CLOSED $G_2$-STRUCTURES ON UNIMODULAR LIE ALGEBRAS WITH NON-TRIVIAL CENTER

ANNA FINO, ALBERTO RAFFERO, AND FRANCESCA SALVATORE

ABSTRACT. We characterize the structure of a seven-dimensional Lie algebra with non-trivial center endowed with a closed $G_2$-structure. Using this result, we classify all unimodular Lie algebras with non-trivial center admitting closed $G_2$-structures, up to isomorphism, and we show that six of them arise as the contactization of a symplectic Lie algebra. Finally, we prove that every semi-algebraic soliton on the contactization of a symplectic Lie algebra must be expanding, and we determine all unimodular Lie algebras with center of dimension at least two that admit semi-algebraic solitons, up to isomorphism.

1. INTRODUCTION

A $G_2$-structure on a seven-dimensional manifold $M$ is a reduction of the structure group of its frame bundle to the exceptional Lie group $G_2$. This reduction exists if and only if $M$ is orientable and spin [28], and it is characterized by the existence of a 3-form $\varphi \in \Omega^3(M)$ whose stabilizer at each point of $M$ is isomorphic to $G_2$. Every such 3-form $\varphi$ induces a Riemannian metric $g_\varphi$ and an orientation on $M$, and thus a Hodge star operator $\ast_\varphi$.

Since every 7-manifold admitting $G_2$-structures is spin, it also admits almost contact structures. The interplay between these structures and the existence of contact structures on 7-manifolds endowed with special types of $G_2$-structures have been recently investigated in [5, 13].

It is possible to construct examples of compact 7-manifolds with both a $G_2$-structure and a contact structure as follows. In [4], Boothby and Wang showed that an even-dimensional compact manifold $N$ endowed with a symplectic form $\omega_0$ with integral periods is the base of a principal $S^1$-bundle $\pi : M \to N$ having a connection 1-form $\theta$ that defines a contact structure on $M$ and satisfies the structure equation $d\theta = \pi^* \omega_0$. If $N$ is six-dimensional and it also admits a definite 3-form $\rho$ and a non-degenerate 2-form $\tilde{\omega}$ which is taming for the almost complex structure $J$ induced by $\rho$ and one of the two orientations of $N$, then the total space $M$ has a natural $G_2$-structure defined by the 3-form $\varphi = \pi^* \omega \wedge \theta + \pi^* \rho$ (see Sect. 2.1 for the relevant definitions). A special case of this construction occurs when the pair $(\tilde{\omega}, \rho)$ defines an $SU(3)$-structure on $N$, namely when the additional conditions $\tilde{\omega} \wedge \rho = 0$ and $3\rho \wedge J\rho = 2\tilde{\omega}^3$ hold. On the other hand, a $G_2$-structure which is invariant under a free $S^1$-action on a 7-manifold $M$ induces an $SU(3)$-structure on the orbit space $M/S^1$ (see [4]).

A $G_2$-structure is said to be closed if the defining 3-form $\varphi$ satisfies the condition $d\varphi = 0$. If a closed $G_2$-structure $\varphi$ is also coclosed, i.e., $d \ast_\varphi \varphi = 0$, then the induced metric $g_\varphi$ is
Ricci-flat, and the Riemannian holonomy group $\text{Hol}(g,\varphi)$ is isomorphic to a subgroup of $G_2$. In this last case, the $G_2$-structure is called torsion-free.

Examples of 7-manifolds with a closed $G_2$-structure can be obtained taking products of lower dimensional manifolds endowed with suitable geometric structures (cf. e.g. [11, 18, 20, 22, 30]). Furthermore, the $G_2$-structure $\varphi = \pi^*\tilde{\omega} \wedge \theta + \pi^*\rho$ on the total space of the $S^1$-bundle $\pi : M \to N$ is closed whenever the 2-form $\tilde{\omega}$ on $N$ is symplectic and $\rho$ satisfies the condition $d\rho = -\omega_0 \wedge \tilde{\omega}$.

In the literature, many examples of closed (non torsion-free) $G_2$-structures have been obtained on compact quotients of simply connected Lie groups by co-compact discrete subgroups (lattices). The first one was given by Fernández on the compact quotient of a nilpotent Lie group [15]. In the solvable non-nilpotent case, various examples are currently known, and lots of them satisfy additional meaningful conditions that one can impose on a closed $G_2$-structure, see e.g. [16, 22, 26, 31, 32, 33]. In these examples, the closed $G_2$-structure on the Lie group $G$ is left-invariant, and thus it is determined by a $G_2$-structure $\varphi$ on the Lie algebra $g = \text{Lie}(G)$ which is closed with respect to the Chevalley-Eilenberg differential of $g$. Recall that a simply connected Lie group $G$ admits lattices only if $g$ is unimodular [38].

The isomorphism classes of nilpotent and unimodular non-solvable Lie algebras admitting closed $G_2$-structures were determined in [12] and [21], respectively. So far, an analogous classification result for solvable non-nilpotent Lie algebras is missing. In fact, there is a lack of classification results for seven-dimensional solvable non-nilpotent Lie algebras.

In this paper, we focus on seven-dimensional unimodular Lie algebras with non-trivial center admitting closed $G_2$-structures. It follows from [21] that every Lie algebra of this type must be solvable.

In Sect. 3 we characterize the structure of a seven-dimensional Lie algebra $g$ with non-trivial center endowed with a closed $G_2$-structure $\varphi$. In detail, we show that it is the central extension of a six-dimensional Lie algebra $h$ by means of a closed 2-form $\omega_0 \in \Lambda^2 h^*$, and that $\varphi = \tilde{\omega} \wedge \theta + \rho$, where $\theta$ is a 1-form on $g$ satisfying $d\theta = \omega_0$, $\rho$ is a definite 3-form on $h$ such that $d\rho = -\omega_0 \wedge \tilde{\omega}$, and $\tilde{\omega}$ is a symplectic form on $h$ that tames the almost complex structure induced by $\rho$ and a suitable orientation. If the 2-form $\omega_0$ is symplectic, the 1-form $\theta$ is a contact form on $g$ and $(g, \theta)$ is the contactization of $(h, \omega_0)$, see [1]. In this last case, the Lie algebra $g$ admits both a closed $G_2$-structure and a contact structure. This is reminiscent of the construction involving the result of [9] that we sketched above.

As a first consequence of this characterization, we determine all isomorphism classes of nilpotent Lie algebras admitting closed $G_2$-structures that arise as the contactization of a six-dimensional nilpotent Lie algebra (see Corollary 3.4). Furthermore, in Sect. 4 we use the characterization to classify all unimodular Lie algebras with non-trivial center admitting closed $G_2$-structures, up to isomorphism (see Theorem 4.1). In addition to the nilpotent ones considered in [12], we show that there exist eleven non-isomorphic solvable non-nilpotent Lie algebras satisfying the required conditions. Among them, only two can be obtained as the contactization of a six-dimensional symplectic Lie algebra. The simply connected Lie groups corresponding to some of these Lie algebras admit lattices. We use the results of [8] to construct a lattice for two of them (see Remark 4.4). In this way, we obtain new locally homogeneous examples of compact 7-manifolds with a closed $G_2$-structure. Finally,
as a corollary of the classification result, we show that the abelian Lie algebra and a certain 2-step solvable Lie algebra are the only unimodular Lie algebras with non-trivial center admitting torsion-free $G_2$-structures.

A special class of closed $G_2$-structures that has attracted a lot of attention in recent years is given by the Laplacian solitons. A closed $G_2$-structure $\varphi$ on a 7-manifold $M$ is said to be a Laplacian soliton if it satisfies the equation

$$\Delta_\varphi \varphi = \lambda \varphi + \mathcal{L}_X \varphi,$$

for some real constant $\lambda$ and some vector field $X$ on $M$, where $\Delta_\varphi$ denotes the Hodge Laplacian of the metric $g_\varphi$. These $G_2$-structures give rise to self-similar solutions of the $G_2$-Laplacian flow introduced by Bryant in [10], and they are expected to model finite time singularities of this flow (see [35] for an account of recent developments on this topic).

Depending on the sign of $\lambda$, a Laplacian soliton is called expanding ($\lambda > 0$), steady ($\lambda = 0$), or shrinking ($\lambda < 0$). On a compact manifold, every Laplacian soliton which is not torsion-free must be expanding and satisfy (1.1) with $\mathcal{L}_X \varphi \neq 0$, see [34, 36]. The existence of non-torsion-free Laplacian solitons on compact manifolds is still an open problem.

In the non-compact setting, examples of Laplacian solitons of any type are known, see e.g. [6, 22, 23, 25, 31, 32, 33, 39]. In particular, the steady solitons in [6] and the shrinking soliton in [25] are inhomogeneous and of gradient type, i.e., $X$ is a gradient vector field. As for the known homogeneous examples, they consist of simply connected Lie groups $G$ endowed with a left-invariant closed $G_2$-structure satisfying the equation (1.1) with respect to a vector field $X$ defined by a one-parameter group of automorphisms induced by a derivation $D$ of the Lie algebra $\mathfrak{g} = \text{Lie}(G)$. According to [31], these solitons are called semi-algebraic.

In Sect. 5 we consider semi-algebraic solitons on unimodular Lie algebras with non-trivial center. Under a natural assumption on the derivation $D$, we are able to relate the constant $\lambda$ in (1.1) to a certain eigenvalue of $D$ and to the norm of the intrinsic torsion form of the semi-algebraic soliton $\varphi$, namely the unique 2-form $\tau$ such that $d \ast_\varphi \varphi = \tau \wedge \varphi = -\ast_\varphi \tau$. Moreover, we show that $\lambda$ coincides with the squared norm of $\tau$ whenever the Lie algebra is the contactization of a symplectic one (see Corollary 5.3). In this last case, the semi-algebraic soliton must be expanding. We also prove the non-existence of semi-algebraic solitons on certain Lie algebras with one-dimensional center, and we obtain the classification of all unimodular Lie algebras with center of dimension at least two that admit semi-algebraic solitons, up to isomorphism (see Theorem 5.7).

2. Preliminaries

2.1. Definite forms and geometric structures in six and seven dimensions. In this section, we review the properties of the geometric structures defined by a differential form satisfying one of the following conditions.

**Definition 2.1.**

- a 3-form $\rho$ on a six-dimensional vector space $V$ is said to be definite if for each non-zero vector $v \in V$ the contraction $\iota_v \rho$ has rank four;
- a 3-form $\varphi$ on a seven-dimensional vector space $W$ is said to be definite if for each non-zero vector $w \in W$ the contraction $\iota_w \varphi$ has rank six.
Let $V$ be a six-dimensional vector space, choose an orientation $\Omega \in \Lambda^6 V^*$ on it, and let $\rho \in \Lambda^3 V^*$ be a definite 3-form. Then, the pair $(\rho, \Omega)$ gives rise to a complex structure $J = J_{\rho, \Omega}$ on $V$ as follows. Consider the endomorphism $S_\rho : V \to V$ defined via the identity

$$t_\rho \rho \wedge \rho \wedge \eta = \eta(S_\rho(v))\Omega,$$

for all $\eta \in V^*$. Then $S_\rho^2 = P(\rho)\text{Id}_V$ for some quartic polynomial $P(\rho)$, and $\rho$ is definite precisely when $P(\rho) < 0$. The complex structure $J = J_{\rho, \Omega}$ is given by $J := (-P(\rho))^{-1/2}S_\rho$.

If $\omega$ is a non-degenerate 2-form on $V$ such that $\omega \wedge \rho = 0$, and $J$ is the complex structure induced by $\rho$ and the orientation $\omega^3$, then the bilinear form $g := \omega(\cdot, J\cdot)$ is symmetric and non-degenerate. In this case, the pair $(\omega, \rho)$ defines an SU(3)-structure if and only if $g$ is positive-definite and $3\rho \wedge J\rho = 2\omega^3$.

In what follows, we shall denote an SU(3)-structure by the usual notation $(\omega, \psi_+)$. Moreover, we denote by $\psi_- := J\psi_+ = \psi_+(J\cdot, J\cdot, J\cdot)$ the imaginary part of the complex $(3, 0)$-form $\psi_+ + i\psi_-$, and by $\text{vol}_g = \frac{1}{6}\omega^3$ the volume form corresponding to the metric $g$. Finally, the Hodge operator induced by $g$ and the orientation $\text{vol}_g$ will be denoted by $*_g$.

The following identities will be useful in the sequel. The reader may refer to [22, Lemma 3.7] for a proof.

**Lemma 2.2.** Let $(\omega, \psi_+)$ be an SU(3)-structure on a six-dimensional vector space $V$, and let $\alpha \in V^*$. Then,

i) $*_g(\alpha \wedge \psi_-) \wedge \omega = J\alpha \wedge \psi_+ = \alpha \wedge \psi_-;
$

ii) $*_g(\alpha \wedge \psi_-) \wedge \omega^2 = 0;
$

iii) $*_g(\alpha \wedge \psi_-) \wedge \psi_+ = -*_g(\alpha \wedge \psi_+) \wedge \psi_- = \alpha \wedge \omega^2 = 2*_g(J\alpha);
$

iv) $*_g(\alpha \wedge \psi_-) \wedge \psi_+ = -*_g(\alpha \wedge \psi_+) \wedge \psi_+ = -J\alpha \wedge \omega^2 = 2*_g\alpha.$

A definite 3-form $\varphi$ on a seven-dimensional vector space $W$ defines a $G_2$-structure on it. In detail, $\varphi$ gives rise to a symmetric bilinear map

$$b_\varphi : W \times W \to \Lambda^7 W^*, \quad b_\varphi(v, w) = \frac{1}{6} t_v \varphi \wedge t_w \varphi \wedge \varphi,$$

(2.1)

the 7-form $\det(b_\varphi)^{1/9}$ is different from zero, $g_\varphi := \det(b_\varphi)^{-1/9} b_\varphi$ is an inner product on $W$, and $\text{vol}_{g_\varphi} = \det(b_\varphi)^{1/9}$ is the corresponding volume form. The Hodge operator induced by $g_\varphi$ and $\text{vol}_{g_\varphi}$ will be denoted by $*_\varphi$.

Now, consider a seven-dimensional vector space $W$, and let $\varphi$ be a 3-form on it. Choose a non-zero vector $z \in W$ and a complementary subspace $V \subset W$ so that $W \cong V \oplus \mathbb{R}z$. Then, we can write

$$\varphi = \tilde{\varphi} \wedge \theta + \rho,$$

where $\theta \in W^*$ is the dual of $z$, and $\tilde{\varphi} \in \Lambda^2 V^*, \rho \in \Lambda^3 V^*$. The 3-form $\varphi$ on $W$ is definite if and only if the 3-form $\rho$ on $V$ is definite and $\tilde{\varphi}$ is a taming form for the complex structure $J$ induced by $\rho$ and one of the two orientations of $V$, namely $\tilde{\omega}(v, Jv) > 0$ for every non-zero vector $v \in V$.

**Remark 2.3.** If $\varphi = \tilde{\varphi} \wedge \theta + \rho$ is definite, then the 2-form $\tilde{\omega} = t_z \varphi|_V$ on $V$ has rank six, namely it is a non-degenerate 2-form. The pair $(\tilde{\omega}, \rho)$ defines an SU(3)-structure on $V$, up to a suitable normalization, if and only if $\tilde{\omega} \wedge \rho = 0$. When this happens, the vector space $V$ coincides with the $g_\varphi$-orthogonal complement of $\mathbb{R}z \subset W$. 
On the other hand, if \( \varphi \) defines a \( G_2 \)-structure on \( W \), we can consider the six-dimensional subspace \( U := (\mathbb{R}z)^{\perp} \subset W \) and the \( g_\varphi \)-orthogonal splitting \( W = U \oplus \mathbb{R}z \). Then, if we let \( u := |z|_\varphi = g_\varphi(z, z)^{1/2} \) and \( \eta := u^{-2}z^\varphi \), so that \( \eta(z) = 1 \), we have

\[
\varphi = u \omega \wedge \eta + \psi_+ \quad \ast \varphi = \frac{1}{2} \omega \wedge \omega + u \psi_- \wedge \eta, \quad g_\varphi = g + u^2 \eta \otimes \eta,
\]

and the pair \((\omega, \psi_+)\) defines an SU(3)-structure on \( U \) inducing the metric \( g \). In particular, \( \text{vol}_{g_\varphi} = u \text{vol}_{\varphi} \wedge \eta \). Since the vector subspaces \( V \) and \( U \) are isomorphic, there exists an SU(3)-structure on \( V \) corresponding to \((\omega, \psi_+)\) via the identification \( V \cong U \). We shall denote this SU(3)-structure using the same symbols. It follows from the discussion in Remark (2.3) that \( V \) and \( U \) coincide if and only if \( \tilde{\omega} \wedge \rho = 0 \). In such a case, \( \eta \) and \( \theta \) coincide, too.

**Remark 2.4.**

1) The structures \((\tilde{\omega}, \rho)\) and \((\omega, \psi_+)\) on \( V \) are related as follows. On \( W = V \oplus \mathbb{R}z \), we have \( \varphi = \tilde{\omega} \wedge \theta + \rho \) and \( \varphi = u \omega \wedge \eta + \psi_+ \). Thus, \( \tilde{\omega} = \iota_z \varphi = u \omega \). Moreover, since \( \eta(z) = 1 \), we can consider the decomposition \( \eta = \eta_V + \theta \), where \( \eta_V \in V^* \), and see that \( \rho = u \omega \wedge \eta_V + \psi_+ \).

2) Let \( \{e_1, \ldots, e_6, e_7\} \) be a basis of \( W = V \oplus \mathbb{R}z \) with \( V = \text{span}\{e_1, \ldots, e_6\} \) and \( e_7 = z \), and denote by \( \{e^1, \ldots, e^6\} \) the dual basis of \( V^* \). Then, a basis of \( U = (\mathbb{R}e_7)^\perp \) is given by \( \left\{ e_k - \frac{2z(e_k, e_7)}{u^2} e_7, k = 1, \ldots, 6 \right\} \). Consequently, \( \{e^1, \ldots, e^6\} \) is a basis of \( U^* \).

### 2.2. SU(3)- and G2-structures on Lie algebras

We now focus on SU(3)- and G2-structures on Lie algebras. It is well-known that an SU(3)-structure on a six-dimensional Lie algebra \( \mathfrak{g} \) gives rise to a left-invariant SU(3)-structure on every Lie group corresponding to \( \mathfrak{g} \). Conversely, a left-invariant SU(3)-structure \((\omega, \psi_+)\) on a six-dimensional Lie group \( \mathfrak{H} \) is determined by the SU(3)-structure \((\omega, \psi_+)\) on \( \mathfrak{H} \). An analogous correspondence holds for G2-structures on seven-dimensional Lie algebras and left-invariant G2-structures on seven-dimensional Lie groups.

Consider a six-dimensional Lie algebra \( \mathfrak{h} \) endowed with an SU(3)-structure \((\omega, \psi_+)\). The natural action of SU(3) on the space of \( k \)-forms \( \Lambda^k \mathfrak{h}^* \), for \( k = 2, 3 \), gives rise to the following splittings (cf. [7, 11])

\[
\Lambda^2 \mathfrak{h}^* = \mathbb{R} \omega \oplus \Lambda^2_0 \mathfrak{h}^* \oplus \Lambda^2_3 \mathfrak{h}^*, \quad \Lambda^3 \mathfrak{h}^* = \mathbb{R} \psi_+ \oplus \mathbb{R} \psi_- \oplus \Lambda^3_0 \mathfrak{h}^* \oplus \Lambda^3_2 \mathfrak{h}^*,
\]

where the irreducible \( r \)-dimensional SU(3)-modules \( \Lambda^r_\beta \mathfrak{h}^* \) are defined as follows

\[
\Lambda^2_\beta \mathfrak{h}^* = \left\{ \sigma \in \Lambda^2 \mathfrak{h}^* \mid J \sigma = -\sigma \right\}, \quad \Lambda^3_\beta \mathfrak{h}^* = \left\{ \beta \in \Lambda^3 \mathfrak{h}^* \mid J \beta = \beta, \beta \wedge \omega^2 = 0 \right\},
\]

and

\[
\Lambda^3 _0 \mathfrak{h}^* = \left\{ \alpha \wedge \omega \mid \alpha \in \mathfrak{h}^* \right\}, \quad \Lambda^3 _2 \mathfrak{h}^* = \left\{ \gamma \in \Lambda^3 \mathfrak{h}^* \mid \gamma \wedge \omega = 0, \gamma \wedge \psi_\pm = 0 \right\}.
\]

The SU(3)-irreducible decomposition of the space \( \Lambda^4 \mathfrak{h}^* \) follows from that of \( \Lambda^2 \mathfrak{h}^* \) via the identity \( \Lambda^4 \mathfrak{h}^* = \ast \Lambda^2 \mathfrak{h}^* \). Notice that every 2-form \( \sigma \in \Lambda^2_0 \mathfrak{h}^* \) satisfies the identity \( \sigma \wedge \omega = \ast \sigma \), and every 2-form \( \beta \in \Lambda^2_2 \mathfrak{h}^* \) satisfies the identity \( \beta \wedge \omega = - \ast \beta \).

Let \( d \) denote the Chevalley-Eilenberg differential of \( \mathfrak{h} \). In view of the previous decompositions, there exist unique \( w_0^+, w_0^- \in \mathbb{R}, \nu_1, w_1 \in \mathfrak{h}^*, w_2^+, w_2^- \in \Lambda^2_2 \mathfrak{h}^* \), \( w_3 \in \Lambda^3_2 \mathfrak{h}^* \), such
that

\[ d\omega = \frac{3}{2}w_0^+ \psi_+ + \frac{3}{2}w_0^- \psi_- + w_3 + \nu_1 \wedge \omega, \]
\[ d\psi_+ = w_0^+ \omega^2 + w_2^+ \wedge \omega + w_1 \wedge \psi_+, \]
\[ d\psi_- = w_0^- \omega^2 + w_2^- \wedge \omega + Jw_1 \wedge \psi_. \]

By [7, 11], the intrinsic torsion of an SU(3)-structure \((\omega, \psi_+)\) is completely determined by these forms, which are thus called the intrinsic torsion forms of \((\omega, \psi_+)\).

A similar result holds for \(G_2\)-structures, see [10, 18]. Here, we shall focus on seven-dimensional Lie algebras \(\mathfrak{g}\) endowed with a closed \(G_2\)-structure \(\varphi\). In such a case, \(d\varphi = 0\), \(d\) being the Chevalley-Eilenberg differential of \(\mathfrak{g}\), and the intrinsic torsion of \(\varphi\) is determined by the unique 2-form \(\tau \in \Lambda^2 g^* := \{\alpha \in \Lambda^2 g^* | \alpha \wedge \varphi = -\ast_{\varphi} \alpha\}\) such that

\[ d\ast_{\varphi} \varphi = \tau \wedge \varphi. \]

In particular, a closed \(G_2\)-structure is torsion-free if and only if \(\tau = 0\). We shall refer to \(\tau\) as the intrinsic torsion form of the closed \(G_2\)-structure \(\varphi\).

2.3. **The contactization of a symplectic Lie algebra.** Let \(\mathfrak{h}\) be a Lie algebra of dimension \(n \geq 2\) and denote by \([\cdot, \cdot]_{\mathfrak{h}}\) its Lie bracket. Consider a 2-form \(\omega_0 \in \Lambda^2 \mathfrak{h}^*\) that is closed with respect to the Chevalley-Eilenberg differential \(d_{\mathfrak{h}}\) of \(\mathfrak{h}\). The central extension of \((\mathfrak{h}, \omega_0)\) is the real Lie algebra of dimension \(n + 1\) defined as follows: consider the vector space

\[ \mathfrak{g} := \mathfrak{h} \oplus \mathbb{R} z, \]

and endow it with the Lie bracket

\[ [z, \mathfrak{h}] = 0, \quad [x, y] = -\omega_0(x, y)z + [x, y]_{\mathfrak{h}}, \quad \forall x, y \in \mathfrak{h}. \]

(2.2)

It is clear from this definition that the vector \(z\) belongs to the center of \(\mathfrak{g}\). More precisely, the center of \(\mathfrak{g}\) is given by

\[ \mathfrak{z}(\mathfrak{g}) = (\mathfrak{z}(\mathfrak{h}) \cap \text{Rad}(\omega_0)) \oplus \mathbb{R} z, \]

where \(\text{Rad}(\omega_0) := \{x \in \mathfrak{h} | \omega_0(x, y) = 0, \forall y \in \mathfrak{h}\}\). Note that the central extension of \((\mathfrak{h}, \omega_0)\) only depends on the cohomology class \([\omega_0] \in H^2(\mathfrak{h})\). Indeed, different representatives of \([\omega_0]\) give rise to isomorphic central extensions.

In the following, we shall denote by \(\theta \in \mathfrak{g}^*\) the dual covector of \(z\) uniquely defined by the conditions \(\theta(z) = 1\) and \(\theta|_{\mathfrak{h}} = 0\).

Let \(d\) denote the Chevalley-Eilenberg differential of \(\mathfrak{g}\). Then, \(d\theta\) defines an exact 2-form on \(\mathfrak{g}\) that coincides with \(\omega_0\) on \(\mathfrak{h}\) and satisfies \(\iota_z d\theta = 0\). Thus, we can write \(d\theta = \omega_0\) on \(\mathfrak{g}\) by extending \(\omega_0\) to \(\mathfrak{g}\) via the condition \(\iota_z \omega_0 = 0\). When \(\omega_0\) is zero, the Lie algebra \(\mathfrak{g}\) is just the direct sum of \(\mathfrak{h}\) and the abelian Lie algebra \(\mathbb{R}\).

When \((\mathfrak{h}, \omega_0)\) is a symplectic Lie algebra of dimension \(2n\), the previous construction gives rise to a contact Lie algebra \((\mathfrak{g}, \theta)\) of dimension \(2n + 1\) with center \(\mathfrak{z}(\mathfrak{g}) = \mathbb{R} z\). Indeed, \(\theta \in \mathfrak{g}^*\) satisfies the condition \((d\theta)^n \wedge \theta \neq 0\), and so it is a contact form on \(\mathfrak{g}\) (see [11]). Moreover, \(\text{Rad}(\omega_0) = \{0\}\), whence \(\mathfrak{z}(\mathfrak{g}) = \mathbb{R} z\). Notice that \(z\) is the Reeb vector of the contact structure \(\theta\), as \(\theta(z) = 1\). This motivates the following.

**Definition 2.5.** The contact Lie algebra \((\mathfrak{g}, \theta)\) obtained from the symplectic Lie algebra \((\mathfrak{h}, \omega_0)\) via the construction described above is called the contactization of \((\mathfrak{h}, \omega_0)\).
It is easy to characterize contact Lie algebras arising as the contactization of a symplectic Lie algebra, as the next result shows (see also [14]).

**Proposition 2.6.** A contact Lie algebra \((g, \theta)\) is the contactization of a symplectic Lie algebra \((h, \omega_0)\) if and only if the center \(z(g)\) of \(g\) is not trivial.

**Proof.** If \((g, \theta)\) is the contactization of a symplectic Lie algebra \((h, \omega_0)\), then the assertion is true. Conversely, let us assume that \((g, \theta)\) is a contact Lie algebra of dimension \(2n + 1\) with non-trivial center. Then, \(z(g)\) is one-dimensional and it is spanned by the Reeb vector \(z\) of the contact structure \(\theta\) (cf. [3, Prop. 1]). Consequently, we can consider the decomposition \(g = h \oplus \mathbb{R}z\), where the \(2n\)-dimensional subspace \(h := \ker \theta\) is a Lie algebra with respect to the bracket

\[
[x, y]_h := [x, y] - \theta([x, y]) z, \quad x, y \in h.
\]

Let \(\omega_0\) be the 2-form on \(h\) defined as \(\omega_0(x, y) = d\theta(x, y)\), for all \(x, y \in h\). A direct computation shows that \(\omega_0\) is closed with respect to the Chevalley-Eilenberg differential \(d_h\) of \(h\). Moreover, \(\omega_0\) is non-degenerate. Indeed, \((d\theta)^n \wedge \theta\) is a volume form on \(g\) and contracting it with \(z\) gives \((d\theta)^n \neq 0\), as \(\theta(z) = 1\) and \(\iota_z d\theta = -\theta([z, \cdot]) = 0\). Therefore, \((g, \theta)\) is the contactization of the symplectic Lie algebra \((h, \omega_0)\).

\[\Box\]

3. Closed \(G_2\)-structures on central extensions of Lie algebras

In this section, we investigate the structure of a seven-dimensional Lie algebra with non-trivial center endowed with a closed \(G_2\)-structure. We begin with the following.

**Proposition 3.1.** Let \(h\) be a six-dimensional Lie algebra and let \(\omega_0\) be a closed 2-form on it. Assume that \(h\) admits a definite 3-form \(\rho\) and a symplectic form \(\tilde{\omega}\) such that

a) \(\tilde{\omega}\) is a taming form for the almost complex structure \(J_\rho\) on \(h\) induced by \(\rho\) and one of the two orientations of \(h\);

b) \(d\rho = -\tilde{\omega} \wedge \omega_0\).

Then, the seven-dimensional Lie algebra \(g := h \oplus \mathbb{R}z\) obtained as the central extension of \((h, \omega_0)\) is endowed with a closed \(G_2\)-structure defined by the 3-form

\[
\varphi = \tilde{\omega} \wedge \theta + \rho.
\]

**Proof.** The hypothesis on \(\rho\) and \(\tilde{\omega}\) guarantee that the 3-form \(\varphi = \tilde{\omega} \wedge \theta + \rho\) defines a \(G_2\)-structure on \(g = h \oplus \mathbb{R}z\). Moreover, since \(\tilde{\omega}\) is closed and \(\omega_0 = d\theta\), we have

\[
d\varphi = d\tilde{\omega} \wedge \theta + \tilde{\omega} \wedge d\theta + d\rho = \tilde{\omega} \wedge \omega_0 + d\rho = 0.
\]

\[\Box\]

The next result is a converse of Proposition 3.1.

**Proposition 3.2.** Let \(g\) be a seven-dimensional Lie algebra endowed with a closed \(G_2\)-structure \(\varphi\). Assume that the center of \(g\) is not trivial, consider a non-zero vector \(z \in z(g)\) and denote by \(\theta \in g^*\) its dual 1-form. Then, \(g\) is the central extension of a six-dimensional Lie algebra \((h, \omega_0)\), and the closed \(G_2\)-structure can be written as \(\varphi = \tilde{\omega} \wedge \theta + \rho\), where \(\rho\) is a definite 3-form on \(h\), \(\tilde{\omega}\) is a taming symplectic form for \(J_\rho\) and \(d\rho = -\tilde{\omega} \wedge \omega_0\).
Proof. Consider the six-dimensional subspace $\mathfrak{h} := \ker \theta$ of $\mathfrak{g}$. Then, $\mathfrak{h}$ is a Lie algebra with respect to the bracket

$$[x, y]_{\mathfrak{h}} := [x, y] - \theta ([x, y]) z, \quad x, y \in \mathfrak{h},$$

and the 2-form $\omega_0 := d\theta|_{\mathfrak{h} \times \mathfrak{h}}$ on $\mathfrak{h}$ is closed with respect to the Chevalley-Eilenberg differential of $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}})$. In particular, $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R} z$ is the central extension of $(\mathfrak{h}, \omega_0)$. Since $\Lambda^3 \mathfrak{g}^* = (\Lambda^2 \mathfrak{h}^* \oplus \mathbb{R} \theta) \oplus \Lambda^3 \mathfrak{h}^*$, there exist some $\tilde{\omega} \in \Lambda^3 \mathfrak{h}^*$, $\rho \in \Lambda^3 \mathfrak{h}^*$ such that $\varphi = \tilde{\omega} \wedge \theta + \rho$. Clearly, $\rho$ is a definite 3-form on $\mathfrak{h}$ and $\tilde{\omega} = \iota_\varphi$ is a non-degenerate taming 2-form for $J_\rho$. Moreover, $\tilde{\omega}$ is symplectic. Indeed,

$$0 = L_z \varphi = d(\iota_\varphi) = d\tilde{\omega},$$

as $z \in \mathfrak{j}(\mathfrak{g})$. Finally, we have

$$0 = d\varphi = \tilde{\omega} \wedge d\theta + d\rho = \tilde{\omega} \wedge \omega_0 + d\rho,$$

where the last identity follows from $\iota_\varphi d\theta = 0$. \qed

Remark 3.3.

1) It follows from [21] that every seven-dimensional unimodular Lie algebra with non-trivial center admitting closed $G_2$-structures is necessarily solvable. On the other hand, there exist unimodular solvable centerless Lie algebras admitting closed $G_2$-structures, see e.g. [32, Ex. 4.7].

2) Any vector $z \in \mathfrak{j}(\mathfrak{g})$ satisfies $L_z \varphi = 0$. More generally, if $x \in \mathfrak{g}$ satisfies $L_x \varphi = 0$, then $L_x \mathfrak{g}_\varphi = 0$, whence it follows that $\ad_x \in \operatorname{Der}(\mathfrak{g})$ is skew-symmetric. Consequently, if the Lie algebra $\mathfrak{g}$ is completely solvable, namely the spectrum of $\ad_v$ is real for all $v \in \mathfrak{g}$, then every vector $x$ satisfying $L_x \varphi = 0$ must belong to the center of $\mathfrak{g}$.

Before showing a first consequence of Proposition [32], we recall some notations that we will use from now on. Given a Lie algebra of dimension $n$, we write its structure equations with respect to a basis of covectors $(e^1, \ldots, e^n)$ by specifying the $n$-tuple $(dz^1, \ldots, dz^n)$. Moreover, we use the notation $e^{ijk\ldots}$ as a shorthand for the wedge product of covectors $e^i \wedge e^j \wedge e^k \wedge \cdots$.

Corollary 3.4. Let $\mathfrak{g}$ be a seven-dimensional nilpotent Lie algebra endowed with a closed $G_2$-structure $\varphi$. Then $\mathfrak{g}$ is the central extension of a six-dimensional nilpotent Lie algebra $\mathfrak{h}$ admitting symplectic structures. Moreover, $\mathfrak{g}$ is the contactization of a six-dimensional symplectic nilpotent Lie algebra $(\mathfrak{h}, \omega_0)$ if and only if $\mathfrak{g}$ is isomorphic to one of the following:

$$\begin{align*}
n_0 &= (0, 0, e^{12}, e^{13}, e^{15} + e^{24}, e^{16} + e^{34} + e^{25}), \\
n_{10} &= (0, 0, e^{12}, 0, e^{13} + e^{24}, e^{14}, e^{46} + e^{34} + e^{15} + e^{23}), \\
n_{11} &= (0, 0, e^{12}, 0, e^{13}, e^{24} + e^{23}, e^{25} + e^{34} + e^{15} + e^{16} - 3e^{26}), \\
n_{12} &= (0, 0, 0, e^{12}, e^{23}, -e^{13}, 2e^{26} - 2e^{34} - 2e^{16} + 2e^{25}).
\end{align*}$$
Proof. Since a nilpotent Lie algebra has non-trivial center, the first assertion immediately follows from Proposition 3.2. By the classification result of \[12\], a seven-dimensional nilpotent Lie algebra admitting closed G₂-structures is isomorphic to one of the following:

\[ n_1 = (0, 0, 0, 0, 0, 0, 0), \]
\[ n_2 = (0, 0, 0, 0, e^{12}, e^{13}, 0), \]
\[ n_3 = (0, 0, 0, e^{12}, e^{13}, e^{23}, 0), \]
\[ n_4 = (0, 0, e^{12}, 0, 0, e^{13} + e^{24}, e^{15}), \]
\[ n_5 = (0, 0, e^{12}, 0, 0, e^{13} + e^{14} + e^{25}), \]
\[ n_6 = (0, 0, 0, e^{12}, e^{13}, e^{14}, e^{15}), \]
\[ n_7 = (0, 0, 0, e^{12}, e^{13}, e^{14} + e^{23}, e^{15}), \]
\[ n_8 = (0, 0, e^{12}, e^{13}, e^{15} + e^{24}, e^{16} + e^{34}), \]
\[ n_9 = (0, 0, e^{12}, e^{13}, e^{15} + e^{24}, e^{16} + e^{34} + e^{25}), \]
\[ n_{10} = (0, 0, e^{12}, 0, e^{13} + e^{24}, e^{14}, e^{25} + e^{34} + e^{15} + e^{23}), \]
\[ n_{11} = (0, 0, e^{12}, 0, e^{13}, e^{24} + e^{23}, e^{25} + e^{34} + e^{15} + e^{16} - 3e^{26}), \]
\[ n_{12} = (0, 0, 0, e^{12}, e^{23}, -e^{13}, 2e^{26} - 2e^{34} - 2e^{16} + 2e^{25}). \]

By \[29\] Thm. 4.2, a decomposable nilpotent Lie algebra cannot admit any contact structure. Consequently, the Lie algebras \(n_1, n_2, n_3\) cannot be the contactization of any symplectic Lie algebra. Seven-dimensional indecomposable nilpotent Lie algebras admitting contact structures have been classified in \[29\]. Comparing this classification with the one above, we see that \(g\) must be isomorphic to one of \(n_9, n_{10}, n_{11}, n_{12}\). For each of these Lie algebras, \(\mathfrak{z}(n_i) = \mathbb{R}e_7\) and the 2-form \(de^7\) induces a symplectic form on the six-dimensional nilpotent Lie algebra \(\mathfrak{h}_i := \ker h\) with Lie bracket defined as in (3.1). \(\square\)

Let us now consider a seven-dimensional Lie algebra \(g\) with non-trivial center endowed with a closed G₂-structure \(\varphi\). Then, from the previous discussion we can assume that \(g = \mathfrak{h} \oplus \mathbb{R}z\) is the central extension of a six-dimensional Lie algebra \((\mathfrak{h}, \omega_0 := d\theta|_{xh})\), and that \(\varphi = \tilde{\omega} \wedge \theta + \rho\), with \(d\rho = -\tilde{\omega} \wedge \omega_0\) and \(d\tilde{\omega} = 0\). From Sect. \[2\] we also know that \(\mathfrak{h}\) admits an SU(3)-structure \((\omega, \psi_+)\) such that \(\varphi = u\omega \wedge \eta + \psi_+,\) where \(u := |z|_\varphi\) and \(\eta := u^{-2}z^0 = \eta_h + \theta\), for some \(\eta_h \in \mathfrak{h}^*\). In particular, \(\mathfrak{h}\) is the \(g_\varphi\)-orthogonal complement of \(\mathbb{R}z\) in \(g\) if and only if \(\eta_h = 0\). It is worth observing that this setting generalizes the one considered in \[22\] Sect. 6.1, which corresponds to the case when both \(\eta_h = 0\) and \(\omega_0 = 0\), i.e., to the direct sum of Lie algebras \(g = \mathfrak{h} \oplus \mathbb{R}\) endowed with a closed G₂-structure inducing the product metric.

We now investigate the properties of the SU(3)-structure \((\omega, \psi_+)\) on \(\mathfrak{h}\). Since \(u\omega = \tilde{\omega}\), we immediately see that \(d\omega = 0\). Consequently, we have

\[
\begin{align*}
d\psi_+ &= w_2^+ \wedge \omega + w_1 \wedge \psi_+ , \\
d\psi_- &= w_2^- \wedge \omega + Jw_1 \wedge \psi_+ ,
\end{align*}
\]

for some unique \(w_1 \in \mathfrak{h}^*\) and \(w_2^\pm \in \Lambda_3^0 \mathfrak{h}^*\) (cf. Sect. \[22\]).
Lemma 3.5. The 2-form $d\eta \in \Lambda^2\mathfrak{h}^*$ has no component in $\Lambda^3\mathfrak{h}^* = \mathbb{R}\omega$, that is, $d\eta \wedge \omega^2 = 0$. Moreover, the intrinsic torsion forms $w_2$ and $w_1$ are related to the components $(d\eta)_k$ of $d\eta$ in $\Lambda^k\mathfrak{h}^*$, $k = 6,8$, as follows

$$u(\eta)_6 = - \ast_g (w_1 \wedge \psi_+),$$

$$u(\eta)_8 = - w_2^+ .$$

In particular, $w_1 = \frac{\omega}{2} \ast_g (\psi_+ \wedge d\eta)$.  

Proof. The condition $d\varphi = 0$ is equivalent to $d\psi_+ = - u \omega \wedge d\eta$. Since $\omega$ is symplectic and $\omega \wedge \psi_+ = 0$, we get $d\eta \wedge \omega^2 = 0$. Now, according to the SU(3)-irreducible decomposition $\Lambda^2\mathfrak{h}^* = \Lambda_1^2\mathfrak{h}^* \oplus \Lambda_2^2\mathfrak{h}^* \oplus \Lambda_3^2\mathfrak{h}^*$, this implies that $(d\eta)_1 = 0$ and we thus have $d\eta = (d\eta)_6 + (d\eta)_8$, with $(d\eta)_6 \wedge \omega = \ast_g (d\eta)_6$ and $(d\eta)_8 \wedge \omega = - \ast_g (d\eta)_8$, see Section 2.2. Therefore,

$$d\psi_+ = - u \omega \wedge d\eta = - u \ast_g (d\eta)_6 + u \ast_g (d\eta)_8.$$ 

Comparing this expression with the one in (3.2), we obtain the identities relating $w_1$ and $w_2^+$, respectively. Finally, to obtain the expression of $w_1$, it is sufficient to notice that

$$u d\eta \wedge \psi_+ = u (\eta)_6 \wedge \psi_+ = - \ast_g (w_1 \wedge \psi_+) \wedge \psi_+ = - 2 \ast_g w_1,$$

where the last identity follows from Lemma 2.2. \hfill \Box

Since $d\eta \in \Lambda^2\mathfrak{h}^* \oplus \Lambda_3^2\mathfrak{h}^*$, it satisfies the following condition (see [7, Rem. 2.7]):

$$d\eta \wedge \omega = - J \ast_g d\eta. \tag{3.3}$$

In the next lemma we describe the intrinsic torsion form $\tau$ of the closed $G_2$-structure $\varphi = u \omega \wedge \eta + \psi_+$ on $g$ in terms of the torsion forms of the SU(3)-structure $(\omega, \psi_+)$ on $\mathfrak{h}$.

Lemma 3.6. The intrinsic torsion form $\tau \in \Lambda_1^2\mathfrak{g}^*$ of the closed $G_2$-structure $\varphi = u \omega \wedge \eta + \psi_+$ is given by

$$\tau = w_2^- - \ast_g (Jw_1 \wedge \psi_+) - 2u Jw_1 \wedge \eta,$$

and its Hodge dual is

$$\ast_\varphi \tau = u \ast_g w_2^- \wedge \eta - u Jw_1 \wedge \psi_+ \wedge \eta + 2 \ast_g Jw_1.$$ 

Consequently, $|\tau|_g^2 = |w_2^-|_g^2 + 6|w_1|_g^2$.

Proof. Recall that $\tau = - \ast_\varphi d \ast_\varphi \varphi$, where $\ast_\varphi \varphi = \frac{1}{2} \omega^2 + u \psi_- \wedge \eta$. We first compute

$$d \ast_\varphi \varphi = u d\psi_- \wedge \eta - u \psi_- \wedge d\eta = u d\psi_- \wedge \eta - u \psi_- \wedge (d\eta)_6 = u d\psi_- \wedge \eta - 2 \ast_g Jw_1,$$

where we used $(d\eta)_8 \wedge \psi_- = 0$, Lemma 3.5 and the identity $\ast_g (w_1 \wedge \psi_+) \wedge \psi_- = - 2 \ast_g Jw_1$.

Using now the relation between the Hodge operators $\ast_\varphi$ and $\ast_g$ together with the identity $d\psi_- = - \ast_g w_2^- + Jw_1 \wedge \psi_+$, we obtain

$$\tau = - \ast_\varphi d \ast_\varphi \varphi = - \ast_g d\psi_- - 2 u Jw_1 \wedge \eta = w_2^- - \ast_g (Jw_1 \wedge \psi_+) - 2u Jw_1 \wedge \eta.$$ 

From this expression and the relation between $\ast_\varphi$ and $\ast_g$, one can easily compute $\ast_\varphi \tau$. To obtain the norm of $\tau$, it is then sufficient to use the identity $\tau \wedge \ast_\varphi \tau = |\tau|_g^2 \text{vol}_{g_\varphi}$ and the identity [3] of Lemma 2.2. \hfill \Box
We now examine an example of closed $G_2$-structure on the nilpotent Lie algebra $\mathfrak{n}_9$ in the light of the Propositions 3.1 and 3.2.

**Example 3.7.** Consider the six-dimensional nilpotent Lie algebra $\mathfrak{h}$ with structure equations

$$(0, 0, e^{12}, e^{13}, e^{23}, e^{15} + e^{24}).$$

The following 2-forms are symplectic forms on $\mathfrak{h}$:

$$\omega_0 = e^{16} + e^{25} + e^{34}, \quad \tilde{\omega} = -e^{12} - e^{14} - e^{35} + e^{26}.$$ 

Let $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}e_7$ be the contactization of $(\mathfrak{h}, \omega_0)$ with contact form $\theta = e^7$ and Reeb vector $z = e_7$. Then, $\mathfrak{g}$ has structure equations

$$(0, 0, e^{12}, e^{13}, e^{23}, e^{15} + e^{24}, e^{16} + e^{25} + e^{34}),$$

and so it coincides with the Lie algebra $\mathfrak{n}_9$ described in the proof of Corollary 3.3. It is easy to check that $\tilde{\omega} \wedge \omega_0 = -d\rho$, where

$$\rho = e^{124} - e^{125} - e^{136} - e^{234} - e^{345} + e^{456}$$

is a definite 3-form on $\mathfrak{h}$. The almost complex structure $J$ induced by $(\rho, e^{123456})$ on $\mathfrak{h}$ is given by

$$Je_1 = -e_4 - e_5, \quad Je_2 = e_6, \quad Je_3 = e_2 - e_5, \quad Je_4 = e_1 - e_3 - e_6, \quad Je_5 = e_3 + e_6, \quad Je_6 = -e_2.$$ 

The 2-form $\tilde{\omega}$ is a taming form for $J$, as for any non-zero vector $\xi = \xi^k e_k \in \mathfrak{g}$ we have

$$\tilde{\omega}(\xi, J\xi) = \sum_{k=1}^{6} (\xi^k)^2 + \xi^1 \xi^6 + \xi^2 \xi^5 - \xi^3 \xi^6 - \xi^4 \xi^5 > 0.$$ 

Therefore, $\varphi = \tilde{\omega} \wedge e^7 + \rho$ defines a closed $G_2$-structure on $\mathfrak{g}$. The metric induced by $\varphi$ has the following expression

$$g_\varphi = 2^{\frac{1}{2}} \left[ \sum_{i=1}^{7} e^i \circ e^i + (e^1 \circ e^6 + e^2 \circ e^5 - e^3 \circ e^6 - e^4 \circ e^5) + (e^2 - e^4 + e^5) \circ e^7 \right],$$

where $e^i \circ e^k = \frac{1}{2} (e^i \circ e^k + e^k \circ e^i)$.

On the other hand, we can start with the Lie algebra $\mathfrak{g}$ endowed with the closed $G_2$-structure $\varphi$ and consider the $SU(3)$-structure induced by it on the $g_\varphi$-orthogonal complement $u$ of the one-dimensional subspace generated by $e_7 \in \mathfrak{j}(\mathfrak{g})$. We have $u = |e_7|_\varphi = 2^\frac{1}{2}$ and

$$\eta = u^{-2} (e_7)^5 = \frac{1}{2} (e^2 - e^4 + e^5) + e^7.$$ 

The closed $G_2$-structure $\varphi$ can be written as $\varphi = u \omega \wedge \eta + \psi_+$, where the pair

$$\omega = u^{-1} \tilde{\omega} = 2^{-\frac{1}{2}} (-e^{12} - e^{14} - e^{35} + e^{26}),$$

$$\psi_+ = \frac{1}{2} e^{125} - e^{136} + \frac{1}{2} e^{145} - e^{234} - \frac{1}{2} e^{246} + \frac{1}{2} e^{256} - \frac{1}{2} e^{345} + \frac{1}{2} e^{235} + e^{456},$$

defines an $SU(3)$-structure on the vector subspace $u \subset \mathfrak{g}$. Notice also that $\mathfrak{h} := \ker \theta$ is a Lie algebra with respect to the bracket 3.1, and that it is endowed with an $SU(3)$-structure.
\((\omega, \psi)\) whose expression with respect to the basis \(\{e^1, \ldots, e^6\}\) of \(\mathfrak{h}^*\) is the same as the one appearing above (cf. Remark 2.4). The metric induced by \((\omega, \psi)\) on \(\mathfrak{h}\) is
\[
g = 2^{1/2} (e^1 \otimes e^1 + e^3 \otimes e^3 + e^6 \otimes e^6) + \frac{3}{4} 2^{1/2} (e^2 \otimes e^2 + e^4 \otimes e^4 + e^5 \otimes e^5)
\]
\[
+ 2^{1/2} \left[ e^1 \otimes e^6 - e^3 \otimes e^6 + \frac{1}{2} (e^2 \otimes e^4 + e^2 \otimes e^5 - e^4 \otimes e^5) \right],
\]
and we have \(g_\phi = g + u^2 \eta \otimes \eta\).

The results of Proposition 3.1 are also useful to produce examples of seven-dimensional solvable non-nilpotent Lie algebras admitting closed \(G_2\)-structures, as the next example shows.

**Example 3.8.** On the six-dimensional unimodular solvable non-nilpotent Lie algebra \(\mathfrak{g}_{0,70}^0\) with structure equations
\[
(-e^{26} + e^{35}, e^{16} + e^{45}, -e^{46}, e^{36}, 0, 0),
\]
consider the closed 2-forms
\[
\omega_0 = 2e^{34}, \quad \widetilde{\omega} = -e^{13} - e^{24} - e^{56},
\]
and the definite 3-form
\[
\rho = e^{125} - e^{146} + e^{236} - e^{345}.
\]
Then, \(d\rho = -\widetilde{\omega} \wedge \omega_0\), and the almost complex structure \(J\) induced by the pair \((\rho, e^{123456})\) is given by
\[
J e_1 = -e_3, \quad J e_2 = -e_4, \quad J e_3 = e_1, \quad J e_4 = e_2, \quad J e_5 = -e_6, \quad J e_6 = e_5.
\]
In particular, \(\widetilde{\omega}\) is a taming form for \(J\), as for any non-zero vector \(\xi = \xi^k e_k \in \mathfrak{g}_{0,70}^0\) we have
\[
\widetilde{\omega}(\xi, J\xi) = \sum_{k=1}^{6} (\xi^k)^2 > 0.
\]
The pair \((\widetilde{\omega}, \rho)\) defines an SU(3)-structure on \(\mathfrak{g}_{0,70}^0\), since \(\widetilde{\omega} \wedge \rho = 0\) and \(3\rho \wedge J_\rho \rho = 2\widetilde{\omega}^3\).

The central extension of \((\mathfrak{g}_{0,70}^0, \omega_0)\) is given by
\[
\mathfrak{g} = \langle -e^{26} + e^{35}, e^{16} + e^{45}, -e^{46}, e^{36}, 0, 0, 2e^{34} \rangle,
\]
and it is isomorphic to the Lie algebra \(\mathfrak{a}_9\) of Theorem 4.1 below. By Proposition 3.1 we know that the 3-form \(\varphi = \widetilde{\omega} \wedge e^7 + \rho\) defines a closed \(G_2\)-structure on \(\mathfrak{g}\). Notice that the 1-form \(e^7\) does not define a contact structure on \(\mathfrak{g}\), as the closed 2-form \(\omega_0\) is degenerate.

More generally, one can consider the list of all six-dimensional unimodular solvable non-nilpotent Lie algebras admitting symplectic structures [37] and determine which of them admit a structure \((\omega_0, \widetilde{\omega}, \rho)\) satisfying the hypothesis of Proposition 3.1. This allows one to obtain further examples of solvable non-nilpotent Lie algebras admitting closed \(G_2\)-structures.
4. A classification result

In this section, we classify all seven-dimensional unimodular Lie algebras with non-trivial center admitting closed $G_2$-structures, up to isomorphism. Every such Lie algebra must be solvable by the results of [21]. If it is nilpotent, then it is isomorphic to one of the Lie algebras $\mathfrak{n}_1, \ldots, \mathfrak{n}_{12}$ by [12]. To complete the classification, we have to determine which unimodular solvable non-nilpotent Lie algebras with non-trivial center admit closed $G_2$-structures. This is done in the following.

**Theorem 4.1.** Let $\mathfrak{g}$ be a seven-dimensional unimodular solvable non-nilpotent Lie algebra with non-trivial center. Then, $\mathfrak{g}$ admits closed $G_2$-structures if and only if it is isomorphic to one of the following.

- $\mathfrak{s}_1 = (e^{23}, -e^{36}, e^{26}, e^{36}, e^{34} + e^{46}, 0, 0)$,
- $\mathfrak{s}_2 = (e^{16} + e^{35}, -e^{26} + e^{45}, e^{36}, -e^{46}, 0, 0)$,
- $\mathfrak{s}_3 = (-e^{16} + e^{25}, -e^{15} - e^{36}, -e^{45}, e^{35} + e^{46}, 0, 0)$,
- $\mathfrak{s}_4 = (0, -e^{13}, -e^{12}, 0, -e^{46}, -e^{45}, 0)$,
- $\mathfrak{s}_5 = (e^{15}, -e^{25}, -e^{35}, e^{45}, 0, 0, 0)$,
- $\mathfrak{s}_6 = (\alpha e^{15} + e^{25}, -e^{15} + \alpha e^{25}, -\alpha e^{35} + e^{45}, -e^{35} - \alpha e^{45}, 0, 0)$, $\alpha > 0$,
- $\mathfrak{s}_7 = (e^{25}, -e^{15}, e^{45}, -e^{35}, 0, 0, 0)$,
- $\mathfrak{s}_8 = (e^{16} + e^{35}, -e^{26} + e^{45}, e^{36}, -e^{46}, 0, 0, 0, e^{34})$,
- $\mathfrak{s}_9 = (-e^{26} + e^{35}, e^{16} + e^{45}, -e^{46}, e^{36}, 0, 0, e^{34})$,
- $\mathfrak{s}_{10} = (e^{23}, -e^{36}, e^{26}, e^{36}, e^{34} + 2 e^{16} + e^{25} - e^{34} + \sqrt{3} e^{24} + \sqrt{3} e^{35})$,
- $\mathfrak{s}_{11} = (e^{23}, -e^{36}, e^{26}, e^{36} - e^{56}, e^{36} + e^{46}, 0, 2 e^{16} + e^{25} - e^{34} - \sqrt{3} e^{24} - \sqrt{3} e^{35})$.

In particular, $\mathfrak{g}$ is the contactization of a symplectic Lie algebra if and only if it is isomorphic either to $\mathfrak{s}_10$ or to $\mathfrak{s}_{11}$.

**Proof.** Since the central extension of a nilpotent Lie algebra is nilpotent, by Proposition 3.2 we can assume that $\mathfrak{g}$ is the central extension of a six-dimensional unimodular solvable non-nilpotent Lie algebra $(\mathfrak{h}, \omega_0)$ admitting symplectic structures. Recall that $\mathfrak{g}$ is determined by any representative in the cohomology class $[\omega_0] \in H^2(\mathfrak{h})$, up to isomorphism. Moreover, $\mathfrak{h}$ is isomorphic to one of the Lie algebras listed in Table [I] (cf. [19, 37]).

If $\omega_0 = 0$, then $\mathfrak{g}$ is the direct sum of $\mathfrak{h}$ and the abelian Lie algebra $\mathbb{R}$. As a consequence of [24] Thm. C, $\mathfrak{g}$ admits closed $G_2$-structures if and only if $\mathfrak{h}$ admits symplectic half-flat $SU(3)$-structures. Therefore, by [19] Thm. 1.1, $\mathfrak{g}$ must be isomorphic to one of the Lie algebras

- $\mathfrak{s}_1 \cong \mathfrak{g}_{0,38}^{0} \oplus \mathbb{R}$, $\mathfrak{s}_2 \cong \mathfrak{g}_{0,54}^{0} \oplus \mathbb{R}$, $\mathfrak{s}_3 \cong \mathfrak{g}_{0,118}^{0,-1} \oplus \mathbb{R}$, $\mathfrak{s}_4 \cong \mathfrak{c}(1, 1) \oplus \mathfrak{c}(1, 1) \oplus \mathbb{R}$, $\mathfrak{s}_5 \cong \mathfrak{a}_{5,7}^{0,-1,1} \oplus \mathbb{R}$, $\mathfrak{s}_6 \cong \mathfrak{a}_{5,17}^{0,-1,1} \oplus \mathbb{R}$, $\mathfrak{s}_7 \cong \mathfrak{a}_{5,17}^{0,0,1} \oplus \mathbb{R}$.

We can then focus on the case when $\omega_0 \neq 0$ and $\mathfrak{h}$ is one of the Lie algebras appearing in Table [I]. To determine those having a central extension admitting closed $G_2$-structure, we proceed as follows. First, we compute a basis of the second cohomology group $H^2(\mathfrak{h})$ using the structure equations given in Table [I] Then, we consider a non-zero representative $\omega_0$.
of the generic element in $H^2(\mathfrak{h})$, and we look for closed non-degenerate 2-forms $\tilde{\omega} \in \Lambda^2\mathfrak{h}^*$ such that $\tilde{\omega} \wedge \omega_0$ is exact (cf. Proposition 3.2). A computation (with the aid of a computer algebra system, e.g. Maple 2021, when needed) shows that there are no exact 4-forms of this type when $\mathfrak{h}$ is a decomposable Lie algebra not isomorphic to $A_{5,15}^{-1} \oplus \mathbb{R}$ or to $A_{5,18}^0 \oplus \mathbb{R}$. Let us prove this claim, for instance, for the first decomposable Lie algebra appearing in Table I, namely $\mathfrak{g}_{6,13}^{-1,\frac{1}{2},0}$. A basis for its second cohomology group is given by

$$(\{e^{13}\}, \{e^{24}\}, \{e^{56}\}),$$

and we can consider the non-zero representative

$$\omega_0 = f_1 e^{13} + f_2 e^{24} + f_3 e^{56},$$

where $f_1, f_2, f_3 \in \mathbb{R}$ and $f_1^2 + f_2^2 + f_3^2 \neq 0$. The generic closed non-degenerate 2-form $\tilde{\omega}$ on $\mathfrak{g}_{6,13}^{-1,\frac{1}{2},0}$ has the following expression

$$\tilde{\omega} = h_1 e^{13} + h_2 \left( e^{23} - \frac{1}{2} e^{16} \right) + h_3 e^{24} + h_4 e^{26} + h_5 e^{36} + h_6 e^{46} + h_7 e^{56},$$

for some $h_i \in \mathbb{R}$ such that $h_1 h_3 h_7 \neq 0$. Now, we compute

$$\tilde{\omega} \wedge \omega_0 = - (f_1 h_3 + f_2 h_1) e^{1234} - f_1 h_4 e^{1236} + f_1 h_6 e^{1346} + (f_1 h_7 + f_3 h_1) e^{1356} - \frac{1}{2} f_2 h_2 e^{1246} - f_2 h_5 e^{2346} + (f_2 h_7 + f_3 h_3) e^{2456} + f_3 h_2 e^{2356},$$

and we see that this 4-form is exact only if the coefficients of $e^{1234}, e^{1356}, e^{2456}$ vanish, namely

$$\begin{aligned}
    f_1 h_3 + f_2 h_1 &= 0, \\
    f_1 h_7 + f_3 h_1 &= 0, \\
    f_2 h_7 + f_3 h_3 &= 0.
\end{aligned}$$

This is a homogeneous linear system in the variables $f_i$’s whose unique solution under the constraint $h_1 h_3 h_7 \neq 0$ is $f_1 = f_2 = f_3 = 0$. Thus, $\tilde{\omega} \wedge \omega_0$ cannot be exact if $\omega_0 \neq 0$. A similar discussion leads us to ruling out all of the decomposable Lie algebras listed in Table I but $A_{5,15}^{-1} \oplus \mathbb{R}$ and $A_{5,18}^0 \oplus \mathbb{R}$. In the remaining two cases, $\mathfrak{h}$ is the direct sum of a five-dimensional ideal $\mathfrak{k}$ and $\mathbb{R}$. A computation shows that there exist pairs $(\tilde{\omega}, \omega_0)$ satisfying the required conditions only when $\omega_0 \in \Lambda^2\mathfrak{k}^*$. In detail, if $\mathfrak{h} \cong A_{5,15}^{-1} \oplus \mathbb{R}$, then the possible 2-forms are given by

$$\tilde{\omega} = h_1 (e^{14} - e^{23}) + h_2 e^{15} + h_3 e^{24} + h_4 e^{25} + h_5 e^{35} + h_6 e^{45} + h_7 e^{56}, \quad \omega_0 = a e^{24},$$

where $a, h_i \in \mathbb{R}$ and $a h_1 h_7 \neq 0$. If $\mathfrak{h} \cong A_{5,18}^0 \oplus \mathbb{R}$, then the possible 2-forms are given by

$$\tilde{\omega} = k_1 (e^{13} + e^{24}) + k_2 e^{15} + k_3 e^{25} + k_4 e^{34} + k_5 e^{35} + k_6 e^{45} + k_7 e^{56}, \quad \omega_0 = b e^{34},$$

where $b, k_i \in \mathbb{R}$ and $b k_1 k_7 \neq 0$. Since in both cases $\mathfrak{h} = \mathfrak{k} \oplus \mathbb{R}$ and $\omega_0 \in \Lambda^2\mathfrak{k}^*$, all possible central extensions of $(\mathfrak{h}, \omega_0)$ split as the Lie algebra direct sum of a six-dimensional ideal and $\mathbb{R}$. If such an extension admits closed $G_2$-structures, then it must be isomorphic to one of $\mathfrak{s}_1, \ldots, \mathfrak{s}_7$.

We are then left with the indecomposable Lie algebras appearing in Table I. Also in this case, with analogous computations as before, one can check that there are no pairs $(\tilde{\omega}, \omega_0)$
satisfying the required conditions when \( h \) is an indecomposable Lie algebra not isomorphic to one of \( 6_{3,38}, 6_{0,54}, 7_{6,70} \). In the remaining three cases, we claim that \( h \) has a central extension admitting closed \( G_2 \)-structures.

If \( h \cong 6_{0,38} \), then there exist closed non-degenerate 2-forms \( \tilde{\omega} \) such that \( \tilde{\omega} \wedge \omega_0 \) is exact if and only if either \( \omega_0 = a \left( 2e^{16} + e^{25} - e^{34} + \sqrt{3} e^{24} + \sqrt{3} e^{35} \right) \), for some \( a \neq 0 \), or \( \omega_0 = b \left( 2e^{16} + e^{25} - e^{34} - \sqrt{3} e^{24} - \sqrt{3} e^{35} \right) \), for some \( b \neq 0 \). These forms are not cohomologous, so they give rise to non-isomorphic central extensions of \( h \). In the first case, the central extension of \((h, \omega_0)\) is isomorphic to \( s_{10} \), and it admits closed \( G_2 \)-structures. An example is given by

\[
\varphi = e^{123} - 4e^{145} + 2e^{167} - \sqrt{3} e^{247} + e^{256} + e^{257} - e^{346} - e^{347} - \sqrt{3} e^{357}.
\]

In the second case, the central extension of \((h, \omega_0)\) is isomorphic to \( s_{11} \) and it admits closed \( G_2 \)-structures. An example is given by

\[
\varphi = e^{123} - 4e^{145} + 2e^{167} + \sqrt{3} e^{247} - e^{256} + e^{257} + e^{346} - e^{347} + \sqrt{3} e^{357}.
\]

Both \( s_{10} \) and \( s_{11} \) are contactizations, since the 2-form \( \omega_0 \) is non-degenerate.

If \( h \cong 6_{0,54} \), then \( \omega_0 = a e^{34} \), for some \( a \neq 0 \). The central extension of \((h, \omega_0)\) is isomorphic to \( s_8 \), and it admits closed \( G_2 \)-structures. An example is given by

\[
\varphi = e^{147} + e^{237} + e^{567} + e^{125} - e^{136} + \frac{1}{2} \left( e^{146} - e^{236} \right) + \frac{5}{4} e^{246} + e^{345}.
\]

If \( h \cong 6_{0,70} \), then \( \omega_0 = a e^{34} \), for some \( a \neq 0 \). The central extension of \((h, \omega_0)\) is isomorphic to \( s_9 \), and it admits closed \( G_2 \)-structures. An example is given by

\[
\varphi = e^{137} + e^{247} + 2e^{567} - e^{125} + e^{146} - e^{236} + e^{345}.
\]

see also Example 3.8.

To conclude the proof, we first observe that the Lie algebras \( s_8 \) and \( s_9 \) cannot be the contactization of a symplectic Lie algebra. Indeed, in both cases \( \omega_0 \) is a closed degenerate 2-form on the unimodular Lie algebra \( h \), thus every representative of \([\omega_0] \in H^2(h)\) is a degenerate 2-form. Finally, a direct computation shows that the remaining Lie algebras \( s_1, \ldots, s_7 \) do not admit any contact structure. \( \square \)
**Remark 4.2.** Notice that there are some misprints in [37] that have been corrected in Table 1 see also the appendix in [19].
Corollary 4.3. A seven-dimensional Lie algebra with non-trivial center admitting torsion-free $G_2$-structures is either abelian or isomorphic to $\mathfrak{s}_7$.

Proof. Let $\mathfrak{g}$ be a seven-dimensional Lie algebra with non-trivial center endowed with a torsion-free $G_2$-structure $\varphi$. Then, the metric $g_\varphi$ induced by $\varphi$ is Ricci-flat, and thus flat by [2]. Consequently, the results of [38] imply that either $\mathfrak{g}$ is abelian, or $\mathfrak{g}$ splits as a $g_\varphi$-orthogonal direct sum $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{a}$, where $\mathfrak{b}$ is an abelian subalgebra, $\mathfrak{a}$ is an abelian ideal, and the endomorphism $\text{ad}_x$ is skew-adjoint for every $x \in \mathfrak{b}$. In the latter case, $\mathfrak{g}$ is a unimodular 2-step solvable Lie algebra and the eigenvalues of $\text{ad}_x$ are purely imaginary for every $x \in \mathfrak{g}$ (cf. [27, Sect. 2.8]). Among the Lie algebras obtained in Theorem 4.1, the 2-step solvable ones are $\mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4, \mathfrak{s}_5, \mathfrak{s}_6, \mathfrak{s}_7$. The first four Lie algebras in this list do not admit any flat metric, as the following endomorphisms have real spectrum: $\text{ad}_{\mathfrak{e}_6} \in \text{Der}(\mathfrak{s}_2)$, $\text{ad}_{\mathfrak{e}_7} \in \text{Der}(\mathfrak{s}_3)$, $\text{ad}_{\mathfrak{e}_8} \in \text{Der}(\mathfrak{s}_4)$, $\text{ad}_{\mathfrak{e}_9} \in \text{Der}(\mathfrak{s}_5)$. Also the Lie algebra $\mathfrak{s}_6$ can be ruled out, since $\text{ad}_{\mathfrak{e}_6}$ has complex eigenvalues that are not purely imaginary. Finally, the Lie algebra $\mathfrak{s}_7$ admits torsion-free $G_2$-structures. An example is given by the $G_2$-structure

$$\varphi = e^{137} + e^{247} + e^{567} + e^{125} - e^{146} + e^{236} - e^{345},$$

which induces the metric $g_\varphi = \sum_{i=1}^{7} e^i \circ e^i$. \qed

Remark 4.4. The simply connected solvable Lie groups whose Lie algebra is one of $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4, \mathfrak{s}_5, \mathfrak{s}_7$ admit lattices, and this is the case also for the family of Lie algebras $\mathfrak{s}_6 \cong A_{5,17}^{\alpha,\alpha_1} \oplus \mathbb{R}^2$, for certain values of the parameter $\alpha > 0$ (see e.g. [19] and the references therein). We now show that the simply connected Lie groups with Lie algebra $\mathfrak{s}_8$ or $\mathfrak{s}_9$ admit lattices, too. Indeed, since they are both almost nilpotent, it is possible to construct a lattice using the following criterion by [8]. Let $G = \mathbb{R} \ltimes_\mu H$ be an almost nilpotent Lie group with nilradical $H$, and let $\mathfrak{g} = \mathbb{R} \ltimes_D \mathfrak{h}$ be its Lie algebra, where $\mathfrak{h} := \text{Lie}(H)$ and $D \in \text{Der}(\mathfrak{h})$ is such that $\mu(t)|_{\mathfrak{e}_6} = \exp(tD)$. If there exists $0 \neq t_0 \in \mathbb{R}$ and a rational basis $(x_1,\ldots,x_n)$ of $\mathfrak{h}$ such that the coordinate matrix of $\exp(t_0D)$ in such a basis is integer, then $\Gamma := t_0\mathbb{Z} \ltimes_\mu \exp(\mathbb{Z}(x_1,\ldots,x_n))$ is a lattice in $G$.

Let us consider the Lie algebra $\mathfrak{s}_9$ with the basis $(e_1,\ldots,e_7)$ as in Theorem 4.1. We can write $\mathfrak{s}_9 = \mathbb{R} \ltimes_D \mathfrak{h}$, where $\mathfrak{h} = \text{span}_\mathbb{R}(e_1,\ldots,e_5,e_7)$ is a nilpotent Lie algebra with structure equations

\begin{equation}
(4.1) \quad (e^{35}, e^{45}, 0, 0, 0, e^{34}),
\end{equation}

and

$$D = \text{ad}(e_6)|_{\mathfrak{h}} = \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. $$

For $t_0 = 2\pi$, this basis satisfies the criterion of [8] guaranteeing the existence of a lattice in the simply connected Lie group corresponding to $\mathfrak{s}_9$.

Let us now focus on the Lie algebra $\mathfrak{s}_8$ with the basis $(e_1,\ldots,e_7)$ as in Theorem 4.1. We note that $\mathfrak{s}_8 = \mathbb{R} \ltimes_D \mathfrak{h}$, where the structure equations of the nilpotent Lie algebra
Thus, the integer matrix $B$ is the matrix associated with $\exp(t_0D)$ with respect to a suitable basis $(e_1, \ldots, e_5, e_7)$ of $\mathfrak{h}$. Moreover, $\mathfrak{h}$ has rational structure equations $(\varepsilon^{25}, 0, \varepsilon^{45}, 0, 0, \varepsilon^{24})$ in such a basis. The existence of a lattice in the simply connected solvable Lie group with Lie algebra $\mathfrak{s}_5$ then follows.

5. Semi-algebraic solitons on the central extension of a Lie algebra

Let $G$ be a seven-dimensional simply connected Lie group with Lie algebra $\mathfrak{g}$. Consider a derivation $D$ of $\mathfrak{g}$, and denote by $X_D$ the vector field on $G$ induced by the one-parameter group of automorphisms $F_1 \in \text{Aut}(G)$ with derivative $dF_1|_{t_0} = \exp(t_0D) \in \text{Aut}(\mathfrak{g})$. According to [31], a left-invariant closed $G_2$-structure $\varphi$ on $G$ is said to be a semi-algebraic soliton if it satisfies the Laplacian soliton equation (1.1) with respect to some vector field $X_D$ corresponding to a derivation $D \in \text{Der}(\mathfrak{g})$. In this case, $\mathcal{L}_{X_D} \varphi = D^* \varphi$, so that the equation (1.1) can be rewritten as follows

$$\Delta_\varphi \varphi = \lambda \varphi + D^* \varphi,$$

where $A^* \beta(x_1, \ldots, x_k) = \beta(Ax_1, \ldots, x_k) + \cdots + \beta(x_1, \ldots, Ax_k)$, for any $A \in \text{End}(\mathfrak{g})$, $x_1, \ldots, x_k \in \mathfrak{g}$, and $\beta \in \Lambda^k \mathfrak{g}^*$. Notice that $\Delta_\varphi \varphi = dd^* \varphi = d\tau$, $\tau$ being the intrinsic torsion form of $\varphi$. When the $g_\varphi$-adjoint $D^t$ of $D$ is also a derivation of $\mathfrak{g}$, the $G_2$-structure $\varphi$ is called an algebraic soliton.

We now focus on the case when $\mathfrak{g}$ is a unimodular Lie algebra with non-trivial center. By the results of Sect. 3, we can assume that $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}z$ is the central extension of a six-dimensional unimodular Lie algebra $(\mathfrak{h}, \omega_0)$. Moreover, every closed $G_2$-structure $\varphi$ on $\mathfrak{g}$ can be written both as $\varphi = \tilde{\omega} \wedge \theta + \rho$, with $d\rho = -\tilde{\omega} \wedge \omega_0$ and $d\tilde{\omega} = 0$, and as $\varphi = u \omega \wedge \eta + \psi_+$, where $(\omega, \psi_+)$ is an SU(3)-structure on $\mathfrak{h}$, $u := |z|_{\varphi}$ and $\eta := u^{-2}\varphi^\flat = \eta_0 + \theta$, for some $\eta_0 \in \mathfrak{h}^*$. If $\varphi$ is a semi-algebraic soliton, the condition (5.1) is equivalent to a set of equations involving either the forms $(\tilde{\omega}, \rho)$ or the SU(3)-structure $(\omega, \psi_+)$ on $\mathfrak{h}$. In the following, we shall see that it is possible to obtain information on the semi-algebraic soliton $\varphi$ under suitable assumptions. We are interested in the case when $z$ is an eigenvector of $D$, as this happens whenever $\mathfrak{g}$ is the contactization of a symplectic Lie algebra. Indeed, in that case the center of $\mathfrak{g}$ is $\mathfrak{z}(\mathfrak{g}) = \mathbb{R}z$ and it is preserved by any derivation of $\mathfrak{g}$.

Henceforth, we assume that $\varphi$ is a semi-algebraic soliton on the unimodular Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}z$, and that it satisfies the equation $\Delta_\varphi \varphi = \lambda \varphi + D^* \varphi$ with respect to a derivation
Lemma 5.1. Let \((\omega, \psi_+)\) be an SU(3)-structure on a six-dimensional vector space \(V\) and let \(A \in \text{End}(V)\). Then,
\[
A^* \psi_+ \wedge \psi_- = A^* \omega \wedge \omega^2 = \frac{1}{3} \text{tr}(A) \omega^3.
\]

Proof. We can always consider a basis \(\{e_1, \ldots, e_6\}\) of \(V\) which is adapted to the SU(3)-structure \((\omega, \psi_+)\). Then, with respect to the dual basis \(\{e^1, \ldots, e^6\}\) of \(V^*\), we have
\[
\omega = e^{12} + e^{34} + e^{56}, \quad \psi_+ = e^{135} - e^{146} - e^{236} - e^{245}, \quad \psi_- = e^{136} + e^{145} + e^{235} - e^{246}.
\]
Now, a direct computation shows that
\[
A^* \psi_+ \wedge \psi_- = A^* \omega \wedge \omega^2 = 2 \text{tr}(A) e^{123456} = \frac{1}{3} \text{tr}(A) \omega^3.
\]

Proposition 5.2. The constant \(\lambda\) is given by
\[
\lambda = -3c - \frac{1}{2} (|w_2|^2_g + 6 |w_1|^2_g) = -3c - \frac{1}{2} |\tau|^2_\varphi.
\]

Proof. Wedging the first equation of (5.2) by the closed 4-form \(\omega^2\), we obtain
\[
\tilde{D}^* \omega \wedge \omega^2 + (c + \lambda) \omega^3 = -2 d(Jw_1) \wedge \omega^2 = -2 d(Jw_1 \wedge \omega^2) = 0,
\]
as every 5-form on the unimodular Lie algebra \(\mathfrak{h}\) is closed. Then, by Lemma 5.1 we get
\[
\text{tr}(D) = -3(c + \lambda).
\]

Let us now consider the second equation in (5.2). Wedging both sides by \(\psi_-\) and using the compatibility condition \(\omega \wedge \psi_- = 0\), we obtain
\[
(dw_2^2 - d \ast_g (Jw_1 \wedge \psi_+) - 2u (d(Jw_1) \wedge \eta_h - Jw_1 \wedge d\eta)) \wedge \psi_- = \left( \tilde{D}^* \rho + \lambda \rho \right) \wedge \psi_-.
\]
Since $\rho = u \omega \wedge \eta_h + \psi_+$ and $\omega \wedge \psi_- = 0$, the RHS of (5.4) can be rewritten as follows:

$$\left( \widetilde{D}_* \rho + \lambda \rho \right) \wedge \psi_- = u \widetilde{D}_* \omega \wedge \eta_h \wedge \psi_- + \widetilde{D}_* \psi_+ \wedge \psi_- + \lambda \psi_+ \wedge \psi_-$$

$$= u \left( -2 d(J w_1) - (c + \lambda) \omega \right) \wedge \eta_h \wedge \psi_- + \frac{1}{3} \text{tr}(\widetilde{D}) \omega^3 + \frac{2}{3} \lambda \omega^3$$

$$= -2u d(J w_1) \wedge \eta_h \wedge \psi_- + \frac{1}{3} \text{tr}(\widetilde{D}) \omega^3 + \frac{2}{3} \lambda \omega^3,$$

where the second equality follows from the first equation of (5.2), Lemma 5.1 and the normalization condition $\psi_+ \wedge \psi_- = \frac{2}{3} \omega^3$. The summands appearing in the LHS of (5.4) can be rewritten as follows. Since $w_2^- \in \Lambda^2 h^*$, we have

$$dw_2^- \wedge \psi_- = d(w_2^- \wedge \psi_+) - w_2^- \wedge d\psi_- = -w_2^- \wedge (-g(w_2^- + J w_1 \wedge \psi_+)) = |w_2|^2 g \text{vol}_g.$$  

Since every 5-form on $h$ is closed, we get

$$-d(g(J w_1 \wedge \psi_+) \wedge \psi_-) = g(J w_1 \wedge \psi_+) \wedge d\psi_- = g(J w_1 \wedge \psi_+) \wedge (-g(w_2^- + J w_1 \wedge \psi_+))$$

$$= J w_1 \wedge 2g(J w_1) = 2 |J w_1|^2 g \text{vol}_g = 2 |w_1|^2 g \text{vol}_g,$$

where we used the identity (iv) of Lemma 2.2. Finally, by Lemma 3.5 and the identity iii of Lemma 2.2, we have

$$2u J w_1 \wedge d\eta \wedge \psi_- = -2 J w_1 \wedge g(w_1 \wedge \psi_+) \wedge \psi_- = 4 J w_1 \wedge g(J w_1) = 4 |w_1|^2 g \text{vol}_g.$$  

Hence, the equation (5.4) becomes

$$\left( |w_2^-|^2 g + 6 |w_1|^2 g \right) \text{vol}_g = \frac{1}{3} \text{tr}(\widetilde{D}) \omega^3 + \frac{2}{3} \lambda \omega^3.$$  

Recalling that $\text{vol}_g = \frac{1}{3} \omega^3$, we have

$$\frac{1}{2} \left( |w_2^-|^2 g + 6 |w_1|^2 g \right) = \text{tr}(\widetilde{D}) + 2 \lambda.$$  

Now, the thesis follows combining this identity with $\text{tr}(\widetilde{D}) = -3(c + \lambda)$ and recalling that $|w_2^-|_{g}^2 + 6 |w_1|_{g}^2 = |\tau|_{\varphi}^2$ (cf. Lemma 3.6). \qed

As a consequence of Proposition 5.2 we have the following.

**Corollary 5.3.** Let $(\mathfrak{g}, \theta)$ be the contactization of a symplectic unimodular Lie algebra $(\mathfrak{h}, \omega_0)$, and let $\varphi$ be a semi-algebraic soliton on $\mathfrak{g}$ such that $\Delta_\varphi \varphi = \lambda \varphi + D^* \varphi$, for some $D \in \text{Der}(\mathfrak{g})$. Then,

$$\lambda = |w_2^-|_{g}^2 + 6 |w_1|_{g}^2 = |\tau|_{\varphi}^2,$$

and $\varphi$ is expanding.

**Proof.** Since $\mathfrak{g}$ is the contactization of a symplectic Lie algebra $(\mathfrak{h}, \omega_0)$, we have $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R} z$ and $\mathfrak{z}(\mathfrak{g}) = \mathbb{R} z$. In particular, $Dz = cz$, for some $c \in \mathbb{R}$. Therefore, by Proposition 5.2, the constant $\lambda$ is given by

$$\lambda = -3(c - \frac{1}{2} (|w_2^-|_{g}^2 + 6 |w_1|_{g}^2) = -3 c - \frac{1}{2} |\tau|_{\varphi}^2.$$  

Recall that $\omega_0 = d\theta$ on $\mathfrak{g}$. Since $D \in \text{Der}(\mathfrak{g})$, we see that

$$D^* \omega_0 = D^* (d\theta) = d(D^* \theta) = d(\alpha + c \theta) = d\alpha + c \omega_0.$$  

On the other hand, \( \omega_0 \) is a non-degenerate 2-form on the unimodular Lie algebra \( \mathfrak{h} \). Consequently,

\[
\frac{1}{3} \text{tr}(\tilde{D}) \omega_0^3 = \tilde{D}^* \omega_0 \wedge \omega_0^2 = D^* \omega_0 \wedge \omega_0^2 = (d\alpha + c \omega_0) \wedge \omega_0^2 = c \omega_0^3,
\]
as every 5-form on \( \mathfrak{h} \) is closed. Now, from the proof of Proposition \ref{prop:3-form}, we know that

\[ 3c = \text{tr}(\tilde{D}) = -3c - 3\lambda, \]
whence \(-2c = \lambda\). Therefore, we have \( \lambda = |w_{2g}^2|_g + 6 |w_1|_g^2 = |\tau|_\varphi^2 \).

To conclude the proof, we observe that \( \lambda = 0 \) if and only if \( \varphi \) is torsion-free. By Corollary \ref{cor:torsion-free} torsion-free \( G_2 \)-structures do not occur on the contactization of any symplectic unimodular Lie algebra. Thus, \( \lambda > 0 \) and \( \varphi \) is expanding. \( \square \)

The previous result applies, for instance, to the nilpotent Lie algebra \( \mathfrak{n}_{12} \) endowed with the closed \( G_2 \)-structure considered in [17 Thm, 3.6].

**Example 5.4.** Consider the nilpotent Lie algebra \( \mathfrak{n}_{12} \), and let \( (e^1, \ldots, e^7) \) be the basis of \( \mathfrak{n}_{12}^* \) for which the structure equations are the following

\[
\begin{pmatrix}
0, 0, 0, \frac{\sqrt{3}}{6} e^{12}, \frac{\sqrt{3}}{12} e^{13} - \frac{1}{4} e^{23}, -\frac{\sqrt{3}}{12} e^{23} - \frac{1}{4} e^{13}, \frac{\sqrt{3}}{12} e^{16} - \frac{1}{4} e^{15} + \frac{\sqrt{3}}{12} e^{25} + \frac{1}{4} e^{26} - \frac{\sqrt{3}}{6} e^{34}
\end{pmatrix}.
\]

Recall that \( \mathfrak{n}_{12} \) is the contactization of a six-dimensional symplectic nilpotent Lie algebra (cf. Corollary \ref{cor:nilpotent}). The 3-form

\[
\varphi = e^{167} + e^{257} + e^{347} + e^{135} - e^{124} - e^{236} - e^{456}
\]
defines a closed \( G_2 \)-structure on \( \mathfrak{n}_{12} \) inducing the metric \( g_\varphi = \sum_{i=1}^{7} e^i \odot e^i \). The corresponding intrinsic torsion form is \( \tau = \frac{1}{2} (e^{56} - e^{37}) \). A computation shows that \( \varphi \) is an expanding algebraic soliton solving the equation \( \Delta_\varphi \varphi = \lambda \varphi + D^* \varphi \) with \( \lambda = \frac{1}{2} |\tau|_{\varphi}^2 \) and

\[
D = -\frac{1}{8} \text{diag} (1, 1, 0, 2, 1, 1, 2) \in \text{Der}(\mathfrak{n}_{12}).
\]

In addition to \( \mathfrak{n}_{12} \), also the non-abelian nilpotent Lie algebras \( \mathfrak{n}_2, \ldots, \mathfrak{n}_7 \) admit (semi-)algebraic solitons (see [39]). However, it is currently not known whether semi-algebraic solitons occur on the nilpotent Lie algebras \( \mathfrak{n}_8, \mathfrak{n}_9, \mathfrak{n}_{10} \) and \( \mathfrak{n}_{11} \).

Using Corollary \ref{cor:semi-algebraic} and Proposition \ref{prop:3-form}, we can show that semi-algebraic solitons do not exist on \( \mathfrak{n}_9 \).

**Proposition 5.5.** The nilpotent Lie algebra \( \mathfrak{n}_9 \) does not admit any semi-algebraic soliton.

**Proof.** As we observed in Corollary \ref{cor:nilpotent} the Lie algebra \( \mathfrak{n}_9 \) is the contactization of the nilpotent Lie algebra \( \mathfrak{h} = (0, 0, e^{12}, e^{13}, e^{23}, e^{15} + e^{24}) \) endowed with the symplectic form \( \omega_0 = e^{16} + e^{34} + e^{25} \). In particular, \( \mathfrak{j}(\mathfrak{n}_9) = \mathbb{R} e_7 \).

By Proposition \ref{prop:3-form} every closed \( G_2 \)-structure on \( \mathfrak{n}_9 \) can be written as \( \varphi = \tilde{\omega} \wedge e^7 + \rho \), where \( \rho \) is a definite 3-form on \( \mathfrak{h} \), \( \tilde{\omega} \) is a taming symplectic form for \( J_\rho \) and \( d\rho = -\tilde{\omega} \wedge \omega_0 \) (see Example \ref{ex:semi-algebraic} for an explicit case). This \( G_2 \)-structure is a semi-algebraic soliton solving \( \Delta_\varphi \varphi = \lambda \varphi + D^* \varphi \), for some \( \lambda \in \mathbb{R} \) and some \( D \in \text{Der}(\mathfrak{n}_9) \), if and only if the equations \ref{eq:5} are satisfied. By Corollary \ref{cor:semi-algebraic} we must have \( D e_7 = c e_7 \) and \( \lambda = -2c = |\tau|_{\varphi}^2 > 0 \). Moreover, the first equation in \ref{eq:5} constrains the 2-form on \( \mathfrak{h} \)

\[
\beta := \tilde{D}^* \tilde{\omega} + (c + \lambda) \tilde{\omega} = \tilde{D}^* \tilde{\omega} - c \tilde{\omega}
\]
to be exact. We shall show that if this last condition holds for any derivation \( D \) and any symplectic form \( \tilde{\omega} \) on \( n_9 \), then \( \tilde{\omega} \wedge \omega_0 \) cannot be exact.

The matrix associated with the generic derivation \( D \in \text{Der}(n_9) \) with respect to the basis \((e_1, \ldots, e_7)\) is given by

\[
D = \begin{pmatrix}
    h_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 2h_1 & 0 & 0 & 0 & 0 & 0 \\
    h_2 & h_3 & 3h_1 & 0 & 0 & 0 & 0 \\
    h_4 & h_5 & h_3 & 4h_1 & 0 & 0 & 0 \\
    h_6 & h_7 & -h_2 & 0 & 5h_1 & 0 & 0 \\
    h_8 & h_9 & h_7 - h_4 & -h_2 & h_3 & 6h_1 & 0 \\
    h_{10} & h_{11} & h_9 - h_6 & h_7 - 2h_4 & -h_5 - h_2 & -h_3 & 7h_1
\end{pmatrix},
\]

where \( h_j \in \mathbb{R} \). In particular, \( c = 7h_1 \) and we can assume that \( h_1 < 0 \).

The generic closed 2-form \( \tilde{\omega} \) on \( h \) is

\[
\tilde{\omega} = f_1 e^{12} + f_2 e^{13} + f_3 e^{14} + f_4 (e^{15} + e^{24}) + f_5 (e^{16} + e^{34}) + f_6 e^{23} + f_7 e^{25} + f_8 (e^{26} - e^{35}),
\]

and it is non-degenerate if and only if \( f_5 f_7 - f_3 f_8^2 \neq 0 \). Now, we have

\[
\beta = \tilde{D}^* \tilde{\omega} - 7h_1 \tilde{\omega}
\]

\[
= (h_3 f_2 - 4 h_1 f_1 + h_5 f_3 - h_4 f_4 + h_7 f_4 + h_9 f_5 - h_2 f_6 - h_6 f_7 - h_8 f_8) e^{12}
\]

\[
+ (h_3 f_3 - 3 h_1 f_2 - h_2 f_4 - 2 h_4 f_5 + f_5 h_7 + h_6 f_8) e^{13} + (h_3 f_5 - h_1 f_4 - h_2 f_8) (e^{15} + e^{24})
\]

\[
+ (h_3 f_4 - h_5 f_5 - 2 h_1 f_6 - h_2 f_7 - f_8 h_4 + 2 h_7 f_8) e^{23}
\]

\[
- 2h_1 f_3 e^{14} + h_1 f_8 e^{26} - h_1 f_8 e^{35}.
\]

Since the space of exact 2-forms on \( h \) is generated by \( e^{12}, e^{13}, e^{15} + e^{24}, e^{23} \), and since \( h_1 < 0 \), we see that \( \beta \) is exact if and only if \( f_3 = 0 = f_8 \). Consequently, \( \tilde{\omega} \) is non-degenerate if and only if \( f_5 f_7 \neq 0 \). This last constraint implies that \( \tilde{\omega} \wedge \omega_0 \) cannot be exact, since the space of exact 4-forms is spanned by

\[
e^{1245}, e^{1256}, e^{1236}, e^{1246} + e^{1345}, e^{1256} - e^{2345}, e^{1356} - e^{2346}.
\]

\[\square\]

**Remark 5.6.** Using a similar argument involving equations (5.2) as in the proof of the last proposition, one can show that semi-algebraic solitons do not occur on the solvable non-nilpotent Lie algebras \( s_8 \) and \( s_9 \).

By [12] and Theorem 1.1, we know that a seven-dimensional unimodular Lie algebra with one-dimensional center admitting closed \( G_2 \)-structures is isomorphic to one of \( n_8, n_9, n_{10}, n_{11}, n_{12}, s_2, s_3, s_4, s_8, s_9, s_{10}, s_{11} \). By Proposition 5.5 and Remark 5.6, the only ones which may admit semi-algebraic solitons are \( n_8, n_{10}, n_{11}, n_{12}, s_2, s_3, s_4, s_{10}, s_{11} \).

Examples of expanding semi-algebraic solitons are known on \( n_{12} \) (see Example 5.4), and on the Lie algebras \( s_2 \) and \( s_4 \) (see [22] Prop. 6.5). In the remaining cases, it is still not known whether semi-algebraic solitons exist. However, if there is any, it must be expanding. This follows from Proposition 5.3 when the Lie algebra is one of \( n_{10}, n_{11}, n_{12}, s_{10}, s_{11} \), while it follows from a direct computation involving equations (5.2) when the Lie algebra is one of \( n_8, s_2, s_3 \) and \( s_4 \).
When the center of the Lie algebra is at least two-dimensional, we have the following classification result.

**Theorem 5.7.** Let $\mathfrak{g}$ be a seven-dimensional unimodular Lie algebra with $\dim \mathfrak{z}(\mathfrak{g}) \geq 2$ admitting closed $G_2$-structures. Then, $\mathfrak{g}$ admits semi-algebraic solitons if and only if it is isomorphic to one of $\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_3, \mathfrak{n}_4, \mathfrak{n}_5, \mathfrak{n}_6, \mathfrak{n}_7, \mathfrak{s}_5, \mathfrak{s}_6, \mathfrak{s}_7$.

**Proof.** If $\mathfrak{g}$ is nilpotent, then it must be isomorphic to one of $\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_3, \mathfrak{n}_4, \mathfrak{n}_5, \mathfrak{n}_6, \mathfrak{n}_7$, by the classification of [12]. Every $G_2$-structure $\varphi$ on the abelian Lie algebra $\mathfrak{n}_1$ is torsion-free, and thus it solves the equation $\Delta \varphi \varphi = \lambda \varphi + D^* \varphi$ with $\lambda = 0$ and $D = 0 \in \text{Der}(\mathfrak{n}_1)$. In the remaining cases, the existence of semi-algebraic solitons is known from [39].

We can then focus on the case when $\mathfrak{g}$ is solvable non-nilpotent. By Theorem 4.1, $\mathfrak{g}$ must be isomorphic to one of $\mathfrak{s}_1, \mathfrak{s}_5, \mathfrak{s}_6, \mathfrak{s}_7$. Examples of semi-algebraic on $\mathfrak{s}_5$ and $\mathfrak{s}_6$ were given in [22] Prop. 6.5. By Corollary 4.3, the Lie algebra $\mathfrak{s}_7$ admits torsion-free $G_2$-structures, which are semi-algebraic solitons with $\lambda = 0$ and $D = 0 \in \text{Der}(\mathfrak{s}_7)$.

To conclude the proof, we must show that the Lie algebra $\mathfrak{s}_1$ does not admit any semi-algebraic soliton. Let us assume by contradiction that $\varphi$ is a semi-algebraic soliton on $\mathfrak{s}_1$. Then, as $\mathfrak{s}_1 \cong \mathfrak{g}_{6,38} \oplus \mathbb{R}$, we can write $\varphi = \tilde{\omega} \wedge e^t + \rho$, where $e^t$ spans $\mathbb{R}$, and $\tilde{\omega}$ and $\rho$ are closed forms on $\mathfrak{g}_{6,38}$. In particular, we have

$$\tilde{\omega} = f_1 \left( 2 e^{16} + e^{25} - e^{34} \right) + f_2 e^{23} + f_3 \left( e^{24} + e^{35} \right) + f_4 e^{26} + f_5 e^{36} + f_6 e^{46} + f_7 e^{56},$$

$$\rho = p_1 e^{123} + p_2 \left( e^{124} + e^{135} \right) + p_3 e^{126} + p_4 e^{136} + p_5 \left( e^{146} - e^{235} \right) + p_6 \left( e^{156} + e^{244} \right) + p_7 e^{236} + p_8 e^{246} + p_9 e^{256} + p_{10} e^{346} + p_{11} e^{356} + p_{12} e^{456},$$

where the $f_i$'s and $p_j$'s are real parameters. The symmetric bilinear form $b_\varphi$ induced by $\varphi$ as in (2.1) satisfies $b_\varphi(e_1, e_1) = -2p_2^2 f_1 e^{1234567}$ and $b_\varphi(e_4, e_4) = -p_2 f_1 p_{12} e^{1234567}$. Since $b_\varphi$ is definite, we must have $p_2 p_{12} f_1 \neq 0$.

The generic derivation $D \in \text{Der}(\mathfrak{s}_1)$ has the following expression with respect to the basis $(e_1, \ldots, e_7)$ of $\mathfrak{s}_1$

$$D = \begin{pmatrix}
  h_1 & h_2 & h_3 & 0 & 0 & h_4 & h_5 \\
  0 & \frac{1}{2} h_1 & h_6 & 0 & 0 & h_2 & 0 \\
  0 & -h_6 & \frac{1}{2} h_1 & 0 & 0 & h_3 & 0 \\
  0 & h_7 & h_8 & \frac{1}{2} h_1 & h_6 & h_9 & 0 \\
  0 & -h_8 & h_7 & -h_6 & \frac{1}{2} h_1 & h_{10} & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & h_{11} & h_{12} \\
\end{pmatrix},$$

where $h_i \in \mathbb{R}$. Since $\varphi$ is a semi-algebraic soliton, there is some $\lambda \in \mathbb{R}$ such that the 3-form $D^* \varphi + \lambda \varphi$ on $\mathfrak{s}_1$ is exact. Under the constraint $p_2 p_{12} f_1 \neq 0$, this implies that $\lambda = 0$ and that $\mathfrak{s}(\mathfrak{s}_1) = \text{span}_\mathbb{R}(e_1, e_7) \subset \ker(D)$. By Proposition 5.2, we then have $|\tau|_\varphi = 0$, i.e., the $G_2$-structure $\varphi$ is torsion-free. However, $\mathfrak{s}_1$ does not carry any torsion-free $G_2$-structure by Corollary 4.3. \qed
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Dipartimento di Matematica “G. Peano”, Università degli Studi di Torino, Via Carlo Alberto 10, 10123 Torino, Italy

Email address: annamaria.fino@unito.it

Email address: alberto.raffero@unito.it

Email address: francesca.salvatore@unito.it