A unified treatment of Ising model magnetizations

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Abstract. We show how the spontaneous bulk, surface and corner magnetizations in the square lattice Ising model can all be obtained within one approach. The method is based on functional equations which follow from the properties of corner transfer matrices and vertex operators and which can be derived graphically. In all cases, exact analytical expressions for general anisotropy are obtained. Known results, including several for which only numerical computation was previously possible, are verified and new results related to general anisotropy and corner angles are obtained.

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1. Introduction

The calculation of order parameters in solvable statistical models has by now a rather long history. It began with Onsager’s result for the bulk magnetization of the square lattice Ising model [1, 2]. Quite some time later, McCoy and Wu obtained the order at a (10) surface of the same system [3, 4], by which the general study of surface critical behaviour began. Again it took some time until it was realised that this behaviour depends on the shape of the surface and that an interesting phenomenon, namely continuously varying exponents, appears at corners [5, 6]. Thus Ising models with corners have been studied over the last 12 years [7] but so far there has been only one completely analytical calculation, namely for a square lattice with a 90° corner formed by two (10) surfaces [8].

The calculations in all these cases can be (and have been) done using row transfer matrices. One then has to find, for example, the matrix element of a spin operator between two asymptotically degenerate eigenvectors of this matrix. However, while this is rather simple for the (10) surface [9, 10], especially in the anisotropic (Hamiltonian) limit [11], it involves the solution of an integral equation for the bulk and the corner problem [10]. Thus an alternative and possibly simpler method is desirable.

Such a technique has become available in the form of Baxter’s corner transfer matrix (CTM) [12]. In fact, the bulk order parameters in most of the more complicated solvable models have been obtained through it. In this method, a square lattice is divided into quadrants and the CTM is the partition function of one such quadrant with the variables at the outer boundary held fixed. One then can calculate directly the expectation value of the central spin in the form of a trace. A basic feature is the simple structure of the CTM eigenvalue spectrum in the thermodynamic limit [12] which leads to simple infinite product formulae for the order parameters [13].

While in the early CTM calculations the spectrum was used directly a somewhat different approach has been developed recently by the Kyoto group [14 – 16]. In this approach, so-called vertex operators play a fundamental rôle. These are half-infinite lines of vertices in vertex models, or lines of couplings in an Ising model, and hence can be viewed as half-infinite row transfer matrices. If they differ from the lattice couplings only in their anisotropy (spectral parameter), then they have simple commutation relations with each other and with the CTMs. This can be used to “rotate” such lines in a lattice and to obtain in this way functional equations (also known as q-difference equations) for order parameters or even correlation functions. (Actually the idea of moving lines about in a general way was introduced by Baxter for the eight vertex model [17], a concept called “Z-invariance”.) Such equations can be solved in some cases in a proper elliptic functions parametrization and this leads to the same type of infinite product formulae as are observed in cases where direct calculation is possible.

So far, this technique has been used mainly for obtaining some bulk order parameters, e.g. in ABF models [18, 19]. Only recently, vertex models with straight surfaces have been considered, and the spontaneous magnetization at a free end of the XYZ spin chain has been obtained [20]. An additional ingredient needed here are boundary Yang-Baxter equations and the corresponding K-matrices.
In this paper we want to consider the square lattice Ising model and show how, using this approach, the various types of order parameters can all be obtained in a unified way. We recover all previously known results and derive several new ones. Although the Ising model is a special case of the eight-vertex or the ABF models, we think that it is useful and warranted to treat it completely on its own.

Thus we will first outline the technique of forming the functional equations in a spin formulation and show how to recover once more the Onsager formula. Then we describe how the procedure has to be modified for a system with a free (11) surface, i.e., one cut diagonally with respect to the couplings. It turns out that, instead of complicated Yang-Baxter boundary equations, one needs only a simple elementary relation here. The structure of the functional equations remains essentially the same as in the bulk and we obtain from it the magnetization in the first and second row. We then apply the procedure to a 90° corner formed by two (11) surfaces and obtain the magnetization at three different corner points. The solution for this corner is especially interesting in the critical region, because the corner exponent $\beta_c$ depends on the anisotropy. We obtain this dependence, which previously was inferred by rescaling onto an isotropic system [5–7], directly from our analytical formulae.

Finally we show that the method also works for a lattice with a (11) surface and a layered structure of the couplings perpendicular to it. An appropriate choice of the layering then produces (10) surfaces, so that these can also be treated. In particular, one can obtain the other type of 90° corner in this way and derive the corresponding magnetization formula. This completes our unified treatment. A concluding section contains some additional remarks and an outlook.

2. Basic constructions

2.1 Parametrisation. We consider an Ising model defined on a regular square lattice taken on the diagonal, and parametrise the couplings using multiplicative rapidity variables $\zeta_1$, $\zeta_2$ attached to lines running in the horizontal and vertical directions. Except in section 6, we do not account for the horizontal and vertical rapidities individually, simply writing $\zeta = \zeta_1/\zeta_2$.

\begin{align*}
K(\zeta): & \begin{array}{c}
\zeta_2 \\
\zeta_1
\end{array} \\
L(\zeta): & \begin{array}{c}
\zeta_2 \\
\zeta_1
\end{array}
\end{align*}

Figure 2.1: Ising couplings $K$ and $L$.

There are two types of coupling as shown in figure 2.1. Our parametrisation uses the standard formulae [12]

\begin{align*}
\sinh 2K &= -i \sinh(iu), & \sinh 2L &= -i \sinh(iI' - iu) = i \sinh(iu)/k, \\
\cosh 2K &= \cosh(iu), & \cosh 2L &= \cosh(iI' - iu) = i \sinh(iu)/k.
\end{align*} (2.1)
Ising model magnetizations

where \( sn, cn \) are the usual Jacobian elliptic functions, of modulus

\[ k = 1/(\sinh 2K \sinh 2L). \]

This parameter, already introduced by Onsager, measures the temperature. The quarter periods of the elliptic functions are \( I \) and \( I' \), where \( I = I(k) \) is the elliptic integral of the first kind and \( I' = I(k') \) with \( k' = \sqrt{1-k^2} \). The anisotropy of the couplings is measured by the variable \( u \), or equivalently by

\[ \zeta = \exp(-\pi u/2I). \]

Furthermore we will use the quantities

\[ q = \exp(-\pi I'/I), \quad x = q^{1/2}, \]

In these variables the crossing symmetry (rotation through 90°) \( L(u) = K(I' - u) \) becomes \( L(\zeta) = K(x/\zeta) \). In our calculations we shall particularly need an expression for \( \tanh L(\zeta) \) in terms of infinite products. It is

\[
\tanh L(\zeta) = \frac{(q^{1/2}/\zeta; q^2)(q^{3/2}/\zeta; q^2)(-q^{3/2}/\zeta; q^2)(-q^{1/2}/\zeta; q^2)}{(-q^{1/2}/\zeta; q^2)(-q^{3/2}/\zeta; q^2)(q^{3/2}/\zeta; q^2)(q^{1/2}/\zeta; q^2)}, \tag{2.2}
\]

where \((z; p)\) is the infinite product

\[ (z; p) = \prod_{n=0}^{\infty} (1 - zp^n). \]

The identification (2.2) follows from standard elliptic function identities, together with the fact that the Jacobi theta function \( H(u) \) has the infinite product expansion

\[ H(iu) = iz^{1/4}q^{-1/2}(q^2/z; q^2)(q^2; q^2), \]

where \( z = \exp(-\pi u/I) \). We note for later use the relation

\[
\frac{H(iu)}{H(iu+I)} = k^{1/2} \frac{sn(iu)}{cn(iu)} = i \frac{(z; q^2)(q^2/z; q^2)}{(-z; q^2)(-q^2/z; q^2)}, \quad z = \exp(-\pi u/I). \tag{2.3}
\]

2.2 Star-triangle relation. Integrability depends on special properties of the Boltzmann weights, particularly the star-triangle relation shown in figure 2.2. The filled circle represents a spin which is summed out, while the scalar factor is determined by the normalisation of the Boltzmann weights. The condition on the rapidities in the three couplings is given by Baxter [12]; taking into account the present notation it is

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{star-triangle.png}
\caption{Star-triangle relation.}
\end{figure}
\[ \zeta_3 = \frac{\zeta_1}{\zeta_2}. \]  

(2.4)

This relation is important in choosing appropriate couplings when deriving functional equations. For vertex operators, we shall use the star-triangle relation in an iterated form shown in figure 2.3. From figure 2.2 we see that \( H \) is of type \( L, M \) of type \( K \); this is discussed further in section 2.4. Notice that in each pair of operations the individual scalar factors \( R \) are mutually reciprocal, and therefore cancel.

![Figure 2.3: Iterated star-triangle relations.](image)

2.3 Corner transfer matrices. A CTM is defined to be the partition function of a quadrant of the lattice, as shown in figure 2.4. The direction of transfer (action as an operator) is shown by an arrow. There are two sectors, Neveu-Schwarz (\( NS \)) and Ramond (\( R \)) according as the centre of the lattice is taken at a single spin site or not; thus \( A^{(NS)}(\zeta) \) is diagonal with respect to the central spin \( \epsilon \). We fix the boundary spins to +.

![Figure 2.4: Corner transfer matrices for the \( NS \) and \( R \) sectors.](image)

The really remarkable properties of the CTMs emerge in the thermodynamic limit. In this limit, we expect that the entries of the CTMs will scale as the partition function per coupling raised to the power of the number of couplings, and we assume that this factor is divided out to give a convenient finite limit.
These normalised infinite CTMs are generated by spin chain operators $H^{(NS)}$ and $H^{(R)}$:

$$A^{(NS)}(\zeta) = \zeta H^{(NS)}, \quad A^{(R)}(\zeta) = \zeta H^{(R)}.$$ 

From this it is evident that the CTMs are symmetric and that they satisfy the group property

$$A^{(NS)}(\zeta_1)A^{(NS)}(\zeta_2) = A^{(NS)}(\zeta_1\zeta_2), \quad A^{(R)}(\zeta_1)A^{(R)}(\zeta_2) = A^{(R)}(\zeta_1\zeta_2). \quad (2.5)$$

The eigenvectors of $H^{(NS)}$ and $H^{(R)}$ span the Neveu-Schwarz and Ramond sector, respectively. Moreover, the Neveu-Schwarz sector divides into two subsectors labelled by the value of the central spin $\epsilon$. Since it is straightforward to diagonalise $H^{(NS)}$ and $H^{(R)}$ using fermions [21, 22], analytic formulae may be given using properties of elliptic functions. In particular, the eigenvalue spectra are equally spaced, although we do not make explicit use of these facts in this paper.

Finally we note that one may define CTMs for the other three quadrants of the lattice, North-East, South-East, South-West. They are simply related to the one we have defined (North-West) by

$$A_{\text{NE}}(\zeta) = A'_{\text{NW}}(x/\zeta), \quad A_{\text{SE}}(\zeta) = A_{\text{NW}}(\zeta), \quad A_{\text{SW}}(\zeta) = A'_{\text{NW}}(x/\zeta).$$

provided that $\zeta$ retains its meaning as the ratio of horizontal and vertical rapidities. The prime denotes the transpose (which reverses the direction of transfer); this is the form in which we construct $A_{\text{NE}}$ from $A_{\text{NW}}$ in the non-symmetric case needed in section 6.

2.4 Vertex operators. The central technique of this paper is to employ “vertex operators” which map between (intertwine) the Neveu-Schwarz and Ramond sectors. We are concerned with the physical interpretation of vertex operators as semi-infinite row transfer matrices $V_\epsilon(\zeta)$ and $W_\epsilon(\zeta)$, and in the functional relations which they satisfy. The two types of operators are shown in figure 2.5. Since we shall often rotate these operators to other orientations, it is important to note that they retain their underlying definition as $V_\epsilon(K, L)$, $W_\epsilon(K, L)$, from which they derive their dependence on $\zeta$ via (2.1). One sees immediately from the crossing symmetry that the vertex operators satisfy

$$V'_\epsilon(\zeta) = W_\epsilon(\zeta/x/\zeta), \quad W'_\epsilon(\zeta) = V_\epsilon(\zeta/x/\zeta).$$

![Figure 2.5: Vertex operators $V_\epsilon(\zeta)$ and $W_\epsilon(\zeta)$.](image-url)

Consider now the left hand side of figure 2.6, which shows the composition of the action of $A^{(NS)}(\zeta)$ with $V_\epsilon(\zeta)$. For a finite system this simply gives a Ramond
corner, but in the infinite case we must be more careful; $A^{(NS)}(\zeta)$ maps the Neveu-Schwarz sector into itself, after which $V_\epsilon(\zeta)$ maps it to the Ramond sector. So, on the right-hand side of the equation, we have inserted a trivial operator which makes the same mapping just by re-labelling the spins (the dashed lines indicate simple identification). However this operator is the anisotropic limit of $V_\epsilon(\zeta)$ at $\zeta = 1$, so we obtain the functional relation

$$V_\epsilon(\zeta) A^{(NS)}(\zeta) = A^{(R)}(\zeta) V_\epsilon(1).$$

By eliminating $V_\epsilon(1)$ using the group property of CTMs, we obtain the general “boost property”

$$V_\epsilon(\zeta_1 \zeta_2) A^{(NS)}(\zeta_1) = A^{(R)}(\zeta_1) V_\epsilon(\zeta_2). \quad (2.6a)$$

This boost property of CTMs has been the subject of some investigation [23, 24], but its real utility only emerged recently [14 – 16].

One can write a similar relation for the second type of vertex operator, which we shall denote $W_\epsilon(\zeta)$. The picture is shown in figure 2.7, and an analogous argument gives

$$W_\epsilon(\zeta_1 \zeta_2) A^{(R)}(\zeta_1) = A^{(NS)}(\zeta_1) W_\epsilon(\zeta_2). \quad (2.6b)$$

Alternatively one can note that the two equations (2.6) are the transpose of each other. However, in the non-symmetric case of section 6, this will relate the action of $A_{NE}$ and $A_{NW}$ on the vertex operators.
Now consider the products $V_\varepsilon(\zeta)W_\varepsilon(\zeta')$ and $W_\varepsilon(\zeta)V_\varepsilon(\zeta')$ which stand on the left hand side of figure 2.8. Using the star-triangle relation in the form shown in figure 2.3, we may move couplings $H$ or $M$ repeatedly from right to left, thereby interchanging the Ising couplings, and with them the rapidity variables. Because of the fixed spin boundary conditions this will eventually cease to have any further effect beyond multiplication by the simple factor $e^M$. Thus we have “commutation relations”, although it is the rapidities rather than the operators which are interchanged. This is required by the fact that they intertwine the Neveu-Schwarz and Ramond sectors.

Consider first the product $F_\varepsilon(\zeta, \zeta') = V_\varepsilon(\zeta)W_\varepsilon(\zeta')$, we have

$$Re^M(\zeta/\zeta') F_\varepsilon(\zeta, \zeta') = \sum_{\varepsilon'} e^{H(\zeta/\zeta')\varepsilon\varepsilon'} F_{\varepsilon'}(\zeta', \zeta'), \quad (2.7a)$$

where $R$ comes from the first application of the star-triangle relation.

Figure 2.8: Commutation of vertex operators.

Typically we shall require combinations of the form $F_+(\zeta, \zeta') \pm F_-(\zeta, \zeta')$, for which we have

$$Re^M(\zeta/\zeta')(F_+(\zeta, \zeta') + F_-(\zeta, \zeta')) = \cosh H(\zeta/\zeta')(F_+(\zeta', \zeta) + F_-(\zeta', \zeta))$$

$$Re^M(\zeta/\zeta')(F_+(\zeta, \zeta') - F_-(\zeta, \zeta')) = \sinh H(\zeta/\zeta')(F_+(\zeta', \zeta) - F_-(\zeta', \zeta))$$

When we are dealing with expectation values rather than operators, we shall take quotients of such factors, as a result of which $R$ will play no further part. From the star-triangle relation one sees that

$$\tanh H(\zeta) = \tanh L(\zeta),$$

the formula for which was given in (2.2).

The case of the product $F_{\varepsilon\varepsilon'}(\zeta, \zeta') = W_\varepsilon(\zeta)V_{\varepsilon'}(\zeta')$ is similar, and we find

$$e^M(\zeta/\zeta') F_{\varepsilon\varepsilon'}(\zeta, \zeta') = e^{M(\zeta/\zeta')\varepsilon\varepsilon'} F_{\varepsilon\varepsilon'}(\zeta', \zeta), \quad (2.7b)$$
For this formula \( \tanh M(\zeta) = \tanh K(\zeta) \). We shall, however, only use this relation in the case that \( \epsilon = \epsilon' \), for which we have the simple result that the rapidities may be interchanged without introducing any factors.

3. **Bulk magnetization**

As a preliminary to the calculation of surface and corner magnetizations in later sections, we will develop the method, and the underlying assumptions, by deriving the Onsager result for the bulk magnetization of the Ising model.

\[
F_{\epsilon}(\zeta_1, \zeta_2) = \text{trace}(A^{(R)}(x) V_{\epsilon}(\zeta_2) A^{(NS)}(x) W_{\epsilon}(\zeta_1)),
\]

\[
F'_{\epsilon}(\zeta_1, \zeta_2) = \text{trace}(A^{(NS)}(x) W_{\epsilon}(\zeta_2) A^{(R)}(x) V_{\epsilon}(\zeta_1)),
\]

where we have combined pairs of CTMs, using the group property (2.5), to eliminate the bulk rapidity. The relation between \( F \) and \( F' \) is a simple reflection about the horizontal axis. This interchanges the couplings \( K \) and \( L \), which is equivalent to \( \zeta \to x/\zeta \), so

\[
F_{\epsilon}(\zeta_1, \zeta_2) = F'_{\epsilon}(x/\zeta_1, x/\zeta_2).
\]

This can be obtained directly from the definitions since \( \text{trace}(X') = \text{trace}(X) \) for any matrix \( X \).

It is easy to see that the dependence on \( \zeta_1 \) and \( \zeta_2 \) is only through the ratio \( \zeta_1/\zeta_2 \). This is because these expectation values are matrix elements of a spin operator between asymptotically degenerate maximal eigenvectors of the horizontal row transfer matrices above and below the spin. But, because of the star-triangle relation, all eigenvectors of the row transfer matrices except the line of dislocation are independent of rapidity while the latter can only depend on the ratio \( \zeta_1/\zeta_2 \) [12]. For the homogeneous case the magnetization is

\[
M_0 = \left( \frac{F_+(\zeta, \zeta) - F_-(\zeta, \zeta)}{F_+(\zeta, \zeta) + F_-(\zeta, \zeta)} \right) = \left( \frac{F'_+(\zeta, \zeta) - F'_-(\zeta, \zeta)}{F'_+(\zeta, \zeta) + F'_-(\zeta, \zeta)} \right),
\]
Therefore we define the function $G(\zeta_1/\zeta_2)$ as

$$G(\zeta_1/\zeta_2) = \left( \frac{F_+(\zeta_1, \zeta_2) - F_-(\zeta_1, \zeta_2)}{F_+(\zeta_1, \zeta_2) + F_-(\zeta_1, \zeta_2)} \right),$$

similarly for $G'(\zeta_1/\zeta_2)$, and

$$M_0 = G(1) = G'(1).$$

Finally, we shall require an inversion relation, namely

$$G(\zeta)G(-\zeta) = 1,$$

from which the normalisation of $G(\zeta)$ may be determined. It follows from

$$F_\epsilon(-\zeta_1, \zeta_2) = \rho \epsilon F_\epsilon(\zeta_1, \zeta_2),$$

where $\rho = i^{n_K-n_L}$ and $n_K$, $n_L$ are the number of bonds of type $K$ and $L$ in the vertex operator $W_\epsilon(\zeta_1)$, regarded now as finite but large. To derive it, note that the transformation $\zeta_1 \to -\zeta_1$ induces $K \to K+i\pi/2$, $L \to L-i\pi/2$ in the operator $W_\epsilon(\zeta_1)$ of figure 3.1, and the Boltzmann weights transform as $\exp(\pm K) \to \pm(i)\exp(\pm K)$, $\exp(\pm L) \to \pm(-i)\exp(\pm L)$. $F_\epsilon(\pm\zeta_1, \zeta_2)$ are sums over configurations of the system. For the ground state term the relation is obvious; after replacing $\zeta_1$ by $-\zeta_1$, $\rho$ accounts for the phase factors $(\pm i)$ and the sign changes if $\epsilon = -1$ since one pair of spins are not aligned in that case. But all other terms in the sum differ from the ground state only by having an even number of further spin pairs of $W_\epsilon(\zeta_1)$ which are not aligned; so every term in $F_\epsilon$ satisfies the same relation. From that, the result (3.3) follows.

3.2 Functional equation. Deriving the functional ($q$-difference) equation involves quite a few steps which we describe in detail. Start with the function $F'_\epsilon(\zeta_1, \zeta_2)$, and perform the following steps.
(i) Rotate the vertex operator $V_\epsilon(\zeta_1)$ $180^\circ$ anticlockwise, using the boost property together with the cyclic property of the trace, to get the left hand picture of figure 3.2. As a result, $\zeta_1 \to x\zeta_1$.

(ii) Interchange the rapidities using (2.7b), to give equality with the right hand picture.

(iii) Rotate $W_\epsilon(x\zeta_1)$ $180^\circ$ anticlockwise to get the lattice which defines $F_\epsilon(x^2\zeta_1, \zeta_2)$. As a result we find

\[ F'_\epsilon(\zeta_1, \zeta_2) = F_\epsilon(x^2\zeta_1, \zeta_2). \]  \hspace{1cm} (3.4)

For the second relation we proceed as in figure 3.3:

(i) Starting with $F_\epsilon(\zeta_1, \zeta_2)$, rotate the vertex operator $W_\epsilon(\zeta_1)$ $180^\circ$ anticlockwise to get the left hand picture of figure 3.3. Again we have $\zeta_1 \to x\zeta_1$.

(ii) Interchange the rapidities, this time using (2.7a), to get to the the right hand picture. This requires that we use the coupling $H(\zeta_2/\zeta_1)$.

(iii) Rotate $V_\epsilon(x\zeta_1)$ $180^\circ$ anticlockwise to recover the lattice which defines $F'_\epsilon(x^2\zeta_1, \zeta_2)$.

Putting all of this together, we get the functional equations

\[ RF_\epsilon(\zeta_1, \zeta_2) = \sum_\xi e^{H(\zeta_2/x\zeta_1)\xi}\epsilon' \epsilon F'_{\epsilon}(x^2\zeta_1, \zeta_2). \]  \hspace{1cm} (3.5)

Finally, by substituting either of (3.4, 3.5) into the other, we get uncoupled functional equations for either of the two functions.

Changing to the functions $G(\xi), G'(\xi)$ removes the dependence on the factor $R$ which comes from the star-triangle relation, and gives functional equations in a single variable. We also set $x = q^{1/2}$ to get

\[ G(\xi) = \tanh H(1/q^{1/2}\xi)G(q^2\xi), \]

\[ G'(\xi) = \tanh H(1/q^{3/2}\xi)G'(q^2\xi). \]
3.3 Solution of functional equation. We shall only discuss the solution for \( G(\xi) \). Using the formula (2.2), we obtain the basic functional equation,

\[
\frac{G(\xi/q^2)}{G(\xi)} = \frac{(\xi/q; q^2)(q^3/\xi; q^2)(-\xi; q^2)(-q^2/\xi; q^2)}{(-\xi/q; q^2)(-q^3/\xi; q^2)(\xi/q^2)(q^2/\xi; q^2)}.
\]

For the purpose of solving functional equations, note that the functions

\[
f(\xi) = (p^{\alpha+2\xi}; p^2, q^2), \quad g(\xi) = (p^\alpha/\xi; p^2, q^2),
\]

satisfy

\[
\frac{f(\xi/p^2)}{f(\xi)} = (p^\alpha\xi; q^2), \quad \frac{g(\xi/p^2)}{g(\xi)} = \frac{1}{(p^\alpha/\xi; q^2)}.
\]

Here we employ the double infinite products

\[
(z; p_1, p_2) = \prod_{n_1, n_2 = 0}^{\infty} (1 - zp_1^{n_1}p_2^{n_2}).
\]

In the present case this gives

\[
G(\xi) = \frac{(q\xi; q^2, q^2)(-q^3/\xi; q^2, q^2)(-q^2\xi; q^2, q^2)(q^2/\xi; q^2, q^2)}{(-q\xi; q^2, q^2)(q^3/\xi; q^2, q^2)(q^2\xi; q^2, q^2)(-q^2/\xi; q^2, q^2)}.
\]

Notice that a possible normalising factor depending on \( k \) is fixed (to unity) by the inversion relation \( G(\xi)G(-\xi) = 1 \) of equation (3.3). Substituting now into (3.2), we arrive at the formula

\[
M_0 = G(1) = \frac{(-q^3; q^2, q^2)(q; q^2, q^2)}{(q^3; q^2, q^2)(-q; q^2, q^2)},
\]

where half the terms already cancelled. Noting further that the remaining ratios fit into the patterns of (3.6), we arrive at Onsager’s original result

\[
M_0 = \frac{(q; q^2)}{(-q; q^2)} = (1 - k^2)^{1/8}.
\]

So, although the form \((1 - k^2)^{1/8}\) is simple, the infinite product carries a message about the underlying structure of the problem. It is interesting to note that Yang derived precisely this infinite product in his original paper in which he proved the Onsager formula [2].

3.4 Uniqueness of solution. The fundamental assumption of the method is that the function \( G(\xi) \) is meromorphic in \( \xi \) and that its analytic properties are determined by the analytic properties of the Ising couplings as parametrized in (2.1). That is, it is a doubly periodic function of \( u \). Even so, the functional equations by themselves determine their solution only to within a “pseudo-constant”; that is, a solution of
\[ G(\xi) = G(q^2\xi). \] Such a function is doubly periodic in the variable \( u \) so it is fully determined by the structure of its poles and zeros.

Our basic hypothesis is that the analytic structure of \( G(\xi) \), which is actually the quotient of two functions \( F_{\pm} \), is completely determined by the factor \( \tanh H \) which occurs in the equation for that quotient. In this case the solutions will only consist of the infinite product of those factors, which also will automatically impose the normalisation up to a constant depending only on \( k \). This constant was determined from the inversion relation (3.3). The calculations, and the arguments employed, are exactly parallel to those of [15] for the spontaneous polarization of the eight-vertex model. This approach also has some precedent, although not so explicitly stated, in Baxter’s book [12]. For example, the working of pp231-236, to obtain the free energy of the eight-vertex model, is exactly the solution of a functional equation (10.8.24) of the type we have just solved, and the solution is the infinite product of just those terms which determine the relation.

In the case just cited [12], arguments are presented that the solution actually has the appropriate meromorphic properties. Actually, for the Ising model it is possible to construct an explicit analytic expression for \( G(\xi) \) as an integral [25], using the fermionic description of the CTMs [21, 22], and check that it does have the correct structure.

For the results which we present in the remainder of this paper we know of no such integral expressions, but there are many points of contact with previous results, some analytic, many numerical, and in all cases we have carefully verified that there is exact agreement. It is also the case that all of our results can be viewed as special cases of a single function. But the result for a (10) surface has long been known [3], moreover one of our results for a (10) corner has recently been obtained independently by solving integral equations [8]. Each of these checks is for the values of the meromorphic function obtained here along a smooth curve in the complex plane, which is very strong evidence for the validity of our approach.

4. Surface magnetizations

In this section we generalize the method to a lattice with a (11) surface. This will allow us to calculate the magnetizations in the first and second surface row.

4.1 Boundary reflection. The essence of the bulk calculation is a rotation of vertex operators through a total of 360°. For a system with a boundary this is not possible, but instead one can use a procedure in which the vertex operators are rotated back and forth, being “reflected” from the boundary after a change of the rapidity. This mechanism forms the basis of all further calculations.

We are interested in simple free boundaries and we now show how this case can be handled in an elementary way. Consider the system shown in figure 4.1. Free boundaries require that we sum over the edge spins. The effect of this is manifested through terms of the form

\[ \sum_{\sigma} e^{(K\sigma_1 + L\sigma_2)\sigma} = 2(\cosh K \cosh L + \sigma_1 \sigma_2 \sinh K \sinh L), \]
where $\sigma$ is a boundary spin, and $\sigma_1$, $\sigma_2$ the two second row spins joined directly to it by $K$ and $L$. Now the crucial point is that the right hand side is completely symmetric with respect to $K$, $L$ and also $\sigma_1$, $\sigma_2$. The interchange of all pairs $K$, $L$ in a vertex operator which is adjacent to a free edge is effected by $\zeta \rightarrow x/\zeta$; this gives the following simple construction:

*a vertex operator adjacent to a free edge can have its rapidity “reflected” off the boundary via the transformations $V_\epsilon(\zeta) \leftrightarrow V_\epsilon(x/\zeta)$, $W_\epsilon(\zeta) \leftrightarrow W_\epsilon(x/\zeta)$.*

This avoids the complication of having alternating rapidities and “double row transfer matrices” which are normally associated with the use of boundary reflections and $K$ matrices [26].

![Diagram of System with a Free Boundary](image)

**Figure 4.1**: System with a free boundary.

From this reflection property follow immediately two symmetries of the expectation value shown in figure 4.1:

$$F_\epsilon(\zeta_1, \zeta_2) = F_\epsilon(x/\zeta_1, \zeta_2) = F_\epsilon(\zeta_1, x/\zeta_2). \quad (4.1)$$

Finally, we note that the inversion relation (3.3) still holds, since the ground state configuration, even with free boundaries, still has all spins except $\epsilon$ fixed to $+$, and the argument of section 3 then goes through.

### 4.2 Functional equations

We now derive functional equations for the system shown in figure 4.1. In order that we can re-use these equations for a corner, we temporarily replace $x$ by an arbitrary constant $p$ in the CTM $A^{(NS)}(x/\zeta)$, dealing instead with $A^{(NS)}(p/\zeta)$. The boundary reflection property, however, still involves the variable $x$.

We shall also consider directly the function $G(\zeta_1, \zeta_2)$, as for the bulk magnetization. For the first functional equation, we require the following steps (see figure 4.2):

(i) Reflect the rapidity $\zeta_1$ off the right boundary; $\zeta_1 \rightarrow x/\zeta_1$.  
(ii) Rotate the right vertex operator $180^\circ$ anti-clockwise so that the two are adjacent; $x/\zeta_1 \rightarrow px/\zeta_1$.  
(iii) Interchange the rapidities by multiplying $G$ with the first factor $\tanh H(\zeta_1 \zeta_2 / px)$.  
(iv) Reflect the rapidity $px/\zeta_1$ off the left boundary; $px/\zeta_1 \rightarrow \zeta_1 / p$.  
(v) Interchange a second time multiplying with the second factor $\tanh H(\zeta_1 / p \zeta_2)$.  
(vi) Rotate back $180^\circ$ to get the original system with $\zeta_1 \rightarrow \zeta_1 / p^2$. 
From these steps we obtain

\[ G(\zeta_1, \zeta_2) = \tanh H(\zeta_1 \zeta_2 / px) \tanh H(\zeta_1 / p \zeta_2) G(\zeta_1 / p^2, \zeta_2). \]  

(4.2a)

4.3 Solution of the functional equations. We have quite a number of relations which must be solved in a consistent way. The first point to notice is that there are two factors which determine the solution of the functional equations; one depends on the ratio \( \zeta_1 / \zeta_2 \) and the other on the product \( \zeta_1 \zeta_2 \). Therefore we use the Ansatz

\[ G(\zeta_1, \zeta_2) = \Phi(\zeta_1 / \zeta_2) \Psi(\zeta_1 \zeta_2). \]  

(4.3)

Imposing the symmetries (4.1) on this Ansatz gives

\[ \Phi(1/\xi) = \Phi(\xi), \quad \Psi(x/\xi) = \Psi(x\xi), \quad \Psi(\xi) = \Phi(\xi/x), \]  

(4.4)

so we need only find a single function \( \Phi(\xi) \). Equations (4.2) give two functional equations for each of \( \Phi(\xi) \) and \( \Psi(\xi) \), however it is easy to check that they are all equivalent to the single equation

\[ \Phi(\xi) = \tanh H(\xi/p) \Phi(\xi/p^2). \]  

(4.5)

Using (3.6) and requiring that \( \Phi(\xi) \Phi(-\xi) = 1 \), we obtain

\[ \Phi(\xi) = \frac{(pq^{1/2}/\xi; p^2, q^2)_{-pq^{3/2}/\xi; p^2, q^2}(-pq^{3/2}/\xi; p^2, q^2)(pq^{1/2}/\xi; p^2, q^2)}{(-pq^{1/2}/\xi; p^2, q^2)(pq^{3/2}/\xi; p^2, q^2)(pq^{3/2}/\xi; p^2, q^2)(-pq^{1/2}/\xi; p^2, q^2)} \]  

(4.6)
Recall that, in order to use these results for an edge, we must set
\[ p = x = q^{1/2}. \]

4.4 First row magnetization. The result for \( G(\zeta_1, \zeta_2) \) gives directly the magnetization in the second row, which will be discussed below. But it also contains the result for the edge, because upon rotating \( V_\epsilon \) and \( W_\epsilon \) into a vertical position, as shown in figure 4.3, the spin \( \epsilon \) appears in the first row. For a homogeneous lattice, the rapidities then have to equal \( x/\zeta \). Their original values before the rotation thus must be \( \zeta_1 = 1, \zeta_2 = x \), as indicated also in the figure. As a result, the edge magnetization is

\[ M_1 = \Phi(1)\Phi(x). \]

![Figure 4.3: Magnetization in first row.](image-url)

For both functions \( \Phi \), the double infinite products can be simplified into single infinite products as in section 3, giving

\[ M_1 = \frac{(q;q^2)^2(q^{1/2};q)}{(-q;q^2)^2(-q^{1/2};q)}. \]

According to (3.7) the first factor equals \((1 - k^2)^{1/4}\). The second one differs from it by the substitution \( q \to q^{1/2} \), which corresponds to a Landen transformation \( k \to 2k^{1/2}/(1 + k) \) of the modulus [27]. This leads to

\[ M_1 = (1 - k)^{1/2}, \quad (4.7) \]

which corresponds exactly with the result in [28], which was obtained for the isotropic case. We see, however, that \( M_1 \) is independent of the anisotropy, as it must be, since it is the matrix element of a spin operator between the boundary state and the maximal eigenvector of the row transfer matrix, both of which are independent of \( \zeta \). This result also shows directly the surface exponent \( \beta_s = 1/2 \).

4.5 Second row magnetization. This is obtained by setting \( \zeta_1 = \zeta_2 = \zeta \) in \( G(\zeta_1, \zeta_2) \), so that

\[ M_2 = \Phi(1)\Phi(\zeta^2/x), \]
which does depend on the anisotropy. The dependence can, however, be simplified very much. By reducing the double products to single products using (3.6), then further to theta functions by (2.3), one first shows that

$$\Phi(\zeta^2/x) = -i(k')^{1/2} \frac{\text{sn}(i(u+I'/2))}{\text{cn}(i(u+I'/2))}.$$

Using addition theorems for elliptic functions and substituting the Ising couplings from (2.1) finally leads to

$$M_2 = (1-k)^{1/2} \coth(K+L).$$

This implies a simple relation between $M_2$ and $M_1$ which can also be derived directly [29]. The expression (4.8) is symmetric in $K$ and $L$, and $M_2$ exceeds $M_1$ for all finite positive $K$, $L$, as it should. Only in the anisotropic limit, where one diagonal line of couplings becomes dominant, one has $M_2 \to M_1$. This can already be seen from the initial expression for $M_2$ by setting $\zeta = 1$ and using $\Phi(x) = \Phi(1/x)$.

In the isotropic case ($\zeta^2 = x$) one has

$$M_2^{iso} = (1-k^2)^{1/2} = M_0^4,$$

which was already noted in [28]. An analogous relation between bulk and surface quantities was found in [20] for the eight-vertex model. In that case the edge polarization (in the isotropic case) is the square of the bulk polarization. One can show that this result implies (4.9) at the decoupling point.

The dependence of $M_2$ on the anisotropy is physically not obvious, but it is exactly what one expects from an eigenvector calculation, for which we would calculate the matrix element between the maximal eigenvector of the row transfer matrix and the vector obtained by acting on the boundary state with the row transfer matrix. However, such a calculation would be technically quite intricate compared with the present method.

5. Corner magnetization

In this section we consider $90^\circ$ corners formed by (11) surfaces and calculate the magnetization for three different spin positions.

![Figure 5.1: Truncated corner with free boundaries.](image)
5.1 **Truncated corner.** We first consider the situation, shown in figure 5.1, namely a corner to which two vertex operators are attached, so that the tip is missing. This is the analogue of the edge geometry of figure 4.1, and all considerations of the previous section can be taken over immediately. The only difference is that now the vertex operators are rotated only through one CTM, so that in deriving the formula for \( G(\zeta_1, \zeta_2) \) one must set

\[
p = \zeta.
\]

The boundary reflection property, however, still involves the variable \( x \). Therefore the equations (4.2, 4.3, 4.4, 4.6) are unchanged and the magnetization of the centre spin \( \epsilon \) has the form

\[
 M_2(\zeta) = \Phi(1) \Phi(q^{1/2}/\zeta^2).
\] (5.1)

We postpone a discussion of the anisotropy dependence and first look at the isotropic case, \( \zeta = x^{1/2} = q^{1/4} \). Then

\[
 M_2^{iso} = \frac{(q^{3/4}; q^{1/2}, q^2)^4(-q^{7/4}; q^{1/2}, q^2)^4}{(-q^{3/4}; q^{1/2}, q^2)^4(q^{7/4}; q^{1/2}, q^2)^4}
 = \frac{(q^{3/4}; q^2)^4(q^{5/4}; q^2)^4}{(q^{3/4}; q^2)^4(q^{5/4}; q^2)^4}
\]

The products can be expressed as a ratio of Jacobian elliptic functions with the help of (2.3),

\[
 \frac{(q^{3/4}; q^2)(q^{5/4}; q^2)}{(-q^{3/4}; q^2)(-q^{5/4}; q^2)} = -(k')^{1/2} \frac{\text{sn}(3iI'/4)}{\text{cn}(3iI'/4)}.
\]

Using Landen transformations to obtain the special values of \( \text{sn} \) and \( \text{cn} \) finally leads to the result

\[
 M_2^{iso} = \left( \frac{1 - \sqrt{k}}{1 + \sqrt{k}} \right) \left( \sqrt{k} + \sqrt{1 + k} \right)^2.
\] (5.2)

\[\begin{align*}
\text{Figure 5.2: Neveu-Schwarz and Ramond corners.}
\end{align*}\]

5.2 **Neveu-Schwarz corner.** This corner, shown on the left of figure 5.2, is obtained by adding a single spin with the coupling \( K \) to the truncated corner. The magnetization \( M_2 \) is not changed by this extra coupling while the magnetization \( M_1 = \langle \epsilon \rangle \) is given by

\[
 M_1(\zeta) = \tanh K(\zeta) \cdot M_2(\zeta).
\] (5.3)
Again, this simplifies very much for the isotropic case \( \zeta = x^{1/2} = q^{1/4} \). One can then either reduce the products introduced by \( \tanh K(\zeta) \) as before, or simply substitute \( \tanh K(\zeta) = (\sqrt{k} + \sqrt{1+k})^{-1} \) to obtain

\[
M_{1}^{iso} = \left( \frac{1 - \sqrt{k}}{1 + \sqrt{k}} \right) \left( \sqrt{k} + \sqrt{1+k} \right).
\] (5.4)

5.3 Ramond corner. This geometry is shown on the right of figure 5.2, where it is also indicated that one can obtain it from figure 5.1 by rotating \( W_\epsilon(\zeta_1) \) through the quadrant. Therefore the magnetization \( M_3 \) of the spin \( \epsilon \) follows by putting \( \zeta_1 = 1, \zeta_2 = \zeta \) in \( G(\zeta_1, \zeta_2) \) so that

\[
M_3(\zeta) = \Phi(\zeta)\Phi(\zeta/x),
\]

The isotropic case is readily extracted since the double infinite products appear in a form which enables their reduction to single products by (3.6), leading to

\[
M_3^{iso} = (1 - k).
\] (5.5)

This formula was already given in [28]. By contrast, the other analytical results are new. In particular those for \( M_1^{iso} \) and \( M_2^{iso} \) have been verified against the numerical calculations carried out previously in [28, 30].

5.4 Critical behaviour. From the results (5.2, 5.4, 5.5) one reads off directly the corner exponent \( \beta_c = 1 \) of the isotropic system. However, for the corner considered here, \( \beta_c \) is not a constant but depends on the anisotropy (as it depends in general on the opening angle of the corner). This dependence can be calculated in a straightforward way from the exact result (5.1). While there is no general simplification of the double infinite products in that case, it is not difficult to find the asymptotic form near the critical point \( (k \to 1) \) and hence evaluate \( \beta_c \).

The function \( \Phi(\zeta) \) consists of factors which have the form

\[
\phi(\xi, \eta) = \frac{(\xi; p^2, q^2)(-\eta; p^2, q^2)}{(-\xi; p^2, q^2)(\eta; p^2, q^2)}.
\]

For \( k \to 1 \) one has [27] \( I \to \infty, I' \approx \pi/2 \), so that \( q \) as well as \( \zeta \), and therefore the quantities \( p, \xi, \eta, \) all approach one. Therefore we set

\[
q = e^{-\epsilon}, \quad p = e^{-\alpha \epsilon}, \quad \xi = e^{-2x \epsilon}, \quad \eta = e^{-2y \epsilon},
\]

with \( \epsilon \approx -\pi^2 / \ln(1 - k) \to 0 \) for \( k \to 1 \), whereupon

\[
\ln \phi(\xi, \eta) = \sum_{m,n=0}^{\infty} \ln \left\{ \frac{\tanh(m + n\alpha + x)\epsilon}{\tanh(m + n\alpha + y)\epsilon} \right\}.
\]

For the asymptotic form we replace the double sum with a double integral and apply the approximation

\[
\frac{\tanh(s + \Delta)}{\tanh(s)} \approx 1 + \frac{2\Delta}{\sinh 2s},
\]
Then
\[ \ln \phi(\xi, \eta) \approx \frac{2(y - x)}{\alpha \epsilon} \int_0^\infty ds \int_0^\infty dt \frac{ds \, dt}{\sinh 2(s + t)} = \frac{\pi^2}{8 \epsilon} \left( \frac{x - y}{\alpha} \right). \]

More generally we see that
\[ \sum_{i=1}^n \ln \phi(\xi_i, \eta_i) \approx \frac{\pi^2}{8 \alpha \epsilon} \sum_{i=1}^n (x_i - y_i). \]

Applying this to \( \Phi(\zeta) \) we have
\[ \Phi(\zeta) \approx \exp \left( -\frac{\pi^2}{8 \alpha \epsilon} \right), \]  \hspace{1cm} (5.6)

which is dependent on \( p \) via \( \alpha \), but not explicitly dependent on the argument \( \zeta \). Setting finally \( p = \zeta \) fixes \( \alpha = u/\pi \). With the value of \( \epsilon \) given above it follows that the corner exponent is
\[ \beta_c = \frac{1}{4 \alpha} = \frac{\pi}{4 u}, \]  \hspace{1cm} (5.7)

Since one is near \( k = 1 \), the parameter \( u \) can be expressed from (2.1) as
\[ u = \arctan \left( \frac{1}{\sinh 2L_c} \right), \]
where \( L_c \) is the critical coupling across the corner. This is precisely the result one obtains by rescaling the anisotropic system onto an isotropic one with effective opening angle \( \theta = 2u \), together with the conformal prediction \( \beta_c = \pi/2 \theta \) for the isotropic case [6, 7]. The present direct calculation is the first analytical proof of this argument. Because \( \Phi(\zeta) \) in (5.6) does not depend explicitly on \( \zeta \), the same result for \( \beta_c \) follows from \( M_3 \), as it should.

Finally, a treatment of corners with other opening angles would just lead to other values of \( p \). For example, the straight surface corresponds to \( p = x = q^{1/2} \), so that \( \alpha = 1/2 \) and one recovers the surface exponent \( \beta_s = 1/2 \). Thus (5.7) is the general result and \( u \) indeed plays the rôle of an effective angle.

5.5 Magnetization curves. In order to give an impression how the corner magnetization varies with temperature for different values of the anisotropy, we have calculated \( M_1 \) numerically from equation (5.3). The results are shown in figure 5.3 for five values of the ratio \( L/K \) of the couplings (rather than \( \zeta \)). One can see that for large \( L/K \) the magnetization curve rises only slowly as the temperature is decreased. This mirrors the almost one-dimensional character of a system with strong couplings across the corner. Conversely, \( M_1 \) increases rapidly for small \( L/K \). The variation of the critical behaviour as described by \( \beta_c \) is just a particular aspect of this general pattern.
6. Surfaces in the principal direction

The two previous sections have treated surfaces and corners formed by edges taken in the diagonal (11) direction. In this section we shall show how to treat edges in the principal (10), (01) directions using a layered system.

Figure 6.1: Layered system with a free boundary.

6.1 Layered system. Consider the system shown in figure 6.1. It is essentially the same as figure 4.1, except that we introduce vertical rapidities \( p_i \) which alternate between columns. The meaning of the CTMs and vertex operators which appear in the figure are exactly as before, except that the couplings are calculated using the
ratio of horizontal to vertical rapidities, viz

\[
K = K(\zeta/p), \quad \zeta \in \{\zeta, \zeta_1, \zeta_2\}, \\
L = L(\zeta/p), \quad p \in \{p_1, p_2, p_3, p_4\}.
\]

We are particularly interested in some special choices of the \(p\)'s which produce (10) and (01) surfaces. These are shown in figures 6.2 and 6.3. Consider for example figure 6.2(a). To get it from the general layered system we must choose \(p_1\) and \(p_4\) so that the couplings which belong to their columns take the limits \(K(\zeta/p) \to \infty\), \(L(\zeta/p) \to 0\). For this we must set \(p_1 = p_4 = \zeta/x\). This effectively removes half of the spins in the lattice, since each pair which becomes identified by our choice is then equivalent to a single spin. Then we choose \(p_2 = p_3 = 1\) since we do not wish to alter the couplings in the other two columns. Finally we must set \(\zeta_1 = \zeta_2 = \zeta\) to get the homogeneous lattice with the (10) surface shown in figure 6.2(a). These specializations will be made after the general solution has been determined.

\[
\text{Figure 6.2: Surfaces with (10) and (01) free boundaries.}
\]

Similar considerations give figure 6.2(b). So for figure 6.2 we need.

\[
\begin{align*}
\text{(6.1a)} & \quad p_2 = p_3 = 1, \quad p_1 = p_4 = \zeta/x, \\
\text{(6.1b)} & \quad p_1 = p_4 = 1, \quad p_2 = p_3 = \zeta.
\end{align*}
\]

\[
\text{Figure 6.3: Corners with (10) and (01) free boundaries.}
\]

Figure 6.3 shows two types of corner, 90° and 270°. For these the appropriate values are

\[
\begin{align*}
\text{(6.1c)} & \quad p_2 = p_4 = 1, \quad p_1 = \zeta/x, \quad p_3 = \zeta, \\
\text{(6.1d)} & \quad p_1 = p_3 = 1, \quad p_2 = \zeta, \quad p_4 = \zeta/x.
\end{align*}
\]
We write $A(\zeta; p_3, p_4)$, $B(\zeta; p_1, p_2)$, $W_\epsilon(\zeta_1; p_1, p_2)$, $V_\epsilon(\zeta_2; p_3, p_4)$ for the operators shown in figure 6.1. Just as in section 2, these operators are related by crossing symmetry. Remembering that the adjoint reverses the direction of transfer, and that the crossing symmetry must now be written as $\zeta/p \rightarrow xp/\zeta$, we see that

$$B(\zeta; p, p') = A'(x/\zeta; 1/p, 1/p'),$$
$$V_\epsilon(\zeta; p, p') = W_\epsilon'(x/\zeta; 1/p, 1/p').$$

We need the boost properties and commutation relations for these operators. A short derivation is given in the appendix; here we state just the properties used in this section. Since we only have to rotate the operator $W_\epsilon(\zeta_1; p_1, p_2)$ through $180^\circ$, we require the composition of two boosts. It is

$$A^{(R)}(\zeta; p_3, p_4)B^{(R)}(\zeta; p_1, p_2)W_\epsilon(\zeta_1; p_1, p_2) = W_\epsilon(x_{\zeta_1}; p_3, p_4)A^{(NS)}(\zeta; p_3, p_4)B^{(NS)}(\zeta; p_1, p_2).$$

(6.2)

Next, we need the commutation relations (2.7) for the vertex operators which have now acquired extra rapidities $p_i$. The important point is that the $180^\circ$ rotation changes the vertical rapidities $p_1, p_2$ to $p_3, p_4$, so they play no further rôle in the commutation because they will match for the two operators which must be multiplied; this means that equations (2.7) actually remain valid with the same choice of couplings, which depend only on the horizontal rapidities.

The last ingredient is the “boundary reflection”. Recall that this is simply the interchange of adjacent pairs $K(\zeta/p)$, $L(\zeta/p')$ which are joined to a common spin on the free boundary, since the sum is symmetric in both the variables and the other pair of spins. This means that we want to make the replacements $\zeta/p \rightarrow xp'/\zeta$, $\zeta/p' \rightarrow xp/\zeta$. At first sight this brings the difficulty that the vertical rapidities have changed place. But we observe that the replacement may also be written as $\zeta/p \rightarrow (xp'/\zeta)/p$, $\zeta/p' \rightarrow (xp'/\zeta)/p'$, which is equivalent to a change in only the horizontal rapidity by $\zeta \rightarrow (xp'/\zeta)$. Thus we have the simple rule:

- a vertex operator adjacent to a free edge can have its rapidity “reflected”
- off the boundary via the transformations $V_\epsilon(\zeta_1; p_1, p_2) \leftrightarrow V_\epsilon(x_{p_1p_2}/\zeta; p_1, p_2)$,
- $W_\epsilon(\zeta_1; p_3, p_4) \leftrightarrow W_\epsilon(x_{p_3p_4}/\zeta; p_3, p_4)$.

6.2 Functional equations. We now derive functional equations for the system shown in figure 6.1. We shall re-use the figures and steps which were used in section 4, except that we retain the original meaning of $x = q^{1/2}$ in the CTMs. For the first functional equation, we require the following steps (see figure 4.2):

(i) Reflect the rapidity $\zeta_1$ off the right boundary: $\zeta_1 \rightarrow xp_1p_2/\zeta_1$.
(ii) Rotate the right vertex operator $180^\circ$ anti-clockwise so that the two are adjacent: $x_{p_1p_2}/\zeta_1 \rightarrow x^2p_1p_2/\zeta_1$.
(iii) Interchange the rapidities by multiplying $G$ with the factor $\tanh H(\zeta_2/x^2p_1p_2)$.
(iv) Reflect the rapidity $x^2p_1p_2/\zeta_1$ off the left boundary: $x^2p_1p_2/\zeta_1 \rightarrow p_3p_4\zeta_1/x_{p_1p_2}$.
(v) Interchange a second time multiplying with the factor $\tanh H(p_3p_4\zeta_1/x_{p_1p_2})$. 

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As before we need only find a single function $\Phi(\xi)$ equivalent to (4).
The boundary reflection property imposes symmetries which are a generalisation of (4.1).
These equations, and equations (6.5) below, generalize those of section 4, to which they reduce if all $p_i = 1$.

6.3 Solution of the equations. We follow the procedure of section 4 by using the Ansatz

$$G(\zeta_1, \zeta_2) = \Phi(\zeta_1/\zeta_2)\Psi(\zeta_1\zeta_2).$$

The boundary reflection property imposes symmetries which are a generalisation of (4.1), namely $G(\zeta_1, \zeta_2) = G(xp_1p_2/\zeta_1, \zeta_2) = G(\zeta_1, x\zeta p_1p_2p_4/\zeta_2)$, which manifest themselves as

$$\Phi(1/p_3p_4\xi) = \Phi(p_1p_2\xi), \quad \Psi(xp_1p_2/\xi) = \Psi(xp_3p_4\xi), \quad \Psi(\xi) = \Phi(xp_1p_2/\xi).$$

As before we need only find a single function $\Phi(\xi)$, since equations (6.3) are all equivalent to

$$\Phi(\xi) = \tanh H(x\xi/r^2)\Phi(\xi/r^2).$$

The solution may be written down by reference to equations (4.5, 4.6), whereupon one sees that it is only required to make the substitutions $p \to r$, $\xi \to x\xi/r = q^{1/2}\xi/r$ in order to obtain

$$\Phi(\xi) = \frac{(r^2/\xi; r^2, q^2)(-q^2\xi; r^2, q^2)(-qr^2/\xi; r^2, q^2)(q\xi; r^2, q^2)}{(-r^2/\xi; r^2, q^2)(q^2\xi; r^2, q^2)(q\xi; r^2, q^2)(-q^2\xi; r^2, q^2)}.$$

The actual solution for any particular surface or corner is obtained by substituting this result, together with the special values of the parameters $p_i$, into (6.4, 6.5).

6.4 90° corner. The relevant parameters for this corner are given in (6.1c) from which $r^2 = x = q^{1/2}$. For the magnetisation, we have

$$M_c(\zeta) = \Phi(1)\Phi(1/\zeta), \quad r = q^{1/4}.\tag{6.7}$$

In the isotropic case, $\zeta = q^{1/4}$, we have already evaluated both products; $\Phi(1)$ in section 4.4 and $\Phi(q^{-1/4})$ in section 5.1. Reusing those results, we have

$$\Phi(1) = \sqrt{1-k}, \quad \Phi(q^{-1/4}) = \left(\frac{1 - \sqrt{k}}{1 + \sqrt{k}}\right)^{1/2} \left(\sqrt{k} + \sqrt{1+k}\right),$$
giving
\[ M_c^{iso} = (1 - \sqrt{k})(\sqrt{k + 1 + k}). \]  
(6.8)

We can show that the general formula (6.7) agrees with the known result of Kaiser and Peschel [8, 10],
\[ M_c = 1 - (\coth K - 1)(\coth L - 1)/2. \]  
(6.9)

Using the simple identity \((\coth K - 1)^2 = 2(\cosh 2K - \sinh 2K)/(\cosh 2K - 1)\), it may be written in terms of half-angles as
\[ M_c(\zeta) = (1 - k) \frac{\text{dn}(iu/2)^2 - k \text{cn}(iu/2)^2}{\text{dn}(iu/2)\text{dn}(iu/2) + ik \text{sn}(iu/2)\text{cn}(iu/2)}. \]  
(6.10a)

To see that this is the same as (6.7), first note that when we set \(r^2 = q^{1/2}\) in \(\Phi(1/\zeta)\), the double products simplify to elliptic functions as follows:
\[ \Phi(1/\zeta) = \frac{(q^{1/2}; q^{1/2}; q^{1/2})(-q^{1/2}; q^{1/2}; q^{1/2})(-q^{3/2}; q^{1/2}; q^{1/2})(-q^{1/2}; q^{1/2}; q^{1/2})}{(-q^{1/2}; q^{1/2}; q^{1/2})(q^{1/2}; q^{1/2}; q^{1/2})(q^{3/2}; q^{1/2}; q^{1/2})(-q^{1/2}; q^{1/2}; q^{1/2})} \]
\[ = \frac{(q^{1/2}; q^{2})(q^{1/2}; q^{2})(q^{3/2}; q^{2})(q^{1/2}; q^{2})}{(-q^{1/2}; q^{2})(-q^{2}; q^{2})(-q^{3/2}; q^{2})(q^{1/2}; q^{2})} \]
\[ = -k' \frac{\text{sn}(i(u/2 + I'/2)) \text{sn}(i(u/2 + I'))}{\text{cn}(i(u/2 + I'/2)) \text{cn}(i(u/2 + I'))}. \]

Using addition theorems to eliminate the quarter period \(I'\) gives the form
\[ M_c(\zeta) = (1 - k) \frac{\text{cn}(iu/2) \text{dn}(iu/2) - i(1 + k) \text{sn}(iu/2)}{\text{dn}(iu/2)(\text{cn}(iu/2) - i \text{sn}(iu/2) \text{dn}(iu/2))}. \]  
(6.10b)

After a fair amount of further calculation, one can show that the two formulae (6.10a, b) agree.

6.5 Surface. Using (6.1a) we have now \(r^2 = x^2 = q\), and for the magnetisation,
\[ M_s(\zeta) = \Phi(1)\Phi(1/\zeta), \quad r = q^{1/2}. \]  
(6.11a)

Substituting \(r^2 = q\) in the formula for \(\Phi\) and reducing the double products to single products gives
\[ \Phi(1/\zeta) = \frac{(q^{1/2}; q^{2})(q/\zeta; q^{2})}{(-q^{1/2}; q^{2})(-q/\zeta; q^{2})} = k'^{1/2} \text{nd}(iu/2). \]  
(6.11b)

It is easy to reconcile equations (6.11a, b) with the formula of McCoy and Wu [3] for the magnetization at a \((10)\) surface, namely
\[ M_s = \left(\frac{\cosh 2K - \cosh 2L}{\cosh 2K - 1}\right)^{1/2}, \]  
(6.12)
where $\hat{L}$ is the dual coupling; $\cosh 2\hat{L} = \coth 2L$. One simply substitutes from (2.1), and uses half angle formulae for the Jacobian elliptic functions, to obtain

$$M_s = (1 - k^2)^{1/2} \text{nd}(iu/2).$$

Finally, we note that choosing $\zeta_1 = \zeta_2 = \zeta'$ different from the bulk rapidity $\zeta$ leads to the system shown in figure 6.4. By summing out the spins in the first row one then obtains a $(10)$ surface with modified couplings in the boundary, a problem treated in detail by Au-Yang [31].

![Figure 6.4: System with a modified (10) boundary.](image)

6.6 270° corner. Using (6.1d) we have now $r^2 = x^3 = q^{3/2}$, and for the magnetisation,

$$M_c = \Phi(1)\Phi(q^{1/2}/\zeta), \quad r = q^{3/4}.$$ 

However, there does not appear to be any simple reduction to Jacobian elliptic functions in this case. This is evident from the fact that the two bases in the doubly infinite products, $q^{3/4}$ and $q^2$, are not related by Landen transformation in the simple way they were for the $90^\circ$ and $180^\circ$ cases. However, the asymptotic analysis of the section 5.4 is easily applied since the function $\Phi(\xi)$ of this section is essentially the same as there, and one sees that the critical exponent is

$$\beta_c = 1/3,$$

independent of the anisotropy, as it must be.

7. Conclusion

We have presented an approach which permits the calculation of order parameters in Ising models with various kinds of simple boundaries. The method uses CTMs which already on geometrical grounds are the natural tool for the shapes studied. But while a previous attempt to use them was only partly successful [30], the vertex operator technique leads to a very elegant and compact solution. Moreover it has a marked geometrical character which makes it easy to use. We have deliberately presented it in some detail, because so far the method is not widely known. By contrast, we have not addressed questions of the infinite lattice limit which is always involved implicitly when CTMs are introduced.
The results obtained from the functional equations all have the form of infinite products which reflect the spectrum of the CTMs as well as the matrix elements with the boundary state vectors. In general the formulae are quite complicated. But they are not difficult to derive, and we have seen that amazing simplifications may be achieved which can lead to very simple final expressions for the order parameters. It can in fact be quite difficult to find these reductions, and in a way, when found, the simple forms obscure the origin and structure of the results.

One can think of many applications to the Ising model beyond those studied in this paper. Here we have not studied two-point correlation functions, or indeed more general correlation functions, although they are solutions of similar (but more complex) functional equations. Other surface configurations could also be studied: for example, by adding more vertex operators at the boundaries one can in principle treat spins in deeper layers. Again, the layered system in section 6 has only been used for simple special cases related to a particular problem, and certainly not investigated in any generality. In this context one should remember that CTMs were themselves developed by considering inhomogeneous systems, as were vertex operators and their functional equations. Now, in this paper, inhomogeneities have been used to rotate the surface from the (11) to the (10) and (01) directions.

The method of \(q\)-difference equations is not confined to the Ising model, having been used in papers on the eight vertex model [15, 32], the \(Z_{n+1} \times Z_{n+1}\) generalized Heisenberg model [33], and for the XYZ chain with a boundary [20]. However the method is presented here in the language of spins for the first time, and this simplifies matters considerably compared with using ABF models, or taking the eight-vertex model at the decoupling point. As such it applies naturally to the chiral Potts model, although there are formidable difficulties in that case due to the fact that the rapidities lie on a high-genus Riemann surface.

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Appendix

Group property. We need to extend the group property (2.5) to include alternating vertical rapidities. The derivation is the same for both Neveu-Schwarz and Ramond sectors, and for the NW and NE corners. Here we choose \(A^{(NS)}(\zeta; p, p')\) and show that

\[
A^{(NS)}(\zeta_1; p, p')A^{(NS)}(\zeta_2) = A^{(NS)}(\zeta_1\zeta_2; p, p'),
\]

where \(A^{(NS)}(\zeta) = A^{(NS)}(\zeta; 1, 1)\). A simple derivation follows from the observation that \(A^{(NS)}(\zeta; p, p')\) may be regarded as the infinite product of vertical semi-infinite row transfer matrices [12],

\[
A^{(NS)}(\zeta_1; p, p') = \cdots W_\epsilon(xp'/\zeta_1)V_\epsilon(xp/\zeta_1)W_\epsilon(xp'/\zeta_1)V_\epsilon(xp/\zeta_1).
\]
Multiplying on the right by $A^{(NS)}(\zeta_2)$ and invoking (2.6) repeatedly (to commute the $V$ and $W$ through $A$) we arrive at the desired result. One can write it also with a scalar normalisation factor, but we assume that such factors are absorbed into the definition of the CTMs, as for the original group property (2.5).

**Boost property.** We sketch the derivation of the boost property (6.2). As in section 2, it is trivial to see that

\[
V_\epsilon(\zeta; p_1, p_2)A^{(NS)}(\zeta; p_1, p_2) = A^{(R)}(\zeta; p_1, p_2)V_\epsilon(1),
\]

\[
W_\epsilon(\zeta; p_1, p_2)A^{(R)}(\zeta; p_1, p_2) = A^{(NS)}(\zeta; p_1, p_2)W_\epsilon(1).
\]

We have also the original boost property (2.6) for $V_\epsilon(\zeta)$ and $W_\epsilon(\zeta)$, which may be used, together with (A1), to “lift” (A2) to

\[
V_\epsilon(\zeta_1 \zeta_2; p_1, p_2)A^{(NS)}(\zeta_1; p_1, p_2) = A^{(R)}(\zeta_1; p_1, p_2)V_\epsilon(\zeta_2),
\]

\[
W_\epsilon(\zeta_1 \zeta_2; p_1, p_2)A^{(R)}(\zeta_1; p_1, p_2) = A^{(NS)}(\zeta_1; p_1, p_2)W_\epsilon(\zeta_2).
\]

Alternatively, one can view this as an expression of $Z$-invariance [17] applied to the rotation of a rapidity line through $90^\circ$; in this process the alternating vertical rapidities present in the horizontal row are absorbed into the CTM.

Taking the transpose of (A2) gives the analogous property for $B^{(NS)}(\zeta; p, p')$:

\[
B^{(NS)}(\zeta_1; p_1, p_2)W_\epsilon(\zeta_1 \zeta_2; p_1, p_2) = W_\epsilon(\zeta_2)B^{(R)}(\zeta_1; p_1, p_2),
\]

\[
B^{(R)}(\zeta_1; p_1, p_2)V_\epsilon(\zeta_1 \zeta_2; p_1, p_2) = V_\epsilon(\zeta_2)B^{(NS)}(\zeta_1; p_1, p_2).
\]

Notice the factor $x$ which comes from the crossing symmetry by which we obtain the $B^{(NS)}(\zeta; p, p')$ corner from the $A^{(NS)}(\zeta; p, p')$. The required result (6.2) now follows from the composition of (A2) with (A3).

**References**

[1] G. S. Rushbrooke: On the theory of regular solutions, *Nuovo Cim. Supplement*, 6, (1949) 251–263 — see discussion p. 261.

[2] C. N. Yang: The spontaneous magnetization of a two-dimensional Ising model, *Phys. Rev.*, 85, (1952) 808–816.

[3] B. M. McCoy and T. T. Wu: Theory of Toeplitz determinants and the spin correlations of the two-dimensional Ising model: IV, *Phys. Rev.*, 162, (1967) 436–475.

[4] B. M. McCoy and T. T. Wu, *The two-dimensional Ising model*, Harvard University Press; Cambridge Mass. (1967).

[5] J. L. Cardy: Critical behaviour at an edge, *J. Phys. A: Math. Gen.*, 16, (1983) 3617–3628.

[6] M. N. Barber, I. Peschel and P. A. Pearce: Magnetization at corners in two-dimensional Ising models, *J. Stat. Phys.*, 37, (1984) 497–527.
[7] F. Iglói, I. Peschel and L. Turban: Inhomogeneous systems with unusual critical behaviour, Adv. Phys., 42, (1993) 683–740.

[8] D. B. Abraham and F. Latrémolière: Corner spontaneous magnetization, Phys. Rev., E50, (1994) R9–R11.
— : Corner spontaneous magnetization, J. Stat. Phys., 81, (1995) 539–559.

[9] D. B. Abraham: On the transfer matrix for the two-dimensional Ising model, Studies in Applied Mathematics, 50, (1971) 71–88.

[10] C. Kaiser and I. Peschel: Surface and corner magnetizations in the two-dimensional Ising model, J. Stat. Phys., 54, (1989) 567–579.

[11] I. Peschel: Surface magnetization in inhomogeneous two-dimensional Ising lattices, Phys. Rev., B30, (1984) 6783–6784.

[12] R. J. Baxter, Exactly Solved Models in Statistical Mechanics, Academic Press (1982).

[13] R. J. Baxter: Exactly Solved Models in Statistical Mechanics, Integrable systems in Statistical Mechanics VII, eds. G. M. d.Ariano, A. Monterosi and M. G. Rasetti, World Scientific, Singapore (1985) 5–63.

[14] B. Davies, O. Foda, M. Jimbo, T. Miwa and A. Nakayashiki: Diagonalization of the XXZ Hamiltonian by vertex operators, Comm. Math. Phys., 151, (1993) 89–153.

[15] M. Jimbo, T. Miwa and A. Nakayashiki: Difference equations for the correlation functions of the eight-vertex model, J. Phys. A: Math. Gen., 26, (1993) 2199–2209.

[16] B. Davies: Infinite Dimensional Symmetry of corner transfer matrices, Confronting the Infinite, ed. A. L. Carey et. al., World Scientific, Singapore (1995) 175–192.

[17] R. J. Baxter: Solvable eight-vertex model on an arbitrary planar lattice, Phil. Trans. Roy. Soc. Lond., 289, (1978), 315-346.

[18] G. E. Andrews, R. J. Baxter and P. J. Forrester: Eight-vertex SOS model and generalized Rogers-Ramanujan type identities, J. Stat. Phys., 35, (1984) 193–266.

[19] E. Date, M. Jimbo, A. Kuniba, T. Miwa and M. Okado: Exactly solvable SOS models: Local height probabilities and theta function identities, Nucl. Phys. B, 290, (1987) 231–273.

[20] M. Jimbo, R. Kedem, H. Konno, T. Miwa and R. Weston: Difference equations in spin chains with a boundary, Nucl. Phys. B, 448, (1995) 429–456.

[21] B. Davies: Corner transfer matrices for the Ising model, Physica A, 154, (1989) 1–20.

[22] T. T. Truong and I. Peschel: Diagonalisation of finite-size corner transfer matrices and related spin chains, Z. Phys. B, 75, (1989) 119–125.

[23] K. Sogo and M. Wadati: Boost operator and its application to quantum Gelfand-Levitan equation for Heisenberg-Ising chain with spin one-half, Prog. Theor. Phys., 69, (1983) 431–450.
[24] H. B. Thacker: Corner transfer matrices and Lorentz invariance on a lattice, *Physica D*, **18**, (1986) 348–359.

[25] O. Foda, M. Jimbo, K. Miki, T. Miwa and A. Nakayashiki: Vertex operators in solvable lattice models, *J. Math. Phys.*, **35**, (1994) 13–46.

[26] C. M. Yung and M. T. Batchelor: Integrable vertex and loop models on the square lattice with open boundaries via reflection matrices, *Nucl. Phys. B*, **435**, (1995) 430–462.

[27] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions*, Dover, 8th ed. (1972).

[28] I. Peschel: Some more results for the Ising square lattice with a corner, *Phys. Lett. A.*, **110**, (1985) 313–315.

[29] A. Pelizzola: Exact boundary magnetization of the layered Ising model on triangular and honeycomb lattices, *Modern Physics Letters*, **B10**, (1996) 145–151.

[30] B. Davies and I. Peschel: Corner transfer matrices and corner magnetization for the Ising model, *J. Phys. A: Math. Gen.*, **24**, (1991) 1293–1306.

[31] H. Au-Yang: Thermodynamics of an anisotropic boundary of a two dimensional Ising model, *J. Math. Phys.*, **14**, (1973) 937–946.

[32] M. Yu. Lashkevich: Equations for correlation functions of eight-vertex model: Ferromagnetic and disordered phases, *Modern Physics Letters*, **B10**, (1996) 101–116.

[33] Y. Koyama: Staggered polarization of vertex models with $U_p(sl(n))$ symmetry, *Comm. Math. Phys.*, **164**, (1994), 277–291.