Reopening a cold case, inspector Echelon, high-ranking in the Row Operations Center, is searching for a lost linear map, known to be nilpotent. When a partially decomposed matrix is unearthed, he reconstructs its reduced form, finding it singular. But were its roots nilpotent?

1. Early in the Investigation

In teaching Linear Algebra, the first topic often is row reduction [1, 7], including Row Reduced Echelon Form (RREF); its applicability is broad and growing. Another topic, surprisingly popular with beginning students, is nilpotent matrices. One naturally wonders about their intersection. For instance, one would expect to find a book exercise asking:

What can be said about the RREF of a nilpotent matrix?

In the early days of the Covid-19 pandemic, as test delivery went remote, demand grew for new, Internet-resistant problems. A limited literature search for the Nilpotent-RREF connection came up short, suggesting potential for take-home final exam questions, hence the note at hand. We’ll first explore examples sufficient to settle the $3 \times 3$ case, then consider the general situation. The upshot is the row reduction eliminates all traces of nilpotence.

2. Stumbling On Evidence

We refer to [1, 2] for general background on RREF and rank. Recall that a square matrix $M$ is nilpotent if some power of $M$, say $M^k$, is the zero matrix; the smallest such $k$ is called the nilpotent index or just index of $M$. For instance, the rightmost matrix in (2) below is nilpotent, of index 3. Indeed, every strictly upper-triangular matrix (square, with zeros on
and below the diagonal) is nilpotent. Examining each general type of $3 \times 3$ matrix in RREF, we’ll apply row operations to obtain a nilpotent matrix.

Using the notation of [11], the $3 \times 3$ matrices of RREF in rank 1 are these:

$$
\begin{pmatrix}
1 & a & b \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 & c \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
$$

(1)

In (1) the entries $a, b, c$ are fixed but unspecified and unrestricted constants.

When working with matrices and their components we’ll follow a left to right and up to down convention. Thus first means leftmost, etc. We enumerate the rows of a matrix using Roman numberals. Hence II is the second row from the top.

The second and third matrices in (1) are strictly upper triangular, hence nilpotent. Call the first matrix $F$. In $F$, if $b = 0$, interchange rows $I$ and $III$ to obtain a strictly lower triangular matrix, hence a nilpotent matrix. If $b \neq 0$, perform $III \rightarrow III - \frac{1}{b}I$ (subtract $\frac{1}{b}$ times row $I$ from row $III$ and make that the new row $III$) to obtain

$$
\begin{pmatrix}
1 & a & b \\
0 & 0 & 0 \\
-\frac{1}{b} & -\frac{a}{b} & -1
\end{pmatrix}.
$$

This matrix squares to zero, hence is nilpotent.

Next, we present the RREFs of $3 \times 3$ matrices with rank 2:

$$
\begin{pmatrix}
1 & 0 & a \\
0 & 1 & b \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
1 & a & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}.
$$

(2)

The third matrix is strictly upper triangular, hence nilpotent. For the second matrix, exchange rows $I$ and $III$, then perform $I \rightarrow I - (a)II$ to obtain

$$
\begin{pmatrix}
0 & 0 & -a \\
0 & 0 & 1 \\
1 & a & 0
\end{pmatrix}.
$$

This matrix cubes to zero, hence is nilpotent. As with all $3 \times 3$ matrices of rank 2, its square does not vanish, so it is nilpotent of index 3.

So far we have been far from systematic. The leftmost matrix in (2), call it $T$, takes a bit more doing. It can be row reduced to the following matrix, which is nilpotent of index 3:
We can get from $T$, the leftmost matrix in (2) to (3) by the following row operations:

$$II \leftrightarrow III; \ II \rightarrow II - \frac{b}{a}I; \ I \rightarrow (-1)I; \ III \rightarrow III - \frac{b-1}{b}II.$$ (4)

This tacitly assumes that $a, b$ are both nonzero. If both $a$ and $b$ are zero, row swaps turn $T$ into a strictly lower triangular, and hence nilpotent matrix.

If $b = 0$ and $a \neq 0$ then (3) is still nilpotent and row equivalent to (3), even though the steps we took to get there involve a zero denominator. In case $a = 0$, perform row reduction steps

$$II \rightarrow III; \ I \rightarrow II; \ II \rightarrow II - bIII,$$

obtaining the following nilpotent matrix:

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & -b & b^2 \\ 0 & 1 & b \end{pmatrix}.$$ 

But (3) and (4) and all the row manipulations beg the question: how did we come up with these constructs?

3. No Basis For An Investigation

the facts which you have brought me are so indefinite
that we have no basis for an investigation

Sherlock Holmes
in The Adventure of the Dancing Men

We have a good working basis, however, on which to start.

Sherlock Holmes
in A Study In Scarlet

Consider the matrix

$$T = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \end{pmatrix}.$$ 

Recall that the (right) null space of $T$ is the solution set of the linear system $Tv = 0$. One easily checks that the span of the vector $\vec{w} \equiv \begin{pmatrix} -a \\ -b \\ 1 \end{pmatrix}^t$ gives all solutions of this linear system. We will form a basis for $\mathbb{R}^3$ by extending the one element set $\{\vec{w}\}$ and use that to build a nilpotent matrix whose RREF is $T$. Using the familiar notation $\vec{e}_2 \equiv \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^t$ and analogs for the canonical basis of $\mathbb{R}^3$, we write $\vec{u} \equiv \vec{e}_2$ and $\vec{v} = \vec{e}_3$. Then, if $a$ is a nonzero
scalar, \(\{\vec{u}, \vec{v}, \vec{w}\}\) is a basis for \(\mathbb{R}^3\). There is a unique linear transformation \(H\) on \(\mathbb{R}^3\) with the properties \(H\vec{u} = \vec{v}\); \(H\vec{v} = \vec{w}\); \(H\vec{w} = 0\); we summarize these as follows:

\[
\vec{u} \rightarrow \vec{v} \rightarrow (-a \quad -b \quad 1)^t \rightarrow 0.
\]

(This is not an exact sequence, and not even trying to be one.) Let’s find \(M\), the matrix representation of \(H\).

The second column of \(M\) is the vector \(M\vec{e}_2\), which is already prescribed: it is \(\vec{e}_3\). The third column of \(M\) is the vector \(M\vec{e}_3\), prescribed as \(\vec{w}\). What about the first column of \(M\)? It is \(M\vec{e}_1\), but what’s that? We can express \(\vec{e}_1\) in the basis \(\{\vec{u}, \vec{v}, \vec{w}\}\) as follows:

\[
(-a)\vec{e}_1 = \begin{pmatrix} -a \\ -b \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},
\]

which we can rewrite as \(\vec{e}_1 = \frac{-1}{a} (\vec{w} - \vec{v} + b\vec{u})\). Thus

\[
M\vec{e}_1 = \frac{-1}{a} (M\vec{w} - M\vec{v} + bM\vec{u}) = \frac{-1}{a} (\vec{0} - \vec{w} + b\vec{v}) = \begin{pmatrix} -1 \\ -\frac{b}{a} \\ 1-\frac{b}{a} \end{pmatrix},
\]

which matches the first column of (3), and thereby reproduces (3).

This approach un-begs one question while begging another. The vector space basis procedure here is guaranteed to produce a nilpotent matrix, but how did we know that this matrix will have the requisite RREF, namely \(T\)? We know that \(M\) and \(T\) have the same null space: \(\text{span}\{\vec{w}\}\). We now quote [7, chp. 2, p. 58] :

**Corollary** (Hoffman-Kunze). Let \(A\) and \(B\) be \(m \times n\) matrices over the field \(F\). Then \(A\) and \(B\) are row-equivalent if and only if they have the same row space.

In our context, relating row equivalence to the null space is needed, and such a relation is implicit in the literature, e.g., [7] again, or [2, VFSLS]. A recent posting [5] gives:

**Corollary.** The null space of a matrix \(M\) determines the RREF and the row space of \(M\). Hence if two matrices of the same size have the same null space, they are row equivalent.

**Remark.** Everyone knows many famous theorems and some famous lemmas, but there is a dearth of famous corollaries. In fact, the two best known to us are not mathematical, emanating from the Monroe Doctrine.

Thus, since the nilpotent matrix \(M\) has the right (right) null space, it has the right RREF as well.

### 4. General Impressions

Never trust to general impressions, 

\[\ldots,\text{ but concentrate yourself upon details.}\]

Sherlock Holmes

in *A Case of Identity*
I have had no proof yet of the existence

Sherlock Holmes

in The Sign Of The Four

Theorem. Every singular matrix is row equivalent to a nilpotent matrix.

Proof. Let $M$ be a singular $n \times n$ matrix and take a basis of the (right) null space of $M$, $\{\vec{k}_1, \ldots, \vec{k}_\ell\}$, where $\ell$ is the nullity of $M$. As $M$ is singular, $\ell$ is greater or equal to 1. If $\ell = n$ then $M$ is the zero vector, which is nilpotent, and we are done; assume $\ell$ is smaller than $n$. Extend $\{\vec{k}_1, \ldots, \vec{k}_\ell\}$ to $\{\vec{z}_1, \ldots, \vec{z}_{n-\ell}, \vec{k}_1, \ldots, \vec{k}_\ell\}$, a basis of the vector space of all $n \times 1$ columns. We consider a linear map that annihilates the basis vectors $\vec{k}_j$ and “shifts” each basis vector $\vec{z}_i$ to the next one, except for $\vec{z}_{n-\ell}$, which is shifted to $\vec{k}_1$. This corresponds to a matrix $N$ with the following properties:

$$N\vec{z}_{i-1} = \vec{z}_i; \quad N\vec{z}_{n-\ell} = \vec{k}_1,$$

for all suitable values of $i$. The matrix $N$ is nilpotent of index $n - \ell + 1$, with null space spanned by $\{\vec{k}_1, \ldots, \vec{k}_\ell\}$. Thus $M$ and $N$ share a null space. Hence, by the companion corollary of the Hoffman-Kunze Corollary, they have the same RREF and are thereby row equivalent. \qed

5. What’s It All About? The Aftermath

Nilpotency figures in the deepest moments of a first course in linear algebra [10]. It is particularly accessible to beginning students. Experience indicates that they latch onto the subject, with curiosity and enthusiasm; ditto for RREF. Yet, in the literature, the two seldom interact. Why? The theorem may give a clue. In fact, our discussion shows that it’s not about nilpotency at all. It’s about the null space.

6. Late Inspiration

In the course of the 2019-2020 academic-pandemic year this author developed the habit of staying up late, delving into the literature, mathematical, fictional and non-fictional. Trying to compose Internet-lookup-resistant take-home final exam questions, he stumbled on the nilpotency-RREF pairing. At the same (late) time, he was reminded of the stories of Jorge Luis Borges, where mathematical points figure into detective stories. In Death and The Compass [4, 12] the mystery hinges on the twists of mathematical reckoning, its inverse and converse, and consequences; an equilateral triangle figures prominently. Recently we noted a new paper about the Morley triangle [6]. Morley’s is perhaps the most famous of all equilateral triangles. Could a Borges-like story revolve around the Morley triangle, he wondered. In his half-dormant state he recalled Borges’ The Garden of Forking Paths, where a secret message is transmitted by the first letter of the name of a person cited prominently in a news story. Could the nilpotency of a matrix serve a similar purpose? More generally, could a mathematical theorem give rise to a mystery story? (More generally still, is there
a functor from the category of mathematics to the category of mystery stories?) That led to the present note. In the vein of the pandemic teachers were asked to exercise particular understanding and accommodation with students. In the same vein one hopes that the reader will do similarly with the would-be \textit{lockdown literato} responsible for this pandemic-produced essay.

\textbf{Note:} The cartoon included in this essay is by Frank Cotham. It appeared in the New Yorker magazine in 2007, and is included in the Cartoon Bank. Permission for use was obtained by arrangement with Condé Nast.

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