The return of the four- and five-dimensional preons

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Abstract

We prove the existence of $\frac{3}{4}$-BPS preons in four- and five-dimensional gauged supergravities by explicitly constructing them as smooth quotients of the AdS$_4$ and AdS$_5$ maximally supersymmetric backgrounds, respectively. This result illustrates how the spacetime topology resurrects a fraction of supersymmetry previously ruled out by the local analysis of the Killing spinor equations.

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1. Introduction

Given a $d$-dimensional supergravity theory with $n$ (real) supercharges, it is an interesting problem to determine the existence of backgrounds preserving the next-to-largest possible number $N_p < n$ of supersymmetries. Depending on the vagaries of the spinor representations and of the Killing spinor equations—i.e., whether the space of Killing spinors is real, complex or quaternionic—the number $N_p$ may be $n - 1$, $n - 2$ or $n - 4$. In any one of these cases, the resulting background is called a supergravity (or BPS) preon, generalizing a notion introduced in [1, 2] in the context of eleven-dimensional supergravity, and recently reviewed in [3]. The name derives from the Pati–Salam model [4], and just as the Pati–Salam preons are constituents of both quarks and leptons, supergravity preons are meant to give rise to other BPS backgrounds; although a precise mechanism has not been proposed. The argument for the existence of preons uses the correspondence between the supersymmetry broken by a BPS state and the rank of the ‘central’ charge in the relevant superalgebra. The algebraic manifestation of preons is simply the fact that a rank-$r$ central charge can be written as a linear combination of $r$ rank-1 central charges, whose corresponding BPS states are supposed to be the preons.

To the best of our knowledge, however, the geometric realization in supergravity of the addition of central charges is still an open problem, and moreover its existence has been brought into question by recent results ruling out supergravity preons in ten and eleven dimensions.
where $n = 32$ and $N_p = 31$. Such nonexistence results are usually obtained in two steps: in the first step one proves that a preonic background always admits locally the maximal number of supersymmetries, whence if it exists at all, it must be the quotient of a maximally supersymmetric background by a discrete symmetry group. In the second step one then uses the classification of maximally supersymmetric solutions [9] to show that no such quotients exist. For eleven-dimensional supergravity preons, these two steps are described in [5] and [8], respectively.

The investigations in four and five dimensions flow along similar lines. For the case of gauged four- and five-dimensional supergravities with matter, where $N_p/n = 3/4$, it has been shown that $1/4$-BPS states are locally maximally supersymmetric backgrounds of the minimal theory, where there is no matter content and gauge fields are flat [10–12]. It therefore remains to investigate whether one can obtain $1/4$-BPS backgrounds by quotienting the maximally supersymmetric backgrounds, which for the $d$-dimensional theory are given by $\text{AdS}_d$.

In this paper, using results of [13], we will show that remarkably such quotients do indeed exist. Indeed, we will exhibit a family (parametrized by the positive real numbers) of discrete groups acting freely and properly discontinuously via isometries on $\text{AdS}_d$, preserving the spin structure, and leaving invariant a $1/4$-dimensional subspace of the Killing spinors. These groups are discrete subgroups of the one-parameter subgroup labelled (6) in the classification of [13, section 4.1.1]. Such quotients have already appeared briefly in [14, section B.2] and are also the subject of an ongoing study [15] on discrete lightcone quantization in the AdS/CFT correspondence.

This paper is organized as follows. In section 2, we will introduce the relevant quotients of $\text{AdS}_d$ and prove that they are smooth. In fact, we will work with the hyperboloid model of $\text{AdS}_d$, which is already an infinite cyclic quotient by the centre, and quotient it further by a non-central $\mathbb{Z}$-subgroup. In section 3, we will show that such quotients preserve $1/4$ of the supersymmetry. We will do this in two ways: one purely representation theoretic using the Bár cone construction, and other by explicitly calculating the Killing spinors relative to an adapted coordinate system. Finally, in section 4 we offer some concluding remarks.

2. A family of smooth quotients of AdS

Let $\mathbb{R}^{d-1}$ denote the $(d+1)$-dimensional pseudo-Euclidean space with global coordinates $X_0, X_1, \ldots, X_d$ and flat metric given by

$$ds^2 = -dX_0^2 - dX_d^2 + \sum_{i=1}^{d-1} dX_i^2.$$  \hspace{1cm} (1)

The hyperboloid

$$X_0^2 + X_d^2 - \sum_{i=1}^{d-1} X_i^2 = \ell^2$$  \hspace{1cm} (2)

describes a Lorentzian manifold with constant negative curvature. The scalar curvature is readily calculated to be

$$R = -\frac{d(d - 1)}{\ell^2},$$  \hspace{1cm} (3)

whence it is locally isometric to $\text{AdS}_d$ with radius of curvature $\ell$. It is in fact the quotient of $\text{AdS}_d$ by the action of the centre of the isometry group and will be hereafter referred to, somewhat loosely, as the hyperboloid model for $\text{AdS}_d$. The virtue of this model is that the
isometries of AdS$_d$ are realized linearly in the embedding space, namely as $O(2, d-1)$ acting linearly on $\mathbb{R}^{2,d-1}$.

Let $k > 0$ and consider the element $\gamma \in SO(2, d-1)$ defined by

$$
\begin{pmatrix}
X_d \\
X_0 \\
X_1 \\
X_2 \\
X_i
\end{pmatrix} \mapsto
\begin{pmatrix}
X_d + k(X_2 - X_0) \\
X_0 + k(X_d - X_1) \\
X_1 + k(X_2 - X_0) \\
X_2 + k(X_d - X_1) \\
X_i
\end{pmatrix},
$$

for $2 < i < d$. The element $\gamma$ is in the image of the exponential map

$$\gamma = \exp(kZ),$$

where $Z \in so(2, d-1)$ is given by

$$Z = (dX_0 - dX_2) \wedge (dX_1 - dX_2),$$

whence a compact set $K$ on the hyperboloid is the intersection of the hyperboloid with a closed and bounded subset of $\mathbb{R}^{d+1}$. It is therefore enough to show that any ball of finite radius centred at $p$ contains finitely many points of the orbit $G_k \cdot p$. To see this, let $p$ have coordinates $X = (X_0, X_1, \ldots, X_d)$ and consider the Euclidean distance between $p$ and the point $\gamma^N \cdot p$ in its $G_k$-orbit, which is given by

$$||\gamma^N X - X||^2 = 2N^2k^2((X_2 - X_0)^2 + (X_d - X_1)^2).$$

Although the expression in the parentheses can be arbitrarily small on the hyperboloid, it is always positive, hence for any $k > 0$ and any finite radius $L$, there will be some $N_0$, depending on the point $p$, for which

$$||\gamma^N X - X|| > L \quad \text{for all} \quad N > N_0.$$

Therefore, at most $N_0$ points in the orbit will lie inside the ball.

In summary, $G_k$ acts freely and properly discontinuously on the hyperboloid and hence the quotient is smooth and locally isometric to AdS$_d$. As shown in [16, section 2.4], a $\mathbb{Z}$-quotient always preserves the spin structure, hence we can ask how much supersymmetry such a quotient preserves. In the following section, we answer this question and show that it preserves $\frac{d}{2}$ of the supersymmetry of AdS$_d$, for $d = 4, 5$.

3. Supersymmetry

As shown in [10, 11], the supersymmetries of the maximally supersymmetric backgrounds of four- and five-dimensional gauged supergravities are in one-to-one correspondence with geometric Killing spinors; that is, spinor fields $\epsilon$ obeying

$$\nabla_a \epsilon = i \lambda \Gamma_a \epsilon,$$

The factor of $i$ is due to our Clifford algebra conventions: $\Gamma_A \Gamma_B + \Gamma_B \Gamma_A = +2\eta_{AB}$, with $\eta$ mostly plus in Lorentzian signature.
where the Killing constant $\lambda$ is related to the scalar curvature by
\begin{equation}
R = 4\lambda^2 d(d - 1),
\end{equation}
whence it can be either real or pure imaginary. Maximal supersymmetry is attained by the spaces of constant curvature and for these theories it is $\text{AdS}_d$, for which $\lambda$ is pure imaginary. In either of these theories, preons will exist if there exists some discrete subgroup of isometries of $\text{AdS}_d$, for $d = 4, 5$, leaving invariant a $\frac{d}{2}$-dimensional subspace of the Killing spinors.

In this section, we will present two proofs that the quotients described above of $\text{AdS}_d$, for $d = 4, 5$, do indeed preserve $\frac{d}{2}$ of the supersymmetry. The first proof uses representation theory and the cone construction and is computationally very simple. On the other hand, the second proof is more elementary but computationally more involved and follows from an explicit calculation of the Killing spinors relative to a coordinate system adapted to the group action.

3.1. The cone construction and representation theory

As discussed above, the supersymmetries in the $\text{AdS}_d$ vacuum are in one-to-one correspondence with the geometric Killing spinors, and those of the quotient by a group $G$ (assumed to preserve the spin structure) are precisely the $G$-invariant Killing spinors. In this section, we will show how to reinterpret this in terms of representation theory.

The main technical tool is the cone construction [17, 18] relating geometric Killing spinors (9) on a manifold $(M, g)$ to parallel spinors on its cone, a manifold one dimension higher with metric $dr^2 + 4\lambda^2 r^2 g$. If $g$ is Riemannian and $\lambda$ real, the cone metric is again Riemannian. We are interested however in the case of $g$ Lorentzian and $\lambda$ pure imaginary, whence the cone has signature $(2, d - 1)$. Since $\text{AdS}_d$ has constant curvature, its cone is an open subset of the pseudo-Euclidean space $\mathbb{R}^{2,d-1}$ with the standard flat metric, corresponding to those vectors with negative norm.

Relative to flat coordinates for $\mathbb{R}^{2,d-1}$ and the corresponding global orthonormal frame, parallel spinors are in one-to-one correspondence with constant $\text{Spin}(2, d - 1)$ spinors. In the case of odd $d$, the spinors are chiral: the choice of chirality corresponding to a choice of sign in the Killing constant $\lambda$ in equation (9). For even $d$, there is up to equivalence a unique spinor representation, and the sign of the Killing constant enters in the choice of embedding $\text{Cl}(d - 1, 1) \subset \text{Cl}(2, d - 1)$ between Clifford algebras. This is explained in detail in [17, 18] and in a supergravity context in the forthcoming paper [19].

The isometry group of the simply-connected $\text{AdS}_d$ is the universal covering group $\tilde{\text{SO}}(2, d - 1)$ of $SO(2, d - 1)$. As discussed, for example, in [13, 16], the action of the spin cover $\tilde{\text{Spin}}(2, d - 1)$ on the Killing spinors is not effective, but factors through the action of $\text{Spin}(2, d - 1)$, which is an infinite cyclic quotient. Let $\tilde{G} < \tilde{\text{SO}}(2, d - 1)$ be a discrete subgroup acting freely and properly discontinuously on $\text{AdS}_d$. The resulting quotient $\text{AdS}_d/\tilde{G}$ is smooth and moreover will be spin if $\tilde{G}$ lifts isomorphically to a subgroup of $\tilde{\text{Spin}}(2, d - 1)$, which we also denote by $\tilde{G}$. Assume this is so and let $G < \text{Spin}(2, d - 1)$ denote its projection onto $\text{Spin}(2, d - 1)$. Then, the Killing spinors on the quotient $\text{AdS}_d/\tilde{G}$ are precisely the $G$-invariant Killing spinors on $\text{AdS}_d$. Now, as discussed in [20], the cone construction is equivariant under the action of isometries. Therefore, the $G$-invariant Killing spinors on $\text{AdS}_d$ are precisely the $G$-invariant parallel spinors on $\mathbb{R}^{2,d-1}$; in other words, the $G$-invariant subspace in the (chiral) spinor representation of $\text{Spin}(2, d - 1)$.

The groups $G_k$ in question are in the image of the exponential map, hence their image in the Spin groups lies in the identity component. For $\text{AdS}_4$, this is the identity component of $\text{Spin}(2, 3)$ which is isomorphic to $\text{Sp}(4, \mathbb{R})$. The spinor representation is therefore real and
four dimensional, and we will show that $G_k$ fixes a three-dimensional subspace. For AdS$_5$, on
the other hand, it is the identity component of Spin(2, 4), which is isomorphic to SU(2, 2).

The relevant spinorial representation is therefore complex and four dimensional, and we will
show that $G_k$ fixes a three-dimensional complex subspace. The calculation follows mutatis
mutandis the approach of [13, section 6.1.4].

In fact, we will work with any $d > 2$. As usual when working with the spin groups, it
is convenient to embed them in the Clifford algebra. The image $\hat{\gamma} \in \text{Spin}(2, d - 1)$ of the
generator $\gamma$ in (5) is given by the exponential

$$\hat{\gamma} = \exp \left( \frac{1}{2}k(\Gamma_0 - \Gamma_2)(\Gamma_1 - \Gamma_d) \right) \in \text{Spin}(2, d - 1) \subset \Cl(2, d - 1)$$

of the Lie algebra element (6). Using the fact that both $\Gamma_0 - \Gamma_2$ and $\Gamma_1 - \Gamma_d$ anticommute
and square to zero, we see that

$$\hat{\gamma} = 1 + \frac{1}{2}k(\Gamma_0 - \Gamma_2)(\Gamma_1 - \Gamma_d).$$

Let $V$ denote the relevant irreducible Clifford representation of $\Cl(2, d - 1)$. For even $d$
there are actually two inequivalent representations, but the cone construction chooses one of
them \textit{ab initio}. Let us introduce four subspaces of $V$ given by

$$V_{\pm \pm} = \ker(\Gamma_0 \pm \Gamma_2) \cap \ker(\Gamma_1 \pm \Gamma_d)$$

with uncorrelated signs. It is plain to see that

$$V = V_{++} \oplus V_{+-} \oplus V_{-+} \oplus V_{--}$$

and that each subspace is $\frac{1}{4}$-dimensional. Relative to this decomposition, the matrix of $\hat{\gamma}$ is
given by

$$\hat{\gamma} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\Phi & 0 & 0 & 1
\end{pmatrix},$$

where $\Phi := \frac{1}{2}k(\Gamma_0 - \Gamma_2)(\Gamma_1 - \Gamma_d) : V_{++} \rightarrow V_{--}$ has no kernel. Therefore, there is a
$\frac{3}{4}$-dimensional invariant subspace given by

$$V^{G_k} = V_{--} \oplus V_{+-} \oplus V_{-+}.$$  

When $d$ is odd, we have to restrict further to chiral spinors, but it is clear that the above
discussion still holds if we substitute for $V$ the relevant chiral spinor representation from
the start. In either case, $G_k$ preserves a $\frac{3}{4}$-dimensional subspace of the relevant spinorial
representation.

### 3.2. Explicit construction of the Killing spinors

We will now explicitly solve for the Killing spinors on the hyperboloid relative to a coordinate
system adapted to the action of the group $G_k$. The calculation is aided by representing spinors
as exterior forms as described, for example, in [21–23]. Although the discussion can be made
general, we will focus for definiteness on the four- and five-dimensional cases of interest.

Let $\Cl(2, 2)$ be the Clifford algebra of the pseudo-Euclidean space $\mathbb{R}^{2,2}$. It is well known
that the unique irreducible representation of $\Cl(2, 2)$ is isomorphic to the exterior algebra $\Lambda P$, for
$P \subset \mathbb{R}^{2,2}$ any isotropic plane. More concretely, one builds the isomorphism as follows.
Given $P$ one chooses a complementary isotropic plane $P'$, so that $\mathbb{R}^{2,2} = P \oplus P'$. Under the
inner product, $P'$ is naturally isomorphic to the dual $P^*$ of $P$. The action of the Clifford algebra
on $\Lambda P$ is uniquely defined by declaring gamma matrices in $P$ to act via the wedge product and those in $P'$ to act by the contraction with the corresponding element of $P'^+$. Moreover, as the Clifford group acts transitively on the space of isotropic planes, all choices of $P$ yield equivalent representations.

In this paper, however, we are interested in $C\ell(1, 3)$, and in $\mathbb{R}^{1,3}$ there are no isotropic planes. Nevertheless, we can still identify spinors of $C\ell(1, 3)$ with exterior algebra elements in $\Lambda\mathbb{R}^{1,3}$ and those in $P$ to the exterior algebra $\Lambda P$ may trivialize the spinor bundle and effectively represent a spinor $\epsilon$. The resulting complex Clifford algebra is denoted by $C\ell^{\mathbb{C}}$. We complexify $\mathbb{R}^{1,3}$ to $\mathbb{C}^4$ extending the inner product to a complex bilinear form. The resulting complex Clifford algebra is denoted by $C\ell^{\mathbb{C}}$, where we no longer keep track of the signature because there is no such notion for a complex inner product.$^3$ We may now choose complementary isotropic (complex) planes $\pi$ and let $e_\pi$ make $\Lambda\mathbb{P}$ into an irreducible representation of $C\ell^{\mathbb{C}}$. We may also extend this to an irreducible representation of $C\ell(1, 4)$ on the same space by defining $\Gamma_4 := \pm i\Gamma_0\Gamma_1\Gamma_2\Gamma_3$. The choice of sign merely reflects the fact that $C\ell(1, 4)$ has two inequivalent irreducible representations.

There are two spin structures on the hyperboloid models for AdS$_4$ and AdS$_5$, due to the noncontractible timelike circles $X_0^2 + X_3^2 = \text{constant}$. Choosing the trivial spin structure, we may trivialize the spinor bundle and effectively represent a spinor $\epsilon$ as a function from the hyperboloid to the exterior algebra $\Lambda\mathbb{C}^2$ generated by $e^1$ and $e^2$:

$$\epsilon = f_0 1 + f_1 e^1 + f_2 e^2 + f_3 e^{12},$$

where $e^{12} = e^1 \wedge e^2$ and $f_i$, for $i = 0, 1, 2, 3$, are complex-valued functions.

### 3.2.1. Killing spinors for AdS$_4$

We start by considering the case of AdS$_4$. Let us parametrize the hyperboloid with coordinates $t, x, r, \rho$ and set

$$X_4 = \frac{\ell}{2} \sinh \rho (r + r^{-1}) \cos \ell t - r x \sin \ell t,$$

$$X_0 = \frac{\ell}{2} \sinh \rho ((r + r^{-1}) \sin \ell t + r x \cos \ell t),$$

$$X_1 = \frac{\ell}{2} \sinh \rho (r - r^{-1}) \cos \ell t - r x \sin \ell t,\quad$$

$$X_2 = \frac{\ell}{2} \sinh \rho (r - r^{-1}) \sin \ell t + r x \cos \ell t,$$

$$X_3 = \ell \sin \rho,$$

for $x, \rho \in \mathbb{R}$, $t \in [0, 2\pi)$ and $r > 0$. In these coordinates, the action of $\gamma$ consists in shifting the $x$ coordinate: $x \mapsto x + 2k$.

On pulling back the metric of $\mathbb{R}^{2,3}$ to the hyperboloid we find the following metric:

$$\ell^{-2} ds^2 = d\rho^2 + \cosh^2 \rho \left(-\left( dt + \frac{1}{2} r^2 dx \right)^2 + \frac{1}{4} r^4 dx^2 + r^2 dr^2 \right).$$
We therefore take the following orthonormal coframe:
\[
\begin{align*}
\theta^0 &= \ell \cosh \rho (dt + \frac{1}{2} \rho^2 \, dx) \\
\theta^1 &= \frac{1}{2} \ell r^2 \cosh \rho \, dx \\
\theta^2 &= \ell \, d\rho \\
\theta^3 &= \ell r^{-1} \cosh \rho \, dr.
\end{align*}
\] (21)

In this basis, the nonvanishing components of the spin connection are given by
\[
\begin{align*}
\omega_0^{02} &= \ell^{-1} \tanh \rho \\
\omega_0^{13} &= \ell^{-1} \sech \rho \\
\omega_0^{12} &= \ell^{-1} \tanh \rho \\
\omega_1^{12} &= 2 \ell^{-1} \sech \rho \\
\omega_3^{23} &= -\ell^{-1} \tanh \rho.
\end{align*}
\] (22)

The Killing spinor equation is given by
\[
(\partial_\mu + \frac{i}{2} \omega_\mu \Gamma^{\gamma \gamma_2} + \frac{1}{2} \ell^{-1} \Gamma_\mu) \epsilon = 0,
\] (23)

and it is straightforward but tedious to show that the space of the Killing spinors is spanned over \(\mathbb{C}\) by \(\epsilon_i\), for \(i = 1, \ldots, 4\), where
\[
\begin{align*}
\epsilon_1 &= 2r \left( \cosh \frac{\rho}{2} - i \sinh \frac{\rho}{2} \right) (1 + e^{i\ell x}) + 2r \left( \sinh \frac{\rho}{2} - i \cosh \frac{\rho}{2} \right) (e^1 - e^2) \\
\epsilon_2 &= 2e^{i\ell x} \left( \cosh \frac{\rho}{2} + i \sinh \frac{\rho}{2} \right) (1 - 2e^{i\ell x} \left( \sinh \frac{\rho}{2} + i \cosh \frac{\rho}{2} \right) e^2 \\
\epsilon_3 &= 2e^{-i\ell x} \left( \cosh \frac{\rho}{2} - i \sinh \frac{\rho}{2} \right) e^1 + 2e^{-i\ell x} \left( \sinh \frac{\rho}{2} - i \cosh \frac{\rho}{2} \right) e^{12}. \\
\epsilon_4 &= \frac{2}{r} (1 - i r^2 x) \left( \cosh \frac{\rho}{2} - i \sinh \frac{\rho}{2} \right) (1 + 2r \left(1 - i r^2 x\right) \left( \sinh \frac{\rho}{2} - i \cosh \frac{\rho}{2} \right) e^1 \\
&\quad - \frac{2}{r} (1 - i r^2 x) \left( \sinh \frac{\rho}{2} - i \cosh \frac{\rho}{2} \right) e^2 - 2r \left(1 - i r^2 x\right) \left( \cosh \frac{\rho}{2} - i \sinh \frac{\rho}{2} \right) e^{12x}. \\
\end{align*}
\] (24)

Note that although \(\epsilon_4\) depends linearly on \(x\), the other three basis elements are independent of \(x\). Hence, under the identification \(x \sim x + 2k\), \(\epsilon_i\), for \(i = 1, 2, 3\), remain globally well defined whereas \(\epsilon_4\) does not, resulting in one-fourth of the supersymmetry being broken.

### 3.2.2. Killing spinors for \(\text{AdS}_5\).

The calculation for \(\text{AdS}_5\) is very similar. We parametrize the hyperboloid now by coordinates \(t, x, r, \phi, \theta\) as follows:
\[
\begin{align*}
X_5 &= \frac{\ell}{2} \cosh \rho ((r + r^{-1}) \cos t - r x \sin t) \\
X_0 &= \frac{\ell}{2} \cosh \rho ((r + r^{-1}) \sin t + r x \cos t) \\
X_1 &= \frac{\ell}{2} \cosh \rho ((-r + r^{-1}) \cos t - r x \sin t) \\
X_2 &= \frac{\ell}{2} \cosh \rho ((-r + r^{-1}) \sin t + r x \cos t) \\
X_3 &= \ell \sinh \rho \cos \phi \\
X_4 &= \ell \sinh \rho \sin \phi,
\end{align*}
\] (25)

where now \(x \in \mathbb{R}, t, \phi \in [0, 2\pi)\) and \(\rho, r > 0\). Because of the condition \(\rho > 0\) this chart does not cover all of the hyperboloid: there is a codimension-2 hyperboloid missing. In these coordinates, the action of \(\rho\) again consists in shifting the \(x\) coordinate: \(x \mapsto x + 2k\).

The induced metric on the hyperboloid is now given by
\[
\ell^{-2} ds^2 = d\rho^2 + \cosh^2 \rho \left( -(dt + \frac{1}{2} r^2 \, dx)^2 + \frac{1}{2} r^4 \, dx^2 + r^{-2} \, dr^2 + \sinh^2 \rho \, d\phi^2 \right). 
\] (26)
We therefore take the following orthonormal coframe:
\[
\begin{align*}
\theta^0 &= \ell \cosh \rho \left(dt + \frac{1}{2} r^2 \, dx\right) \\
\theta^1 &= \frac{1}{2} \ell r^2 \cosh \rho \, dx \\
\theta^2 &= \ell \, d\rho \\
\theta^3 &= \ell r^{-1} \cosh \rho \, dr.
\end{align*}
\]
(27)

In this basis, the nonvanishing components of the spin connection are given by
\[
\begin{align*}
\omega_{01}^{02} &= \ell^{-1} \tanh \rho \\
\omega_{01}^{03} &= \ell^{-1} \text{sech} \rho \\
\omega_{13}^{01} &= 2 \ell^{-1} \text{sech} \rho \\
\omega_{32}^{01} &= -\ell^{-1} \tanh \rho \\
\omega_{24}^{01} &= -\ell^{-1} \text{sech} \rho \\
\omega_{34}^{23} &= -\ell^{-1} \tanh \rho
\end{align*}
\]
(28)

whereas the Killing spinor equation is again given by (23). The form of the connection is such that the Killing spinor equations for AdS₅ consist of the Killing spinor equations for AdS₄ and an extra equation involving the new coordinate ψ. Hence, the Killing spinors of AdS₅ have the form
\[
\psi = \sum_{i=1}^{4} \alpha_i(\psi) \epsilon_i,
\]
(29)

where \( \epsilon_i \) are given in (24) and are subject to the equation
\[
\left(\frac{\partial}{\partial \psi} + \frac{1}{2} \omega_\psi^{24} \Gamma_{24} + \frac{1}{2} \ell^{-1} \Gamma_{\psi}\right) \psi = 0,
\]
(30)

which merely fixes the ψ-dependence of the functions \( \alpha_i(\psi) \). In summary, the space of the Killing spinors for AdS₅ is spanned over \( \mathbb{C} \) by \( \psi_i \), for \( i = 1, 2, 3, 4 \), given by
\[
\begin{align*}
\psi_1 &= e^{i\psi/2} \epsilon_1, \\
\psi_2 &= e^{i\psi/2} \epsilon_2, \\
\psi_3 &= e^{i\psi/2} \epsilon_3 \\
\psi_4 &= e^{i\psi/2} \epsilon_4.
\end{align*}
\]
(31)

where the sign corresponds to the choice of irreducible representation of \( \text{Cℓ}(1, 4) \) and where \( \epsilon_i \) are again given in (24).

It would appear that \( \psi_i \) are not globally well defined, as they are antiperiodic when one takes \( \psi \mapsto \psi + 2\pi \). However, this is merely an artefact of the choice of basis. To see this, make the \( \text{Spin}(1, 4) \) gauge transformation
\[
\psi_i \mapsto \psi'_i = e^{-\frac{1}{2} \Gamma_{24}} \psi_i,
\]
(32)

which induces a rotation by angle \( \varphi \) in the (24) plane at the level of the coframes. Under this gauge transformation, the Killing spinors are now 2π-periodic in \( \varphi \) and, as a bonus, have a well-defined limit as \( \rho \to 0 \):
\[
\begin{align*}
\psi'_1 &= 2 \ell \left(\cosh \frac{\rho}{2} - i e^{\pm i \psi} \sinh \frac{\rho}{2}\right) \left(1 + e^{12}\right) + 2 \ell \left(e^{\pm i \psi} \sinh \frac{\rho}{2} - i \cosh \frac{\rho}{2}\right) \left(e^1 - e^2\right) \\
\psi'_2 &= 2 e^{i\varphi} \left(\cosh \frac{\rho}{2} + i e^{\pm i \psi} \sinh \frac{\rho}{2}\right) \left(1 - 2 e^{i\varphi} \left(e^{\pm i \psi} \sinh \frac{\rho}{2} + i \cosh \frac{\rho}{2}\right) e^2\right) \\
\psi'_3 &= 2 e^{-i\varphi} \left(\cosh \frac{\rho}{2} - i e^{\pm i \psi} \sinh \frac{\rho}{2}\right) e^1 + 2 e^{-i\varphi} \left(e^{\pm i \psi} \sinh \frac{\rho}{2} - i \cosh \frac{\rho}{2}\right) e^{12} \\
\psi'_4 &= \frac{2}{r} (1 - i \varphi^2) \left(\cosh \frac{\rho}{2} - i e^{\pm i \psi} \sinh \frac{\rho}{2}\right) 1 - 2 \frac{2}{r} (1 + i \varphi^2) \left(e^{\pm i \psi} \sinh \frac{\rho}{2} - i \cosh \frac{\rho}{2}\right) e^1 \\
&\quad - 2 \frac{2}{r} (1 - i \varphi^2) \left(e^{\pm i \psi} \sinh \frac{\rho}{2} - i \cosh \frac{\rho}{2}\right) e^2 \\
&\quad - 2 \frac{2}{r} (1 + i \varphi^2) \left(\cosh \frac{\rho}{2} - i e^{\pm i \psi} \sinh \frac{\rho}{2}\right) e^{12}.
\end{align*}
\]
(33)
Again we note that whereas $\psi'_i$, for $i = 1, 2, 3$, do not depend on $x$, $\psi'_4$ depends linearly on $x$, whence it is not invariant under shifting $x$. As before, we conclude that precisely one-fourth of the supersymmetry is broken.

4. Conclusion

In this paper, we have shown that, contrary to expectations, there exist backgrounds of gauged four- and five-dimensional supergravities which are $\frac{1}{4}$-BPS. This was done by exhibiting a family of $\mathbb{Z}$-quotients of the hyperboloid model for AdS$_d$ preserving a $\frac{1}{d}$-dimensional subspace of the Killing spinors. In terms of the simply-connected AdS$_d$, these constitute a family of $(\mathbb{Z} \oplus \mathbb{Z})$-quotients, parametrized by the positive real numbers.

This result was unexpected because, as shown in [10, 11], such backgrounds are locally maximally supersymmetric, and the similar result in ten- and eleven-dimensional supergravities [5, 6] was shown to survive even when the spacetime is not simply connected [8]. In the cases studied here, however, the supersymmetry is broken by global effects and to our knowledge these are the first known cases where a fraction of supersymmetry previously discarded by a local analysis of the Killing spinor equations is resurrected by the topology of the underlying spacetime. It should also sound a note of caution concerning the use of the holonomy algebra to rule out supersymmetry fractions.

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