Rocollements were introduced originally by Beilinson, Bernstein and Deligne to study the derived categories of perverse sheaves, and nowadays become very powerful in understanding relationship among three algebraic, geometric or topological objects. The purpose of this series of papers is to study recollements in terms of derived module categories and homological ring epimorphisms, and then to apply our results to both representation theory and algebraic $K$-theory.

In this paper we present a new and systematic method to construct recollements of derived module categories. For this aim, we introduce a new ring structure, called the noncommutative tensor product, and give necessary and sufficient conditions for noncommutative localizations which appears often in representation theory, topology and $K$-theory, to be homological. The input of our machinery is an exact context which can be easily obtained from a rigid morphism that exists in very general circumstances. The output is a recollement of derived module categories of rings in which the noncommutative tensor product of an exact context plays a crucial role. Thus we obtain a large variety of new recollements from commutative and noncommutative localizations, ring epimorphisms and extensions.

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1 Introduction

Recollements were first introduced by Beilinson, Berstein and Deligne in 1982 in order to describe the derived categories of perverse sheaves over singular spaces, by using derived versions of Grothendieck’s six functors (see [16, 6]). Later, recollements of derived categories were employed to study stratifications of the derived categories of modules over blocks of the Bernstein-Gelfand-Gelfand category $O$ (see [14]). Further, recollements were used by Happel to establish a relationship among finitistic dimensions of finite-dimensional algebras (see [17]). Recently, they become of great interest in understanding the derived categories of the endomorphism rings of infinitely generated tilting modules (see [5, 2, 8]). It turns out that recollements are
Recollements of derived module categories have an intimate connection with homological ring epimorphisms ([15], [20], [8], [25]) which play a crucial role in many branches of mathematics. Recall that a ring epimorphism $R \to S$ is said to be homological if $\text{Tor}_i^R(S,S) = 0$ for all $i > 0$. In commutative algebra, homological ring epimorphisms often appear as localizations which are one of the fundamental tools in algebraic geometry. In representation theory, homological ring epimorphisms have been used to study perpendicular categories, sheaves and stratifications of derived module categories of rings (see [15], [14], [8]), and to construct infinitely generated tilting modules (see [1]). In algebraic $K$-theory, Neeman and Ranicki have employed homological noncommutative localizations, a special class of homological ring epimorphisms, to establish a useful long exact sequence of algebraic homomorphisms, a bimodule and a special element of the bimodule, such that they are linked by an exact sequence. The output is a recollement of derived module categories of rings in which the noncommutative coproducts of rings, dual extensions and endomorphism rings. Under a Tor-vanishing condition, we give a constructive method to produce new homological noncommutative localizations and recollements of derived categories of rings. Roughly speaking, the input of our machinery is a quadruple consisting of two ring homomorphisms, a bimodule and a special element of the bimodule, such that they are linked by an exact sequence. The output is a recollement of derived module categories of rings in which the noncommutative tensor products play an essential role. As a consequence, we apply our general results to ring epimorphisms, (commutative and noncommutative) localizations and extensions, and get a large class of new recollements of derived module categories. This kind of recollements was already applied to study the Jordan-Hölder theorem for stratifications of derived module categories in [8] and will be used to investigate relationships among homological or $K$-theoretical properties of three algebras (see [9], [10]).

Now, let us explain our results more explicitly. First of all, we introduce some notation.

Let $R$, $S$ and $T$ be associative rings with identity, and let $\lambda : R \to S$ and $\mu : R \to T$ be ring homomorphisms. Suppose that $M$ is an $S$-$T$-bimodule together with an element $m \in M$. We say that the quadruple $(\lambda, \mu, M, m)$ is an exact context if the following sequence

$$0 \longrightarrow R \xrightarrow{(\lambda, \mu)} S \oplus T \xrightarrow{(m)} M \longrightarrow 0$$

is an exact sequence of abelian groups, where $-m$ and $m$ denote the right and left multiplication by $m$ maps, respectively. If $M = S \otimes_R T$ and $m = 1 \otimes 1$ in an exact context $(\lambda, \mu, M, m)$, then we simply say that the pair $(\lambda, \mu)$ is exact. Exact contexts can be easily constructed from rigid morphisms in an additive category (see Section 3 below).

Given an exact context $(\lambda, \mu, M, m)$, we introduce, in Section 4 a new multiplication $\circ$ on the abelian group $T \otimes_R S$, so that $T \otimes_R S$ becomes an associative ring with identity and that the following two maps

$$\rho : S \to T \otimes_R S, \quad s \mapsto 1 \otimes s \quad \text{for} \quad s \in S, \quad \text{and} \quad \phi : T \to T \otimes_R S, \quad t \mapsto t \otimes 1 \quad \text{for} \quad t \in T$$

are ring homomorphisms (see Lemma 4.4). Furthermore, if both $S$ and $T$ are $R$-algebras over a commutative ring $R$ and if the pair $(\lambda, \mu)$ is exact, then this new ring structure on $T \otimes_R S$ coincides with the usual tensor product of the $R$-algebras $T$ and $S$ over $R$. Due to this reason, the new ring $(T \otimes_R S, \circ)$ is called the noncommutative tensor product of the exact context $(\lambda, \mu, M, m)$, and denoted by $T \otimes_R S$ in this paper. Note that if $(\lambda, \mu)$ is an exact pair, then the ring $T \otimes_R S$, together with $\rho$ and $\phi$, is actually the coproduct of the $R$-rings $S$ and $T$ (via the ring homomorphisms $\lambda$ and $\mu$) over $R$, and further, if $\lambda$ is a ring epimorphism, then $T \otimes_R S$ is isomorphic to the endomorphism ring of the $T$-module $T \otimes_R S$ (see Remark 5.2).
Let
\[ B := \begin{pmatrix} S & M \\ 0 & T \end{pmatrix}, \quad C := \begin{pmatrix} T \boxtimes_R S & T \boxtimes_R S \\ T \boxtimes_R S & T \boxtimes_R S \end{pmatrix}. \]

Let \( \beta : M \to T \otimes_R S \) be the unique \( R \)-\( R \)-bimodule homomorphism such that \( \phi = (m)\beta \) and \( \rho = (-m)\beta \) (see Section 4.1). We define a ring homomorphism
\[
\theta := \begin{pmatrix} \rho & \beta \\ 0 & \phi \end{pmatrix} : B \to C.
\]

First of all, this ring homomorphism is of particular interest in representation theory: The map \( \theta \) can be regarded as the noncommutative localization of \( B \) at a homomorphism between finitely generated projective \( B \)-modules, and therefore it is a ring epimorphism with \( \text{Tor}^B_1(C, C) = 0 \) (see Section 5.1 and [29]), and yields a fully faithful exact functor \( \theta_* : C\text{-Mod} \to B\text{-Mod} \), called the restriction functor, between the category of all left \( C \)-modules and the one of all left \( B \)-modules. Moreover, the map \( \theta \) plays a fundamental role in stratifications of derived categories and in algebraic \( K \)-theory (see [8, 23, 27]).

Generally speaking, \( \theta \) is not always homological in the sense of Geigle and Lenzing (see [15]). In [8], there is a sufficient condition for \( \theta \) to be homological. Concisely, if \( \lambda : R \to S \) is an injective ring epimorphism with \( \text{Tor}^R_1(S, S) = 0 \) and if \( T \) is the endomorphism ring of the \( R \)-module \( S/R \) with \( \mu : R \to T \) the ring homomorphism defined by \( r \mapsto (x \mapsto xr) \) for \( r \in R \) and \( x \in S/R \), then \( B \) is isomorphic to the endomorphism ring of the \( R \)-module \( S \oplus S/R \). For \( \theta \) to be homological, we assume in [8] that \( R \) has projective dimension at most 1. In general context, it seems not much to be known about the map \( \theta \) being homological. So, the following general questions arise:

**Questions.** Let \((\lambda, \mu, M, m)\) be an exact context.

1. When is \( \theta : B \to C \) homological, or equivalently, when is the derived functor \( D(\theta_*) : \mathcal{D}(C) \to \mathcal{D}(B) \) fully faithful?

2. If \( \theta \) is homological, is the Verdier quotient of \( \mathcal{D}(B) \) by \( \mathcal{D}(C) \) equivalent to the derived module category of a ring? or does \( \mathcal{D}(B) \) admit a recollement of derived module categories of rings \( R \) and \( C \)?

The present paper will provide necessary and sufficient conditions to these questions. Here, we will assume neither that \( \lambda \) is injective, nor that \( R \) is homological (compare with [11, 8]). Furthermore, we allow some flexibilities for the choice of the ring homomorphism \( \mu : R \to T \) and the bimodule \( M \). Our main result in this paper can be formulated as follows.

**Theorem 1.1.** Let \((\lambda, \mu, M, m)\) be an exact context. Then:

1. The following assertions are equivalent:
   a. The ring homomorphism \( \theta : B \to C \) is homological.
   b. \( \text{Tor}^R_i(T, S) = 0 \) for all \( i \geq 1 \).

Moreover, if the pair \((\lambda, \mu)\) is exact and \( \lambda \) is homological, then each of the above is equivalent to

1. The ring homomorphism \( \phi : T \to T \boxtimes_R S \) is homological.

2. If one of the above assertions in (1) holds, then there exists a recollement among the derived module categories of rings:

\[
\mathcal{D}(T \boxtimes_R S) \xrightarrow{\rho} \mathcal{D}(B) \xleftarrow{\beta} \mathcal{D}(R).
\]

If, in addition, the projective dimensions of \( rS \) and \( T_R \) are finite, then the above recollement can be restricted to a recollement of bounded derived module categories:

\[
\mathcal{D}^b(T \boxtimes_R S) \xrightarrow{\rho} \mathcal{D}^b(B) \xleftarrow{\beta} \mathcal{D}^b(R).
\]
Note that $\mathcal{D}(B)$ is always a recollement of $\mathcal{D}(T)$ and $\mathcal{D}(S)$, in which the derived category $\mathcal{D}(R)$ of the given ring $R$ is missing. However, Theorem 1.1 provides us with a different recollement for $\mathcal{D}(B)$. A remarkable feature of this recollement is that it contains $\mathcal{D}(R)$ as a member, and thus provides a way to understand properties of the ring $R$ through those of the rings closely related to $S$ and $T$. This idea will be discussed in detail in the forthcoming papers [9] [10] of this series.

The homological condition (b) in Theorem 1.1 can be satisfied in many cases. For instance, in commutative algebra, we may take $\lambda : R \to S$ to be a localization, and in non-commutative case, we refer to the general examples in Section 4.2.

A realization of Theorem 1.1 occurs in noncommutative localizations which have played an important role in topology (see [27]).

Given a ring homomorphism $\lambda : R \to S$, we may consider $\lambda$ as a complex $Q^*$ of left $R$-modules with $R$ and $S$ in degrees $-1$ and $0$, respectively. Then there is a distinguished triangle $R \xrightarrow{\lambda} S \xrightarrow{\pi} Q^* \to R[1]$ in the homotopy category $\mathcal{H}(R)$ of the category of all $R$-modules. This triangle induces a canonical ring homomorphism from $R$ to the endomorphism ring of $Q^*$ in $\mathcal{H}(R)$, and therefore yields a ring homomorphism $\lambda'$ from $R$ to the endomorphism ring of $Q^*$ in $\mathcal{D}(R)$, which depends on $\lambda$ (see Section 5.2 for details). Let $S' := \text{End}_{\mathcal{D}(R)}(Q^*)$. Observe that if $\lambda$ is injective, then $Q^*$ can be identified in $\mathcal{D}(R)$ with the $R$-module $S/R$, and consequently, the map $\lambda' : R \to S'$ coincides with the induced map $R \to \text{End}_R(S/R)$ by the right multiplication.

Further, let $\Lambda := \text{End}_{\mathcal{D}(R)}(S \oplus Q^*)$, and let $\pi^*$ be the following induced map

$$\text{Hom}_{\mathcal{D}(R)}(S \oplus Q^*, \pi) : \text{Hom}_{\mathcal{D}(R)}(S \oplus Q^*, S) \to \text{Hom}_{\mathcal{D}(R)}(S \oplus Q^*, Q^*)$$

which is a homomorphism of finitely generated projective $\Lambda$-modules. Let $\lambda_{\pi^*} : \Lambda \to \Lambda_{\pi^*}$ stand for the noncommutative localization of $\Lambda$ at $\pi^*$ (“universal localization” in terminology of Cohn and Schofield [13] [29]).

If $\lambda$ is a ring epimorphism such that $\text{Hom}_R(S, \text{Ker}(\lambda)) = 0$, then we show in Section 5.2 that the pair $(\lambda, \lambda')$ is exact. So, applying Theorem 1.1 to $(\lambda, \lambda')$, we get the following corollary.

**Corollary 1.2.** If $\lambda : R \to S$ is a homological ring epimorphism such that $\text{Hom}_R(S, \text{Ker}(\lambda)) = 0$, then the following assertions are equivalent:

1. The noncommutative localization $\lambda_{\pi^*} : \Lambda \to \Lambda_{\pi^*}$ of $\Lambda$ at $\pi^*$ is homological.
2. The ring homomorphism $\phi : S' \to S' \otimes_R S$ is homological.
3. $\text{Tor}^R_i(S', S) = 0$ for any $i \geq 1$.

In particular, if one of the above assertions holds, then there exists a recollement of derived module categories:

$$\xymatrix{ \mathcal{D}(\text{End}_{S'}(S' \otimes_R S)) & \mathcal{D}(\Lambda) & \mathcal{D}(R). }$$

As an application of Corollary 1.2, we obtain the following result which not only generalizes the first statement of [8, Corollary 6.6 (1)] since we do not require that the ring epimorphism $\lambda$ is injective, but also gives a way to get derived equivalences of rings (see [28] for definition).

**Corollary 1.3.** Let $\lambda : R \to S$ be a homological ring epimorphism such that $\text{Hom}_R(S, \text{Ker}(\lambda)) = 0$. Then we have the following:

1. If $\pi S$ has projective dimension at most 1, then $\lambda_{\pi^*} : \Lambda \to \Lambda_{\pi^*}$ is homological.
2. The ring $\Lambda_{\pi^*}$ is zero if and only if there is an exact sequence $0 \to P_1 \to P_0 \to R S \to 0$ of $R$-modules such that $P_i$ is finitely generated and projective for $i = 0, 1$. In this case, the rings $R$ and $\Lambda$ are derived equivalent.

As another application of Corollary 1.2, we have the following result in which we do not impose any restriction on the projective dimension of $R S$. 
Corollary 1.4. Suppose that $R \subseteq S$ is an extension of rings, that is, $R$ is a subring of the ring $S$ with the same identity. Let $S'$ be the endomorphism ring of the $R$-module $S/R$ and $B := \left( \begin{array}{cc} S & \text{Hom}_R(S, S/R) \\ 0 & S' \end{array} \right)$.

1. If the left $R$-module $S$ is flat, then there exists a recollement of derived module categories:

$$
\mathcal{D}(S' \boxtimes_R S) \rightarrow \mathcal{D}(B) \rightarrow \mathcal{D}(R)
$$

where $S' \boxtimes_R S$ is the noncommutative tensor product of an exact context.

2. If $S$ is commutative and the inclusion $R \rightarrow S$ is homological, then the ring $S'$ is commutative and there exists a recollement of derived module categories:

$$
\mathcal{D}(S' \otimes_R S) \rightarrow \mathcal{D}(B) \rightarrow \mathcal{D}(R)
$$

where $S' \otimes_R S$ is the usual tensor product of $R$-algebras.

Let us remark that, in commutative algebra, there is a lot of ring extensions satisfying the ‘homological’ assumption of Corollary 1.4 (2). For example, if $R$ is a commutative ring and $\Phi$ is a multiplicative subset of $R$ (that is, $\emptyset \neq \Phi$ and $st \in \Phi$ whenever $s, t \in \Phi$), then the ordinary localization $R \rightarrow \Phi^{-1}R$ of $R$ at $\Phi$ is always homological. Further, if $f : R \rightarrow R'$ is a homomorphism from the ring $R$ to another commutative ring $R'$, then the image of a multiplicative subset of $R$ under $f$ is again a multiplicative set in $R'$. So, as a consequence of Corollary 1.4 (2), we obtain the following result which may be of its own interest in commutative algebra.

Corollary 1.5. Suppose that $R$ is a commutative ring with $\Phi$ a multiplicative subset of $R$. Let $S$ be the localization $\Phi^{-1}R$ of $R$ at $\Phi$, with $\lambda : R \rightarrow S$ the canonical ring homomorphism. If the map $\lambda$ is injective (for example, if $R$ is an integral domain), then there exists a recollement of derived module categories:

$$
\mathcal{D}(\Psi S') \rightarrow \mathcal{D}(\text{End}_R(S \oplus S/R)) \rightarrow \mathcal{D}(R)
$$

where $S' := \text{End}_R(S/R)$, and $\Psi$ is the image of $\Phi$ under the induced map $R \rightarrow S'$ given by the right multiplication.

Observe that the recollements in Corollaries 1.4 and 1.5 occur in the study of infinitely generated tilting modules (see [1] and [8]).

The contents of this paper are outlined as follows. In Section 2, we fix notation and recall some definitions and basic facts which will be used throughout the paper. In particular, we shall recall the definitions of noncommutative localizations, coproducts of rings and recollements, and prepare several lemmas for our proofs. In Section 3 we introduce the notion of exact contexts. To construct exact contexts, we introduce rigid morphisms or hypercyclic bimodules, and show that rigid morphisms exist almost everywhere in representation theory. For example, all kinds of approximations are rigid morphisms. Thus, exact contexts exist rather abundantly. In Section 4 we define the so-called noncommutative tensor products of exact contexts, which will characterize the left parts of recollements constructed in Section 5. Also, we provide examples to demonstrate that noncommutative tensor products cover many well-known constructions in noncommutative algebra. In Section 5 we prove Theorem 1.1 and all of its corollaries mentioned in Section 1. Finally, in Section 6 we give several examples to explain the necessity of some assumptions in our results.

In the second paper [9] of this series, we shall consider the algebraic $K$-theory of recollements, and establish a long Mayer-Vietoris sequence of higher algebraic $K$-groups for homological Milnor squares. In the third paper [10], we shall study relationships among finitistic dimensions of three algebras involved in a recollement. This will extend an earlier result of Happel and a recent result by Xu.
2 Preliminaries

In this section, we shall recall some definitions, notation and basic results which are closely related to our proofs.

2.1 Notation and basic facts on derived categories

Let \( C \) be an additive category.

Throughout the paper, a full subcategory \( B \) of \( C \) is always assumed to be closed under isomorphisms, that is, if \( X \in B \) and \( Y \in C \) with \( Y \cong X \), then \( Y \in B \).

Given two morphisms \( f : X \to Y \) and \( g : Y \to Z \) in \( C \), we denote the composite of \( f \) and \( g \) by \( fg \) which is a morphism from \( X \) to \( Z \). The induced morphisms \( \text{Hom}_C(Z, f) : \text{Hom}_C(Z, X) \to \text{Hom}_C(Z, Y) \) and \( \text{Hom}_C(f, Z) : \text{Hom}_C(Y, Z) \to \text{Hom}_C(X, Z) \) are denoted by \( f^* \) and \( f_* \), respectively.

We denote the composition of a functor \( F : C \to D \) between categories \( C \) and \( D \) with a functor \( G : D \to \mathcal{E} \) between categories \( D \) and \( \mathcal{E} \) by \( GF \) which is a functor from \( C \) to \( \mathcal{E} \). The kernel and the image of the functor \( F \) are denoted by \( \text{Ker}(F) \) and \( \text{Im}(F) \), respectively.

Let \( \mathcal{Y} \) be a full subcategory of \( C \). By \( \text{Ker}(\text{Hom}_C(\_, \mathcal{Y})) \) we denote the full subcategory of \( C \) which is left orthogonal to \( \mathcal{Y} \), that is, the full subcategory of \( C \) consisting of the objects \( X \) such that \( \text{Hom}_C(X, Y) = 0 \) for all objects \( Y \) in \( \mathcal{Y} \). Similarly, \( \text{Ker}(\text{Hom}_C(\mathcal{Y}, \_)) \) stands for the right orthogonal subcategory in \( C \) with respect to \( \mathcal{Y} \).

Let \( \mathcal{K}(C) \) be the category of all complexes over \( C \) with chain maps, and \( \mathcal{K}(C) \) the homotopy category of \( \mathcal{K}(C) \). When \( C \) is abelian, the derived category of \( C \) is denoted by \( \mathcal{D}(C) \), which is the localization of \( \mathcal{K}(C) \) at all quasi-isomorphisms. It is well known that both \( \mathcal{K}(C) \) and \( \mathcal{D}(C) \) are triangulated categories. For a triangulated category, its shift functor is denoted by \([1]\) universally.

If \( T \) is a triangulated category with small coproducts (that is, coproducts indexed over sets exist in \( T \)), then, for each object \( U \) in \( T \), we denote by \( \text{Tria}(U) \) the smallest full triangulated subcategory of \( T \) containing \( U \) and being closed under small coproducts. We mention the following properties related to \( \text{Tria}(U) \):

Let \( F : T \to T' \) be a triangle functor of triangulated categories, and let \( \mathcal{Y} \) be a full subcategory of \( T' \). We define \( F^{-1}\mathcal{Y} := \{ X \in T \mid F(X) \in \mathcal{Y} \} \). Then

1. If \( \mathcal{Y} \) is a triangulated subcategory, then \( F^{-1}\mathcal{Y} \) is a full triangulated subcategory of \( T \).
2. Suppose that \( T \) and \( T' \) admit small coproducts and that \( F \) commutes with coproducts. If \( \mathcal{Y} \) is closed under small coproducts in \( T' \), then \( F^{-1}\mathcal{Y} \) is closed under small coproducts in \( T \). In particular, for an object \( U \in T \), we have \( F(\text{Tria}(U)) \subseteq \text{Tria}(F(U)) \).

In this paper, all rings considered are assumed to be associative and with identity, and all ring homomorphisms preserve identity. Unless stated otherwise, all modules are referred to left modules.

Let \( R \) be a ring. We denote by \( R\text{-Mod} \) the category of all unitary left \( R \)-modules. By our convention of the composite of two morphisms, if \( f : M \to N \) is a homomorphism of \( R \)-modules, then the image of \( x \in M \) under \( f \) is denoted by \( (x)f \) instead of \( f(x) \). The endomorphism ring of the \( R \)-module \( M \) is denoted by \( \text{End}_R(M) \).

As usual, we shall simply write \( \mathcal{C}(R) \), \( \mathcal{K}(R) \) and \( \mathcal{D}(R) \) for \( \mathcal{C}(R\text{-Mod}) \), \( \mathcal{K}(R\text{-Mod}) \) and \( \mathcal{D}(R\text{-Mod}) \), respectively, and identify \( R\text{-Mod} \) with the subcategory of \( \mathcal{D}(R) \) consisting of all stalk complexes concentrated in degree zero. Further, we denote by \( \mathcal{D}^b(R) \) the full subcategory of \( \mathcal{D}(R) \) consisting of all complexes which are isomorphic in \( \mathcal{D}(R) \) to bounded complexes of \( R \)-modules

Let \( (X^\bullet, d_{X^\bullet}) \) and \( (Y^\bullet, d_{Y^\bullet}) \) be two chain complexes over \( R\text{-Mod} \). The mapping cone of a chain map \( h^\bullet : X^\bullet \to Y^\bullet \) is usually denoted by \( \text{Con}(h^\bullet) \). In particular, we have a triangle \( X^\bullet \xrightarrow{h^\bullet} Y^\bullet \to \text{Con}(h^\bullet) \to X^\bullet[1] \) in \( \mathcal{K}(R) \), called a distinguished triangle. For each \( n \in \mathbb{Z} \), we denote by \( H^n(-) : \mathcal{D}(R) \to R\text{-Mod} \) the \( n \)-th cohomology functor. Certainly, this functor is naturally isomorphic to the Hom-functor \( \text{Hom}_{\mathcal{D}(R)}(R, -[n]) \).

The Hom-complex \( \text{Hom}_R^p(X^\bullet, Y^\bullet) \) of \( X^\bullet \) and \( Y^\bullet \) is defined to be the complex \( (\text{Hom}_R^p(X^\bullet, Y^\bullet), d_{X^\bullet, Y^\bullet})_{n \in \mathbb{Z}} \) with

\[
\text{Hom}_R^n(X^\bullet, Y^\bullet) := \prod_{p \in \mathbb{Z}} \text{Hom}_R(X^p, Y^{p+n})
\]
and the differential $d_{X,Y}^n$ of degree $n$ given by

$$(f^p)_{p \in \mathbb{Z}} \mapsto (f^p d_{Y,X}^{\bullet} + (-1)^n d_X^n f^p \cdot 1)_{p \in \mathbb{Z}}$$

for $(f^p)_{p \in \mathbb{Z}} \in \text{Hom}_R(X^n, Y^n)$. For example, if $X \in R$-Mod, then we have

$$\text{Hom}_R^\bullet(X, Y) = (\text{Hom}_R(X^y), \text{Hom}_R(X^x, \cdot))_{n \in \mathbb{Z}};$$

if $Y \in R$-Mod, then

$$\text{Hom}_R^\bullet(X, Y) = (\text{Hom}_R(X^{-n}, Y), (-1)^{n+1} \text{Hom}_R(d_{X, Y}^{-n}, Y))_{n \in \mathbb{Z}}.$$ For simplicity, we denote $\text{Hom}_R^\bullet(X, Y)$ and $\text{Hom}_R^\bullet(X, \cdot)$ by $\text{Hom}_R(X, Y)$ and $\text{Hom}_R(X, \cdot)$, respectively. Note that $\text{Hom}_R(X, Y)$ is also isomorphic to the complex $(\text{Hom}_R(X^{-n}, Y), \text{Hom}_R(d_{X, Y}^{-n}, Y))_{n \in \mathbb{Z}}$.

Moreover, it is known that $H^n(\text{Hom}_R^\bullet(X, Y)) \simeq \text{Hom}_{\mathcal{C}(R)}(X^\bullet, Y^\bullet[n])$ for any $n \in \mathbb{Z}$.

Let $Z^\bullet$ be a chain complex over $R$-Mod. Then the tensor complex $Z^\bullet \otimes_R X^\bullet$ of $Z^\bullet$ and $X^\bullet$ over $R$ is defined to be the complex $(Z^\bullet \otimes_R X^\bullet, \partial_{Z,X}^n)_{n \in \mathbb{Z}}$ with

$$Z^\bullet \otimes_R X^\bullet := \bigoplus_{p \in \mathbb{Z}} Z^p \otimes_R X^{n-p}$$

and the differential $\partial_{Z,X}^n$ of degree $n$ given by

$$z \otimes x \mapsto (z) d_{Z,X}^n \otimes x + (-1)^n z \otimes (x) d_{X, Y}^{n-p}$$

for $z \in Z^p$ and $x \in X^{n-p}$. For instance, if $X \in R$-Mod, then $Z^\bullet \otimes_R X = (Z^p \otimes_R X, d_{Z,X}^n \otimes 1)_{n \in \mathbb{Z}}$. In this case, we denote $Z^\bullet \otimes_R X$ simply by $Z^\bullet \otimes_R X$.

The following result establishes a relationship between Hom-complexes and tensor complexes.

Let $S$ be an arbitrary ring. Suppose that $X^\bullet = (X^n, d_X^n)$ is a bounded complex of $R$-$S$-bimodules. If $RX^n$ is finitely generated and projective for all $n \in \mathbb{Z}$, then there is a natural isomorphism of functors:

$$\text{Hom}_R(X^\bullet, R) \otimes_R - \rightarrow \text{Hom}_R^\bullet(X^\bullet, -) : \mathcal{C}(R) \rightarrow \mathcal{C}(S).$$

To prove this, we note that, for any $R$-$S$-bimodule $X$ and any $R$-module $Y$, there is a homomorphism of $S$-modules: $\delta_{X,Y} : \text{Hom}_R(X,R) \otimes_R Y \longrightarrow \text{Hom}_R(X,Y)$ defined by $f \otimes y \mapsto [x \mapsto (x)f(y)]$ for $f \in \text{Hom}_R(X,R)$, $y \in Y$ and $x \in X$, which is natural in both $X$ and $Y$. Moreover, the map $\delta_{X,Y}$ is an isomorphism if $RX^n$ is finitely generated and projective. For any $Y^\bullet \in \mathcal{C}(R)$ and any $n \in \mathbb{Z}$, it is clear that

$$\text{Hom}_R(X^\bullet, R) \otimes_R Y^\bullet = \bigoplus_{p \in \mathbb{Z}} \text{Hom}_R(X^{-p}, R) \otimes_R Y^{n-p} \quad \text{and} \quad \text{Hom}_R^\bullet(X^\bullet, Y^\bullet) = \bigoplus_{p \in \mathbb{Z}} \text{Hom}_R(X^p, Y^{n+p})$$

since $X^\bullet$ is a bounded complex. Now, we define $\Delta_{X,Y}^n := \sum_{p \in \mathbb{Z}} (-1)^{p(n-p)} \delta_{X^{-p}, Y^{n-p}}$, which is a homomorphism of $S$-modules from $\text{Hom}_R(X^\bullet, R) \otimes_R Y^\bullet$ to $\text{Hom}_R^\bullet(X^\bullet, Y^\bullet)$. Then, one can check that $\Delta_{X,Y}^n := (\Delta_{X,Y}^n, \cdot)_{n \in \mathbb{Z}}$ is a chain map from $\text{Hom}_R(X^\bullet, R) \otimes_R Y^\bullet$ to $\text{Hom}_R^\bullet(X^\bullet, Y^\bullet)$. Since $RX^n$ is finitely generated and projective for each $p \in \mathbb{Z}$, the map $\delta_{X^{-p}, Y^{n-p}}$ is an isomorphism, and so is the map $\Delta_{X,Y}^n$. This implies that

$$\Delta_{X,Y}^n : \text{Hom}_R(X^\bullet, R) \otimes_R Y^\bullet \longrightarrow \text{Hom}_R^\bullet(X^\bullet, Y^\bullet)$$

is an isomorphism in $\mathcal{C}(S)$. Since the homomorphism $\delta_{X,Y}$ is natural in the variables $X$ and $Y$, it can be checked directly that

$$\Delta_{X,Y}^n : \text{Hom}_R(X^\bullet, R) \otimes_R - \longrightarrow \text{Hom}_R^\bullet(X^\bullet, -)$$

defines a natural isomorphism of functors from $\mathcal{C}(R)$ to $\mathcal{C}(S)$. 


In the following, we shall recall some definitions and basic facts about derived functors defined on derived module categories. For details and proofs, we refer to [7, 19].

Let \( \mathcal{K}(R)_p \) (respectively, \( \mathcal{K}(R)_f \)) be the smallest full triangulated subcategory of \( \mathcal{K}(R) \) which
(i) contains all the bounded above (respectively, bounded below) complexes of projective (respectively, injective) \( R \)-modules, and
(ii) is closed under arbitrary direct sums (respectively, direct products).

Note that \( \mathcal{K}(R)_p \) is contained in \( \mathcal{K}(R)_{\text{Proj}} \), where \( R \)-Proj is the full subcategory of \( R \)-Mod consisting of all projective \( R \)-modules. Moreover, the composition functors
\[
\mathcal{K}(R)_p \hookrightarrow \mathcal{K}(R) \to \mathcal{D}(R) \quad \text{and} \quad \mathcal{K}(R)_f \hookrightarrow \mathcal{K}(R) \to \mathcal{D}(R)
\]
are equivalences of triangulated categories. This means that, for each complex \( X^\bullet \) in \( \mathcal{D}(R) \), there exists a complex \( pX^\bullet \in \mathcal{K}(R)_p \) together with a quasi-isomorphism \( pX^\bullet \to X^\bullet \), as well as a complex \( iX^\bullet \in \mathcal{K}(R)_f \) together with a quasi-isomorphism \( X^\bullet \to iX^\bullet \). In this sense, we shall simply call \( pX^\bullet \) the projective resolution of \( X^\bullet \) in \( \mathcal{K}(R) \). For example, if \( X \) is an \( R \)-module, then we can choose \( pX \) to be a deleted projective resolution of \( pX \).

Furthermore, if either \( X^\bullet \in \mathcal{K}(R)_p \) or \( Y^\bullet \in \mathcal{K}(R)_f \), then \( \text{Hom}_{\mathcal{K}(R)}(X^\bullet, Y^\bullet) \simeq \text{Hom}_{\mathcal{D}(R)}(X^\bullet, Y^\bullet) \), and this isomorphism is induced by the canonical localization functor from \( \mathcal{K}(R) \) to \( \mathcal{D}(R) \).

For any triangulated functor \( H : \mathcal{K}(R) \to \mathcal{K}(S) \), there is a total left-derived functor \( \mathbb{L}H : \mathcal{D}(R) \to \mathcal{D}(S) \) defined by \( X^\bullet \mapsto H(pX^\bullet) \), a total right-derived functor \( \mathbb{R}H : \mathcal{D}(R) \to \mathcal{D}(S) \) defined by \( X^\bullet \mapsto H(iX^\bullet) \). Observe that, if \( H \) preserves acyclicity, that is, \( H(X^\bullet) \) is acyclic whenever \( X^\bullet \) is acyclic, then \( H \) induces a triangle functor \( D(H) : \mathcal{D}(R) \to \mathcal{D}(S) \) defined by \( X^\bullet \mapsto H(X^\bullet) \). In this case, we have \( \mathbb{L}H = \mathbb{R}H = D(H) \) up to natural isomorphism, and \( D(H) \) is then called the derived functor of \( H \).

Let \( M^\bullet \) be a complex of \( R \)-\( S \)-bimodules. Then the functors
\[
M^\bullet \otimes_S^L \mathbb{R}Hom_R(M^\bullet, -) : \mathcal{H}(R) \to \mathcal{H}(S)
\]
form a pair of adjoint triangle functors. Denote by \( M^\bullet \otimes_S^L - \) the total left-derived functor of \( M^\bullet \otimes_S^L - \), and by \( \mathbb{R}Hom_R(M^\bullet, -) \) the total right-derived functor of \( \mathbb{R}Hom_R(M^\bullet, -) \). It is clear that \( (M^\bullet \otimes_S^L - , \mathbb{R}Hom_R(M^\bullet, -)) \) is an adjoint pair of triangle functors. Further, the corresponding counit adjunction
\[
e : M^\bullet \otimes_S^L \mathbb{R}Hom_R(M^\bullet, -) \to \mathbb{I}d_{\mathcal{D}(R)}
\]
is given by the composite of the following canonical morphisms in \( \mathcal{D}(R) \): \( M^\bullet \otimes_S^L \mathbb{R}Hom_R(M^\bullet, X^\bullet) = M^\bullet \otimes_S^L \mathbb{R}Hom_R(M^\bullet, X^\bullet) \)
\[
\text{Hom}_{\mathcal{K}(R)}(M^\bullet, X^\bullet) = M^\bullet \otimes_S^L \left( \mathbb{R}Hom_R(M^\bullet, X^\bullet) \right) \to M^\bullet \otimes_S^L \mathbb{R}Hom_R(M^\bullet, X^\bullet) \to X^\bullet \to \mathbb{R}Hom_R(M^\bullet, X^\bullet).
\]

Similarly, we have a corresponding unit adjunction \( \varepsilon : \mathbb{R}Hom_R(M^\bullet, -) \to \mathbb{I}d_{\mathcal{D}(S)} \) for \( X^\bullet \in \mathcal{D}(R) \), which is given by the following composites for \( Y^\bullet \in \mathcal{D}(S) \):
\[
Y^\bullet \to \mathbb{R}Hom_R(M^\bullet, Y^\bullet) \Rightarrow \mathbb{R}Hom_R(M^\bullet, M^\bullet \otimes_S^L Y^\bullet) \Rightarrow \mathbb{R}Hom_R(M^\bullet, (M^\bullet \otimes_S^L Y^\bullet)) = \mathbb{R}Hom_R(M^\bullet, M^\bullet \otimes_S^L Y^\bullet).
\]

For \( X^\bullet \in \mathcal{D}(R) \) and \( n \in \mathbb{Z} \), we have \( \mathbb{R}Hom_R(M^\bullet, X^\bullet[n]) = \mathbb{R}Hom_R(M^\bullet, X^\bullet[n]) \simeq \text{Hom}_{\mathcal{K}(R)}(M^\bullet, (X^\bullet[n]) \simeq \text{Hom}_{\mathcal{K}(R)}(M^\bullet, X^\bullet[n]) \simeq \text{Hom}_{\mathcal{K}(R)}(M^\bullet, X^\bullet[n]) \).

Let \( T \) be another ring and \( N^\bullet \) a complex of \( S-T \)-bimodules. If \( \mathbb{S}N^\bullet \in \mathcal{K}(S)_p \), then
\[
M^\bullet \otimes_S^L N^\bullet \otimes_T^L - \Rightarrow (M^\bullet \otimes_S^L N^\bullet) \otimes_T^L - : \mathcal{D}(T) \to \mathcal{D}(R)
\]
In fact, since \( \mathbb{S}N^\bullet \in \mathcal{K}(S)_p \), by assumption, we have \( N^\bullet \otimes_S^L (\mathbb{S}W^\bullet) \in \mathcal{K}(S)_p \) for \( W^\bullet \in \mathcal{D}(T) \). It follows that \( M^\bullet \otimes_S^L (N^\bullet \otimes_T^L W^\bullet) = M^\bullet \otimes_S^L (N^\bullet \otimes_T^L (\mathbb{S}W^\bullet)) = M^\bullet \otimes_S^L (N^\bullet \otimes_T^L (\mathbb{S}W^\bullet)) \simeq (M^\bullet \otimes_S^L N^\bullet) \otimes_T^L (\mathbb{S}W^\bullet) = (M^\bullet \otimes_S^L N^\bullet) \otimes_T^L W^\bullet = (M^\bullet \otimes_S^L N^\bullet) \otimes_T^L W^\bullet \).

### 2.2 Homological ring epimorphisms and recollements

Let \( \lambda : R \to S \) be a homomorphism of rings.
We denote by $\lambda_* : S\text{-Mod} \to R\text{-Mod}$ the restriction functor induced by $\lambda$, and by $D(\lambda_*): \mathcal{D}(S) \to \mathcal{D}(R)$ the derived functor of the exact functor $\lambda_*$. We say that $\lambda$ is a ring epimorphism if the restriction functor $\lambda_* : S\text{-Mod} \to R\text{-Mod}$ is fully faithful. It is proved that $\lambda$ is a ring epimorphism if and only if the multiplication map $S \otimes_R S \to S$ is an isomorphism as $S\text{-}S$-bimodules if and only if, for any two homomorphisms $f_1, f_2 : S \to T$ of rings, the equality $\lambda f_1 = \lambda f_2$ implies that $f_1 = f_2$. This means that, for a ring epimorphism, we have $X \otimes_S Y \simeq X \otimes_R Y$ and $\text{Hom}_R(Y, Z) \simeq \text{Hom}_S(Y, Z)$ for all right $S$-modules $X$, and for all $S$-modules $Y$ and $Z$. Note that, for a ring epimorphism $\lambda : R \to S$, if $R$ is commutative, then so is $S$.

Following [15], a ring epimorphism $\lambda : R \to S$ is called homological if $\text{Tor}_i^R(S, S) = 0$ for all $i > 0$. Note that a ring epimorphism $\lambda$ is homological if and only if the derived functor $D(\lambda_*): \mathcal{D}(S) \to \mathcal{D}(R)$ is fully faithful. This is also equivalent to saying that $\lambda$ induces an isomorphism $S \otimes_R S \simeq S$ in $\mathcal{D}(S)$. Moreover, for a homological ring epimorphism, we have $\text{Tor}_i^R(X, Y) \simeq \text{Tor}_i^R(X, Y)$ and $\text{Ext}^i_S(Y, Z) \simeq \text{Ext}^i_R(Y, Z)$ for all $i \geq 0$ and all right $S$-modules $X$, and for all $S$-modules $Y$ and $Z$ (see [15] Theorem 4.4).

Clearly, if $\lambda : R \to S$ is a ring epimorphism such that either $RS$ or $SR$ is flat, then $\lambda$ is homological. In particular, if $R$ is commutative and $\Phi$ is a multiplicative subset of $R$, then the canonical ring homomorphism $R \to \Phi^{-1} R$ is homological, where $\Phi^{-1} R$ stands for the (ordinary) localization of $R$ at $\Phi$.

As a generalization of localizations of commutative rings, noncommutative (“universal” in Cohen’s terminology) localizations of arbitrary rings were introduced in [13] (see also [29]) and provide a class of ring homomorphisms $\Sigma$-inverting, that is, if $S$ is a ring such that there exists a $\Sigma$-inverting homomorphism $\varphi : R \to S$, then there exists a unique homomorphism $\psi : R \to S$ of rings such that $\varphi = \lambda \psi$.

Clearly, if $\lambda : R \to S$ is a ring epimorphism with $\text{Tor}_i^R(R_S, R_S) = 0$.

Following [23], the $\lambda_* : R \to R_S$ in Lemma 2.1 is called the noncommutative localization of $R$ at $\Sigma$. One should be aware that $R_S$ may not be flat as a right or left $R$-module. Even worse, the map $\lambda_* S$ in general is not homological (see [24]). Thus it is a fundamental question when $\lambda_* S$ is homological.

Next, we recall the definition of coproducts of rings defined by Cohn in [12], and point out that noncommutative localizations are preserved by taking coproducts of rings.

Let $R_0$ be a ring. An $R_0$-ring is a ring $R$ together with a ring homomorphism $\lambda_R : R_0 \to R$. An $R_0$-homomorphism from an $R_0$-ring $R$ to another $R_0$-ring $S$ is a ring homomorphism $f : R \to S$ such that $\lambda_S = \lambda_R f$. Then we can form the category of $R_0$-rings with $R_0$-rings as objects and with $R_0$-morphisms as morphisms. Clearly, epimorphisms of this category are exactly ring epimorphisms starting from $R_0$.

The coproduct of a family $\{R_i \mid i \in I\}$ of $R_0$-rings is defined to be an $R_0$-ring $R$ together with a family $\{\rho_i : R_i \to R \mid i \in I\}$ of $R_0$-homomorphisms such that, for any $R_0$-ring $S$ with a family of $R_0$-homomorphisms $\{\tau_i : R_i \to S \mid i \in I\}$, there exists a unique $R_0$-homomorphism $\delta : R \to S$ such that $\tau_i = \rho_i \delta$ for all $i \in I$.

It is well known that the coproduct of a family $\{R_i \mid i \in I\}$ of $R_0$-rings always exists. We denote this coproduct by $\sqcup_{R_0} R_i$. Note that if $I = \{1, 2\}$, then $R_1 \sqcup_{R_0} R_2$ is the push-out in the category of $R_0$-rings. This implies that if $\lambda_{R_1} : R_0 \to R_1$ is a ring epimorphism, then so is the homomorphism $\rho_2 : R_2 \to R_1 \sqcup_{R_0} R_2$. Moreover, $R_0 \sqcup_{R_0} R_1 = R_1 = R_1 \sqcup_{R_0} R_0$ for every $R_0$-ring $R_1$, where $\lambda_{R_0} : R_0 \to R_0$ is the identity.

In general, the coproduct of two $R_0$-algebras may not be isomorphic to their tensor product over $R_0$. For example, given a field $k$, the coproduct over $k$ of the polynomial rings $k[x]$ and $k[y]$ is the free ring $k[x, y]$ in two variables $x$ and $y$, while the tensor product over $k$ of $k[x]$ and $k[y]$ is the polynomial ring $k[x, y]$.

The following result is taken from [8] Lemma 6.2] and will be used later.
Lemma 2.2. Let $R_0$ be a ring, $\Sigma$ a set of homomorphisms between finitely generated projective $R_0$-modules, and $\lambda_\Sigma: R_0 \to R_1 := (R_0)_\Sigma$ the noncommutative localization of $R_0$ at $\Sigma$. Then, for any $R_0$-ring $R_2$, the coproduct $R_1 \sqcup_{R_0} R_2$ is isomorphic to the noncommutative localization $(R_2)_\Delta$ of $R_2$ at the set $\Delta := \{ R_2 \otimes_{R_0} f \mid f \in \Sigma \}$.

Finally, we recall the notion of recollements of triangulated categories, which was first defined in [6] to study “exact sequences” of derived categories of coherent sheaves over geometric objects.

Definition 2.3. Let $\mathcal{D}$, $\mathcal{D}'$ and $\mathcal{D}''$ be triangulated categories. We say that $\mathcal{D}$ is a recollement of $\mathcal{D}'$ and $\mathcal{D}''$ if there are six triangle functors among the three categories:
3 Definitions of rigid morphisms and exact contexts

In this section we introduce the notion of rigid morphisms in an additive category, which occur almost everywhere in the representation theory of algebras, and which will be used to construct exact contexts.

Let $C$ be an additive category. An object $X^\cdot$ in $\mathcal{C}(C)$ is rigid if $\text{Hom}_{\mathcal{C}(C)}(X^\cdot, X^\cdot[1]) = 0$.

**Definition 3.1.** A morphism $f^\cdot : Y^\cdot \to X^\cdot$ in $\mathcal{C}(C)$ is said to be rigid if the object $Z^\cdot$ in a distinguished triangle $Y^\cdot \xrightarrow{f^\cdot} X^\cdot \to Z^\cdot \to Y^\cdot[1]$ is rigid, or equivalently, the mapping cone $\text{Con}(f^\cdot)$ of $f^\cdot$ is rigid in $\mathcal{C}(C)$.

A morphism $f : Y \to X$ in $C$ is said to be rigid if $f$, considered as a morphism from the stalk complex $Y$ to the stalk complex $X$, is rigid, or equivalently, the complex $\text{Con}(f) : 0 \to Y \xrightarrow{f} X \to 0$ is rigid in $\mathcal{C}(C)$.

Note that the rigidity of a morphism $f^\cdot$ does not depend on the choice of the triangle which extends $f^\cdot$.

If we consider a rigid morphism $f$ in $C$ as a two-term complex over $C$, then $f$ is positively self-orthogonal in $\mathcal{K}(C)$, that is, $\text{Hom}_{\mathcal{K}(C)}(f, f[n]) = 0$ for all $n > 0$.

Clearly, a morphism $f : Y \to X$ in $C$ is rigid if and only if $\text{Hom}_C(Y, X) = \text{End}_C(Y) f + f \text{End}_C(X)$. Thus, the zero map $Y \to X$ is rigid if and only if $\text{Hom}_C(Y, X) = 0$, and any isomorphism $Y \to X$ is always rigid.

Let us give some non-trivial examples of rigid morphisms, which show that rigid morphisms exist in very general circumstances.

(i) For an additive category $C$, if $f : Y \to X$ is a morphism in $C$ such that the induced map $\text{Hom}_C(Y, f) : \text{Hom}_C(Y, X) \to \text{Hom}_C(Y, X)$ (respectively, $\text{Hom}_C(f, X) : \text{Hom}_C(Y, X) \to \text{Hom}_C(Y, X)$) is surjective, then $\text{Hom}_C(Y, X) = \text{End}_C(Y) f$ (respectively, $\text{Hom}_C(Y, X) = f \text{End}_C(X)$), and therefore $f$ is rigid. Thus all approximations in the sense of Auslander-Smalø (see [4]) are rigid morphisms.

This type of rigid morphisms includes the following three cases:

(a) Let $A$ be an Artin algebra, and let $0 \to Z \xrightarrow{f} Y \xrightarrow{g} X \to 0$ be an almost split sequence in $A$-mod. Then both $f$ and $g$ are rigid since both $\text{Hom}_A(Y, g)$ and $\text{Hom}_A(f, Y)$ are surjective. For the definition of almost split sequences, we refer the reader to [3].

(b) The covariant morphisms defined in [11] are rigid. Recall that a morphism $f : Y \to X$ in an additive category $C$ is called covariant if the induced map $\text{Hom}_C(X, f) : \text{Hom}_C(X, Y) \to \text{Hom}_C(X, X)$ is injective and the induced map $\text{Hom}_C(Y, f) : \text{Hom}_C(Y, Y) \to \text{Hom}_C(Y, X)$ is a split epimorphism of $\text{End}_C(Y)$-modules.

(c) Let $S$ be a ring with identity. If $Y$ is a quasi-projective $S$-module (that is, for any surjective homomorphism $Y \to X$, the induced map $\text{Hom}_S(Y, Y) \to \text{Hom}_S(Y, X)$ is surjective), then we may take a submodule $Z$ of $Y$ and consider the canonical map $f : Y \to X := Y/Z$. Clearly, we have $\text{Hom}_S(Y, X) = \text{End}_S(Y) f$, and therefore $f$ is rigid. Dually, if $X$ is a quasi-injective $S$-module, that is, for any injective homomorphism $g : Y \to X$, the induced map $\text{Hom}_S(X, Y) \to \text{Hom}_S(Y, X)$ is surjective, then, for any submodule $Y$ of $X$, we have $\text{Hom}_S(Y, X) = \mu \text{End}(X)$, where $\mu$ is the inclusion of $Y$ into $X$. This means that $\mu$ is rigid. In particular, every surjective homomorphism from a projective module to a module is rigid, and every injective homomorphism from a module to an injective module is rigid.

(ii) Let $R \subseteq S$ be an extension of rings, that is, $R$ is a subring of the ring $S$ with the same identity. Then the canonical map $\pi : S \to S/R$ of $R$-modules is rigid.

In fact, for any $f \in \text{Hom}_R(S, S/R)$, we choose an element $s \in S$ such that $(s)\pi = (1) f$, and denote by $\cdot s : S \to S$ the right multiplication by $s$ map. Then the map $f - (\cdot s)\pi$ sends $1 \in S$ to zero. Thus there exists a unique homomorphism $g \in \text{End}_R(S/R)$ such that $f = (\cdot s)\pi + \pi g$. This implies that

$$\text{Hom}_R(S, S/R) = \text{End}_R(S)\pi + \pi \text{End}_R(S/R).$$

Since $\text{End}_R(S) \subseteq \text{End}_R(S)$, we have $\text{Hom}_R(S, S/R) = \text{End}_R(S)\pi + \pi \text{End}_R(S/R)$. Thus the map $\pi$ is rigid.

We should observe that not every nonzero homomorphism is rigid. For example, the right multiplication by $x$ map: $k[X]/(X^2) \to k[X]/(X^2)$ is not rigid in $k[X]/(X^2)$-Mod, where $k$ is a field and $X := X + \{X^2\}$ is the coset of $X$ in $k[X]$. In general, an element $x$ in the radical of an Artin algebra $A$, considered as the right multiplication by $x$ map from $AA$ to itself, is never rigid.
Motivated by the rigid morphisms, we introduce the notion of the so-called hypercyclic bimodules.

Let $S$ and $T$ be two rings with identity, and let $M$ be an $S$-$T$-bimodule. An element $m \in M$ is called a hypergenerator if $M = Sm + mT$. In this case, $M$ is said to be hypercyclic.

Hypercyclic bimodules and rigid morphisms are intimately related in the following way: If $f : Y \to X$ is a rigid morphism in an additive category $C$, then $f$ is a hypergenerator of the $\End_C(Y) \otimes \End_C(X)$-bimodule $\Hom_C(Y, X)$.

If $M$ is hypercyclic with $m$ a hypergenerator, then we may define a map
\[
\zeta : S \oplus T \to M \quad (s, t) \mapsto sm - mt \text{ for } s \in S \text{ and } t \in T,
\]
and get an exact sequence of abelian groups
\[
0 \to K \to S \oplus T \xrightarrow{\zeta} M \to 0,
\]
where $K := \{(s, t) \in S \oplus T \mid sm = mt\}$ is a subring of the ring $S \oplus T$. Let $p$ and $q$ be the canonical projections from $K$ to $S$ and $T$, respectively. Then $S$ and $T$ can be considered as $K$-$K$-bimodules, and therefore the above sequence is actually an exact sequence of $K$-$K$-bimodules.

Thus, for each rigid morphism $f : Y \to X$, there is an exact sequence
\[
0 \to R \xrightarrow{(p, q)} \End_C(Y) \oplus \End_C(X) \xrightarrow{(f, f)} \Hom_C(Y, X) \to 0
\]
of $R$-$R$-bimodules, where $R := \{(s, t) \in \End_C(Y) \oplus \End_C(X) \mid sf = ft\}$ is a subring of the ring $\End_C(Y) \oplus \End_C(X)$.

Now, we give the definition of exact contexts.

**Definition 3.2.** Let $R, S$ and $T$ be rings with identity, let $\lambda : R \to S$ and $\mu : R \to T$ be ring homomorphisms, and let $M$ be an $S$-$T$-bimodule with $m \in M$. The quadruple $(\lambda, \mu, M, m)$ is called an exact context if
\[
(*) \quad 0 \to R \xrightarrow{\lambda \otimes \mu} S \oplus T \xrightarrow{m \otimes 1} M \to 0
\]
is an exact sequence of abelian groups, where $-m$ and $m$- stand for the right and left multiplication by $m$ maps, respectively. In this case, we also say that $(M, m)$ is an exact complement of $(\lambda, \mu)$.

If $(\lambda, \mu, S \otimes_R T, 1 \otimes 1)$ is an exact context, then we say simply that $(\lambda, \mu)$ is an exact pair.

Note that the sequence $(*)$ is exact in the category of abelian groups if and only if
1. The $S$-$T$-bimodule $M$ is hypercyclic with $m$ as a hypergenerator, and
2. the ring homomorphism $R \xrightarrow{\lambda \otimes \mu} S \oplus T$ induces a ring isomorphism from $R$ to $K$.

Note that the quadruple $(\lambda, \mu, M, m)$ is an exact context if and only if the following diagram
\[
\begin{array}{ccc}
R & \xrightarrow{\lambda} & S \\
\mu \downarrow & & \downarrow m \\
T & \xrightarrow{m} & M
\end{array}
\]
is both a push-out and a pull-back in the category of $R$-$R$-bimodules.

Let $(\lambda, \mu, M, m)$ be an exact context. Then, from $(\ddagger)$ we see that, for an $S$-$T$-bimodule $N$ with an element $n \in N$, the pair $(N, n)$ is an exact complement of $(\lambda, \mu)$ if and only if there exists a unique isomorphism $\omega : M \to N$ of $R$-$R$-bimodules such that $(sm)\omega = sn$ and $(mt)\omega = nt$ for all $s \in S$ and $t \in T$. Clearly, $\omega$ preserves hypergenerators, that is $(m)\omega = n$. In general, $\omega$ has not to be an isomorphism of $S$-$T$-bimodules, that is, $M$ and $N$ may not be isomorphic as $S$-$T$-bimodules (see the examples in Subsection 4.2.1).

Next, we mention several examples of exact contexts.
(1) Let $M$ be a hypercyclic $S$-$T$ bimodule with $m$ a hypergenerator. Then the pair $(p, q)$ of ring homomorphisms $p: K \to S$ and $q: K \to T$ together with $(M, m)$ forms an exact context. So rigid morphisms always provide us with a class of exact contexts. Conversely, every exact context appears in this form. In fact, for a given exact context $(\lambda, \mu, M, m)$, we may define $B = \begin{pmatrix} S & M \\ 0 & T \end{pmatrix}$ and consider the canonical map $\phi$ from the first column to the second column of $B$ defined by $\cdot m$. It is easy to see that this $\phi$ is rigid and the induced exact context is precisely the given one. So, rigid morphisms describe exact contexts.

(2) Suppose that $R \subseteq S$ is an extension of rings. Let $\lambda: R \to S$ be the inclusion with $\pi: S \to S/R$ the canonical surjection. Define $S': = \text{End}_R(S/R)$ and

$$\lambda': R \to S': \ r \mapsto (x \mapsto xr) \text{ for } r \in R \text{ and } x \in S/R.$$ 

Then $\text{Hom}_R(S, S/R)$ is an $S$-$S'$ bimodule, and the quadruple $(\lambda, \lambda', \text{Hom}_R(S, S/R), \pi)$ is an exact context since the following diagram

$$\begin{array}{ccc}
0 & \xrightarrow{\lambda} & R \\
& \xrightarrow{\cdot \pi} & S \\
& \downarrow{\lambda'} & \downarrow{\pi} \\
0 & \xrightarrow{\cdot \pi} & \text{Hom}_R(S, S/R) \\
& \downarrow{\lambda} & \downarrow{\text{Hom}_R(R, S/R)} \\
& \xrightarrow{\lambda} & 0
\end{array}$$

is commutative and the sequence of $R$-$R$ bimodules

$$0 \to R \xrightarrow{(\lambda, \lambda')} S \oplus S' \xrightarrow{\begin{pmatrix} \pi \\ -r \end{pmatrix}} \text{Hom}_R(S, S/R) \to 0.$$ 

is exact. In general, the exact context presented here is different from the one induced from the rigid morphism $\pi$, and the pair $(\lambda, \lambda')$ may not be exact, because either $S \simeq \text{End}_R(S)$ as rings or $S \otimes_R S' \simeq \text{Hom}_R(S, S/R)$ as $S$-$S'$ bimodules may fail.

A more general construction of exact contexts from a (not necessarily injective) ring homomorphism will be discussed in Lemma 5.9.

(3) Milnor squares, defined by Milnor in [21, Sections 2 and 3], also provide a class of exact contexts. Recall that a Milnor square is a commutative diagram of ring homomorphisms

$$\Lambda \xrightarrow{i_1} \Lambda_1 \xrightarrow{j_1} \Lambda' \xrightarrow{i_2} \Lambda_2 \xrightarrow{j_2} \Lambda'$$

satisfying the following two conditions:

(M1) The ring $\Lambda$ is the pull-back of $\Lambda_1$ and $\Lambda_2$ over $\Lambda'$, that is, given a pair $(\lambda_1, \lambda_2) \in \Lambda_1 \oplus \Lambda_2$ with $\langle \lambda_1 \rangle j_1 = \langle \lambda_2 \rangle j_2 \in \Lambda'$, there is one and only one element $\lambda \in \Lambda$ such that $(\lambda)i_1 = \lambda_1$ and $(\lambda)i_2 = \lambda_2$.

(M2) At least one of the two homomorphisms $j_1$ and $j_2$ is surjective.

Clearly, $\Lambda'$ can be regarded as an $\Lambda_1$-$\Lambda_2$ bimodule via the ring homomorphisms $j_1$ and $j_2$. Let 1 be the identity of $\Lambda'$. Then $j_1$ and $j_2$ are exactly the multiplication maps $\cdot 1$ and $1 \cdot$, respectively.

Now, we claim that the pair $(i_1, i_2)$ together with $(\Lambda', 1)$ forms an exact context. Indeed, it follows from the condition (M2) that $\Lambda'$ is hypercyclic with 1 as a hypergenerator. With the help of the condition (M1), the following sequence

$$0 \to \Lambda \xrightarrow{(i_1, i_2)} \Lambda_1 \oplus \Lambda_2 \xrightarrow{\begin{pmatrix} j_1 \\ -j_2 \end{pmatrix}} \Lambda' \to 0$$

is an exact sequence of $\Lambda$-$\Lambda$ bimodules. This verifies the claim.

Even more, the pair $(i_1, i_2)$ is exact. Without loss of generality, assume that $j_2$ is surjective. Then, by (M1), the map $i_1$ is also surjective and $i_2$ induces an isomorphism $\text{Ker}(i_1) \simeq \text{Ker}(j_2)$ of $\Lambda$-$\Lambda$ bimodules. Now,
we can check that the map $\Lambda_1 \otimes_A \Lambda_2 \to \Lambda'$, defined by $\lambda_1 \otimes \lambda_2 \mapsto (\lambda_1)j_1(\lambda_2)j_2$ for $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$, is an isomorphism of $\Lambda_1$-$\Lambda_2$-bimodules. Actually, this follows from the following isomorphisms:

$$\Lambda_1 \otimes_A \Lambda_2 \cong (\Lambda/\text{Ker}(i_1)) \otimes_A \Lambda_2 \cong \Lambda_2/(\text{Ker}(i_2)\Lambda_2) \cong \Lambda'.$$

Similarly, we can check that the pair $(i_2,i_1)$ is also exact with $\Lambda_2 \otimes_A \Lambda_1 \cong \Lambda'$ as $\Lambda_2$-$\Lambda_1$-bimodules.

### 4 Noncommutative tensor products of exact contexts

In this section, we shall define a new ring for each exact context. This is the so-called noncommutative tensor product which includes the notion of coproducts of rings, usual tensor products and so on. These noncommutative tensor products can be constructed from both Morita context rings and strictly pure extensions, and will play a crucial role in construction of recollements of derived module categories in the next section.

#### 4.1 Definition of noncommutative tensor products

From now on, let $\lambda : R \to S$ and $\mu : R \to T$ be two arbitrary but fixed ring homomorphisms. Unless stated otherwise, we always assume that $(\lambda,\mu,M,m)$ is an exact context.

First, we characterize when the pair $(\lambda,\mu)$ in the exact context is exact. Recall that we have the following exact sequence of $R$-$R$-bimodules:

$$\xymatrix{ 0 \ar[r] & R \ar[r]^{(\lambda,\mu)} & S \oplus T \ar[r]^{(m)} & M \ar[r] & 0.}$$

According to $(\ast)$, there exist two unique homomorphisms

$$\alpha : M \to S \otimes_R T, \ x \mapsto s_x \otimes 1 + 1 \otimes t_x \quad \text{and} \quad \beta : M \to T \otimes_R S, \ x \mapsto 1 \otimes s_x + t_x \otimes 1,$$

where $x \in M$ and $(s_x,t_x) \in S \oplus T$ with $x = s_x m + mt_x$, such that the following two diagrams

$$\xymatrix{ R \ar[r]^{(\lambda,\mu)} \ar@{=}[d] & S \oplus T \ar[r]^{(m)} \ar[d] & M \ar[d]^{\alpha} \ar[l] \ar[r] & \xymatrix{ R \ar[r]^{(\lambda,\mu)} & S \oplus T \ar[d]^{\nu} & \quad \quad \quad \xymatrix{ S \oplus T \ar[r]^{(\nu')} & S \otimes_R T \ar[l] & M \ar[l]^{\alpha}} & \xymatrix{ R \ar[r]^{(\lambda,\mu)} & S \oplus T \ar[r]^{(m)} & M \ar[l]^{\beta}} }$$

are commutative, where

$$\lambda' = \lambda \otimes_R T : T \to S \otimes_R T, \ t \mapsto 1 \otimes t \quad \text{and} \quad \mu' = S \otimes_R \mu : S \to S \otimes_R T, \ s \mapsto s \otimes 1,$$

$$\rho = \mu \otimes S : S \to T \otimes_R S, \ s \mapsto 1 \otimes s \quad \text{and} \quad \phi = T \otimes \lambda : T \to T \otimes_R S, \ t \mapsto t \otimes 1$$

for $s \in S$ and $t \in T$. Note that $(x)\alpha$ and $(x)\beta$ are independent of different choices of $(s_x,t_x)$ in $S \oplus T$.

Further, let

$$\gamma : S \otimes_R T \to M, \ s \otimes t \mapsto smt.$$  

Clearly, $\alpha$ and $\beta$ are homomorphisms of $R$-$R$-bimodules, $\gamma$ is a homomorphism of $S$-$T$-bimodules and $\alpha\gamma = \text{Id}_M$. In particular, $\alpha$ is injective and $\gamma$ is surjective.

**Lemma 4.1.** The following statements are equivalent:

1. The pair $(\lambda,\mu)$ is an exact pair.
2. The map $\gamma$ is an isomorphism.
3. $\text{Coker}(\lambda) \otimes_R \text{Coker}(\mu) = 0$.
4. $(M/mT) \otimes_R (M/Sm) = 0$. 

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Proof. Note that $\gamma$ is a homomorphism of $S$-$T$-bimodules and 

$$(s(1 \otimes 1))\gamma = (s \otimes 1)\gamma = sm \quad \text{and} \quad ((1 \otimes 1)t)\gamma = (1 \otimes t)\gamma = mt$$

for $s \in S$ and $t \in T$. This implies that the following diagram

$$\begin{array}{cccc}
R & \xrightarrow{(\lambda, \mu)} & S \oplus T & \xrightarrow{\left(\begin{array}{c}
\mu' \\
-\lambda'
\end{array}\right)} & S \otimes T \\
\downarrow{\gamma} & & \downarrow{\gamma} & & \\
0 & \xrightarrow{(\lambda, \mu)} & S \oplus T & \xrightarrow{\left(\begin{array}{c}
m \\
\cdot
\end{array}\right)} & M & \rightarrow & 0
\end{array}$$

is commutative, where the second row is assumed to be exact. Consequently, (1) and (2) are equivalent.

According to (6), we know that $\text{Coker}(\lambda) \simeq M/mT$ and $\text{Coker}(\mu) \simeq M/Sm$ as $R$-$R$-bimodules. Thus (3) and (4) are equivalent.

Now, we verify the equivalences of (2) and (3).

In fact, since $\alpha \gamma = \text{Id}_M$, the map $\gamma$ is an isomorphism if and only if $\alpha$ is a surjection, while the latter is equivalent to that the map

$$\xi := \left(\begin{array}{c}
\mu' \\
-\lambda'
\end{array}\right) : S \oplus T \rightarrow S \otimes T$$

is a surjection by (†). Therefore, it is enough to show that $\xi$ is surjective if and only if $\text{Coker}(\lambda) \otimes_R \text{Coker}(\mu) = 0$. To check this condition, we consider the following two complexes

$$\text{Con}(\lambda) : 0 \rightarrow R \xrightarrow{\lambda} S \rightarrow 0 \quad \text{and} \quad \text{Con}(\mu) : 0 \rightarrow R \xrightarrow{\mu} T \rightarrow 0$$

of $R$-$R$-bimodules, where both $S$ and $T$ are of degree 0, and calculate the tensor complex of them over $R$:

$$\text{Con}(\lambda) \otimes_R^\ast \text{Con}(\mu) : 0 \rightarrow R \otimes_R R \xrightarrow{\left(\begin{array}{c}
\lambda \otimes_R - R \otimes_R \mu
\end{array}\right)} S \otimes_R R \oplus R \otimes_R T \xrightarrow{\left(\begin{array}{c}
\gamma \\
\gamma_{\otimes T}
\end{array}\right)} S \otimes_R T \rightarrow 0$$

where $R \otimes_R R$ is of degree $-2$. If we identify $R \otimes_R R$, $S \otimes_R R$ and $R \otimes_R T$ with $R$, $S$ and $T$, respectively, then $\text{Con}(\lambda) \otimes_R^\ast \text{Con}(\mu)$ is precisely the complex:

$$0 \rightarrow R \xrightarrow{\left(\begin{array}{c}
\lambda \\
-\mu
\end{array}\right)} S \oplus T \xrightarrow{\left(\begin{array}{c}
\mu' \\
-\lambda'
\end{array}\right)} S \otimes_R T \rightarrow 0$$

which is isomorphic to the following complex

$$0 \rightarrow R \xrightarrow{\left(\begin{array}{c}
\lambda \\
-\mu
\end{array}\right)} S \oplus T \xrightarrow{\xi} S \otimes_R T \rightarrow 0.$$

It follows that $\xi$ is surjective if and only if $H^0(\text{Con}(\lambda) \otimes_R^\ast \text{Con}(\mu)) = 0$. Since

$$H^0(\text{Con}(\lambda) \otimes_R^\ast \text{Con}(\mu)) \simeq H^0(\text{Con}(\lambda)) \otimes_R H^0(\text{Con}(\mu)) \simeq \text{Coker}(\lambda) \otimes_R \text{Coker}(\mu),$$

the map $\xi$ is surjective if and only if $\text{Coker}(\lambda) \otimes_R \text{Coker}(\mu) = 0$. Thus $\gamma$ is an isomorphism if and only if $\text{Coker}(\lambda) \otimes_R \text{Coker}(\mu) = 0$. This shows the equivalences of (2) and (3). □

Remark 4.2. By the equivalences of (1) and (2) in Lemma 4.1, if the pair $(\lambda, \mu)$ is exact, then it admits a unique complement $(S \otimes_R T, 1 \otimes 1)$ up to isomorphism (preserving hypergenerators) of $S$-$T$-bimodules.

A sufficient condition to guarantee the isomorphism of $\gamma$ is the following result.

Corollary 4.3. If either $\lambda : R \rightarrow S$ or $\mu : R \rightarrow T$ is a ring epimorphism, then $\gamma : S \otimes_R T \rightarrow M, s \otimes t \mapsto smt$ is an isomorphism of $S$-$T$-bimodules.
Proof. Suppose that $\lambda$ is a ring epimorphism. Then, for any $S$-module $X$, the map $\lambda \otimes X : R \otimes_R X \to S \otimes_R X$ is an isomorphism. This implies that $\text{Coker}(\lambda) \otimes_R X = 0$. Since $\text{Coker}(\mu) \simeq M/Sm$ as $R$-modules by (2) and since $M/Sm$ is an $S$-module, we have $\text{Coker}(\lambda) \otimes_R \text{Coker}(\mu) \simeq \text{Coker}(\lambda) \otimes_R (M/Sm) = 0$. By Lemma 4.1, the map $\gamma$ is an isomorphism.

Similarly, if $\mu$ is a ring epimorphism, then $\gamma$ is an isomorphism. □

As examples of exact pairs, we see from Corollary 4.3 that the rigid morphisms from an almost split sequence always provide us with exact pairs.

Next, we shall introduce the so-called noncommutative tensor products $T \boxtimes_R S$ of the exact context $(\lambda, \mu, M, m)$. That is, we endow $T \otimes_R S$ with an associative multiplication $\circ : (T \otimes_R S) \times (T \otimes_R S) \to T \otimes_R S$, under which it becomes an associative ring with the identity $1 \otimes 1$.

Let

$$\delta := \gamma \beta : S \otimes_R T \to T \otimes_R S, \ s \otimes t \mapsto 1 \otimes s_{\text{sm}} + t_{\text{sm}} \otimes 1$$

for $s \in S$ and $t \in T$, where the pair $(s_{\text{sm}}, t_{\text{sm}}) \in S \otimes T$ is chosen such that $s_{\text{sm}} = s_{\text{sm}} m + m t_{\text{sm}}$. Then $\delta$ is a homomorphism of $R$-$R$-bimodules such that $(s \otimes 1) \delta = 1 \otimes s$ and $(1 \otimes t) \delta = t \otimes 1$.

The multiplication $\circ$ is induced by the following homomorphisms:

$$(T \otimes_R S) \otimes_R (T \otimes_R S) \xrightarrow{\sim} T \otimes_R (S \otimes_R T) \otimes_R S \xrightarrow{T \otimes \delta \otimes S} T \otimes_R (T \otimes_R T) \otimes_R S \xrightarrow{\mu_T \otimes_S} T \otimes_R S$$

where $\mu_T : T \otimes_R T \to T$ and $\mu_S : S \otimes_R S \to S$ are the multiplication maps. More precisely, for $(t_i, s_i) \in T \otimes_R S$ with $i = 1, 2$, we have

$$(t_1 \otimes s_1) \circ (t_2 \otimes s_2) := t_1 (s_1 \otimes t_2) \delta s_2 = t_1 (1 \otimes s_{1t_2m} + t_{1t_2m} \otimes 1) s_2.$$

The following lemma reveals a crucial property of this multiplication.

Lemma 4.4. The following statements are true.

1. With the multiplication $\circ$, the abelian group $T \otimes_R S$ becomes an associative ring with the identity $1 \otimes 1$.
2. The maps $\rho : S \to T \otimes_R S$ and $\phi : T \to T \otimes_R S$ are ring homomorphisms. In particular, $T \otimes_R S$ can be regarded as an $S$-$T$-bimodule via $\rho$ and $\phi$.
3. The map $\beta : M \to T \otimes_R S$ is a homomorphism of $S$-$T$-bimodules such that $(m)\beta = 1 \otimes 1$.

Proof. (1) It suffices to show that the multiplication $\circ$ is associative and that $1 \otimes 1$ is the identity of $T \otimes_R S$.

To check the associativity of $\circ$, we take elements $t_i \in T$ and $s_i \in S$ for $1 \leq i \leq 3$, and choose two pairs $(x, y)$ and $(u, v)$ in $S \times T$ such that

$$s_1 m t_2 = x m + m y \quad \text{and} \quad s_2 m t_3 = u m + m v.$$

On the one hand, $$((t_1 \otimes s_1) \circ (t_2 \otimes s_2)) \circ (t_3 \otimes s_3) = (t_1 (s_1 \otimes t_2) \delta s_2) \circ (s_3 \otimes t_3) = (t_1 (1 \otimes y + y \otimes 1) s_2) \circ (t_3 \otimes s_3) = \gamma (s_3 \otimes t_3) \delta.$$

On the other hand, $$((t_1 \otimes s_1) \circ (t_2 \otimes s_2)) \circ (t_3 \otimes s_3) = (t_1 \otimes s_1) \circ (t_2 (s_2 \otimes t_3) \delta s_3) = (t_1 \otimes s_1) \circ (t_2 (1 \otimes y + y \otimes 1) s_3) = (t_1 \otimes s_1) \circ (t_2 (1 \otimes u + u \otimes 1) s_3) = (t_1 \otimes s_1) \circ (t_2 \delta s_3) = (t_1 \otimes s_1) \circ (t_2 \delta s_3).$$

So, to prove that $$(t_1 \otimes s_1) \circ (t_2 \otimes s_2) \circ (t_3 \otimes s_3) = (t_1 \otimes s_1) \circ ((t_2 \otimes s_2) \circ (t_3 \otimes s_3)),$$

it is enough to verify that $$(x s_2 \otimes t_3) \delta + y (s_2 \otimes t_3) \delta = (s_1 \otimes t_2) \delta u + (s_1 \otimes t_2 v) \delta.$$
This shows that the multiplication \( \circ \) is associative.

Note that \((t_1 \otimes s_1) \circ (1 \otimes 1) = t_1(s_1 \otimes 1) = t_1 \otimes s_1\) and \((1 \otimes 1) \circ (t_1 \otimes s_1) = (1 \otimes t_1) \delta s_1 = (t_1 \otimes 1)s_1 = t_1 \otimes s_1\). Thus \((T \otimes R) S, \circ\) is an associative ring with the identity \(1 \otimes 1\).

(2) Since \((s_1) \rho \circ (s_2) \rho = (1 \otimes s_1) \circ (1 \otimes s_2) = (s_1 \otimes 1) \delta s_2 = (1 \otimes s_1) s_2 = 1 \otimes s_1 s_2 = (s_1 s_2) \rho\), the map \(\rho : S \to T \otimes R S\) is a ring homomorphism. Similarly, we can show that \(\phi : T \to T \otimes R S\) is also a ring homomorphism.

(3) Clearly, by the definition of \(\beta\), we have \((m) \beta = 1 \otimes 1\). It remains to check that \(\beta\) is a homomorphism of \(S\)-\(T\)-bimodules, or equivalently, that

\[
(sat) \beta = (s) \rho \circ (a) \beta \circ (t) \phi
\]

for \(s \in S, a \in M\) and \(t \in T\).

To check this, we pick up \(s_a \in S\) and \(t_a \in T\) such that \(a = s_a m + m t_a\). Then \((sat) \beta = (s_a m t + m t_a t) \beta = (s_a m t) \beta + (s a m t) \delta + (s \otimes t a) \delta = (1 \otimes s a) \circ (t \otimes 1) + (1 \otimes s) \circ (t_a \otimes 1) = (1 \otimes s) \circ (1 \otimes s_a) \circ (t \otimes 1) + (1 \otimes s) \circ (t_a \otimes 1) \circ (t \otimes 1) = (s) \rho \circ (a) \beta \circ (t) \phi.\]

Thanks to Lemma \[4.2\], the ring \((T \otimes R) S, \circ\) will be called the noncommutative tensor product of the exact context \((\lambda, \mu, M, m)\), denoted simply by \(T \boxtimes R S\) if the exact context \((\lambda, \mu, M, m)\) is clear.

We should note that the ring \(T \boxtimes R S\) is not the usual tensor product of two \(R\)-algebras: First, the ring \(R\) is not necessarily commutative, this means that the usual tensor product of \(R\)-algebras on the abelian group \(T \otimes R S\) does not make sense. Second, even if the ring \(R\) is commutative, we cannot ensure that the product has to coincide with the usual tensor product because the image of \(\lambda : R \to S\) does not have to be in the center of \(S\). This means that \(S\) is not necessarily an \(R\)-algebra. Nevertheless, the ring \(T \boxtimes R S\) does generalize the usual tensor product of \(R\)-algebras in the following sense:

Let \(R\) be a commutative ring. Suppose that \(S\) and \(T\) are \(R\)-algebras via \(\lambda\) and \(\mu\), respectively, that is, the images of \(\lambda\) and \(\mu\) are contained in the centers of \(S\) and \(T\), respectively. If \((\lambda, \mu)\) is an exact pair, then the noncommutative tensor product \(T \boxtimes R S\) coincides with the usual tensor product \(T \otimes R S\) of \(R\)-algebras \(T\) and \(S\).

In fact, by our notation, we have \(M = S \otimes R T, \gamma = Id_{S \otimes R T}\) and \(\delta = \beta : S \otimes R T \to T \otimes R S\), where \(\beta\) is determined uniquely by the diagram:

\[
\begin{array}{ccc}
R & (\lambda, \mu) & S \otimes R T \\
\downarrow \downarrow \downarrow \downarrow & \downarrow \downarrow \downarrow \downarrow & \downarrow \downarrow \downarrow \downarrow \\
S \otimes T & (\rho, \phi) & T \otimes R S \\
\beta
\end{array}
\]

However, since \((\lambda, \mu)\) is exact, we can check that the switch map \(\omega : S \otimes R T \to T \otimes R S\), defined by \(s \otimes t \mapsto t \otimes s\) for \(s \in S \) and \(t \in T\), also makes the above diagram commutative, that is, \((\rho, \phi) \omega = (\beta, \beta)\). This implies that \(\beta = \omega\). Thus the multiplication \(\circ : (T \boxtimes R S) \times (T \boxtimes R S) \to T \boxtimes R S\) coincides with the usual tensor product of \(R\)-algebras \(T\) and \(S\) over \(R\).

### 4.2 Examples of noncommutative tensor products

In this section, we present two general receipts for constructing noncommutative tensor products, which show that noncommutative tensor products cover a large variety of interesting algebras.

#### 4.2.1 From Morita context rings

Let \((A, C, X, Y, f, g)\) be an arbitrary but fixed Morita context, that is, \(A\) and \(C\) are rings with identity, \(X\) is an \(A\)-\(C\)-bimodule, \(Y\) is a \(C\)-\(A\)-bimodule, \(f : X \otimes_C Y \to A\) is a homomorphism of \(A\)-\(A\)-bimodules and \(g : Y \otimes_A X \to C\)
is a homomorphism of $C$-$C$-bimodules, such that
\[(x_1 \otimes y_1)f \cdot x_2 = x_1(y_1 \otimes x_2)g \quad \text{and} \quad (y_1 \otimes x_1)g \cdot y_2 = y_1(x_1 \otimes y_2)f\]
for $x_i \in X$ and $y_i \in Y$ with $i = 1, 2$. For simplicity, we denote by $x_1y_1$ and $y_1x_1$ the elements $(x_1 \otimes y_1)f$ and $(y_1 \otimes x_1)g$, respectively.

Given a Morita context $(A, C, X, Y, f, g)$, we can define the Morita context ring $\Gamma := \begin{pmatrix} A & X \\ Y & C \end{pmatrix}$, where the multiplication is given by
\[
\begin{pmatrix} a_1 & x_1 \\ y_1 & c_1 \end{pmatrix} \begin{pmatrix} a_2 & x_2 \\ y_2 & c_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + x_1y_2 & a_1x_2 + x_1c_2 \\ y_1a_2 + c_1y_2 & c_1c_2 + y_1x_2 \end{pmatrix}
\]
for $a_i \in A$, $c_i \in C$, $x_i \in X$ and $y_i \in Y$.

Let
\[R := \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}, \quad S := \begin{pmatrix} A & X \\ 0 & C \end{pmatrix}, \quad T := \begin{pmatrix} A & 0 \\ Y & C \end{pmatrix}, \quad M := \begin{pmatrix} A & X \\ Y & C \end{pmatrix}, \quad m := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M,
\]
and let $\lambda : R \rightarrow S$ and $\mu : R \rightarrow T$ be the canonical inclusions. Note that the $S$-$T$-bimodule structure on $M$ is induced from the ring structure of the Morita context ring $\Gamma$. Since $R = S \cap T$ and $M = S + T$, the quadruple $(\lambda, \mu, M, m)$ is an exact context. So we can consider the noncommutative tensor product $T \boxtimes_R S$ of this exact context. In fact, the multiplication in $T \boxtimes_R S$ can be described explicitly as follows:

We identify $R$-$\text{Mod}$ with the product $A$-$\text{Mod} \times C$-$\text{Mod}$. In this sense, $rS = (A \oplus X) \times C$ and $T_R = (A \oplus Y) \times C$. It follows that the following homomorphism
\[T \otimes_R S \rightarrow \begin{pmatrix} A & X \\ Y & C \oplus (Y \otimes_A X) \end{pmatrix} =: \Lambda,
\]
defined by
\[
\begin{pmatrix} a_1 & 0 \\ y_1 & c_1 \end{pmatrix} \otimes \begin{pmatrix} a_2 & x_2 \\ 0 & c_2 \end{pmatrix} \mapsto \begin{pmatrix} a_1a_2 & a_1x_2 \\ y_1a_2 & (c_1c_2 + y_1x_2) \end{pmatrix},
\]
is an isomorphism of abelian groups. Via this isomorphism, we identify $T \otimes_R S$ with $\Lambda$ and translate the multiplication of $T \boxtimes_R S$ into the one of $\Lambda$. By calculation, this multiplication on $\Lambda$ is exactly given by the following formula:
\[
\begin{pmatrix} a_1 & x_1 \\ y_1 & (c_1, y \otimes x) \end{pmatrix} \circ \begin{pmatrix} a_2 & x_2 \\ y_2 & (c_2, y' \otimes x') \end{pmatrix} = \begin{pmatrix} a_1a_2 + x_1y_2 & a_1x_2 + x_1y'x' \\ y_1a_2 + c_1y_2 + (yx)y_2 & (c_1c_2, y_1x_2 + (c_1y') \otimes x' + y \otimes (xc_2) + y \otimes (xy')x') \end{pmatrix},
\]
where $x, x' \in X$ and $y, y' \in Y$. Thus $T \boxtimes_R S = \Lambda$. In this sense, the associated homomorphisms $\rho : S \rightarrow T \boxtimes_R S$, $\phi : T \rightarrow T \boxtimes_R S$ and $\beta : M \rightarrow T \otimes_R S$ are given by
\[
\begin{pmatrix} a_1 & x_1 \\ y_1 & (c_1, y \otimes x) \end{pmatrix} \mapsto \begin{pmatrix} a_1 & 0 \\ 0 & (c_1, y \otimes x) \end{pmatrix}, \quad \begin{pmatrix} a_1 & x_1 \\ y_1 & (c_1, 0) \end{pmatrix} \mapsto \begin{pmatrix} a_1 & 0 \\ y_1 & (c_1, 0) \end{pmatrix}, \quad \begin{pmatrix} a_1 & x_1 \\ y_1 & (c_1, 0) \end{pmatrix} \mapsto \begin{pmatrix} a_1 & x_1 \\ y_1 & (c_1, 0) \end{pmatrix},
\]
respectively. Note that both $\rho$ and $\phi$ are ring homomorphisms. However, $\beta$ does not have to be a ring homomorphism in general. Actually, it is a ring homomorphism if and only if $Y \otimes_A X = 0$. Moreover, it follows from the multiplication of $\Lambda$ that the map
\[
\pi : \Lambda \rightarrow \Gamma, \quad \begin{pmatrix} a_1 & x_1 \\ y_1 & (c_1, y \otimes x) \end{pmatrix} \mapsto \begin{pmatrix} a_1 & x_1 \\ y_1 & c_1 + yx \end{pmatrix},
\]
is a surjective ring homomorphism such that $\beta \pi = Id_M$. Further, let $e := \begin{pmatrix} 1 & 0 \\ 0 & (0,0) \end{pmatrix} \in \Lambda$. Then $e^2 = e$, $e\Lambda e = A$, $\Lambda e = A \oplus Y$, $e\Lambda = A \oplus X$, $\Lambda e\Lambda = \begin{pmatrix} A & X \\ Y & Y \otimes A \end{pmatrix}$ and $\Lambda/(\Lambda e\Lambda) = C$. This also implies that the canonical multiplication map $\Lambda e \otimes_A e\Lambda \rightarrow \Lambda e\Lambda$ is an isomorphism of $\Lambda$-$\Lambda$-bimodules. So, if $\text{Tor}^A_i(Y,X) = 0$ for all $i > 0$, then the canonical surjective map $\Lambda \rightarrow \Lambda/\Lambda e\Lambda$ is homological.

For each $i \geq 1$, we have $\text{Tor}^P_i(T,S) \simeq \text{Tor}^A_i(Y,X)$. Thus $\text{Tor}^P_i(T,S) = 0$ if and only if $\text{Tor}^A_i(Y,X) = 0$.

Let us give some examples to illustrate how the choices of structure maps in the Morita contexts influence the noncommutative tensor products of exact contexts in the above construction.

Let $k$ be a field, and let $A = C = X = Y = k$. Now we take two different kinds of structure maps $f : X \otimes_R Y \rightarrow A$ and $g : Y \otimes_R X \rightarrow C$ as follows:

(i) Let $f$ and $g$ be the canonical isomorphism $k \otimes_k k \xrightarrow{\cong} k$. Then the Morita context ring is the matrix ring $M_2(k)$ of $2 \times 2$ matrices over $k$. In this case, the noncommutative tensor product of the corresponding exact context is

$$\Lambda := T \otimes_R S = \begin{pmatrix} k & k \\ k & k \oplus k \end{pmatrix}$$

with the multiplication given by

$$\begin{pmatrix} a_1 & x_1 \\ y_1 & (c_1, x) \end{pmatrix} \circ \begin{pmatrix} a_2 & x_2 \\ y_2 & (c_2, x') \end{pmatrix} \mapsto \begin{pmatrix} a_1a_2 + x_1y_2 \\ y_1a_2 + c_1y_2 + x_2 \\ a_1x_2 + x_1c_2 + x_1x' \\ c_1c_2 + c_1x' + xc_2 + xx' \end{pmatrix},$$

where $x, x', a_i, c_i, x_i, y_i \in k$ for $i = 1, 2$. Actually, this ring is Morita equivalent to $k \times k$ since $e_2 := \begin{pmatrix} 0 & 0 \\ 0 & (0,0) \end{pmatrix}$

$$= \begin{pmatrix} 0 & 0 \\ 0 & (1,-1) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & (0,1) \end{pmatrix} = e'_2 + e''_2$$

and $e_2 \Lambda = \Lambda e'_2$.

(ii) Let $f$ and $g$ be the zero homomorphism $k \otimes_k k \rightarrow k$. Then the Morita context ring, denoted by $M_2(k)_0$, has the vector space $M_2(k)$ and admits a new multiplication:

$$\begin{pmatrix} a_1 & x_1 \\ y_1 & c_1 \end{pmatrix} \circ \begin{pmatrix} a_2 & x_2 \\ y_2 & c_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + c_1y_2 \\ y_1a_2 + c_1y_2 + c_1c_2 \\ a_1x_2 + x_1c_2 \\ c_1c_2 \end{pmatrix}. $$

Note that $M_2(k)_0$ can be identified with the following quiver algebra with relations

$$1 \xrightarrow{\alpha} 2, \quad \alpha \beta = \beta \alpha = 0.$$ 

In this case, the noncommutative tensor product $T \boxtimes_R S$ can be calculated analogously and turns out to be isomorphic to the quiver algebra of the same quiver as the above, but with only one zero relation: $\alpha \beta = 0$. Clearly, this noncommutative tensor product $T \boxtimes_R S$ is a quasi-hereditary algebra and has $M_2(k)_0$ as its quotient algebra, as the foregoing general fact indicated.

Note that the noncommutative tensor products in both (i) and (ii) are not derived equivalent to the coproduct $S \sqcup_R T$ of $\lambda$ and $\mu$. In fact, $\lambda$ and $\mu$ are independent of the choices of structure maps $f$ and $g$, and moreover, $S \sqcup_R T$ is given by the following quiver algebra

$$1 \xrightarrow{\alpha} 2, \quad \beta \alpha = \alpha \beta = 0.$$ 

which is infinite-dimensional and hereditary. Note that if a $k$-algebra is derived equivalent to another finite-dimension $k$-algebra, then the algebra itself must be finite-dimensional. Since the noncommutative tensor products in both (i) and (ii) are finite-dimensional, they are not derived equivalent to $S \sqcup_R T$.  

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4.2.2 From strictly pure extensions

An extension $D \subseteq C$ of rings is said to be strictly pure if $C$ has an ideal $X$ such that there exists a splitting $C = D \oplus X$ of $D$-$D$-bimodules. Such a kind of extensions was used by Waldhausen to compute the algebraic $K$-theory of generalized free products in [32].

Now, let $\lambda : R \to S$ and $\mu : R \to T$ be two arbitrary strictly pure extensions. We shall construct an exact context $(\lambda, \mu, M, m)$ from the pair $(\lambda, \mu)$. First of all, we fix two split decompositions of $R$-$R$-bimodules:

$$S = R \oplus X \quad \text{and} \quad T = R \oplus Y$$

where $X$ and $Y$ are ideals of $S$ and $T$, respectively, and define $M := R \oplus X \oplus Y$, the direct sum of abelian groups. Next, we endow $M$ with a ring structure such that $S$ and $T$ are subrings of $M$. Here, we define a multiplication on $M$ as follows:

$$(r_1 + x_1 + y_1)(r_2 + x_2 + y_2) := r_1r_2 + (r_1x_2 + x_1r_2 + x_1x_2) + (r_1y_2 + y_1r_2 + y_1y_2)$$

for $r_i \in R$, $x_i \in X$ and $y_i \in Y$ with $i = 1, 2$. In particular, we have $x_1y_1 = 0 = y_1x_1$ in $M$. One can check that, under this multiplication, $M$ is a ring with identity $1$, and contains both $S$ and $T$ as subrings. Since the intersection of $S$ and $T$ in $M$ is equal to $R$ and since $M = S + T$, we see that the quadruple $(\lambda, \mu, M, 1)$ is an exact context. Clearly, $\text{Tor}^R_1(T, S) = 0$ if and only if $\text{Tor}^R_1(Y, R) = 0$.

Now, we calculate the noncommutative tensor product $T \boxtimes_R S$ of the exact content $(\lambda, \mu, M, 1)$.

Actually, as $R$-$R$-bimodules, we have

$$T \boxtimes_R S = R \oplus X \oplus Y \oplus Y \boxtimes_R X.$$  

In this case, the map $\gamma : S \boxtimes_R T \to M$ is given by $s \otimes t \mapsto st$ for $s \in S$ and $t \in T$, and the map $\beta : M \to T \boxtimes_R S$ is exactly the canonical inclusion. It follows that $\delta : S \boxtimes_R T \to T \boxtimes_R S$ is defined as follows:

$$(r + x) \otimes (r' + y) \mapsto rr' + ry + xr'$$

for $r, r' \in R$, $x \in X$ and $y \in Y$. In particular, we have $(x \otimes y)\delta = 0$. Now, we can check that the multiplication $\circ : (T \boxtimes_R S) \times (T \boxtimes_R S) \to T \boxtimes_R S$ is actually given by

$$(r_1 + x_1 + y_1 \boxtimes x_3) \circ (r_2 + x_2 + y_2 \boxtimes x_4)$$

$$= r_1r_2 + (r_1x_2 + x_1r_2 + x_1x_2) + (r_1y_2 + y_1r_2 + y_1y_2) + (y_1 \boxtimes y_2 + y_1 \boxtimes (x_3x_2) + (r_1y_4) \boxtimes x_4 + (y_1y_4) \boxtimes x_4 + y_3 \boxtimes (x_3x_2))$$

where $r_1, r_2 \in R$, $x_i \in X$ and $y_i \in Y$ for $1 \leq i \leq 4$. Here, we have $x_1 \circ y_2 = 0$ and $y_1 \circ x_2 = y_1 \boxtimes x_2$. Moreover, the following map

$$\pi : T \boxtimes_R S \to M, \quad r_1 + x_1 + y_1 \boxtimes x_3 \mapsto r_1 + x_1 + y_1$$

is a surjective ring homomorphism with $\beta \pi = \text{Id}_M$. Note that $\beta$ may not be a ring homomorphism in general.

In the following, we show that noncommutative tensor products induced from strictly pure extensions cover the trivially twisted extensions in [33].

Let $A$ be an Artin algebra, and let $A_0, A_1$ and $A_2$ be three Artin subalgebras of $A$ with the same identity. We say that $A$ decomposes as a twisted tensor product of $A_1$ and $A_2$ over $A_0$ (see [35]) if the following three conditions hold:

1. $A_0$ is a semisimple $k$-algebra such that $A_1 \cap A_2 = A_0$ and $A = A_0 \oplus \text{rad}(A)$ as a direct sum of $A_0$-$A_0$-bimodules, where $\text{rad}(A)$ denotes the Jacobson radical of $A$.
2. The multiplication map $\sigma : A_2 \otimes_{A_0} A_1 \to A$ is an isomorphism of $A_2$-$A_1$-bimodules.
3. $\text{rad}(A_1) \text{rad}(A_2) \subseteq \text{rad}(A_2) \text{rad}(A_1)$.

Now, we assume that $A$ decomposes as a twisted tensor product of $A_1$ and $A_2$ over $A_0$. Then we always have the following decompositions of $A_0$-$A_0$-bimodules:

$$A_1 = A_0 \oplus \text{rad}(A_1) \quad \text{and} \quad A_2 = A_0 \oplus \text{rad}(A_2),$$
where $A_0$ is a common semisimple subalgebra of $A$, $A_1$ and $A_2$. If $\text{rad}(A_1)\text{rad}(A_2) = 0$, then $A$ is called the \textit{trivially twisted tensor product} of $A_1$ and $A_2$ over $A_0$.

Let $A$ be the trivially twisted tensor product of $A_1$ and $A_2$ over $A_0$. Then we may take
\[
R := A_0, \quad S := A_1, \quad T := A_2, \quad X := \text{rad}(A_1), \quad Y := \text{rad}(A_2),
\]
and let $\lambda : R \to S$ and $\mu : R \to T$ be the inclusions. Clearly, both $\lambda$ and $\mu$ are strictly pure. By the foregoing discussion, $M := R \oplus X \oplus Y$ is a ring and $(\lambda, \mu, M, 1)$ is an exact context. So the noncommutative tensor product $T \boxtimes_R S$ of this exact context can be defined. Since $XY = \text{rad}(A_1)\text{rad}(A_2) = 0$ in $A$, the multiplication of the noncommutative tensor product $T \boxtimes_R S$ implies that the map $\sigma : T \boxtimes_R S \to A$ is actually an isomorphism of rings. Thus $A \simeq T \boxtimes_R S$ as rings.

We do not know whether all twisted tensor products of Artin algebras can be realized as the noncommutative tensor products of some exact contexts.

## 5 Recollements arising from exact contexts

In this section, we shall give a procedure to construct recollements of derived module categories of rings from exact contexts.

Throughout this section, we assume that $(\lambda : R \to S, \mu : R \to T, M, m)$ is an exact context.

### 5.1 Proof of Theorem 1.1

In the following, we shall first show that noncommutative tensor products $T \boxtimes_R S$ can be used to describe noncommutative localizations.

Let
\[
B := \left( \begin{array}{cc} S & M \\ 0 & T \end{array} \right), \quad C := \left( \begin{array}{cc} T \boxtimes_R S & T \boxtimes_R S \\ T \boxtimes_R S & T \boxtimes_R S \end{array} \right).
\]

We define a ring homomorphism
\[
\theta := \left( \begin{array}{cc} \rho & \beta \\ 0 & \phi \end{array} \right) : B \to C.
\]

See Section 4.1 for notation.

Furthermore, let
\[
e_1 := \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right), \quad e_2 := \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) \in B \quad \text{and} \quad \varphi : Be_1 \to Be_2, \quad \left( \begin{array}{cc} s & 0 \\ 0 \end{array} \right) \mapsto \left( \begin{array}{cc} sm & 0 \\ 0 \end{array} \right) \quad \text{for} \quad s \in S.
\]

Then $\varphi$ is a homomorphism of finitely generated projective $B$-modules. If we identify $\text{Hom}_B(Be_1, Be_2)$ with $M$, then $\varphi$ corresponds to the element $m \in M$. Let us now look at the noncommutative localization $\lambda_\varphi : B \to B_\varphi$ of $B$ at $\varphi$.

\textbf{Lemma 5.1.} Up to isomorphism, the map $\theta : B \to C$ is the noncommutative localization of $B$ at $\varphi$.

\textit{Proof.} We first recall a characterization of $B_\varphi$ in terms of generators and relations in [30].

Let $\Lambda$ be the ring defined by the following generators and relations:

\begin{enumerate}
  \item $a_\varphi = 1$;
  \item $a_s + a_y = a_{s+y}$ for $y \in M$;
  \item $a_sm a_x = a_{sx}$ for $s \in S$;
  \item $a_t a_{mt} = a_{st}$ for $t \in T$.
\end{enumerate}

Define $\rho_\varphi : S \to \Lambda, \ s \mapsto a_{sm}$, $\rho_T : T \to \Lambda, \ t \mapsto a_{mt}$ and $\rho_M : M \to \Lambda, \ s \mapsto a_s$ for $s \in S, \ t \in T$ and $x \in M$. Then $\rho_\varphi$ and $\rho_T$ are ring homomorphisms. Moreover, by [30] Theorem 2.4, the noncommutative localization $\lambda_\varphi : B \to B_\varphi$ is (isomorphic to) the following map
\[
\left( \begin{array}{cc} \rho_\varphi & \rho_M \\ 0 & \rho_T \end{array} \right) : \left( \begin{array}{cc} S & M \\ 0 & T \end{array} \right) \to \left( \begin{array}{cc} \Lambda & \Lambda \\ \Lambda & \Lambda \end{array} \right).
\]
Let 
\[ \omega : T \boxtimes_R S \rightarrow \Lambda, \ t \otimes s \mapsto (t)\rho_T(s)\rho_S = \alpha_m a_sm \]
for \( t \in T \) and \( s \in S \). In the following, we shall show that \( \omega \) is a ring isomorphism such that \( \rho \omega = \rho_S, \beta \omega = \rho_M \) and \( \phi \omega = \rho_T \). Thus, up to isomorphism, the map \( \theta \) can be regarded as the noncommutative localization of \( B \) at \( \mathfrak{p} \). This also means that the noncommutative tensor product of an exact context can be described by generators and relations.

Now, we show that \( \omega \) is a ring homomorphism. Clearly, \( (1 \otimes 1)\omega = a_ma_m = a_m \) by the relation (3). To show that \( \omega \) preserves multiplications, that is,
\[ ((t_1 \otimes s_1) \circ (t_2 \otimes s_2))\omega = (t_1 \otimes s_1)(t_2 \otimes s_2)\omega. \]
for \( s_1 \in S \) and \( t_i \in T \) for \( i = 1, 2 \), we pick up \( u \in S \) and \( v \in T \) such that \( s_1mt_2 = um + mv \). Then \( ((t_1 \otimes s_1) \circ (t_2 \otimes s_2))\omega = (t_1(1 \otimes u + v \otimes s_2))\omega = (t_1 \otimes us_2 + t_1v \otimes s_2)\omega = (t_1)\rho_T(us_2)\rho_S + (t_1)\rho_T(s_2)\rho_S = (t_1)\rho_T(u)\rho_S(s_2)\rho_S + (t_1)\rho_T(v)\rho_T(s_2)\rho_S = (t_1)\rho_T((u)v)\rho_T(s_2)\rho_S = (t_1)\rho_T(u)\rho_T(s_2)\rho_S. \) Note that \( (t_1 \otimes s_1)(t_2 \otimes s_2)\omega = (t_1)(t_2)\rho_T(s_1)\rho_S(s_2)\rho_S \). So it is sufficient to prove that \( (u)\rho_S + (v)\rho_T = (s_1)\rho_T(s_2)\rho_T \), or equivalently, that \( a_{sm} + a_{mv} = a_{sm}a_{mt_2} \). Actually, due to the relations (2) and (3), we obtain
\[ a_{sm} + a_{mv} = a_{sm+mv} = a_{sm}a_{mt_2} = a_{sm}a_{mt_2}. \]
Thus \( \omega \) is a ring homomorphism.

Next, we show that \( \omega \) is a bijection. In fact, the element \( 1 \otimes 1 \) is the identity of \( T \boxtimes_R S \) and \( (m)\beta = 1 \otimes 1 \) by Lemma 4.4(3). Moreover, for any \( s \in S, t \in T \) and \( x \in M \), we have \( (sm)\beta \circ (x)\beta = (s)\rho \circ (m)\beta \circ (x)\beta = (s)\circ (x)\beta = (sx)\beta \) and \( (x)\beta \circ (mt)\beta = (x)\rho \circ (m)\beta \circ (t)\phi = (x)\beta \circ (t)\phi = (xt)\beta \). This implies that there exists a unique ring homomorphism \( \psi : \Lambda \rightarrow T \boxtimes_R S \) sending \( a_s \) to \( (x)\beta \). Now, we check that \( \omega \psi = Id_{T \boxtimes_R S} \) and \( \psi \omega = Id_M \). Indeed, the former follows from
\[ (t \otimes s)\psi \omega = (a_{mt}a_{sm})\psi = (mt)\beta \circ (sm)\beta = (m)\beta \circ (t)\phi \circ (s)\rho \circ (m)\beta = (t)\rho \circ (s)\rho = (t \otimes 1) \circ (1 \otimes s) = t \otimes s, \]
while the latter follows from
\[ ((x)\beta)\psi \omega = (tx \otimes 1 + 1 \otimes s_x)\omega = a_{mt}a_m + a_m a_{sm} = a_{mt} + a_{sm} = a_s, \]
where \( s_x \in S \) and \( t_x \in T \) such that \( x = s_xt + mt_x \). Thus \( \omega \) is a ring isomorphism.

Note that \( \beta \omega = \rho_M \) by \( (\rho) \). Since \( (s)\rho \omega = (1 \otimes s)\omega = a_m a_{sm} = a_{sm} = (s)\rho_S \) and \( (t)\phi \omega = (t \otimes 1)\omega = a_{mt}a_m = a_mt = (t)\rho_T \), we see that \( \rho \omega = \rho_S \) and \( \phi \omega = \rho_T \).

**Remark 5.2.** (1) If \( (\lambda, \mu) \) is an exact pair, then it follows from Lemma 5.1 and [29] Theorem 4.10, p. 59] that the noncommutative tensor product \( T \boxtimes_R S \), together with the ring homomorphisms \( \rho : S \rightarrow T \boxtimes_R S \) and \( \phi : T \rightarrow T \boxtimes_R S \), is the coproduct \( S \boxplus_R T \) of the \( R \)-rings \( S \) and \( T \) over \( R \) (via the ring homomorphisms \( \lambda : R \rightarrow S \) and \( \mu : R \rightarrow T \)), that is the push-out in the category of \( R \)-rings. In this case, the map \( \theta : B \rightarrow C \) is actually given by the following:
\[
\begin{pmatrix}
S & S \boxtimes_R T \\
0 & T
\end{pmatrix} \rightarrow \begin{pmatrix}
T \boxtimes_R S & T \boxtimes_R S \\
T \boxplus_R S & T \boxplus_R S
\end{pmatrix}
\]
\[
\begin{pmatrix}
s_1 \\
0
\end{pmatrix} \rightarrow \begin{pmatrix}
(s_1)\rho \\
0
\end{pmatrix}
\]
\[
\begin{pmatrix}
s_2 \\
t_1
\end{pmatrix} \rightarrow \begin{pmatrix}
(s_2)\rho(t_2)\phi \\
(t_1)\phi
\end{pmatrix}
\]
for \( s_1 \in S \) and \( t_i \in T \) with \( i = 1, 2 \).

In fact, since \( (\lambda, \mu) \) is an exact pair, we have \( M = S \boxtimes_R T, \alpha = Id_M \) and \( \delta = \beta : S \boxtimes_R T \rightarrow T \boxtimes_R S \) (see Section 4.4 for notation). Further, \( \delta \) is equal to the following map

\[ S \boxtimes_R T \rightarrow T \boxtimes_R S, \quad s_2 \otimes t_2 \mapsto (s_2)\rho \circ (t_2)\phi. \]
In general, for an exact context, its noncommutative tensor product may not be isomorphic to the coproduct of the $R$-rings $S$ and $T$.

(2) If $\lambda$ is a ring epimorphism, then $T \boxtimes_R S \simeq \text{End}_T(T \otimes_R S)$ as rings.

Actually, in this case, the pair $(\lambda, \mu)$ is an exact pair by Corollary 4.3 and Lemma 4.1. It follows from (1) that $S \otimes_R T = T \boxtimes_R S$. Further, the ring homomorphism $\phi : T \to T \boxtimes_R S$ is a ring epimorphism. Thus $T \boxtimes_R S \simeq \text{End}_{T \boxtimes_R S}(T \boxtimes_R S) \simeq \text{End}_T(T \boxtimes_R S) = \text{End}_T(T \otimes_R S)$ as rings.

From now on, let $P^*$ be the complex

$$0 \to Be_1 \xrightarrow{\varphi} Be_2 \to 0$$

in $\mathcal{C}(B)$ with $Be_1$ and $Be_2$ in degrees -1 and 0, respectively, that is $P^* = \text{Con}(\varphi)$. Further, let $P^{**} := \text{Hom}_B(P^*, B)$ which is isomorphic to the complex

$$0 \to e_2B \xrightarrow{\varphi} e_1B \to 0$$

in $\mathcal{C}(B^{op})$ with $e_2B$ and $e_1B$ in degrees 0 and 1, respectively.

Note that $Be_1$ and $Be_2$ are also right $R$-modules via $\lambda : R \to S$ and $\mu : R \to T$, respectively, and that the map $-m : S \to M$ is a homomorphism of $S$-$R$-bimodules. Thus $\varphi$ is actually a homomorphism of $B$-$R$-bimodules. This implies that $P^*$ is a bounded complex over $B \boxtimes_R R^{op}$, and that there is a distinguished triangle in $\mathcal{K}(B \boxtimes_R R^{op})$:

$$Be_1 \xrightarrow{\varphi} Be_2 \to P^* \to Be_1[1].$$

By Lemma 5.1, the ring homomorphism $\theta : B \to C$ is a ring epimorphism, and therefore the restriction functor $\theta_\ast : C\text{-Mod} \to B\text{-Mod}$ is fully faithful. Now, we define a full subcategory of $\mathcal{D}(B)$:

$$\mathcal{D}(B)_{C\text{-Mod}} := \{X^* \in \mathcal{D}(B) \mid H^n(X^*) \in C\text{-Mod} \text{ for all } n \in \mathbb{Z}\}.$$  

Clearly, we have $X[n] \in \mathcal{D}(B)_{C\text{-Mod}}$ for all $X \in C\text{-Mod}$ and all $n \in \mathbb{Z}$. Also, by [8] Proposition 3.3 (3), we have

$$\mathcal{D}(B)_{C\text{-Mod}} = \text{Ker}\{\text{Hom}_{\mathcal{D}(B)}(\text{Tria}(P^*), -)\} = \{X^* \in \mathcal{D}(B) \mid \text{Hom}_{\mathcal{D}(B)}(P^*, X^*[n]) = 0 \text{ for all } n \in \mathbb{Z}\},$$

or equivalently,

$$\mathcal{D}(B)_{C\text{-Mod}} = \{X^* \in \mathcal{D}(B) \mid H^n(\text{Hom}_B^*(P^*, X^*)) = 0 \text{ for all } n \in \mathbb{Z}\}.$$

The following result is taken from [8] Proposition 3.6 (a) and (b) (4-5). See also [23] Theorem 0.7 and Proposition 5.6.

**Lemma 5.3.** Let $i_\ast$ be the canonical embedding of $\mathcal{D}(B)_{C\text{-Mod}}$ into $\mathcal{D}(B)$. Then there is a recollement

$$\mathcal{D}(B)_{C\text{-Mod}} \xrightarrow{i_\ast} \mathcal{D}(B) \xleftarrow{\iota} \text{Tria}(P^*)$$

such that $i_\ast$ is the left adjoint of $i_\ast$. Moreover, the map $\theta : B \to C$ is homological if and only if $H^n(i_\ast i_\ast(B)) = 0$ for all $n \neq 0$. In this case, the derived functor $D(\theta_\ast) : \mathcal{D}(C) \to \mathcal{D}(B)_{C\text{-Mod}}$ is an equivalence of triangulated categories.

To realize $\text{Tria}(P^*)$ in Lemma 5.1 by the derived module category of a ring, we first show that $P^*$ is a self-orthogonal complex in $\mathcal{D}(B)$. Recall that a complex $X^*$ in $\mathcal{D}(B)$ is called self-orthogonal if $\text{Hom}_{\mathcal{D}(B)}(X^*, X^*[n]) = 0$ for any $n \neq 0$.
Lemma 5.4. The following statements are true:

(1) \( \text{End}_{\mathcal{D}(B)}(P^*) \simeq R \) as rings.

(2) The complex \( P^* \) is self-orthogonal in \( \mathcal{D}(B) \), that is \( \text{Hom}_{\mathcal{D}(B)}(P^*, P^*[n]) = 0 \) for any \( n \neq 0 \).

(3) There exists a recollement of triangulated categories:

\[
\begin{array}{ccc}
\mathcal{D}(B)_{C_{\text{Mod}}} & \xrightarrow{i_*} & \mathcal{D}(B) & \xrightarrow{j_*} & \mathcal{D}(R) \\
\downarrow{i} & & \downarrow{j} & & \downarrow{j_*} \\
\end{array}
\]

where \( i_* \) is the canonical embedding and

\[
j : \mathcal{D}^+ R^n \rightarrow j_* : \text{Hom}_B^*(P^*, -) \simeq \mathcal{D}^+ R^n - , \quad j_* := \mathcal{R}\text{Hom}_R(P^*, -).
\]

Proof. (1) Note that \( P^* \) is a bounded complex over \( B \) consisting of finitely generated projective \( B \)-modules. It follows that \( \text{End}_{\mathcal{D}(B)}(P^*) \simeq \text{End}_{\mathcal{D}(B)}(P^*) \) as rings. Since \( \text{Hom}_R(B e_i, Be_j) = 0 \), we clearly have \( \text{End}_{\mathcal{D}(B)}(P^*) = \text{End}_{\mathcal{D}(B)}(P^*) \). Moreover, if \( \text{End}_B(B e_i) \) and \( \text{End}_B(B e_j) \) are identified with \( S \) and \( T \), respectively, then we can identify \( \text{End}_{\mathcal{D}(B)}(P^*) \) with \( K := \{(s, t) \in S \cdot T \mid sm = mt\} \) which is a subring of \( S \cdot T \). Since \( (\lambda, \mu, M, m) \) is an exact context, we see that \( R \simeq K \) as rings. Thus \( \text{End}_{\mathcal{D}(B)}(P^*) \simeq R \) as rings.

(2) It is clear that \( \text{Hom}_{\mathcal{D}(B)}(P^*, P^*[n]) = \text{Hom}_{\mathcal{D}(B)}(P^*, P^*[n]) = 0 \) for all \( n \in \mathbb{Z} \) with \( |n| \geq 2 \). Since \( \text{Hom}_R(B e_i, Be_j) = 0 \), we get \( \text{Hom}_{\mathcal{D}(B)}(P^*, P^*[n]) = 0 \). Observe that \( \text{Hom}_{\mathcal{D}(B)}(P^*, P^*[1]) = 0 \) if and only if \( \text{Hom}_R(B e_i, Be_j) = 0 \). If we identify \( \text{Hom}_R(B e_i, Be_j), \text{End}_B(B e_i) \) and \( \text{End}_B(B e_j) \) with \( M, S \) and \( T \), respectively, then the latter condition is equivalent to that the map

\[
\begin{pmatrix}
m \\ -m
\end{pmatrix} : S \cdot T \rightarrow M, \quad (s, t) \mapsto sm - mt, \quad \text{for } s \in S, t \in T,
\]

is surjective. Clearly, this is guaranteed by the definition of exact contexts. Thus (2) holds.

(3) The idea of our proof is motivated by [19]. Since \( P^* \) is a complex of \( B-R \)-bimodules, the total left-derived functor \( P^* \otimes_R^L \cdot : \mathcal{D}(R) \rightarrow \mathcal{D}(B) \) and the total right-derived functor \( \mathcal{R}\text{Hom}_R(P^*, -) : \mathcal{D}(B) \rightarrow \mathcal{D}(R) \) are well defined. Moreover, since \( P^* \) is a bounded complex of finitely generated projective \( B \)-modules, the functor \( \text{Hom}_B^*(P^*, -) : \mathcal{D}(B) \rightarrow \mathcal{D}(R) \) preserves acyclicity, that is, \( \text{Hom}_B^*(P^*, W^* \cdot) \) is acyclic whenever \( W^* \in \mathcal{C}(B) \) is acyclic. This automatically induces a derived functor \( \mathcal{D}(B) \rightarrow \mathcal{C}(R) \), which is defined by \( W^* \mapsto \text{Hom}_B^*(P^*, W^*) \). Therefore, we can replace \( \mathcal{R}\text{Hom}_R(P^*, -) \) with the Hom-functor \( \text{Hom}_B^*(P^*, -) \) up to natural isomorphism.

Now, we claim that the functor \( P^* \otimes_R^L \cdot \) is fully faithful and induces a triangle equivalence from \( \mathcal{D}(R) \) to \( \text{Tria}(P^*) \).

To prove this claim, we first show that the functor \( P^* \otimes_R^L \cdot : \mathcal{D}(R) \rightarrow \mathcal{D}(B) \) is fully faithful. Let

\[
\mathcal{V} := \{ Y^* \in \mathcal{D}(R) \mid P^* \otimes_R^L \cdot : \mathcal{D}(R) \rightarrow \mathcal{D}(B) \}
\]

Clearly, \( \mathcal{V} \) is a full triangulated subcategory of \( \mathcal{D}(R) \). Since \( P^* \otimes_R^L \cdot \) commutes with arbitrary direct sums and since \( P^* \) is compact in \( \mathcal{D}(B) \), we see that \( \mathcal{V} \) is closed under arbitrary direct sums in \( \mathcal{D}(R) \).

In the following, we shall show that \( \mathcal{V} \) contains \( R \). It is sufficient to prove that

(a) \( P^* \otimes_R^L \cdot \) induces an isomorphism of rings from \( \text{End}_{\mathcal{D}(R)}(R) \) to \( \text{End}_{\mathcal{D}(R)}(P^* \otimes_R^L R) \), and

(b) \( \text{Hom}_{\mathcal{D}(R)}(P^* \otimes_R^L R, P^* \otimes_R^L R[n]) = 0 \) for any \( n \neq 0 \).

Since \( P^* \otimes_R^L R \simeq P^* \) in \( \mathcal{D}(B) \), we know that (a) is equivalent to that the right multiplication map \( R \rightarrow \text{End}_{\mathcal{D}(R)}(P^*) \) is an isomorphism of rings, and that (b) is equivalent to \( \text{Hom}_{\mathcal{D}(R)}(P^* \otimes_R^L R, P^* \otimes_R^L R[n]) = 0 \) for any \( n \neq 0 \). Actually, (a) and (b) follow directly from (1) and (2), respectively. This shows \( R \in \mathcal{V} \).

Thus we have \( \mathcal{V} = \mathcal{D}(R) \) since \( \mathcal{D}(R) = \text{Tria}(R) \). Consequently, for any \( Y^* \in \mathcal{D}(R) \), there is the following isomorphism:

\[
P^* \otimes_R^L \cdot : \mathcal{D}(R) \rightarrow \mathcal{D}(R), \quad P^* \otimes_R^L R[n] \rightarrow \text{Hom}_{\mathcal{D}(R)}(P^* \otimes_R^L R, P^* \otimes_R^L R[n]) \text{ for all } n \in \mathbb{Z}.
\]
Now, fix $N^\bullet \in \mathcal{D}(R)$ and consider

$$\mathcal{D}_{N^\bullet} := \{ X^\bullet \in \mathcal{D}(R) \mid P^\bullet \otimes_R - : \text{Hom}_{\mathcal{D}(R)}(X^\bullet, N^\bullet[n]) \cong \text{Hom}_{\mathcal{D}(B)}(P^\bullet \otimes_R X^\bullet, P^\bullet \otimes_R N^\bullet[n]) \text{ for all } n \in \mathbb{Z} \}.$$ 

Then, one can check that $\mathcal{D}_{N^\bullet}$ is a full triangulated subcategory of $\mathcal{D}(R)$, which is closed under arbitrary direct sums in $\mathcal{D}(R)$. Since $R \in \mathcal{D}_{N^\bullet}$ and $\mathcal{D}(R) = \text{Tri}a(R)$, we get $\mathcal{D}_{N^\bullet} = \mathcal{D}(R)$. Consequently, for any $M^\bullet \in \mathcal{D}(R)$, we have the following isomorphism:

$$P^\bullet \otimes_R - : \text{Hom}_{\mathcal{D}(R)}(M^\bullet, N^\bullet[n]) \cong \text{Hom}_{\mathcal{D}(B)}(P^\bullet \otimes_R M^\bullet, P^\bullet \otimes_R N^\bullet[n])$$

for all $n \in \mathbb{Z}$. This means that $P^\bullet \otimes_R - : \mathcal{D}(R) \to \mathcal{D}(B)$ is fully faithful.

Recall that $\text{Tri}a(P^\bullet)$ is the smallest full triangulated subcategory of $\mathcal{D}(B)$, which contains $P^\bullet$ and is closed under arbitrary direct sums in $\mathcal{D}(B)$. It follows that the image of $\mathcal{D}(R)$ under $P^\bullet \otimes_R -$ is $\text{Tri}a(P^\bullet)$ (see the property (2) in Section 2.4) and that $P^\bullet \otimes_R -$ induces a triangle equivalence from $\mathcal{D}(R)$ to $\text{Tri}a(P^\bullet)$.

Note that $\text{Hom}^\bullet_B(P^\bullet, -)$ is a right adjoint of $P^\bullet \otimes_R -$. This means that the restriction of the functor $\text{Hom}^\bullet_B(P^\bullet, -)$ to $\text{Tri}a(P^\bullet)$ is the quasi-inverse of the functor $P^\bullet \otimes_R - : \mathcal{D}(R) \to \text{Tri}a(P^\bullet)$. In particular, $\text{Hom}^\bullet_B(P^\bullet, -)$ induces an equivalence of triangulated categories:

$$\text{Tri}a(P^\bullet) \xrightarrow{\sim} \mathcal{D}(R).$$

Furthermore, it follows from [8] Proposition 3.3 (3) that

$$\mathcal{D}(R)_{C_{-Mod}} = \{ X^\bullet \in \mathcal{D}(B) \mid \text{Hom}_{\mathcal{D}(B)}(P^\bullet, X^\bullet[n]) = 0 \text{ for all } n \in \mathbb{Z} \} = \text{Ker}(\text{Hom}^\bullet_B(P^\bullet, -)).$$

Therefore, we can choose $j_1 = P^\bullet \otimes_R -$ and $j_1^! = \text{Hom}^\bullet_B(P^\bullet, -)$.

Since $P^\bullet$ is a bounded complex of $B_R$-bimodules with all of its terms being finitely generated and projective as $B$-modules, there exists a natural isomorphism of functors (see Section 2.4):

$$P^{**} \otimes_B - \xrightarrow{\sim} \text{Hom}^\bullet_B(P^\bullet, -) : C(B) \to \mathcal{C}(R).$$

This implies that the former functor preserves acyclicity, since the latter always admits this property. It follows that the functors $P^{**} \otimes_B -$ and $P^{**} \otimes_B -$ are naturally isomorphic, and therefore $j_1^! \simeq P^{**} \otimes_B -$. Clearly, the functor $P^{**} \otimes_B -$ has a right adjoint $\mathbb{R}\text{Hom}_B(P^{**}, -)$. This means that the functor $j_1^!$ can also have $\mathbb{R}\text{Hom}_B(P^{**}, -)$ as a right adjoint functor (up to natural isomorphism). However, by the uniqueness of adjoint functors in a recollement, we see that $j_1$ is naturally isomorphic to $\mathbb{R}\text{Hom}_B(P^{**}, -)$. Thus, we can choose $j_1 = \mathbb{R}\text{Hom}_B(P^{**}, -)$. This finishes the proof of (3). \hfill \Box

**Lemma 5.5.** The following statements hold true:

1. $i_*i^!(Be_1) \simeq i_*i^!(Be_2)$ in $\mathcal{D}(B)$.
2. $H^n(i_*i^!(Be_1)) \simeq \begin{cases} \text{Tor}_n^R(T, S) \oplus \text{Tor}_n^R(T, S) & \text{if } n \leq 0, \\
0 & \text{if } n > 0, \end{cases}$
3. If $\lambda : R \to S$ is homological, then $i_*i^!(Be_1) \simeq Be_2 \otimes_R S$ in $\mathcal{D}(B)$.

**Proof.** We keep the notation introduced in Lemma 5.4.

1. Applying the triangle functor $i_*i^! : \mathcal{D}(B) \to \mathcal{D}(B)$ to the distinguished triangle:

$$P^\bullet[-1] \to Be_1 \xrightarrow{\varphi} Be_2 \to P^\bullet$$

in $\mathcal{D}(B)$, we obtain another distinguished triangle in $\mathcal{D}(B)$:

$$i_*i^!(P^\bullet)[-1] \to i_*i^!(Be_1) \xrightarrow{i_*i^!(\varphi)} i_*i^!(Be_2) \to i_*i^!(P^\bullet).$$

Since the composition functor $i^*_1 j_1 : \mathcal{D}(R) \to \mathcal{D}(B)_{C_{-Mod}}$ is zero in the recollement $\ast$, we clearly have $i^!(P^\bullet) \simeq i^*_1 j_1(R) = 0$. Thus $i_*i^!(\varphi) : i_*i^!(Be_1) \to i_*i^!(Be_2)$ is an isomorphism.
(2) First, we show that if $n > 0$ or $n < -1$, then

$$H^n(i_*i^*(Be_1)) \simeq \text{Tor}^R_{-n}(T,S) \oplus \text{Tor}^R_{-n}(T,S)$$

where $\text{Tor}^R_{-n}(T,S) := 0$ for $n > 0$.

In fact, let $\epsilon : j_! i^! \to \text{Id}_{\mathcal{D}(B)}$ and $\eta : \text{Id}_{\mathcal{D}(B)} \to i_* i^!$ be the counit and unit adjunctions with respect to the adjoint pairs $(j_!, j^!)$ and $(i^!, i_*)$ in the recollement $(\ast)$, respectively. Then, for any $X^* \in \mathcal{D}(B)$, there is a canonical chain map in $\mathcal{D}(B)$:

$$j_! j^!(X^*) \xrightarrow{\xi} X^* \xrightarrow{\eta} j_! j^!(X^*) \to j_! j^!(X^*)[1].$$

In particular, we have the following chain triangle in $\mathcal{D}(B)$:

$$j_! j^!(Be_1) \xrightarrow{\xi} Be_1 \xrightarrow{\eta} i_! i^!(Be_1) \to j_! j^!(Be_1)[1].$$

Note that $j^!(Be_1) = \text{Hom}_B(P^*, Be_1) \simeq S[-1]$ as complexes of $R$-modules. In the following, we always identify $\text{Hom}_B(P^*, Be_1)$ with $S[-1]$. Under this identification, we obtain the following triangle in $\mathcal{D}(B)$:

$$P^* \otimes_R S[-1] \xrightarrow{\xi} Be_1 \xrightarrow{\eta} i_! i^!(Be_1) \to P^* \otimes_R S.$$

Now, for each $n \in \mathbb{Z}$, we apply the $n$-th cohomology functor $H^n : \mathcal{D}(B) \to \text{B-Mod}$ to this triangle, and conclude that if $n > 0$ or $n < -1$, then $H^n(i_* i^*(Be_1)) \simeq H^n(P^* \otimes_R S)$. Moreover, we have

$$P^* = T \oplus (0 \to S \xrightarrow{m} M \to 0) = T \oplus \text{Con}(\cdot ; m) \in \mathcal{D}(\mathcal{R}^{op}).$$

Since $(\lambda, \mu, M, m)$ is an exact context, it follows from the diagram $(\ast)$ that the chain map $(\lambda, m) : \text{Con}(\mu) \to \text{Con}(\cdot ; m)$ is a quasi-isomorphism. This implies that

$$\text{Con}(\cdot ; m) \simeq \text{Con}(\mu) \text{ in } \mathcal{D}(\mathcal{R}^{op}).$$

Thus $P^* \simeq T \oplus \text{Con}(\mu)$ in $\mathcal{D}(\mathcal{R}^{op})$ and $P^* \otimes_R S \simeq (T \otimes_R S) \oplus (\text{Con}(\mu) \otimes_R S)$ in $\mathcal{D}(\mathbb{Z})$. In particular, we have

$$H^n(P^* \otimes_R S) \simeq H^n(T \otimes_R S) \oplus H^n(\text{Con}(\mu) \otimes_R S)$$

for all $n \in \mathbb{Z}$. Applying the functor $- \otimes_R S$ to the canonical triangle

$$R \xrightarrow{\mu} T \xrightarrow{\eta} \text{Con}(\mu) \xrightarrow{\varepsilon} R[1]$$

in $\mathcal{D}(\mathcal{R}^{op})$, we obtain another triangle $S \to T \otimes_R S \to \text{Con}(\mu) \otimes_R S \to S[1]$ in $\mathcal{D}(\mathbb{Z})$. This implies that if $n > 0$ or $n < -1$, then $H^n(T \otimes_R S) \simeq H^n(\text{Con}(\mu) \otimes_R S)$, and therefore

$$H^n(i_* i^*(Be_1)) \simeq H^n(P^* \otimes_R S) \simeq H^n(T \otimes_R S) \oplus H^n(T \otimes_R S) \simeq \text{Tor}^R_{-n}(T,S) \oplus \text{Tor}^R_{-n}(T,S).$$

Next, we shall show that $H^{-1}(i_* i^*(Be_1)) \simeq \text{Tor}^R_{-1}(T,S) \oplus \text{Tor}^R_{-1}(T,S)$.

Indeed, we have the following two homomorphisms:

$$\sigma : S \otimes_R S \to S, \quad s_1 \otimes s_2 \mapsto s_1 s_2,$n
$$\varphi_1 : S \otimes_R S \to M \otimes_R S, \quad s_1 \otimes s_2 \mapsto s_1 m \otimes s_2$$

for $s_1, s_2 \in S$, and can identify $Be_1 \otimes_R S$ and $Be_2 \otimes_R S$ with $\begin{pmatrix} S & S \otimes S \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} M \otimes_R S \\ T \otimes_R S \end{pmatrix}$ as $B$-modules, respectively. Then there is a chain map in $\mathcal{C}(B)$:

$$P^* \otimes_R S[-1] : \xymatrix{ 0 \ar[r] & (S \otimes_R S) \ar[r]^-{\varphi_1} & (M \otimes_R S) \ar[r] & 0 \\ Be_1 : \xymatrix{ 0 \ar[r] & (S \otimes R S) \ar[r]^-{\varphi_1} & 0 \ar[r] & 0 } }$$

\vspace{1cm}

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Let $\rho S$ be a deleted projective resolution of the module $\rho S$ with $\tau : \rho S \to S$ a quasi-isomorphism. Recall that $j^!(Be_1) = \text{Hom}_B(P^*, Be_1) = S[-1]$. Then the counit $\varepsilon_{Be_1} : j_i j^!(Be_1) \to Be_1$ is just the composite of the following homomorphisms:

$$j_i j^!(Be_1) = P^* \otimes^L_R \text{Hom}_B^*(P^*, Be_1) = P^* \otimes^L_R (\rho S)[-1] \xrightarrow{(1 \otimes \tau[-1])} P^* \otimes^L_R S[-1] \xrightarrow{\varepsilon^*} Be_1.$$ 

Further, let $h^*$ be the following chain map:

$$\begin{array}{ccc}
P^* : & 0 & \to \begin{pmatrix} S \\ 0 \end{pmatrix} \xrightarrow{(\sigma)} \begin{pmatrix} M \\ T \end{pmatrix} \to 0 \\
h^* & \downarrow & \\
Be_1[1] : & 0 & \to \begin{pmatrix} S \\ 0 \end{pmatrix} \xrightarrow{0} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to 0
\end{array}$$

Then we have a commutative diagram:

$$\begin{array}{ccc}
P^* \otimes^L_R (\rho S)[-1] & \xrightarrow{(1 \otimes \tau[-1])} & P^* \otimes^L_R S[-1] \\
\downarrow & \uparrow & \downarrow \\
Be_1[1] \otimes^L_R (\rho S)[-1] & \xrightarrow{(1 \otimes \tau[-1])} & Be_1[1] \otimes^L_R S[-1] \\
\downarrow & \uparrow & \downarrow \\
0 & \xrightarrow{0} & 0 \\
\end{array}$$

This implies that the following diagram

$$(**) \hspace{1cm} P^* \otimes^L_R \text{Hom}_B(P^*, Be_1) \xrightarrow{h^* \otimes^L_1} Be_1[1] \otimes^L_R \text{Hom}_B(P^*, Be_1)$$

$$\hspace{1cm} \xrightarrow{(1 \otimes \tau[-1])} \begin{pmatrix} S \\ 0 \end{pmatrix}$$

is commutative in $\mathcal{D}(B)$. Since we have the following distinguished triangle

$$Be_2 \to P^* \xrightarrow{h^*} Be_1[1] \xrightarrow{\eta_{Be_1}} Be_2[1]$$

in $\mathcal{K}(B \otimes^L_R R^\otimes)$, there is a homomorphism

$$\xi : Be_2 \otimes^L_R \text{Hom}_B(P^*, Be_1)[1] \to i_* i^*(Be_1)$$

in $\mathcal{D}(B)$ and a complex $W \in \mathcal{D}(B)$ such that $**$ is completed to the following commutative diagram:

$$\begin{array}{ccc}
P^* \otimes^L_R \text{Hom}_B(P^*, Be_1) & \xrightarrow{h^* \otimes^L_1} & Be_1[1] \otimes^L_R \text{Hom}_B(P^*, Be_1) \\
\downarrow & \uparrow & \downarrow \\
W[1] & \xrightarrow{W} & W[1] \\
\end{array}$$

with rows and columns being distinguished triangles in $\mathcal{D}(B)$. Note that such a homomorphism $\xi$ is unique. In fact, this follows from

$$\text{Hom}_{\mathcal{D}(B)}(P^*[1] \otimes^L_R \text{Hom}_B(P^*, Be_1), i_* i^*(Be_1)) = \text{Hom}_{\mathcal{D}(B)}(j_!(S), i_* i^*(Be_1)) = 0.$$
Now, we obtain the following triangle in \( \mathcal{D}(B) \):
\[
W \xrightarrow{\psi} Be_2 \otimes_R S \xrightarrow{\xi} i_lei^*(Be_1) \to W[1]
\]
where \( \psi := \zeta(\varphi[1] \otimes \mathbb{L}) \). This yields a long exact sequence of abelian groups:
\[
H^{-1}(W) \xrightarrow{H^{-1}(\psi)} H^{-1}(Be_2 \otimes_R S) \xrightarrow{H^{-1}(\xi)} H^{-1}(i_lei^*(Be_1)) \to H^0(W) \xrightarrow{H^0(\psi)} H^0(Be_2 \otimes_R S)
\]
In the sequel, we show that the map \( H^0(\psi) : H^0(W) \to H^0(Be_2 \otimes_R S) \) is always injective. Note that \( H^0(\psi) \) is the composite of \( H^0(\xi) : H^0(W) \to H^0(Be_1 \otimes_R S) \) with
\[
H^0(\varphi[1] \otimes \mathbb{L}^-1) : H^0(Be_1 \otimes_R S) \to H^0(Be_2 \otimes_R S),
\]
and that \( H^0(Be_1 \otimes_R S) = Be_1 \otimes_R S = S \otimes_R S \) and \( H^0(Be_2 \otimes_R S) = Be_2 \otimes_R S \). On the one hand, applying the functor \( H^0 \) to the triangle
\[
W \xrightarrow{\zeta} Be_1 \otimes_R S \xrightarrow{(1 \otimes \mathbb{L}^-1)(\cdot)} Be_1 \to W[1],
\]
we obtain a short exact sequence
\[
0 \to H^0(W) \xrightarrow{H^0(\xi)} S \otimes_R S \xrightarrow{\sigma} S \to 0,
\]
where \( Be_1 \) is identified with \( S \). This implies that \( H^0(\xi) : H^0(W) \xrightarrow{\cong} \ker(\sigma) \). On the other hand, we can identify \( H^0(\xi) : H^0(W) \xrightarrow{\cong} \ker(\sigma) \) with the map \( \varphi \otimes_R S : Be_1 \otimes_R S \to Be_2 \otimes_R S \) induced from \( \varphi : Be_1 \to Be_2 \). Consequently, \( H^0(\psi) \) is the composite of \( H^0(\varphi) : H^0(W) \to S \otimes_R S \) with \( \varphi \otimes_R S : Be_1 \otimes_R S \to Be_2 \otimes_R S \). Thus \( H^0(\psi) \) is injective if and only if so is the restriction of \( \varphi \otimes_R S \) to \( \ker(\sigma) \), while the latter is also equivalent to saying that the restriction of the map \( \varphi_1 = (m) \otimes_R S : S \otimes_R S \to M \otimes_R S \) to \( \ker(\sigma) \) is injective. Hence, we need to show that \( \ker(\phi_1) \cap \ker(\sigma) = 0 \).

In fact, for the ring homomorphism \( \lambda : R \to S \), the sequence \( 0 \to \ker(\sigma) \to S \otimes_R S \to \sigma \to 0 \) always splits in the category of \( R \)-\( S \)-bimodules since the composite of \( \lambda \otimes_R S : R \otimes_R S \to S \otimes_R S \) with \( \sigma \) is an isomorphism of \( R \)-\( S \)-bimodules. It follows that \( \lambda \otimes_R S \) is injective, \( \text{Im}(\lambda \otimes_R S) \cap \ker(\sigma) = 0 \) and \( S \otimes_R S = \ker(\sigma) \oplus \text{Im}(\lambda \otimes_R S) \). Now, we apply the tensor functor \( \otimes_R S \) to the diagram \( (\xi) \), which is a push-out and pull-back diagram in the category of \( R \)-\( R \)-bimodules, and obtain another diagram

\[
\begin{array}{ccc}
R \otimes_R S & \xrightarrow{\lambda \otimes_R S} & S \otimes_R S \\
\mu \otimes_R S & & \\
T \otimes_R S & \xrightarrow{(m) \otimes_R S} & M \otimes_R S \\
\phi_1 & & \\
\end{array}
\]

which is a push-out and pull-back diagram in the category of \( R \)-\( S \)-bimodules. This implies that the map \( \lambda \otimes_R S \) induces an isomorphism from \( \ker(\mu \otimes_R S) \) to \( \ker(\phi_1) \). In particular, we have \( \ker(\phi_1) \subseteq \text{Im}(\lambda \otimes_R S) \).

It follows from \( \text{Im}(\lambda \otimes_R S) \cap \ker(\sigma) = 0 \) that \( \ker(\phi_1) \cap \ker(\sigma) = 0 \).

Thus \( H^0(\psi) : H^0(W) \to H^0(Be_2 \otimes_R S) \) is injective. Consequently, the map \( H^{-1}(\xi) \) is surjective and \( H^{-1}(i_lei^*(Be_1)) \cong \text{Coker}(H^{-1}(\psi)) \). Observe that \( H^{-1}(\psi) \) is the composite of the isomorphism \( H^{-1}(\xi) : H^{-1}(W) \xrightarrow{\cong} H^{-1}(Be_1 \otimes_R S) \) with the map
\[
H^{-1}(\varphi[1] \otimes \mathbb{L}^-1) : H^{-1}(Be_1 \otimes_R S) \to H^{-1}(Be_2 \otimes_R S).
\]
Therefore, we have
\[
H^{-1}(i_lei^*(Be_1)) \cong \text{Coker}(H^{-1}(\psi)) \cong \text{Coker}(H^{-1}(\varphi[1] \otimes \mathbb{L}^-1)).
\]
So, to show that \( H^{-1}(i_lei^*(Be_1)) \cong \text{Tor}_1^R(T, S) \oplus \text{Tor}_1^R(T, S) \), it suffices to prove that
\[
\text{Coker}(H^{-1}(\varphi[1] \otimes \mathbb{L}^-1)) \cong \text{Tor}_1^R(T, S) \oplus \text{Tor}_1^R(T, S).
\]
Recall that $Be_1 = S$, $Be_2 = M \oplus T$ and $\varphi = (-m, 0) : S \to M \oplus T$ in $R^{op}$-Mod. Moreover, we have $H^{-1}(Be_1 \otimes_R^L S) = \text{Tor}^R_1(S, S)$ and $H^{-1}(Be_2 \otimes_R^L S) = \text{Tor}^R_1(M \oplus T, S)$. In this sense, the map $H^{-1}(\varphi[1] \otimes^L 1)$ is actually given by

$$(\text{Tor}^R_1(-m, S), 0) : \text{Tor}^R_1(S, S) \to \text{Tor}^R_1(M, S) \oplus \text{Tor}^R_1(T, S)$$

Thus Coker $(H^{-1}(\varphi[1] \otimes^L 1)) \simeq \text{Coker}(\text{Tor}^R_1(-m, S)) \oplus \text{Tor}^R_1(T, S)$.

Finally, we show that Coker $(\text{Tor}^R_1(-m, S)) \simeq \text{Tor}^R_1(T, S)$.

In fact, since the quadruple $(\lambda, \mu, M, m)$ is an exact context, we have the following exact sequence of $R$-$R$-bimodules:

$$0 \to R \xrightarrow{(\lambda, \mu)} S \oplus T \xrightarrow{(m, -m)} M \to 0.$$

Applying $\text{Tor}^R_1(-, S)$ for $i = 0, 1$ to this sequence, we obtain a long exact sequence of abelian groups:

$$0 = \text{Tor}^R_1(R, S) \to \text{Tor}^R_1(S, S) \oplus \text{Tor}^R_1(T, S) \xrightarrow{(\text{Tor}^R_1(m, S), -\text{Tor}^R_1(m, S))} \text{Tor}^R_1(M, S) \to R \otimes R S \xrightarrow{(\lambda \otimes_R S, \mu \otimes_R S)} S \otimes_R S \otimes_R T.$$

Since $\lambda \otimes_R S : R \otimes_R S \to S \otimes_R S$ is injective, the map

$$(\text{Tor}^R_1(-m, S), -\text{Tor}^R_1(m, S)) : \text{Tor}^R_1(S, S) \oplus \text{Tor}^R_1(T, S) \to \text{Tor}^R_1(M, S)$$

is an isomorphism, which gives rise to Coker $(\text{Tor}^R_1(-m, S)) \simeq \text{Tor}^R_1(T, S)$. It follows that

$H^{-1}(i_* i^*(Be_1)) \simeq \text{Coker}(H^{-1}(\varphi[1] \otimes^L 1)) \simeq \text{Coker}(\text{Tor}^R_1(-m, S)) \oplus \text{Tor}^R_1(T, S) \simeq \text{Tor}^R(T, S) \oplus \text{Tor}^R_1(T, S)$.

Hence, we have shown that $H^n(i_* i^*(Be_1)) \simeq \text{Tor}^R_n(T, S) \oplus \text{Tor}^R_n(T, T)$ for any $n \in \mathbb{Z}$. This finishes the proof of (2).

(3) Note that if $\lambda$ is homological, then both $1 \otimes \tau[-1] : Be_1[1] \otimes^L_R \text{Hom}_B(P^*, Be_1) \to Be_1[1] \otimes_R S[-1]$ and $\sigma : S \otimes_R S \to S$ are isomorphisms. This implies that the morphism

$$(1 \otimes \tau[-1]) \begin{pmatrix} \sigma \\ 0 \end{pmatrix} : Be_1[1] \otimes^L_R \text{Hom}_B(P^*, Be_1) \to Be_1$$

is an isomorphism. Thus $W \simeq 0$ and $i_* i^*(Be_1) \simeq Be_2 \otimes_R S$ in $\mathcal{D}(B)$. This shows (3). □

For exact pairs, we establish the following result which will be used in the proof of Theorem 1.1.

**Lemma 5.6.** Suppose that $(\lambda, \mu)$ is an exact pair and that $\lambda$ is homological. Then:

(1) $\text{Tor}^R_i(S, T) = 0$ for all $i > 0$.

(2) Given a commutative diagram of ring homomorphisms:

$$\begin{array}{ccc}
R & \xrightarrow{\lambda} & S \\
\mu \downarrow & & \downarrow f \\
T & \xrightarrow{g} & \Gamma,
\end{array}$$

the following statements are equivalent:

(a) The ring homomorphism $g : T \to \Gamma$ is homological.

(b) The ring homomorphism

$$\theta_{f, g} : B \to M_2(\Gamma), \quad \begin{pmatrix} s_1 & s_2 \otimes t_2 \\ 0 & t_1 \end{pmatrix} \mapsto \begin{pmatrix} (s_1) f & (s_2) f(t_2) g \\ 0 & (t_1) g \end{pmatrix}, \quad s_i \in S, t_i \in T, i = 1, 2$$

is homological.
Proof. (1) Let $Q^*$ be the mapping cone of $\lambda$. Then there is a distinguished triangle in $\mathcal{D}(R)$:

$$R \xrightarrow{\lambda} S \rightarrow Q^* \rightarrow R[1].$$

Since $\lambda$ is homological, it follows from [15, Theorem 4.4] that $\lambda$ induces the following isomorphisms

$$S \xrightarrow{\sim} S \otimes_R L \xrightarrow{S \otimes_R A} S \otimes_R S$$

in $\mathcal{D}(S)$. This implies that $S \otimes_R L Q^* = 0$ in $\mathcal{D}(S)$, and therefore $S \otimes_R L Q^* = 0$ in $\mathcal{D}(R)$. Let $\mu^* := (\mu^i)_{i \in \mathbb{Z}}$ be the chain map defined by $\mu_1 := \mu$, $\mu^0 := \mu^i$ and $\mu^i = 0$ for $i \neq -1, 0$. Since $(\lambda, \mu)$ is an exact pair, we see that $\mu^* : Q^* \rightarrow Q^* \otimes_R T$ is an isomorphism in $\mathcal{D}(R)$. It follows that $S \otimes_R (Q^* \otimes_R T) \simeq S \otimes_R L Q^* = 0$ in $\mathcal{D}(S)$. Now, applying $S \otimes_R$ to the triangle $T \xrightarrow{\lambda} S \otimes_R T \rightarrow Q^* \otimes_R T \rightarrow T[1]$, we obtain $S \otimes_R T \simeq S \otimes_R (S \otimes_R T)$ in $\mathcal{D}(S)$ (and also in $\mathcal{D}(R)$). This yields that $\text{Tor}^R_i(S, T) \simeq \text{Tor}^R_i(S, S \otimes_R T)$ for all $i \geq 0$. As $S \otimes_R T$ is a left $S$-module and $\lambda$ is homological, it follows that $\text{Tor}^R_i(S, S \otimes_R T) = \text{Tor}^R_i(S, S \otimes_R T) = 0$ for all $i > 0$, and therefore $\text{Tor}^2(S, T) = 0$. This shows (1).

(2) Set $\Lambda := M_2(\Gamma)$. Let $e_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in B$, and let $e := (e_2)\theta_{f,g} \in \Lambda$. Then we have $e = e^2$, $\text{End}_\Lambda(\Lambda e) \simeq \Gamma$ and $\text{End}_B(\Lambda e) \simeq T$. Observe that $\Lambda e$ is a projective generator for $\Lambda\text{-Mod}$. Then, by Morita theory, the tensor functor $e\Lambda \otimes_A : \Lambda\text{-Mod} \rightarrow \Gamma\text{-Mod}$ is an equivalence of module categories, which can be canonically extended to a triangle equivalence $D(e\Lambda \otimes_A -) : \mathcal{D}(\Lambda) \rightarrow \mathcal{D}(\Gamma)$.

It is clear that $e_2 B \otimes_B \Lambda \simeq e_2 \cdot \Lambda = e\Lambda$ as $T\cdot\Lambda$-bimodules, where the left $T$-module structure of $e\Lambda$ is induced by $g : T \rightarrow \Gamma$. Thus the following diagram of functors between module categories

$$\begin{array}{c}
\Lambda\text{-Mod} \\
\xrightarrow{e\Lambda \otimes_A -} \\
\Gamma\text{-Mod}
\end{array}
\xrightarrow{\left(\theta_{f,g}\right)_*} \\
\begin{array}{c}
\Lambda\text{-Mod} \\
\xrightarrow{e_2 B \otimes_B -} \\
T\text{-Mod}
\end{array}
$$

is commutative, where $(\theta_{f,g})_*$ and $g_*$ stand for the restriction functors induced by the ring homomorphisms $\theta_{f,g}$ and $g$, respectively. Since all of the functors appearing in the diagram are exact, we can pass to derived module categories and get the following commutative diagram of functors between derived module categories:

$$\begin{array}{ccc}
\mathcal{D}(\Lambda) & \xrightarrow{D(e\Lambda \otimes_A -)} & \mathcal{D}(\Gamma) \\
\downarrow{D((\theta_{f,g})_*)} & & \downarrow{D(g_*)} \\
\mathcal{D}(B) & \xrightarrow{D(e_2 B \otimes_B -)} & \mathcal{D}(T)
\end{array}
$$

where the functor $D(e\Lambda \otimes_A -)$ in the upper row is a triangle equivalence.

Note that $\theta_{f,g} : B \rightarrow \Lambda$ (respectively, $g : T \rightarrow \Gamma$) is homological if and only if the functor $D((\theta_{f,g})_*)$ (respectively, $D(g_*)$) is fully faithful. This means that, to prove that (a) and (b) are equivalent, it is necessary to establish some further connection between $D((\theta_{f,g})_*)$ and $D(g_*)$ in the diagram (†).

Actually, the triangle functor $D(e_2 B \otimes_B -)$ induces a triangle equivalence from $\text{Tri}a(Be_2) \rightarrow \mathcal{D}(T)$. This can be obtained from the following classical recollement of derived module categories:

$$\begin{array}{cc}
\mathcal{D}(S) & \xrightarrow{S \otimes_R L} \\
\downarrow{\mathcal{D}(B)} & \xrightarrow{D(e_2 B \otimes_B -)} & \mathcal{D}(T)
\end{array}
$$

which arises from the triangular structure of the ring $B$. 

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Suppose that the image \( \text{Im}(D((\theta_{f,g})_*)) \) of the functor \( D((\theta_{f,g})_*) \) belongs to \( \text{Tria}(Be_2) \). Then we can strengthen the diagram (\( \dagger \)) by the following commutative diagram of functors between triangulated categories:

\[
\begin{array}{ccc}
D((\theta_{f,g})_*) & \xrightarrow{D(e\Lambda \otimes \Lambda -)} & D(\Gamma) \\
\mathcal{D}(\Lambda) & \xrightarrow{\mathcal{D}(\theta_{f,g})_*)} & \mathcal{D}(\Gamma) \\
\mathcal{D}(B) & \xrightarrow{\text{Tria}(Be_2)} & \mathcal{D}(T)
\end{array}
\]

This implies that \( D((\theta_{f,g})_*) \) is fully faithful if and only if so is \( D(g_*) \), and therefore \( \theta_{f,g} \) is homological if and only if \( g \) is homological.

So, to finish the proof of Lemma 5.5.1(2), it suffices to prove that \( \text{Im}(D((\theta_{f,g})_*)) \subseteq \text{Tria}(Be_2) \). In the following, we shall concentrate on proving this inclusion.

In fact, it is known that \( \mathcal{D}(\Lambda) = \text{Tria}(\Lambda e) \) and \( D((\theta_{f,g})_*) \) commutes with small coproducts since it admits a right adjoint. Therefore, according to the property (2) in Section 2.1 in order to check the above inclusion, it is enough to prove that \( \Lambda e \in \text{Tria}(Be_2) \) when considered as a \( B \)-module via \( \theta_{f,g} \). If we identify \( e_2 B \otimes - \) with the left multiplication functor by \( e_2 \), then \( \Lambda e \in \text{Tria}(Be_2) \) if and only if \( Be_2 \otimes \Lambda e \xrightarrow{\sim} \Lambda e \) in \( \mathcal{D}(B) \). Clearly, the latter is equivalent to that \( \text{Tor}_n^T(Be_2, e_2 \cdot (\Lambda e)) = 0 \) for any \( n > 0 \) and the canonical multiplication \( Be_2 \otimes_T e_2 \cdot (\Lambda e) \rightarrow \Lambda e \) is an isomorphism.

Set \( M := S \otimes_R T \) and write \( B \)-modules in the form of triples \((X, Y, h)\) with \( X \in T\text{-Mod}, Y \in S\text{-Mod} \) and \( h : M \otimes_T X \rightarrow Y \) a homomorphism of \( S \)-modules. The morphisms between two modules \((X, Y, h)\) and \((X', Y', h')\) are pairs of morphisms \((\alpha, \beta)\), where \( \alpha : X \rightarrow X' \) and \( \beta : Y \rightarrow Y' \) are homomorphisms in \( T\text{-Mod} \) and \( S\text{-Mod} \), respectively, such that \( \beta h = (M \otimes_T \alpha)h' \).

With these interpretations, we rewrite \( \Lambda e = (\Gamma, \Gamma, \delta_T) \in B\text{-Mod} \), where \( \delta_T : M \otimes_T \Gamma \rightarrow \Gamma \) is defined by \( (s \otimes t) \otimes \gamma \mapsto (s)f(t)g\gamma \) for \( s \in S, t \in T \) and \( \gamma \in \Gamma \). Then \( e_2 \cdot (\Lambda e) = e\Lambda e \simeq \Gamma \) as left \( T \)-modules, and \( Be_2 \simeq M \otimes_T \Gamma \) as right \( T \)-modules. Consequently, we have

\[
Be_2 \otimes_T e_2 \cdot (\Lambda e) \simeq Be_2 \otimes_T \Gamma \simeq (\Gamma, M \otimes_T \Gamma, 1) \quad \text{and} \quad \text{Tor}_n^T(Be_2, e_2 \cdot (\Lambda e)) \simeq \text{Tor}_n^T(M \otimes T, \Gamma) \simeq \text{Tor}_n^T(M, \Gamma)
\]

for any \( n > 0 \). This implies that the multiplication map \( Be_2 \otimes_T e_2 \cdot (\Lambda e) \rightarrow \Lambda e \) is an isomorphism if and only if so is the map \( \delta_T \). It follows that \( Be_2 \otimes^L_T e_2 \cdot (\Lambda e) \simeq \Lambda e \) in \( \mathcal{D}(B) \) if and only if \( \delta_T \) is an isomorphism of \( S \)-modules and \( \text{Tor}_n^T(M, \Gamma) = 0 \) for any \( n > 0 \).

In order to verify the latter conditions just mentioned, we shall prove the following general result:

For any \( \Gamma \)-module \( W \), if we regard \( W \) as a left \( T \)-module via \( g \) and an \( S \)-module via \( f \), then the map \( \delta_W : M \otimes_T W \rightarrow W \), defined by \((s \otimes t) \otimes w \mapsto (s)f(t)g \gamma w \) for \( s \in S, t \in T \) and \( w \in W \), is an isomorphism of \( S \)-modules, and \( \text{Tor}_i^T(M, W) = 0 \) for any \( i > 0 \).

To prove this general result, we fix a projective resolution \( V^* \) of \( S_R \):

\[
\cdots \rightarrow V^n \rightarrow V^{n-1} \rightarrow \cdots \rightarrow V^1 \rightarrow V^0 \rightarrow S_R \rightarrow 0
\]

with \( V^i \) a projective right \( R \)-module for each \( i \). By (1), we have \( \text{Tor}_j^R(S, T) = 0 \) for any \( j > 0 \). It follows that the complex \( V^\bullet \otimes_T T \) is a projective resolution of the right \( T \)-module \( M \). Thus the following isomorphisms of complexes of abelian groups:

\[
(V^\bullet \otimes_T T)\otimes_T W \simeq V^\bullet \otimes_R (T \otimes_T W) \simeq V^\bullet \otimes_R W
\]

imply that \( \text{Tor}_i^T(M, W) \simeq \text{Tor}_i^R(S, W) \) for any \( i > 0 \). Recall that \( W \) admits an \( S \)-module structure via the map \( f \). Moreover, it follows from \( \lambda f = \mu g \) that the \( R \)-module structure of \( W \) endowed via the ring homomorphism \( \mu g \) is the same as the one endowed via the ring homomorphism \( \lambda f \). Then, it follows from \( \lambda \) being a homological ring epimorphism that the multiplication map \( S \otimes_R W \rightarrow W \) is an isomorphism of \( S \)-modules and that \( \text{Tor}_i^R(S, W) = 0 \) for all \( i > 0 \) (see [15], Theorem 4.4)). Therefore, for any \( i > 0 \), we have \( \text{Tor}_i^T(M, W) \simeq \text{Tor}_i^R(S, W) = 0 \). Note that

\[
M \otimes_T W = (S \otimes_T T) \otimes_T W \simeq S \otimes_R (T \otimes_T W) \simeq S \otimes_R W \simeq W
\]
as $S$-modules. Thus the map $\delta_W$ is an isomorphism of $S$-modules. So the above-mentioned general result follows.

Now, by applying the above general result to the ring $\Gamma$, we can show that $\delta_T$ is an isomorphism and $\text{Tor}^T_n(M, \Gamma) = 0$ for any $n > 0$. This completes the proof of Lemma 5.6 (2). □

**Proof of Theorem 1.1 (1)**

Let $(\lambda, \mu, M, m)$ be a given exact context, where $\lambda: R \to S$ and $\mu: R \to T$ are ring homomorphisms. By Lemma 5.3, the map $\theta$ is homological if and only if $H^n(\lambda^* (B)) = 0$ for all $n \neq 0$. However, by Lemma 5.5, we see that $H^n(\lambda^* (B)) \simeq H^n(\lambda^* (B_{e_1})) \oplus H^n(\lambda^* (B_{e_1})) \simeq \bigoplus_{i=1}^4 \text{Tor}^R_{-n}(T, S)$ for each $n \in \mathbb{Z}$. Thus (a) and (b) are equivalent. This shows the first part of Theorem 1.1 (1).

Assume that $(\lambda, \mu)$ is an exact pair such that $\lambda$ is homological. Let $\Lambda := T \otimes_R S$ be the noncommutative tensor product of $(\lambda, \mu, M, m)$ (see Lemma 4.4), and $C := M_2(\Lambda)$. Note that we have the following commutative diagram of ring homomorphisms:

$$
\begin{array}{ccc}
R & \xrightarrow{\lambda} & S \\
\mu \downarrow & & \downarrow \\
T & \xrightarrow{\theta} & \Lambda
\end{array}
$$

and that the map $\theta_{p, \phi}$ defined in Lemma 5.6 (b) is equal to $\theta: B \to C$ by Remark 5.2 (1). It follows from Lemma 5.6 (2) that the statements (a) and (c) in Theorem 1.1 are equivalent. This finishes the proof of Theorem 1.1 (1). □

Combining Theorem 1.1 (1) with Lemmas 5.3 and 5.4 (3), we have the following result.

**Corollary 5.7.** If one of the assertions in Theorem 1.1 (1) holds, then there is a recollement of derived module categories:

$$
\begin{array}{ccc}
\mathcal{D}(C) & \xrightarrow{\theta} & \mathcal{D}(B) \\
\mathcal{D}(R) & \xleftarrow{j_i} & \mathcal{D}(R)
\end{array}
$$

where $D(\theta_\ast)$ is the restriction functor induced by $\theta: B \to C$, and where

$$j_i = B \mathcal{P} \otimes_R \mathcal{L} \quad \text{and} \quad j^i = \text{Hom}_B(\mathcal{P}^\ast, -) \simeq R \mathcal{P} \otimes \mathcal{L} -.
$$

To prove Theorem 1.1 (2), we first establish the following result which describes relationships among projective dimensions of special modules over different rings. For an $R$-module $X$, we denote the projective dimension of $X$ by $\text{proj.dim}_R(X)$.

**Corollary 5.8.** Assume that one of the assertions in Theorem 1.1 (1) holds. Then we have the following:

1. $\text{proj.dim}_R(S) \leq \max \{1, \text{proj.dim}(\beta C)\}$ and $\text{proj.dim}(\beta C) \leq \max \{2, \text{proj.dim}(R S) + 1\}$. In particular, $\text{proj.dim}(R S) < \infty$ if and only if $\text{proj.dim}(\beta C) < \infty$.

2. $\text{proj.dim}(T R) \leq \max \{1, \text{proj.dim}(C B)\}$ and $\text{proj.dim}(C B) \leq \max \{2, \text{proj.dim}(T R) + 1\}$. In particular, $\text{proj.dim}(T R) < \infty$ if and only if $\text{proj.dim}(C B) < \infty$.

**Proof.** Note that the ring homomorphisms $\rho^{op}: R^{op} \to T^{op}$ and $\lambda^{op}: R^{op} \to S^{op}$, together with $(M, m)$ form an exact context. So it is sufficient to show (1) because (2) can be shown similarly.

We first show that $\text{proj.dim}_R(S) \leq \max \{1, \text{proj.dim}(\beta C)\}$.

To see this inequality, we use the recollement given in Corollary 5.7. Clearly, there is a triangle in $\mathcal{D}(B)$:

$$P^\ast \otimes_R \mathcal{P} \to B \xrightarrow{\theta} C \to P^\ast \otimes_R \mathcal{P} \otimes \mathcal{L} [1].$$

This implies that $\text{Con}(\theta)$ is isomorphic in $\mathcal{D}(B)$ to the complex $P^\ast \otimes_R \mathcal{P} \otimes \mathcal{L} [1]$. Since $\text{Con}(\lambda) \simeq \text{Con}(m \ast)$ in $\mathcal{D}(R)$ by the diagram (2) in Section 3, we see that $P^\ast \otimes \mathcal{P} \ast \simeq S \oplus \text{Con}(m \ast) \simeq S \oplus \text{Con}(\lambda)$ in $\mathcal{D}(R)$, and therefore

$$\text{Con}(\lambda) \simeq P^\ast \otimes_R S \oplus P \otimes_R \text{Con}(\lambda) \quad \text{in} \quad \mathcal{D}(B).$$

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As $P^* \otimes_{R}^L \mathcal{D}(R) \rightarrow \mathcal{D}(B)$ is fully faithful, we have

$$\text{Hom}_{\mathcal{D}(R)}(S,Y[n]) \simeq \text{Hom}_{\mathcal{D}(B)}(P^* \otimes_{R}^L S, P^* \otimes_{R}^L Y[n])$$

for every $Y \in R\text{-Mod}$ and $n \in \mathbb{N}$.

Suppose that $\text{proj.dim}(B) < \infty$, and let $s := \max \{1, \text{proj.dim}(B)\}$. We claim that $\text{Hom}_{\mathcal{D}(R)}(S,Y[n]) = 0$ for any $n > s$, and therefore $\text{proj.dim}(R) \leq s$. Since $P^* \otimes_{R}^L S$ is a direct summand of $\text{Con}(\theta)$ in $\mathcal{D}(B)$, it is enough to show that $\text{Hom}_{\mathcal{D}(B)}(\text{Con}(\theta), P^* \otimes_{R}^L Y[n]) = 0$ for any $n > s$.

Recall that $\text{Con}(\theta)$ is the complex $0 \rightarrow B \rightarrow C \rightarrow 0$ with $B$ and $C$ in degrees $-1$ and $0$, respectively. Then $\text{Con}(\theta)$ is isomorphic in $\mathcal{D}(B)$ to a bounded complex

$$X^* : 0 \rightarrow X^{-s} \rightarrow X^{1-s} \rightarrow \cdots \rightarrow X^{-1} \rightarrow X^0 \rightarrow 0$$

such that $X^i$ are projective $B$-modules for all $0 \leq i \leq s$. Let $\rho Y$ be a deleted projective resolution of $Y$ in $R\text{-Mod}$. Then $\text{Hom}_{\mathcal{D}(B)}(\text{Con}(\theta), P^* \otimes_{R}^L Y[n]) \simeq \text{Hom}_{\mathcal{D}(B)}(X^*, P^* \otimes_{R}^L Y[n]) = \text{Hom}_{\mathcal{D}(B)}(X^*, P^* \otimes_{R}^L (\rho Y)[n]) \simeq \text{Hom}_{\mathcal{D}(B)}(X^*, P^* \otimes_{R}^L \rho Y[n]) = 0$ for any $n > s$, where the last equality is due to the observation that all positive terms of the complex $P^* \otimes_{R}^L (\rho Y)$ are zero. This verifies the claim and shows that $\text{proj.dim}(R) \leq s$.

Next, we show that $\text{proj.dim}(B) \leq \max \{2, \text{proj.dim}(R) + 1\}$.

Suppose that $\text{proj.dim}(R) = m < \infty$, and let

$$M^* : 0 \rightarrow M^{-m} \rightarrow M^{1-m} \rightarrow \cdots \rightarrow M^{-1} \rightarrow M^0 \rightarrow 0$$

be a deleted projective resolution of $\rho S$, where $M^i$ are projective $R$-modules for all $-m \leq i \leq 0$. Then $P^* \otimes_{R}^L S = P^* \otimes_{R}^L M^* \text{ in } \mathcal{D}(B)$. Note that $\text{Con}(\lambda)$ is isomorphic in $\mathcal{D}(R)$ to a complex of the form:

$$\widetilde{M}^* : 0 \rightarrow M^{-m} \rightarrow M^{1-m} \rightarrow \cdots \rightarrow M^{-1} \oplus R \rightarrow M^0 \rightarrow 0.$$

In particular, $\widetilde{M}^i = 0$ for $i > 0$ or $i < - \max \{1, m\}$. Then $P^* \otimes_{R}^L \text{Con}(\lambda) = P^* \otimes_{R}^L \widetilde{M}^*$ in $\mathcal{D}(B)$. Thus

$$\text{Con}(\theta) \simeq (P^* \otimes_{R}^L M^*) \oplus (P^* \otimes_{R}^L \widetilde{M}^*) \text{ in } \mathcal{D}(B).$$

Recall that $P^*$ is the two-term complex $0 \rightarrow B e_1 \rightarrow B e_2 \rightarrow 0$ over $B$ with $B e_1$ and $B e_2$ in degrees $-1$ and $0$, respectively. This implies that $\text{Con}(\theta)$ is isomorphic in $\mathcal{D}(B)$ to a complex of the following form:

$$N^* : 0 \rightarrow N^{-t} \rightarrow N^{1-t} \rightarrow \cdots \rightarrow N^{-1} \rightarrow N^0 \rightarrow 0,$$

where $t := \max \{2, m + 1\}$ and $N^i$ are projective $B$-modules for all $-t \leq i \leq 0$. Now, let $X \in B\text{-Mod}$. Then

$$\text{Hom}_{\mathcal{D}(B)}(\text{Con}(\theta), X[n]) \simeq \text{Hom}_{\mathcal{D}(B)}(N^*, X[n]) \simeq \text{Hom}_{\mathcal{D}(B)}(N^*, X[n]) = 0$$

for any $n > t$. Applying $\text{Hom}_{\mathcal{D}(B)}(\cdot, X[n])$ to the canonical distinguished triangle

$$\text{Con}(\theta)[{-1}] \rightarrow B \rightarrow C \rightarrow \text{Con}(\theta)$$

in $\mathcal{D}(B)$, we have $\text{Ext}^{n}_{B}(C,X) \simeq \text{Hom}_{\mathcal{D}(B)}(C,X[n]) = 0$ for any $n > t$. This shows $\text{proj.dim}(R) \leq t$. \[\square\]

**Proof of Theorem 1.1 (2)**

Let $\Lambda := T \otimes_{R} S$ and $C := M_2(\Lambda)$. Then the $\Lambda$-$C$-bimodule $(\Lambda, \Lambda)$ induces an equivalence of module categories:

$$(\Lambda, \Lambda) \otimes_C - : C\text{-Mod} \rightarrow \Lambda\text{-Mod}.$$ 

In view of derived module categories, we obtain a triangle equivalence

$$(\Lambda, \Lambda) \otimes C - : \mathcal{D}(C) \xrightarrow{\simeq} \mathcal{D}(\Lambda).$$
Now, assume that one of the assertions in Theorem 1.1(1) holds. By the above equivalence, we know from Corollary 5.7 that there exists a recollement of derived module categories:

\[ \mathcal{D}(A) \xrightarrow{G} \mathcal{D}(B) \xrightarrow{j} \mathcal{D}(R) \]

where \( G := (\Lambda, \Lambda) \otimes_B \mathbb{L} \) and \( j_1 = B^* \otimes_R \mathbb{L} \). This shows the first part of Theorem 1.1(2).

By [26, Theorem 3], the recollement in Corollary 5.7 can be restricted to a recollement at \( \mathcal{D}^- \)-level:

\[ \mathcal{D}^-(C) \xrightarrow{D(\theta)_1} \mathcal{D}^-(B) \xrightarrow{j} \mathcal{D}^-(R) \]

if and only if the image of the object \( C \in \mathcal{D}(C) \) under the functor \( D(\theta)_1 \) is isomorphic to a bounded complex of projective \( B \)-modules, that is \( \text{proj.dim}(\mathcal{D}(C)) < \infty \). Furthermore, this \( \mathcal{D}^- \)-level recollement can be restricted to \( \mathcal{D}^b \)-level

\[ \mathcal{D}^b(C) \xrightarrow{D(\theta)_1} \mathcal{D}^b(B) \xrightarrow{j} \mathcal{D}^b(R) \]

provided that \( \text{proj.dim}(C_B) < \infty \). However, by Corollary 5.8 we see that \( \text{proj.dim}(\mathcal{D}(C)) < \infty \) if and only if \( \text{proj.dim}(\mathcal{D}(R)) = \infty \), and that \( \text{proj.dim}(\mathcal{D}(B)) = \infty \) if and only if \( \text{proj.dim}(\mathcal{D}(R)) = \infty \). Identifying \( \mathcal{D}^b(C) \) with \( \mathcal{D}^b(\Lambda) \) up to equivalence, we finish the proof of the second part of Theorem 1.1(2). \( \square \)

### 5.2 Proofs of Corollaries

In this section, we shall prove all corollaries of Theorem 1.1 which were mentioned in the introduction.

All notation introduced in the previous sections will be kept. As in Section 1, we fix a ring homomorphism \( \lambda : R \rightarrow S \), and let

\[ (**) \quad R \xrightarrow{\lambda} S \xrightarrow{\pi} Q^* \xrightarrow{\nu} R[1] \]

be the distinguished triangle in the homotopy category \( \mathcal{K}(R) \) of \( R \), where the complex \( Q^* \) stands for the mapping cone of \( \lambda \). Now, we set \( S' := \text{End}_{\mathcal{D}(R)}(Q^*) \) and define \( \lambda' : R \rightarrow S' \) by \( r \mapsto f^* \) for \( r \in R \), where \( f^* \) is the chain map with \( f^{-1} := r \), \( f^0 := \cdot (r) \lambda \) and \( f^i = 0 \) for \( i \neq 0, -1 \). Here, \( \cdot r \) and \( \cdot (r) \lambda \) stand for the right multiplication maps by \( r \) and \( (r) \lambda \), respectively. These data can be recorded in the following commutative diagram:

\[
\begin{array}{ccc}
R & \xrightarrow{\lambda} & S \\
\downarrow \cdot r & & \downarrow (r) \lambda \\
R & \xrightarrow{\lambda} & S \\
\end{array}
\]

The map \( \lambda' \) is called the ring homomorphism *associated to* \( \lambda \). If \( \lambda \) is injective, then we shall identify \( Q^* \) with \( S/R \) in \( \mathcal{D}(R) \), and further, identify \( \lambda' \) with the induced map \( R \rightarrow \text{End}_{\mathcal{D}}(S/R) \) by the right multiplication map.

Recall that \( \Lambda \) denotes the ring \( \text{End}_{\mathcal{D}(R)}(S \oplus Q^*) \) and that \( \pi^* \) is the induced map

\[ \text{Hom}_{\mathcal{D}(R)}(S \oplus Q^*, \pi) : \text{Hom}_{\mathcal{D}(R)}(S \oplus Q^*, S) \rightarrow \text{Hom}_{\mathcal{D}(R)}(S \oplus Q^*, Q^*) \]

Let \( \lambda_{\pi^*} : \Lambda \rightarrow \Lambda_{\pi^*} \) stand for the noncommutative localization of \( \Lambda \) at \( \pi^* \).

Note that \( \text{Hom}_{\mathcal{D}(R)}(S, Q^*) \) is an \( S \)-\( S' \)-bimodule containing \( \pi \). Now we define a homomorphism of \( S \)-\( S' \)-bimodules:

\[ \gamma : S \otimes_R S' \rightarrow \text{Hom}_{\mathcal{D}(R)}(S, Q^*), \quad s \otimes f \mapsto (s)(\pi f) \]
for \( s \in S \) and \( f \in S' \). This induces the following ring homomorphism:

\[
\tau := \left( \begin{array}{cc} \sigma & \gamma \\ 0 & 1 \end{array} \right) : \left( \begin{array}{c} S \\ 0 \end{array} \right) \otimes_{R} \left( \begin{array}{c} S' \\ 0 \end{array} \right) \longrightarrow \left( \begin{array}{c} \text{End}_{R}(S) \\ \text{Hom}_{\mathcal{D}(R)}(S, Q^*) \end{array} \right) \text{ for } S' \text{ is a quasi-isomorphism. This implies that all of their cohomologies are isomorphic. Note that } \sigma \text{ is exact. As a generalization of this result, we show the following statement in which } \lambda \text{ is not necessarily injective and } Q^* \text{ may have non-zero cohomology in two degrees.}

**Lemma 5.9.** Assume that \( \text{Hom}_{\mathcal{D}(R)}(S, \text{Ker}(\lambda)) = 0. \) Then the quadruple \((\lambda, \lambda', \text{Hom}_{\mathcal{D}(R)}(S, Q^*), \pi)\) is an exact context. If \( \lambda \) is additionally a ring epimorphism, then \((\lambda, \lambda')\) is an exact pair. In this case, both \( \gamma \) and \( \tau \) are isomorphisms.

**Proof.** Applying \( \text{Hom}_{\mathcal{D}(R)}(\cdot, Q^*) \) to the triangle \((**)\), we have the following long exact sequence:

\[
\text{Hom}_{\mathcal{D}(R)}(S, Q^*) \xrightarrow{\text{Hom}_{\mathcal{D}(R)}(\pi)} \text{Hom}_{\mathcal{D}(R)}(R, Q^*) \text{ is surjective, as desired.}
\]

Since \( \text{Hom}_{\mathcal{D}(R)}(S, \text{Ker}(\lambda)) = 0, \) the map \( \text{Hom}_{\mathcal{D}(R)}(S, \lambda) \) is injective. As \( \text{Hom}_{\mathcal{D}(R)}(S[1], S) \simeq \text{Hom}_{\mathcal{D}(R)}(S, S[-1]) \simeq \text{Ext}^{-1}(S, S) = 0, \) we obtain \( \text{Hom}_{\mathcal{D}(R)}(S[1], Q^*) = 0 \) by applying \( \text{Hom}_{\mathcal{D}(R)}(S[1], -) \) to the triangle \((**)\). Thus the above map \( \nu_* \) is injective.

Next, we show that \( \lambda_* = \text{Hom}_{\mathcal{D}(R)}(\lambda, Q^*) : \text{Hom}_{\mathcal{D}(R)}(S, Q^*) \longrightarrow \text{Hom}_{\mathcal{D}(R)}(R, Q^*) \) is surjective. In fact, the following diagram:

\[
\begin{array}{ccc}
\text{Hom}_{\mathcal{D}(R)}(S, Q^*) & \xrightarrow{\lambda_*} & \text{Hom}_{\mathcal{D}(R)}(R, Q^*) \\
\uparrow & & \uparrow \simeq \\
\text{Hom}_{\mathcal{D}(R)}(S, Q^*) & \xrightarrow{\text{Hom}_{\mathcal{D}(R)}(\lambda, Q^*)} & \text{Hom}_{\mathcal{D}(R)}(R, Q^*)
\end{array}
\]

is commutative, where the vertical maps are the canonical localization maps from \( \mathcal{D}(R) \) to \( \mathcal{D}(R) \). Since \( \text{Hom}_{\mathcal{D}(R)}(\lambda, S) : \text{Hom}_{\mathcal{D}(R)}(S, S) \longrightarrow \text{Hom}_{\mathcal{D}(R)}(R, S) \) is surjective, we see that \( \text{Hom}_{\mathcal{D}(R)}(\lambda, Q^*) \) is surjective. This implies that \( \lambda_* \) is surjective, as desired.

Let \( \pi : S \longrightarrow \text{Hom}_{\mathcal{D}(R)}(S, Q^*) \) stand for the right multiplication by \( \pi \) map. Then we have the following exact commutative diagram

\[
\begin{array}{c}
0 \longrightarrow \text{Ker}(\lambda) \longrightarrow \text{Ker}(\lambda) \longrightarrow R \\
\downarrow \simeq \downarrow \lambda' \downarrow \lambda \\
0 \longrightarrow \text{Hom}_{\mathcal{D}(R)}(R[1], Q^*) \longrightarrow \text{Ker}(\lambda) \longrightarrow S \longrightarrow \text{Coker}(\lambda) \longrightarrow 0
\end{array}
\]

where the isomorphisms follow from the fact that \( \text{Hom}_{\mathcal{D}(R)}(R[n], Q^*) \simeq H^{-n}(Q^*) \) for \( n \in \mathbb{Z} \). Consequently, the chain map

\[
(\lambda', \pi) : \text{Con}(\lambda) \longrightarrow \text{Con}(\pi_*)
\]

is a quasi-isomorphism. This implies that all of their cohomologies are isomorphic. Note that \( \pi_* \) is exactly the left multiplication by \( \pi \) map. Thus the square in the middle of the above diagram is a pull-back and push-out diagram, and therefore the quadruple \((\lambda, \lambda', \text{Hom}_{\mathcal{D}(R)}(S, Q^*), \pi)\) is an exact context. Assume that \( \lambda \) is a ring epimorphism. By Corollary 4.3, the map \( \gamma : S \otimes_{R} S' \longrightarrow \text{Hom}_{\mathcal{D}(R)}(S, Q^*) \) is an isomorphism of \( S-S' \)-bimodules. This implies that \((\lambda, \lambda')\) is an exact pair by Lemma 4.1. It remains to show that the ring homomorphism \( \tau \) is an isomorphism. In fact, since \( \lambda \) is a ring epimorphism, we have \( S \simeq \text{End}_{S}(S) = \text{End}_{R}(S) \) as rings. Thus \( \sigma : S \longrightarrow \text{End}_{R}(S) \) is an isomorphism. Note that the composite of \( \sigma \) with the valuation map

\[
\text{Hom}_{R}(\lambda, S) : \text{End}_{R}(S) \longrightarrow \text{Hom}_{R}(R, S) = S : f \longrightarrow (1)f \text{ for } f \in \text{End}_{R}(S)
\]

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which sends the map sending coincides with the identity map of . Note that is homological. In particular, if one of the above is homological if and only if the noncommutative localization is homological. Moreover, by Corollary the pair is an isomorphism of rings:

\[ \Lambda := \text{End}_{\mathcal{P}(R)}(S \oplus Q^*) \cong B := \left( \begin{array}{cc} S & S \otimes_R S' \\ 0 & S' \end{array} \right) \]

Lemma 5.9. Moreover, if then the quadruple is an exact context by Lemma 5.9 that the pair is exact. It follows from Lemma 5.1 that \( \lambda_\varphi \) coincides with \( \theta : B \to C := M_2(S \otimes_R S') \). This means that \( \lambda_\pi \) is homological if and only if \( \theta \) is homological. Note that \( S \otimes_R S' \simeq S \sqcup R S' \simeq \text{End}_R(S' \otimes_R S) \) as rings by Remark 5.2. Since \( \lambda \) is homological, Corollary 1.2 follows immediately from Theorem 1.1.

Remark 5.10. The equivalences of (1) and (3) in Corollary 1.2 can be obtained under a weaker assumption, instead of the ‘homological’ assumption on \( \lambda \). Precisely, we have the following result:

If \( \text{Hom}_R(S, \text{Ker}(\lambda)) = 0 \) and \( \text{End}_R(S) = \text{End}_R(S) \), then the map \( \lambda_\pi : \Lambda \to \Lambda_\pi \) is homological if and only if \( \text{Tor}^R_1(S', S) = 0 \) for each \( i \geq 1 \).

Proof. If \( \text{Hom}_R(S, \text{Ker}(\lambda)) = 0 \), then the quadruple \( (\lambda, \lambda', \text{Hom}_R(S, Q^*), \pi) \) is an exact context by Lemma 5.9. Moreover, if \( \text{End}_R(S) = \text{End}_R(S) \), then the map \( \text{Hom}_R(\lambda, S) : \text{Hom}_R(S, S) \to \text{Hom}_R(R, S) \) is an isomorphism, which leads to

\[ \text{Hom}_R(S, S) = 0 \quad \text{and} \quad \Lambda \cong \left( \begin{array}{c} S \\ 0 \end{array} \right) \text{Hom}_R(S, Q^*) \right). \]

Now, the above-mentioned result follows immediately from Lemma 5.1 and Theorem 1.1 (1).

Combining Corollary 1.2 with Lemma 2.2, we get the following criterion for \( \lambda_\pi \) to be homological.

Corollary 5.11. Let \( \Sigma \) be a set of homomorphisms between finitely generated projective \( R \)-modules. Suppose that \( \lambda_\Sigma : R \to R_\Sigma \) is homological such that \( \text{Hom}_R(R_\Sigma, \text{Ker}(\lambda_\Sigma)) = 0 \). Set \( S := R_\Sigma, \lambda := \lambda_\Sigma \) and \( \Phi := \{ S \otimes_R f \mid f \in \Sigma \} \). Then the noncommutative localization \( \lambda_\pi : \Lambda \to \Lambda_\pi \) at \( \pi \) is homological if and only if the noncommutative localization \( \lambda_\Phi : S' \to S'_\Phi \) of \( S' \) at \( \Phi \) is homological. In particular, if one of the above equivalent conditions holds, then there is a recollement of derived module categories:

\[ \mathcal{D}(S'_\Phi) \leftarrowarrow \mathcal{D}(\Lambda) \rightarrowarrow \mathcal{D}(R). \]

As a consequence of Corollary 5.11, we obtain the following result which can be used to adjudge whether a noncommutative localization of the form \( \lambda_\pi : \Lambda \to \Lambda_\pi \) is homological or not.

Corollary 5.12. Let \( F \subseteq D \) be an arbitrary extension of rings. Let \( \omega : D \to D/F \) be the canonical surjection of \( F \)-modules. Set \( R := \left( \begin{array}{cc} D & D \\ 0 & F \end{array} \right) \) and \( S := M_2(D) \). Let \( \lambda : R \to S \) be the canonical inclusion, and let \( \pi : S \to S/R \) be the canonical surjection. Then the noncommutative localization \( \lambda_\pi : \Lambda \to \Lambda_\pi \) of \( \Lambda \) at \( \pi \) is homological if and only if the noncommutative localization \( \lambda_\omega : E \to E_\omega \) of \( E \) at \( \omega \) is homological, where \( E := \text{End}_F(D \oplus D/F), \) and \( \omega^* : \text{Hom}_F(D \oplus D/F, D) \to \text{Hom}_F(D \oplus D/F, D/F) \) is the homomorphism of \( E \)-modules induced by \( \omega \).
Proof. Since $Q^*$ can be identified with $S/R$ in $\mathcal{D}(R)$, we have $S' = \text{End}_R(S/R)$. Thus the map $\lambda' : R \to S'$ is given by the right multiplication. Set $e_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $e_{12} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R$. Furthermore, let $\varphi : R e_1 \to R e_2$ and $\varphi' : S'(e_1) \lambda' \to S'(e_2) \lambda'$ be the right multiplication maps of $e_{12}$ and $(e_{12}) \lambda'$, respectively.

It follows from Lemma 5.13 (see also [29] Theorem 4.10) and $D \square F = D$ that $\lambda : R \to S$ is the non-commutative localization of $R$ at $\varphi$. In particular, $\lambda$ is a ring epimorphism. Since $S \simeq e_1 R \oplus e_1 R$ as right $R$-modules, the embedding $\lambda$ is even homological. Note that $S' \otimes_R \varphi$ can be identified with $\varphi'$. By Corollary 5.11 the map $\lambda_{\varphi} : \Lambda \to \Lambda_{\varphi}$ is homological if and only if the map $\lambda_{\varphi'} : S' \to S'_{\varphi}$ is homological.

Clearly, $R/Re_1 R \simeq F$ as rings. So, every $F$-module can be regarded as an $R$-module. In particular, the $F$-module $D \oplus D/F$ can be considered as an $R$-module. Further, one can check that the map

$$\alpha : D \oplus D/F \to S/R, \quad (d, t + F) \mapsto \begin{pmatrix} 0 & 0 \\ d & t \end{pmatrix} + R$$

for $d, t \in D$, is an isomorphism of $R$-modules. Thus $S' \simeq E$ as rings. Under this isomorphism, $\varphi'$ corresponds to $\varphi^*$, and therefore $S'_{\varphi'} \simeq E_{\varphi'}$ as rings. It follows that $\lambda_{\varphi'} : S' \to S'_{\varphi'}$ is homological if and only if so is $\lambda_{\varphi^*} : E \to E_{\varphi'}$. This finishes the proof. □

Before starting with the proof of Corollary 5.13 we introduce a couple of more definitions and notation.

Recall from [28] that a complex $U^*$ in $\mathcal{D}(R)$ is called a tilting complex if $U^*$ is self-orthogonal, isomorphic in $\mathcal{D}(R)$ to a bounded complex of finitely generated projective $R$-modules, and $\text{Tria}(U^*) = \mathcal{D}(R)$. It is well known that if $U^*$ is a tilting complex over $R$, then $\mathcal{D}(R)$ is equivalent to $\mathcal{D}(\text{End}_{\mathcal{D}(R)}(U^*))$ as triangulated categories (see [28] Theorem 6.4). In this case, $R$ and $\text{End}_{\mathcal{D}(R)}(U^*)$ are called derived equivalent. We refer the reader to [18] for some new advances in constructions of derived equivalences.

If $I$ is an index set, we denote by $U^*(I)$ the direct sum of $I$ copies of $U^*$ in $\mathcal{D}(R)$.

The following result generalizes some known results in the literature. See, for example, [15] Theorem 4.14, [11] Theorem 3.5 (5) and [44] Lemma 3.1 (3), where the ring homomorphism $\lambda : R \to S$ is required to be injective. We shall use this generalization to prove Corollary 5.13.

Lemma 5.13. Let $\lambda : R \to S$ be a ring homomorphism, and let $I$ be an arbitrary nonempty set. Define $U^* := S \oplus Q^*$. Then $\text{Hom}_{\mathcal{D}(R)}(U^*, U^*(I)[n]) = 0$ for any $0 \neq n \in \mathbb{Z}$ if and only if the following conditions hold:

1. $\text{Hom}_R(S, \text{Ker}(\lambda)) = 0$ and
2. $\text{Ext}_R^i(S, S(I)) = 0 = \text{Ext}_R^{i+1}(S, R(I))$ for any $i \geq 1$.

In particular, the complex $U^*$ is a tilting complex in $\mathcal{D}(R)$ if and only if $\text{Hom}_R(S, \text{Ker}(\lambda)) = 0$, $\text{Ext}_R(S, S) = 0$ and there is an exact sequence: $0 \to P_1 \to P_0 \to R S \to 0$ of $R$-modules, such that $P_i$ is finitely generated and projective for $i = 0, 1$.

Proof. Recall that we have a distinguished triangle $\ast \ast \ast : R \xrightarrow{\lambda} S \xrightarrow{\pi} Q^* \xrightarrow{v} R[1]$ in $\mathcal{K}(R)$.

First of all, we mention two general facts: Let $I$ be an arbitrary nonempty set.

(a) By applying $\text{Hom}_{\mathcal{D}(R)}(-, S(I))$ to $\ast \ast \ast$, one can prove that $\text{Hom}_{\mathcal{D}(R)}(Q^*, S(I)[i]) \simeq \text{Hom}_{\mathcal{D}(R)}(S, S(I)[i])$ for $i \in \mathbb{Z} \setminus \{0\}$ and $\text{Hom}_{\mathcal{D}(R)}(Q^*, S(I)) \simeq \text{Ker}(\text{Hom}_R(\lambda, S(I)))$.

(b) By applying $\text{Hom}_{\mathcal{D}(R)}(-, R(I))$ to $\ast \ast \ast$, one can show that $\text{Hom}_{\mathcal{D}(R)}(Q^*, R(I)[j]) \simeq \text{Hom}_{\mathcal{D}(R)}(S, R(I)[j])$ for $j \in \mathbb{Z} \setminus \{0, 1\}$.

Next, we show the necessity of the first part of Lemma 5.13

Suppose that $\text{Hom}_{\mathcal{D}(R)}(U^*, U^*(I)[n]) = 0$ for any $n \neq 0$. Then $\text{Ext}_R(S, S(I)) \simeq \text{Hom}_{\mathcal{D}(R)}(Q^*, S(I)[i]) = 0$ for any $i \geq 1$, and $\text{Hom}_{\mathcal{D}(R)}(S, Q^*[-1]) = 0$. Consequently, the map $\text{Hom}_R(S, \lambda) : \text{Hom}_R(S, R) \to \text{Hom}_R(S, S)$ is
injective. This means that the condition (1) holds. Further, applying $\text{Hom}_{\mathcal{D}(R)}(S, -)$ to the triangle $R^l \xrightarrow{\lambda^l} S^l \xrightarrow{\pi^l} Q^l \xrightarrow{\eta^l} R^l[1]$, we get $\text{Ext}^1_R(S, R^l) \simeq \text{Hom}_{\mathcal{D}(R)}(S, Q^l[i]) = 0$. Thus, the conditions (1) and (2) in Lemma 5.13 are satisfied.

In the following, we shall show the sufficiency of the first part of Lemma 5.13. Assume that the conditions (1) and (2) in Lemma 5.13 hold true. Then, it follows from (a) and (b) that $\text{Hom}_{\mathcal{D}(R)}(Q^*, S^l[n]) = 0 = \text{Hom}_{\mathcal{D}(R)}(Q^*, R^l[m + 1])$ for $n \in \mathbb{Z} \setminus \{0\}$ and $m \in \mathbb{Z} \setminus \{-1, 0\}$. Applying $\text{Hom}_{\mathcal{D}(R)}(Q^*)$ to the triangle $R^l \xrightarrow{\lambda^l} S^l \xrightarrow{\pi^l} Q^l \xrightarrow{\eta^l} R^l[1]$, one can show that $\text{Hom}_{\mathcal{D}(R)}(Q^*, Q^l[m]) = 0$ for $m \in \mathbb{Z} \setminus \{-1, 0\}$. Furthermore, we shall show that the condition (1) in Lemma 5.13 implies also that $\text{Hom}_{\mathcal{D}(R)}(Q^*, Q^l[-1]) = 0$: Clearly, $\text{Hom}_R(S, \text{Ker}(\lambda^l)) \simeq \text{Hom}_R(S, \text{Ker}(\lambda))/I$ where $\text{Ker}(\lambda)/I$ stands for the direct product of $I$ copies of $\text{Ker}(\lambda)$. Since $\text{Ker}(\lambda)/I$ contains $\text{Ker}(\lambda/I)$ as a submodule, we infer that $\text{Hom}_R(S, \text{Ker}(\lambda/I)) = 0$ and $\text{Hom}(\text{Hom}_R(S, \lambda/I)) \simeq \text{Hom}_R(S, \text{Ker}(\lambda/I)) = 0$. Now, it follows from the following exact commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Hom}_{\mathcal{D}(R)}(Q^*, Q^l[-1]) & \xrightarrow{(\lambda^l)^*} & \text{Hom}_{\mathcal{D}(R)}(Q^*, R^l) & \xrightarrow{(\lambda^l)^*} & \text{Hom}_{\mathcal{D}(R)}(Q^*, S^l) & \xrightarrow{\pi_*} & \text{Hom}_{\mathcal{D}(R)}(Q^*, S^l) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{Ker}(\text{Hom}_R(S, \lambda/I)) & \longrightarrow & \text{Hom}_R(S, R^l) & \xrightarrow{(\lambda^l)^*} & \text{Hom}_R(S, S^l) & \longrightarrow & \text{Hom}_R(S, S^l)
\end{array}
\]

that $\text{Ker}(\text{Hom}_R(S, \lambda/I)) \simeq \text{Hom}_R(S, \text{Ker}(\lambda/I)) = 0$, and therefore $\text{Hom}_{\mathcal{D}(R)}(Q^*, Q^l[-1]) = 0$. Thus,

$\text{Hom}_{\mathcal{D}(R)}(Q^*, Q^l[n]) = 0$ for $n \neq 0$.

It remains to prove that $\text{Hom}_{\mathcal{D}(R)}(S, Q^l[n]) = 0$ for $n \neq 0$. Actually, applying $\text{Hom}_{\mathcal{D}(R)}(S, -)$ to the triangle $R^l \xrightarrow{\lambda^l} S^l \xrightarrow{\pi^l} Q^l \xrightarrow{\eta^l} R^l[1]$, we have the following long exact sequence:

\[
\cdots \longrightarrow \text{Hom}_{\mathcal{D}(R)}(S, S^l[j]) \longrightarrow \text{Hom}_{\mathcal{D}(R)}(S, Q^l[j]) \longrightarrow \text{Hom}_{\mathcal{D}(R)}(S, R^l[j + 1]) \xrightarrow{(\lambda^l)^*} \text{Hom}_{\mathcal{D}(R)}(S, S^l[j + 1]) \longrightarrow \cdots
\]

for $j \in \mathbb{Z}$. Since $\text{Hom}_{\mathcal{D}(R)}(S, S^l[r]) = 0$ for $r \neq 0 \in \mathbb{Z}$ and $\text{Hom}_{\mathcal{D}(R)}(S, R^l[t]) = 0$ for $t \in \mathbb{Z} \setminus \{0, 1\}$, we see that $\text{Hom}_{\mathcal{D}(R)}(S, Q^l[j]) = 0$ for $j \in \mathbb{Z} \setminus \{-1, 0\}$ and that $\text{Hom}_{\mathcal{D}(R)}(S, Q^l[-1]) \simeq \text{Ker}(\text{Hom}_R(S, \lambda/I)) = 0$. It follows that $\text{Hom}_{\mathcal{D}(R)}(S, Q^l[n]) = 0$ for $n \neq 0$. Hence $\text{Hom}_{\mathcal{D}(R)}(U^*, U^*[l]) = 0$ for any $n \neq 0$. This finishes the proof of the sufficiency.

As to the second part of Lemma 5.13 we observe the following: The complex $U^*$ over $R$ is a generator of $\mathcal{D}(R)$, that is, $\text{Tria}(U^*) = \mathcal{D}(R)$, since $R \in \text{Tria}(U^*)$ by the triangle $(+u)$. Moreover, the complex $U^*$ is a tilting complex in $\mathcal{D}(R)$ if and only if it is self-orthogonal, and $p_S$ has a projective resolution of finite length consisting of finitely generated projective $R$-modules. Furthermore, if $p_S$ has finite projective dimension and $\text{Ext}^i_R(S, R^l) = 0$ for any $i \geq 1$, then $p_S$ does have projective dimension at most 1. Thus, by the first part of Lemma 5.13, we can show the second part of Lemma 5.13.

**Proof of Corollary 1.13**

(1) By Corollary 1.2, the map $\lambda^l : \Lambda \to \Lambda^l$ is homological if and only if $\text{Tor}^R_0(S', S) = 0$ for all $j \geq 1$. Now, we assume that $R_S$ has projective dimension at most 1. Then $\text{Tor}^R_i(S', S) = 0$ for all $i \geq 2$. This implies that $\Lambda^l$ is homological if and only if $\text{Tor}^R_1(S', S) = 0$. Since $B_{e_2} = S' \oplus S \otimes R S'$ as right $R$-modules, it suffices to show that $\text{Tor}^R_1(B_{e_2}, S) = 0$.

In fact, from Lemma 5.4(3), we obtain a triple $(j^i, j^j, j^k)$ of adjoint functors. Let $\eta : \text{Id}_{\mathcal{D}(B)} \to j^* j^*$ be the unit adjunction with respect to the adjoint pair $(j^i, j^j)$. Then we have the following fact:

For any $X^* \in \mathcal{D}(B)$, there exists a canonical triangle in $\mathcal{D}(B)$:

$$i_* i^*(X^*) \xrightarrow{\eta} X^* \xrightarrow{\eta} j_* j^*(X^*) \xrightarrow{\eta} i_* i^*(X^*)[1].$$
where $j_*j^!(X^*) = \mathbb{R}\text{Hom}_R(P^*, \text{Hom}_B^*(P^*, X^*))$. For the other triple $(i^*, i_*, i^!)$ of adjoin triangle functors, we refer the reader to Lemma 5.4 (3).

Let

$$0 \longrightarrow P^{-1} \overset{\delta}{\longrightarrow} P^0 \longrightarrow R S \longrightarrow 0$$

be a projective resolution of $R S$ with all $P^j$ projective $R$-modules. This exact sequence gives rise to a triangle $P^{-1} \to P^0 \to S \to P^{-1}[1]$ in $\mathcal{D}(R)$. Then we see from the recollement $(\ast)$ in Lemma 5.4 (3) that there is the following exact commutative diagram:

$$
\begin{array}{cccc}
& i_*i^!(Be_2 \otimes_R P^{-1}) & \longrightarrow & Be_2 \otimes_R P^{-1} \quad \eta_{Be_2 \otimes_R P^{-1}} \quad j_*j^!(Be_2 \otimes_R P^{-1}) & \quad i_*i^!(Be_2 \otimes_R P^{-1})[1] \\
& j_*j^!(Be_2 \otimes_R P^{0}) & \longrightarrow & Be_2 \otimes_R P^{0} \quad \eta_{Be_2 \otimes_R P^{0}} \quad j_*j^!(Be_2 \otimes_R P^{0}) & \quad i_*i^!(Be_2 \otimes_R P^{0})[1] \\
i_*i^!(Be_2 \otimes_R P^{-1})[1] & \longrightarrow & Be_2 \otimes_R P^{-1}[1] \quad j_*j^!(Be_2 \otimes_R P^{-1})[1] & \quad i_*i^!(Be_2 \otimes_R P^{-1})[2] \\
& i_*i^!(Be_2 \otimes_R S) & \longrightarrow & Be_2 \otimes_R S \quad \eta_{Be_2 \otimes_R S} \quad j_*j^!(Be_2 \otimes_R S) & \quad i_*i^!(Be_2 \otimes_R S)[1] \\
i_*i^!(Be_2 \otimes_R P^{-1})[1] & \longrightarrow & Be_2 \otimes_R P^{-1}[1] \quad j_*j^!(Be_2 \otimes_R P^{-1})[1] & \quad i_*i^!(Be_2 \otimes_R P^{-1})[2] \\
& i_*i^!(Be_2 \otimes_R S) & \longrightarrow & Be_2 \otimes_R S \quad \eta_{Be_2 \otimes_R S} \quad j_*j^!(Be_2 \otimes_R S) & \quad i_*i^!(Be_2 \otimes_R S)[1] \\
\end{array}
$$

Since $i_*i^!(Be_1) \simeq Be_2 \otimes_R S$ in $\mathcal{D}(B)$ by Lemma 5.5 (3), we know that $j_*j^!(Be_2 \otimes_R S) \simeq j_*j^!i_*i^!(Be_1) = 0$, due to $j^!*i_* = 0$ in the recollement $(\ast)$. It follows that $j_*j^!(1 \otimes \delta)$ is an isomorphism, and so is $H^0(j_*j^!(1 \otimes \delta))$.

Suppose that $H^0(\eta_P): P \to H^0(j_*j^!(P))$ is injective for any projective $B$-module $P$. Then $H^0(\eta_{Be_2 \otimes_R P^{-1}})$ is injective since $P^{-1}$ is projective. It follows from the isomorphism $H^0(j_*j^!(1 \otimes \delta))$ that the map $1 \otimes \delta: Be_2 \otimes_R P^{-1} \to Be_2 \otimes_R P^0$ is injective. This implies that $\text{Tor}^R_1(Be_2, S) = 0$, as desired.

Thus, in the following, we shall prove that $H^0(\eta_P): P \to H^0(j_*j^!(P))$ is injective for any projective $B$-module $P$.

First, we point out that $H^0(\eta_P)$ is injective if and only if $\text{Hom}_{\mathcal{D}(B)}(B, P) \overset{j}{\longrightarrow} \text{Hom}_{\mathcal{D}(R)}(j^!(B), j^!(P))$ is injective. To see this, we consider the following composite of maps:

$$
\omega^0_{X^*}: \text{Hom}_{\mathcal{D}(B)}(B, X^*[n]) \overset{j}{\longrightarrow} \text{Hom}_{\mathcal{D}(R)}(j^!(B), j^!(X^*[n])) \overset{\sim}{\longrightarrow} \text{Hom}_{\mathcal{D}(B)}(B, j_*j^!(X^*)[n])
$$

for each $n \in \mathbb{Z}$, where the second map is an isomorphism induced by the adjoint pair $(j^!, j_*)$. Then, one can check directly that $\omega^0_{X^*} = \text{Hom}_{\mathcal{D}(B)}(B, \eta_{X^*[n]})$. It is known that the $n$-th cohomology functor $H^n(-): \mathcal{D}(B) \to B\text{-Mod}$ is naturally isomorphic to the Hom-functor $\text{Hom}_{\mathcal{D}(B)}(B, -[n])$. So, under this identification, the map $\omega^0_{X^*}$ coincides with $H^n(\eta_{X^*}): H^n(X^*) \to H^n(j_*j^!(X^*))$. It follows that $H^0(\eta_P)$ is injective if and only if so is the map $\text{Hom}_{\mathcal{D}(B)}(B, P) \overset{j}{\longrightarrow} \text{Hom}_{\mathcal{D}(R)}(j^!(B), j^!(P))$.

Second, we claim that if $\text{Hom}_{\mathcal{D}(B)}(i_*i^!(B), P) = 0$, then $\text{Hom}_{\mathcal{D}(B)}(B, P) \overset{j}{\longrightarrow} \text{Hom}_{\mathcal{D}(R)}(j^!(B), j^!(P))$ is injective.

Let $\varepsilon: j_*j^! \to \text{Id}_{\mathcal{D}(B)}$ be the counit adjunction with respect to the adjoint pair $(j_!, j^!)$. Then, for each $X^* \in \mathcal{D}(B)$, there exists a canonical triangle in $\mathcal{D}(B)$:

$$
j_*j^!(X^*) \overset{\varepsilon X^*}{\longrightarrow} X^* \longrightarrow i_*i^!(X^*) \longrightarrow j_*j^!(X^*)[1].
$$

Now, we consider the following morphisms:

$$
\text{Hom}_{\mathcal{D}(B)}(B, X^*[m]) \overset{j}{\longrightarrow} \text{Hom}_{\mathcal{D}(R)}(j^!(B), j^!(X^*[m])) \overset{\sim}{\longrightarrow} \text{Hom}_{\mathcal{D}(B)}(j_*j^!(B), X^*[m])
$$

for any $m \in \mathbb{Z}$, where the last map is an isomorphism given by the adjoint pair $(j_!, j^!)$. One can check that the composite of the above two morphisms is the map $\text{Hom}_{\mathcal{D}(B)}(\varepsilon_B, X^*[m])$. This means that, to show that
Hom_{\mathcal{D}(B)}(B, P) \xrightarrow{j'} Hom_{\mathcal{D}(B)}(j'(B), j'(P)) is injective, it suffices to show that Hom_{\mathcal{D}(B)}(e_B, P) is injective. For this aim, we apply Hom_{\mathcal{D}(B)}(-, P) to the triangle

\[ j:j'(B) \xrightarrow{\varepsilon_B} B \longrightarrow i_i^*(B) \longrightarrow j:j'(B)[1] \]

and get the following exact sequence of abelian groups:

\[
\begin{array}{ccc}
\Hom_{\mathcal{D}(B)}(i_i^*(B), P) & \longrightarrow & \Hom_{\mathcal{D}(B)}(B, P) \xrightarrow{\Hom_{\mathcal{D}(B)}(e_B, P)} \Hom_{\mathcal{D}(B)}(j:j'(B), P).
\end{array}
\]

If Hom_{\mathcal{D}(B)}(i_i^*(B), P) = 0, then Hom_{\mathcal{D}(B)}(e_B, P) is injective, and therefore the map \( j': \Hom_{\mathcal{D}(B)}(B, P) \rightarrow \Hom_{\mathcal{D}(B)}(j':(B), j'(P)) \) is injective, as desired.

Third, we show that if Hom_R(S, S') = 0, then Hom_{\mathcal{D}(B)}(i_i^*(B), P) = 0 for any projective B-module P.

In fact, due to Lemma [5,5] (1) and (2), From Lemma 5.4 (3), we see that the ring \( Q \subseteq \mathcal{D}(B) \) is isomorphic to a quotient of \( \mathcal{D}(B) \), which implies that \( \Hom_{\mathcal{D}(B)}(i_i^*(B), P) = 0 \) if and only if \( \Hom_{\mathcal{D}(B)}(Be_2 \otimes_R S, P) = 0 \). Now, we consider the following isomorphisms

\[
\Hom_{\mathcal{D}(B)}(Be_2 \otimes_R S, P) \simeq \Hom_R(S, \Hom_B(Be_2, P)) \simeq \Hom_R(S, e_2 P) \simeq \Hom_R(S, e_2 P).
\]

Since \( e_2 B \simeq S' \) as \( R \)-modules, we have \( \Hom_R(S, e_2 P) = 0 \). Recall that \( P \in \Add(B) \) and \( e_2 P \in \Add(S) \). Thus there is an index set \( I \) such that \( e_2 P \) is a direct summand of \( S' \) and \( S' \) is isomorphic to \( \Hom_R(S, S' \otimes_R S) \). Hence \( \Hom_R(S, S' \otimes_R S) = 0 \) and \( \Hom_{\mathcal{D}(B)}(i_i^*(B), P) = 0 \), as desired.

Now, it remains to show that \( \Hom_R(S, S'[n]) = 0 \). In the following, we shall prove a stronger statement, namely, \( \Hom_{\mathcal{D}(R)}(S, S'[n]) = 0 \) for any \( n \in \mathbb{Z} \).

Since \( \lambda \) is a ring epimorphism with \( \text{Tor}^R(S, S) = 0 \), we know from [29] Theorem 4.8 that

\[
\text{Ext}_R^1(S, S'^{(l)}) = \text{Ext}_R^1(S, S'^{(l)}) = 0
\]

for any set \( I \). As \( R \) is of projective dimension at most \( 1 \), we can apply Lemma [5,13] to the complex \( U^*: = S \oplus Q^* \), and get \( \Hom_{\mathcal{D}(R)}(U^*, U'^*[m]) = 0 \) for \( m \neq 0 \). This implies that \( \Hom_{\mathcal{D}(R)}(Q^*, U'^*[m]) = 0 \) for \( m \neq 0 \), and that

\[
H^0(\,	ext{Hom}_R(Q^*, Q^*) \simeq \Hom_{\mathcal{D}(R)}(Q^*, U'^*[m]) = \begin{cases} 0 & \text{if } m \neq 0, \\ S' & \text{if } m = 0. \end{cases}
\]

Thus the complex \( \text{Hom}_R(Q^*, Q^*) \) is isomorphic in \( \mathcal{D}(R) \) to the stalk complex \( S' \). On the one hand, by the adjoint pair \( (\cdot \otimes_R - , \text{Hom}_R(\cdot, -)) \) of the triangle functors, we have

\[
\Hom_{\mathcal{D}(R)}(S, S'[n]) \simeq \Hom_{\mathcal{D}(R)}(S, \text{Hom}_R(Q^*, Q^*)[n]) \simeq \Hom_{\mathcal{D}(R)}(S, \text{Hom}_R(Q^*, Q^*)[n]) \simeq \text{Hom}_R(Q^* \otimes_R S, Q^*[n])
\]

for any \( n \in \mathbb{Z} \). On the other hand, since \( \lambda \) is homological by assumption, the homomorphism \( \lambda \otimes_R S: R \otimes_R S \rightarrow S \otimes_R S \) is an isomorphism in \( \mathcal{D}(R) \). It follows from the triangle \( R \otimes_R S \xrightarrow{\lambda \otimes_R S} S \otimes_R S \rightarrow Q^* \otimes_R S \rightarrow R \otimes_R S[1] \) that \( Q^* \otimes_R S = 0 \). Hence \( \Hom_{\mathcal{D}(R)}(S, S'[n]) \simeq \Hom_R(Q^* \otimes_R S, Q^*[n]) = 0 \) for any \( n \in \mathbb{Z} \).

Thus, we have proved that, for any projective \( B \)-module \( P \), the homomorphism \( H^0(\eta_P) : P \rightarrow H^0(j', j'(P)) \) is injective in \( B\text{-Mod} \). This finishes the proof of Corollary [1,4] (1).

(2) From Lemma [5,4] (3), we see that the ring \( \Lambda_{\mathcal{D}} \) is zero if and only if the functor \( j' \) induces a triangle equivalence from \( \mathcal{D}(B) \) to \( \mathcal{D}(R) \). This is equivalent to the statement that \( j'(B) \) is a tilting complex over \( R \). Note that \( j'(B) \simeq U^*[n] \). Thus, the ring \( \Lambda_{\mathcal{D}} \) is zero if and only if \( U^* \) is a tilting complex over \( R \). Now, Corollary [1,4] (2) follows directly from Lemma [5,13].

Proof of Corollary [1,4]
(1) Let \( \lambda : R \to S \) be the inclusion, \( \pi : S \to S/R \) the canonical surjection and \( \lambda' : R \to S' \) the induced map by right multiplication. Since \( \lambda \) is injective, we know from Lemma 5.9 that \( (\lambda, \lambda', \text{Hom}_R(S, S/R), \pi) \) is an exact context. If \( R \) is flat, then \( \text{Tor}^R_i(S', S) = 0 \) for all \( i \geq 1 \). Now, (1) follows from Theorem 1.1.

(2) Since \( \lambda \) is an injective ring epimorphism, the pair \( (\lambda, \lambda') \) is exact by Lemma 5.9. Since the ring \( R \) is commutative and \( \lambda \) is homological, the ring \( S' \) is also commutative by [Lem. 6.5 (5)]. Consequently, the noncommutative tensor product \( S' \otimes_R S \) coincides with the usual tensor product \( S' \otimes_R S \) of \( S' \) and \( S \) over \( R \) (see Section 4.4).

By Lemma 5.6, we know that \( \text{Tor}^R_i(S, S') = 0 \) for any \( i > 0 \). Since \( R, S \) and \( S' \) are commutative rings, we have \( \text{Tor}^R_2(S', S) \simeq \text{Tor}^R_2(S, S') = 0 \). Thus the assertion (3) in Corollary 1.2 is satisfied. It follows from Corollary 1.2 that the ring homomorphism \( S' \otimes_R \lambda : S' \to S' \otimes_R S \) is a ring epimorphism. This implies that \( \text{End}_{S'}(S' \otimes_R S) \simeq \text{End}_{S' \otimes_R S}(S' \otimes_R S) \simeq S \otimes_R S' \) as rings. Note that \( S \simeq \text{End}_R(S) \) as rings and that \( \text{Hom}_R(S/R, S) = 0 \) since \( \lambda \) is a ring epimorphism. Thus \( \Lambda \simeq \text{End}_R(S \otimes_R S') \simeq B \) as rings. Now, (2) is an immediate consequence of Corollary 1.2. 

\[ \square \]

**Proof of Corollary 1.5**

For a commutative ring \( R \) and a multiplicative set \( \Phi \) of \( R \), the localization map \( \lambda : R \to S := \Phi^{-1}R \) is always homological since \( R \) is flat. Therefore, by Corollary 1.4(2), it suffices to show that \( \text{Hom}_R(S/R, S) = 0 \) and that \( S' \otimes_R S \) is isomorphic to \( \Psi^{-1}S' \). Actually, the former follows from the fact that \( \lambda \) is a ring epimorphism. To check the latter, we verify that the well defined map

\[
\alpha : S' \otimes_R \Phi^{-1}R \longrightarrow \Psi^{-1}S', \quad y \otimes \frac{r}{x} \mapsto \frac{(r)\lambda'y}{(x)\lambda'}
\]

for \( y \in S' \), \( r \in \Phi \), \( x \in \Phi \), is an isomorphism of rings, where \( \lambda' : R \to S' \) is the right multiplication map. Clearly, \( \alpha \) is surjective. To see that this map is injective, we note that the map

\[
\beta : \Psi^{-1}S' \longrightarrow S' \otimes_R \Phi^{-1}R, \quad \frac{y}{(x)\lambda'} \mapsto y \otimes \frac{1}{x}
\]

for \( y \in S' \) and \( x \in \Phi \), is a well defined ring homomorphism with \( \alpha \beta = 1 \). Observe that \( \alpha \) preserves the multiplication of \( S' \otimes_R S \). This finishes the proof of Corollary 1.5. 

\[ \square \]

**6 Examples**

Now we present a few examples to show that some conditions in our results cannot be dropped or weakened.

(1) The condition that \( \lambda : R \to S \) is a homological ring epimorphism in Corollary 1.2 cannot be weakened to that \( \lambda : R \to S \) is a ring epimorphism.

Let \( R = \left( \begin{array}{ccc} k & 0 & 0 \\ k[x]/(x^2) & k & 0 \\ k[x]/(x^2) & k[x]/(x^2) & k \end{array} \right) \), where \( k \) is a field and \( k[x] \) is the polynomial algebra over \( k \) in one variable \( x \). Let \( S \) be the 3 by 3 matrix ring \( M_3(k[x]/(x^2)) \). Then the inclusion \( \lambda \) of \( R \) into \( S \) is a noncommutative localization of \( R \), and therefore a ring epimorphism. Further, we have \( \text{Tor}^R_1(S, S) = 0 \neq \text{Tor}^R_1(S, S) \) (see [24]). Thus \( \lambda \) is not homological. So, \( R \) cannot have projective dimension less than or equal to 1. Moreover, one can check that the ring homomorphism \( \lambda' : R \to S' \) associated to \( \lambda \) is an isomorphism of rings. Recall that the pair \( (\lambda, \lambda') \) is exact by Lemma 5.9 and further, that \( S' \otimes_R S \simeq S \cup R S' = S \) as rings by Remark 5.2. In this sense, we have \( \phi = (\lambda')^{-1} \lambda : S' \to S \). Consequently, \( \phi \) is not homological. However, due to Remark 5.10 the map \( \lambda_{\pi^*} : \lambda \to \lambda_{\pi^*} \) is homological since \( \text{Tor}^R_1(S', S) \simeq \text{Tor}^R_1(S', R) = 0 \) for each \( i \geq 1 \). Hence, without the ‘homological’ assumption on \( \lambda \), the conditions (1) and (2) in Corollary 1.2 are not equivalent.

(2) The condition that \( \lambda \) is homological does not guarantee that the noncommutative localization \( \lambda_{\pi^*} : \Lambda \to \Lambda_{\pi^*} \) of \( \Lambda \) at \( \pi^* \) in Corollary 1.2 is always homological.

In the following, we shall use Corollary 5.12 to produce a counterexample.
Now, take $F = \{ \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} | a, b \in k \}$ and $D = \begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$ with $k$ a field. Then one can verify that the extension $\lambda : R \to S$, defined in Corollary 5.12, is homological, and that the canonical map $\omega : D \to D/F$ is a split epimorphism in $F$-Mod, and therefore $D \simeq D \oplus D/F$ as $F$-modules. Let $e$ be the idempotent of $E$ corresponding the direct summand $F$ of the $F$-module $D \oplus D/F$. Then $E\omega \simeq E/EeE \simeq M_2(k)$. Furthermore, the noncommutative localization $\lambda_\omega : E \to E\omega$ of $E$ at $\omega^*$ is equivalent to the canonical surjection $\tau : E \to E/EeE$. Since $\text{Ext}_{E}(E/EeE, E/EeE) \neq 0$, the map $\tau$ is not homological. This implies that $\lambda_\omega$ is not homological, too. Thus $\lambda_\omega : \Lambda \to \Lambda_\omega$ is not homological by Corollary 5.12, that is, the restriction functor $D((\Lambda_\omega)_*) : \mathcal{D}(\Lambda_\omega) \to \mathcal{D}(\Lambda)$ is not fully faithful. In addition, one can check that, for this extension, the $R$-module $\gamma S$ has infinite projective dimension.

(3) In Corollary 1.3(1), we assume that the projective dimension of $\gamma S$ is at most 1. But there does exist an injective homological ring epimorphism $\lambda : R \to S$ such that the projective dimension of $\gamma S$ is greater than 1 and that $\lambda_\omega : \Lambda \to \Lambda_\omega$ is homological.

Let $R$ be a Prüfer domain which is not a Matlis domain. Recall that a Matlis domain is an integral domain $R$ for which the projective dimension of the fractional field $Q$ of $R$ as an $R$-module is at most 1. In this case, the inclusion $\lambda : R \to Q$ is an injective homological ring epimorphism. By Corollary 1.2, the map $\lambda_\omega : \Lambda \to \Lambda_\omega$ is homological.

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