Comparative analysis of models of limiting supports in the study of structurally nonlinear oscillations of elastically supported bar from mobile load

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Abstract. The method of modeling oscillations of an elastically supported strained bar with movement limiters at the ends under the action of a moving variable load is being improved and developed. Two models of limiting support are used. A computational algorithm was constructed. The model is formulated in terms of generalized functions, which makes it possible to effectively model pointwise in space and instantaneous in time forces. When solving partial differential equations, the boundary conditions are included in the original equations. To solve the differential equation, piecewise linear basis functions are used. They remain constant during the calculation. Numerical studies of the behavior of a dynamic system depending on the variation of a wide range of parameters using a computer program in Matlab have been performed. The developed method will allow analyzing the dynamic behavior of a number of structurally nonlinear technical systems.

1. Introduction

Project designers in their engineering practice are faced with the need to adequately model the oscillations of structurally nonlinear building structures and various mechanical systems under moving loads. The nonlinear behavior of such dynamic systems is determined by switching on new work connections and then by switching them off, closing and opening gaps, and using movement limiters of the elements of the structure. The skill of reliable forecasting allows reasonably assign the optimal parameters of systems based on practical needs and actual application conditions.

The use of analytical calculations for studying the behavior of structurally nonlinear systems under the action of a mobile load is impossible, because moments of time of switching on and off one-way work connections is preliminarily unknown. These moments can be determined only by numerically solving the problem. And even more difficult task is to determine the conditions for switching off these work connections. In this case, the computational scheme of the system and the boundary conditions of the mathematical model should change. In recent years structurally nonlinear problems have been solved mainly by numerical finite-element method in various formulations. The most
difficult problems are the dynamical ones (constrained oscillation) even without taking into account the contact deformation. Numerical solution of structurally nonlinear dynamic problems was obtained at time discretization using direct iterative methods. We note some work in this direction [1] – [3]. A review of works on this subject shows that there is currently no satisfactory solution to these problems. No system studies have been conducted to identify the main patterns of behavior of such systems.

As a computational scheme for the majority of structurally nonlinear systems, when modeling their dynamic behavior, a strained bar elastically supported on a number of intermediate supports with movement limiters at its ends under the influence of a moving variable load figure 1 can be taken. It is assumed that the force moving along a bar at a constant speed, changing according to the harmonic law: \( P(t) = P_0 + A\sin(\omega t) \). When the load moves along the bar, the gaps \( f \) between the ends of the bar and the limiting supports may close. End sections structurally serve to smoothly transfer the load when driving in and driving off the bar.

![Figure 1. The analytical model of an elastically supported system with movement limiters.](image)

A mathematical model of the motions of an elastically supported bar is described in detail in [1].

The development of a numerical algorithm for modeling structurally nonlinear oscillations of an elastically supported strained bar with supports at the ends under a moving load is associated with a number of computational difficulties. The nonlinear nature of the boundary conditions and the uncertainty of the time of their multiple changings, the use of elastic bonds of great stiffness for modeling limiting supports significantly complicate the construction of a stable computational algorithm. At each moment of changing the boundary conditions, the solution may lose its stability and quickly accumulate errors in the overall computational process. The use of elastic bonds of great stiffness as a model of limiting supports leads to a significant increase in the requirements for the time integration step to ensure computational stability. The authors' first works in this direction [5], [6], [7] were performed for simplified problem statements.

Later in [1], a computational algorithm was proposed for solving boundary value problems for partial differential equations with dynamically changing boundary conditions for modeling structurally nonlinear oscillations of an elastically supported strained system. The algorithm is based on the formulation of a mathematical model in the form of a system of partial differential equations for generalized functions and the inclusion of boundary conditions in the model equation. The algorithm does not require the subordination of the basis functions to the boundary conditions and was implemented using the Legendre polynomials as the basis. As a model of extreme limiting supports, an elastic bond of great stiffness was used. After the closure of the gap, it is included in the interaction with the elastically supported bar. This interaction is accompanied only by the elastic deformation of the support without energy loss, as well as the spread of disturbance from the interaction with the support along the bar.

This algorithm allowed building a stable computational model. However, when trying to increase the accuracy of a numerical solution by increasing the number of basis functions, the numerical model became unstable. Also, the model became unstable with a significant increase in the stiffness of the extreme limiting supports. In the work, presented on May 18, 2018 at the 23rd International Conference "MECHANIKÁ – 2018" in Druskininkai, Lithuania, Chebyshev polynomials were used as
the basis functions to improve the computational schemes for the implementation of practical problems of modeling structural nonlinear oscillations of building structures. The use of such a basis has led to the fact that the computational model has remained stable with a larger number of basic functions and thus made it possible to increase the accuracy of the calculations. Also, the computational model remained stable for substantially large values of the stiffness of the limiting supports.

In [8], a computational algorithm based on the use of a basis of continuous piecewise linear functions was proposed. Such a computational model remains stable with an arbitrary number of basis functions, which made it possible to carry out the calculation with the necessary accuracy. Also, this model is stable even for any values of stiffness of extreme limiting supports.

This paper is a development of the research described above, in which, instead of the models of limiting supports of elastic bond of a great but finite stiffness, two other models are used. It allows describing both an absolutely elastic rebound of the ends of the bar when the gaps are closed, and an absolutely inelastic touch-down with the complete absorption of energy by a limiting support. When using the first model, the speed of the bar end is reversed instantaneously (about time scale of the task) on the opposite. When using the second one, with an absolutely inelastic touch-down with an end's stop.

2. Mathematical model

A model of a thin-walled elastic bar with free ends was adopted as a design scheme. In this model the bar is exposed by a moving load, forces in intermediate elastic bonds, forces of interaction with the left and right end sections and forces from the limiting supports, which are included in the overall oscillatory system work.

In accordance with the theory of flexural-torsional oscillations of a thin-walled elastic bar, described in [9], a mathematical model of the change in flexural deformations of an elastic thin-walled bar \( v(t,x) \), measured from the horizontal axis passing through the center of mass of the bar in the equilibrium position in the absence of an external load, if variables \( u = (u_1, u_2, u_3)^T = (v, \frac{\partial v}{\partial t}, \frac{\partial^2 v}{\partial x^2})^T \) are entered, can be written in the form of an initial-boundary value problem for a system of partial differential equations with 2nd order spatial variables:

\[
\begin{align*}
\frac{\partial u}{\partial t} + A \frac{\partial^2 u}{\partial x^2} + Bu &= f \\
\left( u(t=0,x) = d(x) \right. \\
\left. \frac{\partial u_s}{\partial x}(t,\pm \frac{L}{2}) = 0, \frac{\partial u_s}{\partial x}(t,\pm \frac{L}{2}) = 0 \right)
\end{align*}
\]

(1)

Here

\[
A = \begin{pmatrix}
0 & 0 & 0 \\
0 & EJ & \frac{EJ}{m} \\
0 & -1 & 0
\end{pmatrix} ; B = \begin{pmatrix}
0 & -1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} ; f = \begin{pmatrix}
0 \\
g + \sum_{i=1}^{S} \frac{f_{is}}{m} + \sum_{i=1}^{S} \frac{f_{is}}{m} \\
0
\end{pmatrix}
\]

\( m \) – bar linear density,
\( EJ \) – flexural stiffness of a thin-walled bar,
\( f_{is}(t,x), s = 1:S \) – distribution of external forces acting on the bar, including from the side of intermediate supports, end sections, etc. The distributions \( f_{is}(t,x) \) are not equal to zero within a
certain neighborhood of a point \( x_s \), the “spot” of the contact of the support (or external load) with the bar, and are equal to zero outside this spot. Further we will use the notation \( f^*_k(t) = \int f_k(t,x)dx \) – for the full force acting from the support on the bar, \( f_k(t,x) \) – distribution of forces acting on the bar, from the side of limiting supports. Limiting supports are located along the bar. Each distribution \( f_k(t,x) \), like the distribution \( f_s(t,x) \), is non-zero within a certain neighborhood of the points \( x_s, k=1:K \) and is zero outside this neighborhood. We have singled out these forces as a separate summand for the purposes of our further exposition. Limiting supports, as well as intermediate supports, are elastic springs, but with stiffness substantially greater than the stiffness of the springs of the intermediate supports.

The boundary conditions at the ends of the bar correspond to the conditions of the bar with free ends.

The initial conditions correspond to the strain profile \( \mu(x) \) of the elastic bar in the equilibrium position.

Further we assume that all distributions of forces are represented by sufficiently smooth functions. That is all necessary derivatives are continuous.

### 3. Mathematical model in terms of generalized functions

The model (1) is written is written using distributions \( f_s(t,x) \), \( f_k(t,x) \) of the forces applied to the thin-walled elastic bar. It would be desirable to formulate this model in terms of the forces applied at a point. In [5]–[7] limiting support simulated elastic supports with a very great stiffness. In contact with the span of these supports, the interaction time is very short. In this paper, we formulate a mathematical model with infinitely stiff limiting supports and their instantaneous interaction with the bar. This can be done using the technique of generalized functions described in [9]. The movement of the bar on the time interval \( G_0 = 0 \cdots T \) can be schematically represented as follows. Build a grid with nodes \( t_\gamma, \gamma = 1:Y \) on the segment \( G \). Thus, we divide this segment into subdomains \( G_\gamma = [t_\gamma, t_{\gamma+1}], \gamma = 1:Y-1 \). We assume that on the interval \( G_\gamma \) the motion of the bar is modeled by the system of equations (1) with boundary conditions corresponding to the conditions of the bar with free ends. At the moment of time \( t_{\gamma+1} \), the limiting supports are instantly “worked out”, as described above. Further, in the time interval \( G_{\gamma+1} \), the motion proceeds in a similar way.

Since the right side of the system of equations (1) is represented by smooth functions, it can be assumed that the solution does not have discontinuities of the derivatives entering into the equations.

Let \( \tilde{u} = [\tilde{u}_1(t,x), \tilde{u}_2(t,x), \tilde{u}_3(t,x)]^T \) the solution to the original problem (1). Build functions:

\[
\mathbf{u} = \begin{cases}
\tilde{u}, & t \in [0,\infty), x \in \left[-\frac{T}{4}, \frac{T}{4}\right] \\
0 & \text{for other values } (t,x).
\end{cases}
\]

We will show that \( \mathbf{u}(t,x) \), considered as generalized functions from \( D' \) for an arbitrary infinitely differentiable testing function \( \varphi(t,x) \in D \) from the space of basic functions, satisfies the equation
\[
\begin{align*}
\left( \frac{\partial u}{\partial t} + A \frac{\partial^2 u}{\partial x^2} + Bu - d\delta(t) - b, \varphi \right) &= 0 \\
\mathbf{b} &= \left( \sum_{k} \theta(u_{\ast} - u_{k})(u_{\ast} - u_{k})\delta(t - t_{r})\delta(x - x_{k}) \right) \\
&\quad - g \sum_{l} f_{m} \delta(x - x_{l}) - \sum_{k} \theta(u_{\ast} - u_{k})u_{k}\delta(t - t_{r})\delta(x - x_{k}) \\
&\quad \frac{\partial u}{\partial x} \delta(x - \frac{1}{2}) - \frac{\partial u}{\partial x} \delta(x - \frac{3}{2}) \\
&\quad \frac{\partial u}{\partial x} \delta(x + \frac{1}{2}) - \frac{\partial u}{\partial x} \delta(x + \frac{3}{2}) \\
\end{align*}
\] (2)

Let us prove the validity of this statement, for example, for the second equation of system (2).

Indeed, by definition, for an arbitrary infinitely differentiable testing function \( \varphi(t, x) \in D \) from the space of basic functions

\[
I = \left( \frac{\partial u}{\partial t} + \frac{EJ}{m} \frac{\partial^2 u_{3}}{\partial x^2}, \varphi \right) = -\int u_{2} \frac{\partial \varphi}{\partial t} \, dx + \frac{EJ}{m} \int u_{2} \frac{\partial^2 \varphi}{\partial x^2} \, dx
\]

Perform integration by parts: once by time and twice by spatial variable

\[
I = \left[ \int u_{2} \frac{\partial \varphi}{\partial t} \right]_{0}^{\infty} \, dx + \frac{EJ}{m} \int \left( u_{2} \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial u_{2}}{\partial x} \varphi \right)_{x=0}^{\infty} \, dx
\]

Given that \( u_1, u_2, u_3 \) satisfies the equations, the initial and boundary conditions of the model (1) and \( \varphi(t = \infty, x) = 0 \), we get

\[
I = \left( \frac{\partial u_{2}}{\partial t} + \frac{EJ}{m} \frac{\partial^2 u_{3}}{\partial x^2}, \varphi \right) = \frac{1}{m} \int \left( -mg \right)_{x=0}^{\infty} \, dx + \frac{EJ}{m} \int \left( u_{2} \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial u_{2}}{\partial x} \varphi \right)_{x=0}^{\infty} \, dx
\]

Let \( f_{s}(t, x) \) corresponds to the density of forces acting on the bar from the side of the support (or active load) and \( f'_{s}(t) = \int f_{s}(t, x) \, dx \) is the total force acting on the bar from the side of this support. The function \( f_{s}(t, x) \) is zero everywhere, except the neighborhood \( x_{i} - \Delta x, x_{i} + \Delta x \) of the point \( x_{i} \), corresponding to the location of this support. Then \( \int f_{s}(t, x) \, dx = \int \int_{x_{i}-\Delta x}^{x_{i}+\Delta x} f_{s}(t, x) \, dx \, dt \). Passing in this expression to the limit \( \Delta x \to 0 \) at so that the value \( \int_{x_{i}-\Delta x}^{x_{i}+\Delta x} f_{s}(t, x) \, dx \) remains constant, we get

\[
\int_{x_{i}-\Delta x}^{x_{i}+\Delta x} f_{s}(t) \varphi(t, x) \, dx = \left( f'_{s}(t) \right) \varphi(t, x) \delta(x - x_{i}), \varphi \). Thus, the concentrated forces are modeled using \( \delta \) – functions whose amplitude is equal to the magnitude of the acting force. Now consider the limiting supports. It is completely similar, if the limiting support is presented as a spring with great stiffness, then \( \int f_{s} \varphi \, dx = \int f'_{s}(t) \varphi(t, x_{j}) \, dt = \left( f'_{s}(t) \right) \varphi(t, x_{j}), \varphi \). Let \( u_{\ast} \) be the vertical coordinate of the upper part of the limiting support. If \( u_{i} > u_{\ast} \), then the bar does not touch the limiting support and the total force from the side of the support is zero. Let at the moment of time \( t_{i} \) the bar is in contact with the limiting support. The force \( f'_{s}(t) \) begins to act on the bar over time \( \Delta t \). If the interaction of the limiting support with the bar is inelastic, then as a result of the support reaction, the
speed of the contacting part of the bar becomes equal 0 and the position is equal $u'$. If we will increase the spring stiffness, the time of interaction of the limiting support with the bar $\Delta t$ will be reduced, but the result will remain unchanged – almost instantaneous change in the speed of this point from the current value $u_z(t_j)$ to zero. In the limit of an absolutely stiff limiting support, it is possible to consider the equality $\int_0^t \frac{f(t)}{m}(t)\varphi(t,x_i)dt = -u_z(t_j)\varphi(t_j,x_i) = \left( -u_z(t)\delta(t-t_j)\delta(x-x_i)\varphi \right)$ be correct. As a result, we obtain the second equation of system (2). Similarly, we obtain other equations of system (2).

If the interaction of the limiting support with the bar is absolutely elastic, then in model (2) the summand $\theta(u' - u_i)u_2$ should be replaced by $\theta(u' - u_i)(u_2 - |u_2|)$.

Thus, the system of equations (2), taking into account the possibility of describing concentrated forces, is a mathematical model of the motion of an elastically supported bar. And the task is to find smooth functions $u_1$, $u_2$, $u_3$ that would satisfy the model equations for an arbitrary finite infinitely differentiable testing function $\varphi(t,x) \in D$ from the space of basic functions. We note that there are no requirements for the fulfillment of boundary conditions for the functions $u_1$, $u_2$, $u_3$. The boundary conditions are already taken into account in the equations of model (2). The system of equations (2) for generalized functions $u_1$, $u_2$, $u_3$ and is taken as the basis for constructing a computational algorithm.

4. Computational algorithm

The computational algorithm is based on the system of equations (2). We take into account that:

$$
\left( \frac{\partial^2 u}{\partial x^2}, \varphi \right) = \int \left[ -A \frac{\partial u}{\partial x} \frac{\partial \varphi}{\partial x} dx + A \frac{\partial \varphi}{\partial x} \left( t, \frac{i}{2} \right) - A \frac{\partial \varphi}{\partial x} \left( t, -\frac{i}{2} \right) \right] dt
$$

We construct a grid of nodes $x_i, i = 1 \div I$ on a segment $[-\frac{I}{2}, \frac{I}{2}]$ so that $x_i = -\frac{i}{2}, x_i = \frac{i}{2}$ and the position of each of the limiting supports coincides with some grid node $x_i$. The grid may be equally or unequally spaced. This grid splits the segment $[-\frac{I}{2}, \frac{I}{2}]$ into intervals. We associate with each node $x_i$ a function $H_i(x)$ that is equal 1 at a point $x_i$, equal 0 at all other grid points and linear on each interval. The solution to problem (2) $u(t,x)$ will be sought in the form $u(t,x) = \sum H_i(x)u'(t) = H' u'$.

In the last equality we used the summation agreement: if in the expression the index occurs twice, then it is assumed to be summed over all values of this index. As testing functions $\varphi(t,x) \in D$ from the space of basic functions we take $\varphi'(t,x) = \psi(t)H'(x)$ numbered by the index $j = 1 \div I$.

Functions $H_j$ are linear combinations of functions $H_j(x)$ that satisfy the conditions $H_j', H_j = \delta_i'$. Substituting into equation (2) and taking into account the remark made above (2), we obtain the system of equations

$$
\left( \frac{du'}{dt} - D' u' - f' - d' \delta(t) - \sum b' \delta(t-t_j) \psi \right) = 0.
$$

In this equality, we used the summation agreement: $D' u' = \sum D' u'$. The arrays $D'$, $f'$, $d'$, $b'(u')$ are easy to calculate, in particular,
Due to the arbitrariness of the testing function, the system of equations (4) is equivalent to the Cauchy problem for the ODE system

\[
\frac{du'}{dt} - D'u' - f' = 0
\]

\[
u'(t=0) = d'
\]

At the same time, at time points \( t_j \), the solution \( u'(t) \) abruptly changes by \( b' \). In formulas (4), we explicitly noted that the jump \( b' \) depends on the values of the solution \( u' \). As follows from (4), the magnitude of this jump is not zero, only if the node \( x_j \), which corresponds to the component \( u'(t) \), coincides with the position of one of the restrictive supports \( x_j = x_k \). But even in this case, this jump is not equal \( 0 \), only if \( u' > u_i(x_j) \), that is, the corresponding point of the bar has dropped below the level of the location of the limiting support. Solving by one or another numerical method the Cauchy problem (4), we define \( u'(t) \). This allows with using formulas \( u(t,x) = \sum H_i(x)u'(t) \) to calculate the solution \( u(t,x) \).

5. Numerical studies

To perform numerical studies of nonlinear oscillations of an elastically supported system, a computational scheme was used. Its parameters described in detail in [1] and further used in [8]. Using the constructed model, a series of computational experiments was carried out. First of all, the question, If the constructed model is a limiting case of models in which limiting supports were represented by springs with great stiffness, was investigated. For this, the deformed state of the bar was calculated from the model constructed with absolutely elastic limiting supports when the initial vertical displacements of the bar were set to \( 2 \) cm from the equilibrium position. Simultaneously, the calculation was performed with the same parameters and initial conditions for the model with limiting supports in the form of elastic bonds of great stiffness at \( 5 \) values of the stiffness of the elastic bond.

At a fixed moment of time, over \( 20 \) sections of an elastic bar, the mean square difference between the solutions obtained for models with an absolutely elastic limiting support and an elastic support of a great but finite stiffness is calculated. Figure 2. Dependence of the logarithm of the mean square error on the logarithm of the stiffness coefficient of the elastic limiting support. shows, on a logarithmic scale, the dependence of the difference between these solutions on the stiffness in the model with the elastic limiting support of a great but finite stiffness. The given dependence clearly demonstrates the convergence of the solution using models with elastic limiting supports to the solution of the model with an absolutely elastic support with an increase in the stiffness of elastic bonds.
Figure 2. Dependence of the logarithm of the mean square error on the logarithm of the stiffness coefficient of the elastic limiting support.

Figure 3 shows the deformation profile of an elastically supported bar at a time moment of 2.4 seconds, calculated from a model with an elastic limiting support of a great but finite stiffness at the indicated 5 values of elastic bond. The nature of the convergence of solutions obtained by a model with an elastic limiting support of a great but finite stiffness to the solution of a model with an absolutely elastic limiting support with an increase in the stiffness coefficient of the elastic supports can be determined using the figure. The greatest differences between the solutions are observed in the locations of the limiting supports. But with an increase in the stiffness coefficient of the elastic supports, this difference monotonously decreases. Thus, it is possible to consider the convergence not only in the Euclidean norm, but also in uniform norm or, in other words, uniform convergence.

Figure 3. The convergence of the solution when using the model with elastic limiting supports to the solution of the model with an absolutely elastic support for the deformed type of the bar.

Using the constructed model, a number of computational experiments were carried out to simulate the movement with a constant speed of a changing load on an elastically supported dynamic system. In so doing the number of intermediate supports, their stiffness, the size of the gap between the limiting support and the ends of the bar, as well as the amplitude of change of the moving load were varied. Below is a graphic illustration of some of the results. Figure 4 shows graphs of the flexural deformations of an elastic bar supported on from 5 to 30 elastic bonds with a step of 5 at time moment of 2.4 sec at a speed of moving load of 7 m/sec.
Figure 4. Flexural deformations of an elastic bar supported on from 5 to 30 elastic bonds with a step of 5 at time moment of 2.4 sec.

Figure 5 shows the graphs of flexural deformations of an elastic bar supported by 5 elastic bonds with a change in stiffness from 1200 to 120000 kN / m at time moment of 2.4 sec.

Figure 5. Flexural deformations of an elastic bar supported by 5 elastic bonds with a change in stiffness from 1200 to 120000 kN / m at time moment of 2.4 sec.

Figure 6 shows graphs of flexural deformations of an elastic bar supported on 5 elastic bonds with a change in the gap from 5 to 15 cm at a time moment of 2.4 seconds.
6. Results

1. Theoretical and methodological foundations of numerical modeling of the behavior of constructive nonlinear elastically supported carrying systems of transportation constructions under the action of various loads were developed, including loads localized in respect to space and time (“instantaneous”), and also systems with boundary conditions that non-linearly change as the system moves. The developed approach is based on the formulation of a mathematical model in terms of differential equations for generalized functions.

2. On the basis of theoretical and methodological results computational algorithms for modeling the behavior of the indicated constructive nonlinear elastically supported carrying systems of transportation constructions were developed. The constructed algorithms allow to numerically simulate the behavior of these systems, described by partial differential equations of the 4th order, using piecewise linear basis functions. Moreover, there is no need to subordinate the basis functions to non-linearly varying boundary conditions. The boundary conditions are already taken into account in a weak form in the equations of the model.

3. Using the developed computational algorithms a universal computing system was created.

4. The constructed computing system can be used to forecast the behavior of particular samples of constructive nonlinear elastically supported carrying systems of transportation constructions. It makes possible to assign the optimal parameters of the carrying systems based on practical needs and actual application conditions.

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