Renormalon-based resummation for QCD observables

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A method of evaluation of spacelike QCD observables \( \mathcal{D}(Q^2) \) is presented, motivated by the renormalon structure of these quantities.

I. INTRODUCTION

The theory of renormalons, and its use in the evaluation of QCD observables, has been known for some time \[1\], and it has remained an area of active interest; for some new ideas that have emerged in this area more recently, see Refs. \[2\] \[4\].

Borel transforms of (the leading-twist part of) the spacelike observables have specific renormalon structures, which include poles, cuts and branching points in the Borel plane. On the other hand, in the large-momentum transfers \( Q \to \infty \), some new ideas that have emerged in this area more recently, see Refs. \[2\] \[4\], and it has remained an area of active interest; for some new ideas that have emerged in this area more recently, see Refs. \[2\] \[4\].

In this presentation we summarize the method of Ref. \[5\] where \( \tilde{D} \) is introduced, which in principle contains the entire information on all the expansion coefficients of the original observable \( \mathcal{D}(Q^2) \), but is renormalization scale independent only at the one-loop level, and agrees with \( \mathcal{D}(Q^2) \) at one-loop level. Motivated by a specific renormalization scale dependence of the Borel transform \( B[\mathcal{D}](b) \), a large-\( \beta_0 \) type of ansatz is made for \( B[\mathcal{D}](b) \). This leads to the correct ("dressed") structure of the Borel transform \( B[\mathcal{D}](b) \) of the original observable. Subsequently, in Sec. \[III\] a Neubert-type of the characteristic (distribution) function \( G_D(t) \) for the original \( \mathcal{D}(Q^2) \) is obtained from the simple Borel transform \( B[\mathcal{D}](b) \). This renormalon-based characteristic function permits evaluation (resummation) of the original observable \( \mathcal{D}(Q^2) \). As a specific illustration, the method is applied to the evaluation of the (leading-twist) massless Adler function and the related (timelike) decay ratio of the \( \tau \) lepton semihadronic decays. At the end, the presented results are summarized.

II. THE METHOD

The perturbation expansion of the considered spacelike observable is

\[
\mathcal{D}(Q^2)_{pt} = \sum_{n \geq 0} d_n(\kappa) a(\kappa Q^2)^{n+1},
\]

where \( \mu^2 \equiv \kappa Q^2 \) is the renormalization scale, and \( a(\mu^2) \equiv \alpha_s(\mu^2)/\pi \). The coupling \( a(\mu^2) \) satisfies the renormalization group equation (RGE)

\[
\frac{da(\mu^2)}{d\ln \mu^2} = -\beta_0 a(\mu^2)^2 - \beta_1 a(\mu^2)^3 - \beta_2 a(\mu^2)^4 - \ldots
\]

We can reorganize the power expansion (1) into expansion in the logarithmic derivatives where

\[
\tilde{d}_{n+1}(\mu^2) = \frac{(-1)^n}{\beta_0^n n!} \left( \frac{d}{d\ln \mu^2} \right)^n a(\mu^2),
\]

(where \( n = 0, 1, \ldots \)), which coincide with the powers \( a(\mu^2)^{n+1} \) only at the one-loop level. We thus obtain the expansion

\[
\mathcal{D}(Q^2)_{pt} = \sum_{n \geq 0} \tilde{d}_n(\kappa) \tilde{d}_{n+1}(\kappa Q^2).
\]

The new expansion coefficients \( \tilde{d}_n \) are unique functions of the coefficients \( d_j (j \leq n) \), and contain all the information about them; these relations can also be inverted, and have similar structure

\[
d_n = \sum_{s=0}^{n-1} k_s (n+1-s) \tilde{d}_{n-s},
\]

where \( n = 1, 2, \ldots \), and \( k_0(m) = 0 \). An auxiliary quantity \( \tilde{D} \) can be introduced, which is the power expansion with the coefficients \( \tilde{d}_n \)

\[
\tilde{D}(Q^2; \kappa) = \sum_{n \geq 0} \tilde{d}_n(\kappa) a(\kappa Q^2)^{n+1}.
\]

It has some renormalization scale (\( \kappa \))-dependence when going beyond the one-loop level. The “reorganized” coefficients \( \tilde{d}_n(\kappa) \) have a significantly simpler (one-loop-type) renormalization scale dependence than the original coefficients \( d_n \)

\[
\frac{d}{d\ln \kappa} \tilde{d}_n(\kappa) = n \beta_0 \tilde{d}_{n-1}(\kappa) \quad (n \geq 1),
\]

and \( \tilde{d}_0 \) is \( \kappa \)-independent. As a consequence, the Borel transform of the auxiliary quantity \( \tilde{D} \)

\[
B[\tilde{D}](u, \kappa) = \sum_{n=0}^{\infty} \frac{\tilde{d}_n(\kappa)}{n! \beta_0^n} u^n
\]

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has the simple one-loop-type (or: large-\(\beta_0\)-type) renormalization scale dependence

\[
B[\tilde{D}](u; \kappa) = \kappa^w B[\tilde{D}](u).
\]

This suggests that the Borel transform \(B[\tilde{D}](u)\) has a one-loop (large-\(\beta_0\)) type renormalon structure (poles):

\[
B[\tilde{D}](u) \sim 1/(p \pm u)^{k}, \ln(1 \pm u/p),
\]

where \(p\) and \(k\) are positive integers. Such ansätze for \(B[\tilde{D}](u)\) will be used to generate the coefficients \(d_n\), and thus via Eqs. (3) the coefficients \(d_n\) of the power expansion of the full \(D(Q^2)\) observable. However, an important question is whether these (large-\(\beta_0\))-type ansätze for \(B[\tilde{D}](u)\) give us correctly behaved \(d_n\) coefficients of \(D(Q^2)\), i.e., whether the Borel transform \(B[\tilde{D}](u; \kappa)\) has the (full-loop) renormalon structure expected theoretically. It can be shown numerically that this is really the case, and the reader is referred for details to Ref. [5].

### III. APPLICATION TO THE MASSLESS ADLER FUNCTION

The Adler function \(D(Q^2)\) is the logarithmic derivative of the quark current-current correlator. In the massless limit, the vector and axial vector channels coincide, and the perturbation expansion (4) of this quantity is known exactly up to order \(a^4\) [7,9]. Further, the leading-\(\beta_0\) (LB) parts \(\tilde{d}^{(\text{LB})}_{\nu} = d^{(\text{LB})}_n\) of the coefficients are known to all orders \(n\), and thus the LB Borel transform \(B[\tilde{D}](u)^{(\text{LB})}\) of the (massless) Adler function is known [10]: it has simple pole \((k = 1)\) at \(u = 2\) [the leading infrared (IR) renormalon], and double poles \((k = 2)\) at \(u = 3, 4, \ldots\) (IR renormalons) and at \(u = -1, -2, \ldots\) (ultraviolet (UV) renormalons).

#### A. The Borel transform of \(\tilde{D}\) of Adler

The first ansatz for the Borel \(B[\tilde{D}](u)\) includes the first two IR renormalon poles, and the first ultraviolet (UV) pole \(u = -1\):

\[
B[\tilde{D}](u)^{(4P)} = \exp \left( K u \right) \pi \left\{ \frac{1}{1 - u} + \tilde{\alpha}(-1) \ln \left( 1 - \frac{u}{2} \right) \right\}
+ \frac{\tilde{d}^{\text{IR}}_{3,2}}{(3 - u)^2} + \frac{\tilde{d}^{\text{UV}}_{1/2}}{(1 + u)^2},
\]

which has four parameters: \(K, \tilde{d}^{\text{IR}}_{3,2}, \tilde{d}^{\text{IR}}_{1/2}\) and \(\tilde{d}^{\text{UV}}_{1/2}\). The values of these four parameters can be determined by requiring that the values of the first four (exactly known) perturbation expansion coefficients \(d_n\) \((n = 0, 1, 2, 3)\) be correctly reproduced.

In practice, this ansatz is made in a specific renormalization scheme, the Lambert MiniMOM (LMM) [33], because in that scheme the IR-safe (and holomorphic) QCD coupling was constructed \(a(Q^2) \rightarrow A(Q^2)\) [16], which at high \(Q^2\) practically coincides with the underlying pQCD coupling \(a(Q^2)\) (in LMM), reproduces the correct semihadronic \(\pi\)-decay ratio \(r_\tau \approx 0.20\), and behaves as \(A(Q^2) \sim Q^2\) when \(Q^2 \rightarrow 0\) as suggested by large-volume lattice data on gluon and ghost propagator dressing functions in the Landau gauge [13-15]. This QCD variant is called \(\Delta\)QCD, because the spectral (discontinuity) function \(\rho_A(\sigma) \equiv \ln A(Q^2) = -\sigma - i\epsilon\) in the low-\(\sigma\) regime \((0 \leq \sigma \lesssim 1 \text{ GeV}^2)\) is parametrized by three Dirac-delta functions, while \(\rho_A(\sigma)\) for higher \(\sigma\) coincides with its underlying pQCD version \(\rho_0(\sigma)\). The reason that the Borel transform (11) is made in a renormalization scheme where a known holomorphic IR-safe QCD coupling \(A(Q^2)\) is available, will become clear in the next Section [11,13].

The parameter \(\tilde{\alpha}\), appearing at the \(u = 2\) “pole term with \(k = 0\) multiplicity” in Eq. (11), is not independent, because of the knowledge of the subleading part of the \(D = 4\) Wilson coefficient (we refer for details to [2,9]). In the LMM scheme, the obtained value is \(\tilde{\alpha}_{\text{LMM}} = -0.14 \pm 0.12\).

For comparison, the Adler function is constructed also in another renormalization scheme, called Lambert scheme: it has a given value of the \(c_2\) parameter [55], and \(c_n = c_2^{n-1}/c_2^{n-2}\) for \(n \geq 3\). The \(c_2 = -4.9\) Lambert scheme was used in the construction of the \(2\sigma\) QCD model [17] which has a holomorphic and IR-safe coupling. In this \(c_2 = -4.9\) Lambert scheme, we can now require that the first four coefficients are the exact ones (in that scheme), and that \(d_4\) coefficient corresponds to that obtained in the LMM case; therefore, now five parameters can be fixed, and the ansatz in the Lambert scheme is

\[
B[\tilde{D}](u)^{(5P)} = \exp \left( K u \right) \pi \left\{ \frac{1}{2 - u} + \tilde{\alpha}(-1) \ln \left( 1 - \frac{u}{2} \right) \right\}
+ \frac{\tilde{d}^{\text{IR}}_{3,2}}{(3 - u)^2} + \frac{\tilde{d}^{\text{IR}}_{1/2}}{(3 - u)} + \frac{\tilde{d}^{\text{UV}}_{1/2}}{(1 + u)^2}.
\]

We are interested in the Adler function in this \(c_2 = -4.9\) Lambert scheme, because in this scheme an IR-safe (and holomorphic) QCD coupling \(A(Q^2)\) was constructed [17], which at high \(Q^2\) practically coincides with the underlying pQCD coupling \(a(Q^2)\) and reproduces the correct \(r_\tau \approx 0.20\); however, at \(Q^2 \rightarrow 0\) the coupling is nonzero, \(0 < A(0) < \infty\), in contrast with the aforementioned
3δ AQCD coupling[16]. This QCD variant is called 2δ AQCD, because its spectral function \( \rho_\Delta(\sigma) \equiv \text{Im} A(Q^2 = -\sigma - i\epsilon) \) in the low-\( \sigma \) regime is parametrized by two Dirac-delta functions.

For comparison, the mentioned five-parameter Borel transform can also be applied in the \( \overline{\text{MS}} \) scheme (five-loop, with \( c_n = 0 \) for \( n \geq 5 \), in the same way, and the parameters are fixed.

The results are given in Table. The \( \tilde{\alpha} \) parameters are:
\[
\tilde{\alpha}_{\text{LMM}} = -0.14 \pm 0.12; \quad \tilde{\alpha}_{\text{Lamb.}} = -0.10 \pm 0.14; \quad \tilde{\alpha}_{\text{MS}} = -0.255 \pm 0.010.
\]

### B. Characteristic function of the Adler function

The characteristic (or: distribution) function \( D(tQ^2) \) of a spacelike observable \( D(Q^2) \) is usually defined as such a function of \( t > 0 \) that

\[
D_{\text{res}}(Q^2) = \int_{t_0}^{+\infty} \frac{dt}{t} F_D(t) a(tQ^2)
\]

represents the (leading-twist) resummation of \( D(Q^2) \). Taylor expansion of the coupling \( a(tQ^2) \) in \( \ln(tQ^2) \) around \( \ln Q^2 \) then implies that the moments of \( F_D(t) \) are precisely the coefficients \( \tilde{a}_n \) appearing in the auxiliary quantity \( D(tQ^2) \)

\[
(-\beta_0)^n \int_0^{+\infty} \frac{dt}{t} F_D(t) \ln^n \left( \frac{t}{\kappa} \right) = \tilde{a}_n(\kappa),
\]

where \( n = 0, 1, \ldots \) Using these relations, with \( \kappa = 1 \), and the expansion [8] in powers of \( u \) for the Borel transform \( B[D](u) \), one obtains

\[
B[D](u) = \int_0^{+\infty} \frac{dt}{t} F_D(t) t^{-u}
\]

Hence \( B[D](u) \) is the Mellin transform of \( F_D(t) \). The inverse Mellin then gives the characteristic function \( F_D(t) \) in terms of \( B[D](u) \) (cf. [18] for application in the large-\( \beta_0 \) (one-loop) context)

\[
F_D(t) = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} du \ B[D](u) t^u,
\]

For the Borel transforms [11] and [13], this inverse Mellin transform can be performed explicitly [5], and the result has the form

\[
D(Q^2)_{\text{res}} = \int_0^1 \frac{dt}{t} G_D^{(-)}(t) a(t e^{-K} Q^2) + \int_1^{+\infty} \frac{dt}{t} G_D^{(+)}(t) a(t e^{-K} Q^2) + \int_0^1 \frac{dt}{t} G_D^{(\text{SL})}(t) \left[ a(t e^{-K} Q^2) - a(e^{-K} Q^2) \right],
\]

where the (characteristic) functions \( G_D^{(-)}(t) \) and \( G_D^{(\text{SL})}(t) \) involve the parameters of the mentioned Borel transforms [11] and [13], and powers of \( t \) and \( \ln t \), cf. [5].

### C. Numerical evaluation

If the running coupling \( a(Q^2) \) is holomorphic (analytic) in the complex \( Q^2 \) plane excluding the timelike axis \( [a(Q^2) \to A(Q^2)] \) [36], it is IR-safe (finite when \( Q^2 \to 0 \)), and thus the integration Eq. [18] can be performed. The problem of analyticity of QCD running couplings was addressed systematically already in the nineties [19, 21], with a QCD variant called Analytic Perturbation Theory (APT) (for extensions and reviews, cf. [22, 23]). Several versions of QCD holomorphic couplings have been applied in evaluations of various QCD quantities [24, 27, 37].

Two recently constructed QCD variants with holomorphic couplings \( A(Q^2) \), the aforementioned 2δ AQCD [17] and 3δ AQCD [16], fulfill several phenomenological constraints of the low-\( Q^2 \) QCD \((|Q^2| \lesssim 1 \text{ GeV}^2)\) as mentioned earlier. The integrals in Eq. [18] can be performed in both variants \((a \to A)\) without ambiguity because of the IR-safety of such couplings.

On the other hand, in pQCD in the usual schemes such as \( \overline{\text{MS}} \), the running coupling \( a(Q^2) \) is not holomorphic and not IR safe; it has Landau singularities for positive small values of \( Q^2 \), which makes the evaluation of the integrals in [18] ambiguous. To avoid this ambiguity, one may take the generalized principal value of these integrals, i.e., the integration is slightly shifted above the real positive axis, \( a(t e^{-K} Q^2) \to a(t e^{-K} Q^2 + i\epsilon) \), and the real part of the result is taken. Taking instead the imaginary part and dividing by \( \pi \) \([\pm (1/\pi) \text{Im} \ldots\] gives us a measure of ambiguity of such a result.

The results of this evaluation, for positive values of \( Q^2 \), are presented in Fig. 1. When the two holomorphic ver-

![FIG. 1: The radiative Adler function resummed with the characteristic function according to Eq. (18) (where \( a \to A \)), as a function of \( Q \equiv \sqrt{Q^2} \), for positive \( Q^2 \): in 3δ AQCD (in the LMM renormalization scheme), and 2δ AQCD (in the Lambert \( c_2 = -4.9 \) renormalization scheme). Included for comparison is the resummed pQCD Adler function \( D(Q^2)_{\text{pQCD}} \), in the (five-loop) \( \overline{\text{MS}} \) scheme, using modification of Eq. (18) as described in the text. All the three frameworks correspond to \( \alpha_s(M_{Z}^2; \overline{\text{MS}}) = 0.1185 \).](https://example.com/fig1.png)
also note that the two holomorphic results in the Figure start differing at \( Q < 0.5 \text{ GeV} \); this is so because the 2\( \delta \) AQCD coupling \( A(Q^2) \) tends to a positive finite value when \( Q^2 \to 0 \), and the 3\( \delta \) AQCD coupling tends to zero (as \( \sim Q^2 \)) when \( Q^2 \to 0 \).

### 4. SUMMARY

- A method of evaluation of spacelike QCD observables \( D(Q^2) \) was developed, motivated by the renormalon structure of these quantities.
- A related auxiliary quantity \( \bar{D}(Q^2) \) was introduced, which is renomalization scale independent only at the one-loop level, and agrees with \( D(Q^2) \) at one-loop level.
- A large-\( \beta_0 \)-type renormalon-motivated ansatz is made for the Borel transform \( B[D](u) \) of \( D(Q^2) \). This leads to a correctly “dressed” Borel transform \( B[D](u) \) of the considered observable \( D(Q^2) \).
- Subsequently, a Neubert-type characteristic (distribution) function, \( G_D^{(\pm)}(t) \) and \( G_D^{SL}(t) \), is obtained for the considered observable \( D(Q^2) \) as the inverse Mellin transform of the Borel transform of \( \bar{D}(Q^2) \).
- As an illustration, the method is applied to the massless Adler function and the related decay ratio of the \( \tau \) lepton semihadronic decays.

| scheme | \( \tilde{K} \) | \( \tilde{d}_{2,1}^{IR} \) | \( \tilde{d}_{3,2}^{IR} \) | \( \tilde{d}_{2,1}^{IR} \) | \( \tilde{d}_{2,2}^{UV} \) |
|--------|--------------|----------------|----------------|----------------|----------------|
| LMM    | -0.7704536  | -1.830666      | 11.0498        | -              | 0.00588513    |
| Lamb.  | 0.2228125   | 4.745825       | -1.04837       | -5.89714       | 0.0276003     |
| NS     | 0.5190386   | 1.108265       | -0.481538      | -0.511642      | -0.0117704    |

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[34] LMM [16] is the lattice MiniMOM (MM) scheme [11, 12] rescaled to the conventional MS scale, i.e., $\Lambda_{MM} \rightarrow \Lambda$, i.e., in the leading order it coincides with the MS scheme, but has different scheme ($\beta$) coefficients $\beta_n (n \geq 2)$.

[35] We recall that the scheme parameters are: $c_n \equiv \beta_n/\beta_0$, for $n \geq 2$. For convenience, the leading scheme parameter A here (and in the LMM scheme) is such that the scaling is in the MS convention; i.e., the scheme is characterized only by the parameters $c_2, c_3, \ldots$.

[36] This means holomorphic in the generalized spacelike regime, $Q^2 \in C \setminus (-\infty, -M_{th}^2]$, where $M_{th} \lesssim 0.1$ GeV is a threshold scale comparable with the lightest meson mass.

[37] Yet another approach is to apply the requirement of the holomorphic behavior directly to QCD spacelike observables, cf. Refs. [25] [33].