Quasi-stationary binary inspiral. I. Einstein equations for the two Killing vector spacetime

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Abstract

The inspiral of a binary system of compact objects due to gravitational radiation is investigated using the toy model of two infinitely long lines of mass moving in a fixed circular orbit. The two Killing fields in the toy model are used, according to a formalism introduced by Geroch, to describe the geometry entirely in terms of a set of tensor fields on the two-manifold of Killing vector orbits. Geroch’s derivation of the Einstein equations in this formalism is streamlined and generalized. The explicit Einstein equations for the toy model spacetime are derived in terms of the degrees of freedom which remain after a particular choice of gauge.

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I. INTRODUCTION

A pair of compact objects (black holes or neutron stars) in binary orbit about one another is stable in Newtonian gravity. In general relativity, however, the system will emit gravitational radiation, causing the bodies to spiral in towards one another. The early stages of this process, where the gravitational interaction is weak everywhere, can be treated with the post-Newtonian approximation, while the final merger of the objects can be modeled using supercomputers. In order to determine the waveform in an intermediate phase, where the rate of energy loss due to gravitational radiation is low, but other strong-field effects may be important, and also to provide accurate initial data for supercomputer calculations, it is useful to employ an approximation scheme based on the fact that the orbits are decaying only slowly. Over some range of time, the physical spacetime should be approximated by a spacetime in which the orbits do not decay. For elliptical orbits this spacetime will be periodic, with the period equal to the orbital period of the objects. If the orbits are circular, this discrete symmetry becomes a continuous symmetry and the spacetime is stationary. Finding this spacetime is thus an essentially three-dimensional problem, rather than a four-dimensional one.

Solving this three-dimensional problem is still numerically intensive, however, so in order to examine the consequences of this approximation scheme in a computationally simpler environment, we consider initially a toy model which exhibits an additional translational symmetry perpendicular to the orbital plane. The desired spacetime then has two Killing vectors, and we have only a two-dimensional problem.

This, the first in a series of papers on this project, is concerned with formulating the Einstein equations in the presence of these two Killing symmetries. Paper II examines, in the simpler model of a non-linear scalar field in 2+1 dimensions, the details of the radiation-balanced boundary conditions which must be imposed “at infinity” to specify a stationary, energy-conserving solution to a radiative system. Subsequent papers will apply this method of radiation balance to the gravitational fields of co-orbiting lines of mass and ultimately to localized sources, thus removing one at a time the simplifications of scalar field theory and translational invariance.

In this paper, we describe a toy model for binary inspiral that has such a two-dimensional symmetry group. The model problem consists of two infinitely long lines of mass (e.g., cosmic strings) orbiting one another at fixed angular velocity. Using a formalism introduced by Geroch, we describe the geometry of the toy model spacetime in terms of a set of tensor fields on the two-dimensional manifold of Killing vector orbits. We also derive explicit expressions for the Einstein equations for the spacetime in terms of the degrees of freedom which remain after a particular choice of gauge. Future work will solve these equations numerically for a reasonable set of boundary conditions.

The plan of this paper is as follows:

In Sec. II, we begin our analysis by describing Minkowski spacetime in a “co-rotating”

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1To maintain this equilibrium, the energy lost in gravitational waves is presumably balanced by gravitational radiation coming in from infinity. [1].
coordinate system corresponding to two co-orbiting cosmic strings. We show that, even for this simple spacetime, the two Killing vector fields (KVFs) defined by the strings do not select a preferred coordinate system consisting of two Killing coordinates and two coordinates on an orthogonal subspace.

In Sec. III, we describe a formalism which can be used to simplify the discussion of spacetimes with two KVFs, even in the absence of a system of mutually orthogonal Killing and non-Killing coordinates. The formalism was initially developed by Geroch [3], and we generalize his derivation of the vacuum Einstein equations by deriving expressions for different projections of the Einstein tensor. We also show how the various components of the Einstein tensor are related by the contracted Bianchi identities, and we show explicitly how to recover the four-geometry from Geroch’s more specialized objects. The discussion in Sec. III applies to any spacetime with two commuting Killing vector fields; it is not specialized to the co-rotating cosmic string spacetime considered in the rest of the paper.

In Sec. IV, we return our focus to the co-rotating cosmic string spacetime by discussing the gauge choices available within Geroch’s formalism. We describe some desirable gauge-fixings, and we enumerate the independent degrees of freedom which remain.

In Sec. V, we derive explicit expressions for the components of the Einstein tensor in terms of the independent functions needed to describe the geometry. These equations, when supplemented by a description of the stress-energy of the cosmic strings and a set of boundary conditions [4], set the stage for a numerical solution of the Einstein equations, which will be performed in the future.

Finally, in Sec. VI, we summarize the results of our paper and discuss how they will be used as the starting point for future work.

The first two appendices contain proofs of formulas used in Sec. III: Appendix A contains a proof of the expression (3.20) for the covariant derivative of a Killing vector field, and Appendix B contains a detailed derivation of the projected components of the Ricci tensor, which we state in Sec. III. The third and fourth appendices consider ancillary subjects: Appendix C describes how to obtain from the quantities defined by Geroch the components of the four-metric in a coordinate basis rather than the non-coordinate one used in the text, and recovers the normal form given by Petrov [5] as a consequence of a particular coordinate choice. Finally, Appendix D demonstrates the correspondence between the formulas contained in Sec. III and those given by Geroch in Appendix A of [3], the latter being a special case of the former.

Note: Throughout this paper, we will follow the sign conventions of [6]. Abstract indices are denoted by lower case Latin letters $a, b, \ldots$ from the beginning of the alphabet, while spacetime coordinate indices are denoted by lower case Greek letters $\mu, \nu, \ldots$. The Killing vectors are labeled by upper case Latin letters $A, B, \ldots$, and the two-dimensional coordinate indices on the space of orbits of the Killing vectors by lower case Latin letters $i, j, \ldots$ from the middle of the alphabet.
II. MINKOWSKI SPACETIME IN CO-ROTATING COORDINATES

As mentioned in the previous section, we wish to describe a spacetime which has two infinitely long cosmic strings orbiting one another at a fixed angular velocity $\Omega$. In a numerical determination of the spacetime geometry, one seeks to fix the coordinate (i.e., gauge) information completely, and thus calculate the minimum number of quantities necessary to define the geometry. It is desirable, of course, to choose a gauge which takes advantage of the symmetries of the problem. In this case, those symmetries are described by two KVFs. One of these, $K^a_1$, corresponds to the translational invariance along the strings, while the other, $K^a_0$, tells us that the spacetime is unchanged if we move forward in time while rotating about the axis by a proportional amount. The desired coordinate system would seem to consist of Killing coordinates $x^0$ and $x^1$, supplemented by coordinates $x^2$ and $x^3$ on an orthogonal subspace. (Indeed, this is what is done in the case of stationary, axisymmetric spacetimes, which also admit two commuting Killing vectors.) However, in the case at hand, the “co-rotational” Killing vector $K^a_0$ is not surface-forming, and there is no subspace orthogonal to the Killing vectors. This is illustrated by describing Minkowski spacetime in coordinates tailored to the symmetries exhibited by co-rotating cosmic strings.

The Minkowski metric $g_{ab}$, written in standard cylindrical polar coordinates $(t, z, \rho, \phi)$, gives rise to the line element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + dz^2 + d\rho^2 + \rho^2 d\phi^2.$$  

We can work in a reference frame which rotates with a fixed angular velocity $\Omega$ by defining a “co-rotating” angle

$$\varphi := \phi - \Omega t$$

and transforming to coordinates $(t, z, \rho, \varphi)$. In these coordinates the line element takes the form

$$ds^2 = -(1 - \Omega^2 \rho^2)dt^2 + dz^2 + d\rho^2 + \rho^2(d\varphi + \Omega dt)^2,$$  

which can be expanded to yield

$$ds^2 = -(1 - \Omega^2 \rho^2)dt^2 + dz^2 + d\rho^2 + \rho^2 d\varphi^2 + 2\Omega \rho^2 d\varphi dt.$$  

If we limit consideration of the symmetries of the spacetime to those described by the two commuting KVFs

$$K^a_0 := \left(\frac{\partial}{\partial t}\right)_\varphi \equiv \left(\frac{\partial}{\partial t}\right)_\phi + \Omega \frac{\partial}{\partial \phi};$$

$$K^a_1 := \frac{\partial}{\partial z},$$

These are cosmic strings with an actual curvature singularity, and not only a deficit angle. Two conical singularities surrounded by flat spacetime could scatter gravitationally, but could not orbit one another.
we see that \( t \) and \( z \) are the corresponding Killing coordinates for the metric written in the form (2.4). The presence of a \( d\varphi \ dt \) term in Eq. (2.4) means that the coordinate pairs \( (t,z) \) and \( (\rho,\varphi) \) have not split the spacetime into orthogonal subspaces. In fact it is impossible to base such a split on these two Killing vectors, for while \( K^a_0 \) is clearly surface forming, \( K^a_1 \) is not, as calculation of

\[
\epsilon^{abcd} K_{0b} \nabla_c K_{0d}
\]

(2.6) clearly shows. The other Killing vector might save us, if Eq. (2.6) had a vanishing projection along \( K^a_1 \), but since

\[
c_0 := \epsilon^{abcd} K_{0a} K_{1b} \nabla_c K_{0d} = 2\Omega \neq 0,
\]

(2.7) the group of symmetries is not orthogonally transitive and the two-dimensional subspaces of the tangent space at each point orthogonal to \( K^a_0 \) and \( K^a_1 \) are not integrable.

If the symmetry group were orthogonally transitive, we could define a coordinate system made of two Killing coordinates \( \{x^A\mid A = 0, 1\} \) and two coordinates \( \{x^i\mid i = 2, 3\} \) on an orthogonal subspace. In that case, the metric would be block diagonal and defined by two \( 2 \times 2 \) symmetric matrices: (i) the matrix of inner products

\[
\lambda_{AB} := g_{ab} K^a_A K^b_B
\]

(2.8) and (ii) the \( ij \)-components of the projection tensor

\[
\gamma_{ab} := g_{ab} - \lambda^{AB} K^a_A K^b_B.
\]

(2.9)

Examination of the metric (2.4) shows that the matrix of components \( \{g_{\mu\nu}\} \) [with respect to the co-rotating coordinates \( \{x^\mu\} := \{x^A, x^i\} = \{t, z; \rho, \varphi\} \) is not block-diagonal, so that

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu \neq \lambda_{AB} dx^A dx^B + \gamma_{ij} dx^i dx^j.
\]

(2.10)

However, the quantities \( \{\lambda_{AB}\} \) and \( \gamma_{ij} \) are still useful in the construction in Sec. III, so we will examine their form for co-rotating flat spacetime to keep them in mind as an example.

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3 A division based on the coordinate pairs \( (t,\varphi) \) and \( (\rho,z) \) does split the spacetime into orthogonal subspaces. However, although \( \varphi \) is a Killing coordinate in Minkowski space, it will not be in the cosmic string spacetime with which we are concerned, and so such a coordinate splitting is not of interest to us.

4 In the language of differential forms, \( K_0 = -dt + \Omega \rho^2 d\varphi \) and \( K_0 \wedge dK_0 = -2\Omega \rho \ dt \wedge d\rho \wedge d\varphi \). Equivalently, \(* (K_0 \wedge dK_0) = 2\Omega dz\), where * denotes the duality operator (see, e.g., p. 88 of [3]).

5 Again, working with differential forms, \( K_1 = dz \) and \( K_0 \wedge K_1 \wedge dK_0 = -2\Omega \rho \ dt \wedge dz \wedge d\rho \wedge d\varphi \). Equivalently, \(* (K_0 \wedge K_1 \wedge dK_0) = c_0 = 2\Omega\).

6 \( \{\lambda^{AB}\} \) is the matrix inverse of \( \{\lambda_{AB}\} \).
The matrix of inner products is

\[
\{\lambda_{AB}\} = \begin{pmatrix} -1 + \Omega^2 \rho^2 & 0 \\ 0 & 1 \end{pmatrix}.
\]

(2.11)

The determinant

\[
\lambda := \det\{\lambda_{AB}\} = -(1 - \Omega^2 \rho^2)
\]

(2.12)

is less than zero for \(\rho < 1/\Omega\), greater than zero for \(\rho > 1/\Omega\), and equal to zero for \(\rho = 1/\Omega\).

The surface on which \(K^a_0 = (\partial/\partial t)_\varphi\) is null, defined by \(\rho = 1/\Omega\), is known as the “light cylinder.”

For \(\rho < 1/\Omega\), \(K^a_0\) is timelike, while for \(\rho > 1/\Omega\), \(K^a_0\) is spacelike.

In terms of \(\{x^\mu\} = (t,z,\rho,\varphi)\), the projection tensor \(\gamma_{ab}\) has components

\[
\{\gamma_{\mu\nu}\} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \rho^2(1 - \Omega^2 \rho^2)^{-1} & 0 \end{pmatrix}.
\]

(2.13)

As we shall describe in Sec. III, \(\gamma_{ab}\) can be thought of as a metric on the space \(S\) of Killing vector orbits, whose line element [in terms of the coordinates \(\{x^i\} := (\rho,\varphi)\)] is

\[
d\Sigma^2 := \gamma_{ij} dx^i dx^j = d\rho^2 + \rho^2(1 - \Omega^2 \rho^2)^{-1} d\varphi^2.
\]

(2.14)

Note that this metric has signature \((++\) for \(\rho < 1/\Omega\), \((+-\) for \(\rho > 1/\Omega\), and is degenerate for \(\rho = 1/\Omega\). The light cylinder \(\rho = 1/\Omega\) can thus be thought of as a “signature change surface” in \(S\). The determinant

\[
\gamma := \det\{\gamma_{ij}\} = \rho^2(1 - \Omega^2 \rho^2)^{-1}
\]

(2.15)

diverges when \(\rho = 1/\Omega\), which is exactly when the matrix \(\{\lambda_{AB}\}\) becomes non-invertible.

### III. SPACETIMES WITH TWO COMMUTING KILLING VECTOR FIELDS

In this section, we describe a general formalism (originally developed by Geroch [3]) that can be used to simplify the discussion of spacetimes admitting two commuting KVFs, even in the absence of a system of mutually orthogonal Killing and non-Killing coordinates. We present a new derivation of the projected form of the Einstein equations (and the Bianchi identities which relate various components of the Einstein tensor), and we show how to reconstruct the original four-geometry, given only the values of certain tensor fields on the two-dimensional space of Killing vector orbits. The analysis that we give in this section is completely general. In particular, we do not restrict attention to the case of the two co-rotating cosmic string spacetime, which we consider in the rest of the paper.

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7This is because an object which sat at \(\rho = 1/\Omega\), with constant \(z\) and \(\varphi\), would be moving at the speed of light.
A. Preliminaries

Let \((\mathcal{M}, g_{ab})\) be a four-dimensional manifold \(\mathcal{M}\) with Lorentzian metric \(g_{ab}\), which admits two commuting Killing vector fields \(K^a_A\) \((A = 0, 1)\). Killing's equation \(\mathcal{L}_{K_A} g_{ab} = 0\) is equivalent to

\[
\nabla_a K^b_a = -\nabla_b K^a_a ,
\]

while commutivity of the vector fields \([K_A, K_B]^a = 0\) is equivalent to

\[
K^b_A \nabla_b K^a_B = K^b_B \nabla_b K^a_A .
\]

In addition,

\[
R^a_{bcd} K^b_A = \nabla_b \nabla_c K^d_A ,
\]

which is valid for any Killing vector. Since we will not assume that the KVFs are orthogonally transitive (i.e., that the two-dimensional subspaces orthogonal to \(K^0_A\) and \(K^1_A\) are integrable), one or both of the quantities

\[
c_A := \epsilon_{abcd} K^a_0 K^b_1 \nabla^c K^d_A
\]

can be non-zero somewhere in \(\mathcal{M}\).

Given \(g_{ab}\) and \(K^a_A\), we can construct the symmetric matrix of inner products

\[
\lambda_{AB} := g_{ab} K^a_A K^b_B .
\]

If the determinant

\[
\lambda := \text{det} \{\lambda_{AB}\}
\]

is non-zero, then we can further define a projection tensor

\[
\gamma_{ab} := g_{ab} - \lambda^{AB} K_{Aa} K_{Bb} ,
\]

where \(\{\lambda^{AB}\}\) denotes the inverse matrix to \(\{\lambda_{AB}\}\). \(\gamma_{ab}\) is orthogonal to the KVFs, and it can be interpreted as a metric on the two-dimensional space \(\mathcal{S}\) of Killing vector orbits. In fact, as shown by Geroch \[3\], any tensor field \(T^{a_1\cdots a_n}_{b_1\cdots b_m}\) on \(\mathcal{M}\) that: (i) is orthogonal to the KVFs

\[
K_{Aa_1} T^{a_1\cdots a_n}_{b_1\cdots b_m} = 0 , \ldots , K_{Aa_n} T^{a_1\cdots a_n}_{b_1\cdots b_m} = 0 ,
K^b_{A} T^{a_1\cdots a_n}_{b_1\cdots b_m} = 0 , \ldots , K^b_{A} T^{a_1\cdots a_n}_{b_1\cdots b_m} = 0 ,
\]

and (ii) has vanishing Lie derivatives

\[
\mathcal{L}_{K_A} T^{a_1\cdots a_n}_{b_1\cdots b_m} = 0
\]

can be thought of as a tensor field on \(\mathcal{S}\). In particular, since
\begin{equation}
\mathcal{L}_{K^c} \lambda_{AB} = 0 \quad \text{and} \quad \mathcal{L}_{K^c} c_A = 0 ,
\end{equation}

\(\lambda_{AB}\) and \(c_A\) are scalar fields on \(S\).

The metric-compatible covariant derivative operator \(D_a\) on \(S\) is given by

\begin{equation}
D_a T^{b_1 \ldots b_n}_{\ldots e_m} := \gamma^d_{\ldots e_1} \ldots \gamma^b_{\ldots e_n} \gamma^f_1 \ldots \gamma^f_m \nabla_d T^{e_1 \ldots e_n}_{f_1 \ldots f_m} ,
\end{equation}

where \(T^{a_1 \ldots a_n}_{b_1 \ldots b_m}\) is any tensor field on \(M\) satisfying Eqs. (3.8) and (3.9), and the two-dimensional Levi-Civita tensor \(\epsilon_{ab}\) can be written as

\begin{equation}
\epsilon_{ab} = |\lambda|^{-1/2} \epsilon_{abcd} K^c_0 K^d_1 .
\end{equation}

If we define \(\epsilon\) in terms of \(\lambda\) and its absolute value via

\begin{equation}
\lambda = \epsilon |\lambda| ,
\end{equation}

(so that \(\epsilon = 1\) corresponds to two spacelike KVFs, and \(\epsilon = -1\) to one spacelike and one timelike KVF), then

\begin{equation}
\epsilon^{ab} \epsilon_{ac} = -\epsilon \delta^b_c .
\end{equation}

Moreover, if we define a Levi-Civita symbol \(\epsilon^{AB}\) so that \(\epsilon^{01} = \epsilon^{-1} |\lambda|^{-1/2}\), then Eqs. (3.14) and (3.12) can be rewritten as

\begin{equation}
c_A = \frac{1}{2} \epsilon |\lambda|^{1/2} \epsilon_{abcd} \epsilon^{CD} K^c_C K^b_D \nabla^e K^d_A
\end{equation}

and

\begin{equation}
\epsilon_{ab} = \frac{1}{2} \epsilon \epsilon_{abcd} \epsilon^{AB} K^c_A K^d_B ,
\end{equation}

respectively, which do not explicitly involve the indices 0 and 1. The presence of \(|\lambda|^{1/2}\) in Eq. (3.15) implies that \(c_A\) transforms as a covariant vector density of weight +1, rather than as a covariant vector, with respect to the index \(A\). Similarly, the absence of any \(\lambda\) factors in Eq. (3.16) implies that \(\epsilon_{ab}\) carries no density weight.

**B. Projected Ricci tensor**

Given the definitions of the previous subsection, we are now ready to calculate the projected components of the four-dimensional Ricci tensor \(R_{bd} := R^c_{bcde}\). This is the first (and most involved) step leading to the projected form of the Einstein equations \(G_{ab} = 8\pi T_{ab}\). Since

\begin{equation}
G_{ab} := R_{ab} - \frac{1}{2} \mathfrak{R} g_{ab} ,
\end{equation}

it follows that
\[ R_{ab} = 8\pi \left( T_{ab} - \frac{1}{2} T g_{ab} \right), \]  

where \( T := T_{ab}g^{ab} \) denotes the trace of the stress-energy tensor. Thus, knowing the projections

\[ R_{AB} := K^c_A K^d_B R_{cd} \]  
\[ \widehat{R}_{ab} := \gamma^e_a \gamma^d_b R_{cd} \]  

of the Ricci tensor is equivalent to knowing the projections of the left-hand side of the Einstein equations.

In [3], Geroch derived the projected form of the Einstein equations for vacuum spacetimes admitting one timelike and one spacelike Killing vector field. In this paper, we extend Geroch’s derivation in the following ways:

(i) we consider non-vacuum spacetimes by allowing a non-zero stress-energy tensor \( T_{ab} \);
(ii) we allow the KVFs to have either “signature”—i.e., they can both be spacelike \((\epsilon = 1)\), or one can be spacelike and the other timelike \((\epsilon = -1)\);
(iii) we take advantage of the index notation to treat both KVFs simultaneously.

In addition, our derivation is somewhat simpler than Geroch’s in the sense that, instead of introducing the symmetric matrix of twist vectors \( \omega_{AB} \) [(A8) of [3]] and their projections \( \nu_{AB} \) [(A10) of [3]], we make repeated use of the expression

\[ \nabla_a K_{ab} = -\frac{1}{2} \epsilon^{-1} |\lambda|^{-1/2} \epsilon_{ab} c_A - \lambda^{BC} K_{B[a} D_{b]} \lambda_{CA} \]  

for the covariant derivative of the KVFs. [The use of Eq. (3.20) greatly simplifies calculations involving one or two derivatives of the KVFs.] A proof of Eq. (3.20) can be found in Appendix A.

The projected components of the four-dimensional Ricci tensor are worked out in detail in Appendix B. The final results, which we simply state here, are

\[ R_{AB} = -\frac{1}{2} D^a D_a \lambda_{AB} + \frac{1}{4} (\lambda^{-1} D^a \lambda) D_a \lambda_{AB} \]  
\[ -\frac{1}{4} \lambda_{AB} \lambda^{-1} (D^a \lambda \lambda^{CD}) D_a \lambda_{CD} - \frac{1}{2} \lambda^{-1} c_{ACB}; \]  

\[ \widehat{R}_{ab} = -\frac{1}{2} \epsilon^{-1} |\lambda|^{-1/2} \epsilon_{bc} D^c c_A; \]  

\[ \widehat{R}_{ab} = -\frac{1}{2} \lambda^{-1} D_a D_b \lambda + \frac{1}{4} \lambda^{-1} (D_a \lambda \lambda^{AB}) (D_b \lambda_{AB}) \]  
\[ + \frac{1}{4} \lambda^{-2} (D_a \lambda) (D_b \lambda) + \frac{1}{2} \gamma_{ab} (\mathcal{R} + \lambda^{-1} \lambda^{AB} c_A c_B). \]  

These quantities are considered in Appendix D for the sake of identifying out results with Geroch’s.
C. Trace and trace-free parts of the projected Ricci tensor

For reasons which shall become clearer in Sec. V, it is convenient to split the projected components \( R_{AB} \) and \( \hat{R}_{ab} \) of the Ricci tensor into their trace and trace-free parts with respect to \( \{\lambda^{AB}\} \), \( \gamma^{cd} \), and the projection operators

\[
P_{AB}^{CD} := \delta^C_A \delta^D_B - \frac{1}{2} \lambda_{AB} \lambda^{CD}, \quad (3.22a)
\]

\[
P_{ab}^{cd} := \gamma^c(a) \gamma^d(b) - \frac{1}{2} \gamma_{ab} \gamma^{cd}. \quad (3.22b)
\]

The projection operators satisfy

\[
P_{AB}^{CD} P_{EF}^{CD} = P_{AB}^{EF}, \quad (3.23a)
\]

\[
\lambda^{AB} P_{AB}^{CD} = \gamma^{ab} P_{ab}^{cd} = \gamma^{cd} P_{ab}^{cd} = P_{ab}^{cd} = 0, \quad (3.23b)
\]

\[
P_{ab}^{cd} K_a^e = P_{ab}^{cd} K_b^e = P_{ab}^{cd} K_{Ac} = P_{ab}^{cd} K_{Ad} = 0. \quad (3.23c)
\]

Using these results together with Eqs. (3.21) and (B29), it immediately follows that

\[
P_{AB}^{CD} R_{CD} = P_{AB}^{CD} \left[ -\frac{1}{2} D^a D_a \lambda_{CD} + \frac{1}{4} (\lambda^{-1} D^a \lambda) D_a \lambda_{CD} - \frac{1}{2} \lambda^{-1} \gamma_{CD} \right]; \quad (3.24a)
\]

\[
\lambda^{AB} R_{AB} = -\frac{1}{2} \lambda^{-1} D^a D_a \lambda + \frac{1}{4} (\lambda^{-1} D^a \lambda) \lambda^{-1} D_a \lambda - \frac{1}{2} \lambda^{-1} \lambda^{AB} \gamma_{CD}; \quad (3.24b)
\]

\[
P_{ab}^{cd} \hat{R}_{cd} = \frac{1}{2} \lambda^{-1} D_c D_d \lambda + \frac{1}{4} (\lambda^{-1} D^a \lambda) (D_d \lambda)^{AB} + \frac{1}{4} \lambda^{-2} (D_c \lambda) (D_d \lambda); \quad (3.24c)
\]

\[
\gamma^{ab} \hat{R}_{ab} = -\frac{1}{2} \lambda^{-1} D^a D_a \lambda + \frac{1}{4} \lambda^{-1} (D^a \lambda) (D_a \lambda)^{AB} + \frac{1}{4} \lambda^{-2} (D^a \lambda) (D_a \lambda) + \mathcal{R} + \lambda^{-1} \lambda^{AB} \gamma_{CD}. \quad (3.24d)
\]

D. Contracted Bianchi identities

Of course, all ten Einstein equations implied by Eq. (3.21) are not independent; they are related by the four contracted Bianchi identities

\[
\nabla^b G_{ab} = 0. \quad (3.25)
\]

In this section, we express the various projections of Eq. (3.25) in terms of the projections

\[
G_{AB} := K^c_A K^d_B G_{cd} \quad (3.26a)
\]

\[
\hat{G}_{Ab} := K^c_A \gamma^d_b G_{cd} \quad (3.26b)
\]

\[
\hat{G}_{ab} := \gamma^c_a \gamma^d_b G_{cd} \quad (3.26c)
\]

of the Einstein tensor. Since \( \mathcal{L}_{K_A} G_{ab} = 0 \), the symmetric matrix \( \{G_{AB}\} \) of scalar fields, the pair \( \{\hat{G}_{Ab}\} \) of covector fields, and the symmetric tensor field \( \hat{G}_{ab} \) all live on the two-manifold \( S \). This means, in particular, that
\[ K_A^a \hat{G}_{ab} = K_A^b \hat{G}_{ab} = K_A^b \hat{G}_{Ab} = 0 \]  
(3.27)

and, due to the vanishing of the Lie derivatives \( \mathcal{L}_{K_C} G_{AB} \) and \( \mathcal{L}_{K_B} \hat{G}_{Ab} \),

\[ K_C^c \nabla_c G_{AB} = 0; \quad (3.28a) \]

\[ K_C^c \nabla_c \hat{G}_{Ab} = -\hat{G}_{Ac} \nabla_b K_C^c. \quad (3.28b) \]

First, the projection of the contracted Bianchi identity along a Killing vector is given by

\[ 0 = K_A^a \nabla^b G_{ab} = \nabla^b (K_A^a G_{ab}) = \nabla^b (\hat{G}_{Ab} + \lambda^{BC} K_B^b G_{AC}) = \nabla^b \hat{G}_{Ab} = D^b \hat{G}_{Ab} + \hat{G}_{Ac} \nabla_b \gamma_c^b = D^b \hat{G}_{Ab} + \frac{1}{2} \hat{G}_{Ab} \lambda^{-1} D^b \lambda, \]  
(3.29)

where we used Eq. (3.28a), (B3), and the antisymmetry of \( \nabla^a K^b \) in \( a \) and \( b \).

Second, the projection orthogonal to both Killing vectors is

\[ 0 = \gamma_a^c \nabla^b G_{cb} = \gamma_a^c \nabla^b (\gamma_c^b G_{cb}) - G_{cb} \nabla^b \gamma_c^c \]

\[ = \gamma_a^c \nabla^b (\hat{G}_{db} + \lambda^{AB} K_B^b \hat{G}_{Ad}) + G_{cb} \lambda^{AB} K^b_A \nabla^b K_B^d] \]

\[ = \gamma_a^c \nabla^b (\hat{G}_{db} - \lambda^{AB} \hat{G}_{Ab} \nabla_d K_B^b + \lambda^{AB} G_{Ab} \nabla^b K_B^d] \]

\[ = \gamma_a^c \nabla^b \hat{G}_{db} - 2 \hat{G}_{Ab} \lambda^{AB} \nabla_d K_B^b + \frac{1}{2} G_{AB} D_a \lambda^{AB}. \]  
(3.30)

The first term in Eq. (3.30) can be written as

\[ \gamma_a^d \nabla^b \hat{G}_{db} = D^b \hat{G}_{ab} + \gamma_a^d \hat{G}_{dc} \nabla^b \gamma_c^c \]

\[ = D^b \hat{G}_{ab} + \frac{1}{2} \hat{G}_{ab} \lambda^{-1} D^b \lambda, \]  
(3.31)

where we again used Eq. (3.30), (B3) to simplify \( \nabla^b \gamma_c^c \). Using Eq. (3.20) to replace the second term gives the result

\[ 0 = \gamma_a^c \nabla^b G_{cb} = D^b \hat{G}_{ab} + \frac{1}{2} \hat{G}_{ab} \lambda^{-1} D^b \lambda + \frac{1}{2} G_{AB} D_a \lambda^{AB} \]

\[ + \epsilon^{-1} |\lambda|^{-1/2} \epsilon_a^b \lambda^{AB} \hat{G}_{Ab} \epsilon_B^c. \]  
(3.32)

Defining the linear differential operator

\[ D_a = D_a + \frac{1}{2} (\lambda^{-1} D_a \lambda) \times, \]  
(3.33)

the contracted Bianchi identities can thus be written as

\[ D^b \hat{G}_{Ab} = 0; \quad (3.34a) \]

\[ D^b \hat{G}_{ab} = -\frac{1}{2} G_{AB} D_a \lambda^{AB} \]

\[ - \epsilon^{-1} |\lambda|^{-1/2} \epsilon_a^b \lambda^{AB} \hat{G}_{Ab} \epsilon_B^c. \]  
(3.34b)
The two relations \((3.34a)\) among the four off-block-diagonal components \(\{G_{Ai}\}\) of the Einstein tensor show why setting those components to zero only allows us to eliminate two degrees of freedom \(\{c_A\}\) from the problem.

The two components of Eq. \((3.34b)\) tell us that only four of the remaining six components (the three \(G_{AB}\) and the three components \(G_{ij}\) of \(\hat{G}_{ab}\)) of the Einstein tensor are independent. Since Eqs. \((3.34a)\), \((3.34b)\) are algebraic, rather than differential, in the three projected components \(G_{AB}\), Eq. \((3.34b)\) can be solved to give two of those components in terms of the other eight.

Also, Eq. \((3.34b)\) can be rewritten, by substituting the form of \(\hat{G}_{Ab} = \hat{R}_{Ab}\) given by Eq. \((3.21b)\), as

\[
D^b \hat{G}_{ab} = -\frac{1}{2} G_{AB} D_a \lambda^{AB} + \frac{1}{2} \lambda^{-1} \lambda^{AB} c_A D_a c_B. \tag{3.35}
\]

E. Recovering the four-geometry

As shown in Sec. \(\text{III A}\), given a spacetime \((\mathcal{M}, g_{ab})\) admitting two commuting KVFs \(\{K^a_A\}\), one can define a number of tensor fields that live on the two-dimensional space \(S\) of Killing vector orbits. In particular, we defined the two-metric \(\gamma_{ab}\), the symmetric matrix of inner products \(\{\lambda_{AB}\}\), and the two scalar fields \(\{c_A\}\). In this section, we complete our general discussion of spacetimes admitting two KVFs by doing the converse. That is, we show how to reconstruct the four-geometry \((\mathcal{M}, g_{ab})\) given only \(\{\lambda_{AB}\}\), \(\{c_A\}\), and the metric components \(\{\gamma_{ij}\}\) of \(\gamma_{ab}\) with respect to a coordinate system \(\{x^i | i = 2, 3\}\) on \(S\). The goal is to: (i) construct a basis \(\{e^a_{\mu} | \mu = 0, 1, 2, 3\}\) on \(\mathcal{M}\); (ii) determine the commutation coefficients of this basis; (iii) specify the metric components \(\{g_{\mu\nu}\}\) of \(g_{ab}\) with respect to this basis. As we shall see below, if the scalar fields \(c_A\) are non-zero, then there is no preferred coordinate basis on \(\mathcal{M}\). (Appendix \(\text{C}\) describes the freedom in choosing a coordinate basis on \(\mathcal{M}\) and the form the metric takes in such a basis.) However, there is always a preferred non-coordinate basis on \(\mathcal{M}\), in terms of which the metric components \(\{g_{\mu\nu}\}\) are block-diagonal.

(i) Let \(\{x^i | i = 2, 3\}\) be any coordinate system on \(S\), and let \(\{\gamma_{ij}\}\) denote the components of \(\gamma_{ab}\) with respect to these coordinates—i.e.,

\[
\gamma_{ab} = \gamma_{ij} (dx^i)_a (dx^j)_b. \tag{3.36}
\]

Then we can use the two contravariant vectors

\[
e^a_i := \gamma_{ij} g^{ab} (dx^b)_j \quad (i = 2, 3) \tag{3.37a}
\]

along with the two Killing vector fields

\[
e^a_A := K^a_A \quad (A = 0, 1) \tag{3.37b}
\]

\(^9\)This only works if the appropriate derivatives of \(\lambda^{AB}\) are non-vanishing. Otherwise, we have two differential relations among the components of the Einstein tensor.
to define a basis \( \{ e^a_\mu | \mu = 0, 1, 2, 3 \} \) on \( \mathcal{M} \).

Although the Killing vector fields commute with everything (i.e., \( [e_A, e_\mu]^a = 0 \)), and
\[
[e_i, e_j]^a := e^b_i D_b e^a_j - e^b_j D_b e^a_i = \gamma^a_b [e_i, e_j]^b = 0 ,
\]
the basis vectors \( \{ e^a_i | i = 2, 3 \} \) need not commute on \( \mathcal{M} \). Thus, \( \{ e^a_\mu | \mu = 0, 1, 2, 3 \} \) is not necessarily a coordinate basis on \( \mathcal{M} \).

(ii) Given a basis \( \{ e^a_\mu | \mu = 0, 1, 2, 3 \} \) on \( \mathcal{M} \), the commutation coefficients \( \{ C^a_\mu \sigma \} \) are defined by
\[
[e_\mu, e_\nu]^a =: C^a_\mu \sigma e^a_\sigma .
\]
Since, as mentioned in (i),
\[
[e_A, e_\mu]^a = 0 \quad \text{and} \quad \gamma^a_b [e_i, e_j]^b = 0 ,
\]
it follows that
\[
[e_i, e_j]^a = C^{AB}_i e^A_i e^B_j
\]
define the only possible non-vanishing commutation coefficients, which are \( \{ C^{AB}_i \} \).

To calculate these coefficients, invert Eq. (3.41):
\[
C^{AB}_i = [e_i, e_j]^a e^A_i e^B_j = [e_i, e_j]^a g_{ab} \lambda^{AB} K^b_B .
\]
Since \( g_{ab} e_i^a K^b_B = 0 \) and \( \nabla_a K_B^B = -\nabla_b K_B^a \), it follows that
\[
[e_i, e_j]^a g_{ab} K^b_B = -2 e_i^a e_j^b \nabla_a K_{AB} .
\]
Now use Eq. (3.20) to expand \( \nabla_a K_{AB} \). The final result is
\[
C^{AB}_i = \epsilon^{-1} \ |\lambda|^{-1/2} \epsilon_{ij} \lambda^{AB} c_B ,
\]
where
\[
\epsilon_{ij} := e^a_i e^b_j e_{ab}
\]
are the components of the Levi-Civita tensor on \( \mathcal{S} \). Thus, we see that \( c_A = 0 \) are the necessary and sufficient conditions for \( \{ e^a_\mu | \mu = 0, 1, 2, 3 \} \) to be a coordinate basis on \( \mathcal{M} \).

(iii) Determining the metric components \( g_{\mu\nu} \) with respect to the basis \( \{ e^a_\mu | \mu = 0, 1, 2, 3 \} \) is relatively straightforward. Using the definitions (3.37a) and (3.37b), it follows that
\[
\begin{align*}
g^{AB} &:= g_{ab} e^A_a e^B_b = \lambda^{AB} , \\
g_{AB} &:= g_{ab} e^a_i e^b_j = 0 , \\
g_{ij} &:= g_{ab} e^a_i e^b_j = \gamma_{ij} ,
\end{align*}
\]
so the metric components
\[
\{ g_{\mu\nu} \} = \begin{pmatrix} \{ \lambda^{AB} \} & 0 \\ 0 & \{ \gamma_{ij} \} \end{pmatrix}
\]
are block diagonal.
IV. FURTHER GAUGE FIXING

Within the formalism of Sec. [III], there are still choices to be made in defining a basis. There is of course the choice of a basis (coordinate or otherwise) on the two-manifold $S$, and we will discuss possible coordinate choices in Sec. [IVB]. But it is also possible, by considering linear combinations of the Killing vectors, to describe the same spacetime with different values for $\{\lambda_{AB}\}$ and $\{c_A\}$, as we show in Sec. [IVA].

A. Relabeling the Killing vectors

The properties of the vectors $\{K^a_A\}$ which allow us to perform the construction of Sec. [III] are that they obey Killing’s equation

$$\nabla^{(a}K^{b)}_A = 0,$$ (4.1a)

and that they commute with one another

$$K^b_A\nabla_bK^a_B - K^b_B\nabla_bK^a_A = 0.$$ (4.1b)

If we define a new pair of vectors $\{K^a_{A'}\}$ to be a linear combination of the first two:

$$K^a_{A'} = L^A_{A'}K^a_B,$$ (4.2)

then the new set of vectors will also be commuting Killing vectors if and only if

$$0 = \nabla^{(a}K^{b)}_{A'} = K^{(a}_B\nabla^{b)}L^A_{B'}A'$$ (4.3a)

and

$$0 = K^b_{A'}\nabla_bK^a_{B'} - K^b_{B'}\nabla_bK^a_{A'}$$
$$= K^b_{A'}\nabla_bL^C_{B'}A'C - K^b_{B'}\nabla_bL^C_{A'}A'C.$$ (4.3b)

Clearly a sufficient condition is that $\{L^a_{A'B}\}$ be constants. It is also straightforward to show that if neither $\lambda_{AB}$ nor $\lambda_{A'B'}$ is degenerate, it is also a necessary condition.\[10\]

Under this global $GL(2, \mathbb{R})$ symmetry, $\lambda_{AB}$ transforms as a second-rank covariant tensor, $\lambda$ as a scalar density of weight two, and $c_A$ as a covariant vector density of weight one:

$$\lambda_{A'B'} = L^A_{A'}L^B_{B'}\lambda_{CD}$$ (4.4a)

$$\lambda' = (\det L)^2\lambda$$ (4.4b)

$$c^\prime_A = (\det L)L^B_{A'}c_B.$$ (4.4c)

In particular, the value of

\[10\]This means that the linear combinations must be constant except on the surfaces of signature change, and we are not interested in discontinuous transformations.
\[ I = \lambda^{-1} \lambda^{AB} c_A c_B \]  

(4.5)

at a given point cannot be changed by the transformation (4.2), nor can the sign of \( \lambda \).

We can use these transformations to bring \( \lambda_{AB} \) and \( c_A \) into a convenient form at one point in the two-manifold \( S \)—i.e., on one of the Killing vector orbits. (In the case of two equal-mass orbiting cosmic strings, where there is an additional discrete rotational symmetry which exchanges the strings, a special point is the fixed point of that rotation, which is the rotational axis.) The desired form depends on the invariant signs of \( I \) and \( \lambda \):

(i) If \( \lambda > 0 \), \( \lambda_{AB} \) is positive definite, and thus \( I \) must also be positive (or else the system of KVFs would be orthogonally transitive). We can choose the Killing vectors so that \( \lambda_{00} = \delta_{00} \) at our desired point, and use the residual \( SO(2) \) symmetry (which preserves that form of \( \lambda_{AB} \)) to rotate \( c_A \) so that \( c_1 = 0 \) and \( c_0 = \sqrt{I} \).

(ii) As mentioned in (i), \( \lambda > 0 \) and \( I < 0 \) is not allowed.

(iii) If \( \lambda < 0 \) we can bring \( \lambda_{AB} \) into a Lorentz form \( \lambda_{AB} = \eta_{AB} \). If \( I > 0 \), we define the KVFs so that \( \lambda_{00} = -1 \) and \( \lambda_{11} = 1 \) at our chosen point, and then use the residual \( SO(1,1) \) symmetry to set \( c_1 = 0 \) and \( c_0 = \sqrt{I} \).

(iv) If \( \lambda < 0 \) and \( I < 0 \), we instead define the KVFs so that \( \lambda_{00} = 1 \) and \( \lambda_{11} = -1 \) at the chosen point, and then use the residual \( SO(1,1) \) symmetry to set \( c_1 = 0 \) and \( c_0 = \sqrt{-I} \).

So, ignoring the special case where \( I = 0 \), we always have the freedom to set \( c_1 = 0 \) at a point. Note that in the case of a vacuum spacetime, where Eq. (3.21b) tells us that the \( \{c_A\} \) are constants, this means that \( c_1 \) vanishes everywhere.

**B. Coordinate choices on the two-manifold**

The description in terms of the two-manifold \( S \) of Killing vector orbits has been entirely coordinate-independent, as emphasized by the use of abstract index notation. Thus we need to make a choice of coordinates on \( S \) to complete the specification of a basis on the four-manifold \( M \). In vacuum spacetimes with orthogonal transitivity \( (c_A = 0) \), (3.24b) implies that

\[ D^a D_a \sqrt{\pm \lambda} = 0, \]  

(4.6)

so (in the absence of signature change) \( \sqrt{|\lambda|} \) is a harmonic coordinate on \( S \). Using \( \sqrt{|\lambda|} \), along with its harmonic conjugate, leads to a two-metric described only by a single conformal factor, and effectively reduces the number of degrees of freedom in \( \lambda_{AB} \) from three to two. This is used, for example, to define the \( \rho \) and \( z \) (Weyl) coordinates in a general stationary, axisymmetric, vacuum spacetime.

However, in the case at hand, \( \lambda \) does not provide us with a harmonic coordinate, and thus this method does not work. We can use either a set of conformal coordinates or a set based on \( \lambda \), as described in the following sections, but not both.

1. **Conformal coordinates**

Since any two-manifold is conformally flat, we are of course free to choose coordinates on \( S \) so that the metric is determined only by a single conformal factor \( \Phi(x^i) \):
TABLE I. Degrees of freedom in the absence of $\zeta$-symmetry. This table enumerates the independent elements of $\{c_A\}$ and $\{\lambda_{AB}\}$ and the components $\{\gamma_{ij}\}$ of the two-metric for various coordinate choices on the two-manifold $S$ of Killing vectors. The fields $\{c_A\}$ become constants in vacuo, or more generally when the off-block-diagonal components $\{T_{Ai}\}$ of the stress-energy tensor vanish, so both the $c_A$ and total counts are considered separately “w/matter” (actually $T_{Ai} \neq 0$) and “in vacuo” (actually $T_{Ai} = 0$). The different columns correspond to the $\lambda$-based coordinates described in Sec. [IV B 2], the conformal coordinates described in Sec. [IV B 1], and, for comparison, the coordinates which can be defined in vacuum ($T_{ab} = 0$) spacetimes in the presence of orthogonal transitivity, which are both $\lambda$-based and conformal. (The counting for the geodesic polar coordinates defined in Sec. [IV B 3] is the same as for conformal coordinates, with the conformal factor $\Phi$ replaced by the metric component $\gamma_{33}$.)

|                     | $\lambda$-based | Conformal        | Weyl$^a$ |
|---------------------|------------------|------------------|----------|
| $c_A$ (w/matter)    | 2 fields $c_0,c_1$ | 2 fields $c_0,c_1$ | N/A$^a$  |
| $c_A$ (in vacuo)    | 1 constant $c_0$  | 1 constant $c_0$  | 0        |
| $\lambda_{AB}$      | 2 fields $\lambda_{01}, \lambda_{11}$ | 3 fields $\lambda_{00}, \lambda_{01}, \lambda_{11}$ | 2 fields $\lambda_{01}, \lambda_{11}$ |
| $\gamma_{ij}$       | 2 components $\gamma_{22}, \gamma_{33}$ | 1 conf. fact. $\Phi$ | 1 conf. fact. $\Phi$ |
| Total w/matter       | 6 DOF            | 6 DOF            | N/A$^a$  |
| Total in vacuo       | 4 DOF + 1 param. | 4 DOF + 1 param. | 3 DOF    |

$^a$Weyl coordinates, which are $\lambda$-based and conformal, are only possible in vacuum spacetimes with orthogonal transitivity, and are included here for comparison only.

$$d\Sigma^2 = \Phi \left[ (dx^2)^2 - \epsilon(dx^3)^2 \right].$$

In this case, we would need two fields (constants in vacuo) $c_0$ and $c_1$, to determine the commutation coefficients for the non-coordinate basis described in Sec. [III E], plus three more fields $\lambda_{00}, \lambda_{01}$ and $\lambda_{11}$ to determine the $\{g_{AB}\}$ block of the metric, plus a single field $\Phi$ to determine the $\{g_{ij}\}$ block of the metric, for a total of six degrees of freedom [four in vacuo if Eq. (3.21b) are imposed a priori], as summarized in Table I.

Note, however, that since the form of the flat metric depends on the signature of the two-manifold $S$ (Euclidean for $\epsilon < 0$ and Lorentzian for $\epsilon > 0$), we would need to use different coordinate patches on either side of the signature change, and there would be no meaningful relations between the definitions of $\Phi$ on either side.

### 2. $\lambda$-based coordinates

Although $\sqrt{|\lambda|}$ is not a harmonic coordinate in our case, we can still reduce the number of independent components in the metric by basing a coordinate system on it. We set $x^2 = \lambda$ [in the absence of Eq. (1.13), there is no reason to work with the square root], and choose $x^3$ so that the metric is diagonal, which we can always do in two dimensions.

We expect $\lambda$ to act as a radial coordinate, since as a geometric quantity it must be preserved by the discrete rotational symmetry which exchanges the two strings. In particular, this means that constant-$\lambda$ surfaces must be closed. [There is also the analogy of co-rotating flat spacetime (Sec. II), where $\lambda$ is related to the traditional radial coordinate $\rho$ by Eq. (2.12).] Because of the choices made in Sec. [IV A], we know that $\lambda = -1$ at the origin (the fixed...
FIG. 1. Surfaces of constant $\lambda$ on the two-manifold $\mathcal{S}$ and the surfaces orthogonal to those which are defined to have constant $\psi$. At the center of rotation, $\lambda = -1$; the innermost level surface is for $\lambda = -0.9$, and the $\lambda$ values increase by 0.1 up to the outermost one, which is $\lambda = 0$ (the light cylinder). The constant-$\psi$ curves which are drawn are $\psi = 0$ (the positive $x$-axis), $\psi = \pi/8$, $\psi = \pi/4$, $\psi = 3\pi/8$, and $\psi = \pi/2$ (the positive $y$-axis). This diagram was made by superposing two low-mass Levi-Civita cosmic strings (mass-per-unit-length $2C = 0.2$ in geometrical units) assumed to move in Newtonian orbit about one another. (See Sec. IV B 3.) The coordinates drawn on the axes are the Cartesian analogues of polar coordinates $\rho$ and $\phi$ in which the metric can be written in the form (4.11). The cosmic strings are located at $y = 0$, $x = \pm0.189$, and $\lambda$ becomes a bad coordinate about 0.02 from each of the strings. It should thus be necessary, in a $\lambda$-based coordinate system, to model the strings with boundary conditions placed at least that far from the strings.

point of the discrete rotational symmetry). And the light cylinder is, by definition, the surface on which $\lambda = 0$. Considering these concentric surfaces of constant $\lambda$ (Fig. 1) we can draw surfaces everywhere orthogonal to these, which are surfaces of constant $x^3 = \psi$. If we call the one passing through one cosmic string $\psi = 0$ and the one passing through the other string $\psi = \pi$, then $\psi$ is an angular coordinate with period $2\pi$. This coordinate can be defined so that the discrete symmetry discussed above corresponds to a rotation by $\pi$ in $\psi$. The metric on $\mathcal{S}$ can be written in terms of two independent components:

$$d\Sigma^2 = \gamma_{22} d\lambda^2 + \gamma_{33} d\psi^2. \quad (4.8)$$

For example, the two-metric (2.14) for co-rotating flat spacetime can be written in this form (the $\varphi$ used in Sec. II is the same as the present $\psi$) with

$$\gamma_{22} = \frac{1}{4\Omega^2(1 + \lambda)} \quad (4.9a)$$
\[
\gamma_{33} = -\frac{1 + \lambda}{\Omega^2 \lambda}. \tag{4.9b}
\]

We have not yet completely defined the coordinates on the two-manifold; the labeling of the constant \( \psi \) surfaces is still to be specified. Put another way, we can make a redefinition \( \psi' = f(\psi) \) which preserves all of the properties thus far discussed of the coordinate system, so long as \( f(0) = 0 \) and \( f(\psi + \pi) = f(\psi) + \pi \). An obvious way to finish that specification is to choose a particular value of \( \lambda \) and decree that equal intervals in \( \psi \) sweep out equal distance along that constant-\( \lambda \) curve, or \( \gamma_{33,\psi} = 0 \) for that value of \( \lambda \). Given the specifications we made at the origin in Sec. [V A], the most convenient place to make this definition is in the limit \( \lambda \to -1 \), assuming that limit exists.

Counting the degrees of freedom needed to specify the metric, we again have two fields (or constants) \( c_0 \) and \( c_1 \); now we only need (say) \( \lambda_{11} \) and \( \lambda_{01} \) as functions of \( \lambda \) and \( \psi \) to specify the matrix \( \{\lambda_{AB}\} \), since we know

\[
\lambda_{00} = \frac{\lambda + (\lambda_{01})^2}{\lambda_{11}}; \tag{4.10}
\]

and finally we need the two diagonal components \( \gamma_{22} \) and \( \gamma_{33} \) to specify the metric on \( S \). Again, we have six independent degrees of freedom, or four if the \( c_A \) are taken to be constant. (See Table I.)

3. Geodesic polar coordinates

The use of \( \lambda \) as a coordinate has a number of potential hazards. In addition to a coordinate singularity at the origin, it may also fail to be monotonic as one moves out from the origin to infinity. For example, consider the co-rotating spacetime obtained by superposing two low-mass Levi-Civita cosmic strings [8], each with mass-per-unit length \( 2C \) in gravitational units, where \( C \) is a small dimensionless parameter. This spacetime will not solve Einstein’s equations (it includes only Coulomb effects and not gravitomagnetic or radiative ones) but it can be written in our formalism. There exist coordinates on the two-manifold in which the two-metric is

\[
d\Sigma^2 = d\rho^2 - \frac{\rho^2}{\lambda} d\varphi^2, \tag{4.11}
\]

and in terms of those coordinates, the strings are located at \( \rho = R \) and \( \varphi = 0, \pi \) and the matrix of Killing vector inner products is given by\[11\]

\[
\{\lambda_{AB}\} = \begin{pmatrix}
-[(\rho_+ \rho_-)^{2C} - \Omega^2 \rho^2] & 0 \\
0 & (\rho_+ \rho_-)^{-2C}
\end{pmatrix}, \tag{4.12}
\]

so that \( \lambda = -1 + (\rho_+ \rho_-)^{-2C} \Omega^2 \rho^2 \), where \( \rho_{\pm} \) are the Cartesian distances from each string, whose product is

\[11\]Note that if \( C = 0 \), this just reduces to [2.11].
\[(\rho_+\rho_-)^2 = (\rho^2 + R^2)^2 - 4R^2\rho^2 \cos^2 \varphi.\]  

(4.13)

This means that sufficiently close to either of the strings, \(\lambda\) becomes arbitrarily large, producing isolated (for small \(C\)) regions around the strings which violate the assumption that \(\lambda\) increases monotonically away from the origin. We may be able to solve this problem by removing those regions from the two-manifold and modelling the strings by conditions on the resulting boundaries.

Note that the coordinates \(\rho\) and \(\varphi\) in which the metric has been written have no such problems. One way to generalize these coordinates to arbitrary spacetimes would be to require \(\gamma_{22} \equiv 1\) and \(\gamma_{23} \equiv 0\) everywhere. This can be achieved, for instance, by defining polar coordinates at the origin and then requiring constant-\(\varphi\) lines to be geodesics along which \(\rho\) is an affine parameter, whence the name geodesic polar coordinates. Then, just as in the case of conformal coordinates, we will need only one function \((\gamma_{33}\) or equivalently \(-\gamma_{33}\lambda/\rho^2\)) to define the two-metric, and three more to define the matrix \(\lambda_{AB}\).

C. An additional discrete symmetry

This section does not actually concern gauge-fixing, but it does describe a way in which the number of independent degrees of freedom can be further reduced under certain circumstances. Consider a transformation which changes the sign of the Killing vector \(K_a^1\) and simultaneously reverses the orientation of the spacetime \(\mathcal{M}\). Under such a transformation, \(c_1, \lambda_{01}, R_{01}\), and \(\{R_{1i}\}\) will change sign, but the other parts of those objects, such as \(c_0, \lambda_{00}, \lambda_{11}\), etc., will not. (In general, any object or component will be transformed to \((-1)^{\zeta}\) times itself, where \(\zeta\) is the number of times 1 appears as an index.) Call this transformation \(\zeta\)-reflection.

Now suppose we have a solution to the Einstein equations determined by some stress-energy distribution and boundary conditions. Note that the auxiliary conditions on \(\lambda_{AB}\) and \(c_A\) at a single point in \(\mathcal{S}\), which we used to define the Killing vector labels, are unchanged by \(\zeta\)-reflection (since they set \(\lambda_{01}\) and \(c_1\) to zero at that point). Thus \(\zeta\)-reflection of our initial solution must also satisfy the Einstein equations, only with \(\zeta\)-reflected stress-energy and boundary conditions. If the stress-energy and boundary conditions are sent into themselves by \(\zeta\)-reflection (i.e., if their \(\zeta\)-odd components all vanish), then we have two solutions to the same boundary-value problem, which are \(\zeta\)-reflections of one another. Assuming that the boundary conditions used are sufficient to specify a unique solution, that means that we are actually talking about one solution which is taken into itself by \(\zeta\)-reflection, which means that the \(\zeta\)-odd quantity \(\lambda_{01}\) vanishes everywhere in this case.\(^{14}\) (See Table II.)

When are the stress-energy and boundary conditions going to be \(\zeta\)-even? The \(\zeta\)-odd parts \(T_{01}\) and \(T_{1i}\) of the stress-energy can be thought of as energy fluxes and shears along the string, and setting them to zero would seem to be reasonable. Considering the boundary conditions at infinity as describing linearized radiation on a cylindrical background, and thinking qualitatively in something like the transverse, traceless gauge describing radiation

\(^{12}\)Note that the angular coordinate \(\varphi\) in this system will not in general be the same as the \(\psi\)
TABLE II. Degrees of freedom in the presence of $\zeta$-symmetry. This table enumerates the independent components of the quantities describing the four-geometry, just as in Table I, but this time after we have assumed that the spacetime is unchanged by the $\zeta$-reflection defined in Sec. IV C, and so for example $\lambda_{01} \equiv 0$. (The counting for the geodesic polar coordinates defined in Sec. IV B 3 is the same as for conformal coordinates, with the conformal factor $\Phi$ replaced by the metric component $\gamma_{33}$.)

|                      | $\lambda$-based | Conformal | Weyl$^a$ |
|----------------------|------------------|-----------|----------|
| $c_A$ (w/matter)     | 1 field $c_0$    | 1 field $c_0$ | N/A$^a$ |
| $c_A$ (in vacuo)     | 1 constant $c_0$ | 1 constant $c_0$ | 0        |
| $\lambda_{AB}$       | 1 field $\lambda_{11}$ | 2 fields $\lambda_{00}$, $\lambda_{11}$ | 1 field $\lambda_{11}$ |
| $\gamma_{ij}$        | 2 components $\gamma_{22}$, $\gamma_{33}$ | 1 conf. fact. $\Phi$ | 1 conf. fact. $\Phi$ |
| Total w/matter       | 4 DOF            | 4 DOF     | N/A$^a$  |
| Total in vacuo       | 3 DOF + 1 param. | 3 DOF + 1 param. | 2 DOF |

$^a$See footnote, Table I.

Moving radially outward, the “plus” polarization will involve the components $h_{\phi\phi}$ and $h_{zz}$ of the metric perturbation, which are $\zeta$-even quantities, while the “cross” polarization will involve the $\zeta$-odd component $h_{z\phi}$. Thus $\zeta$-even boundary values are those which involve only one polarization.

V. THE DIFFERENTIAL EQUATIONS FOR CO-ROTATING COSMIC STRINGS

A. General considerations

1. Number of degrees of freedom

In a general spacetime, the Einstein equations are ten second-order, non-linear, partial differential equations for the ten independent metric components, as functions of the four spacetime coordinates. As described in Sec. IV B, after all of the gauge degrees of freedom have been fixed out of a two-Killing-vector vacuum spacetime (and $G_{\lambda \beta}$ have been set to zero a priori), there remain four independent functions of the two coordinates on the two-dimensional space of Killing vector orbits. There are three Einstein equations involving $G_{\lambda \beta}$ (conveniently divided into the two involving the trace-free part $P_{\lambda \beta}^{CD}G_{CD} = P_{\lambda \beta}^{CD}R_{CD}$ defined in $\lambda$-based coordinates.

13 Equation (3.12) tells us that changing the sign of one of the Killing vectors must change the orientation of either $\mathcal{M}$ or $\mathcal{S}$.

14 The same can be concluded about $c_1$, but if the stress-energy is $\zeta$-even, it is already guaranteed to be a constant by (3.21b).
and one involving the trace $\lambda^{AB}G_{AB} = -\gamma^{ab}\hat{R}_{ab}$ and three more involving $\hat{G}_{ab}$ (two with $P_{ab}^{cd}\hat{G}_{cd} = P_{ab}^{cd}\hat{R}_{cd}$ and one with $\gamma^{ab}\hat{G}_{ab} = -\lambda^{AB}R_{AB}$) for a total of six.\footnote{The contracted Bianchi identities, considered in Sec. [III.D], mean that only four of those six components are independent. This is discussed further in Sec. [V.B.]} If the spacetime is assumed to be unchanged by $\zeta$-reflection (see Sec. [IV.C]), there are only three independent degrees of freedom in the gauge-fixed metric. Since $\hat{G}_{01} \equiv 0$ in that case, there are only five Einstein equations involving those three functions of two variables.\footnote{The “missing” equation is one of the two involving $P_{AB}^{CD}G_{CD} = P_{AB}^{CD}R_{CD}$.}

### 2. Order of the equations

An important practical consideration for a numerical solution to the Einstein equations is their order—i.e., the highest number of derivatives appearing in each equation. The Einstein equations for a generic spacetime are second-order, but we will see that not all of the equations enumerated in Sec. [V.A.1] are actually second-order when $\lambda$ is used as one of the coordinates. This is because both of the components

$$D_i \lambda \equiv \partial_i x^2 = \delta_i^2 \quad (5.1)$$

are constant, and thus second covariant derivatives of $\lambda$ do not translate into second derivatives of the functions of $\lambda$ and $\psi$ which define the metric:

$$D_i D_j \lambda = -\mathcal{G}^{2}_{ji}, \quad (5.2)$$

where

$$\mathcal{G}^{k}_{ij} := \frac{\gamma^{kl}}{2} (\gamma_{il,j} + \gamma_{jl,i} - \gamma_{ij,l}) \quad (5.3)$$

is a Christoffel symbol for the metric $\gamma_{ab}$ in the $\{x^i\}$ coordinates, and only involves first derivatives of metric components.

Thus, the only terms in Eq. (3.24) which can involve second derivatives of metric components are the scalar curvature (of $\gamma_{ab}$) $\mathcal{R}$ and the trace-free (on $A$ and $B$) part of $D_a D_b \lambda_{AB}$. Defining the notation $\cong$ to mean “equal up to first (and lower) derivative terms,” we see that the six Einstein equations divide into three second-order equations involving

$$P_{AB}^{CD}G_{CD} = P_{AB}^{CD}R_{CD} \cong -\frac{1}{2} P_{AB}^{CD} D^a D_a \lambda_{CD}$$

$$\cong -\frac{1}{2} D^a D_a \lambda_{AB} \quad (5.4a)$$

$$\lambda^{AB}G_{AB} = -\gamma^{ab}\hat{R}_{ab} \cong -\mathcal{R} \quad (5.4b)$$

and three first-order equations involving
\[ \gamma^{ab} \tilde{G}_{ab} = \lambda^{AB} R_{AB} \cong 0 \]  
(5.5a)  
\[ P^{cd} \tilde{G}_{cd} = P^{cd} \tilde{R}_{cd} \cong 0. \]  
(5.5b)  

If the spacetime is assumed to be \( \zeta \)-symmetric, \( P^{CD} R_{CD} \) vanishes automatically, and there are only two second-order equations and three first-order ones.

**B. Explicit forms**

1. \( \lambda \)-based coordinates

To determine the explicit differential equations obeyed by functions defining the metric, it is convenient to use as the four independent degrees of freedom

\[
\begin{align*}
X(\lambda, \psi) &:= \lambda_{01} \quad (5.6a) \\
Z(\lambda, \psi) &:= \lambda_{11} \quad (5.6b) \\
P(\lambda, \psi) &:= \gamma_{22} \quad (5.6c) \\
h(\lambda, \psi) &:= -\lambda \gamma_{33}/\gamma_{22}. \quad (5.6d)
\end{align*}
\]

These definitions are chosen in part because they are all non-singular at the light cylinder \( \lambda = 0 \) in co-rotating flat spacetime [cf. Eq. (5.9)]:

\[
\begin{align*}
X(\lambda, \psi) &= 0 \quad (5.7a) \\
Z(\lambda, \psi) &= 1 \quad (5.7b) \\
P(\lambda, \psi) &= [4\Omega^2(1 + \lambda)]^{-1} \quad (5.7c) \\
h(\lambda, \psi) &= 4(1 + \lambda)^2. \quad (5.7d)
\end{align*}
\]

Also \( X \) is a \( \zeta \)-odd quantity, while the other three are \( \zeta \)-even.\(^{17}\)

Substituting the expressions

\[
\begin{align*}
\lambda_{00} &= (\lambda + X^2)Z^{-1} \quad (5.8a) \\
\lambda_{01} &= X \quad (5.8b) \\
\lambda_{11} &= Z \quad (5.8c) \\
\gamma_{22} &= P \quad (5.8d) \\
\gamma_{23} &= 0 \quad (5.8e) \\
\gamma_{33} &= -\lambda^{-1} hP \quad (5.8f) \\
c_0 &\equiv 2\Omega \quad (5.8g) \\
c_1 &\equiv 0 \quad (5.8h)
\end{align*}
\]

\(^{17}\)Note that while this does mean \( X \cong 0 \) in a \( \zeta \)-symmetric spacetime, the non-linearity of the Einstein equations will couple the \( \zeta \)-odd and \( \zeta \)-even parts of the metric.
into Eqs. (3.24b) and (3.24c), a straightforward but lengthy algebraic calculation gives the first-order equations

\(- \gamma^{ab} \tilde{G}_{ab} = \lambda^{AB} R_{AB} = -\frac{1}{4} \lambda^{-1} h^{-1} P^{-1} h_{\lambda} - 2 \lambda^{-2} Z Q^{2} + \frac{1}{2} \lambda^{-2} P^{-1}, \) \hspace{1cm} (5.9a)

\(\tilde{G}_{23} = \tilde{R}_{23} = -\frac{1}{2} (1 + \lambda^{-1} X^2) Z^{-2} Z_{\psi} - \frac{1}{2} \lambda^{-1} X_{\lambda} X_{\psi} + \frac{1}{2} \lambda^{-1} P^{-1} (Z_{\lambda} X_{\psi} + Z_{\psi} X_{\lambda}) + \frac{1}{4} \lambda^{-1} P^{-1} P_{\psi} + \frac{1}{4} \lambda^{-1} Z^{-1} Z_{\psi}; \) \hspace{1cm} (5.9b)

\(\tilde{G}_{22} - \gamma^{33} \tilde{G}_{33} = \frac{1}{2} \tilde{R}_{22} - \gamma^{33} \tilde{R}_{33} = -\frac{1}{2} (1 + \lambda^{-1} X^2) P^{-1} Z^{-2} Z_{\lambda}^{2} - \frac{1}{2} \lambda(1 + \lambda^{-1} X^2) h^{-1} P^{-1} Z^{-2} Z_{\psi}^{2} + \lambda^{-1} P^{-1} Z^{-1} X_{\lambda} X_{\psi} + h^{-1} P^{-1} P_{\lambda} + \frac{1}{2} \lambda^{-1} P^{-1} Z^{-1} Z_{\lambda}, \) \hspace{1cm} (5.9c)

where we have defined the shorthand

\(h_{\lambda} := \frac{\partial h}{\partial \lambda}, \quad X_{\psi \psi} := \frac{\partial^2 X}{\partial \psi^2}, \quad \text{etc.} \) \hspace{1cm} (5.10)

The contracted Bianchi identities derived in Sec. III D mean that two of the second-order expressions \(G_{AB}\) can be written in terms of other components of \(G_{ab}\). Specifically, we can use Eq. (3.34b) to solve for \(G_{00}\) and \(G_{11}\) as linear combinations of \(G_{01}\), \(\{G_{ij}\}\), and \(\{G_{ij,k}\}\). The explicit forms of those expressions are not needed to solve the Einstein equations (since they simply show that \(G_{00}\) and \(G_{11}\) vanish identically in a vacuum). However, the corresponding expressions for \(T_{00}\) and \(T_{11}\) show that those components are completely determined by the other components of the stress-energy tensor \(T_{ab}\) (and their derivatives) due to conservation of energy.

If the spacetime is assumed to be unchanged by \(\zeta\)-symmetry (cf. Sec. IV C), then \(G_{01} \equiv 0\) automatically, and the only independent Einstein equations are \(G_{ij} = 8\pi T_{ij}\), with the components of \(G_{ij}\) given by Eq. (5.9). (Note that those expressions are then further simplified by the condition \(X \equiv 0\).) In that case those three first-order equations, together with an appropriate choice of boundary conditions, determine the three functions \(h(\lambda, \psi)\), \(P(\lambda, \psi)\), and \(Z(\lambda, \psi)\).

If \(\zeta\)-symmetry is not imposed, we have one other independent component of the Einstein tensor, which we take to be \(P_{01}^{AB} G_{AB} = P_{01}^{AB} R_{AB}\). Again, a bit of algebra converts Eq. (3.24a) into

\[ P_{01}^{AB} G_{AB} = G_{01} - \frac{1}{2} X_{\lambda}^{AB} G_{AB} = -\frac{1}{2} P^{-1} X_{\lambda} + \lambda^{-1} P^{-1} P_{\psi} + \frac{1}{2} \lambda^{-1} P^{-1} X X_{\lambda}^{2} - \frac{1}{2} h^{-1} P^{-1} X X_{\psi}^{2} - \lambda^{-1} P^{-1} Z^{-1} X_{\lambda} X_{\psi} + \frac{1}{4} h^{-1} P^{-1} P_{\lambda} X_{\psi} + \frac{1}{4} \lambda^{-2} h^{-1} P_{\psi} X_{\psi} + \frac{1}{2} (1 + \lambda^{-1} X^2) P^{-1} Z^{-2} X Z_{\lambda}^{2} - \frac{1}{2} \lambda(1 + \lambda^{-1} X^2) h^{-1} P^{-1} Z^{-2} X Z_{\psi}^{2} + \frac{1}{2} \lambda^{-1} P^{-1} X_{\lambda}. \]
\[-\frac{1}{2} \lambda^{-1} P^{-1} Z^{-1} X Z + \frac{1}{2} X \gamma^{ab} \tilde{G}_{ab}. \]  

(5.11)

If the spacetime is not assumed to be \( \zeta \)-symmetric, the four functions \( h(\lambda, \psi), P(\lambda, \psi), \) \( Z(\lambda, \psi), \) and \( X(\lambda, \psi) \) are determined by the three first-order partial differential equations (PDEs) corresponding to Eq. (5.9) and the second-order PDE coming from Eq. (5.11), along with an appropriate set of boundary conditions.

**2. Geodesic polar coordinates**

The convenient degrees of freedom for defining the metric in geodesic polar coordinates are

\[
\begin{align*}
X(\rho, \varphi) &:= \lambda_{01} \quad (5.12a) \\
Z(\rho, \varphi) &:= \lambda_{11} \quad (5.12b) \\
\lambda(\rho, \varphi) &:= \lambda_{00} \lambda_{11} - \lambda_{01}^2 \quad (5.12c) \\
F(\rho, \varphi) &:= -\lambda \gamma_{33} / \rho^2. \quad (5.12d)
\end{align*}
\]

Now the light cylinder is a surface in \( \rho, \varphi \) space defined by \( \lambda(\rho, \varphi) = 0 \), and the flat-space forms of the other three quantities are well-behaved on that surface [cf. Eqs. (4.11), (4.12), with \( C = 0 \)]:

\[
\begin{align*}
X(\rho, \varphi) &= 0 \quad (5.13a) \\
Z(\rho, \varphi) &= 1 \quad (5.13b) \\
\lambda(\rho, \varphi) &= -(1 - \Omega^2 \rho^2) \quad (5.13c) \\
F(\rho, \varphi) &= 1. \quad (5.13d)
\end{align*}
\]

Note that the quantities \( X \) and \( Z \) are the same as those defined in \( \lambda \)-based coordinates. Substituting the expressions

\[
\begin{align*}
\lambda_{00} &= (\lambda + X^2)Z^{-1} \quad (5.14a) \\
\lambda_{01} &= X \quad (5.14b) \\
\lambda_{11} &= Z \quad (5.14c) \\
\gamma_{22} &= 1 \quad (5.14d) \\
\gamma_{23} &= 0 \quad (5.14e) \\
\gamma_{33} &= -\lambda^{-1} \rho^2 F \quad (5.14f) \\
c_0 &\equiv 2\Omega \quad (5.14g) \\
c_1 &\equiv 0 \quad (5.14h)
\end{align*}
\]

into Eqs. (3.24b) and (3.24c), another straightforward algebraic calculation gives the equations
\(-\gamma^{ab}\hat{G}_{ab} = \lambda^{AB}R_{AB} = -\frac{1}{2}\lambda^{-1}\lambda_{\rho\rho} + \frac{1}{2}\rho^{-2}F^{-1}\lambda_{\varphi\varphi} + \frac{1}{2}\lambda^{-2}\lambda_{\rho}^2 - \frac{1}{4}\lambda^{-1}F^{-1}\lambda_{\rho}F_{\rho}\) (5.15a)

\[\hat{G}_{23} = \hat{R}_{23} = -\frac{1}{2}\lambda^{-1}\lambda_{\rho\varphi} - \frac{1}{2}(1 + \lambda^{-1}X^2)Z^{-2}Z_{\rho}Z_{\varphi} - \frac{1}{2}\lambda^{-1}X_{\rho}X_{\varphi} + \frac{1}{4}\lambda^{-1}F^{-1}F_{\rho}\lambda_{\varphi} (5.15b)\]

\[-\frac{1}{4}\rho^{-2}Z^2\Omega^2;\]

\[\hat{G}_{ij} - \hat{G}_{33} = \hat{R}_{ij} - \hat{R}_{33}\]

\[-\frac{1}{2}\lambda^{-1}\lambda_{\rho\rho} - \frac{1}{2}\rho^{-2}F^{-1}\lambda_{\varphi\varphi} + \frac{1}{4}\lambda^{-1}F^{-1}\lambda_{\rho}F_{\rho} + \frac{1}{4}\rho^{-2}F^{-2}\lambda_{\varphi}F_{\varphi} (5.15c)\]

\[-\frac{1}{2}\lambda^{-1}Z^{-1}(\lambda_{\rho}Z_{\rho} + \lambda_{\varphi}Z_{\varphi}) + \frac{1}{2}\lambda^{-1}Z^{-1}X(Z_{\rho}X_{\varphi} + Z_{\varphi}X_{\rho}) + \frac{1}{2}\rho^{-1}\lambda^{-1}\lambda_{\rho};\]

which again uses the shorthand

\[h_{\rho} := \frac{\partial h}{\partial \rho}, X_{\varphi} := \frac{\partial^2 X}{\partial \varphi^2}, \text{ etc.} (5.16)\]

Note that these equations are now second order in \(\lambda(\rho, \varphi)\) but not in the other dependent variables. As in the previous section, these three equations, along with appropriate boundary conditions, are enough to specify the three functions \(\lambda, F,\) and \(Z\) if the spacetime is assumed to be even under \(\zeta\)-symmetry (in which case \(X\) vanishes everywhere). If not, we need a fourth, \(\zeta\)-odd, equation derived from Eq. (5.24a):

\[P_{0i}^{AB}G_{AB} = G_{0i} - \frac{1}{2}X\lambda^{AB}G_{AB} (5.17)\]

\[-\frac{1}{2}X_{\rho\rho} + \frac{1}{2}\lambda\rho^{-2}F^{-1}X_{\varphi\varphi} + \frac{1}{2}\lambda^{-1}XX_{\rho}^2 - \frac{1}{2}\rho^{-2}F^{-1}XX_{\rho}^2 - \lambda^{-1}Z^{-1}XX_{\rho}X_{\rho}\]

\[+\rho^{-2}F^{-1}Z^{-1}XX_{\rho}X_{\rho} + \frac{1}{2}\lambda^{-1}\lambda_{\rho}X_{\rho} - \frac{1}{4}F^{-1}F_{\rho}X_{\rho} - \frac{1}{4}\rho^{-2}\lambda F^{-2}F_{\rho}X_{\rho}\]

\[+\frac{1}{2}(1 + \lambda^{-1}X^2)Z^{-2}XX_{\rho}^2 - \frac{1}{2}\lambda(1 + \lambda^{-1}X^2)\rho^{-2}F^{-1}Z^{-2}XX_{\rho}^2 - \frac{1}{2}\lambda^{-1}Z^{-1}XX_{\rho}X_{\rho}\]

\[+\frac{1}{2}\rho^{-2}F^{-1}Z^{-1}XX_{\rho}X_{\rho} + \frac{1}{2}\rho^{-1}\lambda^{-1}\lambda_{\rho};\]

For a spacetime without \(\zeta\)-symmetry, the four functions \(\lambda(\rho, \varphi), F(\rho, \varphi), Z(\rho, \varphi),\) and \(X(\rho, \varphi)\) are determined by the four second-order PDEs corresponding to Eqs. (5.15) and (5.17), along with an appropriate set of boundary conditions.

VI. CONCLUSIONS

To summarize, we have used the two Killing vectors present in the spacetime of a pair of co-rotating cosmic strings to help us find the simplest set of quantities (four functions of
two variables; three if the spacetime is assumed to be $\zeta$-symmetric as defined in Sec. IV C) needed to describe the geometry of that spacetime. Since we learned in Sec. II that it is not possible (due to a lack of orthogonal transitivity) to define a subspace everywhere orthogonal to both Killing vectors, we instead worked on the two-dimensional space $S$ of Killing vector orbits.

In Sec. III, we derived the components of the Einstein tensor in terms of tensor fields on the two-manifold $S$, thereby streamlining and extending the derivation by Geroch of the vacuum Einstein equations. We also expressed the contracted Bianchi identities in terms of those components, and showed that two components of the Einstein tensor (and not just their derivatives) were (generically) simply linear combinations of the other components and their derivatives.

In Sec. IV, we described further gauge fixing possible within the Geroch formalism; in particular, the choice of coordinates on the two-manifold $S$. One possibility was to choose coordinates so that the two-geometry was described only by a single conformal factor; however, that was seen to be inconvenient due to the fact that $S$ had a Lorentzian signature outside of the “light cylinder”. A more convenient set of coordinates was found, which used the determinant $\lambda$ of the inner products of the Killing vectors as a radial coordinate. This reduced the number of independent scalar fields needed to describe the four-geometry. A third possibility is to mimic polar coordinates by defining lines of constant angular coordinate to be spacelike geodesics in $S$ radiating out from the origin and using the distance along those curves as a radial coordinate. With any of these coordinate choices, the spacetime geometry was found to be described by one parameter (representing the frequency of the fixed rotation) and four independent functions of the two coordinates on $S$, as detailed in Table I. If, as described in Sec. IV C, a further discrete symmetry (called $\zeta$-symmetry) was imposed upon the spacetime, the number of independent degrees of freedom was reduced to three, as detailed in Table II.

Finally, in Sec. V, we found explicit forms, in the $\lambda$-based coordinate system and the geodesic polar coordinate system, for the Einstein equations in terms of the four independent functions described in the previous section. This meant finding expressions for four components of the Einstein tensor.

It was shown that, in $\lambda$-based coordinates, three of those components led to first-order partial differential equations—i.e., they contained no second derivatives of the functions which described the metric. The fourth independent component, which was second order, vanished identically if the $\zeta$-symmetry of Sec. IV C was imposed. The Einstein equations were then written as a set of four (homogeneous in vacuo, inhomogeneous in the presence of matter) partial differential equations, three of them first-order and one second-order, for four functions of two variables. If the sources and boundary conditions uniquely specify a solution, and neither of them breaks the discrete $\zeta$-symmetry, the system of PDEs further simplifies to become three first-order equations for three functions of two variables.

In geodesic polar coordinates, which quasi-Newtonian descriptions of orbiting cosmic

\footnote{Four of the ten components of the Einstein tensor $G_{ab}$ had been set to zero \textit{a priori}, and two more could be determined by the Bianchi identities derived in Sec. III D.}
strings indicate may be better behaved, the situation was analogous, except that second
derivatives of one of the functions appeared in all of the Einstein equations.

To conclude, we note two aspects of the problem that were outside the scope of this
paper, but which will be addressed in future papers in this series:

(i) First, we said very little about the sources appearing in the Einstein equations. The
spacetime is taken to be vacuum away from the cosmic strings, but the stress-energy of the
strings themselves (and in fact whether they are better described by a distributional stress-
energy or a set of boundary conditions on a small circle surrounding each string) is not
discussed in this paper. However, since we always kept track of which components appeared
in expressions which were to be set to zero to give the vacuum Einstein equations, we will
be able to insert any sources once we have a stress-energy tensor.

(ii) Second, we left aside the issue of boundary conditions, both at infinity and at the
axis of rotation. In order to keep the system in equilibrium, we expect to need a balance
of incoming and outgoing radiation in the exterior boundary condition. How to implement
such a condition in our problem is a subject for further research \[1,4\]. In addition, there will
be complications in imposing boundary conditions at the axis of rotation (which corresponds
to $\lambda = -1$ in our coordinates), since the $\lambda, \psi$ coordinate system is badly singular there, as
the metric components \[(4.9)\] for co-rotating flat spacetime show. This, along with the fact
that $\lambda$ is not single-valued near the strings in the quasi-Newtonian spacetime discussed in
Sec. IV B3, implies that the geodesic polar coordinate system will prove more convenient.

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APPENDIX A: CALCULATION OF THE COVARIANT DERIVATIVE
OF THE KILLING VECTOR FIELDS

To calculate the covariant derivative of the Killing vector fields, we start by noting that
Killing's equation \[(3.1)\] implies that $\nabla_a K_{ab}$ is antisymmetric in its $a$ and $b$ indices. Thus,
$\nabla_a K_{ab}$ is completely specified by its contractions with the tensor fields $\epsilon^{ab}$, $\gamma^a_K^b$, and
$K^a_C K^b_D$.

(i) The contraction of $\nabla_a K_{ab}$ with $\epsilon^{ab}$ is straightforward. Using Eqs. \[(3.4)\] and \[(3.12)\], it
follows that

$$\epsilon^{ab} \nabla_a K_{ab} = |\lambda|^{-1/2} \epsilon^{abcd} K_{ce} K_{1d} \nabla_a K_{ab} = |\lambda|^{-1/2} c_A . \quad (A1)$$
(ii) To obtain the contraction of $\nabla_a K_{ab}$ with $\gamma_c^b K_D^a$, it is convenient to note that

$$K_D^b \nabla_a K_{ab} = \frac{1}{2} D_a \lambda_{AD} . \tag{A2}$$

Thus,

$$\gamma_c^b K_D^a \nabla_a K_{ab} = \frac{1}{2} D_c \lambda_{AD} . \tag{A3}$$

(iii) Similarly, by using Eq. (A2), we find

$$K_C^b K_D^a \nabla_a K_{ab} = 0 . \tag{A4}$$

Since

$$-\frac{1}{2} \epsilon^{-1} |\lambda|^{-1/2} \epsilon_{ab} c_A - \lambda^{BC} K_B[a D_b] \lambda_{CA} \tag{A5}$$

has the same three contractions with $\epsilon^{ab}$, $\gamma_c^b K_D^a$, and $K_C^b K_D^a$ as does $\nabla_a K_{ab}$, we have proven the equality (3.20).

**APPENDIX B: CALCULATION OF THE PROJECTED COMPONENTS OF THE RICCI TENSOR**

The following three subsections contain proofs of Eqs. (3.21a), (3.21b), and (3.21c). These results were stated without proof in the main text.

1. **Proof of Eq. (3.21a)**

Using the definition $R_{bd} := R^c_{bcd}$ and Eqs. (3.3), (A2), and (3.4), it follows that

$$R_{AB} := K_A^a K_B^b R_{ab} = R^c_{abc} K_A^a K_B^b = (\nabla_c \nabla_b K_A^c) K_B^b$$

$$= \nabla_c (K_B^b \nabla_b K_A^c) - (\nabla_c K_B^b) (\nabla_b K_A^c)$$

$$= -\frac{1}{2} \nabla_c D^c \lambda_{AB} + (\nabla_b K_{Ac}) (\nabla_b K_B^c) . \tag{B1}$$

The first term on the right-hand side (RHS) of Eq. (B1) can be evaluated by writing $D^c \lambda_{AB}$ as $\gamma_d^c D^d \lambda_{AB}$, and then differentiating $\gamma_d^c$ and $D^d \lambda_{AB}$ separately. The result is

$$-\frac{1}{2} \nabla_c D^c \lambda_{AB} = -\frac{1}{4} (\lambda^{-1} D_d \lambda) D^d \lambda_{AB} - \frac{1}{2} D_d D^d \lambda_{AB} , \tag{B2}$$

where we used

---

19 This result follows by differentiating Eq. (3.7).
\[ \nabla_c \gamma^c_d = \nabla_c (\delta^c_d - \lambda^{AB} K^c_A K^d_B) = -\lambda^{AB} K^c_A \nabla_c K^d_B = \frac{1}{2} \lambda^{AB} D_d \lambda_{AB} = \frac{1}{2} \lambda^{-1} D_d \lambda. \]  

(B3)

The second term on the RHS of Eq. (B1) can be evaluated by using Eq. (3.20) to expand the covariant derivatives of the Killing vector fields. The two cross-terms, which are proportional to \( \epsilon_{bc} K^c_d \), vanish. Only the \( \epsilon_{bc} \epsilon^{bc} \) and \( K^c_d K^b_d \) terms remain. Explicitly,

\[
(\nabla_b K_{Ac})(\nabla^b K^c_B) = \frac{1}{4} \epsilon^{-2} |\lambda|^{-1} \epsilon_{bc} \epsilon^{bc} c_{ACB} \]

+ \( \lambda^{CD} \lambda^{EF} K^c_C [b (D_c \lambda_{DA})] K^d_E (D^c \lambda_{FB}) \)

= \(-\frac{1}{2} \lambda^{-1} c_{CB} + \frac{1}{2} \lambda^{DF} (D_c \lambda_{DA})(D^c \lambda_{FB}) \).

(B4)

Thus

\[
R_{AB} = -\frac{1}{4} (\lambda^{-1} D^a \lambda) D_a \lambda_{AB} - \frac{1}{2} D^a D_a \lambda_{AB} \]

= \(-\frac{1}{2} \lambda^{-1} c_{AB} + \frac{1}{2} \lambda^{CD} (D^a \lambda_{AC})(D_a \lambda_{BD}) \).

(B5)

If desired, the RHS of this last expression can be rewritten using the identity

\[
\lambda^{CD} (D^a \lambda_{AC})(D_a \lambda_{BD}) = (\lambda^{-1} D^a \lambda) D_a \lambda_{AB} \]

= \(-\frac{1}{2} \lambda^{AB} \lambda^{-1} (D^a \lambda^{CD}) D_a \lambda_{CD} \),

which holds for any invertible 2 \( \times \) 2 matrix \( \lambda_{AB} \). Substituting Eq. (B6) into Eq. (B5) yields Eq. (3.21a).

2. Proof of Eq. (3.21b)

To obtain the projected components \( \tilde{R}_{\alpha \beta} := K^c_{\alpha} \gamma^{d}_{\beta} R_{cd} \), we start by writing

\[
\tilde{R}_{Ac} = K^a_{A} \gamma^{b}_{c} R_{ab} = R^d_{\alpha \beta} K^a_{A} \gamma^{b}_{c} \]

= \( \gamma^a_{c} \nabla^b \nabla_a K_{Ab} \).

(B7)

We then use Eq. (3.20) to expand \( \nabla_a K_{Ab} \):

\[
\gamma^a_{c} \nabla^b \nabla_a K_{Ab} = \gamma^a_{c} \nabla^b \left( -\frac{1}{2} \epsilon^{-1} |\lambda|^{-1/2} \epsilon_{ab} c_A \right) \]

= \(-\frac{1}{2} \gamma^a_{c} \nabla^b (\epsilon^{-1} |\lambda|^{-1/2} \epsilon_{ab} c_A) \)

= \(-\gamma^a_{c} \nabla^b (\lambda^{BC} K^a_{B[a} D^b) \lambda_{CA} \).

(B8)
A straightforward calculation shows that the 2nd term on the RHS of Eq. (B8) vanishes, while the 1st term can be calculated by first writing 

\[ \epsilon^{-1} |\lambda|^{-1/2} \epsilon \gamma_b^d \epsilon_{ab} c_A \] 

and then differentiating \( \gamma_b^d \) and \( \epsilon^{-1} |\lambda|^{-1/2} \epsilon_{cd} c_A \) separately. The result of this differentiation is that the first term equals

\[
1st \ term = \frac{1}{4} \epsilon^{-1} |\lambda|^{-1/2} (\lambda^{-1} D^d \lambda) \epsilon_{cd} c_A
\]

\[
- \frac{1}{2} D^d (\epsilon^{-1} |\lambda|^{-1/2} \epsilon_{cd} c_A),
\]

where we used Eq. (B3) and definition (B11). Moreover, by differentiating each of the factors of \( \epsilon^{-1} |\lambda|^{-1/2} \epsilon_{cd} c_A \) separately, we find

\[
- \frac{1}{2} D^d (\epsilon^{-1} |\lambda|^{-1/2} \epsilon_{cd} c_A)
\]

\[
= \frac{1}{4} \epsilon^{-1} |\lambda|^{-1/2} (\lambda^{-1} D^d \lambda) \epsilon_{cd} c_A - \frac{1}{2} \epsilon^{-1} |\lambda|^{-1/2} \epsilon_{cd} D^d c_A,
\]

where we used

\[
D^d \epsilon_{cd} = \frac{1}{2} \epsilon_{cd} \epsilon^{-1} D^d \epsilon
\]

to eliminate the derivatives of \( \epsilon \) and \( \epsilon_{cd} \). Finally, by combining Eqs. (B9) and (B10), and recalling that the 2nd term on the RHS of Eq. (B8) vanishes, we get Eq. (3.21b).

3. Proof of Eq. (3.21c)

To obtain the final set of projected components \( \hat{R}_{ab} := \gamma_c^a \gamma_b^d R_{cd} \), we proceed in a manner similar to Geroch [3], and consider the Riemann tensor \( \mathcal{R}^{a}_{bcd} \) of the two-dimensional metric \( \gamma_{ab} \):

\[
\mathcal{R}^{a}_{bcd} v^b := 2 D[v, D_d v^a].
\]

Here \( D_a \) is the covariant derivative operator compatible with \( \gamma_{ab} \) [see Eq. (3.11)], and \( v^a \) is an arbitrary vector field on \( \mathcal{S} \)—i.e., it satisfies

\[
K_{Aa} v^a = 0 \quad \text{and} \quad \mathcal{L}_{K_A} v^a = 0.
\]

In particular, \( K_{Aa} v^a = 0 \) implies \( \nabla_a (K_{Ab} v^b) = 0 \), so that

\[
K^b_A \nabla_a v_b = -v^b \nabla_a K_A b,
\]

while \( \mathcal{L}_{K_A} v^a = 0 \) is equivalent to

\[20\] If we allow \( \epsilon \) to change discontinuously from \(-1\) to \(+1\), \( D_a \epsilon \) and \( D_a \epsilon_{bc} \) do not vanish. They are related by Eq. (B11).
Equations (314) and (315) will be needed below to relate \( \mathcal{R}^a_{bcd} \) to the four-dimensional Riemann tensor

\[
R^a_{bcd}v^b := 2\nabla_c \nabla_d v^a .
\]

Using the symmetry properties of the Riemann tensor, it follows that Eq. (312) is equivalent to

\[
\mathcal{R}_{abcd} v^d = 2D_{[a} D_{b]} v_{c} .
\]

Using Eq. (3.11) to evaluate the RHS, we find

\[
2D_{[a} D_{b]} v_{c} = 2\gamma^d_{[a} \gamma^f_{b]} \gamma^c_{g} \nabla_d (\gamma^g_{e} \gamma^h_{f} \nabla_g v_{h})
= 2\gamma^d_{[a} \gamma^f_{b]} \gamma^c_{g} \left[ (\nabla_d \gamma^h_{e}) \gamma^f_{f} \nabla_g v_{h} + \gamma^g_{e} (\nabla_d \gamma^h_{f}) \nabla_g v_{h} + \gamma^g_{e} \gamma^h_{f} \nabla_d \nabla_g v_{h} \right] .
\]

The last term above can be rewritten as

\[
2\gamma^d_{[a} \gamma^f_{b]} \gamma^h_{c} \nabla_d \nabla_g v_{h} = \gamma^d_{[a} \gamma^g_{b]} \gamma^h_{c} R_{dgh} f v^f = \gamma^d_{[a} \gamma^g_{b]} \gamma^h_{c} R_{efgh} v^d ,
\]

while the first two terms can be rewritten as

1st two terms = \( 2\gamma^d_{[a} \gamma^g_{b]} \gamma^h_{c} \nabla_d \nabla_g v_{h} \)

= \( -2\lambda^{AB} \left[ \gamma^d_{[a} \gamma^g_{b]} \gamma^h_{c} (\nabla_d K_{Ac}) K^a_{B} \nabla_g v_{h} + \gamma^d_{[a} \gamma^g_{b]} \gamma^h_{c} (\nabla_d K_{Af}) K^h_{B} \nabla_g v_{h} \right] \)

= \( -2\lambda^{AB} \left[ \gamma^d_{[a} \gamma^g_{b]} \gamma^h_{c} (\nabla_d K_{Af}) (\nabla_g K_{Bh}) v^g - \gamma^d_{[a} \gamma^g_{b]} \gamma^h_{c} (\nabla_d K_{Af}) (\nabla_g K_{Bh}) v^h \right] \),

where we used Eqs. (314) and (315) to obtain the last equality. After a little more “index gymnastics,” we find

\[
1st \ two \ terms = 2\lambda^{AB} \gamma^e_{[a} \gamma^g_{b]} \gamma^h_{c} \gamma^d_{g} \left[ (\nabla_e K_{Af}) (\nabla_g K_{Bh}) + (\nabla_e K_{Ag}) (\nabla_f K_{Bh}) \right] v^d .
\]

Thus, \( v^d \) is arbitrary,

\[
\mathcal{R}_{abcd} = \gamma^e_{[a} \gamma^g_{b]} \gamma^h_{c} \gamma^d_{g} \left[ R_{efgh} + 2\lambda^{AB} (\nabla_e K_{Af}) (\nabla_g K_{Bh}) + 2\lambda^{AB} (\nabla_e K_{Ag}) (\nabla_f K_{Bh}) \right] .
\]

Now contract Eq. (322) with \( \gamma^ac \). The left-hand side is simply \( \mathcal{R}_{bd} \), while the 1st term on the RHS is given by

\[
\gamma^f_{b} \gamma^h_{d} \gamma^g_{e} R_{efgh} = \gamma^f_{b} \gamma^h_{d} \left( \delta^g_{e} - \lambda^{AB} K^a_{A} K^g_{B} \right) R_{efgh}
= \gamma^a_{g} \gamma^c_{d} \left( R_{ac} - \lambda^{AB} K^e_{A} \nabla_a \nabla_e K_{Bc} \right)
= \gamma^a_{g} \gamma^c_{d} \left( R_{ac} - \lambda^{AB} \nabla_a (K^e_{A} \nabla_e K_{Bc}) + \lambda^{AB} (\nabla_a K^e_{A}) \nabla_e K_{Bc} \right)
= \gamma^a_{g} \gamma^c_{d} \left( R_{ac} + \frac{1}{2} \lambda^{AB} (D_a D_c \lambda_{AB}) + \lambda^{AB} (\nabla_a K^e_{A}) \nabla_e K_{Bc} \right) .
\]
By substituting Eq. (B23) into Eq. (B22), and using Eq. (3.20) to expand the covariant derivatives of the Killing vector fields, we find

\[ R_{bd} = \gamma_b^{a} \gamma_d^{c} R_{ac} + \frac{1}{4} \lambda^{AB} D_b D_d \lambda_{AB} + \frac{1}{4} D_b (\lambda^{-1} D_d \lambda) \]

\[ -\frac{1}{2} \gamma_{bd} \lambda^{-1} \lambda^{AB} c_{ACB} , \]

which is equivalent to

\[ \hat{R}_{ab} := \gamma_a^{c} \gamma_b^{d} R_{cd} = R_{ab} - \frac{1}{4} \lambda^{AB} D_a D_b \lambda_{AB} \]

\[ -\frac{1}{4} D_a (\lambda^{-1} D_b \lambda) + \frac{1}{2} \gamma_{ab} \lambda^{-1} \lambda^{AB} c_{ACB} . \]

If desired, the RHS of this last expression can be simplified further using the identity

\[ R_{abcd} = R_{\gamma(a[c} \gamma_{d]b)} , \]

which holds in two dimensions. In particular,

\[ R_{ab} = \frac{1}{2} R \gamma_{ab} \]

(i.e., the two-dimensional Einstein tensor vanishes), so that

\[ \hat{R}_{ab} = -\frac{1}{4} \lambda^{AB} D_a D_b \lambda_{AB} - \frac{1}{4} D_a (\lambda^{-1} D_b \lambda) \]

\[ +\frac{1}{2} \gamma_{ab} (R + \lambda^{-1} \lambda^{AB} c_{ACB}) . \]

In addition, it will prove useful to use the identity

\[ \lambda^{AB} D_a D_b \lambda_{AB} = \lambda^{-1} D_a D_b \lambda \]

\[ -\lambda^{-1} (D_a \lambda \lambda^{AB})(D_b \lambda_{AB}) \]

to convert Eq. (B28) into the final form, Eq. (3.21c).

APPENDIX C: COORDINATE BASES ON THE FOUR-MANIFOLD

In Sec. IIIE we demonstrated that the metric tensor components and commutation coefficients in a non-coordinate basis on the four-manifold \( \mathcal{M} \) can be obtained from a set of fields on the two-manifold \( S \) of Killing vector orbits: the scalar fields \( \{ \lambda_{AB} \} \) and \( \{ c_A \} \) and the components \( \{ \gamma_{ij} \} \) in a coordinate system \( \{ x^i \} \) of the metric tensor \( \gamma_{ab} \). For those uncomfortable with this definition of “recovering the four-geometry,” we discuss in this appendix the definition of a coordinate basis on \( \mathcal{M} \), and the components of the metric in that basis.

In assigning coordinates \( \{ x^\mu | \mu = 0, 1, 2, 3 \} \) to every point in the spacetime \( \mathcal{M} \), we have two things at our disposal:
The coordinates \( \{x^i | i = 2, 3\} \) on the manifold \( S \) of Killing vector orbits, which can be used to assign values of \( \{x^A\} \) to each Killing trajectory, and hence to each point along each Killing trajectory. This definition means that the basis one-forms \( e^i_\alpha = (dx^i)_\alpha \) on \( S \) are also two of the four basis one-forms on \( M \).

The Killing vectors \( \{K^\alpha_A | A = 0, 1\} \), which can be used to assign values of \( \{x^A\} \) relative to some origin, along each Killing vector orbit (which is itself a two-dimensional subspace of \( M \)). This definition means that the directional derivative along the Killing vector \( K^\alpha_A \) is \( \partial / \partial x^A \), and gives us two basis vectors \( e^\alpha_A = K^\alpha_A \) on \( M \).

However, the assignment of all four coordinates to each point on \( M \) is not complete until we state where on each Killing trajectory the origin \( x^0 = x^1 = 0 \) is, and in particular how that origin is carried from one Killing trajectory to the next. Identifying these surfaces of constant \( x^A \) is equivalent to defining the two basis one-forms \( \tilde{e}^A_\alpha = (dx^A)_\alpha \). We know their components along the Killing vectors are \( \tilde{e}^A_i K^i_A = \partial x^A / \partial x^B = \delta^A_B \), but their components orthogonal to the Killing vectors give us two as yet unknown one-forms \( \{\beta^A_\alpha\} \) on \( S \):

\[
\beta^A_\alpha := (dx^A)_\alpha - \lambda^{AB} K_{BA}.
\]

The specification of the \( \{\beta^A_\alpha\} \) completes the definition of a coordinate system on \( M \) (up to an overall additive constant in \( x^A \)), and we can explicitly write the implied definitions of the complementary basis one-forms \( \{\tilde{e}^A_i\} \) and vectors \( \{e^a_i\} \), as well as the covariant and contravariant components of the metric tensor. To do this, we first note that since the directional derivative along \( \tilde{e}^A_i \) is \( \partial / \partial x^i \), we know that \( \tilde{e}^A_i e^j_\alpha = \delta^A_j \). This means that any vector on \( S \) will have the same components along \( e^a_i \) as it does along \( e^\alpha_i = \gamma_{ij} g^{ab} (dx^j)_b \), the latter form being defined independent of the specification of \( \{\beta^A_\alpha\} \). The components \( \beta^A_i = e^a_i \beta^A_a \) are thus defined in a non-circular way, and it is reasonable to write the basis forms and vectors and metric tensor components in terms of \( \{\beta^A_i\} \):

\[
\begin{align*}
\tilde{e}^A_\alpha &= \lambda^{AB} K_{BA} + \beta^A_\alpha e^i_\alpha & (C2a) \\
e^\alpha_i &= e^\alpha_i - \beta^A_i K^i_A & (C2b) \\
\tilde{g}_{AB} &= \lambda_{AB} & (C2c) \\
\tilde{g}_{ij} &= -\lambda_{AB} \beta^B_j & (C2d) \\
\tilde{g}_{ij} &= \gamma_{ij} + \lambda_{AB} \beta^B_i \beta^B_j & (C2e) \\
\tilde{g}^{AB} &= \lambda^{AB} + \gamma^{ij} \beta^A_i \beta^B_j & (C2f) \\
\tilde{g}^{ij} &= \gamma^{ij} \beta^B_j & (C2g) \\
\tilde{g}^{ij} &= \gamma^{ij} \\
\end{align*}
\]

We know that \( \{\beta^A_\alpha\} \) depend on the choice of constant-\( \{x^A\} \) surfaces, but we have not so far specified what values are allowed for the functions \( \{\beta^A_i\} \), nor have we made use of the parameters \( \{e_A\} \) which we know prevent us from setting \( \{\beta^A_a\} \) to zero. We do both now by noting that we found in Sec. III E that

\[21\text{We use the tilde to distinguish this coordinate basis from the non-coordinate basis in Sec. III E.}\]
\[ [e_i, e_j]^a = \epsilon^{-1} |\lambda|^{-1/2} \epsilon_{ij} \lambda^{AB} c_B K_A^a \quad (C3) \]

[cf. Eq. (3.44)]. On the other hand, Eq. (C2b), along with the condition that \( \{\tilde{e}^a_i\} \) be part of a coordinate basis on \( \mathcal{M} \), tells us that

\[ [e_i, e_j]^a = (\beta^A_{j,i} - \beta^A_{i,j}) K_A^a. \quad (C4) \]

Combining the two gives

\[ \epsilon^{ab} \partial_a \beta^A_b = - |\lambda|^{-1/2} \lambda^{AB} c_B, \quad (C5) \]

which is the equation which must be obeyed by \( \{\beta^A_a\} \). Of course, this condition does not completely specify the one-forms \( \{\beta^A_a\} \); the left-hand side is unchanged by the “gauge transformation”

\[ \beta^A_a \rightarrow \beta^A_a + \partial_a \xi^A \quad (C6) \]

where \( \{\xi^A\} \) are arbitrary scalar fields on \( S \). Not coincidentally, we see from the definition (C1) that this is exactly the change induced in \( \{\beta^A_a\} \) by a displacement (not constant over \( S \))

\[ x^A \rightarrow x^A + \xi^A \quad (C7) \]

in the coordinates along the Killing trajectories.

One possible gauge fixing would be to define geodesic polar coordinates (Sec. IV B 3) on \( S \) and then impose the condition \( \beta^A_\rho = 0 \), which allows \( \beta^A_\rho \) to be defined by integrating (C3) with respect to \( \rho \). We see from Eq. (C2d) and Eq. (C2e) that \( \tilde{g}^A_\rho \equiv 0 \) and \( \tilde{g}_i^\rho = \delta_i^\rho \), which is just the normal form (29.1) specified in [5] for the metric in a coordinate basis on a four-manifold with a two-dimensional Abelian symmetry group.

So, in order to completely specify the components of the metric tensor in a coordinate basis, we must be provided, in addition to the quantities listed in Sec. I I I E, with four additional functions \( \{\beta^A_i\} \) to replace the two functions \( \{c_A\} \). However, the information in those functions beyond that given by \( \{c_A\} \) is simply gauge information needed to state which definition of a constant-\( x^A \) surface we are using.

**APPENDIX D: CORRESPONDENCE WITH THE FORMULAS OF [3]**

While the derivation, in Sec. III, of the components of the Einstein tensor \( G_{ab} \) does not exactly parallel that used by Geroch in [3] to find the vacuum Einstein equations, the purpose of this appendix is to point out the counterparts of the equations in Geroch’s Appendix A, where they exist, and to present the equivalent equations in our notation when they do not. One thing to note is that in many cases two or three equations in Geroch can be written as a single equation in our notation, since we use the indices \( A, B \), etc. to discuss the behavior of quantities like the matrix \( \{\lambda_{AB}\} \) of inner products, while Geroch lists the components \( \lambda_{00}, \lambda_{01} \) and \( \lambda_{11} \) separately.
Geroch’s conditions (A1),(A2) for a tensor to be on $S$ are our Eqs. (3.8),(3.9). Note that we write the Killing vectors as \{K_a^A | A = 0, 1\}, while he writes them as $\xi^a$ and $\xi^a_0$. Geroch’s definition (A3) of $\lambda_{AB}$ is our Eq. (3.3), and his (A4) is our Eq. (3.6). Note that the determinant $\lambda$ is written in his notation as $-\tau^2/2$ (since he assumes it to be negative). His definition (A5) is our Eq. (3.7), with our $\gamma_{ab}$ being the same as his $h_{ab}$. His (A6) is our Eq. (3.12), or, written without resort to the use of explicit values for the $A$ and $B$ indices, Eq. (3.16). His definition (A7) of the covariant derivative on $S$ becomes our Eq. (3.11). As noted in Sec. III B, we do not define a matrix of twist vectors in our derivation, but we can write Geroch’s definitions (A8) of the twists compactly as

$$\omega^{a}_{AB} = \epsilon^{abcd} K_{(Ab} \nabla c K_{B)d}. \quad \text{(D1)}$$

[We have here defined the convention that symmetrization (and antisymmetrization) operate independently on abstract indices like $a, b, \ldots$ and concrete indices like $A, B, \ldots$.] Geroch’s definitions (A9) of $c_0$ and $c_1$ are our Eq. (3.4) or—more elegantly written—Eq. (3.15). The projections (A10) of the twists are of course written

$$\nu^{a}_{AB} = \gamma^{a}_{bc} \omega^{b}_{cAB}. \quad \text{(D2)}$$

Geroch makes the statements, between displayed equations (A10) and (A11), that the vacuum Einstein equations indicate that $c_0$ and $c_1$ are constants and that the twists are curl-free. In the non-vacuum case, the former statement becomes our Eq. (3.21b), and the latter becomes

$$\nabla \omega_{AB} = -\epsilon^{abcd} K_{(Ab} \nabla c K_{B)d}. \quad \text{(D3)}$$

The curls (A11) of the projected twists are written in our notation as

$$D_{[a} \nu_{AB]b} = \epsilon^{abcd} R^c_{e} K^{d}_{(A} K^{e}_{B)}. \quad \text{(D4)}$$

where we have used $\lambda^{AB}$ to raise the second index on $\epsilon_{AB}$. Geroch’s (A12) is one component of our Eq. (A2); his (A13) is the one pair of equations which cannot be written in a compact form because they only involve the norms of the Killing vectors ($\lambda_{00}$ and $\lambda_{11}$) and not their inner product ($\lambda_{01}$), and similarly only the diagonal elements of the twist matrix $\omega^{a}_{AB}$. This is because each is basically a result from the formalism with only one Killing vector \cite{9}. There is, however, an analogous equation

$$\nabla_{a} K_{CB} = \frac{3}{4} \left( - \frac{1}{2} \lambda^{AB} \epsilon_{abcd} K^{c}_{(A} \omega^{d}_{BC)} + \frac{\epsilon_{(A}^c \epsilon_{B)c}^e \epsilon_{C}^d}{2\lambda} \epsilon_{ab} \right). \quad \text{(D5)}$$

for the covariant derivative of a Killing vector in terms of the twists. [Of course, by deriving Eq. (3.20) initially, our derivation has circumvented these steps.] Geroch’s equations (A14) can also be derived from Eq. (D5), and written as

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Geroch’s first Einstein equations (A15) are equivalent to the trace-free equation (3.24a); specifically, they arise from setting to zero in turn the components of \( \epsilon_{(A}^C R_{B)C} \). The fact that only two of these three equations are independent is apparent from the fact that they involve only the trace-free part of the matrix \( \{ R_{AB} \} \). Geroch obtains a third independent equation (A16) by calculating \( D^a D_a \lambda_{00} \); one can equivalently look at \( \lambda^{AB} D^a D_a \lambda_{AB} \), which leads to the additional equation (3.24b). The three independent equations, which Geroch writes as (A18), are just the components of Eq. (3.21a). Geroch obtains a fourth Einstein equation by starting from an equation (A20) for the Riemann tensor on \( S \) which can be written in our compact notation as

\[
R_{abcd} = \gamma_{[a}^e \gamma_{b]}^f \gamma_{c}^g \gamma_{d}^h \left[ R_{efgh} + 4 \lambda^{AB} (\nabla_e K_{A(f)} (\nabla_g) K_{Bh}) \right].
\]

Now, Geroch contracts with \( \gamma_{ab} \) on both pairs of indices to obtain the equation (A21), which is equivalent to our Eq. (3.24d). In so doing, however, he misses the equation (3.24d) for the trace-free part of \( \hat{R}_{ab} \). For, while it is true that all three terms in Eq. (D7) have the symmetries of the Riemann tensor and are thus completely characterized by their trace on the two-dimensional manifold \( S \), when the first term on the right-hand side is traced over one set of indices to give \( \gamma_{a}^c \gamma_{b}^d R_{cd} \), this manifestly trace-only quantity is split into \( \gamma_{a}^c \gamma_{b}^d R_{cd} \) and another piece, each of which may individually have non-vanishing trace-free parts.

Note that while Geroch only provided four of the six block-diagonal vacuum Einstein equations, the contracted Bianchi identities described in Sec. [IIIB] mean that only four of these six equations are independent anyway. However, the identities are algebraic only in the components \( G_{AB} \), and the “missing” equations are for \( P_{cd}^a \hat{R}_{ab} = P_{cd}^a \hat{G}_{ab} \). The statement about those components implied by the Bianchi identity (3.34b) (setting the off-block-diagonal components of the Einstein tensor to zero identically) is

\[
D^b P_{ab} \hat{R}_{ab} = D^b (\gamma_{ab} \lambda^{AB} R_{AB}) - \frac{1}{2} G_{AB} D_a \lambda^{AB},
\]

which seems to imply that two of Geroch’s four equations are higher-order than they need to be. (That is, they are implied by the derivatives of equations he leaves out.)
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