Entire solutions to equivariant elliptic systems
with variational structure

Nicholas D. Alikakos* and Giorgio Fusco†

Abstract

In the present paper we consider the system $\Delta u - W_u(u) = 0$, where $u : \mathbb{R}^n \to \mathbb{R}^n$, for a class of potentials $W : \mathbb{R}^n \to \mathbb{R}$ that possess several global minima and are invariant under a general finite reflection group $G$. We establish existence of nontrivial entire solutions connecting the global minima of $W$ along certain directions at infinity.

1 Introduction

We consider the system

\begin{equation}
\Delta u - W_u(u) = 0, \quad \text{for } u : \mathbb{R}^n \to \mathbb{R}^n,
\end{equation}

where $W : \mathbb{R}^n \to \mathbb{R}$ and the gradient $W_u := (\frac{\partial W}{\partial u_1}, \ldots, \frac{\partial W}{\partial u_n})^\top$; the system above is the Euler–Lagrange equation corresponding to the action

\begin{equation}
J(u) = \int_{\mathbb{R}^n} \left\{ \frac{1}{2} |\nabla u|^2 + W(u) \right\} \, dx.
\end{equation}

One of the obstructions in the study of (1.1) is that the action is infinite for the class of solutions we are interested in, for dimensions $n \geq 2$.

We begin by introducing and explaining the hypotheses on the potential $W$.

**Hypothesis 1** ($N$ nondegenerate global minima). The potential $W$ is of class $C^2$, satisfying $W = 0$ on $A = \{a_1, \ldots, a_N\}$ and $W > 0$ in $\mathbb{R}^n \setminus A$. Furthermore, $\partial^2 W(u) \geq c^2 \text{Id}$ for $|u - a_i| \leq r_0$, with $r_0 > 0$ fixed, and for $i = 1, \ldots, N$.

We recall some of the known examples that have been studied in the past. The case $n = 1, N = 2$ is textbook material and the corresponding solution is known as the heteroclinic connection. In the work of Dang, Fife, and Peletier [8], a saddle solution was constructed on the plane under a symmetry assumption on $W$. There, the solution is scalar but can be trivially embedded in our setup by taking the second component equal to zero, resulting in a

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*Department of Mathematics, University of Athens, Panepistemiopolis, 15784 Athens, Greece and Institute for Applied and Computational Mathematics, Foundation of Research and Technology – Hellas, 71110 Heraklion, Crete, Greece; e-mail: nalinko@math.uoa.gr

†Dipartimeno di Matematica Pura ed Applicata, Università degli Studi dell’Aquila, Via Vetoio, 67010 Coppito, L’Aquila, Italy; e-mail: fusco@univaq.it
problem of the form (1.1) with \( n = 2, N = 2 \). A genuine vector extension of the Dang–Fife–Peletier solution can be found in Alama, Bronsard, and Gui [1]. In [5], Bronsard, Gui, and Schatzman constructed a solution for \( n = 2, N = 3 \), while recently in [19], Gui and Schatzman constructed one for \( n = 3, N = 4 \); these last two solutions are known as the triple-junction solution on the plane and the quadruple-junction solution in space respectively and they were also obtained under symmetry hypotheses on \( W \). They are related to the geometric evolution of interfaces evolving by mean curvature and satisfying Plateau angle conditions; see the work of Mantegazza, Novaga, and Tortorelli [23] and the recent works of Freire [14, 15], Mazzeo and Sáez [24], Schnürer and Schulze [30], Schnürer et al. [31], and Ikota and Yanagida [22]. These solutions have been formally linked to the singular limit in sharp-interface evolution problems (see Bronsard and Reitich [6] for the case \( n = 2 \) and Rubinstein, Sternberg, and Keller [27] for related previous work) and were subsequently studied in [5, 19]. The \( \Gamma \)-limit of a rescaled action \( J \) over a bounded domain under a volume constraint has been studied by Baldo in [4]. For rigorous linking results for the evolution problem, see the work of Sáez Trumper [28, 29].

Hypothesis 2 (Symmetry). The potential \( W \) is invariant under a finite reflection group \( G \) in \( \mathbb{R}^n \) (Coxeter group), that is,

\[
W(ugu) = W(u), \quad \text{for all } g \in G \text{ and } u \in \mathbb{R}^n.
\]

Moreover, we assume that \( W(u) \geq \max_{\partial C_0} W \), for \( u \) outside a certain bounded, \( G \)-invariant convex set \( C_0 \).

We seek equivariant solutions to system (1.1), that is, solutions satisfying

\[
u(gx) = gu(x), \quad \text{for all } g \in G \text{ and } x \in \mathbb{R}^n.
\]

For notation and algebraic background we refer to [18]. In the past, the following groups have been employed: in \([8]\) \( G = \mathcal{H}_2^2 \), the dihedral group on the plane, with four elements, in \([5]\) \( G = \mathcal{H}_2^3 \), the group of symmetries of the equilateral triangle, with six elements, and in \([19]\) \( G = T^* \), the group of symmetries of the tetrahedron, with twenty four elements.

Hypothesis 3 (Location and number of global minima). Let \( F \subset \mathbb{R}^n \) be a fundamental region of \( G \). We assume that \( \bar{F} \) (the closure of \( F \)) contains a single global minimum of \( W \), say \( a_1 \), and let \( \text{Stab}(a_1) \) be the subgroup of \( G \) that leaves \( a_1 \) fixed. Then,

\[
N := \frac{|G|}{|\text{Stab}(a_1)|}.
\]

We give here some examples. For \( \mathcal{H}_2^3 \) on the plane, we can take as \( F \) the sector. If \( a_1 \in F \), then \( N = 6 \), while if \( a_1 \) is on the walls, then \( N = 3 \). Finally, if \( a_1 \) is placed at the origin, then \( N = 1 \). In higher dimensions, we have more options since we can place \( a_1 \) in the interior of \( F \), in the interior of a face, on an edge, and so on. For example, if \( G = W^* \), the group of symmetries of the cube in three-dimensional space, then \( |G| = 48 \). If the cube is situated with its center at the origin and its vertices at the eight points \((\pm 1, \pm 1, \pm 1)\), then we can take as \( F \) the simplex generated by \( s_1 = e_1 + e_2 + e_3 \), \( s_2 = e_2 + e_3 \), and \( s_3 = e_3 \), where the \( e_i \)'s are the standard basis vectors. We have then the following options:
(i) On the edge $s_3$, $N = 6$.
(ii) On the edge $s_1$, $N = 8$.
(iii) On the edge $s_2$, $N = 12$.
(iv) In the interior of a face, $N = 24$.
(v) In the interior of the fundamental region, $N = 48$.
(vi) At the origin, $N = 1$.

**Hypothesis 4 (Q-monotonicity).** Let

\[
D := \text{Int} \left( \bigcup_{g \in \text{Stab}(a_1)} g \bar{F} \right).
\]

We restrict ourselves to potentials $W$ for which there is a function $Q : \bar{D} \to \mathbb{R}$ with the following properties:

\begin{align*}
(1.7a) & \quad Q \text{ is convex}, \\
(1.7b) & \quad Q(u) > 0 \text{ and } Q_u(u) \neq 0, \text{ on } \bar{D} \setminus \{a_1\}, \\
(1.7c) & \quad Q(u + a_1) = |u| + o(|u|),
\end{align*}

and

\[
(1.8) \quad Q_u(u) \cdot W_u(u) \geq 0, \text{ in } D \setminus \{a_1\}.
\]

For $n = 1$ and odd symmetry, for a double-well potential $W$, and $D = \{u > 0\}$, $Q$-monotonicity implies that $W$ is monotone in $D$ along the ray emanating from $a_1$. For $G = H^3_2$ on the plane, $F$ the $\frac{\pi}{3}$ sector, and $a_1 = (1,0)$, it can be verified that the triple-well potential

\[ W(u_1, u_2) = |u|^4 + 2u_1u_2^2 - \frac{2}{3}u_1^3 - |u|^2 + \frac{2}{3} \]

satisfies the $Q$-monotonicity condition in $D = \{(r, \theta) \mid r > 0, \theta \in (-\frac{\pi}{3}, \frac{\pi}{3})\}$, with $Q(u) = |u - a_1|$, where $u = (u_1, u_2)$. For $n = 3$, $G = T^*$, $F$ the simplicial cone generated by

\[
\left( \sqrt{\frac{2}{3}}, 0, \frac{1}{\sqrt{3}} \right), \quad (0, \sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}), \quad (0, 0, \frac{1}{\sqrt{3}}),
\]

and

\[
a_1 = \left( \sqrt{\frac{2}{3}}, 0, \frac{1}{\sqrt{3}} \right),
\]

we can take as an example the quadruple-well potential

\[ W(u_1, u_2, u_3) = |u|^4 - \frac{4}{\sqrt{3}}(u_1^2 - u_2^2)u_3 - \frac{2}{3}|u|^2 + \frac{5}{9}, \]

with $Q(u) = |u - a_1|$, where $u = (u_1, u_2, u_3)$, and $D$ the simplicial cone generated by

\[
(0, \sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}), (0, -\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}), \text{ and } \left( \sqrt{\frac{2}{3}}, 0, -\frac{1}{\sqrt{3}} \right).
\]
Next we explain how the $Q$-monotonicity is utilized in the proof. Let $u$ be a $C^2$ solution to (1.1); then, utilizing the smoothness of convex functions \[12\],
\[
\Delta Q(u(x)) = \text{tr} \left\{ (\partial^2 Q)(\nabla u)(\nabla u)^\top \right\} + Q_u(u(x)) \Delta u(x).
\]
If now $u$ has the property \[1.10\]
\[
u : \bar{D} \to \bar{D},
\]
then, from (1.9) and convexity it follows that
\[
\Delta Q(u(x)) \geq Q_u(u(x)) \Delta u(x) \geq Q_u(u(x)) \cdot W_u(u(x)) \geq 0.
\]
Subharmonicity then provides a global first estimate on $u$. Hence, a key step is to show that the candidate solution $u$ is a positive map, that is, it satisfies the positivity property (1.10).

We now proceed with the statement of the main results.

**Theorem 1.1.** Under Hypotheses 1–4, there exists an equivariant classical solution to system (1.1) such that:

(i) $|u(x) - a_1| \leq Ke^{-kd(x,\partial D)}$, for some positive constants $k, K$ and for $x \in D$,

(ii) $u(D) \subset D$.

In particular, $u$ connects the $N = |G|/|\text{Stab}(a_1)|$ global minima of $W$:
\[
\lim_{\lambda \to +\infty} u(\lambda ga_1) = ga_1, \text{ for all } g \in G.
\]

The proof of Theorem 1.1 is based on a family of constrained minimization problems
\[
\text{(1.12a)} \quad \min_{J_{\Omega_R}} \text{, where } J_{\Omega_R}(u) = \int_{\Omega_R} \left\{ \frac{1}{2} |\nabla u|^2 + W(u) \right\} \text{dx},
\]
under a constraint enforcing the desirable behavior at infinity,
\[
\text{(1.12b)} \quad |u(x) - a_1| \leq \bar{q}, \text{ for } x \in C_R \subset D \cap \Omega_R.
\]
Here, $\bar{q}$ is a fixed positive number with $0 < \bar{q} \leq r_0$ (cf. Hypothesis 1), \{\Omega_R\} is an appropriate class of homothetic symmetric domains (for example, $\Omega_R = \{ x \in \mathbb{R}^n \mid |x| < R \}$) over which the action $J_{\Omega_R}$ is finite, and $C_R$ is a ball $B(x_R, L) \subset D \cap \Omega_R$ with $x_R = \frac{R}{2} x_0$ and $L > 0$, fixed, independent of $R$, and sufficiently large. Problems (1.12) provide a family of minimizers \{u_R\}; we seek then to construct the solution by taking the limit, that is,
\[
\text{(1.13)} \quad u(x) = \lim_{R \to \infty} u_R(x).
\]
For carrying out this procedure successfully, we need uniform estimates in $R$. The following estimates are obtained in the course of the proof of Theorem 1.1.
Theorem 1.2 (Uniform $R$-estimates). Under Hypotheses 1–4, there is an $R_0 > 0$, such that, for $R > R_0$, there exists an equivariant local minimizer $u_R$ of $J_{\Omega_R}$ which is a positive map and satisfies the estimates

$$(1.14a) \quad J_{\Omega_R}(u) \leq c_0 R^{n-1},$$

$$(1.14b) \quad |u_R(x) - a_1| \leq Ke^{-kd(x,\partial D_R)}, \text{ for } x \in D_R := D \cap \Omega_R,$$

where $k, K$ are positive constants independent of $R$.

The estimate (1.14b) is inherited in the limit (1.13). Note that this estimate excludes the trivial solution $u \equiv 0$.

Our proof consists of a geometric part and a PDE part. The geometric part is concerned with positivity; it utilizes the gradient flow

$$\left\{ \begin{array}{l} u_t = \Delta u - W_u(u), \text{ in } \Omega_R \times (0, \infty), \\ \partial u / \partial n = 0, \text{ on } \partial \Omega_R \times (0, \infty). \end{array} \right.$$ (1.15)

in the Sobolev space of equivariant maps $W^{1,2}_E(\Omega_R; \mathbb{R}^n)$. We establish that the set of positive maps

$$(1.16) \quad \{ f \in W^{1,2}(\Omega_R; \mathbb{R}^n) \mid f(D \cap \Omega_R) \subset \bar{D} \}$$

is strongly (positively) invariant under the flow (1.15). With the help of this invariance, we establish that the minimization problem (1.12) for $R > R_0$ has a solution that satisfies the Euler–Lagrange equation $\Delta u - W_u(u) = 0$ in $\Omega_R$, which is positive as well. We remark that the gradient flow is not used further in the rest of the paper. Concerning positivity we also note that the property $u : \bar{F} \to \bar{F}$ ($F$-positivity) together with $G$-equivariance, implies $u : \bar{D} \to \bar{D}$ ($D$-positivity), but not vice versa. Theorem 1.1 always guarantees the existence of an $F$-positive (and hence, $D$-positive), $G$-equivariant solution. In the event, however, that $D$ is the fundamental region of a symmetry subgroup $\mathcal{H}_D$ of $G$, then it also provides another potentially distinct solution with less symmetry that is $D$-positive (but not necessarily $F$-positive) and $\mathcal{H}_D$-equivariant (but not necessarily $G$-equivariant). For example, for the triple-junction problem on the plane, the relevant group is $\mathcal{H}_3^2$ and in this case, if $a_1$ is placed on an edge of $F$ (cf. Hypothesis 3), $D$ is the $2\pi$ sector and Theorem 1.1 provides a solution that is $\mathcal{H}_3^2$-equivariant; here there is no $\mathcal{H}_D$. On the other hand, for the group of symmetries of the square $\mathcal{H}_4^3$, and under analogous conditions for $a_1$, we obtain via Theorem 1.1 in addition to an $\mathcal{H}_4^3$-equivariant solution, an $\mathcal{H}_2^3$-equivariant solution. We remark in passing that the normality of Stab($a_1$) in $G$ is not a necessary condition for $D$ to be a fundamental region of a reflection subgroup of $G$.

The PDE part of the proof is concerned with estimate (1.14b). By $D$-positivity (1.10),

$$\Delta Q(u_R(x)) \geq 0, \text{ in } D \cap \Omega_R.$$ (1.17)
On the other hand, by the nondegeneracy condition in Hypothesis 1

\[(1.18) \quad \Delta p_R(x) \geq c^2 p_R(x), \text{ in } C_R, \]

where
\[p_R(x) := |u(x) - a_1| (\leq \bar{q} \text{ in } C_R).\]

In the case where \(Q(u) = |u - a_1|\), estimate (1.17) provides a first global bound on \(p_R(x)\) in \(D \cap \Omega_R\) while estimate (1.18) implies a stronger exponential bound on \(p_R(x)\) in \(C_R\). By alternating between (1.17) and (1.18), we can keep improving the range of validity of the exponential estimate and we can establish that it holds on a large domain, independent of \(R\). For general \(Q\) though, we need a global change of coordinates first.

Our understanding of the problem owes a great deal to the fundamental works [5, 1]. However, our approach is generally different and, in particular, our assumptions are not generalizations of the ones in those works. In [5] and [19] they proceed via Dirichlet problems and build up a higher-dimensional object out of lower-dimensional solutions. We instead construct a nontrivial solution via Neumann problems, which can later be dissected (not done in the present paper). The paper [2] contains some seeds of the present work.

Symmetry is a rather restrictive assumption. A possible approach for relaxing it could be to establish the stability of the constructed solution(s) in the class of general compact perturbations. This is reasonable for at least some of these solutions which enjoy extra minimality properties (as, for example, the triple-junction solution). Sáez Trumper [28, 29] has extended the results in [5] to the nonsymmetric case utilizing the gradient flow. Finally, in light of [2], uniqueness should not be expected in general.

The scalar problem related to (1.1), for \(u : \mathbb{R}^n \to \mathbb{R}\), and without any symmetry hypotheses on the potential, has been the object of intensive investigation for many years, with the De Giorgi conjecture and the related contributions at the center of this activity (see the expository article of Farina and Valdinoci [13]). It is well known that the vector nature of (1.1) is of significance. On the physical side, we note that for describing coexistence of three or more phases \((N \geq 3)\), a vector-order parameter \(u\) is needed. A triple-well in \(\mathbb{R}^2\) or a quadruple-well in \(\mathbb{R}^3\) would be appropriate for modeling coexistence of three or four phases correspondingly, with the origin \(x = 0\) representing the coexistence point. On the geometric side, the rescaled solution \(u_\varepsilon(x) := u(x/\varepsilon)\) in the triple- and quadruple-well cases is expected to converge, as \(\varepsilon \to 0\), to the solution of the corresponding partitioning problem (see Baldo [4]). The boundaries of the partitioning sets form a system of minimal surfaces meeting each other along free-boundary curves called ‘liquid edges’, and liquid edges meet at ‘supersingular’ points which coincide with the coexistence points mentioned above (cf. Dierkes et al. [9, §4.10.7]).

The paper is structured as follows. In Section 2 we establish the strong positivity property of the semigroup that (1.15) generates. In Section 3 we introduce the Q-coordinate system and in Section 4 we state and prove the comparison lemmas related to estimates (1.17) and (1.18). Finally, in Section 5 we give the proofs of Theorems 1.1 and 1.2 together.
2 The positivity property

2.1 Algebraic preliminaries

Let $G$ be a Coxeter group, that is, a finite effective subgroup of the orthogonal group $O(\mathbb{R}^n)$, generated by a set of reflections. The reflection with respect to the hyperplane $P_r = \{ x \in \mathbb{R}^n \mid \langle x, r \rangle = 0 \}$ is the linear transformation

$$S_r u = u - 2\langle u, r \rangle r,$$

for $u \in \mathbb{R}^n$ and $|r| = 1$,

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product in $\mathbb{R}^n$. Every finite subgroup of $O(\mathbb{R}^n)$ has a fundamental region, that is, a subset $F \subset \mathbb{R}^n$ with the following properties:

(i) $F$ is open and convex,

(ii) $F \cap g(F) = \emptyset$ if $\text{Id} \neq g \in G$, where Id is the identity,

(iii) $\mathbb{R}^n = \bigcup \{ g(F) \mid g \in G \}$.

The two unit vectors $\pm r$ that are perpendicular to $P_r$ are called roots of $G$. The set of all roots of $G$ is denoted by $\Delta$. For a fixed $t \in \mathbb{R}^n$ such that $\langle t, r \rangle \neq 0$ for every root $r$, the set of roots of $G$ is partitioned into two subsets

$$\Delta^+ = \{ r \in \Delta \mid \langle t, r \rangle > 0 \}, \quad \Delta^- = \{ r \in \Delta \mid \langle t, r \rangle < 0 \}.$$

A subset $\Pi$ of $\Delta^+$ that is minimal with respect to the property that every $r \in \Delta^+$ is a linear combination of elements of $\Pi$, with all coefficients nonnegative, is called a t-base for $\Delta$. It can be shown that the t-base is unique and that it is also a basis for $\mathbb{R}^n$, hence $\Pi = \{ r_1, \ldots, r_n \}$.

The roots $r_i$, $i = 1, \ldots, n$, are called fundamental and the corresponding reflections $S_{r_i}$ are the fundamental reflections and generate $G$. Moreover,

$$F = \bigcap_{i=1}^n \{ u \in \mathbb{R}^n \mid \langle u, r_i \rangle \geq 0 \}.$$

The hyperplanes $P_r$ which determine the boundary of $F$ are called the walls of $F$. It can be shown that

$$\langle r_i, r_j \rangle \leq 0, \text{ for } i \neq j.$$

Given an open set $\Omega \subset \mathbb{R}^n$ invariant under $G$, that is, $g(\Omega) = \Omega$ for all $g \in G$, we define as a fundamental domain of $G$ in $\Omega$ the set $F_\Omega := \Omega \cap F$.

2.2 Parabolic flows and the equivariant class

In the following, for simplicity we assume hypotheses that are sufficient for our purposes but stronger than we actually need. We take $W$ to be a $C^2$ potential satisfying the global bound

$$|\partial_{u_i u_j}^2 W(u)| < C, \text{ in } \mathbb{R}^n.$$

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Hypothesis (2.4) can be imposed without loss of generality because of the \textit{a priori} pointwise bound (5.2). Moreover, we assume that the potential is invariant under \( G \), that is,

\begin{equation}
W : \mathbb{R}^n \rightarrow \mathbb{R}^n \cup \{0\}, \text{ with } W(gu) = W(u) \text{ for all } g \in G \text{ and } u \in \mathbb{R}^n.
\end{equation}

For a domain \( \Omega \) invariant under \( G \) and with smooth boundary, we introduce the following evolution problem in the class of maps \( u : \Omega \rightarrow \mathbb{R}^n \) that are \( G \)-equivariant.

\begin{equation}
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u - W_u(u), \text{ in } \Omega \times (0, \infty), \\
\frac{\partial u}{\partial n} &= 0, \text{ on } \partial \Omega \times (0, \infty), \text{ where } \frac{\partial}{\partial n} \text{ is the normal derivative,} \\
u(x,0) &= u_0(x), \text{ in } \Omega.
\end{aligned}
\end{equation}

We will consider (2.6) in the equivariant Sobolev space \( W^{1,2}_E(\Omega;\mathbb{R}^n) \) and denote the solution by \( u(x,t;u_0) \).

A related evolution problem is obtained by considering the equation on a fundamental domain \( F_\Omega \) with boundary conditions on the walls induced by the symmetries of the group.

\begin{equation}
\begin{aligned}
\frac{\partial \hat{u}}{\partial t} &= \Delta \hat{u} - W_u(\hat{u}), \text{ in } F_\Omega \times (0, \infty), \\
\langle \hat{u}, r_i \rangle &= 0 \text{ and } \frac{\partial \hat{u}}{\partial r_i} = \langle \frac{\partial \hat{u}}{\partial r_i}, r_i \rangle, \text{ for } x \in \partial F_\Omega \cap P_i, \text{ and for } i = 1, \ldots, n, \\
\frac{\partial \hat{u}}{\partial n} &= 0, \text{ on } \partial F_\Omega \setminus \partial F, \\
\hat{u}(x,0) &= u_0(x), \text{ on } F_\Omega.
\end{aligned}
\end{equation}

Clearly, solutions of (2.6) with sufficient smoothness satisfy (2.7), and conversely, solutions of (2.7) can be extended to solutions of (2.6) via equivariance by utilizing the invariance of the Laplacian under orthogonal transformations and the generation of the group by the fundamental reflections.

### 2.3 Invariance of the class of positive maps

**Definition.** For \( \Omega = B_R := \{ x \in \mathbb{R}^n \mid |x| < R \} \), we define the set of maps

\begin{equation}
P := \{ v \in W^{1,2}_E(B_R;\mathbb{R}^n) \mid v(F_{BR}) \subset \bar{F} \}
\end{equation}

which we call \textit{positive}. A positive map is called \textit{strongly positive} if

\begin{equation}
v(F_{BR}) \subset F.
\end{equation}

The set of strongly positive maps is denoted by \( P_0 \).

**Theorem 2.1.** Suppose \( W \) satisfies properties (2.4) and (2.5). Then, (2.6) with \( \Omega = B_R \) leaves the positive class \( P \) invariant, that is,

\[ P \ni u_0 \mapsto u(\cdot,t;u_0) \in P, \]

and, moreover,

\[ (P \cap \{ v \in W^{1,2}_E(B_R;\mathbb{R}^n) \mid v(F_{BR}) \notin \partial F \}) \ni u_0 \mapsto u(\cdot,t;u_0) \in P_0, \text{ for } t > 0. \]
We begin with a lemma.

**Lemma 2.2.** Let \( u : B_R \to \mathbb{R}^n \) be an equivariant map. Then, \( u \) is a positive map if and only if

\[
(2.10) \quad u(\mathcal{P}_r^+ \cap B_R) \subset \mathcal{P}_r^+, \text{ for all roots } r.
\]

Here, \( \mathcal{P}_r^+ = \{ x \in \mathbb{R}^n \mid \langle x, r \rangle \geq 0 \} \).

**Proof.** Suppose that (2.10) holds. Then, by (2.2),

\[
\begin{align*}
 u(F_{B_R}) &= u\left( \bigcap_{i=1}^{n} \mathcal{P}_{r_i}^+ \cap B_R \right) \\
 &\subset \bigcap_{i=1}^{n} u(\mathcal{P}_{r_i}^+ \cap B_R) \subset \bigcap_{i=1}^{n} \mathcal{P}_{r_i}^+ = F.
\end{align*}
\]

Hence, \( u \) is positive.

Conversely, suppose that \( u \) is a positive equivariant map on \( B_R \). Then, equivalently, \( u_e \) defined by

\[
(2.11) \quad u_e(x) := \begin{cases} 
 u(x), & \text{for } x \in B_R \\
 0, & \text{for } x \in \mathbb{R}^n \setminus B_R 
\end{cases}
\]

is a positive equivariant map on \( \mathbb{R}^n \). For any \( g \in G \), we have from equivariance and positivity,

\[
(2.12) \quad u_e(g(F)) = g(u_e(F)) \subset g(F).
\]

Now pick a root \( r \) and take an \( x \in \mathcal{P}_r^+ \) and fix it. There is a \( g \in G \), denoted by \( g_x \), such that \( x \in g_x(F) \) and since \( g_x(F) \) is also a fundamental domain and \( \Pi \) is a positive \( t \)-base,

\[
(2.13) \quad g_x(F) \subset \mathcal{P}_r^+.
\]

Thus, by (2.12), \( u_e(\mathcal{P}_r^+) \subset \mathcal{P}_r^+ \), and so (2.10) follows. \( \square \)

We continue with the

**Proof of Theorem 2.1.** Consider (2.6) with \( \Omega = B_R \) and \( u_0 \in W^{1,2}_{E}(B_R; \mathbb{R}^n) \). By the regularizing property of the equation, the solution is classical for \( t > 0 \), and by (2.4), it is global in time (see [21]). By taking the inner product of the equation with \( r \), an arbitrary root, we obtain from (2.6)

\[
(2.14) \quad \begin{cases} 
 \frac{\partial \phi(u)}{\partial t} = \Delta \phi(u) - \frac{\langle W_E(u), r \rangle}{\phi(u)} \phi(u), \text{ in } B_R^+ \times (0, \infty), \\
 \phi(u(x, 0)) = \phi(u_0(x)),
\end{cases}
\]

where

\[
\phi(u) := -\langle u, r \rangle, \text{ for } u \in \mathbb{R}^n.
\]

For establishing positivity, it is sufficient, by Lemma 2.2, to show that \( \phi(u_0(x)) \leq 0 \) implies \( \phi(u(x, t)) \leq 0 \), for \( t \geq 0 \).
We will accomplish this by applying the maximum principle to (2.14). Fix a \( \tau > 0 \). By smoothness, \( \phi(u(x,t)) = 0 \) on \( \mathcal{P}_r \cap [\tau, \infty) \) and

\[
\frac{\partial}{\partial n} \phi(u(x,t)) = 0, \quad \text{on } (\partial B^+_R \setminus \mathcal{P}_r) \times [\tau, \infty) \quad (\text{cf. (2.7)})
\]

By the smoothness of \( W \) and the symmetry of \( u \) we have

\[
(2.15) \quad \langle W_u(u), r \rangle = \langle u - \xi, r \rangle \langle W_{uu}(\xi + \text{Proj}(u - \xi) + \hat{\delta}r), r \rangle,
\]

for \( \xi \in \mathcal{P}_r \) and \( 0 \leq \hat{\delta} \leq \langle u - \xi, r \rangle \), where \( \text{Proj} \) stands for the projection on the plane \( \mathcal{P}_r \). Thus, the coefficient of \( \phi(u) \) in (2.14) is bounded (actually continuous) on \( B^+_R \), hence both the maximum principle and the boundary lemma hold and render that \( \phi(u(x,t)) \leq 0 \), for \( t \geq \tau \) (note that the corner is not an issue since \( \phi(u) = 0 \) there).

Finally, the strong maximum principle applies for \( t \geq \tau > 0 \) to render that

\[ \phi(u(x,t)) < 0, \quad \text{in } B^+_R \times (0, \infty), \]

unless \( \phi(u(x,\tau)) \equiv 0 \), hence unless \( \phi(u_0(x)) \equiv 0 \). But the hypothesis \( u_0(F_{B_R}) \not\subseteq \partial F \) excludes this second option. \( \square \)

In [3], utilizing a different technique, we establish the positivity property for the more general evolution problem

\[
(2.16) \quad \begin{cases}
\frac{\partial u}{\partial t} = \Delta u - W_u(u), \quad \text{in } K \times (0, t_0), \\
\langle u, \xi_i \rangle = 0 \quad \text{and} \quad \frac{\partial u}{\partial \xi_i} = \langle \frac{\partial u}{\partial \xi_i}, \xi_i \rangle r_i, \quad \text{in } (\mathcal{P}_{\xi_i} \cap B_R) \times (0, t_0), \\
\frac{\partial u}{\partial n} = 0, \quad \text{on } (\partial B_R \setminus \partial K_{B_R}) \times (0, t_0).
\end{cases}
\]

for \( K_{B_R} = K \cap B_R \), \( K \) a simplex, not necessarily related to a reflection group, defined via

\[ K := \bigcap_{i=1}^n \{ u \in \mathbb{R}^n \mid \langle u, \xi_i \rangle \geq 0 \}, \]

where \( \{ \xi_i, \ldots, \xi_n \} \) is a set of linearly independent vectors satisfying the inequalities

\[ \langle \xi_i, \xi_j \rangle \leq 0, \quad \text{for } i \neq j \quad (\text{cf. (2.3)}), \]

and with a potential \( W \) satisfying

\[ \langle W_u(u), \xi_i \rangle = 0, \quad \text{for } u \in \mathcal{P}_{\xi_i} \quad \text{and} \quad i = 1, \ldots, n. \]

Note that first-order derivatives can be included in (2.16) without harm.
3 The coordinate system

**Lemma 3.1.** Suppose that $Q : \bar{D} \to \mathbb{R}$ is a function with the following properties:

\begin{align}
(3.1a) & \quad Q \text{ is convex}, \\
(3.1b) & \quad Q(u) > 0 \text{ and } Q_u(u) \neq 0, \text{ on } \bar{D} \setminus \{a_1\}, \\
(3.1c) & \quad Q(u + a_1) = |u| + o(|u|).
\end{align}

Then, for each $\nu \in S^{n-1}$, the ODE system

\begin{equation}
(3.2) \quad \dot{u} = \frac{Q_u(u)}{\langle Q_u(u), Q_u(u) \rangle}, \text{ for } u \in \mathbb{R}^n \setminus \{a_1\},
\end{equation}

has a unique global solution $\tilde{u} : (0, \infty) \to \mathbb{R}^n$, such that

\begin{equation}
(3.3) \quad \lim_{q \to 0^+} \frac{\tilde{u}(q; \nu)}{|\tilde{u}(q; \nu)|} = \nu.
\end{equation}

Moreover, the maps defined through the solution

$$
(q, \nu) \mapsto \tilde{u}(q; \nu), \text{ from } \mathbb{R}^n \times S^{n-1} \to \mathbb{R}^n \setminus \{a_1\}, \\
\nu \mapsto \tilde{u}(q; \nu), \text{ from } S^{n-1} \to \{ u \mid Q(u) = q \}, \text{ for all } q \in (0, \infty),
$$

are global diffeomorphisms onto their range.

**Proof.** Here, for simplicity, we present the proof under the stronger hypothesis

$$
Q(u + a_1) = |u|, \text{ for } |u| \leq r_0 \text{ with } r_0 > 0 \text{ and small.}
$$

For the general case we refer to [3].

From (3.2) we have that

$$
\frac{d}{dq} Q(u(q)) = 1.
$$

This implies that the left extremum of the interval of existence of $u$ is $q = 0$ and, furthermore, that

\begin{equation}
(3.4) \quad \lim_{q \to 0^+} u(q) = a_1.
\end{equation}

Moreover, for $|u - a_1| \leq r_0$,

$$
\frac{d}{dq} \frac{u(q)}{|u(q)|} = \frac{1}{|u| \langle Q_u, Q_u \rangle} \left( Q_u - \frac{u}{|u|} \langle Q_u, u \rangle \right) = 0,
$$

hence, the existence of the limit in (3.3) follows. Finally, note the identity

\begin{equation}
(3.5) \quad Q(\tilde{u}(q; \nu)) = q.
\end{equation}

The rest of the lemma follows by standard ODE facts. \qed
Lemma 3.2. Consider the mapping \((q, \nu) \mapsto \bar{u}(q; \nu)\) as defined in Lemma 3.1. Then, for any fixed vector \(t \perp \nu\), the quadratic form
\[
\omega(\alpha, \beta) = -\langle \bar{u}_{qq}, \bar{u}_q \rangle \alpha^2 + \langle \bar{u}_{q\nu t}, \bar{u}_\nu t \rangle \beta^2 - 2\langle \bar{u}_{q\nu t}, \bar{u}_q \rangle \alpha \beta,
\]
for \(\alpha, \beta \in \mathbb{R}\) is positive semidefinite.

Proof. From (3.5), differentiating in \(q\), we obtain
\[
\langle Q_{uu}, \bar{u}_q \rangle = 1.
\]
On the other hand, differentiating in \(\nu\), we obtain
\[
\langle Q_{u\nu}, \bar{u}_\nu t \rangle = 0 \iff \langle \bar{u}_{q\nu t}, \bar{u}_\nu t \rangle = 0; \tag{3.8a}
\]
\[
\langle \bar{u}_{q\nu t}, \bar{u}_\nu t \rangle + \langle \bar{u}_q, \bar{u}_{\nu \nu}(t, t) \rangle = 0. \tag{3.8b}
\]
Now, differentiating (3.7) in \(q\) yields
\[
\langle Q_{u\nu}, \bar{u}_q \rangle + \langle Q_{u}, \bar{u}_{qq} \rangle = 0 \iff \langle \bar{u}_{qq}, \bar{u}_q \rangle = -\frac{\langle Q_{u\nu}, \bar{u}_q \rangle}{\langle \bar{u}_q, \bar{u}_q \rangle}, \tag{3.9a}
\]
while differentiating in \(\nu\) yields
\[
\langle Q_{uu}, \bar{u}_\nu t \rangle = 0 \iff \langle \bar{u}_{q\nu t}, \bar{u}_\nu t \rangle = 0; \tag{3.9b}
\]
Finally, differentiating (3.8) in \(q\) yields
\[
\langle Q_{uu}, \bar{u}_q \rangle + \langle Q_{u}, \bar{u}_{q\nu t} \rangle = 0 \iff \langle \bar{u}_{q\nu t}, \bar{u}_q \rangle = -\frac{\langle Q_{u\nu}, \bar{u}_q \rangle}{\langle \bar{u}_q, \bar{u}_q \rangle}. \tag{3.9c}
\]
while differentiating in \(\nu\) yields
\[
\langle Q_{uu}, \bar{u}_\nu t \rangle = 0 \iff \langle \bar{u}_{q\nu t}, \bar{u}_\nu t \rangle = 0; \tag{3.9d}
\]
The convexity of \(Q\) implies
\[
\langle Q_{uu}v, v \rangle \geq 0, \text{ for all } v \in \mathbb{R}^n. \tag{3.10}
\]
From this, (3.8b), and (3.9d), we obtain
\[
\langle \bar{u}_{q\nu t}, \bar{u}_\nu t \rangle \geq 0, \tag{3.11}
\]
while from (3.10) and (3.9) we obtain
\[
-\langle \bar{u}_{qq}, \bar{u}_q \rangle \geq 0. \tag{3.12}
\]
From (3.10), by the same argument that proves the Schwarz inequality, we have
\[
\langle Q_{uu}v, w \rangle^2 \leq \langle Q_{uu}v, v \rangle \langle Q_{uu}w, w \rangle, \text{ for all } v, w \in \mathbb{R}^n. \tag{3.13}
\]
Thus, from (3.9) and (3.13),
\[
-\langle \bar{u}_{qq}, \bar{u}_q \rangle \langle \bar{u}_{q\nu t}, \bar{u}_\nu t \rangle - \langle \bar{u}_{q\nu t}, \bar{u}_q \rangle^2 \geq 0,
\]
from which, via (3.11) and (3.12), we obtain (3.6). \(\square\)
The diffeomorphism defined in Lemma 3.1 associates a ‘polar’ representation to a given map \(v(x)\) in the following way:

\[
(3.14) \quad v(x) \leftrightarrow (q^v(x), \nu^v(x)), \quad \text{where} \quad v(x) = \tilde{u}(q^v(x); \nu^v(x)).
\]

Dropping the superscripts, we calculate

\[
(3.15) \quad \frac{\partial}{\partial x_i} v_x = \tilde{u}_q q_i(x) + \tilde{u}_\nu \nu_{x_i}(x),
\]

and thus, we obtain that for \(v \in W^{1,2}(\Omega; \mathbb{R}^n)\) there holds

\[
q \in W^{1,2}(\Omega; \mathbb{R}) \quad \text{and} \quad \nu \in W^{1,2}(\Omega; S^{n-1}).
\]

Given a \(C^2\) potential \(W : \mathbb{R}^n \to \mathbb{R}\), we define a \(C^2\) function \(V : (0, \infty) \times S^{n-1} \to \mathbb{R}\) via

\[
(3.16) \quad V(q, \nu) := W(\tilde{u}(q; \nu)).
\]

Therefore, the action takes the form

\[
J_\Omega(\nu) = \int_\Omega \left\{ \frac{1}{2} |\nabla v|^2 + W(v) \right\} \, dx
= \int_\Omega \left\{ \frac{1}{2} (\langle \tilde{u}_q, \tilde{u}_q \rangle |\nabla q(x)|^2 + \sum_{j=1}^n \langle \tilde{u}_\nu, \nu_{x_j} \rangle \langle \tilde{u}_\nu, \nu_{x_j} \rangle) + V(q, \nu) \right\} \, dx.
\]

For a fixed \(v \in W^{1,2}(\Omega; \mathbb{R}^n)\) and an open set \(A \in \Omega\), we introduce the following functionals:

\[
(3.17) \quad K_A(\rho) := \int_A \frac{1}{2} \left\{ \langle \tilde{u}_q(\rho, \nu), \tilde{u}_q(\rho, \nu) \rangle |\nabla \rho|^2 + \sum_{j=1}^n \langle \tilde{u}_\nu(\rho, \nu) \nu_{x_j}, \tilde{u}_\nu(\rho, \nu) \nu_{x_j} \rangle \right\} \, dx,
\]

\[
(3.18) \quad V_A(\rho) := \int_A V(\rho, \nu) \, dx,
\]

\[
(3.19) \quad E_A(\rho) := K_A + V_A.
\]

We remark that \((1.8)\) in Hypothesis 4 implies, via \((3.16)\) and \((3.2)\),

\[
(3.20) \quad \frac{\partial V}{\partial q}(q, \nu) \geq 0.
\]

On the other hand, just Hypothesis 1 implies

\[
(3.21) \quad \frac{\partial V}{\partial q}(q, \nu) \geq c^2 \langle \tilde{u}_q, \tilde{u}_q \rangle p, \quad \text{for} \quad 0 \leq p \leq q \leq r_0.
\]
4 Comparison lemmas

Let $A \subset \Omega$ be an open, connected set with Lipschitz boundary. In the lemma next, we utilize the following as a comparison problem

$$
\begin{align*}
\Delta \phi &= 0, \text{ in } A, \\
\phi &= g, \text{ on } \partial A,
\end{align*}
$$

for a function $g : \partial A \rightarrow \mathbb{R}$.

Lemma 4.1 below requires global hypotheses (as shown in parenthesis) and provides global estimates.

**Lemma 4.1** (Hypotheses 1, 3, 4). Let $u \in W^{1,2}(\Omega; \mathbb{R}^n)$ with the following properties:

(a) $J_\Omega(u) < \infty$,
(b) $u(\Omega) \subset D$,
(c) $q^u|_{\partial A} \leq g$.

Then, there is a $v \in W^{1,2}(\Omega; \mathbb{R}^n)$ such that

(i) $q^v \leq \phi$, a.e. in $A$, and $\nu^v = \nu^u$, a.e. in $A$,
(ii) $v|_{\Omega \setminus A} = u|_{\Omega \setminus A}$,
(iii) $J_\Omega(v) \leq J_\Omega(u)$.

**Proof.** Consider $\tilde{\rho} \rightarrow K_A(\tilde{\rho})$ in (3.17) as a functional in $\tilde{\rho}$, for a fixed $\nu = \nu^u$, and minimize it in the class of $W^{1,2}(A; \mathbb{R})$ functions with Dirichlet values

$$
\tilde{\rho} = q^u, \text{ on } \partial A.
$$

Since $Q$ is locally Lipschitz, there exists a minimizer $\rho$ which satisfies

$$
\int_A \left\{ |\nabla \rho|^2 + \sum_{j=1}^n \langle \tilde{u}_{q\nu}(\rho, \nu), \tilde{u}_\nu(\rho, \nu) \rangle \nu_{x_j} \right\} \eta \, dx
\quad + \quad \int_A \langle \tilde{u}_q(\rho, \nu), \tilde{u}_\nu(\rho, \nu) \rangle \nabla \rho \nabla \eta \, dx = 0, \text{ for all } \eta \in W^{1,2}_0(A).
$$

Taking $\omega = \omega_j$ in (3.6), with $\alpha = \rho_{x_j}$, $\beta = 1$, and $t = \nu_{x_j}$, we obtain, for $\eta \geq 0$,

$$
\left( \sum_{j=1}^n \omega_j \right) \eta = \left( - \langle \tilde{u}_{qq}, \tilde{u}_q \rangle |\nabla \rho|^2 + \sum_{j=1}^n \langle \tilde{u}_{q\nu}(\rho, \nu), \tilde{u}_\nu(\rho, \nu) \rangle \nu_{x_j} \right) \eta \geq 0.
$$

Subtracting (4.4) from (4.3), after integrating gives

$$
\int_A \nabla \rho \nabla (\langle \tilde{u}_q, \tilde{u}_q \rangle \eta) \, dx \leq 0, \text{ for } \eta \in W^{1,2}_0(A) \text{ with } \eta \geq 0.
$$
or, equivalently, $\Delta \rho(\tilde{u}_q, \tilde{\bar{u}}_q) \leq 0$, or,

\begin{equation}
\Delta \rho \leq 0, \text{ in } W^{1,2}_0(A).
\end{equation}

By the maximum principle, (c) and (1) imply (See Figure 1)

\begin{equation}
\rho \leq \phi, \text{ a.e. in } A.
\end{equation}

Define

\begin{equation}
\begin{cases}
q^u(x) = \min\{\rho(x), q_u(x)\}, & \text{for } x \in A, \\
q^u(x) = q_u(x), & \text{for } x \in \Omega \setminus A, \\
\nu^u(x) = \nu_u(x), & \text{for } x \in \Omega.
\end{cases}
\end{equation}

Statements (i) and (ii) follow by (4.7). For (iii) we argue as follows. Let

$$J_{\Omega}(v) = J_A(v) + J_{\Omega \setminus A}(u).$$

But,

$$J_A(v) = J_{\Sigma}(v) + J_{A \setminus \Sigma}(v) \quad (\text{where } \Sigma := \{x \in A \mid q^u > \rho(x)\})$$

$$= J_{\Sigma}(v) + J_{A \setminus \Sigma}(u) \quad (\text{since } v = u \text{ on } \Sigma \setminus A)$$

$$= K_{\Sigma}(\rho) + V_{\Sigma}(\rho) + J_{A \setminus \Sigma}(u)$$

$$\leq K_{\Sigma}(q^u) + V_{\Sigma}(q^u) + J_{A \setminus \Sigma}(u) \quad (\text{by the definition of } \rho \text{ and (3.20)})$$

$$= J_{\Sigma}(u) + J_{A \setminus \Sigma}(u)$$

$$= J_A(u).$$

Next, we utilize the following as a comparison problem

\begin{equation}
\begin{cases}
\Delta \psi = c^2 \psi, & \text{in } A, \\
\psi = h, & \text{on } \partial A,
\end{cases}
\end{equation}

for a function $h : \partial A \to \mathbb{R}$ and $c$ as in Hypothesis 1.

The following lemma is based only on local properties.
Lemma 4.2 (Hypothesis 1). Let \( u \in W^{1,2}(\Omega; \mathbb{R}^n) \) with the following properties:

(a) \( J_{\Omega}(u) < \infty \),

(b) \( q^v(x) \leq \bar{q} < r_0 \), for \( x \in A \) (cf. Hypothesis 1),

(c) \( q^u(x) \leq h(x) \leq \bar{q} \), for \( x \in \partial A \).

Then, there is a \( v \in W^{1,2}(\Omega; \mathbb{R}^n) \) such that

(i) \( q^v \leq \psi \), a.e. in \( A \), and \( \nu^v = \nu^u \), a.e. in \( A \),

(ii) \( v|_{\Omega \setminus A} = u|_{\Omega \setminus A} \),

(iii) \( J_{\Omega}(v) \leq J_{\Omega}(u) \).

Proof. Let \( \rho \) be a minimizer of \( \tilde{\rho} \mapsto E_A(\tilde{\rho}) \) in (3.19) for \( \nu = \nu^u \) fixed. Then, as in the proof of Lemma 4.1,

\[
\int_A \{ \nabla \rho \nabla (\tilde{u}_q(\rho, \nu), \tilde{u}_q(\rho, \nu)) \eta \} + V_q(\rho, \nu) \eta \, dx \leq 0, \quad \text{for } \eta \in W^{1,2}_0(A) \text{ with } \eta \geq 0,
\]

or, equivalently,

\[
\Delta \rho (\tilde{u}_q, \tilde{u}_q) + V_q(\rho, \nu) \leq 0, \quad \text{in } W^{1,2}_0(A).
\]

Utilizing (3.21) and the maximum principle, we obtain

\[
\rho \leq \psi, \quad \text{a.e. in } A.
\]

The remaining of the proof is analogous to the proof of Lemma 4.1.

\[\square\]

5 Proof of Theorems 1.1 and 1.2

We begin by introducing various sets and relevant notation.

Let \( D_{BR} = D \cap B_R \), with \( D \) as in Hypothesis 4 and \( B_R = B(0; R) \), and let

\[
P_{DB_R} = \{ u \in W^{1,2}_E(B_R; \mathbb{R}^n) \mid u(D_{DB_R}) \subset \bar{D} \}.
\]

Fix a \( \bar{q} \), with \( 0 < \bar{q} < r_0 \), where \( r_0 \) is as in Hypothesis \( \mathbb{I} \). Fix an \( x_0 \) in the interior of \( D \), set \( x_R = \frac{R}{2} x_0 \), with \( B(x_R; L) \) standing for the ball with center \( x_R \) and radius \( L > 0 \), \( L, l \) fixed, independent of \( R \), and to be selected later; \( B(x_R; L + l) \subset D_{BR} \) for \( R > R_0 \). Furthermore, let

\[
PU^c = \{ u \in P_{DB_R} \mid q^u(x) \leq \bar{q}, \quad \text{for } x \in B(x_R; L) \}.
\]

The proof will be presented in three steps.

Step 1. There exists a minimizer \( u_R \in W^{1,2}_E(B_R; \mathbb{R}^n) \) of \( J_{BR} \) on \( PU^c \), for \( R > R_0 \). Moreover, there exist constants \( b > 0, C > 0 \), independent of \( R \), such that

\[
|u_R(x)| < b, \quad \text{for } x \in B_R, \quad \text{and } J_{BR}(u_R) < CR^{n-1}.
\]
Define

\[ u_{\text{aff}}(x) := \begin{cases} 
  d(x; \partial D) \alpha_1, & \text{for } x \in D_{B_R} \text{ and } d(x; \partial D) \leq 1 \\
  \alpha_1, & \text{for } x \in D_{B_R} \text{ and } d(x; \partial D) \geq 1
\end{cases} \tag{5.3} \]

and extend it equivariantly on \( B_R \). Clearly, \( u_{\text{aff}} \in P U^c \) for \( R \geq R_0 \). By the nonnegativity of \( W \) and a simple calculation,

\[ 0 \leq \inf_{P^{U^c}} J_{B_R} < J_{B_R}(u_{\text{aff}}) < CR^{n-1}, \tag{5.4} \]

for some constant \( C \) independent of \( R \).

Let \( \{ u_k \} \) be a minimizing sequence. By Hypothesis 2, without loss of generality we can assume that \( u_k(x) \in C_0 \), and therefore, \( |u_k(x)| < b \), for some \( b > 0 \) independent of \( R \), for \( x \in B_R \).

We have the following easy estimates

\[ \frac{1}{2} \int_{B_R} |\nabla u_k|^2 \, dx < J_{B_R}(u_{\text{aff}}) < CR^{n-1} \text{ and } \int_{B_R} |u_k|^2 \, dx < C(R), \tag{5.5} \]

where \( C(R) \) denotes a constant depending on \( R \). By standard arguments, we obtain, along possibly a subsequence,

\[ u_k \to u_R, \text{ a.e.}, \tag{5.6} \]

where \( u_R \) is a minimizer. Clearly, \( q^{u_R}(x) \leq \bar{q} \) and \( |u_R(x)| < b \) on \( \overline{B(x_R; L)} \).

This finishes the proof of Step 1.

**Step 2.** There exist \( L_0(\bar{q}, b, N) > 0 \) and \( \delta(\bar{q}, b, N) > 0 \), independent of \( R \), such that for \( L > L_0 \) and \( R > R_0 \) we have the estimates

\[ q^{u_R}(x) < \bar{q}(1 - k\delta), \text{ on } B(x_R; L + \delta) \setminus B(x_R; L - \delta), \]

for some \( k = k(\bar{q}, b, N) > 0 \), independent of \( R \).

We introduce the following three comparison functions \( \Psi_I, \Psi_{II} \), and \( \Psi_{III} \).

\[
\begin{aligned}
\begin{cases}
\frac{\partial^2 \Psi_I}{\partial r^2} + \frac{N - 1}{r} \frac{\partial \Psi_I}{\partial r} - c^2 \Psi_I = 0, & \text{for } r = |x - x_R| \text{ and } x \in B(x_R; L), \\
\Psi_I(0) = 0, & \Psi_I(L) = \bar{q};
\end{cases}
\end{aligned}
\]

\[
\begin{aligned}
\begin{cases}
\frac{\partial^2 \Psi_{II}}{\partial r^2} + \frac{N - 1}{r} \frac{\partial \Psi_{II}}{\partial r} = 0, & \text{for } r = |x - x_R| \text{ and } x \in B(x_R; L + l) \setminus B(x_R; L), \\
\Psi_{II}(L) = \bar{q}, & \Psi_{II}(L + l) = b;
\end{cases}
\end{aligned}
\]

\[
\begin{aligned}
\begin{cases}
\frac{\partial^2 \Psi_{III}}{\partial r^2} + \frac{N - 1}{r} \frac{\partial \Psi_{III}}{\partial r} = 0, & \text{for } r = |x - x_R| \text{ and } x \in B(x_R; L + \delta) \setminus B(x_R; L - \delta), \\
\Psi_{III}(L - \delta) = \Psi_I(L - \delta), & \Psi_{III}(L + \delta) = \Psi_{II}(L - \delta);
\end{cases}
\end{aligned}
\]

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where $L, l, \delta$ are positive numbers to be specified.

By standard facts about Bessel functions,

\begin{equation}
\frac{\partial \Psi_1}{\partial r}(L) = c\bar{q}(1 + o(1)), \text{ as } L \to \infty, \text{ with } o(1) \to 0, \text{ as } L \to \infty.
\end{equation}

By an explicit calculation we have the estimates

\begin{equation}
\frac{b - \bar{q}}{l} \leq \frac{\partial \Psi_{II}}{\partial r}(L) \leq (b - \bar{q}) \frac{(L + l)^{N-1}}{lL^{N-1}},
\end{equation}

\begin{equation}
\frac{1}{2} \left( \frac{b - \bar{q}}{l} + c\bar{q}(1 + o(1)) \right) + O(\delta) \leq \frac{\partial \Psi_{III}}{\partial r}(L) \leq \frac{1}{2} \left( (b - \bar{q}) \frac{(L + l)^{N-1}}{lL^{N-1}} + c\bar{q}(1 + o(1)) \right) + O(\delta),
\end{equation}

with $O(\delta)$ uniformly in $L$ and $l$.

We need to satisfy the inequalities

\begin{equation}
\frac{\partial \Psi_{II}}{\partial r}(r) \leq \frac{\partial \Psi_{III}}{\partial r}(r) \leq \frac{\partial \Psi_1}{\partial r}(r), \text{ for } r = |x - x_R|, \ x \in B(x_R; L + \delta) \setminus B(x_R; L - \delta),
\end{equation}

which will follow from certain estimates. It is sufficient to satisfy

\begin{equation}
\frac{1}{2} \left( \frac{b - \bar{q}}{l} + c\bar{q}(1 + o(1)) \right) + O(\delta) > (b - \bar{q}) \frac{(L + l)^{N-1}}{lL^{N-1}},
\end{equation}

\begin{equation}
\frac{1}{2} \left( (b - \bar{q}) \frac{(L + l)^{N-1}}{lL^{N-1}} + c\bar{q}(1 + o(1)) \right) + O(\delta) < c\bar{q}(1 + o(1)).
\end{equation}

We let now $l = L$. Clearly, given a fixed $\lambda \in (0, \frac{1}{2})$ and taking $\delta$ small, there exists an $L_0$, which determines $R_0$, such that for $L \geq L_0$,

\[
\frac{\partial \Psi_1}{\partial r} - \frac{\partial \Psi_{III}}{\partial r} \geq \left( \frac{1}{2} - \lambda \right)c\bar{q} \quad \text{and} \quad \frac{\partial \Psi_{III}}{\partial r} - \frac{\partial \Psi_{II}}{\partial r} \geq \left( \frac{1}{2} - \lambda \right)c\bar{q}.
\]

From these it follows, by taking $\delta$ smaller if necessary, that

\[
\bar{q} \left( 1 - \left( \frac{1}{2} - \lambda \right)c\delta \right) \geq \Psi_{III}, \text{ on } B(x_R; L + \delta) \setminus B(x_R; L - \delta).
\]

The proof of Step 2 is completed by applying in succession Lemma 4.2 once for $\Omega = B_R$, $A = B(x_R; L)$, and Lemma 4.1 twice, for $\Omega = B_R$, $A = B(x_R; L + l) \setminus B(x_R; L)$, and also for $\Omega = B_R$, $A = B(x_R; L + \delta) \setminus B(x_R; L - \delta)$.

**Step 3. (Conclusion)** The estimate provided by Step 2 above has two immediate consequences. First, it implies that there is a minimizer $u_R$ of $J_{BR}$ which does not realize the pointwise constraint $q^{u_R}(x) \leq \bar{q}$ on $B(x_R; L)$. Second, by reapplying the estimate on displaced balls, we conclude that the constraint ‘propagates’ on a much larger set,

\begin{equation}
q^{u_R}(x) \leq \bar{q}, \text{ on } D_{BR} \setminus N(\partial D_{BR}),
\end{equation}

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where $N(\partial D_{B_R})$ is a neighborhood of $\partial D_{B_R}$ determined by $L$, independent of $R$, and the angles of the simplex $D$. In particular, (5.13) holds on a set independent of $R$. Next, we switch to the parabolic flow and apply Theorem 2.1 with $F_{B_R}$ replaced by $D_{B_R}$; $\{a_1\}$ is in the interior of $D$. Thus, by (5.13), $u_R(D_{B_R}) \nsubseteq \partial D$, and so, by strong positivity, $u(\cdot, t; u_R)$ maps $D_{B_R}$ into $D$ for any $t > 0$. By choosing $t > 0$ and small, via (5.13) we secure that

$$q^{u(\cdot, t; u_R)}(x) < \bar{q}, \text{ on } \overline{B(x_R; L)},$$

that is, the open pointwise constraint is satisfied. Fix such a $t$. By the gradient property

$$J_{B_R}(u(\cdot, t; u_R)) \leq J_{B_R}(u_R),$$

and since $u(\cdot, t; u_R)$ is free of all constraints on $D_{B_R}$, it satisfies the Euler–Lagrange equation

$$\Delta u - W_u(u) = 0, \text{ on } D_{B_R}, \text{ for } R > R_0,$$

together with boundary conditions analogous to those in (2.7). By the comments following (2.7), $u$ can be extended to a solution of (5.16) on $B_R$, satisfying Neumann conditions on $\partial B_R$.

Finally, we pass to the limit along a subsequence in $R$ and capture a function

$$u(x) = \lim_{R_n \to \infty} u(\cdot, t; u_{R_n}).$$

By (5.13) and the comments following it, we conclude that $u(x)$ is a nontrivial solution, that is, $u \not\equiv 0$, to

$$\Delta u - W_u(u) = 0, \text{ on } \mathbb{R}^n.$$

By taking the limit in (5.13), we obtain the estimate

$$q^u(x) < \bar{q}, \text{ on } D \setminus N(\partial D).$$

Consider now one of the hyperplanes determining $D$, $P_{r_i} = \{x \in \mathbb{R}^n | \langle x, r_i \rangle = 0\}$. By (5.19), we can assume that there is an $\eta > 0$ such that $q^u(x) < \bar{q}$ for $d(x, P_{r_i}) \geq \eta$. By Lemma 4.2

$$q^u(x) \leq \psi(x), \text{ in } \{x \in \mathbb{R}^n | d(x, P_{r_i}) \geq \eta\},$$

where

$$\begin{cases} \Delta \psi = c^2 \psi, & \text{in } \{x \in \mathbb{R}^n | d(x, P_{r_i}) \geq \eta\}, \\ \psi = \bar{q}, & \text{on } \{x \in \mathbb{R}^n | d(x, P_{r_i}) = \eta\}. \end{cases}$$

By rotating coordinates, we may assume that the hyperplane $P_{r_i}$ coincides with $\{x \in \mathbb{R}^n | x_1 = 0\}$ in the new coordinate system. We simply note that

$$\psi(x) = \bar{q}e^{-c(x_1 - \eta)}$$

satisfies (5.21).

This concludes the proofs of Theorems 1.1 and 1.2.
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