Weak measurement and control of entanglement generation

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In this paper we show how weak measurement and local feedback can be used to control entanglement generation between two qubits. To do this, we make use of a decoherence free subspace (DFS). Weak measurement and feedback can be used to drive the system into this subspace rapidly. Once within the subspace, feedback can generate entanglement rapidly, or turn off entanglement generation dynamically. We also consider, in the context of weak measurement, some of differences between purification and generating entanglement.

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Entanglement is a fundamental resource used in many quantum information applications [1]. This paper considers how weak measurements and local feedback can be used to control the generation of entanglement between two qubits. There are three main points that this paper makes. First, it is possible to speed up the generation of entanglement using feedback. Second, it is possible to start and stop entanglement production. Third, the protocol for increasing the rate purification is not necessarily the same as increasing the rate of entanglement.

Control of entanglement generation can be achieved with the aid of a decoherence free subspace (DFS) [2,3,4,5] in which the measurement no longer produces any useful information about the state of the quantum system - a measurement free subspace. Weak measurement and feedback is used to (i) force the system into the subspace, and (ii) to purify the system using the structure of the subspace. The technique proposed here is analogous of existing single qubit rapid state reduction protocols in each of the subspaces defined by the measurement [6,7,8]. The second process is particularly interesting because it takes the system from a classically correlated state to a maximally entangled Bell state, generating entanglement more rapidly than would be achieved without feedback. Using the structure of the DFS, entanglement generation may also be switched on and off dynamically, without turning off the measurement interaction (an important factor in many solid state qubits). This control may be achieved by applying local Hamiltonian feedback alone. Together local unitary control and weak measurement allow rapid generation of entanglement and control of the level of entanglement which is ultimately generated.

Weak measurement makes it possible to modify the evolution continuously via Hamiltonian feedback, where the Hamiltonian feedback applied to the system depends on the measurement record [9,10,11,12,13,14,15,16,17,18,19]. Hamiltonian feedback during measurement not only affects the final state of the system, but also the measurement process itself. For example it can affect the rate of state reduction/purification. In a protocol described by Jacobs for a single qubit, the average rate of state reduction (as measured by the purity) for a single qubit can be maximized by feedback [6]. This process is known as rapid state reduction [6], or as rapid purification [6]. Jacobs’ protocol is deterministic, but other protocols exist which are stochastic and minimize the average time for a single qubit to reach a given purity [8]. Combes and Jacobs demonstrated that similar feedback can be applied to higher dimensional systems to increase the average increase in purity [8]. More recently Wiseman has proved that these protocols are optimal [10]. These methods have also been applied to bipartite systems [20].

In this paper we consider two continuously monitored qubits. In contrast to previous protocols which aim to increase the rate of purification of the system, we concentrate on speeding up the generation of entanglement between the two qubits. These two goals are compatible. It is possible to view the entanglement generating technique as a modification of the previously existing purification protocols [6,7], applied to an logical qubit.

Generation of entanglement via measurement should also be contrasted with generation through Hamiltonian control alone [21,22]. In this paper the weak measurement is responsible for entanglement generation between the systems which are not directly interacting via an entangling Hamiltonian, and the control presented here is closed loop, rather than open loop.

Weak measurement is modeled by a stochastic master equation (SME). The SME is obtained by introducing an ancilla system, weakly coupled to the system of interest. The auxiliary system then undergoes a measurement giving a stochastic result, and is then traced out. This leaves only the system of interest, ρ, and a stochastic measurement record r(t) which can be used to construct an estimate of the state of the system, ρ. This is often referred to as an ‘unraveling’ of the master equation [6,14,17,23,24].

The stochastic master equation (SME) which governs the evolution of the density operator ρ in the presence of

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a weak measurement of a Hermitian observable, \( y \), in the absence of a driving Hamiltonian is given by

\[
dp = -k[y, [y, \rho]] dt + \sqrt{2k}(yp + ry - 2\langle y \rangle \rho) dW, \tag{1}
\]

where \( k \) is the measurement strength. The first term in this equation describes the familiar drift towards the measurement axis. The second term in the equation is weighted by \( dW \), a Wiener increment with \( dW^2 = dt \). This term describes the update of knowledge of the density matrix conditioned on the measurement record [13].

We consider a specific case: the observable, \( \hat{y} = ZZ \), which is continuously monitored on the system. \( Z(X,Y) \) is short hand for the Pauli matrix, \( \sigma_3(\sigma_1,\sigma_2) \) and the tensor product is implied (ie \( ZZ = Z \otimes Z \)). This parity measurement may arise when the measurement device is coupled to both qubits \([23, 26, 27, 28]\). Theoretically it is arguably the simplest measurement which can be performed on the system, only giving information about whether the spins are aligned or anti-aligned. It is also the type of measurement envisioned for error-correction codes \([29]\). The measurement record,

\[
dr(t) = \langle y \rangle dt + \frac{dW}{\sqrt{8k}} \tag{2}
\]

will depend not on the reduced state of one qubit or the other, but on the correlations between the two qubits.

In this paper we limit ourselves to applying only local Hamiltonian feedback to one qubit or the other. This is a purely practical constraint; many quantum systems exhibit local unitary control.

As a measurement of \( y = ZZ \) is made, the system evolves according to the stochastic master equation:

\[
d\rho = -k[ZZ, [ZZ, \rho]] dt + \sqrt{2k}(ZZ\rho + \rho ZZ - 2\langle ZZ \rangle \rho) dW. \tag{3}
\]

The density matrix can be expanded in the Pauli basis

\[
\rho = \sum_{i,j=x,y,z} r_{ij} \sigma_i \sigma_j, \tag{4}
\]

in which \( d \) is the dimension of the system (that is \( d = 4 \)) and the coefficients, \( r_{ij} \), may be found by

\[
r_{ij} = \text{Tr}(\sigma_i \sigma_j \rho),
\]

which should not be confused with the measurement record, \( r(t) \).

Previous schemes using weak measurement and feedback have often concentrated on the purity of the system. The purity of the system is given by \( P(\rho) = \text{Tr}(\rho^2) \). A completely pure state has a purity of \( 1/d \), and a completely mixed state has a purity of \( 1/d \). For a state which is confined to a particular subspace of the system, but completely mixed within that subspace, the purity is given by \( 1/d_* \), where \( d_* \) is the dimension of the subspace.

In the case of a measurement of \( ZZ \) the total purity of the system evolves according to the equation,

\[
dP = 8k \sum_{ij} (r_{ZZ} \sigma_i \sigma_j - r_{ZZ} r_{ij})^2 - \sum_{mn} (1 - r_{ZZ}^2) r_{mn}^2 \langle y \rangle \rho dt + 4\sqrt{2k} \sum_{ij} (r_{ZZ} \sigma_i \sigma_j - r_{ZZ} r_{ij}) r_{ij} = \sum_{mn} r_{mn}^2 r_{ZZ} dW
\]

where \( m \) and \( n \) range over all the Pauli matrices which anti-commute with \( ZZ \), and \( i \) and \( j \) range over all the Pauli matrices which commute with \( ZZ \).

In order to quantify the correlations between the qubits, we will use the value \( R_2^2 = \sum_{i,j=x,y,z} r_{ij}^2 \). \( R_2^2 \) has a maximum value of 3 when the system is in a maximally entangled state (a pure Bell state). \( R_2^2 \) has a minimum of 0 when there are no classical correlations between the states of the two qubits. \( R_2^2 \leq 1 \) for a product state. For a mixed state, increasing \( R_2^2 \) to its maximum value leads to an increase in both purity and entanglement in the qubits. \( R_2^2 \) is invariant under single qubit rotations of either system. It is similar to purity, for which rapid purification protocols are available. It is simple to calculate, monitor and to obtain analytic results for.

We now show how this two-qubit SME can be applied to decoherence free subspaces (DFS). DFS were introduced as a way to protect fragile quantum information passively from the effects of an interaction with the environment [2, 3, 4, 5]. Under simple assumptions about symmetries of the coupling between the system and the environment, there are subspaces of the overall Hilbert space which remain unaffected by the interaction of the system and its environment. Therefore any information encoded in this subspace is protected from decoherence.

We now consider the system studied in this paper, where the interaction is given by \( y = ZZ \). In this case, there are clearly two degenerate eigenvalues, +1 and −1. The corresponding measurement free subspaces are given by \( D_+ = \text{Span}\{|00\rangle, |11\rangle\} \) and \( D_- = \text{Span}\{|01\rangle, |10\rangle\} \). Consider the stochastic master equation (1) acting on a state, \( \rho \), restricted to a DFS (either \( D_+ \) or \( D_- \)). A DFS is found by finding degenerate eigenstates of error generators, \( F_{ax} \), as well as a measurement, \( y \). Then by definition each of the basis vectors of the DFS are degenerate eigenstates of \( y: y|i\rangle = c_y|i\rangle \). This means that \( [y, \rho] = 0 \) and that for states restricted to the DFS \( \langle y \rangle \rho = \text{Tr}[y|\rho\rangle] = c_y \). Therefore if \( \rho \) is restricted to a DFS whose error generators include the measurement operator, \( dp = 0 \). There is no change in the conditional density matrix \( \rho \) according to the stochastic master equation.

Once restricted to a measurement free subspace, further measurement of \( y = ZZ \) yields no useful information about the system. However, by applying local rotations to the system, it may be rotated outside the DFS where measurement provides further useful information. If Hadamard (H) gates (a local rotation by \( \pi \) around \( X + Z)/\sqrt{2} \)) are applied locally to each of the two qubits then the system will be rotated out of the DFS. Since \( H \otimes H \otimes Z \otimes Z \otimes H \otimes H \) commutes with \( Z \otimes Z \), we will always be able to rotate back to the original subspace. Therefore it is possible to turn the measurement of the system on and off by rotating it into and out of the DFS, even when the measurement apparatus is always interacting with the system. This can be performed using local rotations alone - Hadamard is a local unitary operation.

Often the goal of measurement is to prepare an exper-
mental system in a given state, for example: to prepare states of a given purity. The protocol described here could be applied to a large ensemble of qubit pairs. If the purity were monitored continuously, then the measurement could be turned off dynamically once it reached the desired value (the time of first passage) $t$. Eventually all the qubits will have reached the threshold, and would be held in their DFS. This would allow a large number of systems to be prepared simultaneously.

We now show how the system can be viewed as two encoded qubits. Each encoded qubit has a state whose evolution depends on the collective behaviour of both physical qubits. The first encoded qubit represents the extent to which information is found within one subspace, $D_+$, or the other $D_-$. The second encoded qubit represents the information protected within the DFS. We will then adapt Jacobs’ protocol for a single qubit to apply to physical qubits, but to the encoded qubits.

First, we will describe in more detail how the system can be viewed as two encoded qubits. There is a natural way to divide the space to represent an encoded qubit. Elements of the commutant of $ZZ$ (eg. $X \otimes X$) leave the system inside the DFS (eg. $D_+$). Therefore they can be considered as operating on an encoded qubit which is immune from measurement of $y = ZZ$.

Similarly physical operations which do not commute with $ZZ$ can be considered as rotations of a separate encoded qubit. For example, $X \otimes Z$ could be considered an encoded operation applying an $x$-rotation to the first of the two encoded qubits. In fact, it is possible to construct an entire encoded Bloch sphere. Each

$$H_{ZZ} = Z_L \otimes I_L, \quad H_{XZ} = X_L \otimes I_L, \quad H_{YI} = Y_L \otimes I_L, \quad (6)$$

can be considered as designating an encoded Bloch sphere on the first encoded qubit. This qubit represents the extent to which the the system is confined to the DFS, with the states $|0_L\rangle$ and $|1_L\rangle$ representing states which are entirely confined to one of the spaces or the other.

We apply Jacobs’ single qubit protocol to the encoded qubit $|0\rangle$. Jacobs showed that the fastest average rate of purification was obtained when the Bloch vector was rotated away from the measurement axis. In this case, the encoded qubit is confined to the $X \otimes Z$ axis, by measuring $Z \otimes Z$ and applying local feedback of rotations around the $Y \otimes I$ axis. That is, feedback is only applied locally, to the first of the two physical qubits.

By applying Hamiltonian feedback it is possible to continually rotate the state of the encoded qubit so that it lies along the $X \otimes Z$ axis. In this case, the average rate of purification is largest on average for the encoded qubit. Purifying the first qubit is equivalent (up to a local rotation) to confining the system to a decoherence free subspace. So rapidly purifying the state of the first encoded qubit rapidly forces the system into the DFS.

Another advantage of Jacobs’ protocol is that the rate of purification is deterministic. If the feedback is perfectly applied, the purity of the system can be predicted. A large number of systems undergoing purification to the DFS would all be purified at the same rate, and would achieve the same level of purity at the same time.

Once the system has been purified to the DFS, it is in a classically correlated state. Further purification takes the state to a maximally entangled Bell state. This process is achieved more rapidly with feedback than without.

The states described in this section represent the information encoded within the DFS. In order to operate of the state of a protected qubit, we choose a separate set of encoded operations:

$$K_{XX} = I_L \otimes X_L, \quad K_{XY} = I_L \otimes Y_L, \quad K_{IZ} = I_L \otimes Z_L. \quad (7)$$

Here each rotation is chosen from the commutant, $A'$. In other words, every operation commutes with $y$.

If it was possible to measure $X \otimes X$ directly, and apply Hamiltonian feedback, $I \otimes Z$, to the second physical qubit then we would be able to apply Jacobs’ protocol to rotate the state onto the $X \otimes Y$ axis. Seen in the encoded space, this is simply another application of Jacobs’ rapid purification method applied to the second encoded qubit. Measurement of $X \otimes X$ is equivalent to Hadamard gates applied in parallel to both physical qubits, followed by measurement of $Z \otimes Z$. Local Hamiltonian feedback can be applied by performing $x$-rotations on the second physical qubit to rotate the state the $ZY$ axis. This scheme requires only local unitary operations, and measurement of $Z \otimes Z$.

This is shown in Figure I. In this case, we are going from a classically correlated state in the DFS to a maximally entangled Bell state, and so this process could be viewed as rapid entanglement of the two qubits. It is evident that the average rate of increase in purity with feedback is faster than without feedback.

The protocols which provide the fastest rate of increase of purity are not necessarily the same as those that pro-
vide the fastest rate of increase of entanglement. This is most easily seen by considering the two encoded qubits. Consider the simplest case, when the encoded qubits are separable, and the second encoded qubit is not in a pure state. In this case, to increase the purity of the system a measurement should be made to the second encoded qubit. However, this does not necessarily lead to an entangled state. In particular if the first encoded qubit is in the state $|0_Z\rangle$ (that is $(Z \otimes Z) = 1$), and the second encoded qubit is being purified along the $I \otimes Z$ axis, then this scheme gives the greatest increase in purity, but the entanglement of the system stays the same (ie. 0). Conversely if the second qubit is not quite pure, but has $(X \otimes X) \approx 1$ and the first qubit has $(Y \otimes I) = 1$ then measurements on the first encoded qubit will not change the purity of the system, however they can be used (as described in this paper) to force the two physical qubits towards an entangled state such as $(Z \otimes Z) = \pm 1$. In that case, the entanglement of the system increases, but the purity remains the same. Therefore the protocols which should be applied to increase the level of entanglement in a system are not necessarily the same as those which have been proposed to increase the level of purity.

In this paper we have considered how it is possible to use local measurement and an always-on measurement to control the entanglement of a bipartite quantum system. First we considered if it is possible to turn off the entangling effect of measurement using local rotations alone. We also showed that it is possible to start and stop entanglement production, using well known decoherence free subspaces. We showed how it was possible to guide the system into the decoherence free subspace using feedback. With feedback this operation proceeds faster than is achieved without feedback.

Once in the decoherence free subspace, we showed that it is possible to speed up the generation of entanglement using feedback. We achieved this by taking the system from a classically correlated state to a maximally entangled Bell state. This process proceeds faster, on average, when it is possible to apply local qubit feedback based on the measurement record. The feedback required is simple. It depends on only two parameters, and can be applied using local Hamiltonians alone. We refer to this process as rapid entanglement. Both of these processes could be seen as applying existing purification protocols to encoded qubits, rather than physical qubits. However, as we showed, purification is quite different from generating entanglement. In particular it is possible to speed up the generation of entanglement (whether using feedback or not) without purifying the system.

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