A NOTE ON EQUIMULTIPLE DEFORMATIONS

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Abstract. While the tangent space to an equisingular family of curves can be described by the sections of a twisted ideal sheaf, this is no longer true if we only prescribe the multiplicity which a singular point should have. However, it is still possible to compute the dimension of the tangent space with the aid of the equimultiplicity ideal. In this note we consider families $\mathcal{L}_m = \{(C, p) \in |L| \times S \mid \text{mult}_p(C) = m\}$ for some linear system $|L|$ on a smooth projective surface $S$ and a fixed positive integer $m$, and we compute the dimension of the tangent space to $\mathcal{L}_m$ at a point $(C, p)$ depending on whether $p$ is a unitangential singular point of $C$ or not. We deduce that the expected dimension of $\mathcal{L}_m$ at $(C, p)$ in any case is just $\dim |L| - \frac{m(m+1)}{2} + 2$. The result is used in the study of triple-point defective surfaces in [ChM06a] and [ChM06b].

The paper is based on considerations about the Hilbert scheme of curves in a projective surface (see e.g. [Mum66], Lecture 22) and about local equimultiple deformations of plane curves (see [Wah74]).

Definition 1

Let $T$ be a complex space. An embedded family of curves in $S$ with section over $T$ is a commutative diagram of morphisms

$$
\begin{array}{ccc}
C & \xrightarrow{\varphi} & T \times S \\
\sigma \downarrow & & \downarrow \\
T & \xrightarrow{\varphi} & \\
\end{array}
$$

where $\text{codim}_{T \times S}(C) = 1$, $\varphi$ is flat and proper, and $\sigma$ is a section, i.e. $\varphi \circ \sigma = \text{id}_T$. Thus we have a morphism $\mathcal{O}_T \to \varphi_* \mathcal{O}_C = \varphi_*(\mathcal{O}_{T \times S}/\mathcal{J}_C)$ such that $\varphi_* \mathcal{O}_C$ is a flat $\mathcal{O}_T$-module.
The family is said to be **equimultiple of multiplicity** $m$ **along the section** $\sigma$ if the ideal sheaf $\mathcal{J}_C$ of $C$ in $\mathcal{O}_{T \times S}$ satisfies

$$\mathcal{J}_C \subseteq \mathcal{J}^m_{\sigma(T)} \quad \text{and} \quad \mathcal{J}_C \not\subseteq \mathcal{J}^{m+1}_{\sigma(T)},$$

where $\mathcal{J}_{\sigma(T)}$ is the ideal sheaf of $\sigma(T)$ in $\mathcal{O}_{T \times S}$.

**Remark 2**

Note that the above notion commutes with base change, i.e. if we have an equimultiple embedded family of curves in $S$ over $T$ as above and if $\alpha : T' \rightarrow T$ is a morphism, then the fibre product diagram

$$\begin{array}{ccc}
T' \times S & \longrightarrow & T \times S \\
\downarrow \phi' & & \downarrow \phi \\
C' & \longrightarrow & C \\
\downarrow \sigma' & & \downarrow \sigma \\
T' & \longrightarrow & T
\end{array}$$

gives rise to an embedded equimultiple family of curves over $T'$ of the same multiplicity, since locally it is defined via the tensor product.

**Example 3**

Let us denote by $T_\varepsilon = \text{Spec}(\mathbb{C}[\varepsilon])$ with $\varepsilon^2 = 0$. Then a family of curves in $S$ over $T_\varepsilon$ is just a Cartier divisor of $T_\varepsilon \times S$, that is, it is given on a suitable open covering $S = \bigcup_{\lambda \in \Lambda} U_\lambda$ by equations

$$f_\lambda + \varepsilon \cdot g_\lambda \in \mathbb{C}[\varepsilon] \otimes_{\mathbb{C}} \Gamma(U_\lambda, \mathcal{O}_S) = \Gamma(U_\lambda, \mathcal{O}_{T \times S}),$$

which glue together to give a global section $\{f_\lambda\}_{\lambda \in \Lambda}$ in $H^0(C, \mathcal{O}_C(C))$, where $C$ is the curve defined locally by the $f_\lambda$ (see e.g. [Mum66], Lecture 22).

A section of the family through $p$ is locally in $p$ given as $(x, y) \mapsto (x_\alpha, y_\beta) = (x + \varepsilon \cdot a, y + \varepsilon \cdot b)$ for some $a, b \in \mathbb{C}[x, y] = \mathcal{O}_{S, p}$.

**Example 4**

Let $H$ be a connected component of the Hilbert scheme $\text{Hilb}_S$ of curves in $S$, then $H$ comes with a universal family

$$\pi : \mathcal{H} \longrightarrow H : (C, p) \mapsto C. \quad (1)$$
Let us now fix a positive integer $m$ and set

$$
\mathcal{H}_m = \{(C, p) \in H \times S \mid C \in H, \text{mult}_p(C) = m\}.
$$

Then $\mathcal{H}_m$ is a locally closed subvariety of $H \times S$, and (1) induces via base change a flat and proper family $\mathcal{F}_m = \{(C, q) \in \mathcal{H}_m \times S \mid C_p = (C, p) \in \mathcal{H}_m, q \in C\}$ which has a distinguished section $\sigma$

\[
\xymatrix{ \mathcal{F}_m \ar[r] & \mathcal{H}_m \times S \\
\mathcal{H}_m \ar[u] \ar[ru] & }
\]

sending $C_p = (C, p)$ to $(C_p, p) \in \mathcal{F}_m$. Moreover, this family is equimultiple along $\sigma$ of multiplicity $m$ by construction.

**Example 5**

Similarly, if $|L|$ is a linear system on $S$, then it induces a universal family

$$
\pi : \mathcal{L} = \{(C, p) \in |L| \times S \mid p \in C\} \longrightarrow |L| : (C, p) \mapsto C.
$$

If we now fix a positive integer $m$ and set

$$
\mathcal{L}_m = \{(C, p) \in |L| \times S \mid C \in |L|, \text{mult}_p(C) = m\}.
$$

Then $\mathcal{L}_m$ is a locally closed subvariety of $|L| \times S$, and (3) induces via base change a flat and proper family $\mathcal{G}_m = \{(C, q) \in \mathcal{L}_m \times S \mid C_p = (C, p) \in \mathcal{L}_m, q \in C\}$ which has a distinguished section $\sigma$

\[
\xymatrix{ \mathcal{G}_m \ar[r] & \mathcal{L}_m \times S \\
\mathcal{L}_m \ar[u] \ar[ru] & }
\]

sending $C_p = (C, p)$ to $(C_p, p) \in \mathcal{G}_m$. Moreover, this family is equimultiple along $\sigma$ of multiplicity $m$ by construction.

We may interpret $\mathcal{L}_m$ as the family of curves in $|L|$ with $m$-fold points together with a section which distinguishes the $m$-fold point. This is important if the $m$-fold point is not isolated or if it splits in a neighbourhood into several simpler $m$-fold points.

Of course, since (3) can be viewed as a subfamily of (1) we may view (4) in the same way as a subfamily of (2).
Definition 6
Let \( t_0 \in T \) be a pointed complex space, \( C \subset S \) a curve, and \( p \in C \) a point of multiplicity \( m \). Then an embedded (equimultiple) deformation of \( C \) in \( S \) over \( t_0 \in T \) with section \( \sigma \) through \( p \) is a commutative diagram of morphisms

where the right hand part of the diagram is an embedded (equimultiple) family of curves in \( S \) over \( T \) with section \( \sigma \). Sometimes we will simply write \((\varphi, \sigma)\) to denote a deformation as above.

Given two deformations, say \((\varphi, \sigma)\) and \((\varphi', \sigma')\), of \( C \) over \( t_0 \in T \) as above, a morphism of these deformations is a morphism \( \psi : C' \to C \) which makes the obvious diagram commute:

This gives rise to the deformation functor

\[ \text{Def}_{\varphi \in C/S}^{\text{sec,em}} : \text{(pointed complex spaces)} \to \text{(sets)} \]

of embedded equimultiple deformations of \( C \) with section through \( p \) from the category of pointed complex spaces into the category of sets,
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where for a pointed complex space \( t_0 \in T \)

\[
\text{Def}_{p \in C/S}^{sec, em}(t_0 \in T) = \{ \text{isomorphism classes of embedded equimultiple deformations } (\varphi, \sigma) \text{ of } C \text{ in } S \text{ over } t_0 \in T \text{ with section through } p \}.
\]

Moreover, forgetting the section we have a natural transformation

\[
\text{Def}_{p \in C/S}^{sec, em} \to \text{Def}_{C/S},
\]

where the latter is the deformation functor

\[
\text{Def}_{C/S} : (\text{pointed complex spaces}) \to (\text{sets})
\]

of embedded deformations of \( C \) in \( S \) given by

\[
\text{Def}_{C/S}(t_0 \in T) = \{ \text{isomorphism classes of embedded deformations of } C \text{ in } S \text{ over } t_0 \in T \}.
\]

**Example 7**

According to Example 3 a deformation of \( C \) in \( S \) over \( T_\varepsilon \) along a section through \( p \) is given by

- local equations \( f + \varepsilon \cdot g \) such that \( f \) is a local equation for \( C \) and the \( \frac{\partial f}{\partial \varepsilon} \) glue to a global section of \( \mathcal{O}_C(C) \),
- together with a section which in local coordinates in \( p \) is given as

\[
\sigma : (x, y) \mapsto (x, y, a + \varepsilon \cdot b)
\]

for some \( a, b \in \mathbb{C} \{x, y\} \).

If we forget the section it is well known (see e.g. [Mum66], Lecture 22) that two such deformations are isomorphic if and only if they induce the same global section of \( \mathcal{O}_C(C) \) and this one-to-one correspondence is functorial so that we have an isomorphism of vector spaces

\[
\text{Def}_{C/S}(T_\varepsilon) \cong H^0(C, \mathcal{O}_C(C)).
\]

Considering the natural transformation from (5) we may now ask what the image of \( \text{Def}_{p \in C/S}^{sec, em}(T_\varepsilon) \) in \( H^0(C, \mathcal{O}_C(C)) \) is. These are, of course, the sections which allow a section \( \sigma \) through \( p \) along which the deformation is equimultiple, and according to Lemma 8 we thus have an epimorphism

\[
\text{Def}_{p \in C/S}^{sec, em}(T_\varepsilon) \twoheadrightarrow H^0(C, \mathcal{J}_{Z/C}(C)),
\]
where \( \mathcal{J}_{Z/C} \) is the restriction to \( C \) of the ideal sheaf \( \mathcal{J}_Z \) on \( S \) given by

\[
\mathcal{J}_{Z,q} = \begin{cases} 
\mathcal{O}_{S,q}, & \text{if } q \neq p, \\
\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle + \langle x, y \rangle^m, & \text{if } q = p,
\end{cases}
\]

(6)

here \( f \) is a local equation for \( C \) in local coordinates \( x \) and \( y \) in \( p \).

It remains the question what the dimension of the kernel of this map is, that is, how many different sections such an isomorphism class of embedded deformations of \( C \) in \( S \) over \( T_\varepsilon \) through \( p \) can admit.

J. Wahl showed in [Wah74], Proposition 1.9, that locally the equimultiple deformation admits a unique section if and only if \( C \) in \( p \) is not unitangential. If \( C \) is unitangential we may assume that locally in \( p \) it is given by \( f = y^m + \text{h.o.t.} \). If we have an embedded deformation of \( C \) in \( S \) which along some section is equimultiple of multiplicity \( m \), then locally it looks like

\[
f + \varepsilon \cdot \left( a \cdot \frac{\partial f}{\partial x} + b \cdot \frac{\partial f}{\partial y} + h \right)
\]

with \( h \in \langle x, y \rangle^m \). However, since \( \frac{\partial f}{\partial x} \in \langle x, y \rangle^m \) the deformation is equimultiple along the sections \( (x, y) \mapsto (x + \varepsilon \cdot (c + a), y + \varepsilon \cdot b) \) for all \( c \in \mathbb{C} \). Thus in this case the kernel turns out to be one-dimensional, i.e. there is a one-dimensional vector space \( \mathcal{K} \) such that the following sequence is exact:

\[
0 \rightarrow \mathcal{K} \rightarrow \text{Def}^{\text{sec,em}}_{T_\varepsilon}(C/S) \rightarrow H^0(C, \mathcal{J}_{Z/C}(C)) \rightarrow 0.
\]

(7)

**Lemma 8**

Let \( f + \varepsilon \cdot g \) be a first-order infinitesimal deformation of \( f \in \mathbb{C}\{x, y\} \), \( m = \text{ord}(f) \), \( a, b \in \mathbb{C}\{x, y\} \), and \( x_a = x + \varepsilon \cdot a, y_b = y + \varepsilon \cdot b \). Then \( f + \varepsilon \cdot g \) is equimultiple along the section \( (x, y) \mapsto (x_a, y_b) \) if and only if

\[
g - a \cdot \frac{\partial f}{\partial x} - b \cdot \frac{\partial f}{\partial y} \in \langle x, y \rangle^m.
\]

In particular, \( f + \varepsilon \cdot g \) is equimultiple along some section if and only if

\[
g \in \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) + \langle x, y \rangle^m.
\]
Proof: If \( a, b \in \mathbb{C}\{x, y\} \) and \( h \in \langle x, y \rangle^m \) then by Taylor expansion and since \( \varepsilon^2 = 0 \) we have
\[
f + \varepsilon \cdot \left( a \cdot \frac{\partial f}{\partial x} + b \cdot \frac{\partial f}{\partial y} + h \right) = f(x_a, y_b) + \varepsilon \cdot h(x_a, y_b),
\]
where \( f(x_a, y_b), h(x_a, y_b) \in \langle x, y \rangle^m \), i.e. the infinitesimal deformation
\[
f + \varepsilon \cdot \left( a \cdot \frac{\partial f}{\partial x}(x_a, y_b) + b \cdot \frac{\partial f}{\partial y}(x_a, y_b) + h \right)
\]
is equimultiple along \((x, y) \mapsto (x_a, y_b)\).

Conversely, if \( f + \varepsilon \cdot g \) is equimultiple along \((x, y) \mapsto (x_a, y_b)\) then
\[
f(x, y) + \varepsilon \cdot g(x, y) = F(x_a, y_b) + \varepsilon \cdot G(x_a, y_b)
\]
with \( F(x_a, y_b), G(x_a, y_b) \in \langle x, y \rangle^m \). Again, by Taylor expansion and since \( \varepsilon^2 = 0 \) we have
\[
f(x, y) = f(x_a, y_b) - \varepsilon \cdot \left( a \cdot \frac{\partial f}{\partial x}(x_a, y_b) + b \cdot \frac{\partial f}{\partial y}(x_a, y_b) \right)
\]
and
\[
\varepsilon \cdot g(x, y) = \varepsilon \cdot g(x_a, y_b).
\]
Thus
\[
F(x_a, y_b) = f(x_a, y_b)
\]
and
\[
\langle x, y \rangle^m \ni G(x_a, y_b) = g(x_a, y_b) - a \cdot \frac{\partial f}{\partial x}(x_a, y_b) - b \cdot \frac{\partial f}{\partial y}(x_a, y_b).
\]

\[\blacksquare\]

Example 9
If we fix a curve \( C \subset S \) and a point \( p \in C \) such that mult\(_p\)(\( C \)) = \( m \), i.e. if using the notation of Example 4 we fix a point \( C_p = (C, p) \in \mathcal{H}_m \), then the diagram
\[
\begin{array}{ccc}
S & \cong & \{C_p\} \times S \\
\uparrow & & \uparrow \sigma \\
C & \cong & \{(C_p, q) \mid q \in C\} \\
\downarrow & & \downarrow \sigma \\
t_0 & \rightarrow & C_p \\
\end{array}
\]
\[\text{(8)}\]
is an embedded equimultiple deformation of \( C \) in \( S \) along the section \( \sigma \) through \( p \). Moreover, any embedded equimultiple deformation of \( C \) in \( S \) with section through \( p \) as a family is up to isomorphism induced via
(1) in a unique way and thus factors obviously uniquely through (8).
This means that every equimultiple deformation of $C$ in $S$ through $p$
is induced up to isomorphism in a unique way from (8).
We now want to examine the tangent space to $\mathcal{H}_m$ at a point $C_p = (C, p)$, which is just

$$T_{C_p}(\mathcal{H}_m) = \text{Hom}_{\text{loc-}K-\text{Alg}}(\mathcal{O}_{\mathcal{H}_m, C_p}, C[\varepsilon]) = \text{Hom}(T_\varepsilon, (\mathcal{H}_m, C_p)),$$

where $(\mathcal{H}_m, C_p)$ denotes the germ of $\mathcal{H}_m$ at $C_p$. However, a morphism

$$\psi : T_\varepsilon \rightarrow (\mathcal{H}_m, C_p)$$
gives rise to a commutative fibre product diagram

$$\begin{array}{ccc}
T_\varepsilon \times_{\mathcal{H}_m} \mathcal{F}_m & \xrightarrow{\phi'} & \mathcal{F}_m \\
\sigma' \downarrow & & \downarrow \sigma \\
T_\varepsilon & \xrightarrow{\psi} & \mathcal{H}_m
\end{array}$$

sending the closed point of $T_\varepsilon$ to $C$. Thus $(\phi', \sigma') \in \text{Def}_{p \in C/S}^{\text{sec,em}}(T_\varepsilon)$ is an embedded equimultiple deformation of $C$ in $S$ with section through $p$. The universality of (8) then implies that up to isomorphism each one is of this form for a unique $\phi'$, and this construction is functorial. We thus have

$$T_{C_p}(\mathcal{H}_m) \cong \text{Def}_{p \in C/S}^{\text{sec,em}}(T_\varepsilon),$$

and hence (7) gives the exact sequence

$$0 \rightarrow \mathcal{K} \rightarrow T_{C_p}(\mathcal{H}_m) \rightarrow H^0(C, \mathcal{J}_{Z/C}(C)) \rightarrow 0.$$

In particular,

$$\dim_C (T_{C_p}(\mathcal{H}_m)) = \begin{cases} 
\dim_C H^0(C, \mathcal{J}_{Z/C}(C)) - 2, & \text{if } C \text{ is unitangential}, \\
\dim_C H^0(C, \mathcal{J}_{Z/C}(C)) - 1, & \text{else}.
\end{cases}$$

**Example 10**
If we do the same constructions replacing in (8) the family (2) by (4) we get for the tangent space to $\mathcal{L}_m$ at $C_p = (C, p)$ the diagram of exact
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sequences

\[
0 \rightarrow \mathcal{K} \rightarrow T_{C_p}(\mathcal{H}_m) \rightarrow H^0(C, \mathcal{J}_Z/C(C)) \rightarrow 0
\]

\[\begin{array}{c}
\uparrow \\
0 \rightarrow \mathcal{K} \rightarrow T_{C_p}(\mathcal{L}_m) \rightarrow H^0(S, \mathcal{J}_Z(C)) / H^0(\mathcal{O}_S) \rightarrow 0.
\end{array}\]

In order to see this consider the exact sequence

\[
0 \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_S(C) \rightarrow \mathcal{O}_C(C) \rightarrow 0
\]

induced from the structure sequence of \(C\). This sequence shows that the tangent space to \(|L|\) at \(C\) considered as a subspace of the tangent space \(H^0(C, \mathcal{O}_C(C))\) of \(H\) at \(C\) is just \(H^0(S, \mathcal{O}_S(C)) / H^0(S, \mathcal{O}_S)\) – that is, a global section of \(\mathcal{O}_C(C)\) gives rise to an embedded deformation of \(C\) in \(S\) which is actually a deformation in the linear system \(|L|\) if and only if it comes from a global section of \(\mathcal{O}_S(C)\), and the constant sections induce the trivial deformations. This construction carries over to the families (2) and (4).

In particular we get the following proposition.

**Proposition 11**

*Using the notation from above let \(C\) be a curve in the linear system \(|L|\) on \(S\) and suppose that \(p \in C\) such that \(\text{mult}_p(C) = m\).*

*Then the tangent space of \(\mathcal{L}_m\) at \(C_p = (C, p)\) satisfies*

\[
\dim_C(T_{C_p}(\mathcal{L}_m)) = \begin{cases} 
\dim_C H^0(S, \mathcal{J}_Z(C)) - 2, & \text{if } C \text{ is unitangential,} \\
\dim_C H^0(S, \mathcal{J}_Z(C)) - 1, & \text{else.}
\end{cases}
\]

*Moreover, the expected dimension of \(T_{C_p}(\mathcal{L}_m)\) and thus of \(\mathcal{L}_m\) at \(C_p\) is just*

\[
\expdim_{C_p}(\mathcal{L}_m) = \expdim_C(T_{C_p}(\mathcal{L}_m)) = \dim |L| - \left(\frac{m + 1}{2}\right)m + 2.
\]

*For the last statement on the expected dimension just consider the exact sequence*

\[
0 \rightarrow H^0(S, \mathcal{J}_Z(C)) \rightarrow H^0(S, \mathcal{O}_S(L)) \rightarrow H^0(S, \mathcal{O}_Z)
\]

*and note that the dimension of \(H^0(S, \mathcal{J}_Z(C))\), and hence of \(T_{C_p}(C)\), attains the minimal possible value if the last map is surjective.*
expected dimension of $H^0(S, \mathcal{J}_Z(C))$ hence is
\[
\expdim_C H^0(S, \mathcal{J}_Z(C)) = \dim |L| + 1 - \deg(Z),
\]
and it suffices to calculate $\deg(Z)$. If $C$ is unitangential we may assume that $C$ locally in $p$ is given by $f = y^m + h.o.t.$, so that
\[
\mathcal{O}_{Z,p} = C\{x, y\}/\langle y^{m-1}\rangle + \langle x, y\rangle^m,
\]
and hence $\deg(Z) = \frac{(m+1)m}{2} - 1$. If $C$ is not unitangential, then we may assume that it locally in $p$ is given by an equation $f$ such that $f_m = \jet_m(f) = x^\mu \cdot y^\nu \cdot g$, where $x$ and $y$ do not divide $g$, but $\mu$ and $\nu$ are at least one. Suppose now that the partial derivatives of $f_m$ are not linearly independent, then we may assume $\frac{\partial f_m}{\partial x} \equiv \alpha \cdot \frac{\partial f_m}{\partial y}$ and thus
\[
\mu yg \equiv \alpha \nu xg + \alpha xy \cdot \frac{\partial g}{\partial y} - xy \cdot \frac{\partial g}{\partial x},
\]
which would imply that $y$ divides $g$ in contradiction to our assumption. Thus the partial derivatives of $f_m$ are linearly independent, which shows that
\[
\deg(Z) = \dim_C \left(C\{x, y\} / \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle + \langle x, y \rangle^m\right) = \frac{(m+1)m}{2} - 2.
\]

**Example 12**

Let us consider the Example 5 in the case where $S = \mathbb{P}^2$ and $L = \mathcal{O}_{\mathbb{P}^2}(d)$. We will show that $\mathcal{L}_m$ is then smooth of the expected dimension. Note that $\pi(\mathcal{L}_m)$ will only be smooth at $C$ if $C$ has an ordinary $m$-fold point, that is, if all tangents are different.

Given $C_p = (C, p) \in \mathcal{L}_m$ we may pass to a suitable affine chart containing $p$ as origin and assume that $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$ is parametrised by polynomials
\[
F_\underline{a} = f + \sum_{i+j=0}^{d} a_{i,j} \cdot x^i y^j,
\]
where $f$ is the equation of $C$ in this chart. The closure of $\pi(\mathcal{L}_m)$ in $|L|$ locally at $C$ is then given by several equations, say $F_1, \ldots, F_k \in \mathbb{C}[a_{i,j}|i + j = 0, \ldots, d]$, in the coefficients $a_{i,j}$. We get these equations by eliminating the variables $x$ and $y$ from the ideal defined by
\[
\langle \frac{\partial^{i+j} F_{\underline{a}}}{\partial x^i y^j} | i + j = 0, \ldots, m - 1 \rangle.
\]
And $\mathcal{L}_m$ is locally in $C_p$ described by the equations

$$F_1 = 0, \ldots, F_k = 0, \quad \frac{\partial^{i+j} F_a}{\partial x^i y^j} = 0, \quad i + j = 0, \ldots, m - 1.$$ 

However, the Jacoby matrix of these equations with respect to the variables $x, y, a_{i,j}$ contains a diagonal submatrix of size $\frac{m(m+1)}{2}$ with ones on the diagonal, so that its rank is at least $\frac{m(m+1)}{2}$, which – taking into account that $|L| = \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)))$ – implies that the tangent space to $\mathcal{L}_m$ at $C_p$ has codimension at least $\frac{m(m+1)}{2} - 1$ in the tangent space of $\mathcal{L}$. By Proposition 11 we thus have

$$\dim_{C_p}(\mathcal{L}_m) \leq \dim_{C_T} C_p(\mathcal{L}_m) \leq \dim_{C_T} C_p(\mathcal{L}) - \frac{m \cdot (m + 1)}{2} + 1$$

$$= \dim(\mathcal{L}) - \frac{m \cdot (m + 1)}{2} + 1$$

$$= \dim |L| - \frac{m \cdot (m + 1)}{2} + 2$$

$$= \expdim_{C_p}(\mathcal{L}_m) \leq \dim_{C_p}(\mathcal{L}_m),$$

which shows that $\mathcal{L}_m$ is smooth at $C_p$ of the expected dimension.

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