Coherent states for Landau levels: algebraic and thermodynamical properties

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Abstract

This work addresses a study of coherent states for a physical system governed by a Hamiltonian operator describing the motion, in two dimensional space, of spinless electrons subjected to a perpendicular magnetic field $B$ coupled with a harmonic potential $\frac{1}{2}M\omega_0^2r^2$. The underlying $su(1,1)$ Lie algebra and Barut-Girardello coherent states are constructed and discussed. Further, the Berezin - Klauder - Toeplitz quantization, also known as coherent state (or anti-Wick) quantization, is also performed. The thermodynamics of such a quantum electron gas system is elaborated and analyzed.

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1 Introduction

The system of charged quantum particle interacting with a constant magnetic field is undoubtedly one of the most investigated problems in quantum mechanics, mainly inspired by condensed matter physics and quantum optics. A family of coherent states (CS) adapted to such a system was first proposed in [1]. In [2], the behavior of the transverse motion of electrons in an external uniform magnetic field $B$ was considered. A complete set of CS wave packets was constructed. These states are non spreading packets of minimum uncertainty that follow the classical motion. They are the eigenstates of two non-Hermitian operators that annihilate the zero-angular-momentum ground state. The CS basis was used for the calculation of the partition function. The Landau diamagnetism and the de Haas-van Alphen oscillations were contained in this state setting.

Afterwards, one can also mention some alternative constructions proposed in [3] and [4]. In metal and other dense electronic systems, the electrons occupy many Landau levels. Furthermore, the kinetic energy levels of electrons in two-dimensional gas correspond to Landau levels. In particular in [5], the generalized Gazeau-Klauder CS were extended to systems with more than one degree of freedom. There, three different types of these generalized CS with symmetries of harmonic oscillator, $su(2)$ and $su(1,1)$ for the Hamiltonian corresponding to two-dimensional spinless electrons confined by an isotropic harmonic potential in the presence of a constant magnetic field were considered. CS also play an important role in non-equilibrium statistical physics that describes the evolution towards thermodynamic equilibrium for quantum systems with equidistant energy spectra set in thermostat [6].

Besides, the CS were investigated to obtain Landau diamagnetism for a free electron gas [7]. In [8] generalized Klauder-Perelomov [6] and Gazeau-Klauder [9] CS of Landau levels were constructed using two different representations for the Lie algebra $h_4$. In [10], the Landau levels were reorganized into two different hidden symmetries, namely $su(2)$ and $su(1,1)$. The representation of $su(1,1)$ by the Landau levels then led to the construction of the Barut-Girardello CS (BGCS) [11]. Moreover, Dehghani et al [12] developed Glauber two-variable CS in different representations based on the action of unitary displacement operator corresponding to Weyl-Heisenberg algebra on the Klauder-Perelomov CS of $su(1,1)$ and $su(2)$ algebras, for Landau levels, minimizing the Heisenberg uncertainty relation. More recently, Bergeron et al [13] investigated the consistency of CS quantization, (also named Berezin-Klauder-Toeplitz or anti-Wick quantization), and led to the conclusion that the predictions issued from this type of quantization and those from the canonical quantization are compatible on a physical level, for non-relativistic systems, even if these two quantization techniques are not mathematically equivalent.

This work aims at considering Landau levels generated by a physical system engendered by a Hamiltonian operator describing the motion, in two dimensional space, of spinless electrons subjected to a perpendicular magnetic field $B$ coupled with a harmonic potential $\frac{1}{2}M\omega_0^2 r^2$. The underlying $su(1,1)$ Lie algebra and BGCS are constructed and discussed. Further, the Berezin-Klauder-Toeplitz quantization is performed through this paper. The thermodynamics of such a quantum electron gas system is elaborated and analyzed.

The paper is organized as follows. In section 2, we start with the study of a spinless electron gas on the $(x, y)$-space in a magnetic field $B$ with an isotropic harmonic potential by using step and orbit-center coordinate operators. In section 3, the $su(1,1)$ representation
in the Hamiltonian quantum states is studied. Furthermore, follows a discussion on the
BGCS of the Lie algebra $su(1,1)$ defined on the Hilbert subspaces, the mean values of
$SU(1,1)$ group generators, the probability density and the time dependence of the BGCS.
Some quantum mechanical features deduced from the quantization of a complex plane using
these states are investigated in section 4. The results issued from this procedure are exploited
to derive quadrature operator mean values and some quantum optical properties. In section
5, main statistical properties for the quantum gas in the thermodynamic equilibrium with
a reservoir (thermostat) at temperature $T$ are computed and analyzed. The last section is
devoted to concluding remarks.

2 Hamiltonian operator of an electron in a uniform
magnetic field with a harmonic potential

Consider a system of spinless electrons $(M, e)$ living on the $(x, y)$-space in a magnetic field $B$
along the $z$-direction. Recall that the eigenstates and eigenvalues of such a system were inves-
tigated for the first time by Landau [7]. When a harmonic confining potential is introduced
and the Coulomb interactions are neglected, this system is described by the Fock-Darwin
Hamiltonian [14, 15]

\[
\mathcal{H} = \frac{1}{2M} \left( p + \frac{e}{c} A \right)^2 + \frac{M\omega_0^2}{2} r^2
\]

(1)

where $p$ is the canonical momentum and $A$ is the vector potential. We study the problem
by considering the transverse motion of the electrons on the $(x, y)$-space.

In the symmetric gauge

\[
A = \left( -\frac{B}{2} y, \frac{B}{2} x \right),
\]

(2)

the classical Hamiltonian $\mathcal{H}$ in (1) becomes

\[
\tilde{\mathcal{H}}(p, r) \equiv \tilde{\mathcal{H}} = \frac{1}{2M} \left[ (p_x - \frac{eB}{2c} y)^2 + (p_y + \frac{eB}{2c} x)^2 \right] + \frac{M\omega_0^2}{2} (x^2 + y^2).
\]

(3)

The canonical quantization of this system is obtained by introducing the coordinate and
momentum operators, denoted here by $\hat{R}_i, \hat{P}_i$, respectively, which satisfy

\[
[\hat{R}_i, \hat{P}_j] = i\hbar \delta_{ij},
\]

(4)

with $\hat{R}_i = \hat{X}, \hat{Y}$, $\hat{P}_i = \hat{P}_x, \hat{P}_y$, $i = 1, 2$. Denote the Hamiltonian operator associated with
(3) by $\hat{H}$ and solve the eigenvalue problem

\[
\hat{H} \Psi = \mathcal{E} \Psi
\]

(5)

by performing the following change of variables:

\[
\hat{Z} = \hat{X} + i\hat{Y}, \quad \hat{P}_z = \frac{1}{2}(\hat{P}_x - i\hat{P}_y)
\]

(6)
and considering the set of step operators defined by \[2\]
\[
\pi_+ = 2\hat{P}_z + i\frac{M\Omega}{2}\hat{Z}, \quad \pi_- = 2\hat{P}_z - i\frac{M\Omega}{2}\hat{Z},
\]
(7)
where the motion along the z-axis is free, with \(\Omega = \sqrt{\omega_c^2 + 4\omega_0^2}\) and the cyclotron frequency \(\omega_c = \frac{eB}{Mc}\). These operators satisfy the commutation relation
\[
[\pi_-, \pi_+] = 2M\Omega\hbar.
\]
(8)

Next, consider the orbit-center coordinate operators given by
\[
\mathcal{X}_+ = \left(\hat{X} - \frac{\hat{\pi}_y}{M\Omega}\right) + i\left(\hat{Y} + \frac{\hat{\pi}_x}{M\Omega}\right) = \frac{1}{2}\hat{Z} + \frac{2i}{M\Omega}\hat{P}_z
\]
(9)
and
\[
\mathcal{X}_- = \left(\hat{X} - \frac{\hat{\pi}_y}{M\Omega}\right) - i\left(\hat{Y} + \frac{\hat{\pi}_x}{M\Omega}\right) = \frac{1}{2}\hat{Z} - \frac{2i}{M\Omega}\hat{P}_z
\]
(10)
carrying out the following commutation rules
\[
[\mathcal{X}_3, \mathcal{X}_\pm] = \pm\hbar\mathcal{X}_\pm, \quad [\mathcal{X}_+, \mathcal{X}_-] = 2l^2, \quad [\mathcal{X}_\pm, \mathcal{I}] = 0, \quad [\mathcal{X}_3, \mathcal{I}] = 0, \quad \mathcal{X}_3 = L_z,
\]
(11)
where \(l := \sqrt{\frac{\hbar}{M\Omega}}\) is taken as the classical radius of the ground-state’s Landau orbit for the frequency \(\Omega\). Also, we have
\[
[\pi_3, \pi_\pm] = \pm\hbar\pi_\pm, \quad [\pi_\pm, \mathcal{I}] = 0, \quad [\pi_3, \mathcal{I}] = 0, \quad \pi_3 = L_z.
\]
(12)

Physically, \(\pi_3\) is a constant of the motion of a free particle, as defined in \[16\], along the z-axis; \(\mathcal{X}_\pm\) step only the angular momentum \(m\) and not the energy \[2\]. Besides, the operators \(\mathcal{X}_\pm\) and \(\pi_\pm\) commute through the following relations
\[
[\mathcal{X}_\pm, \pi_\pm] = 0, \quad [\mathcal{X}_\pm, \pi_\mp] = 0.
\]
(13)

Finally, the Hamiltonian \[1\] can be rewritten as
\[
\tilde{H} = \frac{1}{2}\left[\frac{\pi_+\pi_-}{2M} \left(1 + \frac{\omega_c}{\Omega}\right) + \frac{M\Omega^2}{2} \left(1 - \frac{\omega_c}{\Omega}\right) \mathcal{X}_- \mathcal{X}_+ + \hbar\Omega\right]
\]
(14)
with the eigenstates \(|n, m\rangle\) determined by two quantum numbers: \(n\), associated to the energy, and \(m\), to the z-projection of the angular momentum, solving the time-independent Schrödinger equation \(\tilde{H}|n, m\rangle = \mathcal{E}_{n,m}|n, m\rangle\) with the corresponding eigenvalues
\[
\mathcal{E}_{n,m} = \hbar\Omega \left(n + \frac{1}{2}\right) - \frac{\hbar}{2}(\Omega - \omega_c)m.
\]
(15)
3 Representation of su(1, 1) algebra by the quantum Hamiltonian states

3.1 Hilbert space representation

Rewrite now the step and orbit-center coordinate operators \( \hat{\pi}_\pm, \hat{\chi}_\pm \) with the help of dimensionless variables as follows:

\[
\hat{\pi}_+ = l \sqrt{2} \left[ \frac{i}{4l^2} \hat{Z} + \frac{1}{\hbar} \hat{P}_z \right], \quad \hat{\pi}_- = l \sqrt{2} \left[ -\frac{i}{4l^2} \hat{Z} + \frac{1}{\hbar} \hat{P}_z \right],
\]

\[
\hat{\chi}_+ = \frac{\sqrt{2}}{l} \left[ \frac{1}{4} \hat{Z} + \frac{i l^2}{\hbar} \hat{P}_z \right], \quad \hat{\chi}_- = \frac{\sqrt{2}}{l} \left[ \frac{1}{4} \hat{Z} - \frac{i l^2}{\hbar} \hat{P}_z \right],
\]

where \( l \) is defined as previously, so that they satisfy the following canonical commutation relations

\[
[\hat{\pi}_-, \hat{\pi}_+] = 1 = [\hat{\chi}_+, \hat{\chi}_-].
\]

Introduce the Lie algebra su(1, 1) \([17]\) corresponding to the \( SU(1, 1) \) group, spanned by the three group generators \( \{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3\} \):

\[
[\mathcal{K}_1, \mathcal{K}_2] = -i \mathcal{K}_3, \quad [\mathcal{K}_2, \mathcal{K}_3] = i \mathcal{K}_1, \quad [\mathcal{K}_3, \mathcal{K}_1] = i \mathcal{K}_2.
\]

Use the raising and lowering generators \( \mathcal{K}_\pm = \mathcal{K}_1 \pm i \mathcal{K}_2 \) to form the second-order differential operators given by

\[
\mathcal{K}_+ := \hat{\pi}_+ \hat{\chi}_- - \hat{\pi}_- \hat{\chi}_+, \quad \mathcal{K}_- := \hat{\pi}_- \hat{\chi}_+ - \hat{\pi}_+ \hat{\chi}_-, \quad \mathcal{K}_3 := \frac{1}{2} (\hat{\pi}_+ \hat{\pi}_- + \hat{\chi}_- \hat{\chi}_+) - \frac{1}{4} (\hat{\pi}_+ \hat{\chi}_+ + \hat{\pi}_- \hat{\chi}_-)
\]

which satisfy the commutation relations of the Lie algebra su(1, 1):

\[
[\mathcal{K}_+, \mathcal{K}_-] = -2 \mathcal{K}_3, \quad [\mathcal{K}_3, \mathcal{K}_\pm] = \pm \mathcal{K}_\pm.
\]

Then consider the Hilbert space \( \mathcal{H} = \text{span}\{ |n, m\rangle \}_{n \geq 0, 0 \leq m \leq n} = \bigoplus_{m=0}^{\infty} \mathcal{H}_m \), where \( \mathcal{H}_m \) is the subspace defined by \( \mathcal{H}_m := \text{span}\{ |n, m\rangle \}_{n \geq m \geq 0} \). These operators act as follows:

\[
\hat{\pi}_+ |n-1, m-1\rangle = \sqrt{n} |n, m\rangle, \quad \hat{\pi}_- |n, m\rangle = \sqrt{n} |n-1, m-1\rangle,
\]

\[
\hat{\chi}_+ |n, m\rangle = \sqrt{(n-m)} |n, m+1\rangle, \quad \hat{\chi}_- |n, m\rangle = \sqrt{(n-m+1)} |n, m-1\rangle,
\]

\[
\mathcal{K}_+ |n-1, m\rangle = \sqrt{n(n-m)} |n, m\rangle, \quad \mathcal{K}_- |n, m\rangle = \sqrt{n(n-m)} |n-1, m\rangle, \quad \mathcal{K}_3 |n, m\rangle = \frac{1}{2} (2n-m+1) |n, m\rangle.
\]

The lowest state \( |m, m\rangle \) is annihilated by the lowering operator \( \mathcal{K}_- \), i.e.

\[
\mathcal{K}_- |m, m\rangle = 0
\]

while the arbitrary states \( |n, m\rangle \) are constructed by means of the first equation of (24):

\[
|n, m\rangle = \sqrt{\frac{\Gamma(m+1)}{\Gamma(n-m+1)\Gamma(n+1)}} \mathcal{K}_3^{n-m} |m, m\rangle.
\]
3.2 CS for the $su(1, 1)$ algebra

Introducing the BGCS \[11\] in the Hilbert subspace $\mathcal{H}_m$ as eigenstates of the lowering generator $K_-$ of the Lie algebra $su(1, 1)$:

$$
K_- |z\rangle_m = z |z\rangle_m
$$

(27)

where $z$ is an arbitrary complex variable with $z = \rho e^{i\phi}, 0 \leq \rho < \infty, 0 \leq \phi < 2\pi$, we can represent the eigenstates $|z\rangle_m$ as the superposition of the complete orthonormal basis $|n, m\rangle$ of $\mathcal{H}_m$:

$$
|z\rangle_m = \sum_{n=m}^{+\infty} \langle n, m|z\rangle_m |n, m\rangle.
$$

(28)

Using (27), (28), the third equation in (24) and the orthonormality relation

$$
\sum_{n=m}^{+\infty} |n, m\rangle \langle n, m| = I_{\mathcal{H}_m}
$$

(29)

we get

$$
\langle n, m|z\rangle_m = \frac{z}{\sqrt{n(n-m)}} \langle n-1, m|z\rangle_m
$$

(30)

which can be recursively transformed into

$$
\langle n, m|z\rangle_m = \sqrt{\frac{\Gamma(m+1)}{\Gamma(n-m+1)\Gamma(n+1)}} z^{n-m} \langle m, m|z\rangle_m.
$$

(31)

Coming back to the relation (27), the following expansion of the state $|z\rangle_m$ in the Hilbert subspace $\mathcal{H}_m$:

$$
|z\rangle_m = \langle m, m|z\rangle_m \sum_{n=m}^{+\infty} z^{n-m} \sqrt{\frac{\Gamma(m+1)}{\Gamma(n-m+1)\Gamma(n+1)}} |n, m\rangle
$$

(32)

holds. The normalization factor $|\langle m, m|z\rangle_m|$ can be obtained by normalizing to unity the states $|z\rangle_m$ and using the relation \[18\]

$$
\sum_{n=0}^{+\infty} \frac{x^n}{n!\Gamma(n+\nu+1)} = \frac{1}{x^\nu} I_\nu(2x)
$$

(33)

where $I_\nu(2x)$ is the modified Bessel function of order $\nu$. It follows

$$
|\langle m, m|z\rangle_m| = \sqrt{\frac{|z|^m}{I_m(2|z|)\Gamma(m+1)}}
$$

(34)

and finally, the eigenstates $|z\rangle_m$ become

$$
|z\rangle_m = \frac{|z|^{m/2}}{\sqrt{I_m(2|z|)}} \sum_{n=m}^{+\infty} \frac{z^{n-m}}{\sqrt{\Gamma(n-m+1)\Gamma(n+1)}} |n, m\rangle
$$

(35)
where $I_m(2|z|)$ is the modified Bessel function of the first kind given by [18]:

$$I_m(2|z|) = \sum_{n=0}^{+\infty} \frac{|z|^{2n+m}}{\Gamma(n+1)\Gamma(n+m+1)}. \quad (36)$$

These states satisfy the following resolution of the identity [11]:

$$\int_{C} |z\rangle_m \langle z| d\varrho(z) = I_{\delta_m} \quad (37)$$

which, in this case, holds on $\mathcal{H}_m$ where $d\varrho(z)$ is an appropriate measure. Indeed, performing the integrals over the whole complex plane, where $z = \rho e^{i\phi}, \rho \in [0, \infty), \phi \in [0, 2\pi)$ and taking

$$d\varrho(z) = \frac{2}{\pi} I_m(2|z|) K_m(2|z|) d^2z, \quad d^2z = d(Re z)d(Im z), \quad (38)$$

the relation [29] defined on $\mathcal{H}_m$ leads to the resolution of the identity [37] by using the following integral relation:

$$4 \int_{0}^{\infty} \rho^{2n-m+1} K_m(2\rho)d\rho = \Gamma(n-m+1)\Gamma(n+1) \quad (39)$$

where $K_m(2\rho)$ given by

$$K_m(2\rho) = \frac{\pi}{2} \frac{I_{-m}(2\rho) - I_m(2\rho)}{\sin(m\pi)} \quad (40)$$

is the modified Bessel function of the second kind [18]. The relation [39] is provided by the following integral [19]:

$$\int_{0}^{\infty} dx \ x^\mu K_\nu(ax) = 2^{\mu-1}a^{-\mu-1}\Gamma\left(\frac{1+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\mu-\nu}{2}\right), \quad [Re(\mu+1\pm\nu) > 0, Re a > 0]. \quad (41)$$

The states $|z\rangle_m$ form an overcomplete basis for any allowed value of $m$. Indeed, from [32], setting $z = \rho e^{i\phi}$, where $\rho \in [0, \infty), \phi \in [0, 2\pi)$, it follows that

$$\int_{C} |z\rangle_m \langle z'| d\varrho(z) = 4 \sum_{n=m}^{+\infty} \int_{0}^{\infty} K_m(2\rho) \frac{\rho^{2n-m+1}d\rho|n, m\rangle\langle n, m|}{\Gamma(n-m+1)\Gamma(n+1)} \quad (42)$$

and using [39], we get on the Hilbert subspace $\mathcal{H}_m$:

$$\int_{C} |z\rangle_m \langle z'| d\varrho(z) = \sum_{n=m}^{+\infty} |n, m\rangle\langle n, m| = I_{\delta_m}. \quad (43)$$

The BGCS [35] are not orthogonal as expected. Indeed, given two vectors $|z\rangle_m$ and $|z'\rangle_{m'}$ on the Hilbert subspaces $\mathcal{H}_m$ and $\mathcal{H}_{m'} (m \neq m')$, respectively, we get the following expression

$$m\langle z'|z\rangle_m = \left(\frac{|z'|}{\sqrt{I_m(2|z'|)I_m(2|z|)}}\right)^{m/2} \frac{I_m\left(2\sqrt{|z'|}\right)}{I_m(2|z|)} \quad (44)$$
The overcompleteness of the BGCS $|z\rangle_m$ on $\mathcal{H}_m$ suggests to discuss their relation with the reproducing kernels \[20\].

Define the quantity $K(z, z') := m\langle z'|z\rangle_m$ on $\mathcal{H}_m$. Using the facts

$$K(z, z') := m\langle z'|z\rangle_m$$

and setting $n = m + \nu$, it follows that

$$m\langle z'|z\rangle_m = \left(\frac{|z'|}{\bar{z}'z}\right)^{m/2} \frac{I_m(2\sqrt{\bar{z}'z})}{\sqrt{I_m(2|z'|)I_m(2|z|)}} =: K(z', z).$$

(47)

$K(z, z')$ is well a reproducing kernel. Indeed,

**Proposition 3.1** The following properties

(i) hermiticity

$$K(z, z') = \overline{K(z', z)},$$

(48)

(ii) positivity

$$K(z, z) > 0,$$

(49)

(iii) idempotence

$$\int_{\mathbb{C}} d\varrho(z'')K(z, z'')K(z'', z') = K(z, z')$$

(50)

are satisfied by the function $K$ on $\mathcal{H}_m$.

**Proof.** (i) and (ii) follow from (47) and (44), respectively. The left-hand side of (50) is written as:

$$\int_{\mathbb{C}} d\varrho(z'')K(z, z'')K(z'', z') = \int_{\mathbb{C}} d\varrho(z'') \left[ \frac{|z''|}{\bar{z''}z''} \right]^{m/2} \times$$

$$\times \frac{I_m(2\sqrt{\bar{z''}z})}{\sqrt{I_m(2|z''|)I_m(2|z|)}} \frac{I_m(2\sqrt{\bar{z'}z'})}{\sqrt{I_m(2|z'|)I_m(2|z|)}}$$

$$= \int_{\mathbb{C}} \left[ \frac{K_m(2|z'|)I_m(2|z''|)}{\sqrt{I_m(2|z''|)I_m(2|z|)}} \right] d^2z'' \times$$

$$\times \left(\frac{|z'|}{\bar{z}'z}\right)^{m/2} \left(\frac{|z''|}{\bar{z''}z''}\right)^{m/2} \frac{I_m(2\sqrt{\bar{z''}z})}{\sqrt{I_m(2|z''|)I_m(2|z|)}} \frac{I_m(2\sqrt{\bar{z'}z'})}{\sqrt{I_m(2|z'|)I_m(2|z|)}}$$

(51)
where the relation

\[ \int_{\mathbb{C}} d\varrho(z) = \int_{\mathbb{C}} d^2z \frac{2}{\pi} K_m(2 |z|) I_m(2 |z|) = 1 \]  

(52)
is used.

From (43), for any \(|\Psi\rangle \in \mathfrak{H}_m\), we have

\[ |\Psi\rangle = \int_{\mathbb{C}} d\varrho(z) \Psi(z) |z\rangle_m \]  

(53)
where \(\Psi(z) := m\langle z|\Psi\rangle\). The following reproducing property

\[ \Psi(z) = \int_{\mathbb{C}} d\varrho(z') \mathcal{K}(z, z') \Psi(z') \]  

(54)
is also verified.

The Hilbert space \(\mathfrak{H}_m := \text{span}\{|n, m\rangle\}_{n \geq m}, m \geq 0\) can be represented as the Hilbert space of analytic functions in the variable \(z\). Given a normalized state \(|\Phi\rangle = \sum_{k=m}^{+\infty} C_k |k, m\rangle, C_k \in \mathbb{C}\) on \(\mathfrak{H}_m\), we obtain

\[ m\langle \bar{z}|\Phi\rangle = \frac{|z|^{m/2}}{\sqrt{I_m(2 |z|)}} \sum_{n=m}^{+\infty} \frac{C_n z^{n-m}}{\sqrt{\Gamma(n-m+1)\Gamma(n+1)}} \]  

(55)
such that the entire functions

\[ f(z; m) = \frac{\sqrt{I_m(2 |z|)}}{|z|^{m/2}} m\langle \bar{z}|\Phi\rangle = \sum_{n=m}^{+\infty} \frac{C_n z^{n-m}}{\sqrt{\Gamma(n-m+1)\Gamma(n+1)}} \]  

(56)
are analytic over the whole \(z\) plane. Then, from (43), we can write

\[ |\Phi\rangle = \int_{\mathbb{C}} d\varrho(z) \frac{|z|^{m/2}}{\sqrt{I_m(2 |z|)}} f(\bar{z}; m) |z\rangle_m \]  

(57)
and express the scalar product of two states \(|\Phi_1\rangle\) and \(|\Phi_2\rangle\) given on \(\mathfrak{H}_m\) by the formula

\[ \langle \Phi_1|\Phi_2\rangle = \int_{\mathbb{C}} d\varrho(z) \frac{|z|^m}{I_m(2 |z|)} f_1(\bar{z}; m) f_2(\bar{z}; m). \]  

(58)
3.3 Mean values

Using equation \((35)\), the mean value of a physical observable \(\mathcal{O}\) with respect to the BGCS \(|z\rangle_m\) is obtained as:

\[
m\langle z|\mathcal{O}|z\rangle_m \equiv \langle \mathcal{O} \rangle_{z,m} = \frac{|z|^m}{I_m(2|z|)} \sum_{n,k=m}^{+\infty} z^{n-m} z^{k-m} \sqrt{\frac{1}{\Gamma(n-m+1)\Gamma(n+1)}} \times \sqrt{\frac{1}{\Gamma(k-m+1)\Gamma(k+1)}} \langle k, m|\mathcal{O}|n, m \rangle.
\]

(59)

Setting \(n = m + \nu, k = m + \upsilon\), respectively, \((59)\) can be rewritten as follows:

\[
\langle \mathcal{O} \rangle_{z,m} = \frac{|z|^m}{I_m(2|z|)} \sum_{\nu, \upsilon = 0}^{+\infty} \sqrt{\frac{1}{\Gamma(\nu + 1)\Gamma(\nu + m + 1)\Gamma(\nu + 1)\Gamma(\nu + m + 1)}} \langle \nu, m|\mathcal{O}|\nu, m \rangle.
\]

(60)

The computation of the mean values of the \(\mathcal{K}_i, (i = 1, 2, 3)\), from \((24)\) and \((60)\) gives

\[
\langle \mathcal{K}_- \rangle_{z,m} = \frac{1}{I_m(2|z|)} \sum_{\nu = 0}^{+\infty} \frac{|z|^{2\nu + m}}{\Gamma(\nu + 1)\Gamma(\nu + m + 1)} = z,
\]

(61)

and

\[
\langle \mathcal{K}_+ \rangle_{z,m} = \frac{1}{I_m(2|z|)} \sum_{\nu = 0}^{+\infty} \frac{|z|^{2\nu + m}}{\Gamma(\nu + 1)\Gamma(\nu + m + 1)} = \bar{z}.
\]

(62)

We can conclude, therefore, that \(\langle \mathcal{K}_- \rangle_{z,m}\) and \(\langle \mathcal{K}_+ \rangle_{z,m}\) are mutually conjugated.

Since the generators \(\mathcal{K}_+, \mathcal{K}_-\) are given by \(\mathcal{K}_\pm = \mathcal{K}_1 \pm i\mathcal{K}_2\), we obtain from \((61)\) and \((62)\) together the following expressions:

\[
\langle \mathcal{K}_1 \rangle_{z,m} = \frac{1}{2} \langle \mathcal{K}_- + \mathcal{K}_+ \rangle_{z,m} = \frac{1}{2}(z + \bar{z}) = Re z,
\]

(63)

\[
\langle \mathcal{K}_2 \rangle_{z,m} = \frac{i}{2} \langle \mathcal{K}_- - \mathcal{K}_+ \rangle_{z,m} = \frac{i}{2}(z - \bar{z}) = -Im z.
\]

(64)

In order to compute the mean values of the generator \(\mathcal{K}_3\) and its second power \(\mathcal{K}_3^2\), it is useful to evaluate the sum \(S_n\), with \(n = 0, 1, 2, \ldots\), given by (see \([24]\) in Appendix):

\[
S_n = \sum_{\nu = 0}^{+\infty} \frac{(x^2)^\nu}{\Gamma(\nu + 1)\Gamma(\nu + n + 1)} \nu^n.
\]

(65)

Then, from \((24)\), we obtain

\[
\langle \mathcal{K}_3 \rangle_{z,m} = |z| \frac{I_{m+1}(2|z|)}{I_m(2|z|)} + \frac{m + 1}{2}
\]

(66)

and

\[
\langle \mathcal{K}_3^2 \rangle_{z,m} = |z|^2 \frac{I_{m+2}(2|z|)}{I_m(2|z|)} + (m + 2)|z| \frac{I_{m+1}(2|z|)}{I_m(2|z|)} + \left(\frac{m + 1}{2}\right)^2.
\]

(67)
Besides, the number operator $N$ diagonalizing the basis vectors $\{|\nu, m\rangle, \nu \geq 0\}$ for the number states,

$$N|\nu, m\rangle = \nu|\nu, m\rangle,$$  \hspace{1cm} (68)

is used to compute the photon number distribution as follows:

$$|\langle \nu, m|z\rangle_m|^2 = \frac{|z|^{2\nu+m}}{I_m(2|z|)\Gamma(\nu+1)\Gamma(\nu+m+1)}.$$  \hspace{1cm} (69)

Exploiting (68) and (67), we are able to compute explicitly the mean values of the number operator and its second power to obtain:

$$\langle N\rangle_{z,m} = \langle K_3 - \frac{m+1}{2}\rangle_{z,m} = |z|\frac{I_{m+1}(2|z|)}{I_m(2|z|)},$$  \hspace{1cm} (70)

$$\langle N^2\rangle_{z,m} = \left(\frac{K_3^2 - (m+1)K_3 + \left(\frac{m+1}{2}\right)^2}{I_m(2|z|)}\right)_{z,m} = |z|^2\frac{I_{m+2}(2|z|)}{I_m(2|z|)} + |z|\frac{I_{m+1}(2|z|)}{I_m(2|z|)}.$$  \hspace{1cm} (71)

Then, it becomes straightforward to extend the calculation to the intensity correlation defined as in [23]:

$$g^{(2)}_{z,m} = \frac{\langle N^2\rangle_{z,m} - \langle N\rangle_{z,m}^2}{\langle N\rangle_{z,m}^2} = \frac{I_m(2|z|)I_{m+2}(2|z|)}{[I_{m+1}(2|z|)]^2}.$$  \hspace{1cm} (72)

More specifically, for two interesting limiting cases of the $|z|$ variable, i.e. for $|z| \ll 1$ and $|z| \gg 1$, using the approximations for the Bessel modified function $I_m(x)$ [22]

$$I_m(x) \approx \frac{1}{\Gamma(m+1)}\left(\frac{x}{2}\right)^m,$$  \hspace{1cm} (73)

respectively

$$I_m(x) = \frac{e^x}{\sqrt{2\pi x}}\left[1 + O\left(\frac{1}{x}\right)\right],$$  \hspace{1cm} (74)

we get for the intensity correlation function, with

$$I_m(2|z|) \approx \frac{|z|^m}{\Gamma(m+1)}, \ I_{m+1}(2|z|) \approx \frac{|z|^{m+1}}{\Gamma(m+2)}, \ I_{m+2}(2|z|) \approx \frac{|z|^{m+2}}{\Gamma(m+3)},$$  \hspace{1cm} (75)

the following expressions:

$$g^{(2)}_{z,m} \approx \frac{m+1}{m+2},$$  \hspace{1cm} (76)

$$g^{(2)}_{z,m} \approx 1,$$  \hspace{1cm} (77)

respectively. Thus, for small values of $|z|$, the intensity correlation function is smaller than unity, for all values of $m$. The corresponding BGCS $|z\rangle_m$ exhibit sub-Poissonian statistics behaviour, while for large $|z|$, these states tend to have Poissonian statistics. In addition, the photon-number distribution (69) is sub-Poissonian [23].

10
3.4 Probability density and time evolution

From the quantity

\[ m(z'|z)_m = \left( \frac{|z'|z}{z'z} \right)^{m/2} \frac{1}{\sqrt{I_m(2|z|)I_m(2|z|)}} \sum_{\nu=0}^{+\infty} \frac{(\sqrt{z'}z)^{2\nu+m}}{\Gamma(\nu+1)\Gamma(\nu+m+1)} \]  

(78)

given a normalized state \(|z_0\rangle_m\), let us define the probability density as follows:

\[ z \mapsto \varrho_{z_0}(z) := |m(z|z_0)_m|^2 = \frac{I_m(2\sqrt{z_0\bar{z}})I_m(2\sqrt{\bar{z}_0z})}{I_m(2|z|)I_m(2|z_0|)}. \]

(79)

The time evolution behavior of \( \varrho_{z_0}(z) \) is then provided by

\[ z \mapsto \varrho_{z_0}(z, t) := |m(z|e^{-\frac{i\hbar}{\hbar}}|z_0\rangle_m|^2. \]

(80)

By acting the evolution operator \( U(t) = e^{-\frac{i}{\hbar}\tilde{H}t} \) on the state \(|z_0\rangle_m\), we get

\[ |z_0; t\rangle_m = e^{-\frac{i}{\hbar}\tilde{H}t}|z_0\rangle_m = e^{-\frac{i}{2}[n(\Omega+\omega_c)+\Omega]t}|z_0(t)\rangle_m. \]

(81)

where \( z_0(t) := z_0e^{-\frac{i}{2}[\Omega-\omega_c]t} \).

Then, we obtain

\[ \varrho_{z_0}(z, t) := |m(z|e^{-\frac{i}{\hbar}\tilde{H}t}|z_0\rangle_m|^2 = \frac{I_m(2\sqrt{z_0(t)\bar{z}})I_m(2\sqrt{\bar{z}_0z})}{I_m(2|z|)I_m(2|z_0(t)|)}. \]

(82)

It comes that the time dependence of a given BGCS \(|z\rangle_m\) is provided by

\[ |z; t\rangle_m = e^{-\frac{i}{\hbar}\tilde{H}t}|z\rangle_m = e^{-\frac{i}{2}[n(\Omega+\omega_c)+\Omega]t}|z(t)\rangle_m, \quad z(t) := z_0e^{-\frac{i}{2}[\Omega-\omega_c]t}. \]

(83)

4 Quantization with the \( su(1, 1) \) coherent states

As is proved in Section 2, the \( su(1,1) \) CS family resolves the unity. As an immediate consequence, we establish in this section the correspondence (quantization) between classical and quantum quantities. For more details in the quantization procedure see \[24, 25\] and references listed therein.

4.1 Quantization of elementary classical observables

The Berezin-Klauder-Toeplitz quantization of elementary classical variables \( z \) and \( \bar{z} \) is realized via the maps \( z \mapsto A_z \) and \( \bar{z} \mapsto A_{\bar{z}} \) defined by
in the Hilbert space, which give, by using the equations (24), (61) and (62), the following relations:

\[ A_z := \int_C |z\rangle_m \langle z|d\varrho(z), \quad A_{\bar{z}} := \int_C \bar{z}|z\rangle_m \langle z|d\varrho(z) \] (84)

and

\[ A_z = \int_C \langle \mathcal{K}_- \rangle_{z,m} |z\rangle_m \langle z|d\varrho(z) = \sum_{n=m}^{+\infty} \mathcal{K}_- |n+1,m\rangle \langle n+1,m| \]
\[ = \sum_{n=m}^{+\infty} \sqrt{(n - m + 1)(n + 1)|n,m\rangle \langle n + 1,m|} \] (85)

and

\[ A_{\bar{z}} = \int_C \langle \mathcal{K}_+ \rangle_{z,m} |z\rangle_m \langle z|d\varrho(z) = \sum_{n=m}^{+\infty} \mathcal{K}_+ |n-1,m\rangle \langle n-1,m| \]
\[ = \sum_{n=m}^{+\infty} \sqrt{n(n - m)|n,m\rangle \langle n - 1,m|} \] (86)

with the matrix elements:

\[ (A_z)_{n,k} = \frac{2}{\pi} \int_0^{2\pi} K_m(\varrho \rho^{n-m+k+2} d\varrho \times \int_0^{2\pi} e^{-i[(k-m)-(n+1-m)]\varphi} d\varphi \times \frac{1}{\sqrt{\Gamma(n - m + 1)\Gamma(n + 1)\Gamma(k - m + 1)\Gamma(k + 1)}}, \] (87)

\[ (A_{\bar{z}})_{n,k} = \frac{2}{\pi} \int_0^{2\pi} K_m(\varrho \rho^{n-m+k+2} d\varrho \times \int_0^{2\pi} e^{-i[(k+1-m)-(n-m)]\varphi} d\varphi \times \frac{1}{\sqrt{\Gamma(n - m + 1)\Gamma(n + 1)\Gamma(k - m + 1)\Gamma(k + 1)}} \] (88)

respectively.

Their commutator \([A_z, A_{\bar{z}}]\) well gives the result

\[ [A_z, A_{\bar{z}}] = \sum_{n=m}^{+\infty} 2\mathcal{K}_3 |n,m\rangle \langle n,m|, \] (89)

keeping in mind the \(su(1,1)\) commutation rules [21].

Consider now the usual phase space conjugate coordinates \((q, p)\) through \(z = \frac{q+ip}{\sqrt{2}}\). Then, we get for the classical position and momentum functions \(q\) and \(p\) the Hilbert space operators:

\[ Q = A_q := \frac{1}{\sqrt{2}} (A_z + A_{\bar{z}}), \] (90)

and
\[ P = A_p := \frac{1}{i\sqrt{2}}(A_z - A_{\bar{z}}) \]  

(91)

leading to a commutation rule reflecting the \( su(1, 1) \) algebra, given by

\[ [Q, P] = i \sum_{n=m}^{+\infty} 2\mathcal{K}_3|n, m\rangle\langle n, m|. \]  

(92)

Furthermore, using (85) and (86), we get

\[
A_{\frac{1}{2}(z+\bar{z})} = \frac{1}{2} \left[ \sum_{n=m}^{+\infty} \sqrt{(n-m+1)(n+1)}|n, m\rangle\langle n+1, m| 
+ \sum_{n=m}^{+\infty} \sqrt{n(n-m)}|n, m\rangle\langle n-1, m| \right],
\]

and

\[
A_{\frac{1}{2}(z-\bar{z})} = \frac{i}{2} \left[ \sum_{n=m}^{+\infty} \sqrt{(n-m+1)(n+1)}|n, m\rangle\langle n+1, m| 
- \sum_{n=m}^{+\infty} \sqrt{n(n-m)}|n, m\rangle\langle n-1, m| \right].
\]

(93)

(94)

Other interesting results emerging from this context are the following mean values given by:

\[ m\langle z|A_z|z\rangle_m = z, \quad m\langle z|A_{\bar{z}}|z\rangle_m = \bar{z}, \]  

(95)

\[ m\langle z|A_z^2|z\rangle_m = z^2, \quad m\langle z|A_{\bar{z}}^2|z\rangle_m = \bar{z}^2, \]  

(96)

\[ m\langle z|A_z A_{\bar{z}}|z\rangle_m = |z|^2, \quad m\langle z|A_{\bar{z}} A_z|z\rangle_m = |z|^2 + 2\langle \mathcal{K}_3 \rangle_{z,m}. \]  

(97)

### 4.2 Relevant statistical and quantum optical properties

Now, we exploit the results issued from the quantization procedure performed above to derive quadrature operator mean values and some quantum optical properties of the constructed BGCS \( |z\rangle_m \).
4.2.1 Mean values

The mean values of the operators $Q$, $P$, $Q^2$ and $P^2$ are derived in the BGCS $|z\rangle_m$ using (95)-(97) as

$$m\langle z|Q|z\rangle_m = \langle Q \rangle_{z,m} = q, \quad m\langle z|P|z\rangle_m = \langle P \rangle_{z,m} = p,$$

(98)

$$m\langle z|Q^2|z\rangle_m = \langle Q^2 \rangle_{z,m} = q^2 + \langle K_3 \rangle_{z,m}, \quad m\langle z|P^2|z\rangle_m = \langle P^2 \rangle_{z,m} = p^2 + \langle K_3 \rangle_{z,m}.$$

(99)

As well stated indeed in [24], these formulas give CS a quite classical face, although they are rigorously quantal! The physical meaning of the variable $x$ in the Schrödinger position representation of the wave is that of a sharp position. Unlike $x$ in the Schrödinger representation, the variables $p$ and $q$ represent mean values in the CS. For this reason both values can be specified simultaneously at the same time, something that could not be done if instead they both had represented sharp eigenvalues.

From the properties (98) and (99) follow the dispersions:

- For $|z| \ll 1$ by using (66) and the approximation (75) for the Bessel modified function

$$\langle Q^2 \rangle_{z,m} = \langle Q \rangle_{z,m}^2 = \frac{2|z|^2 + (m+1)^2}{2(m+1)}$$

(100)

$$\langle P^2 \rangle_{z,m} = \langle P \rangle_{z,m}^2 = \frac{2|z|^2 + (m+1)^2}{2(m+1)}$$

(101)

yielding

$$\langle Q^2 \rangle_{z,m} \langle P^2 \rangle_{z,m} = \left[ \frac{2|z|^2 + (m+1)^2}{2(m+1)} \right]$$

(102)

which reduces for both conditions $|z| \to 0$ and $m = 0$ to $\langle Q^2 \rangle_{z,m} \langle P^2 \rangle_{z,m} \approx \frac{1}{4}$, i.e. the saturation of the Heisenberg inequality. This is one of the most important features exhibited by CS: their quantal face is the closest possible to its classical counterpart.

- For $|z| \gg 1$ by using (66) and the approximation (74) for the Bessel modified function

$$\langle Q^2 \rangle_{z,m} = \langle Q \rangle_{z,m}^2 = \frac{|z| + \frac{m+1}{2}}{2}$$

(103)

$$\langle P^2 \rangle_{z,m} = \langle P \rangle_{z,m}^2 = \frac{|z| + \frac{m+1}{2}}{2}$$

(104)
providing

\[(\Delta Q)_{z,m}^2 (\Delta P)_{z,m}^2 \simeq \left[ |z| + \frac{m + 1}{2} \right]^2. \tag{105}\]

It is worth noticing that, in all cases, \((\Delta Q)_{z,m}^2 = (\Delta P)_{z,m}^2\) as in standard CS. This is a universal property exhibited by CS.

### 4.2.2 Signal-to-quantum-noise ratio (SNR)

For a normalized state \(\ket{\phi}\), in terms of the self-adjoint quadrature operator \(Q\), the SNR is defined as [26]

\[
\sigma_{\phi} := \frac{\langle Q \rangle_{\phi}^2}{(\Delta Q)_{\phi}^2}
\]

characterizing the quality of the measurements with respect to the noise. Indeed a low SNR indicates that the noise dominates the measurements in the state \(\ket{\phi}\). In opposite, a high SNR implies that the measurements are relatively clean in the state \(\ket{\phi}\). For instance, in the case of \(q = \sqrt{2}|z| \cos \phi\), the SNR in the BGCS \(\ket{z}_m\) is given as follows:

- For \(|z| \ll 1\) by using (100),

\[
\sigma_{z,m} := \frac{\langle Q \rangle_{z,m}^2}{(\Delta Q)_{z,m}^2} = \frac{q^2}{(\Delta Q)_{z,m}^2} \simeq \frac{q^2}{(m + 1)^2} \frac{4|z|^2 \cos^2 \phi}{2|z|^2 + (m + 1)^2}
\]

with \(\sigma_{z,m} \to 0\) as \(|z| \to 0\) indicating an absence of noise in the measurements;

- For \(|z| \gg 1\) by using (103),

\[
\sigma_{z,m} := \frac{\langle Q \rangle_{z,m}^2}{(\Delta Q)_{z,m}^2} = \frac{q^2}{(\Delta Q)_{z,m}^2} \simeq \frac{q^2}{2|z| + m + 1}
\]

attesting the presence of noise in the measurements for \(\cos \phi \neq 0\).

### 4.2.3 Mandel parameter

Several parameters can be introduced to characterize the statistical properties, and the most popular one is the Mandel parameter [28, 31], denoted here by \(Q\), known as a convenient noise-indicator of a non-classical field, which is frequently used to measure the deviation from Poisson distribution, and thus to distinguish quantum process from classical ones [28]-[35].

The Mandel parameter \(Q\) is defined as [28]
\[ Q \equiv \frac{(\Delta N)^2 - \langle N \rangle}{\langle N \rangle} \]
\[ = \frac{2\langle I \rangle}{T} \int_0^T dt_2 \int_0^{t_2} dt_1 [1 + \lambda(t_1)] - \langle I \rangle T, \]  
where:

- \( \langle N \rangle \) is the average counting number;
- \((\Delta N)^2\) is the corresponding square variance;
- \(\langle I \rangle = \langle N \rangle / T\) is the steady-state photon-counting rate expressed in units of cps;
- \(\lambda(\tau) = \langle \Delta I(t)\Delta I(t+\tau) \rangle / \langle I(t) \rangle \langle I(t+\tau) \rangle\) is the normalized two-time correlation of intensity fluctuations \((\Delta I(t) = I(t) - \langle I(t) \rangle)\) of time difference equal to the time \(\tau\) [27].

Moreover, the Mandel parameter \(Q\)

\[ Q = \frac{(\Delta N)^2}{\langle N \rangle} - 1 \equiv F - 1 \]

is closely related to the normalized variance, also called the quantum Fano factor \(F\) [32], given by \(F = (\Delta N)^2 / \langle N \rangle\), of the photon distribution. For \(F < 1 (Q < 0)\), the emitted light is referred to as sub-Poissonian; \(F = 1, Q = 0\) corresponds to the Poisson distribution, whereas for \(F > 1, (Q > 0)\) the light is called super-Poissonian [28–36]. Thus, in this case, using (70) and (71), we have in the BGCS \(|z\rangle_m\):

- For \(|z| \ll 1\)

\[ (\Delta N)_{z,m}^2 \simeq |z|^2 \frac{|z|^2}{(m+2)(m+1)} + |z| \frac{|z|}{m+1} - |z|^2 \frac{|z|^2}{(m+1)^2} \]
\[ \langle N \rangle_{z,m} \simeq |z| \frac{|z|}{m+1} \]  
providing \(F < 1\) and

\[ Q \simeq - \frac{|z|^2}{(m+1)(m+2)} < 0 \]

indicating that the BGCS \(|z\rangle_m\) have sub-Poissonian statistics as discussed in the subsection [3.3].

- For \(|z| \gg 1\)

\[ (\Delta N)_{z,m}^2 \simeq |z|, \quad \langle N \rangle_{z,m} \simeq |z| \]  

(113)
giving \( \mathcal{F} \simeq 1 \) such that

\[
Q \simeq 0
\]  

which implies that the BGCS \( |z\rangle_m \) have Poissonian statistics for \(|z| \gg 1\) as also noticed in the subsection 3.3.

### 4.3 The quantum energy operator

We now deal with the energy operator for the one-dimensional quantum harmonic oscillator. In this case, the classical Hamiltonian \( H = (q^2 + p^2)/2 = |z|^2 \) should be mapped to the Hilbert space operator

\[
A_{|z|^2} := \int_C |z|^2 |z\rangle_m \langle z| \cdot \frac{d\rho(z)}{d\rho(z)} = \sum_{n=m}^{+\infty} (n - m + 1)(n + 1)|n, m\rangle\langle n, m|
\]

\[
= A_z A_\bar{z} = A_\bar{z} A_z + \sum_{n=m}^{+\infty} 2\mathcal{K}_3 |n, m\rangle\langle n, m|
\]  

which, using (90) and (91), is re-expressed as

\[
A_{|z|^2} = \frac{Q^2 + P^2}{2} + \sum_{n=m}^{+\infty} \mathcal{K}_3 |n, m\rangle\langle n, m|
\]

\[
= \frac{Q^2 + P^2}{2} + \frac{1}{2} \sum_{n=m}^{+\infty} (2n - m + 1)|n, m\rangle\langle n, m|
\]

\[
= \frac{Q^2 + P^2}{2} + \frac{1}{2} [A_z, A_\bar{z}].
\]  

It is worth noticing that, using the relations \( q^2 = |z|^2 + \frac{z^2 + \bar{z}^2}{2} \) and \( p^2 = |z|^2 - \frac{z^2 + \bar{z}^2}{2} \), and the quantized values of \( z^2 \) and \( \bar{z}^2 \):

\[
A_z^2 = \sum_{n=m}^{+\infty} \sqrt{(n - m + 2)(n + 2)(n - m + 1)(n + 1)}|n, m\rangle\langle n + 2, m|,
\]  

\[
A_{\bar{z}}^2 = \sum_{n=m}^{+\infty} \sqrt{(n - m + 2)(n + 2)(n - m + 1)(n + 1)}|n + 2, m\rangle\langle n, m|,
\]  

we obtain

\[
A_{q^2} = \sum_{n=m}^{+\infty} (n - m + 1)(n + 1)|n, m\rangle\langle n, m| +
\]

\[
+ \frac{1}{2} \times \left[ \sum_{n=m}^{+\infty} \sqrt{(n - m + 2)(n + 2)(n - m + 1)(n + 1)} \times 
\]

\[
\{ |n, m\rangle\langle n + 2, m| + |n + 2, m\rangle\langle n, m| \}\]
\[ Q^2 + \sum_{n=m}^{+\infty} K_3 |n, m\rangle\langle n, m| + |2, 0\rangle\langle 0, 0| + \sqrt{3} |3, 1\rangle\langle 1, 1| \quad (119) \]

and

\[ A_{p^2} = \sum_{n=m}^{+\infty} (n - m + 1)(n + 1)|n, m\rangle\langle n, m| - \frac{1}{2} \times \left[ \sum_{n=m}^{+\infty} \sqrt{(n - m + 2)(n + 2)(n - m + 1)(n + 1)} \times \right.

\left. \{ |n, m\rangle\langle n + 2, m| + |n + 2, m\rangle\langle n, m| \} \right]

\[ = P^2 + \sum_{n=m}^{+\infty} K_3 |n, m\rangle\langle n, m| - |2, 0\rangle\langle 0, 0| - \sqrt{3} |3, 1\rangle\langle 1, 1|. \quad (120) \]

From (119) and (120), comparing the operators \( Q^2, A_{q^2} \) and \( P^2, A_{p^2} \), respectively, one remarks that they differ by a multiple of the commutator \( [A_z, A_{\bar{z}}] \) given in (89) plus orthogonal projectors. Besides, summing the expressions (119) and (120) yields the relation (116) determining the operator \( A_{|z|^2} \). Finally, the commutators between \( A_{|z|^2} \) and \( A_z, A_{\bar{z}}, A_{z^2}, A_{\bar{z}^2} \) are calculated as follows:

\[ [A_{|z|^2}, A_z] = - \sum_{n=m}^{+\infty} (2n - m + 3) \sqrt{(n - m + 1)(n + 1)} |n, m\rangle\langle n + 1, m|; \quad (121) \]

\[ [A_{|z|^2}, A_{\bar{z}}] = \sum_{n=m}^{+\infty} (2n - m + 1) \sqrt{n(n - m)} |n, m\rangle\langle n - 1, m| \]

\[ = 2 \sum_{n=m}^{+\infty} \sqrt{n(n - m)} K_3 |n, m\rangle\langle n - 1, m|; \quad (122) \]

\[ [A_{|z|^2}, A_{z^2}] = -2 \sum_{n=m}^{+\infty} [2n - m + 4] \sqrt{(n - m + 2)(n + 2)(n - m + 1)(n + 1)} \]

\[ \times |n, m\rangle\langle n + 2, m|; \quad (123) \]

\[ [A_{|z|^2}, A_{\bar{z}^2}] = 2 \sum_{n=m}^{+\infty} [2n - m + 4] \sqrt{(n - m + 2)(n + 2)(n - m + 1)(n + 1)} \]

\[ \times |n + 2, m\rangle\langle n, m|. \quad (124) \]
5 Thermal properties of the BGCS

This section furnishes a description of the statistical properties of the BGCS for the studied model. Consider a quantum gas of the system in the thermodynamic equilibrium with a reservoir (thermostat) at temperature $T$, which satisfies the quantum canonical distribution. The corresponding normalized density operator for a fixed number $m$ is given by

$$
\rho_m = \frac{1}{Z} \sum_{\nu=0}^{\infty} e^{-\beta E_{n,\nu}} |\nu, m\rangle \langle \nu, m|, \quad E_{n,\nu} = \hbar \Omega \left(n + \frac{1}{2}\right) - \frac{\hbar}{2} (\Omega - \omega_c) (n - \nu),
$$

(125)

where the partition function $Z$ is taken as the normalization constant.

The diagonal elements of the density operator in the BGCS representation are given by

$$
m \langle z | \rho_m | z \rangle_m = \frac{1}{Z} I_m(2 |z|) \sum_{\nu=0}^{\infty} e^{-\beta E_{n,\nu}} \frac{|z|^{2\nu}}{\Gamma(\nu + 1) \Gamma(\nu + m + 1)}. \quad (126)
$$

Note that the quantity $m \langle z | \rho_m | z \rangle_m$ is analogous to the “semi-classical” phase space distribution function $\mu(x, p) = \langle z | \rho | z \rangle$ associated to the density matrix $\rho$ (here $\rho_m$) of the system which is normalized as $\int dxdp/2\pi \hbar \mu(x, p) = 1$, and often referred to as the Husimi distribution [37].

Then, from [33] and [125] together with $E_{n,\nu} = \frac{\hbar}{2} [n(\Omega + \omega_c) + \Omega] + \frac{\hbar}{2}(\Omega - \omega_c)\nu$, we deduce

$$
m \langle z | \rho_m | z \rangle_m = \frac{1}{Z} e^{-\frac{\hbar}{2}[n(\Omega + \omega_c) + \Omega]} e^{\frac{\hbar}{2}\omega_c \nu} \frac{I_m(2 |z| e^{-\frac{\hbar}{2}(\Omega - \omega_c)\nu})}{I_m(2 |z|)}.
$$

(127)

The normalization of the density operator leads to

$$
\text{Tr} \rho_m = \int \mathcal{D} \rho(z) m \langle z | \rho | z \rangle_m = 1.
$$

(128)

By the use of the following integral relation [22]:

$$
\int_0^\infty dx \, x^{-\lambda} K_{\mu}(ax) I_{\nu}(bx) = \frac{b^\nu \Gamma \left(\frac{1}{2} - \frac{1}{2} \lambda + \frac{1}{2} \mu + \frac{1}{2} \nu\right) \Gamma \left(\frac{1}{2} - \frac{1}{2} \lambda - \frac{1}{2} \mu + \frac{1}{2} \nu\right)}{2^{\lambda + 1} \Gamma(\nu + 1) a^{-\lambda + \nu + 1}} \times F \left(\frac{1}{2} - \frac{1}{2} \lambda + \frac{1}{2} \mu + \frac{1}{2} \nu, 1 - \frac{1}{2} \lambda + \frac{1}{2} \mu + \frac{1}{2} \nu, v + 1; \frac{b^2}{a^2}\right);
$$

[Re($v + 1 - \lambda \pm \mu$) $> 0, a > b],
$$

(129)

we obtain

$$
Z = e^{-\frac{\hbar}{2}[n(\Omega + \omega_c) + \Omega]} F \left(m + 1, 1; m + 1; e^{-\frac{\hbar}{2}(\Omega - \omega_c)}\right),
$$

(130)

where $F$ corresponds to the hypergeometric function satisfying the following property [38]:

$$
F(c, \mu; c; x) = F(\mu, c; c; x) = (1 - x)^{-\mu}.
$$

(131)

It comes that the partition function $Z$ takes the form:

$$
Z = e^{-\frac{\hbar}{2}[n(\Omega + \omega_c)]} \left\{2 \sinh \left(\frac{\beta \hbar}{4} [\Omega - \omega_c]\right)\right\}^{-1}.
$$

(132)
and the diagonal elements of the density matrix \([127]\) can be written as

\[
m\langle z | \rho_m | z \rangle_m = 2e^{\frac{\beta h}{4} |\Omega - \omega_c|}(m-1) \sinh \left\{ \frac{\beta h}{4} |\Omega - \omega_c| \right\} \frac{I_m(2|z|e^{-\frac{\beta h}{4} |\Omega - \omega_c|})}{I_m(2|z|)}.
\] (133)

This above expression becomes, in the case of a strong magnetic field, i.e. \(\omega_0 \ll \omega_c\),

\[
m\langle z | \rho_m | z \rangle_m = e^{\frac{\beta h}{4} \omega_0^2 (m-1)} \frac{I_m(2|z|e^{-\frac{\beta h}{4} \omega_0^2})}{\omega_c I_m(2|z|)}.
\] (134)

Equations (133) and (134) suggest that we look for another thermodynamical aspect of the studied model in the BGCS representation. Indeed CS are also relevant to the concept of Wehrl entropy \(W\) \([40, 39, 41]\) defined as

\[
W := - \int \frac{dx dp}{2\pi \hbar} \mu(x, p) \ln \mu(x, p)
\] (135)

where \(\mu(x, p) = \langle z | \rho | z \rangle\). Evaluating the Wehrl entropy in terms of the distribution function \(m\langle z | \rho_m | z \rangle_m\) in (133), we obtain

\[
W = - \int C d\varrho(z) \ln[m\langle z | \rho_m | z \rangle_m]
\]

\[
= - \ln \left\{ \frac{2 \sinh \left\{ \frac{\beta h}{4} |\Omega - \omega_c| \right\} - \frac{\beta h}{4} |\Omega - \omega_c| (m-1)}{2e^{\frac{\beta h}{4} |\Omega - \omega_c|}(m-1) \sinh \left\{ \frac{\beta h}{4} |\Omega - \omega_c| \right\}} \right\} \times
\]

\[
\times \int \frac{d^2 z}{\pi} \frac{2K_m(2|z|)}{I_m(2|z|)} \frac{I_m(2|z|e^{-\frac{\beta h}{4} (\Omega - \omega_c)})}{I_m(2|z|)} \ln \left\{ \frac{I_m(2|z|e^{-\frac{\beta h}{4} (\Omega - \omega_c)})}{I_m(2|z|)} \right\}.
\] (136)

Assuming that the argument of the logarithm function under the integral is dominated by the region \(|z| \ll 1\), we get approximately

\[
\int \frac{d^2 z}{\pi} \frac{2K_m(2|z|)}{I_m(2|z|)} \frac{I_m(2|z|e^{-\frac{\beta h}{4} (\Omega - \omega_c)})}{I_m(2|z|)} \ln \left\{ \frac{I_m(2|z|e^{-\frac{\beta h}{4} (\Omega - \omega_c)})}{I_m(2|z|)} \right\}
\]

\[
\simeq -m \frac{\beta h}{4} |\Omega - \omega_c| \times \frac{e^{\frac{\beta h}{4} (\Omega - \omega_c)(1-m)}}{2 \sinh \left\{ \frac{\beta h}{4} |\Omega - \omega_c| \right\}}.
\] (137)

Therefore, the related Wehrl entropy is approximated to

\[
W \simeq - \ln \left[ 1 - e^{-\frac{\beta h}{4} \omega_0^2} \right] = -1 + W_{HO}
\] (138)

where \(W_{HO} = 1 - \ln \left[ 1 - e^{-\frac{\beta h}{2} \omega_0^2} \right]\) is the conventional harmonic oscillator Wehrl entropy \([39]\) with frequency \(\Omega_{-} := \frac{\Omega - \omega_c}{2}\). Besides, setting

\[
m\langle z | \bar{\rho} | z \rangle_m = \left( \frac{2\pi I^2}{A} \right) m\langle z | \rho | z \rangle_m
\] (139)
where \( l_{-} := \sqrt{\frac{\hbar}{M \Omega}} \) and \( A = \pi R^2 \) with \( R \) the radius of the cylindrical body considered in [2], and evaluating

\[
\tilde{W} = -\frac{A}{2\pi l_{-}^2} \int_{C} d\varrho(z) \langle z|\tilde{\rho}|z\rangle_m \ln [\langle z|\tilde{\rho}|z\rangle_m] \tag{140}
\]

as in (136)-(138), we get

\[
\tilde{W} \simeq -\ln \left[ 1 - e^{-\beta \frac{\hbar}{2}\Omega} \right] - \ln \left( \frac{2\pi l_{-}^2}{A} \right) \equiv -1 + W_{\text{calc}} \tag{141}
\]

where \( W_{\text{calc}} = 1 - \ln \left[ 1 - e^{-\beta \Omega} \right] - \ln \left( \frac{2\pi l_{-}^2}{A} \right) \) is the Wehrl entropy calculated for Landau’s diamagnetism for a spinless electron in a uniform magnetic field; the thermal harmonic oscillator [42] frequency is denoted by \( \Omega \), and \( l_{-} = \sqrt{\frac{\hbar}{M \Omega}} \). Hence, the present physical model is an approximation of the problem of a thermal harmonic oscillator with frequency \( \Omega_{-} := \frac{\Omega - \omega_c}{2} \).

In the case of a strong magnetic field, from (134) we get

\[
W \simeq \left[ \beta \frac{\hbar \omega_0^2}{2\omega_c} - \ln \left( \beta \frac{\hbar \omega_0^2}{\omega_c} \right) \right] \times \frac{\beta \frac{\hbar \omega_0^2}{\omega_c}}{2 \sinh \left( \frac{\beta \hbar \omega_0^2}{2\omega_c} \right)}. \tag{142}
\]

The diagonal expansion of the density operator can be performed for this physical model, as investigated in [23] in the constructed BGCS as follows:

\[
\rho_m = \int_{C} d\varrho(z) P_m(z)|\nu, m\rangle \langle \nu, m|, \tag{143}
\]

where the function \( P_m(z) \) must be determined. To this end, we determine at first the diagonal elements of the density operator \( \rho_m \) in the basis of the number states, namely \( \{|\nu, m\rangle, \nu \geq 0\} \), by setting:

\[
\langle \nu, m|\rho_m|\nu, m\rangle = \int_{C} \frac{d^2 z}{\pi} K_m(2|z|) I_m(2|z|) P_m(z) \langle \nu, m|z\rangle_m \langle z|\nu, m\rangle. \tag{144}
\]

By the use of \( \sum_{\nu=0}^{\infty} |\nu, m\rangle \langle \nu, m| = 1 \) and (125), they are given by

\[
\langle \nu, m|\rho_m|\nu, m\rangle = \frac{1}{Z} e^{-\beta E_{n,v}} = \left[ 1 - e^{-\beta \frac{\hbar}{2}(\Omega - \omega_c)} \right] e^{-\beta \frac{\hbar}{2}(\Omega - \omega_c)v}. \tag{145}
\]

By use of the integral (41), we obtain for the function \( P_m(z) \) the following expression:

\[
P_m(z) = \left[ e^{\frac{\beta \hbar}{2}(\Omega - \omega_c)} - 1 \right] \frac{e^{\beta \hbar}(\Omega - \omega_c)m K_m(2|z|) e^{\frac{\beta \hbar}{2}(\Omega - \omega_c)}}{K_m(2|z|)} \tag{146}
\]

which is normalized as

\[
\int_{C} d\varrho(z) P_m(z) = 1. \tag{147}
\]
Then, the diagonal representation of the normalized density operator takes the form

$$
\rho_m = [e^{\frac{\beta h}{2}(\Omega - \omega_c)} - 1]e^{\frac{\beta h}{2}(\Omega - \omega_c)m} \int_C d\varrho(z) \frac{K_m(2|z|e^{\frac{\beta h}{2}(\Omega - \omega_c)})}{K_m(2|z|)} |z\rangle_m \langle z|.
$$

(148)

Therefore, given an observable $O$, one obtains its mean value, i.e., its thermal average as follows:

$$
\langle O \rangle_m = \text{Tr}(\rho_m O) = \int_C d\varrho(z) P_m(z) \langle z|O|z\rangle_m.
$$

(149)

In this manner, the thermal expectation value of the number operator $N$, by using (70), (130) and (131) together, is

$$
\langle N \rangle_m = \text{Tr}(\rho_m N) = \int_C d\varrho(z) P_m(z) \langle z|N|z\rangle_m = \frac{1}{e^{\frac{\beta h}{2}(\Omega - \omega_c)} - 1}.
$$

(150)

For the operators $Q$ and $P$, setting $q = \sqrt{2}|z| \cos \phi$ and $p = \sqrt{2}|z| \sin \phi$, we obtain from (98)

$$
\langle Q \rangle_m = 0, \quad \langle P \rangle_m = 0.
$$

(151)

In the same vein, from (71), (130) and (131) together, the thermal expectation value of the square of the number operator becomes

$$
\langle N^2 \rangle_m = \text{Tr}(\rho_m N^2) = \int_C d\varrho(z) P_m(z) \langle z|N^2|z\rangle_m = \frac{1}{e^{\frac{\beta h}{2}(\Omega - \omega_c)} - 1} + \frac{1}{(e^{\frac{\beta h}{2}(\Omega - \omega_c)} - 1)^2}.
$$

(152)

One remarks that both thermal expectation values $\langle N \rangle_m$ and $\langle N^2 \rangle_m$ are independent of the Bargmann index given here by $m$. One can therefore define the thermal intensity correlation function, which is also independent of the index $m$, by

$$
\langle g \rangle_m \equiv \frac{\langle N^2 \rangle - \langle N \rangle}{\langle N \rangle^2} = \langle g^2 \rangle = 2.
$$

(153)

In the case of $Q^2$ and $P^2$, by use of (99), we have

$$
\langle Q^2 \rangle_m = [e^{\frac{\beta h}{2}(\Omega - \omega_c)} - 1]e^{-\beta h(\Omega - \omega_c)} \times F\left(m + 2, 2; m + 1; e^{-\frac{\beta h}{2}(\Omega - \omega_c)}\right) \times (m + 1)
\quad + (e^{\frac{\beta h}{2}[\Omega - \omega_c]} - 1)^{-1} + \frac{m + 1}{2},
$$

$$
\langle P^2 \rangle_m = [e^{\frac{\beta h}{2}(\Omega - \omega_c)} - 1]e^{-\beta h(\Omega - \omega_c)} \times F\left(m + 2, 2; m + 1; e^{-\frac{\beta h}{2}[\Omega - \omega_c]}\right) \times (m + 1)
\quad + (e^{\frac{\beta h}{2}[\Omega - \omega_c]} - 1)^{-1} + \frac{m + 1}{2}.
$$

(154)

which are not index $m$ independent.
6  Concluding remarks

In this work, we have investigated the Fock-Darwin Hamiltonian describing a spinless electron gas, subject to a perpendicular magnetic field $B$ and confined in a harmonic potential $\frac{1}{2}M\omega_0^2r^2$. Thanks to the set of introduced step and orbit-center coordinate operators, we have shown that the studied system possesses $su(1,1)$ Lie algebra. As a consequence, CS have been constructed as the eigenstates of the $SU(1,1)$ group generator $K_-$. The mean values of $SU(1,1)$ group generators, the probability density and the time dependence of the BGCS have been discussed. By using these CS, the Berezin - Klauder - Toeplitz quantization has been performed. In this framework, mean values for quadrature operators have been computed highlighting the similitude between the BGCS quantal behavior and its classical counterpart. Furthermore, the relevance of this quantization results in deriving some important quantum optical properties such as the SNR and the Mandel parameter has been put into evidence. Statistical properties of an electron gas in the thermodynamic equilibrium with a reservoir (thermostat) at temperature $T$, satisfying the quantum canonical distribution, have been investigated and discussed. Finally, with the density matrix provided in the BGCS representation, we have noticed, via the calculation of the Wehrl entropy, that the studied system can be identified to a model of a thermal harmonic oscillator.

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References

[1] I. A. Malkin and V. I. Man’ko, Zh. Eksp. Teor. Fiz. 55 1014 (1968)

[2] A. Feldman and A. H. Kahn, Phys. Rev. B 1 4584 (1970)

[3] J. P. Gazeau, P. Y. Hsiao and A. Jellal, Phys. Rev. B 65 094427 (2002)

[4] D. Schuch and M. Moshinsky, J. Phys. A: Math. Gen. 36 6571 (2003)

[5] J. P. Gazeau and M. Novaes, J. Phys. A: Math. Gen. 36 199 -212 (2003)

[6] A. M. Perelomov, Generalized Coherent States and Their Applications, Springer-Verlag, Berlin (1986)

[7] L. D. Landau, Z. Phys. 64 629 (1930)

[8] H. Fakhri, Phys. Lett. A 313 243-251 (2003)

[9] J. P. Antoine, J. P. Gazeau, P. Monceau, J. R. Klauder and K. A. Penson, J. Math. Phys. 42 2349-2387 (2001)

[10] H. Fakhri, J. Phys. A: Math. Gen. 37 5203-5210 (2004)

[11] A. O. Barut and L. Girardello, Commun. Math. Phys. 21 41 (1971)
[12] A. Dehghani, H. Fakhri and B. Mojaveri, J. Math. Phys. 53 123527 (2012)
[13] H. Bergeron, J. P. Gazeau and A. Youssef, Phys. Lett. A 377 598-605 (2013)
[14] V. Fock, Z. Phys. 47 446 (1928)
[15] C. G. Darwin, Proc. Camb. Phil. Soc. 27 86 (1930)
[16] M.H. Johnson and B.A. Lippmann, Phys. Rev. 76 828 (1949)
[17] R. Gilmore, Lie Groups, Lie algebras, and Some of their Applications, Wiley, New York (1974)
[18] W. Magnus, F. Oberhettinger and R. P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, Springer-Verlag, New York (1966)
[19] G. N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press, Cambridge (1995)
[20] S. T. Ali, J. P. Antoine and J. P. Gazeau, Coherent States, Wavelets and their Generalizations, Springer-Verlag, New York (2000)
[21] D. Popov, J. Phys. A: Math. Gen. 34 1-14 (2001)
[22] I. S. Gradshtein, Table of integrals, series and products 5th ed Academic Press (1994)
[23] C. Brif and Y. Ben-Aryeh, Quantum Opt. 6 391-6 (1994)
[24] J. P. Gazeau, Coherent States in Quantum Physics, Wiley-VCH, Berlin (2009)
[25] I. Aremua, J. P. Gazeau and M.N. Hounkonnou, J. Phys. A: Math. Theor. 45 335302 (2012)
[26] K. A. Penson and A. I. Solomon, J. Math. Phys. 40 2354 (1999)
[27] H. J. Kimble, M. Dagenais and L. Mandel, Phys. Rev. Lett. 39 691 (1977)
[28] L. Mandel, Opt. Lett. 4 205 (1979)
[29] R. Short and L. Mandel, Phys. Rev. Lett. 51 384 (1983)
[30] F. Diedrich and H. Walther, Phys. Rev. Lett. 58 203 (1987)
[31] L. Mandel and E. Wolf, Optical coherence and quantum optics, Cambridge University Press, Cambridge (1995)
[32] J. Bajer and A. Miranowicz, J. Opt. B 2 L10 (2000)
[33] F. Treussart, R. Alléaume, V. Le Floc’h, L. T. Xiao, J.-M. Courty and J. F. Roch, Phys. Rev. Lett. 89 093601 (2002)
[34] B. Lounis and W. E. Moerner, Nature 407 491 (2000)
[35] G. Li, T. C. Zhang, Y. Li and J. M. Wang, Phys. Rev. A 71 023807 (2005)
[36] X.-Z. Zhang, Z.-H. Wang, H. Li, Q. Wu, B.-Q. Tang, F. Gao and J.-J. Xu, *Chin. Phys. Lett.* **25** 3976 (2008)

[37] K. Husimi, *Proc. Phys. Math. Soc. Japan* **22** 264 (1940)

[38] A. Nikiforov and V. Ouvarov, *Éléments De la Théorie des Fonctions Speciales*, Mir, Moscow (1976) p 237

[39] A. Anderson and J. J. Halliwell, *Phys. Rev. D* **48** 2753 (1993)

[40] A. Wehrl, *Rep. Math. Phys.* **16** 353 (1979)

[41] E.H. Lieb, *Commun. Math. Phys.* **62** 35 (1978)

[42] F. Pennini, A. Plastino and S. Curilef, *Phys. Rev. B* **71** 024420 (2005)