Finiteness of Selmer Groups and Deformation Rings

Chandrashekhar Khare, Ravi Ramakrishna

1 Introduction

Consider a continuous, absolutely irreducible mod $p$ representation

$$\bar{\rho} : G_\mathbb{Q} \rightarrow GL_2(k),$$

with $k$ a finite field of characteristic $p$, and $G_\mathbb{Q}$ the absolute Galois group of the field of rational numbers $\mathbb{Q}$ and $p$ an odd prime.

Suppose now that $\bar{\rho}$ is modular, that is $\bar{\rho}$ arises from a Hecke eigenform $f$. Then we know there exists a representation $\rho_f : G_\mathbb{Q} \rightarrow GL_2(O_f)$ where $O_f$ is the ring of integers of some finite extension of $\mathbb{Q}_p$ whose reduction mod a uniformiser is $\bar{\rho}$. One can associate to $\rho_f$ a Selmer group that is defined via Galois cohomology. The calculation of the order of this Selmer group played a key role in the proof of the Shimura-Taniyama-Weil conjecture.

In this paper we do not assume $\bar{\rho}$ is modular. For instance, if $\bar{\rho}$ is even, i.e. $\text{det}(\bar{\rho}(c)) = 1$ where $c \in G_\mathbb{Q}$ is a complex conjugation, then $\bar{\rho}$ is necessarily not modular. The point of this paper is to find, for $\bar{\rho}$ odd or even, a deformation $\rho : G_\mathbb{Q} \rightarrow GL_2(O)$ of $\bar{\rho}$ where $O$ is again the ring of integers of some finite extension of $\mathbb{Q}_p$ such that the Selmer group associated to $\rho$ is finite.

We remark that in many cases deformations $\rho$ of $\bar{\rho}$ had been constructed in [R2], but finiteness of the associated Selmer did not follow from that work.

In the last section we show how to remove the Generalized Riemann Hypothesis from Theorem 2 of [R1] and construct irreducible potentially semistable 2 dimensional $p$-adic Galois representations that are deformations of $\bar{\rho}$, when $\bar{\rho}$ is odd, and are ramified at infinitely many primes. This method also applies to show how many even mod $p$ Galois representations can be deformed to characteristic zero representations ramified at infinitely many primes.
We now consider the specifics of our situation. Once and for all we fix $\rho$ satisfying the hypotheses below. These hypotheses imply others that we need but use only implicitly.

- $\overline{\rho}$ and $Ad^0(\overline{\rho})$ are absolutely irreducible Galois representations.
- The (prime to $p$) Artin conductor $N(\overline{\rho})$ of $\overline{\rho}$ is minimal amongst its twists.
- We assume, as we may do, that the Serre weight $k(\overline{\rho})$ of $\overline{\rho}$ is between 2 and $p + 1$.
- If $\overline{\rho}$ is even then for the decomposition group $G_p$ above $p$ we assume that $\overline{\rho}|_{G_p}$ is not twist equivalent to $\left( \begin{array}{cc} \chi & 0 \\ 0 & 1 \end{array} \right)$ or twist equivalent to the indecomposable representation $\left( \begin{array}{cc} \chi^{p-2} & * \\ 0 & 1 \end{array} \right)$ where $\chi$ is the mod $p$ cyclotomic character.
- If $\overline{\rho}$ is odd we assume $\overline{\rho}|_{G_p}$ is not twist equivalent to the trivial representation or the indecomposable unramified representation given by $\left( \begin{array}{cc} 1 & * \\ 0 & 1 \end{array} \right)$.
- $p \geq 7$ and the order of the projective image of $\overline{\rho}$ is a multiple of $p$.

If one excludes the second part of the last condition, then $\overline{\rho}$ is allowed to have image of order prime to $p$. In this case it is easy to see that there is a unique ‘Teichmüller’ deformation of $\overline{\rho}$ to $W(k)$ that has image isomorphic to that of $\overline{\rho}$.

Let $S$ be the set of containing $p$, $\infty$ and the primes at which $\overline{\rho}$ is ramified. Note that the primes other than $p$ and $\infty$ at which $Ad^0(\overline{\rho})$ and $\overline{\rho}$ are ramified are the same by our assumptions. Consider the minimal deformation ring $R_S$ (also called $R_\phi$ in many references) of $\overline{\rho}$ and the universal minimal deformation $\rho^{univ}_S : G_Q \to GL_2(R_S)$. The ring $R_S$, like all the rings that will occur in the course of this paper, is a complete Noetherian local ring (CNL) with residue field $k$ that is a $W(k)$-algebra, and all maps between such rings will be maps of $W(k)$-algebras that induce the identity on residue fields.

The universal deformation has the property that it is unramified at the primes outside $S$ and satisfies the following conditions:
• If \( \ell \neq p \), and \( p(\ell) \) has order \( c \) with \( p \nmid c \) then \( \rho_{S_{\psi}}(I_{\psi}) \) has the same order \( c \).

• If \( \ell \neq p \) is such that \( p(\ell) \) has order \( c \) and \( p \neq c \), then \( \rho_{\psi}(\psi(I_{\psi})) \) has order \( c \) and an unramified quotient of rank one. (Note that if \( p \mid c \) then \( c = p \)).

• There is also, for \( \bar{\rho} \) odd, a condition at \( p \) described in the ‘Local at \( p \) considerations’ section of [R2]. This is the condition of ordinariness or flatness etc. For \( \bar{\rho} \) even there is no local condition imposed at \( p \).

• We fix determinants of our deformations. There is an integer \( k \) with \( 2 \leq k \leq p \) and a character \( \zeta \) of \( G_p \) such that \( \bar{\rho}(\psi(I_{\psi})) \) is isomorphic to either \( \psi^{k-1} \otimes \psi^{p(k-1)} \) or \( \begin{pmatrix} \chi^{k-1} & * \\ 0 & 1 \end{pmatrix} \) where \( \psi \) is a fundamental character of level 2. We write the global character \( det(\overline{\rho}) = \alpha \chi^{k-1} \) and fix the determinants of deformations we consider to be \( \overline{\alpha} \).

Then it is expected, at least for \( \overline{\rho} \) odd, that \( R_S \) is a finite, flat \( W(k) \)-module: the finiteness of \( R_S \) over \( Frac(W(k)) \) is predicted by the Fontaine-Mazur conjectures (cf. [FM], but also see Conjecture 22 below).

We need the following definition/lemma to state our main theorem.

**Definition 1** Let \( S' \) be a finite set of primes disjoint from \( S \), such that for \( s' \in S' \), \( s' \) is not \( \pm 1 \mod p \) and \( p(Frob_{s'}) \) has eigenvalues with ratio \( s' \). There is a \( W(k) \)-algebra \( R_{S,S'} \) that is the universal ring for the (representable) deformation problem which to any \( W(k) \)-algebra \( A \), with maximal ideal \( m_A \), assigns the isomorphism classes of continuous representations \( G_Q \to GL_2(A) \) of fixed determinant as above that are

• minimal at \( S \)

• unramified outside \( S \cup S' \),

• mod \( m_A \), the maximal ideal of \( A \), equal to \( \overline{p} \).

Denote by \( \rho_{S_{\psi}}^{univ} \) the corresponding universal representation \( \rho_{S_{\psi}}^{univ} : G_Q \to GL_2(R_{S_{\psi}}) \). For any \( \alpha \subset S' \) consider the deformation problem which to any \( W(k) \)-algebra \( A \) assigns the isomorphism classes of continuous representations \( \rho : G_Q \to GL_2(A) \) of fixed determinant as above that are

• minimal at \( S \)
• unramified outside \( S \cup S' \),

• \( \text{mod } m_A \), the maximal ideal of \( A \), \textbf{equal} to \( \overline{p} \)

• at primes \( s' \in \alpha \), \( \rho|_{G_{s'}} \) is up to twist of the form \( \left( \begin{array}{cc} \varepsilon & * \\ 0 & 1 \end{array} \right) \).

Then this deformation problem is representable (see lemma below) by a CNL \( W(k) \)-algebra \( R_{S \cup S'}^{\text{new}} \), and we denote by \( \rho_{S \cup S'}^{\text{new}} \) the corresponding universal representation \( \rho_{S \cup S'}^{\text{new}} : G_Q \rightarrow GL_2(R_{S \cup S'}^{\text{new}}) \). There is a canonical surjection \( R_{S \cup S'} \rightarrow R_{S \cup S'}^{\text{new}} \) for all \( \alpha \subset S' \). We denote the kernel of the map \( R_{S \cup S'} \rightarrow R_{\overline{S} \cup \overline{S}'}^{\text{new}} \) by \( \wp_{S'} \).

Below is the implicit representability claim in Definition 1.

**Lemma 2** The deformation problem where we require the properties of Definition 1 above is representable by a CNL \( W(k) \)-algebra \( R_{S \cup S'}^{\text{new}} \), and we denote by \( \rho_{S \cup S'}^{\text{new}} \) the corresponding universal representation \( \rho_{S \cup S'}^{\text{new}} : G_Q \rightarrow GL_2(R_{S \cup S'}^{\text{new}}) \). There is a surjection \( R_{S \cup S'} \rightarrow R_{S \cup S'}^{\text{new}} \) whose kernel we denote by \( \wp_{S'} \).

**Proof.** We simply sketch the proof as it is quite standard. Using Mazur’s work ([Ma2]) we know \( R_{S \cup S'} \) exists. Mazur also showed for the prime \( p \) (see [Ma1] for instance) that one could impose the condition of ordinariness at \( p \). Then one had a closed subfunctor of the original representable deformation functor. Here we are imposing for each \( s' \in \alpha \) a condition almost identical to ordinariness. That \( s' \) is not 1 mod \( p \) and \( \overline{p}|_{G_{s'}} \) is of the form above is sufficient to guarantee that a corresponding closed subfunctor of the original deformation functor exists (as these are deformation conditions in the sense of Sections 23 to 25 of [Ma1]). Letting \( s' \) run through \( \alpha \) and taking the intersection of these closed subfunctors, we are done.

The following theorem is the main result of this paper (see Theorem 12 below):

**Theorem 3** There is a finite set of primes \( S' = \{s'_1, \ldots, s'_n \} \) not congruent to \( \pm 1 \text{ mod } p \), at which \( \overline{p} \) has eigenvalues with ratio \( s'_i \) such that \( R_{S \cup S'}^{\text{new}} \simeq W(k) \). We can choose the set of primes \( S' \) so that the universal deformation \( \rho_{S \cup S'}^{\text{new}} : G_Q \rightarrow GL_2(W(k)) \) corresponding to \( R_{S \cup S'}^{\text{new}} \simeq W(k) \) is ramified at all the primes of \( S' \).

4
Remarks:
1) The existence of $S'$ so that $R_{S'\cup S}^{S'-new} \cong W(k)$ follows directly from the methods of [R2]. However in [R2] it was not known that the representation $\rho_{S'\cup S}^{S'-new}$ was ramified at the primes of $S'$. For $s' \in S'$, the corresponding versal local representations are indeed ramified, but it is not automatic that $\rho_{S'\cup S}^{S'-new}$ is ramified at all primes of $S'$. Indeed, for some prime $s' \in S'$ we might have had that $\rho_{S'\cup S}^{S'-new}\mid |G_s'$ was equal, up to twist, to $\left( \begin{array}{cc} \varepsilon & 0 \\ 0 & 1 \end{array} \right)$. In the present work we obtain ramification at these primes by choosing $S'$ to be (possibly) larger than the “auxiliary set” of [R2]. The main dichotomy that this paper addresses is that, on the one hand the deformation problem which assigns to $S'$ as above lifts of $\rho$ that are ramified at all primes in $S'$ is not representable (being ramified at a prime is not a condition in the sense of Section 23 of [Ma1]), while on the other hand it is not obvious that the universal representation $\rho_{S'\cup S}^{S'-new}$ is ramified at all primes in $S'$.

2) For (many) odd $\rho$, the recent important work of Taylor, cf. [T1], implies that $\rho_{S'\cup S}^{S'-new}\mid$ is ramified at all primes in $S'$ for all $S'$ such that $R_{S'\cup S}^{S'-new} \cong W(k)$. Taylor’s methods are partly geometric and very different from the methods of this paper. The methods of this paper are purely Galois cohomological and work for $\rho$ odd and even uniformly, while Taylor’s work does not address the case of $\rho$ even. In [T2] the results of [R2] are extended to the case of $p = 3, 5$: then our methods do prove an analog of Theorem 3 for $p = 5$.

Using Theorem 1 we can deduce easily (see Theorem 15 below):

**Theorem 4** The tangent space $\varphi_{S'}/\varphi_{S'}^2$ is finite.

**Remark:**
The tangent space above in Theorem 4 is called a Selmer group because of the cohomological interpretation of its dual as in Proposition 1.2 of [W] (see also proof of Theorem 15), hence the title of the paper. When $\overline{\rho}$ is even this Selmer group is not the one considered by Bloch and Kato, but it is the one that is more pertinent when studying deformation rings. In the case of $\overline{\rho}$ even, in some particular cases, deformations of $\overline{\rho}$ to characteristic 0 were produced, prior to [R2], in [R4]. Using particular features of these lifts, the finiteness of the associated Selmer groups was proven in [R4] and [B].
We recall that an irreducible component of a ring $R$ is a quotient $R/P$ for $P$ a minimal prime ideal.

**Corollary 5** The prime ideal $\wp_{S'}$ is minimal. Thus the deformation ring $R_{S \cup S'}$ contains an irreducible component isomorphic to $W(k)$.

We will also give an application of Theorem 4 to a conjecture of Fontaine-Mazur (see Proposition 23 below).

We thank the anonymous referee for careful reading of the original manuscript and many helpful suggestions. In particular, the referee suggested a major simplification of the results of section 3.

## 2 Auxiliary sets

**Definition 6** A finite set of primes $S'$ disjoint from $S$ is said to be auxiliary if for all primes $s' \in S'$, $s'$ is not $\pm 1$ mod $p$, $\overline{\rho}(\text{Frob}_{s'})$ has eigenvalues with ratio $s'$, and $R_{S \cup S'}^{S'-\text{new}}$ (defined in Definition 4) is $\simeq W(k)$.

**Definition 7** If $\rho : G_{Q} \to GL_{2}(W(k)/p^n)$ is a continuous representation that is a lift of $\overline{\rho}$, with determinant fixed as in the introduction, then $\rho$ is special at $s'$, if $s'$ is not $\pm 1$ mod $p$ and if $\rho|_{G_{s'}}$ can be conjugated to (we will also say for simplicity, is of the form)

$$
\begin{pmatrix}
\varepsilon & * \\
0 & 1
\end{pmatrix}
$$

up to twist. The $*$ may be trivial.

The $*$ can be genuinely nontrivial if and only if $\rho$ is ramified.

Recall from [R2] and [T2] that when studying a global deformation problem, we prescribe a local deformation condition at each of the primes at which ramification may occur. In particular, for a local problem at $v$ we choose a class of deformations of $\overline{\rho}|_{G_v}$ to $W(k)$ which we call $C_v$. For $v \in S$, $C_v$ consists of deformations of $\overline{\rho}|_{G_v}$ to $W(k)$ that are minimal in the sense of the introduction. Another way to think about this is that $C_v$ is the set of $W(k)$-valued points on a certain smooth quotient of the unrestricted local deformation ring. The surjective map onto this smooth quotient induces a surjective map on the (mod $p$) tangent spaces to these rings. Since the dual of the tangent space of the unrestricted local deformation ring is $H^1(G_v, Ad^0(\overline{\rho}))$, the
smooth quotient gives rise to a subspace of $H^1(G_v, \text{Ad}^0(\overline{\rho}))$ which we denote $\mathcal{N}_v$.

For each possible $\overline{\rho}|_{G_v}$ one needs to compute $C_v$ and $\mathcal{N}_v$. In particular, to find a characteristic zero deformation as in [R2] and [T2], one needs that

$$\dim_k H^1(G_v, \text{Ad}^0(\overline{\rho})) = \dim_k \mathcal{N}_v + \dim_k H^2(G_v, \text{Ad}^0(\overline{\rho})) + \delta$$

where $\delta = 0$ except when $\overline{\rho}$ is odd and $v = p$, in which case $\delta = 2$. The calculations of $\mathcal{N}_v$ and $C_v$ are laborious and not included here. See [R2]. Note, however, that the excluded situations of the introduction are those for which the above equation does not hold. (More precisely, at present we do not see how to choose $C_v$ and $\mathcal{N}_v$ appropriately in these cases). For $\overline{\rho}$ odd and not in an excluded case of the introduction, $C_p$ consists of certain potentially semistable representations. Locally, and therefore globally, these representations are semistable up to a twist by a character of finite order. For $\overline{\rho}$ even the local deformation ring at $G_p$ is smooth and $C_p$ is taken to be the $(W(k)$-valued points of) the entire deformation ring. Throughout this paper, for $s'$ a prime in an auxiliary set, we will insist $\overline{\rho}$ is special at $s'$ and $C_{s'}$ will consist of the special $W(k)$-valued points of the local at $s'$ deformation ring. In this case elements of $H^1(G_{s'}, \text{Ad}^0(\overline{\rho}))$ are trivial on wild inertia, so we may consider them as functions on tame inertia, which is topologically generated by $\sigma_{s'}$ corresponding to Frobenius and $\tau_{s'}$ corresponding to (tame) inertia. $\mathcal{N}_{s'}$ is spanned by the 1-cohomology class given by

$$g(\sigma_{s'}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad g(\tau_{s'}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$  

Also see the discussion after Lemma 2 of [R3].

Recall from section 6 of [R3] that there is a subfield $\tilde{k}$ of $k$ which is the minimal field of definition of the representations of $G_Q$ on $\text{Ad}^0(\overline{\rho})$ and $\text{Ad}^0(\overline{\rho})^*$. Denote the descents to $\tilde{k}$ of $\text{Ad}^0(\overline{\rho})$ and $\text{Ad}^0(\overline{\rho})^*$ by $\tilde{\text{Ad}}^0(\overline{\rho})$ and $\tilde{\text{Ad}}^0(\overline{\rho})^*$ respectively.

**Lemma 8** Let $\overline{\rho}: G_Q \to GL_2(k)$ be given with $S$ defined as usual. Assume $\overline{\rho}$ satisfies the hypotheses of the introduction and $X \supset S$ is a finite set such that $X \setminus S$ consists of special primes. For $n \geq 2$, let $\rho_n : G_X \to GL_2(W(k)/p^n)$ be a deformation of $\overline{\rho}$ unramified outside $X$. Let $\psi_i$ ($i = 1, \ldots, m$) be finitely many $\tilde{k}$-linearly independent elements in $H^1(G_X, \tilde{\text{Ad}}^0(\overline{\rho}))$ and $\phi_j (j =
1, · · · , r) be finitely many \(K\)-linearly independent elements in \(H^1(G_X, \widetilde{Ad}^0(\mathfrak{p}))\). Let \(Q(Ad^0(\mathfrak{p}))\) be the field fixed by the kernel of the action of \(G_Q\) on \(Ad^0(\mathfrak{p})\). Let \(\mathbf{K} = Q(Ad^0(\mathfrak{p}), \mu_p)\) be the field obtained by adjoining the \(p\)th roots of unity to \(Q(Ad^0(\mathfrak{p}))\). We denote by \(K_{\psi_i}\) and \(K_{\phi_j}\) the fixed fields of the kernels of the restrictions of \(\psi_i, \phi_j\) to \(G_K\), the absolute Galois group of \(\mathbf{K}\). Let \(\mathbf{P}_n\) be the fixed field of the kernel of the restriction of the projectivisation of \(\rho_n\) to \(G_K\). Then each of the fields \(K_{\psi_i}, K_{\phi_j}, P_n\) and \(K(\mu_p^n)\) is linearly disjoint over \(K\) with the compositum of the others.

**Proof.** (We actually require in 3) above that \(\phi_j \not\in \mathcal{N}^L_{\mathfrak{p}w}\), but since \(\phi_j\) is unramified at \(w\), it is an exercise to see this is equivalent to \(\phi_j|_{G_w} \neq 0\).

The proof is quite standard and is essentially contained in proof of Lemma 1.2 of [T2]. The main differences are

- In [T2] \(w\) is chosen only so that \(\psi_i|_{G_w} \neq 0\) and \(\phi_j|_{G_w} \neq 0\).
- \(\rho_n\) and condition 1) do not figure there.

We take care of these in the proof we sketch below (as in Lemma 2 of [R1]). Observe the field fixed by kernel of the action of \(G_Q\) on \(Ad^0(\mathfrak{p})^*\) is contained in \(\mathbf{K}\). The following facts are in section 7 of [R3]. See also Section 1 of [T2].

- \(K_{\phi_j}\) and \(K_{\psi_i}\) are Galois over \(Q\) and linearly disjoint over \(\mathbf{K}\)
- \(\text{Gal}(K_{\psi_i}/\mathbf{Q})\) injects into \(Ad^0(\mathfrak{p})\) and the map is \(\text{Gal}(\mathbf{K}/\mathbf{Q})\) equivariant. Also the short exact sequence
  
  \[
  1 \to \text{Gal}(K_{\psi_i}/\mathbf{K}) \to \text{Gal}(K_{\psi_i}/\mathbf{Q}) \to \text{Gal}(\mathbf{K}/\mathbf{Q}) \to 1
  \]
  splits.
- \(\text{Gal}(K_{\phi_j}/\mathbf{K})\) injects into \(Ad^0(\mathfrak{p})^*\) and the map is \(\text{Gal}(\mathbf{K}/\mathbf{Q})\) equivariant. Also the short exact sequence
  
  \[
  1 \to \text{Gal}(K_{\phi_j}/\mathbf{K}) \to \text{Gal}(K_{\phi_j}/\mathbf{Q}) \to \text{Gal}(\mathbf{K}/\mathbf{Q}) \to 1
  \]
  splits.

8
The $\text{Gal}(K/Q)$ equivariant isomorphisms $\text{Gal}(K_\phi/K) \to \text{Ad}^0(\overline{p})^*$ and $\text{Gal}(K_\psi/K) \to \text{Ad}^0(\overline{p})$ give $\tilde{k}$ structures to $\text{Gal}(K_\psi/K)$ and $\text{Gal}(K_\phi/K)$ making these maps isomorphisms of $\tilde{k}[\text{Gal}(K/Q)]$-modules. We have $\text{Gal}(K_\psi/K) \simeq \tilde{\text{Ad}}^0(\overline{p})$ and $\text{Gal}(K_\phi/K) \simeq \tilde{\text{Ad}}^0(\overline{p})$ as $\tilde{k}[\text{Gal}(K/Q)]$-modules. Let $\tilde{\rho}_t : G_Q \to PGL_2(W(\tilde{k}))/p^t$ for $t = 1, \cdots, n$ be the mod $p^t$ reduction of the projectivisation $\tilde{\rho}_n$ of $\rho_n$. Then we claim that the image of $\tilde{\rho}_t$ contains $PSL_2(W(\tilde{k}))/p^t$. This is true for $t = 1$ by assumptions on $\overline{p}$ and then follows inductively for all $t$ by the proof of Lemma 3 on IV-23 of Serre’s book [S] (here we use $p > 3$).

Thus we have an exact sequence

$$1 \to \tilde{\text{Ad}}^0(\overline{p}) \to \text{Image}(\tilde{\rho}_t) \to \text{Image}(\tilde{\rho}_{t-1}) \to 1.$$ 

It is easy to see, using the assumption that $p > 3$, that this is a non-split exact sequence for all $t \geq 2$.

From this and the argument at the end of proof of Lemma 1.2 of [T2] it follows easily that each of $K_{\phi_1}, K_{\psi_1}, P_n$ and $K(\mu_{p^n})$ is linearly disjoint over $K$ with the compositum of the others. After this getting a $w$ that satisfies the conditions of the lemma is an application of Chebotarev’s theorem using the (strong) linear disjointness of $K_{\phi_1}, K_{\psi_1}, P_n$ and $K(\mu_{p^n})$ over $K$.

**Definition 9** For a finite set $X$ containing $S$ we define the Selmer group $H^1_{N_x}(G_X, \text{Ad}^0(\overline{p}))$ to be the kernel of the map

$$H^1(G_X, \text{Ad}^0(\overline{p})) \to \bigoplus_{v \in X} H^1(G_v, \text{Ad}^0(\overline{p}))/N_v$$

and the dual Selmer group $H^1_{N_x}(G_X, \text{Ad}^0(\overline{p})^*)$ to be the kernel of the map

$$H^1(G_X, \text{Ad}^0(\overline{p}^*)) \to \bigoplus_{v \in X} H^1(G_v, \text{Ad}^0(\overline{p})^*)/N_v^\perp$$

where $N_v^\perp \subset H^1(G_v, \text{Ad}^0(\overline{p})^*)$ is the annihilator of $N_v \subset H^1(G_v, \text{Ad}^0(\overline{p}))$ via local duality.

**Fact 10** For our given $\overline{p}$ an auxiliary set exists.

**Proof.** We only give a brief sketch here. We refer the reader to [R2] and [T2] for details. There is a Galois cohomological criterion (realisable!) for auxiliary sets to exist, namely $H^1_{N_x}(G_{S\cup S'}, \text{Ad}^0(\overline{p})^*) = 0$. Recall for any $X \supseteq S$ we get, from the Poitou-Tate exact sequence, the exact sequence

$$H^1(G_X, \text{Ad}^0(\overline{p})) \to \bigoplus_{v \in X} H^1(G_v, \text{Ad}^0(\overline{p}))/N_v \to H^1_{N_v}(G_X, \text{Ad}^0(\overline{p})^*)^\vee$$
\[ \rightarrow H^2(G_X, \text{Ad}^0(\overline{\rho})) \to \oplus_{v \in X} H^2(G_v, \text{Ad}^0(\overline{\rho})). \]

If \( H^1_{\mathcal{N}_v}(G_X, \text{Ad}^0(\overline{\rho})^*) \) is trivial then one can construct a deformation \( \rho \) of \( \overline{\rho} \) to \( W(k) \) such that \( \rho|_{G_v} \in \mathcal{C}_v \) for all \( v \in X \) by deforming from mod \( p^n \) to mod \( p^{n+1} \). One does this by observing the obstruction to deforming lies in \( H^2(G_X, \text{Ad}^0(\overline{\rho})) \) and since \( H^1_{\mathcal{N}_v}(G_X, \text{Ad}^0(\overline{\rho})^*) = 0 \) this can be analyzed locally. The local obstructions can be made to vanish by ‘moving’ from our given mod \( p^n \) deformation to an unobstructed mod \( p^n \) deformation. That this is possible follows from the surjectivity of

\[ H^1(G_X, \text{Ad}^0(\overline{\rho})) \to \oplus_{v \in X} H^1(G_v, \text{Ad}^0(\overline{\rho}))/\mathcal{N}_v. \]

In fact the set of all such deformations ends up being a power series ring over \( W(k) \) in \( \dim \left( H^1_{\mathcal{N}_v}(G_X, \text{Ad}^0(\overline{\rho})) \right) \) variables.

Remarks:
1) As all primes \( s' \) in our auxiliary sets are special for \( \overline{\rho} \) we will always have that \( H^0(G_{s'}, \text{Ad}^0(\overline{\rho})) \) and \( H^1(G_{s'}, \text{Ad}^0(\overline{\rho}))/\mathcal{N}_{s'} \) are one dimensional. See the Section 3 of [R3].
2) By Proposition 1.6 of [W] \( H^1_{\mathcal{N}_v}(G_X, \text{Ad}^0(\overline{\rho})^*) \) and \( H^1(G_{s'}, \text{Ad}^0(\overline{\rho}))/\mathcal{N}_{s'} \) have the same dimension. Thus when we have annihilated the Selmer group, we have also annihilated the dual Selmer group and the power series ring referred to in the proof above is just \( W(k) \).
3) It follows from [R2] and [T2] that any set \( S' \), such that the primes in \( S' \) are not \( \pm 1 \) mod \( p \), and are special for \( \overline{\rho} \), and such that \( R_{S' - \text{new}}^{S' - \text{new}} = W(k) \), has cardinality at least \( n_{\overline{\rho}} = \dim H^1_{\mathcal{N}_v}(G_S, \text{Ad}^0(\overline{\rho})^*) \). We call these \( S' \) minimal auxiliary sets.

### 3 Ramified auxiliary sets

We begin with a definition.

**Definition 11** An auxiliary set \( S' \) such that the corresponding representation \( \rho_{S' - \text{new}}^{S' - \text{new}} \) is ramified at all primes of \( S' \) is said to be a ramified auxiliary set.

The purpose of this section is to prove that ramified auxiliary sets do exist. We state the main theorem of this section which is essentially a restatement of Theorem 3.
Theorem 12 For our given $\overline{\rho}$ there are infinitely many ramified auxiliary sets.

Consider the unique deformation $\rho_{S',new}^{S'} : G_{\mathbb{Q}} \to GL_2(W(k))$. If it is ramified at all primes in $S'$ we are done. If not, we write $S' = S'_{\text{ram}} \cup S'_{\text{un}}$ as the disjoint union of primes that are ramified in $\rho_{S',new}^{S'}$ and primes that are unramified in $\rho_{S',new}^{S'}$. We assume $S'_{\text{un}}$ is nonempty and if we can discard any primes from $S'_{\text{un}} \subset S'$ the set remains auxiliary we do so. Choose $s' \in S'_{\text{un}}$. We will replace $s'$ by two primes $s_1$ and $s_2$ such that $S'' = S' \cup \{s_1, s_2\}\{s'\}$ is an auxiliary set and $\rho_{S'',new}^{S'}$ is ramified at $\{s_1, s_2\} \cup S'_{\text{ram}}$. Thus we replace $S'$ by an auxiliary set that has one more element, but the set of unramified primes for the new representation is at least one smaller than for $S'$. Repeating this process leads to a ramified auxiliary set.

Let $n$ be an integer such that $\rho_{S',new}^{S'}$, the mod $p^n$ reduction of $\rho_{S',new}^{S'}$, is ramified at all primes of $S'_{\text{ram}}$. We will consider the set $S'\{s'\}$. The new deformation we construct will, mod $p^n$, be equal to $\rho_{S',new}^{S'}$ and hence ramified at all primes of $S'_{\text{ram}}$.

Proposition 13 We assume $S \cup S'\{s'\}$ is not auxiliary. Let $\psi$ span the Selmer group for $S \cup S'\{s'\}$. There exist primes $s_1$ and $s_2$ such that

1) $\rho_{S',new}^{S',n-1}$ is special at $s_1$ but $\rho_{S',new}^{S'}$ is not special at $s_1$.
2) $\rho_{S',new}^{S',n-1}$ is special at $s_2$.
3) The map $H^1(G_{S\cup S'\{s\}}, Ad^0(\overline{\rho})) \to \bigoplus_{v \in \mathfrak{S}\{s\}} H^1(G_v, Ad^0(\overline{\rho}))/\mathfrak{N}_v$ has one dimensional kernel spanned by $\psi$
4) The map $H^1(G_{S\cup S'\{s_2\}}, Ad^0(\overline{\rho})) \to \bigoplus_{v \in \mathfrak{S}\{s\}} H^1(G_v, Ad^0(\overline{\rho}))/\mathfrak{N}_v$ has dimensional kernel spanned by $\psi$

Proof: By the first two remarks at the end of the section, the Selmer group for $S \cup S'$ is one dimensional as is the dual Selmer group for $S \cup S'$. Let $\psi$ and $\phi$ span the Selmer group and dual Selmer group for $S \cup S'\{s'\}$ respectively. Use Lemma 8 to choose a prime $s_1$ such that

- 1) is satisfied,
- $\psi|_{G_{s_1}} = 0$
- $\phi|_{G_{s_1}} \neq 0$. 

11
This choice insures that the Selmer group and dual Selmer group for $S \cup S' \cup \{s_1\}\{s'\}$ are one dimensional and that (the inflation of) $\psi$ spans this Selmer group. The map 3) is surjective by Proposition 1.6 of [W]. (We take $L_{s_1}$ to be all of $H^1(G_{s_1}, Ad^0(\overline{\rho}))$ and for $v \neq s_1$ we take $L_v$ to be $N_v$). That the kernel of 3) is one dimensional also follows from Proposition 1.6 of [W].

Let $\tilde{\phi}$ span the Selmer group for $S \cup S' \cup \{s_1\}\{s'\}$. Now choose $s_2$ to satisfy the three bulleted items below.

• 2) is satisfied,

• $\psi|_{G_{s_2}}$ is not zero in $H^1(G_{s_2}/I_{s_2}, Ad^0(\overline{\rho}))$,

• $\phi|_{G_{s_2}} \neq 0$,

• $\tilde{\phi}|_{G_{s_2}} \neq 0$.

By construction $\phi$ and $\tilde{\phi}$ are independent so Lemma 8 allows us to choose $s_2$ as above. This gives that the sets $S' \cup \{s_2\}\{s'\}$ and $S' \cup \{s_1, s_2\}\{s'\}$ are both auxiliary. The Selmer group map for $S \cup S' \cup \{s_2\}\{s'\}$ is then an isomorphism. An application of Proposition 1.6 of [W] gives us 4) and the proof is done.

We have that

• The sets $S'$, $S' \cup \{s_2\}\{s'\}$, and $S'' = S' \cup \{s_1, s_2\}\{s'\}$ are all auxiliary.

• $\rho_{S \cup S', n-1}^{S' - \text{new}}$ is minimal at all primes of $S$, special at all primes in $S''$, and unramified outside $S \cup S''$. Since $R_{S \cup S', n-1}^{S' - \text{new}}$ and $R_{S \cup S', n-1}^{S'' - \text{new}}$ are both isomorphic to $W(k)$ we see the deformations $\rho_{S \cup S', n-1}^{S' - \text{new}}$ and $\rho_{S \cup S', n-1}^{S'' - \text{new}}$ are equal.

• $\rho_{S \cup S', n-1}^{S'' - \text{new}}$ is unramified at $s'$ and not special at $s_1$.

Using these facts it is routine to see there is an $h \in H^1(G_{S \cup S'}, Ad^0(\overline{\rho}))$ such that $\rho_{S \cup S', n}^{S'' - \text{new}} = (I + p^{n-1}h)\rho_{S \cup S', n-1}^{S' - \text{new}}$.

**Proposition 14** $\rho_{S \cup S', n-1}^{S'' - \text{new}}$ is ramified at both $s_1$ and $s_2$.

**Proof.** We have

$$\rho_{S \cup S', n}^{S'' - \text{new}} = (I + p^{n-1}h)\rho_{S \cup S', n-1}^{S' - \text{new}}.$$
Note that $h|_{G_v} \in N_v$ for all $v \in S \cup S' \setminus \{ s' \}$ since both sides are the mod $p^n$ reductions of elements of $C_v$ for these $v$.

Suppose the left hand side is unramified at $s_i$ for $i = 1$ or $2$. Then $h$ inflates from $H^1(G_{S_i \cup S'} \setminus \{ s_i \}, \text{Ad}^0(\bar{\rho}))$ and is in the kernel of the map 4) or 3) of Proposition 13. But the kernels of these maps are spanned by (the inflation of) $\psi$ which is trivial at $s_1$. Then the right hand side above is not special at $s_1$, so the left hand side is also not special, a contradiction. Thus $\rho_{S_i \cup S'}^{\text{new}}$ is ramified at $s_1$ and $s_2$.

Now replace the auxiliary set $S'$ by the new auxiliary set $S'' = S' \cup \{ s_1, s_2 \} \setminus \{ s' \}$ and repeat the process. The fact that there are infinitely many ramified auxiliary sets follows from an inspection of the proof. This finishes the proof of Theorem 12.

### 4 Selmer groups

Consider a ramified auxiliary set $S' = \{ s'_1, \ldots, s'_n \}$. Then the corresponding representation $\rho_{S_i \cup S'}^{\text{new}} : G_Q \to GL_2(W(\kappa))$, that arises from the isomorphism $R_{S_i \cup S'}^{\text{new}} \simeq W(\kappa)$, is ramified at all the primes in $S'$. Such an $S'$ exists by Theorem 12. We have a surjection $R_{S_i \cup S'} \to R_{S_i \cup S'}^{\text{new}} \simeq W(\kappa)$: let $\varphi_{S'}$ be the kernel. For each $i$ let $m_i$ be the largest integer such that $\rho_{S_i \cup S'}^{\text{new}} |_{G_{s'_i}}$ is unramified. By choice we have $m_i < \infty$.

**Theorem 15** If $S'$ is a ramified auxiliary set, the tangent space $\varphi_{S'}/\varphi_{S'}^2$ is a finite abelian group.

**Proof.** We see by standard arguments as in Proposition 1.2 of [W], or [Ma1], that we can identify the dual of $\varphi_{S'}/\varphi_{S'}^2$ as a Selmer group, i.e., as the kernel of the map

$$H^1(G_{S_i \cup S'}, \text{Ad}^0(\rho_{S_i \cup S'}^{\text{new}}) \otimes Q_p/\mathbb{Z}_p) \to \prod_{v \in S} \frac{H^1(G_v, \text{Ad}^0(\rho_{S_i \cup S'}^{\text{new}}) \otimes Q_p/\mathbb{Z}_p)}{L_v}.$$

For $s'_i \in S'$, note the representation space of $\rho_{S_i \cup S'}^{\text{new}} |_{G_{s'_i}}$ has a one dimensional stable subspace. Let $Z \subset \text{Ad}^0(\rho_{S_i \cup S'}^{\text{new}})$ be the (upper triangular) one dimensional subspace of nilpotent elements that are preserved under conjugation by $\rho_{S_i \cup S'}^{\text{new}} |_{G_{s'_i}}$.

We define $L_v$ as follows:
• For $v \in S$, $v \neq p$, we define $L_v$ to be the kernel of the map

$$H^1(G_v, Ad^0(\rho_{S\cup S'}^{S'-new}) \otimes \mathbb{Q}_p/\mathbb{Z}_p) \rightarrow H^1(I_v, Ad^0(\rho_{S\cup S'}^{S'-new}) \otimes \mathbb{Q}_p/\mathbb{Z}_p).$$

• For $s' \in S'$ we define $L_{s'}$ to be the kernel of the map

$$H^1(G_{s'}, Ad^0(\rho_{S\cup S'}^{S'-new}) \otimes \mathbb{Q}_p/\mathbb{Z}_p) \rightarrow H^1(G_{s'}, (Ad^0(\rho_{S\cup S'}^{S'-new})/Z) \otimes \mathbb{Q}_p/\mathbb{Z}_p).$$

• If $\overline{\rho}|_{G_p}$ is odd, reducible, and flat, $L_p$ is defined as in the flat case.

• If $\overline{\rho}|_{G_p}$ is odd and not reducible then we define $L_p$ as in the flat case using Fontaine-Lafaille modules of appropriate filtration length.

• In all other (necessarily reducible) odd cases define $L_p$ as in the ordinary situation.

• If $\overline{\rho}$ is even then $L_p$ is defined to be all of $H^1(G_p, Ad^0(\rho_{S\cup S'}^{S'-new}) \otimes \mathbb{Q}_p/\mathbb{Z}_p)$.

Recall we identify the dual of $\varphi_{S'}/\varphi_{S'}^2$ with the kernel of the map

$$H^1(G_{S\cup S'}, Ad^0(\rho_{S\cup S'}^{S'-new}) \otimes \mathbb{Q}_p/\mathbb{Z}_p) \rightarrow \prod_{v \in S'} \frac{H^1(G_v, Ad^0(\rho_{S\cup S'}^{S'-new}) \otimes \mathbb{Q}_p/\mathbb{Z}_p)}{L_v}. $$

Using the fact that $R_{S\cup S'}^{S'-new}$ is isomorphic to $W(k)$, we deduce that the kernel of the map

$$H^1(G_{S\cup S'}, Ad^0(\rho_{S\cup S'}^{S'-new}) \otimes \mathbb{Q}_p/\mathbb{Z}_p) \rightarrow \prod_{v \in S\cup S'} \frac{H^1(G_v, Ad^0(\rho_{S\cup S'}^{S'-new}) \otimes \mathbb{Q}_p/\mathbb{Z}_p)}{L_v}$$

is trivial. Thus the dual of $\varphi_{S'}/\varphi_{S'}^2$ injects into

$$\prod_{s' \in S'} \frac{H^1(G_{s'}, Ad^0(\rho_{S\cup S'}^{S'-new}) \otimes \mathbb{Q}_p/\mathbb{Z}_p)}{L_{s'}}.$$

The following lemma completes the proof of Theorem 13.

**Lemma 16** Let $s' \in S'$ and let $m$ be the largest integer such that $\rho_{S\cup S', m}^{S'-new}|_{G_{s'}}$ is unramified.

1) $(Ad^0(\rho_{S\cup S'}^{S'-new}) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{I_{s'}} \supseteq Ad^0(\rho_{S\cup S'}^{S'-new}) \otimes p^{-m}\mathbb{Z}/\mathbb{Z}$ and

$$\frac{(Ad^0(\rho_{S\cup S'}^{S'-new}) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{I_{s'}}}{Ad^0(\rho_{S\cup S'}^{S'-new}) \otimes p^{-m}\mathbb{Z}/\mathbb{Z}} \simeq \mathbb{Z} \otimes \mathbb{Q}_p/\mathbb{Z}_p.$$
2) The inflation map

\[ H^1(G_{s'}/I_{s'}, (\text{Ad}^0(\rho_{SS'}^{S'-\text{new}}) \otimes Q_p/Z_p)^1) \rightarrow H^1(G_{s'}/\text{Ad}^0(\rho_{SS'}^{S'-\text{new}}) \otimes Q_p/Z_p) \]

is an isomorphism.

3) \( H^1(G_{s'}/\text{Ad}^0(\rho_{SS'}^{S'-\text{new}}) \otimes Q_p/Z_p) \cong W(k)/p^n \)

4) For all \( s' \in S' \) we have that \( L_{s'} \) is trivial.

**Proof.** First observe since \( s' \) is not \( 1 \mod p \) that \( H^r(G_{s'}/I_{s'}, Z \otimes Q_p/Z_p) \) and \( H^r(G_{s'}/I_{s'}, Z \otimes p^{-m}Z/Z) \) are trivial for \( r = 0, 1 \).

1) This is a routine computation that uses ramification first occurs \( \mod p^{m+1} \).

2) Using inflation-restriction, it suffices to prove \( H^1(I_{s'}/\text{Ad}^0(\rho_{SS'}^{S'-\text{new}}) \otimes Q_p/Z_p) \) is trivial. This last group is isomorphic to \( (\text{Ad}^0(\rho_{SS'}^{S'-\text{new}}) \otimes Q_p/Z_p)_{I_{s'}} \) which in turn is isomorphic to \( W(k)(-1) \otimes Q_p/Z_p \) which is trivial as \( s' \) is not \( 1 \mod p \).

3) We have the exact sequence

\[ 0 \rightarrow \text{Ad}^0(\rho_{SS'}^{S'-\text{new}}) \otimes p^{-m}Z/Z \rightarrow (\text{Ad}^0(\rho_{SS'}^{S'-\text{new}}) \otimes Q_p/Z_p)^1 \rightarrow Z \otimes Q_p/Z_p \rightarrow 0 \]

of \( G_{s'}/I_{s'} \) modules. Since \( s' \) is not \( \pm 1 \mod p \) we see \( H^1(G_{s'}/I_{s'}, Z \otimes Q_p/Z_p) = 0 \) for \( i = 0, 1 \). Taking \( G_{s'}/I_{s'} \) cohomology we conclude

\[ H^1(G_{s'}/I_{s'}, (\text{Ad}^0(\rho_{SS'}^{S'-\text{new}}) \otimes p^{-m}Z/Z)) \cong H^1(G_{s'}/I_{s'}, (\text{Ad}^0(\rho_{SS'}^{S'-\text{new}}) \otimes Q_p/Z_p)^1). \]

Since

\[ \text{Ad}^0(\rho_{SS'}^{S'-\text{new}}) \otimes p^{-m}Z/Z \cong W(k)(-1)/p^m \oplus W(k)/p^m \oplus W(k)(1)/p^m \]

as \( G_{s'}/I_{s'} \) modules, the conclusion follows from 2) and a routine calculation.

4) We take \( G_{s'} \) cohomology of the sequence

\[ 0 \rightarrow Z \otimes Q_p/Z_p \rightarrow \text{Ad}^0(\rho_{SS'}^{S'-\text{new}}) \otimes Q_p/Z_p \rightarrow \text{Ad}^0(\rho_{SS'}^{S'-\text{new}})/Z \otimes Q_p/Z_p \rightarrow 0. \]

The computation of \( G_{s'} \) invariants is routine and we get

\[ 0 \rightarrow 0 \rightarrow W(k)/p^m \rightarrow W(k) \otimes Q_p/Z_p \rightarrow H^1(G_{s'}/Z \otimes Q_p/Z_p) \rightarrow H^1(G_{s'}/\text{Ad}^0(\rho_{SS'}^{S'-\text{new}}) \otimes Q_p/Z_p) \rightarrow H^1(G_{s'}/\text{Ad}^0(\rho_{SS'}^{S'-\text{new}})/Z \otimes Q_p/Z_p) \rightarrow \ldots \]
We will prove the map \( W(\mathbf{k}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow H^1(G_{s'}, Z \otimes \mathbb{Q}_p/\mathbb{Z}_p) \) is surjective. This will give injectivity of

\[
H^1(G_{s'}, Ad^0(\rho_{S_\cup S'}^{s'-new}) \otimes \mathbb{Q}_p/\mathbb{Z}_p) \rightarrow H^1(G_{s'}, Ad^0(\rho_{S_\cup S'}^{s'-new})/Z \otimes \mathbb{Q}_p/\mathbb{Z}_p)
\]

and we will be done.

Recall \((Z \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{I_{s'}} = Z \otimes \mathbb{Q}_p/\mathbb{Z}_p\) and consider the inflation-restriction sequence

\[
0 \rightarrow H^1(G_{s'}/I_{s'}, (Z \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{I_{s'}}) \rightarrow H^1(G_{s'}, Z \otimes \mathbb{Q}_p/\mathbb{Z}_p) \\
\rightarrow H^1(I_{s'}, Z \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{G_{s'}/I_{s'}} \rightarrow 0.
\]

The last (fifth) term is 0 as it is actually an \( H^2 \) of the group \( G_{s'}/I_{s'} \) which has cohomological dimension 1. We know \( H^1(G_{s'}/I_{s'}, (Z \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{I_{s'}}) \) is trivial so

\[
H^1(G_{s'}, Z \otimes \mathbb{Q}_p/\mathbb{Z}_p) \rightarrow H^1(I_{s'}, Z \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{G_{s'}/I_{s'}}
\]

is an isomorphism. Since \( I_{s'} \) acts trivially on \( Z \) we see \( H^1(I_{s'}, Z \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{G_{s'}/I_{s'}} \) is just the \( G_{s'}/I_{s'} \) equivariant elements of \( Hom(I_{s'}, Z \otimes \mathbb{Q}_p/\mathbb{Z}_p) \). By our choice of \( s' \) the actions of \( G_{s'}/I_{s'} \) on (the abelianisation of) \( I_{s'} \) and on \( Z \otimes \mathbb{Q}_p/\mathbb{Z}_p \) are compatible so we need only find \( Hom(I_{s'}, Z \otimes \mathbb{Q}_p/\mathbb{Z}_p) \) which is just \( Hom(\mathbb{Z}_p, W(\mathbf{k}) \otimes \mathbb{Q}_p/\mathbb{Z}_p) \) which in turn is \( W(\mathbf{k}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \).

We have proved \( H^1(I_{s'}, Z \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{G_{s'}/I_{s'}} \) is isomorphic to \( W(\mathbf{k}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \). Thus the map we needed to prove surjective is just the multiplication by \( p^m \) map on \( W(\mathbf{k}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \) which is clearly onto. The proposition and Theorem 15 are now proved.

**Remarks:**

1) Note that the proof of Theorem 15 crucially uses, and is more or less a formal consequence of the fact, that \( \rho_{S_\cup S'}^{s'-new}|_{G_{s_i}} \) is ramified for all \( s_i \in S' \).

2) It also follows from 3) and the proof that if we denote by \( \varphi_\alpha \) the kernel of the map \( R_{S_\cup S'}^{\alpha-new} \rightarrow R_{S_\cup S'}^{s'-new} \), for any \( \alpha \subset S' \), then \( \varphi_\alpha/\varphi_\alpha^2 \) is isomorphic to a subgroup of \( \prod_{s_i \not\in \alpha} W(\mathbf{k})/p^m \).

**Corollary 17** The prime ideal \( \varphi_{S'} \) is minimal and thus \( \text{Spec}(R_{S_\cup S'}) \) has an isolated component that is isomorphic to \( W(\mathbf{k}) \).
Proof. Consider the localisation $R_{\psi,s'}$ of $R_{S\cup S'}$ at $\psi_s'$. Then as $R_{S\cup S'}/\psi_s'$ is of characteristic 0, we see that $p$ is invertible in $R_{\psi,s'}$, and thus that $\psi_s'/\psi_s'^2$ is zero as it follows from Theorem 13 that it is a $p$-torsion group. But then as $\psi_s'$ is a proper ideal, and $R_{\psi,s'}$ is Noetherian, we conclude that $\psi_s'$ is 0 in $R_{\psi,s'}$. Hence $\psi_s'$ is a minimal prime ideal of $R_{S\cup S'}$, i.e., $\text{Spec}(R_{S\cup S'}')$ is an isolated component of $\text{Spec}(R_{S\cup S'})$.

**Corollary 18** If $R_{S\cup S'}$ is a complete intersection ring, then it is finite flat over $W(k)$.

**Proof.** We can arrange for an epimorphism $\phi : W(k)[[T_1, \ldots, T_d]] \to R_{S\cup S'}$ with kernel $I$ contained in the ideal $J = (T_1, \ldots, T_d)$, and further the composite map $W(k)[[T_1, \ldots, T_d]] \to R_{S\cup S'} \to R_{S\cup S'}'$ given by $T_i \to 0$. Let $I$ be generated by $(r_1, \ldots, r_e)$ for some integer $e \geq 0$. As $\phi(J)/\phi(J)^2$ is finite by Theorem 13 the images of $r_i$ in $J/J^2$ span a full sublattice of $J/J^2 \simeq W(k)^d$. Thus $e$ is at least $d$ and the corollary follows.

Let $\gamma_s'$ be a generator of the unique $\mathbb{Z}_p$-quotient of $I_s'$. Consider the universal representation $\rho_{\gamma_s'}^{\text{univ}} : G_\mathbb{Q} \to \text{GL}_2(R_{S\cup S'})$. Because of the nature of the primes $s' \in S'$ (in particular none of them is congruent to $\pm 1 \pmod{p}$) and the structure of tame inertia groups recalled above we see that $\rho_{\gamma_s'}^{\text{univ}}(\gamma_s')$ is well-defined (on choosing a prime of $\mathbb{Q}$ above $s'$) and of the form

$$
\begin{pmatrix}
1 & x_{s'} \\
0 & 1
\end{pmatrix}.
$$

Consider the principal ideal $(\Pi_{s' \in S'} x_{s'})$ and choose a generator $x_{s'}$ for it. Note that $x_{s'}$ is not nilpotent as $S'$ is a ramified auxiliary set.

**Corollary 19** $R_{S\cup S'}[\frac{1}{x_{s'}}] \simeq \text{Frac}(W(k))$.

**Proof.** Consider the representation $G_\mathbb{Q} \to \text{GL}_2(R_{S\cup S'}[\frac{1}{x_{s'}}])$, the composition of $\rho_{\gamma_s'}^{\text{univ}}$ with the map $R_{S\cup S'} \to R_{S\cup S'}[\frac{1}{x_{s'}}]$. In this representation we deduce from the structure of tame inertia, and the fact that $x_{s'}$ is not a zero-divisor in $R_{S\cup S'}[\frac{1}{x_{s'}}]$, that for $s' \in S'$ the ratio of the eigenvalues of the image of a lift of $\text{Frob}_{s'}$ is $s'$. Now upon using the facts that $R_{S\cup S'}^{\text{new}} = W(k)$ and $0 \neq (x_{s'}) \subset (p)$ in $R_{S\cup S'}^{\text{new}}$, the corollary follows.
We end this section by indicating an application of Theorem 15 to a conjecture of Fontaine-Mazur. Conjecture 2c of [FM] in our situation translates as saying that there are only finitely many morphisms $R_{S \cup S'} \to W(k)$ (taking $K = W(k)$ in the notation of loc. cit.). By a use of the finiteness theorem of Hermite-Minkowski this is equivalent to saying that a morphism $\pi : R_{S \cup S'} \to W(k)$ cannot be the nontrivial limit of morphisms $\pi_i : R_{S \cup S'} \to W(k)$, i.e., if for any $n$, $\pi_i \equiv \pi \pmod{p^n}$ for $i$ large enough, then $\pi_i$ is eventually constant. In this situation we provide an application of Theorem 15:

**Proposition 20** If $\pi : R_{S \cup S'} \to W(k)$ and $\ker(\pi)/\ker(\pi)^2$ is a finite abelian group, then any sequence of morphisms $\pi_i : R_{S \cup S'} \to W(k)$ that tends to $\pi$ is eventually constant. Thus for a ramified auxiliary set $S'$, the morphism $R_{S \cup S'} \to R_{S \cup S'}^\text{new} \simeq W(k)$ cannot be the non trivial limit of morphisms $\pi_i : R_{S \cup S'} \to W(k)$.

**Proof:** The second part follows from Theorem 15 and the first. We prove the first part. Let $p^n$ be the exponent of $\ker(\pi)/\ker(\pi)^2$ and we claim that if $\pi_i \equiv \pi \pmod{p^{n+1}}$, then $\pi_i = \pi$. If not there is a $n' \geq n + 1$ such that $\pi_i \equiv \pi \pmod{p^{n'}}$ but $\pi_i$ is not congruent to $\pi \pmod{p^{n'+1}}$. But then $\pi_i - \pi : \phi/\phi^2 \to p^{n'}W(k)/p^{2n'}W(k)$ is surjective. This contradicts the fact that $p^n$ is the exponent of $\ker(\pi)/\ker(\pi)^2$.

## 5 Odds and ends

### 5.1 Parity

We prove a proposition that is needed in [K].

**Proposition 21** We can choose an auxiliary set $\tilde{S}$ so that its cardinality is of any given parity.

**Proof.** We sketch the proof. Consider any auxiliary set $S'$. We may assume that it is not of the desired parity. We add a prime $w_1$ not congruent to $\pm 1 \pmod{p}$ that is special for $\mathfrak{f}$. If $S' \cup \{w_1\}$ is an auxiliary set, take $\hat{S} = S' \cup \{w_1\}$ and we are done. Otherwise the Selmer group $H^1_{\mathfrak{n}_0}(G_{S \cup S' \cup \{w_1\}}, Ad^0(\mathfrak{f}))$ is one dimensional as is the dual Selmer group $H^1_{\mathfrak{n}_0^*}(G_{S \cup S' \cup \{w_1\}}, Ad^0(\mathfrak{f}^*))$. Let $\psi$ span the Selmer group and $\phi$ span the dual Selmer group. Simply choose
a prime $w_2$ such that $\psi$ is trivial at $G_{w_2}$ and $\phi$ does not map to zero in $H^1(G_{w_2}, Ad^0(\mathfrak{p})^*)$. We then see that the dual Selmer group $H^1_{LS}(G_{S \cup S'} \cup \{w_1, w_2\}, Ad^0(\mathfrak{p})^*)$ is one dimensional. Now annihilate this dual Selmer group by choosing an appropriate $w_3$ and take $\tilde{S}$ to be the auxiliary set $S' \cup \{w_1, w_2, w_3\}$ and we are done.

5.2 Finer structure of deformation rings

We make the following conjecture for any ramified auxiliary set $S' = \{s'_1, \ldots, s'_n\}$ that is guaranteed to exist by Theorem [12].

**Conjecture 22** The ring $R_{S \cup S'}$ is a finite flat complete intersection over $W(k)$.

**Remark:**

The recent work of Taylor, cf. [T1], will imply this for odd $\mathfrak{p}$ in many cases (using G. Böckle’s arguments in the appendix to [K]). There is some evidence in the even case in [R4] and [B].

**Conjecture 23** In the situation of Theorem [13], the abelian group $\varphi_{S'}/\varphi_{S'}^2$ that was proved to be a subgroup of $\bigoplus_i W(k)/(p^{m_i})$ in Theorem [13] is in fact isomorphic to $\bigoplus_i W(k)/(p^{m_i})$.

Let $\sigma_{s'_i}$ be a lift of Frobenius in the Galois group of the maximal tame extension of $\mathbb{Q}_{s'_i}$ as before. Denote by $\alpha_{s'_i}$ the ratio of the eigenvalues of $\rho_{s'_i}^{univ}(\sigma_{s'_i})$. The $\alpha_{s'_i}$ are well-defined and independent of choice of the lift $\sigma_{s'_i}$. Using the results of [Bo], and because of our assumption that $s'_i$ is not $\pm 1 \mod p$ we have:

**Proposition 24** The kernel of the surjective map $R_{S \cup S'} \to R_{S \cup S'}^{'new}$ is the ideal generated by $(\alpha_{s'_1} - s'_1, \ldots, \alpha_{s'_n} - s'_n)$.

5.3 Infinite ramification

In [R1] it was proved that there exist odd 2 dimensional continuous, irreducible $p$-adic Galois representations ramified at infinitely many primes. Theorem 2 of that paper proved, assuming the GRH, the existence of odd 2 dimensional $p$-adic Galois representations ramified at infinitely many primes.
that were crystalline at $p$. Here we remove the GRH hypothesis and prove there exist even 2 dimensional $p$-adic Galois representations ramified at infinitely many primes.

**Theorem 25** Let $\overline{\rho} : G_{\mathbb{Q}} \to GL_2(k)$ satisfy the hypotheses of the introduction.

a) If $\overline{\rho}$ is even then there exists a deformation $\rho : G_{\mathbb{Q}} \to GL_2(W(k))$ of $\overline{\rho}$ ramified at infinitely many primes.

b) If $\overline{\rho}$ is odd then there exists a deformation $\rho : G_{\mathbb{Q}} \to GL_2(W(k))$ of $\overline{\rho}$ ramified at infinitely many primes that is potentially semistable at $p$.

**Remark:**

For $E_{/\mathbb{Q}}$ an elliptic curve without complex multiplication it is a well known result of Serre that for almost all $p$, the $p$-torsion of $E$ gives rise to a $\overline{\rho}$ to which part $b)$ of the theorem applies.

**Proof.** Note for $\overline{\rho}$ in part $b)$ that $C_p$ consists of either

- deformations of $\overline{\rho} |_{G_p}$ of fixed positive rational integral weight that are after a fixed finite twist ordinary

- or deformations that after a fixed finite twist come from irreducible Fontaine-Lafaille modules of fixed filtration length.

We will construct our global representation as a $p$-adic limit of global representations whose restrictions to $G_p$ all lie in $C_p$. Since each of the two classes above is closed under limits we get that our limit representation is either potentially ordinary or potentially crystalline, hence potentially semistable.

Choose an auxiliary set $S'_0$ for $\overline{\rho}$. We have deformation $\rho_{S'_0 \cup S'_0 \cup \{q_1\}}^{s'_0 \text{-new}}$ of $\overline{\rho}$ corresponding to the isomorphism $R_{S'_0 \cup S'_0 \cup \{q_1\}}^{s'_0 \text{-new}} \simeq W(k)$. We proceed to inductively construct a sequence of characteristic zero representations ramified at more and more primes whose $p$-adic limit is the desired $\rho$.

Let $n = 1$. As in Lemma 8 choose a prime $q_1$ unramified in $\rho_{S'_0 \cup S'_0 \cup \{q_1\}}^{s'_0 \text{-new}}$ and $\rho_{S'_0 \cup S'_0 \cup \{q_1\}}^{s'_0 \text{-new}}$ is special at $q_1$ (in particular $q_1 \not\equiv \pm 1 \mod p$) but $\rho_{S'_0 \cup S'_0 \cup \{q_1\}}^{s'_0 \text{-new}}$ is not special at $q_1$. If $S'_0 \cup \{q_1\}$ is auxiliary then there is the unique deformation $\rho_{S'_0 \cup S'_0 \cup \{q_1\}}^{s'_0 \cup \{q_1\} \text{-new}}$ of $\overline{\rho}$ corresponding to the isomorphism $R_{S'_0 \cup S'_0 \cup \{q_1\}}^{s'_0 \cup \{q_1\} \text{-new}} \simeq W(k)$. This unique deformation cannot be unramified at $q_1$. For if it were, the eigenvalues of $\rho_{S'_0 \cup S'_0 \cup \{q_1\}}^{s'_0 \cup \{q_1\} \text{-new}}(Frob_{q_1})$ would have ratio $q_1$. Being unramified at
\( q_1 \) this characteristic zero deformation of \( \tilde{\rho} \) would necessarily be \( \rho^{S_0}_{S_0 \cup S_0'} \). But the eigenvalues of \( \rho^{S_0}_{S_0 \cup S_0'}(\text{Frob}_{q_1}) \) do not have ratio \( q_1 \) by choice.

Suppose then that \( S_0' \cup \{q_1\} \) is not an auxiliary set. Following Lemma 8 we can find a prime \( q_2 \) such that:

- \( S_0'' = S_0' \cup \{q_1, q_2\} \) is auxiliary.

Then \( \rho^{S_0''}_{S_0''} \) is the unique deformation of \( \tilde{\rho} \) corresponding to the isomorphism \( R^{S_0''}_{S_0''} \simeq W(k) \). It is ramified at at least one of \( q_1, q_2 \), for otherwise we would have \( \rho^{S_0''}_{S_0''} = \rho^{S_0}_{S_0} \) and \( \rho^{S_0}_{S_0}(\text{Frob}_{q_1}) \) and \( \rho^{S_0}_{S_0}(\text{Frob}_{q_2}) \) would have eigenvalues with ratio \( q_1 \) and \( q_2 \) respectively, a contradiction to how these primes were chosen.

If we are in the first case put \( S'_1 = S_0' \cup \{q_1\} \). If we are in the second case put \( S'_1 = S_0'' = S_0' \cup \{q_1, q_2\} \). In either case \( S'_1 \) is auxiliary, \( \rho^{S_0''}_{S_0''} = \rho^{S_0'}_{S_0'} \) and \( \rho^{S_0'}_{S_0'} \) is ramified at at least one prime of \( S_1' \backslash S_0' \). Continuing the induction we get a sequence of deformations \( \rho^{S_0'}_{S_0'} \) such that \( \rho^{S_0'}_{S_0'} \) is ramified at at least one prime of \( S_k' \backslash S_{k-1}' \) and this sequence converges \( p \)-adically. The limit deformation is ramified at infinitely many primes.

Remarks:

1) If \( \tilde{\rho} \) is odd and has Serre weight between 2 and \( p \), one sees from the proof that we can arrange for the \( \rho \) of the theorem to also be crystalline.

2) Fontaine and Mazur have conjectured in [FM] that all irreducible \( p \)-adic representations of the absolute Galois group of a number field \( F \) that are potentially semistable at all primes above \( p \) and have finite ramification arise in the étale cohomology of some variety over \( F \). Part b) of Theorem 2 shows that hypotheses of finite ramification and potential semistability are independent.

3) Assuming \( \tilde{\rho} \) is modular, Theorem 1 of [K] can be used to give another proof of part b) of Theorem 2.

4) Mestre has shown in [Me] that \( SL_2(F_7) \) is a regular extension of \( Q(T) \). Specialization gives us an infinite supply of \( SL_2(F_7) \) extensions of \( Q \). It is an exercise in local class field theory to see that none of these representations, when restricted to \( G_7 \), are twists of \( \left( \begin{array}{cc} \chi & 0 \\ 0 & 1 \end{array} \right) \) or the indecomposable
representation \( \begin{pmatrix} \chi_{p-2} & * \\ 0 & 1 \end{pmatrix} \). Thus part a) of Theorem 25 applies in all these situations.

6 References

[B] Böckle, G., A local-to-global principle for deformations of Galois representations, J. Reine Angew. Math. 509 (1999), 199–236.

[Bo] Boston, N., Families of Galois representations—increasing the ramification, Duke Math. J. 66 (1992), 357–367.

[Ca] Carayol, H., Formes modulaires et représentations galoisiennes avec valeurs dans un anneau local complet, in p-adic monodromy and the Birch and Swinnerton-Dyer conjecture, 213–237, Contemp. Math., 165, AMS, 1994.

[DDT] Darmon, H., Diamond, F., Taylor, R. Fermat’s last theorem in Elliptic curves, modular forms & Fermat’s last theorem (Hong Kong) 1993, p. 2-140.

[Di] Dickson, L. E., Linear Groups, B. G. Teubner 1901.

[FLT] Modular forms and Fermat’s last theorem, edited by Cornell, G., Silverman, J.H., and Stevens, G. Springer-Verlag, New York, 1997.

[FM] Fontaine, J.-M., Mazur, B., Geometric Galois representations, Elliptic curves, modular forms, and Fermat’s last theorem (Hong Kong, 1993), 41–78, Ser. Number Theory, I, Internat. Press, Cambridge, MA, 1995.

[Ma1] Mazur, B., An introduction to the deformation theory of Galois representations, in [FLT].

[Ma2] Mazur, B., Deforming Galois representations, in Galois Groups over \( \mathbb{Q} \), eds. Y. Ihara, K. Ribet, J. -P. Serre.

[Me] Mestre, J.-F., Construction d’extensions régulières de \( \mathbb{Q}(T) \) à groupe de Galois \( SL_2(\mathbb{F}_7) \) et \( \tilde{\Lambda}_{12} \), C.R. Acad. Sci. Paris 319 (1994), 781–782.

[Mi] Milne, J., Arithmetic Duality Theorems, Academic Press, Inc. 1986.

[K] Khare, C., On isomorphisms between deformation rings and Hecke rings, preprint.
[R1] Ramakrishna, R., *Infinitely ramified representations*, Annals of Mathematics 151 no. 2 (2000), 793–815.

[R2] Ramakrishna, R., *Deforming Galois Representations and the Conjectures of Serre and Fontaine-Mazur*, Annals of Mathematics 156 no. 1 (2002), 115–154.

[R3] Ramakrishna, R., *Lifting Galois Representations*, Inventiones Mathematicae 138 (1999) 537–562.

[R4] Ramakrishna, R., *Deforming an even representation*, Inventiones Mathematicae 132 (1998), 563–580.

[R5] Ramakrishna, R., *Deforming an even representation II, raising the level*, Journal of Number Theory, 72 (1998), 92-109.

[S] Serre, J.-P., Abelian and $l$-adic representations and elliptic curves, W. A. Benjamin Inc., 1968.

[T1] Taylor, R., *Remarks on a conjecture of Fontaine and Mazur*, Journal de l’Institut de Math. de Jussieu, 1 (2002). 1-19.

[T2] Taylor, R., *On icosahedral Artin representations II*, to appear in the American Journal of Mathematics.

[Wa] Washington, L., *Galois cohomology*, in [FLT].

[W] Wiles, A., *Modular elliptic curves and Fermat’s last theorem*, Ann. of Math. (2) 141 (1995), no. 3, 443–551.

*Addresses of the authors:*

CK: School of Mathematics, TIFR, Homi Bhabha Road, Mumbai 400 005, INDIA. e-mail: shekhar@math.tifr.res.in;
155 S 1400 E, Dept of Math, Univ of Utah, Salt Lake City, UT 84112, USA: e-mail: shekhar@math.utah.edu

RR: Department of Mathematics, Cornell University, Malott Hall, Ithaca, NY 14853, USA. e-mail: ravi@math.cornell.edu