Stability and the index of biharmonic hypersurfaces in a Riemannian manifold

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Abstract
In this paper, we give an explicit second variation formula for a biharmonic hypersurface in a Riemannian manifold similar to that of a minimal hypersurface. We then use the second variation formula to compute the normal stability index of the known biharmonic hypersurfaces in a Euclidean sphere and to prove the nonexistence of unstable proper biharmonic hypersurface in a Euclidean space or a hyperbolic space, which adds another special case to support Chen’s conjecture on biharmonic submanifolds.

Keywords The second variations of biharmonic hypersurfaces · Stable biharmonic hypersurfaces · The index of biharmonic hypersurfaces · The index of biharmonic torus · Constant mean curvature hypersurfaces

Mathematics Subject Classification 58E20

1 Stability and the index of minimal hypersurfaces

It is well known that minimal hypersurfaces $M^n \to (N^{m+1}, h)$ in a Riemannian manifold are critical points of the area functional on hypersurfaces, i.e.,

$$\frac{d}{dt} \left( \text{Area}(M_t) \right)_{t=0} = -\int_M fH \, dv_g = 0.$$

This is equivalent to the statement that the mean curvature $H = \frac{1}{m} \text{Tr} A$ of the hypersurface of $M$ vanishes identically, where $A$ is the shape operator of the hypersurface.

As it is also well known that a critical point may not give a local minimum of the area functional. To have a better understanding of minimal hypersurfaces as the critical points of a functional, one needs to know the second variation that leads to the concepts of the stability and the index of minimal hypersurfaces.

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Recall (see, e.g., [1] and [13]) that a minimal hypersurface is stable if the second variation of the area functional is always nonnegative for any normal variation with compact support, i.e.,

$$\frac{d^2}{dt^2} (\text{Area}(M_t))_{t=0} \geq 0.$$ 

For a complete orientable minimal hypersurface $M^m \to (N^{m+1}, h)$ in a Riemannian manifold, there is a unit normal vector field $\xi$ along $M$ so that any section $V$ of the normal bundle with compact support can be written as $V = f \xi$ for a function $f$ with compact support in $M$, and the second variation of the area functional with the $V$ as variation vector field can be written as:

$$\frac{d^2}{dt^2} (\text{Area}(M_t))_{t=0} = \int_M \{ |\nabla f|^2 - (\text{Ric}^N(\xi, \xi) + |A|^2 f^2) \} \, dv_g,$$  

where $|A|^2$ is the squared norm of the second fundamental form of the hypersurface, and $\text{Ric}^N(\xi, \xi) = \sum_{i=1}^m (\text{R}^N(\xi, e_i) e_i, \xi) = \sum_{i=1}^m R^N(\xi, e_i, \xi, e_i)$ is the Ricci curvature in the direction $\xi$.

Note that by using the divergence theorem: $\int_M f \Delta f \, dv_g = -\int_M |\nabla f|^2 \, dv_g$, we can rewrite (1) as

$$\frac{d^2}{dt^2} (\text{Area}(M_t))_{t=0} = q_M(f) = -\int_M f J(f) \, dv_g \geq 0,$$  

where $J(f) = \Delta f + (|A|^2 + \text{Ric}^N(\xi, \xi)) f$ is called the Jacobi operator on the minimal hypersurface.

Recall (see, e.g., [1]) that the index of a minimal hypersurface $M$, denoted by $\text{Ind}(M)$, is the maximum dimension of any subspace $V$ of $C^\infty_0(M)$ on which $q_M(f)$ is negative, i.e.,

$$\text{Ind}(M) = \text{Max} \{ \dim V : V \subset C^\infty_0(M) | q_M(f) < 0, \forall f \in V \}.$$ 

In particular, the index of a minimal hypersurface $M \subset S^{m+1}$ in a Euclidean sphere is the largest dimension of subspace $V \subset C^\infty_0(M)$ on which the quadratic form satisfies

$$q_M(f) = -\int_M f [\Delta f + (|A|^2 + m) f] \, dv_g < 0.$$  

The following are some well-known facts about the index of minimal hypersurfaces in a sphere:

- For a compact minimal hypersurface $M$ in $S^{m+1}$, $\text{Ind}(M) \geq 1$ and with $\text{Ind}(M) = m$ holds if and only if $M$ is a totally geodesic equator $S^m \subset S^{m+1}$ (see [20]);
- For a compact non-totally geodesic minimal hypersurface $M^m \to S^{m+1}$, $\text{Ind}(M) \geq m + 3$ (see [22] for $m = 2$ and [21] for the general case);
- For the minimal Clifford torus $S^p(\sqrt{\frac{p}{m}}) \times S^q(\sqrt{\frac{q}{m}}) \subset S^{m+1}$ with $p + q = m$, the index $\text{Ind}(M) = m + 3$;
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- It has been a conjecture which is still open (see, e.g., [1, 2]) that any compact non-totally geodesic minimal hypersurface \( M^m \to S^{m+1} \) with \( \text{Ind}(M) = m + 3 \) is a Clifford torus.

Biharmonic hypersurfaces are generalizations of minimal hypersurfaces. A biharmonic hypersurface in a Riemannian manifold can be characterized as an isometric immersion \( M^m \to (N^{m+1}, h) \) whose mean curvature function \( H \) solves the following equation (see [8, 9, 11], and [6] for the case when the ambient space is a space form, and [16] for the general case):

\[
\begin{cases}
\Delta H - H|A|^2 + H\text{Ric}^N(\xi, \xi) = 0, \\
2A(\text{grad} H) + \frac{m}{2}\text{grad} H^2 - 2H(\text{Ric}^N(\xi))^\top = 0,
\end{cases}
\]

where \( \text{Ric}^N : T_qN \to T_qN \) denotes the Ricci operator of the ambient space defined by \( \langle \text{Ric}^N(Z), W \rangle = \text{Ric}^N(Z, W) \) and \( A \) is the shape operator of the hypersurface with respect to the unit normal vector \( \xi \).

It is clear from (4) that any minimal hypersurface is automatically a biharmonic hypersurface. So it is a custom to call a biharmonic hypersurface which is not minimal a proper biharmonic hypersurface. For more study of biharmonic maps and biharmonic submanifolds, we refer the reader to a recent book [18] and the references therein.

Some work on general second variations of bienergy, stability, and indices of biharmonic maps and submanifolds has been done in [3, 10, 12, 14, 17, 19]. In this paper, we derive an explicit second variation formula for a biharmonic hypersurface in a Riemannian manifold with respect to normal variations. We then use the second variation formula to compute the normal stability index of the known biharmonic hypersurfaces in a Euclidean sphere, and to prove the nonexistence of unstable proper biharmonic hypersurface in a Euclidean space or a hyperbolic space, which adds another special case to support Chan’s conjecture on biharmonic submanifolds.

2 Stability and the index of biharmonic hypersurfaces

In light of the ideas from the study of stability and the index of minimal hypersurfaces, we define a proper biharmonic hypersurface \( M \to (N^{m+1}, h) \) to be **normally stable** if the second variation of the bienergy functional is always nonnegative for any normal variation with compact support. With this, we have

**Theorem 2.1** A complete orientable biharmonic hypersurface \( \phi : M^m \to (N^{m+1}, h) \) of a Riemannian manifold is normally stable if and only if for any compactly supported function \( f \) on \( M \), we have
When the following second variation formula for a general biharmonic map

\[ Q(f) = \frac{d^2}{dt^2} E_2(\phi_t)|_{t=0} \]

\[ = \int_M [(f(A)^2 - \text{Ric}^N(\xi, \xi)) - \Delta f]^2 dv_g \]

\[ + \int_M |mf\nabla H - 2(\text{Ric}^N(\xi))^T + 2A(\nabla f)|^2 dv_g \]

\[ + \int_M mf^2 H((\nabla^N_{\xi} \text{Ric}^N)(\eta, \xi)) - 2\text{Tr} \text{R}^N(\xi, \eta, \nabla^N_{\xi}(\cdot))dv_g \]

\[ - \int_M 4mf^2 \text{HTr} \text{R}^N(\xi, A(\cdot), \xi, \cdot)dv_g \geq 0. \]

**Proof** The following second variation formula for a general biharmonic map \( \phi : (M^m, g) \to (N^n, h) \) between two Riemannian manifolds was derived by Jiang in [10]:

\[ \frac{d^2}{dt^2} E_2(\phi_t)|_{t=0} = \int_M [J^\phi(V)]^2 + \text{R}^N(\nabla, V, \tau(\phi), V, \tau(\phi))dv_g \]

\[ - \sum_{i=1}^m \int_M \langle V, (\nabla^N_{d\phi(i)}\text{R}^N)(d\phi(e_i), \tau(\phi))V \rangle \]

\[ + \sum_{i=1}^m \langle \nabla^N_{d\phi(i)}\text{R}^N(d\phi(e_i), V)d\phi(e_i) \rangle \]

\[ + 2\text{R}^N(d\phi(e_i), V)\nabla^N_{e_i} \tau(\phi) + 2\text{R}^N(d\phi(e_i), \tau(\phi))\nabla^N_{e_i} V \rangle dv_g. \]

When \( \phi : M^m \to (N^{m+1}, h) \) is an orientable biharmonic hypersurface, we consider the normal variation with variation vector field \( V = f\xi \) and use \( \tau(\phi) = mH\xi \), identify \( d\phi(e_i) = e_i \), and a straightforward computation to have

\[ \text{R}^N(\nabla, V, \tau(\phi), V, \tau(\phi)) = m^2 f^2 H^2 \text{R}^N(\xi, \xi, \xi, \xi) = 0, \]

\[ \sum_{i=1}^m \langle V, (\nabla^N_{d\phi(e_i)}\text{R}^N)(d\phi(e_i), V)d\phi(e_i) \rangle = mfH \sum_{i=1}^m \langle \xi, (\nabla^N_{e_i}\text{R}^N(e_i, f\xi)e_i) \rangle \]

\[ = mf^2 H[-(\nabla^N_{\xi} \text{Ric}^N)(\eta, \xi)) + 2\text{Tr} \text{R}^N(\xi, \eta, \nabla^N_{\xi}(\cdot))], \]

\[ \sum_{i=1}^m \langle V, (\nabla^N_{d\phi(e_i)}\text{R}^N)(d\phi(e_i), \tau(\phi))V \rangle = mf^2 H \sum_{i=1}^m \langle \xi, (\nabla^N_{e_i}\text{R}^N(e_i, \xi) \xi) \rangle = 0, \]

\[ \sum_{i=1}^m \langle V, 2\text{R}^N(d\phi(e_i), V)\nabla^N_{e_i} \tau(\phi) \rangle = 2mf^2 H \sum_{i=1}^m \text{R}^N(\xi, \nabla^N_{e_i}\xi, e_i, \xi) \]

\[ = 2mf^2 \text{HTr} \text{R}^N(\xi, A(\cdot), \xi, \cdot), \] and

\[ \sum_{i=1}^m \langle V, 2\text{R}^N(d\phi(e_i), \tau(\phi))\nabla^N_{e_i} V \rangle = 2mfH \sum_{i=1}^m \langle \xi, \text{R}^N(e_i, \xi)\nabla^N_{e_i}(f\xi) \rangle \]

\[ = 2mf^2 \text{HTr} \text{R}^N(\xi, A(\cdot), \xi, \cdot). \]

On the other hand, using the formula (see, e.g., [15])
\[ J^\phi(V) = J^\phi(f\xi) = fJ^\phi(\xi) - (\Delta f)\xi - 2\nabla^\phi_v f, \]  
(12)

and a further computation (see also [16]), we obtain

\[ J^\phi(\xi) = -\sum_{i=1}^m \left( (\nabla^\phi_{e_i} \nabla^\phi_{e_i} - \nabla^\phi_{\mu_{e_i,e_i}})\xi - R^N(d\phi(e_i), \xi)d\phi(e_i) \right) \]
\[ = (|A|^2 - \text{Ric}^N(\xi, \xi))\xi + m\nabla H - 2(\text{Ric}^N(\xi))^T, \]
\[ 2\nabla^\phi_v f = -2A(\nabla f). \]
(14)

It follows from (13), (14) and (12) that

\[ |J^\phi(V)|^2 = |J^\phi(f\xi)|^2 \]
\[ = [f(|A|^2 - \text{Ric}^N(\xi, \xi)) - \Delta f]^2 + |mf\nabla H - 2(\text{Ric}^N(\xi))^T + 2A(\nabla f)|^2. \]
(15)

Substituting (7)–(11) and (15) into (6), we obtain

\[ Q(f) = \frac{d^2}{dt^2} E_2(\phi_N)|_{t=0} \]
\[ = \int_M \left\{ f(|A|^2 - \text{Ric}^N(\xi, \xi)) - \Delta f \right\} dv_g \]
\[ + \int_M |mf\nabla H - 2(\text{Ric}^N(\xi))^T + 2A(\nabla f)|^2 dv_g \]
\[ + \int_M mf^2 H(\nabla^N \text{Ric}^N(\xi, \xi)) - 2\text{Tr} R^N(\xi, \cdot, \xi, \nabla^N(\cdot)) dv_g \]
\[ - \int_M 4mf^2 H\text{Tr} R^N(\xi, A(\cdot, \cdot, \xi, \cdot)) dv_g, \]
(16)

from which and the definition of the stability of a biharmonic hypersurface we obtain the theorem. \[ \square \]

**Corollary 2.2** An orientable biharmonic hypersurface \( \phi : M^m \to (N^{m+1}(c), h) \) in a space form of constant sectional curvature \( c \) is normally stable if and only if the stability inequality \( Q(f) \geq 0 \) holds for any compactly supported smooth function \( f \) on \( M \), where

\[ Q(f) = \int_M \left\{ f(|A|^2 - cm) - \Delta f \right\} dv_g. \]

In particular, (i) any biharmonic hypersurface in a Euclidean space or a hyperbolic space is normally stable, and (ii) the normal stability quadratic form for a biharmonic hypersurface \( M^m \to S^{m+1} \) in a Euclidean sphere is given by

\[ Q(f) = \int_M \left\{ f(|A|^2 - m) - \Delta f \right\} dv_g. \]
(17)

**Proof** The corollary follows from Eq. (16) and the following identities for a space form \( N^{m+1}(c) \) of constant sectional curvature \( c \):

\[ \text{ Springer} \]
Remark 1  For stable minimal surfaces in Euclidean space $\mathbb{R}^3$, we have a well-known result of do-Carmo-Peng: Any complete oriented and stable minimal surface $\phi : M^2 \to \mathbb{R}^3$ is a plane. On the other hand, we know that catenoid is complete and oriented minimal surfaces in $\mathbb{R}^3$, so it is unstable. In contrast, our corollary above says that there is no unstable biharmonic hypersurface in a Euclidean space or a hyperbolic space. This adds another special case to support Chen’s conjecture on biharmonic submanifolds which can be stated as there exists no proper biharmonic submanifold in a Euclidean space. See [18] for a more detailed account on Chen’s conjecture on biharmonic submanifolds.

3 The stability index of biharmonic hypersurfaces in $S^{m+1}$

Again, following the idea of the index of minimal hypersurfaces, we define the **normal index** of a proper biharmonic hypersurface $M \to (N^{m+1}, h)$ to be the maximum dimension of any subspace $V$ of $C_0^\infty(M)$ on which $Q(f)$ defined in (16) is negative, i.e.,

$$\text{Ind}(M) = \text{Max} \{ \dim V : V \subset C_0^\infty(M) | Q(f) < 0, \forall f \in V \}.$$  

About biharmonic hypersurfaces in a sphere, we know that

- a hypersurface $\varphi : (M^m, g) \to S^{m+1}$ with nonzero constant mean curvature is biharmonic if and only if the squared norm of the shape operator is constant (see [10] or directly using (4));
- the only known proper biharmonic hypersurfaces in a sphere are ([5, 10]): $S^n(\frac{1}{\sqrt{2}})$ and $S^{p}(\frac{1}{\sqrt{2^p}}) \times S^{m-p}(\frac{1}{\sqrt{2^p}})$ for $p \neq \frac{m}{2}$, or an open part of one of these two. It has been a conjecture ([4]) which is still open that there is no other proper biharmonic hypersurface in a sphere than open parts of these two.

In this section, we will use the normal stability form (17) to compute the normal index of the known proper biharmonic hypersurfaces in a sphere.

**Theorem 3.1** (i) For $1 \leq p < q = m - p$, the normal stability index of the proper biharmonic hypersurface $S^{p}(\frac{1}{\sqrt{2}}) \times S^{q}(\frac{1}{\sqrt{2}}) \to S^{m+1}$ is

$$\text{Ind}(S^{p}(\frac{1}{\sqrt{2}}) \times S^{q}(\frac{1}{\sqrt{2}})) = \begin{cases} 1, & \text{for } 1 \leq p l t q \leq 2p, \\ p + 2, & \text{for } 2p \vdash l t q. \end{cases}$$  

(18)

(ii) The normal stability index of the biharmonic hypersurface $S^n(\frac{1}{\sqrt{2}}) \to S^{m+1}$ is

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It follows that the mean curvature of the hypersurface \( H = (m - 2p)/m \) is nonzero constant since \( p \neq m/2 \), \( |A|^2 = m \), and \( |A(\nabla f)|^2 = |\nabla f|^2 \). Substituting these into the stability index form (17) yields

\[ Q(f) = \int_{T^{p,q}} \{(\Delta f)^2 - 4f \Delta f - 4(q - p)^2 f^2\} \, dv_g. \tag{20} \]

For \( 1 \leq p < q = m - p \leq m - 1 \), the Laplacian on \( S^p(\frac{1}{\sqrt{2}}) \times S^q(\frac{1}{\sqrt{2}}) \) has spectrum:

\[ \cdots < \cdots < \cdots < \lambda_2 = -2q < \lambda_1 = -2p < \lambda_0 = 0, \]

For an eigenfunction \( f \) with eigenvalue \( \nu = -2p \), i.e., \( \Delta f = -2pf \), (20) reads

\[ Q(f) = \int_{T^{p,q}} [4p^2 + 8p - 4(q - p)^2] \, f^2 \, dv_g \]

\[ = -4 \int_{T^{p,q}} (q^2 - 2pq - 2p) f^2 \, dv_g. \]

It follows that \( Q(f) < 0 \) if and only if the quadratic function \(-4(q^2 - 2pq - 2p) < 0\), which is equivalent to \( q > p + \sqrt{p^2 + 2p} \). One can further check that \( q > p + \sqrt{p^2 + 2p} \) is equivalent to \( q > 2p \) since there is no integer within \((2p, p + \sqrt{p^2 + 2p})\).

On the other hand, for an eigenfunction \( f \) of the second nonzero eigenvalue \( \nu = -2q \), i.e., \( \Delta f = -2qf \), (20) reads

\[ Q(f) = \int_{T^{p,q}} [4q^2 + 8q - 4(q - p)^2] f^2 \, dv_g \]

\[ > \int_{T^{p,q}} 4(2q + p^2) f^2 \, dv_g > 0. \]

Similarly, we can check that \( Q(f) \) is positive on any other eigenspace of the Laplacian on \( S^p(\frac{1}{\sqrt{2}}) \times S^q(\frac{1}{\sqrt{2}}) \).

Finally, since \( T^{p,q} = S^p(\frac{1}{\sqrt{2}}) \times S^q(\frac{1}{\sqrt{2}}) \) is compact, we have the following Sturm–Liouville’s decomposition

\[ C^\infty(T^{p,q}) = \bigoplus_{i=0}^\infty E_{\lambda_i}, \]
where $E_{\lambda}$ denotes the eigenspace of the Laplacian on $S^p \left( \frac{1}{\sqrt{2}} \right) \times S^q \left( \frac{1}{\sqrt{2}} \right)$ with respect to the eigenvalue $\lambda$. For details of the orthogonal decomposition of eigenspaces of Laplacian on compact manifold, we refer the reader to Chavel [7] (Theorem III.9.1).

From this, together with the above discussion, we conclude that for $1 \leq p < q = m - p \leq 2p$, the largest subspace of smooth functions on the biharmonic hypersurface $S^p \left( \frac{1}{\sqrt{2}} \right) \times S^q \left( \frac{1}{\sqrt{2}} \right)$ on which $Q(f) < 0$ is $E_{\lambda_0}$, the eigenspaces of the Laplacian on $S^p \left( \frac{1}{\sqrt{2}} \right) \times S^q \left( \frac{1}{\sqrt{2}} \right)$ corresponding the eigenvalues $\lambda_0 = 0$. Since $E_{\lambda_0} = \mathbb{R}$ has dimension 1, we obtain the first case in the index formula (18). For the case $1 \leq p < q = m - p$ and $q > 2p$, the largest subspace of smooth functions on the biharmonic hypersurface $S^p \left( \frac{1}{\sqrt{2}} \right) \times S^q \left( \frac{1}{\sqrt{2}} \right)$ on which $Q(f) < 0$ is $E_{\lambda_0} \oplus E_{\lambda_1}$, where $E_{\lambda_0}$ and $E_{\lambda_1}$ are the eigenspaces of the Laplacian on $S^p \left( \frac{1}{\sqrt{2}} \right) \times S^q \left( \frac{1}{\sqrt{2}} \right)$ corresponding the eigenvalues $\lambda_0 = 0$, $\lambda_1 = -2p$. Since the subspace $E_{\lambda_0} \oplus E_{\lambda_1}$ has dimension 1 + $(p + 1) = p + 2$, we obtain the biharmonic index of $S^p \left( \frac{1}{\sqrt{2}} \right) \times S^q \left( \frac{1}{\sqrt{2}} \right)$ for the case $1 \leq p < q = m - p$ and $q > 2p$, which complete the proof of Statement (i).

For Statement (ii), first note that the proper biharmonic hypersurface $S^m \left( \frac{1}{\sqrt{2}} \right) \rightarrow S^{m+1}$ is totally umbilical with $|A|^2 = m, H = -1$, and hence $A(\nabla f) = -\nabla f$. It follows that the index form (17) in this case reads

$$Q(f) = \int_{S^n \left( \frac{1}{\sqrt{2}} \right)} \{ (\Delta f)^2 + 4|\nabla f|^2 - 4m^2 f^2 \} \, dv_g$$

$$= \int_{S^n \left( \frac{1}{\sqrt{2}} \right)} \{ (\Delta f)^2 - 4f \Delta f - 4m^2 f^2 \} \, dv_g,$$

where the second equality was obtained by using the divergence theorem.

Using the eigenvalues of the Laplacian on $S^n \left( \frac{1}{\sqrt{2}} \right)$:

$$\cdots < \lambda_3 < \lambda_2 < \lambda_1 = -2m < \lambda_0 = 0.$$

One can check that for any eigenfunction function $f$ of the eigenvalue $\lambda_1 = -2m$, we have $\Delta f = -2mf$, and hence (21) becomes

$$Q(f) = \int_{S^n \left( \frac{1}{\sqrt{2}} \right)} [(\Delta f)^2 - 4f \Delta f - 4m^2 f^2] \, dv_g = 8m \int_{S^n \left( \frac{1}{\sqrt{2}} \right)} f^2 \, dv_g > 0.$$

Similarly, one can check that $Q(f)$ is positive on any other eigenspace, so the only subspace of $C^\infty \left( S^n \left( \frac{1}{\sqrt{2}} \right) \right)$ on which $Q(f) < 0$ is $\mathbb{R}$ which has dimension one. Thus, we have

$$\text{Ind} \left( S^n \left( \frac{1}{\sqrt{2}} \right) \right) = 1.$$

\[\square\]

**Remark 2** Note that Statement (ii) in Theorem 3.1 was proved in [12] in a quite different way.
We end the paper with the following table which gives the normal indices of biharmonic hypersurfaces of spheres in small dimensions.

| Ambient sphere | Biharmonic hypersurface | Index |
|----------------|-------------------------|-------|
| $S^4$          | $S^1\left(\frac{1}{\sqrt{2}}\right) \times S^2\left(\frac{1}{\sqrt{2}}\right)$ | 1     |
| $S^5$          | $S^1\left(\frac{1}{\sqrt{2}}\right) \times S^3\left(\frac{1}{\sqrt{2}}\right)$ | 3     |
| $S^6$          | $S^1\left(\frac{1}{\sqrt{2}}\right) \times S^4\left(\frac{1}{\sqrt{2}}\right)$ | 3     |
|                | $S^2\left(\frac{1}{\sqrt{2}}\right) \times S^3\left(\frac{1}{\sqrt{2}}\right)$ | 1     |
| $S^7$          | $S^1\left(\frac{1}{\sqrt{2}}\right) \times S^5\left(\frac{1}{\sqrt{2}}\right)$ | 3     |
|                | $S^2\left(\frac{1}{\sqrt{2}}\right) \times S^4\left(\frac{1}{\sqrt{2}}\right)$ | 1     |
| $S^8$          | $S^1\left(\frac{1}{\sqrt{2}}\right) \times S^6\left(\frac{1}{\sqrt{2}}\right)$ | 3     |
|                | $S^3\left(\frac{1}{\sqrt{2}}\right) \times S^5\left(\frac{1}{\sqrt{2}}\right)$ | 4     |
|                | $S^4\left(\frac{1}{\sqrt{2}}\right) \times S^4\left(\frac{1}{\sqrt{2}}\right)$ | 1     |
| $S^9$          | $S^1\left(\frac{1}{\sqrt{2}}\right) \times S^7\left(\frac{1}{\sqrt{2}}\right)$ | 3     |
|                | $S^2\left(\frac{1}{\sqrt{2}}\right) \times S^6\left(\frac{1}{\sqrt{2}}\right)$ | 4     |
|                | $S^3\left(\frac{1}{\sqrt{2}}\right) \times S^5\left(\frac{1}{\sqrt{2}}\right)$ | 1     |
| $S^{10}$       | $S^1\left(\frac{1}{\sqrt{2}}\right) \times S^8\left(\frac{1}{\sqrt{2}}\right)$ | 3     |
|                | $S^2\left(\frac{1}{\sqrt{2}}\right) \times S^7\left(\frac{1}{\sqrt{2}}\right)$ | 4     |
|                | $S^3\left(\frac{1}{\sqrt{2}}\right) \times S^6\left(\frac{1}{\sqrt{2}}\right)$ | 1     |
|                | $S^4\left(\frac{1}{\sqrt{2}}\right) \times S^5\left(\frac{1}{\sqrt{2}}\right)$ | 1     |

It follows from the table and Theorem 3.1 that for any natural number $k$ except $k = 2$, there exists a proper biharmonic hypersurface in a sphere $S^m$ with $m$ depending on $k$ whose index is $k$.

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