Einstein Manifolds in Ashtekar Variables:

Explicit Examples

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Abstract

We show that all solutions to the vacuum Einstein field equations may be mapped to instanton configurations of the Ashtekar variables. These solutions are characterized by properties of the moduli space of the instantons. We exhibit explicit forms of these configurations for several well-known solutions, and indicate a systematic way to get new ones. Some interesting examples of these new solutions are described.
1. Introduction

We present some solutions to the vacuum Einstein equations based upon the Ashtekar
variables\[1\]. These variables are convenient for implementing the canonical description
of the Einstein field equations. The variables are $SO(3)$ gauge fields for Riemannian
manifolds, and we shall show that the classical solutions of the field equations correspond
to the instanton sector of the gauge fields. Not every instanton configuration can be used
to define Einstein manifolds. In this note, we present the conditions under which this
definition will be possible, and work out some explicit examples which demonstrate the
utility of such an approach.

Ashtekar’s variables\[1\] can be obtained from the $3 + 1$ decomposition of the Einstein-
Hilbert action,

$$
\frac{1}{16\pi G} \int 4R \sqrt{g} d^4x
$$

through a series of canonical transformations\[2\]. The canonical pair of variables consists
of the complex Ashtekar potentials

$$
A_{ia} = iK_{ia} - \frac{1}{2} \epsilon_{abc} \omega_{i}^{bc}
$$

and the densitized triad of weight 1,

$$
\tilde{\sigma}^{ia} = \sqrt{3} g \sigma^{ia}
$$

The canonical variables obey the Poisson bracket relations

$$
\{ A_{ia}(\vec{x}), \tilde{\sigma}^{jb}(\vec{y}) \}_{PB} = \frac{i}{2} \delta^{j}_{a} \delta^{3}(\vec{x} - \vec{y})
$$

$$
\{ A_{ia}(\vec{x}), A_{jb}(\vec{y}) \}_{PB} = 0
$$

$$
\{ \tilde{\sigma}^{ia}(\vec{x}), \tilde{\sigma}^{jb}(\vec{y}) \}_{PB} = 0
$$

for all space-time points $\vec{x}, \vec{y}$ on the constant-$x^0$ 3-dimensional hypersurface $M^3$. In the
above, a factor of $16\pi G$ has been suppressed on the right hand side. For concreteness, we
suppose that $M^3$ carries the signature $+++$ . Our convention will be such that, unless
otherwise stated, lower case Latin indices run from 1 to 3, while upper case and Greek
indices run from 0 to 3. In the above, $\omega$ is the torsionless spin connection compatible with
the triads:

$$
d\sigma^{a} + \omega^{a}_{b} \wedge \sigma^{b} = 0
$$
and modulo the constraint which generates triad rotations,

\[ K_{ia} = \sigma^j a K_{ij} \]  \hspace{1cm} (1.6)

where \( K_{ij} \) is the extrinsic curvature.

In terms of the Ashtekar variables, the constraints generating local \( SO(3) \) gauge transformations, or triad rotations, which leave the spatial metric

\[ g^{ij} = \sigma^i a \sigma^j a \]  \hspace{1cm} (1.7)

invariant can be written in the form Gauss’ law:

\[ G_a \equiv \frac{2}{i} (D_i \tilde{\sigma}^i)_a \simeq 0 \]  \hspace{1cm} (1.8)

Ashtekar showed \[1\] that, modulo Gauss law constraints, the usual “supermomentum” and “superhamiltonian” constraints of ADM\[3\] achieve remarkable simplifications when expressed in terms of the new variables. Indeed, the “supermomentum” constraint

\[ -2\pi_i^j |_j \simeq 0 \]  \hspace{1cm} (1.9)

is proportional to

\[ \mathcal{H}_i \equiv \frac{2}{i} \tilde{\sigma}^i a F_{ija} \simeq 0 \]  \hspace{1cm} (1.10)

while the “superhamiltonian” constraint

\[ \frac{\sqrt{g}}{16\pi G} \left( \text{tr} K^2 - 3 R - (\text{tr} K)^2 \right) \simeq 0 \]  \hspace{1cm} (1.11)

is equivalent to

\[ \epsilon_{abc} \tilde{\sigma}^i a \tilde{\sigma}^j b F_{ija}^c \simeq 0 \]  \hspace{1cm} (1.12)

The quantity \( K \) is the extrinsic curvature given by

\[ K^{ij} = -\frac{16\pi G}{\sqrt{g}} \left( \pi^{ij} - \frac{1}{2} \pi g^{ij} \right) \]  \hspace{1cm} (1.13)

The presence of a cosmological term

\[ S_C = \frac{2\lambda}{16\pi G} \int \sqrt{g} d^4 x \]  \hspace{1cm} (1.14)
in the action modifies the usual “superhamiltonian” constraint in that one will need to add a new term:

\[
\frac{2\lambda}{16\pi G} \sqrt{3g}
\]

to the left hand side of (1.11), and \( H \) in (1.12) becomes

\[
H = \epsilon_{abc} \tilde{\sigma}^i a^i j^b F_{ij}^c + \Lambda \epsilon_{abc} \epsilon_{ijk} \tilde{\sigma}^i a^j b^k c
\]

with \( \Lambda = \lambda/3 \).

In the case of metrics with Euclidean signature, one should drop all factors of \( i \) in (1.2) and (1.4), and may further assume that the Ashtekar variables are all real. The “superhamiltonian” constraint in the ADM formalism for Euclidean signature becomes

\[
\mathcal{H}_E = \frac{\sqrt{g}}{16\pi G} (-\text{tr} K^2 - (3R) + (\text{tr} K)^2 + 2\lambda) \approx 0
\]

(1.16)

Modulo Gauss law constraints, \( \mathcal{H}_E \) is still proportional to \( \epsilon_{abc} \tilde{\sigma}^j b^i F_{ija} \). A short computation gives

\[
-16\pi G \mathcal{H}_{Ej} = 2\sqrt{g} (K^i_j |i - K^i_i |j) = 2\tilde{\sigma}^i a F_{ija}
\]

(1.17)

2. Classification Scheme for Solution Space of Constraints

In this section, we exhibit a classification scheme of the solutions of Ashtekar’s constraints, and discuss its connection to results appearing in the literature.

It is known that all solutions of the Einstein field equations in 4D can be classified according to the canonical forms of the Riemann-Christoffel curvature tensor. Such a scheme was first given by Petrov[4] and then further extended by Penrose[5] in the context of spinors and null tetrads.

In the ADM formalism, the “supermomentum” and “superhamiltonian” constraints are projections of the Einstein field equations tangentially and normally to the three-dimensional hypersurface \( \mathcal{M}^3 \), on which the initial data compatible with the constraints is specified. The solutions to the constraints when stacked up according to their \( x^0 \)-evolution by the Hamiltonian are then the solutions to the field equations. The natural question to ask is if a similar classification scheme can be set up in the phase space defined by the Ashtekar variables. What we will do in this section is present one such scheme.

In the ADM formalism, the metric is assumed to be non-degenerate. Ashtekar’s formulation allows for both degenerate and non-degenerate metrics. This is because the
relevant constraints (1.8), (1.10), and (1.15) do not involve the inverse of the momenta \( \tilde{\sigma}^{ia} \). For non-degenerate metrics, the magnetic field of the Ashtekar connection (1.2),

\[
B^{ia} = \frac{1}{2} \varepsilon^{ijk} F_{jk}^a,
\]

can be expanded in terms of the densitized triad \( \tilde{\sigma}^{ia} \), and the most general solution to the three-dimensional diffeomorphism constraint (1.10) is

\[
B^{ia} = \tilde{\sigma}^{ib} S^a_b \tag{2.1}
\]

with \( S \) being a symmetric \( 3 \times 3 \) \( \tilde{\sigma} \)-dependent matrix. Observe that (2.1) is a solution of the diffeomorphism constraint even for the case of degenerate metrics.

The “superhamiltonian” constraint (1.13) becomes an algebraic relation:

\[
(\det \tilde{\sigma}) (S^a_a + \lambda) = 0 \tag{2.2}
\]

which has the solution

\[
\text{tr} S = S^a_a = -\lambda \tag{2.3}
\]

for non-degenerate metrics. (2.2) will not fix \( \text{tr} S \) when the metric is degenerate. It is intriguing to note the apparent shift in the specification of the dynamical degree of freedom from \( \det \tilde{\sigma} \) to \( \text{tr} S \) in the case of degenerate metrics. Metrics which become degenerate at certain points in space-time may well be important in topology changing situations in classical and quantum gravity.

For space-times with Lorentzian signature, the Ashtekar gauge potential \( A \) is complex, and hence so is \( S \). Complex symmetric matrices can be classified according to the number of independent eigenvectors and eigenvalues, according Table 1.

Since \( S_{ab} \) is gauge invariant, one may classify the matrix in terms of the roots of its characteristic polynomial. These can in turn be expressed by:

\[
C_1 = - \text{tr} S = \lambda \\
C_2 = \frac{1}{2} \left( C_1 \text{tr} S + \text{tr} S^2 \right) \tag{2.4} \\
C_3 = - \frac{1}{3} \left( C_2 \text{tr} S + C_1 \text{tr} S^2 + \text{tr} S^3 \right)
\]

The Bianchi identity for the magnetic field associated with \( A \) further implies the consistency condition:

\[
[D_i (S \cdot \tilde{\sigma}^i)]_a = 0 \tag{2.5}
\]

or

\[
\tilde{\sigma}^{ia} (D_i S)_{ab} = 0 \tag{2.6}
\]
when one takes into account (1.8).

There have been attempts to obtain metric-independent gravity theories by expressing \( \bar{\sigma}^i \) in terms of \( B^i \) [6]. However, in view of the displayed classification scheme, this is not the most natural way to proceed. For instance, the scheme of [6] will not work for the simple \( F = 0 \) sector, which has \( S = 0 \) for finite momenta. We shall elaborate on the significance of the cases when \( S \) is degenerate later on. When \( S \) is invertible, we do obtain the results of [6], with

\[
\bar{\sigma}^{ia} = (S^a_b)^{-1} B^{ib}
\]

and the constraints

\[
\begin{align*}
B^{ia} (D_i S)_{ab} &= 0 \\
S^{-1}_{ab} &= S^{-1}_{ba} \\
(\det B) \left[ (\tr S^{-1})^2 - \tr ((S^{-1})^2) + 2 \lambda \det S^{-1} \right] &= 0
\end{align*}
\]

(2.7)

As noted by the authors of [6], these are seven equations on the nine complex components of \( S^{-1} \), and the solutions should give the two unconstrained field degrees of freedom associated with general relativity in 4D. When \( S \) is degenerate, though, as we will show, there could arise phases with fewer degrees of freedom.

It should be emphasized that, as in the Petrov classification scheme, types II, III and N do not occur for space-times with Euclidean signatures. This is because the corresponding Ashtekar variables are all real, so that \( S \) is real and symmetric, and there are always three distinct eigenvectors.

For the case when there is only one eigenvalue, the three roots (2.4) are not independent:

\[
\tr S^3 = (\tr S)(\tr S^2) = \frac{1}{9}(\tr S)^3 = -\frac{\lambda^3}{9}
\]

(2.8)

When two of the eigenvalues are the same, the relationship among the roots is

\[
6 \left[ \tr S^3 + \lambda \tr S^2 - \frac{2\lambda^3}{9} \right] = \left[ \tr S^2 - \frac{\lambda^2}{3} \right]^3
\]

(2.9)

The initial value data thus falls into distinct classes with strikingly distinct properties. For instance, type I has three \( \vec{x} \)-dependent eigenvalues for \( S \), whose sum is restricted to \(-\lambda\), while for type O one has only one \( \vec{x} \)-independent eigenvalue \(-\lambda/3\). This mismatch in the allowable fluctuations is highly suggestive of distinct phases in the theory. For example, we
may show[7] that type O ($S_{ab} = -(\lambda/3)\delta_{ab}$) can be identified with an unbroken topological quantum field theory (TQFT), describing a topological phase in quantum gravity.

The classification scheme described so far becomes equivalent to the usual Petrov classification for non-degenerate metrics. In this case,

$$S_{ab} = R_{\tilde{0}a\tilde{b}} - R_{0a0b} \quad (2.10)$$

### 3. Equations of Motion and Anti-Instantons

In this section, we exhibit the manifestly covariant equations of motion for the Ashtekar variables and discuss the implications. We choose to work explicitly with metrics of Euclidean signature and use $SO(3)$ instead of $SU(2)$ gauge potentials, but we will indicate the necessary modifications for metrics of Lorentzian signature. It will become clear as we go along, that there are Einstein manifolds that cannot be described globally by $SU(2)$ Ashtekar potentials, but can be described by $SO(3)$ connections. This has to do with the fact that not all $SO(3)$ connections can be lifted to be $SU(2)$ connections with integer second Chern class, but all $SU(2)$ connections can be thought of as $SO(3)$ connections with the first Pontrjagin class being a multiple of four. We will furnish examples of such manifolds below.

In working with metrics of Euclidean signature, we should drop all $i$'s, starting with Eqn. (1.1). In the spatial gauge, we have

$$e_{A\mu} = \begin{pmatrix} N & 0 \\ e_{aj}N^j & e_{ai} \end{pmatrix} \quad (3.1)$$

Eqn. (3.1) in no way compromises the values of the lapse and shift functions $(N, N^j)$, and is compatible with the ADM decomposition of the metric:

$$ds^2 = \mp e_0^2 + e_1^2 + e_2^2 + e_3^2$$

$$= \mp N^2(dx^0)^2 + g_{ij}(dx^1 + N^i dx^0)(dx^j + N^j dx^0) \quad (3.2)$$

where $e_A$ is the 1-form $e_{A\mu} dx^\mu$ and the $+(-)$ sign is to be used for metrics of Euclidean (Lorentzian) signature. On the constant $x^0$-hypersurface $\mathcal{M}^3$, $e_0$ vanishes, and we may write

$$F_a = e^0 \wedge T_{ab} e^b + \frac{1}{2} e_{a}^b \epsilon_{bcd} e^c \wedge e^d \quad (3.3)$$
which gives Eqn. (2.1). \( T_{ab} \) however must be chosen carefully because Eqn. (3.3) implies that on \( \mathcal{M}^3 \)
\[
F_{0ia} = T_{ab} (e^0 e^b_i - e^0_i e^b_0) + S_{ab} e^{bcd} e_{c0} e_{d} \\
= N T_{ab} e^b_i + S_{ab} e^{bcd} e_{c0} e_{d} 
\]
(3.4)

For Riemannian manifolds, apart from a boundary term that does not contribute to the equations of motion, the Hamiltonian in the Ashtekar formalism is[1]:
\[
H = \int_{\mathcal{M}^3} d^3 x \sum \left( \epsilon_{abc} \tilde{\sigma}^{ia} \tilde{\sigma}^{jb} F_{ij}^c + \frac{\lambda}{3} \epsilon_{abc} \epsilon_{ijk} \tilde{\sigma}^{ia} \tilde{\sigma}^{jb} \tilde{\sigma}^{kc} \right) \\
+ 2 N^i \tilde{\sigma}^{ja} F_{ija} - 2 A_{0a} (D_i \tilde{\sigma}^i)^a 
\]
(3.5)

and the evolution equation for \( A_{ia} \) on \( \mathcal{M}^3 \) gives
\[
\dot{A}_{ia} = \{ A_{ia}, H \}_{PB} \\
= N \epsilon_{abc} \tilde{\sigma}^{ia} \tilde{\sigma}^{jb} F_{ij}^c + \frac{1}{2} \lambda N \epsilon_{ijk} \epsilon_{abc} \tilde{\sigma}^{ia} \tilde{\sigma}^{jb} \tilde{\sigma}^{kc} \\
+ \partial_i A_{0a} - \epsilon_a^{bc} A_{0b} A_{ic} - N^i F_{ija} 
\]
(3.6)

With the use of Eqn. (2.1) and assuming non-degenerate metrics, we can rewrite Eqn. (3.4) as
\[
F_{0ia} = - NS_{ab} e^b_i + N \epsilon_{ia} (S^c_e + \lambda) \\
+ S_{ab} e^{bcd} e_{c0} e_{d} 
\]
(3.7)

The second term vanishes because of the “superhamiltonian” constraint and comparing with Eqn. (3.4) we observe that the consistent choice for \( T_{ab} \) is:
\[
T_{ab} = - S_{ab} 
\]
(3.8)

Thus we have
\[
F_a = S_{ab} \left( - e^0 \wedge e^b_i + \frac{1}{2} e^b_i \wedge e^c \wedge e^d \right) 
\]
(3.9)

Similarly, for the evolution of \( \tilde{\sigma}^{ia} \), we have
\[
\dot{\tilde{\sigma}}^{ia} = \{ \tilde{\sigma}^{ia}, H \}_{PB} \\
= \epsilon^a_{bc} [D_j(N \tilde{\sigma}^j)] b \tilde{\sigma}^{jc} + \epsilon^a_{bc} N \tilde{\sigma}^{ic} (D_j \tilde{\sigma}^j)^b \\
+ [D_j(N^j \tilde{\sigma}^i)]^a - (\partial_j N^i) \tilde{\sigma}^{ja} \\
- N (D_j \tilde{\sigma}^j)^a + A_{0c} \epsilon^a_{bc} \tilde{\sigma}^{ib} 
\]
(3.10)
It is not difficult to show that the equations of motion for the Ashtekar variables can then be succinctly written as

\[ F_a = S_{ab} \Sigma^b \]  
(3.11a)

\[ (D \Sigma)^a = 0 \]  
(3.11b)

with

\[ S_{ab} = S_{ba} \]  
(3.12a)

\[ \text{tr} S = -\lambda \]  
(3.12b)

Here,

\[ \Sigma^a \equiv -e^0 \wedge e^a + \frac{1}{2} e^a_{bc} e^b \wedge e^c \]  
(3.13)

and \( D \) is the covariant derivative with respect to the Ashtekar connection 1-form. The nine \( x^0 \)-evolution equations for \( A_{ia} \) are contained in Eqn. (3.11a) while the twelve equations in Eqn. (3.11b) can be split off into the set of three equations:

\[ *[(D \Sigma)^a]_{\mathcal{M}^3} = 0 \]  
(3.14)

which is equivalent to the set of Gauss Law constraints, and the nine equations:

\[ [*(D \Sigma)^a]_{\mathcal{M}^3} = 0 \]  
(3.15)

which, modulo the Gauss Law constraints, are equivalent to the \( x^0 \)-evolution equations for \( \tilde{\sigma}^{ia} \), Eqn. (3.10). Ashtekar’s transcription of the “supermomentum” and “superhamiltonian” constraints of general relativity takes the simple form of (3.12a,b). (See also [8] for an alternative derivation of the equations of motion using self-dual two-forms as fundamental variables and a discussion of gravitational instantons as \( SU(2) \) rather than \( SO(3) \) gauge fields.)

We shall now examine the meaning of the equations of motion. Firstly, observe that \( \Sigma^a \) is explicitly anti-self-dual:

\[ *\Sigma^a = -\Sigma^a \]  
(3.16)

Since \( F_a \) is the product of a zero form \( S \) with the 2-form \( \Sigma \), Eqn. (3.10) implies that

\[ *F_a = -F_a \]  
(3.17)
As a result, all Einstein manifolds correspond to anti-instantons of the Ashtekar potentials. However, the converse is not always true. In general, the curvature of an arbitrary anti-instanton can be expanded in terms of $\Sigma^a$ via

$$F_a = Y_{ab} \Sigma^b$$  \hspace{1cm} (3.18)

But the quantity $Y$ will have to satisfy Eqn. (3.12) and Eqn. (3.11) before the anti-instanton can correspond to an Einstein manifold.

The twelve equations in Eqn. (3.11) suggest that the 1-form $A_a$ can be expressed in terms of the vierbein $e_A$. This is indeed the case, for the solution to Eqn. (3.11) is precisely

$$A_a = \omega_0 a - \frac{1}{2} \epsilon_a^{bc} \omega_{bc}$$ \hspace{1cm} (3.19)

where $\omega_{AB}$ can be determined uniquely from $e_A$ through

$$de_A + \omega_{AB} \wedge e^B = 0$$ \hspace{1cm} (3.20)

Eqn. (3.19) says that, apart from a factor of 2, $A_a$ is the anti-self-dual part of the spin-connection and so the curvature 2-form of $A_a$ can be expressed as

$$F_a = R_{0a} - \frac{1}{2} \epsilon_a^{bc} R_{bc}$$ \hspace{1cm} (3.21)

where $R_{AB}$ is the curvature 2-form of the spin-connection. It is then not difficult to show that Eqn. (3.11) is satisfied if and only if

$$S_{ab} = R_{0a0b} - R_{0a0b} = R_{0a0b} - R_{0a0b}$$ \hspace{1cm} (3.22)

and so the constraints Eqn. (3.12) imply that

$$R_{ABCD} = R_{ABCD}$$ \hspace{1cm} (3.23)

and the Ricci scalar becomes

$$R = 4\lambda$$ \hspace{1cm} (3.24)

These equations are completely equivalent to the pure gravity field equations defining Einstein manifolds.

Dimension four is the lowest dimension for which the Riemann curvature tensor assumes its full complexity. It is also the dimension which has the peculiarity that the
curvature 2-form can be decomposed into parts taking values in the \((\pm)\) eigenspaces \(\Lambda^2_\pm\) of the Hodge duality operator. The Riemann curvature tensor, having four indices, can be dualized on the left or on the right, so that it can be viewed as a \(6 \times 6\) mapping of \(\Lambda^2 \to \Lambda^2\): 

\[
\begin{pmatrix}
A & C^+ \\
C^- & B
\end{pmatrix}
\]

(3.25)

where in components,

\[
A_{ab} \equiv + \left( R_{0a0b} + R_{0a\tilde{0}b} \right) + \left( R_{\tilde{0}a0b} + R_{\tilde{0}a\tilde{0}b} \right)
\]

(3.26)

and \(B\) and \(C\) are defined similarly according to the signs of the following:

\[
A \sim (+, +, +, +), \; B \sim (+, -, -, -), \; C^+ \sim (+, -, +, -), \; C^- \sim (+, +, -, +)
\]

(3.27)

It is easy to check that \(A(B)\) is self-dual (anti-self-dual) with respect to both left and right duality operations, while \(C^+(C^-)\) is self-dual (anti-self-dual) under left duality and anti-self-dual (self-dual) under right duality operations. A metric is Einstein if and only if \(C^\pm = 0\), i.e. when Eqn. (3.25) assumes a block diagonal form. In view of Eqs. (3.17) and (3.21), \(F_a\) is the doubly anti-self-dual part of the curvature and apart from a multiplicative factor, \(S\), can be identified with \(B\) when the equations of motion are satisfied. In this context, for Einstein manifolds, the Ashtekar formulation is the realization of Proposition 2.2 of [9] in the canonical framework. However, it should be emphasized that it is the remarkable simplification of the constraints provided by Ashtekar that makes the non-perturbative quantization scheme viable. While it appears that only half of the non-vanishing components of the Riemann curvature tensor is contained in \(F_a\), the equations of motion are completely equivalent to Einstein’s field equations for non-degenerate metrics. Actually, \(A\) and \(B\) interchange under a reversal of orientation because a reversal of orientation changes the definition of self- and anti-self-duality.

While not all Einstein manifolds have anti-self-dual Riemann or Weyl tensors, a manifold is Einstein only if the curvature tensor constructed from the anti-self-dual part of the spin connection is anti-self-dual. It is precisely this property which allows for the description of all Einstein manifolds in terms of anti-instantons of the Ashtekar variables.

As a corollary, we note that for Einstein manifolds, the Weyl 2-form is

\[
W_{AB} = R_{AB} - \frac{\lambda}{3} e_A \wedge e_B
\]

(3.28)
so the anti-self-dual part of the Weyl 2-form $W_a^-$ becomes

$$W_a^- = R_{0a} - \frac{1}{2} \epsilon_{0 b c} R^{b c} + \frac{\lambda}{3} \left[ -\epsilon_{0 a} \wedge e_a + \frac{1}{2} \epsilon_a^{b c} \epsilon_{b} \wedge e_{c} \right]$$

$$= F_a + \frac{\lambda}{3} \Sigma_a \quad (3.29)$$

so an Einstein manifold is conformally flat or self-dual (half-flat when $\lambda = 0$) if and only if

$$F_a = -\frac{\lambda}{3} \Sigma_a$$

(3.30)

or

$$S_{ab} = -\frac{\lambda}{3} \delta_{ab}$$

(3.31)

According to our classification, this situation corresponds precisely to type $O$.

It is possible to eliminate $S_{ab}$ from the equations of motion. We have

$$\Sigma^a \wedge \Sigma^b = -2 \delta^{ab} \quad (*1)$$

(3.32)

where $(*)1$ is the 4-volume element. So from the equations of motion Eqn. (3.11)

$$S_{ab} = -\frac{1}{4} \ast (F_a \wedge \Sigma_b + \Sigma_a \wedge F_b)$$

(3.33)

and the equations of motion can be written as

$$F_a = -\frac{1}{2} [\ast (F_a \wedge \Sigma_b)] \Sigma^b$$

(3.34a)

$$(D \Sigma)^a = 0$$

(3.34b)

$$\epsilon_a^{bc} F_b \wedge \Sigma_c = 0$$

(3.34c)

$$F_a \wedge \Sigma_a = -2 \lambda \ast (1)$$

(3.34d)

4. Invariants and the Ashtekar variables

Unlike other fields, the gravitational field describes the dynamics of space-time. Any viable classical and quantum theory of the gravitational field must therefore be able to take into account not just the local description of curvature, but also the large scale global and topological aspects of the structure of space-time. We shall see how the Ashtekar variables can be used to capture the global invariants in 4D, especially those associated with Einstein manifolds.
As we have discussed in section 2, a specification of the initial value data is equivalent to a specification of the characteristic classes of $S$ which is compatible with the constraints. We may take the gauge-invariant quantities on $\mathcal{M}^3$ to be $\text{tr} \, S = -\lambda$, $\text{tr} \, S^2$, and $\text{tr} \, S^3$, from which we can reconstruct the characteristic classes of $S$. Their integrals over $\mathcal{M}^3$ should reflect global properties of $\mathcal{M}^3$.

It is not difficult to show that when the equations of motion are satisfied,

$$\text{tr} \, S = -\lambda \tag{4.1a}$$
$$\text{tr} \, S^2 = \frac{1}{8} \left\{ \left( R_{AB\tilde{C}\tilde{D}} - R_{A\tilde{B}\tilde{C}\tilde{D}} \right) R^{A\tilde{B}\tilde{C}\tilde{D}} \right\} \tag{4.1b}$$
$$\text{tr} \, S^3 = -\frac{1}{16} \left\{ \left( R_{ABC\tilde{D}} - R_{A\tilde{B}\tilde{C}\tilde{D}} \right) R^{CDE\tilde{F}} R_{E\tilde{F}AB} \right\} \tag{4.1c}$$

Thus their integrals over compact, closed 4-manifolds $\mathcal{M}^4$ give

$$\int_{\mathcal{M}^4} (\text{tr} \, S) = -\lambda V = -\frac{\lambda}{6} \int \Sigma^a \wedge \Sigma_a \tag{4.2}$$

where $V$ is the volume of $\mathcal{M}^4$, and

$$\int_{\mathcal{M}^4} (\text{tr} \, S^2) = 2\pi^2 \left( 2\chi(\mathcal{M}^4) - 3\tau(\mathcal{M}^4) \right)$$
$$= -\frac{1}{2} \int F_a \wedge F^a \tag{4.3}$$
$$= -2\pi^2 P_1$$

where $\chi(\mathcal{M}^4)$ and $\tau(\mathcal{M}^4)$ are the Euler characteristic and signature of $\mathcal{M}^4$, while $P_1$ is the Pontrjagin number of the $SO(3)$ Ashtekar connection. Finally,

$$\int_{\mathcal{M}^4} (\text{tr} \, S^3) = -\frac{1}{2} \int_{\mathcal{M}^4} S_{ab} F^a \wedge F^b \tag{4.4}$$

Observe that the signature $\tau(\mathcal{M}^4)$ depends on the orientation of $\mathcal{M}^4$. Indeed,

$$\tau(\mathcal{M}^4) = \dim H_+^2 - \dim H_-^2$$
$$= b_2^+ - b_2^- \tag{4.5}$$

where $H^2_\pm$ are the self-dual and anti-self-dual subspaces of the second cohomology group, and $b_2^\pm$ are the corresponding Betti numbers. Reversing the orientation interchanges self-dual and anti-self-dual 2-forms, so that

$$\tau(\overline{\mathcal{M}^4}) = -\tau(\mathcal{M}^4) \tag{4.6}$$
where $\overline{\mathcal{M}^4}$ has the opposite orientation relative to $\mathcal{M}^4$. Reversing the orientation changes the spin connections in general, and thus the Ashtekar connections via Eqn. (3.19). For example, consider

$$de^A = -\omega^A_B \wedge e^B$$

A transformation of the form $(e^0, e^a) \to (-e^0, e^a)$ reverses the orientation, though it does not change the metric $ds^2$. The new spin connections become

$$\omega_{0a} \to -\omega_{0a}; \quad \omega_{ab} \to \omega_{ab}$$

so that the Ashtekar connections transform as

$$A_a = \omega_{0a} - \frac{1}{2} \epsilon_a^{bc} \omega_{bc}$$

$$\to A_a - 2\omega_{0a}$$

The Pontrjagin numbers of the Ashtekar connections with respect to the two different orientations are

$$P_1^+ = 3\tau(\mathcal{M}^4) - 2\chi(\mathcal{M}^4)$$

$$P_1^- = 3\tau(\overline{\mathcal{M}^4}) - 2\chi(\overline{\mathcal{M}^4}) = -3\tau(\mathcal{M}^4) - 2\chi(\mathcal{M}^4)$$

Since $P_1^\pm$ are the Pontrjagin numbers of the anti-self-dual Ashtekar connections,

$$P_1^\pm \leq 0$$

An immediate consequence is the Hitchin bound for compact, closed Einstein manifolds

$$|\tau| \leq \frac{2}{3}\chi$$

(4.10)

For compact, closed Einstein manifolds with Euclidean signatures,

$$\chi(\mathcal{M}^4) = \frac{1}{32\pi^2} \int R_{ABCD} R^{ABCD}$$

$$= \frac{1}{32\pi^2} \int (R_{ABCD})^2 \geq 0$$

with the equality holding only if $\mathcal{M}^4$ is flat. Moreover $\tau(\mathcal{M}^4)$ and $\chi(\mathcal{M}^4)$ can be computed from the $SO(3)$ Ashtekar connections through

$$\tau(\mathcal{M}^4) = \frac{1}{6} (P_1^+ - P_1^-)$$

$$\chi(\mathcal{M}^4) = -\frac{1}{4} (P_1^+ + P_1^-)$$

(4.11a)

(4.11b)
If the Einstein manifold possesses an orientation reversing diffeomorphism, then \( P_1^+ = P_1^- \), and \( \tau = 0 \). The vanishing or non-vanishing of the signature has important physical implications. For according to the index theorem for the spin complex for closed, compact Riemannian manifolds,

\[
\begin{align*}
    n_+ - n_- &= -\frac{1}{24} P_1 (T(M^4)) \\
                 &= -\frac{1}{8} \tau(M^4)
\end{align*}
\]

where \( n_\pm \) are the number of \( \pm 1 \) chirality zero-frequency solutions of the Dirac equation. \( P_1 (T(M^4)) \) is the Pontrjagin number of the tangent bundle, \( i.e. \) of the \( SO(4) \) spin connection, and is related to the \( \tau(M^4) \) by the Hirzebruch signature theorem:

\[
P_1 (T(M^4)) = 3\tau(M^4)
\]

Thus

\[
\tau(M^4) = 0 \mod 8
\]

for spin manifolds, since \( n_+ - n_- \) must be an integer. An orientable manifold \( (W_1 = 0) \) has a spin structure iff \( W_2 = 0 \). Here \( W \) refer to the Stiefel-Whitney class. A simply-connected, compact, closed manifold of dimension four has a spin structure iff its intersection form is even, and this spin structure is unique\[11\]. Actually, for the case of simply-connected, compact, closed, \( smooth \) four-manifolds, the intersection form, and hence the topology via Freedman’s theorem, is determined by \( \tau \) and \( \chi \), and whether the intersection form is even (i.e. \( W_2 = 0 \)) or odd. This can be explained as follows: Indefinite intersection forms are determined by their rank, signature, and type (even or odd). The rank of the intersection form is the second Betti number. But

\[
b_2 = b_2^+ + b_2^- = \chi - 2
\]

for simply-connected, compact, closed four-manifolds. \( \tau \) is the signature of the intersection form. Although there are many definite intersection forms of the same rank and signature, Donaldson’s theorem \[12\] asserts that differentiable four-manifolds with definite intersection forms must be of the standard type \( \bigoplus_{\pm} (1) \). So specification of \( P_1^\pm \) and whether the manifold is spin \( (W_2 = 0) \) or not corresponds to a complete specification of the intersection form of a smooth, simply-connected, compact, closed four-manifold. Freedman’s theorem \[13\] asserts that given an even (odd) intersection form, there is exactly one (two,
distinguished by their $\mathbb{Z}_2$-valued Kirby-Siebenmann invariant) simply-connected, closed, compact, topological four-manifold representing that form.

Before we proceed to specific illustrations, we remark that the third invariant Eqn. (4.1c), which involves the explicit form of $S$ could provide a new differential invariant for Einstein manifolds, since the intersection form has already been accounted for by Eqn. (4.3), at least for the case when they are smooth, simply-connected, closed, and compact. See also [7] for a discussion of BRST-invariants of four-dimensional gravity in Ashtekar variables.
5. Examples of Einstein manifolds in Ashtekar variables

A. Known Solutions

The formalism developed in the previous sections provides a coherent framework to discuss explicit Einstein manifolds in the context of Ashtekar variables. Every known solution of the Einstein field equations

\[ R_{\mu\nu} = \lambda g_{\mu\nu} \]  

(5.1)

can be put in the form of Eqns. (3.11). In fact, when the field equations are satisfied, we can use Eqns. (3.12) to obtain the Ashtekar connection, and compute \( S \) via Eqn. (3.11a).

It will be convenient to introduce the 1-forms \( \Theta_a \), where \( \Phi_a = -2 \Theta_a \) obeys the Maurer-Cartan equation for \( SO(3) \):

\[ d\Phi_a + \frac{1}{2} \epsilon_{abc} \Phi_b \wedge \Phi_c = 0 \]  

(5.2)

We can choose the four-dimensional polar coordinates as \((R, \theta, \phi, \psi)\), where for fixed \( R \), \( 0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi \), and \( 0 \leq \psi < 4\pi \). Next introduce

\[ x^1 + ix^2 = R \cos(\theta/2) \exp \frac{i}{2}(\psi + \phi) \]  

\[ x^3 + ix^0 = R \sin(\theta/2) \exp \frac{i}{2}(\psi - \phi) \]  

(5.3)

Then \( \Theta_a \) can be written in terms of the Euler angles \( \theta, \phi, \psi \) on \( S^3 \) as

\[ \Theta_1 = \frac{1}{2} (\sin \psi d\theta - \sin \theta \cos \psi d\phi) \]  

\[ \Theta_2 = \frac{1}{2} (\cos \psi d\theta - \sin \theta \sin \psi d\phi) \]  

\[ \Theta_3 = \frac{1}{2} (d\psi + \cos \theta d\phi) \]  

(5.4)

We concentrate first on solutions with \( S_{ab} = -(\lambda/3)\delta_{ab} \). As we have explained, these solutions correspond to the conformally self-dual sector of Einstein manifolds. It is known that for \( \lambda > 0 \), \( S^4 \) and \( CP_2 \) are the only compact, closed, simply-connected four-manifolds which are conformally self-dual[14].
(a) $S^4$ with the de Sitter metric

The metric for this space is given by

$$ds^2 = \left[1 + \left(\frac{R}{a}\right)^2\right]^{-2} \left[dR^2 + R^2 (\Theta_1^2 + \Theta_2^2 + \Theta_3^2)\right]$$

while the vierbein is expressed as

$$e_A = \left\{ \frac{dR}{1 + \left(\frac{R}{a}\right)^2}, \frac{R\Theta_a}{1 + \left(\frac{R}{a}\right)^2} \right\}$$

The corresponding Ashtekar connections then have the form:

$$A_a = \omega_{0a} - \frac{1}{2} \epsilon_a{}^{bc} \omega_{bc}$$

$$= - \frac{2\Theta_a}{\left(1 + \left(\frac{R}{a}\right)^2\right)}$$

giving

$$F_a = dA_a + \frac{1}{2} \epsilon_a{}^{bc} A_b \wedge A_c$$

$$= - \frac{4}{a^2} \left(-e_0 \wedge e_a + \frac{1}{2} \epsilon_a{}^{bc} e_b \wedge e_c\right)$$

Thus,

$$F_a = S_{ab} \Sigma^b$$

with

$$S_{ab} = - \frac{4}{a^2} \delta_{ab} = - \frac{\lambda}{3} \delta_{ab}$$

so that

$$\lambda = \frac{12}{a^2} > 0$$

and the diameter of the four sphere is related to $\lambda$ by

$$a = \sqrt{\frac{12}{\lambda}}$$

Suppose that we now reverse the orientation, by for example defining the vierbein field to be

$$\tilde{e}_A = \left\{ -\frac{dR}{1 + \left(\frac{R}{a}\right)^2}, \frac{R\Theta_a}{1 + \left(\frac{R}{a}\right)^2} \right\}$$
The Ashtekar connections then change to the form:

\[ \mathbf{A}_a = -\frac{2}{a^2} \frac{R^2 \Theta_a}{1 + (\frac{R}{a})^2} \]  

(5.13)

although

\[ \mathbf{F}_a = -\frac{4}{a^2} \Sigma_a \]  

(5.14)

so that \( S \) is unchanged. \( S^4 \) has an orientation reversing diffeomorphism. By using the explicit form for the Ashtekar connections, one obtains that the first Pontrjagin number equals \(-4\), and is preserved under the reversal. Thus Eqns. (5.14a) and (5.14b) yield

\[ \tau (S^4) = 0 \]  

(5.15a)

\[ \chi (S^4) = 2 \]  

(5.15b)

Note that for \( S^4 \), the SO(3) Ashtekar connections give \( P_{1}^{\pm} = 0 \) mod 4, and so can be lifted to an \( SU(2) \) connection, with second Chern class

\[ c_2 = -\frac{P_1}{4} \]

(5.16)

Actually, the Ashtekar connections given by Eqns. (5.7) and (5.13) are precisely the BPST (anti-)instanton solutions[15]. Since the intersection form has rank

\[ \text{rank} (Q) = b_2 = \chi - 2 = 0 \]  

(5.17)

\( S^4 \) has \( Q = \emptyset \).

The dimension of the moduli space for a single anti-instanton on \( S^4 \) is known to be five[9]. The parameters correspond to the size and location of the (anti-)instanton. For the Ashtekar connections, however, diffeomorphism invariance collapses this space entirely, since the solution must now be translationally invariant, and the size is fixed by the cosmological constant, according to Eqn. (5.11). \( S^4 \) is not only conformally self-dual, but it is also conformally flat. That this is so is also evident in the Ashtekar context because \( S_{ab} = \lambda/3\delta_{ab} \) implies, by Eqn. (3.29) that \( \mathcal{W}_a^- = 0 \). But \( S \) is unchanged by orientation reversal, so \( \mathcal{W}_a^+ = 0 \) also. Hence \( \mathcal{W}_a^{\pm} = 0 \), and \( S^4 \) is conformally flat.
(b) $\text{CP}_2$ and the Fubini-Study Metric

The two dimensional complex projective space is described by the Fubini-Study metric:

$$ds^2 = \frac{dR^2}{(1 + \frac{\lambda}{6} R^2)^2} + \frac{(R\Theta_1)^2}{(1 + \frac{\lambda}{6} R^2)^2} + \frac{(R\Theta_2)^2}{(1 + \frac{\lambda}{6} R^2)^2} + \frac{(R\Theta_3)^2}{(1 + \frac{\lambda}{6} R^2)^2}$$

(5.18)

We may choose the vierbeins as

$$e_A = \begin{cases} dR \\ \frac{R\Theta_1}{(1 + \frac{\lambda}{6} R^2)^{\frac{1}{2}}} \\ \frac{R\Theta_2}{(1 + \frac{\lambda}{6} R^2)^{\frac{3}{2}}} \\ \frac{R\Theta_3}{(1 + \frac{\lambda}{6} R^2)^{\frac{5}{2}}} \end{cases}$$

(5.19)

in which case the Ashtekar variables are:

$$A_1 = \frac{-2\Theta_1}{(1 + \frac{\lambda}{6} R^2)^{\frac{1}{2}}} \quad A_2 = \frac{-2\Theta_2}{(1 + \frac{\lambda}{6} R^2)^{\frac{3}{2}}}$$

and

$$A_3 = \frac{(-2 - \frac{\lambda}{6} R^2) \Theta_3}{(1 + \frac{\lambda}{6} R^2)^{\frac{5}{2}}}$$

(5.20)

These equations yield $F_a = S_{ab}\Sigma^{ab}$, with $S_{ab} = -(\lambda/3)\delta_{ab}$. The solution is therefore again of Type O. However, the Pontrjagin index is found to equal

$$P_1 = \frac{1}{4\pi} \int F_a \wedge F_a = -3$$

(5.21)

As a result, the Ashtekar connections cannot be realized in a globally well-defined manner as an $SU(2)$ gauge potential.

Like $S^4$, $\text{CP}_2$ is conformally flat, since $S$ is of Type O, but unlike $S^4$, it does not have an orientation reversing diffeomorphism. Under a reversal, we obtain $\overline{\text{CP}}_2$, which is described by the same metric, but the vierbein becomes $(-e_0,e_a)$. In which case, the Ashtekar potentials become

$$\overline{A}_1 = \overline{A}_2 = 0$$

while

$$\overline{A}_3 = -\frac{\lambda R^2 \Theta_3}{2(1 + \frac{\lambda}{6} R^2)}$$

(5.22)

giving

$$\overline{F}_1 = \overline{F}_2 = 0$$

$$\overline{F}_3 = d\overline{A}_3$$

$$= -\lambda (e_0 \wedge e_3 + e_1 \wedge e_2)$$

$$= -\lambda \Sigma_3$$

(5.23)
Thus $CP_2$ is described by Ashtekar potentials of a non-abelian anti-instanton, whereas $\overline{CP}_2$ is described by those of an abelian anti-instanton. The corresponding Pontrjagin index is found to be

$$\overline{P}_1 = -9 \quad (5.24)$$

which is different from that of Eqn.(5.21). Accordingly, the Euler characteristic and signature are given by

$$\chi(CP_2) = \chi(\overline{CP}_2) = 3 \quad (5.25)$$

while

$$\tau(CP_2) = -\tau(\overline{CP}_2) = 1 \quad (5.26)$$

From previous studies [11], we already know that $CP_2$ cannot support abelian instantons, while $\overline{CP}_2$ can support only one such object. The Ashtekar potential is simply that unique abelian anti-instanton.

The matrix $S_{ab}$ for $\overline{CP}_2$ is of the form

$$\overline{S} = \text{diag} \ (0, 0, -\lambda) \quad (5.27)$$

and so the solution is of Type D.

This example shows how the Ashtekar variables provide a more natural context in which to study the topological and differential invariants of a 4-manifold.

(c) The Schwarzschild-de Sitter solution

The Schwarzschild-de Sitter metric in Euclidean space is given by

$$ds^2 = \left( 1 - \frac{2M}{r} - \frac{\lambda}{3} r^2 \right) d\tau^2 + \left( 1 - \frac{2M}{r} - \frac{\lambda}{3} r^2 \right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta \, d\phi^2 \quad (5.28)$$

where $M$ is $G/c^2$ times the mass. Taking the vierbein fields to be

$$e_A = \left\{ \left( 1 - \frac{2M}{r} - \frac{\lambda}{3} r^2 \right)^{\frac{1}{2}} d\tau, \left( 1 - \frac{2M}{r} - \frac{\lambda}{3} r^2 \right)^{-\frac{1}{2}} dr, r \, d\theta, \ r \sin \theta \, d\phi \right\} \quad (5.29)$$

yields the Ashtekar potentials

$$A_1 = \left( \frac{M}{r^2} - \frac{\lambda}{3} r \right) d\tau + \cos \theta \, d\phi$$

$$A_2 = -\left( 1 - \frac{2M}{r} - \frac{\lambda}{3} r^2 \right)^{\frac{1}{2}} \sin \theta \, d\phi \quad (5.30)$$

$$A_3 = \left( 1 - \frac{2M}{r} - \frac{\lambda}{3} r^2 \right)^{\frac{1}{2}} d\theta$$
and the matrix $S$:

$$S = \text{diag} \left( \frac{-2M}{r^3} - \frac{\lambda}{3}, \frac{M}{r^3} - \frac{\lambda}{3}, \frac{M}{r^3} - \frac{\lambda}{3} \right)$$

This solution is therefore of Type D generally, for $M \neq 0$, and of Type O in the limit of vanishing mass. One may view the mass term as a parameter which breaks the system out of the Type O sector. When $\lambda \to 0$, we recover the usual Schwarzschild solution.

Reversing the orientation gives

$$\overline{A}_1 = - \left( \frac{M}{r^2} - \frac{\lambda}{3} r \right) d\tau - \cos \theta d\phi = -A_1$$

with

$$\overline{A}_2 = A_2 \quad \text{and} \quad \overline{A}_3 = A_3$$

while the form of $S$ is preserved.

The Pontrjagin index for $\lambda = 0$ can be computed to give

$$P_1 = -\frac{1}{2\pi^2} \int \text{tr} \left( S^2 \right) (*)1$$

$$= -\frac{1}{\pi^2} \int_{\phi=0}^{2\pi} \int_{0}^{\pi} \int_{r=2M}^{\infty} \int_{\tau=0}^{4\pi M} \frac{3M^2}{r^4} d\tau \wedge dr \wedge d\theta \wedge d\phi$$

$$= -4$$

$$= \overline{P}_1$$

The radius $2M$ is the usual event horizon, and we have also used the periodicity in the Euclidean time interval of $8\pi M$ inherent in the Schwarzschild metric. From Eqn. (5.33), we conclude that $\chi = 2$, and $\tau = 0$, in agreement with the standard result.

Finally, note that a general Type D metric with zero cosmological constant can be characterized by $S = \text{diag}(-2\alpha, \alpha, \alpha)$. If $\alpha > 0$, $S$ is gauge equivalent to

$$S_{ab} = \frac{1}{3} \phi^2 \delta_{ab} - \phi_a \phi_b$$

since this form can be diagonalized to

$$\text{diag} \left( -\frac{2}{3} \phi^2, \frac{1}{3} \phi^2, \frac{1}{3} \phi^2 \right)$$

For the Schwarzschild solution, $\phi^2 = 3M/r^3$. In isotropic coordinates, with

$$r \equiv \left( 1 + \frac{M}{2\rho} \right)^2 \rho$$
the quantity $\phi$ above takes on the value

$$\phi^a = (3M)^{\frac{1}{2}} (\rho)^{-\frac{3}{2}} \left( 1 + \frac{M}{2\rho} \right)^{-3} \rho^a$$

yielding for the Ashtekar magnetic field

$$B^{ia} = \frac{M}{\rho^3 \left( 1 + \frac{M}{2\rho} \right)^6} \left( \delta^{ab} - \frac{3\rho^a \rho^b}{\rho^2} \right) \tilde{\sigma}_b^i$$ (5.35)

This establishes the gauge-equivalence between the Schwarzschild solution in Ashtekar variables in our general formalism and the solution exhibited in [10].

(d) The Eguchi-Hanson metric

The Eguchi-Hanson metric [17] with a cosmological constant can be written as

$$ds^2 = \left[ 1 - \left( \frac{a}{R} \right)^4 - \frac{\lambda}{6} R^2 \right]^{-1} dR^2 + R^2 \left( \Theta_1^2 + \Theta_2^2 \right) + R^2 \left[ 1 - \left( \frac{a}{R} \right)^4 - \frac{\lambda}{6} R^2 \right] \Theta_3^2$$ (5.36)

We can choose the vierbein fields to be

$$e_A = \left\{ \left[ 1 - \left( \frac{a}{R} \right)^4 - \frac{\lambda}{6} R^2 \right]^{-\frac{i}{2}} dR, R \Theta_1, R \Theta_2, \left[ 1 - \left( \frac{a}{R} \right)^4 - \frac{\lambda}{6} R^2 \right]^{\frac{i}{2}} R \Theta_3 \right\}$$ (5.37)

which then implies that the Ashtekar potentials are given by

$$A_1 = -2 \left[ 1 - \left( \frac{a}{R} \right)^4 - \frac{\lambda}{6} R^2 \right]^{\frac{i}{2}} \Theta_1$$

$$A_2 = -2 \left[ 1 - \left( \frac{a}{R} \right)^4 - \frac{\lambda}{6} R^2 \right]^{\frac{i}{2}} \Theta_2$$ (5.38)

$$A_3 = -2 \left[ 1 + \left( \frac{a}{R} \right)^4 - \frac{\lambda}{12} R^2 \right]^{\frac{i}{2}} \Theta_3$$

The corresponding matrix $S_{ab}$ takes the form

$$S = \text{diag} \left( \frac{4a^4}{R^6} - \frac{\lambda}{3}, \frac{4a^4}{R^6} - \frac{\lambda}{3}, \frac{8a^4}{R^6} - \frac{\lambda}{3} \right)$$ (5.39)

The Eguchi-Hanson metric is therefore of Type D when $a \neq 0$, and of Type O when $a = 0$, so this parameter causes the system to break out of the Type O sector.
When we apply a reversal, we get another manifold, $\overline{EH}$, with the Ashtekar potentials taking the form:

$$\overline{A}_1 = \overline{A}_2 = 0$$

while

$$\overline{A}_3 = -\frac{\lambda}{2}R^2\Theta_3$$

The field strengths are now controlled by the matrix

$$\overline{S} = \text{diag} (0, 0, -\lambda)$$

Like in the case of $\overline{CP}_2$, this matrix is not invertible, and it is described by an abelian anti-instanton.

However, the Eguchi-Hanson manifold has a boundary of real projective 3-space, $RP_3[17]$. The abelian instanton of Eqn. (5.40) does not depend on the parameter $a$, and furthermore, it is anti-self-dual relative to $\overline{EH}$ for arbitrary $\lambda$ and $a$. In the limit $\lambda \to 0$, $S$ becomes zero, and $\overline{EH}$ becomes half-flat. As we shall see below, the Eguchi-Hanson metric can be obtained as limiting cases of two different classes of explicit solutions, one from the $F = 0$ sector, and the other from the abelian anti-instanton sector.

**B. New Solutions**

The above examples illustrate the procedure for determining the appropriate anti-instanton configuration of the Ashtekar variables once the metric is known. But, the formalism can be used to go the other way and yield new solutions to the Einstein field equations. We shall illustrate the method below by examining a few explicit examples.

Before we do so, recall that the matrix $S$ for Riemannian manifolds is real-symmetric. Solutions are characterized by $\text{tr } S^2$ and $\text{tr } S^3$, which can be further divided into classes relative to a sign change under orientation reversal. This distinction had been utilized in the examples presented so far, and will continue to be significant in the solutions we will be discussing below.

**$F = 0$ sector and hyperkähler manifolds**

We first examine the case where the Ashtekar field strength vanishes. When this happens, the metric is half-flat; i.e. the Riemann curvature is self-dual. $S$ vanishes also,
and for simply-connected manifolds, we may set the connection to be zero globally as well. The equations of motion reduce to

$$d \Sigma_a = 0 \quad (5.42)$$

so that the anti-self-dual $$\Sigma_a$$ is now also closed. As a result,

$$b_2^- = 3 \quad (5.43)$$

Since the Ashtekar curvature vanishes, we obtain

$$0 = P_1 = 3\tau(\mathcal{M}) - 2\chi(\mathcal{M}) \quad (5.44)$$

so that $$\tau$$ takes on the maximal value of the Hitchin bound:

$$\tau(\mathcal{M}) = \frac{2}{3}\chi(\mathcal{M}) \quad (5.45)$$

But we also have the relation

$$\chi(\mathcal{M}) = b_2 + 2 = b_2^+ + b_2^- + 2 = b_2^+ + 5$$

so that finally

$$\tau(\mathcal{M}) \equiv b_2^+ - b_2^- = b_2^+ - 3 \quad (5.46)$$

These relations may be solved to give the following characteristic numbers for simply-connected compact Einstein manifolds in the $$F = 0$$ sector:

$$b_2^+ = 19 \quad b_2^- = 3 \quad \tau = 16 \quad \chi = 24 \quad (5.47)$$

It is known that $$K3$$ manifolds and the 4-torus are the only compact manifolds without boundary admitting metrics of self-dual Riemann curvature [18]. The 4-torus is not simply-connected, and has $$\tau = \chi = 0$$, since its metric is flat. So, choosing the convention that $$\tau(K3) = -16$$, we can identify the simply-connected compact half-flat manifolds without boundary as $$\overline{K3}$$. They have the intersection form [11] from [12]:

$$\mathcal{Q} = \bigoplus^3 \left[ \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \bigoplus^2 E_8 \quad (5.48)$$

The Pontrjagin index for $$\overline{K3}$$ can be computed to be

$$\overline{P}_1 = -3\tau - 2\chi = -96 \quad (= 0 \mod 4) \quad (5.49)$$
As a result, the \( SO(3) \) Ashtekar connection can be lifted to an \( SU(2) \) connection. The metric therefore possesses an \( SU(2) \) holonomy, and is therefore hyperkähler\[14\]. Such metrics have been used to formulate conditions for unbroken supersymmetry in the compactification of superstrings\[19\]. In our present context, these metrics are associated with the unbroken topological field theory of the moduli space of flat connections\[7\].

Note that although \( F = 0 \) and \( S = 0 \) for \( K3 \), the corresponding values for \( K3 \) need not be trivial. It has been calculated that these surfaces are parametrized by 58 parameters\[20\]. According to Eqn. (5.49), these must be associated with an Ashtekar connection with Pontrjagin number \(-96\).

We shall now construct explicitly half-flat Einstein manifolds which are not necessarily simply-connected, or without boundary. They will have \( F = 0 \), but \( F \neq 0 \).

We begin by supposing that the vierbein is of the form:

\[
e_A = \{-a(R)dR, f(R)\Theta_1, g(R)\Theta_2, h(R)\Theta_3\}
\]

This yields

\[
A_1 = \left\{ \frac{f'}{a} - \frac{(g^2 + h^2 - f^2)}{gh} \right\} \Theta_1
\]

\[
A_2 = \left\{ \frac{g'}{a} - \frac{(h^2 + f^2 - g^2)}{fh} \right\} \Theta_2
\]

\[
A_3 = \left\{ \frac{h'}{a} - \frac{(f^2 + g^2 - h^2)}{fg} \right\} \Theta_3
\]

where primes denote differentiation with respect to \( R \). Further simplification can be achieved by assuming that \( f = g \). Setting \( A_\alpha = 0 \) locally, we need to solve

\[
\frac{f'}{a} = \frac{h}{f}
\]

and

\[
\frac{h'}{a} + \frac{h^2}{f^2} = 2
\]

Combining these two equations gives

\[
\frac{(h^2)'}{(f^2)'} + \frac{h^2}{f^2} = 2
\]

With \( u \equiv h^2 \) and \( v \equiv f^2 \), this equation reduces to

\[
(uv)' = (v^2)'
\]

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which has as solution
\[ h^2 = f^2 + \frac{b}{f^2} \]  
with \( b \) being an integration constant. The metric is therefore given by
\[ ds^2 = a^2 dR^2 + f^2 (\Theta_1^2 + \Theta_2^2) + h^2 \Theta_3^2 \]
where \( a = \frac{(f^2)'}{2h} \), and \( h \) is given Eqn. (5.52). The function \( f \) is an arbitrary function of \( R \).

If we now reverse the orientation, the metric is invariant, but the vierbein changes to \( e_A = (-e_0, e_i) \). The Ashtekar potentials become
\[ \bar{\mathcal{A}}_1 = -\frac{2h}{f} \Theta_1 \]
\[ \bar{\mathcal{A}}_2 = -\frac{2h}{f} \Theta_2 \]
\[ \bar{\mathcal{A}}_3 = -2 \left( 2 - \frac{h^2}{f^2} \right) \Theta_3 \]
assuming that the relations among \( h, a \) and \( f \) continue to hold. A short computation then fixes the matrix \( S \) to be:
\[ S = \text{diag} \left( -\frac{4b}{f^6}, -\frac{4b}{f^6}, \frac{8b}{f^6} \right) \]
(5.55)
Thus the equations of motion still hold, but the solution is now of Type D when \( b \neq 0 \).

For compact manifolds without boundaries,
\[ \bar{P}_1 = -\frac{1}{4\pi^2} \int \bar{F}_i \wedge \bar{F}_i \]
\[ = 12b^2 \int (f^{-8})'dR \]
(5.56)
assuming that the variables \( \theta, \phi \) and \( \psi \) are the coordinates of a 3-sphere for fixed values of \( R \). By choosing the appropriate function \( f \), one can obtain self-dual Einstein manifolds with non-trivial values of the Pontrjagin number. For the special case of \( f = R \), and \( b = -a^4 \), we recover the \( \lambda \to 0 \) limit of the Eguchi-Hanson metric discussed above. Recall that our convention is such that \( \overline{EH} \) with \( \lambda = 0 \) is half-flat.
Abelian anti-instantons and Kähler-Einstein manifolds

When \( S = \text{diag}(0, 0, -\lambda) \), the Ashtekar potential is described by an abelian anti-instanton. In this gauge, the only non-vanishing component of the field-strengths is \( F_3 \), and the equations of motion reduce to

\[
\begin{align*}
    dA_3 &= F_3 = -\lambda \Sigma_3 \\
    d\Sigma_1 &= A_3 \wedge \Sigma_2 \\
    d\Sigma_2 &= -A_3 \wedge \Sigma_1 
\end{align*}
\]  

(5.57)

We now suppose that the manifold can support a complex structure, and define

\[
\Sigma^+ \equiv \Sigma_1 + i \Sigma_2
\]

(5.58)

in which case,

\[
d\Sigma^+ + i A_3 \wedge \Sigma^+ = 0
\]

(5.59)

Furthermore, let us define

\[
\begin{align*}
    \Omega^1 &\equiv -e^0 + i e^3 \\
    \Omega^2 &\equiv e^1 + i e^2
\end{align*}
\]

(5.60)

Then,

\[
\Sigma^3 = \frac{i}{2} \Omega^\alpha \wedge \overline{\Omega}^\alpha \quad \alpha = 1, 2
\]

(5.61)

is closed, by Eqn. (5.57). Therefore, Einstein manifolds which are endowed with a complex structure, and are described by abelian Ashtekar anti-instantons can be identified as Kähler manifolds, with \( \Sigma_3 \) as the Kähler form.

We now construct explicit solutions for this class of manifolds. We shall assume that the vierbein fields are of the form Eqn. (5.50), and that the Ashtekar potentials satisfy Eqn. (5.51). The equations Eqn. (5.57), it can be checked, are then satisfied. We shall now suppose, for simplicity, that

\[
A_3 = c(R) \Theta_3
\]

(5.62)

so that

\[
F_3 = c' dR \wedge \Theta_3 + 2c \Theta_1 \wedge \Theta_2
\]

(5.63)

To satisfy the gauge condition on \( S \) as specified above, \( i.e. \ \text{diag}(0, 0, -\lambda) \), we must have

\[
c' = -\lambda a \quad 2c = -\lambda f g
\]

(5.64)
It is easy to check that for \( f = g \), Eqn. (5.64) implies that both \( A_1 \) and \( A_2 \) vanish, and we are left with the condition
\[
\frac{h'}{a} - \frac{(2f^2 - h^2)}{f^2} = c
\]  
(5.65)
Substituting for \( f^2 \) from Eqn. (5.64) gives
\[
-\lambda (h^2c)' = \frac{2}{3} (c^3)' + 2 (c^2)'
\]
The solution is
\[
h^2 = -\frac{2}{3\lambda} \left\{ c(c + 3) + \frac{b}{c} \right\}
\]
\[
a^2 = \frac{(c')^2}{\lambda^2 h^2}
\]
\[
f^2 = g^2 = -\frac{2c}{\lambda}
\]
The function \( c \) is an arbitrary function of \( R \), while \( b \) is an integration constant.

Upon reversal of orientation, the new Ashtekar variables are
\[
\overline{A}_1 = -\frac{2f'}{a} \Theta_1
\]
\[
\overline{A}_2 = -\frac{2f'}{a} \Theta_2
\]
\[
\overline{A}_3 = \left( -\frac{c}{3} - 2 + \frac{2b}{3c^2} \right) \Theta_3
\]  
(5.67)
The connections above now describe a non-abelian anti-instanton, with the corresponding \( S \) matrix given by
\[
\mathbf{S} = \text{diag} \left( -\frac{\lambda}{3} \left[ 1 - \frac{2b}{c^3} \right], -\frac{\lambda}{3} \left[ 1 - \frac{2b}{c^3} \right], -\frac{\lambda}{3} \left[ 1 + \frac{4b}{c^3} \right] \right)
\]  
(5.68)
As a result, the solution is of Type D for \( b \neq 0 \), and of Type 0 when \( b = 0 \).

The corresponding Pontrjagin numbers are
\[
\mathcal{P}_1 = \frac{1}{4\pi^2} \int F_a \wedge F_a
\]
\[
= -\int (c^2)' dR
\]  
(5.69)
for the case of the abelian anti-instanton, and
\[
\overline{\mathcal{P}}_1 = \frac{1}{4\pi^2} \int \overline{F}_a \wedge \overline{F}_a
\]
\[
= -\int \left( \frac{c^2}{3} + \frac{8b}{9c} - \frac{4b^2}{3c^4} \right)' dR
\]  
(5.70)
for the non-abelian case.

If we let

\[ c = -\frac{\lambda R^2}{2 \left(1 + \frac{\lambda}{6} R^2\right)}, \quad b = 0 \]

as an example, we reproduce the expressions for \( CP_2 \) and \( \overline{CP}_2 \). Another example, with \( b \neq 0 \) is obtained by setting

\[ c = -\frac{\lambda}{2} R^2, \quad b = -\frac{3}{4} \lambda^2 a^4 \]

This ansatz gives us the Eguchi-Hanson space and the configuration for \( EH \) discussed in the last section.

6. Matrix \( S \) as an order parameter

We have seen how \( S \) can play an effective role as an order parameter characterizing the Type O sector. This sector corresponds classically to conformally self-dual Einstein manifolds. Actually, we can go further with this hypothesis by studying it in the abelian anti-instanton sector. We have already discussed several explicit examples of Einstein manifolds which belong to this sector.

Suppose that \( S \) is of rank one and can be expressed as

\[ S_{ab} = \pm \phi_a \phi_b \]  

(6.1)

where \( \phi_a \) is a triplet of phenomenological real scalar fields. It is then gauge-equivalent to the form \( S = \text{diag}(0, 0, \pm \phi^2) \) and we may assume that

\[ ||\phi|| = \sqrt{\pm \lambda} \]

(6.2)

where the sign in Eqn. (6.1) is chosen in accordance with whether \( \lambda \) is negative or positive. In the \( U \)-gauge, with \( \phi_a = ||\phi|| \delta_a^3 \), and \( A_{1,2} = 0 \), we have simply the condition

\[ (D\phi)_a = 0 \]  

(6.3)

But Eqns. (6.2) and (6.3) are gauge and diffeomorphism invariant statements, and are therefore valid in arbitrary \( SO(3) \) gauges and coordinate systems. The situation is therefore identical to that of a system possessing a symmetry based on the group \( SO(3) \), which is broken down to \( SO(2) \) by the order parameter \( \phi_a \) acquiring a non-vanishing vacuum expectation value equal to the cosmological constant. The matrix \( S \) is non-invertible, and in this phase the gravitational fields are ordered dynamically in such a way as to break the local \( SO(3) \) Ashtekar symmetry.
7. Concluding Remarks

We have presented in this paper several examples which illustrate the methods to be used in obtaining solutions to the Einstein equations with Ashtekar variables. The examples have been chosen to bring out those features which are particularly transparent within this context. Among these are the properties of Einstein manifolds under orientation reversals and their relations to abelian anti-instantons, the role of the cosmological constant in fixing the the type of Einstein manifolds, and finally, a perspective on spontaneous breaking of the local $SO(3)$ symmetry. We hope to amplify upon some of the physical implications of these features, especially in a quantum context, in the near future.
Figure Caption

Fig. 1 Classification of the initial data according to $S$. 
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