Hybrid DDP in Clutter (CHDDP): Trajectory Optimization for Hybrid Dynamical System in Cluttered Environments

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Abstract—We present an algorithm for obtaining an optimal control policy for hybrid dynamical systems in cluttered environments. To the best of our knowledge, this is the first attempt to have a locally optimal solution for this specific problem setting. Our approach extends an optimal control algorithm for hybrid dynamical systems in the obstacle-free case to environments with obstacles. Our method does not require any preset mode sequence or heuristics to prune the exponential search of mode sequences. By first solving the relaxed problem of getting an obstacle-free, dynamically feasible trajectory and then solving for both obstacle-avoidance and optimality, we can generate smooth, locally optimal control policies. We demonstrate the performance of our algorithm on a box-pushing example in a number of environments against the baseline of randomly sampling modes and actions with a Kinodynamic RRT.

I. INTRODUCTION

We consider the problem of controlling a hybrid system with discrete modes and continuous inputs in a cluttered environment. Many robotics systems for real-world applications are hybrid in nature. For example, while operating an autonomous car, the system must choose an appropriate (discrete) gear and control the (continuous) acceleration. When pushing a block on a surface, the end-effector must choose which (discrete) face to push, with each face having its own (continuous) dynamics. Obstacle-avoidance must also be ensured in both of these cases.

Optimal control of hybrid systems is a crucial aspect of robotic navigation and manipulation. The problem characteristics make it quite difficult to solve efficiently. The discreteness of modes makes the optimization problem non-convex, and makes the control input space exponential in the length of the planning horizon. The presence of obstacles, on the other hand, makes the feasible state-space non-convex.

Although many robotic systems have hybrid nature, our specific problem has not been considered previously in the literature. Recently, the optimal hybrid control problem in the obstacle-free setting, was solved using Differential Dynamic Programming (DDP) [15]. They relaxed the hybrid constraint and transformed the discrete controls into a convex combination of weights. We discuss other related work in Section II, and additional background material in Section III.

Our key insight is that locally optimal methods for hybrid systems in clutter often get stuck in poor local minima because simultaneously finding a feasible mode sequence and moving away from the obstacles is challenging. We address these two challenges by decoupling them and solving the problem in two stages: 1) we generate a collision-free and dynamically feasible trajectory and then 2) we solve for a collision-free, optimal policy using the mode sequence found in the first stage. We describe our approach in detail in Section IV.

Our major contribution is the ability to produce locally optimal control policies for hybrid systems with both state and input constraints. To the best of our knowledge, our algorithm is the first approach for this specific problem domain. Our method does not require a mode sequence to be specified beforehand, nor does it require any heuristics to prune an exponential search space of discrete mode transitions. We achieve wider applicability for the previously introduced approach for hybrid control [15] by non-trivially extending it to handle the presence of clutter.

We compare our approach CHDDP (Constrained Hybrid DDP) to the Kinodynamic RRT (KDRRT) [11] which randomly samples modes and actions without any notion of solution quality or optimality. We show qualitative and quantitative results over several box-pushing problems in Section V. Finally, we conclude in Section VI with a discussion of some limitations and questions for future research.

II. RELATED WORK

The problem of obtaining locally optimal trajectories for a hybrid control system has been addressed in numerous previous works [2, 14, 21], but incorporating obstacles is a highly non-trivial challenge. The feasible set of solutions is non-convex because of the obstacles. The objective function is non-convex because of the hybrid nature of the dynamics.
When dynamic constraints is present, shooting methods like t is the current control input, x as the following:

subclass of such systems in which the mode transition can be dynamical systems indexed by discrete modes. We consider a A. Problem Statement open-loop policies as opposed to closed-loop feedback laws. RRT [11] can handle both hybridness and clutter, but the Sampling-based planning methods such as the kinodynamic challenging to extend these approaches to the hybrid domain. of a steering function for hybrid dynamic systems makes it solution [16, 19] are good at handling obstacles, although the lack trajectory planning methods based on optimiza-

non-convex. Trajectory planning methods based on optimiza-

Real-time feedback control has been produced by reducing the search space into a much smaller set of mode sequences [8]. When dynamic constraints is present, shooting methods like DDP could converge faster than direct methods like MIP. Relaxing the hybrid constraint by representing discrete modes as a probability distribution has enabled shooting methods to solve hybrid problem without requiring heuristics [15].

In addition to the hybrid nature of the dynamics system, the existence of obstacles makes the feasible state-space non-convex. Trajectory planning methods based on optimization [16, 19] are good at handling obstacles, although the lack of a steering function for hybrid dynamic systems makes it challenging to extend these approaches to the hybrid domain. Sampling-based planning methods such as the kinodynamic RRT [11] can handle both hybridness and clutter, but the quality of solutions can be arbitrarily poor and they are only open-loop policies as opposed to closed-loop feedback laws.

III. PRELIMINARIES

A. Problem Statement

A hybrid dynamical system [15] is a set of continuous dynamical systems indexed by discrete modes. We consider a subclass of such systems in which the mode transition can be made arbitrarily. Within each mode, the dynamics is defined as the following:

\[ x_{t+1} = f_{a_t}(x_t, u_t) \]  

(1)

where \( t \) is the current timestep, \( a_t \) is the current mode, \( u_t \in U \) is the current control input, \( x_t \in X \) is the continuous state of the system, and \( f_{a_t} : X \times U \to X \) is a continuous function that represents the dynamics of the current mode. The set of all dynamics functions is denoted as \( F \). Each mode \( a_t \) is chosen among \( N_a \) possible modes and has bounded control inputs, i.e. \( u_t \in [\underline{u}_t, \overline{u}_t] \). The configuration space \( X \subset X \) is a subspace of \( X \) which represents the system kinematics, e.g. rigid body pose, or joint angles for a manipulator. The set of all configurations in collision with the environment and in self-collision (if applicable) is denoted as \( X_{\text{obs}} \) and the complement of this is the free space \( X_{\text{free}} = X \setminus X_{\text{obs}} \). When we say \( x_t \in X_{\text{free}} \) or \( x_t \in X_{\text{obs}} \), we mean that the configuration embedded in \( x_t \) is in \( X_{\text{free}} \) or \( X_{\text{obs}} \).

A trajectory \( \tau \) is defined as a sequence of state-mode-control tuples \( \{(x_t, a_t, u_t)\}_{t=1}^T \). Given a hybrid dynamical system and an environment with obstacles, our objective is to find a locally optimal trajectory \( \tau^* \) that minimizes the total cost

\[ \tau^* = \arg \min_{\tau} l_T(x_T) + \sum_{t=0}^{T-1} l_{a_t}(x_t, u_t) \]  

(2)

subject to dynamics, collision constraints and specified start and goal states \( x_s \) and \( x_f \) respectively. Here, \( l_{a_t}(x_t, u_t) \) is the immediate cost for executing control \( u_t \) under mode \( a_t \), and \( l_T(x_T) \) is the final cost.

B. Differential Dynamic Programming

We briefly review DDP [9, 20] and explain how we extend it to hybrid systems.

The optimal cost-to-go, a.k.a. value of a state \( x \) at timestep \( t \) can be found by recursively choosing actions that minimize the total cost:

\[ V_t(x) = \min_u l(x, u) + V_{t+1}(f(x, u)). \]  

(3)

Given a nominal trajectory \( (\bar{x}, \bar{u}) \), let \( Q \) be the increase of \( V_t \) (3) caused by a change of trajectory \( (\delta x, \delta u) \):

\[ Q(\delta x, \delta u) := l(\bar{x} + \delta x, \bar{u} + \delta u) + V_{t+1}(f(\bar{x} + \delta x, \bar{u} + \delta u)) - (l(\bar{x}, \bar{u}) + V_{t+1}(f(\bar{x}, \bar{u}))) \]
Then, a locally optimal change can be made by choosing \( \delta u \) as the solution to the following problem:

\[
\delta u^* = \arg \min_{\delta u} Q(\delta x, \delta u) \\
\approx \arg \min_{\delta u} \frac{1}{2} \delta u^\top Q_{uu} \delta u + Q_{u,a}^\top \delta u + \delta u^\top Q_{ux} \delta u.
\]

(4)

The second equation comes from a local quadratic approximation of \( Q \), which gives the following solution:

\[
\delta u^* = -Q_{uu}^{-1} Q_u - Q_{u,a}^\top Q_{ux} \delta x
\]

(5)

The first term is the changed nominal control and the second term is a linear feedback control law. \( Q_{uu}, Q_u, Q_{ux} \) can be computed recursively from \( V_T(x_T) \) [9] during a backward pass. DDP iterates between a backward pass and a forward rollout until the change in nominal control is small.

C. DDP for Hybrid System (HDDP)

The classic DDP method can only solve smooth problems. It has been extended to solve switched hybrid control problems [15], where the control \( \hat{u} \) contains both continuous action \( u \) and discrete action \( a \). They replaced the discrete control with a continuous vector \( p \), each element of which represents the probability of choosing one specific discrete mode, i.e. \( \hat{u} = [u \ p] \), with \( p_a \in [0, 1], \forall a \). Then they used DDP to solve the convexified, smooth dynamics:

\[
\hat{f}(x, \hat{u}) = \sum_a p_a f_a(x, u).
\]

(6)

The following term is added to the cost function

\[
c_{ST}(x, u, p) = C_{ST} \sum_a \left\{ \phi(p_a) \frac{1-p_a}{1-p_{th}} \right\} \text{if } p_a < p_{th}
\]

where \( \phi(\cdot) \) is the pseudo-Huber loss, \( p_{th} = 1/N_a \) and \( C_{ST} \) is a coefficient that grows along with the number of DDP iterations. A large \( C_{ST} \) drives each \( p_a \) to either zero or one, so that when DDP converges, the solution to (6) has a fixed sequence of modes, which is also a solution to the original hybrid problem (1).

Constraints on inputs, i.e. \( \forall a, 0 \leq p_a \leq 1 \) and \( \sum_a p_a = 1 \) are handled [20] by solving Equation 4 for \( \delta u \) as a constrained quadratic program.

IV. Approach

Extending obstacle-free hybrid control approaches [7, 10, 15] to a cluttered setting is nontrivial. Our key insight is that locally optimal approaches for hybrid systems in clutter often get stuck in poor local minima because simultaneously finding a feasible mode sequence and moving away from obstacles is challenging. We address these two challenges by decoupling them and running DDP twice to solve the problem:

1) We generate a collision-free geometric path using some path-planning algorithm and obtain a dynamically feasible trajectory using DDP without considering obstacles.

2) Using the mode sequence found from step 1, we solve for a collision-free, locally optimal policy using DDP with (efficient) collision checking.

This decoupling is non-trivial and at the heart of our approach. We discuss it in detail here.

A. Generating a collision-free, dynamically feasible nominal trajectory

During the DDP forward passes, whenever the state violates collision constraints, i.e. is in \( \mathcal{X}_{obs} \), we project it back to \( \mathcal{X}_{free} \), which requires many collision checks and could be computationally expensive. To reduce the computation, we approximate the collision-free space as a union of convex polytopes using IRIS (Iterative Regional Inflation by Semidefinite programming) [4] (See Figure 2b for illustration). Instead of collision checking, the constraint check is performed by checking whether a given point \( x \) belongs to one of the convex regions, and if not, it returns the point projected to the closest convex region. Note that this union of convex polytopes is a conservative approximate of \( \mathcal{X}_{free} \), which is often beneficial for our purpose as long as the space reduction is not too significant.
While DDP without clutter or hybrid dynamics is often capable of finding an optimal solution even from a bad initial trajectory, this is typically not true in the case of a hybrid system in clutter. It often gets stuck in a local minima that does not reach the goal, because it fails to get away from an obstacle or to find the right sequence of modes. In fact, it is critical that we start with a nominal trajectory that has minimal collision while accomplishing the task.

Starting from a collision-free path\(^1\) found by a path-planning algorithm such as visibility graph or RRT (mentioned subsequently), we first use HDDP to solve for a feasible state-action-mode sequence that closely tracks the reference path \(\xi = \{x_0, x_1, \cdots x_T\}\) such that every \(x_t\) along the path is at least \(\epsilon\) away from any obstacle. We do not impose collision avoidance constraints here, but as the reference path is at least \(\epsilon\) away from any obstacle, the resulting state-action trajectory is likely to be collision free.

\[
\min_{u_0, p_0, x_1, u_1, p_1, \cdots, x_T, p_T} \sum_t \phi(x_t - \hat{x}_t) + |u_t|^2
\]

subject to \(x_{t+1} = f(x_t, u_t, p_t), u_t \in [a_t, \bar{a}_t], \forall t,\)

Here \(\phi(\cdot)\) denotes the soft-abs function (pseudo-Huber loss). We use the soft-abs function on the state deviation because its gradient will not diminish outside of a given range, so the optimization can continue to minimize the error until it is within the given range. We use a quadratic cost on the magnitude of the continuous action, since we prefer smaller \(u_t\), but it does not have to be zero.

We use some path-planning algorithm to get an initial collision-free path. For problems that can be projected to two dimensions, we use a visibility graph planning method [13] to get multiple feasible paths in different homotopy classes [1]. This heuristic provides efficient and good quality feasible paths covering different subspaces of the search space, thereby also increasing the probability of a successful solution by CHDDP. We show in Figure 3 how this initialization method leads to multiple good quality CHDDP solutions. If CHDDP fails to generate a feasible solution with any geometrically feasible paths obtained from the homotopy stage, or if the path planning must happen in higher dimensional spaces, we use the probabilistically complete sampling-based RRT (Rapidly-exploring Random Trees) algorithm [11].

B. Optimizing with a fixed mode sequence

After getting a feasible state-action trajectory, we use it to initialize the second optimization, where we optimize the original cost function with collision-free constraints:

\[
\min_{u_0, x_1, \cdots, x_T} l_T(x_T) + \sum_{t=0}^{T-1} l_{a_t}(x_t, u_t)
\]

subject to \(x_{t+1} = f(x_t, u_t, p_t), x_t \in \mathcal{X}_{free}, u_t \in [a_t, \bar{a}_t], \forall t,\)

where \(p_t\) is obtained from Sub-problem I. Our CHDDP algorithm modifies the HDDP method to handle the collision avoidance constraint explicitly. The first forward pass gives us a feasible trajectory (the initial trajectory). During the backward pass, we perform collision checking on the updated trajectory, and shrink the step size until the new trajectory is collision-free. Repeating this process, the forward passes will always produce feasible trajectories. As mentioned earlier, the collision checking is very efficient with IRIS regions as it effectively involves checking affine inequalities.

In this Sub-problem, we only update the trajectory in terms of \(\{x_t, u_t\}\), while keeping the choice of mode \(p_t\) unchanged. Otherwise, the trajectory would vary rapidly during the first few iterations and might converge to some different local minima, which could be really bad due to the existence of obstacles. Practically, we do this by using the cost coefficient \(C_{ST}\) from the ending of Sub-problem I, which is already a large number.

We outline our overall algorithm for Hybrid DDP in Clutter (CHDDP) in Algorithm 1. After getting one or more geometrically feasible paths \(\xi\), we run HDDP to get a feasible \(\tau\) and then optimize this with CHDDP.

V. Experiments

We demonstrate the effectiveness of our approach with the problem setting of pushing an unknown box. We use a similar setup to that of previous work our approach is based on [15] - please refer to them for details of the dynamics, control modes and the objective function. Here we only state the differences between our model and theirs, for the sake of brevity. The state-space of our system is a subset of the 2D Special Euclidean Group (\(SE(2)\)), and each state is represented

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\(\xi\) refers to path as a sequence of states with no associated action. This path may not be kinodynamically feasible.
as \( x = (x, y, \theta) \) where the co-ordinates refer to the centre of the box, and \( \theta \) is the rotation of the box about the centre. We use a motion cone of \([-70^\circ, 70^\circ]\). Also, rather than having a continuous variation of the point being pushed, which would actually cause a tangent friction force, we push a single point on the edge at a time.

The box pushing example is the only one that we run tests on, however it is equivalent to the popular Dubin’s car model [6] with gears, another widely applicable hybrid control system. Our experiments are on different sets of obstacles and start and goal poses for the box.

We have already shown in Figure 3 how multiple geometrically feasible paths in various homotopy classes can be used to initialize CHDDP for good quality locally optimal solutions. For several additional problems, we compare our CHDDP against a baseline trajectory obtained from kinodynamic RRT with random sampling of modes and actions. We show quantitative results in Figure 4, where we compare the costs (as per our objective function) of the trajectories returned by KDRRT and CHDDP. We average over 50 runs of KDRRT for each problem, while we deterministically initialize CHDDP with the geometrically shortest homotopy path, if more than one exists for the problem. We also visualize some solution trajectories in Figure 5. For all of the examples, the solutions produced by CHDDP are of significantly higher quality than those from KDRRT. Since CHDDP is the first attempt to have a locally optimal approach for this specific problem domain, we do not really have any other baseline method to directly compare our performance against.

As we have mentioned before, the core of our approach is independent of the method used to generate an initial solution. The visibility graph method for generating homotopy paths is a means of obtaining initial feasible solutions that are short, while having good clearance (due to obstacle inflation). However, should it not be applicable for the specific problem, CHDDP is also able to derive good quality trajectories and locally optimal control policies when initialized by the Kinodynamic RRT. We show in Figure 6 how even quite jagged and poor quality KDRRT trajectories are improved considerably by CHDDP.

VI. CONCLUSION

We proposed CHDDP, a method of efficiently obtaining locally optimal solutions for a hybrid control problem in a cluttered environment, which is a computationally hard problem. We do this by dividing the overall problem into feasible sub-problems. We approximate the feasible state space with a union of convex collision-free regions, which also
makes collision checking much more efficient. For generating initial solutions for the hybrid DDP method, we generate high quality geometrically feasible paths in multiple homotopy classes. We solve the difficult hybrid DDP problem in two phases, each of which is more tractable than the joint problem. This allows us to extend the capability of recently proposed work to solve for hybrid control trajectories in the presence of obstacles.

In our current approach, the hybrid DDP is used to generate the nominal state-action-mode trajectory without considering collision constraints. Therefore the feedback control gain does not take the collisions into account. This is a problem that could be addressed in potential future work.

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