TESTING FOR BREAKS IN VARIANCE STRUCTURES WITH SMOOTH CHANGES

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Abstract: The problem of detecting variance breaks in the case of smooth time-varying variance structure is studied. It is highlighted that the tests based on (piecewise) constant specification of the variance are not able to distinguish between smooth non constant variance and the case where an abrupt change is present. Consequently, a new procedure for detecting variance breaks taking into account for smooth changes of the variance is proposed. The finite sample properties of the tests introduced in the paper are investigated by Monte Carlo experiments. The theoretical outputs are illustrated using U.S. macroeconomic data.

Keywords: Unconditionally heteroscedastic errors; Variance breaks; CUSUM test.

1 Introduction

In the time series analysis literature, a considerable attention has been paid to the test of abrupt variance breaks (see Inclan and Tiao (1994), Berkes et al. (2004) or Sansó et al. (2004) in the univariate case among others, and Aue et al. (2009) in the
multivariate case). These tests are based on the assumption of constant variance under the null hypothesis, which is sometimes restrictive in the sense that continuous changes of the variance are not taken into account.

In time series modelling it is common to reduce the time range of the data so that the smooth variance change become negligible. For high frequency data (daily financial data for example) it is in general easy to define relatively large samples lengths for which the variance could be approximated by a constant. Therefore the tools for detecting variance breaks based on the constant variance hypothesis under the null may be applied directly in such a case. In such setting Berkes et al. (2004) proposed a test to detect abrupt changes for GARCH processes. Nevertheless for low frequency data (for instance annual, quarterly or monthly macroeconomic data) there are some subperiods of potential interest for applied investigations that exhibit fast smooth changes. As a consequence such situation makes difficult to form subsamples with approximately constant variance. In order to exemplify, let us consider the quarterly foreign direct investment in U.S. in millions of dollars from 1946-10-01 to 2014-01-01. The series plotted in Figure 1 shows a global increasing of the variance. Clearly if one is interested in studying, let us say, the period beginning in the early 90's to the end of the sample, the possible smooth changes of the unconditional variance cannot be neglected.

The aim of this work is to investigate the test for a variance break in presence of smooth changes. It is first established that the tests based on the assumption of constant variance tend to reject spuriously the hypothesis of no variance break in such a case as the sample size increases. In practice this may lead to make a confusion between the case where at least a variance break is present and the case where the variance is only subject to smooth changes. As a consequence we propose a testing procedure that is able to improve the detection of variance breaks. Following the approach of Dahlhaus (2012, p361) the smooth changes of the variance are captured
using polynomial regressions of low orders to correct the test statistics.

The structure of the paper is as follows. In Section 2 we show that testing for a variance break while smooth changes are present can lead to erroneous conclusions. In Section 3 a polynomial correction of the test statistic is proposed. In Section 4 we carried out numerical experiments which show substantial improvements of the control of type I errors when the polynomial correction is applied. The outputs of the paper are illustrated using U.S. macroeconomic data sets.

The following general notations will be used. Independently, identically distributed is abbreviated by i.i.d.. The convergence in distribution is denoted by ⇒ and the symbol \( \overset{p}{\to} \) denotes the convergence in probability. If \((X_n)\) is a sequence of random variables, then \(X_n = O_p(1)\) means that \(X_n\) is bounded in probability and \(X_n = o_p(1)\) means that \(X_n \overset{p}{\to} 0\). We denote by \([\cdot]\) the usual integer part of a real number. If a lower bound of a sum exceeds the upper bound then the sum is set equal to zero. Throughout the paper the constant \(M > 0\) may take possibly different values.

\section{Unreliability of the tests based on constant variance structure}

In this section it is underlined that the standard approach for testing for a variance break may be misleading if the studied sample (or subsample) is built so that smooth changes cannot be neglected. For the sake of conciseness, we illustrate this only in the case where the full sample is considered, although similar arguments can be used if unsuitable subsamples are taken.
Let us consider the process \((x_t)\) given by

\[
x_t = a_1 x_{t-1} + \cdots + a_m x_{t-m} + u_t
\]

\[
u_t = h_t \epsilon_t,
\]

where \(x_t, t = 1, \ldots, n\) are observed random variables and \(\epsilon_t\) i.i.d. centered random variables with unit variance. It is assumed that there exists an estimator \(\hat{\theta} = (\hat{a}_1, \ldots, \hat{a}_m)'\) for the parameters vector \(\theta_0 = (a_1, \ldots, a_m)'\) which is such that \(\sqrt{n}(\hat{\theta} - \theta_0) = O_p(1)\). For instance \(\sqrt{n}\)-asymptotically normal estimators of the parameters giving the conditional mean are provided in Xu and Phillips (2008). The residuals are defined by \(\hat{u}_t = x_t - \hat{a}_1 x_{t-1} - \cdots - \hat{a}_m x_{t-m}\). Of course if the \(x_t\)'s are uncorrelated, the \(u_t\)'s can be directly used in the statistics introduced below. The following assumptions delineate the framework of non constant variance structure for the errors.

**Assumption A1:** Smooth time varying variance with no break.

(i) We assume that \(h_t := g(\frac{t}{n})\) where \(g(\cdot)\) is a measurable deterministic function on the interval \((0, 1]\), such that \(g(r) > 0\) and \(\sup_{r \in (0, 1]} |g(r)| < \infty\).

(ii) The function \(g(\cdot)\) satisfies a Lipschitz condition on \((0, 1]\).

(iii) The process \((\epsilon_t)\) is such that \(E(\epsilon_t^{4\delta}) < \infty\) with \(\delta > 1\).

**Assumption A1': Time varying variance with a break.** Suppose that conditions (i) and (iii) of A1 are fulfilled and that \(g(\cdot)\) is not continuous but satisfies a Lipschitz condition piecewise on two sub-intervals that partition \((0, 1]\).

Since the rescaling device developed by Dahlhaus (1997) is used for the definition of the \(h_t\)'s, \((x_t)\) should be written in a triangular form. However the double subscript
is not used to keep the notations simple. The assumption $A1$ allows to consider the realistic case where the variance evolves in a smooth way. The assumption $A1'$ allows for an abrupt change for the variance in addition to the time-varying smooth variance structure. In this paper we develop tests to detect a variance break in a context where smooth changes are present (i.e. $H_0$: $A1$ holds vs. $H_1$: $A1'$ holds).

The standard situation for the null hypothesis is retrieved when $g(.)$ is taken constant. In order to detect the presence of abrupt breaks if the $u_t$'s are i.i.d. Gaussian, Inclan and Tiao (1994) proposed the following statistic:

$$S = \sup_k |\sqrt{n/2}D_k|, \quad k = 1, \ldots, n,$$

where $D_k = \hat{C}_k \hat{C}_n - \frac{k}{n}$, $\hat{C}_k = \sum_{t=1}^k \hat{u}_t^2$. Sansó et al. (2004) proposed a corrected test statistic in the non Gaussian case:

$$\tilde{S} = \sup_k |n^{-\frac{3}{2}}\hat{B}_k|, \quad k = 1, \ldots, n, \quad \text{with} \quad \hat{B}_k = \frac{\hat{C}_k - \frac{k}{n}\hat{C}_n}{\sqrt{\hat{\eta} - (n^{-1}\hat{C}_n)^2}},$$

and $\hat{\eta} = n^{-1} \sum_{t=1}^n \hat{u}_t^4$. Under the assumption of a constant variance and other additional conditions, it is shown that the statistics (2.1) and (2.2) converge in distribution to $\sup_s |W(s)|$ where $W(s) := B(s) - sB(1)$ is a Brownian bridge, and $B(\cdot)$ being a standard Brownian motion. Of course all the results obtained in this paper for statistics taking into account the non Gaussian case also hold when the errors are actually independent and Gaussian distributed. Sansó et al. (2004) have also proposed a statistic that can take into account nonlinearities, which are typical in financial data. However the non Gaussian case is adopted in the sequel since it provides a large enough framework to handle macro-economic data. The following proposition shows that the usual tests are not valid in our non standard framework.

\textbf{Proposition 1.} Under $A1$, we have

$$\tilde{S} = o_p(n^{\frac{3}{2}}) + Mn^{\frac{1}{2}},$$

5
where \( M > 0 \) is a constant.

From Proposition 1 it turns out that if the smooth changes of the variance are not taken into account correctly, the null hypothesis of no variance break tend to be rejected spuriously by the usual tests as \( n \to \infty \).

In order to apply the classical approach for testing for variance breaks, usually subsamples where the variance is satisfactorily approximated by a constant are considered. We focus on subsamples of length \( q = \lfloor n^\gamma \rfloor \) for some \( \gamma \in (0,1) \) to illustrate this point. Let a sequence \( \hat{r}_n \in (0,1) \) and introduce the following statistic:

\[
\tilde{S}_\gamma^{\hat{r}_n} = \sup_k |q^{-\frac{1}{2}} \hat{B}_{k,\hat{r}_n}^{\gamma}|, \quad k = 1, \ldots, q,
\]

with \( \hat{B}_{k,\hat{r}_n}^{\gamma} = \frac{\hat{C}_{k,\hat{r}_n}^{\gamma} - \frac{1}{q} \hat{C}_{q,\hat{r}_n}^{\gamma}}{\sqrt{\hat{\eta}_{\hat{r}_n}^{\gamma} - (q^{-1}\hat{C}_{q,\hat{r}_n}^{\gamma})^2}} \),

(2.3)

with \( \hat{C}_{k,\hat{r}_n}^{\gamma} = \sum_{t=[\hat{r}_n] \cdot 1}^{[\hat{r}_n] + k} \hat{u}_t^2 \) and \( \hat{\eta}_{\hat{r}_n}^{\gamma} = q^{-1} \sum_{t=[\hat{r}_n] \cdot 1}^{[\hat{r}_n] + q} \hat{u}_t^4 \). Therefore the \( \tilde{S}_\gamma^{\hat{r}_n} \) statistic is computed at fractions \( \hat{r}_n \) of the original sample with a subsample of length \( q \). Note that \( \hat{r}_n \) should be chosen adequately in view of the sample size, \( [\hat{r}_n n] + q < n \). For mathematical convenience the increasing sequence \( \hat{r}_n \) is such that the subsample middle \( r^0 \) is fixed. Also it is assumed that a possible variance break necessarily occurs in \( r^0 \). Note that the above setting can be replaced by the assumption that \( \hat{r}_n \) is increasing, so that the abrupt change is present in all subsamples as \( q \to \infty \) for power results. The terms \( \gamma \) and \( \hat{r}_n \) may be viewed as parameters for calibrating the subsamples of interest. The following proposition gives the asymptotic behavior of the \( \tilde{S}_\gamma^{\hat{r}_n} \) statistic.

**Proposition 2.** Suppose that \( 0 < \gamma \leq \frac{2}{3} \). Then under A1 we have as \( q \to \infty \),

\[
\tilde{S}_\gamma^{\hat{r}_n} \Rightarrow \sup_{s \in (0,1]} |W(s)|.
\]

The proof of Proposition 2 is given in the Appendix. The following result ensures the consistency of the test based on the standard statistic and subsamples where the variance can be approximated by a constant.
Proposition 3. Under A1’ we have $\tilde{S}_{\gamma}^{\gamma} = Mn^2 + O_p(1)$, where $M > 0$ is a constant.

The above results give a testing procedure which corresponds to the common practice consisting in selecting a subsample where the smooth changes in the variance structure can be neglected, so that the classical tests may be applied directly. Indeed it is well known that the processes given by assumption A1 can be viewed as approximately stationary (see e.g. Dahlhaus (2012)).

In general it is clear that marked smooth changes may lead to select too small subsamples with almost constant variance under the null of no variance breaks. Indeed, although Proposition 2 and 3 ensure a good implementation of the classical tests as $n \to \infty$ for suitable subsamples, the lengths of low frequency economic series are too small in many cases. Hence the detection of variance breaks may become intractable and could lead to size distortions problems. On the other hand the approximate constant variance may be questionable when too large subsamples are selected, so that we can loose the control of the type I error in view of Proposition 1. Note also that the practitioner may be interested in analyzing the data on larger samples than those that allow to neglect the smooth variance changes. In the next section a procedure for testing a variance break in presence of marked smooth changes is proposed.

3 Testing for variance break handling smooth changes in the variance structure

Assume that under A1 we can write

$$g^2(r) = \sum_{i=0}^{p} \alpha_{i,r^0}(r - r^0)^i + o((r - r^0)^p),$$

where $r^0$ is the breakpoint under the null hypothesis.
for some \( p > 0 \) and for any \( r, r^0 \in (0, 1) \). In the same way as before a subsample of length \( q = [n^\gamma] \), is taken. For a potentially better precision, we use \( r^0 := (2[\hat{r}_n^\gamma] + \frac{q}{n})/2 \), the subsample middle, and the coefficients are estimated by ordinary least squares (OLS) from the following equation:

\[
u_t^2 = \sum_{i=0}^{p} \alpha_{i,r^0} \left( \frac{t}{n} - r^0 \right)^i + \xi_t,\]

where \( \xi_t = u_t^2 - g^2 \left( \frac{t}{n} \right) \) is the error term and \( t = [\hat{r}_n^\gamma] + 1, \ldots, [\hat{r}_n^\gamma] + q \). As a reduced subsample size is considered, we can think that a relatively small order \( p \) describes satisfactorily the smooth time varying variance structure. Let \( \hat{\alpha}_{i,r^0} \) denote the (OLS) estimators and \( \hat{g}^2(r) \) the estimated variance. It is shown in Lemma 5.1 that \( \hat{\alpha}_{i,r^0} \) is a consistent estimator of \( \alpha_{i,r^0} \), so that a smooth approximation of the variance structure is available. Suppose that \( g^2(r) > c > 0 \), which implies that \( \hat{g}^2(r) > c > 0 \) for large enough \( q \). Define the test statistic:

\[
\tilde{S}_{r_n}^\gamma = \sup_k |q^{-\frac{1}{2}} \tilde{B}_{k,r_n}^\gamma|, \quad k = 1, \ldots, q, \quad \text{with} \quad \tilde{B}_{k,r_n}^\gamma = \frac{\tilde{C}_{k,r_n}^\gamma - \frac{k}{q} \tilde{C}_{q,r_n}^\gamma}{\sqrt{\tilde{\eta}_{r_n}^\gamma - (q^{-1}\tilde{C}_{q,r_n}^\gamma)^2}},
\]

and \( \tilde{\eta}_{r_n}^\gamma = q^{-1} \sum_{t=[\hat{r}_n^\gamma] + 1}^{[\hat{r}_n^\gamma] + q} \hat{g}^{-4} \left( \frac{t}{n} \right) u_t^4 \). To use a statistic corrected from the smooth changes of the variance under the null hypothesis. For \( p = 0 \), it is better to use the simple tests described in the previous section. The following propositions give the asymptotic behavior of the statistic \( \tilde{S}_{r_n}^\gamma \).

**Proposition 4.** Suppose that \( A1 \) holds true, then as \( q \to \infty \),

\[
\tilde{S}_{r_n}^\gamma \Rightarrow \sup_{s \in (0,1)} |W(s)|.
\]

**Proposition 5.** Suppose that \( A1' \) holds true, then as \( q \to \infty \), \( \tilde{S}_{r_n}^\gamma = M n^{\frac{\gamma}{2}} + O_p(1) \), where \( M > 0 \) is a constant.

Using Proposition 4 and 5, we can construct a valid test to detect variance breaks taking into account the smooth changes of the variance. For a suitable polynomial
of order $p$ the test consists in rejecting the null hypothesis at the asymptotic level 5\% if the test statistic $\bar{S}_{p,n}^2$ exceeds the usual critical value of the supremum of a standard Brownian bridge.

4 Monte Carlo experiments

In the sequel, we denote by $Q_{\text{mod}}$ the modified test subject to polynomial regression correction and with polynomial order selection by AIC criteria. The standard test proposed in Sansó et al. (2004) is denoted by $Q_{\text{std}}$. In this section the finite sample properties of the $Q_{\text{mod}}$ and $Q_{\text{std}}$ tests is examined by simulations. We consider two data generating processes:

\begin{alignat}{2}
DGP1 & : & u_t & = h_t \epsilon_t, \\
DGP2 & : & x_t & = 0.4x_{t-1} + u_t \\
& & u_t & = h_t \epsilon_t,
\end{alignat}

where the process $\epsilon_t$ is i.i.d. and follows the standard logistic distribution. In DGP1 the $u_t$’s are directly observed. The autoregressive parameter in DGP2 is estimated by OLS. The residuals are then used to build the different statistics. Note that the errors $(u_t)$ have non constant unconditional variance if the $h_t$’s change over time.

We carried out experiments with different settings for the variance structure. An extract which reflects the outputs we obtained is provided. We consider:

\begin{align}
& h^2(t) = -2.7 + 1.5 \exp \left(1 + \left(\frac{t}{n}\right)\right) + 0.2 \sin \left(5 \pi \left(\frac{t}{n}\right)\right) + f(t), \\
& f(t) = \alpha 1_{\{t \geq [\kappa n]\}}, \quad \kappa = 0.5, \quad t = 1, \cdots, n \quad \text{and} \quad \alpha = 0, 1, 2, \cdots, 5.
\end{align}

The first term in (4.2) gives a global increasing behavior for the variance structure, while the second describes a cyclical behavior often observed in practice. The term $\alpha$
is used for the empirical power study. For each experiment $N = 1000$ independents trajectories are simulated using DGP1 and DGP2. Samples of length $n = 50$, $n = 100$ and $n = 200$ are simulated. In all our experiments the level of the tests is 5%.

4.1 The behavior of the studied tests under the null hypothesis

We study the empirical size of the tests, that is testing for a variance break in presence of smooth changes. To this aim we set $\alpha = 0$ in (4.2). The results are provided in Tables 1 and 2. Assuming that the finite sample size of the test is 5%, and noting that $N = 1000$ replications are performed, the relative rejection frequencies should be between the significant limits 3.65% and 6.35% with probability 0.95. Tables 1 and 2 reveal that, when the unconditional variance is not constant, the standard test spuriously rejects the null hypothesis as the sample size becomes large. On the other hand, it can be seen that the $Q_{mod}$ test improves substantially the control of the type I errors.

| Test Statistics | $n$ | 50  | 100 | 200 |
|-----------------|----|-----|-----|-----|
| $Q_{std}$       |    | 24.0| 57.4| 90.2|
| $Q_{mod}$       |    | 1.0 | 2.9 | 5.5 |

Table 1: Empirical size (in %) of the tests under DGP1.
Table 2: Empirical size (in %) of the tests under DGP2.

4.2 The behavior under the alternative hypothesis

In the empirical power of this section, we examine the ability of $Q_{mod}$ test to detect an abrupt volatility break. We simulate $N = 1000$ independent trajectories using the data generating processes presented in (4.1) with break at level $\kappa = 0.5$, taking $\alpha = 1, 2, 3, 4, 5$ in (4.2). Tables 3 and 4 show the empirical powers of the $Q_{mod}$ test. As expected, the rejections rates increase as $\alpha$ and $n$ are increased. Nevertheless we note a low power, although the $Q_{mod}$ have some ability to detect breaks. This is the price to pay for controlling the type I errors.

Table 3: Empirical power (in %) of the $Q_{mod}$ test under DGP1.
Figure 1: The quarterly foreign direct investment in U.S. in millions of dollars from 1946-10-01 to 2014-01-01 (n= 250) on the left, and their first differences on the right. Data source: The research division of the federal reserve bank of Saint Louis, code ROWFDIQ027S, www.research.stlouis.org.

Figure 2: The OLS residuals for the foreign direct investment data on the left, and their squares on the right.
Figure 3: The monthly real M2 money stock in U.S. in billions of dollars from 1959-01-01 to 2014-01-01 (n= 694) on the left, and their first differences on the right. Data source: The research division of the federal reserve bank of Saint Louis, code: M2REAL, www.research.stlouis.org.

Figure 4: The OLS residuals of the real M2 money stock data on the left, and their squares on the right.
### Table 4: Empirical power (in %) of the $Q_{mod}$ test under DGP2.

| Break length $n$ | 50  | 100 | 200 |
|-----------------|-----|-----|-----|
| $\alpha = 1$    | 2.0 | 3.5 | 7.2 |
| $\alpha = 2$    | 2.4 | 5.5 | 10.0|
| $\alpha = 3$    | 2.1 | 6.0 | 14.0|
| $\alpha = 4$    | 3.2 | 6.0 | 18.0|
| $\alpha = 5$    | 3.4 | 9.0 | 19.4|

5 Illustrative examples

Now we turn to several applications of the test developed above to real data sets for which it is reasonable to suppose at least smooth non constant variance. The standard test is also used for comparison. We investigate two macroeconomic data sets: the first differences of the monthly real M2 money stock in billions of dollars (hereafter noted M2) and the first difference series of the quarterly foreign direct investment in the U.S. in millions of dollars from October 1946 to January 2014 (called FDI hereafter). The two studied series are plotted in Figures [1] and [3]. The data are available seasonally adjusted from the website of the research division of the federal reserve bank of Saint Louis (www.research.stlouisfed.org). Note that such series are often included in many applied works.

In order to study the variance structure of residuals, we fitted AR models to the M2 and FDI series. It appears reasonable to assume that the variance of the residuals of these series is not constant in time, but rather have an increasing behavior. We aim to test if in addition to smooth time varying behaviors, abrupt breaks are present. The $Q_{mod}$ test is then applied to the residuals of the M2 and FDI series. The outputs are compared with those of the standard test in table 5. We first remark that
the $Q_{std}$ test statistic exceeds the predetermined boundary 1.33 which corresponds to the asymptotic critical values of the supremum of a standard Brownian Bridge (see table 1 of Sansó et al. (2004)) in all investigated cases. As a consequence the presence of a break in the variance structure is detected using the standard test. In view of our results, this is possibly due to neglected smooth time-varying variance. Now eliminating the effect of possible smooth changes, it appears that for the M2 serie the value of $Q_{mod}$ is lower than the asymptotic value 1.33 so that the null hypothesis of no variance break cannot be rejected. On the other hand we can see that the $Q_{mod}$ exceeds the predetermined boundary for the FDI. The results in table 5 reveal that the outputs for the $Q_{std}$ and $Q_{mod}$ are quite different.

|       | $Q_{std}$ | $Q_{mod}$ | AIC-Order |
|-------|-----------|-----------|-----------|
| M2    | 4.14      | 1.26      | 3         |
| FDI   | 2.39      | 1.45      | 3         |

Table 5: The $Q_{mod}$ and $Q_{std}$ tests based on residuals from the first-difference of foreign direct investment and real M2 money stock in U.S. series.

Proofs

Recall that we defined $\hat{u}_t = x_t - \hat{a}_1x_{t-1} - \cdots - \hat{a}_m x_{t-m}$ the residuals obtained from $\hat{\theta}$. From the Mean Value Theorem it is easy to see that $n^{-\frac{1}{2}} \sum_{t=1}^{n} \hat{u}_t^2 = n^{-\frac{1}{2}} \sum_{t=1}^{n} u_t^2 + o_p(1)$, and hence the possibly unobserved process $(u_t)$ will be used for our asymptotic derivations without loss of generality. Define $C_k = \sum_{t=1}^{k} u_t^2$, $B_k = \frac{C_k - \frac{1}{2} C_n}{\sqrt{\eta (n^{-1} C_n)^2}}$, and $\eta = n^{-1} \sum_{t=1}^{n} u_t^4$. Recall also that the general constant $M > 0$ may take different values.
Proof of Proposition 1. First using Phillips and Xu (2006), Lemma 1, we write for any $s \in (0, 1]$

$$n^{-1} \sum_{t=1}^{[ns]} u_t^2 = \int_0^s g^2(r)dr + o_p(1), \text{ and } n^{-1} \sum_{t=1}^n u_t^4 = E(\epsilon_t^4) \int_0^1 g^4(r)dr + o_p(1). \quad (5.1)$$

Noting that

$$|n^{-\frac{1}{2}} B_{[ns]}| = \left| \frac{n^{-\frac{1}{2}} (C_{[ns]} - [ns] C_n)}{\sqrt{\eta - (n^{-1} C_n)^2}} \right|$$

$$= n^{-\frac{1}{2}} \left| C_{[ns]} - \frac{[ns]}{n} C_n \right| \times \left| \eta - (n^{-1} C_n)^2 \right|^{-\frac{1}{2}}$$

$$= n^{-\frac{1}{2}} \left| \sum_{t=1}^{[ns]} u_t^2 - \frac{[ns]}{n} \sum_{t=1}^n u_t^2 \right| \times \left[ n^{-1} \sum_{t=1}^n u_t^4 - \left( n^{-1} \sum_{t=1}^n u_t^2 \right)^2 \right]^{-\frac{1}{2}},$$

from (5.1), we obtain

$$|n^{-\frac{1}{2}} B_{[ns]}| = n^{\frac{1}{2}} \left| \int_0^s g^2(r)dr - s \int_0^1 g^2(r)dr \right|$$

$$\times \left[ E(\epsilon_1^4) \int_0^1 g^4(r)dr - \left( \int_0^1 g^2(r)dr \right)^2 \right]^{-\frac{1}{2}} + o_p(\sqrt{n}), \quad (5.2)$$

For the first term on the right hand side of (5.2) we have

$$\sup_{s \in (0, 1]} \left| \int_0^s g^2(r)dr - s \int_0^1 g^2(r)dr \right| > 0$$

provided that $g(\cdot)$ is not constant, while the second term is clearly equal to a strictly positive constant. Hence we obtain

$$\sup_k |n^{-\frac{1}{2}} \hat{B}_k| = n^{\frac{1}{2}} M + o_p(\sqrt{n}),$$

which proves Proposition 1. □

Proof of Proposition 2. We compare the statistic defined by (2.3) to the statistic calculated from a subsample based on the constant variance assumption, defined as

$$\hat{S}_{\gamma n}^\gamma = \sup_k |q^{-\frac{1}{2}} \hat{B}_{k, \gamma n}^\gamma|, \quad \text{with } \hat{B}_{k, \gamma n}^\gamma = \frac{\hat{C}_{k, \gamma n}^\gamma - \frac{k}{q} \hat{C}_{q, \gamma n}^\gamma}{\sqrt{\hat{\eta}_{\gamma n} - (q^{-1} \hat{C}_{q, \gamma n}^\gamma)^2}}, \quad k = 1, \ldots, q,$$
where \( \hat{C}^\gamma_{k, \hat{r}_n} = \sum_{i=[\hat{r}_n]+1}^{[\hat{r}_n]+k} g^2(\frac{[\hat{r}_n]}{n}) \epsilon_i^2 \) and \( \hat{\eta}_n^\gamma = q^{-1} \sum_{t=[\hat{r}_n]+1}^{[\hat{r}_n]+q} g^4(\frac{[\hat{r}_n]}{n}) \epsilon_t^4 \).

There are two parts of the proof of proposition 2, we study the nominator and the denominator in (2.3) separately. For the nominator, we have

\[
\frac{1}{\sqrt{q}} \left( C^\gamma_{k, \hat{r}_n} - \frac{k}{q} C^\gamma_{q, \hat{r}_n} \right) - \frac{1}{\sqrt{q}} \left( \hat{C}^\gamma_{k, \hat{r}_n} - \frac{k}{q} \hat{C}^\gamma_{q, \hat{r}_n} \right) = \frac{1}{\sqrt{q}} \sum_{t=[\hat{r}_n]+1}^{[\hat{r}_n]+k} \left[ g^2(\frac{t}{n}) - g^2(\frac{[\hat{r}_n]}{n}) \right] \epsilon_t^2 - \frac{k}{q} \sum_{t=[\hat{r}_n]+1}^{[\hat{r}_n]+q} g^2(\frac{t}{n}) - g^2(\frac{[\hat{r}_n]}{n}) \epsilon_t^2 \leq \frac{1}{\sqrt{q}} \sum_{t=[\hat{r}_n]+1}^{[\hat{r}_n]+k} \left[ g^2(\frac{t}{n}) - g^2(\frac{[\hat{r}_n]}{n}) \right] \epsilon_t^2 + \frac{k}{q^2} \sum_{t=[\hat{r}_n]+1}^{[\hat{r}_n]+q} g^2(\frac{t}{n}) - g^2(\frac{[\hat{r}_n]}{n}) \epsilon_t^2 \leq Mq^{1-\frac{3}{2}} \left( \frac{1}{\sqrt{q}} \sum_{t=[\hat{r}_n]+1}^{[\hat{r}_n]+k} (\epsilon_t^2 - 1) + \frac{k}{q^2} \sum_{t=[\hat{r}_n]+1}^{[\hat{r}_n]+q} (\epsilon_t^2 - 1) + 2k \right),
\]

where the last inequality follows from the Lipschitz condition, then it follows from the Donsker's functional central limit theorem that for all 0 < \( \gamma < \frac{2}{3} \),

\[
\left| \frac{1}{\sqrt{q}} \left( C^\gamma_{k, \hat{r}_n} - \frac{k}{q} C^\gamma_{q, \hat{r}_n} \right) - \frac{1}{\sqrt{q}} \left( \hat{C}^\gamma_{k, \hat{r}_n} - \frac{k}{q} \hat{C}^\gamma_{q, \hat{r}_n} \right) \right| = o_p(1). \tag{5.3}
\]

For the denominator we introduce

\[
\tau^2 = \hat{\eta}_n^\gamma - (q^{-1} \hat{C}^\gamma_{q, \hat{r}_n})^2 = \frac{1}{q} \sum_{t=[\hat{r}_n]+1}^{[\hat{r}_n]+q} g^4(\frac{t}{n}) \epsilon_t^4 - \left[ \frac{1}{q} \sum_{t=[\hat{r}_n]+1}^{[\hat{r}_n]+q} g^2(\frac{t}{n}) \epsilon_t^2 \right]^2
\]

and

\[
\tau^2 = \hat{\eta}_n^\gamma - (q^{-1} \hat{C}^\gamma_{q, \hat{r}_n})^2 = \frac{1}{q} \sum_{t=[\hat{r}_n]+1}^{[\hat{r}_n]+q} g^4(\frac{[\hat{r}_n]}{n}) \epsilon_t^4 - \left[ \frac{1}{q} \sum_{t=[\hat{r}_n]+1}^{[\hat{r}_n]+q} g^2(\frac{[\hat{r}_n]}{n}) \epsilon_t^2 \right]^2. 
\]

Using the Lipschitz condition and the law of large numbers, we obtain

\[
\left| \frac{1}{q} \sum_{t=[\hat{r}_n]+1}^{[\hat{r}_n]+q} g^2(\frac{t}{n}) - g^2(\frac{[\hat{r}_n]}{n}) \epsilon_t^2 \right| \leq Mq(\frac{\gamma-1}{\gamma}) \left| \frac{1}{q} \sum_{t=[\hat{r}_n]+1}^{[\hat{r}_n]+q} \epsilon_t^2 \right| = o_p(1). \tag{5.4}
\]

Similarly, we can write

\[
\left| \frac{1}{q} \sum_{t=[\hat{r}_n]+1}^{[\hat{r}_n]+q} g^4(\frac{t}{n}) \epsilon_t^4 - \frac{1}{q} \sum_{t=[\hat{r}_n]+1}^{[\hat{r}_n]+q} g^4(\frac{[\hat{r}_n]}{n}) \epsilon_t^4 \right| = o_p(1). \tag{5.5}
\]
From (5.4) and (5.5), we have that
\[ \tau^2 - \bar{\tau}^2 = o_p(1). \tag{5.6} \]

In view of (5.3) and (5.6), we deduce that \( q^{-\frac{1}{2}}\hat{B}_{\kappa, \hat{\tau}}^\gamma \) and \( q^{-\frac{1}{2}}\hat{B}_{\kappa, \hat{\tau}}^\gamma \) have the same asymptotic behavior. The rest of the proof follows the same arguments as in the proof of Proposition 2 in Sansó et al. (2004) and considering \( q^{-\frac{1}{2}}\hat{B}_{\kappa, \hat{\tau}}^\gamma \). \( \square \)

**Proof of Proposition 3.** Under the alternative hypothesis, the variance can be written as \( g^2(\frac{1}{n}) = v(\frac{1}{n}) + \alpha \mathbb{I}_{\{t \geq \lfloor n\kappa \rfloor\}} \), where \( \lfloor n\kappa \rfloor \) is the break location with \( \kappa \in (0, 1) \).

The function \( v(.) \) satisfies a Lipschitz condition with \( \sup_{r \in (0, 1)} |v(r)| < \infty \). Note that under the alternative hypothesis, the break point is located on the subsample \([\hat{\tau}_n, n] + 1, \hat{\tau}_n + q\] so that there exists \( l \in (0, 1) \) such that \( \lfloor n\kappa \rfloor \) can be written as \( \lfloor n\kappa \rfloor = [\hat{\tau}_n, n] + \lfloor lq \rfloor + 1 \). We have
\[
\left| q^{-\frac{1}{2}}\hat{B}_{\kappa, \hat{\tau}}^\gamma \right| = q^{-\frac{1}{2}} \left| \frac{C_{\kappa, \hat{\tau}}^\gamma - \frac{k}{q} C_{\hat{\tau}, \hat{\tau}}^\gamma}{\sqrt{n_{\hat{\tau}} - (q^{-1} C_{\hat{\tau}, \hat{\tau}}^\gamma)^2}} \right|
\]
\[
= q^{-\frac{1}{2}} \left| C_{\kappa, \hat{\tau}}^\gamma - \frac{k}{q} C_{\hat{\tau}, \hat{\tau}}^\gamma \right| \left| \frac{n_{\hat{\tau}} - (q^{-1} C_{\hat{\tau}, \hat{\tau}}^\gamma)^2}{q^{-1} \sum_{t = [\hat{\tau}_n, n] + 1}^{[\hat{\tau}_n, n] + k} u_t^2 - \frac{k}{q} \sum_{t = [\hat{\tau}_n, n] + 1}^{[\hat{\tau}_n, n] + q} u_t^2} \right|^{\frac{1}{2}}
\]
\[
= \frac{1}{\sqrt{q}} \left| \sum_{t = [\hat{\tau}_n, n] + 1}^{[\hat{\tau}_n, n] + k} v \left( \frac{t}{n} \right) \epsilon_t^2 - \frac{k}{q} \sum_{t = [\hat{\tau}_n, n] + 1}^{[\hat{\tau}_n, n] + q} v \left( \frac{t}{n} \right) \epsilon_t^2 \right| + \frac{\alpha}{\sqrt{q}} \left| \sum_{t = \lfloor n\kappa \rfloor}^{[\hat{\tau}_n, n] + k} \epsilon_t^2 - \frac{k}{q} \sum_{t = \lfloor n\kappa \rfloor}^{[\hat{\tau}_n, n] + q} \epsilon_t^2 \right|^{\frac{1}{2}}
\]
\[
\times \left\{ \frac{1}{q} \sum_{t = [\hat{\tau}_n, n] + 1}^{[\hat{\tau}_n, n] + q} v \left( \frac{t}{n} \right) + \alpha \mathbb{I}_{\{t \geq \lfloor n\kappa \rfloor\}} \right\}^2 - \left\{ \frac{1}{q} \sum_{t = [\hat{\tau}_n, n] + 1}^{[\hat{\tau}_n, n] + q} v \left( \frac{t}{n} \right) + \alpha \mathbb{I}_{\{t \geq \lfloor n\kappa \rfloor\}} \right\}^2 \right\}^{\frac{1}{2}}
\]
\[
:= |d_1 + d_2| \times |d_3 - d_4|^{-\frac{1}{2}} := D_1 \times \frac{1}{D_2}.
\]

From the same arguments used to prove equation (5.3), it is easy to see that \( d_1 = O_p(1) \). Let \( k = \lfloor sq \rfloor \) where \( s \in (0, 1) \), so by applying the Donsker’s functional central
Thus, by the limit theorem and the law of large numbers, we have

\[ d_2 := \frac{\alpha}{\sqrt{q}} \left[ \sum_{t=[r_n]+1}^{[r_n]+[sq]} \epsilon_t^2 - \frac{[sq]}{q} \sum_{t=[r_n]+1}^{[r_n]+[sq]} \epsilon_t^2 \right] \]

\[ = \alpha \left[ \left( \frac{1}{\sqrt{q}} \sum_{t=[r_n]+1}^{[r_n]+[sq]} \epsilon_t^2 - \frac{1}{\sqrt{q}} \sum_{t=[r_n]+1}^{[r_n]+[sq]} \epsilon_t^2 \right) \right] - \frac{[sq]}{\sqrt{q}} \left[ \frac{1}{q} \sum_{t=[r_n]+1}^{[r_n]+[sq]} \epsilon_t^2 - \frac{1}{q} \sum_{t=[r_n]+1}^{[r_n]+[sq]} \epsilon_t^2 \right] \]

\[ = \alpha l(s - 1) \sqrt{q} + O_p(1). \]

Thus,

\[ \sup_{s \in (0,1)} D_1 = \sup_{s \in (0,1)} \alpha l(1 - s) \sqrt{q} + O_p(1) = M \sqrt{q} + O_p(1). \]

Now let us evaluate the probability limit of \( D_2 \). Using the same arguments as for equations (5.4) and (5.5), we have

\[ d_3 := \frac{1}{q} \sum_{t=[r_n]+1}^{[r_n]+[sq]} \left( v \left( \frac{t}{n} \right) + \alpha \mathbb{1}_{\{t \geq [rn]\}} \right)^2 \]

\[ = \frac{1}{q} \sum_{t=[r_n]+1}^{[r_n]+[sq]} v^2 \left( \frac{t}{n} \right) \epsilon_t^4 + \frac{\alpha^2}{q} \sum_{t=[r_n]+1}^{[r_n]+[sq]} \mathbb{1}_{\{t \geq [r_n]+[sq]\}} \epsilon_t^4 + \frac{2\alpha}{q} \sum_{t=[r_n]+1}^{[r_n]+[sq]} \mathbb{1}_{\{t \geq [r_n]+[sq]\}} v \left( \frac{t}{n} \right) \epsilon_t^4 \]

\[ = \frac{1}{q} \sum_{t=[r_n]+1}^{[r_n]+[sq]} v^2 \left( \frac{t}{n} \right) \epsilon_t^4 + \frac{\alpha^2}{q} \sum_{t=[r_n]+1}^{[r_n]+[sq]} \epsilon_t^4 - \frac{\alpha^2}{q} \sum_{t=[r_n]+1}^{[r_n]+[sq]} \epsilon_t^4 \]

\[ + \frac{2\alpha}{q} \sum_{t=[r_n]+1}^{[r_n]+[sq]} v \left( \frac{t}{n} \right) \epsilon_t^4 - \frac{2\alpha}{q} \sum_{t=[r_n]+1}^{[r_n]+[sq]} v \left( \frac{t}{n} \right) \epsilon_t^4 \]

\[ = E(\epsilon_t^4) \left[ v^2 (\hat{r}) + \alpha (1 - l) (\alpha + 2v(\hat{r})) \right] + o_p(1). \]

Similarly, it can be shown that

\[ d_4 := \left[ \frac{1}{q} \sum_{t=[r_n]+1}^{[r_n]+[sq]} v \left( \frac{t}{n} \right) + \alpha \mathbb{1}_{\{t \geq [rn]\}} \epsilon_t^2 \right]^2 = [v(\hat{r}) + \alpha (1 - l)]^2 + o_p(1). \]

Consequently, we can see that \( D_2 \) is asymptotically constant and finally we have

\[ \sup_k \left| q^{-\frac{1}{2}} B_{k,r_n}^{(\gamma)} \right| = M \sqrt{q} + O_p(1). \]
The following lemma is used to prove the asymptotic consistency of polynomial regression estimators described in (3.1).

Lemma 5.1. Suppose that A1 holds true, then as $q \to \infty$

$$\hat{\alpha}_{j,r} - \alpha_{j,r} = o_p(1), \quad \text{for all} \quad 0 < j \leq p.$$  

Proof of Lemma 5.1. The model (3.1) can be expressed in matrix notation as follows: $U = X\Lambda + \xi$, where

$$U = \begin{pmatrix} u_{[r,n]+1}^2 \\ \vdots \\ u_{[r,n]+q}^2 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & \frac{[r,n]+1}{n} - r^0 & \cdots & \left( \frac{[r,n]+1}{n} - r^0 \right)^p \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{[r,n]+q}{n} - r^0 & \cdots & \left( \frac{[r,n]+q}{n} - r^0 \right)^p \end{pmatrix},$$

$$\Lambda = \begin{pmatrix} \alpha_{0,r^0} \\ \vdots \\ \alpha_{p,r^0} \end{pmatrix} \quad \text{and} \quad \xi = \begin{pmatrix} \xi_{[r,n]+1} \\ \vdots \\ \xi_{[r,n]+q} \end{pmatrix}.$$ 

The least squares estimate of $\Lambda$ is given by

$$\hat{\Lambda} = (X'X)^{-1}X'U = (X'X)^{-1}X'(X\Lambda + \xi) = \Lambda + (X'X)^{-1}X'\xi,$$

so it follows that

$$\hat{\Lambda} - \Lambda = (X'X)^{-1}X'\xi = \left( \frac{X'X}{q} \right)^{-1} \left( \frac{X'\xi}{q} \right).$$

It is clear that $\left( \frac{X'X}{q} \right)^{-1} = O(1)$. To finish the proof we only need to show that $\left( \frac{X'\xi}{q} \right) = o_p(1)$. By definition we have

$$\frac{X'\xi}{q} = \begin{pmatrix} \frac{1}{q} \sum_{t=[r,n]+1}^{[r,n]+q} X^0_t (u_t^2 - E(u_t^2)) \\ \frac{1}{q} \sum_{t=[r,n]+1}^{[r,n]+q} X^1_t (u_t^2 - E(u_t^2)) \\ \vdots \\ \frac{1}{q} \sum_{t=[r,n]+1}^{[r,n]+q} X^q_t (u_t^2 - E(u_t^2)) \end{pmatrix}.$$
where \( X_t^j = (\frac{t}{n} - r_0)^j \), \( j = 0, \ldots, p \) and \( \xi_t = u_t^2 - g^2(\frac{t}{n}) \). Note that \( E[\xi_t^2] = E[(u_t^2 - g^2(\frac{t}{n}))^2] = g^4(\frac{t}{n})[E(\epsilon_t^4) - 1] < \infty \). Thus, by applying Corollary 3.9 in White (1984), we get \( \frac{1}{q} \sum_{t=[\tilde{r}_n]+1}^{[\tilde{r}_n]+q} [u_t^2 - g^2(\frac{t}{n})] = o_p(1) \), which completes the proof. \( \square \)

**Proof of Proposition 4.** We compare the statistic defined by (3.2) to the statistic defined as

\[
S_{\tilde{r}_n}^\gamma = \sup_k |q^{-\frac{1}{2}} B_{k,\tilde{r}_n}^\gamma|, \quad \text{with} \quad B_{k,\tilde{r}_n}^\gamma = \frac{C_{k,\tilde{r}_n} - k \bar{C}_{k,\tilde{r}_n}^\gamma}{\sqrt{\eta_{\tilde{r}_n} - (q^{-1}C_{q,\tilde{r}_n}^\gamma)^2}}, \quad k = 1, \ldots, q,
\]

where \( C_{k,\tilde{r}_n}^\gamma = \sum_{t=[\tilde{r}_n]+1}^{[\tilde{r}_n]+k} \epsilon_t^2 \) and \( \eta_{\tilde{r}_n} = q^{-1} \sum_{t=[\tilde{r}_n]+1}^{[\tilde{r}_n]+q} \epsilon_t^4 \).

There are two parts of the proof of proposition 4. We study the numerator and the denominator in (3.2) separately. For the nominator, we have

\[
\left| \frac{1}{\sqrt{q}} \left( C_{k,\tilde{r}_n}^\gamma - \frac{k}{q} \bar{C}_{q,\tilde{r}_n}^\gamma \right) \right| - \frac{1}{\sqrt{q}} \left( C_{k,\tilde{r}_n}^\gamma - \frac{k}{q} \bar{C}_{q,\tilde{r}_n}^\gamma \right)
\]

\[
\leq \frac{1}{\sqrt{q}} \left| \sum_{t=[\tilde{r}_n]+1}^{[\tilde{r}_n]+k} \left( \frac{g^2(\frac{t}{n})}{\bar{g}^2(\frac{t}{n})} - 1 \right) \epsilon_t^2 \right| - \frac{k}{q^2} \left| \sum_{t=[\tilde{r}_n]+1}^{[\tilde{r}_n]+q} \left( \frac{g^2(\frac{t}{n})}{\bar{g}^2(\frac{t}{n})} - 1 \right) \epsilon_t^2 \right|
\]

\[
\leq \sup_{[\tilde{r}_n]+1 \leq t \leq [\tilde{r}_n]+q} \left| \frac{g^2(\frac{t}{n})}{\bar{g}^2(\frac{t}{n})} - 1 \right| \left( \frac{1}{\sqrt{q}} \sum_{t=[\tilde{r}_n]+1}^{[\tilde{r}_n]+k} \epsilon_t^2 \right) + \frac{k}{q^2} \left( \sum_{t=[\tilde{r}_n]+1}^{[\tilde{r}_n]+q} \epsilon_t^2 \right)
\]

\[
\leq \sup_{[\tilde{r}_n]+1 \leq t \leq [\tilde{r}_n]+q} \left| \frac{g^2(\frac{t}{n}) - \bar{g}^2(\frac{t}{n})}{\bar{g}^2(\frac{t}{n})} \right| \left( \frac{1}{\sqrt{q}} \sum_{t=[\tilde{r}_n]+1}^{[\tilde{r}_n]+k} (\epsilon_t^2 - 1) \right) + \frac{k}{q^2} \left( \sum_{t=[\tilde{r}_n]+1}^{[\tilde{r}_n]+q} (\epsilon_t^2 - 1) \right) + \frac{2k}{\sqrt{q}}.
\]
We consider a large enough \( n \) such that \( \hat{g}^2(\frac{t}{n}) > c > 0 \). Then

\[
\sup_{t=[\tilde{r}_n]+1 \leq t \leq [\tilde{r}_n]+q} \frac{|g^2(\frac{t}{n}) - \hat{g}^2(\frac{t}{n})|}{\hat{g}^2(\frac{t}{n})} \leq \frac{1}{c} \sup_{t=[\tilde{r}_n]+1 \leq t \leq [\tilde{r}_n]+q} \left| \frac{g^2(t/n) - \hat{g}^2(t/n)}{\hat{g}^2(t/n)} \right|
\]

\[
\leq \frac{1}{c} \sup_{t=[\tilde{r}_n]+1 \leq t \leq [\tilde{r}_n]+q} \sum_{i=0}^{p} \left( \alpha_{i,r^0} - \hat{\alpha}_{i,r^0} \right) \left( \frac{t}{n} - r^0 \right)^i + o \left( \left( \frac{t}{n} - r^0 \right)^p \right)
\]

\[
\leq \frac{1}{c} \sup_{t=[\tilde{r}_n]+1 \leq t \leq [\tilde{r}_n]+q} \sum_{i=0}^{p} \left| \alpha_{i,r^0} - \hat{\alpha}_{i,r^0} \right| \left( \frac{t}{n} - r^0 \right)^i + o \left( n^{p(\gamma-1)} \right)
\]

\[
= o(n^{(\gamma-1)}) + o(n^{p(\gamma-1)}), \quad (5.7)
\]

where the last equality follows from Lemma 5.1. Therefore, it follows from (5.7), the Donsker Theorem's and the law of large numbers that

\[
\left| \frac{1}{\sqrt{q}} \left( \hat{C}_{k,r_n}^{\gamma} - \frac{k}{q} \hat{C}_{q,r_n}^{\gamma} \right) - \frac{1}{\sqrt{q}} \left( C_{k,r_n}^{\gamma} - \frac{k}{q} C_{q,r_n}^{\gamma} \right) \right| = o_p(1), \quad (5.8)
\]

for all \( 0 < \gamma < \frac{2}{3} \).

For the denominator we introduce

\[
\bar{\tau}^2 = \eta_{r_n}^{\gamma} - (q^{-1} C_{q,r_n}^{\gamma})^2 = \frac{1}{q} \sum_{t=[\tilde{r}_n]+1}^{[\tilde{r}_n]+q} \frac{g^4(t/n)}{g^4(t/n)} \epsilon_t^4 - \left( \frac{1}{q} \sum_{t=[\tilde{r}_n]+1}^{[\tilde{r}_n]+q} \frac{g^2(t/n)}{g^2(t/n)} \epsilon_t^2 \right)^2
\]

and

\[
\tau^2 = \eta_{r_n}^{\gamma} - (q^{-1} C_{q,r_n}^{\gamma})^2 = \frac{1}{q} \sum_{t=[\tilde{r}_n]+1}^{[\tilde{r}_n]+q} \epsilon_t^4 - \left( \frac{1}{q} \sum_{t=[\tilde{r}_n]+1}^{[\tilde{r}_n]+q} \epsilon_t^2 \right)^2.
\]

We have

\[
\left| \frac{1}{q} \sum_{t=[\tilde{r}_n]+1}^{[\tilde{r}_n]+q} \frac{g^2(t/n)}{g^2(t/n)} \epsilon_t^2 - \frac{1}{q} \sum_{t=[\tilde{r}_n]+1}^{[\tilde{r}_n]+q} \epsilon_t^2 \right| \leq \frac{1}{q} \sum_{t=[\tilde{r}_n]+1}^{[\tilde{r}_n]+q} \left| \frac{g^2(t/n)}{g^2(t/n)} - 1 \right| \epsilon_t^2
\]

\[
\leq \sup_{t=[\tilde{r}_n]+1 \leq t \leq [\tilde{r}_n]+q} \left| \frac{g^2(t/n) - \hat{g}^2(t/n)}{\hat{g}^2(t/n)} \right| \times \frac{1}{q} \sum_{t=[\tilde{r}_n]+1}^{[\tilde{r}_n]+q} \epsilon_t^2.
\]
Using (5.7) and the law of large numbers, we get
\[
\left| \frac{1}{q} \sum_{t=[r_{n}] + 1}^{[r_{n}] + q} g^2 \left( \frac{t}{n} \right) \epsilon_t^2 - \frac{1}{q} \sum_{t=[r_{n}] + 1}^{[r_{n}] + q} \epsilon_t^2 \right| = o_p(1).
\]
Similarly, we write
\[
\left| \frac{1}{q} \sum_{t=[r_{n}] + 1}^{[r_{n}] + q} g^4 \left( \frac{t}{n} \right) \epsilon_t^4 - \frac{1}{q} \sum_{t=[r_{n}] + 1}^{[r_{n}] + q} \epsilon_t^4 \right| = o_p(1),
\]
which implies that
\[
\bar{r}^2 - \tau^2 = o_p(1). \tag{5.9}
\]
As a result, from (5.8) et (5.9), we deduce that
\[
| q^{-\frac{1}{2}} \tilde{B}_{k,\bar{r}_n}^\gamma - q^{-\frac{1}{2}} B_{k,\bar{r}_n}^\gamma | = o_p(1),
\]
and that \( q^{-\frac{1}{2}} \tilde{B}_{k,\bar{r}_n}^\gamma \) and \( q^{-\frac{1}{2}} B_{k,\bar{r}_n}^\gamma \) have the same asymptotic behavior. The rest of the proof follows the same arguments as in the proof of Proposition 2 in Sansó et al. (2004) and considering \( q^{-\frac{1}{2}} B_{k,\bar{r}_n}^\gamma \). \( \square \)
References

[1] Aue, A., Hörmann, S., Horváth, L., and Reimherr, M. (2009). Break detection in the covariance structure of multivariate time series models. *The Annals of Statistics* 37, 4046-4087.

[2] Berkes, I., Horváth, L., and Kokoszka, P. (2004). Testing for parameter constancy in $GARCH(p,q)$ models. *Statistics and Probability Letters* 70, 263-273.

[3] Dahlhaus, R. (1997). Fitting time series models to nonstationary processes. *The Annals of Statistics* 25, 1-37.

[4] Dahlhaus, R. (2012). Locally stationary processes. *Handbook of statistics* 30, Chap. 13 (Eds T. S. Rao, S. S. Rao, C. R. Rao). Elsevier: New York, 351-412.

[5] Inclan, C., and Tiao, G.C. (1994). Use of cumulative sums of squares for retrospective detection of changes of variance. *Journal of the American Statistical Association* 89, 913-923.

[6] Phillips, P.C.B., et Xu, K.L. (2006). Inference in autoregression under heteroskedasticity. *Journal of Time Series Analysis* 27, 289-308.

[7] Sansó, A., Aragó, V., and Carrion, J.L. (2004). Testing for changes in the unconditional variance of financial time series. *Revista de Economia Financiera* 4, 32-53.

[8] White, H. (1984). Asymptotic Theory for Econometricians. New York: Academic Press.

[9] Xu, K. L., and Phillips, P. C. (2008). Adaptive estimation of autoregressive models with time-varying variances. *Journal of Econometrics* 142, 265-280.