CMC-1 trinoids in hyperbolic 3-space and metrics of constant curvature one with conical singularities on the 2-sphere

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Abstract: CMC-1 trinoids (i.e. constant mean curvature one immersed surfaces of genus zero with three regular embedded ends) in hyperbolic 3-space $H^3$ are irreducible generically, and the reducible ones have been classified. However, the reducible case has not yet been fully treated, so here we give an explicit description of CMC-1 trinoids in $H^3$ that includes the reducible case.

Key words: Constant mean curvature; spherical metrics; conical singularities; trinoids.

1. Introduction. Let $H^3$ denote the hyperbolic 3-space of constant sectional curvature $-1$.

A CMC-1 trinoid in $H^3$ is a complete immersed constant mean curvature one surface of genus zero with three regular embedded ends. There are CMC-1 trinoids with horospherical ends (i.e. regular embedded ends which are asymptotic to a horosphere). However, an irreducible trinoid admits only catenoidal ends. The last two authors [9] gave a classification of those CMC-1 trinoids in $H^3$ that are irreducible. In particular, they showed that the moduli space of irreducible CMC-1 trinoids in $H^3$ (i.e. the quotient space of such immersions by the rigid motions of $H^3$) corresponds to a certain open dense subset of the set of irreducible spherical (i.e. constant curvature 1) metrics with three conical singularities (see Section 2). The paper [9] also investigated the reducible case, but had not obtained a complete classification there.

After that, Bobenko, Pavlyukevich, and Springborn [1] developed a representation formula for CMC-1 surfaces in $H^3$ in terms of holomorphic spinors and derived explicit parametrizations for irreducible CMC-1 trinoids in $H^3$ in terms of hypergeometric functions. The crucial step in [1] was a direct reduction of the ordinary differential equation that produces CMC-1 trinoids into a Fuchsian differential equation with three regular singularities, and we call this BPS-reduction. On the other hand, Daniel [2] gave an alternative proof of the classification theorem for irreducible CMC-1 trinoids, by applying Riemann’s classical work on minimal surfaces in $R^3$ bounded by three straight lines.

After the work [9] on the irreducible case, Furuta and Hattori [4] gave a full classification of spherical metrics with three conical singularities, using a purely geometric method. Later, Eremenko [3] proved it using hypergeometric equations. In this paper, using the argument in [3] and the BPS-reduction, we describe a complete classification of reducible CMC-1 trinoids in $H^3$.

2. Preliminaries. Let $M^2$ be a 2-manifold, and consider a CMC-1 immersion $f: M^2 \to H^3$. The existence of such an immersion implies orientability of $M^2$. By the existence of isothermal coordinates, there is a unique complex structure on $M^2$ such that the metric $ds^2_f$ induced by $f$ is conformal (i.e. $ds^2_f$ is Hermitian). In this situation, there exists a holomorphic immersion (called a null lift of $f$)

$$F: \tilde{M}^2 \to SL(2, C)$$

defined on the universal cover $\tilde{M}^2$ of $M^2$ so that:

- $F$ is a null holomorphic map, namely, $F_z := dF/dz$ is of rank less than 2 on each local complex coordinate $(U; z)$ of $\tilde{M}^2$. 

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• If \( f \circ \pi = \tilde{\pi} \circ F \), where \( \pi : M^2 \to M^2 \) is the covering projection and

\[
\tilde{\pi} : SL(2, C) \to H^3 = SL(2, C)/SU(2)
\]

is the canonical projection.

Then there exist a meromorphic function \( g \) and a holomorphic 1-form \( \omega \) on \( M^2 \) such that

\[
F^{-1}dF = \begin{pmatrix}
g & -g^2 \\
1 & -g
\end{pmatrix} \omega,
\]

and the first fundamental form \( ds^2_f \) of \( f \) satisfies

\[
ds^2_f = (1 + |g|^2)^2 |\omega|^2.
\]

The second fundamental form of \( f \) is given by

\[
h := -Q - ds^2_f \quad (Q := \omega dg),
\]

where the holomorphic 2-differential \( Q \) on \( M^2 \) is called the Hopf differential of \( f \). The set of zeros of \( Q \) corresponds to the set of umbilics of \( f \). We set

\[
F = \begin{pmatrix}
F_{11} & F_{12} \\
F_{21} & F_{22}
\end{pmatrix}.
\]

Since \( \det(df) = 0 \), one can easily show via (1) that

\[
g = -\frac{dF_{12}}{dF_{11}} = -\frac{dF_{12}}{dF_{21}}.
\]

With \( \pi_1(M^2) \) denoting the covering transformation group on the universal cover \( \tilde{M}^2 \), for each \( \tau \in \pi_1(M^2) \), there exists \( \rho(\tau) \in SU(2) \) such that

\[
F \circ \tau = F \rho(\tau),
\]

which gives a representation (i.e. a group homomorphism) \( \rho : \pi_1(M^2) \to SU(2) \) satisfying

\[
g \circ \tau^{-1} = a_{11}g + a_{12} \quad \text{and} \quad \rho(\tau) = (a_{ij})_{i,j=1,2}.
\]

**Definition 1.** A representation \( \rho : \pi_1(M^2) \to SU(2) \) is called reducible if \( \rho(\pi_1(M^2)) \) is abelian and otherwise is called irreducible. A CMC-1 immersion \( f : M^2 \to H^3 \) is called irreducible (reducible) if the induced representation \( \rho \) is irreducible (reducible).

The meromorphic function (cf. [8])

\[
G := \frac{dF_{11}}{dF_{21}} = \frac{dF_{12}}{dF_{22}}
\]

is well-defined on \( M^2 \), and is called the hyperbolic Gauss map of \( f \).

We now consider a CMC-1 immersion \( f \) satisfying the following properties:

(a) The metric \( ds^2_f \) induced by \( f \) is complete and of finite total curvature.

By (a), there exists a closed Riemann surface \( M^2 \) such that \( M^2 \) is bi-holomorphic to \( M^2 \setminus \{p_1, \ldots, p_n\} \), where \( p_1, \ldots, p_n \) are distinct points of \( M^2 \) called the ends of \( f \). Then, the Hopf differential \( Q \) has at most a pole at each of \( p_1, \ldots, p_n \).

Now, we suppose the second condition:

(b) All the ends \( p_1, \ldots, p_n \) of \( f \) are properly embedded, namely, there is a neighborhood \( U_j \) of \( p_j \) in \( M^2 \) such that the restriction \( f|_{U_j \setminus \{p_j\}} \) is a proper embedding, for each \( j = 1, \ldots, n \).

Then, the condition (b) implies that \( G \) has at most a pole at each end \( p_j \) (\( j = 1, \ldots, n \)), namely, the ends \( p_1, \ldots, p_n \) are all regular ends.

**Definition 2** [7]. Let \( M^2 \) be a closed Riemann surface. Let \( ds^2 \) be a conformal metric on \( M^2 \setminus \{p_1, \ldots, p_n\} \), where \( p_1, \ldots, p_n \) are distinct points. Then \( ds^2 \) has a conical singularity of order \( \mu_j \) at \( p_j \) if \( \mu_j > -1 \) and \( ds^2/|z|^{2\mu_j} \) is positive definite at \( p_j \), where \( z \) is a local coordinate so that \( z = 0 \) at \( p_j \). \( 2\pi(1 + \mu_j) \) is called the conical angle of \( ds^2 \) at \( p_j \).

We set \( M^2 = S^2 \) and consider conformal metrics that have exactly three conical singularities at \( 0, 1, \infty \) on \( S^2 = C \cup \{\infty\} \). We denote by \( M_3(S^2) \) the set of such metrics having constant curvature 1 on \( M^2 := C \setminus \{0, 1\} \), namely, \( M_3(S^2) \) can be identified with the moduli space of conformal metrics of constant curvature 1 with three conical singularities. We fix a metric \( ds^2 \in M_3(S^2) \) and then there exists a developing map

\[
g : \tilde{M}^2 \to S^2 = C \cup \{\infty\}
\]

so that \( ds^2 = 4dgd\tilde{g}/(1 + |\tilde{g}|^2)^2 \), where \( \tilde{M}^2 \) is the universal cover of \( M^2 := C \setminus \{0, 1\} \). Then there is a representation ([9, (2.15) and Lemma 2.2])

\[
\rho : \pi_1(M^2) \to SU(2)
\]

satisfying (5). The metric \( ds^2 \) is called irreducible if \( \rho \) is irreducible.

We return to the previous situation of CMC-1 surfaces. Let \( K \) be the Gaussian curvature of the CMC-1 immersion \( f \). Then

\[
ds^2_f := (-K)ds^2_f = \frac{4dgd\tilde{g}}{(1 + |\tilde{g}|^2)^2}.
\]

This relation implies that \( ds^2_f \) has constant curvature 1 whenever \( ds^2_f \) is positive definite. Moreover [8],

\[
ds^2_f ds^2_f = 4QQ.
\]
implies that $d\sigma_j^2$ has a conical singularity at a zero $q$ of $Q$, and the conical order of $d\sigma_j^2$ at $q$ equals $Q$’s order there. The condition (a) implies that $d\sigma_j^2$ has also a conical singularity at each end $p_j$.

**Definition 3.** Let $f : M^2 \to H^3$ be a CMC-1 immersion satisfying conditions (a) and (b). Then $f$ is called a CMC-1 n-noid if $M^2$ is conformally equivalent to the 2-sphere $S^2$. An end $p$ of a CMC-1 n-noid is called catenoidal if $Q$ has a pole of order 2 at $p$. A CMC-1 n-noid is called catenoidal if all ends are catenoidal.

Let $f$ be a CMC-1 n-noid. When $n = 1$, $f$ is congruent to the horosphere. When $n = 2$, $f$ is congruent to a catenoid cousin or a warped catenoid cousin (cf. [6]).

So it is natural to consider the case $n = 3$. Since the three ends are embedded, the Osserman-type inequality [8] implies $\deg(G) = 2$. We call a CMC-1 3-noid a trinoid (or a CMC-1 trinoid). We denote by $\mathcal{M}_3(H^3)$ the set of congruence classes of trinoids. We now fix a trinoid $f$. As shown in [5], there are only two possibilities:

(i) $Q$ has poles of order 2 at $p_1$, $p_2$, $p_3$.

(ii) $Q$ has at most poles of order 2 at $p_1$, $p_2$, $p_3$, but at least one of the $p_i$ has a pole of order 1.

As CMC-1 trinoids satisfying (i) are catenoidal, irreducible trinoids are catenoidal (see [9]). CMC-1 immersions satisfying (ii) have been classified in [5, Theorems 4.5–4.7]. So from now on we consider just the case (i). Without loss of generality we may assume $p_1 = 0$, $p_2 = 1$, $p_3 = \infty$. As mentioned above, the metric $d\sigma_j^2$ given by (7) has conical singularities at the zeros of $Q$ and the three ends $p_1$, $p_2$, $p_3$. We denote by $\beta_j(> -1)$ the order of $d\sigma_j^2$ at $p_j$, and by

$$B_j := \pi(1 + \beta_j)(> 0) \quad (j = 1, 2, 3)$$

the half of the conical angle of $d\sigma_j^2$ at $p_j$ ($j = 1, 2, 3$).

The group $\rho(M^2)$ is generated by three monodromy matrices $\rho_1$, $\rho_2$, $\rho_3$ which represent loops surrounding $z = 0$, 1, $\infty$. Each $\rho_j$ ($j = 1, 2, 3$) has eigenvalues $-\exp(\pm iB_j)$. Then we have (cf. [9])

$$2Q = c_jz^2 + (c_2 - c_3 - c_1)z + c_1 \quad (z^2 - 1)^2,$$

where $c_j := -\beta_j(\beta_j + 2)/2$ does not vanish by (i) (i.e. $B_j \neq \pi$) for $j = 1, 2, 3$, and

$$\frac{(c_1)^2 + (c_2)^2 + (c_3)^2}{2} \neq c_1c_2 + c_2c_3 + c_3c_1.$$

We denote by $q_1$, $q_2$ the two roots of the equation

$$c_3z^2 + (c_2 - c_1 - c_1)z + c_1 = 0.$$

Since $c_3 \neq 0$, the Hopf differential $Q$ has exactly two zeros at $q_1$ and $q_2$. In fact, (9) is equivalent to the condition $q_1 \neq q_2$ (i.e. the discriminant of (10) does not vanish). As shown in [9], the condition (b) implies that $G$ does not branch at the three ends 0, 1, $\infty$, but has exactly two branch points at $q_1$, $q_2$. Since $G$ is of degree 2 and has the ambiguity of Möbius transformations, we may set (cf. [9])

$$G := z + \frac{(q_1 - q_2)^2}{2(2z - q_1 - q_2)}.$$

Take a solution $F : T^2 \to SL(2, C)$ of the ordinary differential equation

$$dFF^{-1} = \left(\begin{array}{cc} G & -G^2 \\ 1 & -G \end{array}\right) \frac{Q}{dG}.$$

If the image $\rho(\pi_1(M^2))$ of the representation $\rho$ of $F$ is conjugate to a subgroup of SU(2), then $f = \tilde{\pi}(Fa)$ gives a CMC-1 trinoid for a suitable choice of $a \in SL(2, C)$ (cf. (4)). We denote by $\mathcal{M}_{B_1,B_3}(H^3)$ (resp. $\mathcal{M}_{B_1,B_3}(S^2)$) the congruence classes of trinoids $f$ satisfying (i) (resp. of the metrics $d\sigma^2$ of constant curvature 1) such that $d\sigma_j^2$ (resp. $d\sigma_j^2$) has conical angle $2B_j(\neq 2\pi)$ at each $p_j$.

**Fact 1** [9]. For each $B_1$, $B_2$, $B_3 \in (0, \infty)$, $\mathcal{M}_{B_1,B_3}(H^3)$ (resp. $\mathcal{M}_{B_1,B_3}(S^2)$) consists of a unique irreducible element if it satisfies (9) (resp. no condition) and

$$\cos^2B_1 + \cos^2B_2 + \cos^2B_3 + 2\cosB_1\cosB_2\cosB_3 < 1.$$

Conversely, any irreducible trinoids (resp. any irreducible metrics in $\mathcal{M}_3(S^2)$) are so obtained.

In particular, there is a unique catenoidal trinoid $f$ such that

- the hyperbolic Gauss map $G$ is given by (11),
- the Hopf differential $Q$ is given by (8),
- $d\sigma_j^2$ has conical angle $2B_j$ at each end $p_j$.

Fig. 1, left (resp. right) is an irreducible trinoid (resp. a cutaway view of an irreducible trinoid) for $B_1 = B_2 = B_3(= B)$ with $B < \pi$ (resp. $B > \pi$).

**Remark 1.** Since the hyperbolic Gauss map $G$ changes under a rigid motion of $H^3$, the above trinoid $f$ is uniquely determined without the ambiguity of isometries of $H^3$ (cf. [9, Appendix B]). After [9], Bobenko, Pavlyukevich and Springborn [1] gave a different proof, whose underlying idea also
appears in the next section. Also, Daniel [2] gave an alternative proof of this fact (see the introduction).

For each $B_j$ ($j = 1, 2, 3$) there exists a unique real number $B_j \in [0, \pi]$ such that $\cos B_j = \cos B_j$, since $\cos t = \cos (2\pi - t)$ for $t \in [0, 2\pi)$. By definition, it holds that $B_j \geq B_j$. Without loss of generality, we may assume that $B_1 \leq B_2 \leq B_3$. We now set $B_1 := B_1$, and for $j = 2, 3,$

$$B_j := \begin{cases} B_j & \text{if } B_j + B_3 \leq \pi, \\ \pi - B_j & \text{if } B_j + B_3 > \pi. \end{cases}$$

Then we have that

$$0 \leq B_1 + B_2$, $B_1' + B_2'$, $B_2 + B_2' \leq \pi,$

and the condition (13) is equivalent to

$$\cos^2 B_1 + \cos^2 B_2 + \cos^2 B_3 + 2 \cos B_1 \cos B_2 \cos B_3 < 1,$$

which is equivalent to the condition

$$\frac{\cos^2 B_1 + \cos^2 B_2 + \cos^2 B_3}{2} \cos \frac{-B_1' + B_2' + B_3'}{2} \times \cos \frac{B_1' - B_2' + B_3'}{2} \cos \frac{B_1' + B_2' - B_3'}{2} < 0.$$}

By (14), this then reduces to the condition

$$B_1' + B_2' + B_3' > \pi.$$}

The condition (13) (or equivalently (15)) implies $B_j \notin \pi \mathbf{Z}$ ($j = 1, 2, 3$), and is the same condition as in [9], [4] or [3] that there exists an irreducible metric in $\mathcal{M}_3(S^2)$ with three conical angles $2B_1$, $2B_2$, $2B_3$.

3. Reducible trinoids. Let $\alpha$ be a $2 \times 2$ matrix valued meromorphic 1-form on $C \cup \{\infty\}$. Consider an ordinary differential equation

$$dEE^{-1} = \alpha,$$

which is called a Fuchsian differential equation if it admits only regular singularities. For example, the equation (12) with $G$, $Q$ satisfying (11) and (8) is a Fuchsian differential equation with regular singularities at $z = 0$, $1$, $\infty$, $q_1$, $q_2$. Let $p_1, \ldots, p_n \in C \cup \{\infty\}$ be the regular singularities of the equation (16). We denote by $\tilde{M}^2$ the universal cover of $M^2 := C \cup \{\infty\} \setminus \{p_1, \ldots, p_n\}$.

Then there exists a solution $E : \tilde{M}^2 \rightarrow \text{GL}(2, C)$ of (16). Since $\alpha$ is defined on $M^2$, there exists a representation $\gamma : \pi_1(M^2) \rightarrow \text{GL}(2, C)$ such that $E \circ \tau = E\gamma(\tau)$. Let

$$\text{GL}(2, C) \ni a \mapsto [a] \in \text{PGL}(2, C) = \text{PSL}(2, C)$$

be the canonical projection. Then

$$h_1 := -E_{12}/E_{11}, \quad h_2 := -E_{22}/E_{21}$$

satisfy (see (5) for the definition of $*$$)$

$$h_1 \circ \tau^{-1} = \gamma(\tau) \ast h_i \quad (\tau \in \pi_1(M^2), \ i = 1, 2),$$

where $E = (E_{jk})_{j, k = 1, 2}$. Thus the functions $h_i$ ($i = 1, 2$) induce a common group homomorphism $[\gamma] : \pi_1(M^2) \rightarrow \text{PGL}(2, C)$ which is called the monodromy representation of the equation (16). In particular, the representation $[\alpha]$ for $F$ as in (12) is just the monodromy representation.

Definition 4. Let $r(z), s(z)$ be meromorphic functions on $C \cup \{\infty\}$ and

$$X'' + rX' + sX = 0$$

be an ordinary differential equation with regular singularities at $z = p_1, \ldots, p_n$, where $X' = dX/dz$. Then there exists a pair of solutions $w_1, w_2 : \tilde{M}^2 \rightarrow C$ which are linearly independent, and $\{w_1, w_2\}$ is called a fundamental system of solutions. There exists a representation $\gamma : \pi_1(M^2) \rightarrow \text{GL}(2, C)$ for each fundamental system $\{w_1, w_2\}$, such that

$$(w_1 \circ \tau, w_2 \circ \tau) = (w_1, w_2)\gamma(\tau),$$

where $(w_1, w_2)$ is a row vector. As a monodromy of the function $-w_2/w_1$, the induced homomorphism $[\gamma] : \pi_1(M^2) \rightarrow \text{PGL}(2, C)$ is called the monodromy representation of the equation (17).

To give a complete classification of trinoids, the following reduction given in [1] is crucial: Let $F$ be a null lift of the catenoidal trinoid $f$ whose hyperbolic Gauss map $G$ and Hopf differential $Q$ are given by (11) and (8), respectively. In the expression (12), we can write

$$\begin{pmatrix} G & -G^2 \\ 1 & -G \end{pmatrix} \frac{Q}{dG} = \begin{pmatrix} P_1P_2 & (P_1)^2 \\ -(P_2)^2 & -P_1P_2 \end{pmatrix} dz,$$
where $P_i := \frac{p_i^0}{z} + \frac{p_i^1}{z-1} + p_i^{\infty}$ and $p_1^0, p_1^1, p_1^{\infty}$ $(i = 1, 2)$ are constants depending only on $B_1, B_2, B_3$. In [1], the matrix $\Phi := D^{-1} F$ is defined by

$$D := \sqrt{z-1} \begin{pmatrix} \frac{p_1}{1} & \alpha_1 z + \beta_1 \\ -p_2 & \alpha z + \beta_2 \end{pmatrix},$$

where $\alpha_j, \beta_j$ $(j = 1, 2)$, $k$ and $\vartheta$ are all real constants depending only on $B_1, B_2, B_3$. Then there exist $2 \times 2$ matrices $A_0, A_1$ with real coefficients such that

$$d\Phi^{-1} = \left( A_0 + \frac{A_1}{z-1} \right) dz. \tag{18}$$

We call (18) the BPS-reduction of (12). (This reduction does not work if $f$ has a horospherical end, but such trinoids would be in the case (ii) mentioned before.) By (4), it holds that $\Phi \circ \tau = \Phi \rho(\tau)$ for each $\tau \in \pi_1(M^2)$. Obviously (18) has three regular singularities at $z = 0, 1, \infty$. Since $A_0$ and $A_1$ are both constant real matrices, it is well-known that there exist real numbers $a, b, c$ such that the monodromy representation of the ordinary differential equation (called the hypergeometric equation)

$$z(1-z)X'' + (c - (a + b + 1)z)X' - abX = 0 \tag{19}$$

is conjugate to that of (18) (i.e. [9]). On the other hand, if we express $F$ as in (2), then $X = F_{11}, F_{12}$ satisfy the ordinary differential equation (cf. [5, p. 32])

$$X'' - (\log(\mathcal{G}'))' X' + \mathcal{Q} X = 0, \tag{20}$$

where $Q = \mathcal{Q}(z) dz^2$ and $\mathcal{G}' = d\mathcal{G}/dz$. Thus the monodromy representation of (20) with respect to $(F_{11}, F_{12})$ is equal to that of $F$. In particular, the monodromy representation of (20) is conjugate to that of (19). Hence, these two ordinary differential equations have the same exponent (i.e. the difference of the two solutions of the indicial equation) at each regular singularity. Since (20) has the exponent $B_1/\pi, B_2/\pi, B_3/\pi$ at $z = 0, 1, \infty$, respectively, we have

$$\pm B_1 = \pi(1-c), \quad \pm B_2 = \pi(a-b), \quad \pm B_3 = \pi(c-a-b),$$

which is the same set of relations as in [3, (4)]. This implies the classification of catenoidal trinoids reduces to that of metrics in $\mathcal{M}_3(S^2)$. In particular, the classification results for reducible metrics in $\mathcal{M}_3(S^2)$ given in Furutani-Hattori [4] and Eremenko [3, Theorem 2] yield the following assertion.

**Theorem.** Suppose $B_j/\pi$ is an integer, and $B_j \neq \pi$ $(j = 1, 2, 3)$. Then $\mathcal{M}_{B_{1}, B_{2}, B_{3}}(H^3)$ (resp. $\mathcal{M}_{B_{1}, B_{2}, B_{3}}(S^2)$) is non-empty if and only if $B_j/\pi, B_2/\pi, B_3/\pi$ satisfy (9) (resp. no condition) and one of the following two conditions:

(C1) $B_2, B_3 \notin \mathbb{Z}$, but either $|B_2 - B_3|/\pi$ or $(B_2 + B_3)/\pi$ is an integer $m$ of opposite parity from $B_j/\pi$, and $\pi m \leq B_j - \pi$. In this case, $\mathcal{M}_{B_{1}, B_{2}, B_{3}}(H^3)$ (resp. $\mathcal{M}_{B_{1}, B_{2}, B_{3}}(S^2)$) is 1-dimensional.

(C2) $B_2, B_3 \in \mathbb{Z}$, and $(B_2 + B_3)/\pi$ is odd, and each of $B_1, B_2, B_3$ is less than the sum of the others. In this case, $\mathcal{M}_{B_{1}, B_{2}, B_{3}}(H^3)$ (resp. $\mathcal{M}_{B_{1}, B_{2}, B_{3}}(S^2)$) is 3-dimensional.

**Corollary 1.** A catenoidal trinoid $f$ is irreducible if and only if at least one of $B_j/\pi, B_2/\pi, B_3/\pi$ are all non-integers, and $f$ is reducible if and only if at least one of $B_1/\pi, B_2/\pi, B_3/\pi$ is an integer.

**Proof.** A trinoid $f$ is irreducible if the representation $\rho$ as in (4) is irreducible. The representation $\rho$ coincides with that of the corresponding metric in $\mathcal{M}_3(S^2)$. The corresponding assertion for metrics in $\mathcal{M}_3(S^2)$ is proved in [9, Lemma 3.1]. □

**Remark 2.** Reducibility is equivalent to at least one of $B_1/\pi, B_2/\pi, B_3/\pi$ being an integer. This cannot be proved purely algebraically, as there are diagonal matrices $\rho_1, \rho_2, \rho_3$ in $\text{SU}(2)$ with $\rho_1 \rho_2 \rho_3 = \text{id}$ so that no eigenvalues of $\rho_1, \rho_2, \rho_3$ are $\pm 1$.

**Remark 3.** Eremenko [3, Theorem 2] asserts the uniqueness of $d\sigma^2 \in \mathcal{M}_{B_{1}, B_{2}, B_{3}}(S^2)$ with prescribed conical angles. This is correct in the irreducible case, but if $B_1 \in \pi \mathbb{Z}$, then the metric has a non-trivial deformation preserving its conical angles: A metric $d\sigma^2 \in \mathcal{M}_3(S^2)$ has the same conical angles as those of $d\sigma^2$ if and only if each developing map of $d\tau^2$ is given by $k = a \ast h$ for suitable $a \in \text{SL}(2, C)$, where $h$ is a developing map of $d\sigma^2$. So $\mathcal{M}_{B_{1}, B_{2}, B_{3}}(S^2)$ can be identified with the set

$$\{ \hat{\pi}(a); \ a(\text{Im} \rho) a^{-1} \subset \text{SU}(2), \ a \in \text{SL}(2, C) \} \subset H^3,$$

where $\hat{\pi} : \text{SL}(2, C) \to H^3$ is the canonical projection and $\text{Im} \rho$ is the image of $\rho$ as in (6). Then $\mathcal{M}_{B_{1}, B_{2}, B_{3}}(S^2) = H^3$ if $B_1/\pi$ $(j = 1, 2, 3)$ are all integers, and $\mathcal{M}_{B_{1}, B_{2}, B_{3}}(S^2)$ is a geodesic line in $H^3$ if one of $B_j/\pi$ $(j = 1, 2, 3)$ is not an integer (cf. [9]). A metric $d\sigma^2$ in $\mathcal{M}_{B_{1}, B_{2}, B_{3}}(S^2)$ is called symmetric if the metric is invariant under an anti-holomorphic involution. We denote by $\mathcal{M}_{B_{1}, B_{2}, B_{3}}(S^2)$ the subset
consisting of symmetric metrics in \( \mathcal{M}_{B_i,B_i,B_i}(S^2) \). If \( B_j/\pi \) \((j = 1, 2, 3)\) are all integers, \( \mathcal{M}_{B_i,B_i,B_i}(S^2) \) consists of a hyperbolic plane in \( H^3 \). If one of \( B_j/\pi \) \((j = 1, 2, 3)\) is not an integer, \( \mathcal{M}_{B_i,B_i,B_i}(S^2) \) coincides with \( \mathcal{M}_{B_j,B_k,B_l}(S^2) \) (cf. [9]). A metric \( d\sigma^2 \) in \( \mathcal{M}_{B_i,B_i,B_i}(S^2) \) with conical angles \( 2B_i, 2B_j \) and \( 2B_k \) can be regarded as a doubling of the generalized spherical triangle with angles \( B_1, B_2 \) and \( B_3 \). Using this, Furuta-Hattori [4] gave two operations in \( \mathcal{M}_3(S^2) \) for distinct \( \{i,j,k\} = \{1,2,3\} \):

\[
(B_i,B_j,B_k) \mapsto (B_i + \pi, B_j + \pi, B_k),
(B_i,B_j,B_k) \mapsto (\pi - B_i, B_j + \pi, B_k),
\]

with the second operation allowed only when \( B_i < \pi \). The first operation is attaching a closed hemisphere in \( S^2 \) to the edge \( B_jB_i \) of the spherical triangle \( \Delta B_jB_iB_k \). The second operation is attaching a geodesic 2-gon of equi-angles \( \pi - B_i \) to the edge \( B_jB_i \) so that the initial vertex \( B_i \) becomes an interior point of an edge of the new triangle. Conditions \( (C_1) \) and \( (C_2) \) are invariant under these two operations. Moreover, the three angles \((B_1,B_2,B_3)\) satisfying conditions \( (C_1) \) and \( (C_2) \) are obtained from a given initial data \((B'_1,B'_2,B'_3)\) by these two operations. Furuta-Hattori proved this using a geometric argument. On the other hand, Eremenko found \( (C_1) \) and \( (C_2) \) from the viewpoint of hypergeometric equations. We remark that spherical triangles of arbitrary angles \( B_1, B_2, B_3 \in (0, \infty) \) were investigated by Felix Klein in 1933 (see the end of [9]). The trinoid shown in Fig. 2 is not symmetric, although there does exist a symmetric trinoid with the same conical angles and dihedral symmetry.

Finally, we group the surfaces by the signatures of \( c_1, c_2, c_3 \). For example, a trinoid \( f \) is said to be of type \((+,+,+)\) if \( c_1, c_2, c_3 \) are all positive, and of type \((-,-,+\) if one of \( c_1, c_2, c_3 \) is negative and the other two are positive, etc. As remarked in [6], by numerical experiment, it seems that the four types \((+,+,+), (-,-,+), (-,-,-)\) have distinct regular homotopy types (see Fig. 3). Surfaces of type \((+,+,+)\) have absolute total curvature less than \( 8\pi \), and it seems that only surfaces in this class can be embedded.

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