$W_{1+\infty}$ Dynamics of Edge Excitations in the Quantum Hall Effect

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Quantum Hall universality classes can be classified by $W_{1+\infty}$ symmetry. We show that this symmetry also governs the dynamics of quantum edge excitations. The Hamiltonian of interacting electrons in the fully-filled first Landau level is expressed in terms of $W_{1+\infty}$ generators. The spectra for both the Coulomb and generic short-range interactions are thus found algebraically. We prove the one-dimensional bosonization of edge excitations in the limit of large number of particles. Moreover, the subleading corrections are given by the higher-spin $W_{1+\infty}$ generators, which measure the radial fluctuations of the electron density. The resulting spectrum for the Coulomb interaction contains a logarithmic enhancement, in agreement with experimental observations. The spectrum for generic short-range interactions is subleading and reproduces the classical capillary frequencies. These results are also extended to fractional filling by using symmetry arguments.

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1. Introduction

In a recent series of papers [1] [2] [3] [4], we have studied the many-body states at the plateaus of the quantum Hall effect [5] by means of an effective field theory with $W_{1+\infty}$ dynamical symmetry.

The effective field theory approach, developed by Landau, Ginsburg, Wilson and others [6], does not attempt to solve the microscopic many-body dynamics, but rather it guesses the macroscopic physics generated by this dynamics. The variables of the effective field theory are the relevant low-energy (long-distance) degrees of freedom, which are characterized by a specific symmetry. They describe universal properties, which are independent of the microscopic details. This approach is well suited for the quantum Hall effect, where experiments yield very precise and universal values of the Hall conductivity.

The main input for our effective field theory is the Laughlin theory of incompressible quantum fluids [7]; this predicts that the many-body ground states at the plateaus have uniform density and a gap for density waves. We have shown that planar incompressible fluids are completely characterized by a dynamical symmetry [8]. This means two basic kinematical properties:

i) each state of the system can be mapped into any other one by a symmetry transformation, such that the classical configuration space (or the quantum Hilbert space) carries a representation of the classical (quantum) symmetry algebra;

ii) the conserved quantities (quantum numbers) characterizing the excitations are given by a set of symmetry generators, which are simultaneously diagonal (the Cartan subalgebra); their spectrum is found by algebraic means.

The standard example of effective field theory with dynamical symmetry is conformal field theory in $(1+1)$ dimensions [9], which describes the long-distance physics of critical phenomena in two space dimensions, and also non-Fermi liquid behaviour of interacting fermions in $(1+1)$ dimensions [10].

In the quantum Hall effect, the study of representations of the $W_{1+\infty}$ dynamical symmetry has given both the relevant degrees of freedom and their quantum numbers. We have thus obtained an algebraic description [2] of anyons, collective excitations of fractional charge, spin and statistics [11], and a classification of quantum Hall universality classes [4], which corresponds to the known hierarchies [12]. This exact algebraic description accounts for the precision of the measured Hall conductivities at the plateaus.
These results on the kinematical data of incompressible fluids are reviewed in section 2, setting the language for the new results discussed afterwards. Let us briefly summarize them. At the classical level, which is relevant for large samples, the incompressible fluid is characterized by the dynamical symmetry under area-preserving diffeomorphisms, obeying the $\mathfrak{w}_\infty$ algebra [13]. This is because all possible configurations of a droplet of uniform density can be obtained by deformations which preserve the area. Furthermore, the relevant degrees of freedom are given by the small fluctuations of a large droplet, say of the shape of a disk, which are localized at the edge - the so-called edge waves [14]. The quantization of these waves yields a (1+1)-dimensional effective field theory, whose Hilbert space consists of a set of unitary, irreducible, highest-weight representations of $W_{1+\infty}$, the quantum version of the $\mathfrak{w}_\infty$ algebra [13].

The $W_{1+\infty}$ algebra is linear and infinite dimensional, like the Virasoro conformal algebra. It involves an infinite number of conserved currents $V_n^i$, having any positive (integer) conformal spin $(i + 1)$. Its representations are characterized by the quantum numbers corresponding to the eigenvalues of $V_0^i$. The basic quantum numbers are the electric charge $V_0^0$ and the spin $V_0^1$ of the excitations. The corresponding currents $V_n^0$ and $V_n^1$ generate the Abelian Kac-Moody and Virasoro subalgebras of $W_{1+\infty}$, respectively [3]. The basic operator is not the Hamiltonian, but the Kac-Moody current on the edge, out of which all the Hilbert space can be constructed. Therefore, we can identify effective theories of edge excitations without specifying completely the dynamics of the incompressible fluid. Moreover, the kinematical data are sufficient to explain the Hall conduction experiments [3] [12].

In this paper, we incorporate the dynamics of quantum edge excitations into the framework of $W_{1+\infty}$ symmetry. Dynamics is important in other types of experiments, where excitations are induced by temperature or other probes. Here, the effective field theory is still useful to predict the thermodynamics and the dynamical response.

In section 3, we discuss the Hamiltonian of edge excitations. We first show how to use the $\mathfrak{w}_\infty$ algebra to describe the dynamics of the classical, chiral, incompressible fluid, and rederive the capillary frequencies of its eigenoscillations [15]. Next, we study the general microscopic quantum Hamiltonian for interacting electrons in the fully filled Landau level (filling fraction $\nu = 1$). It contains two competing contributions: the two-body repulsive interaction and the one-body confining potential, which holds the electrons together in the droplet. For large droplets of radius $R \rightarrow \infty$, this microscopic dynamics produces an effective dynamics for the edge excitations, which can be entirely expressed
in terms of the generators of the $W_{1+\infty}$ algebra, owing to the dynamical symmetry. The confining potential expands linearly in the generators $V_0^i$ of the Cartan subalgebra, while the two-body interaction is represented quadratically in the $V_k^i$. Both pieces in the effective Hamiltonian are infinite series, which can be organized according to a natural expansion in powers of $1/R$. Only a few leading terms are retained in the thermodynamic limit $R \to \infty$. Specifically, the leading contribution to the interaction term is shown to be quadratic in the Kac-Moody generators $V_n^0$, for a large variety of microscopic interactions. This result is the proof of the one-dimensional bosonization of edge excitations, which has been often used in the literature \cite{16} \cite{14}, but has only been confirmed by numerical studies so far \cite{17}. Moreover, the $W_{1+\infty}$ algebra describes the subleading corrections and the associated physical picture.

In section 4, we use the $W_{1+\infty}$ algebra to diagonalize the Hamiltonian derived in section 3, and compute the spectrum of quantum edge excitations. The confining potential always contributes to the leading order $O(1/R)$, yielding a scale invariant spectrum $\mathcal{E}(q) = vq$ (with $q \equiv \Delta J/R$ the edge momentum) \cite{2}. The interactions can contribute to the same order and modify the spectrum, or, otherwise, be negligible. The long-range interactions are of the first type. The short-range interactions, with range of the order of the magnetic length $\ell$ (the ultra-violet cutoff), belong to the second class.

We prove that the three-dimensional Coulomb interaction $V(|x - y|) = e^2/|x - y|$ modifies the spectrum into $\mathcal{E}(q) \propto q \log(q)$, due to the appearance of an infra-red logarithmic singularity. This result was anticipated by several analytic and numerical approximations \cite{18} \cite{19} \cite{20} \cite{17}. Although this spectrum is no more scale invariant, we stress that it can be still described by conformal field theory techniques: the kinematical $W_{1+\infty}$ structure of the Hilbert space of edge excitations can in fact be used for diagonalizing the Hamiltonian.

Next, we discuss the contribution to the spectrum given by a generic short-range interaction, with Gaussian fall-off for $|x| \gg \ell$. This contribution has the same form of the classical capillary frequency - a further indication of the semiclassical nature of the incompressible fluid \cite{3}. The capillary frequency depends on one phenomenological parameter, which represents the edge tension of the one-dimensional boundary. This parameter is found to be a particular moment of the generic short-range interaction.

The capillary energy, of order $O(1/R^3)$, is subdominant with respect to the leading contribution of the confining potential and Coulomb interaction. This shows that the spectrum of edge excitations is universal, i.e., independent of short-range effects. Universality implies experimental predictions that are parameter-free \cite{2}.
We conclude section 4 by extending our results to the Laughlin incompressible fluids with fractional filling $\nu = 1/3, 1/5, \ldots$. These are described by other representations of $W_{1+\infty}$ [4], and a Hamiltonian which is form-invariant.

Finally, in the conclusions we discuss the experiments that confirm the spectrum of edge excitations in the presence of Coulomb interactions [21, 22].

2. Dynamical symmetry and kinematics of incompressible fluids

In this section, we shall review the dynamical symmetry of chiral, two-dimensional, incompressible fluids and indicate how this leads uniquely to the construction of the Hilbert spaces of edge excitations. Next, we shall use their kinematical data to classify quantum Hall universality classes, without specifying the form of the Hamiltonian.

2.1. Classical fluids

A classical incompressible fluid is defined by its distribution function

$$\rho(z, \bar{z}, t) = \rho_0 \chi_{S_A(t)}, \quad \rho_0 \equiv \frac{N}{A}, \quad (2.1)$$

where $\chi_{S_A(t)}$ is the characteristic function for a surface $S_A(t)$ of area $A$, and $z = x + iy, \bar{z} = x - iy$ are complex coordinates on the plane. Since the particle number $N$ and the average density $\rho_0$ are constant, the area $A$ is also constant. The only possible change in response to external forces is in the shape of the surface. The shape changes at constant area can be generated by area-preserving diffeomorphisms of the two-dimensional plane. Thus, the configuration space of a classical incompressible fluid can be generated by applying these transformations to a reference droplet.

Next we recall the Liouville theorem, which states that canonical transformations preserve the phase-space volume. Area-preserving diffeomorphisms are, therefore, canonical transformations of a two-dimensional phase space. In order to use the formalism of canonical transformations, we treat the original coordinate plane as a phase space, by postulating non-vanishing Poisson brackets between $z$ and $\bar{z}$. We do this by defining the dimensionless Poisson brackets

$$\{f, g\} \equiv \frac{i}{\rho_0} (\partial f \bar{\partial} g - \bar{\partial} f \partial g), \quad (2.2)$$

where $\partial \equiv \partial/\partial z$ and $\bar{\partial} \equiv \partial/\partial \bar{z}$, so that

$$\{z, \bar{z}\} = \frac{i}{\rho_0}, \quad (2.3)$$
Note that the Poisson brackets select a preferred chirality, because they are not invariant under the two-dimensional parity transformation $z \rightarrow \bar{z}$, $\bar{z} \rightarrow z$; in the quantum Hall effect, the parity breaking is due to the external magnetic field.

Area-preserving diffeomorphisms, i.e., canonical transformations, are usually defined in terms of a generating function $L(z, \bar{z})$ of both “coordinate” and “momentum”, as follows:

$$
\delta z = \{L, z\}, \quad \delta \bar{z} = \{L, \bar{z}\}.
$$

A basis of (dimensionless) generators is given by

$$
L_{n,m}^{(cl)} \equiv \rho_0 \frac{n+m}{2} z^n \bar{z}^m.
$$

These satisfy the classical $w_\infty$ algebra

$$
\left\{ L_{n,m}^{(cl)}, L_{k,l}^{(cl)} \right\} = -i (mk - nl) L_{n+k-1, m+l-1}^{(cl)}.
$$

Let us now discuss how $w_\infty$ transformations can be used to generate the configuration space of classical excitations above the ground state. These configurations have a classical energy due to the inter-particle interaction and the external confining potential, whose specific form is not needed here. Let us assume a generic convex and rotation-invariant energy function, such that the minimal energy configuration $\rho_{GS}$ has the shape of a disk of radius $R$:

$$
\rho_{GS}(z, \bar{z}) = \rho_0 \Theta \left( R^2 - z\bar{z} \right),
$$

where $\Theta$ is the Heaviside step function. The classical "small excitations" around this ground state configuration are given by the infinitesimal deformations of $\rho_{GS}$ under area-preserving diffeomorphisms,

$$
\delta \rho_{n,m} \equiv \left\{ L_{n,m}^{(cl)}, \rho_{GS} \right\}.
$$

Using the Poisson brackets, we obtain

$$
\delta \rho_{n,m} = i \left( \rho_0 R^2 \right)^{\frac{n+m}{2}} (m-n) e^{i(n-m)\theta} \delta \left( R^2 - z\bar{z} \right).
$$

These correspond to density fluctuations localized on the sharp boundary (parametrized by the angle $\theta$) of the classical droplet. Due to the dynamics provided by the energy function, they will propagate on the boundary with a frequency $\omega_k$ dependent on the angular momentum $k \equiv (n-m)$, thereby turning into edge waves. These are the eigenoscillations of the classical incompressible fluid.
Another type of excitations are classical vortices in the bulk of the droplet, which correspond to localized holes or dips in the density. The absence of density waves, due to incompressibility, implies that any localized density excess or defect is transmitted completely to the boundary, where it is seen as a further edge deformation. For each given vorticity in the bulk, we can then construct the corresponding basis of edge waves in a fashion analogous to (2.8). Thus, the configuration space of the excitations of a classical incompressible fluid (of a given vorticity) is spanned by infinitesimal $w_\infty$ transformations. This is the property of dynamical symmetry described in the introduction.

2.2. Quantum theory at the edge

The quantum * version of the chiral, incompressible fluids is given by the Laughlin theory of the plateaus of the quantum Hall effect \[7\]. The simplest example of such a macroscopic quantum state is the fully filled Landau level (filling fraction $\nu = 1$). Generically, it possesses three types of excitations. First, there are gapless edge excitations, which are the quantum descendants of the classical edge waves described before. These are particle-hole excitations across the Fermi surface represented by the edge of the droplet; therefore, they are called neutral. They are also gapless because their energy, of $O(1/R)$, vanishes for $R \to \infty$ \[14\]. Second, there are localized quasi-particle and quasi-hole excitations, which have a finite gap. These are the quantum analogs of the classical vortices and correspond to the anyon excitations with fractional charge, spin and statistics \[7\]. As in the classical case, they manifest themselves as charged excitations at the edge, owing to incompressibility. The third type of excitations are two-dimensional density waves in the bulk, the magnetoplasmons and (for $\nu < 1$) the magnetophonons \[5\][23]. These have higher gaps and are not included in our effective field theory approach.

In the previous section, we have explained the connection between the classical edge waves and the generators of the algebra $w_\infty$ of area-preserving diffeomorphisms. In the quantum theory, there is a corresponding relation between edge excitations and the generators of the quantum version of $w_\infty$, called $W_{1+\infty}$ \[13\]. This algebra is obtained by replacing the Poisson brackets (2.2) with quantum commutators: $i\{ , \} \to [ , ]$, and by taking the thermodynamic limit \[2\].

In this limit, the radius of the droplet grows as $R \propto \ell \sqrt{N}$, where $\ell = \sqrt{2/\ell}B$ is the magnetic length and $B$ the magnetic field. Quantum edge excitations, instead, are confined

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* Throughout this paper we shall use units such that $c = 1$, $\hbar = 1$. 

6
to a boundary annulus of finite size $O(\ell)$. In order to construct an effective field theory for these low-lying excitations, the quantization should be followed by the limit of second-quantized operators into finite expressions localized on the boundary*. Let us consider, for example, the case of the fully-filled first Landau level. In ref. [2], it was shown that the boundary limit amounts to a straightforward expansion in powers of $1/R \propto 1/\ell\sqrt{N}$.

The quantum incompressible fluid, which corresponds to a Fermi sea of electrons, becomes effectively a Dirac sea. Correspondingly, the (2+1)-dimensional electrons on the boundary become Weyl fermions, i.e., relativistic, charged, chiral fermions in (1+1)-dimensions [9].

The field operator of electrons becomes**

$$F_R(\theta) = \frac{1}{\sqrt{R}} \sum_{k=\infty}^{\infty} e^{i(k-1/2)\theta} b_k , \quad (|z| = R , \ t = 0) , \quad (2.10)$$

where $\theta$ parametrizes the circular boundary, $b_k$ and $b_k^\dagger$ are fermionic Fock space operators satisfying $\{b_l, b_k^\dagger\} = \delta_l,k$, and $k$ is the angular momentum measured with respect to the ground state value.

The generators of the quantum algebra $W_{1+\infty}$ are represented in this Fock space by the bilinears

$$V^j_n = \int_0^{2\pi} \frac{d\theta}{2\pi} : F^\dagger(\theta) e^{-in\theta} g^j_n (i\partial_\theta) F(\theta) : \quad (2.11)$$

In this expression, $F(\theta) = F_R(\theta) e^{i\theta/2}\sqrt{R}$ is the canonical form of the Weyl field operator of conformal field theory. The factor $g^j_n (i\partial_\theta)$ is a $j$-th order polynomial in $i\partial_\theta$, whose form specifies the basis of operators and guarantees the hermiticity $(V^j_n)^\dagger = V_{-n}^j$. The coefficients $p(k, n, j)$ are also $j$-th order polynomials in $k$ to be specified later (see Appendix A). The $W_{1+\infty}$ algebra reads

$$[ V^i_n , V^j_m ] = (jn - im) V^{i+j-1}_{n+m} + q(i, j, m, n) V^{i+j-3}_{n+m} + \ldots + c^i(n) \delta^{i,j} \delta_{n+m,0} . \quad (2.12)$$

Here, $i + 1 = h \geq 1$ represents the “conformal spin” of the generator $V^i_n$, while $-\infty < n < +\infty$ is the angular momentum (the Fourier mode on the circle). The first

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* We refer to our previous papers for a complete discussion of quantization and the limit of the second-quantized operators to the boundary [4][4].

** Hereafter, we choose units such that $\ell = 1$. 

7
term on the right-hand-side of (2.12) reproduces the classical \( w_\infty \) algebra (2.6) by the correspondence \( e^{(cl)}_{i-n,i} \to V^i_n \) and identifies \( W_{1+\infty} \) as the algebra of “quantum area-preserving diffeomorphisms”. The additional terms are quantum operator corrections with polynomial coefficients \( q(i, j, n, m) \), due to the algebra of higher derivatives \([13]\). Moreover, the \( c \)-number term \( c^i(n) \) is the quantum anomaly, a relativistic effect due to the renormalization of operators acting on the infinite Dirac sea. It is diagonal in the spin indices for our choice of basis for the \( f^i_k \) (see Appendix A). Finally, the normal ordering \( \{ : : \} \) of the Fock operators takes care of the renormalization \([9]\).

Let us analyse the generators \( V^0_n \) and \( V^1_n \) of lowest conformal spin. From (2.11) we see that the \( V^0_n \) are Fourier modes of the fermion density evaluated at the edge \( |z| = R \); thus, \( V^0_0 \) measures the edge charge. Instead, the \( V^1_n \) are vector fields which generate angular momentum transformations on the edge, such that \( V^1_0 \) measures the angular momentum of edge excitations. Their algebra is given by

\[
[V^0_n, V^0_m] = c \ n \ \delta_{n+m,0},
\]

and

\[
[V^1_n, V^0_m] = -m \ V^0_{n+m},
\]

\[
[V^1_n, V^1_m] = (n - m) \ V^1_{n+m} + \frac{c}{12} \ (n^3 - n) \ \delta_{n+m,0},
\]

with \( c = 1 \). These equations show that the \( V^0_n \) and \( V^1_n \) operators satisfy an Abelian Kac-Moody algebra and a Virasoro algebra, respectively \([9]\).

Besides this explicit example for \( \nu = 1 \), leading to a theory with \( c = 1 \), it has been shown in general that the algebra (2.12) is the unique quantization of the \( w_\infty \) algebra in the \((1+1)\)-dimensional field theory on the circle \([24]\). For unitary theories, the Virasoro central charge \( c \) can be any \emph{positive integer} \([25]\). Moreover, there is a one-to-one correspondence between unitary irreducible representations of the Kac-Moody algebra (2.13) and those of the \( W_{1+\infty} \) algebra (2.12)*. This means that the operators \( V^0_n \), representing the charge modes on the edge, are the basic building blocks of the Hilbert space.

Following the standard procedure of two-dimensional conformal field theory, we define a \( W_{1+\infty} \) theory as the Hilbert space given by a set of irreducible, highest-weight representations of the \( W_{1+\infty} \) algebra, closed under the \emph{fusion rules} for making composite excitations \([9]\). Any representation contains an infinite number of states, corresponding to all the neutral excitations above a bottom state, the so-called \emph{highest weight} state. This

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* Apart from special cases, not relevant here \([26] \ [27] \).
can be, for example, the ground-state $|\Omega\rangle$ corresponding to the incompressible quantum fluid. The excitations can be written as

$$|k, \{n_1, n_2, \ldots, n_s\} \rangle = V_{-n_1}^0 V_{-n_2}^0 \cdots V_{-n_s}^0 |\Omega\rangle, \quad n_1 \geq n_2 \geq \cdots \geq n_s > 0,$$

while the positive modes ($n_i < 0$) annihilate $|\Omega\rangle$. Here $k = \sum_j n_j$ is the total angular momentum of the edge excitation.

Moreover, any charged edge excitation, together with its tower of neutral excitations, also forms an irreducible, highest-weight representation of $W_{1+\infty}$. The states in this representation have the same form of (2.15), but the bottom state $|Q\rangle$ now represents a quasi-particle inside the droplet. The charge and spin of the quasi-particle are given by the eigenvalues of the operators* $(-V_0^0)$ and $V_0^1$ respectively:

$$V_0^0 |Q\rangle = Q |Q\rangle, \quad V_0^1 |Q\rangle = J |Q\rangle.$$

Actually, all the operators $V_i^0$ are simultaneously diagonal and assign other quantum numbers to the quasi-particle, $V_i^0 |Q\rangle = m_i(Q) |Q\rangle$, $i \geq 2$, which are known polynomials in the charge $Q$ (see Appendix A). These quantum numbers measure the radial moments of the charge distribution of a quasi-particle (see eq. (3.18) below); their fixed functional form indicates the rigidity of density modulations of the quantum incompressible fluid.

All $W_{1+\infty}$ representations, together with their quantum numbers, were recently obtained in ref. [23]. In ref. [1], we used this representation theory to construct an algebraic classification of universality classes of incompressible quantum fluids, which was found to encompass the general hierarchy of plateaus in the quantum Hall effect [12]. The incompressible quantum fluids are characterized by the following kinematical data:

i) the integer central charge $c = m, m \in \mathbb{Z}$, of the $W_{1+\infty}$ algebra;

ii) the (fractional) charges $Q$ of quasi-particle excitations;

iii) their (fractional) spin $J$ (corresponding to fractional statistics $\theta/\pi = 2J$);

iv) the value of the filling fraction $\nu$, given by the Hall conductivity as $\sigma_{xy} = (e^2/h) \nu$.

Here we wish to stress that the $W_{1+\infty}$ dynamical symmetry characterizes completely these kinematical properties of incompressible quantum fluids, without reference to a specific Hamiltonian. Note that fractional statistics is also independent of the dynamics, being a

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* The minus sign is due to the fact that $V_0^0$ measures the charge on the edge. Due to overall charge conservation, the charge of a quasi-particle in the bulk has the opposite sign of its edge counterpart.
static property of correlators on the circle, \(i.e.,\) in the plane. Moreover, these informations are sufficient to determine \(\sigma_{xy},\) and thus to describe the Hall conduction experiments. Actually, the Hall current is computable from the chiral anomaly of the \((1+1)\)-dimensional edge theory \([2]\), which is encoded in the charge commutator eq. (2.13) (see also \([1]\)).

Let us finally mention two more kinematical data of the Hilbert space which were used extensively by X.-G. Wen and other authors to identify and characterize the edge field theory \([14][28]\). These are:

i) the number of neutral edge states, \(i.e.,\) the degeneracy of the states (2.15) at fixed angular momentum \(k\), which was used to match the edge theory to numerical spectra of interacting electrons in the lowest Landau level;

ii) the topological order, \(i.e.,\) the degeneracy of the ground state on a toroidal geometry, which is a consequence of the fusion rules for Virasoro representations \([9]\).

In summary, the behaviour of quantum incompressible fluids at long distances is basically characterized by the above kinematical data, which are expressions of the structure of the Hilbert space in terms of representations of the \(W_{1+\infty}\) algebra. These data specify the universality classes for quantum Hall experiments, where the dynamics (spectrum) of edge excitations does not play any relevant role.

The characterization of the Jain sub-hierarchy \([29]\), which describes the plateaus most prominently observed in the experiments, remains an open issue.

3. \(W_{1+\infty}\) dynamics of edge excitations

In the previous section, we have restricted ourselves to kinematical considerations. In the following, we are going to discuss the dynamics of the edge excitations. We start from the classical problem, the spectrum of eigenfrequencies \(\omega_k\) of eigenoscillations.

3.1. Classical capillary waves

The dynamics of the classical incompressible fluid is fixed by a single parameter \(\tau\), with dimension of action, whose role is to translate between the purely kinematical, dimensionless Poisson brackets (2.2) and the actual Poisson brackets (with dimension \((1/\text{action})\))

\[
\partial_t \delta \rho_{n,m} = \frac{1}{\tau} \{ \delta \rho_{n,m}, H \} = i \omega_k \delta \rho_{n,m}, \quad k = n - m .
\]

The parameter \(\tau\) is not generic, but depends on the specific model under consideration. In our case, it will be given by the classical limit of the Laughlin incompressible fluids.
First we study a Hamiltonian which contains only a rotational invariant confining potential \( V(|x|) \). From eq. (3.1), we obtain the eigenfrequencies,

\[
\omega_k = \frac{V'(R)}{2R\tau \rho_0} k ,
\]

where the prime denotes differentiation with respect to the radial variable \( r = |x| \). Note the sign correlation between \( k \) and \( \omega_k \), which means that the edge waves propagate only in one direction along the one-dimensional edge, \textit{i.e.}, they are \textit{chiral}.

For applications to the quantum Hall effect, one should consider a Hamiltonian also containing the two-body repulsive interaction. A known method for handling the interaction at the classical level is the \textit{hydrodynamic approximation} \cite{15,30}. The two-body interaction is replaced by the one-body effect of a \textit{boundary pressure} arising from the imperfect cancellation of the repulsive forces near the edge of the sample. We expect this approximation to be good for inter-particle interactions which are not too long ranged, as in a liquid. This expectation will be confirmed by the quantum dynamics discussed below.

We thus have the following decomposition of the boundary force:

\[
V'(R) = V_{cp}'(R) + V_p'(R) .
\]

The pressure contribution to the frequencies \( \omega_k \) is known as \textit{capillary frequency} \cite{15}, and can be computed as follows. The radial force due to the interactions is given by \( F_p(r) = -p'(r)/\rho_0 \), where \( p(r) \) is the pressure. Therefore, we have \( V_p'(R) = p'(R)/\rho_0 \). Let us consider an harmonic edge oscillation parametrized by \( r = R + \epsilon(\theta) \), with \( \epsilon(\theta) = \epsilon_0 \sin(k\theta) \), and expand

\[
p(r) = p(R) + p'(R)\epsilon .
\]

We now use the following representation of the boundary pressure:

\[
p = -\frac{\alpha}{R_c} = -\alpha \frac{|\partial_\theta x \wedge \partial_\theta^2 x|}{|\partial_\theta x|^3} = -\alpha \frac{2(\partial_\theta \epsilon)^2 + r \left( r - \partial_\theta^2 \epsilon \right)}{\left( r^2 + (\partial_\theta \epsilon)^2 \right)^{\frac{3}{2}}} .
\]

where \( \alpha \) is the \textit{edge tension}, which is positive for repulsive interactions, and \( R_c \) represents the local radius of curvature. Expanding this expression up to first order in \( \epsilon \) we find,

\[
-p(R + \epsilon) = \frac{\alpha}{R} - \frac{\alpha}{R^2} \left( \epsilon + \partial_\theta^2 \epsilon \right) + O\left( \frac{\epsilon^2}{R^3} \right) .
\]
Using $\partial_\theta^2 \epsilon = -k^2 \epsilon$, we obtain
\[ V'_p(R) = -\frac{\alpha}{R^2 \rho_0} (k^2 - 1) \, , \tag{3.7} \]
and the frequencies
\[ \omega_k = \frac{V'_{cp}(R)}{2R^2 \tau \rho_0} k - \frac{\alpha}{2R^3 \tau \rho_0^2} k(k^2 - 1) \, . \tag{3.8} \]

As a last step, it remains to determine the value of $\tau$ relevant to the classical limit of the Laughlin incompressible quantum fluids. Consider the Lagrangian for planar electrons of mass $m$ in an external magnetic field $B$,
\[ L = \sum_{i=1}^N \left( \frac{m}{2} \dot{x}_i^2 + e \mathbf{x}_i \cdot \mathbf{A}(\mathbf{x}_i) \right) \, , \tag{3.9} \]
in the symmetric gauge $\mathbf{A}(\mathbf{x}) = \frac{B}{2} (-y, x)$, with $B > 0$. The Laughlin fluids involve electrons in the first Landau level only. Thus, we can project out the higher Landau levels in the limit of large $B$. As was shown in [31], this projection can be implemented by taking the limit $m \to 0$ of (3.9). The residual dynamics is then governed by the following Lagrangian:
\[ L_R = \lim_{m \to 0} L = \frac{eB}{2} \sum_{i=1}^N \mathbf{x}_i \wedge \dot{\mathbf{x}}_i \, . \tag{3.10} \]
This Lagrangian is of first order in time derivatives, which implies that the projection on a subspace of constant (kinetic) energy amounts to a phase space reduction from a $4N$-dimensional phase-space to a $2N$-dimensional phase-space. The new symplectic structure is given by [31]
\[ \{ z_i, \bar{z}_j \}_{PB} = i \frac{2}{eB} \delta_{ij} \, , \tag{3.11} \]
and shows that the original coordinate plane of the electrons behaves indeed as a phase space for motions at constant kinetic energy. By comparing (3.11) with (2.2) we obtain the explicit expression for the parameter $\tau$:
\[ \tau = \frac{eB}{2 \rho_0} \, . \tag{3.12} \]
Inserting this in (3.8), we obtain the final result for the dispersion relation of the classical edge waves:
\[ \omega_k = \frac{v}{R^2} k - \frac{\alpha}{eBR^2 \rho_0} k(k^2 - 1) \, , \quad v = \frac{V'_{cp}(R)}{eB} \, . \tag{3.13} \]
This formula can be also obtained by solving the hydrodynamic equations in the presence of a magnetic field [15][20]. Here, we showed that it can be derived in a much simpler way after realizing that the kinematics of the incompressible classical fluid is specified by $w_\infty$ transformations*.

Equation (3.13) has some properties which are worth stressing and must be kept in mind for the quantum dynamics. First, there are zero modes, which correspond to symmetries: the $k = 0$ radial breathing mode of the droplet is clearly absent in an incompressible fluid, as is evident from (2.3). In addition, a translation invariant two-body interaction gives rise to a vanishing capillary frequency for the modes $k = \pm 1$, which have $\epsilon(\theta) \propto \sin \theta$ and thus correspond to an overall translation of the droplet. Another general property of the classical fluid is the degeneracy of its eigenoscillations, as it is also apparent from eqs. (2.3) and (3.1): although a generic $w_\infty$ deformation of the droplet is parametrized by two integers $(n, m)$, the frequency depends on their difference $k$ only. This is the consequence of the sharp boundary of the classical droplet, which forbids infinitesimal radial fluctuations. On the contrary, the quantum incompressible fluid has a smooth boundary, and radial fluctuations produce a hierarchy of dynamical effects, which are described by the action of the quantum algebra $W_{1+\infty}$.

Finally, we note that $\omega_k$ contains two phenomenological parameters $v$ and $\alpha$, in addition to the generic parameters $\rho_0$, $A = \pi R^2$ and $B$. For large droplets, the confining potential term dominates over the capillary frequency and the dispersion relation becomes of relativistic nature, with effective light-velocity $v$.

3.2. Quantum hamiltonian in the first Landau level

In the following, we shall discuss the quantum Hamiltonian for the incompressible fluid at filling $\nu = 1$. We consider the microscopic model of planar electrons confined to the first Landau level. This projection quenches the kinetic energy [32], so that the Hamiltonian contains only the confining potential $V_{CP}(|x|)$ and the repulsive two-body interaction $V_I(|x - y|),$

$$H = H_{CP} + H_I = \int d^2x \ V_{CP}(x) \ \rho(x) + \frac{1}{2} \int d^2x \int d^2y \ \rho(x) \ \rho(y) \ V_I(x - y) \ \rho(y), \quad (3.14)$$

* Note, however, that the hydrodynamic equations also give a higher, gapful branch of frequencies for compression waves corresponding to the magneto-plasmon.
where \( \rho(x) \equiv \Psi^\dagger(x)\Psi(x) \) is the density operator, constructed from the field operator

\[
\Psi(x) = \sum_{j=0}^{\infty} a_j \psi_j(x). \tag{3.15}
\]

Here, \( a_j \) and \( a_j^\dagger \) are fermionic Fock space annihilators and creators, satisfying the usual anticommutation rules \( \{a_l, a_k^\dagger\} = \delta_{l,k} \), with all other anticommutators vanishing, and \( \psi_j(x) \) are the first Landau level wave functions

\[
\psi_j(x) = \frac{1}{\ell \sqrt{\pi N^j}} \left( \frac{z}{\ell} \right)^j e^{-|z|^2/2\ell^2}. \tag{3.16}
\]

In this equation, we have restored the magnetic length \( \ell \), for the sake of completeness.

In the following, we shall write the Hamiltonian in terms of quantum \( W_{1+\infty} \) operators, which will be used later to find the spectrum of quantum edge excitations. The operators producing the quantum analogs of the classical deformations \( \delta_{n,m} \rho \) have been defined in ref. [1], by quantizing the classical generators (2.5) in the first Landau level:

\[
L_{n,m} = \int d^2x \, \Psi^\dagger(x) \left( b^\dagger \right)^n (b)^m \Psi(x), \tag{3.17}
\]

These operators satisfy a “non-relativistic” version of the quantum algebra \( W_{1+\infty} \) [1] [33]. The operators \( L_{n,m} \) are mutually commuting, as in the classical case, and can be simultaneously diagonalized. The physical meaning of their eigenvalues can be inferred from the expectations values

\[
\langle L_{n,n} \rangle = (-1)^n 2\pi n! \int_0^\infty dr \, r \, L_n(r^2) \langle \rho(r^2) \rangle, \tag{3.18}
\]

where the \( L_n(x) \) are the Laguerre polynomials [34]. Therefore, they measure certain radial moments of the density distribution in any given quantum state. In order to describe quantum numbers of excitations, they have to be normal ordered by subtracting the ground-state expectation values, \( L_{n,n} \rightarrow : L_{n,n} : = L_{n,n} - \langle \Omega | L_{n,n} | \Omega \rangle \), where \( |\Omega\rangle \) is the filled Fermi sea

\[
|\Omega\rangle = a_0^\dagger \cdots a_N^\dagger |0\rangle, \tag{3.19}
\]

of \( (N+1) \) electrons. Given that \( \langle \Omega | L_{n,n} | \Omega \rangle = O(N^{n+1}) \), the \( L_{n,m} \) operators, as well as their “non-relativistic” algebra, undergo a severe renormalization in the large \( N \) limit. It
can be shown that specific linear combinations $\sum_{n=0}^{i} \lambda_n \mathcal{L}_{n-k,n}$ have a finite limit as the normal-ordered boundary operators $V_k^i$ introduced in (2.11), satisfying the $W_{1+\infty}$ algebra (2.12). Therefore, we shall expand the Hamiltonian in the basis of the operators $V_k^i$.

Let us first consider the one-body term $H_{CP}$. As discussed in ref. [2], for $\nu = 1$ the geometrical (semi)-classical picture of edge waves has a simple correspondence in the Fock space of fermions. Edge waves are particle-hole excitations above the filled Fermi sea (3.19) with angular momentum $k = \Delta J \sim O(1)$, $|k| \ll \sqrt{N}$ [2]. The boundary field theory is constructed by operators which measure these transitions and remain finite for large $N$.

Let us consider the Fock expression of $H_{CP}$ and expand the confining potential in a power series:

$$V_{CP}(|x|) = \gamma_0 + \gamma_1 |x|^2 + \gamma_2 |x|^4 + \cdots ,$$

$$H_{CP} = \sum_{k=0}^{\infty} \left[ \gamma_0 + \gamma_1 (k+1) + \gamma_2 (k^2 + 3k + 2) + \cdots \right] a_k^\dagger a_k .$$

Following ref. [2], it is convenient to redefine the Fock operators by shifting the angular momentum index,

$$b_r = a_{N+r} , \quad b_k^\dagger = a_k^\dagger ,$$

such that the first empty state has index $r = 1$. For large $N$, the excitations of electrons from the deep inside of the droplet have high energy and can be neglected. Thus, we can extend the summation in (3.21) from $k \in [-N, \infty)$ to $k \in (-\infty, +\infty)$, i.e., consider a relativistic Dirac sea. We can now compare $H_{CP}$ in (3.21) to the explicit Fock space expression of the $W_{1+\infty}$ operators (2.11) (see ref. [2] and appendix A),

$$V_{n}^0 = \sum_{r=-\infty}^{\infty} :b_{r-n}^\dagger b_r: ,$$

$$V_{n}^1 = \sum_{r=-\infty}^{\infty} \left( r - \frac{n+1}{2} \right) :b_{r-n}^\dagger b_r: ,$$

$$V_{n}^2 = \sum_{r=-\infty}^{\infty} \left( r^2 - (n+1)r + \frac{(n+1)(n+2)}{6} \right) :b_{r-n}^\dagger b_r: ,$$

and obtain the desired expansion,

$$H_{CP} - \langle \Omega | H_{CP} | \Omega \rangle = \gamma_0 V_0^0 + \gamma_1 \left( V_0^1 + \left( N + \frac{3}{2} \right) V_0^0 \right)$$

$$+ \gamma_2 \left( V_0^2 + 2 (N+2) V_0^1 + \left( (N+2)^2 - \frac{1}{3} \right) V_0^0 \right) + \cdots .$$
In general, confining potentials growing as \( V_{CP} = O(|x|^{2i}) \) involve all operators \( V_0^j \) with \( j \leq i \).

We consider the confining potential \( V_{CP} \) to be generated dynamically at the edge of the sample. Therefore, we expect a self-tuning of the parameters \( \gamma_i \) for large \( N \equiv R^2 \), such that the normal-ordered energy is finite when written in terms of the quantum numbers \( \langle V_0^i \rangle \) of edge excitations. We thus consider the normal-ordered Hamiltonian

\[
H_{CP} = \alpha_0 (V_R)_0^0 + \alpha_1 (V_R)_0^1 + \alpha_2 (V_R)_0^2 + \ldots , \quad N \to \infty ,
\]

where the \((V_R)_0^i\) are the generators of \( W_{1+\infty} \) properly normal-ordered on the cylinder \((R\theta,t)\). They are related to the \( V_0^i \) of eq. (2.11), conventionally defined on the conformal plane, by well-known transformation rules [35] (see Appendix A). The resulting expression is

\[
H_{CP} = \alpha_0 V_0^0 + \frac{\alpha_1}{R} (V_0^1 - \frac{c}{24}) + \frac{\alpha_2}{R^2} \left( V_0^2 - \frac{1}{12} V_0^0 \right) + O \left( \frac{1}{R^3} \right) , \quad R \equiv \sqrt{N} .
\]

Note that the \((V_R)_0^i\) are dimensionful, because their eigenvalues are of order \( O( (k/R)^i ) = O(q^i) \) for states of angular momentum \( k \) and edge momentum \( q \), as explained in section 4. A sensible thermodynamic limit of the edge theory is achieved if the phenomenological coefficients \( \alpha_i \) are asymptotically independent of \( R \) (modulo logarithmic corrections). In this case, the higher-spin operators \( V_0^i \quad (i \geq 2) \) measure the deviation from the massless, critical spectrum \( E_k = vk/R = vq \).

3.3. Two-body interactions

The interaction term \( H_I \) of the Hamiltonian (3.14) can be treated similarly. In Fock space, it reads

\[
H_I = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=-m}^{n} \frac{1}{2} (M(n,m;k) - M(n,m;n-m-k) ) \ a_{m+k}^\dagger a_{n-k}^\dagger a_n a_m ,
\]

\[
M(n,m;k) \equiv \frac{1}{2} \int d^2 z_1 \int d^2 z_2 \ \psi_{n-k}^*(z_1, \bar{z}_1) \psi_n(z_1, \bar{z}_1) \ V_I(|z_1 - z_2|) \cdot \\
\psi_{m+k}^*(z_2, \bar{z}_2) \psi_m(z_2, \bar{z}_2) ,
\]

where we have explicitly anti-symmetrized the matrix element. We study again \( H_I \) in the large \( N \) limit for transitions corresponding to edge excitations, which involve indices.
\( n = N + r \) and \( m = N + s \) with \(|r|, |s|, |k| \sim O(1)\). In these cases, the “transition density” describing each virtual electron in (3.27) can be approximated as follows:

\[
\rho_{s,k}(z, \bar{z}) \equiv \psi_{N+s+k}^*(z, \bar{z}) \psi_{N+s}(z, \bar{z}) \simeq \frac{1}{\pi \sqrt{2\pi N}} e^{ik\theta} \exp \left( -\frac{(|z|^2 - (N + s + k)^2)}{2N} \right),
\]

(3.28)

where \( z = |z| e^{i\theta} \). The phase of this expression is the chiral edge wave of momentum \( k \). The modulus has the shape of a Gaussian, with center located at \( |z| \simeq R + (2s + k)/4R \), and spread \( \Delta |z| = O(1) \). This shows that all virtual electrons participating in edge transitions are localized on a boundary annulus of the size of the ultra-violet cut-off. We can develop some physical intuition on the form of the matrix element by thinking of the interaction between the densities of the two electrons through their phases (index \( k \)) and modulus (radial) fluctuations (indices \( r, s \)).

Consider in the first place the (three-dimensional) Coulomb potential*

\[
V(|z_1 - z_2|) = \frac{e^2}{|z_1 - z_2|}.
\]

(3.29)

The two integrals in (3.27) are effectively restricted to the annular region. One can distinguish two contributions: when \(|z_1 - z_2| > 1\), i.e., the relative angle is \(|\theta| = |\theta_1 - \theta_2| > 1/R\), and its complement. In the first case, the two densities (3.28) are far apart, so that their radial fluctuations of \( O(1/R) \) can be neglected. This allows the further approximation,

\[
\rho_{s,k}(z, \bar{z}) \simeq \frac{1}{2\pi R} e^{ik\theta} \Theta \left( (|z| - R) (R + 1 - |z|) \right).
\]

(3.30)

Therefore, the matrix element has only one non-trivial integration over the relative angle:

\[
M(k) \equiv M(N + r, N + s; k) \simeq \frac{e^2}{2R} \int_{1/R}^{\pi} \frac{d\theta}{2\pi} \frac{\cos(k\theta)}{|\sin \frac{\theta}{2}|}.
\]

(3.31)

In the other region, \(|z_1 - z_2| < 1\), the integrals are fully two-dimensional. Nevertheless, the ultra-violet behaviour of (3.29) is integrable and the result is subleading with respect to (3.31). The integral in (3.31) can be computed by a recursion relation in \( k \), and reads**,

\[
M(k) = M(|k|) \simeq M(0) - \frac{e^2}{\pi R} \sum_{l=1}^{k} \frac{1}{2l - 1}.
\]

(3.32)

* We shall take the dielectric constant equal to one.

** We have also obtained this result by more accurate two-dimensional integrations.
The Coulomb matrix element \( M(k) \) exhibits a logarithmic behaviour for \( k \gg 1 \),

\[
M(k) = M(0) - \frac{e^2}{2\pi R} \log k, \quad k \gg 1, \tag{3.33}
\]

which is accessible within the validity of the edge wave approximation \( k \ll R \). This behaviour is due to the infrared singularity of the electrostatic energy of charges in the annular region. Note that this singularity cannot be screened by the ion background, which ensures charge neutrality over distances larger than \( R \).

For interactions that are not too long-ranged, both phase and radial fluctuations of \( \rho_{s,k}(z,\bar{z}) \) could be relevant in (3.27), and give an involved dependence of the matrix element on \( (r,s,k) \). We can establish a general result in the opposite regime of (very) short-range interactions, whose range is of the order of the ultra-violet cut-off. A convenient way to expand these potentials is given by the Gaussian basis of Ref. \[32\]

\[
V_I(|z_1 - z_2|) = \int_0^\infty dt \, f(t) \, e^{-t|z_1 - z_2|^2}. \tag{3.34}
\]

For short-range interactions, \( f(t) \) has a compact support over values of \( t \simeq O(1) \). The matrix element of the Gaussian interaction,

\[
M_t(N + r, N + s; k) \equiv \frac{1}{2} \int d^2z_1 \int d^2z_2 \, \psi^*_N+r-k(z_1,\bar{z}_1) \psi_N+r(z_1,\bar{z}_1) \, e^{-t|z_1 - z_2|^2} \cdot \psi^*_N+s+k(z_2,\bar{z}_2) \, \psi_N+s(z_2,\bar{z}_2), \tag{3.35}
\]

can be evaluated for \( N \to \infty \) using the saddle-point technique, which is equivalent to the approximation in (3.28) (for details see Appendix B). The resulting expression is

\[
M_t(N + r, N + s; k) \simeq \frac{1}{\sqrt{N}} \frac{1}{4\sqrt{\pi}} \sqrt{\frac{1}{t(1+t)}} \left\{ 1 + \frac{1}{4N} \left[ -\frac{1}{t} \left( k^2 - \frac{1}{4} \right) \right. \right. \\
+ 2 \left( k(r-s-k) - r^2 - s^2 - \frac{r + s}{2} \right) \\
\left. \left. + \frac{1}{1+t} \left( k^2 - 2k(r-s) + 2(r^2 + s^2) + t(r+s)^2 \right) \right] \right\}. \tag{3.36}
\]

One can verify that this expression is symmetric under the interchange of the two particles, \( (r,s,k) \leftrightarrow (s,r,-k) \). Moreover, the anti-symmetric matrix element in (3.27) becomes

\[
\frac{1}{2} (M_t(k) - M_t(r-s-k)) = -\frac{1}{R^3 32\sqrt{\pi}} \frac{(1+2t)}{(t(1+t))^{3/2}} \left[ k^2 - (r-s-k)^2 \right]. \tag{3.37}
\]
We can now insert this result back into the Hamiltonian (3.27) and exploit again the antisymmetry in the indices $k$ and $(r - s - k)$, to obtain a matrix element which depends on $k$ only, as in the case of the Coulomb interaction. Actually, eq. (3.37) has the unique index structure, at most quadratic in $(r, s, k)$, which has the above symmetries.

The next step is to rewrite the Hamiltonian in terms of $W_{1+\infty}$ generators, as done for $H_{CP}$ (see eqs. (3.23)-(3.25)). For the Coulomb and short-range interactions, equation (3.27) takes the form

$$H_I = \sum_{r=-N}^{\infty} \sum_{s=-N}^{\infty} \sum_{k=-N-s}^{N+r} \frac{\lambda_k}{2} b_{s+k}^\dagger b_{r-k}^\dagger b_r b_s ,$$

where the $b$'s are the Fock operators with shifted index (3.22), and $\lambda_k/2$ is the matrix element in the limit of edge excitations, (3.32) and (3.37). Due to the explicit antisymmetrization of (3.27), however, these matrix elements are determined up to a $k$-independent additive term, which vanishes in any expectation value. This freedom will be useful later.

We manipulate eq. (3.38) by letting $N \to \infty$ in the summation extrema and by replacing the four Fermi operators with a pair of $V_0^k$ operators (3.23). Moreover, we introduce the normal ordering $:\quad$ with respect to the incompressible ground state (3.19). This subtracts a one-body operator (assigned to $H_{CP}$) and a constant from (3.38). The first subtraction gives,

$$H_I = \sum_{k=-\infty}^{\infty} \frac{\lambda_k}{2} : V_0^k V_0^k : .$$

The second subtraction requires some knowledge of $W_{1+\infty}$ representations. Let us recall from Section 2.2, eq. (2.15), that the ground state is the highest weight state of a $W_{1+\infty}$ representation, which satisfies the conditions $V_k^i|\Omega\rangle = 0$, for $k \geq 0$ and $i \geq 0$. Therefore, the normal ordered expression is found to be*:

$$H_I = \frac{\lambda_0}{2} (V_0^0)^2 + \sum_{k=1}^{\infty} \lambda_k V_0^0 V_k^0 .$$

An important property of the microscopic Hamiltonian (3.27) is the translation invariance of two-body interactions. This is enforced on the edge approximation (3.40) as a further normal ordering condition, which fixes the up to now free additive constant in $\lambda_k$.

---

* Note that finite-size effects are absent in this formula, in particular $(V_R)_k^0 = V_k^0$ (see appendix A).
As pointed out in section 3.2, the edge excitations with \( k = \pm 1 \) correspond to infinitesimal overall translations of the droplet. At the quantum level, they are generated by \( V_{\pm 1}^0 \). Given that the translation of any state should cost no energy, we require

\[
[ H_I , V_{\pm 1}^0 ] = 0 \quad \rightarrow \quad \lambda_1 = 0 , \quad \text{(translation invariance)} .
\] (3.41)

In conclusion, the Hamiltonian for edge excitations with the Coulomb and short-range interactions is given by (3.40), with

\[
\lambda_k = -\frac{2e^2}{\pi R} \sum_{l=2}^{k} \frac{1}{2l-1} \simeq -\frac{e^2}{\pi R} \log k , \quad \left( \lambda_0 = \frac{2e^2}{\pi R} \right) , \quad \text{(Coulomb interaction)},
\]

\[
\lambda_k = -\frac{k^2-1}{8\sqrt{\pi R^3}} \int_{0}^{\infty} dt \, f(t) \, \frac{(1+2t)}{(t(1+t))^{3/2}} , \quad \text{(short-range interactions)}.
\] (3.42)

Let us make a few remarks on the result (3.40) for the edge Hamiltonian. This form for \( H_I \) has been assumed in phenomenological and semiclassical descriptions of the edge excitations \[16\] [14] [19]: a one-dimensional chiral boson \( \varphi(R\theta - vt) \), which expresses the fluctuations of the density at the edge by \( \rho(\theta) = -\frac{1}{2\pi} \partial_\theta \varphi(\theta) \) \[2\], is introduced. Moreover, its Hamiltonian is taken to be

\[
H_I^{\text{(boson)}} = \frac{1}{2} \int_{0}^{2\pi} d\theta_1 \int_{0}^{2\pi} d\theta_2 \rho(\theta_1) \, V_I^{(eff)}(|\theta_1 - \theta_2|) \, \rho(\theta_2) .
\] (3.43)

Given that the chiral boson carries a representation of the Kac-Moody algebra (2.13), we can express \( \rho(\theta) = \sum_k \exp(ik\theta) \, V_k^0 \) and recover the operator form in (3.40). In this description, as well as in the semiclassical limit, the radial degree of freedom is completely neglected, thus the matrix element can only depend on the Fourier mode \( k \) on the edge. This bosonization of the edge waves in the quantum Hall effect was predicted by Stone \[16\] and later verified numerically \[17\].

Our derivation can be regarded as the first proof of the bosonization, for both the Coulomb and generic short-range interactions. Moreover, the \( W_{1+\infty} \) dynamical symmetry encodes the physical picture and the subleading corrections. The bosonization formula (3.40) and the operators \( V_k^0 \) describe the leading semiclassical effect of phase fluctuations at the edge. Corrections can have the following structure: equation (3.18) shows that the \( V_i^0, \ i > 0 \), measure the radial fluctuations of the density at the edge (3.28), which appear as subleading corrections in the \( 1/R \) expansion of the matrix elements. These corrections
can be expressed in terms of the $V^i_k$, $i > 0$, via their Fock expressions (3.23). For example, the first subleading correction to (3.40) has the form

$$H_I^{(sub)} = \sum_{k > 0} \eta_k \left( V^0_{-k} V^1_k + V^1_{-k} V^0_k \right)$$

$$= \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \rho(\theta_1) V_I^{(sub)}(|\theta_1 - \theta_2|) : \rho(\theta_2)^2 : .$$ \hspace{1cm} (3.44)

The bosonization of edge waves in the presence of the Coulomb interaction is rather easy to understand, because radial fluctuations in (3.28), of size $O(1/R)$, are subleading for long-range $O(R)$ interactions. The bosonization of short-range interactions is, instead, due to the semiclassical nature of the incompressible fluid. As we show in the next section, their spectrum actually reproduces the classical capillary frequencies $\omega_k$ of section 3.1. Although a priori radial excitations could be important at short distances, they are quenched by incompressibility, which is enforced by Fermi statistics (see eq. (3.37)).

In conclusion, the bosonization of edge excitations is a semiclassical effect. Here we provided both a physical picture and a technique to derive it from the microscopic quantum theory. Moreover, we indicated the generic form and the origin of subleading corrections to the bosonization formula.

4. The spectrum of edge excitations

Up to now we have derived the generic form of the Hamiltonian $H = H_{CP} + H_I$ governing the dynamics of quantum edge excitations for $\nu = 1$, and we have shown that it can be written entirely in terms of $W_{1+\infty}$ generators. The piece $H_{CP}$, describing the effect of the confining potential, is a linear combination of the generators $V^i_0$ of the Cartan subalgebra, see eq. (3.26). The piece $H_I$, describing the two-body interactions, is quadratic, see eq. (3.40).

The diagonalization of $H$ follows in principle from the algebraic properties of the operators $V^i_k$ only, owing to the dynamical $W_{1+\infty}$ symmetry. In practice, however, the diagonalization is rendered difficult by the fact that

$$[ V^i_0 , H_I ] \neq 0 , \quad i \geq 2 .$$ \hspace{1cm} (4.1)

This implies that there is no general basis of quantum edge excitations, which diagonalizes simultaneously all terms of the confining potential and the interaction. Actually, each
contribution to the Hamiltonian is diagonal in a known basis: the operators \((V^0_0, V^1_0, H_1)\) are diagonal in the standard *bosonic* basis \((2.13)\); the Cartan subalgebra \((V^i_i, i \geq 0)\), is diagonal in the *fermionic* basis to be discussed later. Therefore, we shall analyse each interaction separately, choosing the most appropriate basis according to the dominant operators in the \((1/R)\) expansion.

4.1. Coulomb interaction

The Coulomb interaction \(V(x) = e^2/|x|\) gives rise to an interacting Hamiltonian of order \(O(1/R)\), eqs. \((3.40)\) \((3.42)\). In this case, the generators \(V^i_i, i \geq 2\), in the expansion of \(H_{CP}\) lead to subdominant contributions to the spectrum for large \(R\) and can be dropped. The only term which must be retained is the leading \(1/R\) contribution, which is proportional to \(V^1_0\). We thus consider the Hamiltonian:

\[
H^{(\text{Coulomb})} = \alpha_0 V^0_0 + \frac{\alpha_1}{R} \left( V^1_0 - \frac{c}{24} \right) + \frac{\lambda_0}{2} (V^0_0)^2 + \sum_{k=1}^{\infty} \lambda_k V^0_{-k} V^0_k + O\left( \frac{1}{R^2} \right), \tag{4.2}
\]

where the conformal central charge for \(\nu = 1\) is \(c = 1\), and \(\lambda_k \simeq -\left( e^2/\pi R \right) \log k, k \gg 1\). A contribution to the confining potential is given by the Coulomb interaction of the electrons with the neutralizing ion background, which has also a droplet shape. This determines \[20\] the phenomenological quantity \(\alpha_1 = \left( e^2/\pi \right) \log \gamma R\), where \(\gamma\) is a non-universal parameter which depends on the details of the droplet edge *. Other contributions to the confining potential can be included in \(\gamma\).

Let us now analyse the spectrum of \((4.2)\) for edge excitations above the ground state \(|\Omega\rangle\). Since \(V^0_0 = 0\) for all these excitations, it drops out of eq. \((4.2)\). The Hamiltonian is diagonal in the bosonic basis \((2.13)\), because the Kac-Moody generators satisfy the ladder relations \([H, V^0_{-k}] \propto V^0_{-k}\) (see eqs. \((2.13)\) \((2.14)\)). The c-number term in \((4.2)\),

\[
E_0 = -\frac{e^2}{24\pi R} \log \gamma R, \tag{4.3}
\]

represents the finite-size Casimir energy of the ground state \([2]\).

* Note that the term proportional to \(V^1_0\) in \((4.2)\) does have the correct \(1/R\) dependence anticipated in \((3.26)\), but also a logarithmic correction, due to the Coulomb infra-red singularity discussed in the previous section.
The fundamental excitations in the bosonic basis are the one-boson states $V_0^0 |\Omega\rangle, k > 0$, whose energy is given by

$$\mathcal{E}_k \equiv E_k - E_0 = \frac{e^2}{\pi} \frac{k}{R} \log \left( \frac{\gamma R}{k} \right), \quad 1 \ll k \ll R. \quad (4.4)$$

This spectrum has been previously found by using various approximations: phenomenological classical hydrodynamics [18][20], edge bosonization [19] and a Feynman-like ansatz [20]. Here, we gave a complete proof of this result, free of unnecessary assumptions, in the framework of the $W_{1+\infty}$ dynamical symmetry. In the next section, we discuss how this spectrum has been confirmed experimentally.

A virtue of our approach is the complete knowledge of the Hilbert space of quantum edge excitations. Given that we have found the form of the Hamiltonian to leading order $O(1/R)$, we can find the complete spectrum of all many-body states (2.15). Due to the ladder property of the operators $V_k^0$ with respect to the Hamiltonian (4.2), the energy of the many-boson states is simply additive in the one-boson components,

$$H^{(\text{Coulomb})} | k, \{n_1, n_2, \ldots, n_s\} \rangle = \mathcal{E}_{\{n_1, \ldots, n_s\}} | k, \{n_1, n_2, \ldots, n_s\} \rangle,$$

$$\mathcal{E}_{\{n_1, \ldots, n_s\}} = \sum_{i=1}^{s} \mathcal{E}_{n_i}. \quad (4.5)$$

This property of the many-body spectrum was first shown by the numerical studies of Ref. [17]. Actually, it is the characteristic property of the one-dimensional bosonization of quantum edge excitations, which we have proved in this paper, and can be considered a success of the conformal field theory approach to the quantum Hall effect. Although the logarithmic deviation from the linear spectrum breaks scale and conformal invariance, the conformal field theory techniques remain still valid. In fact, the Hilbert space carrying the representations of the Virasoro algebra (angular momentum) also diagonalizes the non-conformal Hamiltonian (1.2). Actually, the $W_{1+\infty}$ algebra is richer than the Virasoro algebra, because it includes more operators. These can be used to construct non-conformal but still diagonalizable Hamiltonians. This is one of the major points we would like to emphasize here.
4.2. Short-range interactions

It is also interesting to study the dynamics of edge excitations in the presence of short-range interactions, which we have parametrized with Gaussians

$$V(x) = \int_0^\infty dt \ f(t) \ e^{-t|x|^2}, \quad (4.6)$$

and weight functions $f(t)$ of compact support centered around $t = 1$.

We can add this interaction to the Hamiltonian $(4.2)$, to describe short-distance deformations of the Coulomb interaction due to microscopic effects, like the thickness of the layer. Using the matrix elements of the short-range interaction in eq. $(3.42)$, we obtain the following contribution to the energy of the one-boson states $(4.4)$,

$$\Delta E_k = -\frac{\pi \alpha}{2R^3} \ k \ (k^2 - 1), \quad \alpha = \frac{1}{4\pi^{3/2}} \int_0^\infty dt \ f(t) \ \frac{1 + 2t}{(t(1+t))^{3/2}}. \quad (4.7)$$

This spectrum coincides with the classical capillary frequency $(3.13)$, for $\rho_0 = eB\nu/(2\pi) = \nu/(\ell^2\pi)$, and $\nu = 1$, in units $\ell = 1$. It identifies the quantum expression for the edge tension $\alpha$ as a “moment” of the interaction. This result shows that the quantum incompressible fluid behaves semiclassically in the presence of short-range interactions, as anticipated in the previous section.

Note that the result $(4.7)$ includes the Haldane short-range interaction $[36]$, $H_I(|x_1 - x_2|) = a \ \Delta \delta^{(2)}(x_1 - x_2)$, which has often been used in numerical and approximate analyses to stabilize the $\nu = 1/3$ Laughlin incompressible fluid. In the present case, instead, it can be considered as a residual interaction for edge excitations of the stable $\nu = 1$ ground state. The derivative of the delta function can be regularized by a Gaussian with weight function $f(t) = \lim_{t_o \to \infty} a \left(8t^2/\pi \right) (2 + t\partial_t) \delta(t - t_o)$, leading to the spectrum

$$E_k = -\frac{a}{\pi \sqrt{\pi} R^3} \ k \ (k^2 - 1), \quad (4.8)$$

in agreement with previous results $[17][20]$.

Another important property of eq. $(4.7)$ is that the short-range contribution to the spectrum $(4.4)$ is of $O(1/R^3)$ and thus negligible for large $R$. This proofs the universality of the dynamics of edge excitations: in the thermodynamic limit, short-range interactions become irrelevant for the dynamics of edge excitations, which is only governed by the long-distance physics. The spectrum $(4.4)$ yields further universal quantities, which add
to the kinematical data of section 2; these are combinations of finite-size energies which are parameter-free (see also ref. [2]).

Finally, we would like to discuss the excitation spectrum in the case of short-range interactions only. Since the experiments [21] confirm the Coulomb spectrum (4.4), this is a rather academic issue. However, it allows us to clarify some aspects of the $W_{1+\infty}$ dynamics associated with the higher spin operator $V_0^2$, which measures the leading radial fluctuations of edge excitations. In this case, we can consider the following Hamiltonian

$$H^{(sr)} = \alpha_0 V_0^0 + \frac{\alpha_1}{R} \left( V_0^1 - \frac{1}{24} \right) + \frac{\alpha_2}{R^2} \left( V_0^2 - \frac{1}{12} V_0^0 \right) + O \left( \frac{1}{R^3} \right).$$  (4.9)

The term proportional to $V_0^2$ produces subleading effects which break the leading scale-invariant spectrum of conformal field theory studied in [3]. The appropriate basis of edge excitations which diagonalizes $V_0^2$ (as well as all higher spin $W_{1+\infty}$ operators) is obtained from the fermionic Fock operators,

$$b_{n_1}^\dagger b_{m_1} \cdots b_{n_s}^\dagger b_{m_s} |\Omega\rangle, \quad n_1 > \cdots > n_s > 0 \geq m_1 > \cdots > m_s, \quad \sum_i (n_i - m_i) = k.$$  (4.10)

These are multiple particle-hole excitations with total angular momentum $k$. The operators $V_0^i$ are diagonal in this basis because the pair ($b_{n}^\dagger b_{m}$) is a ladder operator for them (see eqs. (2.11)(3.23)). The spectrum of one particle-hole excitation $b_{n}^\dagger b_{m} |\Omega\rangle$ follows from (3.23):

$$E_{nm} = E_{nm} - E_0 = \frac{\alpha_1}{R} (n - m) + \frac{\alpha_2}{R^2} (n - m) (n + m - 1).$$  (4.11)

Since the indices $(n, m)$ can be traced back to the ones of $w_{\infty}$ classical deformations $\delta_{nm\rho}$ in (2.9), we find that quantum radial fluctuations indeed break the classical degeneracy at fixed $k = (n - m)$. Note that this degeneracy is still present in the leading conformal spectrum. For a given value of $k$, the non-degenerate eigenvalues are distributed in the interval

$$\frac{\alpha_1}{R} k - \frac{\alpha_2}{R^2} k (k - 1) \leq E_{nm} \leq \frac{\alpha_1}{R} k + \frac{\alpha_2}{R^2} k (k - 1).$$  (4.12)

Note, however, that the breaking of conformal invariance generated by $V_0^2$ is different from the one induced by the Coulomb interaction, because the fermionic and bosonic bases are different. The fermionic splitting of the conformal spectrum is characteristic of another $(1 + 1)$-dimensional model with $W_{1+\infty}$ symmetry: the Calogero-Sutherland model [37]. In this case, it has been show [38] that the two-body interaction can be written entirely in terms of the generator $V_0^2$. In the quantum Hall effect, instead, the interaction cannot be written in terms of the generators $V_0^i$ only.
4.3. Charged excitations

We can easily extend the previous discussion to compute the spectrum of charged edge excitations. Let us consider states with \( q \) electrons \((q \in \mathbb{Z})\) added to the boundary and, correspondingly, \( q \) vortices in the interior of the droplet. These are the highest-weight states \(|Q\rangle\) for other \( W_{1+\infty} \) representations, with weights given by (2.16). In particular, the weights of \( V_0^0 \) and \( V_0^1 \) are given by \( Q = q \) and \( J = q^2/2 \), respectively, and determine the normal-ordered energy of these states by eq. (4.2),

\[
H|Q = q\rangle = E_{0,q}|Q\rangle, \quad E_{0,q} = \alpha_0 q + \frac{e^2}{\pi R} \left( \frac{q^2}{2} - \frac{1}{24} \right) \log \gamma R + \frac{e^2}{\pi R} q^2. \quad (4.13)
\]

Clearly, this is only a contribution to the energy gap of these states. The main contribution comes from the physics at the core of the vortices in the deep interior of the droplet, which cannot be computed in the boundary theory [7]. However, eq. (4.13) gives the universal part of this gap [2].

The edge excitations on top of both the charged and neutral states have the same structure. The energy levels for \( Q \neq 0 \) are only shifted by a \( Q \)-dependent constant. In particular, the Coulomb interaction yields the spectrum (4.4) for the energy differences \((E_{k,q} - E_{0,q})\).

4.4. Filling \( \nu = 1/(2p + 1) \)

We can use the \( W_{1+\infty} \) dynamical symmetry to generalize the previous results for the spectrum of edge excitations to the Laughlin fluids at fractional filling \( \nu = 1/3, 1/5, 1/7, \ldots \). As discussed in section 2.2, these excitations are given by other representations of the \( W_{1+\infty} \) algebra with central charge \( c = 1 \) [2][4][39]. In reference [2], we have shown that these representations have one free parameter, which is fixed by the value of the Hall conductivity. The operators \( V_k^0 \), normalized to measure the physical charge, satisfy the Kac-Moody algebra,

\[
[ V_n^0, V_m^0 ] = \frac{n}{2p+1} \delta_{n+m,0}, \quad \left( \nu = \frac{1}{2p+1}, p = 1, 2, \ldots \right)
\]

and the spectrum of anyon highest weights is given by \( Q = q/(2p + 1), J = q^2/(4p + 2) \), with \( q \in \mathbb{Z} \) [2][4].

Lacking a microscopical derivation of the Coulomb Hamiltonian for fractional filling, we shall use an argument based on the \( W_{1+\infty} \) dynamical symmetry. Namely, we shall
argue that the Hamiltonian for edge excitations must be form invariant, i.e., must take the same form \(^{(*)} (4.2)\) in terms of the \(W_{1+\infty}\) operators for any value of \(\nu = 1/(2p + 1)\). Indeed, the structure of edge excitations, i.e., of the \(W_{1+\infty}\) highest-weight representations, is identical for all these values of \(\nu\). Therefore, we can repeat the analysis after \((4.2)\), and use the commutators \((4.14)\) and \((2.14)\) to compute the corresponding spectrum of one-boson neutral edge excitations,

\[
\mathcal{E}_k = E_k - E_0 = \frac{e^2\nu}{\pi} \frac{k}{R} \log \left( \frac{\eta R}{k} \right), \quad 1 \ll k \ll R, \quad \left( \nu = \frac{1}{2p + 1} \right), \quad (4.15)
\]

where the parameter \(\eta\) may depend on \(\nu\). This result is again semiclassical \([19][20]\). It is possible, although more difficult, to generalize our results to the complete hierarchy of incompressible quantum fluids corresponding to \(W_{1+\infty}\) theories with \(c > 1\) \([4]\).

5. Conclusions

In this paper, we further developed the \(W_{1+\infty}\) effective field theory for the plateaus of the quantum Hall effect.

We first reviewed how the \(W_{1+\infty}\) representations describe the kinematical data of the Laughlin incompressible quantum fluids, like the fractional charges and statistics of anyon excitations. Secondly, we expressed the general Hamiltonian for edge excitations in terms of the \(W_{1+\infty}\) generators for both cases of Coulomb and short-range interactions, and obtained its spectrum. We proved the one-dimensional bosonization of edge excitations by showing that the leading term in the \(1/R\) expansion of the Hamiltonian contains the Kac-Moody and Virasoro operators only. Moreover, for large \(R\), the spectrum is independent of short-range effects and thus possesses universal features.

Furthermore, we showed that the infinite tower of higher spin operators \(V^i_k\), \(i \geq 2\), parametrizes the subleading corrections to the bosonization, which arise from radial fluctuations of the edge excitations. This is the physical picture for the “(2+1)-dimensional nature” of the \(W_{1+\infty}\) algebra recently remarked in Ref. \([27]\).

Finally, we briefly discuss the experiment \([21]\) of radio-frequency resonance, which confirmed the finite size spectra \((1.4)\) and \((1.15)\), for the case of the Coulomb interaction. Low dissipation excitations (called edge magnetoplasmons there) were observed as sharp

\(^{*}\) Note that this corresponds to a complicated microscopic expression, which incorporates the idea of “attaching flux tubes to electrons” \([23]\), see also \([38]\).
resonance peaks in the transmitted signal, at the plateaus for \( \nu = 1, 2 \) and \( 2/3 \). The observed resonance frequencies \( \omega_i = E_i/h \) match the formulae (4.4) and (4.13), and verify both their logarithmic functional form and their \( \nu \) dependence (see table I and eq. (7) of ref. [21]). Let us also quote the recent experiment with very precise time resolution [22], which clearly establishes the one-dimensional and chiral nature of the edge excitations.

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Appendix A.

$W_{1+\infty}$ algebra and Weyl fermion representation

In this appendix, we derive explicit expressions for the $W_{1+\infty}$ algebra, the form of the generators $(V_R)^i_n$ defined on the cylinder geometry, and the realization of the $W_{1+\infty}$ algebra in terms of a Weyl fermion.

$W_{1+\infty}$ algebra

The $W_{1+\infty}$ algebra can be written in a compact first-quantized form [25],

$$[V(z^r f(D)), V(z^s g(D))] = V(z^{r+s} f(D + s)g(D)) - V(z^{r+s} f(D)g(D + r))$$

$$+ c \Psi(z^r f(D), z^s g(D)) , \tag{A.1}$$

where the central extension is

$$\Psi(z^r f(D), z^s g(D)) \equiv \delta_{r+s,0} \sum_{j=1}^r f(-j)g(r-j) \equiv -\Psi(z^s g(D), z^r f(D)) , \quad r > 0 . \tag{A.2}$$

In these expressions, the $W_{1+\infty}$ operators $V(z^r f(D))$ are specified by the corresponding first-quantized differential expression $z^r f(D)$, where $f(D)$ is an entire function of $D \equiv z \frac{\partial}{\partial z}$. In the text, we introduced the operators $V^i_k$, characterized by mode index $k \in \mathbb{Z}$ and conformal spin $h = i + 1 \geq 1$: the two definitions are related by

$$V^i_k \equiv V(z^k f^i_k(D)) , \tag{A.3}$$

where $f^i_k(D)$ are specific $i$-th order polynomials. The basis for these polynomials is chosen by requiring that the central extension (A.2) is diagonal in the spin indices. This can be done uniquely, because $\Psi$ has the property of a symplectic metric. The generators of lower spin are thus found to be:

$$V^0_n = V(-z^n) ,$$

$$V^1_n = V\left(-z^n \left(D + \frac{1}{2}(n+1)\right)\right) ,$$

$$V^2_n = V\left(-z^n \left(D^2 + (n+1)D + \frac{6n^2 + 15n + 11}{60}D + \frac{(n+3)(n+2)(n+1)}{20}\right)\right) ,$$

$$V^3_n = V\left(-z^n \left(D^3 + \frac{3}{2}(n+1)D^2 + \frac{6n^2 + 15n + 11}{10}D + \frac{(n+3)(n+2)(n+1)}{20}\right)\right) , \tag{A.4}$$
where $V_k^0$ and $V_k^1$ are, by definition, the Kac-Moody and Virasoro generators, respectively [3]. Some of the algebraic relations are:

\[
\begin{align*}
[V_n^0, V_m^0] &= n \delta_{n+m,0}, \\
[V_n^1, V_m^0] &= -m V_{n+m}^0 , \\
[V_n^1, V_m^1] &= (n - m) V_{n+m}^1 + \frac{c}{12} (n^3 - n) \delta_{n+m,0}, \\
[V_n^2, V_m^0] &= -2m V_{n+m}^1 , \\
[V_n^2, V_m^1] &= (n - 2m) V_{n+m}^2 - \frac{1}{6} (m^3 - m) V_{n+m}^0 , \\
[V_n^2, V_m^2] &= (2n - 2m) V_{n+m}^3 + \frac{n-m}{15} (2n^2 + 2m^2 - nm - 8) V_{n+m}^1 \\
&\quad + \frac{c}{180} \frac{n(n^2-1)(n^2-4)}{180} \delta_{n+m,0} .
\end{align*}
\] (A.5)

Any highest weight representation is formed by all the states obtained by applying $W_{1+\infty}$ generators with negative moding to the highest weight state $|Q\rangle$, which satisfies,

\[V_n^i |Q\rangle = 0 , \quad \forall \ n > 0 , \ i \geq 0 , \] (A.6)

and is the eigenstate of $V_0^i$,

\[V_0^i |Q\rangle = m_i(Q) |Q\rangle . \] (A.7)

We shall only discuss the unitary, irreducible representations for $c = 1$, which describe the Laughlin incompressible fluids with $\nu = 1/(2p + 1)$, $p = 0, 1, 2, \cdots$ [2]. For these representations, the $m_i(Q)$ are $(i + 1)$-th polynomials in $Q$, which are given by the generating function,

\[V (e^{-x D}) |Q\rangle = \frac{e^{xQ} - 1}{e^x - 1} |Q\rangle . \] (A.8)

For example, by expanding in powers of $x$, and using (A.4), one finds $m_0(Q) = Q$ and $m_1(Q) = J = Q^2/2$. The eigenvalues of $V_0^0/(2p + 1)$ and $V_0^1$ measure the physical charge and the angular momentum of the states, respectively [2] [4]. For $c = 1$, all the states in a unitary irreducible representation can obtained by using the Kac-Moody generators only [25]. These states have the form,

\[|k, \{n_1, n_2, \ldots, n_s\} \rangle = V_{-n_1}^0 V_{-n_2}^0 \cdots V_{-n_s}^0 |Q\rangle , \quad n_1 \geq n_2 \geq \cdots \geq n_s > 0 , \] (A.9)

where $k = Q^2/2 + \sum_{i=1}^s n_i$ is their angular momentum.
In the quantum Hall effect, fields and operators are naturally defined on the boundary circle \([0 \leq \theta < 2\pi]\), \textit{i.e.}, on a compact space. In the mathematical literature, however, they are conventionally considered in an unbounded space. There is a conformal mapping between these two spaces, which are the \textit{cylinder} \((u = \tau - iR\theta)\) and the \textit{conformal plane} \((z)\), after inclusion of the euclidean time \(\tau\),

\[
z = \exp\left(\frac{u}{R}\right) = \exp\left(\frac{\tau}{R} - i\theta\right) .
\]  

(A.10)

For example, the operators (A.4) define the \(W_{1+\infty}\) currents \(V^i(z)\) on the conformal plane as follows,

\[
V^i(z) \equiv \sum_n V^i_n z^{-n-1} .
\]  

(A.11)

On the other hand, the Hamiltonian of edge excitations should be expressed in terms of the \(W_{1+\infty}\) generators \((V_R)_n^i\) on the cylinder. The latter can be obtained by transforming the \(V^i(z)\) under the conformal mapping (A.10), as follows. The generator \(G\) of infinitesimal conformal transformations \(z = u + \epsilon(u)\), with \(\epsilon(u) = \sum_n \epsilon_n u^{n+1}\), is parametrized by the Virasoro operators,

\[
G = \sum_n \epsilon_n V^1_n .
\]  

(A.12)

Therefore, the infinitesimal transformation of the \(W_{1+\infty}\) currents, \(V^i(u) = V^i(z) + \delta \epsilon \tilde{V}^i(z)\), are computed from the algebra (A.3),

\[
\delta \epsilon \tilde{V}^0 \equiv [G,V^0] = \epsilon \partial V^0 + (\partial \epsilon)V^0 ,
\]

\[
\delta \epsilon \tilde{V}^1 \equiv [G,V^1] = \epsilon \partial V^1 + (\partial \epsilon)V^1 + \frac{c}{12} \partial^3 \epsilon ,
\]

\[
\delta \epsilon \tilde{V}^2 \equiv [G,V^2] = \epsilon \partial V^2 + (\partial \epsilon)V^2 + \frac{1}{6} (\partial^3 \epsilon) V^0 .
\]  

(A.13)

Note that these transformations contain the differential operator \([\epsilon \partial + h(\partial \epsilon)]\) applied to the current, plus additional terms. If the latter are absent, as in the case of \(V^0(z)\), the infinitesimal transformation can be integrated to yield the finite form

\[
\Phi(u) = \left(\frac{dz}{du}\right)^h \Phi(z(u)) .
\]  

(A.14)

Those fields \(\Phi(z)\) which transform homogeneously are called \textit{primary} \([9]\). The current \(V^i\), \(i > 0\), have additional terms which, nevertheless, vanish for \(\epsilon = \epsilon_{-1} + \epsilon_0 z + \epsilon_1 z^2\), \textit{i.e.}, for
global conformal transformations generated by \((V^1_{11}, V^0_0, V^1_1)\). There is only one function \(S(z(u), u)\) of the conformal mapping, which vanishes under the finite global transformations: the Schwartzian derivative [9]

\[
S(f(z), z) \equiv \frac{f'''(f')}{{f'(f')}}^2, \tag{A.15}
\]

where the prime denotes differentiation. Owing to this property, all infinitesimal transformations in (A.13) can be integrated, and yield

\[
V^0_0(u) = \frac{dz}{du} V^0_0(z), \\
V^1_0(u) = \left(\frac{dz}{du}\right)^2 V^1_0(z) + \frac{c}{12} S(z, u), \\
V^2_0(u) = \left(\frac{dz}{du}\right)^3 V^2_0(z) + \frac{1}{6} \frac{dz}{du} S(z, u) V^0_0(z).
\tag{A.16}
\]

The \(W_{1+\infty}\) currents on the cylinder are thus found by using the mapping (A.10),

\[
V^0_0^R(u) = \frac{z}{R} V^0_0(z), \\
V^1_0^R(u) = \frac{1}{R^2} \left(z^2 V^1_0(z) - \frac{1}{24}\right), \\
V^2_0^R(u) = \frac{1}{R^3} \left(z^3 V^2_0(z) - \frac{z}{12} V^0_0(z)\right).
\tag{A.17}
\]

Using the definition

\[
(V^j_R)_0 \equiv \int_0^{2\pi i R} \frac{du}{-2\pi i} V^j_R(u), \tag{A.18}
\]

we obtain the relation between the zero modes in the two geometries

\[
(V^0_R)_0 = V^0_0, \quad (V^1_R)_0 = \frac{1}{R} \left(V^1_0 - \frac{c}{24}\right), \quad (V^2_R)_0 = \frac{1}{R^2} \left(V^2_0 - \frac{1}{12} V^0_0\right). \tag{A.19}
\]

**Weyl fermion representation of \(W_{1+\infty}\)**

In the thermodynamic limit [2], the field operator of electrons in the lowest Landau level can be evaluated at the boundary of the droplet, and mapped into the \((1 + 1)\)-dimensional field of the chiral, relativistic Weyl fermion [3],

\[
F_R(\theta) = \frac{1}{\sqrt{R}} \sum_{k=-\infty}^{\infty} e^{i(k-\frac{1}{2})\theta} b_k, \tag{A.20}
\]
with Neveu-Schwarz boundary conditions on the circle (the $b_k$ are the fermionic Fock operators). The vacuum of this theory corresponds to the ground state of the $\nu = 1$ incompressible fluid $|\Omega\rangle$, which satisfies,

$$b_l|\Omega\rangle = 0 \ , \ l > 0 \ , \quad b_l^\dagger|\Omega\rangle = 0 \ , \ l \leq 0 \ . \quad (A.21)$$

The Hilbert space of a Weyl fermion consists of an infinity of $c = 1 \ W_{1+\infty}$ representations (A.6)-(A.9), all those corresponding to $Q \in \mathbb{Z}$. For example, $|\Omega\rangle$ is the highest weight state (A.6) with $Q = 0$.

The Weyl fermion is a primary conformal field (A.14) with $h = 1/2$, and on the plane $z$ takes the form ($\tau = 0$),

$$F(z) = \left( \frac{du}{dz} \right)^{1/2} F_R(\theta) = \sum_l e^{i\theta} b_l , \quad F(z) = \left( \frac{du}{dz} \right)^{1/2} F_R^\dagger(\theta) = \sum_l e^{-i(l-1)\theta} b_l^\dagger \ . \quad (A.22)$$

The representation of $W_{1+\infty}$ operators on this Hilbert space is obtained by sandwiching the first-quantized expressions (A.3) between the field operators,

$$V_n^i \equiv \oint \frac{dz}{2\pi i} : F(\theta) \ z^n \ f_n^i(D) \ F(\theta) : = \oint \frac{dz}{2\pi i} : \bar{F}(\theta) \ z^n \ g_n^i(D) \ F(\theta) : , \quad (A.23)$$

where $g_n^i = (-1)^{i+1} f_n^i$, and the integration is carried clockwise over the unit circle. Since the anticommutator of fields is a delta function in Fock space, the $V_n^i$ defined above clearly represent the algebra (A.1). One can verify that they also have the eigenvalues (A.8) when acting on $Q$ fermion states. In eq. (A.23), they are also written in the canonical form ($\bar{F}F$) of quantum field theory, which identifies the polynomials $g_n^i$ defined in section 2.2.

Moreover, the normal ordering prescription : w.r.t. the ground state (A.21) is obtained by writing the annihilation operators $b_l$ ($l > 0$) and $b_l^\dagger$ ($l \leq 0$) to the right-hand side of the creation operators $b_l$ ($l \leq 0$) and $b_l^\dagger$ ($l > 0$).

From the expressions (A.4), we obtain the Fock space expression of the generators,

$$V_k^0 = \sum_l : b_{l-k}^\dagger \ b_l : ,$$

$$V_k^1 = \sum_l \left( l - \frac{k+1}{2} \right) : b_{l-k}^\dagger \ b_l : , \quad (A.24)$$

$$V_k^2 = \sum_l \left( l^2 - (k+1)l + \frac{(k+1)(k+2)}{6} \right) : b_{l-k}^\dagger \ b_l : ,$$
and the form of the currents (A.11) on the plane,

\[ V^0(z) = : \bar{F} F : , \]
\[ V^1(z) = \frac{1}{2} : \partial_z \bar{F} F : - \frac{1}{2} : \bar{F} \partial_z F : , \]  
\[ V^2(z) = - : \partial_z \bar{F} \partial_z F : + \frac{1}{6} : \partial^2_z (\bar{F} F) : . \]  

(A.25)

Let us finally remark that the different form (A.17) of the \( W_{1+\infty} \) currents on the plane and the cylinder is due to a normal ordering effect \[35\]. Actually, we can recover (A.17) in the fermionic case \( (c = 1) \) as follows. We apply the conformal mapping (A.10) to each fermion field in (A.25), paying attention to the different normal ordering in the plane and the cylinder. For example,

\[ V^0_R(u) = : \bar{F}_R(u) F_R(u) : \equiv \lim_{u_1, u_2 \to u} \left( \bar{F}_R(u_1) F_R(u_2) - \frac{1}{u_1 - u_2} \right) \]
\[ = \frac{dz}{du} V^0(z) + \lim_{u_1, u_2 \to u} \left[ \left( \frac{dz_1}{du_1} \frac{dz_2}{du_2} \right)^{1/2} \frac{1}{z_1 - z_2} - \frac{1}{u_1 - u_2} \right] = \frac{z}{R} V^0(z) . \]  

(A.26)

Proceeding similarly for the other currents, we indeed recover (A.17).
Appendix B.

The Gaussian interaction two-body matrix element when $N \to \infty$

In this appendix, we evaluate the two-body matrix element for the Gaussian interaction (3.35) in the limit $N \to \infty$. It is given by the expression

$$M(N + r, N + s; k) = \frac{1}{2} \int_0^\infty d^2 z_1 d^2 z_2 \frac{z_1^{N+s} z_1^{N+s+k}}{\sqrt{(N + s)! (N + s + k)!}} e^{-|z_1 - z_2|^2} \frac{z_2^{N+r} z_2^{N+r-k}}{\sqrt{(N + r)! (N + r - k)!}} e^{-t|z_1 - z_2|^2}.$$ (B.1)

The idea is to evaluate exactly the angular integrations in (B.1) and use the saddle-point technique to calculate the integrals that involve the modulus of the coordinates, when $N \to \infty$. We first rewrite (B.1) upon replacing $z_1 = x e^{i \theta}$, $z_2 = y$, with $x, y \geq 0$:

$$M(N + r, N + s; k) = \frac{2}{C(N, s, r, k)} \int_0^\infty dx \int_0^\infty dy x^{2N+2s+k+1} y^{2N+2r-k+1} e^{- (1+t)(x^2+y^2)} I_k(2txy),$$

$$C(N, s, r, k) \equiv \sqrt{(N + s)! (N + s + k)! (N + r)! (N + r - k)!},$$

$$I_k(z) = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{ik\theta} z \cos \theta.$$ (B.2)

For $N \to \infty$, the most important contribution to the radial integrations in (B.2) arise from values of $x, y \sim O(\sqrt{N})$. Therefore, for values of $t$ that are not too small, one can assume that the argument of the imaginary Bessel function $I_k(z)$ in (B.2) satisfies $z = 2txy \gg 1$. In this regime, one may use the asymptotic form [34]:

$$I_k(z) \simeq \frac{e^z}{\sqrt{2\pi z}} \left( 1 - \frac{1}{2z} (k^2 - 1/4) + O \left( \frac{1}{z^2} \right) \right), \quad z = 2txy \gg 1.$$ (B.3)

Replacing (B.3) in (B.2), one obtains

$$M(N + r, N + s; k) = \frac{1}{C(N, s, r, k)} \frac{1}{\sqrt{\pi t}} \int_0^\infty dx \int_0^\infty dy e^{\Phi(x,y)},$$

$$\Phi(x, y) \equiv -(1+t)(x^2+y^2) + 2txy + (2N + 2s + k + 1/2) \ln x + (2N + 2r - k + 1/2) \ln y.$$ (B.4)
One then finds the two-dimensional saddle-point of (B.4) \((x_0, y_0)\):

\[
\begin{align*}
 x_0^2 &= N + 1/4 + \frac{2s + k + t(r + s)}{2(1 + t)} + O\left(\frac{1}{N}\right) , \\
y_0^2 &= N + 1/4 + \frac{2r - k + t(r + s)}{2(1 + t)} + O\left(\frac{1}{N}\right).
\end{align*}
\]

Therefore, the saddle-point value of (B.4) is given by:

\[
M(N + r, N + s; k) = \frac{1}{C(N, s, r, k) \sqrt{\pi t}} \cdot \frac{1}{\sqrt{\pi t}} \cdot \left\{ e^{\Phi(x_0, y_0)} 2\pi \left( \frac{\partial^2 \Phi}{\partial x \partial y} \right)_0^{-1/2} - \frac{(k^2 - 1/4)}{4t} \left( \begin{array}{c} r \rightarrow r - 1/2 \\ s \rightarrow s - 1/2 \end{array} \right) \right\}.
\]

The value of the Jacobian is given by:

\[
\det \left( \frac{\partial^2 \Phi}{\partial x \partial y} \right)_0 = 16(1 + t) + O\left(\frac{1}{N^2}\right).
\]

Replacing now (B.5) and (B.7) into (B.6), and retaining terms up to \(O(1/N^{3/2})\), one finds

\[
M(N + r, N + s; k) = \frac{1}{\sqrt{N}} \frac{1}{4\sqrt{\pi} \sqrt{t(1 + t)}} \left( 1 - \frac{k^2 - 1/4}{4t N} \right) \cdot \left\{ 1 + \frac{1}{2N} \left( k(r - s - k) - r^2 - s^2 - \frac{r + s}{2} - \frac{5}{24} \right) + \frac{k^2 - 2k(r - s) + 2(r^2 + s^2) + t(r + s)^2}{4N(1 + t)} \right\}.
\]

which is formula (3.36).
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