On the local Bump-Friedberg $L$-function II

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Abstract

Let $F$ be a $p$-adic field with residue field of cardinality $q$. To each irreducible representation of $GL(n,F)$, we attach a local Euler factor $L^{BF}(q^{-s},q^{-t},\pi)$ via the Rankin-Selberg method, and show that it is equal to the expected factor $L(s + t + 1/2,\phi_{\pi})L(2s,\Lambda^2 \circ \phi_{\pi})$ of the Langlands’ parameter $\phi_{\pi}$ of $\pi$. The corresponding local integrals were introduced in [B-F], and studied in [M15]. This work is in fact the continuation of [M15]. The result is a consequence of the fact that if $\delta$ is a discrete series representation of $GL(2m,F)$, and $\chi$ is a character of Levi subgoup $L = GL(m,F) \times GL(m,F)$, trivial on $GL(m,F)$ embedded diagonally, then $\delta$ is $(L,\chi)$-distinguished if and only if it admits a Shalika model, a result which was only established for $\chi = 1$ before.

1 Introduction

Let $F$ be a $p$-adic field with residue field of cardinality $q$. In [M15], we attached to any irreducible representation $\pi$ of $GL(n,F)$, and any character $\alpha$ of $F^*$ an Euler factor $L^{\text{lin}}(s,\alpha,\pi)$ (denoted $L^{\text{lin}}(\pi,\chi_{\alpha},s)$ in [ibid.]). It is defined as the gcd of a family of local integrals $\Psi(s,\alpha,W,\Phi)$ for $W$ in the Whittaker model of (the induced representation of Langlands’ type above) $\pi$ and $\Phi$ a Schwartz map on $F^{[n+1/2]}$. These integrals were introduced in [B-F] where the corresponding global integrals were studied. More precisely, in [B-F], the corresponding global integrals were considered as maps of two complex variables, whereas in [M15], the character $\alpha$ is fixed, and the integrals are viewed as maps of the complex variable $s$. In particular, if $\alpha$ is an unramified character $|.|$, writing $L^{\text{lin}}(s,t,\pi)$ for $L^{\text{lin}}(s,\alpha,\pi)$ and $\Psi(s,t,W,\Phi)$ for $\Psi(s,\alpha,W,\Phi)$, it is not obvious that the map $L^{\text{lin}}(s,t,\pi)$ is rational in $q^{-s}$ and $q^{-t}$. Here, we consider the integrals $\Psi(s,t,W,\Phi)$ as maps of the variables $s$ and $t$, and show that they belong to $\mathbb{C}[q^{-s},q^{-t}]$, and admit a gcd in a certain sense, that we denote $L^{BF}(s,t,\pi)$, which is the inverse of an element of $\mathbb{C}[q^{-s},q^{-t}]$ with constant term equal to 1. We show that $L^{BF}(s,t,\pi)$ admits a functional equation, and that it is equal to the Galois factor $L(s + t + 1/2,\phi_{\pi})L(2s,\Lambda^2 \circ \phi_{\pi})$ (Theorem 3.3), where $\phi_{\pi}$ is the Langlands’ parameter of $\pi$. This result follows from the results of [M15] about $L^{\text{lin}}$, and from the new main ingredient, which is that if $\delta$ is a discrete series representation of $GL(2m,F)$, and $\chi$ is any character of the bloc diagonal Levi $L = GL(m,F) \times GL(m,F)$, trivial on $GL(m,F)$ embedded diagonally, then $\delta$ is $(L,\chi)$-distinguished if and only if it admits a Shalika model (Theorem 3.4), this result was only known for $\chi = 1$ before.

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2 Preliminaries

2.1 Notations

We denote by $F$ a $p$-adic field, by $\mathcal{O}$ its ring of integers, by $\varpi$ its uniformiser, and by $q$ the cardinality of its residue field. We denote by $|.|$ the absolute value of $F$ normalised by $|\varpi| = q^{-1}$. We denote by $G_n$ the group $GL(n, F)$ of invertible elements of the algebra $\mathcal{M}(n, F)$ (which we denote by $\mathcal{M}_n$). We denote by $\nu(g)$ or simply by $|g|$ the positive real number $|\det(g)|$ when $g \in G_n$. The group $A_n$ will be the diagonal torus of $G_n$, contained in the Borel subgroup $B_n$ of upper triangular matrices of $G_n$. We denote by $N_n$ the unipotent radical of $B_n$ (the matrices of $B_n$ with 1 on the diagonal). We denote by $P_n$ the mirabolic subgroup of $G_n$, i.e. the group of matrices with last row equal to $(0, \ldots, 0, 1)$. We set $K_n = GL(n, \mathcal{O})$. We denote by $w_n$ the element of the symmetric group $S_n$ naturally embedded in $G_n$, defined by

$$
\begin{pmatrix}
1 & 2 & \ldots & m - 1 & m & m + 1 & m + 2 & \ldots & 2m - 1 & 2m \\
1 & 3 & \ldots & 2m - 3 & 2m - 1 & 2 & 4 & \ldots & 2m - 2 & 2m
\end{pmatrix}
$$

when $n = 2m$ is even, and by

$$
\begin{pmatrix}
1 & 2 & \ldots & m - 1 & m & m + 1 & m + 2 & \ldots & 2m & 2m + 1 \\
1 & 3 & \ldots & 2m - 3 & 2m - 1 & 2m + 1 & 2 & \ldots & 2m - 2 & 2m
\end{pmatrix}
$$

when $n = 2m + 1$ is odd. We denote by $L_n$ the standard Levi subgroup of $G_n$ which is $G_{[n+1]/2}\times G_{[n/2]}$ embedded by the map $(g_1, g_2) \mapsto \text{diag}(g_1, g_2)$. If $\alpha$ is a smooth character of $F^\times$, we denote by $\psi_\alpha$ the character of $L_n$ defined as

$$
\psi_\alpha : \text{diag}(g_1, g_2) \mapsto \alpha(\det(g_1))/\alpha(\det(g_2)).
$$

We denote by $H_n$ the group $w_n^{-1}L_nw_n$, by $h(g_1, g_2)$ the matrix $w_n^{-1}\text{diag}(g_1, g_2)w_n$ of $H_n$ (with $\text{diag}(g_1, g_2) \in L_n$), and by $\chi_\alpha$ the character of $H_n$ defined as

$$
\chi_\alpha : h(g_1, g_2) \mapsto \alpha(\det(g_1))/\alpha(\det(g_2)).
$$

We denote by $d_n$ the matrix $\text{diag}(1, -1, 1, -1, \ldots)$ of $G_n$, the group $H_n$ is the subgroup of $G_n$ fixed by the involution $g \mapsto d_ngd_n$. We denote by $\delta_n$ the character

$$
\delta_n = \chi_{|.|} : h(g_1, g_2) \mapsto |g_1|/|g_2|
$$

of $H_n$, and we denote by $\chi_n$ (resp. $\mu_n$) the character of $H_n$ equal to $\delta_n$ when $n$ is odd (resp. even), and trivial when $n$ is even (resp. odd). Hence $\chi_n|_{H_{n-1}} = \mu_{n-1}$, and $\mu_n|_{H_{n-1}} = \chi_{n-1}$. If $C$ is a subset of $G_n$, we sometimes denote by $C^n$ the set $C \cap H_n$. When $n = 2m$ is even, we denote by $U_{(m,m)}$ the subgroup of $G_n$ of matrices

$$
u(x) = \begin{pmatrix}
I_m & x \\
0 & I_m
\end{pmatrix}
$$

for $x \in \mathcal{M}_m$, and we denote by $S_n$ the Shalika subgroup of $G_n$ of matrices of the form $u(x)\text{diag}(g, g)$, for $g \in G_m$ and $x \in \mathcal{M}_m$. 

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We fix until the end of this work, a nontrivial smooth character \( \theta \) of \((F,+)\), which defines by the formula \( \theta(n) = \sum_{i=1}^{n-1} n_{i,i+1} \) a character still denoted \( \theta \) of \( N_n \). For \( n \) even, we will denote by \( \Theta \) the character of \( S_n \) given by the formula

\[
\Theta(u(x)\text{diag}(g,g)) = \theta(T_r(x)).
\]

All the representations of closed subgroups of \( G_n \) that we will consider will be smooth and complex. We will use the product notation of \([B-Z]\) for normalised parabolic induction.

### 2.2 Representations of Whittaker type and their derivatives

Here, for convenience of the reader, we reproduce up to small modifications, Section 2.2 of \([MIL]\), which is a compilation of well-known results about representations of Whittaker type and their derivatives. Those results are extracted from \([B-Z], Z, R, J-S]\, we refer to \([MIL]\) for the details.

**Definition 2.1.** Let \( a < b \) be integers, and \( r \) a positive integer, and set \( n = r(b-a) \). If \( \rho \) is a cuspidal representation of \( G_r \), then the representation \( \nu^r \rho \times \cdots \times \nu^{b-1} \rho \) has a unique irreducible quotient which we denote by \( \delta([a,b],\rho) \). We call such a representation of \( G_n \) a discrete series representation.

If \( \pi \) is a representation of \( G_n \) admitting a central character, we denote it by \( \omega_z \).

**Definition 2.2.** Let \( \pi \) be a representation of \( G_n \), such that \( \pi \) is a product of discrete series \( \delta_1 \times \cdots \times \delta_i \) of smaller linear groups, we say that \( \pi \) is of Whittaker type.

If the discrete series \( \delta_i \) are ordered such that \( \text{Re}(\omega_{\delta_i}) \geq \text{Re}(\omega_{\delta_{i+1}}) \), we say that \( \pi \) is (induced) of Langlands' type. It follows from \([S]\) that \( \pi \) has a unique irreducible quotient \( L(\pi) \) (its Langlands’ quotient) which determines \( \pi \), and that any irreducible representation of \( G_n \) is the Langlands’ quotient of a representation of Langlands’ type.

We now define the Whittaker model of a representation of Whittaker type. We denote by \( \text{Ind} \) the smooth induction functor, and by \( \text{ind} \) the compact smooth induction functor.

**Proposition 2.1.** Let \( \pi \) be a representation of Whittaker type, then \( \text{Hom}_{G_n}(\pi,\theta) \) is of dimension 1, hence the space of intertwining operators \( \text{Hom}_{G_n}(\pi,\text{Ind}_{N_n}^{G_n}(\theta)) \) is of dimension 1. The image of the (unique up to scaling) intertwining operator from \( \pi \) to \( \text{Ind}_{N_n}^{G_n}(\theta) \) is called the Whittaker model of \( \pi \), we denote it by \( W(\pi,\theta) \). When \( \pi \) is of Langlands’ type, the \( G_n \)-module \( W(\pi,\theta) \) is isomorphic to \( \pi \), and we set \( W(L(\pi),\theta) = W(\pi,\theta) \), so this defines the Whittaker model of any irreducible representation.

If \( \pi \) is an irreducible representation of \( G_n \), we denote by \( \pi^\vee \) its (smooth) contragredient. If \( \pi = \delta_1 \times \cdots \times \delta_i \) is a representation of Whittaker type of \( G_n \), and \( w \) is the anti-diagonal matrix of \( G_n \) with only ones on the second diagonal, then \( \tilde{\pi} : g \mapsto \pi(g^{-1}) \) is of Whittaker type, isomorphic to \( \delta_1^\vee \times \cdots \times \delta_i^\vee \). In particular, if \( \pi \) is of Langlands' type, then \( \tilde{\pi} \) as well, and \( L(\tilde{\pi}) = L(\pi)^\vee \). Moreover for \( W \) in \( W(\pi,\theta) \), then \( \tilde{W} : g \mapsto W(wg^{-1}) \) belongs to \( W(\tilde{\pi},\theta^{-1}) \). Denoting by \( \text{Alg}(H) \) the category of smooth representations of \( H \) for \( H = G_n \) or \( P_n \), we now move on to derivatives.

**Definition 2.3.** We denote by \( \Phi^+ : \text{Alg}(P_n) \to \text{Alg}(P_{n+1}) \), \( \Phi^- : \text{Alg}(P_n) \to \text{Alg}(P_{n-1}) \), \( \Psi^+ : \text{Alg}(G_n) \to \text{Alg}(P_{n+1}) \), \( \Psi^- : \text{Alg}(P_n) \to \text{Alg}(G_{n-1}) \) the functors defined in 3.2. of \([B-Z]\). If \( \pi \) is a smooth \( P_n \)-module, we denote by \( \pi^{(k)} \) the \( G_{n-k} \)-module \( (\Phi^-)^{k-1}\Psi^-(\pi) \).
If \( \pi \) is a representation of \( G_n \), we will sometimes consider \( \pi|_{P_n} \) without referring to \( P_n \), for example we will write \( \pi^{(k)} \) for \( (\pi|_{P_n})^{(k)} \). The main property of representations of \( P_n \) is that they admit a natural filtration.

**Proposition 2.2.** The functors \( \Psi^- \Psi^+ \) and \( \Phi^- \Phi^+ \) are the identity of \( \text{Alg}(G_n) \) and \( \text{Alg}(P_n) \) respectively. For any \( P_n \)-module \( \tau \), and \( k \in \{1, \ldots, n\} \), the \( P_n \)-module

\[
\tau_k = (\Phi^+)^{k-1}(\Phi^-)^{k-1}(\tau)
\]

is submodule of \( \tau \), and one has \( \tau_{k+1} \subset \tau_k \), with \( \tau_k/\tau_{k+1} \simeq (\Phi^+)^{k-1}\Psi^+(\tau^{(k)}) \).

For representations of Whittaker type, the filtration can be made explicit.

**Proposition 2.3.** Let \( \delta = \delta([0,k-1],\rho) \) be a discrete series of \( G_n \), with \( \rho \) a cuspidal representation of \( G_r \), and \( n = kr \). Then for \( l \in \{1, \ldots, n\} \), the \( G_{k-1} \)-module \( \delta(l) \) is null, unless \( l = ir \) a multiple of \( r \), in which case \( \delta(l) = \delta([i,k-1],\rho) \) when \( i \leq k-1 \), and \( \delta(n) = 1 \). If \( \pi = \delta_1 \cdots \delta_l \) is a representation of \( G_n \) of Whittaker type, and \( l \in \{1, \ldots, n\} \), then \( \pi^{(k)} \) has a filtration with factors the modules \( \delta_1^{(a_1)} \cdots \delta_l^{(a_l)} \), for non negative integers \( a_i \) such that \( \sum_{i=1}^l a_i = k \).

Representations of Whittaker type of \( G_n \) all have the same irreducible \( P_n \)-submodule:

**Proposition 2.4.** Let \( \pi \) be a representation of Whittaker type of \( G_n \), and \( W(\pi,\theta) \) its Whittaker model, then \( \text{Hom}_{\mathcal{N}_n}(W(\pi,\theta),\theta) \) is of dimension 1. The space \( W(\pi,\theta)|_{P_n} \) contains \( \text{ind}_{\mathcal{N}_n}^P(\theta) \) (which appears with multiplicity one, as the bottom piece of the filtration).

The asymptotics of Whittaker functions are controlled by the exponents of the corresponding representation.

**Definition 2.4.** Let \( \pi = \delta_1 \cdots \delta_l \) be a representation of \( G_n \) of Whittaker type. Let \( k \) be an element of \( \{0, \ldots, n\} \), such that \( \pi^{(k)} \) is not zero, and let \( a_1, \ldots, a_n \) be a sequence of non-negative integers, such that \( \sum_{i=1}^l a_i = k \), and \( \delta_1^{(a_1)} \cdots \delta_l^{(a_l)} \) is nonzero. We will say that the central character of the \( G_{n-k} \)-module \( \delta_1^{(a_1)} \cdots \delta_l^{(a_l)} \) is a \( k \)-exponent of \( \pi \). If \( \pi^{(k)} = 0 \), the family of \( k \)-exponents of \( \pi \) is empty.

The following result is extracted from the proof of Theorem 2.1 of [M11] (see the “stronger statement” in [loc. cit.]).

**Proposition 2.5.** Let \( \pi \) be a representation of \( G_n \) of Whittaker type. For \( k \in \{1, \ldots, n\} \), let \( (c_{k,i_k})_{i_k=1,\ldots,r_k} \) be the family of \( (n-k) \)-exponents of \( \pi \), then for every \( W \) in \( W(\pi,\theta) \), the map \( W(z_1, \ldots, z_n) \) is a linear combination of functions of the form

\[
c_{\pi}(t(z_n)) \prod_{k=1}^{n-1} c_{k,i_k}(t(z_k)) |z_k|^{(n-k)/2} \varphi(t(z_k))^{m_k} \phi_k(t(z_k)),
\]

where \( z_k = \text{diag}(t(z_k)I_{i_k}, I_{n-k}) \), for \( i_k \) between 1 and \( r_k \), non negative integers \( m_k \), and functions \( \phi_k \) in \( C^\infty_c(F) \).
2.3 Local Langlands correspondence and local factors

We refer to Section 7 of [B-H] for the vocabulary and assertions concerning the Weil-Deligne representations of the Weil group of $F$ and their local constants. Let $W_F$ be the Weil group of $F$. If $\phi$ is a semi-simple Weil-Deligne representation of $W_F$, we denote by $L(s, \phi)$ its Artin $L$-function, which satisfies $L(s, \phi_1 \oplus \phi_2) = L(s, \phi_1)L(s, \phi_2)$ for any semi-simple Weil-Deligne representations $\phi_1$ and $\phi_2$ of $W_F$. On the other hand, if $\pi$ and $\pi'$ are irreducible representations of $G_n$ and $G_{n'}$ respectively, we denote by $L(s, \pi, \pi')$ the local factor attached to the pair $(\pi, \pi')$ in [I-PS-S]. We will denote by $L(s, \pi)$ the factor $L(s, \pi, 1)$ where $1$ is the trivial representation of the trivial group $G_0$. It is a theorem from [H-T] and [H] that there is a bijection $\phi : \pi \mapsto \phi_\pi$ from the set of irreducible representations of $G_n$ (up to isomorphism) to the set of semi-simple Weil-Deligne representations of $W_F$ of dimension $n$ (up to isomorphism), which satisfies amongst other properties, that if $\pi$ and $\pi'$ are irreducible representations of $G_n$ and $G_{n'}$ respectively, the one has

$$L(s, \pi, \pi') = L(s, \phi_\pi \otimes \phi_{\pi'}) .$$

The map $\phi$ is called the Langlands correspondence, and if $\pi$ is an irreducible representation of $G_n$, we will say that $\phi_\pi$ is the Langlands’ parameter of $\pi$. If $\phi$ is a semi-simple Weil-Deligne representation of $W_F$, we will denote by $\Lambda^2 \circ \phi$ its exterior-square, which is again a semi-simple Weil-Deligne representation of $W_F$. If $\phi_1, \ldots, \phi_t$ are semi-simple Weil-Deligne representations of $W_F$. Because

$$\Lambda^2 \circ (\oplus_{i=1}^t \phi_i) = \oplus_{k=1}^t \Lambda^2 \circ \phi_k \oplus \bigoplus_{1 \leq i < j \leq t} \phi_i \otimes \phi_j ,$$

we deduce the formula

$$L(s, \Lambda^2 \circ (\oplus_{i=1}^t \phi_i)) = \prod_{k=1}^t L(s, \Lambda^2 \circ \phi_k) \prod_{1 \leq i < j \leq t} L(s, \phi_i \otimes \phi_j).$$

Notice that it is well known that if $\pi = L(\delta_1, \ldots, \delta_t)$ is an irreducible representation of $G_n$, then

$$\phi_\pi = \oplus_{i=1}^t \phi_{\delta_i} .$$

3 Distinguished discrete series

Let $H$ be a closed subgroup of $G_n$, and $\chi$ a character of $H$, we recall that a representation $\pi$ of $G_n$ is said to be $(H, \chi)$-distinguished if the space $Hom_H(\pi, \chi)$ is nonzero. In this section, which is the core of the paper, we show that if $\delta$ is a discrete series representation of $G_n$ (with $n$ even), and $\psi$ is a character of $L_n$ trivial on the diagonal embedding of $G_n/2$ in $L_n$, then $\delta$ is $(L_n, \psi)$ if and only if it admits a Shalika model. This was known for $\psi$-trivial (the proof adapting easily to unitary $\psi$, see Remark [5,1]), the whole point is to extend the result to non unitary characters $\psi$.

**Proposition 3.1.** Let $\rho$ be a cuspidal representation of $G_{r}$, with $r$ a positive even integer, then $\rho$ is $(H_{r}, \chi_{\alpha})$-distinguished if and only if it is $(S_{r}, \Theta)$-distinguished.

**Proof.** We work with $L_r$ rather than $H_r$. Of course $\rho$ is $(L_r, \psi_{\alpha})$-distinguished, so we take a nonzero element $L$ in $Hom_{L_r}(\rho, \psi_{\alpha})$. From [D10], Theorem 4.4, (ii) (see [K-T] for $\alpha = 1$), we know that for any $v$ in the space of $\rho$, the relative coefficient

$$\psi_{L,v} : g \in G_r \mapsto L(\rho(g)v)$$

is non-zero. Therefore, $\psi_{L,v}$ is nonzero for all $\chi_{\alpha}$.
belongs to $C_c^\infty(L_r\backslash G_r, \psi_{\alpha})$, and $v \mapsto \psi_{L,v}$ is a $G_r$-module injection of $\rho$ in $C_c^\infty(L_r\backslash G_r, \psi_\alpha)$. We set $m = r/2$. Using the Iwasawa decomposition $L_rU_{(m,m)}K_r$, we see that the map

$$f_{L,v} : x \mapsto \psi_{L,v}(u(x))$$

belongs to $C_c^\infty(M_m)$. We denote by $S_L$ the linear form on $\delta$, defined by

$$S_L(v) = \int_{x \in M_m} f_{L,v}(x)\theta^{-1}(x)dx \quad (1)$$

for $v \in \rho$. We claim that $S_L$ is a nonzero Shalika functional on $\rho$. Indeed, $S_L(v) = 0$ for all $v$ in $\rho$ means that $\int_{x \in M_m} f_{L,v}(x)\theta^{-1}(x)dx$ for all $\rho$. Replacing $v$ by $\text{diag}(g, I_r)$ for $g \in G_r$, we deduce that

$$\alpha(\det(g))\int_{x \in M_m} f_{L,v}(g^{-1}x)\theta^{-1}(x)dx = |\det(g)|^m\alpha(\det(g))\int_{x \in M_m} f_{L,v}(x)\theta^{-1}(gx)dx = 0$$

for all $g \in G_r$. Hence the Fourier transform of $f_{L,v}$ is zero on $G_m$, hence on $M_m$ by smoothness of $f_{L,v}$ and density of $G_m$ in $M_m$. So the map $f_{L,v}$ must be zero, and $f_{L,v}(I_m) = L(v) = 0$, and this for any $v \in \rho$. It follows from simple change of variable in (1) that $S_L$ is $\Theta$-invariant under $S_r$.

**Corollary 3.1** (of the proof). Let $\rho$ be a cuspidal representation of $G_r$, then $\text{Hom}_{H_r}(\rho, \chi_\alpha)$ is of dimension at most 1.

**Proof.** We saw that the map $L \mapsto S_L$ in the proof above is injective. Our claim now follows from the uniqueness of Shalika functionals for irreducible representations of $G_r$ ([J-R], Proposition 6.1). \hfill \Box

**Remark 3.1.** It is not clear that Theorem 3.1 extends easily to discrete series as in [M14], for $\alpha \neq 1$. Indeed, the relative coefficients are not in $L^2(H_n\backslash G_n)$ as soon as $\alpha$ is not unitary (in fact if $C$ is an $(H_n, \chi_\alpha)$-relative coefficient, with $\alpha$ not unitary, then $|C|^2$ is not a map on $H_n\backslash G_n$).

Before we study the case of discrete series, we also need to clear up a misunderstanding in Theorem 3.1 of [M14]. In the statement of this theorem, the integer $n$ must be $\geq 2$ (it is also tacitly assumed in its proof). Indeed, for $n = 1$, we have $H_1 = G_1$, all characters of $G_1$ are cuspidal, so a character $\mu$ of $G_1$ can be by $(H_1, \chi)$-distinguished, and this if and only if $\mu = \chi$.

**Proposition 3.2.** Suppose that $\rho$ is a cuspidal representation of $G_r$ ($r \geq 1$), and $d$ is a positive integer, and set $n = dr$. Let

$$\pi = \nu^{(1-d)/2} \rho \times \cdots \times \nu^{(d-1)/2} \rho.$$ 

Then $\dim_{\mathbb{C}}(\text{Hom}_{H_n}(\pi, \chi_\alpha)) \leq 1$, and if $d$ is odd, $\dim(\text{Hom}_{H_n}(\pi, \chi_\alpha)) = 1$ implies that $\rho$ is $(H_r, \chi_\alpha)$-distinguished (in particular $r$ is even except if $r = 1$, in which case $\rho = \alpha$).
Proof. The reader would benefit from reading Section 3 of [M15] until the discussion before Theorem 3.2 of [ibid.] before reading this proof. We suppose that \( \pi \) is distinguished. In particular, as \( Z_n \subset H_n \), the central character \( \omega_p \) is of order \( d \), and \( \rho \) is unitary. We set \( M = M_{(r_{1}, ..., r_{l})} \) the standard Levi subgroup of \( G = G_n \) such that \( R = \nu^{(1-d)/2} \rho \otimes \cdots \otimes \nu^{(d-1)/2} \rho \) is a representation of \( M \), and \( P = P_{(r_{1}, ..., r_{l})} \) the corresponding parabolic subgroup of \( G \). We also set \( H = H_n \), and \( \chi = \chi_\alpha \). A system of representatives \( \mathcal{R}(P\setminus G/H) \) of the double quotient \( P\setminus G/H \) is determined in Section 3.1 of [M15]. To every \( s \) in \( \mathcal{R}(P\setminus G/H) \), a standard parabolic subgroup \( P_s \subset P \) of \( G \), and its standard Levi subgroup \( M_s \) is associated, and we denote by \( H_{M_s} \) the intersection \( M \cap sHs^{-1} \), and by \( \chi_s \) the character \( \chi(s^{-1}.s) \) of \( H_{M_s} \). Then, in the discussion before Theorem 3.2 of [loc. cit.], for any \( (H, \chi) \)-invariant linear \( L \) form on \( \pi \), and any \( s \) in \( \mathcal{R}(P\setminus G/H) \), an \( (H_{M_s}, \chi_s) \)-linear form \( L_s \) is defined on the normalised Jacquet module \( r_{M_s} R \), with the property that if \( L_s \) is zero for every \( s \) in \( \mathcal{R}(P\setminus G/H) \), then \( L \) is zero. In particular, if \( \pi \) is \((H, \chi)\)-distinguished, and \( L \) is a nonzero linear form in \( \text{Hom}_H(\pi, \chi) \), then \( L_s \) is nonzero for at least one \( s \). As the representation \( R \) of \( M \) is cuspidal, this implies first that \( M_s = M \). Thanks to Section 3.2 of [M15], it also implies that there are \( l \) disjoint couples \( i_k < j_k \) \((k = 1, \ldots, l \) with \( 2l \leq d \) \) in \( \{1, \ldots, d\} \), and natural integers \( n_i^- \) such \( n_i^+ \) with \( r = n_i^- + n_i^+ \) for each \( i \in I = \{1, \ldots, d\} \) \( \cup \{k, j_k\} \), such that \( H_{M_s} \) is of the form

\[
\{\text{diag}(g_1, \ldots, g_d) \in M, g_{i_k} = g_{j_k} \text{ for } k = 1, \ldots, l, \text{ and } g_i \in M_{(n_i^-, n_i^+)} \text{ for } i \in I\}.
\]

Moreover, thanks to Theorem 3.1 of [M14], when \( r \geq 2 \), for every \( i \in I \), we must have \( n_i^- = n_i^+ \). As \( R = \nu^{(1-d)/2} \rho \otimes \cdots \otimes \nu^{(d-1)/2} \rho \) is \((H_{M_s}, \chi_s)\)-distinguished, and as \( \rho \) is unitary, this implies that \( l = d \) when \( l \) is even, and \( i_k = k \) and \( j_k = d + 1 - k \) for each \( k \in \{1, \ldots, d/2\} \), and \( l = d - 1 \) when \( l \) is odd, \( i_k = k \) and \( j_k = d + 1 - k \) for each \( k \in \{1, \ldots, (d-1)/2\} \). Moreover if \( r \geq 2 \), then \( r \) is even and \( \rho \) is \((M_{(r/2, r/2)}, \mu_{\alpha})\)-distinguished \((i.e. (H_r, \chi_\alpha)\)-distinguished), and when \( r = 1 \), then \( \rho = \alpha \). In particular, in all cases, this implies that there is a unique \( s \in \mathcal{R}(P\setminus G/H) \) such that \( L_s \) is nonzero, thus the map \( L \mapsto L_s \) is injective. As \( L_s \) then lives in a 1-dimensional space thanks to Corollary 3.3, this proves the result.

\[\square\]

**Corollary 3.2.** Let \( c \) be a positive integer, and \( r \) be a positive integer. Let \( \rho \) be a cuspidal representation of \( G_r \), and \( \delta' = \delta(c, \rho) \). Then the representation \( \tau = \nu^{-c/2} \delta' \times \nu^{c/2} \delta' \) satisfies \( \dim_{\mathbb{C}}(\text{hom}_{H_n}(\tau, \chi_\alpha)) \leq 1 \).

**Proof.** It is obvious, because setting \( d = 2c \), the representation \( \tau \) is a quotient of \( \pi = \nu^{(1-d)/2} \rho \times \cdots \times \nu^{(d-1)/2} \rho \). The result now follows from Proposition 3.2.

\[\square\]

Let \( n = n_1 + \cdots + n_t \), and \( \pi_i \) be an irreducible representation of \( G_{n_i} \) for every \( i \), and let \( \pi = \pi_1 \times \cdots \times \pi_t \). According to Chapter 3 of [G-J], there is \( r_\pi \in \mathbb{R} \) such that for any coefficient \( f \) of \( \pi \), a map of the form \( g \in G_n \mapsto \langle v^\vee, \pi(g)v \rangle \) for \( v \in V \) and \( v^\vee \in V'^\vee \), and any \( \Phi \in C_\infty(M_n) \), the integral \( Z(s, \Phi, f) = \int_{G_n} f(g) \Phi(\nu(g)^s)dg \) converges absolutely for \( \text{Re}(s) > r_\pi \). These zeta integrals in fact belong to \( \mathbb{C}(q^{-s}) \), and span a fractional ideal of \( \mathbb{C}(q^\mathbb{Z}) \) containing \( 1 \). One denotes by \( L(s, \pi) \) the unique Euler factor which is a generator of this ideal, it satisfies the relation \( L(s, \pi) = \prod_{i=1}^{t} L(s, \pi_i) \). Notice that this notation is coherent with that of Section 2.3 as it is proved in Section 5 of [JPS-S] that if \( \pi \) is irreducible, the Godement-Jacquet factor \( L(s, \pi) \) and the Rankin-Selberg factor \( L(s, \pi, 1) \) are equal. We now recall the following result from [FJ].
Proposition 3.3. Let $2m = n = n_1 + \cdots + n_t$ be an even integer, and $\pi_i$ be an irreducible representation of $G_{n_i}$ for every $i$. If $\pi = \pi_1 \times \cdots \times \pi_t$ is such that $\text{Hom}_{S_n}(\pi, \theta) \neq 0$. Take a nonzero element $L$ of $\text{Hom}_{S_n}(\pi, \theta)$, and denote by $S_L(\pi, \Theta)$ the space of maps from $G_n$ to $C$ of the form $S_{L,v} : g \mapsto L(\pi(g)v)$ for $v \in V$. Then there is $r \in \mathbb{R}$, such that for any $S$ in $S_L(\pi, \Theta)$ the integral

$$I(s, \alpha, S) = \int_{G_m} S(\text{diag}(g, I_m))\alpha(\text{det}(g))\nu(g)^s \, dg$$

is absolutely convergent. Moreover, these integrals in fact belong to $\mathbb{C}(q^{-s})$, and span a fractional ideal of $\mathbb{C}[q^{\pm s}]$ equal to $L(s + 1/2, \pi \otimes \alpha)\mathbb{C}[q^{\pm s}]$. In particular, the map

$$\Lambda_{\pi,L} : v \mapsto I(0, \alpha, S_{L,v})/L(1/2, \pi \otimes \alpha)$$

is a nonzero element of $\text{Hom}_{L_n}(\pi, \psi_{\alpha})$.

Proof. The assumptions in [FJ] are that $\pi$ is unitary and irreducible, and the proof is given in the archimedean case. However, we adapted their arguments to the $p$-adic case in [M14], but the absolute convergence of $I(S, s, \alpha)$ was not shown in this reference, because we were dealing with linear forms invariant under $S_n \cap P_n$ rather than $S_n$, hence we couldn’t use the asymptotic expansion of Shalika functions from [J-R]. We thus give a proof for the case at hand, referring to [M14] and [J-R], where only minor modifications are needed.

Let $S$ belong to $S_L(\pi, \Theta)$. First, an asymptotic expansion of the restriction of $S$ to the torus $A_n$ is given in [J-R], Theorem 6.1. Their proof assumes $\pi$ irreducible, but all the arguments work for $\pi$ of finite length (as the Jacquet modules of $\pi$ are also of finite length). Hence, we get the absolute convergence of $I(s, \alpha, S)$, for $Re(s)$ larger than a certain real number $r$ independent of $S$, as in [J-R] after Lemma 6.1. We can now write for $Re(s) > r$, the equality $I(s, \alpha, S) = \sum_{k \epsilon \mathbb{Z}} c_k(\alpha, S)q^{-ks}$, where $c_k(\alpha, S) = \int_{|g|=q^{-k}} S(\text{diag}(g, I_r))\alpha(\text{det}(g)) \, dg$. The statement now follows from Proposition 4.2 of [M14], which is for $\alpha = 1$, but valid for any $\alpha$. 

We are now able to prove the main result of this section.

Theorem 3.1. Let $n > 0$ be an even integer. The representation $\delta(d, \rho)$ is $(H_n, \chi_{\alpha})$-distinguished if and only if it is $(S_n, \Theta)$-distinguished, or equivalently if and only if $\rho$ is $(H_r, \chi_{\alpha})$-distinguished.

Proof. First, we notice that if $\delta(d, \rho)$ is $(H_n, \chi_{\alpha})$-distinguished or $(S_n, \Theta)$-distinguished, then its central character (which is that of $\rho$), is trivial, so we assume that the $\omega_\rho$ is trivial so that $\delta(k, \rho)$ is unitary for any $k$. By Proposition 3.3 if $\delta(d, \rho)$ is $(S_n, \Theta)$-distinguished, then it is $(H_n, \chi_{\alpha})$-distinguished. For the converse, there are two cases to consider, which are proved differently. We thus suppose that $\delta(d, \rho)$ is $(H_n, \chi_{\alpha})$-distinguished.

If $d$ is odd (hence $r$ is even), as $\delta(d, \rho)$ is a quotient of $\nu^{1-d/2}\rho \times \cdots \times \nu^{(d-1)/2}\rho$, then by Proposition 4.2, the representation $\rho$ is $(H_r, \chi_{\alpha})$-distinguished, hence it is $(S_r, \Theta)$-distinguished according to Proposition 3.1. This in turn implies that $\delta(d, \rho)$ is $(S_n, \Theta)$-distinguished according to Theorem 6.1. of [M14]. So this proves the theorem when $d$ is odd.

If $d = 2c$ is even, we write $\delta' = \delta(c, \rho)$, hence $\tau = \nu^{-c/2}\delta' \times \nu^{c/2}\delta'$ is $(S_n, \Theta)$-distinguished according to Proposition 3.8 of [M15], and we denote by $L$ a nonzero...
Shalika functional on $\tau$. It follows from Section 3 of [4], that $\pi$ is of length 2, with irreducible quotient $\delta = \delta(d, \rho)$, and a unique irreducible submodule that we denote by $\sigma$, which is the Langlands quotient of $\nu^{c/2}g' \times \nu^{-c/2}g'$. We suppose that the quotient $\delta$ is not $(S_n, \Theta)$-distinguished, and we will obtain a contradiction. In this situation $\sigma$ must be $(S_n, \Theta)$-distinguished, as $L_{\sigma}$ is nonzero. Now, according to Proposition 3.3, $\Lambda_{\pi, L}$ is a nonzero linear form on $\pi$ which is $(H_n, \chi_{\alpha})$-invariant, and $\Lambda_{\pi, L}$ is a nonzero linear form on $\pi$ which is $(H_n, \chi_{\alpha})$-invariant. Finally, it is known that $L(s, \pi \otimes \alpha) = L(s, \sigma \otimes \alpha)$ ([3], Theorem 3.4), hence the linear form $\Lambda_{\pi, L}$ extends $\Lambda_{\sigma, L}$, in particular $\Lambda_{\pi, L}|_{\sigma} = \Lambda_{\sigma, L}$ is nonzero. But $\delta$ is $(H_n, \chi_{\alpha})$-distinguished, and any linear form on $\pi$ which descends to $\delta$ vanishes on $\sigma$, this contradicts Proposition 3.2 and ends the proof. \hfill \Box

4. The local Bump-Friedberg $L$-function

For simplicity, from now on, we will assume that $\alpha$ is an unramified character of the form $|.|^t$ for some $t \in \mathbb{C}$. This is the situation considered in [B-F]. We will denote by abuse of notation $\chi_t$ the character $\chi_{|.|^t}$ of $H_n$.

4.1 Definition

For $n \in \mathbb{N} - \{0\}$, we set $m = (n + 1)/2$. Let $\pi$ be a representation of $G_n$ of Whittaker type. Let $W$ belong to $W(\pi, \theta)$, and $\Phi$ belong to $\mathcal{C}_c^\infty(F^m)$, $s$ and $t$ be complex numbers, and for $h(h_1, h_2) \in H_n$, we denote by $l_n(h(h_1, h_2))$ the bottom row of $h_2$ when $n$ is even, and the bottom row of $h_1$ when $n$ is odd. We consider the following integrals, the convergence of which will be addressed just after

$$\Psi(s, t, W, \Phi) = \int_{N_{\pi} \backslash H_n} W(h)\Phi(l_n(h))\chi_t(h)\chi_n(h)^{1/2}|h|^s dh$$

and

$$\Psi(0)(s, t, W) = \int_{N_{\pi} \backslash P_{\pi, \infty}} W(h)\chi_t(h)\mu_n^{1/2}(h)|h|^{s-1/2} dh$$

$$= \int_{N_{\pi} \backslash G_{n-1} \backslash G_{n-1}} W(h)\chi_t(h)\chi_{n-1}^{1/2}(h)|h|^{s-1/2} dh.$$

Remark 4.1. Notice the difference of notations with [M15]. In [ibid.], we denote by $\Psi(W, \Phi, \chi_{|.|^t}, s)$ the integral denoted by $\Psi(s, t, W, \Phi)$ here. Now we introduce the Rankin-Selberg integrals which will be used to define the local Bump-Friedberg $L$-factor.

Proposition 4.1. Let $\pi$ be a representation of $G_n$ of Whittaker type. Let $W$ belong to $W(\pi, \theta)$, $\Phi$ belong to $\mathcal{C}_c^\infty(F^m)$, and $\epsilon_k = 0$ when $k$ is even and 1 when $k$ is odd. If for all $k$ in $\{1, \ldots, n\}$ (resp. in $\{1, \ldots, n-1\}$), we have $Re(s) > -[Re(c_k) + \epsilon_k Re(t + 1/2)]/k$, the integral $\Psi(s, t, W, \phi)$ (resp. $\Psi(0)(s, t, W)$) converges absolutely. It admits meromorphic extension to $\mathbb{C} \times \mathbb{C}$ as elements of $\mathbb{C}(q^{-s}, q^{-t})$.

Proof. Let $B_n^\sigma$ be the standard Borel subgroup of $H_n$. The integral

$$\Psi(s, t, W, \Phi)$$
will converge absolutely as soon as the integrals
\[ \int_{A_n} W(a)\phi(l_n(a))\chi_n^{1/2}(a)\chi_t(a)|a|^s\delta_{B^*_n}(a)d^*a \]
will do so for any \( W \in W(\pi, \theta) \), and any \( \Phi \in \mathcal{C}_{\infty}^{\infty}(F^m) \). But, according to Proposition 2.5 and writing \( z = z_1 \ldots z_n \), this will be the case if for every \( k \) and each integral \( \phi \)

**Proof.**

Let \( \Phi \) be an Euler factor \( 1/PBF(q^{-s}, q^{-t}, \pi) \) with \( PBF(q^{-s}, q^{-t}, \pi) \) in \( L[q^{\pm s}] \), where \( LBF(s, t, \pi) \) is an Euler factor \( 1/PBF(q^{-s}, q^{-t}, \pi) \) with \( PBF(q^{-s}, q^{-t}, \pi) \) in \( L[q^{\pm s}] \) and \( PBF(0, q^{-t}, \pi) = 1 \). The \( PBF(q^{-s}, q^{-t}, \pi) \) belongs to \( O_t[q^{-s}] = \mathbb{C}[q^{-s}] \), and for \( W \in W(\pi, \theta) \) and \( \Phi \in \mathcal{C}_{\infty}(F^m) \), the integral \( PBF(q^{-s}, q^{-t}, \pi) \) belongs to \( O_t[q^{\pm s}] \).

**Proposition 4.2.** Let \( \pi \) be a representation of \( G_n \) of Whittaker type. The \( L_t \)-vector space spanned by the integrals \( \Psi(s, t, W, \Phi) \) when \( W \) and \( \Phi \) vary in \( W(\pi, \theta) \) and \( \mathcal{C}_{\infty}(F^m) \) is a fractional ideal of \( L[q^{\pm s}] \) of the form \( LBF(s, t, \pi)L_t[q^{\pm s}] \), where \( LBF(s, t, \pi) \) is an Euler factor \( 1/PBF(q^{-s}, q^{-t}, \pi) \) with \( PBF(q^{-s}, q^{-t}, \pi) \) in \( L[q^{\pm s}] \) and \( PBF(0, q^{-t}, \pi) = 1 \), which is uniquely determined. In fact, the \( PBF(q^{-s}, q^{-t}, \pi) \) belongs to \( O_t[q^{-s}] = \mathbb{C}[q^{-s}] \), and for \( W \in W(\pi, \theta) \) and \( \Phi \in \mathcal{C}_{\infty}(F^m) \), the integral \( PBF(q^{-s}, q^{-t}, \pi) \) belongs to \( O_t[q^{\pm s}] \).

**Proof.** Changing \( \Psi(s, t, W, \Phi) \) by \( \Psi(s, t, \rho(h_0)W, \rho(h_0)\Phi) \) (where \( \rho \) denotes right translation) multiplies \( \Psi(s, t, W, \Phi) \) by \( \chi_n(h_0)\chi_t(h_0)|h_0|^{-s} \). This implies that the integrals \( \Psi(s, t, W, \Phi) \) span a fractional ideal of \( L[q^{\pm s}] \). Moreover, it is shown in the proof of Proposition 4.8. of [MT13], that one can choose \( W \) and \( \Phi \) such that \( \Psi(s, t, W, \Phi) = 1 \) for all \( s \) and \( t \). This implies the existence of \( PBF(q^{-s}, q^{-t}, \pi) \). Now, thanks to Corollary 4.1.4.
$P^{BF}(q^{-s}, q^{-t}, \pi)$ divides the polynomial $Q \in O_t[q^{-s}]$ in the ring $L_t[q^\pm s]$. As both have constant term equal to 1, this first implies that $P^{BF}$ divides $Q$ in $L_t[q^{-s}]$, and as $O_t[q^{-s}]$ is a unique factorisation domain (because $O_t = \mathbb{C}[q^{-t}]$ is), this also implies that $P^{BF}$ belongs to $O_t[q^{-s}]$. Finally, for all $W$ and $\Phi$ we have $P^{BF}(q^{-s}, q^{-t}) \Psi(s, t, W, \Phi) \in L_t[q^\pm s]$, so we can write it $\sum_{k=-N}^{N} a_k(q^{-t}) q^{ks}$ for some $N \geq 0$, with $a_k \in \mathbb{C}(X)$. According to Corollary 4.1, the integral $\Psi(s, t, W, \Phi)$ has no singularities of the form \( \{s \in \mathbb{C} \times \{t = t_0\}\}$ (because the same holds for the Tate $L$-factors $L(c_k, sk + \epsilon_k(t + 1/2))$), in particular the $a_k$’s belong to $O_t$.

\[\square\]

### 4.2 Results on a specialisation of the local Bump-Friedberg factor

This specialisation in the variable $t$, which is an Euler factor with respect to $q^{-s}$, denoted $L^{lin}(s, t, \pi)$ (we fix the value of $t$ in $L^{BF}(s, t, \pi)$) is studied in [M15]. As we will see, it is not really defined as as specialisation, but rather as the gcd of a the family of specialised integrals, and for this reason, it is not obvious at all that $L^{lin}(s, t, \pi)$ is rational as a map of $q^{-s}$ and $q^{-t}$. Hence, we will use results about this specialisation as an intermediate step, but they will not appear in our final statements. In fact, in general, we will see that except maybe for a finite number of values of $t$, the equality $L^{lin}(s, t, \pi) = L^{BF}(s, t, \pi)$ is true (the equality is probably true for all values of $t$, but we did not check it, it is for example the case for discrete series representations).

We start by recalling a special case of Proposition 4.8 of [M15].

**Proposition 4.3.** Let $n$ be a positive integer, and $\pi$ be a representation of $G_n$ of Whittaker type. For fixed $t \in \mathbb{C}$, the integrals $\Psi(s, t, W, \Phi)$ generate a (necessarily principal) fractional ideal $I(t, \pi)$ of $\mathbb{C}[q^\mp s]$ when $W$ and $\Phi$ vary in their respective spaces, and $I(t, \pi)$ has a unique generator which is an Euler factor which we denote by $L^{lin}(s, t, \pi)$ (and denoted by $L^{lin}(\pi, \chi_t, s)$ in [M15]).

The integrals $\Psi(0)(s, t, W)$ (which are equal to 1 by convention if $n = 1$) generate a (necessarily principal) fractional ideal $I(0)(t, \pi)$ of $\mathbb{C}[q^s, q^{-s}]$ when $W$ and $\Phi$ vary in their respective spaces, and $I(0)(t, \pi)$ has a unique generator which is an Euler factor, which we denote by $L^{lin}(s, t, \pi)$ (and denoted by $L^{lin}(0)(\pi, \chi_t, s)$ in [M15]).

For $t \in \mathbb{C}$, we set

$$L^{lin,(0)}(s, t, \pi) = L^{lin}(s, t, \pi)/L^{lin}(0)(s, t, \pi)$$

(denoted $L^{rad(\chi_t)}(\pi, \chi_t, s)$ in [M15]). The following property is a consequence of Proposition 4.9 and Corollary 4.2 of [M15].

**Proposition 4.4.** Let $n$ be a positive integer, and $\pi$ be a representation of $G_n$ of Whittaker type. Fix $t \in \mathbb{C}$, then $L^{lin,(0)}(s, t, \pi)$ is an Euler factor with simple poles.

We also recall Theorem 5.2 of [M15].

**Theorem 4.1.** Let $n$ be a positive integer, and $\pi = L(\tau)$ be an irreducible representation of $G_n$, with $\tau = \delta_1 \times \cdots \times \delta_r$ induced of Langlands’ type. Then, for $t \in [-1/2, 0]$, we have

$$L^{lin}(s, t, \pi) = \prod_{i=1}^{r} L^{lin}(s, t, \delta_i) \prod_{1 \leq i < j \leq r} L(2s, \delta_i, \delta_j).$$
4.3 Equality with the Galois factor for discrete series

We recall a consequence of Proposition 4.14 of [M15] in the special case of discrete series.

As a convention, when \( n = 0 \), and \( \pi \) is the trivial representation of \( G_0 \), we set

\[
L^{\text{lin}}(s, t, \pi) = 1.
\]

**Proposition 4.5.** Let \( r \) and \( d \) be positive integers (and set \( n = dr \)), and \( \rho \) be a cuspidal representation of \( G_r \). Then one has

\[
L^{\text{lin}}(s, t, \delta(d, \rho)) = L^{\text{lin},(0)}(s, t, \delta(d, \rho))L^{\text{lin}}(s, t, \nu^{1/2}\delta(d - 1, \rho)).
\]

In order to compute the factor \( L^{\text{lin},(0)}(s, t, \delta(d, \rho)) \), we recall Corollary 4.3 of [M15].

**Proposition 4.6.** Let \( n \) be a positive integer and \( t \in \mathbb{C} \). If \( \delta \) be a discrete series representation of \( G_n \), then the factor \( L^{\text{lin},(0)}(s, t, \delta) \) has a pole at zero if and only if \( \delta \) is \((H_n, \chi_t^{-1})\)-distinguished.

Now, we have the following consequence of Theorem 3.1.

**Proposition 4.7.** Let \( n \geq 2 \) be a positive integer, and \( \delta \) be a discrete series representation of \( G_n \). Then the factor \( L^{\text{lin},(0)}(s, t, \delta) \) is equal to \( L^{\text{lin},(0)}(s, 0, \delta) \) for any \( t \in \mathbb{C} \).

**Proof.** Both are Euler factors with simple poles, hence it suffices to prove that they have the same poles. The result follows at once from Theorem 3.1 and Proposition 4.6. \( \square \)

We can now prove the following.

**Theorem 4.1.** Let \( n \) be a positive integer and \( t \in \mathbb{C} \), and \( \delta \) be a discrete series representation of \( G_n \). Then the factor \( L^{\text{lin}}(s, t, \delta) \) is equal to \( L(s + t + 1/2, \phi_{\delta})L(2s, \Lambda^2 \circ \phi_{\delta}) \).

**Proof.** Write \( \delta = \delta(d, \rho) \), for \( d \in \mathbb{N} - \{0\} \) dividing \( n \), and \( \rho \) a cuspidal representations of \( G_r = G_{n/d} \). If \( r \geq 2 \), by Propositions 4.5 and 4.7 we obtain the equalities

\[
L^{\text{lin}}(s, t, \delta(d, \rho)) = \prod_{k=0}^{d-1} L^{\text{lin},(0)}(s, t, \nu^{k/2}\delta(d - k, \rho))
\]

\[
= \prod_{k=0}^{d-1} L^{\text{lin},(0)}(s, 0, \nu^{k/2}\delta(d - k, \rho)) = L^{\text{lin}}(s, 0, \delta(d, \rho)).
\]

Now, by Theorem 5.4 of [M15], we have \( L^{\text{lin}}(s, 0, \delta) = L(s + 1/2, \phi_{\delta})L(2s, \Lambda^2 \circ \phi_{\delta}) \), but \( L(s + 1/2, \delta) \) is equal to 1 according to [G-J], so we obtain the equality \( L^{\text{lin}}(s, t, \delta) = L(s + t + 1/2, \delta)L(2s, \Lambda^2 \circ \phi_{\delta}) \) when \( r \geq 2 \).

When \( r = 1 \), by Proposition 4.5 we have

\[
L^{\text{lin}}(s, t, \delta(d, \rho)) = \prod_{k=0}^{d-1} L^{\text{lin},(0)}(s, t, \nu^{k/2}\delta(d - k, \rho)).
\]

The factor \( L^{\text{lin},(0)}(s, t, \nu^{(d-1)/2}\delta(1, \rho)) \) is equal to

\[
L^{\text{lin},(0)}(s, t, \nu^{(d-1)/2}\rho) = L^{\text{lin}}(s, t, \nu^{(d-1)}\rho) = L(s + t + d/2, \rho),
\]

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so
\[ L^{\text{lin}}(s,t,\delta(d,\rho)) = L(s+t+d/2,\rho) \prod_{k=0}^{d-2} L^{\text{lin},(0)}(s,t,\delta(d-k,\rho)). \]

For \( k \leq d-2 \), the factor \( L^{\text{lin},(0)}(s,t,\delta(d-k,\rho)) \) do not depend on \( t \) by Proposition 4.7. As we know from Theorem 5.4 of [M15] that
\[ L^{\text{lin}}(s,0,\delta) = L(s+1/2,\phi_\delta)L(2s,\Lambda^2 \circ \phi_\delta), \]
and as \( L(s+1/2,\delta(d,\rho)) = L(s+d/2,\rho) \) we deduce that \( L(2s,\Lambda^2 \circ \phi_\delta) \) must be equal to
\[ \prod_{k=0}^{d-2} L^{\text{lin},(0)}(s,t,\nu_k/2\delta(d-k,\rho)), \]
and this proves the result in this case. \( \square \)

We will see in the next paragraph that this implies that \( L^{BF}(s,t,\delta) \) is equal to \( L(2s,\Lambda^2 \circ \phi_\delta) \) for any discrete series \( \delta \) of \( G_n \) (and in fact for any irreducible representation).

### 4.4 Equality with the Galois factor for irreducible representations

We first notice a consequence of the results of the preceding section. If \( \pi \) is an irreducible representation of \( G_n \), we denote by \( L^{\text{Gal}}(s,t,\pi) = L(s+t+1/2,\phi_\pi)L(2s,\Lambda^2 \circ \phi_\pi). \)

**Proposition 4.8.** Let \( \pi \) be an irreducible representation of \( G_n \), then for \( (s,t) \in \mathbb{C} \times [-1/2,0] \), we have
\[ L^{\text{lin}}(s,t,\pi) = L^{\text{Gal}}(s,t,\pi). \]

**Proof.** We saw in Section 2.3 that if \( \pi = L(\delta_1 \times \cdots \times \delta_r) \), then
\[ L^{\text{Gal}}(s,t,\pi) = \prod_{k=1}^r L^{\text{Gal}}(s,t,\delta_k) \prod_{1 \leq i < j \leq r} L(2s,\phi_{\delta_i} \otimes \phi_{\delta_j}). \]

The result now follows from Theorems 4.1 and 4.2. \( \square \)

We will use the following elementary fact about polynomials in two variables.

**Lemma 4.1.** Let \( P \) be a polynomial in \( \mathbb{C}[X,Y] \), and suppose that it does not vanish on a set of the form \( \mathbb{C} \times A \), where \( A \) is infinite, then \( P \) is of the form \( P(X,Y) = P(0,Y) \).

**Proof.** Write \( P \) as \( P = \sum_{k=0}^d a_k(Y)X^k \), then for any \( y \in A \), by the fundamental theorem of algebra, one has \( P(X,y) = a_0(y) \). As \( A \) is infinite, the result follows. \( \square \)

**Proposition 4.9.** Let \( \pi \) be an irreducible representation of \( G_n \), for \( n \geq 1 \). Then the polynomial
\[ P^{BF}(s,t,\pi) = \frac{1}{L^{BF}(s,t,\pi)} \]
divides the polynomial
\[ P^{\text{Gal}}(s,t,\pi) = \frac{1}{L(s+t+1/2,\phi_\pi)L(2s,\Lambda^2 \circ \phi_\pi)} \]
in \( \mathbb{C}[q^{-s},q^{-t}] \).
Proof. For any \( W \) in \( W(\pi, \theta) \) and \( \Phi \in C_c^\infty(F'''') \), the Laurent polynomial \( R(s, t) = P^{Gal}(s, t, \pi)\Psi(s, t, W, \Phi) \in L_1[q^{\pm s}] \) has no singularities for \( (s, t) \) in \( C \times [-1/2, 0] \), according to Proposition 4.8. We can always write \( R \) as a quotient \( U/V \), with \( U \) and \( V \in \mathbb{C}[q^{-s}, q^{-t}] \), and co-prime. In particular, \( U \) and \( V \) have a finite number of zeroes in common (it follows for example from Bézout identities in \( L_1[q^{-s}] \) and \( L_1[q^{-s}] \)). In particular, \( V \) does not vanish on a set of the form \( C \times A \), where \( A \) is infinite (take \( A \) the complementary set of the projection on the second coordinate of the set of common zeroes of \( U \) and \( V \)), and is thus of the form \( V = V(q^{-t}) \) according to Lemma 4.1. This implies that \( R \) belongs to \( L_1[q^{-s}] \), and by definition of \( P^{BF}(s, t, \pi) \), it in turn implies that \( P^{BF}(s, t, \pi) \) divides \( P^{Gal}(s, t, \pi) \) in \( L_1[q^{-s}] \), hence in \( \mathcal{O}_q[q^{-s}] \) as they both have constant term equal to 1.

Let \( \pi \) be an irreducible representation of \( G_n \), for fixed \( t \), we set

\[
P^{lin}(s, t, \pi) = \frac{1}{L^{lin}(s, t, \pi)}.
\]

Proposition 4.10. For fixed \( t \), \( P^{lin}(s, t, \pi) \) divides \( P^{BF}(s, t, \pi) \) in \( \mathbb{C}[q^{-s}] \).

Proof. Fix \( t = t_0 \). According to the last part of Proposition 4.2 for any \( W \) in \( W(\pi, \theta) \) and any \( \Phi \in C_c^\infty(F^{([n+1]/2)}) \) the element

\[
P^{BF}(q^{-s}, q^{-t_0}, \pi)\Psi(s, t_0, W, \Phi)
\]

belongs to \( \mathbb{C}[q^{\pm s}] \), and the proposition follows.

We can now state the main result of this section.

Theorem 4.3. Let \( \pi \) be an irreducible representation of \( G_n \), then the two factors \( L^{Gal}(s, t, \pi) \) and \( L^{BF}(s, t, \pi) \) are equal.

Proof. For \( (s, t) \in C \times [-1/2, 0] \), we have \( P^{Gal}(s, t, \pi) = P^{lin}(s, t, \pi) \) by Proposition 1.8. By Propositions 1.9 and 4.10 we know that for fixed \( t \), \( P^{lin}(s, t, \pi) \) divides \( P^{BF}(s, t, \pi) \), and \( P^{BF}(s, t, \pi) \) divides \( P^{Gal}(s, t, \pi) \). In particular, for \( (s, t) \in C \times [-1/2, 0] \), \( P^{BF}(s, t, \pi) \) must be equal to \( P^{Gal}(s, t, \pi) \), so this equality is in fact true for all \( s \) and \( t \).

We can also prove the following almost everywhere equality (which should be true everywhere).

Proposition 4.11. Let \( \pi \) be an irreducible representation of \( G_n \). Then there is a finite (maybe empty) set \( A_\pi \) of \( C^* \), such that for \( q^{-t} \in C^* - A_\pi \), the two factors \( L^{lin}(s, t, \pi) \) and \( L^{BF}(s, t, \pi) \) are equal in \( \mathbb{C}(q^{-s}) \).

Proof. By definition of \( L^{BF}(s, t, \pi) \), there is a finite set \( I \), Whittaker maps \( W_i \in W(\pi, \theta) \), Schwartz functions \( \Phi_i \in C_c^\infty(F^{([n+1]/2)}) \), and rational maps \( \lambda_i \in L_t \), such that

\[
L^{BF}(s, t, \pi) = \sum_{i \in I} \lambda_i(t)\Psi(s, t, W_i, \Phi_i).
\]

Let \( A_\pi \) be the union of the possible poles of the \( \lambda_i \)'s, it is a finite set, and for \( q^{-t_0} \in C^* - A_\pi \), we can specialise the equality and obtain that \( L^{BF}(s, t_0, \pi) \) belongs to the \( \mathbb{C}[q^{\pm s}] \)-submodule of \( L_s \) spanned by the integrals \( \Psi(s, t, W, \Phi) \) for \( W \in W(\pi, \theta) \) and \( \Phi \in C_c^\infty(F^{([n+1]/2)}) \), hence \( L^{BF}(s, t_0, \pi) \) divides \( L^{lin}(s, t_0, \pi) \). This together with Proposition 4.10 implies that \( L^{BF}(s, t_0, \pi) = L^{lin}(s, t_0, \pi) \), which is our claim.
We can finally obtain the functional equation of the local Bump-Friedberg L-factor. For \( \Phi \in \mathcal{C}_c^\infty (F[\lfloor \frac{n}{2} \rfloor]), \) we denote by \( \Phi^\theta \) the Fourier transform of \( \Phi \) with respect to a \( \theta \)-self-dual Haar measure of \( F[\lfloor \frac{n}{2} \rfloor]. \)

**Corollary 4.2.** Let \( \pi \) be an irreducible representation of \( G_n. \) Then there is a unit \( \epsilon(s,t,\pi,\theta) \) of \( \mathbb{C}[q^{\pm s},q^{\pm t}], \) such that for all \( W \in W(\pi,\theta) \) and \( \Phi \in \mathcal{C}_c^\infty (F[\lfloor \frac{n}{2} \rfloor]), \) one has

\[
\Psi(1 - s, -1/2 - t, W, \widehat{\Phi}^\theta) = \frac{\epsilon(s,t,\pi,\theta)\Psi(s,t,W,\Phi)}{L^{BF}(1 - s, -1/2 - t, \pi^\vee)}.
\]

**Proof.** According to Proposition 4.11 of [M15] and Proposition 4.11 there is a finite (maybe empty) set \( A_\pi \) of \( \mathbb{C}^* \), such that for \( q^{-t} \in \mathbb{C}^* - A_\pi, \) there is a unit \( \epsilon(s,t,\pi,\theta) \) of \( \mathbb{C}[q^{\pm s}] \) such that for all \( W \in W(\pi,\theta) \) and \( \Phi \in \mathcal{C}_c^\infty (F[\lfloor \frac{n}{2} \rfloor]), \) one has

\[
\Psi(1 - s, -1/2 - t, W, \widehat{\Phi}^\theta) = \frac{\epsilon(s,t,\pi,\theta)\Psi(s,t,W,\Phi)}{L^{BF}(1 - s, -1/2 - t, \pi^\vee)}.
\]

As both quotients

\[
\frac{\Psi(1 - s, -1/2 - t, W, \widehat{\Phi}^\theta)}{L^{BF}(1 - s, -1/2 - t, \pi^\vee)}
\]

and

\[
\frac{\Psi(s,t,W,\Phi)}{L^{BF}(s,t,\pi)}
\]

belong to \( \mathbb{C}[q^{\pm s},q^{\pm t}] \), the map \( \epsilon(s,t,\pi,\theta) \) extends to an element \( \epsilon(s,t,\pi,\theta) \) of \( \mathbb{C}(q^{-s},q^{-t}) \) such that Equation (2) is satisfied. It remains to show that \( \epsilon(s,t,\pi,\theta) \) is a unit of \( \mathbb{C}(q^{-s},q^{-t}) \). For \( q^{-t} \notin A_\pi, \) there is \( \alpha \in \mathbb{C}^* \), and \( m_t \in \mathbb{Z} \), such that \( \epsilon(s,t,\pi,\theta) = \alpha q^{m_t s} \), and this implies that \( \alpha \) extends to an element \( \alpha(t) \) of \( \mathbb{C}(q^{-t}) \) and that \( m_t \) does not depend on \( t \). Suppose that \( \alpha(t_0) = \infty \), this would imply that \( \Psi(s,t_0,W,\Phi)/L^{BF}(s,t_0,\pi) \) is equal to 0 for all \( W \in W(\pi,\theta) \) and \( \Phi \in \mathcal{C}_c^\infty (F[\lfloor \frac{n}{2} \rfloor]), \) but as we can choose them such that \( \Psi(s,t_0,W,\Phi) = 1 \), this would imply that \( p^{BF}(s,t_0,\pi) = 0 \). This would in turn imply, according to Proposition 4.9 that \( p^{Gal}(s,t_0,\pi) = 0 \), which is absurd. Hence \( \alpha(t) \) has no pole, and we prove in the exact same manner that it has no zeros. This implies that \( \alpha(t) \) is of the form \( \alpha q^{-lt} \) for some \( l \in \mathbb{Z} \), and we are done. \( \square \)

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