Information entropy and its application in heat conduction

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Abstract. In this paper, heat conduction problems are analyzed from the view of information entropy. The variation law of temperature with time in the lumped-heat-capacity system can be expressed in the form of exponential distribution. It is proved that exponential distribution is the solution of a conditional extreme value problem with information entropy as an objective function. The general analysis procedure of heat conduction problems from the view of information entropy is proposed. Fundamental solution of heat conduction equation can be written in the form of normal distribution, and normal distribution can be derived from the maximum entropy principle.

1. Introduction

The concept of information entropy was firstly put forward by the American electrical engineer Shannon C E [1] in 1948. It is the generalization of physics entropy (thermodynamic entropy) and they are both the description of randomness and uncertainty of systems [2,3]. Jaynes E T [4] proposed the maximum entropy principle to determine the probability distribution in 1957. In 1967, Burg J P put forward the maximum entropy spectral estimation in time series analysis based on the maximum entropy principle [5]. Li X S [6] adopted the maximum entropy principle in structural optimization design. In this paper, we will study the application of information entropy in heat conduction.

2. Lumped-heat-capacity system and information entropy

The information entropy can be calculated if the probability density is given. For one-dimensional random variable, suppose its probability density is \( f(x) \), the information entropy is defined as [7]

\[
S[f(x)] = -\int_{-\infty}^{+\infty} f(x) \ln f(x) dx \tag{1}
\]

It can be seen from the above expression that information entropy is a functional of probability density.

According to [8], the analytical solution of temperature field based on lumped parameter method is

\[
\Theta = \theta_0 - \frac{t - t_{\infty}}{t_0 - t_{\infty}} = \exp\left(-\frac{hA}{\rho cV} \tau \right) = \exp\left(-\frac{\tau}{\tau_c} \right) \tag{2}
\]

Where \( \tau_c \) is called the time constant of the lumped-heat-capacity system.

Transient heat flow can be expressed as
\[ \Phi = -\rho c V \frac{dt}{d\tau} = (t_0 - t_\infty)hA \exp(-\frac{t}{\tau_c}) \quad (3) \]

Let \((t_0 - t_\infty)hA = \Phi_0\), then dimensionless transient heat flow can be written
\[ \frac{\Phi}{\Phi_0} = \exp(-\frac{hA}{\rho c V} \tau) = \exp(-\frac{\tau}{\tau_c}) \quad (4) \]

From formulas (2) and (4), we can see that dimensionless temperature difference and transient heat flow have the same variation law and they are both in exponential attenuation with time.

Divided by \(\tau_c\) of both sides of formulas (2) and (4), we can get
\[ f(\tau) = \frac{1}{\tau_c} \exp(-\frac{\tau}{\tau_c}), \tau > 0 \quad (5) \]

According to [9], \(f(\tau)\) obeys exponential distribution with parameter \(\tau_c\), where time constant \(\tau_c\) represents mathematical expectation.

The quantity of heat exchange can be obtained by integrating equation (3)
\[ Q = \int_0^\tau \Phi d\tau = (t_0 - t_\infty)\rho c V [1 - \exp(-\frac{hA}{\rho c V} \tau)] \quad (6) \]

Let \((t_0 - t_\infty)\rho c V = Q_\infty\), then dimensionless quantity of heat exchange can be expressed as
\[ \frac{Q}{Q_\infty} = 1 - \exp(-\frac{hA}{\rho c V} \tau) = 1 - \exp(-\frac{\tau}{\tau_c}) \quad (7) \]

Formula (7) is distribution function of exponential distribution with parameter \(\tau_c\).

From the above discussion, we can find the variation law of physical quantities in lumped-heat-capacity system is closely related with exponential distribution.

Now we prove that exponential distribution with parameter \(\mu\) is the solution of the following conditional extreme value problem.
\[
\begin{aligned}
\max_{f(x)} S[f(x)] &= -\int_0^\infty f(x) \ln f(x) dx \\
\text{s.t.} \quad &\int_0^\infty f(x)dx = 1 \\
&\int_0^\infty xf(x)dx = \mu
\end{aligned}
\quad (8)
\]

Proof: Introduce the auxiliary function
\[ H = -f(x) \ln f(x) + \lambda_1 f(x) + \lambda_2 x f(x) \quad (9) \]

Where \(\lambda_1\) and \(\lambda_2\) are the Lagrange multipliers.

Introduce the auxiliary functional
\[ S^* = \int_0^\infty \{-f(x) \ln f(x) + \lambda_1 f(x) + \lambda_2 x f(x)\} dx \quad (10) \]

By Euler-Lagrange Equation, we can get
\[-\ln f(x) - 1 + \lambda_1 + \lambda_2 x = 0 \quad (11)\]

So

\[f(x) = e^{\lambda_1 x - \lambda_2 x} \quad (12)\]

Substituting formula (12) in limiting conditions, we can finally get

\[f(x) = \frac{1}{\mu} e^{\frac{x}{\mu}} \quad (13)\]

From the above discussion, we can get the general analysis procedure of heat conduction problems from the view of information entropy:

- Expressing physical quantities in the dimensionless form;
- Normalization of physical quantities, i.e., transforming physical quantities into a probability density function;
- Finding the conditional extreme value problem corresponding to probability density function with information entropy as an objective function.

It is easy to verify that exponential distribution is the solution of the following initial value problem of ODE.

\[
\begin{aligned}
    y' + \frac{1}{\mu} y &= 0 \\
    y(0) &= \frac{1}{\mu}
\end{aligned}
\]  

(14)

That is to say, initial value problem (14) is equivalent to conditional extreme value problem (8). According to [10,11], there is no classical variational principle corresponding to initial value problem (14) because of the appearance of the first-order derivative. The introduction of information entropy concept may bring some new ideas to solve this problem.

### 3. Fundamental solution of heat conduction equation and information entropy

The solution of the following definite solution problem is defined as fundamental solution of one-dimensional heat conduction equation

\[
\begin{aligned}
    u_t - a^2 u_{xx} &= 0, \quad -\infty < x < +\infty, t > 0, \\
    u(x,0) &= \delta(x), \quad -\infty < x < +\infty
\end{aligned}
\]  

(15)

Where \( \delta(x) \) is one-dimensional Dirac Delta function.

By Fourier transform method, fundamental solution of one-dimensional heat conduction equation can be expressed as

\[u(x,t) = \frac{1}{2a\sqrt{\pi t}} e^{\frac{x^2}{4a^2 t}}, \quad -\infty < x < +\infty, t > 0\]  

(16)

Let \( 2a^2 t = \sigma^2 \), then formula (16) can be rewritten as

\[f(x) = \frac{1}{\sqrt{2\pi \sigma}} e^{\frac{x^2}{2\sigma^2}} \]

(17)

Formula (17) is one-dimensional normal distribution whose mathematical expectation is zero and
variance is $\sigma^2$. Variance $\sigma^2$ is proportional to time $t$.

By variational method [12], we can prove that normal distribution (17) is the solution of the following conditional extreme value problem.

$$\max_{f(x)} S[f(x)] = -\int_{-\infty}^{+\infty} f(x) \ln f(x) dx$$

$$\text{s.t.} \quad \int_{-\infty}^{+\infty} f(x) dx = 1$$
$$\int_{-\infty}^{+\infty} x^2 f(x) dx = \sigma^2$$

Substituting formula (17) into equation (1), we have

$$S[f(x)] = \ln \sqrt{2\pi e\sigma^2} = \frac{1}{2} + \ln \sqrt{2\pi} + \ln \sigma^2$$

Therefore, information entropy of normal distribution is proportional to its variance $\sigma^2$.

The above discussion can also be extended to multi-dimensional heat conduction problems.

4. Conclusion

- The variation law of temperature and transient heat flow in lumped-heat-capacity system can be expressed in the form of exponential distribution.
- The general analysis procedure of heat conduction problems from the view of information entropy is proposed.
- The fundamental solution of heat conduction equation can be written in the form of normal distribution, and normal distribution can be derived from the maximum entropy principle.

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