KOSZUL ALGEBRAS AND SHEAVES OVER PROJECTIVE SPACE

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Abstract. We are going to show that the sheafification of graded Koszul modules $K_{\Gamma}$ over $\Gamma_n = K[x_0, x_1, \ldots, x_n]$ form an important subcategory $\hat{K}_{\Gamma}$ of the coherents sheaves on projective space, $\text{Coh}(P^n)$. One reason is that any coherent sheaf over $P^n$ belongs to $\hat{K}_{\Gamma}$ up to shift.

More importantly, the category $K_{\Gamma}$ allows a concept of almost split sequence obtained by exploiting Koszul duality between graded Koszul modules over $\Gamma$ and over the exterior algebra $\Lambda$. This is then used to develop a kind of relative Auslander-Reiten theory for the category $\text{Coh}(P^n)$, with respect to this theory, all but finitely many Auslander-Reiten components for $\text{Coh}(P^n)$ have the shape $ZA_{\infty}$. We also describe the remaining ones.

1. Introduction

The aim of the paper is to apply the methods of non commutative algebra, and very particularly, of finite dimensional algebras to the study of sheaves on projective space as well as to some non commutative generalizations. The starting point for this approach was the paper by Bernstein Gelfand Gelfand in which they proved that there exists a derived equivalence of categories: $\text{mod}_{\Lambda} \cong D^b(\text{Coh}(P^n))$, $[BGG]$ between the stable category $\text{mod}_{\Lambda}$ of the exterior algebra $\Lambda = K < x_0, x_1, \ldots, x_n > / < x_i^2, x_ix_j + x_jx_i >$ and the derived category of bounded complexes of coherent sheaves over projective space.

It is of interest to know which modules over the exterior algebra correspond to particular sheaves over projective space, for example: which modules correspond to locally free sheaves? which to torsion free sheaves?

The answer to the first question was given by J. Bernstein and S. Gelfand $[BG]$ who proved that vector bundles over $P^n$ correspond to "nice" modules over the exterior algebra. We will give here a different characterization of the modules over the exterior algebra corresponding to the locally free sheaves on projective space.

2. Some basic facts on vector bundles over projective space

For the convenience of the reader, and to fix notation, we recall in this section standard theorems of algebraic geometry on sheaves and vector bundles over projective space.

We will consider graded quiver algebras $\Gamma = KQ/I$, where $K$ is a field, $Q$ a finite connected quiver and $I$ an homogeneous ideal in the grading given by path length,
I \subset J^2$, where $J$ is the ideal generated by the arrows. Let $Gr \Gamma$ be the category of graded modules and degree zero maps. In order to prove the main results it seems to be essential to assume $\Gamma$ is noetherian hence; we will assume it. We will denote by $gr \Gamma$ the full subcategory of all finitely generated graded $\Gamma-$modules. We will say that a graded modules $M$ is torsion if it is a sum of submodules of finite length, in particular, if $M$ is finitely generated it is torsion when it is of finite length. For a given module $M$ we will denote by $t(M)$ the sum of all submodules of finite length.

Given two graded $\Gamma-$ modules $X$ and $Y$ we denote by $\text{Hom}_\Gamma(X,Y)_0$ the set of all maps $f : X \rightarrow Y$ such that $f(\xi_i) \subset Y_{i+n}$ by $\text{Hom}_{Gr \Gamma}(X,Y)$ the maps in degree zero and $\text{Hom}_\Gamma(X,Y) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_\Gamma(X,Y)_i$

Let $QGr \Gamma$ be the quotient category with the same objects as $Gr \Gamma$, denote by $\pi : Gr \Gamma \rightarrow QGr \Gamma$ the quotient functor, the maps in $QGr \Gamma$ are defined as $\text{Hom}_{QGr \Gamma}(\pi(X),\pi(Y))_0 = \lim_{\to} \text{Hom}_{Gr \Gamma}(X',Y/Y'_0)$ where the limit is taken over all pairs $(X',Y')$ of submodules of $X$ and $Y$, respectively, such that $X/X'$ and $Y'$ are torsion. We define $\text{Hom}_{QGr \Gamma}(\pi(X),\pi(Y)) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{QGr}(X',Y/Y'_n)$.

Given a graded module $X$ we define the truncated module $X_{\geq n}$ as follows: $(X_{\geq n})_j = 0$ if $j < n$ and $(X_{\geq n})_j = X_j$ if $j \geq n$. The following result is well known $[S2], [M2]$ :

**Proposition 1.** Let $\Gamma = KQ/I$ be a noetherian graded quiver algebra. Then if $X$ is a graded finitely generated module and $Y$ is an arbitrary module, then we have isomorphisms: $\text{Hom}_{QGr \Gamma}(\pi(X),\pi(Y))_m = \lim_{\to} \text{Hom}_{QGr}(X_{\geq n},Y)_m$ for all $m$.

It is well known $[P], [S2], QGr \Gamma$ has injective envelopes and the extension groups $\text{Ext}^k_{QGr \Gamma}(\pi(X),\pi(Y))_m$ are obtained as derived functors of $\text{Hom}_{QGr \Gamma}(\pi(X),\pi(Y))_m$, it follows: $\text{Ext}^k_{QGr \Gamma}(\pi(X),\pi(Y))_m = \lim_{\to} \text{Ext}^k(X_{\geq n},Y)_m$.

**Theorem 1.** (Serre [Se]) Let $k$ be a field, $\Gamma = K[x_0,x_1,...,x_r]/I$ be the quotient of the polynomial algebra module an homogeneous ideal $I$. Let $X$ be the subscheme of $P_r$ defined by $I$ and $O_X$ the sheaf of regular functions. Let $\text{Coh}(X)$ be the category of coherent sheaves, $O_X(n)$ the $n-$th– power of the twisting sheaf. Define a functor, $\Gamma_* : \text{Coh}(X) \rightarrow Qgr \Gamma$ by $\Gamma_*(F) = \oplus H^0(X,F \otimes O_X(d))$, where $H^0(X,F)$ denotes the global sections of the sheaf $F$. The functor is an equivalence of categories with inverse: $\sim : Qgr \Gamma \rightarrow \text{Coh}(X)$ defined as follows:

The set $\{X_f\}_{f \in \Gamma}$ with $X_f = \{x \in X \mid f(x) \neq 0\}$ forms a basis of open sets for the topology of $X$. The structure sheaf $O$ is defined by $O(X_f) = \Gamma [f^{-1}]_0$, the degree zero part of $\Gamma [f^{-1}]$, which consists of all rational functions $g/h$ in $K(X)$ having no pole on $X_f$, the polynomials $g,h$ homogeneous with $\deg(g) = \deg(h)$. If $M$ is a graded module define the sheaf $\tilde{M}$ by $\tilde{M}(X_f) = (\Gamma [f^{-1}] \otimes M)_0 = M [f^{-1}]_0$.

For a coherent sheaf $F$ cohomology groups $H^q(F)$ were defined by Čech, we refer to the reader to Serre ‘s paper for definitions and for the proof of the isomorphisms: $H^q(F) \cong H^q(X,F)$, where $H^q(X, -)$ denotes the derived functor of the global sections functor: $H^0(X, -)$.

The groups $H^q(X,F)$ can be determined in terms of the extension groups as follows:

Let $M$ be a graded $\Gamma-$ module such that $\tilde{M} = F$. It was proved in $[Se], H^q(X,F) = \text{Ext}^q_{QGr \Gamma}(O_X,F)_0$, the degree zero part of the extension group.
Let \( J = (x_0, x_1, ..., x_r) / I \). Then \( \operatorname{Ext}^q_{\mathcal{O}_X}(O_X, F)_0 \cong \lim_{\to} \operatorname{Ext}^q_{\Gamma}(J^k, M)_0 \). Define \( H^q(X, F) = \bigoplus_{n \in \mathbb{Z}} H^n(X, F(n)) \), where \( F(n) = F \otimes O_X(n) \) and \( F(\sim) = \widetilde{M} \) the \( n \)-th shift of \( M \).

With these definitions \( H^q(X, F) = \bigoplus_{n \in \mathbb{Z}} \lim_{\to} \operatorname{Ext}^q_{\Gamma}(J^k, M[n])_0 = \lim_{\to} \operatorname{Ext}^q_{\Gamma}(J^k, M) \).

In particular, \( \Gamma_*(F) = H^0(X, F) = \operatorname{Hom}_{\mathcal{O}_X}(\pi(\Gamma), \pi(M)) \).

Using these isomorphisms Serre duality \([H2]\) has the following form:

**Theorem 2.** Let \( \Gamma = K[x_0, x_1, ..., x_r] / I \) be the quotient of the polynomial algebra by an homogeneous ideal and \( M \) a finitely generated graded module. Then there exists a natural isomorphism:

\[
\operatorname{Hom}_K(\lim\operatorname{Ext}^q_{\Gamma}(J^k, M[-n])_0, K) \cong \lim\operatorname{Ext}^r_{\Gamma}(M[-n]_{\geq k}, \Gamma[-(r + 1)])_0.
\]

**Definition 1.** A sheaf \( F \) is locally free if and only if \( X \) can be covered by open sets \( U \) for which \( F \mid_U \) is a free \( \mathcal{O}_X \mid_U \) module.

We will need the following well known characterization of algebraic vector bundles:

**Proposition 2.** \([L]\) The functor which associates the sheaf of modules of regular sections to a vector bundle \( E \) is an equivalence of categories between the category of algebraic vector bundles over \( X \) and the category of locally free sheaves of finite rank on \( X \).■

The following result will be also needed:

**Theorem 3.** \((\text{Serre } [Se])\) Let \( X = P_r \) be projective space. Then a coherent sheaf \( F \) is locally free if and only if \( H^n(X, F(-n)) = 0 \) for \( n >> 0 \) and \( 0 < q < r \).■

We can prove now the following:

**Proposition 3.** Let \( \Gamma = K[x_0, x_1, ..., x_r] \) be the polynomial algebra and \( F = \pi(M) = \widetilde{M} \) the sheaf associated to a finitely generated module \( M \). Then \( F \) is locally free if and only if \( \lim\operatorname{Ext}^q_{\Gamma}(M_{\geq 2t}, \Gamma[s])_0 = 0 \) for \( s >> 0 \) and \( 0 < j \leq r \).

**Proof.** It is easy to see \( M[-n]_{\geq k} = M_{k-n}[-n] \).

By Serre duality:

\[
\operatorname{Hom}_K(\lim\operatorname{Ext}^q_{\Gamma}(J^k, M[-n])_0, K) \cong \lim\operatorname{Ext}^r_{\Gamma}(M[-n]_{\geq k}, \Gamma[-(r + 1)])_0
\]

\[
\cong \lim\operatorname{Ext}^r_{\Gamma}(M_{k-n}[-n], \Gamma[-(r + 1)])_0 \cong \lim\operatorname{Ext}^l_{\Gamma}(M_{k-n}[-n], \Gamma[n - (r + 1)])_0.
\]

Now it follows \( H^n(X, F(-n)) = 0 \) for \( n >> 0 \) and \( 0 \leq q < r \), if and only if \( \lim\operatorname{Ext}^r_{\Gamma}(M_{\geq 2t}, \Gamma[s])_0 = 0 \) for \( 0 < j \leq r \) and \( s >> 0 \). ■

3. **Koszul Algebras.**

In this section we recall some basic facts on Koszul algebras, the results given here were proved in \([BGS]\), \([ADL]\), \([GM1]\), \([GM2]\). See also the references given there.

**Definition 2.** Let \( \Lambda = KQ/I \) be a graded quiver algebra. A finitely generated module \( M \) with minimal graded projective resolution: \( ... \to P_k \to P_{k-1} \to ... \to P_1 \to P_0 \to M \to 0 \) with each \( P_j \) finitely generated is called quasi Koszul if and only if \( J\Omega^k(M) = \Omega^k M \cap J^2 P_{k-1} \) for any \( k \geq 1 \).
The module $M$ is called weakly Koszul if and only if $J^j\Omega^k(M) = \Omega^kM \cap J^{j+1}P_{k-1}$ for any $j \geq 0$ and any $k \geq 1$.

The module $M$ is Koszul if and only if each $P_j$ is finitely generated in degree $j$.

A graded algebra is called Koszul if and only if all graded simple (generated in degree zero) are Koszul.

**Definition 3.** Let $\Lambda$ be a graded quiver algebra and $M$ a module with minimal graded injective co-resolution: $0 \to M \to I_0 \to I_1 \to \ldots I_k \to \ldots$, with each $I_k$ finitely cogenerated.

1) We say that $M$ is quasi co-Koszul if for any $j > 0$ there exist an epimorphism: $soc^2I_{j-1}/socI_{j-2} \to soc^1\Omega^{-j}(M) \to 0$.

2) If for any pair of positive integers $j$ and $k$ there are epimorphisms: $soc^{k+2}I_{j-1}/soc^{k+1}I_{j-2} \to soc^{k+1}\Omega^{-j}(M)/soc^k\Omega^{-j}(M) \to 0$.

Then we say $M$ is weakly co-Koszul.

3) If for all non negative integers $j$ the injective module $I_j$ is cogenerated in degree $j$, then we say $M$ is co-Koszul.

**Remark 1.** Weakly Koszul (co-Koszul) modules were called strongly quasi Koszul (quasi co-Koszul) in [GM1], [GM2].

We will denote by $K\Lambda$ the category of Koszul modules and degree zero maps, by $\Gamma$ the Yoneda algebra, $\Gamma = \bigoplus Ext^k_{\Lambda}(\Lambda_0, \Lambda_0)$ and by $F : Gr\Lambda \to Gr\Gamma^{op}$ the ext functor: $F(M) = \bigoplus Ext^k_{\Lambda}(M, \Lambda_0)$.

**Theorem 4.** Let $\Lambda = KQ/I$ be a Koszul algebra with Yoneda algebra $\Gamma$. Then the following statements hold:

1) The algebra $\Gamma$ is Koszul with Yoneda algebra $\bigoplus_{k \geq 0} Ext^k_{\Lambda}(\Lambda_0, \Lambda_0) \cong \Lambda$.

2) If $K\Lambda$ and $K\Gamma^{op}$ denote the category of Koszul $\Lambda$ and $\Gamma^{op}$ modules, respectively, then the ext functor induces a duality $F: K\Lambda \to K\Gamma^{op}$ such that $F(J^kM[k]) \cong \Omega^kF(M)[k]$ and $J^kF(M)[k] \cong F(\Omega^kM[k])$.■

We are interested in the following examples of Koszul algebras:

1) The polynomial algebra $\Gamma = K[t_0, t_1, \ldots t_r]$ is a Koszul algebra with Yoneda algebra the exterior algebra $\Lambda = K < t_0, x_1, \ldots x_r > / (t_i^2, t_ix_j + x_jt_i)$.

2) If $\Gamma$ is a Koszul $K-$algebra and $G$ a finite group of automorphisms of $\Gamma$ such that $charK \parallel G \parallel$ and $\Lambda$ is the Yoneda algebra of $\Gamma$, then $G$ acts on $\Lambda$ and the skew group algebra $\Lambda * G$ is Koszul with Yoneda algebra $\Lambda * G$. [M4]

In the examples above the exterior algebra $\Lambda$ is selfinjective and given a finite group of automorphisms of the $K-$ algebra $\Lambda$ such that the $charK \parallel G \parallel$. Then the skew group algebra $\Lambda * G$ is selfinjective if and only if $\Lambda$ is [RR].

The following theorem characterizes selfinjective Koszul algebras, it was proved first in the connected case by P. S. Smith:

**Theorem 5.** [S1], [M3] Let $\Lambda$ be a finite dimensional indecomposable Koszul algebra with Yoneda algebra $\Gamma$. Then the following statements are equivalent:

1) The algebra $\Lambda$ is selfinjective with radical $J$ such that $J^{r+1} \neq 0$ and $J^{r+2} = 0$.

2) i) The graded $\Gamma^{op}$ simple have projective dimension $r + 1$.

ii) For any graded simple module $S$ the equality $Ext^i_{\Gamma^{op}}(S, \Gamma) = 0$ for $i \neq r + 1$.

iii) The functor $Ext^{r+1}_{\Gamma^{op}}(-, \Gamma)$ induces a bijection between the graded $\Gamma$ and $\Gamma^{op}$ simple modules.■
Definition 4. We will call a graded algebra $\Gamma$ satisfying conditions i), ii), iii) Artin Shelter regular [AS].

Remark 2. These algebras were called generalized Auslander regular in [GMT], [M2], [M3].

Another example of Artin Shelter regular Koszul algebra is the preprojective algebra $\Gamma = KQ/I$ of a non Dynkin bipartite graph $Q$, its Yoneda algebra is the trivial extension algebra $\Lambda = KQ \triangleright D(Q)$. [M1], [GMT].

We will need the following results from [M5], [MM].

Theorem 6. Let $\Lambda = KQ/I$ be a Koszul algebra and $M$ and $N$ two Koszul modules. Then for any pair of integers $k$ and $l$, with $k \geq 0$, the following two statements are true:

i) If $\text{Ext}^k(M, N[l])_0 \neq 0$, then $k \geq -l$.

ii) If $k \geq -l$, then there exists a vector space isomorphism:

$$\text{Ext}^k(M, N[l])_0 \cong \text{Ext}^{k+l}_{\Gamma_{op}}(F(N)[l], F(M))_0.$$  

Theorem 7. Let $\Gamma$ be an infinite dimensional Koszul algebra with Yoneda algebra $\Lambda$ and let $F : \text{Gr} \Gamma \to \text{Gr} \Lambda_{op}$ be Koszul duality $F(M) = \bigoplus_{k \geq 0} \text{Ext}^k(M, \Gamma_0)$. Then for any pair of Koszul $\Gamma-$modules $M$ and $N$, any integer $p$ and an integer $n \geq 0$, there exist a functorial isomorphism:

$$\lim_{n} \text{Ext}^n_{\Gamma}(J^{k}M, N)_p \cong \lim_{n} \text{Ext}^n_{\Lambda_{op}}(\Omega^{k+p}F(N), \Omega^{k}F(M))_{-p}.$$  

Corollary 1. Under the conditions of the theorem if we assume in addition $\Lambda$ selfinjective the isomorphism becomes:

$$\lim_{n} \text{Ext}^n_{\Gamma}(J^{k}M, N)_p \cong \text{Ext}^n_{\Lambda_{op}}(\Omega^{p}F(N), F(M))_{-p}.$$  

4. Main results

Let $\Gamma = K[x_0, x_1, \ldots x_r]$ be the polynomial algebra, $\Lambda = K < x_0, x_1, \ldots x_r > / (x_i^2, x_ix_j + x_jx_i)$ the exterior algebra, $\pi : \text{Gr} \Gamma \to Q\text{Gr} \Gamma$ the quotient functor. We are interested in finitely generated graded modules $M$ such that $\pi(M)$ corresponds to a locally free sheaf under Serre’s equivalence. By the approximation proposition [M3], given a finitely generated $\Gamma-$module $M$ there exists an integer $k$ such that $M_{\geq k}[k] = N$ is Koszul. We know [P], [S2] that for two graded $\Gamma-$modules $X$ and $Y$ there exists an integer $k$ such that $X_{>k} \cong Y_{>k}$ if and only if $\pi(X) \cong \pi(Y)$. It follows $\pi(M) \cong \pi(N_{>-k})$. Hence we may assume $M$ is Koszul up to shifting, this is: there exists some integer $k$ such that $M[k]$ is Koszul. Using Koszul duality $F : K_{\Lambda} \to K_{\Gamma_{op}}$ we want to characterize the Koszul $\Lambda-$modules $N$ such that $\pi F(N)$ is locally free.

Definition 5. Let $\Gamma$ be an indecomposable noetherian Artin Shelter regular Koszul algebra of global dimension $r + 1$. Let $\pi : \text{Gr} \Gamma \to Q\text{Gr} \Gamma$ be the quotient functor and $M$ a finitely generated module. We say that $\pi(M)$ is locally free if $\lim_{n} \text{Ext}^n_{\Gamma}(M_{\geq t}, \Gamma[s])_0 = 0$ for $s \gg 0$ and $0 < j \leq r$. (See also [BGK]).

Observe that if $\pi(M)$ is locally free and $k$ is an integer, then $\pi(M[k])$ is locally free:

We have: $M[k]_{\geq t} = M_{\geq k+t}[k]$. Set $s = s - k$ and $k + t = t'$. Then there exists a chain of isomorphisms:
We remarked above that for any finitely generated module $M$ there exists some integer $k$ such that $M_{\geq k}$ is Koszul. We are interested in Koszul $\Lambda^\text{op}$-modules $M$ such that $\pi(F(M))$ is locally free.

**Theorem 8.** Let $\Lambda$ be an indecomposable selfinjective Koszul algebra with $J^{r+1} \neq 0$ and $J^{r+2} = 0$. Assume the Yoneda algebra $\Gamma$ is noetherian. Let $F : K_\Lambda \to \text{K}  \Gamma^\text{op}$ be Koszul duality. Then a Koszul $\Lambda-\text{module}$ $N$ has the property that $\pi(F(N))$ is locally free if and only if $\text{Ext}^1_\Lambda(\Omega^s \Lambda_0 [s], N)_0 = 0$ for $0 < j \leq r$ and $s >> 0$.

**Proof.** By Theorem 7, we have natural isomorphisms:

\[
\lim_\rightarrow \text{Ext}^1_\Lambda(J^k F(N), \Gamma[s])_0 \cong \lim_\rightarrow \text{Ext}^1_\Lambda(J^k F(N), \Gamma)s \cong \lim_\rightarrow \text{Ext}^1_\Lambda(\Omega^{k+s} F^{-1}(\Gamma), \Omega^{k} N)_{-s} \cong \lim_\rightarrow \text{Ext}^1_\Lambda(\Omega^{k+s} \Lambda_0 [s], \Omega^k N)_0
\]

\[
\cong \text{Ext}^1_\Lambda(\Omega^s \Lambda_0 [s], N)_0.
\]

It follows $\pi(F(N))$ is locally free if and only if $\text{Ext}^1_\Lambda(\Omega^s \Lambda_0 [s], N)_0 = 0$ for $0 < j \leq r$ and $s >> 0$. ■

**Corollary 2.** Let $N$ be a Koszul $\Lambda$-module. Then for any positive integer $k$ the module $\Omega^k N [k]$ is such that $\pi(F(\Omega^k N [k]))$ is locally free if and only if $\pi(F(N))$ is locally free.

**Proof.** $\text{Ext}^1_\Lambda(\Omega^s \Lambda_0 [s], \Omega^k N [k])_0 \cong \text{Ext}^1_\Lambda(\Omega^s \Lambda_0 [s-k], N)_0 = 0$ for $s >> 0$. ■

**Lemma 1.** Let $\Lambda = KQ/I$ be a graded quiver algebra, let $M$ and $N$ be graded modules generated in degree $l$ and $k$, respectively, with $l > k$. Then there is a natural isomorphism:

\[
\text{Ext}^1_\Lambda(M, N)_0 \cong \text{Ext}^1_\Lambda(M, J^{l-k} N)_0.
\]

**Proof.** The exact sequence: $0 \to J^{l-k} N \to N \to N/J^{l-k} N \to 0$ induces an exact sequence:

\[
H \text{om}_\Lambda(M, N/J^{l-k} N)_0 \to \text{Ext}^1_\Lambda(M, J^{l-k} N)_0 \Rightarrow \text{Ext}^1_\Lambda(M, N)_0.
\]

Where $H \text{om}_\Lambda(M, N/J^{l-k} N)_0 = 0$.

We must prove $\delta$ is an epimorphism.

Let $x \in \text{Ext}^1_\Lambda(M, N)_0$. Then the exact sequence $x : 0 \to N \to E \to M \to 0$ induces an exact commutative diagram:

\[
0 \to N_{\geq l} \to E_{\geq l} \to M \to 0
\]

\[
0 \to N \to E \to M \to 0
\]

With $N_{\geq l} \cong J^{l-k} N$ and we have proved $\delta$ is an epimorphism. ■

**Proposition 4.** Let $\Lambda$ be an indecomposable selfinjective Koszul algebra with $J^{r+1} \neq 0$ and $J^{r+2} = 0$. Assume the Yoneda algebra $\Gamma$ is noetherian and let $F : K_\Lambda \to \text{K} \Gamma^\text{op}$ be the Koszul duality. Let $0 \to X \to Y \to Z \to 0$ be an exact sequence of Koszul $\Lambda-\text{modules}$ and degree zero maps. Then the following statements hold:

1) If $\pi(F(X))$ and $\pi(F(Z))$ are locally free, then $\pi(F(Y))$ is locally free.

2) If $\pi(F(X))$ and $\pi(F(Z))$ are locally free, then $\pi(F(Y))$ is locally free.
Proof. 1) The exact sequence \(0 \to X \to Y \to Z \to 0\) of Koszul \(\Lambda\)-modules induces an exact sequence:

\[
\text{Ext}^i_{\Lambda}(\Omega^s \Lambda_0 [s], X)_0 \to \text{Ext}^i_{\Lambda}(\Omega^s \Lambda_0 [s], Y)_0 \to \text{Ext}^i_{\Lambda}(\Omega^s \Lambda_0 [s], Z)_0,
\]

by hypothesis, \(\text{Ext}^i_{\Lambda}(\Omega^s \Lambda_0 [s], X)_0 = \text{Ext}^i_{\Lambda}(\Omega^s \Lambda_0 [s], Z)_0 = 0\) for \(0 < i \leq r\) and \(s \gg 0\).

Hence: \(\text{Ext}^i_{\Lambda}(\Omega^s \Lambda_0 [s], Y)_0 = 0\) for \(0 < i \leq r\) and \(s \gg 0\).

2) Let \(k\) be an integer \(k \geq r + 1\) and \(L\) a torsion free \(\Gamma^{-}\)module.

It was proved in \([M2]\), \(\text{Ext}^k_{\Lambda}(L, \Gamma) = 0\), hence: \(\text{Ext}^k_{\Lambda}(L, \Gamma)_j = 0\) for all \(j\) and \(k \geq r + 1\).

If \(F(X)\) is torsion free, then using lemma 1 and theorem 6, we have a chain of natural isomorphisms:

\[
\text{Ext}^i_{\Lambda}(\Omega^s \Lambda_0 [s], X)_0 \cong \text{Ext}^i_{\Lambda}(\Omega^{k+s-1} \Lambda_0 [s], X)_0 \cong \text{Ext}^i_{\Lambda}(\Omega^{k+s-1} \Lambda_0 [s], J^{k-1}X)_0
\]

\[
\cong \text{Ext}^i_{\Lambda}(\Omega^{k-1} F(X)[s], J^{k+s-1} F(\Lambda_0))_0 \cong \text{Ext}^i_{\Lambda}(\Omega^{k-1} F(X)[s], F(\Lambda_0))_0
\]

\[
\cong \text{Ext}^i_{\Lambda}(F(X), F(\Lambda_0))_s = \text{Ext}^i_{\Lambda}(F(X), \Gamma)_s = 0 \text{ for } k \geq r + 1.
\]

Assume \(F(X)\) has torsion and let \(S_1\) be a simple in the socle, then there is an integer \(k\) such that \(J^k F(X) \cong F(d^k X) \cong S_1 \oplus W_1\). Applying \(F^{-1}\), there is an isomorphism: \(\Omega^s X \cong P_1 \oplus X_1\), but since \(P_1\) is projective injective \(k = 0\) and \(S_1\) appears at the top of \(F(X)\). We have proved \(F(X) \cong S \oplus W\) with \(S\) semisimple and \(W\) torsion free, hence \(X \cong P \oplus X'\) with \(P\) projective and \(X'\) is such that \(\pi(F(X'))\) is locally free.

We have an exact sequence:

\[
\text{Ext}^i_{\Lambda}(\Omega^s \Lambda_0 [s], X)_0 \to \text{Ext}^i_{\Lambda}(\Omega^s \Lambda_0 [s], Z)_0 \to \text{Ext}^i_{\Lambda}(\Omega^s \Lambda_0 [s], X)_0.
\]

By hypothesis and the above remark, for \(0 < i \leq r\) and \(s \gg 0\) we have:

\[
\text{Ext}^i_{\Lambda}(\Omega^s \Lambda_0 [s], Y)_0 = 0 \cong \text{Ext}^i_{\Lambda}(\Omega^s \Lambda_0 [s], X'_0).
\]

Hence: \(\text{Ext}^i_{\Lambda}(\Omega^s \Lambda_0 [s], Z)_0 = 0\) for \(0 < i \leq r\) and \(s \gg 0\). \(\blacksquare\)

**Proposition 5.** Let \(\Lambda\) be an indecomposable selfinjective Koszul algebra with \(J^{r+1} \neq 0\) and \(J^{r+2} = 0\). Assume the Yoneda algebra \(\Gamma\) is noetherian and let \(f: K_{\Lambda} \to K_{\Lambda^{op}}\) be the Koszul duality. Then the following statement holds:

If \(X\) is a Koszul \(\Lambda^{-}\)module such that \(\pi(F(X))\) is locally free, then \(JX[1]\) is such that \(\pi(F(JX[1]))\) is locally free.

Proof. Let \(X\) be a Koszul \(\Lambda^{-}\)module such that \(\pi(F(X))\) is locally free, let \(0 \to \Omega(X) \to P \to X \to 0\) be exact with \(P\) the projective cover of \(X\). It induces an exact sequence: \(0 \to \Omega(X)[1] \to JP[1] \to JX[1] \to 0\). By Corollary 2, \(\pi(F(\Omega(X)[1]))\) is locally free.

If \(P\) is projective generated in degree zero and \(M\) is a Koszul module, then from the exact sequence: \(0 \to JP[1] \to P[1] \to S[1] \to 0\) we obtain an exact sequence:

\[
0 \to \text{Hom}_{\Lambda}(M, JP[1])_0 \to \text{Hom}_{\Lambda}(M, P[1])_0 \to \text{Hom}_{\Lambda}(M, S[1])_0 \to \text{Ext}^i_{\Lambda}(M, JP[1])_0 \to 0
\]

and isomorphisms:

\[
\text{Ext}^i_{\Lambda}(M, S[1])_0 \cong \text{Ext}^i_{\Lambda}(M, JP[1])_0.
\]

Since \(M\) is generated in degree zero and \(\Omega^{j-1} M\) in degree \(j - 1\), it follows \(\text{Hom}_{\Lambda}(M, S[1])_0 = 0\) and \(\text{Ext}^i_{\Lambda}(M, S[1])_0 = 0\). Therefore \(\text{Ext}^i_{\Lambda}(M, JP[1])_0 = 0\) for \(j > 0\).

Letting \(M\) be equal to \(\Omega^s \Lambda_0 [s]\) we get \(\text{Ext}^i_{\Lambda}(\Omega^s \Lambda_0 [s], JP[1])_0 = 0\) for all \(s > 0\) and all \(i > 0\).
We have proved \( \pi(F(JP[1])) \) is locally free. It follows by 2) of previous proposition \( \pi(F(JX[1])) \) is locally free. ■

We will consider now the case \( r = 1 \).

**Theorem 9.** Let \( \Gamma \) be an Artin Shelter regular Koszul algebra of global dimension 2 and assume \( \Gamma \) is noetherian indecomposable with Yoneda algebra \( \Lambda \), let \( F : K_{\Lambda} \to K_{\Gamma\text{op}} \) be the Koszul duality. A Koszul \( \Lambda \)– module \( M \) is such that \( \pi(F(M)) \) is locally free if and only if \( M \cong \Omega^s S[s] \) for a simple module \( S \) and some integer \( s \geq 0 \).

**Proof.** A Koszul \( \Lambda \) – module \( M \) is such that \( \pi(F(M)) \) is locally free if and only if \( \text{Ext}_1^{\Lambda}(\Omega^s \Lambda_0[s], M)_0 = 0 \) for \( s >> 0 \). Equivalently, if and only if the maps module projectives \( \text{Hom}_{\Lambda}(\Omega^{s+1} \Lambda_0[s], M)_0 = 0 \) for \( s >> 0 \).

Assume \( M = \Omega^t S[t] \), with \( S \) a simple \( \Lambda \)– module and let \( s > t \).

Then \( \text{Hom}_{\Lambda}(\Omega^{s+1} \Lambda_0[s], \Omega^t S[t])_0 = \text{Hom}_{\Lambda}(\Omega^{s-t+1} \Lambda_0[s-t], S)_0 \), but \( \Omega^{s-t+1} S[s-t] \) is generated in degree 1, hence; \( \text{Hom}_{\Lambda}(\Omega^{s-t+1} \Lambda_0[s-t], S)_0 = 0 \).

Conversely, assume \( M \) is not isomorphic to \( \Omega^t S[s] \) for any simple \( S \) and any \( s \).

There is an isomorphism: \( \text{Hom}_{\Lambda}(\Omega^{s+1} \Lambda_0[s], M)_0 = \text{Hom}_{\Lambda}(\Lambda_0, \Omega^{-s-1} M[-s])_0 \).

If the Loewy length of \( M \) is two, then \( \text{soc} M \) is generated in degree one. We have an exact sequence: \( 0 \to M \to P \xrightarrow{\delta} \Omega^{-1} M \to 0 \) with \( P \) generated in degree \(-1\). Since \( \Omega^{-1} M \) is not projective, \( \rho(\text{soc} P) = 0 \) and Loewy length of \( \Omega^{-1} M \leq 2 \).

If \( \Omega^{-1} M \) is not simple, then \( \text{soc}(\Omega^{-1} M)[−1] \) is generated in degree 1. It follows by induction \( \text{soc}(\Omega^{-1} M)[−s] \) is generated in degree 1, hence; \( \Omega^{-s-1} M[−s] \) has socle in degree zero and \( \text{Hom}_{\Lambda}(\Lambda_0, \Omega^{-s-1} M[−s])_0 \neq 0 \) for all \( s \).

In the general case we have the following remarks:

**Proposition 6.** Let \( \Gamma \) be an indecomposable Artin Shelter regular Koszul algebra of global dimension \( r + 1 \) with Yoneda algebra \( \Lambda \), let \( F : K_{\Lambda} \to K_{\Gamma\text{op}} \) be the Koszul duality. Let \( N \) be a Koszul \( \Lambda \)– module such that \( \pi(F(N)) \) is locally free. Then there exists some integer \( s \) such that \( \Omega^{-s} N[−s] \) is not Koszul.

**Proof.** Assume for all \( s \geq 0 \) the module \( \Omega^{-s} N[−s] \) is Koszul. The fact \( \pi(F(N)) \) is locally free implies for \( s >> 0 \) and \( 0 < j \leq r \) there are isomorphisms:

\[
\text{Hom}_{\Lambda}(\Omega^{s+j} \Lambda_0[s], N)_0 = \text{Hom}_{\Lambda}(\Omega \Lambda_0, \Omega^{-s} N[−s])_0 = \text{Hom}_{\Lambda}(\Lambda_0, \Omega^{-s-j} N[−s])_0 = 0
\]

By hypothesis the module \( X = \Omega^{-s-j} N[−s−j] \) is Koszul. Consider the exact sequence: \( 0 \to X \to P \xrightarrow{\delta} \Omega^{-1} X \to 0 \) with \( P \) the injective envelope of \( X \). Since \( \Omega^{-1} X[−1] \) is Koszul then \( \text{soc} X \) is generated in degree \( r \).

If \( j = r \), then \( X[j] \) has socle generated in degree zero and \( \text{Hom}_{\Lambda}(\Lambda_0, X[j]) = \text{Hom}_{\Lambda}(\Lambda_0, \Omega^{-s-j} N[−s])_0 \neq 0 \) is a contradiction. ■

We have the following characterization of the Koszul \( \Lambda \)– modules \( M \) with \( \pi(F(M)) \) locally free:

**Theorem 10.** Let \( \Gamma \) be an indecomposable noetherian Artin Shelter regular Koszul algebra of global dimension \( r + 1 \) with Yoneda algebra \( \Lambda \), let \( F : K_{\Lambda} \to K_{\Gamma\text{op}} \) be the Koszul duality. Then an indecomposable Koszul module \( M \) is such that \( \pi(F(M)) \) is locally free if and only if there exists an integer \( s > 0 \) such that \( \Omega^{-s} M[−s] \geq 1 = 0 \).

**Proof.** Assume \( \pi(F(M)) \) is locally free and suppose for all \( s > 0 \) the module \( \Omega^{-s} M[−s] \geq 1 \neq 0 \).

We know by \([MZ]\), there exists an integer \( s \) such that \( \Omega^{-s} M[−s] \) is weakly co-Koszul.
Let \( X = \Omega^{-s}M[-s] \) and let \( 0 \to X \to Q \to \Omega^{-1}X \to 0 \) be exact with \( Q \) the injective envelope of \( X \). Since \( X \) is weakly co-Koszul the epimorphism: \( Q/socQ \to \Omega^{-1}X \to 0 \) induces an epimorphism: \( soc^2Q/socQ \to soc\Omega^{-1}X \to 0 \) this means that if \( X \) has a cogenerator in degree \( k \), then \( \Omega^{-1}X \) has a cogenerator in degree \( k - 1 \) and \( \Omega^{-1}X[-1] \) has a cogenerator in degree \( k \).

The module \( M \) is generated in degree zero, then it has cogenerators in degree \( k \) with \( k \leq r \) and \( \Omega^{-1}M[-1] \) has cogenerators in degree \( \leq r \), it follows by induction \( X \) has cogenerators in degree \( \leq r \), by hypothesis \( \Omega^{-s}M[-s] \not= 0 \), then \( \Omega^{-s}M[-s] \) has a cogenerator in degree \( k \) with \( 1 \leq k \leq r \). It follows for all \( t \geq 0 \) the module \( \Omega^{-t-s}M[-t-s] \) has a cogenerator in lowest degree \( 1 \leq k \leq r \). Then \( \Omega^{-k}\Omega^{-s}M[-(s+t)] \) has a cogenerator in degree zero and it follows:

\[
\text{Hom}_\Lambda(A_0, \Omega^{-k}\Omega^{-s}M[-(s+t)])_0 \not= 0,
\]

hence; \( \text{Ext}^k(\Omega\ell \Lambda_0[t], M)_0 \not= 0 \) for all \( \ell \geq s \) a contradiction.

Assume now there exists an \( s \) such that \( \Omega^{-s}M[-s] \not= 0 \) and set \( X = \Omega^{-s}M[-s] \).

As above, if \( X \) has a cogenerator in lowest degree \( k \), then \( \Omega^{-1}X[-1] \) has a cogenerator in lowest degree \( \leq k \). It follows for all \( t \geq 0 \) the module \( \Omega^{-s}M[-s-t] \) is not locally free.

Hence; for all \( 0 < j \leq r \) the module \( \Omega^{-j}\Omega^{-s}M[-(s+t)] \) has cogenerators in degree less than zero. Therefore: \( \text{Hom}_\Lambda(A_0, \Omega^{-j}\Omega^{-s}M[-(s+t)])_0 = 0 \) for all \( t \geq 0 \) and \( 0 < j \leq r \).

We have proved \( \pi(F(M)) \) is locally free.

**Corollary 3.** Let \( \Lambda \) and \( \Gamma \) be as in the theorem. Let \( M \) be a Koszul and quasi co-Koszul \( \Lambda - \) module. Then \( \pi(F(M)) \) is locally free if and only if \( M \) is simple.

**Proof.** It is clear that if \( M \) is simple, then \( F(M) \) is projective and \( \pi(F(M)) \) locally free.

Assume \( M \) is not simple. Then \( M \) has cogenerators in degree \( k \) with \( r \geq k > 0 \), hence: \( M_{\geq 1} \not= 0 \).

As in the proof of the theorem, \( \Omega^{-s}M[-s] \) has cogenerators in degree \( k > 0 \). Therefore: \( \text{Hom}_\Lambda(A_0, \Omega^{-k}\Omega^{-s}M[-s])_0 \not= 0 \) for all \( s > 0 \).

It follows \( \pi(F(M)) \) is not locally free.

We can improve now the characterization of the \( \Lambda - \)modules corresponding to locally free sheaves.

**Theorem 11.** Let \( \Gamma \) be an indecomposable noetherian Artin Schelter regular Koszul algebra of global dimension \( r + 1 \) with Yoneda algebra \( \Lambda \), let \( F : K_\Lambda \to K_{\Gamma_{yn}} \) be Koszul duality. Then an indecomposable Koszul module \( M \) is such that \( \pi F(M) \) is locally free if and only if there exists a non negative integer \( t \) such that \( \Omega^{-t}M[-t] \) is co-Koszul.

**Proof.** Assume there exists an integer \( t \geq 0 \) such that \( \Omega^{-t}M[-t] \) is co-Koszul. By hypothesis, \( \Omega^{-t}M[-t] \) is cogenerated in degree zero, this means \( \Omega^{-t}M[-t] \geq 1 = 0 \). Applying Theorem 10, it follows \( \pi F(M) \) is locally free.

We assume now \( \pi F(M) \) is locally free, by Theorem 10, there exists an integer \( t \geq 0 \) such that \( \Omega^{-t}M[-t] \geq 1 = 0 \). Let \( S \) be a simple in the socle of \( \Omega^{-t}M[-t] \). By hypothesis, \( S \) is generated in degree \( k \leq 0 \). Suppose \( k < 0 \), then \( \Omega^kS \) is generated in degree \( k + t \) and \( M[-t] \) is generated in degree \( t \) with \( k + t < t \). It follows \( \text{Hom}_\Lambda(\Omega^kS, M[-t])_0 = 0 \).
We have isomorphisms: \( \text{Hom}_A(S, \Omega^{-t}M[-t])_0 = \text{Hom}_A(S, \Omega^{-t}M[-t])_0 \cong \text{Hom}_A(\Omega^jS, \Omega^{-t}M[-t])_0 \) and \( \text{Hom}_A(S, \Omega^{-t}M[-t])_0 \neq 0 \). Since the natural inclusion: \( j : S \to \Omega^{-t}M[-t] \) is not zero, we have reached a contradiction, proving \( \text{soc}\Omega^{-t}M[-t] \) is generated in degree zero.

Let \( s > t \). It is clear \( \Omega^{-s}M[-s] \geq 0 \), hence; \( \text{soc}\Omega^{-s}M[-s] \) is generated in degree zero. It follows \( \Omega^{-t}M[-t] \) is co-Koszul.

**Theorem 12.** Let \( \Gamma \) and \( \Lambda \) be Koszul algebras satisfying the conditions of Theorem 11 and \( F : \text{K}_{\Lambda} \to \text{K}_{\Gamma} \) be Koszul duality. Let \( M \) be a Koszul \( \Lambda \)-module such that \( \pi F(M) \) is locally free. Then there exists a semisimple module \( S \) finitely generated in degree zero, a non negative integer \( t \) and an epimorphism: \( \Omega^tS[t] \to M \to 0 \).

**Proof.** Let \( t \) be a non negative integer such that \( \Omega^{-t}M[-t] \) is co-Koszul and let \( S \) be the socle of \( \Omega^{-t}M[-t] \), the inclusion: \( j : S \to \Omega^{-t}M[-t] \) induces a non zero map: \( \Omega^tj : \Omega^tS \to M[-t] \). We want to prove that \( \Omega^tj \) is an epimorphism.

Assume \( \Omega^tj \) is not an epimorphism, then there exists a simple \( S' \) generated in degree zero and a non zero map \( h : M[-t] \to S'[-t] \) such that \( h \Omega^tj = 0 \). Therefore: \( (\Omega^{-t}h)j = 0 \) and \( \text{Ker} \Omega^{-t}h \supseteq \text{soc}\Omega^{-t}M[-t] \).

Let \( T \) be a simple module contained in \( \text{soc}(\text{Im}\Omega^{-t}h) \), taking the pull back of the inclusion \( i \) we obtain a commutative exact diagram:

\[
\begin{array}{cccc}
0 & \to & \text{Ker}\Omega^{-t}h/\text{soc}\Omega^{-t}M[-t] & \to & W & \to & T & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \text{Ker}\Omega^{-t}h/\text{soc}\Omega^{-t}M[-t] & \to & \Omega^{-t}M[-t]/\text{soc}\Omega^{-t}M[-t] & \to & \text{Im}\Omega^{-t}h & \to & 0
\end{array}
\]

Since \( \Omega^{-t}M[-t] \) is cogenerated in degree zero \( W_k \neq 0 \) implies \( k < 0 \). Therefore: \( W \) has all its generators in negative degrees. It follows \( T \) is generated in negative degree, but this is a contradiction since \( \text{soc}(\text{Im}\Omega^{-t}h) \subseteq \text{soc}\Omega^{-t}S'[-t] \) and \( \text{soc}\Omega^{-t}S'[-t] \) is generated in degree zero.

5. The Auslander Reiten quiver of \( K_\Lambda \).

In this section, we study the Auslander Reiten quiver of the category \( K_\Lambda \) of Koszul modules over a selfinjective Koszul algebra \( \Lambda \). We recall from [GMRSZ], that if \( M \) is an indecomposable Koszul module, then there exists an Auslander sequence: \( 0 \to \sigma(M) \to E \to M \to 0 \) in \( K_\Lambda \) ending at \( M \). This sequence is constructed as follows: we first look at the Auslander Reiten sequence ending at \( M \) in \( gr \Lambda : 0 \to \tau(M) \to F \to M \to 0 \) and then, if we define \( \sigma(M) = (\tau M)_{\geq 0} \) and \( E = F_{\geq 0} \), it turns out that \( \sigma(M) \) is an indecomposable Koszul module, and we also get that the induced sequence is an Auslander-Reiten sequence in \( K_\Lambda \). Thus \( K_\Lambda \) has left A-R sequences. Also note that \( \sigma(M) \) has Loewy length exactly two, hence it is also a \( \Lambda/J^2 \)-module. We also mention that if \( 0 \to A \to B \to C \to 0 \) is an A-R sequence in \( K_\Lambda \), then it is also an A-R sequence in the category of graded modules finitely generated in degree zero, \( gr\Lambda \). This will be used through the section.

**Lemma 2.** Let \( 0 \to A \to B \to C \to 0 \) be a non split exact sequence in \( K_\Lambda \). Then the induced sequence \( 0 \to \sigma(A) \to \sigma(B) \to \sigma(C) \to 0 \) is also exact in \( K_\Lambda \).

**Proof.** Since for each \( i \geq 0 \) we have \( J^iA = J^iB \cap A \), we know from [MZ], that the exactness of \( 0 \to A \to B \to C \to 0 \) implies that we have an induced exact sequence: \( 0 \to \tau(A) \to \tau(B) \to \tau(C) \to 0 \). The lemma follows immediately from the definition of \( \sigma \).

The main tool that will be used very often, is the following result from [GMRSZ]:
Proposition 7. Include proofs only for our more restricted needs. Reiten sequences. They are part of the finite dimensional algebra folklore and we exclude proofs only for our more restricted needs.

\[ f_1 \leq P_0 \]

Lemma 3. Let \( \mathcal{L}_\Lambda \) denote the full subcategory of \( gr_{0\Lambda} \) consisting of the modules \( M \) having a linear presentation, that is a projective presentation of the form: \( P_1 \to P_0 \to M \to 0 \) where \( P_1 \) is generated in degree 1 and \( P_0 \) is generated in degree zero. Then \( \Lambda/J^2 \otimes_\Lambda - : \mathcal{L}_\Lambda \to gr_{0\Lambda}/J^2 \) is an equivalence of categories. \( \blacksquare \)

The following results are true in a more general setting for certain subcategories of the graded module category of a finite dimensional algebra having left Auslander-Reiten sequences. They are part of the finite dimensional algebra folklore and we include proofs only for our more restricted needs.

Proposition 7. Let \( M \) be an indecomposable non-projective Koszul \( \Lambda \)-module and let \( 0 \to \sigma(M) \xrightarrow{f} E \xrightarrow{g} M \to 0 \) be the A-R sequence in \( K_\Lambda \) ending at \( M \). Let \( f = [f_1, f_2, ... f_k] \) and \( g = [g_1, g_2, ... g_k]^T \), where \( E = E_1 \oplus E_2 \oplus ... E_k \), and for each \( 1 \leq i \leq k \), \( E_i \) is indecomposable in \( K_\Lambda \). Then, for each \( 1 \leq i \leq k \), the maps \( f_i \) and \( g_i \) are irreducible in \( K_\Lambda \).

Proof. First note that, by construction, the sequence \( 0 \to \sigma(M) \xrightarrow{f} E \xrightarrow{g} M \to 0 \) is also an A-R sequence in \( gr_{0\Lambda} \). We show, more generally, that the maps are in fact irreducible in \( gr_{0\Lambda} \).

Assume now we have a factorization of \( g_i \) in \( gr_{0\Lambda} \): \( \xrightarrow{i} j \xrightarrow{h,g_i} \) and that \( h \) is a not splittable epimorphism.

Then we have an induced factorization

\[
\begin{bmatrix}
E_i \\
E'_i
\end{bmatrix} \xrightarrow{[g_i,g'_i]} M
\]

where \( E'_i = \bigoplus_{i \neq s} E_s \). Let us show that \( [h,g'_i] \) is not a splittable epimorphism. If it is, there exists a morphism \( \begin{bmatrix} s \\ t \end{bmatrix} : M \to X \oplus E'_i \), such that \( [h,g'_i] \begin{bmatrix} s \\ t \end{bmatrix} = 1_M \), and we have \( hs + g'_i t = 1_M \). The composition \( g'_i t \) is an endomorphism of \( M \), and since \( g'_i \) is not a splittable epimorphism, the image is in the radical of \( \text{End}M \), hence \( g'_i t \) is nilpotent. Therefore \( hs = 1 - g'_i t \) is invertible in \( \text{End}M \), so \( h \) is a splittable epimorphism contradicting our assumption on \( h \). This means that we can lift \( [h,g'_i] \) to \( E_i \oplus E'_i \), and we obtain the following commutative diagram with exact rows:

\[
\begin{array}{cccc}
0 & \to & \sigma(M) & \to & E_i \oplus E'_i & \xrightarrow{[g_i,g'_i]} & M & \to 0 \\
\downarrow & & \downarrow & \quad \downarrow & \quad & \quad & \quad & \\
0 & \to & K & \to & X \oplus E'_i & \xrightarrow{[h,g'_i]} & M & \to 0 \\
\downarrow & & \downarrow & \quad \downarrow & \quad & \quad & \quad & \\
0 & \to & \sigma(M) & \to & E_i \oplus E'_i & \xrightarrow{[g_i,g'_i]} & M & \to 0
\end{array}
\]
In the composite diagram

\[
\begin{array}{c}
0 \rightarrow \sigma(M) \rightarrow E_i \oplus E'_i \xrightarrow{[g_i,g'_i]} M \rightarrow 0 \\
\downarrow \alpha \quad \downarrow \begin{bmatrix} a & j \\ c & d \end{bmatrix} \\
0 \rightarrow \sigma(M) \rightarrow E_i \oplus E'_i \xrightarrow{[g_i,g'_i]} M \rightarrow 0
\end{array}
\]

we observe that \( \alpha \) cannot be nilpotent, otherwise a simple argument would show that \( 1_M \) factors through \( E_i \oplus E'_i \), contradicting our assumptions. Since \( \sigma(M) \) is indecomposable, \( \alpha \) must be invertible. Thus \( \begin{bmatrix} a & j \\ c & d \end{bmatrix} \) is invertible, so the matrix \( \begin{bmatrix} j & 0 \\ 0 & 1 \end{bmatrix} \) is a splittable monomorphism. So there exists a map \( \begin{bmatrix} x & y \\ z & w \end{bmatrix} \) that is a left inverse of \( \begin{bmatrix} j & 0 \\ 0 & 1 \end{bmatrix} \), and we obtain \( xj = 1 \), so that \( j \) is a splittable monomorphism proving that \( g_i \) is an irreducible morphism. We show in a similar way that each \( f_i \) is also an irreducible morphism. ■

It turns out that each irreducible morphism in \( \text{gr}_{00} \Lambda \) is either a monomorphism or an epimorphism. For instance, assume that we have a map \( f : E \rightarrow M \) that is irreducible in \( \text{gr}_{00} \Lambda \). If \( f \) is not onto, then, since \( X = \text{Im} f \) is again generated in degree zero, we have a factorization

\[
E \xrightarrow{f} M \xrightarrow{\sigma(M)} E_i \oplus E'_i \xrightarrow{[g_i,g'_i]} M \rightarrow 0
\]

in degree zero, we have a factorization \( X \xrightarrow{s} E \xrightarrow{f} M \) and this implies that \( s \) is a splittable monomorphism, hence \( f \) is also a monomorphism. This implies that if \( 0 \rightarrow \sigma(M) \xrightarrow{i} E \xrightarrow{\sigma(M)} M \rightarrow 0 \) is an A-R sequence in \( K_{\Lambda} \), and \( f = [f_1,f_2,...f_k] \) and \( g = [g_1,g_2,...g_k]^T \), then for each \( i \), the morphisms \( f_i \) and \( g_i \) are either monomorphisms or epimorphisms.

We recall the following definition. Assume that the subcategory \( \mathcal{C} \) has left Auslander-Reiten sequences, and let \( M \in \mathcal{C} \). The cone of \( M \) is the subquiver of the A-R quiver of \( \mathcal{C} \), consisting of \( M \) and its predecessors. Let now \( M \) be a non projective Koszul module over an indecomposable selfinjective Koszul algebra of Loewy length \( r + 1 \), for example the exterior algebra \( \Lambda = \bigwedge^{r+1} V \), where \( V \) is a \( K \)-vector space of dimension \( r + 1 \) and \( r \geq 2 \). We will describe the cone of \( M/J \) in \( K_{\Lambda} \), by analyzing the cone of \( M/J^2 M \) in the A-R quiver of \( \text{gr}_{\Lambda/J^2} \). Namely, using the Auslander-Reiten quiver of \( \text{gr}_{\Lambda/J^2} \) and \( [GMRSZ] \), we see that two of the components of the A-R quiver of \( \text{gr}_{\Lambda/J^2} \) are of the form \( Z \Delta \), where \( \Delta \) is the separated quiver of \( \Lambda/J^2 \). For example in the case \( \Lambda \) is the exterior algebra the separated quiver of \( \Lambda/J^2 \) is the quiver • ⇒ • with two vertices and \( r + 1 \) arrows from the first vertex to the second one. There are the "preprojective" component

\[
Y_0 = \Lambda/J^2
\]

and the "preinjective" component

\[
X_0 = K
\]
Lemma 4. Let Λ be a Koszul algebra and let \( f : M \to N \) be a map in \( \text{gr}_0\Lambda \). Then the induced map \( \overrightarrow{f} : M/J^2M \to N/J^2N \) is a monomorphism if \( f \) is a monomorphism and an epimorphism if \( f \) is an epimorphism.

Proof. The statement about epimorphisms is trivial. Assume \( f \) is a monomorphism, so we have an exact sequence \( 0 \to M \to N \to C \to 0 \). It is easy to show that \( J^iM = J^iN \cap M \) for all \( i \geq 0 \). The result follows from [GM1].

Proposition 8. Let Λ be a selfinjective Koszul algebra, \( M \) an indecomposable non projective Koszul module, and let \( 0 \to \sigma(M) \xrightarrow{f} E \xrightarrow{g} M \to 0 \) be the A-R sequence in \( K_\Lambda \) ending at \( M \), where the maps are \( f = [f_1, f_2, \ldots, f_k] \) and \( g = [g_1, g_2, \ldots, g_k]^T \).

Then:
  
  i) The induced sequence \( 0 \to \sigma(M) \xrightarrow{\overrightarrow{f}} E/J^2E \xrightarrow{\overrightarrow{g}} M/J^2M \to 0 \) is the A-R sequence ending at \( M/J^2M \) in \( \text{gr}_{0\Lambda}/J^2 \).

  ii) The number of indecomposable summands of \( E \), equals the number of indecomposable summands of \( E/J^2E \).

  iii) The irreducible morphisms \( f_i, g_i \) are monomorphisms (epimorphisms) if and only if the induced maps \( \overrightarrow{f_i}, \overrightarrow{g_i} \) are monomorphisms (epimorphisms).

Proof. i) Assume \( X \to M/J^2M \) is not a nonsplittable epimorphism in \( \text{gr}_{0\Lambda}/J^2 \). Using the equivalence mentioned earlier, this map is induced by a homomorphism \( h : Y \to M \) in \( L_\Lambda \) that is not a splittable epimorphism. Since \( h \) can be lifted to \( E \), the result follows since the equivalence \( L_\Lambda \cong \text{gr}_{0\Lambda}/J^2 \) restricted to \( K_\Lambda \) implies the indecomposability of \( M/J^2M \).

ii) and iii) follow immediately from our previous remarks.

As an immediate application of this result we see that if \( M \) is a Koszul module of Loewy length 2, then the A-R sequence in \( K_\Lambda \) ending at \( M \) coincides with the one in \( \text{gr}_{0\Lambda}/J^2 \). Let us assume now that \( M \) is an indecomposable non projective module in \( K_\Lambda \). Then \( M/J^2M \) is again an indecomposable \( \Lambda/J^2 \)-module, so by analyzing the A-R quiver of \( \Lambda/J^2 \), we see that \( M/J^2M \) either belong to the preprojective or preinjective components of \( \text{gr}_{0\Lambda}/J^2 \) or is regular. We show that one of these possibilities cannot exist:

Lemma 5. Let \( \Lambda \) be an indecomposable selfinjective Koszul algebra of Loewy length \( r + 1 \), and \( M \) an indecomposable non projective Koszul module. Then \( M/J^2M \) can not lie in the preprojective component of \( \text{gr}_{0\Lambda}/J^2 \).

Proof. If it does, it follows from the previous result that \( \sigma(M) \) is also in the preprojective component.

This component has two kinds of \( \tau \)-orbits, one of them is the \( \tau \)-orbit of \( P/J^2P \), with \( P \) an indecomposable projective. If \( \sigma(M) \) is in the orbit of \( P/J^2P \), then by induction, \( P/J^2P \) is the \( \sigma \) of a Koszul module, therefore it must be a Koszul but we know that if \( r > 1 \) then \( P/J^2P \) is not Koszul, and we obtain a contradiction. Assume \( \sigma(M) \) is in the orbit of \( Y_1 = \tau^{-1}_{\Lambda/J^2}(S) \) with \( S \) a simple module generated in degree zero. This implies by induction \( Y_1 \) is Koszul. On the other hand, it is easy to verify that over \( \Lambda \), the module \( Y_1 \) can not be a Koszul module since this would imply that \( \sigma(Y_1) \) is simple and this is impossible.

We have enough to describe the shape of the Auslander Reiten quiver of \( K_\Lambda \), where \( \Lambda \) is selfinjective Koszul. Using our previous results and the preceding remarks, it is not hard to see that, if \( M \) an indecomposable non projective Koszul module,
then the cones of $M$ in $K_\Lambda$ and of $M/J^2M$ in $gr_{0\Lambda/J^2}$ are isomorphic via an isomorphism that takes monomorphisms into monomorphisms and epimorphisms into epimorphisms. Moreover, since the $\Lambda-$modules $soc^2P = I$, where $P$ is an indecomposable projective injective and the simple $S$ are Koszul, and the injective modules of $\Lambda/J^2$ are precisely the modules $I$, the entire preinjective component of $gr_{0\Lambda/J^2}$ consists of Koszul $\Lambda-$modules. Thus the two kinds of $\tau-$orbits in this component are in fact, $\sigma-$orbits. Putting together these facts gives us the main result of this section:

**Theorem 13.** Let $\Lambda$ be an indecomposable selfinjective Koszul algebra of Loewy length $r + 1$, with $r > 1$. The $A-R$ quiver of $K_\Lambda$ has connected components containing the indecomposable projective modules, a component that coincides with the preinjective component of $gr_{0\Lambda/J^2}$, and all the remaining connected components are full subquivers of a quiver of type $ZA_\infty$.

Since the preinjective component of $K_\Lambda$ consists only of Koszul modules of Loewy length at most two, it turns out that each non projective Koszul module of Loewy length three or higher lies in a "regular" component. The following result yields examples of modules in the regular component of $K_\Lambda$ lying at the mouths of these components (see also [GMRSZ], 3.2and 3.6).

**Proposition 9.** Let $\Lambda$ be as in the previous theorem, let $M$ be an indecomposable non projective Koszul $\Lambda-$module and assume that $M$ has no cogenerators in degree 1. Let $0 \rightarrow \sigma(M) \xrightarrow{f} E \xrightarrow{g} M \rightarrow 0$ be the Auslander-Reiten sequence in $K_\Lambda$ ending at $M$. Then, the middle term $E$ is indecomposable.

**Proof.** Assume that $E$ decomposes into $k > 1$ indecomposable summands. Then $f = [f_1, f_2, ..., f_k]$ and $g = [g_1, g_2, ..., g_k]^T$. It is clear that each composition $g_i f_i$ is nonzero. On the other hand, it was proved in [GMRSZ], that $\tau M$, and therefore $\sigma M$ too, are cogenereted in degree 1. Hence we obtain a contradiction to the fact that $M$ has no cogenerators in degree 1. $\blacksquare$

6. Construction of locally free sheaves.

In this section we will prove that for an indecomposable noetherian Artin Shelter regular Koszul algebra $\Gamma$ of global dimension $r + 1$ the category of coherent sheaves $Qgr\Gamma$ has relative right almost split sequences. Moreover, we will prove that the category of locally free sheaves has relative right almost split sequences [AR]. When we specialize to $\Gamma = K[x_0, x_1, ..., x_r]$ with $r \geq 2$ we will construct indecomposable vector bundles on $P_r$ of arbitrary large rank.

**Lemma 6.** Let $\Gamma$ be a noetherian Artin Shelter regular Koszul algebra, $\pi : gr\Gamma \rightarrow Qgr\Gamma$ the quotient functor and let $M$ be $\Gamma-$module with $\pi(M)$ indecomposable in $Qgr\Gamma$. Then $\text{End}_{Qgr\Gamma}(\pi(M))$ is local.

**Proof.** Let $M$ be a graded torsion free module with $\pi(M)$ indecomposable. We know there exists an integer $k$ such that $M_{\geq k}[k] = N$ is Koszul. Since $\pi(N[-k]) = \pi(M)$, it follows $N$ is indecomposable.

Hence; we may assume $M$ is Koszul up to shifting.

Let $L$ be a torsion free module we have:

$$\text{Hom}_{Qgr\Gamma}(\pi(M), \pi(L))^0 = \bigcup_{k \geq 0} \text{Hom}_{gr\Gamma}(J^kM, L)^0.$$
To see this, observe first that if \( k < l \), then \( J^k M \subset J^k M \) and the exact sequence:
\[
0 \to J^l M \to J^k M \to J^k M / J^l M \to 0
\]
induces an exact sequence:
\[
\text{Hom}_{\text{gr}}(J^k M / J^l M, L)_0 \to \text{Hom}_{\text{gr}}(J^k M, L)_0 \to \text{Hom}_{\text{gr}}(J^l M, L)_0,
\]
with \( \text{Hom}_{\text{gr}}(J^k M / J^l M, L)_0 = 0 \).

It follows, \( \text{Hom}_{\text{gr}}(J^k M, L)_0 \subset \text{Hom}_{\text{gr}}(J^l M, L)_0 \) and
\[
\lim_{k \geq 0} \text{Hom}_{\text{gr}}(J^k M, L)_0 = \bigcup_{k \geq 0} \text{Hom}_{\text{gr}}(J^k M, L)_0.
\]

Assume \( M \) is Koszul and torsion free.

Then \( \text{End}_{\text{gr}}(M) = \bigcup_{k \geq 0} \text{Hom}_{\text{gr}}(J^k M, M)_0 \).

Let \( f \in \text{Hom}_{\text{gr}}(J^k M, M)_0 \). Then \( \text{Im} f \subset J^k M \). Set \( f' \) to be the restriction of \( f \) to \( J^k M \).

Let \( \Lambda \) be the Yoneda algebra of \( \Gamma \) and let \( F : K_{\Lambda^\text{op}} \to K_{\Gamma^\text{op}} \) be the Koszul duality. We have natural isomorphisms:
\[
\text{Hom}_{\Lambda^\text{op}}(\Omega^k F^{-1}(M), \Omega^k F^{-1}(M))_0 \cong \text{Hom}_{\Gamma^\text{op}}(J^k M, J^k M)_0 \cong \\
\text{Hom}_{\Lambda^\text{op}}(F^{-1}(M), F^{-1}(M))_0 \cong \text{Hom}_{\Gamma^\text{op}}(M, M)_0.
\]

It follows, \( \text{End}_{\Gamma^\text{op}}(J^k M)_0 \) is local and \( f' \) is either nilpotent or invertible.

Then \( f \) is either nilpotent or invertible.

**Proposition 10.** Let \( \Gamma \) be an indecomposable noetherian Artin Schelter regular Koszul algebra of global dimension \( r + 1 \), with Yoneda algebra \( \Lambda \). Let \( F : K_{\Lambda^\text{op}} \to K_{\Gamma^\text{op}} \) be the Koszul duality. Then the category of non simple Koszul \( \Gamma^\text{op} \)-modules has right almost split sequences.

**Proof.** Let \( M \) be an indecomposable non simple Koszul \( \Gamma^\text{op} \)-module. Then there exists a non projective Koszul \( \Lambda^\text{-module} \) \( N \) such that \( F(N) = M \).

Let \( 0 \to \tau(N) \to E \to N \to 0 \) be the almost split sequence in \( \text{gr} \Lambda \). The module \( \tau(N) \) is generated in degree \( -r + 1 \) and \( \tau(N) \uparrow [-r + 1] \) is Koszul.

It was proved in [GMRSZ], \( J^{-r-1} \tau(N) \) is Koszul \( 0 \to \tau(N)_{\geq 0} \to E_{\geq 0} \to N \to 0 \) is an almost split sequence in \( K_{\Lambda} \) and \( \tau(N)_{\geq 0} = J^{-r-1} \tau(N) \) is indecomposable.

Applying Koszul duality \( F \) we obtain an almost split sequence in \( K_{\Gamma^\text{op}} \):
\[
0 \to F(\tau(N)) \to F(E_{\geq 0}) \to F(J^{-r-1} \tau(N)) \to 0.
\]

Let \( \Gamma \) be a noetherian Artin Schelter regular Koszul algebra and \( \pi : gr_{\Gamma^\text{op}} \to Q_{\text{gr}_{\Gamma^\text{op}}} \) the quotient functor. Denote by \( \widehat{K}_{\Gamma^\text{op}}[n] \) the subcategory of \( Q_{\text{gr}_{\Gamma^\text{op}}} \) defined as:
\[
\widehat{K}_{\Gamma^\text{op}}[n] = \{ \pi(M[n]) | M \text{ is a Koszul } \Gamma^\text{-module} \}
\]

it follows by Proposition 5, the category \( Q_{\text{gr}_{\Gamma^\text{op}}} \) is equal to \( \bigcup_{n \in Z} \widehat{K}_{\Gamma^\text{op}}[n] \). With this notation we have the following:

**Definition 6.** An object \( \widehat{M} \) in \( Q_{\text{gr}_{\Gamma^\text{op}}} \) has a relative right almost split sequence if given an integer \( n \) such that \( \widehat{M} \in \widehat{K}_{\Gamma^\text{op}}[n] \), then there exist \( \widehat{L}, \widehat{N} \in \widehat{K}_{\Gamma^\text{op}}[n] \) and a short exact sequence:
\[
0 \to \widehat{M} \to \widehat{L} \to \widehat{N} \to 0
\]
which is an almost split sequence in \( \widehat{K}_{\Gamma^\text{op}}[n] \).

**Theorem 14.** Let \( \Gamma \) be an indecomposable noetherian Artin Schelter regular Koszul algebra. Then the category of sheaves \( Q_{\text{gr}_{\Gamma^\text{op}}} \) has relative right almost split sequences.
Proof. Let \( \tilde{M} \) be an indecomposable object in \( Qgr\Gamma^{op} \) and \( \pi : gr\Gamma^{op} \to Qgr\Gamma^{op} \) be the quotient functor. Then there exists an indecomposable torsion free \( \Gamma \)-module \( M \) such that \( \pi(M) = \tilde{M} \).

Since for some integer \( k \) the truncated module \( M_{\geq k} \) is Koszul and \( \pi(X[-k]) = \pi(M) \), hence; \( X \) is indecomposable.

By proposition 10, there exists an almost split sequence in \( K_{\Gamma^{op}} \) of the form:
\[
0 \to X \to Y \to Z \to 0
\]
and an exact sequence:
\[
0 \to X[-k] \to Y[-k] \to Z[-k] \to 0
\]
The functor \( \pi \) is exact \([P], [S2] \).
Hence; there exists an exact sequence:
\[
0 \to \pi(X[-k]) \xrightarrow{\pi(i)} \pi(Y[-k]) \xrightarrow{\pi(p)} \pi(Z[-k]) \to 0
\]
Assume there exits \( h : \pi(Y[-k]) \to \pi(X[-k]) \) such that \( h\pi(i) = 1 \), where \( h \in \cup Hom(J^iY[-k],X[-k])) \). Then the sequence:
\[
0 \to J^iX[-k] \to J^iY[-k] \to J^iZ[-k] \to 0
\]
splits. It follows
the sequence:
\[
0 \to \Omega^iF^{-1}(Z)[-k] \to \Omega^iF^{-1}(Y)[-k] \to \Omega^iF^{-1}(X)[-k] \to 0
\]
Then the sequence:
\[
0 \to F^{-1}(Z)[-k] \to F^{-1}(Y)[-k] \to F^{-1}(X)[-k] \to 0
\]
It follows
0 \to X[-k] \to Y[-k] \to Z[-k] \to 0

We must prove the sequence is almost split in \( \hat{K}_{\Gamma^{op}} [-k] \).
Let \( h : \pi(W)[-k] \to \pi(Z)[-k] \) be a non splittable map in \( \hat{K}_{\Gamma^{op}} [-k] \). The map \( h \) belongs to \( \cup Hom(J^iW[-k],Z[-k])) \), then \( h : J^iW[-k] \to J^iZ[-k] \) does not split.

Applying the functor \( F^{-1} \) we obtain the following exact diagram:
\[
\begin{array}{c}
0 \to \\
\Omega^iF^{-1}(Z)[-k] \to \\
\Omega^iF^{-1}(Y)[-k] \to \\
\Omega^iF^{-1}(X)[-k] \to 0
\end{array}
\]
Applying \( \Omega^{-1} \) to the diagram and using the fact
\[
0 \to F^{-1}(Z)[-k] \to F^{-1}(Y)[-k] \to F^{-1}(X)[-k] \to 0
\]
is almost split in \( K_{\Lambda} [-k] \) we obtain an exact diagram:
\[
\begin{array}{c}
0 \to \\
F^{-1}(Z)[-k] \to \\
F^{-1}(Y)[-k] \to \\
F^{-1}(X)[-k] \to 0
\end{array}
\]
Applying \( \Omega^i \) to the diagram we obtain and extension of the map
\[
\begin{array}{c}
\Omega^iF^{-1}(Z)[-k] \to \\
\Omega^iF^{-1}(W)[-k] \to
\end{array}
\]
Applying the functor \( F \) we obtain a lifting of the map \( h : J^iW[-k] \to J^iZ[-k] \) to \( g : J^iW[-k] \to Y[-k] \). Finally, applying the functor \( \pi \) to the diagram we obtain a lifting \( \pi(g) : \pi(W)[-k] \to \pi(Y)[-k] \) of \( h \) as claimed.

We have proved \( 0 \to \pi(M) \xrightarrow{\pi(i)} \pi(L) \xrightarrow{\pi(p)} \pi(N) \to 0 \) is a relative almost split sequence in \( Qgr\Gamma^{op} \).

Corollary 4. Let \( Coh(slP_r) \) be the category of coherent sheaves on projective space \( slP_r \). Then \( Coh(slP_r) \) has relative right almost split sequences.
Let $\Gamma$ be an Artin Schelter regular Koszul algebra, $L$ and $M$ finitely generated Koszul $\Gamma^{\text{op}}$-module such that $\pi(L[n]) \cong \pi(M[m])$, and assume $n \geq m$. Then there exits an integer $k$ such that $L[n]_{\geq k} = L_{\geq n+k}[n] \cong M[m]_{\geq k} = M_{m+k}[m]$ or $J^{n+k}L[n + k] = J^{m+k}M[m + k]$.

Applying the duality $F^{-1}$ we obtain: $\Omega^{n+k}F^{-1}L[n + k] \cong \Omega^{m+k}F^{-1}M[m + k]$.

Therefore: $\Omega^{n-m}F^{-1}L[n - m] \cong F^{-1}M$. Applying $F$ it follows: $J^{n-m}L[n - m] \cong M$.

The following lemma will be needed in the next proposition:

**Lemma 7.** Let $\Lambda$ be an indecomposable graded selfinjective quiver algebra of Loewy length $r + 1$. Then for any non negative integer $k$ we have an isomorphism: $(D(\Omega^k\Lambda)^*)^s [r + 1] \cong \Omega^k\Lambda_0$.

**Proof.** Let $S$ be a graded simple generated in degree zero. Then By [M3], $D(S)$ is also generated in degree zero. Let $P$ be the projective cover of $D(S)$ applying the functor $(-)^* = \text{Hom}_\Lambda(-, \Lambda)$ to the exact sequence: $P \to D(S) \to 0$ we obtain an exact sequence: $0 \to (D(S))^* \to P^*$ with $P^*$ an indecomposable projective generated in degree zero of Loewy length $r + 1$. Therefore $(D(S))^*[r+1]$ is a simple module generated in degree zero.

The Nakayama equivalence: $S \to (D(S))^*[r+1]$ induces a bijection of the graded simple modules generated in degree zero, hence $\Lambda_0 \cong (D(\Lambda_0))^*[r+1]$.

Applying $(D(-))^*[r+1]$ to the minimal projective resolution of $\Lambda_0$ we obtain a minimal projective resolution of $(D(\Lambda_0))^*[r+1]$. It follows: $(D(\Omega^k\Lambda_0))^*[r+1] \cong \Omega^k\Lambda_0$. $\blacksquare$

**Theorem 15.** Let $\Gamma$ be an indecomposable noetherian Artin Shelter regular Koszul algebra of global dimension $r + 1$. Then the category of locally free sheaves of finite rank has relative right almost split sequences.

**Proof.** Let $\Lambda$ be the Yoneda algebra of $\Gamma$ and $F : K_{\Lambda} \to K_{\Gamma^{\text{op}}}$ Koszul duality. Let $\tilde{M} \in Qgr\Gamma^{\text{op}}$ be an indecomposable locally free sheaf, we may assume $\tilde{M} = \pi(M)$, where $\pi : gr\Gamma^{\text{op}} \to Qgr\Gamma^{\text{op}}$ is the quotient functor and $M$ is an indecomposable module.

Then there exists an indecomposable non projective Koszul module $X$ and some integer $n$ with $F(X[n]) = M$.

Let $0 \to \tau(X) \to E \to X \to 0$ be the almost split sequence in $\text{gr}\Lambda$ and $0 \to J^{-1}\tau(X) \to E_{\geq 0} \to X \to 0$ the almost split sequence in $K_{\Lambda}$.

Since $\pi(F(X))$ is locally free $\text{Ext}_\Lambda^j(\Omega^s\Lambda_0[s], X)_0 = 0$ for $0 < j \leq r$ and $s >> 0$.

There is a chain of isomorphisms:

$\text{Hom}\Lambda(\Omega^{s+j}\Lambda_0[s], D(X^*) [-r - 1])_0 \cong \text{Hom}\Lambda(\Omega^{s+j}\Lambda_0, D(X^*) [-r - s - 1])_0 \cong \text{Hom}\Lambda((D(\Omega^{s+j}\Lambda_0))^*, X[-r - s - 1])_0 \cong \text{Hom}\Lambda((D(\Omega^{s+j}\Lambda_0))^* [r + 1], X[-s])_0 \cong \text{Hom}\Lambda(\Omega^{s+j}\Lambda_0, X[-s])_0 \cong \text{Hom}\Lambda(\Omega^{s+j}\Lambda_0[s], X)_0 = 0$ for $s >> 0$.

But $\tau(X) \cong \Omega^2D(X^*)$, hence; $\tau(X)[r + 1] \cong \Omega^2D(X^*)[r - 1]$ [2]

It follows by Corollary 2 $\tau(X)[r + 1]$ is such that $\pi(F(\tau(X)[r + 1]))$ is locally free.

By Proposition 5 $J^{-1}\tau(X)$ is such that $\pi(F(J^{-1}\tau(X)))$ is locally free and by Proposition 4 $\pi(F(E_{\geq 0}))$ is locally free.

It follows $0 \to \pi(F(X)) \to \pi(F(E_{\geq 0})) \to \pi(F(\tau(X))) \to 0$ is a relative almost split sequence of locally free sheaves, hence; $0 \to \pi(F(X)[n]) \to \pi(F(E_{\geq 0}[n]) \to \pi(F(\tau(X))[n]) \to 0$ is relative almost split. $\blacksquare$
Corollary 5. The category of vector bundles $V(P_r)$ on projective space $P_r$ has relative right almost split sequences.\[\]

Let $\Gamma$ and $\Lambda$ be like in the theorem. The almost split sequences in $Qgr\Gamma^{op}$ are related with the almost split sequences in $Gr\Lambda/J_2$, using the results of Section 3 we have the following version of the main theorem of Section 3:

Theorem 16. Let $\Gamma$ be an indecomposable noetherian Artin Schelter regular Koszul algebra of global dimension $r$ with $r > 2$, with Yoneda algebra $\Lambda$, let $F: K\Lambda \rightarrow K\Gamma^{op}$ be the Koszul duality and let $M$ be an indecomposable Koszul $\Lambda$–module. Then the relative Auslander Reiten component of $\pi F(M)$ is either contained in the nonegative part of a quiver of type $Z\Delta$, where $\Delta$ is the separated quiver of $\Gamma/J^2$ or of the form:

$$\pi F M \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow$$

If $\pi(F(M))$ is locally free, then all the sheaves in the diagram are locally free.

For all the modules $M$ in the nonegative part of a quiver of type $Z\Delta$, where $\Delta$ is the separated quiver of $\Gamma/J^2$ the sheaf $\pi(F(M))$ is locally free.\[\]

As a corollary we have:

Theorem 17. Let $\Gamma = K[x_0, x_1, ..., x_r]$ be the polynomial algebra with $r > 1$ and $\Lambda = K<x_0, x_1, ..., x_r>/(x_i^2, x_i x_j + x_j x_i)$ the exterior algebra, let $F: K\Lambda \rightarrow K\Gamma^{op}$ be the Koszul duality and let $M$ be an indecomposable Koszul $\Lambda$–module. Then the relative Auslander Reiten component of $\pi F(M)$ is either contained in the non negative part of the quiver $Z\Delta$, where $\Delta$ is the quiver: $\bullet \Rightarrow \bullet$ with $r + 1$ arrows, or of the form:

$$\pi F M \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow$$

If $\pi(F(M))$ is locally free, then all the sheaves in the diagram are locally free.

If $M$ is contained in the non negative part of the quiver $Z\Delta$, where $\Delta$ is the quiver: $\bullet \Rightarrow \bullet$ with $r + 1$ arrows, then $\pi(F(M))$ is locally free.\[\]

Corollary 6. There are indecomposable vector bundles on $P_r$, with $r > 1$, of arbitrary high rank.

Proof. If we have an exact sequence of vector bundles: $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ it is clear that $rk\ Y = rk\ X + rk\ Z$, where $rk\ Y$ denotes the rank of $Y$.

Let $M$ be an indecomposable Koszul $\Lambda$–module and assume $Hom\Lambda(\Lambda_0 [-1], M)_0 = 0$, for example $M = J[1]$. Then the relative Auslander Reiten component of $\pi F(M)$ is of the form:
For any \( j \geq 0 \) the following equality hold:
\[
\text{rk}(\pi F(\sigma^j M_i)) = \text{rk}(\pi F(\sigma^{j+1} M_i)) + \text{rk}(\pi F(\sigma^j M_{i-1}))
\]
Assume for all \( j \geq 0 \) there is an equality:
\[
\text{rk}(\pi F(\sigma^j M_i)) = \text{rk}(\pi F(\sigma^j M_{i-1})) + \text{rk}(\pi F(\sigma^{i+j} M))
\]
We have the following commutative diagram:
\[
\begin{array}{ccc}
\pi F(\sigma^{j+1} M_{i-1}) & \rightarrow & \pi F(\sigma^j M_i) \\
| & & | \\
\pi F(\sigma^j M_i) & \rightarrow & \pi F(\sigma^{j+1} M_i) \\
| & & | \\
\pi F(\sigma^{j+1} M_i) & \rightarrow & \pi F(\sigma^j M_{i+1})
\end{array}
\]
Where the sum of the two terms in the middle is the middle term of an exact sequence. It follows:
\[
\text{rk}(\pi F(\sigma^j M_i)) + \text{rk}(\pi F(\sigma^{j+1} M_i)) = \text{rk}(\pi F(\sigma^j M_{i+1})) + \text{rk}(\pi F(\sigma^{j+1} M_{i-1}))
\]
Hence, \( \text{rk}(\pi F(\sigma^j M_{i+1})) = \text{rk}(\pi F(\sigma^j M_i)) + \text{rk}(\pi F(\sigma^{i+j} M)) - \text{rk}(\pi F(\sigma^j M_{i-1})) = \text{rk}(\pi F(\sigma^j M_{i+1})) + \text{rk}(\pi F(\sigma^{i+j} M)) \)
It follows: \( \text{rk}(\pi F(\sigma^j M_i)) < \text{rk}(\pi F(\sigma^j M_{i+1})) \) for all \( j \geq 0 \) and \( i \geq 0 \).■

REFERENCES

[ADL] Agoston, I., Dlab, V., Lukás, E. Homological duality and quasi hereditary, Can. J. Math. 48 (1996), 897-917.

[AS] Artin, M.; Shelter W.; Graded algebras of dimension 3, Advances in Mathematics, 66 (1987), 172-216.

[ABPRS] Auslander, M.; Bautista, R.; Platzeck, M.I.; Reiten, I.; Smalo, S., Canadian J. Math. 31 (1979), No. 5, 942-960.

[AR] Auslander, M.; Reiten, I., Representation Theory of Artin algebras III, Almost split sequences, Comm. in Algebra 3, (1975), 239-294.

[BGK] Baranovsky, V.; Ginzburg, V.; Kuznetsov, A.; Quiver varieties and a noncommutative \( P_2 \), preprint (2001).

[Be] Belinson, A. Coherent Sheaves on \( P_n \) and problems of linear algebra. Funkts. Anal. Prilozh. 12, No. 3 (1978), English. Trans: Funct. Anal. Appl. 12 (1979), 214-216.

[BGS] Belinson, A.; Ginzburg, V.; Soergel, W., Koszul duality patterns in representation theory, J. Amer. Math. Society, 9, no. 2, (1996), 473-527.

[Bo] Bondal, A., I., Representation of Associative Algebras and Coherent Sheaves. Izv. Akad. Nauk SSSR, Ser. Mat. 53, No. 1 (1989), 25-44. English translation: Math USSR, Izv. 34 (1990), 23-42.

[DM] Dowbor, P.; Meltzer, H., On equivalences of Bernstein-Gelfand-Gelfand, Beilinson and Happel. Comm. in Algebra 20 (9), (1992), 2513-2532.
[G] Gelfand, S., I., Sheaves on $P_n$ and problems of linear algebra. Appendix, In. C. Okonek, M. Schneider and H. Spindler, Vector bundles on complex projective spaces, Moscow, Mir (1984), 278-305.

[GMRSZ] Green, E., L.; Martínez-Villa, R.; Reiten, I.; Solberg, Ø.; Zacharia, D., On modules with linear presentations, J. Algebra 205, (1998), no-2, 578-604.

[GM1] Green, E., L.; Martínez-Villa, R., Koszul and Yoneda algebras, Representation Theory of Algebras, 247-297, CMS Conf. Proc., 18, Amer. Math. Soc. (1996).

[GM2] Green, E., L.; Martínez-Villa, R., Koszul and Yoneda algebras II, Algebras and Modules II, 227-244, CMS Conf. Proc., 24, Amer. Math. Soc. (1998).

[GMT] Guo, J., Y.; Martínez-Villa, R.; Takane, M., Koszul generalized Auslander regular algebras, Algebras and Modules, II, 263-283, CMS Conf. Proc., 24, Amer. Math. Soc. (1998).

[H1] Hartshorne, R., Algebraic Vector Bundles on Projective Space: A Problem List, Topology Vol. 18, 117-128, (1979).

[H2] Hartshorne, R., Algebraic Geometry, Springer, Graduate Texts in Math. 52, (1997).

[L] Le Potier, J. Lectures on Vector Bundles, Cambridge studies in advanced mathematics, 54, (1997)

[M1] Martínez-Villa, R., Applications of Koszul algebras: the preprojective algebra, Representation Theory of Algebras, 487-504, CMS Conf. Proc., 18, Amer. Math. Soc. (1996).

[M2] Martínez-Villa, R., Serre Duality for Generalized Auslander Regular Algebras, Contemporary Math. 229, (1998) 237-263.

[M3] Martínez-Villa, R., Graded, Selfinjective, and Koszul algebras, J. Algebra 215 (1999), no.1, 34-72.

[M4] Martínez-Villa, R., Skew group algebras and their Yoneda algebras, J. Okayama U. Vol. 43, 1-10, (2001)

[M5] Martínez-Villa, R. Koszul algebras and the Gorenstein condition, Representation of Algebras, Lecture Notes in Pure and Applied Math. Math. Vol. 224, Dekker, (2001)

[MM] Martínez-Villa, R.; Martsinkovsky, A., Sheaf Cohomology and Tate Cohomology for Koszul algebras, preprint, (2001).

[MZ] Martínez-Villa, R.; Zacharia, D.; Approximations with modules having linear resolutions, preprint, (2001).

[P] Popescu, N., Abelian Categories with Applications to Ring and Modules, Academic Press (1973).

[Pr] Priddy, S., Koszul resolutions, Trans. Amer. Math. Soc. 152 (1970), 39-60.

[RR] Reiten, I.; Riedtmann, C., Skew Group Algebras in the Representation Theory of Artin Algebras, J. of Algebra 92, 224-282, (1985)

[R] Rudakov, A., Helices and Vector Bundles, LMS Lect. Notes Series 148, Cambridge, 1990.

[Sc] Schneider, M., Holomorphic Vector Bundles on $P_n$, Séminaire Bourbaki, 31 annés, 1978/1979, no. 530.

[Se] Serre, J., P., Faisceaux algébriques cohérents, Ann. Math. 61 (1955), 197-278.
[S1] Smith, P., Some finite-dimensional algebras related to elliptic curves, Representation Theory of Algebras and related topics, 315-348, CMS Conf. Proc., 19, Amer. Math. Soc. (1996).

[S2] Smith, P., Non-Commutative Algebraic Geometry, preprint, University of Washington.

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