Selfdecomposability and Semi-selfdecomposability in Subordination of Cone-parameter Convolution Semigroups

Ken-iti SATO

(Communicated by A. Tani)

Abstract. Extension of two known facts concerning subordination is made. The first fact is that, in subordination of 1-dimensional Brownian motion with drift, selfdecomposability is inherited from subordinator to subordinated. This is extended to subordination of cone-parameter convolution semigroups. The second fact is that, in subordination of strictly stable cone-parameter convolution semigroups on $\mathbb{R}^d$, selfdecomposability is inherited from subordinator to subordinated. This is extended to semi-selfdecomposability.

1. Introduction

A subset $K$ of $\mathbb{R}^N$ is called a cone if it is a non-empty closed convex set which is closed under multiplication by nonnegative reals and contains no straight line through 0 and if $K \neq \{0\}$. Given a cone $K$, we call $\{\mu_s : s \in K\}$ a $K$-parameter convolution semigroup on $\mathbb{R}^d$ if it is a family of probability measures on $\mathbb{R}^d$ satisfying

$$\mu_{s_1} * \mu_{s_2} = \mu_{s_1 + s_2} \quad \text{for} \quad s_1, s_2 \in K,$$

$$\mu_{ts} \rightarrow \delta_0 \quad \text{as} \quad t \downarrow 0, \quad \text{for} \quad s \in K,$$

where $\delta_0$ is delta distribution located at 0 $\in \mathbb{R}^d$. Convergence of probability measures is understood as weak convergence. It follows from (1.1) and (1.2) that $\mu_0 = \delta_0$.

Subordination of a cone-parameter convolution semigroup is defined as follows. Let $K_1$ and $K_2$ be cones in $\mathbb{R}^{N_1}$ and $\mathbb{R}^{N_2}$, respectively. Let $\{\mu_u : u \in K_2\}$ be a $K_2$-parameter convolution semigroup on $\mathbb{R}^d$ and $\{\rho_s : s \in K_1\}$ a $K_1$-parameter convolution semigroup on $\mathbb{R}^{N_2}$ supported on $K_2$ (that is, $\text{Supp}(\rho_s) \subseteq K_2$). Define a probability measure $\sigma_s$ on $\mathbb{R}^d$ by

$$\sigma_s(B) = \int_{K_2} \mu_u(B) \rho_s(du) \quad \text{for} \quad B \in \mathcal{B}(\mathbb{R}^d),$$

where $\mathcal{B}(\mathbb{R}^d)$ is the class of Borel sets in $\mathbb{R}^d$. Then $\{\sigma_s : s \in K_1\}$ is a $K_1$-parameter convolution semigroup on $\mathbb{R}^d$. This procedure to get $\{\sigma_s : s \in K_1\}$ is called subordination of $\{\mu_u : u \in K_2\}$ by $\{\rho_s : s \in K_1\}$. Convolution semigroups $\{\mu_u : u \in K_2\}, \{\rho_s : s \in K_1\},$
and \( \{ \sigma_s : s \in K_1 \} \) are respectively called subordinand, subordinating (or subordinator), and subordinated.

Cone-parameter convolution semigroups on \( \mathbb{R}^d \) and their subordination are introduced in Pedersen and Sato [11]. Their basic properties are proved in Theorems 2.8, 2.11, and 4.4 of [11]. A number of examples are given there. In Barndorff-Nielsen, Pedersen, and Sato [1], several models leading to \( \mathbb{R}_+ \)-parameter convolution semigroups supported on \( \mathbb{R}_+ \) are discussed, including some financial models. Here \( \mathbb{R}_+ = [0, \infty) \) and \( \mathbb{R}_+^N = (\mathbb{R}_+)^N \).

In \( \mathbb{R}_+ \)-parameter case, any convolution semigroup on \( \mathbb{R}^d \) corresponds to a unique (in law) Lévy process. For a general cone \( K \), any \( K \)-parameter Lévy process \( \{ X_s : s \in K \} \) on \( \mathbb{R}^d \) defined in Pedersen and Sato [12] induces a \( K \)-parameter convolution semigroup \( \{ \mu_s \} \) on \( \mathbb{R}^d \) as \( \mu_s = L(X_s) \), the law of \( X_s \). But, for a given \( K \)-parameter convolution semigroup on \( \mathbb{R}^d \), neither existence nor uniqueness (in law) of a \( K \)-parameter Lévy process which induces the semigroup can be proved in general, as is shown in [12]. The existence is proved when \( d = 1 \), when \( K \) is isomorphic to \( \mathbb{R}_+^N \), or when \( \mu_s \) does not have Gaussian part for any \( s \). The non-existence is proved for the canonical \((d\text{-dimensional Gaussian})\) \( \mathcal{S}_d^+ \)-parameter convolution semigroup defined in [12] for \( d \geq 2 \), where \( \mathcal{S}_d^+ \) is the cone of \( d \times d \) symmetric nonnegative-definite matrices. Concerning the uniqueness, some sufficient conditions for the uniqueness and for the non-uniqueness are given in [12]. For example, if \( \{ \mu_s \} \) is an \( \mathbb{R}_+^2 \)-parameter convolution semigroup on \( \mathbb{R} \) such that the Gaussian part of \( \mu_s \) is nonzero for any \( s \neq 0 \), then the corresponding \( \mathbb{R}_+^2 \)-parameter Lévy process on \( \mathbb{R} \) is not unique in law. Subordination of a \( K_2 \)-parameter Lévy process on \( \mathbb{R}^d \) by a \( K_1 \)-parameter Lévy process on \( \mathbb{K}_2 \) results in a new \( K_1 \)-parameter Lévy process on \( \mathbb{R}_+^d \), as is shown in Pedersen and Sato [12] and earlier, in the case \( K_2 = \mathbb{R}_+ \) and \( K_1 = \mathbb{R}_+ \), in Barndorff-Nielsen, Pedersen, and Sato [1]. It induces subordination of a cone-parameter convolution semigroup. But subordination of a cone-parameter convolution semigroup is not always accompanied by subordination of a cone-parameter Lévy process.

In this paper we give some results on inheritance of selfdecomposability, semi-selfdecomposability, and some related properties from subordinating to subordinated in subordination of cone-parameter convolution semigroups. Applications to distributions of type \( \text{mult}G \) are given.

Semi-selfdecomposable distributions were introduced by Maejima and Naito [8]. Their probabilistic representations were given by Maejima and Sato [9]. Their remarkable continuity properties were discovered by Watanabe [19]. Recent papers of Kondo, Maejima, and Sato [5] and Lindner and Sato [7] studied them in stationary distributions of some generalized Ornstein–Uhlenbeck processes.
2. One-dimensional Gaussian subordinands

Let $G_{a,\gamma}$ denote Gaussian distribution on $\mathbb{R}$ with variance $a \geq 0$ and mean $\gamma \in \mathbb{R}$, where $G_{0,\gamma} = \delta_{\gamma}$. A $K$-parameter convolution semigroup $\{\mu_u : u \in K\}$ is called 1-dimensional Gaussian if, for each $u \in K$, $\mu_u$ is $G_{a,\gamma}$ with some $a$ and $\gamma$.

A distribution $\mu$ on $\mathbb{R}^d$ is said to be selfdecomposable if, for each $b > 1$, there is a distribution $\mu'$ on $\mathbb{R}^d$ such that
\begin{equation}
\hat{\mu}(z) = \hat{\mu}(b^{-1}z)\hat{\mu}'(z), \quad z \in \mathbb{R}^d.
\end{equation}
Here $\hat{\mu}(z)$ and $\hat{\mu}'(z)$ are the characteristic functions of $\mu$ and $\mu'$, respectively. If $\mu$ is selfdecomposable, then $\mu$ is infinitely divisible.

Noting that selfdecomposability is equivalent to semi-selfdecomposability with span $b$ for all $b > 1$ (see Section 3 for the definition) and using Theorem 15.8 of [15], we see that an infinitely divisible distribution $\mu$ on $\mathbb{R}^d$ with Lévy measure $\nu$ is selfdecomposable if and only if
\begin{equation}
\nu(b^{-1}B) \geq \nu(B) \quad \text{for } b > 1 \text{ and } B \in B(\mathbb{R}^d \setminus \{0\}).
\end{equation}
The condition (2.2) holds if and only if $\nu$ has a polar representation
\begin{equation}
\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) r^{-1} k_\xi(r) dr \quad \text{for } B \in B(\mathbb{R}^d \setminus \{0\}),
\end{equation}
where $S = \{\xi : |\xi| = 1\}$, the unit sphere in $\mathbb{R}^d$, $\lambda$ is a measure on $S$, and $k_\xi(r)$ is a nonnegative function measurable in $\xi$ and decreasing in $r > 0$ (Theorem 15.10 of [15]). We are using the word decrease in the wide sense allowing flatness.

**Theorem 2.1.** Let $K_1$ and $K_2$ be cones in $\mathbb{R}^N_1$ and $\mathbb{R}^N_2$, respectively. Let $\{\mu_u : u \in K_2\}$ be a 1-dimensional Gaussian $K_2$-parameter convolution semigroup (subordinand), $\{\rho_s : s \in K_1\}$ a $K_1$-parameter convolution semigroup supported on $K_2$ (subordinating), and $\{\sigma_s : s \in K_1\}$ the subordinated $K_1$-parameter convolution semigroup on $\mathbb{R}$. Fix $s \in K_1$. If $\rho_s$ is selfdecomposable, then $\sigma_s$ is selfdecomposable.

We stress that the Gaussian distribution $\mu_u$ is not necessarily centered. For the centered Gaussian (that is strictly 2-stable), the result is largely extended in Theorem 3.1 in Section 3. Historically, Halgreen [4] raised a question equivalent to asking whether the statement of Theorem 2.1 for $K_1 = K_2 = \mathbb{R}_+$ is true. After 22 years, Theorem 1.1 of Sato [16] answered this question affirmatively. The theorem above is an extension of it. In order to prove the theorem, we prepare a lemma.

**Lemma 2.2.** Let $f(r)$ be a nonnegative decreasing function of $r > 0$ satisfying $\int_0^\infty (r+1) r^{-1} f(r) dr < \infty$. Let $a \geq 0$ and $\gamma \in \mathbb{R}$. Then, for every $b > 1$ and $B \in B(\mathbb{R} \setminus \{0\})$,
\begin{equation}
\int_0^\infty G_{ra,\gamma}(b^{-1}B)r^{-1} f(r) dr \geq \int_0^\infty G_{ra,\gamma}(B)r^{-1} f(r) dr.
\end{equation}
PROOF. Let \( \{X_t : t \in \mathbb{R}_+\} \) be the Lévy process with distribution \( G_{a,y} \) at time 1. Let \( \{Z_t : t \in \mathbb{R}_+\} \) be a selfdecomposable subordinator with Lévy measure \( r^{-1} f(r)dr \) and drift 0. Let \( \{Y_t : t \in \mathbb{R}_+\} \) be the Lévy process on \( \mathbb{R} \) obtained by subordination of \( \{X_t\} \) by \( \{Z_t\} \). Then Theorem 30.1 of [15] tells us that the Lévy measure \( \nu \) of \( \{Y_t\} \) is expressed as

\[
\nu(B) = \int_0^\infty G_{ra,yr}(B)r^{-1} f(r)dr, \quad B \in \mathcal{B}(\mathbb{R} \setminus \{0\}).
\]

If \( a > 0 \), then Theorem 1.1 of [16] establishes that \( Y_t \) has a selfdecomposable distribution for any \( t \geq 0 \). If \( a = 0 \), then \( \{X_t\} \) is a trivial Lévy process (that is, \( X_t = \gamma t \), nonrandom) and \( Y_t = \gamma Z_t \), which has a selfdecomposable distribution. In any case, \( \{Y_t\} \) is selfdecomposable.

Hence \( \nu(B) \geq \nu(b^{-1}B) \), which is exactly (2.4).

PROOF OF THEOREM 2.1. Let \( \nu_{\mu u}, \nu_{\rho s}, \text{and} \nu_{\sigma s} \) denote the Lévy measures of \( \mu_u, \rho_s, \) \text{and} \( \sigma_s \), respectively. We have \( \mu_u = G_{a_u,y_u} \) with some \( a_u \geq 0 \) and \( y_u \in \mathbb{R} \). These \( a_u \) and \( y_u \) are continuous functions of \( u \) (Theorem 2.8 of [11]). Since \( \mu_u \) has Lévy measure 0, Theorem 4.4 of [11] says that

\[
\nu_{\sigma s}(B) = \int_{K_2} G_{a_u,y_u}(B)\nu_{\rho s}(du), \quad B \in \mathcal{B}(\mathbb{R} \setminus \{0\}).
\]

Assume that \( \rho_s \) is selfdecomposable. Then \( \nu_{\rho s} \) is expressed as in the right-hand side of (2.3) with \( d = N_2 \). Since \( \text{Supp}(\rho_s) \subseteq K_2 \), it follows from Skorohod’s theorem [17] (or Lemma 4.1 of [11]) that the measure \( \lambda \) is supported on \( S \cap K_2 \) and that

\[
\int_{S \cap K_2} \lambda(d\xi) \int_0^\infty (r \land 1)r^{-1}k_{\xi}(r)dr < \infty.
\]

For any \( b > 1 \) and \( B \in \mathcal{B}(\mathbb{R} \setminus \{0\}) \) we have

\[
\nu_{\sigma s}(b^{-1}B) = \int_{K_2} G_{a_u,y_u}(b^{-1}B)\nu_{\rho s}(du) = \int_{S \cap K_2} \lambda(d\xi) \int_0^\infty G_{a_{\xi},y_{\xi}}(b^{-1}B)r^{-1}k_{\xi}(r)dr = I \quad (\text{say}).
\]

Notice that \( k_{\xi}(r) \) is decreasing in \( r \) and satisfies \( \int_0^\infty (r \land 1)r^{-1}k_{\xi}(r)dr < \infty \) for \( \lambda \)-almost every \( \xi \) and that \( a_{\xi} = ra_{\xi} \) and \( y_{\xi} = r y_{\xi} \) (see Proposition 2.7 of [11]). Thus we can apply Lemma 2.2 to obtain

\[
I \geq \int_{S \cap K_2} \lambda(d\xi) \int_0^\infty G_{a_{\xi},y_{\xi}}(B)r^{-1}k_{\xi}(r)dr = \nu_{\rho s}(B).
\]

This means that \( \sigma_s \) is selfdecomposable. \( \square \)

REMARK 2.3. Let \( K \) be a cone and let \( \{\mu_s : s \in K\} \) be a \( K \)-parameter convolution semigroup on \( \mathbb{R}^d \). Let \( \mu_0 \in K \). If \( \mu_{s_0} \) is selfdecomposable, then \( \mu_{ts_0} \) is selfdecomposable for all \( t \geq 0 \) since \( \mu_{ts_0} = \mu_{s_0}^t \), the \( t \) th convolution power of \( \mu_{s_0} \) (Proposition 2.7 of [11]), but
µ_{s_1} may not be selfdecomposable for some s_1 \in K \setminus \{t s_0 : t \geq 0\}. This follows from Sections 2 and 3 of [11].

**Remark 2.4.** In Theorem 2.1 let K_1 = K_2 = \mathbb{R}_+ and replace “Gaussian” by “α-stable (not necessarily strictly α-stable),” where α \in (0, 2]. Then the statement for α = 2 is exactly Theorem 1.1 of [16]. The statement for α \in (1, 2) is not true, which is pointed out by Kozubowski [6] using Theorem 2.1(v) of Ramachandran [13]. It is not known whether the statement for α \in (0, 1] is true.

**Remark 2.5.** If µ is selfdecomposable, then the distribution µ′ in (2.1) is uniquely determined by µ and b, and µ′ is also infinitely divisible. For nonnegative integers m we define L_m(\mathbb{R}^d) as follows: L_0(\mathbb{R}^d) is the class of selfdecomposable distributions on \mathbb{R}^d; for m \geq 1, L_m(\mathbb{R}^d) is the class of µ ∈ L_0(\mathbb{R}^d) such that, for every b > 1, µ′ in (2.1) belongs to L_{m-1}(\mathbb{R}^d). Thus we get a strictly decreasing sequence of subclasses of the class ID(\mathbb{R}^d) of infinitely divisible distributions on \mathbb{R}^d. We define L_∞(\mathbb{R}^d) as the intersection of L_m(\mathbb{R}^d), m = 0, 1, 2, . . . . It is not known even in the case K_1 = K_2 = \mathbb{R}_+ whether Theorem 2.1 is true with “selfdecomposable” replaced by “of class L_m” for m ∈ \{1, 2, . . . , ∞\}.

**Remark 2.6.** Let d \geq 2. Theorem 2.1 cannot be generalized to d-dimensional Gaussian. If [µ_u : u \in \mathbb{R}_+] is an \mathbb{R}_+-parameter convolution semigroup (subordinand) induced by d-dimensional Brownian motion with nonzero drift and \{ρ_t : t \in \mathbb{R}_+\} is an \mathbb{R}_+-parameter convolution semigroup supported on \mathbb{R}_+ (subordinating) of Thorin class (of generalized gamma convolutions, in other words) satisfying some additional condition, then the subordinated \mathbb{R}_+-parameter convolution semigroup \{σ_t : t \in \mathbb{R}_+\} on \mathbb{R}^d is not selfdecomposable for any t > 0. This fact was noticed by Takano [18] and Grigelionis [3]. Recall that the Thorin class is a subclass of the class of selfdecomposable distributions. This σ_t supplies an example of an infinitely divisible non-selfdecomposable distribution whose one-dimensional projections are selfdecomposable, since we can apply Theorem 1.1 of [16] to one-dimensional projections of [µ_u : u \in \mathbb{R}_+]. The first example of a distribution with this projection property was constructed in Sato [14].

**Remark 2.7.** It is not known even in the case K_1 = K_2 = \mathbb{R}_+ whether Theorem 2.1 is true with “selfdecomposable” replaced by “semi-selfdecomposable”, which will be defined in the next section.

3. **Inheritance of semi-selfdecomposability**

A distribution on \mathbb{R}^d is called semi-selfdecomposable if there are b > 1 and µ′ ∈ ID(\mathbb{R}^d) such that

\begin{equation}
\hat{µ}(z) = \hat{µ}(b^{-1}z)\hat{µ}'(z), \quad z \in \mathbb{R}^d.
\end{equation}
The \( b \) in this definition is called a span of \( \mu \); it is not uniquely determined by \( \mu \). The class of semi-selfdecomposable distributions on \( \mathbb{R}^d \) having \( b \) as a span is denoted by \( L_0(b^{-1}, \mathbb{R}^d) \). If \( \mu \in L_0(b^{-1}, \mathbb{R}^d) \), then \( \mu \) is infinitely divisible and the distribution \( \mu' \) is uniquely determined by \( \mu \) and \( b \). For any positive integer \( m \) we inductively define

\[
L_m(b^{-1}, \mathbb{R}^d) = \{ \mu \in L_0(b^{-1}, \mathbb{R}^d) : \mu' \in L_{m-1}(b^{-1}, \mathbb{R}^d) \}.
\]

Then \( L_m(b^{-1}, \mathbb{R}^d) \) is a subclass of \( L_{m-1}(b^{-1}, \mathbb{R}^d) \). In fact we can prove that the former is a strict subclass of the latter (see Remark 3.1 of [10]). Further we define \( L_\infty(b^{-1/\alpha}, \mathbb{R}^d) \) as the intersection of \( L_m(b^{-1}, \mathbb{R}^d) \) for \( m = 0, 1, \ldots \).

Let \( 0 < \alpha \leq 2 \). A distribution \( \mu \) on \( \mathbb{R}^d \) is called strictly \( \alpha \)-semistable if \( \mu \in ID(\mathbb{R}^d) \) and if there is a real number \( b > 1 \) such that

\[
\hat{\mu}(z)^{b^\alpha} = \hat{\mu}(bz), \quad z \in \mathbb{R}^d,
\]

or, equivalently, \( \hat{\mu}(z)^{b^{-\alpha}} = \hat{\mu}(b^{-1}z), z \in \mathbb{R}^d \). In this case we say that the \( \alpha \)-semistable distribution \( \mu \) has a span \( b \), which is not uniquely determined by \( \mu \). If \( \mu \) is strictly \( \alpha \)-semistable on \( \mathbb{R}^d \) with a span \( b \), then it is easy to see that \( \mu \in L_\infty(b^{-1}, \mathbb{R}^d) \), since we have

\[
\hat{\mu}(z) = \hat{\mu}(z)^{b^{-\alpha}} \hat{\mu}(z)^{1-b^{-\alpha}} = \hat{\mu}(b^{-1}z) \hat{\mu}(z)^{1-b^{-\alpha}}.
\]

For description and examples of Lévy measures of semi-selfdecomposable and semistable distributions, see Sections 14 and 15 of [15].

The statement of Remark 2.3 is true also for “semi-selfdecomposable with a span \( b \)” and “strictly \( \alpha \)-semistable with a span \( b \)” in place of “selfdecomposable”.

**Theorem 3.1.** Let \( K_1 \) and \( K_2 \) be cones in \( \mathbb{R}^{N_1} \) and \( \mathbb{R}^{N_2} \), respectively. Let \( \{\mu_u : u \in K_2\} \) be a \( K_2 \)-parameter convolution semigroup on \( \mathbb{R}^d \) (subordinand), \( \{\rho_s : s \in K_1\} \) a \( K_1 \)-parameter convolution semigroup supported on \( K_2 \) (subordinating), and \( \{\sigma_s : s \in K_1\} \) the subordinated \( K_1 \)-parameter convolution semigroup on \( \mathbb{R}^d \). Suppose that there are \( 0 < \alpha \leq 2 \) and \( b > 1 \) such that, for every \( u \in K_2 \), \( \mu_u \) is strictly \( \alpha \)-semistable with a span \( b^{1/\alpha} \). Fix \( s \in K_1 \). Then the following statements are true.

(i) Let \( m \in \{0, 1, \ldots, \infty\} \). If

\[
\rho_s \in L_m(b^{-1}, \mathbb{R}^{N_2}) \tag{3.3}
\]

then

\[
\sigma_s \in L_m(b^{-1/\alpha}, \mathbb{R}^d) \tag{3.4}
\]

(ii) Let \( 0 < \alpha' \leq 1 \). If

\[
\rho_s \text{ is strictly } \alpha' \text{-semistable with a span } b \tag{3.5}
\]

then

\[
\sigma_s \text{ is strictly } \alpha \alpha' \text{-semistable with a span } b^{1/\alpha} \tag{3.6}
\]
Note that strictly 1-semistable distributions supported on a cone are delta distributions.
This theorem is an extension of Theorem 4.10 of Pedersen and Sato [11] to the “semi” case.
We prepare a lemma. This is an analogue of Lemma 4.11 of [11] and the proof is almost the same.

**Lemma 3.2.** Let \( K_2 \) be a cone in \( \mathbb{R}^{N_2} \). Suppose that \( \rho \) is in \( L_0(b^{-1}, \mathbb{R}^{N_2}) \) and that \( \text{Supp}(\rho) \subseteq K_2 \). Let \( \rho' \) be defined by \( \hat{\rho}(z) = \hat{\rho}(b^{-1}z)\hat{\rho}(z), z \in \mathbb{R}^{N_2} \). Then \( \text{Supp}(\rho') \subseteq K_2 \).

**Proof of Theorem 3.1.** Let us prove assertion (i) for \( m = 0 \). Assume that \( \rho_s \in L_0(b^{-1}, \mathbb{R}^{N_2}) \). Define \( \rho''_s \) as \( \rho''_s(z) = \hat{\rho}_s(b^{-1}z) \). Then
\[
\hat{\rho}_s(z) = \hat{\rho}''_s(z)\hat{\rho}'(z)
\]
and thus \( \rho_s = \rho''_s \ast \rho' \). Lemma 3.2 tells us that \( \rho''_s \) is supported on \( K_2 \). Clearly \( \rho''_s \) is also supported on \( K_2 \). Hence
\[
\hat{\sigma}_s(z) = \int K_2 \hat{\mu}_u(z)\rho_s(du) = \int K_2 \int K_2 \hat{\mu}_{u_1 + u_2}(z)\rho''_s(du_1)\rho'_s(du_2)
\]
\[
= \int K_2 \int K_2 \hat{\mu}_{u_1}(z)\hat{\mu}_{u_2}(z)\rho''_s(du_1)\rho'_s(du_2)
\]
\[
= \int K_2 \hat{\mu}_{b^{-1}u_1}(z)\rho_s(du_1) \int K_2 \hat{\mu}_{u_2}(z)\rho'_s(du_2).
\]
Using Proposition 2.7 of [11] and the assumption that \( \mu_u \) is strictly \( \alpha \)-semistable with a span \( b^{1/\alpha} \), we have
\[
\hat{\mu}_{b^{-1}u}(z) = \hat{\mu}_u(b^{-1}z) = \hat{\mu}_u(b^{-1/\alpha}z).
\]
It follows that
\[
\hat{\sigma}_s(z) = \hat{\sigma}_s(b^{-1/\alpha}z) \int K_2 \hat{\mu}_u(z)\rho'_s(du).
\]
Since \( \int K_2 \hat{\mu}_u(z)\rho'_s(du) \) is subordination of \( \{\mu_u\} \) by \( \{\rho'_s\} : t \in \mathbb{R}_+ \), we see that \( \int K_2 \hat{\mu}_u(z)\rho'_s(du) \) is infinitely divisible. This shows that \( \sigma_s \in L_0(b^{-1/\alpha}, \mathbb{R}^d) \).

Next, we assume that (i) is true for a fixed \( m \in \{0, 1, \ldots\} \). We claim that (i) is true for \( m + 1 \). Suppose that \( \rho_s \in L_{m+1}(b^{-1}, \mathbb{R}^{N_2}) \). Then \( \hat{\rho}_s(z) = \hat{\rho}_s(b^{-1}z)\hat{\rho}'(z) \) with \( \rho'_s \in L_m(b^{-1}, \mathbb{R}^{N_2}) \). We have (3.7) since \( L_{m+1}(b^{-1}, \mathbb{R}^{N_2}) \subseteq L_0(b^{-1}, \mathbb{R}^{N_2}) \). Now \( \int K_2 \hat{\mu}_u(z)(\rho'_s)^{(t)}(du) \) is subordination such that \( (\rho'_s)^{(t)} \) is in \( L_m(b^{-1}, \mathbb{R}^{N_2}) \). Hence \( \int K_2 \hat{\mu}_u(z)(\rho'_s)^{(t)}(du) \) is the characteristic function of a distribution in \( L_m(b^{-1/\alpha}, \mathbb{R}^d) \). It follows that \( \sigma_s \in L_{m+1}(b^{-1/\alpha}, \mathbb{R}^d) \), which shows (i) for \( m + 1 \).

Assertion (i) for \( m = \infty \) is a consequence of that for finite \( m \).

To prove (ii), assume (3.5). Let us show (3.6), that is,
\[
\hat{\sigma}_s(z)b^{\nu'} = \hat{\sigma}_s(b^{1/\alpha}z).
\]
Using
\[ \hat{\rho}_b(\varphi'(z)) = \hat{\rho}_s(z)b \]
and
\[ \hat{\mu}_b(\varphi(z)) = \hat{\mu}(b^{1/\alpha}z) , \]
we obtain
\[ \hat{\sigma}_s(z)b^{1/\alpha} = \hat{\sigma}_b(z) = \int_{K_2} \hat{\mu}_b(z)\rho_s(du) = \int_{K_2} \hat{\mu}_b(z)\rho_s(du) = \hat{\sigma}_s(b^{1/\alpha}z) , \]
completing the proof.

**Application to distributions of type multG.** Following Barndorff-Nielsen and Pérez-Abreu [2], we say that a probability measure \( \sigma \) on \( \mathbb{R}^d \) is of type \( \text{multG} \) if
\[ \sigma = L(Z_{1/2}X) , \]
where \( X \) is a standard Gaussian on \( \mathbb{R}^d \), \( Z \) is an \( S^+_d \)-valued infinitely divisible random variable, \( Z_{1/2} \) is the nonnegative-definite symmetric square root of \( Z \), and \( X \) and \( Z \) are independent. Here, as in Section 1, \( S^+_d \) is the class of \( d \times d \) symmetric nonnegative-definite matrices and elements of \( \mathbb{R}^d \) are considered as column \( d \)-vectors. Regarding the lower triangle \((s_{jk})_{j \leq k, k=1} \in S^+_d \) of \( s = (s_{jk})_{j=1}^{d} \in S^+_d \) as a \( d(d+1)/2 \)-vector, \( S^+_d \) is identified with a cone in \( \mathbb{R}^{d(d+1)/2} \). The \( S^+_d \)-parameter convolution semigroup \( \{ \mu_s : s \in S^+_d \} \) on \( \mathbb{R}^d \) where \( \mu_s \) is \( d \)-dimensional Gaussian with mean vector 0 and covariance matrix \( s \) is called the canonical \( S^+_d \)-parameter convolution semigroup ([11]). The following fact is known (Theorem 4.7 of [11] and its proof).

**Proposition 3.3.** Let \( \{ \mu_u : u \in S^+_d \} \) be the canonical \( S^+_d \)-parameter convolution semigroup (subordinand), \( \{ \rho_t : t \in \mathbb{R}_+ \} \) an \( \mathbb{R}_+ \)-parameter convolution semigroup on \( \mathbb{R}^{d(d+1)/2} \) supported on \( S^+_d \) (subordinating), and \( \{ \sigma_t : t \in \mathbb{R}_+ \} \) the subordinated \( \mathbb{R}_+ \)-parameter convolution semigroup on \( \mathbb{R}^d \). Then \( \sigma_t \) (or, more generally, \( \sigma_t \)) is of type multG. Conversely, any distribution on \( \mathbb{R}^d \) of type multG is expressible as \( \sigma_t \) of such an \( \mathbb{R}_+ \)-parameter convolution semigroup \( \{ \sigma_t : t \in \mathbb{R}_+ \} \). The correspondence of the two representations of a distribution of type multG is that \( \rho_1 = L(Z) \).

We can show the following.

**Proposition 3.4.** Let \( \sigma \) be a distribution of type multG, that is, let \( \sigma = L(Z_{1/2}X) \), where \( X \) is a standard Gaussian on \( \mathbb{R}^d \), \( Z_{1/2} \) is the nonnegative-definite symmetric square root of \( S^+_d \)-valued infinitely divisible random variable \( Z \), and \( X \) and \( Z \) are independent.

(i) Let \( m \in \{0, 1, \ldots, \infty\} \) and \( b > 1 \). If \( L(Z) \in L_m(b^{-1}, \mathbb{R}^{d(d+1)/2}) \), then \( \sigma \in L_m(b^{-1/2}, \mathbb{R}^d) \).
(ii) Let $0 < \alpha' \leq 1$ and $b > 1$. If $\mathcal{L}(Z)$ is strictly $\alpha'$-semistable with a span $b$, then $\sigma$ is strictly $2\alpha'$-semistable with a span $b^{1/2}$.

PROOF. Recall that a distribution $\mu$ is strictly $\alpha$-stable if and only if it is strictly $\alpha$-semistable with a span $b$ for all $b > 1$. Apply Theorem 3.1 combined with Proposition 3.3.

References

[1] O. E. Barndorff-Nielsen, J. Pedersen and K. Sato, Multivariate subordination, selfdecomposability and stability, Adv. Appl. Probab. 33 (2001), 160–187.
[2] O. E. Barndorff-Nielsen and V. Pérez-Abreu, Extensions of type $G$ and marginal infinite divisibility, Theory Probab. Appl. 47 (2003), 202–218.
[3] B. Grigelionis, On subordinated multivariate Gaussian Lévy processes, Acta Appl. Math. 96 (2007), 233–246.
[4] C. Halgreen, Self-decomposability of the generalized inverse Gaussian and hyperbolic distributions, Zeit. Wahrsch. Verw. Gebiete 47 (1979), 13–17.
[5] H. Kondo, M. Maejima and K. Sato, Some properties of exponential integrals of Lévy processes and examples, Elect. Comm. in Probab. 11 (2006), 291–303.
[6] T. J. Kozubowski, A note on self-decomposability of stable process subordinated to self-decomposable subordinator, Stat. Probab. Let. 73 (2005), 343–345 and 74 (2005), 89–91.
[7] A. Lindner and K. Sato, Continuity properties and infinite divisibility of stationary distributions of some generalized Ornstein-Uhlenbeck processes, Ann. Probab. 37 (2009), 250–274.
[8] M. Maejima and Y. Naito, Semi-selfdecomposable distributions and a new class of limit theorems, Probbab. Theory Relat. Fields 112 (1998), 13–31.
[9] M. Maejima and K. Sato, Semi-Lévy processes, semi-selfsimilar additive processes, and semi-stationary Ornstein-Uhlenbeck type processes, J. Math. Kyoto Univ. 43 (2003), 609–639.
[10] M. Maejima, K. Sato and T. Watanabe, Operator semi-selfdecomposability, $(C, Q)$-decomposability and related nested classes, Tokyo J. Math. 22 (1999), 473–509.
[11] J. Pedersen and K. Sato, Cone-parameter convolution semigroups and their subordination, Tokyo J. Math. 26 (2003), 503–525.
[12] J. Pedersen and K. Sato, Relations between cone-parameter Lévy processes and convolution semigroups, J. Math. Soc. Japan 56 (2004), 541–559.
[13] B. Ramachandran, On geometric-stable laws, a related property of stable processes, and stable densities of exponent one, Ann. Inst. Statist. Math. 49 (1997), 299–313.
[14] K. Sato, Multivariate distributions with selfdecomposable projections, J. Korean Math. Soc. 35 (1998), 783–791.
[15] K. Sato, Lévy Processes and Infinitely Divisible Distributions, Cambridge Univ. Press, 1999.
[16] K. Sato, Subordination and self-decomposability, Stat. Probab. Let. 54 (2001), 317–324.
[17] A. V. Skorohod, Random Processes with Independent Increments, Kluwer Academic Pub., 1991.
[18] K. Takano, On mixtures of the normal distribution by the generalized gamma convolutions, Bull. Fac. Sci. Ibaraki Univ. Ser. A 21 (1989), 29–41; Correction and addendum, 22 (1990), 49–52.
[19] T. Watanabe, Absolute continuity of some semi-selfdecomposable distributions and self-similar measures, Probab. Theory Relat. Fields, 117 (2000), 387–405.

Added April 17, 2008. For $\mathbb{R}_+$-parameter convolution semigroups the following papers discuss related results:

G. S. Choi, S. Y. Joo and Y. K. Kim, Subordination, self-decomposability and semi-stability, Comm. Korean Math.
Soc. 21 (2006), 787–794; G. S. Choi, Some results on subordination, selfdecomposability and operator semi-stability, Stat. Probab. Let. 78 (2008), 780–784.

Present Address:
HACHIMAN-YAMA 1101–5–103, TENPAKU-KU, NAGOYA, 468–0074 JAPAN.
e-mail: ken-iti.sato@nifty.ne.jp