Non-existence of points rational over number fields on Shimura curves

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Abstract

Jordan, Rotger and de Vera-Piquero proved that Shimura curves have no points rational over imaginary quadratic fields under a certain assumption. In this article, we expand their results to the case of number fields of higher degree. We also give counterexamples to the Hasse principle on Shimura curves.

1 Introduction

Let $B$ be an indefinite quaternion division algebra over $\mathbb{Q}$, and $d(B)$ its discriminant. Fix a maximal order $\mathcal{O}$ of $B$. A QM-abelian surface by $\mathcal{O}$ over a field $F$ is a pair $(A, i)$ where $A$ is a 2-dimensional abelian variety over $F$, and $i : \mathcal{O} \hookrightarrow \text{End}_F(A)$ is an injective ring homomorphism satisfying $i(1) = id$ (cf. [2, p.591]). Here, $\text{End}_F(A)$ is the ring of endomorphisms of $A$ defined over $F$. We assume that $A$ has a left $\mathcal{O}$-action. Let $M^B$ be the Shimura curve over $\mathbb{Q}$ associated to $B$, which parameterizes isomorphism classes of QM-abelian surfaces by $\mathcal{O}$ (cf. [3, p.93]). We know that $M^B$ is a proper smooth curve over $\mathbb{Q}$. For an imaginary quadratic field $k$, we have $M^B(k) = \emptyset$ under a certain assumption ([3, Theorem 6.3], [5, Theorem 1.1]). We expand this result to the case of number fields of higher degree in this article. The method of the proof is based on the strategy in [3], and the key is to control the field of definition of the QM-abelian surface corresponding to a rational point on $M^B$.

We also give counterexamples to the Hasse principle on $M^B$ over number fields. We will discuss the relevance to the Manin obstruction in a forthcoming article.

For a prime number $q$, let $\mathcal{B}(q)$ be the set of isomorphism classes of indefinite quaternion division algebras $B$ over $\mathbb{Q}$ such that

$$
\begin{cases}
B \otimes_{\mathbb{Q}} \mathbb{Q}((\sqrt{-q})) \neq M_2(\mathbb{Q}((\sqrt{-q}))) & \text{if } q \neq 2, \\
B \otimes_{\mathbb{Q}} \mathbb{Q}((\sqrt{-1})) \neq M_2(\mathbb{Q}((\sqrt{-1}))) \text{ and } B \otimes_{\mathbb{Q}} \mathbb{Q}((\sqrt{-2})) \neq M_2(\mathbb{Q}((\sqrt{-2}))) & \text{if } q = 2.
\end{cases}
$$

For positive integers $N$ and $e$, let

$$
\mathcal{C}(N, e) := \left\{ \alpha^e + \overline{\alpha}^e \in \mathbb{Z} \mid \alpha \in \mathbb{C} \text{ is a root of } T^2 + sT + N \text{ for some } s \in \mathbb{Z}, s^2 \leq 4N \right\},
$$

$$
\mathcal{D}(N, e) := \left\{ a, a \pm N^\frac{e}{2}, a \pm 2N^\frac{e}{2}, a^2 - 3N^e \in \mathbb{R} \mid a \in \mathcal{C}(N, e) \right\}.
$$
Here, $\overline{\alpha}$ is the complex conjugate of $\alpha$. If $e$ is even, then $D(N, e) \subseteq \mathbb{Z}$. For a subset $D \subseteq \mathbb{Z}$, let

$$\mathcal{P}(D) := \{ \text{prime divisors of some of the integers in } D \setminus \{0\} \}.$$

For a number field $k$ and a prime $q$ of $k$ of residue characteristic $q$, let

- $\kappa(q)$: the residue field of $q$,
- $N_q$: the cardinality of $\kappa(q)$,
- $e_q$: the ramification index of $q$ in $k/\mathbb{Q}$,
- $f_q$: the degree of the extension $\kappa(q)/\mathbb{F}_q$,
- $S(k, q)$: the set of isomorphism classes of indefinite quaternion division algebras $B$ over $\mathbb{Q}$ such that any prime divisor of $d(B)$ belongs to

$$\begin{cases} \mathcal{P}(D(N_q, e_q)) \cup \{q\} & \text{if } B \otimes_{\mathbb{Q}} k \cong M_2(k) \text{ and } e_q \text{ is even}, \\ \mathcal{P}(D(N_q, 2e_q)) \cup \{q\} & \text{if } B \otimes_{\mathbb{Q}} k \not\cong M_2(k). \end{cases}$$

Note that $S(k, q)$ is a finite set. The main result of this article is:

**Theorem 1.1.** Let $k$ be a number field of even degree, and $q$ a prime number such that

- there is a unique prime $q$ of $k$ above $q$,
- $f_q$ is odd (and so $e_q$ is even), and
- $B \in \mathcal{B}(q) \setminus S(k, q)$.

Then $M^B(k) = \emptyset$.

**Remark 1.2.** (1) By [7, Theorem 0], we have $M^B(\mathbb{R}) = \emptyset$.

(2) If $k$ is of odd degree, then $k$ has a real place, and so $M^B(k) = \emptyset$.

## 2 Canonical isogeny characters

In this section, we review canonical isogeny characters associated to QM-abelian surfaces, which were introduced in [3, §4]. Let $K$ be a number field, $\overline{K}$ an algebraic closure of $K$, $G_K = \text{Gal}(\overline{K}/K)$ the absolute Galois group of $K$, $\mathcal{O}_K$ the ring of integers of $K$, $(A, i)$ a QM-abelian surface by $\mathcal{O}$ over $K$, and $p$ a prime divisor of $d(B)$. Then the $p$-torsion subgroup $A[p](\overline{K})$ of $A$ has exactly one non-zero proper left $\mathcal{O}$-submodule, which we shall denote by $C_p$. Then $C_p$ has order $p^2$, and is stable under the action of $G_K$. Let $\mathfrak{P}_\mathcal{O} \subseteq \mathcal{O}$ be the unique left ideal of reduced norm $p\mathbb{Z}$. 

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In fact, $\mathcal{P}_O$ is a two-sided ideal of $O$. Then $C_p$ is free of rank 1 over $O/\mathcal{P}_O$. Fix an isomorphism $O/\mathcal{P}_O \cong \mathbb{F}_{p^2}$. The action of $G_K$ on $C_p$ yields a character
\[ \varrho_p : G_K \rightarrow \text{Aut}_O(C_p) \cong \mathbb{F}_{p^2}^\times. \]
Here, $\text{Aut}_O(C_p)$ is the group of $O$-linear automorphisms of $C_p$. The character $\varrho_p$ depends on the choice of the isomorphism $O/\mathcal{P}_O \cong \mathbb{F}_{p^2}$, but the pair $\{ \varrho_p, (\varrho_p)^p \}$ is independent of this choice. Either of the characters $\varrho_p, (\varrho_p)^p$ is called a canonical isogeny character at $p$. We have an induced character
\[ \varrho^{ab}_p : G^{ab}_K \rightarrow \mathbb{F}_{p^2}^\times, \]
where $G^{ab}_K$ is the Galois group of the maximal abelian extension $K^{ab}/K$.

For a prime $\ell$ of $K$, let $\mathcal{O}_{K,\ell}$ be the completion of $\mathcal{O}_K$ at $\ell$, and
\[ r_p(\ell) : \mathcal{O}_K^\times \rightarrow \mathbb{F}_{p^2}^\times \]
the composition
\[ \mathcal{O}_{K,\ell}^\times \xrightarrow{\omega_{\ell}} G^{ab}_K \xrightarrow{\varrho^{ab}_p} \mathbb{F}_{p^2}^\times. \]
Here $\omega_{\ell}$ is the Artin map.

**Proposition 2.1** ([3 Proposition 4.7 (2)]). If $\ell \nmid p$, then $r_p(\ell)^{12} = 1$.

Fix a prime $\mathcal{P}$ of $K$ above $p$. Then we have an isomorphism $\kappa(\mathcal{P}) \cong \mathbb{F}_{p^2}$ of finite fields. Let $\ell_{\mathcal{P}} := \gcd(2, f_{\mathcal{P}}) \in \{1, 2\}$.

**Proposition 2.2** ([3 Proposition 4.8]).

1. There is a unique element $c_{\mathcal{P}} \in \mathbb{Z}/(p^{\ell_{\mathcal{P}}} - 1)\mathbb{Z}$ satisfying $r_p(\mathcal{P})(u) = \text{Norm}_{\kappa(\mathcal{P})/\mathbb{F}_{p^2}}(\tilde{u})^{-c_{\mathcal{P}}}$ for any $u \in \mathcal{O}_{K,\mathcal{P}}^\times$. Here, $\tilde{u} \in \kappa(\mathcal{P})$ is the reduction of $u$ modulo $\mathcal{P}$.

2. \[ \frac{2c_{\mathcal{P}}}{\ell_{\mathcal{P}}} \equiv c_{\mathcal{P}} \mod (p - 1). \]

**Corollary 2.3.** For any prime number $l \neq p$, we have $r_p(\mathcal{P})(l^{-1})^2 = l^{c_{\mathcal{P}}f_{\mathcal{P}}}$ mod $p$.

**Proof.**
\[ r_p(\mathcal{P})(l^{-1})^2 = (\text{Norm}_{\kappa(\mathcal{P})/\mathbb{F}_{p^2}}(l^{-1})^{c_{\mathcal{P}}})^2 = \text{Norm}_{\mathbb{F}_{p^2}/\mathbb{F}_{p_{\mathcal{P}}}}(l)^{2c_{\mathcal{P}}} = l^{\frac{2c_{\mathcal{P}}}{\ell_{\mathcal{P}}}} \equiv l^{c_{\mathcal{P}}f_{\mathcal{P}}} \mod p. \]

For a prime number $l$, the action of $G_K$ on the $l$-adic Tate module $T_l A$ yields a representation
\[ R_l : G_K \rightarrow \text{Aut}_O(T_l A) \cong \mathcal{O}_l^\times \subseteq B_l^\times, \]
where $\text{Aut}_O(T_l A)$ is the group of automorphisms of $T_l A$ commuting with the action of $O$, and $\mathcal{O}_l = O \otimes \mathbb{Z}_l$, $B_l = B \otimes \mathbb{Q}_l$. Let $\text{Nrd}_{B_l/\mathbb{Q}_l}$ be the reduced norm on $B_l$. Let $\mathfrak{M}$ be a prime of $K$, and $F_{\mathfrak{M}} \in G_K$ a Frobenius element at $\mathfrak{M}$. For each $e \geq 1$, there is an integer $a(F_{\mathfrak{M}}^e) \in \mathbb{Z}$ satisfying
\[ \text{Nrd}_{B_l/\mathbb{Q}_l}(T - R_l(F_{\mathfrak{M}}^e)) = T^2 - a(F_{\mathfrak{M}}^e)T + (N_{\mathfrak{M}})^e \in \mathbb{Z}[T] \]
for any $l$ prime to $\mathfrak{M}$. 

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Proposition 2.4 ([3, Proposition 5.3]). (1) We have $a(F_{2M}^e)^2 \leq 4(N_{2M})^e$ for any positive integer $e$.

(2) Assume $M \nmid p$. Then

$$a(F_{2M}^e) \equiv q_p(F_{2M}^e) + (N_{2M})^e q_p(F_{2M}^e)^{-1} \mod p$$

for any positive integer $e$.

Let $\alpha_{2M}, \overline{\alpha}_{2M} \in \mathbb{C}$ be the roots of $T^2 - a(F_{2M})T + N_{2M}$. Then $\alpha_{2M} + \overline{\alpha}_{2M} = a(F_{2M})$ and $\alpha_{2M}\overline{\alpha}_{2M} = N_{2M}$. We see that the roots of $T^2 - a(F_{2M}^e)T + (N_{2M})^e$ are $\alpha_{2M}^e, \overline{\alpha}_{2M}^e$. Then $\alpha_{2M}^e + \overline{\alpha}_{2M}^e = a(F_{2M}^e)$. We have the following corollary to Proposition 2.4(1) (for $e = 1$):

**Corollary 2.5.** We have $a(F_{2M}) \in \mathcal{C}(N_{2M})$ for any positive integer $e$.

For a later use, we give the following lemma:

**Lemma 2.6.** Let $m$ be the residue characteristic of $M$. The the following conditions are equivalent:

(i) $m \mid a(F_{2M})$.
(ii) $m \mid a(F_{2M}^e)$ for a positive integer $e$.
(iii) $m \mid a(F_{2M}^e)$ for any positive integer $e$.

**Proof.** For each $e \geq 1$, there is a polynomial $P_e(S, T) \in \mathbb{Z}[S, T]$ such that $(S + T)^e = S^e + T^e + STP_e(S + T, ST)$. Then $a(F_{2M})^e = a(F_{2M})^e + N_{2M}P_e(a(F_{2M}), N_{2M})$. Since $m \mid N_{2M}$, we have $m \mid a(F_{2M})$ if and only if $m \mid a(F_{2M}^e)$.

\[\square\]

**3 Proof of the main result**

Now we prove Theorem 1.1. Suppose that the assumption of Theorem 1.1 holds. Assume that there is a point $x \in M^B(k)$. When $B \otimes_k k \not\cong M_2(k)$, let $K_0$ be a quadratic extension of $k$ satisfying $B \otimes_k K_0 \cong M_2(K_0)$. Let

$$K := \begin{cases} k & \text{if } B \otimes_k k \cong M_2(k), \\ K_0 & \text{if } B \otimes_k k \not\cong M_2(k). \end{cases}$$

Note that the degree $[K : \mathbb{Q}]$ is even. Then $x$ is represented by a QM-abelian surface $(A, i)$ by $O$ over $K$ (see [3, Theorem 1.1]). Since $B \not\in S(k, q)$, there is a prime divisor $p$ of $d(B)$ such that $p \not= q$ and $p$ does not belong to

$$\begin{cases} \mathcal{P}(\mathcal{D}(N_q, e_q)) & \text{if } B \otimes_k k \cong M_2(k), \\ \mathcal{P}(\mathcal{D}(N_q, 2e_q)) & \text{if } B \otimes_k k \not\cong M_2(k). \end{cases}$$

Fix such $p$, and let

$$q_p : G_K \longrightarrow \mathbb{F}_p^\times$$

be a canonical isogeny character at $p$ associated to $(A, i)$.
By Proposition 2.1, the character $\varphi_p^{12}$ is unramified outside $p$. Then it is identified with a character $3_k(p) \rightarrow \mathbb{F}_p^{\times}$, where $3_k(p)$ is the group of fractional ideals of $K$ prime to $p$. When $B \otimes Q k \not\cong M_2(k)$, we may assume that $q$ is ramified in $K/k$ by replacing $K_0$ if necessary. In any case, let $Q$ be the unique prime of $K$ above $q$. Note that $Q$ is the unique prime of $K$ above $q$, and so $qO_K = Q^{e_Q}, (N_Q)^{e_Q} = (qJ_Q)^{e_Q} = q^{[K:Q]}$. Then by Corollary 2.3, we have

$$\varphi_p^{12}(P_Q) = \varphi_p^{12}(Q^{e_Q}) = \varphi_p^{12}(qO_K) = \varphi_p^{12}(1, \ldots, 1, q, \ldots, q, \ldots) = \varphi_p^{12}(q^{-1}, \ldots, q^{-1}, 1, \ldots, 1, \ldots) = \prod_{\psi \mid p} r_p(\psi)^{12}(q^{-1}) \equiv \prod_{\psi \mid p} q^{6e_Q} \psi = q^{6[K:Q]} \mod p.$$

Here, $(1, \ldots, 1, q, \ldots, q, \ldots)$ (resp. $(q^{-1}, \ldots, q^{-1}, 1, \ldots, 1, \ldots)$) is the idèle of $K$ whose components above $p$ are 1 and the others $q$ (resp. whose components above $p$ are $q^{-1}$ and the others 1), and $\mathcal{P}$ runs through the primes of $K$ above $p$. On the other hand, we have

$$a(F_Q^{e_Q}) \equiv \varphi_p(F_Q^{e_Q}) + (N_Q)^{e_Q} \varphi_p(F_Q^{e_Q})^{-1} = \varphi_p(F_Q^{e_Q}) + q^{[K:Q]} \varphi_p(F_Q^{e_Q})^{-1} \mod p.$$ 

by Proposition 2.4(2). Let $\varepsilon := q^{-\frac{[K:Q]}{2}} \varphi_p(F_Q^{e_Q}) \in \mathbb{F}_p^\times$. Then

$$\varepsilon^{12} = 1 \quad \text{and} \quad a(F_Q^{e_Q}) \equiv (\varepsilon + \varepsilon^{-1})q^{\frac{[K:Q]}{2}} \mod p.$$ 

Therefore

$$a(F_Q^{e_Q}) \equiv 0, \pm q^{\frac{[K:Q]}{2}}, \pm 2q^{\frac{[K:Q]}{2}} \mod p \quad \text{or} \quad a(F_Q^{e_Q})^2 \equiv 3q^{[K:Q]} \mod p.$$ 

By Corollary 2.5, we have $a(F_Q^{e_Q}) \in C(N_Q, e_Q)$. We also have

$$N_Q = N_q \quad \text{and} \quad e_Q = \begin{cases} e_q & \text{if } B \otimes Q k \cong M_2(k), \\ 2e_q & \text{if } B \otimes Q k \not\cong M_2(k). \end{cases}$$ 

Then

$$a(F_Q^{e_Q}), a(F_Q^{e_Q}) \pm q^{\frac{[K:Q]}{2}}, a(F_Q^{e_Q}) \pm 2q^{\frac{[K:Q]}{2}}, a(F_Q^{e_Q})^2 - 3q^{[K:Q]} \in D(N_Q, e_Q).$$

Since $p \not\in \mathcal{P}(D(N_q, e_Q))$, we have

(1) $a(F_Q^{e_Q}) = 0, \pm q^{\frac{[K:Q]}{2}}, \pm 2q^{\frac{[K:Q]}{2}}$, or

(2) $a(F_Q^{e_Q})^2 = 3q^{[K:Q]}$.

[Case (1)]. In this case, we have $q \mid a(F_Q^{e_Q})$. Then by Lemma 2.6 we have $q \mid a(F_Q)$. Since $f_Q(= f_q)$ is odd, we obtain $B \otimes Q \mathbb{Q}(\sqrt{-q}) \cong M_2(\mathbb{Q}(\sqrt{-q}))$ or $q = 2$ and $B \otimes Q \mathbb{Q}(\sqrt{-1}) \cong M_2(\mathbb{Q}(\sqrt{-1}))$ (see [3, Theorem 2.1, Propositions 2.3 and 5.1 (1)]). This contradicts $B \in B(q)$.

[Case (2)]. In this case, $q = 3$ and $[K : \mathbb{Q}]$ is odd, which is a contradiction.

Therefore we conclude $M^B(k) = \emptyset$. □
| $(N, e)$ | $\mathcal{C}(N, e)$ | $\mathcal{D}(N, e)$ | $\mathcal{P}(\mathcal{D}(N, e))$ |
|---|---|---|---|
| $(2, 2)$ | $0, -3, -4$ | $0, \pm 1, \pm 2, -3, \pm 4, -5, -6, -7, -8, -12$ | $2, 3, 5, 7$ |
| $(2, 4)$ | $1, \pm 8$ | $0, 1, -3, \pm 4, 5, -7, \pm 8, 9, \pm 12, \pm 16, -47$ | $2, 3, 5, 7, 47$ |
| $(2, 6)$ | $0, 9, -16$ | $0, 1, -7, \pm 8, 9, \pm 16, 17, -24, 25, -32, 64, -111, -192$ | $2, 3, 5, 7, 17, 37$ |
| $(2, 8)$ | $-31, 32$ | $0, 1, -15, 16, -31, 32, -47, 48, -63, 64, 193, 256$ | $2, 3, 5, 7, 19, 47, 193$ |
| $(2, 10)$ | $0, 57, -64$ | $0, -7, 25, \pm 32, 57, \pm 64, 89, -96, 121, -128, 177, 1024, -3072$ | $2, 3, 5, 7, 11, 19, 59, 89$ |
| $(2, 12)$ | $-47, \pm 128$ | $0, 17, -47, \pm 64, 81, -111, \pm 128, -175, \pm 192, \pm 256, 4096, -10079$ | $2, 3, 5, 7, 17, 37, 47, 10079$ |
| $(2, 14)$ | $0, -87, -256$ | $0, 41, -87, \pm 128, 169, -215, \pm 256, -343, -384, -512, 16384, -41583, -49152$ | $2, 3, 5, 7, 13, 29, 41, 43, 83, 167$ |
| $(2, 16)$ | $449, 512$ | $0, -63, 193, 256, 449, 512, 705, 768, 961, 1024, 4993, 65536$ | $2, 3, 5, 7, 19, 47, 193, 449, 4993$ |
| $(3, 2)$ | $-2, 3, -5, -6$ | $0, 1, 2, \pm 3, 4, -5, \pm 6, -8, \pm 9, -11, -12, -18, -23$ | $2, 3, 5, 11, 23$ |
| $(3, 4)$ | $7, -9, -14, 18$ | $0, -2, 4, -5, 7, \pm 9, -11, -14, 16, \pm 18, -23, 25, \pm 27, -32, 36, -47, 81, -162, -194$ | $2, 3, 5, 7, 11, 23, 47, 97$ |
| $(3, 6)$ | $10, 46, -54$ | $0, -8, 10, -17, 19, -27, 37, -44, 46, -54, 64, -71, 73, -81, 100, -108, 729, -2087$ | $2, 3, 5, 7, 11, 19, 23, 37, 71, 73, 2087$ |
| $(3, 8)$ | $34, -81, -113, 162$ | $0, -32, 34, -47, 49, \pm 81, -113, 115, -128, \pm 162, -194, 196, \pm 243, -275, 324, 6561, -6914, -13122, -18527$ | $2, 3, 5, 7, 11, 17, 23, 47, 97, 113, 191, 3457$ |
| $(3, 10)$ | $243, 475, -482, -486$ | $0, 4, -11, 232, -239, \pm 243, 475, \pm 482, \pm 486, 718, -725, \pm 729, 961, -968, -972, 48478, 55177, 59049, -118098$ | $2, 3, 5, 7, 11, 19, 23, 31, 239, 241, 359, 2399, 24239$ |
| $(3, 12)$ | $658, -1358, 1458$ | $0, -71, 100, -629, 658, 729, -800, -1358, 1387, 1458, -2087, 2116, 2187, -2816, 2916, 249841, 531441, -1161359$ | $2, 3, 5, 7, 11, 17, 19, 23, 37, 47, 71, 73, 97, 433, 577, 1009, 1151, 2087$ |
| $(3, 14)$ | $2187, 2515, 3022, -4374$ | $0, 328, 835, -1352, -1859, \pm 2187, 2515, 3022, \pm 4374, 4702, 5209, \pm 6561, 6889, 7396, -8748, 4782969, -5216423, -8023682, -9565938$ | $2, 3, 5, 7, 11, 13, 23, 41, 43, 83, 167, 337, 503, 673, 1511, 2351, 5209, 24023$ |
| $(3, 16)$ | $-353, -6561, -11966, 13122$ | $0, -353, 1156, -5405, 6208, \pm 6561, -6914, -11966, 12769, \pm 13122, -13475, -18527, \pm 19683, -25088, 26244, 14044993, 43046721, -86093442, -129015554$ | $2, 3, 5, 7, 11, 17, 23, 31, 47, 97, 113, 191, 193, 353, 383, 2113, 3457, 30529, 36671$ |
4 Counterexamples to the Hasse principle

We have computed the sets \( \mathcal{C}(N,e), \mathcal{D}(N,e), \mathcal{P}(\mathcal{D}(N,e)) \) in several cases as seen in Table II. Then we obtain the following counterexamples to the Hasse principle on \( M^B \) over number fields:

**Proposition 4.1.** (1) Let \( d(B) = 39 \), and let \( k = \mathbb{Q}(\sqrt{2}, \sqrt{-13}) \) or \( \mathbb{Q}(\sqrt{-2}, \sqrt{13}) \). Then \( B \otimes_{\mathbb{Q}} k \cong M_2(k) \), \( M^B(k) = \emptyset \) and \( M^B(k_v) \neq \emptyset \) for any place \( v \) of \( k \). Here, \( k_v \) is the completion of \( k \) at \( v \).

(2) Let \( L \) be the subfield of \( \mathbb{Q}(\zeta_9) \) satisfying \( [L : \mathbb{Q}] = 3 \), and let \( (d(B), k) = (62, L(\sqrt{-39})) \) or \( (86, L(\sqrt{-15})) \). Then \( B \otimes_{\mathbb{Q}} k \not\cong M_2(k) \), \( M^B(k) = \emptyset \) and \( M^B(k_v) \neq \emptyset \) for any place \( v \) of \( k \).

**Proof.** (1) The prime number 3 (resp. 13) is inert (resp. ramified) in \( \mathbb{Q}(\sqrt{-13}) \). Then \( B \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{-13}) \cong M_2(\mathbb{Q}(\sqrt{-13})) \), and so \( B \otimes_{\mathbb{Q}} k \cong M_2(k) \).

Applying Theorem 1.1 to \( q = 2 \), we obtain \( M^B(k) = \emptyset \). In fact, \((e_q, f_q) = (4, 1)\) where \( q \) is the unique prime of \( k \) above \( q = 2 \), and the prime divisor 13 of \( d(B) \) does not belong to \( \mathcal{P}(\mathcal{D}(2,4)) \cup \{2\} \) (see Table I). Since 3 (resp. 13) splits in \( \mathbb{Q}(\sqrt{-2}) \) (resp. \( \mathbb{Q}(\sqrt{-1}) \)), we have \( B \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{-2}) \not\cong M_2(\mathbb{Q}(\sqrt{-2})) \) (resp. \( B \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{-1}) \not\cong M_2(\mathbb{Q}(\sqrt{-1})) \)).

By [3, p.94], we have \( M^B(\mathbb{Q}(\sqrt{-13})_w) = \emptyset \) for any place \( w \) of \( \mathbb{Q}(\sqrt{-13}) \) (cf. [4]). Therefore \( M^B(k_v) \neq \emptyset \) for any place \( v \) of \( k \).

(2) For a field \( F \) of characteristic \( \neq 2 \) and two elements \( a, b \in F^\times \), let

\[
\left( \frac{a,b}{F} \right) = F + F e + F f + F e f
\]

be the quaternion algebra over \( F \) defined by

\[
e^2 = a, \quad f^2 = b, \quad e f = - f e.
\]

For a prime number \( p \), let \( e_p, f_p, g_p \) be the ramification index of \( p \) in \( k/\mathbb{Q} \), the degree of the residue field extension above \( p \) in \( k/\mathbb{Q} \), and the number of primes of \( k \) above \( p \) respectively.

Let \( (d(B), k) = (62, L(\sqrt{-39})) \) (resp. \( (86, L(\sqrt{-15})) \)). First, we prove \( B \otimes_{\mathbb{Q}} k \not\cong M_2(k) \). We see \( B \cong \left( \frac{62,13}{\mathbb{Q}} \right) \) (resp. \( \left( \frac{86,5}{\mathbb{Q}} \right) \)) by [6, §3.6 g]). We have \((e_2, f_2, g_2) = (1,3,2)\). Let \( v \) be place of \( k \) above 2. By the same argument as in the proof of [1, Proposition 8.1], we have \( B \otimes_{\mathbb{Q}} k_v \not\cong M_2(k_v) \). Therefore \( B \otimes_{\mathbb{Q}} k \not\cong M_2(k) \).

Applying Theorem 1.1 to \( q = 3 \), we obtain \( M^B(k) = \emptyset \). In fact, \((e_q, f_q) = (6, 1)\) where \( q \) is the unique prime of \( k \) above \( q = 3 \), and the prime divisor 31 (resp. 43) of \( d(B) \) does not belong to \( \mathcal{P}(\mathcal{D}(3,12)) \cup \{3\} \). Since 31 (resp. 43) splits in \( \mathbb{Q}(\sqrt{-3}) \), we have \( B \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{-3}) \not\cong M_2(\mathbb{Q}(\sqrt{-3})) \).

By [5, Table 1], we have \( M^B(\mathbb{Q}(\sqrt{-39})_w) \neq \emptyset \) (resp. \( M^B(\mathbb{Q}(\sqrt{-15})_w) \neq \emptyset \)) for any place \( w \) of \( \mathbb{Q}(\sqrt{-39}) \) (resp. \( \mathbb{Q}(\sqrt{-15}) \)). Therefore \( M^B(k_v) \neq \emptyset \) for any place \( v \) of \( k \).

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References

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