Sum Rule of Quantum Uncertainties: Coupled Harmonic Oscillator System with Time-Dependent Parameters

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Abstract

The uncertainties $\langle \Delta x \rangle^2$ and $\langle \Delta p \rangle^2$ are analytically derived in $N$-coupled harmonic oscillator system when spring and coupling constants are arbitrarily time-dependent and each oscillator is in arbitrary excited state. When $N = 2$, it is shown that those uncertainties are just arithmetic average of uncertainties of two single harmonic oscillators. However, this arithmetic property is not generally maintained when $N \geq 3$. This property is recovered in $N$-coupled oscillator system if and only if $(N - 1)$ quantum numbers are equal. Generalization of our results to more general quantum system is briefly discussed.
Uncertainty [1–4] and entanglement [5–7] are two major characteristics of quantum mechanics. These properties make quantum mechanics to be different from classical mechanics. Quantum uncertainty provides a limit on the precision of measurement for incompatible observables. Most typical expression of uncertainty relation is $\Delta x \Delta p \geq \hbar/2$, where $\Delta$ means a standard deviation. Recently, different expressions of uncertainty relations were studied such as entropic uncertainty relations [8, 9] in the context of quantum information and generalized uncertainty principle [10] in the context of Planck scale physics. Even though entanglement is also studied from the beginning of quantum mechanics [5], it is extensively explored for last few decades with development of quantum technology. It is used as a physical resource in various quantum information processing such as quantum teleportation [11, 12], superdense coding [13], quantum cloning [14], quantum cryptography [15, 16], quantum metrology [17], and quantum computer [18, 19]. Furthermore, many experimentalists have tried to realize such quantum information processing in the laboratory for last few decades. As a result, quantum cryptography and quantum computer seems to approaching to the commercial level [20, 21].

Although these two phenomena seem to be distinct properties of quantum mechanics, there is some, albeit unclear, connection between them because of the fact that both are strongly dependent on the interaction between subsystems. For example, the uncertainty of a given system was computed in Ref. [22, 23] to discuss on the effect of the rest of universe [24]. It was shown that ignoring the rest of universe appears as an increasing of uncertainty and entropy in the system in which we are interested. In other words, if the system we are interested in is one of subsystems in the whole system and it interacts with other subsystems, its uncertainty and entanglement monotonically increase with increasing the interaction strength. More specifically, let us consider the two coupled harmonic oscillator system, whose Hamiltonian is

$$H_2 = \frac{1}{2} (p_1^2 + p_2^2) + \frac{1}{2} \left[ k_0(x_1^2 + x_2^2) + J(x_1 - x_2)^2 \right].$$  \hspace{1cm} (1)

If we assume that the two oscillators, say $A$ and $B$, were in each ground state, the uncertainty and entanglement of formation [25] (EoF) are given by [26]

$$(\Delta x \Delta p)_{A,B}^2 = \frac{1}{4} \left( \frac{1 + \xi}{1 - \xi} \right)^2 \quad \mathcal{E}_F = -\ln(1 - \xi) - \frac{\xi}{1 - \xi} \ln \xi$$  \hspace{1cm} (2)

where $h = 1$ and $\xi = \left\{ (\sqrt{k_0} + 2J - \sqrt{k_0})/(\sqrt{k_0} + 2J + \sqrt{k_0}) \right\}^2$. It is manifest to show that
both $(\Delta x \Delta p)^2_{A,B}$ and $\mathcal{E}_F$ increase with increasing the coupling constant $J$. Thus, uncertainty and entanglement are, in this case, implicitly related to each other via $\xi$. In Ref. [27] quantum uncertainty is used to provide a sufficient criterion for inseparability for continuous variable systems. In Ref. [28] it was shown that the uncertainty relation for all eigenstates in the single harmonic oscillator system is saturated in the plot with respect to Gaussianity.

So far EoF cannot be exactly computed in the coupled harmonic oscillator system except ground state because of non-Gaussian nature of exciting states\(^1\). Since EoF and uncertainty exhibit similar behavior as Eq. (2) shows, one may use the uncertainty as a measure of entanglement after rescaling appropriately when EoF cannot be computed exactly. In this reason it is important to examine the uncertainty for the arbitrary excited states in the coupled harmonic oscillator system.

In there any other similarity between EoF and uncertainty? EoF is believed to have the additivity property\[^{[30]}\], even though still not solved completely. For mixed states EoF is generally defined by a convex-roof method\[^{[25, 31]}\] as follows:

$$
\mathcal{E}_F(\rho) = \min \sum_i p_i \mathcal{E}_F(\rho_i),
$$

where the minimum is taken over all possible ensembles of pure states with $\sum_i p_i = 1$. Let $\rho^{(i)} (i = 1, 2)$ be two bipartite density matrix and $\rho = \rho^{(1)} \otimes \rho^{(2)}$. If we regard $\rho$ as a bipartite state, where $\rho^{(1)}$ and $\rho^{(2)}$ belong to each party, Eq. (3) guarantees $\mathcal{E}_F(\rho) \leq \mathcal{E}_F(\rho^{(1)}) + \mathcal{E}_F(\rho^{(1)})$. The additivity conjecture of EoF is that the equality always holds. Many examples were demonstrated in Ref. [32]. In this paper we will show that uncertainty in the coupled harmonic system also has particular additive property, which we call sum rule. We will present this sum rule in the coupled harmonic oscillator system with arbitrary time-dependent parameters.

We start with simple single harmonic oscillator Hamiltonian with arbitrary time-dependent frequency: $H_1 = \frac{p^2}{2} + \frac{1}{2} \omega^2(t)x^2$. This simple model is important to study on the squeezed states, which appear in various branches of physics such as quantum optics\[^{[33, 36]}\] and cosmology\[^{[37, 40]}\]. The time-dependent Schrödinger equation (TDSE) of this system was examined in detail in Ref. [41–44]. The linearly independent solutions $\psi_n(x, t) (n = 0, 1, \cdots)$

\(^1\) The Rényi-$\alpha$ entropies of few non-Gaussian states have been derived in Ref. [29].
are expressed in a form

\[ \psi_n(x, t) = e^{-iE_n\tau(t)} \frac{1}{\sqrt{2\pi n!}} \left( \frac{\omega'}{\pi} \right)^{1/4} H_n(\sqrt{\omega'}x) e^{-\frac{v^2}{2}} \]

(4)

where \( \omega' = \frac{\omega(0)}{b^2} \) and

\[ v = \omega' - i\frac{\dot{b}}{b} \quad E_n = \left( n + \frac{1}{2} \right) \omega(0) \quad \tau(t) = \int_0^t \frac{ds}{b^2(s)}. \]

(5)

In Eq. (4) \( H_n(z) \) is \( n \)th-order Hermite polynomial and \( b(t) \) satisfies the nonlinear Ernakov equation

\[ \ddot{b} + \omega^2(t)b = \frac{\omega^2(0)}{b^3} \]

(6)

with \( b(0) = 1 \) and \( \dot{b}(0) = 0 \). As Eq. (4) exhibits, \( b(t) \) plays a role as a scaling of the frequency. Solution of the Ernakov equation was discussed in Ref. [43, 45–47]. If \( \omega(t) \) is time-independent, \( b(t) \) is simply one. If \( \omega(t) \) is instantly changed as

\[ \omega(t) = \begin{cases} \omega_i & t = 0 \\ \omega_f & t > 0, \end{cases} \]

(7)

then \( b(t) \) becomes

\[ b(t) = \sqrt{\frac{\omega_i^2 - \omega_f^2}{2\omega_f^2} \cos(2\omega_f t) + \frac{\omega_f^2 + \omega_i^2}{2\omega_f^2}}. \]

(8)

Of course, more general case of \( \omega(t) \) the nonlinear Ernakov equation should be solved numerically or approximately.

The \( d \)-dimensional Wigner distribution function[24, 48] is defined in terms of the phase space variables in a form

\[ W(x, p : t) = \frac{1}{\pi^d} \int dz e^{-2i[p \cdot z]_t} \Psi^*(x + z : t)\Psi(x - z : t) \]

(9)

where \( x = (x_1, x_2, \cdots, x_d) \) and \( p = (p_1, p_2, \cdots, p_d) \). The Wigner distribution function is used to compute the expectation values. For example, the expectation value of \( f(x_1, p_1) \) can be computed by

\[ \langle f(x_1, p_1) \rangle = \int dx dp f(x_1, p_1) W(x, p : t). \]

(10)

Also, the Wigner distribution function has information on the substate of density matrix \( \rho(x, x' : t) = \Psi(x : t)\Psi^*(x' : t) \). If \( \rho_A(x_1, x'_1 : t) = \text{Tr}_{2,3,\cdots,d} \rho(x, x' : t) \), the purity function of \( \rho_A \) can be computed as

\[ P_A(t) \equiv \text{Tr}\rho_A^2 = 2\pi \int dx_1 dp_1 W^2(x_1, p_1 : t), \]

(11)
Then, the Wigner distribution function for \( H \) where \( W(x, t) \) yield an uncertainty
\[
\int_{-\infty}^{\infty} dx e^{-px^2 + 2px} H_m(ax + b) H_n(cx + d)
\]
(12)
\[
\frac{\sqrt{\pi} e^{\frac{2}{p}}} {p} \sum_{k=0}^{\min(m,n)} \binom{m}{k} \binom{n}{k} k! \left(1 - \frac{a^2}{p}\right) \frac{m-k}{2} \left(1 - \frac{c^2}{p}\right) \frac{n-k}{2} \left(\frac{2ac}{p}\right)^k
\times H_{m-k} \left(\frac{b + \frac{aq}{p}}{\sqrt{1 - \frac{a^2}{p}}} \right) H_{n-k} \left(\frac{d + \frac{cq}{p}}{\sqrt{1 - \frac{c^2}{p}}} \right).
\]

Then, the Wigner distribution function for \( H_1 \) can be written in a form
\[
W_n(x, p : t) = \frac{1}{\pi} \exp \left[ -\omega' x^2 - \frac{1}{\omega'} \left( p + \frac{i}{b} x \right)^2 \right]
\times \sum_{k=0}^{n} \binom{n}{k} (-1)^{k} \frac{2^{n-k}}{(n-k)!} \left[ \omega' x^2 + \frac{1}{\omega'} \left( p + \frac{i}{b} x \right)^2 \right]^{n-k}
= \frac{1}{n!\pi} \exp \left[ -\omega' x^2 - \frac{1}{\omega'} \left( p + \frac{i}{b} x \right)^2 \right] U \left( -n, 1, 2 \left[ \omega' x^2 + \frac{1}{\omega'} \left( p + \frac{i}{b} x \right)^2 \right] \right),
\]
where \( U(a, b, z) \) is a confluent hypergeometric function. It is straightforward to show
\[
\int dx dp W_n(x, p : t) = 2\pi \int dx dp W_n^2(x, p : t) = 1,
\]
which guarantees \( \psi_n(x, t) \) is pure state. Using the Wigner distribution function it is straightforward to show that for non-negative integer \( m \), \( \langle x^{2m+1} \rangle = \langle p^{2m+1} \rangle = 0 \) and
\[
\langle x^{2m} \rangle = \frac{2^n (m + n)!}{m!n!\sqrt{\pi} \omega^m} \Gamma \left( \frac{2m + 1}{2} \right) {}_2F_1 \left( -n, -n : -n - m : 1/2 \right)
\]
(14)
\[
\langle p^{2m} \rangle = \frac{2^n (m + n)!}{m!n!\sqrt{\pi} \omega^m} \Gamma \left( \frac{2m + 1}{2} \right) \left[ \omega' + \frac{1}{\omega'} \left( \frac{i}{b} \right)^2 \right]^m {}_2F_1 \left( -n, -n : -n - m : 1/2 \right),
\]
where \( \Gamma(z) \) and \( {}_2F_1(a, b : c : z) \) are gamma and hypergeometric functions. Thus, the uncertainties for \( x \) and \( p \) are
\[
(\Delta x)^2 = \frac{n + \frac{1}{\omega'}}{\omega'} \quad (\Delta p)^2 = \left( n + \frac{1}{\omega'} \right) \left[ \omega' + \frac{1}{\omega'} \left( \frac{i}{b} \right)^2 \right],
\]
(15)
which yield an uncertainty
\[
(\Delta x \Delta p)^2 = \left( n + \frac{1}{\omega'} \right)^2 \left[ 1 + \frac{1}{\omega'^2} \left( \frac{i}{b} \right)^2 \right].
\]
(16)
Now let us consider the Hamiltonian \( H \) again when \( k_0 \) and \( J \) are arbitrarily time-dependent. It is not difficult to show that the Hamiltonian is diagonalized by introducing normal coordinates \( y_1 = (x_1 + x_2)/\sqrt{2} \) and \( y_2 = (-x_1 + x_2)/\sqrt{2} \), and their conjugate momenta \( \pi_1 \) and \( \pi_2 \) with normal mode frequencies \( \omega_1 = \sqrt{k_0} \) and \( \omega_2 = \sqrt{k_0 + 2J} \). If two oscillators are \( n^{th} \) and \( m^{th} \) states, in the following we will show that the uncertainties for \( x_j \) and \( p_j \) \((j = 1, 2)\) are just arithmetic mean of two single oscillators, that is

\[
(\Delta x_1)^2 = (\Delta x_2)^2 = \frac{1}{2} \left[ \frac{2n + 1}{2\omega_1'} + \frac{2m + 1}{2\omega_2'} \right]
\]

\[
(\Delta p_1)^2 = (\Delta p_2)^2 = \frac{1}{2} \left[ \frac{2n + 1}{2} \left\{ \omega_1' + \frac{1}{\omega_1'} \left( \frac{\ddot{b}_1}{b_1} \right)^2 \right\} + \frac{2m + 1}{2} \left\{ \omega_2' + \frac{1}{\omega_2'} \left( \frac{\ddot{b}_2}{b_2} \right)^2 \right\} \right]
\]

where \( \omega_j' = \omega_j(0)/b_j^2 \) \((j = 1, 2)\), and \( b_j \) satisfy their own nonlinear Ermakov equations \( \ddot{b}_j + \omega_j^2(t)b_j = \frac{\omega_j^2(0)}{b_j^2} \) with \( \ddot{b}_j(0) = 0 \) and \( b_j(0) = 1 \).

In order to show Eq. (17) we start with solutions of TDSE for \( H_2 \) in terms of \( y_j \), which is

\[
\psi_{n,m}(x_1, x_2 : t) = \frac{1}{\sqrt{2^{(n+m)}n!m!}} \left( \frac{\omega_1'\omega_2'}{\pi^2} \right)^{1/4} H_n(\sqrt{\omega_1'}y_1)H_m(\sqrt{\omega_2'}y_2) \times \exp \left[ -i(E_{n,1}\tau_1 + E_{m,2}\tau_2) - \frac{1}{2} \left( v_1y_1^2 + v_2y_2^2 \right) \right],
\]

where \( E_{m,j} = (m + \frac{1}{2})\omega_j(0) \), \( \tau_j = \int_0^t \frac{ds}{\omega_j(s)} \), and \( v_j = \omega_j' - i\frac{\dot{b}_j}{b_j} \). Now, let us compute the Wigner distribution functions of \( H_2 \) system by choosing \( \Psi(x : t) = \psi_{n,m}(x_1, x_2 : t) \) in Eq. (9). If we change Eq. (18) into the original coordinates \( x_j \) and \( p_j \), and inserting it to Eq. (9), the computation of the Wigner distribution function is highly complicated. However, we can escape this difficulty. Since \( y_j \)'s are orthogonal normal modes, they preserve inner product and 2-dimensional volume elements. Thus, the Wigner distribution function for \( H_2 \) are simply reduced to

\[
W_{n,m}(x_1, x_2 : p_1, p_2 : t) = W_n(y_1, \pi_1 : t) \bigg|_{\omega' \rightarrow \omega_1', \dot{b} \rightarrow b_1} \times W_m(y_2, \pi_2 : t) \bigg|_{\omega' \rightarrow \omega_2', \dot{b} \rightarrow b_2},
\]

where \( W_n \) is a Wigner distribution function of the single harmonic oscillator given in Eq. (13).

At this stage we want to digress little bit. Sometimes we need to derive the lower-dimensional reduced Wigner distribution function to explore the properties of reduced
quantum state. Although, however, we can compute the 2-dimensional Wigner distribution function quickly by making use of normal mode, derivation of reduced 1-dimensional Wigner distribution function is very complicated problem. For example, let us consider $W_{n,m}(x_1, p_1 : t) \equiv \int dx_2 dp_2 W_{n,m}(x_1, x_2 : p_1, p_2 : t)$. The difficulty arises due to the fact that $dx_2 dp_2$ is not invariant measure in the normal modes. Thus, we should compute the reduced Wigner distribution function by making use of original coordinates and conjugate momenta. After long and tedious calculation it is possible to show

\[
W_{n,m}(x_1, p_1 : t) = \frac{\sqrt{4\omega'_1\omega'_2}}{\pi} \sum_{k=0}^{n} \sum_{\ell=0}^{m} \left( \begin{array}{c} n \\ k \end{array} \right) \left( \begin{array}{c} m \\ \ell \end{array} \right) \frac{(-1)^{k+\ell}}{(n-k)!(m-\ell)!} 2^{(n+m)-(k+\ell)} (20)
\]

\[
\times \left( -\frac{\partial}{\partial \mu_1} \right)^{n-k} \left( -\frac{\partial}{\partial \mu_2} \right)^{m-\ell} \frac{1}{\sqrt{\Omega(\mu_1, \mu_2 : t)}} \exp \left[ -2 \Theta(x_1, p_1 : \mu_1, \mu_2 : t) \right] \bigg|_{\mu_1=\mu_2=1},
\]

where

\[
\Omega(\mu_1, \mu_2 : t) = \omega'_1\omega'_2(\mu_1^2 + \mu_2^2) + \left[ \omega'^2_1 + \omega'^2_2 + \left( \frac{\dot{b}_1}{b_1} - \frac{\dot{b}_2}{b_2} \right)^2 \right] \mu_1\mu_2
\]

\[
\Theta(x_1, p_1 : \mu_1, \mu_2 : t) = \omega'_1 \left[ \omega'^2_2 x_1^2 + \left( p_1 + \frac{\dot{b}_2}{b_2} x_1 \right)^2 \right] \mu_1^2\mu_2 + \omega'_2 \left[ \omega'^2_1 x_1^2 + \left( p_1 + \frac{\dot{b}_1}{b_1} x_1 \right)^2 \right] \mu_1^2\mu_2.
\]

Thus, the reduced Wigner distribution function for $n = m = 0$ is easily computed by

\[
W_{0,0}(x_1, p_1 : t) = \frac{1}{\pi} \sqrt{\frac{4\omega'_1\omega'_2}{\Omega(1,1 : t)}} e^{-2\Theta(x_1,p_1,1,1 : t)/\Omega(1,1 : t)}.
\]

(22)

The purity function $P_{0,0}^A(t) = \text{tr} \rho^2_{0,0}(x_1, x'_1 : t)$, where $\rho_{0,0}(x_1, x'_1 : t)$ is an effective state of $A$-oscillator derived by taking a partial trace to $\rho_{0,0}(x_1, x_2 : x'_1, x'_2 : t) = \psi_{0,0}(x_1, x_2 : t)\psi^*_{0,0}(x'_1, x'_2 : t)$ over $B$-oscillator, is

\[
P_{0,0}^A(t) = 2\pi \int dx_1 dp_1 W_{0,0}^2(x_1, p_1 : t) = 2\sqrt{z}
\]

(23)

where $z = \omega'_1\omega'_2/\Omega(1,1 : t)$. From Eq. (20) one can show directly $\int dx_1 dp_1 W_{n,m}(x_1, p_1 : t) =$
1 by making use of simple binomial formula. Furthermore, it is possible to show

\[
2\pi \int dx_1dp_1W_{m,n}(x_1,p_1:t) = 4\sqrt{\omega_1^2\omega_2^2} \sum_{k,k'=0}^{n} \sum_{\ell,\ell'=0}^{m} \binom{n}{k} \binom{m}{\ell} \binom{m}{\ell'} (-1)^{k+k'+\ell+\ell'} \frac{2^{2(n+m)-(k+k'+\ell+\ell')}}{(n-k)!(n-k')!(m-\ell)!(m-\ell')} \times
\]

\[
\left( -\frac{\partial}{\partial \mu_1} \right)^{n-k} \left( -\frac{\partial}{\partial \nu_1} \right)^{n-k'} \left( -\frac{\partial}{\partial \mu_2} \right)^{m-\ell} \left( -\frac{\partial}{\partial \nu_2} \right)^{m-\ell'} \frac{1}{\sqrt{\Gamma(\mu_1, \mu_2; \nu_1, \nu_2)}} \bigg|_{\mu_1=\mu_2=\nu_1=\nu_2=1}
\]

where

\[
\Gamma(\mu_1, \mu_2; \nu_1, \nu_2) = \omega'_1\omega'_2 \left[ \mu^2_1\nu^2_2(\mu_2 + \nu_2)^2 + \mu^2_2\nu^2_2(\mu_1 + \nu_1)^2 \right] + \mu_1\mu_2\nu_1\nu_2(\mu_1 + \nu_1)(\mu_2 + \nu_2) \left[ \omega'^2_1 + \omega'^2_2 + \left( \frac{b_1}{b_1} - \frac{b_2}{b_2} \right)^2 \right].
\]

If we define the ratios

\[
\gamma_n = \frac{P_{n,0}(t)}{P_{0,0}(t)} \quad \delta_n = \frac{P_{n,n}(t)}{P_{0,n}(t)}
\]

they are summarized at Table I.

| $n$ | $\gamma_n$ | $\delta_n$ |
|-----|------------|------------|
| 1   | $\frac{1}{4}(3 - 4z)$ | $\frac{1}{16}(9 - 40z + 144z^2)$ |
| 2   | $\frac{1}{64}(41 - 104z + 144z^2)$ | $\frac{1}{4096}(1681 - 19344z + 256608z^2 - 1440000z^3 + 2822400z^4)$ |
| 3   | $\frac{1}{256}(147 - 540z + 1488z^2 - 1600z^3)$ | too long |

Table I: The ratios $\gamma_n$ and $\delta_n$ for $n = 1, 2, 3$

The time-dependence of $\gamma_n$ and $\delta_n$ is plotted in Fig. 1(a) and Fig. 1(b) when $k_0(0) = J(0) = 1$ and $k_0(t) = J(t) = 2$ ($t > 0$). As expected the figures exhibit that the effective states for $A$-oscillator is more and more mixed with increasing $n$. The remarkable fact these figures show is that the state $\rho_{n,n}$ is more mixed that $\rho_{n,0}$.

Now, let us return to discuss on the uncertainties. From Eq. (19) it is easy to show that $\langle y_j^{2m+1} \rangle = \langle \pi_j^{2m+1} \rangle = 0$, and $\langle y_j^{2m} \rangle$ and $\langle \pi_j^{2m} \rangle$ are equal to $\langle x^{2m} \rangle$ and $\langle p^{2m} \rangle$ in Eq. (14) with changing $\omega_j \rightarrow \omega_j'$ and $b \rightarrow b_j$. Using this fact and the normal modes it is easy to show Eq. (17).
FIG. 1: (Color online) The time-dependence of the ratios is plotted in Fig. 1(a) ($\gamma_n$) and Fig. 1(b) ($\delta_n$) when $k_0(0) = J(0) = 1$ and $k_0(t) = J(t) = 2$ ($t > 0$). As expected the figures exhibit that the effective states for $A$-oscillator is more and more mixed with increasing $n$.

In order to check whether the property of arithmetic average for uncertainties is maintained or not in multi-coupled harmonic oscillator system we consider the three-coupled harmonic oscillator system, whose Hamiltonian is

$$H_3 = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \frac{1}{2} \left[ k_0(t)(x_1^2 + x_2^2 + x_3^2) + J(t) \left\{ (x_1 - x_2)^2 + (x_1 - x_3)^2 + (x_2 - x_3)^2 \right\} \right].$$

(27)

The normal mode coordinates of $H_3$ is $y_1 = (x_1 + x_2 + x_3)/\sqrt{3}$, $y_2 = (x_1 - x_2)/\sqrt{2}$, and $y_3 = (x_1 + x_2 - 2x_3)/\sqrt{6}$ with normal mode frequencies $\omega_1 = \sqrt{k_0}$ and $\omega_2 = \omega_3 = \sqrt{k_0 + 3J} \equiv \omega$. If three oscillators are in $n^{th}$, $m^{th}$, and $\ell^{th}$ states respectively, the 3-dimensional Wigner distribution function can be computed in a form

$$W_{n,m}(x_1, x_2, x_3 : p_1, p_2, p_3 : t) = W_n(y_1, \pi_1 : t) \bigg|_{\omega' \rightarrow \omega'_1, b \rightarrow b_1} \times W_m(y_2, \pi_2 : t) \times W_\ell(y_3, \pi_3 : t)$$

(28)

where $\pi_j$ are conjugate momenta of $y_j$ and $W_n$ is a Wigner distribution function of the single harmonic oscillator given in Eq. [13]. Of course, $b_1(t)$ and $b(t)$ are solutions for Ermakov equations for $\omega_1$ and $\omega$, and $\omega'_1 = \omega_1(0)/b_1^2(t)$ and $\omega' = \omega(0)/b^2(t)$. Then, it is
straightforward to show

\[(\Delta x_1)^2 = (\Delta x_2)^2 = \frac{1}{3} \left[ \frac{2n + 1}{2\omega'_1} + \frac{3(2m + 1) + (2\ell + 1)}{4\omega'} \right] \tag{29}\]

\[(\Delta x_3)^2 = \frac{1}{3} \left[ \frac{2n + 1}{2\omega'_1} + \frac{2\ell + 1}{2\omega'} \right] \]

\[(\Delta p_1)^2 = (\Delta p_2)^2 = \frac{1}{3} \left[ \frac{2n + 1}{2} \left\{ \omega'_1 + \frac{1}{\omega'_1} \left( \frac{b_1}{b_1} \right)^2 \right\} + \frac{3(2m + 1) + (2\ell + 1)}{4} \left\{ \omega' + \frac{1}{\omega'} \left( \frac{b}{b} \right)^2 \right\} \right] \]

\[(\Delta p_3)^2 = \frac{1}{3} \left[ \frac{2n + 1}{2} \left\{ \omega'_1 + \frac{1}{\omega'_1} \left( \frac{b_1}{b_1} \right)^2 \right\} + \frac{2\ell + 1}{2} \left\{ \omega' + \frac{1}{\omega'} \left( \frac{b}{b} \right)^2 \right\} \right]. \]

Thus, the property of the arithmetic average in uncertainties is not maintained when \(N = 3\) except \(m = \ell\).

Finally, let us consider the \(N\)-coupled harmonic oscillator system, whose Hamiltonian is

\[H_N = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + \frac{1}{2} \sum_{i=1}^{N} x_i^2 + J(t) \sum_{i<j}^{N} (x_i - x_j)^2 \tag{30}\]

It is diagonalized by introducing the normal mode coordinates \(y_1 = (x_1 + x_2 + \cdots + x_N)/\sqrt{N}\) and \(y_j = (x_1 + x_2 + \cdots + x_{j-1} - (j-1)x_j)/\sqrt{J(j-1)}\) \((j = 2, 3, \cdots, N)\) with normal mode frequencies \(\omega_1 = \sqrt{k_0}\) and \(\omega_2 = \omega_3 = \cdots = \omega_N = \sqrt{k_0 + NJ} \equiv \omega). If \(N\) oscillators are \(n_1, n_2, \cdots, n_N\) states, the \(N\)-dimensional Wigner distribution function can be written in a form

\[W_{n_1,n_2,\cdots,n_N}(x, p : t) = W_{n_1}(y_1, \pi_1 : t) \bigg|_{\omega' = \omega'_1, b = b_1} \times \prod_{j=2}^{N} W_{n_j}(y_j, \pi_j : t), \tag{31}\]

where \(\pi_j\) are conjugate momenta of \(y_j\) and \(W_n\) is a Wigner distribution function of the single harmonic oscillator given in Eq. \[13\]. Then it is straightforward to show

\[(\Delta x_j)^2 = \frac{1}{N} \left[ \frac{2n_1 + 1}{2\omega'_1} + \frac{1}{2\omega'} \left\{ \frac{2N(j-1)}{j} n_j + 2N \sum_{k=j+1}^{N} \frac{n_k}{k(k-1) + (N-1)} \right\} \right] \tag{32}\]

\[(\Delta p_j)^2 = \frac{1}{N} \left[ \frac{2n_1 + 1}{2} \left\{ \omega'_1 + \frac{1}{\omega'_1} \left( \frac{b_1}{b_1} \right)^2 \right\} \right. \]

\[+ \left. \frac{1}{2} \left\{ \frac{2N(j-1)}{j} n_j + 2N \sum_{k=j+1}^{N} \frac{n_k}{k(k-1) + (N-1)} \right\} \left\{ \omega' + \frac{1}{\omega'} \left( \frac{b}{b} \right)^2 \right\} \right]. \]

One can show that Eq. \[32\] reproduces Eq. \[17\] and Eq. \[29\] when \(N = 2\) and \(N = 3\) if the quantum numbers \(n_1, n_2, \) and \(n_3\) are replaced by \(n, m, \) and \(\ell\). If \(n_2 = n_3 = \cdots = n_N, \) one can
show that \((\Delta x_j)^2\) and \((\Delta p_j)^2\) are independent of \(j\) and arithmetic average of uncertainties for each oscillator.

In this paper we compute the uncertainties \((\Delta x)^2\) and \((\Delta p)^2\) analytically in the \(N\)-coupled harmonic oscillator system. When \(N = 2\), it is shown that those uncertainties are just arithmetic average of uncertainties of two single harmonic oscillators. However, this property is not generally maintained when \(N \geq 3\). This property is recovered in \(N\)-coupled oscillator system only when \((N - 1)\) quantum numbers are equal.

Our calculation can be generalized to more general case. For example, let us consider a Hamiltonian

\[
\tilde{H}_3 = \frac{1}{2} (p_1^2 + p_2^2 + p_3^2) + \frac{1}{2} \left[ k_0(t) \left( x_1^2 + x_2^2 + x_3^2 \right) + J_{12}(t)(x_1 - x_2)^2 + J_{13}(t)(x_1 - x_3)^2 + J_{23}(t)(x_2 - x_3)^2 \right].
\]

In this case the normal mode coordinates are

\[
y_1 = \frac{1}{\sqrt{3}} (x_1 + x_2 + x_3) \quad \quad \quad (34)
\]

\[
y_+ = A_+(-J_{12} + J_{23} - \zeta)x_1 + A_+(J_{12} - J_{13} + \zeta)x_2 + A_+(J_{13} - J_{23})x_3
\]

\[
y_- = A_-(J_{12} - J_{23} + \zeta)x_1 + A_-(J_{12} - J_{13} - \zeta)x_2 + A_-(J_{13} - J_{23})x_3
\]

with \(\zeta = \sqrt{J_{12}^2 + J_{13}^2 + J_{23}^2 - (J_{12}J_{13} + J_{12}J_{23} + J_{13}J_{23})}\) and

\[
A_{\pm} = \frac{1}{J_{13} - J_{23}} \sqrt{\frac{2\zeta \pm (J_{13} + J_{23} - 2J_{12})}{6\zeta}}. \quad \quad \quad (35)
\]

In this case the normal mode frequencies are \(\omega_1 = \sqrt{k_0}\) and \(\omega_{\pm} = \sqrt{k_0 + J_{12} + J_{13} + J_{23} \pm \zeta}\).

If the three oscillators are \(n^{th}\), \(m^{th}\), and \(\ell^{th}\) exciting states, our procedure yields

\[
(\Delta x_1)^2 = \frac{1}{3} \frac{2n + 1}{2\omega_1'} + A_+^2 u_+^2 \frac{2m + 1}{2\omega_+'} + A_-^2 \frac{2\ell + 1}{2\omega_-'} \quad \quad \quad (36)
\]

\[
(\Delta x_2)^2 = \frac{1}{3} \frac{2n + 1}{2\omega_1'} + A_+^2 v_+^2 \frac{2m + 1}{2\omega_+'} + A_-^2 v_-^2 \frac{2\ell + 1}{2\omega_-'}
\]

\[
(\Delta x_3)^2 = \frac{1}{3} \frac{2n + 1}{2\omega_1'} + (J_{13} - J_{23})^2 \left[ A_+^2 \frac{2m + 1}{2\omega_+'} + A_-^2 \frac{2\ell + 1}{2\omega_-'} \right],
\]

where \(u_{\pm} = -J_{12} + J_{23} \pm \zeta, v_{\pm} = J_{12} - J_{13} \pm \zeta,\) and \(\omega_j' = \omega_j/b_j(t)\) \((j = 1, \pm)\). Of course \(b_j\)'s are the scaling factors of \(\omega_j\). Similarly, the uncertainties \((\Delta p_j)^2\) can be computed explicitly by following the same procedure.
The quantum information processing with continuous variables attracts much attention from the aspect of both theory and experiment\[50, 51\]. The quantum uncertainties are closely connected to inseparability criterion of the continuous variable quantum system\[27, 52\]. Furthermore, the distillation protocols to maximally entangled state have been already suggested in Ref. \[53, 54\]. We hope our results on explicit expressions of uncertainties may give valuable tools to the various continuous variable quantum information processing.

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