A particle method for continuous
Hegselmann-Krause opinion dynamics

Bruce Boghosian*, Christoph Börgers*, Natasa Dragovic*, Anna Haensch*†, and Arkadz Kirshtein*

Abstract. We derive a differential-integral equation akin to the Hegselmann-Krause model of opinion dynamics [R. Hegselmann and U. Krause, JASSS, vol. 5, 2002], and propose a particle method for solving the equation. Numerical experiments demonstrate second-order convergence of the method in a weak sense. We also show that our differential-integral equation can equivalently be stated as a system of differential equations. An integration-by-parts argument that would typically yield an energy dissipation inequality in physical problems then yields a concentration inequality, showing that a natural measure of concentration increases monotonically.

keywords: opinion dynamics; Hegselmann-Krause model; bounded confidence model; particle method

AMS subject classification: 91D30, 65M75

1 Introduction.

People’s opinions and beliefs are influenced in complex ways by families, friends, colleagues, media, as well as politicians and other mega-influencers [1, 5, 6, 9, 11]. In recent decades, attempts have been made to understand aspects of this process using mathematical modeling and computational simulation; for surveys on opinion dynamics, see for instance [2, 3, 24, 27, 29, 30].

Many models of opinion dynamics are based on the assumption that we are influenced more easily by people whom we almost agree with to begin with than by those whose views starkly differ from ours. A similar but more general phenomenon is known as biased assimilation among psychologists — our tendency to filter and interpret information in such a way that it supports our preconceived notions [21]. Models of opinion dynamics based on this assumption are known as bounded confidence models [3, 10, 25]. A popular example is due to Hegselmann and Krause [17, 18], building on earlier work by Krause [19, 20]. It has been studied extensively in the literature (see for instance [7, 22, 23]), and will be our starting point here. The Weissbuch-Deffuant model [33] is very close to that of Hegselmann and Krause; while Hegselmann and Krause assume that each opinion holder responds to all nearby opinions simultaneously, Weissbuch and Deffuant assume random encounters between pairs of opinion holders with similar views. For other bounded confidence models, see [4, 14, 31].
The original Hegselmann-Krause model is discrete in both time and opinion space. Similar models that are continuous in time [28], opinion space [32], or opinion space and time [15] have been proposed as well. We are particularly interested in fully continuous models, since we plan, in future work, to explore the response of candidates to a dynamic electorate. In a previous paper, we have already discussed the response of candidates to a static electorate [8]. We want to describe candidate dynamics in opinion space by ordinary differential equations, and find that easiest to do in clean and natural ways if the opinion dynamics of the electorate are described fully continuously.

We note that “continuous” does not mean the same to all authors in this field. For instance, in Lorenz’s earlier papers [22,23], the dynamics are discrete in both opinion space and time. The word “continuous” appears in the titles of both papers, but it indicates merely that the opinions can take arbitrary real values. To us, by contrast, a “fully continuous” model is one in which a continuum of agents changes opinions in continuous time. We note that fully continuous models in our sense were studied by Lorenz in [24].

In this paper, we derive a fully continuous version of the Hegselmann-Krause model. We start with a time-continuous, space-discrete model. In contrast with many of the existing time-continuous models [28], we don’t interpret particles as agents, but as agent clusters of different sizes. Our time-continuous model has a natural space-time-continuous analogue, a differential-integral equation. The time-continuous model that we start out with can then be interpreted as a numerical method for the differential-integral equation, a particle method. We numerically test the speed of convergence of this method. We note that particle methods are a natural choice for the numerical simulation of bounded confidence models because biased assimilation tends to result in the formation of clusters of like-minded individuals — groups of friends confirming and equalizing each others’ opinions on Facebook or over dinner, for instance — causing accuracy issues for numerical methods based on fixed grids.

We also observe that the differential-integral equation can be translated into a system of partial differential equations without any integrals, somewhat reminiscent of the Poisson-Nernst-Planck model of electro-diffusion: The density (of individuals in opinion space, or of charged particles in physical space) moves in a velocity field that itself is determined by the density via Poisson-like differential equations. In our model, we show that an integration-by-parts argument that would lead to an energy dissipation inequality in physical systems leads to a mass concentration inequality here.

2 A time-continuous model.

2.1 Opinion space and opinion holder distributions.

We assume that any individual’s opinions can be characterized by a single real number $x$. In politics, one could think of this as the “left-right axis”, with values of $x$ on the left side of the axis corresponding to “left” views, and values on the right side to “right” ones. This is a gross simplification that captures some aspect of the truth, since political views on different issues are correlated: If you tell us your thoughts about immigration policy and about allowing organized prayer in schools, we cannot be sure how you feel about a single-payer healthcare system, but we do have a guess.
Tony Blair has suggested a re-interpretation of the one-dimensional axis as “open” (in favor of immigration, multi-culturalism, globalism) and “closed” (in favor of restricting immigration, culturally conservative, primarily focused on one’s own country) [13]. The interpretation of the one-dimensional axis does not, of course, affect our abstract modeling.

We refer to the $x$-axis as “the opinion axis” or “opinion space”. A space-continuous model typically uses a time-dependent density of opinion holders,

$$f(x,t), \quad x \in \mathbb{R}, \quad t \geq 0.$$ 

The time $t$ could still tick discretely in such a model, but we are primarily interested in space-time-continuous models in which $t$ flows continuously. We always assume $f(x,t) \geq 0$ and normalize so that

$$\int_{-\infty}^{\infty} f(x,t) \, dx = 1 \quad \text{for all } t.$$ 

More generally and abstractly, the opinion holder distribution $f$ could be a time-dependent Borel probability measure on $\mathbb{R}$; however, the only measures without densities that we’ll talk about in this paper are weighted sums of Dirac measures.

### 2.2 Particle representation of opinion distributions.

Let $X_1, X_2, \ldots, X_n \in \mathbb{R}$, and assume for now that the $X_i$ are the only opinions represented in the electorate. If $w_i$ is the fraction of individuals who hold opinion $X_i$, then the “density” of opinions altogether is the distribution

$$\sum_{i=1}^{n} w_i \delta(x - X_i),$$

where $\delta$ denotes the Dirac delta distribution. The condition that this be a probability measure becomes $\sum_{i=1}^{n} w_i = 1$.

Any Borel probability measure $\mu$ on the real line can be approximated arbitrarily well, in the distributional sense, by a weighted sum of delta functions in the form (1). In fact, let $m \geq 1$ be an integer, $\Delta x > 0$ a real number, and define, for all integers $i$ with $-m+1 \leq i \leq m-1$,

$$w_i = \frac{\mu\left(\left(i - \frac{1}{2}\right)\Delta x, \left(i + \frac{1}{2}\right)\Delta x\right)}{\sum_{k=-m+1}^{m-1} \mu\left(\left(k - \frac{1}{2}\right)\Delta x, \left(k + \frac{1}{2}\right)\Delta x\right)}.$$

Then

$$\sum_{i=-m+1}^{m-1} w_i \delta(x - i\Delta x),$$

converges weakly to $\mu$ if $m \to \infty$ and $\Delta x \to 0$ in such a way that $m\Delta x \to \infty$.

Any weighted sum of delta functions in the form (1) can in turn be approximated arbitrarily well by a smooth density. For instance, the smooth probability density

$$\sum_{i=1}^{n} \frac{w_i}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x - X_i)^2}{2\sigma^2}}$$

converges to (1), in the distributional sense, as $\sigma \to 0$. 

3
2.3 The dynamics of conformist opinion holders.

We assume that the opinion holders in the \( i \)-th cluster, that is, opinion holders with opinion \( X_i \), consider a weighted average of the opinions of others, in the form

\[
\frac{\sum_{j=1}^{n} \eta(|X_i - X_j|)w_j X_j}{\sum_{j=1}^{n} \eta(|X_i - X_j|)w_j}
\]

where

\[
\eta: [0, \infty) \rightarrow [0, 1]
\]

is a decreasing function with \( \lim_{z \to \infty} \eta(z) = 0 \), called the interaction function. The further \( X_i \) is removed from \( X_j \), the less will the \( j \)-th cluster affect the opinion of the \( i \)-th cluster. We then assume that \( X_i \) drifts towards a weighted average of opinions (including their own), where nearby opinions are weighed more strongly than ones far from \( X_i \):

\[
\frac{dX_i}{dt} = \alpha \left( \frac{\sum_{j=1}^{n} \eta(|X_i - X_j|)w_j X_j}{\sum_{j=1}^{n} \eta(|X_i - X_j|)w_j} - X_i \right).
\]

(5)

Here \( \alpha > 0 \) is a parameter determining how eager the opinion holders are to fall in line with those who already hold opinions similar to theirs. We will take \( \alpha = 1 \). This is just a matter of choosing time units. Using this, and simplifying a bit, (5) becomes

\[
\frac{dX_i}{dt} = \sum_{j=1}^{n} \frac{\eta(|X_i - X_j|)w_j}{\sum_{\ell=1}^{n} \eta(|X_i - X_{\ell}|)w_{\ell}} (X_j - X_i).
\]

(6)

The model is most closely analogous to that of Hegselmann and Krause if the interaction function \( \eta \) is taken to be the indicator function of an interval \([0, \varepsilon]\) with \( \varepsilon > 0 \). However, we use

\[
\eta(z) = e^{-z/\nu}
\]

where \( \nu > 0 \) is a parameter determining how broad-minded the opinion holders are. Larger \( \nu \) means greater broad-mindedness.

Setting \( w_j = 1 \) in (6), our equation simplifies to [3, eq. (6)]. A time-discrete version of the model of [3] also appears in [10, eqs. (3) and (4)]. A time-discrete model including weights can be found for instance in [7, eq. (2)]. If \( \eta \) is taken to be an indicator function, our model becomes a time-continuous version of that in [7].

2.4 Examples.

We first assume that the initial opinion distribution has the density

\[
f_0(x) = \frac{1}{2} \left( \frac{e^{-2(x+1)^2}}{\sqrt{\pi/2}} + \frac{e^{-2(x-1)^2}}{\sqrt{\pi/2}} \right).
\]

The graph of this function is shown in Fig. 1.
Figure 1: An opinion distribution with two distinct “camps”, a “left” one and a “right” one.

We approximate this distribution by a weighted sum of 399 Dirac delta functions, as described by equations (2) and (3) with \( m = 200 \) and \( \Delta x = 3/m \), approximating

\[
\mu\left([\left(k - \frac{1}{2}\right)\Delta x, \left(k + \frac{1}{2}\right)\Delta x]\right) = \int_{(k-1/2)\Delta x}^{(k+1/2)\Delta x} f_0(x) dx
\]

by

\[
f_0(k\Delta x) \Delta x.
\]

We compute the time evolution as described by eq. (6), using \( \eta(z) = e^{-2z} \), and using the midpoint method with \( \Delta t = 0.04 \). At each time \( t > 0 \), this results in a weighted sum of Dirac delta functions approximating the opinion distribution. We approximate this sum by a smooth probability density as defined in (4) with \( \sigma = 0.1 \). The upper panel of Fig. 2 shows the resulting densities at times \( t = 0 \) (blue), 5 (black), and 10 (red). The lower panel shows the same time evolution as a surface plot. The two initial clusters tighten, but they also move towards each other, and eventually they merge into one cluster at the center.

For the initial opinion distribution

\[
f_0(x) = \frac{1}{3} \left( \frac{e^{-5(x+1)^2}}{\sqrt{\pi/5}} + \frac{e^{-5x^2}}{\sqrt{\pi/5}} + \frac{e^{-5(x-1)^2}}{\sqrt{\pi/5}} \right)
\]

we obtain the time evolution shown in Fig. 3. A feature of some interest is that the three clusters start out with equal amplitude, but by time 10, the middle cluster has a lower amplitude than the outlying ones. A closer inspection of the computed density shows that this effect is mostly attributable to less tightening in the central cluster, not to migration of individuals out of the central cluster: The percentage of individuals between \( x = -0.5 \) and \( x = 0.5 \) is nearly exactly the same at time 10 as at time 0. At approximately time 30, the three clusters merge into one. This calculation was carried out with a bit less resolution: \( m = 100 \) (so 199 Dirac delta functions), again \( \Delta x = 3/m \), and \( \Delta t = 0.1 \), and \( \sigma = 0.1 \).

### 2.5 Concentration of the opinion holder density.

We saw in our numerical results that multiple clusters in our model always appear to be transient, eventually collapsing into a single cluster. We will give the easy proof that this must always happen.
Figure 2: If the initial opinion distribution is that shown in Fig. 1, the initial clusters tighten at first, but also move towards one another, and eventually merge. The upper panel shows $f$ at times 0 (blue), 5 (black), and 10 (red). The lower panel shows the time evolution as a 3D surface plot.

Figure 3: Tightening and eventual collapse of three initial clusters.

**Proposition 1.** Assume that the $X_i$, $1 \leq i \leq n$, obey eq. (6). Then $\min_{1 \leq i \leq n} X_i$ is increasing, $\max_{1 \leq i \leq n} X_i$ is decreasing, and

$$\lim_{t \to \infty} \left( \max_{1 \leq i \leq n} \, X_i - \min_{1 \leq i \leq n} \, X_i \right) = 0.$$ 

**Proof.** Without loss of generality, assume $X_1 = \min_{1 \leq i \leq n} X_i$ and $X_n = \max_{1 \leq i \leq n} X_i$. (This is
merely a matter of notation.) We have

\[
\frac{dX_1}{dt} = \sum_{j=1}^{n} \frac{\eta(|X_j - X_1|)w_j}{\sum_{\ell=1}^{n} \eta(|X_\ell - X_1|)w_\ell} (X_j - X_1)
\]

\[
\geq \frac{\eta(|X_n - X_1|)w_n}{\sum_{\ell=1}^{n} \eta(|X_\ell - X_1|)w_\ell} (X_n - X_1)
\]

\[
\geq \frac{\eta(|X_n - X_1|)w_n}{n \sum_{\ell=1}^{n} \eta(|X_\ell - X_1|)w_\ell} (X_n - X_1)
\]

\[
= \frac{w_n(X_n - X_1)}{n}.
\]  

(7)

Similarly,

\[
\frac{dX_n}{dt} = \sum_{j=1}^{n} \frac{\eta(|X_j - X_n|)w_j}{\sum_{\ell=1}^{n} \eta(|X_\ell - X_n|)w_\ell} (X_j - X_n)
\]

\[
\leq \frac{\eta(|X_1 - X_n|)w_1}{\sum_{\ell=1}^{n} \eta(|X_\ell - X_n|)w_\ell} (X_1 - X_n)
\]

\[
\leq \frac{\eta(|X_1 - X_n|)w_1}{n \sum_{\ell=1}^{n} \eta(|X_\ell - X_n|)w_\ell} (X_1 - X_n)
\]

\[
= -\frac{w_1(X_1 - X_n)}{n}.
\]  

(8)

Subtracting (7) from (8), we find

\[
\frac{d}{dt} (X_n - X_1) \leq -(w_1 + w_n)(X_n - X_1).
\]

Since all the weights \(w_j\) are positive, this implies the assertion.

\[
\square
\]

For much less straightforward results about convergence to consensus in higher-dimensional spaces, see [26, Section 2].

3 A space-time-continuous model

3.1 Differential-integral formulation.

Let now \(f = f(x,t)\) be a continuous opinion holder density. The analogues of

\[
\sum_{j=1}^{n} \eta(|X_j - X_j|)w_jX_j \quad \text{and} \quad \sum_{\ell=1}^{n} \eta(|X_\ell - X_\ell|)w_\ell
\]

are

\[
\int_{-\infty}^{\infty} |z| f(x-z,t)(x-z) \, dz \quad \text{and} \quad \int_{-\infty}^{\infty} \eta(|z|) f(x-z,t) \, dz.
\]

We will derive a continuous evolution equation using arguments similar to those often used to derive conservation equations such as convection or diffusion equations. Consider an interval \([a,b]\). At \(a\), opinion holders are moving right with velocity

\[
\frac{\int_{-\infty}^{\infty} |z| f(a-z,t)(a-z) \, dz}{\int_{-\infty}^{\infty} \eta(|z|) f(a-z,t) \, dz} - a = -\frac{\int_{-\infty}^{\infty} |z| f(a-z,t)z \, dz}{\int_{-\infty}^{\infty} \eta(|z|) f(a-z,t) \, dz}.
\]
At \( b \), they are similarly moving right with velocity

\[
\frac{\int_{-\infty}^{\infty} \eta(|z|) f(b-z,t)\,dz}{\int_{-\infty}^{\infty} \eta(|z|) f(b-z,t)\,dz}.
\]

Now think about a short time interval of duration \( \Delta t \). The fraction of opinion holders entering \([a, b]\) through \( a \) in the time interval \([t, t+\Delta t]\) is about

\[
-f(a,t) \frac{\int_{-\infty}^{\infty} \eta(|z|) f(a-z,t)\,dz}{\int_{-\infty}^{\infty} \eta(|z|) f(a-z,t)\,dz} \Delta t.
\]

The fraction exiting through \( b \) is similarly

\[
-f(b,t) \frac{\int_{-\infty}^{\infty} \eta(|z|) f(b-z,t)\,dz}{\int_{-\infty}^{\infty} \eta(|z|) f(b-z,t)\,dz} \Delta t.
\]

It follows that

\[
\frac{d}{dt} \int_{a}^{b} f(x,t)\,dx = \int_{a}^{b} f_i(x,t)\,dx = f(b,t) \frac{\int_{-\infty}^{\infty} \eta(|z|) f(b-z,t)\,dz}{\int_{-\infty}^{\infty} \eta(|z|) f(b-z,t)\,dz} - f(a,t) \frac{\int_{-\infty}^{\infty} \eta(|z|) f(a-z,t)\,dz}{\int_{-\infty}^{\infty} \eta(|z|) f(a-z,t)\,dz}.
\]

Since this holds for any choice of \([a, b]\), we conclude:

\[
f_i(x,t) = \left( \frac{\int_{-\infty}^{\infty} \eta(|z|) f(x-z,t)\,dz}{\int_{-\infty}^{\infty} \eta(|z|) f(x-z,t)\,dz} \right) f(x,t).
\]

(9)

The particle model presented in Section 2 can be viewed as a numerical method for solving eq. (9), with initial condition \( f(x,0) = f_0(x) \) and zero boundary conditions at \( \pm \infty \).

Equation (9) is closely related to others that have appeared in the literature, for instance [10, eqs. (7) and (8)]. We take the velocity at opinion space location \( x \) at time \( t \) to be

\[
\frac{\int_{-\infty}^{\infty} \eta(|z|) f(x-z,t)\,dz}{\int_{-\infty}^{\infty} \eta(|z|) f(x-z,t)\,dz} \frac{\int_{-\infty}^{\infty} \eta(|y-x|) f(y,t)(y-x)\,dy}{\int_{-\infty}^{\infty} \eta(|y-x|) f(y,t)\,dy}
\]

(10)

while the velocity in [10, eq. (8)], in the same notation, is

\[
\int_{-\infty}^{\infty} \eta(|y-x|) f(y,t)(y-x)\,dy.
\]

(11)

In (10), a weighted average of the differences \( y-x \) is taken, while (11) will be larger if \( x \) is surrounded by many nearby agents, smaller if it isn’t. Which is more accurate depends on how opinion dynamics work — we assume that all agents are equally eager to conform, even those surrounded by only few other agents, whereas [10, eq. (8)] implicitly assumes that those surrounded by many agents are more eager to conform than those surrounded by few agents.
3.2 Differential formulation.

In our numerical experiments, we always use \( \eta(z) = e^{-z/\nu} \) with \( \nu > 0 \). This assumption was not crucial until now, but will be here; we could not do the following computation with a general \( \eta \). There are two integrals in eq. (9),

\[
g(x,t) = \int_{-\infty}^{\infty} e^{-|z|/\nu} f(x-z,t) \, dz \quad \text{and} \quad h(x,t) = \int_{-\infty}^{\infty} z e^{-|z|/\nu} f(x-z,t) \, dz.
\]

The strategy is to write eq. (9) as

\[
f_t(x,t) = \left( \frac{h(x,t)}{g(x,t)} f(x,t) \right)_x
\]

and then add supplementary differential equations for \( g \) and \( h \). We note for later reference that

\[
\int_{-\infty}^{\infty} g(x,t) \, dx = 2\nu \quad \text{for all } t.
\]

In fact, \( g(\cdot,t) = \eta \ast f(\cdot,t) \) and \( \frac{h}{2\nu} \) is a probability density. We refer to \( g \) as the \textit{locally averaged opinion holder density}. On the other hand,

\[
\int_{-\infty}^{\infty} h(x,t) \, dx = 0 \quad \text{for all } t.
\]

The ratio \( h(x,t)/g(x,t) \) is a weighted average of positions, with a weight that is larger for views that are more commonly held and for views that are closer to \( x \).

3.2.1 Supplementary differential equation for \( g \).

We differentiate \( g \) with respect to \( x \), assuming sufficient smoothness of \( f \), and use

\[
\frac{\partial}{\partial x} f(x-z,t) = -\frac{\partial}{\partial z} f(x-z,t).
\]

We obtain:

\[
g_x(x,t) = \int_{-\infty}^{\infty} e^{-|z|/\nu} \frac{\partial}{\partial x} f(x-z,t) \, dz = -\int_{-\infty}^{\infty} e^{-|z|/\nu} \frac{\partial}{\partial z} f(x-z,t) \, dz.
\]

We integrate by parts to move the \( z \)-derivative to the exponential term, using the following formula, which will be used several times in this section:

\[
\frac{d}{dz} \left( e^{-|z|/\nu} \right) = -\frac{1}{\nu} \text{sign}(z) e^{-|z|/\nu}.
\]

We find

\[
g_x(x,t) = -\frac{1}{\nu} \int_{-\infty}^{\infty} \text{sign}(z) e^{-|z|/\nu} f(x-z,t) \, dz.
\]
We differentiate with respect to $x$ again, using this formula:

$$\frac{d}{dz} \left( \text{sign}(z) e^{-|z|/\nu} \right) = 2\delta(z) - \frac{e^{-|z|/\nu}}{\nu}$$

(18)

where $\delta$ is the Dirac delta function. This follows from the product rule, which is in fact rigorously applicable here, and from the fact that $\text{sign}(z) \cdot \text{sign}(z) = 1$ for all $z \neq 0$. We find

$$g_{xx}(x,t) = -\frac{2}{\nu} f(x,t) + \frac{1}{\nu^2} g(x,t).$$

We re-arrange a bit like this:

$$-g_{xx} + \frac{g}{\nu^2} = \frac{2}{\nu} f.$$

This is the supplementary differential equation for $g$.

### 3.2.2 Supplementary differential equation for $h$.

We will use similar reasoning for $h$, and for this purpose we first record that

$$\frac{d}{dz} \left( z e^{-|z|/\nu} \right) = e^{-|z|/\nu} - \frac{1}{\nu} |z| e^{-|z|/\nu}.$$  

(19)

This follows from the product rule, using also that $z \text{sign}(z) = |z|$. Differentiation under the integral sign, using (15), then integration by parts, now yields

$$h_x(x,t) = \int_{-\infty}^{\infty} \left( e^{-|z|/\nu} - \frac{1}{\nu} |z| e^{-|z|/\nu} \right) f(x - z, t) \, dz =$$

$$g(x,t) - \frac{1}{\nu} \int_{-\infty}^{\infty} |z| e^{-|z|/\nu} f(x - z, t) \, dz.$$

We differentiate again with respect to $x$, using the following twin of formula (19):

$$\frac{d}{dz} \left( |z| e^{-|z|/\nu} \right) = \text{sign}(z) e^{-|z|/\nu} - \frac{1}{\nu} z e^{-|z|/\nu}.$$  

(20)

We obtain, also using eq. (17):

$$h_{xx}(x,t) = g_x(x,t) - \frac{1}{\nu} \int_{-\infty}^{\infty} \text{sign}(z) e^{-|z|/\nu} f(x - z) \, dz + \frac{1}{\nu^2} h(x,t) = 2g_x(x,t) + \frac{1}{\nu^2} h(x,t).$$

We re-arrange a bit to obtain

$$-h_{xx} + \frac{1}{\nu^2} h = -2g_x.$$

This is the supplementary differential equation for $h$. 

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3.2.3 Summary of the differential formulation.

We have re-written eq. (9) as follows:

\[
f_t = \left( \frac{h}{g} \right)_x, \tag{21}
\]

\[-g_{xx} + \frac{g}{\nu^2} = \frac{2}{\nu}f, \tag{22}\]

\[-h_{xx} + \frac{h}{\nu^2} = -2g_x. \tag{23}\]

3.2.4 Concentration of the locally averaged opinion holder density.

We define

\[H(x,t) = \int_{-\infty}^x h(s,t) \, ds.\]

Note \(H_x = h\) and \(H(-\infty, t) = 0\). Integrating eq. (23), we obtain

\[-H_{xx} + \frac{H}{\nu^2} = -2g + C\]

with \(C\) independent of \(x\). Assuming that \(H_{xx} = h_x\) and \(g\) vanish at \(x = -\infty\), we conclude \(C = 0\), so

\[-H_{xx} + \frac{H}{\nu^2} = -2g. \tag{24}\]

Multiply both sides of (21) by \(H\), integrate with respect to \(x\), and then integrate by parts on the right-hand side:

\[
\int_{-\infty}^\infty H(x,t)f_t(x,t) \, dx = -\int_{-\infty}^\infty \frac{h^2(x,t)}{g(x,t)}f(x,t) \, dx. \tag{25}\]

We’ll re-write the left-hand side of eq. (25) now. First, use eq. (22):

\[
\int_{-\infty}^\infty H(x,t)f_t(x,t) \, dx = \int_{-\infty}^\infty H(x,t) \left( -\frac{\nu}{2} g_{xx} + \frac{g}{2\nu} \right) \, dx.
\]

Integrating by parts twice, we obtain

\[
\int_{-\infty}^\infty \left( -\frac{\nu}{2} H_{xx} + \frac{1}{2\nu} H \right) g_t \, dx,
\]

and using (24), this is

\[
\int_{-\infty}^\infty -\nu g(x,t)g_t(x,t) \, dx = -\nu \frac{d}{dt} \int_{-\infty}^\infty g(x,t)^2 \, dx.
\]

In summary, canceling minus signs, (25) becomes

\[
\frac{d}{dt} \|g\|_2^2 = \frac{2}{\nu} \int_{-\infty}^\infty \frac{h^2(x,t)}{g(x,t)}f(x,t) \, dx. \tag{26}\]
Positivity of \( f \) implies positivity of \( g \), and therefore
\[
\frac{d}{dt} \|g\|_{L^2}^2 > 0. \tag{27}
\]
Recall from eq. (13) that the \( L^1 \)-norm of \( g \) is equal to \( 2\nu \) for all time. The square of the \( L^2 \)-norm is a measure of concentration of \( g \). This is reflected by the fact that if \( X \) is a random number with probability density \( \frac{g}{2\nu} \), then
\[
\|g\|_{L^2}^2 = 2\nu E(g(X)),
\]
and therefore \( E(g(X)) \) rises as \( t \) increases. When \( X \) is drawn with density \( g \), the expected value of \( g(X) \) gets larger with time. This means that \( g \) becomes increasingly concentrated.

4 Numerical convergence tests.

As an example, we test convergence for the initial condition
\[
f_0(x) = \frac{1}{3} \left( \frac{e^{-5(x+1)^2}}{\sqrt{\pi/5}} + \frac{e^{-5x^2}}{\sqrt{\pi/5}} + \frac{e^{-5(x-1)^2}}{\sqrt{\pi/5}} \right).
\]
We track the \( X_j \) up to time \( t = 1 \). We use a time step \( \Delta t \) and assume that \( 1/\Delta t \) is an integer. We initialize the \( X_j \) at
\[
X_j(0) = -3 + j\Delta x, \quad j = 1, 2, \ldots, \frac{6}{\Delta x} - 1,
\]
assuming that \( 6/\Delta x \) is an integer.

4.1 Weak convergence of the computed opinion holder density.

We compute approximations for \( f(x, t) \), \( x = j\Delta x \), \( j \) integer, \(-3 < x < 3\), using eq. (4), where \( \sigma = 0.1 \). We denote these approximations by \( f_{\Delta x, \Delta t}(x, t) \), and will test whether they converge to some limit as \( \Delta x \) and \( \Delta t \) are simultaneously reduced. We cannot test convergence to an exact solution, since we have no analytic expression for an exact solution.

Fixing \( \sigma \) independently of \( \Delta x \) amounts to testing for a form of weak convergence. The computed distribution is a sum of \( \delta \)-functions, but we test for convergence of the convolution with a Gaussian.

We define
\[
E_{\Delta x, \Delta t} = \max \left\{ \left| f_{\Delta x, \Delta t}(x, 1) - f_{\Delta x, \Delta t}(x, 1) \right| : x = j\Delta x, \quad j \text{ integer}, \quad -3 < x < 3 \right\}.
\]
If there is second-order convergence as \( \Delta x \) and \( \Delta t \) simultaneously tend to zero, one should expect
\[
\frac{E_{\Delta x, \Delta t}}{E_{\Delta x/2, \Delta t/2}} \approx 4
\]
for small \( \Delta x \) and \( \Delta t \). Table 1 confirms that this is indeed the case.
Table 1: Numerical test confirming second order convergence of the approximation obtained by convolving the computed sum of delta functions with a Gaussian, as both $\Delta x$ and $\Delta t$ are refined; see text for details.

| $\Delta x$ | 0.06 | 0.03 | 0.015 |
|------------|------|------|-------|
| $\Delta t$ | 0.1  | 0.05 | 0.025 |
| $E_{\Delta x, \Delta t}/E_{\Delta x/2, \Delta t/2}$ | 4.01 | 3.98 | 4.00 |

4.2 Dependence of the error on the time step.

The most accurate calculation underlying Table 1 uses

$$\Delta x = \frac{0.06}{2^4} = 0.00375, \quad \Delta t = \frac{0.1}{2^4} = 0.00625.$$ 

To test the importance of $\Delta t$ for the overall accuracy, we compare the results of this computation with results obtained using the same $\Delta x$, but coarser $\Delta t$. We define

$$F_{\Delta t} = \max \left\{ |f_{\Delta x=0.00375, \Delta t=0.00625}(x, 1) - f_{\Delta x=0.00375, \Delta t}(x, 1)| : x = 0.00375j, \right.\]

$$\left. j \text{ integer, } -3 < x < 3 \right\}.$$ 

Table 2 shows the dependence of $F_{\Delta t}$ on $\Delta t$, confirming second order convergence.

| $\Delta t$ | 0.1  | 0.05 | 0.025 | 0.0125 |
|------------|------|------|-------|--------|
| $F_{\Delta t}$ | $2.04 \times 10^{-5}$ | $5.12 \times 10^{-6}$ | $1.23 \times 10^{-6}$ | $2.47 \times 10^{-7}$ |
| $F_{\Delta t}/F_{\Delta t/2}$ | 3.98 | 4.17 | 4.98 |

4.3 Dependence of the error on the spatial mesh size.

To test the importance of $\Delta x$ for the overall accuracy, we compare the results of the fine computation using $\Delta x = 0.00375$ and $\Delta t = 0.00625$ with results obtained using the same $\Delta t$, but coarser $\Delta x$. We define

$$G_{\Delta x} = \max \left\{ |f_{\Delta x=0.00375, \Delta t=0.00625}(x, 1) - f_{\Delta x, \Delta t}(x, 1)| : x = j\Delta x, \right.\]

$$\left. j \text{ integer, } -3 < x < 3 \right\}.$$ 

Table 2 shows the dependence of $G_{\Delta x}$ on $\Delta x$, again indicating second order convergence.

5 Summary and discussion.

We began with a time-continuous version of Hegselmann-Krause dynamics, similar to equations that have been proposed in the literature previously, but with weighted particles, which we think of as representing clusters of agents, not individuals. The weights
Table 3: Error as a function of $\Delta x$; see text for details.

| $\Delta x$ | 0.06 | 0.03 | 0.015 | 0.0075 |
|------------|------|------|-------|--------|
| $G_{\Delta x}$ | $9.58 \times 10^{-4}$ | $2.36 \times 10^{-4}$ | $5.65 \times 10^{-5}$ | $1.13 \times 10^{-5}$ |
| $G_{\Delta x}/G_{\Delta x/2}$ | 4.06 | 4.18 | 5.00 |

have a numerical advantage — instead of needing many agents in a part of opinion space populated by many opinion holders, we can use fewer but heavier particles.

The time-continuous model suggests a fully continuous macroscopic model, which we formulated first as a single integral-differential equation, then — for the special case of an exponential interaction function — as a system of differential equations.

The time-continuous model (discretized using the midpoint method) can be viewed as a particle method for the fully continuous model. We demonstrated, numerically, the second-order convergence of this method in a weak sense, meaning that the convolution of the solution with a mollifier is computed with second-order accuracy.

In our numerical computations, all opinion holders eventually arrive at consensus. This could be counter-acted by adding diffusion (spontaneous random small changes in opinions) in the model, as some authors have proposed (see for instance [4, 15]). We have refrained from doing that here because it would raise, in our context, the question how to incorporate diffusion in the particle method. One possibility would be a method similar to Chorin’s random walk method for viscous fluid dynamics [12, 16].

Our model starts with the original Hegselmann-Krause model [17]. In a later paper [18], Hegselmann and Krause suggested that individuals might respond not to the arithmetic average (or, in our modification of the model, weighted arithmetic average) of opinions in their vicinity, but to a different kind of average — geometric averages for instance. We have not yet thought about what would happen if we followed this interesting suggestion in our model.

In future work, we plan to use the method presented in this paper to explore the interaction of candidate dynamics with voter opinion dynamics. We have taken a first step in that direction in [8].

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