MAXIMAL OPERATOR IN DUNKL-FOFANA SPACES

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Abstract. We generalize Wiener amalgam spaces by using Dunkl translation instead of the classical one, and we give some relationship between these spaces, Dunkl-Lebesgue spaces and Dunkl-Morrey spaces. We prove that the Hardy-Littlewood maximal function associated with the Dunkl operator is bounded on these generalized Dunkl-Morrey spaces.

1. Introduction

For $1 \leq p, q \leq \infty$, the amalgam of $L^q$ and $L^p$ on the real line is the space $(L^q, L^p)$ of complex values functions $f$ on $\mathbb{R}$ which are locally in $L^q$ and such that the function $y \mapsto \|f \chi_{I(y,1)}\|_{L^p}$ belongs to $L^p(\mathbb{R})$, where $\chi_{I(y,1)}$ denotes the characteristic function of the open interval $I(y,1) = (y - 1, y + 1)$ and $\|\cdot\|_{L^q}$ the usual Lebesgue norm in $L^q$.

We recall that for $q < \infty$

$$\|f \chi_{I(y,1)}\|_{L^q} \leq \left( \int_{I(0,1)} |f(u+y)|^q \, du \right)^{\frac{1}{q}} = \| (-y)^q f \chi_{I(0,1)} \|_{L^q},$$

where $(\circ g)(x) = g(x - y)$ is the classical translation.

In the last 30 years, many works in Harmonic analysis has been focusing to the generalization in the setting of Dunkl analysis, classical Fourier analysis result. The Dunkl translation (see Section 2) is an important tool in this task. It is natural to seek to know what become Wiener amalgam spaces once we replace classical translation by Dunkl one.

Taking in the definition of Wiener amalgam space the generalized translation as defined in [22], we give the analogue of the Wiener amalgam spaces and look at some subspaces of this spaces as well as the boundedness of the Hardy-Littlewood maximal operator associated with the Dunkl operator. For this purpose we define the function spaces using the harmonic analysis, associated with the Dunkl operator on $\mathbb{R}$. The generalized shift operators we are considering are associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. For the basic properties of the Dunkl analysis, we refer the reader to [6, 18, 22] and the references therein. The Lebesgue measure on the real line will be denoted by $dx$. For any
Lebesgue measurable subset $E$ of $\mathbb{R}$, $|E|$ stands for its Lebesgue measure. We denote by $\mathcal{E}$ the space of functions of class $C^\infty$ and by $C_c$ the space of continuous functions with compact supports.

This paper is organized as follow:

Section 2 is devoted for prerequisite on Dunkl analysis on the real line while Section 3 deal with the definition and properties of Dunkl-amalgam spaces. We introduce Fofana spaces associated to the Dunkl-amalgam in Section 4, and compare this space with Lebesgue spaces and Morrey-Dunkl spaces. In Section 5, we stated our results for Dunkl-type Hardy-Littlewood maximal function and give their proofs in Section 6.

The letter $C$ will be used for non-negative constants independent of the relevant variables, and this constant may change from one occurrence to another. Constant with subscript, such as $C_\kappa$, do not change in different occurrences but depend on the parameters mentioned in it.

We propose the following abbreviation $A \lesssim B$ for the inequalities $A \leq CB$, where $C$ is a positive constant independent of the main parameters. If $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$.

2. Prerequisite on Dunkl analysis on the real line

For a real parameter $\kappa > -1/2$, we consider the Dunkl operator, associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$:

$$\Lambda_\kappa f(x) = \frac{df}{dx}(x) + \frac{2\kappa + 1}{x} \left( \frac{f(x) - f(-x)}{2} \right).$$

For $\lambda \in \mathbb{C}$, the Dunkl kernel denoted $\mathcal{E}_\kappa(\lambda)$ (see [5]), is the unique solution of the initial value problem

$$\Lambda_\kappa f(x) = \lambda f(x), \quad f(0) = 1, \quad x \in \mathbb{R}.$$

It is given by the formula

$$\mathcal{E}_\kappa(\lambda x) = j_\kappa(i\lambda x) + \frac{\lambda x}{2(\kappa + 1)} j_{\kappa+1}(i\lambda x), \quad x \in \mathbb{R},$$

where

$$j_\kappa(z) = 2^\kappa \Gamma(\kappa + 1) \frac{J_\kappa(z)}{z^\kappa} = \Gamma(\kappa + 1) \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{n! 2^{2n} \Gamma(n + \kappa + 1)}$$

is the normalized Bessel function of the first kind and of order $\kappa$. Notice that $\Lambda_{-1/2} = \frac{d}{dx}$ and $\mathcal{E}_{-1/2}(\lambda x) = e^{\lambda x}$. It is also proved (see [18]) that $|\mathcal{E}_\kappa(ix)| \leq 1$ for every $x \in \mathbb{R}$.

In the rest of the paper, we fix $\kappa > -1/2$, and let $\mu$ be the weighted Lebesgue measure on $\mathbb{R}$, given by

$$(2.1) \quad d\mu(x) := (2^{\kappa+1} \Gamma(\kappa + 1))^{-1} |x|^{2\kappa+1} dx.$$
For every $1 \leq p \leq \infty$, we denote by $L^p(\mu)$ the Lebesgue space associated to the measure $\mu$, while $L^0(\mu)$ stands for the complex vector space of equivalent classes (modulo equality $\mu$-almost everywhere) of $\mu$-measurable complex-valued functions on $\mathbb{R}$. For $f \in L^p(\mu)$, we denoted by $\|f\|_p$ the classical norm of $f$.

The Dunkl kernel $\mathcal{E} = \mathcal{E}_\kappa$ gives rise to an integral transform on $\mathbb{R}$ denoted by $\mathcal{F} = \mathcal{F}_\kappa$ and called Dunkl transform (see [3]). For $f \in L^1(\mu)$,

$$\mathcal{F}f(\lambda) = \int_{\mathbb{R}} \mathcal{E}(-i\lambda x)f(x)d\mu(x), \quad \lambda \in \mathbb{R}. $$

We have the following properties of the Dunkl transform given in [3] (see also [6]).

**Theorem 2.1.** (1) Let $f \in L^1(\mu)$. If $\mathcal{F}(f)$ is in $L^1(\mu)$, then we have the following inversion formula:

$$f(x) = c \int_{\mathbb{R}} \mathcal{E}(i xy)\mathcal{F}(f)(y)d\mu(y).$$

(2) The Dunkl transform has a unique extension to an isometric isomorphism on $L^2(\mu)$.

**Definition 2.2.** Let $y \in \mathbb{R}$ be given. The generalized translation operator $f \mapsto \tau_y f$ is defined on $L^2(\mu)$ by the equation

$$\mathcal{F}(\tau_y f)(x) = \mathcal{E}(i xy)\mathcal{F}(f)(x); \quad x \in \mathbb{R}$$

Mourou proved in [17] that the translation operator has the following properties:

**Theorem 2.3.** (1) The operator $\tau_x$, $x \in \mathbb{R}$, is a continuous linear operator from $\mathcal{E}(\mathbb{R})$ into itself,

(2) For $f \in \mathcal{E}(\mathbb{R})$ and $x, y \in \mathbb{R}$

(a) $\tau_x f(y) = \tau_y f(x)$,

(b) $\tau_0 f = f$,

(c) $\tau_x \circ \tau_y = \tau_y \circ \tau_x$.

For $x \in \mathbb{R}$, let $B(x, r) = \{y \in \mathbb{R} : \max \{0, |x| - r\} < |y| < |x| + r\}$ if $x \neq 0$ and $B_0 := B(0, r) = [-r, r[$. We have $\mu(B_r) = b_\kappa r^{2\kappa+2}$ where $b_\kappa = [2^{\kappa+1}(\kappa + 1)\Gamma(\kappa + 1)]^{-1}$, and for $f \in L^1_{loc}(\mu)$, the following analogue of the Lebesgue differentiation theorem (see [20]).

\[
\lim_{r \to 0} \frac{1}{\mu(B_r)} \int_{B_r} |\tau_x f(y) - f(x)|d\mu(y) = 0 \quad \text{for a. e. } x \in \mathbb{R}
\]

and

\[
\lim_{r \to 0} \frac{1}{\mu(B_r)} \int_{B_r} \tau_x f(y)d\mu(y) = f(x) \quad \text{for a. e. } x \in \mathbb{R}.
\]
For \( x \in \mathbb{R} \) and \( r > 0 \), the map \( y \mapsto \tau_x \chi_{B_r}(y) \) is supported in \( B(x, r) \) and
\[
0 \leq \tau_x \chi_{B_r}(y) \leq \min \left\{ 1, \frac{2C_\kappa}{2\kappa + 1} \left( \frac{r}{|x|} \right)^{2\kappa+1} \right\}, \quad y \in B(x, r),
\]
as proved in \cite{14}.

Let \( f \) and \( g \) be two continuous functions on \( \mathbb{R} \) with compact support. We define the generalized convolution \( *_\kappa \) of \( f \) and \( g \) by
\[
f *_\kappa g(x) = \int_{\mathbb{R}} \tau_x f(-y)g(y)d\mu(y).
\]
The generalized convolution \( *_\kappa \) is associative and commutative \cite{18}. We also have the following results.

**Proposition 2.4** (see Soltani \cite{20}).

1. For all \( x \in \mathbb{R} \), the operator \( \tau_x \) extends to \( L^p(\mu) \), \( p \geq 1 \), and
\[
\| \tau_x f \|_p \leq 4 \| f \|_p
\]
for all \( f \in L^p(\mu) \).

2. Assume that \( p, q, r \in [1, \infty] \) and satisfy \( \frac{1}{r} + \frac{1}{q} = 1 + \frac{1}{p} \). Then the generalized convolution defined on \( C_c \times C_c \), extends to a continuous map from \( L^p(\mu) \times L^q(\mu) \) to \( L^r(\mu) \), and we have
\[
\| f *_\kappa g \|_r \leq 4 \| f \|_p \| g \|_q.
\]

It is also proved in \cite{13} that if \( f \in L^1(\mu) \) and \( g \in L^p(\mu) \), \( 1 \leq p < \infty \), then
\[
\tau_x(f * g) = \tau_x f * g = f * \tau_x g, \quad x \in \mathbb{R}.
\]

3. **Dunkl-Wiener amalgam spaces**

For \( 1 \leq q, p \leq \infty \), the generalized amalgam space \((L^q, L^p)(\mu)\) is defined by
\[
(L^q, L^p)(\mu) = \left\{ f \in L^0(\mu) : \| f \|_{q,p} < \infty \right\},
\]
where
\[
\| f \|_{q,p} = \begin{cases} 
\| \int_{\mathbb{R}} (\tau_y f(y)\chi_{B_1}(x))d\mu(x) \|_{L^p_q}^{\frac{1}{q}} & \text{if } q < \infty, \\
\| f\chi_{B(y,1)} \|_{L^p_\infty} & \text{if } q = \infty,
\end{cases}
\]
and the \( L^p(\mu) \)-norm is taken with respect to the \( y \) variable. We recover the classic Wiener amalgam spaces by taking the Lebesgue measure instead of \( \mu \), and the classical translation instead of that of Dunkl.

It is easy to see that for \( 1 \leq q \leq p \leq \infty \), \((L^q, L^p)(\mu)\) is a complex subspace of the space \( L^0(\mu) \). Notice also that \((L^\infty, L^\infty)(\mu) = L^\infty(\mu) \).

In fact, since
\[
|f\chi_{B(y,1)}| \leq |f|, \quad y \in \mathbb{R},
\]
we have \( \|f\|_{+\infty,+\infty} \leq \|f\|_{+\infty} \).

Conversely, assuming that \( \|f\|_{+\infty} > 0 \) (since otherwise there is nothing to prove), let \( 0 < r < \|f\|_{+\infty} \). We have

\[
\mu(\{ x \in \mathbb{R} / |f(x)| > r \}) > 0.
\]

Thus, there is a compact set \( K \subset \{ x \in \mathbb{R} / |f(x)| > r \} \) satisfying \( \mu(K) > 0 \). Since \( K \subset \bigcup_{i=1}^{n} B(y_i, \frac{1}{i}) \) for a finite sequence \( y_1, y_2, \ldots, y_n \in K \), there exists \( 1 \leq i_0 \leq n \) such that \( \mu(K \cap B(y_{i_0}, \frac{1}{2})) > 0 \). It is easy to see that

\[
B(y_{i_0}, \frac{1}{2}) \subset B(y, 1), \quad y \in B(y_{i_0}, \frac{1}{2}).
\]

Thus

\[
B(y_{i_0}, \frac{1}{2}) \subset \{ y \in \mathbb{R} / \|f \chi_B(y, 1)\|_{\infty} > r \},
\]

so that

\[
r < \|f\|_{+\infty,+\infty}.
\]

This implies that \( \|f\|_{+\infty} \leq \|f\|_{+\infty,+\infty} \).

Hölder inequality is also valid in these spaces.

**Proposition 3.1.** Let \( 1 \leq q_1, p_1 \leq \infty \) and \( 1 \leq q_2, p_2 \leq \infty \) such that

\[
\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q} \leq 1 \quad \text{and} \quad \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} \leq 1.
\]

If \( f \in (L^{q_1}, L^{p_1})(\mu) \) and \( g \in (L^{q_2}, L^{p_2})(\mu) \), then \( fg \in (L^{q}, L^{p})(\mu) \).

Moreover, we have

\[
\|fg\|_{q,p} \leq \|f\|_{q_1,p_1} \|g\|_{q_2,p_2}.
\]

**Proof.** Let \( f \in (L^{q_1}, L^{p_1})(\mu) \) and \( g \in (L^{q_2}, L^{p_2})(\mu) \).

We suppose that \( q_1 < \infty \) and \( q_2 < \infty \). For almost every \( y \in \mathbb{R} \), the maps \( x \mapsto |f(x)| (\tau_y \chi_{B(0,1)}(-x))^\frac{1}{q_1} \) and \( x \mapsto |g(x)| (\tau_y \chi_{B(0,1)}(-x))^\frac{1}{q_2} \) are respectively in \( L^{q_1}(\mu) \) and \( L^{q_2}(\mu) \) so that, applying Hölder inequality in Lebesgue space, we have

\[
\left[ \int_{\mathbb{R}} (|f(x)| |g(x)|)^q (\tau_y \chi_{B(0,1)}(-x)) d\mu(x) \right]^{\frac{1}{q}} \leq \left[ \int_{\mathbb{R}} |f(x)|^{q_1} \tau_y \chi_{B(0,1)}(-x) d\mu(x) \right]^{\frac{1}{q_1}}
\]

\[
\times \left[ \int_{\mathbb{R}} |g(x)|^{q_2} \tau_y \chi_{B(0,1)}(-x) d\mu(x) \right]^{\frac{1}{q_2}}.
\]

Taking the \( L^p(\mu) \)-norm of both sides and applying once more the Hölder inequality on the term in the right hand side, we obtain the desired result.
We suppose known that some $q_i = \infty$ let say $q_1$, and $q_2 < \infty$. Then $q = q_2$ and the result follows from the following inequality
\[
\left[ \int_{\mathbb{R}} (|f(x)| g(x)) q_2 (\tau_y \chi B_0(1)) d\mu(x) \right] \frac{1}{q_2} \leq \left\| f \chi B(y,1) \right\| \left[ \int_{\mathbb{R}} |g(x)| q_2 (\tau_y \chi B_0(1)) d\mu(x) \right] \frac{1}{q_2},
\]
by taking the $L^p(\mu)$-norm of both sides and applying Hölder inequality for $p_i$ indexes.

In the case $q_1 = q_2 = \infty$, the result is immediate. \(\square\)

**Proposition 3.2.** Let $1 \leq q \leq p \leq \infty$. The maps $L^0(\mu) \ni f \mapsto \| f \|_{q,p}$ define a norm in $(L^q, L^p)(\mu)$.

**Proof.** Let $f$ be a $\mu$-measurable function such that $\| f \|_{q,p} = 0$.

1. Suppose $q < \infty$, then
   \[
   0 = \| f \|_{q,p} = \left\| \int_{\mathbb{R}} (\tau_y |f|^q) \chi B_1(x) d\mu(x) \right\|_p
   \]
   implies that there exist a $\mu$-null set $N$ such that for all $y \in \mathbb{R} \setminus N$, we have
   \[
   0 = \int_{\mathbb{R}} \tau_y |f(x)|^q \chi B_1(x) d\mu(x) = \int_{\mathbb{R}} |f(x)|^q \tau_y \chi B_1(x) d\mu(x).
   \]
   Thus for any $y \in \mathbb{R} \setminus N$, there exists a subset $N_y \subset \mathbb{R}$ with $\mu(N_y) = 0$, such that for $x \in \mathbb{R} \setminus N_y$
   \[
   |f|^q \tau_y \chi B_1(x) = 0.
   \]
   Fix $y \in \mathbb{R} \setminus N$. For every $\eta \leq 1$ we have $B_\eta \subset B_1$, so that that $\tau_{-y}(\chi B_1 - \chi B_\eta) \geq 0$. Hence
   \[
   \frac{1}{\mu(B_\eta)} \int_{B_\eta} (\tau_y |f|^q)(x) d\mu(x) = \frac{1}{\mu(B_\eta)} \int_{\mathbb{R}} |f|^q(x) \tau_{-y} \chi B_\eta(x) d\mu(x)
   \]
   \[
   \leq \frac{1}{\mu(B_\eta)} \int_{\mathbb{R}} |f|^q(x) \tau_{-y} \chi B_1(x) d\mu(x) = 0.
   \]
   It comes from (2.3) that
   \[
   0 = \lim_{\eta \to 0} \frac{1}{\mu(B_\eta)} \int_{B_\eta} (\tau_y |f|^q)(x) d\mu(x) = |f|^q(y).
   \]
   Thus $f = 0 \mu$-a.e.
For the triangular inequality, we have
\[
\|f + g\|_{q,p} = \left\| \int_{\mathbb{R}} \left( |f(x) + g(x)| (\tau_{(-x)} \chi_{B_1})(x) \right)^{\frac{q}{2}} d\mu(x) \right\|^{\frac{1}{q}}_p
\]
for \( f, g \in (L^q, L^p)(\mu) \), and the result follows by applying respectively the triangular inequality in \( L^q(\mu) \) and \( L^p(\mu) \).

(2) If \( q = p = \infty \) then there is nothing to prove since \( (L^\infty, L^\infty)(\mu) = L^\infty(\mu) \).

Positive homogeneity is immediate. \( \square \)

We have the following result. The proof, which is an adaptation of [7, Proposition 1.2.8], is given for completeness.

**Proposition 3.3.** Let \( 1 \leq p, q \leq \infty \). The space \((L^q, L^p)(\mu), \| \cdot \|_{q,p}\) is a complex Banach space.

**Proof.** Let \((f_n)_{n \geq 1}\) be a sequence of elements of \((L^q, L^p)(\mu)\) such that
\[
(3.1) \quad \sum_{n \geq 1} \|f_n\|_{q,p} < +\infty
\]

1st case: we suppose that \( q < \infty \).

We have
\[
\sum_{n \geq 1} \|f_n\|_{q,p} = \sum_{n \geq 1} \left\| \int_{\mathbb{R}} \left( |f_n| \chi_{B(y,1)}(x) \tau_{-y} \chi_{B_1}(x) \right)^{\frac{q}{2}} d\mu(x) \right\|^{\frac{1}{q}}_p \leq \sum_{n \geq 1} \left\| f_n \chi_{B(y,1)} \tau_{-y} \chi_{B_1} \right\|_q^{\frac{1}{p}} < \infty.
\]

Since \( L^p(\mu) \) is a Banach space, there exists a measurable subset \( N \subset \mathbb{R} \) with \( \mu(N) = 0 \) such that
\[
\sum_{n \geq 1} \left\| f_n \chi_{B(y,1)} \tau_{-y} \chi_{B_1} \right\|_q < +\infty \quad y \in \mathbb{R} \setminus N.
\]

Hence the series \( \sum_{n \geq 1} f_n \chi_{B(y,1)} \tau_{-y} \chi_{B_1} \) converges in the Banach space \( L^q(\mu) \) for all \( y \in \mathbb{R} \setminus N \). Put
\[
E = \{ x \in \mathbb{R} \mid \exists y \in \mathbb{R} \setminus N \text{ such that } \chi_{B(y,1)}(x) \tau_{-y} \chi_{B_1}(x) \neq 0 \}
\]
and
\[
f(x) = \begin{cases} 
\sum_{n \geq 1} f_n(x) & \text{if } x \in E, \\
0 & \text{if } x \notin E.
\end{cases}
\]
On one hand we have:

\[ \left\| f \right\|_{q,p} = \left\| \left[ \int_{\mathbb{R}} \left| f(x) \chi_{B(y,1)}(x) \tau_{-y} \chi_{B_1}(x) \right|^q d\mu(x) \right]^{\frac{1}{q}} \right\|_p \]

\[ = \left\| \left[ \int_{\mathbb{R}} \left\{ \sum_{n \geq 1} f_n(x) \tau_{-y} \chi_{B_1}(x) \right\}^q d\mu(x) \right]^{\frac{1}{q}} \right\|_p \]

\[ \leq \left\| \left[ \int_{\mathbb{R}} \left\{ \sum_{n \geq 1} f_n(x) \tau_{-y} \chi_{B_1}(x) \right\} d\mu(x) \right]^{\frac{1}{q}} \right\|_p \]

On the other hand, we have

\[ \left\| f - \sum_{i=1}^{n} f_i \right\|_{q,p} = \left\| \left[ \int_{\mathbb{R}} \left\{ \sum_{i>n} f_i(x) \tau_{-y} \chi_{B_1}(x) \right\}^q d\mu(x) \right]^{\frac{1}{q}} \right\|_p \]

\[ \leq \left\| \left[ \int_{\mathbb{R}} \left\{ \sum_{i>n} f_i(x) \tau_{-y} \chi_{B_1}(x) \right\} d\mu(x) \right]^{\frac{1}{q}} \right\|_p \leq \sum_{i>n} \left\| f_i \right\|_{q,p} . \]

It comes from Estimate (5.1), that \( \lim_{n \to \infty} \sum_{i>n} \left\| f_i \right\|_{q,p} = 0. \)

2nd case: we suppose \( q = \infty. \)

Since \( \sum_{n \geq 1} \left\| f_n \right\|_{+ \infty,p} = \sum_{n \geq 1} \left\| f \chi_{B(y,1)} \right\|_{+ \infty,p} < +\infty, \) there exists a \( \mu \)-measurable subset \( N \) of \( \mathbb{R} \) such that

\[ \left\{ \begin{array}{l} \mu(N) = 0 \\
\sum_{n \geq 1} \left\| f_n \chi_{B(y,1)} \right\|_{+ \infty} < +\infty \quad \forall \ y \in \mathbb{R} \setminus N. \end{array} \right. \]

Hence for all \( y \in \mathbb{R} \setminus N, \) the series \( \sum_{n \geq 1} f_n \chi_{B(y,1)} \) converges a.e on \( \mathbb{R}. \)

Since for all \( y_1, y_2 \in \mathbb{R} \setminus N \) such that \( B(y_1,1) \cap B(y_2,1) \neq \emptyset, \) we have

\[ \sum_{n \geq 1} f_n(x) \chi_{B(y_1,1)}(x) = \sum_{n \geq 1} f_n(x) \chi_{B(y_2,1)}(x) \]
for $\mu$-almost every $x \in B(y_1, 1) \cap B(y_2, 1)$, we put $f(x) = \sum_{n \geq 1} f_n(x)$.

The function $f$ is well defined almost everywhere on $\mathbb{R}$ and

$$
\|f\|_{+\infty,p} = \left\|\|f\|_{\infty} \right\|_p = \left\|\left\|\sum_{n \geq 1} f_n \chi_{B(y_1)} \right\|_{+\infty} \right\|_p \sum_{n \geq 1} \|f_n\|_{+\infty,p} < \infty.
$$

We also have

$$
\|f\chi_{B(y_1)} - \sum_{i=1}^n f_i \chi_{B(y_1)}\|_{+\infty} = \left\|\sum_{i>n} f_i \chi_{B(y_1)}\right\|_{+\infty} \leq \sum_{i>n} \|f_i \chi_{B(y_1)}\|_{+\infty}
$$

for $\mu$-almost every $y \in \mathbb{R}$ and for all positive integers $n$. Thus

$$
\left\|\sum_{i=1}^n f_i \right\|_{+\infty,p} \leq \sum_{i>n} \|f_i\|_{+\infty,p},
$$

which tends to zero as $n$ goes to infinity. \hfill \Box

It comes from the definition of the Dunkl convolution that for $1 \leq q < \infty$, we have

$$
\int_{B_1} (\tau_y |f|^q(x)d\mu(x) = \left(\int_{\mathbb{R}} (\tau_y |f|^q(x)\chi_{B_1}(-x)d\mu(x) \right) = |f|^q \ast \chi_1(y).
$$

This observation couple with Young inequality in Dunkl Lebesgue spaces, allow us to prove the following result.

**Proposition 3.4.** Let $1 \leq q \leq s \leq p \leq \infty$. We have

$$
\|f\|_{q,p} \leq 4^{\frac{1}{q}} \|f\|_s \mu(B_1)^{\frac{1}{q} - \frac{1}{s} + \frac{1}{q}}.
$$

**Proof.** Let $1 \leq q \leq s \leq p < \infty$. We have

$$
\frac{ps}{qs - pq + ps} \geq 1 \quad \text{and} \quad \frac{1}{q} + \frac{ps}{qs - pq + ps} = \frac{1}{q} + 1
$$

so that

$$
\|f\|_{q,p}^q = \|f\|_{q}* \chi_{B_1}\|_{q}^q \leq 4 \|f\|_s^q \mu(B_1)^{\frac{q-s+pq-ps}{pq}}.
$$

according to Proposition 2.4. \hfill \Box

The above result allows us to say that $L^s(\mu) \subset (L^q, L^p)(\mu)$ provided $1 \leq q \leq s \leq p < \infty$.

We also have, as in the classical case, that the family of spaces $(L^q, L^p)(\mu)$ is decreasing with respect to the $q$ power. More precisely we have the following result.

**Proposition 3.5.** Let $1 \leq q_1 \leq q_2 \leq p \leq \infty$. For any locally integrable function $f$, we have that

$$
\|f\|_{q_1,p} \leq \mu(B_1)^{\frac{1}{q_1} - \frac{1}{q_2}} \|f\|_{q_2,p}
$$


Proof. Let \( f \in L^0(\mu) \).
We suppose that \( q_1 < q_2 < +\infty \). Hölder inequality, and [3, Theorem 6.3.3] allow us to write that
\[
\left[ \int_{\mathbb{R}} \tau_y |f|^{q_1}(x) \chi_{B_1}(x) d\mu(x) \right]^{\frac{1}{q_1}} \leq \left[ \int_{\mathbb{R}} |f|^{q_2}(x) \tau_{-y} \chi_{B_1}(x) d\mu(x) \right]^{\frac{1}{q_2}} \left[ \int_{\mathbb{R}} \tau_{-y} \chi_{B_1}(x) d\mu(x) \right]^{\frac{1}{q_2}}
\]
\[
\leq \left[ \int_{\mathbb{R}} |f|^{q_2}(x) \tau_{-y} \chi_{B_1}(x) d\mu(x) \right]^{\frac{1}{q_2}} (\mu(B_1))^{\frac{1}{q_1} - \frac{1}{q_2}}
\]
\[
= (\mu(B_1))^{\frac{1}{q_1} - \frac{1}{q_2}} \left[ \int_{\mathbb{R}} \tau_y |f|^{q_1}(x) \chi_{B_1}(x) d\mu(x) \right]^{\frac{1}{q_2}},
\]
so that taking the \( L^p(\mu) \)-norm of both sides leads to the desire result.

Suppose \( q_1 < q_2 = +\infty \). We assume that \( \|f\|_{\infty,p} < \infty \) since otherwise there is nothing to prove. We have
\[
\left[ \int_{\mathbb{R}} \tau_y |f|^{q_1}(x) \chi_{B_1}(x) d\mu(x) \right]^{\frac{1}{q_1}} = \left[ \int_{B(y,1)} |f|^{q_1}(x) \tau_{-y} \chi_{B_1}(x) d\mu(x) \right]^{\frac{1}{q_1}} \leq \|f\chi_{B(y,1)}\|_{\infty} \left[ \int_{\mathbb{R}} \tau_{-y} \chi_{B_1}(x) d\mu(x) \right]^{\frac{1}{q_1}}
\]
\[
= (\mu(B_1))^{\frac{1}{q_1}} \|f\chi_{B(y,1)}\|_{\infty}.
\]
We also conclude in this case by taking the \( L^p(\mu) \)-norm of both sides. \( \square \)

4. Fofana spaces associated to Dunkl-amalgam
Let \( 1 \leq q \leq \alpha \leq p \leq \infty \). The space \((L^q, L^p)^\alpha(\mu)\) is defined by
\[
(L^q, L^p)^\alpha(\mu) = \left\{ f \in L^0(\mu) : \|f\|_{q,p,\alpha} < \infty \right\},
\]
where
\[
\|f\|_{q,p,\alpha} = \sup_{r>0} (\mu(B_r))^{\frac{1}{q_1} - \frac{1}{q_2}} \frac{1}{r} \|f\xi_{B_r}\|
\]
and for \( r > 0 \)
\[
r \|f\|_{q,p} = \begin{cases} \left\| \left[ \int_{\mathbb{R}} (\tau_y |f|^q \chi_{B_r}(x)) d\mu(x) \right]^{\frac{1}{q'}} \right\|_p & \text{if } q < \infty \\ \|f\chi_{B(y,r)}\|_{\infty} & \text{if } q = \infty. \end{cases}
\]
It is easy to see that for \( q \leq \alpha \leq p \) the space \((L^q, L^p)^\alpha(\mu)\) is a complex vector subspace of \((L^q, L^p)(\mu)\). Notice that \((L^q, L^\infty)^\alpha(\mu)\) is the Dunkl-type Morrey space as defined in [3].
From the definition of \((L^q, L^p)^{\alpha}(\mu)\) spaces and the proof of Proposition 3.2, we deduce that the map \(L^0(\mu) \ni f \mapsto \|f\|_{q,p,\alpha}\) define a norm in \((L^q, L^p)^{\alpha}(\mu)\).

These spaces have been introduced in the classical case; i.e., taking in the above definition the measure of Lebesgue instead of \(\mu\), and the classical translation instead of that of Dunkl, by I. Fofana in [10] where he proved that the space is non trivial if and only if \(q \leq \alpha \leq p\) (this is why we are referring to these spaces as Fofana’s spaces). Thus we will always assume that this condition is fulfilled. Some important properties of Fofana’s spaces in the classical case are resumed in the following (see [8]).

**Proposition 4.1.** Let \(1 \leq q \leq \alpha \leq p \leq \infty\).

1. \(\left((L^q, L^p)^{\alpha}(dx), \|\cdot\|_{(L^q, L^p)^{\alpha}(dx)}\right)\) is a complex Banach space.
2. There exists \(C > 0\) such that
   \[\|f\|_{(L^q, L^p)^{\alpha}(dx)} \leq \|f\|_{(L^q, L^p)^{\alpha}(dx)} \leq C \|f\|_{L^\alpha(dx)} .\]
3. If \(q \leq q_1 \leq \alpha \leq p_1 \leq p\) then
   \[\|f\|_{(L^q, L^p)^{\alpha}(dx)} \leq \|f\|_{(L^{q_1}, L^{p_1})^{\alpha}(dx)} .\]
4. If \(\alpha \in \{q, p\}\) then \((L^q, L^p)^{\alpha}(dx) = L^\alpha(dx)\) with equivalent norms.

In the case of harmonic analysis associated to Dunkl operator, it follows from Proposition 3.4 that for \(1 \leq q \leq \alpha \leq p \leq \infty\),

\[\|f\|_{q,p,\alpha} \leq 4^{\frac{1}{q}} \|f\|_{\alpha},\]

since Relation 3.3 is valid for any ball \(B_r\) and \(\mu(B_r) > 0\) for all \(r > 0\). This proves that \(L^\alpha(\mu)\) is continuously embedded in \((L^q, L^p)^{\alpha}(\mu)\). We also deduce from Proposition 3.5 that for \(1 \leq q_1 \leq q_2 \leq \alpha \leq p \leq \infty\), we have

\[\|f\|_{q_1,p,\alpha} \leq \|f\|_{q_2,p,\alpha} .\]

5. **Norm inequalities for Hardy-Littlewood type operator**

Let \(f \in L^0(\mu)\). The Dunkl-type Hardy-Littlewood maximal function \(Mf\) is defined by

\[Mf(x) = \sup_{r>0} (\mu(B(0, r)))^{-1} \int_{B(0, r)} \tau_x f(y) d\mu(y) \; x \in \mathbb{R}.\]

It is proved in [21] that \(M\) is bounded on \(L^p(\mu)\) for \(1 < p \leq \infty\) and it is of weak type \((1,1)\).

We have the following norms inequalities for the Dunkl-type Hardy-Littlewood maximal function, in the setting of Dunkl-Fofana spaces.
Theorem 5.1. Let $1 < q \leq \alpha \leq p \leq \infty$. There exists a constant $C > 0$ such that
\[
\|Mf\|_{q,p,\alpha} \leq C \|f\|_{q,p,\alpha}, \quad f \in L^q_{\text{loc}}(\mu).
\]

Let $1 \leq q \leq \alpha \leq \infty$. For a locally $\mu$-measurable function $f$, we put
\[
\|f\|_{(L^{1,\infty},L^p)_{\alpha}(\mu)} = \sup_{r>0}(\mu(B(0, r)))^{\frac{1}{\alpha}} \|f\|_{\frac{1}{\alpha}} \|f\|_{1,\infty}(\mu)^{\frac{1}{\alpha}} \|f\|_{p},
\]
where $L^{1,\infty}(\mu)$ denote the weighted weak-Lebesgue space, consists of $f \in L^0(\mu)$ satisfying the condition
\[
\|f\|_{L^{1,\infty}(\mu)} = \sup_{\lambda>0} \lambda \mu(\{ x \in \mathbb{R} : |f(x)| > \lambda \}) < \infty.
\]

We have the following weak-type inequality, when $q = 1$.

Theorem 5.2. Let $1 \leq \alpha \leq p \leq \infty$. There exists $C > 0$ such that
\[
\|Mf\|_{(L^{1,\infty},L^p)_{\alpha}(\mu)} \leq C \|f\|_{1,p,\alpha}, \quad f \in L^1_{\text{loc}}(\mu).
\]

Notice that $(\mathbb{R}, |.|, \mu)$ is a space of homogeneous type in the sense of Coifman and Stein.

Let $X$ be a set. A map $d : X \times X \to [0, \infty)$ is called a quasi-distance on $X$ if the following conditions are satisfied:

(1) for every $x$ and $y$ in $X$, $d(x, y) = 0$ if and only if $x = y$,

(2) for every $x$ and $y$ in $X$, $d(x, y) = d(y, x)$,

(3) there exists a constant $\epsilon$ such that $d(x, y) \leq \epsilon(d(x, z) + d(z, y))$
for every $x$, $y$ and $z$ in $X$.

Let $x \in X$ and $r > 0$. The ball of center $x$ and radius $r$ is the subset $I(x, r) = \{ y : d(x, y) < r \}$ of $X$. Fix $x \in X$. The family $\{ I(x, r) \}_{r>0}$ form a basis of neighborhoods of $x \in X$ for the topology induced by $d$.

Let $X$ be a set endowed with a quasi-distance $d$ and a non-negative measure $\mu$ defined on a $\sigma$-algebra of subsets of $X$ which contains the $d$-open subsets and the balls $I(x, r)$. Assume that there exists a finite constant $C$, such that
\[
0 < \mu(I(x, 2r)) \leq C \mu(I(x, r)) < \infty \quad x \in X \quad r > 0,
\]
holds for every $x \in X$ and $r > 0$. A set $X$ with a quasi-distance $d$ and a measure $\mu$ satisfying the above conditions, will be called a space of homogeneous type and denoted by $(X, d, \mu)$. If $C'_\mu$ is the smallest constant $C$ for which (5.1) holds, then $D_\mu = \log_2 C'_\mu$ is called the doubling order of $\mu$. It is known (see [19]) that for all balls $I_2 \subset I_1$
\[
\frac{\mu(I_1)}{\mu(I_2)} \leq C_\mu \left( \frac{r(I_1)}{r(I_2)} \right)^{D_\mu}.
\]
where $r(I)$ denote the radius of the ball $I$ and $C_\mu = C'_\mu(2\epsilon)^{D_\mu}$, $\epsilon \geq 1$ being a constant such that
\[
d(x, y) \leq \epsilon(d(x, z) + d(z, y)), \quad x, y, z \in X.
\]
As proved in [24], the last assumption implies that there exists two constants $C_{\mu} > 0$ and $\delta_\mu > 0$ such that for all balls $I_2 \subset I_1$ of $X$

\begin{equation}
\frac{\mu(I_1)}{\mu(I_2)} \geq C_{\mu} \left( \frac{r(I_1)}{r(I_2)} \right)^{\delta_\mu}.
\end{equation}

For $y \in \mathbb{R}$ and $r > 0$, we have the following relation between $\mu(B(y, r))$ and $\mu(I(y, r))$, when $\mu$ is the weighted Lebesgue measure defined in Relation (2.1).

**Lemma 5.3.** Let $x \in \mathbb{R}$. For all $r > 0$ we have $\mu(B(x, r)) \leq 2\mu(I(x, r))$.

**Proof.** Let $x \in \mathbb{R}$ and $r > 0$. We have that

\[ B(x, r) = \begin{cases} 
| - |x| - r, 0 \cup |x| + r[ & \text{if } |x| \leq r \\
| - |x| - r, -|x| + r[ \cup |x| - r, |x| + r] & \text{if } |x| > r 
\end{cases} \]

(1) If $x \geq r$ then

\[ \mu(B(x, r)) = c_{\kappa}(\kappa + 1)^{-1}[(x + r)^{2\kappa+2} - (x - r)^{2\kappa+2}] \]

so that $\mu(B(x, r)) = 2\mu(I(x, r))$.

(2) We suppose now that $x \leq r$.

(a) For $0 < x < r$, we have $\mu(B(x, r)) = c_{\kappa}(\kappa + 1)^{-1}(x + r)^{2\kappa+2}$.

and

\[ \mu(I(x, r)) = \frac{c_{\kappa}}{2\kappa + 2}[(r - x)^{2\kappa+2} + (x + r)^{2\kappa+2}], \]

which leads to $\mu(B(x, r)) < 2\mu(I(x, r))$.

(b) Suppose $-r < x < 0$. We have $\mu(B(x, r)) = \mu(B(-x, r))$ and

\[ \mu(I(x, r)) = \frac{c_{\kappa}}{2\kappa + 2}[(r - x)^{2\kappa+2} + (x + r)^{2\kappa+2}] = \mu(I(-x, r)), \]

so that $\mu(B(-x, r)) < 2\mu(I(-x, r))$.

(c) Suppose $x < -r$. We have $\mu(B(x, r)) = c_{\kappa}(\kappa + 1)^{-1}[(|x| + r)^{2\kappa+2} - (|x| - r)^{2\kappa+2}] = \mu(B(-x, r))$ and

\[ \mu(I(x, r)) = \frac{c_{\kappa}}{2\kappa + 2}[(r - x)^{2\kappa+2} + (x + r)^{2\kappa+2}] \]

Thus $2\mu(I(x, r)) > 2\mu(I(-x, r)) = 2\mu(B(-x, r)) = 2\mu(B(x, r))$.

This completes the proof.
It comes from the above lemma that there exists $C > 0$ such that for all $x \in \mathbb{R}$ and $r > 0$, we have
\begin{equation}
\mu(I(x, 2r)) \leq C \mu(I(x, r)),
\end{equation}
thanks to \cite{4}.

In \cite{4}, the space $(L^q, L^p)^\alpha(X, d, \mu)$ is defined for $1 \leq q \leq \alpha \leq p \leq \infty$ as the set of $\mu$-measurable functions $f$ satisfying $\|f\|_{(L^q, L^p)^\alpha(X)} < \infty$, where $\|f\|_{(L^q, L^p)^\alpha(X)} = \sup_{r > 0} r \|f\|_{(L^q, L^p)^\alpha(X)}$ with
\begin{equation}
r \|f\|_{(L^q, L^p)^\alpha(X)} = \left\{ \begin{array}{ll}
\int_X \mu(I(y, r))^{\frac{1}{q} - \frac{1}{2}} \|f|_{I(X,y,r)}\|_{L^p(X,d,\mu)}^p \, d\mu(y) \right\}^{\frac{1}{p}} & \text{if } p < \infty \\
\sup_{y \in X} \mu(I(y, r))^{\frac{1}{q} - \frac{1}{2}} \|f|_{I(X,y,r)}\|_{L^p(X,d,\mu)} & \text{if } p = \infty
\end{array}\right.
\end{equation}

It is proved in \cite{4} Theorem 2.9 that if $\alpha \in \{q, p\}$ then
\begin{equation}
\| \cdot \|_\alpha \lesssim \| \cdot \|_{(L^q, L^p)^\alpha(\mathbb{R})}.
\end{equation}

We have the following result when $\mu$ is a de measure defined by Relation \cite{2,1}.

Lemma 5.4. Let $x \in \mathbb{R}$. For all $r > 0$
\begin{equation}
\mu(I(0, r)) \leq 2\mu(I(x, r)).
\end{equation}

Proof. Let’s fix $x \in \mathbb{R}$ and $r > 0$.

(1) We suppose that $x < r$. We have
\begin{equation}
2\mu(I(x, r)) = c_\alpha(\kappa + 1)^{-1}[(x + r)^{2\kappa + 2} + (x - r)^{2\kappa + 2}]
\end{equation}
as we can see in the proof of Lemma \cite{5,3} Thus, $\mu(I(0, r)) \leq 2\mu(I(x, r))$.

(2) We suppose now $x \geq r$. According to the proof of Lemma \cite{5,3} once more, we have
\begin{equation}
2\mu(I(x, r)) = c_\alpha(\kappa + 1)^{-1}[(x + r)^{2\kappa + 2} - (x - r)^{2\kappa + 2}]
\end{equation}

Put $A = (x + r)^{2\kappa + 2} - (x - r)^{2\kappa + 2} - r^{2\kappa + 2}$

$\phi(t) = r^{2\kappa + 2}[(t + 1)^{2\kappa + 2} - 1]$. Hence $2\mu(I(x, r)) \leq \mu(I(0, r)).$

\hfill \Box

It comes from the above lemma and a result in \cite{16} that for $1 \leq q \leq p \leq \infty$ and $\mu$ as in \cite{2,1}, the spaces $(L^q, L^p)^\alpha(\mu) \subset (L^q, L^p)^\alpha(\mathbb{R}, | \cdot |, \mu)$. More precisely, we have $\| \cdot \|_{(L^q, L^p)^\alpha(\mathbb{R}, | \cdot |, \mu)} \leq \| \cdot \|_{(L^q, L^p)^\alpha(\mu)}$. However, we still have the following analogue of the Inequality \cite{5,6} in $(L^q, L^p)^\alpha(\mu)$ space.

Proposition 5.5. Let $1 \leq q \leq \alpha \leq p \leq \infty$. If $\alpha \in \{q, p\}$ then $(L^q, L^p)^\alpha(\mu) = L^\alpha(\mu)$. 

\hfill \Box
Proof. Let \(1 \leq q \leq \alpha \leq p \leq \infty\). It comes from Inequality (4.1) that \(L^\alpha(\mu) \subset (L^q, L^p)\alpha(\mu)\). Thus all we have to prove is the reverse inclusion.

Let \(f \in (L^q, L^p)\alpha(\mu)\). We have

\[\|f\|_{(L^q, L^p)\alpha(\mathbb{R})} \leq \|f\|_{q,p,\alpha} < \infty.\]

Hence \(f \in (L^q, L^p)\alpha(\mathbb{R}, |\cdot|, \mu)\). The result follows from Inequality (5.6). \(\square\)

For a \(\mu\)-measurable function \(f\) in the space of homogeneous type \((X, d, \mu)\), we define the Hardy-Littlewood maximal function \(M_\mu f\) by

\[M_\mu f(x) = \sup_{r>0} \mu(I(x, r))^{-1} \int_{I(x,r)} |f(y)| \, d\mu(y).\]

The following result is an extension of the analogue on Euclidean space given in [8]. Its proof is just an adaptation of the one given there.

**Proposition 5.6.** Let \(1 < q \leq \alpha \leq p \leq \infty\). There exists \(C > 0\) such that

\[\|M_\mu f\|_{(L^q, L^p)\alpha(X)} \leq C \|f\|_{(L^q, L^p)\alpha(X)}\]

for all \(f \in (L^q, L^p)\alpha(X, d, \mu)\).

For the proof, we will use the following well known lemma. The proof is given just for completeness.

**Lemma 5.7.** There exists a constant \(C > 0\) such that for all \(r > 0\),

\[M_\mu \chi_{I(y,r)}(x) \leq C \frac{\mu(I(y,r))}{\mu(I(y,d(x,y)))}, \quad x \neq y \in X.\]

**Proof.** Let \(r > 0\), \(x, y \in X\). We have

\[M_\mu \chi_{I(y,r)}(x) = \sup_{\rho>0} \mu(I(x, \rho))^{-1} \int_{I(x,\rho)} \chi_{I(y,\rho)}(z) \, d\mu(z) = \sup_{\rho>0} \frac{\mu(I(y, r) \cap I(x, \rho))}{\mu(I(x, \rho))} \leq 1.\]

Let’s fix \(\rho > 0\). If \(d(x, y) \leq \rho\) then the result is immediate. Assume \(\rho > d(x, y)\). We also assume that \(I(x, \rho) \cap I(y, \rho) \neq \emptyset\), since otherwise there is nothing to prove.

\[\|f\|_{q,p,\alpha} = \sup_{r>0} (\mu(B_r))^{\frac{1}{q} - \frac{1}{p}} r \|f\|_{q,p} ,\]

We have \(d(x, y) \leq c(d(x, u) + d(u, y)) < c(\rho + r) \leq 2c \max(\rho, r)\), where \(u \in I(x, r) \cap I(y, \rho)\). It comes from the doubling condition that

\[\mu((I(x, d(x, y))) \approx \mu(I(y, d(x, y))) \leq \begin{cases} \mu(I(y, 2c\rho)) \leq C\mu(I(y, r)) & \text{if } \rho \leq r \\ \mu(I(x, 2c\rho)) \leq C\mu(I(x, \rho)) & \text{if } \rho > r \end{cases}.\]

This end the proof. \(\square\)
Proof of Proposition 5.6. For all $y \in X$ and $r > 0$, we have

\[(5.7) \quad \int_{I(y,r)} M_{\mu}(f)^{q}(x)d\mu(x) \lesssim \int_{X} |f(x)|^{q} M_{\mu}(\chi_{I(y,r)})(x)d\mu(x),\]

thanks to \cite{23} Theorem 5.1.

Let’s fix $y \in X$ and $r > 0$. It comes from \cite{23} that

$$\|M_{\mu}(f)\chi_{I(y,r)}\|_{L^{q}(X,d\mu)}^{q} \lesssim \int_{I(y,2cr)} |f(x)|^{q} M_{\mu}(\chi_{I(y,r)})(x)d\mu(x) + \int_{I(y,2cr)^{c}} |f(x)|^{q} M_{\mu}(\chi_{I(y,r)})(x)d\mu(x) = I + II.$$

But, $I \lesssim \|f\chi_{I(y,2cr)}\|_{L^{q}(X,d\mu)}^{q}$. Furthermore, it comes from Lemma \cite{5.7} that,

$$II \lesssim \sum_{k=1}^{\infty} \int_{12^{k}cr \leq d(x,y) < 2^{k+1}cr} |f(x)|^{q} \frac{\mu(I(y,r))}{\mu(I(y,2^{k}cr))} d\mu(x).$$

Therefore

\[(5.8) \quad \|M_{\mu}(f)\chi_{I(y,r)}\|_{L^{q}(X,d\mu)}^{q} \lesssim \|f\chi_{I(y,2cr)}\|_{L^{q}(X,d\mu)}^{q} + \sum_{k=1}^{\infty} \mu(I(y,2^{k}cr)) \|f\chi_{I(y,2^{k+1}cr)}\|_{L^{q}(X,d\mu)}^{q}.\]

Multiplying both sides of \cite{5.8} by $\mu(I(y,r))^{q(\frac{1}{n} - \frac{1}{p} - \frac{1}{q})}$ and using the reverse doubling condition \cite{5.4}, we obtain

$$\mu(I(y,r))^{q(\frac{1}{n} - \frac{1}{p} - \frac{1}{q})} \|M_{\mu}(f)\chi_{I(y,r)}\|_{L^{q}(X,d\mu)}^{q} \lesssim \mu(I(y,2cr))^{q(\frac{1}{n} - \frac{1}{p} - \frac{1}{q})} \|f\chi_{I(y,2cr)}\|_{L^{q}(X,d\mu)}^{q} + \sum_{k=1}^{\infty} \left( \frac{1}{2^{k}d_{\mu}(\frac{1}{n} - \frac{1}{p})} \right)^{k} \mu(I(y,2^{k+1}cr))^{q(\frac{1}{n} - \frac{1}{p} - \frac{1}{q})} \|f\chi_{I(y,2^{k+1}cr)}\|_{L^{q}(X,d\mu)}^{q},$$

for all $y \in X$ and $r > 0$. Thus, taking the $L^{q}$-norms of the above estimation leads to

$$r \|M_{\mu}f\|_{(L^{q},L^{p})_{\alpha}(X)}^{q} \lesssim \|f\|_{q,p,\alpha}^{q} + \sum_{k=1}^{\infty} \left( \frac{1}{2^{k}d_{\mu}(\frac{1}{n} - \frac{1}{p})} \right)^{k} 2^{k+1}Kr \|f\|_{(L^{q},L^{p})_{\alpha}(X)}^{q}.$$
The result follows from the definition of the \((L^q, L^p)^\alpha(X)\) spaces and the convergence of the series \(\sum_{k=1}^{\infty} \left(\frac{1}{2^k\alpha(1+p^{-1})}\right)^k\).

6. Proof of Theorems 5.1 and 5.2

For the proof of these theorems, we need some lemmas.

**Lemma 6.1.** Let \(x \in \mathbb{R}, r > 0\). For all \(y \in \mathbb{R}\),

\[ M_\mu(\tau_x \chi_{I(0,r)})(y) = M_\mu(\chi_{I(x,r)})(y). \]

**Proof.** Let \(x \in \mathbb{R}, r > 0\). For \(y \in \mathbb{R}\),

\[
\int_{I(y,\rho)} \tau_x \chi_{I(0,r)}(z) d\mu(z) = \int_{\mathbb{R}} \chi_{I(0,r)}(z) \tau_x \chi_{I(y,\rho)}(z) d\mu(z) = \int_{\mathbb{R}} \tau_x \chi_{I(z,r)}(0) \chi_{I(y,\rho)}(z) d\mu(z) = \int_{I(y,\rho)} \chi_{I(x,r)}(z) d\mu(z).
\]

The result follows. \(\square\)

**Lemma 6.2.** Let \(r > 0\) and \(y \in \mathbb{R}\). Then

\(\chi_{B(y,r)} \leq \chi_{I(y,3r)} + \chi_{I(-y,3r)}\).

**Proof.** Let \(r > 0\) and \(y \in \mathbb{R}\). We have \(B(y,r) \subset I(y,3r) \cup I(-y,3r)\). \(\square\)

For all locally \(\mu\)-integrable function \(f\), we put

\[ \tilde{M}f(x) = \sup_{\rho > 0} \frac{1}{\mu(B(x,\rho))} \int_{B(x,\rho)} |f(z)| d\mu(z) \quad x \in \mathbb{R}. \]

We have the following result:

**Lemma 6.3.** Let \(f \in L^1_{\text{loc}}(\mu)\). For all \(y \in \mathbb{R}\),

\[ Mf(y) \approx \tilde{M}f(y) \approx M_\mu f(y). \]

**Proof.** Let \(f \in L^1_{\text{loc}}(\mu)\). There exist constants \(C_1 > 0\) and \(C_2 > 0\) not depending on \(f\), such that

\[ (6.1) \quad Mf(y) \leq C_1 \tilde{M}f(y) \leq C_2 M_\mu f(y), \]

for all \(y \in \mathbb{R}\) as we can see in [17]. We also have (see [16]) that there exists a constant \(C > 0\) not depending on \(f\) such that

\[
\int_{I(y,\rho)} |f(x)| d\mu(x) \leq C \int_{B(0,\rho)} \tau_y |f(x)| d\mu(x)
\]

for all \(\rho > 0\) and \(y \in \mathbb{R}\). Since we have

\[
\frac{1}{\mu(I(y,\rho))} \int_{I(y,\rho)} |f(z)| d\mu(z) \leq \frac{2C}{\mu(B(0,\rho))} \int_{B(0,\rho)} \tau_y |f(x)| d\mu(x)
\]

according to Lemma 5.4, the result follows from Relation (6.1). \(\square\)
Proof of Theorem 5.1. We can suppose that $1 < q < \alpha < p \leq \infty$, since otherwise the space is trivial or equal to Lebesgue space. Let $f \in (L^q, L^p)^\alpha(\mu)$. Let us fix $y \in \mathbb{R}$ and $r > 0$. It comes from Lemma 6.3, Lemma 6.1 and Lemma 6.2 and the triangular inequality for $L^q(\mu)$-norm that

$$
\left[ \int_{\mathbb{R}} (\tau_y (Mf)^q) \chi_{B_r}(x) d\mu(x) \right]^{\frac{1}{q}} 
\lesssim \left[ \int_{\mathbb{R}} (\tilde{M}f)^q(x) \tau_y \chi_{B_r}(x) d\mu(x) \right]^{\frac{1}{q}} \lesssim \left[ \int_{\mathbb{R}} [\tilde{M}f(x) \chi_{B(y,r)}(x)]^q d\mu(x) \right]^{\frac{1}{q}}
$$

$$
\lesssim \left[ \int_{\mathbb{R}} [\tilde{M}f(x) \chi_{I(-y,3r)}(x) + \tilde{M}f(x) \chi_{I(y,3r)}(x)]^q d\mu(x) \right]^{\frac{1}{q}}
$$

$$
\lesssim \left( \int_{\mathbb{R}} [\tilde{M}f(x) \chi_{I(-y,3r)}(x)]^q d\mu(x) \right)^{\frac{1}{q}} + \left( \int_{\mathbb{R}} [\tilde{M}f(x) \chi_{I(y,3r)}(x)]^q d\mu(x) \right)^{\frac{1}{q}}.
$$

Making appeal to [4, Theorem 2.35], we have

$$
\left( \int_{\mathbb{R}} [\tilde{M}f(x) \chi_{I(-y,3r)}(x)]^q d\mu(x) \right)^{\frac{1}{q}} \lesssim \left( \int_{\mathbb{R}} |f(x)|^q \tilde{M} \chi_{I(-y,3r)}(x) d\mu(x) \right)^{\frac{1}{q}}
$$

and

$$
\left( \int_{\mathbb{R}} [\tilde{M}f(x) \chi_{I(y,3r)}(x)]^q d\mu(x) \right)^{\frac{1}{q}} \lesssim \left( \int_{\mathbb{R}} |f(x)|^q \tilde{M} \chi_{I(y,3r)}(x) d\mu(x) \right)^{\frac{1}{q}}.
$$

It follows that

$$
\left[ \int_{\mathbb{R}} (\tau_y (Mf)^q) \chi_{B_r}(x) d\mu(x) \right]^{\frac{1}{q}} \lesssim \left( \int_{\mathbb{R}} |f(x)|^q \tilde{M} \chi_{I(-y,3r)}(x) d\mu(x) \right)^{\frac{1}{q}} + \left( \int_{\mathbb{R}} |f(x)|^q \tilde{M} \chi_{I(y,3r)}(x) d\mu(x) \right)^{\frac{1}{q}}.
$$
But, using the fact that Dunkl translation commute with Dunkl Max-
imal function (see [11]), we have

\[
\int_{\mathbb{R}} |f|^q(x)\widetilde{M}\chi_{I(-y,3r)}(x)d\mu(x) \lesssim \int_{\mathbb{R}} |f|^q(x)M_{\mu}\chi_{I(-y,3r)}(x)d\mu(x) \\
\lesssim \int_{\mathbb{R}} |f|^q(x)M_{\mu}(\tau_y\chi_{I(0,3r)})(x)d\mu(x) \\
\lesssim \int_{\mathbb{R}} |f|^q(x)\tau_y(M\chi_{I(0,3r)})(x)d\mu(x) \\
\lesssim \int_{\mathbb{R}} \tau_y|f|^q(x)(M\chi_{I(0,3r)})(x)d\mu \\
\lesssim \int_{\mathbb{R}} \tau_y|f|^q(x)(M\mu\chi_{I(0,3r)})(x)d\mu(x)
\]

according to Lemma 6.3 and Lemma 6.1. That is

\[
\left(\int_{\mathbb{R}} |f|^q(x)\widetilde{M}\chi_{I(-y,3r)}(x)d\mu(x)\right)^{\frac{1}{q}} \lesssim \left(\int_{\mathbb{R}} \tau_y|f|^q(x)(M\mu\chi_{I(0,3r)})(x)d\mu(x)\right)^{\frac{1}{q}}
\]

The term on the right hand side of (6.3) is less or equal to

\[
C \left( \int_{I(0,2r)} \tau_y |f(x)|^q M_\mu \chi_{I(0,3r)}(x) d\mu_k(x) \right)^{\frac{1}{q}} + \left( \int_{I(0,2r)^c} \tau_y |f(x)|^q M_\mu \chi_{I(0,3r)}(x) d\mu(x) \right)^{\frac{1}{q}} = I + II.
\]

Since \(M_\mu \chi_{I(0,3r)}(x) = \sup_{\rho > 0} \frac{\mu(I(0,3r) \cap I(x,\rho))}{\mu(I(x,\rho))} \leq 1\) for all \(x \in \mathbb{R}\), we have

\[
I \leq \left( \int_{I(0,2r)} \tau_y |f(x)|^q d\mu(x) \right)^{\frac{1}{q}}
\]
and

\[
II = \left( \sum_{i=1}^{\infty} \int_{2^i r \leq |x| < 2^{i+1} r} |f(x)|^q M_{\mu} \chi_{I(0,3r)}(x) d\mu(x) \right)^{\frac{1}{q}}
\]

\[
\leq \sum_{i=1}^{\infty} \left( \int_{2^i r \leq |x| < 2^{i+1} r} |f(x)|^q \frac{\mu(I(0, r))}{\mu(I(0, |x|))} d\mu(x) \right)^{\frac{1}{q}}
\]

\[
\leq \sum_{i=1}^{\infty} \frac{\mu(I(0, r))^{\frac{1}{q}}}{\mu(I(0, 2^i r))^{\frac{1}{q}}} \left( \int_{I(0,2^{i+1} r)} |f(x)|^q d\mu(x) \right)^{\frac{1}{q}},
\]

thanks to Lemma 5.7. Therefore

\[
\left( \int_{\mathbb{R}} |f|^q(x) \tilde{M}_{\chi_{I(-y,3r)}}(x) d\mu(x) \right)^{\frac{1}{q}}
\]

\[
\leq \left( \int_{I(0,2r)} \tau_y |f(x)|^q d\mu(x) \right)^{\frac{1}{q}} + \sum_{i=1}^{\infty} \frac{\mu(I(0, r))^{\frac{1}{q}}}{\mu(I(0, 2^i r))^{\frac{1}{q}}} \left( \int_{I(0,2^{i+1} r)} \tau_y |f(x)|^q d\mu(x) \right)^{\frac{1}{q}}.
\]

Using the same arguments as above, we obtain

\[
\left( \int_{\mathbb{R}} |f|^q(x) \tilde{M}_{\chi_{I(y,3r)}}(x) d\mu(x) \right)^{\frac{1}{q}}
\]

\[
\leq \left( \int_{I(0,2r)} \tau_{-y} |f(x)|^q d\mu(x) \right)^{\frac{1}{q}} + \sum_{i=1}^{\infty} \frac{\mu(I(0, r))^{\frac{1}{q}}}{\mu(I(0, 2^i r))^{\frac{1}{q}}} \left( \int_{I(0,2^{i+1} r)} \tau_{-y} |f(x)|^q d\mu(x) \right)^{\frac{1}{q}}.
\]

Taking these two estimates in Relation (6.2), we have

\[
\left[ \int_{\mathbb{R}} (\tau_y (Mf)^q) \chi_{B_r}(x) d\mu(x) \right]^{\frac{1}{q}}
\]

\[
\leq \left( \int_{I(0,2r)} \tau_y |f(x)|^q d\mu(x) \right)^{\frac{1}{q}} + \sum_{i=1}^{\infty} \frac{\mu(I(0, r))^{\frac{1}{q}}}{\mu(I(0, 2^i r))^{\frac{1}{q}}} \left( \int_{I(0,2^{i+1} r)} \tau_y |f(x)|^q d\mu(x) \right)^{\frac{1}{q}}
\]

\[
+ \left( \int_{I(0,2r)} \tau_{-y} |f(x)|^q d\mu(x) \right)^{\frac{1}{q}} + \sum_{i=1}^{\infty} \frac{\mu(I(0, r))^{\frac{1}{q}}}{\mu(I(0, 2^i r))^{\frac{1}{q}}} \left( \int_{I(0,2^{i+1} r)} \tau_{-y} |f(x)|^q d\mu(x) \right)^{\frac{1}{q}}.
\]
Since $\mu$ is symmetric invariant measure, the $L^p(\mu)$-norm of both sides in the above relation yields

\[
\left\| \left[ \int_{\mathbb{R}} (\tau_{(\cdot)} (Mf)^q) \chi_{B_r(x)} d\mu(x) \right]^{\frac{1}{q}} \right\|_p
\leq \left[ \int_{\mathbb{R}} \left( \int_{I(0,2r)} \tau_y |f(x)|^q d\mu_k(x) \right)^{\frac{p}{q}} d\mu_k(y) \right]^{\frac{1}{p}}
+ \sum_{i=1}^{\infty} \left( \frac{\mu(I(0,r))^{\frac{1}{q}}}{\mu(I(0,2^i r))^{\frac{1}{q}}} \right) \left( \int_{\mathbb{R}} \left( \int_{I(0,2^{i+1}r)} \tau_y |f(x)|^q d\mu(x) \right)^{\frac{p}{q}} d\mu(y) \right)^{\frac{1}{p}}
\]

thanks to Minkowski inequality for integral. Multiplying both sides of the inequality by $(\mu(I(0,r)))^{\frac{1}{p} - \frac{1}{q} - \frac{1}{p}}$, we obtain

\[
(\mu(I(0,r)))^{\frac{1}{p} - \frac{1}{q} - \frac{1}{p}} \left\| \left[ \int_{\mathbb{R}} (\tau_{(\cdot)} (Mf)^q) \chi_{B_r(x)} d\mu(x) \right]^{\frac{1}{q}} \right\|_p
\leq (\mu(I(0,r)))^{\frac{1}{p} - \frac{1}{q} - \frac{1}{p}} \left[ \int_{\mathbb{R}} \left( \int_{I(0,2r)} \tau_y |f(x)|^q d\mu(x) \right)^{\frac{p}{q}} d\mu(y) \right]^{\frac{1}{p}}
+ \sum_{i=1}^{\infty} \left( \frac{\mu(I(0,r))^{\frac{1}{q}}}{\mu(I(0,2^i r))^{\frac{1}{q}}} \right) \left( \int_{\mathbb{R}} \left( \int_{I(0,2^{i+1}r)} \tau_y |f(x)|^q d\mu(x) \right)^{\frac{p}{q}} d\mu(y) \right)^{\frac{1}{p}} = III + IV.
\]

(6.4)

\[
III \leq (\mu(I(0,2r)))^{\frac{1}{p} - \frac{1}{q} - \frac{1}{p}} \left[ \int_{\mathbb{R}} \left( \int_{I(0,2r)} \tau_y |f(x)|^q d\mu(x) \right)^{\frac{p}{q}} d\mu(y) \right]^{\frac{1}{p}} \lesssim \|f\|_{q,p,\alpha}.
\]
as \( \frac{1}{q} - \frac{1}{p} \) < 0 and \( \mu \) is doubling. For the second term, we have

\[
IV \lesssim \sum_{i=1}^{\infty} \left( \frac{1}{2^{\beta_i(\frac{1}{q} - \frac{1}{p})}} \right)^i \left( \mu(I(0, r)) \right)^{\frac{1}{q} - \frac{1}{p}} \left( \mu(I(0, 2^{i+1}r)) \right)^{\frac{1}{q} - \frac{1}{p}} \left( \int_{I(0, 2^{i+1}r)} \frac{d\mu(x)}{\tau_y |f(x)|^q} \right)^{\frac{p}{q}} \frac{1}{p}
\]

thanks to Estimate (5.4).

Hence

\[
(6.5) \quad IV \lesssim \sum_{i=1}^{\infty} \left( \frac{1}{2^{\beta_i(\frac{1}{q} - \frac{1}{p})}} \right)^i \|f\|_{q,p,\alpha} \lesssim \|f\|_{q,p,\alpha},
\]

since the series \( \sum_{i=1}^{\infty} \left( \frac{1}{2^{\beta_i(\frac{1}{q} - \frac{1}{p})}} \right)^i \) converges. From Relations (6.4) and (6.5), we deduce that

\[
(\mu(I(0, r)))^{\frac{1}{q} - \frac{1}{p} - \frac{1}{p}} \left\| \int_{\mathbb{R}} (\tau_{\cdot}(Mf)^q) \chi_{B_r}(x) d\mu(x) \right\|_{p}^{\frac{1}{q}} \lesssim \|f\|_{q,p,\alpha}.
\]

We conclude by taking the supremum in the right hand side over all \( r > 0 \). \( \square \)

**Proof of Theorem 7.2.** Let \( 1 < \alpha < p < +\infty \) and \( f \in (L^1, L^p)^\alpha(\mu) \). There exists a constant \( C > 0 \) such that \( Mf(x) \leq C\overline{Mf}(x) \) for all \( x \in \mathbb{R} \), according to Lemma 6.3. It follows that

\[
(6.6) \quad \|Mf\|_{(L^1, +\infty, L^p)^\alpha(\mu)} \leq C \|\overline{Mf}\|_{(L^1, +\infty, L^p)^\alpha(\mu)}.
\]

Let \( y \in \mathbb{R} \) and \( r > 0 \).

\[
(6.7) \quad \|\overline{Mf}\tau_{-y}\chi_{B_r}\|_{L^1, +\infty} \leq \|\overline{Mf}\chi_{B(y, r)}\|_{L^1, +\infty}
\]

since \( \tau_{-y}\chi_{B_r} \leq 1 \) and the support of \( \tau_{-y}\chi_{B_r} \) is a subset of \( B(y, r) \). Let \( \lambda > 0 \). We have

\[
\mu \left( \left\{ x \in B(y, r) : \overline{Mf}(x) > \lambda \right\} \right) \leq \mu \left( \left\{ x \in I(-y, 3r) : \overline{Mf}(x) > \lambda \right\} \right) + \mu \left( \left\{ x \in I(y, 3r) : \overline{Mf}(x) > \lambda \right\} \right),
\]

thanks to Lemma 6.2. But, as we can see in the proof of [4] Theorem 2.35,

\[
\mu \left( \left\{ x \in I(-y, 3r) : \overline{Mf}(x) > \lambda \right\} \right) \lesssim \frac{1}{\lambda} \int_{\mathbb{R}} |f(x)\overline{Mf}(x)\chi_{I(-y, 3r)}(x) d\mu(x)
\]
and
\[ \mu \left( \left\{ x \in I(y, 3r) : \widetilde{M}f(x) > \frac{\lambda}{2} \right\} \right) \lesssim \frac{1}{\lambda} \int_R |f|(x) \widetilde{M} \chi_{I(y,3r)}(x) d\mu(x) \]

It follows that
\[ \lambda \mu \left( \left\{ x \in B(y, r) : \widetilde{M}f(x) > \lambda \right\} \right) \lesssim \int_R |f|(x) \widetilde{M} \chi_{I(-y,3r)}(x) d\mu(x) \]
\[ + \int_R |f|(x) \widetilde{M} \chi_{I(y,3r)}(x) d\mu(x), \]

so that taking the supremum over all \( \lambda > 0 \), leads to
\[ \left\| (\widetilde{M}f) \chi_{B(y,r)} \right\|_{L^{1,\infty}(\mu)} \lesssim \int_R |f|(x) \widetilde{M} \chi_{I(-y,3r)}(x) d\mu(x) \]
\[ + \int_R |f|(x) \widetilde{M} \chi_{I(y,3r)}(x) d\mu(x). \]

Taking this Estimate in Relation (6.7) yields
\[ \left\| (\widetilde{M}f) \tau_{-y} \chi_{B_r} \right\|_{L^{1,\infty}(\mu)} \lesssim \int_R |f|(x) \widetilde{M} \chi_{I(-y,3r)}(x) d\mu(x) \]
\[ + \int_R |f|(x) \widetilde{M} \chi_{I(y,3r)}(x) d\mu(x). \]

Using the same argument as in the proof of Theorem 5.1, we have:
\[ \int_R |f|(x) \widetilde{M} \chi_{I(-y,3r)}(x) d\mu(x) \]
\[ \lesssim \int_{I(0,2r)} \tau_y |f(x)| d\mu(x) + \sum_{i=1}^{\infty} \frac{\mu(I(0,r))}{\mu(I(0,2^ir))} \int_{I(0,2^{i+1}r)} \tau_y |f(x)| d\mu(x). \]
and
\[ \int_R |f|(x) \widetilde{M} \chi_{I(y,3r)}(x) d\mu(x) \]
\[ \lesssim \int_{I(0,2r)} \tau_{-y} |f(x)| d\mu(x) + \sum_{i=1}^{\infty} \frac{\mu(I(0,r))}{\mu(I(0,2^ir))} \int_{I(0,2^{i+1}r)} \tau_{-y} |f(x)| d\mu(x). \]
Taking the above estimations in Relation (6.8), we have

\[
\| (\tilde{M} f) \tau_y \chi_{B_r} \|_{L^{1, +\infty}(\mu)} \lesssim \left( \int_{I(0, 2r)} \tau_y |f(x)| \, d\mu(x) + \sum_{i=1}^{\infty} \frac{\mu(I(0, r))}{\mu(I(0, 2^i r))} \int_{I(0, 2^{i+1} r)} \tau_y |f(x)| \, d\mu(x) \right) + \left( \int_{I(0, 2r)} \tau_{-y} |f(x)| \, d\mu(x) + \sum_{i=1}^{\infty} \frac{\mu(I(0, r))}{\mu(I(0, 2^i r))} \int_{I(0, 2^{i+1} r)} \tau_{-y} |f(x)| \, d\mu(x) \right).
\]

We end the prove as we did in Theorem 5.1. \(\square\)

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