Double Affine Bundles

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Abstract

A theory of double affine and special double affine bundles, i.e. differential manifolds with two compatible (special) affine bundle structures, is developed as an affine counterpart of the theory of double vector bundles. The motivation and basic examples come from Analytical Mechanics, where double affine bundles have been recognized as a proper geometrical tool in a frame independent description of many important systems. Different approaches to the (special) double affine bundles are compared and carefully studied together with the problems of double vector bundle models and hulls, duality, and relations to associated phase spaces, contact structures, and other canonical constructions.

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1 Introduction

Double structures appear in geometry in a natural way, as the result of iteration of some functors. For example, iterations of the tangent and cotangent functors result in $T^*T M$, $TT M$, $T^*T^* M$, and a fundamental role played by these objects in Analytical Mechanics is well known. These are canonical examples of double vector bundles \footnote{Research financed by the Polish Ministry of Science and Higher Education under the grant No. N201 005 31/0115.}, i.e. manifolds with two compatible vector bundle structures. The Lagrangian mechanics of a system with the configuration manifold $M$ is based on a canonical isomorphism between the double vector bundles $T^*T M$ and $TT^* M$. On the other hand, the Hamiltonian formulation of a dynamics is based on an isomorphism between $T^*T^* M$ and $TT^* M$ and
the Legendre transformation makes use of the canonical isomorphism of $T^*E$ and $T^*E^*$ for $E$ being a vector bundle ($[T]$). Natural generalizations of these isomorphisms lead to a very useful understanding of a Lie algebroid structure as a certain morphism of double vector bundle structures. There are many other instances in geometry, where double vector bundles emerge in a natural way ($[KU]$).

On the other hand, we encounter in Physics many situations, where we are forced to replace vector-like objects by affine ones, in order to obtain frame independent (gauge independent) formulations of our theories. For example, Newtonian mechanics, relativistic mechanics of a charged particle and nonrelativistic mechanics of nonautonomous systems require affine-like objects. Lagrangians (Hamiltonians) are no longer functions on tangent (cotangent) bundles, but sections of affine bundles ($[TU, U, GU1, GU2]$). Geometrical objects which are suitable for these situations are provided by the geometry of affine values (AV-geometry) ($[U, GU1, GU2]$). Like in the case of vector bundles, we have to work with iteration of functors which give objects with two compatible affine bundle structure.

The aim of this work is to develop a consistent theory of double objects in the category of affine bundles, which generalize canonical objects known from the AV-geometry. The basic example ($[GU1]$) is the affine phase bundle used in relativistic mechanics of a charged particle (for details see Section 5). We concentrate on purely mathematical problems, as examples of applications being the starting point of our investigations one can find in the papers cited above. We present a systematic introduction to double affine objects in both: double affine bundles and special double affine bundles settings proving several theorems describing mutual relations of the introduced objects. Our approach to double affine structures corresponds to the novel approach to double vector bundles presented in ($[GR]$).

The paper is organized as follows:

In Section 2 we give two equivalent definitions of a double affine bundle. One uses compatibility conditions for two affine bundle structures, while the other is given in terms of local trivializations. In Section 3 we introduce the notion of a model double vector bundle and a vector hull of a double affine bundle. From the point of view of applications, the most important is the notion of a special affine bundle and consequently, a special double affine bundle. We should view a special affine bundle as a generalization of the product $E \times \mathbb{R}$, where $E$ is a vector bundle, with functions on $E$ interpreted as sections of the trivial bundle $E \times \mathbb{R} \to E$. Like vector bundles, special affine bundles form a category with duality (which is no longer true for just affine bundles). We get also duality theorems, similar to the ones known in the category of double vector bundles ($[M, KU]$). In Section 5 we analyze the case of double bundles which appear as a result of an application of the phase functor to a special affine bundle. This generalizes the well-known structures of the cotangent of a vector bundle $T^*E$. The canonical example, the contact bundle $CA$ of a special double affine bundle, is given in Section 6. It is obtained by applying the contact functor to a special affine bundle $A$. There is a canonical isomorphism of $CA$ and $CA^\#$, where $A^\#$ is the special affine dual of $A$, which can be interpreted as a functorial version of the universal Legendre transformation of Analytical Mechanics. This section contains also an interesting description of affine duality as being encoded in a single geometrical object - the double affine dual $B.A$ -
derived from the cotangent bundle $T^*A$ and its canonical symplectic structure. The last section is devoted to natural generalizations of all these concepts to $n$-affine bundles.

2 Basic examples and definitions

We assume that the reader has a basic knowledge about affine spaces, which can be found in many books on linear algebra and linear geometry.

An affine space is usually defined as a triple $(A, V, +)$, where $+ : A \times V \to A$ is a free and transitive action of a vector space $V$ on a manifold $A$. Within this definition the particular vector space $V$ is fixed and the affine combinations map $\text{aff}_+ = \text{aff} : A \times A \times \mathbb{R} \to A$,

$$\text{aff}(a, b; \lambda) := b + \lambda \cdot [b, a]_V,$$

where $[b, a]_V \in V$ is the unique vector such that $b + [b, a]_V = a$, does not determine the affine structure. Indeed, for any linear automorphism $\phi : V \to V$, the action $+': A \times V \to A$ defined by $a' + v = a + \phi(v)$ gives the same map $\text{aff}_+ = \text{aff}_+$. We prefer to work with objects whose structure is completely encoded in the affine combinations map. In order to do this, we formally define an affine space as an abstract class of a triple $(A, V, +)$ as above subject to the following equivalence relation:

$$(A, V, +) \sim (A', V', +')$$

if and only if $A = A'$ and $\text{aff}_+ = \text{aff}_+$. Let $(A, V, +)$ be a representant of an affine space and consider the following relation on $A \times A$:

$$(a, b) \sim (a', b') \iff \text{aff}(a, b'; 1/2) = \text{aff}(b, a'; 1/2),$$

(1)

It follows immediately that it is an equivalence relation and its equivalence classes, denoted by $[a, b]$, $a, b \in A$, form a vector space, denoted by $\mathcal{V}(A)$, which is isomorphic to $V$. Within this isomorphism, the action of $\mathcal{V}(A)$ reads as

$$a + [a, b] = b.$$  

(2)

The structure maps of $\mathcal{V}(A)$ can be expressed in terms of the map $\text{aff}$,

$$\lambda \cdot [a, b] := [a, \text{aff}(b, a; \lambda)], \quad [a, b] + [a, c] = 2 \cdot [a, \text{aff}(b, c; 1/2)],$$

hence the vector space $\mathcal{V}(A)$ is defined independently on the choice of a representant $(A, V, +)$. This shows that the considered affine space has a canonical representant $(A, \mathcal{V}(A), +)$. It allows us to write simply $(A, \text{aff})$ instead of the class of $(A, V, +)$. The vector space $\mathcal{V}(A)$ is called the model vector space of $(A, \text{aff})$.

We begin with a definition of a double affine bundle formulated in terms of natural maps for affine bundles. Then we shall prove that this notion has also a nice description in local coordinate systems as in (GU1).

**Definition 2.1.** A double affine bundle $A = (A; A_1, A_2; M)$ is a commuting diagram of four affine bundles

$$A \xrightarrow{\pi_2} A_2 \quad \pi_1 \quad \pi_1' \quad \pi_2'$$

$$A_1 \quad \pi_2'$$

such that

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where $[b, a]_V \in V$ is the unique vector such that $b + [b, a]_V = a$, does not determine the affine structure. Indeed, for any linear automorphism $\phi : V \to V$, the action $+': A \times V \to A$ defined by $a' + v = a + \phi(v)$ gives the same map $\text{aff}_+ = \text{aff}_+$. We prefer to work with objects whose structure is completely encoded in the affine combinations map. In order to do this, we formally define an affine space as an abstract class of a triple $(A, V, +)$ as above subject to the following equivalence relation:

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such that
(i) $(\pi_i, \pi'_i), i = 1, 2,$ are morphisms of affine bundles,

(ii) $(\pi_1, \pi_2) : A \rightarrow A_1 \times_M A_2$ is surjective.

For $x, y \in A$ being in the same fiber of $\pi_i$ and $\lambda \in \mathbb{R}$, let $\text{aff}_i(x, y; \lambda) \in A$ denote the affine combination of $x$ and $y$ with weights $\lambda$ and $1 - \lambda$, respectively.

(iii) For each $\lambda \in \mathbb{R}, i = 1, 2,$

$$\text{aff}_i(-, -; \lambda) : A \times_{\pi_i} A \rightarrow A$$

is a morphism of affine bundles.

These conditions need a few explanation remarks.

**Remark 1.** The condition (ii) means that the morphisms $(\pi_i, \pi'_i), i = 1, 2,$ are fiberwise surjective maps. Condition (i) does not imply (ii). For example, let us take $M = \{m\}$ to be a single point, $A = \mathbb{R}^2$ and let $\pi_1 = \pi_2$ be the projection on the first factor $A_1 = A_2 = \mathbb{R}$. Then $\pi_1$ (and so $\pi_2$) is a morphism of affine bundles but it is constant on each fiber, so the map $(\pi_1, \pi_2) : A \rightarrow A_1 \times_M A_2$ is not surjective. Let us mention that in a definition of a double vector bundle it is enough to have a commuting diagram of four vector bundles and assume that homotheties of $\pi_1$ and $\pi_2$ commute (see [GR]). Then automatically the analog of (ii) is satisfied.

**Remark 2.** Let us explain the condition (iii), say for $i = 1$. Consider the horizontal inclusions

$$A \times A \xrightarrow{\pi_2 \times \pi_2} A \times_{\pi_1} A \xleftarrow{\tau} A \xrightarrow{\pi_2} A$$

$$A_2 \times A_2 \xleftarrow{\tau} A_2 \times_{\pi'_1} A_2 \xrightarrow{\pi_2} A_2$$

where $\tau$ is the restriction of $\pi_2 \times \pi_2$ to $A \times_{\pi_1} A$. We shall show that $A \times_{\pi_1} A$ is an affine subbundle of $(A \times A, \pi_2 \times \pi_2)$. The condition (i) implies that the fibers of $\tau$ are closed for taking affine combinations. Indeed, let $p = (p_1, p_2), q = (q_1, q_2)$ be in a fiber $F$ of $\tau$ and let $\lambda \in \mathbb{R}$. We want to show that $\text{aff}_2(p, q; \lambda)$ lies in $F$, i.e.

$$\pi_1(\text{aff}_2(p_1, q_1; \lambda)) = \pi_1(\text{aff}_2(p_2, q_2; \lambda)).$$

We use the fact that $\pi_1$ is an affine morphism and get

$$\pi_1(\text{aff}_2(p_1, q_1; \lambda)) = \text{aff}_2(\pi_1(p_1), \pi_1(q_1); \lambda) = \text{aff}_2(\pi_1(p_2), \pi_1(q_2); \lambda) = \pi_1(\text{aff}(p_2, q_2; \lambda)).$$

(4)

First we prove that the canonical map $\pi := (\pi_1, \pi_2) : A \rightarrow A_1 \times_M A_2$ is an affine bundle projection with respect to both affine structures on $A$. Since $\pi_1$ is a morphism of affine bundles which is surjective along the fibers (by condition (ii)), each fiber $F_{a_1, a_2} = (\pi_1, \pi_2)^{-1}(a_1, a_2), (a_1, a_2) \in A_1 \times_M A_2$, is an affine subspace of $\pi_2^{-1}(a_2)$ and the dimension of $F_{a_1, a_2}$ does not depend on $(a_1, a_2)$. From symmetry, $F_{a_1, a_2}$ has also
another affine space structure inherited from $\text{aff}_1$. We shall show later (Lemma 2.1) that these two structures coincide. Moreover, the fibers $F_{a_1, a_2}$ depend smoothly on $a_1$ and $a_2$, hence $\pi$ is a projection of a locally trivial fibration, and because the fibers are affine spaces, it is an affine bundle projection. Hence for any contractible open set $U \subset M$, $A_{U} := (\pi_1 \circ \pi_2)^{-1}(U)$ admits a trivialization $A_U \simeq (A_1 \times_U A_2) \times F$ for a fixed affine space $F$. Note that we do not claim here, that $A$ is locally trivial as a double affine bundle, what we prove in the Theorem 2.1. It follows now easily that $\tau$ is also a locally trivial fibration, and because of (3), it is an affine subbundle, as we claimed. If (ii) is not satisfied then it may happen that some of the fibers of $\tau$ are empty, as it is the case for the data from Remark 1.

**Remark 3.** Condition (iii) can be written in a form of interchange law

$$\text{aff}_2(\text{aff}_1(x_1, x_2; \lambda), \text{aff}_1(y_1, y_2; \lambda); \mu) = \text{aff}_1(\text{aff}_2(x_1, y_1; \mu), \text{aff}_2(x_2, y_2; \mu); \lambda).$$

Note that the affine structure on $A_1$ (resp. $A_2$) is determined by affine combinations $\text{aff}_2$ (resp. $\text{aff}_1$) on $A$. This is so because $\pi_1$ (resp. $\pi_2$) is a morphism of affine bundles. Later on we shall write simply $\text{aff}_2$ (resp. $\text{aff}_1$) for affine combinations on $A_1$ (resp. $\pi_2$).

Another approach to double affine bundles is possible. In the paper ([GU1]) double affine bundles are defined by means of gluing trivial double affine bundles

$$U_{\alpha} \times K_1 \times K_2 \times K \xrightarrow{pr_2} U_{\alpha} \times K_2$$

We glue such trivial double affine bundles by means of a family of isomorphisms of trivial double affine bundles $\phi_{\alpha, \beta} : (U_{\alpha} \cap U_{\beta}) \times K_1 \times K_2 \times K \rightarrow$. Let us take affine coordinates $(y^i), (z^a), (c^u)$ on affine spaces $K_1$, $K_2$ and $K$, respectively. Then, being an isomorphism of trivial double affine bundles means that the change of coordinates $\phi_{\alpha, \beta} : (x, y, z, c) \mapsto (x', y', z', c')$ has the form

$$x' = x'(x),$$

$$y^j' = \alpha_0^j(x) + \sum_i \alpha_i^j(x)y^i,$$

$$z^a' = \beta_0^a(x) + \sum_b \beta_b^a(x)z^b,$$

$$c^w' = \gamma_{00}^w(x) + \sum_i \gamma_{i0}^w(x)y^i + \sum_j \gamma_{0j}^w(x)z^j + \sum_{i,b} \gamma_{ij}^w(x)y^i z^j + \sum w \sigma_{w}^u(x)c^w.$$  

We shall prove that these two approaches are equivalent. Without loss of generality we assume throughout this paper that the manifold $M$ is connected.

**Theorem 2.1.** Let $A = (A_1, A_2; M)$ be a double affine bundle as in (3). Then, there exist an open covering $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ of $M$, affine spaces $K_1$, $K_2$, $K$ and diffeomorphisms

$$\phi_{\alpha} : A_{|U_{\alpha}} \rightarrow U_{\alpha} \times K_1 \times K_2 \times K$$

inducing isomorphisms of $A_{|U_{\alpha}}$ with trivial double affine bundle, such that the gluing $\phi_{\beta} \circ \phi_{\alpha}^{-1}$ over $U_{\alpha} \cap U_{\beta}$ has the form (7).
We have already observed that for each \((a_1, a_2) \in A_1 \times_M A_2\), the intersection of the fibers \(\pi_1^{-1}\{a_1\} \cap \pi_2^{-1}\{a_2\} \subset A\) carries two structures of an affine space. From the lemma below it follows that these structures coincide.

**Lemma 2.1.** If a manifold \(A\) has two structures of an affine space, determined by affine combinations \(\text{aff}_1\) and \(\text{aff}_2\) satisfying the interchange law \((5)\), then \(\text{aff}_1 = \text{aff}_2\).

**Proof.** Let us fix a point \(\theta \in A\) and carry the structure of the model vector space of \((A, \text{aff}_i)\), denoted by \(\mathcal{V}_i(A)\), \(i = 1, 2\), to \(A\) using isomorphisms

\[
I_{\theta,i} : A \to \mathcal{V}_i(A), \quad a \mapsto [\theta, a]_i,
\]

where \([a, b]_i\) denotes the vector from \(a \in A\) to \(b \in A\) with respect to the affine structure \(\text{aff}_i\) on \(A\). We shall show that these vector space structures coincide. In view of [GR, Proposition 3.1], it is enough to verify that

\[
\lambda \cdot \mu \cdot a = \mu \cdot \lambda \cdot a,
\]

where \(a \in A\), \(\lambda, \mu \in \mathbb{R}\), and \(\lambda \cdot a\) stands for the scalar multiplication with respect to \(a\)th vector space structure on \(A\), i.e. \(\lambda \cdot a = \text{aff}_i(a, \theta; \lambda)\). Short calculations show that

\[
\mu \cdot \lambda \cdot a = \text{aff}_2(\text{aff}_1(a, \theta; \lambda), \theta; \mu) = \text{aff}_1(\text{aff}_2(a, \theta; \mu), \theta; \lambda) = \lambda \cdot \mu \cdot a.
\]

\[\square\]

Now we are ready to prove Theorem 2.1. We shall show that, by making suitable choices of ‘zero-sections’, one can equip any double affine bundle with a compatible structure of a double vector bundle. This can be seen as a construction of the model double vector bundle of \(A\), denoted by \(\mathcal{W}(A)\). A detailed and equivalent but shorter description of \(\mathcal{W}(A)\) we postpone to the next section.

For simplicity, let us work first with the case of \(M = \{m\}\) being a single point. Let us fix \(\theta \in A\) and denote \(\theta_i = \pi_i(\theta), i = 1, 2\), which will later play the role of zero in the corresponding vector spaces.

Note that, thanks to the condition (ii), the morphism \(\pi_2 : A \to A_2\) is surjective on each fiber of \(\pi_1 : A \to A_1\). Hence \(\pi_2^{-1}(\{\theta_2\}) \to A_1\) is an affine subbundle of \(\pi_1 : A \to A_1\). We claim that it is possible to find a section \(\sigma_1\) of this subbundle such that its image \(\sigma_1(A_1)\) is an affine subspace of \((\pi_2^{-1}(\{\theta_2\}), \text{aff}_2)\), i.e. \(\sigma_1\) is a morphism of affine bundles from \(A_1\) to \((A, \text{aff}_2)\). Indeed, consider the affine map

\[
\bar{\sigma}_1 : \pi_2^{-1}(\{\theta_2\}) \to A_1, \quad \bar{\sigma}_1 = \pi_1|_{\pi_2^{-1}(\{\theta_2\})}
\]

between the fibers over \(\theta_2\) and \(m\), respectively. Let \(W \subset \mathcal{V}_2(\pi_2^{-1}(\{\theta_2\}))\) be any complementary vector subspace to the kernel of the linear part \(\bar{\sigma}_1^*\) of \(\bar{\sigma}_1\). Here \(\mathcal{V}_i(A)\) \((i = 1, 2)\) stands for the model vector bundle of the affine bundle \((A, \text{aff}_i)\). The restriction of \(\bar{\sigma}_1^*\) to \(W\) is a linear isomorphism from \(W\) to \(\mathcal{V}_2(A_1)\), the model vector bundle of \(A_1 \to M\). We define \(\sigma_1\) in an obvious way such that \(\sigma_1(\theta_1) = \theta\) and the image \(\sigma_1(A_1)\) is equal to
the affine subspace $\theta +_i W$ of $\pi^{-1}_2(\{\theta_i\})$, where $+_i$, $i = 1, 2$, stands for the action of the model of $(A, \text{aff}_i)$ on $A$. We shall later use $\sigma_1$ to get a vector bundle structure on $A \to A_1$.

Let us work now over an arbitrary manifold $M$ and let $U$ be a small neighbourhood of a given point in $M$. Without loss of generality we may assume that $M = U$. From a smooth section $\theta : M \to A$ of the fibration $\pi_1 \circ \pi_2 : A \to M$ one get the sections $\theta_i \in \text{Sec}_M(A_i)$, $\theta_i = \pi_i \circ \theta$, and find analogously the sections $\sigma_i \in \text{Sec}_A(A)$, ($i = 1, 2$), which are morphisms of the affine bundles, and moreover $\pi_2(\sigma_1(A_1))$ lies in the image of the "zero section" of $\pi'_1 : A_2 \to M$, i.e. $\pi_2(\sigma_1(A_1)) \subset \theta_2(M)$. Similarly, $\pi_1(\sigma_2(A_2))$ is a subset of the image of the "zero section" of $\pi'_2 : A_1 \to M$.

The choices of $\sigma_1, \sigma_2$ put the vector bundle structures on the affine bundles $\pi_i : A \to A_i$, $i = 1, 2$. We shall check that these two structures give a double vector bundle structure. From (GR, Theorem 3.1) it suffices to check that

$$\lambda \cdot \mu \cdot a = \mu \cdot \lambda \cdot a.$$  \hspace{1cm} (8)

for $\lambda, \mu \in \mathbb{R}$. We have

$$\mu \cdot a = \text{aff}_i(a, \sigma_i(a_i); \mu),$$

where $a_i = \pi_i(a)$, hence

$$\lambda \cdot \mu \cdot a = \text{aff}_i(\text{aff}_2(a, \sigma_2(a_2)); \mu, \sigma_1(\text{aff}_2(a_1, \theta_1); \mu); \lambda) = \text{aff}_i(\text{aff}_2(a, \sigma_2(a_2)); \mu, \text{aff}_2(\sigma_1(a_1), \theta; \mu); \lambda).$$  \hspace{1cm} (9)

We used the fact that $\sigma_1$ is an affine bundle morphism. We get a similar formula for $\mu \cdot \lambda \cdot a$ from which we get (3) by the interchange law (3). This shows that locally any double affine bundle can be given a compatible double vector structure. From the local decomposition theorem of double vector bundles ([KU, GR]) we get local identifications $\phi_a : A_{U_\alpha} \to U_\alpha \times K_1 \times K_2 \times K$. Because $\phi_{U_\beta} \circ \phi_{U_\alpha}^{-1}$ is a morphism of double affine bundles, it has the form as in (7).

We shall call a coordinate system $(x, y^i, z^a, c^a)$ induced from a local decomposition of $A$ as in Theorem 2.1 an adapted coordinate system for $A$.

Let $l \in \text{Sec}_M(E^*)$ be a linear function on a vector bundle $\pi : E \to M$ such that $0 \neq l_m \in E^*_m$ for any $m \in M$. Then any level set of $l$, $A = \{x \in E : l(x) = c\}$, $c \in \mathbb{R}$, is an affine bundle modelled on the kernel of $l$. We shall denote it by $E^d_c$, when $c = 1$, and by $E^{d=1}_c$, in general. This construction can be even partially reversed: any affine bundle $A$ is canonically embedded into a vector bundle called the vector hull of $A$ ([GM, GU2]). One can ask about a similar passage from double vector bundles to double affine bundles. Let us assume that $(D; D_1, D_2; M)$ is a double vector bundle and let $l_1$ be a linear function on $D_1$. Then the pullback $\tilde{l}_1$ of $l_1$ with respect to the projection $D \to D_1$ is a linear function on $D$ with respect to the vector bundle structure on $D \to D_2$, because it is a composition of the morphism $\pi_1$ and $l_1$, if we treat a linear function on the total space as a morphism to the trivial bundle $M \times \mathbb{R} \to M$. We can interpret $\tilde{l}_1$ as an element of $\text{Sec}_{D_2}(D^*\times D^*_2)$, where we denote the dual of $D$ as a vector bundle over $D_2$ by $D^{\times D_2}$. In the graded approach to double vector bundles, as in (GR), one can simply view $l_1$ and $l_2$ as functions of degrees (0, 1) and (1, 0), respectively, on the total space $D$. We have the following
Theorem 2.2. Let \((D; D_1, D_2; M)\) be a double vector bundle and let \(l_i\) be fiberwise non-zero linear functions on \(D_i, i = 1, 2\). Let \(\bar{l}_1 \in \text{Sec}_{D_2}(D^* \times D_2)\) be the corresponding pullback of \(l_1\), and similarly for \(\bar{l}_2\). Let \(c_1, c_2 \in \mathbb{R}\), \(A = \{x \in D : \bar{l}_1(x) = c_1, \bar{l}_2(x) = c_2\}\) and \(A_i = D_1^2; c_i\). Then

\[
\begin{array}{ccc}
A & \overset{\pi_2}{\longrightarrow} & A_2 \\
\downarrow{\pi_1} & & \downarrow{\pi'_1} \\
A_1 & \overset{\pi'_2}{\longrightarrow} & M
\end{array}
\]

is a double affine bundle, where \(\pi_i, i = 1, 2\), are the restrictions of the projections of \(D\) onto \(D_i\).

Proof.- Because a restriction of a linear map to an affine subspace is an affine map, the condition \((i)\) in the definition of a double affine bundle is satisfied. Similarly, the interchange law of \((iii)\) holds, because it is so for double vector bundles. The condition \((ii)\) follows immediately from the definition of \(A\).

If \(A'\) is a submanifold of the total space \(A\) of a double affine bundle \(A = (A; A_1, A_2; M)\) such that \(A' = (A'; \pi_1(A'), \pi_2(A'); \pi'_1(\pi_2(A')))\) is a double affine bundle with the structure maps induced from \(A\), then \(A'\) is called a \textit{double affine subbundle} of \(A\). In the situation described by the theorem above, when \(c_1 = c_2 = 1\), we shall refer to \(A = (A; A_1, A_2; M)\) as to a double affine subbundle of \(D = (D; D_1, D_2; M)\) given in \(D\) by means of linear functionals \(l_1, l_2\).

If \(\phi\) is a surjective vector bundle morphism from \(E_1 \to M\) to \(E_2 \to M\), which is identity on the base \(M\), and if \(s \in \text{Sec}_M(E_2)\), then the preimage \(\phi^{-1}(s(M))\) is an affine subbundle of \(E_1\). In the case of double structures we have the following.

Proposition 2.1. Let \((D; D_1, D_2; M), (D'; D'_1, D'_2; M)\) be double vector bundles and let \(\phi\) be a surjective morphism of double vector bundles from \(D\) to \(D'\) such that \(\phi|_M = \text{Id}|_M\). Let \(s\) be a section of the fibration \(D' \to M\) and let us assume that the core of \(D'\) is trivial. Then \(\phi^{-1}(s(M))\) is a double affine subbundle of \(D\).

The assumption of the triviality of the core bundle in the above proposition is essential as shows the following example.

Example. Let \(D = M \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}\) be a trivial double vector bundle constructed from three trivial bundles of rank 1. Consider a double bundle morphism \(\phi\),

\[
\phi(m; x; y; z) := (m; xy)
\]

from \(D\) to the trivial double vector bundle \(M \times \mathbb{R}\) with the core of rank 1. Then for the section \(s = 1_M\), the set \(\phi^{-1}(s(M)) = \{(m; x, y; z) : xy = 1\}\) is not a double affine subbundle of \(D\). Moreover, in the following example

\[
A = \{(m; x; y; z) \in D : x + y + z = 1\}
\]

is a double affine subbundle of \(D\) which cannot be presented in a form \(\phi^{-1}(s(M))\) for any double vector bundle morphism \(\phi\) from \(D\).
3 Canonical constructions

Let us recall that any affine space $A$ is modelled on a vector space $V(A)$ which consists of vectors $[a, b] \in V(A)$, $a, b \in A$, where by definition $[a, b] = [a', b']$ if and only if $\text{aff}(a, b'; 1/2) = \text{aff}(b, a'; 1/2)$. The vector space $V(A)$ acts on $A$ according the rule $a + [a, b] = b$. Moreover, any affine space $A$ can be canonically immersed into a vector space $\hat{A}$ called the vector hull of $A$. This immersion satisfies a universal property for affine functions ([GU2, GM]). The hull of $A$ has the dimension greater by one than $A$ and can be defined as a factor vector space of a free vector space with basis $\{x_a : a \in A\}$ modulo a subspace spanned by vectors of the form $x_{\text{aff}(a,b),\lambda} - \lambda x_a - (1 - \lambda)x_b$, $\lambda \in \mathbb{R}$, $a, b \in A$ ([GU2]). There is a unique linear function $s : \hat{A} \to \mathbb{R}$ which assigns 1 to each class of $x_a$, $a \in A$. The subset of $\hat{A}$ defined by equations $s = 1$, and respectively $s = 0$, can be canonically identified with $A$ and $V(A)$, respectively. These notions are extended naturally (fiberwise) to the case of affine bundles.

A similar constructions can be performed with double affine bundles. For notation, if $\pi : A \to M$ is an affine bundle then $V(\pi) : V(A) \to M$ stands for its model vector bundle. If $A$ has several affine structures, we add a subscript to $V$ to indicate which structure is considered. If $\phi : A \to B$ is an affine bundle morphism then the linear part of $\phi$ is denoted by $\phi^v : V(A) \to V(B)$. The induced morphism between the vector hulls is denoted by $\hat{\phi} : \hat{A} \to \hat{B}$. For an affine function $y$ on the total space $A$ of $\pi$, $\hat{y}$ and $y$ denote the extension of $y$ to a linear function on $\hat{A}$ and the linear part of $y$, respectively. If $(x; y^j)$ is an adapted local coordinate system for $\pi : A \to M$ then $(x; y^j)$ is the induced coordinate systems for $V(A)$. For the rest of this section $A = (A; A_1, A_2; M)$ is a double affine bundle as in (3).

3.1 Model double vector bundle

Let us consider the model vector bundles of the vertical affine bundles of $\mathbf{A}$ and the induced morphism $(\pi_1^v, \pi_2^v)$ between them:

$$
\begin{array}{ccc}
V_1(A) & \xrightarrow{\pi_1^v} & V(A_2) \\
\downarrow V_1(\pi_1) & & \downarrow V(\pi_1^) \\
A_1 & \xrightarrow{\pi_2^} & M
\end{array}
$$

It will turn out that $\pi_1^v : V_1(A) \to V(A_2)$ is still an affine bundle (canonically) and $V_1(\pi_1)$ is an affine bundle morphism. Let us consider its linear part $V_1(\pi_1)^v$:

$$
\begin{array}{ccc}
V_2V_1(A) & \xrightarrow{V_2(\pi_1^v)} & V(A_2) \\
\downarrow V_1(\pi_1)^v & & \downarrow V(\pi_1^) \\
V(A_1) & \xrightarrow{V(\pi_1^)} & M
\end{array}
$$

We shall show that $\mathcal{V}(A) := V_2V_1(A)$ is a double vector bundle and call it the model double vector bundle of $\mathbf{A}$. Moreover, an analogous construction, relying on taking
first the model vector bundles with respect to the horizontal and then vertical affine structures of \([\mathcal{V}_1]\), gives an isomorphic double vector bundle \(\mathcal{V}_1 \mathcal{V}_2(A)\). The proof will use a local coordinate description of \(A\).

Let \((x; y, z; c)\) and \((x'; y'; z'; c')\) be two adapted local coordinate systems for \((A; A_1, A_2; M)\) related as in \([7]\). Then \((x; y, z; c)\) and \((x'; y', z'; c')\) are the induced local coordinate systems for \(\mathcal{V}_1(A)\) related by

\[
\begin{align*}
x' &= x'(x), \\
y' &= \alpha_0^j(x) + \sum_i \alpha_i^j(x)y^i, \\
z'^a &= \sum_b \beta_b^a(x)z^b, \\
c'^a &= \sum_b \left(\gamma_{0b}^a(x) + \sum_i \gamma_{ib}^a(x)y^i\right)z^b + \sum_w \sigma_w^a(x)c^w.
\end{align*}
\]

From this description one easily recognizes that \(\mathcal{V}_1(A)\) is an affine bundle with the base manifold \(\mathcal{V}(A_2)\) and the fiber coordinates \((z, y)\). For the model vector bundle \(\mathcal{V}(A)\), one finds that the induced coordinate system \((x, y, z; c)\) transforms according to:

\[
\begin{align*}
x' &= x'(x), \\
y' &= \sum_i \alpha_i^j(x)y^i, \\
z'^a &= \sum_b \beta_b^a(x)z^b, \\
c'^a &= \sum_i \gamma_{ib}^a(x)y^i z^b + \sum_w \sigma_w^a(x)c^w,
\end{align*}
\]

where all the indices start from 1. We can do the same construction in the reversed order getting the same coordinates and the same transformation rules which shows that indeed \(\mathcal{V}(A)\) is a double vector bundle and \(\mathcal{V}_2 \mathcal{V}_1(A) \cong \mathcal{V}_1 \mathcal{V}_2(A)\).

One can give a more geometric description of the affine structure of the bundle \(\pi^v_2 : \mathcal{V}_1(A) \rightarrow \mathcal{V}(A_2)\) as follows. Suppose we are given two vectors: \(v = [a_1, b_1]_1, w = [a_2, b_2]_1, \) \(v, w \in \mathcal{V}_1(A)\) lying in the same fiber of \(\pi^v_2\). Because \(\pi^{-1}_1(\{u_1\}) \cap \pi^{-1}_2(\{u_2\}) \neq \emptyset\) for any \((u_1, u_2) \in A_1 \times_M A_2\), we can find \(a'_2, b'_2\) such that \(\pi_2(a_1) = \pi_2(a'_2), \pi_1(a_2) = \pi_1(a'_2)\) and \([a'_2, b'_2]_1 = [a_2, b_2]_1\). Then necessarily \(\pi_2(b_2) = \pi_2(b'_2)\) and we can define

\[\text{aff}(v, w; \lambda) := [\text{aff}_2(a_1, a'_2; \lambda), \text{aff}_2(b_1, b'_2; \lambda)]_1.\]

It makes sense, since \(\pi_1\) is an affine morphism and so the head and the tail of the above vector lie in the same fiber \(\pi_1\). One can check that this construction gives the same affine structure as the one described in local coordinates, and so it does not depend on the particular choices of \(a_i\)’s and \(b_i\)’s.

### 3.2 Vector hull of a double affine bundle

Now we are going to describe a construction of the vector hull of a double affine bundle. First we take the vector hulls of the affine bundles \(\pi_1 : A \rightarrow A_1\) and \(\pi'_1 : A_2 \rightarrow M\) and
get the induced morphism \((\hat{\pi}_2, \pi'_2)\) between them:

\[
\begin{array}{c}
\hat{A}^1 \xrightarrow{\pi_2} \hat{A}^2 \\
\downarrow \pi_1 \quad \downarrow \pi'_1 \\
A^1 \xrightarrow{\pi_2'} M
\end{array}
\]

We still use the same letters for projections from the total space of a vector hull on its base. We shall show later that \(\hat{\pi}_2 : \hat{A}^1 \to \hat{A}^2\) is an affine bundle and \((\pi_1, \pi'_1)\) is an affine bundle morphism. Now we take the vector hulls with respect to the horizontal affine bundle structures and get the induced morphism \((\hat{\pi}_1, \pi'_1)\):

\[
\begin{array}{c}
\hat{A} \xrightarrow{\hat{\pi}_2} \hat{A}^2 \\
\downarrow \hat{\pi}_1 \quad \downarrow \pi'_1 \\
\hat{A}^1 \xrightarrow{\pi_2'} M
\end{array}
\]

We shall show that this is a double vector bundle and that the construction is symmetric i.e. we obtain a canonically isomorphic object if we take the hull with respect to the second \(\text{aff}_2\) structure in the first step, and then apply the hull with respect to \(\text{aff}_1\). It will be called the hull of a double affine bundle \(A\) and denoted by \(\hat{A}\).

Recall that if \(A\) is an affine space and \((y^j)\) is a system of affine coordinates on \(A\), then \((\tilde{y}^j, s)\) is the induced system of linear coordinates on the vector hull \(\hat{A}\) of \(A\), where \(s\) is the unique extension of the constant function \(1_A\) on \(A\) to a linear function on \(\hat{A}\). If \(A\) is an affine bundle over \(M\) and \((x; y^j)\) is an adapted local coordinate systems on \(A\), which transforms as

\[
y'^j = \alpha^j_0(x) + \sum_i \alpha^j_i(x) y^i,
\]

then the corresponding coordinates on the vector bundle hull \(\hat{A}\) transform in the following way:

\[
\tilde{y}'^j = \alpha^j_0(x)s + \sum_i \alpha^j_i(x) \tilde{y}^i, \quad s' = s.
\]

Let us go now to a double affine bundle setting. With local coordinates \((x; y^j, \tilde{z}^a; \tilde{c}^u; s)\) on \(\hat{A}^1\) and then with \((x; \tilde{y}^j, \tilde{z}^a; \tilde{c}^u; s, t)\) on \(\hat{A}\) one finds that the corresponding transformations are the linearizations of (7), that is

\[
\begin{align*}
\tilde{y}'^j &= \alpha^j_0(x)t + \sum_i \alpha^j_i(x) \tilde{y}^i, \\
\tilde{z}'^a &= \beta^a_0(x)s + \sum_b \beta^a_b(x) \tilde{z}^b, \\
\tilde{c}'^u &= \gamma^u_0(x)s + \sum_i \gamma^u_i(x) \tilde{y}^i s + \sum_b \gamma^u_b(x) \tilde{z}^b t + \sum_{i,b} \gamma^u_{i,b}(x) \tilde{y}^i \tilde{z}^b + \sum_w \sigma^u_w(x) \tilde{c}^w.
\end{align*}
\]  

One can get a more compact formula by putting \(\tilde{y}^0 := t, \tilde{z}^0 = s\).
Note that the main objects associated with a double affine bundle \( \mathbf{A} \) are canonically embedded into the hull of \( \mathbf{A} \). In particular, we find that the cores of \( \mathbf{\hat{A}} \) and \( \mathcal{V}(\mathbf{A}) \) are isomorphic.

**Proposition 3.1.** Let \( (A, A_1, A_2; M) \) be given in a double vector bundle \( (D; D_1, D_2; M) \) by means of linear functions \( l_1, l_2 \) as in Theorem 2.2 (with \( c_1 c_2 \neq 0 \)). Then the vector hull of \( A \) is canonically isomorphic to \( D \). Moreover, the model double vector bundle of \( A \) is a double vector subbundle of \( D \) given by equations \( l_1 = 0 = l_2 \).

**Proof.** Without loss of generality we may assume that \( c_1 = c_2 = 1 \). A similar fact is well known in the categories of vector spaces and vector bundles: if \( l \in V^* \), \( l \neq 0 \), is a linear function on a vector space \( V \) then \( \mathbf{A} = V^\mathbf{l} \) is an affine subspace of \( V \) and its vector hull \( \mathbf{\hat{A}} \) is naturally isomorphic to \( V \). Let us apply it to the vector bundle \( \mathbf{D}^\mathbf{l}_1 \rightarrow A_1 = \mathbf{D}^\mathbf{l}_1 \) and its affine subbundle given by the equation \( \mathbf{\tilde{l}}_2 = 1 \), \( A = (\mathbf{D}^\mathbf{l}_1)^\mathbf{l}_2 \rightarrow A_1 \). It follows that the vector hull \( \mathbf{\hat{A}}_1 \rightarrow A_1 \) is canonically isomorphic with the vector bundle \( \mathbf{D}^\mathbf{l}_1 \rightarrow A_1 \). The total space \( \mathbf{D}^\mathbf{l}_1 \) is also an affine subbundle of \( \mathbf{D} \rightarrow A_2 \) as a subset of points satisfying \( \mathbf{\tilde{l}}_1 = 1 \). Hence the vector hull of \( \mathbf{D}^\mathbf{l}_1 \rightarrow A_2 \) is \( \mathbf{D} \rightarrow A_2 \). Reassuming, we have performed, as in definition, the construction of the hull of the double affine bundle \( \mathbf{A} \), and eventually arrived at the double vector bundle \( \mathbf{D} \), what justifies the first assertion. The second one is clear when we compare the transformations of (12) and (13) with \( s = t = 0 \).

\[ \Box \]

## 4 Special double affine bundles and duality

A **special affine space** \( \mathbf{A} = (A, v_A) \) is an affine space \( A \) with a distinguished non-zero element \( v_A \in \mathcal{V}(A) \) in the model vector space. A vector space with a distinguished non-zero element is also called **special**. It is known (GU, GU1) that special affine spaces, in contrast to ordinary affine spaces, have well-defined dual objects in the same category. Let \( \mathbf{I} = (\mathbb{R}, 1) \) be the special vector space \( \mathbb{R} \) with the distinguished element 1. The dual of \( \mathbf{A} = (A, v_A) \) is, by definition, the space of morphisms from \( A \) to \( \mathbf{I} \) and is denoted by \( A^\# \). It consists of all affine maps from \( A \) to \( \mathbb{R} \) whose linear part preserve the distinguished elements. We shall call them **special** affine maps. The model vector space of \( A^\# \) is

\[ \mathcal{V}(A^\#) \simeq \{ \phi : A \rightarrow \mathbb{R} \mid \phi \text{ is an affine map, } \phi^v(v_A) = 0 \}, \]

where \( \phi^v : \mathcal{V}(A) \rightarrow \mathbb{R} \) is the linear part of \( \phi \). Hence \( \mathbf{A}^\# = (A^\#, 1_A) \) is a special affine space with the constant function \( 1_A \in \mathcal{V}(A^\#) \) as a distinguished element. In finite dimensions we have a true duality: \( (\mathbf{A}^\#)^\# \simeq \mathbf{A} \) for a special affine space \( \mathbf{A} = (A, v_A) \). The above notions and statements can be automatically extended to the case of affine bundles.
On the other hand, if we have a double vector bundle

$$
\begin{array}{ccc}
D & \xrightarrow{\pi_2} & D_2 \\
\downarrow{\pi_1} & & \downarrow{\pi'_1} \\
D_1 & \xrightarrow{\pi'_2} & M
\end{array}
$$

with the core bundle $D_3 \to M$, which we shortly denote by $D = (D; D_1, D_2; M)$, then the total space $D^{\ast D_1}$ of the dual bundle to $\pi_1 : D \to D_1$ has a double vector bundle structure

$$
\begin{array}{ccc}
D^{\ast D_1} & \to & D_3^{\ast} \\
\downarrow & & \downarrow \\
D_1 & \to & M
\end{array}
$$

with the core bundle isomorphic to $D_2^\ast$, which we call the vertical dual of $D$ ([KU, M]) and denote by $D^V$. Similarly we can form a horizontal dual of $D$ which we denote by $D^H = (D^{\ast D_2}; D_3^\ast; D_2; M)$.

We are going to join both concepts and define a category where dualities of double affine bundles live in.

Let us recall that if $A$ is an affine space then the space of affine maps $A^\dagger = \text{Aff}(A, \mathbb{R})$ is an example of a special vector space with the constant function $1_A$ as a distinguished element. If $V = (V, v)$ is a special vector space then $V^\dagger = \{ \phi \in V^* : \phi(v) = 1 \} \subset V^*$ is an affine subspace of codimension 1.

For the rest of this section, $A = (A; A_1, A_2; M)$ is as in (3), and $D = (D; D_1, D_2; M)$ is the hull of $A$ with the core denoted by $D_3$. We shall also call $D_3$ the core of $A$.

**Definition 4.1.** The objects of the category of special double affine bundles are double affine bundles (3) equipped with a distinguished nowhere vanishing section $\sigma \in \text{Sec}_M(D_3)$. Morphisms are assumed to preserve the distinguished elements, i.e. the induced map between the cores is a morphism of special vector bundles.

**Proposition 4.1.** A nowhere vanishing section $\sigma \in \text{Sec}_M(D_3)$ induces sections $\sigma_i \in \text{Sec}_{A_i}(V_i(A)), i = 1, 2$, what turns $\pi_i : A \to A_i$ into a special affine bundle.

**Proof.** If $(D; D_1, D_2; M)$ is a double vector bundle with the core $D_3$ then the pullback of the core bundle, $(\pi'_2)^\ast(D_3)$, with respect to $\pi'_2 : D_1 \to M$ is a vector subbundle of $\pi_1 : D \to D_1$, because it can be identified with the kernel of the morphism $\pi_2$ ([M]).

Given a section $\sigma$ as in the hypothesis, we get the section

$$
\tilde{\sigma}_1 := (\pi'_2)^\ast(\sigma) \in \text{Sec}_{D_1}((\pi'_2)^\ast(D_3)) \subset \text{Sec}_{D_1}(D)
$$

and similarly $\tilde{\sigma}_2 \in \text{Sec}_{D_2}(D)$. Now assume that $(D; D_1, D_2; M)$ is the hull of $A$ and define $\sigma_1 := \tilde{\sigma}_1|_{A_1}$. It is straightforward to check that the image $\sigma_1(A_1)$ is a subset of $V_i(A) \subset D$, the total space of the model vector bundle of $\pi_1 : A \to A_1$. We view $\sigma_1$ as a special section for this affine bundle.

\[\square\]
The core bundle of the double vector bundle \((TE; E, TM; M)\), associated with a vector bundle \(E \to M\), is canonically isomorphic to \(E\). Indeed, it consists of vertical vectors along the zero section. The construction \(\sigma \mapsto \tilde{\sigma}_1\) from the proposition above coincides with the well-known one of the vertical lift \(Sec_M(E) \to Sec_E(TE)\).

**Theorem 4.1.** Let \(A = (A; A_1, A_2; M)\) be a special double affine bundle and \(D_3 = (D_3, \sigma)\) be its core. Then

\[
A^{\#A_1} \to D_3^\dagger \\
\downarrow \quad \quad \downarrow \\
A_1 \to M
\]

is also a special double affine bundle with the core equal to \((A_2, 1_{A_2})\).

There is even a simpler description of duals to a special double affine bundle. Let us assume that a special double affine bundle \(A\) is given in \(D\) by means of linear functions \(l_1, l_2\) on the bundles \(D_1, D_2\), respectively. Let \(\sigma \in Sec_M(D_3)\) be the distinguished nowhere vanishing section for \(A\). Let us consider \(\sigma\) as a linear function \(l_3\) on \(D_3^\dagger\). We shall prove the following

**Theorem 4.2.** The double affine bundle determined by the linear functions \(l_1, l_2\) and \(l_3\) in the vertical dual vector bundle \((D^{*D_1}; D_1, D_3^\dagger; M)\) is canonically isomorphic to the double affine bundle from Theorem 4.1. Moreover, \(l_2\) is the distinguished section of the core bundle \(D_2^\dagger \to M\) of \(D^{*D_1}\) which corresponds to \(1_{A_2} \in Sec_M(A_2^\dagger)\).

**Proof.** (of Theorems 4.1 and 4.2) We can assume that \(A\) is given in \(D\) by means of linear functions \(l_1, l_2\). We shall recognize \(A^{\#A_1}\) as the subset

\[
A^{\#A_1} = \{ \varphi \in D^{*D_1} : \tilde{l}_1(\varphi) = \tilde{l}_3(\varphi) = 1 \} \tag{15}
\]

where \(l_3\) is the linear function on \(D_3^\dagger\) associated with the section \(\sigma \in Sec_M(D_3)\) and \(\tilde{l}_3\) is the pullback of \(l_3\) with respect to the bundle projection \(D^{*D_1} \to D_3^\dagger\). We have \(l_3(\varphi_m) = \langle \varphi_m, \sigma(m) \rangle\) for \(m \in M\) and \(\varphi_m \in (D_3^\dagger)_m\), hence

\[
D_3^\dagger = \{ \varphi_m \in D_3^\dagger : \langle \varphi_m, \sigma(m) \rangle = 1, m \in M \} = (D_3^\dagger)^{2A_1}.
\]

Let \(\tilde{\sigma}_1 \in Sec_{D_1}(D), \sigma_1 = \tilde{\sigma}_1|_{A_1}\), be the distinguished elements induced from the section \(\sigma\) of the core bundle, as in the proof of Proposition 4.1. For \(\varphi \in D^{*D_1}\), which lies in a fiber over \(x \in D_1\), we have

\[
\langle \tilde{l}_3, \varphi \rangle = \langle \varphi, \tilde{\sigma}_1(x) \rangle,
\]

hence \(\varphi \in A^{\#A_1} \subset D^{*D_1}\) if and only if \(\langle \tilde{l}_3, \varphi \rangle = \langle \varphi, \sigma_1(x) \rangle = 1\) for \(x \in A_1 \subset D_1\), what proves (15). This way, in the vertical dual bundle \(D^V = (D^{*D_1}; D_1, D_3^\dagger; M)\) we have distinguished the affine subbundles

\[
A^{\#A_1} \subset D^{*D_1} \to D_3^\dagger \supset D_3^\dagger \\
\downarrow \quad \quad \downarrow \\
A_1 \subset D_1 \to M
\]
Moreover, the core of $A^{\#A_1}$ is the core of $D^V$ and is equal to $D_2^A \simeq \text{Aff}(A_2, \mathbb{R}) = A_2^1$. It is equipped with the section $l_2 \in \text{Sec}_M(D_2^*)$, which corresponds to $1_{A_2}$, as $l_{2|A_2} = 1_{A_2}$.

Recall that, if $(A, v_A)$ is a special affine bundle, then its adjoint is the special affine bundle $(A, -v_A)$. If $(A, \sigma)$ is a special double affine bundle, where $\sigma$ is the distinguished section of the core bundle, then its adjoint is $(A, -\sigma)$. We shall denote with $D' = (D; D_2, D_1; M)$ and $A' = (A; A_2, A_1; M)$ the flips of $D$ and $A$, respectively. We consider the flip as an operation on double vector (or affine) bundles relying on switching the horizontal and vertical arrows in the diagrams representing them. If $A$ is special then so is $A'$ with the same distinguished section of the core bundle. Let $\pi_D : D \to D_1 \times_M D_2$ denotes the canonical fibration.

**Proposition 4.2.** With the assumptions of Theorem 4.1, the special affine bundles $A^{\#A_1} \to D_3^1$ and the adjoint of $A^{\#A_2} \to D_3^1$ are in duality. The special double affine bundle $A^{HVH}$ is naturally isomorphic to the adjoint of $A'$.

**Proof.** Let $D$ be as usual, the hull of $A$. For $\Phi \in D^{\ast D_1}$ and $\Psi \in D^{\ast D_2}$ such that $\pi_{D V}(\Phi) = (d_1, \varphi) \in D_1 \times_M D_3^*$ and $\pi_{D A}(\Psi) = (\varphi, d_2) \in D_3^* \times_M D_2$ the difference

$$\langle \Phi, \Psi \rangle = \Phi(x) - \Psi(x),$$

where $x \in D$ is any element lying over $(d_1, d_2) \in D_1 \times_M D_2$, does not depend on the choice of $x$ and defines a non-degenerate pairing between $D^{\ast D_1}$ and $D^{\ast D_2}$ ([KU]). We shall restrict the pairing $\langle \cdot, \cdot \rangle$ to the affine subbundles $A^{\#A_1}$ and $A^{\#A_2}$. The sections $l_2$, $l_1$ of the core bundles $D_2^*$, $D_1^*$ of the double vector bundles $D^{\ast D_1}$, $D^{\ast D_2}$, respectively, induce the sections $\bar{l}_2 \in \text{Sec}_{D_3^*}(D^{\ast D_1})$, $\bar{l}_1 \in \text{Sec}_{D_3^*}(D^{\ast D_2})$. After the restriction to $D_3^j \subset D_3^*$ we get the distinguished sections of the model bundles of $A^{\#A_j} \to D_3^1$, $j = 1, 2$, which we still denote by $l_2, \bar{l}_1$. The sum $\Phi + \bar{l}_2(\varphi)$ of two elements in the same fiber of $D^{\ast D_1} \to D_3^1$, evaluated on $x \in D$, gives $\Phi(x) + l_2(d_2)$. Since $l_{j|A_j} = 1_{A_j}$, and $d_j \in A_j$, for $j = 1, 2$, we have

$$\langle \Phi + \bar{l}_2(\varphi), \Psi \rangle = \langle \Phi, \Psi \rangle + 1 = \langle \Phi, \Psi - \bar{l}_1(\varphi) \rangle,$$

what justifies the first statement of our proposition.

If we encode the special double affine bundle $A$ as in the theorem in the form $(D; l_1, l_2; l_3)$ (the special double affine subbundle of $D$ determined by the linear functions $l_1$, $l_2$ on the bases of the side bundles and a linear function $l_3$ on the dual to the core bundle), then the horizontal dual $A^H = A^{\#A_2}$ goes with $(D^H; l_3, l_2, l_1)$. Consequently, $A^{HV}$ corresponds to $(D^{HV}; l_3, l_1, l_2)$ and finally $A^{HVH}$ to $(D^{HVH}; l_2, l_1, l_3)$. Moreover, it is known that the double vector bundle $D^{HVH}$ obtained from $D$ by first taking horizontal dual, then vertical, and again horizontal dual, is naturally isomorphic to the flip $D'$ of $D$ ([KU, MM]). The isomorphism $\alpha : D^{HVH} \to D'$ is the identity on the side bundles $D_2$ and $D_1$, but is minus the identity on the core $D_3$. Let us restrict $\alpha$ to the double affine subbundle $A^{HVH}$. The distinguished section $\sigma$ (corresponding to $l_3$ above) of the core bundle is moved to $-\sigma$, hence $\alpha$ induces an isomorphism between $A^{HVH}$ and the adjoint of $A'$.

$\square$
5 The affine phase bundle

Let $\zeta : Z \to M$ be a bundle of affine values (AV-bundle, in short). By definition ([GU2]), it is a special affine bundle of rank 1. It follows that the model vector bundle of $\zeta$ is the trivial vector bundle $M \times \mathbb{R} \to M$. The action of the model bundle on the total space $Z$ allows us to consider $\zeta$ as $(\mathbb{R}, +)$-principal bundle with the base $M$. The affine phase bundle ([GU2]) $P\zeta : PZ \to M$ is defined as follows. Let us consider the following equivalence relation on the set of pairs $(m, \sigma)$, $m \in M$, $\sigma \in \text{Sec}(\zeta)$:

$$(m, \sigma) \sim (m', \sigma') \quad \text{if and only if} \quad m = m' \quad \text{and} \quad d_m(\sigma - \sigma') = 0.$$ 

Here we identify $\sigma - \sigma'$ with a section of the model bundle $M \times \mathbb{R} \to M$, i.e. a function on $M$. The equivalence classes of $\sim$ are the elements of $PZ$. It is an affine bundle modelled on $\mathbb{T}^*M$. The class of $(m, \sigma)$ is denoted by $d_m \sigma$ and called an affine differential of $\sigma$ at $m$.

Let $\eta : A \to M$ now be a special affine bundle with the distinguished nowhere vanishing section $v_A \in \text{Sec}_M(\mathcal{V}(A))$ and let $\zeta : A \to A$, $A = A/\langle v_A \rangle$, be the associated AV-bundle. We shall use the convention of ([GU2]), that the distinguished section of $\zeta$ is $-v_A(m)$ at points $a \in A$ lying over $m \in M$. The total space of the affine phase bundle $P\zeta : PA \to A$, has also another structure of an affine bundle, over the base $A^\# = A^#/\langle 1_A \rangle$, what makes $PA$ a canonical example of a double affine bundle depicted in the diagram

$$
\begin{array}{ccc}
PA & \xrightarrow{\rho^\#} & A^# \\
\downarrow P\zeta & & \downarrow \eta^# \\
A & \xrightarrow{\eta} & M
\end{array}
$$

In this section, basing on the example of $PA$, we are going to describe canonical objects associated with double affine bundles. We shall give a clear picture of the hull, the model double vector bundles and the duals of $PA$.

We are going first to describe the double affine structure on $PA$. Let us start with the projection in the affine bundle $P^\#\zeta : PA \to A^#$. Let $\omega_a$ be any element (an affine covector) of $PA$, $a \in A$, $\eta(a) = m \in M$, $a = \zeta(a)$, $(P\zeta)(\omega_a) = a$. Let us write it in a form $\omega_a = d_a \sigma$, for a section $\sigma \in \text{Sec}_A(A)$. It is possible to find an affine section representing $\omega_a$, i.e. we may assume that $\sigma$ is an affine morphism from $\eta : A \to M$ to $\eta : A \to M$. The space of affine sections of $\zeta : A \to A$ is in 1-1 correspondence $\sigma \mapsto f_\sigma$ with the set of special affine maps on $A$, and hence with $\text{Sec}_M(A^#)$ ([U]), where $f_\sigma : A \to \mathbb{R}$ is defined by

$$f_\sigma(x) \cdot v_A(m') = [\sigma(x), x] \in \mathcal{V}(A_{m'})$$

for $x \in A$, $m' = \eta(x)$. We have $f_\sigma(x + v_A(m')) = f_\sigma(x) + 1$, so $f_\sigma$ is a special affine map. Moreover, if $\omega_a = d_a \sigma'$ for an affine section $\sigma'$, then $d_a(\sigma - \sigma') = 0$, hence $f_\sigma - f_{\sigma'}$ is constant on $A_m$. Thus we get a well-defined map $P^\#\zeta : PA \to A^#$, $\omega_a \mapsto f_{\sigma|A_m} + \mathbb{R} \cdot 1_A \in A_{m}^#$. We are going now to define the affine structure in a fiber of $P^\#\zeta$. Let us fix $m \in M$, $a_1, a_2 \in A_m$, $f_0 \in A_m^#$ and consider the fiber $F = (P^\#\zeta)^{-1}(f_0)$

$$
\begin{array}{ccc}
PA & \xrightarrow{\rho^\#} & A^# \\
\downarrow P\zeta & & \downarrow \eta^# \\
A & \xrightarrow{\eta} & M
\end{array}
$$

\[16\]
and the set $\sum$ of all affine sections $\sigma \in \text{Sec}_A(A)$ such that $d_{a}\sigma \in F$ at some point $a \in A_m$. The set $\sum$ does not depend on the choice of $a$ from $A_m$. Let us choose a section $\sigma_0 \in \sum$. Then any element $\omega \in F$ is of the form

$$\omega = d_{a}\sigma_0 + \alpha,$$

for an $a \in A_m$ and a unique $\alpha \in T^*_m M$, where $T^*_m M$ is considered as a subspace of the model space $T^*_A A \simeq \mathcal{V}(P, A)$ via the pullback map with respect to the projection $A \to M$. Indeed, the difference of two affine forms from $F$, $\omega - d_{a}\sigma_0 \in T^*_A A$, is zero on any vector tangent to the fiber $\eta^{-1}(m)$ and hence it is the pullback of a 1-form on $M$.

Let us define the affine combinations map in $F$ by the formula

$$\text{aff}(d_{a_{1}}\sigma_0 + \alpha_1, d_{a_{2}}\sigma_0 + \alpha_2; \lambda) = d_{a}\sigma_0 + (\lambda(\alpha_1 + (1 - \lambda)\alpha_2)), \quad (18)$$

where $a = \text{aff}(a_{1}, a_{2}; \lambda)$ and $\alpha_i \in T^*_m M$, $i = 1, 2$. It follows easily from the above discussion that this definition does not depend on the choice of $\sigma_0$. Indeed, if $\sigma, \sigma' \in \sum$ are such that $d_{a}\sigma = d_{a}\sigma'$ for $i = 1, 2$, then $d_{a}\sigma = d_{a}\sigma'$ for any affine combination $a = \text{aff}(a_{1}, a_{2}; \lambda)$, because the function $\sigma - \sigma' : A \to \mathbb{R}$ is affine.

A vector space $V$ with two distinguished non-zero elements $v_0 \in V$ and $\alpha_0 \in V^*$ such that $\alpha_0(v_0) = 0$ is called a bispecial vector space ($\text{[GU2]}$). If $(A, v_A)$ is a special affine space then its vector hull $\hat{A}$ is canonically a bispecial vector space with distinguished elements $v_A \in \mathcal{V}(A) \subset \hat{A}$ and the unique function $\alpha_A \in (\hat{A})^*$ for which $A$ in $\hat{A}$ is defined by the equation $\alpha_A = 1$. Now we assume that $(\eta, v_A), \eta : A \to M$, is a special affine bundle. Then $\hat{A}$ is a bispecial vector bundle over $M$. We shall use the letter $E$ for $\hat{A}$. Let us analyze the cotangent bundle $T^*E$. The action $\phi$ of $(\mathbb{R}, +)$, $\phi_t : v_m \mapsto v_m + t \cdot v_A(m)$ on $E$, can be lifted to an $(\mathbb{R}, +)$-action, say $\psi_1$, on $T^*E$ by means of pullback

$$(\psi_1)_t := (\phi_{-t})^*. \quad (19)$$

We have also another $(\mathbb{R}, +)$-action on $T^*E$ given by

$$\psi_2(t, \omega_v) = \omega_v + t \cdot d_v \alpha_A \quad (20)$$

for $v \in \hat{A}, \omega_v \in T^*_v E, t \in \mathbb{R}$. Obviously the actions $\psi_1$ and $\psi_2$ commute and give rise to an action $\psi = (\psi_1, \psi_2)$ of $\mathbb{R} \times \mathbb{R}$ on $T^*E$. We shall show that the orbits of $\psi$ form the vector hull of $PA$.

**Theorem 5.1.** Let $\eta : (A, v_A) \to M$ be a special affine bundle and let $\psi$ be the canonical action of $\mathbb{R} \times \mathbb{R}$ on $T^*E$ defined above. The orbit space of $\psi$, denoted by $S^*A = T^*E / \mathbb{R} \times \mathbb{R}$, has a well-defined structure of a double vector bundle

$$S^*A \longrightarrow E^* \longrightarrow E \longrightarrow M \quad (21)$$

inherited from $(T^*E; E, E^*; M)$. It is canonically isomorphic to the double vector hull of the double affine phase bundle $PA$ depicted in $\text{[10]}$. 

\[17\]
Let us denote by \((x^a), 1 \leq a \leq m\), the local coordinates on \(M\), by \((y^i), 0 \leq i \leq n+1\), the coordinates in the fibers of \(E \to M\), so that \(A \subset E\) is described by the equation \(y^{n+1} = 1\) and \(y^i(v_A) = \delta^i_0\). Denote the conjugate momenta in \(T^*E\) by \(p_a\) and \(\pi_i\). The action \(\psi_{st}\) of \((s,t) \in \mathbb{R} \times \mathbb{R}\) reads as

\[
\psi_{st}^*(y^0) = y^0 + s, \quad \psi_{st}^*(\pi_{n+1}) = \pi_{n+1} + t
\]

and \(\psi_{st}^*(y^i) = y^i, \psi_{st}^*(\pi_j) = \pi_j\) for \(i \neq 0\) and \(j \neq n+1\). The structure of a double vector bundle on \(T^*E\) is given by the two commuting homoteties \(h_1^i : T^*E \to T^*E,\)

\[
h_1^i(x^a, y^i, p_a, \pi_i) = (x^a, y^i, t \cdot p_a, t \cdot \pi_i), \quad h_2^i(x^a, y^i, p_a, \pi_i) = (x^a, t \cdot y^i, t \cdot p_a, \pi_i), \quad t \in \mathbb{R}.
\]

Note that if \(\omega\) and \(\omega'\) are in the same orbit of \(\psi\) then the same holds for \(h^i_1(\omega)\) and \(h^i_2(\omega')\), \(i = 1, 2\). Hence we get well-defined commuting homogeneous structures \(\tilde{h}^1, \tilde{h}^2\) on the quotient space \(T^*E / \mathbb{R} \times \mathbb{R}\). This proves the first assertion of the theorem.

Let us compute images of the projections \(\tilde{h}^1_0\) and \(\tilde{h}^2_0\). The image of \(\tilde{h}^1_0\) consists of null 1-forms on \(E\) modulo the translations in the direction of \(v_A\), hence

\[
\tilde{h}^1_0(S^*A) \simeq E / \langle v_A \rangle = E.
\]

The image of \(\tilde{h}^2_0\) consists of orbits of \(\omega_v \in T^*_vE\) with \(v \in M \subset E\) such that \(\omega_v|_{T_vM}\) is zero, hence the image is clearly isomorphic with \(E^*/\langle \alpha_A \rangle \simeq E^*_m\). Here we identified \(M\) with the image of the zero section of \(E \to M\).

Let us define the linear functions \(l_1 : E \to \mathbb{R}\) and \(l_2 : E^* \to \mathbb{R}\) by

\[
l_1(v + \mathbb{R} \cdot v_A(m)) := \alpha_A(v), \quad l_2(f + \mathbb{R} \cdot \alpha_A) := f(v_A(m)),
\]

for \(v \in E_m,\ f \in \text{Aff}(A_m, \mathbb{R}) \simeq E^*_m\). The functions \(l_1, l_2\) are well defined because \(\alpha_A(v_A) = 0\). The pullbacks of \(l_i, i = 1, 2, \bar{l}_i : S^*A \to \mathbb{R}\) are

\[
\bar{l}_1([\omega_v]) = \alpha_A(v), \quad \bar{l}_2([\omega_v]) = \langle \omega_v, (X_A)_v \rangle,
\]

where \([\omega_v]\) is the class of \(\omega_v \in T^*_vE\) in \(S^*A\) and the vector field \(X_A \in \text{Vect}(E)\) is the vertical lift of \(v_A \in \text{Sec}(E)\). Note that the subset of \(E\) given by the equation \(l_1 = 1\) is \(\overline{A}\) and the subset of \(E^*\) given by \(l_2 = 1\) is \(\overline{A^\#}\).

Let us recall that the vector hull of \(\zeta : PA \to \overline{A}\) is isomorphic with the reduced cotangent bundle \(\bar{T}^*A = (GU \cup [GU]) \cup [GU2]\), the orbit space of the \((\mathbb{R}, +)\)-action on \(T^*A\) induced by translations along \(v_A\). Although there is no canonical embedding of \(T^*A\) into \(T^*E\), we shall show that we do have such an embedding of \(\bar{T}^*A\) into the factor space \(S^*A\). Indeed, \(T^*A \simeq (T^*E)|_{\overline{A}} / \mathbb{R} \cdot d\alpha_A\), hence

\[
\bar{T}^*A \simeq (T^*E)|_{\overline{A}} / \mathbb{R} \times \mathbb{R} \hookrightarrow (T^*E)/\mathbb{R} \times \mathbb{R} = S^*A.
\]
Within this embedding, $\tilde{T}^*A$ is given in $S^*A$ by the equation $\tilde{l}_1 = 1$. Moreover, it is known (see [GU2]) that $PA$ can be naturally identified with the following affine subbundle of $\tilde{T}^*A$:

$$PA = \{[\omega_v] \in \tilde{T}^*A : \langle \omega_v, (X_A)_v \rangle = 1\},$$

hence $PA$ is given in $S^*A$ as

$$PA = \{[\omega] \in S^*A : \tilde{l}_1([\omega]) = 1 = \tilde{l}_2([\omega])\}.$$

We claim that $PA$ is isomorphic to the double affine bundle associated with the double vector bundle $S^*A$ and the functions $l_1$ and $l_2$. One easily checks that the projections from $S^*A$ onto $E$ and $E^*$ correspond to the projections from $PA$ onto $A$ and $A^\#$, respectively, and moreover, the affine combinations in $PA$ are compatible with the homogeneous structures $\tilde{h}^1, \tilde{h}^2$ on $S^*A$, what proves our claim. The isomorphism $S^*A \simeq \tilde{PA}$ follows now from Proposition 3.1.

\[\square\]

**Proposition 5.1.** The model double vector bundle of the double phase bundle $PA$ is canonically isomorphic to the double vector bundle $(T^*F; F, F^*; M)$ with $F = \mathcal{V}(A)$.

**Proof.** From Proposition 3.1 the model double vector bundle $\mathcal{V}(PA)$ is recognized in $S^*A$ as the set of orbits $[\omega] \in S^*A$ such that $\tilde{l}_1([\omega]) = 0 = \tilde{l}_2([\omega])$, i.e.

$$\mathcal{V}(PA) = \{[\omega_v] \in S^*A : \omega_v \in T_v^*E, v \in \mathcal{V}(A) \subset E, \langle \omega_v, (X_A)_v \rangle = 0\}.$$

We want to find a canonical isomorphism $\mathcal{V}(PA) \simeq T^*\mathcal{V}(A)$. Note that $\mathcal{V}(A) \simeq \mathcal{V}(A)$, naturally. We shall find an injection $\iota : T^*\mathcal{V}(A) \rightarrow S^*A$. It can be defined as the following composition

$$T^*\mathcal{V}(A) \xrightarrow{\iota} T^*\mathcal{V}(A) \simeq (T^*E)_{\mathcal{V}(A)}/\mathbb{R} \xrightarrow{(T^*E)_{\mathcal{V}(A)}/\mathbb{R} \times \mathbb{R}} S^*A \simeq T^*E/\mathbb{R} \times \mathbb{R},$$

where $T^*\mathcal{V}(A) \xrightarrow{\iota} T^*\mathcal{V}(A)$ is the induced pullback relation with respect to the bundle projection $\mathcal{V}(\zeta) : \mathcal{V}(A) \rightarrow \mathcal{V}(A)$. It is easy to check that $\iota$ is an injective function, preserves the homogeneous structures and gives an isomorphism of $T^*\mathcal{V}(A)$ with the image of $\iota$ which is clearly $\mathcal{V}(PA)$.

\[\square\]

The core of the double affine bundle $PA$ is $T^*M$. To put a special double affine structure on $PA$, suppose that there exist a nowhere vanishing 1-form $\omega_M \in \Omega^1(M)$ on $M$. We are going to investigate the duals of the special double affine bundle $(PA, \omega_M)$. For, we need first to describe the dual of the vector bundle $S^*A \rightarrow E$. Let us consider the subbundle

$$T^\text{hor}E := \{X_v \in T_vE : \langle d_v\alpha_A, X_v \rangle = 0, v \in E\}$$

of $TE$. We call $T^\text{hor}E$ the set of horizontal vectors. The $(\mathbb{R}, +)$-action $\phi$ on $E$ induced by translations along $v_A$ can be lifted to an action on $TE$. It moves horizontal vectors into horizontal ones, so we can consider the orbit space of this action restricted to $T^\text{hor}E$:

$$SA := \{[X] : X \in T^\text{hor}E\},$$

where $[X]$ denotes the orbit of $X \in T^\text{hor}E$ of $(\mathbb{R}, +)$-action on $TE$ induced by translations along $v_A$. 

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Proposition 5.2. The vertical dual of the double vector bundle \((S^*A; E, E^*; M)\) is canonically isomorphic to

\[
\begin{array}{c}
S^*A\rightarrow TM \\
\downarrow \\
E\rightarrow M \\
\end{array}
\]

Its core bundle is the model vector bundle \(V(A)\) of \(A\).

**Proof.** The natural pairing \(\langle \cdot, \cdot \rangle : TE \times TE \rightarrow \mathbb{R}\) satisfies

\[
\langle d_v \alpha_A, X_v \rangle = 0
\]

for \(X_v \in T^\text{hor}_v E \subset T_v E\) and is invariant with respect to \((\mathbb{R}, +)\)-actions on \(TE\) and \(T^*E\) induced by translations along \(v_A\). Thus it induces a well-defined and non-degenerate pairing \(S^*A \times SA \rightarrow \mathbb{R}\). It is also clear that the vector bundle structures \(TE \rightarrow E\) and \(TE \rightarrow TM\) can be passed to, respectively, \(SA \rightarrow E\) and \(SA \rightarrow TM\). If \(X_v \in T^\text{hor}_v E\) lies in the intersections of the kernels of the morphisms \(T^\text{hor}_E \rightarrow TM\) and \(T^\text{hor}_E \rightarrow E\), then \(v\) lies in the line spanned by \(v_A(m)\), for some \(m \in M\) and \(X_v\) is tangent to a fiber of \(V(A) \rightarrow M\). The isomorphism of the core bundle with \(V(A)\) comes from the identification of the vector bundle \(E\) with the subbundle \((VE)_M \subset TE\) of vertical vectors, restricted to \(M \subset E\).

\[\Box\]

By virtue of Theorem 4.2, the vertical dual of the special double affine bundle \((PA, \omega_M)\) is a double affine subbundle of \(SA\) described by means of linear functions \(l_1\) and \(l_3\), where \(l_1\) is given in (22) and \(l_3\) is the linear function on \(TM\) associated with \(\omega_M\). The subset of \(SA\) described by the pullbacks of linear functions \(l_1, l_3\) is equal to

\[
P^*A := \{[X_v] : v \in A \subset E, X_v \in T_v A \subset T_v E, \langle (\eta^*\omega_M)v, X_v \rangle = 1\},
\]

where \(\eta^*\omega_M\) is the pullback of \(\omega_M\) to a 1-form on \(A\). The set \(P^*A\) is an affine subbundle of the reduced tangent bundle \(\bar{T}A \rightarrow A\) consisting of those orbits \([X_a] \in \bar{T}A, X_a \in T_a A\), such that \(\langle (\eta^*\omega_M)_a, X_a \rangle = 1\). The dual bundle to \(E^*\) is clearly \(V(A)\) and the linear function \(l_2\) on \(E^*_X\) corresponds to \(v_A \in \text{Sec}(V(A))\). We summarize the above discussion as follows.

Proposition 5.3. The vertical and horizontal duals of the special double affine bundle \((PA, \omega_M)\) are naturally isomorphic to

\[
\begin{array}{c}
P^*A \rightarrow (TM)^{\omega_M} \\
\downarrow \\
A \rightarrow M \\
\end{array}
\]

and

\[
\begin{array}{c}
P^*A^\# \rightarrow A^\# \\
\downarrow \\
(TM)^{\bar{\omega}_M} \rightarrow M,
\end{array}
\]

respectively, where \((TM)^{\omega_M} = \{X \in TM : \langle \omega_M, X \rangle = 1\}\) is an affine subbundle of \(TM\). The core bundles are the special vector bundles \((V(A), v_A)\) and \((V(A^\#), 1_A)\), respectively.
The form of the vertical dual of \((\mathcal{P}A, \omega_M)\) is already justified. Let us consider the canonical isomorphism \(\beta_E\) from \(T^*E\) to (the flip of) \(T^*E^*\) of double vector bundles, which in the adapted coordinate systems \((x_a; y_i; p_b; \pi_j)\) and \((x_a; \xi_i; q_b; z_j)\) on \(T^*E\) and \(T^*E^*\), respectively, has the following form:

\[
\beta_E^*(x_a) = x_a, \quad \beta^*_E(\xi_i) = \pi_i, \quad \beta^*_E(q_b) = -p_b, \quad \beta_E^*(z_j) = y_j.
\]

It has been first discovered by Tulczyjew for \(E = TM\) (11). It is the identity on the side bundles \(E, E^*\), and minus the identity on the core \(T^*M\). It induces an isomorphism of the double vector bundles \(S^*A\) and the flip of \(S^*A^\#\), and also between the special double affine subbundles \((\mathcal{P}A, \omega_M)\) and the flip of \((\mathcal{P}A^\#, -\omega_M)\). What we need now is the flip of the vertical dual of \((\mathcal{P}A^\#, -\omega_M)\), the form of which we derive from the diagram on the left in the proposition.

\[\square\]

We shall visualize the theorems from this section in two diagrams for which we shall give now some explanation. Let us first assume that \(f : Z_1 \to Z_2\) is a morphisms between AV-bundles with the base map \(f : M_1 \to M_2\). We have an induced relation, called the phase lift of \(f\) and denoted by \(\mathcal{P}f : \mathcal{P}Z_2 \longrightarrow \mathcal{P}Z_1\). It consists of all pairs \((d_{(m)}\alpha_2, d_m\alpha_1)\) in \(\mathcal{P}Z_2 \times \mathcal{P}Z_1\) such that \(f \circ \alpha_1 = \alpha_2 \circ f\), where \(\alpha_i \in \text{Sec}(Z_i), i = 1, 2\).

In case of trivial bundles, \(Z_i = M_i \times \mathbb{R}\), the relation \(\mathcal{P}f\) coincides with the phase lift \(T^*f : T^*M_2 \longrightarrow T^*M_1\). We shall apply the phase "functor" to the following sequence of AV-bundles:

\[
\begin{array}{ccc}
A & \longrightarrow & A \times \mathbb{R} \\
\downarrow q & & \downarrow \iota \\
\downarrow \iota & & \downarrow E \\
A & \longrightarrow & E
\end{array}
\]

The right arrow \(\iota\) is the injection morphism of trivial bundles. The left arrow \(q\) is the affine bundle morphism, \(q(a, r) = a - r \cdot v_A(m)\), for \(r \in \mathbb{R}\) and \(a \in A\) lying over \(m \in M\). Since \(\mathcal{P}(A \times \mathbb{R}) \simeq T^*A\), we get the following sequence of phase bundles and relations between them:

\[
T^*E \overset{\mathcal{P}\iota}{\longrightarrow} T^*A \overset{(\mathcal{P}q)^{-1}}{\longrightarrow} \mathcal{P}A
\]

A covector \(\beta_v \in T^*_aE\) is in the relation \(\mathcal{P}\iota\) with a covector \(\alpha_a \in T^*_aA\) if and only if \(v \in A \subset E, a = v\), and \(\beta_a|_{T^*_aA} = \alpha_a\). The relation \(\mathcal{P}q : \mathcal{P}A \longrightarrow T^*A\) consists of pairs \((d_\zeta \sigma, d_\sigma f_\sigma)\), where \(\zeta(a) = \sigma, \sigma \in \text{Sec}_A(A)\), and \(f_\sigma\) is given by (17). Hence \(\mathcal{P}A\) is realized in \(T^*A\) as in (24). The sequence (26) can be seen as a process of generalized reduction. One can also pass from \(T^*E\) to \(\mathcal{P}A\) but with the first step being the reduction of \(T^*E\) to \(S^*A\). Then we can consider \(\mathcal{P}A\) as a reduction of \(S^*A\). Note that the induced relations preserve also the second canonical (affine or vector bundle) structure. Thus we get a sequence of generalized morphisms (relations) of double affine bundles as it is depicted in the first diagram below. The dual picture of this diagram is given on the second one.
We can here distinguish two types of relations: injections (or inverses of injections) and projections. The relation \( T^*E \rightarrow T^*A \) is a composition of the inverse of the injection \( (T^*E)|_A \rightarrow T^*E \) and the projection onto \( (T^*E)|_A/\mathbb{R} \simeq T^*A \), so it is of mixed type. Similarly for the reduction \( T^*A \rightarrow PA \). The reduction \( T^*E \rightarrow S^*A \rightarrow PA \) is the composition of the projection and the inverse of the injection of \( PA \) into \( S^*A \). We have put labels \((i,j)\) on some of the arrows in the diagrams to indicate how many injections \((i)\) and projections \((j)\) it involves.

### 6 Contact bundles and double affine duals

In the previous section we considered the double affine phase bundle \( PA \) in the category of special double affine bundles. In order to do this we needed to distinguish a section
of the core bundle which in our case meant to choose a 1-form $\omega_M$ on the base manifold $M$. However, there are purely canonical examples of special affine bundles. The affine contact bundle $CA$ ([GU2]) is such a one. Let us describe its double affine structure and the duals of $CA$.

Let us first recall what the affine contact bundle is. Assume that $\zeta : (Z, 1_M) \to M$ is an $AV$-bundle. A first-jet of a section $\sigma \in \text{Sec}(\zeta)$ at $m \in M$ is the class of $\sigma$ subject to the following equivalence relation:

$$\sigma \sim \sigma' \iff \sigma(m) = \sigma'(m) \text{ and } d_m(\sigma - \sigma') = 0.$$ 

We shall denote such a class by $c_m \sigma$. Note that $\sigma - \sigma'$ is a section of the model bundle $M \times \mathbb{R} \to M$ and so it can be seen as a function on $M$, hence the differential of $\sigma - \sigma'$ makes sense. The affine contact space $CZ$ is a collection of $c_m \sigma$ as $m$ varies through $M$ and $\sigma \in \text{Sec}(Z)$. It is an affine bundle over $Z$ (modelled on $T^*M$) and an $AV$-bundle over $PZ$:

$$\begin{align*}
CZ \xrightarrow{\zeta} PZ, \\
Z \xrightarrow{\zeta} M.
\end{align*}$$

It is also an affine bundle over $M$ modelled on $T^*M \oplus \mathbb{R}$, with the affine structure defined by

$$c_m \sigma_2 - c_m \sigma_1 = (d_m(\sigma_2 - \sigma_1), \sigma_2(m) - \sigma_1(m)). \quad (27)$$

In other words, $CA$ is the first-jet bundle of $\zeta$ ([S]).

Now let $(\eta, v_A)$, $\eta : A \to M$, $v_A \in \text{Sec}(\eta)$, be a special affine bundle. The affine contact bundle of $A$, denoted by $C\zeta : CA \to A$, is by definition the affine contact bundle of the $AV$-bundle $\zeta : A \to A$. The contact manifold $CA$ is naturally fibred over $A$, and also over $A^\#$ with the projection $CA \to A^\#$ given by $c_{z^2} \sigma \mapsto f_\sigma$, where $f_\sigma \in A^\#_m$ is defined in ([L]).

$$Z \xrightarrow{\zeta} M \quad \text{and} \quad A \xrightarrow{\eta} M.$$ 

The image of the projection $CA \to A \times_M A^\#$ is $\{(a, f) : a \in A, f \in A^\#_m, m \in M, f(a) = 0\}$, hence the condition $(ii)$ in the definition of a double affine bundle is not satisfied. However, we do get a special double affine bundle structure on $CA$ but fibred over $A$ and $A^\#$:

$$\begin{align*}
CA \xrightarrow{c^\# \zeta} A^\#.
\end{align*}$$

Since the core of the affine double bundle $PA$ is isomorphic to $T^*M$, the core bundle of $CA$ is the special vector bundle $(T^*M \oplus \mathbb{R}, (0, 1_M))$. There is a canonical isomorphism $\kappa : CA \to CA^\#$ ([U]), which is a morphism of double affine bundles. It is identity on the
bases \( A, A^\# \) and induces also isomorphism of \( PA \) and \( PA^\# \) of double affine bundles:

Let us analyze the vertical dual of (28). For any \( AV \)-bundle \( \zeta : Z \to M \), the affine dual of \( C\zeta : CA \to M \) is \( \bar{\zeta} : \bar{T}Z \to M \), where by definition \((\ref{U})\) \( \bar{T}Z \) is the space of orbits of the following diagonal action \( \Delta \) on \( T\bar{T}Z \):

\[
(t, v) \mapsto (\phi_t)_*(v + t \cdot X_Z(z)),
\]

for \( v \in T_zZ, z \in Z, t \in \mathbb{R} \), where \( \phi_t : Z \to Z, z \to z + t \) is the canonical action of \( \mathbb{R} \) on \( Z \), and \( X_Z \in \text{Sec}(T\bar{T}Z) \) is the fundamental vector field of this action. In the adapted coordinate system \((x_a, s; \hat{x}_a, \hat{s})\) on \( T\bar{T}Z \) we have \( X_Z = -\partial_s \) and the diagonal actions read as

\[
(t, (x_a, s; \hat{x}_a, \hat{s})) \mapsto (x_a, s + t; \hat{x}_a, \hat{s} - t).
\]

The model vector bundle of \( \bar{T}\zeta \) is the reduced tangent bundle \( \bar{T}\zeta : \bar{T}Z \to M \). Since the vector field \( X_Z \) is invariant it can be seen as a section of \( \bar{T}\zeta \). We shall consider \( \bar{T}Z \) as a spacial affine bundle over \( M \) with the distinguished section given by \( X_Z \). Let us treat \( CZ \) as an affine hyperbundle of \( T^*Z \) under the canonical embedding \( c_m \sigma \mapsto \omega \in T^*_Z \), defined by \( \langle \omega, v \rangle X_Z(z) = v - v' \), where \( z = \sigma(m) \) and \( v' \in T_zZ \) is the vertical projection of \( v \in T_zZ \) onto the tangent space to the image of \( \sigma \). Then

\[
CZ \simeq \{ \omega_z \in T^*Z : \langle \omega_z, X_Z(z) \rangle = 1, z \in Z \}.
\]

We have an obvious bi-affine special pairing \( CZ \times \bar{T}Z \to \mathbb{R} \) induced by the one of cotangent and tangent bundles, which gives the desired isomorphism \( (CZ)^\# \simeq T\bar{T}Z \) of special affine bundles over \( M \). One can prove that \( T\zeta \) is the bundle of affine derivations on \( Z \) with values in \( Z \) denoted by \( \text{Aff}_M(PZ, Z) \) \((\ref{GU2})\).

Now we apply the above observations to the \( AV \)-bundle \( \zeta : A \to A \) and find that, according to Theorem \((\ref{L})\) the vertical dual of \( \bar{A} \) is

\[
\begin{array}{c}
\bar{A} \\
\downarrow \tau \\
A \\
\end{array}
\xymatrix{
\bar{T}A \ar[r]^\eta \ar[d]_\tau & T\bar{T}Z \ar[d]_{\tau_M} \\
\tau_A & TM \\
\end{array}
\]

since \( D^1_3 = (T^*M \oplus \mathbb{R})^\dagger = TM \). Note that the map \( T\eta : \bar{T}A \to TM \) factors to a map from \( \bar{T}A \) which is an affine bundle over \( TM \). The core of \( \bar{A} \) is \( (A^\#)^\dagger \simeq \mathcal{V}(A) \) which
is a special vector bundle. Indeed, as $\text{Aff}(A^\#; \mathbb{R}) = (A^\#)^\dagger \simeq \hat{A}$, the elements of $(A^\#)^\dagger$ correspond to the vectors $v \in \hat{A}$ such that the special element $1_A$ of $(\hat{A})^*$ annihilates $v$, and so $v \in \mathcal{V}(A)$. The described isomorphism preserve also the distinguished elements: the constant function equal 1 on $A^\#$ and $v_A \in \text{Sec}(\mathcal{V}(A))$. This way we have discovered another canonical object in the category of special double affine bundles.

In a vector bundle setting, the cotangent bundle $T^*E$ is a very interesting and intriguing object. Recall that, if $E$ is an $n$-vector bundle and $\Delta_i$, $i = 1, \ldots, n$, are the Euler vector fields corresponding to the $n$-vector bundle structures on $E$, then $T^*E$ is canonically an $(n + 1)$-vector bundle, whose vector bundle structures are encoded in the Euler vector fields $T^*\Delta_i$, which is by definition the phase lift of $\Delta_i$, and the natural cotangent vector bundle structure. Moreover, the canonical symplectic structure on $T^*E$ gives the pairings between $E$ and the other bases of the side bundles of $T^*E$ ([GR]). In the following we are going to find a similar passage but in an affine setting.

Let $(\eta : A \rightarrow M, v_A)$ be a special affine bundle, $E := \hat{A}$ be its vector hull and let $\alpha_A \in \text{Sec}(E^*)$ be the distinguished section. We shall consider $\alpha_A$ and $v_A$ as linear functions on $E$ and $E^*$, respectively, and then as the functions on $T^*E$ via pullbacks with respect to the canonical projections $T^*E \rightarrow E$ and $T^*E \rightarrow E^*$. Let us define $\mathbb{B}A$ as a double affine subbundle $(T^*E; E, E^*; M)$ determined by the following linear functions on $E$ and $E^*$: $\alpha_A$ and (evaluation on) $v_A$. One easily finds that

$$\mathbb{B}A = \{\omega_x \in T^*_xE : x \in A \subset E, \langle \omega_x, X_A(x) \rangle = 1 \},$$

where $X_A \in \text{Sec}_E(TE)$ is the vertical lift of $v_A$. As $\mathbb{B}A$ is a canonical double affine bundle with side bundles being dual special affine bundles $A$ and $A^\#$, we will call it the double affine dual bundle of $A$ (or $A^\#$).

Note that $\mathbb{B}A$ can also be described as the set of first jets $j^1_A\sigma$ of the bundle $E \rightarrow E$ at points $a \in \underline{A} \subset \underline{E}$. It is also possible to present $\mathbb{B}A$ as a middle step of the reduction from $T^*E$ to $\mathcal{P}A$:

$$\begin{array}{c}
\mathbb{B}A \\
\downarrow \pi_A \\
A
\end{array} \xrightarrow{\pi_A^\#} \begin{array}{c}
\mathcal{P}A \\
\downarrow \pi_A \end{array} \xrightarrow{\pi_A^\#} \begin{array}{c}
\mathbb{B}A \\
\downarrow \pi_A \end{array} \xrightarrow{\pi_A^\#} \begin{array}{c}
\mathcal{P}A \\
\downarrow \pi_A \end{array} \xrightarrow{\pi_A^\#}
$$

The projections in the diagram

$$\begin{array}{c}
\mathbb{B}A \\
\downarrow \pi_A \\
A
\end{array} \xrightarrow{\pi_A^\#} \begin{array}{c}
\mathcal{P}A \\
\downarrow \pi_A \end{array} \xrightarrow{\pi_A^\#} \begin{array}{c}
\mathbb{B}A \\
\downarrow \pi_A \end{array} \xrightarrow{\pi_A^\#} \begin{array}{c}
\mathcal{P}A \\
\downarrow \pi_A \end{array} \xrightarrow{\pi_A^\#}
$$

read as $\pi_A(\omega_x) = x$ for $\omega_x \in T^*_xE \cap \mathbb{B}A$, $\pi_A^\#(\omega_x) : A_m \rightarrow \mathbb{R}$ is the composition

$$A_m \xrightarrow{\omega} E_m \xrightarrow{y_i} \mathbb{R},$$

where $x \in E_m$ and $\omega_x$ is restricted to the tangent space to the fiber $E_m$. Let $(y_i)$ and $(\xi_j)$ be local bases of sections of $E^*$ and $E$, respectively, such that $y_i(\xi_j) = \delta_{ij}$.
δ_{ij}. We shall consider them also as linear functions on dual bundles. Let \( \beta_E : T^*E \to T^*E^* \) be the Tulczyjew isomorphism \(^{(25)}\), which in the adapted coordinate systems \((x_a; y_i; p_b; \pi_j)\) and \((x_a; \xi_i; q_b; z_j)\) on \( T^*E \) and \( T^*E^* \), respectively, reads as \( \beta_E(x_a; y_i; p_b; \pi_j) = (x_a; \pi_i; -p_b; y_j) \). The restriction of \( \beta_E \) to \( \mathbb{B}A \) induces an isomorphism of double affine bundles \( \mathbb{B}A \) and \( (\text{the flip of}) \ \mathbb{B}A^\# \):

\[
\begin{align*}
T^*E & \quad \xrightarrow{\beta_E} \quad T^*E^* \\
\mathbb{B}A & \quad \xrightarrow{\beta} \quad \mathbb{B}A^\#
\end{align*}
\]

The space \( \mathbb{B}A \) is preserved by the action \( \psi = (\psi_1, \psi_2) \) (defined in \(^{(19)}\), \(^{(20)}\)) of \( \mathbb{R} \times \mathbb{R} \) on \( T^*E \). We now consider \( \psi = \psi^A \) (respectively, \( \psi^{A^\#} \)) as an action on \( \mathbb{B}A \) (respectively, \( \mathbb{B}A^\# \)). The orbit space of \( \psi \), \( \mathbb{B}A/\mathbb{R} \times \mathbb{R} \), is canonically identified with the affine phase bundle \( PA \subset T^*A \) \(^{(252)}\). The isomorphism \( \beta_E \) moves the orbits of \( \psi^A \) into the orbits of \( \psi^{A^\#} \), and so it induces an isomorphism of the orbit spaces, \( PA \) and \( PA^\# \). Moreover, the orbit space of \( \psi^2_A \), \( \mathbb{B}A/\{0\} \times \mathbb{R} \), can be naturally identified with the affine contact bundle \( CA \). Indeed, thanks to our convention of the distinguished section of the model bundle of an AV-bundle, \( CA \) is identified with the affine hyperbundle of \( T^*A \) of those covectors \( \omega \in T^*_aA \), \( a \in A \), such that \( \langle \omega, X_A(a) \rangle = 1 \), where \( X_A \) is the vertical lift of \( v_A \). Beside, each orbit of \( \psi^A_2 \) (respectively, \( \psi^A_1 \)) is moved by \( \beta_E \) to an orbit of \( \psi^{A^\#}_1 \) (respectively, \( \psi^{A^\#}_2 \)). It turns out that the orbit spaces of \( \psi_1 \) and \( \psi_2 \) are canonically isomorphic:

**Theorem 6.1.** There is a canonical isomorphism

\[
\tau : \mathbb{B}A/\{0\} \times \mathbb{R} \to \mathbb{B}A/\mathbb{R} \times \{0\}.
\]

The composition

\[
CA \simeq \mathbb{B}A/\{0\} \times \mathbb{R} \xrightarrow{\tau} \mathbb{B}A/\mathbb{R} \times \{0\} \xrightarrow{\beta} \mathbb{B}A^\#/\{0\} \times \mathbb{R} \simeq CA^\#
\]

is an isomorphism of double affine bundles. However, it moves the distinguished section \((0, 1_A)\) of the model vector bundle \( T^*A \oplus \mathbb{R} \) into the section \((0, -1_{A^\#})\) and so it establishes an isomorphism of special affine bundles

\[
\bar{CA} \simeq CA^\#,
\]

where \( \bar{A} = (A, -v_A) \) is the adjoint of \( A \).
Proof.- Let us choose the local basis \((y_i)\) and its dual \((\xi_i)\) of sections of \(E^*\) and \(E\), respectively, in such a way that \(y_{n+1} = \alpha_A\) and \(\xi_0 = \nu_A\). Let us define \(\tau\) in the adapted coordinates \((x_a, y_i; p_b; \pi_j)\) on \(T^*E\) by the following formula

\[
\tau(x_a; y_0, \ldots, y_n, 1; p_b; 1, \pi_1, \ldots, \pi_n, \mathbb{R}) = (x_a; \mathbb{R}, y_1, \ldots, y_n, 1; p_b; 1, \pi_1, \ldots, \pi_n, -y_0 - \sum_{i=1}^{n} y_i \pi_i),
\]

where the argument of \(\tau\) is an orbit of the action \(\psi_2\) of an element from \(\mathbb{B}A\) and similarly the value of \(\tau\) is an orbit of \(\psi_1\). Let us check that \(\tau\) is defined independently of the choice of coordinates. For another choice of local sections \((y'_i)\) and \((\xi'_i)\) we have

\[
\pi'_i = \sum_i a_{ij} \pi_j,
\]

for a matrix \((a_{ij})\) with entries in \(C^\infty(M)\) which satisfies: \(a_{00} = a_{n+1,n+1} = 1\), \(a_{0j} = 0\) for \(j > 0\), \(a_i,i+1 = 0\) for \(1 \leq i \leq n + 1\). These conditions on the matrix \((a_{ij})\) come from the fact that the transition maps preserve the special elements \(\nu_A\) and \(\alpha_A\). Let us calculate \(\pi'_{n+1} \circ \tau\). We have

\[
\pi'_{n+1} \circ \tau = \sum_{j=0}^{N} a_{n+1,j} \pi_j \circ \tau + \pi_{n+1} \circ \tau = \sum_{j=0}^{n} a_{n+1,j} \pi_j - y_0 - \sum_{i=1}^{n} y_i \pi_i
\]

\[
= \sum_{j,k} a_{n+1,j} \alpha_i k \pi_k - \pi_{n+1} - y_0 - \sum_{i,j,k} a_{ij} y_j' \alpha_i k \pi_k' + y_0 \pi_0 + y_{n+1} \pi_{n+1}
\]

\[
= \pi_{n+1} - \sum_{j=0}^{n+1} y_j' \pi_j' = -y_0 - \sum_{j=1}^{n} y_j' \pi_j',
\]

thanks to \(\pi_0 = 1\) and \(y_{n+1} = 1\). The other equalities \(y'_i \circ \tau = y'_i\) for \(i \neq 0\), \(\pi'_i \circ \tau = \pi'_i\) for \(j \neq n + 1\) and \(p'_a \circ \tau = p'_a\) follow immediately. It is also clear that \(\beta \circ \tau: CA \to \hat{CA}^\#\) has the desired properties.

\[\square\]

7 \(n\)-affine bundles

A notion of a double affine bundle has an obvious generalization for a manifold which is equipped with more than two affine structures. One can view an \(n\)-tuple affine bundle \(\mathcal{A}\) (\(n\)-affine, in short), \(n \in \mathbb{N}\), as an object which is build by means of gluing trivial ones, which by definition are of the form \(\mathcal{A}_A := U_A \times K\), where \((U_a)\) is a covering of the base manifold \(M\), \(K = \prod_{\varepsilon} K_{\varepsilon}\), the product over \(\varepsilon \in \{0, 1\}^n\), \(\varepsilon \neq 0^n\), and \(K_{\varepsilon}\) are fixed vector spaces. The manifold \(\mathcal{A}_A\) has \(n\) natural affine structures over the bases \(\mathcal{A}_{A,i} = U_A \times \prod_{\varepsilon(i) = 0} K_{\varepsilon}\) and each of \(\mathcal{A}_{A,i}\) is an \((n-1)\)-affine bundle. In order to preserve these affine structures, the gluing maps \(\phi_{A,B}: U_{A,B} \times K \to U_B \times K\), where \(U_{A,B} = U_A \cap U_B\), have to preserve the "filtration" of the structure sheaf. We shall explain this condition more precisely. Let us assign the degree \(\varepsilon \in \mathbb{Z}^n\) to linear coordinates on \(K_{\varepsilon}\) and degree 0 to functions on \(U_A\) and let \(\mathcal{F}_A\) be the algebra generated by \(C^\infty(U_A)\) and linear functions on \(K_{\varepsilon}\) as \(\varepsilon\) varies in \(\{0, 1\}^n\), \(\varepsilon \neq 0^n\). The algebra \(\mathcal{F}_A\) is a subalgebra
of smooth functions on $A$. Let $\mathcal{F}_{\alpha,\mu}$ be the subspace of those elements in $\mathcal{F}_\alpha$ which are of degree less or equal $\mu \in \{0,1\}^n$ with respect to the product (partial) order on $\mathbb{Z}^n$. The condition for preserving the affine structures is that the pullback of a function $f \in \mathcal{F}_{\beta,\mu}$ restricted to $U_\alpha$ with respect to $\phi_{\alpha,\beta}$, $f|_{U_\alpha \times K} \circ \phi_{\alpha,\beta}$, is a function of degree less or equal $\mu$, for $\mu \in \{0,1\}^n$. In case $n = 2$ we recover that this condition says that gluing transformations are of the form (7), where the coordinates $y^j$, $z^a$, $c^u$ have been assigned degrees $(0,1)$, $(1,0)$ and $(1,1)$, respectively. Note that the algebra

$$\mathcal{A} := \{ f \in C^\infty(A) : f|_{A_\alpha} \in \mathcal{F}_\alpha \}$$

is not graded but filtered by $\mathbb{Z}^n$, i.e. for $\mu \in \mathbb{Z}^n$ there are distinguished subspaces $A_\mu$ of $\mathcal{A}$ such that $A_{\mu_1} \subset A_{\mu_2}$ whenever $\mu_1$ is less or equal $\mu_2$ with respect to the product partial order on $\mathbb{Z}^n$ and $\bigcup_\mu A_\mu = \mathcal{A}$.

Alternatively, one can define an $n$-affine bundle $A$ as a subset of an $n$-vector bundle $E$ determined by $n$ functions $l_1, \ldots, l_n$, where the degree of $l_i$ is $\varepsilon_i \in \mathbb{Z}^n$, $\varepsilon_i(j) = \delta_{ij}$, by setting

$$A = \{ x \in E : l_i(x) = 1 \text{ for } i = 1, \ldots, n \},$$

with obvious $n$-affine structures inherited from $E$. Here we consider $E$ as a $\mathbb{Z}^n$-graded manifold, as in ([GR]). Its structure sheaf is generated only by functions of degrees from $\{0,1\}^n$.

Let us assume now that $A$ is an $n$-affine bundle given in an $n$-vector bundle $E$ by linear functions $l_1, \ldots, l_n$, where the degree of $l_i$ is $\varepsilon_i \in \mathbb{Z}^n$, $\varepsilon_i(j) = \delta_{ij}$. Let us consider the $(n + 1)$-vector bundle $T^*E$ as $\mathbb{Z}^{n+1}$-graded manifold, as in ([GR]). If $(y^i_\alpha)$ is the adapted coordinate system on $E$, where the degree of $y^i_\alpha$ is $\alpha \in \mathbb{Z}^n$, $\alpha(k) \in \{0,1\}$, for $k = 1, \ldots, n$, then the induced coordinate system on $T^*E$ is of the form $(y^i_\alpha, p^i_\alpha)$, where $p^i_\alpha$ are the corresponding momenta which have the degree equal to $|p^i_\alpha| = -\alpha + 1^{n+1} \in \mathbb{Z}^n$, where $1^k \in \mathbb{Z}^k$ denotes the vector of ones and we assumed that $\alpha(n + 1) = 0$. To get an $(n + 1)$-affine subbundle of $T^*E$ out of the linear functions $l_1, \ldots, l_n$, we need to choose additionally a linear function of degree $\varepsilon_{n+1} \in \mathbb{Z}^{n+1}$. Let us recall that for any $n$-vector bundle $E$, the core $C$ of $E$ (sometimes called ultracore ([M])) is defined as $C = \cap_{i=1}^n \ker \pi_i$, where $\pi_i : E \to E_i$, $i = 1, \ldots, n$, are the side bundles of $E$. In the adapted local coordinate system $(y^i_\alpha)$ on $E$, the core $C$ is given by

$$C = \{ x \in E : y^i_\alpha(x) = 0 \text{ for } \alpha \neq 1^n \text{ and } \alpha \neq 0^n \}.$$

The core $C$ is a vector bundle over $M$, the total base of $E$. There is a canonical action of the core bundle on the fiber bundle $E \to M$. In the adapted local coordinate system, for $c \in C$ and $v \in E$ lying over the same point $m \in M$, it reads as

$$y^i_\alpha(c + v) := \begin{cases} y^i_\alpha(c) + y^i_\alpha(v) & \text{for } \alpha = 1^n, \\ y^i_\alpha(v) & \text{otherwise.} \end{cases}$$

Note that $\pi_j(c + v) = \pi_j(v)$ for any $j = 1, 2, \ldots, n$ and $v \in A \subset E$ if and only if $c + v \in A$. The canonical projection $T^*E \to C^*$ can be defined as the restriction of $\omega : T_vE \to \mathbb{R}$ to the tangent space at $v \in E$ of the orbit $v + C_m$ under the natural identification $T_v(v + C_m) \simeq C_m$, where $C_m$ is a fiber of $C \to M$ over $m$. A function $l_{n+1}$
on $T^*E$ of degree $\varepsilon_{n+1}$ has a local form $\sum_j f_j p^*_j$, where $\alpha = 1^n \in \mathbb{Z}^n$ and $f_j \in C^\infty(M)$. Hence $l_{n+1}$ is a pullback of a linear function on $C^* \to M$ with respect to the mentioned projection $T^*E \to C^*$ and so it corresponds to a nowhere vanishing section of the core bundle $C \to M$. This way we arrived at a definition of a special $n$-affine bundle.

**Definition 7.1.** A special $n$-affine bundle is an $n$-affine bundle together with a nowhere vanishing section of the core bundle.

Let $A$ be a special $n$-affine bundle given in $E$ by means of $l_1, \ldots, l_{n+1}$ as above, where $l_{n+1}$ determines the special section of the core bundle. We can consider $l_{n+1}$ as a function on the total space $T^*E$ thanks to the canonical projection $T^*E \to C^*$. We define the $(n + 1)$-affine dual bundle $\mathbb{B}A$ as an $(n + 1)$-affine subbundle of $T^*E$ by

$$\mathbb{B}A = \{ \omega \in T^*E : l_i(\omega) = 1, \text{for } i = 1, \ldots, n + 1 \}.$$ 

Let us denote $A_i = \{ x \in E_i : l_j(x) = 1 \text{ for } j \neq i, 1 \leq j \leq n + 1 \}, i = 1, \ldots, n$, the bases of the side bundles of $A$. The affine bundle $A \to A_i$ is a special corank one subbundle of $E \to E_i$. The special section is induced from $l_{n+1}$ and the mentioned canonical action of the core bundle $C \to M$ on $A$. Hence $A^{\#A_i} \to A_i$ is an affine corank 1 subbundle of $E^{**E_i} \to E_i$, which we know is one of the side bundles of $T^*E$. Moreover, the total space $A^{\#A_i}$ has also another $(n-1)$-affine bundle structures, which are denoted by $aff_j^\#$, $j \neq i, 1 \leq j \leq n$, and are induced from the vector bundle structure of $T^*E \to E^{**E_j}$, whose Euler vector field is the phase lift of the Euler vector field for $E \to E_j$ ([GR]).

**Theorem 7.1.** The bases of the side bundles of the $(n + 1)$-affine bundle $\mathbb{B}A$ are $A$ and its duals $A^{\#A_1}, \ldots, A^{\#A_n}$. Moreover, $(A^{\#A_i}, aff_j^\#)$ is a special affine bundle which is dual to the adjoint of $(A^{\#A}, aff_i^\#), i \neq j$.

**Proof.** The first statement follows from the above discussion. The second one is an immediate consequence of Proposition [22] since we can restrict our consideration to the special double affine bundle $(A, aff_i, aff_j)$ with the induced from $l_{n+1}$ section of the core bundle. \hfill \square

Reassuing, a special $n$-affine bundle $A$ gives rise to an $(n + 1)$-affine bundle $\mathbb{B}A$. The duals of $A$ can be recognized as the bases of the side bundles of $\mathbb{B}A$. The associated pairings are derived from the canonical symplectic structure on $T^*E$, the $(n + 1)$-tuple hull of $\mathbb{B}A$. We postpone to a separate paper the problem, how the contact structure on an abstract $n$-affine bundle $A$ determines the pairings between some $(n - 1)$-affine bundles related to the side bundles of $A$ (compare with Theorem 6.1 in [GR]).

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