An Itô Formula for Isochron Maps in Separable Hilbert Space  
(preprint)

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Abstract

In this note, we prove an Itô formula for the isochron map of a reaction-diffusion system. This follows the proof of a new result which states that the second derivative of the isochron map of a reaction-diffusion system is trace class. This result, in turn, is a corollary of Proposition 2.3, which guarantees that the first and second Fréchet derivatives of the flow of a reaction-diffusion system with respect to initial conditions are trace class.

Keywords: Isochrons • Itô’s Lemma • Reaction-diffusion equations

1 Setup

Consider an autonomous reaction-diffusion system

$$\partial_t x = V(x) = D\Delta x + N(x),$$

where $N$ is a Nemytskii polynomial operator and $D$ is a diagonal matrix with non-negative (possibly zero) diagonal entries. Supposing solutions to (1) can be a priori bounded in $C_b(S^1; \mathbb{R}^n)$, then unique, global-in-time solutions exist in $C_b(S^1; \mathbb{R}^n) \subset L^2(S^1; \mathbb{R}^n)$, where $S^1$ is the circle. For the rest of the document we let $H := L^2(S^1; \mathbb{R}^n)$ and $E := C_b(S^1; \mathbb{R}^n)$. Let the flow of (1) be $(t, x) \mapsto \phi_t(x)$, expressed using Duhamel’s principle as

$$\phi_t(x) = e^{tD\Delta}x + \int_0^t e^{(t-s)D\Delta}N(\phi_s(x))\,ds.$$  

Suppose (1) has an attracting limit cycle $\Gamma = \{\gamma_t\}_{t \in \mathbb{R}}$ of period $T > 0$, and let $B(\Gamma)$ be its basin of attraction. That is, for $s, t \in \mathbb{R}$,

$$\phi_t(\gamma_s) = \gamma_{s+t}, \quad \gamma_{s+T} = \gamma_s.\footnote{We will often consider solutions as objects in $L^2(S^1; \mathbb{R}^n)$, to make use of its Hilbert space structure. We also remark that our main arguments extend to bounded spatial domains of higher dimension and with other boundary conditions, but we here work on the circle for simplicity.}$$

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and for any $x \in B(\Gamma)$ there is an $s \in [0,T)$ such that 
\[
\|\phi_t(x) - \gamma_{t+s}\| \xrightarrow{t \to \infty} 0.
\]

We now define the isochron map and its derivatives – see [3] for further details. The isochron map is an operator $\pi : B(\Gamma) \to [0,T)$ defined for $x \in B(\Gamma)$ as the unique number $\pi(x) \in [0,T)$ such that 
\[
\lim_{t \to \infty} \|\phi_t(x) - \gamma_{t+\pi(x)}\| = 0.
\]
The isochron map is well-defined on $B(\Gamma)$ by the definition of the basin of attraction and the continuity of the flow in a neighbourhood of $\Gamma$. The isochron map may also be characterized as the unique operator such that 
\[
\begin{align*}
\pi(\phi_T(x)) &= \pi(x), \\
\pi(\gamma_t) &= t \mod T,
\end{align*}
\]
for $x \in B(\Gamma)$ and $t \in \mathbb{R}$. The characterization (2), (3), of the isochron map is essential to our arguments.

We can now state the main goal of this paper, which is to prove the following theorem, stated colloquially for now. Theorem 3.1 is a more precise formulation, presented with proof.

**Theorem.** Let $(X_t)_{t \geq 0}$ be an Itô diffusion generated by a stochastic perturbation of (1) with non-trace class noise. Then, $\pi(X_t)$ satisfies the Itô formula (13).

Suppose that the flow of (1) is twice continuously Fréchet differentiable \(^2\) with respect to any initial condition $x_0 \in B(\Gamma)$, with derivatives denoted by 
\[
x \mapsto D\phi_t(x_0)[x], \quad (x, y) \mapsto D^2\phi_t(x_0)[x, y].
\]

Section 4.2 in [4] can be modified to prove that $\pi$ is $C^2$ in a $\delta$-neighbourhood of $\Gamma$, denoted $\Gamma_\delta$, for sufficiently small $\delta > 0$. However, it must be remarked that [4] works in a “frame of reference” such that $\Gamma$ is a one dimensional manifold of fixed points. To put our setup into this framework, we must act on (1) by a one-parameter group transformation corresponding to $\{\gamma_t\}_{t \in \mathbb{R}}$, so that the dynamics “move with” the limit cycle.

Having obtained the fact that $\pi$ is $C^2$ in $\Gamma_\delta$ (in the uniform topology), we may conclude that $\pi$ is $C^2$ in the interior of $B(\Gamma)$ using (2). For any $x \in B(\Gamma)$ and $\delta > 0$, there exists $k \in \mathbb{N}$ such that $\phi_{kT}(x) \in \Gamma_\delta$. Moreover, by taking possibly larger $k \in \mathbb{N}$, for any $z \in E$ we have $\phi_{kT}(x + \Delta z) \in \Gamma_\delta$ for sufficiently small $\Delta > 0$. Hence 
\[
\lim_{h \to 0} \frac{\pi(x + \Delta z) - \pi(x)}{h} = \lim_{h \to 0} \frac{\pi(\phi_{kT}(x + \Delta z)) - \pi(\phi_{kT}(x))}{h} = \lim_{h \to 0} \frac{\pi'(\phi_{kT}(x))D\phi_{kT}(x)\Delta z + "O(h)"}{h} = \pi'(\phi_{kT}(x))D\phi_{kT}(x)z.
\]
A similar argument applies to higher derivatives of $\pi$, given their existence in some $\Gamma_\delta$.

\(^2\)In the uniform topology, hence also the $L^2(S^1; \mathbb{R}^n)$ topology; this is the case if $N$ is $C^3$. 

2 Useful Properties of the Isochron Map

The following proposition is the main result of this section. Its proof follows from Proposition 2.3.

**Proposition 2.1.** The second Fréchet derivative of \( \pi \) is such that for any \( x_0 \in B(\Gamma) \) there is an orthonormal basis \( \{e_k\}_{k \in \mathbb{N}} \) of \( H \) satisfying

\[
\sum_{k \in \mathbb{N}} \left| \pi''(x_0)[e_k, e_k] \right| < \infty. \tag{5}
\]

Moreover, the map

\[
x_0 \mapsto \sum_{i \in \mathbb{N}} \pi''(x_0)[B(x_0)e_i, B(x_0)e_i]
\]

is continuous in \( B(\Gamma) \).

**Remark 2.2.** The notion of “trace” used here is, in general, not equivalent to the usual definition of the trace. That is, for a bounded linear operator on a Hilbert space \( A : H \to H \), we do not necessarily have that

\[
\sum_{k \in \mathbb{N}} \langle Ae_k, e_k \rangle = \sum_{k \in \mathbb{N}} \| Ae_k \|. \tag{6}
\]

However, in many situations (6) does in fact hold – for instance, when \( H \) admits an orthonormal basis of eigenfunctions of \( A \). ▲

Differentiating (2) twice at \( x_0 \in B(\Gamma) \) yields

\[
\pi''(x_0)[x, y] = \pi''(\phi_T(x_0))[D\phi_T(x_0)[x], D\phi_T(x_0)[y]] + \pi'(\phi_T(x_0))[D^2\phi_T(x_0)[x, y]]. \tag{7}
\]

It follows that

\[
\sum_{k \in \mathbb{N}} \left| \pi''(x_0)[e_k, e_k] \right| \leq \sum_{k \in \mathbb{N}} \left| \pi''(\phi_T(x_0))[D\phi_T(x_0)[e_k], D\phi_T(x_0)[e_k]] \right|
+ \sum_{k \in \mathbb{N}} \left| \pi'(\phi_T(x_0))[D^2\phi_T(x_0)[e_k, e_k]] \right|
\leq \| \pi''(\phi_T(x_0)) \| \sum_{k \in \mathbb{N}} \| D\phi_T(x_0)[e_k] \|^2
+ \| \pi'(\phi_T(x_0)) \| \sum_{k \in \mathbb{N}} \| D^2\phi_T(x_0)[e_k, e_k] \|. \tag{8}
\]

Hence, Proposition 2.1 immediately follows as a corollary to the following result.

**Proposition 2.3.** The maps \( x \mapsto D\phi_t(x_0)[x] \) and \( x \mapsto D^2\phi_t(x_0)[x, x] \) are such that for some orthonormal basis \( \{e_k\}_{k \in \mathbb{N}} \) of \( H \) we have

\[
\sum_{k \in \mathbb{N}} \| D\phi_t(x_0)[e_k] \| < \infty \quad \text{and} \quad \sum_{k \in \mathbb{N}} \| D^2\phi_t(x_0)[e_k, e_k] \| < \infty. \tag{9}
\]
Proof. We consider the evolution equations of the operators in (1), written in mild form as

\[ D\phi_t(x_0)[x] = e^{t\Delta} D\phi_0(x_0)[x] + \int_0^t e^{(t-s)\Delta} N'(\phi_s(x_0)) D\phi_s(x_0)[x] \, ds, \]

\[ D^2\phi_t(x_0)[x, y] = e^{t\Delta} D^2\phi_0(x_0)[x, y] + \int_0^t \int_0^t e^{(t-s)\Delta} N''(\phi_s(x_0)) D^2\phi_s(x_0)[x, y] \, ds \, ds \]

where \( N' \) and \( N'' \) are the first and second Fréchet derivatives of \( N \).

Now, we observe that \( D\phi_0(x_0)[x] = x \), so that

\[
\sum_{k \in \mathbb{N}} \| D\phi_t(x_0)[e_k] \| \leq \sum_{k \in \mathbb{N}} \| e^{t\Delta} e_k \| + \sum_{k \in \mathbb{N}} \int_0^t \| e^{(t-s)\Delta} N'(\phi_s(x_0)) \| \| D\phi_s(x_0)[e_k] \| \, ds \\
\leq \sum_{k \in \mathbb{N}} \| e^{t\Delta} e_k \| + C_0 \int_0^t \sum_{k \in \mathbb{N}} \| D\phi_s(x_0)[e_k] \| \, ds
\]

for some \( C_0 > 0 \). Since \( \Lambda_t \) has finite trace, the first sum is finite. We can therefore apply Grönwall’s inequality to obtain, for arbitrary \( s > t \),

\[
\sum_{k \in \mathbb{N}} \| D\phi_t(x_0)[e_k] \| \leq K_s e^{C_0 t} < \infty. \quad (11)
\]

Furthermore, (11) implies a local integrability of the sum,

\[
\sum_{k \in \mathbb{N}} \int_0^t \| D\phi_s(x_0)[e_k] \| \, ds = \int_0^t \sum_{k \in \mathbb{N}} \| D\phi_s(x_0)[e_k] \| \, ds \leq \frac{K_s}{C_0} e^{C_0 t}.
\]

Similarly, observe that

\[
\sum_{k \in \mathbb{N}} \| D\phi_t(x_0)[e_k] \|^2 \leq \sum_{k \in \mathbb{N}} \| e^{t\Delta} e_k \|^2 + 2C_0 \sum_{k \in \mathbb{N}} \left( \| e^{t\Delta} e_k \| \int_0^t \| D\phi_s(x_0)[e_k] \| \, ds \right) \\
+ C_0^2 \sum_{k \in \mathbb{N}} \left( \int_0^t \| D\phi_s(x_0)[e_k] \| \, ds \right)^2.
\]

The first sum is bounded, since \( e^{t\Delta} \) is Hilbert-Schmidt. The second sum is bounded, due to (11) and the fact that for any two summable sequences \((a_k)_{k \in \mathbb{N}}\), \((b_k)_{k \in \mathbb{N}}\) we have

\[
\sum_{k \in \mathbb{N}} a_k b_k \leq \left( \sum_{k \in \mathbb{N}} a_k \right) \left( \sum_{k \in \mathbb{N}} b_k \right).
\]

Therefore, for arbitrary \( s > 0 \),

\[
\sum_{k \in \mathbb{N}} \| D\phi_t(x_0)[e_k] \|^2 \leq K_s + 2K_s e^{C_0 t} + C_0^2 \sum_{k \in \mathbb{N}} \left( \int_0^t \| D\phi_s(x_0)[e_k] \| \, ds \right)^2 \\
\leq R_s + C_0^2 \int_0^t \sum_{k \in \mathbb{N}} \| D\phi_s(x_0)[e_k] \|^2 \, ds,
\]

\[ \]
for \( R_s := K_s(1 + 2e^{C_0t}) \). Applying Grönwall’s inequality again, we find that \( \sum_{k \in \mathbb{N}} \| D\phi_s(x_0)[e_k] \|^2 \) is finite and locally integrable.

Once we have the first two bounds in (9), we can similarly obtain the third: since \( D^2\phi_0(x_0)[x, x] = x \), we have

\[
\sum_{k \in \mathbb{N}} \| D^2\phi_t(x_0)[e_k, e_k] \| \leq \sum_{k \in \mathbb{N}} \| e^\Delta e_k \| + C_1 \int_0^t \sum_{k \in \mathbb{N}} \| D^2\phi_s(x_0)[e_k, e_k] \| ds
\]

\[
+ C_1 \int_0^t \sum_{k \in \mathbb{N}} \| D\phi_s(x_0)[e_k] \|^2 ds.
\]

By our previous results, the first and third sum in the above are finite. We therefore have, for arbitrary \( s > 0 \),

\[
\sum_{k \in \mathbb{N}} \| D^2\phi_t(x_0)[e_k, e_k] \| \leq L_s + C \int_0^t \sum_{k \in \mathbb{N}} \| D^2\phi_s(x_0)[e_k, e_k] \| ds
\]

for another constant \( L_s > 0 \). We may therefore apply Grönwall’s inequality again, to obtain the final bound.

\[ \square \]

3 An Itô formula for the isochron map

We now prove an Itô formula for the isochron map \( \pi : B(\Gamma) \rightarrow [0, T) \). Let \( (X_t)_{t \geq 0} \) be an Itô diffusion on \( H := L^2(S^1; \mathbb{R}^n) \), satisfying a stochastic version of (11),

\[
dX = V(X) \, dt + \sigma DB(X) \, dW. \tag{12}
\]

The noise amplitude is controlled by \( \sigma \geq 0 \), and \( W = (W_t)_{t \geq 0} \) is a cylindrical Wiener process on \( H \). We let \( x \mapsto B(x) \) be a locally Lipschitz map taking values in the space of bounded linear operators on \( H \), and do not require it to be trace class. Notice that the noise only acts on the components of the system which are diffusive. Since the isochron projection is only guaranteed to be trace class for the system components with diffusion, this is necessary for the arguments presented here.

The solution theory of (12) can be worked out following [1], with some modifications. However, to the author’s knowledge, the solution theory for (12) when \( D \) has some zero diagonal entries has only been worked out in the setting of trace class noise [2]. In this section we simply assume the existence and uniqueness of mild solutions to (12) with non-trace class noise. The existence theory of (12) is rigorously worked out in Appendix A.

Note that, unless restrictions are placed on \( x \mapsto B(x) \), the noise in (12) may push the solution out of \( B(\Gamma) \) in finite time. We therefore define the exit time

\[ \tau := \inf \{ t > 0 : \varphi_t(x) \in \partial B(\Gamma) \}. \]

An Itô formula for the evolution of \( \pi(X_t) \) for \( t < \tau \) is proven in [4]. However, in that work it was assumed that \( B(x) \) is a trace class operator for all \( x \) in our phase space. Since we have shown above that \( \sum_{i \in \mathbb{N}} \| \pi''(x)[e_i, e_i] \| < \infty \) for any \( x \in B(\Gamma) \), this assumption is no longer needed. In particular, we are able to take \( W \) as a cylindrical Wiener process, and \( B \) a Nemytskii polynomial operator multiplying the identity. Otherwise, the proof of the following is essentially due to [4].
Theorem 3.1. Suppose that $\{12\}$ has a unique mild solution $(X_t)_{t \geq 0}$ in $C_b(S^1; \mathbb{R}^n)$ for any initial condition $X_0 \in B(\Gamma)$. Then,

$$
\pi(X_{t_{i+1}^*}) = \pi(X_0) + \sum_{i=1}^M \left( \pi'(X_{t_i^*}) [X_{t_{i+1}^*} - X_{t_i^*}] + \pi''(w_i) [X_{t_{i+1}^*} - X_{t_i^*}, X_{t_{i+1}^*} - X_{t_i^*}] \right),
$$

(14)

where $w_i = \alpha_i X_{t_i^*} + (1 - \alpha_i) X_{t_{i+1}^*}$ for some $\alpha_i \in (0, 1)$. Since $(X_t)_{t \geq 0}$ is a mild solution of $\{12\}$,

$$
X_{t_{i+1}^*} - X_{t_i^*} = \left[ e^{(t_{i+1}^* - t_i^*)D\Delta} - I \right] X_0 + \int_{t_i^*}^{t_{i+1}^*} e^{(t - t_{i+1}^*)D\Delta} N(X_s) ds
$$

+ $\sigma \int_{t_i^*}^{t_{i+1}^*} e^{(t - t_{i+1}^*)D\Delta} B(X_s) dW_s.
$$

So (14) becomes

$$
\pi(X_{t_{i+1}^*}) - \pi(X_0) = \sum_{i=1}^M \pi'(X_{t_i^*}) [Y_i] + \sum_{i=1}^M \pi'(X_{t_i^*}) [Z_i] + \sigma \sum_{i=1}^M \pi'(X_{t_i^*}) [W_i^{\Delta,B}]
$$

$$
+ \sum_{i=1}^M \pi''(w_i) [Y_i + Z_i, Y_i + Z_i] + 2\sigma \sum_{i=1}^M \pi''(w_i) [Y_i + Z_i, W_i^{\Delta,B}]
$$

$$
+ \sigma^2 \sum_{i=1}^M \pi''(w_i) [W_i^{\Delta,B}, W_i^{\Delta,B}]
$$

(15)

$$
= I + II + III + IV + V + VI.
$$

We will prove that, as the mesh of the partition $\{t_i\}_{i=1}^M$ tends to zero, we have

$$
(I + II) \to \int_0^{t_{i+1}^*} \pi'(X_s) V(X_s) ds,
$$

$$
III \to \int_0^{t_{i+1}^*} \pi'(X_s) B(X_s) dW_s,
$$

$$
IV \to 0, \quad V \to 0, \quad VI \to \int_0^{t_{i+1}^*} \sum_{i \in \mathbb{N}} \pi''(X_s) [B(X_s) e_i, B(X_s) e_i] ds.
$$

We first handle the nonzero limits.

For a fixed partition $\{t_i\}_{i=1}^M$ with mesh size $h > 0$, choose arbitrary $i \in \{1, \ldots, M\}$ and let $(\hat{X}_t)_{t \in [t_i, t_{i+1}]}$ be given by

$$
\hat{X}_{t_i} = X_{t_i}, \quad \hat{X}_t = e^{(t-t_i)D\Delta} X_{t_i} + \int_{t_i}^{t} e^{(t-s)D\Delta} N(\hat{X}_s) ds \quad \text{for} \quad s \in [t_i, t_{i+1}].
$$
For the limit of \((I + II)\), we apply Taylor's theorem to \(\pi(X_{t^*})\) twice to obtain
\[
\pi'(\hat{X}_{t^*}) \left[ \hat{X}_{t_{i+1}} - \hat{X}_{t^*} \right] + \pi''(Y_i) \left[ \hat{X}_{t_{i+1}} - \hat{X}_{t^*}, W_i - \hat{X}_{t^*} \right] = \pi(\hat{X}_{t_{i+1}}) - \pi(\hat{X}_{t^*}).
\]
In the above, \(W_i, Y_i\) are defined for some \(a_1, a_2 \in [0, 1]\) as
\[
W_i := a_1 \hat{X}_{t^*} + (1 - a_1) \hat{X}_{t_{i+1}}, \quad Y_i := a_2 \hat{X}_{t^*} + (1 - a_2)W_i.
\]
By Grönwall's inequality and the Lipschitz property of \(N\), we can show that
\[
\|X_s - \bar{X}_s\| \sim O(h^2) \quad \text{for} \quad t \in [t_i, t_{i+1}].
\]
It then follows that
\[
\left\| \hat{X}_{t_{i+1}} - \hat{X}_{t^*} \right\| \leq K_0 h \sup_{s \in [t_i, t_{i+1}]} \left\| \hat{X}_s - X_s \right\| + K_1 h^2
\]
for some constants \(K_0, K_1 > 0\). Noting that
\[
\lim \sup_{h \to 0} h^{-1/2}\left\| (e^{h\Delta} - I)X_t \right\| = 0,
\]
it follows that
\[
\| \hat{X}_{t^*} - \hat{X}_{t_{i+1}} \| \sim o(h^{1/2}) \quad \text{and} \quad \| W_i - \hat{X}_{t^*} \| \sim o(h^{1/2}).
\]
Hence, each summand in \((I + II)\) can be estimated as
\[
\pi'(X_{t^*}) \left[ (e^{(t_{i+1} - t^*)\Delta} - I)X_{t^*} + \int_{t_i}^{t_{i+1}} e^{(t_{i+1} - s)\Delta} N(X_s) \, ds \right] = \pi''(Y_i) \left[ \hat{X}_{t_{i+1}} - \hat{X}_{t^*}, W_i - \hat{X}_{t^*} \right]
\]
\[
= O(h^2) + \pi(\hat{X}_{t_{i+1}}) - \pi(\hat{X}_{t^*}) + o(h) + O(h^2) + \pi(X_{t_{i+1}}) - \pi(X_{t^*})
\]
\[
= o(h) + h\pi'(X_{t_{i+1}})V(X_{t_{i+1}}),
\]
where in the last line we have taken \(\pi(X_{t_{i+1}}) = \pi(X_{t^*}) + \pi'(X_{t^*})V(X_{t^*})h + O(h^2)\), and absorbed the \(O(h^2)\) terms in \(o(h)\). Summing over \(\{t_i\}_{i=1}^M\) and letting \(h \to 0\) yields the correct convergence.

For the limit of \(III\), note that for small \(h > 0\) we have
\[
\int_{t_k^*}^{t_{k+1}^*} e^{(t_{k+1} - s)\Delta} B(X_s) \, dW_s = B(X^*_t) [W_{t_{k+1}} - W_{t^*_k}] + O(h^2).
\]
Hence, observing that \(K = t/h\), we have
\[
III = \sum_{k=0}^{K} \pi'(X_{t^*_k}) \left[ \int_{t_k^*}^{t_{k+1}^*} e^{(t_{k+1} - s)\Delta} B(X_s) \, dW_s \right]
\]
\[
= \sum_{k=0}^{K} \pi'(X_{t^*_k}) B(X_{t^*_k}) [W_{t_{k+1}} - W_{t^*_k}] + K O(h^2)
\]
\[
\longrightarrow \int_0^t \pi'(X_s) B(X_s) \, dW_s.
\]
To obtain $VI$, we write the cylindrical Wiener process $W = (W_t)_{t \geq 0}$ as

$$W_t = \sum_{k \in \mathbb{N}} e_k e_k^t,$$  \hfill (17)

where $\{\beta_k^e\}_{k \in \mathbb{N}}$ is a collection of independent identically distributed $\mathbb{R}$-valued Brownian motions, and $\{e_k\}_{k \in \mathbb{N}}$ is an orthonormal basis of $E$. The sum in (17) does not converge in $E$, but a larger Hilbert (Banach) space into which $E$ is densely embedded by a Hilbert-Schmidt ($\gamma$-radonifying) operator.

First note that

$$\pi''(w_k) \left[ \int_{t_k^*}^{t_{k+1}^*} e^{(t_{k+1}^*-s)D\Delta} B(X_s) \sum_{j \in \mathbb{N}} e_j d\beta_j^s, \int_{t_k^*}^{t_{k+1}^*} e^{(t_{k+1}^*-s)D\Delta} B(X_s) \sum_{j \in \mathbb{N}} e_j d\beta_j^s \right]$$

is equivalent in mean square to

$$\int_{t_k^*}^{t_{k+1}^*} \sum_{j \in \mathbb{N}} \pi''(w_k) \left[ e^{(t_{k+1}^*-s)D\Delta} B(X_s) e_j, e^{(t_{k+1}^*-s)D\Delta} B(X_s) e_j \right] ds.$$ 

This follows from the fact that integrals and infinite sums commute with the action of bounded linear operators, and an application of Itô’s isometry (note then that we must be in the Hilbert space setting; the Banach space setting does not allow the Itô isometry in a sufficiently strong form). Then, a Taylor expansion of $e^{(t_{k+1}^*-s)D\Delta}$ about $s = t_{k+1}$ yields

$$e^{(t_{k+1}^*-s)D\Delta} = I + D\Delta e^{uD\Delta} (t_{k+1}^* - s)$$

for some $u \in [s, t_{k+1}]$. Hence

$$\int_{t_k^*}^{t_{k+1}^*} \sum_{j \in \mathbb{N}} \pi''(w_k) \left[ e^{(t_{k+1}^*-s)D\Delta} B(X_s) e_j, e^{(t_{k+1}^*-s)D\Delta} B(X_s) e_j \right] ds$$

equals

$$\int_{t_k^*}^{t_{k+1}^*} \sum_{j \in \mathbb{N}} \left( \pi''(w_k) \left[ B(X_s) e_j, B(X_s) e_j \right] + 2\pi''(w_k) \left[ D\Delta e^{uD\Delta} B(X_s) e_j, B(X_s) e_j \right] (t_{k+1}^* - s) \right. \right.$$

$$\left. + \pi''(w_k) \left[ D\Delta e^{uD\Delta} B(X_s) e_j, D\Delta e^{uD\Delta} B(X_s) e_j \right] (t_{k+1}^* - s)^2 \right) ds$$

$$= \left( t_{k+1}^* - t_k^* \right) \sum_{j \in \mathbb{N}} \pi''(w_k) \left[ B(X_{t_k^*}) e_j, B(X_{t_k^*}) e_j \right] + O((t_{k+1}^* - t_k^*)^2).$$

Summing over $k \in \{1, \ldots, K\}$ and letting $h \to 0$, we have convergence of $VI$.

To handle the zero limits, we remark that the terms $IV$ and $V$ tend to zero as $h \to 0$ as a consequence of (16). \hfill $\Box$

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A Solution theory of stochastic nerve axon equations

Our approach to the solution theory of (12) is very similar to that worked out in the appendix of [6], along with several arguments in [1]. However, we must tailor our approach to allow for \( N \) to be defined only on \( E := C_b(S^1; \mathbb{R}^n) \), rather than all of \( H := L^2(S^1; \mathbb{R}^n) \). In this approach, we use the following version of the implicit function theorem, a proof of which is found in [1].

**Theorem A.1** (Lemma 9.2 of [1]). Let \( E_i, E_o \) be Banach spaces such that \( E_i \) embeds densely and continuously into \( E_o \). Let \( \Xi \) be an open subset of a (possibly different) Banach space, and take a map \( F : \Xi \times E_o \to E_o \) such that \( F(\Xi \times E_i) \subset E_i \). If there exists \( \alpha \in [0, 1) \) such that

\[
\| F(\xi, x) - F(\xi, y) \|_{E_i} \leq \alpha \| x - y \| \quad \text{for} \quad \xi \in \Xi \quad \text{and} \quad x, y \in E_i,
\]

\[
\| F(\xi, x) - F(\xi, y) \|_{E_o} \leq \alpha \| x - y \| \quad \text{for} \quad \xi \in \Xi \quad \text{and} \quad x, y \in E_o,
\]

then there exists a unique map \( \varphi : \Xi \to E_o \) such that

\[ \varphi(\xi) = F(\xi, \varphi(\xi)). \]

Moreover, if \( F : \Xi \times E_o \to E_o \) is continuous with respect to \( \xi \in \Xi \), then \( \varphi : \Xi \to E_o \) is continuous. Finally, if \( F \) is continuously Fréchet differentiable in both of its arguments, then \( \varphi : \Xi \to E_o \) is continuously Fréchet differentiable.

We use the implicit function theorem stated above to prove the existence and uniqueness of solutions to (12) in \( C_b(S^1; \mathbb{R}^n) \subset L^2(S^1; \mathbb{R}^d) \). We note that the boundedness of our spatial domain \( O \) is essential in the proof for the inclusion \( C_b \subset L^2 \) to hold.

**Theorem A.2.** Suppose that either (i) \( x \mapsto N(x) \) and \( x \mapsto B(x) \) are globally Lipschitz, or (ii) \( x \mapsto N(x) \) and \( x \mapsto B(x) \) are locally Lipschitz, and solutions of (12) can be a priori bounded in supremum norm. Then, there exists a unique, global in time mild solution of (12) in \( E := C_b(S^1; \mathbb{R}^d) \) for any initial condition \( X_0 \in H \).

**Proof.** First, suppose that \( x \mapsto N(x) \) and \( x \mapsto B(x) \) are globally Lipschitz. For arbitrary \( s \geq 0 \) and \( \delta > 0 \), let \( \mathcal{K}_{s, \delta} \) be the collection of all predictable \( E \)-valued stochastic processes \( (Z_t)_{t \in [s, s+\delta]} \) such that

\[ \| Z \|_{s, \delta}^2 := \sup_{t \in [s, s+\delta]} \mathbb{E} \left[ \| Z_t \|^2 \right] \]

is finite. Equipped with the norm in (18), \( \mathcal{K}_{s, \delta} \) is a Banach space. Now, for \( Z \in \mathcal{K}_{s, \delta}, t \in [s, s+\delta] \), and \( X_0 \in L^2(\Omega; H) \), define

\[
\mathcal{J}_{s, \delta}[X_0, Z](t) := e^{(t-s)D\Delta}X_0 + \int_s^t e^{(t-r)D\Delta}N(Z_s) \, ds + \int_s^t e^{(t-r)D\Delta}B(Z_s) \, dW_s.
\]

For any \( X_0 \in L^2(\Omega; H) \) and \( Z \in \mathcal{K}_{s, \delta} \), it can be shown that \( \mathcal{J}_{s, \delta}[X_0, Z] \) is a stochastic process in \( \mathcal{K}_{s, \delta} \). This follows from the (spatial) smoothing property of the semigroup \( e^{tD\Delta} \) and the assumed global Lipschitz property of \( N \) and \( B \). Moreover, it is straightforward to prove that
for each $r > 0$ there exists $\delta > 0$ and $\alpha \in [0, 1)$ such that for any $s \in (0, r]$, any $Z_1, Z_2 \in \mathcal{X}_{s, \delta}$ and any $X_0 \in L^2(\Omega; H)$ we have
\[
\| J_{s, \delta}[X_0, Z_1] - J_{s, \delta}[X_0, Z_2] \|_{s, \delta} \leq \alpha \| Z_1 - Z_2 \|_{s, \delta}.
\]
Therefore, applying Theorem [A.1] we find that there exists a unique map $\varphi : L^2(\Omega; H) \to \mathcal{X}_{s, \delta}$ such that
\[
\varphi[X_0](t) = J_{s, \delta}[X_0, \varphi(X_0)](t) = e^{(t-s)D\Delta}X_0 + \int_0^t e^{(t-r)D\Delta}N(\varphi[X_0](r)) \, dr + \int_s^t e^{(t-r)D\Delta}B(\varphi[X_0](r)) \, dW_r.
\]
We conclude that $\varphi[X_0]$ is the unique solution to (12) with initial condition $X_0$. Since $\delta > 0$ is arbitrary, the solution exists up to an arbitrarily large time.

Now, suppose that $x \mapsto N(x)$ and $x \mapsto B(x)$ are only locally Lipschitz. Then, since solutions to (12) can be a priori uniformly bounded means that a cutoff argument can be applied. The dynamics may be studied only in $O$, and $x \mapsto N(x)$ and $x \mapsto B(x)$ can again be treated as globally Lipschitz. We omit the finer details of this argument. \qed

Remark A.3. If $\Gamma$ is locally asymptotically attracting under the flow of (1) with basin of attraction $B(\Gamma)$, then the solutions of (12) can be a priori bounded in any bounded domain $O$ such that $\Gamma \subset O \subset B(\Gamma)$ by choosing the diffusion coefficient $B(x)$ to be zero on $\partial O$ and positive, continuous, on $O$. Of course, one is not usually free to choose the diffusion coefficient of (12), and the applicability of our solution theory is dictated by the application at hand. In many circumstances, Theorem [A.2] can probably be improved. ▲

References

[1] G. Da Prato and J. Zabczyk, Stochastic equations in infinite dimensions, Cambridge university press, 2014.

[2] K. Eichinger, M.V. Gnann, and C. Kuehn, Multiscale analysis for traveling-pulse solutions to the stochastic FitzHugh-Nagumo equations, arXiv preprint arXiv:2002.07234 (2020).

[3] J. Guckenheimer, Isochrons and phaseless sets, Journal of Mathematical Biology 1, no. 3 (1975), pp. 259–273.

[4] J. MacLaurin, Metastability of waves and patterns subject to spatially-extended noise, arXiv preprint arXiv:2006.12627 (2020).

[5] A. Pazy, Semigroups of linear operators and applications to partial differential equations, Vol. 44 Springer Science & Business Media, 2012.

[6] S. Peszat and J. Zabczyk, Strong Feller property and irreducibility for diffusions on Hilbert spaces, The Annals of Probability (1995), pp. 157–172.