PULLBACK ATTRACTORS FOR 2D NAVIER–STOKES EQUATIONS WITH DELAYS AND THE FLATTENING PROPERTY

Dedicated to Professor Tomás Caraballo on occasion of his Sixtieth Birthday

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Abstract. This paper treats the existence of pullback attractors for a 2D Navier–Stokes model with finite delay formulated in [Caraballo and Real, J. Differential Equations 205 (2004), 271–297]. Actually, we carry out our study under less restrictive assumptions than in the previous reference. More precisely, we remove a condition on square integrable control of the memory terms, which allows us to consider a bigger class of delay terms. Here we show that the asymptotic compactness of the corresponding processes required to establish the existence of pullback attractors, obtained in [García-Luengo, Marín-Rubio and Real, Adv. Nonlinear Stud. 13 (2013), 331–357] by using an energy method, can be also proved by verifying the flattening property – also known as “Condition (C)”. We deal with dynamical systems in suitable phase spaces within two metrics, the $L^2$ norm and the $H^1$ norm. Moreover, we provide results on the existence of pullback attractors for two possible choices of the attracted universes, namely, the standard one of fixed bounded sets, and secondly, one given by a tempered condition.

1. Introduction. The importance of physical models for fluid mechanic problems including delay terms is related, for instance, to real applications where devices to control properties of fluids (temperature, velocity, etc.) are inserted in domains and make a local influence on the behaviour of the system (e.g., cf. [24] for a wind-tunnel model).

The study of Navier–Stokes models including delay terms – existence, uniqueness, stationary solutions, exponential decay, and other asymptotic properties such as the existence of attractors – was initiated in the references [3, 4, 5], and after that, many different questions, as dealing with unbounded domains, and models (for instance, in three dimension for modified terms) have been addressed (e.g., cf. [17, 28, 32, 30, 25, 31, 18, 26, 27] among others).

Generally, in all finite delay frameworks the assumptions for the delay terms used to involve estimates in $L^2$ spaces, which in turn means some restrictive conditions on the operators and on the function driving the delayed time. However, as long as

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the solution for the problem (without delay) in dimension two is continuous in time, it seemed natural to develop a theory just considering a phase space only requiring continuity in time. In this sense, in [14] we treated a relaxation on the assumptions for the delay operator, removing conditions related to the control of the $L^2$ norm of the delay terms (e.g., cf. conditions (IV) and (V) in [3, 4, 5, 17]). Although this implies to restrict the phase space to continuous functions instead of square integrable in time, the delay functions driving the delayed time within this theory can be taken just measurable, without any additional assumption as continuity or $C^1$ with bounded derivative, as usually in the literature.

In this more general framework for the delay operators, in [14] we also established the existence of minimal pullback for the associated processes in both $L^2$ and $H^1$ norms, by proving the asymptotic compactness via an energy method (see also [15] for related results). In this sense, to verify asymptotic compactness one can either proceed directly, or make use of a splitting of the solutions into high and low components. Such a splitting is a very common technique in the study of the qualitative behaviour of solutions for PDE problems, in particular when considering the long-time behaviour of dynamics, as in the construction of invariant manifolds [8, 19] and inertial manifolds [12, 9], the squeezing property [10, 35], the notion of ‘determining modes’ [11, 20], and the theory of attractors [23]. In the context of proofs of the existence of attractors it was formalised by Ma, Wang, and Zhong [23] as their ‘Condition (C)’. A more descriptive terminology, ‘the flattening property’, was coined by Kloeden and Langa [21], and we adopt this terminology here. However, it is worth making the observation that this is not so much a ‘property’ as a (powerful) technique for obtaining the asymptotic compactness of a flow, be it autonomous or non-autonomous.

In this paper we are able to prove that the processes satisfy the flattening property in $L^2$ and $H^1$, and as a consequence we obtain the asymptotic compactness. It is worth pointing out that, while in the case of $L^2$ a direct proof of asymptotic compactness is no harder than a proof of the flattening property, in $H^1$ a proof of asymptotic compactness via the flattening property is significantly shorter than as provided in [14], that is based on an energy method. This is due to the fact that there are stronger estimates available for the nonlinear term in $H^1$ than in $L^2$ (see estimates (3) and (4) below).

The structure of the paper is as follows. Section 2 contains some preliminaries, including the functional setting of the problem. Section 3 is devoted to recalling standard results from the theory of pullback attractors (within the framework of time-dependent universes of sets), such as existence and comparison, and the flattening property in $C([-h,0]; Z)$, the space of continuous functions from $[-h,0]$ into $Z$, with $Z$ a Banach space. The analysis in the space $L^2$ is carried out in Section 4, where we conclude the existence of minimal pullback attractors for a universe not only of fixed bounded sets but also for a set of tempered universes. Finally, in Section 5 we establish additional attraction, namely, in the $H^1$ norm instead of $L^2$ as in Section 4. Different families of universes (tempered and non-tempered) are introduced and we obtain the flattening property in $H^1$, which implies the asymptotic compactness of the corresponding process in this norm. We finish with the existence of minimal pullback attractors and comparison among them under suitable additional assumptions.
2. **Statement of the problem.** Let $\Omega \subset \mathbb{R}^2$ be an open bounded set with smooth enough boundary $\partial \Omega$, and consider an arbitrary initial time $t \in \mathbb{R}$, and the following functional Navier-Stokes problem:

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p &= f(t) + g(t, u_t) \quad \text{in } \Omega \times (t, \infty), \\
\text{div } u &= 0 \quad \text{in } \Omega \times (t, \infty), \\
u u &= 0 \quad \text{on } \partial \Omega \times (t, \infty), \\
u u(x, \tau + s) &= \phi(x, s), \quad x \in \Omega, \ s \in [-h, 0],
\end{aligned}
\]

where we assume that $\nu > 0$ is the kinematic viscosity, $u = (u_1, u_2)$ is the velocity field of the fluid, $p$ is the pressure, $f$ is a non-delayed external force field, $g$ is another external force containing some hereditary characteristics, and $\phi(x, s - \tau)$ is the initial datum in the interval of time $[\tau - h, \tau]$, where $h > 0$ is the time of memory effect. For each $t \geq \tau$, we denote by $u_t$ the function defined on $[-h, 0]$ by the relation $u_t(s) = u(t + s), \ s \in [-h, 0]$.

To set our problem in the abstract framework, we consider the usual spaces in the variational theory of the Navier-Stokes equations:

$\mathcal{V} = \{ u \in (C_0^\infty(\Omega))^2 : \text{div } u = 0 \}$, $H = \text{the closure of } \mathcal{V} \text{ in } (L^2(\Omega))^2 \text{ with the norm } | \cdot |$, and inner product $(\cdot, \cdot)$, where for $u, v \in (L^2(\Omega))^2$,

\[(u, v) = \sum_{j=1}^{2} \int_{\Omega} u_j(x)v_j(x) \, dx,
\]

$V = \text{the closure of } \mathcal{V} \text{ in } (H_0^1(\Omega))^2 \text{ with the norm } \| \cdot \|$ associated to the inner product $(\cdot, \cdot)$, where for $u, v \in (H_0^1(\Omega))^2$,

\[(u, v) = \sum_{i,j=1}^{2} \int_{\Omega} \frac{\partial u_i}{\partial x_i} \frac{\partial v_j}{\partial x_j} \, dx.
\]

We will use $\| \cdot \|$ for the norm in $V'$ and $(\cdot, \cdot)$ for the duality product between $V'$ and $V$. We consider every element $h \in H$ as an element of $V'$, given by the equality $\langle h, v \rangle = (h, v)$ for all $v \in V$. It follows that $V \subset H \subset V'$, where the injections are dense and continuous, and, in fact, compact.

Define the operator $A : V \to V'$ as $\langle Au, v \rangle = ((u, v))$ for all $u, v \in V$. Let us denote $D(A) = \{ u \in V : Au \in H \}$. By the regularity of $\partial \Omega$, one has $D(A) = (H^2(\Omega))^2 \cap V$, and $Au = -P \Delta u$ for all $u \in D(A)$ is the Stokes operator ($P$ is the ortho-projector from $(L^2(\Omega))^2$ onto $H$). On $D(A)$ we consider the norm $| \cdot |_{D(A)}$ defined by $|u|_{D(A)} = |Au|$. Observe that on $D(A)$ the norms $\| \cdot \|_{(H^2(\Omega))^2}$ and $| \cdot |_{D(A)}$ are equivalent (see [7] or [34]), and $D(A)$ is compactly and densely injected in $V$.

By standard spectral theory, since it will be useful in the paper, let us denote by $\{w_j\}_{j \geq 1}$ a Hilbert basis of $H$ formed by ortho-normalized eigenfunctions of the Stokes operator $A$ with corresponding eigenvalues $\{\lambda_j\}_{j \geq 1}$ being $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ and $\lim_{j \to \infty} \lambda_j = \infty$.

Let us define

\[b(u, v, w) = \sum_{i,j=1}^{2} \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j \, dx,
\]

for every functions $u, v, w : \Omega \to \mathbb{R}^2$ for which the right-hand side is well defined.

In particular, $b$ has sense for all $u, v, w \in V$, and is a continuous trilinear form on $V \times V \times V$. 


Some useful properties concerning $b$ that we will use in the next sections are the following (see [33] or [35]):

\begin{align}
    b(u, v, w) &= -b(u, w, v) \quad \forall u, v, w \in V, \\
    b(u, v) &= 0 \quad \forall u, v \in V, \\
    |b(u, v, w)| &\leq 2^{-1/2}|u|^{1/2}|v|^{1/2}|w|^{1/2} \quad \forall u, v, w \in V,
\end{align}

and there exists a constant $C_1 > 0$, only dependent on $\Omega$, such that

\begin{equation}
    |b(u, v, w)| \leq C_1|u|^{1/2}|Au|^{1/2}|v| \quad \forall u \in D(A), v \in V, w \in H. \tag{4}
\end{equation}

Now, we establish some appropriate assumptions on the term in (1) containing the delay.

Let $Z$ be a Banach space. We denote by $C_Z = C([-h, 0]; Z)$ the space of continuous functions from $[-h, 0]$ into $Z$, with the norm

\[ |\varphi|_{C_Z} = \max_{s \in [-h, 0]} |\varphi(s)|_Z. \]

Let us consider over the delay operator from (1) that is well defined as $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$, and it satisfies the following assumptions:

(I) for all $\xi \in C_H$, the function $\mathbb{R} \ni t \mapsto g(t, \xi) \in (L^2(\Omega))^2$ is measurable,

(II) $g(t, 0) = 0$, for all $t \in \mathbb{R}$,

(III) there exists $L_g > 0$ such that for all $t \in \mathbb{R}$, and for all $\xi, \eta \in C_H$,

\[ |g(t, \xi) - g(t, \eta)| \leq L_g|\xi - \eta|_{C_H}. \]

Observe that (I) – (III) imply that given $T > \tau$ and $u \in C([\tau - h, T]; H)$, the function $g_u : [\tau, T] \to (L^2(\Omega))^2$ defined by $g_u(t) = g(t, u_t)$ for all $t \in [\tau, T]$, is measurable and, in fact, belongs to $L^\infty(\tau, T; (L^2(\Omega))^2)$.

It is worth pointing out that no condition involving $L^2$ norms of the memory term in $g$ is assumed (in opposition to conditions (IV) and (V) in [3, 4, 5, 17]).

Assume that $\phi \in C_H$, and $f \in L^2_{loc}(\mathbb{R}; V')$.

**Definition 1.** A weak solution of (1) is a function $u \in C([\tau - h, \infty); H)$ such that $u \in L^2(\tau, T; V)$ for all $T > \tau$, with $u(t) = \phi(t - \tau)$ for all $t \in [\tau - h, \tau]$, and such that for all $v \in V$,

\begin{equation}
    \frac{d}{dt}(u(t), v) + \nu(Au(t), v) + b(u(t), u(t), v) = \langle f(t), v \rangle + g(t, u_t, v), \tag{5}
\end{equation}

where the equation must be understood in the sense of $\mathcal{D}'(\tau, \infty)$.

**Remark 1.** If $u$ is a weak solution of (1), then from (5) we deduce that for any $T > \tau$, one has $u' \in L^2(\tau, T; V')$, and the following energy equality holds:

\[ |u(t)|^2 + 2\int_s^t \|u(r)\|^2 \, dr = |u(s)|^2 + 2\int_s^t \left[ \langle f(r), u(r) \rangle + \langle g(r, u_r), u(r) \rangle \right] \, dr \forall \tau \leq s \leq t. \]

A notion of more regular solution is also suitable for problem (1).

**Definition 2.** A strong solution of (1) is a weak solution $u$ of (1) such that $u \in L^2(\tau, T; D(A)) \cap L^\infty(\tau, T; V)$ for all $T > \tau$.

**Remark 2.** If $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^2)$ and $u$ is a strong solution of (1), then $u' \in L^2(\tau, T; H)$ for all $T > \tau$, and so $u \in C([\tau, \infty); V)$. In this case the following energy equality holds:
3. Abstract results on minimal pullback attractors. Pullback $\hat{D}_0$-flattening property. Now, we present a summary of some results from [13] about the existence of minimal pullback attractors (see also [1, 2, 29]). In particular, we assume that the process $U$ is closed (see Definition 3 below).

Consider given a metric space $(X,d_X)$, and let us denote $\mathbb{R}^2_\tau = \{(t,\tau) \in \mathbb{R}^2 : \tau \leq t\}$. A process $U$ on $X$ is a mapping $\mathbb{R}^2_\tau \times X \ni (t,\tau,x) \mapsto U(t,\tau)x \in X$ such that $U(t,\tau)x = x$ for any $(t,\tau) \in \mathbb{R} \times X$, and $U(t,\tau)x = U(t,\tau)x$ for any $\tau \leq t \leq t$ and all $x \in X$.

**Definition 3.** Let $U$ be a process on $X$.

(a) $U$ is said to be continuous if for any pair $\tau \leq t$, the mapping $U(t,\tau) : X \to X$ is continuous.

(b) $U$ is said to be closed if for any $\tau \leq t$, and any sequence $\{x_n\} \subset X$, if $x_n \to x \in X$ and $U(t,\tau)x_n \to y \in X$, then $U(t,\tau)x = y$.

**Remark 3.** It is clear that every continuous process is closed.

Let us denote by $\mathcal{P}(X)$ the family of all nonempty subsets of $X$, and consider a family of nonempty sets $\hat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$.

**Definition 4.** We say that a process $U$ on $X$ is pullback $\hat{D}_0$-asymptotically compact if for any $t \in \mathbb{R}$ and any sequences $\{\tau_n\} \subset (-\infty,t]$ and $\{x_n\} \subset X$ satisfying $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all $n$, the sequence $\{U(t,\tau_n)x_n\}$ is relatively compact in $X$.

Denote

$$\Lambda(\hat{D}_0,t) = \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} U(t,\tau)D_0(\tau)} \quad \forall t \in \mathbb{R},$$

where $\overline{\cdots}^X$ is the closure in $X$. We denote by $\text{dist}_X(O_1,O_2)$ the Hausdorff semi-distance in $X$ between two sets $O_1$ and $O_2$, defined as

$$\text{dist}_X(O_1,O_2) = \sup_{x \in O_1} \inf_{y \in O_2} d_X(x,y) \quad \text{for } O_1, O_2 \subset X.$$

Let $\mathcal{D}$ be a nonempty class of families parameterized in time $\hat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$. The class $\mathcal{D}$ will be called a universe in $\mathcal{P}(X)$.
Definition 5. A process $U$ on $X$ is said to be pullback $\mathcal{D}$-asymptotically compact if it is pullback $\mathcal{D}$-asymptotically compact for any $\hat{D} \in \mathcal{D}$.

It is said that $\mathcal{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is pullback $\mathcal{D}$-absorbing for the process $U$ on $X$ if for any $t \in \mathbb{R}$ and any $\hat{D} \in \mathcal{D}$, there exists a $\tau_0(t, \hat{D}) \leq t$ such that

$$U(t, \tau)D(\tau) \subset D_0(t) \quad \forall \tau \leq \tau_0(t, \hat{D}).$$

Remark 4. Observe that if $\mathcal{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is pullback $\mathcal{D}$-absorbing for the process $U$ on $X$, and $U$ is pullback $\mathcal{D}_0$-asymptotically compact, then $U$ is also $\mathcal{D}$-asymptotically compact.

With the above definitions, we may establish the main result of this section (cf. [13, Theorem 3.11]).

Theorem 2. Consider a closed process $U : \mathbb{R}_+^2 \times X \to X$, a universe $\mathcal{D}$ in $\mathcal{P}(X)$, and a family $\mathcal{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ which is pullback $\mathcal{D}$-absorbing for $U$, and assume also that $U$ is pullback $\mathcal{D}_0$-asymptotically compact.

Then, the family $\mathcal{A}_\mathcal{D} = \{A_\mathcal{D}(t) : t \in \mathbb{R}\}$ defined by

$$A_\mathcal{D}(t) = \bigcup_{\hat{D} \in \mathcal{D}} \Lambda(\hat{D}, t)^X$$

has the following properties:

(a) for any $t \in \mathbb{R}$, the set $A_\mathcal{D}(t)$ is a nonempty compact subset of $X$, and $A_\mathcal{D}(t) \subset \Lambda(\mathcal{D}_0, t)$,

(b) $A_\mathcal{D}$ is pullback $\mathcal{D}$-attracting, i.e., $\lim_{r \to -\infty} \text{dist}_X(U(t, \tau)D(\tau), A_\mathcal{D}(t)) = 0$ for all $\hat{D} \in \mathcal{D}$, and any $t \in \mathbb{R}$,

(c) $A_\mathcal{D}$ is invariant, i.e., $U(t, \tau)A_\mathcal{D}(\tau) = A_\mathcal{D}(t)$ for all $(t, \tau) \in \mathbb{R}^2$,

(d) if $\mathcal{D}_0 \in \mathcal{D}$, then $A_\mathcal{D}(t) = \Lambda(\mathcal{D}_0, t) \subset D_0(t)$ for all $t \in \mathbb{R}$.

The family $A_\mathcal{D}$ is minimal in the sense that if $\mathcal{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is another family of closed sets such that for any $\hat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$

$$\lim_{\tau \to -\infty} \text{dist}_X(U(t, \tau)D(\tau), C(t)) = 0,$$

then $A_\mathcal{D}(t) \subset C(t)$.

Remark 5. Under the assumptions of Theorem 2, the family $A_\mathcal{D}$ is called the minimal pullback $\mathcal{D}$-attractor for the process $U$.

If $A_\mathcal{D} \in \mathcal{D}$, then it is the unique family of closed subsets in $\mathcal{D}$ that satisfies (b)-(c).

A sufficient condition for $A_\mathcal{D} \in \mathcal{D}$ is to have that $\mathcal{D}_0 \in \mathcal{D}$, the set $D_0(t)$ is closed for all $t \in \mathbb{R}$, and the family $\mathcal{D}$ is inclusion-closed (i.e., if $\hat{D} \in \mathcal{D}$, and $\hat{D}' = \{D'(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ with $D'(t) \subset D(t)$ for all $t$, then $\hat{D}' \in \mathcal{D}$).

We will denote by $\mathcal{D}_F(X)$ the universe of fixed nonempty bounded subsets of $X$, i.e., the class of all families $\hat{D}$ of the form $\hat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with $D$ a fixed nonempty bounded subset of $X$.

Now, it is easy to conclude the following result.

Corollary 1. Under the assumptions of Theorem 2, if the universe $\mathcal{D}$ contains the universe $\mathcal{D}_F(X)$, then both attractors, $A_{\mathcal{D}_F(X)}$ and $A_\mathcal{D}$, exist, and the following relation holds:

$$A_{\mathcal{D}_F(X)}(t) \subset A_\mathcal{D}(t) \quad \forall t \in \mathbb{R}.$$
Remark 6. It can be proved (see [29]) that, under the assumptions of the preceding corollary, if for some \( T \in \mathbb{R} \), the set \( \bigcup_{t \leq T} D_0(t) \) is a bounded subset of \( X \), then

\[
A_{D_0(X)}(t) = A_D(t) \quad \forall t \leq T.
\]

Now, we introduce a notion which is a slight modification of Ma, Wang, and Zhong’s “Condition (C)” [23] (renamed the “flattening property” by Kloeden and Langa [21]), after Definition 2.24 in the book by Carvalho, Langa, and Robinson [6], where \( P_\varepsilon \) need not be a projection operator.

Definition 6. Assume that \( Z \) is a Banach space and that a process \( U \) on \( C_Z \) is well-defined. Let be \( \tilde{D}_0 = \{ D_0(t) : t \in \mathbb{R} \} \subset \mathcal{P}(C_Z) \) a given family. We will say that the process \( U \) on \( C_Z \) satisfies the pullback \( \tilde{D}_0 \)-flattening property if for any \( t \in \mathbb{R} \), and \( \varepsilon > 0 \), there exist \( \tau_\varepsilon < t \), a finite dimensional subspace \( Z_{\varepsilon} \) of \( Z \), and a continuous mapping \( P_\varepsilon : Z \to Z_{\varepsilon} \), all depending on \( \tilde{D}_0 \), \( t \) and \( \varepsilon \), such that

\[
\{ P_\varepsilon U(t, \tau) \phi : \tau \leq \tau_\varepsilon, \phi \in D_0(\tau) \}
\]

is bounded in \( C_Z \), and equi-continuous on \([-h, 0]\) with values in \( Z \), and

\[
| (I - P_\varepsilon) U(t, \tau) \phi |_{C_Z} < \varepsilon \quad \text{for any } \tau \leq \tau_\varepsilon, \phi \in D_0(\tau),
\]

where by notational abuse \( (P_\varepsilon U(t, \tau) \phi)(\theta) = P_\varepsilon((U(t, \tau) \phi)(\theta)) \).

Similarly to the results in [23], [21] and [16] (see also [6]), we will see that in order to show that a process \( U \) is pullback \( \tilde{D}_0 \)-asymptotically compact, it is enough to verify the pullback \( \tilde{D}_0 \)-flattening property given in the definition above.

Proposition 1. Assume that \( Z \) is a Banach space, \( U \) is a well-defined process on \( C_Z \), and \( \tilde{D}_0 = \{ D_0(t) : t \in \mathbb{R} \} \subset \mathcal{P}(C_Z) \) is a given family such that \( U \) satisfies the pullback \( \tilde{D}_0 \)-flattening property. Then, the process \( U \) is pullback \( \tilde{D}_0 \)-asymptotically compact.

Proof. Let be fixed \( t \in \mathbb{R} \), a sequence \( \{ \tau_n \} \subset (-\infty, t] \) such that \( \tau_n \to -\infty \), and a sequence \( \{ \phi_n \} \subset C_Z \) such that \( \phi_n \in D_0(\tau_n) \) for all \( n \), be fixed. We must show that \( \{ U(t, \tau_n) \phi_n \} \) is relatively compact in \( C_Z \).

For a fixed integer \( k \geq 1 \), by the pullback \( \tilde{D}_0 \)-flattening property, there exist \( N_k \geq 1 \), a finite dimensional subspace \( Z_k \) of \( Z \), and a continuous mapping \( P_k : Z \to Z_k \), such that \( \{ P_k U(t, \tau_n) \phi_n : n \geq N_k \} \) is a bounded and equi-continuous subset of \( C_{Z_k} \), and therefore a relatively compact subset of \( C_Z \), and \( | (I - P_k) U(t, \tau_n) \phi_n |_{C_Z} \leq 1/(2k) \) for all \( n \geq N_k \). Thus, \( \{ U(t, \tau_n) \phi_n : n \geq 1 \} \) can be covered by a finite number of balls in \( C_Z \) of radius \( 1/k \). As \( k \) is arbitrary, it is not difficult to check that \( \{ U(t, \tau_n) \phi_n : n \geq 1 \} \) possesses a Cauchy subsequence in \( C_Z \). Since \( C_Z \) is complete, this subsequence is convergent, whence \( \{ U(t, \tau_n) \phi_n : n \geq 1 \} \) is relatively compact in \( C_Z \).

Finally, we recall an abstract result that allows us to compare two attractors for a process under appropriate assumptions (see [13, Theorem 3.15]).

Theorem 3. Let \( \{(X_i, d_{X_i})\}_{i=1,2} \) be two metric spaces such that \( X_1 \subset X_2 \) with continuous injection, and for \( i = 1, 2 \), let \( D_i \) be a universe in \( \mathcal{P}(X_i) \), with \( D_1 \subset D_2 \). Assume that we have a map \( U \) that acts as a process in both cases, i.e., \( U : \mathbb{R}_+^2 \times X_1 \to X_i \) for \( i = 1, 2 \) is a process.
For each $t \in \mathbb{R}$, let us denote

$$A_i(t) = \bigcup_{\hat{D}_i \in \mathcal{D}_i} \Lambda_i(\hat{D}_i, t)^{X_i} \quad i = 1, 2,$$

where the subscript $i$ in the symbol of the omega-limit set $\Lambda_i$ is used to denote the dependence of the respective topology.

Then, $A_1(t) \subset A_2(t)$ for all $t \in \mathbb{R}$.

If in addition

(i) $A_1(t)$ is a compact subset of $X_1$ for all $t \in \mathbb{R}$, and

(ii) for any $\hat{D}_2 \in \mathcal{D}_2$ and any $t \in \mathbb{R}$, there exist a family $\hat{D}_1 \in \mathcal{D}_1$ and a $t_{\hat{D}_1}^* \leq t$ (both possibly depending on $t$ and $\hat{D}_2$), such that $U$ is pullback $\hat{D}_1$-asymptotically compact, and for any $s \leq t_{\hat{D}_1}^*$ there exists a $\tau_s \leq s$ such that $U(s, \tau)D_2(\tau) \subset D_1(s)$ for all $\tau \leq \tau_s$,

then $A_1(t) = A_2(t)$ for all $t \in \mathbb{R}$.

4. Existence of minimal pullback attractors in $H$ norm. The goal of this section is to establish the existence of minimal pullback attractors for a suitable process $U$ on $C_H$ associated to problem (1). This result was obtained in [14] by applying an energy method. Now, we will make use of the pullback flattening property. As pointed out in the Introduction, in the phase space $C_H$, the proofs of the flattening property and the asymptotic compactness of this process are in fact very similar and no extra benefit is appreciated, in contrast with Section 5 as shown below for the $C_V$-norm.

We first define the following process $U$ on $C_H$ associated to (1) (cf. [14, Proposition 4.1]).

**Proposition 2.** Let $f \in L^2_{loc}(\mathbb{R}; V')$, and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ satisfying (I)–(III), be given. Then, the bi-parametric family of maps $U(t, \tau) : C_H \to C_H$, with $\tau \leq t$, given by

$$U(t, \tau)\phi = u_t,$$

where $u = u(\cdot; \tau, \phi)$ is the unique weak solution of (1), defines a continuous process on $C_H$.

Concerning the existence of a pullback absorbing family for the process $U$ on $C_H$ defined above, we have the following results which were also established in [14]. We recall them for the sake of clarity.

**Lemma 1.** Under the assumptions of Proposition 2, consider $\mu \in (0, 2\nu \lambda_1)$ and $\phi \in C_H$. Then the following estimates hold for the solution to (1) for all $t \geq \tau$

$$|u_t|_{C_H}^2 \leq e^{\mu h} e^{-(\mu - 2e^{\mu h} L_g)(t-\tau)}|\phi|_{C_H}^2$$

$$+ e^{\mu h} (2\nu - \mu \lambda_1^{-1})^{-1} \int_\tau^t e^{-(\mu - 2e^{\mu h} L_g)(t-s)}\|f(s)\|_s^2 \, ds, \quad (8)$$

$$\nu \int_\tau^t \|u(s)\|^2 \, ds \leq |u(\tau)|^2 + \nu^{-1} \int_\tau^t \|f(s)\|_s^2 \, ds + 2L_g \int_\tau^t |u(s)|_{C_H}^2 \, ds. \quad (9)$$

From now on we will assume that

$$\text{there exists } 0 < \mu < 2\nu \lambda_1 \text{ such that } 2e^{\mu h} L_g < \mu, \quad (10)$$
and

\[ \int_{-\infty}^{0} e^{(\mu - 2e^{\mu h} L_g) s} \|f(s)\|^2 \, ds < \infty. \] (11)

**Remark 7.** The above conditions will be the key for the uniform asymptotic estimates in what follows. Observe that when \( L_g << 1 \) these are essentially the same assumptions as in the case without delay (i.e. \( g \equiv 0 \)). In the current case when \( g \) exists and therefore \( L_g > 0 \), the above relation among the elements \( h, \lambda_1, L_g, \nu \) and the integrability condition on \( f \) indicate some balance such that the resulting system will be dissipative.

Once the estimate (8) has been obtained, we introduce the following tempered universe in \( P(C_H) \).

**Definition 7.** For any \( \sigma > 0 \) we will denote by \( \mathcal{D}_\sigma(C_H) \) the class of all families of nonempty subsets \( \hat{D} = \{D(t) : t \in \mathbb{R}\} \subset P(C_H) \) such that

\[ \lim_{\tau \to -\infty} \left( e^{\sigma \tau} \sup_{v \in D(\tau)} |v|^2 \right) = 0. \]

Accordingly to the notation introduced in the previous section, \( \mathcal{D}_F(C_H) \) will denote the class of families \( \hat{D} = \{D(t) = D : t \in \mathbb{R}\} \) with \( D \) a fixed nonempty bounded subset of \( C_H \).

Observe that for any \( \sigma > 0 \), \( \mathcal{D}_F(C_H) \subset \mathcal{D}_\sigma(C_H) \) and that the universe \( \mathcal{D}_\sigma(C_H) \) is inclusion-closed.

For short, we will denote

\[ \sigma = \mu - 2e^{\mu h} L_g. \] (12)

**Corollary 2.** Under the assumptions of Proposition 2, if moreover conditions (10) and (11) are satisfied, then the family \( \hat{D}_{0,\mu} = \{D_{0,\mu}(t) : t \in \mathbb{R}\} \), with \( D_{0,\mu}(t) = \mathcal{B}_{C_H}(0, \rho_{\mu}(t)) \), the closed ball in \( C_H \) of center zero and radius \( \rho_{\mu}(t) \), where

\[ \rho_{\mu}^2(t) = 1 + e^{\mu h} (2\nu - \mu \lambda_1^{-1})^{-1} \int_{-\infty}^{t} e^{-\sigma (t-s)} \|f(s)\|^2 \, ds, \]

is pullback \( \mathcal{D}_{\sigma}(C_H) \)-absorbing for the process \( U \) defined by (7). Moreover, \( \hat{D}_{0,\mu} \in \mathcal{D}_{\sigma}(C_H) \).

Now, we establish several estimates for the process \( U \) when the initial time is sufficiently shifted in a pullback sense, that will be used in order to prove the pullback flattening property.

**Lemma 2.** Under the assumptions of Corollary 2, for any \( t \in \mathbb{R} \) and \( \hat{D} \in \mathcal{D}_{\sigma}(C_H) \), there exist \( \tau \leq \tau_1(\hat{D}, t, h) < t - 2h \) and functions \( \{\rho_i\}_{i=1}^{2} \) depending on \( t \) and \( h \), such that for any \( \tau \leq \tau_1(\hat{D}, t, h) \) and any \( \phi \in D(\tau) \),

\[ |u(r; \tau, \phi^\tau)|^2 \leq \rho_1^2(t) \quad \forall r \in [t - 2h, t], \] (13)

\[ \nu \int_{t-h}^{t} \|u(\theta; \tau, \phi^\tau)\|^2 \, d\theta \leq \rho_2^2(t), \] (14)

\[ \int_{t-h}^{t} \|u(\theta; \tau, \phi^\tau)\|^2 \, d\theta \leq \rho_3^2(t), \] (15)
where
\[
\rho_1^2(t) = 1 + e^{\mu h}(2\nu - \mu \lambda_{1}^{-1})^{-1}e^{-\sigma_{\nu}(t-2h)}\int_{-\infty}^{t} e^{\sigma_{\nu} \theta} ||f(\theta)||^2_* d\theta,
\]
\[
\rho_2^2(t) = (1 + 2L_{g} h)\rho_1^2(t) + \nu^{-1} \int_{t-h}^{t} ||f(\theta)||^2_* d\theta,
\]
\[
\rho_3^2(t) = (4\nu + 2\nu^{-1} \rho_1^2(t)) \rho_2^2(t) + 4 \int_{t-h}^{t} ||f(\theta)||^2_* d\theta + 4L_{g}^2 \lambda_{1}^{-2} h \rho_3^2(t).
\]

**Proof.** The first two estimates (13) and (14) follow directly from (8) and (9).

Now, from (2), (3), (5), and the fact that $A$ is an isometric isomorphism, we obtain
\[
||u'(\theta)||_* \leq \nu ||u(\theta)|| + 2^{-1/2} ||u(\theta)|| ||u(\theta)|| + ||f(\theta)||_* + \lambda_{1}^{-1/2} |g(\theta, u_\theta)|, \quad \text{a.e. } \theta > \tau,
\]
which combined with properties (II) and (III) of $g$, implies that
\[
\int_{t-h}^{t} ||u(\theta)||^2_* d\theta \leq 4\nu^2 \int_{t-h}^{t} ||u(\theta)||^2 d\theta + 2 \int_{t-h}^{t} |u(\theta)|^2 ||u(\theta)|| d\theta
\]
\[
+ 4 \int_{t-h}^{t} ||f(\theta)||^2_* d\theta + 4L_{g}^2 \lambda_{1}^{-1} \int_{t-h}^{t} |u_\theta|^2_{C,h} d\theta
\]
for all $\tau \leq t - h$.

Thus, from this last inequality we can conclude the third estimate (15). \qed

In order to prove the pullback flattening property for the process $U$ on $C_H$, we need the following auxiliary result, which can be obtained from [14, Proposition 4.2], taking into account Remark 4.

**Lemma 3.** Under the assumptions of Corollary 2, the process $U$ on $C_H$ defined by (7) is pullback $\mathcal{D}_{\sigma_{\nu}}(C_H)$-asymptotically compact, i.e., for any $t \in \mathbb{R}$, $\hat{D} \in \mathcal{D}_{\sigma_{\nu}}(C_H)$, and sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{\phi^{\tau_n}\} \subset C_H$ such that $\tau_n \rightarrow -\infty$ and $\phi^{\tau_n} \in D(\tau_n)$ for all $n$, the sequence $\{u(\cdot; \tau_n, \phi^{\tau_n})\}$ is relatively compact in $C([t-h, t]; H)$.

**Remark 8.** While in the non-delayed case (see [16]), since the process is defined on the Hilbert space $H$, the pullback asymptotic compactness guarantees the pullback flattening property (it is known that both are equivalent in any uniformly convex Banach space, see [6]), it does not happen in the delayed case, where the phase space is $C_H$.

We also need the next corollary of Lemma 3, that can be proved similarly to [16, Corollary 20].

**Corollary 3.** Under the assumptions of Corollary 2, for any $\varepsilon > 0$, $t \in \mathbb{R}$, and $\hat{D} \in \mathcal{D}_{\sigma_{\nu}}(C_H)$, there exists $\delta = \delta(\varepsilon, t, \hat{D}) \in (0, h)$, such that
\[
\nu^{-1} \left|\left|u(\xi; \tau, \phi^{\tau})\right|^2 - |u(\xi - s; \tau, \phi^{\tau})|^2 \right| < \varepsilon/3
\]
\[
\forall \xi \in [t-h, t], s \in [0, \delta], \tau \leq \tau_1(\hat{D}, t, h), \phi^{\tau} \in D(\tau),
\]
where $\tau_1(\hat{D}, t, h)$ is given in Lemma 2.

In particular,
\[
\int_{\xi - \delta}^{\xi} ||u(\theta; \tau, \phi^{\tau})||^2 d\theta < \varepsilon \quad \forall \xi \in [t-h, t], \tau \leq \tau_1(\hat{D}, t, h), \phi^{\tau} \in D(\tau).
\]
Proof. Consider \( t \in \mathbb{R} \) and \( \hat{D} \in \mathcal{D}_{\sigma}(C_H) \). For simplicity we split the proof in two parts.

Step 1. It holds that
\[
\frac{1}{\nu} \|u(t, \tau, \phi^\tau)\|^2 - \|u(t - s, \tau, \phi^\tau)\|^2 < \frac{\varepsilon}{3} \quad \forall s \in [0, \delta], \quad \tau \leq \tau_1(\hat{D}, t, h), \quad \phi^\tau \in D(\tau),
\]
(18)
\[
\int_{t-\delta}^t \|u(\theta; \tau, \phi^\tau)\|^2 d\theta < \varepsilon \quad \forall \tau \leq \tau_1(\hat{D}, t, h), \quad \phi^\tau \in D(\tau).
\]
(19)
Indeed reasoning as for the obtention of (9), we observe that for any \( \delta \in (0, h) \), \( \tau \leq t - h \), and \( \phi^\tau \in D(\tau) \),
\[
\nu \int_{t-\delta}^t \|u(\theta; \tau, \phi^\tau)\|^2 d\theta \leq \|u(t - \delta; \tau, \phi^\tau)\|^2 - \|u(t; \tau, \phi^\tau)\|^2 + \nu^{-1} \int_{t-\delta}^t \|f(\theta)\|^2 d\theta + 2L_g \int_{t-\delta}^t |u_\theta|^2 d\theta.
\]
Therefore, taking into account that \( f \in L^2_{\text{loc}}(\mathbb{R}; V') \), and the estimate (13), property (19) is certainly a consequence of (18).

We prove now (18) by a contradiction argument. If (18) were not true, there would exist \( \varepsilon > 0 \), \( t \in \mathbb{R} \), a family \( \hat{D} \in \mathcal{D}_{\sigma}(C_H) \), and sequences \( \{\tau_n\} \subset (-\infty, t] \) with \( \tau_n \to -\infty \), \( \{s_n\} \) with \( 0 \leq s_n \leq h/n \), and \( \{\phi^{\tau_n}\} \) with \( \phi^{\tau_n} \in D(\tau_n) \) for all \( n \), such that
\[
\nu^{-1} \|u(t; \tau_n, \phi^{\tau_n})\|^2 - \|u(t - s_n; \tau_n, \phi^{\tau_n})\|^2 \geq \frac{\varepsilon}{3} \quad \forall n \geq 1,
\]
which is in contradiction with Lemma 3.

Step 2. The general statements (17) and (16) hold. Namely we just extend to the whole interval \([t - h, t]\) the previously proved inequalities (19) and (18) using again the estimate (13) and uniform continuity arguments and the compactness of \([t - h, t]\).

We will also use the following result, whose proof is analogous to that of [22, Lemma 12].

Lemma 4. If \( f \in L^2_{\text{loc}}(\mathbb{R}; V') \) satisfies conditions (10) and (11), then, for any \( t \in \mathbb{R} \),
\[
\lim_{\rho \to \infty} e^{-\rho t} \int_{-\infty}^t e^{\rho s} \|f(s)\|^2 ds = 0.
\]

Proposition 3. Under the assumptions of Corollary 2, for any \( \varepsilon > 0 \), \( t \in \mathbb{R} \), and \( \hat{D} \in \mathcal{D}_{\sigma}(C_H) \), there exists \( m = m(\varepsilon, t, \hat{D}) \in \mathbb{N} \) such that the projection \( P_m : H \mapsto \text{span}[w_1, \ldots, w_m] \) satisfies
\[
\{P_m U(t, \tau)D(\tau) : \tau \leq \tau_1(\hat{D}, t, h)\} \text{ is bounded in } C_H,
\]
and equi-continuous on \([-h, 0]\) with values in \( H \),
\[
\| (I - P_m) U(t, \tau) \phi^\tau \|_{C_H} < \varepsilon \quad \text{for any } \tau \leq \tau_1(\hat{D}, t, h), \quad \phi^\tau \in D(\tau),
\]
(20)
where \( \tau_1(\hat{D}, t, h) \) is given in Lemma 2.

In particular, the process \( U \) on \( C_H \) satisfies the pullback \( \hat{D} \)-flattening property for any \( \hat{D} \in \mathcal{D}_{\sigma}(C_H) \).
Proof. Let $\varepsilon > 0$, $t \in \mathbb{R}$, and $\mathcal{D} \in \mathcal{D}_\sigma(C_H)$ be fixed.

Since $P_m$ is non-expansive, from (13) we deduce the boundedness in $C_H$ of the set $\{P_m U(t, \tau)|D(\tau) : \tau \leq \tau_1(\mathcal{D}, t, h)\}$, for all $m \geq 1$.

Now, let us fix $\tau \leq \tau_1(\mathcal{D}, t, h)$, $\phi^* \in D(\tau)$, and $\theta_1, \theta_2 \in [-h, 0]$, with $\theta_2 > \theta_1$. Define $u(r) = u(r; \tau, \phi^*)$ and $q_m(r) = u(r) - P_m u(r)$. Observe that by the choice of the basis $\{w_j\}_{j \geq 1}$ and (15), for any $m \geq 1$ we have

$$
\begin{align*}
&\left|(P_m U(t, \tau)\phi^*)(\theta_2) - (P_m U(t, \tau)\phi^*)(\theta_1)\right| \\
\leq &\lambda_m^{1/2}||P_m U(t, \tau)\phi^*)(\theta_2) - (P_m U(t, \tau)\phi^*)(\theta_1)||_s \\
\leq &\lambda_m^{1/2}||u(t + \theta_2) - u(t + \theta_1)||_s \\
\leq &\lambda_m^{1/2}\int_{t + \theta_1}^{t + \theta_2} ||u'(s)||_s \, ds \\
\leq &\lambda_m^{1/2}|\theta_2 - \theta_1|^{1/2} \rho_1(t).
\end{align*}
$$

From this we deduce that the set $\{P_m U(t, \tau)|D(\tau) : \tau \leq \tau_1(\mathcal{D}, t, h)\}$ is equi-continuous on $[-h, 0]$ with values in $H$, for all $m \geq 1$.

Now we check (20). For simplicity we start verifying it for the final time $t$, i.e. given $\tau \leq \tau_1(\mathcal{D}, t, h)$ and $\phi^* \in D(\tau)$, $\|[I - P_m U(t, \tau)\phi^*]|0\| = |q_m(t)| < \varepsilon$.

Indeed, using the energy equality, for each $m \geq 1$ one has

$$
\begin{align*}
\frac{1}{d} \frac{d}{d\tau} |q_m(r)|^2 + \nu |q_m(r)|^2 &\leq b(u(r), u(r), q_m(r)) \\
&= \langle f(r), q_m(r) \rangle + \langle g(r, u_r), q_m(r) \rangle \\
&\leq 2\nu \|f(r)\|_s^2 + 2\nu \|q_m(r)\|_s^2 + \frac{2t^2}{\nu \lambda_1} |u_r|_{C^2_H}^2 + \frac{\nu}{4} \|q_m(r)\|_s^2, \quad \text{a.e. } r > \tau
\end{align*}
$$

where we have used Young’s inequality and the assumptions (II) and (III) of $g$.

Observing that by (2), (3), and Young’s inequality,

$$
\begin{align*}
|b(u(r), u(r), q_m(r))| &\leq 2^{1/2}|u(r)||u(r)||q_m(r)| \\
&\leq \frac{1}{2\nu}|u(r)|^2 \|u(r)\|^2 + \frac{\nu}{4} \|q_m(r)\|^2,
\end{align*}
$$

we obtain a.e. $r > \tau$

$$
\frac{d}{d\tau} |q_m(r)|^2 + \nu |q_m(r)|^2 \leq 4\nu^{-1}\|f(r)\|_s^2 + 4L_2^2(\nu \lambda_1)^{-1} |u_r|_{C^2_H}^2 + \nu^{-1} \|u(r)\|^2 \|u(r)\|^2.
$$

Therefore, since $\|q_m(r)\|^2 \geq \lambda_{m+1}|q_m(r)|^2$, we deduce that a.e. $r > \tau$

$$
\frac{d}{d\tau} |q_m(r)|^2 + \nu \lambda_{m+1} |q_m(r)|^2 \leq 4\nu^{-1}\|f(r)\|_s^2 + 4L_2^2(\nu \lambda_1)^{-1} |u_r|_{C^2_H}^2 + \nu^{-1} |u(r)|^2 \|u(r)\|^2
$$

Thus, multiplying this last inequality by $e^{\nu \lambda_{m+1} t}$, integrating in $[t - h, t]$, and again taking into account (13), we obtain (after multiplying by $e^{-\nu \lambda_{m+1} t}$)

$$
\begin{align*}
|q_m(t)|^2 &\leq e^{-\nu \lambda_{m+1} h} |q_m(t - h)|^2 + \frac{\rho_1^2(t)}{\nu} e^{-\nu \lambda_{m+1} t} \int_{t-h}^t e^{\nu \lambda_{m+1} r} \|u(r)\|^2 \, dr \\
&\quad + \frac{4}{\nu} e^{-\nu \lambda_{m+1} t} \int_{t-h}^t e^{\nu \lambda_{m+1} r} \|f(r)\|^2 \, dr + \frac{4L_2^2 \rho_1^2(t)}{\nu^2 \lambda_1 \lambda_{m+1}}.
\end{align*}
$$

The desired inequality $|q_m(t)| < \varepsilon$ holds if we estimate by $\varepsilon/4$ the four terms in the RHS in (21). Let us start with the second one, the most complicated. We claim that
there exists \( m = m(\varepsilon, t, \hat{D}) \in \mathbb{N} \) such that for any \( \tau \leq \tau_1(\hat{D} , t, h) \) and \( \phi^\tau \in D(\tau) \),
\[
e^{-\nu \lambda_{m+1} t} \int_{t-h}^{t} e^{\nu \lambda_{m+1} r} \| u(r, \tau, \phi^\tau) \|^2 \, dr < \frac{\varepsilon}{4 p^2(t)}.
\] (22)

Take \( \delta = \delta(\frac{\varepsilon}{8 p^2(t)}, t, \hat{D}) \in (0, h) \) as in Corollary 3. Then, using (14), for each \( m \geq 1 \) we have
\[
e^{-\nu \lambda_{m+1} t} \int_{t-h}^{t} e^{\nu \lambda_{m+1} r} \| u(r) \|^2 \, dr
\]
\[
= e^{-\nu \lambda_{m+1} t} \int_{t-h}^{t-\delta} e^{\nu \lambda_{m+1} r} \| u(r) \|^2 \, dr + e^{-\nu \lambda_{m+1} t} \int_{t-h}^{t} e^{\nu \lambda_{m+1} r} \| u(r) \|^2 \, dr
\]
\[
\leq e^{-\nu \lambda_{m+1} \delta} \int_{t-h}^{t-\delta} \| u(r) \|^2 \, dr + \int_{t-h}^{t} \| u(r) \|^2 \, dr
\]
\[
\leq e^{-\nu \lambda_{m+1} \delta} \rho_2^2(t) + \int_{t-h}^{t} \| u(r) \|^2 \, dr.
\]

By taking now \( m = m(\varepsilon, t, \hat{D}) \) such that \( e^{-\nu \lambda_{m+1} \delta} \rho_2^2(t) < \frac{\varepsilon}{8 p^2(t)} \), jointly with (17) in Corollary 3, we conclude (22).

Actually, the other three terms in the RHS in (21) can be also controlled by \( \varepsilon/4 \) by using Lemma 4, and since \( |q_m(t-h)|^2 \leq |u(t-h)|^2 \leq \rho_1^2(t) \) and \( \lambda_m \to \infty \) as \( m \to \infty \), so it suffices to take eventually a bigger value \( m \). So, we conclude \( |q_m(t)| < \varepsilon \).

Finally, observe that (20) holds by applying the above arguments uniformly in the interval \([t-h, t]\). Indeed, all the estimates for (21) with \( t \) replaced by \( \xi \in [t-h, t] \) hold in the same uniform manner, since \( \delta \) in Corollary 3 can be chosen in such a way for the whole interval \([t-h, t]\). \( \square \)

From the previous results, we obtain the existence of minimal pullback attractors for the process \( U \) on \( C_H \) (see [14, Theorem 4.1]).

**Theorem 4.** Assume that \( f \in L^2_{loc}(\mathbb{R}; V') \), and \( g : \mathbb{R} \times C_H \to (L^2(\Omega))^2 \) satisfying the assumptions (I)–(III), (10) and (11), are given. Then, there exist the minimal pullback \( \mathcal{D}_F(C_H) \)-attractor \( \mathcal{A}_{\mathcal{D}_F(C_H)} \), and the minimal pullback \( \mathcal{D}_{\sigma_\mu}(C_H) \)-attractor \( \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)} \), for the process \( U \) defined by (7). The family \( \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)} \) belongs to \( \mathcal{D}_{\sigma_\mu}(C_H) \), and the following relation holds:
\[
\mathcal{A}_{\mathcal{D}_F(C_H)}(t) \subset \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)}(t) \subset \overline{B}_{C_H}(0, \rho_\mu(t)) \quad \forall t \in \mathbb{R}.
\]

**Remark 9.** If, additionally, we assume that
\[
\sup_{\tau \leq 0} \int_{-\infty}^{t} e^{-\sigma_\mu(r-s)} \| f(s) \|_2^2 \, ds < \infty,
\]
where \( \sigma_\mu \) is given by (12), then, taking into account Remark 6, we deduce that
\[
\mathcal{A}_{\mathcal{D}_F(C_H)}(t) = \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)}(t) \quad \forall t \in \mathbb{R}.
\]

5. **Regularity of pullback attractors and attraction in \( V \) norm.** Now, we will make use again of the pullback flattening property in order to establish the existence of minimal pullback attractors in the \( C_V \) norm, improving the results of the previous Section 4. In this case, a proof of the pullback flattening property allows us to obtain the pullback asymptotic compactness of the process in a shorter way than by using an energy method as in [14].
First, we define some new phase spaces. For any $\tilde{h} \in [0, h]$, let us denote
\[
C_{H}^{\tilde{h},V} = \{ \varphi \in C_H : \varphi|_{[-\tilde{h},0]} \in B([-\tilde{h},0]; V) \},
\]
where $B([-\tilde{h},0]; V)$ is the space of bounded functions from $[-\tilde{h},0]$ into $V$. The space $C_{H}^{\tilde{h},V}$ is a Banach space with the norm
\[
\| \varphi \|_{\tilde{h},V} = |\varphi|_{C_H} + \sup_{\theta \in [-\tilde{h},0]} \| \varphi(\theta) \|.
\]
Observe that the space $C_V = C([-h,0]; V)$ is a Banach subspace of $C_{H}^{h,V}$.

**Proposition 4.** Assume that $f \in L_{loc}^2(\mathbb{R}; (L^2(\Omega))^2)$, and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ satisfying the assumptions (I)–(III), are given. Then, for any $\tilde{h} \in [0, h]$, the family of mappings $U(t,\tau)|_{C_{H}^{\tilde{h},V}}$, with $\tau \leq t$, is a well defined continuous process on $C_{H}^{\tilde{h},V}$.

**Proof.** It follows from Theorem 1 above and from [14, Proposition 5.2].

From now on we assume that $f \in L_{loc}^2(\mathbb{R}; (L^2(\Omega))^2)$ satisfies that
\[
\text{there exists } 0 < \mu < 2\nu\lambda_1 \text{ such that } 2e^{(h)\nu}L_g < \mu,
\]
and
\[
\int_{-\infty}^{0} e^{\sigma_u s} |f(s)|^2 \, ds < \infty,
\]
where $\sigma_u$ is given by (12).

The following result is similar to Lemma 2 in Section 4, but now we are able to establish estimates for the process $U$ in higher norms, thanks to the regularity result (a) in Theorem 1 and the energy equality (6). It can be obtained analogously to [14, Lemma 5.2].

**Lemma 5.** Under the assumptions of Proposition 4, if moreover conditions (23) and (24) are satisfied, then for any $t \in \mathbb{R}$ and $\hat{D} \in D_{\sigma_u}(C_H)$, there exist $\tau_2(\hat{D}, t, h) < t - 2h - 1$ and non-negative functions $\{R_i\}_{i=1}^4$ depending on $t$ and $h$, such that for any $\tau \leq \tau_2(\hat{D}, t, h)$ and any $\phi^\tau \in D(\tau)$, it holds
\[
|u(r; \tau, \phi^\tau)|^2 \leq R_1^2(t) \quad \forall r \in [t - 2h - 1, t],
\]
\[
|u(r; \tau, \phi^\tau)|^2 \leq R_2^2(t) \quad \forall r \in [t - h, t],
\]
\[
\nu \int_{t-h}^{t} |A u(\theta; \tau, \phi^\tau)|^2 \, d\theta \leq R_3^2(t),
\]
\[
\int_{t-h}^{t} |u'(\theta; \tau, \phi^\tau)|^2 \, d\theta \leq R_4^2(t),
\]
where
\[
R_1^2(t) = 1 + e^{(h)\nu}(2\nu(\lambda_1 - \mu))^{-1}e^{-\sigma_u(t-2h-1)} \int_{-\infty}^{t} e^{\sigma_u s} |f(s)|^2 \, ds,
\]
\[
R_2^2(t) = \nu^{-1} \left( (1 + (2(\nu(\lambda_1) - 1)L_g^3 + 4L_g^3) R_1^2(t) + (4 + 2(\nu(\lambda_1) - 1)^{-1}) \int_{t-h}^{t} |f(\theta)|^2 \, d\theta \right)
\]
\times \exp\left\{ \frac{2}{\nu} C^{(\nu)} R_1^2(t) \left[ (1 + (2(\nu(\lambda_1) - 1)L_g^3) R_1^2(t) + (2(\nu(\lambda_1) - 1)^{-1}) \int_{t-h}^{t} |f(\theta)|^2 \, d\theta \right) \right\},
\]
\[
R_3^2(t) = R_3^2(t) + 2C^{(\nu)} h R_1^2(t) R_2^2(t) + 4L_g^2 \nu^{-1} h R_3^2(t) + 4\nu^{-1} \int_{t-h}^{t} |f(\theta)|^2 \, d\theta,
\]
\[
R_4^2(t) = R_4^2(t) + 2C^{(\nu)} h R_1^2(t) R_2^2(t) + 4L_g^2 \nu^{-1} h R_4^2(t) + 4\nu^{-1} \int_{t-h}^{t} |f(\theta)|^2 \, d\theta.
\]
\[ R_2^2(t) = \nu R_2^2(t) + 4L_3^2 h R_2^2(t) + 2C_1^2 \nu^{-1} R_2^2(t) R_3^2(t) + 4 \int_{t-h}^{t} |f(\theta)|^2 d\theta, \]

with \( C(\nu) = 27C_4^4(4\nu^3)^{-1} \).

**Remark 10.** Under the assumptions of Lemma 5, \( \lim_{t \to -\infty} e^{\sigma \tau} R_2^2(t) = 0 \).

Now, we introduce the following universes in \( \mathcal{D}(c_{H}^{\hat{h}, V}) \).

**Definition 8.** For any \( \sigma > 0 \) and \( \hat{h} \in [0, h] \), we will denote by \( \mathcal{D}_{\sigma}^{\hat{h}, V}(C_H) \) the class of families \( \bar{D} = \{ D(t) : t \in \mathbb{R} \} \in \mathcal{D}_{\sigma}(C_H) \) such that for any \( t \in \mathbb{R} \) and for any \( \varphi \in D(t) \), it follows that \( \varphi|_{[-\hat{h}, 0]} \in B([-\hat{h}, 0]; V) \).

Analogously, we will denote by \( \mathcal{D}_{\sigma}^{\hat{h}, V}(C_H) \) the class of families \( \bar{D} = \{ D(t) = D : t \in \mathbb{R} \} \) with \( D \) being a fixed nonempty bounded subset of \( C_H \) such that for any \( \varphi \in D \), it holds that \( \varphi|_{[-\hat{h}, 0]} \in B([-\hat{h}, 0]; V) \).

Finally, we will denote by \( \mathcal{D}_{\sigma}(C_H^{\hat{h}, V}) \) the class of families \( \bar{D} = \{ D(t) = D : t \in \mathbb{R} \} \) with \( D \) a fixed nonempty bounded subset of \( C_H^{\hat{h}, V} \).

**Remark 11.** The following chain of inclusions for the universes in the above definition and the universes introduced in Section 4 hold

\[ \mathcal{D}_{\sigma}(C_H^{\hat{h}, V}) \subset \mathcal{D}_{\sigma}^{\hat{h}, V}(C_H) \subset \mathcal{D}_{\sigma}^{\hat{h}, V}(C_H) \subset \mathcal{D}_{\sigma}(C_H), \]

and

\[ \mathcal{D}_{\sigma}(C_H^{\hat{h}, V}) \subset \mathcal{D}_{\sigma}^{\hat{h}, V}(C_H) \subset \mathcal{D}_{\sigma}(C_H) \subset \mathcal{D}(C_H), \]

for all \( \sigma > 0 \) and any \( \hat{h} \in [0, h] \). It must also be pointed out that \( \mathcal{D}_{\sigma}^{\hat{h}, V}(C_H) \) is also inclusion-closed. Finally, it is clear that if \( 0 \leq \hat{h}_1 < \hat{h}_2 \leq h \), then

\[ \mathcal{D}_{\sigma}(C_H^{\hat{h}_2,V}) \subset \mathcal{D}_{\sigma}(C_H^{\hat{h}_1,V}), \]

\[ \mathcal{D}_{\sigma}(C_H^{\hat{h}_2,V}) \subset \mathcal{D}_{\sigma}(C_H^{\hat{h}_1,V}), \]

\[ \mathcal{D}_{\sigma}(C_H^{\hat{h}_2,V}) \subset \mathcal{D}_{\sigma}(C_H^{\hat{h}_1,V}). \]

The following result is immediate.

**Proposition 5.** Under the assumptions of Lemma 5, for any \( \hat{h} \in [0, h] \), the family

\[ \hat{D}_{0, \mu, \hat{h}} = \{ B_{C_H}(0, R_1(t)) \cap C_H^{\hat{h}, V} : t \in \mathbb{R} \} \]

is pullback \( \mathcal{D}_{\sigma}^{\hat{h}, V}(C_H) \)-absorbing for the process \( U : \mathbb{R}_d^2 \times C_H^{\hat{h}, V} \to \mathbb{R}_d^2 \). Moreover, \( \hat{D}_{0, \mu, \hat{h}} \) belongs to \( \mathcal{D}_{\sigma}^{\hat{h}, V}(C_H) \).

Now, we will prove some sort of pullback flattening property for the family of processes \( U : \mathbb{R}_d^2 \times C_H^{\hat{h}, V} \to \mathbb{R}_d^2 \), with \( \hat{h} \in [0, h] \) (note that we are not considering the process \( U \) restricted to \( \mathbb{R}_d^2 \times \mathcal{C}_V \)). Nevertheless, we will be able to obtain the pullback asymptotic compactness reasoning as in the proof of Proposition 1.

Analogously to Lemma 4, we have the following result.

**Lemma 6.** If \( f \in L_2^2(\mathbb{R}; (L^2(\Omega))^2) \) satisfies conditions (23) and (24), then, for any \( t \in \mathbb{R} \)

\[ \lim_{t \to \infty} e^{-\rho t} \int_{-\infty}^{t} e^{\rho s} |f(s)|^2 ds = 0. \]
Proposition 6. Under the assumptions of Lemma 5, for any \( \varepsilon > 0 \) and \( t \in \mathbb{R} \), there exists \( m = m(\varepsilon, t) \in \mathbb{N} \) such that for any \( \hat{D} \in \mathcal{D}_{\sigma_n}(C_H) \), the projection \( P_m : V \rightarrow V_m := \text{span}[w_1, \ldots, w_m] \) satisfies the following properties:

\[
\{ P_m U(t, \tau) D(\tau) : \tau \leq \tau_2(\hat{D}, t, h) \} \text{ is bounded in } C_V,
\]

and equi-continuous on \([-h, 0] \) with values in \( V \),

where \( \tau_2(\hat{D}, t, h) \) is given in Lemma 5.

Proof. Let \( \varepsilon > 0 \), \( t \in \mathbb{R} \), and \( \hat{D} \in \mathcal{D}_{\sigma_n}(C_H) \) be fixed.

Since \( \{ w_j \}_{j \geq 1} \) is a special basis, \( P_m \) is non-expansive in \( V \). From this and the second estimate in Lemma 5, we deduce the boundedness in \( C_V \) of the set \( \{ P_m U(t, \tau) D(\tau) : \tau \leq \tau_2(\hat{D}, t, h) \} \), for all \( m \geq 1 \).

Now, let us fix \( \tau \leq \tau_2(\hat{D}, t, h) \), \( \phi^\tau \in D(\tau) \), and \( \theta_1, \theta_2 \in [-h, 0] \), with \( \theta_2 > \theta_1 \). Let us denote again \( u(r) = u(r; \tau, \phi^\tau) \) and \( q_m(r) = u(r) - P_m u(r) \). By the choice of the basis \( \{ w_j \}_{j \geq 1} \) and the fourth estimate in Lemma 5, for any \( m \geq 1 \) one has

\[
\| (P_m U(t, \tau) \phi^\tau)(\theta_2) - (P_m U(t, \tau) \phi^\tau)(\theta_1) \| \\
\leq \lambda_m^{1/2} \| (P_m U(t, \tau) \phi^\tau)(\theta_2) - (P_m U(t, \tau) \phi^\tau)(\theta_1) \| \\
\leq \lambda_m^{1/2} |u(t + \theta_2) - u(t + \theta_1)| \\
\leq \lambda_m^{1/2} \int_{t+\theta_1}^{t+\theta_2} |u'(s)| \, ds \\
\leq \lambda_m^{1/2} |\theta_2 - \theta_1|^{1/2} R_4(t).
\]

Then, from this we deduce that the set \( \{ P_m U(t, \tau) D(\tau) : \tau \leq \tau_2(\hat{D}, t, h) \} \) is equi-continuous on \([-h, 0] \) with values in \( V \), for any \( m \geq 1 \).

On the other hand, taking into account (4) and Lemma 5, for each \( m \geq 1 \) one has

\[
\frac{1}{2} \frac{d}{dr} \| q_m(r) \|^2 + \nu |Aq_m(r)|^2 \\
= -b(u(r), u(r), Aq_m(r)) + (f(r), Aq_m(r)) + (g(r, u_r), Aq_m(r)) \\
\leq \nu \| Aq_m(r) \|^2 + \frac{2}{\nu} (|f(r)|^2 + L_g^2 R_1^2(t)) + \frac{C^2}{\nu} R_1(t) R_2^2(t) |A u(r)|
\]
a.e. \( t - h < r < t \).

Therefore, since \( |Aq_m(r)|^2 \geq \lambda_{m+1} \| q_m(r) \|^2 \), we deduce that

\[
\frac{d}{dr} \| q_m(r) \|^2 + \nu \lambda_{m+1} \| q_m(r) \|^2 \\
\leq 4 \nu^{-1} (|f(r)|^2 + L_g^2 R_1^2(t)) + 2C^2 \nu^{-1} R_1(t) R_2^2(t) |A u(r)|
\]
a.e. \( t - h < r < t \).

Thus, multiplying this last inequality by \( e^{\nu \lambda_{m+1} t} \), integrating in \( [t - h, t] \), and taking into account Lemma 5, we obtain

\[
e^{\nu \lambda_{m+1} t} \| q_m(t) \|^2 \\
\leq e^{\nu \lambda_{m+1} (t-h)} \| q_m(t-h) \|^2 + \frac{4}{\nu} \int_{t-h}^t e^{\nu \lambda_{m+1} r} |f(r)|^2 \, dr
\]
Theorem 5. A minimal pullback satisfies (23) and (24). Then, for any attractor in the $C_A$-attractor for any $\tilde{\phi}$.

Remark 12. (i) Observe that, thanks to the regularity property (a) in Theorem 1, we neither need to restrict the process $U$ nor to define a universe in $P(C_V)$ in order to obtain the asymptotic compactness of the process $U : \mathbb{R}_+^2 \times C_{\tilde{h},V} \to C_{\tilde{h},V}$, for any $\tilde{\phi} \in [0, h]$.

(ii) It is worth pointing out that in this way the proof of the asymptotic compactness is much shorter than [14, Lemma 5.3].

As an immediate consequence of Proposition 1 and Proposition 6 we have the following

Lemma 7. Under the assumptions of Lemma 5, for any $\tilde{h} \in [0, h]$, the process $U : \mathbb{R}_+^2 \times C_{\tilde{h},V} \to C_{\tilde{h},V}$ is pullback $D_{\sigma_V}(C_{\tilde{h}})$-asymptotically compact.

Combining all the above statements, we obtain the existence of minimal pullback attractors for the process $U : \mathbb{R}_+^2 \times C_{\tilde{h},V} \to C_{\tilde{h},V}$, for any $\tilde{h} \in [0, h]$. Moreover, we improve the attractor of the pullback attractor $A_{D_{\sigma_V}(C_{\tilde{h}})}$, so that it actually attracts in the $C_V$ norm (see [14, Theorem 5.1]).

Theorem 5. Let $g$ satisfying (I)–(III) be given. Assume that $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^2)$ satisfies (23) and (24). Then, for any $\tilde{h} \in [0, h]$, the process $U$ on $C_{\tilde{h},V}$ possesses a minimal pullback $D_{\sigma_V}(C_{\tilde{h}})$-attractor $A_{D_{\sigma_V}(C_{\tilde{h}})}$, a minimal pullback $D_{F}(C_{\tilde{h},V})$-attractor $A_{D_{F}(C_{\tilde{h},V})}$, and a minimal pullback $D_{F}(C_{\tilde{h},V})$-attractor $A_{D_{F}(C_{\tilde{h},V})}$. Besides, the following relations hold:

\[
\begin{align*}
A_{D_{F}(C_{\tilde{h},V})}(t) & \subset A_{D_{F}(C_{\tilde{h},V})}(t) \\
& \subset A_{D_{F}(C_{\tilde{h},V})}(t) \\
& \subset A_{D_{F}(C_{\tilde{h},V})}(t) = A_{D_{\sigma_V}(C_{\tilde{h}})}(t) \\
& \subset C_V \quad \forall t \in \mathbb{R}, \\
\end{align*}
\]

and for any family $\hat{D} \in D_{\sigma_V}(C_{\tilde{h}})$,

\[
\lim_{\tau \to -\infty} \text{dist}_{C_V}(U(t, \tau) \hat{D}(\tau), A_{D_{\sigma_V}(C_{\tilde{h}})}(t)) = 0 \quad \forall t \in \mathbb{R}.
\]
Finally, if moreover $f$ satisfies
\[
\sup_{s \leq 0} \left( e^{-\sigma_\mu s} \int_{-\infty}^{s} e^{\sigma_\mu \theta} |f(\theta)|^2 d\theta \right) < \infty,
\] (26)
then all attractors in (25) coincide, and this family is tempered in $C_V$, in the sense that
\[
\lim_{t \to -\infty} \left( e^{\sigma_\mu t} \sup_{v \in A_{D_\sigma_\mu (C_H)}(t)} \|v\|^2_{C_V} \right) = 0,
\]

Remark 13. (i) Observe that, under the assumptions of Theorem 5, one has $A_{D^h_{\sigma_\mu}}(C_H) \equiv A_{D^h_{\sigma_\mu}}(C_H)$ for any $\hat{h} \in [0, h]$, i.e., the pullback attractor $A_{D^h_{\sigma_\mu}}(C_H)$ is independent of $\hat{h}$. Actually, if $f$ also satisfies (26), then $A_{D^h_{\sigma_\mu}}(C_H) \equiv A_{D^h_{\sigma_\mu}}(C_H)$, and $A_{D^h_{\sigma_\mu}}(C_H) \equiv A_{D^h_{\sigma_\mu}}(C_H)$.

(ii) Observe that since $\hat{D}_0, \mu, h \in D_{\sigma_\mu}(C_H)$, and that for each $t \in \mathbb{R}$, $D_0, \mu, h(t)$ is closed in $C_{\sigma_\mu}(C_H)$, from Remark 5 and Remark 11, we deduce that $A_{D^h_{\sigma_\mu}}(C_H) \in D_{\sigma_\mu}(C_H)$.

Remark 14. We can also consider, for each $0 \leq \hat{h} \leq h$, the class $D_{\sigma_\mu}(C_{\hat{h}})$ of all families $D = \{D(t) : t \in \mathbb{R}\} \subset P(C_{\hat{h}})$ such that
\[
\lim_{\tau \to -\infty} \left( e^{\sigma_\mu \tau} \sup_{v \in D(\tau)} \|v\|^2_{C_{\hat{h}}(V)} \right) = 0.
\]
For this universe we have the chain of inclusions
\[
D_{\sigma_\mu}(C_{\hat{h}}) \subset D_{\sigma_\mu}(C_{\hat{h}}) \subset D_{\sigma_\mu}(C_{\hat{h}}) \subset D_{\sigma_\mu}(C_{\hat{h}}).
\]
Under the assumptions of Theorem 5, we deduce the existence of the minimal pullback $D_{\sigma_\mu}(C_{\hat{h}})$-attractor $A_{D_{\sigma_\mu}(C_{\hat{h}})}$. Moreover, this pullback attractor satisfies
\[
A_{D_{\sigma_\mu}(C_{\hat{h}})}(t) \subset A_{D_{\sigma_\mu}(C_{\hat{h}})}(t) \subset A_{D_{\sigma_\mu}(C_H)(t)} \forall t \in \mathbb{R}.
\]
In fact, if assumption (26) is satisfied, then $A_{D_{\sigma_\mu}(C_{\hat{h}})} \equiv A_{D_{\sigma_\mu}(C_H)}$.

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