Multi-dimensional reflected BSDEs driven by $G$-Brownian motion with diagonal generators

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Abstract

We consider the well-posedness problem of multi-dimensional reflected backward stochastic differential equations driven by $G$-Brownian motion ($G$-BSDEs) with diagonal generators. Two methods, including the penalization method and the Picard iteration argument, are provided to prove the existence and uniqueness of the solutions. We also study its connection with the obstacle problem of a system of fully nonlinear PDEs.

Keywords: $G$-expectation, $G$-Brownian motion, multi-dimensional BSDEs, fully nonlinear PDEs.

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1 Introduction

Pardoux and Peng [20] first introduced the nonlinear backward stochastic differential equations (BSDEs) taking the following form

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s,$$

where the solution is a pair of adapted processes $(Y, Z)$. The BSDE theory has attracted a great deal of attention since it has wide applications in partial differential equations (PDEs), stochastic control and financial mathematics (e.g., [6, 21, 22]). One of the most important extensions of BSDE theory is the reflected BSDEs (see [4]), which means that the solution $Y$ of the equation is forced to stay above a given process, called the obstacle. To this end, a non-decreasing process should be added to push the solution upward, which behaves in a minimal way such that it satisfies the Skorohod condition. For some further developments, one may refer to [1, 5, 7, 12, 13, 26, 29] and the references therein.

It is worth pointing out that all the papers listed above are considered under the classical expectation framework. Recently, Peng systematically introduced a time-consistent nonlinear expectation theory, i.e., the $G$-expectation theory, see [23, 24, 25], which is a useful tool to study the financial problems under volatility uncertainty and the probabilistic representation for fully nonlinear PDEs. In this framework, a new type of Brownian motion with independent and stationary increments, called $G$-Brownian motion, was constructed and the corresponding Itô's integral was established. Then, Hu et al. [8] investigated the following type of backward stochastic differential equations driven by $G$-Brownian motion ($G$-BSDEs):

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)dB_s - \int_t^T Z_s dB_s - (K_T - K_t).$$

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Compared with the classical case, there exists an additional non-increasing \(G\)-martingale \(K\) in this equation and the quadratic variation process of \(B\) is not deterministic. The authors obtained the existence and uniqueness result of the above \(G\)-BSDE. The comparison theorem, Feymann-Kac formula and Girsanov transformation were established in the companion paper \cite{9}. One important feature is that the solution \(Y\) in \cite{8} is required to be one-dimensional. Liu \cite{19} extended the results to the multi-dimensional case where the generators are assumed to be diagonal with respect to the \(z\)-term. After that, Hu et al. \cite{11} consider the multi-dimensional case with diagonally quadratic generators.

The reflected \(G\)-BSDE with a lower obstacle \(S\) has been established by Li, Peng and Soumana Hima \cite{16}, where the solution is a triple of processes \((Y, Z, A)\) with dynamics

\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d(B)_s - \int_t^T Z_s dB_s + (A_T - A_t),
\]

such that \(Y_t \geq S_t\) and \(\{-\int_0^t (Y_s - S_s)dA_s\}_{t \in [0, T]}\) is a non-increasing \(G\)-martingale. In fact, the non-decreasing process \(A\) can be regarded as the discrepancy between the non-decreasing process aiming to push the solution \(Y\) upward and the non-increasing \(G\)-martingale appearing in \(G\)-BSDEs. Due to the appearance of the non-increasing \(G\)-martingale, the process \(A\) in the reflected \(G\)-BSDE with an upper obstacle is not monotone, which makes this kind of reflected \(G\)-BSDEs significantly different from the lower obstacle case. Applying a variant comparison theorem, Li and Peng \cite{15} shows that the solution constructed by the penalization method is the maximal one. For the double obstacles case, i.e., the solution lies between two given processes, we may refer to the paper \cite{17}. It should be pointed out that the first component \(Y\) of reflected \(G\)-BSDEs in \cite{15, 16, 17} is one-dimensional.

The objective of this paper is to investigate the multi-dimensional reflected \(G\)-BSDEs. The main difficulty in the multi-dimensional case is that, due to the nonlinearity of \(G\)-expectation, the linear combination of \(G\)-martingales is no longer a \(G\)-martingale. To overcome this difficulty, we make use of a diagonal structure so that the \(i\)-th component of the equation can be independently considered under appropriate circumstances. More precisely, let \(x^i\) be the \(i\)-th component of a vector \(x \in \mathbb{R}^k\), \(i = 1, \cdots, k\). The multi-dimensional reflected \(G\)-BSDE is of the following type: for \(i = 1, \cdots, k\),

\[
\begin{cases}
Y_i^t = \xi^i + \int_t^T f^i(s, Y_s, Z_s)ds + \int_t^T g^i(s, Y_s, Z_s)d(B)_s - \int_t^T Z_s dB_s + (A_T^i - A_t^i), \\
Y_i^t \geq S_i^t, \ 0 \leq t \leq T, \\
\{-\int_0^t (Y_s^i - S_s^i)dA_s^i\}_{t \in [0, T]}\text{ is a non-increasing } G\text{-martingale,}
\end{cases}
\]

where \(Y = (Y^1, \cdots, Y^k)^T\). Note that the \(z\)-term of the \(i\)-th components of generators \(f^i, g^i\) here only depend on \(z^i\), which is the point where diagonal refers to.

The existence and uniqueness result is established by two different approaches. The first one provided in Subsection 3.2 is the penalization method, a scheme that is frequently used for constructing the solutions of reflected equations. By a suitable modification to our new situation, this construction is still valid. Roughly speaking, the \(Y\)-term of the solutions to multi-dimensional reflected \(G\)-BSDEs can be approximated by a family of solutions to multi-dimensional \(G\)-BSDEs. However, unlike the one-dimensional case, the comparison theorem does not apply in general since a kind of structure condition is needed for the multi-dimensional case (see Theorems 2.4 and 3.8). To tackle this, we shall make use of a method of linearization to prove the convergence of the approximation sequence. The second one given in Subsection 3.3 is the Picard iteration method motivated by the study of the standard multi-dimensional \(G\)-BSDEs in \cite{19}. We make use of a contraction argument for the \(Y\)-term to derive the local well-posedness result, in virtue of the a priori estimates for reflected \(G\)-BSDEs. The global situation is then obtained by a backward iteration of the local ones. As an application, we provide a probabilistic representation for solutions of a system of fully nonlinear PDEs with obstacle constraints, based on the construction via penalization.

This paper is organized as follows. In Section 2, we recall some basic notions of \(G\)-expectation, multi-dimensional \(G\)-BSDEs and one-dimensional reflected \(G\)-BSDEs. Section 3 is devoted to the study of well-posedness of multi-dimensional reflected \(G\)-BSDEs, based on the Picard iteration and the penalization method, respectively. We formulate our probabilistic representation for fully nonlinear PDE systems with obstacles in Section 4.
2 Preliminaries

We first recall some basic results about $G$-expectation, multi-dimensional $G$-BSDEs and one-dimensional reflected $G$-BSDEs, which are needed in the sequel and the readers may refer to the papers \[3, 16, 17, 19, 23, 24, 25, 27\] for more details. For convenience, every element $x \in \mathbb{R}^k$ is identified as a column vector with $l$-th component $x^l$, and the corresponding Euclidian norm and Euclidian scalar product are denoted by $| \cdot |$ and $\langle \cdot , \cdot \rangle$, respectively. For two vectors $a$ and $b$ in $\mathbb{R}^k$, we say that $a \geq b$ if $a^l \geq b^l$, for each $1 \leq l \leq k$.

2.1 $G$-expectation

Let $\Omega = C_0([0, \infty); \mathbb{R})$, the space of real-valued continuous functions starting from the origin, be endowed with the distance

$$
\rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i} [\max_{t \in [0,1]} |\omega^1_t - \omega^2_t|] \wedge 1, \text{ for } \omega^1, \omega^2 \in \Omega.
$$

Let $B$ be the canonical process on $\Omega$. Set

$$
L_{ip}(\Omega) := \{ \varphi(B_{t_1}, \ldots, B_{t_n}) : n \in \mathbb{N}, \ t_1, \ldots, t_n \in [0, \infty), \ \varphi \in C_b, Lip(\mathbb{R}^n) \},
$$

where $C_b, Lip(\mathbb{R}^n)$ denotes the set of bounded Lipschitz functions on $\mathbb{R}^n$. Let $G : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$
G(a) = \frac{1}{2} (\sigma^2 a^+ - \sigma^2 a^-),
$$

for $0 < \sigma^2 \leq \bar{\sigma}^2 < \infty$. The (conditional) $G$-expectation for $\xi \in L_{ip}(\Omega)$ can be calculated as follows. Assume

$$
\xi = \varphi(B_{t_1}, B_{t_2}, \ldots, B_{t_n}).
$$

Then, for $t \in [t_{k-1}, t_k)$, $k = 1, \ldots, n$, we define

$$
\hat{\mathbb{E}}_t[\varphi(B_{t_1}, B_{t_2}, \ldots, B_{t_n})] = u_k(t, B_{t}; B_{t_1}, \ldots, B_{t_{k-1}}),
$$

where, for any $k = 1, \ldots, n$, $u_k(t, x; x_1, \ldots, x_{k-1})$ is a function of $(t, x)$ parameterized by $(x_1, \ldots, x_{k-1})$ such that it solves the following fully nonlinear PDE defined on $[t_{k-1}, t_k) \times \mathbb{R}$:

$$
\partial_t u_k + G(\partial_x^2 u_k) = 0
$$

with terminal conditions

$$
u_k(t_k, x_1, \ldots, x_{k-1}) = u_{k+1}(t_k, x_1, \ldots, x_{k-1}, x), \ k < n
$$

and $u_n(t_n, x_1, \ldots, x_{n-1}) = \varphi(x_1, \ldots, x_{n-1}, x)$. The $G$-expectation of $\xi$ is defined by $\hat{\mathbb{E}}[\xi] := \hat{\mathbb{E}}_0[\xi]$. We call $(\Omega, L_{ip}(\Omega), \hat{\mathbb{E}})$ the $G$-expectation space.

For each $p \geq 1$, the completion of $L_{ip}(\Omega)$ under the norm $\| \xi \|_{L_{ip}^p} := \left( \hat{\mathbb{E}}[|\xi|^p]\right)^{1/p}$ is denoted by $L_{ip}^p(\Omega)$. The conditional $G$-expectation $\hat{\mathbb{E}}_t[\cdot]$ can be extended continuously to the completion $L_{ip}^p(\Omega)$. For each fixed $T \geq 0$, set $\Omega_T = \{ \omega_{NT} : \omega \in \Omega \}$. We may define $L_{ip}(\Omega_T)$ and $L_{ip}^p(\Omega_T)$ similarly. Besides, Denis et al. \[3\] proved that the $G$-expectation has the following representation.

**Theorem 2.1** (\[3\]). There exists a weakly compact set $\mathcal{P}$ of probability measures on $(\Omega, \mathcal{B}(\Omega))$ such that

$$
\hat{\mathbb{E}}[\xi] = \sup_{P \in \mathcal{P}} E_P[\xi] \text{ for all } \xi \in L_{ip}^1(\Omega).
$$

$\mathcal{P}$ is called a set that represents $\hat{\mathbb{E}}$. 

3
Let $\mathcal{P}$ be a weakly compact set that represents $\hat{E}$. For this $\mathcal{P}$, we define the capacity

$$c(A) := \sup_{P \in \mathcal{P}} P(A), \ A \in \mathcal{B}(\Omega).$$

A set $A \in \mathcal{B}(\Omega)$ is called polar if $c(A) = 0$. A property holds “quasi-surely” (q.s.) if it holds outside a polar set. In the following, we do not distinguish two random variables $X$ and $Y$ if $X = Y$, q.s.

For $\xi \in L_{ip}(\Omega_T)$, let $\mathcal{E}(\xi) = \hat{E}[(\sup_{t \in [0,T]} E_t[\xi])]$ and $\mathcal{E}$ is called the $G$-evaluation. For $p \geq 1$ and $\xi \in L_{ip}(\Omega_T)$, define $\|\xi\|_{p,\mathcal{E}} = [E(\|\xi\|^p)]^{1/p}$ and denote by $L_{ip}^p(\Omega_T)$ the completion of $L_{ip}(\Omega_T)$ under $\| \cdot \|_{p,\mathcal{E}}$.

The following theorem can be regarded as Doob’s maximal inequality under $G$-expectation.

**Theorem 2.2** ([27]). For any $\alpha \geq 1$ and $\delta > 0$, $L_{ip}^{p+\delta}(\Omega_T) \subset L_{ip}^0(\Omega_T)$. More precisely, for any $1 < \gamma < \beta := (\alpha + \delta)/\alpha$, $\gamma \leq 2$, we have

$$\|\xi\|_{\alpha,\mathcal{E}} \leq \gamma^\gamma (\|\xi\|_{L_{ip}^{p+\delta}}^{\alpha} + 14^{1/\gamma} C_{\beta/\gamma} \|\xi\|_{L_{ip}^{p+\delta}}^{(\alpha+\delta)/\gamma}), \ \forall \xi \in L_{ip}(\Omega_T),$$

where $C_{\beta/\gamma} = \sum_{i=1}^{\infty} i^{-\beta/\gamma}, \ \gamma^* = \gamma/(\gamma - 1)$.

For $T > 0$ and $p \geq 1$, the following spaces will be frequently used in this paper.

- $M_{ip}^0(0,T) := \{ \eta : \eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega)1_{(t_j,t_{j+1})}(t) \}$, where $\xi_j \in L_{ip}(\Omega_{t_j})$, $t_0 \leq \cdots \leq t_N$ is a partition of $[0,T]$.
- $M_{ip}^p(0,T)$ is the completion of $M_{ip}^0(0,T)$ under the norm $\|\eta\|_{M_{ip}^p} := (\hat{E}[\int_0^T |\eta_t|^p dt])^{1/p}$.
- $H_{ip}^p(0,T)$ is the completion of $M_{ip}^0(0,T)$ under the norm $\|\eta\|_{H_{ip}^p} := (\hat{E}[\int_0^T |\eta_t|^p dt])^{1/p}$.
- $S_{ip}^p(0,T) = \{ h(t, B_{t\wedge t}, \ldots, B_{t\wedge T}) : t_1, \ldots, t_n \in [0,T], h \in C_b(L_{ip}(\mathbb{R}^{n+1})) \}$.
- $S_{ip}^p(0,T)$ is the completion of $S_{ip}^p(0,T)$ under the norm $\|\eta\|_{S_{ip}^p} = (\hat{E}[\sup_{t \in [0,T]} |\eta_t|^p])^{1/p}$.
- $A_{ip}^p(0,T)$ is the collection of processes $K \in S_{ip}^p(0,T)$ such that $K$ is a non-increasing $G$-martingale with $K_0 = 0$.

We denote by $\langle B \rangle$ the quadratic variation process of the $G$-Brownian motion $B$. For two processes $\eta \in M_{ip}^p(0,T)$ and $\zeta \in H_{ip}^p(0,T)$, Peng [25] and Li and Peng [18] established the $G$-Itô integrals $\int_0^T \eta_t d\langle B \rangle_t$ and $\int_0^T \zeta_t dB_t$.

### 2.2 Multi-dimensional G-BSDEs

We denote by $M_{ip}^p(0,T; \mathbb{R}^k)$ the set of $k$-dimensional stochastic process $X = (X^1, \ldots, X^k)$ such that $X^l \in M_{ip}^0(0,T)$, $1 \leq l \leq k$, and we also define $S_{ip}^p(0,T; \mathbb{R}^k)$, $H_{ip}^p(0,T; \mathbb{R}^k)$, $A_{ip}^p(0,T; \mathbb{R}^k)$ and $L_G^p(\Omega_T; \mathbb{R}^k)$ similarly. Consider the following type of $k$-dimensional $G$-BSDE with diagonal generators on the interval $[0,T]$:

$$Y_l^t = \xi^l + \int_t^T f^l(s,Y_s,Z^l_s)ds + \int_t^T g^l(ds,Y_s,Z^l_s)d\langle B \rangle_s - \int_t^T Z^l_sdB_s - (K^l_T - K^l_t), \ 1 \leq l \leq k.$$

Here by *diagonal* we mean that in the generators

$$f^l(t,\omega, y, z^l), g^l(t,\omega, y, z^l) : [0,T] \times \Omega_T \times \mathbb{R}^k \times \mathbb{R} \rightarrow \mathbb{R}, \ \forall 1 \leq l \leq k,$

the $z$ parts of the $l$-th components $f^l, g^l$ only depend on $z^l$. We impose that

(A1) there is some constant $\beta > 2$ such that for each $y \in \mathbb{R}^k$, $z \in \mathbb{R}$, $f^l(\cdot, \cdot, y, z), g^l(\cdot, \cdot, y, z) \in M_G^0(0,T)$, $1 \leq l \leq k$;

(A2) there exists some $L > 0$ such that, for each $1 \leq l \leq k$, $y_1, y_2 \in \mathbb{R}^k, z_1, z_2 \in \mathbb{R},$

$$|f^l(t, y_1, z_1) - f^l(t, y_2, z_2)| + |g^l(t, y_1, z_1) - g^l(t, y_2, z_2)| \leq L(|y_1 - y_2| + |z_1 - z_2|).$$

4
Theorem 2.3 ([19]). Suppose $\xi \in L^\beta_G(\Omega_T; \mathbb{R}^k)$ and (A1)-(A2) are satisfied for some $\beta > 2$. Then for any $2 \leq \alpha < \beta$, the G-BSDE (2.1) has a unique solution $(Y, Z, K) \in S^\alpha_G(0, T; \mathbb{R}^k) \times H^\alpha_G(0, T; \mathbb{R}^k) \times A^\alpha_G(0, T; \mathbb{R}^k)$. Moreover, $Y \in M^\alpha_G(0, T; \mathbb{R}^k)$.

Then, we present the comparison theorem for multi-dimensional G-BSDEs.

Theorem 2.4 ([19]). Given two G-BSDEs on the interval $[0, T]$:

$$ Y^l_t = \xi^l + \int_t^T f^l(s, Y_s, Z^l_s)ds + \int_t^T g^l(s, Y_s, Z^l_s)d(B)_s + V^l_T - V^l_t - \int_t^T Z^l_sdB_s - (K^l_T - K^l_t), $$

and

$$ \tilde{Y}^l_t = \tilde{\xi}^l + \int_t^T \tilde{f}^l(s, \tilde{Y}_s, \tilde{Z}^l_s)ds + \int_t^T \tilde{g}^l(s, \tilde{Y}_s, \tilde{Z}^l_s)d(B)_s + \tilde{V}^l_T - \tilde{V}^l_t - \int_t^T \tilde{Z}^l_sdB_s - (\tilde{K}^l_T - \tilde{K}^l_t), $$

where $1 \leq l \leq k$. For any $1 \leq l \leq k$, suppose that $f^l(t, y, z), \tilde{f}^l(t, \tilde{y}, \tilde{z}), g^l(t, y, z), \tilde{g}^l(t, \tilde{y}, \tilde{z})$ satisfy (A1)-(A2), $\xi^l, \tilde{\xi}^l \in L^\beta_G(\Omega_T)$ and $V^l_T, \tilde{V}^l_T$ are RCLL (right-continuous with left limits) such that $\mathbb{E}[\sup_{t \in [0, T]} |V^l_T|^\beta] \leq \infty$, for some $\beta > 2$. Assume the following conditions hold:

(i) for any $j' \in \mathbb{R}$ and $y, \tilde{y} \in \mathbb{R}^k$ satisfying $y^j \geq \tilde{y}^j$ for $j \neq l$ and $y^l = \tilde{y}^l$, it holds that $f^l(t, y, z') \geq \tilde{f}^l(t, \tilde{y}, \tilde{z}')$, $g^l(t, y, z') \geq \tilde{g}^l(t, \tilde{y}, \tilde{z}')$, for $1 \leq l \leq k$;

(ii) $\xi \geq \tilde{\xi}$ and $V^l_t - \tilde{V}^l_t$ is non-decreasing, $1 \leq l \leq k$.

Then $Y_t \geq \tilde{Y}_t$ for each $t \in [0, T]$, q.s.

2.3 One-dimensional reflected G-BSDEs

Consider the following parameters: the generators $f$ and $g$, the obstacle process $\{S_t\}_{t \in [0, T]}$ and the terminal value $\xi$. We assume that the generators

$$ f(t, \omega, y, z), g(t, \omega, y, z) : [0, T] \times \Omega_T \times \mathbb{R}^2 \to \mathbb{R}, $$

satisfy:

(H1) for any $y, z, f(\cdot, \cdot, y, z), g(\cdot, \cdot, y, z) \in M^\beta_G(0, T)$ with $\beta > 2$

(H2) $|f(t, \omega, y, z) - f(t, \omega, y', z')| + |g(t, \omega, y, z) - g(t, \omega, y', z')| \leq L(|y - y'| + |z - z'|)$ for some $L > 0$.

The lower obstacle $S$ is bounded from above by some given generalized G-Itô process. More precisely, it satisfies the following condition:

(H3) $S_t \leq I_t$ for any $t \in [0, T]$, where $I$ is a generalized G-Itô process:

$$ I_t = I_0 + \int_0^t b^l(s)ds + \int_0^t \sigma^l(s)dB_s + K_t^l, $$

with $I_0 \in \mathbb{R}, b^l \in M^\beta_G(0, T), \sigma^l \in H^\alpha_G(0, T), K^l \in A^\alpha_G(0, T)$.

The terminal value $\xi$ satisfies

(H4) $\xi \in L^\beta_G(\Omega_T)$ and $\xi \geq S_T$, q.s.

A triple of processes $(Y, Z, A)$ is called a solution of reflected G-BSDE with a lower obstacle $S$, terminal value $\xi$ and generators $f, g$, if for some $2 \leq \alpha \leq \beta$ the following properties hold:

(a) $(Y, Z, A) \in S^\alpha_G(0, T)$ and $Y_t \geq S_t, 0 \leq t \leq T$;
where $\beta > 2$.

Proposition 2.7 (\cite{16, 17}). Suppose that $(\xi, f, g, S)$ satisfy (H1)-(H4). Then, the reflected G-BSDE with parameters $(\xi, f, g, S)$ has a unique solution $(Y, Z, A) \in \mathcal{S}_G^\alpha(0, T)$, for any $2 \leq \alpha < \beta$. Besides, there exists a constant $C := C(\alpha, T, L, \sigma) > 0$ such that for $2 \leq \alpha \leq \beta$,

$$|Y_t|^\alpha \leq C\hat{E}_t[|\xi|^\alpha + \int_t^T (|f(s, 0, 0)|^\alpha + |g(s, 0, 0)|^\alpha + |b(s)|^\alpha + |\sigma(s)|^\alpha)ds + \sup_{s \in [t, T]} |I_s|^\alpha].$$

Then, we present some a priori estimates. It is worth pointing out that in the following two propositions, we do not need to assume that $(Y, Z, A)$ and $(Y^{(i)}, Z^{(i)}, A^{(i)}), i = 1, 2$, are the solutions of reflected G-BSDEs, i.e., the condition (c) is not needed.

Proposition 2.6 (\cite{16}). Let $f, g$ satisfy (H1) and (H2). Assume

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d(B)_s - \int_t^T Z_sdB_s + (AT - A_t),$$

where $(Y, Z, A) \in \mathcal{S}_G^\alpha(0, T)$ with $2 \leq \alpha \leq \beta$. Then, there exists a constant $C := C(\alpha, T, L, \sigma) > 0$ such that for each $t \in [0, T]$,

$$\hat{E}_t[\int_t^T |Z_s|^2ds] \leq C\{\hat{E}_t[\sup_{s \in [t, T]} |Y_s|^\alpha] + (\hat{E}_t[\sup_{s \in [t, T]} |Y_s|^\alpha])^{1/2}(\hat{E}_t[\int_t^T h_sds])^{1/2}\},$$

$$\hat{E}_t[|AT - A_t|^\alpha] \leq C\{\hat{E}_t[\sup_{s \in [t, T]} |Y_s|^\alpha] + \hat{E}_t[\int_t^T h_sds]\},$$

where $h_s = |f(s, 0, 0)| + |g(s, 0, 0)|$.

Proposition 2.7 (\cite{16}). For $i = 1, 2$, let $\xi^{(i)} \in L_\alpha^G(\Omega_T), f^{(i)}, g^{(i)}$ satisfy (H1) and (H2) for some $\beta > 2$. Assume

$$Y_t^{(i)} = \xi^{(i)} + \int_t^T f^{(i)}(s, Y_s^{(i)}, Z_s^{(i)})ds + \int_t^T g^{(i)}(s, Y_s^{(i)}, Z_s^{(i)})d(B)_s - \int_t^T Z_s^{(i)}dB_s + (A_t^{(i)} - A_t^{(0)}),$$

where $(Y^{(i)}, Z^{(i)}, A^{(i)}) \in \mathcal{S}_G^\alpha(0, T)$ for some $2 \leq \alpha \leq \beta$. Set $\hat{Y}_t = Y_t^{(1)} - Y_t^{(2)}, \hat{Z}_t = Z_t^{(1)} - Z_t^{(2)}$. Then, there exists a constant $C := C(\alpha, T, L, \sigma)$ such that

$$\hat{E}[\int_0^T |\hat{Z}|^2ds] \leq C_\alpha \{\hat{E}[\sup_{t \in [0, T]} |\hat{Y}_t|^\alpha]\}^{1/2} \sum_{i=1}^2 (\hat{E}[\sup_{t \in [0, T]} |Y_t^{(i)}|^\alpha])^{1/2}

+ \hat{E}[(\int_0^T h_s^{(i)}ds)^{1/2}] + \hat{E}[\sup_{t \in [0, T]} |\hat{Y}_t|^\alpha],$$

where $h_s^{(i)} = |f^{(i)}(s, 0, 0)| + |g^{(i)}(s, 0, 0)|$.

Proposition 2.8 (\cite{16, 17}). Let $(\xi^{(1)}, f^{(1)}, g^{(1)}, S^{(1)})$ and $(\xi^{(2)}, f^{(2)}, g^{(2)}, S^{(2)})$ be two sets of data satisfying (H1)-(H4). Let $(Y^{(i)}, Z^{(i)}, A^{(i)}) \in \mathcal{S}_G^\alpha(0, T)$ be the solutions of the reflected G-BSDEs with data $(\xi^{(i)}, f^{(i)}, g^{(i)}, S^{(i)}), i = 1, 2$ respectively, with $2 \leq \alpha \leq \beta$. Set $\hat{Y}_t = Y_t^{(1)} - Y_t^{(2)}, \hat{S}_t = S_t^{(1)} - S_t^{(2)}$, $\hat{\xi} = \xi^{(1)} - \xi^{(2)}$. Then, there exists a constant $C := C(\alpha, T, L, \sigma) > 0$ such that

$$|\hat{Y}_t|^\alpha \leq C\{\hat{E}_t[|\hat{\xi}|^\alpha] + \int_t^T |\hat{h}|_s^\alpha ds + (\hat{E}_t[\sup_{s \in [t, T]} |\hat{S}_s|^\alpha])^{1/2}\hat{E}_t[\sup_{r \in [t, T]} |\hat{\Psi}|_r^{\alpha - 1}]\}.$$
where \( \hat{h}_s = |f^{(1)}(s, Y_s^{(2)}, Z_s^{(2)}) - f^{(2)}(s, Y_s^{(2)}, Z_s^{(2)})| + |g^{(1)}(s, Y_s^{(2)}, Z_s^{(2)}) - g^{(2)}(s, Y_s^{(2)}, Z_s^{(2)})| \) and

\[
\Psi_{t,T} = \sum_{i=1}^2 \mathbb{E}_t \left[ \sup_{s \in [t,T]} |\xi^{(i)}|^\alpha + \sup_{s \in [0,T]} |I^{(i)}|^\alpha \right. \]

\[
+ \int_0^T \left( |f^{(i)}(s,0,0)|^\alpha + |g^{(i)}(s,0,0)|^\alpha + |\sigma^{(i)}(s)|^\alpha \right) ds \right].
\]

The following result is the comparison theorem for reflected G-BSDEs.

**Theorem 2.9** ([16, 17]). Let \((\xi^{(1)}, f^{(1)}, g^{(1)}, S^{(1)})\) and \((\xi^{(2)}, f^{(2)}, g^{(2)}, S^{(2)})\) be two sets of parameters satisfying (H1)-(H4). We furthermore assume the following:

(i) \( \xi^{(1)} \geq \xi^{(2)}, \text{ q.s.}; \)

(ii) \( f^{(1)}(t, y, z) \geq f^{(2)}(t, y, z), g^{(1)}(t, y, z) \geq g^{(2)}(t, y, z), \forall (y, z) \in \mathbb{R}^2; \)

(iii) \( S_{t}^{(1)} \geq S_{t}^{(2)}, 0 \leq t \leq T, \text{ q.s.}; \)

Let \((Y^{(i)}, Z^{(i)}, A^{(i)})\) be the solutions of the reflected G-BSDE with parameters \((\xi^{(i)}, f^{(i)}, g^{(i)}, S^{(i)}), i = 1, 2, \text{ respectively}.\) Then, we have \( Y_{t}^{(1)} \geq Y_{t}^{(2)}, 0 \leq t \leq T, \text{ q.s.}. \)

### 3 Multi-dimensional reflected G-BSDEs

In this paper, we shall consider the \( k \)-dimensional reflected G-BSDE with diagonal generators in the following form:

\[
Y_t^l = \xi^l + \int_t^T f^l(s, Y_s^l, Z_s^l)ds + \int_t^T g^l(s, Y_s^l, Z_s^l)d(B)_s - \int_t^T Z_s^l dB_s + (A_T^l - A_t^l), 1 \leq l \leq k,
\]

where \( f^l, g^l \) satisfy (A1) and (A2) as in Subsection 2.2, \( 1 \leq l \leq k. \) We call \((Y, Z, A) = (\langle Y^l, 1 \leq l \leq k \rangle, \langle Z^l, 1 \leq l \leq k \rangle, \langle A^l, 1 \leq l \leq k \rangle)\) a solution for the reflected G-BSDE with generators \( f, g, \) terminal value \( \xi \) and lower obstacle \( S, \) if it satisfies the following conditions:

(1) \((Y^l, Z^l, A^l) \in \mathcal{S}_G^k(0,T) \) and \( Y_t^l \geq S_t^l, t \in [0,T], \) where \( 2 \leq \alpha \leq \beta, 1 \leq l \leq k (\beta \text{ is the order of integrability for the parameters});

(2) \( Y_t^l = \xi^l + \int_t^T f^l(s, Y_s^l, Z_s^l)ds + \int_t^T g^l(s, Y_s^l, Z_s^l)d(B)_s - \int_t^T Z_s^l dB_s + A_T^l - A_t^l, 1 \leq l \leq k; \)

(3) \(- \int_0^T (Y_s^l - S_s^l)dA_s^l \) is a non-increasing G-martingale, \( 1 \leq l \leq k. \)

We need to propose the following assumptions on the obstacle \( S \) and terminal value \( \xi; \) there exists some \( \beta > 2 \) such that

(A3) for any \( 1 \leq l \leq k, S_t^l \leq I_t^l, t \in [0,T], \) where \( I^l \) is a generalized G-Itô process:

\[
I_t^l = I_0^l + \int_0^t b^l(s)ds + \int_0^t \sigma^l(s)dB_s + K_t^l,
\]

with \( I_0 \in \mathbb{R}, b^l \in \mathcal{M}_{G}^\beta(0,T), \sigma^l \in \mathcal{H}_{G}^\beta(0,T), K^l \in \mathcal{A}_{G}^\beta(0,T); \)

(A4) for any \( 1 \leq l \leq k, S_t^l \leq \xi^l \) and \( \xi^l \in L_G^\beta(\Omega_T). \)

For simplicity, we denote

\( \phi(t, y, z) = (\phi^1(t, y, z^1), \cdots, \phi^k(t, y, z^k))^T, \)

for any \( t \in [0,T], y = (y^1, \cdots, y^k)^T, z = (z^1, \cdots, z^k)^T \in \mathbb{R}^k \) and any function \( \phi^l : [0,T] \times \Omega_T \times \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}, 1 \leq l \leq k. \) We denote by \( \mathcal{S}_G^k(0,T; \mathbb{R}^k) \) the collection of \( k \)-dimensional triples \((Y, Z, A) = ((Y^1, Z^1, A^1), \cdots, (Y^k, Z^k, A^k))^T, \) for \((Y^l, Z^l, A^l) \in \mathcal{S}_G^k(0,T), 1 \leq l \leq k. \)

We first introduce the main result in this paper:

**Theorem 3.1.** Assume that \((\xi, f, g, S)\) satisfy Assumptions (A1)-(A4). Then, the multi-dimensional reflected G-BSDE (3.1) has a unique solution \((Y, Z, A) \in \mathcal{S}_G^k(0,T; \mathbb{R}^k). \)
3.1 A priori estimates

In this subsection, we present some useful a priori estimates for multi-dimensional reflected G-BSDEs. For simplicity, we only consider the case that \( g^l \equiv 0, l = 1, \ldots, k \), but similar results still hold for the general case. In the sequel, \( C \) will always be a universal constant which may change from line to line and \( 0 \) represents the \( k \)-dimensional zero vector.

**Proposition 3.2.** Let \( (Y, Z, A) \) be the solution of reflected G-BSDE with parameters \((\xi, f, S)\) satisfying (A1)-(A4). Then, for any \( 2 \leq \alpha \leq \beta \), there exists a constant \( C \) depending on \( T, G, k, L \) and \( \alpha \) such that

\[
|Y_t|^{\alpha} \leq C \mathbb{E}_t[|\xi|^{\alpha} + \sup_{s \in [t, T]} |I_s|^{\alpha} + \int_t^T (|f(s, 0, 0)|^{\alpha} + |b(s)|^{\alpha} + |\sigma(s)|^{\alpha}) ds].
\]

**Proof.** By Theorem 2.5, for any \( 1 \leq l \leq k \) and \( t \leq r \leq T \), we have

\[
|Y_t|^l \leq C \mathbb{E}_r[|\xi|^l + \sup_{s \in [t, T]} |I_s|^l + \int_t^r (|\hat{f}^l(s)|^l + |b^l(s)|^l + |\sigma^l(s)|^l) ds]
\]

\[
\leq C \mathbb{E}_r[|\xi|^l + \sup_{s \in [t, T]} |I_s|^l + \int_t^T (|f(s, 0, 0)|^l + |b(s)|^l + |\sigma(s)|^l) ds + \int_r^T |Y_s|^l ds],
\]

where \( \hat{f}^l(s) = f^l(s, Y_s^1, \ldots, Y_s^l, 0, Y_s^{l+1}, \ldots, Y_s^k, 0) \). Summing up over \( l \), we obtain that

\[
|Y_t|^\alpha \leq C \mathbb{E}_t[|\xi|^\alpha + \sup_{s \in [t, T]} |I_s|^\alpha + \int_t^T (|f(s, 0, 0)|^\alpha + |b(s)|^\alpha + |\sigma(s)|^\alpha) ds + C \int_T^T \mathbb{E}_r[|Y_s|^\alpha] ds.
\]

Taking conditional expectations on both sides of the above equation, we have

\[
\mathbb{E}_t[|Y_t|^\alpha] \leq C \mathbb{E}_t[|\xi|^\alpha + \sup_{s \in [t, T]} |I_s|^\alpha + \int_t^T (|f(s, 0, 0)|^\alpha + |b(s)|^\alpha + |\sigma(s)|^\alpha) ds + C \int_T^T \mathbb{E}_r[|Y_s|^\alpha] ds.
\]

It follows from the Gronwall inequality that

\[
\mathbb{E}_t[|Y_t|^\alpha] \leq C \mathbb{E}_t[|\xi|^\alpha + \sup_{s \in [t, T]} |I_s|^\alpha + \int_t^T (|f(s, 0, 0)|^\alpha + |b(s)|^\alpha + |\sigma(s)|^\alpha) ds],
\]

which is the desired result by letting \( r = t \). \( \square \)

**Proposition 3.3.** Assume that \( ^{\xi}, ^{f}, ^{S} \) satisfy (A1)-(A4), \( 1 \leq l \leq k, i = 1, 2 \). Let \( (^{\xi}, ^{f}, ^{S}) \) be the solution to reflected G-BSDEs with parameters \((^\xi, ^f, ^S)\), \( i = 1, 2 \), for some \( 2 \leq \alpha \leq \beta \). Set \( \hat{Y}_t = ^{1}Y_t - ^{2}Y_t \). Then there exists a constant \( C \) depending on \( T, G, k, L \) and \( \alpha \) such that

\[
|\hat{Y}_t|^\alpha \leq C (\mathbb{E}_t[|^{\xi}|^\alpha + \int_t^T |^{f}_s|^\alpha ds + (\mathbb{E}_t[\sup_{s \in [t, T]} |^{S}_s|^\alpha])^{\frac{\alpha}{\alpha - 1}}]^{\frac{\alpha - 1}{\alpha}}),
\]

where \( ^{\xi} = ^{1}\xi - ^{2}\xi, ^{f}_s = ^{1}f(s, ^{2}Y_s, ^{2}Z_s) - ^{2}f(s, ^{2}Y_s, ^{2}Z_s), ^{S}_t = ^{1}S_t - ^{2}S_t \) and

\[
^{\Psi}_{t,T} = \sum_{i=1}^2 \mathbb{E}_t[|^{\xi}|^\alpha + \sup_{s \in [t, T]} |^{I}_s|^\alpha + \int_t^T |^{i}f(s, 0, 0)|^\alpha + |^{i}b(s)|^\alpha + |^{i}\sigma(s)|^\alpha + |^{i}Y_s|^\alpha ds].
\]

**Proof.** For any \( 1 \leq l \leq k \), by Proposition 2.8, we have for any \( s \in [t, T] \),

\[
|^{Y}_s|^\alpha \leq C (\mathbb{E}_s[|^{\xi}|^\alpha + \int_s^T |^{f}_l|^\alpha ds] + (\mathbb{E}_t[\sup_{u \in [s, T]} |^{S}_u|^\alpha])^{\frac{\alpha}{\alpha - 1}}]^{\frac{\alpha - 1}{\alpha}}
\]

\[
\leq C (\mathbb{E}_s[|^{\xi}|^\alpha + \int_s^T |^{f}_l|^\alpha ds + \int_s^T |^{Y}_r|^\alpha dr] + (\mathbb{E}_t[\sup_{u \in [s, T]} |^{S}_u|^\alpha])^{\frac{\alpha}{\alpha - 1}}]^{\frac{\alpha - 1}{\alpha}}.
\]
where \( f_l = f^l(r, 1Y^{(l)}_r, 2Z^l_r) - 2f^l(r, 4Y^{(l)}_r, 3Z_r^l), Y^{(l)} = (1Y^1, \ldots, 1Y^{l-1}, 2Y^l, 1Y^{l+1}, \ldots, 1Y^k) \) and
\[
\Psi_{s,T} = \sum_{i=1}^N \hat{\mathcal{E}}_s[|\xi|^\alpha] + \sup_{u \in [s,T]} |f^l_u|^\alpha + \int_s^T |\alpha f^l(r)|^\alpha + |\beta u|^\alpha + |\gamma u|^\alpha + |\delta u|^\alpha dr,
\]
and \( f^l(r) = f^l(r, 1Y^{(l)}_r, \ldots, 1Y^{l-1}_r, 0, 1Y^{l+1}_r, \ldots, 1Y^k_r, 0) \). Summing up over \( l \), we obtain that for any \( s \geq t \),
\[
|\hat{Y}_s|^\alpha \leq C\{\hat{\mathcal{E}}_s[|\xi|^\alpha] + \int_t^T |\hat{f}_r|^\alpha dr\} + \hat{\mathcal{E}}_s[\sup_{u \in [t,T]} |\hat{S}_u|^\alpha] \frac{\Psi_{t,T}}{\Psi_{t,T}} + C \int_t^T \hat{\mathcal{E}}_s[|\hat{Y}_r|^\alpha] dr.
\]
Taking conditional expectations on both sides implies that
\[
\hat{\mathcal{E}}_t[|\hat{Y}_s|^\alpha] \leq C\{\hat{\mathcal{E}}_t[|\hat{\xi}|^\alpha] + \int_t^T |\hat{f}_r|^\alpha dr\} + \hat{\mathcal{E}}_t[\sup_{u \in [t,T]} |\hat{S}_u|^\alpha] \frac{\Psi_{t,T}}{\Psi_{t,T}} + C \int_t^T \hat{\mathcal{E}}_t[|\hat{Y}_r|^\alpha] dr.
\]
Applying the Gronwall inequality, we have for any \( s \geq t \)
\[
\hat{\mathcal{E}}_t[|\hat{Y}_s|^\alpha] \leq C\{\hat{\mathcal{E}}_t[|\hat{\xi}|^\alpha] + \int_t^T |\hat{f}_r|^\alpha dr\} + \hat{\mathcal{E}}_t[\sup_{u \in [t,T]} |\hat{S}_u|^\alpha] \frac{\Psi_{t,T}}{\Psi_{t,T}}.
\]
Letting \( s = t \), we get the desired result. \( \square \)

3.2 Construction via penalization method

For any \( 1 \leq l \leq k \), consider the following \( k \)-dimensional G-BSDEs parameterized by \( n = 1, 2, \ldots, \)
\[
Y^{l,n}_t = \xi^l + \int_t^T f^l(s, Y^{n}_s, Z^{l,n}_s)ds + n \int_t^T (Y^{l,n}_s - S^{l}_s)^{-} ds - \int_t^T Z^{l,n}_s dB_s - (K^{l,n}_T - K^{l,n}_t), \tag{3.2}
\]
where \( Y^n = (1Y^{1,n}, \ldots, 1Y^{k,n}) \). Let \( L^{l,n}_t = \int_0^t n(Y^{l,n}_s - S^{l}_s)^{-} ds \). It is easy to check that \( L^{l,n} \) is a non-decreasing process and
\[
Y^{l,n}_t = \xi^l + \int_t^T f^l(s, Y^{n}_s, Z^{l,n}_s)ds - \int_t^T Z^{l,n}_s dB_s - (K^{l,n}_T - K^{l,n}_t) + (L^{l,n}_T - L^{l,n}_t). \tag{3.3}
\]
The objective is to prove that for any \( 1 \leq l \leq k \), the processes \( (Y^{l,n}, Z^{l,n}, A^{l,n}) \) converge to \( (Y^l, Z^l, A^l) \), which is the solution to the reflected G-BSDEs, where \( A^{l,n} = L^{l,n} - K^{l,n} \). The proof is similar with the one-dimensional case studied in [16]. The main difference comes from the appearance of \( Y^{l,n} \) in the generator \( f \) in Equation (3.2) with \( j \neq l \), which leads to the modifications on the proofs for the estimate of \( Y^n, Y^{l,n} - Y^n \) and also results in the lack of comparison theorem (which needs a kind of monotonicity condition as (i) of Theorem 2.4).

**Lemma 3.4.** There exists a constant \( C \) depending on \( \alpha, T, L, k, G \), but not on \( n \), such that for \( 2 \leq \alpha < \beta \) and \( 1 \leq l \leq k \),
\[
\hat{\mathcal{E}}[\sup_{t \in [0,T]} |Y^{l,n}_t|^\alpha] \leq C, \quad \hat{\mathcal{E}}[|K^{l,n}_T|^\alpha] \leq C, \quad \hat{\mathcal{E}}[|L^{l,n}_T|^\alpha] \leq C, \quad \hat{\mathcal{E}}[\int_0^T |Z^{l,n}_t|^2 dt]^\frac{\alpha}{2} \leq C.
\]

**Proof.** For any \( r > 0 \), set \( \hat{Y}^{l,n}_t = |\hat{Y}^{l,n}_t|^2, \hat{Y}^{l,n}_t = Y^{l,n}_t - I^l_t \) and \( \hat{Z}^{l,n}_t = Z^{l,n}_t - \sigma^l(t) \). Note that for each
\( t \in [0,T] \), \((Y_{t}^{1,n} - I_{t}^1)(Y_{t}^{l,n} - S_{t}^l) \leq 0 \). Applying Itô's formula to \( \hat{Y}_{t}^{\alpha/2}e^{rt} \) yields that

\[
\begin{align*}
(Y_{t}^{1,n})^\alpha e^{rt} + \int_{t}^{T} r e^{rs}(\hat{Y}_{s}^{1,n})^{\alpha/2} ds + \int_{t}^{T} \frac{\alpha}{2} e^{rs}(\hat{Y}_{s}^{1,n})^{\alpha/2-1}(\hat{Z}_{s}^{l,n})^{2} d(B)_{s} \\
= |\xi|^\alpha e^{rt} + \alpha(1 - \frac{\alpha}{2}) \int_{t}^{T} e^{rs}(\hat{Y}_{s}^{1,n})^{\alpha/2-1}(\hat{Z}_{s}^{l,n})^{2} d(B)_{s} \\
+ \int_{t}^{T} \alpha e^{rs}(\hat{Y}_{s}^{1,n})^{\alpha/2-1}(f_{s}^{(l,n)} + b^{l}(s)) ds + \int_{t}^{T} \alpha e^{rs}(\hat{Y}_{s}^{1,n})^{\alpha/2-1}\hat{Y}_{s}^{1,n} dL_{s}^{n} \\
- \int_{t}^{T} \alpha e^{rs}(\hat{Y}_{s}^{1,n})^{\alpha/2-1}(\hat{Z}_{s}^{l,n}d\hat{B}_{s} + \hat{Y}_{s}^{1,n}dK_{s}^{l,n} - \hat{Y}_{s}^{1,n}dK_{s}^{l,l}) \\
\leq |\xi|^\alpha e^{rt} + \alpha(1 - \frac{\alpha}{2}) \int_{t}^{T} e^{rs}(\hat{Y}_{s}^{1,n})^{\alpha/2-1}(\hat{Z}_{s}^{l,n})^{2} d(B)_{s} \\
+ \int_{t}^{T} \alpha e^{rs}(\hat{Y}_{s}^{1,n})^{\alpha/2-1}(f_{s}^{(l,n)} + b^{l}(s)) ds - (M_{T}^{l,n} - M_{t}^{l,n}),
\end{align*}
\]

where \( f_{s}^{(l,n)} = f(s, Y_{s}^{1,n} + I_{s}^{1}, \ldots, \hat{Y}_{s}^{k,n} + I_{s}^{k}, \hat{Z}_{s}^{l,n} + \sigma^{l}(s)) \) and

\[
M_{t}^{l,n} = \int_{t}^{T} \alpha e^{rs}(\hat{Y}_{s}^{1,n})^{\alpha/2-1}(\hat{Y}_{s}^{1,n} + \hat{Z}_{s}^{l,n}d\hat{B}_{s} + \hat{Y}_{s}^{1,n}dK_{s}^{l,n} + \hat{Y}_{s}^{1,n}dK_{s}^{l,l})
\]

is a G-martingale. By the assumption on \( f_{l} \) and the Young inequality, we have

\[
\begin{align*}
\int_{t}^{T} \alpha e^{rs}(\hat{Y}_{s}^{1,n})^{\alpha/2-1}(|f_{s}^{(l,n)}| + b^{l}(s)) ds \\
\leq \int_{t}^{T} \alpha e^{rs}(\hat{Y}_{s}^{1,n})^{\alpha/2-1}(|f_{s}^{(l,n)}| + L(|Y_{s}^{1,n}| + |\sigma^{l}(s)|) + \sum_{j=1}^{k} |\hat{Y}_{s}^{j,n}| + \sum_{j=1}^{k} |I_{j}^{l}| + b^{l}(s)) \\
\leq \int_{t}^{T} e^{rs}(|f_{s}^{(l,n)}|^{\alpha} + |b^{l}(s)|^{\alpha} + L^{\alpha}(|\sigma^{l}(s)|^{\alpha} + \sum_{j=1,j\neq l}^{k} |\hat{Y}_{s}^{j,n}|^{\alpha} + \sum_{j=1}^{k} |I_{j}^{l}|^{\alpha})) ds \\
+ \frac{\alpha(\alpha - 1)}{4} \int_{t}^{T} e^{rs}(\hat{Y}_{s}^{1,n})^{\alpha/2-1}(\hat{Z}_{s}^{l,n})^{2} d(B)_{s} + C(\alpha, k, L, \alpha) \int_{t}^{T} e^{rs}(\hat{Y}_{s}^{1,n})^{\alpha/2} ds.
\end{align*}
\]

where \( C(\alpha, k, L, \alpha) = 2(k+1)(\alpha - 1) + \alpha L + \frac{\alpha L^{2}}{2(\alpha - 1)} \). Set \( r = C(\alpha, k, L, \alpha) + 1 \). We obtain that

\[
|\hat{Y}_{t}^{1,n}| e^{rt} + M_{T}^{l,n} - M_{t}^{l,n} \\
\leq |\xi| e^{rt} + \int_{t}^{T} e^{rs}(|f_{s}^{(l,n)}|^{\alpha} + |b^{l}(s)|^{\alpha} + L^{\alpha}(|\sigma^{l}(s)|^{\alpha} + \sum_{j=1,j\neq l}^{k} |\hat{Y}_{s}^{j,n}|^{\alpha} + \sum_{j=1}^{k} |I_{j}^{l}|^{\alpha})) ds \\
\leq C|\xi| + \int_{t}^{T} (|f_{s}^{(l,n)}|^{\alpha} + |b^{l}(s)|^{\alpha} + L^{\alpha}(|\sigma^{l}(s)|^{\alpha} + \sum_{j=1,j\neq l}^{k} |\hat{Y}_{s}^{j,n}|^{\alpha} + \sum_{j=1}^{k} |I_{j}^{l}|^{\alpha})) ds.
\]

Taking conditional expectations on both sides implies that

\[
|\hat{Y}_{t}^{1,n}| \leq C \hat{E}[|\xi|] + \int_{t}^{T} (|f_{s}^{(l,n)}|^{\alpha} + |b^{l}(s)|^{\alpha} + L^{\alpha}(|\sigma^{l}(s)|^{\alpha} + \sum_{j=1,j\neq l}^{k} |\hat{Y}_{s}^{j,n}|^{\alpha} + \sum_{j=1}^{k} |I_{j}^{l}|^{\alpha})) ds.
\]

Summing up over \( l \), we have

\[
|\hat{Y}_{t}^{n}| \leq C \hat{E}[|\xi|] + \int_{t}^{T} (|f_{s}(s, 0, 0)|^{\alpha} + |b(s)|^{\alpha} + L^{\alpha}(|\sigma^{l}(s)|^{\alpha} + \sum_{j=1,j\neq l}^{k} |\hat{Y}_{s}^{j,n}|^{\alpha} + \sum_{j=1}^{k} |I_{j}^{l}|^{\alpha})) ds.
\]

Taking expectations on both sides indicates that

\[
\hat{E}[|\hat{Y}_{t}^{n}|^{\alpha}] \leq C \hat{E}[|\xi|] + \int_{t}^{T} (|f_{s}(s, 0, 0)|^{\alpha} + |b(s)|^{\alpha} + L^{\alpha}(|\sigma^{l}(s)|^{\alpha} + \sum_{j=1,j\neq l}^{k} |\hat{Y}_{s}^{j,n}|^{\alpha} + \sum_{j=1}^{k} |I_{j}^{l}|^{\alpha})) ds + C \int_{t}^{T} \hat{E}(|\hat{Y}_{s}^{n}|^{\alpha}) ds.
\]

10
Applying the Gronwall inequality, we get

$$\hat{\mathbb{E}}[|Y^n_t|^\alpha] \leq C\hat{\mathbb{E}}[|\xi|^\alpha + \int_0^T (|f(s, 0, 0)|^\alpha + |b(s)|^\alpha + |\sigma(s)|^\alpha + |l_s|^\alpha)ds].$$

Recalling Equation (3.5) and Theorem 2.2, there exists a constant $C$ independent of $n$ such that for any $2 \leq \alpha < \beta$, $\hat{\mathbb{E}}[\sup_{t \in [0, T]} |Y_t^n|^\alpha] \leq C$. Consequently, $\hat{\mathbb{E}}[\sup_{t \in [0, T]} |Y_t^n|^\alpha] \leq C$. By Proposition 2.6, we have

$$\hat{\mathbb{E}}[\int_0^T |Z_{n,t}^l|^2 ds] \leq C\{\hat{\mathbb{E}}[\sup_{s \in [0, T]} |Y_s^{l,n}|^\alpha] + (\hat{\mathbb{E}}[\sup_{s \in [0, T]} |Y_s^{l,n}|^\alpha])^{1/2} (\hat{\mathbb{E}}[\int_0^T |f(l,n)(s, 0, 0)|ds])^{1/2}\},$$

$$\hat{\mathbb{E}}[|L_{-T}^l - K_{T}^l|^\alpha] \leq C\{\hat{\mathbb{E}}[\sup_{s \in [0, T]} |Y_s^{l,n}|^\alpha] + \hat{\mathbb{E}}[\int_0^T |f(l,n)(s, 0, 0)|ds]\},$$

where $f(l,n)(s, 0, 0) = f^l(s, Y_s^{1,n}, \cdots, Y_s^{l-1,n}, 0, Y_s^{l+1,n}, \cdots, Y_s^{k,n}, 0)$. Therefore, we get the desired uniform estimates for $K^{l,n}$, $L^l$, and $Z^n$, respectively.

Based on the uniform estimates obtained in Lemma 3.4, by a similar analysis as the proof of Lemma 4.4 in [17], we can obtain the following convergence property for $(Y_t^{l,n} - S_t^l)^-$, $1 \leq l \leq k$. The only difference is in (12) of [17]. In our cases, $U = +\infty$, $f^{x,0}(s) = f^l(s, Y_s^x, 0) + m_s^x$ and $Y^n$ here is $k$-dimensional while it is 1-dimensional in [17].

**Lemma 3.5.** For any $2 \leq \alpha < \beta$ and $1 \leq l \leq k$, we have

$$\lim_{n \to \infty} \hat{\mathbb{E}}[\sup_{t \in [0, T]} |(Y_t^{l,n} - S_t^l)|^\alpha] = 0.$$

**Remark 3.6.** Actually, Lemma 4.3 in [16] also indicates the uniform convergence property similar as in Lemma 3.5 for the 1-dimensional case. However, the proof needs the comparison theorem for G-BSDEs. In the multi-dimensional case, since we do not assume that the generator $f$ satisfy the monotonicity property (i) as in Theorem 2.4, $Y^{l,n}$ may not increasing in $n$. Therefore, the method used in proving Lemma 4.3 in [16] cannot be applied to the multi-dimensional case.

We then show that $(Y^n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $S_G^\alpha(0, T)$.

**Lemma 3.7.** For any $2 \leq \alpha < \beta$, we have

$$\lim_{n, m \to \infty} \hat{\mathbb{E}}[\sup_{t \in [0, T]} |Y_t^n - Y_t^m|^\alpha] = 0.$$

**Proof.** For any $r > 0$, set $\hat{Y}_t = Y_t^{1,n} - Y_t^{l,m}, \hat{Z}_t = Z_t^{l,n} - Z_t^{l,m}, \hat{K}_t = K_t^{l,n} - K_t^{l,m}, \hat{L}_t = L_t^{l,n} - L_t^{l,m}, \hat{Y}_t = |\hat{Y}_t|^2$ and $\hat{f}_t = f^l(t, Y_t^{l,n}, Z_t^{l,n}) - f^l(t, Y_t^{l,m}, Z_t^{l,m})$. By applying Itô’s formula to $|Y_t^{l}|^{\alpha/2}e^{rt}$, we get

$$|Y_t^{l}|^{\alpha/2}e^{rt} + \int_t^T re^{rs}|Y_s^{l}|^{\alpha/2}ds + \frac{\alpha}{2}\int_t^T e^{rs}|Y_s^{l}|^{\alpha/2-1}(\hat{Z}_s^l)^2d(B)_s$$

$$= \alpha(1 - \frac{\alpha}{2})\int_t^T e^{rs}|Y_s^{l}|^{\alpha/2-2}(\hat{Z}_s^l)^2d(B)_s + \int_t^T e^{rs}|Y_s^{l}|^{\alpha/2-1}\hat{Y}_s^l\hat{L}_s^l$$

$$+ \int_t^T e^{rs}|Y_s^{l}|^{\alpha/2-1}\hat{Y}_s^l\hat{f}_sds - \int_t^T e^{rs}|Y_s^{l}|^{\alpha/2-1}(\hat{Y}_s^l\hat{Z}_s^l)dB_s + \hat{Y}_s^l d\hat{K}_s$$

$$\leq \alpha(1 - \frac{\alpha}{2})\int_t^T e^{rs}|Y_s^{l}|^{\alpha/2-2}(\hat{Z}_s^l)^2d(B)_s + \int_t^T e^{rs}|Y_s^{l}|^{\alpha/2-1}\hat{f}_sds - (M_T - M_t)$$

$$- \int_t^T e^{rs}|Y_s^{l}|^{\alpha/2-1}(\hat{Y}_s^{l,n} - \hat{Y}_s^{l,m})dL_s^l - \int_t^T e^{rs}|Y_s^{l}|^{\alpha/2-1}(Y_s^{l,m} - S_t^l)dB_s - \int_t^T e^{rs}|Y_s^{l}|^{\alpha/2-1}(Y_s^{l,m} - S_t^l)dL_s^{l,n},$$
where $M_t = \int_t^T \alpha e^{r^s} |\hat{Y}^t|^\alpha \frac{d\alpha}{\alpha} ds \leq \int_t^T e^{r^s} \sum_{j=1, j \neq l}^k |\hat{Y}^j|^\alpha ds + \frac{\alpha(\alpha - 1)}{4} \int_t^T e^{r^s} |\hat{Y}^t|^\alpha (\hat{Z}^t_s)^2 d\langle B \rangle_s + ((k-1)(\alpha - 1) + \alpha L + \alpha L^2 \alpha \alpha - 1) \int_t^T e^{r^s} |\hat{Y}^t|^\alpha ds.$

Let $r = 1 + (k-1)(\alpha - 1) + \alpha L + \frac{\alpha L^2 \alpha \alpha - 1}{2}$. By the above analysis, we have

$$|\hat{Y}^t|^\alpha = \frac{Ct}{\hat{E}} \left( \int_t^T (m+n) |\hat{Y}^t|^\alpha \frac{d\alpha}{\alpha} + \int_t^T \sum_{j=1, j \neq l}^k |\hat{Y}^j|^\alpha ds \right) \leq C \hat{E} \left( \int_t^T (m+n) |\hat{Y}^t|^\alpha \frac{d\alpha}{\alpha} + \int_t^T \sum_{j=1, j \neq l}^k |\hat{Y}^j|^\alpha ds \right) \leq C \hat{E} \left( \int_t^T (m+n) |\hat{Y}^t|^\alpha \frac{d\alpha}{\alpha} + \int_t^T |\hat{Y}^t|^\alpha ds \right).$$

Summing up yields

$$|\hat{Y}^t|^\alpha \leq C \hat{E} \left( \int_t^T (m+n) |\hat{Y}^t|^\alpha \frac{d\alpha}{\alpha} + \int_t^T |\hat{Y}^t|^\alpha ds \right).$$

Taking expectations on both sides, we have

$$\hat{E}[|\hat{Y}^t|^\alpha] \leq C \hat{E} \left( \int_t^T (m+n) |\hat{Y}^t|^\alpha \frac{d\alpha}{\alpha} + \int_t^T |\hat{Y}^t|^\alpha ds \right).$$

Applying the Gronwall inequality, it follows that

$$\hat{E}[|\hat{Y}^t|^\alpha] \leq C \hat{E} \left( \int_t^T (m+n) |\hat{Y}^t|^\alpha \frac{d\alpha}{\alpha} + \int_t^T |\hat{Y}^t|^\alpha ds \right).$$

For any $1 \leq \gamma < \beta/\alpha$, by the H"older inequality, we have for any $1 \leq l \leq k$,

$$\hat{E}[\left( \int_t^T (m+n) |\hat{Y}^t|^\alpha \frac{d\alpha}{\alpha} \right)^\gamma] \leq C \hat{E} \left( \sup_{s \in [0,T]} |\hat{Y}^t|^\alpha, \gamma \right) \left( \sup_{s \in [0,T]} |\hat{Y}^s|^\alpha \right)^\gamma \left( \sup_{s \in [0,T]} |L_t^{m,n}| \gamma \right) \frac{\alpha}{\alpha} (\hat{E}[|L_t^{m,n}| \gamma])^{\frac{\alpha}{\alpha}} + \left( \hat{E}[\sup_{s \in [0,T]} |(Y_s^{m,n} - S_s^l)| \gamma] \right)^\gamma (\hat{E}[|L_t^{m,n}| \gamma])^{\frac{\alpha}{\alpha}} + (\hat{E}[\sup_{s \in [0,T]} |(Y_s^{m,n} - S_s^l)| \gamma]) \frac{\alpha}{\alpha},$$

which converges to 0 when $m, n$ go to infinity by Lemma 3.4 and Lemma 3.5. Therefore, for any $2 \leq \alpha < \beta$, we have

$$\lim_{n,m \to \infty} \hat{E}[|Y_t^n - Y_t^m|^\alpha] = 0.$$
Now, we are in a position to prove the main result, i.e., the existence and uniqueness of solutions to multi-dimensional reflected G-BSDEs.

**Proof of Theorem 3.1.** The uniqueness for $Y$ is a direct consequence of Proposition 3.3. Note that for each fixed $1 \leq l \leq k$, $(Y^l, Z^l, A^l)$ can be seen as the solution to the 1-dimensional reflected G-BSDE with parameters $(\xi^l, \tilde{f}^l, S^l)$, where

$$
\tilde{f}^l(t, y, z) = f^l(t, Y^l_t, \cdots, Y^l_{t-1}, y, Y^l_{t+1}, \cdots, Y^l_T, z).
$$

By the uniqueness for 1-dimensional reflected G-BSDEs, we obtain the uniqueness for the multi-dimensional case.

For the existence, by Lemma 3.7, there exists some $Y \in S^2_G(0, T)$ such that

$$
\lim_{n \to \infty} \mathbb{E}[\sup_{t \in [0,T]} |Y_t - Y_t^n|^\alpha] = 0.
$$

Besides, Lemma 3.5 implies that for any $1 \leq l \leq k$, $Y^l_t \geq S^l_t$, $t \in [0, T]$. It remains to prove $(Z^{l,n}, A^{l,n})$ converges to $(Z^l, A^l)$ and $-\int_0^T (Y^l_s - S^l_s) dA^l_s$ is a non-increasing G-martingale. The proof is similar with the one of Theorem 5.1 in [16], so we omit it.

As a byproduct of the penalization construction, we have the following comparison theorem for multi-dimensional reflected G-BSDEs.

**Theorem 3.8.** Let $(\xi, f, S)$ and $(\tilde{\xi}, \tilde{f}, \tilde{S})$ be two sets of data. Suppose that the coefficients $f, \tilde{f}, \xi, \tilde{\xi}, \tilde{S}$ satisfy $(A1) - (A4)$. Assume the following conditions hold:

(i) for each $1 \leq l \leq k$, $f^l(t, y, z^l) \geq \tilde{f}^l(t, \tilde{y}, \tilde{z}^l)$ if $z^l \in \mathbb{R}$ and $y, \tilde{y} \in \mathbb{R}^k$ satisfying $y^l \geq \tilde{y}^l$ for $j \neq l$ and $y^l = \tilde{y}^l$;

(ii) $\xi \geq \tilde{\xi}$;

(iii) $S_t \geq \tilde{S}_t$, $0 \leq t \leq T$, $\mathbb{Q}$-a.s.

Suppose that $(Y, Z, A)$ and $(\tilde{Y}, \tilde{Z}, \tilde{A})$ are the solutions to the reflected G-BSDE with the above two sets of parameters, respectively. Then, we have

$$
Y_t \geq \tilde{Y}_t, \quad 0 \leq t \leq T, \quad \mathbb{Q} \text{-a.s.}
$$

**Proof.** Consider the following G-BSDEs parameterized by $n = 1, 2, \cdots$,

$$
\tilde{Y}^{l,n}_t = \tilde{\xi}^l + \int_t^T \tilde{f}^l(s, Y^l_s, Z^{l,n}_s) ds + n \int_t^T (\tilde{Y}^{l,n}_s - \tilde{S}^l_s)^- ds - \int_t^T \tilde{Z}^{l,n}_s dB_s
$$

$$
- (\tilde{K}^{l,n}_t - \tilde{K}^{l,n}_t), \quad 1 \leq l \leq k.
$$

Similar analysis as the proof of Theorem 3.1, we can show that $\lim_{n \to \infty} \mathbb{E}[\sup_{t \in [0,T]} |\tilde{Y}_t - \tilde{Y}^{n}_t|^\alpha] = 0$, where $2 \leq \alpha < \beta$. Note that $(Y, Z, A)$ is the solution of the reflected G-BSDE with parameters $(\xi, f, S)$ and $Y_t \geq \tilde{S}_t$, $0 \leq t \leq T$. Thus we have

$$
Y^l_t = \xi^l + \int_t^T f^l(s, Y^l_s, Z^l_s) ds + \int_t^T n(Y^l_s - S^l_s)^- ds - \int_t^T Z^l_s dB_s + (A^l_T - A^l_t).
$$

According to Theorem 2.4, we have $Y^l_t \geq \tilde{Y}^{l,n}_t$, for each $n \in \mathbb{N}$. Now letting $n \to \infty$, we conclude that $Y^l_t \geq \tilde{Y}^l_t$, as desired.

**Remark 3.9.** To guarantee the comparison theorem to hold for multi-dimensional BSDEs with diagonal generators, the condition (i) in Theorem 2.4 (i.e., the condition (i) in Theorem 3.8) is usually imposed. Indeed, in the linear conditional expectation case, it is necessary and sufficient for the comparison theorem of multi-dimensional BSDEs with diagonal generators to hold.
To be more detailed, on the classical probability space, suppose that the generators \( f^l(t, y, z^l) \) and \( \bar{f}^l(t, \bar{y}, \bar{z}^l) \), for \( 1 \leq l \leq k \), are diagonal and satisfy standard assumptions of BSDEs (see [20]). Consider the following two \( k \)-dimensional BSDEs on \([0, T]):\n
\[
Y^l_t = \xi^l + \int_t^T f^l(s, Y_s, Z^l_s) \, ds - \int_t^T Z^l_s \, dB_s,
\]

and

\[
\bar{Y}^l_t = \bar{\xi}^l + \int_t^T \bar{f}^l(s, \bar{Y}_s, \bar{Z}^l_s) \, ds - \int_t^T \bar{Z}^l_s \, dB_s,
\]

where \( 1 \leq l \leq k \). Then the following are equivalent:

(i) for any stopping time \( \tau \leq T, \xi, \bar{\xi} \in L^2(F^\tau, \mathbb{R}^k) \) such that \( \xi \geq \bar{\xi} \), the solution \((Y, Z)\) and \((\bar{Y}, \bar{Z})\) to the above two BSDEs, respectively, on \([0, \tau]\) satisfy \( Y_t \geq \bar{Y}_t, \ t \in [0, \tau]\);

(ii) for any \( z^l \in \mathbb{R} \) and \( y, \bar{y} \in \mathbb{R}^k \) satisfying \( y^l \geq \bar{y}^l \) for \( j \neq l \) and \( y^l = \bar{y}^l \), it holds that \( f^l(t, y, z^l) \geq \bar{f}^l(t, \bar{y}, \bar{z}^l) \), for \( 1 \leq l \leq k \).

It can be proved by a similar argument as in Theorem 2.2 in [11] as follows:

**Step 1.** Assume (i) hold. From Theorem 2.1 in [11], we know that (i) is equivalent to: for each \( y, \bar{y} \in \mathbb{R}^k \) and \( z, \bar{z} \in \mathbb{R}^k \),

\[
-4(y^-, f(t, y^+ + \bar{y}, z) - \bar{f}(t, \bar{y}, \bar{z}))_{\mathbb{R}^k} \leq 2 \sum_{i=1}^k I_{\{y^i < 0\}} |z^i - \bar{z}^i|^2 + C|y^-|^2. \tag{3.9}
\]

Then for any fixed \( l \), we take \( \delta y \in \mathbb{R}^k \) satisfying \( \delta y \geq 0 \) and \( (\delta y)^l = 0 \). We then take, for any \( \varepsilon > 0 \),

\[
y = \delta y - \varepsilon e^l.
\]

Plug this into (3.9) and let \( z = \bar{z} \), we get

\[
-4\varepsilon [f^l(t, \delta y + \bar{y}, \bar{z}^l) - \bar{f}^l(t, \bar{y}, \bar{z}^l)] \leq C\varepsilon^2.
\]

Divide on both sides by \(-\varepsilon\), we get

\[
4[f^l(t, \delta y + \bar{y}, \bar{z}^l) - \bar{f}^l(t, \bar{y}, \bar{z}^l)] \geq -C\varepsilon.
\]

Letting \( \varepsilon \rightarrow 0 \), we get (ii).

**Step 2.** Now assume (ii) hold. For each \( l \), we take \( \delta y^+ \in \mathbb{R}^k \) satisfying \( (\delta y^+)^j = (y^+)^j, j \neq l \) and \( (\delta y^+)^l = 0 \). From (ii),

\[
-4(y^-, f(t, y^+ + \bar{y}, z) - \bar{f}(t, \bar{y}, \bar{z}))_{\mathbb{R}^k}
= \sum_{i=1}^k -4(y^-)^l [f^l(t, y^+ + \bar{y}, \bar{z}^l) - \bar{f}^l(t, \bar{y}, \bar{z}^l)]
= \sum_{i=1}^k -4(y^-)^l [(f^l(t, y^+ + \bar{y}, \bar{z}^l) - f^l(t, \delta y^+ + \bar{y}, \bar{z}^l)) + (f^l(t, \delta y^+ + \bar{y}, \bar{z}^l) - \bar{f}^l(t, \bar{y}, \bar{z}^l))]
\leq \sum_{i=1}^k -4(y^-)^l [f^l(t, y^+ + \bar{y}, \bar{z}^l) - f^l(t, \delta y^+ + \bar{y}, \bar{z}^l)]
\]

14
Then applying the Lipschitz assumption, we further obtain
\[-4(y^-, f(t, y^+ + \bar{y}, z) - \bar{f}(t, \bar{y}, \bar{z}))_{R^k} \leq \sum_{i=1}^{k} 4C(y^-)l(|y^+ - \delta t y^+| + |z^I - \bar{z}^I|)\]
\[= \sum_{i=1}^{k} 4C(y^-)(|y^I|^I + |z^I - \bar{z}^I|)\]
\[= \sum_{i=1}^{k} 4C(y^-)|z^I - \bar{z}^I|\]
\[\leq \sum_{i=1}^{k} 2|z^I - \bar{z}^I|^2 I_{\{y^I < 0\}} + C((y^-)^2)\]
\[= 2 \sum_{i=1}^{k} |z^I - \bar{z}^I|^2 I_{\{y^I < 0\}} + C|y^-|^2.\]
The proof is complete.
Moreover, a counterexample when such a condition is violated can be found in Example 3.2 in [29].

3.3 Picard iteration method

In this subsection, we will give another proof of Theorem 3.1, based on a fixed point argument. We denote by \(\mathcal{M}_G(a, b; R^k), \mathcal{S}_G(a, b; R^k)\) and \(\mathcal{S}_0(a, b; R^k)\) the corresponding spaces for the stochastic processes defined on the time interval \([0, T]\).

We have the following well-posedness result on the local solution to reflected G-BSDE (2.1).

**Theorem 3.10.** Assume that (A1)-(A4) hold for some \(\beta > 2\). Then there exists a constant \(0 < \delta \leq T\) depending only on \(T, G, k, \beta\) and \(L\) such that for any \(h \in (0, \delta), t \in [0, T-h]\) and given \(\zeta \in L_G^2(\Omega_{t+h}; R^k)\) such that \(\zeta \geq S_{t+h}\), the G-BSDE with parameters \((\zeta, f, S)\) on the interval \([t, t+h]\) admits a unique solution \((Y, Z, A) \in \mathcal{S}_G^2(t, t+h; R^k)\) for each \(2 \leq \alpha < \beta\). Moreover, \(Y \in \mathcal{M}_G^2(t, t+h; R^k)\).

To prove Theorem 3.10, let us consider the following G-BSDE with reflection \((S)\) on the interval \([t, t+h]\), for any \(h \in (0, \delta), t \in [0, T-h]\):

\[
\begin{align*}
Y_{s}^{U,I} &= \zeta^I + \int_{s}^{t+h} f(t, Y_{t}^{U,I}, Z_{t}^{U,I})dt - \int_{s}^{t+h} Z_{r}^{U,I}dB_{r} + (A_{s+h}^{U,I} - A_{s}^{U,I}), 1 \leq I \leq k, \\
Y_{s}^{U,I} &\geq S_{s}^{I}, s \in [t, t+h], 1 \leq I \leq k, \\
\{- \int_{s}^{t} (Y_{r}^{U,I} - S_{r}^{I})dA_{r}^{U,I}\}_{s \in [t, t+h]} \text{ is a non-increasing G-martingale,} 1 \leq I \leq k,
\end{align*}
\]

where \(U \in \mathcal{M}_G^2(t, t+h; R^k), \zeta \in L_G^2(\Omega_{t+h}; R^k)\) and

\[f^{I,U}(t, y^I, z^I) = f(t, U_t^I, \ldots, U_{t-1}^I, y^I, U_t^{I+1}, \ldots, U_k^I, z^I).
\]

We denote \(X^U = (X^{U,1}, \ldots, X^{U,k})\) for \(X = Y, Z, A\).

**Lemma 3.11.** Given \(U \in \mathcal{M}_G^2(t, t+h; R^k)\) and \(\zeta \in L_G^2(\Omega_{t+h}; R^k)\), for any \(2 \leq \alpha < \beta\), the reflected G-BSDE (3.10) has a unique solution \((Y^U, Z^U, K^U)\) in \(\mathcal{S}_G^2(t, t+h; R^k)\), and moreover, \(Y^U \in \mathcal{M}_G^2(t, t+h; R^k)\).

**Proof.** Fix any \(I \geq 1\). From Lemma 3.3 in [19], we know that \(f^{I,U}(s, y^I, z^I) \in \mathcal{M}_G^2(t, t+h)\) for each \(y^I \in R, z^I \in R^d\). Then applying Theorem 2.5 to the \(I\)-th components of reflected G-BSDE (3.10), we get the unique solution \((Y_t^{U,I}, Z_t^{U,I}, A_t^{U,I})\), which constitutes the unique solution to (3.10) when \(I\) runs through 1 to \(k\).

Next we show that \(Y_t^{U,I} \in \mathcal{M}_G^2(t, t+h)\). By Proposition 2.5, we have for all \(s \in [t, t+h]\) that

\[|Y_s^{U,I}|^\beta \leq C_E_s||\zeta|^\beta + \int_{t}^{t+h} (|f^{I,U}(r, 0, 0)|^\beta + |b^{I,U}(r)|^\beta + |\sigma^{I,U}(r)|^\beta)dr + \sup_{r \in [t, t+h]} |I_r^{U,I}|^\beta =: \rho_s^I.
\]

Then following the same steps as those in Lemma 3.2 in [19], we can derive the desired result. □
Thus, we can define a map \( \Gamma : U \to \Gamma(U) \) from \( M^\beta_G(t, t + h; \mathbb{R}^k) \) to \( M^\beta_G(t, t + h; \mathbb{R}^k) \) by
\[
\Gamma(U) := Y^U, \quad \text{for each } U \in M^\beta_G(t, t + h; \mathbb{R}^k).
\]
In the following, we show the map \( \Gamma \) is a contraction if we take sufficiently small \( h \).

**Lemma 3.12.** There exists some constant \( \delta > 0 \) depending only on \( T, G, k, \beta \) and \( L \) such that for each \( h \in (0, \delta] \),
\[
\|Y^U - \bar{Y}^U\|_{M^\beta_G(t, t + h; \mathbb{R}^k)} \leq \frac{1}{2}\|U - \bar{U}\|_{M^\beta_G(t, t + h; \mathbb{R}^k)}, \quad \text{for each } U, \bar{U} \in M^\beta_G(t, t + h; \mathbb{R}^k).
\]

**Proof.** For each fixed \( 1 \leq l \leq k \), by applying Proposition 2.8 to \( (Y^U,Y^\bar{U}) \), we obtain that
\[
|Y_{s}^{U,l} - Y_{s}^{\bar{U},l}|^\beta \leq C\bar{E}[\int_{s}^{t+h} |\hat{h}_{r}^{l}|^\beta dr], \quad \forall s \in [t, t + h],
\]
where
\[
\hat{h}_{s}^{l} = |f_{s,l}(s, Y_{s}^{U,l}, Z_{s}^{U,l}) - f_{s,l}(s, Y_{s}^{\bar{U},l}, Z_{s}^{\bar{U},l})|.
\]
By Assumption (A2), we then have
\[
\bar{E}[|Y_{s}^{U,l} - Y_{s}^{\bar{U},l}|^\beta] \leq C\bar{E}[\int_{s}^{t+h} |U_{r} - \bar{U}_{r}|^\beta dr], \quad \forall s \in [t, t + h].
\]
Summing over \( l \), we deduce that for \( s \in [t, t + h] \),
\[
\bar{E}[|Y_{s}^{U} - Y_{s}^{\bar{U}}|^\beta] \leq C \sum_{l=1}^{k} \bar{E}[|Y_{s}^{U,l} - Y_{s}^{\bar{U},l}|^\beta] \leq C\bar{E}[\int_{t}^{t+h} |U_{r} - \bar{U}_{r}|^\beta dr].
\]
Thus,
\[
\|Y^U - \bar{Y}^U\|_{M^\beta_G(t, t + h; \mathbb{R}^k)} \leq \left| \int_{t}^{t+h} \bar{E}[|Y_{s}^{U} - Y_{s}^{\bar{U}}|^\beta] ds \right|^\frac{1}{\beta} \leq C h^\frac{1}{\beta} \|U - \bar{U}\|_{M^\beta_G(t, t + h; \mathbb{R}^k)}.
\]
Then we can take \( \delta > 0 \) small enough such that, for each \( h \in (0, \delta] \),
\[
\|Y^U - \bar{Y}^U\|_{M^\beta_G(t, t + h; \mathbb{R}^k)} \leq \frac{1}{2}\|U - \bar{U}\|_{M^\beta_G(t, t + h; \mathbb{R}^k)}.
\]
The proof is complete. \( \square \)

Now we can state the proof for the existence and uniqueness of local solutions as follows.

**The proof of Theorem 3.10.** To prove the result, we shall use a fixed point argument for the map \( \Gamma : M^\beta_G(t, t + h; \mathbb{R}^k) \to M^\beta_G(t, t + h; \mathbb{R}^k) \). We first prove the existence. From Lemma 3.12, we see that \( \Gamma \) is a contraction, and thus has a unique fixed point \( Y \in M^\beta_G(t, t + h; \mathbb{R}^k) \) such that \( \Gamma(Y) = Y \). Moreover, for \( U = Y \), from Lemma 3.11 we know that there exists solution \( (Y, Z, A) \in S^\beta_G(t, t + h; \mathbb{R}^k) \) that solves (3.10). Combining the above analysis, we deduce that \( (Y, Z, A) \) solves (3.10), which can be written as follows: for \( 1 \leq l \leq k \), it holds on \( [t, t + h] \) that
\[
Y_{s}^{l} = \zeta_{t}^{l} + \int_{s}^{t+h} f(r, Y_{r}, Z_{r}^{Y,l}) dr - \int_{s}^{t+h} Z_{r}^{Y,l} dB_{r} + (A_{t+h}^{Y,l} - A_{s}^{Y,l}),
\]
with
\[
Y_{s}^{l} \geq S_{s}^{l} \quad \text{and} \quad -\int_{t}^{s} (Y_{r}^{l} - S_{r}^{l}) dA_{r} \text{ being non-increasing G-martingale.}
\]
That means that \( (Y, Z, A) \) is the solution of reflected G-BSDE with parameters \( (\zeta, f, S) \) on time interval \( [t, t + h] \).
We then consider the uniqueness. If \((Y', Z', A') \in \mathcal{S}_G^\alpha(t, t + h; \mathbb{R}^k)\) is also the solution of reflected \(G\)-BSDE with parameters \((\zeta, f, S)\) on time interval \([t, t+h]\), then \(Y' \in M_G^\beta(t, t+h; \mathbb{R}^k)\) and \(\Gamma(Y') = Y'\), which means that \(Y'\) is the fixed point of \(\Gamma\). So \(Y = Y'\). Then for any \(1 \leq l \leq k\), \((Y^l, Z^l, A^l)\) and \((Y'^l, Z'^l, A'^l)\) are both solutions of (3.10) with \(U = Y\). Applying Lemma 3.11 implies that \((Z^l, A^l) = (Z', A')\).

By a backward iteration on local solutions, we finally have the following new proof for the global well-posedness Theorem 3.1 for reflected \(G\)-BSDE (2.1).

Proof. We choose the constant \(\delta\) determined in Theorem 3.10 and take an integer \(m\) large enough such that \(m\delta \geq T\). We can then take \(h = \frac{T}{m}\) and apply Theorem 3.10 to deduce that the reflected \(G\)-BSDE with parameters \((\xi, f, S)\) on \([T - h, T]\) admits a unique solution \((Y^{(m)}, Z^{(m)}, A^{(m)}) \in \mathcal{S}_G^\alpha(T - h, T; \mathbb{R}^k)\).

Choosing \(T - h\) as the terminal time and \(Y^{(m)}_{T - h}\) as the terminal value, we apply Theorem 3.10 again to obtain that the reflected \(G\)-BSDE (3.1) with parameters \((Y^{(m)}_T, f, S)\) on \([T - 2h, T - h]\) has a unique solution \((Y^{(m-1)}, Z^{(m-1)}, A^{(m-1)}) \in \mathcal{S}_G^\alpha(T - 2h, T - h; \mathbb{R}^k)\). Finally we obtain a sequence \((Y^{(i)}, Z^{(i)}, K^{(i)})_{i \leq m}\) by repeating this procedure. Let us take

\[
Y_t = \sum_{i=1}^{m} Y_t^{(i)} I_{[(i-1)h, ih]}(t) + Y_T^{(m)} I_{[T]}(t), \quad Z_t = \sum_{i=1}^{m} Z_t^{(i)} I_{[(i-1)h, ih]}(t) + Z_T^{(m)} I_{[T]}(t)
\]

and

\[
A_t = A_t^{(i)} + \sum_{j=1}^{i-1} A_{j}^{(j)} + \frac{1}{m} \sum_{j=1}^{m} A_{j}^{(j)}, \quad \text{for } t \in [(i-1)h, ih), \ i = 1, \cdots, \ m \quad \text{and} \quad A_T = A_T^{(m)} + \sum_{j=1}^{m-1} A_{j}^{(j)},
\]

where we make the convention that \(\sum_{j=1}^{0} A_{j}^{(j)} = 0\). Then it is easy to see that \((Y, Z, K) \in \mathcal{S}_G^\alpha(0, T; \mathbb{R}^k)\) is a solution to reflected \(G\)-BSDE with parameters \((\xi, f, S)\) and \(Y \in M_G^\beta(0, T; \mathbb{R}^k)\).

The uniqueness result on the whole interval follows from the one on each small time interval, which completes the proof. \(\square\)

4 Multi-dimensional obstacle problems for fully nonlinear PDEs

In this section, we establish the connection between the multi-dimensional reflected \(G\)-BSDEs and the system of fully nonlinear PDEs with obstacle constraints. Roughly speaking, the solution of a multi-dimensional reflected \(G\)-BSDE in a Markovian setting coincides with the unique viscosity solution to a system of fully nonlinear PDE with obstacle constraints.

For this purpose, for any fixed \(t \in [0, T]\) and \(\eta \in L^p_G(\Omega; \mathbb{R}^n)\) with \(p \geq 2\), let us first consider the following \(G\)-SDEs:

\[
\begin{aligned}
&dx^{t, \eta}_s = b(s, X^{t, \eta}_s)ds + h(s, X^{t, \eta}_s)d(B)_s + \sigma(s, X^{t, \eta}_s)dB_s, \quad s \in [t, T], \\
&X^{t, \eta}_t = \eta,
\end{aligned}
\]

where \(b, h, \sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n\) are deterministic continuous functions satisfying

(Ai) there exists a positive constant \(L\) such that, for any \(t \in [0, T]\) and any \(x, x' \in \mathbb{R}^n\)

\[
|b(t, x) - b(t, x')| + |h(t, x) - h(t, x')| + |
\sigma(t, x) - \sigma(t, x')| \leq L|x - x'|.
\]

By a similar analysis as in Chapter 5 of [25], the \(G\)-SDE (4.1) has a unique solution \(X^{t, \eta} \in M_G^\beta(t, T; \mathbb{R}^n)\). Furthermore, the following estimates hold.
Proposition 4.1 ([25]). Let \( \eta, \eta' \in L^p_G(\Omega; \mathbb{R}^n) \) and \( p \geq 2 \). Then we have, for each \( \delta \in [0, T-t] \),

\[
\hat{E}_t \left[ \sup_{s \in [t, t+\delta]} |X_{s,t}^{\eta} - X_{s,t}^{\eta'}|^p \right] \leq C|\eta - \eta'|^p,
\]

\[
\hat{E}_t \left[ |X_{t+\delta,t}^{\eta}|^p \right] \leq C(1 + |\eta|^p),
\]

\[
\hat{E}_t \left[ \sup_{s \in [t, t+\delta]} |X_{s,t}^{\eta} - \eta|^p \right] \leq C(1 + |\eta|^p)\delta^{p/2},
\]

where the constant \( C \) depends on \( L, G, p, n \) and \( T \).

Now, let us consider the following \( k \)-dimensional reflected G-BSDEs on the interval \( [t, T] \):

\[
\begin{cases}
-dY_{s,t}^{i,\eta,i} = f^i(s, X_{s,t}^{i,\eta}, Y_{s,t}^{i,\eta}, Z_{s,t}^{i,\eta,i}) \, ds + g^i(s, X_{s,t}^{i,\eta}, Y_{s,t}^{i,\eta}, Z_{s,t}^{i,\eta,i}) \, dB_s \\
\quad -Z_{s,t}^{i,\eta,i} dB_i + dA_{s,t}^{i,\eta,i},
\end{cases}
\]

where \( 1 \leq i \leq k \) and the functions \( \phi^i : \mathbb{R}^n \to \mathbb{R} \), \( f^i, g^i : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R} \to \mathbb{R} \), \( l^i : [0, T] \times \mathbb{R}^n \to \mathbb{R} \) are continuous, deterministic functions satisfying the following assumptions:

(Aii) there exists a constant \( L > 0 \) such that for any \( 1 \leq i \leq k \), \( x, x_1, x_2, y, z, z_1, z_2 \in \mathbb{R} \)

\[
|\phi^i(x) - \phi^i(x_2)| \leq L|x_1 - x_2|, \quad |f^i(t, x_1) - f^i(t, x_2)| \leq L|x_1 - x_2|,
\]

\[
|f^i(t, x_1, y_1, z_1) - f^i(t, x_2, y_2, z_2)| \leq L(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|),
\]

\[
|g^i(t, x_1, y_1, z_1) - g^i(t, x_2, y_2, z_2)| \leq L(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|);
\]

(Aiii) for any \( 1 \leq i \leq k \), \( x, x_1, x_2 \in \mathbb{R}^n \) and \( t \in [0, T] \), \( l^i(t, x) \leq \bar{l}^i(t, x) \) and \( l^i(T, x) < \bar{l}^i(t, x) \), where \( \bar{l}^i \) belongs to the space \( C_L^{1,2}([0, T] \times \mathbb{R}^n) \), which is the collection of all functions of class \( C^{1,2}([0, T] \times \mathbb{R}^n) \) whose partial derivatives of order less than or equal to 2 and itself are continuous in \( t \) and Lipschitz continuous in \( x \);

(Aiv) for \( 1 \leq i \leq k \), for any \( y, \bar{y} \in \mathbb{R}^k \) satisfying \( y^j \geq \bar{y}^j \) for \( j \neq i \) and \( y^i = \bar{y}^i \), it holds \( f^i(t, y, z) \geq f^i(t, \bar{y}, z) \) and \( g^i(t, y, z) \geq g^i(t, \bar{y}, z) \).

By Theorem 3.1, for any \( \eta \in L^\beta_G(\Omega; \mathbb{R}^n) \) with \( \beta > 2 \), there exists a unique solution \( (Y_{t,t}^{\eta}, Z_{t,t}^{\eta}, A_{t,t}^{\eta}) \in S^\beta_G([t, T]; \mathbb{R}^k) \) to reflected G-BSDE (4.2), where \( 2 \leq \alpha < \beta \). For each fixed \( (t, x) \in [0, T] \times \mathbb{R}^n \) and \( 1 \leq i \leq k \), we define:

\[
u^i(t, x) := Y_{t,t}^{i,x,i}.
\]

Note that \( u^i \) is a deterministic function. By Proposition 3.2, 3.3 and 4.1, it is easy to check that there exists a constant \( C \) depending on \( L, G, n, k \) and \( T \), such that

\[
|u(t, x)| \leq C(1 + |x|),
\]

\[
|u(t, x) - u(t, x')| \leq C|x - x'|.
\]

Proposition 4.2. The function \( u : [0, T] \times \mathbb{R}^n \to \mathbb{R}^k \) is continuous.

Proof. It remains to prove that \( u \) is continuous in \( t \). For simplicity, we assume that \( g \equiv 0 \). For any \( t \geq 0 \), we define

\[
Y_{s,t}^{i,x} := Y_{t,t}^{i,x}, \quad X_{s,t}^{i,x} := x, \quad A_{s,t}^{i,x} := 0, \quad Z_{s,t}^{i,x} := 0, \quad l(s, X_{s,t}^{i,x}) := l(t, x), \quad s \in [0, t].
\]

For \( 1 \leq i \leq k \), it is easy to check that \( Y_{t,t}^{i,x;i} \) is the first component of solution to reflected G-BSDE on the interval \( [0, T] \) with terminal value \( \phi^i(X_{t,t}^{i,x;i}) \), generator \( f_{t,t}^{i,x;i} \) and obstacle process \( \bar{S}_{t,t}^{i,x;i} \), where

\[
\bar{f}_{t,t}^{i,x;i}(s, y, z) = I_{[0,t]}(s)f^i(s, X_{s,t}^{i,x}, y, z), \quad \bar{S}_{s,t}^{i,x;i} = l(t, x)I_{[0,t]}(s) + l(s, X_{s,t}^{i,x})I_{[t,T]}(s).
\]
Set

\[ I^x_t = \tilde{I}(t, x) I_{[0, t]}(s) + \tilde{I}(s, X^t_s) I_{[t, T]}(s), \]
\[ b^x_t = (\partial_s \tilde{I}(s, X^t_s) + \langle b(s, X^t_s), D_x \tilde{I}(s, X^t_s) \rangle) I_{[t, T]}(s), \]
\[ \sigma^x_t = \langle \sigma(s, X^t_s), D_x \tilde{I}(s, X^t_s) \rangle I_{[t, T]}(s). \]

For \( 0 \leq t_1 \leq t_2 \leq T \), by Proposition 3.3, we have

\[ |u(t_1, x) - u(t_2, x)|^2 = |Y^{t_1, x}_0 - Y^{t_2, x}_0|^2 \]
\[ \leq C \tilde{E}[|\phi(X^{t_1, x}_0) - \phi(X^{t_2, x}_0)|^2] + C \tilde{E}[\int_0^T |\tilde{f}^{t_1, x}(s, Y^{t_1, x}_s, Z^{t_1, x}_s) - \tilde{f}^{t_2, x}(s, Y^{t_2, x}_s, Z^{t_2, x}_s)|^2 ds] =: I \] (4.4)

Similar as the proof of Proposition 4.5 in [19], we have

\[ I \leq C(1 + |x|^2)(t_2 - t_1). \]

Applying Proposition 4.1 yields that \( \Psi_{0, T} \leq C(1 + |x|^2) \). We directly calculate that

\[ \tilde{E}[\sup_{s \in [0, T]} |S^{t_1, x}_s - S^{t_2, x}_s|^2] \]
\[ \leq C \tilde{E}[|\tilde{l}(t_1, x) - \tilde{l}(t_2, x)|^2 + \sup_{s \in [t_1, t_2]} |\tilde{l}(s, X^{t_1, x}_s) - \tilde{l}(s, x)|^2] + C \tilde{E}[\sup_{s \in [t_1, t_2]} |\tilde{l}(s, X^{t_1, x}_s) - \tilde{l}(s, X^{t_2, x}_s)|^2] \]
\[ \leq 2 \sup_{s \in [t_1, t_2]} |\tilde{l}(s, x)| + C \tilde{E}[\sup_{s \in [t_1, t_2]} |X^{t_1, x}_s - x|^2] + C \tilde{E}[\sup_{s \in [t_1, t_2]} |X^{t_1, x}_s - X^{t_2, x}_s|^2] \]
\[ \leq 2 \sup_{s \in [t_1, t_2]} |\tilde{l}(s, x)| + C \tilde{E}[\sup_{s \in [t_1, t_2]} |X^{t_1, x}_s - x|^2] + C \tilde{E}[\sup_{s \in [t_1, t_2]} |X^{t_1, x}_s - X^{t_2, x}_s|^2] \]
\[ \leq 2 \sup_{s \in [t_1, t_2]} |\tilde{l}(s, x)| + C(1 + |x|^2)(t_2 - t_1) + C \tilde{E}[|X^{t_1, x}_s - x|^2] \]
\[ \leq 2 \sup_{s \in [t_1, t_2]} |\tilde{l}(s, x)| + C(1 + |x|^2)(t_2 - t_1). \]

Then, combining all the above analysis, we obtain the desired result. □

**Remark 4.3.** The continuity property of \( u \) only needs Assumptions (Ai)-(Aiii). The Assumption (Aiv) is used to guarantee that \( u^i, 1 \leq i \leq k \) can be approximated by certain monotone sequences in the proof of Theorem 4.6 below.

Consider the following system of parabolic PDEs with obstacle constraints:

\[
\begin{cases}
\min \{u^i(t, x) - l^i(t, x), -\partial_t u^i(t, x) - F^i(D^2 u^i, D_x u^i, u, x, t)\} = 0, & (t, x) \in (0, T) \times \mathbb{R}^n, \\
u^i(T, x) = \phi^i(x),& x \in \mathbb{R}^n; 1 \leq i \leq k,
\end{cases}
\]

where

\[
F^i(A, p, r, x, t) := G(\sigma^T(t, x) A \sigma(t, x) + 2\langle p, h(t, x) \rangle + 2g^i(t, x, r, \langle \sigma(t, x), p \rangle))
\]
\[+ \langle b(t, x), p \rangle + f^i(t, x, r, \langle \sigma(t, x), p \rangle), \]

for \( (A, p, r, x, t) \in \mathcal{S}(n) \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \times [0, T], \)
and \( S(n) \) is the set of symmetric \( n \times n \)-matrices.

In the following, we will show that the value function defined in (4.3) is the unique solution to the above PDE. Note that \( u^t \) is continuous but may be not differentiable. Therefore, when we call \( u^t \) a solution to (4.5), it means a viscosity solution. Let us first introduce the definition of viscosity solution to the PDE (4.5), which needs the following definitions of sub-jets and super-jets. For more details, we may refer to the paper [2].

**Definition 4.4.** Let \( u \in C((0,T) \times \mathbb{R}^n) \) and \( (t,x) \in (0,T) \times \mathbb{R}^n \). We denote by \( \mathcal{P}^{2,+}u(t,x) \) (the “parabolic superjet” of \( u \) at \( (t,x) \)) the set of triples \( (p,q,X) \in \mathbb{R} \times \mathbb{R}^n \times S(n) \) satisfying

\[
    u(s,y) \leq u(t,x) + p(s-t) + \langle q, y-x \rangle + \frac{1}{2} \langle X(y-x), y-x \rangle + o(|s-t| + |y-x|^2).
\]

Similarly, we define \( \mathcal{P}^{2,-}u(t,x) \) (the “parabolic subjet” of \( u \) at \( (t,x) \)) by \( \mathcal{P}^{2,-}u(t,x) := -\mathcal{P}^{2,+}(-u)(t,x) \).

**Definition 4.5.** Let \( u \in C([0,T] \times \mathbb{R}^n; \mathbb{R}^k) \). It is called an:

(i) **viscosity subsolution** of (4.5) if for each \( 1 \leq i \leq k \), \( u^i(T,x) \leq \phi^i(x), \, x \in \mathbb{R}^n \), and at any point \( (t,x) \in (0,T) \times \mathbb{R}^n \), for any \( (p,q,X) \in \mathcal{P}^{2,+}u^i(t,x) \),

\[
    \min \left( u^i(t,x) - l^i(t,x), -p - F^i(X, u(t,x), x, t) \right) \leq 0;
\]

(ii) **viscosity supersolution** of (4.5) if for each \( 1 \leq i \leq k \), \( u^i(T,x) \geq \phi^i(x), \, x \in \mathbb{R}^n \), and at any point \( (t,x) \in (0,T) \times \mathbb{R}^n \), for any \( (p,q,X) \in \mathcal{P}^{2,-}u^i(t,x) \),

\[
    \min \left( u^i(t,x) - l^i(t,x), -p - F(X, u(t,x), x, t) \right) \geq 0;
\]

(iii) **viscosity solution** of (4.5) if it is both a viscosity subsolution and supersolution.

**Theorem 4.6.** The function \( u = (u^1, \cdots, u^k) \) defined by (4.3) is the unique viscosity solution to the system of parabolic PDE (4.5).

**Proof.** For each fixed \( (t,x) \in [0,T] \times \mathbb{R}^n \), for any \( m = 1,2, \cdots \) and \( 1 \leq i \leq k \), we first consider the following penalized G-BSDEs:

\[
    Y_{s}^{m,t,x,i} = \phi^i(X_{T}^{s,t,x}) + \int_{s}^{T} f^i(r, X_{r}^{t,x}, Y_{r}^{m,t,x}, Z_{r}^{m,t,x,i}) dr + \int_{s}^{T} g^i(r, X_{r}^{t,x}, Y_{r}^{m,t,x}, Z_{r}^{m,t,x,i}) dB_{r} + m \int_{s}^{T} (Y_{r}^{m,t,x,i} - l^i(r, X_{r}^{t,x}))^- dr - \int_{s}^{T} Z_{r}^{m,t,x,i} dB_{r} - (K_{T}^{m,t,x,i} - K_{s}^{m,t,x,i}).
\]

By Theorem 4.7 in [19], the function \( u^m = (u^{m,1}, \cdots, u^{m,k}) \) defined as

\[
    u^{m,i}(t,x) := Y_{t}^{m,t,x,i}, \quad (t,x) \in [0,T] \times \mathbb{R}^n, \quad 1 \leq i \leq k,
\]

is the viscosity solution of the parabolic PDE

\[
    \begin{cases}
    \partial_{t}u^{m,i}(t,x) + F^i(D_{x}^{2}u^{m,i}, D_{x}u^{m,i}, u^{m,i}, x, t) \\
    + m(u^{m,i}(t,x) - l^i(t,x))^- = 0, \quad (t,x) \in (0,T) \times \mathbb{R}^n, \quad 1 \leq i \leq k, \\
    u^{m,i}(T,x) = \phi^i(x), \quad x \in \mathbb{R}, \quad 1 \leq i \leq k.
    \end{cases}
\]

Besides, by the construction of the solution of reflected G-BSDEs via penalization and Theorem 2.4, we know that for each \( 1 \leq i \leq k \) and \( (t,x) \in \mathbb{R}^n \), \( u^{m,i}(t,x) \) converges monotonically up to \( u^i(t,x) \) as \( m \) approaches infinity. Recalling Proposition 4.5 in [19] and Proposition 4.2, \( u^{m,i}, \, u^i \in C([0,T] \times \mathbb{R}^n) \). Thus it follows from the Dini theorem that \( u^{m,i} \) converges uniformly to \( u^i \) on compact sets.

We first prove that \( u \) is a viscosity subsolution of (4.5). Let \( (t,x) \) be a point such that \( u^i(t,x) > l^i(t,x) \) and let \( (p,q,X) \in \mathcal{P}^{2,+}u^i(t,x) \). By Lemma 6.1 in [2], we may find sequences

\[
    m_{j} \rightarrow \infty, \quad (i_j, x_j) \rightarrow (t,x), \quad (p_{j}, q_{j}, X_{j}) \in \mathcal{P}^{2,+}u^{m_{j},i}(t_j, x_j),
\]

at any point \( (t,x) \in (0,T) \times \mathbb{R}^n \), with the following properties:

\[
    \begin{align*}
    u(t,x) &\leq \liminf_{j \rightarrow \infty} u^{m_{j},i}(t_j, x_j) - l^i(t_j, x_j) + \frac{1}{2} \left| Y_{t}^{m_{j},t,x,i} - Y_{t}^{m_{j-1},t,x,i} \right| + o(1), \\
    l^i(t_j, x_j) &\leq \limsup_{j \rightarrow \infty} u^{m_{j},i}(t_j, x_j) - l^i(t_j, x_j) + \frac{1}{2} \left| Y_{t}^{m_{j},t,x,i} - Y_{t}^{m_{j-1},t,x,i} \right| + o(1).
    \end{align*}
\]
such that \((p_j, q_j, X_j) \rightarrow (p, q, X)\). Note that \(u^m\) is a viscosity solution to Equation (4.6), hence a subsolution. We have, for any \(j\),
\[-p_j - F^i(X_j, q_j, u^{m; j}(t_j, x_j), x_j, t_j) - m_j(u^{m; j}(t_j, x_j) - l^i(t_j, x_j))^- \leq 0.\]
Since \((t, x)\) is a point at which \(u^i(t, x) > h^i(t, x)\), it follows from the uniform convergence of \(u^{m; j}\) that \(u^{m; j}(t_j, x_j) > l^i(t_j, x_j)\) for large enough \(j\). Letting \(j\) go to infinity in the above inequality, we obtain
\[-p - F^i(X, q, u(t, x), x, t) \leq 0,\]
which implies that \(u\) is a subsolution of (4.5).

It remains to prove that \(u\) is a supersolution of (4.5). For any \(1 \leq i \leq k\) and \((t, x) \in [0, T] \times \mathbb{R}^n\), let \((p, q, X) \in \mathcal{P}^{2}\), \(u^i(t, x) \geq l^i(t, x)\). Then it is sufficient to prove that
\[-p - F^i(X, q, u(t, x), x, t) \geq 0.\]

By a similar analysis as above, there exist sequences
\[m_j \rightarrow \infty, (t_j, x_j) \rightarrow (t, x), (p_j, q_j, X_j) \in \mathcal{P}^{2} - u^{m; j}(t_j, x_j),\]
such that \((p_j, q_j, X_j) \rightarrow (p, q, X)\). Since \(u^m\) is a viscosity solution to Equation (4.6), hence a supersolution, we have, for any \(j\),
\[-p_j - F^i(X_j, q_j, u^{m; j}(t_j, x_j), x_j, t_j) - m_j(u^{m; j}(t_j, x_j) - l^i(t_j, x_j))^- \geq 0.\]
Letting \(j\) approach infinity, we get the desired result.

The uniqueness of viscosity solutions to (4.5) follows from classical arguments. See, e.g., [2] and [28]. So we omit it and the proof is complete. 

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Declarations

Conflict of Interest  The authors declared that they have no conflict of interest.

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