Optical Hall conductivity in 2DEG and graphene QHE systems

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Abstract
We have revealed from a numerical study that the Hall plateaus are retained in the optical Hall conductivity $\sigma_{xy}(\omega)$ in the ac ($\sim$ THz) regime in both of the ordinary two-dimensional electron gas and graphene in the quantum Hall regime, although the plateau height in ac deviates from integer multiples of $e^2/h$. The effect remains unexpectedly robust against a significant strength of disorder, which we attribute to an effect of localization. We predict the ac Hall plateaus are observable through the Faraday rotation with the rotation angle characterized by the fine-structure constant $\alpha$. In this paper we clarify the relationship between plateau structures and the disorder strength by performing numerical calculation.

Key words: quantum Hall effects, optical Hall conductivity, Faraday rotation, graphene

1. Introduction
While the quantum Hall effect (QHE) enjoys a long history over almost three decades, QHE physics still harbors avenues that have not been fully explored. Here we wish to bring to light that the optical Hall effect is indeed a new avenue, which does harbor an unexpected plateau structures. One motivation for the study is that recent experimental spectroscopy in the THz regime are becoming so advanced that optical measurements (e.g., Faraday rotation in magnetic fields) begin to be feasible for QHE systems in the THz regime.\textsuperscript{1, 2} Theoretically, we pose here a fundamental question of how QHE, a topological phenomenon\textsuperscript{3, 4}, evolves into the optical Hall conductivity, especially in the THz ($\sim$ cyclotron energy) regime.

Another motivation is the recent emergence of physics of graphene with its "massless Dirac particle", whose anomalous QHE has been experimentally observed\textsuperscript{5, 6}. For graphene QHE system, optical properties begin to be studied, among which are experimental transmission spectra\textsuperscript{7}, or theoretical examination of the cyclotron emission\textsuperscript{8}. One interest about graphene is the $n = 0$ Landau level (LL) in graphene QHE, which should be unusual since the level consists of a mixture of electrons and holes around the Dirac point. Special features of $n = 0$ LL are intensively discussed, including its robustness against long-ranged random magnetic fields or corrugations of graphene\textsuperscript{9, 10, 11}.

Motivated by these, we have theoretically calculated the optical Hall conductivity $\sigma_{xy}(\omega)$ in the quantum Hall regime with the Kubo formula for both the ordinary 2DEG and graphene. The effect of long-ranged potential disorder is taken into account with the exact diagonalization method to incorporate the effects of localization due to disorders, since the ac conductivity, even for 2DEG, has only been dealt with a phenomenological (Drude) formalism\textsuperscript{13} or with Maxwell’s equations\textsuperscript{14}. We shall unravel from the numerical study the following: (i) The plateaus in $\sigma_{xy}(\omega)$ in the optical (THz) region, although not quantized, are unexpectedly robust against disorder up to a significant degree of disorder. We attribute the unexpected robustness to an effect of localization. (ii) For graphene, the optical Hall plateaus are again retained in the THz regime, while the cyclotron resonance structure reflects the graphene Landau levels that are not uniformly spaced due to the ‘massless Landau’ band. (iii) Experimentally the step structure in $\sigma_{xy}(\omega)$ should be measured as steps in the Faraday rotation, for which we can estimate $\Theta_H \simeq \alpha \simeq 7 \text{ mrad}$, which is well within the experimental resolution\textsuperscript{15} and may be called the fine-structure constant $\alpha$ seen as a rotation, while $\alpha$ has been visualized from transmission in\textsuperscript{15}. In this paper we present the details of our calculations and clarify the relationship between plateau structures and the disorder strength by extensive numerical calculation, which has not been done in our related work\textsuperscript{16}.

2. Optical Hall conductivity in 2DEG
Let us first look at the optical Hall conductivity in the QHE in 2DEG as realized in GaAs/AlGaAs with a Hamiltonian, $H_0 = \frac{1}{2m^*}(\mathbf{p} + e\mathbf{A})^2$. To include the effect of disorder, we employ the exact diagonalization method for disorder described by randomly placed scatterers with a potential $V(\mathbf{r}) = \sum_j u_j \exp(-|\mathbf{r} - \mathbf{R}_j|^2/\ell^2)/(2\pi\ell^2)$, where $d$ is the range of the potential, while the strength of the potential, each placed at $\mathbf{R}_j$, is assumed to take $u_j = \pm u$ with random signs so that the density of states broadens symmetrically in energy. We adopt $\ell = 0.7\ell$, which is comparable to the magnetic length $\ell = \sqrt{\hbar/eB}$. The degree of disorder can be characterized by $\Gamma = 2u(n_{\text{imp}}/2\pi(\ell^2 + d^2))^{1/2}$, which is a measure of the Landau level broadening\textsuperscript{12, 13} with $n_{\text{imp}}$ being the density of impurities.
We have diagonalized the Hamiltonian $H_0 + V$ by representing $V$ with the basis of the free Hamiltonian $H_0$, $\psi_{n,k} = [2^n n! \sqrt{\pi}]^{-1/2} \exp \left[ -\frac{1}{2} (\frac{k^2}{\ell^2})^2 \right] H_0^{\ell} e^{-ik \cdot r}$, where $H_0$ is the Hermite polynomial and $L$ the system size.

To calculate the optical conductivity in the QHE system we use the Kubo formula,

$$\sigma_{opt}(\omega) = \frac{e^2 \hbar}{i \pi} \int d\epsilon \frac{f(\epsilon)}{\hbar \omega} \left[ \text{Tr} \left( j_x \text{Im}G(\epsilon) j_y (G^*(\epsilon + i\hbar \omega) - G^*(\epsilon)) \right) - \text{Tr} \left( j_x (G^*(\epsilon) - G^*(\epsilon - i\hbar \omega)) j_y \text{Im}G(\epsilon) \right) \right], \quad (1)$$

where $\alpha, \beta = x, y$, $f(\epsilon)$ the Fermi distribution, and $G^* \equiv G(\epsilon \pm i\delta)$ with a positive infinitesimal $\delta$. At $T = 0$ which we assume here, the formula is reduced to

$$\sigma_{xy}(\omega) = \frac{i e^2}{L^2} \sum_{\epsilon\in k_{\ell}} \sum_{\epsilon'\in k_{\ell}} \frac{1}{\epsilon_b - \epsilon_a} \times \left( \frac{j_{ab} j_{ba}}{\epsilon_b - \epsilon_a - \hbar \omega} + \frac{j_{ab} j_{ba}}{\epsilon_b - \epsilon_a + \hbar \omega} \right), \quad (2)$$

where $\epsilon_a$ is the eigenenergy, and $\epsilon_F$ the Fermi energy. The current matrix elements, $j_{ab}$, are calculated through the current matrices between the free Landau levels,

$$j_{ab}^{\rho} = i \hbar \omega_c / 2m^* \left( \sqrt{n_0 \delta_{n,-1,n'} - \sqrt{n_0 + 1} \delta_{n+1,n'}} \right),$$

$$j_{ab}^{\rho'} = \hbar \omega_c / 2m^* \left( \sqrt{n_0 \delta_{n,-1,n'} + \sqrt{n_0 + 1} \delta_{n+1,n'}} \right), \quad (3)$$

where $n, n'$ are the Landau indices and $\omega_c$ the cyclotron frequency. Here we have retained 7 Landau levels in the basis, which is sufficient in the energy range considered here for $L = 15\ell$. For the ensemble average we have taken 5000 random configurations for the impurity potential.

Figure 1 shows the results for 2DEG. We plot $\sigma_{xy}$ on an $(\omega, \epsilon_F)$ plane, where the $\omega = 0$ cross section corresponds to the familiar static Hall conductivity, which is quantized into $e^2/h$. We immediately notice two features: (i) $\sigma_{xy}(\omega)$ for a fixed value of $\epsilon_F$ exhibits a resonance structure around the cyclotron frequency (as observed in the experiment). (ii) Away from the resonance, if we look $\sigma_{xy}(\omega)$ for a fixed frequency, a step-like structure is preserved as a function of the Fermi energy $\epsilon_F$.

Although the step heights are not quantized exactly, the flatness is surprisingly retained as seen in Fig.1 (iii) The density of states (DOS) shown in Fig.1(b) broadens with the disorder strength, where the levels having smaller Landau indices tend to be destroyed faster. By contrast, ac as well as static Hall conductivity $\sigma_{xy}(\epsilon_F)$ in Fig.1(c) are preserved with $\Gamma$ much more slowly and the Hall step structure remains. As compared with the self-consistent Born approximation, which with the broadening of DOS and the width of the plateau-to-plateau transition are solely determined by $\Gamma$, the present exact diagonalization approach includes the localization effect and the associated plateau structure in $\sigma_{xy}(\epsilon_F)$.

The step structure is in fact a quantum effect (outside the Drude picture). In the dc QHE, the localization is the cause of the plateaus in the Aoki-Ando picture. In the ac QHE, the localization is the cause of the plateau structure in $\sigma_{xy}(\epsilon_F)$.

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3. Optical Hall conductivity in graphene

We now turn to the optical Hall conductivity in graphene. Here we again adopt the exact diagonalization method for the disorder potential introduced by randomly placed scatterers. When the range of the random potential is much larger than the lattice constant in graphene, the scattering between K and K' points in the Brillouin zone is suppressed, so that we can assume the random term takes a diagonal form in the Dirac Hamiltonian as

\[ H_0 + V = v_F \begin{pmatrix} 0 & \pi^- & 0 & 0 \\ \pi^+ & 0 & 0 & 0 \\ 0 & 0 & \pi^- & 0 \\ 0 & 0 & \pi^+ & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} V(\mathbf{r}). \]  

(5)

So we adopt the Dirac model as in Refs. [20, 9] to obtain wave functions and conductivity in the presence of disorder. After diagonalizing the Hamiltonian \( H_0 + V \) by retaining 9 Landau levels with the system size \( L = 15 \ell \), we have obtained eigenstates in the presence of disorder in the same manner as in the 2DEG QHE system. Then we have calculated the conductivity in the presence of disorder.

Here we again adopt the exact diagonalization method for the optical Hall conductivity. The optical Hall conductivity \( \sigma_{xy}(\epsilon_F, \omega) \) with the Kubo formula eqn [2] with the current matrix for graphene \([21, 8]\):

\[ J_{\text{int}} = \frac{v_F}{\hbar} C_n \left[ \text{sgn}(n)a_{\ell}^{0} - \text{sgn}(n)a_{\ell}^{-1} \right], \]

(6)

with \( C_n = 1/(n+1) \) or \( 1/\sqrt{2} \) (otherwise), which has a selection rule, \( \ell |\ell'| = n + 1 \), peculiar to graphene.

In the result, Fig 2 we notice several features distinct from the result for the ordinary QHE system (Fig 1):

(i) The optical Hall conductivity \( \sigma_{xy}(\epsilon_F, \omega) \) exhibits a more complex structure, which reflects the Landau levels, \( \text{sgn}(n) \sqrt{|\epsilon|/\hbar} \), with \( \omega_c = v_F \sqrt{|2eB/\hbar|} \), that are not uniformly spaced for the massless Dirac dispersion. Thus a series of resonances appear around many allowed transitions, \( |n - |n'|| = \pm 1 \).

(ii) Away from these resonances, we again observe that step-like structures remain in the optical Hall conductivity, as clearly seen in Fig 2. Due to the electron-hole symmetry, \( \sigma_{xy}(\epsilon_F, \omega) \) is odd in \( \epsilon_F \) throughout, so the step structure is symmetric as well.

When we more precisely look at the result, DOS in Fig 2 broadens with the disorder strength, where the nonuniform level spacing causes \( n \neq 0 \) Landau levels merge faster than \( n = 0 \) Landau level does, and a trace of \( n = 0 \) Landau level is visible even for \( \Gamma = 0.7 \hbar \omega_c \). In the static Hall conductivity \( \sigma_{xy}(\epsilon_F) \), \( n \neq 0 \) Hall steps are smeared as soon as Landau levels are merged while the step associated with \( n = 0 \) Landau level is relatively robust.

The present ac result indicates that the step in \( \sigma_{xy}(\epsilon_F, \omega) \) associated with \( n = 0 \) Landau level exhibits robustness against disorder in the ac regime as well, which we take to be the effect of localization, while the plateau-to-plateau transition is related to the persistent delocalized states which, for \( n = 0 \), cannot float due to the electron-hole symmetry.

4. Faraday rotation

Finally let us mention the experimental feasibility. We propose that the ac Hall steps should be observable through accurate Faraday-rotation measurements in the THz to far-infrared spectroscopy. The Faraday-rotation angle \( \Theta_H \) is directly connected to the optical Hall conductivity via

\[ \Theta_H = \frac{1}{2} \arg \left( \frac{f_c(\omega)}{f_s(\omega)} \right) \]

\[ = \frac{1}{2} \arg \left( \frac{n_0 + n_1 + (i\sigma_{xy} \omega_0)}{n_0 + n_1 + (\sigma_{xy} - i\sigma_{xy})/(\omega_0)} \right) \]

\[ \sim \frac{1}{(n_0 + n_1)\epsilon_0^2} \sigma_{xy}(\omega), \]  

(7)
where $t_\pm$ is the transmission coefficients for circularly polarized light, $c$ is the velocity of the light, $n_0(n_s)$ is the refractive index of air (substrate), and we have assumed $n_0 + n_s \gg \sigma_{xy}(\omega)/(c\varepsilon_0)$ on the last line. In the QHE regime, the Faraday rotation angle is proportional to $\sigma_{xy}(\omega)$, so that the step structure in $\sigma_{xy}(\omega)$ should be observed as jumps in Faraday-rotation measurements. We can estimate the size of the jumps $\Delta \Theta_H$ by putting $\sigma_{xy} \sim e^2/h$ (when $\omega$ is well below the resonance), so that

$$\Delta \Theta_H \sim \frac{1}{(n_0 + n_s)c\varepsilon_0} \frac{e^2}{h} \sim \frac{2}{n_0 + n_s} \alpha \sim 7 \text{ mrad}, \quad (8)$$

where $\alpha = e^2/(2\varepsilon_0hc)$ is the fine-structure constant. So the steps in the Faraday-rotation angle should be of the order of the fine structure constant. Recently Shimano et al. have achieved an experimental resolution of $\sim 1$ mrad [2], so that the present effect is well within the experimental feasibility. Nair et al. [15] have seen the fine-structure constant from visual transparency of graphene, and the present proposal amounts to the fine-structure constant seen from a rotation.

To summarize, we have revealed that the optical Hall conductivity in both the usual 2DEG QHE and graphene QHE systems has plateau structures that persist even in ac regimes for significant strengths of disorder. We wish to thank Ryo Shimano, Yohei Ikebe, Seigo Tarucha, Takeo Kato and Takashi Oka for illuminating discussions. TM was supported by Grant-in-Aid for the Japan Society for Promotion of Science (JSPS) Research Fellows. This work has been supported in part by Grants-in-Aid for Scientific Research, No.20340098 (YH, HA), 20654034 (YH) from JSPS and Nos.220029004, 20046002 (YH) from MEXT.

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