NEW GENERAL DECAY RESULTS FOR A VON KARMAN PLATE EQUATION WITH MEMORY-TYPE BOUNDARY CONDITIONS

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Abstract. In this paper we consider a von Karman plate equation with memory-type boundary conditions. By assuming the relaxation function \( k_i (i = 1, 2) \) with minimal conditions on the \( L^1(0, \infty) \), we establish an optimal explicit and general energy decay result. In particular, the energy result holds for \( H(s) = s^p \) with the full admissible range \( [1, 2) \) instead of \( [1, 3/2) \). This result is new and substantially improves earlier results in the literature.

1. Introduction. In this paper, we consider the following von Karman plate equation

\[
\begin{align*}
\left\{ \begin{array}{l}
u_{tt} + \Delta^2 u &= \alpha[u, F(u)], & \text{in } \Omega \times \mathbb{R}^+, \\
\Delta^2 F(u) &= -[u, u], & \text{in } \Omega \times \mathbb{R}^+,
\end{array} \right.
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^2 \) is a bounded domain with smooth boundary \( \Gamma, \Gamma = \Gamma_0 \cup \Gamma_1, \Gamma_0 \cap \Gamma_1 \neq \emptyset, \Gamma_0 \) and \( \Gamma_1 \) have positive measures and \( x = (x, y) \). The function \( u(x, y, t) \) represents the transversal displacement, \( F(u(x, y, t)) \) is the Airy stress function. The constant \( \alpha \) is positive, the von Karman bracket \([\omega, \phi]\) is given by

\[ [\omega, \phi] = \omega_{xx} \phi_{yy} + \omega_{yy} \phi_{xx} - 2 \omega_{xy} \phi_{xy}. \]

We consider the following boundary conditions

\[
\begin{align*}
F(u) &= \frac{\partial F(u)}{\partial \nu} = 0, & \text{on } \Gamma \times \mathbb{R}^+, \\
u &= \frac{\partial u}{\partial \nu} = 0, & \text{on } \Gamma_0 \times \mathbb{R}^+, \\
\frac{\partial u}{\partial \nu} + \int_0^t g_1(t-s)B_1u(s)ds &= 0, & \text{on } \Gamma_1 \times \mathbb{R}^+, \\
u - \int_0^t g_2(t-s)B_2u(s)ds &= 0, & \text{on } \Gamma_1 \times \mathbb{R}^+,
\end{align*}
\]

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and initial conditions
\[ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \]  
(7)
where \( \nu = (\nu_1, \nu_2) \) is the unit outward normal to \( \Gamma \), and
\[ B_1 u = \Delta u + (1 - \mu)B_1 u, \quad B_2 u = \frac{\partial}{\partial \nu} \Delta u + (1 - \mu)B_2 u, \]
here
\[ B_1 u = 2\nu_1\nu_2 u_{xy} - \nu_1^2 u_{yy} - \nu_2^2 u_{xx}, \quad B_2 u = \frac{\partial}{\partial \nu} \left[(\nu_1^2 - \nu_2^2)u_{xy} + \nu_1\nu_2(u_{yy} - u_{xx})\right], \]
and \( \mu \in (0, 1/2) \) is Poisson’s ration and \( \tau = (-\nu_2, \nu_1) \) is the unit tangent positively oriented on \( \Gamma \). Then system (1)-(7) describes the transversal displacement of a thin vibrating plate subjected to the boundary viscoelastic damping.

In the past decades, the mathematical analysis of Kirchhoff plates, which contain global existence, uniqueness and stability, with different boundary feedbacks was considered by many authors, see, for instance, Chueshov and Lasiecka [5, 6], Favini et al. [7], Komornik [12], Lagnese [13, 14], Lasiecka [15], Muñoz Rivera [18], Jorge Silva et al. [8, 9]. For system (1)-(7), if \( \alpha = 0 \), Santos and Junior [29] established the energy decays exponentially (polynomially) if the kernel \( g \) decays exponentially (polynomially). But they considered the following assumptions
\[ k_i'(t) \leq -ak_i(t), \quad k_i''(t) \geq -bk_i(t), \quad i = 1, 2, \]
where \( a, b > 0 \) are constants and \( k_i \) are the resolvent kernel of \(-\frac{g_i}{g_i(0)}\). Mustafa and Abusharkh [23] by assuming \( u_0 = 0 \) on one part of boundary and considering a more general assumption
\[ k_i''(t) \geq H(-k_i'(t)), \quad i = 1, 2, \]
extend the decay result in Santos and Junior [29], where \( H(t) > 0 \) is strictly increasing and strictly convex near the origin with \( H(0) = H'(0) = 0 \), was introduced by Alabau-Boussouira and Cannarsa [1]. They also considered a special case \( H(t) = t^p \) and obtained the decay result for \( p \in [0, 3/2) \). When \( \alpha \neq 0 \), Park and Park [27] proved the global existence and regularity of solution and established exponential stability. Park [24], using the same assumption on the kernel in Mustafa and Abusharkh [23], established a general decay result of energy and also extended the result to the case \( u_0 \neq 0 \) on one part of boundary. For more results on memory-type of von Karman equation, we refer to [4, 10, 11, 19, 20, 26, 25, 28, 30].

In this paper, we consider the von Karman plate equation with memory-type boundary conditions (1)-(7), by assuming the relaxation function \( k_i \) (\( i = 1, 2 \)) with minimal conditions on the \( L^1(0, \infty) \), i.e., \( k_i''(t) \geq \eta(t)H_2(-k_i'(t)) \), where \( H_1 \) and \( H_2 \) are two linear or strictly increasing and strictly convex functions of class \( C^2(\mathbb{R}^+) \). We establish an optimal explicit and general energy decay result. In particular, the energy result holds for \( H(s) = s^p \) with the full admissible range \([1, 2)\) instead of \([1, 3/2)\). Hence our results extend and improve the stability results in [23] and in [24, 27]. The arguments in this paper mainly rely on Lyapunov functional method and some properties of convex function developed by Alabau-Boussouira and Cannarsa [1] and Lasiecka and Tataru [16].

The paper is organized as follows. In Section 2, we give some assumptions used in this paper. In Section 3, we state our main results. The proof of stability result will be given in Section 4.
2. Preliminaries. In this paper, we use the standard Lebesgue and Sobolev spaces with their usual scalar products and norms. We denote the norm of Banach space $X$ by $\| \cdot \|_X$. For convenience, we write $\| \cdot \|_{L^2(\Omega)}$ and $\| \cdot \|_{L^2(\Gamma_1)}$ by $\| \cdot \|$ and $\| \cdot \|_{\Gamma_1}$, respectively. We define the spaces $W$ and $\tilde{W}$ by

$$W = \left\{ \omega \in H^2(\Omega) : \omega = \frac{\partial \omega}{\partial \nu} = 0 \text{ on } \Gamma_0 \right\},$$

$$\tilde{W} = \left\{ \omega \in H^2(\Omega) : \omega = \frac{\partial \omega}{\partial \nu} = 0 \text{ on } \Gamma_1 \right\}.$$

The bilinear symmetric form $a(\omega, \phi)$ is defined by

$$a(\omega, \phi) = \omega_{xx}\phi_{xx} + \omega_{yy}\phi_{yy} + \mu(\omega_{xx}\phi_{yy} + \omega_{yy}\phi_{xx}) + 2(1 - \mu)\omega_{xy}\phi_{xy}.$$ 

Since $\Gamma_0 \neq 0$, we have that there exist two positive constants $c_1$ and $c_2$ such that

$$c_1 \| u \|_{H^2(\Omega)} \leq \int_{\Omega} a(u, u) \, dx \leq c_2 \| u \|_{H^2(\Omega)}, \quad (8)$$

i.e., $\int_{\Omega} a(u, u) \, dx$ is equivalent to the $H^2(\Omega)$ norm on $W$, see [5]. Then combining Sobolev imbedding theorem, we get for $u \in W$,

$$\| u \|^2 \leq c_p \int_{\Omega} a(u, u) \, dx, \quad \| u \|^2_{\Gamma_1} \leq \bar{c}_p \int_{\Omega} a(u, u) \, dx, \quad (9)$$

where $c_p > 0$ and $\bar{c}_p > 0$ are embedding constants.

For von Karman bracket and Airy stress function, we have the following two lemmas, see [3, 5, 7].

Lemma 2.1. For $\omega \in H^2(\Omega)$, $\phi \in \tilde{W}$ and $v \in W$,

$$\int_{\Omega} [\omega, \phi] v \, dx = \int_{\Omega} [\omega, v] \phi \, dx.$$

The following lemma plays a crucial role in proving our result, which can be found in [5]. This is very deep result of [7].

Lemma 2.2. If $\omega \in H^2(\Omega)$, then $[\omega, F(\omega)] \in L^2(\Omega)$ and

$$\| [\omega, F(\omega)] \| \leq c \| \omega \|_{H^2(\Omega)} \| F(\omega) \|_{W^{2, \infty}(\Omega)}, \quad \| F(\omega) \|_{W^{2, \infty}(\Omega)} \leq c \| \omega \|^2_{H^2(\Omega)}.$$

The following lemma is very useful in the proof of our main result, one can find in [27].

Lemma 2.3. Let $\omega, \phi \in H^4(\Omega)$ and $\mu \in \mathbb{R}$. Then

$$\int_{\Omega} (\Delta^2 \omega) \phi \, dx = \int_{\Omega} a(\omega, \phi) \, dx + \int_{\Gamma} \left[ (B_2\omega) \phi - (B_1\omega) \frac{\partial \omega}{\partial \nu} \right] \, d\Gamma, \quad (10)$$

and

$$\int_{\Omega} (m \cdot \nabla \omega) \Delta^2 \omega \, dx$$

$$= \int_{\Omega} a(\omega, \omega) \, dx + \frac{1}{2} \int_{\Gamma} m \cdot \nu \left[ \omega_{xx}^2 + \omega_{yy}^2 + 2\mu\omega_{xx}\omega_{yy} + 2(1 - \mu)\omega_{xy}^2 \right] \, d\Gamma$$

$$+ \int_{\Gamma} \left[ (B_2\omega) m \cdot \nabla \omega - (B_1\omega) \frac{\partial}{\partial \nu} (m \cdot \nabla \omega) \right] \, d\Gamma. \quad (11)$$
Taking the derivative of (5) and (6) with respect to $t$, we get
\[
\mathcal{B}_1 u = -\frac{1}{g_1(0)} \left[ \frac{\partial u}{\partial \nu} + g'_i * \mathcal{B}_1 u \right], \quad \mathcal{B}_2 u = \frac{1}{g_2(0)} [u_t + g'_2 * \mathcal{B}_2 u].
\]
Volterra’s inverse operator implies that
\[
\mathcal{B}_1 u = -\frac{1}{g_1(0)} \left[ \frac{\partial u}{\partial \nu} + k_1 * \frac{\partial u}{\partial \nu} \right], \quad \mathcal{B}_2 u = \frac{1}{g_2(0)} [u_t + k_2 * u_t]
\]
where $k_i (i = 1, 2)$ are the resolvent kernels of $(-g'_i / g_i(0))$ satisfying for $t \geq 0$, $k_i(t) + \frac{1}{g_i(0)} (g'_i * k_i)(t) = -\frac{1}{g_i(0)} g'_i(t)$.

Denoting $\alpha_i = \frac{1}{g_i(0)} (i = 1, 2)$ and assuming $u_0 = 0$ on $\Gamma_1$ in this paper, we have
\[
\mathcal{B}_1 u = -\alpha_1 \left[ \frac{\partial u}{\partial \nu} + k_1(0) \frac{\partial u}{\partial \nu} + k'_1 * \frac{\partial u}{\partial \nu} \right], \quad \text{(12)}
\]
and
\[
\mathcal{B}_2 u = \alpha_2 \left[ u_t + k_2(0) u_t + k'_2 * u \right]. \quad \text{(13)}
\]
In the following, we use boundary conditions (12)-(13) instead of (5) and (6).

Here, we introduce our assumptions.
(A1) There exists a fixed point $x_0 \in \mathbb{R}^2$ and some constant $\delta > 0$ such that for $m(x) = x - x_0$,
\[
\Gamma_0 = \{ x \in \Gamma : m(x) \cdot \nu(x) \leq 0 \}, \quad \Gamma_1 = \{ x \in \Gamma : m(x) \cdot \nu(x) \geq \delta \}. \quad \text{(14)}
\]
As in [17, 21, 22], we make the following assumptions on the kernels $k_1$ and $k_2$.
(A2) The functions $k_i (i = 1, 2) : \mathbb{R}^+ \to \mathbb{R}^+$ are nonincreasing and twice differentiable functions satisfying for any $t \geq 0$,
\[
k_i(0) > 0, \quad \lim_{t \to +\infty} k_i(t) = 0, \quad k'_i(t) \leq 0. \quad \text{(15)}
\]
(A3) There exist two $C^1$ functions $H_i : \mathbb{R}^+ \to \mathbb{R}^+$ which are linear or are strictly increasing and strictly convex functions of class $C^2(\mathbb{R}^+)$ on $(0, r]$, $r \leq -k'_i(0)$, with $H_i(0) = H'_i(0) = 0$, such that
\[
k''_i(t) \geq \eta_i(t) H_i(-k'_i(t)), \quad \forall \ t \geq 0, \quad \text{(16)}
\]
where $\eta_i(t)$ are $C^1$ nonincreasing continuous functions.

**Remark 1.** (1) For $i = 1, 2$, by $\lim_{t \to +\infty} k_i(t) = 0$ and $(-k'_i(t))$ is nonincreasing and nonnegative, we arrive at $\lim_{t \to +\infty} (-k'_i(t)) = 0$. Hence we can get for some $t_1 \geq 0$ large,
\[
-k'_i(t_1) = r \Rightarrow -k'_i(t) \leq r, \quad \forall \ t \geq t_1.
\]
As $(-k'_i)$ is nonincreasing, $-k'_i(0) > 0$ and $-k'_i(t_1) > 0$, we know $-k'_i(t) > 0$ for any $t \in [0, t_1]$ and for any $t \in [0, t_1]$,
\[
0 < -k'_i(t_1) \leq -k'_i(t) \leq -k'_i(0), \quad 0 < \eta_i(t_1) \leq \eta_i(t) \leq \eta_i(0).
\]
There exist some positive constants $a_i$ and $b_i$ such that for any $t \in [0, t_1]$,
\[
a_i \leq \eta_i(t) H_i(-k'_i(t)) \leq b_i.
\]
Then for any \( t \in [0, t_1] \),
\[
k''_i(t) \geq \eta_i(t)H_i(-k'_i(t)) \geq a_i = \frac{a_i}{k'_i(0)}k'_i(0) \geq \frac{a_i}{k'_i(0)}k'_i(t),
\]
which gives us for any \( t \in [0, t_1] \),
\[
k'_i(t) \geq -dk'_i(t), \quad (17)
\]
where \( d \) is a positive constant.

(2) The following Jensen’s inequality is critical to prove our main result.

Let \( J \) be a convex increasing function on \([a, b]\), \( f : \Omega \to [a, b] \) and \( h \) are integrable functions on \( \Omega \) such that \( h(x) \geq 0 \) and \( \int_\Omega h(x)dx = k > 0 \), then Jensen’s inequality states that
\[
\int_\Omega J^{-1}(f(x))h(x)dx \leq kJ^{-1}\left[\frac{1}{k} \int_\Omega f(x)h(x)dx\right].
\]

3. **Main results.** For completeness, we give the well-posedness result in the following theorem proved in [27].

**Theorem 3.1.** Assume that (A1) and (A2) hold. Let \((u_0, u_1) \in (H^4(\Omega) \cap W) \times W\), then problem (1)-(7) admits a unique solution \( u \) satisfying
\[
u \in L^\infty(0, T; H^4(\Omega) \cap W) \cap W^1,\infty(0, T; W) \cap W^2,\infty(0, T; L^2(\Omega)).
\]

The total energy of the system is defined by
\[
E(t) = \frac{1}{2}\|u\|^2 + \frac{1}{2} \int_\Omega a(u, u)dx + \frac{\alpha}{4}\|\Delta F(u)\|^2 + \frac{\alpha_1}{2} k_1(t) \left\| \frac{\partial u}{\partial \nu} \right\|^2_{\Gamma_1} - \int_{\Gamma_1} k'_1 \circ \frac{\partial u}{\partial \nu} d\Gamma + \frac{\alpha_2}{2} \left\| u_i \right\|^2_{\Gamma_1} + \int_{\Gamma_1} k'_2 \circ u d\Gamma,
\]
where
\[
(k \circ u)(t) = \int_0^t k(t-s)[u(t) - u(s)]^2 ds.
\]

The stability result is stated in the following theorem.

**Theorem 3.2.** Let (A1)-(A3) hold. Under the assumptions of Theorem 3.1, the energy \( E(t) \) satisfies
\[
E(t) \leq \mu_2 H_4^{-1} \left( \mu_1 \int_{K^{-1}(r)} \eta(s)ds \right), \quad \forall t > K^{-1}(r), \quad (19)
\]
where the positive constant \( \mu_1, \mu_2 > 0 \) and
\[
H_4(t) = \int_0^t \frac{1}{sH_0(s)}ds, \quad H_0(t) = \min\{H'_1(t), H'_2(t)\},
\]
and \( \eta(t) = \min\{\eta_1(t), \eta_2(t)\} \), \( K(t) = \max\{-k'_1(t), -k'_2(t)\} \). In particular, when \( H_i(t) = ct^{p_i} \) \((i = 1, 2)\), then the total energy \( E(t) \) satisfies that for any \( t > 0 \),
\[
E(t) \leq \begin{cases} c_1 e^{-ct} \int_0^t \eta(s)ds, & \text{if } p = 1, \\ c_3 \left( 1 + \int_0^t \eta(s)ds \right)^{-\frac{1}{p-2}}, & \text{if } 1 < p < 2, \end{cases} \quad (20)
\]
where \( c_1, c_3 \) and \( c_2 \leq 1 \) are positive constants, and \( p = \max\{p_1, p_2\} \).
Remark 2. Our result is obtained under a much larger class of kernels that guarantee the optimal decay rates of the energy. In particular, the energy result holds for $H(s) = s^p$ with the full admissible range $[1, 2]$ instead of $[1, 3/2)$. It must to be pointed out that, if the viscoelastic term is as internal feedback, Lasiecka and Wang [17] provides the proof for optimal decay rates of second order systems in the full admissible range $[1, 2)$.

In the sequel we give some examples to illustrate explicit formulas for the decay rates of the energy.

Example 1. For $k_1^\prime(t) = k_2^\prime(t) = -ae^{-ct}$ with $0 < q < 1$ and $a > 0$, we have $k_i^\prime(t) = H_i(-k_i^\prime(t))$ ($i = 1, 2$) where $H_1(t) = H_2(t) = \frac{q t}{\ln(a/t)^{\frac{1}{q}}-1}$. Because

$$H_1^\prime(t) = H_2^\prime(t) = \frac{(1-q) + q\ln\left(\frac{q}{t}\right)}{\ln\left(\frac{q}{t}\right)^{\frac{1}{q}}}$$

and

$$H_1^{\prime\prime}(t) = H_2^{\prime\prime}(t) = \frac{(1-q)\left[\ln\left(\frac{q}{t}\right) + \frac{1}{q}\right]}{\ln\left(\frac{q}{t}\right)^{\frac{1}{q}+1}},$$

we know that the functions $H_1$ and $H_2$ satisfies (A3) on $(0, r]$ for any $0 < r < 1$. Therefore

$$E(t) \leq c_1 e^{-c_2 t^2}.$$  

Example 2. For $k_1^\prime(t) = k_2^\prime(t) = -\frac{1}{(t+e)\ln(t+e)}$ with $p > 1$, we obtain $k_i^\prime(t) = k_2^\prime(t) = \frac{[\ln(t+e) + p]}{(t+e)^2[\ln(t+e)]^{p+1}}$. Obviously,

$$k_i^\prime(t) = \frac{[\ln(t+e) + p]}{(t+e)^{\frac{1}{p}}\ln(t+e)}[-k_i^\prime(t)].$$

It follows from $(20)_1$ that

$$E(t) \leq c_1 \exp\left(-c_2 \int_0^t \frac{[\ln(t+e) + p]}{(t+e)^{\frac{1}{p}}\ln(t+e)} ds\right) = \frac{c_1}{((t+e)^{\frac{1}{p}}[\ln(t+e)]^{p+2})}.$$  

Since $c_2 \leq 1$, this is slower rate than $[-k_i^\prime(t)]$. In addition,

$$k_i^\prime(t) = \frac{[\ln(t+e) + p]}{(t+e)^{\frac{1}{p}}\ln(t+e)}[-k_i^\prime(t)]^{1+\frac{1}{\alpha}}.$$  

We infer from $(20)_2$ that for large $t$

$$E(t) \leq c_3 \left(1 + \int_0^t \frac{[\ln(t+e) + p]}{(t+e)^{\frac{1}{p}}\ln(t+e)} ds\right)^{-p} \leq \frac{c_3}{((t+e)^{\frac{1}{p}}[\ln(t+e)]^{p+2})}.$$  

This is the same rate as $[-k_i^\prime(t)]$.

Example 3. Let $k_1^\prime(t) = -ae^{-ct},$ $\alpha > 0,$ and $k_2^\prime(t) = -\frac{b}{(1+t)^{\mu}},$ $\mu > 1,$ $a$ and $b$ are two positive constants. Then there exists a certain $c_1 > 0$ such that for $t > t_1,$

$$E(t) \leq \frac{c_1}{(1+t)^{\mu}}.$$  

Example 4. Consider $k_1^\prime(t) = -ae^{-ct},$ $\alpha > 0,$ and $k_2^\prime(t) = -be^{-(1+t)^{\mu}},$ $0 < \mu < 1,$ $a$ and $b$ are two positive constants. Then there exists two positive constants $c_1$ and $c_2$ such that for $t > t_1,$

$$E(t) \leq c_1 e^{-c_2(1+t)^{\mu}}.$$
4. **Proof of main result.** In this section, we will give the proof of Theorem 3.2, which is divided into the following two subsections.

4.1. **Technical lemmas.**

**Lemma 4.1.** The total energy functional $E(t)$ is nonincreasing and satisfies for any $t \geq 0$,

$$E'(t) \leq -\frac{\alpha_1}{2} \left\| \frac{\partial u_t}{\partial t} \right\|^2_{\Gamma_1} + \int_{\Gamma_1} k^{\prime\prime}_1 \circ \frac{\partial u_t}{\partial t} \, d\Gamma - \frac{\alpha_2}{2} \left[ \left\| u_t \right\|^2_{\Gamma_1} + \int_{\Gamma_1} k^{\prime\prime}_2 \circ u_t \, d\Gamma \right]. \quad (21)$$

**Proof.** See, for example, [24] or [23].

**Lemma 4.2.** Define the functional $\Phi(t)$ by

$$\Phi(t) := \int_{\Omega} (m \cdot \nabla u) u_t \, dx.$$

Under the assumptions of Theorem 3.2, for any $\varepsilon > 0$, we have the following estimate:

$$\Phi'(t) \leq -\| u_t \|^2 - \left( 1 - \frac{\varepsilon}{2} \right) \int_{\Omega} a(u, u) \, dx - \frac{\alpha}{2} \| \Delta F(u) \|^2 + \frac{1}{2} \int_{\Gamma_1} (m \cdot \nu)|u_t|^2 d\Gamma$$

$$- \frac{1}{2} \left( 1 - \frac{\varepsilon}{\delta} \right) \int_{\Gamma_1} (m \cdot \nu) a(u, u) d\Gamma - \frac{\alpha}{2} \int_{\Gamma_0} (m \cdot \nu)|\Delta F(u)|^2 d\Gamma$$

$$+ \frac{2\alpha^2}{\varepsilon} \left[ \left\| \frac{\partial u_t}{\partial t} \right\|^2_{\Gamma_1} + k_1^2(t) \left\| \frac{\partial u}{\partial t} \right\|^2_{\Gamma_1} + C_{\delta_1} \int_{\Gamma_1} h_1 \circ \frac{\partial u}{\partial t} d\Gamma \right]$$

$$+ \frac{2\alpha^2}{\varepsilon} \left[ \left\| u_t \right\|^2_{\Gamma_1} + k_2^2(t) \left\| u \right\|^2_{\Gamma_1} + C_{\delta_2} \int_{\Gamma_1} h_2 \circ u d\Gamma \right], \quad (22)$$

where for $i = 1, 2$,

$$C_{\delta_i} = \int_0^\infty \frac{k_i''(s)}{k_i'(s) - \delta_i k_i'(s)} \, ds, \quad k_i(t) = k_i''(s) - \delta_i k_i'(s),$$

and $0 < \delta_i < 1$.

**Proof.** Recalling $u = \frac{\partial u}{\partial t} = 0$ on $\Gamma_0$, we know that

$$B_1 u = B_2 u = 0, \quad \text{on } \Gamma_0,$$

and

$$\frac{\partial}{\partial \nu} (m \cdot \nabla u) = (m \cdot \nu) \Delta u, \quad \text{on } \Gamma_0,$$

$$u_{xx}^2 + u_{yy}^2 + 2\mu u_{xx} u_{yy} + 2(1-\mu) u_{xy}^2 = (\Delta u)^2 \quad \text{on } \Gamma_0,$$

since

$$u_{xx} u_{yy} - (u_{xy})^2 = 0 \quad \text{on } \Gamma_0.$$

Then by using (1)-(2) and (10)-(11), we obtain

$$\Phi'(t) = -\| u_t \|^2 - \left( 1 - \frac{\varepsilon}{2} \right) \int_{\Omega} a(u, u) \, dx - \frac{\alpha}{2} \| \Delta F(u) \|^2 + \frac{1}{2} \int_{\Gamma_1} (m \cdot \nu)|u_t|^2 d\Gamma$$

$$- \frac{1}{2} \int_{\Gamma_1} (m \cdot \nu) a(u, u) d\Gamma - \frac{\alpha}{2} \int_{\Gamma_0} (m \cdot \nu)|\Delta F(u)|^2 d\Gamma$$

$$+ \int_{\Gamma_1} (B_1 u) \frac{\partial}{\partial \nu} (m \cdot \nabla u) d\Gamma - \int_{\Gamma_1} (B_2 u) (m \cdot \nabla u) d\Gamma. \quad (23)$$
Replacing (26)-(27) in (24), we infer for any $\varepsilon > 0$,

$$\int_{\Gamma_1} (B_1u) \frac{\partial}{\partial \nu} (m \cdot \nabla u) d\Gamma - \int_{\Gamma_1} (B_2u)(m \cdot \nabla u) d\Gamma$$

$$\leq \frac{\varepsilon}{2} \left( \left\| \frac{\partial}{\partial \nu} (m \cdot \nabla u) \right\|_{\Gamma_1}^2 + \left\| m \cdot \nabla u \right\|_{\Gamma_1}^2 \right) + \frac{1}{2\varepsilon} \left( \|B_1u\|_{\Gamma_1}^2 + \|B_2u\|_{\Gamma_1}^2 \right),$$

which, along with $m \cdot \nu \geq \delta$ and the trace theory, implies for any $\varepsilon > 0$,

$$\int_{\Gamma_1} (B_1u) \frac{\partial}{\partial \nu} (m \cdot \nabla u) d\Gamma - \int_{\Gamma_1} (B_2u)(m \cdot \nabla u) d\Gamma$$

$$\leq \frac{\varepsilon}{2} \left( c \int_{\Omega} a(u, u) d\Omega + \frac{c}{\delta} \int_{\Gamma_1} (m \cdot \nu)a(u, u) d\Gamma \right)$$

$$+ \frac{1}{2\varepsilon} \left( \|B_1u\|_{\Gamma_1}^2 + \|B_2u\|_{\Gamma_1}^2 \right),$$

(24)

Noting that $k' \ast u = (-k' \circ u) + [k(t) - k(0)]u$, where $k \circ u = \int_0^t k(t-s)(u(t) - u(s)) ds$, then we get from (13)

$$B_2u(t) = \alpha_2[u_t(t) + k_2(t)u(t) + (-k_2' \circ u)(t)].$$

By using Young’s inequality, we obtain

$$\|B_2u\|_{\Gamma_1}^2 \leq 4\alpha_2^2 \left[ \|u_t\|_{\Gamma_1}^2 + k_2^2(t)\|u\|_{\Gamma_1}^2 + \int_{\Gamma_1} (-k_2' \circ u)^2 d\Gamma \right].$$

(25)

By Hölder’s inequality, we have

$$(-k_i' \circ u)^2 = \left( \int_0^t (-k_i'(t-s))(u(t) - u(s)) ds \right)^2$$

$$= \left( \int_0^t \frac{-k_i'(t-s)}{\sqrt{k_i''(s) - \delta_i k_i'(s)}} \frac{1}{\sqrt{k_i''(s) - \delta_i k_i'(s)}} \sqrt{k_i''(s) - \delta_i k_i'(s)}(u(t) - u(s)) ds \right)^2$$

$$\leq \left( \int_0^t \frac{|k_i'(s)|^2}{k_i''(s) - \delta_i k_i'(s)} ds \right) \int_0^t (k_i''(s) - \delta_i k_i'(s))(u(t) - u(s))^2 ds$$

$$\leq C\delta_i (h_i \circ u),$$

which together with (25) gives us that

$$\|B_2u\|_{\Gamma_1}^2 \leq 4\alpha_2^2 \left[ \|u_t\|_{\Gamma_1}^2 + k_2^2(t)\|u\|_{\Gamma_1}^2 + C\delta_i \int_{\Gamma_1} (h_2 \circ u) d\Gamma \right].$$

(26)

Similarly, we can get

$$\|B_1u\|_{\Gamma_1}^2 \leq 4\alpha_2^2 \left[ \left\| \frac{\partial u}{\partial \nu} \right\|_{\Gamma_1}^2 + k_1^2(t)\left\| \frac{\partial u}{\partial \nu} \right\|_{\Gamma_1}^2 + C\delta_i \int_{\Gamma_1} \left( h_1 \circ \frac{\partial u}{\partial \nu} \right) d\Gamma \right].$$

(27)

Replacing (26)-(27) in (24), we infer for any $\varepsilon > 0$,

$$\int_{\Gamma_1} (B_1u) \frac{\partial}{\partial \nu} (m \cdot \nabla u) d\Gamma - \int_{\Gamma_1} (B_2u)(m \cdot \nabla u) d\Gamma$$

$$\leq \frac{\varepsilon}{2} \left( c \int_{\Omega} a(u, u) d\Omega + \frac{c}{\delta} \int_{\Gamma_1} (m \cdot \nu)a(u, u) d\Gamma \right)$$

$$+ \frac{1}{2\varepsilon} \left( \|B_1u\|_{\Gamma_1}^2 + \|B_2u\|_{\Gamma_1}^2 \right),$$

(28)
Lemma 4.3. The functional $\Psi(t)$ defined by

$$\Psi(t) := \int_0^t k_1(t-s) \left( \left\| \frac{\partial u}{\partial \nu}(s) \right\|_{\Gamma_1}^2 + k_2(t-s) \left\| u(s) \right\|_{\Gamma_1}^2 \right) ds + \int_0^t k_2(t-s) \left\| u(t) \right\|_{\Gamma_1}^2 ds,$$

satisfies for any $t > 0$,

$$\Psi(t) \leq \frac{1}{2} \int_{\Gamma_1} k_1 \circ \frac{\partial u}{\partial \nu} d\Gamma + \frac{1}{2} \int_{\Gamma_1} k_2 \circ u d\Gamma + 3k_1(0) \left\| \frac{\partial u}{\partial \nu}(t) \right\|_{\Gamma_1}^2 + 3k_2(0) \left\| u(t) \right\|_{\Gamma_1}^2. \quad (28)$$

Proof. A direct computation gives us

$$\Psi(t) = \int_0^t k_1(t-s) \left( \left\| \frac{\partial u}{\partial \nu}(s) \right\|_{\Gamma_1}^2 + k_1(0) \left\| \frac{\partial u}{\partial \nu}(t) \right\|_{\Gamma_1}^2 \right) ds + \int_0^t k_2(t-s) \left\| u(t) \right\|_{\Gamma_1}^2 ds$$

$$+ \int_0^t k_2(t-s) \left\| u(s) \right\|_{\Gamma_1}^2 ds$$

$$= \int_0^t k_1(t-s) \int_{\Gamma_1} \left( \frac{\partial u}{\partial \nu}(s) - \frac{\partial u}{\partial \nu}(t) \right)^2 d\Gamma ds + k_1(t) \left\| \frac{\partial u}{\partial \nu}(t) \right\|_{\Gamma_1}^2$$

$$+ 2 \int_{\Gamma_1} \frac{\partial u}{\partial \nu}(t) \int_0^t k_1'(t-s) \left( \frac{\partial u}{\partial \nu}(s) - \frac{\partial u}{\partial \nu}(t) \right) ds$$

$$+ \int_0^t k_2(t-s) \int_{\Gamma_1} [u(s) - u(t)]^2 d\Gamma ds + k_2(t) \left\| u(t) \right\|_{\Gamma_1}^2$$

$$+ 2 \int_{\Gamma_1} u(t) \int_0^t k_2'(t-s) [u(s) - u(t)] ds d\Gamma.$$

It follows from Young’s and Hölder’s inequalities that

$$2 \int_{\Gamma_1} u(t) \int_0^t k'(t-s) [u(s) - u(t)] ds d\Gamma$$

$$\leq 2k(0) \int_{\Gamma_1} u^2(t) d\Gamma$$

$$+ \frac{1}{2k(0)} \int_{\Gamma_1} \left( \int_0^t \sqrt{k'(t-s)} \sqrt{-k'(t-s)[u(s) - u(t)] ds} \right)^2 d\Gamma$$

$$\leq 2k(0) \left\| u(t) \right\|_{\Gamma_1}^2 + \int_0^t k'(s) ds \int_0^t k'(t-s) \left\| u(s) - u(t) \right\|_{\Gamma_1}^2 ds.$$

For $i = 1, 2$, noting the fact

$$k_i(t) \leq k_i(0) \quad \text{and} \quad \int_0^t k_i'(s) ds \geq \frac{1}{2}.$$
we obtain (28).

4.2. Proof of Theorem 3.2.

Proof. Here we define the functional $L(t)$ by

$$L(t) := NE(t) + \Phi(t),$$

where $N > 0$ is a constant will be chosen later. It is easy to verify that one can take $N$ large (if needed) such that

$$L(t) \sim E(t).$$

Recalling $k''_i = \delta_i k'_i + h_i$, we infer from (21) and (22) that for any $\varepsilon > 0$,

$$L'(t) \leq -\left(\frac{\alpha_1}{2}N - \frac{2\alpha_1^2}{\varepsilon}\right) \left\| \frac{\partial u}{\partial \nu} \right\|_{L^1}^2 - \left(\frac{\alpha_2}{2}N - \frac{2\alpha_2^2}{\varepsilon}\right) \left\| u_t \right\|_{L^1}^2$$

$$-\left\| u_t(t) \right\|^2 - \left(1 - \frac{\varepsilon}{2}\right) \int_{\Omega} a(u, u) dx - \frac{\alpha}{2} \left\| \Delta F(u) \right\|^2$$

$$-\frac{1}{2} \left(1 - \frac{\varepsilon}{\delta}\right) \int_{\Gamma_1} (m \cdot \nu) a(u, u) d\Gamma - \frac{\alpha}{2} \int_{\Gamma_0} (m \cdot \nu) |\Delta F(u)|^2 d\Gamma$$

$$+ \frac{2\alpha_1^2}{\varepsilon} k_1^2(t) \left\| \frac{\partial u}{\partial \nu} \right\|_{L^1}^2 + \frac{2\alpha_2^2}{\varepsilon} k_2^2(t) \left\| u \right\|_{L^2}^2$$

$$- \left(\frac{\alpha_2}{2}N - cC_{\delta_i}\right) \int_{\Gamma_1} h_2 \circ ud\Gamma - \left(\frac{\alpha_1}{2}N - cC_{\delta_i}\right) \int_{\Gamma_1} h_1 \circ \frac{\partial u}{\partial \nu} d\Gamma$$

$$- \frac{\alpha_1}{2}N \delta_i \int_{\Gamma_1} k'_i \circ \frac{\partial u}{\partial \nu} d\Gamma - \frac{\alpha_2}{2}N \delta_2 \int_{\Gamma_1} k'_2 \circ ud\Gamma,$$

where $R = \max \{m(x) \cdot \nu(x) : x \in \Gamma_1\}$.

Taking $\varepsilon > 0$ small enough such that

$$1 - \frac{\varepsilon}{2c} > 0 \text{ and } 1 - \frac{\varepsilon}{\delta} > 0.$$

For $i = 1, 2$, since $-k'_i > 0$ and $k''_i > 0$, we obtain that for each $s \in [0, \infty)$,

$$\lim_{\delta_i \to 0} \frac{\delta_i [k'_i(s)]^2}{k''_i(s) - \delta_i k'_i(s)} ds = 0 \text{ and } \frac{\delta_i [k'_i(s)]^2}{k''_i(s) - \delta_i k'_i(s)} < -k'_i(s).$$

By using Lebesgue dominated convergence theorem, we have

$$\lim_{\delta_i \to 0} \delta_i C_{\delta_i} = \lim_{\delta_i \to 0} \left(\int_0^\infty \frac{\delta_i [k'_i(s)]^2}{k''_i(s) - \delta_i k'_i(s)} ds \right) = 0.$$

Thus there exist $0 < \gamma_i < 1$ such that if $\delta_i < \gamma_i$ then

$$\delta_i C_{\delta_i} < \frac{1}{4c}.$$

At last take $N$ large enough so that

$$\frac{\alpha_1}{2}N - \frac{2\alpha_1^2}{\varepsilon} > 4k_1(0), \quad \frac{\alpha_2}{2}N - \frac{2\alpha_2^2}{\varepsilon} > 4k_2(0),$$

and choose $\delta_i > 0$ satisfying

$$\delta_1 = \frac{1}{2\alpha_1 N} < \gamma_1 \text{ and } \delta_2 = \frac{1}{2\alpha_2 N} < \gamma_2,$$

which gives us

$$\frac{\alpha_1}{2}N - cC_{\delta_i} > 0 \text{ and } \frac{\alpha_2}{2}N - cC_{\delta_2} > 0.$$
In [6], the following fact was proved
\[ \|\Delta F(u)\|_{H^2(\Omega)}^2 \leq c\|u\|_{H^2(\Omega)}^2 \leq c\|u\|_{H^2(\Omega)}^2 \text{ for some } \eta > 0. \]

Then it follows from (8) that
\[ -\frac{\alpha}{2} \int_{\Gamma_0} (\mathbf{m} \cdot \mathbf{n}) |\Delta F(u)|^2 d\Gamma \leq c\alpha \|u\|_{H^2(\Omega)}^2 \leq c\alpha (\int_{\Omega} a(u, u) d\mathbf{x})^2 \leq c\alpha E^2(0), \]
which implies that \( \int_{\Omega} a(u, u) d\mathbf{x} \) is bounded. Then it allows us to take \( \alpha \) so small such that
\[ -\frac{1}{4} (1 - \frac{\varepsilon}{c}) \int_{\Omega} a(u, u) d\mathbf{x} + c\alpha (\int_{\Omega} a(u, u) d\Omega) \leq 0. \]

Noting that \( \lim_{t \to +\infty} k_1(t) = 0 \) and using (9) and the trace theorem, we can get that there exist a constant \( \beta_1 > 0 \) such that for large \( t_1 > 0, \)
\[ L(t) \leq -\beta_1 \left( \|u_t\|^2 + \int_{\Omega} a(u, u) dx + \|\Delta F(u)\|^2 \right) - 4k_1(0) \left\| \frac{\partial u}{\partial \nu} \right\|^2_{\Gamma_1} - 4k_2(0) \|u_t\|^2_{\Gamma_1} - \frac{1}{4} \int_{\Gamma_1} k_1' \circ \frac{\partial u}{\partial \nu} d\Gamma - \frac{1}{4} \int_{\Gamma_1} k_2' \circ ud\Gamma. \] (31)

It follows from (17) and (21) that for any \( t \geq t_1, \)
\[ \int_0^{t_1} (-k_1'(s)) \int_{\Gamma_1} \left[ \frac{\partial u}{\partial \nu}(t) - \frac{\partial u}{\partial \nu}(t - s) \right]^2 d\Gamma ds \leq \frac{1}{d} \int_0^{t_1} k_1''(s) \int_{\Gamma_1} \left[ \frac{\partial u}{\partial \nu}(t) - \frac{\partial u}{\partial \nu}(t - s) \right]^2 d\Gamma ds \leq -cE'(t), \]
and
\[ \int_0^{t_1} (-k_2'(s)) \int_{\Gamma_1} [u(t) - u(t - s)]^2 d\Gamma ds \leq \frac{1}{d} \int_0^{t_1} k_2''(s) \int_{\Gamma_1} [u(t) - u(t - s)]^2 d\Gamma ds \leq -cE'(t). \]

Then by (31), we conclude that there exists a constant \( m > 0 \) such that
\[ L(t) \leq -mE(t) - c \int_0^t k_1'(s) \int_{\Gamma_1} \left[ \frac{\partial u}{\partial \nu}(t) - \frac{\partial u}{\partial \nu}(t - s) \right]^2 d\Gamma ds \]
\[ -c \int_0^t k_2'(s) \int_{\Gamma_1} [u(t) - u(t - s)]^2 d\Gamma ds \]
\[ \leq -mE(t) - 2cE'(t) - c \int_0^t k_1'(s) \int_{\Gamma_1} \left[ \frac{\partial u}{\partial \nu}(t) - \frac{\partial u}{\partial \nu}(t - s) \right]^2 d\Gamma ds \]
\[ -c \int_{t_1}^t k_2'(s) \int_{\Gamma_1} [u(t) - u(t - s)]^2 d\Gamma ds. \] (32)

Define the functional \( F(t) := L(t) + 2cE(t) \sim E(t) \), we get from (32) that
\[ F'(t) \leq -mE(t) - c \int_{t_1}^t k_1'(s) \int_{\Gamma_1} \left[ \frac{\partial u}{\partial \nu}(t) - \frac{\partial u}{\partial \nu}(t - s) \right]^2 d\Gamma ds \]
\[ -c \int_{t_1}^t k_2'(s) \int_{\Gamma_1} [u(t) - u(t - s)]^2 d\Gamma ds. \] (33)
We will consider the following two cases.

**Case 1.** The particular case $H_i(t) = \alpha t^{p_i}$ $(i = 1, 2)$.

(I) $p_1 = p_2 = 1$

We multiply (33) by $\eta(t) = \min\{\eta_1(t), \eta_2(t)\}$, and use (18) and (A2)-(A3) to obtain

$$\eta(t)F'(t) \leq -q\eta(t)E(t) - cE'(t), \quad \forall \ t \geq t_1.$$ 

As $\eta(t)$ is a nonincreasing continuous function and $\eta'(t) \leq 0$ for a.e. $t$, we get

$$(\eta F + cE)'(t) \leq (\eta F'(t) + cE'(t)) \leq -m\eta(t)E(t), \quad a.e. \ t \geq t_1.$$ 

Noting that $\eta F + cE \sim E$, we infer that there exist two positive constants $c_1, c_2 > 0$,

$$E(t) \leq c_1 e^{-c_2 \int_{t_1}^{t} \eta(s) ds}.$$ 

(II) $1 \leq p_i < 2$ and $p_1 \neq p_2$.

Define $\mathcal{E}(t)$ by

$$\mathcal{E}(t) = L(t) + \Psi(t).$$

Combining (28) and (31), we get $\mathcal{E}(t) \geq 0$ and for any $t \geq t_1$,

$$\mathcal{E}'(t) \leq -\beta_1 \left( \|u(t)\|^2 + \int_{\Omega} a(u, u) dx + \|\Delta F(u)\|^2 \right) - k_1(0) \left\| \frac{\partial u}{\partial \nu} \right\|^2_{\Gamma_1}$$

$$-k_2(0) \|u_{t_1}\|^2_{\Gamma_1} + \frac{1}{4} \int_{\Gamma_1} k_1' \circ \frac{\partial u}{\partial \nu} d\Gamma + \frac{1}{4} \int_{\Gamma_1} k_2' \circ u d\Gamma.$$ 

Therefore there exists some constant $\beta_2 > 0$,

$$\mathcal{E}'(t) \leq -\beta_2 \mathcal{E}(t), \quad \text{for all } t \geq t_1.$$ 

This gives

$$\beta_2 \int_{t_1}^{t} E(s) ds \leq \mathcal{E}(t_1) - \mathcal{E}(t) \leq \mathcal{E}(t_1).$$

Hence

$$\int_{0}^{\infty} E(s) ds < \infty. \quad (34)$$

Denote

$$I_1(t) = \int_{t_1}^{t} \int_{\Gamma_1} \left[ \frac{\partial u}{\partial \nu}(t) - \frac{\partial u}{\partial \nu}(t-s) \right]^2 d\Gamma ds$$

and

$$I_2(t) = \int_{t_1}^{t} \int_{\Gamma_1} \left[ \frac{\partial u}{\partial \nu}(t-s) \right] d\Gamma ds.$$ 

Obviously,

$$I_1(t) \leq c \int_{t_1}^{t} E(s) ds \quad \text{and} \quad I_2(t) \leq c \int_{t_1}^{t} E(s) ds.$$ 

Without loss of generality assuming $t_1$ so large that $I_i(t_1) > 0$, we obtain

$$0 < I_i(t_1) \leq I_i(t) < \infty, \quad \forall \ t \geq t_1.$$
By (21) and (A2)-(A3) and using Jensen’s inequality, we have from (31) that for some constant $q > 0$,

$$
L'(t) \leq -qE(t) + \frac{cI_1(t)}{I_1(t)} \int_{\Gamma_1} \frac{1}{(p_1^*)^{p_1 - 1}} \left[ \frac{\partial u}{\partial \nu} \right] d\Gamma \\
+ \frac{cI_2(t)}{I_2(t)} \int_{\Gamma_1} \left[ (p_2^*)^{p_2 - 1} \circ u \right] d\Gamma \\
\leq -qE(t) + cI_1 \left[ \frac{1}{I_1(t)} \int_{\Gamma_1} \frac{1}{(p_1^*)^{p_1 - 1}} \left[ \frac{\partial u}{\partial \nu} d\Gamma \right] \right]^{\frac{1}{p_1}} \\
+ cI_2 \left[ \frac{1}{I_2(t)} \int_{\Gamma_1} (p_2^*)^{p_2 - 1} \circ ud\Gamma \right]^{\frac{1}{p_2}} \\
\leq -qE(t) + c \left[ \frac{1}{\eta_1(t)} \int_{\Gamma_1} \frac{k''}{\eta_1} \circ \frac{\partial u}{\partial \nu} d\Gamma \right]^{\frac{1}{p_1}} \\
+ c \left[ \frac{1}{\eta_2(t)} \int_{\Gamma_1} \frac{k''}{\eta_2} \circ ud\Gamma \right]^{\frac{1}{p_2}} \\
\leq -qE(t) + c \left[ \frac{1}{\eta_1(t)} \int_{\Gamma_1} \frac{k''}{\eta_1} \circ \frac{\partial u}{\partial \nu} d\Gamma \right]^{\frac{1}{p_1}} \\
+ c \left[ \frac{1}{\eta_2(t)} \int_{\Gamma_1} \frac{k''}{\eta_2} \circ ud\Gamma \right]^{\frac{1}{p_2}} \cdot (35)
$$

Multiplying (35) by $E^{p-1}(t)$ ($p = \max \{p_1, p_2\}$) and using (18), we see that

$$(LE^{p-1})'(t) \leq L'(t)E^{p-1}(t)$$

$$\leq -qE^p(t) + c \left[ \frac{-E'(t)}{\eta_1(t)} \right]^{\frac{1}{p_1}} E^{p-1}(t) + c \left[ \frac{-E'(t)}{\eta_2(t)} \right]^{\frac{1}{p_2}} E^{p-1}(t).$$

By Young’s inequality, we get for any $\varepsilon > 0$,

$$
(LE^{p-1})'(t) \leq -qE^p(t) + \varepsilon E^{p\left(\frac{p-1}{p_1}\right)}(t) + \frac{c}{\varepsilon} \left[ \frac{-E'(t)}{\eta_1(t)} \right]^{\frac{1}{p_1}} \\
+ \varepsilon E^{p\left(\frac{p-1}{p_2}\right)}(t) + \frac{c}{\varepsilon} \left[ \frac{-E'(t)}{\eta_2(t)} \right]^{\frac{1}{p_2}}.
$$

Since $\frac{p\left(\frac{p-1}{p_1}\right)}{p_1} \geq p$, taking $\varepsilon < \frac{1}{2}q$, we obtain

$$
(LE^{p-1})'(t) \leq -q'E^p(t) - \frac{c}{\eta_1(t)} E'(t) - \frac{c}{\eta_2(t)} E'(t). 
$$

Define $F(t) = \eta LE^{p-1} + cE \sim E$. Multiplying (36) by $\eta(t)$ we infer

$$
F'(t) \leq -q'\eta(t)E^p(t),
$$

where $q' = \frac{q}{q-1}$. Then we deduce that there exists some constant $q_0 > 0$ such that

$$
F'(t) \leq -q_0\eta(t)E^p(t),
$$

from which we get there exists a positive constant $c_3$ such that

$$
E(t) \leq c_3 \left( 1 + \int_{t_1}^t \eta(s)ds \right)^{-\frac{1}{q-1}}.
$$

Combining (I) and (II) and using the boundedness of $\eta(t)$ and $E(t)$, we can get (20).
**Case 2.** The general case.

Define

\[ I_1(t) := q \int_{t_1}^{t} \int_{\Gamma_1} \left[ \frac{\partial u}{\partial \nu}(t) - \frac{\partial u}{\partial \nu}(t-s) \right]^2 d\Gamma ds, \]

and

\[ I_2(t) := q \int_{t_1}^{t} \int_{\Gamma_1} [u(t) - u(t-s)]^2 d\Gamma ds. \]

Noting (34), it allows us to take \( 0 < q < 1 \) such that for \( i = 1, 2, \)

\[ I_i(t) < 1, \quad \forall \ t \geq t_1. \quad (37) \]

Without loss of generality, we assume that \( I_i(t) > 0 \) for all \( t \geq t_1. \) Moreover, we define

\[ \pi_1(t) := \int_{t_1}^{t} k''(s) \int_{\Gamma_1} \left[ \frac{\partial u}{\partial \nu}(t) - \frac{\partial u}{\partial \nu}(t-s) \right]^2 d\Gamma ds, \]

and

\[ \pi_2(t) := \int_{t_1}^{t} k''(s) \int_{\Gamma_1} [u(t) - u(t-s)]^2 d\Gamma ds. \]

It follows from (21) that \( \pi_i(t) \leq \frac{c}{q} E(t) \) \( (i = 1, 2). \) Since \( H_i(t) \) is strictly convex on \( (0, r] \) and \( H_i(0) = 0, \) we have

\[ H_i(\theta x) \leq \theta H_i(x), \quad i = 1, 2, \quad 0 \leq \theta \leq 1, \quad x \in (0, r]. \]

Performing Jensen’s inequality and using (16) and (37), we will see that

\[
\begin{align*}
\pi_1(t) & = \frac{1}{q I_1(t)} \int_{t_1}^{t} I_1(t) (k''(s)) q \int_{\Gamma_1} \left[ \frac{\partial u}{\partial \nu}(t) - \frac{\partial u}{\partial \nu}(t-s) \right]^2 d\Gamma ds \\
& \geq \frac{1}{q I_1(t)} \int_{t_1}^{t} I_1(t) \eta_1(s) H_1(-k''(s)) q \int_{\Gamma_1} \left[ \frac{\partial u}{\partial \nu}(t) - \frac{\partial u}{\partial \nu}(t-s) \right]^2 d\Gamma ds \\
& \geq \frac{\eta_1(t)}{q I_1(t)} \int_{t_1}^{t} H_1(I_1(t)(-k''(s))) q \int_{\Gamma_1} \left[ \frac{\partial u}{\partial \nu}(t) - \frac{\partial u}{\partial \nu}(t-s) \right]^2 d\Gamma ds \\
& \geq \frac{\eta_1(t)}{q} H_1 \left( \frac{1}{I_1(t)} \int_{t_1}^{t} I_1(t)(-k''(s)) q \int_{\Gamma_1} \left[ \frac{\partial u}{\partial \nu}(t) - \frac{\partial u}{\partial \nu}(t-s) \right]^2 d\Gamma ds \right) \\
& = \frac{\eta_1(t)}{q} H_1 \left( q \int_{t_1}^{t} (-k''(s)) \int_{\Gamma_1} \left[ \frac{\partial u}{\partial \nu}(t) - \frac{\partial u}{\partial \nu}(t-s) \right]^2 d\Gamma ds \right) \\
& \geq \frac{\eta_1(t)}{q} \overline{H}_1 \left( q \int_{t_1}^{t} (-k''(s)) \int_{\Gamma_1} \left[ \frac{\partial u}{\partial \nu}(t) - \frac{\partial u}{\partial \nu}(t-s) \right]^2 d\Gamma ds \right), \quad (38)
\end{align*}
\]

where \( \overline{H}_1, \) which is strictly convex and increasing function on \( (0, \infty) \) of class \( C^2, \)

is called the extension of \( H_1. \) By (38), we have

\[ \int_{t_1}^{t} (-k''(s)) \int_{\Gamma_1} \left[ \frac{\partial u}{\partial \nu}(t) - \frac{\partial u}{\partial \nu}(t-s) \right]^2 d\Gamma ds \leq \frac{1}{q} \overline{H}_1^{-1} \left( \frac{q \pi_1(t)}{\eta_1(t)} \right). \]

Similarly,

\[ \int_{t_1}^{t} (-k''(s)) \int_{\Gamma_1} [u(t) - u(t-s)]^2 d\Gamma ds \leq \frac{1}{q} \overline{H}_2^{-1} \left( \frac{q \pi_2(t)}{\eta_2(t)} \right). \]
Then for any $t \geq t_1$, from (33), we shall see below,

$$F'(t) \leq -mE(t) + c\overline{H}_1^{-1}\left(\frac{q\pi_1(t)}{\eta_1(t)}\right) + c\overline{H}_2^{-1}\left(\frac{q\pi_2(t)}{\eta_2(t)}\right). \quad (39)$$

Let’s denote by:

$$H_0(t) = \min\{\overline{H}'_1, \overline{H}'_2\}.$$

For $\varepsilon_0 < r$, we define the function $\mathcal{K}_1(t)$

$$\mathcal{K}_1(t) = \overline{H}_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) F(t) + E(t) \sim E(t).$$

Since $E'(t) \leq 0$, $\overline{H}_0' > 0$ and $\overline{H}_0'' > 0$, we infer from (39) that

$$\mathcal{K}_1'(t) = \varepsilon_0 \frac{E'(t)}{E(0)} H_0 \left(\varepsilon_0 \frac{E(t)}{E(0)}\right) F(t) + H_0 \left(\varepsilon_0 \frac{E(t)}{E(0)}\right) F'(t) + E'(t)$$

$$\leq -mE(t)H_0 \left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + cH_0 \left(\varepsilon_0 \frac{E(t)}{E(0)}\right) H_0^{-1}\left(\frac{q\pi_1(t)}{\eta_1(t)}\right)$$

$$+ cH_0 \left(\varepsilon_0 \frac{E(t)}{E(0)}\right) \overline{H}_2^{-1}\left(\frac{q\pi_2(t)}{\eta_2(t)}\right). \quad (40)$$

We denote the conjugate function of the convex function $\overline{H}_i$ by $\overline{H}_i^*$, see Arnold [2], then

$$\overline{H}_i^*(s) = s(\overline{H}_i)^{-1}(s) - (\overline{H}_i)[(\overline{H}_i)^{-1}(s)],$$

satisfies Young’s inequality,

$$AB_i \leq \overline{H}_i^*(A) + \overline{H}_i(B).$$

With $A = \overline{H}_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)$ and $B = \overline{H}_i^{-1}\left(\frac{q\pi_i(t)}{\eta_i(t)}\right)$, and using $\overline{H}_i^*(s) \leq s(\overline{H}_i)^{-1}(s)$ and (40), we have

$$\mathcal{K}_1'(t) \leq -mE(t)H_0 \left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + cH_0 \left(\varepsilon_0 \frac{E(t)}{E(0)}\right) (\overline{H}_1^{-1})^{-1}\left(H_0 \left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\right)$$

$$+ c\overline{H}_2^* \left(H_0 \left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\right) + c\frac{q\pi_1(t)}{\eta_1(t)}$$

$$\leq -mE(t)H_0 \left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + cH_0 \left(\varepsilon_0 \frac{E(t)}{E(0)}\right) (\overline{H}_1^{-1})^{-1}\left(H_0 \left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\right)$$

$$+ c\frac{q\pi_1(t)}{\eta_1(t)} + cH_0 \left(\varepsilon_0 \frac{E(t)}{E(0)}\right) (\overline{H}_2^{-1})^{-1}\left(H_0 \left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\right)$$

$$+ c\frac{q\pi_2(t)}{\eta_2(t)}$$

$$\leq -mE(0) - c\varepsilon_0 \frac{E(t)}{E(0)} H_0 \left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\left(\frac{\pi_1(t)}{\eta_1(t)} + \frac{\pi_2(t)}{\eta_2(t)}\right). \quad (41)$$
We multiply (41) by \( \eta(t) = \min\{\eta_1(t), \eta_2(t)\} \) to obtain
\[
\eta(t)K'_1(t) \leq -(mE(0) - \varepsilon_0)\eta(t)\frac{E(t)}{E(0)}H_0 \left( \frac{\varepsilon_0 E(t)}{E(0)} \right) + cq(\pi_1(t) + \pi_2(t))
\]
\[
\leq -(mE(0) - \varepsilon_0)\eta(t)\frac{E(t)}{E(0)}H_0 \left( \frac{\varepsilon_0 E(t)}{E(0)} \right) - cE'(t). \tag{42}
\]
Now we define the functional \( K_2(t) \) by
\[
K_2(t) = \eta(t)K_1(t) + cE(t).
\]
We find that there exist constants \( \beta_5 > 0 \) and \( \beta_6 > 0 \) such that
\[
\beta_5 K_2(t) \leq E(t) \leq \beta_6 K_2(t). \tag{43}
\]
Choosing a suitable \( \varepsilon_0 > 0 \), we obtain from (42) that for a constant \( \zeta > 0 \),
\[
K'_2(t) \leq -\zeta \eta(t)\frac{E(t)}{E(0)}H_0 \left( \frac{\varepsilon_0 E(t)}{E(0)} \right) := -\zeta \eta(t)H_3 \left( \frac{E(t)}{E(0)} \right), \tag{44}
\]
where \( H_3(t) = tH'_0(\varepsilon_0 t) \). It follows from \( 0 \leq \varepsilon_0 \frac{E(t)}{E(0)} < r \) that for any \( t > 0 \)
\[
H_0 \left( \frac{\varepsilon_0 E(t)}{E(0)} \right) = \min \left\{ \overline{P}'_1 \left( \frac{\varepsilon_0 E(t)}{E(0)} \right), \overline{P}'_2 \left( \frac{\varepsilon_0 E(t)}{E(0)} \right) \right\}
\]
\[
= \min \left\{ H'_1 \left( \frac{\varepsilon_0 E(t)}{E(0)} \right), H'_2 \left( \frac{\varepsilon_0 E(t)}{E(0)} \right) \right\}.
\]
Denote \( R(t) = \frac{\beta_5 K_2(t)}{E(0)} \). From (43), we obtain
\[
R(t) \sim E(t). \tag{45}
\]
Since \( H'_5(t) = H_0(\varepsilon_0 t) + \varepsilon_0 tH'_0(\varepsilon_0 t) \), then, recalling the strict convexity of \( H_0 \) on \( (0, r] \), we conclude \( H'_5(t), H_3(t) > 0 \) on \( (0, 1] \). By (44), we obtain for any \( t \geq t_1 \),
\[
R'(t) \leq -\zeta_1 \eta(t)H_3(R(t)), \tag{46}
\]
where \( \zeta_1 \) is a positive constant. Integrating (46) over \( (t_1, t) \), we have
\[
\int_{t_1}^{t} \frac{-R'(s)}{H_3(R(s))}ds \geq \zeta_1 \int_{t_1}^{t} \eta(s)ds \Rightarrow \int_{\varepsilon_0 R(t)}^{\varepsilon_0 R(t_1)} \frac{1}{sH'_0(s)}ds \geq \zeta_1 \int_{t_1}^{t} \eta(s)ds.
\]
Since \( H_4 \), defined by
\[
H_4(t) = \int_{t}^{r} \frac{1}{sH_0(s)}ds,
\]
is strictly decreasing on \( (0, r] \) and \( \lim_{t \to +\infty} H_4(t) = +\infty \), we get that
\[
R(t) \leq \frac{1}{\varepsilon_0} H_4^{-1} \left( \zeta_1 \int_{t_1}^{t} \eta(s)ds \right). \tag{47}
\]
Then (19) follows from (45) and (47). This ends of the proof. \( \square \)

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