Six-Point One-Loop $\mathcal{N} = 8$ Supergravity NMHV Amplitudes and their IR behaviour

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Abstract

We present compact formulas for the box coefficients of the six-graviton NMHV one-loop amplitudes in $\mathcal{N} = 8$ supergravity. We explicitly demonstrate that the corresponding box integral functions, with these coefficients, have the complete IR singularities expected of the one-loop amplitude. This is strong evidence for the conjecture that $\mathcal{N} = 8$ one-loop amplitudes may be expressed in terms of scalar box integral functions. This structure, although unexpected from a power counting viewpoint, is analogous to the structure of $\mathcal{N} = 4$ super-Yang-Mills amplitudes. The box-coefficients match the tree amplitude terms arising from recursion relations.
1 Introduction

Maximal $\mathcal{N} = 8$ supergravity [1] is a remarkable theory, rich in symmetries, however, the knowledge of its perturbative expansion is relatively poor compared to $\mathcal{N} = 4$ super-Yang-Mills. In determining the one-loop amplitudes of $\mathcal{N} = 4$ SYM, a key observation is that the amplitudes may be expressed in terms of scalar box-integral functions with rational coefficients [2, 3],

$$ A^{1\text{-loop}} = \sum_a \hat{c}_a I_a. \quad (1.1) $$

Considerable progress has recently been made in determining the coefficients, $\hat{c}_a$, using a variety of methods including those based on unitarity [2, 3, 4, 5] and those inspired by the weak-weak duality [6] between $\mathcal{N} = 4$ Yang-Mills and a twistor string theory [7, 8, 9, 10].

String theory, inspires a relation between the Yang-Mills amplitudes and those of gravity at tree level,

$$ \text{gravity} \sim \left( \text{gauge theory} \right) \times \left( \text{gauge theory} \right), \quad (1.2) $$

which arises from the heuristic relation

$$ \text{closed string} \sim \left( \text{left-moving open string} \right) \times \left( \text{right-moving open string} \right). \quad (1.3) $$

This has a concrete realisation in the Kawai, Lewellen and Tye (KLT) relations [11] which express gravity tree amplitudes in terms of quadratic products of Yang-Mills tree amplitudes. Even in low energy effective field theories for gravity [12] the KLT-relations can be seen to link effective operators [13], and KLT-relations also hold regardless of massless matter content [14].

At one-loop level, string theory would suggest such a relation within the loop momentum integrals. Such relations would not be expected to persist in the amplitude after integration have been performed. The first definite calculation of a one-loop $\mathcal{N} = 4$ amplitude was performed by Green, Schwarz and Brink [15], who obtained the four point one-loop amplitude

$$ A^{1\text{-loop}}(1, 2, 3, 4) = st \times A^{\text{tree}}(1, 2, 3, 4) \times I_4(s, t). \quad (1.4) $$

Here $I_4(s, t)$ denotes the scalar box integral with attached legs in the order 1234 and $s$, $t$ and $u$ and the usual Mandelstam variables. The $A^{1\text{-loop}}$ and $A^{\text{tree}}$ are the color-ordered partial amplitudes. Similarly they computed the one-loop $\mathcal{N} = 8$ amplitude to be \footnote{In this, and in the following, we suppress a factor of $(\kappa/2)^{(n-2)}$ in the $n$-point tree amplitude and a factor of $(\kappa/2)^n$ in the $n$-point one-loop amplitude.}

$$ M^{1\text{-loop}}(1, 2, 3, 4) = stu \times M^{\text{tree}}(1, 2, 3, 4) \times \left( I_4(s, t) + I_4(s, u) + I_4(t, u) \right), \quad (1.5) $$

so that, like the $\mathcal{N} = 4$ Yang-Mills amplitude, the $\mathcal{N} = 8$ amplitude can be expressed in terms of scalar box-functions.

From a power counting analysis, the similarities of the four-point amplitudes, are not expected to extend to higher point functions. In evaluating loop amplitudes in a gauge theory one must perform integrals over the loop momenta, $\ell^\mu$, with polynomial numerator $P(\ell^\mu)$. In a Yang-Mills theory, the loop momentum polynomial is generically of degree $n$ in
a \( n \)-point loop. The \( \mathcal{N} = 4 \) one-loop amplitudes have a considerable simplification where the loop momentum integral is of degree \( n - 4 \) \[2\]. Consequently, the amplitudes can be expressed as a sum of scalar box integrals with rational coefficients, as follows from a Passarino-Veltman reduction \[16\].\(^2\) The equivalent power counting arguments for supergravity \[17\] give the loop momentum polynomial of an \( n \)-point amplitude as having degree

\[
2(n - 4)
\]

consistent with the heuristic relation eq. \((1.2)\). Performing a Passarino-Veltman reduction for \( n > 4 \) would lead one to express the amplitude as a sum of tensor box integrals with non-trivial integrands of degree \( n - 4 \). These tensor integrals would be expected to reduce to scalar boxes and triangle, bubble and rational functions.

Despite this power counting argument, there is evidence that the one-loop amplitudes of \( \mathcal{N} = 8 \) can be expressed simply as a sum over scalar box integrals analogous to the \( \mathcal{N} = 4 \) case \((1.1)\). Triangle or bubble functions do not appear in any computation. Neither do factorisation properties of the physical amplitudes demand the presence of these functions. In ref. \[18\] the five and six point amplitudes were computed and an all-\( n \) form of the supergravity “Maximally Helicity Violating” (MHV) amplitudes was presented. The simplification is peculiar to \( \mathcal{N} = 8 \) and explicitly does not apply for \( \mathcal{N} < 8 \) supergravities \[19, 17, 20\]. The similarity between \( \mathcal{N} = 4 \) and \( \mathcal{N} = 8 \) amplitudes is also apparent in the two-loop four point amplitude \[21, 22\]. These forms for the amplitude consist only of box functions. Recently, it was conjectured in \[23\] from factorisation arguments that all one-loop \( \mathcal{N} = 8 \) supergravity amplitudes can be expressed as only box functions and the coefficients of the boxes were given for the six-point NMHV one-loop amplitude.

In this letter we will explore this conjecture further. First, we present the box coefficients in an equivalent but more compact form. Next, we will show that boxes contain the entire IR singularities expected in the one-loop amplitude. Other integral functions such as scalar triangle functions generically contain IR singularities. If such functions were present, IR singularities would have to cancel between these functions alone. Since the five-point amplitude only contains box integral functions these other functions would also have to conspire to not contribute to any of the soft or factorisation limits or generate UV singularities. This provides strong evidence for the absence of other integral functions for general \( n \)-point amplitudes.

In Yang-Mills theory, an analysis of the IR singularities led to compact forms of the tree amplitudes. We will also use the box-coefficients to determine forms of the amplitude and compare these to the forms derived recently using recursive techniques \[24, 25\].

\section{Box Coefficients of \( M^{1\text{-loop}}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+) \)}

Quadruple cuts were used in \[10\] to compute the box coefficients of \( \mathcal{N} = 4 \) gauge theory algebraically from the tree amplitudes. As shown in \[23\], this technique, together with the

\(^2\)The Passarino-Veltman reduction expresses an \( n \)-point integral with loop momentum polynomial of degree \( m \) as a sum of \( n - 1 \)-point integrals with loop momentum polynomials of degree \( m - 1 \). For \( \mathcal{N} = 4 \), since the loop momentum polynomial for the \( n \)-point is only of degree \( n - 4 \), repeated Passarino-Veltman reductions express the amplitude as a sum of scalar boxes.
KLT relations allows the computation of box-coefficients in $\mathcal{N} = 8$ supergravity amplitudes. In particular box coefficients were given for the “next-to-MHV” (NMHV) six-point amplitude $M(1^-2^-3^-4^+5^+6^+)$, the simplest non-trivial non-MHV amplitude.

We will use these methods and compute the explicit form of the box coefficients. We will present the box coefficients in a very compact form, which allows to point out similarities of $\mathcal{N} = 8$ supergravity and $\mathcal{N} = 4$ super-Yang-Mills.

The six-point NMHV amplitude contains two types of box integral functions: the one-mass box and the “two mass-easy” box,

$$I_4^{(ab)cdef}$$

and may be expressed in terms of these as

$$M^{1-\text{loop}}(1^-2^-3^-4^+5^+6^+)_{\text{boxes}} = c_T \left( \sum_{(abcdef) \in P'_6} \hat{c}^{(abc)def} I_4^{(abc)def} + \sum_{(abcdef) \in P''_6} \hat{c}^{a(bc)(de)f} I_4^{a(bc)(de)f} \right).$$

Here the sum runs over the permutations $P'_6$ and $P''_6$ of indices $\{123456\}$ modulo symmetries of the integral functions $I_4^{(abc)def}$ and $I_4^{a(bc)(de)f}$ respectively. The dimensionally regulated integral functions $I_4^{(abc)def}$ and $I_4^{a(bc)(de)f}$ are symmetric in $a$, $b$ and $c$ and under the exchange of $d$ and $f$. The integral function $I_4^{a(bc)(de)f}$ are symmetric under the exchange of $b$ and $c$ and of $e$ and $f$ independently.

The box coefficients $\hat{c}$ have been computed in [23] in terms of $\mathcal{N} = 4$ box coefficients $\hat{c}_{\mathcal{N}=4}$. For example the two mass hard coefficient can be expressed as

$$\hat{c}^{a(bc)(de)f} = 2s_{bc}s_{de} \times \sum_{i=ns,s} \hat{c}_{\mathcal{N}=4,i}^{a(bc)(de)f} \hat{c}_{\mathcal{N}=4,i}^{a(bc)(ed)f},$$

where the label $i$ can take the values “singlet” (s) or “non-singlet” (ns), as we will discuss presently.

Six point box coefficients contain two terms which we label as non-singlet or singlet

$$\hat{c} = \hat{c}_{\text{ns}} + \hat{c}_{\text{s}}.$$  

The two terms arise from different helicity structures in the cut in three-particle channels, as illustrated in fig. 1. The singlet term corresponds to the two cut legs having the same helicity on one side of the cut and has contributions only from gluons/gravitons crossing the cut. The non-singlet term has its cut legs having opposite helicity on one side of the cut. For this
Figure 1: Examples of non-singlet and singlet contributions to a two-mass hard and a one-mass box integral. The dashed line indicates the cut to which the singlet and non-singlet description refer.

configuration all terms in the $\mathcal{N} = 4/\mathcal{N} = 8$ multiplet contribute. Note that the designation of singlet/non-singlet depends on which channel we are considering.

It turns out that two-mass as well as the single mass box coefficients can be expressed in terms of a single generating function in each case. Specifically, the two-mass hard coefficients can be generated from

$$\hat{G}_0 \equiv \frac{i}{2} [a b][b c][c a] [b c] \langle d e \rangle \langle e f \rangle \langle c d \rangle \langle c f \rangle \langle a | K_{abc} | d \rangle \langle a | K_{abc} | e \rangle \langle b | K_{abc} | e \rangle \langle c | K_{abc} | f \rangle,$$

(2.4)

where we are using the usual spinor products $\langle j l \rangle \equiv \langle j^- | l^+ \rangle = \bar{u}-(k_j)u_+(k_l)$ and $[j l] \equiv \langle j^+ | l^- \rangle = \bar{u}_+(k_j)u_-(k_l)$, and where $\langle i | K_{abc} | j \rangle$ denotes $\langle i^+ | K_{abc} | j^+ \rangle$ with $K_{abc}^\mu = k_a^\mu + k_b^\mu + k_c^\mu$ and $s_{ab} = (k_a + k_b)^2$. We are using a spinor helicity basis for the graviton polarisation tensors [26, 27] where $\epsilon_{\mu\nu}^\pm \equiv \epsilon_\mu^\pm \epsilon_\nu^\pm$ and $\epsilon_\mu^\pm$ are the usual Yang-Mills polarisation vectors [28].

For the coefficients $\hat{c}^{a-(b-c)(d+e)j^+}$ and $\hat{c}^{a+(b+c)(d-e)j^-}$ there is only a singlet contribution and we have

$$\hat{c}^{a-(b-c)(d+e)j^+} = \hat{G}_0,$$
$$\hat{c}^{a+(b+c)(d-e)j^-} = \hat{G}_0^\ast.$$

(2.5)

The definition of $\hat{G}_0^\ast$ is to parity conjugate $\hat{G}_0$ by $\langle a b \rangle \leftrightarrow [a b]$ and $\langle i | K_{abc} | j \rangle \rightarrow \langle j | K_{abc} | i \rangle$.  

See ref. [3] for definitions and conventions.

In the following we will use the symbols $\hat{G}_i$ and $\hat{H}_i$ for the functions $\hat{G}_i[a, b, c, d, e, f]$ and $\hat{H}_i[a, b, c, d, e, f]$, unless the argument is given explicitly.
The remaining box coefficients are sums of a non-singlet and a singlet contribution respectively,

\[ \hat{c}^{a^+(b^+c^-)(d^+e^+)}f^- = \hat{G}_1 \equiv \left( \langle a|K_{abc}|f \rangle \right)^2 \hat{G}_0 + \left( \langle b c|d e \rangle \right)^2 \hat{G}_0^*, \]

\[ \hat{c}^{a^-(b^-c^+)(d^-e^+)}f^+ = \hat{G}_2 \equiv \left( \langle c|K_{abc}|d \rangle \right)^2 \hat{G}_0 + \left( \langle a b|e f \rangle \right)^2 \hat{G}_0^*, \]

\[ \hat{c}^{a^+(b^+c^-)(d^-e^-)}f^- = \hat{G}_3 \equiv \left( \langle c|K_{abc}|b \rangle \right)^2 \hat{G}_0 + \left( \langle a c|d f \rangle \right)^2 \hat{G}_0^*, \]

\[ \hat{c}^{a^-(b^-c^+)(d^-e^-)}f^+ = \hat{G}_4 \equiv \left( \langle c|K_{abc}|f \rangle \right)^2 \hat{G}_0 + \left( \langle a b|d e \rangle \right)^2 \hat{G}_0^*, \]

\[ \hat{c}^{a^+(b^+c^-)(d^-e^-)}f^- = \hat{G}_5 \equiv \left( \langle f|K_{abc}|c \rangle \right)^2 \hat{G}_0 + \left( \langle a b|d e \rangle \right)^2 \hat{G}_0^*. \] (2.6)

Note that the various terms have the structure implied by (2.2), which can be verified by comparing to the expressions of the super-Yang-Mills box coefficients given in [3].

The one-mass box coefficients can be generated from

\[
\hat{H}_0 \equiv \frac{1}{2} \frac{[d e]^2 [e f]^2 (K_{abc}^2)^7}{[a b]} \left( [a b] \langle b c | c | K_{abc} | f \rangle \langle d a \rangle + [a b] [b c] \langle c d | a | K_{abc} | f \rangle \right).
\] (2.7)

Again, there are two box coefficients with only a “singlet” term,

\[ \hat{c}^{(a^+b^-c^-)d^+e^+}f^+ = \hat{H}_0, \]

\[ \hat{c}^{(a^+b^+c^-)d^-e^-}f^- = \hat{H}_0^*. \] (2.8)

The remaining box coefficients are sums of non-singlet and singlet terms

\[ \hat{c}^{(a^+b^-c^-)d^-e^-}f^+ = \hat{H}_1 \equiv \left( \langle f|K_{abc}|c \rangle \right)^2 \hat{H}_0^* + \left( \langle b c|d e \rangle \right)^2 \hat{H}_0^*, \]

\[ \hat{c}^{(a^+b^-c^-)d^+e^+}f^- = \hat{H}_2 \equiv \left( \langle e|K_{abc}|d \rangle \right)^2 \hat{H}_0^* + \left( \langle b c|d f \rangle \right)^2 \hat{H}_0^*, \]

\[ \hat{c}^{(a^+b^-c^-)d^-e^-}f^- = \hat{H}_3 \equiv \left( \langle d|K_{abc}|a \rangle \right)^2 \hat{H}_0^* + \left( \langle b c|e f \rangle \right)^2 \hat{H}_0^*, \]

\[ \hat{c}^{(a^+b^-c^-)d^+e^+}f^+ = \hat{H}_4 \equiv \left( \langle c|K_{abc}|d \rangle \right)^2 \hat{H}_0^* + \left( \langle a b|e f \rangle \right)^2 \hat{H}_0^*, \]

\[ \hat{c}^{(a^+b^-c^-)d^-e^-}f^- = \hat{H}_5 \equiv \left( \langle c|K_{abc}|f \rangle \right)^2 \hat{H}_0 + \left( \langle a b|d f \rangle \right)^2 \hat{H}_0^*, \]

\[ \hat{c}^{(a^+b^-c^-)d^+e^+}f^- = \hat{H}_6 \equiv \left( \langle c|K_{abc}|d \rangle \right)^2 \hat{H}_0 + \left( \langle a b|d e \rangle \right)^2 \hat{H}_0^*. \] (2.9)

One observes that the singlet and the non-singlet terms are proportional to the generating functions \( \hat{H}_0 \) or \( \hat{H}_0^* \). As above this pattern of the box-coefficients resembles the pattern for
\( \mathcal{N} = 4 \) super-Yang-Mills six-point box coefficients [3]. Furthermore, the relations
\[
\hat{c}_{\mathcal{N}=4}^{(abc)def} = \hat{c}_{\mathcal{N}=4}^{(def)abc} = \hat{c}_{\mathcal{N}=4}^{a(bc)(de)} = \hat{c}_{\mathcal{N}=4}^{d(ef)(ab)}
\]
(2.10)
hold for \( \mathcal{N} = 4 \) box coefficients. As shown in [29] three particle factorisation properties imply these relations. Similarly, in the \( \mathcal{N} = 8 \) theory the following identities
\[
\hat{c}^{(abc)def} + \hat{c}^{(abc)edf} = \hat{c}^{a(bc)(de)} + \hat{c}^{b(ac)(de)} + \hat{c}^{c(ab)(de)},
\]
(2.11)
hold. They can be viewed as symmetrisations of the super-Yang-Mills relations.

It is convenient to define \( F \)-functions which are rescaled scalar box integrals [3]
\[
I^a_{bc}(def) = -\frac{2r}{s_af} F_a^{(bc)(de)f}, \quad I^{(abc)def} = -\frac{2r}{s_de s_ef} F^{(abc)def}.
\]
(2.12)

We shall use the convention that coefficients of the integrals \( I \) are denoted \( \hat{c} \) whilst the coefficients of the \( F \) functions are denoted \( c \). Following this convention, we define
\[
H_i = -\frac{2}{s_de s_ef} \hat{H}_i, \quad G_i = -\frac{2}{s_af K_{abc}^2} \hat{G}_i.
\]
(2.13)

### 3 IR Singularities

Gravity amplitudes contain soft divergences. At one-loop, the expected form of the soft divergence is [30]
\[
M_{\epsilon}^{\text{one-loop}}(1, 2, \ldots, n) = i \epsilon \kappa^2 \left( \sum_{i<j} \frac{s_{ij} \ln(-s_{ij})}{2\epsilon} \right) \times M_{\text{tree}}^{(1, 2, \ldots, n)}.
\]
(3.1)

This applies for supersymmetric as well as non-supersymmetric theories.

Contributions to these IR singularities can arise from both box and triangle integral functions. We will argue below, that no triangle functions contribute to the IR singularities (3.1), since all singularities are generated from box functions. Furthermore, we will show, that the coefficient of the \( \ln(-K_{abc})/\epsilon \) singularity vanishes among box functions. These facts give strong support for the conjecture that triangle functions are absent in \( \mathcal{N} = 8 \) one-loop amplitudes.

The box integrals contain IR singularities
\[
I^{(abc)def}_{\epsilon} = -\frac{2}{s_de s_ef} \left[ \ln(-s_{de}) + \ln(-s_{ef}) - \ln(-K_{abc}^2) \right],
\]
\[
I^{a(bc)(de)f}_{\epsilon} = -\frac{2}{s_af K_{abc}^2} \left[ \ln(-s_{af}) + 2 \ln(-K_{abc}^2) - \ln(-s_{bc}) - \ln(-s_{de}) \right].
\]
(3.2)

When inserted into (2.1) their contributions to the IR of the six point amplitude, that is (3.1) with \( n = 6 \), can be computed.
For example we find that 26 terms contribute to the coefficient of $\ln(-s_{12})$ in the six-point amplitude. Explicitly they are

\[-\frac{1}{2} \left( G_6[5, 4, 3, 1, 2, 6] + G_5[6, 4, 3, 1, 2, 5] + G_5[4, 5, 3, 1, 2, 6] + G_5[6, 5, 3, 1, 2, 4] \\
+ G_5[5, 6, 3, 1, 2, 4] + G_5[4, 6, 3, 1, 2, 5] \right) \\
- \frac{1}{2} \left( G_0[3, 1, 2, 4, 5, 6] + G_0[3, 1, 2, 4, 6, 5] + G_0[3, 1, 2, 6, 5, 4] \right) \\
- \frac{1}{2} \left( G_1[4, 1, 2, 5, 6, 3] + G_1[5, 1, 2, 4, 6, 3] + G_1[6, 1, 2, 5, 4, 3] \right) \\
- \frac{1}{2} \left( H_0^*[4, 5, 6, 1, 2, 3] + H_0^*[4, 5, 6, 2, 1, 3] \right) \\
+ \left( H_1[3, 5, 6, 1, 2, 4] + H_1[3, 4, 6, 1, 2, 5] + H_1[3, 5, 4, 1, 2, 6] + H_1[3, 5, 6, 2, 1, 4] \\
+ H_1[3, 4, 6, 2, 1, 5] + H_1[3, 5, 4, 2, 1, 6] \right) \\
+ \frac{1}{2} \left( G_4[2, 3, 4, 5, 6, 1] + G_4[2, 3, 5, 4, 6, 1] + G_4[2, 3, 6, 5, 4, 1] + G_4[1, 3, 4, 5, 6, 2] \\
+ G_4[1, 3, 5, 4, 6, 2] + G_4[1, 3, 6, 5, 4, 2] \right). \tag{3.3}
\]

Note that the various terms appear with coefficients $\pm 1/2$ and $1$.

Although it is not easy to analyse equation (3.3) analytically, it can be verified using computer algebra, that it is of the correct form

\[ s_{12} \times M_{\text{tree}}. \tag{3.4} \]

The expression for $M_{\text{tree}}$ was determined independently using the KLT-relation [11]

\[ M_{6\text{tree}}^\text{tree}(1, 2, 3, 4, 5, 6) = -is_{12}s_{45}A_{6\text{tree}}^\text{tree}(1, 2, 3, 4, 5, 6)(s_{35}A_6^{\text{tree}}(2, 1, 5, 3, 4, 6) \\
+ (s_{34} + s_{35})A_6^{\text{tree}}(2, 1, 5, 4, 3, 6)) + \mathcal{P}(2, 3, 4), \tag{3.5}
\]

where $\mathcal{P}(2, 3, 4)$ represents the sum over permutations of legs $2, 3, 4$ and $A_i^{\text{tree}}$ are the tree-level $i$-point colour-ordered gauge theory partial amplitudes [31].

We have verified that the box-coefficients yield the correct coefficient of $\ln(-s_{ab})$ for all choices of $s_{ab}$.

A further check is to test the coefficient of $\ln(-K_{abc}^2)/\epsilon$ which should be zero. Explicitly the coefficient of $\ln(-K_{123}^2)/\epsilon$ is

\[ \left( H_0[1, 2, 3, 4, 5, 6] + H_0[1, 2, 3, 4, 6, 5] + H_0[1, 2, 3, 5, 4, 6] + H_0^*[4, 5, 6, 1, 2, 3] \\
+ H_0^*[4, 5, 6, 1, 3, 2] + H_0^*[4, 5, 6, 2, 1, 3] \right) \\
- \left( G_0[1, 2, 3, 4, 5, 6] + G_0[1, 2, 3, 4, 6, 5] + G_0[1, 2, 3, 6, 5, 4] + G_0[2, 1, 3, 4, 5, 6] \\
+ G_0[2, 1, 3, 4, 6, 5] + G_0[2, 1, 3, 6, 5, 4] + G_0[3, 2, 1, 4, 5, 6] + G_0[3, 2, 1, 4, 6, 5] + G_0[3, 2, 1, 6, 5, 4] \right). \tag{3.6}
\]
It can be shown to vanish by application of the identities (2.11). By analogous reasoning, all \( \ln(-K_{ab}/\epsilon) \) terms also cancel in the sum over boxes. We conclude that the box functions give the full IR singularity structure.

The above expression (3.3) displays features of the amplitude, not present in the KLT-form. For example when considering the twistor structure of NMHV amplitudes, the tree amplitudes is expected to have “coplanar” support in twistor space. This can be tested by acting on the expression for \( M^{\text{tree}} \) with the differential operator \( K_{ijkl} \),

\[
K_{ijkl} = \langle ij \rangle \epsilon^{\hat{a}\hat{b}} \frac{\partial}{\partial \lambda^i_k} \frac{\partial}{\partial \lambda^l_{\hat{b}}} + \text{perms}. \tag{3.7}
\]

For gravity amplitudes, one has to act multiple times with the operator \( K_{ijkl} \) in order to annihilate the amplitude. For the six-point amplitude it was shown in ref. [23] that \( K^3_{ijkl} M^{\text{tree}} = 0 \). This annihilation was rather involved to show using the form of the amplitude generated by the KLT-relations. One reason for this is that individual terms in the expression for the KLT tree amplitude are not annihilated individually by \( K^3_{ijkl} \) but combine to zero at the final stage. In the expression for the tree generated from the IR singularities, each term is individually annihilated by \( K^3_{ijkl} \) because the box-coefficients of the NMHV amplitudes are generically coplanar [5, 32, 33]. This form for the tree amplitude is thus much closer to a “CSW”-type expression for the amplitude.

The CSW formulation [34] of tree level amplitudes in terms of MHV Parke-Taylor amplitudes [35], interprets these amplitudes as vertices in twistor space and uses this to construct amplitudes with any helicity configuration. Such constructions can be generalised to one-loop MHV calculations [36, 37, 38] and for other particle types [39, 40]. However it still remains a challenge to give a CSW formulation for gravity tree amplitudes [41, 42]. Compact tree level forms, such as the above, are build up from individual terms which independently are annihilated by \( K^n_{ijkl} \), hence such an expression is a potential starting point for formulating a CSW construction for gravity tree amplitudes.

4 Connection to Recursion Relations

The requirement, that the coefficients of the IR singularities reproduce the tree amplitude, can surprisingly be used to generate tree amplitudes which are often in a very compact form. This was observed in [5] and used in [43] to obtain a compact form of one of the eight-point amplitudes with four negative helicities in \( \mathcal{N} = 4 \) Yang-Mills. By examining the general structure of these IR relations, Britto, Cachazo and Feng proposed [44] a recursion relation to evaluate tree amplitudes in Yang-Mills theory. These recursive relations have been extended to include graviton scattering [24] where alternate expression for the MHV graviton amplitudes [26] were found, and in [25] where the recursion relations were used to give an explicit form of the six-point NMHV amplitude. In this section we shall relate the box-coefficients to the later computation.

Our starting point is the IR relation for the coefficient of \( \ln(-s_{34}) \) within the amplitude
\[ M^{1\text{-loop}}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+), \]
\[
\sum_{i \in S_{\text{tot}}} a_i c_i = s_{34} \times M^{\text{tree}},
\]
where the sum of \( S_{\text{tot}} \) is over boxes which have a \( \ln(-s_{34})/\epsilon \) singularity. Specifically,
\[
\sum_{i \in S_{\text{tot}}} a_i c_i = \frac{1}{2} \left( G_0[3, 1, 2, 5, 6, 4] + G_1[4, 1, 2, 5, 6, 3] \right) \]
\[
+ \frac{1}{2} \left( G_3[4, 1, 5, 2, 6, 3] + G_3[4, 2, 5, 1, 6, 3] + G_3[4, 1, 6, 2, 5, 3] + G_3[4, 2, 6, 1, 5, 3] \right) \]
\[
+ \frac{1}{2} \left( G_2[3, 4, 5, 6, 1] + G_4[1, 3, 4, 5, 6, 2] \right) - \frac{1}{2} \left( G_5[6, 4, 3, 1, 2, 5] + G_5[5, 4, 3, 1, 2, 6] \right) \]
\[
- \frac{1}{2} \left( G_5[6, 3, 4, 2, 5, 1] + G_3[6, 3, 4, 1, 5, 2] + G_3[5, 3, 4, 2, 6, 1] + G_3[5, 3, 4, 1, 6, 2] \right) \]
\[
- \frac{1}{2} \left( G_2[1, 3, 4, 2, 5, 6] + G_2[2, 3, 4, 1, 5, 6] + G_2[1, 3, 4, 2, 6, 5] + G_2[2, 3, 4, 1, 6, 5] \right) \]
\[
+ \left( H_4[1, 2, 6, 3, 4, 5] + H_4[1, 2, 5, 3, 4, 6] \right) + \left( H_2[2, 5, 6, 3, 4, 1] + H_2[1, 5, 6, 3, 4, 2] \right) \]
\[
+ \left( H_5[1, 2, 6, 4, 3, 5] + H_5[1, 2, 5, 4, 3, 6] \right) + \left( H_3[2, 5, 6, 4, 3, 1] + H_3[1, 5, 6, 4, 3, 2] \right). \]

Although this does give a mechanism for generating the tree amplitude it is not optimally compact. Using the identities (2.11), we can reduce this to a smaller expression which will also generate the tree amplitude,
\[
\sum_{i \in S} a_i c_i = \frac{1}{2} \left( G_0[3, 1, 2, 5, 6, 4] + G_1[4, 1, 2, 5, 6, 3] \right) \]
\[
+ \frac{1}{2} \left( G_3[4, 1, 5, 2, 6, 3] + G_3[4, 2, 5, 1, 6, 3] + G_3[4, 1, 6, 2, 5, 3] + G_3[4, 2, 6, 1, 5, 3] \right) \]
\[
+ \frac{1}{2} \left( H_4[1, 2, 6, 3, 4, 5] + H_4[1, 2, 5, 3, 4, 6] \right) + \frac{1}{2} \left( H_2[2, 5, 6, 3, 4, 1] + H_2[1, 5, 6, 3, 4, 2] \right) \]
\[
+ \frac{1}{2} \left( H_5[1, 2, 6, 4, 3, 5] + H_5[1, 2, 5, 4, 3, 6] \right) + \frac{1}{2} \left( H_3[2, 5, 6, 4, 3, 1] + H_3[1, 5, 6, 4, 3, 2] \right). \]

alogous to expressions in the Yang-Mills case [43].

In Yang-Mills theory, it was noticed that, if we split the box-coefficients into singlet and non-singlet terms, the sum contains sub-sets which individually sum to the tree amplitudes. This same simplification also arises here and we have
\[
\sum_{i \in S} a_i c_i = \sum_{i \in S - S'} a_i c_i = \frac{1}{2} \times s_{34} \times M^{\text{tree}},
\]
with
\[
\sum_{i \in S'} a_i c_i = \frac{1}{2} G_1^{ns}[4, 1, 2, 5, 6, 3] \\
+ \frac{1}{2} \left( G_3^{s}[4, 1, 5, 2, 6, 3] + G_3^{s}[4, 2, 5, 1, 6, 3] + G_3^{s}[4, 1, 6, 2, 5, 3] + G_3^{s}[4, 2, 6, 1, 5, 3] \right) \\
+ \frac{1}{2} \left( H_4^{ns}[1, 2, 6, 3, 4, 5] + H_4^{ns}[1, 2, 5, 3, 4, 6] \right) + \frac{1}{2} \left( H_5^{s}[2, 5, 6, 3, 4, 1] + H_5^{s}[1, 5, 6, 3, 4, 2] \right) \\
+ \frac{1}{2} \left( H_6^{s}[1, 2, 6, 4, 3, 5] + H_6^{s}[1, 2, 5, 4, 3, 6] \right) + \frac{1}{2} \left( H_3^{ns}[2, 5, 6, 4, 3, 1] + H_3^{ns}[1, 5, 6, 4, 3, 2] \right). 
\]

This subset corresponds to one quarter of the terms in the full sum. The terms in this sum are precisely the box-coefficients arising from boxes with identical helicity structure in the 34 two-particle cut. Specifically it corresponds to contributions with intermediate helicity structure.

This subset of contributions corresponds to contributions where the trivalent vertex attached to leg three is MHV and the subset of contributions \( S - S' \) is when the trivalent vertex attached to leg four is MHV.

This summation of terms, i.e., eqn. (4.5) corresponds precisely to the terms arising from using the recursive methods of Britto, Cachazo and Feng [44] applied to graviton scattering [24, 25]. Specifically in ref. [25] the NMHV tree amplitude was written in the form
\[
D_1 + D_1|_{1 \leftrightarrow 2} + \bar{D}_1 + \bar{D}_1|_{1 \leftrightarrow 2} + D_2 + D_2|_{1 \leftrightarrow 2} + D_2|_{5 \leftrightarrow 6} + D_2|_{1,5 \leftrightarrow 2,6} + D_3 + D_3|_{1 \leftrightarrow 2} + \bar{D}_3 + \bar{D}_3|_{5 \leftrightarrow 6} + D_6,
\]

where each term arises from a recursive diagram. This expression identifies term-by-term with eq. (4.5) with
\[
D_1 = H_1^{ns}[1, 5, 6, 2, 3, 4]/s_{34}, \quad D_2 = G_3^{s}[4, 2, 5, 1, 6, 3]/s_{34}, \\
D_3 = H_2^{s}[2, 5, 6, 1, 4, 3]/s_{34}, \quad D_6 = G_1^{ns}[4, 1, 2, 5, 6, 3]/s_{34}.
\]

We have checked that this expression agrees numerically with the expression for the six-point tree amplitude obtained using the KLT-relation.

By considering, the coefficient of \( \ln(-s_{12}) \) we can also deduce that the following subset of
terms gives $s_{12} \times \ln(-s_{12})$

\[
\left( H^{1}_{0}[4, 5, 6, 1, 2, 3] \right) + \left( H^{ns}_{1}[3, 5, 6, 1, 2, 4] + H^{ns}_{1}[3, 4, 6, 1, 2, 5] + H^{ns}_{1}[3, 5, 4, 1, 2, 6] \right)
+ \left( G^{ns}_{4}[1, 3, 4, 5, 6, 2] + G^{ns}_{4}[1, 3, 5, 4, 6, 2] + G^{ns}_{4}[1, 3, 6, 5, 4, 2] + G^{s}_{4}[2, 3, 4, 5, 6, 1] + G^{s}_{4}[2, 3, 5, 4, 6, 1] + G^{s}_{4}[2, 3, 6, 5, 4, 1] \right),
\]

(4.8)

which corresponds to the contributions with intermediate helicity structure

This alternate expression also contains thirteen terms and presumably arises by taking legs 1 and 2 as the reference legs for the recursion relations.

5 Conclusions

We have presented a form for the box-coefficients of the one-loop NMHV six-graviton amplitude in $\mathcal{N} = 8$ supergravity, which makes its relation to $\mathcal{N} = 4$ super-Yang-Mills amplitudes manifest. The coefficients have a very similar structure to those in $\mathcal{N} = 4$ super-Yang-Mills being a sum of a singlet and a non-singlet term with all terms obtained from a generating function. The IR singularities contained in these box-functions were shown to give the entire IR structure of the amplitude. This is strong evidence that the six-point amplitudes, like those of $\mathcal{N} = 4$ super-Yang-Mills, are composed entirely of scalar box-functions with rational coefficients.

We have used the box coefficients to generate expressions for the tree amplitude. These expressions have a better twistor space structure than those generated via the KLT relations and could prove to be related to an underlying CSW type formulation of gravity scattering amplitudes.

Given the absence of integral functions beyond scalar box integrals within the five and six-point amplitudes, it becomes difficult to see how these functions can appear in higher point amplitudes whilst still satisfying factorisation and soft limit constraints. Hence, it seems increasingly likely that $\mathcal{N} = 8$ supergravity one-loop amplitudes are composed only of box integral functions.

This simplification is unexpected from power counting arguments, which are based on the known symmetries of $\mathcal{N} = 8$ supergravity. One might suspect, this implies the existence of further symmetries and additional constraints on the scattering amplitudes. It seems promis-
ing, although a challenge, to utilise the simplification of the one-loop amplitudes to determine the ultra-violet behaviour of higher loop scattering amplitudes in $\mathcal{N} = 8$ supergravity.

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