MEAN-SQUARE CONVERGENCE OF A SEMI-DISCRETE SCHEME FOR STOCHASTIC NONLINEAR MAXWELL EQUATIONS

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ABSTRACT. In this paper, we propose a semi-implicit Euler scheme to discretize the stochastic nonlinear Maxwell equations with multiplicative Itô noise, which is implicit in the drift term and explicit in the diffusion term of the equations, in order to suit itô product. Uniform bounds with high regularities of solutions for both the continuous and the discrete problems are obtained, which are crucial properties to derive the mean-square convergence with certain order. Allowing sufficient spatial regularity and utilizing the energy estimate technique, the convergence order $\frac{1}{2}$ in mean-square sense is obtained.

KEY WORDS: mean-square convergence order, semi-discrete scheme, stochastic nonlinear Maxwell equations, regularity

1. INTRODUCTION

Stochastic Maxwell equations play an important role in stochastic electromagnetism and statistical radiophysics fields. Some articles (see, e.g., [1, 2, 6]) introduced randomness into Maxwell equations in order to strengthen the correspondence between theoretical results and the real-life situations. In [7], problems about how to account, rigorously, for uncertainties in classical macroscopic electromagnetic interactions between fields and systems of linear material were discussed. [15] considered the problem about how to use the spectral representation to describe the random electromagnetic fields, which are coupled by Maxwell’s equations with a random source term. [3] dealt with the mathematical analysis of stochastic problems arising in the theory of electromagnetic in complex media, including well-posedness, controllability and homogenization. Assuming the existence of magnetic charges or monopoles, consider the following generalized symmetrized stochastic nonlinear Maxwell equations driven by multiplicative Itô noise,

$$
\begin{aligned}
\varepsilon \frac{\partial}{\partial t} E - \nabla \times H &= -J_e(t, x, E, H) - J_e(t, x, E, H) \cdot \dot{W}, \\
\mu \frac{\partial}{\partial t} H + \nabla \times E &= -J_m(t, x, E, H) - J_m(t, x, E, H) \cdot \dot{W}, \\
E(0, x) &= E_0(x), H(0, x) = H_0(x), \quad \forall x \in D, \\
\mathbf{n} \times E &= 0,
\end{aligned}
$$

(1.1)

where $D \subset \mathbb{R}^d$ with $d = 3$ is a bounded domain, $T \in (0, \infty)$, and the function $J : [0, T] \times D \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is a smooth nonlinear function satisfying

$$
|J(t, x, u, v)| \leq L(1 + |u| + |v|),
$$

(1.2)

and

$$
|J(t, x, u_1, v_1) - J(s, x, u_2, v_2)| \leq L(|t - s| + |u_1 - u_2| + |v_1 - v_2|),
$$

(1.3)

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for all $x \in D$, $t,s \in [0,T]$, $u,v,u_1,v_1,u_2,v_2 \in \mathbb{R}^d$ and some constant $L > 0$. Here $| \cdot |$ denotes the Euclidean norm, and $J$ could be $J_e$, $J_c$, $J_m$ or $J_m^e$.

Recently, more and more attention has been paid to the numerical analysis of stochastic Maxwell equations. In [4], the author considered the stochastic Maxwell equations (1.1) driven by a color noise and investigated the finite element method for these equations and furthermore obtained the $L^2$ error estimates. In [5], the authors considered the two-dimensional Maxwell equations through a random source term and constructed a new numerical method based on Wiener chaos expansion. Due to the superiorities of multi-symplectic methods, many researchers have studied the stochastic multi-symplectic methods to stochastic Maxwell equations. [3] first proposed a stochastic multi-symplectic method for stochastic Maxwell equations with additive noise by using stochastic variational principle. The further analysis of preservation of physical properties of stochastic Maxwell equations with additive noise via stochastic multi-symplectic methods was investigated in [9]. More recently, [10] designed an innovative stochastic multi-symplectic method to three-dimensional stochastic Maxwell equations with multiplicative noise based on wavelet interpolation technique. This method has been applied successfully to solve a three-dimensional stochastic electromagnetic fields problem with periodic boundary condition.

The main difficulty in dealing with stochastic partial differential equations is the presence of unbounded differential operators and stochastic integrals. Here we strongly use the fact that the equation is semilinear so that we can write it in the abstract form:

\[
\begin{aligned}
\begin{cases}
\text{du}(t) = [Mu(t) + F(t,u(t))]dt + B(t,u(t))dW(t), & t \in (0, T],
\text{u}(0) = u_0,
\end{cases}
\end{aligned}
\]

whose solution can be written in an integral form containing a bounded linear semigroup instead of the unbounded differential operator,

\[
u(t) = S(t)u_0 + \int_0^t S(t-s)F(s,u(s))ds + \int_0^t S(t-s)B(s,u(s))dW(s) \text{ a.s.},
\]

with $M$ being the Maxwell operator and generating the unitary $C_0$-semigroup $S(t) = e^{tM}$; see Section 2 for the procedure of rewriting stochastic Maxwell equations (1.1) into the abstract form (1.4). Most of the analysis is made on this mild solution form (1.5) of the equation. In this way we require the minimal regularity assumptions on the solutions. We first establish the uniform boundedness of the solution in $L^p(\Omega; \mathcal{D}(M^k))$-norm for a given integer $k \in \mathbb{N}$ with $\mathcal{D}(M^k)$ being the $k$-th power of the operator $M$. Thanks to the mild solution (1.5) and the estimates for stochastic convolutions, we get

\[
\begin{aligned}
\sup_{t \in [0,T]} \|u(t)\|_{L^p(\Omega; \mathcal{D}(M^k))} \leq C(1 + \|u_0\|_{L^p(\Omega; \mathcal{D}(M^k))}),
\end{aligned}
\]

where the positive constant $C$ may depend on $p$, $T$, and $\|Q^1\|_{HS(U,H^{k+\gamma}(D))}$ with $\gamma > d/2$. After establishing the convergence order of the semigroup $S(t)$ and identity operator $Id$ with respect to time $t$ in Lemma 3.1, the Hölder continuity of the solution in $L^2(\Omega; \mathcal{D}(M^{k-1}))$-norm is derived, i.e.,

\[
\begin{aligned}
\mathbb{E}\|u(t) - u(s)\|_{\mathcal{D}(M^{k-1})}^2 \leq C|t - s|,
\end{aligned}
\]

where the positive constant $C$ may depend on $p$, $T$, $\|Q^1\|_{HS(U,H^{k+\gamma}(D))}$, and $\|u_0\|_{L^p(\Omega; \mathcal{D}(M^k))}$. 

The main goal of this work is to propose and study a semi-discretization in the temporal direction of (1.4) which inherits a uniform estimate in $\mathcal{D}(M)$-norm,
\[
u^{n+1} = u^n + \tau Mu^{n+1} + \tau F(t_{n+1}, u^{n+1}) + B(t_n, u^n)\Delta W^{n+1}, \quad n \geq 0.
\] (1.8)
We show the existence of an $\{\mathcal{T}_n; 0 \leq n \leq N\}$-adapted discrete solution $\{u^n; n \in \mathbb{N}\}$, and the iterates $\{u^n; n \in \mathbb{N}\}$ satisfy
\[
\max_{1 \leq n \leq N} \mathbb{E}\|u^n\|^p_{\mathcal{D}(M)} \leq C(1 + \|u_0\|^p_{L^p(\Omega; \mathcal{D}(M))}).
\] (1.9)
In order to derive this result, we take $\mathbb{H}$-inner product of (1.8) with $u^{n+1}$ and $Mu^{n+1} - Mu^n$, respectively. It is important to note that the appearance of terms $\|u^{n+1} - u^n\|^2_{\mathbb{H}}$ in the right-hand side of (4.3), and $\|Mu^{n+1} - Mu^n\|^2_{\mathbb{H}}$ in the right-hand side of (4.4) is crucial to obtain the boundedness result (1.9), which could absorb the difficulty caused by the stochastic term. It shows the stability of the iterates $\{u^n; n \in \mathbb{N}\}$ for the scheme (1.8).

It is important to understand how the numerical methods approximate the solutions of (1.1) and the first step is to analyze the error. In the second part of this work, we are interested in the mean-square convergence for iterates $\{u^n; n \in \mathbb{N}\}$ of (1.8). To the best of our knowledge, however, there has been no work in the literature which analyzes the convergence of numerical method for stochastic Maxwell equations. A relevant prerequisite for this purpose is to provide strong stability results (1.6) for the original problem (1.4), and (1.9) for the discretization (1.8). Define the local truncation error of scheme (1.8) by
\[
\delta^{n+1} := u(t_{n+1}) - u(t_n) - \tau Mu(t_{n+1}) - \tau F(t_{n+1}, u(t_{n+1})) - B(t_n, u(t_n))\Delta W^{n+1},
\] (1.10)
where $u(t)$ means the solution of stochastic nonlinear Maxwell equations (1.4). Using the classical energy technique, we find the relationship between the global error in mean-square sense and the local truncation error in mean and mean-square senses, i.e.,
\[
\mathbb{E}\|e^{n+1}\|^2_{\mathbb{H}} \leq \mathbb{E}\|e^n\|^2_{\mathbb{H}} + C\tau \left( \mathbb{E}\|e^n\|^2_{\mathbb{H}} + \mathbb{E}\|e^{n+1}\|^2_{\mathbb{H}} \right) + C\mathbb{E}\|\delta^{n+1}\|^2_{\mathbb{H}} + \frac{C}{\tau} \mathbb{E}\|\mathbb{E}(\delta^{n+1}|\mathcal{T}_n)\|^2_{\mathbb{H}},
\] (1.11)
where $e^n = u(t_n) - u^n$. It states that the global mean-square convergence order depends only on the local truncation error in mean and mean-square senses for sufficiently small time step size. Via replacing the expression of $u(t_{n+1}) - u(t_n)$ in (1.10) by the strong solution of (1.4), we get the estimates of local truncation error $\delta^{n+1}$,
\[
\mathbb{E}\|\delta^{n+1}\|^2_{\mathbb{H}} \leq C\tau^2, \quad \mathbb{E}\|\mathbb{E}(\delta^{n+1}|\mathcal{T}_n)\|^2_{\mathbb{H}} \leq C\tau^3,
\] (1.12)
which leads to
\[
\max_{0 \leq n \leq N} \left( \mathbb{E}\|e^n\|^2_{\mathbb{H}} \right)^{1/2} \leq C\tau^{1/2},
\] (1.13)
where the positive constant $C$ may depend on the Lipschitz coefficients of $F$ and $B$, $T$, $\|u_0\|_{L^2(\Omega; \mathcal{D}(M^2))}$ and $\|Q^{1/2}\|_{HS(U,H^2+\gamma(D))}$, but independent of $\tau$ and $n$. The estimate (1.13) means that the mean-square convergence order of (1.8) is of 1/2.

The outline of this paper is as follows. In Section 2, some preliminaries are collected and an abstract formulation of (1.1) is set forth. In Section 3, we analyze the regularities of the solution of stochastic nonlinear Maxwell equations (1.1), including the uniform boundedness and Hölder continuity. In Section 4, a semi-implicit Euler scheme in temporal direction is proposed and the mean-square convergence order is derived.
2. Preliminaries and Framework

For the coefficients of equations (1.1) we suppose that
\[ \mu, \varepsilon \in L^\infty(D), \mu, \varepsilon \geq \delta > 0, \] (2.1)
for a constant \( \delta > 0 \). The basic Hilbert space we work with is \( \mathbb{H} = L^2(D)^3 \times L^2(D)^3 \) with the inner product defined by
\[ \left\langle \left(\frac{E_1}{H_1}, \frac{E_2}{H_2}\right) \right\rangle_{\mathbb{H}} := \int_D (\varepsilon |E|^2 + \mu |H|^2) dx. \]

By our assumption on \( \mu \) and \( \varepsilon \), this inner product is obviously equivalent to the standard inner product on \( L^2(D)^6 \). The norm induced by this inner product corresponds to the electromagnetic energy of the physical system
\[ \left\| \frac{E}{H} \right\|_{\mathbb{H}} = \left( \int_D (\varepsilon |E|^2 + \mu |H|^2) dx \right)^{\frac{1}{2}}. \]

If there’s no external source, the electromagnetic energy of (1.4) is a conserved quantity, i.e., \( \|u(t)\|_{\mathbb{H}} = \|u_0\|_{\mathbb{H}} \).

The Maxwell operator is defined by
\[ M = \begin{pmatrix} 0 & \varepsilon^{-1} \nabla \times \\ -\mu^{-1} \nabla \times & 0 \end{pmatrix} \] (2.2)
with domain
\[ \mathcal{D}(M) = \left\{ \left(\frac{E}{H}\right) \in \mathbb{H} : M \left(\frac{E}{H}\right) = \left(\varepsilon^{-1} \nabla \times H, -\mu^{-1} \nabla \times E\right) \in \mathbb{H}, \partial_D n \times E = 0 \right\} \] (2.3)
where the curl-spaces are defined by
\[ H(\text{curl}, D) := \{ v \in L^2(D)^3 : \nabla \times v \in L^2(D)^3 \}, \]
and
\[ H_0(\text{curl}, D) := \{ v \in H(\text{curl}, D) : \partial_D n \times v = 0 \}. \]

The corresponding graph norm is \( \|v\|_{\mathcal{D}(M)} := (\|v\|_{\mathbb{H}}^2 + \|Mv\|_{\mathbb{H}}^2)^{\frac{1}{2}}. \) The Maxwell operator \( M \) defined in (2.2) with domain (2.3) is closed, skew-adjoint on \( \mathbb{H} \), and thus generates a unitary \( C_0 \)-group \( S(t) = e^{itM} \) on \( \mathbb{H} \) in the view of Stone’s theorem (see for instance [12, Theorem II.3.24]).

A frequently used property for Maxwell operator \( M \) is: \( \langle Mu, u \rangle_{\mathbb{H}} = 0, \forall u \in \mathcal{D}(M) \). We refer interested readers to [14, Chapter 3] and references therein for more introduction about Maxwell operator. Recursively, we could define the domain \( \mathcal{D}(M^k) = \{ u \in \mathcal{D}(M^{k-1}) : M^{k-1}u \in \mathcal{D}(M) \} \) for the \( k \)-th power of the operator \( M \), \( k \in \mathbb{N} \), with \( \mathcal{D}(M^0) = \mathbb{H} \), given the norm
\[ \|v\|_{\mathcal{D}(M^k)} := \left( \|v\|_{\mathbb{H}}^2 + \|M^k v\|_{\mathbb{H}}^2 \right)^{\frac{1}{2}}, \forall v \in \mathcal{D}(M^k), \]
which is a Hilbert space. In fact, the norm \( \| \cdot \|_{\mathcal{D}(M^k)} \) corresponds to the scalar product
\[ \langle u, v \rangle_{\mathcal{D}(M^k)} = \langle u, v \rangle_{\mathbb{H}} + \langle M^ku, M^kv \rangle_{\mathbb{H}}, \forall u, v \in \mathcal{D}(M^k), \]
and thus $\mathcal{D}(M^k)$ is a pre-Hilbert space. If $\{v_\ell\}_{\ell \in \mathbb{N}}$ is a Cauchy sequence for $\| \cdot \|_{\mathcal{D}(M^k)}$, then $\{v_\ell\}_{\ell \in \mathbb{N}}$ and $\{M^k v_\ell\}_{\ell \in \mathbb{N}}$ are Cauchy sequences in $\mathbb{H}$. Since $\mathbb{H}$ is complete, $v_\ell \to v$ and $M^k v_\ell \to v^k$ in $\mathbb{H}$. The closedness of operator $M$ leads to $v^k = M^k v$, i.e., $v_\ell \to v$ in $\mathcal{D}(M^k)$ which is thus a Hilbert space. Moreover, it can be shown that $\|u\|_{\mathcal{D}(M^{k_1})} \leq C \|u\|_{\mathcal{D}(M^{k_2})}$, $\forall \ u \in \mathcal{D}(M^{k_2})$, $k_1 \leq k_2$.

Let $F : [0, T] \times \mathbb{H} \to \mathbb{H}$ be a Nemytskij operator associated to $J_e, J_m$, defined by

$$F(t,u)(x) = \left( -\varepsilon^{-1} J_e(t,x,E(t,x),\mathbf{H}(t,x)) \right) \bigg|_{x \in D}, u = \begin{pmatrix} E \\ H \end{pmatrix} \in \mathbb{H}. \tag{2.4}$$

Thanks to (1.2) and (1.3), the operator $F$ satisfies

$$\|F(t,u)\|_{\mathbb{H}} \leq C(1 + \|u\|_{\mathbb{H}}), \tag{2.5}$$
$$\|F(t,u) - F(s,v)\|_{\mathbb{H}} \leq C(|t-s| + \|u - v\|_{\mathbb{H}}), \tag{2.6}$$
for all $t, s \in [0, T]$, and $u, v \in \mathbb{H}$. Here the positive constant $C$ may depend on $\delta$, the volume $|D|$ of domain $D$, and the constant $L$ in (1.2) and (1.3). In fact,

$$\|F(t,u)\|_{\mathbb{H}} = \left( \int_{D} |\varepsilon^{-1} J_e|^2 + \mu |\mu^{-1} J_m|^2 \ dx \right)^{\frac{1}{2}} \leq \delta^{-\frac{1}{2}} \left( \int_{D} 2L^2 (1 + |E| + |H|^2) \ dx \right)^{\frac{1}{2}} \leq \delta^{-\frac{1}{2}} \left[ (6L^2 |D|)^{\frac{1}{2}} + \left( \int_{D} (\varepsilon |E|^2 + \mu |H|^2) \ dx \right)^{\frac{1}{2}} \right] \leq C(1 + \|u\|_{\mathbb{H}}),$$

and the proof of (2.6) is similar as above.

Let $Q$ be a symmetric, positive definite operator with finite trace. The driven stochastic process $W(t)$ is a standard $Q$-Wiener process with respect to the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$, which can be represented as

$$W(t) = \sum_{i=1}^{\infty} Q^{\frac{1}{2}} e_i \beta_i(t), \ t \in [0, T], \tag{2.7}$$

where $\{\beta_i(t)\}_{i \in \mathbb{N}}$ is a family of independent real-valued Brownian motions and $\{e_i\}_{i \in \mathbb{N}}$ is an orthonormal basis of the space $U = L^2(D)$.

For diffusion term, we introduce the Nemytskij operator $B : [0, T] \times \mathbb{H} \to HS(U_0, \mathbb{H})$ by

$$(B(t,u)v)(x) = \left( \begin{array}{c} -\varepsilon^{-1} J_e(t,x,E(t,x),\mathbf{H}(t,x))v(x) \\ -\mu^{-1} J_m(t,x,E(t,x),\mathbf{H}(t,x))v(x) \end{array} \right), \tag{2.8}$$

where $x \in D$, $u = \begin{pmatrix} E \\ H \end{pmatrix} \in \mathbb{H}$, and $v \in U_0 := Q^{\frac{1}{2}} U$. Here $HS(U, H)$ denotes the separable Hilbert space of all Hilbert-Schmidt operators from one separable Hilbert space $U$ to another separable Hilbert space $H$, equipped with the scalar product

$$\langle \Gamma_1, \Gamma_2 \rangle_{HS(U, H)} = \sum_{j=1}^{\infty} \langle \Gamma_1 \eta_j, \Gamma_2 \eta_j \rangle_H,$$
and the corresponding norm
\[ \| \Gamma \|_{HS(U,H)} = \left( \sum_{j=1}^{\infty} \| \Gamma \eta_j \|_{H}^2 \right)^{\frac{1}{2}}, \]
where \( \{ \eta_j \}_{j \in \mathbb{N}} \) is an orthonormal basis of \( U \).

Thanks to (1.2) and (1.3), we have
\[ \| B(t,u) \|_{HS(U,H^\gamma)} \leq C \| Q^{\frac{1}{2}} \|_{HS(U,H^\gamma(D))} (1 + \| u \|_{H}^2)^{\frac{1}{2}}, \quad (2.9) \]
\[ \| B(t,u) - B(s,v) \|_{HS(U,H^\gamma)} \leq C \| Q^{\frac{1}{2}} \|_{HS(U,H^\gamma(D))} (|t-s| + \| u - v \|_{H}), \quad (2.10) \]
for all \( t,s \in [0, T], u,v \in \mathbb{H} \). Here the positive constant \( C \) may depend on \( \delta \), the volume \( |D| \) of domain \( D \), and the constant \( L \) in (1.2) and (1.3). In fact,
\[ \| B(t,u) \|_{HS(U,H^\gamma)}^2 = \sum_{j=1}^{\infty} \| B(t,u)Q^{\frac{1}{2}} \eta_j \|_{H}^2 \]
\[ = \sum_{j=1}^{\infty} \int_D \varepsilon^{-1} |J^\gamma_{\mu} \eta_j \eta^s_j(x)|^2 + \mu^{-1} |J^\gamma_{\mu} \eta_j \eta^s_j(x)|^2 \ dx \]
\[ \leq 6L^2 \delta^{-1} \sum_{j=1}^{\infty} \| Q^{\frac{1}{2}} \eta_j \|_{L^{\infty}(D)}^2 \int_D (1 + |E|^2 + |H|^2) \ dx \]
\[ \leq 6L^2 \delta^{-1} \| Q^{\frac{1}{2}} \|_{HS(U,H^\gamma(D))}^2 |D| + \delta^{-1} \int_D \varepsilon |E|^2 + \mu |H|^2 \ dx \]
\[ \leq C \| Q^{\frac{1}{2}} \|_{HS(U,H^\gamma(D))}^2 (1 + \| u \|_{H}), \]
where we have used the Sobolev embedding \( H^\gamma(D) \subset L^{\infty}(D) \) for any \( \gamma > d/2, d = 3 \) in this paper.

At this point, we introduce the abstract form of stochastic nonlinear Maxwell equations in finite dimensional space \( \mathbb{H} \):
\[ \begin{cases} d u(t) = [M(u(t) + F(t,u(t)))] dt + B(t,u(t))dW(t), \ t \in (0, T], \\ u(0) = u_0, \end{cases} \quad (2.11) \]
where \( M, F, B \) and \( W \) are defined as above, and
\[ u = \begin{pmatrix} E(t,x) \\ H(t,x) \end{pmatrix}, \quad u_0 = \begin{pmatrix} E_0(x) \\ H_0(x) \end{pmatrix}. \]

Now we look at the existence and uniqueness of the mild solution of stochastic nonlinear Maxwell equations (2.11) under certain conditions on the original functions \( J_c, J_m, J_{c}, J_{m}, Q \) and initial data; see [1] for the well-posedness of stochastic Maxwell equations in complex media given conditions directly on \( F \) and \( B \). Moreover, using the Burkholder-Davis-Gundy-type inequality we present an priori estimation on \( \sup_{t \in [0,T]} \| u(t) \|_{L^p(\Omega, \mathbb{H})} \) in Theorem 2.1 and on \( \mathbb{E} \left( \sup_{t \in [0,T]} \| u(t) \|_{H^\gamma}^2 \right) \) in Corollary 2.1; see [12] for the estimations about stochastic integrals and stochastic convolutions.

**Theorem 2.1.** Suppose conditions (1.2) and (1.3) are fulfilled, let \( W(t), t \in [0, T] \) be a Q-Wiener process with \( Q^{\frac{1}{2}} \in HS(U,H^\gamma(D)) \) for \( \gamma > d/2 \), and let \( u_0 \) be an \( \mathcal{F}_0 \)-measurable \( \mathbb{H} \)-valued random
variable satisfying \( \|u_0\|_{L^p(\Omega; \mathbb{H})} < \infty \) for some \( p \geq 2 \). Then stochastic Maxwell equations (2.11) have a unique mild solution given by

\[
u(t) = S(t)u_0 + \int_0^t S(t-s)F(s,u(s))ds + \int_0^t S(t-s)B(s,u(s))dW(s), \, \mathbb{P}\text{-a.s.}, \tag{2.12}
\]

for each \( t \in [0, T] \). Moreover, there exists a constant \( C := C(p,T,\|Q^\frac{1}{2}\|_{HS(U,H^p(D))}) \in (0, \infty) \) such that

\[
\sup_{t \in [0,T]} \|u(t)\|_{L^p(\Omega; \mathbb{H})} \leq C(1 + \|u_0\|_{L^p(\Omega; \mathbb{H})}). \tag{2.13}
\]

**Proof.** Under conditions (1.2) and (1.3), we see that from (2.6) and (2.10) \( F \) and \( B \) are both globally Lipschitz functions, the existence and uniqueness of the mild solution (2.12) follows from \( \|$ \text{Theorem 12.4.7} \text{, or [11, Theorem 7.2] for general stochastic evolution equation. Using the Burkholder-Davis-Gundy-type inequality [11, Theorem 4.36], linear growth (2.5) and (2.9) of } F \text{ and } B, \text{ we have}$

\[
\mathbb{E}\|u(t)\|_{L^p}^p \lesssim \mathbb{E}\|S(t)u_0\|_{\mathbb{H}}^p + \mathbb{E}\int_0^t \|S(t-s)F(s,u(s))\|_{\mathbb{H}}^p ds
\]

\[
+ \mathbb{E}\int_0^t \|S(t-s)B(s,u(s))dW(s)\|_{\mathbb{H}}^p
\]

\[
\lesssim \mathbb{E}\|u_0\|_{\mathbb{H}}^p + \int_0^t (1 + \mathbb{E}\|u(s)\|_{\mathbb{H}}^p) ds + \mathbb{E}\int_0^t \|B(s,u(s))\|_{HS(U_0,\mathbb{H})}^2 ds
\]

\[
\lesssim \|u_0\|_{L^p(\Omega; \mathbb{H})}^p + \int_0^t (1 + \mathbb{E}\|u(s)\|_{\mathbb{H}}^p) ds + \|Q^\frac{1}{2}\|_{HS(U,H^p(D))}^2 \int_0^t (1 + \mathbb{E}\|u(s)\|_{\mathbb{H}}^p) ds,
\]

where notation \( A \lesssim B \) means that there exists a positive constant such that \( A \leq CB \).

By Gronwall’s inequality, there exists a positive constant \( C := C(p, T, \|Q^\frac{1}{2}\|_{HS(U,H^p(D))}) \) such that

\[
\mathbb{E}\|u(t)\|_{\mathbb{H}}^p \leq C(1 + \|u_0\|_{L^p(\Omega; \mathbb{H})}^p), \forall t \in [0, T].
\]

Therefore, we complete the proof. \( \square \)

**Corollary 2.1.** Under the same conditions of Theorem 2.7 there exists a constant \( C := C(p, T, u_0, Q) \) such that

\[
\mathbb{E}\left( \sup_{t \in [0,T]} \|u(t)\|_{\mathbb{H}}^p \right) \leq C. \tag{2.14}
\]

**Proof.** The main step to derive (2.14) from the mild solution (2.12) is that we need to deal with the stochastic integral

\[
\mathbb{E}\left[ \sup_{t \in [0,T]} \left\| \int_0^t S(t-s)B(s,u(s))dW(s) \right\|_{\mathbb{H}}^p \right].
\]

By using the Burkholder-Davis-Gundy-type inequality for stochastic convolution [11, Proposition 7.3], we have

\[
\mathbb{E}\left[ \sup_{t \in [0,T]} \left\| \int_0^t S(t-s)B(s,u(s))dW(s) \right\|_{\mathbb{H}}^p \right] \lesssim \mathbb{E}\int_0^T \|S(t-s)B(s,u(s))\|_{HS(U_0,\mathbb{H})}^p ds
\]

\[
\lesssim \|Q^\frac{1}{2}\|_{HS(U,H^p(D))}^2 \int_0^T (1 + \mathbb{E}\|u(s)\|_{\mathbb{H}}^p) ds \leq C(p, T, u_0, Q),
\]
where we use the result of Theorem 2.1 in the last step.

**Remark 2.1.** If we apply Itô formula to the functional $\mathcal{H}(u) = \|u\|_{H^2}^2$, we may get the evolution of the electromagnetic energy of system (2.11). In fact, the first and the second order derivatives of $\mathcal{H}(u)$ are

$$D\mathcal{H}(u)(\psi) = 2\langle u, \psi \rangle_{H}, \quad D^2\mathcal{H}(u)(\psi, \phi) = 2\langle \psi, \phi \rangle_{H}.$$  

Itô formula (see for instance [11, Theorem 4.32]) gives us

$$\mathcal{H}(u(t)) = \mathcal{H}(u_0) + \int_0^t 2\langle u(s), F(s, u(s)) \rangle_{H} + \text{tr}(\langle B(s, u(s))Q^1, (B(s, u(s))Q^2) \rangle_{H}) \, ds + 2\int_0^t \langle u(s), B(s, u(s))dW(s) \rangle_{H}. \quad (2.15)$$

We observe that if $F = 0$ and $B$ is a constant operator, then the average energy $\mathbb{E}\mathcal{H}(u(t))$ grows linearly with respect to time $t$, see [9, Theorem 2.1] for the analysis of stochastic Maxwell equations with additive noise.

3. **Regularities of the solution of stochastic Maxwell equations**

This section is devoted to the regularity analysis for the solution of stochastic Maxwell equations (1.1) or (2.11), including the uniform boundedness of the solution in $L^2(\Omega; \mathcal{D}(M^k))$-norm and Hölder continuity of the solution in $L^2(\Omega; \mathcal{D}(M^{k-1}))$-norm for a given integer $k \in \mathbb{N}$.

First, we present the assumptions on coefficients of equations (1.1) in order to get enough space regularity of the solution.

**Assumption 3.1.** Assume the coefficients $\mu, \epsilon \in C_b^k(D)$, and $\mu, \epsilon \geq \delta > 0$.

By this assumption, we know that for any integer $\ell \leq k$,

$$\|\partial_\epsilon^\ell \mu\|_{L^\infty(D)} + \|\partial_\epsilon^\ell \mu\|_{L^\infty(D)} \leq K_1, \quad (3.1)$$

$$\|\partial_\epsilon^\ell \epsilon^{-1}\|_{L^\infty(D)} + \|\partial_\epsilon^\ell \mu^{-1}\|_{L^\infty(D)} \leq K_2, \quad (3.2)$$

where $K_2$ depends on $\delta$ and $K_1$.

**Assumption 3.2.** Assume function $J : [0, T] \times D \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is a smooth enough nonlinear function with bounded derivatives, i.e., $J \in C_b^{1,k+1, k+1}([0, T] \times D \times \mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$. Here $J$ could be $J$, $J^\epsilon$, $J_m$ or $J_m^\epsilon$.

Throughout this paper $C_b^{\ell,m,n,n}$ denotes the set of vector-valued continuously differential functions $\Phi : [0, T] \times D \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ with uniformly bounded partial derivatives $\partial_\epsilon^\ell \Phi$, $\partial_\Phi^m \Phi$ and $\partial_\epsilon^\ell \partial_\Phi^m \Phi$ for $\ell_1 \leq \ell$, $m_1 \leq m$ and $n_1 + n_2 \leq n$.

**Assumption 3.3.** Assume that the operator $Q^\pm \in HS(U, H^{k+\gamma}(D))$.

It follows from Assumptions 3.1 and 3.2 that the drift term $F$ satisfies

$$\|F(t, u)\|_{\mathcal{D}(M^\ell)} \leq C(1 + \|u\|_{\mathcal{D}(M^\ell)}), \quad (3.3)$$

$$\|F(t, u) - F(s, v)\|_{\mathcal{D}(M^\ell)} \leq C|t - s| + \|u - v\|_{\mathcal{D}(M^\ell)}, \quad (3.4)$$

where $0 \leq \ell \leq k$, and $u, v \in \mathcal{D}(M^\ell)$. Here the constant $C$ may depend on $\delta$, $K_1$, $K_2$ in (3.1) and (3.2), the volume $|D|$ of the domain $D$, and the derivative bounds $L$ of functions $J$ and $J_m$. We only
present the proof of (3.3) in the case \( \ell = 1 \), for other cases and inequality (3.4) could be proved by the same approach,

\[
\|MF(t,u)\|_{L^2} = \left( \int_D \eps^{-1} |\nabla \times (\mu^{-1}J_m^0)|^2 \, dx + \mu^{-1} |\nabla \times (\eps^{-1}J_e)|^2 \, dx \right)^{\frac{1}{2}} 
\leq \delta^{-\frac{1}{2}} \left[ \int_D \delta^{-2} (|\nabla \times J_m^0|^2 + |\nabla \times J_e|^2) + K_2^2 (|J_m^0|^2 + |J_e|^2) \, dx \right]^{\frac{1}{2}} 
\leq \delta^{-\frac{1}{2}} \left[ (6L^2|D|)^{\frac{1}{2}} + \left( 6L^2 K_1 \int_D \mu^{-1} |\nabla \times E|^2 + \eps^{-1} |\nabla \times H|^2 \, dx \right)^{\frac{1}{2}} \right] 
+ \delta^{-\frac{1}{2}} K_2 \left[ (6L^2|D|)^{\frac{1}{2}} + \left( 6L^2 \delta^{-1} \int_D \eps |E|^2 + \mu |H|^2 \, dx \right)^{\frac{1}{2}} \right] 
\leq C(1 + \|u\|_{\mathcal{G}(M)}) .
\]

Under Assumptions 3.1, 3.2 and 3.3, we get that for diffusion term \( B \) and \( 0 \leq \ell \leq k \), and for any \( \gamma > d/2 \),

\[
\|B(t,u)\|_{H^\gamma(U_0, \mathcal{G}(M^\ell))} \leq C \|Q^\frac{1}{2}\|_{H^\gamma(U, H^{1+\gamma}(D))} (1 + \|u\|_{\mathcal{G}(M^\ell)})^{\frac{1}{2}}, \tag{3.5}
\]

\[
\|B(t,u) - B(s,v)\|_{H^\gamma(U_0, \mathcal{G}(M^\ell))} \leq C \|Q^\frac{1}{2}\|_{H^\gamma(U, H^{1+\gamma}(D))} (|t-s| + \|u-v\|_{\mathcal{G}(M^\ell)}), \tag{3.6}
\]

where \( t, s \in [0, T] \), and \( u, v \in \mathcal{G}(M^\ell) \). Here the constant \( C \) may depend on \( \delta, K_1, K_2 \) in (3.1) and (3.2), the volume \( |D| \) of the domain \( D \), and the derivative bounds \( L \) of functions \( J'_e \) and \( J''_m \). We just present the proof of (3.5) in the case \( \ell = 1 \), for other cases and inequality (3.6) could be proved by the same approach,

\[
\sum_{j=1}^{\infty} \|BQ^\frac{1}{2}e_j\|_{L^2}^2 = \sum_{j=1}^{\infty} \int_D \eps^{-1} |\nabla \times (\mu^{-1}J_m^0Q^\frac{1}{2}e_j)|^2 + \mu^{-1} |\nabla \times (\eps^{-1}J'_eQ^\frac{1}{2}e_j)|^2 \, dx 
\leq \delta^{-1} \sum_{j=1}^{\infty} \int_D \delta^{-2} \|Q^\frac{1}{2}e_j\|_{L^\infty(D)}^2 (|\nabla \times J'_m|^2 + |\nabla \times J'_e|^2) \, dx 
+ \delta^{-1} \sum_{j=1}^{\infty} \int_D (K_2^2 \|Q^\frac{1}{2}e_j\|_{L^\infty(D)}^2 + \delta^{-2} \|Q^\frac{1}{2}e_j\|_{L^\infty(D)}^2 (|J'_m|^2 + |J'_e|^2) \, dx 
\leq 6L^2 \delta^{-3} \|Q^\frac{1}{2}\|_{H^\gamma(U, H^{1+\gamma}(D))}^2 \int_D (1 + |\nabla \times E|^2 + |\nabla \times H|^2) \, dx 
+ \delta^{-1} (K_2^2 \|Q^\frac{1}{2}\|_{H^\gamma(U, H^{1+\gamma}(D))}^2 + \delta^{-2} \|Q^\frac{1}{2}\|_{H^\gamma(U, H^{1+\gamma}(D))}^2 \int_D (1 + |E|^2 + |H|^2) \, dx 
\leq C \|Q^\frac{1}{2}\|_{H^\gamma(U, H^{1+\gamma}(D))}^2 (1 + \|u\|_{\mathcal{G}(M)}) ,
\]

where we have used the Sobolev embedding \( H^\gamma(D) \subset L^\infty(D) \) for any \( \gamma > d/2 \).

3.1. Uniform boundedness of the solution. We are now ready to establish establish the uniform boundedness of the solution of stochastic Maxwell equations (2.11) in \( L^p(\Omega; \mathcal{G}(M^\ell)) \)-norm, which is stated in the following theorem.
Proposition 3.1. Let Assumptions 3.1-3.3 be fulfilled, and suppose that \( u_0 \) be an \( \mathcal{F}_0 \)-measurable \( \mathbb{H} \)-valued random variable satisfying \( \|u_0\|_{L^p(\Omega; \mathcal{F}(M^k))} < \infty \) for some \( p \geq 2 \). Then the mild solution (2.12) satisfies
\[
\sup_{t \in [0,T]} \|u(t)\|_{L^p(\Omega; \mathcal{F}(M^k))} \leq C(1 + \|u_0\|_{L^p(\Omega; \mathcal{F}(M^k))}),
\] (3.7)
where the positive constant \( C \) may depend on \( p, T, \) and \( \|Q^{\frac{1}{2}}\|_{HS(U, H^{k+\gamma}(D))} \).

Proof. Under Assumptions 3.1-3.3 we see that from (3.3) and (3.5) \( F \) and \( B \) are linear growth functions. Using the Burkholder-Davis-Gundy-type inequality for stochastic integrals \([11, \text{ Theorem 4.36}]\), we have for the mild solution (2.12),
\[
E\|u(t)\|_{\mathcal{F}(M^k)}^p \lesssim E\|S(t)u_0\|_{\mathcal{F}(M^k)}^p + E \int_0^t \|S(t-s)F(s, u(s))\|_{\mathcal{F}(M^k)}^p ds
+ E \int_0^t \|S(t-s)B(s, u(s))dW(s)\|_{\mathcal{F}(M^k)}^p ds
\lesssim\|u_0\|_{L^p(\Omega; \mathcal{F}(M^k))}^p + \int_0^t (1 + E\|u(s)\|_{\mathcal{F}(M^k)}^p) ds + \left[ E \int_0^t \|B(s, u(s))\|_{HS(U, \mathcal{F}(M^k))}^2 ds \right]^{\frac{p}{2}}
\lesssim\|u_0\|_{L^p(\Omega; \mathcal{F}(M^k))}^p + \int_0^t (1 + E\|u(s)\|_{\mathcal{F}(M^k)}^p) ds
+ \|Q^{\frac{1}{2}}\|_{HS(U, H^{k+\gamma}(D))} \int_0^t (1 + E\|u(s)\|_{\mathcal{F}(M^k)}^p) ds.
\]

By Gronwall’s inequality, there exists a positive constant \( C := C(p, T, \|Q^{\frac{1}{2}}\|_{HS(U, H^{k+\gamma}(D))}) \) such that
\[
E\left( \sup_{t \in [0,T]} \|u(t)\|_{\mathcal{F}(M^k)}^p \right) \leq C(1 + \|u_0\|_{L^p(\Omega; \mathcal{F}(M^k))}), \forall t \in [0,T].
\]
Therefore, we complete the proof. \( \square \)

Corollary 3.1. Under the same conditions of Proposition 3.1 there exists a constant \( C := C(p, T, u_0, Q) \) such that
\[
E\left( \sup_{t \in [0,T]} \|u(t)\|_{\mathcal{F}(M^k)}^p \right) \leq C.
\] (3.8)

Proof. The main step to derive (3.8) from the mild solution (2.12) is that we need to deal with the stochastic integral
\[
E \left[ \sup_{t \in [0,T]} \left\| \int_0^t S(t-s)B(s, u(s))dW(s) \right\|_{\mathcal{F}(M^k)}^p \right].
\]

By using the Burkholder-Davis-Gundy-type inequality for stochastic convolution \([11, \text{ Proposition 7.3}]\), we have
\[
E \left[ \sup_{t \in [0,T]} \left\| \int_0^t S(t-s)B(s, u(s))dW(s) \right\|_{\mathcal{F}(M^k)}^p \right] \lesssim E \int_0^T \|S(t-s)B(s, u(s))\|_{HS(U, \mathcal{F}(M^k))}^p ds
\lesssim \|Q^{\frac{1}{2}}\|_{HS(U, H^{k+\gamma}(D))} \int_0^T (1 + E\|u(s)\|_{\mathcal{F}(M^k)}^p) ds
\leq C(p, T, u_0, Q),
\]
3.2. H"{o}lder continuity of the solution. In this subsection, we shall obtain the H"{o}lder continuity of the solution of stochastic Maxwell equations (2.11) in $L^2(\Omega; \mathcal{D}(M^{k-1}))$-norm. To this end, we first give an very useful lemma.

**Lemma 3.1.** For the semigroup $\{S(t); t \geq 0\}$ on $\mathbb{H}$, it holds that

$$
\|S(t) - Id\|_{\mathcal{L}(\mathcal{D}(M); \mathbb{H})} \leq Ct, \tag{3.9}
$$

where the constant $C$ does not depend on $t$.

**Proof.** We start from the system

$$
\begin{cases}
\frac{\partial u(t)}{\partial t} = Mu(t), & t \in (0, T],\\
u(0) = u_0.
\end{cases} \tag{3.10}
$$

Thus

$$
\frac{\partial}{\partial t} \|u(t)\|^2_{\mathbb{H}} = 2 \left\langle M \frac{\partial u(t)}{\partial t}, u(t) \right\rangle_{\mathbb{H}} = 2 \left\langle Mu(t), u(t) \right\rangle_{\mathbb{H}} = 0,
$$

leads to

$$
\|u(t)\|_{\mathbb{H}} = \|S(t)u_0\|_{\mathbb{H}} = \|u_0\|_{\mathbb{H}},
$$

which means $\|S(t)\|_{\mathcal{L}(\mathbb{H}; \mathbb{H})} = 1$.

Similarly, consider

$$
\frac{\partial}{\partial t} \|Mu(t)\|^2_{\mathbb{H}} = 2 \left\langle M^2 \frac{\partial u(t)}{\partial t}, Mu(t) \right\rangle_{\mathbb{H}} = 2 \left\langle M^2u(t), Mu(t) \right\rangle_{\mathbb{H}} = 0,
$$

which leads to $\|S(t)\|_{\mathcal{L}(\mathcal{D}(M); \mathcal{D}(M))} = 1$.

The assertion in this lemma is equivalent to

$$
\|u(t) - u_0\|_{\mathbb{H}} = \|(S(t) - Id)u_0\|_{\mathbb{H}} \leq C\|u_0\|_{\mathcal{D}(M)t}.
$$

In fact, we can conclude from (3.10) that

$$
\left\langle u(t) - u_0, u(t) \right\rangle_{\mathbb{H}} = \left\langle \int_0^t Mu(s)ds, u(t) \right\rangle_{\mathbb{H}},
$$

where the term in left-hand side is

$$
\frac{1}{2} \left( \|u(t)\|^2_{\mathbb{H}} - \|u_0\|^2_{\mathbb{H}} + \|u(t) - u_0\|^2_{\mathbb{H}} \right) = \frac{1}{2} \|u(t) - u_0\|^2_{\mathbb{H}},
$$

and the term in right-hand side can be estimated by

$$
\left\langle \int_0^t Mu(s)ds, u(t) \right\rangle_{\mathbb{H}} \leq \int_0^t \|Mu(s)\|_{\mathbb{H}}\|u(t)\|_{\mathbb{H}}ds
\leq \|u_0\|_{\mathbb{H}} \int_0^t \|u(s)\|_{\mathcal{D}(M)}ds \leq C\|u_0\|_{\mathcal{D}(M)t}^2.
$$

Therefore we complete the proof. □
**Proposition 3.2.** Under the same assumption of Proposition 3.1, we have for $0 \leq t, s \leq T$,

\[
\mathbb{E}\|u(t) - u(s)\|_{\mathcal{D}(M^{k-1})}^p \leq C|t - s|^{p/2},
\]

where the positive constant $C$ may depend on $p, T$, $\|Q^{1/2}_{H^k(H^{k+\gamma}(D))}$, and $\|u_0\|_{L^p(\Omega; \mathcal{D}(M^k))}$.

**Proof.** From equation (2.12), we have

\[
u(t) - u(s) = (S(t - s) - I)u(s) + \int_s^t S(t - r)F(r, u(r))\,dr
\]

for the first term $I$, we have

\[
I = \mathbb{E}\|(S(t) - S(s))u(s)\|_{\mathcal{D}(M^{k-1})}^p \leq \|S(t) - S(s)\|_{\mathcal{L}(\mathcal{D}(M^k), \mathcal{D}(M^{k-1}))}^p \mathbb{E}\|u(s)\|_{\mathcal{D}(M^k)}^p
\]

\[
\leq C|t - s|^p \|u(s)\|_{L^p(\Omega; \mathcal{D}(M^k))}^p,
\]

where we use the estimate $\|S(t) - I\|_{\mathcal{L}(\mathcal{D}(M), \mathcal{D}(M))} \leq Ct$ (see Lemma 3.1) in the last step. From Proposition 3.1, we have

\[
I \leq C(1 + \|u_0\|_{L^p(\Omega; \mathcal{D}(M^k))}^p)(t - s)^p.
\]

For the second term $II$, it holds

\[
II = \mathbb{E}\left\|\int_s^t S(t - r)F(r, u(r))\,dr\right\|_{\mathcal{D}(M^{k-1})}^p
\]

\[
\lesssim (t - s)^{p-1} \int_s^t \mathbb{E}\|S(t - r)F(r, u(r))\|_{\mathcal{D}(M^{k-1})}^p \,dr
\]

\[
\leq (t - s)^{p-1} \int_s^t \mathbb{E}\|S(t - r)\|_{\mathcal{L}(\mathcal{D}(M^{k-1}), \mathcal{D}(M^{k-1}))}^p \|F(r, u(r))\|_{\mathcal{D}(M^{k-1})}^p \,dr
\]

\[
= (t - s)^{p-1} \int_s^t \mathbb{E}\|F(r, u(r))\|_{\mathcal{D}(M^{k-1})}^p \,dr
\]

\[
\leq (t - s)^{p-1} \int_s^t \mathbb{E}(1 + \|u(r)\|_{\mathcal{D}(M^{k-1})}^p) \,dr \leq C(t - s)^p,
\]

where in the last step, we utilize the estimate $\sup_{t \in [0, T]} \mathbb{E}\|u(t)\|_{\mathcal{D}(M^{k-1})}^p \leq C(1 + \|u_0\|_{L^p(\Omega; \mathcal{D}(M^{k-1}))})$ with the constant $C := C(p, T, \|Q^{1/2}_{H^k(H^{k+\gamma}(D))})$. 

Using the Burkholder-Davis-Gundy-type inequality for stochastic integrals [11, Theorem 4.36], we obtain:

\[ III \lesssim \left( \int_s^t \mathbb{E}\left[ \left\| S(t-r)B(r,u(r)) \right\|_{\mathcal{HS}(U_0,\mathcal{D}(M_k^{-1}))}^2 \right] dr \right)^{p/2} \]

\[ \leq \left( \int_s^t \left\| S(t-r) \right\|_{\mathcal{D}(M_k^{-1})}^2 \mathbb{E}\left[ \left\| B(r,u(r)) \right\|_{\mathcal{HS}(U_0,\mathcal{D}(M_k^{-1}))}^2 \right] dr \right)^{p/2} \]

\[ \leq \| Q_1^2 \|^p_{\mathcal{HS}(U,H_k^{-1}+\gamma(D))} \left( \int_s^t \mathbb{E}(1 + \left\| u(r) \right\|_{\mathcal{D}(M_k^{-1})}^2) dr \right)^{p/2} \leq C(t-s)^{p/2}. \]

Combining equations (3.15), (3.16) and (3.17) and based on the assumption \( u_0 \in \mathcal{D}(M^k) \), we obtain the first result.

To get the second assertion, we take the expectation to the both sides of equation (3.13), it yields

\[ \mathbb{E}(u(t) - u(s)) = \mathbb{E}((S(t-s) - I)u(s)) + \mathbb{E} \left( \int_s^t S(t-r)F(r,u(r)) dr \right) \] (3.18)

Therefore, similar as (3.13) and (3.16) we get

\[ \| \mathbb{E}(u(t) - u(s)) \|_{\mathcal{D}(M_k^{-1})} \leq \| \mathbb{E}((S(t-s) - I)u(s)) \|_{\mathcal{D}(M_k^{-1})} + \int_s^t \| \mathbb{E}(S(t-r)F(r,u(r))) \|_{\mathcal{D}(M_k^{-1})} dr \]

\[ \leq \mathbb{E}((S(t-s) - I)u(s))_{\mathcal{D}(M_k^{-1})} + \mathbb{E} \int_s^t (S(t-r)F(r,u(r)))_{\mathcal{D}(M_k^{-1})} dr \]

\[ \leq C(t-s). \] (3.19)

Therefore we finish the proof. \( \square \)

4. TEMPORAL SEMIDISCRETIZATION

In this section, we apply semi-implicit Euler scheme to discretize stochastic Maxwell equations (2.11) in temporal direction, and investigate the convergence order in mean-square sense of this scheme. For the time interval \([0, T]\), we introduce a uniform partition with step size \( \tau = \frac{T}{N} \):

\[ 0 = t_0 < t_1 < \ldots < t_N = T. \] (4.1)

Applying the semi-implicit Euler scheme to equation (2.11) in temporal direction, we have

\[ u^{n+1} = u^n + \tau M u^{n+1} + \tau F(t_{n+1},u^{n+1}) + B(t_n,u^n) \Delta W^{n+1}, \]

\[ u^0 = u_0, \] (4.2)

where the increment \( \Delta W^{n+1} \) is given by

\[ \Delta W^{n+1} := W(t_{n+1}) - W(t_n) = \sum_{j=1}^{\infty} (\beta_j(t_{n+1}) - \beta_j(t_n)) Q^\frac{1}{2} e_j. \]
Recall $u^n = \begin{pmatrix} E^n \\ H^n \end{pmatrix}$, then scheme (4.2) is equivalent to

$$
\begin{align*}
&\varepsilon E^{n+1} = \varepsilon E^n + \tau \nabla \times H^{n+1} - \tau J_e(t_{n+1}, E^{n+1}, H^{n+1}) - J_e'(t_{n+1}, E^n, H^n) \Delta W^{n+1}, \\
&\mu H^{n+1} = \mu H^n - \tau \nabla \times E^{n+1} - \tau J_m(t_{n+1}, E^{n+1}, H^{n+1}) - J_m'(t_{n+1}, E^n, H^n) \Delta W^{n+1},
\end{align*}
$$

(4.3)

4.1. Properties of the discrete solution. In this subsection, we will show that there exists a $\mathcal{D}(M)$-valued $\{ \mathcal{F}_n \}_{0 \leq n \leq N}$-adapted discrete solution $\{ u^n; n = 0, 1, \ldots, N \}$ for scheme (4.2) or $\{ E^n, H^n; n = 0, 1, \ldots, N \}$ for scheme (4.3).

**Lemma 4.1.** For a fixed $T = t_N > 0$, let $p \geq 2$ and $\tau \leq \tau^*$ with $\tau^* := \tau^*(\|u_0\|_{L^p(\Omega; \mathcal{D}(M))}, T, p)$. There exists a $\mathcal{D}(M)$-valued $\{ \mathcal{F}_n \}_{0 \leq n \leq N}$-adapted discrete solution $\{ u^n; n = 0, 1, \ldots, N \}$ of the scheme (4.2), and a constant $C := C(p, T, \| \Omega \|_{H^1(U, H^1(\mathcal{D}))}^2) > 0$ such that

$$
\max_{1 \leq n \leq N} \mathbb{E}\| u^n \|^p_{\mathcal{D}(M)} \leq C(1 + \| u_0 \|_{L^p(\Omega; \mathcal{D}(M))}^p).
$$

(4.4)

**Proof.** Step 1: Existence and $\{ \mathcal{F}_n \}_{0 \leq n \leq N}$-adaptedness. Fix a set $\Omega' \subset \Omega$, $\mathbb{P}(\Omega') = 1$ such that $W(t, \omega) \in U$ for all $t \in [0, T]$ and $\omega \in \Omega'$. In the following, let us assume that $\omega \in \Omega'$. The existence of iterates $\{ u^n; n = 0, 1, \ldots, N \}$ follows from a standard Galerkin method and Brouwer’s theorem, in combining with assertion (4.4).

Define a map

$$
\Lambda : \mathcal{D}(M) \times U \to \mathcal{P}(\mathcal{D}(M))
$$

$$
(u^n, \Delta W^{n+1}) \to \Lambda(u^n, \Delta W^{n+1})
$$

where $\mathcal{P}(\mathcal{D}(M))$ denotes the set of all subsets of $\mathcal{D}(M)$, and $\Lambda(u^n, \Delta W^{n+1})$ is the set of solutions $u^{n+1}$ of (4.2). By the closedness of the graph of $\Lambda$ and a selector theorem [13, Theorem 3.1], there exists a universally and Borel measurable mapping $\lambda : \mathcal{D}(M) \times U \to \mathcal{D}(M)$ such that $\lambda(s_1, s_2) \in \Lambda(s_1, s_2)$ for all $(s_1, s_2) \in \mathcal{D}(M) \times U$. Therefore, $\mathcal{F}_{n+1}$-measurability of $u^{n+1}$ follows from the Doob-Dynkin lemma.

Step 2: Case $p = 2$ for (4.4). We apply $\langle \cdot, u^{n+1} \rangle_{\mathbb{H}}$ into both sides of (4.2) and get

$$
\begin{align*}
&\frac{1}{2} \left( \| u^{n+1} \|^2_{\mathbb{H}} - \| u^n \|^2_{\mathbb{H}} \right) + \frac{1}{2} \| u^{n+1} - u^n \|^2_{\mathbb{H}} \\
&= \tau \langle F(t_{n+1}, u^{n+1}), u^{n+1} \rangle_{\mathbb{H}} + \langle B(t_n, u^n) \Delta W^{n+1}, u^{n+1} \rangle_{\mathbb{H}} \\
&\leq C \tau (1 + \| u^{n+1} \|_{\mathbb{H}}) \| u^{n+1} \|_{\mathbb{H}} + \| B(t_n, u^n) \Delta W^{n+1} \|^2_{\mathbb{H}} \\
&+ \frac{1}{4} \| u^{n+1} - u^n \|^2_{\mathbb{H}} + \langle B(t_n, u^n) \Delta W^{n+1}, u^n \rangle_{\mathbb{H}},
\end{align*}
$$

(4.5)

which gives

$$
\begin{align*}
&\frac{1}{2} \left( \| u^{n+1} \|^2_{\mathbb{H}} - \| u^n \|^2_{\mathbb{H}} \right) + \frac{1}{4} \| u^{n+1} - u^n \|^2_{\mathbb{H}} \\
&\leq C \tau \| u^{n+1} \|^2_{\mathbb{H}} + \| B(t_n, u^n) \Delta W^{n+1} \|^2_{\mathbb{H}} + \langle B(t_n, u^n) \Delta W^{n+1}, u^n \rangle_{\mathbb{H}}.
\end{align*}
$$

(4.6)
Next we apply $\langle \cdot, Mu^{n+1} - Mu^n \rangle_H$ into both sides of (4.2) and get
\[
\frac{1}{2} \left( \|Mu^{n+1}\|_H^2 - \|Mu^n\|_H^2 \right) + \frac{1}{2} \|Mu^{n+1} - Mu^n\|_H^2 \\
= -\langle F(t_{n+1}, u^{n+1}), Mu^{n+1} - Mu^n \rangle_H - \frac{1}{\tau} \langle B(t_n, u^n)\Delta W^{n+1}, Mu^{n+1} - Mu^n \rangle_H \\
= I + II.
\]
For the term $I$, using the skew adjoint property of operator $M$ and (4.2), we get
\[
I = \langle MF(t_{n+1}, u^{n+1}), u^{n+1} - u^n \rangle_H \\
= \langle MF(t_{n+1}, u^{n+1}), \tau Mu^{n+1} + \tau F(t_{n+1}, u^{n+1}) + B(t_n, u^n)\Delta W^{n+1} \rangle_H \\
\leq \tau \|MF(t_{n+1}, u^{n+1})\|_H \|Mu^{n+1}\|_H + \langle MF(t_{n+1}, u^{n+1}), B(t_n, u^n)\Delta W^{n+1} \rangle_H \\
\leq \tau \|M\| \|Mu^{n+1}\|_H^2 + \langle MF(t_{n+1}, u^{n+1}), B(t_n, u^n)\Delta W^{n+1} \rangle_H.
\]
Similarly, for the term $II$, we get
\[
II = \frac{1}{\tau} \langle M(B(t_n, u^n)\Delta W^{n+1}), u^{n+1} - u^n \rangle_H \\
= \frac{1}{\tau} \langle M(B(t_n, u^n)\Delta W^{n+1}), \tau Mu^n + \tau F(t_{n+1}, u^{n+1}) + B(t_n, u^n)\Delta W^{n+1} \rangle_H \\
= \langle M(B(t_n, u^n)\Delta W^{n+1}), Mu^{n+1} - Mu^n \rangle_H + \langle M(B(t_n, u^n)\Delta W^{n+1}), Mu^n \rangle_H \\
+ \langle M(B(t_n, u^n)\Delta W^{n+1}), F(t_{n+1}, u^{n+1}) \rangle_H \\
\leq \frac{1}{4} \|Mu^{n+1} - Mu^n\|_H^2 + \|M(B(t_n, u^n)\Delta W^{n+1})\|_H^2 + \langle M(B(t_n, u^n)\Delta W^{n+1}), Mu^n \rangle_H \\
- \langle B(t_n, u^n)\Delta W^{n+1}, MF(t_{n+1}, u^{n+1}) \rangle_H.
\]
Substituting (4.8) and (4.9) into (4.7), we have
\[
\frac{1}{2} \left( \|Mu^{n+1}\|_H^2 - \|Mu^n\|_H^2 \right) + \frac{1}{4} \|Mu^{n+1} - Mu^n\|_H^2 \\
\leq \tau \|M\| \|Mu^{n+1}\|_H^2 + \|M(B(t_n, u^n)\Delta W^{n+1})\|_H^2 + \langle M(B(t_n, u^n)\Delta W^{n+1}), Mu^n \rangle_H.
\]
Summing (4.6) and (4.10) together leads to
\[
\frac{1}{2} \left( \|u^{n+1}\|_{S(M)}^2 - \|u^n\|_{S(M)}^2 \right) + \frac{1}{4} \|u^{n+1} - u^n\|_{S(M)}^2 \\
\leq \tau \|M\| \|u^{n+1}\|_{S(M)}^2 + \|B(t_n, u^n)\Delta W^{n+1}\|_{S(M)}^2 + \langle B(t_n, u^n)\Delta W^{n+1}, u^n \rangle_{S(M)}.
\]
After applying expectation on both sides of (4.11), one arrives at
\[
\frac{1}{2} \left( \mathbb{E}\|u^{n+1}\|_{S(M)}^2 - \mathbb{E}\|u^n\|_{S(M)}^2 \right) + \frac{1}{4} \mathbb{E}\|u^{n+1} - u^n\|_{S(M)}^2 \\
\leq \tau \|M\| \mathbb{E}\|u^{n+1}\|_{S(M)}^2 + C\mathbb{E}\|\Delta W^{n+1}\|_{HS(U,H^1)}^2 + \tau(1 + \mathbb{E}\|u^n\|_{S(M)}^2).
\]
The discrete Gronwall’s lemma then leads to the assertion of this lemma in case $\tau \leq \tau^*$ is chosen.
Step 3: Case \( p > 2 \) for \((4.4)\). In order to show assertion \((4.4)\), we employ an inductive argument. To obtain the result for \( p = 4 \), we multiply \((4.11)\) by \( \|u^{n+1}\|^2_{\mathcal{G}(M)} \), and use the identity \( (a - b)a = \frac{1}{2}(a^2 - b^2 + (a - b)^2) \) where \( a, b \in \mathbb{R} \), to get

\[
\frac{1}{4}\left(\|u^{n+1}\|^4_{\mathcal{G}(M)} - \|u^n\|^4_{\mathcal{G}(M)}\right) + \frac{1}{4}(\|u^{n+1}\|^2_{\mathcal{G}(M)} - \|u^n\|^2_{\mathcal{G}(M)})^2 + \frac{1}{4}||u^{n+1} - u^n||^2_{\mathcal{G}(M)}\|u^{n+1}\|^2_{\mathcal{G}(M)} \\
\leq C\tau\|u^{n+1}\|^2_{\mathcal{G}(M)} + C\tau\|u^{n+1}\|^4_{\mathcal{G}(M)} + \|B(t_n,u^n)\Delta W^{n+1}\|^2_{\mathcal{G}(M)}\|u^{n+1}\|^2_{\mathcal{G}(M)} \\
+ \langle B(t_n,u^n)\Delta W^{n+1}, u^n \rangle_{\mathcal{G}(M)}\|u^{n+1}\|^2_{\mathcal{G}(M)}.
\]

(4.12)

After applying expectation on both sides of the above inequality and using the linear growth property of \( B \), one gets

\[
\frac{1}{4}\left(\mathbb{E}\|u^{n+1}\|^4_{\mathcal{G}(M)} - \mathbb{E}\|u^n\|^4_{\mathcal{G}(M)}\right) + \frac{1}{8}\mathbb{E}(\|u^{n+1}\|^2_{\mathcal{G}(M)} - \|u^n\|^2_{\mathcal{G}(M)})^2 \\
\leq C\tau\mathbb{E}\|u^{n+1}\|^2_{\mathcal{G}(M)} + C\tau\mathbb{E}\|u^{n+1}\|^4_{\mathcal{G}(M)} + \mathbb{E}\|Q^2\|^2_{\mathcal{H}(U,H^1(\mathbb{D}))}\mathbb{E}(\|u^n\|^2_{\mathcal{G}(M)}(1 + \mathbb{E}\|u^n\|^2_{\mathcal{G}(M)}))
\]

(4.13)

The discrete Gronwall’s lemma then leads to the assertion for \( p = 4 \) in case \( \tau \leq \tau^* \) is chosen.

Using the case when \( p = 2 \) and \( p = 4 \), it is easy to check that the following holds true

\[
\mathbb{E}\|u^n\|^3_{\mathcal{G}(M)} \leq \frac{1}{2}\left(\mathbb{E}\|u^n\|^2_{\mathcal{G}(M)} + \mathbb{E}\|u^n\|^4_{\mathcal{G}(M)}\right) \leq C,
\]

which leads to the assertion for \( p = 3 \).

By repeating the above procedure, we could show that the assertion holds for general \( p \geq 2 \). Thus we complete the proof. \( \square \)

4.2. Mean-square convergence order. In this subsection we investigate the convergence order in mean-square sense of semidiscretization \((4.2)\) via the truncation error approach.

Denote by \( \delta^{n+1} \) the truncation error of the semi-implicit Euler scheme, i.e.,

\[
\delta^{n+1} := u(t_{n+1}) - u(t_n) - \tau Mu(t_{n+1}) - \tau F(t_{n+1},u(t_{n+1})) - B(t_n,u(t_n))\Delta W^{n+1},
\]

(4.14)

where \( u(t) \) means the solution of stochastic Maxwell equations \((2.11)\). The estimate of this truncation error is stated in the following lemma.

Lemma 4.2. Let Assumptions \((3.1)\) and \((3.2)\) be fulfilled with \( k = 2 \), and suppose that \( u_0 \) is an \( \mathcal{F}_0 \)-measurable random variable satisfying \( \|u_0\|_{L^2(\Omega;\mathcal{G}(M^2))} < \infty \). Then we have

\[
\mathbb{E}\|\delta^{n+1}\|^2_{\mathcal{H}} \leq C\tau^2, \quad \mathbb{E}\|\mathbb{E}(\delta^{n+1}\|\mathcal{F}_{t_n})\|^2_{\mathcal{H}} \leq C\tau^3,
\]

(4.15)

where the positive constant \( C \) depends on the Lipschitz coefficients of \( F \) and \( B \), \( T \), \( \|u_0\|_{L^2(\Omega;\mathcal{G}(M^2))} \) and \( \|Q^2\|_{\mathcal{H}(U,H^2(\mathbb{D}))} \)
Proof. By replacing the expression
\[
 u(t_{n+1}) - u(t_n) = \int_{t_n}^{t_{n+1}} Mu(s)ds + \int_{t_n}^{t_{n+1}} F(s, u(s))ds + \int_{t_n}^{t_{n+1}} B(s, u(s))dW(s)
\]
into (4.14), we have
\[
\delta^{n+1} = \int_{t_n}^{t_{n+1}} (Mu(s) - Mu(t_{n+1}))ds + \int_{t_n}^{t_{n+1}} (F(s, u(s)) - F(t_{n+1}, u(t_{n+1})))ds \\
+ \int_{t_n}^{t_{n+1}} (B(s, u(s)) - B(t_n, u(t_n)))dW(s).
\]
(4.16)

Then,
\[
\mathbb{E}\|\delta^{n+1}\|_{\mathbb{H}}^2 \leq \mathbb{E}\left(\int_{t_n}^{t_{n+1}} \left(\|Mu(s) - Mu(t_{n+1})\|_{\mathbb{H}}^2 + \|F(s, u(s)) - F(t_{n+1}, u(t_{n+1}))\|_{\mathbb{H}}^2 + \|B(s, u(s)) - B(t_n, u(t_n))\|_{\mathbb{H}}^2\right)ds\right) \\
=: I + II + III.
\]

Using Hölder inequality to the first term I leads to
\[
I \leq \tau \mathbb{E} \int_{t_n}^{t_{n+1}} \|Mu(s) - Mu(t_{n+1})\|_{\mathbb{H}}^2 ds \\
\leq \tau \int_{t_n}^{t_{n+1}} \mathbb{E}\|u(s) - u(t_{n+1})\|^2_{\mathcal{D}(M)} ds.
\]
(4.17)

Based on Proposition 3.2, it holds
\[
I \leq C \tau^3,
\]
(4.18)

where C depends on the coefficients F and B, T, \(\|Q^1\|_{HS(U, H^{2+\gamma}(D))}\) and \(\|u_0\|_{L^2(\Omega; \mathcal{D}(M^2))}\).

For the second term II, similarly, by Proposition 3.2 and continuous differentiability of F with respect to t, we have
\[
II \leq \tau \mathbb{E} \int_{t_n}^{t_{n+1}} \|F(s, u(s)) - F(t_{n+1}, u(t_{n+1}))\|_{\mathbb{H}}^2 ds \\
\leq \tau \mathbb{E} \int_{t_n}^{t_{n+1}} \|F(s, u(s)) - F(s, u(t_{n+1}))\|_{\mathbb{H}}^2 ds \\
+ \tau \mathbb{E} \int_{t_n}^{t_{n+1}} \|F(t_{n+1}, u(t_{n+1})) - F(s, u(t_{n+1}))\|_{\mathbb{H}}^2 ds \\
\leq C \tau \int_{t_n}^{t_{n+1}} \mathbb{E}\|u(s) - u(t_{n+1})\|^2_{\mathbb{H}} + \|\partial_s F(\theta, u(t_{n+1}))(t_{n+1} - s)\|^2_{\mathbb{H}} ds \\
\leq C \tau^3,
\]
(4.19)

where C depends on the coefficients F and B, T, \(\|Q^1\|_{HS(U, H^{1+\gamma}(D))}\) and \(\|u_0\|_{L^2(\Omega; \mathcal{D}(M))}\).
By the infinite dimensional Itô isometry formula, for the third term $III$ we get,

$$III = \mathbb{E} \int_{t_n}^{t_{n+1}} \| (B(s, u(s)) - B(t_n, u(t_n))) \|^2_{HS(U_0, H)} \, ds$$

$$\leq \mathbb{E} \int_{t_n}^{t_{n+1}} \| (B(s, u(s)) - B(s, u(t_n))) \|^2_{HS(U_0, H)} \, ds$$

$$+ \mathbb{E} \int_{t_n}^{t_{n+1}} \| (B(s, u(t_n)) - B(t_n, u(t_n))) \|^2_{HS(U_0, H)} \, ds$$

$$\leq C \| Q^\frac{1}{2} \|_{HS(U_0, H^1(D))}^2 \mathbb{E} \int_{t_n}^{t_{n+1}} \| u(s) - u(t_n) \|^2_H \, ds$$

$$+ \mathbb{E} \int_{t_n}^{t_{n+1}} \| \partial_t B(\theta_1, u(t_n))(s - t_n) \|^2_{HS(U_0, H)} \, ds$$

$$\leq C \tau^2,$$

where $C$ depends on the coefficients $F$ and $B$, $T$, $\| Q^\frac{1}{2} \|_{HS(U_0, H^1+\gamma(D))}$ and $\| u_0 \|_{L^2(\Omega; \mathcal{D}(M))}$. Combining the above equations, we can obtain the first assertion.

In the similar way, we can prove that

$$\mathbb{E} \| (\delta^{n+1} | \mathcal{F}_{t_n}) \|^2_{\mathbb{H}}$$

$$\leq \mathbb{E} \left( \int_{t_n}^{t_{n+1}} M(u(s) - u(t_n+1)) \, ds \right)^2 + \mathbb{E} \left( \int_{t_n}^{t_{n+1}} (F(s, u(s)) - F(t_n+1, u(t_n+1))) \, ds \right)^2$$

$$\leq \mathbb{E} \left( \int_{t_n}^{t_{n+1}} M(u(s) - u(t_n+1)) \, ds \right)^2 + \mathbb{E} \left( \int_{t_n}^{t_{n+1}} (F(s, u(s)) - F(t_n+1, u(t_n+1))) \, ds \right)^2$$

$$\leq I + II \leq C \tau^3,$$

where $C$ depends on the coefficients $F$ and $B$, $T$, $\| Q^\frac{1}{2} \|_{HS(U_0, H^2+\gamma(D))}$ and $\| u_0 \|_{L^2(\Omega; \mathcal{D}(M^2))}$. Thus, we have finished the proof.

Denote by $e^n = u(t_n) - u^n$, then the main result of this paper is stated in the following theorem.

**Theorem 4.1.** Under the same assumption of Lemma 4.2 we have

$$\max_{0 \leq n \leq N} \left( \mathbb{E} \| e^n \|^2_{\mathbb{H}} \right)^{1/2} \leq C \tau^\frac{1}{4},$$

where the positive constant $C$ may depend on the Lipschitz coefficients of $F$ and $B$, $T$, $\| u_0 \|_{L^2(\Omega; \mathcal{D}(M^2))}$ and $\| Q^\frac{1}{2} \|_{HS(U_0, H^2+\gamma(D))}$ but independent of $\tau$ and $n$.

**Proof.** Subtracting equation (4.2) from (4.14), it leads to

$$e^{n+1} - e^n = \delta^{n+1} + \tau M e^{n+1} + \tau \left( F(t_{n+1}, u(t_{n+1})) - F(t_{n+1}, u^{n+1}) \right)$$

$$+ \left( B(t_n, u(t_n)) - B(t_n, u^n) \right) \Delta W^{n+1}.$$  (4.21)
Taking the $\mathbb{H}$-inner product of the above equality with $e^{n+1}$, we get

$$
\langle e^{n+1} - e^n, e^{n+1} \rangle_{\mathbb{H}} = \langle \delta^{n+1}, e^{n+1} \rangle_{\mathbb{H}} + \tau \langle Me^{n+1}, e^{n+1} \rangle_{\mathbb{H}} + \tau \langle F(t_{n+1}, u(t_{n+1})) - F(t_{n+1}, u^{n+1}), e^{n+1} \rangle_{\mathbb{H}} + \left( \langle B(t_n, u(t_n)) - B(t_n, u^n) \rangle \Delta W^{n+1}, e^{n+1} \right)_{\mathbb{H}}
$$

(4.22)

Noticing that $e^{n+1} = \frac{1}{2}(e^{n+1} - e^n) + \frac{1}{2}(e^{n+1} + e^n)$, for the left-hand side of the above equation, we have

$$
\langle e^{n+1} - e^n, e^{n+1} \rangle_{\mathbb{H}} = \frac{1}{2} \| e^{n+1} \|_{\mathbb{H}}^2 - \frac{1}{2} \| e^n \|_{\mathbb{H}}^2 + \frac{1}{2} \| e^{n+1} - e^n \|_{\mathbb{H}}^2.
$$

(4.23)

For the first term in the right-hand side of (4.22), it follows from $ab \leq a^2 + \frac{1}{4}b^2$ that

$$
\langle \delta^{n+1}, e^{n+1} \rangle_{\mathbb{H}} = \langle \delta^{n+1}, e^{n+1} - e^n \rangle_{\mathbb{H}} + \langle \delta^{n+1}, e^n \rangle_{\mathbb{H}} \leq \| \delta^{n+1} \|_{\mathbb{H}} \| e^{n+1} - e^n \|_{\mathbb{H}} + \langle \delta^{n+1}, e^n \rangle_{\mathbb{H}} \leq \| \delta^{n+1} \|_{\mathbb{H}}^2 + \frac{1}{8} \| e^{n+1} - e^n \|_{\mathbb{H}}^2 + \langle \delta^{n+1}, e^n \rangle_{\mathbb{H}}.
$$

(4.24)

After applying expectation on both sides of the above inequality, we have

$$
\mathbb{E} \langle \delta^{n+1}, e^{n+1} \rangle_{\mathbb{H}} \leq \mathbb{E} \| \delta^{n+1} \|_{\mathbb{H}}^2 + \frac{1}{8} \mathbb{E} \| e^{n+1} - e^n \|_{\mathbb{H}}^2 + \mathbb{E} \langle \delta^{n+1}, \mathbb{E}(\delta^{n+1}|\mathcal{F}_t_n), e^n \rangle_{\mathbb{H}} \leq \mathbb{E} \| \delta^{n+1} \|_{\mathbb{H}}^2 + \frac{1}{8} \mathbb{E} \| e^{n+1} - e^n \|_{\mathbb{H}}^2 + \mathbb{E} \| \mathbb{E}(\delta^{n+1}|\mathcal{F}_t_n) \|_{\mathbb{H}}^2 + \frac{1}{8} \mathbb{E} \| e^n \|_{\mathbb{H}}^2 \leq C \tau^2 + \mathbb{E} \| e^n \|_{\mathbb{H}}^2 + \frac{1}{8} \mathbb{E} \| e^{n+1} - e^n \|_{\mathbb{H}}^2,
$$

(4.25)

where in the last step, we utilize the results on the estimates for truncation error $\delta^{n+1}$ in Lemma 4.2.

For the second term in the right-hand side of (4.22), utilizing the skew-adjointness of the Maxwell operator $M$, it holds

$$
\langle Me^{n+1}, e^{n+1} \rangle_{\mathbb{H}} = 0.
$$

(4.26)

For the third and forth terms in the right-hand side of (4.22), utilizing the global Lipschitz properties of $F$ and $B$, respectively, we obtain

$$
\tau \langle F(t_{n+1}, u(t_{n+1})) - F(t_{n+1}, u^{n+1}), e^{n+1} \rangle_{\mathbb{H}} \leq \tau \| F(t_{n+1}, u(t_{n+1})) - F(t_{n+1}, u^{n+1}) \|_{\mathbb{H}} \| e^{n+1} \|_{\mathbb{H}} \leq C \tau \| e^{n+1} \|_{\mathbb{H}}^2,
$$

(4.27)
and

\[
\left\langle \left( B(t_n, u(t_n)) - B(t_n, u^n) \right) \Delta W^{n+1}, e^{n+1} \right\rangle_{\mathbb{H}}
\]

\[
\begin{aligned}
&= \left\langle \left( B(t_n, u(t_n)) - B(t_n, u^n) \right) \Delta W^{n+1}, e^n \right\rangle_{\mathbb{H}} \\
&\quad + \left\langle \left( B(t_n, u(t_n)) - B(t_n, u^n) \right) \Delta W^{n+1}, e^n \right\rangle_{\mathbb{H}} \\
&\leq \left\| \left( B(t_n, u(t_n)) - B(t_n, u^n) \right) \Delta W^{n+1} \right\|^2_{\mathbb{H}} + \frac{1}{8} \left\| e^{n+1} - e^n \right\|^2_{\mathbb{H}} \\
&\quad + \left\langle \left( B(t_n, u(t_n)) - B(t_n, u^n) \right) \Delta W^{n+1}, e^n \right\rangle_{\mathbb{H}}.
\end{aligned}
\]  

(4.28)

After applying expectation on both sides of the above inequality (4.28) and using the global Lipschitz property of $B$, we get

\[
\mathbb{E} \left\langle \left( B(t_n, u(t_n)) - B(t_n, u^n) \right) \Delta W^{n+1}, e^{n+1} \right\rangle_{\mathbb{H}}
\]

\[
\leq \mathbb{E} \left\| \left( B(t_n, u(t_n)) - B(t_n, u^n) \right) \Delta W^{n+1} \right\|^2_{\mathbb{H}} + \frac{1}{8} \mathbb{E} \left\| e^{n+1} - e^n \right\|^2_{\mathbb{H}}
\]

\[
\leq \| Q^1 \|_{HS(U, H^2(\Omega))}^2 \tau \mathbb{E} \left\| e^n \right\|_{\mathbb{H}}^2 + \frac{1}{8} \mathbb{E} \left\| e^{n+1} - e^n \right\|^2_{\mathbb{H}}.
\]  

(4.29)

Substituting (4.23), (4.25), (4.26), (4.27) and (4.29) into (4.22), leads to

\[
\frac{1}{2} \mathbb{E} \left\| e^{n+1} \right\|^2_{\mathbb{H}} - \frac{1}{2} \mathbb{E} \left\| e^n \right\|^2_{\mathbb{H}} + \frac{1}{2} \mathbb{E} \left\| e^{n+1} - e^n \right\|^2_{\mathbb{H}}
\]

\[
\leq C \tau^2 + \tau \mathbb{E} \left\| e^n \right\|^2_{\mathbb{H}} + \frac{1}{4} \mathbb{E} \left\| e^{n+1} - e^n \right\|^2_{\mathbb{H}} + C \tau \mathbb{E} \left\| e^{n+1} \right\|^2_{\mathbb{H}} + \| Q^1 \|_{HS(U, H^2(\Omega))} \tau \mathbb{E} \left\| e^n \right\|^2_{\mathbb{H}}.
\]  

(4.30)

The discrete Gronwall’s lemma leads to the assertion in case $\tau \leq \tau^*$ is chosen.

Thus, the proof is completed. \qed

Remark 4.1. If a $\theta$-method is applied to discretize stochastic Maxwell equations (2.11) in the temporal direction, i.e.,

\[
u^{n+1} = u^n + \theta \tau Mu^{n+1} + (1 - \theta) \tau Mu^n + \theta \tau F(t_{n+1}, u^{n+1}) + (1 - \theta) \tau F(t_n, u^n) + B(t_n, u^n) \Delta W^{n+1},
\]

then via the same procedure as Theorem 4.7, we could derive the result of mean-square convergence order $1/2$, i.e.,

\[
\max_{0 \leq n \leq N} \left( \mathbb{E} \left\| e^n \right\|_{\mathbb{H}}^2 \right)^{1/2} \leq C \tau^2,
\]  

(4.32)

where the positive constant $C$ may depend on the Lipschitz coefficients of $F$ and $B$, $T$, $\| u_0 \|_{L^2(\Omega)}$ and $\| Q^1 \|_{HS(U, H^2(\Omega))}$, but independent of $\tau$ and $n$. The key character in the proof is the appearance of the positive term $\| e^{n+1} - e^n \|^2_{\mathbb{H}}$ in the right-hand side of (4.30), which could absorb the difficulty caused by the stochasticity of the continuous and discrete systems.
5. Conclusions

In this paper, we consider a semi-implicit discretization in temporal direction for stochastic nonlinear Maxwell equations. First, we establish the regularity properties of the continuous and discrete problems. Then based on these regularity properties and utilizing the energy estimate technique, the mean-square convergence order $1/2$ is derived.

Future work will include the study for the full discretization of the stochastic Maxwell equations, in which the error estimates in spatial direction depend on the enough smoothness of the noise covariance and the initial data. Besides, due to the high dimensions and stochasticity of stochastic Maxwell equations, the computational implement is an important and technical issue. In order to approximate this problem efficiently and effectively, some techniques such as splitting approach may be employed, and thus the analysis of the effect on the convergence order induced by these techniques also constitutes future work.

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