Renormalon structure in compactified spacetime

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It is pointed out that the location of renormalon singularities in theory on a circle compactified spacetime $\mathbb{R}^{d-1} \times S^1$ (with a small radius $R \Lambda \ll 1$) differs from that on the non-compactified spacetime $\mathbb{R}^d$. The argument proceeds under the assumption that a loop integrand of a renormalon diagram is volume independent, i.e., it is not modified by the compactification, as it is often the case for large $N$ theories with twisted boundary conditions. We find that the Borel singularity is generally shifted by $-1/2$ in the Borel $u$-plane, where the renormalon ambiguity of $\mathcal{O}(\Lambda^k)$ is changed to $\mathcal{O}(\Lambda^{k-1}/R)$ due to the circle compactification $\mathbb{R}^d \rightarrow \mathbb{R}^{d-1} \times S^1$. The result is general for any dimension $d$ and is independent of details of the quantities under consideration. As an example, we study the $\mathbb{C}P^{N-1}$ model on $\mathbb{R} \times S^1$ with $\mathbb{Z}_N$ twisted boundary conditions in the large $N$ limit.

Subject Index \hspace{1cm} B06, B32, B35
1. Introduction

In perturbation theory of quantum field theory, perturbative series are typically divergent due to the factorial growth of perturbative coefficients. There are two sources of this growth. One of the sources is the factorial growth of the number of the Feynman diagrams at the \( n \)th order as \( \sim n! \). The other typically originates from a single Feynman diagram whose amplitude grows factorially \( \sim \beta_0^n n! \), and is related to the beta function of the theory, where \( \beta_0 \) is the first coefficient of the beta function. The latter one is known as the renormalon \(^1,^2\). Such divergences of perturbative series imply that accuracy of perturbative predictions is limited. Based on the so-called Borel procedure (which is used to obtain a finite result from divergent series), the first one induces the imaginary ambiguity of \( O(\mathcal{e}^{-2S_I}) \) in terms of the one instanton action \( S_I = (4\pi)^{d/2}N/(2\lambda) \) in \( d \)-dimensional spacetime, and the second one does \( O(\mathcal{e}^{-2S_I/(N\beta_0)}) \), called the renormalon ambiguity, where \( \lambda \) denotes the ‘t Hooft coupling (defined as \( \lambda = g^2N \) from a conventional coupling \( g \)).

It is believed that the perturbative ambiguities disappear after the nonperturbative contributions are added. It has been pointed out that the first kind of the ambiguity is canceled against the ambiguity associated with the instanton-anti-instanton calculation \(^3^,^4\). Here, the semiclassical configuration plays an important role, where the two-instanton action \( 2S_I \) has the same contribution to the path integral as the first kind of the perturbative ambiguity. On the other hand, it is hardly known how the renormalon ambiguity is cured. A clear exposition is only known in the \( O(N) \) non-linear sigma model on two-dimensional spacetime, where a nonperturbative condensate, which appears in the context of the operator product expansion, cancels the renormalon ambiguity \(^2,^7^–^1^1\).

Recently, there are active attempts to seek the semiclassical object which cancels the renormalon ambiguity, while expecting the scenario analogous to the cancellation mechanism of the first kind of the perturbative ambiguity. Since the instanton action is not compatible with the renormalon ambiguity by the factor \( N\beta_0 \), another configuration is needed. A recent idea is to find such a configuration by the \( S^1 \) compactification of the spacetime as \( \mathbb{R}^d \rightarrow \mathbb{R}^{d-1} \times S^1 \) (see \(^1^2\) and references therein). In some theories on the compactified spacetime, the semiclassical solution which may be able to cancel the renormalon ambiguity is found and it is called the bion. In this scenario, the ambiguity associated with the bion calculation is expected to cancel the renormalon ambiguity in the theory on \( \mathbb{R}^{d-1} \times S^1 \) first, and then smooth connection to \( R \rightarrow \infty \) is assumed, while the \( S^1 \) radius \( R \) is first taken to be small \( \Lambda R \ll 1 \) so that the semiclassical treatment is justified, where \( \Lambda \) denotes the dynamical scale. In Refs. \(^1^3^–^1^7\), it is claimed that the bion ambiguity is consistent with the renormalon ambiguity.

So far, however, there is no explicit confirmation that the bion truly cancels the renormalon ambiguity. To examine the validity of the bion scenario, it is of great importance to clarify the renormalon structure on the compactified spacetime because, as mentioned, this scenario expects the cancellation of the ambiguities of the renormalon and bion calculation first on the compactified spacetime. The purpose of this paper is to present some insight on the renormalon structure of the theory on the compactified spacetime.

The understanding of the renormalon structure on the compactified spacetime is as follows. In a recent paper \(^1^8\), we have studied the renormalon ambiguity in the supersymmetric
\( \mathbb{C}P^{N-1} \) model on \( \mathbb{R} \times S^1 \) with \( \mathbb{Z}_N \) twisted boundary conditions. At leading order of a systematic expansion in \( 1/N \), we found that the renormalon ambiguity changes from that for \( \mathbb{R}^2 \) by studying the photon condensate and its gradient flow extension. Furthermore, a more recent work on \( SU(N) \) QCD on \( \mathbb{R}^3 \times S^1 \) with adjoint fermions \( [19] \), in which the bion analysis has been carried out \( [13, 14] \), showed a parallel result that the renormalon ambiguity is changed due to the compactification, where the so-called large-\( \beta_0 \) approximation combined with large \( N \) limit is used to study the gluon condensate. Both of the works of Refs. \( [18, 19] \) indicated that the Borel singularities are shifted by \(-1/2\) in the Borel \( u \)-plane (whose definition is explained shortly) due to compactification. Rephrasing this, the renormalon ambiguity in \( \mathbb{R}^d \) of \( \mathcal{O}(\Lambda_k) \) is changed to \( \mathcal{O}(\Lambda_k^{-1}/R) \) in \( \mathbb{R}^{d-1} \times S^1 \); we have the renormalon ambiguity peculiar to the compactified spacetime.

In this paper, as a generalization of our previous works, we present a general mechanism to explain the shift of the Borel singularity, or renormalon ambiguity. Our argument proceeds under the assumption that a loop integrand of a renormalon diagram is not modified by the circle compactification, although we consider sufficiently small radius \( \Lambda R \ll 1 \). In other words, we assume that a loop integrand exhibits the so-called volume independence. This feature would be general in the large \( N \) limit with certain twisted boundary conditions \( [21–27] \). Under this assumption, we study a renormalon diagram in the compactified spacetime \( \mathbb{R}^{d-1} \times S^1 \), not restricting the dimension \( d \). It tells us that the Borel singularity is shifted by \(-1/2\) in the Borel \( u \)-plane due to the compactification, independently of the dimension of spacetime or details of physical quantities under consideration. The origin of this shift can be easily and clearly understood by effective reduction of the dimension of the momentum integration, as shall be explained. We also treat, as an explicit example, the \( \mathbb{C}P^{N-1} \) model on \( \mathbb{R} \times S^1 \) with the \( \mathbb{Z}_N \) twisted boundary conditions in the large \( N \) limit, where we observe the shift of a Borel singularity for an observable defined from the gradient flow \( [28, 29] \).

Let us clarify the definitions adopted in this paper to study factorially divergent series. For perturbative series,

\[
\lambda \sum_{n=0}^{\infty} d_n \left( \frac{\beta_0 \lambda}{(4\pi)^{d/2}} \right)^n,
\]

we define its Borel transform as

\[
B(u) = \sum_{n=0}^{\infty} \frac{d_n}{n!} u^n,
\]

and correspondingly the Borel integral is given by

\[
\frac{(4\pi)^{d/2}}{\beta_0} \int_0^{\infty} du B(u) e^{-(4\pi)^{d/2} u/(\beta_0 \lambda)}.
\]

In our definition, a pole singularity of the Borel transform at \( u = u_0 > 0 \) gives the ambiguity in the Borel integral of order \( e^{-(4\pi)^{d/2} u_0/(\beta_0 \lambda)} = e^{-2S_I u_0/(N\beta_0)} \). Thus, our definition is convenient to grasp the ambiguity in units of \( e^{-S_I} \); it is enough to focus on the value \( u_0 \)

\footnote{We considered the large \( N \) limit with \( RA \ll 1 \) kept fixed.}

\footnote{Roughly speaking, the large-\( \beta_0 \) approximation considers the so-called renormalon diagrams while encoding the asymptotic freedom of theory by hand.}

\footnote{The pioneering work \( [20] \), which studied the same theory in the similar setup to Ref. \( [19] \), claimed the absence of renormalon for small \( N \).}
and not necessary to pay attention to dimension $d$. (This definition coincides with those in Refs. [18, 19].) As we mentioned, we point out the shift of the Borel singularity by $-1/2$ in the compactified spacetime, compared to the case of the non-compactified spacetime.

This paper is organized as follows. In Sec. 2 we explain the general mechanism how the shift of the Borel singularity occurs with the circle compactification of spacetime. We assume volume independence of the loop integrand of a renormalon diagram. In Sec. 3 as an example, we study the $\mathbb{C}P^{N-1}$ model on $\mathbb{R} \times S^1$ with the $\mathbb{Z}_N$ twisted boundary conditions in the large $N$ limit. The effective action for auxiliary fields exhibit volume independence, and it renders the loop integrands of the renormalon diagrams volume independent; hence the above mechanism indeed works. We note that due to the volume independence, our large $N$ calculation essentially reduces to the one in Ref. [30]. We also note that the volume independence of this model has been already clarified in Ref. [24], although the clarification of the renormalon structure is the novel point in this paper. Sec. 4 is devoted to the conclusions.

2. Renormalon structure in compactified spacetime

In asymptotically free theory on the non-compactified spacetime $\mathbb{R}^d$, a typical form from which a renormalon ambiguity appears is given by

$$\int \frac{d^dp}{(2\pi)^d} F(p)\lambda(p^2e^{-C}),$$

where $\int d^dp$ is typically a loop integral and $C$ is a constant. We encounter Eq. (2.1) in analyzing renormalon using the leading logarithmic approximation, the large-$\beta_0$ approximation [31–33], and the large $N$ approximation [2, 7–11]. Here $\lambda$ denotes the running coupling which satisfies the renormalization group equation,

$$\mu^2 d \lambda(\mu^2) = \frac{-\beta_0}{(4\pi)^{d/2}} \lambda^2(\mu^2) \quad \text{with} \quad \beta_0 > 0,$$

whose solution is given by

$$\lambda(\mu^2) = \frac{(4\pi)^{d/2}}{\beta_0} \frac{1}{\log(\mu^2/\Lambda^2)},$$

with a renormalization group invariant (dynamical) mass scale $\Lambda^2 = \mu^2e^{-(4\pi)^{d/2}/[\beta_0\lambda(\mu^2)]}$. When the asymptotic form of $F(p)$ in the infrared (IR) region is given by

$$F(p) \simeq (p^2)^\alpha,$$

the Borel singularity arises at

$$u = \alpha + \frac{d}{2},$$

from perturbative expansion of Eq. (2.1), which gives the renormalon ambiguity of $\mathcal{O}(\Lambda^{2\alpha+d})$.

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4 We consider a ultraviolet (UV) convergent quantity and hence the behavior of $F(p)$ in the UV region is not the same as that in the IR region.

5 This can be easily seen by repeating the subsequent argument but without compactification.
Suppose that, in asymptotically free theory on the compactified spacetime $\mathbb{R}^{d-1} \times S^1$, we have
\[ \sum_{n=-\infty}^{\infty} \frac{1}{2\pi R} \int \frac{d^{d-1}p}{(2\pi)^{d-1}} F(p, p_d = n/R) \lambda(p^2 e^{-C}), \tag{2.6} \]
as a perturbative contribution. Here $p_d$ denotes the Kaluza-Klein (KK) momentum along $S^1$ and thus is discrete $p_d = n/R$, whereas $p$ denotes the (continuous) momentum on $\mathbb{R}^{d-1}$. As in Eq. (2.6), we assume that the loop integrand/summand is not modified from the infinite volume case (2.1). That is, we assume the volume independence of the integrand/summand. In the next section, as an example where we have Eq. (2.6), we study the $\mathbb{C}P^N$ model on $\mathbb{R} \times S^1$ with the $\mathbb{Z}_N$ twisted boundary conditions, where it is shown that the large $N$ limit and the $\mathbb{Z}_N$ twisted boundary conditions play an essential role in realizing the volume independence of the loop integrand/summand (24).

We analyze the renormalon ambiguity involved in Eq. (2.6). To analyze the IR renormalon, it is sufficient to focus on the IR region by introducing a UV cutoff $q$ to the momentum $p^2 < q^2$. We take the UV cutoff as $\Lambda \ll q \ll R^{-1}$. Then, due to $p^2 = p^2 + (n/R)^2$, only the $n = 0$ term and the range $0 < p^2 < q^2$ have to be considered:
\[ \frac{1}{2\pi R} \int_{p^2 < q^2} \frac{d^{d-1}p}{(2\pi)^{d-1}} (p^2)^\alpha \lambda(p^2 e^{-C}). \tag{2.7} \]
We study the Borel transform [defined in Eq. (1.2)] corresponding to Eq. (2.7), which is obtained as
\[ B(u) = \frac{1}{2\pi R} \int_{p^2 < q^2} \frac{d^{d-1}p}{(2\pi)^{d-1}} (p^2)^\alpha \left( \frac{\mu^2 e^{-C}}{p^2} \right)^u. \tag{2.8} \]
Here, we note that the perturbative expansion of Eq. (2.7) in terms of $\lambda(\mu^2)$ is obtained through
\[ \lambda(p^2 e^{-C}) = \lambda(\mu^2) \sum_{k=0}^{\infty} \log^k \left( \frac{\mu^2 e^{-C}}{p^2} \right) \left[ \frac{\beta_0 \lambda(\mu^2)}{(4\pi)^{d/2}} \right]^k. \tag{2.9} \]
The Borel transform (2.8) is easily evaluated as
\[ B(u) = (\mu^2 e^{-C})^u \frac{1}{(4\pi)^{(d-1)/2}} \frac{1}{\Gamma((d-1)/2)} \frac{1}{2\pi R} \frac{q^{2\alpha + d - 1 - 2u}}{\alpha + (d - 1)/2 - u}. \tag{2.10} \]
This possesses a single pole at
\[ u = \alpha + \frac{d - 1}{2} > 0. \tag{2.11} \]
The Borel singularity is shifted by $-1/2$ compared to the infinite volume case as shown in Eq. (2.5). As a result, the renormalon ambiguity appears as
\[ \frac{(4\pi)^{d/2}}{\beta_0} \int_0^{\infty} du \frac{B(u) e^{-(4\pi)^{d/2}u/|\beta_0 \lambda(\mu)|}}{\Gamma((d-1)/2)} \frac{1}{2\pi R} \lambda^{2\alpha + d - 1}, \tag{2.12} \]
where only the renormalon ambiguity is shown. Eqs. (2.11) and (2.12) are the main results of this argument.

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6 We assume $2\alpha + d - 1 > 0$ for IR finiteness of Eq. (2.7).
As one can see from Eq. (2.12), the renormalon ambiguity is independent of the artificial momentum cutoff \( q \). In fact, this cutoff independence holds in a broader sense. Let us take the UV cutoff as \( \Lambda \ll R^{-1} < q \) instead of \( \Lambda \ll q \ll R^{-1} \). In this case, we obtain the Borel transform,

\[
B(u) = \frac{1}{2\pi R} \int_{p^2 < q^2} \frac{d^{d-1}p}{(2\pi)^{d-1}} (p^2)^{\alpha} \left( \frac{\mu^2 e^C}{p^2} \right)^u + \sum_{|n/R|<q, \ n \neq 0} \frac{1}{2\pi R} \int_{p^2 < q^2 - (n/R)^2} \frac{d^{d-1}p}{(2\pi)^{d-1}} (p^2)^{\alpha} \left( \frac{\mu^2 e^C}{p^2} \right)^u. \tag{2.13}
\]

The first line is the same as Eq. (2.8). For the second line, since \( p^2 = p^2 + (n/R)^2 \) always has a non-zero positive value larger than \( 1/R^2 \gg \Lambda^2 \), the integrals never become singular for any \( u \). Thus, we do not have additional singularities.

As we observed, \( p_d = n/R \) with \( |n| \geq 1 \) does not give any renormalon singularities. This is because the compactification radius \( 1/R \gg \Lambda \) plays a role of an IR cutoff for this sector. Hence, only the lowest KK mode (with \( n = 0 \)) can give the renormalon singularities and should be focused, where we have Eq. (2.7). This is nothing but Eq. (2.1) with replacement of \( d \rightarrow d - 1 \) (apart from the overall factor \( 1/(2\pi R) \)). This replacement is the origin of the shift. Thus, the shift is simply understood as the reduction of the dimension of momentum integration [cf. Eqs. (2.5) and (2.11)].

### 3. Renormalon of \( \mathbb{C}P^{N-1} \) model on \( \mathbb{R} \times S^1 \) with \( \mathbb{Z}_N \) twisted boundary conditions

As an example where the mechanism in Sec. 2 applies, we consider the \( \mathbb{C}P^{N-1} \) model on \( \mathbb{R} \times S^1 \) with the \( \mathbb{Z}_N \) twisted boundary conditions. The action of this model in terms of the homogeneous coordinate \( z^A (A = 1, \ldots, N) \) obeying the constraint \( \bar{z}^A z^A = 1 \) is defined by

\[
S = \int d^2 x \frac{N}{\lambda_0} \left( \partial_\mu \bar{z}^A \partial_\mu z^A - j_\mu j_\mu \right) + S_{\text{top}}, \tag{3.1}
\]

with the current \( j_\mu \),

\[
j_\mu = \frac{1}{2i} \left( \bar{z}^A \partial_\mu z^A - z^A \partial_\mu \bar{z}^A \right). \tag{3.2}
\]

The topological term is given by

\[
S_{\text{top}} = \int d^2 x \frac{i\theta}{2\pi} \epsilon_{\mu\nu} \partial_\mu j_\nu, \tag{3.3}
\]

where \( \epsilon_{xy} = -\epsilon_{yx} = +1 \). Here and hereafter, summation over the repeated indices is always understood. It is convenient to adopt the following action with auxiliary fields to carry out

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7 However, we restrict the UV cutoff to be \( q \ll Q \), where \( Q \) denotes the typical scale of an observable (which is not explicitly considered here) such that the expansion of \( F \) in the low energy region is justified. \( F \) is generally a function of \( p \) and \( Q \). For instance, for the photon (or gluon) condensate defined in the gradient flow (as we consider in Sec. 3), the typical scale is \( Q^2 = t^{-1} \), where \( t \) is the flow time.
the large $N$ expansion \[34\]:

\[
S' = S + \int d^2x \frac{N}{\lambda_0} [(A_\mu + j_\mu)(A_\mu + j_\mu) + f (\bar{z}^A z^A - 1)] - \int d^2x \frac{i\theta}{2\pi} \epsilon_{\mu\nu} \partial_\mu (A_\nu + j_\nu)
\]

\[
= \int d^2x \frac{N}{\lambda_0} [-f + \bar{z}^A (-D_\mu D_\mu + f) z^A] - \int d^2x \frac{i\theta}{2\pi} \epsilon_{\mu\nu} \partial_\mu A_\nu,
\]

(3.4)

where $D_\mu = \partial_\mu + iA_\mu$. To respect $U(1)$ gauge symmetry of the model, $A_\mu$ behaves as a gauge field under the transformation $z^A \rightarrow gz^A$ with $g \in U(1)$:

\[
A_\mu \rightarrow A_\mu - \frac{1}{i} g^{-1} \partial_\mu g.
\]

(3.5)

We impose the following $\mathbb{Z}_N$ twisted boundary conditions along the $S^1$ direction for $z^A$:

\[
z^A(x, y + 2\pi R) = e^{2\pi i m_A R} z^A(x, y),
\]

(3.6)

where $(x, y) \in \mathbb{R} \times S^1$ and

\[
m_A = \frac{A}{NR} \quad \text{for } A = 1, \ldots, N - 1,
\]

(3.7)

\[
m_N = 0.
\]

(3.8)

The other fields, $A_\mu$ and $f$, satisfy the periodic boundary conditions. In what follows, we analyze the renormalon ambiguity in this model by using $1/N$ expansion. As we shall see, renormalon diagrams of this model possess the same integrands as the infinite volume case.

### 3.1. Volume independence of effective action

As pointed out in Ref. \[24\], the effective action for the auxiliary fields $S_{\text{eff}}[A_\mu, f]$ exhibits the volume independence due to the $\mathbb{Z}_N$ twisted boundary conditions and the large $N$. We illustrate this point aiming at self-contained explanation. After integrating out $z^A$, the effective action is obtained as

\[
S_{\text{eff}}[A_\mu, f] = -\int d^2x \frac{N}{\lambda_0} f + \sum_A \text{Tr} \ln(-D_\mu D_\mu + f),
\]

(3.9)

where the topological term should be treated separately. We first calculate the effective potential, which is obtained as $S_{\text{eff}} = V_2 \cdot V_{\text{eff}}(A_{\mu 0}, f_0)$; the fields with the subscript 0 denote the constant values at the saddle point; $V_2$ represents the volume of two-dimensional spacetime. $V_{\text{eff}}$ is explicitly given by

\[
V_{\text{eff}}(A_{\mu 0}, f_0) = -\frac{N}{\lambda_0} f_0 + \sum_A \int \frac{dp_y}{2\pi} \frac{1}{2\pi R} \sum_{p_\nu} \ln \left[ (p_x + A_{x 0})^2 + (p_y + m_A + A_{y 0})^2 + f_0 \right].
\]

(3.10)

Here, the KK momentum $p_y$ is discrete:

\[
p_y = \frac{n}{R}, \quad n \in \mathbb{Z}.
\]

(3.11)

By using the formula,

\[
\frac{1}{2\pi R} \sum_{n=-\infty}^{\infty} F(n/R) = \sum_{n=-\infty}^{\infty} \int \frac{dp_y}{2\pi} e^{ip_y 2\pi R n} F(p_y),
\]

(3.12)
we can rewrite the infinite sum by the infinite sum of the integrals, where the momentum shift \( p_y \to p_y - m_A - A_{y0} \) is allowed. By this, we obtain

\[
V_{\text{eff}}(A_{\mu 0}, f_0) = -\frac{N}{\lambda_0} f_0 + \sum_A \sum_{n=-\infty}^{\infty} e^{-i(m_A + A_{\mu 0})2\pi Rn} \int \frac{d^2p}{(2\pi)^2} e^{ip_y 2\pi Rn} \ln(p^2 + f_0) .
\] (3.13)

It is important to note that the sum over \( A \) yields

\[
\sum_A e^{-im_A 2\pi Rn} = \sum_{j=0}^{N-1} \left( e^{-2\pi i/N} \right)^j = \begin{cases} N & \text{for } n = 0 \mod N , \\ 0 & \text{for } n \not= 0 \mod N . \end{cases}
\] (3.14)

Thus, in the sum \( \sum_{n=-\infty}^{\infty} \) in Eq. (3.13), only \( n = Nm \) with integers \( m \) can contribute. Then, we have

\[
V_{\text{eff}}(A_{\mu 0}, f_0) = -\frac{N}{\lambda_0} f_0 + N \sum_{m=-\infty}^{\infty} e^{-iA_{\mu 0} 2\pi Rn} \int \frac{d^2p}{(2\pi)^2} e^{ip_y 2\pi Rn} \ln(p^2 + f_0) ,
\] (3.15)

where the \( m = 0 \) term is the same contribution as the infinite volume case, whereas the \( m \neq 0 \) terms are peculiar to the compactified spacetime. However, for \( m \neq 0 \) since we have the oscillating factor \( e^{ip_y 2\pi RNm} \) in the integrand, these integrals vanish in the large \( N \) limit where \( RN \to \infty \). Hence, we obtain the same effective potential as the infinite volume case [30],

\[
V_{\text{eff}}(A_{\mu 0}, f_0) = V_{\text{eff}, \infty}(A_{\mu 0}, f_0) = -\frac{N}{4\pi} f_0 \left[ \log \left( f_0 / \Lambda^2 \right) - 1 \right] .
\] (3.16)

In Appendix A, we present the explicit result of the \( m \neq 0 \) terms and one can give an explicit proof that this contribution is negligible for large \( N \) in a parallel manner to Appendix B of Ref. [18]. (This contribution is exponentially suppressed as \( \sim e^{-N} \).)

In obtaining Eq. (3.16), we apply dimensional regularization to the \( m = 0 \) term in Eq. (3.15), where the dimension is set to be \( 2 \to d = 2 - 2\epsilon \), and accomplish the renormalization of the bare coupling in the \( \overline{\text{MS}} \) scheme as

\[
\lambda_0 = \left( \frac{e^{\gamma_E} \mu^2}{4\pi} \right)^\epsilon \lambda(\mu^2) \left[ 1 + \frac{\lambda(\mu^2)}{4\pi} \frac{1}{\epsilon} \right]^{-1} .
\] (3.17)

The structure of the renormalization is not modified from the infinite volume case, and the theory is indeed asymptotically free:

\[
\mu^2 \frac{d}{d\mu^2} \lambda(\mu^2) = -\frac{\beta_0}{4\pi} \lambda^2(\mu^2) \quad \text{with } \beta_0 = 1 .
\] (3.18)

The \( \Lambda \) scale used in Eq. (3.16) is defined as \( \Lambda^2 = \mu^2 e^{-4\pi/|\beta_0\lambda(\mu^2)|} \). From Eq. (3.16), the saddle point is given by

\[
f_0 = \Lambda^2 ,
\] (3.19)

as in the infinite volume case. On the other hand, \( A_{y0} \) is not determined and this moduli parameter should be integrated[8].

8 The integration range is determined as follows. Noting that the theory is invariant under \( g \in U(1) \) satisfying the non-trivial boundary condition,

\[
g(x, y + 2\pi R) = e^{2\pi i/N} g(x, y) ,
\] (3.20)
Based on the same reasoning as above, thanks to the $\mathbb{Z}_N$ twisted boundary condition and large $N$, the effective action for the fluctuation of the fields, $A_\mu = A_{\mu 0} + \delta A_\mu$ and $f = f_0 + \delta f$, reduces to the same form as the infinite volume case. We show it to the quadratic order [18]:

$$S_{\text{eff}}[\delta A_\mu, \delta f]_{\text{quadratic}} = \frac{N}{4\pi} \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_y} \left[ \frac{1}{2} (p^2 \delta_{\mu \nu} - p_\mu p_\nu) L^A_{\infty}(p) \delta A_\mu(p) \delta A_\nu(-p) - \frac{1}{2} e^{\delta f}(p) \delta f(p) \delta f(-p) \right],$$

(3.22)

where we define

$$\delta A_\mu(x, y) = \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_y} e^{ip_x x + ip_y y} \delta \tilde{A}_\mu(p), \quad \delta f(x, y) = \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_y} e^{ip_x x + ip_y y} \delta f(p),$$

(3.23)

and

$$L^A_{\infty}(p) = \frac{2\sqrt{p^2 + 4\Lambda^2}}{p^2 \sqrt{p^2}} \log \left( \frac{\sqrt{p^2 + 4\Lambda^2} + \sqrt{p^2}}{\sqrt{p^2 + 4\Lambda^2} - \sqrt{p^2}} \right) - \frac{4}{p^2},$$

(3.24)

$$L^{\delta f}_{\infty}(p) = \frac{2}{\sqrt{p^2 (p^2 + 4\Lambda^2)}} \log \left( \frac{\sqrt{p^2 + 4\Lambda^2} + \sqrt{p^2}}{\sqrt{p^2 + 4\Lambda^2} - \sqrt{p^2}} \right).$$

(3.25)

In Appendix A we also present the explicit result of the effective action including the finite volume contributions, which are omitted here. We again note that these terms are shown to be exponentially suppressed $\sim e^{-N}$ in the large $N$ limit in a parallel manner to Appendix B of Ref. [18].

### 3.2. Renormalon

To calculate the propagators of the auxiliary fields, we add the gauge fixing term,

$$S_{gf} = \frac{N}{4\pi} \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_y} \frac{1}{2} p_\mu p_\nu L^{\delta A}_{\infty}(p) \delta \tilde{A}_\mu(p) \delta \tilde{A}_\nu(-p),$$

(3.26)

to the effective action Eq. (3.22). Then, the propagators read

$$\langle \tilde{A}_\mu(p) \tilde{A}_\nu(q) \rangle = \frac{4\pi}{N} \delta_{\mu \nu} \frac{1}{p^2 L^{\delta A}_{\infty}(p)} 2\pi \delta(p_x + q_x) 2\pi R \delta_{p_y + q_y, 0},$$

(3.27)

$$\langle \tilde{f}(p) \tilde{f}(q) \rangle = -\frac{4\pi}{N} \frac{1}{L^{\delta f}_{\infty}(p)} 2\pi \delta(p_x + q_x) 2\pi R \delta_{p_y + q_y, 0}.$$

(3.28)

These are the leading order results of the two point functions in $1/N$. Since they are obtained from the volume independent effective action, these results are of course volume independent. It is worth noting that, however, they do not contain renormalons.\footnote{The shift of $A_\mu$ induced by an element $e^{ig/(RN)} \in U(1)$, $A_\mu \rightarrow A_\mu - 1/(RN)$, reduces to an equivalent theory. Thus, the integral over $\int_0^1 d(A_{\mu 0} RN)$ should be considered. As long as the quantity to be integrated over this moduli parameter is independent of $A_{\mu 0}$, this integral has no apparent effect.\footnote{Regarding the gauge field propagator, since it is gauge dependent, this result itself does not have physical meaning.}} To see this, we
consider the expansion of $1/L_{\infty}^A(p)$ and $1/L_{\infty}^{\delta f}(p)$ in high energy region $\Lambda^2/p^2 \ll 1$ so that the perturbative expansion in the asymptotically free theory works:

\[
\frac{1}{L_{\infty}^A(p)} = p^2 \left\{ \frac{\Lambda^2}{8\pi} \frac{\lambda(p^2 e^{-2})}{p^2} + \frac{3\lambda^2(p^2 e^{-2})}{16\pi^2} + O(\Lambda^4/p^4) \right\}, \tag{3.29}
\]

\[
\frac{1}{L_{\infty}^{\delta f}(p)} = p^2 \left\{ \frac{\lambda(p^2)}{8\pi} + \frac{\Lambda^2}{p^2} \left[ \frac{\lambda(p^2)}{4\pi} - \frac{\lambda^2(p^2)}{16\pi^2} \right] + O(\Lambda^4/p^4) \right\}. \tag{3.30}
\]

Since $\Lambda^2 = \mu^2 e^{-4\pi/[\beta_0 \lambda(\mu^2)]}$ is zero in perturbative evaluation, these quantities are evaluated in perturbation theory (PT) as\[10\]

\[
\left. \frac{1}{L_{\infty}^A(p)} \right|_{PT} = p^2 \frac{\lambda(p^2 e^{-2})}{8\pi}, \quad \left. \frac{1}{L_{\infty}^{\delta f}(p)} \right|_{PT} = p^2 \frac{\lambda(p^2)}{8\pi}. \tag{3.31}
\]

These results do not contain renormalon; they are truncated at $O(\lambda)$. If one uses a general renormalization scale $\mu$ in accordance with the concept of the fixed order perturbation theory, the infinite sums in $\lambda(\mu^2)$ appear through Eq. (2.9) but they can be unambiguously resummed.

Renormalons appear when the propagator containing the running coupling as in Eq. (3.31) is involved in a loop integrand. As simple examples, let us consider the condensates, $\langle f(x)f(x) \rangle$ and $\langle F_{\mu\nu}(x)F_{\mu\nu}(x) \rangle$:

\[
\langle f(x)f(x) \rangle = \Lambda^4 - \frac{4\pi}{N} \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_y} \left. \frac{1}{L_{\infty}^{\delta f}(p)} \right|_{p^2 < q^2}, \tag{3.32}
\]

\[
\langle F_{\mu\nu}(x)F_{\mu\nu}(x) \rangle = \frac{4\pi}{N} \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_y} \left. \frac{2}{L_{\infty}^A(p)} \right|_{p^2 < q^2}. \tag{3.33}
\]

Since the condensates are UV divergent, we introduce a UV cutoff $q$ to define them. The perturbative evaluations of these quantities are given by\[11\]

\[
\left. \langle f(x)f(x) \rangle \right|_{PT} = -\frac{4\pi}{N} \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_y} \left. p^2 \frac{\lambda(p^2)}{8\pi} \right|_{p^2 < q^2, \text{expansion in } \lambda(\mu)}, \tag{3.34}
\]

\[
\left. \langle F_{\mu\nu}(x)F_{\mu\nu}(x) \rangle \right|_{PT} = \frac{4\pi}{N} \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_y} \left. p^2 \frac{\lambda(p^2 e^{-2})}{4\pi} \right|_{p^2 < q^2, \text{expansion in } \lambda(\mu)}, \tag{3.35}
\]

where we explicitly show that their integrands should be expanded in $\lambda(\mu)$\[12\]. Now, we indeed encounter Eq. (2.6), assumed in the general argument in Sec. 2, the loop integrands are not modified from the infinite volume case, but the integration measure is modified to that of

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\[10\] The coefficients of $(\Lambda^2/p^2)^k$ are regarded as Wilson coefficients, and they are calculated in perturbation theory as they are given in perturbative series in $\lambda$.

\[11\] The perturbative results of the condensates are the same as the loop integrations of the perturbative expressions of the integrands. We also note that the perturbative expression is reliable only in high energy region. Using such a result in low energy region, which is not justified, is a cause of a renormalon ambiguity.

\[12\] If the integrands are not expanded in $\lambda(\mu)$, the integrals are ill-defined since they contain the poles around $p^2 \sim \Lambda^2$, which are related to the renormalon ambiguities.
the compactified space. Following the result in Sec. 2, the renormalon ambiguities are given by
\[ \langle f(x)f(x) \rangle_{\text{renormalon}} = \mp i\pi \frac{1}{N} \frac{1}{2\pi R} \Lambda^3, \] (3.36)
\[ \langle F_{\mu\nu}(x)F_{\mu\nu}(x) \rangle_{\text{renormalon}} = \pm i\pi \frac{2e^3}{N} \frac{1}{2\pi R} \Lambda^3. \] (3.37)

As already noted in Sec. 2, the renormalon ambiguities are independent of the UV cutoff. These renormalon ambiguities are peculiar to the compactified spacetime, since they depend on \( R \).

To give an example of a UV finite observable which possesses a renormalon ambiguity, we consider the gradient flow \[ 28, 29 \]. The flow equation is given by
\[ \partial_t B_\mu(t, x) = \partial_\nu G_{\nu\mu}(t, x) + \alpha_0 \partial_\mu \partial_x B_\mu(t, x), \quad B_\mu(t = 0, x) = A_\mu(x), \] (3.38)
where \( G_{\mu\nu}(t, x) = \partial_\mu B_\nu(t, x) - \partial_\nu B_\mu(t, x) \) is the field strength of the flowed gauge field; \( \alpha_0 \) is a constant regarded as a gauge parameter; \( t \) is called the flow time, whose mass dimension is \( -2 \). The flowed gauge field is obtained as
\[ B_\mu(t, x) = A_\mu_0 + \int d^2 x' \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_\nu \exp[ip(x-x')] \left[ \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) e^{-tp^2} + \frac{p_\mu p_\nu}{p^2} e^{-\alpha_0 tp^2} \right] \delta A_\mu(x'). \] (3.39)

Using the flowed gauge field, we can construct an observable \( \langle G_{\mu\nu}(t, x)G_{\mu\nu}(t, x) \rangle \),
\[ \langle G_{\mu\nu}(t, x)G_{\mu\nu}(t, x) \rangle = \frac{4\pi}{N} \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_\nu \frac{2}{\mathcal{L}_0^{\infty}(p)} e^{-2tp^2}, \] (3.40)
where the gaussian damping factor makes this quantity UV finite. In perturbation theory, it is given by
\[ \langle G_{\mu\nu}(t, x)G_{\mu\nu}(t, x) \rangle|_{\text{PT}} = \frac{4\pi}{N} \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_\nu \frac{2}{\mathcal{L}_0^{\infty}(p)} e^{-2tp^2} \left| \text{expansion in } \lambda(\mu) \right. \] (3.41)
To analyze the renormalon in this quantity, we introduce a UV cutoff \( \Lambda^2 \ll q^2 \ll t^{-1} \) as done in Sec. 2. Since the integrand has the same behavior as that of the photon condensate \[ 3.35 \] in the IR region due to \( e^{-2tp^2} \simeq 1 \), we have the same renormalon ambiguity as \( \langle F_{\mu\nu}(x)F_{\mu\nu}(x) \rangle \):
\[ \langle G_{\mu\nu}(t, x)G_{\mu\nu}(t, x) \rangle_{\text{renormalon}} = \pm i\pi \frac{2e^3}{N} \frac{1}{2\pi R} \Lambda^3. \] (3.42)

As seen from the above reasoning, it is fairly general that the leading renormalon ambiguity of the photon (or gluon) condensate defined by the gradient flow, which is a UV finite observable, is the same as that of the photon (or gluon) condensate defined with the UV cutoff.

In Eq. 3.42, only the leading renormalon ambiguity of \( \langle G_{\mu\nu}(t, x)G_{\mu\nu}(t, x) \rangle \) is shown. By considering the expansion of \( e^{-2tp^2} \) in \( tp^2 \) at higher order, we obtain the renormalon ambiguity beyond this order of the form
\[ \langle G_{\mu\nu}(t, x)G_{\mu\nu}(t, x) \rangle_{\text{renormalon}} = \pm i\pi \left( c_0 \frac{1}{R} \Lambda^3 + c_1 t \frac{1}{R} \Lambda^5 + c_2 t^2 \frac{1}{R} \Lambda^7 + \ldots \right), \] (3.43)
according to the argument in Sec. 2 where \( c_0, c_1, c_2, \ldots \) denote the constants. In the Borel \( u \)-plane, these renormalon ambiguities correspond to the singularities at \( u = 3/2, 5/2, 7/2, \ldots \).
These positions are different from the infinite volume case, where the singularities are located at $u = 2, 3, 4, \ldots$.

We finally note that the emergence of the renormalon ambiguities is indeed an artifact of perturbation theory. By seeing Eq. (3.40), which is not evaluated in perturbation theory, one can see that this quantity can be unambiguously evaluated because any divergence is not found in this expression. It indicates that the renormalon ambiguities found above are cured after nonperturbative effects are properly added.

4. Conclusions

In this paper, we presented a general argument that the renormalon structure is significantly affected by the circle compactification of the spacetime as $\mathbb{R}^d \to \mathbb{R}^{d-1} \times S^1$. The assumptions of this argument are that (i) the $S^1$ radius is sufficiently small, $R \Lambda \ll 1$, and that (ii) a loop integrand of a renormalon diagram is not modified by the compactification. The latter property is often realized in the large $N$ theories with twisted boundary conditions. Under these assumptions, we showed that a shift of the renormalon singularity generally occurs due to the circle compactification $\mathbb{R}^d \to \mathbb{R}^{d-1} \times S^1$. In particular, the singularity is shifted by $-1/2$ in the Borel $u$-plane regardless of the dimension of spacetime $d$ and details of the quantities under consideration. This can be easily understood by the reduction of the dimension of the loop momentum integral, which stems from the fact that only the lowest KK mode can give renormalon singularity as we discussed.

As an example, we studied the $\mathbb{C}P^{N-1}$ model on $\mathbb{R} \times S^1$ with the $\mathbb{Z}_N$ twisted boundary conditions in the large $N$ limit. In this model, the loop integrands of the renormalon diagrams indeed exhibit volume independence, and the above shift is explicitly shown by studying the photon condensate which is defined by the gradient flow and is a UV finite observable. As already mentioned, the preceding works [18, 19] had provided examples where this mechanism applies in the large $N$ approximation: the supersymmetric $\mathbb{C}P^{N-1}$ model on $\mathbb{R} \times S^1$, and $SU(N)$ QCD with adjoint fermions on $\mathbb{R}^3 \times S^1$.

We finally emphasize that the volume independence of the effective action does not indicate the volume independence of the renormalon structure, as we observed in the example of the $\mathbb{C}P^{N-1}$ model. In this example, the volume independent effective action gave the two-point functions or propagators which do not contain renormalon ambiguity. The renormalon ambiguity arises when these propagators (determined from the effective action) are included as loop integrands. Such quantities do not show volume independence any more, and the renormalon structure is not kept intact.

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[13] Here, we mean by a renormalon diagram that its loop integrand possesses a one-loop running coupling.

[14] The latter model requires a slight modification. This is because this model partially breaks the gauge symmetry due to compactification and the effective action is not completely same as that on $\mathbb{R}^4$. However, the difference can be removed by a shift of loop momentum of a renormalon diagram.
A. Finite volume corrections

The effective action is given by

$$V_{\text{eff}}(A_\mu^0, f_0) = V_{\text{eff}, \infty}(A_\mu^0, f_0) + V_{\text{eff, finite}}(A_\mu^0, f_0)$$  \hspace{1cm} (A1)

with

$$V_{\text{eff, finite}}(A_\mu^0, f_0) = -\frac{N}{\pi} \sum_{m \neq 0} e^{-iA_{\omega}2\pi Rm} \frac{\sqrt{f_0}}{2\pi RN|m|} K_1(\sqrt{f_0}2\pi RN|m|),$$ \hspace{1cm} (A2)

where $K_\nu(z)$ denotes the modified Bessel function of the second kind.

Including the finite volume effects, the effective action is given by

$$S_{\text{eff}} = \frac{N}{4\pi} \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_y} \left\{ \frac{1}{2} (p^2 \delta_{\mu\nu} - p_\mu p_\nu) \left[ L_{\infty}^A(p) + L_{\text{finite}}^A(p) \right] \tilde{\delta}A_\mu(p) \tilde{\delta}A_\nu(-p) \right. \right.
\left. \left. - \frac{1}{2} \left[ L_{\infty}^f(p) + L_{\text{finite}}^f(p) \right] \tilde{\delta}f(p) \tilde{\delta}f(-p) \right.
\left. \left. - \left( \delta_{\mu y} - \frac{p_\mu p_y}{p^2} \right) L_{\text{mix}}^\mu(p) \left[ \tilde{\delta}A_\mu(p) \tilde{\delta}f(-p) + \tilde{\delta}f(p) \tilde{\delta}A_\mu(-p) \right] \right\} \right\}$$ \hspace{1cm} (A3)

with

$$L_{\text{finite}}^A(p) = \int_0^1 dx \sum_{m \neq 0} e^{-iA_{\omega}2\pi Rm} e^{ip_x2\pi Rm} (2x - 1)^2$$
$$\times \frac{2\pi RN|m|}{\sqrt{\Lambda^2 + x(1-x)p^2}} K_1(\sqrt{\Lambda^2 + x(1-x)p^22\pi RN|m|})$$
$$- \frac{2i}{p_y} \int_0^1 dx \sum_{m \neq 0} e^{-iA_{\omega}2\pi Rm} e^{ip_x2\pi Rm} (2x - 1)$$
$$\times 2\pi RNmK_0(\sqrt{\Lambda^2 + x(1-x)p^22\pi RN|m|}),$$ \hspace{1cm} (A4)

$$L_{\text{finite}}^f(p) = \int_0^1 dx \sum_{m \neq 0} e^{-iA_{\omega}2\pi Rm} e^{ip_x2\pi Rm}$$
$$\times \frac{2\pi RN|m|}{\sqrt{\Lambda^2 + x(1-x)p^2}} K_1(\sqrt{\Lambda^2 + x(1-x)p^22\pi RN|m|}),$$ \hspace{1cm} (A5)

and

$$L_{\text{mix}}^\mu(p) = i \int_0^1 dx \sum_{m \neq 0} e^{-iA_{\omega}2\pi Rm} e^{ip_x2\pi Rm} 2\pi RNmK_0(\sqrt{\Lambda^2 + x(1-x)p^22\pi RN|m|}).$$ \hspace{1cm} (A6)

We note that these results are consistent with the gauge invariance; the tensors before $\tilde{\delta}A_\mu$ are transverse.

References
[31] D. J. Broadhurst and A. L. Kataev, “Connections Between Deep Inelastic and Annihilation Processes at Next-to-Next-to-Leading Order and Beyond,” *Phys. Lett. B315* (1993) 179–187, \texttt{arXiv:hep-ph/9308274 [hep-ph]}.

[32] P. Ball, M. Beneke, and V. M. Braun, “Resummation of $(\beta_0 \alpha_s)^N$ Corrections in QCD: Techniques and Applications to the Tau Hadronic Width and the Heavy Quark Pole Mass,” *Nucl. Phys. B452* (1995) 563–625, \texttt{arXiv:hep-ph/9502300 [hep-ph]}.

[33] M. Beneke and V. M. Braun, “Naive Nonabelianization and Resummation of Fermion Bubble Chains,” *Phys. Lett. B348* (1995) 513–520, \texttt{arXiv:hep-ph/9411229 [hep-ph]}.

[34] S. Coleman, *Aspects of Symmetry*, Cambridge University Press, Cambridge, U.K., 1985.