Geometric Singularities and Enhanced Gauge Symmetries

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Using “Tate’s algorithm,” we identify loci in the moduli of F-theory compactifications corresponding to enhanced gauge symmetry. We apply this to test the proposed F-theory/heterotic dualities in six dimensions. We recover the perturbative gauge symmetry enhancements of the heterotic side and the physics of small $SO(32)$ instantons, and discover new mixed perturbative/non-perturbative gauge symmetry enhancements. Upon further toroidal compactification to 4 dimensions, we derive the chain of Calabi-Yau threefolds dual to various Coulomb branches of heterotic strings.

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1. Introduction

One of the key observations in the recent advances in understanding non-perturbative aspects of string theory has been the appreciation that singular geometries of string compactification can be re-interpreted in terms of solitonic states which become light. For example, in type IIA/B string compactification on $K3$ the A-D-E singularities are associated with massless/tensionless particles/strings $[1,2,3]$, while the conifold singularities of Calabi-Yau manifolds are associated with massless hypermultiplets $[4,3]$. A similar story also turns out to be true for small instantons of $SO(32)$ heterotic strings $[5,6]$.

A better understanding of non-perturbative aspects of string theory clearly requires an extensive knowledge of the physical reinterpretation of geometric singularities. It is the aim of this paper to find a geometry/physics dictionary for a limited series of geometric singularities arising in string compactification. We will mainly concentrate on compactification to $d = 6$ with $N = 1$ supersymmetry (or equivalently $d = 4$, $N = 2$) but the methods have a wider range of validity.

The starting point is the type IIA/heterotic dualities observed for $d = 4$, $N = 2$ $[7,8]$ and, more precisely, their extensions $[9]$ as F-theory/heterotic dualities in $d = 6$, $N = 1$. In particular, we will concentrate on the physical interpretation of the singularities of the hypermultiplet moduli in F-theory/heterotic dualities in $d = 6$. Since a hypermultiplet in $d = 6$, $N = 1$ remains a hypermultiplet upon further toroidal compactification to $d = 4$, our results hold for that case as well. Using a method known as “Tate’s algorithm” $[10]$, we explicitly calculate the geometric singularities of the F-theory compactification which lead to enhanced gauge symmetries, and are able to classify these singularities. In particular, we will find the singularity realization of all classical gauge groups, including the non-simply laced $B$ and $C$ series and $F_4$ and $G_2$ gauge groups. On the heterotic side, some of the singularities are mapped to partial restoration of perturbative gauge symmetries upon partial ‘un-Higgsing’. The match between these descriptions provides further strong evidence for the proposed dualities. This also shows how the singularities encode certain matter representations of the gauge group.

These results have a number of applications. In particular, we can uncover certain aspects of non-perturbative gauge symmetry enhancements on the heterotic side by using the dictionary we develop for singularities of F-theory compactifications. This includes recovering the physical interpretation of small $SO(32)$ instantons $[11]$ as well as discovering new mixed perturbative/non-perturbative gauge symmetry enhancements for certain heterotic compactifications. In another direction, having identified the loci of enhanced gauge
symmetries, and upon further compactification to 4 dimensions, we will get new branches corresponding to the Coulomb branch of these enhanced gauge symmetries. In this way, we will be able to derive the chain of Calabi-Yau threefolds which correspond to various choices of Coulomb branches on the heterotic side. In this way we begin to map out and in fact derive the web of $d = 4, N = 2$ heterotic/type II dualities. These results explain (and extend) some previous results on this topic [11,12].

The organization of this paper is as follows: In section 2, we review the F-theory/heterotic duality of [13,9]. In section 3, we present the basic idea for finding the detailed map of the enhanced gauge symmetry loci. This is facilitated by the work of Tate, which we review (and extend). In section 4, we compare the loci of perturbative enhanced gauge symmetries on the F-theory and heterotic sides and find a perfect match. In section 5, we apply the dictionary developed to the case of small instantons of $SO(32)$ heterotic strings. In section 6, we consider loci of enhanced gauge symmetries which are mixtures of perturbative and non-perturbative gauge groups on the heterotic side. In section 7, we consider further toroidal compactification to 4 dimensions, where the F-theory/heterotic duality turns into $N = 2$ type IIA/heterotic duality. We identify the new Coulomb branches of perturbative heterotic strings on the type IIA side by deriving the dual chains of Calabi-Yau threefolds. Tate’s algorithm turns out to be particularly helpful in enabling us to derive these results. In section 8, we present our conclusions.

As this work was being completed, we received a paper with some overlapping results [14]. While the bulk of the work here, and in particular the section on the F-theory realization of small $SO(32)$ instantons was completed independently, we have used (in §3) some of the ideas of [14] concerning monodromy.

2. Review of F-theory/heterotic dualities

In this section we review compactifications of F-theory on elliptic Calabi-Yau manifolds and the corresponding heterotic duals [13,9]. F-theory can be compactified on elliptic Calabi-Yau manifolds. Such compactifications can be interpreted as type IIB vacua where two things happen: 1-The coupling constant $\tau$, which is to be identified with the complex modulus of elliptic fiber, varies over space. 2-It is allowed to undergo $SL(2, \mathbb{Z})$ monodromies, which are conjectured to be a symmetry of type IIB strings [15]. The first fact implies that, at least in the regions where the coupling is weak ($\tau_2 >> 1$) we can use a perturbative string description. However, the second fact is a marked departure from
type IIB perturbative string vacua. This is also similar to M-theory compactifications: In the same way, we can think of M-theory compactifications as type IIA compactifications where the coupling is allowed to vary over the space and also make jumps. It is the latter fact which makes the geometric M-theory description more powerful and the type IIA perspective more limited.

An elliptic Calabi-Yau can be described in the Weierstrass form
\[
y^2 = x^3 + xf + g
\] (2.1)
which describes the elliptic fibration (parameterized by \((y, x)\) subject to the above equation) over the base \(B\), where \(f\) and \(g\) are functions on the base.

At some divisors \(D_i\) the torus (fiber) degenerates. These divisors are given by the zero loci of the discriminant
\[
\Delta = 4f^3 + 27g^2. \tag{2.2}
\]

Singularities of the manifold are coded in the polynomials \(f, g\) and determine the gauge group and the matter content of the F-theory compactification. One can get a singular locus by adjusting various coefficients in the polynomials \(f\) and \(g\). On the heterotic side this process typically corresponds to “un-Higgsing” by turning off charged fields and restoring some gauge symmetry. In this paper, we compare these two mechanisms and establish the precise dictionary between various singularities of elliptic fibrations and gauge symmetry enhancement.

The types of singularities of elliptic fibrations were classified by Kodaira. His results are summarized in the table below. \(\text{ord}(X)\) denotes the order of the zero of a polynomial \(X\) at the discriminant locus.

| \(\text{ord}(f)\) | \(\text{ord}(g)\) | \(\text{ord}(\Delta)\) | fiber type | singularity type |
|-----------------|-----------------|-----------------|-----------|---------------|
| \(\geq 0\)      | \(\geq 0\)      | 0               | smooth    | none          |
| 0               | 0               | \(n\)           | \(I_n\)   | \(A_{n-1}\)   |
| \(\geq 1\)      | 1               | 2               | \(II\)    | none          |
| 1               | \(\geq 2\)      | 3               | \(III\)   | \(A_1\)       |
| \(\geq 2\)      | 2               | 4               | \(IV\)    | \(A_2\)       |
| 2               | \(\geq 3\)      | \(n+6\)         | \(I_n^*\) | \(D_{n+4}\)   |
| \(\geq 2\)      | 3               | \(n+6\)         | \(I_n^*\) | \(D_{n+4}\)   |
| \(\geq 3\)      | 4               | 8               | \(IV^*\)  | \(E_6\)       |
| 3               | \(\geq 5\)      | 9               | \(III^*\) | \(E_7\)       |
| \(\geq 4\)      | 5               | 10              | \(II^*\)  | \(E_8\)       |
When \( \text{ord}(f) \geq 4 \) and \( \text{ord}(g) \geq 6 \), the singularity of the manifold is so bad that it generically destroys the triviality of the canonical bundle.

If the base \( B = \mathbb{P}^1 \) then \( f \) and \( g \) are functions of one variable only, say \( z_1 \). This corresponds to compactification of F-theory on \( K3 \), which has been conjectured to be dual to heterotic compactification on \( T^2 \) [13]. The duality of this F-theory compactification with type I has been recently discussed in [16]. In F-theory on \( K3 \), the singularity type of Kodaira exactly matches the conjectured enhanced gauge symmetry of the eight-dimensional theory. This, in fact, can be verified in the F-theory language for the \( A_n \) case by the realization that it corresponds to \( n + 1 \) parallel 7-branes [13]. Upon further compactification on \( T^2 \) and using the equivalence with type IIA on \( K3 \), this realizes the mechanism of gauge symmetry enhancement for singular \( K3s \) suggested in [3,17]. For the other types of singularities, string-string duality in six dimensions requires that we identify the Weierstrass A-D-E classification with the corresponding gauge symmetry enhancement [1].

The meaning of the singularity type when the base is more than one dimensional is a priori less clear. We can still continue to use Kodaira’s terminology of A-D-E for the singularity type even when the dimension of \( B \) is bigger than one, but the physical implication of the singularity turns out to be more intricate. The main aim of this paper is to address this issue. We will mainly concentrate on the case \( B = F_n \), where \( F_n \) is the rational ruled surface, but our methods are more general.

F-theory on the elliptic CY 3-fold over \( F_n \) is conjectured to be dual to the \( E_8 \times E_8 \) heterotic string on \( K3 \) with instanton numbers \((12 + n, 12 - n)\) in the two \( E_8s \) [3]. We can better map the moduli of this duality if we can identify loci on each side which correspond to enhanced gauge symmetries. It turns out to be relatively easy to identify the loci corresponding to enhanced gauge symmetries on the F-theory side, for any given group, as we will explain below. However, it turns out to require somewhat more work to identify the matter representations involved. To do that rigorously one can consider further compactification to 4 dimensions, upon which F-theory becomes equivalent to type IIA on the CY 3-fold. One can then study the wrapping of the D-branes in the type IIA theory to see the gauge group [3,18] and check the matter representation as well. This approach has been considered for some examples in the recent work [19]. Instead of this direct approach, we use the duality with heterotic strings to develop the matter multiplet dictionary on the F-theory side.

As mentioned above we will mainly concentrate on the Hirzebruch surface \( F_n \) as a base for the Calabi-Yau threefold. The Hirzebruch surface is a \( \mathbb{P}^1 \) fibration over \( \mathbb{P}^1 \).
characterized by one integer $n$. We choose our convention so that $z_1$ is the coordinate along $\mathbb{P}^1$ fiber, while $z_2$ is the coordinate along the base. To describe the CY 3-fold over $F_n$ it is most convenient to expand the functions $f(z_1, z_2)$ and $g(z_1, z_2)$ of (2.1) in powers of $z_1$:

$$xf(z_1, z_2) = x \sum_{i=0}^{I} z_1^i f_{8+n(4-i)}(z_2),$$

$$g(z_1, z_2) = \sum_{j=0}^{J} z_1^j g_{12+n(6-j)}(z_2),$$

(2.3)

where the subscripts on $f$ and $g$ denote the degree of the polynomial in $z_2$ and where $I \leq 8$ is the largest value with $8 + n(4 - I) \geq 0$ and $J \leq 12$ is the largest value with $12 + n(6 - J) \geq 0$. The correlation between the degree of the polynomials and the power of $z_1$ follows from the fact that $(z_1, z_2)$ parameterize $F_n$.

It has been argued in the second reference in [9] that the “middle polynomials”, i.e. the coefficients of $xz_1^4$ and $z_1^6$ (corresponding to $i = 4$ and $j = 6$) in (2.3), correspond to the moduli of the $K3$ on which the heterotic theory is compactified. Furthermore, it was argued that polynomials of lower degree in $z$ (i.e., $xz_1^i$ for $i < 4$ and $z_1^j$ for $j < 6$) control the moduli of one $E_8$, based near $z_1 = 0$, with instanton number $12 + n$. Polynomials of higher degree in $z$ ($i > 4$ and $j > 6$) control the other $E_8$, based near $z_1 = \infty$, with instanton number $12 - n$. The zeroes of $g_{12+n}$ and $g_{12-n}$ (the coefficients of $z_1^5$ and $z_1^7$ in (2.3)) were conjectured, when all lower/higher terms are set to zero, to correspond to $12 + n$ small instantons in the $z_1 = 0$ $E_8$ and $12 - n$ small instantons in the $z_1 = \infty$ $E_8$.

We wish to map the perturbative enhanced gauge symmetries of the heterotic side onto the F-theory moduli. On the F-theory side, a perturbative enhanced gauge symmetry of the heterotic theory should already be part of the gauge symmetry in 8 dimensions. In other words, if we consider the size of the second $\mathbb{P}^1$ to be big then, as a function of $z_1$, we should get a singularity on the F-theory side, reflecting the existence of gauge symmetry already in 8-dimensions. We thus restrict our attention to the singularities at $z_1 = \text{const}$. We will be mainly focusing on $E_8 \times E_8$ heterotic strings. In this case, the gauge symmetries are localized on the F-theory side on two points, $z_1 = 0, \infty$. With no loss of generality, we will focus on the gauge symmetries coming from the $E_8$ at $z_1 = 0$ with instanton number $12 + n$. Singularities at $z_2 = \text{const}$ (or more general singular loci that cannot be represented as $z_1 = \text{const}$) correspond to non-perturbative gauge symmetry enhancement, as they are localized on a point in the $z_2$ space which is ‘visible’ to both the heterotic strings and F-theory.
Before getting to a more detailed match of the moduli of the $E_8$ bundle, let us see how the total count of the dimension of the moduli space of $E_8$ bundles works. The dimension of the hypermultiplet moduli space for $E_8$ with $12 + n$ instantons is $30n + 112$. Each F theory complex structure modulus leads to one hypermultiplet. Counting the terms in (2.3) with $i < 4$ and $j < 6$, we find $31n+114$, but $n+2$ of these are associated with reparameterizations $z_1 \rightarrow az_1 + P_n(z_2)$, yielding the correct number of hypermultiplets. Requiring the unbroken part of the gauge group to be at $z_1 = 0$ fixes the reparameterizations corresponding to shifts by $P_n(z_2)$. We then only have to subtract one (corresponding to rescaling $z_1$) from the total count of the parameters in the polynomials to obtain the dimension of the moduli of the bundles.

3. Basic Idea and Tate’s algorithm

Before getting to specific cases, we wish to give the general picture: We start with an A-D-E type singularity at $z_1 = 0$. We then consider a $\mathbb{P}^1$ fibration of this, given by the coordinate $z_2$. Then, depending on what this fibration is, we will in general ‘break’ some of the symmetries that we start with. We thus expect that, if we start with an A-D-E singularity, we end up with a gauge symmetry which is a subgroup of that of the singularity type. The ‘breaking’ is actually very restrictive: As we go around a point on the $z_2$ plane where the fiber degenerates, we come back to the same singularity in the fiber up to a monodromy action on the singularity. If the monodromy action is given by a Weyl group element, i.e. if it is an inner automorphism, we can undo it by a gauge transformation in the fiber. However, if the monodromy action is not given by a Weyl group element, i.e. if it is an outer automorphism, it cannot be undone by a gauge transformation. As we shrink the cycle to zero size, we end up orbifolding the gauge group by an outer automorphism, which thus means that we have reduced gauge symmetry \[14\].

It is well known that the actions of outer automorphisms are realizable as automorphisms of the A-D-E Dynkin diagrams. The main ones we will encounter in this paper are given by

\[
\begin{align*}
A_{2n-1} &\rightarrow C_n \\
D_n &\rightarrow B_{n-1} \\
E_6 &\rightarrow F_4 \\
D_4 &\rightarrow G_2,
\end{align*}
\] (3.1)
where all but the last one are involutions and the last one is the triality automorphism of $D_4$. We shall find below that all of these cases are essentially realized. In fact, the best way to phrase them in the singularity language is as follows: Suppose we resolve the singularity of the Calabi-Yau manifold. Then we get the A-D-E Dynkin diagram corresponding to the intersection diagram of vanishing 2-cycles. As we fiber this space over another parameter $z_2$ and go over non-trivial cycles, in general the vanishing cycles come back to themselves but may undergo monodromy. If these vanishing cycles are exchanged according to an outer automorphism of the Dynkin diagram, the actual gauge group will be smaller as indicated above. In fact, the distinction between the types of A-D-E singularities occurring in elliptic fibrations according to whether the vanishing cycles mix, which we call non-split, or do not mix, which we call split is implicit in the work of Tate [10]. Generically the A-D-E fibrations will undergo outer automorphisms, i.e. we have the non-split case, which means that for a generic fibration of the A-D-E type we will get the right hand side above as the gauge symmetry. However, if we put some extra conditions on the fibration, in accord with Tate’s algorithm, we can avoid this breaking and get back the original group. This will be seen in the examples below.

It is important to keep in mind that a given A-D-E singularity may not correspond to an enhanced gauge symmetry in the F-theory. One example is $E_8$ fiber singularities, which seem to correspond to small instantons but not $E_8$ gauge symmetry [9]. We will find that all the other cases noted above, on the other hand, do occur as expected. There are other outer automorphisms which do not occur, however. For example the non-split $A_{2k}$ singularity, which would be expected to give $Sp(k)$ gauge symmetry, does not seem to give rise to conventional gauge symmetries. Presumably this is similar to the situation which arises when $E_8$ instantons shrink to zero size [21,22,23].

We will now review (and extend) Tate’s algorithm.

3.1. Tate’s algorithm

Tate’s algorithm [10] gives a procedure for computing the Kodaira type of a singular fiber in an elliptic fibration (as well as the “split/non-split” distinction mentioned in the previous subsection), at a generic point along any divisor $\Sigma$ in the base $B$ of the fibration.

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1 There is a subtle argument for this having to do with the fact that the outer automorphism acting on the Lie algebra has order 4 [20].
The basic idea is a refinement of the method already mentioned, and proceeds by studying the order of vanishing of the coefficients of the defining equation along $\Sigma$.

In order to carry this program out in an efficient manner, it is necessary to begin with a more general form of the Weierstrass equation. We take the form to be

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6,$$

(3.2)

where the $a_i$’s are locally defined polynomial functions on the base (or more generally, sections of line bundles). The traditional Weierstrass form (2.1) can be obtained from (3.2) by completing the square in $y$ and then completing the cube in $x$.

The algorithm makes reference to several quantities derived from the coefficients $a_i$, defined by

$$b_2 = a_1^2 + 4a_2$$
$$b_4 = a_1 a_3 + 2a_4$$
$$b_6 = a_3^2 + 4a_6$$
$$b_8 = b_2 a_6 - a_1 a_3 a_4 + a_2 a_3^2 - a_4^2$$
$$\Delta = -b_2^2 b_8 - 8b_4^3 - 27b_6^2 + 9b_2 b_4 b_6.$$ 

(3.3)

The discriminant $\Delta$ is the same one used earlier, up to a numerical factor. For later reference, we also record the results of completing the square and the cube:

$$f = -\frac{1}{48}(b_2^2 - 24b_4)$$
$$g = -\frac{1}{864}(-b_2^3 + 36b_2 b_4 - 216b_6).$$

(3.4)

In carrying out the algorithm, we let $\{\sigma = 0\}$ be a local defining equation for $\Sigma$, and we examine various divisibility conditions of the form $'\sigma^k$ divides $a_j'$. When such a condition holds, we define $a_{j,k} = a_j / \sigma^k$, a notation which will be used throughout this discussion. We similarly define $b_{j,k} = b_j / \sigma^k$ whenever it makes sense.

The algorithm now proceeds roughly as follows: locate the singularity in the fibers over $\Sigma$, make a change of coordinates in $(x, y)$ to put the singularity in a convenient location, blow up the singularity, and then repeat. At each stage in this process, after the change of coordinates has been made, the coefficients in the equation will be divisible by certain powers of the local defining equation $\sigma$. This can be used to characterize which branch of the algorithm should be used.
For example, in the first step of the algorithm we ask whether the fibers over $\Sigma$ are actually singular, i.e., whether $\sigma$ divides $\Delta$. If so, we change coordinates to put the singularity at $(x, y) = (0, 0)$. When the singularity is located there, we have $\sigma$ dividing $a_3$, $a_4$ and $a_6$, and generically the Kodaira fiber is of type $I_1$ (no enhanced gauge symmetry). The discriminant mod $\sigma^2$ can be calculated as

$$\Delta \equiv -\sigma b_2^3 a_{6,1} \pmod{\sigma^2}.$$

Any worsening of the singularities is thus indicated by either $b_2 \equiv 0 \pmod{\sigma}$, or by $a_{6,1} \equiv 0 \pmod{\sigma}$. The first case leads to the branch in the algorithm giving fibers of Kodaira types $\Pi$, $\Pi$, etc., while the second case leads to $I_n$ type fibers.

To see how the split/non-split distinction arises in this algorithm, we need to proceed a bit further along the $I_n$ branch. So suppose that in addition to the previous conditions, $\sigma^2$ divides $a_6$. Then the singular point of the fiber is also a singular point of the total space of the fibration, and we should blow up the origin in $(x, y, \sigma)$ space to begin resolving the singularity. The leading order terms in the equation are

$$y^2 + a_1 xy + a_{3,1} \sigma y = a_2 x^2 + a_{4,1} \sigma x + a_{6,2} \sigma^2; \quad (3.5)$$

if this quadratic is nonsingular, then blowing up the origin in $(x, y, \sigma)$ resolves the singularity which was an $A_1$. We have in this case Kodaira type $I_2$, and gauge group $SU(2)$. The discriminant mod $\sigma^3$ can be calculated as

$$\Delta = -\sigma^2 b_2 b_{8,2} \pmod{\sigma^3},$$

where $b_{8,2}$ coincides with the discriminant of the quadratic equation (3.3) (a cubic in the coefficients). To get worse singularities, either $b_2$ or $b_{8,2}$ should vanish mod $\sigma$.

The split/non-split distinction arises at the next branch in the algorithm. If in addition to the previous assumptions, we assume $b_{8,2}$ is divisible by $\sigma$, but $b_2$ is not, then (3.5) is a quadratic equation of rank precisely two, and by a change of coordinates we may assume that (3.5) involves $x$ and $y$ alone, i.e., that the entire equation takes the form

$$y^2 + a_1 xy + a_{3,2} \sigma^2 y = a_2 x^2 + a_{4,2} \sigma^2 x + a_{6,2} \sigma^2.$$

The exceptional divisor of the blowup map is defined by

$$y^2 + a_1 xy - a_2 x^2 \equiv 0 \pmod{\sigma} \quad (3.6)$$
which consists of two lines for each specific numerical value of the coefficients $a_{j,k}$. However, since those coefficients will actually depend on the other parameters in the base, it may not be possible to individually define those two lines globally along $\Sigma$. In order for it to be possible, we need a factorization

$$y^2 + a_1 xy - a_2 x^2 \equiv (y - rx)(y - sx) \pmod{\sigma}$$

for some functions $r$ and $s$ on the base. If this factorization exists, then after a change of coordinates in $y$ we may assume that $r \equiv 0 \pmod{\sigma}$, i.e., that $\sigma$ divides $a_2$. This case gives Kodaira fiber $I_3^s$ and gauge group $SU(3)$. (We use the superscripts $ns$ and $s$ on Kodaira fibers to distinguish between the non-split and split cases.) If the factorization does not exist, then we are in the ‘non-split’ situation with a singularity of type $A_2$; this is the case in which the gauge symmetry is unconventional. (The Kodaira fiber is $I_3^{ns}$.) Further down this branch of the algorithm one encounters the $Sp(k)$ gauge groups.

Rather than continuing through the algorithm step by step, we have summarized the conditions it entails in table 2.
Table 2: Tate’s Algorithm

| type  | group   | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_6$ | $\Delta$ |
|-------|---------|-------|-------|-------|-------|-------|----------|
| $I_0$ | —       | 0     | 0     | 0     | 0     | 0     | 0        |
| $I_1$ | —       | 0     | 0     | 1     | 1     | 1     | 1        |
| $I_2$ | $SU(2)$ | 0     | 0     | 1     | 1     | 2     | 2        |
| $I_3$ | unconven.| 0     | 0     | 2     | 2     | 3     | 3        |
| $I_4$ | unconven.| 0     | 1     | 1     | 2     | 3     | 3        |
| $I_{2k}$ | $Sp(k)$ | 0     | 0     | $k$   | $k$   | 2$k$  | 2$k$     |
| $I_{2k}^n$s | $SU(2k)$ | 0 | 1 | $k$ | $k$ | 2$k$ | 2$k$ |
| $I_{2k+1}$ | unconven.| 0 | 0 | $k+1$ | $k+1$ | 2$k+1$ | 2$k+1$ |
| $I_{2k+1}$ | $SU(2k+1)$ | 0 | 1 | $k$ | $k+1$ | 2$k+1$ | 2$k+1$ |
| $II$  | —       | 1     | 1     | 1     | 1     | 1     | 2        |
| $III$ | $SU(2)$ | 1     | 1     | 1     | 1     | 1     | 2        |
| $IV^n$s | unconven.| 1     | 1     | 1     | 2     | 2     | 4        |
| $IV^s$ | $SU(3)$ | 1     | 1     | 1     | 2     | 3     | 4        |
| $I_0^n$s | $G_2$ | 1     | 1     | 2     | 2     | 3     | 6        |
| $I_0^s$ | $SO(7)$ | 1     | 1     | 2     | 2     | 4     | 6        |
| $I_0^*$ | $SO(8)^*$ | 1 | 1 | 2 | 2 | 4 | 6 |
| $I_1^n$s | $SO(9)$ | 1 | 1 | 2 | 3 | 4 | 7 |
| $I_1^*$ | $SO(10)$ | 1 | 1 | 2 | 3 | 5 | 7 |
| $I_2^n$s | $SO(11)$ | 1 | 1 | 3 | 3 | 5 | 8 |
| $I_2^*$ | $SO(12)^*$ | 1 | 1 | 3 | 3 | 5 | 8 |
| $I_{2k-3}^n$s | $SO(4k+1)$ | 1 | 1 | $k$ | $k+1$ | 2$k$ | 2$k+3$ |
| $I_{2k-3}^s$ | $SO(4k+2)$ | 1 | 1 | $k$ | $k+1$ | 2$k+1$ | 2$k+3$ |
| $I_{2k-2}^n$s | $SO(4k+3)$ | 1 | 1 | $k+1$ | $k+1$ | 2$k+1$ | 2$k+4$ |
| $I_{2k-2}^s$ | $SO(4k+4)^*$ | 1 | 1 | $k+1$ | $k+1$ | 2$k+1$ | 2$k+4$ |
| $IV^n$s | $F_4$ | 1 | 2 | 2 | 3 | 4 | 8 |
| $IV^s$ | $E_6$ | 1 | 2 | 2 | 3 | 5 | 8 |
| $III^*$ | $E_7$ | 1 | 2 | 3 | 3 | 5 | 9 |
| $II^*$ | $E_8$ | 1 | 2 | 3 | 4 | 5 | 10 |
| non-min | — | 1 | 2 | 3 | 4 | 6 | 12 |
Table 2, con.: Tate’s Algorithm, Discriminant and Next Branches

| type         | $\Delta \mod \sigma$ | next branches |
|--------------|------------------------|---------------|
| $I_0$        | $\Delta$              | $I_1$         |
| $I_1$        | $-b_2^2 a_{6,1}$       | $I_2$, $II$   |
| $I_2$        | $-b_2^2 b_{8,2}$       | $I_3^{n_s}$, $III$ |
| $I_3^{n_s}$  | $-b_3^2 a_{6,3}$       | $I_4^{n_s}$, $(I_3^s)$ |
| $I_3^s$      | $-a_1^6 a_{6,3}$       | $I_4^s$, $IV^s$ |
| $I_{2k}^{n_s}$| $-b_2^2 b_{8,2k}$      | $I_{2k+1}^{n_s}$, $I_{2k}^s$ |
| $I_{2k}^s$   | $-a_1^4 b_{8,2k}$      | $I_{2k+1}^s$, $(I_{2k}^s-4)$ |
| $I_{2k+1}^{n_s}$| $-b_2^2 a_{6,2k+1}$   | $I_{2k}^{n_s}$, $(I_{2k+1}^s)$ |
| $I_{2k+1}^s$ | $-a_1^4 a_{6,2k+1}$   | $I_{2k+2}^{s}$, $I_{2k-3}^s$ |
| $II$         | $-432 a_{6,1}^2$       | $III$         |
| $III$        | $-64 a_{1,1}^3$        | $IV^{n_{ns}}$ |
| $IV^{n_{ns}}$| $-27 b_{6,2}^3$        | $IV^s$        |
| $IV^s$       | $-27 a_{3,2}^4$        | $I_0^{n_{ns}}$ |
| $I_0^{n_{ns}}$| $\Delta/\sigma^6$     | $I_0^{s_{ss}}$ |
| $I_0^{s_{ss}}$| $16 a_{4,2}^2 (a_{2,1}^2 - 4a_{4,2})$ | $I_0^{s_{s}}$, $I_1^{n_{ss}}$ |
| $I_0^{s_{ss}}$| $16 a_{4,2}^2 (a_{2,1}^2 - 4a_{4,2})$ | $(I_1^{n_{ss}})$ |
| $I_1^{n_{ns}}$| $-16 a_{3,1}^3 b_{6,4}$ | $I_1^{s_{s}}$, $IV^{s_{ns}}$ |
| $I_1^{s_{ns}}$| $-16 a_{3,1}^3 a_{3,2}^2$ | $I_2^{n_{ns}}$, $IV^{s_{s}}$ |
| $I_2^{n_{ns}}$| $16 a_{2,1}^2 (a_{4,3}^2 - 4a_{2,1} a_{6,5})$ | $I_2^{s_{s}}$, $III^*$ |
| $I_2^{s_{ns}}$| $16 a_{2,1}^2 (a_{4,3}^2 - 4a_{2,1} a_{6,5})$ | $I_2^{n_{ns}}$, $II_{2k-3}$ |
| $I_{2k-3}^{s_{ns}}$| $-16 a_{2,1}^3 b_{6,2k}$ | $I_{2k-3}^{s_{s}}$, non-min |
| $I_{2k-3}^{s_{ns}}$| $-16 a_{2,1}^3 a_{3,2}^2$ | $I_{2k-2}^{n_{ns}}$, non-min |
| $I_{2k-2}^{n_{ns}}$| $16 a_{2,1}^2 (a_{4,3}^2 - 4a_{2,1} a_{6,2k+1})$ | $I_{2k-2}^{s_{ns}}$, non-min |
| $I_{2k-2}^{s_{ns}}$| $16 a_{2,1}^2 (a_{4,3}^2 - 4a_{2,1} a_{6,2k+1})$ | $I_{2k-1}^{n_{ns}}$, non-min |
| $IV^{s_{ns}}$  | $-27 b_{6,4}^2$        | $IV^{s_{s}}$ |
| $IV^{s_{s}}$  | $-27 a_{3,2}^4$        | $III^*$       |
| $III^*$       | $-64 a_{3,3}^3$        | $II^*$        |
| $II^*$        | $-432 a_{6,5}^2$       | non-min       |
| non-min       | $\Delta/\sigma^{12}$  | —             |

This table is to be interpreted as follows: If upon change of coordinates in $(x, y)$, the coefficients in the equation are divisible by the given powers of $\sigma$, but the coefficients
are otherwise generic, then the Kodaira fiber has the stated type and we predict the enhanced gauge symmetry stated in the second column. The * next to the $SO(8), SO(12)$ and $SO(4k + 4)$ cases signifies that in addition to the conditions specified in table 2 a factorization condition must be satisfied to obtain the enhanced gauge symmetry: For $SO(8)$ the polynomial

$$X^2 + a_{2,1}X + a_{4,2}$$

should factor modulo $\sigma$, while for the other $SO(4k + 4)$ cases (including $SO(12)$) the polynomial

$$a_{2,1}X^2 + a_{4,k+1}X + a_{6,2k+1}$$

should factor modulo $\sigma$. The order of vanishing of the discriminant is also shown. Worse singularities occur if $\hat{\Delta} = \Delta/\sigma^{\text{ord}(\Delta)}$ satisfies $\hat{\Delta} \equiv 0 \mod \sigma$ or if a factorization condition is satisfied, as in the $SO(4k + 4)$ case noted above. The second portion of the table exhibits $\hat{\Delta}$, and indicates which branch of the algorithm is to be followed when worse singularities occur.

We will see in section 7 how the conditions stated in the table are used to describe the resolution of singularities by blowing up. Let us comment here a bit further about the “factorization conditions” which lead to the distinctions between non-split and split cases. These distinctions, as well as the gauge groups in the second column, are not explicitly present in Tate’s paper, although the rest of the algorithm is.

1) In the case of $I_n$, the polynomial whose factorization is at issue takes the form

$$Y^2 + a_1 XY - a_2 X^2$$

and (as we already saw in the case $n = 3$) whenever it factors (mod $\sigma$) we can make a change of coordinates to make one of the factors be $Y$, i.e. to make $\sigma$ divide $a_2$.

2) In the cases of types IV, IV*, and $I_{2k-3}^*$, the polynomial whose factorization is at issue takes the form

$$Y^2 + a_{3,k} Y - a_{6,2k}$$

and similarly in this case, a change of coordinates allows us to take one of the factors (mod $\sigma$) to be $Y$.

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2 The final entry in the table labeled “non-min” refers to elliptic fibrations whose singularities are sufficiently bad as to destroy the “trivial canonical bundle” property of the total space of the fibration—in the mathematics literature, these are called non-minimal Weierstrass fibrations.
3) In the case of type $I_0^*$, the polynomial which needs to be factored is

$$X^3 + a_{2,1}X^2 + a_{4,2}X + a_{6,3}.$$  

There are three cases: no factorization (non-split), full factorization as a product of three linear factors (split), or factorization into a linear factor and a quadratic factor. We refer to this last case as “semi-split” and denote the corresponding Kodaira fiber as $I_{0}^{*ss}$. The Dynkin diagram for the singularity is $D_4$; the non-split case corresponds to the quotient by $S_3$, yielding the $G_2$ diagram, whereas the semi-split case corresponds to the quotient by $Z_2$, yielding the $B_3$ diagram. When there is a linear factor, a coordinate change can be used to make this factor be $X$, but in the split case only one of the three factors can be so shifted and we must formulate the condition in terms of whether a polynomial can be factored mod $\sigma$.

4) Finally, in the case of type $I_{2k-2}$, the polynomial we must consider is

$$a_{2,1}X^2 + a_{4,k+1}X + a_{6,2k+1}.$$ 

Even when this factors, we cannot make a compensating shift of coordinates. For one thing, the factorization involves a factorization of $a_{2,1}$ mod $\sigma$, which is itself a function (or a section of a line bundle) on the curve $\Sigma$. In fact, the flexibility to choose this factorization differently allows for a number of different types of these singularities, as we shall see in the examples.

4. Higgs Branches

We now begin our detailed comparison of the F-theory and heterotic loci of enhanced gauge symmetries. We start with the enhanced gauge symmetries which are realized perturbatively on the heterotic side. We focus on the $E_8$ with $12 + n$ instantons and obtain a dictionary of the correspondence between F-theory geometric singularities and gauge symmetry. The dictionary, of course, also directly applies to the $E_8$ with $12 - n$ instantons as well as to situations with additional tensor multiplets.

$E_8$ with $12 + n$ instantons has a $30n + 112$ dimensional space of hypermultiplet moduli associated with the gauge bundle. $E_8$ is generically broken on this moduli space, with subgroups un-Higgsed on various subspaces. An enhanced gauge symmetry $G \subset E_8$ corresponds to restricting the $12 + n$ instantons to sit in $H \subset E_8$ which is the commutant of $G$:
$G \times H \subset E_8$ is a maximal subgroup. The dimension of the subspace of enhanced $G$ gauge symmetry, corresponding to the number of $G$ neutral hypermultiplet moduli, is given by

$$\dim \mathcal{M}(G) = c_2(H)(12 + n) - \dim H$$

(4.1)

where $c_2(H)$ is the dual Coxeter number of $H$; this is the dimension in units of hypermultiplets (one quarter of the real dimension) for $12 + n$ instantons embedded in $H$.

In addition to the above neutral hypermultiplets, on the subspace with enhanced $G$ gauge symmetry there are massless matter hypermultiplets transforming in representations $R_i$ of $G$. The number $N_i$ of matter fields in representation $R_i$ of $G$ is given by an index theorem to be

$$N_i = (12 + n)\text{index}(S_i) - \dim(S_i),$$

(4.2)

where $S_i$ is the representation of $H$ entering in the decomposition $adj(E_8) = \sum_i (R_i, S_i)$ and the last term is a gravitational contribution which takes into account the compactification on $K3$. “Relaxing” the instantons to instead lie in $H' \supset H$ corresponds to breaking $G$ to commutant $G' \subset G$ by the Higgs mechanism, giving an expectation value to some of the $G$ charged matter.

The entire moduli space has an intricate structure, with a variety of subspaces with enhanced gauge symmetry corresponding to the different possible Higgs mechanisms. We will organize our discussion by following two different chains of the Higgs mechanism: one starting with unbroken $E_7$, corresponding to instantons in $H = SU(2)$, and one starting from unbroken $SO(12)$, corresponding to instantons in $H = SU(2) \times SU(2)$. Upon Higgsing, as will be discussed, these two connect at various places. One could consider, more generally, Higgs chains starting from instantons in $H \subset E_8$ consisting of more $SU(2)$ factors. The number of possible factors depends on $n$ because, as follows from (4.2), each $SU(2)$ factor must have at least 4 instantons.

The intricate structure of various enhanced gauge symmetries will be exactly matched by the geometric singularities of F-theory, with the dimensions of enhanced gauge symmetry subspaces perfectly matching the dimensions of the moduli spaces of compactifications with various singularities in F-theory. Moreover, we will use this dictionary to deduce on the F-theory side which matter representations are present and how they are encoded in

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3 This and the next equation apply for $H$ simple. When, as will be the case in some of the examples which follow, $H$ is a product of simple factors, these formulae are modified in an obvious fashion.
the singularity. It is natural to expect that the matter comes from intersecting loci of singu-
larities, i.e. at points on the $z_2$ plane where the fiber has a worse singularity, which are re-
lected as extra zeroes of the discriminant. For example, such is the case in the context of singularities realizable as intersecting D-branes [3]. We find that this is indeed the case, at least for simply laced gauge groups. In the case where the unbroken gauge group is non-simply laced, it seems difficult to ‘localize’ the matter at the intersection points of the singularity.
The portion of the web of vacua that we have explored is summarized in table 3 and diagram 1 below. The details are provided in the rest of this section. The superscripts s, ns, and ss in table 3 stand for split, nonsplit, and semi-split, in the terminology of §3.

Table 3: Various Higgs Branches

| Type   | Group     | Matter content | Dim(\(\mathcal{M}\)) |
|--------|-----------|----------------|-----------------------|
| \(E_7\) | \(E_7\)   | \((\frac{n}{2} + 4)56\) | 2n + 21               |
| \(E_6\) | \(E_6\)   | \((n + 6)27\) | 3n + 28               |
| \(E_6^{ns}\) | \(F_4\) | \((n + 5)26\) | 4n + 34               |
| \(D_5^s\) | \(SO(10)\) | \((n + 4)16 + (n + 6)10\) | 4n + 33               |
| \(D_5^{ns}\) | \(SO(9)\) | \((n + 5)9 + (n + 4)16\) | 5n + 39               |
| \(D_4^s\) | \(SO(8)\) | \((n + 4)(8_c + 8_s + 8_v)\) | 6n + 44               |
| \(D_4^{ss}\) | \(SO(7)\) | \((n + 3)7 + (2n + 8)8\) | 7n + 48               |
| \(D_4^{ns}\) | \(G_2\) | \((3n + 10)7\) | 9n + 56               |
| \(A_3^s\) | \(SU(4)\) | \((n + 2)6 + (4n + 16)4\) | 8n + 51               |
| \(A_3^{ns}\) | \(SO(5)\) | \((n + 1)5 + (4n + 16)4\) | 9n + 53               |
| \(A_1 \times A_1\) | \(SO(4)\) | \(n(2,2) + (4n + 16)[(1,2) + (2,1)]\) | 10n + 54              |
| \(A_2^s\) | \(SU(3)\) | \((6n + 18)3\) | 12n + 66              |
| \(A_1\) | \(SU(2)_2\) | \((6n + 16)2\) | 18n + 83              |
| \(A_1\) | \(SU(2)\) | \((8n + 32)2 + (n − 1)3\) | 11n + 54              |
| \(D_6^s\) | \(SO(12)\) | \(\frac{5}{2}32 + (\frac{4 + n - r}{2})32' + (n + 8)12\) | 2n + 18               |
| \(D_6^{ns}\) | \(SO(11)\) | \(\frac{5}{2} + 2)32 + (n + 7)11\) | 3n + 26               |
| \(A_5^s\) | \(SU(6)\) | \(\frac{5}{2}20 + (16 + r + 2n)6 + (2 + n - r)15\) | 3n - r + 21           |
| \(A_5^{ns}\) | \(Sp(3)\) | \((16 + 2n + \frac{3}{2}r)6 + (n + 1 - r)14 + \frac{3}{2}r14'\) | 4n + 23 - 2r          |
| \(A_4^s\) | \(SU(5)\) | \((3n + 16)5 + (2 + n)10\) | 5n + 36               |
| \(A_4^s\) | \(SU(3)_2\) | \((6n + r + 34)3 + (r - 2)6 + (n + 1 - r)8\) | 4n + 22 - r           |
4.1. Unbroken $E_7$ gauge symmetry

There is a subspace of the Higgs moduli space with unbroken $E_7$ when the $12 + n$ instantons are in commutant $H = SU(2)$. The dimension of this subspace, according to (4.1), is $2n + 21$. In addition to these neutral hypermultiplets, it follows from (4.2) that there are $(n + 8) \frac{1}{2}$-hypermultiplets in the 56 of $E_7$. The codimension of this space of enhanced $E_7$ is given by the Higgs mechanism to be $(n + 8)(\frac{1}{2})(56) = 28n + 91$, leading to the expected total dimension of $30n + 112$.

Consider now F-theory with an $E_7$ geometric singularity. According to table 1, this is the case when $\text{ord}(f) \geq 3$, and $\text{ord}(g) > 5$. The F-theory moduli associated with the $E_8$ with $12 + n$ instantons satisfying these conditions are the terms $\text{ord}(f) = i = 3$ and $\text{ord}(g) = j = 5$ in (2.3), i.e. the terms $f_{8+n}$ and $g_{12+n}$. The number of moduli associated with these terms, subtracting one, as always, to account for the rescaling of $z_1$ mentioned earlier, is $(13 + n) + (9 + n) - 1 = 2n + 21$. This is exactly the dimension found above for enhanced $E_7$ gauge symmetry, which is a strong check of the proposed F-theory/heterotic duality.

The $E_7$ matter can be seen by considering the discriminant on the $E_7$ locus:

$$\Delta = z_1^9(4f_{8+n}^3(z_2) + o(z_1)). \quad (4.3)$$
The zero locus of the discriminant consists of the $z_1 = 0$ locus and the other branch, which intersects the $z_1 = 0$ locus at $(n + 8)$ points. From the point of view of type IIB string theory, the $z_1 = 0$ locus describes a 7-brane with $E_7$ vector fields propagating inside its world volume. The other branch has an interpretation as $(n + 8)$ 7-branes intersecting the 7-brane located at $z_1 = 0$. We see a nice match between extra zeroes of the discriminant, which coincide with the zeroes of $f_{8+n}$, and the number of $\frac{1}{2}56$ hypermultiplets. We are thus led to conclude that each charged $\frac{1}{2}$-hypermultiplet is localized at a zero of $f_{8+n}$.

4.2. Unbroken $E_6$ gauge symmetry

There is a subspace of the moduli space with unbroken $E_6$ gauge symmetry when the $12+n$ instantons of the heterotic theory are embedded in commutant $SU(3) \subset E_8$. Starting from the theory of the previous subsection, relaxing the instantons from commutant $SU(2)$ to $SU(3)$ corresponds to Higgs breaking of $E_7$ to $E_6$ by giving an expectation value to two fields in the 56. It follows either from the massless matter of the previous subsection and the Higgs mechanism or from (4.1) and (4.2) applied to the commutant $SU(3)$ that the dimension of the $E_6$ locus is $3n + 28$ and that there are $n + 6$ matter hypermultiplets in the 27 of $E_6$.

Let us compare this with the dimension of the locus in the F-theory moduli space with an $E_6$ singularity. To obtain an $E_6$ singularity, using table 2 and putting the equation in the Weierstrass form, one finds that one has to first relax the restriction on $g(z_1, z_2)$ in (2.3), allowing a term $g_{2n+12}(z_2)$ in (2.3) (ord$(g) = 4$). So the moduli of an $E_6$ singularity correspond to the terms in $f_{n+8}$, $g_{n+12}$ and $g_{2n+12}$. However, to actually obtain $E_6$ gauge symmetry, one finds that the $E_6$ singularity in the fiber should not be generic, but should be of the ‘split’ form in table 2. This implies that there are $(n + 6)$ constraints on the coefficients of $g_{2n+12}(z_2)$, namely it should have double zeroes: $g_{2n+12}(z_2) = q_{n+6}^2(z_2)$. Counting the number of moduli in $q_{n+6}$, $g_{n+12}$, and $f_{n+8}$, we find $n + 7 + n + 13 + n + 9 - 1 = 3n + 28$, in exact agreement with the expected number from the heterotic side.

Further, we can again see the charged matter content from the zeroes of the discriminant:

$$\Delta = z_1^8 (27q_{n+6}^4 + o(z_1)).$$

(4.4)

The $n + 6$ 27s of $E_6$ are localized at the zeroes of $q_{n+6}$. 

19
4.3. Unbroken $F_4$

The heterotic theory has an unbroken $F_4$ gauge symmetry when the $12 + n$ instantons lie in commutant $H = G_2 \subset E_8$. This theory can be obtained from that of the previous subsection by giving an expectation value to one of the fields in the $27$. It follows either from the Higgs mechanism or from (4.1) and (4.2) applied to the commutant $F_4$ that there are $4n + 34$ neutral hypermultiplet moduli and $n + 5$ hypermultiplets in the $26$ of $F_4$.

Using Tate’s algorithm, we find that $F_4$ gauge symmetry corresponds to the generic $E_6$ singularity, relaxing the condition on $g_{2n+12}$ found in the previous subsection. This relaxing, corresponding to splitting the double zeroes of $g_{2n+12}$ in the $E_6$ case, must have the interpretation of breaking $E_6$ to $F_4$ by giving expectation values to the $F_4$ singlet components of the $27$s of $E_6$ ($27 \to 26 + 1$). Counting the dimension of the generic $E_6$ singularity, corresponding to the terms in $f_{n+8}$, $g_{n+12}$, and $g_{2n+12}$, we find $n + 9 + n + 13 + 2n + 13 - 1 = 4n + 34$, in precise agreement with the expected dimension for unbroken $F_4$ gauge symmetry.

In this example, it is difficult to localize the expected $n + 5$ hypermultiplets in the $26$ of $F_4$ at the extra zeroes of the discriminant. This seems to be the case for all the non-simply laced gauge symmetries we will find below as well.

4.4. Unbroken $SO(11)$

There is a locus of unbroken $SO(11)$, which can be obtained by starting from the above $E_7$ locus and giving expectation values to components of two $56$s. The massless hypermultiplet content consists of $3n + 26$ singlet moduli, $n + 7$ hypermultiplets in the $11$, and $\frac{1}{2}(n + 4)$ in the $32$; this follows from the $E_7$ spectrum and the Higgs mechanism or from (4.1) and (4.2) applied to the commutant, which is $SO(5)$.

In the F-theory this should correspond to compactification on an elliptic Calabi-Yau with a generic $D_6$ singularity. The condition for such a singularity is that $\text{ord}(f) = 2$, $\text{ord}(g) \geq 3$ (or the signs $=$, $\geq$ being permuted), with coefficients chosen to cancel the $z_1^6$ and $z_1^7$ terms in $\Delta$. This requires $f_{2n+8} \sim s_{n+4}^2$, $g_{3n+12} \sim s_{n+4}^3$, and $g_{2n+12} \sim f_{n+8}s_{n+12}$. So such a manifold is specified by independent functions

$$ g_{12+n}, \ f_{8+n}, \ s_{4+n}, $$

as follows immediately from Tate’s algorithm. The resulting locus has dimension $3n + 26$, which exactly agrees with the heterotic string prediction.
4.5. Unbroken $SO(10)$

There is a locus of unbroken $SO(10)$ which can be obtained from the $E_6$ of subsec. 4.2 by giving expectation values to two of the 27 hypermultiplet flavors or from the $SO(11)$ of subsec. 4.4 by giving an expectation value to one of the 11 flavors. It follows from the Higgs mechanism applied to either route or from (4.1) and (4.2) applied to the $SO(10)$ commutant, which is $SU(4)$, that there are $4n + 33$ neutral hypermultiplets, $(n + 4)\ 16$s, and $n + 6\ 10$s.

On the F-theory side, according to table 2, to obtain $SO(10)$ gauge symmetry the fiber should have a ‘split’ $D_5$ singularity. The condition for a non-split $D_5$ singularity is that $\text{ord}(f) = 2$, $\text{ord}(g) \geq 3$ (or the signs $=$, $\geq$ being permuted), with coefficients chosen to cancel the $z_1^6$ term in $\Delta$. That means that the polynomials $g_{3n+12}$ and $f_{2n+8}$ should be related via $g_{3n+12} \sim h_{n+4}^3$ and $f_{2n+8} \sim h_{n+4}^2$. The condition of ‘splitness’ in table 2 implies that in addition we should take $g_{12+2n} = q_{n+6}^2 - f_{8+n} h_{4+n}$. So the moduli of the split $D_5$ singularity correspond to the terms in $h_{n+4}$, $q_{n+6}$, $g_{n+12}$, and $f_{n+8}$, for a total of $4n + 33$ hypermultiplet moduli, exactly as expected from the heterotic side. The discriminant on this locus is

$$\Delta = z_1^7 h_{4+n}^3 (q_{6+n}^2 + o(z_1)). \quad (4.5)$$

In this example, we can localize the matter at the extra zeroes of the discriminant. The $n + 4\ 16$s are localized at the zeroes of $h$ and the $n + 6\ 10$s are localized at the zeroes of $q_{n+6}$.

4.6. Unbroken $SO(9)$

The locus of unbroken $SO(9)$ gauge symmetry is reached by starting either from the $F_4$ of subsec. 4.3 and giving an expectation value to a field in the 26 or starting from the $SO(10)$ of the previous subsection and giving an expectation value to a field in the 10. The massless matter content is obtained either by the Higgs mechanism applied to either route or by applying (4.1) and (4.2) to the commutant $SO(7)$: There are $5n + 39$ neutral hypermultiplets, giving the dimension of the $SO(9)$ locus, $n + 5$ hypermultiplets in the 9 and $n + 4$ in the 16.

On the F-theory side, we see from table 2 that a manifold with a generic (‘nonsplit’) $D_5$ singularity should yield $SO(9)$ gauge group. Relaxing the “split” condition of the previous subsection, means that the moduli are now the terms in $h_{n+4}$, $g_{2n+12}$, $g_{n+12}$, and $f_{n+8}$, for a total of $5n + 39$, exactly as expected from the heterotic side.
4.7. Unbroken $SO(8)$

The locus of unbroken $SO(8)$ is obtained from the $SO(9)$ above by giving an expectation value to a field in the $9$. The massless matter content is obtained either from the Higgs mechanism or from (4.1) and (4.2) applied to the commutant, which is also $SO(8)$. There are $6n + 44$ hypermultiplets which are $SO(8)$ singlets, giving the dimension of the $SO(8)$ locus, and $n + 4$ hypermultiplets in the $(8_v + 8_s + 8_c)$.

Tate’s algorithm implies that the F-theory side should be compactified on an elliptic Calabi-Yau with a $D_4$ singular fiber satisfying certain additional restrictions, as discussed after table 2. These restrictions tell us that $f_{2n+8}(z_2)$ and $g_{3n+12}(z_2)$ only contribute $2n + 10$ independent parameters instead of $5n + 22$ – they can be parameterized as $f_{2n+8}(z_2) \sim h_{n+4}^2$ and $g_{3n+12}(z_2) \sim q_{n+4}^3$. This is a relaxation of the above $D_5$ singularity, for which $h_{n+4} = q_{n+4}$. Counting moduli, we have the terms in $q_{n+4}$ in addition to those of the previous subsection, leading to a total of $6n + 44$ exactly as expected on the heterotic side. The discriminant in this parameterization can be written as follows

$$\Delta = z_1^6 ((h_{n+4}^2 + q_{n+4}^2)(h_{n+4}^2 + \omega q_{n+4}^2)(h_{n+4}^2 + \omega^2 q_{n+4}^2) + o(z_1)),$$

where $\omega^3 = 1$. The zeroes of each bracket correspond to the charged matter multiplets. There is a $Z_3$ symmetry $q \rightarrow \omega q$ exchanging the various factors in the discriminant – this presumably is a consequence of $SO(8)$ triality. It is quite natural to associate the $n + 4$ matter fields in each of the three eight dimensional representations of $SO(8)$ with each of the factors in the discriminant.

4.8. Unbroken $SO(7)$

Continuing from the $SO(8)$ above, the Higgs mechanism implies that the $SO(7)$ locus has $7n + 48$ singlet hypermultiplets, $n + 3$ in the $7$, and $2n + 8$ in the $8$. This massless spectrum can also be obtained from (4.1) and (4.2) applied to the commutant $SO(9)$.

On the F-theory side, unbroken $SO(7)$ again corresponds to a $D_4$ singularity but with a different restriction, namely $g_{3n+12} = f_{2n+8} q_{n+4}$, which follows from table 2. The moduli of this locus thus correspond to the terms in $f_{2n+8}$, $q_{n+4}$, $f_{n+8}$, $g_{2n+12}$, and $g_{n+12}$, for a total of $7n + 48$, exactly as expected from the heterotic side. The discriminant locus is given by the zeroes of

$$\Delta = z_1^6 (f_{2n+8} q_{n+4}^2 + o(z_1)) .$$

The zeroes of $f_{2n+8}$ correspond to the spinors, while there is no such simple statement for vectors – as in all the examples of non-simply laced groups, the origin of the matter does not seem to be completely localized.
4.9. Unbroken $G_2$

There is a locus of unbroken $G_2$, which is obtained from the $SO(7)$ locus above by giving an expectation value to a hypermultiplet in the $8$. The massless hypermultiplet content consists of $9n + 56$ singlet moduli and $(3n + 10)$ hypermultiplets in the $7$ of $G_2$. This spectrum is obtained either from the above $SO(7)$ spectrum and the Higgs mechanism or from (4.1) and (4.2) applied to the commutant $F_4$.

In F-theory, $G_2$ finally corresponds to generic $D_4$ singularity. Compared to $SO(7)$, this corresponds to trading the terms in $q_{n+4}$ for terms in $g_{3n+12}$, introducing $2n + 8$ extra moduli, for a total of $9n + 56$ moduli, exactly as expected from the heterotic side. There are $(3n + 10)$ $7$s which again are not localizable in any obvious way.

4.10. Unbroken $SU(4)$

The locus of unbroken $SU(4) \cong SO(6)$ is obtained by starting from the unbroken $SO(7)$ locus discussed above and giving an expectation value to a field in the $7$. The massless hypermultiplet content consists of $8n + 51$ singlet moduli, $n + 2$ hypermultiplets in the $6$, and $4n + 16$ in the $4$; this follows from the $SO(7)$ spectrum and the Higgs mechanism or from (4.1) and (4.2) applied to the commutant, which is $SO(10)$.

It follows from table 2 that in order to get $SU(4)$ gauge symmetry we need to take a compactification manifold with a ‘split’ $A_3$ fiber singularity. A generic $A_3$ singularity implies various relations between the polynomials in (2.3):

$$f_{8+4n} \sim h_{4+2n}^2, \quad f_{8+3n} \sim h_{4+2n}H_{4+n}$$

$$g_{12+6n} = h_{4+2n}^3, \quad g_{12+5n} = -h_{4+2n}^2H_{4+n}$$

$$g_{12+4n} = -f_{8+2n}h_{4+2n} + \frac{1}{12}h_{4+2n}H_{4+n}^2$$

$$g_{12+3n} = \frac{1}{216}H_{4+n}^3 + \frac{1}{6}f_{8+2n}H_{4+n} - f_{8+n}h_{4+2n}^2.$$  

To get the $SU(4)$ enhanced gauge symmetry one needs a ‘split’ $A_3$ singularity which implies that one must impose the additional constraint

$$h_{2n+4} = h_{n+2}^2.$$  

23
The moduli of the $SU(4)$ locus thus correspond to the terms in $h_{n+2}$, $H_{n+4}$, $f_{2n+8}$, $f_{n+8}$, $g_{2n+12}$, and $g_{n+12}$ for a total of $8n + 51$, exactly as expected on the heterotic side. The discriminant on this locus is equal to

$$\Delta = z_1^4 (h_{2+n}^2 P_{16+4n} + o(z_1)),$$  \hspace{1cm} (4.6)

where $P_{16+4n}$ is some polynomial constructed from the $f$s and $g$s. The $h_{2+n}$ factor in $\Delta$ is responsible for the presence of $n + 2$ antisymmetric tensors in the $SU(4)$ theory, while the zeroes of $P_{16+4n}$ yield $4n + 16$ 4s. Just as in the case of all other simply laced groups, the matter seems localized.

### 4.11. Unbroken $SO(5)$

The locus of unbroken $SO(5) \cong Sp(2)$ can be obtained by starting from the unbroken $SU(4)$ locus discussed above and giving an expectation value to a field in the 6. The massless hypermultiplet content consists of $9n + 53$ singlet moduli, $n + 1$ hypermultiplets in the 5, and $4n + 16$ in the 4; this follows from the $SU(4)$ spectrum and the Higgs mechanism or from (4.1) and (4.2) applied to the commutant, which is $SO(11)$.

It follows from table 2 that on the F-theory side one obtains $SO(5)$ gauge symmetry from the presence of a generic $A_3$ singularity. Starting from the split $A_3$ singularity this corresponds to relaxing the condition $h_{2n+4} = h_{n+2}^2$. The moduli thus correspond to the terms of the previous subsection but with $h_{n+2}$ traded for $h_{2n+4}$, yielding $n + 2$ additional moduli for a total of $9n + 53$ – exactly as expected from the heterotic side.

### 4.12. Unbroken $SO(4)$

There is a locus of unbroken $SO(4) \cong SU(2) \times SU(2)$, which can be reached from the $SO(5)$ locus discussed above by giving an expectation value to a field in the 5. The massless hypermultiplet content consists of $10n + 54$ singlet moduli, $n$ hypermultiplets in the $(2, 2)$, and $4n + 16$ in the $(1, 2) + (2, 1)$; this follows from the $SO(5)$ spectrum and the Higgs mechanism or from (L1) and (L2) applied to the commutant, which is $SO(12)$.

In F-theory, $SO(4)$ appears when the discriminant locus has two irreducible components of $D_u = D_v + nD_s$ (following the notation of [9]) with an $A_1$ singularity on each. The codimension of this configuration is $2(12n + 29) - 4n = 20n + 58$, which agrees with that

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4 This codimension can be obtained by applying “the deficit argument” introduced in section 6.
expected from the heterotic side (recall the total dimension for the bundle associated with this $E_8$ is $30n + 112$). If $n = 0$, this corresponds to two independent theories. However for $n > 0$ these two components will intersect at $D_u^2 = n$ points leading to $n$ hypermultiplets in the $(2, 2)$, agreeing with the expected result of the heterotic side.

4.13. Unbroken $SU(3)$

The locus of unbroken $SU(3)$ can be obtained by starting either from the unbroken $SU(4)$ locus and giving an expectation value to two hypermultiplets in the 4 or by starting from the $G_2$ locus and giving an expectation value to a 7 hypermultiplet. The massless hypermultiplet content consists of $12n + 66$ singlet moduli and $(6n + 18)$ hypermultiplets in the 3. This follows from either the $SU(4)$ or $G_2$ matter content discussed above and the Higgs mechanism or from (4.1) and (4.2) applied to the commutant, which is $E_6$.

It follows from table 2 that a split $A_2$ singularity must yield $SU(3)$ gauge symmetry in F-theory. The conditions on the polynomials for such a split $A_2$ singularity are

$$g_{12+6n} = h_{2+n}^5, \quad f_{8+4n} = h_{2+n}^4, \quad g_{12+5n} = -Q_{6+2n}^3, \quad h_{2+2n} = h_{2+n}^2f_{8+2n} + \frac{1}{12}Q_{6+2n}^2. \quad (4.7)$$

The moduli of the $SU(3)$ locus thus correspond to the terms in $h_{n+2}$, $Q_{2n+6}$, $f_{2n+8}$, $f_{n+8}$, $g_{3n+12}$, $g_{2n+12}$, and $g_{n+12}$, for a total of $12n + 66$ exactly as expected above. The discriminant on this locus is given by

$$\Delta = z_1^3h_{n+2}^4P_{16+5n}. \quad (4.8)$$

Just as in the $SU(4)$ case, it is natural to associate $(n+2)$ antisymmetric tensors (which are the same as fundamentals for $SU(3)$) to the zeroes of $h_{n+2}$ and $5n+16$ more hypermultiplets of 3s to the zeroes of $P_{16+5n}$, for a total of $6n + 18$ 3s, in agreement with the heterotic side.

4.14. Unbroken $SU(2)$

The locus of unbroken $SU(2)$ can be obtained by starting either from the unbroken $SU(3)$ locus discussed above and giving an expectation value to two hypermultiplets in the 3, or by starting from the unbroken $SO(4)$ locus discussed above and giving an expectation value to two fields in the $(1, 2)$ (or $(2, 1)$). The massless hypermultiplet content consists of $18n + 83$ singlet moduli, and $6n + 16$ hypermultiplets in the 2, this follows from the above
SU(3) or SO(4) matter content and the Higgs mechanism or from (4.1) and (4.2) applied to the commutant, which is $E_7$.

A generic $A_1$ singularity yields an $SU(2)$ gauge group. The condition that the discriminant $\Delta$ and its derivative vanish at $z_1 = 0$ implies that the polynomials $f$ and $g$ must satisfy the relations

$$g_{12+6n} \sim h_{4+2n}^3, \quad f_{8+4n} \sim h_{4+2n}^2, \quad g_{12+5n} = -f_{8+3n}h_{4+2n}.$$

The moduli thus correspond to the terms in $h_{2n+4}$, $f_{3n+8}$, $f_{2n+8}$, $f_{n+8}$, $g_{4n+12}$, $g_{3n+12}$, $g_{2n+12}$, and $g_{n+12}$ for a total of $18n + 83$, exactly as expected from the heterotic side. The discriminant on the $SU(2)$ locus is equal to

$$\Delta = z_1^2(h_{4+2n}^2P_{6n+16} + o(z_1)). \quad (4.9)$$

The zeroes of $h_{2n+4}$ give $(n + 2)$ antisymmetric tensors (which are singlets for $SU(2)$), while the $(6n + 16)$ doublets of $SU(2)$ are localized at the zeroes of $P_{6n+16}$.

4.15. Unbroken $SU(2)_2$

There is an unbroken, level two $SU(2)_2$ locus (for $n > 0$) which can be obtained from the $SO(4)$ locus discussed above by giving an expectation to a field in the $(2, 2)$. It follows from the $SO(4)$ spectrum and the Higgs mechanism that the dimension of the $SU(2)_2$ locus is $11n + 54$ hypermultiplet moduli and there are $(8n + 32)$ hypermultiplets in the $2$ and $(n − 1)$ in the $3$. This spectrum also follows from (4.1) and (4.2) applied to the commutant, which is $SO(13)$.

In F-theory one obtains $SU(2)_2$ by smoothing a configuration which yields $SO(4)$. We remind the reader that $SO(4)$ appears when the discriminant has two rational components $D_u$ (in the notation of [1]) which necessarily intersect each other in $n$ points. Such a configuration has codimension $20n + 58$. Smoothing introduces $n$ complex parameters leading to codimension $19n + 58$ for $SU(2)_2$ in agreement with the Higgs mechanism. The smooth curve $C$ has genus $(n − 1)$ which is the number of adjoints $3$ in agreement with the topological theory arguments [18] briefly discussed below in section 6.
4.16. Unbroken $SO(12)$

The heterotic theory has an unbroken $SO(12)$ when the $12+n$ instantons are embedded in the commutant $SO(4) \cong SU(2) \times SU(2)$. A new feature here is that there are different $SO(12)$ loci corresponding to the different choices of how the $12+n$ instantons are distributed in the two $SU(2)$ factors. Because it follows from (4.2) that each $SU(2)$ must have at least four instantons, we will parameterize the choices by putting $4+r$ instantons in the first $SU(2)$ and $8+n−r$ in the second, with $r = 0\ldots n+4$. It follows from the obvious generalization of (4.1) and (4.2) for commutant $H = SU(2) \times SU(2)$ with these instanton numbers that there are $2n+18$ singlet hypermultiplet moduli, giving the dimension of each $SO(12)$ locus, $n+8$ hypermultiplets in the $\mathbf{12}$, $\frac{1}{2}r$ in the $\mathbf{32}$, and $\frac{1}{2}(4+n−r)$ in the $\mathbf{32}'$.

On the F-theory side, table 2 (and the discussion after it) says that we will find $SO(12)$ gauge symmetry enhancement by starting with a $D_6$ fiber singularity and imposing some additional restrictions. The generic $D_6$ singularity requires

$$f_{8+2n} = s_{n+4}^2, \ g_{3n+12} = s_{n+4}^3, \ g_{12+2n} = −s_{n+4}f_{n+8}. \quad (4.10)$$

Working through Tate’s algorithm as described in §3, we see that there are different ways we can further restrict to obtain $SO(12)$ gauge symmetry, parameterized by a single integer $r$. This is in agreement with the freedom of choosing how to divide the $12+n$ instantons between the two $SU(2)$s in $E_8$ on the heterotic side. The basic condition (in the notation of §3) is that the polynomial

$$a_{2,1}X^2 + a_{4,3}X + a_{6,5}$$

should factorize as

$$(p_{4+n−r}X + q_{8+n−r}) \ (t_rX + u_{r+4}) \ .$$

Transforming to Weierstrass form, this implies relations of the form

$$s_{n+4} \sim p_{4+n−r}t_r, \ g_{n+12} \sim q_{8+n−r}u_{r+4}, \ f_{8+n} \sim q_{8+n−r}t_r + p_{4+n−r}u_{r+4} \ . \quad (4.11)$$

The parameters in $p_{4+n−r}, q_{8+n−r}, t_r, u_{r+4}$, after subtracting two degrees of freedom which can be absorbed in reparameterizations, give us a $2n+18$ dimensional $SO(12)$ locus, as expected from the heterotic side.
The discriminant looks like

\[ \Delta \sim z^{8} p_{4+n-r}^{2} r^{2} P_{8+n}^{2}. \]

We can associate the \( r \) \( 1/2 \) \( 32s \) with the zeroes of \( t \), the \((4 + n - r) \) \( 1/2 \) \( 32' \)'s with the zeroes of \( p \), and the vectors with the zeroes of \( P_{8+n} \).

Note that giving an expectation value to a \( 12 \) hypermultiplet Higgses to the \( SO(11) \) locus discussed above. The distinction between the different \( SO(12) \) loci, labeled above by \( r \), is lost upon Higgsing to \( SO(11) \), corresponding to the fact that the commutant is enhanced to \( SO(5) \), which is simple.

4.17. Unbroken \( SU(6) \)

There are different loci of unbroken \( SU(6) \), corresponding to the different ways of distributing the \( 12 + n \) instantons among the two factors in the commutant, which is \( SU(2) \times SU(3) \). The choices can again be labeled by putting \( 4 + r \) instantons in the first factor and \( 8+n-r \) in the second, \( 0 \leq r \leq n+2 \). These \( SU(6) \) loci can be obtained from the above \( SO(12) \) loci by the Higgs mechanism, giving an expectation value to the appropriate component of a \( 32' \). The massless hypermultiplet content consists of \( 3n + 21 \) singlet moduli, \((16 + 2n + r)\) hypermultiplets in the \( 6 \), \( 2+n-r \) in the \( 15 \) and \( \frac{1}{2}r \) in the \( 20 \).

In F-theory, one obtains \( SU(6) \) gauge group from a split \( A_{5} \) singularity. It follows from Tate’s algorithm that the most generic such singularity is specified by polynomials

\[ f_{4+2n} = h_{2+n}^{2}, \quad f_{8+n}, \quad s_{4+n}, \quad f_{4}. \]

This is the same as generic \( A_{5} \) except for the constraint that

\[ f_{2n+4}(z_{2}) = h_{n+2}^{2}(z_{2}). \] \hfill (4.12)

The discriminant locus looks like

\[ \Delta = z_{1}^{6} h_{n+2}^{4} P_{2n+16}. \] \hfill (4.13)

This locus has dimension \( 3n + 21 \) and corresponds to the \( r = 0 \) case of the heterotic string, with the \( 2n + 16 \) \( 6s \) localized at the zeroes of \( P_{2n+16} \) and the \( 2 + n \) \( 15s \) localized at the zeroes of \( q_{n+2} \).
Natural candidates for the F-theory duals to the heterotic theories with \( r \neq 0 \) are the special ‘split’ \( A_5 \) singularities which satisfy the constraints (4.11) of the \( r \)th \( SO(12) \) locus and in which
\[
h_{n+2} = t_r \tilde{h}_{2+n-r},
\]
with \( t_r \) as in subsection 4.16. The independent polynomials are \( \tilde{h}_{2+n-r}, t_r, q_{8+n-r}, p_{4+n-r} \) and \( u_{4+r} \). Remembering to subtract the two reparameterizations, we see that such a theory occurs at dimension \( 2n + 18 + n + 3 - r = 3n + 21 - r \), in agreement with the series of heterotic \( SU(6) \) theories parameterized by \( r \).

4.18. Unbroken \( SU(5) \)

Unbroken \( SU(5) \) can be obtained either from the \( SO(10) \) by Higgsing with a 16 or from \( SU(6) \) by Higgsing with two fundamentals. (All of the \( SU(6) \) theories, labeled by \( r \), Higgs to the same \( SU(5) \) theory.) The massless matter spectrum consists of \( 5n + 36 \) hypermultiplet moduli, \( 3n + 16 \) in the 5 and \( n + 2 \) in the 10. This spectrum can be obtained from that of \( SO(10) \) or \( SU(6) \) and the Higgs mechanism, or by applying (4.1) and (4.2) to the commutant, which is also \( SU(5) \).

It follows from table 2 that in F-theory one obtains \( SU(5) \) gauge group by compactifying on a manifold with a split \( A_4 \) singularity. It follows from Tate’s algorithm that such a singularity is given by specifying five polynomials \( h_{2+n}, H_{4+n}, q_{6+n}, f_{8+n}, g_{12+n} \). The other \( f \)s and \( g \)s are specified in terms of these, e.g.
\[
g_{12+6n} \sim h_{2+n}^6, \quad f_{8+4n} \sim h_{2+n}^4, \ldots
\]
The dimension of the \( SU(5) \) locus is thus \( 5n + 36 \), exactly as expected from the heterotic side. The discriminant locus looks like
\[
\Delta \sim z_1^5 h_{2+n}^4 P_{16+3n}.
\]
The \( 3n + 16 \) zeroes of \( P \) yield the 5s while the zeroes of \( h_{2+n} \) correspond to the antisymmetric tensor 10s.
4.19. Unbroken Sp(3)

As above, there are theories labeled by an integer \( r \) corresponding to the different ways of distributing the \( 12 + n \) instantons among the factors in the commutant, which is \( SU(2) \times G_2 \). These Sp(3) loci can be obtained from \( SU(6) \) by giving an expectation value to one of the 15 hypermultiplets. The hypermultiplet content consists of \( 4n + 23 - 2r \) singlet moduli, \( 16 + 2n + \frac{3}{2}r \) hypermultiplets in the 6, \( n + 1 - r \) in the 14, and \( \frac{1}{2}r \) in the three index antisymmetric 14'.

In F-theory, one can obtain this family of theories by taking the \( r \)th \( SU(6) \) theory of subsection 4.17 and relaxing the condition (4.12) on \( f_{2n+4} \) to be

\[
f_{2n+4} = \tilde{h}_{2n+4-2r} t_r^2.
\]

For \( r = 0 \), this is the generic \( A_5 \) singularity. The independent polynomials going into specifying the \( Sp(3) \) theories are \( \tilde{h}_{2n+4-2r}, t_r, q_{8+n-r}, p_{4+n-r}, \) and \( u_{4+r} \). The dimension increases by \( n + 2 - r \) with respect to the \( r \)th \( SU(6) \) locus, hence it is given by \( 4n + 23 - 2r \), in agreement with the expected result from the heterotic side.

4.20. Unbroken SU(3)_2

The above \( Sp(3) \) theory can be broken to a level 2 \( SU(3) \) by giving an expectation value to the 14'. The dimension of the \( SU(3)_2 \) locus is \( 4n + 22 - r \) and there are \( 34 + 6n + r \) hypermultiplets in the 3, \( r - 2 \) in the 6, and \( n + 1 - r \) in the 8. This spectrum is obtained either from this Higgsing or by applying (4.1) and (4.2) to the commutant, which is \( SU(3) \times G_2 \).

One can also obtain \( SU(3)_2 \) from the \( SU(3) \times SU(3) \) theory in the same way as one obtains \( SU(2)_2 \) from \( SO(4) = SU(2) \times SU(2) \). The theory with \( SU(3) \times SU(3) \) appears at codimension \( 2(18n + 46) - 9n = 27n + 92 \) (this follows from the “deficit argument” discussed in sect. 6) when the discriminant has two \( A_2 \) components \( D_u \) intersecting in \( n \) points. Such a theory has \( n \) hypermultiplets in the (3, 3) and \( 3n + 18 \) hypermultiplets in the (3, 1) + (1, 3). Smoothing \( n + 2 - r \) double points one adds \( n + 2 - r \) parameters to end up with codimension \( 26n + r + 90 \), exactly as expected from the heterotic side. This smoothing corresponds to Higgsing \( n + 2 - r \) of the \( n \) mixed (3, 3) hypermultiplets. The remaining \( r - 2 \) mixed representations decompose as (3) + (6) reproducing the spectrum of \( SU(3)_2 \). For \( r = 2 \) the genus of the smooth curve is \( n - 1 \) in agreement with the number of adjoints 8. For \( r > 2 \) the resulting curve is not smooth, but it has exactly \( n + 1 - r \) holomorphic 1-differentials which in the topological theory correspond to adjoints. The remaining \( r - 2 \) double points thus correspond to the matter in the 3 + 6.
5. Small Instantons

The $E_8 \times E_8$ heterotic string with instanton numbers $(16, 8)$ embedded in the two $E_8$s has been conjectured [9] to be equivalent to the $SO(32)$ heterotic string. In fact, this has recently been established via T-duality [24]. Note that both theories have generic unbroken $SO(8)$ gauge symmetry; in the $E_8 \times E_8$ theory that is the generic unbroken gauge symmetry associated with the $E_8$ with 8 instantons, while in the $SO(32)$ theory it is the generic unbroken gauge group with 24 instantons. The $SO(32)$ string is known to develop an enhanced $Sp(1) \simeq SU(2)$ non-perturbative gauge symmetry whenever an instanton collapses to zero size [5]. When $k$ collapse at the same point in $K3$, an $Sp(k)$ gauge symmetry develops non-perturbatively. In light of the above correspondence, we should be able to find these non-perturbative enhanced gauge symmetries in the context of F-theory on the elliptic fibration over $F_4$. We will demonstrate in this section that this is so.

We have seen that the singularities of F-theory in $z_1$ correspond to perturbative enhanced gauge symmetry of the heterotic theory. Singularities of F-theory at $z_2 = \text{const.}$, on the other hand, should be interpreted as non-perturbative gauge symmetries in the dual heterotic picture. Therefore, we should recover the results of [5] by studying degenerations of this sort in the $n = 4$ model.

The case of a single small instanton leads to an enhanced non-perturbative $Sp(1)$ and should thus correspond to an $A_1$ singularity located at a point on the base, say $z_2 = 0$. We expand $f$ and $g$ as

$$f = w_1^2 f_0 + w_1^3 f_4(z_2) + \cdots, \quad g = w_1^3 g_0 + w_1^4 g_4(z_2) + \cdots,$$  \hspace{1cm} (5.1)

where, for convenience, we expand in terms of $w_1 = 1/z_1$ around $w_1 = 0$. The discriminant takes the form

$$\Delta = w_1^6 (D_0 + w_1 D_4(z_2) + \cdots).$$  \hspace{1cm} (5.2)

Note that our conditions for an unbroken $SO(8)$ perturbative gauge symmetry are satisfied at $w_1 = 0$. We now impose the condition for an $A_1$ singularity, i.e. that $\Delta$ develops a double zero, in $z_2$. This condition implies that $f$ and $g$ must satisfy

$$f(w_1, 0) = 3h^2(w_1), \quad g(w_1, 0) = -2h^3(w_1)$$
$$g'(w_1, 0) = -h f'(w_1, 0)$$  \hspace{1cm} (5.3)
for some function $h_3(w_1)$ of degree 3. In this process, we trade the 7+10+6+9 parameters in $f, g, f', g'$ for the 4+6 parameters in $h$ and $f'$. After subtracting one to account for the arbitrary choice of the location of the singularity at $z_2 = 0$, we see that the codimension for an enhanced non-perturbative $Sp(1)$ is 21.

In addition to there being an enhanced non-perturbative gauge symmetry, the perturbative gauge symmetry is also automatically enhanced on the locus (5.3). To see that, note that the condition that the discriminant (5.2) have a second order zero at $z_2 = 0$ requires the constant term $D_0$ to vanish, yielding a $D_5$ singularity at $w_1 = 0$. Therefore, according to our dictionary, the perturbative $SO(8)$ gauge symmetry is automatically extended to $SO(9)$. To summarize, we are finding the generic unbroken $SO(8)$ enhanced to an unbroken $SO(9) \times Sp(1)$, with $SO(9)$ perturbative and $Sp(1)$ non-perturbative, at codimension 21.

The above result is perfect for the conjectured equivalence of $n = 4$ F-theory with the $SO(32)$ heterotic string on $K3$! Shrinking one instanton to zero size gives an enhanced non-perturbative $Sp(1)$ and the remaining 23 instantons can only break $SO(32)$ to $SO(9)$. A single small instanton in this model is expected to lead to $SO(9) \times SU(2)$ as the generic unbroken gauge group with matter $\frac{1}{2} (9, 2)$ and $\frac{23}{2} (1, 2)$ hypermultiplets. The codimension for this enhanced gauge symmetry is 21, precisely as found above.

Consider, more generally, the condition for an enhanced non-perturbative $Sp(k)$ in the $n = 4$ F-theory. We saw in sect. 4 that, for all $n$, a perturbative enhanced $Sp(k)$ gauge symmetry corresponds to an $A_{2k-1}$ singularity in $z_1$. Therefore, we should adjust the moduli so as to obtain an $A_{2k-1}$ singularity in $z_2$ to obtain an enhanced non-perturbative $Sp(k)$ gauge symmetry.

To find the conditions and the codimension for an $A_{2k-1}$ singularity at $z_2 = 0$, note that the derivatives of the functions $f$ and $g$ in (5.1) with respect to $z_2$ have degree in $z_1$ given by $\text{deg}(f^{(r)}(z_1, 0)) = 6 - \lfloor(r + 3)/4 \rfloor$ and $\text{deg}(g^{(r)}(z_1, 0)) = 9 - \lfloor(r + 3)/4 \rfloor$, where $\lfloor \rfloor$ denotes the integer part. It can thus be seen that an $A_{2k-1}$ singularity is enhanced to an $A_{2k+1}$ singularity at codimension $22 - k$. Iterating this, the codimension for an $A_{2k-1}$ singularity at $z_2 = 0$, and thus a non-perturbative $Sp(k)$, is $\frac{1}{2} k(45 - k) - 1$.

As in the above $k = 1$ case, the singularity at $z_1 = \infty (w_1 = 0)$ is also automatically enhanced, corresponding to a larger perturbative gauge group. It is easily seen in the

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5 By induction, it can be shown that a $A_{2k-1}$ singularity in $z_1$ occurs at codimension $-2nk^2 + 15nk + 30k - n - 1$. This agrees with the codimension for an enhanced perturbative $Sp(k)$ with matter $16 + 2n(4 - k)$ fundamentals and $n + 1$ two-index antisymmetric tensors.
first few cases that the singularity at \( w_1 = 0 \) is precisely that which we found for a perturbative \( SO(8 + k) \) gauge symmetry. For example, an \( A_3 \) singularity in \( z_2 \) requires \( D_4(z_2) = (\text{const})z_2^4 \) in (5.2) which leads automatically to a \( D_5 \) singularity at \( w_1 = 0 \) with the extra condition for \( SO(10) \) gauge symmetry; an \( A_5 \) singularity in \( z_2 \) leads to a \( D_6 \) singularity at \( w_1 = 0 \), our condition for \( SO(11) \) gauge symmetry; etc. To summarize, we have an enhanced \( SO(8 + k) \times Sp(k) \) gauge symmetry, with \( SO(8 + k) \) perturbative and \( Sp(k) \) non-perturbative, at codimension \( \frac{1}{2}k(45 - k) - 1 \). The matter content for this theory is hypermultiplets in the \( \frac{1}{2}(8 + k, 2k) \), \( \frac{1}{2}(24 - k)(1, 2k) \), and \( (1, k(2k - 1) - 1) \). The codimension for this enhanced gauge symmetry, by the Higgs counting, is \( \frac{1}{2}k(45 - k) - 1 \), agreeing with the codimension found above for an \( A_{2k - 1} \) singularity in \( z_2 \). Again, this enhanced gauge symmetry and matter content perfectly agrees with the expected result, based on the considerations of [3], for the generic unbroken gauge group and matter content for \( k \) small instantons of the \( SO(32) \) heterotic theory at the same point in \( K3 \)!

### 6. Mixing Perturbative and Non-Perturbative Gauge Symmetry

In this section we focus on the \( n = 0 \) case, which corresponds to the symmetric \((12, 12)\) heterotic compactification on \( K3 \). For \( n = 0 \), the Hirzebruch surface \( F_0 = \mathbb{P}^1 \times \mathbb{P}^1 \) is a product of two projective lines. The Kähler class decomposes as \( k = k_1a + k_2b \) where \( k_1 \) and \( k_2 \) are the areas of these \( \mathbb{P}^1 \)s and \( a \) and \( b \) are their dual 2-cocycles. The string coupling constant is given by

\[
\frac{1}{\lambda^2} = \exp(2\phi) = \frac{k_1}{k_2}.
\]

There is a manifest \( \mathbb{Z}_2 \) symmetry exchanging the factors, which was interpreted in [25] as the strong-weak coupling duality of the \((12, 12)\) heterotic theory proposed in [26]. This duality exchanges gauge symmetries which arise perturbatively in the heterotic theory with ones which arise non-perturbatively. In this section we will discuss new enhanced gauge symmetries which are mixtures of being perturbative and non-perturbative and are naturally seen in \( F \) theory. Some cases of these mixed gauge symmetries were also seen in the orientifold analysis of [27].

In Section 4, we found the perturbative enhanced gauge symmetry of the heterotic theory from the singularities along \( z_1 = 0 \) (or more generally, the section \( z_1 = P_n(z_1) \)). In Section 5, we considered singularities along \( z_2 = \text{const} \), finding the non-perturbative enhanced gauge symmetry associated with small \( SO(32) \) instantons. In the present section,
we will consider singularities along a more general curve $\Sigma_{p,q}$ given by $F_{p,q}(z_1, z_2) = 0$, where $(p, q)$ denotes the degree of the polynomial in $z_1$ and $z_2$, respectively. Once we have chosen to call the gauge symmetry corresponding to $(1, 0)$ “perturbative,” we should call the gauge symmetry corresponding to $(0, 1)$ “non-perturbative”. The more general $(p, q)$ singularity corresponds to enhanced gauge symmetry which is a mixture of perturbative and non-perturbative.

At this stage, it is useful to recall the constraint of six dimensional anomaly factorization, discussed in [28]: the anomaly polynomial should factorize as

$$I = (R^2 - \sum_a u_a F_a^2)(R^2 - \sum_a v_a F_a^2), \quad (6.2)$$

where $a$ runs over the different gauge groups. A given gauge group and matter content thus has an associated $(u, v)$, which enter in the gauge kinetic terms [29,26]. The level one perturbative gauge groups and matter content associated with the $(12, 12)$ heterotic theory have $(u, v) = (2, 0)$, while the non-perturbative gauge groups have $(u, v) = (0, 2)$ [28]. (Here we are normalizing $(u, v)$ in (6.2) by defining $F_a^2$ to be the trace normalized by the index of the representation; for example, $F_a^2 = \text{tr} F_a^2$ for $SU(N)$ and $\frac{1}{2} \text{tr} F_a^2$ for $SO(N)$, with tr in the fundamental representation in both cases).

Consider first the case of a $\Sigma_{1,1}$ given by $z_1 z_2 = 0$, with $A_1$ singularities along both $z_1 = 0$ and $z_2 = 0$. With only the $A_1$ singularity along $z_1 = 0$, we would expect a perturbative enhanced $SU(2)_{(1,0)}$ with 16 fundamental hypermultiplets and $(u, v) = (2, 0)$. With only the $A_1$ singularity along $z_2 = 0$, we would expect a non-perturbative $SU(2)_{(0,1)}$, also with 16 fundamental hypermultiplets, and $(u, v) = (0, 2)$. With both singularities, we expect to find an enhanced $SU(2)_{(1,0)} \times SU(2)_{(0,1)}$ with matter given by a single $(2,2)$ field and 14 fields in the $(1,2) + (2,1)$. Indeed, this matter content is the unique solution of the above anomaly factorization condition for which $SU(2)_{(1,0)}$ and $SU(2)_{(0,1)}$ are coupled, as they should be because $z_1 = 0$ and $z_2 = 0$ intersect at a point, and which properly reduces to $SU(2)_{(0,1)}$ or $SU(2)_{(1,0)}$ when the singularity along $z_1$ or $z_2$ is smoothed. The fact that there is a matter field in the $(2,2)$ also follows from the intersecting D-brane picture.

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6 This theory and matter content can be obtained from the $U(16) \times U(16)$ of [27] upon Higgsing (up to $U(1)$s). The perturbative $SU(2)$ comes from the $SO(32)$ 9 branes upon Higgsing by Wilson lines and the non-perturbative $SU(2)$ is associated with a small instanton [3] or Type-1 5 brane. The $(2, 2)$ field comes from the 5 − 9 sector.
discussed in [3]. The codimension for the enhanced $SU(2) \times SU(2)$ with this matter content is $4 + 56 - 6 = 54$.

In F-theory, we interpret enhanced gauge symmetry as coming from coinciding 7-branes wrapped around components of discriminant. It is natural that the gauge symmetry associated with $A_1$ singularities along $z_1 = 0$ and $z_2 = 0$ should be two copies of the $SU(2)$ theory which are coupled by a single $(2, 2)$ matter field, corresponding to the intersection at a point. Consider now the codimension in F-theory for having $A_1$ singularities along both $z_1 = 0$ and $z_2 = 0$. Near $z_1 = z_2 = 0$, an $A_1$ singularity of the discriminant $\Delta(z_1, z_2)$ in either $z_1$ or $z_2$ is obtained at codimension 29, as in section 4.14. Having an $A_1$ singularity in both $z_1$ and $z_2$ is almost just the sum of the two conditions, but the vanishing of $\Delta$ to second order in both $z_1$ and $z_2$ kills the four terms with $\Delta \sim 1, z_1, z_2, z_1z_2$ twice\footnote{This addition of the independent codimensions and subtraction of the twice counted terms is the “deficit argument” referred to in the previous section.}. So the codimension is $29 + 29 - 4 = 54$, in agreement with the above expected result!

The $SU(2)_{(1,0)} \times SU(2)_{(0,1)}$ theory can be Higgsed by giving expectation values to the fields in the $(2, 1)$ or $(1, 2)$, breaking the theory to the purely non-perturbative $SU(2)_{(0,1)}$ or perturbative $SU(2)_{(1,0)}$ theories, respectively, discussed before. In terms of $F$ theory, such Higgsing corresponds to smoothing the $A_1$ singularity along $z_1$ or $z_2$. Another possibility is to give an expectation value to the $(2, 2)$ field, which breaks $SU(2)_{(1,0)} \times SU(2)_{(0,1)}$ to the diagonally embedded $SU(2)_{(1,1)}$ with 28 fundamentals (coming from the fourteen $(2, 1) + (1, 2)$). Because this theory is obtained by Higgsing from one with factorized anomaly, it of course also has a factorized anomaly. The coefficients in (6.2) are $(u, v) = (2, 2)$. In terms of $F$ theory, this latter Higgsing corresponds to smoothing $\Sigma_{(1,1)}$ from $z_1z_2 = 0$ to $z_1z_2 = \epsilon$. The codimension for an $A_1$ singularity along this smoothed surface is 53, which agrees with the codimension for $SU(2)_{(1,1)}$ with 28 doublets.

So F-theory predicts a new enhanced $SU(2)$ which is neither purely perturbative nor purely non-perturbative. It could not have been seen from the arguments of [26] because the $\mathbb{Z}_2$ strong-weak duality maps $(1, 1)$ to itself.

We can extend the above analysis in two directions. One is to consider other singularities, which would produce other enhanced gauge groups. For example, consider $Sp(n) \times Sp(m)$, corresponding to an $A_{2n-1}$ singularity at $z_1 = 0$ and a $A_{2m-1}$ singularity at $z_2 = 0$. The matter content is a field in the $(2n, 2m)$, 16 $- 2m$ fields in the $(2n, 1)$, 16 $- 2m$ fields in the $(1, 2m)$, a field in the $(n(2n - 1) - 1, 1)$ and a field in the
The fundamental only under each of the SU under each of the is the unique solution of the anomaly factorization equation for $29(2\text{ pq} + 2)$. This enhanced gauge group and matter content is $4n m + 2n (16 - 2m) + 2m (16 - 2n) + n (2n - 1) - 1 + m (2m - 1) - 1 - n (2n + 1) - m (2m + 1) = 30(n + m) - 4nm - 2$. In F-theory an $A_{2n-1}$ singularity has codimension $30n - 1$ (as remarked in footnote 6). So an $A_{2n-1}$ along $z_1 = 0$ and $A_{2m-1}$ along $z_2 = 0$ has codimension $30(n + m) - 2 - 4nm$, where the last term corresponds to the fact that the terms $a_{ij}$ with $\Delta \sim \sum_{i=0}^{2n-1} \sum_{j=0}^{2m-1} a_{ij} z_1^i z_2^j$ were killed twice. So the F theory gives the correct codimension.

Another extension of the above ideas is to more perturbative and (or) non-perturbative gauge groups. In F-theory it is clear that we can, more generally, have a “grid” $\Sigma_{p,q}$, given by $\prod_{i=1}^{p} (z_1 - a_i) \prod_{j=1}^{q} (z_2 - b_j) = 0$, with $p$ singularities along $z_1 = a_i$, with $a_i$ constants and $q$ singularities along $z_2 = b_j$, with $b_j$ constants. With $A_1$ singularities for each line of the grid, the enhanced gauge group is $\prod_{i=1}^{p} SU(2)(i)_{(1,0)} \times \prod_{j=1}^{q} SU(2)(j)_{(0,1)}$ with $(2^{(i)}, 2^{(j)})$ matter fields coupling each $SU(2)(i)_{(1,0)}$ to each $SU(2)(j)_{(0,1)}$, corresponding to the vertices of the grid, and $16 - 2q$ matter fields $2^{(i)}$ transforming transforming as fundamentals only under each of the $SU(2)(i)_{(1,0)}$ gauge groups and $16 - 2p$ matter fields $2^{(j)}$ transforming as fundamentals only under each of the $SU(2)(j)_{(0,1)}$ gauge groups. It is easily verified that this is the unique solution of the anomaly factorization equation for $p$ perturbative $SU(2)$s with $(u_i, v_i) = (2, 0)$ and $q$ non-perturbative $SU(2)$s with $(u_j, v_j) = (0, 2)$. The codimension for this enhanced gauge group and matter content is $4pq + 2p(16 - 2q) + 2q(16 - 2p) - 3(p + q) = 29(p + q) - 4pq$. This agrees with the codimension computed in F theory: Again, each of the $p + q$ $A_1$ singularities occur at codimension 29 but $4pq$ terms in $\Delta$ are killed twice.

By giving expectation values to the fields in the various $(2^{(i)}, 2^{(j)})$, it is possible to Higgs to a variety of different gauge groups which are neither purely perturbative nor purely non-perturbative. In F-theory this corresponds to smoothing the various intersections of the “grid” as with $z_1 z_2 = 0$ deformed to $z_1 z_2 = \epsilon$. This leads to more general diagrams of intersecting singularities in the $z_1, z_2$ plane. Generally, by this Higgsing, we can get gauge groups $\prod_{i} SU(2)(p_i, q_i)$ with matter given by $p_i q_j + q_i p_j$ fields transforming as a $(2^{(i)}, 2^{(j)})$ fundamental under both $SU(2)(p_i, q_i)$ and $SU(2)(p_j, q_j)$, $16(p_i + q_i) - 4p_i q_i$ matter fields transforming only as the fundamental $2^{(i)}$ under $SU(2)(p_i, q_i)$, and $(p_i - 1)(q_i - 1)$ matter fields transforming only as an adjoint $3^{(i)}$ of $SU(2)(p_i, q_i)$. Because the $\prod_{i} SU(2)(p_i, q_i)$
theory with this matter content was obtained by Higgsing from a theory which satisfies the anomaly factorization condition, it of course also satisfies this condition, as can be directly verified. The gauge group $SU(2)_{(p_i,q_i)}$ has $(u_i,v_i) = (2p_i,2q_i)$. The above matter content is the unique solution of the anomaly factorization condition with only fundamentals and adjoints and with these values of $(u,v)$.

As an extreme case, we can smooth all of the intersections, Higgsing to a single $SU(2)$ with $N_F = 16(p + q) - 4pq$ fundamentals and $N_A = (p - 1)(q - 1)$ adjoints. This theory has a factorized anomaly with

$$(u,v) = (2p,2q).$$

The codimension for this $SU(2)$ is $2[16(p + q) - 4pq] + 3(p - 1)(q - 1) - 3 = 29(p + q) - 5pq$. The completely smoothed grid corresponds to a surface $\Sigma$ of genus $(p - 1)(q - 1)$. It is interesting to note that, when $p = 1$, this mixed perturbative/non-perturbative spectrum for $E_8$ with 12 instantons coincides with the perturbative $SU(2)$ spectrum discussed in sect. 4 for $E_8$ with $12 + 2q$ instantons.

In $F$ theory, we can easily explain why $u/v = p/q$: Following [9], the gauge coupling of the six-dimensional theory is proportional to the integral

$$\frac{1}{g^2} = \int_C k = pk_1 + qk_2 \propto (pe^\phi + qe^{-\phi})$$

of the Kähler class $k$ over the compact part $C$ of the 7-brane world-volume. On the other hand, it follows from supersymmetry that $g^{-2} \propto (ue^{\phi} + ve^{-\phi})$, so $u/v = p/q$. To explain the coefficient of 2 in (6.3), one would have to understand the anomaly factorization property in terms of D-branes. Assuming (6.3), we know a priori that the above $SU(2)$ gauge theory appears when two 7-branes wrap around a smooth $(p,q)$ curve $C$. On the other hand, compactifying on $T^2$ down to 4 dimensions one can use the results of [18] which predicts genus($C$) massless hypermultiplets in the adjoint representation in this situation. With a little triumph, one notices that indeed genus($C$) = $(p - 1)(q - 1)$, as it should be for consistency. Also, by the above “deficit argument” the codimension of the corresponding locus is $29(p + q) - 5pq$, in agreement with the above Higgs mechanism codimension.

Note that the above extensions are not unrelated. $SU(2)_{(m,n)}$ is obtained by deforming away from a configuration of $m$ copies of $SU(2)_{(1,0)}$ intersecting $n$ copies of $SU(2)_{(0,1)}$. On the other hand, bringing together all $n$ parallel 7-branes of such configuration, one ends up with $Sp(n) \times Sp(m)$ theory on the singular $(1,1)$ curve $zw = 0$. When $n = m$ one can further break to diagonal $Sp(n)$ by deforming to a smooth $zw = \epsilon$. 37
One can easily consider other types of singularities leading to a variety of gauge groups. For example, in the perturbative heterotic theory in codimension 64 one finds an \( \text{SO}(7) \) gauge theory with \( N_F = 3 \) hypermultiplets in (7) and \( N_S = 8 \) hypermultiplets in (8). This \( \text{SO}(7) \) can be Higgsed down to a theory with the exceptional \( G_2 \) gauge group and with \( N'_F = 10 \) hypermultiplets in (7) living in codimension 56. Both theories have \((u,v) = (2,0)\). Let us look for \( \text{SO}(7) \) and \( G_2 \) gauge theories with factorizable anomaly with \((u,v) = (2p,2q)\). In codimension \( 64(p+q) - 18pq \) one finds an \( \text{SO}(7) \) theory with \( N_A = (p-1)(q-1) \) adjoints (21), \( N_F = 3(p+q) - pq \) and \( N_S = 8(p+q) - 4pq \). In codimension \( 56(p+q) - 14pq \) one finds a \( G_2 \) theory with \( N'_A = (p-1)(q-1) \) adjoints and \( N_F = 10(p+q) - 4pq \) fundamentals. The relations \( N'_F = N_F + N_S + N_A - 1 \) and \( N_A = N'_A \) guarantee that these two theories are connected by the Higgs mechanism for all \((p,q)\). Again, following the lines of Section 4, one obtains these models in F-theory from a constrained \( D_4 \) singularity along a smooth \((p,q)\) curve. Both codimensions turn out to be consistent with such an interpretation, as does the number of adjoints \( N_A \) which is always given by the genus of the curve. Again, it is interesting to note that for \( p = 1 \) the matter spectrum of these mixed perturbative/non-perturbative theories coincides with that of the perturbative theory with \( 12 + 2q \) instantons.

It is very interesting that, unlike the above example of \( \text{SU}(2)_{(m,n)} \), these theories cannot be obtained by Higgsing from intersecting perturbative and non-perturbative gauge groups. For instance, there is no appropriate \( \text{SO}(7)_{(1,0)} \times \text{SO}(7)_{(0,1)} \) theory with factorizable anomaly form. In the case of intersecting \( A_n \) type singularities, the fact that they can be replaced by D-branes implies that we must have a conventional interpretation of the resulting singularities. There is no such reason in the \( D_4 \) case and evidently we are finding that there must be new physics going on when such singularities intersect. In fact at the intersection of the \( D_4 \) singularities there is a vanishing 2-cycle which signals the appearance of a tensionless string (coming from a 3-brane wrapped around the vanishing cycle). This is similar to the occurrence of tensionless strings in strong coupling transitions in heterotic string theory \[22\].

7. Coulomb Branch and Duality Chains

7.1. Resolving the singularities

In our earlier discussion of Tate’s algorithm, we indicated what restrictions on the coefficients in the defining equation would lead to which kinds of singularities in the total
space, but we did not explain how the singularity type is determined or how the singularities are resolved. We will now complete those tasks.

We work with coordinates $x$, $y$, $\sigma$ on the total space, and wish to recast our conditions on the coefficients in the Weierstrass equation as being conditions which determine which monomials are allowed to occur in that equation. In other words, if we write the equation in the form

$$\sum c_{i,j,k} x^{i+1} y^{j+1} \sigma^{k+1} = 0,$$

then we are searching for conditions on $(i, j, k)$ which describe which monomials are allowed. The first conditions are that $i \geq -1$, $j \geq -1$ and $k \geq -1$; we search for other conditions of this form.

Phrasing the problem in this way makes contact with the methods of toric geometry (see \[30\] for a review for physicists). The conditions which are natural from the toric point of view are expressed in terms of vectors $v$ with integer entries, with the condition given by

$$v \cdot (i, j, k) \geq -1.$$

The initial conditions mentioned above correspond to the coordinate vectors $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, and any additional conditions (with nonnegative entries in $v$) automatically correspond to blowups from the toric point of view. We can resolve the singularity if we find enough such vectors.

For example, the condition associated to $v = (1, 1, 1)$ can be written as $i + j + k \geq -1$ and it implies that each allowed monomial $x^{i+1} y^{j+1} \sigma^{k+1}$ has degree at least two. In other words, this is precisely the condition for a singularity to appear at the origin. To see the connection to the corresponding blowup, rewrite a monomial in the form

$$x^{i'} y^{j'} \sigma^{k'} = (x/\sigma)^{i'} (y/\sigma)^{j'} \sigma^{i'+j'+k'}$$

(with the form of this determined by the vectors $(1, 0, 0)$, $(0, 1, 0)$, $(1, 1, 1)$ used to measure the exponents), and then introduce $(x/\sigma, y/\sigma, \sigma)$ as coordinates on the blowup.

When applying a condition given by a vector $v = (\alpha, \beta, \gamma)$ to a Weierstrass equation, it should be applied separately to the terms in the equation, treating each $a_j$ as a polynomial in $\sigma$. The corresponding condition will always take the form `$\sigma^k$ divides $a_j$'. For example, the term $a_3(\sigma)y$ has monomials of the form $x^{-1+1} y^{0+1} \sigma^{c+1}$ and the condition would imply $-\alpha + c \gamma \geq -1$ which gives a minimum divisibility for $a_3(\sigma)$.
We will now work our way through Tate’s algorithm, exhibiting the conditions in toric terms (to the greatest extent possible), and describing the corresponding blowups. The first branch of the algorithm we consider follows the sequence $I_2, I_3^{n_s}, I_4^{n_s}, \ldots$. At the $n^{th}$ step, the condition in Tate’s algorithm is that $\sigma^{[n+1/2]}$ divides $a_3$ and $a_4$, and $\sigma^n$ divides $a_6$. It is easy to see that this condition is reproduced by the vector $v_{2k-1} = (k, k, 1)$ when $n = 2k$, and that it cannot be expressed in toric terms when $n = 2k + 1$.

If we perform the blowups corresponding to $v_1, v_3, \ldots, v_{2k-1}$ in turn, we arrive at a coordinate chart involving $x_k = x/\sigma^k$, $y_k = y/\sigma^k$ and $\sigma$, and the Weierstrass equation has become

$$y_k^2 + a_1 x_k y_k + a_{3,k} y_k = x_k^3 \sigma^k + a_{2,k} x_k^2 + a_{4,k} x_k + a_{6,2k}.$$  \hspace{1cm} (7.1)

The exceptional divisor of the most recent blowup is described by $\sigma = 0$ within this coordinate chart; for generic coefficients, it is an irreducible nonsingular quadratic equation. If one additional power of $\sigma$ divides each of $a_{3,k}, a_{4,k}$ and $a_{6,2k}$, then the exceptional divisor consists of two lines (which will generally experience monodromy as the coefficients are varied); if in addition $\sigma^2$ divides $a_{6,2k}$ then there is a singular point at the origin, leading to the next blowup $v_{2k+1}$. Thus, in the generic case we will have found $2k - 2$ non-split exceptional divisors from the first $k - 1$ blowups, and a $k^{th}$ (split) divisor from $v_{2k-1}$, giving the case $A_{2k}^{n_s}$ (with predicted gauge group $Sp(k)$). When there is an additional power of $\sigma$ dividing those three coefficients, the last step also has 2 non-split exceptional divisors and we find the singularity type $A_{2k}^{n_s}$ (predicting unconventional gauge symmetry). Finally, when $\sigma^2$ divides $a_{6,2k}$, we must iterate the algorithm again.

The next branch of the algorithm we will follow is $I_2, I_3^{s}, I_4^{s}, \ldots$. The condition in Tate’s algorithm for $I_{2k+1}^{s}$ can be given torically by the vector $v_{2k} = (k, k + 1, 1)$. (Note that it is possible to give a uniform description of the vectors we have used so far as $v_n = ([n+1/2], [n+2/2], 1)$.) After blowing up $v_1, v_2, \ldots, v_{2k-1}$, the relevant coordinate chart is again given by $(x_k, y_k, \sigma)$, with equation again given by (7.1). The condition for $I_{2k+1}^{s}$ implies that there is one additional power of $\sigma$ dividing each of $a_2, a_4$ and $a_6$, so the exceptional divisor is described by

$$y_k^2 + a_1 x_k y_k = 0.$$  \hspace{1cm} (7.1)

We use odd subscripts for compatibility with a later branch in the algorithm.
The vector $v_{2k}$ now gives a toric blowup which in more conventional terms could be described as blowing up the locus $\{y_k = \sigma = 0\}$. One of the relevant coordinate charts is given by $(x_k, y_{k+1}, \sigma)$, in which the Weierstrass equation becomes

$$y_{k+1}^2 \sigma + a_1 x_k y_{k+1} + a_{3,k} y_{k+1} = x_k^3 \sigma^{k-1} + a_{2,1} x_k^2 + a_{4,k+1} x_k + a_{6,2k+1},$$

with the exceptional divisor for this blowup given by $\sigma = 0$ in this coordinate chart. If the singularities are no worse than this, the blowups terminate with an irreducible exceptional divisor. If they are worse, then following this branch of the algorithm we have that $\sigma$ divides $a_{3,k}$ and $a_{6,2k+1}$, and we should blow up $\{x_k = \sigma = 0\}$. Doing so leads back to (7.4) (with the next value of $k$), and the algorithm then repeats. It is easy to see that the total process has produced, in the case of $I_n^s$, precisely $n-1$ split exceptional divisors, so that the singularity type is $A_{n-1}^s$ and the predicted gauge group is $SU(n)$.

In order to describe the remaining branches of the algorithm efficiently, we will introduce a bit more toric language. Toric geometry teaches us that the combinatorics of a toric resolution of singularities are essentially determined by the convex hull of the vectors $v$ used to define the allowed monomials. Moreover, any integer vectors which lies in the interior of a codimension one face of that convex hull represents a toric divisor which does not meet the hypersurface defined by the vanishing of the generic allowed polynomial. Thus, we can describe the divisors in a toric blowup of the hypersurface by specifying the vectors $v$ which determine the convex hull (i.e. the vertices of the convex hull), as well as all other integer vectors which do not lie in the interior of codimension one faces (these will lie on edges of the convex hull). There are no points in the interior of the convex hull itself, since we are only resolving singularities which do not disturb the triviality of the canonical bundle of the space.

We introduce the vectors $w_k = (k+1, k+2, 2)$, and some additional vectors $u_1 = (4, 6, 3), u_2 = (3, 5, 2), u_3 = (6, 9, 4), u_4 = (5, 7, 3)$. Then the toric description of these resolutions can be summarized by the data in the following table.
Table 4: Toric Data

| Type     | Gauge Group | Vertices | Edge-Vectors |
|----------|-------------|----------|--------------|
| $I^*_0^{ns}$ | $G_2$       | $w_1$    | $v_2$        |
| $I^*_0^{ss}$ | $SO(7)$    | $v_3$, $w_1$ | $v_2$      |
| $I^*_{k-1}^{ns}$ | $SO(2k + 5)$ | $w_1$, $w_k$ | $v_2$, $v_{k+1}$, $w_2$, ..., $w_{k-1}$ |
| $I^*_{k-1}^{s}$ | $SO(2k + 6)$ | $v_{k+2}$, $w_1$, $w_k$ | $v_2$, $v_{k+1}$, $w_2$, ..., $w_{k-1}$ |
| $IV^{*ns}$ | $E_6$       | $u_1$, $u_2$, $v_4$ | $v_3$, $w_1$, $w_2$ |
| $IV^{*s}$  | $E_7$       | $w_3$, $u_3$ | $v_4$, $w_1$, $u_1$, $u_2$, $u_4$ |

Note that we have included data for the case of $SO(4k)$, $k > 2$, even though we were not able to give a toric description of the conditions for $SO(4k)$ gauge symmetry. In fact, by imposing the further divisibility condition `$\sigma^{2k+2}$ divides $a_6$', we can force the factorization of the corresponding polynomial in Tate’s algorithm. This does not give a general polynomial with the corresponding gauge symmetry, but it does give some polynomials with that gauge group. (This method does not work for $SO(8)$ – the corresponding condition would give $SO(7)$ instead).

There are several additional cases in Tate’s algorithm which are not included in this toric analysis – cases II, III, IV, and (as already indicated) case $I^*_2^{ns}$. In each of these cases, even if the corresponding condition on the monomials can be described by vectors $v_j$ those vectors will not have integer entries.

7.2. Coulomb Branch and Duality Chains

We now wish to apply Tate’s algorithm and the resolution of singularities to determine chains of F-theory models (and the corresponding chains of Calabi–Yau manifolds), related by extremal transitions, with the transitions mapping to the Higgs mechanism on the heterotic side. The starting point is the F-theory model given in [4] which describes the dual of the $E_8 \times E_8$ heterotic string with the instantons distributed as $(12 + n, 12 – n)$. We describe this model in terms of the corresponding Weierstrass equation with coefficients $f$ and $g$ given by (2.3). For our present purposes, we need to allow the more general form of the Weierstrass equation, which we can write as

$$
\sum c_{ijkl} x^{i+1} y^{j+1} z_1^{k+1} z_2^{\ell+1} = 0.
$$
The conditions which make this a Weierstrass equation can be written as

\[ 2(i + 1) + 3(j + 1) \leq 6, \quad (7.2) \]

while the conditions which restrict the degrees of the polynomials in \( z_1 \) and \( z_2 \) can be written as

\[
\begin{align*}
k + 1 & \leq 12 - 4(i + 1) - 6(j + 1) \\
\ell + 1 & \leq (12 - 4(i + 1) - 6(j + 1)) + n(6 - 2(i + 1) - 3(j + 1) - (k + 1)).
\end{align*} 
\quad (7.3)
\]

These three conditions can be recast in the vector form of the previous subsection, yielding the vectors

\[ e_5 = (-2, -3, 0, 0) \]
\[ e_6 = (-4, -6, -1, 0) \]
\[ e_7 = (-2n - 4, -3n - 6, -n, -1) \]

such that the corresponding conditions take the form \( v \cdot (i, j, k, \ell) \geq -1 \). The standard coordinate vectors \( e_1, e_2, e_3, e_4 \) in \( R^4 \) should be adjoined to these conditions. (They guarantee that the exponents in all the monomials are all nonnegative.)

The set of all monomials \( x^{i+1}y^{j+1}z_1^{k+1}z_2^{\ell+1} \) which satisfy the conditions

\[ e_\alpha \cdot (i, j, k, \ell) \geq -1, \quad \alpha = 1, \ldots, 7 \]

forms a so-called reflexive polyhedron [11], which is the condition needed to ensure that the generic hypersurface of this type is Calabi–Yau. This was checked explicitly by Candelas and Font [12], who found the additional vectors \( e_\alpha, \alpha > 7 \), which must be adjoined to the defining ones in order to completely describe the dual polyhedron of the polyhedron of monomials. We will not reproduce those points here, but we note that for comparison with [12], one must use the following change of basis:

\[ e_1 \leftrightarrow (0, 0, -1, 0) \]
\[ e_2 \leftrightarrow (0, 0, 0, -1) \]
\[ e_3 \leftrightarrow (0, -1, 2, 3) \]
\[ e_4 \leftrightarrow (-1, 0, 2, 3) \]

(The vectors on the right side are in the notation of [12].) Note that varying the value of \( n \) varies the “top” of the reflexive polyhedron in the terminology of [12].
If the coefficients in the polynomial are generic, then the gauge group is the one determined in \cite{9}, associated to the $E_8$ factor with $12 - n$ instantons. For special values of the coefficients, however, there will be additional singularities of the Calabi–Yau space along the curve $z_1 = 0$, which will correspond to gauge symmetry enhancement in the other $E_8$ factor. The corresponding polynomials describe the Coulomb branch for such a gauge group, related to the original one by an extremal transition.

Calculating these Coulomb branches is a fairly simple matter given all of the technology we have developed. For each choice of group in our first chain, we can give a toric description of the corresponding moduli space, other than the $SO(8)$ case. (In the $SO(12)$ case, the toric moduli space is only a subspace of the full moduli, but non-toric deformations can be expected to make up the difference.) This is done by adding to the reflexive polyhedron spanned by $\{e_\alpha\}$ certain vectors from the $u$‘s, $v$‘s and $w$‘s determined in the previous subsection. (We are implicitly adding a fourth component of 0 to each of those vectors, e.g., $v_n$ now denotes $([n+1], [n+2], 1, 0)$.) The results are summarized in table 5.
Table 5: Chains of Type IIA Duals

| $H$ | Points to Add | $a_H$ | $b_H$ |
|-----|---------------|-------|-------|
| $SU(2)$ | $v_1$ | 32 | 24 |
| $SU(3)$ | $v_1, v_2$ | 54 | 36 |
| $G_2$ | $v_2, w_1$ | 54 | 36 |
| $SU(4)$ | $v_1, v_2, v_3$ | 76 | 44 |
| $SO(7)$ | $v_2, v_3, w_1$ | 76 | 44 |
| $SU(5)$ | $v_1, v_2, v_3, v_4$ | 100 | 50 |
| $SO(10)$ | $v_2, v_3, v_4, w_1, w_2$ | 124 | 52 |
| $SO(11)$ | $v_2, v_4, w_1, w_2, w_3$ | 124 | 52 |
| $E_6$ | $v_3, v_4, w_1, w_2, u_1, u_2$ | 162 | 54 |
| $E_7$ | $v_4, w_1, w_3, u_1, u_2, u_3, u_4$ | 224 | 56 |
| $Sp(2)$ | $v_1, v_3$ | 64 | 40 |
| $SO(9)$ | $v_2, v_3, w_1, w_2$ | 96 | 48 |
| $F_4$ | $v_3, w_1, w_2, u_1$ | 96 | 48 |

(The data in the table corresponds to the “bottoms” of the reflexive polyhedra in [12]; the quantities $a_H, b_H$, defined in [12], are included to facilitate easy comparison of results. Note also that “pt’” of [12] is already included in our reflexive polyhedra, as $e_3$, so we have not included it in the “points to add.” This agrees with [12] in every particular, other than in the identification of the gauge groups for some of these spaces.)

8. Conclusion

We have seen how the enhanced gauge symmetry loci in the complex moduli space of F-theory (type IIA) compactifications on Calabi-Yau threefolds get mapped to the enhanced perturbative, non-perturbative and mixed gauge symmetries of the hypermultiplet moduli for heterotic compactifications on $K3$ ($K3 \times T^2$). On the F-theory side Tate’s algorithm proved very helpful in identifying enhanced gauge symmetry loci. Using this detailed map we have identified the Calabi-Yau threefolds dual to the various possible Coulomb branches.

10 Candelas and Font attempted to assign a simply-laced gauge group to each branch of the moduli space. Remarkably, there is always a choice of such a group which produces the correct dimension of the moduli space. However, the methods of this paper indicate that the actual gauge groups are non-simply-laced in several instances.
of heterotic compactification on $K3 \times T^2$. This provides us with a systematic method for mapping out the web of type IIA/heterotic dualities in $d = 4$, $N = 2$ theories.

As far as the matter representations are concerned, more work needs to be done to verify the structure of the matter which follows from the duality. In principle this should be possible to study: Similar cases have been analyzed recently in [19] by studying the D-branes of type IIA on Calabi-Yau manifolds. In this paper, as far as the matter representations go, we have limited ourselves to what matter a given singularity must encode in order to be compatible with duality. In many cases (especially the simply laced cases) we have found evidence that the matter is localized at the zeroes of certain polynomials, extending some of the observations in [1].

One can ask whether one can map the F-theory moduli to the heterotic moduli in a more detailed fashion, even away from the enhanced gauge symmetry points. In particular it is natural to wonder how the polynomial degrees of freedom we have found on the F-theory side map to the moduli of bundles on the heterotic side. Progress in this direction, as well as a heterotic explanation of the localization of matter at the zeroes of the polynomials that we have found, has been recently made [32]. One can also ask whether we can map the $N = 2$, $d = 4$ Coulomb phase on the heterotic side in a more detailed way to the Coulomb branch on the type IIA side. Given that we have identified the relevant Calabi-Yau manifold, one would simply have to study the Kahler moduli space of this manifold. However, it is convenient to use mirror symmetry to find the relevant Calabi-Yau in the type IIB setup: This can be done easily as our Calabi-Yau manifolds are nicely characterized by toric data and thus Batyrev’s construction easily applies to identify the mirror [31]. We would thus study the complex structure of this mirror Calabi-Yau and identify it with the Coulomb branch of the heterotic side. In particular it should be possible to go to the weak coupling limit of the heterotic string, as in [33][34], and find the field theory analogues. Many of these results will be new even as far as field theory is concerned, for example $F_4$ with matter. Note that given the dictionary we have developed in identifying the various Coulomb branches, and given mirror symmetry, we have thus managed to derive the dictionary for a large number of cases for the Coulomb branch. In fact, it is quite suggestive that in the description of $N = 2$ Yang-Mills theories with non-simply laced gauge groups [35], groups appear with a correspondence which is very similar to what we have found with the outer automorphisms (3.1) of Dynkin diagrams. In [35] this corresponds to exchanging the long and short roots.
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