1. Introduction. Imposing the uncertainty upon the optimization problems is an interesting research topic. The uncertainty may be interpreted as randomness or fuzziness. The randomness occurring in the optimization problems is categorized as the stochastic optimization problems. The books written by Birge and Louveaux [2], Kall [11], Prékopa [13], Stancu-Minasian [16] and Vajda [17] provide many interesting ideas and useful techniques for tackling the stochastic optimization problems. On the other hand, the fuzziness occurring in the optimization problems is categorized as the fuzzy optimization problems. The collection of papers on fuzzy optimization edited by Słowiński [14] and Delgado et al. [8] gives many interesting topics. The fusion of randomness and fuzziness occurring in the optimization problems is even a challenge research topic. The book edited by Słowiński and Teghem [15] gives the comparisons between fuzzy optimization and stochastic optimization for the multiobjective programming problems. Inuiguchi and Ramík [9] also gives a brief review of fuzzy optimization and a comparison with stochastic optimization in portfolio selection problem.

In the stochastic optimization problems, the coefficients of the problem are assumed as random variables with known probability distributions. On the other hand, in the fuzzy optimization problems, the coefficients of the problem are assumed as fuzzy numbers with known membership functions. However, the specifications of probability distributions and membership functions in the stochastic optimization problems and fuzzy optimization problems, respectively, are very
subjective. For example, many researchers adopt the Gaussian (normal) distributions with different parameters in the stochastic optimization problems, and the bell-shaped or S-shaped membership functions in the fuzzy optimization problems. These specifications may make the optimization problems more complicated to be solved. Therefore interval-valued optimization problems can provide an alternative choice for considering the uncertainty into the optimization problems. That is to say, the coefficients in the interval-valued optimization problems are assumed as bounded closed intervals. Although the specifications of bounded closed intervals may still be judged as subjective viewpoint, we may argue that the bounds of uncertain data (i.e., determining the bounded closed intervals to bound the possible observed data) are easier to be handled than specifying the probability distributions and membership functions in stochastic optimization and fuzzy optimization problems, respectively.

The duality theorems and optimality conditions for interval-valued optimization problems have been studied in Wu [18, 19, 20, 21] using the so-called Hukuhara derivative. Recently, Chalco-Cano et al. [4] and Osuna-Gomez et al. [12] extend to study the optimality conditions by using the so-called generalized Hukuhara derivative. Also, Jayswal et al. [10] studied the duality theorems and optimality conditions by using the concept of generalized convexity. Bhurjee and Panda [1] study the efficient solution of interval optimization problem by using the parametric form of interval-valued functions. Many other interesting articles regarding the interval-valued optimization problems can also be consulted from the references therein.

In this paper, we are going to solve the interval-valued optimization problems based on the concept of null set. Let \( I \) denote the space of all bounded closed intervals. We also assume that \( I \) is endowed with the interval addition and scalar multiplication as follows. Given \( A = [a_L, a_U] \) and \( B = [b_L, b_U] \) in \( I \), the interval addition is defined to be the set addition
\[
A \oplus B = \{ a + b : a \in A \text{ and } b \in B \}.
\]
Then it is clear to see that
\[
A \oplus B = [a_L + b_L, a_U + b_U].
\]
For \( k \in \mathbb{R} \) and \( A \in I \), the scalar multiplication is defined by \( kA = \{ ka : a \in A \} \). Then it is clear to see that
\[
kA = \begin{cases} 
ka_L, ka_U & \text{if } k \geq 0, \\
ka_U, ka_L & \text{if } k < 0.
\end{cases}
\]
We also we have
\[
A \odot A = [a_L, a_U] \odot [a_L, a_U] = [a_L, a_U] \odot [-a_U, -a_L] = [a_L - a_U, a_U - a_L].
\]
This says that the difference between a bounded closed interval and itself cannot be the zero element. In other words, each element in \( I \) cannot have the additive inverse element. This also says that the set \( I \) cannot form a real vector space.

Although \( I \) cannot be a real vector space, the concept of interval addition and scalar multiplication that are described above can still be used to define the concept of convex cone given in Definition 2.4 below. Based on the convex cone, we can define two different types of partial orderings on \( I \) given in Definition 3.2 below. In order to define the partial orderings via the convex cone, two kinds of difference called substraction \( A \ominus B \) and Hukuhara difference \( A \ominus H B \) between any two closed bounded intervals \( A \) and \( B \) will be defined, which are given in Definition 2.1 below.
Since the difference \( A \ominus A \) cannot be the zero element for any \( A \in \mathcal{I} \) as described above, we define the so-called null set
\[
\Omega = \{ A \ominus A : A \in \mathcal{I} \}
\]
that can be regarded as a kind of “zero element” of \( \mathcal{I} \) for the purpose of defining two partial orderings \( A \preceq B \) based on the substraction \( A \ominus B \) and \( A \preceq_H B \) based on the Hukuhara difference \( A \ominus_H B \). Using these two partial orderings, the solution concepts of interval-valued optimization problems can be defined.

In order to solve the interval-valued optimization problem, we are going to transform it into a vector optimization problem. Given an interval-valued function \( f : U \to \mathcal{I} \) defined on a vector space \( U \), we are going to minimize \( f \) subject to some constraints. This is categorized as the interval-valued optimization problem. The solution concepts of interval-valued optimization problems can be defined as described above. Let \( \mathcal{L} : \mathcal{I} \to \mathcal{V} \) be a function from \( \mathcal{I} \) into another vector space \( \mathcal{V} \). Then we can consider the composition function \( \mathcal{L} \circ f : U \to \mathcal{V} \) of functions \( \mathcal{L} \) and \( f \). In this case, we are going to minimize the vector-valued function \( \mathcal{L} \circ f \) subject to the same constraints. This transformed problem is categorized as the vector optimization problem. Considering the vector optimization problem to solve the interval-valued optimization problem is a new attempt according to the limited knowledge of the author. By providing a suitable linear functional \( \phi : \mathcal{V} \to \mathbb{R} \), we are going to solve the scalar optimization problem \( \phi(\mathcal{L} \circ f) : U \to \mathbb{R} \) subject to the same constraints. The solution concepts of vector optimization problem and scalar optimization problem are defined in the conventional way. The key issue is to show that the optimal solution of the scalar optimization problem is also the optimal solution of the original interval-valued optimization problem. This will be presented in section 5.

In section 2, we present many properties in the space of all bounded closed intervals that will be used in the further study. In section 3, by introduce an vector-valued function defined on the space of all bounded closed intervals, we shall present the order-preserving properties that will be used to study the optimal solutions. In section 4, we formulate a vector optimization problem that can be regarded as an auxiliary problem for the purpose of solving the original interval-valued optimization problems. In section 5, we apply the scalarization technique to solve the vector optimization problem proposed in section 4. Finally, in section 6, we apply the proposed methodology to solve the interval-valued linear programming problems.

2. Interval spaces. Throughout the paper, let \( \mathcal{I} \) be the set of all bounded and closed intervals in \( \mathbb{R} \). Since any real number \( x \in \mathbb{R} \) can be regarded as a bounded closed interval \([x, x]\), it means that \( \mathbb{R} \) is contained in \( \mathcal{I} \). Given \( A = [a_L^L, a_L^U] \) and \( B = [b_L^L, b_L^U] \) in \( \mathcal{I} \), the interval addition is given by
\[
A \oplus B = [a_L^L + b_L^L, a_L^U + b_L^U]
\]
and the scalar multiplication in \( \mathcal{I} \) is given by
\[
kA = \begin{cases} 
[ka_L^L, ka_U^L] & \text{if } k \geq 0 \\
[ka_U^L, ka_L^L] & \text{if } k < 0.
\end{cases}
\]
By the above definition, we have
\[
-A = -[a_L^L, a_L^U] = [-a_U^L, -a_L^L].
\]
The concepts of substraction and difference between any two bounded closed intervals should be clarified below.

**Definition 2.1.** Let \( A \) and \( B \) be two bounded closed intervals.
- The *substraction* between \( A \) and \( B \) is denoted and defined by
  \[
  A \ominus B = A + (-B) = [a^L - b^U, a^U - b^L].
  \]
- If there exists a bounded closed interval \( C \) such that \( A = B \oplus C \). Then we say that \( C \) is the *Hukuhara difference* between \( A \) and \( B \). In this case, we write \( C = A \ominus_H B \).

It is clear that the substraction \( A \ominus B \) always exists. However, the Hukuhara difference \( A \ominus_H B \) does not necessarily exist in general.

Suppose that \( A = B \oplus C \). Then
\[
[a^L, a^U] = [b^L + c^L, b^U + c^U].
\]
Therefore we obtain \( c^L = a^L - b^L \) and \( c^U = a^U - b^U \), which says that if \( a^L - b^L \leq a^U - b^U \), i.e., \( a^L - a^U \leq b^L - b^U \), then the Hukuhara difference \( A \ominus_H B \) exists and
\[
A \ominus_H B = [a^L - b^L, a^U - b^U] \neq A \ominus B.
\]

Now we have
\[
A \ominus A = [a^L, a^U] \ominus [a^L, a^U] = [a^L, a^U] \oplus [-a^L, -a^U] = [a^L - a^U, a^U - a^L],
\]
which says that the additive inverse element in \( \mathcal{I} \) does not exist. One of the reasons is that the concept of “zero element” of \( \mathcal{I} \) is not defined. This also says that \( \mathcal{I} \) cannot form a vector space under the above interval addition and scalar multiplication.

The following set
\[
\Omega = \{ A \ominus A : A \in \mathcal{I} \}
\]
is called the *null set* of \( \mathcal{I} \), which can be regarded as a kind of “zero element” of \( \mathcal{I} \).

We also see that the true zero element of \( \mathcal{I} \) is \([0, 0] \), since it is clear that \( A \oplus [0, 0] = A \) for any \( A \in \mathcal{I} \). This also says that \( A \ominus_H A = [0, 0] \).

**Remark 1.** We have the following observations.
- It is clear that
  \[
  \Omega = \{ [-k, k] : k \geq 0 \} = \{ k[-1, 1] : k \geq 0 \},
  \]
since
  \[
  A \ominus A = [a^L, a^U] \ominus [a^L, a^U] = [a^L, a^U] \oplus [-a^U, -a^L] = [a^L - a^U, a^U - a^L],
  \]
  where \( a^U - a^L \geq 0 \).
- \( \omega \in \Omega \) implies \( -\omega = \omega \).
- Since \( [x, x] \in \mathcal{I} \), it follows that \( [x, x] \ominus [x, x] = [0, 0] \in \Omega \).
- \( \lambda \Omega = \Omega \) for \( \lambda \in \mathbb{R} \) with \( \lambda \neq 0 \).
- \( \Omega \) is closed under the interval addition and substraction; that is, \( \omega_1 \ominus \omega_2 \in \Omega \) and \( \omega_1 \ominus \omega_2 \in \Omega \) for any \( \omega_1, \omega_2 \in \Omega \).

Since the null set \( \Omega \) can be regarded as a kind of “zero element”, we can propose the concept of almost identical concept for elements in \( \mathcal{I} \).

**Definition 2.2.** Given any \( A, B \in \mathcal{I} \), we say that \( A \) and \( B \) are *almost identical* if and only if there exist \( \omega_1, \omega_2 \in \Omega \) such that \( A \ominus \omega_1 = B \ominus \omega_2 \). In this case, we write \( A \equiv B \).
If $A \otimes H B$ exists for $A, B \in I$, then, by definition, there exists $C \in I$ such that $A = B \oplus C$. However, for $A \oplus B = C$, which always exists, we cannot have $A = B \oplus C$. We can just have $A \gtrless B \oplus C$. Indeed, since $A \ominus B = C$, by adding $B$ on both sides, we obtain $A \oplus \omega = B \oplus C$, where $\omega = B \ominus B \in \Omega$. This says that $A \gtrless B \oplus C$.

**Proposition 1.** Given any $A, B \in I$, we have the following properties.

(i) $A \gtrless B$ if and only if there exists $\omega \in \Omega$ such that $A = B \oplus \omega$ or $B = A \oplus \omega$.

(ii) $A \preceq \omega$ for some $\omega \in \Omega$ if and only if $A \in \Omega$.

Proof. To prove part (i), since the zero interval $[0, 0]$ is in $\Omega$, it is clear to see that if $A = B \oplus \omega$ or $B = A \oplus \omega$ then $A \gtrless B$. On the other hand, for $\omega \in \Omega$, we have $\omega = [-k, k]$ for some $k \geq 0$. Suppose that $A \gtrless B$. Then there exist $\omega_1 = [-k_1, k_1]$ and $\omega_2 = [-k_2, k_2]$ such that $A \oplus \omega_1 = B \oplus \omega_2$, i.e., $a^L - k_1 = b^L - k_2$ and $a^U + k_1 = b^U + k_2$. We consider the following cases.

- If $k_1 \geq k_2$, then we can obtain $A \oplus \omega = B$, where $\omega$ is taken by $\omega = (-(k_1 - k_2), (k_1 - k_2))$.

- If $k_1 < k_2$, then we can obtain $A = B \oplus \omega$, where $\omega$ is taken by $\omega = (-(k_2 - k_1), (k_2 - k_1))$.

To prove part (ii), it is clear that $A \in \Omega$ implies $A \gtrless \omega$ for some $\omega \in \Omega$. Now we assume that $A \oplus \omega_1 = \omega \oplus \omega_2$ for some $\omega_1, \omega_2 \in \Omega$. Since $\omega \oplus \omega_2 \in \Omega$, we can say that $[a^L, a^U] \oplus [-k_1, k_1] = [-k_2, k_2]$ for some $k_1, k_2 \geq 0$, which implies $a^L = k_1 - k_2$ and $a^U = k_2 - k_1$. Since $a^L \leq a^U$, it follows that $k_1 \leq k_2$ and $A \in \Omega$. This completes the proof.

**Proposition 2.** The following statements hold true.

(i) If $A \oplus B \in \Omega$, then $A \gtrless B$.

(ii) If $A \gtrless B$, then there exists $\omega \in \Omega$ such that $A \oplus B \oplus \omega \in \Omega$.

Proof. To prove part (i), we have $A \oplus (-B) = \omega_1$ for some $\omega_1 \in \Omega$, which implies $A \oplus (-B) \oplus B = \omega_1 \oplus B$ by adding $B$ on both sides. This shows that $A \oplus \omega_2 = \omega_1 \oplus B$ for $\omega_1, \omega_2 \in \Omega$.

To prove part (ii), since $A \gtrless B$, we have $A \oplus \omega_2 = \omega_1 \oplus B$ for some $\omega_1, \omega_2 \in \Omega$. By adding $-B$ on both sides, we obtain $A \oplus B \oplus \omega_2 = \omega_1 \oplus \omega_3 \in \Omega$, where $\omega_3 = B \oplus B \in \Omega$. This completes the proof.

**Definition 2.3.** We consider the vector-valued function $\mathcal{L} : I \to V$ from $I$ into a vector space $V$.

- We say that $\mathcal{L}$ is additive if and only if $\mathcal{L}(A \oplus B) = \mathcal{L}(A) + \mathcal{L}(B)$.
- We say that $\mathcal{L}$ is homogeneous if and only if $\mathcal{L}(\lambda A) = \lambda \mathcal{L}(A)$ for $\lambda \in \mathbb{R}$.
- We say that $\mathcal{L}$ is positively homogeneous if and only if $\mathcal{L}(\lambda A) = \lambda \mathcal{L}(A)$ for $\lambda \geq 0$.
- We say that $\mathcal{L}$ is linear if and only if it is additive and homogeneous.
Example 1. We define the function $L : \mathcal{I} \to \mathbb{R}$ by $L([a_L, a_U]) = a_L + a_U$. Since 
\[ L([a_L, a_U] \oplus [b_L, b_U]) = L([a_L + b_L, a_U + b_U]) = a_L + a_U + b_L + b_U \]
and
\[ L(\lambda [a_L, a_U]) = \begin{cases} L(\left[ \lambda a_L, \lambda a_U \right]) & \text{if } \lambda \geq 0 \\ L(\left[ \lambda a_U, \lambda a_L \right]) & \text{if } \lambda < 0 \end{cases} \]
the function $L$ is linear. We can also define the function $L : \mathcal{I} \to \mathbb{R}^2$ by $L([a_L, a_U]) = (a_L, a_U)$. Then we can show that $L$ is additive and positively homogeneous.

Since $-\omega = \omega$ for any $\omega \in \Omega$, we can obtain the following interesting results that will be used for the further discussion.

Proposition 3. Consider the function $L : \mathcal{I} \to V$ from $\mathcal{I}$ into a vector space $V$.

(i) Suppose that $-L(\omega) = L(-\omega)$ for any $\omega \in \Omega$. Then $L(\omega) = \theta_V$ for any $\omega \in \Omega$, where $\theta_V$ is the zero element of vector space $V$.

(ii) Suppose that $L(\omega) = \theta_V$ for any $\omega \in \Omega$, and that $L$ is additive. Then 
\[ L(A \oplus B) = L(A) - L(B) \]
for any $A, B \in \mathcal{I}$.

(iii) Suppose that $L$ is additive and the Hukuhara difference $A \ominus_H B$ exists for $A, B \in \mathcal{I}$. Then 
\[ L(A \ominus_H B) = L(A) - L(B). \]

Although the interval space $\mathcal{I}$ is not a vector space, we can also consider the concept of convexity based on the interval addition and scalar multiplication.

Definition 2.4. Let $\mathcal{C}$ be a subset of $\mathcal{I}$. We say that $\mathcal{C}$ is convex if and only if $\lambda A \oplus (1 - \lambda)B \in \mathcal{C}$ for $A, B \in \mathcal{C}$ and $\lambda \in [0, 1]$. We say that $\mathcal{C}$ is a cone if and only if $\lambda A \in \mathcal{C}$ for $A \in \mathcal{C}$ and $\lambda > 0$. A cone $\mathcal{C}$ is said to be a convex cone if and only if it is also convex.

We remark that the cone $\mathcal{C}$ does not necessarily contains $\{0, 0\}$ since $\lambda \neq 0$. From Remark 1, it is easy to see that the null set $\Omega$ is a convex cone.

Example 2. We take 
\[ \mathcal{C} = \{ [c_L, c_U] \in \mathcal{I} : c_L + c_U \geq 0 \}. \]
For $A = [a_L, a_U], B = [b_L, b_U] \in \mathcal{C}$ and $0 < \lambda < 1$, we have 
\[ \lambda A = [\lambda a_L, \lambda a_U] \]
and 
\[ \lambda A \oplus (1 - \lambda)B = [\lambda a_L + (1 - \lambda)b_L, \lambda a_U + (1 - \lambda)b_U], \]
which say that $\mathcal{C}$ is a convex cone. We also see that $\Omega \subset \mathcal{C}$.

Proposition 4. Let $\mathcal{C}$ be a convex cone in $\mathcal{I}$. Then $A \oplus B \in \mathcal{C}$ for any $A, B \in \mathcal{C}$.
Proof. Since $\mathfrak{C}$ is a cone, we have $\frac{1}{2}A, \frac{1}{2}B \in \mathfrak{C}$. Also, since $\mathfrak{C}$ is convex, we have
\[
\frac{1}{2} (A \oplus B) = \frac{1}{2} A \oplus \frac{1}{2} B \in \mathfrak{C}.
\]
Then
\[
A \oplus B = 2 \cdot \frac{1}{2} (A \oplus B) \in \mathfrak{C},
\]
since $\mathfrak{C}$ is a cone. This completes the proof.

The following interesting and useful result is clear.

**Proposition 5.** We consider the function $\mathcal{L} : \mathcal{I} \to V$ from $\mathcal{I}$ into a vector space $V$. Let $\mathfrak{C}$ be a convex cone in $\mathcal{I}$. If $\mathcal{L}$ is additive and positively homogeneous, then the set $\mathcal{L}(\mathfrak{C}) = \{ \mathcal{L}(A) : A \in \mathfrak{C} \}$ is a convex cone in the vector space $V$.

3. **Order-preserving.** In this section, we shall consider many kinds of partial orderings, and present the order-preserving properties under a transformation.

**Definition 3.1.** Let $\mathcal{F}$ be a subset of $\mathcal{I}$. Given any $A \in \mathcal{I}$, we write $A \in \Omega \mathcal{F}$ if and only if $A \in \mathcal{F}$ for some $F \in \mathcal{F}$.

**Definition 3.2.** Let $\mathfrak{C}$ be a convex cone in $\mathcal{I}$. For $A, B \in \mathcal{I}$, we define two binary relations on $\mathcal{I}$ as follows:
\[
A \preceq B \text{ if and only if } B \ominus A \in \Omega \mathfrak{C}
\]
and
\[
A \preceq_H B \text{ if and only if } B \ominus_H A \text{ exists and } (B \ominus_H A) \ominus \omega \in \mathfrak{C} \text{ for some } \omega \in \Omega.
\]

**Proposition 6.** Let $\mathfrak{C}$ be a convex cone in $\mathcal{I}$. We have the following properties.

(i) Suppose that $\Omega \subseteq \mathfrak{C}$. Then the binary relation $\preceq$ is reflexive.

(ii) The binary relation $\preceq$ is transitive.

(iii) Given any $A, B \in \mathcal{I}$ and $\lambda > 0$, if $A \preceq B$ then $\lambda A \preceq \lambda B$; that is, the binary relation $\preceq$ is compatible with scalar multiplication.

(iv) Given any $A, B, D, E \in \mathcal{I}$, if $A \preceq B$ and $D \preceq E$ then $A \oplus D \preceq B \oplus E$; that is, the binary relation $\preceq$ is compatible with interval addition.

Proof. To prove part (i), we have $A \ominus A \in \Omega \subseteq \mathfrak{C}$ for any $A \in \mathcal{I}$ by the assumption, which shows $A \preceq A$ by Definition 3.2.

To prove part (ii), suppose that $A \preceq B \preceq D$. We want to show $A \preceq D$. By definition, we have $B \ominus A \in \Omega \mathfrak{C}$ and $D \ominus B \in \Omega \mathfrak{C}$ for some $C_1, C_2 \in \mathfrak{C}$, i.e.,
\[
B \ominus A \ominus \omega_1 = C_1 \ominus \omega_2 \text{ and } D \ominus B \ominus \omega_3 = C_2 \ominus \omega_4
\]
for some $\omega_i \in \Omega$ for $i = 1, \cdots, 4$. By adding the above two equalities side by side, we obtain
\[
(D \ominus A) \ominus (\omega_3 \ominus \omega_1 \ominus \omega_5) = (C_1 \ominus C_2) \ominus \omega_4 \ominus \omega_2,
\]
where $B \ominus B = \omega_2$. From Proposition 4, we also see that $C_1 \ominus C_2 \in \mathfrak{C}$. Since $\Omega$ is closed under the interval addition, the equality (3) says that $A \preceq D$, which shows the transitivity.

To prove part (iii), since $A \preceq B$ and $\lambda > 0$, we have $B \ominus (-A) \ominus \omega_1 = C \ominus \omega_2$ for some $\omega_1, \omega_2 \in \Omega$ and $C \in \mathfrak{C}$. Since $\mathfrak{C}$ is a cone and $\lambda \Omega = \Omega$ for $\lambda > 0$, we have
\[
\lambda B \ominus (-\lambda A) \ominus \lambda \omega_1 = \lambda C \ominus \lambda \omega_2 \in \mathfrak{C} \ominus \Omega,
\]
which shows that $\lambda A \preceq \lambda B$. 

Proposition 7. Let $\mathcal{C}$ be a convex cone in $I$. We have the following properties.

1. Suppose that $[0, 0] \in \mathcal{C}$. Then the binary relation $\preceq_H$ is reflexive.
2. The binary relation $\preceq_H$ is transitive.
3. Given any $A, B, C, D \in I$ and $\lambda > 0$, if $A \preceq_H B$ then $\lambda A \preceq_H \lambda B$; that is, the binary relation $\preceq_H$ is compatible with scalar multiplication.
4. Given any $A, B, C, D \in I$, if $A \preceq_H B$ and $C \preceq_H D$ then $A \oplus C \preceq_H B \oplus D$; that is, the binary relation $\preceq_H$ is compatible with set addition.

Proof. To prove part (i), since $A = A \oplus [0, 0]$, we see that

$$A \oplus_H A = [0, 0] \in \mathcal{C}$$

by the definition of $\oplus_H$. This shows $A \preceq_H A$.

To prove part (ii), suppose that $A \preceq_H B \preceq_H C$. We want to show $A \preceq_H C$. We first have that $D = B \ominus_H A$ and $E = C \ominus_H B$ exist, and that $D \oplus \omega_1 \in \mathcal{C}$ and $E \oplus \omega_2 \in \mathcal{C}$ for some $\omega_1, \omega_2 \in \Omega$. We also have $B = D \oplus A$ and $C = E \oplus B$. Let $F = D \oplus \omega_1 \oplus E \oplus \omega_2$. Using Proposition 4, we see that $F \in \mathcal{C}$. Then

$$C = E \oplus B = E \oplus (D \oplus A) = (D \oplus E) \oplus A,$$

which says that $C \ominus_H A = D \oplus E$ exists. We also have

$$(C \ominus_H A) \oplus \omega_1 \oplus \omega_2 = D \oplus \omega_1 \oplus E \oplus \omega_2 = F \in \mathcal{C},$$

Since $\Omega$ is closed under the interval addition, it follows that $A \preceq_H C$, which shows that the binary relation $\preceq_H$ is transitive.

To prove part (iii), since $A \preceq_H B$, by definition, we have that $C = B \ominus_H A$ exists and $C \oplus \omega \in \mathcal{C}$ for some $\omega \in \Omega$. Then $B = A \oplus C$ that implies $\lambda B = \lambda A \oplus \lambda C$ since $\lambda > 0$. This shows that $\lambda C = \lambda B \ominus_H \lambda A$. Since $\mathcal{C}$ is a cone, we see that $\lambda C \oplus \lambda \omega \in \mathcal{C}$, i.e.,

$$(\lambda B \ominus_H \lambda A) \oplus \lambda \omega = \lambda C \oplus \lambda \omega \in \mathcal{C},$$

which says that $\lambda A \preceq_H \lambda B$ since $\lambda \omega \in \mathcal{C}$.

To prove part (iv), by definition, we first have that $E = B \ominus_H A$ and $F = D \ominus_H C$ exist, and that $E \oplus \omega_1 \in \mathcal{C}$ and $F \oplus \omega_2 \in \mathcal{C}$ for some $\omega_1, \omega_2 \in \Omega$. We also have $B = E \oplus A$ and $D = F \oplus C$. Let $G = E \oplus \omega_1 \oplus F \oplus \omega_2$. Using Proposition 4, we see that $G \in \mathcal{C}$. Then

$$B \oplus D = (E \oplus A) \oplus (F \oplus C) = (A \oplus C) \oplus (E \oplus F),$$

which says that

$$(B \oplus D) \ominus_H (A \oplus C) = E \oplus F$$

exists.

We also have

$$(B \oplus D) \ominus_H (A \oplus C) \oplus \omega_1 \oplus \omega_2 = E \oplus \omega_1 \oplus F \oplus \omega_2 = G \in \mathcal{C}.$$
Let \( \mathcal{C} \) be a subset of \( \mathcal{I} \). We define
\[
\mathcal{C}^- = \left\{ C^- \in \mathcal{I} : \text{there exists } C \in \mathcal{C} \text{ such that } C \oplus C^- \in \Omega \right\}.
\]
Let us recall that \(-\mathcal{C} = \{-C : C \in \mathcal{C}\}\). It is clear that \(-\mathcal{C} \subseteq \mathcal{C}^-\) since \(-C \oplus C \in \Omega\).

From part (ii) of Proposition 1, we also see that
\[
\mathcal{C}^- = \left\{ C^- \in \mathcal{I} : \text{there exist } C \in \mathcal{C} \text{ and } \omega \in \Omega \text{ such that } C \oplus C^- \stackrel{\Omega}{=} \omega \right\}.
\]

**Example 3.** Continued from Example 2, we see that \([c^L, c^U] \in \mathcal{C}\) if and only if \(c^L + c^U \geq 0\). For \([c^L, c^U] \in \mathcal{C}^-\), by definition, we have
\[
[c^L, c^U] \oplus [\bar{c}^L, \bar{c}^U] = [c^L + \bar{c}^L, c^U + \bar{c}^U] = \omega
\]
for some \(\omega \in \Omega\). It suffices to say that there exists \(k_1, k_2 \in \mathbb{R}_+\) such that
\[
[c^L + \bar{c}^L, c^U + \bar{c}^U] \oplus [-k_1, k_1] = [-k_2, k_2],
\]
which implies \(c^L + \bar{c}^L - k_1 = -k_2\) and \(c^U + \bar{c}^U + k_1 = k_2\), i.e., \(c^L + \bar{c}^L = k_1 - k_2\) and \(c^U + \bar{c}^U = k_2 - k_1\). This says that
\[
c^L + \bar{c}^L = -c^U - \bar{c}^U \leq 0.
\]
Therefore we obtain
\[
\mathcal{C}^- = \left\{ [\bar{c}^L, \bar{c}^U] : \bar{c}^L + \bar{c}^U \leq 0 \right\}.
\]

The following useful result is not hard to prove.

**Proposition 8.** Let \( \mathcal{C} \) be a subset of \( \mathcal{I} \). Then \( \mathcal{C}^- \oplus \Omega = \mathcal{C}^-\).

Since \([0, 0] \in \Omega\) and \(C = C \oplus [0, 0] \in \mathcal{C} \oplus \Omega\) for \(C \in \mathcal{C}\), it is clear that \(\mathcal{C} \subseteq \mathcal{C} \oplus \Omega\).

Therefore we propose the following concept.

**Definition 3.3.** Let \( \mathcal{C} \) be a convex cone in \( \mathcal{I} \).

- We say that \( \mathcal{C} \) is pointed-like if and only if \(A \in \mathcal{C} \cap \mathcal{C}^-\) implies \(A \in \Omega\).
- We say that \( \mathcal{C} \) is strongly pointed-like if and only if \(A \in (\mathcal{C} \oplus \Omega) \cap \mathcal{C}^-\) implies \(A \in \Omega\).

Since \(\mathcal{C} \subseteq \mathcal{C} \oplus \Omega\), it is clear that if \(\mathcal{C}\) is strongly pointed-like then it is also pointed-like.

**Remark 2.** From part (ii) of Proposition 1, we also see that \(\mathcal{C}\) is pointed-like if and only if \(A \in \mathcal{C} \cap \mathcal{C}^-\) implies \(A \stackrel{\Omega}{=} \omega\) for some \(\omega \in \Omega\), and that \(\mathcal{C}\) is strongly pointed-like if and only if \(A \in (\mathcal{C} \oplus \Omega) \cap \mathcal{C}^-\) implies \(A \stackrel{\Omega}{=} \omega\) for some \(\omega \in \Omega\).

**Example 4.** Continued from Example 3, we want to claim that \(\mathcal{C}\) is strongly pointed-like. For \([a^L, a^U] \in (\mathcal{C} \oplus \Omega) \cap \mathcal{C}^-\), we have
\[
[a^L, a^U] = [c^L, c^U] \oplus [-k, k]
\]
and \(a^L + a^U \leq 0\) for some \([c^L, c^U] \in \mathcal{C}\) and \(k \geq 0\). This says that \(c^L + c^U \geq 0\), \(a^L = c^L - k\) and \(a^U = c^U + k\), which implies \(a^L + a^U \geq 0\). Therefore we must have \(a^L + a^U = 0\), i.e., \(a^L = -a^U\), which says that \([a^L, a^U] \in \Omega\). Therefore we conclude that \(\mathcal{C}\) is strongly pointed-like, and is also pointed-like.

**Definition 3.4.** We say that the binary relation \(\preceq\) on \(\mathcal{I}\) is antisymmetric-like if and only if \(A \preceq B\) and \(B \preceq A\) implies \(A \stackrel{\Omega}{=} B\) for any \(A, B \in \mathcal{I}\).

**Proposition 9.** Let \( \mathcal{C} \) be a convex cone in \( \mathcal{I} \). Suppose that \( \mathcal{C} \) is strongly pointed-like. Then we have the following properties.
Proof. To prove part (i), suppose that \( A \preceq B \) and \( B \preceq A \). By definition, we have

\[
B \oplus A \oplus \omega_i = C_i \oplus \omega_i
\]

for some \( \omega_i \in \Omega, \ i = 1, \ldots, 4, \) and \( C_1, C_2 \in \mathcal{C} \). Therefore we obtain

\[
(A \oplus B \oplus \omega_3) \oplus (C_1 \oplus \omega_2) = (A \oplus B \oplus \omega_3) \oplus (B \oplus A \oplus \omega_1)
\]

\[
\quad = (A \oplus A) \oplus (B \oplus B) \oplus \omega_1 \oplus \omega_3 \in \Omega.
\]

This shows that \( A \oplus B \oplus \omega_3 \oplus \omega_2 \in \mathcal{C}^-, \) i.e., \( A \oplus B \oplus \omega_3 \oplus \omega_2 \in (\mathcal{C} \oplus \Omega) \cap \mathcal{C}^- \), which says that \( A \oplus B \oplus \omega_3 \oplus \omega_2 \in \Omega \), since \( \mathcal{C} \) is strongly pointed-like. Therefore we obtain

\[
A \oplus B \oplus \omega_3 \oplus \omega_2 = \omega_5
\]

for some \( \omega_5 \in \Omega \). By adding \( B \) on both sides, we obtain

\[
A \oplus \omega_3 \oplus \omega_2 = B \oplus \omega_5,
\]

where \( B \oplus B = \omega_5 \in \Omega \). This shows that \( A \mathrel{\Omega} B \), since \( \Omega \) is closed under the interval addition.

To prove part (ii), suppose that \( A \preceq \bar{\omega}_1 \) and \( \bar{\omega}_2 \preceq A \) for some \( \bar{\omega}_1, \bar{\omega}_2 \in \Omega \). Then, by definition,

\[
\bar{\omega}_1 \oplus A \oplus \bar{\omega}_3 = C_1 \oplus \bar{\omega}_4 \quad \text{and} \quad A \oplus \bar{\omega}_2 \oplus \bar{\omega}_5 = C_2 \oplus \bar{\omega}_6 \in \mathcal{C} \oplus \Omega
\]

for some \( \bar{\omega}_i \in \Omega, \ i = 3, \ldots, 6, \) and \( C_1, C_2 \in \mathcal{C} \). Let \( \bar{\omega}_7 = A \oplus A \). Then we obtain

\[
(A \oplus \bar{\omega}_2 \oplus \bar{\omega}_3) \oplus (C_1 \oplus \bar{\omega}_4) = (A \oplus \bar{\omega}_2 \oplus \bar{\omega}_5) \oplus (\bar{\omega}_1 \oplus A \oplus \bar{\omega}_3)
\]

\[
\quad = \bar{\omega}_1 \oplus \bar{\omega}_3 \oplus \bar{\omega}_2 \oplus \bar{\omega}_7 \oplus \bar{\omega}_5 \in \Omega,
\]

since \( -\bar{\omega}_2 \in \Omega \) and \( \Omega \) is closed under the interval addition. This shows that

\[
A \oplus \bar{\omega}_2 \oplus \bar{\omega}_5 \oplus \bar{\omega}_4 \in \mathcal{C}^-.
\]

Therefore we have

\[
A \oplus \bar{\omega}_2 \oplus \bar{\omega}_5 \oplus \bar{\omega}_4 \in (\mathcal{C} \oplus \Omega) \cap \mathcal{C}^-,
\]

which says that

\[
A \oplus \bar{\omega}_2 \oplus \bar{\omega}_5 \oplus \bar{\omega}_4 \in \Omega,
\]

since \( \mathcal{C} \) is strongly pointed-like. Therefore we obtain \( A \mathrel{\Omega} \omega \) for some \( \omega \in \Omega \). This completes the proof. \( \square \)

**Proposition 10.** Let \( \mathcal{C} \) be a convex cone in \( \mathcal{I} \). Suppose that \( \mathcal{C} \) is pointed-like. Then we have the following properties.

(i) The binary relation \( \preceq_H \) is antisymmetric-like.

(ii) \( \omega_1 \preceq_H \omega_2 \) and \( \omega_2 \preceq_H \omega_1 \) for some \( \omega_1, \omega_2 \in \Omega \) imply \( \omega \mathrel{\Omega} \omega \) for some \( \omega \in \Omega \).

Proof. To prove part (i), suppose that \( A \preceq_H B \) and \( B \preceq_H A \). Then, by definition, we first have that \( C = B \ominus_H A \) and \( D = A \ominus_H B \) exist, and that \( (A \ominus_H B) \oplus \omega_1 \in \mathcal{C} \) and \( (B \ominus_H A) \oplus \omega_2 \in \mathcal{C} \) for some \( \omega_1, \omega_2 \in \Omega \). We also have \( B = A \oplus C \) and \( A = B \oplus D \). Then

\[
A = B \oplus D = (A \oplus C) \oplus D = A \oplus (C \oplus D).
\]

This shows that \( C \oplus D = [0, 0] \in \Omega \). Therefore we obtain

\[
[(A \ominus_H B) \oplus \omega_1] \oplus [(B \ominus_H A) \oplus \omega_2] = C \oplus D \oplus \omega_1 \oplus \omega_2 = [0, 0] \oplus \omega_1 \oplus \omega_2 \in \Omega.
\]
This shows that \((B \oplus_H A) \oplus \omega_2 \in \mathcal{C}^-\), since \((A \oplus_H B) \oplus \omega_1 \in \mathcal{C}\). In other words, we have
\[
(B \oplus_H A) \oplus \omega_2 \in \mathcal{C}^- \cap \mathcal{C}.
\]
Since \(\mathcal{C}\) is pointed-like, we have \((B \oplus_H A) \oplus \omega_2 \in \Omega\), i.e., \(C \oplus \omega_2 \in \Omega\), which says that \(C \oplus \omega_2 = \omega_3\) for some \(\omega_3 \in \Omega\). Since \(B = A \oplus C\), we have
\[
B \oplus \omega_2 = A \oplus C \oplus \omega_2 = A \oplus \omega_3.
\]
This shows that \(\frac{A}{\Omega} = B\).

To prove part (ii), suppose that \(A \preceq_H \omega_1\) and \(\omega_2 \preceq H \omega_1\) for some \(\omega_1, \omega_2 \in \Omega\). By definition, we have that \(E = \omega_1 \ominus_H A\) and \(F = A \ominus_H \omega_2\) exist, and that \(E \oplus \omega_1 \in \mathcal{C}\) and \(F \oplus \omega_2 \in \mathcal{C}\) for some \(\omega_1, \omega_2 \in \Omega\). We also have \(\omega_1 = A \oplus E\) and \(A = \omega_2 \oplus F\), which imply
\[
\omega_1 = A \oplus E = \omega_2 \oplus F = E.
\]
Therefore we obtain
\[
\[(\omega_1 \ominus_H A) \oplus \omega_1] \oplus [(A \ominus_H \omega_2) \oplus \omega_2] \oplus \omega_2 = E \oplus \omega_1 \oplus F \oplus \omega_2 \oplus \omega_2 = \omega_1 \oplus \omega_2 \oplus \omega_1 \in \Omega.
\]
This shows that \((\omega_1 \ominus_H A) \oplus \omega_1 \in \mathcal{C}^-\), since \((A \ominus_H \omega_2) \oplus \omega_2 \in \mathcal{C}\). In other words, we have
\[
(\omega_1 \ominus_H A) \oplus \omega_1 \in \mathcal{C}^- \cap \mathcal{C}.
\]
Since \(\mathcal{C}\) is pointed-like, we obtain \((\omega_1 \ominus_H A) \oplus \omega_1 \in \Omega\), i.e., \(E \oplus \omega_1 \in \Omega\), which says that \(E \oplus \omega_1 = \omega_2\) for some \(\omega_2 \in \Omega\). Since \(\omega_1 = A \oplus E\), we obtain
\[
\omega_1 \oplus \omega_1 = A \oplus E \oplus \omega_1 = A \oplus \omega_2,
\]
which says that \(\frac{A}{\Omega} = \omega_1\). This completes the proof. \(\Box\)

**Proposition 11.** Let \(\preceq\) be an arbitrary binary relation on \(\mathcal{I}\). If \(\preceq\) is compatible with the interval addition and scalar multiplication, then the following set
\[
\mathcal{C} = \{A \in \mathcal{I} : A \succeq \omega \text{ for some } \omega \in \Omega\}
\]
is a convex cone. If we further assume that \(A \succeq \omega_1\) and \(\omega_2 \succeq A\) for some \(\omega_1, \omega_2 \in \Omega\) implies \(A \succeq \omega\) for some \(\omega \in \Omega\), then \(\mathcal{C}\) is also strongly pointed-like.

**Proof.** Suppose that \(A \in \mathcal{C}\), i.e., \(A \succeq \omega\) for some \(\omega \in \Omega\). Then \(\lambda A \succeq \lambda \omega\) for \(\lambda > 0\) by the compatibility with scalar multiplication. Since \(\Omega\) is a cone, i.e., \(\lambda \omega \in \Omega\), it shows that \(\lambda A \in \mathcal{C}\). Suppose that \(A, B \in \mathcal{C}\), i.e., \(A \succeq \omega_1\) and \(B \succeq \omega_2\) for some \(\omega_1, \omega_2 \in \Omega\). Therefore, by the compatibility with interval addition and scalar multiplication, we have
\[
\lambda A \oplus (1 - \lambda)B \succeq \lambda \omega_1 \oplus (1 - \lambda) \omega_2 \in \Omega,
\]
since \(\Omega\) is convex. This shows that \(\mathcal{C}\) is a convex cone.

Under the further assumption, we want to show that \(\mathcal{C}\) is also strongly pointed-like. For \(A \in (\mathcal{C} \bowtie \Omega) \cap \mathcal{C}^-\), we have \(A \in \mathcal{C}^-\) and \(A = C \oplus \omega_1\) for some \(C \in \mathcal{C}\) and \(\omega_1 \in \Omega\). Therefore we have \(C \succeq \omega_2\) for some \(\omega_2 \in \Omega\), which implies \(C \oplus \omega_1 \succeq \omega_1 \oplus \omega_2\) by adding \(\omega_1\) on both sides. This shows
\[
A \succeq \omega_1 \oplus \omega_2. \tag{4}
\]
On the other hand, since \(A \in \mathcal{C}^->\), by definition, there exist \(B \in \mathcal{C}\) and \(\omega_3, \omega_4 \in \Omega\) such that \(A \oplus B \oplus \omega_3 = \omega_4\). Since \(B \in \mathcal{C}\), we also have \(B \succeq \omega_5\) for some \(\omega_5 \in \Omega\), which implies \(A \oplus B \oplus \omega_3 \succeq A \oplus \omega_3 \oplus \omega_5\) by adding \(A \oplus \omega_3\) on both sides, i.e.,
\[
\omega_4 \succeq A \oplus \omega_3 \oplus \omega_5. \tag{5}
\]
By adding \( \omega_3 \oplus \omega_5 \) on both sides of (4), we can obtain
\[
A \oplus \omega_3 \oplus \omega_5 \succeq \omega_1 \oplus \omega_2 \oplus \omega_3 \oplus \omega_5.
\] (6)
Since \( \Omega \) is closed under the interval addition, using the further assumption, it follows that (5) and (6) implies \( A \oplus \omega_3 \oplus \omega_5 \oplus \omega_6 \succeq \omega_6 \) for some \( \omega_6 \in \Omega \), which also says that \( A \oplus \omega \) for some \( \omega \in \Omega \). Using Remark 2, we complete the proof. \( \square \)

Let \( L : I \to V \) be a function from \( I \) into a vector space \( V \). The kernel of \( L \) is defined by
\[
\ker L = \{ A : L(A) = \theta_V \},
\]
where \( \theta_V \) is the zero element of vector space \( V \). It is obvious that
\[
L(\omega) = \theta_V \text{ for any } \omega \in \Omega \text{ if and only if } \Omega \subseteq \ker L.
\]

Let \( \mathcal{C} \) be a convex cone in \( I \). Proposition 5 says that if \( L \) is additive and positively homogeneous, then \( L(\mathcal{C}) \) is a convex cone in the vector space \( V \). Therefore we can define two binary relations \( \preceq \) and \( \preceq_H \) on \( L(I) \subseteq V \) as follows:
\[
L(A) \preceq L(B) \text{ if and only if } L(B) - L(A) \in L(\mathcal{C}).
\] (7)
and
\[
L(A) \preceq_H L(B) \text{ if and only if } L(B) - L(A) \in L(\mathcal{C}) \text{ and } B \ominus_H A \text{ exists.}
\] (8)
It is obvious that \( L(A) \preceq_H L(B) \) implies \( L(A) \preceq L(B) \).

The difference between \( \preceq \) and \( \preceq_H \) will be more clear when we investigate the order-preserving properties.

**Proposition 12.** Let \( L : I \to V \) be an additive and positively homogeneous function from \( I \) into a vector space \( V \), and let \( \mathcal{C} \) be a convex cone in \( I \). Then we have the following properties.

(i) Suppose that \([0,0] \in \mathcal{C} \). Then the binary relation \( \preceq_H \) is reflexive in \( L(I) \).

(ii) The binary relation \( \preceq_H \) is transitive.

(iii) The binary relation \( \preceq_H \) is compatible with scalar multiplication in \( L(I) \).

(iv) The binary relation \( \preceq_H \) is compatible with interval addition in \( L(I) \).

**Proof.** We first have that \( L(\mathcal{C}) \) is a convex cone in \( V \) by Proposition 5. To prove part (i), by the additivity, we have
\[
L([0,0]) = L([0,0] \oplus [0,0]) = L([0,0]) + L([0,0]),
\]
which implies \( \theta_V = L([0,0]) \in L(\mathcal{C}) \) is the unique zero element of vector space \( V \). Then we have \( L(A) - L(A) = \theta_V \in L(\mathcal{C}) \) for any \( A \in I \). From the proof of part (i) of Proposition 7, we see that \( A \ominus_H A \) exists. Therefore we obtain \( L(A) \preceq_H L(A) \).

To prove part (ii), suppose that \( L(A) \preceq_H L(B) \preceq_H L(C) \). Then \( B \ominus_H A \) and \( C \ominus_H B \) exist. From the proof of part (ii) of Proposition 7, we see that \( C \ominus_H A \) exists. We also have \( L(B) - L(A) \in L(\mathcal{C}) \) and \( L(C) - L(B) \in L(\mathcal{C}) \). Since \( L(\mathcal{C}) \) is a convex cone in the vector space \( V \), it follows that \( L(C) - L(A) \in L(\mathcal{C}) \) by adding them together, which says that \( L(A) \preceq_H L(C) \).

To prove part (iii), suppose that \( L(A) \preceq_H L(B) \) and \( \lambda > 0 \). Then, by definition, \( L(B) - L(A) \in L(\mathcal{C}) \) and \( B \ominus_H A \) exists. From the proof of part (iii) of Proposition 7, we see that \( \lambda A \ominus_H \lambda B \) exists. Since \( L(\mathcal{C}) \) is a convex cone in the vector space \( V \), it follows that \( \lambda L(B) - \lambda L(A) \in L(\mathcal{C}) \), i.e., \( L(\lambda B) - L(\lambda A) \in L(\mathcal{C}) \) by the positive homogeneity, which says that
\[
\lambda L(A) \preceq_H L(\lambda B) \preceq_H \lambda L(B).
\]
To prove part (iv), suppose that $\mathcal{L}(A) \preceq_H \mathcal{L}(B)$ and $\mathcal{L}(C) \preceq_H \mathcal{L}(D)$. Then $B \ominus_H A$ and $D \ominus_H C$ exist. From the proof of part (iv) of Proposition 7, we see that $(B \ominus D) \ominus_H (A \ominus C)$ exists. We also have $\mathcal{L}(B) - \mathcal{L}(A) \in \mathcal{L}(\mathcal{C})$ and $\mathcal{L}(D) - \mathcal{L}(C) \in \mathcal{L}(\mathcal{C})$. Since $\mathcal{L}(\mathcal{C})$ is a convex cone in the vector $V$, we obtain

$$\mathcal{L}(B) + \mathcal{L}(D) - \mathcal{L}(A) - \mathcal{L}(C) \in \mathcal{L}(\mathcal{C}).$$

By the additivity, we obtain

$$\mathcal{L}(B + D) - \mathcal{L}(A + C) \in \mathcal{L}(\mathcal{C}),$$

which says that

$$\mathcal{L}(A) + \mathcal{L}(C) = \mathcal{L}(A + C) \preceq_H \mathcal{L}(B + D) = \mathcal{L}(B) + \mathcal{L}(D).$$

This completes the proof.

**Proposition 13.** Let $\mathcal{L}: \mathcal{I} \rightarrow V$ be an additive and positively homogeneous function from $\mathcal{I}$ into a vector space $V$, and let $\mathcal{C}$ be a convex cone in $\mathcal{I}$. Then the binary relation $\preceq$ is transitive and compatible with the interval addition and scalar multiplication on $\mathcal{L}(\mathcal{I})$. If we further assume that $\theta_V \in \mathcal{L}(\mathcal{C})$, then the binary relation $\preceq$ is reflexive.

**Proof.** Since $\mathcal{L}(\mathcal{C})$ is a convex cone in the vector space $V$ by Proposition 5, the conventional argument is available for proving the desired results.

**Proposition 14.** (Order-Preserving) Let $\mathcal{L}: \mathcal{I} \rightarrow V$ be an additive and positively homogeneous function from $\mathcal{I}$ into a vector space $V$, and let $\mathcal{C}$ be a convex cone in $\mathcal{I}$. Suppose that $\Omega \subseteq \ker \mathcal{L}$. Then we have the following results.

(i) $A \preceq B$ implies $\mathcal{L}(A) \preceq \mathcal{L}(B)$.

(ii) Suppose that $\ker \mathcal{L} \subseteq \mathcal{C}$. Then $\mathcal{L}(A) \preceq \mathcal{L}(B)$ implies $A \preceq B$.

**Proof.** To prove part (i), by definition, we have $(B \ominus A) \oplus \omega_1 \in \mathcal{C} \ominus \Omega$ for some $\omega_1 \in \Omega$, which also says that $(B \ominus A) \oplus \omega_1 = C \ominus \omega_2$ for some $C \in \mathcal{C}$ and $\omega_2 \in \Omega$. By adding $A$ on both sides, we obtain

$$B \ominus \omega_3 \ominus \omega_1 = A \ominus C \ominus \omega_2,$$

where $\omega_3 = A \ominus A \in \Omega$. Using the additivity, from part (i) of Proposition 3, we have

$$\mathcal{L}(B) = \mathcal{L}(B \ominus \omega_3 \ominus \omega_1) = \mathcal{L}(A \ominus C \ominus \omega_2) = \mathcal{L}(A) + \mathcal{L}(C),$$

which implies

$$\mathcal{L}(B) - \mathcal{L}(A) = \mathcal{L}(C) \in \mathcal{L}(\mathcal{C}).$$

This shows that $\mathcal{L}(A) \preceq \mathcal{L}(B)$.

To prove part (ii), part (ii) of Proposition 3 says that $\mathcal{L}(A) \preceq \mathcal{L}(B)$ implies $\mathcal{L}(B \ominus A) \in \mathcal{L}(\mathcal{C})$. Therefore there exists $C \in \mathcal{C}$ such that $\mathcal{L}(B \ominus A) = \mathcal{L}(C)$, i.e., $\mathcal{L}(B \ominus A) - \mathcal{L}(C) = \theta_V$, that is the zero element of the vector space $V$. Using part (ii) of Proposition 3 again, we obtain $B \ominus A \ominus C \in \ker \mathcal{L}$, which says that $B \ominus A \ominus C = D$ for some $D \in \ker \mathcal{L}$. By adding $C$ on both sides, we obtain $(B \ominus A) \oplus \omega = C \ominus D$, where $\omega = C \ominus C$. Since $\ker \mathcal{L} \subseteq \mathcal{C}$, from Proposition 4, it follows that

$$(B \ominus A) \oplus \omega = C \ominus D \in \mathcal{C},$$

which says that $A \preceq B$. This completes the proof.

**Proposition 15.** (Order-Preserving) Let $\mathcal{L}: \mathcal{I} \rightarrow V$ be an additive and positively homogeneous function from $\mathcal{I}$ into a vector space $V$, and let $\mathcal{C}$ be a convex cone in $\mathcal{I}$. Suppose that $\Omega \subseteq \ker \mathcal{L}$. Then we have the following results.
(i) $A \preceq_H B$ implies $\mathcal{L}(A) \preceq_H \mathcal{L}(B)$.
(ii) Suppose that $\ker \mathcal{L} \subseteq \mathfrak{C}$. Then $\mathcal{L}(A) \preceq_H \mathcal{L}(B)$ implies $A \preceq_H B$.

Proof. To prove part (i), by definition, we have that $B \oplus_H A$ exists and $(B \oplus_H A) \oplus \omega \in \mathfrak{C}$ for some $\omega \in \Omega$, which implies $\mathcal{L}(B \oplus_H A) \in \mathcal{L}(\mathfrak{C})$ since $\mathcal{L}(\omega) = \theta_V$. From part (iii) of Proposition 3, we have $\mathcal{L}(B) - \mathcal{L}(A) \in \mathcal{L}(\mathfrak{C})$. This says that $\mathcal{L}(A) \preceq_H \mathcal{L}(B)$.

To prove part (ii), by definition, we have $\mathcal{L}(B) - \mathcal{L}(A) = \mathcal{L}(C)$ for some $C \in \mathfrak{C}$ and $B \oplus_H A$ exists. Using part (iii) of Proposition 3, we have $\mathcal{L}(B \oplus_H A) = \mathcal{L}(C)$, i.e., $\mathcal{L}(B \oplus_H A) - \mathcal{L}(C) = \theta_V$. Using part (ii) of Proposition 3, we obtain $(B \oplus_H A) \ominus C \in \ker \mathcal{L}$, which also says that $(B \oplus_H A) \ominus C = D$ for some $D \in \ker \mathcal{L}$. By adding $C$ on both sides, we have $(B \oplus_H A) \ominus \omega = C \oplus D$, where $\omega = C \ominus C \in \Omega$. Since $\ker \mathcal{L} \subseteq \mathfrak{C}$, using Proposition 4, it follows that $(B \oplus_H A) \ominus \omega \in \mathfrak{C}$, which also says that $A \preceq_H B$. This completes the proof.

4. Vector optimization problems. Let $\mathcal{L} : \mathcal{I} \to V$ be an additive and positively homogeneous function from $\mathcal{I}$ into a vector space $V$, and let $\mathfrak{C}$ be a convex cone in $\mathcal{I}$. Then $\mathcal{L}(\mathfrak{C})$ is a convex cone in $V$. Given a subset $\mathcal{F}$ of $\mathcal{I}$, we can define the concepts of minimal elements of $\mathcal{F}$ and $\mathcal{L}(\mathcal{F})$, respectively, as follows.

- Based on the binary relation $\preceq$ in (1), an element $A^* \in \mathcal{F}$ is called a minimal element of $\mathcal{F}$ if and only if $A \preceq A^*$ for $A \in \mathcal{F}$ implies $A^* \preceq A$. We denote by $\text{MIN}_{\mathcal{C}}(\mathcal{F})$, the set of all minimal elements of $\mathcal{F}$ according to $\preceq$. Based on the binary relation $\preceq_H$ in (2), we can similarly define the concept of H-minimal element and the set $\text{H-MIN}_{\mathcal{C}}(\mathcal{F})$.

- Based on the binary relation $\ll$ in (7), an element $B^* \in \mathcal{L}(\mathcal{F})$ is called a minimal element of $\mathcal{L}(\mathcal{F})$ if and only if $B \ll B^*$ for $B \in \mathcal{L}(\mathcal{F})$ implies $B^* \ll B$. We denote by $\text{MIN}_{\mathcal{L}(\mathcal{I})}(\mathcal{L}(\mathcal{F}))$ the set of all minimal elements of $\mathcal{L}(\mathcal{F})$ according to $\ll$. Based on the binary relation $\ll_H$ in (8), we can similarly define the concept of H-minimal element and the set $\text{H-MIN}_{\mathcal{L}(\mathcal{I})}(\mathcal{L}(\mathcal{F}))$.

Since the binary relation $\preceq$ is not a total ordering (it may just be a partial ordering by referring to Proposition 6), if we consider the concept of minimal element $A^*$ in the sense of $A^* \preceq A$ for all $A \in \mathcal{F}$ then, frequently, the minimal element $A^*$ does not exist. It is too strong to consider $A^* \preceq A$ for all $A \in \mathcal{F}$ when the the binary relation $\preceq$ is just a partial ordering. We also remark that if the the binary relation $\preceq$ is antisymmetric-like, then $A^* \in \mathcal{F}$ is a minimal element of $\mathcal{F}$ if and only if $A \preceq A^*$ for $A \in \mathcal{F}$ implies $A = A^*$ by Definition 3.4. Moreover, if the binary relation $\preceq$ happens to be antisymmetric, then $A^* \in \mathcal{F}$ is a minimal element of $\mathcal{F}$ if and only if $A \preceq A^*$ for $A \in \mathcal{F}$ implies $A = A^*$. The same situation also applies to the binary relation $\preceq_H$ regarding the H-minimal element.

Proposition 16. Let $\mathcal{L} : \mathcal{I} \to V$ be an additive and positively homogeneous function from $\mathcal{I}$ to $V$, and let $\mathfrak{C}$ be a convex cone in $\mathcal{I}$. Let $\mathcal{F}$ be a subset of $\mathcal{I}$ and $A^* \in \mathcal{F}$. Suppose that $\Omega \subseteq \ker \mathcal{L} \subseteq \mathfrak{C}$. Then $A^* \in \text{H-MIN}_{\mathcal{C}}(\mathcal{F})$ if and only if $\mathcal{L}(A^*) \in \text{H-MIN}_{\mathcal{L}(\mathcal{I})}(\mathcal{L}(\mathcal{F}))$.

Proof. Let $A^* \in \text{H-MIN}_{\mathcal{C}}(\mathcal{F})$. Suppose that there exists $B \in \mathcal{L}(\mathcal{F})$ such that $B \ll_H \mathcal{L}(A^*)$, where $B = \mathcal{L}(A)$ for some $A \in \mathcal{F}$; that is, we have $\mathcal{L}(A) \ll_H \mathcal{L}(A^*)$. We want to claim that $\mathcal{L}(A^*) \ll_H \mathcal{L}(A) = B$. From part (ii) of Proposition 15, we see that $A \preceq_H A^*$, which also says that $A^* \preceq_H A$ by the definition of minimal element. Using part (i) of Proposition 15, we have $\mathcal{L}(A^*) \preceq_H \mathcal{L}(A) = B$. This
shows that $L(A^*)$ is an H-minimal element in $L(F)$ with respect to the convex cone $L(\mathcal{C})$.

Conversely, for $L(A^*) \in H\text{-MIN}_{L(\mathcal{C})}(L(F))$, suppose that there exists $A \in F$ such that $A \preceq_H A^*$. We want to claim that $A^* \preceq_H A$. By part (i) of Proposition 15, we have $L(A) \preceq_H L(A^*)$. Since $L(A^*) \in H\text{-MIN}_{L(\mathcal{C})}(L(F))$, we have $L(A^*) \preceq_H L(A)$. By part (ii) of Proposition 15, we also have $A^* \preceq_H A$. This shows that $A^*$ is an minimal element of $F$. This completes the proof.

Proposition 17. Let $L : \mathcal{I} \to V$ be an additive and positively homogeneous function from $\mathcal{I}$ to $V$, and let $\mathcal{C}$ be a convex cone in $\mathcal{I}$. Let $F$ be a subset of $\mathcal{I}$ and $A^* \in F$. Suppose that $\Omega \subseteq \ker L \subseteq \mathcal{C}$. Then $A^* \in \text{MIN}_C(F)$ if and only if $L(A^*) \in \text{MIN}_{L(\mathcal{C})}(L(F))$.

Proof. Applying Proposition 14 to the proof of Proposition 16, we can similarly obtain the desired result.

Let $U$ be another vector space. By referring to Chalco-Cano et al. [4], Osuna-Gomez et al. [12] and Wu [18, 19, 20, 21], the interval-valued function and interval-valued optimization problem are considered below. The function $f : U \to \mathcal{I}$ defined on $U$ is called an interval-valued function. Now we consider the following interval-valued optimization problem:

$$\text{(IOP)} \quad \min \quad f(u)$$

subject to $u \in G$, where $G$ is a subset of $U$. Let

$$F \equiv f(G) = \{f(u) : u \in G\} \subseteq \mathcal{I}$$

be the set of all objective values of problem (IOP), and let $\mathcal{C}$ be a convex cone in $\mathcal{I}$.

- We say that $u^*$ is an optimal solution of problem (IOP) if and only if $f(u^*) \in \text{MIN}_\mathcal{C}(\mathcal{F})$.
- We say that $u^*$ is an H-optimal solution of problem (IOP) if and only if $f(u^*) \in H\text{-MIN}_\mathcal{C}(\mathcal{F})$.

In order to solve problem (IOP), we are going to introduce an auxiliary optimization problem that is solvable by the well-known techniques.

Let $L : \mathcal{I} \to V$ be a function from $\mathcal{I}$ into a vector space $V$. Then we can consider the composition function $L \circ f : U \to \mathcal{I}$ of functions $L$ and $f$. Now we consider the following vector optimization problem:

$$\text{(VOP)} \quad \min \quad (L \circ f)(u)$$

subject to $u \in G$. It is clear that the set of all objective values of (VOP) is $L(F)$.

- We say that $u^*$ is an optimal solution of problem (VOP) if and only if $(L \circ f)(u^*) \in \text{MIN}_{L(\mathcal{C})}(L(F))$.
- We say that $u^*$ is an H-optimal solution of problem (VOP) if and only if $(L \circ f)(u^*) \in H\text{-MIN}_{L(\mathcal{C})}(L(F))$.

Considering the vector optimization problem to solve the interval-valued optimization problem is a new attempt according to the limited knowledge of the author.

Proposition 18. Let $L : \mathcal{I} \to V$ be an additive and positively homogeneous function from $\mathcal{I}$ to $V$, and let $\mathcal{C}$ be a convex cone in $\mathcal{I}$. Suppose that $\Omega \subseteq \ker L \subseteq \mathcal{C}$.
(i) \( u^* \) is an optimal solution of problem (IOP) if and only if \( u^* \) is an optimal solution of problem (VOP).

(ii) \( u^* \) is an H-optimal solution of problem (IOP) if and only if \( u^* \) is an H-optimal solution of problem (VOP).

**Proof.** Since the feasible sets of problems (IOP) and (VOP) are identical, the result follows from Propositions 16 and 17 immediately by taking \( A^* = f(u^*) \).

Inspired by Proposition 18, in order to solve problem (IOP), it suffices to solve problem (VOP), where the domain \( U \) and range \( V \) of the objective function \( L \circ f \) in problem (VOP) are all vector spaces. Therefore we can apply the well-known techniques in vector optimization problem to solve problem (VOP). For example, the scalarization technique in vector optimization will be invoked in this paper to solve problem (VOP).

5. **Scalarization.** In order to present the scalarization results, we first present the following interesting and useful result.

**Proposition 19.** Let the function \( L : \mathcal{I} \to V \) be additive and positively homogeneous. Let \( \mathcal{C} \) be a convex cone and be pointed-like in \( \mathcal{I} \). If \( \ker L = \Omega \), then \( L(\mathcal{C}) \) is a pointed convex cone in the vector space \( V \).

**Proof.** Since \( \mathcal{C} \) is a convex cone, we see that \( L(\mathcal{C}) \) is a convex cone in \( V \) from Proposition 5. We remain to claim that \( L(\mathcal{C}) \cap -L(\mathcal{C}) = \{0\} \), where \( 0 \) is the zero element of the vector space \( V \). Suppose that \( \bar{c} \in L(\mathcal{C}) \cap -L(\mathcal{C}) \). Then \( \bar{c} = L(A) = -L(B) \) for some \( A, B \in \mathcal{C} \). It means that \( L(A + B) = L(A) + L(B) = 0 \), which implies \( A + B \in \Omega \) by the assumption of \( \ker L = \Omega \). It also shows that \( A \in \mathcal{C} \), since \( B \in \mathcal{C} \). In other words, we have \( A \in \mathcal{C} \cap \mathcal{C}^- \), i.e., \( A \equiv \omega \) for some \( \omega \in \Omega \), since \( \mathcal{C} \) is pointed-like. Therefore we have \( A \oplus \omega_1 = \omega \oplus \omega_2 \) for some \( \omega_1, \omega_2 \in \Omega \). Since \( \Omega \) is closed under the interval addition, it follows that \( A \oplus \omega_1 \in \Omega \). Using the assumption of \( \ker L = \Omega \), we obtain \( \bar{c} = L(A) = L(A) + \theta_V = L(A) + L(\omega_1) = L(A \oplus \omega_1) = \theta_V \).

This completes the proof.

**Remark 3.** Let the function \( L : \mathcal{I} \to V \) be additive and positively homogeneous. Let \( \mathcal{C} \) be a convex cone and be pointed-like in \( \mathcal{I} \). Suppose that \( \ker L = \Omega \). From Proposition 19 and (8), the binary relation \( \preceq_H \) is antisymmetric. Then it is clear to see that \( B^* \in L(\mathcal{F}) \) is an H-minimal element of \( L(\mathcal{F}) \) if and only if \( B \preceq_H B^* \) for \( B \in L(\mathcal{F}) \) implies \( B = B^* \).

Let the function \( L : \mathcal{I} \to V \) be additive and positively homogeneous, and let \( \mathcal{C} \) be a convex cone in \( \mathcal{I} \). Then \( \mathcal{C} = L(\mathcal{C}) \) is a convex cone in \( V \) by Proposition 5. We write \( V' \) to denote the set of all linear functionals from \( V \) to \( \mathbb{R} \). The dual cone of \( \mathcal{C} \) is defined by \( \mathcal{C}' = \{ \phi \in V' : \phi(\bar{c}) \geq 0 \text{ for all } \bar{c} \in \mathcal{C} \} \).

Now we are in a position to present the scalarization results.
Theorem 5.1. Let the function $L : I \rightarrow V$ be additive and positively homogeneous. Let $C$ be a convex cone and be pointed-like in $I$. Assume that ker $L = \Omega \subseteq C$. If there exists a linear functional $\phi \in C^*$ and an element $u^* \in G$ such that
\[ \phi((L \circ f)(u^*)) < \phi((L \circ f)(u)) \quad \text{for all } u \in G \setminus \{u^*\}, \] then $u^*$ is an $H$-optimal solution of problem (IOP).

Proof. Suppose that $u^*$ is not an $H$-optimal solution of problem (VOP). We are going to lead a contradiction. By definition, we have $(L \circ f)(u^*) \not\in \text{H-MIN}_{C}(L(F))$, which says that $(L \circ f)(u^*)$ is not an $H$-minimal element of $L(F)$ based on the binary relation $\preceq_H$. By Remark 3, there exists $u \in G$ such that $(L \circ f)(u) \neq (L \circ f)(u^*)$ and $(L \circ f)(u) \preceq_H (L \circ f)(u^*)$, i.e., $(L \circ f)(u^*) - (L \circ f)(u) \in L(C) = \mathcal{E}$ according to (8). We also see that $u \neq u^*$. Since $\phi \in C^*$, we obtain
\[ \phi((L \circ f)(u^*)) - \phi((L \circ f)(u)) = \phi((L \circ f)(u^*)) - (L \circ f)(u)) \geq 0, \]
which contradicts (9). This contradiction says that $u^*$ is an $H$-optimal solution of problem (VOP). Using part (ii) of Proposition 18, we complete the proof.

Theorem 5.2. Let the function $L : I \rightarrow V$ be additive and positively homogeneous. Let $C$ be a convex cone and be pointed-like in $I$. Assume that ker $L = \Omega \subseteq C$. If there exists a linear functional $\phi \in C^*$ and an element $u^* \in G$ such that
\[ \phi((L \circ f)(u^*)) < \phi((L \circ f)(u)) \quad \text{for all } u \in G \setminus \{u^*\}, \]
then $u^*$ is an optimal solution of problem (IOP).

Proof. The result follows from the similar arguments of Theorem 5.1 by considering the binary relation $\preceq$ and using part (i) of Proposition 18.

Given $\phi \in C^*$, we consider the following real-valued (scalar) optimization problem
\[ (\text{SAOP}) \quad \min_{u \in G} \phi((L \circ f)(u)) \]
subject to $\phi((L \circ f)(u))$. Then we have the following interesting result.

Theorem 5.3. Let the function $L : I \rightarrow V$ be additive and positively homogeneous. Let $C$ be a convex cone and be pointed-like in $I$. Assume that ker $L = \Omega \subseteq C$. If $u^*$ is a unique optimal solution of problem (SAOP), then $u^*$ is both an optimal solution and $H$-optimal solution of problem (IOP).

Proof. The desired result follows from Theorems 5.1 and 5.2 immediately.
Proof. From the proof of Theorem 5.1, there exists \( u \in G \) such that \((\mathcal{L} \circ f)(u) \neq (\mathcal{L} \circ f)(u^*)\) and \((\mathcal{L} \circ f)(u) \leq_H (\mathcal{L} \circ f)(u^*)\), i.e.,

\[
\theta_V \neq (\mathcal{L} \circ f)(u^*) - (\mathcal{L} \circ f)(u) \in \mathcal{L}(\mathcal{C}) = \mathcal{C}
\]

according to (8). Since \( \phi \notin \mathcal{I}_V \), we first present a particular problem. Incorporating programming problems. In order to realize the essential idea of general problem, we consider the following interval-valued linear programming problem (VOP). Using part (ii) of Proposition 18, we complete the proof.

**Theorem 5.5.** Let the function \( \mathcal{L} : \mathcal{I} \to V \) be additive and positively homogeneous. Let \( \mathcal{C} \) be a convex cone and be pointed-like in \( \mathcal{I} \). Assume that \( \ker \mathcal{L} = \Omega \subseteq \mathcal{C} \). If there exists a linear functional \( \phi^\circ \in \mathcal{C}_V^\circ \) and an element \( u^* \in G \) such that

\[
\phi^\circ ((\mathcal{L} \circ f)(u^*)) \leq \phi^\circ ((\mathcal{L} \circ f)(u)) \text{ for all } u \in G,
\]

then \( u^* \) is an optimal solution of problem (IOP).

**Proof.** The result follows from the similar arguments of Theorem 5.4 by considering binary relation \( \leq \) and using part (i) of Proposition 18.

Given \( \phi^\circ \in \mathcal{C}_V^\circ \), we consider the following real-valued (scalar) optimization problem

\[
\text{(SOP)} \quad \min \quad \phi^\circ ((\mathcal{L} \circ f)(u)) \quad \text{subject to} \quad u \in G.
\]

Then we have the following interesting result.

**Theorem 5.6.** Let the function \( \mathcal{L} : \mathcal{I} \to V \) be additive and positively homogeneous. Let \( \mathcal{C} \) be a convex cone and be pointed-like in \( \mathcal{I} \). Assume that \( \ker \mathcal{L} = \Omega \subseteq \mathcal{C} \). If \( u^* \) is an optimal solution of problem (SOP\(^0\)), then \( u^* \) is both an optimal solution and \( H\)-optimal solution of problem (IOP).

**Proof.** The desired result follows from Theorems 5.4 and 5.5 immediately.

### 6. Interval-valued linear programming problems

From Example 4, we see that the following set

\[
\mathcal{C} = \{ [c_L, c_U] \in \mathcal{I} : c_L + c_U \geq 0 \}
\]

is a convex cone and is pointed-like in \( \mathcal{I} \) satisfying \( \Omega \subseteq \mathcal{C} \). This special type of convex cone \( \mathcal{C} \) will be taken in this section for studying the interval-valued linear programming problems. In order to realize the essential idea of general problem, we first present a particular problem.

#### 6.1. Particular problem

Let \( A_i = [a_i^L, a_i^U] \) be bounded closed intervals for \( i = 1, \cdots, n \). We consider the following interval-valued linear programming problem

\[
\text{(ILP)} \quad \min \quad f(x_1, \cdots, x_n) = x_1 A_1 \oplus x_2 A_2 \oplus \cdots \oplus x_n A_n \quad \text{subject to} \quad x = (x_1, \cdots, x_n) \in G \subseteq \mathbb{R}^n \text{ and } x \in \mathbb{R}^n_+.
\]

where \( G \) is a feasible set consisting of linear constraints.

Let \( \mathcal{L} : \mathcal{I} \to \mathbb{R}^2 \) be defined by

\[
\mathcal{L} ([a^L, a^U]) = (-a^L - a^U, a^L + a^U).
\]

It is easy to see that \( \mathcal{L} \) is additive and positively homogeneous. We also have

\[
\bar{\mathcal{C}} = \mathcal{L}(\mathcal{C}) = \{ (-(c_L - c_U), c_L + c_U) \in \mathbb{R}^2 : c_L + c_U \geq 0 \} \subseteq \{ (-x, x) \in \mathbb{R}^2 : x \geq 0 \}.
\]
For any $\omega = [-k, k] \in \Omega$, we have $L(\omega) = (0, 0)$ that is the zero element of $\mathbb{R}^2$. If $L([a^L, a^U]) = (0, 0)$, then $-a^L = a^U$, which says that $[a^L, a^U] \in \Omega$. Therefore we obtain $\ker L = \Omega$, i.e., $\ker L = \Omega \subseteq \mathcal{C}$.

Now we introduce the following bi-objective optimization problem

\[
\text{(BOP)} \quad \min_{\mathbf{x}} \quad (L \circ f)(x_1, \ldots, x_n)
\]

subject to $\mathbf{x} = (x_1, \ldots, x_n) \in G \subseteq \mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}_+$,

where the objective function $L \circ f$ is given by

\[
(L \circ f)(x_1, \ldots, x_n) = (g_1(x_1, \ldots, x_n), g_2(x_1, \ldots, x_n)) \in \mathbb{R}^2
\]

with

\[
g_1(x_1, \ldots, x_n) = -(a_1^L + a_1^U) x_1 - \cdots - (a_n^L + a_n^U) x_n
\]

and

\[
g_2(x_1, \ldots, x_n) = (a_1^L + a_1^U) x_1 + \cdots + (a_n^L + a_n^U) x_n = -g_1(x_1, \ldots, x_n).
\]

We are going to apply the scalarization to solve problem (BOP) by taking the linear functional $\phi^o : \mathbb{R}^2 \to \mathbb{R}$ as

\[
\phi^o(x, y) = \frac{1}{2} x + \frac{3}{2} y + k,
\]

where $k > 0$ is a constant. It is clear that $\phi^o(c) > 0$ for all $c \in \mathcal{C}$ and $\phi^o \in \mathcal{C}^+_\omega$. We also see that

\[
(\phi^o \circ L)(([a^L, a^U])) = \phi^o(-a^L - a^U, a^L + a^U) = a^L + a^U + k.
\]

Now we consider the following linear programming problem

\[
\text{(LP)} \quad \min_{\mathbf{x}} \quad \phi^o ((L \circ f)(x_1, \ldots, x_n))
\]

subject to $\mathbf{x} = (x_1, \ldots, x_n) \in G \subseteq \mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}_+$,

where the real-valued linear objective function is given by

\[
\phi^o ((L \circ f)(x_1, \ldots, x_n)) = \phi^o(g_1(x_1, \ldots, x_n), g_2(x_1, \ldots, x_n))
\]

\[
= \frac{1}{2} \cdot g_1(x_1, \ldots, x_n) + \frac{3}{2} \cdot g_2(x_1, \ldots, x_n) + k
\]

\[
= g_2(x_1, \ldots, x_n) + k
\]

\[
= (a_1^L + a_1^U) x_1 + \cdots + (a_n^L + a_n^U) x_n + k.
\]

Using Theorem 5.6, if $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$ is an optimal solution of linear programming problem (LP), then $\mathbf{x}^*$ is both an optimal solution and $H$-optimal solution of problem (ILP).

It is clear that solving the linear programming problem (LP) is equivalent to solving the following linear programming problem

\[
\text{(LP)} \quad \min_{\mathbf{x}} \quad (a_1^L + a_1^U) x_1 + \cdots + (a_n^L + a_n^U) x_n
\]

subject to $\mathbf{x} = (x_1, \ldots, x_n) \in G \subseteq \mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}_+$,
6.2. General Problem. Given any fixed $0 \neq \sigma_i \in \mathbb{R}$ for $i = 1, \ldots, n$, we consider the following generalized interval-valued linear programming problem

\[
\text{(GILP)} \quad \min \quad f(x_1, \ldots, x_n) = \sigma_1 A_1 x_1 \oplus \sigma_2 A_2 x_2 \oplus \cdots \oplus \sigma_n A_n x_n
\]

subject to \( x = (x_1, \ldots, x_n) \in G \subseteq \mathbb{R}^n \) and \( x \in \mathbb{R}_+^n \).

where \( G \) is a feasible set consisting of linear constraints.

Given \( A = [a^L, a^U] \in \mathcal{I} \), let \( a = a^L + a^U \). Given any fixed \( 0 \neq \alpha_i \in \mathbb{R} \) for \( i = 1, \ldots, m \), let \( \mathcal{L} : \mathcal{I} \rightarrow \mathbb{R}^m \) be defined by

\[
\mathcal{L} \left([a^L, a^U]\right) = (\alpha_1 a, \alpha_2 a, \ldots, \alpha_m a).
\]

For any other \( B = [b^L, b^U] \in \mathcal{I} \), let \( b = b^L + b^U \). Then we have

\[
\mathcal{L} \left([a^L, a^U]\right) + \mathcal{L} \left([b^L, b^U]\right) = (\alpha_1 a, \alpha_2 a, \ldots, \alpha_m a) + (\alpha_1 b, \alpha_2 b, \ldots, \alpha_m b)
\]

\[
= (\alpha_1 (a + b), \ldots, \alpha_m (a + b)) = \mathcal{L} \left([a^L + b^L, a^U + b^U]\right) = \mathcal{L} \left([a^L + b^L, a^U + b^U]\right)
\]

and, for \( \lambda \geq 0 \),

\[
\lambda \mathcal{L} \left([a^L, a^U]\right) = \lambda (\alpha_1 a, \ldots, \alpha_m a) = (\lambda \alpha_1 a, \ldots, \lambda \alpha_m a) = \mathcal{L} \left([\lambda a^L, \lambda a^U]\right)
\]

which shows that \( \mathcal{L} \) is additive and positively homogeneous. We also see that

\[
\mathcal{E} = \mathcal{L}(\mathcal{C}) = \{(\alpha_1 c, \ldots, \alpha_m c) \in \mathbb{R}^m : c^L + c^U \geq 0\}
\]

\[
\subset \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x \geq 0\}.
\]

For any \( \omega = [-k, k] \in \Omega \), we have \( \mathcal{L}(\omega) = (0, \ldots, 0) \) that is the zero element of \( \mathbb{R}^m \). If \( \mathcal{L}([a^L, a^U]) = (0, \ldots, 0) \), then we must have \(-a^L = a^U\), which says that \([a^L, a^U] \in \Omega \). Therefore we obtain \( \ker \mathcal{L} = \Omega \), i.e., \( \ker \mathcal{L} = \Omega \subseteq \mathcal{E} \).

The objective function of problem (GILP) is re-written as

\[
f(x_1, \ldots, x_n) = \sigma_1 A_1 x_1 \oplus \sigma_2 A_2 x_2 \oplus \cdots \oplus \sigma_n A_n x_n
\]

\[
= [g^L(x_1, \ldots, x_n), g^U(x_1, \ldots, x_n)].
\]

Let

\[
g(x_1, \ldots, x_n) = g^L(x_1, \ldots, x_n) + g^U(x_1, \ldots, x_n).
\]

Now we introduce the following multi-objective optimization problem

\[
\text{(MOP)} \quad \min \quad (\mathcal{L} \circ f)(x_1, \ldots, x_n)
\]

subject to \( x = (x_1, \ldots, x_n) \in G \subseteq \mathbb{R}^n \) and \( x \in \mathbb{R}_+^n \),

where the objective function \( \mathcal{L} \circ f \) is given by

\[
(\mathcal{L} \circ f)(x_1, \ldots, x_n) = (\alpha_1 g(x), \ldots, \alpha_m g(x)) \in \mathbb{R}^m.
\]

We are going to apply the scalarization to solve problem (MOP) by taking the linear functional \( \phi^\circ : \mathbb{R}^m \rightarrow \mathbb{R} \) as

\[
\phi^\circ(x_1, \ldots, x_m) = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_m x_m + k,
\]

where \( k > 0 \) is a constant and \( \beta_i \neq 0 \) for \( i = 1, \cdots, m \) are also constants. Then we have

\[
(\phi^\circ \circ \mathcal{L}) \left([a^L, a^U]\right) = \phi^\circ (\alpha_1 a, \alpha_2 a, \ldots, \alpha_m a)
\]

\[
= \alpha_1 \beta_1 a + \cdots + \alpha_m \beta_m a + k = \gamma (a^L + a^U) + k,
\]

where

\[
\gamma = \alpha_1 \beta_1 + \cdots + \alpha_m \beta_m \in \mathbb{R}.
\]
Here we take $\beta_i$ for $i = 1, \ldots, m$ such that $\gamma \geq 0$. Then we can show that $\phi^\circ(\bar{c}) > 0$ for all $\bar{c} \in \mathcal{C}$, which also says that $\phi^\circ \in \mathcal{C}^*_\gamma$.

Now we consider the following linear programming problem

$$\text{(LP)} \quad \text{min} \quad \phi^\circ((\mathcal{L} \circ f)(x_1, \ldots, x_n))$$

subject to $x = (x_1, \ldots, x_n) \in G \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n_+$,

where the real-valued linear objective function is given by

$$\phi^\circ((\mathcal{L} \circ f)(x_1, \ldots, x_n)) = \phi^\circ(\alpha_1 g(x), \ldots, \alpha_m g(x))$$

$$= \alpha_1 \beta_1 g(x) + \cdots + \alpha_m \beta_m g(x) + k$$

$$= \gamma \left( g^L(x_1, \ldots, x_n) + g^U(x_1, \ldots, x_n) \right) + k$$

Using Theorem 5.6, if $x^* = (x^*_1, \ldots, x^*_n)$ is an optimal solution of linear programming problem (LP$^\circ$), then $x^*$ is both an optimal solution and H-optimal solution of problem (GILP).

It is clear that solving the linear programming problem (LP$^\circ$) is equivalent to solving the following linear programming problem

$$\text{(LP)} \quad \text{min} \quad g^L(x_1, \ldots, x_n) + g^U(x_1, \ldots, x_n)$$

subject to $x = (x_1, \ldots, x_n) \in G \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n_+$,

where the constants $\gamma$ and $k$ can be ignored. We define the following sets of indices:

$$P = \{i : \sigma_i \geq 0\} \quad \text{and} \quad N = \{j : \sigma_j < 0\}.$$

Then $P \cup N = \{1, 2, \ldots, n\}$. Therefore we obtain

$$g^L(x_1, \ldots, x_n) = \sum_{i \in P} \sigma_i a_i^L x_i + \sum_{j \in N} \sigma_j a_j^U x_j$$

and

$$g^U(x_1, \ldots, x_n) = \sum_{i \in P} \sigma_i a_i^U x_i + \sum_{j \in N} \sigma_j a_j^U x_j,$$

which says that problem (LP) is given by

$$\text{(LP)} \quad \text{min} \quad \sigma_1 (a_1^L + a_1^U) x_1 + \sigma_2 (a_2^L + a_2^U) x_2 + \cdots + \sigma_n (a_n^L + a_n^U) x_n$$

subject to $x = (x_1, \ldots, x_n) \in G \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n_+$,

Therefore, in order to solve the generalized interval-valued linear programming problem (GILP) by obtaining the optimal solution and H-optimal solution, we can simply solve the conventional linear programming problem (LP) shown above.

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