Star integrals and unbiased estimators

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ABSTRACT
We review in brief the development and implementation of the Star integral, a tool yielding measurements of correlations much superior to conventional methods. A version for use in pion interferometry is explained. We also show how effects of non-poissonian overall multiplicity distributions may be eliminated if desired and quote results eliminating statistical biases arising in correlation measurements within small samples.

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1 Point distributions

Distributions are fundamental to almost any branch of the exact sciences. A point distribution is characterized by the fact that the object under scrutiny has no intrinsic structure or size; it is a point rather than a field. The coordinates of this point may be discrete or continuous, real or complex, embedded in a single- or multidimensional space. Physically, whenever the objects (“particles”) have a size small in comparison with the embedding space, the assumption of a point distribution is justified.

Typical examples of point distributions are galaxies in the sky and pions in phase space; in these cases, the embedding space is continuous. If \( X_i \) are the coordinates of \( N \) particles measured in a particular “event” (as measured by the detector, a region of the sky, a throw of \( N \) dice, . . . ), the number of such particles at the point \( x \) is

\[
\hat{\rho}_1(x) = \sum_{i_1=1}^{N} \delta(x - X_{i_1}).
\]  

In general, the simultaneous behavior ( = correlation) of \( q \) of these particles for a given event \( a \) is given by

\[
\hat{\rho}_q^a(x_1, \ldots, x_q) = \sum_{i_1 \neq i_2 \neq \ldots \neq i_q}^{N} \delta(x_1 - X_{i_1}^a) \delta(x_2 - X_{i_2}^a) \cdots \delta(x_q - X_{i_q}^a).
\]  

Note that \( \hat{\rho}_q \) is a nonnegative integer while the coordinates \( X \) are continuous. Meaningful results are extracted by averaging over many events, to yield the \( q \)-tuple density

\[
\rho_q = \langle \hat{\rho}_q^a \rangle = N_{ev}^{-1} \sum_{a=1}^{N_{ev}} \hat{\rho}_q^a,
\]  

which is pseudocontinuous (actually, a rational number with denominator \( N_{ev} \)). Allowing the variables to range over some restricted domain \( \Omega \), one measures moments of point distributions

\[
\xi_q(\Omega) = \int_{\Omega} \rho_q(x_1, \ldots, x_q) \, dx_1 \ldots dx_q = \langle n^{[q]} \rangle_{\Omega}
\]  

which contain information on the correlation strength in this particular region \( \Omega \). Here \( n^{[q]} \equiv n!/(n-q)! = n(n-1) \cdots (n-q+1) \), so that \( \xi_q \) is actually a factorial moment of the distribution.

Below, we explore one particular type of domain \( \Omega \) which leads to the so-called Star integral. The reason for this particular choice of \( \Omega \) is that it maximizes the amount of information extracted from a given sample of events while restricting the amount of computer time to a minimum \( \square \).
2 Star integral moments

Conventional measurements of correlations proceed first to discretize the continuous variable \( X \) (“binning the data”) and then to find \( \xi_q \) by averaging over all events the counts \( n_{m}^{[q]} \) in every bin,

\[
\xi_{q}^{\text{conv}} = \left\langle \sum_{\text{bins} \ m} n_{m}^{[q]} \right\rangle. \tag{5}
\]

By contrast, the Star integral belongs to the class of correlation integrals, where distances between pairs of points \( X_{i_{1}i_{2}} \equiv |X_{i_{1}} - X_{i_{2}}| \) are computed directly, before binning \[3\]. For the Star integral, the domain \( \Omega \) is given by the sum of all spheres of radius \( \epsilon \) centered around each of the \( N \) particles in the event. The number of particles (“sphere count”) within each of these spheres is, not counting the particle at the center \( X_{i_{1}} \),

\[
\hat{n}(X_{i_{1}}, \epsilon) \equiv \sum_{i_{2}=1}^{N} \Theta(\epsilon - X_{i_{1}i_{2}}), \quad i_{2} \neq i_{1}, \tag{6}
\]

and the factorial moment of order \( q \) is

\[
\xi_{q}^{\text{Star}}(\epsilon) = \left\langle \sum_{i_{1}} \hat{n}(X_{i_{1}}, \epsilon)^{[q-1]} \right\rangle. \tag{7}
\]

The obvious similarity to the conventional moment of Eq. (3) should not disguise the fact that \( \sum_{i_{1}} \) is a sum over particles rather than bins.

The Star factorial moment of Eq. (7) can be derived rigorously \[1\] from Eq. (2) using for \( \Omega \) the equivalent implicit definition

\[
\xi_{q}^{\text{Star}}(\epsilon) = \int \rho_{q}(x_{1}, \ldots, x_{q}) \Theta_{12} \Theta_{13} \ldots \Theta_{1q} \, dx_{1} \ldots dx_{q} \tag{8}
\]

with the theta functions \( \Theta_{1j} \equiv \Theta(\epsilon - |x_{1} - x_{j}|) \) restricting all \( q-1 \) coordinates \( x_{j} \) to within a distance \( \epsilon \) of \( x_{1} \).

Correlation measurements can be made either as a function of the sphere size \( \epsilon \) — useful in looking for self-similarity and fractal structure — as a function of the distance from a fixed center coordinate, as has been the case traditionally. This is implemented in the “differentials” of Section 3 below.

The superiority of the Star integral moments of Eq. (8) over the conventional \( \xi_{q}^{\text{Conv}} \) arises because the artificial discretization inherent in the latter has two bad effects \[3\]. First, it leads to instabilities in the measured quantities \( n_{m}^{[q]} \) as the bin size is varied and, second, it often sorts particles into different bins when in fact they are quite close together, thus unwittingly throwing away information. By contrast, \( \xi_{q}^{\text{Star}} \) is much more stable and has smaller errors, especially if the coordinates in use live in a two- or three-dimensional space.

In the literature on galaxy correlations \[4\] and in the characterization of strange attractors \[5\], an approach similar to our Star integral has been used for some time, utilizing, however, not the factorial but the ordinary form

\[
\left\langle \frac{1}{N^{2}} \sum_{i_{1}} \hat{n}(X_{i_{1}}, \epsilon)^{q-1} \right\rangle. \tag{9}
\]

3
We believe that this is an ad hoc approximation to the exact factorial form of Eq. (7), necessarily breaking down when $\hat{n}$ becomes small. Current measurements in these fields may therefore be suffering from distortion at small $\epsilon$.

In order to eliminate, among other things, the overall multiplicity, it has become customary in high energy physics to measure normalized factorial moments \cite{2}. The denominator used for such normalization should be made up of the uncorrelated background, $\rho_q^b$. While it can be implemented in a number of ways, we prefer the “vertical” normalization, in which $\rho_q^b$ is integrated over exactly the same domain $\Omega$ as the inclusive density $\rho_q$ in the numerator. Thus for the Star integral, the normalized moment is

$$F_q^{\text{Star}}(\epsilon) \equiv \frac{\xi_{q}^{\text{Star}}}{\xi_{q}^{\text{norm}}} = \frac{\int \rho_q(x_1, \ldots, x_q) \Theta_{12} \Theta_{13} \ldots \Theta_{1q} \, dx_1 \ldots dx_q}{\int \rho_1(x_1) \ldots \rho_1(x_q) \Theta_{12} \Theta_{13} \ldots \Theta_{1q} \, dx_1 \ldots dx_q}, \quad (10)$$

We have shown rigorously \cite{1} that the denominator $\xi_{q}^{\text{norm}}$ is given by the following double event average: with $X_{i_1 i_2}^{ab} \equiv |X_{i_1}^a - X_{i_2}^b|$ measuring the distance between two particles taken from different events $a$ and $b$,

$$\xi_{q}^{\text{norm}}(\epsilon) = \left\langle \sum_{i_1} \left( \sum_{i_2} \Theta(\epsilon - X_{i_1 i_2}^{ab}) \right) \right\rangle^q \equiv \left\langle \sum_{i_1} \langle \hat{n}_b(X_{i_1}^a, \epsilon) \rangle^q \right\rangle, \quad (11)$$

where the outer event average and sum over $i_1$ are taken over the center particle taken from event $a$, each of which is used as the center of sphere counts $\hat{n}_b(X_{i_1}^a, \epsilon)$ taken over other events $b$ in the inner event average. We thus see the natural emergence of the heuristic procedure of normalization known as “event mixing” \cite{4, 3}.

Apart from the double event average and the appearance of the ordinary power $q - 1$ rather than the factorial power $[q - 1]$, the similarities between the numerator Eq. (7) and denominator Eq. (11) are obvious. Both do sphere counts around a given center particle at $X_{i_1}$; the numerator $\xi_{q}^{\text{Star}}$ does so within the same event $a$, while the denominator $\xi_{q}^{\text{norm}}$ inserts this center particle into all other events $b$ to perform a similar count there. This is shown schematically in Figure 1.

## 3 Cumulants and differentials

Cumulants are combinations of correlation functions constructed in such a way as to become zero whenever any one or more of the points $x$ becomes statistically independent of the others. This is done so as to strip away the combinatorial background from the correlation measurements,

$$C_2(x_1, x_2) = \rho_2(x_1, x_2) - \rho_1(x_1)\rho_1(x_2), \quad (12)$$

$$C_3(x_1, x_2, x_3) = \rho_3(x_1, x_2, x_3) - \rho_1(x_1)\rho_2(x_2, x_3) - \rho_1(x_2)\rho_2(x_3, x_1) - \rho_1(x_3)\rho_2(x_1, x_2) + 2\rho_1(x_1)\rho_1(x_2)\rho_1(x_3) \quad \text{etc.} \quad (13)$$
Figure 1: Sphere counts. On the left is shown a typical event $a$, with the particles denoted as circles. For each particle $i_1$ of $a$, count all other particles within a sphere of radius $\epsilon$; this yields $\hat{n}(X_{i_1}, \epsilon)$ of Eq. (10) used in the numerator of the Star integral moments and cumulants. On the left is shown a different event $b$, with particles denoted as squares. For normalization and cumulants, the same center particle is inserted at $X_{i_1}$ into event $b$ and a count performed to yield $\hat{n}_b(X_{i_1}, \epsilon)$. Performing an event average over all $b$-events, one obtains the normalization $\xi_{\text{norm}}^q$ as in Eq. (11) and cumulants as in Eq. (18).

Integrating these over the Star integral domain, we can find the normalized cumulants

$$K_{\text{Star}}^q(\epsilon) \equiv \frac{f_q(\epsilon)}{\xi_{\text{norm}}^q(\epsilon)},$$

with

$$f_q(\epsilon) \equiv \int C_q(x_1, \ldots, x_q) \Theta_{12} \Theta_{13} \ldots \Theta_{1q} \, dx_1 \ldots dx_q$$

the unnormalized (Star) factorial cumulant. The latter can be written entirely in terms of the sphere counts introduced previously; writing in shorthand

$$a = \sum_j \Theta(\epsilon - X_{ij}^a) = \hat{n}(X_{i_1}^a, \epsilon), \quad j \neq i$$

$$b = \sum_j \Theta(\epsilon - X_{ij}^{ab}) = \hat{n}_b(X_{i_1}^a, \epsilon)$$

and defining for convenience the “$i$-particle cumulant” $\hat{f}_q(i)$ by

$$\left\langle \sum_i \hat{f}_q(i) \right\rangle = f_q,$$

we find

$$\hat{f}_2(i) = a - \langle b \rangle,$$

$$\hat{f}_3(i) = a^{[2]} - \langle b^{[2]} \rangle - 2a \langle b \rangle + 2\langle b \rangle^2.$$
etc. In Section 6 below, we shall show that these $\hat{f}_q$ as well as the normalization $\xi_q^{\text{norm}}$ must be corrected for a remaining statistical bias. This correction should become important for small data samples.

Besides counting the number of combinations of $q$ particles lying inside a sphere of radius $\epsilon$, a second useful form for Star moments and cumulants are the so-called differential moments: Here, one defines not only a maximum distance $\epsilon_t$ but a minimum also, $\epsilon_{t-1}$ ($t$ can define a sequence of such distances). For a given combination of $q-1$ particles around a center particle at $X_{i_1}$, at least one of these must lie inside the spherical shell bounded by radii $\epsilon_{t-1}$ and $\epsilon_t$, while the others are restricted only by the maximum distance $\epsilon_t$. This is illustrated in Figure 2.

Figure 2: Sphere counts for differentials. Given the center particle at $X_{i_1}$, at least one other particle must be in the shell bounded by radii $\epsilon_{t-1}$ and $\epsilon_t$ to count. For $q = 2$, this reduces to subtracting the sphere count for $\epsilon_{t-1}$ from that for $\epsilon_t$. Higher orders are also easily calculated.

This definition leads rigorously \[1\] to simple and efficient prescriptions for measurements. For $q = 2$, the unnormalized differential moment is, with $\Delta \hat{\xi}_q(\epsilon_t) = \langle \sum_{i_1} \Delta \hat{\xi}_q(i_1, \epsilon_t) \rangle$

$$\Delta \hat{\xi}_2(i_1, t) = \hat{n}(X_{i_1}^a, \epsilon_t) - \hat{n}(X_{i_1}^a, \epsilon_{t-1}) \equiv a_t - a_{t-1}, \tag{21}$$

the latter defining the shortened notation. For higher orders, we find

$$\Delta \hat{\xi}_q(i_1, t) = \hat{n}(X_{i_1}^a, \epsilon_t)^{[q-1]} - \hat{n}(X_{i_1}^a, \epsilon_{t-1})^{[q-1]} \equiv a_t^{[q-1]} - a_{t-1}^{[q-1]}, \tag{22}$$

i.e. just the difference of $[q-1]$th factorial powers of two sphere counts. Equivalent forms for the (differential) normalizations $\Delta \xi_q^{\text{norm}}$ and differential cumulants $\Delta \hat{f}_q$ are easily found,
leading to the normalized differential moments and cumulants

\[ \Delta F_q(t) = \frac{\langle \sum_i a_i^{[q-1]} - a_{i-1}^{[q-1]} \rangle}{\langle \sum_i b_i^{q-1} - b_{i-1}^{q-1} \rangle}, \]

\[ \Delta K_q(t) = \frac{\langle \sum_i \hat{f}_q(i, \epsilon_i) - \hat{f}_q(i, \epsilon_{i-1}) \rangle}{\langle \sum_i b_i^{q-1} - b_{i-1}^{q-1} \rangle}, \]

in obvious notation. The point is that these quantities can all be measured in terms of the two types of sphere counts, \( \hat{n} \) within the same \( a \)-event and \( \hat{n}_b \) within the other \( b \)-events; see Eqs. (16)–(17).

4 Eliminating effects of the overall multiplicity distribution

If there are \( N \) particles within the total phase space (sky region) \( \Omega_{tot} \) considered, the normalized factorial moment for this whole region is \( F_q(\Omega_{tot}) = \langle N^{[q]} \rangle / \langle N \rangle^q \), which is unity only when the overall multiplicity distribution is poissonian. All measurements of \( F_q, K_q \) and their differentials thus implicitly contain correlations arising from the non-poissonian nature of the overall multiplicity distribution. This is as it should be, of course, but it may sometimes be desirable to eliminate this dependence on global effects (for example when the multiplicity distribution is artificial, such as in centrality cuts in heavy ion collisions). One way of achieving this is to modify all the formulae of the preceding sections by changing the event-by-event counts to

\[ \hat{\rho}_q(x) \rightarrow \hat{h}_q(x) = \frac{1}{\hat{N}} \sum_{i=1}^{N} \delta(x - X_{i}) = \frac{\hat{\rho}_q(x)}{\int_{\Omega_{tot}} \hat{\rho}_q(x) d\mathbf{x}}, \]

where \( \hat{N} \) is the event’s multiplicity within the total domain \( \Omega_{tot} \), and \( \hat{\rho}_q \) to

\[ \hat{h}_q(x_1, \ldots, x_q) = \frac{1}{\hat{N}^{[q]}_{\hat{N}}} \sum_{i_1 \neq i_2 \neq \ldots \neq i_q}^{N} \delta(x_1 - X_{i_1}) \delta(x_2 - X_{i_2}) \ldots \delta(x_q - X_{i_q}) \]

\[ = \frac{\hat{\rho}_q(x_1, \ldots, x_q)}{\int_{\Omega_{tot}} \hat{\rho}_q(x_1, \ldots, x_q) d\mathbf{x}_1 \ldots d\mathbf{x}_q}. \]

The event average \( h_q = \langle \hat{h}_q \rangle \) satisfies the requirements of a joint probability (normalization to unity and correct projection properties). These changes then propagate to yield, for example

\[ F_q(\epsilon) = \left( \frac{1}{\hat{N}_a} \sum_{i_1} \frac{a_{i_1}^{[q-1]}}{\langle N_{a-1}^{[q]} \rangle} \right) / \left( \frac{1}{\hat{N}_a} \sum_{i_1} \frac{b_{i_1}^{q-1}}{\langle N_b \rangle^{q-1}} \right), \]

7
which yields \( F_q(\Omega_{tot}) = 1 \) for any overall multiplicity distribution. Analogously, the individual terms in the cumulant functions are divided by their respective integrals, so that, for example,

\[
C_3(x_1, x_2, x_3) \rightarrow c_3(x_1, x_2, x_3) = h_3 - \sum_{(3)} h_1 h_2 + 2 h_1 h_1 h_1
\]  

yielding after normalization (cf. Eq. (24))

\[
K_3(\epsilon) = \frac{\left\langle \frac{1}{N_a} \sum_i \left( \frac{a^{[2]}}{(N_a - 1)^{[2]}} - \frac{2}{N_a - 1} \frac{b^{[2]}}{N_b^{[2]}} \right) \right\rangle}{\left\langle \frac{1}{N_a} \sum_i \frac{b}{N_b^{[2]}} \right\rangle}.
\]  

Statistical independence is understood for these cumulants to mean a factorization of the probabilities \( h_q \) rather than of the densities \( \rho_q \).

## 5 Bose-Einstein moments and cumulants

One great advantage of correlation integrals in general is that they allow the use of variables which are functions of two or more particles \([6]\), while conventional binning is usually done in terms of single-particle variables only.

Bose-Einstein correlations are a prime example of the use of relative coordinates: the quantum mechanical interference of identical particles manifests itself in a rise of the two-particle correlation function

\[
k_2(p_1, p_2) = \frac{\rho_2(p_1, p_2)}{\rho_1(p_1) \rho_1(p_2)} - 1
\]  

at small relative momenta \( q = p_1 - p_2 \). Other formulations \([7]\) test the correlation in terms of the one-dimensional variable \( Q^2 = -(p_1 - p_2)^2 \), the relative four-momentum. The correlation integral formalism can be utilized for both these variables to yield moments and cumulants of higher order \([6]\). Taking \( Q^2 \) as an example, one first integrates out the unneeded degrees of freedom in both \( \rho_2 \) and the normalization \( \rho_1 \rho_1 \) \([8]\),

\[
\rho_2(Q^2) = \int d^3 p_1 d^3 p_2 \rho_2(p_1, p_2) \delta[Q^2 + (p_1 - p_2)^2],
\]  

\[
\rho_1 \otimes \rho_1(Q^2) = \int d^3 p_1 d^3 p_2 \rho_1(p_1) \rho_1(p_2) \delta[Q^2 + (p_1 - p_2)^2],
\]  

which, using the delta function form of Eq. (2), translates into the measurement prescriptions

\[
\rho_2(Q^2) = \left\langle \sum_{i \neq j} \delta[Q^2 - (Q_{ij}^{ab})^2] \right\rangle,
\]  

\[
\rho_1 \otimes \rho_1(Q^2) = \frac{1}{N_{ev}^{[2]}} \sum_{a \neq b} \sum_{i,j} \delta[Q^2 - (Q_{ij}^{ab})^2] = \left\langle \sum_i \left\langle \sum_j \delta[Q^2 - (Q_{ij}^{ab})^2] \right\rangle \right\rangle,
\]  

(34)
where \((Q^{ab}_{ij})^2 = -(P^a_i - P^b_j)^2\) measures the relative four-momentum between particles \(i\) and \(j\) taken from two different events \(a\) and \(b\). Here, too, we see the direct emergence of the event mixing prescription as the appropriate method of normalizing Bose-Einstein correlation functions.

For measurement of higher orders, one must first make an ansatz how the \(q\) three-momenta are to be combined into a single variable, the choice of which depends on physical arguments of the specific system and the signal being sought. One possibility is to sum all \(q(q-1)/2\) pairs of relative four-momenta to give a measure of the overall \(q\)-particle four-momentum \([9]\), e.g. for \(q = 3\),

\[
Q^2 = -(p_1 - p_2)^2 - (p_1 - p_3)^2 - (p_2 - p_3)^2;
\]

(35)

this amounts to a GHP-type topology of the correlation integral in the four-momenta \([1, 3]\).

(The \(Q^2\) defined in this way is merely the \(q\)-particle invariant mass minus a constant.)

Moments are found by formulas analogous to Eq. (31) above, while cumulants are constructed directly from Eqs. (12)ff. inserted into

\[
C_q(Q^2) = \int d^3p_1 \ldots d^3p_q C_q(p_1, \ldots, p_q) \delta[Q^2 + \sum_{\alpha<\beta=1}^q (p_\alpha - p_\beta)^2],
\]

(36)
i.e. the expansion of \(C_q\) in terms of the \(\rho_q\) must be done before projection of the three-momenta onto \(Q^2\).

6 Biased and unbiased estimators

The use of the Star integral (or other forms such as the form used above for Bose-Einstein correlations) permits much more accurate measurements and hence will likely reveal more detailed structure of the underlying dynamics. Greater accuracy requires, however, that possible biases be understood on a higher level than before. One such bias arising generally in the measurement of correlations has to do with the theory of estimators \([10]\).

To understand this, we must go back to the basics of sampling theory. For a given quantity of interest (“statistic”) \(U\), there ideally exists an infinite set of measurements \(\hat{U}\); this is termed the population of such measurements. A statistical average based on the whole population would yield the “true” value \(\bar{U}\) of this quantity.

In practice, the size of the set of measurements carried out is limited, corresponding to a single sample of \(N_{ev}\) measurements taken out of the total population. Many such samples \(N\) could theoretically be taken, each one yielding a sample average \(\langle U \rangle_s\), the set of which in itself forms a distribution, the sampling distribution. While there is no way to ascertain where within this distribution the \(\langle U \rangle_s\) obtained from a particular sample will fall, at least one can test whether the average of this sampling distribution coincides with \(\bar{U}\). Surprisingly, such a sampling average

\[
\{U\} = \lim_{N \to \infty} \sum_s \langle U \rangle_s / N
\]

(37)
does not necessarily coincide with $\bar{U}$ except in the (for the experimentalist uninteresting) case $N_{ev} \to \infty$. When it does not, one looks for a modification, say $e(U)$; which is called an unbiased estimator of $U$ if it fulfills the condition

$$\{e(U)\} = \bar{U} \quad \text{for all values of } N_{ev}. \quad (38)$$

The age-old problem of finding suitable estimators has been extensively investigated and we merely quote the results. It has been shown that the inclusive density $\rho_q$ we have been using in the previous sections is an unbiased estimator,

$$\{\rho_q(x_1, \ldots, x_q)\} = \hat{\rho}_q(x_1, \ldots, x_q); \quad (39)$$

in addition, we note that the sampling average of a single event inclusive density $\hat{\rho}_q$ as defined in Eq. (2) also yields $\bar{\rho}$ since Eq. (38) is valid for samples consisting of single events, $N_{ev} = 1$,

$$\{\hat{\rho}_q(x_1, \ldots, x_q)\} = \hat{\rho}_q(x_1, \ldots, x_q). \quad (40)$$

However, whenever two or more event averages are involved, the naive product of sample densities yields a biased estimator, for example $\{\rho_1(x_1)\rho_1(x_2)\} \neq \hat{\rho}_1(x_1)\hat{\rho}_1(x_2)$ and $\{\rho_2(x_1, x_2)\rho_1(x_3)\} \neq \hat{\rho}_2(x_1, x_2)\hat{\rho}_1(x_3)$, so that all normalizations and higher-order cumulants discussed in the previous sections must be corrected to yield unbiased estimators of their corresponding expectation values.

Consider for example the product of two single-particle densities. Let $A$ be the number of events used for averaging; when one corrects interferometry or conventional factorial moments, $A$ will be equal to $N_{ev}$; for the Star integral, the usual choice is $A = N_{ev} - 1$ or, when a faster inner event loop is desired, $A$ can be made much less than $N_{ev}$ [1]. Now, using Eq. (2), we have

$$\{\rho_1(x_1)\rho_1(x_2)\} = \left\{\frac{1}{A^2} \sum_{e_1, e_2} \hat{\rho}_{e_1}^1(x_1)\hat{\rho}_{e_2}^2(x_2)\right\} \neq \left\{\frac{1}{A} \sum_{e_1} \hat{\rho}_{e_1}^1(x_1)\right\} \left\{\frac{1}{A} \sum_{e_2} \hat{\rho}_{e_2}^2(x_2)\right\},$$

the second part being the desired true value $\bar{\rho}_1(x_1)\bar{\rho}_1(x_2)$. The reason why $\{\rho_1\rho_1\}$ does not yield the true value lies in the “diagonal terms” $e_1 = e_2$ in the double sum above which prevent the desired factorization, as $\{\hat{\rho}_{e_1}^1\hat{\rho}_{e_2}^2\} \neq \{\hat{\rho}_{e_1}^1\}\{\hat{\rho}_{e_2}^2\}$ unless $e_1$ and $e_2$ refer to two different (and hence independent) events. Clearly, the desired unbiased estimator is given by the double sum restricted to unequal events, since

$$\left\{\frac{1}{A(A-1)} \sum_{e_1 \neq e_2} \hat{\rho}_{e_1}^1\hat{\rho}_{e_2}^2\right\} = \frac{1}{A(A-1)} \sum_{e_1 \neq e_2} \{\hat{\rho}_{e_1}^1\} \{\hat{\rho}_{e_2}^2\} = \hat{\rho}_1\hat{\rho}_1. \quad (41)$$

In general, therefore, the unbiased estimator for the product of $q$ single-particle densities is given by

$$\frac{1}{A^{[q]}} \sum_{e_1 \neq e_2 \neq \ldots \neq e_q} \hat{\rho}_{e_1}^1\hat{\rho}_{e_2}^2 \cdots \hat{\rho}_{e_q}^q. \quad (42)$$
Implementing these, we find the unbiased estimators for the Star integral normalization 

\[ \hat{\xi}_2^{\text{norm}} = \langle b \rangle, \]

\[ \hat{\xi}_3^{\text{norm}} = \langle b \rangle^2 - \frac{\kappa_2(b, b)}{(A - 1)}, \]  

\[ \hat{\xi}_4^{\text{norm}} = \langle b \rangle^3 - \frac{3\langle b \rangle \kappa_2(b, b)}{(A - 1)} + \frac{2\kappa_3(b, b, b)}{(A - 1)[2]}, \]

\[ \hat{\xi}_5^{\text{norm}} = \langle b \rangle^4 - \frac{6\langle b \rangle^2 \kappa_2(b, b)}{(A - 1)} + \frac{8\langle b \rangle^3 \kappa_3(b, b, b) + 3\kappa_2(b, b)}{(A - 1)[2]} - \frac{6\kappa_4(b, b, b, b) + 9\kappa_2^2(b, b)}{(A - 1)[3]}, \]

where

\[ \kappa_2(U, V) = \langle UV \rangle - \langle U \rangle \langle V \rangle, \]  

\[ \kappa_3(U, V, W) = \langle UVW \rangle - \sum_{\langle 3 \rangle} \langle UV \rangle \langle W \rangle + 2 \langle U \rangle \langle V \rangle \langle W \rangle, \]

\[ \kappa_4(U, V, W, X) = \langle UVWX \rangle - \sum_{\langle 4 \rangle} \langle U \rangle \langle VWX \rangle - \sum_{\langle 3 \rangle} \langle UV \rangle \langle WX \rangle + 2 \sum_{\langle 6 \rangle} \langle U \rangle \langle V \rangle \langle WX \rangle - 6 \langle U \rangle \langle V \rangle \langle W \rangle \langle X \rangle, \]

where the sums indicate the number of combinations to be taken and \( U, V, \ldots \) is any statistic of interest; for example \( \kappa_2(b, b)[2] = \langle bb[2] \rangle - \langle b \rangle \langle b[2] \rangle \). Note that (for the Star integral) the second order normalization does not need a correction; this is because the first event sum over \( e_1 \) is always pulled out in calculating the \( a \)-quantities; only the sums over \( e_2, e_3 \ldots \) must be made explicitly unequal.

For the cumulants, we need the more general statement: if \( \rho_{q_1}, \rho_{q_2} \ldots \rho_{q_K} \) are densities of order \( q_1, q_2, \ldots q_K \), the unbiased estimator of their product is given by

\[ \frac{1}{A[K]} \sum_{e_1 \neq e_2 \neq \ldots \neq e_K} \hat{\rho}_{q_1} \hat{\rho}_{q_2} \cdots \hat{\rho}_{q_K}. \]

Implementing these, we find the unbiased estimators for the \( i \)-particle cumulants to be

\[ \hat{f}_2(i) = a - \langle b \rangle, \]

\[ \hat{f}_3(i) = a[2] - \langle b[2] \rangle - 2a\langle b \rangle + 2\langle b \rangle^2 - \frac{2}{A - 1} \kappa_2(b, b), \]

\[ \hat{f}_4(i) = a[3] - \langle b[3] \rangle - 3a[2]\langle b \rangle - 3a\langle b[2] \rangle + 6\langle b \rangle \langle b[2] \rangle + 6a\langle b \rangle^2 - 6\langle b \rangle^3 \]

\[ + \frac{6}{A - 1} \left\{ (3\langle b \rangle - a) \kappa_2(b, b) - \kappa_2(b, b[2]) \right\} - \frac{12}{(A - 1)[2]} \kappa_3(b, b, b), \]

\[ \hat{f}_5(i) = a[4] - \langle b[4] \rangle - 4a[3]\langle b \rangle - 4a\langle b[3] \rangle \]
\[-6a^{[2]}b^{[2]} + 8b^{[3]} + 12a^{[2]}b^{[2]} + 6b^{[2]}b^{[2]} + 24a(b^{[2]} - 36b^{[2]}b^{[2]} - 24b^{[2]} + 24b^{[4]} - \frac{2}{A - 1}\left\{\left(6a^{[2]} - 18b^{[2]} - 36a\langle b\rangle + 72\langle b\rangle^{2}\right)\kappa_2(b, b) + 4\kappa_2(b, b^{[3]}) + 3\kappa_2(b^{[2]}, b^{[2]}) + (12a - 36\langle b\rangle)\kappa_2(b, b^{[2]})\right\}\right]
\frac{24}{(A - 1)^{[2]}}\left\{3\kappa_2^2(b, b) + (8\langle b\rangle - 2a)\kappa_3(b, b, b) - 3\kappa_3(b, b, b^{[2]})\right\}
\frac{72}{(A - 1)^{[3]}}\left\{2\kappa_4(b, b, b, b) + 3\kappa_2^2(b, b)\right\}.
\tag{54}
\]

For very small samples, when the inner event average sum $\sum_b$ cannot be taken strictly over $b \neq a$, corrections must also be made for equal-event terms $[10]$. These are very important for small samples found e.g. in fixed-$N$ cuts and cosmic ray data. When full event mixing is implemented for Bose-Einstein correlations and conventional factorial moments or cumulants, similar bias corrections are mandatory $[10]$.

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