Bounces of a marble and Zeno’s paradox

R De Luca¹, M Di Mauro¹,² and A Naddeo²

¹Dipartimento di Fisica “E.R. Caianiello”, University of Salerno, Fisciano (SA), 84084, Italy
²Istituto Nazionale di Fisica Nucleare, Sezione di Napoli, Napoli, 80126, Italy

Abstract. The coefficient of restitution of the bounces of a marble on a table is here studied and measured in terms of the total bouncing time. Interesting links arise with mathematics (geometric series) as well as philosophy (Zeno’s paradox), which make our proposal appealing for an interdisciplinary teaching-learning sequence. The relevance of this work for high school and undergraduate students is discussed with emphasis on students’ misconceptions when dealing with infinite processes.

1. Introduction
Convergent infinite series [1] arise as a mathematical tool in the solution of a lot of problems in physics. The issue of convergence of an infinite process and its application to concrete physics phenomena has been widely recognized to be a difficult one to deal with by high school as well as undergraduate students. Indeed recent studies pointed out students’ difficulties in blending mathematics and physics [2,3].

In this contribution we propose an interdisciplinary teaching-learning sequence aimed to put together physics, mathematics and philosophy. The sequence is integrated by the proposal of simple experimental activities, where students have the opportunity to learn how to carry out measurements and perform data analysis in a concrete context. The designed activities are based on the use of built-in sensors and video cameras currently available on smartphones, according to the Bring Your Own Device (BYOD) paradigm. The core of our proposal is the study of the bouncing of a marble on a table by focusing on the progression of time intervals separating successive bounces. We find that, by measuring the total time of bouncing \( \Delta t \), the coefficient of restitution \( \epsilon \) can be estimated. In fact, in an inelastic collision the kinetic energy is not conserved, therefore the speed decreases. There is a simple relation between speeds \( v_i \) and \( v_f \), before and after the collision, respectively, which defines \( \epsilon \):

\[ v_f = \epsilon v_i \quad (\epsilon < 1) \quad [4] \]

By measuring the initial height \( h_0 \) from which the marble is released, the coefficient of restitution \( \epsilon \) can be expressed in terms of the total bouncing time as:

\[ \epsilon = \frac{\Delta t - \frac{2h_0}{g}}{\Delta t + \frac{2h_0}{g}} \tag{1} \]

where \( g \) is the gravity acceleration. The derivation of Eq. (1) allows us to talk about kinematics, mathematics and philosophy in a simple way and, then, to perform experiments. Indeed, obtaining the total bouncing time \( \Delta t \) requires the sum of a geometric series [1] and we show how, for the particular value \( \epsilon = \frac{1}{2} \), this is related to the solution of Zeno’s paradox of Achilles and the turtle [5]. Finally we propose an experimental activity based on the BYOD paradigm, whose aim is to estimate the coefficient.
of restitution $\varepsilon$ starting from the measurement of the total bouncing time $\Delta t$ according to equation (1). We point out that, in general, the coefficient of restitution exhibits a slight dependence on velocity [6] and, hence, on the initial release height. But this is really a tiny effect, which can be neglected as we assume in our initial proposal. Further approximations are a vertical movement of the marble without spinning effects, a negligible time of contact with the floor at each bounce as well as a negligible effect of air drag and buoyancy.

2. The kinematics of bouncing bodies

In this Section we deal with successive bounces of a body (e.g. a marble) on the ground, and study the progression of the time intervals separating successive bounces. We show that, if the coefficient of restitution is assumed to be the same for all bounces, these times are in a geometric progression with ratio $\varepsilon$. This suggests a possible measure of the coefficient of restitution in terms of time intervals. In fact, we shall find that the coefficient of restitution is simply related to the total bouncing time. Alternative proposals of calculation and experimental measurement of the coefficient of restitution are reported in the literature (see for instance [7-9] and references therein).

Let us start by assuming that the body is initially at rest when released. At $t = 0$ the body will be at height $h_0$ with respect to the ground, so that its potential energy (assuming that the zero is on the ground) is $U_0 = mg h_0$. Conservation of energy allows us to write that $U_0$ will be equal to the kinetic energy immediately before the collision with the ground:

$$mgh_0 = \frac{1}{2}m v_i^2 = K_i$$  \hspace{1cm} (2)

After the collision the kinetic energy becomes $K_f = \frac{1}{2} m v_f^2 = \varepsilon^2 K_i$ and the body will reach a height $h_1 < h_0$, which can be determined again by requiring conservation of energy:

$$K_f = \frac{1}{2} m v_f^2 = mgh_1$$  \hspace{1cm} (3)

These two relations allow one to relate the coefficient of restitution to the ratio of the heights:

$$\varepsilon = \frac{v_f}{v_i} = \sqrt{\frac{K_f}{K_i}} = \sqrt{\frac{h_1}{h_0}}$$  \hspace{1cm} (4)

The coefficients of restitution can be related with the times between collisions by means of the kinematic relation:

$$y(t) = h_0 - \frac{1}{2} gt^2$$  \hspace{1cm} (5)

where $y$ is the height of the body, measured from the ground. The time $t_0$ from the release of the body to the first collision is given by $y(t_0) = 0$, so that:

$$h_0 = \frac{1}{2} gt_0^2 \iff t_0^2 = \frac{2h_0}{g}$$  \hspace{1cm} (6)

The time needed to reach the height $h_1$ after the first collision can be computed by considering that the body will start with velocity $v_f$ and be decelerated by gravity, so that one has to use the following kinematic relation:

$$y(t) = v_f t - \frac{1}{2} gt^2$$  \hspace{1cm} (7)

whose derivative with respect to time is:

$$v(t) = v_f - gt$$  \hspace{1cm} (8)

Time $t_1$ is defined by the condition $v(t_1) = 0$, so that one gets:

$$v_f - gt_1 = 0 \iff t_1 = \frac{v_f}{g}$$  \hspace{1cm} (9)
But equation (3) gives $v_f^2 = 2gh_1$, so we get
\[ v_f^2 = \frac{2h_1}{g}. \] 
Therefore we express the coefficient of restitution as
\[ \epsilon = \frac{h_1}{h_0} = \frac{t_1}{t_0} \]
which gives the following relation between bouncing times:
\[ t_1 = \epsilon t_0 \]
By iterating the same procedure for each bounce and assuming the same coefficient of restitution for all the bounces we therefore get:
\[ t_{n+1} = \epsilon t_n \]
that defines a geometric progression with ratio $\epsilon$. Finally we get the relation
\[ t_n = \epsilon^n t_0 \]
which is the basis for the results of the following sections.

3. Computing the total bouncing time: the geometric series
The aim of this Section is to exploit the result obtained in equation (14) to compute the total time of bouncing. We shall see that the total time is finite even if there is an infinite number of bounces, thanks to the fact that the ratio $\epsilon$ is less than one \([1]\). This point is difficult to understand for most students. Indeed a widespread misconception among students is that a series with an infinite number of terms always diverges. The total bouncing time is:
\[ \Delta t = t_0 + 2\sum_{n=1}^{\infty} t_n = 2\sum_{n=0}^{\infty} t_n - t_0 = \left[ 2\left( \sum_{n=0}^{\infty} \epsilon^n \right) - 1 \right] \cdot t_0 \]
The series in round parentheses is a geometric series of ratio $\epsilon < 1$, which is known to be convergent. To compute the sum we follow the usual procedure:
\[ S = 1 + \epsilon + \epsilon^2 + \epsilon^3 + ... = 1 + \epsilon \left( 1 + \epsilon + \epsilon^2 + \epsilon^3 + ... \right) = 1 + \epsilon \cdot S \]
or
\[ S = \frac{1}{1 - \epsilon} \]
where the first equality in (16) relies on the fact that the series has an infinite number of terms. In this way the well-known expression for the sum of a geometric series has been recovered. By putting this result in equation (15), we get the total bouncing time
\[ \Delta t = \frac{1 + \epsilon}{1 - \epsilon} t_0 \]
It is a finite quantity, because the infinite number of bounces is balanced by the fact that they take less and less time. This is the key observation in order to understand the phenomenon and overcome the misconception that infinite bounces give infinite time.

Notice that the limiting case $\epsilon \to 1$ gives an infinite total time, which corresponds to the case of elastic bounces: in this situation the ball never stops as expected (at odds with the previous case here the balance does not take place). Solving the above equation with respect to $\epsilon$ we finally arrive at
\[ \epsilon = \frac{\Delta t - t_0}{\Delta t + t_0}, \]
which, upon substituting equation (7), coincides with equation (1). Formula (19) allows one to estimate the coefficient of restitution starting from the knowledge of the initial height and the measurement of the total bouncing time.
4. Philosophy: Zeno’s paradox

In this Section we give a geometric interpretation of the sum of the geometric series in equations (16), (17) for the particular case $\epsilon = \frac{1}{2}$ and relate the above results to Zeno’s paradox [5]. In this way students have also the possibility to understand how and why it is not to be considered a paradox at all, thanks to the mathematics of infinite series. According to Zeno’s paradox, Achilles and the turtle run on a path. We assume that Achilles starts running when the turtle reaches the middle point of the path, and that the turtle moves with half the velocity of Achilles. Then in the time Achilles will take to get to the middle point, the turtle will cover another quarter of the path. When Achilles will have covered this quarter, the turtle will cover another half of it, and so on. As a result, Achilles would never reach the slower turtle, and this is the usual statement of the paradox. This situation can be described as follows. We divide the path in two, then divide the second half in two, then divide the second half of the second half in two and so on. Then we clearly get

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n,$$

or

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 1.$$

Let us now consider the geometric series of ratio $\frac{1}{2}$, whose sum is:

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} = \frac{1}{1-\frac{1}{2}} = 2.$$

Thus the above strategy allowed us to compute the sum of the series without manipulating infinite series. This shows that Achilles reaches the turtle in a finite time, despite this takes an infinite number of steps. As for the case of the bouncing marble, this happens by summing an infinite number of steps, which are shorter and shorter.

5. Experiment and data analysis

The set-up described in section 2 can be realized with a marble or a ball, which can be released from a certain height as shown in figure 1. A meter is used to measure $h_0$ while the total bouncing time can be measured by means of a phonometer. Several free apps are available for such a measurement, both on Android and iOS (for instance, Sound Analyzer App for Android [10]). We used the native iOS app, which has the advantage of being very sensitive, so that the time of every single bounce can be measured. As an alternative, one can use his/her own ears and a chronometer and obtain quite accurate measurements as well.
Figure 1. The simple experimental apparatus: the marble and the phonometer.

The experimental results, obtained by performing the measurements with a marble and a phonometer and then carrying out a linear fit analysis of the collected data, are shown in figure 2.

![Experimental data and least square line.](image)

Figure 2. Experimental data and least square line.

The starting point of our analysis is equation (18), which can be conveniently rewritten as:

\[
(\Delta t)^2 = \frac{2h_0(1+\varepsilon)^2}{g(1-\varepsilon)}.
\]
Indeed, by putting $y = (\Delta t)^2$ and $x = h_0$, the coefficients (and their respective errors) of the line $y = ax + b$ are obtained. Finally, from the knowledge of the slope $a$ and its error $\Delta a$, we get the coefficient of restitution and its error

\begin{equation}
\epsilon = \sqrt{\frac{ag}{2} - 1},
\end{equation}

\begin{equation}
\Delta \epsilon = \frac{\Delta a}{\sqrt{\frac{ag}{2} + 1}}. \left(\frac{ag}{2} + 1\right)^{-\frac{1}{2}}.
\end{equation}

Equation (24) has been obtained by means of the standard formula for the propagation of errors [11] $\Delta \epsilon = \left|\frac{de}{da}\right| \Delta a$. The experimental data and the least squares line are shown in figure 2. The growth of the error bars with height is due to various factors, such as air resistance (as also noticed in [7]), irregularities in the bouncing surfaces [12] and human errors in measuring times. Further uncertainties can be introduced by the release of the marble, which may give rise to unwanted spinning. In fact a careful initial release is crucial in order to improve the experimental conditions by minimizing lateral displacements of the marble. A detailed discussion of the effects of the errors on the experimental data has been given in [12].

The estimated value for the coefficient of restitution is $\epsilon = 0.850 \pm 0.002$.

6. Conclusions
A simple and common setup made of a marble bouncing on a table has been proposed, which allows us to estimate the coefficient of restitution $\epsilon$ in terms of the initial height $h_0$ from which the ball is released and of the total bouncing time $\Delta t$. This issue is the core of an interdisciplinary teaching-learning sequence, aimed to put together physics, mathematics and philosophy. Indeed, we show how a kinematics topic can be related to a strictly mathematical subject, namely geometric series, and then provide a geometric interpretation of the sum of such series. Finally the argument utilized in Zeno’s paradox is proven to be false. The focus of our proposal is on the convergence of infinite processes and its application to concrete examples borrowed from different fields, in order to make high school as well as undergraduate students to tackle and overcome their difficulties in blending mathematics and physics [2,3].

7. References
[1] Stewart J 2002 Calculus, Fifth Edition (Pacific Grove (CA): Brooks Cole)
[2] Bing T J and Redish E F 2007 AIP Conference Proceedings 883 26
[3] Redish E F 2015 Science & Education 43 75
[4] Halliday D, Resnick R and Walker J 2005 Fundamentals of Physics, Seventh Edition (New York: Wiley & Sons)
[5] Huggett N 2018 Zeno’s Paradoxes The Stanford Encyclopedia of Philosophy ed E N Zalta (Stanford (CA): Stanford University)
[6] Falcon E, Laroche C, Fauve S and Coste C 1998 Eur Phys J B 3 45
[7] Bernstein A D 1977 Am J Phys 45 41
[8] guiar C E and Laudares F 2003 Am J Phys 71 499
[9] Maynes K, Compton M and Baker B 2005 The Physics Teacher 43 352
[10] https://play.google.com/store/apps/details?id=com.dom.audioanalyzer&hl=en
[11] Taylor J R 1997 Introduction To Error Analysis (Herndon (VA): University Science Books)
[12] Heckel M, Glielmo A, Gunkelmann N and Poschel T 2016 Phys. Rev. E 93 032901