Supermultiplets of AdS Black Holes
in 2+1 Dimensions

Sharmanthie Fernando and Freydoon Mansouri

Physics Department, University of Cincinnati, Cincinnati, OH 45221

Abstract

We construct super AdS black holes in 2+1 dimensions in terms of Chern Simons gauge theory of $N = (1, 1)$ super AdS group coupled to a (super)source. We take the source to be a super AdS state specified by its Casimir invariants. We show that the corresponding space-time is a supermultiplet of AdS space-times related to each other by supersymmetry transformations. We give explicit expressions for the masses and the angular momenta of the black holes in a supermultiplet. With one exception, for $N = (1, 1)$ one pair of extremal black holes can be accommodated in such all-black hole supermultiplets. The requirement that the source be a unitary representation leads to a discrete tower of excited states which provide a microscopic model for the super black hole.

A conventional method of searching for signs of supersymmetry in black hole solutions is to look for Killing spinors. Many works along these lines already exist in the literature. We cite a representative few here [1]-[9], from which more references can be traced. One way to see whether a given black hole solution admits Killing spinors is identify it with the bosonic part of an appropriate supergravity theory [1]-[9]. Then by requiring that the fermions as well as their variations vanish, one arrives at Killing spinor equation(s). The asymptotic supersymmetries depend on the number of non-trivial solutions of these equations consistent with the black hole topology. For example, in asymptotically flat space-times, a typical supermultiplet consists of a black hole and a number of ordinary particles all with the same mass. In contrast to the familiar situation in particle physics, where we have Supermultiplets consisting of particles only, in this approach there is no systematic way of looking for supermultiplets consisting of black holes only. The main purpose of the present work is to show

*email address: fernando@physung.phy.uc.edu
†email address: Mansouri@uc.edu
that in \(2 + 1\) dimensions it is possible to construct a theory which permits macro-
scopic solutions consisting of all AdS black hole supermultiplets [10]. It involves the
Chern Simons gauge theory of the \((1,1)\) super AdS group [11, 12] co upled to a super
AdS state (source). As we shall see below, to be able to accommodate the structure
of the solution which emerges from such a theory it becomes necessary to broaden
the standard notions of classical geometries to include some quantum mechanical
elements.

The theory which we will describe below is the supersymmetric generalization of a
theory [13] which has been used recently to provide both a microscopic and a macro-
scopic description of the BTZ black hole [14]. Since the concepts used in reference [13],
in which the gauge group is the AdS group, are essential for the understanding of
the present work, we begin with a brief summary of the results of that work. This
will also allow us to establish our notation and to describe our unconventional use
of Chern Simons theory. In the same sense as in reference [13], we take the Chern
Simons theory to be defined on a manifold \(M\) with topology \(R \times \Sigma\), where \(\Sigma\) is a two
dimensional space. Moreover, we consider the theory to be an explicit realization of
the Mach Principle, so that in the absence of sources the field strengths vanish and the
topology is trivial (no punctures). In this way, we associate non-trivial topologies to
the presence of sources [13, 16]. We then show that the information encoded in such
a theory identifies the physical (metrical) space-time as the space \(M_q\) of \(0 + 1\) dimen-
sional fields which represent the source. Taking the source to be a super AdS state,
we find that the emerging physical space-time has a multilayered structure, distinct
from the manifold \(M\) over which the gauge theory is defined. The layers are con-
ected to each other by supersymmetry transformations. As a result, for \(N = (1,1)\)
and appropriate ranges of Casimir invariants of the source state, the physical space-
time becomes a super AdS (super)multiplet consisting of four AdS black holes.

Let us now consider the details. It will be recalled that the anti-de Sitter space in
\(2+1\) dimensions can be viewed as a subspace of a flat 4-dimensional space with the
line element

\[
 ds^2 = dX_A dX^A = dX_0^2 - dX_1^2 - dX_2^2 + dX_3^2
\]

It is determined by the constraint

\[
 (X_0)^2 - (X_1)^2 - (X_2)^2 + (X_3)^2 = l^2
\]

where \(l\) is a real constant. The set of transformations which leave the line ele-
ment invariant form the anti-de Sitter group \(SO(2,2)\) which is locally isomorphic to
\(SL(2,R) \times SL(2,R)\). From here on by anti-de Sitter group we shall mean its universal
covering group.

With \(a = 0,1,2\), we can write the AdS algebra in two convenient forms [13]:

\[
 J_a = J_a^+ + J_a^- \quad \Pi^a = J_a^+ - J_a^-
\]
Setting
\[ \epsilon^{012} = 1; \quad \eta^{ab} = (1, -1, -1) \] (4)
the commutation relations in, say, \( J_\pm a \) basis will take the form,
\[ [J_\pm a, J_\pm b] = -i\epsilon^{\pm c}_{ab} J_\pm c; \quad [J_\pm a, J_\mp b] = 0 \] (5)
The Casimir operators are then given by
\[ j^2_\pm = \eta^{ab} J_\pm a J_\pm b \] (6)
Alternatively, we can take a combination of these with eigenvalues corresponding to the parameters of the BTZ solution:
\[ M = l^2(\Pi_a \Pi_a + l^{-2} J^a J_a) = 2(j_+^2 + j_-^2) \]
\[ J/l = 2\Pi_a J_a = 2(j_+^2 - j_-^2) \] (7)
Unless stated otherwise, we will use the same symbols for operators and their eigenvalues. As explained elsewhere \cite{[13]}, an irreducible representation associated with an AdS black hole can be labeled either with the pair \((M, J)\) or the pair \((j_+, j_-)\). It is also possible to label these representations by eigenvalues which are proportional to the Horizon radii \(r_\pm\) of the AdS black hole \cite{[13]}.

To write down the Chern Simons action, we begin by expressing the connection in \( SL(2, R) \times SL(2, R) \) basis
\[ A_\mu = \omega^a_\mu J_a + e^a_\mu \Pi_a = A_\mu^+ a J_a^+ + A_\mu^- a J_a^- \] (8)
where
\[ A_\mu^\pm a = \omega^a_\mu \pm l^{-1} e^a_\mu \] (9)
Eq. (9) should be viewed as definitions of \( e \) and \( \omega \) in terms of the two \( SL(2, R) \) connections. The covariant derivative will have the form
\[ D_\mu = \partial_\mu - iA_\mu = \partial_\mu - iA_\mu^+ a J_a^+ - iA_\mu^- a J_a^- \] (10)
Then the components of the field strength are given by
\[ [D_\mu, D_\nu] = -iF^{+ a}_{\mu \nu} J_a^+ - iF^{- a}_{\mu \nu} J_a^- = -iF^{+ a}_{\mu \nu} [A^+] - iF^{- a}_{\mu \nu} [A^-] \] (11)
For a simple or a semi-simple group, the Chern Simons action has the form
\[ I_{cs} = \frac{1}{4\pi} Tr \int_M A \wedge \left( dA + \frac{2}{3} A \wedge A \right) \] (12)
where \( Tr \) stands for trace and
\[ A = A_\mu dx^\mu = A^+ + A^- \] (13)
So, the Chern Simons action with $SL(2, R) \times SL(2, R)$ gauge group will take the form

$$I_{cs} = \frac{1}{4\pi} Tr \int_M \left[ \frac{1}{a_+} A^+ \wedge \left( dA^+ + \frac{2}{3} A^+ \wedge A^+ \right) + \frac{1}{a_-} A^- \wedge \left( dA^- + \frac{2}{3} A^- \wedge A^- \right) \right]$$

Here the quantities $a_{\pm}$ are, in general, arbitrary coefficients, reflecting the semisimplicity of the gauge group. Up to an overall normalization, only their ratio is significant.

In the presence of a source (or of sources) in $M$, any á priori choice of the coefficients $a_{\pm}$ reduces the class of allowed holonomies [13]. We will, therefore, keep the coefficients $a_{\pm}$ as free parameters in the sequel.

As stated above, the manifold $M$ has the topology $R \times \Sigma$ with $R$ representing $x^0$. Then subject to the constraints

$$F_a^\pm[A^\pm] = \frac{1}{2} \eta_{ab} \epsilon^{ij} \left( \partial_i A^\pm_j - \partial_j A^\pm_i + \epsilon_{cd} A^\pm_i A^\pm_j \right) = 0$$

the Chern Simons action for $SO(2, 2)$ will take the form

$$2\pi I_{cs} = \frac{1}{a_+} \int_R dx^0 \int_{\Sigma} dx^2 \left( -\epsilon^{ij} \eta_{ab} A^\pm_i A^\pm_j + A_0^+ F_a^+ \right) + \frac{1}{a_-} \int_R dx^0 \int_{\Sigma} dx^2 \left( -\epsilon^{ij} \eta_{ab} A^\pm_i A^\pm_j + A_0^- F_a^- \right)$$

where $i, j = 1, 2$.

To introduce interactions, we follow an approach which has been successful in coupling sources to Poincaré and super Poincaré Chern Simons theories [15, 16] and take a source for the present problem to be an irreducible representation of anti-de-Sitter group characterized by Casimir invariants $M$ and $J$. Within the representation, the states are further specified by the phase space variables of the source $\Pi^A$ and $q^A$, $A = 0, 1, 2, 3$, subject to anti-de Sitter constraints. The relevant irreducible representations of the AdS group have been discussed in reference [13]. Here we note that to allow for the possibility of quantizing the Chern Simons theory consistently, we must require that our sources be represented by unitary representations of AdS group. Since the AdS group in $2+1$ dimensions can be represented in the $SL(2, R) \times SL(2, R)$ form, the unitary representations of $SO(2, 2)$ can be constructed from those of $SL(2, R)$. The latter group has four series of unitary representations all of which are infinite dimensional [17]. Of these, the relevant representations for our purposes turn out to be the discrete series bounded from below [13]. For this series, the states in an irreducible representation of $SL(2, R)$ are specified by the eigenvalues of its Casimir operator $j^2$ (see Eq. 6) and, e.g., the element $J_0$, where we have suppressed the superscripts $\pm$ distinguishing our two $SL(2, R)$’s. Thus, we have

$$j^2|F, m >= F(F - 1)|F, m >$$
\[ J_0 |F, m> = (F + m)|F, m> \]

In these expressions
\[ F = \text{real number} \geq 0; \quad m = 0, 1, 2, \ldots \] (17)

So, for this series, the eigenvalues of the Casimir invariants of \( SL(2,R) \times SL(2,R) \) can be written as,
\[ j^2_{\pm} = F^2_{\pm} - F \] (18)

It follows that the infinite set of states can, in a somewhat redundant notation, be specified as
\[ |j^2_{\pm}, F_{\pm} + m_{\pm}>, \quad m_{\pm} = 0, 1, 2, \ldots \] (19)

Clearly, the integers \( m_{\pm} \) are not necessarily equal. Using these states, we can construct the discrete series of the unitary representations of \( SO(2,2) \). A typical state will have the following labels:
\[ |M, J> = |j^2_{\pm}, j^2_{\mp}, F_{\pm} + m_{\pm}, F_{\mp} + m_{\mp}> \] (20)

To be able to identify the labels \( M \) and \( J \) with the corresponding labels in the AdS black hole, we must require that \( F_{\pm} \geq 1 \) [13]. It would then follow that \( |J/l| \leq |M| \), as required for having a black hole solution.

With this background, let us now consider the coupling of sources to the AdS Chern Simons theory. It is given by
\[ I_s = \int_C d\tau \left[ \Pi_A \partial_\tau q^A - (A^{+a} J^+_a + A^{-a} J^-_a) + \lambda \left( q^A q_A - l^2 \right) \right] \]
\[ + \int_C d\tau \left[ \lambda_+ (J^{+a} J^+_a - l^2 j^2_+) + \lambda_- (J^{-a} J^-_a - l^2 j^2_-) \right] \] (21)

In this expression, \( C \) is a path in \( M \), \( \tau \) is a parameter along \( C \), and \( J^a_\pm \) play the role of c-number generalized angular momenta which transform in the same way as the corresponding generators which label the source. The quantities \( \lambda \) and \( \lambda_\pm \) are Lagrange multipliers. The first constraint in this action ensures that \( q^A(\tau) \) satisfy the AdS constraint. As explained in previous occasions [13, 14, 15], it is not the manifold \( M \) over which the gauge theory is defined but the space of \( q_A \)'s which give rise to the classical space-time. The last two constraints identify the source being coupled to the Chern Simons theory as an anti-de Sitter state with invariants \( j^+ \) and \( j_- \). These constraints are crucial in relating the invariants of the source to the asymptotic observables of the coupled theory via Wilson loops.

The total action for the theory is given by:
\[ I = I_{cs} + I_s \] (22)

It is easy to check that in this theory the components of the field strength still vanish everywhere except at the location of the sources. So, the analog of Eq. 15 become
\[ \epsilon^{ij} F_{ij}^\pm a = 2\pi a_{\pm} j^\pm a \delta^2(\vec{x}, \vec{x}_0) \] (23)
In particular, fixing the gauge so that $SO(2, 2)$ symmetry reduces to $SO(2) \times SO(2)$, we get

$$\epsilon^{ij} F_{ij}^\pm = 2\pi a_\pm F_\pm \delta^2(\vec{x}, \vec{x}')$$  \hspace{1cm} (24)

where $F_\pm$ are the invariant labels of the state as in Eq. 18. All other components of the field strength vanish. We thus see that because of the constraints appearing in the action given by Eq. 22, the strength of the sources corresponding to the maximal compact subgroup of the gauge group become related to their Casimir invariants. These invariants, in turn, determine the asymptotic observables of the theory. Since such observables must be gauge invariant, they are expressible in terms of Wilson loops, and a Wilson loop about our source can only depend on, e.g., $j_+$ and $j_-$. 

From the data on the manifold $M$ given above, one can determine the properties of the emerging space-time by solving Eq 24. The only non-vanishing components of the gauge potential are given by

$$A_\theta^\pm = 2a_\pm F_\pm$$  \hspace{1cm} (25)

where $\theta$ is an angular variable. In particular, using Eq. 9, we can write

$$\omega_\theta^0/l = (j_+ - j_-) = r_-/l$$  \hspace{1cm} (26)

$$\epsilon_\theta^0 = (j_+ + j_-) = r_+/l$$  \hspace{1cm} (27)

Although these are components of a connection which is a pure gauge, they give rise to non-trivial holonomies around the source. More explicitly, we have

$$W[e] = exp^{\gamma} \epsilon_\theta^0 \Pi_0$$  \hspace{1cm} (28)

$$W[\omega] = exp^{\gamma} \omega_\theta^0 J_0$$  \hspace{1cm} (29)

Here, $\gamma$ is a loop around the source.

The reduction from Eq. 23 to Eq. 24 with $SO(2) \times SO(2)$ as left over symmetry relies on diagonalizing compact subgroup generators. Although this can work for black hole solutions as described in reference [13], the natural left over symmetry from the point of view of black holes is $SO(1, 1) \times SO(1, 1)$. Starting from Eq. 23, one can carry out the gauge fixing such that the analog of Eq. 24 will have this non-compact symmetry [18]. The analysis of the holonomies will go through as in the case of the compact subgroup. The main advantage in this case is that it is no longer necessary to carry out a Wick rotation to make contact with the black hole solution [13].

Irrespectively of whether the left over symmetry were compact or non-compact, it was shown in reference [13] how these holonomies lead to a discrete identification subgroup of $SO(2, 2)$, which shows that the manifold $M_q$ of the $0 + 1$ dimensional fields $q^A$ has all the relevant features of the macroscopic AdS black hole solution. The approach used in this reference was to improve and make precise some of the previous
attempts to obtain this kind of identifications \[19, 15, 16, 20\]. As we shall see below, the same holonomies, suitably interpreted, will play a crucial role in establishing the space-time structure of the supersymmetric theory discussed below. To actually obtain the BTZ line element, one must determine a parametrization of \(q^A\)'s consistent with the above holonomy properties. This was carried out in reference \[13\], a typical parametrization for \(r > r_+\) being

\[
q^1 = f \cos \left( \frac{r - r_+ t}{l^2} \right), \\
q^2 = f \sin \left( \frac{r - r_+ t}{l^2} \right), \\
q^0 = \sqrt{f^2 + l^2} \cos \left( \frac{r - r_+ t}{l^2} \right), \\
q^3 = \sqrt{f^2 + l^2} \sin \left( \frac{r - r_+ t}{l^2} \right)
\]

(30)

where

\[
f^2 = \frac{r^2 - r_+^2}{r_+^2 - r_-^2}; \quad r > r_+ \tag{31}
\]

the important point to note here is that the quantities \(q^A\) carry the Casimir invariants \((r_+, r_-)\) of the source state.

We now turn to the supersymmetric generalization of the Chern Simons theory described above and show that the corresponding macroscopic theory consists of a supermultiplet of ordinary space-times and, as a special case, a supermultiplet consisting of black holes only. The simplest way of obtaining the supersymmetric extension of the anti-de Sitter group is to begin with the AdS group in its \(SL(2, \mathbb{R}) \times SL(2, \mathbb{R})\) basis. The \(N = 1\) supersymmetric form of each \(SL(2, \mathbb{R})\) factor is the supergroup \(OSp(1|2; \mathbb{R})\). Thus, one arrives at the \((1,1)\) form of the \(N = 2\) super AdS group. Its algebra is given by

\[
\{J^\pm_a, J^\pm_b\} = -i \epsilon_{ab}^\epsilon J^\pm_c; \quad \{J^\pm_a, Q^\pm_\alpha\} = -\sigma^\alpha_\beta Q^\pm_\beta; \quad \{Q^\pm_\alpha, Q^\pm_\beta\} = -\sigma^\alpha_\beta J^\pm_a \\
\{Q^+_\alpha, Q^-_\beta\} = 0; \quad [J^+, J^-] = 0 \tag{32}
\]

The Casimir invariants are given by

\[
C_\pm = J^2_\pm + \epsilon^{\alpha\beta} Q^\pm_\alpha Q^\pm_\beta \tag{33}
\]

The spinor indices are raised and lowered by antisymmetric metric \(\epsilon^{\alpha\beta}\) defined by \(\epsilon^{12} = -\epsilon_{12} = 1\). The matrices \((\sigma^a)_\alpha^\beta, (a = 0, 1, 2)\), form a representation of \(SL(2, \mathbb{R})\) and satisfy the Clifford algebra

\[
\{\sigma^a, \sigma^b\} = \frac{1}{2} \eta^{ab} \tag{34}
\]
More explicitly, we can take them to be:

\[
\sigma^0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \sigma^1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}; \quad \sigma^2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]  

(35)

It is important to note that the supersymmetry generators of \( OSp(1|2, R) \) do not commute with the Casimir invariant of its \( SL(2, R) \) subgroup. That is,

\[
[j_2^Q, Q] \neq 0
\]

(36)

Since super AdS group is semi-simple, we can construct its irreducible representations by first constructing the irreducible representations of \( OSp(1|2, R) \). Depending on which \( OSp(1|2, R) \) we are considering, the states within any such supermultiplet are the corresponding irreducible representations of \( SL(2, R) \) characterized by the Casimir invariants \( j_+ \) and \( j_- \), respectively. Based on the rationale given for the non-supersymmetric case, the irreducible representations of interest for the present case are those which can be obtained from the unitary discrete series of \( SL(2, R) \) and which are bounded from below. To construct the supermultiplet corresponding to, say, the “plus” generators in Eq. 32, we can take the Clifford vacuum state \( |\Omega^+ \rangle \) to be the \( SL(2, R) \) state with the lowest eigenvalue of \( J_+^o \). In the notation of Eq. 19, this corresponds to an \( m = 0 \) state:

\[
|F_+, m \rangle = |F_+, m = 0 \rangle = |F_+ \rangle
\]

Then, the superpartner of this state, again with \( m = 0 \), is the state \( |F_+ + 1/2 \rangle \) obtained by the application of one of the \( Q \)'s. The corresponding values of \( j_+^2 \) are \( F_+(F_+ - 1) \) and \( (F_+ + 1/2)(F_+ - 1/2) \), respectively. The supermultiplet for the second \( OSp(1|2, R) \) can be constructed in a similar way.

We are now in a position to construct the \((1,1)\) super AdS supermultiplet as a direct product of the two \( OSp(1|2, R) \) doublets. Altogether, there will be four states in the supermultiplet. They will have the following labels:

\[
|F_+, F_- \rangle; \quad |F_+ + 1/2, F_- \rangle; \quad |F_+, F_- + 1/2 \rangle; \quad |F_+ + 1/2, F_- + 1/2 \rangle
\]

(37)

From these, we can also obtain the expressions for the eigenvalues \((M, J)\) of various states within the supermultiplet:

\[
|M_1, J_1 \rangle = |M, J \rangle \\
|M_2, J_2 \rangle = |M + 2F_+ - 1/2, J + 2F_+ - 1/2 \rangle \\
|M_3, J_3 \rangle = |M + 2F_- - 1/2, J - 2F_- + 1/2 \rangle \\
|M_4, J_4 \rangle = |M + 2(F_+ + F_-) - 1, J + 2(F_+ - F_-) \rangle
\]

(38)

these states transform into one another under supersymmetry transformations.

The Chern Simons action for simple and semisimple supergroups has the same structure as that for Lie groups. The only difference is that the trace operation is
replaced by super trace (Str) operation. So, in the $OSp(1|2, R) \times OSp(1|2, R)$ basis the Chern Simons action for the super AdS group has the same form as that given by Eq. 14. But now the expression for connection is given by

$$A^\pm = [A^\pm_a J^\pm_a + \chi^{\pm\alpha}_\mu Q^\pm_\alpha] \, dx^\mu$$  \hspace{1cm} (39)

Just as in the non-supersymmetric case, to have a nontrivial theory, we must couple sources to the Chern Simons action. To do this in a gauge invariant and locally supersymmetric fashion, we must take a source to be an irreducible representation of the super AdS group. As we saw above, such a supermultiplet consists of four AdS states. To couple it to the gauge fields, we must first generalize the canonical variables we used in the AdS theory to their supersymmetric forms:

$$\Pi_A \rightarrow (\Pi_A, \Pi_\alpha) \hspace{1cm} q_A \rightarrow (q_A, q_\alpha)$$  \hspace{1cm} (40)

Then, the source coupling can be written as

$$I_s = \int_C [\Pi_A dq^A + \Pi_\alpha dq^\alpha + (A^+ + A^-) + \text{constraints}]$$  \hspace{1cm} (41)

where again $C$ is a path in $M$. The constraints here include those discussed for the AdS group and, in addition, those which relate the AdS labels of the Clifford vacuum to the Casimir eigenvalues of the super AdS group. The combined action

$$I = I_{cs} + I_s$$  \hspace{1cm} (42)

leads to the constraint equations

$$\epsilon^{ij} F_{ij}^\pm a = 2\pi a^\pm J^\pm a(\vec{x}, \vec{x}_0); \hspace{1cm} \epsilon^{ij} F_{ij}^{\pm\alpha} = 2\pi a^\pm Q^\pm_{\alpha}(\vec{x}, \vec{x}_0)$$  \hspace{1cm} (43)

Up to this point, everything proceeds in parallel with the non-supersymmetric case. Differences begin to show up when one attempts to solve these equations by choosing a gauge again so that the gauge symmetry is reduced to $SO(2) \times SO(2)$:

$$\epsilon^{ij} F_{ij}^0 = 2\pi a^\pm J^0_\alpha(\vec{x}, \vec{x}_0)$$  \hspace{1cm} (44)

Although this equation is identical in form to Eq. 24 for the non-supersymmetric case, there is an essential difference in the underlying physics. In the supersymmetric case, the supermultiplet which we couple to the Chern Simons action consist of four $SO(2, 2)$ states with different values of $F^\pm$. As a result, in the parallel transport of $q^A$ around a close path analogous to the non-supersymmetric case, there will be four sets of holonomies with different values of $(j_+, j_-)$ or, equivalently, $(r_+, r_-)$. Moreover, in the non-supersymmetric case, a single source with Casimir invariants $(r_+, r_-)$ or, equivalently, $(M, J)$ will give rise to an AdS black hole \cite{14} for which the line element is characterized with the corresponding values of $M$ and $J$:

$$ds^2 = -\frac{r^2}{l^2} - M + \frac{J^2}{4l^2} \, dt^2 + \frac{dr^2}{(\frac{r^2}{l^2} - M + \frac{J^2}{4l^2})} + r^2 (d\phi - \frac{J}{2r^2} \, dt)^2$$  \hspace{1cm} (45)

9
In the supersymmetric case, the source is a supermultiplet in which there are four states of differing \((M, J)\) values. Moreover, recall that the explicit parametrization of \(q^A\) given by Eq. 30, depends on the Casimir invariants \((r_+, r_-)\) or, equivalently, on \((M, J)\). Then, depending on which set \((M, J)\) that we choose, we will get a different BTZ solution. Since \(M\) and \(J\) are not invariant under supersymmetry transformations, these solutions are transformed into each other under supersymmetry. This makes it impossible for a single c-number line element of the type given by Eq. 45 to correspond to all the AdS states of a supermultiplet.

The situation here runs parallel to what was encountered in connection with super Poincaré Chern Simons theory [16]. There it was pointed out that standard classical geometries were not capable describing these structures and that one must make use of \textit{nonclassical geometries}. Such geometries can be based on three elements:

1. An algebra such as a Lie algebra or a Lie superalgebra.
2. A line element operator with values in this algebra.
3. A Hilbert space on which the algebra acts linearly.

For the problem at hand, the algebra of interest is the \(N = (1, 1)\) super AdS algebra in \(2 + 1\) dimensions. The corresponding Hilbert space is the representation space of the superalgebra given by Eq. 38. Then, instead of the BTZ line element given above, we begin with a line element operator with values in the \(N = (1, 1)\) superalgebra and assume that its diagonal elements depend on the algebra only through the Casimir operators \((\hat{M}, \hat{J})\) of its \(SO(2,2)\) subalgebra. The “hats” on top of \(M\) and \(J\) are meant to distinguish the operators from the corresponding eigenvalues. Thus, we have

\[
ds^2 = ds^2(\hat{M}, \hat{J})
\]

The matrix element of this operator for each state of the supermultiplet will produce a c-number line element:

\[
<M_k, J_k|ds^2(\hat{M}, \hat{J})|M_k, J_k> = ds^2(M_k, J_k) \tag{46}
\]

In other words, for each state of the supermultiplet, the nonclassical geometry produces a layer of classical space-time. The number of the layers is equal to the dimension of the supermultiplet. Supersymmetry transformations act as messengers linking different layers of this multilayered space-time. For consistency, we must also interpret the quantities \(J_0^\pm\) in Eq. 44 as operators. Acting on different states of the supermultiplet, they will give the corresponding \(F_\pm\) eigenvalues. There will therefore be not one set but four sets of holonomies \(W[e]\) and \(W[\omega]\). Each set will produce the discrete identification subgroup in the corresponding layer of space-time.

Consider, next, the conditions under which every layer of the supermultiplet corresponds to an AdS black hole. For this to be true, we must have

\[
M_k \geq 0; \quad |J_k| \leq lM_k
\]
This, in turn implies that

\[ F_+ \geq F_- \geq 1 \]

In the notation of Eq. 38, for \(|J| = lM\), two layers of the supermultiplet become extreme AdS black holes. The only exception is in the limiting case when \(M = J = 0\), in which case there will be three extremal black holes in the supermultiplet. It is also interesting to note that for an appropriate choice of \(M\) and \(J\) or, equivalently, \(F_+\) and \(F_-\), the same supersource which generates a black hole in one layer can generate a naked singularity in another.

This work was supported, in part by the Department of Energy under the contract number DOE-FGO2-84ER40153. The hospitality of Aspen Center for Physics, where part of this work was carried out, is gratefully acknowledged.

References

[1] P.C. Aichelburg, R. Guven, Phys. Rev. D 24 (1981) 2066
[2] R. Kallosh, A. Linde, T. Ortin, A. Peet, A. Van Proyen Phys. Rev. D 46 (1992) 5278
[3] O. Coussaert, M. Henneaux, eprint no. hep-th/9310194; Phys. Rev. Lett. 72 (1993) 183
[4] J.M. Izquierdo and P.K. Townsend, eprint no. gr-qc/9501018; Class. Quant. Grav. 12 (1995) 895
[5] R. Kallosh, D. Kaster, T. Ortin, T. Torma, Phys. Rev. D 50 (1994) 6374
[6] M.J. Duff, J. Rahmfeld, Phys. Lett. B 345 (1995) 441; eprint no. hep-th/9605083; M.J. Duff, H. Lu, C.N. Pope, Phys. Lett. B 409 (1997) 136
[7] A.R. Steif, Phys. Rev. Lett. 69 (1995) 1849
[8] R. Kallosh, eprint no. hep-th/9503029; Phys. Rev. D 52 (1995) 1234
[9] T. Ortin, eprint no. hep-th/9705095
[10] A preliminary version of this work was reported at the Twenty sixth Global Foundation Conference on High Energy Physics and Cosmology, Miami, FL, December 12-15, 1997; S. Fernando, F. Mansouri, University of Cincinnati preprint UCTP101.98, to appear in the proceedings.
[11] A. Achucarro, P. Townsend, Phys. Lett. B 180 (1986) 35
[12] E. Witten, Nucl. Phys B311 (1988) 46; B 323 (1989) 113
[13] S. Fernando, F. Mansouri, eprint no. hep-th/9804147, Int. Jour. Mod. Phys. A, in press

[14] M. Bañados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. 69 (1992) 1849; M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. D 48 (1993) 1506.

[15] F. Mansouri, M.K. Falbo-Kenkel, Mod. Phys Lett. A 8 (1993) 2503; F. Mansouri, Proceedings of XIIth Johns Hopkins Workshop, ed. Z. Horwath, World Scientific, 1994

[16] Sunme Kim, F. Mansouri, Phys. Lett. B 397 (1997) 81; F. Ardalan, S. Kim, F. Mansouri, Int. Jour. Mod. Phys., A 12 (1997) 1183

[17] B.G. Wybourne, Classical Groups, John Wiley and Sons, 1974. A.O. Barut, C. Fronsdal, Proc. Roy. Soc. (London) A 287 (1965) 532

[18] S. Fernando, F. Mansouri, Proceedings of XXII Johns Hopkins Workshop, ed. L. Brink and R. Marnelius, Goteborg, Sweden, August 20-22, 1998.

[19] K. Koehler, F. Mansouri, C. Vaz, L. Witten, Nucl. Phys. B 348 (1991) 373.

[20] C. Vaz, L. Witten, Phys. Lett. B 327 (1994) 29.