On the Impossibility of Convergence of Mixed Strategies with No-Regret Learning

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Abstract

We study the limiting behavior of the mixed strategies that result from a general class of optimal no-regret learning strategies in a repeated game setting where the stage game is any $2 \times 2$ competitive game (for which all the Nash equilibria of the game are completely mixed) that may be zero-sum or non-zero-sum. We consider optimal no-regret strategies that are mean-based (i.e. information set at each step is the empirical average of the opponent’s realized play) and monotonic (either non-decreasing or non-increasing) in their argument. We show that for any such choice of strategies, the limiting mixed strategies of the players cannot converge almost surely to any Nash equilibrium. This negative result is also shown to hold under a broad class of relaxations of these assumptions, which includes popular variants of Online-Mirror-Descent with optimism and/or adaptive step-sizes. Finally, we conjecture that the monotonicity assumption can be removed, and provide partial evidence for this conjecture. Our results identify the inherent stochasticity in players’ realizations as a critical factor underlying this divergence, and demonstrate a crucial difference in outcomes between using the opponent’s mixtures and realizations to make strategy updates.

Keywords: No-regret Learning, Game Theory, Repeated Games, $2 \times 2$ games, Last-iterate Convergence.

1 Introduction

The mixed strategy Nash equilibrium (NE) is one of the oldest solution concepts central to game theory. A finer understanding of how the NE arises as an outcome of learning behavior in a repeated game setting continues to be an active area of research. Classical research in economics\(^1\) dating back to Brown [1951] and Robinson [1951] as well as recent

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\(^1\)For a recent survey, see Fudenberg and Levine [1998].
work in computer science [Freund et al., 1999] tells us that when both the players in a two-player zero-sum game use strategies based on no-regret learning dynamics [Hannan, 1957, Littlestone et al., 1989, Kalai and Vempala, 2005], then the time-average of their strategies will converge, almost surely, to a Nash equilibrium [Freund et al., 1999]. However, the convergence of the time-averaged mixed actions to a NE does not necessarily imply that the day-to-day behavior, i.e. the sequence of mixed strategies, of these players converges. In the asymptotic sense, the quantity of interest is the tuple of the limiting mixed strategies of both players, also referred to as the last-iterate (e.g. in Daskalakis and Panageas [2018]). Bailey and Piliouras [2018] discovered the following surprising property of the last iterate: When the players in a two-player zero-sum game compete against each other with the popular multiplicative weights update algorithm with certain learning rates, which constitutes a popular no-regret algorithm, then their resulting mixed strategies drift away from any interior NE — in fact, they drift towards the boundary of the strategy space. This intriguing result is derived in an environment where players can play what we term telepathic strategies, i.e. player 1 can observe the exact mixed strategies used by player 2, and vice versa. However, in the traditional repeated game setting, players can only observe the realizations of the opponent’s mixed strategies. One would only expect the oscillation problem to be exacerbated by the ensuing stochastic feedback.

The natural question that arises is whether these last-iterate oscillations are a specific property of the family of multiplicative weights algorithms, or a fundamental consequence of the no-regret property itself. This paper provides substantial evidence that it is the latter, by showing that last-iterate oscillation occurs for a broad, generic class of asymptotically optimal no-regret algorithms in the traditional repeated game setting (where players only observe realizations of each other’s mixed strategies, not the mixed strategies themselves). In this “non-telepathic” scenario, we show that the ensuing stochasticity in realizations is one of the critical ingredients underlying the last-iterate oscillation. Our results suggest that no-regret learning strategies possess certain intrinsic properties by which the two notions—no-regret and convergence of the limiting mixed strategies—inherently conflict with one another.

Our contributions: We consider a repeated $2 \times 2$ game, i.e. a two player game repeatedly played infinitely many times at steps $t = 1, 2, \ldots$, where both the players can play mixtures of two pure strategies each. The repeated game strategy for a player outlines the rule by which she picks her mixed action at step $t$ based on the history up to and including step $(t - 1)$. We will first describe our main result in its most stylized form to help identify the key components responsible for the phenomenon of last-iterate oscillation. We will make three natural assumptions on each player’s repeated game strategy, all of which are ubiquitous to most popular learning dynamics:

1. We assume the player’s strategy to be an optimal no-regret strategy with respect to her utility function, that is, she has an expected average regret of $O(t^{1/2})$ irrespective of the strategy employed by the other player. See Definition 2.3 for formal definitions of no-regret algorithms, optimal or otherwise.

2. We assume that the player’s optimal no-regret strategy is mean-based, i.e. the player
uses only the empirical average of the actions of the other player at step \((t - 1)\) as a sufficient statistic to decide her mixed action\(^2\) at step \(t\). Note, further, that such strategies are self-agnostic, in the sense that they do not use the actual realizations of their own mixed strategies to update their strategy. In general, the player is aware of the step \(t\), and we accordingly allow her rule for mapping empirical averages to mixed strategies to depend on the step \(t\).

3. We assume that the player’s optimal no-regret, mean-based strategy is monotonic in its argument, i.e. the empirical average of her opponent’s actions, at every step \(t \geq 1\). The monotonicity does not need to be strict, and its direction (increasing or decreasing) can vary arbitrarily across rounds. We note that monotonicity in the direction of the player’s best response (in the sense that the larger the relative advantage of a response is, the more likely a player is to use it) is a natural constraint to impose on a rational agent. In this context, time-varying monotonicity constitutes a significantly weaker regularity condition on the player’s strategies. We discuss a possible relaxation of this monotonicity assumption in Section 3.5.

Most popular online learning dynamics, such as Online-Mirror-Descent [Nemirovsky and Yudin, 1983] (or Follow-the-Regularized Leader Shalev-Shwartz et al. [2011]) strategies, can easily be verified to satisfy all three of these assumptions. Our main contribution is to show that if both players deploy strategies satisfying the above three properties, and the stage game possesses a unique completely mixed NE\(^3\), then their mixed actions cannot converge to Nash NE. We denote the unique NE of the stage game by the tuple \((p^*, q^*)\), where \(0 < p^*, q^* < 1\) denote the equilibrium strategies of playing action 1 by players 1 and 2 respectively. In Theorem 3.4, we prove the following statement for any such game (described here informally):

\begin{quote}
If players 1 and 2 use (possibly different) optimal-no-regret, mean-based and monotonic repeated game strategies, then their mixed strategies cannot converge to the NE \((p^*, q^*)\).
\end{quote}

Our proof technique isolates the ensuing stochasticity in either of the player’s realizations as a critical ingredient underlying these last-iterate oscillations. In particular, we prove Theorem 3.4 via contradiction: suppose, instead, that the sequence of mixed strategies \((P_t, Q_t)\) converged to \((p^*, q^*)\), which implies that \(Q_t \to q^*\). We show that this would cause sufficient stochasticity by itself to necessitate \(P_t\) to oscillate with a positive probability as a fundamental consequence of no-regret (together with the mean-based and monotonic properties). The intuition for why stochasticity in realizations is the primary cause of last-iterate

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\(^2\)In other words, the player is agnostic to the ordering in the opponent’s action realizations. In Section 3.4, we weaken this assumption to show the validity of our results with strategies that display recency bias.

\(^3\)We note that any \(2 \times 2\) game that possesses only completely mixed NE, can be shown to possess one unique NE, which is also the unique correlated equilibrium of the game [Phade and Anantharam, 2019]. These games are designated as competitive games by Calvó-Armengol [2006]. These games have been of special interest in the design of experiments to test the performance of NE as a predictor of behavior in games as they have an unambiguous NE. For example, Selten and Chmura [2008], Binmore et al. [2001] refer to these games as completely mixed \(2 \times 2\) games.
oscillations is contained in an elementary “warm-up” argument provided in Theorem 3.3, which shows that the iterates of player 1 oscillate even in an idealized scenario in which player 2 has already converged to his NE strategy, i.e. \( q_t = q^* \) for all \( t \geq 1 \).

The mean-based and monotonic properties described above do not constitute all popular no-regret strategies used in practice; notable exceptions are the family of optimistic no-regret algorithms [Rakhlin and Sridharan, 2013b, Syrgkanis et al., 2015, Daskalakis and Panageas, 2018] and Online-Mirror-Descent algorithms run with data-adaptive step sizes [Hazan and Kale, 2010, Cesa-Bianchi et al., 2007, Erven et al., 2011, Rakhlin and Sridharan, 2013a,b]. However, we prove that such strategies can be reduced in a natural way to mean-based and monotonic strategies in Section 3.4, and consequently show that last-iterate oscillations will continue to arise when these strategies are used. Our negative result for optimistic strategies in particular highlights a contrast to the telepathic setting, in which Daskalakis and Panageas [2018] showed that the last iterate (which is deterministic under telepathic dynamics) will converge to NE when both players use optimistic mirror descent strategies. A complete removal of the mean-based and monotonic assumptions remains an important direction for future work; however, in Section 3.5 we provide partial evidence that the monotonicity assumption in particular can be removed.

**Related work:** While the evolution of the time-averages of players’ strategies as a consequence of multiple players using no-regret dynamics has been an active topic of study for several decades [Brown, 1951, Robinson, 1951, Foster and Vohra, 1997, Fudenberg and Levine, 1998, Freund et al., 1999, Hart and Mas-Colell, 2000, Hart, 2005, Kalai and Vempala, 2005], the properties of the limiting mixed strategies, or the last-iterates, have only been examined more recently. This topic has also seen substantial attention in the related setup of min-max optimization [Daskalakis et al., 2018, Mertikopoulos et al., 2018, Liang and Stokes, 2019, Abernethy et al., 2019, Lei et al., 2020], where the primary goal is to attain a pure-strategy NE of a game with a continuous-pure-strategy set through the use of first-order optimization algorithms, e.g. gradient descent-ascent. This problem has been primarily studied in the deterministic setting, corresponding to the aforementioned telepathic dynamics in the game-theoretic setup. Recently, Daskalakis and Panageas [2018] showed that a modification of the multiplicative weights strategy that incorporates recency bias succeeds in last-iterate convergence in the game-theoretic setup with telepathic dynamics. This type of recency bias, commonly called optimism, has also been shown to successfully converge in min-max optimization when applied to the gradient descent/ascent algorithms [Daskalakis et al., 2018, Mertikopoulos et al., 2018, Liang and Stokes, 2019, Abernethy et al., 2019, Lei et al., 2020]. Moreover, optimistic algorithms have other notable properties, such as leading to faster convergence rates of the time-average of mixed strategies in zero-sum as well as non-zero-sum games [Rakhlin and Sridharan, 2013b, Syrgkanis et al., 2015]. However, we show in Section 3.4 that when stochastic realization-based feedback is considered, optimistic variants on mean-based strategies do not resolve the last-iterate oscillation issue. In fact, the key phenomena that we outlined above manifest in recency-bias-based strategies as well. This illustrates that the issue of last-iterate oscillation runs deeper in the traditional repeated-game setting than in the telepathic setting. We
briefly discuss alternative (non-constructive) strategies that could satisfy the last-iterate-convergence property in Section 5 — these strategies are not no-regret, but satisfy a weaker property of “smoothly calibrated forecasting” [Foster and Hart, 2018].

2 Setup

We consider $2 \times 2$ games, i.e. a two player game where both the players have two pure strategies, namely, action 0 and action 1. Let the payoff matrices for player 1 and player 2 be given by

$$G := \begin{bmatrix} G(0, 0) & G(0, 1) \\ G(1, 0) & G(1, 1) \end{bmatrix} \quad \text{and} \quad H := \begin{bmatrix} H(0, 0) & H(0, 1) \\ H(1, 0) & H(1, 1) \end{bmatrix},$$

respectively. Thus, if player 1 plays action $i \in \{0, 1\}$ and player 2 plays action $j \in \{0, 1\}$, the payoff to player 1 is given by $G(i, j)$ and the payoff to player 2 is given by $H(i, j)$.

We denote by the indicator random variables $\mathbf{I}$ and $\mathbf{J}$ the realizations of the mixed strategies of player 1 and player 2, respectively. We follow the convention of denoting random variables by the bold versions of their corresponding deterministic variables. Let $p := \mathbb{E}[\mathbf{I}]$ and $q := \mathbb{E}[\mathbf{J}]$ be the probabilities with which the two players play action 1, respectively. In general, since the two players will implement their mixed strategies independently, the random variables $\mathbf{I}$ and $\mathbf{J}$ will be independent. Therefore, the expected payoff for player 1 and player 2 corresponding to the choice of mixed strategies $(p, q)$ is given by $G(p, q)$ and $H(p, q)$, respectively, where

$$X(p, q) := (1 - p)(1 - q)X(0, 0) + (1 - p)qX(0, 1) + p(1 - q)X(1, 0) + pqX(1, 1),$$

where $X$ stands for $G$ or $H$. In the repeated game setting, $\{I_t\}_{t \geq 1}$ and $\{J_t\}_{t \geq 1}$ denote the action sequences of the two players and $\{P_t\}_{t \geq 1}$ and $\{Q_t\}_{t \geq 1}$ denote the mixed strategy sequences of the two players. We denote by

$$(\mathbf{I})^t := \{I_s\}_{s=1}^t \quad \text{and} \quad (\mathbf{J})^t := \{J_s\}_{s=1}^t$$

the random sequence of actions up to step $t$. The empirical averages, or time-averages, of the actions of the two players are given by

$$\hat{P}_t := \frac{1}{t} \sum_{s=1}^t I_t \quad \text{and} \quad \hat{Q}_t := \frac{1}{t} \sum_{s=1}^t J_t,$$

respectively. Similarly, the empirical averages, or time-averages, of the mixed strategies of the two players are given by

$$\hat{P}_t := \frac{1}{t} \sum_{s=1}^t P_t \quad \text{and} \quad \hat{Q}_t := \frac{1}{t} \sum_{s=1}^t Q_t.$$


respectively. General repeated game strategies for player 1 and player 2 are given by sequences of functions \( \{f_t\}_{t \geq 1} \) and \( \{g_t\}_{t \geq 1} \), where for every \( t \geq 1 \), \( f_t, g_t : \{0, 1\}^{2(t-1)} \to [0, 1] \) map the history up to step \( t \) to mixed strategies given by

\[
P_t := f_t((I)^{t-1}, (J)^{t-1}) \quad \text{and} \quad Q_t := f_t((I)^{t-1}, (J)^{t-1})
\]

for players 1 and 2 respectively. We will refer to these functions \( f_t \) and \( g_t \) as the strategy functions for players 1 and 2 at step \( t \). Critically, observe that we are not allowing for telepathy in the updates. In other words, the history used by player 1 at step \( t \) does not include \( \{Q_s\}_{s=1}^{t-1} \), and the history used by player 2 at step \( t \) does not include \( \{P_s\}_{s=1}^{t-1} \). This is in agreement with the information structure of the traditional repeated game environment. We now put forward several definitions to identify key sub-classes of these strategies.

**Definition 2.1.** We say that a repeated game strategy for player 1 is self-agnostic if player 1 uses only the action sequence of player 2 to decide her mixed strategy \( P_t \) at step \( t \). With an abuse of notation, the strategy function can be replaced by a function \( f_t : \{0, 1\}^{t-1} \to [0, 1] \) such that the mixed strategy for player 1 at step \( t \) is given by \( P_t = f_t((J)^{t-1}) \).

Note that player 1 is actually aware of her mixed strategies \( P_1, P_2, \ldots, P_{t-1} \) at step \( t \) since she is aware of her strategy functions \( f_1, f_2, \ldots, f_{t-1} \) and, for \( 1 \leq s \leq t - 1 \), \( P_s \) can be determined from \( f_s \) and \( (J)^{s-1} \). Thus, by self-agnostic, we only mean that player 1 is agnostic to the actual realizations of her actions up to that step in the process of choosing her next mixed strategy.

From the point of view of player 1, we now define no-regret strategies, as well as uniformly no-regret strategies against an oblivious opponent. The former is precisely the classical definition of consistency first proposed by Hannan [1957], while the latter is a strictly stronger condition, requiring an effective non-asymptotic guarantee on regret. We will use the stronger uniform no-regret condition to derive our results.

**Definition 2.2.** A self-agnostic repeated game strategy \( \{f_t\}_{t \geq 1} \) is said to be no-regret if

\[
\limsup_{T \to \infty} \frac{1}{T} \left[ \max_{i \in \{0, 1\}} \sum_{t=1}^{T} G(i, J_t) - \sum_{t=1}^{T} G(f_t((J)^{t-1}), J_t) \right] \leq 0,
\]

for all opponent sequences \( \{J_t\}_{t \geq 1} \).

**Definition 2.3.** A self-agnostic repeated game strategy \( \{f_t\}_{t \geq 1} \) is said to be uniformly no-regret if

\[
\limsup_{T \to \infty} \max_{\{J_t\}_{t=1}^{T}} \frac{1}{T} \left[ \max_{i \in \{0, 1\}} \sum_{t=1}^{T} G(i, J_t) - \sum_{t=1}^{T} G(f_t((J)^{t-1}), J_t) \right] \leq 0.
\]

In particular, the strategy achieves a no-regret rate of \((r, c)\) if

\[
\limsup_{T \to \infty} \max_{\{J_t\}_{t=1}^{T}} \frac{1}{T^r} \left[ \max_{i \in \{0, 1\}} \sum_{t=1}^{T} G(i, J_t) - \sum_{t=1}^{T} G(f_t((J)^{t-1}), J_t) \right] \leq c.
\]
Observe that all uniformly no-regret strategies are also no-regret. The converse need not hold — however, algorithms that are no-regret but not uniformly no-regret are typically contrived examples that are unlikely to be used in practice.

The following properties of uniformly no-regret strategies \( \{ f_t \} \) can easily be verified:

1. If a strategy \( \{ f_t \} \) satisfies a uniform no-regret rate of \( (r, c) \), then it satisfies a no-regret rate of \( (r, c') \) for all \( c' \geq c \).

2. If a strategy \( \{ f_t \} \) satisfies a uniform no-regret rate of \( (r, c) \), then it satisfies a no-regret rate of \( (r', 0) \) for all \( r' > r \).

It is well-known (e.g. see [Cesa-Bianchi and Lugosi, 2006, Chapter 3]) that, for any finite constant \( 0 < c < \infty \), the best possible no-regret rate is \( r = 1/2 \). Moreover, several commonly used algorithms, such as those in the Online-Mirror-Descent family [Nemirovsky and Yudin, 1983, Shalev-Shwartz et al., 2011], typically match the optimal no-regret rate for appropriately chosen constant \( c \).

We state two additional assumptions satisfied by no-regret strategies commonly encountered in the literature. We will prove our main results under these assumptions.

**Definition 2.4.** The strategy of player 1 (or 2) is called mean-based if player 1 uses only the empirical averages of player 2 (or 1) as a sufficient statistic to determine her mixed strategy \( P_t \) at step \( t \). In this case, with another abuse of notation, the strategy function can be replaced by a function \( f_t : [0, 1] \to [0, 1] \), such that the mixed strategy for player 1 at step \( t \) is given by \( P_t = f_t(Q_{t-1}) \).

For example, all algorithms in the popular Online-Mirror-Descent framework [Nemirovsky and Yudin, 1983, Shalev-Shwartz et al., 2011] satisfy this assumption. Our results will also apply to variants on the mean-based property that incorporate a recency bias or adaptive step-sizes. These extensions are discussed in Section 3.4.

**Definition 2.5.** The mean-based strategy of player 1 (see Definitions 2.4) is called monotonic if, for every \( t \geq 1 \), \( f_t(\cdot) \) is either non-increasing or non-decreasing in its argument.

Note that this assumption does not require strict monotonicity, and also does not require the direction of the monotonicity to be the same across rounds. We can interpret it as a regularity condition that we impose primarily for technical reasons. We conjecture that even this regularity condition is not required to show that last-iterate oscillations occur, and discuss partial evidence for this conjecture at length in Section 3.5.

In this paper, we will restrict our attention to \( 2 \times 2 \) games for which none of the NE lie on the boundary of the mixed strategy space, i.e. neither player plays a pure-strategy in any NE\(^4\). Phade and Anantharam [2019] characterize the NE of all \( 2 \times 2 \) games, and show that all the NE of a \( 2 \times 2 \) game are completely mixed if and only if it is a competitive game, defined for completeness below.

\(^4\)Note that, although for different reasons, the last-iterate oscillation result shown by Bailey and Piliouras [2018] for multiplicative weights also requires the assumption that all the NE lie in the interior of the strategy space.
Definition 2.6. A $2 \times 2$ game is said to be competitive if either of the following conditions holds:

(a) $G(0, 0) > G(1, 0), G(0, 1) < G(1, 1), H(0, 0) < H(0, 1), H(1, 0) > H(1, 1),$
(b) $G(0, 0) < G(1, 0), G(0, 1) > G(1, 1), H(0, 0) > H(0, 1), H(1, 0) < H(1, 1).$

Moreover, competitive games have a unique NE $(p^*, q^*)$ which is also its unique correlated equilibrium. Henceforth, without loss of generality, we will assume that the payoffs satisfy condition (a) above. Observe that in the special case of zero-sum games, i.e. $H = -G$, the condition (a) would become: $G(0, 0) > G(1, 0), G(0, 1) < G(1, 1), G(0, 0) > G(0, 1)$ and $G(1, 0) < G(1, 1)$. This condition is satisfied by several common zero-sum games, including the matching pennies game which we will use as a running example.

Remark 2.7. Note that the set of zero-sum games with unique NE is strictly contained in the set of competitive games. The results in Phade and Anantharam [RPQY] imply that, for any competitive game, there exists a corresponding zero-sum game$^5$ such that the best-response functions of both the players are exactly the same. It is easy to see that if two games have the same best-response functions, then they have the same NE. However, this does not imply that the players’ behavior (whether on average, or day-to-day) will be the same when such games are played repeatedly. Appendix C illustrates significant qualitative differences in the last-iterate behavior of players playing no-regret strategies for two such “equivalent” games.

We define $R^* := G(p^*, q^*) = G(1, q^*) = G(0, q^*)$ (where the chain of equalities follows because $p^*$ is in the interior and by the definition of a Nash equilibrium).

3 Main results

In this section, we prove in a series of steps that the limiting mixed strategies as an outcome of both players using optimal, mean-based and monotonic no-regret learning cannot converge. In the process, we will show some fundamental properties about no-regret strategies that hold in more general settings and may be of independent interest.

3.1 A fundamental sensitivity to fluctuations

We start by proving a condition that any (not necessarily mean-based or monotonic) self-agnostic, no-regret strategy with a rate $r \geq 1/2$ necessarily satisfies. We show that any such strategy is fundamentally sensitive to stochastic fluctuations in the outcomes of player 2’s strategy, in the sense that such fluctuations will cause player 1 to deviate from her equilibrium strategy by an additive constant with a constant non-zero probability. More precisely:

$^5$We note that the transformation between the non-zero-sum game and corresponding zero-sum game may not be linear, where a linear transformation is one that is obtained by changing the payoff matrices to be $a_1G + b_1$ and $a_2H + b_2$ for player 1 and 2, respectively, where $a_1, b_1, a_2, b_2 \in \mathbb{R}$. 

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is uniformly no-regret with a rate of 

\( \eta = 1/\sqrt{t} \). Here, we set \( t = 10^6 \).

Figure 1: Depiction of the sensitivity of common no-regret strategies in the matching pennies game in terms of \( f_t(\hat{Q}_t) \), as a function of \( \hat{Q}_t \) for \( t = 10^6 \). Figure 1a depicts the extent of sensitivity for two popular no-regret strategies, and Figure 1b illustrates the proof strategy.

we show that there must be infinitely many mappings \( \{ f_t(\cdot) \}_{t \geq 1} \) for player 1 that deviate (with a constant non-zero probability) by at least a constant value (say \( \delta > 0 \)) from her NE strategy \( p^* \) whenever player 2 deviates from his NE strategy and plays action 1 (or 0) for a fraction of steps on the order of \( t^{-(1-r)} \). When the strategy is mean-based, the deviation of player 2 for \( \mathcal{O}(t^{-(1-r)}) \) fraction of steps corresponds to a deviation of \( \mathcal{O}(t^{-(1-r)}) \) in the empirical average of the actions of player 2.

Overall, this shows that if we need our no-regret algorithm to have better regret rates, then we have to make the mappings constituting the no-regret algorithm more fluctuation-sensitive. A depiction of this fluctuation-sensitivity is in Figure 1a for the example of the multiplicative weights algorithm, parameterized to have optimal no-regret rate \( r = 1/2 \), in the matching pennies game. This property of fluctuation-sensitivity is formally defined and proved below.

**Proposition 3.1.** For any self-agnostic repeated game strategy \( \{ f_t \}_{t \geq 1} \) used by player 1 that is uniformly no-regret with a rate of \((r,c), 1/2 \leq r < 1\), and any \( 0 < \delta < (1 - p^*)/3 \), there exists a positive constant \( \alpha \) and an infinite sequence of integer tuples \( \{(t_k, s_k)\}_{k \geq 1} \) such that

\[
0 < s_k \leq \alpha(t_k) r, \text{ for all } k \geq 1, \quad (3.1)
\]

and

\[
\mathbb{E} \left[ f_{t_k} \left( (J'(k))^t_k \right) \right] \geq p^* + 2\delta \text{ for all } k \geq 1, \quad (3.2)
\]
where the expectation is over the random sequence $(J'(k))_{t_k}^t := \{J'_s(k)\}_{s=1}^{t_k}$ defined as below:

$$J'_s(k) = \begin{cases} 
    J^*_s \text{ i.i.d. } \sim \text{Bernoulli}(q^*), & \text{if } 1 \leq s \leq t_k - s_k, \\
    1 & \text{otherwise}.
\end{cases}$$  

(3.3)

Similarly, we also have

$$\mathbb{E} \left[ f_{t_k}((J'(k))_{t_k}^t) \right] \leq p^* - 2\delta \text{ for all } k \geq 1,$$

(3.4)

where the expectation is over the random sequence $(J''(k))_{t_k}^t := \{J''_s(k)\}_{s=1}^{t_k}$ defined as below:

$$J''_s(k) = \begin{cases} 
    J^*_s \text{ i.i.d. } \sim \text{Bernoulli}(q^*), & \text{if } 1 \leq s \leq t_k - s_k, \\
    0 & \text{otherwise}.
\end{cases}$$

(3.5)

In particular, if the self-agnostic repeated game strategy $\{f_t\}_{t \geq 1}$ is optimally uniformly no-regret i.e. $r = 1/2$, then condition 3.1 would be

$$0 < s_k \leq \alpha \sqrt{t_k}, \text{ for all } k \geq 1.$$  

For this case, Figure 1b shows player 1’s response, when she uses the multiplicative weight algorithm, to an opponent who plays the random sequence defined in Equation (3.3). The game that is being played is matching pennies, which is a zero-sum game with $G(0,0) = G(1,1) = 1$ and $G(0,1) = G(1,0) = -1$. Player 1’s strategy, averaged over player 2’s realizations, is shown to deviate sizably from the NE of the matching pennies game, $p^* = 0.5$. This case is important because, even if player 2 remained close to her equilibrium mixed strategy $q^*$ (as is necessary in any hypothetical scenario of last-iterate convergence), there is a non-trivial probability of player 2’s empirical average deviating from $q^*$ by a number on the order of $1/\sqrt{t}$ at step $t$. The sensitivity in an optimal no-regret strategy to pick deviations of this order will allow us to show the oscillation of player 1’s mixed strategies in the subsequent Sections 3.2, 3.3, 3.4 and 3.5. To execute this proof approach, we would use Equation (3.2) if $f_t(\cdot)$ is monotonically increasing in its argument, and Equation (3.4) if $f_t(\cdot)$ is monotonically decreasing in its argument. Henceforth, we will assume that $f_t(\cdot)$ is increasing in its argument without loss of generality.

We conclude this subsection with the proof of Proposition 3.1.

**Proof.** Recall that by the competitive $2 \times 2$ game assumption, we have $G(0,1) < G(1,1)$. Thus, for any $0 \leq p < 1$, we have $G(p,1) < G(1,1)$. Consider $\{f_t\}_{t \geq 1}$ to be any self-agnostic uniformly no-regret strategy with a no-regret rate of $(r,c)$. We note that for any $t$, and any sequence $\{J_s\}_{s=1}^t$, we have

$$\max_{i \in \{0,1\}} \sum_{s=1}^t G(i,J_s) = t \cdot \max \{G(0,\tilde{Q}_t),G(1,\tilde{Q}_t)\}.$$  

Recall that

$$\tilde{Q}_t = \frac{1}{t} \sum_{s=1}^t J_s.$$ 

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Thus, for \( c' > c \), there exists a sufficiently large \( t_0 \) such that for all \( t \geq t_0 \), we have

\[
\frac{1}{tr} \left[ t \cdot \max \{ G(0, \tilde{Q}_t), G(1, \tilde{Q}_t) \} - \sum_{s=1}^{t} G(f_s((J)^{s-1}, J_s)) \right] \leq c',
\]

for any sequence \( \{J_s\}_{s=1}^t \).

Recall that \( R^* \) denotes the Nash equilibrium payoff of player 1. Because the equilibrium \((p^*, q^*)\) is strictly mixed, we have \( R^* = G(p^*, q^*) = G(p^*, 1) < G(1, 1) \). Let \( 0 < \delta < \frac{1-p^*}{3} = \frac{(G(1, 1) - R^*)}{3(G(1, 1) - G(0, 1))} \). and let \( \delta' := \delta(G(1, 1) - G(0, 1)) \). Note that \( 0 < \delta' < \frac{\delta}{3} \). Let \( \alpha := \frac{c'}{(G(1, 1) - R^* - 3\delta')} > 0 \). For any \( t > t_1 := \max \{ t_0, \alpha^{1/r} \cdot (\alpha \delta'/G(1, 1))^{-1/r} \} \), let \( t^*(t) := t - \lfloor \alpha t \rfloor \), where \( \lfloor \cdot \rfloor \) is the floor function. Note that since \( t > \alpha^{1/r} \), we have \( t^*(t) \geq 1 \). Let \( \{J_s^*\} \) be an i.i.d. sequence of Bernoulli\((q^*)\) random variables for \( 1 \leq s \leq t \). Let \( \tilde{Q}_s^* := \frac{1}{s} \sum_{s'=1}^{s} J_{s'}^* \) denote the empirical average of this sequence at step \( s \). We state the following useful lemma.

**Lemma 3.2.** For the sequence \( \{J_s^*\}_{s \geq 1} \) defined above, and any \( t \geq 1 \), we have

\[
\mathbb{E} \left[ \sum_{s=1}^{t} G(f_s((J^*)^{s-1}, J_s^*)) \right] = tR^* \quad \text{and} \quad (3.7a)
\]

\[
\mathbb{E} \left[ \sum_{s=1}^{t} J_s^* \right] = tq^*. \quad (3.7b)
\]

See Appendix A for the proof of this lemma. We define the sequence \( \{J_s'\}_{s=1}^t \) as specified in the statement of Proposition 3.1. In other words, we define \( J_s' = J_s^* \) for \( 1 \leq s \leq t^*(t) \), and \( J_s' = 1 \) for \( t^*(t) < s \leq t \). Then, we can denote the empirical average of this sequence as \( \tilde{Q}_s' := \frac{1}{s} \sum_{s'=1}^{s} J_{s'}' \) for \( 1 \leq s \leq t \). We denote

\[
M := \frac{1}{[\alpha t^r]} \cdot \mathbb{E} \left[ \sum_{s=1}^{[\alpha t^r]} G \left( f_{t^*(t)+s} \left( (J')^{t^*(t)+s-1} \right), 1 \right) \right]. \quad (3.8)
\]

From the definition of uniform no-regret, i.e. Equation (3.6), we have

\[
c' \geq \frac{1}{t^r} \cdot \mathbb{E} \left[ t \cdot \max \{ G(0, \tilde{Q}_t), G(1, \tilde{Q}_t) \} - \sum_{s=1}^{t} G(f_s((J')^{s-1}, J_s')) \right]
\]

\[
\geq \frac{1}{t^r} \cdot \mathbb{E} \left[ t \cdot G(1, \tilde{Q}_t) - \sum_{s=1}^{t} G(f_s((J')^{s-1}, J_s')) \right]
\]

\[
= \frac{1}{t^r} \cdot [t^*(t) \cdot R^* + [\alpha t^r] \cdot G(1, 1) - t^*(t) \cdot R^* - [\alpha t^r] \cdot M]
\]

\[
\geq (G(1, 1) - M) \cdot \alpha - G(1, 1)t^{-r}.
\]

Using the fact that \( \alpha := \frac{c'}{(G(1, 1) - R^* - 3\delta')} \) and \( t > \left( \frac{\alpha \delta'}{G(1, 1)} \right)^{-1/r} \), we get
\[
M \geq G(1, 1) - \frac{c'}{\alpha} - \frac{G(1, 1) \cdot t^{-r}}{\alpha} = R^* + 3\delta' - \frac{G(1, 1) \cdot t^{-r}}{\alpha} \geq R^* + 2\delta'.
\]

Now, using linearity of expectation, and linearity of the payoff function \( G(p, 1) \) in the argument \( p \), we get

\[
M = G(\bar{f}, 1) \quad \text{where} \quad \bar{f} := \frac{1}{[\alpha t^*]} \sum_{s=1}^{[\alpha t^*]} \mathbb{E} \left[ f_{t^*(t)+s} \left( (J')^{t^*(t)+s-1} \right) \right].
\]

Now, since \( G(1, 1) > G(0, 1) \), we note that \( G(p, 1) = G(0, 1) + (G(1, 1) - G(0, 1))p \) is an increasing function in \( p \) and so we get

\[
\bar{J} \geq p^* + \frac{2\delta'}{G(1, 1) - G(0, 1)} = p^* + 2\delta,
\]

from the above inequality on \( M \). Thus, there exists \( s(t) \) such that \( 1 \leq s(t) \leq [\alpha t^*] \) and

\[
\mathbb{E} \left[ f_{t^*(t)+s(t)} \left( (J')^{t^*(t)+s(t)-1} \right) \right] \geq p^* + 2\delta. \tag{3.9}
\]

To write this in the language of Equation (3.2), we observe that

\[
0 < \frac{s(t)}{t^*(t) + s(t)} \leq \frac{[\alpha t^*]}{t^*(t) + [\alpha t^*]} \leq \frac{\alpha t^*}{t} = \alpha t^{r-1} \leq \alpha (t^*(t) + s(t))^{r-1},
\]

and therefore we get,

\[
s(t) \leq \alpha (t^*(t) + s(t))^r. \tag{3.10}
\]

We note that \( t^*(t) \to \infty \) as \( t \to \infty \), and hence we can define an infinite sequence of integer tuples \( \{t_k, s_k\}_{k \geq 1} \) (where we have defined \( t_k = t^*(t_1 + k) + s_k \) and \( s_k = s(t_1 + k) \) as above) such that \( 0 < s_k \leq \alpha (t_k)^r \) and

\[
\mathbb{E} \left[ f_{t_k} \left( (J'(k))^t \right) \right] \geq p^* + 2\delta \quad \text{for all} \quad k = 1, 2, \ldots \tag{3.11}
\]

This is precisely the statement in Equation (3.2), and completes the proof of Proposition 3.1.

\[\square\]

### 3.2 Warmup: Last-iterate oscillation when opponent is already at equilibrium

Equation (3.2) highlights a critical property of any self-agnostic uniformly no-regret algorithm with a regret rate of \((r,c)\): it needs to be sufficiently sensitive to small perturbations
P2’s average frequency of 1’s, number of rounds (t)

0.5

0.6

0.4

0.499

0.501

10 100 10^3 10^4 10^5 10^6 10^7 10^8 10^9

Figure 2: Depiction of constant fluctuations of player 2’s time-averages, i.e. \( \hat{Q}_t \), as a function of \( t \), when player 2 plays \( Q_t \) i.i.d. \( \sim q^* \).

on the order of \( t^{-r} \) in the opponent’s strategy. We can concretize this property to show that the last-iterate oscillates (i.e. does not converge almost surely) when both players use optimal no-regret strategies (i.e. \( r = 1/2 \)) that are mean-based (Definition 2.4) and monotonic (Definition 2.5). Recall that under the mean-based assumption, player 1’s strategy functions are given by

\[
Q_t = \frac{f_t((J)^{i-1})}{f_t(\hat{Q}_{t-1})}
\]

for all \( t \geq 1 \).

In this section, we state and prove a warm-up result that clearly illustrates how the stochasticity in realizations alone can lead to last-iterate oscillation. We consider the idealized case in which player 2 is playing his equilibrium strategy at all steps, i.e. \( \{J_t\}_{t \geq 1} \) i.i.d. \( \sim \text{Bernoulli}(q^*) \). Remarkably, we show that even this simple case necessitates the limiting mixed strategy of player 1 to diverge! The central reason for this is as follows: if player 2 is playing the mixed NE \( q^* \) at all his steps, then the time-averages of his realized actions will fluctuate on the order of \( t^{-1/2} \) infinitely often as well, leading to a \( \delta \)-deviation of player 1 from her NE strategy \( p^* \) with a positive probability via Proposition 3.1. Figure 2 (which should remind the reader of the fluctuations of a symmetric random walk) depicts these recurring fluctuations on the order of \( t^{-1/2} \) of player 2’s time-averaged actions for a typical realization of player 2 playing mixed strategy \( q^* = 1/2 \) at all steps. The ensuing last-iterate oscillation is stated and proved below.

**Theorem 3.3.** Let player 2’s strategy \( \{J_t\}_{t \geq 1} \) be an i.i.d. sequence of Bernoulli(\( q^* \)) random variables. Then, any mean-based and monotonic repeated game strategy \( \{f_t\}_{t \geq 1} \) that has a regret rate of \( (1/2, c) \) causes player 1’s last iterate to diverge in probability, i.e. there exist

---

\(^6\)This is in stark contrast to the setting of telepathic dynamics, in which a simple algorithm like multiplicative weights used in the matching pennies game would trivially lead player 1 to converge to the NE strategy, \( p^* = 1/2 \).

\(^7\)This figure was inspired by Dean P. Foster’s illustration of the law of the iterated logarithm: [https://en.wikipedia.org/wiki/Law_of_the_iterated_logarithm](https://en.wikipedia.org/wiki/Law_of_the_iterated_logarithm)

\(^8\)We prove this theorem without the monotonicity assumption in Proposition F.1.
positive constants \((\delta, \epsilon)\) such that

\[
\limsup_{t \to \infty} \mathbb{P} \left[ |P_t - p^*| \geq \delta \right] \geq \epsilon. \tag{3.12}
\]

The proof of Theorem 3.3 constitutes an application of Markov’s inequality and the central limit theorem, and is provided below.

**Proof.** We start by defining some notation pertinent to mean-based strategies. Let \(Z_t \sim \text{Binomial}(t, q^*)\), for \(t \geq 1\). Let \(Z''_{t,s} \sim Z_{t-s} + s\) and \(Q''_{t,s} = Z''_{t,s}/t\), for \(0 \leq s \leq t, t \geq 1\). Let \(\{(t_k, s_k)\}_{k \geq 1}\) be an infinite sequence and \(0 < \delta < (1 - p^*)/3\) as in Proposition 3.1. Thus, for every \(k \geq 1\), we have

\[
Q''_{t_k, s_k} \overset{d}{=} \frac{1}{t} \sum_{s=1}^{t} J'_t(k),
\]

where \(\overset{d}{=}\) denotes that the two random variables are identical in distribution, and \(J'_t(k)\) are random variables as defined in Proposition 3.1. We thus have

\[
0 < s_k \leq \alpha(t_k)^{1/2}, \text{ for all } k \geq 1,
\]

and

\[
\mathbb{E} \left[ f_{t_k} \left( Q''_{t_k, s_k} \right) \right] \geq p^* + 2\delta, \text{ for all } k \geq 1,
\]

Consider the random variable \(Y := 1 - f_{t_k}(\hat{Q}'_{t_k, s_k})\). Since the range of \(f_t\) is always \([0, 1]\), we have \(Y \geq 0\). Thus, we have \(\mathbb{E}[Y] \leq 1 - (p^* + 2\delta) = (1 - p^*) - 2\delta\). By Markov’s inequality, we have

\[
\mathbb{P} (Y \geq (1 - p^*) - \delta) \leq \frac{(1 - p^*) - 2\delta}{(1 - p^*) - \delta}.
\]

Thus, we get

\[
\mathbb{P} \left( f_{t_k} \left( Q''_{t_k, s_k} \right) > p^* + \delta \right) \geq \epsilon_0, \tag{3.15}
\]

where we define \(\epsilon_0 := \delta / ((1 - p^*) - \delta)\). Note that \(0 < \epsilon_0 < 1/2\) (because we had defined \(0 < \delta < (1 - p^*)/3\)). By the central limit theorem, we know that

\[
\lim_{t \to \infty} \mathbb{P} \left( \hat{Q}_t > q^* + \frac{\gamma}{\sqrt{t}} \right) = 1 - \Phi \left( \frac{\gamma}{\sqrt{q^*(1 - q^*)}} \right),
\]

where \(\Phi(\cdot)\) is the cumulative distribution function of the standard normal distribution. Now, note that the function \(\Phi(\cdot) : \mathbb{R} \to (0, 1)\) in continuous, strictly increasing. Moreover, we have \(\Phi(0) = 1/2\) and \(\lim_{x \to \infty} \Phi(x) = 1\). Hence, there exists \(\gamma_0 > 0\) such that \(1 - \Phi(\gamma_0/\sqrt{q^*(1 - q^*)}) = \epsilon_0/4 < 1/2\). Consequently, there exists \(T'_1(\epsilon_0) > 1\) such that

\[
\mathbb{P} \left( \hat{Q}_t > q^* + \frac{\gamma_0}{\sqrt{t}} \right) \leq (1 - \phi(\gamma_0)) + \frac{\epsilon_0}{4} \leq \frac{\epsilon_0}{2}
\]


for all $t \geq T'_1(\epsilon_0)$. Now, observe that
\[ t_k \hat{Q}_{tk,s_k}'' \overset{d}{=} (t_k - s_k)Q_{tk-s_k} + s_k. \]
Since $t_k - s_k \to \infty$ as $k \to \infty$, there exists a $k_1 > 1$ such that
\[ \mathbb{P} \left( t_k \hat{Q}_{tk,s_k}'' > q^* (t_k - s_k) + s_k + \gamma_0 \sqrt{t_k - s_k} \right) \leq \frac{\epsilon_0}{2} \]
for all $k \geq k_1$. Using the bound $s_k \leq \alpha \sqrt{t_k}$, we get
\[ \mathbb{P} \left( Z_{tk,s_k}'' > q^* \cdot t_k + \beta \sqrt{t_k} \right) \leq \frac{\epsilon_0}{2} \quad \text{(3.16)} \]
for all $k \geq k_1$, where $\beta := \alpha + \gamma_0$. From the union bound and Equations (3.15) and (3.16), we get
\[ \mathbb{P} \left( f_{tk} \left( \frac{Z_{tk,s_k}'}{t_k} \right) \geq p^* + \delta, Z_{tk,s_k}'' \leq q^* \cdot t_k + \beta \sqrt{t_k} \right) \geq \frac{\epsilon_0}{2}. \quad \text{(3.17)} \]
Now, we use the monotonicity assumption (Definition 2.5) to show that this requires
\[ f_{tk} \left( q^* + \frac{\beta}{\sqrt{t_k}} \right) \geq p^* + \delta \text{ for all } k \geq 1. \quad \text{(3.18)} \]
This follows because if Equation (3.18) were not true, we would get $f_{tk}(z/t_k) < p^* + \delta$ for all $z \leq q^* + \beta/\sqrt{t_k}$ by Definition 2.5, which would violate the statement of Equation (3.17).
Applying the monotonicity assumption again gives us
\[ f_{tk}(x) \geq p^* + \delta \text{ for all } x \geq q^* + \frac{\beta}{\sqrt{t_k}} \text{ and for all } k \geq 1. \quad \text{(3.19)} \]
Recall that $P_{tk}$ refers to the individual iterates, and $\hat{Q}_{tk}$ refers to the time-average of iterates. The proof of Theorem 3.3 is now completed by another elementary application of the central limit theorem, as detailed below:
\[ \mathbb{P} \left[ P_{tk} \geq p^* + \delta \right] = \mathbb{P} \left[ f_{tk}(\hat{Q}_{tk-1}) \geq p^* + \delta \right] \]
\[ \geq (i) \mathbb{P} \left[ \hat{Q}_{tk-1} \geq q^* + \frac{\beta}{\sqrt{t_k}} \right] \]
\[ \geq (ii) \frac{1}{2} \cdot \mathbb{P} \left[ W \geq \frac{\beta}{\sqrt{q^*(1-q^*)}} \right] \text{ where } W \sim \mathcal{N}(0,1) \]
\[ = \frac{1}{2} \text{erfc} \left( \frac{\beta}{\sqrt{q^*(1-q^*)}} \right) := \epsilon(\delta) > 0 \text{ for any } \delta < 1/4. \]
Above, inequality (i) follows from Equation (3.19) and inequality (ii) follows from the central limit theorem, where $1/2 < 1$ is chosen as an arbitrary constant. This completes the proof. \qed
3.3 Main result: Last-iterate oscillation when both players use no-regret

The case for which player 2 is already at equilibrium serves to isolate the ramifications of the stochasticity in realizations. While the day-to-day behavior of players is clearly quite different when both players are using no-regret strategies (see Figures 4 and 3b in Section 4 for a comparison), we show below that the inherent stochasticity in realizations continues to be the dominating factor that causes last-iterate oscillations. Our main result is stated below.

**Theorem 3.4.** If both players 1 and 2 use optimal no-regret repeated game strategies that are mean-based (Definition 2.4) and monotonic (Definition 2.5), then the pair of mixed strategies of both the players \((P_t, Q_t)\) cannot converge to the NE \((p^*, q^*)\) almost surely.

The proof of Theorem 3.4 mirrors the proof of the simpler “warm-up” case (Theorem 3.3) by taking a proof-by-contradiction approach: Suppose, instead, that both players did converge almost surely. Then player 2 would have to converge almost surely by definition, and we show that in this scenario player 1 cannot converge because player 2’s realizations are effectively too stochastic to allow it. The primary challenge is in showing this sufficient stochasticity in player 2’s realizations; after all, player 2’s realizations are highly dependent on past outcomes of player 1. To do this, we use the martingale central-limit-theorem [Hall and Heyde, 2014] together with a classical result on the convergence of the time-average of the mixed strategies to NE [Freund et al., 1999, Roughgarden, 2016]. The full proof is provided below.

**Proof.** We will use the approach of proof-by-contradiction. Suppose that we in fact have \((P_t \rightarrow p^*, Q_t \rightarrow q^*)\) a.s.. Note that this trivially implies that \(Q_t \rightarrow q^*\) a.s. To complete the proof by contradiction, we will show that if \(Q_t \rightarrow q^*\) a.s., then we need to have

$$\lim_{t \rightarrow \infty} \mathbb{P}[P_t \geq p^* + \delta] \geq \epsilon(\delta) > 0 \text{ for any } 0 < \delta < \frac{(1 - p^*)}{3}.$$  

Here \(\epsilon(\delta)\) is as defined in the proof of Theorem 3.3. Our first critical ingredient is the following claim, which implies sufficient stochasticity in the iterates under the assumed almost sure convergence of the mixed strategies of player 2. We prove this claim in Appendix B.

**Claim 3.5.** \(Q_t \rightarrow q^*\) a.s. implies that

$$\liminf_{k \rightarrow \infty} \frac{1}{t_k} \left[ \left\{ s \leq t_k : \frac{q^*}{2} \leq Q_s \leq \frac{q^* + 1}{2} \right\} \right] = 1 \text{ almost surely},$$  

(3.20)

for any deterministic sub-sequence \(\{t_k\}_{k \geq 1}\).

Equation (3.20) is reminiscent of a convergence condition that is used by Foster and Hart [2018] for “smoothly calibrated” strategies. As a direct consequence of the above claim, we
have
\[
\liminf_{t \to \infty} \frac{1}{t} \sum_{s=1}^{t} Q_s (1 - Q_s) \geq \min \left\{ \frac{q^* (2 - q^*)}{4}, \frac{(1 + q^*) (1 - q^*)}{4} \right\} =: \gamma(q^*) \text{ a.s.} \quad (3.21)
\]

We consider the filtration \( \{ \mathcal{F}_t := (I_s, J_s)_{s=1}^{t-1} \}_{\geq 1} \) and the stochastic process \( \{ D_t := Z_t - Z_{t-1} - Q_t \}_{\geq 1} \). Recall that we had defined \( Z_t := \sum_{s=1}^{t} J_t \), where \( J_t \sim \text{Ber}(Q_t) \). We observe the following properties of the stochastic process \( \{ D_t \}_{t \geq 1} \):

1. \( D_t \) is a martingale difference sequence with respect to the filtration \( \mathcal{F}_{t-1} \). This is a direct consequence of \( Q_t \) being a deterministic function of \( \mathcal{F}_{t-1} \).

2. \( |D_t| \leq 1 \).

3. Note that because \( 0 < q^* < 1 \), we have \( \gamma(q^*) > 0 \). Therefore, the sum of conditional variances diverges to \( \infty \). In other words, we have
   \[
   \sigma_t^2 := \mathbb{E} [D_t^2 | \mathcal{F}_{t-1}] = \mathbb{E} [(J_t - Q_t)^2 | \mathcal{F}_{t-1}] = Q_t (1 - Q_t)
   \]
   and from Equation (3.21) it is clear that \( \liminf_{t \to \infty} \sum_{s=1}^{t} \sigma_s^2 = \infty \) a.s.

As a consequence of these properties, we can apply the martingale central limit theorem [Hall and Heyde, 2014] in the following form:

**Theorem 3.6** (Martingale CLT, [Hall and Heyde, 2014]). Let \( \{ D_t \}_{t \geq 1} \) be a martingale difference sequence with respect to the filtration \( \mathcal{F}_{t-1} \) such that \( |D_t| \leq 1 \). Further, let \( \{ \sigma_t^2 := \mathbb{E} [D_t^2 | \mathcal{F}_{t-1}] \}_{t \geq 1} \) denote the conditional variance sequence with the property that \( \sum_{t=1}^{\infty} \sigma_t^2 \) diverges to infinity almost surely. Then, we have

\[
\frac{\sum_{s=1}^{t} D_s}{\sqrt{\sum_{s=1}^{t} \sigma_s^2}} \to Z \sim \mathcal{N}(0, 1). \quad (3.22)
\]

Note that Equation (3.22) directly implies

\[
\frac{\sum_{s=1}^{t_k} D_s}{\sqrt{\sum_{s=1}^{t_k} \sigma_s^2}} \to Z \sim \mathcal{N}(0, 1). \quad (3.23)
\]

for any deterministic subsequence \( \{ t_k \}_{k \geq 1} \). Further, note that for any \( k \geq 1 \), we have

\[
\frac{\sum_{s=1}^{t_k} D_s}{\sqrt{t_k}} = \frac{1}{\sqrt{t_k}} \left( Z_{t_k} - \sum_{s=1}^{t_k} Q_s \right) = \sqrt{t_k} (\widehat{Q}_{t_k} - Q_{t_k}).
\]
Consequently, we have for any $\beta' > 0$,

$$
\lim_{k \to \infty} \mathbb{P} \left( \sqrt{t_k} (\hat{Q}_{t_k} - \overline{Q}_{t_k}) \geq \beta' \right) = \lim_{k \to \infty} \mathbb{P} \left( \frac{\sum_{s=1}^{t_k} D_s}{\sqrt{t_k}} \geq \beta' \right)
\geq \lim_{k \to \infty} \mathbb{P} \left( \frac{\sum_{s=1}^{t_k} D_s}{\sqrt{\sum_{s=1}^{t_k} Q_s(1 - Q_s)}} \geq \frac{\beta'}{\gamma(q^*)} \right)
\geq \text{erfc} \left( \frac{\beta'}{\gamma(q^*)} \right) > 0.
$$

Above, inequality (i) uses Equation (3.20), and inequality (ii) uses the martingale CLT on the subsequence $\{t_k\}_{k \geq 1}$ (as described in Equation (3.23)).

Our next critical observation is a fundamental and well-known [Freund et al., 1999, Roughgarden, 2016] property of the average iterates arising as a consequence of both players using optimal no-regret algorithms. We state this property as a lemma

Lemma 3.7 ([Roughgarden, 2016]). Suppose players 1 and 2 both use optimal no-regret strategies in a competitive game. Then, there exists a universal constant $C > 0$ such that

$$
|\overline{Q}_t - q^*| \leq \frac{C}{\sqrt{t}} \text{ pointwise, for all } t \geq 1. \quad (3.24)
$$

Lemma 3.7 is proved in Appendix D for completeness. Selecting $\beta' := \beta + C$ for any $\beta > 0$ (where $C$ is the constant of choice in Lemma 3.7), we get

$$
\lim_{k \to \infty} \mathbb{P} \left( \sqrt{t_k} (\hat{Q}_{t_k} - q^*) \geq \beta \right) \geq \mathbb{P} \left( \sqrt{t_k} (\hat{Q}_{t_k} - \overline{Q}_{t_k}) \geq \beta' \right) \geq \text{erfc} \left( \frac{\beta + C}{\gamma(q^*)} \right) > 0. \quad (3.25)
$$

Consequently, we can use Equation (3.25) to complete the proof-by-contradiction of Theorem 3.4. In particular, we follow an identical series of steps to the proof of Theorem 3.3 to show that if $(P_t, Q_t) \to (p^*, q^*)$ a.s., we need to have

$$
\lim_{k \to \infty} \mathbb{P} [P_k \geq p^* + \delta] \geq \lim_{k \to \infty} \mathbb{P} \left( \sqrt{t_k} (\hat{Q}_{t_k} - q^*) \geq \beta \right) \geq \text{erfc} \left( \frac{\beta + C}{\gamma(q^*)} \right) > 0,
$$

where $\beta > 0$ is chosen just as in the proof of Theorem 3.3. This provides the desired contradiction. Ultimately, we have shown that if the last iterate of player 2 were assumed to converge, the last iterate of player 1 could not converge (in fact, even in probability). This completes our proof-by-contradiction: at least one of the players cannot converge almost surely to NE.

\[ \square \]

3.4 Beyond the mean-based assumption

In this section, we show that an exact mean-based strategy (Definition 2.4) is not required to produce the last-iterate oscillation phenomenon. In particular, we show that equivalents of Theorem 3.4 hold under two algorithmic variants of mean-based strategies that are ubiquitous in the online learning and games literature.
3.4.1 Oscillation under optimism

One of the most common algorithmic variants of exact mean-based strategies incorporates a form of recency bias, colloquially called “optimism”. We define the broad class of recency-bias strategies below.

**Definition 3.8.** The class of $\ell$-recency bias strategies is defined as below:

$$ f_t(J_{t-1}) := f_t(\hat{Q}_t) $$

where

$$ \hat{Q}_t := \frac{1}{t} \left( \sum_{s=1}^{t} J_s + \sum_{j=1}^{\ell} r_j J_{t-j+1} \right), $$

and \{r_j\}_{j=1}^\ell are positive integers taking values in \{1, \ldots, \ell\}. We continue to assume that $f_t(\cdot)$ is monotonic in its argument for every $t \geq 1$ (similar to Definition 2.5).

Note that the class of 0-recency bias strategies is equivalent to the class of mean-based strategies. Further, 1-recency bias strategies with $r_1 = 1$ constitute the class of optimistic mean-based strategies, since they are using $\sum_{s=1}^{t} J_s + J_t$ as the summary statistic.

As mentioned in Section 1, the study of the last iterate of optimism-based strategies has generated a lot of interest in the optimization literature [Daskalakis et al., 2018, Mertikopoulos et al., 2018, Liang and Stokes, 2019, Abernethy et al., 2019, Lei et al., 2020]; more-over, these strategies are known to cause faster time-averaged convergence [Rakhlin and Sridharan, 2013b, Syrgkanis et al., 2015]. Most recently, it was shown that the last iterate of the players’ strategies in the setting of telepathic dynamics (that arises when players use each others’ mixtures to update their strategies) will converge [Daskalakis and Panageas, 2018]. In the more realistic realization-based model, the following result shows that the ensuing stochasticity alone causes recency-bias-based strategies to diverge.

**Theorem 3.9.** Fix any $\ell > 0$. If both players 1 and 2 use any $\ell$-recency-biased, monotonic, optimal-no-regret strategies \{f_t\}_{t \geq 1} and \{g_t\}_{t \geq 1} respectively, then the pair of the mixed strategies of both the players $(P_t, Q_t)$ cannot converge to $(p^*, q^*)$ almost surely.

The proof of Theorem 3.9 is a relatively simple modification of the proof of Theorem 3.4, and is provided in Appendix E for completeness. It is worth noting that the bounded memory of the recency bias, as well as bounded increments \{r_j\}_{j=1}^\ell are critical to the essence of the proof argument. It is plausible that stronger recency biases that grow with the number of steps $t$ could lead to different last-iterate behavior; however, such stronger recency biases could also break the no-regret property.

3.4.2 Oscillation under stochastic domination (e.g. adaptive step sizes)

The assumption of mean-based strategies (Definition 2.4) is not satisfied by some of the most popular online learning algorithms that retain the worst-case optimal no-regret guarantee, but obtain faster rates under “easier data”. These include variants of Online-Mirror-Descent
that adapt to small cumulative loss or variance [Hazan and Kale, 2010], predictable sequences [Rakhlin and Sridharan, 2013a,b], and stochasticity [Cesa-Bianchi et al., 2007, Erven et al., 2011]; while retaining the usual worst-case bounds. The catch in these algorithms is that the learning rate (or step size) can now be adaptively set, i.e. \( \eta_t := \eta_t((I)^{t-1}, (J)^{t-1}) \).

The adaptive learning rate can be set via the ubiquitous “doubling-trick” [Cesa-Bianchi et al., 2007, Hazan and Kale, 2010] or in a more continuous manner [Erven et al., 2011, Rakhlin and Sridharan, 2013a,b]. Either way, none of these algorithms satisfy the mean-based property as in general, the learning rate function \( \eta_t(\mathcal{F}_{t-1}) \) will depend in some non-linear way on \( \{P_s, Q_s, I_s, J_s\}_{s=1}^{t-1} \).

In spite of this, we can prove that the last-iterate oscillates even for these algorithms by showing that their iterates stochastically dominate those of an algorithm in the Online-Mirror-Descent family, which is mean-based\(^9\). More generally, we now show that our main result of last-iterate oscillation would directly extend to any algorithm whose iterates stochastically dominate those of a mean-based algorithm.

**Corollary 3.10.** Consider any optimal-no-regret strategy for player 1 (or player 2) \( \{f_t\}_{t \geq 1} \) that stochastically dominates some mean-based and monotonic optimal-no-regret strategy \( \{f_t'\}_{t \geq 1} \) in the following sense: for every \( t \geq 1 \) and every history \( ((I)^{t-1}, (J)^{t-1}) \), we have

\[
|f_t((I)^{t-1}, (J)^{t-1})) - p^*| \geq |f_t'(\hat{Q}_{t-1}) - p^*|.
\]

Further, let player 2 (or player 1) follow any optimal-no-regret strategy \( \{g_t\}_{t \geq 1} \). Then, the pair of mixed strategies \( (P_t, Q_t) \) cannot converge almost surely.

Corollary 3.10 automatically implies that the players cannot converge almost surely if even one of them is using any of the aforementioned adaptive variants of Online-Mirror-Descent. Let us see why this is true for the simplest case of adaptive variants of multiplicative weights/Hedge (which is a special case of the Online-Mirror-Descent family). We recall the following critical property that is shared by all adaptive variants of Hedge: for some universal constant \( C > 0 \), we have

\[
\eta_t \geq \frac{C}{\sqrt{t}} \text{ pointwise.}
\]

This property is usually proved to show that such adaptive algorithms retain the \( O(\sqrt{t}) \) regret guarantee in the worst case (while admitting faster rates under more favorable conditions). For a given round \( t \), we define \( Z_{t-1} \) as above, and further define

\[
P_t,\text{Hedge} := f_{t,C}(\hat{Q}_{t-1}) := \frac{\exp\{C\sqrt{t}\hat{Q}_{t-1}\}}{\exp\{C\sqrt{t}\hat{Q}_{t-1}\} + \exp\{C\sqrt{t}(1 - \hat{Q}_{t-1})\}}
\]

\(^9\)It is important to note that the Online-Mirror-Descent family does not comprise all algorithms used in online prediction. In particular, the “parameter-free” online learning paradigm [Chaudhuri et al., 2009, Orabona, 2014, Koolen and Van Erven, 2015, van Erven and Koolen, 2016] is not mean-based or easily reducible to a mean-based family. Understanding whether our techniques could plausibly extend to the parameter-free online learning paradigm is an intriguing direction for future work.
as the \textit{counterfactual iterate} that would have been generated if the algorithm switched to Hedge with learning rate $\eta_t = C/\sqrt{t}$ at round $t$. We note that $|P_t - 1/2| \geq |P_t, \text{Hedge} - 1/2|$, implying that the stochastic dominance property in Equation (3.26) is satisfied.

We conclude this section with the proof of Corollary 3.10, provided below.

\textbf{Proof.} At every round $t$, we can consider the \textit{counterfactual} iterate given by $P'_t := f_t(\hat{Q}_{t-1})$. Using this, we can follow the same proof-by-contradiction strategy as considered in the proof of Theorem 3.4, by observing two facts:

1. The two essential properties of $\hat{Q}_{t-1}$: that it satisfies the conditions for the martingale CLT, and that it is sufficiently close to the average iterate given by $\overline{Q}_{t-1}$, continue to hold here. The former property is a consequence of the assumed convergence that we will contradict, and the latter property is a consequence of both players following optimal-no-regret algorithms.

2. We have $P_t \geq P'_t$ almost surely. Since the proof of Theorem 3.4 uses that $P'_t \geq p^* + \delta$ whenever $\hat{Q}_{t-1} \geq q^* + \frac{\beta}{\sqrt{t}}$, it clearly follows that $P_t \geq p^* + \delta$ under this event as well.

The proof of Corollary 3.10 trivially follows as a consequence of these facts. \hfill $\square$

\section{Beyond the monotonicity assumption: A conjecture}

In Section 3.4, we showed that our results can be extended beyond exact mean-based strategies, allowing their applicability to a wide range of popular online learning algorithms. In this section, we ask whether the other principal assumption of \textit{monotonicity} of the mapping from empirical average to mixed strategy at every round (Definition 2.5) is also relaxable. While this assumption of monotonicity will intuitively be satisfied by any learning agent that aims to maximize its utility\footnote{Note that this is significantly more general than the expected-utility-theory as it is only a qualitative statement, not a quantitative one.}, and can be verified to be satisfied by all known mean-based no-regret algorithms, it is of interest to examine whether it is truly necessary to prove our results. We conjecture below that last-iterate oscillations will continue to hold for mean-based strategies that are not monotonic.

\textbf{Conjecture 3.11.} If both players 1 and 2 use mean-based (not necessarily monotonic) repeated game strategies $\{f_t\}_{t \geq 1}$ and $\{g_t\}_{t \geq 1}$ that are uniformly no-regret and each have a regret rate of $(1/2, c)$, then the pair of mixed strategies $(P_t, Q_t)$ does not converge almost surely.

Conjecture 3.11 turns out to be significantly more difficult to prove than the corresponding result with the additional monotonicity assumption, i.e. Theorem 3.4. The assumption of monotonicity on strategy maps (Definition 2.5) allowed us to link the event of last-iterate oscillations to an inequality relation on the empirical average $\hat{Q}_{t-1}$; therefore, we could lower bound the probability of a last-iterate oscillation by the cumulative distribution function (CDF) of $\hat{Q}_{t-1}$ and invoke limit theorems. In the absence of monotonicity of strategy...
maps, it turns out that we need to reason about the probability mass function (PMF) of $\hat{Q}_{t-1}$ instead. This is a mathematically far more difficult object to study under the general scenario where $Q_t$ is now a complex functional of the history of $(I^{t-1}, J^{t-1})$. Importantly, the realizations of player 2, given by $\{J_t\}_{t \geq 1}$ are not mutually independent, even conditionally on the mixed strategies $\{Q_t\}_{t \geq 1}$. Moreover, martingale structure by itself is insufficient to obtain adequate control on the PMF of $\hat{Q}_t$.

Nevertheless, we can make partial progress on proving Conjecture 3.11. First, we note that the proof of the “warm-up” Theorem 3.3 can be modified to show that player 1 would oscillate in the idealized case where player 2 has already converged to his NE even when the mean-based strategies are non-monotonic (Proposition F.1 in Appendix F). Second, while we know that the realizations $\{J_t\}_{t \geq 1}$ cannot be mutually independent, they are still likely to be “minimally stochastic” across rounds in a certain sense: after all, an independent coin is being tossed on every round to generate the realization of player 2, $J_t$, from his mixed strategy on that round, $Q_t$. Thus, it is reasonable to conjecture that the PMF of $\hat{Q}_t$ is sufficiently “close” to a sum of independent random variables, which we denote by:

\[ Z_t' := \sum_{s=1}^{t} J_s' \]

where

\[ J_s' \sim \text{Ber}(E[Q_s]), J_s' \text{ mutually independent}. \]

More precisely, suppose that we could show that the induced distribution on the empirical average of player 2, i.e. $\hat{Q}_t$, was similar to the normalized sum of independent random variables, defined above by $Z_t'/t$, in the following quantitative sense:

**Definition 3.12.** The mean-based strategies $\{f_t\}_{t \geq 1}$ and $\{g_t\}_{t \geq 1}$ would satisfy the **shaky-hands** property if for any $\gamma > 0$, there exists an $\epsilon > 0$ and a $T > 0$ such that

\[
\frac{\mathbb{P}(t\hat{Q}_t = z)}{\mathbb{P}(Z_t' = z)} \geq \epsilon \text{ for all } z \in [tq^* - \gamma \sqrt{t}, tq^* + \gamma \sqrt{t}], \tag{3.27}
\]

for all $t \geq T$.

This **shaky-hands** property, if true, would posit a minimal amount of stochasticity on the dependent realizations of player 2. The following theorem shows that Conjecture 3.11 is true if the shaky-hands property holds.

**Theorem 3.13.** If the mean-based strategies $\{f_t\}_{t \geq 1}$ and $\{g_t\}_{t \geq 1}$ satisfy the Shaky-hands property (Equation (3.27), and the strategy $\{f_t\}_{t \geq 1}$ of player 1 has a regret-rate of $(1/2, c)$, then player 1’s last-iterates diverge from the equilibrium strategy $p^*$ in probability, i.e. there exist positive constants $(\delta, \epsilon_0)$ such that

\[
\limsup_{t \to \infty} \mathbb{P} [ |P_t - p^*| \geq \delta ] \geq \epsilon_0. \tag{3.28}
\]

Theorem 3.13 shows, in fact, a stronger statement of lack of convergence in probability under the shaky-hands property. This is a valuable result, as it ensures that proving...
Conjecture 3.11 reduces to proving that the shaky-hands property (Equation (3.27)) will be satisfied by an arbitrary mean-based, non-monotonic no-regret algorithm. The proof of Theorem 3.13 is significantly more technically involved than the CLT-based arguments provided in the previous sections, and is provided in Appendix F.

4 Simulations

In this section, we provide empirical evidence for last-iterate oscillations under even suboptimal no-regret strategies. We evaluate three popular no-regret strategies:

1. The standard multiplicative weights update, which is known to lead to last-iterate oscillation even under telepathic dynamics (the deterministic setting) [Bailey and Pilionas, 2018].

2. The optimistic multiplicative weights update, which converges in the last-iterate in the deterministic setting [Daskalakis and Panageas, 2018].

3. The online mirror descent algorithm with the log function as regularizer, often called “log barrier” [Nemirovsky and Yudin, 1983]. This regularizer has been successfully used to establish robustness of fast time-average convergence guarantees in limited-information feedback settings [Foster et al., 2016], and is thus naturally interesting to evaluate.

All of the above algorithms fall under the online-mirror-descent framework, and employ fixed learning rates \( \{\eta_t\}_{t \geq 1} \). The rate of decay of \( \eta_t \) with \( t \) dictates the no-regret rate in all three cases: for any \( r \geq 1/2 \), if \( \eta_t = 1/t^r \), the no-regret rate is equal to \((r, c)\) for a suitable positive constant \( c \). We will evaluate these algorithms with two learning rate choices: \( \eta_t = 1/\sqrt{t} \) (optimal), and \( \eta_t = 1/t^{0.7} \) (suboptimal rate\(^{11} r = 0.7 \)), and study the evolution of player 1’s iterates until \( T := 10^8 \) steps.

Furthermore, we will consider the simplest 2 \times 2 game: the matching pennies game, for which \( G(0, 0) = G(1, 1) = 1 \) and \( G(0, 1) = G(1, 0) = 0 \) (without loss of generality, player 1 is the player who wants the actions to match). Note that the unique mixed-strategy equilibrium of this game is \( p^* = q^* = 1/2 \). We will plot the evolution of the mixed strategies of player 1 with time — since the matching pennies game is symmetric, player 2 has similar behavior.

Figure 3 studies the optimal-no-regret case, and shows the striking difference between the evolution of the mixed strategies when the players use opponents’ mixtures (the deterministic case) as opposed to their realizations (the stochastic case, studied in this paper). Notably, we see in Figure 3a that while multiplicative weights converges to a limit

\(^{11}\)It is easy to verify that the same suboptimal rate \( r = 0.7 \) would also result from the slower choice of learning rate decay, \( \eta_t = 1/t^{0.3} \). We do not evaluate this choice, as the argument in Proposition 3.1 can be employed to show that it results in even more fluctuation-sensitivity than the case of optimal-no-regret; thus, it provably causes last-iterate oscillation. Intuitively, higher learning rates correspond to less randomness in the mixed strategies as a function of the past, and so more fluctuation-sensitivity.
cycle, optimistic multiplicative weights converges to NE quite quickly. The third algorithm, log-barrier Online-Mirror-Descent, also oscillates, but the amplitude of the cycles is much smaller\textsuperscript{12} than for multiplicative weights, and is in fact not visible in the figure. On the other hand, we see in Figure 3b that all three of these algorithms diverge in the last iterate. In fact, they are very rarely close to the equilibrium strategy $p^* = 0.5$! All in all, Figure 3b empirically corroborates Theorems 3.4 and 3.9, and shows in particular that introducing optimism into no-regret strategies does not fix the issue of last-iterate oscillation.

It is also worth examining the differential impact on player 1 as a result of player 2 using an optimal no-regret strategy, as opposed to player 2 playing his fixed NE strategy. In the case of the matching pennies game, the latter case corresponds to player 2 playing $q^* = 0.5$ at every step. Figure 4 depicts the evolution of the mixed strategies of player 1 in this latter case. Comparing the evolution to Figure 3b, it is evident that the mixed strategies diverge in both cases. While the “period” of limiting cycles, if any, seems to be larger in the fixed-strategy case, the amplitude of oscillation is similar in both cases.

Finally, the techniques in our theory have crucially relied on the optimality of the no-regret algorithms used by both players. It is naturally interesting to ask whether relaxing the optimality of the no-regret rate could lead to better results in the evolution of the mixed strategies. Figure 5 provides preliminary empirical evidence that this may not be the case. In particular, Figure 5b shows that the suboptimal variants of all three algorithms continue to lead to last-iterate oscillation in the stochastic setting, although the amplitude of the oscillations does appear to be sizably reduced.

\textsuperscript{12}This likely reflects the increased entropy of the strategies used in the log-barrier algorithm.
Figure 4: Evolution of the iterates of multiplicative weights in the matching pennies game (for player 1) when the optimal-no-regret rate $r = 1/2$ is used, and player 2 is already at NE.

Figure 5: Evolution of the iterates of multiplicative weights in the matching pennies game (for player 1) when the suboptimal-no-regret rate $r = 0.7$ is used.
5 Conclusion and future work

In this paper, we have shown partial but compelling evidence for a fundamental tension between the guarantees of no-regret and last-iterate convergence on uncoupled dynamics that use the opponents’ realizations alone as feedback. Perhaps the most important immediate question to address is whether last-iterate oscillations occur for strategies not satisfying monotonicity; in particular, whether Conjecture 3.11 is true. Additionally, we can ask whether the mean-based nature of the strategies is truly needed for our impossibility result. While we did show that our results are quite robust to inexact versions of the mean-based assumption (through Theorem 3.9 and Corollary 3.10), whether they also hold for strategies that significantly deviate from the class of mean-based strategies is an intriguing question.

Section 4 provided preliminary empirical evidence that last-iterate oscillation can occur even when no-regret strategies with suboptimal rates are used. Our techniques break down in the face of suboptimal no-regret rates, so it is interesting to ponder whether last-iterate oscillation happens for general suboptimal no-regret algorithms, or if it is just a property of the particular algorithms that were simulated.

Finally, while last-iterate convergence may seem like a strong requirement in the realization-based repeated game model, we note that there are non-trivial classes of strategies that can be shown to satisfy it. Of particular interest are the (intractable for large games) smoothly calibrated strategies proposed by Foster and Hart [2018]. These strategies constitute randomized responses to deterministic forecasting, and are conceptually quite different from strategies satisfying the no-regret property. Whether these strategies can be studied more constructively, and from a behavioral game theory standpoint, is an important question for future work.

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Appendices

Before proving the full proofs of statements, we state our convention for notation in the appendix. We designate constants that take a value in $(0, 1)$ by $\epsilon$, and strictly positive, finite constants by $C \in (0, \infty)$. Moreover, we will designate $t_0$ as a lower bound on $t$ above which
all our statements apply.

In general, these constants can depend on the parameters of the game, either directly, or just on the equilibrium strategies \((p^*, q^*)\).

For ease of exposition, we will also sub-script these constants by alphabets \(\{a, b, \ldots\}\), corresponding to the lemmas in which they appear and are used. Thus, for example, in the first lemma the constants will be denoted as \(\{\epsilon_a, C_a\}\) and the lower bound on \(t\) will be denoted as \(t_{0,a}\). While in general we will overload notation within a lemma for our choice of constants, we will be explicit about manipulations when possible.

**A  Proof of Lemma 3.2**

We consider the distribution of mutually independent coin tosses,

\[
J_t \text{ i.i.d. } \sim \text{Bernoulli}(q^*),
\]

and denote the expectation of quantities under this probability distribution by \(\mathbb{E}[\cdot]\). Recall that \(q^*\) is the Nash equilibrium strategy of player 2. By linearity of expectation, it is trivial to show the second statement, i.e.

\[
\mathbb{E} \left[ \sum_{t=1}^{T} J_t \right] = Tq^*.
\]

To show the first statement, recall that \(J_t \perp (J)^{t-1}\) for all \(t \in \{1, \ldots, T\}\) due to mutual independence. Thus, we use the law of iterated expectations to get

\[
\mathbb{E} \left[ \sum_{t=1}^{T} G(f_t((J)^{t-1}), J_t) \right]
= \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{E} \left[ f_t((J)^{t-1}) \cdot G(1, J_t) + (1 - f_t((J)^{t-1}))G(0, J_t) \mid (J)^{t-1} \right] \right]
= \mathbb{E} \left[ \sum_{t=1}^{T} f_t((J)^{t-1}) \cdot G(1, q^*) + (1 - f_t((J)^{t-1})) \cdot G(0, q^*) \right]
= TR^*,
\]

where the last statement follows from \(G(0, q^*) = G(1, q^*) = G(p^*, q^*) = R^*\). This completes the proof.

**B  Proof of Claim 3.5**

Let \(\Omega\) denote the \(\sigma\)-algebra induced by \(\{P_t, Q_t\}_{t \geq 1}\) and the strategies \(\{f_t(\cdot), g_t(\cdot)\}_{t \geq 1}\). Let \(\omega \in \Omega\) denote any realization of this, and \((P_t(\omega), Q_t(\omega))\) denote the corresponding mixed
strategies at time $t$. Then, $Q_t \to q^*$ a.s. implies that
\[ \mathbb{P} \left( \omega \in \Omega : \lim_{t \to \infty} Q_t(\omega) = q^* \right) = 1. \]
Thus, we get
\[ \mathbb{P} \left( \omega \in \Omega : \lim_{k \to \infty} Q_{t_k}(\omega) = q^* \right) = 1 \]
for any subsequence $\{t_k\}_{k \geq 1}$. This implies, by the definition of a limit, that for every $\omega$ such that $\lim_{t \to \infty} Q_t(\omega) = q^*$, there exists finite $t_0(\omega)$ such that
\[ Q_t(\omega) \in \left[ \frac{q^*}{2}, \frac{q^*+1}{2} \right] \text{ for all } t \geq t_0(\omega). \]
Let $k_0(\omega) := \min \{k : t_k \geq t_0(\omega)\}$. This directly gives us
\[ \lim \inf_{k \to \infty} \frac{1}{t_k} \left\{ s \leq t_k : \frac{q^*}{2} \leq Q_s(\omega) \leq \frac{q^*+1}{2} \right\} \geq \lim \inf_{k \to \infty} \frac{t_k - t_{k_0}(\omega)}{t_k} = 1. \]
Thus, we get
\[ \mathbb{P} \left( \omega \in \Omega : \lim \inf_{k \to \infty} \frac{1}{t_k} \left\{ s \leq t_k : \frac{q^*}{2} \leq Q_s(\omega) \leq \frac{q^*+1}{2} \right\} \geq 1 \right) = 1, \]
which completes the proof. \hfill \Box

C Connections between non-zero-sum and zero-sum competitive games

In this section, we recap the connection between a non-zero-sum competitive game and a zero-sum competitive game that share the same best-response functions (and thus NE), that was alluded to in Remark 2.7. The following characterization for competitive games was provided in Phade and Anantharam [2019]:

**Definition C.1.** For any choice of parameters $\alpha, \beta > 0$, we define a competitive game with payoff matrix entries given by
\[
\begin{align*}
G(0, 0) &= -\alpha, G(1, 1) = -1, H(0, 0) = \beta, H(1, 1) = 1 \\
G(0, 1) &= G(1, 0) = H(0, 1) = H(1, 0) = 0.
\end{align*}
\]

The unique Nash equilibrium of this game (which is also a correlated equilibrium) is given by $p^* = \beta/(1 + \beta), q^* = \alpha/(1 + \alpha)$. Moreover, the corresponding zero-sum game with payoff matrix entries for player 1 given by
\[
\begin{align*}
G'(0, 0) &= \frac{1 - \alpha \beta}{1 + \beta}, G'(0, 1) = 1, G'(1, 0) = \frac{1 + \alpha}{1 + \beta}, G'(1, 1) = 0
\end{align*}
\]
is equivalent to the above competitive game in its best-response function and NE. Note that the transformation between payoff matrices $(G, H)$ and $(G', -G')$ is non-affine.
(a) Non-zero-sum game with parameters \((\alpha = 0.111, \beta = 4)\).

(b) Zero-sum “equivalent” with parameters \((\alpha = 0.111, \beta = 4)\).

(c) Non-zero-sum game with parameters \((\alpha = 0.25, \beta = 0.429)\).

(d) Zero-sum “equivalent” with parameters \((\alpha = 0.25, \beta = 0.429)\).

Figure 6: Comparison of the iterates of player 1 using common no-regret algorithms in the non-zero-sum and zero-sum version of the competitive games as specified in Definition C.1 for two choices of parameters \(\alpha, \beta\). While both the non-zero-sum and zero-sum game have identical NE \((p^* = 0.8 \text{ for player 1 in the first case, and } p^* = 0.3 \text{ for player 1 in the second case})\), the iterates as a consequence of no-regret learning exhibit qualitatively different behavior.
As mentioned in Remark 2.7, this equivalence in best-response functions does not translate to an equivalence in actual day-to-day behavior when the players use no-regret algorithms. See Figure 6 for a depiction of the qualitative discrepancies in behavior for two choices of parameters \((\alpha, \beta)\). Understanding these discrepancies at a deeper level remains an intriguing direction for future work.

D Convergence of the empirical average: Proof of Lemma 3.7

In this section, we provide for completeness the proof of Lemma 3.7, which shows almost-sure convergence of the time-average of the mixed strategies of players who deploy no-regret algorithms to the unique mixed NE of a competitive game. Suppose that both the players employ no-regret algorithms with regret rate \(r\). Then, Roughgarden [2016, Proposition 17.9], shows that the product distribution \(P_t \times Q_t\) is an approximate coarse-correlated equilibrium (CCE), in the sense that there exists a universal constant \(C > 0\) such that

\[
E_{(I,J) \sim P_t \times Q_t} [G(I, J)] \leq E_{(I,J) \sim P_t \times Q_t} [G(i', J)] + \frac{C}{T^r} \quad \text{pointwise}
\]

for any unilateral deviation \(i' \in \{0, 1\}\) of player 1, and similarly,

\[
E_{(I,J) \sim P_t \times Q_t} [H(I, J)] \leq E_{(I,J) \sim P_t \times Q_t} [H(I, j')] + \frac{C}{T^r} \quad \text{pointwise}
\]

for any unilateral deviation \(j' \in \{0, 1\}\) of player 2. Next, we observe that for any competitive game, there is a unique CCE, i.e. there is a unique distribution \(\mu\) that satisfies

\[
E_{(I,J) \sim \mu} [G(I, J)] \leq E_{(I,J) \sim \mu} [G(i', J)],
\]

for any unilateral deviation \(i' \in \{0, 1\}\) of player 1, and

\[
E_{(I,J) \sim \mu} [H(I, J)] \leq E_{(I,J) \sim \mu} [H(I, j')],
\]

for any unilateral deviation \(j' \in \{0, 1\}\) of player 2. In fact, this distribution corresponds to the unique NE of this game, i.e. \(\mu = p^* \times q^*\). As a consequence of this uniqueness, and from the conditions on the payoffs of a competitive game given in Definition 2.6, we get that the distribution \(\mu_t := P_t \times Q_t\) is with a distance on the order of \(O(T^r)\) from \(\mu\) in the absolute norm. In particular, when \(r = 1/2\), we get that the time-average of player 2’s actions converges to \(q^*\) at the rate

\[
|Q_t - q^*| \leq \frac{C}{\sqrt{t}} \quad \text{pointwise, for all } t \geq 1.
\]

which is precisely Equation (3.24). This completes the proof of Lemma 3.7.

While the above reasoning holds for all competitive games and all no-regret algorithms, we note that this lemma was first proved by Freund et al. [1999] for the special case of zero-sum games with agents playing the multiplicative weights algorithm.
E Proof of Theorem 3.9

We first define notation pertinent to \( \ell \)-recency-bias strategies. We denote 
\[
Z^\ell_t := \sum_{t' = 1}^t (1 + r'_{t'}) J_{t'},
\]
where we designate \( r'_{t'} = 0 \) for \( t' \leq (t - \ell) \), and \( r'_{t'} = r_{t-t'+1} \) thereafter.

We will essentially mimic the proof-by-contradiction approach of Theorem 3.4. We will suppose that \((P_t, Q_t) \to (p^*, q^*)\) almost surely, and show that if \( Q_t \to q^* \) almost surely, then \( P_t \) must oscillate, which provides the desired contradiction. We state the following claim:

**Claim E.1.** Let \( C \) be the universal constant (well-defined by Lemma 3.7 for any pair of optimal no-regret algorithms for players 1 and 2) such that \( \overline{Q}_t \geq q^* - C/\sqrt{t} \) pointwise. Then, for any deterministic subsequence \( \{t_k\}_{k \geq 1} \) and any \( \beta > 0 \), we have

\[
\lim_{k \to \infty} P(\sqrt{t_k} (\hat{Q}_{t_k} - q^*) \geq \beta) \geq \text{erfc}\left(\frac{\beta + C}{\gamma(q^*)}\right) > 0,
\]

where \( \gamma(\cdot) > 0 \) is defined as in the proof of Theorem 3.4.

Claim E.1 essentially provides a CLT-like statement for the recency-bias-adjusted random sequence \( \{\hat{Q}_{t_k}^\ell\}_{k \geq 1} \), and is proved below.

**Proof.** By the definition of \( \hat{Q}_{t_k}^\ell \), we notice that
\[
\hat{Q}_{t_k} \leq \hat{Q}_{t_k}^\ell \leq \hat{Q}_{t_k} + \frac{\ell^2}{t_k} \text{ a.s.},
\]
which gives us
\[
\sqrt{t_k} (\hat{Q}_{t_k} - q^*) \leq \sqrt{t_k} (\hat{Q}_{t_k}^\ell - q^*) \leq \sqrt{t_k} (\hat{Q}_{t_k} - q^*) + \frac{\ell^2}{\sqrt{t_k}} \text{ a.s.}
\]
Since \( \ell \) is assumed to be a constant that does not grow with \( t \), we can apply the sandwich theorem to get
\[
\sqrt{t_k} (\hat{Q}_{t_k}^\ell - q^*) \to \sqrt{t_k} (\hat{Q}_{t_k} - q^*) \text{ a.s.}
\]
Substituting Equation (3.23) from the proof of Theorem 3.4 (which used the martingale CLT together with the time-averaged convergence property) into the above yields
\[
\lim_{k \to \infty} P(\sqrt{t_k} (\hat{Q}_{t_k}^\ell - q^*) \geq \beta) \geq \text{erfc}\left(\frac{\beta + C}{\gamma(q^*)}\right),
\]
which completes the proof of the claim. \( \square \)
We now use Claim E.1 to complete the proof of Theorem 3.9. We denote \( \{J'_t\}_{t \geq 1} \) to be a sequence of iid Bernoulli\((q^*)\) random variables. Using this notation, we can then write, for any \( 0 \leq s \leq t \),

\[
(Z'')_{t,s}^\ell := \sum_{t'=1}^{t-s} (1 + r'_t) J'_t + \sum_{t'=t-s+1}^{t} (1 + r'_t),
\]

\[
(Z')_{t}^\ell := \sum_{t'=1}^{t} (1 + r'_t) J'_t,
\]

\[
(\tilde{Q}'')_{t,s}^\ell := \frac{(Z'')_{t,s}^\ell}{t},
\]

\[
(\tilde{Q}')_{t}^\ell := \frac{(Z')_{t}^\ell}{t}.
\]

From Proposition 3.1, we know that there exists a sequence \( \{t_k, s_k\}_{k \geq 1} \) such that \( 0 \leq s_k \leq \alpha(t_k)^{1/2} \) for all \( k \geq 1 \), and

\[
\mathbb{E} \left[ f_{t_k}( (\tilde{Q}'')_{t_k,s_k}^\ell) \right] \geq p^* + 2\delta \text{ for all } k \geq 1.
\]

As in the proof of Theorem 3.3, we can use Markov's inequality to get

\[
P \left( f_{t_k}( (\tilde{Q}'')_{t_k,s_k}^\ell) > p^* + \delta \right) \geq \epsilon_0,
\]

where \( \epsilon_0 := \delta/((1-p^*) - \delta) \). Note that \( 0 < \epsilon_0 < 1/2 \), as in the proof of Theorem 3.4.

Next, in an argument similar to the proof of Claim E.1, we can apply the central-limit-theorem to the recency-biased random variable \((\tilde{Q}')_{t}^\ell = (Z')_{t}^\ell/t\). This will yield

\[
P \left( f_{t_k} \left( \frac{(Z'')_{t_k,s_k}}{t_k} \right) \geq p^* + \delta, (Z'')_{t_k,s_k} \leq q^* t_k + \beta \sqrt{t_k} \right) \geq \frac{\epsilon_0}{2}.
\]

Equation (E.2) together with the monotonicity assumption (Assumption 2.5) yields

\[
f_{t_k} \left( q^* + \frac{\beta}{\sqrt{t_k}} \right) \geq p^* + \delta \text{ for all } k \geq 1,
\]

where the justification is the same as in the proof of Theorem 3.3, and \( \beta > 0 \) is chosen in the same way. Accordingly, we get

\[
\lim_{k \to \infty} P[ P_k \geq p^* + \delta ] \geq \lim_{k \to \infty} P[ \sqrt{t_k}(\tilde{Q}_t - q^*) \geq \beta ] \geq \text{erfc} \left( \frac{\beta + C}{\gamma(q^*)} \right),
\]

where the last inequality follows from Equation (E.1). This completes the proof of Theorem 3.9.
Proof of Theorem 3.13

In this section, we provide the proof for our partial result (Theorem 3.13) that shows that last-iterate oscillations can occur even as a consequence of non-monotonic strategies under the conjecture that the shaky-hands condition (Equation (3.27)) holds. We begin by showing a version of the “warm-up” Theorem 3.3 that holds for non-monotonic strategies.

**Proposition F.1.** Let player 2’s strategy \( \{J_t\}_{t \geq 1} \) be an i.i.d. sequence of Bernoulli\((q^*)\) random variables. Then, any mean-based repeated game strategy \( \{f_t\}_{t \geq 1} \) that has a regret rate of \((1/2, c)\) causes player 1’s last iterate to diverge, i.e. there exist positive constants \((\delta, \epsilon)\) such that

\[
\limsup_{t \to \infty} \mathbb{P} [ |P_t - p^*| \geq \delta ] \geq \epsilon. \tag{F.1}
\]

**Proof.** The proof is identical to the proof of Theorem 3.3 until the step

\[
\mathbb{P} \left( f_{t_k} \left( \frac{Z''_{t_k,s_k}}{t_k} \right) \geq p^* + \delta, Z''_{t_k,s_k} \leq q^* \cdot t_k + \beta \sqrt{t_k} \right) \geq \frac{\epsilon_0}{2}.
\]

Since our strategy \( \{f_t\}_{t \geq 1} \) is no longer guaranteed to be monotonic, we can no longer turn the above equation into a deterministic statement on which we can apply the central-limit-theorem. We now use a more specialized argument that controls the ratio of probability mass functions. Recall that we defined the opponent sequence \( \{J'_t\}_{t \geq 1} \) to be an iid sequence of Bernoulli\((q^*)\) random variables. Note that \( Z'_t \equiv t \tilde{Q}_t \), and by definition \( Z''_{t,s} \geq s \) point-wise. We denote \( \beta_0 := \beta/q^* \). Note that \( q^* t + \beta \sqrt{t} = q^* (t + \beta_0 \sqrt{t}) \) for any \( t \). Now, we show that

\[
\min_{s \leq z \leq q^* (t + \beta_0 \sqrt{t})} \frac{\mathbb{P}(Z'_t = z)}{\mathbb{P}(Z''_{t,s} = z)} \geq (1 + \beta_0)^{-\alpha}, \tag{F.2}
\]

for all \( 0 < s \leq \alpha \sqrt{t}, t \geq 1 \). Indeed, we have,

\[
\frac{\mathbb{P}(Z'_t = z)}{\mathbb{P}(Z''_{t,s} = z)} = \frac{\binom{t}{z} (q^*)^{t-z} (1-q^*)^{t-z}}{\binom{t-s}{z-s} (q^*)^{t-s} (1-q^*)^{t-s}} = \frac{t}{z} \cdot \frac{t-1}{z-1} \cdots \frac{t-s+1}{z-s+1} \cdot (q^*)^{s} \geq \left( \frac{q^* t}{z} \right)^s \geq \left( \frac{t}{t + \beta_0 \sqrt{t}} \right)^{\alpha \sqrt{t}} \geq (1 + \beta_0)^{-\alpha} > 0,
\]

where the first inequality follows from \( z \leq t \) and therefore \( \frac{t-s}{z-s} \) is increasing in \( \ell \), and the last inequality follows from the fact that

\[
\left( \frac{t + \beta_0 \sqrt{t}}{t} \right)^{\alpha \sqrt{t}} = \left( 1 + \frac{\beta_0}{\sqrt{t}} \right)^{\alpha \sqrt{t}} \leq (1 + \beta_0)^{\alpha}.
\]

(Note that the function \((1 + \beta_0 / x)^{\alpha x}\) is decreasing in \( x \) for \( x \geq 1 \).)
We are now ready to complete our proof via a simple “change-of-measure” argument and the above lower bound on the ratio of the probability mass functions. From Equations (3.17) and (F.2) and the law of total probability, we get

\[
P \left( f_{t_k} \left( \frac{Z'_{t_k}}{t_k} \right) \geq p^* + \delta, Z'_{t_k} \leq q^* \cdot t_k + \beta \sqrt{t_k} \right) \\
\geq \sum_{z=s_k}^{q^* \cdot t_k + \beta \sqrt{t_k}} P \left( Z'_{t_k} = z \right) \cdot \mathbb{1} \left[ f_{t_k} \left( \frac{z}{t_k} \right) \geq p^* + \delta \right] \\
\geq (1 + \beta_0)^{-\alpha} \cdot \sum_{z=s_k}^{q^* \cdot t_k + \beta \sqrt{t_k}} P \left( Z''_{t_k, s_k} = z \right) \cdot \mathbb{1} \left[ f_{t_k} \left( \frac{z}{t_k} \right) \geq p^* + \delta \right] \\
= (1 + \beta_0)^{-\alpha} \cdot P \left( f_{t_k} \left( \frac{Z''_{t_k}}{t_k} \right) \geq p^* + \delta, Z''_{t_k} \leq q^* \cdot t_k + \beta \sqrt{t_k} \right) \\
\geq \frac{\epsilon_0}{2} (1 + \beta_0)^{-\alpha},
\]

and hence

\[
P \left( f_{t_k} \left( \frac{Z'_{t_k}}{t_k} \right) \geq p^* + \delta \right) \geq \frac{\epsilon_0}{2} (1 + \beta_0)^{-\alpha},
\]

for \( k \geq k_1 \). Since \( P_{t_k} \overset{d}{=} f_{t_k}(\hat{Q}_{t_k}) \), taking \( \epsilon := (\epsilon_0/2)(1 + \beta_0)^{-\alpha} \), we get

\[
P \left[ P_{t_k} \geq p^* + \delta \right] \geq \epsilon,
\]

for all \( k \geq k_1 \). This implies Equation (F.1) and completes the proof of the proposition.

Now, to extend the ideas from the proof of Proposition F.1, we need to show that the probability mass function of the random variable \( Z_t := \sum_{s=1}^t J_s \) is sufficiently similar to the probability mass function of the sum of independent Bernoulli\((q^*)\) random variables, which we denoted above by \( Z'_t \). In other words, it suffices to show that

\[
P(\frac{Z_t = z}{Z'_t = z}) \geq \epsilon \text{ for all } z \in [tq^* - \beta \sqrt{t}, tq^* + \beta \sqrt{t}] \quad \text{(F.3)}
\]

for some universal positive constant \( \epsilon > 0 \). We will do this by defining two sets of intermediate random variables:

1. The random variable \( Y_t := \sum_{s=1}^t J''_s \), where \( J''_s \sim \text{Ber}(\mathbb{E}[Q_t]) \) and \( \{J''_s\}_{t \geq 1} \) are mutually independent.

2. The random variable \( Y'_t := \text{Binomial}(t, \mathbb{E}[Q_t]) \).

We will show that the ratios of the pmfs between \((Z_t, Y_t), (Y'_t, Y'_t)\), and \((Y'_t, Z'_t)\) are lower bounded under the shaky-hands conjecture, i.e. assuming that Equation (3.27) holds. First, we note that Equation (3.27) directly yields

\[
P(\frac{Z_t = z}{Y_t = z}) \geq \epsilon \text{ for all } z \in [tq^* - \beta \sqrt{t}, tq^* + \beta \sqrt{t}].
\]

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It remains to show that the pmf of $Y_t$ is sufficiently similar to the pmf of $Z'_t \sim \text{Binomial}(t, q^*)$. Henceforth, we denote $q_t := \mathbb{E}[Q_t]$ and $\bar{q}_t := \mathbb{E}[\overline{Q}_t]$ as shorthand. We will use a similar proof-by-contradiction approach to the proof of Theorem 3.4 to establish the required pmf control. In other words, we will suppose that $Q_t \rightarrow q^*$ (in fact in probability), and show that we cannot have $P_t \rightarrow p^*$ in probability. Note that $Q_t \rightarrow q^*$ (in probability) obviously requires $q_t \rightarrow q^*$. Moreover, it follows from the property of time-averaged convergence and the definition of a competitive game (Appendix D) that $|\bar{q}_t - q^*| \leq C/\sqrt{t}$ for all $t \geq 1$. In summary, the sequence $\{\mathbb{E}[Q_t]\}_{t \geq 1}$ belongs to the following class of fixed-convergent strategies defined with respect to $q^*$:

$$|q_t - q^*| \leq \frac{\delta}{2} \text{ for all } t \geq t_0$$

$$|\bar{q}_t - q^*| \leq \frac{C}{\sqrt{t}} \text{ for all } t \geq 1,$$

where

$$\bar{q}_t := \frac{1}{t} \sum_{s=1}^{t} q_s.$$  

We also denote the set of such fixed-convergent strategies by $Z_{\delta,t_0,C}$ and denote their truncation to step $t$ by $Z_{\delta,t_0,C}(t)$.

First, we show that the pmf of $Y_t$ is very close to the pmf of $Y'_t \sim \text{Binomial}(t, \bar{q}_t)$ for any fixed-convergent sequence $\{q_t\}_{t \geq 1}$ satisfying Equations (F.4a) and (F.4b) (note that $\bar{q}_t = \sum_{s=1}^{t} q_s$.) This is encapsulated in the following lemma.

**Lemma F.2.** Consider any fixed-convergent sequence $\{q_t\}_{t \geq 1}$, i.e. such that Equations (F.4a) and (F.4b) both hold. Let $Y_t = \sum_{s=1}^{t} J'_s$ where $J'_s \sim \text{Ber}(q_s)$ and $\{J'_s\}_{t \geq 1}$ are mutually independent, and let $Y'_t \sim \text{Binomial}(t, \bar{q}_t)$. Then, there exists positive constant $\epsilon_b$ and integer $t_{0,b}$ such that for all $t \geq t_{0,b}$, we have

$$\frac{\mathbb{P}(Y_t = z)}{\mathbb{P}(Y'_t = z)} \geq \epsilon_b \text{ for all } z \in [tq^* - \beta \sqrt{t}, tq^* + \beta \sqrt{t}].$$

We next show that the pmfs of $Y'_t \sim \text{Binomial}(t, \bar{q}_t)$ and $Z'_t \sim \text{Binomial}(t, q^*)$ are sufficiently close.

**Lemma F.3.** There exists a positive constant $\epsilon_d > 0$ and a sufficiently large $t_{0,d}$ such that for all $t > t_{0,d}$, we have

$$\frac{\mathbb{P}(Y'_t = z)}{\mathbb{P}(Z'_t = z)} \geq \epsilon_d \text{ for all } z \in [q^* t - \beta \sqrt{t}, q^* t + \beta \sqrt{T}].$$

Notice that from Lemma F.2, Lemma F.3 and Equation (3.27), we get

$$\frac{\mathbb{P}(Z_t = z)}{\mathbb{P}(Z'_t = z)} \geq \epsilon \text{ for all } z \in [tq^* - \beta \sqrt{T}, tq^* + \beta \sqrt{T}].$$
for a constant $\epsilon = \epsilon_b \epsilon_d > 0$. In the following two subsections, we prove these two lemmas. This then completes the proof of Theorem 3.13.

It remains to prove Lemmas F.2 and F.3, which are significantly more technical than the martingale CLT that could be applied under the monotonicity assumption. We denote the pdf of the normal distribution $\mathcal{N}(\mu, \sigma^2)$ by $p(\cdot; \mu, \sigma^2)$. To prove these lemmas, we use the following theorem of De-Moivre and Laplace in several sub-lemmas that follow.

**Theorem F.4** (de-Moivre and Laplace, statement from Feller [1957]). Let $X \sim \text{Binomial}(t, q)$ for any $0 < q < 1$, and consider any sequence $\{k_t\}_{t \geq 1}$ such that $k_t^3/t^2 \to 0$ as $t \to \infty$. Then, for every $0 < \epsilon < 1$, there exists a $t_0$ sufficiently large such that for all $t > t_0$, we have

$$1 - \epsilon < \frac{\Pr(X = z)}{p(z; tq, tq(1 - q))} \leq 1 + \epsilon,$$

for all integers $qt - k_t \leq z \leq qt + k_t$.

Note that Theorem F.4 is a much sharper form of asymptotic normality than the typically stated Central Limit Theorem, as it obtains direct control on the probability mass function itself.

**Proof of Lemma F.2**

We consider the constant step index $t_{0,b} := \max\{t_0, 4C^2/\delta^2\}$, and note that by the triangle inequality and the assumed convergence (that we are going to contradict), we have

$$|q_t - \tilde{q}_t| \leq |q_t - q^*| + |q^* - \tilde{q}_t|$$

$$\leq \frac{\delta}{2} + \frac{C}{\sqrt{t}}$$

$$\leq \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

where the last inequality holds for all $t \geq t_{0,b}$. Thus, we have $|q_t - \tilde{q}_t| \leq \delta$, which is useful and required for comparing the pmfs of the random variables $Y_t$ and $Y_t'$.

Consider a fixed $t \geq t_{0,b}$. In general, relating the probability mass function of $Y_t$, which is the Poisson binomial random variable, directly to the binomial distribution is challenging. The following technical lemma characterizes the sequence $\{q_s\}_{s=1}^t$ that minimizes the probability mass function $\Pr(Y_t = z)$ for a fixed choice of $z$. This minimizing sequence takes values $q_s \in \{q_t - \delta, q_t, q_t + \delta\}$, which turns out to be a much simpler form to analyze.

**Sublemma F.5.** Consider any step index $t \geq 1$. Then, for every $z \in \{1, \ldots, t\}$, there exists an even integer $0 \leq n_t(z) \leq t$ such that

$$\Pr(Y_t = z) \geq \Pr(\tilde{Y}_t = z),$$

where $\tilde{Y}_t = \text{Binomial} \left( \frac{n_t(z)}{2}, q^* + \delta \right) + \text{Binomial} \left( \frac{n_t(z)}{2}, q^* - \delta \right) + \text{Binomial} \left( t - n_t(z), q^* \right)$. (Here, the three random variables are independent.)
Proof of sublemma F.5. Let \( \eta_s := q_s - \overline{q}_t \), for \( 1 \leq s \leq t \), denote the deviation of \( q_s \) from the average at time \( t, \overline{q}_t \). Thus we have \( \sum_{s=1}^{t} \eta_s = 0 \), and \( \eta_s \in [-\delta, \delta] \) for all \( s \in \{1, \ldots, t\} \).

Let \( e^t := \{e_1, e_2, \ldots, e_t\} \) where \( e_s \in \{-1, +1\} \) for all \( 1 \leq s \leq t \) represent the unique encoding of the output sequence \( J^t \in \{0, 1\}^t \). Let \( |e^t| := \{|e_s = +1 : 1 \leq s \leq t\}| \) denote the number of positive ones in the vector \( e^t \). Now, we consider \( 1 \leq z \leq t \). We have

\[
\mathbb{P}(Y_t = z) = \sum_{|e^t| = z} \prod_{s=1}^{t} q_s 1\{e_s = +1\} + (1 - q_s) 1\{e_s = -1\})
\]

\[
= \sum_{|e^t| = z} \prod_{s=1}^{t} (\overline{q}_t + \eta_s) 1\{e_s = +1\} + (1 - \overline{q}_t - \eta_s) 1\{e_s = -1\}).
\]

On the other hand, we have

\[
\mathbb{P}(\tilde{Y}_t = z) = \sum_{|e^t| = z} \prod_{s=1}^{t} (\overline{q}_t) 1\{e_s = +1\} + (1 - \overline{q}_t) 1\{e_s = -1\})
\]

\[
= \left(\frac{t}{z}\right) (\overline{q}_t)^z (1 - \overline{q}_t)^{(t-z)}.
\]

Thus, we get

\[
\frac{\mathbb{P}(Y_t = z)}{\mathbb{P}(\tilde{Y}_t = z)} = \left(\frac{t}{z}\right)^{-1} \sum_{|e^t| = z} \prod_{s=1}^{t} \frac{\overline{q}_t + \eta_s}{\overline{q}_t} 1\{e_s = +1\} + \frac{1 - \overline{q}_t - \eta_s}{1 - \overline{q}_t} 1\{e_s = -1\}).
\]

\[
= \left(\frac{t}{z}\right)^{-1} \sum_{|e^t| = z} \prod_{s=1}^{t} \left(1 + \frac{\eta_s}{\overline{q}_t}\right) 1\{e_s = +1\} + \left(1 - \frac{\eta_s}{1 - \overline{q}_t}\right) 1\{e_s = -1\}).
\]

Let \( \tilde{e}_s = \frac{+1}{\overline{q}_t} \) if \( e_s = +1 \) and \( \tilde{e}_s = \frac{-1}{1 - \overline{q}_t} \) if \( e_s = -1 \). Let \( \tilde{e}^t = \{\tilde{e}_1, \ldots, \tilde{e}_t\} \) and let \( |\tilde{e}^t| := |\{\tilde{e}_s = +1/\overline{q}_t : 1 \leq s \leq t\}| \). Then, we get

\[
\frac{\mathbb{P}(Y_t = z)}{\mathbb{P}(\tilde{Y}_t = z)} = \left(\frac{t}{z}\right)^{-1} \sum_{|e^t| = z} \prod_{s=1}^{t} (1 + \tilde{e}_s \eta_s).
\]

We will now try to lower bound the ratio \( \mathbb{P}(Y_t = z)/\mathbb{P}(\tilde{Y}_t = z) \) over \( \eta^t := \{\eta_1, \ldots, \eta_t\} \) such that \( \eta_s \in [-\delta, \delta] \) for all \( 1 \leq s \leq t \) and \( \sum_{s=1}^{t} \eta_s = 0 \). Let \( F \) denote the set of all such vectors \( \eta^t \). Let

\[
P(\eta^t) = \sum_{|e^t| = z} \prod_{s=1}^{t} (1 + \tilde{e}_s \eta_s),
\]

for \( \eta \in F \), and let

\[
\tilde{\eta}^t \in \arg \min_{\eta^t \in F} P(\eta).
\]

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Note that $P(\eta)$ is a multinomial in $\eta_1, \ldots, \eta_t$. We now show that $\tilde{\eta}_t$ satisfies: $\tilde{\eta}_s \in \{-\delta, 0, \delta\}$, for all $1 \leq s \leq t$. First note that if $\tilde{\eta}_s \in \{-\delta, \delta\}$ for all $1 \leq s \leq t$ then we are done. If this does not hold, then without loss of generality let $\tilde{\eta}_t \in (-\delta, \delta)$. Since $\sum_{s=1}^{t} \tilde{\eta}_s = 0$, let us substitute $\tilde{\eta}_t = -\sum_{s=1}^{t-1} \tilde{\eta}_s$. We now argue that $\tilde{\eta}_1 \in \{-\delta, \delta\}$. We have

$$\sum_{|e|=z} \prod_{s=1}^{t} (1 + \tilde{e}_s \tilde{\eta}_s) = \sum_{|e|=z} (1 + \tilde{e}_1 \tilde{\eta}_1)(1 - \tilde{e}_t(\tilde{\eta}_1 + \cdots + \tilde{\eta}_{t-1})) \prod_{s=2}^{t-1} (1 + \tilde{e}_s \tilde{\eta}_s)$$

$$= \sum_{|e|=z} (1 + \tilde{e}_1 \tilde{\eta}_1 - \tilde{e}_t \tilde{\eta}_1 - \tilde{e}_t(\tilde{\eta}_2 + \cdots + \tilde{\eta}_{t-1}) - \tilde{e}_1 \tilde{e}_t \tilde{\eta}_1^2 - \tilde{e}_1 \tilde{e}_2(\tilde{\eta}_2 + \cdots + \tilde{\eta})) H(\tilde{e}_2, \ldots, \tilde{e}_{t-1}),$$

where

$$H(\tilde{e}_2, \ldots, \tilde{e}_{t-1}) = \prod_{s=2}^{t-1} (1 + \tilde{e}_s \tilde{\eta}_s).$$

Note that the above is a quadratic expression in $\tilde{\eta}_1$. We now observe that the coefficient of $\tilde{\eta}_1$ in this expression is zero. Indeed, the coefficient of $\tilde{\eta}_1$ is given by

$$\sum_{|e|=z} H(\tilde{e}_2, \ldots, \tilde{e}_{t-1})(\tilde{e}_1 - \tilde{e}_t) = 0,$$

because of the symmetry in $\tilde{e}_1$ and $\tilde{e}_t$ in the above expression. A quadratic of the form $ax^2 + b$ attains its minimum on an interval $[l, h]$ either at $x = l, h$ or $x = 0$.

This establishes that $\tilde{\eta}_1 \in \{-\delta, 0, \delta\}$, and indeed the same argument works for all $t \in \{1, \ldots, (T - 1)\}$. Moreover, we get $\tilde{\eta}_t \in \{-\delta, 0, \delta\}$, as these are the only choices that can allow $\sum_{s=1}^{t} \tilde{\eta}_s = 0$.

Thus, we have established that $\tilde{\eta}_s \in \{-\delta, 0, \delta\}$ for all $s \in \{1, \ldots, t\}$.

Thus, there must be exact $n_t(z)/2$ values of $s$ corresponding to $\tilde{\eta}_s = \delta$, $n_t(z)/2$ values of $s$ corresponding to $\tilde{\eta}_s = -\delta$, and $(t - n_t(z))$ values of $s$ corresponding to $\tilde{\eta}_s = 0$. Thus, we have shown that

$$\mathbb{P}(Y_t = z) \geq \mathbb{P}(\tilde{Y}_t = z),$$

which completes the proof of the sublemma.

\[ \text{Sublemma F.6. Let } Y_t(n) := \text{Binomial (} \frac{n}{2}, q_t + \delta) + \text{Binomial (} \frac{n}{2}, q_t - \delta) + \text{Binomial (} t - n, q_t) \text{ for any even } n \in \{1, \ldots, t\}. \text{ Then, there exists universal constant } \epsilon_c > 0 \text{ such that for every } z \in [q^t - \beta \sqrt{t}, q^t + \beta \sqrt{t}], \text{ we have}
\]

$$\frac{\mathbb{P}(Y_t(n) = z)}{\mathbb{P}(Y_t' = z)} \geq \epsilon_c > 0.$$

\text{Recall that we defined } Y_t' \sim \text{Binomial}(t, q_t).
Note that Lemma F.6 immediately implies that
\[
\frac{\mathbb{P}(\hat{Y}_t = z)}{\mathbb{P}(Y'_t = z)} = \frac{\mathbb{P}(Y_t(n_t(z)) = z)}{\mathbb{P}(Y_t = z)} \geq \epsilon_c > 0,
\]
and we have thus related the original random variable \( Y_t \) to the Binomial random variable \( Y'_t \) through the constant \( \epsilon_b := \epsilon_c \), which would complete the proof of Lemma F.2. Thus, it only remains to prove Sublemma F.6, which we do below.

**Proof of Sublemma F.6.** Our proof will critically use the classical de-Moivre-Laplace theorem, stated earlier. To see how we can apply the de-Moivre-Laplace theorem to the denominator \( \mathbb{P}(Y'_t = z) \), we fix \( q := \bar{q}_t \). Then, note that since \( z \in [t q^* - \beta \sqrt{t}, t q^* + \beta \sqrt{t}] \) and, from (F.4b), we have \( \bar{q}_t \in \{q^* - C/\sqrt{t}, q^* + C/\sqrt{t}\} \). Thus, we have
\[
z \in [t \bar{q}_t - (C - \beta) \sqrt{t}, t \bar{q}_t + (C + \beta) \sqrt{t}].
\]
Designating \( C_c := (\beta + C) \), we consider the choice of sequence \( \{k_t = C_c \sqrt{t}\}_{t \geq 1} \). This sequence clearly satisfies \( k_t^2 / t^2 \to 0 \), and so we can directly apply the statement of the DeMoivre-Laplace theorem to get
\[
(1 - \epsilon_c) \cdot p(z; t \bar{q}_t, t \bar{q}_t(1 - \bar{q}_t)) \leq \mathbb{P}(Y'_t = z) \leq (1 + \epsilon_c) \cdot p(z; t \bar{q}_t, t \bar{q}_t(1 - \bar{q}_t)),
\]
for all \( t \geq t_{0,c} \) for all \( qt - k_t \leq z \leq qt + k_t \). Further, we will adjust \( t_{0,c} \) such that \( t > t_{0,c} := t_{0,c}/q^* \). Recall that \( p(\cdot; \mu, \sigma^2) \) denotes the pdf of the normal distribution \( \mathcal{N}(\mu, \sigma^2) \).

There are two cases to study depending on the value that \( n \) takes. The first one considers \( n \leq t_{0,c} \). Noting that \( t_{0,c} \) is a constant, in this case we can directly bound the ratio of pmfs.

First, we very crudely lower bound the numerator to get
\[
\mathbb{P}(Y_t(n) = z) = \sum_{0 \leq k_1, k_2 \leq n/2, 2.0 \leq k_3 \leq (t - n), k_1 + k_2 + k_3 = z} \binom{n/2}{k_1} \binom{n/2}{k_2} \binom{t - n}{k_3} (\bar{q}_t + \delta)^{k_1} \cdot (1 - \bar{q}_t - \delta)^{n/2 - k_1} \cdot (\bar{q}_t - \delta)^{k_2} \cdot (1 - \bar{q}_t + \delta)^{n/2 - k_2} \cdot (\bar{q}_t)^{k_3} (1 - \bar{q}_t)^{t - n - k_3} > \left( \frac{t - n}{z - n} \right) (\bar{q}_t + \delta)^{n/2} (\bar{q}_t - \delta)^{n/2} (\bar{q}_t)^{z-n} (1 - \bar{q}_t)^{t-z},
\]
where in the last inequality we considered only the point \( k_1 = k_2 = n/2, k_3 = z - n \) in the sum. (Note that this is a valid point as \( z \leq t \) and \( z - n \geq q^*t - t_{0,c} > 0 \). The latter inequality follows because we assumed that \( q^*T > t_{0,c} \).) On the other hand, for the denominator we have
\[
\mathbb{P}(Y'_t = z) = \binom{t}{z} (\bar{q}_t)^{z} (1 - \bar{q}_t)^{t-z},
\]
\[
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\]
and so we get, after some algebraic simplification,

\[
\frac{\mathbb{P}(Y_t(n) = z)}{\mathbb{P}(Y'_t = z)} > \left(\frac{t-n}{z-n}\right) \left(1 - \frac{\beta^2}{\eta^2}\right)^{n/2} \\
\geq \epsilon_c > 0,
\]

where the constant \(\epsilon_c\) will depend on \(\eta_t, \delta, t_{0,c}\), but not on \(t\). Here, we have critically used \(n \leq t_{0,c}\) to lower bound the term \(\left(1 - \frac{\beta^2}{\eta^2}\right)\) by such a constant, as well as noting that

\[
\frac{(t-n)}{\binom{t}{z}} = \frac{z}{t} \cdot \frac{z-1}{t-1} \cdots \frac{z-n+1}{t-n+1} \\
\geq \left(\frac{z-n+1}{t-n+1}\right)^n \\
\geq (\epsilon_c)^n \geq (\epsilon_c)^{t_{0,c}},
\]

for some constant \(\epsilon_c\) that is close to \(q^*\).

Notice that the above crude argument does not work for the case where \(n > t_{0,c}\), in particular, if it can grow indefinitely as a function of \(t\), is less trivial. For this case, we make the following claim using the de-Moivre-Laplace theorem, under which it suffices to prove the lemma.

**Claim F.7.** There exists a constant \(\epsilon_c \in (0, 1)\) that can depend on \(C_c\), but is independent of \((t, n)\), such that for \(t_{0,c} \leq n \leq t\), we have

\[
\begin{align*}
\mathbb{P}(Y'_t = z) &\leq (1 + \epsilon_c) \cdot p(z; t\eta_t, t\eta_t(1 - \eta_t)) \\
\mathbb{P}(Y_t(n) = z) &\geq (1 - \epsilon_c) \cdot p(z; t\eta_t, t\sigma^2(\eta_t, n, t))
\end{align*}
\]

where

\[
\sigma^2(\eta_t, n, t) := \frac{1}{t} \left(\frac{t}{2} (t\eta_t - \delta)(1 - \eta_t + \delta) + \frac{n}{2} v^+(\eta_t; \delta) + (t - n)\eta_t(1 - \eta_t)\right),
\]

for any \(z \in [tq^* - C_c\sqrt{t}, tq^* + C_c\sqrt{t}]\).

First, notice that Claim F.7 directly gives us our proof for the case where \(n \geq t_{0,c}\). To see this, consider the second case where \(\eta_t > 1/2\). This gives us

\[
\begin{align*}
\frac{\mathbb{P}(Y_t(n) = z)}{\mathbb{P}(Y'_t = z)} &\geq \frac{(1 - \epsilon_c) \cdot p(z; t\eta_t, t\sigma^2(\eta_t, n, t))}{(1 + \epsilon_c) \cdot p(z; t\eta_t, t\eta_t(1 - \eta_t))} \\
&= \frac{(1 - \epsilon_c) \cdot \sqrt{2\pi \cdot t\eta_t(1 - \eta_t)} e^{-\frac{(z-t\eta_t)^2}{2t\eta_t(1 - \eta_t)}}}{(1 + \epsilon_c) \cdot \sqrt{2\pi \cdot t\sigma^2(\eta_t, n, t)} e^{-\frac{(z-t\eta_t)^2}{2t\sigma^2(\eta_t, n, t)}}}.
\end{align*}
\]
First, we note that \( \sigma^2(q_t, n, t) \leq \frac{1}{4} \). Moreover, we know that \( q_t \in [q^* - \delta, q^* + \delta] \), and so we have

\[
\frac{\sqrt{2\pi \cdot t q_t(1 - q_t)}}{\sqrt{2\pi \cdot t \sigma^2(q_t, n, t)}} \geq \epsilon_c > 0,
\]

where \( \epsilon_c \) is a constant that depends only on \( \delta \). Thus, we get

\[
\frac{\mathbb{P}(Y_t(n) = z)}{\mathbb{P}(Y_t' = z)} \geq \epsilon_c \cdot \frac{e^{-\frac{(z-1)^2}{2\sigma^2(q_t, n, t)}}}{e^{-\frac{(z-1)^2}{2\sigma^2(1-\frac{1}{n})}}}.
\]

Finally, we note that \( z \in [tq^* - C_c\sqrt{t}, tq^* + C_c\sqrt{t}] \). Thus, to lower bound the numerator we get

\[
e^{-\frac{(z-1)^2}{2\sigma^2(q_t, n, t)}} \geq e^{-\frac{4C_c^2 t}{2\sigma^2(1-\frac{1}{n})}} \geq \epsilon_c > 0,
\]

where we now use the fact that \( \sigma^2(q_t, n, t) \geq (q_t + \delta)(1 - q_t - \delta) \). Thus, we get \( q_t + \delta \leq q^* + 2\delta < 1 \). Note that this constant \( \epsilon_c \) will depend on \((q^*, \delta, C_c)\), but is independent of \( t \).

For the denominator, we trivially have \( e^{-\frac{(z-1)^2}{2\sigma^2(1-\frac{1}{n})}} \leq 1 \). Putting all of these together, we get

\[
\frac{\mathbb{P}(Y_t(n) = z)}{\mathbb{P}(Y_t' = z)} \geq \epsilon_c > 0,
\]

where \( \epsilon_c \) is the product of all the above constants and thus depends on \((t_0, q^*, C_c, \delta)\), but is independent of \( t \). Thus, given Claim F.7, we have proved Lemma F.6. (A symmetric argument, which we omit, also works for the case \( q_t \leq 1/2 \).)

It only remains to prove this claim, which we do below using the DeMoivre-Laplace theorem.

**Proof of Claim F.7.** As we noted above, Equation (F.6) follows immediately from the statement of Theorem F.4. To prove Equation (F.7), we need to do a little more work, but essentially we can exploit the mixture-of-binomials structure in the random variable \( Y_t(n) := \text{Binomial} \left( \frac{n}{2}, q_t + \delta \right) + \text{Binomial} \left( \frac{n}{2}, q_t - \delta \right) + \text{Binomial} \left( t - n, \frac{q_t}{t} \right) \) for any even \( n \in \{1, \ldots, t\} \).

First, we consider the extreme case where the distribution is “most different” from \( Y_t \), i.e., \( n = t \). In this case, note that \( Y_t(t) = Y_{t,1} + Y_{t,2} \) where \( Y_{t,1} \sim \text{Binomial} \left( \frac{t}{2}, q_t + \delta \right) \) and \( Y_{t,2} \sim \text{Binomial} \left( \frac{t}{2}, q_t - \delta \right) \), and the random variables \( Y_{t,1} \) and \( Y_{t,2} \) are independent. Thus, we get

\[
\mathbb{P}(Y_t(t) = z) = \sum_{y} \mathbb{P}(Y_{t,1} = y) \mathbb{P}(Y_{t,2} = (z - y))
\]

\[
\geq \sum_{y} \mathbb{P}(Y_{t,1} = y) \mathbb{P}(Y_{t,2} = (z - y)).
\]

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Now, observe that \( y \in [t/2 \cdot (q + \delta) - C_c t^{5/9}, t/2 \cdot (q + \delta) + C_c t^{5/9}] \), and because we have assumed that \( z \in [\bar{q} C_C \sqrt{t}, \bar{q} + C C \sqrt{t}] \), we also have \( (z - y) \in [t/2 \cdot (q - \delta) - C_c t^{5/9}, t/2 \cdot (q - \delta) + C_c t^{5/9}] \) for slightly adjusted constant \( C_c \). Moreover, it is easy to verify that the sequence \( \{k_t := C_c t^{5/9}\}_{t \geq 1} \) satisfies the conditions required for application of de-Moivre-Laplace theorem.

We denote \( v^+(q; \delta) := (q + \delta)(1 - q - \delta) \) and \( v^-(q; \delta) := (1 - q + \delta)(q - \delta) \) as the variances of the Bernoulli random variables with parameters \( (q + \delta) \) and \( (1 - q + \delta) \) respectively. Therefore, for large enough \( t \geq t_{0,c} \) (where \( t_{0,c} \) will depend on \( (\epsilon_c, \delta, q, C_c) \), and appropriately chosen constant \( \epsilon_c \in (0, 1) \), and the specified ranges of \( (y, z) \), we get

\[
\mathbb{P}(Y_{t,1} = y) \geq (1 - \epsilon_c) \cdot p \left( y; \frac{t}{2}(q + \delta), \frac{t}{2} v^+(q; \delta) \right) \]

\[
\mathbb{P}(Y_{t,2} = (z - y)) \geq (1 - \epsilon_c) \cdot p \left( (z - y); \frac{t}{2}(q - \delta), \frac{t}{2} v^-(q; \delta) \right),
\]

and so we get

\[
\mathbb{P}(Y_t(t) = z) \geq (1 - \epsilon_c)^2 \sum_{y = \frac{t}{2}((q + \delta) - C_c t^{5/9})}^{\frac{t}{2}((q + \delta) + C_c t^{5/9})} p \left( y; \frac{t}{2}(q + \delta), \frac{t}{2} v^+(q; \delta) \right) \cdot p \left( (z - y); \frac{t}{2}(q - \delta), \frac{t}{2} v^-(q; \delta) \right)
\]

\[
\geq (1 - \epsilon_c)^2 \cdot p \left( z; \frac{t}{2}(q + \delta), \frac{t}{2} v^+(q; \delta) \right) \cdot p \left( (z - y); \frac{t}{2}(q - \delta), \frac{t}{2} v^-(q; \delta) \right) - 2(1 - \epsilon_c)^2 \cdot e^{-C_c t^{1/9}}
\]

\[
= (1 - \epsilon_c)^2 \cdot p \left( z; \bar{q} + t \delta, \frac{t}{2} v^+(q; \delta) + \frac{t}{2} v^-(q; \delta) \right) - 2(1 - \epsilon_c)^2 \cdot e^{-C_c t^{1/9}}
\]

\[
\geq \epsilon_c(1 - \epsilon_c) p \left( z; \bar{q} + t \delta, \frac{t}{2} v^+(q; \delta) + \frac{t}{2} v^-(q; \delta) \right).
\]

Here, inequality (ii) follows for large enough \( t \geq t_{0,c} \) noting that for the specified range of \( z \), we have \( p \left( z; \bar{q} + t \delta, \frac{t}{2} v^+(q; \delta) + \frac{t}{2} v^-(q; \delta) \right) \geq \epsilon_c > 0 \); and also that \( e^{-C_c t^{1/9}} \) goes to 0 as \( t \to \infty \). Inequality (i) follows by noting that

\[
\sum_{y > \frac{t}{2}((q + \delta) + C_c t^{5/9})} p \left( y; \frac{t}{2}(q + \delta), \frac{t}{2} v^+(q; \delta) \right) \leq \mathbb{P}(W > C_c t^{1/18})
\]

\[
\leq e^{-C_c t^{1/9}},
\]

where \( W \) denotes the standard normal random variable, and we have overloaded notation in choices of \( C_c \). Similarly, we have

\[
\sum_{y < \frac{t}{2}((q - \delta) - C_c t^{5/9})} p \left( y; \frac{t}{2}(q - \delta), \frac{t}{2} v^-(q; \delta) \right) \leq e^{-C_c t^{1/9}}.
\]
From this, we get
\[
\frac{1}{2}(\pi + \delta) + C_v t^{v/9} \sum_{y = \frac{1}{2}(\pi + \delta) - C_v t^{v/9}} p \left( y; \frac{t}{2} (\overline{q}_t + \delta), \frac{t}{2} v^+ (\overline{q}_t; \delta) \right) 
\cdot p \left( (z - y); \frac{t}{2} (\overline{q}_t - \delta), \frac{t}{2} v^- (\overline{q}_t; \delta) \right)
\geq p \left( z; \frac{t}{2} (\overline{q}_t + \delta), \frac{t}{2} v^+ (\overline{q}_t; \delta) \right) \cdot p \left( z; \frac{t}{2} (\overline{q}_t - \delta), \frac{t}{2} v^- (\overline{q}_t; \delta) \right) - 2e^{-C_v t^{v/9}},
\]
and plugging this in above gives inequality (i).

Clearly, we have proved Equation (F.7) for this extreme case. Let us now extend this extreme case more generally. Recall that we assumed \( n \geq t_{0,c} \). In this case, by an identical argument to the above, we get
\[
P(Y_{n,1} + Y_{n,2} = z') \geq \epsilon_c (1 - \epsilon_c)^2 \cdot \left( z; n \overline{q}_t, \frac{n}{2} v^+ (\overline{q}_t; \delta) \right).
\]

Thus, we can utilize a similar convolution argument as before to study \( Y_t(n) := Y_{n,1} + Y_{n,2} + Y_{(t-n),3} \), where \( Y_{(t-n),3} \sim \text{Binomial}(n, q_t) \) and is independent from \( \{Y_{n,1}, Y_{n,2}\} \). Thus, we get
\[
P(Y_t(n) = z) = \sum_{z' = n \overline{q}_t - C_v n^{5/9}}^{n \overline{q}_t + C_v n^{5/9}} p(Y_{n,1} + Y_{n,2} = z') p(Y_{(t-n),3} = (z - z')).
\]

There are two cases depending on the value of \( t - n \):

1. \( t - n \leq t_{0,c} \). In this case, because \( Y_{(t-n),3} \in \{0, \ldots, t_{0,c}\} \), we have
   \[
P(Y_{(t-n),3} = (z - z')) \geq (\min\{\overline{q}_t, 1 - \overline{q}_t\})^{(t-n)} \geq (\min\{\overline{q}_t, 1 - \overline{q}_t\})^{t_{0,c}}.
\]

   Similarly, it is easy to verify that the normal pdf \( p((z - z'); (t - n) \overline{q}_t, (t - n) \overline{q}_t (1 - \overline{q}_t)) \) is also bounded above by a constant \( \epsilon_c' \) as \( (z - z') \leq t_{0,c} \). Thus, the ratio of the two is bounded by a universal constant \( \epsilon_c \) for all \( (z - z') \in \{0, \ldots, t_{0,c}\} \).

2. \( t - n \geq t_{0,c} \). Now, we note that \( (z - z') \in [(t - n) \overline{q}_t - C_v \sqrt{n} - C_v \sqrt{t}, (t - n) \overline{q}_t + C_v \sqrt{n} + C_v \sqrt{t}] \), therefore, \( (z - z') \in [(t - n) \overline{q}_t - C_v \sqrt{(t - n)}, (t - n) \overline{q}_t + C_v \sqrt{(t - n)}] \) for slightly adjusted constant \( C_v \). Then, since \( t - n \geq t_{0,c} \) as well, we can apply the de-Moivre-Laplace theorem on the Binomial random variable \( Y_{(t-n),3} \) and show that
   \[
   \frac{p(Y_{(t-n),3} = (z - z'))}{p((z - z'); (t - n) \overline{q}_t, (t - n) \overline{q}_t (1 - \overline{q}_t))} \geq (1 - \epsilon_c)
   \]
   for the specified range on \( (z - z') \).
Thus, in both cases, for appropriately chosen $\epsilon_c > 0$ we get
\[
\mathbb{P}(Y'_t(n) = z) \geq \epsilon_c \sum_{z' = \epsilon c^{-C} \sqrt{n}^{5/9}} \mathbb{P} \left( z' ; n \bar{q}, \frac{n}{2} \nu^+(\bar{q}_l; \delta) + \frac{n}{2} \nu^-(\bar{q}_l; \delta) \right) \cdot 
\]
where the second inequality uses an identical argument as earlier again noting that $n \geq t_{0,c}$. This completes the statement of the claim, and thus completes the proof.

Now that we have proved Claim F.7, we have completed the proof of Sublemma F.6.

**Proof of Lemma F.3**

**Proof.** From the DeMoivre-Laplace theorem, there exists positive constant $\epsilon_d \in (0, 1)$ and a sufficiently large $t_{0,d}$, such that for all $t > t_{0,d}$, we have
\[
\mathbb{P}(Y'_t = z) \geq (1 - \epsilon_d) \cdot p(z; t\bar{q}, t\bar{q}(1 - \bar{q})) ,
\]
\[
\mathbb{P}(Z'_t = z) \leq (1 + \epsilon_d) \cdot p(z; tq', tq' (1 - q')) .
\]
Hence, we have
\[
\frac{\mathbb{P}(Y'_t = z)}{\mathbb{P}(Z'_t = z)} \geq \frac{(1 - \epsilon_d)}{(1 + \epsilon_d)} \cdot \frac{p(z; t\bar{q}, t\bar{q}(1 - \bar{q}))}{p(z; tq', tq' (1 - q'))} = \frac{(1 - \epsilon_d)}{(1 + \epsilon_d)} \cdot \sqrt{\frac{2\pi \cdot t(\bar{q})(1 - \bar{q})}{2\pi \cdot tq'(1 - q')}} \cdot \frac{e^{-\frac{(z-tq)^2}{2tq' (1 - q')}}}{e^{-\frac{(z-tq')^2}{2tq' (1 - q')}}} .
\]
Now, we have
\[
\frac{\sqrt{2\pi \cdot t(\bar{q})(1 - \bar{q})}}{\sqrt{2\pi \cdot tq'(1 - q')}} \geq \max \left\{ \frac{\sqrt{2\pi \cdot (q^* - \delta)(1 - q^* + \delta)}}{\sqrt{2\pi \cdot q^* (1 - q^*)}}, \frac{\sqrt{2\pi \cdot (q^* + \delta)(1 - q^* - \delta)}}{\sqrt{2\pi \cdot q^* (1 - q^*)}} \right\} > 0 ,
\]
for all $t$. This is because $\bar{q} \in [q^* - \delta, q^* + \delta]$ and $\bar{q}(1 - \bar{q})$ being concave over this interval attains its minimum on the boundary. Further, we get
\[
e^{-\frac{(z-tq)^2}{2tq' (1 - q')}} \geq \max \left\{ e^{-\frac{2C^2}{2(\delta - \theta)(1 - q^* + \theta)}}, e^{-\frac{2C^2}{2(\delta - \theta)(1 - q^* - \theta)}} \right\} > 0 .
\]
Note that we have obtained bounds that do not depend on $t$. Thus there exists a positive constant $\epsilon_d$ such that the statement in the lemma holds.
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