ISOPARAMETRIC FOLIATIONS AND EXOTIC SMOOTH STRUCTURES

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Abstract. One of our main theorems shows that all homotopy $n$-spheres ($n \neq 4$) admit the “same” isoparametric foliations, i.e., there are 1-1 correspondences among the sets of equivalence classes of isoparametric foliations on homotopy $n$-spheres, solving in particular the Problem 4.4 raised by Tang and the author in [6]. Moreover, among other results, we establish an approach to detect negative examples to the problem that whether two foliations dividing a manifold into the same disk bundles are equivalent. In particular, we prove that either pseudo-isotopy implies isotopy on $S^4$, or $S^5$ admits nonequivalent isoparametric foliations with two points as the focal submanifolds. Inspired by the proof of the main theorem above, we observe a new relation between the two inertia groups $I_1(M)$ and $I_0(N)$ when $M$ is a hypersurface of $N$. This leads to our second main theorem that for any closed hypersurface $M$ in $S^n$, there exist at least $|\Theta_{n+k}|$ distinct oriented smooth structures on $M^{n-1} \times P^k \times S^1$, where $P^k$ is any product of standard spheres of total dimension $k \geq 0$ and $|\Theta_n|$ is the number of homotopy $n$-spheres.

1. Introduction

A transnormal system $\mathcal{F}$ on a complete Riemannian manifold $M$ is a decomposition of $M$ into complete, injectively immersed connected submanifolds, called leaves, such that every geodesic emanating perpendicularly to one leaf remains perpendicular to all leaves. A singular Riemannian foliation is a transnormal system $\mathcal{F}$ which is also a singular foliation, i.e., such that there are smooth vector fields $X_i$ on $M$ that span the tangent space $T_pL_p$ to the leaf $L_p$ through each point $p \in M$. A leaf of maximal dimension is called a regular leaf, and its codimension is defined to be the codimension of $\mathcal{F}$. Leaves of lower dimensions are called singular leaves. By a foliated diffeomorphism between two foliated manifolds we mean a diffeomorphism maps leaves to leaves, and such foliations are called equivalent. By $M \cong M'$, $(M, \mathcal{F}) \cong (M', \mathcal{F}')$, we mean the manifolds are diffeomorphic, foliated diffeomorphic, respectively.

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A singular Riemannian foliation \((M, \mathcal{F})\) of codimension 1 is called an \textit{isoparametric} foliation if the regular leaves have constant mean curvature. The regular leaves of an isoparametric foliation are called \textit{isoparametric hypersurfaces}, and the singular leaves are called \textit{focal submanifolds}. The study of isoparametric foliations on the unit sphere \(S^n(1)\) originated in 1930’s by E. Cartan and it developed into a very beautiful and valuable theory during the past decades. So far the classification in this case has been almost completed by a lot of contributions (for recent progress and applications see for example in \cite{1, 3, 13, 20, 21} and a survey in \cite{23}).

In \cite{6} Tang and the author turned to study isoparametric foliations on exotic spheres and showed there are no isoparametric foliations on any exotic 4-spheres (if exist). There were also some existence examples presented on Milnor exotic 7-spheres and an example of codimension 1 singular Riemannian foliation with two points as the singular leaves on the Gromoll-Meyer 7-sphere. According to these we proposed the Problem 4.4 asking whether exotic \(n\)-spheres \((n \neq 4)\) always admit isoparametric foliations with the same focal submanifolds as those occurring on \(S^n\).

In \cite{16} Qian and Tang gave a fundamental construction of isoparametric foliations on closed manifolds that admit a decomposition into two linear\(^1\) disk bundles of rank greater than 1 over closed submanifolds. On such manifolds they first constructed a Riemannian metric so that the canonical codimension 1 singular foliation (regular leaves correspond to concentric tubes around the zero sections) becomes a singular Riemannian foliation. Then they constructed a new (bundle-like) Riemannian metric so that the foliation becomes isoparametric. In particular, more examples of isoparametric foliations on exotic spheres analogues to those on standard spheres were obtained. We remark that, conversely, a codimension 1 singular Riemannian foliation on a closed simply connected manifold gives such a decomposition on the manifold (cf. \cite{13}). Therefore, on closed simply connected manifolds isoparametric foliations require no more on the topology than codimension 1 singular Riemannian foliations.

In this paper, we answer affirmatively the Problem 4.4 in \cite{6} mentioned above. In fact, we obtain

\textbf{Theorem 1.1.} All homotopy \(n\)-spheres \((n \neq 4)\) admit the “same” isoparametric foliations, i.e., there are 1-1 correspondences among the sets of equivalence classes of isoparametric foliations on homotopy \(n\)-spheres, each foliation corresponds to the same (two) disk bundles and in particular has the same (diffeomorphic) isoparametric hypersurfaces and focal submanifolds.

Therefore, to study isoparametric foliations on homotopy \(n\)-spheres it suffices to study on \(S^n\). However, the classification of equivalence classes of isoparametric foliations on \(S^n\ (n > 4)\) is far from completed.

\(^{1}\)Throughout this paper, disk bundles are assumed linear, that is, disk bundles of vector bundles.
On the one hand, the decompositions of $S^n$ into two disk bundles are abundant. For example, as generalizations of classical isoparametric hypersurfaces in unit spheres, closed Dupin hypersurfaces divide the sphere into two disk bundles (cf. [22], [7]), while Dupin hypersurfaces are far from classified. Another subtle difficulty is that there are nonequivalent disk bundles (over the same base manifold) with diffeomorphic total spaces (cf. [4]). Manifolds glued from a given disk bundle with such nonequivalent disk bundles along their boundaries by a (same) gluing diffeomorphism will be diffeomorphic, but not foliated diffeomorphic to each other if equipped with the canonical codimension 1 singular Riemannian foliations. In particular, Haefliger and Levine’s examples (cf. [4]) ensure that $S^{11}, S^{12}, S^{13}, S^{15}, S^{16}, S^{17}$ admit isoparametric foliations (under some metrics) not equivalent to the classical isoparametric foliations (under the round metric), with the same (diffeomorphic) isoparametric hypersurfaces (being products of the focal submanifolds) and the same focal submanifolds:

- $(S^7, S^3), (S^8, S^3), (S^9, S^3), (S^{11}, S^4), (S^{11}, S^5), (S^{11}, S^5)$ respectively.

On the other hand, it is not known that whether two foliations dividing the sphere into the same disk bundles are equivalent. About this we propose Problem 2.1 in Section 2 and there, among other results, we establish an approach to detect negative examples to this problem (see Theorem 2.6). In particular, we prove that either $\pi_0(Diff^+(S^4)) = 0$, i.e., pseudo-isotopy implies isotopy on $S^4$, or $S^5$ admits nonequivalent isoparametric foliations with two points as the focal submanifolds (see Corollary 2.8).

Inspired by the proof of Theorem 1.1, we observe a new relation (4.2), in Theorem 4.1, between the two inertia groups $I_1(M)$ and $I_0(N)$ when $M$ is a hypersurface of $N$. Combining this with the original rigid relation (4.1) proven by Levine [10], we see that $M \times S^1$ has the smallest inertia group $I_0$ among all manifolds containing $M$ as a hypersurface. In particular, we obtain

**Theorem 1.2.** Let $M$ be a closed hypersurface embedded in $S^n$. Then for any $k \geq 0$ and any product $P^k := S^{k_1} \times \cdots \times S^{k_l}$ of standard spheres of total dimension $k = \sum_{i=1}^l k_i$ ($k_i \geq 1$) ($P^k$ is a point when $k = 0$), there exist at least $|\Theta_{n+k}|$ distinct oriented smooth structures on $M^{n-1} \times P^k \times S^1$, where $|\Theta_n|$ is the order of the finite abelian group $\Theta_n$ of $h$-cobordism classes of oriented homotopy $n$-spheres.

**Remark 1.3.** It also follows that the group $\Gamma(M^{n-1} \times P^k)$ of concordance classes of orientation-preserving diffeomorphisms of $M^{n-1} \times P^k$ (and hence the mapping class group $\pi_0(Diff^+(M^{n-1} \times P^k))$) has at least $|\Theta_{n+k}|$ elements.

The proof of Theorem 4.1 (and hence Theorem 1.2) is based on an elementary cutting-gluing technique, where the equation (4.3) should be of equal attention as the relations (4.1) and (4.2).
2. Disk bundle decomposition by singular Riemannian foliation

A codimension 1 singular Riemannian foliation \( \mathcal{F} \) on a closed simply connected manifold \( N \) has exactly two closed singular leaves \( M_\pm \) and decomposes \( N \) into two unit disk bundles \( E_\pm \) of the normal vector bundles \( \xi_\pm \) over \( M_\pm \) of rank \( m_\pm > 1 \) (cf. [14]). The decomposition can be described by the following commutative diagram

\[
E := E_+ \sqcup E_-
\]

Here \( E \) is the disjoint union of \( E_\pm \), \( \pi_\pm := \pi|_{E_\pm} : E_\pm \xrightarrow{\cong} N_\pm \) are closed tubular neighborhoods of the singular leaves \( \iota_\pm : M_\pm \to \bar{N} \) with \( s_\pm := \pi_\pm^{-1} \circ \iota_\pm : M_\pm \to E_\pm \) the zero sections of \( E_\pm \), \( \varphi = \pi_\pm^{-1} \circ \iota_\pm|_{\partial E_\pm} : \partial E_+ \to \partial E_- \) is the gluing diffeomorphism for \( E_\varphi \), \( \tilde{\pi} \) is the diffeomorphism from \( E_\varphi \) to \( N \) whose composition with the natural projection \( p \) satisfies \( \tilde{\pi} \circ p = \pi \). The regular leaves of \( \mathcal{F} \) are the images of the concentric tubes around the zero sections of \( E_\pm \), and the singular leaves are the images of the zero sections. The preimage of \( \mathcal{F} \) under \( \tilde{\pi} \) defines a codimension 1 singular Riemannian foliation \( \mathcal{F}_\varphi \) on \( E_\varphi \) with the induced metric by \( \tilde{\pi} \). The leaves of \( \mathcal{F}_\varphi \) are just the concentric tubes (including the tubes of radius 0, the zero sections) in \( E_\pm \). Therefore, the equivalence class of \((N, \mathcal{F})\) can be represented by \((E_\varphi, \mathcal{F}_\varphi)\).

Conversely, given unit disk bundles \( E_\pm \) over complete manifolds \( M_\pm \) and a diffeomorphism \( \varphi : \partial E_+ \to \partial E_- \), the foliation \( \mathcal{F}_\varphi \) consisting of concentric tubes on \( E_\varphi = E_+ \cup_\varphi E_- \) would be a singular Riemannian foliation, provided with a Riemannian metric by a suitable choice of a one-parameter family of metrics on \( \partial E_+ \) in a collar of \( \partial E_+ \) in \( E_+ \), connecting \( g_+|_{\partial E_+} \) and \( \varphi^*(g_-|_{\partial E_-}) \), where \( g_\pm \) are metrics on \( E_\pm \) compatible with the Euclidean metrics (cf. [16]). Moreover, if \( E_\pm \) are of rank greater than 1 and \( M_\pm \) are closed, \((E_\varphi, \mathcal{F}_\varphi)\) can become isoparametric by a more careful choice of the one-parameter family of metrics on \( \partial E_+ \) as shown in [16].

It follows that to study classification of equivalence classes of codimension 1 singular Riemannian (isoparametric) foliations one needs only to study the foliations in the form \((E_\varphi, \mathcal{F}_\varphi)\) determined by pairs of unit disk bundles \( E_\pm \subset \xi_\pm \) with diffeomorphic boundaries and gluing diffeomorphisms \( \varphi : \partial E_+ \to \partial E_- \). Moreover, it is independent of the choices of the vector bundles \( \xi_\pm \) in their bundle-equivalence classes and of the Euclidean metrics. In fact, for any Euclidean bundles \( \xi'_\pm \) equivalent to \( \xi_\pm \), there are vector bundle isomorphisms \( F_\pm : \xi_\pm \to \xi'_\pm \) which are isometries with respect to the Euclidean metrics.

\[2\text{In general, the radius can be a positive constant which can be normalized by a homothetic transformation of the Riemannian metric on } N.\]
Euclidean metrics (cf. [12]) and hence map concentric tubes of $E_{\pm}$ to concentric tubes of $E_{\pm}'$. Set $\psi = F_+ \circ \varphi \circ F_+^{-1}|_{\partial E_+} : \partial E'_+ \to \partial E'_+$. Then the map

$$F : E_\varphi = E_+ \cup_\varphi E_- \to E'_+ \cup_\psi E'_- = E'_\psi$$

defined by $F|_{E_\pm} = F_\pm$ is a foliated diffeomorphism between $(E_\varphi, F_\varphi)$ and $(E'_\psi, F'_\psi)$.

However, it is not known in general that whether the disk bundles $E_{\pm}$ are sufficient to determine the foliation on one manifold. Explicitly, we propose

**Problem 2.1.** For $\varphi_i : \partial E_+ \to \partial E_-$ ($i = 0, 1$) satisfying $E_{\varphi_0} \cong E_{\varphi_1}$, when is $(E_{\varphi_0}, F_{\varphi_0})$ foliated diffeomorphic to $(E_{\varphi_1}, F_{\varphi_1})$?

For instance, it holds when $E_{\varphi_i}$ are closed simply connected 4-manifolds as shown in the classification by the author and Radeschi [3]. Motivated by the discussion above, we observe the following criterion.

**Proposition 2.2.** For $\varphi_i : \partial E_+ \to \partial E_-$ ($i = 0, 1$), $(E_{\varphi_0}, F_{\varphi_0})$ is foliated diffeomorphic to $(E_{\varphi_1}, F_{\varphi_1})$ if and only if there are diffeomorphisms $F_{\pm} \in \text{Diff}(E_{\pm})$ mapping concentric tubes to concentric tubes such that $\varphi_1 = F_- \circ \varphi_0 \circ F_+^{-1}|_{\partial E_+}$.

**Proof.** The equivalence follows directly from the definitions and can be described by the commutative diagram

$$
\begin{array}{ccc}
E & \xrightarrow{F_+ \cup F_-} & E \\
\downarrow p_0 & & \downarrow p_1 \\
(E_{\varphi_0}, F_{\varphi_0}) & \cong & (E_{\varphi_1}, F_{\varphi_1}),
\end{array}
$$

where $p_i$ are the natural projections mapping $x \in \partial E_+$ and $\varphi_i(x) \in \partial E_-$ to the gluing point in $E_{\varphi_i}$, $F$ is the foliated diffeomorphism satisfying $F \circ p_0|_{E_{\pm}} = p_1 \circ F_{\pm}$.\hfill \Box

If $\varphi_1$ is isotopic to $\varphi_0$, by considering a collar $C$ of $\partial E_-$ in $E_-$ (resp. $\partial E_+ \in E_+$) we can extend $\varphi_1 \circ \varphi_0^{-1}$ (resp. $\varphi_1^{-1} \circ \varphi_0$) to a diffeomorphism $F_- \in \text{Diff}(E_-)$ (resp. $F_+ \in \text{Diff}(E_+)$) preserving concentric tubes, and then $\varphi_1 = F_- \circ \varphi_0$ (resp. $\varphi_1^{-1} \circ \varphi_0$), hence $(E_{\varphi_0}, F_{\varphi_0}) \cong (E_{\varphi_1}, F_{\varphi_1})$. This shows

**Corollary 2.3.** The equivalence class of $(E_{\varphi}, F_{\varphi})$ is independent of the choice of $\varphi$ in its isotopy class.

In the rest part of this section, we develop an approach to detect negative examples towards Problem 2.1 according to the criterion in Proposition 2.2.

Let $\pi : E_1 \to B$ be the unit disk bundle of a Euclidean vector bundle $\xi$ over a complete connected manifold $B$. Let $\text{Diff}_c(E_1)$ denote the subgroup $\text{Diff}(E_1, F_c)$ of $\text{Diff}(E_1)$ consisting of foliated diffeomorphisms from $(E_1, F_c)$ to itself, where $F_c$ is
the foliation consisting of concentric tubes $T_t$ of constant radii $t \in [0, 1]$ around the zero section. Let $\text{Isom}_b(E_1)$ be the subgroup of $\text{Diff}_c(E_1)$ consisting of vector bundle isomorphisms preserving the Euclidean metric. Denote by $\text{Isom}_b(\partial E_1)$ bijections on path components, i.e., the foliation consisting of concentric tubes $T_t$ where $| \cdot |$ denotes the norm of the Euclidean metric, and for any $V \in \partial E_1$, we define

$$F_t := (F \circ h_t)/\lambda(t) \in \text{Diff}_c(E_1),$$

where $\lambda : [0, 1] \to [0, 1]$ is the function defined by $F(T_t) = T_{\lambda(t)}$ for any concentric tube $T_t$ of radius $t \in [0, 1]$. Explicitly, we have

$$\lambda(t) = |F(tV)|, \quad \text{for any } V \in \partial E_1,$$

where $| \cdot |$ denotes the norm of the Euclidean metric, and for $t = 0$, $tV = \pi(V)$ means the base point in the zero section $B$. In fact, it is not hard to verify that

$$\lambda(t) = \int_0^t |(\gamma'_{\lambda}(t))^{-1}| dt, \quad \text{for any } V \in \partial E_1,$$

where $\gamma'_{\lambda}(t)$ is the tangent vector of the curve $\gamma_{\lambda}(t) := F(tV)$, and $(\cdot)^{-1}$ means the projection from $T(E_1)$ to the vertical distribution $E_1$. Moreover, $\gamma'_{\lambda}(t) = (F_*)_{tV}(V)$ is the image of $V$ under the tangential map $F_* : T(E_1) \to T(E_1)$ at $tV \in E_1$. Therefore, $\lambda$ is a smooth function with $\lambda'(0) > 0$, $\text{for any } V \in \partial E_1$. Define $F_0 : E_1 \to E_1$ by

$$F_0(V) := ((F_*)_{tV}(V))^{-1}/\lambda'(0), \quad \text{for } V \in E_1.$$

Then since $F_*$ is linear, $F_0$ is linear and preserves lengths, thus $F_0 \in \text{Isom}_b(E_1)$.

Set $f_t := \rho(F_t) = F_t|_{\partial E_1} \in \text{Diff}_c(\partial E_1)$, and $f_0 := \lim_{t \to 0^+} f_t$. It follows that

$$f_0(V) = \lim_{t \to 0^+} F(tV)/\lambda(t) = ((F_*)_{tV}(V))^{-1}/\lambda'(0) = F_0(V), \quad \text{for } V \in \partial E_1.$$ 

Hence $f_0 = \rho(F_0) \in \text{Isom}_b(\partial E_1)$.

In conclusion, we have shown that any $F \in \text{Diff}_c(E_1)$ corresponds uniquely to a path $\{f_t | t \in [0, 1]\}$ in $\text{Diff}_c(\partial E_1)$ with one end $f_0 \in \text{Isom}_b(\partial E_1)$, which also shows that $\text{Isom}_b(\partial E_1) \hookrightarrow \text{Diff}_c(\partial E_1)$ is a bijection on path components. Conversely, given

**Proposition 2.4.** The inclusions $\text{Isom}_b(E_1) \hookrightarrow \text{Diff}_c(E_1)$, $\text{Isom}_b(\partial E_1) \hookrightarrow \text{Diff}_c(\partial E_1)$ are bijections on path components, i.e., $\pi_0(\text{Isom}_b(E_1)) \simeq \pi_0(\text{Diff}_c(E_1))$, $\pi_0(\text{Isom}_b(\partial E_1)) \simeq \pi_0(\text{Diff}_c(\partial E_1))$. In particular, any diffeomorphism in $\text{Diff}_c(\partial E_1)$ (resp. $\text{Diff}_c(E_1)$) is isotopic to one in $\text{Isom}_b(\partial E_1)$ (resp. $\text{Isom}_b(E_1)$).

**Proof.** For $t \in (0, 1]$, let $h_t : E_1 \to E_1$ be the dilatation mapping $V \in E_1$ to $tV$, and for any $F \in \text{Diff}_c(E_1) = \text{Diff}_c(E_1, \mathcal{F}_c)$, we define

$$F_t := (F \circ h_t)/\lambda(t) \in \text{Diff}_c(E_1),$$

where $\lambda : [0, 1] \to [0, 1]$ is the function defined by $F(T_t) = T_{\lambda(t)}$ for any concentric tube $T_t$ of radius $t \in [0, 1]$. Explicitly, we have

$$\lambda(t) = |F(tV)|, \quad \text{for any } V \in \partial E_1,$$

where $| \cdot |$ denotes the norm of the Euclidean metric, and for $t = 0$, $tV = \pi(V)$ means the base point in the zero section $B$. In fact, it is not hard to verify that

$$\lambda(t) = \int_0^t |(\gamma'_{\lambda}(t))^{-1}| dt, \quad \text{for any } V \in \partial E_1,$$

where $\gamma'_{\lambda}(t)$ is the tangent vector of the curve $\gamma_{\lambda}(t) := F(tV)$, and $(\cdot)^{-1}$ means the projection from $T(E_1)$ to the vertical distribution $E_1$. Moreover, $\gamma'_{\lambda}(t) = (F_*)_{tV}(V)$ is the image of $V$ under the tangential map $F_* : T(E_1) \to T(E_1)$ at $tV \in E_1$. Therefore, $\lambda$ is a smooth function with $\lambda'(0) > 0$, $\text{for any } V \in \partial E_1$. Define $F_0 : E_1 \to E_1$ by

$$F_0(V) := ((F_*)_{tV}(V))^{-1}/\lambda'(0), \quad \text{for } V \in E_1.$$ 

Then since $F_*$ is linear, $F_0$ is linear and preserves lengths, thus $F_0 \in \text{Isom}_b(E_1)$.

Set $f_t := \rho(F_t) = F_t|_{\partial E_1} \in \text{Diff}_c(\partial E_1)$, and $f_0 := \lim_{t \to 0^+} f_t$. It follows that

$$f_0(V) = \lim_{t \to 0^+} F(tV)/\lambda(t) = ((F_*)_{tV}(V))^{-1}/\lambda'(0) = F_0(V), \quad \text{for } V \in \partial E_1.$$ 

Hence $f_0 = \rho(F_0) \in \text{Isom}_b(\partial E_1)$.

In conclusion, we have shown that any $F \in \text{Diff}_c(E_1)$ corresponds uniquely to a path $\{f_t | t \in [0, 1]\}$ in $\text{Diff}_c(\partial E_1)$ with one end $f_0 \in \text{Isom}_b(\partial E_1)$, which also shows that $\text{Isom}_b(\partial E_1) \hookrightarrow \text{Diff}_c(\partial E_1)$ is a bijection on path components. Conversely, given
such a path \( \{ f_t | t \in [0, 1] \} \) and a smooth nondecreasing function \( \lambda : [0, 1] \to [0, 1] \) with \( \lambda'(0) > 0 \), one can construct an \( F \in \text{Diff}_c(E_1) \) by: for \( V \in \partial E_1 \), \( F(tV) = \lambda(t)f_0(V) \) for \( t \in (0, 1) \), and \( F(b) = F_0(b) \) for \( b = \pi(V) \in B \), where \( F_0 \) is the isomorphism that restricts to \( f_0 \). Therefore, a retraction of the path \( \{ f_t | t \in [0, 1] \} \) to the constant path \( \{ f_0 \} \) induces a path connecting \( F \in \text{Diff}_c(E_1) \) with \( F_0 \in \text{Isom}_b(E_1) \) in \( \text{Diff}_c(E_1) \). The proof is now complete. □

**Corollary 2.5.** Either the inclusion \( \text{Isom}_b(\partial E_1) \hookrightarrow \text{Diff}_c(\partial E_1) \) is not surjective on path components, i.e., \( \pi_0(\text{Isom}_b(\partial E_1)) \not\subseteq \pi_0(\text{Diff}_c(\partial E_1)) \), or \( \text{Diff}_c(\partial E_1) = \text{Isom}_b(\partial E_1) \).

**Proof.** If \( \pi_0(\text{Isom}_b(\partial E_1)) \not\simeq \pi_0(\text{Diff}_c(\partial E_1)) \), then \( \pi_0(\text{Diff}_c(\partial E_1)) \not\subseteq \pi_0(\text{Isom}_b(\partial E_1)) \) by Proposition 2.4. In particular, any \( f \in \text{Diff}_c(\partial E_1) \) is isotopic to one in \( \text{Diff}_c(\partial E_1) \). Then by considering a collar of \( \partial E_1 \) in \( E_1 \), the isotopy induces an extension of \( f \) to a diffeomorphism \( F \in \text{Diff}_c(E_1) \), proving that \( f = F|_{\partial E_1} \in \text{Diff}_c(\partial E_1) \). □

Now we are ready to consider the possibility of the existence of negative examples towards Problem 2.1. Let \( \varphi_0 : \partial E_+ \to \partial E_- \) be a gluing diffeomorphism and \((E_{\varphi_0}, F_{\varphi_0})\) be the foliation as before. For any \( h_\pm \in \text{Diff}_{E_\pm}(\partial E_\pm) \), it is easily seen that the glued manifold \( E_{h_- \circ \varphi_0 h_+^{-1}} \) is diffeomorphic to \( E_{\varphi_0} \). The foliations \((E_{h_- \circ \varphi_0 h_+^{-1}}, F_{h_- \circ \varphi_0 h_+^{-1}})\) then are candidates for negative examples. In the following we show that the groups \( \pi_0(\text{Isom}_b(\partial E_\pm)), \pi_0(\text{Diff}_{E_\pm}(\partial E_\pm)) \) take the key role in voting for them.

**Theorem 2.6.** With notations as before, we have

1. If there are \( h_\pm \in \text{Diff}_{E_\pm}(\partial E_\pm) \) such that \((E_{h_- \circ \varphi_0 h_+^{-1}}, F_{h_- \circ \varphi_0 h_+^{-1}}) \not\simeq (E_{\varphi_0}, F_{\varphi_0}) \), then either \( \pi_0(\text{Isom}_b(\partial E_\pm)) \not\subseteq \pi_0(\text{Diff}_{E_\pm}(\partial E_\pm)) \) or \( \pi_0(\text{Isom}_b(\partial E_-)) \not\subseteq \pi_0(\text{Diff}_{E_-}(\partial E_-)) \).

2. Consider \( E_+ = E_- \). If \( \pi_0(\text{Isom}_b(\partial E_\pm)) \not\subseteq \pi_0(\text{Diff}_{E_\pm}(\partial E_\pm)) \), then for any \([\varphi_0] \in \pi_0(\text{Isom}_b(\partial E_\pm)) \) and \([\varphi_1] \in \pi_0(\text{Diff}_{E_\pm}(\partial E_\pm)) \backslash \pi_0(\text{Isom}_b(\partial E_\pm)) \), we have \( E_{\varphi_0} \not\simeq E_{\varphi_1} \) but \((E_{\varphi_0}, F_{\varphi_0}) \not\equiv (E_{\varphi_1}, F_{\varphi_1}) \).

**Proof.** (1) We prove this by contradiction. If both \( \pi_0(\text{Isom}_b(\partial E_\pm)) \simeq \pi_0(\text{Diff}_{E_\pm}(\partial E_\pm)) \), then \( \text{Diff}_{E_\pm}(\partial E_\pm) = \text{Diff}_c(\partial E_\pm) \) by Corollary 2.5 whence \((E_{h_- \circ \varphi_0 h_+^{-1}}, F_{h_- \circ \varphi_0 h_+^{-1}}) \simeq (E_{\varphi_0}, F_{\varphi_0}) \) for any \( h_\pm \in \text{Diff}_{E_\pm}(\partial E_\pm) \) by Proposition 2.2.

(2) Under the assumptions, we see that \( h := \varphi_1 \circ \varphi_0^{-1} \) lies in \( \pi_0(\text{Diff}_{E_+}(\partial E_+)) \backslash \pi_0(\text{Isom}_b(\partial E_+)) \), which shows immediately \( E_{\varphi_0} \not\simeq E_{h \circ \varphi_0} = E_{\varphi_1} \). Now we prove the nonequivalence of the foliations by contradiction.

If \((E_{\varphi_0}, F_{\varphi_0}) \simeq (E_{\varphi_1}, F_{\varphi_1}) \), then by Proposition 2.2 there exist \( f_\pm \in \text{Diff}_c(\partial E_\pm) \) such that \( \varphi_1 = f_- \circ \varphi_0 \circ f_+^{-1} \). Hence \( h = f_- \circ \varphi_0 \circ f_+^{-1} \circ \varphi_0^{-1} \). By Proposition 2.4 \([f_\pm] \in \pi_0(\text{Isom}_b(\partial E_\pm)) \) and hence we have \([h] = [f_- \circ \varphi_0 \circ f_+^{-1} \circ \varphi_0^{-1}] \in \pi_0(\text{Isom}_b(\partial E_\pm)) \), the contradiction. □
Remark 2.7. Consider $E_+ = D^n$. Isom$_0(\partial E_+) = \text{Isom}(S^{n-1}) = O(n)$ and Diff$_+(\partial E_+) = \rho(\text{Diff}(D^n)) = \text{Diff}_+(S^{n-1}) \subseteq \text{Diff}(S^{n-1})$. For $n \geq 6$, it is well-known from the pseudo-isotopy theorem of Cerf (cf. [2]) that $\pi_0(\text{Diff}^+(D^n)) = 0$, and thus $\pi_0(\text{Diff}_+(S^{n-1})) \simeq \pi_0(\text{Isom}(S^{n-1})) \simeq \mathbb{Z}_2$. For $n \leq 5$, $\text{Diff}_+(S^{n-1}) = \text{Diff}(S^{n-1})$ because of the exact sequence $\pi_0(\text{Diff}^+(D^n)) \xrightarrow{\partial} \pi_0(\text{Diff}^+(S^{n-1})) \to \Gamma_n$, where the group of twisted n-spheres $\Gamma_n = 0$ for $n \leq 6$. Therefore, for $n \leq 4$, we have also $\pi_0(\text{Diff}_+(S^{n-1})) \simeq \pi_0(\text{Isom}(S^{n-1})) \simeq \mathbb{Z}_2$, since $\text{Diff}(S^{n-1}) \simeq O(n)$ are homotopy equivalent proven by Smale [18] for $n = 3$ and by Hatcher [3] for $n = 4$ (well-known for $n \leq 2$). Nothing is known so far on the homotopy of the group $\text{Diff}^+(D^5) \simeq \text{Diff}^+(S^4)$, even its $\pi_0$.

Now by the disk theorem of Palais [15], every pair $E_{\varphi_0} \cong E_{\varphi_1}$ has the form in (1) of Theorem 2.6, i.e., $\varphi_1 = h_- \circ \varphi_0 \circ h_+^{-1}$ for some $h_+ \in \text{Diff}_{E_+}(\partial E_+)$. It follows that negative examples exist in the case when $E_+ = D^n (n \neq 5)$, and only if $\pi_0(\text{Isom}_0(\partial E_-)) \not\subseteq \pi_0(\text{Diff}_{E_-}(\partial E_-))$. In particular, every foliations $(E_{\varphi_i}, F_{\varphi_i})$ ($i = 0, 1$) with $E_\pm = D^n (n \neq 5)$ are equivalent provided $E_{\varphi_0} \cong E_{\varphi_1}$. In other word, we have shown the following.

Corollary 2.8.  

1. Every homotopy sphere $S^n (n \neq 4, 5)$ and $S^4$ admit exactly one equivalence class of isoparametric foliations with two points as the focal submanifolds.

2. Either the inclusion $O(5) \hookrightarrow \text{Diff}(S^4)$ is a bijection on path components, i.e., $\pi_0(\text{Diff}(S^4)) \simeq \mathbb{Z}_2$, or $S^5$ admits nonequivalent isoparametric foliations with two points as the focal submanifolds.

Proof. The only left stuff is to recall the well-known fact that every homotopy n-sphere except for exotic 4-spheres (if exist) is a twisted n-sphere, the topological n-sphere obtained by gluing two n-disks through a diffeomorphism $\varphi \in \text{Diff}^+(S^{n-1})$. We recall that exotic 4-spheres (if exist) admit no singular Riemannian foliations, of codimension 1 by [6], or of general codimension by [9].

It is well-known that the group $\Gamma(D^n)$ of concordance classes of orientation-preserving diffeomorphisms of $D^n$ is trivial, thus $\Gamma(S^4) = 0$. It follows that the first option in (2) above means that concordance (pseudo-isotopy) implies isotopy on $S^4$, which always holds on closed simply connected manifolds of dimension at least 5 by the pseudo-isotopy theorem of Cerf (cf. [2]).

3. ISOPARAMETRIC FOLIATIONS ON HOMOTOPY SPHERES

Recall that a homotopy n-sphere $\Sigma^n$ is a closed smooth manifold which has the homotopy type of $S^n$. It is well-known (cf. [9]) that $\Sigma^n$ is always homeomorphic to $S^n$.
but not always diffeomorphic in general, in which case \( \Sigma^n \) is called an exotic sphere. For \( n \leq 6 \) and \( n \neq 4 \), there are no exotic \( n \)-spheres. In contrast, there are finitely many exotic \( n \)-spheres in infinitely many dimensions \( n \geq 7 \). Henceforth, it suffices to consider homotopy \( n \)-spheres for \( n \geq 7 \). In this case, the group \( \Theta_n \) of h-cobordism classes of oriented homotopy \( n \)-spheres (always isomorphic to the group \( \Gamma_n \) of oriented twisted \( n \)-spheres) is isomorphic to the mapping class group \( \pi_0 \text{Diff}^+(\mathbb{S}^{n-1}) \) by (cf. [2])

\[
\pi_0 \text{Diff}^+(\mathbb{S}^{n-1}) \rightarrow \Gamma_n \simeq \Theta_n
\]

\[|\phi| \mapsto \Sigma_\phi := D^n \cup \phi D^n.\]

Note that \( \Sigma_\phi \) depends only on the isotopy class of \( \phi \in \text{Diff}^+(\mathbb{S}^{n-1}) \).

**Proof of Theorem 1.1.** Given an isoparametric foliation \( F \) on a homotopy \( n \)-sphere \( \Sigma \), we have the decomposition (2.1) with \( N = \Sigma = \Sigma_+ \cup \Sigma_- \cong E_+ \cup_\varphi E_- \). In the following we fix the orientations and (foliated) diffeomorphisms are assumed orientation-preserving.

It is well-known that any orientation-preserving diffeomorphism is isotopic to one that restricts to the identity on an embedded disk. Thus, for any \( \phi \in \text{Diff}^+(\mathbb{S}^{n-1}) \), we can assume \( \phi : \mathbb{S}^{n-1} = D^n \cup \text{id} D^n \rightarrow D^n \cup \text{id} D^n \) satisfy \( \phi|_{D^n} = \text{id} \). Consider a disk \( D^n \) in \( M := \partial E_- \) and write \( M \) as \( M = M' \cup D^n \). Define a diffeomorphism \( d_\phi \) on \( M \) by setting \( d_\phi|_{D^n} = \phi|_{D^n} \) and identity on \( M' \). Then gluing \( E_- \) with \( E_+ \) by \( d_\phi \circ \varphi \), we get a manifold

\[ E_{d_\phi \circ \varphi} := E_+ \cup_{d_\phi \circ \varphi} E_- \]

It follows easily that \( E_{d_\phi \circ \varphi} \) depends only on the isotopy class of \( \phi \in \text{Diff}^+(\mathbb{S}^{n-1}) \) and hence we have defined a map \( \Phi_\varphi : \pi_0 \text{Diff}^+(\mathbb{S}^{n-1}) \simeq \Theta_n \rightarrow \Phi_\varphi(\Theta_n) \) by mapping \( |\phi| \simeq \Sigma_\phi \) to \( E_{d_\phi \circ \varphi} \). Since the induced map \( (d_\phi)_* \) on homotopy groups \( \pi_*(M) \) and homology groups \( H_*(M) \) is trivial, it follows from van Kampen theorem and Mayer-Vietoris sequence that \( E_{d_\phi \circ \varphi} \) is also a homotopy \( n \)-sphere. We claim that \( E_{d_\phi \circ \varphi} \) is essentially diffeomorphic to the homotopy \( n \)-sphere \( \Sigma_\phi \# \Sigma \). Consequently, the image \( \Phi_\varphi(\Theta_n) = \Theta_n + \Sigma = \Theta_n \), which means that each homotopy \( n \)-sphere can be decomposed into the same disk bundles \( E_\pm \). Moreover, it follows from the discussion in the last section that, the 1-1 correspondence \( \{(\Sigma, F)\} \leftrightarrow \{\tilde{\Sigma}, \tilde{F}\} \) between the sets of equivalence classes of isoparametric foliations on any two homotopy \( n \)-spheres \( \Sigma \) and \( \tilde{\Sigma} = \Sigma_\phi \# \Sigma \) can be represented by

\[ (E_\varphi, F_\varphi) \mapsto (E_{d_\phi \circ \varphi}, F_{d_\phi \circ \varphi}), \]

and the inverse is in the same form if we wrote \( \Sigma = \Sigma_\phi^{-1} \# \tilde{\Sigma} \). It is still left to show that the 1-1 correspondence is well-defined, i.e., if \( (E_{\varphi_0}, F_{\varphi_0}) \cong (E_{\varphi_1}, F_{\varphi_1}) \) then \( (E_{d_{\phi_0} \circ \varphi_0}, F_{d_{\phi_0} \circ \varphi_0}) \cong (E_{d_{\phi_1} \circ \varphi_1}, F_{d_{\phi_1} \circ \varphi_1}) \). By Proposition 2.2 and Corollary 2.3, there are diffeomorphisms \( h_\pm \in \text{Diff}^c(\partial E_\pm) \) such that \( \varphi_1 = h_- \circ \varphi_0 \circ h_-^{-1} \) and without loss of
generality, we can assume $h_-|_{D^n \cap S^{n-1}} = id$. Then $d_\phi \circ h_- = h_- \circ d_\phi$ on $M$ and hence $d_\phi \circ \varphi_1 = h_- \circ d_\phi \circ \varphi_0 \circ h_+^{-1}$, proving $(E_{d_\phi \circ \varphi_0}, \mathcal{F}_{d_\phi \circ \varphi_0}) \cong (E_{d_\phi \circ \varphi_1}, \mathcal{F}_{d_\phi \circ \varphi_1})$.

Now we come back to consider the claim that $E_{d_\phi \circ \varphi} \cong \Sigma_\phi \# E_\varphi$. It requires essentially no more than an alternative explanation of the connected sum $N \# \Sigma_\phi$ by removing a disk in $N$ and gluing the disk back through $\phi$. In fact, we will prove it in a more general setting by (4.3) in Section 4, where replacing Gluing $W$ embedding of identity elsewhere so as to define a diffeomorphism $\Psi_\phi$ on $\partial W$.

That is, we construct an explicit $h$-cobordism between $E_{d_\phi \circ \varphi}$ and $\Sigma_\phi \cong \Sigma_\phi \# S^n$ when $E_\varphi = S^n$. Take an interior disk $D^n$ in $E_-$ and write $E_- = E_-^\prime \cup D^n$. Then $\partial E_-^\prime = M \sqcup S^{n-1}$. Take disks $D_-^{n-1}$ in $M$ and $S^{n-1}$ respectively and connect them by an embedding of $D_-^{n-1} \times [0,1]$ in $E_-^\prime$. Extend $\phi|_{D_-^{n-1}}$ isomorphically to $D_-^{n-1} \times [0,1]$ and by the identity elsewhere so as to define a diffeomorphism $\Psi_\phi$ on $E_-^\prime$, thereby $\Psi_\phi|_M = d_\phi$ and $\Psi_\phi|_{S^{n-1}} = \phi$. Let $D(E_-) = E_- \cup_{id} E_-$ be the double of $E_-$. Then it is an $S^{m_-}$ bundle over $M_-$ bounding a disk bundle, say $W_-$, over $M_-$ of rank $m_- + 1$ (see 19 for an interesting study of the topology and geometry on this double). Rewrite the boundaries $\partial W_-$ and $\partial D^{n+1}$ as

$$\partial W_- = D(E_-) = E_- \cup D^-_n \sqcup S^{n-1} D^n,$$

$$\partial D^{n+1} = S^n = E_\varphi = E_+ \cup \varphi E_-^\prime \sqcup S^{n-1} D^n.$$

Gluing $W_-$ with $D^{n+1}$ along the common part $E_-^\prime$ of their boundaries by the diffeomorphism $\Psi_\phi$ (and smoothing the corners), we get a manifold $W^{n+1} = D^{n+1} \cup_{\Psi_\phi} W_-$ with boundary $\partial W = E_{d_\phi \circ \varphi} \sqcup \Sigma_\phi$.

By van Kampen theorem $W$ is simply connected and thus $H_1(W) = 0$. Since both $E_-$ and $W_-$ contract to $M_-$, we have $H_k(E_-) \simeq H_k(W_-) \simeq H_k(M_-)$ for any $k$ and they vanish when $k \geq n - 1$ since $dim(M_-) < n - 1$. By the Mayer-Vietoris sequence, we have $H_k(E_-^\prime) \simeq H_k(E_-) \simeq H_k(W_-)$ for $k = 1, \ldots, n - 2$, $H_{n-1}(E_-^\prime) \simeq \mathbb{Z}$ and $H_n(E_-^\prime) = 0$. By the Mayer-Vietoris sequence

$$\cdots \to H_k(E_-^\prime) \to H_k(W_-) \to H_k(W) \to H_{k-1}(E_-^\prime) \to H_{k-1}(W_-) \to \cdots,$$

we obtain

$$H_*(W) = \begin{cases} \mathbb{Z} & \text{if } * = 0, n, \\ 0 & \text{otherwise}. \end{cases}$$

Applying the exact sequence of relative homology gives $H_*(W, \Sigma_\phi) = 0$. Therefore, by the $h$-cobordism theorem (cf. 11), $E_{d_\phi \circ \varphi}$ is diffeomorphic to $\Sigma_\phi$.

The proof is now complete. \qed
4. Inertia groups and exotic smooth structures

As in the last section, we always choose a representative $\phi$ of $[\phi] \in \pi_0(Diff^+(\mathbb{S}^{n-1}))$ (identified with $\Sigma_\phi \subset \Theta_n$ by (3.1)) such that $\phi|_{D_+^{n-1}} = id$. Diffeomorphisms and embeddings with codimension zero are orientation-preserving.

Let $M^n$ be a closed oriented manifold. Recall (cf. [10]) that there are two subgroups $I_0(M) \subset \Theta_n$, $I_1(M) \subset \Theta_{n+1}$ called the inertia groups of $M$. $I_0(M)$ consists of all $\Sigma_\phi \in \Theta_n$ such that $M#\Sigma_\phi \cong M$. $I_1(M)$ consists of all $\Sigma_\phi \in \Theta_{n+1}$ such that the diffeomorphism $d_\phi$ of $M$ which differs from the identity only on an $n$-disk in $M$ (identified with $D^n \subset \mathbb{S}^n$), and there coincides with $\phi$, is concordant to the identity. It is not hard to see that for $\Sigma_\phi \in \Theta_n \setminus I_0(M)$, $M#\Sigma_\phi \not\cong M$ is homeomorphic to $M$ (by a radial extension of $\phi$ to $D^n$) and hence gives an exotic oriented smooth structure on $M$. Moreover, different cosets in $\Theta_n/I_0(M)$ give distinct oriented smooth structures on $M$. Therefore, there exist at least $|\Theta_n|/|I_0(M)|$ distinct oriented smooth structures on $M$. $I_0(M)$ is then of great importance in the study of exotic smooth structures on $M$. On the other hand, $I_1(M)$ contributes to the study of the group $\Gamma(M)$ of concordance classes of diffeomorphisms of $M$, i.e., the coset space $\Theta_{n+1}/I_1(M)$ corresponds to a subset of $|\Theta_{n+1}|/|I_1(M)|$ elements in $\Gamma(M)$ (and hence in $\pi_0(Diff^+(M))$).

Levine [10] showed these two inertia groups have a very close relation:

\[(4.1)\quad I_1(M) = I_0(M \times S^1).\]

In the following we relate further $I_1(M)$ with $I_0(N)$ when $M$ is a hypersurface in $N$, which leads to a proof of Theorem 1.2 and would certainly induce more applications.

**Theorem 4.1.** Let $M^{n-1}$ be a closed oriented hypersurface embedded in a closed oriented manifold $N^n$. Then

\[(4.2)\quad I_1(M^{n-1}) \subseteq I_0(N^n).\]

Therefore, $I_0(M^{n-1} \times \mathbb{S}^1) \subseteq I_0(N^n)$. In particular, there exist at least $|\Theta_n|/|I_0(N)|$ distinct oriented smooth structures on $M^{n-1} \times \mathbb{S}^1$, and there exist at least $|\Theta_n|/|I_0(N)|$ elements in $\Gamma(M)$ and $\pi_0(Diff^+(M))$.

**Proof.** Let $\tilde{N}$ be the complementary of $M$ in $N$ with boundary $M \sqcup -M$. Given any $f \in Diff^+(M)$, one gets a closed oriented manifold $N_f$ by gluing $\tilde{N}$ along $M$ through $f$; thus $N_{id} = N$. For $\Sigma_\phi \in \Theta_n$, we denote by $N_\phi$ the manifold $N_{d_\phi}$, where $d_\phi \in Diff^+(M)$ is the diffeomorphism that equals $\phi$ in a disk $D_+^{n-1} \subset M$ and $id$ outside it, as in the definition of $I_1(M)$. We claim that

\[(4.3)\quad N_\phi \cong N#\Sigma_\phi.\]

Consider a collar $C := M \times [0, \infty)$ of $M$ in $\tilde{N}$ and the embedded disk $\tilde{D}^n := D_+^{n-1} \times [0, 1] \cong D^n$ in $C$, where the diffeomorphism $\tilde{d} : D^n \to \tilde{D}^n$ can be chosen so that
it restricts to the identity on the common part \(D_n^{n-1}\) of the boundaries \(\partial \tilde{D}^n\) and \(\partial D^n = \mathbb{S}^{n-1} = D_+^{n-1} \cup D_-^{n-1}\). Otherwise, by the disk theorem, there is a diffeomorphism \(f \in \text{Diff}^+(\mathbb{S}^{n-1})\) isotopic to the identity such that \(f\big|_{D_n^{n-1}} = \tilde{d}^{-1}\big|_{D_n^{n-1}}\). Let \(F \in \text{Diff}^+(D^n)\) be an extension of \(f\). Then we can choose the diffeomorphism \(\tilde{d} \circ F : D^n \to \tilde{D}^n\) which restricts to the identity on \(D_n^{n-1}\). Now since the diffeomorphism \(\tilde{d}_\phi := \tilde{d} \circ \phi \circ \tilde{d}^{-1}|_{\partial \tilde{D}^n} \in \text{Diff}^+(\partial \tilde{D}^n)\) restricts to \(\phi\) on \(D_n^{n-1}\) and \(id\) elsewhere, we can regard the manifold \(N_\phi\) as

\[
N_\phi = (N \setminus \tilde{D}^n) \cup_{\tilde{d}_\phi} \tilde{D}^n.
\]

On the other hand, by the disk theorem we can write the connected sum as

\[
N\# \Sigma_\phi = (N \setminus \tilde{D}^n) \cup_{d_+ \circ d_-|_{\partial D^n}} (\Sigma_\phi \setminus \partial \Sigma_\phi),
\]

where \(d_\pm : D^n \to \overline{D}_\pm^n\) are disks embedded in \(\Sigma_\phi = \overline{D}_+^n \cup \overline{D}_-^n\) with \(d_\pm \circ d_\pm|_{\partial D^n} = \phi\). Then the equation (4.3) follows from

\[
(N \setminus \tilde{D}^n) \cup_{d_+ \circ d_-|_{\partial D^n}} \overline{D}_-^n \cong (N \setminus \tilde{D}^n) \cup_{d_- \circ d_+|_{\partial D^n}} D^n \cong (N \setminus \tilde{D}^n) \cup_{d_\phi} \tilde{D}^n.
\]

Now for any \(\Sigma_\phi \in I_1(M)\), \(N_\phi = N_{d_\phi} \cong N\) since \(d_\phi\) is concordant to \(id\). Then by (4.3) we obtain \(\Sigma_\phi \in I_0(N)\), proving (4.2).

The proof is now complete. \(\Box\)

**Proof of Theorem 1.2.** Let \(M\) be a closed hypersurface embedded in \(\mathbb{S}^n\). Then \(M\) is orientable and we fix its orientation (cf. [12]). For any \(k \geq 0\) and any product \(P^k := \mathbb{S}^{k_1} \times \cdots \times \mathbb{S}^{k_l}\) of standard spheres of total dimension \(k = \sum_{i=1}^l k_i\) (\(k_i \geq 1\) \((P^k\) is a point when \(k = 0\)), we can embed \(M^{n-1} \times P^k\) in \(\mathbb{S}^n \times P^k\). Then by Theorem 1.1 we have \(I_0(M^{n-1} \times P^k \times S^1) \subset I_0(\mathbb{S}^n \times P^k)\). By the theorem of Schultz [17], we obtain \(I_0(M^{n-1} \times P^k \times S^1) = I_0(\mathbb{S}^n \times P^k) = 0\), completing the proof. \(\Box\)

At last we remark that combining the relations (4.1), (4.2) with (4.3) would deduce more applications than what we have showed, once provided with some known \(I_0(N)\).

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