A NOTE ON RELATIVE AMENABILITY

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Abstract. P-E. Caprace and N. Monod isolate the class \( X \) of locally compact groups for which relatively amenable closed subgroups are amenable. It is unknown if \( X \) is closed under group extension. In this note, we exhibit a large, group extension stable subclass of \( X \), which suggests \( X \) is indeed closed under group extension. Along the way, we produce generalizations of the class of elementary groups and obtain information on groups outside \( X \).

1. Introduction

In [1], P-E. Caprace and N. Monod introduce the notion of relative amenability.

Definition 1.1. For a locally compact group \( G \), a closed subgroup \( H \leq G \) is relatively amenable if \( H \) fixes a point in every non-empty convex compact \( G \)-space.

A convex compact \( G \)-space is a convex compact subset of a locally convex topological vector space such that the subset has a continuous affine \( G \) action.

They go on to study the relationship between amenability and relative amenability. In particular, they isolate a large and interesting class of locally compact groups.

Definition 1.2. The class \( X \) is the collection of locally compact groups for which every relatively amenable closed subgroup is amenable.

Theorem 1.3 (Caprace, Monod [1] Theorem 2).

(a) \( X \) contains all discrete groups.
(b) \( X \) contains all groups amenable at infinity.
(c) \( X \) is closed under taking closed subgroups.
(d) \( X \) is closed under taking (finite) direct products.
(e) \( X \) is closed under taking adelic products.
(f) \( X \) is closed under taking directed unions of open subgroups.

Let \( N \trianglelefteq G \) be a closed normal subgroup of a locally compact group \( G \).

(g) If \( N \) is amenable, then \( G \in X \iff G/N \in X \).
(h) If \( N \) is connected, then \( G \in X \iff G/N \in X \).
(i) If \( N \) is open, then \( G \in X \iff N \in X \).
(j) If \( N \) is discrete and \( G/N \in X \), then \( G \in X \).
(k) If \( N \) is amenable at infinity and \( G/N \in X \), then \( G \in X \).

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Two questions concerning the class $\mathcal{X}$ arise.

**Question 1.4** (Caprace, Monod).

1. Is $\mathcal{X}$ stable under group extension?
2. Are there locally compact groups outside of $\mathcal{X}$?

Our note contributes primarily to the study of the former. Indeed, let $\mathcal{Y}$ be the smallest collection of locally compact groups such that

1. $\mathcal{Y}$ contains all compact groups, discrete groups, and connected groups,
2. $\mathcal{Y}$ is closed under group extensions, and
3. $\mathcal{Y}$ is closed under directed unions of open subgroups. That is to say if $G = \bigcup_{i \in I} O_i$ where $\{O_i\}_{i \in I}$ is a directed system of open subgroups of $G$ such that $O_i \in \mathcal{Y}$ for each $i$, then $G \in \mathcal{Y}$.

We prove the following.

**Theorem 1.5.** The class $\mathcal{Y}$ is contained in $\mathcal{X}$ and enjoys the following additional permanence properties:

(a) $\mathcal{Y}$ is closed under taking closed subgroups.
(b) $\mathcal{Y}$ is closed under taking quotients by closed normal subgroups.

This provides evidence the class $\mathcal{X}$ may be stable under group extensions. Furthermore, it gives additional information on potential groups $G \notin \mathcal{X}$, see Remark 5.4 below.

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## 2. Generalities on locally compact groups

All groups are taken to be Hausdorff topological groups. We abbreviate “locally compact” by “l.c.”, “totally disconnected” by “t.d.”, and “second countable” by “s.c.”. We write $H \leq G$, $H \leq_o G$ and $H \leq_{cc} G$ to indicate $H$ is a closed subgroup, an open subgroup and a closed cocompact subgroup of $G$, respectively. We denote by $S(G)$ and $U(G)$ the collection of closed subgroups of $G$ and the collection of compact open subgroups of $G$, respectively.

We make frequent use of the first isomorphism theorem for topological groups (see [5 (5.33)]), whose statement is the following.

> Let $G$ be a locally group, and let $A \leq G$ and $H \leq G$ be closed. If $A$ is $\sigma$-compact and $AH$ is closed, then $AH/H \simeq A/A \cap H$ as topological groups.

An old topological result for t.d.l.c. groups is also frequently invoked.

**Theorem 2.1** (Kakutani, Kodaira, see [5 (8.7)]). If $G$ is a $\sigma$-compact t.d.l.c. group, then there is a compact $K \leq G$ such that $G/K$ is metrizable, hence second countable.
We require a few additional results. A t.d.l.c. group $G$ is said to be a **small invariant neighborhood group**, denoted SIN, if $G$ admits a basis at 1 of compact open normal subgroups. These groups when compactly generated admit a useful characterization.

**Theorem 2.2** (Caprace, Monod [2, Corollary 4.1]). A compactly generated t.d.l.c. group is SIN if and only if it is residually discrete.

The **discrete residual** of a t.d.l.c. group $G$, denoted Res($G$), is the intersection of all open normal subgroups. When $G$ is compactly generated, Theorem 2.2 implies $G/\text{Res}(G)$ is a SIN group.

A l.c. group is **locally elliptic** if every compact subset is contained in a compact subgroup. V. P. Platonov shows in [8] every l.c. group $G$ has a greatest locally elliptic normal subgroup, denoted by $\text{Rad}_{LE}(G)$ and called the **locally elliptic radical** of $G$. Additionally, $\text{Rad}_{LE}(G)$ is closed and the quotient $G/\text{Rad}_{LE}(G)$ has trivial locally elliptic radical.

The following result provides a criterion for $\text{Rad}_{LE}(G)$ to be open in a t.d.l.c. group $G$. The latter condition implies that $G$ belongs to $\mathcal{F}$, as a consequence of Theorem 1.3.

**Proposition 2.3.** Let $G$ be a t.d.l.c. group. If $N \triangleleft G$ is compactly generated, and if $N$ and $G/N$ both have an open locally elliptic radical, then $G$ has an open locally elliptic radical.

**Proof.** The radical $\text{Rad}_{LE}(N)$ is characteristic in $N$, hence contained in $\text{Rad}_{LE}(G)$. Upon replacing $G$ by $G/\text{Rad}_{LE}(N)$, we may therefore assume $N$ is discrete and finitely generated. It follows the centralizer $C_G(N)$ is an open normal subgroup of $G$. Let $Q$ be the inverse image of $\text{Rad}_{LE}(G/N)$ in $G$, so $Q$ is also an open normal subgroup of $G$. It suffices to show $P = Q \cap C_G(N)$ has an open locally elliptic radical.

Notice $P \cap N$ is contained in the center of $P$ and that $P/P \cap N \cong PN/N \leq Q/N$ is locally elliptic by construction. Therefore, $P$ is a central extension of a locally elliptic group. By [3] Proposition 2.3], this implies $P/\text{Rad}_{LE}(P)$ is abelian, with trivial locally elliptic radical. Since every compact open subgroup of an abelian t.d.l.c. group is contained in its locally elliptic radical, we infer $P/\text{Rad}_{LE}(P)$ is discrete, thereby confirming that $\text{Rad}_{LE}(P)$ is open. □

### 3. Elementary Groups

#### 3.1. Preliminaries

The class of elementary groups captures the intuitive idea of t.d.l.c.s.c. groups "built by hand" from profinite and discrete groups. Formally,

**Definition 3.1.** The class of **elementary groups** is the smallest class $\mathcal{E}$ of t.d.l.c.s.c. groups such that

1. $\mathcal{E}$ contains all second countable profinite groups and countable discrete groups.

2. $\mathcal{E}$ is closed under taking group extensions of second countable profinite or countable discrete groups. I.e. if $G$ is a t.d.l.c.s.c. group and $H \triangleleft G$ is a closed normal subgroup with $H \in \mathcal{E}$ and $G/H$ profinite or discrete, then $G \in \mathcal{E}$.
(3) If $G$ is a t.d.l.c.s.c. group and $G = \bigcup_{i \in \mathbb{N}} O_i$ where $(O_i)_{i \in \mathbb{N}}$ is an $\subseteq$-increasing sequence of open subgroups of $G$ with $O_i \in \mathcal{E}$ for each $i$, then $G \in \mathcal{E}$. We say $\mathcal{E}$ is closed under countable increasing unions.

The class $\mathcal{E}$ enjoys robust permanence properties, which supports the thesis $\mathcal{E}$ is exactly the groups “built by hand”.

**Theorem 3.2 ([10] Theorem 3.18).** $\mathcal{E}$ enjoys the following permanence properties:

(a) $\mathcal{E}$ is closed under group extension.

(b) If $G \in \mathcal{E}$, $H$ is a t.d.l.c.s.c. group, and $\psi : H \to G$ is a continuous, injective homomorphism, then $H \in \mathcal{E}$. In particular, $\mathcal{E}$ is closed under taking closed subgroups.

(c) $\mathcal{E}$ is closed under taking quotients by closed normal subgroups.

A number of surprising further permanence properties hold of $\mathcal{E}$; we direct the interested reader to [10].

The class $\mathcal{E}$ admits two canonical rank functions. Here we make use of the somewhat technical to define decomposition rank. To recall this rank, we need to recount the basics of descriptive set theoretic trees; see [6] for an in depth treatment.

Denote the collection of finite sequences of natural numbers by $\mathbb{N}^{<\mathbb{N}}$. For one element sequences $(n)$, we suppress the parentheses and write $n$. For sequences $s := (s_0, \ldots, s_n) \in \mathbb{N}^{<\mathbb{N}}$ and $r := (r_0, \ldots, r_m) \in \mathbb{N}^{<\mathbb{N}}$, we put

$$s \cdot r := (s_0, \ldots, s_n, r_0, \ldots, r_m).$$

We write $s \subseteq r$ if $s$ is an initial segment of $r$. That is to say $n \leq m$ and $s_i = r_i$ for $0 \leq i \leq n$. The empty sequence, $\emptyset$, is considered to be an element of $\mathbb{N}^{<\mathbb{N}}$ and an initial segment of any $t \in \mathbb{N}^{<\mathbb{N}}$. For $\alpha \in \mathbb{N}^{\mathbb{N}}$, we define $\alpha \upharpoonright n := (\alpha(0), \ldots, \alpha(n-1))$, so $\alpha \upharpoonright n \in \mathbb{N}^{<\mathbb{N}}$ for any $n \geq 0$.

**Definition 3.3.** $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is a **tree** if it is closed under taking initial segments. We call the elements of $T$ the **nodes** of $T$. If $s \in T$ and there is no $n \in \mathbb{N}$ such that $s \cdot n \in T$, we say $s$ is a **terminal node** of $T$. An **infinite branch** of $T$ is a sequence $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $\alpha \upharpoonright n \in T$ for all $n$. The **body** of $T$, denoted $[T]$, is the collection of all infinite branches. If $[T] = \emptyset$, we say $[T]$ is well-founded.

For $T$ a well founded tree, there is an ordinal valued rank, denoted $\rho_T$, on the nodes of $T$ defined inductively as follows: If $s \in T$ is terminal, $\rho_T(s) = 0$. For a non-terminal node $s$,

$$\rho_T(s) := \sup \{ \rho_T(s \cdot n) + 1 \mid n \in \mathbb{N} \text{ and } s \cdot n \in T \}.$$

The **rank** of a well founded tree $T$ is defined to be

$$\rho(T) := \sup \{ \rho_T(s) + 1 \mid s \in T \}.$$

In the case $T \neq \emptyset$, it is easy to verify $\rho(T) = \rho_T(\emptyset) + 1$.

The decomposition rank of $G \in \mathcal{E}$ is the rank of a certain well founded tree. Indeed, fix $G$ a t.d.l.c.s.c. group, $U \in \mathcal{U}(G)$, and $D = \{d_n\}_{n \in \mathbb{N}}$ a set of selector functions for $S(G)$. That is a collection of functions $d_n : S(G) \to G$ such that for
all $H \in S(G)$, $\{d_i(H)\}_{n \in \mathbb{N}}$ is dense in $H$; see [6] (12.13)]. For $H \leq G$ and $n \in \mathbb{N}$, put

$$R_n^{(U,D)}(H) := \langle U \cap H, d_0(H), \ldots, d_n(H) \rangle$$

where the $d_i$ are selector functions from $D$. We now define a tree $T_{(U,D)}(G)$ and associated subgroups of $G$ as follows:

- Put $\emptyset \in T_{(U,D)}(G)$ and $G_0 := G$.
- Suppose we have defined $s \in T_{(U,D)}(G)$ and $G_s$. Put $s \setminus n \in T_{(U,D)}(G)$ and $G_{s \setminus n} := \text{Res} \left( R_n^{(U,D)}(G_s) \right)$ if and only if $G_s \neq \{1\}$.

We call $T_{(U,D)}(G)$ the **decomposition tree** of $G$ with respect to $U$ and $D$. It turns out this tree is well founded if and only if $G \in \mathcal{E}$, [10] Theorem 4.7. Additionally, the rank of $T_{(U,D)}(G)$ for $G \in \mathcal{E}$ is independent of the choice of $U$ and $D$, [10] Proposition 4.8. The **decomposition rank** of $G \in \mathcal{E}$ is thus defined to be

$$\xi(G) := \rho \left( T_{(U,D)}(G) \right).$$

A key property of the decomposition rank, which we exploit in this note, is the following.

**Lemma 3.4** ([10] Lemma 4.12). If $G \in \mathcal{E}$ and $G = \bigcup_{i \in \mathbb{N}} O_i$ with $(O_i)_{i \in \mathbb{N}}$ an \(\subseteq\)-increasing sequence of compactly generated open subgroups of $G$, then $\xi(G) = \sup_{i \in \mathbb{N}} \xi(\text{Res}(O_i)) + 1$.

3.2. **Elementary groups in $\mathcal{X}$**. We now argue $\mathcal{E}$ is a subclass of $\mathcal{X}$. To this end, we require a lemma due to Caprace and Monod.

**Lemma 3.5** ([1] Lemma 13). Suppose $G$ is a locally compact group, $H \leq G$, and $O \trianglelefteq_o G$ containing $H$. If $O \in \mathcal{X}$ and $G/H \in \mathcal{X}$, then $G \in \mathcal{X}$.

**Theorem 3.6.** $\mathcal{E} \subseteq \mathcal{X}$.

**Proof.** We first argue by induction on $\xi(H)$ for the following hypothesis: If $G$ is a t.d.l.c.s.c. group, $H \in \mathcal{E}$, and $H \leq_{cc} G$, then $G \in \mathcal{X}$. For the base case, $\xi(H) = 1$, $H = \{1\}$, so $G$ is compact and, plainly, an element of $\mathcal{X}$.

Suppose the induction hypothesis holds up to $\beta$, $\xi(H) = \beta + 1$, and $H \leq_{cc} G$ with $G$ a t.d.l.c.s.c. group. Fix $U \in \mathcal{U}(G)$ and let $\{g_i\}_{i \in \mathbb{N}}$ list a countable dense subset of $G$. Certainly, $G$ is the increasing union of

$$O_n := \langle U, g_0, \ldots, g_n \rangle,$$

and each $O_n$ is open and compactly generated. Additionally,

$$H = \bigcup_{n \in \mathbb{N}} H \cap O_n$$

with $H \cap O_n \leq_{cc} O_n$ for each $n$. Since $O_n$ is compactly generated, it follows $H \cap O_n$ is compactly generated for each $n$, see [7].

Fix $n \in \mathbb{N}$ and consider $H \cap O_n$. Certainly, $\text{Res}(H \cap O_n) \leq O_n$ with $\text{Res}(H \cap O_n) \leq H \cap O_n \leq_{cc} O_n$.

Via Theorem 2.2, $H \cap O_n / \text{Res}(H \cap O_n)$ is a SIN group. Proposition 2.3 implies $O_n / \text{Res}(H \cap O_n)$ has an open locally elliptic radical, so $O_n / \text{Res}(H \cap O_n) \in \mathcal{X}$. 
On the other hand, we have $\xi(\text{Res}(O_n \cap H)) \leq \beta$ by Lemma 3.4. Applying the induction hypothesis, $U \text{Res}(H \cap O_n) \in \mathcal{X}$, and in view of Lemma 3.5, we infer $O_n \in \mathcal{X}$.

We conclude $G$ is a directed union of open subgroups each in $\mathcal{X}$ and, via Theorem 1.3, $G \in \mathcal{X}$. This completes the induction, and we have the hypothesis. The theorem is now in hand: Taking $G \in \mathcal{E}$, $G \sqsubseteq \text{cc} \ G$ and, via our work above, is an element of $\mathcal{X}$. \hfill \Box

4. The class $\mathcal{E}^*$

We relax the second countability assumption on $\mathcal{E}$ by introducing the following:

**Definition 4.1.** The class $\mathcal{E}^*$ is the smallest collection of t.d.l.c. groups such that

1. $\mathcal{E}^*$ contains all profinite groups and discrete groups,
2. $\mathcal{E}^*$ is closed under group extensions of profinite and discrete groups, and
3. $\mathcal{E}^*$ is closed under directed unions of open subgroups.

There is an ordinal rank on $\mathcal{E}^*$. For $G \in \mathcal{E}$, define

- $G \in \mathcal{E}^*_0$ if and only if $G$ is profinite or discrete.
- Suppose $\mathcal{E}^*_0$ is defined. Put $G \in (\mathcal{E}^*_0)^c_{\alpha}$ if and only if there exists $N \leq G$ such that $N \in \mathcal{E}^*_\alpha$ and $G/N \in \mathcal{E}^*_\alpha$. Put $G \in (\mathcal{E}^*_0)^l_{\alpha}$ if and only if $G = \bigcup_{i \in I} H_i$ where $(H_i)_{i \in I}$ is an $\subseteq$-directed set of open subgroups of $G$ and $H_i \in \mathcal{E}^*_\alpha$ for each $i \in I$. Define $\mathcal{E}^*_\alpha := (\mathcal{E}^*_0)^c_{\alpha} \cup (\mathcal{E}^*_0)^l_{\alpha}$.
- For $\lambda$ a limit ordinal, $\mathcal{E}^*_\lambda := \bigcup_{\beta < \lambda} \mathcal{E}^*_\beta$.

Certainly, $\mathcal{E}^* = \bigcup_{\alpha \in \text{ORD}} \mathcal{E}^*_\alpha$, so for $G \in \mathcal{E}^*$, we define

$$\text{rk}(G) := \min\{\alpha \in \text{ORD} \mid G \in \mathcal{E}^*_\alpha\}.$$  

We call $\text{rk}(G)$ the **construction rank** of $G$.

The construction rank has a number of nice properties.

**Lemma 4.2.** Let $G \in \mathcal{E}^*$. Then

(a) $\text{rk}(G)$ is a successor ordinal when $\text{rk}(G)$ is non-zero.
(b) If $G$ is compactly generated and has non-zero rank, then $\text{rk}(G)$ is given by a group extension. I.e. if $\text{rk}(G) = \beta + 1$, there is $H \leq G$ such that $\text{rk}(H) = \beta$ and $\text{rk}(G/H) = 0$.
(c) If $O \leq_o G$, then $O \in \mathcal{E}^*$ and $\text{rk}(O) \leq \text{rk}(G)$.

**Proof.** These follow by transfinite induction on $\text{rk}(G)$. \hfill \Box

**Proposition 4.3.** Let $G \in \mathcal{E}^*$ be $\sigma$-compact and let $K \leq G$ be compact and such that $G/K$ is second countable. Then $G/K \in \mathcal{E}$.

**Proof.** We induct on the construction rank of $G$. As the proposition is immediate if $\text{rk}(G) = 0$, suppose $\text{rk}(G) = \alpha + 1$.

Suppose $\text{rk}(G)$ is given by a directed union, so there is $(O_i)_{i \in I}$ a directed system of open subgroups of $G$ such that $G = \bigcup_{i \in I} O_i$ with $\text{rk}(O_i) \leq \alpha$ for each $i \in I$.

Certainly,

$$G/K = \bigcup_{i \in I} O_i K/K,$$
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and we may find a countable subcover \((O_i K/K)_{i \in \mathbb{N}}\) since \(G/K\) is Lindelöf. One checks this cover may be taken to be an increasing \(\subseteq\)-chain. Further, \(O_i K/K \simeq O_i/(O_i \cap K)\), \(\text{rk}(O_i) \leq \alpha\), and \(O_i \cap K\) is a compact normal subgroup of \(O_i\) whose quotient is second countable. The induction hypothesis implies \((O_i K/K)_{i \in \mathbb{N}} \in \mathcal{E}\), and as \(G/K\) is the countable increasing union of \((O_i K/K)_{i \in \mathbb{N}}\), we conclude \(G/K \in \mathcal{E}\).

Suppose the construction rank of \(G\) is given by a group extension; say \(H \triangleleft G\) is such that \(\text{rk}(H) = \alpha\) and \(\text{rk}(G/H) = 0\). Since \(H\) is \(\sigma\)-compact and \(K\) compact, we have \(HK/K \simeq H/(H \cap K)\), and as \(\text{rk}(H) = \alpha\), the induction hypothesis implies \(H/(H \cap K) \in \mathcal{E}\). On the other hand, \(HK/H \triangleleft G/H\) is closed, and the quotient \((G/H)/(HK/H) \simeq G/HK \simeq (G/K)/(HK/K)\)
is second countable and either discrete or compact. We conclude \(G/K\) is a group extension of \(G/HK\) by \(HK/K\) and, therefore, \(G/K \in \mathcal{E}\). This completes the induction, and we conclude the proposition. □

**Theorem 4.4.** \(\mathcal{E}^* \subseteq \mathcal{X}\).

**Proof.** Take \(G \in \mathcal{E}^*\). Every locally compact group is the directed union of its open compactly generated subgroups. Since \(\mathcal{X}\) is closed under directed unions, we may assume \(G\) is compactly generated and, therefore, \(\sigma\)-compact. Applying Theorem 2.1, there is \(K \triangleleft G\) such that \(G/K\) is second countable. Theorem 4.3 implies \(G/K \in \mathcal{E}\), and by Theorem 3.6, we have that \(G/K \in \mathcal{X}\). Since \(K\) is compact, we deduce from Theorem 1.3 that \(G \in \mathcal{X}\). □

We record a sufficient condition to be a member of \(\mathcal{E}^*\) for later use.

**Lemma 4.5.** Let \(G\) be a t.d.l.c. group whose quasi-center \(QZ(G) := \{g \in G \mid C_G(g)\text{ is open}\}\) is dense. Then \(G \in \mathcal{E}^*\).

**Proof.** Since \(G\) is the directed union of its compactly generated open subgroups, we may assume \(G\) is compactly generated. By [2, Proposition 4.3], every compactly generated t.d.l.c. group with dense quasi-centre is compact-by-discrete and, hence, belongs to \(\mathcal{E}^*\). □

We conclude this section by noting three permanence properties of \(\mathcal{E}^*\).

**Theorem 4.6.** \(\mathcal{E}^*\) enjoys the following permanence properties:

(a) \(\mathcal{E}^*\) is closed under group extension.
(b) If \(G \in \mathcal{E}^*\), then every t.d.l.c. group admitting a continuous, injective homomorphism into \(G\) also belongs to \(\mathcal{E}^*\). In particular, \(\mathcal{E}^*\) is closed under taking closed subgroups.
(c) \(\mathcal{E}^*\) is closed under taking quotients by closed normal subgroups.

**Proof.** The first statement follows just in the case of \(\mathcal{E}\); see the proof of [10, Proposition 3.5].

For (b), let \(G \in \mathcal{E}^*\), let \(H\) be a t.d.l.c. group and let \(\psi: H \to G\) be a continuous, injective homomorphism. Every locally compact group is the directed union of its open compactly generated subgroups. Since \(\mathcal{E}^*\) is closed under directed unions, we may therefore assume \(H\) is compactly generated. Therefore, so is \(P = \psi(H)\).
Since $H$ is $\sigma$-compact, and it has a compact normal subgroup $K$ such that $H/K$ is second countable (see Theorem 2.1). It follows $\psi(K)$ is a compact normal subgroup of $P$ such that $P/\psi(K)$ is second countable. Applying Theorem 4.3, we conclude $P/\psi(K) \in \mathcal{E}$ and, via Theorem 3.2, $H/K \in \mathcal{E}$. Therefore, in view of part (a), $H \in \mathcal{E}^*$ as desired.

For (c), suppose $L \leq_G G$; as above, it suffices to consider the case that $G$ is compactly generated. Applying Theorem 2.1, we find $K \leq_G G$ compact such that $G/K \in \mathcal{E}$. Certainly, $LK/K$ is a closed normal subgroup of $G/K$. Theorem 3.2 implies

$$\frac{G/L}{(LK/L)} \simeq \frac{G/LK}{(LK/K)} \in \mathcal{E}$$

On the other hand, $LK/L \simeq K/(K \cap L)$ with the latter group is compact. We conclude $G/L$ is compact-by-$\mathcal{E}$ and, via (a), belongs to $\mathcal{E}^*$.

5. The class $\mathcal{Y}$

In order to prove Theorem 1.5, we introduce the class $\mathcal{Y}^*$ consisting of those locally compact groups $G$ such that $G/G^\circ \in \mathcal{E}^*$ where $G^\circ$ is the connected component of the identity. By definition, we have that $\mathcal{E}^* \subseteq \mathcal{Y}^*$, hence $\mathcal{Y}^* \subseteq \mathcal{Y}$ since $\mathcal{Y}$ is stable under group extensions and contains all connected l.c. groups. We will eventually show that $\mathcal{Y}^* = \mathcal{Y}$. The proof of this relies on the following.

**Proposition 5.1.** The class $\mathcal{Y}^*$ enjoys the following permanence properties:

(a) $\mathcal{Y}^*$ is closed under directed unions of open subgroups.
(b) $\mathcal{Y}^*$ is closed under taking closed subgroups.
(c) $\mathcal{Y}^*$ is closed under taking group extensions.
(d) $\mathcal{Y}^*$ is closed under taking quotients by closed normal subgroups.

The proof of (c) requires the following subsidiary fact.

**Lemma 5.2.** Let $G$ be a $\sigma$-compact l.c. group, and let $L,P \leq G$ be such that $[P,L] = \{1\}$ and $L \geq G^\circ$. If $P \in \mathcal{E}^*$, then $\overline{PL}/L \in \mathcal{E}^*$.

**Proof.** Let $\psi: G \to G/L$ be the canonical projection. We need to show $\overline{\psi(P)} \in \mathcal{E}^*$; we argue by induction on $\text{rk}(P)$.

For the base case, $\text{rk}(P) = 0$, the group $P$ is either profinite or discrete. For the former case, $\psi(P)$ is compact, hence closed and profinite, so the desired conclusion is clear. For the latter case, the group $P$ is contained in the quasi-center of $G$, hence of $\overline{PL}$, which implies that $\overline{\psi(P)}$ has a dense quasi-center. The conclusion now follows from Lemma 4.5.

Suppose $\text{rk}(P) = \beta + 1$. Let $V \in U(G/L)$ and $U = \psi^{-1}(V)$, so $U \leq_a G$ and $L \leq_{cc} U$. Since $[P,L]$ is trivial, it follows that for each compact subset $\Sigma \subseteq P$, the set $\{usu^{-1} \mid u \in U, s \in \Sigma\}$ remains compact. It follows every compactly generated subgroup of $P$ is contained in a compactly generated subgroup normalized by $U$. We may thus write $P$ as a directed union $P = \bigcup_{n \in I} P_n$ of compactly generated open subgroups, each normalized by $U$. Certainly, $\psi(P_n)$ is normalized by $V$, and

$$\overline{\psi(P)} = \bigcup_n \psi(P_n) \left(V \cap \overline{\psi(P)}\right).$$
Since $\psi(P_n) \leq \psi(P_n)(V \cap \psi(P))$, we conclude $\psi(P) \in \mathcal{E}^*$ provided $\psi(P_n) \in \mathcal{E}^*$ for each $n$.

Fix $n$ and put $Q := P_n$ and $R := Q/L$. Since $Q \leq_o P$ and $Q$ is compactly generated, we have that $\text{rk}(Q) \leq \beta + 1$ and there is $M \leq Q$ such that $\text{rk}(M) \leq \beta$ and $\text{rk}(Q/M) = 0$. Further, $Q \leq R$ and $M \leq R$ because $L$ centralizes $P$. Passing to $R/M$, we have $Q/M \leq R/M$ and $QL/M \leq R/M$. Additionally, $Q/M \in \mathcal{E}^*$ with $\text{rk}(Q/M) = 0$, $(R/M)^{\circ} \leq ML/M$, and $[Q/M, ML/M] = \{1\}$. By the base case and construction of $R$,

$$(R/M)/(ML/M) \in \mathcal{E}^*.$$ 

Therefore, $R/ML \in \mathcal{E}^*$. 

On the other hand, $\text{rk}(M) \leq \beta$, $[M, L] = \{1\}$, $R^{\circ} \leq L$, and $M, L \leq R$. The induction hypothesis thus implies $ML/L \in \mathcal{E}^*$. We now have that $ML/L \leq R/L$, $ML/L \in \mathcal{E}^*$, and

$$(R/L)/(ML/L) \simeq R/ML \in \mathcal{E}^*.$$ 

Hence, $\psi(P_n) \simeq \psi \in \mathcal{E}^*$ finishing the induction. 

\textbf{Proof of Proposition 5.1.} 
(a) Let $G$ be the directed union of $(O_i)_{i \in I}$ with $O_i \leq_o G$ and $O_i \in \mathcal{Y}^*$ for each $i \in I$. For each $i$, we have $G^o = O_i^o \leq O_i$ since $O_i$ is open. So $G/G^o = \bigcup_{i \in I} O_i/O_i^o$, and therefore, $G/G^o \in \mathcal{E}^*$. 

(b) Given $H \leq G$, we have $H/H \cap G^o$ embeds continuously into $G/G^o$. By Theorem 4.6 this implies $H/H \cap G^o \in \mathcal{E}^*$. On the other hand, $H \cap G^o \leq G^o$ with $G^o$ a connected locally compact group. By the celebrated solution to Hilbert’s fifth problem, $G^o$ is pro-Lie. Let $K \leq G^o$ be compact such that $G^o/K$ is a Lie group; we may take $G^o/K$ to be a Lie group over the real numbers. In view of Cartan’s theorem, see [9, LG 5.42], $(H \cap G^o)K/K$ is a Lie group. Putting $J := K \cap H \cap G^o$, we have

$$H^o J/J = (H \cap G^o / J^o) \leq_o (H \cap G^o)/J,$$

so $(H \cap G^o)/H^o J$ is discrete. Since we may find such $J$ inside arbitrarily small neighborhoods of $1$, $(H \cap G^o)/H^o$ is residually discrete. Via Theorem 2.2, we infer $(H \cap G^o)/H^o$ is a directed union of SIN groups and, therefore, in $\mathcal{E}^*$. Since $\mathcal{E}^*$ is closed under group extension, $H/H^o \in \mathcal{E}^*$, so that $H \in \mathcal{Y}^*$ as desired. 

(c) Let $H \leq G$ be such that $H$ and $G/H$ both belong to $\mathcal{Y}^*$. We need to show $G \in \mathcal{Y}^*$. To this end, we may reduce to $G$ compactly generated. Indeed, let $O \leq_o G$ be compactly generated. We see

$$H^o = (O \cap H)^o \leq O \cap H \leq O$$

and $(O \cap H)/H^o \leq_o H/H^o$. So $(O \cap H)/H^o \in \mathcal{E}^*$, and $O \cap H \in \mathcal{Y}^*$. Similarly, $O/(O \cap H) \in \mathcal{Y}^*$. Hence, if $O \in \mathcal{Y}^*$, then $G \in \mathcal{Y}^*$ since $\mathcal{Y}^*$ is closed under directed unions.

Suppose $G$ is compactly generated, put $\tilde{G} := G/(H \cap G^o)$, and let $\pi : G \to \tilde{G}$ be the usual projection. Since $H^o \leq H \cap G^o$, $\tilde{H} := \pi(H)$ is a quotient of $H/H^o$ and, via Theorem 4.6, a member of $\mathcal{E}^*$. Put $\tilde{G}^o = \pi(G^o)$. As $[\tilde{H}, \tilde{G}^o] = \{1\}$ and $\tilde{G}$ is $\sigma$-compact, we apply Lemma 5.2 to conclude $\tilde{H}G^o/\tilde{G} \simeq \tilde{H}G^o/G^o$. 

$\tilde{H}G^o/G^o \simeq \tilde{H}G^o/G^o$.
we indeed have that $\overline{HG^0/G^0} \in \mathcal{E}^*$. 

The neutral component of any locally compact group coincides with the intersection of all its open subgroups. This implies $\overline{H/G^0} = (G/H)^*$, so

$$(G/G^0)/(H/G^0) \simeq G/H \simeq (G/H)/(H/G^0) \in \mathcal{E}^*.$$ 

We conclude $G/G^0$ is an extension of a group in $\mathcal{E}^*$ by another group in $\mathcal{E}^*$ and, thus, a member of $\mathcal{E}^*$. Therefore, $G \in \mathcal{Y}^*$. 

(d) Let $G \in \mathcal{Y}^*$ and $N \trianglelefteq G$. As noticed above, we have $NG^0/N = (G/N)^*$. Therefore, in order to show that $G/N \in \mathcal{Y}^*$, it suffices to show that $G/G^0N \in \mathcal{E}^*$. The latter is isomorphic to a quotient of $G/G^0 \in \mathcal{E}^*$, so the desired conclusion follows from Theorem 4.6.

\[\Box\]

Corollary 5.3. $\mathcal{Y} = \mathcal{Y}^* \subseteq \mathcal{X}$. 

Proof. We have already observed that $\mathcal{Y}^* \subseteq \mathcal{Y}$. Since $\mathcal{Y}^*$ is stable under group extensions and directed unions of open subgroups by Proposition 5.1, the reverse inclusion follows. Taking $G \in \mathcal{Y}^*$, we see $G/G^0 \in \mathcal{E}^* \subseteq \mathcal{X}$ and, via Theorem 1.3, $G \in \mathcal{X}$.

\[\Box\]

Proof of Theorem 1.5. We have $\mathcal{Y} \subseteq \mathcal{X}$ by Corollary 5.3. The permanence properties of $\mathcal{Y}$ follow from Proposition 5.1.

Remark 5.4. The results of this note give new information concerning locally compact $G \not\in \mathcal{X}$. As remarked in [1], we may take $G \not\in \mathcal{X}$ to be a compactly generated t.d.l.c. group. Applying Theorem 2.1, we have a compact $K \trianglelefteq G$ such that $G/K$ is second countable. Theorem 1.3 implies $G/K$ must also lie outside of $\mathcal{X}$. We may thus take $G \not\in \mathcal{X}$ to be a compactly generated t.d.l.c.s.c. group.

By [10, Theorem 7.8], a t.d.l.c.s.c. group $G$ admits a unique maximal closed normal elementary subgroup, denoted $\text{Rad}_E(G)$ and called the \textbf{elementary radical} of $G$. Suppose $G$ is a t.d.l.c.s.c. group outside of $\mathcal{X}$ and fix $U \in \mathcal{U}(G)$. By Theorem 3.6, $\text{Rad}_E(G) \in \mathcal{X}$, and since $U\text{Rad}_E(G) \in \mathcal{E}$, we further have $U\text{Rad}_E(G) \in \mathcal{X}$. In view of Lemma 3.5, it must be the case $G/\text{Rad}_E(G) \not\in \mathcal{X}$. We may thus suppose $G \not\in \mathcal{X}$ has trivial elementary radical, and via [10, Corollary 9.12], $G$ is $[A]$-semisimple. The definition and a discussion of $[A]$-semisimple groups may be found in [4]. Here we merely recall that $[A]$-semisimple groups have a canonical action on a lattice and, in many cases, a non-trivial boolean algebra.

\[\text{References}\]

[1] P.-E. Caprace and N. Monod, Relative amenability. ArXiv:1309.2890 [math.GR], http://arxiv.org/abs/1309.2890.
[2] P.-E. Caprace and N. Monod, Decomposing locally compact groups into simple pieces. Math. Proc. Cambridge Philos. Soc. 150 (2011) (1), 97–128.
[3] P.-E. Caprace and N. Monod, Fixed points and amenability in non-positive curvature. Math. Ann. 356 (2013) (4), 1303–1337.
[4] P.-E. Caprace, C. Reid, and G. Willis, Locally normal subgroups of totally disconnected groups. Part I: General theory. ArXiv:1304.5144 [math.GR], http://arxiv.org/abs/1304.5144.
[5] E. Hewitt and K. Ross, *Abstract harmonic analysis. Vol. I*, vol. 115 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, Springer-Verlag, Berlin, 1979, second edn.

[6] A. Kechris, *Classical descriptive set theory*, vol. 156 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 1995.

[7] A. M. Macbeath and S. Świerczkowski, On the set of generators of a subgroup. *Nederl. Akad. Wetensch. Proc. Ser. A* 62 = *Indag. Math.* 21 (1959), 280–281.

[8] V. P. Platonov, Locally projectively nilpotent subgroups and nilpotents in topological groups. *Izv. Akad. Nauk SSSR Ser. Mat.* 30 (1966), 1257–1274.

[9] J.-P. Serre, *Lie algebras and Lie groups*, vol. 1500 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, 1992, second edn. 1964 lectures given at Harvard University.

[10] P. Wesolek, Elementary totally disconnected locally compact groups. ArXiv:1405.4851 [math.GR], http://arxiv.org/abs/1405.4851.

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