COTANGENT BUNDLE TO THE FLAG VARIETY-I

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ABSTRACT. We show that there is a $SL_n$-stable closed subset of an affine Schubert variety in the infinite dimensional Flag variety (associated to the Kac-Moody group $\hat{SL}_n$) which is a natural compactification of the cotangent bundle to the finite-dimensional Flag variety $SL_n/B$.

1. INTRODUCTION

Let the base field $K$ be the field of complex numbers. Consider a cyclic quiver with $h$ vertices and dimension vector $\underline{d} = (d_1, \cdots, d_h)$:

$$
\begin{array}{cccccc}
1 & \rightarrow & 2 & \rightarrow & \cdots & \rightarrow & h-2 & \rightarrow & h-1 \\
& & & & & & & & h
\end{array}
$$

Denote $V_i = K^{d_i}$. Let

$$Z = \text{Hom}(V_1, V_2) \times \cdots \times \text{Hom}(V_h, V_1), \quad GL_{\underline{d}} = \prod_{1 \leq i \leq h} GL(V_i)$$

We have a natural action of $GL_{\underline{d}}$ on $Z$: for $g = (g_1, \cdots, g_h) \in GL_{\underline{d}}, f = (f_1, \cdots, f_h) \in Z$,

$$g \cdot f = (g_2 f_1 g_1^{-1}, g_3 f_2 g_2^{-1}, \cdots, g_1 f_h g_h^{-1})$$

Let

$$\mathcal{N} = \{ (f_1, \cdots, f_h) \in Z \mid f_h \circ f_{h-1} \circ \cdots \circ f_1 : V_1 \rightarrow V_1 \text{ is nilpotent} \}$$

Note that $f_h \circ f_{h-1} \circ \cdots \circ f_1 : V_1 \rightarrow V_1$ being nilpotent is equivalent to $f_{i-1} \circ f_{i-2} \circ \cdots \circ f_1 : V_i \rightarrow V_i$ being nilpotent. Clearly $\mathcal{N}$ is $GL_{\underline{d}}$-stable. Lusztig (cf. [7]) has shown that an orbit closure in $\mathcal{N}$ is canonically isomorphic to an open subset of a Schubert variety in $\hat{SL}_n/Q$, where $n = \sum_{1 \leq i \leq h} d_i$, and $Q$ is the parabolic subgroup of $\hat{SL}_n$ corresponding to omitting $\alpha_0, \alpha_{d_1}, \alpha_{d_1+d_2}, \cdots, \alpha_{d_1+\cdots+d_{h-1}}$ ($\alpha_i, 0 \leq i \leq n-1$ being the set of simple roots for $\hat{SL}_n$). Corresponding to $h = 1$, we have that $\mathcal{N}$ is in fact the variety of nilpotent elements in $M_{d_1, d_1}(K)$, and thus the above isomorphism identifies $\mathcal{N}$ with an open subset of a Schubert variety $X_{\alpha'}$ in $\hat{SL}_n/G_0$, $G_0$ being the maximal parabolic subgroup of $\hat{SL}_n$ corresponding to “omitting” $\alpha_0$.

Let now $h = 2$

$$Z_0 = \{ (f_1, f_2) \in Z \mid f_2 \circ f_1 = 0, f_1 \circ f_2 = 0 \}$$

Strickland (cf. [10]) has shown that each irreducible component of $Z_0$ is the conormal variety to a determinantal variety in $M_{d_1, d_2}(K)$. A determinantal variety

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in $M_{d_1,d_2}(K)$ being canonically isomorphic to an open subset in a certain Schubert variety in $G_{d_2,d_1+d_2}$ (the Grassmannian variety of $d_2$-dimensional subspaces of $K^{d_1+d_2}$) (cf. [6], the above two results of Lusztig and Strickland suggest a connection between conormal varieties to Schubert varieties in the (finite-dimensional) flag variety and the affine Schubert varieties. This is the motivation for this article. Let $G = SL_n$.

Inspired by Lusztig’s embedding of $N$ in $\hat{SL}_n/Q$, we define a family of maps $\psi_p : T^*G/B \rightarrow \hat{SL}_n/B$, parametrized by polynomials in one variable with coefficients in $\mathbb{C}((t))$, and with 1 as the constant term. For a particular map $\phi$ (analogous to Lusztig’s map) in this family, we find a $\kappa_0 \in \hat{W}$ such that the affine Schubert variety $X(\kappa_0)$ is $G_0$-stable ($G_0$ being as above, the maximal parabolic subgroup of $\hat{SL}_n$ corresponding to “omitting” $\alpha_0$) and show that $\phi$ gives an embedding $T^*G/B \hookrightarrow X(\kappa_0) \subset \hat{SL}_n/B$. We thus obtain a $SL_n$-stable closed subvariety of $X(\kappa_0)$ as a natural compactification of $T^*G/B$ (cf. Theorem 6.4). Let $\pi : \hat{SL}_n/B \rightarrow \hat{SL}_n/G_0$ be the canonical projection. Then we show that $\pi(T^*G/B) = N$, the variety of nilpotent matrices, and that $\pi|_{T^*G/B : T^*G/B \rightarrow N}$ is in fact the Springer resolution.

Following the above ideas, Lakshmibai (cf. [4]) has obtained a stronger result for $T^*G_{d,n}$, the cotangent bundle to the Grassmannian variety $G_{d,n}$. She shows that there is an embedding $\chi$ (analogous to $\phi$) of $T^*G_{d,n}$ inside a Schubert variety $X(\iota) \subset \hat{SL}_n/Q_d$ (where $Q_d$ is the two step parabolic subgroup of $\hat{SL}_n$ corresponding to omitting $\alpha_0, \alpha_d$) such that $X(\iota)$ is in fact a compactification of $T^*G_{d,n}$. The result of [4] has been generalized to $T^*G/P$ in [5], $G/P$ being a cominuscule Grassmannian variety.

It would be interesting to know if the result of [4] could be achieved replacing $P$ with $B$, for a suitable generalization of $\chi$. We show in [7] that this is not possible for any $\psi_p$ in the above family, even when $n = 3$. We think that our result about the embedding $\phi : T^*G/B \rightarrow X(\kappa_0)$ identifying a certain $SL_n$-stable closed subvariety of $X(\kappa_0)$ as a natural compactification of $T^*G/B$ is the best possible in relating $T^*G/B$ and affine Schubert varieties in $\hat{SL}_n/B$.

The results of this paper open up other related problems like, the study of line bundles on $T^*G/B, G = SL_n$ (using the embedding of $T^*G/B$ into $X(\kappa_0)$, and realizing line bundles on $T^*G/B$ as restrictions of suitable line bundles on $X(\kappa_0)$), establishing similar embeddings of the cotangent bundles to partial flag varieties $G/Q$ ($G$ semi-simple and $Q$ a parabolic subgroup) etc. Further, the facts that conormal varieties to Schubert varieties in $G/B$ are closed subvarieties of $T^*G/B$, and that the affine Schubert variety $X(\kappa_0)$ contains a $G$-stable closed subvariety which is a natural compactification of $T^*G/B$, suggest similar compactifications for conormal varieties to Schubert varieties in $G/B$ (by suitable affine Schubert varieties in $\hat{SL}_n/B$); such a realization could lead to important consequences such as a knowledge of the equations of the conormal varieties (to Schubert varieties) as subvarieties of the cotangent bundle. These problems will be dealt with in a subsequent paper.

Regarding results on similar compactifications, we mention Mirkovic-Vybornov’s work (cf. [9]), where the authors construct compactifications of Nakajima’s quiver varieties of type $A$ inside affine Grassmannians of type $A$. Also, Manivel and
Michalek ([8]) have recently studied the local geometry of tangential varieties (which are compactifications of the tangent bundle) to cominuscule Grassmannians.

The sections are organized as follows. In [2] we fix notation and recall affine Schubert varieties. In [3] we introduce the elements $\kappa$ and $\kappa_0$ (in $\hat{W}$, the affine Weyl group), and prove some properties of $\kappa$. In [4] we prove a crucial result on $\kappa$ needed for realizing the embeddings of $\mathcal{N}$ and $T^*SL_n/B$ inside $\hat{SL}_n/G_0$ and $\hat{SL}_n/B$ respectively. In [4] we spell out Lusztig’s isomorphism which identifies $\mathcal{N}$ with an open subset of $X_{G_0}(\kappa)$ (inside $\hat{SL}_n/G_0$). In [6] using the map $\phi : T^*G/B \to \hat{SL}_n/B$ as above and the natural projection $\hat{SL}_n/B \to \hat{SL}_n/G_0$, we recover the Springer resolution of $\mathcal{N}$; we also prove the main result that $\phi$ identifies an $SL_n$-stable closed subvariety of $X(\kappa_0)$ as a compactification of $T^*G/B$. In [7] we show that it is not possible, for any choice in the family $\psi_p$, to realize an affine Schubert variety (in $\hat{SL}_3/B$) as a compactification of the cotangent bundle $T^*SL_3/B$.

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2. Affine Schubert varieties

Let $K = \mathbb{C}, F = K[[t]]$, the field of Laurent series, $A = K[[t]]$. Let $G$ be a semi-simple algebraic group over $K$, $T$ a maximal torus in $G$, $B$ a Borel subgroup, $B \supset T$, and let $B^-$ be the Borel subgroup opposite to $B$. Let $\mathcal{G} = G(F)$. The natural inclusion $K \hookrightarrow A \hookrightarrow F$ induces an inclusion

$$G \twoheadrightarrow G(A) \hookrightarrow \mathcal{G}$$

The natural projection $A \to K$ given by $t \to 0$ induces a homomorphism

$$\pi : G(A) \to G$$

The group $B := \pi^{-1}(B)$ is a Borel subgroup of $\mathcal{G}$.

2.1. Bruhat decomposition: Let $\hat{W} = N(K[t, t^{-1}])/T$, the affine Weyl group of $G$ (here, $N$ is the normalizer of $T$ in $G$). The group $\hat{W}$ is a Coxeter group (cf. [1]). We have that

$$G(F) = \bigcup_{w \in \hat{W}} BwB, G(F)/B = \bigcup_{w \in \hat{W}} BwB \mod B$$

For $w \in \hat{W}$, let $X(w)$ be the affine Schubert variety in $G(F)/B$:

$$X(w) = \bigcup_{r \leq w} BrB \mod B$$

It is a projective variety of dimension $t(w)$.

2.2. Affine Flag variety, Affine Grassmannian: Let $G = SL(n), \mathcal{G} = G(F), G_0 = G(A)$. We say $g \in \mathcal{G}$ is integral if and only if $g \in G_0$, i.e. viewed as $G$-valued meromorphic function on $\mathbb{C}$, it has no poles at $t = 0$. The homogeneous space $\mathcal{G}/B$ is the affine Flag variety, and $\mathcal{G}/G_0$ is the affine Grassmannian. Further,

$$\mathcal{G}/G_0 = \bigcup_{w \in \hat{W}G_0} BwG_0 \mod G_0$$

where $\hat{W}G_0$ is the set of minimal representatives in $\hat{W}$ of $\hat{W}/WG_0$.

Let

$$Gr(n) = \{ A\text{-lattices in } F^n \}$$
Here, by an $A$-lattice in $F^n$, we mean a free $A$-submodule of $F^n$ of rank $n$. Let $E$ be the standard lattice, namely, the $A$-span of the standard $F$-basis $\{e_1, \ldots, e_n\}$ for $F^n$. For $V \in \widehat{\text{Gr}}(n)$, define
\[
\text{vdim}(V) := \dim_K(V/V \cap E) - \dim_K(E/V \cap E)
\]
One refers to $\text{vdim}(V)$ as the virtual dimension of $V$. For $j \in \mathbb{Z}$ denote
\[
\text{Gr}_j(n) = \{ V \in \widehat{\text{Gr}}(n) \mid \text{vdim}(V) = j \}
\]
Then $\text{Gr}_j(n), j \in \mathbb{Z}$ give the connected components of $\widehat{\text{Gr}}(n)$. We have a transitive action of $\text{GL}_n(F)$ on $\widehat{\text{Gr}}(n)$ with $\text{GL}_n(A)$ as the stabilizer at the standard lattice $E$. Further, let $G_0$ be the subgroup of $\text{GL}_n(F)$, defined as,
\[
G_0 = \{ g \in \text{GL}_n(F) \mid \text{ord(det } g) = 0 \}
\]
(here, for a $f \in F$, say $f = \sum a_i t^i$, order $f$ is the smallest $r$ such that $a_r \neq 0$). Then $G_0$ acts transitively on $\text{Gr}_0(n)$ with $\text{GL}_n(A)$ as the stabilizer at the standard lattice $E$. Also, we have a transitive action of $\text{SL}_n(F)$ on $\text{Gr}_0(n)$ with $\text{SL}_n(A)$ as the stabilizer at the standard lattice $E$. Thus we obtain the identifications:
\[
\text{GL}_n(F)/\text{GL}_n(A) \simeq \text{Gr}(n)
\]
\[
G_0/\text{GL}_n(A) \simeq \text{Gr}_0(n), \text{SL}_n(F)/\text{SL}_n(A) \simeq \text{Gr}_0(n)
\]
In particular, we obtain
\[
G_0/\text{GL}_n(A) \simeq \text{SL}_n(F)/\text{SL}_n(A)
\]

2.3. Generators for $\hat{W}$: Recall that the Weyl group $\hat{W} = N(K[t, t^{-1}])/T$. Let $R$ (resp. $R^+$) be the set of roots (resp. positive roots) of $G$ relative to $B$, and let $\delta$ be the basic imaginary root of the affine Kac-Moody algebra of type $\tilde{A}_{n-1}$ given by
(cf. [1])
\[
\delta = \alpha_0 + \theta = \alpha_0 + \cdots + \alpha_{n-1}
\]
The set of real roots of $G$ is given by $\{r\delta + \beta \mid r \in \mathbb{Z}, \beta \in R\}$, and the set of positive roots of $G$ is given by $\{r\delta + \beta \mid r > 0, \beta \in R\} \cup R^+$ (cf. [1]). Following the notation in [1], we shall work with the set of generators for $\hat{W}$ given by $\{s_0, s_1, \ldots, s_{n-1}\}$, where $s_i, 0 \leq i \leq n-1$ are the reflections with respect to $\alpha_i, 0 \leq i \leq n-1$. Note that $\{s_1, 1 \leq i \leq n-1\}$ simply the set of simple roots of $\text{SL}_n$ (with respect to the Borel subgroup $B$). In particular, the Weyl group $W$ of $\text{SL}_n(\mathbb{C})$ is simply the subgroup of $\hat{W}$ generated by $\{s_1, \ldots, s_{n-1}\}$.

2.4. The Affine Presentation: The generators $s_i, 1 \leq i \leq n-1$ have the following canonical lifts to $N(K[t, t^{-1}])$: $s_i$ is the permutation matrix $(a_{ij})$, with $a_{jj} = 1, j \neq i, i+1, a_{i+1i} = 1, a_{i+1i+1} = -1$, and all other entries are 0. A canonical lift for $s_0$ is given by
\[
\begin{pmatrix}
0 & 0 & \cdots & t^{-1} \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & 0 \\
-t & 0 & 0 & 0
\end{pmatrix}
\]
Let $s_0 \in W$ be the reflection with respect to the longest root $\theta$ in $\tilde{A}_{n-1}$ given by $\theta = \alpha_1 + \cdots + \alpha_{n-1}$. Let $L$ (resp. $Q$) be the root (resp. coroot) lattice of
Follows from the equivalence
\[ s_i \mapsto s_i \quad \text{for } 1 \leq i \leq n - 1 \]
\[ s_0 \mapsto s_0 \lambda_{\theta^\vee} \]
where we write \( \lambda_q \) for \((\text{id}, q) \in W \ltimes Q \). In particular, we get \( s_0 s_0 \mapsto \lambda_{\theta^\vee} \), which we use to compute a lift of \( \lambda_{\theta^\vee} \) to \( \mathcal{N}(K[t, t^{-1}]) \):
\[
\begin{pmatrix}
0 & 0 & \cdots & t^{-1} \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-t & 0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & \cdots & -1 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & 0 \\
1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & t
\end{pmatrix}
\]
Consider the element \( w \in W \) corresponding to \((1, i)(i + 1, n) \in S_n\), and observe that \( w(\theta^\vee) = \alpha_i^\vee \), the \( i \)-th simple coroot. It follows that a lift of \( \lambda_{\alpha_i^\vee} = w \lambda_q \cdot w^{-1} \) is given by
\[
w \cdot
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & 0 \\
0 & 0 & \cdots & t
\end{pmatrix}
\cdot \left( \begin{array}{c}
t^{-1} \\
\vdots \\
t^{-1} \\
\vdots \\
t
\end{array} \right)
\]
where in the matrix on the right hand side, the dots are 1, and the off-diagonal entries are 0, i.e., the matrix on the right hand side is the diagonal matrix with \( i, (i + 1) \)-th entries being \( t^{-1}, t \) respectively, and all other diagonal entries being 1.

The (Coxeter) length of \( \lambda_q \) is given by the following formula (cf. [2], §13.1.E(3)):
\[
l(\lambda_q) = \sum_{\alpha \in R^+} |\alpha(q)|, \quad q \in Q
\]
where \( \alpha(q) := \langle \alpha, q \rangle \). The action of \( \lambda_q \) on the root system of \( G \) is determined by the following formulae (cf. [2], §13.1.6):
\[
\lambda_q(\alpha) = \alpha - \alpha(q) \delta, \quad \text{for } \alpha \in R, q \in Q
\]
\[
\lambda_q(\delta) = \delta
\]
In particular, for \( \alpha \in R^+, \lambda_q(\alpha) > 0 \) if and only if \( \alpha(q) \leq 0 \).

**Corollary 2.5.** For \( \alpha \in R^+, q \in Q, l(\lambda_q s_\alpha) > l(\lambda_q) \) if and only if \( \alpha(q) \leq 0 \).

**Proof.** Follows from the equivalence \( w s_\alpha > w \) if and only if \( w(\alpha) > 0 \), applied to \( w = \lambda_q \).

3. The element \( \kappa_0 \)

Our goal is to give a compactification of the cotangent bundle \( T^*G/B \) as a (left) \( SL_n \) stable subvariety of the affine Schubert variety \( X(\kappa_0) \), where \( \kappa_0 \) is as defined
below:

\[ \tau := s_{n-1} \cdots s_2 s_1 s_0 \]
\[ \kappa := \tau^{n-1} \]
\[ \kappa_0 := w' \tau^{n-1} \]

where \( w' \) is the longest element in the Weyl group generated by \( s_1, \ldots s_{n-2} \). We first prove some properties of \( \kappa \) and \( \tau \) which are consequences of the braid relations

\[ s_is_{i+1}s_i = s_{i+1}s_is_{i+1}, 0 \leq i \leq n-2, \]
\[ s_0s_{n-1}s_0 = s_{n-1}s_0s_{n-1} \]

and the commutation relations:

\[ s_is_j = s_js_i, 1 \leq i, j \leq n-1, |i-j| > 1, \]
\[ s_0s_i = s_is_0, 2 \leq i \leq n-2 \]

3.1. Some Facts: Fact 1: \( \tau(\delta) = \delta \)

Fact 2: \( \tau(\alpha_1 + \cdots + \alpha_{n-1}) = 2\delta + \alpha_{n-1} \)

Fact 3: \( \tau(\delta + \alpha_i + \cdots + \alpha_{n-1}) = (r+1)\delta + \alpha_i + \cdots + \alpha_{n-1}, 2 \leq i \leq n-1, r \in \mathbb{Z}_+ \)

Fact 4: \( s_{n-1} \cdots s_{j+1}(\alpha_j) = \alpha_j + \alpha_{j+1} + \cdots + \alpha_{n-1}, j \neq 0, n-1 \)

Fact 5: \( s_{n-1} \cdots s_1(\alpha_0) = \delta + \alpha_{n-1} \)

Fact 6: \( \tau(\alpha_{n-1}) = \delta + \alpha_{n-2} + \alpha_{n-1} \) (a special case of Fact 3 with \( r = 0, i = n-1 \))

Fact 7: \( \tau(\alpha_1) = \alpha_0 + \alpha_{n-1} \)

Fact 8: \( \tau(\alpha_i) = \alpha_{i-1}, i \neq 1, n-1 \)

Fact 9: \( \tau(\alpha_0 + \alpha_{n-1}) = \alpha_{n-2} \)

Remark 3.2. Facts 7, 8, 9 imply that \( (\alpha_{n-1} + \alpha_0, \alpha_{n-2}, \alpha_{n-3}, \ldots, \alpha_1) \) is a cycle of order \( n-1 \) for \( \tau \). In particular, each of these roots is fixed by \( \kappa \).

3.3. A reduced expression for \( \kappa \). Let \( \kappa \) be the element in \( \widehat{W} \) defined as above. We may write \( \kappa = \tau_1 \cdots \tau_{n-1} \), where \( \tau_i \)'s are equal, and equal to \( \tau(= s_{n-1} \cdots s_2 s_1 s_0) \) (we have a specific purpose behind writing \( \kappa \) as above).

Lemma 3.4. The expression \( \tau_1 \cdots \tau_{n-1} \) for \( \kappa \) is reduced.

Proof. Claim: \( \tau_1 \cdots \tau_is_{n-1} \cdots s_{j+1}(\alpha_j), 1 \leq i \leq n-2, 0 \leq j \leq n-2, \tau_1 \cdots \tau_i(\alpha_{n-1}), 1 \leq i \leq n-2 \) are positive real roots.

Note that the Claim implies the required result. We divide the proof of the Claim into the following three cases.

Case 1: To show: \( \tau_1 \cdots \tau_i(\alpha_{n-1}), 1 \leq i \leq n-2 \) is a positive real root.

We have
\[ \tau_1 \cdots \tau_i(\alpha_{n-1}) = \tau_1 \cdots \tau_{i-1}(\delta + \alpha_{n-2} + \alpha_{n-1}) \] (cf. [3.1] Fact 6)
\[ = \tau_1 \cdots \tau_{i-2}(2\delta + \alpha_{n-3} + \alpha_{n-2} + \alpha_{n-1}) \] (cf. [3.1] Fact 3)
\[ = \tau_1 \cdots \tau_{i-k}(k\delta + \alpha_{n-k-1} + \cdots + \alpha_{n-1}), 0 \leq k \leq i-1 \] (cf. [3.1] Fact 3)

Note that \( k \leq i-1 \) implies that \( n-k-1 \geq n-i \geq 2 \), and hence we can apply [3.1] Fact 3. Corresponding to \( k = i-1 \), we obtain \( \tau_1 \cdots \tau_i(\alpha_{n-1}) = \tau_i((i-1)\delta + \alpha_{n-i} + \cdots + \alpha_{n-1}) \). Hence once again using [3.1] Fact 3, we obtain

\[ \tau_1 \cdots \tau_i(\alpha_{n-1}) = i\delta + \alpha_{n-i-1} + \cdots + \alpha_{n-1}, 1 \leq i \leq n-2 \]

(note that for \( 1 \leq i \leq n-2, n-i-1 \geq 1 \)).

Case 2: To show: \( \tau_1 \cdots \tau_is_{n-1} \cdots s_1(\alpha_0), 1 \leq i \leq n-2 \) is a positive real root.

We have
\( \tau_1 \cdots \tau_is_{n-1} \cdots s_1(\alpha_0) \)
\( = \tau_1 \cdots \tau_i(\delta + \alpha_{n-1}) \) (cf. §3.1 Fact 5)
\( = \tau_1 \cdots \tau_{i-1}(2\delta + \alpha_{n-2} + \alpha_{n-1}) \) (cf. §3.1 Fact 6)
\( = \tau_1 \cdots \tau_{i-k}(k+1)\delta + \alpha_{n-k-1} + \cdots + \alpha_{n-1}) \), 0 \leq k \leq i-1 (cf. §3.1 Fact 3)

Note that as in Case 1, for \( k \leq i - 1 \), we have, \( n-k-1 \geq 2 \), and therefore §3.1 Fact 3 holds. Corresponding to \( k = i - 1 \), we have,
\( \tau_1 \cdots \tau_is_{n-1} \cdots s_1(\alpha_0) = \tau_i(\delta + \alpha_{n-i} + \cdots + \alpha_{n-1}) \). Hence once again using §3.1 Fact 3, we obtain
\[
\tau_1 \cdots \tau_is_{n-1} \cdots s_1(\alpha_0) = (i+1)\delta + \alpha_{n-i} + \cdots + \alpha_{n-1}, 1 \leq i \leq n-2
\]
(note that for \( 1 \leq i \leq n-2 \), \( n-i-1 \geq 1 \).

**Case 3:** To show: \( \tau_1 \cdots \tau_is_{n-1} \cdots s_{j+1}(\alpha_j), 1 \leq i \leq n-2, j \neq 0, n-1 \) is a positive
real root.
We have \( \tau_1 \cdots \tau_is_{n-1} \cdots s_{j+1}(\alpha_j) = \tau^i(\alpha_j + \alpha_{j+1} + \cdots + \alpha_{n-1}) \) (cf. §3.1 Fact 4)
\( = \tau^i(\alpha_j) + \cdots + \tau^i(\alpha_{n-2}) + \tau^i(\alpha_{n-1}) \) which is positive because each term is positive
(cf. Case 1 and Remark 3.2).

**Corollary 3.5.** \( \ell(\kappa) = n(n-1) \).

### 3.6. Minimal representative-property for \( \kappa \).

**Lemma 3.7.** \( \kappa(\alpha_i) \) is a real positive root for all \( \alpha \neq 0 \).

**Proof.** For \( 1 \leq i \leq n-2 \), \( \kappa(\alpha_i) = \alpha_i \) is positive from Remark 3.2. Further,
\( \tau_1 \cdots \tau_i(\alpha_{n-1}) \)
\( = \tau_1 \cdots \tau_{i-1}(\alpha_{n-1}) \) (cf. §3.1 Fact 6)
\( = \tau_1 \cdots \tau_{i-k}(k+1)\delta + \alpha_{n-k} + \cdots + \alpha_{n-1}) \), 1 \( \leq k \leq n-1 \) (cf. §3.1 Fact 3)

Note that for \( 1 \leq k \leq n-2, n-k \geq 2 \) and hence §3.1 Fact 3 holds. Corresponding to \( k = n-1 \), we get,
\( \tau_1 \cdots \tau_{n-1}(\alpha_{n-1}) \)
\( = \tau_1((n-2)\delta + \alpha_1 + \cdots + \alpha_{n-1}) \)
\( = n\delta + \alpha_{n-1} \) (cf. §3.1 Facts 1,2).

**Corollary 3.8.** \( \kappa \) is a minimal representative in \( \hat{W}/\hat{W}_{G_0} \).

For \( w \) \( \in \hat{W} \), we shall denote the Schubert variety in \( G/G_0 \) by \( X_{G_0}(w) \).

**Lemma 3.9.** \( X_{G_0}(\kappa) \) is stable for multiplication on the left by \( G_0 \).

**Proof.** It suffices to show that
\[
(*) 
\text{s}_i\kappa \leq \kappa(\text{mod} \hat{W}_{G_0}), 1 \leq i \leq n-1
\]
The assertion (*) is clear if \( i = n-1 \). Observe that \( ws_\alpha = s_{w(\alpha)}w \). In particular, since \( \kappa \) fixes \( \alpha_i \), \( 1 \leq i \leq n-2 \), it follows \( s_i\kappa = \kappa s_i = \kappa(\text{mod} \hat{W}_{G_0}) \), for \( 1 \leq i \leq n-2 \).

**Lemma 3.10.** Let \( \mathcal{P} \) be the parabolic subgroup of \( G \) corresponding to the choice of simple roots \( \{\alpha_1, \cdots, \alpha_{n-2}\} \). The element \( \kappa \) is a minimal representative in \( \hat{W}_P \backslash \hat{W} \).

**Proof.** It is enough to show that \( s_i\kappa > \kappa \), or equivalently, \( \kappa^{-1}(\alpha_i) > 0 \) for \( 1 \leq i \leq n-2 \). This follows from Remark 3.2.
Remark 3.11. For the discussion in [3.3, 3.6] concerning reduced expressions, minimal-representative property and \(G_0\)-stability, we have used the expression for elements of \(\hat{W}\), \(W\) being considered as a Coxeter group. One may as well carry out the discussion using the permutation presentations for elements of \(\hat{W}\).

**Theorem 3.12** (A reduced expression for \(\kappa_0\)). The element \(\kappa_0 = w'\tau^{n-1}\) is the maximal representative of \(\kappa\) in \(\hat{W}_{G_0}W\), i.e. the unique element in \(W\) such that

\[
X(\kappa_0) = G_0\mathcal{B}(\text{mod } \mathcal{B})
\]

In particular, \(X(\kappa_0)\) is (left) \(G_0\)-stable. Let \(w'\) be a reduced expression for the longest element \(w'\) in \(\hat{W}_{G_0}\) and \(\tau\) the reduced expression \(s_{n-1}\cdots s_1s_0\). Then \(w'\tau^{n-1}\) is a reduced expression for \(\kappa_0\).

**Proof.** Observe that \(w = w's_{n-1}\cdots s_1\) is a reduced expression for the longest element \(w\) in \(\hat{W}_{G_0}\), and so \(w'\tau = ws_0\tau^{n-2}\). Lemma 3.10 implies that \(w'\tau^{n-1}\) is a reduced expression. In particular,

\[
l(\kappa_0) = l(w'\kappa) = l(w's_{n-1}\cdots s_1) + l(s_0\tau^{n-2}) = l(w) + l(s_0\tau^{n-2})
\]

It remains to show that \(w'\kappa\) is a maximal representative in \(\hat{W}_{G_0}W\), i.e \(s_iw'\kappa < w'\kappa\), or equivalently \(l(s_iw'\kappa) < l(w'\kappa)\) for \(1 \leq i \leq n - 1\). First note that

\[
l(s_iw'\kappa) = l(s_is_0\tau^{n-2}) \leq l(s_iw) + l(s_0\tau^{n-2})
\]

Now, since \(w\) is the longest element in \(\hat{W}_{G_0}\), it follows \(l(s_iw) < l(w)\) and further

\[
l(s_iw'\kappa) < l(w) + l(s_0\tau^{n-2}) = l(w'\kappa)
\]

\(\square\)

## 4. The Main Lemma

In this section, we prove one crucial result involving \(\kappa\), which we then use to prove the main result.

**Lemma 4.1.** Let \(Y = \sum_{1 \leq i < j \leq n} a_{ij}E_{ij}\), where \(E_{ij}\) is the elementary \(n \times n\) matrix with 1 at the \((i, j)\)-th place and 0’s elsewhere. Let \(Y = \text{Id}_{n \times n} + \sum_{1 \leq i \leq n-1} t^{-1}Y^i\) (note that \(Y^n = 0\)). Assume that \(a_{ii+1} \neq 0, 1 \leq i \leq n-1\). There exist \(g \in G_0, h \in \mathcal{B}\) such that \(g\kappa = Yh\)

**Proof.** Choose \(g\) to be the matrix

\[
g = \begin{pmatrix}
0 & 0 & 0 & \cdots & 1 \\
-1 & 0 & 0 & \cdots & g_{2n} \\
0 & -1 & 0 & \cdots & g_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -1 & g_{nn}
\end{pmatrix}
\]

Note that the lower left corner submatrix (i.e., the \(n - 1 \times n - 1\) submatrix with rows 2-nd through the \((n-1)\)-th of \(g\), and the first \(n-1\) columns of \(g\)) is \(-\text{Id}_{n-1 \times n-1}\), and that determinant of \(g\) equals 1. Hence, we may take \(g_{in}, 2 \leq i \leq n\) as elements in \(K[[t]]\) so that \(g \in G_0\). We shall now show that there exist \(g_{in}, 2 \leq i \leq n, h_{ij}, 1 \leq i, j \leq n\) such that \(h \in \mathcal{B}\), and \(g\kappa = Yh\). We have, \(Y^{-1} = \text{Id}_{n \times n} - t^{-1}Y\). Set

\[
h = (\text{Id}_{n \times n} - t^{-1}Y)g\kappa
\]
We have (by definition of $\nu$ (3), and the choice of lifts for $s_i$ (cf.2.3))

$$\nu = \text{diag}(t, \cdots, t, t^{-(n-1)})$$

Note that since we want $h$ to belong to $B$, each diagonal entry in $h$ (as an element of $K[[t]]$) should have order 0, $h_{ij}, i > j$ should have order $>0$, and $h_{ij}, i < j$ should have order $\geq 0$ (since $h(0)$ should belong to $B$). Now the diagonal entries in $h$ are given by

$$h_{ii} = a_{ii+1}, 1 \leq i \leq n - 1, h_{nn} = t^{-(n-1)}g_{nn}$$

Hence choosing $g_{nn}$ such that order $g_{nn} = n - 1$ (note that since $g \in G_0$, order $g_{ij} \geq 0, 1 \leq i, j \leq n$, so this choice for $g$ is allowed), we obtain that each diagonal entry in $h$ is in $K[[t]]$, with order equal to 0. Also, we have

$$h_{i+1i} = -t, 1 \leq i \leq n - 1,$$

$$h_{ik} = 0, k = i - 2, 3 \leq i \leq n - 1, h_{ik} = a_{ik+1}, 1 \leq i < k \leq n - 1$$

Thus the entries $h_{ik}, k \leq n - 1$ satisfy the order conditions mentioned above. Let us then consider $h_{jn}, 1 \leq j \leq n$. We have

$$(*) \quad h_{jn} = t^{-(n-1)}g_{jn} - \sum_{j+1 \leq k \leq n} t^{-n}a_{jk}g_{kn}, 1 \leq j \leq n$$

We shall choose $g_{in}$ (in $K[[t]]$) so that order of $g_{in}$ equals $i - 1$ (note that this agrees with the above choice of $g_{nn}$ - in the discussion of the diagonal entries in $h$). Let us write

$$g_{in} = \sum g_{in}^{(k)}t^k$$

We shall show that with the above choice of $g_{in}$, the integrality condition on the $h_{in}$'s imposes conditions on $g_{in}^{(k)}, i-1 \leq k \leq n, 1 \leq i \leq n$, leading to a linear system in these $g_{in}^{(k)}$'s (note that, the integrality condition on the $h_{in}$'s, $1 \leq i \leq n$, implies that $h_{in}$'s should belong to $K[[t]]$, with the additional condition that $h_{nn}$ should have order 0 - the latter condition having already been accommodated, since $g_{nn}$ has been chosen to have order $n - 1$). Treating $g_{in}^{(k)}$'s as the unknowns, we show that the resulting linear system has a unique solution, thus proving the choice of $g, h$ with the said properties. We shall now describe this linear system. The linear system will involve $\binom{n}{2}$ equations in $\binom{n}{2}$ unknowns, namely, $g_{in}^{(k)}, i-1 \leq k \leq n, 2 \leq i \leq n$. The linear system is obtained as follows. The lowest power of $t$ appearing on the right hand side of $(*)$ above is $-(n-j)$ (note that order of $g_{kn}$ equals $k - 1$). Hence equating the coefficients of $t^{-(n-j)}, j \leq i \leq n - 1$ on the right hand side of $(*)$ to 0, we obtain

$$(**) \quad g_{jn}^{(i-1)} - \sum_{j+1 \leq k \leq n} a_{jk}g_{kn}^{(i)} = 0, j \leq i \leq n - 1, 1 \leq j \leq n - 1$$

Note that, corresponding to $h_{nn}$, we do not have any conditions, since by our choice of $g_{nn}$ (order of $g_{nn}$ is $n - 1$), we have that $h_{nn} (= t^{-(n-1)}g_{nn})$ is integral. Also, corresponding to $g_{1n}$ (which is equal to 1, by our choice of $g$), we have $g_{1n}^{(i)} = 0, i \geq 1$, and occurs just in one equation, namely, the equation corresponding to the coefficient of $t^{-(n-1)}$ in $h_{1n}$:

$$g_{1n} - a_{12}g_{2n}^{(1)} = 0$$

Rewriting this equation as

$$-a_{12}g_{2n}^{(1)} = -1$$
(there is a purpose behind retaining the negative sign in $-a_{12}g_{2n}^{(1)}$), we arrive at the linear system

$$A_n X = B$$

where $A_n$ is a square matrix of size $\binom{n}{2}$, $X$ is the $\binom{n}{2}$ column matrix $(g_{jn}^k, j - 1 \leq k \leq n, 2 \leq j \leq n)$, and $B$ is the $\binom{n}{2}$ column matrix with the first entry equal to $-1$, and all other entries equal to $0$.

**Claim:** $A_n$ is invertible, and $|A_n| = (-1)^{\binom{n}{2}} \prod_{1 \leq i \leq n-1} a_{i,i+1}^{n-i}$.

Note that Claim implies that $(g_{jn}^{(k)})$, $j - 1 \leq k \leq n, 2 \leq j \leq n$ are uniquely determined, and therefore we may choose $g_{jn}$ as elements in $K[[t]]$ with $(g_{jn}^{(k)})$, $j - 1 \leq k \leq n$ as the solutions of the above linear system, with $(g_{jn}^{(k)}), k > n$ being arbitrary.

We prove the Claim by induction on $n$. We shall first show that $A_{n-1}$ can be identified in a natural way as a submatrix of $A_n$. We want to think of the rows of $A_n$ forming $(n - 1)$ blocks (referred to as row-blocks in the sequel) of size $n - 1, n - 2, \ldots, n - j, \ldots, 1$, namely, the $j$-th block consists of $n - j$ rows given by the coefficients occurring on the left hand side of $(**)$ for $j \geq 2$, and for $j = 1$, the first block consists of $n - 1$ rows given by the coefficients occurring on the left hand side of the following $n - 1$ equations:

$$-a_{12}g_{2n}^{(1)} = -1, \quad -g_{2n}^{(i)} - \sum_{3 \leq k \leq n} a_{2k}g_{kn}^{(i)} = 0, 2 \leq i \leq n - 1$$

Similarly, we want to think of the columns of $A_n$ forming $(n - 1)$ blocks (referred to as column-blocks in the sequel) of size $n - 1, n - 2, \ldots, n - j, \ldots, 1$, namely, the $j$-th block consisting of $n - j$ columns indexed by $g_{jn}^{(i)}, j - 1 \leq i \leq n$. Then indexing the $n - j$ rows in the $j$-th row-block as $j, j + 1, \ldots, n - 1$, the entries in the rows of the $j$-th row-block have the following description:

The non-zero entries in the $i$-th row in the $j$th row-block ($j \geq 2$) are

$$1, -a_{23}, -a_{24}, \ldots, -a_{2i+1}$$

respectively, occurring at the columns indexed by $g_{2n}^{(i-1)}, g_{3n}^{(i)}, \ldots, g_{i+1n}^{(i)}$.

The non-zero entries in the $i$-th row in the first row-block ($j \geq 2$) are

$$-a_{12}, -a_{13}, \ldots, -a_{2i+1}$$

respectively, occurring at the columns indexed by $g_{2n}^{(i)}, g_{3n}^{(i)}, \ldots, g_{i+1n}^{(i)}$.

From this it follows that $A_{n-1}$ is obtained from $A_n$ by deleting the first row in each row-block and the first column in each column-block. For instance, we describe below $A_5$ and $A_4$; for convenience of notation, we denote $b_{ij} = -a_{ij}$. We have,

$$A_5 = \begin{pmatrix}
1 & 0 & 0 & b_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b_{12} & 0 & 0 & b_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & b_{12} & 0 & b_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & b_{12} & 0 & b_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & b_{12} & 0 & b_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & b_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{23} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{23} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{23} & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & b_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{23} & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & b_{23} & 0 & 0 & 0 & 0 & 0 & 0 & b_{23} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & b_{23} & 0 & 0 & 0 & 0 & 0 & b_{23} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & b_{23} & 0 & 0 & 0 & 0 & b_{23} & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & b_{23} & 0 & 0 & 0 & b_{23} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & b_{23} & 0 & 0 & b_{23} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & b_{23} & 0 & b_{23} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & b_{23} & 0 & b_{23} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & b_{23} & 0 & b_{23} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & b_{23} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & b_{23} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$
As rows (respectively columns) of $A_5$, the positions of the first row (respectively, the first column) in each of the four row-blocks (respectively columns-blocks) in $A_5$ are given by 1, 5, 8, 10; deleting these rows and columns in $A_5$, we get $A_4$. These rows and columns are highlighted in $A_5$.

As above, let $b_{ij} = -a_{ij}$. Now expanding $A_n$ along the first row, we have that $|A_n|$ equals $b_{12}|M_1|$, $M_1$ being the submatrix of $A_n$ obtained by deleting the first row and first column in $A_n$ (i.e., deleting the first row (respectively, the first column) in the first row-block (respectively, the first column-block)). Now in $M_1$, in the first row in the second row-block the only non-zero entry is $b_{23}$, and it is a diagonal entry in $M_1$. Hence expanding $M_1$ through this row, we get that $|A_n|$ equals $b_{12}b_{23}|M_2|$, $M_2$ being the submatrix of $A_n$ obtained on deleting the first row (respectively, the first columns) in the first two row-blocks (respectively, the first two column-blocks) in $A_n$. Now in $M_2$, in the first row in the third row-block, the only non-zero entry is $b_{34}$, and it is a diagonal entry in $M_2$. Hence expanding $M_2$ along this row, we get that $|A_n|$ equals $b_{12}b_{23}b_{34}|M_3|$, $M_3$ being the submatrix of $A_n$ obtained by deleting the first rows (respectively, the first columns) in the first three row-blocks (respectively, the first three column-blocks) in $A_n$. Thus proceeding, at the $(n-1)$-th step, we get that $|A_n|$ equals $b_{12}b_{23} \cdots b_{n-1}n|A_{n-1}|$. By induction, we have $|A_{n-1}| = (-1)^{\binom{n-1}{2}} \prod_{1 \leq i \leq n-2} a_{i+1}^{n-1-i}$. Substituting back for $b_{ij}$’s, we obtain $|A_n| = (-1)^{\binom{n}{2}} \prod_{1 \leq i \leq n-1} a_{i+1}^{n-1-i}$. It remains to verify the statement of the claim when $n = 2$ (starting point of induction). In this case, we have

$$g = \begin{pmatrix} 0 & 1 \\ -1 & g_{22} \end{pmatrix}, \kappa = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix},$$

$$\sum^{-1} = \begin{pmatrix} 1 & t^{-1}a_{12} \\ 0 & 1 \end{pmatrix}, h = \begin{pmatrix} a_{12} & t^{-2}a_{12}g_{22} \\ -t & -t^{-1}g_{22} \end{pmatrix}$$

Hence the linear system consists of the single equation

$$-a_{12}g_{22}^{(1)} = -1$$

Hence $A_2$ is the $1 \times 1$ matrix $(-a_{12})$, and $|A_2| = -a_{12}$, as required.

5. Lusztig’s map

Consider $\mathcal{N}$, the variety of nilpotent elements in $g$ (the Lie algebra of $G$). In this section, we spell out (Lusztig’s) isomorphism which identifies $X_{G_0}(\kappa)$ as a compactification of $\mathcal{N}$.

5.1. The map $\psi$: Consider the map

$$\psi: \mathcal{N} \to G/G_0, \psi(N) = (1d + t^{-1}N + t^{-2}N^2 + \cdots)(mod G_0), N \in \mathcal{N}$$

Note that the sum on the right hand side is finite, since $N$ is nilpotent. We now list some properties of $\psi$. 
(i) $\psi$ is injective: Let $\psi(N_1) = \psi(N_2)$. Denoting $\lambda_i := \psi(N_i), i = 1, 2$, we get that $\lambda_2^{-1} \lambda_1$ belongs to $G_0$. On the other hand,

$$\lambda_2^{-1} \lambda_1 = (Id - t^{-1} N_2)(Id + t^{-1} N + t^{-2} N^2 + \cdots)$$

Now $\lambda_2^{-1} \lambda_1$ is integral. It follows that both sides of the above equation equal $Id$. This implies $\lambda_1 = \lambda_2$ which in turn implies that $N_1 = N_2$. Hence we obtain the injectivity of $\psi$.

(ii) $\psi$ is $G$-equivariant: We have

$$\psi(g \cdot N) = \psi(g N g^{-1}) = (Id + t^{-1} gNg^{-1} + t^{-2} gNg^{-1} + \cdots)(mod G_0) = g(Id + t^{-1} N + t^{-2} N^2 + \cdots)g^{-1}(mod G_0) = g(Id + t^{-1} N + t^{-2} N^2 + \cdots)(mod G_0)$$

Proposition 5.2. For $N \in \mathcal{N}, \psi(N)$ belongs to $X_{G_0}(\kappa)$.

Proof. We divide the proof into two cases.

Case 1: Let $N$ be upper triangular, say,

$$N = (n_{ij})_{1 \leq i, j \leq n}$$

where $n_{ij} = 0$, for $i \geq j$; note that $N \in \mathfrak{b}_u, \mathfrak{b}_u$ being the Lie algebra of $B_u$, the unipotent radical of $B$. We may work in the open subset $x_{i+1} \neq 0, 1 \leq i \leq n - 1$ in $\mathfrak{b}_u \setminus \sum_{1 \leq i, j \leq n} x_{ij}E_{ij}$ being a generic element in $\mathfrak{b}_u$. Hence we may suppose that $n_{i+1} \neq 0, 1 \leq i \leq n - 1$. In this case, in view of Lemma 4.1 we have that there exist $g \in G_0, h \in B$ such that $gN = \psi(N)h$. This implies, in view of the $G_0$-stability for $X_{G_0}(\kappa)$ (cf. Lemma 3.9), $\psi(N)$ belongs to $X_{G_0}(\kappa)$.

Case 2: Let $M$ be an arbitrary nilpotent matrix. Then there exists an upper triangular matrix $N$ in the $G$-orbit through $N$. Hence there exists a $g \in G$ such that $M = gNg^{-1}(= g \cdot N)$ with $N$ upper triangular. Now by $G$-equivariance of $\psi$ (cf. (i) above), we have $\psi(M) = g \cdot \psi(N)$. By case 1, $\psi(N) \in X_{G_0}(\kappa)$; this together with the $G_0$-stability for $X_{G_0}(\kappa)$ implies that $\psi(M)$ belongs to $X_{G_0}(\kappa)$. \qed

Theorem 5.3. $X_{G_0}(\kappa)$ is a compactification of $\mathcal{N}$.

Proof. Let $\overline{\mathcal{N}}$ be the closure of $\mathcal{N}$ in $\mathcal{G}/G_0$. Combining the above Proposition with 5.1 (i) and the facts that $\dim \mathcal{N} = n(n-1) = \dim X_{G_0}(\kappa)$ (cf. Corollaries 3.5, 3.8), we obtain $\overline{\mathcal{N}} = X_{G_0}(\kappa)$. \qed

6. Cotangent bundle

In this section, we first recall the Springer resolution. We then construct a family $\psi_p$, parametrized by polynomials $p$ in one variable with coefficients in $\mathbb{C}(t)$ and constant term 1, of maps $\psi_p : T^*G/B \to \mathcal{G}/B$. We show that for a particular choice $\phi$ in the family, we get an embedding of $T^*G/B$ inside $\mathcal{G}/B$. Using the natural projection $\mathcal{G}/B \to \mathcal{G}/G_0$ and the results of §5, we recover the Springer resolution. We then show that $\phi$ identifies an $SL_n$-stable closed subvariety of $X(n_0)$ as a compactification of $T^*G/B$.

The cotangent bundle $T^*G/B$ is a vector bundle over $G/B$, with the fiber at any point $x \in G/B$ being the cotangent space to $G/B$ at $x$; the dimension of $T^*G/B$ equals $2 \dim G/B$. Also, $T^*G/B$ is the fiber bundle over $G/B$ associated to the
It is clear that $p$ is constant term 1. We write $\psi$. Also, it is clear that Springer resolution is proper and birational and is the celebrated

\[ \text{6.2. The Maps } \psi_p. \text{ Let } p(Y) \text{ be a polynomial in } Y \text{ with coefficients in } F, \text{ and constant term 1. We write} \]

\[ \sum_{i \geq 1} p_i(t)Y^i \]

$B$-bundle $G \to G/B$, for the Adjoint action of $B$ on $B_u$ (the Lie algebra of the unipotent radical $B_u$ of $B$). Thus

\[ T^*G/B = G \times^B B_u = G \times B_u / \sim \]

where the equivalence relation $\sim$ is given by $(g, Y) \sim (gb, b^{-1}Yb), g \in G, Y \in B_u, b \in B$.

6.1. Springer resolution. Let $N$ be the variety of nilpotent elements in $g$, the Lie algebra of $G$. Consider the map

\[ \theta : G \times^B B_u \to G/B \times N, \theta((g, Y)) = (gB, gY g^{-1}), g \in G, Y \in B_u \]

We observe the following on the map $\theta$:

(i) $\theta$ is well defined: Let $b \in B$. Consider $(gb, b^{-1}Yb)(\sim (g, b))$. We have,

\[ \theta((gb, b^{-1}Yb)) = (gB, gb(b^{-1}Yb)b^{-1}g^{-1}) = (gB, gY g^{-1}) = \theta((g, Y)) \]

(ii) $\theta$ is injective: Suppose $\theta((g_1, Y_1)) = \theta((g_2, Y_2))$. Then $(g_1B, g_1Y_1, g_1^{-1}) = (g_2B, g_2Y_2 g_2^{-1})$. This implies

\[ g_1B = g_2B, g_1Y_1g_1^{-1} = g_2Y_2g_2^{-1} \]

Hence we obtain

\[ g_1^{-1}g_2 =: b \in B, Y_2 = g_2^{-1}g_1Y_1g_1^{-1}g_2 \]

\[ \therefore g_2 = g_1b, Y_2 = b^{-1}Y_1b \]

\[ \therefore (g_1, Y_1) = (g_1b, b^{-1}Y_1b) = (g_2, Y_2) \]

Thus we get an embedding

\[ \theta : T^*G/B \hookrightarrow G/B \times N \]

The second projection

\[ T^*G/B \to N, (g, Y) \mapsto gY g^{-1} \]

is proper and birational and is the celebrated Springer resolution

6.2. The Maps $\psi_p$. Let $p(Y)$ be a polynomial in $Y$ with coefficients in $F$, and constant term 1. We write

\[ p(Y) = 1 + \sum_{i \geq 1} p_i(t)Y^i \]

It is clear that $p(Y) \in G$. Define the map $\psi_p : G \times^B B_u \to G/B$ by

\[ \psi_p(g, Y) = gp(Y) (mod B), g \in G, Y \in B_u \]

The following calculation shows that $\psi_p$ is well defined: Let $g \in G, b \in B, Y \in B_u$. Then

\[ \psi_p((gb, b^{-1}Yb)) = gb(Id + p_1(t)b^{-1}Yb + p_2(t)b^{-1}Y^2b + \cdots) (mod B) \]

\[ = g(Id + p_1(t)Yb + p_2(t)Y^2b + \cdots) (mod B) \]

\[ = g(Id + p_1(t)Y + p_2(t)Y^2 + \cdots) (mod B) \]

\[ = \psi_p(g, Y) \]

Also, it is clear that $\psi_p$ is $G$-equivariant.
6.3. Embedding of $T^*G/B$ into $G/B$: We consider one particular member $\phi$ of the family $\psi_p$: namely $\phi = \psi_p$ where $p(Y)$ is the polynomial $(1 - t^{-1}Y)^{-1}$; observe that for nilpotent $Y$, the function
\[
p(Y) = (1 - t^{-1}Y)^{-1} = 1 + t^{-1}Y + t^{-2}Y^2 + \ldots
\]
is a polynomial, since the sum on the right hand side is finite. In particular, $\phi : G \times B \to G/B$ is given by
\[
\phi(g, Y) = g(Id + t^{-1}Y + t^{-2}Y^2 + \cdots)(mod B)
\]
In the sequel, we shall denote
\[
\underline{Y} := Id + t^{-1}Y + t^{-2}Y^2 + \cdots
\]
We now list some facts on the map $\phi$:

(i) $\phi$ is well-defined
(ii) $\phi$ is injective: Let $\phi((g_1, Y_1)) = \phi((g_2, Y_2))$. This implies that $g_1 \underline{Y}_1 = g_2 \underline{Y}_2 (mod B)$, where recall that for $Y \in B$, $\underline{Y} = Id + t^{-1}Y + t^{-2}Y^2 + \cdots$. Hence, $g_1 \underline{Y}_1 = g_2 \underline{Y}_2 x$, for some $x \in B$. Denoting $h := g_2^{-1} g_1$, we have, $h \underline{Y}_1 = \underline{Y}_2 x$, and therefore,
\[
x = Y_2^{-1}h \underline{Y}_2 = Y_2^{-1}(h \underline{Y}_2^{-1}h)h = Y_2^{-1} \underline{Y}_1
\]
where $\underline{Y}_1' = h \underline{Y}_1 h^{-1}$. Hence
\[
xh^{-1} = Y_2^{-1} \underline{Y}_1' = (Id - t^{-1}Y_2)(Id + t^{-1}hY_1 h^{-1} + t^{-2}hY_1^2 h^{-1} + \cdots)
\]
Now, since $x \in B$, $h(= g_2^{-1} g_1) \in G$, the left hand side is integral, i.e. it does not involve negative powers of $t$. Hence both sides equal $Id$. This implies
\[
\underline{Y}_2 = \underline{Y}_1', x = h
\]
The fact that $x = h$ together with the facts that $x \in B, h \in G$ implies that
\[
(\ast)
\]
Further, the fact that $\underline{Y}_2 = \underline{Y}_1'$ implies that $\underline{Y}_1 = h^{-1} \underline{Y}_2 h$. Hence
\[
Id + t^{-1}Y_1 + t^{-2}Y_2^2 + \cdots = Id + t^{-1}h^{-1}Y_2 h + t^{-2}h^{-1}Y_2^2 h + \cdots
\]
From this it follows that
\[
(\ast\ast)
\]
Now $(\ast), (\ast\ast)$ together with the fact that $h = g_2^{-1} g_1$ imply that
\[
(g_1, Y_1) = (g_2h, h^{-1} \underline{Y}_2 h) \sim (g_2, Y_2)
\]
From this injectivity of $\phi$ follows.

(iii) $G$-equivariance: It is clear that $\phi$ is $G$-equivariant.

(iv) Springer resolution: Consider the projection $\pi : G/B \to G/G_0$. Let $x \in T^*G/B$, say, $x = (g, Y), g \in G, Y \in B$. We have
\[
\pi((g, Y)) = \phi((g, Y))(mod G_0) = g(Id + t^{-1}Y + t^{-2}Y^2 + \cdots)(mod G_0) = g(Id + t^{-1}Y + t^{-2}Y^2 + \cdots)g^{-1}(mod G_0) = (Id + t^{-1}N + t^{-2}N^2 + \cdots)(mod G_0)
\]
where \( N = gYg^{-1} \) is nilpotent. Hence, in view of Lusztig's isomorphism (cf. Proposition 5.3), we recover the Springer resolution as

\[
\pi|_{T^*G/B} : T^*G/B \to N \to G/G_0
\]

**Theorem 6.4** (Compactification of \( T^*G/B \)). Let \( G = SL_n(\mathbb{C}) \) and \( \phi : G/B \to G/B \) be as in section 5.3. Then \( \phi \) identifies \( \overline{T^*G/B} \) (the closure being in \( G/B \)) with a \( G \)-stable closed subvariety of the affine Schubert variety \( X(\kappa_0) \).

**Proof.** Let \((g_0, Y), g_0 \in G, Y \in \mathfrak{b}_\alpha\). Then \( \phi(g_0, Y) = g_0(Id + t^{-1}Y + t^{-2}Y^2 + \cdots)(mod \mathcal{B}) = g_0\mathcal{Y}(mod \mathcal{B}), \) where \( \mathcal{Y} = Id + t^{-1}Y + t^{-2}Y^2 + \cdots \). Writing \( Y = \sum_{1 \leq i < j \leq n} a_{ij}E_{ij} \) with \( E_{ij} \) as in Lemma 6.1 we may work in the open subset \( x_{ii+1} \neq 0, 1 \leq i \leq n-1 \) in \( \mathfrak{b}_\alpha \), \( \sum_{1 \leq i < j \leq n} x_{ij}E_{ij} \) being a generic element in \( \mathfrak{b}_\alpha \). Then Lemma 4.1 implies that there exist \( g \in G_0, h \in \mathcal{B} \) such that \( g\kappa = Yh \). Hence \( \mathcal{Y} \) belongs to \( X(\kappa_0)(= G_0\mathcal{B}(mod \mathcal{B})) \); hence \( g_0\mathcal{Y} \) is also in \( X(\kappa_0) \) (since \( g_0 \) is clearly in \( G_0 \)).

7. Consequences of \( \psi_p \) for \( T^*G/B \)

In this section, we show that for any polynomial \( p \), the map \( \psi_p \) as defined in 6.2 cannot realize an affine Schubert variety (in \( SL_3/\mathcal{B} \)) as a compactification of the cotangent bundle \( T^*SL_3(K)/\mathcal{B} \).

**Proposition 7.1.** Let \( G \) be the group \( SL_3(K) \) and the \( B \) the Borel subgroup of upper triangular matrices in \( G \). Let \( p \) be a polynomial as in 6.2. Suppose that the associated map \( \psi_p : T^*G/B \to G/B \) is injective. Then there exist \( g \in G, w \in \tilde{W} \) and \( Y \in \mathfrak{b}_\alpha \) such that \( \psi_p(g, Y) \in Bw\mathcal{B} \) and \( l(w) > 6 \).

**Proof.** From 6.2 we may assume \( p(Y) = 1 + \sum_{i \geq 1} p_i(t)Y^i \). We first claim that \( p_1(t) \notin A \). Assume the contrary. For

\[
Z = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

we see that \( Z^2 = 0 \), and so \( p(Z) = 1 + p_1(t)Z \in \mathcal{B} \). In particular, \( \psi_p(Z) = \psi_p(0) \), contradicting the injectivity of \( \psi_p \).

We now write \( p(Y) = 1 - t^{-a}qY - t^{-b}rY^2 \) where

- \( q, r \in A \)
- \( q(0) \neq 0 \)
- \( a \geq 1 \)
  - Either \( r = 0 \) or \( r(0) \neq 0 \).

We now fix \( Y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \) and \( g = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \), so that

\[
gp(Y) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & t^{-a}q \\ -1 & t^{-a}q & t^{-b}r \end{pmatrix}
\]

Our strategy is to find elements \( C, D \in \mathcal{B} \) such that \( Cgp(Y)D \in N(K[t, t^{-1}]) \). We can then identify the Bruhat cell containing \( gp(Y) \), and so identify the minimal
Schubert variety containing \( \psi_p(g,Y) \). The choice of \( C, D \) depends on the values of certain inequalities, which we divide into 4 cases. We draw here a decision tree showing the relationship between the inequalities and the choice \( C, D \).

A rational function in \( t \) is implicitly equated with its Laurent power series at 0. In particular, a rational function \( f \) belongs to \( A \) if and only if \( f \) has no poles at 0, i.e. its denominator is not divisible by \( t \).

1. If \( r = 0 \) or \( b \leq a \), let

\[
C = \begin{pmatrix}
0 & -\frac{qt^a}{q} & -\frac{t^{2a}}{rt^{2a-b}+q^2} \\
0 & 0 & -\frac{1}{q} \\
-\frac{rt^{2a-b}+q^2}{rt^{2a-b}+q^2} & 0 & 0
\end{pmatrix}
\]

\[
D = \begin{pmatrix}
1 & 0 & 0 \\
\frac{qt^a}{rt^{2a-b}+q^2} & 1 & -\frac{rt^{a-b}}{q} \\
\frac{rt^{2a-b}+q^2}{rt^{2a-b}+q^2} & 0 & 1
\end{pmatrix}
\]

We compute

\[
Cgp(Y)D = \begin{pmatrix}
-t^{2a} & 0 & 0 \\
0 & 0 & t^{-a} \\
0 & t^{-a} & 0
\end{pmatrix}
\]

It follows \( gp(Y) \in B\lambda_q s_2 B \), where \( q = -2a\alpha_1^\vee - a\alpha_2^\vee \). We calculate

\[
l(\lambda_q) = |\alpha_1(q)| + |\alpha_2(q)| + |\alpha_1(q) + \alpha_2(q)| \\
= 3a + 0 + 3a \\
= 6a
\]

It follows from lemma \( \ref{lem:invariants} \) that \( l(\lambda_q s_2) > l(\lambda_q) = 6a \geq 6 \).
(2) Suppose $a < b < 2a$. In particular, $a \geq 2, b \geq 3$. Let

$$C = \begin{pmatrix} -rt^{2a-b} + q^2 & qt^a & t^{2a} \\ 0 & -rt^{2a-b} + q^2 & -qt^{b-a} \\ 0 & 0 & \frac{1}{r} \end{pmatrix}$$

$$D = \begin{pmatrix} \frac{1}{t^a} & 0 & 0 \\ \frac{qt^a}{q^2 + t^{2a-b}r} & 1 & 0 \\ \frac{qt^{b-a}}{r} & 1 \end{pmatrix}$$

We compute

$$Cgp(Y)R = \begin{pmatrix} t^{2a} & 0 & 0 \\ 0 & t^{b-2a} & 0 \\ 0 & 0 & t^{-b} \end{pmatrix}$$

It follows $gp(Y) \in B\lambda qB$, where $q = -2a\alpha_1^\vee - b\alpha_2^\vee$. We calculate

$$l(\lambda q) = |\alpha_1(q)| + |\alpha_2(q)| + |\alpha_1(q) + \alpha_2(q)|$$
$$= (4a - b) + (2b - 2a) + (2a + b)$$
$$= 4a + 2b \geq 14$$

(3) If $b = 2a$ and $q^2 + r = 0$, let

$$C = \begin{pmatrix} q & t^a & 0 \\ 0 & q & t^a \\ 0 & 0 & \frac{1}{q^2} \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{t^{2a}}{q^2} & \frac{t^a}{q} & 1 \end{pmatrix}$$

We compute

$$Cgp(Y)D = \begin{pmatrix} 0 & -t^a & 0 \\ -t^a & 0 & 0 \\ 0 & 0 & -t^{-2a} \end{pmatrix}$$

It follows $gp(Y) \in B\lambda q s_1 B$, where $q = -a\alpha_1^\vee - 2a\alpha_2^\vee$. Similar to the first case, we see that $l(\lambda q s_1) > 6$.

(4) Suppose either $b > 2a$, or $b = 2a$ and $r + q^2 \neq 0$. In particular, $r + q^2 b - 2a \neq 0$ and $b \geq 2$. Let

$$C = \begin{pmatrix} -r - q^2 t^{b-2a} & -qt^{b-a} & -qt^{b-a} \\ 0 & -\frac{r}{r + q^2 t^{b-2a}} & \frac{r}{r + q^2 t^{b-2a}} \\ 0 & 0 & \frac{1}{r} \end{pmatrix}$$

$$D = \begin{pmatrix} \frac{1}{qt^{b-a}} & 0 & 0 \\ \frac{qt^{b-a}}{q^2 t^{b-2a} + r} & 1 & 0 \\ \frac{qt^{b-a}}{q^2 t^{b-2a} + r} & \frac{1}{r} & 1 \end{pmatrix}$$
We compute

\[ Cgp(Y)D = \begin{pmatrix} t^b & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^{-b} \end{pmatrix} \]

It follows \( gp(Y) \in B_{-b}B \), where \( q = -b\alpha_1 - b\alpha_2 \). We calculate

\[
l(\lambda_q) = |\alpha_1(q)| + |\alpha_2(q)| + |\alpha_1(q) + \alpha_2(q)| \\
= b + b + 2b \\
= 4b \geq 8
\]

\[ \square \]

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