THE EIGENVALUES OF STOCHASTIC BLOCKMODEL GRAPHS

BY MINH TANG

Johns Hopkins University

We derive the joint limiting distribution for the largest eigenvalues of the adjacency matrix for stochastic blockmodel graphs when the number of vertices tends to infinity. We show that, in the limit, these eigenvalues are jointly multivariate normal with bounded covariances. Our result extends the classical result of Füredi and Komlós on the fluctuation of the largest eigenvalue for Erdős-Rényi graphs.

1. Introduction. The systematic study of eigenvalues of random matrices dates back to the seminal work of Wigner (1955) on the semicircle law for Wigner ensembles of symmetric or Hermitian matrices. A random $n \times n$ symmetric matrix $A = (a_{ij})_{i,j=1}^{n}$ is said to be a Wigner matrix if, for $i \leq j$, the entries $a_{ij}$ are independent mean zero random variables with variance $\sigma_{ij}^{2} = 1$ for $i < j$ and $\sigma_{ii}^{2} = \sigma^{2} > 0$. Many important and beautiful results are known for the spectral properties of these matrices, such as universality of the semi-circle law for bulk eigenvalues (Erdős et al., 2010; Tao and Vu, 2010), universality of the Tracy-Widom distribution for the largest eigenvalue (Soshnikov, 1999), universality properties of the eigenvectors (Tao and Vu, 2012; Knowles and Yin, 2013), and eigenvalue and eigenvector delocalization (Erdős et al., 2009).

In contrast, much less is known about the spectral properties of random symmetric matrices $A = (a_{ij})_{i,j=1}^{n}$ where the entries $a_{ij}$ are independent but not necessarily mean zero random variables with possibly heterogeneous variances. Such random matrices arise naturally in many settings, with the most popular example being perhaps the adjacency matrices of (inhomogeneous) independent edge random graphs. In the case when $A$ is the adjacency matrix for an Erdős-Rényi graph where the edges are i.i.d. Bernoulli random variables, Arnold (1967) and Ding and Jiang (2010) show that the empirical distribution of the eigenvalues of $A$ also converges to a semi-circle law. Meanwhile, the following result of Füredi and Komlós (1981) shows that the largest eigenvalue of $A$ is normally distributed when $\mathbb{E}[a_{ij}] = \mu$ and $\text{Var}[a_{ij}] = \sigma^{2}$ for $i < j$.

**Theorem 1** (Füredi and Komlós (1981)). Let $A = (a_{ij})$ be an $n \times n$ symmetric matrix where the $a_{ij}$ are independent (not necessarily identically distributed) random variables uniformly bounded in magnitude by a constant $C$. Assume that for $i > j$, the $a_{ij}$ have a common expectation $\mu > 0$ and variance $\sigma^{2}$. Furthermore, assume that $\mathbb{E}[a_{ii}] = v$ for all $i$. Then the distribution of $\lambda_{1}(A)$, the largest eigenvalue of $A$, can be approximated in order $n^{-1/2}$ by a normal distribution with mean $(n-1)\mu + v + \sigma^{2}/\mu$ and variance $2\sigma^{2}$, i.e.,

$$(1.1) \quad \lambda_{1}(A) - (n-1)\mu - v \xrightarrow{d} \mathcal{N}\left(\frac{\sigma^{2}}{\mu}, 2\sigma^{2}\right)$$

as $n \to \infty$. Furthermore, with probability tending to 1,

$$(1.2) \quad \max_{i \geq 2} |\lambda_{i}(A)| < 2\sigma \sqrt{n} + O(n^{1/3} \log n).$$
In the case when \( A \) is the adjacency matrix of an Erdős-Rényi graph with edge probability \( p \), Theorem 1 yields
\[
\lambda_1(A) - np \xrightarrow{d} \mathcal{N}(1-p, 2p(1-p))
\]
as \( n \to \infty \).

A natural generalization of Erdős-Rényi random graphs is the notion of stochastic blockmodel graphs (Holland et al., 1983) where, given an integer \( K \geq 1 \), the \( a_{ij} \) for \( i \leq j \) are independent Bernoulli random variables with \( E[a_{ij}] \in S \) for some set \( S \) of cardinality \( K(K+1)/2 \). More specifically, we have the following definition.

**Definition 1.** Let \( K \geq 1 \) be a positive integer and let \( \pi = (\pi_1, \pi_2, \ldots, \pi_K) \) be a non-negative vector in \( \mathbb{R}^K \) with \( \sum_k \pi_k = 1 \). Let \( B \in [0,1]^{K \times K} \) be symmetric. We say that \( (\tau, A) \sim \text{SBM}(\pi, B) \) if the following holds. First, \( \tau = (\tau_1, \ldots, \tau_n) \) and the \( \tau_i \) are i.i.d. random variables with \( P[\tau_i = k] = \pi_k \). Then \( A \in \{0,1\}^{n \times n} \) is a symmetric matrix such that, conditioned on \( \tau \), for all \( i \geq j \) the \( a_{ij} \) are independent Bernoulli random variables with \( E[a_{ij}] = B_{\pi_i, \pi_j} \).

The stochastic blockmodel is among the most popular generative models for random graphs with community structure; the nodes of such graphs are partitioned into blocks or communities, and the probability of connection between any two nodes is a function of their block assignment. The adjacency matrix \( A \) of a stochastic blockmodel graph can be viewed as \( A = E[A] + (A - E[A]) \) where \( E[A] \) is a low-rank deterministic matrix and \( (A - E[A]) \) is a generalized Wigner matrix whose elements are independent mean zero random variables with heterogeneous variances. We emphasize that our assumptions on \( A - E[A] \) distinguish us from existing results in the literature. For example, Péché (2006); Knowles and Yin (2014); Bordenave and Capitaine (2016); Pizzo et al. (2013) consider finite rank additive perturbations of the random matrix \( X \) given by \( \tilde{X} = X + P \) under the assumption that \( X \) is either a Wigner matrix or is sampled from the Gaussian unitary ensembles. Meanwhile, in Benaych-Georges and Nadakuditi (2011), the authors assume that \( X \) or \( P \) is orthogonally invariant; a symmetric random matrix \( H \) is orthogonally invariant if its distribution is invariant under similarity transformations \( H \mapsto W^{-1}HW \) whenever \( W \) is an orthogonal matrix. Finally, in O’Rourke and Renfrew (2014), the entries of \( X \) are assumed to be from an elliptical family of distributions, i.e., the collection \( \{(X_{ij}, X_{ji})\} \) for \( i < j \) are i.i.d. according to some random variable \( (\xi_1, \xi_2) \) with \( E[\xi_1\xi_2] = \rho \).

The characterization of the empirical distribution of eigenvalues for stochastic blockmodel graphs is of significant interest, but there are only a few available results. In particular, Zhang et al. (2014) and Avrachenkov et al. (2015) derived the Stieltjes transform for the limiting empirical distribution of the bulk eigenvalues for stochastic blockmodel graphs, thereby showing that the empirical distribution of the eigenvalues need not converge to a semicircle law. Zhang et al. (2014) and Avrachenkov et al. (2015) also considered the edge eigenvalues, but their characterization depends upon inverting the Stieltjes transform and thus currently does not yield the limiting distribution for these largest eigenvalues. Lei (2016) derived the limiting distribution for the largest eigenvalue of a centered and scaled version of \( A \). More specifically, Lei (2016) showed that there is a consistent estimate \( \tilde{E}[A] = (\tilde{a}_{ij}) \) of \( E[A] \) such that the matrix \( \tilde{A} = (\tilde{a}_{ij}) \) with entries \( \tilde{a}_{ij} = (a_{ij} - \tilde{a}_{ij})/\sqrt{(n-1)\tilde{a}_{ij}(1-\tilde{a}_{ij})} \) has a limiting Tracy-Widom distribution, i.e., \( n^{2/3}(\lambda_1(\tilde{A}) - 2) \) converges to Tracy-Widom.
This paper addresses the open question of determining the limiting distribution of the edge eigenvalues of adjacency matrices for stochastic blockmodel graphs. In particular, we extend the result of Füredi and Komlós and show that, in the limit, these eigenvalues are jointly multivariate normal with bounded covariances.

2. Main results. We present our result in the more general framework of generalized random dot product graph where $E[A]$ is only assumed to be low rank, i.e., we do not require that the entries of $E[A]$ takes on a finite number of distinct values. We first define the notion of a (generalized) random dot product graph (Young and Scheinerman, 2007; Rubin-Delanchy et al., 2017).

**Definition 2** (Generalized random dot product graph). Let $d$ be a positive integer and $p \geq 1$ and $q \geq 0$ be non-negative integers such that $p + q = d$. Let $I_{p,q}$ denote the diagonal matrix whose first $p$ diagonal elements equal 1 and the remaining $q$ diagonal entries equal $-1$. Let $\mathcal{X}$ be a subset of $\mathbb{R}^d$ such that $x^\top I_{p,q} y \in [0,1]$ for all $x, y \in \mathcal{X}$. Let $F$ be a distribution taking values in $\mathcal{X}$.

We say $\mathbf{X}\sim GRDPG(F)$ with signature $(p,q)$ if the following holds. First let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} F$ and set $\mathbf{X} = [X_1 \mid \cdots \mid X_n]^\top \in \mathbb{R}^{n \times d}$. Then $\mathbf{A} \in \{0,1\}^{n \times n}$ is a symmetric matrix such that the entries $\{a_{ij}\}_{i \leq j}$ are independent and

$$\begin{equation}
    a_{ij} \sim \text{Bernoulli}(X_i^\top I_{p,q} X_j).
\end{equation}$$

We therefore have

$$\begin{equation}
    P[A \mid X] = \prod_{i \leq j} (X_i^\top I_{p,q} X_j)^{a_{ij}} (1 - X_i^\top I_{p,q} X_j)^{(1-a_{ij})}.
\end{equation}$$

When $q = 0$, we say that $(\mathbf{A},\mathbf{X}) \sim RDPG(F)$, i.e., $\mathbf{A}$ is a random dot product graph.

**Remark.** Any stochastic blockmodel graph $(\tau, \mathbf{A}) \sim \text{SBM}(\mathbf{\pi}, \mathbf{B})$ can be represented as a (generalized) random dot product graph $(\mathbf{X}, \mathbf{A}) \sim GRDPG(F)$ where $F$ is a mixture of point masses. Indeed, suppose $\mathbf{B}$ is a $K \times K$ matrix and let $\mathbf{B} = U \Sigma U^\top$ be the eigendecomposition of $\mathbf{B}$. Then, denoting by $\nu_1, \nu_2, \ldots, \nu_K$ the rows of $U \Sigma^{1/2}$, we can define $F = \sum_{k=1}^K \pi_k \delta_{\nu_k}$, where $\delta$ is the Dirac delta function. The signature $(p,q)$ is given by the number of positive and negative eigenvalues of $\mathbf{B}$, respectively. Similar constructions show that degree-corrected stochastic blockmodel graphs (Karrer and Newman, 2011) and mixed-membership stochastic blockmodel graphs (Airoldi et al., 2008) are also special cases of generalized random dot product graphs.

**Remark.** We note that non-identifiability is an intrinsic property of generalized random dot product graphs. More specifically, if $(\mathbf{X}, \mathbf{A}) \sim GRDPG(F)$ where $F$ is a distribution on $\mathbb{R}^d$ with signature $(p,q)$, then for any matrix $\mathbf{W}$ such that $\mathbf{W}I_{p,q} \mathbf{W}^\top = I_{p,q}$, we have that $(\mathbf{Y}, \mathbf{B}) \sim RDPG(F \circ \mathbf{W})$ is identically distributed to $(\mathbf{X}, \mathbf{A})$, where $F \circ \mathbf{W}$ denote the distribution of $\mathbf{W} \xi$ for $\xi \sim F$. A matrix $\mathbf{W}$ satisfying $\mathbf{W}I_{p,q} \mathbf{W}^\top = I_{p,q}$ is said to be an *indefinite orthogonal* matrix. For the special case of random dot product graphs where $q = 0$, the condition on $\mathbf{W}$ reduces to that of an orthogonal matrix.

With the above notations in place, we now state our generalization of Füredi and Komlós (1981) for the generalized random dot product graph setting.
Theorem 2. Let $(A, X) \sim \text{GRDPG}(F)$ be a $d$-dimensional generalized random dot product graph with signature $(p, q)$. Let $\Delta = E[XX^\top]$ where $X \sim F$ and suppose that $\Delta I_{p,q}$ has $p + q = d$ simple eigenvalues. Let $P = XI_{p,q}X^\top$ and for $1 \leq i \leq d$, let $\lambda_i$ and $\eta_i$ be the $i$-th largest eigenvalue and associated (unit-norm) eigenvector pair for the matrix $\Delta I_{p,q}$. Also let $\Gamma$ be the $d \times d$ matrix whose elements are

$$
\eta_i = \frac{1}{\lambda_i(\Delta I_{p,q})}E[\xi_i^\top \Delta^{-1/2}XX^\top \Delta^{-1/2}\xi_i(X^\top I_{p,q}\mu - X^\top I_{p,q}\Delta I_{p,q}X)]
$$

Also let $\Gamma$ be the $d \times d$ matrix whose elements are

$$
\Gamma_{ij} = 2(E[\xi_i^\top \Delta^{-1/2}XX^\top \Delta^{-1/2}\xi_jX]I_{p,q}E[\xi_i^\top \Delta^{-1/2}XX^\top \Delta^{-1/2}\xi_jX])
- 2\text{tr}(E[\xi_i^\top \Delta^{-1/2}XX^\top \Delta^{-1/2}\xi_jX^\top I_{p,q}E[\xi_i^\top \Delta^{-1/2}XX^\top \Delta^{-1/2}\xi_jX^\top I_{p,q}])
$$

We then have

$$(\hat{\lambda}_1 - \lambda_1, \hat{\lambda}_2 - \lambda_2, \ldots, \hat{\lambda}_d - \lambda_d) \distr MVN(\eta, \Gamma)$$

as $n \to \infty$.

When $A$ is a $d$-dimensional random dot product graph, Theorem 2 simplifies to the following result.

Corollary 1. Let $(A, X) \sim \text{RDPG}(F)$ be a $d$-dimensional random dot product graph and suppose that $\Delta = E[XX^\top]$ has $d$ simple eigenvalues. Let $P = XX^\top$ and let $\lambda_i(\Delta)$ and $\xi_i$ denote the $i$-th largest eigenvalue and associated (unit-norm) eigenvector of $\Delta$. Let $\mu = E[X]$ and denote by $\eta$ the $d \times 1$ vector with elements

$$
\eta_i = \frac{E[\xi_i^\top XX^\top \xi_i(X^\top \mu - X^\top \Delta X)]}{\lambda_i(\Delta)^2},
$$

and by $\Gamma$ the $d \times d$ matrix whose elements are

$$
\Gamma_{ij} = \frac{2}{\lambda_i(\Delta)\lambda_j(\Delta)}(E[\xi_i^\top XX^\top \xi_jX]^\top E[\xi_i^\top XX^\top \xi_jX])
- \frac{2}{\lambda_i(\Delta)\lambda_j(\Delta)}\text{tr}(E[\xi_i^\top XX^\top \xi_jX^\top X]E[\xi_i^\top XX^\top \xi_jX^\top X])
$$

We then have

$$(\hat{\lambda}_1 - \lambda_1, \hat{\lambda}_2 - \lambda_2, \ldots, \hat{\lambda}_d - \lambda_d) \distr MVN(\eta, \Gamma)$$

as $n \to \infty$.

To illustrate Corollary 1, let $A$ be an Erdős-Rényi graph with edge probability $p$; then $F$ is the Dirac delta measure at $pI/2$ and hence $\Delta = p$, $\xi_1 = 1$, and $\lambda_1(\Delta) = p$. We thus recover the earlier result of Füredi and Komlós that

$$
\hat{\lambda}_i - np \distr N(1 - p, 2p(1 - p)).
$$

When the eigenvalues of $\Delta I_{p,q}$ are not all simple eigenvalues, Theorem 2 can be adapted to yield the following result.
Theorem 3. Let \((X, A) \sim \text{GRDPG}(F)\) be a \(d\)-dimensional generalized random dot product graph on \(n\) vertices with signature \((p, q)\). Let \(P = XI_{p,q}X^\top\) and for \(1 \leq i \leq r\), let \(\hat{\lambda}_i\) and \(\lambda_i\) denote the \(i\)-th largest eigenvalues of \(A\) and \(P\) (in modulus), respectively. Also let \(v_i\) be the unit norm eigenvector satisfying \((X^\top X)^{1/2}I_{p,q}(X^\top X)^{1/2}v_i = \lambda_i v_i\) for \(i = 1, 2, \ldots, d\). Denote by \(\tilde{\eta} = \tilde{\eta}(X)\) the \(d \times 1\) vector with elements

\[
\tilde{\eta}_i = \frac{n}{\lambda_i} \left( \frac{1}{n} \sum_{s=1}^{n} v_i^\top \left( \frac{X^\top X}{n} \right)^{-1/2} X_s^\top X_s \left( \frac{X^\top X}{n} \right)^{-1/2} v_i X_s^\top I_{p,q} \left( \frac{1}{n} \sum_{t=1}^{n} (X_t - X_t^\top I_{p,q} X_s) \right) \right)
\]

and by \(\sigma^2 = \sigma^2(X)\) the \(d \times 1\) vector whose elements are

\[
\sigma_i^2 = 2 \left( \sum_k (X_k^\top (X^\top X)^{-1/2} v_i) (X_k^\top (X^\top X)^{-1/2} v_i)^\top I_{p,q} \left( \sum_l (X_l^\top (X^\top X)^{-1/2} v_i)^\top X_l X_l^\top (X_l^\top (X^\top X)^{-1/2} v_i) \right) \right)
\]

\[
- 2 \text{tr} \left( \sum_k (X_k^\top (X^\top X)^{-1/2} v_i) (X_k^\top (X^\top X)^{-1/2} v_i)^\top X_k X_k^\top I_{p,q} \left( \sum_l (X_l^\top (X^\top X)^{-1/2} v_i)^\top X_l X_l^\top (X_l^\top (X^\top X)^{-1/2} v_i) \right) \right)
\]

We then have, for \(1 \leq i \leq d\),

\[
\frac{1}{\sigma_i}(\hat{\lambda}_i - \lambda_i - \tilde{\eta}_i) \longrightarrow N(0, 1)
\]

as \(n \to \infty\).

The main differences between Theorem 3 and Theorem 2 are that (1) we do not claim that the quantities \(\tilde{\eta}_i\) and \(\sigma_i^2\) in Theorem 3 (which, for \((X, A) \sim \text{GRDPG}(F)\) are functions of the underlying latent positions \(X\)) converge as \(n \to \infty\) and (2) we do not claim that the collection \((\hat{\lambda}_i - \lambda_i)_{i=1}^d\)
in Theorem 3 converges jointly to multivariate normal. The above differences stem mainly from the fact that when the eigenvalues of $\Delta$ are not simple eigenvalues, then $\frac{X_n^TX}{n_n} \to \Delta$ as $n \to \infty$ but $v_i$ does not necessarily converges to $\xi_i$, the corresponding eigenvector of $\Delta^{1/2}I_{p,q} \Delta^{1/2}$, as $n \to \infty$.

3. Proof of Theorem 2 and Theorem 3. Let $u_1, u_2, \ldots, u_d$ be the eigenvectors corresponding to the non-zero eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_d$ of $P$. Similarly, let $\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_d$ be the eigenvectors corresponding to the eigenvalues $\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_d$ of $A$.

A sketch of the proof of Theorem 2 and Theorem 3 is as follows. First we derive the following approximation of $\hat{\lambda}_i - \lambda_i$ by a sum of two quadratic forms $u_i^T(A-P)u_i$ and $u_i^T(A-P)^2u_i$, namely

$$\hat{\lambda}_i - \lambda_i = \frac{\lambda_i}{\hat{\lambda}_i} u_i^T(A-P)u_i + \frac{\lambda_i}{\hat{\lambda}_i^2} u_i^T(A-P)^2u_i + O_P(n^{-1/2}).$$

Now, the term $\frac{\lambda_i}{\hat{\lambda}_i^2} u_i^T(A-P)^2u_i$ is a function of the $n(n+1)/2$ independent random variables $\{a_{rs} - p_{rs}\}_{r \leq s}$ and hence is concentrated around its expectation, i.e.,

$$\frac{\lambda_i}{\hat{\lambda}_i^2} u_i^T(A-P)^2u_i = \mathbb{E}[\lambda_i^{-1} u_i^T(A-P)^2u_i] + O_P(n^{-1/2})$$

where the expectation is taken with respect to $A$. Letting $\tilde{\eta}_i = \mathbb{E}[\lambda_i^{-1} u_i^T(A-P)^2u_i]$, we obtain, after some straightforward algebraic manipulations, the expression for $\tilde{\eta}_i$ in Eq. (2.7). When the eigenvalues of $\Delta I_{p,q}$ are distinct, we derive the limit $\tilde{\eta}_i \xrightarrow{a.s.} \eta_i$ where $\eta_i$ is defined in Eq. (2.3). Next, with $u_{is}$ being the $s$-th element of $u_i$,

$$u_i^T(A-P)u_i = \sum_{r < s} 2u_{is}u_{ir}(a_{rs} - p_{rs}) + \sum_r u_{ir}^2(a_{rr} - p_{rr})$$
is, conditional on $X$, a sum of independent mean 0 random variables and the Lindeberg-Feller central limit theorem yield

$$
\frac{\lambda_i}{\lambda_i \sigma_i} u_i^\top (A - P) u_i \overset{d}{\longrightarrow} N(0, 1)
$$

as $n \to \infty$, where $\sigma_i^2$ is as defined in Eq. (2.8). Thus for each $i \leq d$, $\frac{1}{\sigma_i^2} (\lambda_i - \lambda_i - \tilde{\eta}_i) \to N(0, 1)$ as $n \to \infty$. When the eigenvalues of $\Delta I_{p,q}$ are distinct, then $\sigma_i^2 \overset{a.s.}{\to} \Gamma_{ii}$ as defined in Eq. (2.4). The joint distribution of $(\lambda_i - \lambda_i)_{i=1}^d$ in Theorem 2 then follows from the Cramer-Wold device.

We now provide detailed derivations of Eq. (3.1) through Eq. (3.3).

**Proof of Eq. (3.1)** For a given $i \leq d$, we have

$$
(\hat{\lambda}_i I - (A - P)) \hat{u}_i = A \hat{u}_i - (A - P) \hat{u}_i = P \hat{u}_i = \left( \sum_{j=1}^{r} \gamma_j u_j u_j^\top \right) \hat{u}_i
$$

Now suppose that $\hat{\lambda}_i I - (A - P)$ is invertible; this holds with high probability for sufficiently large $n$. Then multiplying both sides of the above display by $u_i^\top \left( \hat{\lambda}_i I - (A - P) \right)^{-1}$ on the left and using the von Neumann identity $(I - X)^{-1} = I + \sum_{k=1}^{\infty} X^k$ for $\|X\| < 1$, we have

$$
u_i^\top \hat{u}_i = \sum_{j=1}^{d} \gamma_j u_i^\top \left[ \hat{\lambda}_i I - (A - P) \right]^{-1} u_j u_j^\top \hat{u}_i = \sum_{j=1}^{d} \gamma_j \hat{\lambda}_i^{-1} u_i^\top \left[ I - \hat{\lambda}_i^{-1} (A - P) \right]^{-1} u_j u_j^\top \hat{u}_i
$$

$$
= \sum_{j=1}^{d} \gamma_j \hat{\lambda}_i^{-1} u_i^\top \left( I + \sum_{k=1}^{\infty} \hat{\lambda}_i^{-k} (A - P)^k \right) u_j u_j^\top \hat{u}_i
$$

$$
= \frac{\lambda_i}{\lambda_i} u_i^\top \left( I + \sum_{k=1}^{\infty} \hat{\lambda}_i^{-k} (A - P)^k \right) u_i \hat{u}_i + \sum_{j \neq i} \frac{\gamma_j}{\lambda_i} \left( \sum_{k=1}^{\infty} \hat{\lambda}_i^{-k} (A - P)^k \right) u_j u_j^\top \hat{u}_i
$$

We first assume that all of the eigenvalues of $\Delta I_{p,q}$ are simple eigenvalues. The eigenvalues of $P = X^\top I_{p,q} X^\top$ are then well-separated, i.e., $\min_{j \neq i} |\lambda_i - \lambda_j| = O_P(n)$ for $1 \leq i \neq j \leq d$. The Davis-Kahan sin $\Theta$ theorem (Davis and Kahan, 1970; Yu et al., 2015) therefore implies, for some constant $C$,

$$
1 - u_i^\top \hat{u}_i = \frac{1}{2} u_i^2 - \frac{C^2 \|A - P\|^2}{\min\{|\lambda_i - \lambda_{i+1}|, |\lambda_{i-1} - \lambda_{i}|\}^2} = O_P(n^{-1})
$$

$$
| u_i^\top \hat{u}_i | \leq \frac{C \|A - P\|}{\min\{|\lambda_i - \lambda_{i+1}|, |\lambda_{i-1} - \lambda_{i}|\}} = O_P(n^{-1/2}).
$$

We can thus divide both side of the above display by $u_i^\top \hat{u}_i$ to obtain

$$
1 = \frac{\lambda_i}{\lambda_i} + \frac{\lambda_i}{\lambda_i} u_i^\top \left( \sum_{k=1}^{\infty} \hat{\lambda}_i^{-k} (A - P)^k \right) u_i + \sum_{j \neq i} \frac{\gamma_j}{\lambda_i} \left( \sum_{k=1}^{\infty} \hat{\lambda}_i^{-k} (A - P)^k \right) u_j u_j^\top \hat{u}_i
$$

Equivalently,

$$
\hat{\lambda}_i - \lambda_i = \lambda_i u_i^\top \left( \sum_{k=1}^{\infty} \hat{\lambda}_i^{-k} (A - P)^k \right) u_i + \sum_{j \neq i} \frac{\gamma_j}{\lambda_i} (A - P) u_j u_j^\top \hat{u}_i
$$

$$
+ \sum_{j \neq i} \frac{\gamma_j}{\lambda_i} \left( \sum_{k=2}^{\infty} \hat{\lambda}_i^{-k} (A - P)^k \right) u_j u_j^\top \hat{u}_i.
$$
Now $\lambda_i^{-1}\hat{\lambda}_j = O_p(1)$, and by Hoeffding’s inequality, $u_j^\top (A - P) u_i = O_p(1)$. Since $u_j^\top \hat{u}_i = O_p(n^{-1/2})$, we have

$$\sum_{j \neq i} \frac{\lambda_j}{\lambda_i} u_i^\top (A - P) u_j \frac{u_j^\top \hat{u}_i}{u_i^\top \hat{u}_i} = O_p(n^{-1/2}).$$

Next we note that $\| \sum_{k=2}^{\infty} \hat{\lambda}_i^{-k} (A - P)^k \|$ can be bounded as

$$\| \sum_{k=2}^{\infty} \hat{\lambda}_i^{-k} (A - P)^k \| \leq \frac{\| \hat{\lambda}_i^{-2} (A - P)^2 \|}{1 - \| \hat{\lambda}_i^{-1} (A - P) \|} = O_p(\hat{\lambda}_i^{-1}).$$

We thus have

$$\sum_{j \neq i} \lambda_j u_i^\top \left( \sum_{k=2}^{\infty} \hat{\lambda}_i^{-k} (A - P)^k \right) u_j \frac{u_j^\top \hat{u}_i}{u_i^\top \hat{u}_i} = O_p(n^{-1/2}).$$

The above bounds then implies

$$\hat{\lambda}_i - \lambda_i = \lambda_i u_i^\top \left( \sum_{k=1}^{\infty} \hat{\lambda}_i^{-k} (A - P)^k \right) u_i + O_p(n^{-1/2}).$$

Similar to the derivation of Eq. (3.8), we also show that

$$\| \sum_{k=3}^{\infty} \hat{\lambda}_i^{-k} (A - P)^k \| \leq C \| \hat{\lambda}_i^{-3} (A - P)^3 \| \leq C \hat{\lambda}_i^{-3/2}$$

with high probability and thus Eq. (3.9) and Eq. (3.10) imply

$$\hat{\lambda}_i - \lambda_i = \frac{\lambda_i}{\lambda_i} u_i^\top (A - P) u_i + \frac{\lambda_i}{\lambda_i^2} u_i^\top (A - P)^2 u_i + O_p(n^{-1/2}),$$

and Eq. (3.1) is established.

We now consider the case where the $i$-th eigenvalue of $\Delta I_{p,q}$ has multiplicity $r_i \geq 2$. Let $S_i$ be the indices of the $r_i$ eigenvalues $\lambda_j$ of $P = XI_{p,q}X^\top$ that is closest to $n\lambda_i(\Delta I_{p,q})$, i.e.,

$$\max_{j \in S_i} |\lambda_j - n\lambda_i(\Delta I_{p,q})| \leq \min_{k \notin S_i} |\lambda_k - n\lambda_i(\Delta I_{p,q})|; \quad |S_i| = r_i.$$  

Denote by $U_{S_i}$ the $n \times r_i$ matrix whose columns are the eigenvectors corresponding to the $\lambda_j, j \in S_i$. We note that $i \in S_i$ with high probability for sufficiently large $n$. Furthermore, $|\lambda_j - \lambda_k| = O_p(n)$ for all $j \in S_i$ and $k \notin S_i$. Therefore, by the Davis-Kahan theorem, $u_i^\top \hat{u}_k = O_p(n^{-1/2})$ for all $k \notin S_i$. We now consider $u_i^\top \hat{u}_j$ for $j \in S_i, j \neq i$. We note that

$$u_i^\top \hat{u}_j = u_i^\top A \hat{u}_j - u_i^\top P \hat{u}_j = u_i^\top (A - P) U_{S_i} U_{S_i}^\top \hat{u}_j + \frac{u_i^\top (A - P)}{\lambda_j - \lambda_i} U_{S_i} U_{S_i}^\top \hat{u}_j - \lambda_j - \lambda_i$$

$$= \frac{n^{-1/2} u_i^\top (A - P) U_{S_i} U_{S_i}^\top \hat{u}_j}{n^{-1/2}(\lambda_j - \lambda_j) + n^{-1/2}(\lambda_j - \lambda_i)} + \frac{n^{-1/2} u_i^\top (A - P)}{n^{-1/2}(\lambda_j - \lambda_j) + n^{-1/2}(\lambda_j - \lambda_i)}.$$  

By Hoeffding inequality, $u_i^\top (A - P) U_{S_i} = O_p(1)$ with high probability. Since $j \in S_i$, we have $\| (I - U_{S_i} U_{S_i}^\top) \hat{u}_j \| = O_p(n^{-1/2})$ by the Davis-Kahan theorem. We then bound $\lambda_j - \lambda_i$ using the following result of (Cape et al., 2016, Theorem 3.7) (see also O’Rourke et al. (2013, Theorem 23)).
Theorem 4. Let $A$ and be a $n \times n$ symmetric random matrix with $A_{ij} \sim \text{Bernoulli}(P_{ij})$ for $i \leq j$ and the entries $\{A_{ij}\}$ are independent. Denote the $d+1$ largest singular values of $A$ by $0 \leq \sigma_{d+1} < \sigma_d \leq \sigma_{d-1} \leq \cdots \leq \sigma_1$, and denote the $d+1$ largest singular values of $P$ by $0 \leq \sigma_{d+1} < \sigma_d \leq \sigma_{d-1} \leq \cdots \leq \sigma_1$. Suppose that $Y = \max_i \sum_j P_{ij} = \omega(\log^4 n)$, $\sigma_1 \geq CY$, $\sigma_{d+1} \leq cY$ for some absolute constants $C > c > 0$. Then for each $k \in \{1, 2, \ldots, d\}$, there exists some positive constant $c_{k,d}$ such that as $n \to \infty$, with probability at least $1 - n^{-3}$, we have

$$|\lambda_k - \sigma_k| \leq c_{k,d} \log n.$$  

We thus have

$$|u_i^\top \hat{u}_j| = \frac{n^{-1/2}O_2(1)}{n^{-1/2}O_p(\log n) + n^{-1/2}(\lambda_j - \lambda_i)}.$$  

We now analyze $n^{-1/2}(\lambda_j - \lambda_i)$. We can view $P = X I_{p,q} X^\top$ as a kernel matrix with symmetric kernel $h(X_i, X_j) = X_i^\top I_{p,q} X_j$ where $X_i, X_j \sim \mathcal{N}(\mu, \Sigma)$. As $h$ is finite-rank, let $(\phi_i, \lambda_i(I_{p,q} \Delta))_{i=1}^d$ denote the eigenvalues and associated eigenfunctions of the integral operator $K_h : L_2(F) \to L_2(F)$, i.e.,

$$K_h \phi_i(x) := \int h(x, y) \phi_i(y) \, dF(y) = \lambda_i(I_{p,q} \Delta) \phi_i(x).$$  

Then, following Koltchinskii and Giné (2000), let $\Psi_i$ denote the $r_i \times r_i$ random symmetric matrix whose half-vectorization $\operatorname{vech}(\Psi_i)$ is (jointly) distributed multivariate normal with mean $0$ and $r_i(r_i+1)/2 \times r_i(r_i+1)/2$ covariance matrix with entries of the form

$$\text{Cov}(\Psi_i(s,t), \Psi_i(u,v)) = \int \phi_s(y) \phi_t(y) \phi_u(y) \phi_v(y) \, dF(y) - \int \phi_s(y) \phi_t(y) \, dF(y) \int \phi_u(y) \phi_v(y) \, dF(y)$$

for $1 \leq s \leq t \leq r_i, 1 \leq u \leq v \leq r_i$, where, with a slight abuse of notation, the collection $\{\phi_s\}_{s \leq r_i}$ denote the $r_i$ eigenfunctions of $K_h$ associated with the eigenvalue $\lambda_i(I_{p,q} \Delta)$. A simplification of the statement of Theorem 5.1 in Koltchinskii and Giné (2000), to the setting where $h$ is a finite-rank kernel, yields

$$n^{-1/2}(\lambda_j/n - \lambda_i(I_{p,q} \Delta))_{j \in S_i} \to \lambda_i(I_{p,q} \Delta) \times (\lambda_i(\Psi_i))_{1 \leq s \leq r_i}$$

as $n \to \infty$; here we use the notation $\lambda_s(M)$ to denote the $s$-th largest eigenvalue, in modulus, of the matrix $M$. Thus, the joint distribution of $\{n^{-1/2}(\lambda_s - n \lambda_i(\Delta I_{p,q}))\}_{s \in S_i}$ converges to a non-degenerate limiting distribution and hence the limiting distribution of $n^{-1/2}(\lambda_i - \lambda_j)$ is also non-degenerate.

We therefore have

$$|u_i^\top \hat{u}_j| = \frac{n^{-1/2}O_p(1)}{n^{-1/2}O_p(\log n) + n^{-1/2}(\lambda_j - \lambda_i)} = o_p(1); \quad j \in S_i, j \not= i.$$  

Finally, we note that there exists an orthogonal matrix $W$ such that $\|U^\top \hat{U} - W\| = O_p(n^{-1})$. Hence, for any $i \leq d$, $\sum_{j=1}^d (u_i^\top \hat{u}_j)^2 = 1 + O_p(n^{-1})$; hence, from our bounds for $u_i^\top \hat{u}_j$ for $j \not= i$ given above, we have $u_i^\top \hat{u}_i = 1 - o_p(1)$. In summary, when the eigenvalues of $I_{p,q} \Delta$ are not all simple eigenvalues, we have (in place of Eq. (3.5) and Eq. (3.6)), the bounds

$$u_i^\top \hat{u}_i = 1 - o_p(1); \quad u_i^\top \hat{u}_j = o_p(1).$$

Thus Eq.(3.7) still holds and the remaining steps in the derivation of Eq. (3.11) can be easily adapted to yield

$$\hat{\lambda}_i - \lambda_i = \frac{\lambda_i}{\lambda_i} u_i^\top (A - P) u_i + \frac{\lambda_i}{\lambda_i^2} u_i^\top (A - P)^2 u_i + o_p(1).$$
Proof of Eq. (3.2) Let \( Z = \lambda_i^{-1} u_i ^\top (A - P)^2 u_i \). To derive Eq. (3.2), we show the concentration \( Z \) around \( \mathbb{E}[Z] \) (where the expectation is taken with respect to \( A \), conditional on \( P \)) using a log-Sobolev concentration inequality from Boucheron et al. (2003). More specifically, let \( A' = (a'_{rs}) \) be an independent copy of \( A \), i.e., the upper triangular entries of \( A' \) are independent Bernoulli random variables with mean parameters \( \{p_{rs}\}_{r \leq s} \). For any pair of indices \( (r,s) \), let \( A_{(rs)} \) be the matrix obtained by replacing the \( (r,s) \) and \( (s,r) \) entries of \( A \) by \( a'_{ij} \) and let \( Z_{(rs)} = \lambda_i^{-1} u_i ^\top (A_{(rs)} - P)^2 u_i \). Then Theorem 5 of Boucheron et al. (2013) states that

**Theorem 5.** Assume that there exists positive constants \( a \) and \( b \) such that

\[
\sum_{r \leq s} (Z - Z_{(rs)})^2 \leq aZ + b.
\]

Then for all \( t > 0 \),

\[
\mathbb{P}[Z - \mathbb{E}[Z] \geq t] \leq \exp\left( \frac{-t^2}{4a\mathbb{E}[Z] + 4b + 2at} \right),
\]

(3.14)

\[
\mathbb{P}[Z - \mathbb{E}[Z] \leq t] \leq \exp\left( \frac{-t^2}{4a\mathbb{E}[Z]} \right).
\]

(3.15)

The main technical step is then to bound \( \sum_{r \leq s} (Z - Z_{(rs)})^2 \). An identical argument to that in proving Lemma A.6 in Tang et al. (2017) yield that

\[
\sum_{r \leq s} (Z - Z_{(rs)})^2 \leq a\lambda_i^{-1}Z + b.
\]

for some constants \( a \) and \( b \). Theorem 5 therefore implies

\[
|Z - \mathbb{E}[Z]| \leq \sqrt{\mathbb{E}[Z]} \times O_P(n^{-1/2}) = O_P(n^{-1/2}).
\]

as desired.

We now evaluate \( \mathbb{E}[Z] = \mathbb{E}[\lambda_i^{-1} u_i ^\top (A - P)^2 u_i] \). Let \( \zeta_{rs} \) denote the \( rs \)-th entry of \( \mathbb{E}[(A - P)^2] \). We note that \( \zeta_{rs} \) is of the form

\[
\zeta_{rs} = \sum_t \mathbb{E}[(a_{rt} - p_{rt})(a_{st} - p_{st})] = \begin{cases} 0 & \text{if } r \neq s \\ \sum_t p_{rt}(1 - p_{rt}) & \text{if } r = s \end{cases}.
\]

We therefore have,

\[
\mathbb{E}[Z] = \lambda_i^{-1} u_i ^\top \mathbb{E}[(A - P)^2] u_i = \lambda_i^{-1} \sum_{s=1}^n u_{is}^2 \sum_{t=1}^n p_{st}(1 - p_{st})
\]

\[
= \lambda_i^{-1} \sum_{s=1}^n u_{is}^2 \sum_{t=1}^n X_s ^\top I_{p,q} X_t (1 - X_s ^\top I_{p,q} X_t)
\]

Let \( \tilde{\lambda}_i \) and \( \tilde{v}_i \) be an eigenvalue/eigenvector pair for the eigenvalue problem

\[
(X ^\top X)^{1/2} I_{p,q} (X ^\top X)^{1/2} v = \tilde{\lambda} v.
\]

(3.16)
We note that if $\tilde{\lambda}_i$ and $\tilde{v}_i$ satisfies Eq. (3.16) then $\tilde{\lambda}_i$ and $\tilde{u}_i = X(X^\top X)^{-1/2}v_i$ are an eigenvalue/eigenvector pair for the eigenvalue problem

$$X I_{p,q} X^\top u = \tilde{\lambda} u; \quad \tilde{\lambda} \neq 0.$$ 

Conversely, if $\tilde{\lambda}_i$ and $\tilde{u}_i$ are an eigenvalue/eigenvector pair for $XI_{p,q}X^\top$ then $\tilde{\lambda}_i$ and $\tilde{v}_i = (X^\top X)^{-1/2}X^\top u_i$ satisfies Eq. (3.16). In addition, if the vectors $\{v_i\}_{i=1}^d$ are mutually orthonormal then the vectors $\{u_i\}_{i=1}^d$ are also mutually orthonormal. We therefore have

$$\mathbb{E}[Z] = \frac{1}{\tilde{\lambda}_i} \sum_{s=1}^{n} (v_i^\top (X^\top X)^{-1/2} X s)^2 \sum_{l=1}^{n} X_s^\top I_{p,q} X t (1 - X_s^\top I_{p,q} X t)$$

$$= \frac{1}{\tilde{\lambda}_i} \sum_{s=1}^{n} v_i^\top (X^\top X)^{-1/2} X s X_s^\top (X^\top X)^{-1/2} v_i X_s^\top I_{p,q} \left( \sum_{l=1}^{n} (X_t - X_t X_l^\top I_{p,q} X s) \right)$$

$$= \frac{n}{\tilde{\lambda}_i} \left( \frac{1}{n} \sum_{s=1}^{n} v_i^\top \frac{(X^\top X)}{n}^{-1/2} X s X_s^\top \frac{(X^\top X)}{n}^{-1/2} v_i X_s^\top I_{p,q} \left( \frac{1}{n} \sum_{l=1}^{n} (X_t - X_t X_l^\top I_{p,q} X s) \right) \right) = \tilde{\eta}_i.$$ 

By the strong law of large numbers

$$\frac{1}{n} \sum_{t=1}^{n} X_t \rightarrow \mu, \quad \frac{X^\top X}{n} \rightarrow \Delta, \quad \tilde{\lambda}_i \rightarrow \lambda_i (\Delta I_{p,q}).$$ 

as $n \rightarrow \infty$. In addition, when $\lambda_i (\Delta I_{p,q})$ is a simple eigenvalue, then $\lambda_i \rightarrow \xi_i$ as $n \rightarrow \infty$. We therefore have, when $\lambda_i (\Delta I_{p,q})$ is a simple eigenvalue, that

$$\mathbb{E}[Z] \rightarrow \frac{1}{\lambda_i (\Delta I_{p,q})} \mathbb{E}[\xi_i \Delta^{-1/2} XX^\top \Delta^{-1/2} \xi_i (X^\top I_{p,q} \mu - X^\top I_{p,q} \Delta I_{p,q} X)]$$

as desired.

**Proof of Eq. (3.3)** We recall that, conditional on $P$, $u_i^\top (A - P)u_i$ is a sum of mean zero random variables. Therefore, by the Lindeberg-Feller central limit theorem for triangular arrays, we have $\sigma_i^{-1} u_i^\top (A - P)u_i$ converges to standard normal; here $\sigma_i^2 = \text{Var}(u_i^\top (A - P)u_i).$ All that remains is to evaluate $\sigma_i^2$. Since $A - P$ is symmetric, we have

$$\sigma_i^2 = \text{Var} \left[ \sum_{k<l} (a_{kl} - p_{kl}) (u_{ik} u_{il} + u_{il} u_{ik}) + \sum_k (a_{kk} - p_{kk}) u_{ik}^2 \right]$$

$$= \sum_{k<l} 4u_{ik}^2 u_{il}^2 p_{kl} (1 - p_{kl}) + \sum_k p_{kk} (1 - p_{kk}) u_{ik}^4$$

$$= 2 \sum_k \sum_l u_{ik}^2 u_{il}^2 p_{kl} (1 - p_{kl}) - \sum_k p_{kk} (1 - p_{kk}) u_{ik}^4$$

$$= 2 \sum_k \sum_l (X_k^\top (X^\top X)^{-1/2} v_i)^2 (X_l^\top (X^\top X)^{-1/2} v_i)^2 X_k^\top I_{p,q} X_l (1 - X_k^\top I_{p,q} X_l) + o_P(1)$$

$$= 2 \sum_k (X_k^\top (X^\top X)^{-1/2} v_i)^2 X_k^\top I_{p,q} \left( \sum_l (X_l^\top (X^\top X)^{-1/2} v_i)^2 \right) + o_P(1)$$

$$- 2 \text{tr} \left( \sum_k (X_k^\top (X^\top X)^{-1/2} v_i)^2 X_k X_k^\top \right) I_{p,q} \left( \sum_l (X_l^\top (X^\top X)^{-1/2} v_i)^2 X_l X_l^\top \right) I_{p,q}$$
When $\lambda_i(I_{p,q}\Delta)$ is a simple eigenvalue, the strong law of large numbers and Slutsky’s theorem implies,

$$
\sigma_i^2 \to \left(\mathbb{E}[\xi_i^T \Delta^{-1/2}XX^T \Delta^{-1/2}\xi_iX]^T I_{p,q}\mathbb{E}[\xi_i^T \Delta^{-1/2}XX^T \Delta^{-1/2}\xi_iX]\right) - \text{tr}\left(\mathbb{E}[\xi_i^T \Delta^{-1/2}XX^T \Delta^{-1/2}\xi_iX]^T I_{p,q}\mathbb{E}[\xi_i^T \Delta^{-1/2}XX^T \Delta^{-1/2}\xi_iX]\right)_{I_{p,q}} = \Gamma_{ii}
$$

(3.18)

One last application Slutsky’s theorem yield $\frac{\lambda_i}{\lambda_i} u_i^T (A - P) u_i \overset{d}{\to} \mathcal{N}(0, \Gamma_{ii})$ as desired. Finally, we show that if the eigenvalues of $I_{p,q}\Delta$ are all simple eigenvalues, then $(\lambda_i - \lambda_i)_{i=1}^d \to \text{MVN}(\mu, \Gamma)$. More specifically, for any vector $s = (s_1, s_2, \ldots, s_d)$ in $\mathbb{R}^d$, we have

$$
\sum_i s_i(\hat{\lambda}_i - \lambda_i) = \sum_i \frac{s_i \lambda_i}{\lambda_i} u_i^T (A - P) u_i + \sum_i \frac{s_i \lambda_i}{\lambda_i^2} u_i^T (A - P)^2 u_i + o_P(1)
$$

$$
= \sum_i s_i u_i^T (A - P) u_i + \sum_i s_i \mu_i + o_P(1)
$$

$$
= \text{tr}\left((A - P)(\sum_i s_i u_i u_i^T)\right) + \sum_i s_i \mu_i + o_P(1)
$$

Now let $H = \sum_i s_i u_i u_i^T$. Then conditional on $P$, $\text{tr}\left((A - P)H\right)$ is once again a sum of independent mean 0 random variables. Letting $h_{ij}$ be the $ij$-th entry of $H$, we have

$$
\text{Var}\left(\text{tr}\left((A - P)H\right)\right) = 2 \sum_k \sum_l p_{kl}(1 - p_{kl})h_{kl}^2 + o_P(1)
$$

$$
= 2 \sum_k \sum_l p_{kl}(1 - p_{kl})\left(\sum_i s_i u_{ik} u_{il}\right)^2 + o_P(1)
$$

$$
= 2 \sum_i \sum_j s_i s_j \sum_k \sum_l p_{kl}(1 - p_{kl})u_{ik} u_{il} u_{jk} u_{jl} + o_P(1)
$$

which converges to $\sum_i \sum_j s_i s_j \Gamma_{ij}$ as $n \to \infty$ where $\Gamma_{ij}$ is as defined in Eq. (2.4). Thus for any $s$,

$$
\sum_i s_i(\hat{\lambda}_i - \lambda_i) \to \mathcal{N}(s^T \mu, s^T \Gamma^2 s)\] and hence by the Cramer-Wold device, $(\hat{\lambda}_i - \lambda_i)_{i=1}^d \to \text{MVN}(\mu, \Gamma)$.

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**Department of Applied Mathematics and Statistics, Johns Hopkins University,**

3400 N. CHARLES ST.

Baltimore, MD 21218, USA.

E-MAIL: minh@jhu.edu