On the Homology of the Dual de Rham Complex

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Dedicated to Professor Banghe Li on His 80th Birthday

Abstract We study the homology of the dual de Rham complex as functors on the category of abelian groups. We give a description of homology of the dual de Rham complex up to degree 7 for free abelian groups and present a corrected version of the proof of Jean’s computations of the zeroth homology group.

Keywords Dual de Rham complex, polynomial functor, divided powers, derived functors

MR(2010) Subject Classification 18G10

1 Introduction

1.1 Divided Power Functor

Let Ab be the category of abelian groups. Recall the definition of the graded divided power functor (see [9]) $\Gamma_* = \bigoplus_{n \geq 0} \Gamma_n : \text{Ab} \to \text{Ab}$. The graded abelian group $\Gamma_*(A)$ is generated by symbols $\gamma_i(x)$ of degree $i \geq 0$ satisfying the following relations for all $x, y \in A$:

1) $\gamma_0(x) = 1$,
2) $\gamma_1(x) = x$,
3) $\gamma_s(x)\gamma_t(x) = \binom{s + t}{s} \gamma_{s+t}(x)$,
4) $\gamma_n(x + y) = \sum_{s+t=n} \gamma_s(x)\gamma_t(y)$, $n \geq 1$,
5) $\gamma_n(-x) = (-1)^n \gamma_n(x)$, $n \geq 1$.

In particular, the canonical map $A \simeq \Gamma_1(A)$ is an isomorphism. The following additional properties of elements of the abelian group $\Gamma(A)$ will be useful ($x, y \in A, r \geq 1$):

$\gamma_r(nx) = n^r \gamma_r(x), \quad n \in \mathbb{Z}$.
\[ r\gamma_r(x) = x\gamma_{r-1}(x); \]
\[ x^r = r! \gamma_r(x); \]
\[ \gamma_r(x)y^r = x^r\gamma_r(y). \]

A direct computation implies that
\[ \Gamma_r(\mathbb{Z}/n) \simeq \mathbb{Z}/n(r, n^\infty), \]
where \((r, n^\infty)\) is the limit \(\lim_{m \to \infty} (r, n^m)\). The degree 2 component \(\Gamma_2(A)\) of the divided power algebra is the Whitehead functor \(\Gamma(A)\). It is the universal group for homogenous quadratic maps from \(A\) into abelian groups.

### 1.2 Dual de Rham complex

Let \(A\) be an abelian group. For \(n \geq 1\), denote by \(\text{SP}^n\) and \(\Lambda^n\) the \(n\)th symmetric and exterior power functors respectively. For \(n \geq 1\), let \(D^n_i(A)\) and \(C^n_i(A)\) be the complexes of abelian groups defined by

\[ D^n_i(A) = \text{SP}^i(A) \otimes \Lambda^{n-i}(A), \quad 0 \leq i \leq n, \]
\[ C^n_i(A) = \Lambda^i(A) \otimes \Gamma_{n-i}(A), \quad 0 \leq i \leq n, \]

where the differentials \(d_i : D^n_i(A) \to D^n_{i-1}(A)\) and \(d_i : C^n_i(A) \to C^n_{i-1}(A)\) are:

\[ d_i((b_1 \ldots b_i) \otimes b_{i+1} \wedge \cdots \wedge b_n) = \sum_{k=1}^{i} (b_1 \ldots \hat{b}_k \ldots b_i) \otimes b_k \wedge b_{i+1} \wedge \cdots \wedge b_n, \]
\[ d_i(b_1 \wedge \cdots \wedge b_i \otimes X) = \sum_{k=1}^{i} (-1)^k b_1 \wedge \cdots \wedge \hat{b}_k \wedge \cdots \wedge b_i \otimes b_k X \]

for any \(X \in \Gamma_{n-i}(A)\). The complex \(D^n(A)\) is the degree \(n\) component of the classical de Rham complex, first introduced in the present context of polynomial functors in [4] and denoted \(\Omega_n\) in [5]. The dual complexes \(C^n(A)\) were considered in [6]. We will call them the dual de Rham complexes.

The dual de Rham complexes appear naturally in the theory of homology of Eilenberg–Mac Lane spaces. Let \(A\) be a free abelian group. There are well-known natural isomorphisms (see, for example, [1]):

\[ H_n K(A, 1) = \Lambda^n(A), \quad n \geq 1, \]
\[ H_{2n} K(A, 2) = \Gamma_n(A), \quad n \geq 1, \]
\[ H_{2n+1} K(A, 2) = 0, \quad n \geq 0. \]

Consider the path-fibration:
\[ K(A, 2) \to PK(A, 3) \to K(A, 3) \]

and the homology spectral sequence
\[ E^2_{p,q} = H_p K(A, 3) \otimes H_q K(A, 2) \Rightarrow \mathbb{Z}[0]. \]
The dual de Rham complexes can be recognized as natural parts of the $E^3$-term of this spectral sequence. For example, we have the following natural diagrams:

$$
\begin{array}{c}
E_{0,0}^3 \xrightarrow{d_{0,0}^3} E_{2,0}^3 \xrightarrow{d_{2,0}^3} E_{0,4}^3 \\
\Lambda^2(A) \xrightarrow{d_2} A \otimes A \xrightarrow{d_1} \Gamma_2(A), \\
E_{3,0}^3 \xrightarrow{d_{3,0}^3} E_{6,2}^3 \xrightarrow{d_{6,2}^3} E_{3,5}^3 \xrightarrow{d_{3,4}^3} E_{0,6}^3 \\
\Lambda^3(A) \xrightarrow{d_3} \Lambda^2(A) \otimes A \xrightarrow{d_2} A \otimes \Gamma_2(A) \xrightarrow{d_1} \Gamma_3(A).
\end{array}
$$

We will now give a functorial description of certain homology groups of these complexes $C^n(A)$. For some applications of these results in the theory of derived functors one can read [2].

**Proposition 1.1** Let $A$ be a free abelian. Then

1. [5] For any prime number $p$, $H_0C^n(A) = A \otimes \mathbb{Z}/p$, and $H_iC^n(A) = 0$, for all $i > 0$;
2. [6] There is a natural isomorphism

$$H_0C^n(A) \simeq \bigoplus_{p | n, p \text{ prime}} \Gamma_{n/p}(A \otimes \mathbb{Z}/p).$$

We will make use of the following fact from number theory (see [7, Corollary 2]):

**Lemma 1.2** Let $n$ and $k$ be a pair of positive integers and $p$ is a prime number, then

$$\binom{pn}{pk} \equiv \binom{n}{k} \mod p^r,$$

where $r$ is the largest power of $p$ dividing $pnk(n-k)$.

**Proof of Proposition 1.1 (2)** Let $n \geq 2$ and define the map

$$q_n : \Gamma_n(A) \to \bigoplus_{p | n} \Gamma_{n/p}(A \otimes \mathbb{Z}/p)$$

by setting:

$$q_n : \gamma_{i_1}(a_1) \cdots \gamma_{i_t}(a_t) \mapsto \sum_{p | i_k, \text{ for all } 1 \leq k \leq t} \gamma_{i_1/p}(\bar{a}_1) \cdots \gamma_{i_t/p}(\bar{a}_t),$$

where $\bar{a}_k \in A \otimes \mathbb{Z}/p$.

If $(i_1, \ldots, i_t) = 1$, then we set

$$q_n(\gamma_{i_1}(a_1) \cdots \gamma_{i_t}(a_t)) = 0, \quad \text{where } a_k \in A \text{ for all } k.$$

Let us check that the map $q_n$ is well-defined. For that we have to show that

$$q_n(\gamma_{j_1}(x) \gamma_{j_2}(x) \cdots \gamma_{j_t}(x_t)) = q_n\left( \binom{j_1 + j_2}{j_1} \gamma_{j_1+j_2}(x) \cdots \gamma_{j_t}(x_t) \right) \quad \text{(1.1)}$$
\[
q_n(\gamma_{j_1}(x_1 + y_1) \cdots \gamma_{j_t}(x_t)) = \sum_{k+l=j} q_n(\gamma_k(x_1)\gamma_l(y_1) \cdots \gamma_{j_t}(x_t))
\]  
(1.2)

\[
q_n(\gamma_{j_1}(-x_1) \cdots \gamma_{j_t}(x_t)) = (-1)^j q_n(\gamma_{j_1}(x_1) \cdots \gamma_{j_t}(x_t))
\]  
(1.3)

Verification of (1.1). First suppose that \( p \mid j_1 + j_2, \ p \nmid j_1 \). Since for every pair of numbers \( n \geq k \), one has
\[
\frac{n}{(n, k)} \left( \begin{array}{c} n \\ k \end{array} \right),
\]
so we have
\[
\left( \begin{array}{c} j_1 + j_2 \\ j_1 \end{array} \right) = p^s m_1, \quad j_1 + j_2 = p^s m_2, \quad (m_1, p) = (m_2, p) = 1, \quad s \geq 1.
\]
Hence \( \left( \begin{array}{c} j_1 + j_2 \\ j_1 \end{array} \right) m_2/m_1 = p \left( \begin{array}{c} j_1 + j_2 \\ j_1 \end{array} \right) \). Observe that \( \Gamma_{j_1+j_2} (A \otimes \mathbb{Z}/p) \) is a \( p \)-group, hence
\[
\left( \begin{array}{c} j_1 + j_2 \\ j_1 \end{array} \right) \frac{m_2}{m_1} \gamma_{j_1+j_2} (\bar{x}) = 0, \quad \bar{x} \in A \otimes \mathbb{Z}/p
\]
(1.5)

since
\[
\left( \begin{array}{c} j_1 + j_2 \\ j_1 \end{array} \right) \frac{m_2}{m_1} \gamma_{j_1+j_2} (\bar{x}) = p \bar{x} \gamma_{j_1+j_2-1} (\bar{x}) = 0.
\]
The equality (1.5) implies that
\[
q_n \left( \left( \begin{array}{c} j_1 + j_2 \\ j_1 \end{array} \right) \gamma_{j_1+j_2}(x) \cdots \gamma_{j_t}(x_t) \right) - q_n (\gamma_{j_1}(x) \gamma_{j_2}(x) \cdots \gamma_{j_t}(x_t))
\]
\[
= \left( \begin{array}{c} j_1 + j_2 \\ j_1 \end{array} \right) \sum_{p \mid j_1+j_2, \ p \mid j_k, \ k > 2} \gamma_{j_1+j_2} (\bar{x}) \cdots \gamma_{j_t/p}(\bar{x}_t)
\]
\[
- \sum_{p \mid j_k, \ \text{for all} \ 1 \leq k \leq t} \gamma_{j_1/p}(\bar{x}) \gamma_{j_2/p}(\bar{x}) \cdots \gamma_{j_t/p}(\bar{x}_t)
\]
\[
= \left( \begin{array}{c} j_1 + j_2 \\ j_1 \end{array} \right) - \left( \begin{array}{c} j_1 + j_2 \\ p \end{array} \right) \sum_{p \mid j_k, \ \text{for all} \ 1 \leq k \leq t} \gamma_{j_1+j_2} (\bar{x}) \cdots \gamma_{j_t/p}(\bar{x}_t).
\]

Let \( j_1 + j_2 = p^r m, \ (m, p) = 1 \). Lemma 1.2 implies that
\[
\left( \begin{array}{c} j_1 + j_2 \\ j_1 \end{array} \right) = \left( \begin{array}{c} j_1 + j_2 \\ p \end{array} \right) \ mod \ p^r,
\]
where \( r \) is the largest power of \( p \), dividing \( (j_1 + j_2)j_1j_2/p^2 \). Since \( p \mid j_1, \ p \nmid j_2 \), we have
\[
\left( \begin{array}{c} j_1 + j_2 \\ j_1 \end{array} \right) = \left( \begin{array}{c} j_1 + j_2 \\ p \end{array} \right) \ mod \ p^r.
\]
Hence
\[
\left( \left( \begin{array}{c} j_1 + j_2 \\ j_1 \end{array} \right) - \left( \begin{array}{c} j_1 + j_2 \\ p \end{array} \right) \right) \gamma_{j_1+j_2/p}(\bar{x}) = 0, \quad \bar{x} \in A \otimes \mathbb{Z}/p
\]
and the property (1.1) follows.

Verification of (1.2). We have
\[
q_n (\gamma_{j_1}(x_1 + y_1) \cdots \gamma_{j_t}(x_t)) - \sum_{k+l=j} q_n (\gamma_k(x_1)\gamma_l(y_1) \cdots \gamma_{j_t}(x_t))
\]
This implies that the cross-effect

\[ \gamma_{j_1/p}(\bar{x}_1 + \bar{y}_1) \cdots \gamma_{j_t/p}(\bar{x}_t) - \sum_{k+t=j_1 \atop k,p|l,p|j_1, t \geq 2} \sum_{\gamma_{k/p}(\bar{x}_1) \gamma_{l/p}(\bar{y}_1) \cdots \gamma_{j_t/p}(\bar{x}_t)} \]

\[ = \sum_{p|j_t, t \geq 1} \gamma_{j_1/p}(\bar{x}_1 + \bar{y}_1) \cdots \gamma_{j_t/p}(\bar{x}_t) - \sum_{p|k,p|l,p|j_t, t \geq 2} \sum_{\gamma_{k/p}(\bar{x}_1) \gamma_{l/p}(\bar{y}_1) \cdots \gamma_{j_t/p}(\bar{x}_t)} \]

\[ = 0. \]

Verification of (1.3). We have

\[ q_n(\gamma_{j_1}(x_1) \cdots \gamma_{j_t}(x_t)) - (-1)^{j_1} q_n(\gamma_{j_1}(x_1) \cdots \gamma_{j_t}(x_t)) \]

\[ = \sum_{p|j_t, k \geq 1} (\gamma_{j_1/p}(\bar{x}_1) - (-1)^{j_1} \gamma_{j_1/p}(\bar{x}_1)) \cdots \gamma_{j_t/p}(\bar{x}_t) = 0 \]

since

\[ (\gamma_{j_1/p}(\bar{x}_1) - (-1)^{j_1} \gamma_{j_1/p}(\bar{x}_1)) = 0 \]

(we separately check the cases \( p = 2 \) and \( p \neq 2 \)).

We now know that the map \( q_n \) is well-defined. It induces a map

\[ \bar{q}_n : H^0 C^n(A) \to \bigoplus_{p|n} \Gamma_n/p(A \otimes \mathbb{Z}/p), \]

since \( q_n(a) = 0 \) for every \( a \in \text{im}\{A \otimes \Gamma_{n-1}(A) \to \Gamma_n(A)\} \).

Let \( A = \mathbb{Z} \). Then

\[ H_0(\mathbb{Z}) = \text{coker}\{\Gamma_{n-1}(\mathbb{Z}) \otimes \mathbb{Z} \to \Gamma_n(\mathbb{Z})\} \simeq \text{coker}\{\mathbb{Z} \to \mathbb{Z} \} \simeq \mathbb{Z}/n. \]

Let \( n = \prod p_i^{s_i} \) be the prime decomposition of \( n \). Then

\[ \bigoplus_{p_i|n} \Gamma_{n/p_i}(\mathbb{Z}/p_i) = \bigoplus_{p_i|n} \mathbb{Z}/p_i = \mathbb{Z}/n. \]

It follows from definition of the map \( \bar{q}_n \), that

\[ \bar{q}_n : H_0 C^n(\mathbb{Z}) \to \bigoplus_{p_i|n} \Gamma_{n/p_i}(\mathbb{Z}/p_i) \]

is an isomorphism.

For free abelian groups \( A \) and \( B \), one has a natural isomorphism of complexes

\[ C^n(A \oplus B) \simeq \bigoplus_{i+j=n, i,j \geq 0} C^i(A) \otimes C^j(B). \]

This implies that the cross-effect\(^1\) \( C^n(A|B) \) of the functor \( C^n(A) \) is described by

\[ C^n(A|B) \simeq \bigoplus_{i+j=n, i,j > 0} C^i(A) \otimes C^j(B) \]

and its homology \( (H_k C^n)(A|B) = H_k C^n(A|B) \) can be described with the help of Künneth formulas:

\[ 0 \to \bigoplus_{i+j=n, i,j > 0, r+s=k} H_r C^i(A) \otimes H_s C^j(B) \to H_k C^n(A|B) \]

\(^1\) Given a functor \( F : \text{Ab} \to \text{Ab} \), its cross effect is defined as the kernel of the natural map \( F(A|B) = \ker\{F(A \oplus B) \to F(A) \oplus F(B)\} \), \( A, B \in \text{Ab} \).
On the other hand, we have the following decomposition of the cross-effect of $H_0C^n(A)$:

$$H_0C^n(A|B) \simeq \bigoplus_{i+j=n, \; i,j > 0} H_0C^i(A) \otimes H_0C^j(B). \quad (1.6)$$

On the other hand, we have the following decomposition of the cross-effect of the functor $\tilde{\Gamma}^p_{n/p}(A) := \Gamma_{n/p}(A \otimes \mathbb{Z}/p)$:

$$\tilde{\Gamma}^p_{n/p}(A|B) = \bigoplus_{l+k=n/p} \Gamma_l(A \otimes \mathbb{Z}/p) \otimes \Gamma_k(B \otimes \mathbb{Z}/p).$$

Hence

$$\bigoplus_{p|n} \tilde{\Gamma}^p_{n/p}(A|B) = \bigoplus_{p|n} \bigoplus_{l+k=n/p} \Gamma_l(A \otimes \mathbb{Z}/p) \otimes \Gamma_k(B \otimes \mathbb{Z}/p) \quad (1.7)$$

We must now show that the maps $\bar{q}_n$ preserve the decompositions (1.6) and (1.7). This is equivalent to the commutativity of the following diagram:

$$\begin{array}{ccc}
H_0C^i(A) \otimes H_0C^j(B) & \xrightarrow{\bar{q}_i \otimes \bar{q}_j} & \bigoplus_{p|i} \Gamma_{i/p}(A \otimes \mathbb{Z}/p) \otimes \bigoplus_{p|j} \Gamma_{j/p}(B \otimes \mathbb{Z}/p) \\
\downarrow & & \downarrow \varepsilon' \\
H_0(C^i(A) \otimes C^j(B)) & \xrightarrow{\bar{q}_{i+j}} & \bigoplus_{p|(i+j)} \Gamma_{i+j/p}((A \otimes B) \otimes \mathbb{Z}/p)
\end{array} \quad (1.8)$$

The map $\varepsilon' \circ (\bar{q}_i \otimes \bar{q}_j)$ is defined via the natural map

$$\prod \gamma_{i_k}(x_k) \otimes \prod \gamma_{j_k}(y_k) \mapsto \sum_{p|i_k} \sum_{p|j_k} \prod \gamma_{i_k/p}(\tilde{x}_i) \prod \gamma_{j_k/p}(\tilde{y}_i) \in \bigoplus_{p|(i+j)} \Gamma_{i+j/p}((A \otimes B) \otimes \mathbb{Z}/p),$$

$$x_k \in A, \quad y_k \in B, \quad \tilde{x}_i \in A \otimes \mathbb{Z}/p, \quad \tilde{y}_i \in B \otimes \mathbb{Z}/p, \quad \sum j_k = j, \quad \sum i_k = i$$

and the commutativity of the diagram (1.8) follows. This proves that the natural map

$$H_0C^n(A|B) \simeq \bigoplus_{i+j=n, \; i,j > 0} H_0C^i(A) \otimes H_0C^j(B)$$

$$\rightarrow \bigoplus_{p|n} \tilde{\Gamma}^p_{n/p}(A|B) = \bigoplus_{p|n} \bigoplus_{l+k=n/p} \Gamma_l(A \otimes \mathbb{Z}/p) \otimes \Gamma_k(B \otimes \mathbb{Z}/p)$$

induced by $\bar{q}_n$ on cross-effects is an isomorphism, and it follows from this that $\bar{q}_n$ is an isomorphism for all free abelian groups $A$. \hfill \Box

1.3 Derived Functors and Homology

Let $A$ be an abelian group, and $F$ an endofunctor on the category of abelian groups. Recall that for every $n \geq 0$ the derived functor of $F$ in the sense of Dold–Puppe [3] are defined by

$$L_i F(A,n) = \pi_i(FKP_n[r]), \quad i \geq 0$$
where $P_* \to A$ is a projective resolution of $A$, and $K$ is the Dold-Kan transform, inverse to the Moore normalization functor

$$N : \text{Simpl}(\text{Ab}) \to C(\text{Ab})$$

from simplicial abelian groups to chain complexes.

Recall the description of the highest derived functors of the tensor power functor due to Mac Lane [8]. The group $\text{Tor}^n(A)$ is generated by the $n$-linear expressions $\tau_h(a_1, \ldots, a_n)$ (where all $a_i$ live in the subgroup $hA$ of elements of $A$ for which $ha = 0$ ($h > 0$), subject to the so-called slide relations

$$\tau_{hk}(a_1, \ldots, a_i, \ldots, a_n) = \tau_h(ka_1, \ldots, ka_i, a_{i+1}, \ldots, ka_n) \tag{1.9}$$

for all $i$ whenever $hka_j = 0$ for all $j \neq i$ and $ha_i = 0$. The associativity of the derived tensor product functor implies that there are canonical isomorphisms

$$\text{Tor}^n(A) \simeq \text{Tor}(\text{Tor}^{n-1}(A), A), \quad n \geq 2.$$ 

For $n \geq 2$, there is a natural isomorphism:

$$L_{n-1} \otimes^n (A) \simeq \text{Tor}^n(A).$$

The map $\otimes^n \to \text{SP}^n$ induces a natural epimorphism

$$\text{Tor}^n(A) \to L_{n-1} \text{SP}^n(A) \tag{1.10}$$

which sends the generators $\tau_h(a_1, \ldots, a_n)$ of $\text{Tor}^n(A)$ to generators $\beta_h(a_1, \ldots, a_n)$ of

$$\mathcal{S}_n(A) := L_{n-1} \text{SP}^n(A).$$

The kernel of this map is generated by the elements $\tau_h(a_1, \ldots, a_n)$ with $a_i = a_j$ for some $i \neq j$. It is shown by Jean in [6] that

$$L_i \text{SP}^n(A) \simeq (L_{i+1} \text{SP}^n(A) \otimes \text{SP}^{n-(i+1)}(A))/\text{Jac}_{\text{SP}}, \tag{1.11}$$

where $\text{Jac}_{\text{SP}}$ is the subgroup generated by elements of the form

$$\sum_{k=1}^{i+2} (-1)^k \beta_h(x_1, \ldots, x_k, \ldots, x_{i+2}) \otimes x_ky_1 \cdots y_{n-i-2}.$$ 

with $x_i \in hA$ and $y_j \in A$ for all $i, j$.

We will now construct a series of maps:

$$f_i^{n,p} : L_i \text{SP}^{n/p}(A \otimes \mathbb{Z}/p) \to H_i C^n(A)$$

for a free abelian $A$ and $p | n$. We first choose liftings $(x_i)$ to $A$ of a given family of elements $(\bar{x}_i) \in A \otimes \mathbb{Z}/p$. We set

$$f_i^{n,p} : \beta_p(\bar{x}_1, \ldots, \bar{x}_{i+1}) \otimes \bar{x}_{i+2} \cdots \bar{x}_p \mapsto \eta_i(x_1, \ldots, x_p)$$

:= \sum_{t=1}^{i+1} (-1)^t x_1 \wedge \cdots \wedge \hat{x}_t \wedge \cdots \wedge x_{i+1} \otimes \gamma_{p-1}(x_1) \cdots \gamma_{p-1}(x_{i+1})$$

$$\cdot \gamma_p(x_t) \gamma_p(x_{i+2}) \cdots \gamma_p(x_p)$$

$$\in A^i(A) \otimes \Gamma_{n-i}(A), \quad \bar{x}_k \in A \otimes \mathbb{Z}/p, \quad x_k \in A.$$
Proposition 1.3  The maps $f_{i}^{∗, p}$ are well defined for all $i, n, p$.

Proof  We have

\[ \eta_i(px_1, \cdots, x_{\frac{n}{p}}) = \sum_{t=1}^{i+1} (-1)^t px_1 \wedge \cdots \wedge x_t \wedge x_{i+1} \otimes \gamma_{p-1}(x_1) \cdots \gamma_{p-1}(x_t) \gamma_p(x_{i+1}) \gamma_p(x_t) \]

\[ \cdot \gamma_p(x_{i+2}) \cdots \gamma_p(x_{\frac{n}{p}}) \]

\[ \sum_{t=1}^{i+1} (-1)^t x_1 \wedge \cdots \wedge x_t \wedge x_{i+1} \otimes x_t \gamma_{p-1}(x_1) \cdots \gamma_{p-1}(x_t) \gamma_p(x_{i+1}) \gamma_p(x_{i+2}) \cdots \gamma_p(x_{\frac{n}{p}}) \]

\[ \in \text{im}(\Lambda^{i+1}(A) \otimes \Gamma_{n-i-1}(A) \xrightarrow{d_{i+1}} \Lambda^i(A) \otimes \Gamma_{n-i}(A)). \]

One verifies that for every $1 \leq k \leq n/p$, one has

\[ \eta_i(x_1, \ldots, px_k, \ldots, x_{\frac{n}{p}}) \in \text{im}(\Lambda^{i+1}(A) \otimes \Gamma_{n-i-1}(A) \xrightarrow{d_{i+1}} \Lambda^i(A) \otimes \Gamma_{n-i}(A)). \]

It follows that the map $f_{i}^{∗, p} : L_{i}P^{n/p}(A \otimes \mathbb{Z}/p) \rightarrow (\Lambda^i(A) \otimes \Gamma_{n-i}(A))/\text{im}(d_{i+1})$ is well-defined.

The simplest examples of such elements are the following

\[ \eta_1(x_1, x_2) = x_1 \otimes x_1 \gamma_2(x_2) - x_2 \otimes x_2 \gamma_2(x_1) \in A \otimes \Gamma_2(A), \]

\[ \eta_2(x_1, x_2, x_3) = -x_1 \wedge x_2 \otimes x_1 x_2 \gamma_2(x_3) + x_1 \wedge x_3 \otimes x_1 x_3 \gamma_2(x_2) \]

\[ -x_2 \wedge x_3 \otimes x_2 x_3 \gamma_2(x_1) \in \Lambda^2(A) \otimes \Gamma_4(A). \]

By construction, the elements $\eta_i$ lie in $\Lambda^i(A) \otimes \Gamma_{n-i}(A)$. In fact, let us verify that

\[ \eta_i(x_1, \ldots, x_{\frac{n}{p}}) \in \ker(\Lambda^i(A) \otimes \Gamma_{n-i}(A) \xrightarrow{d_{i}} \Lambda^{i-1}(A) \otimes \Gamma_{n-i+1}(A)) \]

Observe that

\[ d_i \eta_i(x_1, \cdots, x_{\frac{n}{p}}) \]

\[ = d_i \sum_{t=1}^{i+1} (-1)^t x_1 \wedge \cdots \wedge x_t \wedge x_{i+1} \otimes \gamma_{p-1}(x_1) \cdots \gamma_{p-1}(x_t) \gamma_p(x_{i+1}) \gamma_p(x_t) \]

\[ \cdot \gamma_p(x_{i+2}) \cdots \gamma_p(x_{\frac{n}{p}}). \]

In this sum, for every pair of indexes $1 \leq r < s \leq i+1$ there occurs a pair of terms $(-1)^{s+r} x_1 \wedge \cdots \wedge x_r \wedge x_{s+1} \otimes x_r \gamma_{p-1}(x_1) \cdots \gamma_{p-1}(x_s) \gamma_p(x_{s+1}) \gamma_p(x_r) \gamma_p(x_{i+2}) \cdots \gamma_p(x_{\frac{n}{p}})$ and $(-1)^{s+r+1} x_1 \wedge \cdots \wedge x_r \wedge x_{s+1} \otimes x_s \gamma_{p-1}(x_1) \cdots \gamma_{p-1}(x_r) \gamma_p(x_{s+1}) \gamma_p(x_r) \gamma_p(x_{i+2}) \cdots \gamma_p(x_{\frac{n}{p}})$ which cancel each other. It follows that the entire sum is equal to zero.

For the same reason, the map

\[ \sum_{k=1}^{i+2} (-1)^k \beta_k(x_1, \ldots, \hat{x}_k, \ldots, x_{i+2}) \otimes x_k y_1 \cdots y_l \]

\[ \mapsto \sum_{k=1}^{i+2} (-1)^k \sum_{m=1}^{i+2} (-1)^m x_1 \wedge \cdots \hat{x}_m \wedge \cdots \wedge x_{i+2} \]
The map $f_i^n$ is an isomorphism for $i > 0$, $n \leq 7$.

**Proof** Given an abelian simplicial group $(G_\bullet, \partial, s_\bullet)$, let $A(G_\bullet)$ be the associated chain complex with $A(G_n) = G_n$, $d_n = \sum_{i=0}^n (-1)^n \partial_i$. Recall that given abelian simplicial groups $G_\bullet$ and $H_\bullet$, the Eilenberg–Mac Lane map

$$g : A(G_\bullet) \otimes A(H_\bullet) \to A(G_\bullet \otimes H_\bullet)$$

is given by

$$g(a_p \otimes b_q) = \sum \left((-1)^{\text{sign}(\mu, \nu)} s_{\nu_q} \cdots s_{\nu_1}(a_p) \otimes s_{\mu_p} \cdots s_{\mu_1}(b_q)\right).$$

For any free abelian group $A$ and infinite cyclic group $B$, we will show that there is a natural commutative diagram with vertical Künneth short exact sequences:

\[
\begin{array}{cccc}
\bigoplus_{i+j=0, r+s=k-1} & L_\cdot SP^i(A \otimes \mathbb{Z}/p) \otimes L_\cdot SP^j(B \otimes \mathbb{Z}/p) & \rightarrow & \bigoplus_{i+j=n, r+s=k} H_\cdot C^i(A) \otimes H_\cdot C^j(B) \\
\downarrow & L_kSP^\#((A \oplus B) \otimes \mathbb{Z}/p) & \rightarrow & H_kC^n(A \oplus B) \\
\bigoplus_{i+j=0, r+s=k-1} & \text{Tor}(L_\cdot SP^i(A), L_\cdot SP^j(B)) & \rightarrow & \bigoplus_{i+j=n, \ r+s=k-1} \text{Tor}(H_\cdot C^i(A), H_\cdot C^j(B)) \\
\end{array}
\]

where all maps are induced by maps $f_{i,p}^n$. Since $B$ is a cyclic, it is enough to consider the case $s = 0$ and summands of the upper square from (1.12)

\[
L_\cdot SP^i(A \otimes \mathbb{Z}/p) \otimes SP^j(B \otimes \mathbb{Z}/p) \xrightarrow{f_{i,p}^n \otimes f_{j,p}^n} H_\cdot C^i(A \otimes \mathbb{Z}/p) \otimes H_\cdot C^j(B \otimes \mathbb{Z}/p)
\]

where the maps $s_r, h_r$ come from Künneth exact sequences. Consider natural projections

$$u_1 : L_rSP^{r+1}(A \otimes \mathbb{Z}/p) \otimes SP^{i-1}(A \otimes \mathbb{Z}/p) \rightarrow L_rSP^i(A \otimes \mathbb{Z}/p)$$
\[ u_2 : L_r \text{SP}^{r+1}((A \otimes B) \otimes \mathbb{Z}/p) \otimes \text{SP}^{i+j-r-1}((A \otimes B) \otimes \mathbb{Z}/p) \to L_r \text{SP}^{i+j}((A \otimes B) \otimes \mathbb{Z}/p) \]

The map \( s_r \) is defined by
\[
s_r : u_1(\beta_p(a_1, \ldots, a_{r+1}) \otimes a_{r+2} \cdot \cdot a_i) \otimes b_1 \cdot \cdot b_j \mapsto u_2(\beta_p(a_1, \ldots, a_{r+2}) \otimes a_{r+2} \cdot \cdot a_i b_1 \cdot \cdot b_j),
\]
\[ a_k \in A \otimes \mathbb{Z}/p, \; b_l \in B \otimes \mathbb{Z}/p. \]

We have
\[
f_r^{\text{ip}, p} \otimes f_0^{\text{ip}, p}(u_1(\beta_p(a_1, \ldots, a_{r+1}) \otimes a_{r+2} \cdot \cdot a_i) \otimes b_1 \cdot \cdot b_j) = \eta_r(a_1, \ldots, a_i) \otimes \gamma_p(b_1) \cdot \cdot \gamma_p(b_j)
\]
and we see that the diagram (1.13) is commutative.

The map \( f^n_1 \) is an isomorphism for a cyclic group \( A \), since both source and target groups are trivial. For \( i = 1, n = 4 \), and cyclic \( B \), we have a natural diagram
\[
\begin{align*}
L_1\text{SP}^2(A \otimes \mathbb{Z}/2) & \to H_1\text{C}^4(A; B) \\
\text{Tor}(A \otimes \mathbb{Z}/2, B \otimes \mathbb{Z}/2) & \to \text{Tor}(H_0\text{C}^2(A), H_0\text{C}^2(B))
\end{align*}
\]
and the isomorphism \( f_1^4 \) follows. The proof is similar for other \( i, n \). The only non-trivial case here is \( i = 1, n = 6 \), for the 2-torsion component of \( H_1\text{C}^6(A) \). In that case, the statement follows from the natural isomorphism
\[
\text{Tor}(\text{SP}^2(A \otimes \mathbb{Z}/2), \mathbb{Z}/2) \to \text{Tor}(\Gamma_2(A \otimes \mathbb{Z}/2), \mathbb{Z}/2)
\]
for every free abelian group \( A \).

**Remark 1.5** For any free abelian group \( A \) and prime number \( p \), there are canonical isomorphisms
\[
L_{n-1}\text{SP}^n(A \otimes \mathbb{Z}/p) \simeq \Lambda^n(A \otimes \mathbb{Z}/p)
\]
\[
L_i\text{SP}^n(A \otimes \mathbb{Z}/p) = \text{coker}\left\{ \Lambda^{i+2}(A \otimes \mathbb{Z}/p) \otimes \text{SP}^{n-i-2}(A \otimes \mathbb{Z}/p) \right\}
\]
\[
\kappa_i^+ \text{A}^{i+1}(A \otimes \mathbb{Z}/p) \otimes \text{SP}^{n-i-1}(A \otimes \mathbb{Z}/p)
\]
where \( \kappa_i \) is the corresponding differential in the \( n \)-th Koszul complex.

When \( i = 1, n = 3 \) and \( A \) a free abelian group there is a natural isomorphism
\[
L_1\text{SP}^3(A \otimes \mathbb{Z}/p) \simeq \mathcal{L}^3(A \otimes \mathbb{Z}/p),
\]
where \( \mathcal{L}^3 \) is the third Lie functor (see [2], for example). Observe however that the natural map
\[
f_1^3 : L_1\text{SP}^4(A \otimes \mathbb{Z}/2) \to H_1\text{C}^8(A)
\]
is not an isomorphism. Indeed, every element of \( L_1\text{SP}^4(A \otimes \mathbb{Z}/2) \) is 2-torsion, whereas \( H_1\text{C}^8(A) \) can contain 4-torsion elements, since its cross-effect \( H_1\text{C}^8(A; B) \) contains \( \text{Tor}(\Gamma_2(A \otimes \mathbb{Z}/2), \Gamma_2(B \otimes \mathbb{Z}/2)) \) as a subgroup. The map \( f_1^3 \) is given by
\[
\beta_2(a, \bar{b}) \otimes \bar{c}d \mapsto a \otimes a \gamma_2(b) \gamma_2(c) \gamma_2(d) - b \otimes b \gamma_2(a) \gamma_2(c) \gamma_2(d), \quad a, b, c, d \in A.
\]
The following table, which is a consequence of Theorem 1.4, gives a complete description of $H_iC^n(A)$ for $n \leq 7$ and $A$ free abelian:

| $q$  | $H_0C^q(A)$ | $H_1C^q(A)$ | $H_2C^q(A)$ | $H_3C^q(A)$ |
|------|-------------|-------------|-------------|-------------|
| 7    | $A \otimes \mathbb{Z}/7$ | 0 | 0 | 0 |
| 6    | $\Gamma_2(A \otimes \mathbb{Z}/3) \oplus \Gamma_3(A \otimes \mathbb{Z}/2)$ | $A^2(A \otimes \mathbb{Z}/3) \oplus L^3(A \otimes \mathbb{Z}/2)$ | $A^3(A \otimes \mathbb{Z}/2)$ | 0 |
| 5    | $A \otimes \mathbb{Z}/5$ | 0 | 0 | 0 |
| 4    | $\Gamma_2(A \otimes \mathbb{Z}/2)$ | $A^2(A \otimes \mathbb{Z}/2)$ | 0 | 0 |
| 3    | $A \otimes \mathbb{Z}/3$ | 0 | 0 | 0 |
| 2    | $A \otimes \mathbb{Z}/2$ | 0 | 0 | 0 |

For example, the isomorphism

$$f : \Lambda^2(A \otimes \mathbb{Z}/2) \to H_1C^4(A)$$

is defined, for representatives $a, b \in A$ of $\bar{a}, \bar{b} \in A \otimes \mathbb{Z}/2$, by

$$f : \bar{a} \otimes \bar{b} \mapsto a \otimes a\gamma_2(b) - b \otimes b\gamma_2(a).$$

Acknowledgements  The author thanks L. Breen for various discussions related to the subject of the paper.

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