Advances in 4D regularizations towards the systematization of calculations in multiple loops

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There is currently a high demand for theoretical predictions for processes at next-to-next-to-leading order (NNLO) and beyond, mainly due to the large amount of data which has already been collected at LHC. This requires practical methods that meet the physical requirements of the models under study. We develop a new approach to Implicit Regularization which simplifies the calculation of amplitudes, including finite parts. The algebraic identities to separate the divergent parts free from the external momenta are used after the Feynman parametrization. These algebraic identities establish a set of scale relations which are always the same and do not need to be calculated in each situation. This procedure unifies the calculations in massive and non-massive models in an unique procedure. We establish a systematization of the calculation of one-loop amplitudes and extend the procedure for higher-loop orders.

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I. INTRODUCTION

The success of Quantum Field Theory is highly attached to the principles of renormalization, which allow, through suitable normalization conditions, the redefinition of the quantities of interest in order to obtain physical meaningful results. This is because the perturbative calculations of transition amplitudes involve divergences. In the high energy regime, one has to deal with the ultraviolet (UV) divergences coming from the locality of fundamental interactions. The calculation of Feynman amplitudes, thus, is intricate with the technical problem of regularizing integrals in a consistent way such that the finite part presents the expected physical content of the amplitude.

The regularization technique to be used should meet some important features of the \(S\)-matrix, like the preservation of unitarity and causality, the preservation of gauge symmetry and supersymmetry, among others. Besides, the renormalization procedure can be more involved depending on the regularization applied. The usual textbook technique is Dimensional regularization [1], which is powerful due to gauge symmetry preservation in all orders of perturbation theory. The dimensional extension, however, causes difficulties when the model under investigation encompasses dimensional dependent mathematical objects, like the \(\gamma_5\) matrix. This is the case of topological and supersymmetric theories. Dimensional Reduction [2] is a way out of this problem for extending only the dimension of the Feynman integrals, but preserving the symmetry group algebra in the original space-time dimension. However, there are some mathematical inconsistencies in the procedure when the calculation is performed beyond one-loop order. The quantum action principle has been used in order to consistently apply Dimensional Reduction beyond one-loop order in supersymmetric models [3]. These difficulties with dimensional extension are the motivating fact...
for several developments of regularization procedures which are carried out in the proper dimension of
the model (good discussions on this topic are carried out in [4], [5] and [6]).

Differential Renormalization (DR) is one of these approaches [7]. It works in the proper dimension
of the theory in coordinate space, and has been proved to be simple and powerful in many applications
[8]-[11]. It consists in the manipulation of singular distributions attributing to them properties of the
regular ones. In the end, these singularities are substituted by renormalized functions and several mass
parameters are introduced in the results. Relations between these parameters are established in order to
preserve symmetries. A further development in order to automatically satisfy symmetries came with the
constrained version of Differential Renormalization (CDR) [12]-[19].

Implicit Regularization [20]-[22], on the other hand, is carried out in momentum-space, also in the
physical dimension of the theory. The basic idea of the Implicit Regularization (IReg) procedure of a
Feynman integral is to consider, before manipulating the integrands, the presence of some implicit reg-
ularization scheme or function. The scheme composes the originally divergent integral and allows the
separation of its part dependent on the regularization from the finite part, which must be independent of
the regularization used. The separation can be done by applying a simple algebraic identity to the inte-
grand, such that the divergent parts are written only in terms of the internal momentum in the loops and
do not need to be evaluated. The independence of the divergent integrals from the external momentum is
a highly desirable feature, since we will only need local counter-terms in the Lagrangian of the model in
order to eliminate any divergences that arise in the perturbative calculus. Furthermore, these divergent
integrals can be written as functions of an arbitrary mass parameter that characterizes the freedom of
separation of the divergent part of an amplitude and plays the role of scale in the renormalization group
equation. Symmetries of the model or phenomenological requirements determine arbitrary parameters
which are surface terms and arise from the procedure. There is a special choice for the parameters that
automatically delivers symmetric amplitudes in anomaly free cases. Fixing these parameters at the begin-
nning of the calculation considerably simplifies the application of the method. This results in a constrained
version of Implicit Regularization (CIReg). Implicit Regularization has been successfully applied to a
wide variety of problems, including non-abelian and supersymmetric models, and calculations beyond the
one-loop order [23]-[35].

The procedure described above is the traditional one for Implicit Regularization. It is important to note
that the procedure is applied after performing the algebra of the symmetry group in the amplitude, which
is decomposed into a combination of integrals. It is in these integrals that the Implicit Regularization is
applied. Each integral, then, is separated into finite and divergent parts. The Feynman parameterization
process is applied only to finite integrals. However, for a single Feynman integral, the expansion to
separate the divergences generates a set of finite integrals, which can be large in case of high degrees of
divergence. In addition, expansions generate high powers of momenta in the numerator and denominator,
which can complicate the calculations quite a bit.

The technique known as Loop Regularization (LORE) [36]-[38] has some aspects in common with Con-
strained Implicit Regularization, such as the use of consistency conditions, which, in practice, eliminate
surface terms that cause violation of symmetries. LORE prescribes that Feynman parameterization is
applied to the amplitude as a whole, like in Dimensional Regularization. Afterwards, the algebra of
the symmetry group is performed and then the integration in momenta. In the case of divergent parts,
consistency conditions are used to write the amplitude in terms of scalar loop integrals. Finally, these
scalar integrals are regularized similarly to the Pauli-Villars procedure and then calculated.

In this paper, we propose a new approach to Implicit Regularization which greatly simplifies the
calculation of the amplitudes, mainly the finite parts. Assuming the action of a regularization, we apply,
as in LORE, Feynman parameterization in the complete amplitude and eliminate the surface terms through
what we call consistency relations, so as to have only scalar divergent integrals. We then make use of
algebraic identities to expand the integrands to obtain basic divergences which are free from the external
momenta. These algebraic identities establish a set of scale relations which are always the same and
do not need to be calculated in each situation. As a byproduct, the scale relations allow to introduce
an arbitrary mass scale that will be useful in the process of renormalization. The results of this new
approach are the same of Constrained Implicit Regularization. However, the procedure also permits the
introduction of local arbitrary parameters for the models they are needed. This approach is also extended
for multiloop calculations, since there is currently a high demand for theoretical predictions for processes
at next-to-next-to-leading order (NNLO) and beyond, mainly due to the large amount of data which has already been collected at LHC. Therefore, practical methods for higher order calculations are being intensively investigated.

The paper is divided as follows: in section II we present a brief review of the traditional Implicit Regularization, with examples to be compared with the new ones; in section III, the new approach is presented, with discussions and comparisons with the cases of the previous section; in section IV, we carry out a systematization of the calculation of one-loop amplitudes; in section V, we present the extension for higher-loop calculations; section VI is left for conclusions and perspectives.

II. A BRIEF REVIEW OF IMPLICIT REGULARIZATION

Implicit Regularization (IReg) can be formulated by a set of rules. The first thing to be done is to assume a regularization is applied to the complete amplitude, so as algebraic manipulations can be carried out in the integrand. We then perform the group algebra and write the momentum-space amplitude as a combination of basic integrals, multiplied by polynomials of the external momentum and typical objects of the symmetry group. We give below examples of basic integrals:

\[ I, I_\mu, I_{\mu\nu} = \int^{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2)( (p-k)^2 - m^2)}, \]  

in which the index \( \Lambda \) in the integrals is to indicate they are regularized. Each one of these basic integrals can be treated following a set of rules. So, a table with their results can be used whenever a new calculation is being performed. The rules of Constrained Implicit Regularization (CIReg) for calculations at one-loop order can be stated as:

1. an amplitude is assumed to be regularized with a technique which is maintained implicit and which has the properties of not modifying neither the integrand nor the dimension of space-time. The first property is to preserve the finite part and the second is a requirement in order to not violate supersymmetry. It is the case of a simple cutoff. The problem of possible violation of symmetries by this technique will be automatically handled by the constraining character of Implicit Regularization;

2. to obtain the divergent part of a basic integral, we apply recursively the identity,

\[ \frac{1}{(p-k)^2 - m^2} = \frac{1}{(k^2 - m^2)} - \frac{p^2 - 2p \cdot k}{(k^2 - m^2)( (p-k)^2 - m^2)}, \]  

so as the divergent part do not have the external momentum \( p \) in the denominator. This will assure local counter-terms. The remaining divergent integrals have the form

\[ \int^{\Lambda} k_{\mu_1} k_{\mu_2} \cdots \frac{1}{(k^2 - m^2)\alpha}, \]  

in which we use \( \int k \) as a simplification of \( \int d^4k/(2\pi)^4 \);

3. The divergent integrals with Lorentz indices must be expressed in terms of divergent scalar integrals and surface terms. For example:

\[ \int^{\Lambda} \frac{k_{\mu} k_{\nu}}{(k^2 - m^2)^3} = \frac{1}{4} \left\{ \eta_{\mu\nu} \int^{\Lambda} \frac{1}{(k^2 - m^2)^2} - \int^{\Lambda} \frac{\partial}{\partial k^\nu} \left( \frac{k_{\mu}}{(k^2 - m^2)^2} \right) \right\}. \]  

The surface terms, that vanish for finite integrals, depend here on the method of regularization to be applied. They are symmetry-violating terms, since the possibility of making shifts in the integrals needs the surface terms to vanish. Non-null surface terms imply that the amplitude depend on the momentum routing choice. In practice, setting them zero from the beginning is equivalent to canceling these surface terms by means of local symmetry-restoring counter-terms;
4. Finally, the divergent part of the integrals is written as a combination of the basic divergences

\[ I_{\log}(m^2) = \int_k^\Lambda \frac{1}{(k^2 - m^2)^2} \quad \text{and} \quad I_{\text{quad}}(m^2) = \int_k^\Lambda \frac{1}{(k^2 - m^2)}, \tag{5} \]

which will require local counter-terms in the process of renormalization.

Let us now present an example to show how the traditional IReg applies to a complete amplitude in order to compare, in the next section, with the new proposed approach. We consider below the vacuum polarization tensor of spinorial QED, which after the use of Feynman rules, is given by

\[ -i \gamma^{\mu \nu} = q^2 \int_k^\Lambda \frac{\text{tr} \{ \gamma^\mu (\not{k} - \not{p} + m) \gamma^\nu (\not{k} + m) \}}{(k^2 - m^2)((k - p)^2 - m^2)}. \tag{6} \]

Following the steps listed above, we first calculate the trace and write the amplitude as a combination of basic integrals. We end up with

\[ -i q \gamma^{\mu \nu} = 4q^2 \left\{ 2 I^{\mu \nu} - p^\mu I^\nu - p^\nu I^\mu - \frac{1}{2} \eta^{\mu \nu} \left[ I_{\text{quad}}(m^2) + I_1 - p^2 I \right] \right\}, \tag{7} \]

where \( I, I_\mu \) and \( I_{\mu \nu} \) are defined in eq. \( 1 \), \( I_{\text{quad}}(m^2) \) is given in \( 5 \) and

\[ I_1 = \int_k^\Lambda \frac{1}{(k - p)^2 - m^2}. \tag{8} \]

The next step is to calculate each one of the integrals. Let us give the directions of the calculations of \( I_{\mu \nu} \). It is a divergent integral and the integrand need to be expanded using \( 2 \) in order to separate the finite part from the regularization dependent one. We have, after discarding the null integrals,

\[ I_{\mu \nu} = \int_k^\Lambda \frac{k_\mu k_\nu}{(k^2 - m^2)^2} - p^\mu \int_k^\Lambda \frac{k_\mu k_\nu}{(k^2 - m^2)^2} + 4 p^\mu p^\nu \int_k^\Lambda \frac{k_\mu k_\nu k_\alpha k_\beta}{(k^2 - m^2)^4} + \]

\[ + p^4 \int_k^\Lambda \frac{k_\mu k_\nu}{(k^2 - m^2)^4} - \int_k^\Lambda \frac{[p^2 - 2 (p \cdot k)]^2 k_\mu k_\nu}{(k^2 - m^2)^4 (k - p)^2 - m^2}. \tag{9} \]

The first term is quadratically divergent and the second and third terms are logarithmically divergent. We use the procedure of \( 4 \) to obtain scalar basic divergences and follow the notation from reference \( 39 \):

\[ \int_k^\Lambda \frac{k_\mu k_\nu}{(k^2 - m^2)^2} = \frac{\eta_{\mu \nu}}{2} \left[ I_{\text{quad}}(m^2) - \nu_2 \right], \tag{10} \]

\[ \int_k^\Lambda \frac{k_\mu k_\nu}{(k^2 - m^2)^2} = \frac{\eta_{\mu \nu}}{4} \left[ I_{\log}(m^2) - \nu_0 \right] \tag{11} \]

and

\[ \int_k^\Lambda \frac{k_\mu k_\nu k_\alpha k_\beta}{(k^2 - m^2)^4} = \frac{\eta_{\mu \nu \alpha \beta}}{24} \left[ I_{\log}(m^2) - \xi_0 \right], \tag{12} \]

in which \( \eta_{\mu \nu \alpha \beta} \equiv \eta_{\mu \nu} \eta_{\alpha \beta} + \eta_{\mu \alpha} \eta_{\nu \beta} + \eta_{\mu \beta} \eta_{\nu \alpha} \) and where \( \nu_0, \nu_2 \) and \( \xi_0 \) are surface terms. These terms are arbitrary and regularization dependent because they are differences between two integrals with the same degree of divergences, as shown in eq. \( 4 \). In this case 0 and 2 corresponds to logarithmic and quadratic divergences, respectively.
The last two integrals in (9) are finite and can be solved. The Feynman parametrization can be applied when necessary, like in the last one. Note that the high power on the momenta in the numerator and the denominator make calculations longer. The final result for \( I_{\mu\nu} \) is given by

\[
I_{\mu\nu} = \frac{\eta_{\mu\nu}}{2} I_{\text{quad}}(m^2) + \frac{1}{12} \left( -p^2 \eta_{\mu\nu} + 4p_\mu p_\nu \right) I_{\text{log}}(m^2) - \frac{\eta_{\mu\nu}}{12} \left( 6\upsilon_2 - 3p^2 \upsilon_0 + 2p^2 \xi_0 \right) - \frac{p_\mu p_\nu \xi_0}{3} + \frac{i}{(4\pi)^2} \left\{ \frac{1}{12p^2} \left[ (p^2 - 4m^2)p^2 \eta_{\mu\nu} - 4(p^2 - m^2)p_\mu p_\nu \right] Z_0(p^2, m^2, m^2, m^2) + \right.
\]
\[
+ \frac{1}{18}(p_\mu p_\nu - p^2 \eta_{\mu\nu}) \right\},
\]

where

\[
Z_k(p^2, m_1^2, m_2^2, m_3^2) = \int_0^1 dz \ z^k \ln \left\{ \frac{p^2 z(1-z) + (m_1^2 - m_2^2)z - m_3^2}{(-m_3^2)} \right\}.
\]

The same procedure is used for calculating the other Feynman integrals. We obtain, for the vacuum polarization tensor,

\[
-i q \Pi^{\mu\nu} = \frac{4}{3} \left( p^2 \eta^{\mu\nu} - p^\mu p^\nu \right) \left\{ I_{\text{log}}(m^2) - \frac{i}{(4\pi)^2} \left[ \frac{(p^2 + 2m^2)}{p^2} Z_0(p^2, m^2) + \frac{1}{3} \right] \right\} + 
\]
\[
+ 4\upsilon_2 \eta^{\mu\nu} - \frac{4}{3} \left\{ \upsilon_0(p^2 \eta^{\mu\nu} - p^\mu p^\nu) - (2p^\mu p^\nu + p^2 \eta^{\mu\nu})(\xi_0 - 2\upsilon_0) \right\},
\]

in which, for economy, we used \( Z_k(p^2, m_1^2, m_2^2, m_3^2) \equiv Z_k(p^2, m^2) \). Note that the amplitude is transversal if \( \upsilon_2 = 0 \) and \( \xi_0 = 2\upsilon_0 \). This approach of Implicit Regularization, in which the surface terms are parametrized and fixed in the end is useful when the model under investigation presents ambiguities as in the cases of chiral anomalies or topological field theories [42]. The Constrained Implicit Regularization, on the other hand, fixes the surface terms to zero from the beginning. This automatically delivers symmetric results, as it has been shown for Abelian and non-Abelian gauge theories and for supersymmetric models.

\section{III. A NEW APPROACH FOR CONSTRAINED IMPLICIT REGULARIZATION}

The basic idea of Constrained Implicit Regularization, as we already discussed, is to assume the presence of a regularization with the aim of using mathematical identities in order to separate the regularization dependent from the finite part. The divergent part is a combination of scalar basic divergences, which are obtained after the use of consistency relations that eliminate surface terms. Here, we propose a new formulation for Constrained Implicit Regularization which maintains the principles of the original procedure, but that simplifies enormously the process of calculation. The rules are listed below:

1. as in the original procedure, a regularization procedure is assumed to be acting in the complete amplitude, with the same desirable characteristics already presented;

2. Feynman parametrization is applied to the complete amplitude. This will assure that the needed shift in the momentum of integration is just a modification in the loop momentum;

3. the algebra of the group of symmetry is carried out;

4. the integrals in the momenta are separated by degree of divergence, all with even powers of the integration momentum in the numerator, of the type

\[
\int^\Lambda \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 + H^2)^n},
\]

being \( H^2 \) function of the external momenta, of the masses and of the Feynman parameters. If factors of \( k^2 \) appear in the numerator, they should be canceled with factors of the denominator by adding and subtracting \( H^2 \);
5. for the divergent parts, the surface terms are eliminated by means of the consistency relations, in order to obtain scalar integrals. For one-loop logarithmically and quadratically divergent integrals, we set, respectively,

$$\int_{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{k_{\mu_1}k_{\mu_2} \cdots k_{\mu_n}}{(k^2 + H^2)^{1+\frac{\alpha}{2}}} = \frac{\eta_{\mu_1,\mu_2} \cdots \eta_{\mu_{n-1},\mu_n}}{(2\pi)^4 (k^2 + H^2)^{\frac{\alpha}{2}}} \left\{ \int_{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 + H^2)^{\frac{\alpha}{2}}} \right\}$$  \hspace{1cm} (17)$$

and

$$\int_{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{k_{\mu_1}k_{\mu_2} \cdots k_{\mu_n}}{(k^2 + H^2)^{1+\frac{\beta}{2}}} = \frac{\eta_{\mu_1,\mu_2} \cdots \eta_{\mu_{n-1},\mu_n}}{(2\pi)^4 (k^2 + H^2)^{\frac{\beta}{2}}} \left\{ \int_{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 + H^2)^{\frac{\beta}{2}}} \right\},$$  \hspace{1cm} (18)

in which \(n\) is even and \(\eta_{\mu_1,\mu_2} \cdots \eta_{\mu_{n-1},\mu_n}\) is the symmetric combination of the products of metric tensors, \(\eta_{\mu_1,\mu_2} \cdots \eta_{\mu_{n-1},\mu_n}\), with coefficient 1. In the expression above, we left the parameters for the surface terms \(\alpha\) and \(\beta\) only for completeness, since in the constrained version of IReg they are fixed null. In order to obtain the relations above, we use recursively the relation,

$$\int_{\Lambda} \frac{\partial}{\partial k^{\mu_n}} \left( \frac{k_{\mu_1} \cdots k_{\mu_{n-1}}}{(k^2 + H^2)^{m-1}} \right) = \int_{\Lambda} \frac{\partial}{\partial k^{\mu_n}} \left( \frac{k_{\mu_1} \cdots k_{\mu_{n-1}}}{(k^2 + H^2)^{m-1}} \right) - 2(m - 1) \int_{\Lambda} \frac{k_{\mu_1} \cdots k_{\mu_n}}{(k^2 + H^2)^{m}},$$  \hspace{1cm} (19)

until the first integral of the second member of the equation is scalar. Apart two integrals, including the scalar one, all the others will be surface terms and are gathered in one parameter. In the equation above, we define \(\mathcal{S}[T_{\mu_1} \cdots T_{\mu_n}]\) as the minimal symmetrization of the tensor \(T\), in the sense that only distinct terms are considered and all of them have coefficient one. For example, \(\mathcal{S}[k_{\mu_p}k_{\mu_n}] = k_{\nu_p}k_{\nu_n} + k_{\nu_p}k_{\mu_n} + k_{\mu_p}k_{\nu_n}\). Important to note that \(m = \frac{n}{2} + 2\) for logarithmic divergences and \(m = \frac{n}{2} + 1\) for quadratic divergences. By the definitions of equation (15), the remaining scalar divergences above are \(I_{\text{log}}(-H^2)\) and \(I_{\text{quad}}(-H^2)\).

6. the next step, which is one of the basic ideas of IReg, is the use of algebraic identities in order to get the divergent integrals free from the external momenta. Here, we use recursively a simpler expansion,

$$\frac{1}{(k^2 + H^2)^{1+\frac{\alpha}{2}}} = \frac{1}{(k^2 - \lambda^2)^{1+\frac{\alpha}{2}}} - \frac{\lambda^2 + H^2}{(k^2 - \lambda^2)(k^2 + H^2)}. \hspace{1cm} (20)$$

We have the advantage of obtaining closed expressions to be used in any calculation:

$$I_{\text{log}}(-H^2) = I_{\text{log}}(\lambda^2) - \frac{i}{16\pi^2} \ln \left( -\frac{H^2}{\lambda^2} \right) \hspace{1cm} (21)$$

and

$$I_{\text{quad}}(-H^2) = I_{\text{quad}}(\lambda^2) - (\lambda^2 + H^2)I_{\text{log}}(\lambda^2) - \frac{i}{16\pi^2} \left[ \lambda^2 + H^2 - H^2 \ln \left( -\frac{H^2}{\lambda^2} \right) \right]. \hspace{1cm} (22)$$

These are called scale relations, which, as a byproduct, introduce an energy scale for the renormalization group, \(\lambda^2\), that can be simply one of the masses of the model. The basic divergences are now factorized out of the integrals in the Feynman parameters, which can be computed. As in the traditional formulation of IReg, the divergent part of the amplitudes is written in terms of these basic divergences: \(I_{\text{log}}(\lambda^2)\), \(I_{\text{quad}}(\lambda^2)\), etc.

This new formulation simplifies a lot the calculations of the finite parts. An additional advantage is related to models which present fields with different masses or non-massive fields. In non-massive models, the traditional formulation requires that a fictitious mass is introduced in the propagator so as the expansion in the integrand can be carried out. At the end of the calculation, the scale relations are
FIG. 1. Diagram which contributes to the self-energy of a massive vector-field. The fermionic fields in the loop have different masses and the momentum routing in the amplitude reads $k + (\alpha - 1)p$ for the propagator with mass $m_1$ and $k + \alpha p$ for the propagator with mass $m_2$. The wavy and solid lines represent the vectorial and fermionic fields, respectively.

used to remove the fictitious mass from the divergent part so that the limit of the mass going to zero can be taken. In the new formulation, the procedure is unified, since all the mass dependence is inside $H^2$.

Let us now carry out an example of calculation which is a little more involved than the QED vacuum polarization tensor.

Let us consider the vector-field self-energy in which the two fermions in the loop have different masses. It is the case of the influence of the heavy quarks, the doublet $(t,b)$, in the corrections to the $W$-boson mass (see, for example, [40]). The corresponding Feynman graph is depicted in Figure 1. Let us also, for pedagogical reasons, assign an arbitrary distribution of momenta in the internal lines. The amplitude is proportional to

$$\Pi^{\mu\nu}(p,m_1,m_2) = \int_k^\Lambda \frac{\text{tr} \left\{ \gamma^\mu [k + (\alpha - 1)p + m_2] \gamma^\nu [k + \alpha p + m_1] \right\}}{[k + \alpha p]^2 - m_1^2} \{[k + (\alpha - 1)p]^2 - m_2^2\}, \quad (23)$$

which after Feynman parametrization, with the shift $k \rightarrow k + (x - \alpha)p$, reads

$$\Pi^{\mu\nu}(p,m_1,m_2) = \int_0^1 dx \int_k^\Lambda \frac{\text{tr} \left\{ \gamma^\mu [k + (x - 1)p + m_2] \gamma^\nu [k + x p + m_1] \right\}}{(k^2 + H^2)^2}, \quad (24)$$

with $H^2 = p^2 x(1 - x) + (m_1^2 - m_2^2)x - m_1^2$. We note that the entire dependence on the $\alpha$ parameter disappeared. In other words, when the total amplitude is Feynman parametrized together, the amplitude is automatically momentum-routing invariant. After calculating the trace, with only even terms in $k^\mu$ remaining, and canceling terms in $k^2$ by adding and subtracting $H^2$, we stay with

$$\Pi^{\mu\nu}(p,m_1,m_2) = 4 \int_0^1 dx \left\{ -\eta^{\mu\nu} \int_k^\Lambda \frac{1}{k^2 + H^2} + 2 \int_k^\Lambda \frac{k^\mu k^\nu}{(k^2 + H^2)^2} + 2 \left( p^2 \eta^{\mu\nu} - p^\mu p^\nu \right) x(1 - x) + (m_1 - m_2) [(m_1 + m_2)x - m_1] \eta^{\mu\nu} \right\} \int_k^\Lambda \frac{1}{(k^2 + H^2)^2}. \quad (25)$$

The first two integrals, which are quadratically divergent, cancel out if we use equation (10) and fix $\nu_2 = 0$, as it is prescribed by constrained IReg. For the remaining $I_{log}(-H^2)$, we use the scale relation
Finally, we obtain (21), so that the amplitude is written as

\[ \Pi^{\mu\nu}(p, m_1, m_2) = 4 \int_0^1 dx \left\{ \frac{1}{2} I_{\log}(\lambda^2) - \frac{i}{16\pi^2} \ln \left( \frac{-H^2}{\lambda^2} \right) \right\} . \]

(26)

Finally, we obtain

\[ \Pi^{\mu\nu}(p, m_1, m_2) = 8(p^2 \eta^{\mu\nu} - p^\mu p^\nu) \left\{ \frac{1}{6} I_{\log}(\lambda^2) - \frac{i}{(4\pi)^2} (\tilde{Z}_1 - \tilde{Z}_2) \right\} + 4(m_1 - m_2) \eta^{\mu\nu} \left\{ - \frac{1}{2} (m_1 - m_2) I_{\log}(\lambda^2) - \frac{i}{(4\pi)^2} [(m_1 + m_2) \tilde{Z}_1 - m_1 \tilde{Z}_0] \right\} , \]

(27)

where \( \tilde{Z}_k \) is a short for \( Z_k(p^2, m_1^2, m_2^2, \lambda^2) \). Since the vector field is massive, the polarization tensor is not transverse. Rather, the amplitude obeys the relation

\[ p_\nu \Pi^{\mu\nu} = 4(m_1 - m_2) p^\mu \left\{ - \frac{1}{2} (m_1 - m_2) I_{\log}(\lambda^2) - \frac{i}{(4\pi)^2} [(m_1 + m_2) \tilde{Z}_1 - m_1 \tilde{Z}_0] \right\} = (m_1 - m_2) T^\mu , \]

(28)

where

\[ T^\mu = \int_k^{\Lambda} \frac{\text{tr} \{ \gamma^\mu (k - p + m_2)(k + m_1) \}}{(k^2 - m_1^2)[(k - p)^2 - m_2^2]} . \]

(29)

This is a very direct and compact calculation. Note that the procedure would be identical in the case of non-massive QED, with the obvious modification of \( H^2 \), with only the transverse part remaining.

The Ward identity in eq. (28) is a particular case of a diagrammatic relation. In Figure 2, we see this diagrammatic relation is independent of regularization. In the abelian case, QED for example, gauge invariance is fulfilled if and only if momentum routing invariance is as well, as we can easily see in Figure 2(a). In the approach we present in this work, the shifts we perform in Feynman parameterization is already an assumption of momentum routing invariance. This already delivers gauge invariant results. Furthermore, when considering the electroweak theory as a whole, we have an additional relation as in Fig. 2(b) due to the change of flavors and we recover QED when \( m_1 \to m_2 = m \). In this case, mass terms have already broken the larger gauge symmetry \( SU(2) \otimes U(1) \) in simply \( U(1) \).

It is important to comment on the fact that some aspects of our approach are similar to procedures adopted in Loop Regularization (LORE). In the case of LORE, after the amplitude is Feynman parametrized, consistency conditions are applied which, in practice, discard surface terms. Such conditions, which are the same of IReg, were determined, in that case, by the requirement that symmetries be respected in specific amplitudes [36] and then generalized. The remaining scalar divergent loop integrals, however, are calculated using a procedure which is similar to the Pauli-Villars regularization [41]. On the other hand, Implicit Regularization is based on the elimination of surface terms and in the expansion of the integrand so as the renormalization needs only local counter-terms. The remaining divergent integrals do not need be explicitly calculated.

In the next section, we present a systematization for calculations of general one-loop divergent amplitudes.

IV. A SYSTEMATIZATION FOR THE CALCULATION OF ONE-LOOP INTEGRALS

We present now a systematization of the calculation of one-loop Feynman integrals in the framework of this new approach for Implicit Regularization. The procedure is very interesting, since the results encompass the finite parts. The methodology we carry out in this section applies to integrals which are part of the amplitude. Since one of the principles adopted in this approach is applying Feynman
parametrization to the amplitude as a whole for assuring the shift takes place as a redefinition in the momentum routing in the loop, care must be taken. It is possible to carry out the symmetry algebra and writing the amplitude as a combination of integrals, and then parameterizing each one of the integrals. However, for the sake of consistency, the parametrization and the shift should be the same in all the integrals which compose the amplitude.

Let us begin with a general one-loop integral with logarithmic degree of divergence, which is written as

$$I_{\mu_1...\mu_n}^{(0)} = \int_k^A \frac{k_{\mu_1}\cdots k_{\mu_n}}{(k^2 - m^2)(p_1 - k)^2 - m_1^2}\cdots[(p_r - k)^2 - m_r^2],$$

with $r = 1 + \frac{n}{2}$. The first step is to carry out the Feynman parametrization. We use

$$\frac{1}{a_1\cdots a_r b} = r! \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \int_0^{1-\sum_{i=1}^{r-1} x_i} dx_r \frac{1}{\sum_{k=1}^r (a_k - b)x_k + b}^{r+1},$$

with $a_k = [(p_k - k)^2 - m_k^2]$ and $b = (k^2 - m^2)$. Considering the denominator is given by $D^{r+1}$, it is possible to rearrange $D$ in order to write

$$D = \left[k - \sum_{k=1}^r p_k x_k\right]^2 + Q^2,$$
from which only the even powers of \( k \) survive, let us call it \( \tilde{N}_{\mu_1 \cdots \mu_n} \). Our logarithmic divergent amplitude is then given by

\[
I^{(0)}_{\mu_1 \cdots \mu_n} = r! \int dX \int_k^\Lambda \frac{\tilde{N}_{\mu_1 \cdots \mu_n}}{(k^2 + Q^2)^{r+1}},
\]

where \( \int dX \) stands for all the integrals in the Feynman parameters. The higher power in \( k \) in \( \tilde{N}_{\mu_1 \cdots \mu_n} \) is responsible for the logarithmic divergence. All the other terms are finite. For the divergent part, we have

\[
I^{(0)\Lambda}_{\mu_1 \cdots \mu_n} = \frac{\eta_{\mu_1 \cdots \mu_n}}{2\pi} \int dX \left\{ \int_k^\Lambda \frac{1}{(k^2 + Q^2)^2} - \alpha \right\},
\]

in which we have used the consistency relation of (17) and the fact that \( r = \frac{n}{2} + 1 \). Next, we use the the scale relation (21) to obtain

\[
I^{(0)\Lambda}_{\mu_1 \cdots \mu_n} = \frac{\eta_{\mu_1 \cdots \mu_n}}{2\pi} \int dX \left\{ I_{\log}(m^2) - \alpha \right\}.
\]

The unique part which is dependent of the Feynman parameters is the one that contains \( Q^2 \). The other can be factorized out of the integral. The final result is given by

\[
I^\Lambda_{\mu_1 \cdots \mu_n} = \frac{\eta_{\mu_1 \cdots \mu_n}}{2\pi} \left\{ \frac{1}{m!} I_{\log}(m^2) + \alpha \right\} - \frac{i}{16\pi^2} Z^{(r,0)},
\]

in which we define the function

\[
Z^{(r,k_1,\cdots,k_r)} = Z^{(r,k_1,\cdots,k_r)}(p_1, \cdots, p_r, m_1^2, \cdots, m_r^2, m^2) \equiv \int dX q_{\mu_1} \cdots q_{\mu_n} x_1^{k_1} \cdots x_r^{k_r} \log \left( \frac{Q^2}{m^2} \right).
\]

We now have to take care of the other finite terms, which appears if \( n \geq 2 \). A typical finite term of (35) is

\[
F^{(l)}_{\mu_1 \cdots \mu_n} = r! \int dX S \left[ A_{\mu_1 \cdots \mu_l q_{\mu_{l+1}} \cdots q_{\mu_n}} \right],
\]

with \( l < n \) even and

\[
A_{\mu_1 \cdots \mu_l} = \int_k^\Lambda \frac{k_{\mu_1} \cdots k_{\mu_l}}{(k^2 + Q^2)^{r+1}} = \frac{i}{16\pi^2} \frac{\Gamma\left(\frac{n-l}{2}\right)}{2^{\frac{n-l}{2}} \Gamma\left(r+\frac{1}{2}\right)} \frac{1}{(Q^2)^{\frac{n-l}{2}}} \eta_{\mu_1 \cdots \mu_l}.
\]

We, then, stay with

\[
F^{(l)}_{\mu_1 \cdots \mu_n} = \frac{i}{16\pi^2} \frac{\Gamma\left(\frac{n-l}{2}\right)}{2^{\frac{n-l}{2}} S \left[ \eta_{\mu_1 \cdots \mu_l} \int dX q_{\mu_{l+1}} \cdots q_{\mu_n} \right]}
\]

\[
= \frac{i}{16\pi^2} \frac{\Gamma\left(\frac{n-l}{2}\right)}{2^{\frac{n-l}{2}}} S \left[ \eta_{\mu_1 \cdots \mu_l} Y^{(l, \frac{n-l}{2})}_{\mu_{l+1} \cdots \mu_n} \right],
\]

where

\[
Q^2 = \sum_{k=1}^r \left[p_k^2 x_k(1 - x_k) + (m^2 - m_k^2)x_k\right] - \sum_{k \neq l} (p_k \cdot p_l)x_k x_l - m^2
\]
where

\[ Y_{\mu_1 \cdots \mu_n}^{(r,l)} = Y_{\mu_1 \cdots \mu_n}^{(r,l)}(p_1, \cdots, p_r, m_1^2, \cdots, m_r^2, m^2) = \int dx_1 \cdots dx_r \frac{q_{\mu_1} \cdots q_{\mu_n}}{(Q^2)^l}. \]

Finally, we can write the general result as

\[ I_{\mu_1 \cdots \mu_n}^{(0)} = \frac{\eta_{\mu_1 \cdots \mu_n}}{2^2} \left\{ \frac{1}{r!} \left[ I_{\log}(m^2) - \alpha_{n+2} \right] - \frac{i}{16\pi^2} Z^{(r,0)} \right\} + \]

\[ + \frac{i}{16\pi^2} \sum_{i=0}^{n-2} \Gamma \left( \frac{n-2i}{2} \right) S \left[ \eta_{\mu_1 \cdots \mu_2} Y_{\mu_2 i+1 \cdots \mu_n}^{(r, n-2i-1)} \right]; \quad r = \frac{n}{2} + 1, \]

in which we substituted \( l \) by \( 2i \), since \( l \) is even.

For a Linearly divergent integral,

\[ I_{\mu_1 \cdots \mu_n}^{(1)} = \int_{k}^{A} \frac{k_{\mu_1} \cdots k_{\mu_n}}{(k^2 - m^2)((p_1 - k)^2 - m_1^2) \cdots ((p_r - k)^2 - m_r^2)}, \]

with \( n \) odd and \( r = \frac{n+1}{2} \), the procedure is very similar, with the result

\[ I_{\mu_1 \cdots \mu_n}^{(1)} = \frac{1}{2^2} \sum_{k=1}^{r} p_{k \mu_1 \cdots \mu_n} \left( \frac{1}{(r+1)!} \left[ I_{\log}(m^2) - \alpha_{2n+2} \right] - \frac{i}{16\pi^2} \right) + \frac{1}{2^2 \Gamma(0)} S \left[ Z_{\mu_1}^{(r,0)} \eta_{\mu_2 \cdots \mu_n} \right] + \]

\[ + \frac{i}{16\pi^2} \sum_{i=0}^{n-2} \Gamma \left( \frac{n-2i+1}{2(2r+1)} \right) S \left[ \eta_{\mu_1 \cdots \mu_2} Y_{\mu_2 i+1 \cdots \mu_n}^{(r, n-2i-1)} \right]; \quad r = \frac{n+1}{2}, \]

in which the last term only appears for \( n \geq 3 \).

In order to complete the systematization of the calculation of one-loop amplitudes, we turn our attention now to quadratically divergent integrals, \( I_{\mu_1 \cdots \mu_n}^{(2)} \). Since the calculation is a little more involved, we will show below the main steps. The integral to be calculated has the same form as (30), but with \( r = \frac{n}{2} \), \( n \) even. After Feynman parametrization, we obtain the expression of (33), which will be broken in three parts: the higher power in \( k \) in the numerator is quadratically divergent; the \( (n-2) \)-th power in \( k \) in the numerator is logarithmically divergent; and the \( (n-4) \)-th power and lower, if they exist, are finite. It is important to note that even the two divergent pieces contribute to the finite result, as it is evident from the calculations of the last section.

Let us begin with the quadratic divergent:

\[ I_{\mu_1 \cdots \mu_n}^{(2,1)} = \left( \frac{n}{2} \right)! \int dX \int_{k}^{A} \frac{k_{\mu_1} \cdots k_{\mu_n}}{(k^2 + Q^2)^{n+2}} = \frac{\eta_{\mu_1 \cdots \mu_n}}{2^2} \int dX \left\{ \int_{k}^{A} \frac{1}{(k^2 + Q^2)^{n+2}} - \beta_{\frac{n+2}{2}} \right\}, \]

in which we made use of (18). Next, we resort to the scale relation of equation (22) to obtain

\[ I_{\mu_1 \cdots \mu_n}^{(2,1)} = \frac{\eta_{\mu_1 \cdots \mu_n}}{2^2} \int dX \left\{ \frac{1}{(n/2)!} I_{\text{quad}}(m^2) - \frac{1}{n/2 + 2} \left[ \sum_{k=1}^{r} \left( \frac{n}{2} p_{k}^2 + \frac{n}{2} + 2 \right) (m^2 - m_k^2) \right] + \right. \]

\[ - \left. \sum_{k \neq l} (p_{k} \cdot p_{l}) \right\} \left[ I_{\log}(m^2) + \frac{i}{16\pi^2} \right] + \frac{1}{16\pi^2} \int dX Q^2 \ln \left( \frac{Q^2}{m^2} \right), \]

where the last integral can be written in terms of \( Z^{(r, k_1, \cdots, k_r)} \) functions.
For the second part, which is logarithmically divergent, we have

\[ I^{(2),2}_{\mu_1 \cdots \mu_n} = \left( \frac{n}{2} \right) ! \int dX \mathcal{S} \left[ q_{\mu_1} q_{\mu_2} \int_k \frac{k_{\mu_3} \cdots k_{\mu_n}}{(k^2 + Q^2)^{\frac{n}{2} + 1}} \right] \]

\[ = \frac{1}{2} \frac{n-2}{2^{n-2}} \int dX [q_{\mu_1} q_{\mu_2} \eta_{\mu_3 \cdots \mu_n}] \left\{ I_{\log}(m^2) - \frac{\alpha}{\pi^2} + \frac{i}{16\pi^2} \ln \left( -\frac{Q^2}{m^2} \right) \right\} \]

\[ = \frac{1}{2} \frac{n-2}{2^{n-2}} \mathcal{S} \left[ \eta_{\mu_3 \cdots \mu_n} \left\{ \frac{1}{(n/2 + 2)!} \left[ 2 \sum_{k=1}^{n/2} p_{k\mu_1} p_{k\mu_2} + \sum_{k \neq l} p_{k\mu_1} p_{l\mu_2} \right] I_{\log}(m^2) - \frac{\alpha}{\pi^2} \right\} + \right. \]

\[ \left. - \frac{i}{16\pi^2} \gamma_{(n/2,0)}(n/2,0) \right] \right\}. \quad (49) \]

The remaining finite part is given by

\[ I^{(2),3}_{\mu_1 \cdots \mu_n} = \left( \frac{n}{2} \right) ! \sum_{l=4}^{n} \int dX \mathcal{S} \left[ q_{\mu_1} \cdots q_{\mu_l} \int_k \frac{k_{\mu_{l+1}} \cdots k_{\mu_n}}{(k^2 + Q^2)^{\frac{l}{2} + 1}} \right] \]

\[ = \frac{i}{16\pi^2} \sum_{l=4}^{n} \frac{\Gamma \left( \frac{l}{2} - 1 \right)}{2^{(n-l)/2}} \int dX \mathcal{S} \left[ q_{\mu_1} \cdots q_{\mu_l} \eta_{\mu_{l+1} \cdots \mu_n} \right] \frac{1}{(Q^2)^{\frac{l}{2} - 1}} \]

\[ = \frac{i}{16\pi^2} \sum_{l=4}^{n/2} \frac{\Gamma \left( i - 1 \right)}{2^{(n-2l)/2}} \mathcal{S} \left[ \eta_{\mu_{2l+1} \cdots \mu_n} \gamma_{\mu_{1 \cdots \mu_{2l}}} \right]. \quad (50) \]

The total result for the quadratically divergent integral is given by

\[ I^{(2)}_{\mu_1 \cdots \mu_n} = I^{(2),1}_{\mu_1 \cdots \mu_n} + I^{(2),2}_{\mu_1 \cdots \mu_n} + I^{(2),3}_{\mu_1 \cdots \mu_n}. \]

V. DIRECTIONS ON HIGHER ORDER CALCULATIONS

There is currently a high demand for theoretical predictions for processes at next-to-next-to-leading order (NNLO) and beyond, mainly due to the large amount of data which has already been collected at LHC. With this aim, new calculation techniques have been developed in recent years, which seek, as far as possible, to preserve the physical dimension of the spacetime [3]. Higher-loop calculations are usually very long and intricate and friendly procedures are welcome. In the context of this new approach for IRReg, we give some directions for future systematization of multi-loop calculations.

In the last section, we obtained general finite parts which are integrals, in the Feynman parameters, which contain factors of \( \ln \left( Q^2 / (-m^2) \right) \) or powers of \( 1/Q^2 \). The challenging part is the one which includes the logarithm. With the aim of applying Feynman parametrization, we will make use of the following mathematical identity:

\[ \ln a = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (a^\varepsilon - 1). \quad (51) \]

In a simple view, we say that \( \ln a \) is equal to the first order coefficient of the expansion of \( a^\varepsilon \) in the limit of small \( \varepsilon \).

So, let us give an example in which the one-loop finite part is an integral of a function with a factor of a logarithm. We consider the two-loop nested self-energy of the electron in QED, which is depicted on Figure [3].

The finite part of the subgraph is given by

\[ i \Sigma^{(1)} = 2q^2 \frac{i}{16\pi^2} \int_0^1 dx \left[ 2m - p(1 - x) \right] \ln \left( -\frac{H^2}{m^2} \right), \quad (52) \]

with \( H^2 = H^2(p^2, m^2) = p^2x(1 - x) - m^2x \). The integral for the two-loop graph is then written as

\[ i \Sigma^{(2)}(p) = -2iq^4 \frac{i}{16\pi^2} \int_0^1 dx \int_k^\Lambda \gamma_{m} \frac{2m + (x - 1)k_m}{(k^2 - m^2)^2(p - k)^2} \ln \left( -\frac{H^2(k^2, m^2)}{m^2} \right). \quad (53) \]
FIG. 3. Diagrammatic representation of the nested contribution to the electron self-energy at two-loop order. The wavy and solid lines represent the photon and fermion propagators, respectively.

Let us define

\[ F = -2iq^4 \frac{i}{16\pi^2} \int_0^1 dx \int_k^\Lambda \frac{\gamma^\alpha(k + m)[2m + (x - 1)\hat{k}](\hat{k} + m)\gamma_\alpha}{(k^2 - m^2)^2(p - k)^2} \left( -\frac{H^2}{m^2} \right)^\varepsilon, \]  

such that \( i\Sigma^{(2)}(p) = F_\varepsilon \), being \( F_\varepsilon \) the first order coefficient in the expansion of \( F \) in powers of \( \varepsilon \). We then write

\[ \left( -\frac{H^2}{m^2} \right)^\varepsilon = \left[ \frac{x(1 - x)}{(-m^2)} \right]^\varepsilon \left( k^2 - \tilde{m}^2 \right)^\varepsilon; \quad \tilde{m}^2 = \frac{m^2}{1 - x}, \]  

to obtain

\[ F = -2iq^4 \frac{i}{16\pi^2} \int_0^1 dx \left[ \frac{x(1 - x)}{(-m^2)} \right]^\varepsilon \int_k^\Lambda \frac{\gamma^\alpha(k + m)[2m + (x - 1)\hat{k}](\hat{k} + m)\gamma_\alpha(k^2 - \tilde{m}^2)}{(k^2 - m^2)^2(k^2 - \tilde{m}^2)^{1-\varepsilon}(p - k)^2}. \]  

In the equation above, we multiplied the numerator and the denominator by a factor of \( (k^2 - \tilde{m}^2) \) for convenience to obtain a well defined Feynman parametrization, from which, we obtain

\[ F = -2iq^4 \frac{i}{16\pi^2} \Gamma(4 - \varepsilon) \int_0^1 dx \int_0^1 du \int_0^{1-u} dv \left[ \frac{x(1 - x)}{(-m^2)(1 - u - v)} \right]^\varepsilon \int_k^\Lambda \frac{\tilde{N}}{(k^2 + Q^2)^{1-\varepsilon}}, \]  

where \( Q^2 = p^2u(1 - u) - \tilde{m}^2(1 - u - v) - m^2v \) and \( \tilde{N} \) is the even part in \( k \) of

\[ N = \gamma^\alpha(\hat{k} + \hat{p}u + m)[2m + (x - 1)(\hat{k} + \hat{p}u)](\hat{k} + \hat{p}u + m)\gamma_\alpha[(k + pu)^2 - \tilde{m}^2]. \]  

The divergent part of \( F \) is the one with quartic terms in \( k \) in the numerator. Let us carry out explicitly the calculation of this part. After performing the Dirac algebra, the quartic part of the numerator is written as

\[ N^{(4)} = 2k^4[4mx - u(x - 1)\hat{p}] - 8k^2u(x - 1)(p \cdot k)\hat{k} \]
\[ = 2(k^2 + Q^2)^2[4mx - u(x - 1)\hat{p}] - 4(k^2 + Q^2)Q^2[4mx - u(x - 1)\hat{p}] \]
\[ + 2Q^4[4mx - u(x - 1)\hat{p}] - 8(k^2 + Q^2)u(x - 1)(p \cdot k)\hat{k} + 8Q^2u(x - 1)(p \cdot k)\hat{k}, \]

in which we added and subtracted \( Q^2 \) to the factors of \( k^2 \). The first and fourth terms will result in logarithmically divergent integrals. For the first, we have

\[ F_1^{(4)} = -4iq^4 \frac{i}{16\pi^2} \Gamma(4 - \varepsilon) \int_{x,u,v} x(1 - x) \left[ \frac{x(1 - x)}{(-m^2)(1 - u - v)} \right]^\varepsilon \int_k^\Lambda \frac{1}{(k^2 + Q^2)^{2-\varepsilon}}. \]
with \( \int_{x,u,v} \) representing the integrals in the Feynman parameters. We then expand the above expression for small \( \varepsilon \) to get the coefficient of the first order term

\[
F^{(4)}_{1\varepsilon} = -4i q^4 \frac{i}{16\pi^2} \int_{x,u,v} v[4mx - u(x-1)\hat{p}] \left\{ 6I^{(2)}_{\log}(-Q^2, m^2) + \right. \\
+ \left. \left[ 6 \ln \left( \frac{x(1-x)}{(1-u-v)} \right) - 11 \right] I_{\log}(-Q^2) \right\},
\]

in which, we use of the definition of the basic logarithmic divergence typical of two-loop order,

\[
I^{(2)}_{\log}(m^2, \lambda^2) = \int_k^A \frac{1}{(k^2 - m^2)^2} \ln \left( \frac{(k^2 - m^2)}{(-\lambda^2)} \right).
\]

Besides the scale relations (21) and (22) we have already used for the one-loop calculations, we can easily obtain, through the same algebraic manipulations, the corresponding two-loop one,

\[
I^{(2)}_{\log}(m^2, \lambda^2) = I^{(2)}_{\log}(\lambda^2) - \frac{i}{16\pi^2} \left\{ \ln \left( \frac{m^2}{\lambda^2} \right) + \frac{1}{2} \ln^2 \left( \frac{m^2}{\lambda^2} \right) \right\},
\]

where \( I^{(2)}_{\log}(\lambda^2) = I^{(2)}_{\log}(\lambda^2, \lambda^2) \). This relation will be used to obtain a divergent part free from the external momenta. But let us first get the result of the other divergent term,

\[
F^{(4)}_4 = 16i q^4 \frac{i}{16\pi^2} \frac{\Gamma(4 - \varepsilon)}{\Gamma(1 - \varepsilon)} \int_{x,u,v} v u (x - 1) \left[ \frac{x(1-x)}{(-m^2)(1-u-v)} \right] \epsilon \gamma^\alpha \gamma^\beta \int_k^A \frac{k_\alpha k_\beta}{(k^2 + Q^2)^{3-\epsilon}}.
\]

For the integral in \( k \), we can write

\[
\int_k^A \frac{k_\alpha k_\beta}{(k^2 + Q^2)^{3-\epsilon}} = \frac{1}{2(2 - \varepsilon)} \left\{ \int_k^A \frac{\eta_{\alpha\beta}}{(k^2 + Q^2)^{2-\epsilon}} - \int_k^A \frac{\partial}{\partial k^\gamma} \frac{k_\alpha}{(k^2 + Q^2)^{2-\epsilon}} \right\},
\]

from which we discard the surface term as prescribed by constrained IReg. After substituting the above relation in \( F^{(4)}_4 \) and picking the first order coefficient of the expansion in powers of \( \varepsilon \), we get

\[
F^{(4)}_{4\varepsilon} = 8i q^4 \frac{i}{16\pi^2} \hat{p} \int_{x,u,v} v u (x - 1) \left\{ 3I^{(2)}_{\log}(-Q^2, m^2) + \left[ 3 \ln \left( \frac{x(1-x)}{(1-u-v)} \right) - 4 \right] I_{\log}(-Q^2) \right\}.
\]

We now get together the two divergent integrals, use the scale relations (21) and (63) and integrate the coefficients of the basic divergences to obtain

\[
F^{(4)}_{1\varepsilon} + F^{(4)}_{4\varepsilon} = -\frac{i}{2} q^4 \frac{i}{16\pi^2} \left\{ 2(\hat{p} + 8m)I^{(2)}_{\log}(m^2) - (3\hat{p} + 32m)I_{\log}(m^2) \right\} + \\
+ 4i q^4 \left\{ \frac{i}{16\pi^2} \right\}^2 \int_{x,u,v} v \left\{ \left[ 4mx \left[ 6 \ln \left( \frac{x(1-x)}{(1-u-v)} \right) - 5 \right] \right. \\
- \left. u(x-1)\hat{p} \left[ 12 \ln \left( \frac{x(1-x)}{(1-u-v)} \right) - 7 \right] \right] \ln \left( -\frac{Q^2}{m^2} \right) + \\
+ 12[2mx - u(x-1)\hat{p}] \ln^2 \left( \frac{Q^2}{m^2} \right) \right\}.
\]

There are still three finite integrals considering the quartic part of the numerator. Besides, we have finite integrals coming from the quadratic and zeroth order in \( k \) terms in the numerator. These calculations are straightforward: the integration in \( k \) is performed, resulting in \( \varepsilon \)-depending powers of \( Q^2 \) in the
denominator; the expansion in powers of $\epsilon$ is performed to take the first order coefficient. The final result for the two-loop amplitude is given by

$$i\Sigma^{(2)}(p) = \frac{i}{2}q^4 \left\{ 2(\phi + 8m)I_{\log}^{(2)}(m^2) - (3\phi + 32m)i_{\log}(m^2) + \frac{1}{16\pi^2} (5\phi + 352m) \right\} +$$

$$+ 4iq^4 \left\{ \frac{i}{16\pi^2} \int_{x,u,v} \ln \left( \frac{Q^2}{m^2} \right) \ln \left( \frac{Q^2}{m^2} \frac{x(1-x)}{(1-u-v)} \right) +$$

$$+ \frac{1}{Q^2} \left[ B \ln \left( \frac{Q^2}{m^2} \frac{x(1-x)}{(1-u-v)} \right) + C \right] + D \right\}. \tag{68}$$

with

$$A = 12(2ux - u(x - 1)),$$
$$B = 2 \{ u^2[p^2(x - 1) + 2m^2(x + 4)] - 2m[2u(u + 3)xp^2 + 3m^2 - x\tilde{m}^2]\}, \tag{69}$$
$$C = 4m(2uxp^2 + m^2 - \tilde{m}^2 x) - u^2[p^2(x - 1) + m^2(x + 4)], \tag{70}$$
$$D = -(u^2p^2 - \tilde{m}^2) \left\{ 4m(ux^2p^2 + m^2) - u^2[p^2(x - 1)p^2 + m^2(x + 3)] \right\}. \tag{71}$$

There are some interesting comments about the two-loop calculation above. First, the basic divergence $I_{\log}^{(2)}(m^2)$ appears naturally, even for massive models, when the expansion in $\epsilon$ is carried out. Second, the momentum integration of the finite part is easily performed with the help of Feynman parametrization. And, last but not least, the elimination of the surface terms is simpler than the traditional procedure adopted in IReg \cite{35}, which needs new relations for each loop-order. The procedure above can be systematized and be of great help in multi-loop phenomenological calculations.

VI. CONCLUDING COMMENTS

In this paper, we have established a new procedure for the application of Implicit Regularization in its constrained version. The constrained version of Implicit Regularization, which fixes all the surface terms to zero, automatically delivers symmetric amplitudes as already demonstrated in a wide variety of articles. This is due to the fact that symmetries such as gauge-invariance are related, in the context of Feynman integrals, to momentum routing invariance in the loops \cite{39}. This new approach uses this fact with the aim of Feynman parametrize the complete amplitude after a regularization is assumed to be implicitly acting in the divergent integral. As it is well known, the usefulness of Feynman parameterization lies in the possibility of making a shift in the momentum of integration. A shift in a divergent integral would have to be compensated with a surface term if the degree of divergence is at least linear. This is why the amplitude should be regularized before Feynman parametrization is carried out. In addition, the regularization prescription must be such that these surface terms are null. This is the case of Dimensional Regularization, which turns the amplitude finite in the extended dimension and thus forces the surface terms to vanish. In the case of the constrained version of Implicit Regularization, this is accomplished with the help of the consistency relations.

The procedure which is presented in this paper enforces momentum routing invariance by shifting the momentum of integration after the Feynman parametrization of the complete amplitude. It also fixes other remaining surface terms to zero. The great simplification in the approach occurs in consequence of the way the divergent part is separated from the finite one. While in the traditional application of IReg the integrand is expanded before Feynman parametrization, which is carried out only in the finite part, here this separation takes place after this step, by using scale relations that are always the same. We then avoid to deal with finite integrals with high powers in the momenta in the numerator and the denominator. Another advantage is the unification of the procedure to be adopted in massive and non-massive models, since the scale relations are in charge of introducing the mass parameter for the basic divergences and for the renormalization group equations. The great simplification in one-loop calculation for this new approach extends for higher-loop orders, as demonstrated in section V. The procedure above can be systematized and be of great help in multi-loop phenomenological calculations. This is part of a future work.
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