Singularity of Mean Curvature Flow of Lagrangian Submanifolds

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Abstract

In this article we study the tangent cones at first time singularity of a Lagrangian mean curvature flow. If the initial compact submanifold $\Sigma_0$ is Lagrangian and almost calibrated by $\text{Re} \Omega$ in a Calabi-Yau $n$-fold $(M, \Omega)$, and $T > 0$ is the first blow-up time of the mean curvature flow, then the tangent cone of the mean curvature flow at a singular point $(X_0, T)$ is a stationary Lagrangian integer multiplicity current in $\mathbb{R}^{2n}$ with volume density greater than one at $X_0$. When $n = 2$, the tangent cone consists of a finite union of more than one 2-planes in $\mathbb{R}^4$ which are complex in a complex structure on $\mathbb{R}^4$.

1 Introduction

Let $M$ be a compact Calabi-Yau manifold of complex dimension $n$ with a Kähler form $\omega$, a complex structure $J$, a Kähler metric $g$ and a parallel holomorphic $(n,0)$-form $\Omega$ of unit length. An immersed submanifold $\Sigma$ in $M$ is Lagrangian if $\omega|_{\Sigma} = 0$. The induced volume form $d\mu_\Sigma$ on a Lagrangian submanifold $\Sigma$ from the Ricci-flat metric $g$ is related to $\Omega$ by

$$\Omega|_{\Sigma} = e^{i\theta} d\mu_\Sigma = \cos \theta d\mu_\Sigma + i \sin \theta d\mu_\Sigma,$$

(1)

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where the phase function $\theta$ is multi-valued and is well-defined up to an additive constant $2k\pi, k \in \mathbb{Z}$. Nevertheless, $\cos \theta$ and $\sin \theta$ are single valued function on $\Sigma$. For any tangent vector $X$ to $M$ a straightforward calculation shows

$$X\theta = -g(H, JX)$$  \hspace{1cm} (2)

where $H$ is the mean curvature vector of $\Sigma$ in $M$ (cf. [HL], [TY]). Equivalently, $H = J\nabla \theta$. The Lagrangian submanifold $\Sigma$ is special, i.e. it is a minimal submanifold, if and only if $\theta$ is constant. When $\theta$ is constant on a Lagrangian submanifold $\Sigma$, the real part of $e^{-i\theta} \Omega$ is a calibration of $M$ with comass one and $\Sigma$ is a volume minimizer in its homology class [HL]. Let $\text{Re} \Omega$ be the real part of $\Omega$. A Lagrangian submanifold is called almost calibrated by $\text{Re} \Omega$ if $\cos \theta > 0$.

Constructing minimal Lagrangian submanifolds is an important but very challenging task. In a compact Kähler-Einstein surface, Schoen and Wolfson [ScW] have shown the existence of a branched surface which minimizes area among Lagrangian competitors in each Lagrangian homology class, by variational method.

For a one-parameter family of immersions $F_t = F(\cdot, t) : \Sigma \rightarrow M$, we denote the image submanifolds by $\Sigma_t = F_t(\Sigma)$. If $\Sigma_t$ evolves along the gradient flow of the volume functional, the first variation of the volume functional asserts that $\Sigma_t$ satisfy a mean curvature flow equation:

$$\left\{ \begin{array}{l}
\frac{d}{dt} F(x, t) = H(x, t) \\
F(x, 0) = F_0(x),
\end{array} \right. \hspace{1cm} (3)$$

When $\Sigma$ is compact the mean curvature flow (3) has a smooth solution for short time $[0, T)$ by the standard parabolic theory. If $\Sigma_0$ is Lagrangian in a Kähler-Einstein ambient space $M$, Smoczyk has shown that $\Sigma_t$ remains Lagrangian for $t < T$ and the phase function $\theta$ evolves by

$$\frac{d\theta}{dt} = \Delta \theta$$  \hspace{1cm} (4)

where $\Delta$ is the Laplacian of the induced metric on $\Sigma_t$ ([Sm1-3], also see [TY] for a derivation of (4)). It then follows that

$$\frac{\partial \cos \theta}{\partial t} = \Delta \cos \theta + |H|^2 \cos \theta.$$  \hspace{1cm} (5)

If the initial Lagrangian submanifold $\Sigma_0$ is almost calibrated, $\Sigma_t$ is almost calibrated, i.e. $\cos \theta > 0$, along a smooth mean curvature flow by the parabolic maximum principle.

It is well-known that if $|A|^2$, where $A$ is the second fundamental form on $\Sigma_t$, is bounded uniformly as $t \rightarrow T > 0$ then (3) admits a smooth solution over $[0, T + \epsilon)$ for some $\epsilon > 0$. When $\max_{\Sigma_t} |A|^2$ becomes unbounded as $t \rightarrow T$, we
say that the mean curvature flow develops a singularity at \( T \). A lot of work has been devoted to understand these singularities (cf. [CL1-2], [E1-2], [H1-3], [HS1-2], [I1], [Wa], [Wh1-3].)

In this paper, we shall study the tangent cones at singularities of the mean curvature flow of a compact Lagrangian submanifold in a compact Calabi-Yau manifold. Especially, we shall focus on the structure of tangent cones of the mean curvature flow where a singularity occurs at the first singular time \( T < \infty \).

To describe the tangent cones, suppose that \((X_0, T)\) is a singular point of the flow (3), i.e. \(|A(x, t)|\) becomes unbounded when \((x, t) \to (X_0, T)\). For an arbitrary sequence of numbers \( \lambda \to \infty \) and any \( t < 0 \), if \( T + \lambda^2 t > 0 \) we set

\[
F_\lambda(x, t) = \lambda(F(x, T + \lambda^2 t) - X_0).
\]

We denote the scaled submanifold by \((\Sigma^\lambda t, d\mu^\lambda t)\). If the initial submanifold is Lagrangian and almost calibrated by \( \Re \Omega \), it is proved in Proposition 2.3 that there is a subsequence \( \lambda_i \to \infty \) such that for any \( t < 0 \), \((\Sigma^\lambda_i t, d\mu^\lambda_i t)\) converges to \((\Sigma^\infty, d\mu^\infty)\) in the sense of measures; the limit \( \Sigma^\infty \) is called a tangent cone arising from the rescaling \( \lambda \), or simply a \( \lambda \) tangent cone at \((X_0, T)\). This tangent cone is independent of \( t \) as shown in Proposition 2.3.

There is also a time dependent scaling which we would like to consider

\[
\tilde{F}(\cdot, s) = \frac{1}{\sqrt{2(T - t)}} F(\cdot, t),
\]

where \( s = -\frac{1}{2} \log(T - t), c_0 \leq s < \infty \). Here we have chosen the coordinates so that \( X_0 = 0 \). Rescaling of this type arises naturally in classification of singularities of mean curvature flows [H2]: assume \( \lim_{t \to T^-} \max_{\Sigma_t} |A|^2 = \infty \), if there exists a positive constant \( C \) such that \( \limsup_{t \to T^-} ((T - t) \max_{\Sigma_t} |A|^2) \leq C \), the mean curvature flow \( F \) has a Type I singularity at \( T \); otherwise it has a Type II singularity at \( T \). Denote \( \Sigma_s \) the rescaled submanifold by \( \tilde{F}(\cdot, s) \). If a subsequence of \( \Sigma_s \) converges in measures to a limit \( \Sigma_\infty \), then the limit is called a tangent cone arising from the time dependent scaling at \((X_0, T)\), or simply a \( t \) tangent cone. In this paper, a tangent cone of the mean curvature flow at \((X_0, T)\) means either a \( \lambda \) tangent cone or a \( t \) tangent cone at \((X_0, T)\).

The main result of this paper is

**Theorem 1.1** Let \((M, \Omega)\) be a compact Calabi-Yau manifold of complex dimension \( n \). If the initial compact submanifold \( \Sigma_0 \) is Lagrangian and almost calibrated by \( \Re \Omega \), and \( T > 0 \) is the first blow-up time of the mean curvature flow (3), and \((X_0, T)\) is a singular point, then the tangent cone of the mean curvature flow at \((X_0, T)\) is a stationary Lagrangian integer multiplicity current in \( \mathbb{R}^{2n} \) with volume density greater than one at \( X_0 \). When \( n = 2 \), the tangent cone consists of a finite union of more than one 2-planes in \( \mathbb{R}^4 \) which are complex in a complex structure on \( \mathbb{R}^4 \).
For symplectic mean curvature flow in Kähler-Einstein surfaces, results similar to Theorem 1.1 was obtained in [CL1]. The authors are grateful to Professor Gang Tian for stimulating conversations.

2 Existence of $\lambda$ tangent cones

This section contains basic formulas and estimates which are essential for this article. First, we will derive a monotonicity formula which has a weight function introduced by the $n$-form Re $\Omega$. Second, we use the monotonicity formula to derive three integral estimates, which roughly say that when averaged over any time interval the mean curvature vector $H_\lambda$ and the phase function $\cos \theta_\lambda$ both tend to 0 in $L^2$ norm over a fixed ball near the singularity, as $\lambda \to \infty$. Another direct consequence of the monotonicity formula is that there is an upper bound of the volume density of the rescaled submanifolds $\Sigma^\lambda_t$, which allows us to extract converging subsequence in measure.

2.1 A weighted monotonicity formula

Let $H(X, X_0, t_0, t)$ be the backward heat kernel on $\mathbb{R}^k$. Let $N_t$ be a smooth family of submanifolds of dimension $n$ in $\mathbb{R}^k$ defined by $F_t : N \to \mathbb{R}^k$. Define

$$\rho(X, t) = \frac{1}{(4\pi(t_0 - t))^{n/2}} \exp\left(-\frac{|X - X_0|^2}{4(t_0 - t)}\right)$$

for $t < t_0$.

A straightforward calculation (cf. [CL1], [H1], [Wa]) shows

$$\frac{\partial}{\partial t} \rho = \left(\frac{n}{2(t_0 - t)} - \frac{H \cdot (X - X_0)}{2(t_0 - t)} - \frac{|X - X_0|^2}{4(t_0 - t)^2}\right) \rho$$

and along $N_t$

$$\Delta \rho = \left(\frac{|X - X_0, \nabla X|^2}{4(t_0 - t)^2} - \frac{\langle X - X_0, \Delta X \rangle}{2(t_0 - t)} - \frac{|
abla X|^2}{2(t_0 - t)}\right) \rho$$

where $\Delta, \nabla$ are on $N_t$ in the induced metric. Let $N_t = \Sigma_t$ be a smooth 1-parameter family of compact Lagrangian submanifolds in a compact Calabi-Yau manifold $(M, \Omega)$ of complex dimension $n$. Note that in the induced metric on $\Sigma_t$

$$|
abla F|^2 = n \quad \text{and} \quad \Delta F = H.$$ 

Therefore

$$\left(\frac{\partial}{\partial t} + \Delta\right) \rho = - \left|H + \frac{(F - X_0)_{\perp}}{2(t_0 - t)}\right|^2 \rho - |H|^2 \rho.$$  

(8)
On $\Sigma_t$ we set 
\[ v = \cos \theta. \]
Denote the injectivity radius of $(M,g)$ by $i_M$. For $X_0 \in M$, take a normal coordinate neighborhood $U$ and let $\phi \in C_0^\infty(B_{2r}(X_0))$ be a cut-off function with $\phi \equiv 1$ in $B_r(X_0)$, $0 < 2r < i_M$. Using the local coordinates in $U$ we may regard $F(x,t)$ as a point in $\mathbb{R}^{2n}$ whenever $F(x,t)$ lies in $U$. We define
\[ \Psi(X_0, t_0, t) = \int_{\Sigma_t} \frac{1}{v} \phi(F) \rho(F, X_0, t, t_0) d\mu_t \]
where $\rho$ is defined by (7) by taking $k = 2n$.

**Proposition 2.1** Let $F_t : \Sigma \to M$ be a smooth mean curvature flow of a compact Lagrangian submanifold $\Sigma_0$ in a compact Calabi-Yau manifold $M$ of complex dimension $n$. Suppose that $\Sigma_0$ is almost calibrated by $\text{Re}\Omega$. Then there are positive constants $c_1$ and $c_2$ depending only on $M$, $F_0$ and $r$ which is the constant in the definition of $\phi$, such that
\[ \partial_t \left( e^{c_1 \sqrt{t_0 - t}} \int_{\Sigma_t} \frac{1}{v} \phi d\mu_t \right) \leq -e^{c_1 \sqrt{t_0 - t}} \int_{\Sigma_t} \frac{1}{v} \phi \rho \left( 2 \frac{|\nabla v|^2}{v^2} + \left| H + \frac{(F - X_0) \perp}{2(t_0 - t)} \right|^2 + \frac{|H|^2}{2} \right) \]
\[ + c_2 e^{c_1 \sqrt{t_0 - t}}. \]

**Proof.** Notice that
\[ \Delta F = H + g^{ij} \Gamma^\alpha_{ij} v_\alpha \]
where $v_\alpha, \alpha = 1, \ldots, n$ is a basis of $T^\perp \Sigma_t$, $g^{ij}$ is the induced metric on $\Sigma_t$ and $\Gamma^\alpha_{ij}$ is the Christoffel symbol on $M$.

Equation (8) reads as
\[ \left( \frac{\partial}{\partial t} + \Delta \right) \rho = -\left( H + \frac{(F - X_0) \perp}{2(t_0 - t)} \right)^2 - |H|^2 + \frac{g^{ij} \Gamma^\alpha_{ij} v_\alpha \cdot (F - X_0)}{t_0 - t} \rho. \]

From (5) we have
\[ \frac{\partial}{\partial t} \frac{1}{v} = \frac{\Delta}{v} - \frac{|H|^2}{v} - \frac{2|\nabla v|^2}{v^3} \]
and
\[ \frac{d}{dt} d\mu_t = -|H|^2 d\mu_t. \]
Moreover
\[ \frac{\partial \phi(F)}{\partial t} = \nabla \phi \cdot H. \]
Now we have
\[
\frac{d}{dt} \int_{\Sigma_t} \frac{1}{v} \phi \rho
\]
\[
= \int_{\Sigma_t} \phi \rho \Delta \frac{1}{v} - \int_{\Sigma_t} \left( \frac{|H|^2}{v} + \frac{2}{v^3} |\nabla v|^2 \right) \phi \rho + \int_{\Sigma_t} \frac{1}{v} \nabla \phi \cdot H \rho
\]
\[
- \int_{\Sigma_t} \frac{1}{v} \phi \left( \Delta \rho + \left( \frac{1}{v} + \frac{1}{v^2} \phi \Delta \rho \right) - \int_{\Sigma_t} \frac{1}{v} \phi \rho \Gamma_{ij}^\alpha v_\alpha \cdot (F - X_0) \right)
\]
\[
- \int_{\Sigma_t} \frac{1}{v^2} \phi \left( \frac{2}{v} |\nabla v|^2 + 1 \right) H + \left( \frac{F - X_0}{2(t_0 - t)} \right)^2 + |H|^2
\]
\[
\leq - \int_{\Sigma_t} \phi \rho \left( \frac{2}{v^2} |\nabla v|^2 + 1 \right) H + \left( \frac{F - X_0}{2(t_0 - t)} \right)^2 + |H|^2
\]
\[
+ \int_{\Sigma_t} \phi \rho \left( \frac{1}{v} + \frac{1}{v^2} \phi \Delta \rho \right) - \int_{\Sigma_t} \frac{1}{v} \phi \rho \Gamma_{ij}^\alpha v_\alpha \cdot (F - X_0)
\]
\[
+ \int_{\Sigma_t} \frac{1}{v^2} \rho \left( \frac{1}{v^2} |H|^2 + \frac{1}{4e^2} |\nabla \phi|^2 \right)
\]
\[
(11)
\]
where we used Cauchy-Schwartz inequality for $\nabla \phi \cdot H$. By Stokes formula
\[
\int_{\Sigma_t} \phi \rho \left( \frac{1}{v} + \frac{1}{v^2} \phi \Delta \rho \right) - \int_{\Sigma_t} \frac{1}{v} \phi \rho \left( \frac{2}{v^2} |\nabla v|^2 + 1 \right) H + \left( \frac{F - X_0}{2(t_0 - t)} \right)^2 + |H|^2
\]
\[
= 2 \int_{\Sigma_t} \frac{1}{v} \nabla \phi \nabla \rho + \int_{\Sigma_t} \frac{1}{v} \rho \Delta \phi.
\]
Since $\phi \in C^\infty_0(B_{2r}(X_0), \mathbb{R}^+)$, we have (cf. Lemma 6.6 in [Il])
\[
\frac{|\nabla \phi|^2}{\phi} \leq 2 \max_{\phi > 0} |\nabla^2 \phi|.
\]
Note that $\nabla \phi \equiv 0$ in $B_r(X_0)$, so $|\rho \Delta \phi|$ and $|\nabla \phi \cdot \nabla \rho|$ are bounded in $B_{2r}(X_0)$. Hence
\[
\int_{\Sigma_t} \frac{1}{v} \rho \Delta \phi + \int_{\Sigma_t} \frac{1}{v} \nabla \phi \cdot \nabla \rho \leq C \int_{\Sigma_t} \frac{1}{v} d\mu_t \leq \frac{C}{\min_{\Sigma_0} v} \text{vol}(\Sigma_0)
\]
\[
(12)
\]
where $C$ depends only on $r$, $\max(|\nabla^2 \phi| + |\nabla \phi|)$. Since $\Gamma_{ij}^\alpha(X_0) = 0$, we may choose $r$ sufficiently small such that
\[
|g^{ij} \Gamma_{ij}^\alpha(F)| \leq C |F - X_0|
\]
in $B_{2r}(X_0)$ for some constant $C$ depending on $M$. We claim
\[
\frac{|g^{ij} \Gamma_{ij}^\alpha v_\alpha \cdot (F - X_0)|}{t_0 - t} \rho(F,t) \leq c_1 \frac{\rho(F,t)}{\sqrt{t_0 - t}} + C.
\]
In fact it suffices to show for any $x$ and $s > 0$
\[
\frac{x^2 e^{-x^2/s}}{s^{n/2}} \leq C \left( 1 + \frac{1}{s^{1/2}} \frac{e^{-x^2/s}}{s^{n/2}} \right).
\]
To see this, let $y = x^2/s$ and then it is easy to verify that
\[ y \leq C \left( s^{n/2} e^y + \frac{1}{s^{1/2}} \right) \]
holds trivially if $y \leq 1/s^{1/2}$ and follows from $y^{n+1} \leq C e^y$ if $y > 1/s^{1/2}$ for some $C$. So (13) is established.

Letting $\epsilon^2 = 1/2$ in (11) and applying (12), (13) to (11) we have
\[
\frac{\partial}{\partial t} \Psi \leq -\int_{\Sigma_t} \frac{1}{v} \phi \rho \left( \frac{2|\nabla v|^2}{v^2} + \left| \mathbf{H} + \frac{(F - \mathbf{X}_0)_{\perp}}{2(t_0 - t)} \right|^2 + \frac{\left| \mathbf{H} \right|^2}{2} \right) + \frac{c_1}{\sqrt{t_0 - t}} \Psi + c_2.
\]
The proposition follows. Q.E.D.

Suppose that $(X_0, T)$ is a singular point of the mean curvature flow (3). We now describe the rescaling process around $(X_0, T)$. For any $t < 0$, we set
\[
F_\lambda(x, t) = \lambda(F(x, T + \lambda^{-2}t) - X_0)
\]
where $\lambda$ are positive constants which go to infinity. The scaled submanifold is denoted by $\Sigma^\lambda_t = F_\lambda(\Sigma, t)$ on which $d\mu^\lambda_t$ is the area element obtained from $d\mu_t$. If $g^\lambda$ is the metric on $\Sigma^\lambda_t$, it is clear that
\[
\begin{align*}
  g^\lambda_{ij} &= \lambda^2 g_{ij}, \\
  (g^\lambda)^{ij} &= \lambda^{-2} g^{ij}.
\end{align*}
\]
We therefore have
\[
\begin{align*}
  \frac{\partial F_\lambda}{\partial t} &= \lambda^{-1} \frac{\partial F}{\partial t} \\
  \mathbf{H}_\lambda &= \lambda^{-1} \mathbf{H} \\
  |\mathbf{A}_\lambda|^2 &= \lambda^{-2} |\mathbf{A}|^2.
\end{align*}
\]
It follows that the scaled submanifold also evolves by a mean curvature flow
\[
\frac{\partial F_\lambda}{\partial t} = \mathbf{H}_\lambda.
\]
Moreover, since
\[
\begin{align*}
  d\mu^\lambda_t(F_\lambda(x, t)) &= \lambda^n d\mu_t(F(x, T + \lambda^{-2}t)) \\
  \Omega|_{\Sigma^\lambda_t}(F_\lambda(x, t)) &= \lambda^n \Omega|_{\Sigma_t}(F(x, T + \lambda^{-2}t))
\end{align*}
\]
we have
\[
\cos \theta_\lambda(F_\lambda(x, t)) = \cos \theta(F(x, T + \lambda^{-2}t)).
\]
2.2 Integral estimates

**Proposition 2.2** Let $(M, \Omega)$ be a Calabi-Yau manifold of complex dimension $n$. If the initial compact submanifold is Lagrangian and is almost calibrated by $\text{Re}\Omega$, then for any $R > 0$ and any $-\infty < s_1 < s_2 < 0$, we have

\[
\int_{s_1}^{s_2} \int_{\Sigma^t \cap B_R(0)} |\nabla \cos \theta_\lambda|^2 d\mu_\lambda^t dt \to 0 \quad \text{as} \quad \lambda \to \infty, \quad (16)
\]

\[
\int_{s_1}^{s_2} \int_{\Sigma^t \cap B_R(0)} |H_\lambda|^2 d\mu_\lambda^t dt \to 0 \quad \text{as} \quad \lambda \to \infty, \quad (17)
\]

and

\[
\int_{s_1}^{s_2} \int_{\Sigma^t \cap B_R(0)} |F_\lambda^t|^2 d\mu_\lambda^t dt \to 0 \quad \text{as} \quad \lambda \to \infty. \quad (18)
\]

**Proof:** For any $R > 0$, we choose a cut-off function $\phi_R \in C^\infty_0(B_R(0))$ with $\phi_R \equiv 1$ in $B_R(0)$, where $B_r(0)$ is the metric ball centered at 0 with radius $r$ in $\mathbb{R}^{2n}$. For any fixed $t < 0$, the mean curvature flow (3) has a smooth solution near $T + \lambda^{-2}t < T$ for sufficiently large $\lambda$, since $T > 0$ is the first blow-up time of the flow. Let $v_\lambda = \cos \theta_\lambda$. It is clear

\[
\int_{\Sigma^t} \frac{1}{v_\lambda} \frac{1}{(0-t)^{n/2}} \phi_R(F_\lambda) \exp \left(-\frac{|F_\lambda|^2}{4(0-t)}\right) d\mu_\lambda^t
\]

\[
= \int_{\Sigma_t^{T+\lambda^{-2}t}} \frac{1}{v_\lambda} \frac{\phi(F_\lambda)}{(T - (T + \lambda^{-2}t))^{n/2}} \exp \left(-\frac{|F(x, T + \lambda^{-2}t) - X_0|^2}{4(T - (T + \lambda^{-2}t))}\right) d\mu_t,
\]

where $\phi$ is the function defined in the definition of $\Phi$. Note that $T + \lambda^{-2}t \to T$ for any fixed $t$ as $\lambda \to \infty$. By the weighted monotonicity formula (9),

\[
\frac{\partial}{\partial t} \left(e^{c_1 \sqrt{t_0 - t}} \Psi\right) \leq c_2 e^{c_1 \sqrt{t_0 - t}},
\]

and it then follows that $\lim_{t \to t_0} e^{c_1 \sqrt{t_0 - t}} \Psi$ exists. This implies, by taking $t_0 = T$ and $t = T + \lambda^{-2}s$, that for any fixed $s_1$ and $s_2$ with $-\infty < s_1 < s_2 < 0$,

\[
e^{c_1 \sqrt{T - (T + \lambda^{-2}s_2)}} \int_{\Sigma_{s_1}^{s_2}} \frac{1}{v_\lambda} \phi_R \left(\frac{1}{0-s_2}\right)^{n/2} \exp \left(-\frac{|F_\lambda|^2}{4(0-s_2)}\right) d\mu_\lambda^{s_2}
\]

\[-e^{c_1 \sqrt{T - (T + \lambda^{-2}s_1)}} \int_{\Sigma_{s_1}^{s_2}} \frac{1}{v_\lambda} \phi_R \frac{1}{(0-s_1)^{n/2}} \exp \left(-\frac{|F_\lambda|^2}{4(0-s_1)}\right) d\mu_\lambda^{s_1}
\]

\[\to 0 \quad \text{as} \quad \lambda \to \infty. \quad (19)
\]

Integrating (9) from $T + \lambda^{-2}s_1$ to $T + \lambda^{-2}s_2$, and letting $T + \lambda^{-2}s = t$, we get

\[-e^{c_1 \sqrt{-\lambda^{-2}s_2}} \int_{\Sigma_{s_2}} \frac{1}{v_\lambda} \phi_R \left(\frac{1}{0-s_2}\right)^{n/2} \exp \left(-\frac{|F_\lambda|^2}{4(0-s_2)}\right) d\mu_\lambda^{s_2}
\]
Now we show the existence of the 2.3 Upper bound on volume density
From (19) and (20) the proposition follows. Q.E.D.

We shall first prove the inequality (21). We shall use
Proof: Suppose that is a compact Lagrangian submanifold in
Proposition 2.3 Suppose that evolves along mean curvature flow and is a compact Lagrangian submanifold in (M, Ω) and is almost calibrated by ReΩ. For any λ, R > 0 and any t < 0,
\[ \mu_t^λ(Σ_t^λ \cap B_R(0)) \leq CR^n, \] (21)
where \( B_R(0) \) is a metric ball in \( \mathbb{R}^{2n} \) and \( C > 0 \) is independent of \( λ \). For any sequence \( λ_k \to \infty \), there is a subsequence \( λ_k \to \infty \) such that \( (Σ_t^λ_k, μ_t^λ_k) \to (Σ^∞, μ^∞) \) in the sense of measure, for any fixed \( t < 0 \), where \( (Σ^∞, μ^∞) \) is independent of \( t \). The multiplicity of \( Σ^∞ \) is finite.

Proof: We shall first prove the inequality (21). We shall use \( C \) below for uniform positive constants which are independent of \( R \) and \( λ \). Straightforward computation shows
\[ μ_t^λ(Σ_t^λ \cap B_R(0)) = \lambda^n \int_{Σ_{t+\lambda^{-2}t} \cap B_{\lambda^{-1}R}(X_0)} dμ_t \]
\[ = R^n(λ^{-1}R)^{-n} \int_{Σ_{t+\lambda^{-2}t} \cap B_{\lambda^{-1}R}(X_0)} dμ_t \]
\[ \leq CR^n \int_{Σ_{t+\lambda^{-2}t} \cap B_{λ^{-1}R}(X_0)} \frac{1}{v_λ (4π)^{n/2}(λ^{-1}R)^n} e^{-\frac{|X-X_0|^2}{4(λ^{-1}R)^2}} dμ_t \]
\[ = CR^n Ψ(X_0, T + (λ^{-1}R)^2 + λ^{-2}t, T + λ^{-2}t). \]
By the weighted monotonicity inequality (9), we have
\[ \mu^\lambda_t(\Sigma_t^\lambda \cap B_R(0)) \leq CR^n \Psi(X_0, T + (\lambda^{-1}R)^2 + \lambda^{-2}t, T/2) + CR^n \]
\[ \leq \frac{\mu_{T/2}(\Sigma_{T/2})}{T^{n/2} \min_{\Omega_0} v} CR^n + CR^n. \]

Since volume is non-increasing along mean curvature flow:
\[ \frac{\partial}{\partial t} \mu_t(\Sigma_t) = -\int_{\Sigma_t} |H| d\mu_t, \]
we have therefore established (21):
\[ \mu^\lambda_t(\Sigma_t^\lambda \cap B_R(0)) \leq CR^n. \]

By (21), the compactness theorem of the measures (c.f. [Si1], 4.4) and a diagonal subsequence argument, we conclude that there is a subsequence \( \lambda_k \to \infty \) such that \( (\Sigma_{t_0}^{\lambda_k}, \mu_{t_0}^{\lambda_k}) \to (\Sigma_t^\infty, \mu_t^\infty) \) in the sense of measures for a fixed \( t_0 < 0 \).

We now show that, for any \( t < 0 \), the subsequence \( \lambda_k \) which we have chosen above satisfies \( (\Sigma_{t_k}^{\lambda_k}, \mu_{t_k}^{\lambda_k}) \to (\Sigma_t^\infty, \mu_t^\infty) \) in the sense of measure. And consequently the limiting submanifold \( (\Sigma_t^\infty, \mu_t^\infty) \) is independent of \( t_0 \). Recall that the following standard formula for mean curvature flows
\[ \frac{d}{dt} \int_{\Sigma_t^\lambda} \phi d\mu_t^\lambda = -\int_{\Sigma_t^\lambda} \left( \phi |H_\lambda|^2 + \nabla \phi \cdot H_\lambda \right) d\mu_t^\lambda \quad (22) \]
is valid for any test function \( \phi \in C_0^\infty(M) \) (cf. (1) in Section 6 in [I2] and [B] in the varifold setting).

Then for any given \( t < 0 \) integrating (22) yields
\[ \int_{\Sigma_{t_k}^{\lambda_k}} \phi d\mu_{t_k}^{\lambda_k} - \int_{\Sigma_{t_0}^{\lambda_k}} \phi d\mu_{t_0}^{\lambda_k} = \int_{t_0}^{t_k} \int_{\Sigma_{t_k}^{\lambda_k}} \left( \phi |H_{\lambda_k}|^2 + \nabla \phi \cdot H_{\lambda_k} \right) d\mu_{t_k}^{\lambda_k} dt \]
\[ \to 0 \quad \text{as} \quad k \to \infty \quad \text{by (17)}. \]

So, for any fixed \( t < 0 \), \( (\Sigma_{t_k}^{\lambda_k}, \mu_{t_k}^{\lambda_k}) \to (\Sigma_t^\infty, \mu_t^\infty) \) in the sense of measures as \( k \to \infty \). We denote \( (\Sigma_t^\infty, \mu_t^\infty) \) by \( (\Sigma^\infty, \mu^\infty) \), which is independent of \( t_0 \).

The inequality (21) yields a uniform upper bound on \( R^{-n} \mu_{t_k}^{\lambda_k}(\Sigma_{t_k}^{\lambda_k} \cap B_R(0)) \), which yields finiteness of the multiplicity of \( \Sigma^\infty \). Q.E.D.

**Definition 2.4** Let \((X_0, T)\) be a singular point of the mean curvature flow of a compact Lagrangian submanifold \( \Sigma_0 \) in a compact Calabi-Yau manifold \( M \). We call \((\Sigma^\infty, d\mu^\infty)\) obtained in Proposition 2.3 a \( \lambda \) tangent cone of the mean curvature flow \( \Sigma_t \) at \((X_0, T)\).
3 Rectifiability of $\lambda$ tangent cones

In this section we shall show that the $\lambda$ tangent cone $\Sigma^\infty$ is $\mathcal{H}^n$-rectifiable, where $\mathcal{H}^n$ is the $n$-dimensional Hausdorff measure.

**Proposition 3.1** Let $M$ be a compact Calabi-Yau manifold of complex dimension $n$. If the initial compact submanifold $\Sigma_0$ is Lagrangian and almost calibrated by $\text{Re}\Omega$, then the $\lambda$ tangent cone $\left(\Sigma^\infty, d\mu^\infty\right)$ of the mean curvature flow at $(X_0, T)$ is $\mathcal{H}^n$-rectifiable.

**Proof:** Let $(\Sigma^k_t, d\mu^k_t) = (\Sigma^\lambda_t, d\mu^\lambda_t)$. We set

$$A_R = \left\{ t \in (-\infty, 0) \mid \lim_{k \to \infty} \int_{\Sigma^k_t \cap B_R(0)} |H_k|^2 d\mu^k_t \neq 0 \right\},$$

and

$$A = \bigcup_{R > 0} A_R.$$

Denote the measures of $A_R$ and $A$ by $|A_R|$ and $|A|$ respectively. It is clear from (17) that $|A_R| = 0$ for any $R > 0$. So $|A| = 0.$

For any $\xi \in \Sigma^\infty$, choose $\xi_k \in \Sigma^k_t$ with $\xi_k \to \xi$ as $k \to \infty$. By the monotonicity identity (17.4) in [Si1], we have

$$\sigma^{-n} \mu_t^k(B_{\sigma}(\xi_k)) = \rho^{-n} \mu_t^k(B_{\rho}(\xi_k)) - \int_{B_{\rho}(\xi_k) \setminus B_{\sigma}(\xi_k)} \frac{|D^+ r|^2}{r^n} d\mu_t^k
- \frac{1}{n} \int_{B_{\rho}(\xi_k)} (x - \xi_k) \cdot H_k \left( \frac{1}{r^\sigma} - \frac{1}{r^\rho} \right) d\mu_t^k,$$

(23)

for all $0 < \sigma \leq \rho$, where $\mu_t^k(B_{\sigma}(\xi_k))$ is the measure of $\Sigma^k_t \cap B_{\sigma}((\xi_k))$, $r = r(x)$ is the distance from $\xi_k$ to $x$, $r = \max\{r, \sigma\}$, and $D^+ r$ denotes the orthogonal projection of $Dr$ (which is a vector of length 1) onto $\left(T_{\xi_k} \Sigma^k_t\right)^\perp$. Choosing $t \notin A$, we have

$$\lim_{k \to \infty} \int_{B_{\rho}(\xi_k)} |H_k|^2 d\mu_t^k = 0.$$

Hölder’s inequality and (21) then lead to

$$\lim_{k \to \infty} \int_{B_{\rho}(\xi_k)} (x - \xi_k) \cdot H_k \left( \frac{1}{r^\sigma} - \frac{1}{r^\rho} \right) d\mu_t^k \leq C \rho \left( \frac{1}{\sigma^n} - \frac{1}{\rho^n} \right) \lim_{k \to \infty} \left( \mu_t^k(B_{\rho}(\xi_k)) \sqrt{\int_{B_{\rho}(\xi_k)} |H_k|^2 d\mu_t^k} \right)$$

$$\leq C \rho^{1+n/2} \left( \frac{1}{\sigma^n} - \frac{1}{\rho^n} \right) \lim_{k \to \infty} \sqrt{\int_{B_{\rho}(\xi_k)} |H_k|^2 d\mu_t^k} = 0.$$

(24)
Letting $k \to \infty$ in (23) and using (24), we obtain
\[
\sigma^{-n} \mu^\infty(B_\sigma(\xi)) \leq \rho^{-n} \mu^\infty(B_\rho(\xi)),
\]
for all $0 < \sigma \leq \rho$. By (21) we know that
\[
\lim_{\rho \to 0} \rho^{-n} \mu^\infty(B_\rho(\xi)) < C < \infty.
\]
Therefore, $\lim_{\rho \to 0} \rho^{-n} \mu^\infty(B_\rho(\xi))$ exists.

We shall show that there exists a positive number $r_0$ such that for any $0 < r < r_0 < 1$ the following density estimate holds
\[
\lim_{\rho \to 0} \rho^{-n} \mu^\infty(B_\rho(\xi)) \geq \frac{1}{4c(n) + 4} > 0 \quad (25)
\]
for some positive constant $c(n)$ which will be determined below. Assume (25) fails to hold. Then there is $\rho_0 > 0$ such that
\[
(2\rho_0)^{-n} \mu^\infty(B_{2\rho_0}(\xi)) < \frac{1}{4c(n) + 4}.
\]
By the monotonicity formula (23) and that $\mu^k_t$ converges to $\mu^\infty$ as measures, there exists $k_0 > 0$ such that, for all $0 < \rho < 2\rho_0$ and $k > k_0$, we have
\[
\rho^{-n} \mu^k_t(B_\rho(\xi)) < \frac{1}{2c(n) + 2}. \quad (26)
\]
Take a cut-off function $\phi_\rho \in C^\infty_0(B_\rho(\xi))$ on the $2n$-dimensional ball $B_\rho(\xi_k)$ so that
\[
\phi_\rho \equiv 1 \quad \text{in} \quad B_{\frac{\rho}{2}}(\xi)
\]
\[0 \leq \phi_\rho \leq 1, \quad \text{and} \quad |\nabla \phi_\rho| \leq \frac{C}{\rho} \quad \text{in} \quad B_\rho(\xi).
\]
From (22), we have
\[
\rho^{-n} \int_{B_\rho(\xi)} \phi_\rho d\mu^k_{t-r^2} - \rho^{-n} \int_{B_\rho(\xi)} \phi_\rho d\mu^k_t
\]
\[\leq C \rho^{-n} \int_{t-r^2}^t \int_{B_\rho(\xi)} |H_k|^2 d\mu^k_s ds + C \rho^{-n-1} \int_{t-r^2}^t \int_{B_\rho(\xi)} |H_k| d\mu^k_s ds
\]
\[\leq C \rho^{-n} \int_{t-r^2}^t \int_{B_\rho(\xi)} |H_k|^2 d\mu^k_s ds + C \rho^{-n-1} \int_{t-r^2}^t \left( \int_{B_\rho(\xi)} |H_k|^2 d\mu^k_s \right)^{1/2} \mu^k_s(B_\rho(\xi))^{1/2} ds
\]
\[\leq C \rho^{-n} \int_{t-r^2}^t \int_{B_\rho(\xi)} |H_k|^2 d\mu^k_s ds + C \rho^{-n/2-1} \int_{t-r^2}^t \left( \int_{B_\rho(\xi)} |H_k|^2 d\mu^k_s \right)^{1/2} ds \quad \text{by} \quad (21)
\]
\[\to 0, \quad \text{as} \quad k \to \infty \quad \text{by} \quad (17).
\]
Here we have used $C$ for uniform positive constants which are independent of $k$ and $\rho$. Therefore, there are constants $\delta_1 > 0$ and $k_1 > 0$ such that for all $\rho$ and $k$ with $0 < \rho < \delta_1$, $0 < r < 1$, and $k > k_1$ the estimate

$$\rho^{-n} \mu_{t-r^2}(B_\rho(\xi)) < \frac{1}{c(n) + 1} < 1$$

holds. Let $d\sigma_{t-r^2}^k$ be the area element of $\partial B_\rho(\xi) \cap \Sigma_{t-r^2}^k$. By the co-area formula, for $0 < r << 1$, for a smooth cut-off function $\phi$ with support in the $2n$-dimensional ball $B_{\delta_1}(0)$ in $\mathbb{R}^{2n}$ with $0 \leq \phi \leq 1$, $\phi \equiv 1$ in $B_{\delta_1/2}(0)$, we have

$$\Phi_k(\xi, t, t - r^2) = \frac{1}{(4\pi r^2)^{n/2}} \int_{\Sigma_{t-r^2}^k} \phi e^{-\frac{|F_k - \xi|^2}{4r^2}} d\mu_{t-r^2}^k \leq \frac{1}{(4\pi)^{n/2} r^n} \int_0^{\delta_1} \int_{\partial B_\rho(\xi) \cap \Sigma_{t-r^2}^k} e^{-\frac{\rho^2}{4r^2}} d\sigma_{t-r^2}^k d\rho \leq \frac{1}{(4\pi)^{n/2} r^n} \int_0^{\delta_1} e^{-\frac{\rho^2}{4r^2}} d\rho \Vol(B_\rho(\xi) \cap \Sigma_{t-r^2}^k) e^{-\frac{\delta_1^2}{4r^2}} \leq \frac{1}{\pi^{n/2} (2r)^n} \Vol(B_{\delta_1}(\xi) \cap \Sigma_{t-r^2}^k) e^{-\frac{\delta_1^2}{4r^2}} \leq C \left(\frac{\delta_1}{2r}\right)^n e^{-\frac{\delta_1^2}{4r^2}} = o(r)$$

by integration by parts and (27). By (21),

$$\frac{1}{\pi^{n/2} (2r)^n} \Vol(B_{\delta_1}(\xi) \cap \Sigma_{t-r^2}^k) e^{-\frac{\delta_1^2}{4r^2}} \leq C \left(\frac{\delta_1}{2r}\right)^n e^{-\frac{\delta_1^2}{4r^2}} = o(r).$$

Letting $y = \rho/2r$ we have

$$\int_0^{\delta_1} e^{-\frac{y^2}{4r^2}} \left(\frac{\rho}{2r}\right)^n d\left(\frac{\rho}{2r}\right)^2 \leq 2 \int_0^{\infty} e^{-y^2} y^{n+1} dy = c(n) < \infty,$$

and there is an explicit formula for $c(n)$ depends on whether $n$ is odd or even. Thus we conclude

$$\Phi_k(\xi, t, t - r^2) \leq 1 + o(r).$$

For any classical mean curvature flow $\Gamma_t$ in a compact Riemannian manifold which is isometrically embedded in $\mathbb{R}^N$, White proves a local regularity theorem (Theorem 3.1 and Theorem 4.1 in [Wh1]): When $\dim \Gamma_t = n$, there is a constant $\epsilon > 0$ such that if the Gaussian density satisfies

$$\lim_{r \to 0} \int_{\Gamma_{t-r^2}} \frac{1}{(4\pi r^2)^{n/2}} \exp \left(-\frac{|y - x|^2}{4r^2}\right) d\mu(y) < 1 + \epsilon,$$
then the mean curvature flow is smooth in a neighborhood of $x$. Combining this regularity result with (28), we are led to choose $r > 0$ sufficiently small and then conclude that

$$\sup_{B_r(x) \cap \Sigma^k_t} |A_k| \leq C$$

and consequently $\Sigma^k_t$ converges strongly in $B_r(x) \cap \Sigma^k_t$ to $\Sigma^\infty_t \cap B_r(x)$, as $k \to \infty$. So $\Sigma^\infty \cap B_r(x)$ is smooth. Smoothness of $\Sigma^\infty \cap B_r(x)$ immediately implies

$$\lim_{r \to 0} \rho^{-n} \mu^\infty(B_\rho(x)) = 1.$$ 

This contradicts (26). Hence we have established (25).

In summary, we have shown that $\lim_{\rho \to 0} \rho^{-n} \mu^\infty(B_\rho(x))$ exists and for $H^n$ almost all $x \in \Sigma^\infty$,

$$\frac{1}{4c(n)} + 4 \leq \lim_{\rho \to 0} \rho^{-n} \mu^\infty(B_\rho(x)) < \infty.$$ 

(28)

Finally, we recall a fundamental theorem of Priess in [P]: if $0 \leq m \leq p$ are integers and $\Omega$ is a Borel measure on $\mathbb{R}^p$ such that

$$0 < \lim_{r \to 0} \frac{\Omega(B_r(x))}{r^m} < \infty,$$

for almost all $x \in \Omega$, then $\Omega$ is $m$-rectifiable. Now we conclude from (28) that $(\Sigma^\infty_t, \mu^\infty_t)$ is $H^n$-rectifiable.

4 Minimality of the $\lambda$ tangent cones

In this section, we will show that the $\lambda$ tangent cone $\Sigma^\infty$ is a stationary integer multiplicity rectifiable current in $\mathbb{R}^{2n}$.

**Theorem 4.1** Let $M$ be a compact Calabi-Yau manifold. If the initial compact submanifold is Lagrangian and is almost calibrated by $\Re \Omega$, then the $\lambda$ tangent cone $\Sigma^\infty$ is a stationary rectifiable Lagrangian current in $\mathbb{R}^{2n}$ with volume density greater than one at $X_0$.

**Proof:** Let $V^k_t$ be the varifold defined by $\Sigma^k_t$. By the definition of varifolds, we have

$$V^k_t(\psi) = \int_{\Sigma_t^k} \psi(x, T\Sigma_t^k) d\mu^k_t$$

for any $\psi \in C^0_0(G^2(\mathbb{R}^{2n}), R)$, where $G^2(\mathbb{R}^{2n})$ is the Grassmanian bundle of all $n$-dimensional planes tangent to $\Sigma^\infty_t$ in $\mathbb{R}^{2n}$. For each smooth submanifold $\Sigma^k_t$, the first variation $\delta V^k_t$ of $V^k_t$ (cf. [A], (39.4) in [Si1] and (1.7) in [I2]) is

$$\delta V^k_t = -\mu^k_t[H_k].$$
By Proposition 2.2, we have that \( \delta V^k_t \to 0 \) at \( t \) as \( k \to \infty \).

Recall that a \( k \)-varifold is a Radon measure on \( G^k(M) \), where \( G^k(M) \) is the Grassmann bundle of all \( k \)-planes tangent to \( M \). Allard’s compactness theorem for rectifiable varifolds (6.4 in [A], also see 1.9 in [I2] and Theorem 42.7 in [Si1]) asserts the following: let \( (V_i, \mu_i) \) be a sequence of rectifiable \( k \)-varifolds in \( M \) with

\[
\sup_{i \geq 1} (\mu_i(U) + |\delta V_i|(U)) < \infty \quad \text{for each } U \subset M.
\]

Then there is a varifold \( (V, \mu) \) of locally bounded first variation and a subsequence, which we also denote by \( (V_i, \mu_i) \), such that (i) Convergence of measures: \( \mu_i \to \mu \) as Radon measures on \( M \), (ii) Convergence of tangent planes: \( V_i \to V \) as Radon measures on \( G^k(M) \), (iii) Convergence of first variations: \( \delta V_i \to \delta V \) as \( TM \)-valued Radon measures, (iv) Lower semicontinuity of total first variations: \( |\delta V| \leq \lim \inf_{k \to \infty} |\delta V_i| \) as Radon measures.

By (iii) in Allard’s compactness theorem, we have

\[
-\mu^\infty[H^\infty] = \delta V^\infty = \lim_{k \to \infty} \delta V^k_t = 0.
\]

Therefore \( \Sigma^\infty \) is stationary. The rescaling process in a neighborhood of \( X_0 \) in \( M \) implies that the metrics \( g^\lambda \) tends to the flat metric on \( \mathbb{R}^{2n} \) and the Kähler 2-form \( \omega^\lambda \) tends to a constant closed 2-form \( \omega_0 \) which is determined by \( \omega_0(0) = \omega(X_0) \). The tangent spaces to \( \Sigma^k_t \) converge to that to \( \Sigma^\infty \) as measures by (ii) in Allard’s compactness theorem. Hence \( \omega^\lambda|_{\Sigma^k_t} \to \omega_0|_{\Sigma^\infty} \). But \( \Sigma^k_t \) is Lagrangian, it follows \( \omega^\lambda|_{\Sigma^k_t} = 0 \) therefore \( \omega_0|_{\Sigma^\infty} = 0 \). Therefore, \( \Sigma^\infty \) is a Lagrangian.

On the other hand, as \( \lambda \to \infty \) in the blow-up process, the holomorphic \((n, 0)\)-form \( \Omega \) converges to a constant holomorphic \((n, 0)\)-form \( \Omega_0 \) on \( \mathbb{R}^n \) determined by \( \Omega_0(0) = \Omega(X_0) \). We write

\[
\text{Re } \Omega_0|_{\Sigma^\infty} = \theta_0 d\mu^\infty,
\]

and from Allard’s compactness theorem

\[
\text{Re } \Omega^\lambda|_{\Sigma^k_t} \to \text{Re } \Omega_0|_{\Sigma^\infty},
\]

and the tangent cone \( \Sigma^\infty \) is of integer multiplicity by the integral compactness theorem of Allard ([A] and [Si1] 42.8). It follows that \( \text{Re } \Omega_0|_{\Sigma^\infty} > 0 \), which implies that the tangent cone \( \Sigma^\infty \) is orientable. Since \( \Sigma^\infty \) is of integer multiplicity, we have that \( d\mu^\infty = \eta(x) \mathcal{H}^n \) where \( \eta(x) \) is a locally \( \mathcal{H}^n \)-integrable positive integer-valued function. So the cone is an integral current (see Definition 27.1 in [Si1]).
We now show that the volume density of $\Sigma^\infty$ at $X_0$ is greater than 1. Otherwise, we would have

$$\lim_{\rho \to 0} \frac{1}{\omega_n \rho^n} \mu^\infty(B_\rho(0)) \leq 1$$

where $\omega_n$ is the volume of the unit $n$-ball in $\mathbb{R}^n$:

$$\omega_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)}.$$

It then follows from (23) that for any $\epsilon > 0$, there are $\delta > 0$ and $k_0 > 0$ such that for any $0 < \rho < 2\delta$ and $k > k_0$,

$$\rho^{-n} \mu^k_0(B_\rho(\xi)) < \omega_n (1 + \epsilon)$$

for any fixed $r > 0$. The choice of $r$ will be based on the following observation. Set

$$\Phi(F, X_0, t_0, t) = \int_{\Sigma_t} \phi(F) \frac{1}{(4\pi(t_0 - t))^{n/2}} e^{-\frac{\rho^2}{4(t_0 - t)}} d\mu_t$$

where $\phi$ is supported in $B_\delta(0)$ and $0 \leq \phi \leq 1$, $\phi \equiv 1$ in $B_{\delta/2}(0)$. Then we have

$$\Phi(F_k, 0, 0, 0 - r^2) \leq \frac{1}{(4\pi r^2)^{n/2}} \int_0^\delta \int_{\partial B_\rho(0) \cap \Sigma_0^{k-r^2}} e^{-\frac{\rho^2}{4r^2}} d\mu^k_0$$

$$\leq \frac{1}{(4\pi r^2)^{n/2}} \int_0^\delta e^{-\frac{\rho^2}{4r^2}} \int_{\partial B_\rho(0) \cap \Sigma_0^{k-r^2}} d\mu^k_0$$

$$\leq \frac{1}{(4\pi r^2)^{n/2}} \int_0^\delta e^{-\frac{\rho^2}{4r^2}} \frac{\rho}{2r^2} \text{Vol}(B_\rho(0) \cap \Sigma_0^{k-r^2}) d\rho$$

$$+ \frac{1}{(4\pi r^2)^{n/2}} e^{-\frac{42}{4r^2}} \text{Vol}(B_\rho(0) \cap \Sigma_0^{k-r^2})$$

$$\leq \left(\frac{1 + \epsilon}{\pi^{n/2}}\right) \int_0^\delta e^{-\frac{\rho^2}{4r^2}} \frac{\rho^{n+1}}{2r^2} d\rho + o(r) \text{ by (29) and (21)}$$

$$= \left(\frac{1 + \epsilon}{\pi^{n/2}}\right) \int_0^\delta e^{-x} x^{n/2} dx + o(r)$$

because $\Gamma(\frac{n}{2} + 1) = \int_0^\infty e^{-x} x^{n/2} dx$. Choosing $r > 0$ sufficiently small, we therefore have

$$\Phi(F, X_0, T, T - \lambda_k^{-2}r^2) = \Phi(F_k, 0, 0, 0 - r^2) \leq 1 + \epsilon.$$

Now by White’s local regularity theorem ([Wh1] Theorem 3.1 and Theorem 4.1, also see [E2]), $(X_0, T)$ could not be a singular point of the mean curvature flow. This is a contradiction. Q.E.D.
5 Flatness of λ-cone in dimension 2

Regularity of the λ tangent cone can be greatly improved in the 2-dimensional case: dim_C M = 2.

**Theorem 5.1** Let (M, Ω) be a compact Calabi-Yau manifold and let Σ₀ be a compact Lagrangian surface in M which is almost calibrated by ReΩ. If 0 < T < ∞ is the first blow-up time of a mean curvature flow of Σ₀ in M, then the λ tangent cone at (X₀, T) consists of a finite union (but more than two) 2-planes in R^4 which are complex in a complex structure on R^4.

**Proof:** We use the same notation as that in the proof of Theorem 4.1, we shall show that θ₀ is constant H² a.e. on Σ∞. To do so, we claim that for any r > 0, ξ₁, ξ₂ ∈ Σₖ ∩ B_r/2(0) the following holds

\[
\left| \frac{1}{\text{Vol}(B_r(ξ₁) ∩ Σₖ)} \int_{B_r(ξ₁)∩Σₖ} \cos θ_k dμ^k - \frac{1}{\text{Vol}(B_r(ξ₂) ∩ Σₖ)} \int_{B_r(ξ₂)∩Σₖ} \cos θ_k dμ^k \right| 
\]

\[
\leq \frac{C_1(r)}{\text{Vol}(B_r(ξ₁) ∩ Σₖ)} \cdot \frac{C_2(r)}{\text{Vol}(B_r(ξ₂) ∩ Σₖ)} \int_{B_r(0)∩Σₖ} |∇ \cos θ_k| dμ^k, \quad (30)
\]

where B_r(ξ_i), i = 1, 2, are the 4-dimensional balls in M. To prove (30), let us first recall the isoperimetric inequality on Σₖ (c.f. [HSp] and [MS]): let B^k(ρ) be the geodesic ball in Σₖ, with radius ρ and center p, then

\[
\text{Vol}(B^k(ρ)) \leq C \left( \text{length}(∂(B^k(ρ))) + \int_{B^k(ρ)} |H_k|^2 dμ^k \right)^2
\]

\[
\leq C \left( \text{length}(∂(B^k(ρ))) + \left( \int_{B^k(ρ)} |H_k|^2 dμ^k \right)^{1/2} \right)^2 \left( \text{Vol}^{1/2}(B^k(ρ)) \right)^2,
\]

for any p ∈ Σₖ, and any ρ > 0, where C does not depend on k, ρ, and p. By Proposition 2.2, we have

\[
\int_{B^k(ρ)} |H_k|^2 dμ^k \to 0 \text{ as } k \to ∞.
\]

So, for k sufficiently large, we obtain:

\[
\text{Vol}(B^k(ρ)) \leq C \left( \text{length}(∂(B^k(ρ))) \right)^2.
\]

In particular, for k sufficiently large, the isoperimetric inequality implies

\[
\text{Vol}(B^k(ρ)) \geq C ρ^2,
\]

where C is a positive constant independent of k, ρ and p.
Suppose that the diameter of $B_r(\xi) \cap \Sigma_k^t$ is $d_k(\xi)$. Then

$$C_r^2 \geq \int_{B_r(\xi) \cap \Sigma_k^t} d\mu_k^t \quad \text{by (21)}$$

$$= \int_0^{d_k(\xi)/2} \int_{\partial B_{\rho}^k(\rho)} d\sigma d\rho \quad \text{for some } \rho \in \Sigma_k^t$$

$$\geq c \int_0^{d_k(\xi)/2} \text{Vol}^{1/2}(B_{\rho}^k(\rho)) d\rho + o(1), \quad o(1) \to 0 \text{ as } k \to \infty$$

$$\geq c \int_0^{d_k(\xi)/2} C\rho d\rho + o(1) \quad \text{by (31)}$$

$$\geq c d_k(\xi)^2 + o(1).$$

We therefore have, for any $\xi$,

$$d_k(\xi) \leq C r + o(1) \quad (32)$$

where the constant $C$ is independent of $\xi$ and $k$.

For any fixed $\eta \in B_r(\xi_2) \cap \Sigma_k^t$ and any $\xi \in B_r(\xi_1) \cap \Sigma_k^t$, we choose a geodesic $l_{\eta \xi}$ connecting $\eta$ and $\xi$, call it a ray from $\eta$ to $\xi$. Take an open tubular neighborhood $U(l_{\eta \xi})$ of $l_{\eta \xi}$ in $\Sigma_k^t$. Within this neighborhood $U(l_{\eta \xi})$, we call the line in the normal direction of the ray $l_{\eta \xi}$ the normal line which we denote by $n(l_{\eta \xi})$. It is clear that

$$\cos \theta_k(\xi) - \cos \theta_k(\eta) = \int_{l_{\eta \xi}} \partial_t \cos \theta_k dl \quad (33)$$

where $dl$ is the arc-length element of $l_{\eta \xi}$. Choose $r$ small enough so that $B_r(\xi_1) \cap \Sigma_k^t$ is contained in $U(l_{\eta \xi_1})$. Keeping $\eta$ fixed and integrating (33) with respect to the variable $\xi$, first along the normal direction $n(l_{\eta \xi_1})$ and then on the ray direction $l_{\eta \xi_1}$, we have

$$\left| \frac{1}{\text{Vol}(B_r(\xi_1) \cap \Sigma_k^t)} \int_{B_r(\xi_1) \cap \Sigma_k^t} \cos \theta_k(\xi) d\mu_k^t - \cos \theta_k(\eta) \right|$$

$$\leq \frac{1}{\text{Vol}(B_r(\xi_1) \cap \Sigma_k^t)} \int_0^{d_k(\xi_1)} \int_{n(l_{\eta \xi_1})} |\nabla \cos \theta_k| dl d\mu_k^t d\rho$$

$$\leq \frac{1}{\text{Vol}(B_r(\xi_1) \cap \Sigma_k^t)} \int_0^{d_k(\xi_1)} \int_{B_{\rho}(0)} |\nabla \cos \theta_k| d\mu_k^t d\rho$$

$$\leq \frac{C r}{\text{Vol}(B_r(\xi_1) \cap \Sigma_k^t)} \int_{B_{\rho}(0)} |\nabla \cos \theta_k| d\mu_k^t, \quad (34)$$

here in the last step we have used (32). From (34), integrating with respect to $\eta$ in $B_r(\xi_2) \cap \Sigma_k^t$ and dividing by $\text{Vol}(B_r(\xi_2) \cap \Sigma_k^t)$, we get the desired inequality (30).
For $i = 1, 2$ Hölder’s inequality and (21) lead to
\[
\int_{B_r(ξ_i) \cap Σ_t^k} |∇ \cos θ_k| \, dμ_t^k \leq Cr \left( \int_{B_r(ξ_i) \cap Σ_t^k} |∇ \cos θ_k|^2 \, dμ_t^k \right)^{1/2}.
\]
The triangle inequality implies $B_r^k(ξ_i) \subset B_r(ξ_i) \cap Σ_t^k$ for $i = 1, 2$; therefore by (31)
\[
\text{Vol}(B_r(ξ_i) \cap Σ_t^k) \geq \text{Vol}(B_r^k(ξ_i)) \geq Cr^2.
\]
Now first letting $k \to ∞$ in (30) and using that the right hand side of (30) tends to 0 by Proposition 2.2, and then letting $r \to 0$, we conclude that $θ$ is constant $H^2$ a.e. on $Σ^∞$.

The $(2,0)$-form $Ω_0$ is fixed by $Ω(X_0)$ hence it has unit length. In the complex structure $J_{X_0}$ on $R^4$, $Ω_0 = dz_1 ∧ dz_2$. We define a new complex structure $J^*$ on $R^4$:
\[
J^*(∂/∂x_1) = θ_0(∂/∂y_1), \quad J^*(∂/∂y_1) = -1/θ_0(∂/∂x_1),
\]
\[
J^*(∂/∂x_2) = 1/θ_0(∂/∂y_2), \quad J^*(∂/∂y_2) = -θ_0(∂/∂x_2).
\]
In $J^*$, the complex coordinates are: $z_1^* = x_1 + √{-1}θ_0^{-1}y_1, z_2^* = θ_0^{-1}x_2 + √{-1}y_2$. Then $Ω_0^* = dz_1^* ∧ dz_2^*$ satisfies that $Re Ω_0^*|Σ^∞ = dμ^∞$.

We can further choose a new complex structure $J'$ on $R^4$ such that $Ω_0'$ is of type $(1,1)$ in $J'$. In fact, if we express $J^*$ in the local coordinates $x_1, θ_0^{-1}y_1, θ_0^{-1}x_2, y_2$ by
\[
J^* = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad \text{with} \quad I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]
then we can take
\[
J' = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.
\]
Therefore $Σ^∞$ is a stationary rectifiable current of type $(1,1)$ with respect to the complex structure $J'$. By Harvey-Shiffman’s Theorem 2.1 in [HS], $Σ^∞$ is a $J'$-holomorphic subvariety of complex dimension one. It then follows that the singular locus $S$ of $Σ^∞$ consists of isolated points.

Without loss of any generality, we may assume $0 ∈ Σ^∞$ where 0 is the origin of $R^4$. In fact, if not, $Σ^∞$ would move to infinity, then we would have
\[
Φ(F, X_0, T, T − λ_k^{-1}r^2) = Φ(F_k, 0, 0, 0 − r^2) \to 0 \text{ as } k \to ∞.
\]
But White’s regularity theorem then implies that $(X_0, T)$ is a regular point. This is impossible.

There is a sequence of points $X_k ∈ Σ_t^k$ satisfying $X_k \to 0$ as $k \to ∞$. By Proposition 2.2, for any $s_1$ and $s_2$ with $−∞ < s_1 < s_2 < 0$ and any $R > 0$, we have
\[
\int_{s_1}^{s_2} \int_{Σ_t^k ∩ B_R(0)} |F_k^i|^2 \, dμ_t^k \, dt \to 0 \text{ as } k \to ∞.
\]
Thus, by (21)

$$\lim_{k \to \infty} \int_{s_1}^{s_2} \int_{\Sigma_k t \cap B_R(0)} \left| (F_k - X_k) \right|^2 d\mu_k^t dt$$

$$\leq 2 \lim_{k \to \infty} \int_{s_1}^{s_2} \int_{\Sigma_k t \cap B_R(0)} \left| F_k \right|^2 d\mu_k^t dt + C(s_2 - s_1)R^2 \lim_{k \to \infty} \left| X_k \right|^2$$

$$= 0.$$

Let us denote the tangent spaces of $\Sigma_k t$ at the point $F_k(x,t)$ and of $\Sigma^\infty$ at the point $F^\infty(x,t)$ by $T_{\Sigma_k t}$ and $T_{\Sigma^\infty}$ respectively. It is clear that

$$(F_k - X_k)^\perp = \text{dist} \left( X_k, T_{\Sigma_k t} \right),$$

and

$$(F^\infty)^\perp = \text{dist} \left( 0, T_{\Sigma^\infty} \right).$$

By Allard’s compactness theorem, we have

$$\int_{s_1}^{s_2} \int_{\Sigma^\infty t \cap B_R(0)} \left| (F^\infty)^\perp \right|^2 d\mu^\infty dt = \int_{s_1}^{s_2} \int_{\Sigma^\infty t \cap B_R(0)} \left| \text{dist} \left( 0, T_{\Sigma^\infty} \right) \right|^2 d\mu^\infty dt$$

$$= \lim_{k \to \infty} \int_{s_1}^{s_2} \int_{\Sigma_k t \cap B_R(0)} \left| \text{dist} \left( X_k, T_{\Sigma_k t} \right) \right|^2 d\mu_k^t dt$$

$$= \lim_{k \to \infty} \int_{s_1}^{s_2} \int_{\Sigma_k t \cap B_R(0)} \left| (F_k - X_k)^\perp \right|^2 d\mu_k^t dt$$

$$= 0.$$

Therefore $F^\infty \equiv 0$. Differentiating $\langle F^\infty, v_\alpha \rangle = 0$, inner product is taken in $\mathbb{R}^4$, leads to

$$0 = \langle \partial_i F^\infty, v_\alpha \rangle + \langle F^\infty, \partial_i v_\alpha \rangle = \langle F^\infty, \partial_i v_\alpha \rangle.$$

Because $\partial_i F^\infty$ is tangential to $\Sigma^\infty$, by Weingarten’s equation we observe

$$\langle h^\infty_{ij}, v_j \rangle = 0 \text{ for all } \alpha, i = 1, 2.$$

Since either $\langle F^\infty, e_1 \rangle \neq 0$ or $\langle F^\infty, e_2 \rangle \neq 0$, we conclude $\det(h^\infty_{ij}) = 0$. Recall $h^\infty_{11} + h^\infty_{22} = 0$. It then follows $h^\infty_{ij} = 0$, for $i, j, \alpha = 1, 2$. Now we conclude that $\Sigma^\infty$ consists of flat 2-planes. Q.E.D.

### 6 Tangent cones from a time dependent scaling

In this section, we consider the tangent cones which arise from the rescaled submanifold $\Sigma_s$ defined by

$$\tilde{F}(\cdot, s) = \frac{1}{\sqrt{2(T - t)}} F(\cdot, t),$$  \hspace{1cm} (35)
where \( s = -\frac{1}{2} \log(T - t) \), \( c_0 \leq s < \infty \). Here we choose the coordinates so that \( X_0 = 0 \). Rescaling of this type was used by Huisken [H2] to distinguish Type I and Type II singularities for mean curvature flows. Denote the rescaled submanifold by \( \tilde{\Sigma}_{s} \). From the evolution equation of \( F \) we derive the flow equation for \( \tilde{F} \)

\[
\frac{\partial}{\partial s} \tilde{F}(x, s) = \tilde{H}(x, s) + \tilde{F}(x, s).
\]  

(36)

It is clear that

\[
\cos \tilde{\theta}(x, s) = \cos \theta(x, t),
\]

\[
|\tilde{H}|^2(x, s) = 2(T - t)|H|^2(x, t),
\]

\[
|\tilde{A}|^2(x, s) = 2(T - t)|A|^2(x, t).
\]

We set \( \tilde{v}(x, t) = \cos \tilde{\theta}(x, s) \).

**Lemma 6.1** Assume that \((M, \Omega)\) is a compact Calabi-Yau manifold and \(\Sigma_t\) evolves by a mean curvature flow in \(M\) with the initial submanifold \(\Sigma_0\) being Lagrangian and almost calibrated by \(\text{Re} \Omega\). Then

\[
\left( \frac{\partial}{\partial s} - \tilde{\Delta} \right) \tilde{v}(x, s) = |\tilde{H}|^2 \tilde{v}(x, s).
\]  

(37)

**Proof:** One can check directly that

\[
\left( \frac{\partial}{\partial s} - \tilde{\Delta} \right) \cos \tilde{\alpha}(x, s) = 2(T - t) \left( \frac{\partial}{\partial t} - \Delta \right) \cos \alpha(x, t).
\]

It follows that

\[
\left( \frac{\partial}{\partial s} - \tilde{\Delta} \right) \tilde{v}(x, s) = 2(T - t) \left( \frac{\partial}{\partial t} - \Delta \right) v(x, t)
\]

\[
\geq 2(T - t) |\tilde{H}|^2 v(x, t)
\]

\[
= |\tilde{H}|^2 \tilde{v}(x, s).
\]

This proves the lemma. Q.E.D.

Next, we shall derive the corresponding weighted monotonicity formula for the scaled flow. By (37), we have

\[
\left( \frac{\partial}{\partial s} - \tilde{\Delta} \right) \frac{1}{\tilde{v}} = -\frac{\tilde{H}^2}{\tilde{v}} - \frac{2|\nabla \tilde{v}|^2}{\tilde{v}^3}.
\]

Let

\[
\tilde{\rho}(X) = \exp \left( -\frac{1}{2} |X|^2 \right),
\]

\[
\Psi(s) = \int_{\tilde{\Sigma}_{s}} \frac{1}{\tilde{v}} \phi \tilde{\rho}(\tilde{F}) d\tilde{\mu}_{s}.
\]

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Lemma 6.2 There are positive constants $c_1$ and $c_2$ which depend on $M$, $F_0$ and $r$ which is the constant in the definition of $\phi$, so that the following monotonicity formula holds

$$\frac{\partial}{\partial s} \exp(c_1 e^{-s}) \Psi(s) \leq -\exp(c_1 e^{-s}) \left( \int_{\Sigma_s} \frac{1}{v} \phi \tilde{\rho}(F) |\tilde{H} + \tilde{F}_\perp|^2 d\tilde{\mu}_s \right. \\
+ \left. \int_{\Sigma_s} \frac{1}{v} \phi \tilde{\rho}(F) \left| \frac{1}{2} \tilde{H}^2 + \frac{2}{v^3} |\tilde{v}||\phi \tilde{\rho}(F)| d\tilde{\mu}_s \right) \\
+ c_2 \exp(c_1 e^{-s}). \quad (38)$$

Proof: Note that

$$\tilde{F}(x, s) = \frac{F(x, t)}{\sqrt{2(T - t)}}$$

$$\tilde{H}(x, s) = \sqrt{2(T - t)}H(x, t),$$

$$|\tilde{\nabla} \tilde{v}|^2(x, s) = 2(T - t)|\nabla v|^2(x, t).$$

By the chain rule

$$\frac{\partial}{\partial s} = 2e^{-2s} \frac{\partial}{\partial t}$$

and the monotonicity inequality (9) for the unscaled submanifold, we obtain the desired inequality. Q.E.D.

Lemma 6.3 Let $(M, \Omega)$ be a compact Calabi-Yau manifold. If the initial compact submanifold $\Sigma_0$ is Lagrangian and almost calibrated by $\text{Re} \Omega$, then there is a sequence $s_k \to \infty$ such that, for any $R > 0$,

$$\int_{\Sigma_{s_k} \cap B_R(0)} |\tilde{\nabla} \cos \tilde{\theta}|^2 d\tilde{\mu}_{s_k} \to 0 \quad \text{as} \quad k \to \infty, \quad (39)$$

$$\int_{\Sigma_{s_k} \cap B_R(0)} |\tilde{H}|^2 d\tilde{\mu}_{s_k} \to 0 \quad \text{as} \quad k \to \infty, \quad (40)$$

and

$$\int_{\Sigma_{s_k} \cap B_R(0)} |\tilde{F}_\perp|^2 d\tilde{\mu}_{s_k} \to 0 \quad \text{as} \quad k \to \infty. \quad (41)$$

Proof: Integrating (38), we have

$$\infty > \int_{s_0}^{\infty} \int_{\Sigma_s} \frac{1}{v} \phi \tilde{\rho}(F) |\tilde{H} + \tilde{F}_\perp|^2 d\tilde{\mu}_s ds \\\n+ \int_{s_0}^{\infty} \left( \int_{\Sigma_s} \frac{1}{v} \phi \tilde{\rho}(F) \left| \frac{1}{2} \tilde{H}^2 + \frac{2}{v^3} |\tilde{v}||\phi \tilde{\rho}(F)| d\tilde{\mu}_s \right) ds.$$

Hence there is a sequence $s_k \to \infty$, such that as $k \to \infty$

$$\int_{\Sigma_{s_k}} \frac{1}{v} \phi \tilde{\rho}(F) \left| \frac{1}{2} \tilde{H}^2 + \frac{2}{v^3} |\tilde{v}||\phi \tilde{\rho}(F)| d\tilde{\mu}_s \to 0,$$
\[
\int_{\Sigma_{s_k}} \frac{2}{\nu |\vec{\nabla} v|^2} \phi \tilde{\rho}(\tilde{F}) d\mu_{s_k} \to 0,
\]
and
\[
\int_{\Sigma_{s_k}} \frac{1}{\nu} \phi \tilde{\rho}(\tilde{F}) |\tilde{H} + \tilde{F}^\perp|^2 d\mu_{s_k} \to 0.
\]
Since \( \tilde{v} \) has a positive lower bound, the proposition now follows. Q.E.D.

The proof of the following lemma is essentially the same as the one for Proposition 3.1, except there are two parameters \( \lambda, t \) for the \( \lambda \) tangent cones but only one parameter \( t \) for the time dependent tangent cones. Note that the alternative proof given in [CL1] using the isoperimetric inequality only works in dimension 2.

**Lemma 6.4** There is a subsequence of \( s_k \), which we also denote by \( s_k \), such that \( (\tilde{\Sigma}_{s_k}, d\mu_{s_k}) \to (\tilde{\Sigma}_\infty, d\mu_\infty) \) in the sense of measures. And \( (\tilde{\Sigma}_\infty, d\mu_\infty) \) is \( \mathcal{H}^n \)-rectifiable.

**Proof:** To show the subconvergence, it suffices to show that, for any \( R > 0 \),
\[
\tilde{\mu}_{s_k} (\tilde{\Sigma}_{s_k} \cap B_R(0)) \leq C R^n,
\]
where \( B_R(0) \) is a metric ball in \( \mathbb{R}^{2n} \), \( C > 0 \) is independent of \( k \). Direct calculation leads to

\[
\begin{align*}
\tilde{\mu}_{s_k} (\tilde{\Sigma}_{s_k} \cap B_R(0)) & = (2(T - t))^{-n/2} \int_{\Sigma_{T - e^{2s_k}} \cap B_{\sqrt{2(T - t)}}(0)} d\mu_t \\
& = R^n \left( \sqrt{2} e^{-s_k} R \right)^{-n} \int_{\Sigma_{T - e^{2s_k}} \cap B_{\sqrt{2} e^{-s_k} R}(0)} d\mu_t \\
& \leq CR^n \int_{\Sigma_{T - e^{2s_k}} \cap B_{\sqrt{2} e^{-s_k} R}(0)} \frac{1}{\nu (4\pi)^{n/2} (\sqrt{2} e^{-s_k} R)^n} e^{-\frac{|x - x_0|^2}{4\sqrt{2} e^{-s_k} R}} d\mu_t \\
& \leq CR^n \Phi \left( 0, T + (\sqrt{2} e^{-s_k} R)^2 - e^{2s_k}, T - e^{2s_k} \right)
\end{align*}
\]

By the monotonicity inequality (9), we have

\[
\begin{align*}
\tilde{\mu}_{s_k} (\tilde{\Sigma}_{s_k} \cap B_R(0)) & \leq CR^n \Phi \left( 0, T + (\sqrt{2} e^{-s_k} R)^2 - e^{2s_k}, T/2 \right) + CR^n \\
& \leq \frac{C \mu_{T/2}(\Sigma_{T/2})}{T^{n/2} \min_{\Sigma_0} \nu} R^n + CR^n.
\end{align*}
\]

Since volume is non-increasing along mean curvature flow, we see

\[
\tilde{\mu}_{s_k} (\tilde{\Sigma}_{s_k} \cap B_R(0)) \leq CR^n.
\]

We now prove that \( (\tilde{\Sigma}_\infty, d\tilde{\mu}_\infty) \) is \( \mathcal{H}^n \)-rectifiable. For any \( \xi \in \tilde{\Sigma}_\infty \), choose \( \xi_k \in \tilde{\Sigma}_{s_k} \) with \( \xi_k \to \xi \) as \( k \to \infty \). By the monotonicity identity (17.4) in [Si1],
we have

\[
\sigma^{-n}\tilde{\mu}_{sk}(B_\sigma(\xi_k)) = \rho^{-n}\tilde{\mu}_{sk}(B_\rho(\xi_k)) - \int_{B_\rho(\xi_k) \setminus B_\sigma(\xi_k)} \frac{|D^\perp r|^2}{r^n} d\mu_{sk} \nonumber \\
- \frac{1}{n} \int_{B_\rho(\xi_k)} (x - \xi_k) \cdot \overline{H}_k \left( \frac{1}{r^2} \frac{1}{r^n} \right) d\mu_{sk},
\]

for all \(0 < \sigma \leq \rho\), where \(\tilde{\mu}_{sk}(B_\sigma(\xi_k))\) is the area of \(\tilde{\Sigma}_{sk} \cap B_\sigma(\xi_k)\), \(r_\sigma = \max\{r, \sigma\}\) and \(D^\perp r\) denotes the orthogonal projection of \(Dr\) (which is a vector of length 1) onto \((T_{\xi_k} \tilde{\Sigma}_{sk})^\perp\). Letting \(k \to \infty\), by Lemma 6.3, we have

\[
\sigma^{-n}\tilde{\mu}_\infty(B_\sigma(\xi)) \leq \rho^{-n}\tilde{\mu}_\infty(B_\rho(\xi)),
\]

for all \(0 < \sigma \leq \rho\). Therefore, \(\lim_{\rho \to 0} \rho^{-n}\tilde{\mu}_\infty(B_\rho(\xi))\) exists and is finite by (42).

By converting \(s\) to \(t\), the argument for the positive lower bound of the volume density in the proof of Proposition 3.1 carries over to the present situation.

We conclude that \(\lim_{\rho \to 0} \rho^{-n}\tilde{\mu}_\infty(B_\rho(\xi))\) exists and for \(H^n\) almost all \(\xi \in \tilde{\Sigma}_\infty\),

\[
0 < C \leq \lim_{\rho \to 0} \rho^{-n}\tilde{\mu}_\infty(B_\rho(\xi)) < \infty.
\]

Priess’s theorem in [P] then asserts the \(H^n\)-rectifiability of \((\tilde{\Sigma}_\infty, d\tilde{\mu}_\infty)\). Q.E.D.

**Definition 6.5** We call \((\tilde{\Sigma}_\infty, d\tilde{\mu}_\infty)\) obtained in Lemma 6.4 a tangent cone of the mean curvature flow \(\Sigma_t\) at \((X_0, T)\) in the time dependent scaling.

With the lemmas established in this section, by using arguments completely similar to those for the \(\lambda\) tangent cones in the previous sections, we can prove

**Theorem 6.6** Let \((M, \Omega)\) be a compact Calabi-Yau manifold. If the initial compact submanifold \(\Sigma_0\) is Lagrangian and almost calibrated by \(R\Omega\) and \(T > 0\) is the first blow-up time of the mean curvature flow, then the tangent cone \(\tilde{\Sigma}_\infty\) of the mean curvature flow at \((X_0, T)\) coming from time dependent scaling is a rectifiable stationary Lagrangian current with integer multiplicity in \(R^{2n}\). Moreover, if \(M\) is of complex 2-dimensional, then \(\tilde{\Sigma}_\infty\) consists of a finitely many (more than 1) 2-planes in \(R^4\) which are complex in a complex structure on \(R^4\).

The result below can also be found in [Wa].

**Corollary 6.7** If the initial compact submanifold \(\Sigma_0\) is Lagrangian and is almost calibrated in a compact Calabi-Yau manifold \((M, \Omega)\), then mean curvature flow does not develop Type I singularity.
Proof: Let $X_0$ be a Type I singularity at $T < \infty$ and set $\lambda = \max_{\Sigma_t} |A|^2$. The $\lambda$ tangent cone $\Sigma_\infty$ is smooth if $T$ is a Type I singularity. Therefore $\Sigma_\infty$ is a smooth minimal Lagrangian submanifold in $\mathbb{C}^n$ by Theorem 6.6. Because $\Sigma_\infty$ is smooth, (18) implies $F_\infty^\perp \equiv 0$ everywhere. The monotonicity identity (23) then implies $\sigma^{-1} \mu(\Sigma_\infty \cap B_\sigma(0))$ is a constant independent of $\sigma$, and the volume density ratio at 0 is one due to the smoothness of $\Sigma_\infty$, so $\Sigma_\infty$ is a flat linear subspace of $\mathbb{R}^{2n}$. But the second fundamental form of $\Sigma_\infty$ has length one at 0 according to the blow-up process, and the contradiction rules out any Type I singularity. Q.E.D.

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