The effective potential of composite fields in weakly coupled QED in a uniform external magnetic field

D.-S. Lee
Department of Physics, National Dong Hwa University, Shoufeng, Hualien 974, Taiwan, ROC

P. N. McGraw, Y. J. Ng
Department of Physics and Astronomy, University of North Carolina, Chapel Hill, North Carolina 27599

I. A. Shovkovy
Physics Department, University of Cincinnati, Cincinnati, OH 45221-0011, USA

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The effective potential for the composite fields responsible for chiral symmetry breaking in weakly coupled QED in a magnetic field is derived. The global minimum of the effective potential is found to acquire a non-vanishing expectation value of the composite fields that leads to generating the dynamical fermion mass by an external magnetic field. The results are compared with those for the Nambu-Jona-Lasinio model.

I. INTRODUCTION

The problem of symmetry breaking and vacuum instability under the influence of external fields has attracted a lot of attention in the past few years [1,2]. Recently, some progress was attained in understanding the mechanism of the so-called catalysis of chiral symmetry breaking [3–10] (for some earlier studies see also [11–13]). This problem provides an especially interesting example of vacuum engineering: i.e., manipulating external fields to alter the symmetry properties of the vacuum. It is also one of a small number of tractable non-perturbative problems in (3+1)-dimensional quantum field theories, and as such may provide insight into the non-perturbative vacuum structure of other theories such as quantum chromodynamics.

Here we continue the study of this effect in weakly coupled quantum electrodynamics (QED). In particular, we derive the expression for the effective potential of composite fields responsible for chiral symmetry breaking by applying the method of [14,15], originally introduced in the gauged Nambu-Jona-Lasinio (NJL) model and QED3.

In this paper we are interested in constructing the effective potential for the composite local fields \( \sigma(x) = \langle 0 | \bar{\psi}(x) \psi(x) | 0 \rangle \) and \( \pi(x) = \langle 0 | \bar{\psi}(x) i \gamma_5 \psi(x) | 0 \rangle \). In general, this problem requires consideration of the corresponding generating functional,

\[
Z(J_s, J_p) \equiv \exp \left[ i W(J_s, J_p) \right] = \int \prod_x (d\psi(x) d\bar{\psi}(x) dA^\mu(x)) \exp \left[ i \int d^4x \left( \mathcal{L} + J_s(x) \bar{\psi}(x) \psi(x) + J_p(x) \bar{\psi}(x) i \gamma_5 \psi(x) \right) \right].
\]

where \( \mathcal{L} \) is the Lagrangian density of massless QED in an external magnetic field. Then the calculation of the effective action reduces to obtaining the Legendre transform of the functional \( W \) with respect to the external sources,

\[
\Gamma(\sigma, \pi) = W(J_s, J_p) - \int d^4x \left[ \int [J_s(x) \sigma(x) + J_p(x) \pi(x)] \right].
\]

Here the sources are functions of fields, which are obtained by inverting the following expressions:

\[
\frac{\delta W}{\delta J_s(x)} = \sigma(x), \quad \frac{\delta W}{\delta J_p(x)} = \pi(x).
\]

The latter, in turn, mean that

\[
\frac{\delta \Gamma}{\delta \sigma(x)} = -J_s(x), \quad \frac{\delta \Gamma}{\delta \pi(x)} = -J_p(x).
\]

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*On leave of absence from Bogolyubov Institute for Theoretical Physics, Kiev 252143, Ukraine*
II. THE EFFECTIVE POTENTIAL FOR COMPOSITE FIELDS

Due to the presence of chiral symmetry, the effective potential is a function of the only invariant, \( \rho = \sqrt{\sigma^2 + \pi^2} \). Thus, it is sufficient to keep only one of the sources to be non-zero (we choose \( J_x \equiv J \), and \( J_p = 0 \)). At the end, the effective potential of one variable \( \sigma \) could be promoted to the complete expression by the substitution \( |\sigma| \to \rho \).

Since the constant source (this is just enough for obtaining the effective potential) enters the expression for the generating functional in Eq. (1) in exactly the same way as a bare mass would \( (m_0 = -J) \), the calculation could be considerably simplified.

Indeed, from the definition of the \( \sigma = \langle 0 | \bar{\psi} \psi | 0 \rangle_J \) and the first relation in Eq. (3), we obtain

\[
\Sigma_0(J) = \int \langle 0 | \bar{\psi} \psi | 0 \rangle_J \frac{dJ}{d\Sigma_0} d\Sigma_0,
\]

where, by definition, \( \Sigma_0 \) is the value of self-energy function at \( p = 0 \), and

\[
W(J) \equiv w(J) \int d^4x.
\]

The chiral condensate is given by the trace of the fermion propagator. In weakly coupled QED, it is adequate to use the so-called lowest Landau level approximation. In this approximation, the fermion propagator (with the momentum dependent dynamical mass \( \Sigma(p) \)) is given by

\[
S^{(LLL)}(p) = i \exp \left( -p^2 \frac{i^2}{2} \right) \frac{\Sigma(p)}{\not{p}^2 - \Sigma^2(p)} \left( 1 - i \gamma^5 \right),
\]

where \( l \equiv 1/\sqrt{|eB|} \) is the magnetic length and, without loss of generality, we assume that \( \text{sign}(eB) = 1 \). Here we have adopted the same notations as in Refs. [4,5]. Also, it is reasonable to use a sharp momentum cut-off at around the scale of magnetic field, i.e. \( \Lambda \sim \sqrt{|eB|} \). The latter means that all the “bare” quantities in such an approach are defined at the characteristic scale of magnetic field.

The main argument in support of such an approximation is the presence of the large Landau gap of order \( \sqrt{|eB|} \) in the energy spectrum for the modes from the higher Landau levels [3], and the absence of any (or the presence of a small one after dynamical fermion mass is generated [4,5]) for the modes in the lowest Landau level. Besides, the infrared region with \( k \ll \sqrt{|eB|} \) has been shown to play a dominant role in the catalysis of chiral symmetry breaking by an external magnetic field [3].

In the lowest Landau level approximation, from Eq. (6) we derive

\[
\langle 0 | \bar{\psi} \psi | 0 \rangle_J = -\frac{1}{4\pi^2l^2} \int \frac{d(k^2)\Sigma(k)}{k^2 + \Sigma^2(k)} \tag{8}
\]

where we switched to Euclidean space. We would like to mention here that the above expression for the chiral condensate can also be obtained by making use of the \( E_p \) representation (see Appendix B in the second paper of Ref. [1]). The Schwinger-Dyson equation for the self-energy function, in the same approximation, reads [5] (compare also with Eq. (106) in the second paper of Ref. [1])

\[
\Sigma(p) = -J + \frac{\alpha}{2\pi^2} \int \frac{d^2k\Sigma(k)}{k^2 + \Sigma^2(k)} \int_0^\infty dq \exp \left( \frac{-q^2}{2} \right) \frac{1}{(k-p)^2 + q} \tag{9}
\]

Note that here we added the bare mass term \( m_0 = -J \), which is absent in [4,5].

To find a non-perturbative solution to the above equation for the self-energy, we use the method of [4] (see Appendix C of the second paper). As is common in solving such integral equations, we approximate it by its linearized version. While looking for a solution in the form \( \Sigma = \Sigma(k^2) \), we can perform the angular integration in Eq. (3) exactly. Therefore, we arrive at

\[
\Sigma(p^2) = -J + \frac{\alpha}{2\pi} \int_0^{L^2} \frac{d(k^2)\Sigma(k^2)}{k^2 + \Sigma_0^2} \int_0^\infty dq \exp \left( \frac{-q^2}{2} \right) \frac{1}{\sqrt{(k^2 + p^2 + q)^2 - 4k^2p^2}} \tag{10}
\]

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Finally, after approximating the kernel by its asymptotes in two regions \( k^2 \ll p^2 \) and \( k^2 \gg p^2 \) and dropping the exponential in the second integral \(^2\) (which is justified in our theory with a sharp cut-off at the scale of magnetic field), we arrive at the following integral equation,

\[
M(x) = -j - \frac{\alpha}{2\pi} \ln(x) \int_0^x \frac{dy}{y} \frac{M(y)}{y + M_0^2} - \frac{\alpha}{2\pi} \frac{1}{x} \int_0^1 dy \ln(y) \frac{M(y)}{y + M_0^2},
\]

where we use the dimensionless quantities

\[
M = \Sigma l, \quad j = Jl, \quad x = p^2l^2, \quad y = k^2l^2.
\]

It is straightforward to check that the integral equation \(^1\) is equivalent to the following differential equation:

\[
\frac{d^2 M(x)}{dx^2} + \frac{1}{x} \frac{dM(x)}{dx} + \frac{\alpha}{2\pi} \frac{M(x)}{x(M(x) + M_0^2)} = 0,
\]

with a solution subject to the following (infrared and ultraviolet) boundary conditions:

\[
(xM(x))|_{x=0} = 0, \quad \left(M(x) - x \ln(x) \frac{dM(x)}{dx}\right)|_{x=1} = -j.
\]

Note also that using the definition of the chiral condensate in Eq. \((8)\), we arrive at the following expression:

\[
\langle 0| \bar{\psi} \psi |0 \rangle_J \approx - \frac{1}{4\pi^2 l^3} \int_0^1 dy \frac{M(y)}{y + M_0^2} = \frac{1}{2\pi \alpha l^3} \left( x \frac{dM(x)}{dx} \right)|_{x=1},
\]

where, to be consistent with the approximations to the Schwinger-Dyson equation, we again used the linearized version instead of the exact relation.

The solution to Eq. \((13)\) that is regular at the origin (in order to satisfy the infrared boundary condition in Eq. \((14)\)) is the hypergeometric function \(^4\):

\[
M(x) = M_0 F(i\nu, -i\nu; 1; -\frac{x}{M_0^2}), \quad \nu = \sqrt{\frac{\alpha}{2\pi}}.
\]

The ultraviolet boundary condition leads to the following relation between \(M_0\) and the external source:

\[
M_0 F(i\nu, -i\nu; 1; -\frac{1}{M_0^2}) = -j.
\]

Now we are in a position to derive the generating functional and the effective potential. In weakly coupled QED, the solution for the self-energy is such that \(M_0 \ll 1\). Therefore, using the asymptotic behavior of the hypergeometric function \(^4\), we obtain

\[
j \approx -M_0 \sqrt{\frac{\tanh(\pi\nu)}{\pi\nu}} \cos(\nu \ln(M_0^2) + \Phi(\nu)),
\]

\[
\langle 0| \bar{\psi} \psi |0 \rangle_J \approx \frac{M_0}{4\pi^2 l^3} \sqrt{\frac{\tanh(\pi\nu)}{\pi\nu}} \sin(\nu \ln(M_0^2) + \Phi(\nu)),
\]

where we have introduced the following function (which is of order \(\nu^2\) in the coupling)

\[
\Phi(\nu) = \arg \left( \frac{\Gamma(1-2i\nu)}{\Gamma^2(1-i\nu)} \right).
\]

\(^1\)We would like to thank V.P. Gusynin for suggesting this approximation.
Making use of the expressions in Eqs. (18) and (19), we calculate the generating functional,

\[ w(j) = -\frac{\tanh(\pi \nu)}{8\pi^3\nu^2 l^4} \int d\mu \sin (\nu \ln (\mu) + \Phi(\nu)) \left[ \cos (\nu \ln (\mu) + \Phi(\nu)) - 2\nu \sin (\nu \ln (\mu) + \Phi(\nu)) \right] . \]

\[ = -\frac{M^2 \tanh(\pi \nu)}{16\pi^3\nu^2 l^4} [\sin (2\nu \ln (M^2_0) + 2\Phi(\nu)) - 2]\].

Then the effective potential \( V(\rho) \) is given by the following parametric representation:

\[ V(M_0) = \int \rho \left. \left( \frac{d\rho}{dM_0} \left( \frac{d}{dM_0} \right)^{-1} \right) \right|_{\rho^{(n)}} = -\frac{M^2 \tanh(\pi \nu)}{16\pi^3\nu^2 l^4} [\sin (2\nu \ln (M^2_0) + 2\Phi(\nu)) + 2\nu] . \]

\[ \rho(\rho_M) = \int \rho \left. \left( \frac{d\rho}{dM_0} \left( \frac{d}{dM_0} \right)^{-1} \right) \right|_{\rho^{(n)}} = \frac{M_0}{4\pi^2\nu l^3} \sqrt{\frac{\tanh(\pi \nu)}{\nu \pi}} [\sin (\nu \ln (M^2_0) + \Phi(\nu))] . \]

Here, as we argued above, we substituted the chiral invariant \( \rho \) instead of \( |\sigma| \).

We notice that the effective potential given by Eqs. (22), (23) is a multivalued function. This, at first sight, unusual property of the potential originates from the presence of an infinite tower of excited resonances with the same quantum numbers as those of \( \sigma \) and \( \pi \) particles. Among all of these branches, we have to choose the only one which corresponds to the stable states.

The gap equation, derived from this effective potential, is equivalent to the equation \( \rho(\rho_M) = 0 \). The latter leads to the equation \( j = 0 \). The latter leads to the infinite set of solutions

\[ M^{(n)}_0 = \exp \left( -\frac{\pi}{4\nu} (2n + 1) \right) , \quad n = 0, 1, 2, \ldots , \]

\[ \rho^{(n)} = \frac{1}{4\pi^2\nu l^3} \sqrt{\frac{\tanh(\pi \nu)}{\nu \pi \nu}} \exp \left( -\frac{\pi}{4\nu} (2n + 1) \right) . \]

Note that the solutions with negative values of \( n \) are all spurious, since they correspond to values of the dynamical fermion mass larger than the cut-off (we remind that \( \Lambda = 1/l \) here). All the solutions in Eq. (24) are local minima of the effective potential. Indeed,

\[ \frac{d^2V(\rho)}{d\rho^2} \bigg|_{\rho^{(n)}} = \frac{1}{7} \left. \left( \frac{d\rho}{dM_0} \left( \frac{d}{dM_0} \right)^{-1} \right) \right|_{\rho^{(n)}} = 8\pi^2\nu^2 l^2 > 0 . \]

However, the only solution that represents the stable composite \( \sigma \) and \( \pi \) particles is that with \( n = 0 \). We could also check that the value of the effective potential at \( \rho^{(0)} \) is lower than any of the values at the other minima, as it should be. In figure 1 the potential is plotted in the physically interesting region in the vicinity of \( \rho^{(0)} \). In this region, the potential as a function of \( \rho \) is approximately given by the following expression:

\[ V(\rho) \approx -2\pi^2\nu^2 l^2 \rho^2 \left[ 1 - \ln \left( \frac{\rho}{\rho^{(0)}} \right)^2 \right] . \]

Clearly, the dynamical fermion mass \( m_{dyn} \) that corresponds to the global minimum \( \rho^{(0)} \) is

\[ m_{dyn} = M^{(0)}_0 / l \approx \sqrt{\frac{eB}{\alpha}} \exp \left( -\frac{\pi}{\sqrt{\alpha}} \right) , \]

which is consistent with the result obtained in Refs. [4,5].

**III. CONCLUSION**

In this paper we derived the effective potential of the composite fields responsible for chiral symmetry breaking in weakly coupled QED in a constant magnetic field. The global minimum of the effective potential is obtained and is found to acquire a non-vanishing expectation value of the composite fields that leads to generating the dynamical fermion mass by an external magnetic field. Comparing the results in QED with those in the NJL model, we
notice a qualitative difference in their effective potentials. While in the NJL model the potential is a single valued function, in QED it is multivalued. The latter property reflects the presence of a tower of excited states with the same quantum numbers as those of the Nambu-Goldstone bosons. From the physical point of view, this difference appears due to the long-range interaction in QED in contrast to the short-range interaction in NJL model.

To calculate the full low-energy effective action, we also need the kinetic terms for these fields. Applying the general method of Refs. [17,18], the kinetic part of the effective Lagrangian density can be expressed in terms of the propagators of composite scalar and pseudoscalar fields, \(\Delta_s(q)\) and \(\Delta_p(q)\):

\[
\mathcal{L}_{\text{kin}} = - \frac{1}{4} \frac{\partial^2 (\Delta^{-1}_s(q))}{\partial q_\mu \partial q_\lambda} \partial_\mu \bar{\sigma}(x) \partial_\lambda \sigma(x) - \frac{1}{4} \frac{\partial^2 (\Delta^{-1}_p(q))}{\partial q_\mu \partial q_\lambda} \partial_\mu \bar{\pi}(x) \partial_\lambda \pi(x).
\]

The propagators, entering the right hand side of this formula, depend on the non-perturbative fermion propagator (the solution to the Schwinger-Dyson equation) and the scalar and pseudoscalar vertex functions [17,18]. The calculation of these vertex functions requires the solution of an additional non-perturbative self-consistency equation analogous to the Schwinger-Dyson equation, which is beyond the scope of the present paper. We expect that the forms of the kinetic terms and the corresponding dispersion relations should be qualitatively similar to those derived in the case of the NJL model [3]. Important points are: 1) there is no mass gap in the pseudoscalar dispersion relation, since the pseudoscalar particle is a Goldstone boson. 2) For the scalar particle there is a mass gap, and we expect that, as in the NJL model, the coefficients of the kinetic terms will be different for the directions parallel and perpendicular to the applied magnetic field. The dependence of the dispersion law on the transverse components is expected to be strongly suppressed, but it is nonetheless extremely important, playing the key role in preventing the chiral symmetry breaking from being washed out.

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FIG. 1. The effective potential at $\nu = 0.2$ in the vicinity of the $n = 0$ minimum. The units are chosen so that $l = 1$. The $x$ axis represents the chiral invariant $\rho$. 
