Important remarks to Wolfenstein’s equation for passing neutrino through the matter

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Abstract

It is supposed that while neutrino passing through the matter a resonance enhancement of neutrino oscillations in the matter appears. It is shown that Wolfenstein’s equation, for neutrino passing through the matter, contains a disadvantage (does not take into account the law of momentum conservation). It leads, for example, to changing of the effective mass of neutrino by the value of $0.87 \cdot 10^{-2}eV$ from the very small value of the energy polarization of the matter caused by neutrino which is equal to $5 \cdot 10^{-12}eV$. After removing this disadvantage (i.e., taking into account this law) we have obtained a solution of this equation. In this solution a very small enhancement of neutrino oscillations in the matter appears due to the smallness of the energy polarization of the matter caused by neutrino.

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1 Introduction

The suggestion that, in analogy with $K^o, \bar{K}^o$ oscillations, there could be neutrino-antineutrino oscillations ($\nu \rightarrow \bar{\nu}$), was made by Pontecorvo [1] in 1957. It was subsequently considered by Maki et al. [2] and Pontecorvo [3] that there could be mixings (and oscillations) of neutrinos of different flavors (i.e., $\nu_e \rightarrow \nu_\mu$ transitions).

The first experiment [4] on the solar neutrinos has shown that there is a deficit of neutrinos, i.e., the solar neutrinos flux detected in the experiment was few times smaller than the flux computed in
the framework of the Sun Standard Model [5]. The subsequent experiments and theoretical computation have confirmed the deficit of the solar neutrinos [6].

The short base reactor and accelerator experiments [7] have shown that there is no neutrino deficit, then this result was interpreted as an indication that the neutrino vacuum angle mixing is very small (subsequent experiments have shown [8] that this vacuum angle is big and near the maximal value). Then the question appears: what is the deficit of the solar neutrinos related with? In 1978 the work by L. Wolfenstein [9] appeared where an equation describing neutrino passing through the matter was formulated (afterwards that equation was named Wolfenstein’s). In the framework of this equation the enhancement of neutrino oscillations in the matter arises via weak interactions (critical remarks to this equation see in [10]). This mechanism of neutrino oscillations enhancement in the matter attracted the attention of neutrino physicists after publications [11] by S. Mikheyev and A. Smirnov where it was shown that in the framework of this equation the resonance enhancement of neutrino oscillations in the matter would take place. Also it is clear that the adiabatic neutrino transitions can arise in the matter if effective masses of neutrinos change in the matter [12].

This work is devoted to discussion of neutrino oscillations in the matter by using the Wolfenstein’s type equation.

2 The neutrino (particle) passing through the matter

Before consideration of a neutrino (particle) passing through the matter it is necessary to gain some understanding of the physical origin of this mechanism. While neutrino passing through the matter there can be two processes- neutrino scattering and polarization of the matter by neutrino. Our interest is related with neutrino elastic interactions in the matter namely with neutrino forward elastic scattering, i.e., polarization of the matter by the passing neutrino. The neutrino passing
through the matter at its forward scattering can be considered by using the following Wolfenstein’s equation [9]

\[
i \frac{d\nu_{Ph}}{dt} = (E + \hat{W})\nu_{Ph} \equiv (\sqrt{p^2 I + \hat{M}^2 + \hat{W}})\nu_{Ph},
\]

where \(p, \hat{M}^2, \hat{W}_i\) are, respectively, the momentum, the (nondiagonal) square mass matrix in vacuum, and the matrix, taking into account neutrino interactions in the matter,

\[
\nu_{Ph} = \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}, \quad \hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

\[
\hat{M}^2 = \begin{pmatrix} m_{\nu_e\nu_e}^2 & m_{\nu_e\nu_\mu}^2 \\ m_{\nu_\mu\nu_e}^2 & m_{\nu_\mu\nu_\mu}^2 \end{pmatrix}, \quad \hat{W} = \begin{pmatrix} \hat{W}_e & 0 \\ 0 & \hat{W}_\mu \end{pmatrix}.
\]

This equation has standard solution which was found by S. P. Mikheyev and A. Ju. Smirnov [11] which come to resonance enhancement of neutrino oscillations in the matter. Now come to consider this decision.

2.1 The standard mechanism of resonance enhancement of neutrino oscillations in the matter and some critical remarks

In the ultrarelativistic limit \((E \simeq p\hat{I} + \frac{\hat{M}^2}{2p})\), the evolution equation for the neutrino wave function \(\nu_{Ph}\) in the matter has the following form [9], [11]:

\[
i \frac{d\nu_{Ph}}{dt} = (p\hat{I} + \frac{\hat{M}^2}{2p} + \hat{W})\nu_{Ph},
\]

where \(p, \hat{M}^2, \hat{W}_i\) are, respectively, the momentum, the (nondiagonal) square mass matrix in vacuum, and the matrix, taking into account neutrino interactions in the matter.

If we suppose that neutrinos in the matter behave analogously to the photon in the matter (i.e., the polarization appears while neutrino passing through the matter) and the neutrino refraction indices are defined by the following expression:

\[
n_i = 1 + \frac{2\pi N}{p^2} f_i(0) = 1 + 2\frac{\pi W_i}{p},
\]
where \( i \) is a type of neutrinos (\( e, \mu, \tau \)), \( N \) is density of the matter, \( f_i(0) \) is a real part of the forward scattering amplitude, then \( W_i \) characterizes the polarization of the matter by neutrinos (i.e. it is the energy of the matter polarization). In reality, as we will see below, there is fundamental difference: photon is a massless particle while neutrino is a massive particle and this distinction is fundamental.

The electron neutrino (\( \nu_e \)) in the matter interacts via \( W^\pm, Z^0 \) bosons and \( \nu_\mu, \nu_\tau \) interact only via \( Z^0 \) boson. These differences in interactions lead to the following differences in the refraction coefficients of \( \nu_e \) and \( \nu_\mu, \nu_\tau \):

\[
\Delta n = \frac{2\pi n_e}{p^2} \Delta f(0),
\]

\[
\Delta f(0) = \sqrt{2G_F} p,
\]

\[
E_{\text{eff}} = \sqrt{p^2 + m^2} + \langle e\nu|H_{\text{eff}}|e\nu\rangle \approx p + \frac{m^2}{2p} + \sqrt{2G_F n_e},
\]

where \( G_F \) is the Fermi constant.

Energy of the matter polarization \( E \) is

\[
E \approx W = \sqrt{2G_F n_e}, \quad W = 7.6 \left( \frac{n_e}{n_0} \right) \cdot 10^{-14} eV. \tag{5'}
\]

where \( G_F \) is the Fermi constant, \( n_e \) is the electron density in the matter. For the Sun

\[
E_{\text{SUN}} \approx 10^{-13} \div 10^{-11} eV. \tag{5''}
\]

Therefore the velocities (or effective masses) of \( \nu_e \) and \( \nu_\mu, \nu_\tau \) in the matter are different. And at the suitable density of the matter this difference can result in resonance enhancement of neutrino oscillations in the matter \[9], [12]. Expression for \( \sin^2 2\theta_m \) in the matter has the following form:

\[
\sin^2 2\theta_m = \sin^2 2\theta \cdot \left[ (\cos 2\theta - \frac{L_0}{L^0})^2 + \sin^2 2\theta \right]^{-1}, \tag{6}
\]
where \( \sin^2 2\theta_m \) and \( \sin^2 2\theta \) characterize neutrino mixings in the matter and vacuum, \( L_0 \) and \( L^0 \) are lengths of neutrino oscillations in vacuum and neutrino refraction length in the matter:

\[
L_0 = \frac{4\pi E_\nu \hbar}{\Delta m^2 c^3} \quad L^0 = \frac{\sqrt{2}\pi \hbar c}{G_F n_e},
\]

(7)

where \( E_\nu \) is neutrino energy, \( \Delta m^2 = m_2^2 - m_1^2 \) - difference between squared neutrino masses, \( c \) is light velocity, \( \hbar \) is Plank constant, \( G_F \) is Fermi constant and \( n_e \) is electron density of the matter.

Probability of \( \nu_e \to \nu_\mu \) neutrino transitions is given by the following expression (\( E \simeq pc \)):

\[
P(E, t, ...) = 1 - \sin^2 2\theta_m \sin \frac{2\pi ct}{L_m},
\]

(8)

where \( L_m = \frac{\sin 2\theta_m}{\sin 2\theta} L_0 \).

At resonance

\[
\cos 2\theta \simeq \frac{L_0}{L^0} \quad \sin^2 2\theta_m \simeq 1 \quad \theta_m \simeq \frac{\pi}{4}.
\]

(9)

The expression (9) for resonance condition can be rewritten in the following form:

\[
\sqrt{2}G_F n_e = \frac{\Delta m^2}{2E_{\nu e}^{\text{res}}} \cos 2\theta,
\]

(10)

or

\[
E_{\nu e}^{\text{res}} = \frac{\Delta m^2 \cos 2\theta}{2W} \quad \rightarrow \Delta m^2 - \frac{2E_{\nu e}^{\text{res}} W}{\cos 2\theta} = 0.
\]

(11)

If we consider \( \nu_e \to \nu_\mu \) and use KamLAND data [13]

\[
tan^2 \theta_{12} = 0.56(+0.10, -0.07)(\text{stat})(+0.1, -0.06)(\text{syst}), \quad \theta = 36.8^o,
\]

\[
\Delta m^2_{12} = 7.58(+0.14, -0.13)(\text{stat}) \pm 0.15(\text{syst}) \times 10^{-5}eV^2,
\]

(12)

then at \( n_e = 65.8n_o \) energy \( W^{\text{Sun}} \) of neutrino polarization is \( W^{\text{Sun}} = 5 \times 10^{-12}eV \) and for \( E_{\nu e}^{\text{res}} \) we obtain

\[
E_{\nu e}^{\text{res}} = 2.14 \times 10^6eV = 2.14 MeV.
\]

(13)

The expressions (9)-(13) mean that when the electron neutrino with energy \( E_\nu = 2.14 \ MeV \) is passing through the Sun matter, the effective mass of electron neutrino becomes equal to the muon neutrino.
mass (see below expressions (23),(23')) and as result there is resonance transition of electron neutrinos into muon neutrinos. In this case changing of the squared mass of neutrino $\nu_1$ is $\Delta m_{12}^2 = 7.58 \times 10^{-5} eV^2$ (we suppose that $m_2 > m_1$), i.e., effective mass of neutrino $\nu_1$ is

$$m_{1,\text{eff}}^2 \simeq m_1^2 + 7.58 \times 10^{-5} eV^2,$$

and (in reality $m_{\nu_e}^{\text{matt}} \simeq m_{\nu_\mu}$, see expression (23),(24))

$$m_{1,\text{matt}}^2 \approx m_2^2.$$  \hspace{1cm} (14)

We see that this additional big mass arises at polarization of the matter by neutrino with energy $W = 5 \times 10^{-12} eV$! It is a very strange result. A primary ultrarelativistic electron neutrino having energy $E_\nu = 2.14 \times 10^6 eV$ interacts with the matter with the energy $W = 5 \times 10^{-12} eV$ and as a result we obtained the mass increase on $\delta m \approx \sqrt{7.58 \times 10^{-5}} = 0.87 \times 10^{-2} eV$. We know that the matter polarization has to move with the velocity equal to the neutrino velocity which generates this polarization. Then the energy of the electron neutrino has to increase on

$$\Delta E_\nu \approx \delta m \gamma,$$

where $\gamma = \frac{E_\nu}{m_\nu}$ and neutrino velocity $v \simeq c$. Why did we come to this result? It is a consequence of the above used approach when we included the full energy of the matter polarization in the neutrino mass. It is possible only at a serious violation of the law of energy-momentum conservation. We can suppose that increasing of neutrino effective mass is accompanied by the decreasing of neutrino velocity, i.e., the energy is conserved but it is not save the situation since the mass increasing is $\delta m \approx \sqrt{7.58 \times 10^{-5}} = 0.87 \times 10^{-2} eV$ while the energy of matter polarization is $W = 5 \times 10^{-12} eV$). It means that the mechanism of resonance enhancement of neutrino oscillations in the matter can be realized at violation the law of energy-momentum conservation. The approach which was considered above becomes physically realizable only when $p^2 \ll M^2$ and then full energy of the matter polarization is transformed in neutrino mass, but the neutrinos produced in weak interactions are relativistic since the neutrino masses are very small.
As it was stressed above the approach which was used should work in a non-relativistic case when $p^2 \ll M^2$ but not in the ultrarelativistic case. Then expression for neutrino energy will have the form
\[ \sqrt{p^2 + M^2} + W = \sqrt{p^2 + M'^2} \rightarrow M'^2 = M^2 + 2W\sqrt{p^2 + M^2} + W^2, \] (17)
then taking into account that $p^2 \ll M^2$, $W^2 \ll M^2$ we obtain
\[ M'^2 \simeq (M + W)^2. \] (18)
Then the Wolfenstein’s equation (1) can be written in the following form:
\[ i \frac{d\nu_{ph}}{dt} = \sqrt{p^2 I + (\hat{M} + \hat{W})^2} \nu_{ph} \equiv \sqrt{p^2 I + (\hat{M}')^2} \nu_{ph} \simeq (\hat{M}' + \frac{p^2}{2M'^2})\nu_{ph}, \] (19)
Then we see that, at low energies we can include the full energy $W$ of the matter polarization in the neutrino mass and expressions (6) ÷ (11) will be replaced by expressions (17) ÷ (19). Then by diagonalization of mass the matrix $M'$ we obtain that the neutrino mixing angle change and at appropriate conditions (at enough large value of $W$) there will take place resonance enhancement of neutrino oscillations in the matter. It is necessary to remark that the full energy of matter polarization $E_{Sun} \approx 10^{-13} \div 10^{-11}$ eV which arise at passing electron neutrino through the Sun is too small to generate a resonance enhancement of neutrino oscillations. Above we supposed that masses can be generate in the framework of the standard weak interactions, i.e., there is no problem with mass generation.

Now we consider common case where the total energy of the matter polarization is included in neutrino kinetic energy and mass.

2.2 The common case of the neutrino passing through the matter

In [14] a common method was developed to avoid the above paradox when from the very small energy arises huge mass change. The example which we consider is very simple, therefore it will be sufficient to take into account the law of momentum conservation besides the law of energy conservation.
Above we have considered the case when full energy of the matter polarization caused by neutrino is included in the mass. If the particle (neutrino) interaction with the matter is the left-right symmetric one then the mass can be generated there (as it takes place in strong and electromagnetic interactions). In this case we have to share the full energy of the matter polarization caused by the particle (neutrino) between the kinetic and mass parts of the particle (neutrino) energy. We will suppose that the weak interactions are the left-right symmetric ones and then we will not consider the problem of mass generation in the weak interactions.

To solve this problem, it is necessary to compute full energy \( W \) of the matter polarization and then taking into account the law of energy-momentum conservation in the vacuum \((p, M)\) and in the matter \((p', M')\), - to distribute this full energy of polarization between the kinetic and mass parts of the particle (neutrino) energy. It coincides with the problem of polaron for a certain interaction (for references see Wikipedia). So in the matter

\[
E' = E + W, \quad (20)
\]
and \((p_W = W v_\nu)\)

\[
p' = p + p_W, \quad (21)
\]

since \(p^2 \gg M^2\) then neutrino is ultrarelativistic particle and \(v_\nu \simeq c\) (\(c\) is the light velocity) then \(p_W \simeq W\).

\[
p' \simeq p + W. \quad (22)
\]

Then the expression for neutrino energy in the matter will have the following form:

\[
\sqrt{p^2 + M^2} + W = \sqrt{p'^2 + M'^2} \rightarrow
\]
\[
\rightarrow M'^2 + p'^2 = M^2 + 2W \sqrt{p^2 + M^2} + W^2, \quad (23)
\]

where \(p' \simeq p + W\). Then using expressions (20), (22) and taking into account that \(p^2 \gg M^2\) from the expression. (23) we obtain

\[
M'^2 - M^2 \simeq W p (\frac{M^2}{p^2}). \quad (24)
\]
If to take into account that \( p^2 \gg M^2 \) and \( W \approx 10^{-12} \), then

\[
M'^2 \simeq M^2 + W p \left( \frac{M^2}{p^2} \right) \simeq M^2.
\] (25)

In this case the Wolfenstein’s equation has the same form as equation (1) since term \( W \) originated from the left-right symmetric interaction is inserted in this equation with the left-right symmetric wave function.

\[
i \frac{d\nu_{Ph}}{dt} = (\sqrt{p'^2 + M'^2} \nu_{Ph} \rightarrow (p' \hat{I} + \frac{M'^2}{2p'})\nu_{Ph} \rightarrow
\]

\[
\rightarrow (p' \hat{I} + \frac{M^2 + W p (\frac{M^2}{p^2})}{2p'})\nu_{Ph},
\] (26)

or taking into account that the term \( W p (\frac{M^2}{p^2}) \) is very small

\[
i \frac{d\nu_{Ph}}{dt} = (p' I + \frac{M^2}{2p'})\nu_{Ph},
\] (27)

where \( p' = (p + W) \). The expression for the neutrino transition probability in this case has the following form:

\[
P(E', t, ...) = 1 - \sin^2 2\theta' \sin \frac{2\pi ct}{L''_o},
\] (28)

where \( E' \simeq p' c \) and \( L''_o = \frac{\sin 2\theta'}{\sin 2\theta} L'_o \), since \( M'^2 \simeq M^2 \) then \( \sin \theta \simeq \sin \theta' \), \( L''_o \approx L'_o \)

\[
L'_0 = \frac{4\pi E'_0 \hbar}{\Delta m^2 c^3}, \quad \sin^2 2\theta' = \sin^2 2\theta.
\] (29)

So, since changing of the neutrino effective mass is very small then changing of the neutrino transition probability arises only owing to the neutrino momentum change. It is necessary to take into account that \( p \gg W \), then this changing will be also very small. We have come to the following conclusion: Taking into account not only the law of neutrino energy conservation but as well as the law of neutrino momentum conservation, the neutrino transition probability in the matter change is very small and noticeable enhancement of the neutrino oscillations in the matter does not appear (i.e., the condition (9) cannot be fulfilled). We see that, the term which generates huge
changing of the neutrino effective mass and leads to the resonance enhancement of neutrino oscillations in the matter, appears since the law of momentum conservation was not taken into account in the original Wolfenstein’s equation.

3 Conclusion

The Wolfenstein’s equation is used to describe the neutrino (particle) passing through the matter. Though this equation was obtained to describe the neutrino passing through the matter (by weak interactions which are left-side interactions) but it is a Schrodinger’s type of equation and therefore this equation is one for left-right symmetric wave function and, correspondingly, it is valid for the left-right symmetric interactions.

It is supposed that while neutrino passing through the matter a resonance enhancement of neutrino oscillations in the matter appears. It is shown that Wolfenstein’s equation, for neutrino passing through the matter, contains a disadvantage (does not take into account the law of momentum conservation). It leads, for example, to changing of the effective mass of neutrino by the value of $0.87 \cdot 10^{-2} eV$ from the very small value of the energy polarization of the matter caused by neutrino which is equal to $5 \cdot 10^{-12} eV$. After removing this disadvantage (i.e., taking into account this law) we have obtained a solution of this equation. In this solution a very small enhancement of neutrino oscillations in the matter appears due to the smallness of the energy polarization of the matter caused by neutrino.

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