Intervened Exponential Distribution: Properties and Applications

Vilayat Ali Bhat¹, Sudesh Pundir¹*

*Corresponding author

1. Pondicherry University, Department of Statistics, Puducherry - 605014. INDIA.
vilayat.stat@gmail.com, and sudeshpundir19@gmail.com

Abstract

This manuscript aims to study the intervention-based probability model. Statistical and reliability properties such as the expressions for, cumulative density function (cdf), mean deviations about mean and median, $r^{th}$ order central and non-central moments, generation functions for moments have been derived. Moreover, the expression for reliability function, hazard rate function, reverse hazard rate function, aging intensity, mean residual life function, stress-strength reliability, and entropy metrics due to Rényi and Shannon are also derived. Monte Carlo simulation study performance of maximum likelihood estimates (MLEs) has been carried out, followed by calculations of average bias ($Abias$), and Mean Square Error ($MSE$). The applicability of the model in real-life situations has been discussed by analyzing the two real-life data sets.

Key Words: Bias; Entropy; Intervention; Mean Square Error; Monte Carlo Simulation.

Mathematical Subject Classification: 60E05, 62E15.

1. Introduction

In reliability and survival analysis, the lifetime of the system or an individual is viewed as, an essential characteristic. This important feature in reliability theory is investigated with traditional lifetime distributions available in the literature, such as Gamma, Exponential, Weibull, Rayleigh, Normal, Log-normal, etc. For a deep and more detailed summary about the lifetime models, one could refer to Barlow and Proschan (1975), Zacks (1992), Marshall and Olkin (2007), etc. In distribution theory, numerous continuous and discrete lifetime distributions along with modified versions are available in the literature. However, Shanmugam (1985) is the pioneer to develop intervention-based models in distribution theory, due to its impressive applications in several areas of statistics and other applied sciences, very useful publications on intervention-based models of Huang and Fung (1989), Scollnik (1995), Dhanavanthan (1998, 2000), Shanmugam et al. (2002), Scollnik (2006), etc. are available in the statistical literature. Besides these few attempts observed in this direction, reliability theory is yet to explore with the idea. The most popular and frequently used distribution among all continuous distributions is considered Exponential distribution having a cluster of applications in reliability theory, survival analysis, engineering sciences, economics, physics, business statistics etc. In history, researchers have developed, different types of modified and generalized models of the Exponential distribution. The explanatory summary on Exponential distribution is given by Balakrishnan (2019). However, the recent developments in distribution theory, in the form of discrete intervened probability models motivated us to explore the continuous lifetime intervened model. In this connection, we have attempted to study the new extension of Exponential distribution namely intervened Exponential distribution developed by Shanmugam et al. (2002), and have also given a nice explanation of the intervention parameter ($\rho$) used in the model. We hope this move
would enlighten distribution theory with new advantageous lifetime probability models such as intervened mixture probability models, generalized intervened probability models, and the intervention-based probability models developed by using different transformations or by adding new parameters are some possible directions. The distribution function (cdf) of intervened Exponential distribution \( I_{\rho,\alpha,\beta} \) along with probability density function (pdf) are given as

\[
F_{I_{\rho,\alpha,\beta}}(z; \xi) = \begin{cases} 
1 - e^{-(z-\alpha)/\rho\beta} - e^{-(z-\alpha)/\beta} & \rho \neq 1. \\
1 - \left(1 + \frac{z-\alpha}{\rho\beta}\right) e^{-(z-\alpha)/\beta} & \rho = 1.
\end{cases}
\]  

(1)

and,

\[
f_{I_{\rho,\alpha,\beta}}(z; \xi) = \begin{cases} 
\frac{e^{-(z-\alpha)/\rho\beta} - e^{-(z-\alpha)/\beta}}{(\rho-1)\beta} & \rho \neq 1. \\
\frac{e^{-(z-\alpha)/\rho\beta}}{\rho\beta} e^{-(z-\alpha)/\beta} & \rho = 1.
\end{cases}
\]  

(2)

where \( z > \alpha > 0 \), and the parametric space \( \xi = \{ (\rho, \alpha, \beta) : \rho > 0, \alpha > 0, \beta > 0 \} \), where \( \rho \) is the intervention parameter, \( \alpha \) is the truncation and \( \beta \) is the rate parameter of the distribution. Also, some derived results mentioned in the article of Shanmugam et al. (2002) are noticed as, the mean \( \mu_z = \alpha + (\rho+1)\beta \), variance \( \sigma_z^2 = (\rho^2+1)\beta^2 \) and the mode \( M_z = \alpha + (\rho\beta/(\rho - 1)) \ln |\rho| \) respectively.

The graphical illustration of pdf given in equation (2) is shown in Fig.1, where as for cdf given in equation (1) it is shown in Fig.2. The set of parametric values for both the plots are taken as, data1 = (\( \rho = 2.018, \alpha = 0.010, \beta = 4.500 \)), data2 = (\( \rho = 1.009, \alpha = 0.050, \beta = 5.055 \)), data3 = (\( \rho = 2.001, \alpha = 0.020, \beta = 6.050 \)), and data4 = (\( \rho = 2.018, \alpha = 0.010, \beta = 8.500 \)). It is quite visible from the graphical plots the pdf is exhibiting different shapes while changing the parametric values.

2. Inferential Study of \( I_{\rho,\alpha,\beta} \)

In this section, some interesting characteristics about \( I_{\rho,\alpha,\beta} \) are discussed in detail, these include mean deviations, central and non-central \( r^{th} \) order moments, different generating functions for moments, etc. are some of the useful measures having considerable importance in distribution theory. Firstly we begin the section by the median \( M_d \) derivation. Let \( z \) be a continuous and absolutely non-negative random variable \((r.v.)\) possessing \( I_{\rho,\alpha,\beta} \), then we proceed the steps for derivation as,

\[
\int_{M_d}^{\infty} f_{I_{\rho,\alpha,\beta}}(z; \xi) dz = 1/2
\]

\[
\Rightarrow \quad \frac{1}{(\rho-1)\beta} \int_{M_d}^{\infty} \left\{ e^{-(z-\alpha)/\rho\beta} - e^{-(z-\alpha)/\beta} \right\} dz = 1/2
\]
After solving the integral and simplifications, the final expression obtained is as,

$$\rho e^{-\frac{(M_d-\alpha)}{\rho \beta}} - e^{-\frac{(M_d-\alpha)}{\beta}} = \frac{(\rho - 1)}{2}$$

(3)

This derived expression for $M_d$ does not reveal the explicit solution, but we can find the approximate value for $M_d$ in R-software by using the `uniroot` function.

### 2.1. Mean Deviations

In statistics, the two prominent deviations are the mean deviation ($D_{\mu_z}$) and median deviation ($D_{M_d}$), which is the average of all the deviations taken from the mean and the respective median. These deviations have the advantage to measure the amount of scatteredness present in the data. Mathematically, the expressions are given by

$$D_{\mu_z} = \int_{-\infty}^{\infty} |z - \mu_z| f(z) dz \quad \text{and} \quad D_{M_d} = \int_{-\infty}^{\infty} |z - M_d| f(z) dz$$

**Theorem 2.1.** If a continuous and absolutely non-negative r.v. $z \sim I_v ED(\rho, \alpha, \beta)$, then the mean deviations about mean ($\mu_z$) and median ($M_d$) are given by

(i) $$D_{\mu_z} = 2 \left\{ (\mu_z - \alpha) - \beta (\rho + 1) + \frac{\beta}{(\rho - 1)} \left[ \rho^2 e^{-\frac{(\mu_z-\alpha)}{\rho \beta}} - e^{-\frac{(\mu_z-\alpha)}{\beta}} \right] \right\}$$

(4)

(ii) $$D_{M_d} = (\mu_z - M_d) + 2 \left\{ (M_d - \alpha) - \beta (\rho + 1) + \frac{\beta}{(\rho - 1)} \left[ \rho^2 e^{-\frac{(M_d-\alpha)}{\rho \beta}} - e^{-\frac{(M_d-\alpha)}{\beta}} \right] \right\}$$

(5)

**Proof.** (i) We have a non-negative r.v., $z \sim I_v ED(\rho, \alpha, \beta)$. Then the expression for $D_{\mu_z}$ is given by,

$$D_{\mu_z} = 2 \left\{ \mu_z F_{I_v, ED}(\mu_z) - \int_{\alpha}^{\mu_z} z f_{I_v, ED}(z; \xi) dz \right\}$$

$$= 2 \left\{ \mu_z F_{I_v, ED}(\mu_z) - \int_{\alpha}^{\mu_z} z \frac{e^{-\frac{(z-\alpha)/\rho \beta}{(\rho - 1)}} - e^{-\frac{(z-\alpha)/\beta}{\beta}}}{(\rho - 1)\beta} dz \right\}$$

$$= 2 \left\{ (\mu_z - \alpha) - \beta (\rho + 1) + \frac{\beta}{(\rho - 1)} \left[ \rho^2 e^{-\frac{(\mu_z-\alpha)}{\rho \beta}} - e^{-\frac{(\mu_z-\alpha)}{\beta}} \right] \right\}$$

Hence, completes proof for part first.

(ii) Again, for a continuous, and non-negative r.v., $z \sim I_v ED(\rho, \alpha, \beta)$, we can write the mathematical expression for median deviation as,

$$D_{M_d} = \mu_z - M_d + 2 \left\{ M_d F_{I_v, ED}(M_d) - \int_{\alpha}^{M_d} z f_{I_v, ED}(z; \xi) dz \right\}$$

$$= (\mu_z - M_d) + 2 \left\{ M_d F_{I_v, ED}(M_d) - \int_{\alpha}^{M_d} z \frac{e^{-\frac{(z-\alpha)/\rho \beta}{(\rho - 1)}} - e^{-\frac{(z-\alpha)/\beta}{\beta}}}{(\rho - 1)\beta} dz \right\}$$

$$= (\mu_z - M_d) + 2 \left\{ (M_d - \alpha) - \beta (\rho + 1) + \frac{\beta}{(\rho - 1)} \left[ \rho^2 e^{-\frac{(M_d-\alpha)}{\rho \beta}} - e^{-\frac{(M_d-\alpha)}{\beta}} \right] \right\}$$

This completes proof for part (ii). □

### 2.2. Moments and Generating Functions

In this subsection, we derive the expressions for generating functions, $r^{th}$ order central and non-central moments for $I_v ED(\rho, \alpha, \beta)$, in the following subsequent theorems.
**Theorem 2.2.** If a continuous r.v., \( z \sim I_v ED(\rho, \alpha, \beta) \), then the \( r \)th central moment \( \mu_r \), and non-central moment \( \mu'_r \) are given by

\[
(i) \quad \mu_r = \frac{1}{(\rho - 1)} \sum_{n=0}^{r} rC_n(-\mu)^{r-n} \beta^n \left\{ \frac{\rho^{n+1} e^{\alpha/\beta}}{\Gamma(n+1, \frac{\alpha}{\beta \rho})} - e^{\alpha/\beta} \Gamma(n+1, \frac{\alpha}{\beta}) \right\}.
\]

\[
(ii) \quad \mu'_r = \frac{\beta^{r+1}}{(\rho - 1)} \left\{ \frac{\rho^{n+1} e^{\alpha/\beta}}{\Gamma(n+1, \frac{\alpha}{\beta \rho})} - e^{\alpha/\beta} \Gamma(n+1, \frac{\alpha}{\beta}) \right\}.
\]

where, \( r = 1, 2, ..., n \) and \( \Gamma(s, x) = \int_x^\infty t^{s-1} e^{-t} dt \) is the upper incomplete gamma function.

**Proof.** (i) The expression for moments about mean for a r.v., \( z \sim I_v ED(\rho, \alpha, \beta) \) is given by

\[
\mu_r = \int_{-\infty}^{\infty} (z - \mu_z)^r f_{I_v ED}(z; \xi) dz
\]

\[
= \frac{1}{(\rho - 1)\beta} \int_{\alpha}^{\infty} (z - \mu_z)^r \left\{ e^{-(z - \alpha)/\beta} - e^{-(z - \alpha)/\beta} \right\} dz.
\]

\[
= \frac{1}{(\rho - 1)\beta} \sum_{n=0}^{r} rC_n(-\mu)^{r-n} \int_{\alpha}^{\infty} z^n \left\{ e^{-(z - \alpha)/\beta} - e^{-(z - \alpha)/\beta} \right\} dz.
\]

\[
= \frac{1}{(\rho - 1)} \sum_{n=0}^{r} rC_n(-\mu)^{r-n} \beta^n \left\{ \frac{\rho^{n+1} e^{\alpha/\beta}}{\Gamma(n+1, \frac{\alpha}{\beta \rho})} - e^{\alpha/\beta} \Gamma(n+1, \frac{\alpha}{\beta}) \right\}.
\]

Hence, completes proof for part (i).

**Proof.** (ii) Since, we know the mathematical expression for non-central moments, for a r.v. \( z \sim I_v ED(\rho, \alpha, \beta) \), is defined by

\[
\mu'_r = \int_{-\infty}^{\infty} z^r f_{I_v ED}(z; \xi) dz
\]

\[
= \frac{\rho^r e^{\alpha/\beta}}{(\rho - 1) \beta} \int_{\alpha}^{\infty} z^r e^{-z/\beta} dz - \frac{e^{\alpha/\beta}}{(\rho - 1) \beta} \int_{\alpha}^{\infty} z^r e^{-z/\beta} dz
\]

\[
= \frac{\beta^{r+1}}{(\rho - 1)} \left\{ \frac{\rho^{n+1} e^{\alpha/\beta}}{\Gamma(n+1, \frac{\alpha}{\beta \rho})} - e^{\alpha/\beta} \Gamma(n+1, \frac{\alpha}{\beta}) \right\}; \quad r = 1, 2, ..., n.
\]

Hence, proved the result.

**Theorem 2.3.** If \( z \sim I_v ED(\rho, \alpha, \beta) \), then, we defined the expressions for, moment generating function \( M_z(t) \), characteristic function \( \phi_z(t) \), and cumulant generating function \( K_z(t) \) as

\[
(i) \quad M_z(t) = \frac{e^{\alpha t}}{(1 - \beta t)(1 - \beta \rho t)}
\]

\[
(ii) \quad \phi_z(t) = \frac{e^{\alpha t}}{(1 - i \beta t)(1 - i \beta \rho t)}
\]

\[
(iii) \quad K_z(t) = \alpha t - \log(1 - \beta t) - \log(1 - \beta \rho t)
\]

**Proof.** (i) We know that, if \( z \sim I_v ED(\rho, \alpha, \beta) \), then mathematical expression for moment generating function is given by

\[
M_z(t) = \int_{\alpha}^{\infty} e^{zt} f_{I_v ED}(z; \xi) dz = \int_{\alpha}^{\infty} e^{zt} \frac{e^{-(z - \alpha)/\beta} - e^{-(z - \alpha)/\beta}}{(\rho - 1) \beta} dz
\]

\[
= \frac{\rho\beta e^{\alpha/\beta}}{\beta(\rho - 1)(1 - \beta \rho t)} - \frac{\beta e^{\alpha/\beta}}{\beta(\rho - 1)(1 - \beta t)}
\]

\[
= e^{\alpha t}[(1 - \beta t)(1 - \beta \rho t)]^{-1}
\]
Hence, proved the result (i).

(ii) To prove part (ii) of the above theorem, similar steps have to be repeated while deriving the expression, but the only difference is that instead of \( t \), we have to proceed with \( ut \) to obtain the resulting equation for characteristic function, i.e

\[
\phi_z(u) = e^{ut[(1 - \xi t)(1 - \xi t)]^{-1}}.
\]

(iii) Since, we know that, if \( z \sim I_e ED(\rho, \alpha, \beta) \), then the cumulant generating function is mathematically defined by:

\[
K_z(u) = \log \{ M_z(t) \} = \alpha t - \log(1 - \beta t) - \log(1 - \beta t)
\]

Hence, proved.

\( \square \)

3. Reliability Characterization

Reliability or Survival function is defined as the failure-free operation of a system, during a particular interval of time. For, many decades ago, this function is meant intrinsic features in performance activity measurements of the system, by considering failure data. Mathematically, the reliability \( R_{I_e ED}(z; \xi) \), for a non-negative, and absolutely continuous r.v., \( z \sim I_e ED(\rho, \alpha, \beta) \) with pdf \( f_{I_e ED}(z; \xi) \), is defined as

\[
R_{I_e ED}(z; \xi) = Pr(Z > z) = 1 - Pr(Z \leq z)
\]

\[
\Rightarrow R_{I_e ED}(z; \xi) = \begin{cases} 
\rho e^{-(z-\alpha)/\rho^\beta - e^{-(z-\alpha)/\beta}} / (\rho^1) \\
(1 + z/\beta) e^{-(z-\alpha)/\beta} 
\end{cases} \quad \rho \neq 1. \\
\rho = 1.
\]

By invariance property of MLEs, we can write the reliability estimate as given by

\[
R_{I_e ED}(z; \xi) = \begin{cases} 
\rho e^{-(z-\alpha)/\rho^\beta - e^{-(z-\alpha)/\beta}} / (\rho^1) \\
(1 + z/\beta) e^{-(z-\alpha)/\beta} 
\end{cases} \quad \rho \neq 1. \\
\rho = 1.
\]

The opposite of the reliability function is hazard rate function, often frequently named failure rate function, maybe constant or the function of the time. In literature, this function is widely acknowledged with alternative sub-titles in demographic and actuarial sciences. Among them, intensity rate, force of mortality, instantaneous force of mortality, and the mortality rate are sharply noticed. For a non-negative continuous r.v., \( z \sim I_e ED(\rho, \alpha, \beta) \) with pdf \( f_{I_e ED}(z; \xi) \) and cdf \( F_{I_e ED}(z; \xi) \), hazard rate function \( h_{I_e ED}(z; \xi) \) is as

\[
\hat{h}_{I_e ED}(z; \xi) = \frac{e^{-(z-\alpha)/\rho^\beta - e^{-(z-\alpha)/\beta}}}{\beta \{\rho e^{-(z-\alpha)/\rho^\beta - e^{-(z-\alpha)/\beta}}\}}
\]

The graphical plots of the reliability and hazard rate functions are shown in Fig.(3) and Fig.(4) respectively,

For both the plots the values of parameters are taken as, data1 = (\( \rho = 2.018, \alpha = 0.010, \beta = 4.500 \)), data2 = (\( \rho = 1.009, \alpha = 0.010, \beta = 4.500 \)).
\( \alpha = 0.050, \beta = 5.055 \), data3 \( = (\rho = 2.001, \alpha = 0.020, \beta = 6.050) \), and data4 \( = (\rho = 2.018, \alpha = 0.010, \beta = 8.500) \). The graphical illustration of hazard rate function as shown in Fig.(4) is clearly showing the increasing trend, which has useful application while analyzing the failure data in reliability.

In reliability theory, the well-known extended concept reverse hazard rate function is described as a hazard rate, but in a reverse direction of time. The ratio of pdf \( f_{I_v ED}(z; \xi) \) upon cdf \( F_{I_v ED}(z; \xi) \) is defined as a reverse hazard rate function. For \( I_v ED \) it is obtained as

\[
\lambda_{I_v ED}^p(z; \xi) = \frac{e^{-(z-\alpha)/\rho\beta} - e^{-(z-\alpha)/\beta}}{\beta \left( (\rho - 1) - \left[ \rho e^{-(z-\alpha)/\rho\beta} - e^{-(z-\alpha)/\beta} \right] \right)}
\]  

Statistical literature revealed, the approximate constant, increasing or decreasing trends of unimodal hazard rate function. Therefore, for the system, aging representation is tedious. Thus, a new reliability function, named aging intensity (A.I) profounded by Jiang et al. (2003) would have the quantitative measurement for aging. Thus A.I for a r.v., \( z \sim I_v ED(\rho, \alpha, \beta) \) is denoted by \( L_{I_v ED}(z; \xi) \) and is defined as

\[
L_{I_v ED}(z; \xi) = \frac{h_{I_v ED}(z; \xi)}{X_{I_v ED}(z; \xi)} = \frac{-zf_{I_v ED}(z; \xi)}{R_{I_v ED}(z; \xi) \ln |R_{I_v ED}(z; \xi)|} ; \quad z > 0.
\]

Thus, for \( I_v ED \) our derived result for A.I are as

\[
L_{I_v ED}(z; \xi) = \frac{z}{\beta} \left[ e^{-(z-\alpha)/\rho\beta} - e^{-(z-\alpha)/\beta} \right] \left[ \ln |\rho - 1| - \ln |\rho e^{-(z-\alpha)/\rho\beta} - e^{-(z-\alpha)/\beta}| \right]
\]

where, \( f_{I_v ED}(.) \) and \( R_{I_v ED}(.) \) are respective pdf and reliability function of model, and the average failure rate function \( X_{I_v ED}(z; \xi) = (1/z) \int_0^z h(z; \xi)dz \).

### 3.1. Shape of Hazard Rate Function

The actual shape behavior of the hazard rate function given in equation (8) is not clear from the graphical plot as shown in Fig.(4). To provide the solid mathematical interpretation regarding the shape of the hazard rate function, the following lemma due to Glaser (1980) is defined as:

**Lemma 3.1.** Suppose for a continuous r.v. \( x \), the density function \( f(x) > 0 \), for all \( x > 0 \), \( f'(x) \) be its derivative, and \( \eta(x) = -f'(x)/f(x) \). Thus, if \( \eta(x) > 0 \), for all \( x > 0 \). The hazard rate function corresponding to \( f(x) \) is increasing function of \( x \).

**Proof.** For complete proof refer to Glaser (1980).

**Theorem 3.2.** The hazard rate function of \( I_v ED(\alpha, \beta, \rho) \) is increasing, when \( \rho \neq 1 \) and \((\alpha, \beta, \rho) > 0 \).

**Proof.** The pdf \( f(z; \xi) \) of \( I_v ED(\alpha, \beta, \rho) \), when \( \rho \neq 1 \), is defined in equation (2). Therefor according to above Lemma 3.1., the function \( \eta(z; \xi) \) of a continuous non-negative r.v. \( z \) is defined as

\[
\eta(z; \xi) = -\frac{f'(z; \xi)}{f(z; \xi)} = \frac{e^{-(z-\alpha)/\rho\beta} - \rho e^{-(z-\alpha)/\beta}}{\rho\beta \left( e^{-(z-\alpha)/\rho\beta} - e^{-(z-\alpha)/\beta} \right)}
\]

After differentiating \( \eta(z; \xi) \) with respect to \( z \) and on simplifying we get

\[
\eta'(z; \xi) = \frac{(\rho - 1)^2}{(\rho\beta)^2} \frac{e^{-(z-\alpha)(\rho+1)/\rho\beta}}{\left( e^{-(z-\alpha)/\rho\beta} - e^{-(z-\alpha)/\beta} \right)^2} > 0
\]

This is true for \((\alpha, \beta, \rho) > 0 \), when \( \rho \neq 1 \). Thus, we have \( \eta'(z; \xi) > 0 \), for all \( z > 0 \), for given pdf \( f(z; \xi) \). Therefore according to Lemma 3.1., the statement of Theorem 3.2. is proved, the hazard rate function of \( I_v ED(\alpha, \beta, \rho) \), when \( \rho \neq 1 \), is increasing for all \( z > 0 \).
3.2. Mean Residual Life Function

Specifically, to determine the distribution, an alternative approach Mean Residual Life \((MRL)\) function does exist. Suppose a system working up to time \(t \geq 0\), then the remaining working hours until it fails i.e beyond time \(t\) is termed as residual life, defined by conditional \(r.v.\) \(Z - t | Z > t\). Precisely, more attractive and interesting story about \(MRL\) function being that, this is advantageous in many branches of science, engineering science, economics, survival analysis, and reliability theory for characterizing the lifetime. Although, it is quite difficult to explore failure rate function without undertaking other measures. It is theoretically obvious, the most prominent among them being \(MRL\) function as they are complementary functions to each other Finkelstein (2008). Thus for a \(r.v.\) \(z \sim I_v ED(\rho, \alpha, \beta)\), the \(MRL\) function \((m_{I_v ED}(z; \xi))\) is obtained as,

\[
m_{I_v ED}(z; \xi) = E[Z - t | Z > t] = \frac{1}{R_{I_v ED}(z; \xi)} \int_z^\infty R_{I_v ED}(z; \xi)dz\]

\[
= \left[\rho e^{-(z-\alpha)/\rho \beta} - e^{-(z-\alpha)/\beta}\right]^{-1} \int_z^\infty \left[\rho e^{-(z-\alpha)/\rho \beta} - e^{-(z-\alpha)/\beta}\right]dz
\]

\[
= \beta \left[\rho^{2} e^{-(z-\alpha)/\rho \beta} - e^{-(z-\alpha)/\beta}\right]\left[\rho e^{-(z-\alpha)/\rho \beta} - e^{-(z-\alpha)/\beta}\right]^{-1}
\]

which is required derived \(MRL\) function for \(I_v ED\).

4. Mean Inactivity Time

Mean Inactivity Time \((MIT)\) have strong applications in different branches of applied sciences, such as, reliability analysis, survival analysis and economics are some areas. If \(z \sim I_v ED(\rho, \alpha, \beta)\), then the mean inactivity time is given by

\[
M_IT(z; \xi) = \frac{1}{\alpha - f_{I_v ED}(z; \xi)} \int_\alpha^z zf_{I_v ED}(z; \xi)dz, \quad z > \alpha.
\]

\[
= \frac{1}{(\rho - 1)\beta F_{I_v ED}(z)} \int_\alpha^z z \left\{e^{-(z-\alpha)/\rho \beta} - e^{-(z-\alpha)/\beta}\right\}dz.
\]

\[
= \frac{1}{(\rho - 1)\beta F_{I_v ED}(z)} \left\{e^{\alpha/\rho \beta} \int_\alpha^z z e^{-(z-\alpha)/\rho \beta}dz - e^{\alpha/\beta} \int_\alpha^z ze^{-(z-\alpha)/\beta}dz\right\}
\]

After integration, and simplifications we get the result as

\[
\Rightarrow M_IT(z; \xi) = \frac{(\rho - 1) [\alpha] + (\rho + 1) \beta - \beta \left[\rho^{2} e^{-(z-\alpha)/\rho \beta} - e^{-(z-\alpha)/\beta}\right]}{(\rho - 1) - \left[\rho e^{-(z-\alpha)/\rho \beta} - e^{-(z-\alpha)/\beta}\right]} \quad \quad \quad (11)
\]

5. Strong Mean Inactivity Time

This newly developed reliability measure strong mean inactivity time \((SMIT)\) function by Kayid and Izadkhah (2014) is useful in reliability and survival analysis. Let, \(z \sim I_v ED(\rho, \alpha, \beta)\), then we derive \(SMIT\) function as

\[
M_{IT}^S(z; \xi) = \frac{1}{\beta F(z)} \int_\alpha^z 2zf_{I_v ED}(z; \xi)dz
\]

\[
= z^{2} - \frac{1}{\beta F(z)} \int_\alpha^z 2zf_{I_v ED}(z; \xi)dz
\]

\[
= z^{2} - \frac{1}{\beta F(z)} \int_\alpha^z 2e^{-(z-\alpha)/\rho \beta} - e^{-(z-\alpha)/\beta} \frac{1}{(\rho - 1)\beta}dz
\]

\[
= z^{2} - \frac{1}{\beta(\rho - 1)F(z)} \sum_{r=0}^{\infty} (-1)^r \frac{e^{\alpha/\rho \beta} - \rho e^{\alpha/\beta}}{r!(r+3)}\left[z^{r+3} - \alpha e^{\alpha+3}\right]
\]

(12)
6. Entropy Measures

In this section, we discuss two popular entropy metrics namely Rényi entropy (see Rényi 1961) and Shannon entropy (see Shannon 1948), which are used to measure the amount of information or uncertainty. Here, we derive Rényi and Shannon entropy mathematically.

6.1. Rényi Entropy

Let a r.v. \( z \sim I_v ED(\rho, \alpha, \beta) \), then the Rényi entropy of order \( \kappa \) is given by

\[
H_R(\kappa) = \frac{1}{1 - \kappa} \log \left\{ \int_{0}^{\infty} (f_{I_v ED}(z; \xi))^{\kappa} \, dz \right\}; \quad \kappa \geq 0, \kappa \neq 1
\]

\[
= \frac{1}{1 - \kappa} \log \left\{ \int_{0}^{\infty} \left( e^{-\frac{(1-z)\rho \beta}{(\rho-1)\beta}} - e^{-\frac{(1-z)\alpha}{\beta}} \right)^{\kappa} \, dz \right\}
\]

\[
= \frac{1}{1 - \kappa} \log \left\{ \frac{\rho}{(\rho-1)^2} \int_{0}^{1} z^{\kappa} [1-z]^{\rho (\kappa+1) - 1} \, dz \right\}
\]

\[
= \frac{1}{1 - \kappa} \log \left\{ \frac{\rho}{(\rho-1)^2} B \left( 2 - \frac{\beta \rho \kappa}{\rho - 1}, \kappa + 1 \right) \right\}
\]

which is required expression of Rényi entropy of order \( \kappa \).

Note: \( B(a, b) = \int_{0}^{1} z^{a-1}(1-z)^{b-1} \, dz \) is the beta function.

6.2. Shannon Entropy

Shannon entropy is the extended result of Rényi entropy, thus for a continuous r.v. \( z \sim I_v ED(\rho, \alpha, \beta) \) this measure is obtained as

\[
H_{I_v ED}(z) = -\int_{0}^{\infty} f_{I_v ED}(z; \xi) \log (f_{I_v ED}(z; \xi)) \, dz.
\]

After substituting the equation (2) of pdf, the simplified result we obtained for Shannon measure of entropy are given by

\[
H_{I_v ED}(z) = \rho \sum_{r=1}^{\infty} \left\{ r [\beta \rho + (r+1)(\rho-1)] [\beta \rho + r(\rho-1)] \right\}^{-1} + \frac{2\beta \rho + \rho - 1}{\beta (\beta \rho + \rho - 1)^2}
\]

7. Order Statistics of \( I_v ED \)

Order statistics are counted among important branches of statistics, having numerous applications in reliability analysis and life testing modeling used to study the system reliability characteristics. Let \( z = \{z_1, z_2, z_3, \ldots, z_n\} \), be a random sample from \( I_v ED(\rho, \alpha, \beta) \) with cdf and its pdf defined in equations (1) and (2) respectively. Let us consider the ordered random sample as \( z_{(1:n)} \leq z_{(2:n)} \leq z_{(3:n)} \leq \ldots \leq z_{(n:n)} \), so that \( z_{(i:n)}; 1 \leq i \leq n \) having the life time of \([n-i+1]\) out-of-n system consisting \( n \) i.i.d components. The pdf of \( i^{th} \) order statistic \( z_{(i:n)}; 1 \leq i \leq n \), is given by:

\[
f_{i:n}(z) = K_1 [F_{I_v ED}(z; \xi)]^{i-1} [1 - F_{I_v ED}(z; \xi)]^{n-i} f_{I_v ED}(z; \xi).
\]

(13)

The joint \((i, j)^{th}\) order density function of \((z_{(i:n)}, z_{(j:n)})\) for \(1 \leq i \leq j \leq n\) is given below:

\[
f_{i:j:n}(z_i, z_j) = K_2 [F_{I_v ED}(z_i)]^{i-1} [F_{I_v ED}(z_j) - F_{I_v ED}(z_i)]^{j-i-1} [1 - F_{I_v ED}(z_j)]^{n-j} f_{I_v ED}(z_i) f_{I_v ED}(z_j).
\]

(14)

where,

\[
K_1 = \frac{n!}{(i-1)!(n-i)!}, \quad \text{and} \quad K_2 = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}.
\]
Thus, for an ordered random sample of \( n \) observations, \( 1^{st} \) order statistic is the smallest observation given by \( z_{(1)} = \min \{ z(1), z(2), z(3), \ldots, z(n) \} \), while the \( n^{th} \) order statistic is the largest sample observation given by \( z_{(n)} = \max \{ z(1), z(2), z(3), \ldots, z(n) \} \), and the median order statistics is the middle observation given by \( z_{m+1} \).

7.1. Density Function of \( 1^{st}, n^{th} \) and Median Order Statistic

Let \( z_1, z_2, z_3, \ldots, z_n \) be \( i.i.d \) r.v. from \( I_v \, \text{ED} \). Then, the density functions for first \((f_{1:n}(.)\)) last \((f_{n:n}(.)\)) and the median \((f_{m+1:n}(.)\) [\( m = \frac{n}{2} \)] are given below:

\[
f_{1:n}(z) = n \left[ 1 - F_{I_v \, \text{ED}}(z; \xi / \beta) \right]^{n-1} f_{I_v \, \text{ED}}(z; \xi / \beta). \tag{15}
\]

Similarly,

\[
f_{n:n}(z) = n \left[ F_{I_v \, \text{ED}}(z; \xi / \beta) \right]^{n-1} f_{I_v \, \text{ED}}(z; \xi / \beta). \tag{16}
\]

and,

\[
f_{m+1:n}(z) = \frac{(2m + 1)!}{(m!)^2} \left[ F_{I_v \, \text{ED}}(z; \xi / \beta) \right]^m \left[ 1 - F_{I_v \, \text{ED}}(z; \xi / \beta) \right]^m f_{I_v \, \text{ED}}(z; \xi / \beta). \tag{17}
\]

7.2. Joint Order Density

The joint \((i, j)^{th}\) order statistic pdf of \( I_v \, \text{ED} \) is given by:

\[
f_{i:j:n}(z(i), z(j)) = \frac{K_2}{\beta^2(\rho - 1)^{n+2}} \left[ (\rho - 1) - \left\{ \frac{e^{-(z(i)-\alpha)/\rho^2} - e^{-(z(i)-\alpha)/\beta}}{\beta} \right\} \right]^{i-1} \cdot \left[ \left\{ \frac{e^{-(z(i)-\alpha)/\rho^2} - e^{-(z(i)-\alpha)/\beta}}{\beta} \right\} - \left\{ \frac{e^{-(z(j)-\alpha)/\rho^2} - e^{-(z(j)-\alpha)/\beta}}{\beta} \right\} \right]^{j-1} \cdot \left[ \frac{e^{-(z(i)-\alpha)/\rho^2} - e^{-(z(j)-\alpha)/\beta}}{\beta} \right] \cdot \left[ \frac{e^{-(z(i)-\alpha)/\rho^2} - e^{-(z(i)-\alpha)/\beta}}{\beta} \right]^{n-i-j} \tag{18}
\]

8. Stress-Strength Reliability

Stress-strength modeling, is another famous system reliability measurement technique, having a wide variety of applications in different areas of statistical sciences and engineering. Suppose a system, having specific strength \( Z_1 \) subjected to common stress \( Z_2 \). Whenever stress exceeds the strength, failure will occur and functions smoothly when \( Z_1 > Z_2 \). Since, \( R = Pr(Z_1 > Z_2) \) is the desired model for system reliability under stress strength modeling.

Now, for independent r.v.s \( Z_1 \sim I_v \, \text{ED}(\alpha, \rho_1, \beta_1) \) and \( Z_2 \sim I_v \, \text{ED}(\alpha, \rho_2, \beta_2) \), having the same parameter \( (\alpha) \), with pdf of \( Z_1 \) and cdf of \( Z_2 \), when \( \rho_i \neq 1 \) (\( i = 1, 2 \)), the derivation for \( R \) is given below:

\[
f_{Z_1}(z) = \frac{e^{-(z-\alpha)/\rho_1\beta_1} - e^{-(z-\alpha)/\beta_1}}{(\rho_1 - 1)\beta_1} \tag{18}
\]

and,

\[
F_{Z_2}(z) = 1 - \frac{\rho_2 e^{-(z-\alpha)/\rho_2\beta_2} - e^{-(z-\alpha)/\beta_2}}{(\rho_2 - 1)} \tag{19}
\]
We have,

\[
R = \int_{\alpha}^{\infty} \left\{ \int_{\alpha}^{z} f_{z_2}(z) dz \right\} f_{z_1}(z) dz = \int_{\alpha}^{\infty} F_{z_2}(z)f_{z_1}(z) dz
\]

\[
= \int_{\alpha}^{\infty} \left\{ 1 - \frac{\rho_2 e^{-(z-\alpha)/\rho_2 \beta_2} - e^{-(z-\alpha)/\beta_2}}{(\rho_2 - 1)\beta_2} \right\} \left\{ \frac{e^{-(z-\alpha)/\rho_1 \beta_1} - e^{-(z-\alpha)/\beta_1}}{(\rho_1 - 1)\beta_1} \right\} dz
\]

\[
= 1 - \beta_2^2 \left\{ \frac{\rho_1 \beta_1 (\rho_1 + \rho_2 + \rho_2^2 \beta_2) + \rho_2 \beta_1 \beta_2 (\rho_1 + 1)(\rho_2 + 1) + \rho_2^2 \beta_2^2}{(\beta_1 + \beta_2)(\beta_1 + \rho_2 \beta_2)(\beta_2 + \rho_1 \beta_1)(\rho_1 \beta_1 + \rho_2 \beta_2)} \right\}
\]

Now, let us consider \( \rho_1 = 1 \) and \( \rho_2 \neq 1 \). Then, in this situation, we have derived the expression for stress-strength reliability, which is given by

\[
R = 1 - \frac{\beta_2^2}{(\rho_2 - 1)} \left( \frac{1}{(\beta_1 + \beta_2)^2} + \frac{\rho_2^3}{(\beta_1 + \rho_2 \beta_2)} \right)
\]

The next possibility is if \( \rho_1 \neq 1 \) but \( \rho_2 = 1 \). Then, in this case, the obtained stress-strength reliability is given by

\[
R = 1 - \frac{\beta_2^2}{(\beta_1 + \beta_2)} \left( \frac{\rho_1 \beta_1 (\rho_1 + \beta_2) + \beta_2 \beta_1 \beta_2 (\rho_1 + 1)(\rho_2 + 1)}{(\beta_1 + \beta_2)^2 (\rho_1 \beta_1 + \beta_2)} \right)
\]

The last possibility is when both \( \rho_1 \) and \( \rho_2 \) are equal to 1. Then, in this case, we have derived the stress-strength reliability given by

\[
R = 1 - \frac{\beta_2^2}{(\beta_1 + \beta_2)^2} - \frac{2\beta_2}{(\beta_1 + \beta_2)^3}
\]

9. Stochastic Ordering

To judge the performances of lifetime distributions stochastic ordering is considered to be the critical and essential technique briefly discussed by Shaked and Shanthikumar (2007). Let \( Z_1 \) and \( Z_2 \), be two r.v.’s each following \( I_v.E_D \) having cdf’s \( F_1(z) \) and \( F_2(z) \), with respective pdf’s \( f_1(z) \) and \( f_2(z) \) respectively. Then, we say \( Z_1 \) is smaller than \( Z_2 \), according to the below-mentioned orderings:

[a1] Stochastic order \((Z_1 \leq_{st} Z_2)\), if \( F_1(z) \geq F_2(z) \) for all \( z \).

[a2] Hazard rate order \((Z_1 \leq_{hr} Z_2)\), if \( H_{Z_1}(z) \geq H_{Z_2}(z) \) for all \( z \).

[a3] Mean residual life order \((Z_1 \leq_{MRL} Z_2)\), if \( M_{Z_1}(z) \geq M_{Z_2}(z) \) for all \( z \).

[a4] Likelihood ratio order \((Z_1 \leq_{LR} Z_2)\), if \( f_1(z)/f_2(z) \) decreasing in \( z \).

Hence, the above stochastic ordering reveal the following implications.

\( Z_1 \leq_{LR} Z_2 \Rightarrow Z_1 \leq_{hr} Z_2 \Rightarrow Z_1 \leq_{MRL} Z_2 \) and \( Z_1 \leq_{hr} Z_2 \Rightarrow Z_1 \leq_{st} Z_2 \)

The following, theorem defines likelihood ratio ordering of \( I_v.E_D \) with respect to their strongest likelihood.

**Theorem 9.1.** Let \( Z_1 \sim I_v.E_D(\alpha, \rho_1, \beta_1) \) and \( Z_2 \sim I_v.E_D(\alpha, \rho_2, \beta_2) \). If \((\rho_1, \beta_1) > (\rho_2, \beta_2)\), then \( Z_1 \leq_{st} Z_2 \).

**Proof.** The likelihood ratio is given by

\[
\phi = \frac{f_{Z_1}(z)}{f_{Z_2}(z)} = \frac{\beta_1 (\rho_1 - 1)}{\beta_2 (\rho_2 - 1)} \left\{ \frac{e^{-z(\rho_1 \beta_1)} - e^{-z(\rho_2 \beta_2)}}{e^{-z(\beta_1 \beta_2)} - e^{-z(\beta_2 \beta_1)}} \right\}
\]

\[
\Rightarrow \phi' = (C/K) \left\{ (\rho_1 \beta_1) \left( e^{-z(\rho_1 \beta_1)} - e^{-z(\rho_2 \beta_2)} \right) \left( \beta_2 e^{-z(\rho_1 \beta_1)} - \beta_1 e^{-z(\rho_2 \beta_2)} \right) - (\rho_2 \beta_2) \left( e^{-z(\rho_1 \beta_1)} - e^{-z(\rho_2 \beta_2)} \right) \right\}
\]

where, \( C = \{(\rho_2 - 1)\{\beta_1^2 \rho_2 \beta_2 (\rho_2 - 1)\}^{-1} \) and \( K = (e^{-z(\rho_1 \beta_1)} - e^{-z(\rho_2 \beta_2)})^2 \). Hence it is clear form \( \phi' \), if \((\rho_1, \beta_1) > (\rho_2, \beta_2) \Rightarrow \phi' > 0 \). Therefore, \((Z_1 \leq_{LR} Z_2)\). By above relationship other statements also hold. \( \square \)
10. Estimation Procedure for Parameters

Let \( z = (z_1, z_2, z_3, ..., z_n) \) be a sample of random observations drawn form \( I_vED \) with some desired parameters \( \rho, \alpha \) and \( \beta \). Let us consider \( \omega = (\rho, \alpha, \beta)^T \) be a \( k \times 1 \) vector of parameters. Then the complete sample log likelihood function when \( \rho \neq 1 \) is defined below:

\[
\log L = \sum_{i=1}^{n} \log \left\{ e^{-(z_\omega)/\rho\beta} - e^{-(z_\omega)/\beta} \right\} - n \log(\rho - 1) - n \log \beta
\]  

(20)

Let us consider, \( G_1 = e^{-\left(\frac{z_\omega}{\rho\beta}\right)} \) and \( G_2 = e^{-\left(\frac{z_\omega}{\beta}\right)} \). Therefore the above equation can be written as,

\[
\log L = \sum_{i=1}^{n} \log \left\{ G_1 - G_2 \right\} - n \log(\rho - 1) - n \log \beta
\]  

(21)

Now differentiate above equation partially with respect to given parameters \( \rho, \alpha \) and \( \beta \) we get.

\[
\frac{\partial \log L}{\partial \rho} = \sum_{i=1}^{n} \frac{(z_i - \alpha)G_1}{\beta \rho^2 [G_1 - G_2]} - \frac{n}{\rho - 1}
\]  

(22)

\[
\frac{\partial \log L}{\partial \alpha} = \sum_{i=1}^{n} \frac{G_1 - \rho G_2}{\beta \rho [G_1 - G_2]} - 0 - 0
\]  

(23)

\[
\frac{\partial \log L}{\partial \beta} = \sum_{i=1}^{n} \frac{(z_i - \alpha) [G_1 - \rho G_2]}{\beta \rho^2 [G_1 - G_2]} - \frac{n}{\beta}
\]  

(24)

On equating the above equations to zero i.e., \( \frac{\partial \log L}{\partial \rho} = 0, \frac{\partial \log L}{\partial \alpha} = 0 \) and \( \frac{\partial \log L}{\partial \beta} = 0 \), the \( MLE_s \) of parameters \( (\rho, \alpha, \beta) \) are obtained say \( (\hat{\rho}, \hat{\alpha}, \hat{\beta}) \). Since the equations (22), (23) and (24) are not in closed form, so alternative approach such as Newton-Raphson technique can be employed to obtain the \( MLE_s \). However log-likelihood equation may be maximized more conveniently with the help of R-software by using the \text{optim} or \text{nls} functions.

For all the three parameters of \( I_vED(\rho, \alpha, \beta) \), the second order partial derivatives of log-likelihood function exists. Hence the obtained Inversion dispersion matrix is given below:

\[
\begin{pmatrix}
\hat{\rho} \\
\hat{\alpha} \\
\hat{\beta}
\end{pmatrix} \sim N \left( \begin{bmatrix}
\rho \\
\alpha \\
\beta
\end{bmatrix}, \begin{bmatrix}
Z_{\rho\rho} & Z_{\rho\alpha} & Z_{\rho\beta} \\
Z_{\alpha\rho} & Z_{\alpha\alpha} & Z_{\alpha\beta} \\
Z_{\beta\rho} & Z_{\beta\alpha} & Z_{\beta\beta}
\end{bmatrix} \right)
\]  

(25)

Therefore,

\[
I^{-1} = -E \begin{bmatrix}
Z_{\rho\rho} & Z_{\rho\alpha} & Z_{\rho\beta} \\
Z_{\alpha\rho} & Z_{\alpha\alpha} & Z_{\alpha\beta} \\
Z_{\beta\rho} & Z_{\beta\alpha} & Z_{\beta\beta}
\end{bmatrix}
\]

where \( Z_{\rho\rho} = \frac{\partial^2 \log L}{\partial \rho^2} \), \( Z_{\alpha\alpha} = \frac{\partial^2 \log L}{\partial \alpha^2} \), \( Z_{\beta\beta} = \frac{\partial^2 \log L}{\partial \beta^2} \), \( Z_{\rho\alpha} = \frac{\partial^2 \log L}{\partial \rho \partial \alpha} \), \( Z_{\rho\beta} = \frac{\partial^2 \log L}{\partial \rho \partial \beta} \) and \( Z_{\alpha\beta} = \frac{\partial^2 \log L}{\partial \alpha \partial \beta} \).

After finding the Inverse dispersion matrix, we can obtain easily asymptotic variances and co-variances for \( MLE_s(\rho, \alpha, \beta) \). Now, to determine the approximate confidence interval of 100(1 - \delta)% by using (25) as given below:

\[
\hat{\rho} \pm y_{\frac{\delta}{2}} \sqrt{Z_{\rho\rho}}, \hat{\alpha} \pm y_{\frac{\delta}{2}} \sqrt{Z_{\alpha\alpha}} \quad \text{and} \quad \hat{\beta} \pm y_{\frac{\delta}{2}} \sqrt{Z_{\beta\beta}}.
\]

where \( y_{\frac{\delta}{2}} \) is defined as upper 100\delta th quantile of standard normal distribution.
10.1. Simulation Study Performance

For $I_v ED(\rho, \alpha, \beta)$, a Monte Carlo simulation study by repeating the process 1000 times has been carried out for different sample sizes, to check the performances of MLEs. Since the quantile function of the proposed model is not in closed form. Thus inverse transformation method is not applicable to generate the data from $I_v ED$. However, in such a situation other data generation techniques namely, the acceptance-rejection algorithm is used to simulate the data. The results are shown in table 1.

|        |        |        |        |        |        |        |        |
|--------|--------|--------|--------|--------|--------|--------|--------|
| $n$    | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\rho}$ | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\rho}$ |
| 025    | 0.16432 | 0.32000 | 41.84011 | 0.02700 | 0.10240 | 1750.595 |
| 075    | 0.05933 | 0.18554 | 02.50047 | 0.00352 | 0.03442 | 06.25237 |
| 125    | 0.03919 | 0.12980 | 00.77050 | 0.00994 | 0.01055 | 00.34922 |
| 175    | 0.02590 | 0.08546 | 00.36669 | 0.00067 | 0.00730 | 00.13446 |
| 250    | 0.01624 | 0.05426 | 00.21414 | 0.00015 | 0.00294 | 00.04586 |
| 400    | 0.01624 | 0.05426 | 00.21414 | 0.00015 | 0.00294 | 00.04586 |

We have obtained the estimates for all parameters with the help of R-software by using the nlm package, corresponding Abias and MSE for parameters are also calculated with different sample sizes ($n = 25, 75, 125, 175, 250, 400$) respectively. The simulation study reveals that, the decrease of absolute Abias and MSE while increasing the sample size. Hence consistency property for $I_v ED$ are satisfied.

11. Applications

This section illustrates, model applicability based on two data sets. The comparison of $I_v ED$ with other pre-existing models available in the literature, such as Exponential distribution (ED) and generalized Exponential distribution (GED) has been performed. In Table 2, and Table 3, the Comparision Criteria – I will provide the key information regarding Akaike, Corrected Akaike, Bayesian and Hannan-Quinn Information Criteria abbreviated as AIC, CAIC, BIC, and HQIC. Whereas the Comparision Criteria – II will provide the information about Cramer-Von Mises (W), Anderson Darling (A), Kolmogorov-Smirnov (KS) statistic and p-value. Thus both the comparisons together will fulfill the model adequacy requirement.

The first data set given by Fuller Jr contains window strength data. By using this data set, the lifespan of the glass-airplane window was predicted by Fuller Jr et al. (1994). The collected data are: 18.83, 20.80, 21.657, 23.03, 23.23, 24.05, 24.321, 25.5, 25.52, 25.8, 26.96, 26.77, 26.78, 27.05, 27.67, 29.9, 31.11, 33.2, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98, 37.08, 37.09, 39.58, 44.045, 45.29, 45.381.
The output results based on data set first is given below:

| Models | Comparison Criteria – I | \( \hat{\alpha} \) | \( \hat{\beta} \) | \( \hat{\rho} \) | AIC | CAIC | BIC | HQIC |
|--------|--------------------------|----------------|----------------|----------------|-----|-----|-----|-----|
| \( I_v ED \) | 18.2147 | 06.1514 | 01.0419 | 215.1429 | 216.0318 | 219.4448 | 216.5452 |
| \( ED \) | 30.8192 | - | - | 276.5464 | 276.6843 | 277.9804 | 277.0139 |
| \( GED \) | 00.0927 | 10.6835 | - | 225.7815 | 226.2101 | 228.6495 | 226.7164 |

| Models | Comparison Criteria – II | \( \hat{\alpha} \) | \( \hat{\beta} \) | \( \hat{\rho} \) | W | A | KS | p-value |
|--------|---------------------------|----------------|----------------|----------------|---|---|---|---------|
| \( I_v ED \) | 04.2968 | 15.0007 | 13.4493 | 567.1858 | 567.7858 | 527.5384 | 569.1708 |
| \( ED \) | 116.593 | - | - | 589.4366 | 589.5319 | 591.2208 | 590.0983 |
| \( GED \) | 0.00888 | 2.39893 | - | 586.1392 | 586.4319 | 589.7076 | 587.4625 |

From the results shown in Table 2 and Table 3 it is clear that \( I_v ED \) performs better than \( ED \) and \( GED \) respectively. Hence, this confirms the model applicability for future data analysis purposes.

12. Conclusion

In this article, we introduced intervened Exponential distribution \( (I_v ED) \) as a lifetime model in reliability and survival analysis. The derived statistical and reliability properties has been presented in this study. We have plotted graphs of different functions, for pdf it exhibits different shapes, whereas the hazard rate function plot shows this could be useful to model the increasing failure rate data sets. Simulation study of \( I_v ED \) based on \( MLE \) were also discussed to judge the performance of the parameters. Finally, we have done the comparison study with the pre-exists models by analyzing two real data sets, and this study revealed that the \( I_v ED \) outperformed among the two well-known models, namely \( ED \) and \( GED \) available in statistical literature.

Acknowledgment

The first author is very grateful to Pondicherry University for providing him with a research fellowship to carry out this work.
References

1. Balakrishnan, K. (2019). *Exponential Distribution: theory, methods and applications*. CRC Press.

2. Barlow, R. E. and Proschan, F. (1975). *Statistical theory of reliability and life testing*. Holt Rinehart and Winston, New York.

3. Dhanavanthan, P. (1998). Compound intervened Poisson distribution. *Biometrical Journal*, 40(5):641–664, doi:10.1002/(SICI)1521-1521.

4. Dhanavanthan, P. (2000). Estimation of the parameters of compound intervened Poisson distribution. *Biometrical Journal*, 42(3):315–320, doi.org/10.1002/1521–4036(200007)42:3<315::AID–BIMJ315>3.0.CO;2–E.

5. Efron, B. (1988). Logistic regression, survival analysis, and the Kaplan-Meier curve. *Journal of the American Statistical Association*, 83(402):414–425, doi:10.1080/01621459.1988.10478612.

6. Finkelstein, M. (2008). *Failure rate modelling for reliability and risk*. Springer Science & Business Media.

7. Fuller Jr, E. R., Freiman, S. W., Quinn, J. B., Quinn, G. D., and Carter, W. C. (1994). Fracture mechanics approach to the design of glass aircraft windows: a case study. In Kloczek, P., editor, *Window and Dome Technologies and Materials IV*, International Society for Optics and Photonics, SPIE,. volume 2286, pages 419 – 430, doi :10.1117/12.187363.

8. Glaser, R. E. (1980). Bathtub and related failure rate characterizations. *Journal of the American Statistical Association*, 75(371):667–672, doi: 10.1080/01621459.1980.10477530.

9. Huang, M. L. and Fung, K. Y. (1989). Intervened truncated Poisson distribution. *Sankhyā: The Indian Journal of Statistics, Series B*, 51(3):302–310, doi: https://www.jstor.org/stable/25052598.

10. Jiang, R., Ji, P., and Xiao, X. (2003). Aging property of unimodal failure rate models. *Reliability Engineering & System Safety*, 79(1):113–116, doi:10.1016/S0951–8320(02)00175–8.

11. Kayid, M. and Izadkhah, S. (2014). Mean inactivity time function, associated orderings, and classes of life distributions. *IEEE Transactions on Reliability*, 63(2):593–602, doi:10.1109/TR.2014.2315954.

12. Marshall, A. W. and Olkin, I. (2007). *Life distributions*. Springer, Verlag New York.

13. Rényi, A. (1961). On measures of entropy and information. In *In Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics*, pages 547–561. University of California Press.

14. Scollnik, D. P. (1995). Bayesian analysis of an intervened Poisson distribution. *Communications in Statistics - Theory and Methods*, 24(3):735–754, doi:10.1080/03610929508831519.

15. Scollnik, D. P. (2006). On the intervened generalized Poisson distribution. *Communications in Statistics - Theory and Methods*, 35(6):953–963, doi:10.1080/03610920600672278.

16. Shaked, M. and Shanthikumar, J. G. (2007). *Stochastic orders*. Springer, New York.

17. Shanmugam, R. (1985). An intervened Poisson distribution and its medical application. *Biometrics*, 41(4):1025–1029, doi:10.2307/2530973.

18. Shanmugam, R., Bartolucci, A. A., and Singh, K. P. (2002). The analysis of neurologic studies using an extended exponential model. *Mathematics and Computers in Simulation*, 59(1-3):81–85, doi:10.1016/S0378–4754(01)00395–0.

19. Shannon, C. E. (1948). A mathematical theory of communication. *The Bell System Technical Journal*, 27(3):379–423, doi:10.1002/j.1538–7305.1948.tb01338.x.

20. Zacks, S. (1992). *Introduction to reliability analysis: probability models and statistical methods*. Springer, New York.

Intervened Exponential Distribution: Properties and Applications