Robustness of Higher Dimensional Nonlocality against dual noise and sequential measurements

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Robustness in the violation of Collins-Linden-Gisin-Masser-Popescu (CGLMP) inequality is investigated from the dual perspective of noise in measurements as well as in states. To quantify it, we introduce a quantity called the area of nonlocal region which reveals a dimensional advantage. Specifically, we report that with the increase of dimension, the maximally violating states show a greater enhancement in the area of nonlocal region in comparison to the maximally entangled states and the scaling of the increment, in this case, grows faster than visibility. Moreover, we examine the robustness in the sequential violation of CGLMP inequality using weak measurements and find that even for higher dimensions, two observers showing a simultaneous violation of the CGLMP inequality as obtained for two-qubit states persists. We notice that the complementarity between information gain and disturbance by measurements is manifested by the decrease of the visibility in the first round and the increase of the same in the second round with dimensions. Furthermore, the amount of white noise that can be added to a maximally entangled state so that it gives two rounds of the violation, decreases with the dimension, while the same does not appreciably change for the maximally violating states.

I. INTRODUCTION

The journey from the Einstein-Podolski-Rosen paradox [1] to Bell theorem [2] via Bohmian mechanics [3] is a fascinating story that contributed towards our present outlook of a physical theory. It asserts that a satisfactory description of nature cannot assume both the assumptions of locality and reality simultaneously. Jointly these two assumptions, known as local-realism has recently been refuted experimentally by loophole-free Bell test [4–6]. Apart from the foundational significance, Bell-Clauser-Horne-Shimony-Holt (CHSH) inequality [7] enables the device-independent certification of randomness [8], secure key distribution [9–11], detection of entanglement [12] etc.

Going beyond the much-studied simplest Bell scenario involving two settings of measurements for two party with two outcomes, denoted by $(2 - 2 - 2)$, new insightful and qualitatively different results have been derived which was otherwise impossible if restricted to the simplest scenario. In particular, violation of local realism is manifested more sharply than $(2 - 2 - 2)$-case via Greenberger-Horne-Zeilinger (GHZ) argument [13], which requires at least a three-qubit system. With a suitable choice of binary observables, it has been shown that maximal violation of Bell inequality persists for a singlet state of arbitrary spins [14], turning down the belief that systems may loose their quantumness with increasing system size, thereby leading to the decrease in violation of Bell inequality [15]. Later, the dimensional advantage in violation of local realism has been established considering a more general choice of observables [16–18], since dichotomic measurements can not exploit the higher dimensional system with full generality. In the bipartite system of arbitrary local dimension, with two choices of non-degenerate measurements, named as $(2 - 2 - d)$-situation, corresponding Bell inequalities have been derived by Collins-Gisin-Linden-Massar-Popescu (CGLMP) [19], which is violated maximally by a nonmaximally entangled state [20] for the specific choices of observables [16–18]. These tight higher dimensional Bell inequalities [21] exhibit enhancement in visibility with the increase of dimension, thereby showing more robustness against noise [16, 19] Here visibility refers to the noise strength upto which a pure state mixed with white noise exhibits violation. It has been shown explicitly that performances of many quantum information processing tasks get enhanced by considering higher dimensional systems. Specifically, compared to qubits, they exhibit greater robustness to noise [19], stronger security in device-dependent quantum key distribution (QKD) [24] and device-independent extraction of random bits [25]. However, dimensional advantage is not always straightforward as noted in a recent work [26] in the context of device-independent QKD – the estimated lower bound on the secure key rate does not improve with increasing the dimension whereas the upper bound on the key rate exhibits the opposite trend.

In another direction, the conventional Bell scenario has been extended where half of a bipartite system is...
possessed by a single observer, called Alice, while the other half is possessed by a series of observers, named as Bobs, who can measure sequentially [27]. In this new scheme of the Bell test, it has been shown that no more than two observers can violate Bell-CHSH inequality if the series of observers measure independently [27, 28]. Such a sequential scenario has also been tested experimentally [29, 30] and is further extended in several situations which include detecting steerable correlation [31, 32], witnessing entanglement [33, 34], testing Bell inequalities other than CHSH [35], identifying genuine entanglement [36], and preparation contextuality [37]. An interesting twist in this situation is that with the slight modification to the independent and unbiased measurement scheme, an unbounded sequence of observers can be found who can certify non-classical correlation with the single observer in another side [27, 28]. Recently, some interesting applications of the sequential scheme like self testing unsharp measurement [39, 40], reusing teleportation channel [41], generating randomness [42] have been proposed, thereby showing its potentiality in quantum technologies. An interesting observation from the above studies is that if one restricts to a particular measurement scheme, i.e., independent and unbiased measurement by the series of observers [28], it is not straightforward to predict the number of sequential observers that would show non-local correlations. As the number of sequential violations depends on the initial strength of the correlation, detection and measurement process in a non-trivial way, it is not well characterized yet. The number is finite and dictated by the trade-off between the disturbance and information gained by measurements. For example, it was found that for maximally entangled two-qubit initially shared state, at most twelve Bobs can detect entanglement with a single Alice employing measurement settings pertinent to the optimal witness operator [33] while two Bobs sharing the same state with Alice can violate CHSH inequality [27, 28] based on optimal measurement settings required for Bell-violation. In the sequential measurement, partial information is extracted which is sufficient for the detection scheme and at the same time, some residual correlation remains for other rounds which gradually diminishes with a longer sequence of Bobs. It also reveals that witnessing entanglement possibly disturbs the state less compared to the situation when the Bell-CHSH test is performed, thereby admitting more robustness of the former scheme against noise. Similarly, measurement-device-independent entanglement witness [43] turns out to be more suitable in the sequential situation than that of the standard entanglement witness [44] as shown through the increased number of Bobs [34].

In the present work, we first investigate the robustness of CGLMP inequality by going beyond the visibility measure of ‘nonlocality’ [16, 19]. Specifically, in addition to white noise in the state, we consider noisy measurement (which we call as weak/unsharp measurement) on the maximally entangled state \(M_E\) as well as on the maximally CGLMP violating states \(M_V\). Such a consideration of dual noise leads to a measure of robustness, dubbed as the ‘area of nonlocal region’ (where nonlocality means the violation of CGLMP inequality), which scales with dimension more sharply than the visibility one. The introduction of noise to the measurement enables the possibility of sequential violation of CGLMP inequality. In particular, we find that the violation by two Bobs’ persists even with the increase of dimension, as found in the two-qubit case with CHSH inequality. In this respect, the pertinent question is how the robustness of CGLMP is reflected in the sequential scenario. It was noticed that in the context of a violation of CGLMP inequality, the visibility decreases with the increase of dimension [16, 19]. However, we observe that if we demand the violation of CGLMP inequality in two rounds of a sequential scheme, the required visibility increases with the dimension for maximally entangled states while surprisingly, it remains constant for maximally violating states. It also demonstrates that the sequential scenario can reveal a kind of robustness which is qualitatively different from the visibility and ‘area of nonlocal region’ obtained for a single round. It is due to the trade-off present in the disturbance by the weak measurements and the information gain via measurements in a sequential scheme.

The paper is organised in the following way. In Sec. II, we briefly discuss the prerequisite of the present work. In Sec. III, the robustness of CGLMP is discussed with a new measure introducing dual noise. For higher dimensional pure states, CGLMP inequality is used to certify entanglement sequentially in Sec. IV and a similar study is carried out for noisy mixed states in Sec. V. We conclude in Sec. VI with a brief discussion.

II. PREREQUISITES: BELL INEQUALITIES IN HIGHER DIMENSIONS AND SEQUENTIAL MEASUREMENT SCHEME

Before presenting our results, let us briefly discuss the CGLMP inequality and sequential scenario of the Bell test.

A. CGLMP inequality

Let Alice and Bob be two observers allowed to perform two \(d\) outcome measurements. If \(A_1\) and \(A_2\) are measurement settings of Alice while \(B_1\) and \(B_2\) are of Bob, which can take values from \([0, d − 1]\), i.e., \(A_{1(2)}, B_{1(2)} = 0, 1, \ldots, d − 1\). The CGLMP inequality
reads as \[ I_d = \sum_{k=0}^{d-1} \left(1 - \frac{2k}{d-1}\right)|f(k) - f(-k-1)| \leq 2, \] (1)

where
\[ f(k) = P(A_1 = B_1 + k) + P(B_1 = A_2 + k + 1) + P(A_2 = B_2 + k) + P(B_2 = A_1 + k). \] (2)

The probabilities of the outcomes of Alice’s measurement, \( A_d \) and Bob’s measurement, \( B_{\theta}, (a, b = 1, 2) \) in \( f(k) \) differ by \( k \) mod \( d \) and can be written as
\[ P(A_1 = B_1 + k) = \sum_{j=0}^{d-1} P(A_d = j, B_b = j + k \mod d). \]

The strongest violation of CGLMP inequality is obtained for a maximally entangled state and a particular class of non-maximally entangled states if the measurements performed by Alice and Bob are in the basis \( \{|k\}_A \} \) and \( \{|l\}_B \) with
\[ |k\rangle_A = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \exp(\frac{2\pi}{d} j (k + \alpha_a)) |j\rangle_A, \] (3)

and
\[ |l\rangle_B = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \exp(\frac{2\pi}{d} j (-l + \beta_b)) |j\rangle_B, \] (4)

where
\[ \alpha_1 = 0, \quad \alpha_2 = 1/2, \quad \beta_1 = 1/4 \quad \text{and} \quad \beta_2 = -1/4. \] (5)

The special thing about the above inequality is that its quantum violation increases with dimension \( d \).

### B. Sequential measurement scenario

The sequential measurement scenario considers an entangled state of two \( d \)-dimensional systems shared in such a way that half of the system is in possession of the observer (say, Alice) and the other half is with several observers (say, \( n \) Bobs, referred as Bob\(_1\), Bob\(_2\), Bob\(_3\), \ldots, Bob\(_n\)). The task of Bob\(_1\) is to pass the system to Bob\(_2\) after performing an unsharp measurement on his part. Similarly, Bob\(_2\) passes the system to Bob\(_3\) after the measurement and so on. In other words, several Bobs measure their part sequentially, hence the name of the sequential measurement scheme. Note that the measurement of each Bob is independent and all the measurement settings of each Bob are equally probable. In the scenario described above, if the measurement statistics between Alice and any Bob, say, Bob\(_k\) Bob\(_k\) (\( k > 1 \)) exhibits a violation of CGLMP, then we call it as ‘sharing of nonlocality’ between them in the sequential scenario.

To know the number of Bob sharing nonlocality of a shared entangled state (say, \( \rho \)) between Alice and \( n \) Bobs, we have to assume that the measurement of Alice and Bob\(_n\) is sharp (i.e., they perform projection measurements on their parts). In contrast, 1, \ldots, \( n - 1 \) Bobs perform unsharp measurements represented by positive-operator valued measurements (POVMs). If measurement settings at Alice are denoted by \( \{ |k\}_A \{ k\} \) and the measurement settings of Bob\(_m\) is represented by
\[ E_{b_m}^l = \lambda_m |l\rangle_B \langle l| + \frac{1 - \lambda_m}{d} \mathbb{1}_d, \] (6)

where \( k, l = 0, 1, 2, d - 1, m = 1, 2, 3, \ldots, n - 1, \lambda_m (0 < \lambda_m \leq 1) \) is the sharpness parameter of Bob\(_m\) and \( \mathbb{1}_d \) is the \( d \)-dimensional identity matrix. The state after the measurements of \( (m - 1) \)th Bob and without any measurement at Alice’s end transforms as
\[ \rho_m = \frac{1}{d} \sum_{l=0}^{d-1} (\mathbb{1}_d \otimes \sqrt{E_{b_{m-1}}^l}) \rho_{m-1} (\mathbb{1}_d \otimes \sqrt{E_{b_{m-1}}^l}), \] (7)

where \( \rho_{m-1} \) is the state before the unsharp measurement performed by Bob\(_{m-1}\). We will use the post measured state \( \rho_m \) and POVM in Eqs. (6) and (7) respectively, when we certify nonlocality via CGLMP inequality in this scenario.

### III. Robustness in CGLMP Violation: Area of Nonlocal Region

The study of the violation of Bell-type inequalities is a major endeavour in studies of nonlocality. Another important aspect is the investigation of robustness in the obtained violation. Typical studies of robustness consist of the addition of noise to the state and tracking the response of violation due to the amount of noise added to the state. However, for the violation of Bell-type inequalities, measurements play as crucial a role as states. Therefore, robustness analysis should also be carried out when noise is added to the measurements as well.

We perform a general robustness analysis when both the state as well as the measurements are simultaneously noisy. In particular, we explore the role of the dimension of the bipartite state whose nonlocal characteristics in terms of violation of CGLMP inequality in Eq. (1) are under investigation. Before that, we briefly discuss the scenario when white noise is mixed with the state, given by
\[ \rho = p |\psi\rangle \langle \psi| + \frac{1 - p}{d^2} \mathbb{1}_{d^2}, \] (8)

where \( |\psi\rangle \) is a bipartite pure state with each party of dimension \( d \), and \( \mathbb{1}_{d^2} \) is the \( d \otimes d \) maximally mixed.
state (white noise). It was observed [19] that when \(|\psi\rangle\) is a maximally entangled state in \(d \otimes d\), given by 
\[
|\psi_{M_{dE}}^E\rangle = \frac{1}{\sqrt{d}} \sum |ii\rangle,
\]
the robustness to noise which can be called as visibility, measured as \(p\), increases with the increase in \(d\). This is in the sense that the maximal white noise that can be added to \(|\psi_{M_{dE}}^E\rangle\) such that the resultant mixed state \(\rho\) violates the CGLMP inequality which increases with dimension \(d\). For a given \(d\), this maximal amount of white noise is denoted by \(1 - p_{\text{min}}\). In other words, for \(|\psi_{M_{dE}}^E\rangle\), as \(p_{\text{min}}\) decreases, \(1 - p_{\text{min}}\) increases with \(d\). Such dimensional advantage of robustness is enhanced when instead of a \(M_E\), |\(\psi\rangle\) is chosen to be the non-maximally entangled state which violates CGLMP inequality maximally [20]. We call such maximally violating states as \(M_V\). The exact form of \(M_V\) up to \(d = 10\) can be found in Ref. [45]. The \(M_V\) offer a greater robustness with \(d\) in comparison to \(M_{E}\). For the convenience of readers, we present the exact form of the maximally violating states in Appendix A.

We now consider the opposite situation where the shared state is noise free and the measurements the measurements are taken to be noisy. The effect operators for the noisy measurements are described by POVMs given in Eq. (6). Note that here we are interested in the first round violation \((m = 1)\), see Eq. (6). Considering \(|\psi\rangle\) to be noiseless \(M_{E}\) and \(M_V\), the amount by which measurements can be made noisy preserving violation of the CGLMP inequality is denoted by \(1 - \lambda_{\text{min}}\), which also increases with increasing \(d\). We observe an exact similar dimensional dependence with noise in the measurements denoted by \(\lambda\), as obtained in the case of noisy states adding white noise to the state or measurements is equivalent. Mathematically, for any pure state \(|\psi\rangle\),
\[
I^Q_{d}(p = 1, \lambda = x) = I^Q_{d}(p = x, \lambda = 1).
\]
Here superscript \(Q\) in Eq. (9) implies that the probabilities associated with the CGLMP expression are calculated for the quantum states and measurements performed by Alice and Bob, stated earlier. For brevity, we do not henceforth use the superscript \(Q\) as we always work with probabilities generated by quantum states and measurements. Things become more interesting and involved when both the state and measurements suffer from noise simultaneously, which we will discuss in the next subsection.

### A. Complementarity of Robustness

We are now going to study the robustness obtained from the violation of CGLMP inequality by considering both the state and the measurements noisy. We again start with a \(d\)-dimensional maximally entangled state, \(|\psi_{M_{dE}}^E\rangle\) as well as \(M_V\), \(|\psi_{M_{dV}}^V\rangle\) (for two-qutrit state, \(M_V\) is of the form, \(\frac{1}{\sqrt{t}}(|00\rangle + \gamma |11\rangle + |22\rangle\), where \(t = 2 + \gamma^2\) and \(\gamma = 0.7923\) [20, 45]). In this general framework, \(p_{\text{min}}\) is a function of the noise in the measurements, which we denote as \(p_{\text{min}}(\lambda)\), and naturally \(\lambda_{\text{min}}\) in turn, becomes a function of the noise added to the state, which is referred to as \(\lambda_{\text{min}}(p)\). For convenience, we drop the min and functional labels, thereby indicating that \(1 - p_{\text{min}}(\lambda)\) and \(1 - \lambda_{\text{min}}(p)\) as \(1 - p\) and \(1 - \lambda\) respectively. We investigate the dual version of robustness by tracking the locus of all the points in the \((1 - \lambda, 1 - p)\)-plane that just crosses the local realist value of 2 by considering \(M_{E}\) and \(M_V\), see Figs. 1 (a) and (b). Note that all noise configurations that fall below the curve lead to the violation of the CGLMP inequality. Therefore, in the \((1 - \lambda, 1 - p)\)-plane, the ratio of the areas under the curve can be considered to be a measure of robustness when both the state and the measurement are affected by noise. Motivated by this observation, we introduce a generalized robustness measure as the area under this curve, which we call the “area of nonlocal region” \((A)\). Mathematically, the area of nonlocal region \((A)\) in the noise plane can be defined as
\[
A = \int_0^{1-p_{\text{min}}(\lambda = 1)} \left(1 - \lambda_{\text{min}}(1 - p)\right) dp.
\]
We then compute \(A\) values for both \(M_{E}\) and \(M_V\), and make a comparative analysis of their respective scalings with \(d\), see Figs. 1 (a), (b), and (c). The \(A\) values for the \(M_{E}\) and \(M_V\) are listed in Table I. Our findings are listed below:

1. The \(A\) values for \(M_V\) are strictly greater than those obtained for \(M_{E}\). Furthermore, the gap of \(A\) values for the \(M_V\) and the \(M_{E}\) grows with \(d\) as clearly discernible from Table I and Fig. 1 (a) - (c).

2. The \(A\) scales much faster with \(d\) for the \(M_V\) in comparison to the \(M_{E}\), see Fig. 1 (c).

3. The gap in the growth between \(M_V\) and \(M_{E}\) in the case of \(A\) grows much faster than that of the visibility.

From the fact that the value of the CGLMP expression is same when either the state or the measurement becomes noisy with the same amount of white noise, \(I_d(p = 1, \lambda = x) = I_d(p = x, \lambda = 1)\) as in Eq. (9), it seems reasonable to assume that robustness can be completely characterized by looking at either \(p\) or \(\lambda\). However, this symmetry does not hold in the general case where both noises are non-vanishing, i.e., \(I_d(p = x, \lambda = y) \neq I_d(p = y, \lambda = x)\). It is reflected by the fact that the curves in Figs. 1 (a) and (b) are non linear. Therefore, in the general case, the CGLMP nonlocality is both a function of the unsharp parameter and the amount of noise in the state in a non trivial way. This leads us to introduce \(A\) as a single-letter formula to analyze robustness that incorporates the effects of both.
Furthermore, note that the white noise paradigm can be motivated by interpreting the noise is arising from a depolarizing noisy channel that is independent of the source state and can be interpreted as a systematic noise involved in the setup. This, as we understand, gives a fair platform for comparing the performance of various states with respect to their robustness against noise. For coloured noise, there is no unique way to ascribe the nonuniformity of the noise, and it can alter the nonlocal properties of various states in completely different ways. It would make our comparative studies hard. This is why we focus on the systematic white noise in both cases. In the general case, one intends to consider an arbitrary separable state as noise. Such noise would depend on the initial shared entangled state. To create a meaningful platform of comparison for various initial shared states ($M_E$, $M_V$ etc.), for each initial state, one has to perform an optimization over the entire set of separable states pinning down on the least disturbing noise for each initial state. This in general is a very hard problem. Nevertheless, we take a particular coloured noise scheme that might be worthwhile to consider, which we call the point noise, where any state $|\psi\rangle$ under consideration is made noisy by

$$\rho = p|\psi\rangle\langle\psi| + (1-p)|k=0\rangle\langle k=0| \otimes |l=0\rangle\langle l=0|.$$  

For the expressions of $|k(l) = 0\rangle$, see Eqs. (3) and (4). Compared to white noise (mixing maximally mixed state), both $M_E$ and $M_V$ show enhanced robustness features with respect to the point noise which can be easily observed from Fig. 2 for $d = 3$. However, the scaling of $1 - p_{\text{min}}$ with $d$ remain qualitatively similar with the hierarchies between $M_V$ and $M_E$ being preserved. For $d = 3$ case, see Fig. 2.

Typically, noise in the system has an adverse effect on the system in the form of lowering the visibility. As shown in this section, the bane can turn out to be boon in disguise if we look at the situation from a different point of view. In the context of sequential measurements, the “white noise” in the measurement actually constitutes a POVM strategy which allows multiple Bobs to share nonlocality, thereby manifesting the robustness from a different perspective, as will be shown in the succeeding section.

**IV. SHARING OF NONLOCALITY IN HIGHER DIMENSION**

In the sharing scenario considered in this section, we deal with the maximally entangled and maximally violating states shared by Alice and Bob, in arbitrary dimension. We will start our discussion from $d = 3$ and a detailed analysis is presented for $M_E$ in $d = 3$ to $d = 5$. We then repeat the investigation for the maximally violating states.
After substituting \( d = 3 \) in Eq. (1), the CGLMP inequality reads as
\[
I_3 = P(A_1 = B_1) + P(B_1 = A_2 + 1) + P(A_2 = B_2) + P(B_2 = A_1) - [P(A_1 = B_1 - 1) + P(B_1 = A_2) + P(A_2 = B_2 - 1) + P(B_2 = A_1 - 1)] \leq 2. \tag{12}
\]
If the shared state is the two-qutrit \( \mathcal{M}_E \), given by
\[
|\psi_{\mathcal{M}_E}^3\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle), \tag{13}
\]
by performing POVM at Bob’s side, and by considering the measurement settings for CGLMP test given in Eqs. (3), (4) and (5) for Alice and Bob, the quantum expression of CGLMP inequality, \( I_3 \) (Eq.(12)) for Alice-Bob\(_1\)-pair reduces to
\[
I_3^1 = \frac{4}{9} (3 + 2\sqrt{3}) \lambda_1, \tag{14}
\]
where the superscript, "1" represents the number of rounds in the sequential scenario. Hence, the non-locality can be demonstrated by showing the violation of CGLMP inequality between Alice and Bob if \( \lambda_1 > 2/((\frac{4}{9})(3 + 2\sqrt{3})) = 0.69615 \) while the optimal quantum value for Alice and Bob\(_1\) is 2.87293 obtained at \( \lambda_1 = 1 \). In a similar fashion, we can find the quantum expressions for Alice-Bob\(_2\) and Alice-Bob\(_3\)-pairs are respectively
\[
I_3^2 = \frac{4\lambda_2}{81} \left[ -2 (\sqrt{3} + 3) \lambda_1 + 12 \sqrt{1 - \lambda_1} \sqrt{2\lambda_1 + 1} + \right.
\left. 4 \sqrt{2\lambda_1 + 1} \sqrt{3 - 3\lambda_1} + 14, \sqrt{3} + 15 \right], \tag{15}
\]
and
\[
I_3^3 = \frac{4\lambda_3}{229} \left[ 4(\sqrt{3} + 6) \left( 2\sqrt{1 - \lambda_2} \sqrt{2\lambda_2 + 1} - \lambda_2 \right) \right.
\left. \times \sqrt{1 - \lambda_1} \sqrt{2\lambda_1 + 1} - 2\lambda_1 (7\sqrt{3} + 15) \lambda_2 \right.
\left. + 2(\sqrt{3} + 6) \sqrt{1 - \lambda_2} \sqrt{2\lambda_2 + 1} - 2(7\sqrt{3} + 15) \lambda_2 \right.
\left. + 4(7\sqrt{3} + 15) (\sqrt{1 - \lambda_1} \sqrt{2\lambda_1 + 1} + \right.
\left. \sqrt{1 - \lambda_2} \sqrt{2\lambda_2 + 1} + 75 + 98\sqrt{3}) \right]. \tag{16}
\]
Considering the situation of minimum violation of \( I_3^1 \) by Alice and Bob\(_1\), the quantum expression of \( I_3^2 \) reduces to be 2.40856\( \lambda_2 \). In this case, the violation of CGLMP inequality for Alice and Bob\(_2\) is possible if \( \lambda_2 > 0.830372 \) while the optimal quantum value is 2.40856 with \( \lambda_2 = 1 \). Substituting the conditions for \( \lambda_1 \) and \( \lambda_2 \), we get that two Bobs surely violate CGLMP inequality. Let us now check whether the third Bob, Bob\(_3\) can also violate CGLMP inequality or not. In this case, the optimal quantum value of \( I_3^3 \) turns out to be 1.83798 < 2 by taking minimum violation condition for Bob\(_2\) and Bob\(_3\). Since optimal quantum value of \( I_3^3 \) is strictly less than 2, we can claim that only two Bobs, Bob\(_1\) and Bob\(_2\), can exhibit nonlocality with Alice by using CGLMP inequality for \( d = 3 \). Notice here that only two Bobs can violate CHSH inequality with Alice if they initially share a two-qubit maximally entangled state [27].

Let us now move to \( d = 4 \) and \( d = 5 \). In this case, \( I_d \) in Eq. (1) reduces to
\[
I_4 = P(A_1 = B_1) + P(B_1 = A_2 + 1) + P(A_2 = B_2) + P(B_2 = A_1) + P(A_1 = B_1) + P(B_1 = A_2) + P(A_2 = B_2) + P(B_2 = A_1) + P(A_1 = B_1) + P(B_1 = A_2) + P(A_2 = B_2) + P(B_2 = A_1) + P(A_1 = B_1) + P(B_1 = A_2) + P(A_2 = B_2) + P(B_2 = A_1) \leq 2.
\]
By following a similar prescription, for a maximally entangled state, \( |\psi_{\mathcal{M}_E}^4\rangle = \frac{1}{2} (|00\rangle + |11\rangle + |22\rangle + |33\rangle) \), we find that Bob\(_1\) starts sharing nonlocality with Alice through the violation of CGLMP when \( \lambda_1 > 0.690551 \) and max \( I_4^1 = 2.89624 \) for \( \lambda_1 = 1 \). Again, if we restrict the situation such that Alice-Bob\(_1\) duo just shows violation, Alice and Bob\(_2\) violate CGLMP when \( \lambda_2 > 0.834603 \) and in the second round, the maximal quantum value is reduced which is 2.39635 (\( \lambda_2 = 1 \)). By taking the minimum violation condition of sharpness parameter for Bob\(_2\) and Bob\(_3\), the optimal quantum value of \( I_3^2 \), given in Table II, again turns out to be less than 2.

| \( d \) | \( \mathcal{A}(\mathcal{M}_E) \) | \( \mathcal{A}(\mathcal{M}_V) \) | Diff. |
|---|---|---|---|
| 3 | 0.05307 | 0.05685 | 7.14% |
| 4 | 0.05317 | 0.06207 | 12.51% |
| 5 | 0.05644 | 0.06595 | 16.85% |
| 6 | 0.0573 | 0.06909 | 23.81% |
| 7 | 0.05792 | 0.07171 | 27.45% |
| 8 | 0.0584 | 0.07382 | 26.40% |
| 9 | 0.05878 | 0.07567 | 28.73% |
| 10 | 0.05906 | 0.07733 | 30.93% |

TABLE I. The \( \mathcal{A} \) values for maximally entangled (\( \mathcal{M}_E \)) as well as maximally violating states (\( \mathcal{M}_V \)), and their percentage differences (labelled as Diff.) from \( d = 3 \) to \( d = 10 \) are given in different columns. The difference grows with \( d \) since the \( \mathcal{A} \) for \( \mathcal{M}_V \) scales much faster with increase in \( d \) than that of \( \mathcal{M}_E \).
For $d = 5,$

$$I_5 = P(A_1 = B_1) + P(B_1 = A_2 + 1) + P(A_2 = B_2 + 1) + P(B_2 = A_1 - 1) + \frac{1}{2} (P(A_1 = B_1 + 1) + P(B_1 = A_2 + 2) + P(A_2 = B_2 + 1) + P(B_2 = A_1 + 1) - [P(A_1 = B_1 - 1) + P(B_1 = A_2 - 1) + P(A_2 = B_2 - 2) + P(B_2 = A_1 - 2)]) \leq 2$$

can be used to obtain violations of CGLMP in a sequential situation with $|\psi_{M_3}^5\rangle = \frac{1}{\sqrt{5}}(|00\rangle + |11\rangle + |22\rangle + |33\rangle + |44\rangle).$ The optimal quantum violation of CGLMP inequality for Bob$_1,$ Bob$_2$ and Bob$_3$ are given in Table II up to $d = 10.$

From Table II, we can see that as the dimension increases, there is an increment of optimal quantum value for Bob$_1$ although the same decreases with the increase of dimension for Bob$_2$ and Bob$_3.$ It also indicates that the trade-off between the information gained by the measurement and the disturbance created by the measurement plays a crucial role in this enterprise.

Since the CGLMP inequality gives the maximum violation for a non-maximally entangled state, let us examine if the initial shared state in a sequential scenario is $M_V$ and whether the situation improves or not. The observation is that although the first round of violation is more and increases faster with $d$ in comparison to the $M_E,$ the measurements disturb the state to such an extent that violation for more than two Bobs still remains an impossibility. Also, note that the third round value of the CGLPM expression decreases on increasing the dimension, so the possibility of getting a simultaneous violation for three rounds is unlikely even if $d$ is increased beyond 10. See Table III for details. Comparing Tables II and III, we observe that the gap between the $I_d^3$ values obtained for $M_V$ and $M_E$ decreases with the increase of dimension. It possibly indicates that the unsharp measurements disturb the $M_V$ more drastically than the $M_E$ in higher dimensions.

### A. Optimality analysis of measurements in the sequential scenario

An important question is whether the POVMs considered by weakening the optimal single round measurement strategy with white noise might not be the optimal one to get the maximal number of Bobs that can sequentially violate the CGLMP inequality with a single Alice. The optimality in the sequential case might be measured by the strategy that maximizes the number of Bobs that violate the CGLMP inequality. However, as one can clearly see, such a definition of optimality leaves a lot of degeneracy in the number of measurement choices that achieve the “optimal” number of Bobs, since the indicator of optimality increases in inte-
Ideally, one would like to perform a “full optimization” to maximize \( n \). The full optimization refers to a maximization over measurement settings of the CGLMP inequality and unsharp parameters of all the \( n \) rounds collectively. Since the measurement settings of the \( n \)th round adaptively depend on the settings of all the previous rounds, the full optimization process becomes realistically intractable. Even if each round is treated independently with an objective to obtain an infinitesimal amount of violation with minimal disturbance to the state, we again run into the domain of infeasibility. This is because if the optimization at every round is carried over arbitrary POVMs, the dimension of the parameter space in the optimization would become very large even if one considers POVMs with a fixed number of outcomes. This again renders the problem intractable. Therefore, we resort to optimization over a simplified setting where we are left with a five-parameter optimization at every round. Specifically, we numerically scan the measurement strategies of the form, the optimal setting used for CGLMP inequality made unsharp with white noise, denoted by (CGLMP + white) to evaluate the optimal setting. In particular, we perform a five-dimensional optimization over \( \alpha, \beta_1, \beta_2 \) and \( \lambda \) to find the violation and the post-measurement state in one round and then perform optimal projective CGLPM measurements to check whether we are getting any violation in the next round. If no, we stop, and if yes, we repeat the same drill of numerical optimization. Our analysis using DIViding RECTangles algorithm [46] for global optimization reveals that for the measurement strategies of the form (CGLMP + white), the best setting is when \( \alpha, \beta_1, \beta_2 \) and \( \lambda \) are chosen to be optimal setting as in the projective (single round) case of the CGLMP inequality.

Lastly we believe that our choice of weak measurements is one of the most “intuitive” options one may consider. If the full optimization, in principle, yields some other strategy, we suspect that this approach might constitute some exotic measurements that might be difficult to implement operationally. In such a situation, we argue that our choice of unsharp measurements is the most practically motivated one.

Now we present an analysis of the level of optimality provided by our choice of the set of weak measurements over which our optimization runs. Given a weak measurement scheme, the optimality of such a strategy involves the study of the tradeoff between the information gain by the measurement and the amount of disturbance it imparts to the state. The amount of disturbance quantifies the quality of measurement \( (F) \) and can be obtained from the post-measurement state \( \rho' \)

\[
\rho' = F\rho + (1-F) \sum_{i=0}^{d-1} P_i \rho P_i^d, \tag{19}
\]

where \( \rho \) is the initial state and \( P_i \)'s are the projectors that have been weakened by the strategy

\[
E_i = \lambda P_i + \frac{1-\lambda}{d} \mathbb{I}_d, \tag{20}
\]

with \( \mathbb{I} \) being the \( d \)-dimensional identity, the same strategy we have used in our work, see Eq. (6). Again, from Eq. (6), we can make the following identification, \( P_i = |l = i\rangle\langle l = i| \). The second quantity of interest measures the amount of information gained in the experiment, which can be interpreted as the precision of the experiment is given by \( G \), where

\[
p(E_i) = G \text{ tr}(P_i\rho) + \frac{1-G}{d}, \tag{21}
\]

where \( p(E_i) \) is the clicking probability of \( E_i \). With \( F \) and \( G \), we have the following information gain vs. disturbance inequality [27]

\[
F^2 + G^2 \leq 1. \tag{22}
\]

The optimal measurement strategy saturates the above inequality. In the \( d = 2 \) case, it was proven that weakening the optimal projective measurements via white noise can saturate the \( F^2 + G^2 \leq 1 \) inequality, thereby demonstrating its optimality. In the absence of the solution for the full optimization, we continue the weakening strategy via white noise in accordance with the optimal setting for \( d = 2 \) to higher dimensions. To test the optimality of our ansatz, for \( d \geq 2 \), we test how close to unity does \( F^2 + G^2 \) goes for our choice of weak measurements. For a general \( d \), we compute

\[
F = 2 \sqrt{(1+(d-1)\lambda)(1-\lambda) + \frac{(d-2)(1-\lambda)}{d}}, \tag{23}
\]

See Appendix B for a detailed calculation. Furthermore, from Eq. (21), we get \( G = \lambda \). We find that for \( d \geq 2 \), \( F^2 + G^2 < 1 \), thereby suggesting that our measurement choices are not optimal. However, to our advantage, our analysis reveals that for low \( d \), the inequality reaches near saturation. For example, for \( d = 3 \) and \( 4 \), we get almost 90% saturation of the inequality for the relevant choice of system parameters, see Fig. 3.

V. ROBUSTNESS IN SEQUENTIAL EXHIBITION OF NONLOCALITY

In Sec. III A, we analyzed how much noise we could add to the state as well as measurements so that it continues to violate the CGLMP inequality. However, the option of using sequential measurements to obtain violations for multiple Bobs with a single Alice opens up a possibility to examine robustness from a new point of view. In this context, we define robustness as the maximal amount of noise that can be added to a state such that the CGLMP inequality can be violated for multiple
rounds, which we claim to be two, since from the previous section, we observed that for the $M_E$ and $M_V$, the maximum number of Bobs that can violate the CGLMP inequality with Alice remains two.

**TABLE IV.** The $q_{\text{min}}$ values for maximally entangled states ($M_E$) and maximally violating states ($M_V$) for $d = 3$ to 10 are reported when we demand violation of CGLMP inequality by two Bobs sequentially with Alice.

| Dimension | $q_{\text{min}}^{M_E}$ | $q_{\text{min}}^{M_V}$ |
|-----------|--------------------------|--------------------------|
| 3         | 0.8773                   | 0.8845                   |
| 4         | 0.8748                   | 0.8872                   |
| 5         | 0.8737                   | 0.8900                   |
| 6         | 0.8736                   | 0.8933                   |
| 7         | 0.8738                   | 0.8963                   |
| 8         | 0.8741                   | 0.8987                   |
| 9         | 0.8748                   | 0.9012                   |
| 10        | 0.8752                   | 0.9034                   |

Let us consider the pure state, $|\psi\rangle$ admixed with white noise, given in Eq. (8), having the visibility, $q$, as an initial state in the sequential scenario. We now demand that if two Bobs have to show a violation of local realism with Alice, both $I_d^q$ and $I_d^V$ have to be greater than 2. We define $q_{\text{min}}$ to be the minimum value of $q$ above which both $I_d^q > 2$ and $I_d^V > 2$. We now compute how the $q_{\text{min}}$ scales with $d$ and compare it with the scaling obtained for $p_{\text{min}}$ as discussed in Sec. III A for both $M_E$ and $M_V$.

Recall that in CGLMP test, we observed an enhanced amount of robustness (as defined in terms of persistence of the violation on the addition of white noise) on increasing $d$ as indicated by lowered values of $p_{\text{min}}$. The maximal amount of white noise that the state can absorb such that the violation persists is simply given by $1 - p_{\text{min}}$. For both $M_E$ and $M_V$, $p_{\text{min}}$ decreases with $d$ [16, 19, 20]. Furthermore, note that we expectedly find $p_{\text{min}} < q_{\text{min}} < 1$.

When robustness is analyzed in the context of sustaining dual-round violation via the use of sequential measurements, we observe a qualitatively different trend. For $M_E$, $q_{\text{min}}$ actually increases with $d$. This implies that robustness actually decreases with $d$ when $M_E$ are employed and we demand CGLMP violations by two Bobs. However, for $M_V$, $q_{\text{min}}$ values do not change significantly on increasing $d$. See Table. IV for details of the $q_{\text{min}}$ values for both $M_E$ and $M_V$. However, in both cases, the gap between $q_{\text{min}}$ and $p_{\text{min}}$ increases with $d$. For a pictorial representation of the situation, see Fig. 4.

The above results explain in part why despite an increase in the first round violation with $d$, one does not get a higher number of Bobs which sequentially violates CGLMP inequality i.e., $I_d^q > 2$ with $k > 2$ for higher dimensional systems. Although the amount of maximal first-round violation grows, the disturbance induced by the measurements is high enough to actually bring down the violation in the second round with $d$ which ultimately leads to the third round becoming non-violating.

**VI. DISCUSSION**

To achieve quantum advantage, manipulating and analysing higher dimensional quantum systems is essential, since, in several quantum information processing tasks, higher dimensional quantum systems turn out to be more beneficial than the qubit pairs. CGLMP inequality is a family of tight Bell inequalities for bipartite systems of arbitrary dimension, which is known to exhibit more robustness against noise with increasing dimension. Therefore, it is interesting to investigate how CGLMP inequality responds if noise is present not only in the state but also in measurement.

We introduced a new measure of robustness which we referred to as the ‘area of nonlocal region’ under consideration of dual noises, both in states and measurements. In particular, this area indicates the region in noise parameter space where violation of CGLMP can be observed. We found that this region grows more rapidly with the increase of dimension with respect to the increase in visibility associated solely with states or measurements.

The introduction of noise in measurement facilitates to invoke of a sequential violation of CGLMP inequality as it retains some residual correlation after obtaining a violation in the first round. Interestingly, we found that the violation of CGLMP inequality by two sequential observers on one side and another observer on the other end persists with dimension. Moreover, the minimum visibility required to achieve double violation in...
the sequential case increases with the increase of dimension, thereby exhibiting the opposite behaviour as compared to the violation obtained for the shared state without unsharp measurement. It indicates that robustness in the sequential measurement scenario is qualitatively distinct from that of the typical Bell test since it involves the disturbance of the state introduced via the measurement. It is interesting to probe further how the double violation obtained in CGLMP inequality enables applications in the context of information processing tasks involving higher dimensional quantum systems.

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Appendix A: The maximally violating states

The expressions for the maximally violating states are given in [45]. We state them here once more for the convenience of the readers.

\[
\begin{align*}
|\psi_{MV}^{d=3}\rangle & = 0.6169 |00\rangle + 0.4888 |11\rangle + 0.6169 |22\rangle, \\
|\psi_{MV}^{d=4}\rangle & = 0.5686 |00\rangle + 0.4204 |11\rangle + 0.4204 |22\rangle + 0.5686 |33\rangle, \\
|\psi_{MV}^{d=5}\rangle & = 0.5368 |00\rangle + 0.3859 |11\rangle + 0.3548 |22\rangle + 0.3859 |33\rangle + 0.5368 |44\rangle, \\
|\psi_{MV}^{d=6}\rangle & = 0.5137 |00\rangle + 0.3644 |11\rangle + 0.3214 |22\rangle + 0.3214 |33\rangle + 0.3644 |44\rangle + 0.5137 |55\rangle, \\
|\psi_{MV}^{d=7}\rangle & = 0.4957 |00\rangle + 0.3493 |11\rangle + 0.3011 |22\rangle + 0.2882 |33\rangle + 0.3011 |44\rangle + 0.3493 |55\rangle + 0.4957 |66\rangle, \\
|\psi_{MV}^{d=8}\rangle & = 0.4812 |00\rangle + 0.3379 |11\rangle + 0.2872 |22\rangle + 0.2679 |33\rangle + 0.2679 |44\rangle + 0.2872 |55\rangle + 0.3379 |66\rangle + 0.4812 |77\rangle, \\
|\psi_{MV}^{d=9}\rangle & = 0.4690 |00\rangle + 0.3288 |11\rangle + 0.2770 |22\rangle + 0.2541 |33\rangle + 0.2474 |44\rangle + 0.2541 |55\rangle + 0.2770 |66\rangle + 0.3288 |77\rangle + 0.4690 |88\rangle, \\
|\psi_{MV}^{d=10}\rangle & = 0.4587 |00\rangle + 0.3212 |11\rangle + 0.2690 |22\rangle + 0.2440 |33\rangle + 0.2334 |44\rangle + 0.2334 |55\rangle + 0.2440 |66\rangle + 0.2690 |77\rangle + 0.3212 |88\rangle + 0.4587 |99\rangle. \\
\end{align*}
\]

Appendix B: Computation of measurement quality index $F$

Following the weak measurement strategy given in Eq. (20), the post measurement state reads as

\[ \rho'(1) = \sum_{i=0}^{d-1} M_i \rho M_i^\dagger, \quad (B1) \]

where $M_i$s are the update operators given by

\[ M_i = \sqrt{E_i} = xP_i + y(1 - P_i) \quad (B2) \]

with

\[ x = \sqrt{\frac{1 + (d - 1)\lambda}{d}}, \quad y = \sqrt{\frac{1 - \lambda}{d}}. \quad (B3) \]

See Eq. (20) for the form of $E_i$s in terms of $P_i$s. Now $\rho'$ can be written as

\[
\rho' = \sum_{i=0}^{d-1} P_i \rho P_i + (2xy + (d - 2)y^2) \sum_{i=j=0}^{d-1} P_i \rho P_j, \\
= (2xy + (d - 2)y^2) \rho + (1 - 2xy - (d - 2)y^2) \sum_{i=0}^{d-1} P_i \rho P_i. \quad (B4)
\]

Now following Eq. (19), we make the following identification for the quality factor of the measurement

\[
F = 2xy + (d - 2)y^2, \\
= \frac{2}{d} \sqrt{(1 + (d - 1)\lambda)(1 - \lambda)} + \frac{(d - 2)(1 - \lambda)}{d}. \quad (B5)
\]
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