Shared values of meromorphic functions on a compact Riemann surface

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Abstract

We prove a new bound on the number of shared values of distinct meromorphic functions on a compact Riemann surface, explain a mistake in a previous paper on this topic, and give a survey of related questions.

1. Introduction

Let $X$ be a compact Riemann surface of genus $g$, and let $p$ and $q$ be distinct nonconstant meromorphic functions on $X$. Let $S$ be the set of all points $\alpha$ in the Riemann sphere $\hat{\mathbb{C}}$ for which the sets $p^{-1}(\alpha)$ and $q^{-1}(\alpha)$ are identical. The main result of [1] asserts that

$$|S| - 4 \cdot (\deg p + \deg q) \leq 2g - 2. \quad (1)$$

However, as we will explain, and as the author of [1] has confirmed via email, the argument in [1] contains a crucial mistake so that it does not prove (1); moreover, currently it is not known whether (1) is always true. We will prove the following inequality, which is stronger than all inequalities in the literature that resemble (1) (other than (1) itself):

**Theorem 1.1:** We have

$$|S| - 2 \cdot \max(\deg p, \deg q) \leq 2g - 2 + \deg p + \deg q. \quad (2)$$

Theorem 1.1 implies the following inequality from [2], which more closely resembles (1):

**Corollary 1.2:** We have

$$|S| - 4 \cdot \max(\deg p, \deg q) \leq 2g - 2. \quad (3)$$
In this note we prove these results, explain the step in the proof in [1] where the mistake occurs, and survey related topics.

2. Proofs

In this section we prove Theorem 1.1 and Corollary 1.2. We maintain the notation from the first two sentences of the introduction.

Lemma 2.1: We have

$$\deg(p - q) \leq \deg p + \deg q.$$  

Proof: Every order-$k$ pole of $p-q$ must be a pole of order at least $k$ in at least one of $p$ or $q$, so the sum of the orders of the poles of $p-q$ is at most the sum of the orders of the poles of $p$ and $q$. Since the sum of the orders of the poles of a meromorphic function on $X$ equals the degree of the function, the conclusion follows. 

Proof of Theorem 1.1: Let $T$ be a finite subset of $S$. Replace $p$, $q$, $T$ by $\mu \circ p$, $\mu \circ q$, and $\mu(T)$ for a suitable Möbius transformation $\mu(z)$, in order to assume that $\infty \notin T$. The Riemann–Hurwitz formula for $p$ asserts that

$$2g - 2 = -2 \deg p + \sum_{\beta \in X} (\text{mult}_p(\beta) - 1),$$

where $\text{mult}_p(\beta)$ is the local multiplicity of $p$ near $\beta$. For any $\alpha \in \hat{C}$ we have $\sum_{\beta \in p^{-1}(\alpha)} \text{mult}_p(\beta) = \deg p$, so that

$$2g - 2 \geq -2 \deg p + \sum_{\alpha \in T} \sum_{\beta \in p^{-1}(\alpha)} (\text{mult}_p(\beta) - 1)$$

$$= -2 \deg p + \sum_{\alpha \in T} (\deg p - |p^{-1}(\alpha)|)$$

$$= -2 \deg p + |T| \deg p - |p^{-1}(T)|. \quad (4)$$

Likewise

$$2g - 2 \geq -2 \deg q + |T| \deg q - |q^{-1}(T)|. \quad (5)$$

Since $p^{-1}(T) = q^{-1}(T)$ by hypothesis, the previous two inequalities imply that

$$2g - 2 \geq (|T| - 2) \cdot \max(\deg p, \deg q) - |p^{-1}(T)|. \quad (6)$$

Since $p^{-1}(T) = q^{-1}(T)$ and $\infty \notin T$, the set $p^{-1}(T)$ is contained in the set of zeroes of the meromorphic function $p-q$. This function is nonzero by hypothesis, so it has at most as
many zeroes as its degree. Thus Lemma 2.1 implies that
\[ |p^{-1}(T)| \leq \deg(p - q) \leq \deg p + \deg q. \]
Combining this with (6) yields
\[ 2g - 2 \geq (|T| - 2) \cdot \max(\deg p, \deg q) - (\deg p + \deg q). \] (7)
Thus
\[ |T| \leq 2 + \frac{2g - 2 + \deg p + \deg q}{\max(\deg p, \deg q)}, \] (8)
so that in particular \( T \) is bounded. It follows that \( S \) is finite, so we may choose \( T \) to be \( S \) in (7) in order to obtain (2), which concludes the proof. \[ \blacksquare \]

**Proof of Corollary 1.2:** Combining (2) with the inequality
\[ \deg p + \deg q \leq 2 \cdot \max(\deg p, \deg q) \]
yields
\[ (|S| - 2) \cdot \max(\deg p, \deg q) \leq 2g - 2 + \deg p + \deg q \leq 2g - 2 + 2 \cdot \max(\deg p, \deg q), \]
and subtracting \( 2 \cdot \max(\deg p, \deg q) \) from both sides yields (3). \[ \blacksquare \]

### 3. The mistake in [1]

We now compare the argument in [1] with our proof of Theorem 1.1. The argument in [1] also uses the Riemann–Hurwitz formula to deduce the inequalities (4) and (5), and proves Lemma 2.1 by a different argument than ours. It then claims that (1) follows from these results and the inequality
\[ 2g - 2 \geq -2 \deg(p - q) + (\deg(p - q) - |p^{-1}(T)|). \] (9)
Equations (4) and (5) have the form \( |T| \leq A + B \cdot |p^{-1}(T)| \) where \( A \) and \( B \) are constants depending on \( g \) and the degrees of \( p \) and \( q \), with \( B > 0 \). Thus, in order to deduce an upper bound on \( |T| \) which depends only on \( g \) and the degrees of \( p \) and \( q \), one needs an upper bound on \( |p^{-1}(T)| \). However, (9) instead yields a lower bound on \( |p^{-1}(T)| \), and Lemma 2.1 does not involve \( |T| \) or \( |p^{-1}(T)| \), so it is not possible to obtain an upper bound on \( |T| \) of the desired form by combining (4), (5), (9) and Lemma 2.1.

It is reasonable to guess that the author of [1] was attempting to produce a similar proof to the one we gave for (2) (except for his different proof of Lemma 2.1), since [1] begins by saying that its proof was suggested by reading [3], and the argument in [3] is the specialization to the case \( X = \hat{C} \) of the argument used in the same author’s previous paper [4] to prove a weaker version of Corollary 1.2.
4. Related results

We conclude with a quick survey of related literature, in which we continue to use the notation from the first two sentences of the introduction.

- Another bound is $|S| \leq 2 + \sqrt{2g+2}$, which improves the best bound deducible from Corollary 1.2 that depends only on $X$ and not on the functions $p$ and $q$ [2].
- Stronger bounds on $|S|$ are known for special classes of Riemann surfaces: for instance, if $X$ is hyperelliptic then $|S| \leq 5$ [2].
- Stronger bounds on $|S|$ are known when at least one element of $S$ has the same preimages counting multiplicities under $p$ as it does under $q$ [2,4].
- Examples of $p$, $q$, $X$ where $S$ is ‘large’ appear in [2–6].
- There are analogous results for holomorphic maps between prescribed compact Riemann surfaces [4], and for holomorphic maps $p, q: X \to \mathbb{P}^n(\mathbb{C})$ having the same preimages of several hyperplanes [7,8].
- Results about ‘unique range sets’ for meromorphic functions on $X$ appear in [9]; by definition, such a set is a finite subset $T$ of $\hat{\mathbb{C}}$ for which any two distinct nonconstant meromorphic functions on $X$ have distinct preimages of $T$.
- If there are three disjoint nonempty finite subsets $T_1, T_2, T_3$ of $\hat{\mathbb{C}}$ such that each $T_i$ has the same preimages under $p$ (counting multiplicities) as it does under $q$, then $g \circ p = g \circ q$ for some nonconstant rational function $g(z)$ [10].
- There are related results in case $X$ is either the complement of a finite subset of a compact Riemann surface [6], an open Riemann surface with a harmonic exhaustion [11,12], or a higher-dimensional algebraic variety [13–15].

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