Lower semicontinuous envelopes in $W^{1,1} \times L^p$

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Abstract

It is studied the lower semicontinuity of functionals of the type $\int_\Omega f(x, u, v, \nabla u)\,dx$ with respect to the $(W^{1,1} \times L^p)$-weak * topology. Moreover in absence of lower semicontinuity, it is also provided an integral representation in $W^{1,1} \times L^p$ for the lower semicontinuous envelope.

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1 Introduction

In this paper we consider energies depending on two vector fields with different behaviours: $u \in W^{1,1}(\Omega; \mathbb{R}^n)$, $v \in L^p(\Omega; \mathbb{R}^m)$, $\Omega$ being a bounded open set of $\mathbb{R}^N$. Let $1 < p \leq +\infty$, for every $(u, v) \in W^{1,1}(\Omega; \mathbb{R}^n) \times L^p(\Omega; \mathbb{R}^m)$ define the functional

$$J(u, v) := \int_\Omega f(x, u(x), v(x), \nabla u(x))\,dx$$  \hspace{1cm} (1)

where $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times N} \to [0, +\infty)$ is a continuous function with linear growth in the last variable and $p$-growth in the third variable (cf. $(H1_p)$ and $(H1_\infty)$ below).

The energies (1), which generalize those considered by [14], [15] and [9], have been introduced to deal with equilibria for systems depending on elastic strain and chemical composition. In this context a multiphase alloy is represented by the set $\Omega$, the deformation gradient is given by $\nabla u$, and $v$ (when $m = 1$) denotes the chemical composition of the system. We also recall that our result may find applications also in the framework of Elasticity, when dealing with Cosserat’s theory, see [19]. In [14], the density $f \equiv f(v, \nabla u)$ is a convex-quasiconvex function, while in our model we also take into account heterogeneities and the deformation, without imposing any convexity restriction.

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We are interested in studying the lower semicontinuity and relaxation of \( (1) \) with respect to the \( L^1 \)-strong \( \times \) \( L^p \)-weak convergence. Clearly, bounded sequences \( \{ u_n \} \subset W^{1,1}(\Omega; \mathbb{R}^n) \) may converge in \( L^1 \), up to a subsequence, to a \( BV \) function. In this paper we restrict our analysis to limits \( u \) which are in \( W^{1,1}(\Omega; \mathbb{R}^n) \). Thus, our results can be considered as a step towards the study of relaxation in \( BV(\Omega; \mathbb{R}^n) \times L^p(\Omega; \mathbb{R}^m) \) of functionals \( (1) \).

We will consider separately the cases \( 1 < p < \infty \) and \( p = \infty \). To this end we introduce for \( 1 < p < +\infty \) the functional

\[
J_p(u, v) := \inf \{ \liminf J(u_n, v_n) : u_n \in W^{1,1}(\Omega; \mathbb{R}^n), v_n \in L^p(\Omega; \mathbb{R}^m), u_n \rightarrow u \text{ in } L^1, v_n \rightharpoonup v \text{ in } L^p \},
\]

for any pair \( (u, v) \in W^{1,1}(\Omega; \mathbb{R}^n) \times L^p(\Omega; \mathbb{R}^m) \), and for \( p = \infty \) the functional

\[
J_\infty(u, v) := \inf \{ \liminf J(u_n, v_n) : u_n \in W^{1,1}(\Omega; \mathbb{R}^n), v_n \in L^\infty(\Omega; \mathbb{R}^m), u_n \rightarrow u \text{ in } L^1, v_n \rightharpoonup^* v \text{ in } L^\infty \},
\]

for any pair \( (u, v) \in W^{1,1}(\Omega; \mathbb{R}^n) \times L^\infty(\Omega; \mathbb{R}^m) \).

For any \( p \in (1, +\infty] \) we will achieve the following integral representation (see Theorems 12 and 14):

\[
J_p(u, v) = \int_\Omega CQf(x, u(x), v(x), \nabla u(x))dx,
\]

where \( CQf \) represents the convex-quasiconvexification of \( f \) defined in [1].

2 Notations and General Facts

In this section we introduce the sets of assumptions we will make to obtain our results. We prove some properties related to convex-quasiconvex functions and we recall several facts that will be useful through the paper.

2.1 Assumptions

Let \( 1 < p < +\infty \), to obtain a characterization of the relaxed functional \( J_p \) in [2], we will make several hypotheses on the continuous function \( f : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times N} \rightarrow [0, +\infty) \). They are inspired by the set of assumptions in [17] for the case with no dependence on \( v \).

(H1) There exists a constant \( C \) such that

\[
\frac{1}{C}(|v|^p + |\xi|) - C \leq f(x, u, v, \xi) \leq C(1 + |v|^p + |\xi|),
\]

for every \( (x, u, \xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times N} \).
(H2\textsubscript{p}) For every compact set \( K \) of \( \Omega \times \mathbb{R}^n \) there exists a continuous function \( \omega_K : \mathbb{R} \rightarrow [0, +\infty) \) with \( \omega_K(0) = 0 \) such that
\[
|f(x, u, v, \xi) - f(x', u', v, \xi)| \leq \omega_K(|x - x'| + |u - u'|)(1 + |v|^p + |\xi|)
\]
for every \( (x, u, v, \xi) \) and \( (x', u', v, \xi) \) in \( K \times \mathbb{R}^m \times \mathbb{R}^{n \times N} \).

Moreover, given \( x_0 \in \Omega \), and \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( |x - x_0| \leq \delta \) then
\[
\forall (u, v, \xi) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times N} \quad f(x, u, v, \xi) - f(x_0, u, v, \xi) \geq -\varepsilon(1 + |v|^p + |\xi|).
\]

In order to characterize the functional \( J_\infty \) defined in (3) we will replace assumptions (H1\textsubscript{p}) and (H2\textsubscript{p}) by the following ones.

(H1\textsubscript{\infty}) Given \( M > 0 \), there exist \( C_M > 0 \) and a bounded continuous function \( G_M : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty) \) such that, if \( |v| \leq M \) then
\[
\forall (x, u, \xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{n \times N} \quad \frac{1}{C_M}G_M(x, u)|\xi| - C_M \leq f(x, u, v, \xi) \leq C_M G_M(x, u)(1 + |\xi|).
\]

(H2\textsubscript{\infty}) For every \( M > 0 \), and for every compact set \( K \) of \( \Omega \times \mathbb{R}^n \) there exists a continuous function \( \omega_{M, K} : \mathbb{R} \rightarrow [0, +\infty) \) with \( \omega_{M, K}(0) = 0 \) such that if \( |v| \leq M \) then
\[
|f(x, u, v, \xi) - f(x_0, u_0, v, \xi)| \leq \omega_{M, K}(|x - x_0| + |u - u_0|)(1 + |\xi|)
\]
for every \( (x, u, \xi), (x_0, u_0, \xi) \in K \times \mathbb{R}^{n \times N} \).

Moreover, given \( M > 0 \), \( x_0 \in \Omega \), and \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( |v| \leq M \) and \( |x - x_0| \leq \delta \) then
\[
\forall (u, \xi) \in \mathbb{R}^n \times \mathbb{R}^{n \times N} \quad f(x, u, v, \xi) - f(x_0, u, v, \xi) \geq -\varepsilon G_M(x, u)(1 + |\xi|),
\]
where the function \( G_M \) is as in (H1\textsubscript{\infty}).

### 2.2 Convex-Quasiconvex Functions

We start recalling the notion of convex-quasiconvex function, presented in [14] (see also [19, Definition 4.1], [15] and [13]). This notion plays, in the context of lower semicontinuity problems where the density depends on two fields \( v, \nabla u \), the same role of the well known notion of quasiconvexity introduced by Morrey for the lower semicontinuity of functionals where the dependence is just on \( \nabla u \).

**Definition 1.** A Borel measurable function \( h : \mathbb{R}^m \times \mathbb{R}^{n \times N} \rightarrow \mathbb{R} \) is said to be **convex-quasiconvex** if there exists a bounded open set \( D \) of \( \mathbb{R}^N \) such that
\[
h(v, \xi) \leq \frac{1}{|D|} \int_D h(v + \eta(x), \xi + \nabla \varphi(x)) \, dx,
\]
for every \( (v, \xi) \in \mathbb{R}^m \times \mathbb{R}^{n \times N} \), for every \( \eta \in L^\infty(D; \mathbb{R}^m) \), with \( \int_D \eta(x) \, dx = 0 \), and for every \( \varphi \in W^{1,\infty}_0(D; \mathbb{R}^n) \).
Remark 2. (i) It can be easily seen that, if $h$ is convex-quasiconvex, then, condition (H1) is true for any bounded open set $D \subset \mathbb{R}^N$.

(ii) We recall that a convex-quasiconvex function is separately convex.

(iii) Through this paper we will work with functions $f$ defined in $\Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times N}$ and when saying that $f$ is convex-quasiconvex we mean the previous definition with respect to the last two variables of $f$.

The following result adapts to the context of $W^{1,1} \times L^p$, i.e. growth conditions expressed by (H1), a well known result due to Marcellini (see Proposition 2.32 in [10] or Lemma 5.42 in [3]). Indeed the following proposition follows as a particular case of [3, Proposition 2.11].

**Proposition 3.** Let $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times N} \to \mathbb{R}$ be a separately convex function in each entry of the variables $(v, \xi)$, verifying the growth condition

$$|f(x, u, v, \xi)| \leq c(1 + |\xi| + |v|^p), \forall (x, u, \xi, v) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^{n \times N} \times \mathbb{R}^n$$

for some $p > 1$.

Then, denoting by $p'$, the conjugate exponent of $p$, there exists a constant $\gamma > 0$ such that

$$|f(x, u, v, \xi) - f(x, u, v', \xi')| \leq \gamma |\xi - \xi'| + \gamma \left(1 + |v|^{p-1} + |v'|^{p-1} + |\xi'|^{1/p'}\right)|v - v'|$$

for every $\xi, \xi' \in \mathbb{R}^{n \times N}$, $v, v' \in \mathbb{R}^m$ and $(x, u) \in \Omega \times \mathbb{R}^n$.

A similar result holds in the case $W^{1,1} \times L^\infty$ (i.e. growth conditions expressed by (H1\infty)).

**Proposition 4.** Let $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times N} \to \mathbb{R}$ be a separately convex function in each entry of the variables $(v, \xi)$, verifying assumption (H1\infty). Then, given $M > 0$ there exists a constant $\beta(M, n, m, N)$ such that

$$|f(x, u, v, \xi) - f(x, u, v', \xi')| \leq \beta(1 + |\xi| + |\xi'|)|v - v'| + \beta|\xi - \xi'|,$$  \hspace{0.5cm} (5)

for every $v, v' \in \mathbb{R}^m$, such that $|v| \leq M$ and $|v'| \leq M$, for every $\xi, \xi' \in \mathbb{R}^{n \times N}$ and for every $(x, u) \in \Omega \times \mathbb{R}^n$.

We introduce the notion of convex-quasiconvexification with respect to the last variables for a function $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times N} \to [0, +\infty)$. This notion is crucial in order to deal with the subsequent relaxation processes.

If $h : \mathbb{R}^m \times \mathbb{R}^{n \times N} \to \mathbb{R}$ is any given Borel measurable function bounded from below, it can be defined the convex-quasiconvex envelope of $h$, that is the largest convex-quasiconvex function below $h$:

$$CQh(v, \xi) := \sup\{g(v, \xi) : g \leq h, \text{ } g \text{ convex-quasiconvex}\}.$$  \hspace{0.5cm} (6)

Moreover, by Theorem 4.16 in [19]

$$CQh(v, \xi) = \inf \left\{ \frac{1}{|D|} \int_D h(v + \eta(x), \xi + \nabla \varphi(x)) \, dx : \right.$$  
$$\left. \eta \in L^\infty(D; \mathbb{R}^m), \int_D \eta(x) \, dx = 0, \varphi \in W_0^{1,\infty}(D; \mathbb{R}^n) \right\}.$$  \hspace{0.5cm} (7)
Consequently given a Carathéodory function $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times N} \rightarrow \mathbb{R}$, by $CQf(x, u, v, \xi)$ we denote the convex-quasiconvexification of $f(x, u, v, \xi)$ with respect to the last two variables.

As for convex-quasiconvexity, condition (7) can be stated for any bounded open set $D \subseteq \mathbb{R}^N$ and it can be also showed that if $f$ satisfies a growth condition of the type $(H_1p)$ then in (3) and (7) the spaces $L^\infty$ and $W_0^{1, \infty}$ can be replaced by $L^p$ and $W_0^{1,1}$, respectively.

The following results will be exploited in the sequel. We omit the proofs since they are very similar to [21 Proposition 2.2], in turn inspired by [10].

**Proposition 5.** Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times N} \rightarrow [0, +\infty)$ be a continuous function satisfying $(H_1p)$ and $(H_2p)$. Let $CQf$ be the convex-quasiconvexification of $f$ in (7). Then $CQf$ satisfies $(H_1p)$, $(H_2p)$ and it is a continuous function.

Analogously it holds

**Proposition 6.** Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, let $\alpha : [0, +\infty) \rightarrow [0, +\infty)$ be a convex and increasing function, such that $\alpha(0) = 0$ and let $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times N} \rightarrow [0, +\infty)$ be a continuous function satisfying the following conditions.

For every $(x, u) \in \Omega \times \mathbb{R}^n$ and for every $(v, \xi) \in \mathbb{R}^m \times \mathbb{R}^{n \times N}$ it results

$$\frac{1}{C}(\alpha(|v|) + |\xi|) - C \leq f(x, u, v, \xi) \leq C(1 + \alpha(|v|) + |\xi|).$$

(8)

For every compact set $K \subset \Omega \times \mathbb{R}^n$ there exists a continuous function $\omega'_K : \mathbb{R} \rightarrow [0, +\infty)$ such that $\omega'_K(0) = 0$ and

$$|f(x, u, v, \xi) - f(x', u', v, \xi)| \leq \omega'_K(|x - x'| + |u - u'|)(1 + \alpha(|v|) + |\xi|),$$

(9)

$\forall (x, u), (x', u') \in K, \forall (v, \xi) \in \mathbb{R}^m \times \mathbb{R}^{n \times N}$.

For every $x_0 \in \Omega$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - x_0| \leq \delta \Rightarrow f(x, u, v, \xi) - f(x_0, u, v, \xi) \geq -\varepsilon(1 + \alpha(|v|) + |\xi|),$$

(10)

$\forall (u, \xi) \in \mathbb{R}^m \times \mathbb{R}^{n \times N}$.

Let $CQf$ be the convex-quasiconvexification of $f$ (see (7)). Then $CQf$ satisfies analogous conditions to (8), (9) and (10). Moreover $CQf$ is a continuous function.

**Remark 7.** We observe that, if from one hand [8], [9], [10] generalize $(H_1p)$ and $(H_2p)$, from the other hand they can be regarded also as a stronger version of $(H_11\infty)$ and $(H_21\infty)$.

In order to provide an integral representation for $\overline{\mathcal{J}}_p$ in (2) and $\overline{\mathcal{J}}_\infty$ in (3) on $W^{1,1} \times L^p$ and $W^{1,1} \times L^\infty$ respectively, we prove some preliminary results. For every $p \in (1, +\infty)$ we introduce the following functionals $J_{CQf} : L^1(\Omega; \mathbb{R}^n) \times L^p(\Omega; \mathbb{R}^m) \rightarrow \mathbb{R} \cup \{+\infty\}$ defined as

$$J_{CQf}(u, v) := \begin{cases} \int_\Omega CQf(x, u(x), v(x), \nabla u(x)) \, dx & \text{if } (u, v) \in W^{1,1} \times L^p, \\ +\infty & \text{otherwise,} \end{cases}$$

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that, since

Proof. Via a diagonal argument (remind that weak $L_p$ metrizable on bounded sets), there exists a sequence $\{u_n, v_n\}$ with $u_n \to u$ in $L^1$, $v_n \rightharpoonup v$ in $L^p$. 

Lemma 8. Let $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \to [0, +\infty)$ be a continuous function. Let $p \in (1, +\infty]$ and consider the functionals $J$ and $CQf$ and their corresponding relaxed functionals $\overline{J}_p$ and $\overline{CQf}$. If $f$ satisfies conditions $(H1_p) - (H2_p)$ (if $p \in (1, +\infty)$), and both $f$ and $CQf$ satisfy $(H1_\infty) - (H2_\infty)$ (if $p = +\infty$), then

\[ \overline{J}_p(u, v) = \overline{CQf}(u, v) \]

for every $(u, v) \in BV(\Omega; \mathbb{R}^n) \times L^p(\Omega; \mathbb{R}^m)$.

Remark 9. We emphasize that in the above lemma, by virtue of Proposition 8 if $p \in (1, +\infty)$ it is enough to assume growth and continuity hypotheses just on $f$ (and not on $CQf$). If $p = +\infty$, by virtue of Proposition 7 we can also only make assumptions of $f$, replacing conditions $(H1_\infty)$ and $(H2_\infty)$ by $[8] - [10]$.

Proof. The argument is close to the proof of [21] Lemma 3.1. First we observe that, since $CQf \leq f$, it results $\overline{CQf} \leq \overline{J}_p$. Next we prove the opposite inequality in the nontrivial case that $\overline{CQf}(u, v) < +\infty$. For fixed $\delta > 0$, we can consider $(u_n, v_n) \in W^{1,1}(\Omega; \mathbb{R}^n) \times L^p(\Omega; \mathbb{R}^m)$ with $u_n \to u$ strongly in $L^1(\Omega; \mathbb{R}^n)$, $v_n \rightharpoonup v$ in $L^p(\Omega; \mathbb{R}^m)$ and such that

\[ \overline{CQf}(u, v) \geq \liminf_n \int_{\Omega} CQf(x, u_n(x), v_n(x), \nabla u_n(x)) \, dx - \delta. \]

Applying the results in [8] and [9], for each $n$ there exists a sequence $\{(u_{n,k}, v_{n,k})\}$ converging to $(u_n, v_n)$ weakly in $W^{1,1}(\Omega; \mathbb{R}^n) \times L^p(\Omega; \mathbb{R}^m)$ such that

\[ \int_{\Omega} CQf(x, u_{n,k}(x), v_{n,k}(x), \nabla u_{n,k}(x)) \, dx = \lim_k \int_{\Omega} f(x, u_{n,k}(x), v_{n,k}(x), \nabla u_{n,k}(x)) \, dx. \]

Consequently

\[ \overline{CQf}(u, v) \geq \liminf_k \int_{\Omega} f(x, u_{n,k}(x), v_{n,k}(x), \nabla u_{n,k}(x)) \, dx - \delta, \]

and

\[ v_{n,k} \rightharpoonup v \text{ in } L^p \text{ as } k \to +\infty \text{ and } n \to +\infty. \]

Via a diagonal argument (remind that weak $L^p$ and weak $^\ast L^\infty$ topologies are metrizable on bounded sets), there exists a sequence $\{(u_{n,k_n}, v_{n,k_n})\}$ satisfying $u_{n,k_n} \to u$ in $L^1(\Omega; \mathbb{R}^n)$, $v_{n,k_n} \rightharpoonup v$ in $L^p(\Omega; \mathbb{R}^m)$ and realizing the double limit in the right hand side of (11). Thus, it results

\[ \overline{CQf}(u, v) \geq \lim_n \int_{\Omega} f(x, u_{n,k_n}(x), v_{n,k_n}(x), \nabla u_{n,k_n}(x)) \, dx - \delta \geq \overline{J}_p(u, v) - \delta. \]

Letting $\delta$ go to 0 the conclusion follows. \qed
2.3 Some Results on Measure Theory

Let \( \Omega \) be a generic open subset of \( \mathbb{R}^N \), we denote by \( \mathcal{M}(\Omega) \) the space of all signed Radon measures in \( \Omega \) with bounded total variation. By the Riesz Representation Theorem, \( \mathcal{M}(\Omega) \) can be identified to the dual of the separable space \( \mathcal{C}_0(\Omega) \) of continuous functions on \( \Omega \) vanishing on the boundary \( \partial \Omega \). The \( N \)-dimensional Lebesgue measure in \( \mathbb{R}^N \) is designated as \( L^N \) while \( H^{N-1} \) denotes the \((N-1)\)-dimensional Hausdorff measure. If \( \mu \in \mathcal{M}(\Omega) \) and \( \lambda \in \mathcal{M}(\Omega) \) is a nonnegative Radon measure, we denote by \( \frac{d\mu}{d\lambda} \) the Radon-Nikodym derivative of \( \mu \) with respect to \( \lambda \). By a generalization of the Besicovich Differentiation Theorem (see [2, Proposition 2.2]), it can be proved that there exists a Borel set \( E \subset \Omega \) such that \( \lambda(E) = 0 \) and
\[
\frac{d\mu}{d\lambda}(x) = \lim_{\rho \to 0^+} \frac{\mu(x + \rho C)}{\lambda(x + \rho C)}
\]
for all \( x \in \text{Supp} \mu \setminus E \) and any open convex set \( C \) containing the origin. (Recall that the set \( E \) is independent of \( C \).)

We also recall the following generalization of Lebesgue-Besicovitch Differentiation Theorem, as stated in [18, Theorem 2.8].

**Theorem 10.** If \( \mu \) is a nonnegative Radon measure and if \( f \in L^1_{\text{loc}}(\mathbb{R}^d; \mu) \) then
\[
\lim_{\varepsilon \to 0^+} \frac{1}{\mu(x + \varepsilon C)} \int_{x + \varepsilon C} |f(y) - f(x)| d\mu(y) = 0,
\]
for \( \mu \)-a.e. \( x \in \mathbb{R}^d \) and for every bounded, convex, open set \( C \) containing the origin.

In particular, if \( v \in L^\infty(\Omega; \mathbb{R}^m) \), then, for \( \mathcal{L}^N \)-a.e. \( x \in \Omega \)
\[
\lim_{\varepsilon \to 0^+} \frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} |v(y) - v(x)| dy = 0. \tag{12}
\]

In the sequel we exploit the Calderón-Zygmund theorem for \( u \in BV \), cf. [3, Theorem 3.83, page 176]
\[
\lim_{\varepsilon \to 0^+} \frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} |u(y) - u(x) - \nabla u(x)(y-x)| dy = 0 \quad \mathcal{L}^N \text{-a.e. } x \in \Omega. \tag{13}
\]

3 Lower semicontinuity in \( W^{1,1} \times L^p \)

This section is devoted to provide a lower bound for the integral representation of \( J_p \) in (2) under assumptions (H1_p) and (H2_p), as stated in Theorem 14. Clearly this is equivalent to prove the lower semicontinuity with respect to the \( L^1 \)-strong \( \times \) \( L^p \)-weak topology of \( \int_{\Omega} CQf(x, u(x), v(x), \nabla u(x)) dx \), when \( (u, v) \in W^{1,1} \times L^p \).

Indeed we prove the following result
Theorem 11. Let $\Omega$ be a bounded open set of $\mathbb{R}^N$, and let $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times N \to [0, +\infty)$ be a continuous function. Assuming that $f$ satisfies hypotheses (H1p) and (H2p), and it is convex-quasiconvex, it results that
\[
\int_{\Omega} f(x, u(x), v(x), \nabla u(x)) \, dx
\]
is lower semicontinuous in $W^{1,1}(\Omega; \mathbb{R}^n) \times L^p(\Omega; \mathbb{R}^m)$, with respect to the $(L^1$-strong $\times L^p$-weak)- convergence.

Proof. The proof is mostly a combination of the theorems in [18] and [14], which used already some ideas from [17]. For convenience of the reader we present here some details, however we may refer to some separate results in the papers mentioned above.

Let
\[
G(u, v) = \int_{\Omega} f(x, u(x), v(x), \nabla u(x)) \, dx.
\]
It’s enough to prove that for every $(u, v) \in W^{1,1}(\Omega; \mathbb{R}^n) \times L^p(\Omega; \mathbb{R}^m)$,
\[
G(u, v) \leq \lim \inf_{n \to \infty} J(u_n, v_n)
\]
for any $u_n \to u$ in $L^1$ with $u_n \in W^{1,1}(\Omega; \mathbb{R}^n)$ and $v_n \rightharpoonup v$ in $L^p$.

Using the same arguments as in [1, Proof of Theorem II.4] (see also [17, Proposition 2.4]) and the density of smooth functions in $L^p$, we can reduce to the case where $u_n \in C_0^\infty(\mathbb{R}^N; \mathbb{R}^n)$ and $v_n \in C_0^\infty(\mathbb{R}^N; \mathbb{R}^m)$.

Moreover, we can also suppose
\[
\lim \inf_{n \to \infty} J(u_n, v_n) = \lim_{n \to \infty} J(u_n, v_n) < +\infty.
\]
Then $J(u_n, v_n)$ is bounded and so, up to a subsequence, $\mu_n := f(x, u_n, v_n, \nabla u_n)dx \rightharpoonup \mu$ in the sense of measures for some positive measure $\mu$.

By the Radon-Nikodym theorem, $\mu = g\mathcal{L}^N + \mu_s$ for some $g \in L^1(\Omega)$, with $\mu_s$ singular with respect to $\mathcal{L}^N$. It will be enough to prove the following inequality:
\[
g(x) \geq f(x, u(x), v(x), \nabla u(x)), \quad \mathcal{L}^N - a.e. \ x \in \Omega.
\]

Indeed, once proved (14), since $\mu_n \rightharpoonup \mu$, by the lower semicontinuity of $\mu$, and since $\mu_s$ is nonnegative
\[
\lim_{n \to +\infty} J(u_n, v_n) = \lim_{n \to +\infty} \int_{\Omega} f(x, u_n(x), v_n(x), \nabla u_n(x)) \, dx
\]
\[
\geq \int_{\Omega} d\mu(x) = \int_{\Omega} g(x) \, dx + \int_{\Omega} d\mu_s(x)
\]
\[
\geq \int_{\Omega} f(x, u(x), v(x), \nabla u(x)) \, dx.
\]

In order to prove (14), we follow the proofs of Theorem 2.1 in [17] and condition (2.3) in [14]. We start freezing the terms $x$ and $u$. This will be achieved through Steps 1 to 5.
By the Besicovitch derivation theorem

$$g(x) = \lim_{\varepsilon \to 0} \frac{\mu(B_\varepsilon(x))}{|B_\varepsilon|} \in \mathbb{R} \quad L^N \text{ a.e. } x \in \Omega.$$  

(15)

Let \( x_0 \) be any element of \( \Omega \) satisfying \( |B_\varepsilon| \), \( |B_\varepsilon| \) and \( \Omega \) (notice that such an \( x_0 \) can be taken in \( \Omega \) up to a set of Lebesgue-measure zero) and let’s prove that \( g(x_0) \leq f(x_0, u(x_0), v(x_0), \nabla u(x_0)) \). First remark that, as noticed before, since \( v_n \to v \in L^p \), we have \( \|v_n\|_{L^p}, \|v\|_{L^p} \leq C \).

**Step 1. Localization.** This part can be reproduced in the same way as in [17], pages 1085-1086. We present some details for the reader’s convenience. We start providing a first estimate for \( g \). Observe that we can choose a sequence \( \varepsilon \to 0^+ \) such that \( \mu(\partial B_\varepsilon(x_0)) = 0 \). Let \( B := B_1(0) \). Applying Proposition 1.203 iii) in [19],

\[
g(x_0) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^N} \mu(B_\varepsilon(x_0)) |B| = \lim_{\varepsilon \to 0} \limsup_{n \to +\infty} \frac{1}{\varepsilon^N |B|} \int_{B_\varepsilon(x_0)} f(y, u_n(y), v_n(y), \nabla u_n(y)) dy
\]

\[
= \lim_{\varepsilon \to 0} \limsup_{n \to +\infty} \frac{1}{|B|} \int_B f(x_0 + \varepsilon x, u_n(x_0 + \varepsilon x), v_n(x_0 + \varepsilon x), \nabla u_n(x_0 + \varepsilon x)) dx
\]

\[
\geq \lim_{\varepsilon \to 0} \limsup_{n \to +\infty} \frac{1}{|B|} \int_B f(x_0 + \varepsilon x, u(x_0) + \varepsilon w_{n,\varepsilon}(x), v_n(x_0 + \varepsilon x), \nabla w_{n,\varepsilon}(x)) dx
\]

where \( w_{n,\varepsilon}(x) = u_n(x_0 + \varepsilon x) - u(x_0) \).

**Step 2. Blow-up.** Next we will “identify the limits” of \( w_{n,\varepsilon} \) and \( u_n(x_0 + \varepsilon \cdot) \) in a sense to be made precise below. Define \( w_0 : B \to \mathbb{R}^n \) such that \( w_0(x) = \nabla u(x_0)x \).

Then

\[
\lim_{\varepsilon \to 0} \limsup_{n \to +\infty} \frac{1}{\varepsilon^N+1} \int_{B_\varepsilon(x_0)} |u(y) - u(x_0) - \nabla u(x_0)(y - x_0)| dy = 0
\]

where we have used [13] in the last identity.

Let \( q \) be the Hölder’s conjugate exponent of \( p \). Since \( L^q \) is separable, consider \( \{\varphi_l\} \) a countable dense set of functions in \( L^q(B) \). Then

\[
\lim_{\varepsilon \to 0} \limsup_{n \to +\infty} \int_B (v_n(x_0 + \varepsilon x) - v(x_0))\varphi_l(x) dx = \lim_{\varepsilon \to 0} \int_B (v(x_0 + \varepsilon x) - v(x_0))\varphi_l(x) dx = 0
\]

where we have used in the last identity the fact that \( x_0 \) is a Lebesgue point for \( v \).

**Step 3. Diagonalization.** Arguing as in [18] and [14] we can use a diagonalization argument to find \( \varepsilon_n \in \mathbb{R}^+ \), \( w_n \in W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^n) \) and \( v_n \in L^p(B; \mathbb{R}^m) \cap L^\infty(B; \mathbb{R}^m) \).
$C_0^\infty(\mathbb{R}^N; \mathbb{R}^m)$, such that $\varepsilon_n \to 0$, $w_n \to w_0$ in $L^1(B; \mathbb{R}^n)$, $v_n \to v(x_0)$ in $L^p(B; \mathbb{R}^m)$ as $n \to +\infty$ and

$$g(x_0) \geq \lim_{n \to +\infty} \frac{1}{|B|} \int_B f(x_0 + \varepsilon_n x, u(x_0) + \varepsilon_n w_n(x), v_n(x), \nabla w_n(x)) \, dx.$$

**Step 4. Truncation.** We show that the sequences $\{w_n\}$ and $\{v_n\}$ constructed in the preceding steps can be replaced by sequences $\{\tilde{w}_n\} \subset W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^n)$ and $\{\tilde{v}_n\} \subset L^p(B; \mathbb{R}^m) \cap C_0^\infty(\mathbb{R}^N; \mathbb{R}^m)$ such that $\|\tilde{w}_n\|_{W^{1,\infty}(B; \mathbb{R}^n)} \leq C$, $\tilde{w}_n \to w_0$ in $L^\infty(B; \mathbb{R}^n)$, $\|\tilde{v}_n\|_{L^p(B; \mathbb{R}^m)} \leq C$, $\tilde{v}_n \to v(x_0)$ in $L^p(B; \mathbb{R}^m)$ and

$$g(x_0) \geq \lim_{k \to \infty} \frac{1}{\mathcal{L}^N(\tilde{B})} \int_{\tilde{B}} f(x_0 + r_n x, u(x_0) + r_n \tilde{v}_n(x), \tilde{v}_n(x), \nabla \tilde{w}_n(x)) \, dx.$$

Let $0 < s < t < 1$ and $\lambda > 1$ and define $\varphi_{s,t}$ a cut-off function such that $0 \leq \varphi_{s,t} \leq 1$, $\varphi_{s,t}(\tau) = 1$ if $\tau \leq s$, $\varphi_{s,t}(\tau) = 0$ if $\tau \geq t$ and $\|\varphi_{s,t}\|_\infty \leq \frac{C}{s}$. Set

$$w_{n,t}^{s,\lambda}(x) := w_0(x) + \varphi_{s,t} \left( \left| w_n(x) - w_0(x) \right| + \frac{|v_n(x)|}{\lambda} \right) (w_n(x) - w_0(x)),
$$

$$v_{n,t}^{s,\lambda}(x) := v(x_0) + \varphi_{s,t} \left( \left| v_n(x) - v_0(x) \right| + \frac{|v_n(x)|}{\lambda} \right) (v_n(x) - v(x_0)).$$

Clearly,

$$\|w_{n,t}^{s,\lambda} - w_0\|_\infty \leq t \quad \text{and} \quad v_{n,t}^{s,\lambda} \to v(x_0) \quad \text{in} \quad L^p \quad \text{as} \quad n \to +\infty. \quad (16)$$

Define

$$h_n(x, s, b, A) := f(x_0 + r_n x, u(x_0) + r_n s, b, A).$$

By the growth conditions there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$c (|b|^p + |A|) - C \leq h_n(x, s, b, A) \leq C (|b|^p + |A| + 1) \quad (17)$$

for some constants $c$, $C > 0$. Consequently there exist $C > 0$ such that

$$-C \leq h_n(x, w_0(x), v(x_0), \nabla w_0(x)) \leq C.$$

Also

$$\int_B h_n(x, w_{n,t}^{s,\lambda}(x), \tilde{v}_{n,t}^{s,\lambda}(x), \nabla u_{n,t}^{s,\lambda}(x)) \, dx =$$

$$= \int_{B \cap \{|w_n(x) - w_0(x)| + \frac{|v_n(x)|}{\lambda} \leq s\}} h_n(x, w_n, v_n, \nabla w_n) \, dx$$

$$+ \int_{B \cap \{s < |w_n(x) - w_0(x)| + \frac{|v_n(x)|}{\lambda} \leq t\}} h_n(x, w_{n,t}^{s,\lambda}, \tilde{v}_{n,t}^{s,\lambda}, \nabla u_{n,t}^{s,\lambda}) \, dx$$

$$+ \int_{B \cap \{|w_n(x) - w_0(x)| + \frac{|v_n(x)|}{\lambda} > t\}} h_n(x, w_0(x), v(x_0), \nabla w_0(x)) \, dx$$

$$:= I_1 + I_2 + I_3.$$
By the growth conditions and the definition of $h_n$, we have that
\[
I_3 \leq C \left\{ x \in B : |w_n(x) - w_0(x)| + \frac{|v_n(x)|}{\lambda} > t \right\}.
\]

On the other hand, if $s < |w_n(x) - w_0(x)| + \frac{|v_n(x)|}{\lambda} < t$ then
\[
\nabla w_{n,t}^s(x) = \nabla u(x_0) + \varphi_{s,t} \left( |w_n(x) - w_0(x)| + \frac{|v_n(x)|}{\lambda} \right) \nabla w_n(x) - \nabla w_0(x) + (w_n(x) - w_0(x)) \otimes \varphi'_{s,t} \left( |w_n(x) - w_0(x)| + \frac{|v_n(x)|}{\lambda} \right) \nabla \left( |w_n(x) - w_0(x)| + \frac{|v_n(x)|}{\lambda} \right).
\]

By \(17\) we have
\[
I_2 \leq C \int_{B \cap \{ s < |w_n(x) - w_0(x)| + \frac{|v_n(x)|}{\lambda} \leq t \}} \left( 1 + |\nabla w_n(x) - \nabla w_0(x)| + |v_n(x) - v(x_0)|^p \right) dx + \frac{C}{t - s} \int_{B \cap \{ s < |w_n(x) - w_0(x)| + \frac{|v_n(x)|}{\lambda} \leq t \}} |w_n(x) - w_0(x)| |\nabla (|w_n(x) - w_0(x)| + \frac{|v_n(x)|}{\lambda})| dx.
\]

We remark that for almost every $t$ and $\lambda$ we have
\[
\lim_{s \to t^{-}} \int_{s < |w_n(x) - w_0(x)| + \frac{|v_n(x)|}{\lambda} < t} \left( 1 + |\nabla w_n(x) - \nabla u(x_0)| + |v_n(x) - v(x_0)|^p \right) dx = 0
\]  \hspace{1cm} (18)

and by the coarea formula
\[
\lim_{s \to t^{-}} \frac{1}{t - s} \int_{B \cap \{ s < |w_n(x) - w_0(x)| + \frac{|v_n(x)|}{\lambda} \leq t \}} |w_n(x) - w_0(x)| \nabla \left( |w_n(x) - w_0(x)| + \frac{|v_n(x)|}{\lambda} \right) dx \leq \lim_{s \to t^{-}} \frac{1}{t - s} \int_{B \cap \{ s < |w_n(x) - w_0(x)| + \frac{|v_n(x)|}{\lambda} \leq t \}} \left( |w_n(x) - w_0(x)| + \frac{|v_n(x)|}{\lambda} \right) dx \leq t \mathcal{H}^{N-1} \left( \left\{ x \in B : |w_n(x) - w_0(x)| + \frac{|v_n(x)|}{\lambda} = t \right\} \right), \hspace{1cm} (19)
\]

Due to the fact that \( \{v_n\} \) is a \( C_0^\infty(\mathbb{R}^N; \mathbb{R}^m) \) sequence, for every $C > 0$, for every $n$ there exists $\lambda_n \in (1, +\infty)$ such that $\lambda_n \leq \lambda_{n+1}$, $\lambda_n \to +\infty$ as $n \to +\infty$. 

and \( \int_B \frac{\nabla |v_n|}{\lambda_n} \, dx \leq C. \) On the other hand by (17)

\[
\int_{B \cap \{|w_n(x) - w_0(x)| + \frac{|v_n(x)|}{\lambda_n} \leq 1\}} \nabla \left( |w_n(x) - w_0(x)| \right) \, dx \\
\leq \int_{B \cap \{|w_n(x) - w_0(x)| + \frac{|v_n(x)|}{\lambda_n} \leq 1\}} (|\nabla w_n(x)| + C) \, dx \\
\leq C \int_B h_n(x, u_n(x), v_n(x), \nabla w_n(x)) \, dx \leq C
\]

since \( \{w_n\} \) and \( \{v_n\} \) are convergent.

Thus

\[
\int_{B \cap \{|w_n(x) - w_0(x)| + \frac{|v_n(x)|}{\lambda_n} \leq 1\}} \left| \nabla \left( |w_n(x) - w_0(x)| + \frac{|v_n(x)|}{\lambda_n} \right) \right| \, dx \leq C.
\]

Recall that \( \int_B \frac{|v_n|^p}{\lambda_n} \, dx \leq C \) and by Hölder’s inequality also \( \int_B \frac{|v_n|}{\lambda_n} \, dx \leq C. \)

Hence, by Lemma 2.6 in [17] there exists

\[
t_n \in \left[ \left( \|w_n - w_0\|_{L^1} + \frac{\|v_n\|_{L^1}}{\lambda_n} \right)^{\frac{1}{p}}, \left\| (\|w_n - w_0\|_{L^1} + \frac{\|v_n\|_{L^1}}{\lambda_n}) \right\| \right]
\]

such that (18) and (19) hold (with \( t = t_n \)), and

\[
t_n \mathcal{H}^{-1} \left( \{ x \in B : |w_n(x) - w_0(x)| + \frac{|v_n(x)|}{\lambda_n} = t_n \} \right) \leq C \ln \left( \frac{C}{\lambda_n} \right) \cdot \left( \|w_n - w_0\|_{L^1} + \frac{\|v_n\|_{L^1}}{\lambda_n} \right)^{\frac{1}{p}}.
\]

According to (18) and (19) we may choose \( 0 < s_n < t_n \) such that

\[
\int_{\{s_n < |w_n(x) - w_0(x)| + \frac{|v_n(x)|}{\lambda_n} \leq t_n\}} (1 + |\nabla w_n(x) - \nabla u(x_0)| + |v_n(x) - v(x_0)|^p) \, dx = O \left( \frac{1}{n} \right),
\]

\[
\frac{1}{t_n - s_n} \int_{B \cap \{s_n < |w_n(x) - w_0(x)| + \frac{|v_n(x)|}{\lambda_n} \leq t_n\}} \left( |w_n(x) - w_0(x)| + \frac{|v_n(x)|}{\lambda_n} \right) \, dx \\
\leq t_n \mathcal{H}^{-1} \left( \{ x \in B : |w_n(x) - w_0(x)| + \frac{|v_n(x)|}{\lambda_n} = t_n \} \right) + O \left( \frac{1}{n} \right)
\]

Set

\[
\bar{w}_n(x) := w_n^{s_n, t_n}(x), \quad \bar{v}_n(x) := v_n^{s_n, t_n}(x)
\]

thus by (18)

\[
\| \bar{w}_n - w_0 \|_\infty \leq t_n \to 0, \quad \bar{v}_n \to v(x_0) \text{ in } L^p \text{ as } n \to \infty.
\]
Using the previous estimates we conclude that

\[ g(x_0) \geq \lim_{n \to \infty} \frac{1}{\mathcal{L}^N(B)} \int_B f(x_0 + r_n x, u(x_0) + r_n w_n(x), v_n(x), \nabla w_n(x)) \, dx \]

\[ \geq \liminf_{n \to \infty} \frac{1}{\mathcal{L}^N(B)} \int_{B \cap \{ |w_n(x) - w_0(x)| + \frac{|v_n(x)|}{\lambda_n} \leq \varepsilon \}} h_n(x, w_n(x), v_n(x), \nabla w_n(x)) \, dx \]

\[ \geq \liminf_{n \to \infty} \frac{1}{\mathcal{L}^N(B)} \int_B h_n(x, \bar{w}_n(x), \bar{v}_n(x), \nabla \bar{w}_n(x)) \, dx - O\left( \frac{1}{n} \right) + \]

\[ - \frac{\ln \left( \frac{\|w_n - w_0\|_{L^1(B)} + \frac{\|v_n\|_{L^1(B)}}{\lambda_n}}{C} \right)^{-\frac{1}{p}}}{C} \]

\[ - C \mathcal{L}^N \left( \left\{ x \in B : |w_n(x) - w_0(x)| + \frac{|v_n(x)|}{\lambda_n} > t_n \right\} \right) \]

\[ = \liminf_{n \to \infty} \frac{1}{\mathcal{L}^N(B)} \int_B h_n(x, \bar{w}_n(x), \bar{v}_n(x), \nabla \bar{w}_n(x)) \, dx, \]

since

\[ t_n \geq \left( \frac{\|w_n - w_0\|_{L^1(B)} + \frac{\|v_n\|_{L^1(B)}}{\lambda_n}}{\lambda_n} \right)^{\frac{1}{p}} \]

and thus

\[ \mathcal{L}^N \left( \left\{ x \in B : |w_n(x) - w_0(x)| + \frac{|v_n(x)|}{\lambda_n} > t_n \right\} \right) \leq \frac{1}{t_n} \left( \frac{\|w_n - w_0\|_{L^1(B)} + \frac{\|v_n\|_{L^1(B)}}{\lambda_n}}{\lambda_n} \right) \]

\[ \leq \left( \frac{\|w_n - w_0\|_{L^1(B)} + \frac{\|v_n\|_{L^1(B)}}{\lambda_n}}{\lambda_n} \right)^{\frac{1}{p}} \to 0. \]

The bound of \( \{ \| \nabla \bar{w}_n \|_{L^1} \} \) follows from \([17]\).

**Step 5.** We now fix in \( f \) the value of \( x \) and \( u \). Indeed, using hypothesis \((H2)\) and the fact that \( \nabla \bar{w}_n \) and \( |\bar{v}_n|^p \) have bounded \( L^1 \) norm, one gets

\[ g(x_0) \geq \limsup_{n \to +\infty} \frac{1}{|B|} \int_B f(x_0 + \varepsilon_n x, u(x_0) + \varepsilon_n \bar{w}_n(x), \bar{v}_n(x), \nabla \bar{w}_n(x)) \, dx \]

\[ \geq \limsup_{n \to +\infty} \frac{1}{|B|} \int_B f(x_0, u(x_0), \bar{v}_n(x), \nabla \bar{w}_n(x)) \, dx. \]

**Step 6.** At this point we are in an analogous context to \([14]\) and the desired inequality follows in the same way. It relies on the slicing method in order to modify \( \bar{v}_n \) and \( \bar{w}_n \) and exploit the convex-quasiconvexity of \( f \), namely it is possible to find new sequences, denoted by \( \bar{v}_n \) and \( \bar{w}_n \) such that

\[ \frac{1}{|B|} \int_B \bar{v}_n(z) \, dz = v(x_0) \text{ and } \bar{w}_j \in w_0 + W^{1,\infty}_0(B; \mathbb{R}^n). \]
4 Relaxation in $W^{1,1} \times L^\infty$

This section is devoted to characterize the relaxed functional $J_\infty$ introduced in [3]. Indeed we prove the following relaxation result

**Theorem 12.** Let $\Omega$ be a bounded open set of $\mathbb{R}^N$, and let $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times N} \to [0, +\infty)$ be a continuous function. Then, assuming that $f$ and $CQf$ satisfy hypotheses $(H1_\infty)$ and $(H2_\infty)$

$$J_\infty(u, v) = \int_{\Omega} CQf(x, u(x), v(x), \nabla u(x)) \, dx,$$

for every $(u, v) \in W^{1,1}(\Omega; \mathbb{R}^n) \times L^\infty(\Omega; \mathbb{R}^m)$.

**Remark 13.**
1) We recall that if hypotheses $(H1_\infty)$ and $(H2_\infty)$ are replaced by $(\mathcal{S} - \mathcal{H})$, Propositions 6 and 8 guarantee the validity of Theorem 12 assuming that only $f$ satisfies $(\mathcal{S} - \mathcal{H})$.

2) We also observe that Theorem 12 can be proven also imposing $(H1_\infty)$ and $(H2_\infty)$ only on the function $f$ but with the further requirement that $f$ satisfies $(\mathcal{S})$.

3) We also stress that if $f$ satisfies $(H1_p)$ and $(H2_p)$ then clearly $J_p(u, v) \leq J_\infty(u, v)$ for every $(u, v) \in BV(\Omega; \mathbb{R}^n) \times L^\infty(\Omega; \mathbb{R}^m)$.

**Proof of Theorem 12.** The thesis will be achieved by double inequality. Clearly the lower bound can be proven as for the case $W^{1,1} \times L^p$, with a proof easier than that of Theorem 11 since it is not necessary ‘truncate’ the $\{v_n\}$ which are already bounded in $L^\infty$. For what concerns the upper bound, we first observe that by virtue of Proposition 8 there is no loss of generality in assuming $f$ already convex-quasiconvex. In order to provide an upper bound for $J_\infty$ we start by localizing our functional. The following procedure is entirely similar to [3, Theorem 4.3]. We define for every open set $A \subset \Omega$ and for any $(u, v) \in BV(\Omega; \mathbb{R}^n) \times L^\infty(\Omega; \mathbb{R}^m)$

$$\overline{F}_\infty(u, v, A) := \inf \left\{ \liminf_n F(u_n, v_n, A) : u_n \to u \in L^1(A; \mathbb{R}^n), v_n \rightharpoonup v \text{ in } L^\infty(A; \mathbb{R}^m) \right\}$$

where

$$F(u, v, A) = \begin{cases} \int_A f(x, u(x), v(x), \nabla u(x)) \, dx & \text{if } (u, v) \in W^{1,1}(A; \mathbb{R}^n) \times L^\infty(A; \mathbb{R}^m), \\ +\infty & \text{in } (L^1(A; \mathbb{R}^n) \setminus W^{1,1}(A; \mathbb{R}^n)) \times L^\infty(A; \mathbb{R}^m). \end{cases}$$

We start remarking that $(H1_\infty)$ implies that for every $u \in BV(\Omega; \mathbb{R}^n)$ and for every $v \in L^\infty(\Omega; \mathbb{R}^m)$ such that $\|v\|_{L^\infty} \leq M$, there exists a constant $C_M$ such that $\overline{F}_\infty(u, v, A) \leq C_M(|A| + |Du|(A))$. Moreover one has

1) $\overline{F}_\infty$ is local, i.e. $\overline{F}_\infty(u, v, A) = \overline{F}_\infty(u', v', A)$, for every $A \subset \Omega$ open, $(u, v), (u', v') \in L^1(A; \mathbb{R}^n) \times L^\infty(A; \mathbb{R}^m)$. 

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2) $\overline{F}_\infty$ is sequentially lower semi-continuous, i.e. $\overline{F}_\infty(u, v, A) \leq \lim \inf_{\epsilon \to 0} \overline{F}_\infty(u_n, v_n, A)$, \( \forall \, A \subset \Omega \) open, \( \forall \, u_n \to u \) in $L^1(A; \mathbb{R}^n)$ and $v_n \overset{\ast}{\rightharpoonup} v$ in $L^\infty(A; \mathbb{R}^m)$.

3) $\overline{F}_\infty(u, v, \cdot)$ is the trace on $A(\Omega) := \{ A \subset \Omega : A \text{ is open} \}$ of a Borel measure in $B(\Omega)$ (the Borelians of $\Omega$).

Condition 1) follows from the fact the adopted convergence doesn’t see sets of null Lebesgue measure. Condition 2) follows by a diagonalization argument, entirely similar to the proof of (ii) in [13]. Condition 3) follows applying De Giorgi-Letta criterium, (cf. [12]) and indeed proving that for any fixed $(u, v) \in BV(\Omega; \mathbb{R}^m) \times L^\infty(\Omega; \mathbb{R}^m)$,

$$\overline{F}_\infty(u, v, A) \leq \overline{F}_\infty(u, v, C) + \overline{F}_\infty(u, v, A \setminus B), \ \forall \, A, B, C \in A(\Omega).$$

We omit the details, since they are very similar to the proof of Theorem 4.3 in [4]. The only difference consists of the fact that one has to deal with both $u$’s and $v$’s and exploit the growth condition (H1\(_\infty\)).

Since $\overline{F}_\infty(u, v) = \overline{F}_\infty(u, v, \Omega)$ and $\overline{F}_\infty(u, v, \cdot)$ is the trace of a Radon measure on the open subsets of $\Omega$, (i.e. $A(\Omega)$) absolutely continuous with respect to $|Du| + L^N$, it will be enough to prove the following inequality

$$\frac{d\overline{F}_\infty(u, v, \cdot)}{dL^N}(x) \leq f(x, u(x), v(x), \nabla u(x)), \ L^N - \text{a.e. } x \in \Omega.$$

The proof of these inequalities follows closely [11, 4] and [5]. Assume first that $(u, v) \in (W^{1,1}(\Omega; \mathbb{R}^n) \cap L^\infty(\Omega; \mathbb{R}^n)) \times L^\infty(\Omega; \mathbb{R}^m)$. Fix a point $x_0 \in \Omega$ such that

$$\frac{d\overline{F}_\infty(u, v, \cdot)}{dL^N}(x_0)$$

exists and is finite, which is also a Lebesgue point of $u$, $v$ and $\nabla u$ and a point of approximate differentiability for $u$. Clearly $L^N$-a.e. $x_0 \in \Omega$ satisfy all the above requirements.

As in [13] (see formula (5.6) therein) we may also assume that

$$\lim_{\epsilon \to 0} \frac{1}{|Q(x_0, \epsilon)|} \int_{Q(x_0, \epsilon)} |u(x) - u(x_0)| (1 + |\nabla u(x)|) dx = 0,$$

$$\lim_{\epsilon \to 0} \frac{1}{|Q(x_0, \epsilon)|} \int_{Q(x_0, \epsilon)} |v(x) - v(x_0)| |\nabla u(x)| dx = 0,$$

(21)

(22)

(23)

(23)

where we used Theorem [10] since $v \in L^1_{loc}(\Omega; \mathbb{R}^m)$ with respect to the measure $|\nabla u|L^N$. Choose a sequence of numbers $\epsilon \in (0, \text{dist}(x_0, \partial \Omega))$. Then, clearly for any sequences $\{u_n\}, u_n \to u$ in $L^1$, $\{v_n\}, v_n \overset{\ast}{\rightharpoonup} v$ in $L^\infty$,

$$\frac{\partial \overline{F}_\infty(u, v, \cdot)}{\partial L^N}(x_0) = \lim_{\epsilon \to 0^+} \frac{\overline{F}_\infty(u, v, B_\epsilon(x_0))}{|B_\epsilon(x_0)|} \leq$$

$$\lim \inf \lim \inf_{\epsilon \to 0^+, n \to +\infty} \frac{1}{|B_\epsilon(x_0)|} \int_{B_\epsilon(x_0)} f(x, u_n(x), v_n(x), \nabla u_n(x)) dx.$$
By virtue of Proposition 2.2 in [2] we can replace the ball $B_\varepsilon(x_0)$ in (23) by a cube of side length $\varepsilon$, and in fact from now on we consider such cubes.

As in Proposition 4.6 of [4], (see also [18] and [15]) we consider the Yosida transforms of $f$, defined as

$$f_\lambda(x, u, v, \xi) := \sup_{(x', u') \in \Omega \times \mathbb{R}^n} \{ f(x', u', v, \xi) - \lambda(|x - x'| + |u - u'|)(1 + |\xi| + |v|) \}$$

for every $\lambda > 0$. Then

(i) $f_\lambda(x, u, v, \xi) \geq f(x, u, v, \xi)$ and $f_\lambda(x, u, v, \xi)$ decreases to $f(x, u, v, \xi)$ as $\lambda \to +\infty$.

(ii) $f_\lambda(x, u, v, \xi) \geq f_\eta(x, u, v, \xi)$ if $\lambda \leq \eta$ for every $(x, u, v, \xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times N}$.

(iii) $|f_\lambda(x, u, v, \xi) - f_\lambda(x', u', v, \xi)| \leq \lambda(|x - x'| + |u - u'|)(1 + |\xi| + |v|)$ for every $(x, u, v, \xi), (x', u', v, \xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times N}$.

(iv) The approximation is uniform on compact sets. Precisely, let $K$ be a compact subset of $\Omega \times \mathbb{R}^n$ and let $\delta > 0$. There exists $\lambda > 0$ such that $f(x, u, v, \xi) \leq f_\lambda(x, u, v, \xi) \leq f(x, u, v, \xi) + \delta(1 + |v| + |\xi|)$ for every $(x, u, v, \xi) \in K \times \mathbb{R}^m \times \mathbb{R}^{n \times N}$.

Let $x_0$ such that (20) and (21) hold, let $\{ \varrho_n \}$ be a sequence of standard symmetric mollifiers and set

$$\begin{cases} u_n := u * \varrho_n, \\ v_n := v. \end{cases}$$

It results that

$$\begin{cases} u_n \to u & \text{in } L^1(Q(x_0, \varepsilon); \mathbb{R}^n), \\ v_n \to v & \text{in } L^\infty(Q(x_0, \varepsilon); \mathbb{R}^m). \end{cases}$$

Fix $\delta > 0$ and let $K := \overline{B(x_0, \frac{\text{dist}(x_0, \partial \Omega)}{2}) \times \overline{B}(0, \|u\|_\infty)}$. By (i) \(\div (iv)\),

$$f(x, u_n(x), v(x), \nabla u_n(x)) \leq f_\lambda(x, u_n(x), v(x), \nabla u_n(x)) \leq$$

$$f(x_0, u(x_0), v(x_0), \nabla u_n(x_0)) + \lambda(|x - x_0| + |u_n(x) - u(x_0)|)(1 + |v| + |\nabla u_n(x)|) \leq$$

$$f(x_0, u(x_0), v(x), \nabla u_n(x)) + \delta(1 + |\nabla u_n(x)| + |v(x)|) + \lambda(|x - x_0| + |u_n(x) - u(x_0)|)$$

$$\times (1 + |\nabla u_n(x)| + |v(x)|).$$
Since $\nabla u_n(x) = (\nabla u * g_n)(x)$,

$$
F^\infty_\infty(u, v, Q(x_0, \varepsilon)) \leq \liminf_{n \to +\infty} \int_{Q(x_0, \varepsilon)} f(x, u_n(x), v(x), \nabla u_n(x)) \, dx 
$$

$$
\liminf_{n \to +\infty} \int_{Q(x_0, \varepsilon)} f(x_0, u(x_0), v(x_0), \nabla u(x_0)) \, dx + 
$$

$$
\limsup_{n \to +\infty} \int_{Q(x_0, \varepsilon)} \delta(1 + |\nabla u_n(x)| + |v(x)|) + \lambda(|x - x_0| + |u_n(x) - u(x_0)|) \times (1 + |\nabla u_n(x)| + |v(x)|) \, dx \leq 
$$

$$
\liminf_{n \to +\infty} \int_{Q(x_0, \varepsilon)} f(x_0, u(x_0), v(x_0), \nabla u(x_0)) \, dx + 
$$

$$
\limsup_{n \to +\infty} \int_{Q(x_0, \varepsilon)} \beta(1 + |\nabla u(x)| + |\nabla u * g_n|) |v(x) - v(x_0)| \, dx + 
$$

$$
\limsup_{n \to +\infty} \int_{Q(x_0, \varepsilon)} \beta |\nabla u * g_n - \nabla u(x_0)| \, dx 
$$

(24)

(25)

(25)

Passing to the limit on the right hand side of (24), exploiting (25) in the third line and applying [15, Lemma 2.5], in the fourth line, we get

$$
F^\infty_\infty(u, v, Q(x_0, \varepsilon)) \leq |Q(x_0, \varepsilon)|((f(x_0, u(x_0), v(x_0), \nabla u(x_0))) + 
$$

$$
\beta(1 + |\nabla u(x_0)|) \int_{Q(x_0, \varepsilon)} |v(x) - v(x_0)| \, dx + 
$$

$$
\beta \limsup_{n \to +\infty} \int_{Q(x_0, \varepsilon)} |\nabla u * g_n| |v(x) - v(x_0)| \, dx + \int_{Q(x_0, \varepsilon)} |\nabla u(x) - \nabla u(x_0)| \, dx + 
$$

$$
(\lambda \varepsilon + \delta)[(1 + C)|Q(x_0, \varepsilon)|] + \lambda \limsup_{n \to +\infty} \int_{Q(x_0, \varepsilon)} |u_n - u(x_0)|(1 + C + |\nabla u_n|) \, dx.
$$
Recalling that $x_0$ is a Lebesgue point for $v$, $\nabla u$ and (20) holds, we have
\[
\limsup_{\varepsilon \to 0^+} \frac{1}{|Q(x_0, \varepsilon)|} \int_{Q(x_0, \varepsilon)} |v(x) - v(x_0)|\,dx = 0,
\]
\[
\limsup_{\varepsilon \to 0^+} \frac{1}{|Q(x_0, \varepsilon)|} \int_{Q(x_0, \varepsilon)} |\nabla u(x) - \nabla u(x_0)|\,dx = 0,
\]
\[
\limsup_{\varepsilon \to 0^+} (\lambda \varepsilon + \delta)(1 + C) \frac{|Q(x_0, \varepsilon)|}{|Q(x_0, \varepsilon)|} = \delta(1 + C).
\]
Moreover by virtue of (21) and arguing as in the estimate of formula (5.11) of [18] we can conclude that
\[
\limsup_{\varepsilon \to 0^+} \frac{\lambda}{|Q(x_0, \varepsilon)|} \limsup_{n \to +\infty} \int_{Q(x_0, \varepsilon)} |u_n - u(x_0)|(1 + C + |\nabla u_n|)\,dx = 0.
\]
Then we can exploit (22) and argue again as done for (5.11) in [18] in order to evaluate
\[
\limsup_{\varepsilon \to 0^+} \frac{\beta}{|Q(x_0, \varepsilon)|} \frac{\lambda}{n \to +\infty} \int_{Q(x_0, \varepsilon)} |\nabla u * \varrho_n||v(x) - v(x_0)|\,dx.
\]
We will apply [18, Lemma 2.5] and the dominated convergence theorem with respect to the measure $|\nabla u|\,dx$, obtaining
\[
\limsup_{n \to +\infty} \int_{Q(x_0, \varepsilon)} |v(x) - v(x_0)||\nabla u_n(x)|\,dx \leq
\]
\[
\limsup_{n \to +\infty} \int_{Q(x_0, \varepsilon)} |v(x) - v(x_0)||\nabla u(x)|\,dx \leq
\]
\[
\int_{Q(x_0, \varepsilon)} |v(x) - v(x_0)||\nabla u(x)|\,dx.
\]
Taking into account that $|Du|((\partial Q(x_0, \varepsilon))) = 0$ for a.e. $\varepsilon$ one obtains from (21) that
\[
\limsup_{\varepsilon \to 0^+} \limsup_{n \to +\infty} \frac{1}{|Q(x_0, \varepsilon)|} \int_{Q(x_0, \varepsilon)} |v(x) - v(x_0)||\nabla u_n(x)|\,dx = 0.
\]
Consequently,
\[
g(x_0) = \frac{\partial F_{\infty}(u, v)(x_0)}{\partial L^N} \leq f(x_0, u(x_0), v(x_0), \nabla u(x_0)) + (1 + C)\delta
\]
Finally, we send $\delta$ to 0 and that concludes the proof, when $(u, v) \in (W^{1, 1}(\Omega; \mathbb{R}^n) \cap L^\infty(\Omega; \mathbb{R}^n)) \times L^\infty(\Omega; \mathbb{R}^m)$. 

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To conclude the proof, we can argue as in [18, Theorem 2.16, Step 4], in turn inspired by [4], introducing the following approximation. Let \( \phi_n \in C_0^1(R^n; R^n) \) be such that \( \phi_n(y) = y \) if \( y \in B_n(0) \), \( \| \nabla \phi_n \|_{L^\infty} \leq 1 \). By [3, Theorem 3.96] \( \phi_n(u) \in W^{1,1}(\Omega; R^n) \cap L^\infty(\Omega; R^n) \) for every \( n \in \mathbb{N} \). Since \( \phi_n(u) \to u \) in \( L^1 \), by the lower semicontinuity of \( J_\infty \) we get

\[
J_\infty(u, v) \leq \liminf_{n \to +\infty} \int_\Omega f(x, \phi_n(u), v, \nabla \phi_n(u)) \, dx.
\]

Arguing in analogy with [4, Theorem 4.9] one can prove that

\[
\limsup_{n \to +\infty} \int_\Omega f(x, \phi_n(u), v, \nabla \phi_n(u)) \, dx \leq \int_\Omega f(x, u, v, \nabla u) \, dx,
\]

and this concludes the proof.

\[\square\]

5 Relaxation in \( W^{1,1} \times L^p \)

This section is devoted to the proof of the following theorem. It relies on Theorem 12 and on some approximation results (see [4]).

**Theorem 14.** Let \( \Omega \) be a bounded open set of \( R^N \), and let \( f : \Omega \times R^n \times R^m \times R^n \times N \to [0, +\infty) \) be a continuous function. Then, assuming that \( f \) satisfies hypotheses \((H1_p)\) and \((H2_p)\)

\[
J_p(u, v) = \int_\Omega CQf(x, u(x), v(x), \nabla u(x)) \, dx,
\]

for every \( (u, v) \in W^{1,1}(\Omega; R^n) \times L^p(\Omega; R^m) \).

**Proof.** The lower bound follows from Theorem 11. For what concerns the upper bound, without loss of generality, by virtue of Lemma 8 and Proposition 5 we may assume that \( f \) is convex-quasiconvex.

Observe first that since \( f \) fulfills \((H1_p)\) and \((H2_p)\), then it satisfies \((H1_\infty)\) and \((H2_\infty)\) in the strong form \((8) - (10)\). Consequently

\[
J_p(u, v, \Omega) \leq J_\infty(u, v, \Omega) \tag{26}
\]

for every \((u, v) \in BV(\Omega; R^n) \times L^\infty(\Omega; R^m)\).

For every positive real number \( \lambda \), let \( \tau_\lambda : [0, +\infty) \to [0, +\infty) \) be defined as

\[
\tau_\lambda(t) = \begin{cases} 
  t & \text{if } 0 \leq t \leq \lambda, \\
  0 & \text{if } t \geq \lambda.
\end{cases}
\]

For every \( v \in L^p(\Omega; R^m) \), define \( v_\lambda := \tau_\lambda(|v|)v \). Clearly \( \int_\Omega |v_\lambda|^pdx \leq \int_\Omega |v|^pdx \) and \( v_\lambda \to v \) in \( L^p(\Omega; R^m) \), as \( \lambda \to +\infty \). By the lower semicontinuity of \( J_p \), (26), and Theorem 12 for every sequence \( \lambda \) such that \( \lambda \to +\infty \) we have that

\[
J_p(u, v) \leq \liminf_{\lambda \to +\infty} J_p(u, v_\lambda) = \liminf_{\lambda \to +\infty} \int_\Omega f(x, u(x), v_\lambda(x), \nabla u(x)) \, dx.
\]
Lebesgue’s dominated convergence Theorem entails that
\[
J_p(u, v, \Omega) = \int_{\Omega} f(x, u(x), v(x), \nabla u(x)) dx,
\]
for every \((u, v) \in W^{1,1}(\Omega; \mathbb{R}^n) \times L^p(\Omega; \mathbb{R}^m)\), and that concludes the proof. □

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