Abstract—We present two constructions for binary self-orthogonal codes. It turns out that our constructions yield a constructive bound on binary self-orthogonal codes. In particular, when the information rate $R = 1/2$, by our constructive lower bound, the relative minimum distance $\delta \approx 0.0595$ (for GV bound, $\delta \approx 0.110$). Moreover, we have proved that the binary self-orthogonal codes asymptotically achieve the Gilbert-Varshamov bound.

Index Terms—Algebraic geometry codes, concatenated codes, Gilbert-Varshamov bound, Reed-Muller codes, self-dual basis, self-orthogonal codes.

I. INTRODUCTION

In coding theory, we are interested in good codes with large length, i.e., we want to find a family of codes with length tending to $\infty$. For a family of linear $[n, k, d]$ codes over $\mathbb{F}_q$, the ratio $R := \lim_{n \to \infty} k/n$ and $\delta := \lim_{n \to \infty} d/n$ denote the information rate and the relative minimum distance, respectively, of the codes. The set $U_q \subseteq [0, 1] \times [0, 1]$ which is defined as follows: a point $(\delta, R) \in \mathbb{R}^2$ with $0 \leq \delta \leq 1$ and $0 \leq R \leq 1$ belongs to $U_q$ if and only if there exists a sequence $\{C_i = [n_i, k_i, d_i]\}_{i \geq 0}$ of codes over $\mathbb{F}_q$ such that

$$n_i \to \infty, \frac{d_i}{n_i} \to \delta \text{ and } \frac{k_i}{n_i} \to R, \text{ as } i \to \infty.$$  

A main coding problem is to determine the domain $U_q$. Manin and Vlăduţ gave a description of $U_q$ through a function $\alpha_q : [0, 1] \to [0, 1]$ which is defined by

$$\alpha_q(\delta) = \sup \{R : (\delta, R) \in U_q, \text{ for } \delta \in [0, 1]\}.$$

It is well-known that the function $\alpha_q$ is continuous and decreasing, see [1].

An $[n, k]$ linear code $C$ over the finite field $\mathbb{F}_q$ is a linear $k$-dimensional subspace of $\mathbb{F}_q^n$. The dual code $C^\perp$ of $C$ is defined as the orthogonal space of $C$, i.e.,

$$C^\perp = \{y \in \mathbb{F}_q^n | \langle x, y \rangle = 0 \text{ for every } x \in C\},$$

where $\langle xy \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$ is the ordinary scalar product of vectors $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{F}_q^n$. A code $C$ is self-orthogonal if $C \subseteq C^\perp$, and self-dual if $C = C^\perp$. It is well-known that there exists a class of long binary self-orthogonal codes which meet the Gilbert-Varshamov bound.

[2]. We employ the method which mentioned in [2], proof that binary self-orthogonal codes also achieve the Gilbert-Varshamov bound. However, this result is not constructive.

To obtain the constructive bound on $R$ and $\delta$, we involved two different ways to construct binary self-orthogonal codes. Both of the two constructions are based on a kind of algebraic geometry codes which achieves the Tsfasman-Vlăduţ-Zink bound. In the Construction A, we concatenate algebraic geometry codes with binary self-orthogonal codes to obtain the desired codes. In the Construction B, we also get the desired codes by considering self-orthogonal algebraic geometry codes and express these algebraic geometry codes into binary self-orthogonal codes by employing the self-dual basis. Using these two constructions, we obtain a lower bound on $R$ and $\delta$. In particular, using Construction B, we get $\delta \approx 0.0595$ when $R = 1/2$ (by Gilbert-Varshamov bound, $\delta \approx 0.110$ when $R = 1/2$).

This correspondence is organized as follows. We first recall some basic results of concatenated codes, Reed-Muller codes, Gilbert-Varshamov bound, and some well-known facts about algebraic geometry codes which are necessary for our purpose. The main description of our two constructions are given in Section III, and we calculate some examples. In Section IV, we have shown that there exists a binary self-orthogonal code achieving the Gilbert-Varshamov bound. The conclusion of this paper is given in the last section.

II. PRELIMINARIES

In this section, we give some fundamental properties about concatenated codes, algebraic geometry codes and Reed-Muller codes. We recall the results in [3], [4] and [5] as follows.

Let $C$ be an $[s, v, w]$ code over $\mathbb{F}_{q^k}$ and, for $i = 1, 2, \ldots, s$, let $\pi_i : \mathbb{F}_{q^k} \to \mathbb{F}_{q^k}$ be an $\mathbb{F}_q$-linear injective map whose image $C_i = \text{im}(\pi_i)$ is an $[n_i, k_i, d_i]$ code over $\mathbb{F}_q$. The image $\pi(C)$ of the following $\mathbb{F}_q$-linear injective map:

$$\pi : C \to \mathbb{F}_{q^k}^{n_1, \ldots, n_s}$$

$$c = (c_1, \ldots, c_s) \mapsto \pi(c) = (\pi_1(c_1), \ldots, \pi_s(c_s))$$

is an $[n_1 + \cdots + n_s, v, k]$ linear concatenated code over $\mathbb{F}_q$.

From the definition of the concatenated code, we know that if two codes $C, C'$ over $\mathbb{F}_{q^k}$ satisfy $C \subseteq C'$, then the two concatenated codes $\pi(C) \subseteq \pi(C')$ over $\mathbb{F}_q$.

Lemma 1: If $\text{im}(\pi_i) = [n_i, k_i, d_i]$ $(1 \leq i \leq s)$ are self-orthogonal codes, then $\pi(C)$ is also a self-orthogonal code.

Proof: Given any two codewords $\pi(c) = (\pi_1(c_1), \ldots, \pi_s(c_s))$ and $\pi(c') = (\pi_1(c'_1), \ldots, \pi_s(c'_s))$
of \( \pi(C) \), where \( c = (c_1, c_2, \ldots, c_s) \) and \( c' = (c'_1, c'_2, \ldots, c'_s) \) are two codewords of \( C \). Then

\[
(\pi(c), \pi(c')) = \sum_{i=1}^{s} (\pi_i(c_i), \pi_i(c'_i)),
\]

where \( (\cdot, \cdot) \) stands for the ordinary scalar product over \( \mathbb{F}_q \). Since \( \text{im}(\pi_i) \) \( (1 \leq i \leq s) \) are self-orthogonal, we have \( \pi_i(c_i) \cdot \pi_i(c'_i) = 0 \) for all \( 1 \leq i \leq s \). Thus \( \pi(c) \cdot \pi(c') = 0 \), therefore, \( \pi(C) \) is a self-orthogonal code.

**Lemma 2:** \((3)\) Suppose the images \( \text{im}(\pi_i) \) \( (1 \leq i \leq s) \) are identical and have parameters \( [n, k, d] \). Then \( \pi(C) \) is an \( [ns, nk, d] \) linear code over \( \mathbb{F}_q \) with the minimum distance at least \( wd \).

From now on, we assume that the images \( \text{im}(\pi_i) \) \( 1 \leq i \leq s \) are identical, and denote as \( \text{im}(\pi_s) \), i.e., \( \pi((c_1, c_2, \ldots, c_s)) = (\pi_s(c_1), \pi_s(c_2), \ldots, \pi_s(c_s)) \) in equation \((1)\).

Next, we review some basic conclusions of algebraic geometry codes.

Let \( X \) be a smooth, projective, absolutely irreducible curve of genus \( g \) defined over \( \mathbb{F}_q \), let \( D \) be a set of \( N \) \( \mathbb{F}_q \)-rational points of \( X \) and let \( G \) be an \( \mathbb{F}_q \)-rational divisor of \( X \) such that \( \text{supp}(G) \cap D = \emptyset \) and \( 2g-2 < \text{deg}(G) < N \), where \( \text{supp}(G) \) and \( \text{deg}(G) \) denote the support and the degree of \( G \), respectively. Then the functional algebraic-geometry code \( C_L(G, D) \) with parameters \( [N, \text{deg}(G) - g + 1, N - \text{deg}(G)] \) can be defined, see \([1]\).

Let \( q = p^l \) be a square. It is known that there exists a family of algebraic curves \( \{X_i\} \) over \( \mathbb{F}_q \) with \( g_i \rightarrow \infty \) attaining the Drinfeld-Vlăduţ bound, i.e.,

\[
\lim_{i \rightarrow \infty} \text{sup}(N(X_i/\mathbb{F}_q)/g_i) = l - 1
\]

where \( N(X_i/\mathbb{F}_q) \) and \( g_i \) are the number of \( \mathbb{F}_q \)-rational points and the genus of \( X_i \), respectively (see \([4]\)). Then, the paper \([6]\) constructs a family of algebraic geometry codes \( T_i = C_L(G_i, D_i) = [N_i, K_i, D_i]_\mathbb{F}_q \) achieving the Tsfasman-Vlăduţ-Zink bound where \( D_i \) contain all \( \mathbb{F}_q \)-rational point except only one rational point \( P \) which is the support of divisor \( G_i \), i.e.,

\[
R_i + \delta_i = 1 - \frac{1}{l - 1}
\]

where

\[
R_i = \lim_{i \rightarrow \infty} \frac{K_i}{N_i} \quad \text{and} \quad \delta_i = \lim_{i \rightarrow \infty} \frac{D_i}{N_i},
\]

denote the information rate and the relative minimum distance, respectively, of the codes.

For the Construction A, we also need some properties of Reed-Muller codes.

Let \( v = (v_1, \ldots, v_m) \) denote a vector which ranges over \( \mathbb{F}_q^m \), and \( f \) is the vector of length \( 2m \) by list of values which are taken by a Boolean function \( f(v_1, \ldots, v_m) \) on \( \mathbb{F}_q^m \).

**Definition 1:** \((3)\) The \( r \)th order binary Reed-Muller code (or RM code) \( \mathcal{R}(r, m) \) of length \( n = 2^m \), for \( 0 \leq r \leq m \), is the set of all vectors \( f \), where \( f(v_1, \ldots, v_m) \) is a Boolean function which is a polynomial of degree at most \( r \).

**Lemma 3:** \((3)\) The \( r \)th order binary Reed-Muller code \( \mathcal{R}(r, m) \) has dimension \( k = \sum_{i=0}^{r} \binom{m}{i} \) and minimum distance \( 2^{m-r} \) for \( 0 \leq r \leq m \), where \( \binom{m}{i} \) are binomial coefficients.

**Lemma 4:** \((3)\) \( \mathcal{R}(m-r-1, m) \) is the dual code of \( \mathcal{R}(r, m) \) with respect to the ordinary scalar product, for \( 0 \leq r \leq m-1 \).

From the definition of Reed-Muller codes, it is easy to known that we have \( \mathcal{R}(r_1, m) \subseteq \mathcal{R}(r_2, m) \) when \( 0 \leq r_1 \leq r_2 \leq m \). By Lemma 4, when \( r \leq \frac{m-1}{2} \), \( \mathcal{R}(r, m) \) is a self-orthogonal code. In particular, when \( m \) is an odd number, \( \mathcal{R}(\frac{m-1}{2}, m) \) is a self-dual code.

Now, let \( m \) go through all positive odd number, we get a family of self-dual Reed-Muller codes \( \mathcal{R}(r, m) \), where \( r = \frac{m-1}{2} \), with parameters \( [2^m, 2^{m-1}, 2^{m-1}/2] \).

At the end of this section, in order to compare our bound with the existed bound, we give the asymptotically Gilbert-Varshamov bound.

**Lemma 5:** (Asymptotic Gilbert-Varshamov Bound) If \( 0 \leq \delta \leq \frac{q-1}{q} \) then

\[
\alpha_q(\delta) \geq 1 - H_q(\delta),
\]

where \( H_q(\delta) \) is q-ary entropy function defined by

\[
H_q(x) = \begin{cases} 
  x \log_q(q-1) - x \log_q x - (1-x) \log_q(1-x), & 0 < x \leq (q-1)/q; \\
  0, & x = 0.
\end{cases}
\]

**Remark 1:** In Fig.1 we show this bound for \( q = 2 \).

### III. CONSTRUCTIONS OF SELF-ORTHOGONAL CODES

In this section, we will present two constructions of binary self-orthogonal codes.

#### A. Construction A

Assume that \( q = 2^{2t} \) in this subsection. Let \( \text{im}(\pi_s) \) be an binary \([n, 2t]\) linear code.

Let \( T_i = [N_i, K_i, D_i] \) be a family of algebraic geometry codes over \( \mathbb{F}_{2^{2t}} \) achieving the Tsfasman-Vlăduţ-Zink bound, i.e.,

\[
R_i + \delta_i = 1 - \frac{1}{2^t - 1}
\]

where

\[
R_i = \lim_{i \rightarrow \infty} \frac{K_i}{N_i} \quad \text{and} \quad \delta_i = \lim_{i \rightarrow \infty} \frac{D_i}{N_i}.
\]

Now we state our first construction.

**Proposition 1:** Let \( C_0 \) be a self-orthogonal code over \( \mathbb{F}_2 \) with parameters \([n, 2t, d]\), take \( C_0 \) as \( \text{im}(\pi_s) \), concatenate the family of algebraic geometry codes \( T_i \) and \( C_0 \) under the map \( \pi = (\pi_s, \pi_s, \ldots, \pi_s) \), then we obtain a family of binary self-orthogonal codes \( C \) with parameters \([nN_i, 2tK_i, dD_i] \).

Moreover, we have asymptotic equation

\[
R + \frac{2t}{d} \delta = \frac{2t}{n} (1 - \frac{1}{2^t - 1})
\]

where \( R \) and \( \delta \) denote the information rate and the relative minimum distance, respectively, of the concatenated codes.

**Proof:** The result follow immediately consequence of the properties of algebraic geometry codes \( T_i \) and concatenated codes.

Now we give some examples to illustrate the result in Proposition 1.

**Example 1:** (RM codes) If we fixed an odd number \( m \geq 3 \), then we get a binary self-dual code \([2^m, 2^{m-1}, 2^{(m+1)/2}] \). Let \( 2t = 2^{m-1} \), then \( \mathcal{F}_{2^{2t}} = \mathcal{F}_{2^{2m-1}} \).
It is well-known that there exists a family of algebraic geometry codes $T_i$ over $F_{q^{2m-1}}$ with parameters $[N_i, K_i, D_i]$ satisfy the equation (4). Then by Proposition 1, we get a family of binary concatenated codes $C_i = [2^m N_i, 2^{m-1} K_i, 2^{m+1} D_i]$. Asymptotically, we have the equation

$$R + 2 \frac{m-1}{n} \delta = \frac{1}{2} \left( 1 - \frac{1}{2^m - 2 - 1} \right)$$  \hspace{1cm} (6)

Thus when we go through all odd number $m \geq 3$, we get a sequence of equations for $R$ and $\delta$.

**Example 2:** (Some special binary self-orthogonal codes)

From [9] and [7], we get several optimal self-orthogonal codes. Using these codes to do the concatenation, we get some equations about $R$ and $\delta$ for small $t$. We list them in Table I. The last column of Table I was calculated by [5].

Using these two examples, we get an asymptotic bound for $\alpha_2(\delta)$.

**B. Construction B**

In this subsection, we will give another construction of binary self-orthogonal codes. Let us first recall the definition of self-dual basis.

Let $\{e_1, \cdots, e_k\}$ be an $F_q$-basis of $F_{q^k}$. A set $\{e'_1, \cdots, e'_k\}$ of $F_{q^k}$ is called the dual basis of $\{e_1, \cdots, e_k\}$ if we have

$$\text{Tr}_{F_{q^k}/F_q}(e_i e'_j) = \delta_{ij} = \begin{cases} 0, & i \neq j; \\ 1, & i = j, \end{cases}$$

(Kronecker symbol). It is well-known that the dual basis always exists. We say that a basis is self-dual if it is its own dual. It is well-known that the self-dual basis always exists when $\text{char}(F_q) = 2$.

Now we consider the finite field $F_{2^t}$, we know that there exists a self-dual $F_2$-basis $\{e_1, \cdots, e_{2t}\}$ of $F_{2^t}$. Then for any element $\alpha$ in $F_{2^t}$, there exists a unique $2t$-tuple vector $\alpha^{(e)} = (\alpha_1, \cdots, \alpha_{2t}) \in F_2^{2t}$ such that $\alpha = \sum_{i=1}^{2t} \alpha_i e_i$. For any two elements $\alpha$ and $\beta$ of $F_{2^t}$, we have

$$\text{Tr}_{F_{2^t}/F_2}(\alpha \beta) = (\alpha^{(e)}, \beta^{(e)}) = \sum_{i=1}^{2t} \alpha_i \beta_i,$$

where $(\cdot)$ stands for the ordinary scalar product over $F_2$. Thus, we have a one-to-one correspondence $\rho$ between $F_{2^t}$ and $F_{2^{2t}}$ such that $\rho(a) = \rho(a_1, \cdots, a_n) = (a_1^{(e)}, \cdots, a_n^{(e)})$, where $a_i^{(e)} (1 \leq i \leq n)$ is a vector of length $2t$ over $F_2$ and $\text{Tr}_{F_{2^t}/F_2}(a \cdot b) = \text{Tr}_{F_{2^t}/F_2}(a_1 b_1 + \cdots + a_n b_n) = \sum_{i=1}^{n} (a_i^{(e)}, b_i^{(e)})$, where $(\cdot)$ stands for the ordinary scalar product over $F_{2^t}$. Thus we have

**Lemma 6:** Let $C$ be a self-orthogonal code over $F_{2^t}$, then $\rho(C)$ is a self-orthogonal code over $F_2$.

**Proof:** For any two codewords $\rho(a)$ and $\rho(b)$ of $\rho(C)$

$$(\rho(a), \rho(b)) = \sum_{i=1}^{n} (a_i^{(e)}, b_i^{(e)}) = \text{Tr}_{F_{2^t}/F_2}(a \cdot b) = 0$$

the last equality holds because $C$ is a self-orthogonal code.

To show our construction, we also need the result of self-orthogonal codes from [11]:

**Lemma 7:** ([11]) Let $q = t^2$ be a square. Then the class of self-orthogonal codes meet the Tsfasman-Vlăduţ-Zink bound.

More precisely, we have the following holds.

1. Let $0 \leq R \leq 1/2$ and $\delta \geq 0$ with $R = 1 - \delta - 1/(1-l)$.

Then there is a sequence $(C_j)_{j \geq 0}$ of linear codes $C_j$ over $F_q$ with parameters $[n_j, k_j, d_j]$ such that the following:

1. all $C_j$ are self-orthogonal codes;
2. $n_j \to \infty$ as $j \to \infty$;
3. $\lim_{j \to \infty} k_j/n_j \geq R$ and $\lim_{j \to \infty} d_j/n_j \geq \delta$.

**Remark 2:** The existence of the self-orthogonal codes in Lemma 7 is constructive. For the detail of the construction of this codes, we refer to [11].

Then by Lemma 7, we know that there exists a class of self-orthogonal codes over $F_{2^{2t}}$ which meet the Tsfasman-Vlăduţ-Zink bound. Now, we give the characterization of our construction.

**Proposition 2:** Let $C_i$ be a family of self-orthogonal codes over $F_{2^{2t}}$ which meets the Tsfasman-Vlăduţ-Zink bound with parameters $[n_i, k_i, d_i]$, i.e.,

$$\lim_{i \to \infty} \frac{k_i}{n_i} + \frac{d_i}{n_i} = 1 - \frac{1}{2^t - 1}.$$

Then $\rho(C_i)$ is a family of self-orthogonal codes over $F_2$ with parameters $[2tn_i, 2tk_i, d_i]$. Moreover, we have equation

$$R + 2t\delta = 1 - \frac{1}{2^t - 1},$$

where $R$ and $\delta$ denote the information rate and relative minimum distance, respectively, of the codes $\rho(C_i)$.

**Example 3:** Using this construction, we get the equations of $R$ and $\delta$ in Table II. The second column of Table II was calculated by [7]. In particular, it is easy to see that when we choose $t = 3$ and $R = 1/2$, we get the best value of $\delta$, $\delta \approx 0.0595$ from (7) (for asymptotic Gilbert-Varshamov bound, we have $\delta \approx 0.110$).

**IV. GILBERT-VARSHAMOV BOUND**

In this section, by mimicking the idea in [2], we give the proof that there exists a family of binary self-orthogonal codes achieving the Gilbert-Varshamov bound. For binary self-orthogonal code, it is easy to know that the weight of every codeword is even. Now we assume that the length of code $n$ is also an even number.
We first introduce two notations. Let $A$ be the set of self-orthogonal codes of length $n$ over $F_2$, and let $A_1$ denote the subset of $A$ consisting of all self-dual codes of length $n$ over $F_2$.

**Lemma 8:** ([2]) Let $n = 2h$ and, let $C$ be an $[n, s]$ binary self-orthogonal code. The number of codes in $A_1$, which contain $C$ is

$$(2^h - 1) (2^h - 1) \cdots (2^h - 1).$$

Let $\sigma_{n,k,s}$, $s \leq k < h$, be the number of self-orthogonal codes $D$ with parameters $[n, k]$ which contain the given code $C$. In the proof of Lemma 8, the authors establish a recursion formula for $\sigma_{n,k,s}$.

$$\sigma_{n,k,s} = \frac{2n - 2k - 1}{2^{k-s+1} - 1}.$$  \hspace{1cm} (8)

Then we have

**Corollary 1:** The number of codes in $A$ of dimension $k$ is

$$\left(\frac{2^n - 2^{k-1} - 1}{2^k - 1}\right)\left(\frac{2^n - 2^{k-2} - 1}{2^k - 1}\right)\cdots\left(\frac{2^n - 2 - 1}{2 - 1}\right).$$ \hspace{1cm} (9)

**Proof:** It is easy to know that every self-orthogonal code containing $\omega$ must contain the trivial code $0$. When $s = 0$, then $\sigma_{n,k,0}$ is the number we require. Using the recursion formula we get (9). \hspace{1cm} ■

**Corollary 2:** Let $v$ be a vector other than $0, 1$ with $wt(v) \equiv 0 \pmod{2}$. The number of codes in $A$ of dimension $k$ which contain $v$ is

$$\left(\frac{2^n - 2^{k-1} - 1}{2^k - 1}\right)\left(\frac{2^n - 2^{k-2} - 1}{2^k - 1}\right)\cdots\left(\frac{2^n - 2 - 1}{2 - 1}\right).$$ \hspace{1cm} (10)

**Proof:** It is easy to know that every self-orthogonal code containing the vector $v$ must contain the code $C$, where $C$ is the linear code with basis $\{v\}$. Then $\sigma_{n,k,1}$ is the number we require. Using the recursion formula we get (10). \hspace{1cm} ■

Using these two Corollaries, we have

**Theorem 1:** Let $r$ be a positive integer such that

$$\left(\frac{n}{2}\right) + \left(\frac{n}{4}\right) + \left(\frac{n}{6}\right) + \cdots + \left(\frac{n}{2(r-1)}\right) < 2^{n-1} - 2^{k-1}.$$ \hspace{1cm} (11)

Then there exists an $[n,k]$ self-orthogonal code with minimum distance at least $2r$.

**Proof:** The theorem is an immediate consequence of Corollaries 1 and 2. \hspace{1cm} ■

**Remark 3:** For any $0 \leq \delta \leq 1/2$, let $r = \lceil \frac{1}{2}\rceil$, then

$$k = \left\lfloor \log_2 \left( \frac{2^n - 1}{\left(\frac{n}{2}\right) + \left(\frac{n}{4}\right) + \cdots + \left(\frac{n}{2(r-1)}\right)} \right) \right\rfloor$$

satisfy (11), i.e., there exists an $[n,k,2r]$ binary self-orthogonal code and asymptotically, we have

$$\frac{k}{n} \to 1 - H_2(\delta).$$ \hspace{1cm} (12)

By Lemma 5, (12) implies that the binary self-orthogonal code meets the Gilbert-Varshamov bound.

**V. CONCLUSION**

Using these two constructions, we get a sequence of equations on $R$ and $\delta$. Then we get a constructive bound on $\sigma_{n,k,1}$ by combining the equations (5) and (7). We draw the figure of this bound in Fig. 1. When $R \to 0$, the constructive bound (5) is better than the constructive bound (7). When $R \to 1/2$, the constructive bound (7) is better than the constructive bound (5).

In Section IV, we proof that binary self-orthogonal codes meet the Gilbert-Varshamov bound, we also show the figure of this bound for self-orthogonal codes in Fig. 1.

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