Robust control approach for handling matched and/or unmatched uncertainties in port-controlled Hamiltonian systems

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Abstract: This study considers the problem of robust interconnection and damping assignment passivity-based control approach (IDA-PBC) for underactuated mechanical systems from two points of view. First, the robustness of IDA-PBC in the presence of non-vanishing matched and unmatched uncertainties is analysed and sufficient conditions are derived to ensure ultimate boundedness of system. Second, the robust control design is provided by adding a new control input to the former controller designed by IDA-PBC, such that asymptotic stability of the closed-loop system in the face of non-vanishing matched uncertainties is fulfilled. Finally, the proposed robust controller is evaluated through simulations of two practical examples, i.e. an inertia wheel pendulum and a single-link elastic joint robot. The simulation outcomes clearly reveal the effectiveness of the presented robust controller.

1 Introduction

Control of mechanical systems has been drawing great attention recently due to their extensive range of applications such as robotic manipulators which are employed in the industrial, medical, and aerospace fields [1–5]. Mechanical systems need high precision control in order to achieve their required performance. However, because of their non-linear dynamics, design of a suitable controller is complex especially when external disturbances and parametric uncertainties exist.

Passivity-based control (PBC) is a control approach for non-linear systems, which is well known in mechanical applications. Interconnection and damping assignment PBC approach (IDA-PBC) method is a new formulation of PBC, which has attracted growing attention in recent years. It was initially introduced in [6–8]. IDA-PBC is applicable for systems expressed in port-controlled-Hamiltonian (PCH) form, which is an important class of non-linear systems presented first in [9]. The main idea of IDA-PBC is to transform a given Hamiltonian system into a desired Hamiltonian one by modifying its potential and kinetic energies and by means of state feedback. The new Hamiltonian function, which is the sum of potential energy (excluding gravitational potential energy) and kinetic energy of a mechanical system, must have a minimum at the desired equilibrium point in order to assure stability. The advantage of IDA-PBC is that the Hamiltonian function can be implemented as an appropriate candidate for Lyapunov function in the stability analysis. IDA-PBC is a thoroughly well known control strategy for stabilisation of a wide class of mechanical systems such as those presented in [1, 10, 11].

Modelling of mechanical systems is always done by some system parameter approximations, as the designer's knowledge about the system is insufficient in many cases or the parameters change unpredictably. This type of uncertainty is called parametric uncertainty. Furthermore, some dynamics of the system may not be considered due to model simplification or their small impact. In addition to parametric uncertainty and unmodelled dynamics, the effect of external disturbances on mechanical systems is unavoidable [12, 13]. These factors together, which are called uncertainties in this paper, can affect the desired performance of the system or may lead to instability. Therefore, robust IDA-PBC problem against uncertainties is an indispensable matter.

Robustness of fully actuated mechanical systems has been studied in [14]. Its main idea is to add an integral action to the non-passive output, which preserves the PCH structure of the system. This idea was first introduced in [15]. In that, asymptotical stability in the presence of constant matched uncertainty and input-to-state stability (ISS) property were addressed.

First studies about the robustness of underactuated systems in the presence of constant uncertainties were done in [16] and a special case of a two degree of freedom mechanical system with one degree of underactuated and constant inertia matrix was considered. After that, Ryalat and Laila [17] proposed a controller for underactuated systems which reject constant matched uncertainty and its main idea was completed in [18]. The robustifying problem of IDA-PBC was studied in [16] by adding an external control loop to the closed-loop system, which preserves the PCH structure of the system. This idea complements the idea of [16]. Robustness analysis of IDA-PBC for an underactuated mechanical system in the face of uncertainties has been considered in [19]. Indeed, Haddad et al. [19] derived some conditions on the upper bounds of uncertain terms which verify the robust performance of IDA-PBC controller. The most recent achievement in the field of robustifying IDA-PBC has been addressed in [20, 21]. In [20], a robust model reference adaptive controller against matched uncertainty for underactuated systems in PCH form has been proposed. This controller improves the robustness of IDA-PBC by rapid reduction of tracking error and increasing the admissible boundaries. Moreover, Ryalat and Laila [21] have studied disturbance rejection problem by applying integral control, which ensures ISS stability; however, the given approach has a complicated structure.

Controller design for underactuated systems is a complex problem itself. This fact together with the effectiveness of IDA-PBC approach in the stabilisation of mechanical systems leads to the robust analysis of IDA-PBC method. A mechanical system may be subjected to non-vanishing uncertainties. In this situation, the system would not have an equilibrium point. Thus, the asymptotic stability of the system would not be guaranteed. Therefore, it is desirable to assure the boundedness of system solutions. For this purpose, in this paper, sufficient conditions are presented to guarantee ultimate boundedness of system in spite of non-vanishing matched and unmatched uncertainties existence. Furthermore, according to lack of achievements in the field of robustifying IDA-PBC for underactuated systems against time-varying uncertainty, this paper considers problem of designing a robust controller in the presence of non-vanishing time-varying
matched uncertainty as another main contribution. The proposed robust controller has variable structure.

The main contributions of this paper are: providing sufficient conditions for ultimate boundedness assurance of system in the presence of non-vanishing matched and unmatched uncertainties and also designing a robust controller against non-vanishing matched uncertainty.

The rest of this paper is organised as follows: In Section 2, some required preliminaries about ultimate boundedness stability is presented. Section 3 describes the target model and provides a brief review of IDA-PBC method. Section 4 describes devoted to ultimate boundedness analysis in the presence of both matched and unmatched uncertainties. Another contribution of this paper which is developing a new control input to robustify the IDA-PBC is discussed in Section 5. Section 6 provides simulations of two practical systems to verify the performance of the proposed robust controller, which is followed by the conclusion in Section 7.

2 Preliminaries

In this section, first, the definitions of boundedness and ultimate boundedness are given and then a theorem, that guarantees these properties, is proposed.

**Definition 1:** The solutions of \( x = f(t, x) \) are: uniformly bounded if there exists a constant \( c \), independent of \( t_0 \), and for every \( a \in (0, c) \), there is \( \beta = \beta(a) > 0 \), independent of \( t_0 \), such that

\[
\| x(t_0) \| \leq a \implies \| x(t) \| \leq \beta, \quad \forall t \geq t_0
\]

Globally uniformly bounded if (1) holds, for arbitrarily large \( a \).

Uniformly ultimately bounded with the ultimate bound \( b \) if there exist positive constants \( b, c \), independent of \( t_0 \), and for every \( a \in (0, c) \), there is \( T = T(a, b) \geq 0 \), independent of \( t_0 \), such that

\[
\| x(t_0) \| \leq a \implies \| x(t) \| \leq b, \quad \forall t \geq t_0 + T
\]

Globally uniformly ultimately bounded if (2) holds, for arbitrarily large \( a \) [22].

**Remark 1:** A non-autonomous system with time-varying external disturbance can be shown in the form of \( x = f(t, x) \). Consider the system \( x = -x + \delta \sin(t) \) with \( x(t_0) = a \), in which \( \delta(t) = \delta \sin(t); (0 < \delta < a) \) is an external disturbance. This system has no equilibrium point and is uniformly ultimately bounded [22].

The Lyapunov analysis can be used to show boundedness of the solution of the state equation, even if the system has no equilibrium point [22–24]. In this regard, the following theorem is given.

**Theorem 1:** Let \( D \subseteq \mathbb{R}^n \) be a domain that contains the origin and suppose \( V(t, x) : [0, \infty) \times D \rightarrow \mathbb{R} \) is the Lyapunov function which is a continuously differentiable function such that \( \forall t \geq t_0 \) and \( \forall x \in D \)

\[
\alpha_1(|| x ||) \leq V(t, x) \leq \alpha_2(|| x ||)
\]

\[
\frac{dV}{dt} + \frac{\partial V}{\partial x} f(t, x) \leq -W(x), \quad \forall || x || \geq \mu > 0
\]

where \( \alpha_1(\cdot) \), \( \alpha_2(\cdot) \) are class \( \mathcal{K} \) functions and \( W(\cdot) \) is a continuous positive-definite function. Moreover, \( dV/\partial x \) is a row vector. Take \( r > 0 \) such that \( B_r = \{ x \in \mathbb{R}^n || x || \leq r \} \) in \( D \) and suppose that \( \mu < \alpha_2(\alpha_1(\alpha_0(r))) \), then there exists a class \( \mathcal{K} \) function \( \beta(\cdot) \) and for every initial state \( x(t_0) \), satisfying \( || x(t_0) || \leq \alpha_1(\alpha_0(r)) \), there is \( T \geq 0 \) [dependent on \( x(t_0) \) and \( a \)] such that the solution of \( x = f(t, x) \) satisfies Definition 1 and

\[
\| x(t) \| \leq \beta(|| x(t_0) ||, t - t_0), \quad \forall t \leq t_0 \leq t \leq t_0 + T
\]

\[
\| x(t) \| \leq b, \quad \forall t \geq t_0 + T
\]

Proof: See [22].

3 Problem statement

Motion equations of an \( n \) degree of freedom underactuated mechanical system in the PCH structure and in the presence of uncertainties can be written as follows (by defining \( V_H := \partial H/\partial q \) and \( V_{\dot{H}} := \partial H/\partial \dot{p} \)):

\[
\begin{bmatrix}
\dot{q} \\
\dot{p}
\end{bmatrix} =
\begin{bmatrix}
0 & I_n \\
-L_n & 0
\end{bmatrix} \begin{bmatrix}
V_H \\
V_{\dot{H}}
\end{bmatrix} + \begin{bmatrix}
0 \\
G(q)
\end{bmatrix} \alpha + \delta(q, t) + \gamma_0 \delta_0(x, t)
\]

\[
y = G(q)^T V_H
\]

with the following Hamiltonian function which is the total energy of the system:

\[
H(q, p) = \frac{1}{2} p^T M^{-1}(q) p + V_0(q)
\]

where \( q \in \mathbb{R}^n \) and \( p \in \mathbb{R}^n \) are generalised position and momenta matrices, respectively. \( x = [p^T, q^T]^T \in \mathbb{R}^n \) is the state vector, \( u \in \mathbb{R}^m \) is the control input, \( y \in \mathbb{R}^m \) is the output vector, \( \delta(q, t) \) and \( \delta_0(x, t) \) are, respectively, matched and unmatched uncertainties resulting from: parametric uncertainties, model simplification, and external disturbances, \( G(q) \in \mathbb{R}^{n \times n} \) is the input matrix with full column rank, \( M = M(q) > 0 \) is the inertia matrix and \( V_0(q) \) is the potential energy. Moreover, in the rest of this paper \( \| . \| \) presents the Euclidean norm.

Considering \( \delta(q, t) = 0 \) and \( \delta_0(x, t) = 0 \) results in the following nominal system:

\[
\begin{bmatrix}
\dot{q} \\
\dot{p}
\end{bmatrix} =
\begin{bmatrix}
0 & I_n \\
-L_n & 0
\end{bmatrix} \begin{bmatrix}
V_H \\
V_{\dot{H}}
\end{bmatrix} + \begin{bmatrix}
0 \\
G(q)
\end{bmatrix} \alpha 
\]

The IDA-PBC approach is an effective method to regulate the position of system (7), so as to achieve the following desired dynamic:

\[
\begin{bmatrix}
\dot{q} \\
\dot{p}
\end{bmatrix} = (J_d(q, p) - R_d(q, p)) \begin{bmatrix}
V_H \\
V_{\dot{H}}
\end{bmatrix} + \begin{bmatrix}
0 \\
G(q)
\end{bmatrix} \alpha 
\]

where \( J_d \) and \( R_d \) are desired interconnection and damping matrices

\[
J_d = J + \frac{1}{M^{0.5}} \begin{bmatrix}
0 & M^{-1} \delta M_d \\
0 & 0
\end{bmatrix}
\]

\[
R_d = \begin{bmatrix}
0 & 0 \\
0 & G K, G^T
\end{bmatrix}
\]

Furthermore, \( K \in \mathbb{R}^{n \times n} \) is a positive and symmetric matrix (i.e. \( K = K^T > 0 \)) and \( J_d(q, p) \) is a free parameter that should be designed. The desired energy function is

\[
H_d(q, p) = \frac{1}{2} p^T M_d^{-1}(q) p + V_0(q)
\]

where \( M_d = M_d > 0 \) and \( V_0(q) \) are desired inertia matrix and potential energy, respectively. It is required that \( V_0(q) \) has an isolated minimum at the equilibrium point \( q \).

**Assumption 1:** It is assumed that the desired system (8) is zero state observable. This means that \( y(t) \equiv 0 \Rightarrow x(t) \equiv 0 \).

Designing a control law that transforms system (7) to the closed-loop system (8) has two essential steps [8]. The first step is shaping the energy of the system by modifying its total energy

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Moreover, solutions of system (14) are ultimately bounded. For more details, see [8, 25].

The following energy shaping term can be achieved by substituting (11) and (12) into (7) and comparing it with (8):

\[
\begin{align*}
\mathbf{u}_u &= \begin{bmatrix} G^T \mathbf{G} \end{bmatrix}^{-1} \begin{bmatrix} \nabla J \end{bmatrix}^T \nabla \mathbf{H} + J_2 \nabla \mathbf{p} \\
&= \begin{bmatrix} G^T \mathbf{G} \end{bmatrix}^{-1} \begin{bmatrix} \nabla J \end{bmatrix}^T \nabla \mathbf{H} + J_2 \mathbf{M}^{-1} \mathbf{G}^T \mathbf{p}
\end{align*}
\]  

(13)

For more details, see [8, 25].

The second term of (11) is selected as [8]

\[
\mathbf{u}_{\mathit{IDA-\text{PBC}}} = \mathbf{u}_a(q, p) + \mathbf{u}_d(q, p)
\]

(11)

The following energy shaping term can be achieved by substituting (11) and (12) into (7) and comparing it with (8):

\[
\begin{bmatrix} \mathbf{q} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} J_1(q, p) - R(q, p) \end{bmatrix}^T \nabla \mathbf{H} + \begin{bmatrix} 0 \\ G^T \mathbf{G} \end{bmatrix} \mathbf{d}(x, t) = \begin{bmatrix} 0 \\ \mathbf{M}^T \mathbf{M} \end{bmatrix} \mathbf{H}_d + \begin{bmatrix} \mathbf{G}^T \mathbf{G} \end{bmatrix} \mathbf{d}(x, t)
\]

(14)

The following proposition gives sufficient conditions that ensure ultimate boundedness of system (14) in the presence of matched uncertainties \( \mathbf{d}(x, t) \neq 0 \).

**Proposition 1:** Consider system (14), assume that the following conditions hold:

\[
\begin{align*}
\| \mathbf{d}(x, t) \| &\leq k_1 \| \mathbf{H}_d \| + \epsilon; \quad k_1 > 0; \quad \epsilon \geq 0 \\
\| \mathbf{H}_d \| &\leq k_2 \| x \|; \quad k_2 > 0
\end{align*}
\]

(15)

such that \( 0 < k_1 < k_{\text{min}}(K_0) \), where \( k_{\text{min}}(K_0) \) is the smallest eigenvalue of the positive-definite and symmetric matrix \( K_0 \). Moreover, \( \mathbf{d}(0, t) \neq 0, \forall t \geq 0 \). It means that \( \mathbf{d}(x, t) \) is non-vanishing matched bounded perturbation. In these situations, the solutions of system (14) are ultimately bounded.

**Proof:** Choose \( H_d \) as the Lyapunov function candidate [i.e. \( V(t, x) = H_d \)]. Taking its time derivative along the trajectories of (14) yields

\[
\begin{align*}
\dot{V}(t, x) &= -\beta_1 \| \mathbf{H}_d \| + e \mathbf{p} \\
&= -\beta_1 \| \mathbf{H}_d \| + k_1 \| x \| \\
&= -\beta_1 \| \mathbf{H}_d \| + k_1 \| x \| \\
&= -\beta_1 \| \mathbf{H}_d \| + k_1 \| x \|
\end{align*}
\]

(16)

Since \( J_2 \) is skew-symmetric (\( J_2^T = -J_2 \)); thus, \( \mathbf{H}_d = \mathbf{H}_d^T \mathbf{J}_1 \mathbf{p} - \mathbf{H}_d \mathbf{J}_1 \mathbf{p} = 0 \). Moreover, \( \mathbf{M} \) and \( \mathbf{M}^{-1} \) are symmetric matrices thus (16) can be rewritten as follows:

\[
\begin{align*}
\dot{V}(t, x) &= -\beta_1 \| \mathbf{H}_d \| + e \mathbf{p} \\
&= -\beta_1 \| \mathbf{H}_d \| + k_1 \| x \| \\
&= -\beta_1 \| \mathbf{H}_d \| + k_1 \| x \|
\end{align*}
\]

(17)

On the other hand, the assumption of system (11) is applied. Second, we aim to design an additional control input in order to preserve the asymptotic stability of the system in spite of time-varying matched uncertainties.

**Proposition 2:** Suppose that \( k_2 \mathbf{d}(x, t) = \mathbf{0} \), one has

\[
\begin{align*}
\dot{V}(t, x) &= -\beta_1 \| \mathbf{H}_d \| + e \mathbf{p} \\
&= -\beta_1 \| \mathbf{H}_d \| + k_1 \| x \| \\
&= -\beta_1 \| \mathbf{H}_d \| + k_1 \| x \|
\end{align*}
\]

(18)

Considering \( \epsilon \| \mathbf{H}_d \| \geq \theta \| \mathbf{H}_d \| \) \( \geq 0 \) which occurs in the region \( \epsilon \| \mathbf{H}_d \| \geq \theta \| \mathbf{H}_d \| \) holds. Moreover, considering (15), \( \| \mathbf{H}_d \| \leq k_1 \| x \| \). Consequently in the region \( \| x \| \geq \epsilon k_2 \mathbf{d} \), then

\[
\begin{align*}
\dot{V}(t, x) &\leq -\beta_1 \| \mathbf{H}_d \| + e \mathbf{p} \\
&= -\beta_1 \| \mathbf{H}_d \| + k_1 \| x \| \\
&= -\beta_1 \| \mathbf{H}_d \| + k_1 \| x \|
\end{align*}
\]

(19)

From (20), it is concluded that conditions of Theorem 1 are completely satisfied, since first \( H_d \) is a positive-definite function, so there exist upper and lower bounds of class \( \mathcal{X} \) functions for it and second \( \| \mathbf{H}_d \| \) is a positive function, and according to Assumption 1 \( \mathbf{y} \equiv 0 \Rightarrow \| \mathbf{H}_d \| \geq 0 \Rightarrow x \equiv 0 \). Therefore, solutions of the closed-loop system (14), which are obtained by applying IDA-PBC control law (11), are ultimately bounded in the presence of matched uncertainties. According to (20), the smaller \( \epsilon \) or the larger \( k_{\theta} \) is, the wider the range of analysis validation is. This means that the ultimate bound will be smaller, and therefore closer to the equilibrium point. In the next section, sufficient conditions for ultimate boundedness of solutions of system (5) in the presence of unmatched uncertainties are presented.

**Proof:** Choose \( H_d \) as the Lyapunov function candidate [i.e. \( V(t, x) = H_d \)]. Taking its time derivative along the trajectories of (14) yields

\[
\begin{align*}
\dot{V}(t, x) &= -\beta_1 \| \mathbf{H}_d \| + e \mathbf{p} \\
&= -\beta_1 \| \mathbf{H}_d \| + k_1 \| x \| \\
&= -\beta_1 \| \mathbf{H}_d \| + k_1 \| x \|
\end{align*}
\]

(16)
\[
g(x, t) = \begin{bmatrix} \delta_1(x, t) \\ \delta_2(x, t) \end{bmatrix}
\]

The following proposition provides sufficient conditions that ensure ultimate boundedness of the closed-loop system (21) in spite of unmatched perturbations.

**Proposition 2:** Consider system (21); assume that the following conditions hold:

\[
\| g(x, t) \| \leq k_1 \| (VH_d)^T \| + \epsilon_1 \quad \text{and} \quad \| i \| \leq k_2 \| x \| \quad \text{for all} \quad t \geq 0
\]

\[
\| VH_d \| \leq k_3 \| (VH_d)^T G \| \leq k_4 \| x \| \quad \text{and} \quad k_3, k_4 > 0
\]

such that \( k_2 \delta < \lambda_{\text{max}}(K_h) \) and \( \delta(x, t) \neq 0, \forall t \geq 0 \) then solutions of system (21) are ultimately bounded in the presence of unmatched perturbations.

**Proof:** Similar to the proof of Proposition 1, \( V(x, t) = H_d \) is chosen and its time derivative along the trajectories of (21) is taken as

\[
\dot{H}_d = (VH_d)^T q + (VH_d)^T p
\]

\[
= (VH_d)^T \frac{M^* - M}{2} (VH_d) + (VH_d)^T \delta_1
\]

\[
= - (VH_d)^T \frac{M^* - M}{2} (VH_d) + (VH_d)^T \delta_2
\]

\[
= - (VH_d)^T G \dot{1} (VH_d) + (VH_d)^T g \delta_1
\]

where \( (VH_d)^T \delta_1 + (VH_d)^T \delta_2 = (VH_d)^T g \delta_2 \). In the case that (22) is satisfied, by assuming \( 0 < \theta < \lambda_{\text{max}}(K_h) \) the following inequality is obtained:

\[
\dot{H}_d \leq -\lambda_{\text{max}}(K_h) \| (VH_d)^T G \|^2 + k_1 \| VH_d \|^2 + \epsilon_1 \| VH_d \|
\]

\[
\leq -\lambda_{\text{max}}(K_h) \| (VH_d)^T G \|^2 + k_3 k_4 \| VH_d \|^2 + \epsilon_1 \| VH_d \|
\]

\[
\leq - \frac{\lambda_{\text{max}}(K_h)}{\lambda_{\text{max}}(K_h) - k_2 \delta} \| (VH_d)^T G \|^2 + \epsilon_1 \| VH_d \|
\]

\[
\leq - \frac{\lambda_{\text{max}}(K_h)}{\lambda_{\text{max}}(K_h) - k_2 \delta} \| (VH_d)^T G \|^2 + \epsilon
\]

Thus, from (29) the following inequality is obtained:

\[
\| \delta(x, t) \| \leq k_3 \| VH_d \| (q) \| + \epsilon \quad \text{for all} \quad t \geq 0
\]

\[
\| VH_d \| \leq k_4 \| x \|
\]

\[
\| VH_d \| \leq k_5 \| VH_d \| \| G \| \quad \text{for all} \quad t \geq 0
\]

\[
\| VH_d \| \leq k_6 \| VH_d \| \| G \| \quad \text{for all} \quad t \geq 0
\]

In brief, the IDA-PBC controller guarantees asymptotical stability of the nominal system; however, it is shown in this paper that in the presence of uncertainties (satisfying the given conditions), it leads to ultimate boundedness of the uncertain system. Therefore, in what follows the objective is to design a robustifying additional control term and adding it to the IDA-PBC controller to preserve the asymptotic stability of the system in the presence of non-vanishing matched uncertainties.

\[
\nu(x) = -\eta(x) \frac{\omega}{\omega} \| + \epsilon
\]

\[
\| VH_d \| \leq - \frac{\lambda_{\text{max}}(q)}{\lambda_{\text{max}}(K_h) - k_2 \delta} \| VH_d \| (q) \| + \epsilon \quad \text{for all} \quad t \geq 0
\]

The following proposition provides sufficient conditions that ensure ultimate boundedness of the closed-loop system (21) in spite of non-vanishing matched uncertainties.

**Proposition 5:** In this section, it is intended to design an additional controller \( \nu \), which is added to \( u_{\text{IDA-PBC}} \) such that the new controller \( u = u_{\text{IDA-PBC}} + \nu \) leads to robust stability in spite of non-vanishing matched uncertainties.

The equation of system (5) in the presence of matched uncertainties (i.e. \( \delta(x, t) \neq 0 \) and \( \delta(x, t) = 0 \)) and new additional control input \( \nu \) has the following form:

\[
\begin{bmatrix}
q \\
p
\end{bmatrix} =
\begin{bmatrix}
0 & I_n \\
-I_a & \frac{0}{\nu(x)} - G(q)
\end{bmatrix}
\begin{bmatrix}
\nu(x) \\
p
\end{bmatrix}
\]

Applying the IDA-PBC controller (11), results in

\[
\begin{bmatrix}
q \\
p
\end{bmatrix} = (J_d(q, p) - R_d(q, p)) VH_d + \begin{bmatrix}
0 \\
G(q)
\end{bmatrix}(\delta(x, t) + \lambda_{\text{max}}(K_h))
\]

Thus, from (29) the following inequality is obtained:

\[
\| VH_d \| \leq - \frac{\lambda_{\text{max}}(K_h)}{\lambda_{\text{max}}(K_h) - k_2 \delta} \| VH_d \| \| G \| \quad \text{for all} \quad t \geq 0
\]

\[
\| VH_d \| \leq k_7 \| VH_d \| \| G \| \quad \text{for all} \quad t \geq 0
\]

\[
\| VH_d \| \leq k_8 \| VH_d \| \| G \| \quad \text{for all} \quad t \geq 0
\]

\[
\| VH_d \| \leq k_9 \| VH_d \| \| G \| \quad \text{for all} \quad t \geq 0
\]

Considering \( \delta(x, t) \neq 0 \) and \( \delta(x, t) = 0 \) which occurs in the region that \( \| (VH_d)^T G \| \leq k_2 \| x \| \) holds. Moreover, considering (22) \( \| VH_d \| \| G \| \leq k_3 \| x \| \) is bounded. Consequently, if the robust controller is chosen as

\[
\nu(x) = -\eta(x) \frac{\omega}{\omega} \| + \epsilon
\]

where \( \eta(x) \) is a non-negative function, then inserting \( (32) \) into \( (31) \) yields to

\[
H_d \leq - \frac{\lambda_{\text{max}}(K_h)}{\lambda_{\text{max}}(K_h) - k_2 \delta} \| VH_d \| (q) \| - \eta(x) \| \omega \| \| + \omega \| \| k_2 \| VH_d \| (q) \| + \epsilon \| \quad \text{for all} \quad t \geq 0
\]

Considering \( \eta(x) \geq k_3 \| VH_d \| (q) \| + \epsilon \) results in the following inequality:
Thus, the robust controller guarantees stability of the system controlled by the proposed robust controller in the presence of non-vanishing matched uncertainties. Taking Assumption 1 it is concluded that (32) guarantees asymptotic stability of the system controlled by the proposed robust controller in spite of non-vanishing uncertainties.

As the second approach of designing $\nu(x)$, suppose that (30) is satisfied with $\omega$.

Thus, according to (29), the following inequality is achieved:

$$ H_d \leq -\lambda_{\text{min}}(K_i) \| (V_p H_d)^T G(q) \| \hat{\nu} + \| \omega \| + k_1 \| (V_p H_d)^T G(q) \| + \| \omega \| $$

Taking

$$ \nu(x) = -\eta(x) \text{sgn}(\omega) $$

where $\eta(x) \geq (k_1 \| (V_p H_d)^T G(q) \| + \varepsilon)$ and $\text{sgn}(\omega)$ is a function whose ith element is $\text{sgn}(\omega_i)$; thus, $\omega^T \text{sgn}(\omega) = \sum |\omega_i| = \| \omega \|_1$; therefore, substituting (37) into (36) yields to (see (38)). Consequently, similar to the previous case, the designed robust controller guarantees asymptotic stability of the system in the face of non-vanishing matched uncertainty. Indeed, this result is obtained because of the discrete structure of robustifying term $\nu(x)$. It should be noted that for single-input systems, $\omega$ is scalar, and thus $\| \omega \| = \| \omega \|_1 = |\omega|$.

Fig. 1 shows the block diagram of the closed-loop system.

Remark 1: The proposed controllers are the functions of the state variables. Having the state variables, for real implementation of the designed controller, it is necessary to discretise the controller and replacing the continuous variables with their sampled versions. Since the choice of sampling time affects the estimation accuracy, by appropriate selection of sampling time the discretised controller leads to acceptable performance.

6 Practical design examples

To investigate the performance of the developed controller, it is applied on two mechanical systems: inertia wheel pendulum and single-link elastic joint robot.

6.1 Inertia wheel pendulum system

Fig. 2 represents the inertia wheel pendulum system. The mechanical equations of this system with external disturbances have the following form [26]:

$$ \begin{bmatrix} I_1 + I_2 & I_2 & 0 & -mgl \sin \theta_1 \\ I_2 & I_2 & -mgl \sin \theta_1 & 0 \\ 0 & -mgl \sin \theta_1 & (I_2 + I_3) \omega_2 & 0 \\ 0 & 0 & 0 & 0 \\ \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\omega}_2 \\ \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -mgl \sin \theta_1 \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left[ u + \delta(x, t) \right] $$

Fig. 1 Block diagram of the closed-loop system

Fig. 2 Inertia wheel pendulum
where \( \theta_1, \theta_2 \) and \( l_1, l_2 \) are respective angles and moments of inertia of the pendulum and disc, \( m \) is the pendulum mass, \( l \) is its length, \( g \) is the gravity constant, and \( u \in R \) is the control input torque. Moreover, \( \delta_t(x, t) \) is the perturbation torque applied to the pendulum.

By changing the coordinates as

\[
\begin{bmatrix}
q_1 \\
q_2
\end{bmatrix} = \begin{bmatrix}
\theta_1 \\
\theta_1 + \theta_2
\end{bmatrix}
\]

(40)

The following description of the system is obtained:

\[
\begin{bmatrix}
l_1 & 0 \\
nl_2 & 0
\end{bmatrix} \begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2
\end{bmatrix} + \begin{bmatrix}
-mgl \sin q_1 \\
0
\end{bmatrix} = \begin{bmatrix}
-1 \\
1
\end{bmatrix} (u + \delta_t(x, t))
\]

(41)

System (41) can be written in PCH form (7) with \( p = [p_1, p_2]^T = [Iq_1, Iq_2]^T \) and

\[
M = \begin{bmatrix}
l_1 & 0 \\
nl_2 & 0
\end{bmatrix}, \quad G = \begin{bmatrix}
-1 \\
1
\end{bmatrix}, \quad V_0(q) = mgl(\cos q_1 - 1)
\]

(42)

By assuming \( M_d = \begin{bmatrix}
a_1 & a_2 \\
a_2 & a_1
\end{bmatrix} > 0 \), the following controller was obtained by IDA-PBC approach [8]:

\[
u_{IDA-PBC} = \gamma_1 \sin q_1 + k_1(q_1 + \gamma_1 q_1) + k_2(q_1 + \gamma_1 q_1)
\]

(43)

where \( \gamma_1, \gamma_2, k_1 \) and \( k_2 \) satisfy the following inequalities:

\[
\gamma_1 > \frac{a_2}{a_1 + a_2} mgl, \quad \gamma_2 > \frac{l_1}{l_2 \gamma_1 - mgl}
\]

(44)

Now, the matched uncertainty is considered with the following structure:

\[
\delta_t(x, t) = \alpha_t(t)p_2 + \alpha_0(t);
\]

(45)

Simulations are done by assuming mgl = 10, \( l_1 = 0.1 \text{ kg m}^2 \), \( l_2 = 0.2 \text{ kg m}^2 \), \( \gamma_1 = 20 \), \( \gamma_2 = 1.5 \), \( u \in R \), \( \alpha_t(t) = 3 \sin(4t) \), \( a_0(t) = 3 \sin(4t) \), \( K_v = 3 \), \( k_2 = 8 \), \( M_d = \begin{bmatrix}1 & -2 \\ -1 & 5\end{bmatrix} \) and with initial condition \( (q_1, q_2, \dot{q}_1, \dot{q}_2) = (1, 0, -1, 0) \). Furthermore, since the discrete structure of \( \kappa(x) \) leads to chattering phenomena, thus, the sign function is approximated with the saturation function with a high slope to avoid chattering [22, 27].

Fig. 2 depicts the angle of pendulum (\( \theta_1 \)) and angle of disc (\( \theta_2 \)) in the closed-loop system and in the case that the system is subjected to uncertain term \( \delta_t(x, t) \). Performance of both IDA-PBC and proposed robust controllers are clearly shown in this figure. As seen in Fig. 3, it is concluded that the proposed robust controller can effectively reject the applied time-varying uncertainty while the IDA-PBC controller does not possess a robust manner and only conserves that states of the system are bounded.

In Fig. 4, the corresponding control inputs of both methods are illustrated.

**6.2 Single-link elastic joint robot**

A single-link elastic joint system is a single-link manipulator with revolute joint actuated with a DC motor (Fig. 5). The elasticity of the joint is modelled as a linear torsional spring with stiffness \( k \). The equations of motion are as follows [28]:

\[
\begin{align*}
Iq_1 + mgl \sin q_1 + k(q_1 - q_2) &= 0 \\
Jq_2 + k(q_2 - q_1) &= u
\end{align*}
\]

(46)

Fig. 3 Time response of state variables of the closed-loop Inertia wheel pendulum in the presence of matched uncertainty

(a) Via IDA-PBC controller, 
(b) Via the proposed controller

Fig. 4 Time response of control input in the first example

Fig. 5 Single-link elastic joint robot

where \( q_1, q_2 \) are the angular positions of the arm and the motor, respectively, and \( q = [q_1, q_2]^T \). Also, \( I \) is the moment of inertia of the arm and \( J \) is that of the motor. Furthermore, \( m \) is the mass of arm and \( l \) is the effective length of it. Moreover, \( u \in R \) is the applied motor torque and the stiffness of spring is assumed as \( k \).

System (46) can be written in PCH form (7) with \( p = [p_1, p_2]^T = [Iq_1, Jq_2]^T \) and

\[
M = \begin{bmatrix}
I & 0 \\
0 & J
\end{bmatrix}, \quad G = \begin{bmatrix}
0 \\
1
\end{bmatrix}, \quad V_0(q) = \frac{1}{2} k(q_1 - q_2)^2 - mgl \cos q_1
\]

(47)

According to IDA-PBC method [8], in order to obtain the energy shaping term \( \psi_o \) of the controller (11) the following condition, in case of underactuated systems, must be satisfied:

\[
G^T \{ \nabla_x h - M_d M^\perp \nabla_y H_4 + JM_d^\perp p \} = 0
\]

(48)

where \( G^\perp \) is a full rank left annihilator of \( G \) (i.e. \( G^\perp G = 0 \)). Equation (48) can be separated into two terms as
\[ G^{-1} \{ q^T M^{-1} p \} - M q \nabla_q (q^T M^{-1} p) + 2 J M^{-1} p = 0 \]  \hspace{1cm} (49)

\[ G^{-1} \{ V_{q_0} - M q \nabla_q V_{q_0} \} = 0 \]  \hspace{1cm} (50)

Such that (49) and (50) correspond to kinetic and potential energy shaping, respectively. Since \( M(q) \) is constant, we can take \( J_1 = 0 \). It is supposed that

\[ M_a = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_1 \end{bmatrix} \]  \hspace{1cm} (51)

Such that \( a_1 > 0 \) and \( a_1 a_2 > a_2^2 \) to ensure the positive definiteness condition of the desired inertia matrix. Considering (47) and (51), condition (50) can be written as follows:

\[ \frac{\alpha}{T} V_{q_i} (V_{q_0}) + \frac{\alpha}{T} V_{q_j} (V_{q_0}) = k (q_i - q_j) + m g l \sin q_i \]  \hspace{1cm} (52)

The general solution of (52) has the following form:

\[ V_{q_0} (q) = - \frac{1}{\alpha} g l \cos \alpha_1 q_i + \frac{1}{2 \alpha_1} \frac{k}{\alpha_1} q_1 - \frac{1}{2 \alpha_1} \frac{k}{\alpha_1} q_1 \]  \hspace{1cm} (53)

\( V_{q_0} (q) \) must have an isolated minimum at the equilibrium point \( q' = (0, 0) \). Therefore, it is necessary that \( V_{q_0} (0) = 0 \) and \( V_{q_0} (0) > 0 \). The condition \( V_{q_0} (0) = 0 \) is satisfied but \( V_{q_0} (0) > 0 \) holds if \( \alpha < 0 \). Thus, the energy shaping term \( u_{es} \) is given as

\[ u_{es} = \left( G^T G \right)^{-1} G^T \{ V_{q_0} H - M \nabla_q H \} \]

\[ = - k (q_i - q_j) - \frac{\alpha}{T} \nabla_q q_i (V_{q_0}) - \frac{\alpha}{T} \nabla_q q_j (V_{q_0}) \]

\[ = k \left( 1 - \frac{a_2 k}{\alpha} \right) q_j + \frac{a_1 m g l}{\alpha} \sin q_i + k \left( 1 + \frac{a_2 k}{\alpha} \right) q_j \]  \hspace{1cm} (54)

The second term of IDA-PBC controller \( u_{ba} \) is given as

\[ u_{ba} = - K_v G^T H = \frac{-K_v}{a_1 a_2 - a_2^2} (-a_1 p_1 + a_1 p_3) \]  \hspace{1cm} (55)

Simulations are carried out by considering the time-varying matched uncertain term as \( \delta_i (x, t) = \alpha (t) p_i + b(t) \) and choosing system parameters as \( l = 0.0059 \text{ km} \), \( J = 0.0021 \text{ kg m}^2 \), \( k = 1.611 \text{ N m} \), \( m = 0.404 \text{ kg} \), \( k(t) = 3 \sin 3t \), \( b(t) = \sin 4t \) and \( l = 0.06 \text{ m} \). The initial condition of the system is \( (q_i, q_j, q_1, q_3) = (\frac{\pi}{4}, 0, \frac{\pi}{3}, 0) \). As seen in Fig. 6, the proposed robust controller \( u = u_{\text{IDA-PBC + v}} \) was able to damp the applied time-varying uncertainty. However, states of the system controlled by IDA-PBC controller are bounded, however, do not converge to the desired equilibrium point.

To investigate the effect of unmatched disturbances, two pulses are applied during \( [1-3] \text{ s} \) with the amplitudes of 3 and 2, respectively. Fig. 7 reveals the robust performance of the proposed robust controller in the face of unmatched uncertainties.

7 Conclusion

In this paper, the robustness problems of IDA-PBC method against non-vanishing matched and unmatched uncertainties were investigated. Sufficient conditions were derived to ensure ultimate boundedness of the system solutions in the presence of non-vanishing matched and unmatched uncertainties. Furthermore, a robust additional control input was designed such that the overall control law had a robust manner which led to asymptotic stability of the uncertain closed-loop system. Finally, the effectiveness of the proposed controller in rejection of non-vanishing perturbations was also shown through simulations for two practical examples.

8 References

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Fig. 6 Time response of state variables of the closed-loop single-link elastic joint robot in the presence of matched uncertainty
(a) Via IDA-PBC controller,
(b) Via the proposed controller

Fig. 7 Time response of state variables of the closed-loop single-link elastic joint robot in the presence of unmatched uncertainty via the proposed controller
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