Fixing D7 Brane Positions by F-Theory Fluxes

Christoph Lüdeling
bctp and PI, University of Bonn

A. Braun, A. Hebecker, CL, R. Valandro,
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Motivation

- F-Theory: Nonperturbative version of type IIB string theory [Vafa;Sen]
- Add two auxiliary dimensions, singularities of compactification manifold encode brane positions
- Recently, lots of interest in F-theory for model building interest [Beasley, Heckman, Vafa, Saulina, Schäfer-Nameki, Bourjaily, Tatar, Watari . . . ]
- Local models do not address global constraints like tadpole cancellation
  - Four-form flux can stabilise moduli, including brane positions
  - Simple example: F-Theory on $K3 \times \tilde{K}3$, where $\tilde{K}3$ is an elliptic fibration over $\mathbb{P}^1$ [Görlich et al.; Lust et al.; Aspinwall, Kallosh; Dasgupta et al.]
  - Includes as special case the type IIB orientifold $K3 \times T^2/\mathbb{Z}_2$ [Angelantonj et al.]
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  \[\text{[Angelantonj et al.]}\]
F-Theory/M-Theory Duality

M-Theory on $X_6 \times T^2$ \quad \xrightarrow{\text{Compactify}} \quad \text{Type IIA on } X_6 \times S^1_A \quad \xrightarrow{T\text{-dualise}} \quad \text{Type IIB on } X_6 \times S^1_B \quad (R_B = 1/R_A)

Fibrewise duality: $X_6 \times T^2 \sim$ elliptically fibred CY$_4$ dual to type IIB on base of fibration
F-Theory/M-Theory Duality

M-Theory on $X_6 \times T^2$ compactify one $S^1$ → Type IIA on $X_6 \times S^1_A$ $T$-dualise along $S^1_A$ → Type IIB on $X_6 \times S^1_B$ ($R_B = 1/R_A$)

$R_B \to \infty$
($R_A \to 0$)

Type IIB on $X_6$ Four noncompact dimensions

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F-Theory/M-Theory Duality

M-Theory on $X_6 \times T^2$ \(\xrightarrow{\text{Compactify one } S^1}\) Type IIA on $X_6 \times S^1_A$ \(\xrightarrow{T\text{-dualise along } S^1_A}\) Type IIB on $X_6 \times S^1_B$ \(R_B = 1/R_A\)

Shrink one $S^1$

M-Theory on $X_6 \times T^2$ \(\xrightarrow{\text{Vol } (T^2) \rightarrow 0}\) M-Theory/F-Theory Duality \(\xrightarrow{\text{Four noncompact dimensions}}\) Type IIB on $X_6$

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F-Theory/M-Theory Duality

M-Theory on $X_6 \times T^2$ -> Compactify one $S^1$ -> Type IIA on $X_6 \times S_A^1$ -> $T$-dualise along $S_A^1$ -> Type IIB on $X_6 \times S_B^1$ ($R_B = 1/R_A$)

Shrink one $S^1$

M-Theory on $X_6 \times T^2$ -> $Vol(T^2) \rightarrow 0$

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K3: Calabi–Yau Two-Fold

- $H^2(K3, \mathbb{R})$ has signature $(3, 19)$
- Holomorphic two-form and Kähler form spanned by three real forms $\omega_i$ with $\omega_i \cdot \omega_j = \delta_{ij}$ and overall volume $\nu$:

$$\omega = \omega_1 + i\omega_2 \quad j = \sqrt{2\nu} \omega_3$$

- K3 is hyperkähler, i.e. $SO(3)$ rotating the $\omega_i \sim$ geometry fixed by positive-norm three-plane $\Sigma \subset H^2(K3, \mathbb{R})$ and $\nu$
- Moduli space has $3 \times 19 + 1 = 58$ dimensions
- Integral basis for $H^2(K3)$ with intersection matrix

$$U \oplus U \oplus U \oplus (-E_8) \oplus (-E_8), \text{ where } U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } E_8 \text{ is Cartan matrix of } E_8$$

$\Rightarrow$ The $\omega_i$ must have components along the $U$ blocks, components along “$E_8$ directions” determine gauge group
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K3: Elliptic Fibration and F-Theory Limit

• For an elliptically fibred $K3$, require integral cycles $B$ and $F$ (base and fibre) with
  • intersection matrix $\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$
  • $B \cdot \omega = F \cdot \omega = 0$

$\Rightarrow (B, F)$ spans a $U$ block, and we can parametrise the Kähler form as

$$j = bB + fF + c^a u_a \quad \text{(where } u_a \cdot \omega = 0\text{)}$$

• F-theory limit: Fibre volume shrinks to zero $\Rightarrow b \to 0$. $K3$ volume is $\nu \sim bf - c^a c^a$, so we have to take $c^a \to 0$ as fast as $\sqrt{b}$ (as intuitively expected)

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Cycles Between Branes

- One leg in the base, one in the fibre torus
- **Shrink to zero** when the branes are moved on top of each other.
- They are topologically a **sphere** $\leftrightarrow$ self-intersection $-2$.
- Cycles meeting at a brane intersect once, cycles encircling $\mathbb{O}$ planes ($\times$) do not intersect.
Shrinking Cycles and Gauge Enhancement

Intersection matrix of shrinking cycles determines gauge group:
Consider e.g. $T^2/\mathbb{Z}_2$ orientifold: One O7, four D7s $\mapsto SO(8)$

\[
\begin{pmatrix}
-2 & 1 & 0 & 0 \\
1 & -2 & 1 & 1 \\
0 & 1 & -2 & 0 \\
0 & 1 & 0 & -2 \\
\end{pmatrix}
\]

In appropriate basis, complex structure of $\hat{\mathcal{K}}3$ is [Braun, Hebecker, Freundl]

\[
\omega = \frac{\alpha}{2} + u e_2 + s \frac{\beta}{2} - \left( u s - \frac{z^2}{2} \right) e_1 + z \hat{E}_I
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Explicit mapping between complex structure and brane positions!
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base complex structure
axiodilaton
brane positions, $z_I = 0$ is $SO(8)^4$ point

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Flux Potential

- Type IIB: Three-form flux $G_3$ on the bulk, two-form gauge flux $F_2$ on the branes can stabilise geometric and brane moduli
- In M-theory, these are combined into four-form flux $G_4$ (brane moduli become four-form geometric moduli)
- Consistency conditions:
  - Flux quantisation: flux needs to be integral
  - Tadpole cancellation (without spacetime-filling M2 branes)
    \[
    \frac{1}{2} \int_{K3 \times \tilde{K3}} G_4 \wedge G_4 = \frac{\chi}{24} = 24
    \]
  - $G_4$ needs to have exactly one leg on each on base and fibre for Lorentz invariance, hence two on each $K3$: $G = G^{I\Lambda} \eta_I \wedge \tilde{\eta}_\Lambda$, but no flux along $B$ or $F$
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Fixing Branes by Fluxes  
String Pheno 2009, Warsaw
Potential

- Flux potential ($\mathcal{V}$ is the volume):

$$\mathcal{V} = \frac{1}{4\mathcal{V}^3} \left( \int_{K3 \times \tilde{K3}} G \wedge \ast G - \frac{\chi}{12} \right)$$

- $K3 \times \tilde{K3}$ is not a proper CY$_4$: Holonomy is $SU(2) \times SU(2)$

- $G_4$ induces map $G : H^2(\tilde{K3}) \to H^2(K3)$ and its adjoint $G^a$ by

$$G\tilde{\eta} = \int_{\tilde{K3}} G \wedge \tilde{\eta} \quad \quad G^a \eta = \int_{K3} G \wedge \eta$$

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$K3 \times \tilde{K3}$ Flux Potential

$$V = -\frac{1}{2(\nu \cdot \tilde{\nu})^3} \left( \sum_j \| G \tilde{\omega}_j \|_\perp^2 + \sum_i \| G^a \omega_i \|_\perp^2 \right)$$

Here $\| \cdot \|_\perp^2$ is the norm orthogonal to the three-plane

- Positive definite potential
- Manifestly symmetric under $SO(3)$
- Minima at $V = 0$:

$$G \tilde{\omega}_j \in \langle \omega_1, \omega_2, \omega_3 \rangle \quad G^a \omega_i \in \langle \tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3 \rangle$$

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Minima: Existence, Flat Directions

- Minkowski minima do not necessarily exist: $G^a G$ must be diagonalisable and positive semi-definite (not guaranteed although $G^a G$ is self-adjoint, since metric is indefinite!)
- Flat directions generally exist and are desired: M-theory moduli become part of 4D vector fields in F-theory limit $\leadsto$ fixing these moduli breaks the gauge group (rank-reducing)
- Flux also induces explicit mass term for three-dimensional vectors
- Vacua can preserve $\mathcal{N} = 4$, $\mathcal{N} = 2$ or $\mathcal{N} = 0$ supersymmetry in four dimensions, depending on the action of $G$ on the three-plane
Stabilisation Strategy

- F-theory limit fixes Kähler form (up to base volume), $j = f F$
- Holomorphic two-form determines shrinking cycles, i.e. gauge enhancement
- To stabilise a desired brane configuration:
  - Identify set of shrinking cycles to obtain desired brane stacks
  - Choose these as part of a basis of $H^2(\widetilde{K3})$ and complete by integral cycles
  - Find an integral block-diagonal flux that satisfies tadpole cancellation condition (strong constraint and computationally costly)
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Examples

We give explicit examples of

- The $T^2/\mathbb{Z}_2$ orientifold with $SO(8)^4$: Four stacks of four D7 branes and one O7 plane each
  - Moving one brane off a stack.
    $\leadsto SO(8)^3 \times SO(6) \times U(1)$ or $SO(8)^3 \times SO(6)$
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Conclusion

- We have a nice geometric picture of D7 brane motion
- We found the flux potential in M-theory and explicit conditions for the existence of minima and gauge symmetry breaking
- Translation to F-theory ⇒ recipe to find fluxes that stabilise a desired situation
- Explicit examples: We can move branes

- Open problem: Numerical scan of matrices is very time-consuming
- Outlook: Generalise to elliptically fibred four-folds to get physically more realistic models, in particular intersecting branes
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