OPTIMAL INVESTMENT AND REINSURANCE TO MINIMIZE THE PROBABILITY OF DRAWDOWN WITH BORROWING COSTS

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Abstract. We study the optimal investment and reinsurance problem in a risk model with two dependent classes of insurance businesses, where the two claim number processes are correlated through a common shock component and the borrowing rate is higher than the lending rate. The objective is to minimize the probability of drawdown, namely, the probability that the value of the wealth process reaches some fixed proportion of its maximum value to date. By the method of stochastic control theory and the corresponding Hamilton-Jacobi-Bellman equation, we investigate the optimization problem in two different cases and divide the whole region into four subregions. The explicit expressions for the optimal investment/reinsurance strategies and the minimum probability of drawdown are derived. We find that when wealth is at a relatively low level (below the borrowing level), it is optimal to borrow money to invest in the risky asset; when wealth is at a relatively high level (above the saving level), it is optimal to save more money; while between them, the insurer is willing to invest all the wealth in the risky asset. In the end, some comparisons are presented to show the impact of higher borrowing rate and risky investment on the optimal results.

1. Introduction. The theory of optimal investment can date back to the seminal works of Merton [25, 26]. From then on, the optimal investment-reinsurance problem has been paid great attention by the scholars all over the world and the stochastic control theory has been widely used in the literature of investment and reinsurance. See, for example, Brown [7], Hipp and Plum [19], Promislow and Young [27], Liang and Bayraktar [22], Zhang et al. [30], Liang and Young [21]. The main popular criteria include minimizing the probability of ruin, maximizing the expected utility of terminal wealth, mean-variance criterion, etc.

Recently, drawdown as a frequently quoted risk metric, is used to measure the decline of portfolio value from its historic higher-water mark. For example, Angoshtari et al. [1] investigated the minimum drawdown probability problem over an infinite-time horizon, and found that the optimal strategy which minimized the probability of ruin also minimized the probability of drawdown before drawdown.

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happens. Besides, Chen et al. [8] and Angoshtari et al. [2] both studied a lifetime investment problem aiming at minimizing the risk of drawdown. They found that the optimal strategy for a random (or finite) maturity setting such as lifetime drawdown was very different from that of the corresponding ruin problem. Furthermore, Han et al. [16, 18] considered the problem of optimal reinsurance which minimized the probability of drawdown for the risk model with thinning dependence/common shock structure. For any other works involving drawdown, we can refer to Grossman and Zhou [15], Cvitanić and Karatzas [9], and Elie and Touzi [11].

The literature mentioned above shares one common characteristic, that is, there is no distinction between borrowing rate and lending rate. But as we know, in the real financial market, the borrowing rate is higher than lending rate to make up for the risk that the lender takes. Very few scholars have mentioned the difference between borrowing and lending in their works. For example, Bayraktar and Young [4] considered the problem of minimizing the probability of lifetime ruin, in which the individual continuously consumed either a constant dollar amount or a constant proportion of the wealth under the higher borrowing rate. By using the stochastic piecewise linear-quadratic control technique, Fu et al. [12] derived the optimal investment strategy and the efficient frontier with higher borrowing rate constraint in the mean-variance framework. According to the borrowing curve they defined, the whole region was divided into two partitions: one was borrowing area and the other was no-borrowing area. Bo and Capponi [6] considered optimal portfolio under the framework of utility maximization for a special financial market, in which the market price of the risky asset was correlated with credit quality and the borrowing rate was higher than the saving rate. However, there is a typical drawback in these three papers, that is, the optimal investment strategy becomes arbitrarily large when wealth is close to 0, which means that the investors on the brink of bankruptcy can also freely borrow money without any limitations. To overcome the disadvantage, Luo [24] considered the minimum ruin probability for an insurance company with opportunities of proportional reinsurance and investment under the limited leverage rate constraint, i.e., the proportion of the borrowed amount to the wealth level is no more than a fixed constant $k$.

In this paper, under the criterion of minimizing the probability of drawdown, we investigate the optimal investment and reinsurance problem with borrowing costs. Two borrowing constraints are considered in our model: (i) the borrowing rate is higher than the lending rate; (ii) the maximum leverage ratio is no more than a fixed constant $1 + k$ ($k \geq 0$), that is, the proportion of the borrowed amount to the current wealth level is no more than a non-negative constant $k$. Based on the technique of stochastic control theory and corresponding Hamilton-Jacobi-Bellman (HJB) equation, we obtain the explicit solutions for optimal investment strategy, reinsurance strategy and the minimum drawdown probability in each case. By some interesting analytic technique, we come to a conclusion that the insurer (i) prefers to borrow money to invest in the risky asset when the current wealth is lower than the borrowing level; (ii) would rather borrow money and maintain the maximum leverage ratio $1 + k$ when the wealth level is less than the high leverage level; (iii) tends to invest money in the risk-free asset when the wealth level is higher than the saving level; (iv) invests all wealth in the risky asset if the wealth level is between the borrowing level and the saving level. Moreover, we see that the higher borrowing rate prompts insurer to be more conservative and the retention level is no more than that without the borrowing costs in the full-investment region. We also find that
risky investment makes the optimal reinsurance strategies become less than those without risky investment due to the risk brought by the uncertain market price of the risky asset.

Comparing with the existing literature, the main contributions of this paper are as follows: (i) we consider an optimal investment and reinsurance problem under the constraints of higher borrowing rate and limited leverage rate; (ii) a more general criterion, that is, minimizing the probability of drawdown is investigated in our paper, which covers the minimizing the probability of ruin as a special case; (iii) the insurance company has two kinds of claims, and the two claim number processes are correlated through a common shock component, which makes the optimization problem more challenging and practical; (iv) the impacts of higher borrowing rate and risky investment on the investment/reinsurance strategies are studied and quantified in our paper, respectively.

The remainder of this paper is organized as follows. In Section 2, the model and the optimization problem are presented. With the higher borrowing rate and limited leverage ratio, explicit expressions for the optimal strategies and the corresponding minimum probability of drawdown are obtained in Sections 3. In Section 4, we present some comparisons and numerical examples to show the impact of borrowing costs and risky investment on the optimal strategies. Finally, we conclude the paper in Section 5.

2. Model and problem formulation. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, with filtration $\mathcal{F} := \{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions, where $\mathbb{P}$ is the real-world probability measure and $\mathcal{F}_t$ stands for the information available until time $t$. Assume that all Brownian motions introduced below are well-defined and adapted processes defined on the complete probability space. The market is supposed to be a perfect financial market, which is frictionless, complete and continuously open.

2.1. The financial market. We assume that the insurer is allowed to invest in a risky asset (stock), whose price process evolves as

$$dS_1(t) = S_1(t) \left( \mu dt + \sigma dB^*(t) \right),$$

where $\mu$ is the return of the risky asset, $\sigma$ is the volatility of the risky asset and $B^*(t)$ is a standard Brownian motion.

In this paper, we suppose that the insurer can invest a non-negative amount in a risk-free asset that earns interest at the constant rate $r$. If the insurer borrows the money, then it pays interest at a higher rate $R > r$. Thus, the function $\psi$ describing the dynamics of risk-free asset is given by

$$\psi(S_0(t)) = \begin{cases} rS_0(t), & \text{if } S_0(t) \geq 0, \\ RS_0(t), & \text{if } S_0(t) < 0. \end{cases}$$

Here, we assume that $\mu > R > r > 0$. The case of $R > \mu > r > 0$ can be reduced to the special case with no-borrowing. Please see our discussion in Remark 7 for the latter case.

2.2. The insurance market. Suppose that there is an insurance company, whose surplus process $R(t)$ is modelled by

$$\begin{align*}
R(t) &= R_0 + ct - C_1(t) - C_2(t), \\
R(0) &= R_0 > 0,
\end{align*}$$
where $c$ is the constant rate of the premium received by the insurance company and the aggregate processes for the two dependent classes of insurance claims are given by:

$$C_1(t) = \sum_{i=1}^{N_1(t)+N(t)} Y_{1i} \quad \text{and} \quad C_2(t) = \sum_{i=1}^{N_2(t)+N(t)} Y_{2i},$$

where $C_1(t)$ and $C_2(t)$ are two compound Poisson processes defined on $(\Omega, \mathcal{F}, \mathbb{P})$. $N_1(t)$, $N_2(t)$ and $N(t)$ are three mutually independent and homogeneous Poisson processes with intensities $\lambda_1$, $\lambda_2$ and $\lambda$, respectively. The claim’s sizes $\{Y_{1i}\}_{i \in \mathbb{N}^+} (\{Y_{2i}\}_{i \in \mathbb{N}^+})$ are independent of $N_1(t)$, $N_2(t)$ and $N(t)$; they are independent and identically distributed positive random variables with a continuous distribution function $F_{Y_1}$ ($F_{Y_2}$) and finite first and second moments $E(Y_1)$ ($E(Y_2)$) and $E(Y_1^2)$ ($E(Y_2^2)$).

Here, we allow the insurance company to continuously reinsure a fraction of its claim with the retention levels $q_i(t)$ ($i = 1, 2$) for two classes of claims, respectively. Then, the corresponding surplus process for the insurer becomes:

$$dR(t) = (c - \delta(q_1, q_2)) \, dt - q_1 dC_1(t) - q_2 dC_2(t), \quad (2.1)$$

where $\delta(q_1, q_2)$ is the reinsurance premium rate at time $t$. With expected value principle, $c = (1 + \theta_1)a_1 + (1 + \theta_2)a_2$, and $\delta(q_1, q_2) = (1 + \eta_1)(1 - q_1)a_1 + (1 + \eta_2)(1 - q_2)a_2$, in which $a_i = (\lambda_i + \lambda)E(Y_i)$ and $b_i^2 = (\lambda_i + \lambda)E(Y_i^2)$ ($i = 1, 2$); $\theta_i$ and $\eta_i$ ($i = 1, 2$) are the insurer’s and reinsurer’s safety loadings of the two classes of the insurance businesses, respectively. Without loss of generality, we assume that $\eta_i > \theta_i$ holds at least for one $i$ ($i = 1, 2$), otherwise the problem becomes trivial. Because the insurer can transfer all the risk to the reinsurer without extra cost and drawdown will never happen.

To get explicit solutions, similar to Yuen et al. [29], Liang and Yuen [23], Deng et al. [10] and Han et al [17], based on the theory of Grandell [13, 14], we deal with the optimization problem of study by using the diffusion approximation of the jump process in (2.1). That is,

$$C_i(t) \approx a_i t - b_i B_i(t), \quad i = 1, 2.$$  

Here, $B_1(t)$ and $B_2(t)$ are standard Brownian motions with the correlation coefficient

$$\rho = \frac{\lambda E(Y_1) E(Y_2)}{\sqrt{(\lambda_1 + \lambda)E(Y_1^2)(\lambda_2 + \lambda)E(Y_2^2)}}.$$

Thus, $E[B_1(t)B_2(t)] = \rho t$ and the continuous-time dynamics of the surplus process for the insurer is finally formulated as

$$\begin{cases}
    dR(t) = [a_1(\theta_1 - \eta_1 + \eta_1 q_1) + a_2(\theta_2 - \eta_2 + \eta_2 q_2)]dt + b_1 q_1 dB_1(t) + b_2 q_2 dB_2(t), \\
    R(0) = R_0 > 0,
\end{cases}$$

or equivalently,

$$\begin{cases}
    dR(t) = [a_1(\theta_1 - \eta_1 + \eta_1 q_1) + a_2(\theta_2 - \eta_2 + \eta_2 q_2)]dt \\
    \quad + \sqrt{b_1^2 q_1^2 + b_2^2 q_2^2 + 2b_1 b_2 q_1 q_2 \rho} dB(t), \\
    R(0) = R_0 > 0,
\end{cases} \quad (2.2)$$

where $B(t)$ is a standard Brownian motion.
Remark 1. Note that even though we are using the diffusion approximation of the compound Poisson model, one can consider the random claim severities \( Y_i \) \((i = 1, 2)\) as existing in reality. In other words, we use the diffusion approximation to obtain the optimal retention strategy for that model, but an insurer could apply the obtained proportional reinsurance strategies to claims within the original compound Poisson model. See Liang and Young [21] for numerical comparisons of the optimal retention strategies under the diffusion and compound Poisson models. They showed that the optimal strategies are close under the two models when the surplus is not too small, even when \( \lambda \) is small. Besides, Liang et al. [20] investigated the connection between the results of compound Poisson model and its diffusion model. They derived that, under an appropriate scaling of the classical risk model, the minimum probability of ruin converges to the minimum probability of ruin under the diffusion approximation.

2.3. The wealth process and drawdown model. We denote the fraction of the wealth invested in the risky asset by \( \pi \) and the fraction \((1 - \pi)\) is then invested in the risk-free asset. We can see that \( \pi < 1 \) means that saving is existent; if \( \pi > 1 \), it means that the insurance company has to borrow money from the market and invest all the wealth in the risky asset. The wealth process with admissible investment and reinsurance strategy satisfies the stochastic differential equation:

\[
\begin{cases}
\quad dX(t) = \left[ \mu \pi X(t) + \psi((1 - \pi)X(t)) + a_1(\theta_1 - \eta_1 + \eta_1 q_1) + a_2(\theta_2 - \eta_2 + \eta_2 q_2) \right] dt \\
\quad + \sqrt{b_1^2 q_1^2 + b_2^2 q_2^2 + 2b_1 b_2 q_1 q_2 \rho dB(t) + \pi \sigma X(t) dB^*(t)} \\
\quad X(0) = x,
\end{cases}
\]

(2.3)

where \( B(t) \) and \( B^*(t) \) are two standard and independent Brownian motions. Define the maximum wealth value \( M(t) \) at time \( t \) by

\[
M(t) := \max \left\{ \sup_{0 \leq s \leq t} X(s), \ M(0) \right\},
\]

where \( M(0) = m > 0 \). Note that we set \( x > m \) to allow the wealth process to have a financial past. When the value of the wealth process reaches \( \alpha \in [0, 1) \) times its maximum value, drawdown occurs. Define the corresponding hitting time by

\[
\tau_\alpha := \inf \{ t \geq 0 : X(t) \leq \alpha M(t) \}.
\]

We can see that, if \( \alpha = 0 \), then we are in the case of minimizing the probability of ruin for the fixed ruin level 0. Besides, if the value of the wealth is large enough, say, at least

\[
a_1(\eta_1 - \theta_1) + a_2(\eta_2 - \theta_2) := x^*,
\]

(2.4)

then the insurer can transfer all the risk and invest all the wealth in the risk-free asset, drawdown will never happen in this case. For this reason, we call \( x^* \) safe level as defined in Angoshtari et al. [1].

In this paper, the insurer can borrow money from the market at the borrowing rate \( R \), and the borrowing amount should not be limitless. That is, the amount borrowed by the insurer should be no more than a fixed proportion \( k \) of the current wealth, where \( k \) is a non-negative constant. Besides, to make the paper concise, we relax the usual constraint of reinsurance strategies from \([0, 1]\) to \([0, +\infty)\), and assume that the insurer can acquire new business (by acting as a reinsurer for other
insurers, see for example, Bäuerle [3]) to manage his/her insurance business risk.
To sum up, the admissible set of \((\pi, q_1, q_2)\) is given in the following definition.

**Definition 2.1.** A strategy \((\pi(\cdot), q_1(\cdot), q_2(\cdot))\) is said to be admissible if the following conditions are satisfied.

(i) \((\pi(\cdot), q_1(\cdot), q_2(\cdot))\) is \(\mathcal{F}_t\)-progressively measurable, and satisfies \(\int_0^t \pi(s)^2 ds < \infty\) and \(\int_0^t q_i(s)^2 ds < \infty\) for \(i = 1, 2\);

(ii) \(q_2(\cdot) \in [0, +\infty)\) for \(i = 1, 2\);

(iii) \(\pi(\cdot) \in (0, 1 + k]\);

(iv) the equation (2.3) for \(X(t)\) has a unique solution.

The set of all admissible strategies is denoted by \(\mathcal{D}\).

Now assume that the insurer is interested in minimizing the probability of drawdown. We denote the minimum probability of drawdown by \(\phi\), which depends on the initial wealth \(D\) and the maximum (past) value \(m\). Specifically, \(\phi\) is the minimum probability of \(\tau_\alpha < \infty\), thus, we derive the objective function

\[
J^{\pi, q_1, q_2}(x, m) = P^{x, m}(\tau_\alpha < \infty) = E^{x, m}(1_{\{\tau_\alpha < \infty\}}).
\]

Here, \(P^{x, m}\) and \(E^{x, m}\) denote the probability and expectation, respectively, conditional on \(X(0) = x\) and \(M(0) = m\). Then, the corresponding value function is given by

\[
\phi(x, m) = \inf_{\pi, q_1, q_2 \in \mathcal{D}} J^{\pi, q_1, q_2}(x, m).
\]

Before defining the HJB equation for the characterization of the value function \(\phi\) and corresponding optimal strategy, we denote \(\mathcal{O} = \{(x, m) \in (R^+)^2 | om \leq x \leq \min(m, x^*)\}\), and

\[
C^{2,1}(\mathcal{O}) = \{\phi(x, m) | \phi(x, \cdot) \text{ is twice continuously differentiable on } \mathcal{O}, \text{ and } \phi(\cdot, m) \text{ is once continuously differentiable on } \mathcal{O}\}.
\]

It follows from the standard arguments that the \(C^{2,1}(\mathcal{O})\) value function \(\phi\) satisfies the following HJB equation:

\[
\inf_{\pi, q_1, q_2 \in \mathcal{D}} A^{\pi, q_1, q_2} \phi(x, m) = 0,
\]

where

\[
A^{\pi, q_1, q_2} \phi(x, m) = [\mu \pi x + \psi((1 - \pi)x) + a_1(\theta_1 - \eta_1 + \eta_1 q_1) + a_2(\theta_2 - \eta_2 + \eta_2 q_2)] \phi_x
+ \frac{1}{2} (q_1^2 b_1^2 + q_2^2 b_2^2 + 2 \rho b_1 b_2 q_1 q_2 + \pi^2 \sigma^2 x^2) \phi_{xx}.
\]

Applying the methods of Chen et al. [8], Angoshtari et al. [1], and Han et al. [17], we can obtain the following verification theorem directly, thus we omit the proof here.

**Theorem 2.2.** (Verification Theorem) Suppose that \(h : \mathcal{O} \to R\) is a bounded, continuous function, which satisfies the following conditions:

(i) \(h(x, \cdot) \in C^2(om, \min(m, x^*))\) is a decreasing convex function;

(ii) \(h(\cdot, m)\) is continuously differentiable, except possibly at \(x^*\);

(iii) \(h_m(m, m) \geq 0\) if \(m < x^*\);

(iv) \(h(om, m) = 1\).
(v) $h(x^*, m) = 0$ if $m \geq x^*$;
(vi) $A^\pi,q_1,q_2 \phi f \geq 0$ for all $\pi, q_1, q_2 \in D$.

Then, $h(x, m) \leq \phi(x, m)$ on $\cO$. Furthermore, suppose that the function $h$ satisfies the conditions mentioned above in such a way that conditions (iii) and (vi) hold with equality for some admissible strategy $(\pi^*, q_1^*, q_2^*)$, which is defined in feedback form $(\pi^*(X(t)), q_1^*(X(t)), q_2^*(X(t)))$. Then, we have $h(x, m) = \phi(x, m)$ on $\cO$, and $(\pi^*(X(t)), q_1^*(X(t)), q_2^*(X(t)))$ is the optimal investment-reinsurance strategy.

3. The main results. To solve the problem, we split it into two different cases: $m \geq x^*$ and $m < x^*$. Recall the safe level $x^*$ defined in (2.4), when $m \geq x^*$, the wealth level will reach $x^*$ before $m$ and $M(t) = m$ holds almost surely for all $t \geq 0$, which leads to a fixed drawdown level $\alpha m$. However, when $m < x^*$, the maximum process $M(t)$ can increase above $m$, i.e., the drawdown level we set is not necessarily a fixed one.

3.1. Minimizing the probability of drawdown when $m \geq x^*$. We start with the case of $m \geq x^*$. Note that when $\alpha m > x^*$, the probability of drawdown $\phi(x, m) = 1$ for all $x \leq \alpha m$, which means that the drawdown appears at the beginning; when $x > \alpha m$, we have $x > x^*$, then the value of the wealth is an increasing process and never hits the boundary $\alpha m$. Thus, the probability of drawdown $\phi(x, m) = 0$. Therefore, in the following context, we shall focus on the optimization problem for the case of $\alpha m \leq x^*$ in detail.

When $X(0) = x > x^*$, the drawdown is impossible; and when $X(0) = x < \alpha m$, the drawdown has occurred. Therefore, we assume that $X(0) = x \in [\alpha m, x^*]$. $X(0) = x \leq x^*$ implies that either $X(t) < x^*$ almost surely, for all $t \geq 0$, or $X(t) = x^*$ for some $t > 0$. Since $m \geq x^*$, $M(t) = m$ holds almost surely for all $t \geq 0$. Therefore, avoiding drawdown is equivalent to avoiding ruin with a fixed ruin level $\alpha m$ when $m \geq x^*$.

For convenience, we denote

\[
\begin{align*}
\hat{f}(\pi(x), q_1(x), q_2(x)) & = [a_1(\theta_1 - \eta_1) + a_2(\theta_2 - \eta_2)]h_x + \{[\mu \pi x + \psi(1 - \pi)x]h_x + \frac{1}{2} \pi^2 \sigma^2 x^2 h_{xx}\} \tag{3.1} \\
& + \{(a_1 \eta_1 q_1 + a_2 \eta_2 q_2)h_x + \frac{1}{2}(b_1^2 q_1^2 + b_2^2 q_2^2 + 2 \rho b_1 b_2 q_1 q_2)h_{xx}\},
\end{align*}
\]

and

\[
\begin{align*}
A & = a_1(\theta_1 - \eta_1) + a_2(\theta_2 - \eta_2) < 0, \\
B & = b_1^2 a_2^2 \eta_2^2 + b_2^2 a_1^2 \eta_1^2 - 2 b_1 b_2 a_1 a_2 \eta_1 \eta_2 \rho > 0. \tag{3.2}
\end{align*}
\]

We will use the notation (3.2) throughout this paper to simplify the expression. The Hessian matrix of $\hat{f}$ is

\[
\begin{pmatrix}
\sigma^2 x^2 h_{xx} & 0 & 0 \\
0 & b_1^2 h_{xx} & \rho b_1 b_2 h_{xx} \\
0 & \rho b_1 b_2 h_{xx} & b_2^2 h_{xx}
\end{pmatrix},
\]

which is positive definite. Then, the $\hat{f}$ is a convex function with respect to $\pi, q_1$ and $q_2$. According to the first-order optimality conditions, the minimizer of the function $\hat{f}(\pi(x), q_1(x), q_2(x))$ is obtained at
we need to discuss the following three cases:

**Case 1:**
If Theorem 2.2 (i) holds, we must have \( \frac{h_x}{h_{xx}} < 0 \). It immediately follows that
\[
\hat{\pi}^r(x) > \hat{\pi}^R(x) > 0.
\]

Because of the constraints of \((q^*_1, q^*_2)\) \(\in D\) and the result
\[
\frac{(a_2 b_1 / a_1 \rho b_2) \eta_2}{(a_2 \rho b_1 / a_1 b_2) \eta_2} = \frac{1}{\rho^2} > 1,
\]
we need to discuss the following three cases:

\[
\begin{aligned}
\text{Case 1: } & \frac{a_2 \rho b_1}{a_1 b_2} \eta_2 < \eta_1 < \frac{a_2 b_1}{a_1 \rho b_2} \eta_2 & (i.e., \hat{q}_1(x) > 0, \hat{q}_2(x) > 0), \\
\text{Case 2: } & \eta_1 \leq \frac{a_2 \rho b_1}{a_1 b_2} \eta_2 & (i.e., \hat{q}_1(x) \leq 0, \hat{q}_2(x) > 0), \\
\text{Case 3: } & \eta_1 \geq \frac{a_2 b_1}{a_1 \rho b_2} \eta_2 & (i.e., \hat{q}_1(x) > 0, \hat{q}_2(x) \leq 0).
\end{aligned}
\]

In each case, we will think about the higher borrowing rate and limited leverage ratio constraints at the same time. Since the proofs in Case 3 are similar to those in Case 2, in the following context, we only need to present the analysis of Case 1 and Case 2 in detail.

**Case 1:** \( \frac{a_2 \rho b_1}{a_1 b_2} \eta_2 < \eta_1 < \frac{a_2 b_1}{a_1 \rho b_2} \eta_2 \).

In this case, we obtain that \( \hat{q}_1(x) > 0 \) and \( \hat{q}_2(x) > 0 \). So it is easy to see that
\( q^*_1(x) = \hat{q}_1(x) \) and \( q^*_2(x) = \hat{q}_2(x) \). Therefore, when \( m \geq x^* \), the whole region is defined as
\[
\Gamma := \{(x, m) | \hat{q}_1(x) > 0, \hat{q}_2(x) > 0\} = [\alpha m, x^*] \times [x^*, +\infty).
\]

Now, based on the value of investment strategy, we define four subregions as follows
\[
\begin{aligned}
\Gamma^{(1)}_1 := & \{(x, m) \in \Gamma | \hat{\pi}^r(x) < 1\}, \\
\Gamma^{(2)}_1 := & \{(x, m) \in \Gamma | \hat{\pi}^R(x) \geq 1 + k\}, \\
\Gamma^{(1)}_2 := & \{(x, m) \in \Gamma | 1 < \hat{\pi}^R(x) < 1 + k\}, \\
\Gamma^{(3)}_3 := & \{(x, m) \in \Gamma | \hat{\pi}^R(x) \leq 1 \leq \hat{\pi}^r(x)\}.
\end{aligned}
\]

**Remark 2.** We call \( \Gamma^{(1)}_1 \) saving region, \( \Gamma^{(2)}_1 \) high leverage borrowing region, \( \Gamma^{(1)}_2 \) borrowing region and \( \Gamma^{(3)}_3 \) full-investment region.

We can check that
\[
\Gamma = \Gamma^{(1)}_1 \cup \Gamma^{(2)}_2 \cup \Gamma^{(1)}_3 \cup \Gamma^{(3)}_3.
\]

From equation (2.6), it is obvious that for any \((x, m) \in \Gamma\), \( A^{\pi_{q_i}:q_i} h(x, m) \) is a continuous function w.r.t. \( \pi \) and \( q_i \) \((i = 1, 2)\) and uniformly in \( q_i \) \((i = 1, 2)\). Thus,
for any \((x, m) \in \Gamma\), there always exist the minimum points \(\pi^*(x)\) and \(q_i^*(x)\) \((i = 1, 2)\) with \(\pi^*(x) \in (0, 1 + k]\) and \(q_i^*(x) \in [0, +\infty)\) \((i = 1, 2)\) such that
\[
\mathcal{A}_{\pi^*, q_1^*, q_2^*} h(x, m) = \inf_{\pi, q_1, q_2 \in D} \mathcal{A}_{\pi, q_1, q_2} h(x, m).
\]

If minimizers \(\pi^*(x)\) and \(q_i^*(x)\) \((i = 1, 2)\) are determined on each region, by considering the following boundary-value problem:
\[
\begin{align*}
    h_x h_{xx} &= f(x), \\
    h(x^*, m) &= 0, h(\alpha m, m) = 1,
\end{align*}
\]
for \(\alpha m \leq x \leq x^* \leq m\), we can then construct the solution which is actually the minimum probability of drawdown according to Theorem 2.2.

There are different functions \(f(x)\) for different subregions. Hence, we give several Lemmas in the following, which will be applied to construct the solution of the HJB equation.

**Lemma 3.1.** If \(h(x, m)\) is a twice continuously differentiable decreasing convex function, which solves the HJB equation (2.5) in \(\Gamma_1^{(1)}\), then we have
\[
\frac{h_{xx}}{h_x} = f_1^r(x),
\]
and the optimal strategies in \(\Gamma_1^{(1)}\) are
\[
\begin{align*}
    \pi^*(x) &= \hat{\pi}^r(x) = -\frac{\mu - r}{x \sigma^2 f_1^r(x)}, \\
    q_1^*(x) &= \hat{q}_1^r(x) = \frac{pb_1 b_2 a_2 \eta_2 - b_2^2 a_1 \eta_1}{b_1^2 b_2^2 (1 - \rho^2) f_1^r(x)}, \\
    q_2^*(x) &= \hat{q}_2^r(x) = \frac{pb_1 b_2 a_1 \eta_1 - b_2^2 a_2 \eta_2}{b_1^2 b_2^2 (1 - \rho^2) f_1^r(x)},
\end{align*}
\]
in which
\[
f_1^r(x) = \frac{b_2^2 (1 - \rho^2)(\mu - r)^2}{2(xA + B)b_1^2 b_2^2 (1 - \rho^2)}.\]

**Proof.** Note that \(\pi^*(x) = \hat{\pi}^r(x), q_1^*(x) = \hat{q}_1^r(x)\) and \(q_2^*(x) = \hat{q}_2^r(x)\) for any \((x, m) \in \Gamma_1^{(1)}\). Substituting \(\hat{\pi}^r, \hat{q}_1^r\) and \(\hat{q}_2^r\) which are shown in (3.3) into the HJB equation (2.5), leads to the equation (3.5). Replacing \(\frac{h_x}{h_{xx}}\) in (3.3) with \(f_1^r(x)\) is easy to obtain the optimal strategies (3.6), which completes the proof.

Similarly, we can get following Lemmas.

**Lemma 3.2.** If \(h(x, m)\) is a twice continuously differentiable decreasing convex function, which solves the HJB equation (2.5) in \(\Gamma_2^{(1)}\), then we have
\[
\frac{h_{xx}}{h_x} = f_2^l(x),
\]
and the optimal strategies in \(\Gamma_2^{(1)}\) are
Proof. For any \((x, m) \in \Gamma_{21}^{(1)}\), we have \(\pi^*(x) = 1 + k\), \(q^*_1(x) = \hat{q}_1(x)\) and \(q^*_2(x) = \hat{q}_2(x)\). Following the same steps as in the proof of Lemma 3.1, we can get the results.

Lemma 3.3. If \(h(x, m)\) is a twice continuously differentiable decreasing convex function, which solves the HJB equation (2.5) in \(\Gamma_{22}^{(1)}\), then we have

\[
\frac{h_{xx}}{h_x} = f_{12}^R(x),
\]

and the optimal strategies in \(\Gamma_{22}^{(1)}\) are

\[
\begin{align*}
\pi^*(x) &= \hat{\pi}^R(x) = -\frac{\mu - R}{x\sigma^2 f_{12}^R(x)}, \\
q_1^*(x) &= \hat{q}_{12}^R(x) = \frac{\rho b_1 b_2 a_2 \eta_2 - b_2^2 a_1 \eta_1}{b_1^2 b_2^2 (1 - \rho^2) f_{12}^R(x)}, \\
q_2^*(x) &= \hat{q}_{22}^R(x) = \frac{\rho b_1 b_2 a_1 \eta_1 - b_1^2 a_2 \eta_2}{b_1^2 b_2^2 (1 - \rho^2) f_{12}^R(x)},
\end{align*}
\]

in which

\[
f_{12}^R(x) = \frac{\frac{b_1^2 b_2^2 (1 - \rho^2)(\mu - R)^2}{\sigma^4} + B}{2(Rx + A)b_1^2 b_2^2 (1 - \rho^2)}. \tag{3.13}
\]

Proof. For any \((x, m) \in \Gamma_{22}^{(1)}\), we can see that \(\pi^*(x) = \hat{\pi}^R(x)\), \(q_1^*(x) = \hat{q}_1(x)\) and \(q_2^*(x) = \hat{q}_2(x)\). Again, the results can be derived along the same way as in the proof of Lemma 3.1.

Lemma 3.4. If \(h(x, m)\) is a twice continuously differentiable decreasing convex function, which solves the HJB equation (2.5) in \(\Gamma_3^{(1)}\), then we have

\[
\frac{h_{xx}}{h_x} = f_1(x), \tag{3.14}
\]

and the optimal strategies in \(\Gamma_3^{(1)}\) are

\[
\begin{align*}
\pi^*(x) &= 1, \\
q_1^*(x) &= \hat{q}_1(x) = \frac{\rho b_1 b_2 a_2 \eta_2 - b_2^2 a_1 \eta_1}{b_1^2 b_2^2 (1 - \rho^2) f_1(x)}, \\
q_2^*(x) &= \hat{q}_2(x) = \frac{\rho b_1 b_2 a_1 \eta_1 - b_1^2 a_2 \eta_2}{b_1^2 b_2^2 (1 - \rho^2) f_1(x)},
\end{align*} \tag{3.15}
\]
in which
\[
f_1(x) = -\frac{\mu x + A}{\sigma^2 x^2} - \frac{\sqrt{(\mu x + A)^2 + \frac{\sigma^2 x^2}{\beta_1^2(1 - \rho^2)}} B}{\sigma^2 x^2},
\]  
(3.16)

Proof. For any \((x, m) \in \Gamma_3^{(1)}\), it is obvious that \(q_1^*(x) = \hat{q}_1(x)\) and \(q_2^*(x) = \hat{q}_2(x)\).

Borrowing the idea in Lemma 3.5 of Luo [24], we can prove that \(\pi^* = 1\). Plugging \(\hat{q}_1(x)\) and \(\hat{q}_1(x)\) into the HJB equation implies
\[
\inf_{\pi \in \{0, 1+k\}} A^{\pi, \hat{q}_1, \hat{q}_2} h(x, m) = 0.
\]

If the minimum on the left-hand side of the HJB equation is attained at the minimizer \(\pi^*(x) > 1\). Then, we can see
\[
\frac{dA^{\pi, \hat{q}_1, \hat{q}_2} h(x, m)}{d\pi} \bigg|_{\pi = \pi^*(x)} = 0,
\]
and \(\pi^*(x) = \hat{\pi}^R(x)\) and thus \(\hat{\pi}^R(x) > 1\), which contradicts to the definition of \(\Gamma_3^{(1)}\).

We can prove that \(\pi^*(x) < 1\) is impossible using the same method. Hence, the minimizer of the left-hand side of the equation is obtained at the point of 1. Then, the results in Lemma 3.4 can be derived by the same analysis in Lemma 3.1. \(\square\)

To construct the solution of the HJB equation, we need to locate the four sub-regions and find the corresponding expressions for the solution firstly, and then we can determine the parameters by smoothness and boundary conditions.

For any \((x, m)\) in region \(\Gamma_1^{(1)}\), we have
\[
\pi^*(x) = \hat{\pi}^r(x) = -\frac{\mu - r}{x \sigma^2 f_1^r(x)}
\]
\[
= -\frac{\mu - r}{x \sigma^2} \frac{2(rx + A)b_1^2 b_2^2 (1 - \rho^2)}{b_1^2 b_2^2 (1 - \rho^2) (\mu - r)^2 + B},
\]
and \(\hat{\pi}^r(x) < 1\) gives \(x > x_1^r\) with
\[
x_1^r = \frac{-2(\mu - r)Ab_1^2 b_2^2 (1 - \rho^2)}{b_1^2 b_2^2 (1 - \rho^2) (\mu - r)^2 + B \sigma^2}. \quad (3.17)
\]

Compared with \(x^*\), it is not difficult to see that \(0 < x_1^r < x^*\) and
\[
\Gamma_1^{(1)} := \left( (x_1^r, x^*) \times [x^*, +\infty) \right) \cap \Gamma.
\]

Due to the definition of the region \(\Gamma_1^{(1)}\), the \(x_1^r\) is denoted as the saving level in Case 1. Because once the wealth level is above \(x_1^r\), the insurer will choose to invest in the risk-free asset.

In region \(\Gamma_2^{(1)}\), we have
\[
\pi^*(x) = \hat{\pi}^R(x) = -\frac{\mu - R}{x \sigma^2 f_1^R(x)}
\]
\[
= -\frac{2(\mu - R)(Rx + A)b_1^2 b_2^2 (1 - \rho^2)}{b_1^2 b_2^2 (1 - \rho^2) (\mu - R)^2 x + B \sigma^2 x},
\]
and \(1 < \hat{\pi}^R(x) < 1 + k\) yields \(x_{12}^R < x < x_{11}^R\), where
We define the function \( x_{i1}^R \) and \( x_{i2}^R \) as the high leverage level and borrowing level, respectively. It means that the insurer chooses to borrow money when the wealth level is no more than \( x \) and the leverage ratio has the maximum value \( 1 + k \). Once the wealth level falls below \( x_{i1}^R \), the leverage ratio maintains the high level \( 1 + k \).

It is plain to see that \( 0 < x_{i1}^R < x_{i2}^R < x^* \) and

\[
\begin{align*}
\Gamma_1^{(1)} &= \left((\alpha m, x_{i1}^R] \times [x^*, +\infty)\right) \cap \Gamma, \\
\Gamma_2^{(1)} &= \left((x_{i1}^R, x_{i2}^R] \times [x^*, +\infty)\right) \cap \Gamma.
\end{align*}
\]

To locate the region \( \Gamma_3^{(1)} \), the relationship between \( x_{i1}^R \) and \( x_{i2}^R \) needs to be determined first, which is shown in the following Lemma.

**Lemma 3.5.** Under the assumption that \( 0 < r < R < \mu \), we have \( x_{i2}^R < x_1^r \).

**Proof.** We define the function \( \kappa_1(p) \) as

\[
\kappa_1(p) := \frac{2(\mu - p)}{b_1^2 b_2^2 (\mu^2 - \rho^2) + B \sigma^2}, \quad p \in (0, \mu).
\]

Differentiating \( \kappa_1(p) \) with respect to \( p \) yields

\[
\kappa_1'(p) = \frac{-2b_1^2 b_2^2 (\mu^2 - \rho^2)(\mu - p)^2 + B \sigma^2}{b_1^2 b_2^2 (\mu^2 - \rho^2) + B \sigma^2},
\]

for \( 0 < p < \mu \), we have \( \kappa_1'(p) < 0 \), and thus \( \kappa_1(p) \) is a strictly decreasing function with respect to \( p \). Then, we can derive that

\[
\kappa_1(R) < \kappa_1(r),
\]

and

\[
x_{i2}^R = -2Ab_1^2 b_2^2 (1 - \rho^2)\kappa_1(R) < -2Ab_1^2 b_2^2 (1 - \rho^2)\kappa_1(r) = x_1^r,
\]

which completes the proof. \( \square \)

For any \((x, m)\) in region \( \Gamma_3^{(1)} \), we have \( \pi^* = 1 \), \( q_1^*(x) = q_1(x) \) and \( q_2^*(x) = q_2(x) \), which lead to \( \Gamma_3^{(1)} := \left((x_{i2}^R, x_1^r) \times [x^*, +\infty)\right) \cap \Gamma. \)

Based on the results in Lemmas 3.1-3.4, we summarize the optimal strategies and the corresponding minimum probability of drawdown for the case of \( \frac{\alpha \rho}{\alpha_1 b_2} \eta_1 < \eta_1 < \frac{\alpha \rho}{\alpha_1 b_2} \eta_2 \) in Theorem 3.6.

**Theorem 3.6.** Suppose that \( \frac{\alpha \rho}{\alpha_1 b_2} \eta_1 < \eta_1 < \frac{\alpha \rho}{\alpha_1 b_2} \eta_2 \). Parameters \( C_{i4} \) \((i = 1, 2, \ldots, 8)\) are given in Appendix A, \( f_{1}^x(v) \), \( f_{1}^{R}(v) \), \( f_{1}^{R}(v) \) and \( f_{1}(v) \) are given by (3.7), (3.10), (3.13) and (3.16). \( \hat{R}^*, \hat{\pi}^*, \hat{q}_{i1}^*, \hat{q}_{i2}^R \) \((i = 1, 2)\), \( \hat{q}_1^*, \hat{q}_2^*, \hat{q}_1^r \) and \( \hat{q}_2^r \) are expressed as in (3.6), (3.9), (3.12) and (3.15). Let \( x_{i1}^R, x_{i2}^R \) and \( x_1^r \) be given by (3.17) and (3.18), respectively. For any \( x \in [\alpha m, x^*] \) and \( m \geq x^r \), the minimum probability of
drawdown for the wealth process (2.3) is
\[ \phi(x, m) = \begin{cases} C_{11} + C_{12} \int_0^x \exp \left\{ \int_0^u f_1(v) \, dv \right\} \, du, & \alpha m \leq x < \alpha m \lor x_{11}^R, \\ C_{13} + C_{14} \int_{x_{11}^R \lor \alpha m}^x \exp \left\{ \int_{x_{11}^R \lor \alpha m}^u f_2(v) \, dv \right\} \, du, & \alpha m \lor x_{11}^R \leq x < \alpha m \lor x_{12}^R, \\ C_{15} + C_{16} \int_{x_{12}^R \lor \alpha m}^x \exp \left\{ \int_{x_{12}^R \lor \alpha m}^u f_1(v) \, dv \right\} \, du, & \alpha m \lor x_{12}^R \leq x < \alpha m \lor x_{11}^R, \\ C_{17} + C_{18} \int_{x_{11}^R \lor \alpha m}^x \exp \left\{ \int_{x_{11}^R \lor \alpha m}^u f_1(v) \, dv \right\} \, du, & \alpha m \lor x_{11}^R \leq x \leq x^*. \end{cases} \]

The optimal investment and reinsurance strategies are
\[ (\pi^*, q_1^*, q_2^*) = \begin{cases} (1 + k, \hat{q}_{11}(x), \hat{q}_{21}(x)), & \alpha m \leq x < \alpha m \lor x_{11}^R, \\ (\hat{\pi}^R(x), \hat{q}_{12}(x), \hat{q}_{22}(x)), & \alpha m \lor x_{11}^R \leq x < \alpha m \lor x_{12}^R, \\ (1, \hat{q}_1(x), \hat{q}_2(x)), & \alpha m \lor x_{12}^R \leq x < \alpha m \lor x_{11}^R, \\ (\hat{\pi}^R(x), \hat{q}_1^*(x), \hat{q}_2^*(x)), & \alpha m \lor x_{11}^R \leq x \leq x^*. \end{cases} \]

**Proof.** See Appendix C.

By the same way, we can derive the optimal results in the other two cases.

**Case 2:** \( \eta_1 \leq \frac{a_2 \rho h}{a_2 \rho h + \alpha_2 \eta_2} \eta_2. \)

In this case, we obtain that \( \hat{q}_1(x) \leq 0 \) and \( \hat{q}_2(x) > 0 \). Then, we have to choose \( q_1^*(x) = 0 \), and thus we derive
\[ q_2^*(x) = \hat{q}_2(x) = -\frac{a_2 \rho h}{b_2} \frac{h_x}{h_{xx}}. \]

For this case, we also need to construct a solution in the region
\[ \Gamma = \{(x, m) | \hat{q}_1(x) \leq 0, \hat{q}_2(x) > 0 \} = [\alpha m, x^*] \times [x^*, +\infty). \]

As shown in Case 1, in order to determine the optimal strategies, we divide the whole region into four parts:
\[ \Gamma = \Gamma_1^{(2)} \cup \Gamma_2^{(2)} \cup \Gamma_3^{(2)} \cup \Gamma_4^{(2)} \]

and we have
\[ \Gamma = \Gamma_1^{(2)} \cup \Gamma_2^{(2)} \cup \Gamma_3^{(2)} \cup \Gamma_4^{(2)}. \]

Similarly, we write the following Lemmas and Theorems without proof.

**Lemma 3.7.** If \( h(x, m) \) is a twice continuously differentiable decreasing convex function, which solves the HJB equation (2.5) in \( \Gamma_1^{(2)} \), then we have
\[ \frac{h_{xx}}{h_x} = f_2^*(x), \]
with
\[ f_2^R(x) = \frac{\frac{b^2_2(\mu-r)^2}{\sigma^2} + a^2_2\eta_2^2}{2(rx + A)b_2^2}. \] (3.21)

**Lemma 3.8.** If \( h(x, m) \) is a twice continuously differentiable decreasing convex function, which solves the HJB equation (2.5) in \( \Gamma_{21}^{(2)} \), then we have
\[ \frac{h_{xx}}{h_x} = f_{21}^R(x), \]
with
\[ f_{21}^R(x) = -\mu x + (\mu - R)kx + A \sqrt{(\mu x + (\mu - R)kx + A)^2 + \frac{a^2_2\eta_2^2}{b_2^2}(1 + k)^2\sigma^2 x^2} \div \frac{(1 + k)^2\sigma^2 x^2}{2}. \] (3.22)

**Lemma 3.9.** If \( h(x, m) \) is a twice continuously differentiable decreasing convex function, which solves the HJB equation (2.5) in \( \Gamma_{22}^{(2)} \), then we have
\[ \frac{h_{xx}}{h_x} = f_{22}^R(x), \]
with
\[ f_{22}^R(x) = \frac{\frac{b^2_2(\mu-r)^2}{\sigma^2} + a^2_2\eta_2^2}{2(Rx + A)b_2^2}. \] (3.23)

**Lemma 3.10.** If \( h(x, m) \) is a twice continuously differentiable decreasing convex function, which solves the HJB equation (2.5) in \( \Gamma_{3}^{(2)} \), then we have
\[ \frac{h_{xx}}{h_x} = f_2(x), \]
with
\[ f_2(x) = -\frac{\mu x + A}{\sigma^2 x^2} - \frac{\sqrt{(\mu x + A)^2 + \frac{a^2_2\eta_2^2}{b_2^2}\sigma^2 x^2}}{\sigma^2 x^2}. \] (3.24)

To construct the solution of the HJB equation, we again need to locate the four subregions and find the corresponding expressions for the solution. Then, we can determine the parameters by smoothness and boundary conditions.

In region \( \Gamma_{1}^{(2)} \), we have
\[ \pi^*(x) = \hat{\pi}^*(x) = -\frac{\mu - r}{x\sigma^2 f_2^R(x)} = -\frac{2(\mu - r)(rx + A)b_2^2}{b_2^2(\mu - r)^2x + a^2_2\eta_2^2\sigma^2 x^2}, \] (3.25)
and \( \hat{\pi}^*(x) < 1 \) gives \( x > x^*_2 \), where
\[ x^*_2 = \frac{-2(\mu - r)Ab_2^2}{b_2^2(\mu^2 - r^2) + a^2_2\eta_2^2\sigma^2}. \] (3.26)
It is not difficult to see that \( 0 < x^*_2 < x^* \) and
\[ \Gamma_{1}^{(2)} := ((x^*_2, x^*) \times [x^*, +\infty)) \cap \Gamma. \]
Besides, using the same way as in Lemma 3.5, we can obtain the following results.

\[ (2.3) \]

In region \( \Gamma_{22}^{(2)} \), we have

\[
\pi^*(x) = \hat{\pi}^R(x) = -\frac{\mu - R}{x\sigma^2 f_2^R(x)} = -\frac{2(\mu - R)(Rx + A)b_2^2}{b_2^2(\mu - R)^2 x + a_2^2 a_2^2 \sigma^2 x}.
\]

(3.27)

and \( 1 < \hat{\pi}^R(x) < 1 + k \) yields \( x_{21}^R < x < x_{22}^R \), where

\[
\begin{aligned}
x_{21}^R &= -\frac{2(\mu - R)Ab_2^2}{b_2^2(\mu^2 - R^2) + a_2^2 a_2^2 \sigma^2 + k(b_2^2(\mu - R)^2 + a_2^2 a_2^2 \sigma^2)}, \\
x_{22}^R &= -\frac{2(\mu - R)Ab_2^2}{b_2^2(\mu^2 - R^2) + a_2^2 a_2^2 \sigma^2}.
\end{aligned}
\]

(3.28)

We can prove that \( 0 < x_{21}^R < x_{22}^R < x^* \) and

\[
\begin{aligned}
\Gamma_{21}^{(2)} &= \left([am, x_{21}^R] \times [x^*, +\infty]\right) \cap \Gamma, \\
\Gamma_{22}^{(2)} &= \left((x_{21}^R, x_{22}^R] \times [x^*, +\infty]\right) \cap \Gamma.
\end{aligned}
\]

(3.29)

Besides, using the same way as in Lemma 3.5, we can obtain the following results.

**Lemma 3.11.** Under the assumption that \( 0 < r < R < \mu \), we have \( x_{22}^R < x_R^* \).

For any \((x, m)\) in \( \Gamma_{22}^{(2)} \), we have \( \pi^*(x) = 1, q_1^*(x) = 0 \) and \( q_2^*(x) = \hat{q}_2(x) \). Hence, from the discussion above, we must have \( \Gamma_{22}^{(2)} := \left([am, x_{22}^R] \times [x^*, +\infty]\right) \cap \Gamma \).

Based on Lemmas 3.7-3.10, we give the optimal strategies and the corresponding minimum probability of drawdown for the case of \( \eta_1 \leq \frac{a_2 b_2}{a_1 b_2} \eta_2 \) in Theorem 3.12.

**Theorem 3.12.** Suppose that \( \eta_1 \leq \frac{a_2 b_2}{a_1 b_2} \eta_2 \). Parameters \( C_{2i} \) \((i = 1, 2, \cdots, 8)\) can be found in Appendix A. \( f_2^R(v) \), \( f_2^R(v) \), \( f_2^R(v) \) and \( f_2(v) \) are given by (3.21), (3.22), (3.23) and (3.24). \( \pi^* \) and \( \pi^* \) are expressed as in (3.25) and (3.27). Let \( x_{21}^R, x_{22}^R \) and \( x_R^* \) be given by (3.26) and (3.28), respectively. For any \( x \in [am, x^*] \) and \( m \geq x^* \), the minimum probability of drawdown for the wealth process (2.3) is

\[
\phi(x, m) = \begin{cases} 
C_{21} + C_{22} \int_{am}^{x} \exp \left\{ \int_{am}^{u} f_2^R(v) dv \right\} du, & \text{ if } am \leq x < am \lor x_{21}^R, \\
C_{23} + C_{24} \int_{x_{21}^R}^{x} \exp \left\{ \int_{x_{21}^R}^{u} f_2^R(v) dv \right\} du, & \text{ if } am \lor x_{21}^R \leq x < am \lor x_{22}^R, \\
C_{25} + C_{26} \int_{x_{22}^R}^{x} \exp \left\{ \int_{x_{22}^R}^{u} f_2(v) dv \right\} du, & \text{ if } am \lor x_{22}^R \leq x < am \lor x_2^*, \\
C_{27} + C_{28} \int_{x_2^*}^{x} \exp \left\{ \int_{x_2^*}^{u} f_2(v) dv \right\} du, & \text{ if } am \lor x_2^* \leq x \leq x^*.
\end{cases}
\]

The optimal investment and reinsurance strategies are

\[
(\pi^*, q_1^*, q_2^*) = \begin{cases} 
(1 + k, 0, \hat{q}_1^R(x)), & am \leq x < am \lor x_{21}^R, \\
(\hat{\pi}^R(x), 0, \hat{q}_2^R(x)), & am \lor x_{21}^R \leq x < am \lor x_{22}^R, \\
(1, q_2(x)), & am \lor x_{22}^R \leq x < am \lor x_2^*, \\
(\hat{\pi}^*(x), 0, \hat{q}_2^*(x)), & am \lor x_2^* \leq x \leq x^*.
\end{cases}
\]
where
\[
\tilde{q}_{11}^R(x) = -\frac{a_2\eta_2}{b_2^2 R_{21}^R(x)}, \quad \tilde{q}_{22}^R(x) = -\frac{a_2\eta_2}{b_2^2 R_{22}^R(x)},
\]
\[
\tilde{q}_2(x) = -\frac{a_2\eta_2}{b_2^2 f_2(x)}, \quad \tilde{q}_3^*(x) = -\frac{a_2\eta_2}{b_2^2 f_3^*(x)}.
\]
(3.30)

Case 3: \(\eta_1 \geq \frac{a b_k}{\alpha_1 b^2} \eta_2\).

In this case, we have \(\tilde{q}_1(x) > 0\) and \(\tilde{q}_2(x) \leq 0\). Before giving the optimal results, we need some auxiliary Lemmas but omit the proofs which are similar to those in Case 2.

Lemma 3.13. The high leverage level \(x_{31}^R\), borrowing level \(x_{32}^R\) and saving level \(x_3^*\) in Case 3 are given by
\[
\begin{align*}
x_{31}^R &= \frac{-2(\mu - R)Ab_1^2}{b_1^2(\mu^2 - R^2) + a_1^2\eta_1^2 \sigma^2 + k(b_1^2(\mu - R)^2 + a_1^2\eta_1^2 \sigma^2)}, \\
x_{32}^R &= \frac{-2(\mu - R)Ab_1^2}{b_1^2(\mu^2 - R^2) + a_1^2\eta_1^2 \sigma^2}, \\
x_3^* &= \frac{-2(\mu - r)Ab_1^2}{b_1^2(\mu^2 - r^2) + a_1^2\eta_1^2 \sigma^2}.
\end{align*}
\]
(3.31)

Lemma 3.14. If \(h(x,m)\) is a twice continuously differentiable decreasing convex function, which solves the HJB equation (2.5) in \(\Gamma\), then we have
\[
\frac{h_{xx}}{h_x} = \begin{cases} 
  f_{31}^R(x), & \alpha m \leq x < \alpha m \lor x_{31}^R, \\
  f_{32}^R(x), & \alpha m \lor x_{31}^R \leq x < \alpha m \lor x_{32}^R, \\
  f_3(x), & \alpha m \lor x_{32}^R \leq x < \alpha m \lor x_3^*, \\
  f_3^*(x), & \alpha m \lor x_3^* \leq x \leq m < x^*, 
\end{cases}
\]

where
\[
\begin{align*}
f_{31}^R(x) &= -\frac{\mu x + (\mu - R)kx + A}{(1 + k)^2 \sigma^2 x^2} - \left(\frac{\sqrt{(\mu x + (\mu - R)kx + A)^2 + \frac{a_1^2\eta_1^2}{b_1^2 (1 + k)^2 \sigma^2} x^2}}{(1 + k)^2 \sigma^2 x^2}\right), \\
f_{32}^R(x) &= \frac{b_1^2 (\mu - R)^2 + a_1^2 \eta_1^2 \sigma^2}{2(\mu x + A)b_1^2 \sigma^2}, \\
f_3(x) &= -\frac{\mu x + A}{\sigma^2 x^2} - \left(\frac{\sqrt{(\mu x + A)^2 + \frac{a_1^2\eta_1^2}{b_1^2} \sigma^2 x^2}}{\sigma^2 x^2}\right), \\
f_3^*(x) &= \frac{b_1^2 (\mu - r)^2 + a_1^2 \eta_1^2 \sigma^2}{2(\mu x + A)b_1^2 \sigma^2}.
\end{align*}
\]
(3.32)

Along the same lines as in Case 2, we summarize the results in the Theorem 3.15.

Theorem 3.15. Suppose that \(\eta_1 > \frac{a b_k}{\alpha_1 b^2} \eta_2\). Parameters \(C_i\), \((i = 1, 2, \ldots, 8)\) can be found in Appendix A. \(f_{12}^R(v), f_{21}^R(v), f_{22}^R(v)\) and \(f_2(v)\) are given by (3.32). Let \(x_{31}^R, x_{32}^R\) and \(x_3^*\) be given by (3.31). For any \(x \in [\alpha m, x^*]\) and \(m \geq x^*\), the minimum probability of drawdown for the wealth process (2.3) is
\[ \phi(x, m) = \begin{cases} 
C_{31} + C_{32} \int_{\alpha m}^{x} \exp \left\{ \int_{\alpha m}^{u} f_{31}^R(v) \, dv \right\} \, du, & \alpha m \leq x < \alpha m \lor x_{31}^R, \\
C_{33} + C_{34} \int_{x_{31}^R \lor \alpha m}^{x} \exp \left\{ \int_{x_{31}^R \lor \alpha m}^{u} f_{32}^R(v) \, dv \right\} \, du, & \alpha m \lor x_{31}^R \leq x < \alpha m \lor x_{32}^R, \\
C_{35} + C_{36} \int_{x_{32}^R \lor \alpha m}^{x} \exp \left\{ \int_{x_{32}^R \lor \alpha m}^{u} f_{3}(v) \, dv \right\} \, du, & \alpha m \lor x_{32}^R \leq x < \alpha m \lor x_{3}, \\
C_{37} + C_{38} \int_{x_{3} \lor \alpha m}^{x} \exp \left\{ \int_{x_{3} \lor \alpha m}^{u} f_{3}^R(v) \, dv \right\} \, du, & \alpha m \lor x_{3} \leq x \leq x^*, 
\end{cases} \]

The optimal investment and reinsurance strategies are

\[ (\pi^*, q_1^*, q_2^*) = \begin{cases} 
(1 + k, \tilde{q}_{31}^R(x), 0), & \alpha m \leq x < \alpha m \lor x_{31}^R, \\
(\hat{\pi}^R(x), \tilde{q}_{32}^R(x), 0), & \alpha m \lor x_{31}^R \leq x < \alpha m \lor x_{32}^R, \\
(1, \tilde{q}_{3}(x), 0), & \alpha m \lor x_{32}^R \leq x < \alpha m \lor x_{3}, \\
(\hat{\pi}^*(x), \tilde{q}_{3}^2(x), 0), & \alpha m \lor x_{3} \leq x \leq x^*, 
\end{cases} \]

where

\[ \hat{\pi}^R(x) = -\frac{\mu - R}{x \sigma^2 f_{31}^R(x)}, \quad \hat{\pi}^*(x) = -\frac{\mu - r}{x \sigma^2 f_{32}^R(x)}, \quad \tilde{q}_{31}^R(x) = -\frac{a_1 \eta_1}{b_1^2 f_{31}^R(x)}, \quad \tilde{q}_{32}^R(x) = -\frac{a_1 \eta_1}{b_1^2 f_{32}^R(x)}, \quad \tilde{q}_{3}(x) = -\frac{a_1 \eta_1}{b_1^2 f_{3}^R(x)}. \]

In view of the results from Cases 1-3, we have the following Lemma stating that the borrowing level in the first case is at most as large as the levels of the second and third cases.

**Lemma 3.16.** We have \( x_{12}^R \leq x_{22}^R \) and \( x_{12}^R \leq x_{32}^R \).

**Proof.** We only give the detailed proof for \( x_{12}^R \leq x_{22}^R \). Along the same way, we can prove that \( x_{12}^R \leq x_{32}^R \). From equation (3.18), we notice that

\[ x_{12}^R = -\frac{2(\mu - R)Ab_1^2b_2^2(1 - \rho^2)}{b_1^2b_2^2(1 - \rho^2)(\mu^2 - R^2) + B\sigma^2} = -\frac{2(\mu - R)Ab_2^2}{b_2^2(\mu^2 - R^2) + \frac{B}{b_1^2(1 - \rho^2)}\sigma^2}. \]

The only difference compared with \( x_{22}^R \) is the denominator. By calculation, we can derive the following inequality

\[ \frac{B}{b_1^2(1 - \rho^2)} - a_2^2 \eta_2^2 = \frac{(b_1 a_2 \eta_2 \rho - b_2 a_1 \eta_1)^2}{b_1^2(1 - \rho^2)} \geq 0, \]

which completes the proof. \( \square \)

The comparison results in Lemma 3.16 are natural consequences. Because there always exists one kind of claim which is fully covered by the reinsurer in Cases 2 and 3, and the reinsurance premium considered in this paper is non-cheap. Therefore, the expensive reinsurance premium makes the wealth process deteriorate and promote the insurer to borrow money at a relatively high borrowing level in order to make profit in the risky investment.

So far, we have solved the optimal control problem with explicit solutions in the case of \( m \geq x^* \). Before ending the Section 3.1, a few Remarks are given to show some implications of the optimal results.
Remark 3. If we set $\alpha = 0$ in Theorems 3.6, 3.12 and 3.15, then we can derive the optimal results in case of minimizing the probability of ruin. Furthermore, when considering only one kind of claim and constraining the reinsurance strategy on the interval $[0, 1]$, we can get the same results as those in Luo [24].

Remark 4. From the results in Theorems 3.6, 3.12 and 3.15, we can see that when the wealth is below the borrowing level, the optimal strategy is heavily leveraged in the risky asset. More exactly, the insurance company tends to borrow a certain amount of money to invest in the risky asset together with all its wealth. Another interesting observation is that when the wealth level increases, the insurance company tends to have a more conservative policy, that is, to invest more in risk-free assets and buy more reinsurance.

Remark 5. Somewhat surprisingly, the model with a higher borrowing rate always converges to the one without borrowing constraints as the borrowing rate goes to the drift of the risky asset. It makes sense that with higher borrowing cost and lower return on risky investment, it is no longer optimal for insurer to borrow money to invest in the risky asset.

3.2. Minimizing the probability of drawdown when $m < x^*$. In Section 3.1, we show the minimum probability of drawdown and the corresponding optimal strategies for the case of $m \geq x^*$. In this subsection, we will consider the problem for the case of $m < x^*$.

The minimizers $\pi^*(x)$ and $q^*_i(x)$ ($i = 1, 2$) determined in each subregion are same to those in Section 3.1. Hence, by considering the following boundary-value problem with $\alpha m \leq x < 0 \leq x^*$,

$$
\begin{align*}
\frac{h_x}{h_{xx}} &= \frac{1}{f(x)}, \\
h(x^*, x^*) &= 0, \\
h_m(m, m) &= 0, \\
q^*_i(x) &= \hat{q}^*_i(x), \\
q^*_2(x) &= \hat{q}^*_2(x). \\
\end{align*}
$$

we can construct the solution which is actually the value function according to Theorem 2.2.

Compared with (3.4), the differences are that there is another boundary condition for the function $h_m(x, m)$ and the whole region $\Gamma$ becomes $\Gamma = [am, m] \times [x, x^*].$

Next, we will show the optimal strategies and the corresponding minimum probability of drawdown in each case.

Case 1: $\frac{a_2 b_1}{a_1 b_2} \eta_2 < \eta_1 < \frac{a_2 b_1}{a_1 b_2} \eta_2$. In this case, we have $\hat{q}_1(x) > 0$ and $\hat{q}_2(x) > 0$. Then $q^*_1(x) = \hat{q}_1(x)$, $q^*_2(x) = \hat{q}_2(x)$. Same to the Case 1 in Section 3.1, we obtain

$$
\begin{align*}
\frac{h_{xx}}{h_x} &= \begin{cases} f_{11}(x), & \alpha m \leq x < \alpha m \vee x^*_1, \\
f_{12}(x), & \alpha m \vee x^*_1 \leq x < am \vee x^*_1, \\
f_1(x), & am \vee x^*_1 \leq x < am \vee x^*_2, \\
\hat{f}_1(x), & am \vee x^*_2 \leq x < m \leq x^*. \\
\end{cases}
\end{align*}
$$

We can also recover the uncertain parameters by boundary and smoothness conditions. The solutions of (3.35) for four cases of $m \in [am \vee x^*_1, x^*], m \in [am \vee x^*_1, x^*_1, \alpha m \vee x^*_1], m \in [am \vee x^*_1, am \vee x^*_1]$ and $m \in [am, am \vee x^*_1]$ are presented respectively in the next Theorem.
Theorem 3.17. Suppose that \( \frac{\alpha_1 \phi_1}{\alpha_2 \phi_2} \eta_2 < \eta_1 < \frac{\alpha_2 \phi_1}{\alpha_1 \phi_2} \eta_2 \). \( f_1(v), f_1^R(v), f_2^R(v) \) and \( f_1^R(v) \) are given by (3.7), (3.10), (3.13) and (3.16). \( \bar{\pi}^R, \bar{\pi}^R, \bar{q}_{i1}^R, \bar{q}_{i2}^R \) (i = 1, 2), \( q_1, q_2, \bar{q}_i \) and \( \bar{q}^*_i \) are expressed as in (3.6), (3.9), (3.12) and (3.15). Let \( x_{11}^R, x_{12}^R \) and \( x_{11}^R \) be given by (3.17) and (3.18), respectively, and \( C_{ij}^R(m) \) (i = 1, 2, \( \cdots \), 8; j = 1, 2, \( \cdots \), 4) be defined in Appendix B.

(i) if \( am \vee x_{11}^R m < x^* \), for any \( x \in [am, m] \), the minimum probability of drawdown for the wealth process (2.3) is given by

\[
\phi(x, m) = \left\{ \begin{array}{ll}
C_{11}^{(1)}(m) + C_{12}^{(1)}(m) & \int_{am}^{x} \exp \left\{ \int_{am}^{u} f_{11}^R(v)dv \right\} du, \quad am \leq x < am \vee x_{11}^R, \\
C_{13}^{(1)}(m) + C_{14}^{(1)}(m) & \int_{x_{12}^R \vee am}^{x} \exp \left\{ \int_{x_{12}^R \vee am}^{u} f_{11}^R(v)dv \right\} du, \quad am \vee x_{12}^R \leq x < am \vee x_{12}^R, \\
C_{15}^{(1)}(m) + C_{16}^{(1)}(m) & \int_{x_{12}^R \vee am}^{x} \exp \left\{ \int_{x_{12}^R \vee am}^{u} f_1(v)dv \right\} du, \quad am \vee x_{12}^R \leq x \leq am \vee x_{11}^R, \\
C_{17}^{(1)}(m) + C_{18}^{(1)}(m) & \int_{x_{11}^R \vee am}^{x} \exp \left\{ \int_{x_{11}^R \vee am}^{u} f_1(v)dv \right\} du, \quad am \vee x_{11}^R \leq x < m < x^*;
\end{array} \right.
\]

(ii) if \( am \vee x_{12}^R \leq m < am \vee x_{11}^R \), for any \( x \in [am, m] \), the minimum probability of drawdown for the wealth process (2.3) is given by

\[
\phi(x, m) = \left\{ \begin{array}{ll}
C_{11}^{(2)}(m) + C_{12}^{(2)}(m) & \int_{am}^{x} \exp \left\{ \int_{am}^{u} f_{11}^R(v)dv \right\} du, \quad am \leq x < am \vee x_{11}^R, \\
C_{13}^{(2)}(m) + C_{14}^{(2)}(m) & \int_{x_{12}^R \vee am}^{x} \exp \left\{ \int_{x_{12}^R \vee am}^{u} f_{11}^R(v)dv \right\} du, \quad am \vee x_{12}^R \leq x < am \vee x_{12}^R, \\
C_{15}^{(2)}(m) + C_{16}^{(2)}(m) & \int_{x_{12}^R \vee am}^{x} \exp \left\{ \int_{x_{12}^R \vee am}^{u} f_1(v)dv \right\} du, \quad am \vee x_{12}^R \leq x \leq m < x_{11}^R; \\
\end{array} \right.
\]

(iii) if \( am \vee x_{11}^R \leq m \leq am \vee x_{12}^R \), for any \( x \in [am, m] \), the minimum probability of drawdown for the wealth process (2.3) is given by

\[
\phi(x, m) = \left\{ \begin{array}{ll}
C_{11}^{(3)}(m) + C_{12}^{(3)}(m) & \int_{am}^{x} \exp \left\{ \int_{am}^{u} f_{11}^R(v)dv \right\} du, \quad am \leq x < am \vee x_{11}^R, \\
C_{13}^{(3)}(m) + C_{14}^{(3)}(m) & \int_{x_{11}^R \vee am}^{x} \exp \left\{ \int_{x_{11}^R \vee am}^{u} f_{11}^R(v)dv \right\} du, \quad am \vee x_{11}^R \leq x \leq m < x_{11}^R; \\
\end{array} \right.
\]

(iv) if \( am \leq m < am \vee x_{11}^R \), for any \( x \in [am, m] \), the minimum probability of drawdown for the wealth process (2.3) is given by

\[
\phi(x, m) = C_{11}^{(4)}(m) + C_{12}^{(4)}(m) \int_{am}^{x} \exp \left\{ \int_{am}^{u} f_{11}^R(v)dv \right\} du.
\]

Furthermore, the corresponding optimal strategies are

\[
(\pi^*, q_1^*, q_2^*) = \left\{ \begin{array}{ll}
(1 + k_1 q_{i1}^R(x), \bar{q}_{i2}^R(x)), & \text{am} \leq x \leq m < am \vee x_{11}^R, \\
(\bar{\pi}^R(x), \bar{q}_{12}^R(x), \bar{q}_{22}^R(x)), & \text{am} \vee x_{11}^R \leq x \leq m < am \vee x_{12}^R, \\
(1, q_1(x), \bar{q}_2(x)), & \text{am} \vee x_{12}^R \leq x \leq m < am \vee x_{11}^R, \\
(\bar{\pi}^R(x), \bar{q}_1(x), \bar{q}_2^R(x)), & \text{am} \vee x_{11}^R \leq x \leq m < x^*.
\end{array} \right.
\]

Proof. See Appendix D. \( \square \)
Case 2: \( \eta_1 \leq \frac{a_2\rho_1}{a_1\beta_2} \eta_2 \).

In this case, we obtain that \( \hat{q}_1(x) \leq 0 \) and \( \hat{q}_2(x) > 0 \). The analysis is similar to Case 1, thus we give the following Theorem directly.

**Theorem 3.18.** Suppose that \( \eta_1 \leq \frac{a_2\rho_1}{a_1\beta_2} \eta_2 \). \( f_2^R(v), f_2^2(v), f_2^3(v) \) and \( f_2(v) \) are given by (3.21), (3.22), (3.23) and (3.24). \( \hat{\pi}^R \) and \( \hat{\pi}^r \) are expressed as in (3.25) and (3.27). Let \( x^R_{11}, x^R_{21}, x^R_{22} \) and \( x^R_2 \) be given by (3.26) and (3.28), \( \hat{q}_{11}^R, \hat{q}_{22}^R, \hat{q}_2 \) and \( \hat{q}_2^R \) be given by (3.30), and \( C_{11}^{(j)}(m) (i = 1, 2, \ldots, 8; j = 1, 2, \ldots, 4) \) be defined in Appendix B. The minimum probability of drawdown is given as in Theorem 3.17 where \( f_1^R(v), f_1^3(v), f_1^2(v), f_1(v), x^R_{11}, x^R_{12}, x^R_1, x^R_{21}, x^R_{22}, x^R_2, C_{11}^{(j)}(m) (i = 1, 2, \ldots, 8; j = 1, 2, \ldots, 4) \) is replaced with \( f_2^R(v), f_2^3(v), f_2^2(v), f_2(v), x^R_{21}, x^R_{22}, x^R_2, C_{21}^{(j)}(m) (i = 1, 2, \ldots, 8; j = 1, 2, \ldots, 4) \) for each case.

Furthermore, the corresponding optimal strategies are

\[
(\pi^*, q_1^*, q_2^*) = \begin{cases} 
(1 + k, 0, \hat{q}_{11}^R(x)), & \text{if } \eta_1 \leq \frac{a_2\rho_1}{a_1\beta_2} \eta_2 \leq \alpha m \lor x^R_3, \\
(\hat{\pi}^R(x), 0, \hat{q}_{22}^R(x)), & \text{if } \alpha m \lor x^R_3 \leq x \leq \alpha m \lor x^R_2, \\
(1, 0, \hat{q}_2(x)), & \text{if } \alpha m \lor x^R_2 \leq x \leq \alpha m \lor x^R_1, \\
(\hat{\pi}^r(x), 0, \hat{q}_2^R(x)), & \text{if } \alpha m \lor x^R_1 \leq x \leq x^*. 
\end{cases}
\]

Case 3: \( \eta_1 \geq \frac{a_2\rho_1}{a_1\beta_2} \eta_2 \).

In this case, we have \( \hat{q}_1(x) > 0 \) and \( \hat{q}_2(x) \leq 0 \). Then we can get the following results.

**Theorem 3.19.** Suppose that \( \eta_1 \geq \frac{a_2\rho_1}{a_1\beta_2} \eta_2 \). \( f_3^R(v), f_3^2(v), f_3^3(v) \) and \( f_3(v) \) are expressed as in (3.32). Let \( x^R_{31}, x^R_{32}, x^R_3 \) be given by (3.31) and \( \hat{\pi}^R, \hat{\pi}^r, \hat{q}_{31}^R, \hat{q}_{32}^R, \hat{q}_3 \) and \( \hat{q}_3^R \) be given by (3.33) and (3.34), and \( C_{31}^{(j)}(m) (i = 1, 2, \ldots, 8; j = 1, 2, \ldots, 4) \) be defined in Appendix B. The minimum probability of drawdown is given as in Theorem 3.17 where \( f_1^R(v), f_1^3(v), f_1^2(v), f_1(v), x^R_{11}, x^R_{12}, x^R_1, C_{11}^{(j)}(m) (i = 1, 2, \ldots, 8; j = 1, 2, \ldots, 4) \) is replaced with \( f_2^R(v), f_2^3(v), f_2^2(v), f_2(v), x^R_{21}, x^R_{22}, x^R_2, C_{21}^{(j)}(m) (i = 1, 2, \ldots, 8; j = 1, 2, \ldots, 4) \) for each case.

Furthermore, the corresponding optimal strategies are

\[
(\pi^*, q_1^*, q_2^*) = \begin{cases} 
(1 + k, \hat{q}_{31}^R(x), 0), & \text{if } \alpha m \leq x \leq \alpha m \lor x^R_3, \\
(\hat{\pi}^R(x), \hat{q}_{32}^R(x), 0), & \text{if } \alpha m \lor x^R_3 \leq x \leq \alpha m \lor x^R_2, \\
(1, \hat{q}_3(x), 0), & \text{if } \alpha m \lor x^R_2 \leq x \leq \alpha m \lor x^R_1, \\
(\hat{\pi}^r(x), \hat{q}_3^R(x), 0), & \text{if } \alpha m \lor x^R_1 \leq x \leq x^*. 
\end{cases}
\]

**Remark 6.** Comparing the expressions for the optimal investment and reinsurance strategies in Sections 3.1 and 3.2, it is clear that the optimal drawdown policy follows the optimal investment strategy since \( \hat{\pi}^R < 0 \) in this case (see (3.3) for details). Under this assumption, the optimal investment strategy \( \hat{\pi}^r \) is as either less than or equal to 1, which means that there is no need to borrow money, and thus the optimal results are identical to that in the case without borrowing costs.

**Remark 7.** All the results in Sections 3.1 and 3.2 are based on the assumption of \( \mu > R > r > 0 \). If \( R > \mu > r > 0 \), we can see that \( \hat{\pi}^R \) will never be the optimal investment strategy since \( \hat{\pi}^R < 0 \) in this case (see (3.3) for details). Under this assumption, the optimal investment strategy \( \hat{\pi}^r \) is as either less than or equal to 1, which means that there is no need to borrow money, and thus the optimal results are identical to that in the case without borrowing costs.
Remark 8. Note that when $k \to +\infty$, we have $x^{R}_{i1} \to 0$ ($i = 1, 2, 3$), and then the regions $\Gamma^{(i)}_{21}$ ($i = 1, 2, 3$) become empty. Specically, the regions $\Gamma^{(i)}_{21}$ and $\Gamma^{(i)}_{22}$ ($i = 1, 2, 3$) are merged into one region $\Gamma^{(i)}_{2} = \{(x, m) \in \Gamma | \pi(x) > 1\}$. Hence, we have $\Gamma = \Gamma^{(i)}_{1} \cup \Gamma^{(i)}_{2} \cup \Gamma^{(i)}_{3}$ ($i = 1, 2, 3$). In each subregion, the optimal reinsurance strategy are exactly the same as those in the case of $k \neq +\infty$.

Remark 9. The insurer hesitates to borrow money, because the higher the borrowing rate is, the more he/she is willing to invest all of the wealth in risky asset when the wealth level is less than the saving level. Borrowing money is necessary to avoid the appearance of drawdown only when the wealth level is below the borrowing level.

4. Effect of higher borrowing rate and risky investment. In this section, we analyze the effect of higher borrowing rate and risky investment on the optimal strategies respectively. To keep things simple, the constraint of limited leverage rate is removed. Also, we only compare the corresponding results under the conditions of $m \geq x^*$ and $\frac{a_2 b_1}{a_1 b_2} \eta_2 < \eta_1 < \frac{a_2 b_1}{a_1 b_2} \eta_2$. The same results can be derived for the other cases by the similar way.

4.1. The effect of higher borrowing rate. To begin with, we provide a Proposition which theoretically demonstrates the effect of higher borrowing rate on the optimal investment strategies in each subregion.

Proposition 1. The optimal investment strategies with and without borrowing costs are same in region $\Gamma^{(1)}_{1}$. But the former becomes smaller than the latter in region $\Gamma^{(1)}_{3}$, and the relationship in region $\Gamma^{(1)}_{2}$ is uncertain and strongly dependent on the parameters.

Proof. See Appendix E.

In the following, we give a special example to show the effect of higher borrowing rate intuitively. Suppose that $\mu = 0.35$, $r = 0.25$, $R = 0.3$, $a_1 = 1$, $a_2 = 2$, $b_1 = 3$, $b_2 = 4$, $\theta_1 = 0.15$, $\theta_2 = 0.12$, $\eta_1 = 0.22$, $\theta_2 = 0.2$, $\sigma = 2$, $\alpha = 0.2$, $m = 1$ and $\rho = 0.5$. In this case, we have $x^0 = -1.9371 < \alpha m = 0.2$. The results are shown in Figure 1.

The notation $\pi^0$ represents the optimal investment strategy without higher borrowing rate. Figure 1 shows that in such a special case, the optimal investment strategy with higher borrowing rate is less than that without constraint in the borrowing and full-investment regions. It makes sense that because of the higher borrowing rate, the insurer becomes more conservative and hesitates to borrow money. Only if the wealth level falls below the borrowing level, borrowing will occur.

Next step, we begin to investigate the effect of higher borrowing rate on the optimal reinsurance strategies. Similarly, we summarize the results in the following Proposition.

Proposition 2. The optimal reinsurance strategies with and without borrowing constraint are the same in region $\Gamma^{(1)}_{1}$. But the former becomes no less than the latter in region $\Gamma^{(1)}_{3}$, and the relationship in region $\Gamma^{(1)}_{2}$ is uncertain, and strongly depends on the parameters.

Proof. See Appendix F.
In this example, we set $\mu = 0.35$, $r = 0.25$, $R = 0.3$, $a_1 = 1$, $a_2 = 2$, $b_1 = 3$, $b_2 = 4$, $\theta_1 = 0.15$, $\theta_2 = 0.12$, $\eta_1 = 0.22$, $\eta_2 = 0.2$, $\sigma = 2$, $\alpha = 0.2$, $m = 1$ and $\rho = 0.5$. In this case, we have $x^0 = 0.4496 > x^R = 0.4074$. The results are shown in Figure 2.

The notations $q_1^0$ and $q_2^0$ stand for the optimal reinsurance strategies without the higher borrowing rate constraint. Figure 2 shows that the optimal reinsurance strategy $(q_1^*, q_2^*)$ is a decreasing and continuous function of $x$. Furthermore, we can observe that $q_i^* > q_i^0$ ($i = 1, 2$) in the full-investment and borrowing regions, which means that, with the higher borrowing rate, the insurer is willing to keep more insurance businesses.

4.2. The effect of risky investment on reinsurance strategies. In this subsection, we want to explore the effect of risky investment on reinsurance strategies, which are compared with those in Han et al. [18]. The results are shown in the next Proposition.
Proposition 3. The optimal reinsurance strategies in case of no risky investment are always greater than those with risky investment.

Proof. See Appendix G.

Similarly, we provide a numerical example to illustrate the comparison results in Proposition 3. Suppose that $\mu = 0.35$, $r = 0.25$, $R = 0.3$, $a_1 = 1$, $a_2 = 2$, $b_1 = 3$, $b_2 = 4$, $\theta_1 = 0.15$, $\theta_2 = 0.12$, $\eta_1 = 0.22$, $\theta_2 = 0.2$, $\sigma = 1$, $\alpha = 0.2$, $m = 1$ and $\rho = 0.5$. The results are shown in Figure 3.

![Figure 3](image-url)

The notations $q_1$ and $q_2$ represent the optimal reinsurance strategies without risky investment. From Figure 3, it is clear that the optimal reinsurance strategy $(q_1^*, q_2^*)$ decreases when the wealth level increases to $x^*$. Furthermore, we can observe that the optimal reinsurance strategies with risky investment are always less than those without risky investment due to the risk of investing in the risky asset.

Remark 10. The results in Proposition 3 show that the optimal reinsurance strategies are less than those in the case of no risky asset. It is a natural consequence since the uncertainty of risky asset makes the insurer keep a smaller retention level to reduce the drawdown risk.

5. Conclusion. We first recap the main results of this paper. Under the criterion of minimizing the probability of drawdown, we consider the optimal investment and proportional reinsurance problem in a diffusion approximation risk model with common shock dependence and borrowing costs. Using the technique of stochastic control theory and the corresponding Hamilton-Jacobi-Bellman equation, we derive the optimal investment-reinsurance strategy and the corresponding minimized probability of drawdown under the constraints of higher borrowing rate and limited leverage rate. By analysis, we find some important conclusions: (i) the behavior of borrowing typically occurs with a lower wealth level and the insurer becomes conservative when the wealth increases, which can also be observed in Young [28], Bayraktar and Young [5] and Luo [24]; (ii) the insurer hesitates to borrow money because of the higher borrowing rate, and tends to invest all of the wealth in risky asset when the wealth is less than the saving level; (iii) in the full-investment region, the retention level is no less than that without the borrowing costs; (iv) the
risky investment makes the optimal reinsurance strategies become less than those without risky investment due to the risk brought by the uncertain market price of the risky asset.

Concerning future related work, there are many remaining interesting problems to discuss. One, related to this paper, is to consider the optimal reinsurance and investment with borrowing costs in some other criteria, such as, maximizing the expected utility of terminal wealth or mean-variance framework. The second is to discuss the same problem with additional penalization on ambiguity. The last but not the least, it is interesting to extend the borrowing rate from one deterministic value to a general Markov regime switching model, that is, the borrowing rate can be varied based on the different wealth levels of the insurance company, which makes the problem more complicated and challenging.

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Appendix A. Parameters in section 3.1. We define $C_{ij}$ ($i = 1, 2, 3; j = 1, 2, \cdots, 8$) as follows

\[
\begin{aligned}
C_{i1} &= 1, \\
C_{i2} &= \left[ \int_{x_{i2}^{R \lor \alpha m}}^{x_{i1}^{R \lor \alpha m}} \exp \left\{ \int_{x_{i1}^{R \lor \alpha m}}^{u} f_{i1}^{R}(v) dv \right\} du \\
&\quad + \int_{x_{i2}^{R \lor \alpha m}}^{x_{i1}^{R \lor \alpha m}} \exp \left\{ \int_{x_{i1}^{R \lor \alpha m}}^{x_{i2}^{R \lor \alpha m}} f_{i1}^{R}(v) dv + \int_{u}^{x_{i2}^{R \lor \alpha m}} f_{i2}^{R}(v) dv \right\} du \\
&\quad + \int_{x_{i1}^{R \lor \alpha m}}^{x_{i2}^{R \lor \alpha m}} \exp \left\{ \int_{x_{i1}^{R \lor \alpha m}}^{x_{i2}^{R \lor \alpha m}} f_{i1}^{R}(v) dv + \int_{u}^{x_{i2}^{R \lor \alpha m}} f_{i2}^{R}(v) dv + \int_{x_{i1}^{R \lor \alpha m}}^{u} f_{i}(v) dv \right\} du \\
&\quad + \int_{x_{i1}^{R \lor \alpha m}}^{u} f_{i}(v) dv \right\} du \right]^{-1}, \\
C_{i3} &= 1 + C_{i2} \int_{x_{i1}^{R \lor \alpha m}}^{x_{i1}^{R \lor \alpha m}} \exp \left\{ \int_{x_{i1}^{R \lor \alpha m}}^{u} f_{i1}^{R}(v) dv \right\} du, \\
C_{i4} &= C_{i2} \exp \left\{ \int_{x_{i1}^{R \lor \alpha m}}^{x_{i1}^{R \lor \alpha m}} f_{i1}^{R}(v) dv \right\}, \\
C_{i5} &= C_{i3} + C_{i4} \int_{x_{i1}^{R \lor \alpha m}}^{x_{i2}^{R \lor \alpha m}} \exp \left\{ \int_{x_{i1}^{R \lor \alpha m}}^{u} f_{i2}^{R}(v) dv \right\} du, \\
C_{i6} &= C_{i4} \exp \left\{ \int_{x_{i1}^{R \lor \alpha m}}^{x_{i2}^{R \lor \alpha m}} f_{i2}^{R}(v) dv \right\}, \\
C_{i7} &= C_{i5} + C_{i6} \int_{x_{i1}^{R \lor \alpha m}}^{x_{i1}^{R \lor \alpha m}} \exp \left\{ \int_{x_{i1}^{R \lor \alpha m}}^{u} f_{i}(v) dv \right\} du, \\
C_{i8} &= C_{i6} \exp \left\{ \int_{x_{i1}^{R \lor \alpha m}}^{x_{i1}^{R \lor \alpha m}} f_{i}(v) dv \right\}. 
\end{aligned}
\]
Appendix B. Parameters in section 3.2. Firstly, we give the expressions for the functions $g_{ij}(x, m)$ ($i = 1, 2, 3; j = 1, 2, 3, 4$) as follows

\[
\begin{aligned}
g_{11}(x, m) &= \int_{am}^{x} \exp \left\{ \int_{am}^{u} f_{i1}^R(v)dv \right\} du, \\
g_{12}(x, m) &= \int_{am}^{am \vee x_1^R} \exp \left\{ \int_{am}^{u} f_{i1}^R(v)dv \right\} du \\
&\quad + \int_{am \vee x_1^R}^{x} \exp \left\{ \int_{am \vee x_1^R}^{u} f_{i1}^R(v)dv \right\} du, \\
g_{13}(x, m) &= \int_{am}^{am \vee x_1^R} \exp \left\{ \int_{am}^{u} f_{i1}^R(v)dv \right\} du \\
&\quad + \int_{am \vee x_1^R}^{am \vee x_1^R} \exp \left\{ \int_{am}^{am \vee x_1^R} f_{i1}^R(v)dv \right\} du \\
&\quad + \int_{am \vee x_1^R}^{x} \exp \left\{ \int_{am \vee x_1^R}^{u} f_{i1}^R(v)dv \right\} du, \\
g_{14}(x, m) &= \int_{am}^{am \vee x_1^R} \exp \left\{ \int_{am}^{u} f_{i1}^R(v)dv \right\} du \\
&\quad + \int_{am \vee x_1^R}^{am \vee x_1^R} \exp \left\{ \int_{am}^{am \vee x_1^R} f_{i1}^R(v)dv \right\} du \\
&\quad + \int_{am \vee x_1^R}^{x} \exp \left\{ \int_{am \vee x_1^R}^{u} f_{i1}^R(v)dv \right\} du \\
&\quad + \int_{am \vee x_1^R}^{am \vee x_1^R} \exp \left\{ \int_{am \vee x_1^R}^{am \vee x_1^R} f_{i1}^R(v)dv \right\} du \\
&\quad + \int_{am \vee x_1^R}^{x} \exp \left\{ \int_{am \vee x_1^R}^{u} f_{i1}^R(v)dv \right\} du + \int_{am \vee x_1^R}^{x} \exp \left\{ \int_{am \vee x_1^R}^{am \vee x_1^R} f_{i1}^R(v)dv \right\} du \\
&\quad + \int_{am \vee x_1^R}^{am \vee x_1^R} \exp \left\{ \int_{am \vee x_1^R}^{am \vee x_1^R} f_{i1}^R(v)dv \right\} du.
\end{aligned}
\]

Then, the functions $C_{ij}^{(i)}(m)(i = 1, 2, 3; j = 1, 2, \cdots, 8)$ are defined as

\[
\begin{aligned}
C_{i1}^{(1)}(m) &= 1, \\
C_{i2}^{(1)}(m) &= -\frac{1}{g_{i4}(x^*, x^*)} \exp \left\{ -\int_{am}^{x^*} k_{i3}(u)du \right\}, \\
C_{i3}^{(1)}(m) &= 1 + C_{i2}^{(1)}(m) \int_{am}^{x_1^R} \exp \left\{ \int_{am}^{u} f_{i1}^R(v)dv \right\} du, \\
C_{i4}^{(1)}(m) &= C_{i2}^{(1)}(m) \exp \left\{ \int_{am}^{x_1^R} f_{i1}^R(v)dv \right\}, \\
C_{i5}^{(1)}(m) &= C_{i3}^{(1)}(m) + C_{i4}^{(1)}(m) \int_{am \vee x_1^R}^{x_1^R} \exp \left\{ \int_{am \vee x_1^R}^{u} f_{i2}^R(v)dv \right\} du, \\
C_{i6}^{(1)}(m) &= C_{i4}^{(1)}(m) \exp \left\{ \int_{am \vee x_1^R}^{x_1^R} f_{i2}^R(v)dv \right\}, \\
C_{i7}^{(1)}(m) &= C_{i5}^{(1)}(m) + C_{i6}^{(1)}(m) \int_{am \vee x_1^R}^{x} \exp \left\{ \int_{am \vee x_1^R}^{u} f_{i1}(v)dv \right\} du, \\
C_{i8}^{(1)}(m) &= C_{i6}^{(1)}(m) \exp \left\{ \int_{am \vee x_1^R}^{x} f_{i1}(v)dv \right\}.
\end{aligned}
\]
The functions $C_i^{(2)}(m)$ ($i = 1, 2, 3; j = 1, 2, \cdots, 6$) are given as follows

$$
\begin{align*}
C_{i1}^{(2)}(m) &= 1, \\
C_{i2}^{(2)}(m) &= \frac{1}{g_{i4}(x^*, x^*)} \exp \left\{ - \left( \int_{m}^{x_i^R} k_{i2}(u) + \int_{x_i^R}^{x_i^*} k_{i4}(u) \right) du \right\}, \\
C_{i3}^{(2)}(m) &= 1 + C_{i2}^{(2)}(m) \int_{\alpha m}^{x_i^R} \exp \left\{ \int_{\alpha m}^{u} f_{i2}^R(\nu) d\nu \right\} du, \\
C_{i4}^{(2)}(m) &= C_{i2}^{(2)}(m) \exp \left\{ \int_{\alpha m}^{x_i^R} f_{i1}^R(\nu) d\nu \right\}, \\
C_{i5}^{(2)}(m) &= C_{i3}^{(2)}(m) + C_{i4}^{(1)}(m) \int_{x_{i1}^R \wedge \alpha m}^{x_i^R} \exp \left\{ \int_{x_{i1}^R \wedge \alpha m}^{u} f_{i2}^R(\nu) d\nu \right\} du, \\
C_{i6}^{(2)}(m) &= C_{i4}^{(2)}(m) \exp \left\{ \int_{x_{i1}^R \wedge \alpha m}^{x_i^R} f_{i2}^R(\nu) d\nu \right\},
\end{align*}
$$

with

$$
\begin{align*}
k_{i3}(u) &= \begin{cases} 
\alpha \left[ \frac{1}{g_{i4}(u, u)} + f_i(\alpha u) \right], & \text{if } x_{i2}^R \leq \alpha m, \\
\alpha \left[ \frac{1}{g_{i4}(u, u)} + f_{i2}^R(\alpha u) \right], & \text{if } x_{i1}^R \leq \alpha m < x_{i2}^R, \\
\alpha \left[ \frac{1}{g_{i4}(u, u)} + f_{i1}^R(\alpha u) \right], & \text{if } \alpha m < x_{i1}^R.
\end{cases}
\end{align*}
$$

The functions $C_{i}^{(3)}(m)$ ($i = 1, 2, 3; j = 1, 2, 3, 4$) are given by

$$
\begin{align*}
C_{i1}^{(3)}(m) &= 1, \\
C_{i2}^{(3)}(m) &= -\frac{1}{g_{i4}(x^*, x^*)} \exp \left\{ - \left( \int_{m}^{x_{i2}^R} k_{i2}(u) + \int_{x_{i2}^R}^{x_i^R} k_{i4}(u) \right) du \right\}, \\
C_{i3}^{(3)}(m) &= 1 + C_{i2}^{(3)}(m) \int_{\alpha m}^{x_{i2}^R} \exp \left\{ \int_{\alpha m}^{u} f_{i1}^R(\nu) d\nu \right\} du, \\
C_{i4}^{(3)}(m) &= C_{i2}^{(3)}(m) \exp \left\{ \int_{\alpha m}^{x_{i2}^R} f_{i1}^R(\nu) d\nu \right\},
\end{align*}
$$
with
\[ k_{i2}(u) = \begin{cases} 
\alpha \left[ \frac{1}{g_{i4}(u,u)} + f_{i2}^R(\alpha u) \right], & \text{if } x_{i1}^R \leq \alpha m, \\
\alpha \left[ \frac{1}{g_{i4}(u,u)} + f_{i1}^R(\alpha u) \right], & \text{if } \alpha m < x_{i1}^R.
\end{cases} \]

The functions \( C_{ij}^4 \) (\( i = 1, 2, 3; j = 1, 2 \)) are given by

\[
\begin{align*}
C_{i1}^4(m) &= 1, \\
C_{i2}^4(m) &= -\frac{1}{g_{i4}(x^*,x^*)} \exp \left\{ - \left( \int_{x_{i1}^R}^{x_{i1}} k_{i1}(u) \, du + \int_{x_{i1}^R}^{u} k_{i2}(u) \, du + \int_{x_{i2}^R}^{x_i} k_{i4}(u) \, du \right) \right\},
\end{align*}
\]

with
\[ k_{i1}(u) = \alpha \left[ \frac{1}{g_{i4}(u,u)} + f_{i1}^R(\alpha u) \right]. \]

**Appendix C. Proof of Theorem 3.6.**

**Proof.** In this Appendix, we are going to give the detailed proof of Theorem 3.6. Firstly, we need to find the solution of the HJB equation (2.5), and then prove that this solution satisfies the conditions in Theorem 2.2. Based on the results in Lemmas 3.1-3.4, we can get the solution, which has the following structure:

\[ \phi(x,m) \]

\[
\begin{align*}
C_{11} + C_{12} &\int_{\alpha m}^{x} \exp \left\{ \int_{\alpha m}^{u} f_{11}^R(v) \, dv \right\} \, du, & \alpha m \leq x < \alpha m \vee x_{11}^R, \\
C_{13} + C_{14} &\int_{x_{11}^R \vee \alpha m}^{x} \exp \left\{ \int_{x_{11}^R \vee \alpha m}^{u} f_{12}^R(v) \, dv \right\} \, du, & \alpha m \vee x_{11}^R \leq x < \alpha m \vee x_{12}^R, \\
C_{15} + C_{16} &\int_{x_{12}^R \vee \alpha m}^{x} \exp \left\{ \int_{x_{12}^R \vee \alpha m}^{u} f_{1}^R(v) \, dv \right\} \, du, & \alpha m \vee x_{12}^R \leq x < \alpha m \vee x_1^*, \\
C_{17} + C_{18} &\int_{x_1^* \vee \alpha m}^{x} \exp \left\{ \int_{x_1^* \vee \alpha m}^{u} f_{1}^R(v) \, dv \right\} \, du, & \alpha m \vee x_1^* \leq x \leq x^*.
\end{align*}
\]

where \( C_{ii}(i = 1, 2, \cdots, 8) \) are the constants to be determined.

For convenience, we assume that \( \alpha m \) is less than \( x_{11}^R \). For the other cases, it can be proved by the same way. From the boundary and smoothness conditions, we can determine \( C_{ii}(i = 1, 2, \cdots, 8) \) in the following steps.

(i) \( \phi_1(\alpha m, m) = 1 \Rightarrow C_{11} = 1, \)

(ii) \( \phi_1(x_{11}^R, m) = \phi_2(x_{11}^R, m) \Rightarrow C_{12} \int_{\alpha m}^{x_{11}^R} \exp \left\{ \int_{\alpha m}^{u} f_{11}^R(v) \, dv \right\} \, du + C_{11} = C_{13}, \)
(iii) \( \phi_2(x_{12}^R, m) = \phi_3(x_{12}^R, m) \Rightarrow C_{14} \int_{x_{11}^R} f_{12}^R(v) \exp \left\{ \int_{x_{11}^R} f_{11}^R(v) \right\} du + C_{13} = C_{15} \),

(iv) \( \phi_3(x_{1}^R, m) = \phi_4(x_{1}^R, m) \Rightarrow C_{16} \int_{x_{12}^R} \exp \left\{ \int_{x_{12}^R} f_1^R(v) \right\} du + C_{15} = C_{17} \),

(v) \( \frac{\partial \phi_1}{\partial x} |_{x = x_{11}^R} = \frac{\partial \phi_2}{\partial x} |_{x = x_{11}^R} \Rightarrow C_{12} \exp \left\{ \int_{x_{12}^R} f_1^R(v) \right\} = C_{14} \),

(vi) \( \frac{\partial \phi_2}{\partial x} |_{x = x_{12}^R} = \frac{\partial \phi_3}{\partial x} |_{x = x_{12}^R} \Rightarrow C_{14} \exp \left\{ \int_{x_{11}^R} f_{11}^R(v) \right\} = C_{16} \),

(vii) \( \frac{\partial \phi_3}{\partial x} |_{x = x_{11}^R} = \frac{\partial \phi_4}{\partial x} |_{x = x_{11}^R} \Rightarrow C_{16} \exp \left\{ \int_{x_{12}^R} f_1^R(v) \right\} = C_{18} \).

By some calculations, we can rewrite \( C_{17} \) and \( C_{18} \) as follows

\[
\begin{align*}
C_{17} &= 1 + C_{12} \cdot g_{13}(x_{1}^R, m), \\
C_{18} &= C_{12} \exp \left\{ \int_{x_{12}^R} f_1^R(v) \right\} + C_{16} \exp \left\{ \int_{x_{12}^R} f_{12}^R(v) \right\} + C_{15} \exp \left\{ \int_{x_{12}^R} f_1^R(v) \right\},
\end{align*}
\]

(C.1)

which are the expressions with respect to \( C_{12} \). Substituting (C.1) back into \( \phi_4(x, m) \) yields

\[
\phi_4(x, m) = 1 + C_{12} \left[ \int_{x_{11}^R} f_{11}^R(v) \exp \left\{ \int_{x_{11}^R} f_{11}^R(v) \right\} du \\
+ \int_{x_{11}^R} \exp \left\{ \left( \int_{x_{11}^R} f_{11}^R(v) + \int_{x_{12}^R} f_{12}^R(v) \right) \right\} du \\
+ \int_{x_{12}^R} f_1(v) \right\} du + \int_{x_{12}^R} f_1(v) + \int_{x_{12}^R} f_1(v) \right\} du \right].
\]

By the boundary condition \( \phi(x^*, m) = 0 \), we derive

\[
C_{12} = - \left[ \int_{x_{11}^R} \exp \left\{ \int_{x_{11}^R} f_{11}^R(v) \right\} du \\
+ \int_{x_{11}^R} \exp \left\{ \left( \int_{x_{11}^R} f_{11}^R(v) + \int_{x_{12}^R} f_{12}^R(v) \right) \right\} du \\
+ \int_{x_{12}^R} \exp \left\{ \left( \int_{x_{11}^R} f_{11}^R(v) + \int_{x_{12}^R} f_{12}^R(v) \right) \right\} du \right].
\]
Proof. In this Appendix, the proof of the Theorem 3.17 is presented in the case of 
\[D.\]

Appendix 

\[\epsilon\]

Thus for other parameters, which have been shown in Appendix A.

From the boundary and smoothness conditions, we can obtain
\[
\phi(x, m)
\]

Putting \(C_{12}\) into (C.1), we can get the explicit expressions for \(C_{17}\) and \(C_{18}\), and these for other parameters, which have been shown in Appendix A.

So far, we have obtained the solution of the HJB equation (2.5). It is easy to verify that \(\phi(x, m)\) satisfies the conditions (i), (ii), (iv), (v) and (vi) in Theorem 2.2. Condition (iii) is moot because \(m \geq x^*\). Thus, we have \(\phi = h\), and \((\pi^*, q_1, q_2)\) given by (3.19) is the optimal strategy.

\[\square\]

Appendix D. Proof of Theorem 3.17.

Proof. In this Appendix, the proof of the Theorem 3.17 is presented in the case of 
\[m \in [am \lor x_1, x^*] \text{ only.}\]

The proofs for \(m \in [am \lor x_1^R, am \lor x_1^R)\), \(m \in [am \lor x_1^R, \alpha m \lor x_1^R)\) and \(m \in [am, am \lor x_1^R]\) can be derived similarly. Based on Lemmas 3.1-3.4 and boundary conditions, the solution of the HJB equation (2.5) has the following form:

\[
\phi(x, m) = \begin{cases} 
C_{11}(m) + C_{12}(m) \int_{am}^{x} \exp \left\{ \int_{am}^{u} f_{12}^R(v) dv \right\} du, & \alpha m \leq x < am \lor x_1^R, \\
C_{13}(m) + C_{14}(m) \int_{am}^{x_1^R} \exp \left\{ \int_{am}^{u} f_{11}^R(v) dv \right\} du, & am \lor x_1^R \leq x < am \lor x_1^R, \\
C_{15}(m) + C_{16}(m) \int_{am}^{x_1^R} \exp \left\{ \int_{am}^{u} f_{11}^R(v) dv \right\} du, & am \lor x_1^R \leq x \leq am \lor x_1^R, \\
C_{17}(m) + C_{18}(m) \int_{am}^{x_1^R} \exp \left\{ \int_{am}^{u} f_{11}^R(v) dv \right\} du, & am \lor x_1^R \leq x \leq m < x^*; \\
\phi_4(x, m)
\end{cases}
\]

where \(C_{1i}(m)(i = 1, 2, \ldots, 8)\) are the functions to be determined. From the boundary and smoothness conditions, we can obtain

(i) \(\phi_1(am, m) = 1 \Rightarrow C_{11}(m) = 1,\)

(ii) \(\phi_2(x_1^R, m) = \phi_2(x_1^R, m)\)

\(\Rightarrow C_{12}(m) = \int_{am}^{x_1^R} \exp \left\{ \int_{am}^{u} f_{12}^R(v) dv \right\} du + C_{11}(m) = C_{11}(m),\)

(iii) \(\phi_3(x_1^R, m) = \phi_3(x_1^R, m)\)

\(\Rightarrow C_{14}(m) = \int_{am}^{x_1^R} \exp \left\{ \int_{am}^{u} f_{12}^R(v) dv \right\} du + C_{13}(m) = C_{13}(m),\)
(iv) \( \phi_3(x_1^*, m) = \phi_4(x_1^*, m) \)

\[
\Rightarrow C^{(1)}_{16}(m) \int_{x_1^*}^{x_1^*} \exp \left\{ \int_{x_1^*}^{u} f_1(v)dv \right\} du + C^{(1)}_{15}(m) = C^{(1)}_{17}(m),
\]

(v) \( \frac{\partial \phi_1}{\partial x} \bigg|_{x=x_1^*} = \frac{\partial \phi_2}{\partial x} \bigg|_{x=x_1^*} \)

\[
\Rightarrow C^{(1)}_{12}(m) \exp \left\{ \int_{x_1^*}^{R} f_1^{R}(v)dv \right\} = C^{(1)}_{14}(m),
\]

(vi) \( \frac{\partial \phi_2}{\partial x} \bigg|_{x=x_1^*} = \frac{\partial \phi_3}{\partial x} \bigg|_{x=x_1^*} \)

\[
\Rightarrow C^{(1)}_{14}(m) \exp \left\{ \int_{x_1^*}^{R} f_1^{R}(v)dv \right\} = C^{(1)}_{16}(m),
\]

(vii) \( \frac{\partial \phi_3}{\partial x} \bigg|_{x=x_1^*} = \frac{\partial \phi_4}{\partial x} \bigg|_{x=x_1^*} \)

\[
\Rightarrow C^{(1)}_{16}(m) \exp \left\{ \int_{x_1^*}^{x_1^*} f_1(v)dv \right\} = C^{(1)}_{18}(m).
\]

By some simplifications, we can rewrite \( C^{(1)}_{17}(m) \) and \( C^{(1)}_{18}(m) \) as follows

\[
\begin{align*}
C^{(1)}_{17}(m) &= 1 + C^{(1)}_{12}(m) \cdot g_{13}(x_1^*, m), \\
C^{(1)}_{18}(m) &= C^{(1)}_{12}(m) \exp \left\{ \int_{a m}^{x_1^*} f_1^{R}(v)dv + \int_{x_1^*}^{R} f_1^{R}(v)dv + \int_{x_1^*}^{x_1^*} f_1(v)dv \right\}.
\end{align*}
\]

(D.1)

In order to obtain the expressions for \( C^{(1)}_{17}(m) \) and \( C^{(1)}_{18}(m) \), we may derive \( C^{(1)}_{12}(m) \) firstly:

(viii) \( \frac{\partial \phi_4(m, m)}{\partial m} = 0 \)

\[
\Rightarrow C^{(1)}_{12}(m)' g_{14}(m, m) + C^{(1)}_{12}(m) g_{14}(m, m) \left( \frac{- \alpha f_1^{R}(m)}{D_{11}(m)} \right) - \alpha = 0
\]

\[
\Rightarrow C^{(1)}_{12}(m)' = \frac{g_{14}(m, m) \alpha f_1^{R}(m) + \alpha}{g_{14}(m, m)} = \alpha \left( f_1^{R}(m) + \frac{1}{g_{14}(m, m)} \right).
\]

To obtain \( C^{(3)}_{12}(m) \), we integrate both sides of this equation from \( m \) to \( x^* \), which yields

\[
\int_{m}^{x^*} \frac{C^{(1)}_{12}(u)'}{C^{(1)}_{12}(u)} du = \int_{m}^{x^*} \alpha \left( f_1^{R}(\alpha u) + \frac{1}{g_{14}(u, u)} \right) du,
\]

and then we have

\[
C^{(1)}_{12}(m) = C^{(1)}_{12}(x^*) \exp \left\{ \int_{m}^{x^*} \alpha \left( f_1^{R}(\alpha u) + \frac{1}{g_{14}(u, u)} \right) du \right\}.
\]

Hence, we only need to derive \( C^{(1)}_{12}(x^*) \). Substituting the expressions for \( C^{(1)}_{17}(m) \) and \( C^{(1)}_{18}(m) \) into \( \phi_4(x, m) \) yields

\[
\phi_4(x, m) = 1 + C^{(1)}_{12}(m) \cdot g_{14}(x, m).
\]
Following the condition \( \phi(x^*, x^*) = 0 \), we have
\[
C_{12}^{(1)}(x^*) = -\frac{1}{g_{14}(x^*, x^*)},
\]
and then
\[
C_{12}^{(1)}(m) = -\frac{1}{g_{14}(x^*, x^*)} \exp \left\{ -\int_{m}^{x^*} \alpha \left( f_{11}(\alpha u) + \frac{1}{g_{14}(u, u)} \right) du \right\}.
\]
Therefore, we can get the explicitly expressions for \( C_{12}^{(1)}(m) \) and \( C_{18}^{(1)}(m) \) by (D.1), and thus for the other parameters, which have been shown in Appendix B.

Overall, the solution of the HJB equation (2.5) has been found. It is easy to verify that \( \phi(x, m) \) is a \( C^{2,1} \) convex function and satisfies the conditions (i)-(vi) in Theorem 2.2. Thus, we have \( \phi = h \), and \( (\pi^*, q_1^*, q_2^*) \) given by (3.36) is the optimal strategy.

**Appendix E. Proof of proposition 1.**

**Proof.** If there is no constraint of higher borrowing rate, we can easily get the optimal investment strategy
\[
\pi^0(x) = -\frac{\mu - r}{\sigma^2 x f_1'(x)} = -\frac{2(\mu - r)(\mu + A) b_1^2 b_2^2 (1 - \rho^2)}{b_1^2 b_2^2 (1 - \rho^2)(\mu - r)^2 x + B\sigma^2 x} = \pi^r(x).
\]
The superscript 0 means that there is no borrowing cost. With the constraint of higher borrowing rate, we have
\[
\pi^*(x) = \begin{cases} 
-\frac{\mu - r}{\sigma^2 x f_1'(x)}, & \text{if } (x, m) \in \Gamma_1^{(1)}, \\
-\frac{\mu - R}{\sigma^2 x f_1'(x)}, & \text{if } (x, m) \in \Gamma_2^{(1)}, \\
1, & \text{if } (x, m) \in \Gamma_3^{(1)}.
\end{cases}
\]
It is clear that the optimal investment strategies with and without borrowing costs are the same in region \( \Gamma_1^{(1)} \), and also the definition of region \( \Gamma_1^{(1)} \) shows that \( \pi^0(x) = \pi^r(x) \) has to be greater than 1. So now we only need to discuss the relationship of optimal strategies in region \( \Gamma_2^{(1)} \). Here, we assume that \( \alpha m < x^R \), otherwise region \( \Gamma_2^{(1)} \) is empty. For convenience, we denote \( \Delta = b_1^2 b_2^2 (1 - \rho^2) > 0 \). It is not difficult to see that \( \pi^0(x) \) and \( \pi^*(x) \) are both decreasing, continuous and convex functions with respect to \( x \) in the interval \((\alpha m, x^*)\). So the only thing we need to do in the following is to explore that whether the intersection point of these two functions exists.

From the equation \( \pi^0(x) = \pi^*(x) \), we have
\[
[\Delta(\mu - r)(\mu - R)\mu + (\mu - R - r)B\sigma^2]x = -A[(\mu - r)(\mu - R)\Delta - B\sigma^2].
\]
If \( \Delta(\mu - r)(\mu - R)\mu + (\mu - R - r)B\sigma^2 = 0 \), then we know that there is no intersection point in region \( \Gamma_2^{(1)} \). Furthermore, notice that \( \pi^0(x^R) > \pi^*(x^R) = 1 \), then we can conclude that inequality \( \pi^0(x) > \pi^*(x) \) holds in region \( \Gamma_2^{(1)} \).

If \( \Delta(\mu - r)(\mu - R)\mu + (\mu - R - r)B\sigma^2 \neq 0 \), we can obtain the intersection point \( x^0 \) as follow
\[
x^0 = \frac{-A[(\mu - r)(\mu - R)\Delta - B\sigma^2]}{\Delta(\mu - r)(\mu - R)\mu + (\mu - R - r)B\sigma^2}.
\]
Then we need to discuss that whether the intersection point $x^0$ locates in region $\Gamma_2^{(1)}$ in the following two cases:

1°: If $x^0 < \alpha m$ or $x^0 > x_1^R$, then we have $\pi^* < \pi^0$;

2°: If $\alpha m < x^0 < x_1^R$, then we have $\pi^* > \pi^0$ for any $x \in (\alpha m, x^0)$ and $\pi^* < \pi^0$ for any $x \in (x^0, x_1^R)$. \hspace{1cm} \square

Appendix F. Proof of proposition 2.

Proof. If there is no higher borrowing rate constraint, the optimal reinsurance strategies are

$$q_1^0 = \frac{2(rx + A)(a_2\eta_2\rho_1 b_2 - a_1\eta_1 b_2)}{b_1^2 b_2^2(1 - \rho^2)\frac{(\mu - r)^2}{\sigma^2} + B},$$

$$q_2^0 = \frac{2(rx + A)(a_1\eta_1 \rho_1 b_1 - a_2 \eta_2 b_1)}{b_1^2 b_2^2(1 - \rho^2)\frac{(\mu - r)^2}{\sigma^2} + B}.$$

By comparing the optimal strategies, we can see that the optimal reinsurance strategies are the same in region $\Gamma_1^{(1)}$ and different in regions $\Gamma_3^{(1)}$ and $\Gamma_3^{(1)}$. Firstly, we consider the relationship between $q_1^0(x)$ and $q_1^*(x)$ in region $\Gamma_3^{(1)}$. Since

$$q_1^* - q_1^0 \geq 0 \Leftrightarrow (a_2\eta_2\rho_1 b_2 - a_1\eta_1 b_2) \left( \frac{(\mu x + A) - \sqrt{(\mu x + A)^2 + \sigma^2 x^2 B}}{B} - \frac{2(rx + A)\sigma^2}{B\sigma^2 + \Delta(\mu - r)^2} \right) \geq 0$$

$$\Leftrightarrow \left( \mu x + A - \left( \frac{(\mu x + A)^2 + \sigma^2 x^2 B}{\Delta} \right) \right) (B\sigma^2 + \Delta(\mu - r)^2) - 2(rx + A)\sigma^2 B \leq 0.$$

By simplifying the inequality, we can get the equivalent inequality as follow:

$$\tilde{\kappa}(x) = A_1 x^2 + B_1 x + C_1 \geq 0, \hspace{1cm} \text{(F.1)}$$

where

$$A_1 = \frac{B\sigma^2}{\Delta} (B\sigma^2 + \Delta(\mu - r)^2)^2 + 4r\mu B\sigma^2 (B\sigma^2 + \Delta(\mu - r)^2) - 4r^2 B^2\sigma^4 > 0,$$

$$B_1 = 4(r + \mu) A B \sigma^2 (B\sigma^2 + \Delta(\mu - r)^2) - 8r AB^2\sigma^4 < 0,$$

$$C_1 = 4A^2 B^2 \sigma^2 (B\sigma^2 + \Delta(\mu - r)^2)^2 - 4A^2 B^2\sigma^4 > 0.$$

Because $\tilde{\kappa}(x)$ is a quadratic function and $B_1^2 - 4A_1 C_1 = 0$, it is obvious that for any $(x, m)$ in region $\Gamma_3^{(1)}$, the inequality (F.1) always holds. Then, we obtain the inequality $q_1^* \geq q_1^0$.

In what follows, we try to derive the relationship in region $\Gamma_2^{(1)}$. It is not difficult to see that $q_1^0(x)$ and $q_1^*(x)$ are both decreasing, continuous and convex functions with respect to $x$ in the interval $(\alpha m, x^*)$. So the only thing we need to do in the following is to determine that whether the intersection point of the two functions exists.

From the equation $q_1^0(x) = q_1^*(x)$, we have

$$[\Delta(\mu^2 - Rr) + B\sigma^2]x = -(2\mu - r - R)A\Delta.$$
It is obvious that $\Delta(\mu^2 - Rr) + B\sigma^2 > 0$ and we can obtain the intersection point $x^0$ as follow

$$x^0 = \frac{-A\Delta(2\mu - r - R)}{\Delta(\mu^2 - Rr) + B\sigma^2}.$$

Then, we derive the results in the following two cases:

1°: If $x^0 < \alpha m$ or $x^0 > x^R_f$, then we have $q_1^* > q_1^0$;

2°: If $\alpha m < x^0 < x^R_f$, then we have $q_1^* < q_1^0$ for any $x \in (\alpha m, x^0)$ and $q_1^* > q_1^0$ for any $x \in (x^0, x^R_f)$.

The analysis above focuses on the reinsurance strategy $q_1^*$, the discussion on the reinsurance strategy $q_2^*$ can be obtained along the same lines. \hfill \Box

**Appendix G. Proof of proposition 3.**

**Proof.** When there is no investment in risky asset, the optimal reinsurance strategies are

$$\begin{align*}
q_1 &= \frac{2(rx + A)(a_2\eta_2 \rho b_1 b_2 - a_1\eta_1 b_2^2)}{B}, \\
q_2 &= \frac{2(rx + A)(a_1\eta_1 \rho b_1 b_2 - a_2\eta_2 b_1^2)}{B}.
\end{align*}$$

In region $\Gamma_1^{(1)}$ which is called saving region in our paper, we can see

$$\begin{align*}
q_1^* &= \frac{2(rx + A)(a_2\eta_2 \rho b_1 b_2 - a_1\eta_1 b_2^2)}{b_1^2 b_2^2 (1 - \rho^2) (\mu - r)^2 + B}, \\
q_2^* &= \frac{2(rx + A)(a_1\eta_1 \rho b_1 b_2 - a_2\eta_2 b_1^2)}{b_1^2 b_2^2 (1 - \rho^2) (\mu - r)^2 + B}.
\end{align*}$$

As is shown above, the numerators are the same, while denominators are not. Hence, it is obvious that $q_1 > q_1^*$ holds in region $\Gamma_1^{(1)}$.

In region $\Gamma_2^{(1)}$ which is called borrowing region, we have

$$\begin{align*}
q_1^* &= \frac{2(Rx + A)(a_2\eta_2 \rho b_1 b_2 - a_1\eta_1 b_2^2)}{b_1^2 b_2^2 (1 - \rho^2) (\mu - R)^2 + B}, \\
q_2^* &= \frac{2(Rx + A)(a_1\eta_1 \rho b_1 b_2 - a_2\eta_2 b_1^2)}{b_1^2 b_2^2 (1 - \rho^2) (\mu - R)^2 + B}.
\end{align*}$$

For convenience, we denote

$$\begin{align*}
C &= \Delta \frac{(\mu - R)^2}{\sigma^2} > 0, \\
D_1 &= a_2\eta_2 \rho b_1 b_2 - a_1\eta_1 b_2^2 < 0,
\end{align*}$$

and we get

$$q_1 - q_1^* = 2D_1 \left( \frac{rx + A}{B} - \frac{R x + A}{C + B} \right) = 2D_1 \frac{(r - R)xB + (rx + A)C}{B(B + C)}.$$

Note that $D_1 < 0$, $r - R < 0$, and $rx + A < 0$, it comes to the conclusion that $q_1 > q_1^*$. 


In region $\Gamma_3^{(1)}$, which is called full-investment region, it is clear that

$$q_1^* = \frac{D_1}{b_1^2 b_2^2 (1 - \rho^2)} \left( -\frac{\sigma^2 x^2}{(\mu x + A) + \sqrt{(\mu x + A)^2 + \frac{\sigma^2 x^2}{b_1^2 b_2^2 (1 - \rho^2)} B}} \right)$$

$$= \frac{D_1}{B} \left( (\mu x + A) - \sqrt{(\mu x + A)^2 + \frac{\sigma^2 x^2}{b_1^2 b_2^2 (1 - \rho^2)} B} \right) .$$

To determine the relationship between $q_1$ and $q_1^*$, we set the following inequality

$$q_1 - q_1^* = \frac{D_1}{B} \left[ 2(rx + A) - (\mu x + A) + \sqrt{(\mu x + A)^2 + \frac{\sigma^2 x^2}{b_1^2 b_2^2 (1 - \rho^2)} B} \right] > 0. \quad (G.1)$$

By simplifying the inequality, one can show that $(G.1)$ is equivalent to

$$(\mu x + A)^2 + \frac{\sigma^2 x^2}{b_1^2 b_2^2 (1 - \rho^2)} B \leq [(\mu - r)x - (rx + A)]^2$$

$$\Leftrightarrow \quad \frac{\sigma^2 x^2}{b_1^2 b_2^2 (1 - \rho^2)} B \leq -2(rx + A) \cdot 2(\mu - r)x$$

$$\Leftrightarrow \quad x \leq \frac{-2(\mu - r)Ab_1^2 b_2^2 (1 - \rho^2)}{\sigma^2 B + 2(\mu - r)rb_1^2 b_2^2 (1 - \rho^2)} = x^\pi .$$

From (3.17), we can directly obtain $x^\pi > x_1^*$. Hence, it is always true that $q_1 > q_1^*$ in region $\Gamma_3^{(1)}$. By the same method, we can see $q_2 > q_2^*$ in each subregion. \[\square\]

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