3, \Sigma_g)$ for some embedding $\Sigma_g \to S^3$, and let $\sigma$ be a generator of this extension. Since $\sigma|S^2_0$ is a rotation of order $2g$ with two fixed points, the fixed point set of $\sigma$ is not empty. By Smith theory the fixed point set of the group $\langle \sigma \rangle$ acting on $S^3$ must be a circle $C$. It follows that the (singular or branching) index of $C \cap S^2_0$ must be $2g$, therefore the index of the whole circle $C$ must be $2g$, that is to say the whole $C$ is the fixed point set $\sigma$. On the other hand, $\sigma^g$ is a $\pi$-rotation on the tube $N_1$ which has two fixed points $x, y$, therefore $x \in C$ which implies that $\sigma$ has fixed points on $N_1$. Since $g > 1$, $\sigma$ sends the whole $N_1$ to $N_2$ which gives a contradiction.

**Proof 2.** The second one is elementary and using linking number only.

Choose an arc $\gamma$ on the boundary of the orbifold $X$, as showed in Figure 8. The pre-image of $\gamma$ consists of $g$ arcs $\gamma_i$, $i = 1, \ldots, g$ on the surface $\Sigma_g$, equivariant under the action of $G$. Let $D$ denote the upper hemisphere of the $2g$-punctured sphere $S^2_0$ described above. Then the boundary of $\gamma_i$ divides $\partial D$ into $2g$ arcs denoted by $\alpha_i$ and $\beta_i$ such that the $\frac{2\pi}{g}$-rotation maps $\alpha_i$ to $\beta_i$; see the right hand side of Figure 8.

Now consider the embedding $\Sigma_g \hookrightarrow S^3$, and let

$$K_1 = \gamma_i \cup \alpha_i, \ K_2 = \gamma_i \cup \beta_i$$

which are knots in $S^3$, see Figure 9.
Suppose the $G$-action extends to a $G$-action on $S^3$, and let $\sigma \in G$ be the generator of $G$ such that the restriction of $\sigma$ to $D$ is the $\frac{\pi}{g}$-rotation. Then we have $\sigma(K_1) = K_2$. The fixed point set of the periodic map $\sigma$ is not empty (it has a fixed point $x$ on $D$). Denote by $K_0$ the circle component of the fixed point set of $\sigma$ containing $x$. Now $\sigma\{K_0, K_1\} = \{K_0, K_2\}$.

Let us compute the mod 2 linking numbers $\text{lk}_2(K_0, K_1)$ and $\text{lk}_2(K_0, K_2)$ to reach a contradiction. Choose a standard embedded $S^2$ in $S^3$, which contains the disk $D$. Project each knot $K_i$ and $K_0$ to this $S^2$; we use this projected diagram to compute the linking numbers $\text{lk}_2(K_0, K_i)$: for each crossing, if the arc $K_0$ goes over the arc of $K_i$, then this crossing contributes 1 to $\text{lk}_2(K_0, K_1)$ and $\text{lk}_2(K_0, K_2)$, and if the arc $K_0$ goes under the arc of $K_i$ then this crossing contributes 0 to $\text{lk}_2(K_0, K_i)$ (this is just an application of the general linking number method, see [Ro] for details).

We notice that the only differences between $\text{lk}_2(K_0, K_1)$ and $\text{lk}_2(K_0, K_2)$ are coming from the crossings of $K_0$ with $\partial D$. There are three cases, shown in Figure 10.

**Case 1.** Both ends of the arc in the disk go over $\partial D$. If the two ends are both over $\alpha_i$ or both over $\beta_i$, then it contribute 0 to both $\text{lk}_2(K_0, K_1)$ and $\text{lk}_2(K_0, K_2)$. If one end goes over $\alpha_i$ and the other goes over $\beta_i$ then this arc contributes 1 to both $\text{lk}_2(K_0, K_1)$ and $\text{lk}_2(K_0, K_2)$.

**Case 2.** Both ends of the arc in the disk go under $\partial D$. In this case the arc contributes 0 to both $\text{lk}_2(K_0, K_1)$ and $\text{lk}_2(K_0, K_2)$.

**Case 3.** One end of the arc in the disk goes over $\partial D$, and the other end goes under $\partial D$. In this case, the arc of $K_0$ must intersect $D$ in its interior, and that means it goes through the only fixed point in $D$. So there is exactly one such arc. If the “over end” is over $\alpha_i$, then this arc contributes 1 to $\text{lk}_2(K_0, K_1)$ but 0 to $\text{lk}_2(K_0, K_2)$. If the “over end” is over $\beta_i$ then this arc contributes 0 to $\text{lk}_2(K_0, K_1)$ but 1 to $\text{lk}_2(K_0, K_2)$. Anyway, it is not the same for the two linking numbers.
From the above, we must have that
\[ lk_2(K_0, K_1) \neq lk_2(K_0, K_2). \]
Since \( \sigma \) is an automorphism of \( S^3 \) which send \( \{ K_0, K_1 \} \) to \( \{ K_0, K_2 \} \), this is a contradiction. So for \( |G| = 2g \) the \( G \)-action on \( \Sigma_g \) does not extend to a \( G \)-action on \( S^3 \).

4 Intuitive view of large symmetries of \( (S^3, \Sigma_g) \)

In this section we will present some extendable group actions on surfaces with “large” symmetry. Also if the existence of such examples often can be derived from the powerful orbifold theory, we would like to see how these symmetries stay in the symmetry of our 3-sphere in a more direct and intuitive way, as we mentioned in the end of the introduction.

**Remark:** In the ten examples below, the first seven are constructed before applying orbifold theory to get the result in Section 2 and 3, and the last three examples was constructed after we knowing results in Sections 2 and 3. The constructions of first nine examples use no information from orbifold theory.

**Example 4.1** For every \( g > 1 \), there is an abelian group \( G \cong \mathbb{Z}_2 \times \mathbb{Z}_{g+1} \) which acts on \( \Sigma_g \) such that the action extends to a \( G \)-action on \( S^3 \), for the standard embedding of \( \Sigma_g \subset S^3 \). When \( g \) is even, we get a cyclic group action of order \( 2g + 2 \) on \( \Sigma_g \) which extends over \( S^3 \).

We embed \( V_g \) in \( S^3 \) as two 3-balls together with \( g + 1 \) handles attached to the equator, see Figure 11 for \( g = 2 \). One can easily see a \( \mathbb{Z}_{g+1} \)-action by a rotation on \( S^3 \) which keeps \( V_g \) invariant. We construct a \( \mathbb{Z}_2 \) involution on \( V_g \) as follows: The involution restricted to each handle is a \( \pi \)-rotation about each line \( l_i \) drawn in Figure 11, and the involution maps each of the two 3-balls to the other one, without any rotation or reflection. One can carefully check the attaching disks to show this is well defined; moreover the involution and the rotation commutate and give an action of \( \mathbb{Z}_{g+1} \times \mathbb{Z}_2 \) on \( (S^3, V_g) \). When \( g \) is even, the composition of the involution and the \( \frac{2\pi}{g+1} \)-rotation is an order \( 2g + 2 \) map.

![Figure 11](image-url)
Another simple and useful way to observe this action is just to think about a pair of pants, on the right hand side of Figure 11. Its neighborhood in $S^3$ is a genus 2 handlebody. This pair of pants consists of two pieces of cloth sewn together along three lines $c_i$. So this is just the same as on the left hand side, letting each piece of cloth correspond to a 3-ball and each $c_i$ to a handle. Now the rotation on the pair of pants is quite obvious and is extendable. The involution just changes the positions of the two pieces of cloth, keeping $c_i$ fixed. This makes the pants inner to outer, and because you can really do this practically, it extends to an involution of $S^3$.

Example 4.2 Now we construct an example of a $\mathbb{Z}_{2g-2}$ action on $V_g$ for $g$ odd. Let us consider $S^3$ as the unit sphere in $\mathbb{C}^2$,

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}.$$ 

There is a solid torus $T \in S^3$,

$$T = \{(z_1, z_2) \in S^3 \mid |z_1| \leq \frac{\sqrt{2}}{2}\}.$$ 

We choose $g-1$ pairs of points $a_k$ and $b_k$, $k = 1, 2, \ldots, g-1$,

$$a_k = \left(\frac{\sqrt{2}}{2} e^{\frac{2(k)}{g-1} \pi i}, \frac{\sqrt{2}}{2} e^{\frac{k}{g-1} \pi i} \right),$$

$$b_k = \left(\frac{\sqrt{2}}{2} e^{\frac{2(k)}{g-1} \pi i}, \frac{\sqrt{2}}{2} e^{\frac{(k+g-1)}{g-1} \pi i} \right).$$

In the disk

$$D_k = \{(re^{\frac{2k}{g-1} \pi i}, z_2) \in S^3 \mid r \geq \frac{\sqrt{2}}{2}\},$$

there is a unique diameter $\gamma_k$ connecting $a_k$ and $b_k$. Let $N_k$ be a neighborhood of $\gamma_k$ in $S^3$; then the solid torus $T$ together with the $(g-1)$ handles $N_k$ gives an embedding of $V_g$ into $S^3$.

Now the $(2g-2)$ order action on $S^3$ can be described like this:

$$\sigma : S^3 \to S^3$$

$$(z_1, z_2) \mapsto (z_1 e^{\frac{2k}{g-1} \pi i}, z_2 e^{\frac{4k}{g-1} \pi i}).$$

This action keeps $T$ invariant and sends each $N_k$ to $N_{k+1(mod(g-1))}$. A rough picture is showed in Figure 12 for $g = 5$. 

![Figure 12](image-url)
Example 4.3 For every \( g > 1 \), we will construct a group \( G \) of order \( 4(g + 1) \) which acts on \( S^3 = V_g \cup V'_g \). We consider \( V_g \) as the neighborhood of a sphere \( S^2 \) with \( g + 1 \) punctured holes. We choose the holes all on the equator, centered at the vertices of a regular \( g + 1 \)-polygon. There is a dihedral group \( D_{g+1} \) acting on \( S^2 \) which keeps the holes invariant (as a set). And there is also a \( \mathbb{Z}_2 \) action changing the inner and outer of \( S^2 \), as described in Example 4.1. So there is a \( D_{g+1} \times \mathbb{Z}_2 \) action on \( V_g \). This group has order \( 4(g + 1) \).

Example 4.4 For every square number \( g > 1 \), we construct a group \( G \) of order \( 4(\sqrt{g} + 1)^2 \) which acts on \( S^3 = V_g \cup V'_g \) (notice that this is greater than \( 4(g + 1) \)).

With \( g = k^2 \), the group \( G \) is a semidirect product

\[
(Z_{k+1} \times Z_{k+1}) \rtimes \varphi (Z_2 \times Z_2).
\]

Writing \( Z_{k+1} \times Z_{k+1} = \langle x, y | xy = yx, x^{k+1} = y^{k+1} = 1 \rangle \), \( Z_2 \times Z_2 = \langle s, t | st = ts, s^2 = t^2 = 1 \rangle \), the semidirect product is given by

\[
\varphi : sxs^{-1} = y, \quad sys^{-1} = x, \quad txt^{-1} = x^{-1}, \quad tyt^{-1} = y^{-1}.
\]

Consider \( S^3 \) as the unit sphere in \( \mathbb{C}^2 \)

\[
S^3 = \{ (z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1 \},
\]

and let

\[
a_j = (e^{\frac{2\pi i}{k+1}}, 0), \quad b_j = (0, e^{\frac{2\pi i}{k+1}}), \quad j = 0, 1, \ldots, k.
\]

Then the \( G \)-action on \( S^3 \) is given by:

\[
x : (z_1, z_2) \mapsto (e^{\frac{2\pi i}{k+1}} z_1, z_2),
\]

\[
y : (z_1, z_2) \mapsto (z_1, e^{\frac{2\pi i}{k+1}} z_2),
\]

\[
s : (z_1, z_2) \mapsto (z_2, z_1),
\]

\[
t : (z_1, z_2) \mapsto (z_1, z_2).
\]

It is easy to check this is a faithful orientation-preserving action.

Notice that this \( G \)-action keeps the set \( \{a_i, b_j\}, i, j = 0, 1, \ldots, k \), invariant. If we join each \( a_i \) and \( b_j \) by a geodesic in \( S^3 \), we get a two-parted graph \( \Gamma \in S^3 \) with \( 2k + 2 \) vertices and \( (k + 1)^2 \) edges. Hence \( \chi(\Gamma) = -k^2 + 1 \). A neighborhood of \( \Gamma \) in \( S^3 \) is a handlebody of genus \( k^2 = g \). Since the \( G \)-action
maps \( \Gamma \) to itself, it induces an action on \( V_g \). Figure 13 gives the picture for \( g = 4 \).

For \( g = 4 \), this example gives a group action of order 36, which is maximal.

**Example 4.5** We view \( S^3 = \{(x, y, z, w) \mid x^2 + y^2 + z^2 + w^2 = 1\} \) and \( S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\} \subset S^2 \). For each finite group \( G \leq O(3) \) which acts on \( S^2 \subset R^3 \), we define \( \tilde{G} \leq SO(4) \) acts on \( S^3 \) which keeps the \( S^2 \) invariant as below: For each \( \sigma \in G \), define \( \tilde{\sigma} : S^3 \to S^3 \) by

\[
\tilde{\sigma}(x, y, z, w) \mapsto \begin{cases} (\sigma(x, y, z), w) & \text{is orientation-preserving;} \\ (\sigma(x, y, z), -w) & \text{is orientation-reversing.} \end{cases}
\]

Now if we choose \( G \) to be the symmetry group of a tetrahedron with vertices on this \( S^2 \), then \( G \cong S_4 \). Let \( \{v_1, v_2, v_3, v_4\} \) be the vertices of this tetrahedron, then \( G \) and \( \tilde{G} \) keeps this vertices set invariant. If we choose 4 holes puncturing this \( S^2 \) at the positions of \( \{v_1, v_2, v_3, v_4\} \), we get a 4-punctured sphere \( X \), its neighborhood in \( S^3 \) is a handlebody \( V_3 = N(X) \). And \( \tilde{G} \) act on \( (V_3, S^3) \). Note that \( \tilde{G} \cong G \cong S_4 \), \( |\tilde{G}| = 24 \), this gives an example for \( g = 3 \) and \( |\tilde{G}| = 24 = 12(g - 1) \).

Similarly we can choose \( G \) to be the symmetry group of a cube or a dodecahedron, we get \( \tilde{G} \cong S_4 \times Z_2 \) for \( g = 5 \) and \( \tilde{G} \cong A_5 \times Z_2 \) for \( g = 11 \). They all satisfies \( |\tilde{G}| = 12(g - 1) \).

**Example 4.6** The quotients \( S^3/I^* \) of \( S^3 \) by the binary icosahedral group is the famous Poincaré homology 3-sphere which is also obtained by identifying pairs of faces of the dodecahedron. The covering \( p : S^3 \to S^3/I^* \) provides a tessellation of \( S^3 \) by dodecahedra. Since the deck group is of order 120 and the symmetry group of the dodecahedron is 60, it is easy to see there is a group \( G \) of order \( 120 \times 60 = 7200 \) acting on this tessellation, and in particular acting on the boundary surface \( \Sigma_g \) of a regular neighborhood of the 1-skeleton of this tessellation. Let \( v_i \) be the number of the \( i \)-dimensional cells of this tessellation; then \( v_3 - v_1 + v_2 - v_3 = \chi(S^3) = 0 \). Clearly \( v_3 = 120 \) and \( v_2 = 120 \times 12/2 = 720 \). So \( v_1 - v_0 = v_2 - v_3 = 600 \), and therefore \( g = 11 \) and \( |\tilde{G}| = 7200 = 12(g - 1) \).

This example can be also considered as the orientation preserving symmetry group of the regular 120-cell in 4-space, in the spirit of the next example.

**Example 4.7** Let \( \Delta \) be the 4-dimensional regular Euclidean simplex and \( \Theta \) be the 4-dimensional Euclidean cube centered at the origin of \( E^4 \) and inscribed in the unit sphere \( S^4 \). The radical projection of their boundaries to \( S^3 \) gives two regular tessellations of \( S^3 \), still denoted as \( \partial \Delta \) and \( \partial \Theta \) respectively.

Using the notions defined in Example 4.6, for \( \partial \Delta \) we have \( v_0 = v_1 = 5 \), \( v_2 = 10 \). Since \( 10 - 5 + 1 = 6 \), the boundary surface of the regular neighborhood of the 1-skeleton of this tessellation is \( \Sigma_6 \). Each 3-dimensional face is a tetrahedron which has a symmetry of order 12, and then it follows that a group \( G \) of \( 60 = 12 \times 5 = 12(g - 1) \) acts on this tessellation, and in particular acts on \( \Sigma_6 \).
For \( \partial \Theta \), since \( \Theta \) is the product of the 3-dimensional cube with an interval, it derived easily that we have \( v_0 = 16, v_1 = 32, v_2 = 24 \) and \( v_3 = 8 \). Since \( 32 - 16 + 1 = 17 \), the boundary surface of the regular neighborhood of the 1-skeleton of this tessellation is \( \Sigma_{17} \). Each 3-dimensional face is a 3-dimensional cube which has a symmetry of order 24, and it follows that a group \( G \) of order \( 24 \times 8 = 192 = 12(g - 1) \) acts on this tessellation, and in particular on \( \Sigma_{17} \).

So in this example we get actions of groups on \( (S^3, \Sigma_g) \) of maximum order \( 12(g - 1) \), for \( g = 6 \) and 17.

**Example 4.8** Let \( M' = P \times S^1 \), where \( P \) is the oriented pair of pants, with the induced orientation on \( \partial P = \{c_1, c_2, c_3\} \), and \( S^1 \) is oriented and represented by a curve \( h \). Now attach 3 solid tori \( N_i \) along the first three boundary tori of \( M' \) so that the meridian of \( N_i \) is identified with a curve of slope \( l_i = 2c_i + h, i = 1, 2, 3 \). Denote the resulting manifold by \( M \), which has the following properties:

1. Recall that the full symmetry group of the pair of pants \( P \) is \( G_P = D_3 \times \mathbb{Z}_2 \), where the \( D_3 \) action on \( P \) is orientation preserving, and the \( \mathbb{Z}_2 \)-action, which exchanges the inner and outer of \( P \) as described in Example 4.3, is orientation reversing. We extend the \( G_P = D_3 \times \mathbb{Z}_2 \) action on \( P \) to \( P \times S^1 \) in an orientation preserving way by matching \( D_3 \) with the identity of \( S^1 \) and \( \mathbb{Z}_2 \) with an orientation reversing involution (reflection) on \( S^1 \). The latter extends over \( M \) since it preserves the set of attaching slopes.

2. The map \( \pi_1(P) \to \pi_1(M) \) is a surjection, therefore the map \( \pi_1(\Sigma_2) \to \pi_1(M) \) is a surjection where \( \Sigma_2 \) is the boundary of the regular neighborhood of \( P \) in \( M \); here all maps are induced by inclusions, and we use the fact that each attaching curve goes over the \( S^1 \) direction only once.

3. \( M \) is a spherical 3-manifold with \( \pi_1(M) = \mathbb{Z}_3 \times D_8^* \), where \( D_8^* \) is the quaternion group of order 8, see [Or], in particular \( |\pi_1(M)| = 24 \).

Now consider the covering \( p : S^3 \to M \). By (2) and Lemma 2.2 the preimage of \( \Sigma_2 \) is connected, therefore it is the surface \( \Sigma_{25} \) by (3) which is invariant under the group of order \( 24 \times 12 = 12(g - 1) \).

**Example 4.9** We perform \( -1 \) surgery on each component of the link given in Figure 14. considering the Wirtinger presentation, the fundamental group of the resulting manifold is \( < x, y, z | x = yz, y = zx, z = xy > \) which is isomorphic to the quaternion group \( Q \) by the map induced by \( x \mapsto i, y \mapsto j, z \mapsto k \). The resulting manifold is the quaternion manifold \( S^3/Q \). Consider a point in front of the paper and a point behind. Through each of the three components of the link choose a string connecting these two points. We get a \( \theta \)-graph. The boundary of a neighborhood of the \( \theta \)-graph is the surface \( \Sigma_2 \). As described before, there is a group of order 12 acting on \( S^3 \) which leaves this surface invariant. This action leaves also the link invariant and extends to the surgered solid tori. Lifting to \( S^3 \) we have a group action of order 96. Since \( \pi_1(\theta) \) is generated by \( x^{-1}z \) and \( x^{-1}y \), it is easy to see the homomorphism \( \pi_1(\theta) \to \pi_1(S^3/Q) \) is surjective. By Lemma 2.2, the lift of \( \Sigma_2 \) is connected and hence is \( \Sigma_9 \). Hence we get an extendable group action on \( \Sigma_9 \), of order \( 12(9 - 1) = 96 \).
Similarly, performing +1 surgery on a trefoil indicated in Figure 15 we obtain the Poincaré homology 3-sphere $P$. Letting $u = yx$, the fundamental group has a presentation $\pi_1(P) = \langle u, x | u^3 = x^5 = (xu)^2 \rangle$. It is isomorphic to the binary icosahedral group $I^*$ by the map induced by $u \mapsto (1 + \sigma i + \delta j)/2, x \mapsto (\sigma + i - \delta k)/2$, where $\sigma = (\sqrt{5} + 1)/2, \delta = (\sqrt{5} - 1)/2$. We can similarly construct a $\theta$-graph, and $\pi_1(\theta)$ is generated by $x^{-1}ux^{-1} \mapsto (\sigma + \delta j + k)/2$ and $ux^{-2} \mapsto (\sigma - \delta j + k)/2$. One can verify that $\pi_1(\theta) \to \pi_1(P)$ is surjective. Then we lift the neighborhood boundary surface of the $\theta$-graph and the group action as before and obtain an extendable action on $\Sigma_{121}$ of order $120 \times 12 = 1440 = 12(g - 1)$.

Example 4.10. Considering the 2:1 map $SO(4) \to SO(3) \times SO(3)$, the preimage of $O \times O$, denoted by $O \times O$, acts on $S^3$ ($O$ denotes the octahedral group, of order 24). It has order $24 \times 24 \times 2 = 1152 = 12(97 - 1)$ and the pre-fundamental domain is a truncated cube [Du2]. If two such domains are adjacent via an octagon, we draw an edge between the centers of them. Then we get a graph, and the boundary of the regular neighborhood of this graph is a surface $\Sigma_{97}$ with an action of $O \times O$ which is obviously extendable.

The preimage of $O \times J$, denoted by $O \times J$, acts on $S^3$. It has order $24 \times 60 \times 2 = 2880 = 12(241 - 1)$ and the pre-fundamental domain is a ‘twice truncated tetrahedron’(or small tetrahedron in Dunbar’s paper). If two such domains are adjacent via a dodecagon, we draw an edge between their centers. Then we get a graph, and the boundary of a regular neighborhood of this graph is a surface $\Sigma_{241}$ with an extendable action of $O \times J$. 
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