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Optimal control and cost-effectiveness analysis for a COVID-19 model with individual protection awareness

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**A B S T R A C T**

This paper is focused on the design of optimal control strategies for COVID-19 and the model containing susceptible individuals with awareness of protection and susceptible individuals without awareness of protection is established. The goal of this paper is to minimize the number of infected people and susceptible individuals without protection awareness, and to increase the willingness of susceptible individuals to take protection measures. We conduct a qualitative analysis of this mathematical model. Based on the sensitivity analysis, the optimal control method is proposed, namely personal protective measures, vaccination and awareness raising programs. It is found that combining the three methods can minimize the number of infected people. Moreover, the introduction of awareness raising program in society will greatly reduce the existence of susceptible individuals without protection awareness. To evaluate the most cost-effective strategy we performed a cost-effectiveness analysis using the ICER method.

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1. Introduction

Since the emergence of the COVID-19 case, it has spread rapidly to the world in a short period of time. More than 200 countries and regions around the world are experiencing the epidemic on a large scale [1]. Statistics as of April 8, 2022 showed that there were 496,738,773 confirmed cases, 6,196,337 confirmed deaths and 432,452,035 recovered cases globally [2]. The transmission characteristics of COVID-19 are very special, with faster spreading speed, wider spreading range and higher spreading risk. Infected people will have symptoms of varying degrees. The mildest one is without any external symptoms (called asymptomatic infection), some just have fever or a mild cough, some will develop into pneumonia and the severe ones may even die. COVID-19 is mainly spread through close person-to-person contact [3], and asymptomatic infected people can also spread the virus to others. COVID-19 spreads most often through close contact and being in close physical distance (within 6 ft) carries a high risk of infection. Respiratory droplets are produced when an infected person coughs, sneezes or talks. If these small droplets are concentrated enough, the susceptible person is likely to be infected with the virus once exposed and then contact with the mouth, nose and eyes may be infected with COVID-19. The spread of the epidemic is very serious and it poses a huge threat to global public health security and economy, so it is necessary to control COVID-19.

Mathematical models play a very important role in the prevention and control of infectious diseases. Many infectious disease models have been developed to investigate the dynamics of disease transmission, such as Ebola [4], measles [5,6], dysentery [7], malaria [8], pneumonia [9], childhood diseases [10] and others. Among them [6,8,9,11] studied the co-dynamics of the two diseases. After the outbreak of the COVID-19, some researchers have established infectious disease

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models based on the transmission characteristics. Some additionally added asymptomatic infected persons [12,13] and some joined the quarantined population [13,14]. Both [15,16] take into account the impact of quarantine on the spread of coronavirus. The authors in [17] constructed a SEIR model of COVID-19, and to conduct stability analysis and numerical simulation of the spread of COVID-19. In [18], a mathematical model for hypertensive or diabetic patients open to COVID-19 is considered along with a set of first-order nonlinear differential equations. And there are also some studies discussing the fractional model of COVID-19 [19]. The Genocchi collocation technique is applied to investigate the proposed fractional mathematical model numerically via Caputo–Fabrizio fractional derivative in [20]. In [21], the COVID-19 transmission dynamics is studied by using CF and ABC non-singular fractional derivatives. The model of COVID-19 disease under fuzzy caputo, random and ABC fractional order derivative is proposed in [22]. More, some researchers have studied stochastic models that fit the dynamics of the spread of the COVID-19 [23].

Some strategies have been implemented to control the COVID-19 such as wearing masks [24], washing hands frequently, maintaining a certain social distance [25], spraying disinfectant in the environment and vaccinating the entire population [26]. These measures are found to be effective in reducing the number of people infected with COVID-19 [27]. In order to propose better control strategies and alleviate the development of the COVID-19 more quickly, many infectious disease model have been added various control methods to investigate the effectiveness of their control measures [28]. An epidemic model with combined vaccination and antiviral controls for COVID-19 pandemic have been studied in [29]. A non-smooth SIR Filippov system is proposed to investigate the impacts of three control strategies (media coverage, vaccination and treatment) on the spread of an infectious disease in [30]. Optimal control approach is often used in the study of control effects, which aims to minimize the number of infected individuals with the least cost [31]. Numerical simulations show that optimal control strategies could effectively reduce the transmission of echinococcosis [32]. Different control strategies are proposed to eradicate the disease in a specific period of time in [33]. Since the outbreak of the COVID-19, numerous mathematical models with optimal control that conform to the law of the COVID-19 transmission have now been developed and studied [34]. The authors in [35] considered setting four different controls to minimize infection spread and control. J.K.K. Asamoah et al. [36] formulated a person-environment-person mathematical model and studied the control effects of isolation, use of masks through media education, proper cough etiquette, fumigation of commercial areas and use of disinfectants. Among the six measures under consideration, maintaining a certain social distancing while coughing was the most cost-effective strategy. The same approach was used by O. Agossou et al. [37] to find the optimal strategy to suppress the disease. The findings suggest that the optimal combination of preventive measures and vaccination is less costly and more effective than a single implementation. In addition, Z. Abbasi et al. [13] aimed to eliminate the infection by isolating and treating the infected. And an impulsive epidemic model was considered to deal with potential population increases due to immigration or travel. A.Kouidere et al. [38] studied an optimal control approach with time delays for state and control variables. The existence of a time delay means that there is a delay in taking preventive measures. These studies that have been done have mainly focused on the transmission dynamics of COVID-19 and the effectiveness of COVID-19 control strategies.

The public’s individual awareness of the COVID-19 and their willingness to implement effective preventive behaviors to reduce the spread of the disease is also an important idea for the COVID-19 control strategy. We can use awareness to produce the effects of social distancing and self-isolation among individuals as a form of indirect control over susceptible individuals. Many mathematical models used to study the impact of individual awareness on the dynamics of disease transmission have been studied. The work by M.Z. Ndii et al. [39] discussed a mathematical model of absence and presence of awareness programs and vector control for malaria. Results of the model showed that the public’s awareness of taking preventive measures can effectively reduce the contact with mosquitoes, and indirectly control the occurrence of diseases. Furthermore, the number of infected people can be reduced with minimal cost by using the Pontryagin maximum principle. Media campaigns are an important tool to raise awareness of personal protection. Kar et al. [40] investigated the effects of awareness on disease transmission. The effects of awareness through media control are taken into account in their model, and the spread of disease can be effectively reduced through media campaigns. We know that there are not many mathematical models to study the impact of awareness on the dynamics of the spread of COVID-19. SS Musa et al. [41] formulated a mathematical model to study the dynamics of the spread of COVID-19 in Nigeria. The model combined awareness programs and different hospitalization strategies for various symptoms to assess the impact of public awareness programs on the dynamics of COVID-19 transmission. Results suggested that awareness raising programs can effectively control and mitigate epidemics. The authors in [42] proposed a deterministic model governed by a system of nonlinear differential equations which consider the intervention of media campaign to increase human awareness, and rapid testing to track the undetected cases in the field. The above article is based on the model established by the COVID-19. On the basis of the above-mentioned model, the model of this paper is established. Therefore, the model in this paper can be considered to be in line with the dynamics of COVID-19 transmission.

In this paper, we investigate a mathematical model that takes into account the impact of individual awareness on the dynamics of COVID-19 transmission. Different from the previous studies above, we divide the traditional susceptible persons into two categories: lack of awareness of protection and presence of awareness of protection. Through the implementation of effective control measures, the goal of minimizing the number of infected persons and converting the susceptible persons without awareness into susceptible persons with awareness to the greatest extent can be achieved. Firstly, the mathematical model is qualitatively analyzed, the basic reproduction number $R_0$ is solved and the stability analysis and bifurcation analysis are carried out. Afterwards, the sensitivity analysis of the model parameters is studied.
in order to find the parameters that have a greater impact on $R_0$. The relevant optimal control methods are proposed for the above parameters, namely personal protective measures, vaccination and awareness raising programs. We use the Pontryagin maximum principle to obtain the optimality system. Numerical simulation results show that a combination of the three methods was more effective than a single method, and can minimize the number of infected people. Moreover, the introduction of awareness raising procedures in society will greatly reduce the existence of susceptible individuals without protection awareness. Finally, we perform a cost-effectiveness analysis using the ICER method, which showed that the only use of vaccination is the most cost-effective strategy and the combination of the personal protective measures, vaccination and awareness raising program is the second most cost-effective strategy.

The subsequent part of the paper is as follows. In Section 2, an uncontrolled infectious disease model with individual disease awareness is proposed, and the dynamic problem of the infectious disease model is analyzed by using differential equation qualitative theory. The boundedness, positive definiteness, basic reproduction number, equilibrium point, stability and bifurcation analysis of the model are also analyzed. Section 3 includes the sensitivity analysis of the parameters of the uncontrolled model to analyze the influence of individual parameters on the disease threshold. Section 4 includes theoretical analysis and numerical simulation of the optimal control. Section 5 includes cost-effectiveness analysis of the optimal control. In Section 6, conclusive evaluations and the outlook for future research are presented.

2. Mathematical modeling and methods

In this section, the dynamics of the spread of COVID-19 are discussed. SEIR can model COVID-19 because it takes into account the existence of an incubation period. A susceptible person who is not immune to the virus may enter an incubation period after contact with an infected person. After that, there is a certain chance of becoming an infected person. After treatment, they will become a recovery population. This is a normal process of the spread of the COVID-19. More details on how SEIR can model COVID-19 are described in [43,44]. With the pandemic of the COVID-19, a group of asymptomatic infected people has emerged. Accordingly, based on the traditional SEIR model, warehouse $A(t)$ is introduced to express the status of asymptomatic infections. In this way, the model can more accurately conform to the current pandemic trend. In the real society, we can see that not everyone can correctly recognize the danger of the epidemic, so we assume that some people are aware of taking preventive measures to avoid infection. This paper divides the total population $N(t)$ during the spread of COVID-19 into six compartments: susceptible individuals who are not aware and aware of COVID-19 ($S_u(t)$ and $S_a(t)$, respectively), exposed/pre-asymptomatic $E(t)$, symptomatic infected $I(t)$, asymptomatic infected $A(t)$ and the recovered $R(t)$ at any time. The population changes in the $S_uS_aEIAR$ dynamic model during the spread of COVID-19 are shown in the flowchart in Fig. 1. Definition of the variables and parameters of the $S_uS_aEIAR$ dynamic model is presented in Table 1.

Here, we make the following hypotheses: We assume that a proportion of the population has a natural awareness of taking preventive measures. Due to the presence of awareness, the infection rate of susceptible individuals with awareness can be reduced to $[1 - \delta u_1]([\beta_1 S_a I + \beta_2 S_a A]$, where $\delta$ represents coefficient of precautions measures for awareness people. And because of some compulsory measures, susceptible people without awareness will also take a little bit of prevention against the virus. But preventive measures are implemented far less than in susceptible individuals with awareness. Suppose that the infection rate of susceptible individuals without awareness can be reduced to $[1 - u_1]([\beta_1 S_u I + \beta_2 S_u A]$. $I(t)$ with symptomatic infection and $A(t)$ with asymptomatic infection have different infectivity which are $\beta_1$ and $\beta_2$ respectively, see [45] for details. After the infected person recovers, they will return to the susceptible population with
Theorem 1. Let the initial conditions \( S_u(0) > 0, S_d(0) > 0, E(0) > 0, I(0) > 0, A(0) > 0 \) and \( R(0) > 0 \). Then, the solutions of the model (1) are non-negative for all \( t > 0 \).

Proof. Assume that \( t_1 = \sup \{ t > 0 : S_u > 0, S_d > 0, E > 0, I > 0, A > 0, R > 0 \} \), \( t_1 \in [0, t] \), where \( \sup \) represents the supremum. Thus, \( t_1 > 0 \).

From the first equation of model (1), we obtain that:

\[
\frac{dS_u}{dt} + (1 - u_1)[\beta_1 S_u I + \beta_2 S_u A] + \mu S_u \geq 0.
\]

It can be re-written in the following form:

\[
\frac{d}{dt}[S_u(t)\exp(\mu t + \int_0^t (1 - u_1)(\beta_1 I(\sigma) + \beta_2 A(\sigma))d\sigma)] = (1 - \tau)\Lambda\exp(\mu t + \int_0^t (1 - u_1)(\beta_1 I(\sigma) + \beta_2 A(\sigma))d\sigma).
\]

This implies that

\[
S_u(t_1)\exp(\mu t_1 + \int_0^{t_1} (1 - u_1)(\beta_1 I(\sigma) + \beta_2 A(\sigma))d\sigma) - S(0) = \int_0^{t_1} (1 - \tau)\Lambda \exp(\mu x + \int_0^x (1 - u_1)(\beta_1 I(\sigma) + \beta_2 A(\sigma))d\sigma)dx.
\]
For $S(0) > 0$, it gives

$$\tilde{S}_u(t) = \exp(-\mu t - \int_0^{t_1} (1 - u_1)\beta_1 I(\sigma) + \beta_2 A(\sigma) d\sigma) \times \int_{t_1}^t (1 - \tau) \Lambda \exp(\mu x + \int_0^x (1 - u_1)\beta_1 I(\sigma) + \beta_2 A(\sigma) d\sigma) dx,$$

which implies that $S_u(t) > 0$.

Regarding other variables, we can deduce that

$$\tilde{S}_u(t) \geq \exp(-\mu t - \int_0^{t_1} (1 - \delta u_1)\beta_1 I(\sigma) + \beta_2 A(\sigma) d\sigma) \times \int_{t_1}^t \Lambda \exp(\mu x + \int_0^x (1 - \delta u_1)\beta_1 I(\sigma) + \beta_2 A(\sigma) d\sigma) dx > 0,$$

$$I(t) \geq I(0) \exp(-\int_0^t (\mu + r_1 + d) d\sigma) > 0,$$

$$A(t) \geq A(0) \exp(-\int_0^t (\mu + r_2) d\sigma) > 0,$$

$$E(t) \geq E(0) \exp(-\int_0^t (\mu + \alpha) d\sigma) > 0,$$

$$R(t) \geq R(0) \exp(-\int_0^t (\mu + r_3) d\sigma) > 0.$$

**Theorem 2.** All solutions of the model (1) are uniformly bounded. The feasible region of the model (1) is positively invariant, written as

$$\Omega = \left\{ (S_u, S_a, E, I, A, R) \in R^6_+ | 0 \leq N(t) \leq \frac{\Lambda}{\mu} \right\}.$$

**Proof.** After taking the first derivative on both sides of Eq. (2), we obtain

$$N'(t) = S_u'(t) + S_a'(t) + E'(t) + I'(t) + A'(t) + R'(t).$$

Substituting from the model (1), we obtain

$$N'(t) = \Lambda - \mu N(t) - dl(t).$$

From Theorem 1, we have $l(t) \geq 0$. Therefore, the following inequality is established:

$$N'(t) + dl(t) \leq \Lambda.$$

In fact, according to the theory of differential inequality [47] we can obtain

$$N(t) \leq N(0)e^{-\mu t} + \frac{\Lambda}{\mu}(1 - e^{-\mu t}).$$

Then, we obtain the solution:

$$\lim_{t \to \infty} \sup N(t) \leq \frac{\Lambda}{\mu},$$

which completes the proof. This means that $N(t)$ is always bounded.

### 2.2. Basic reproduction number

The basic reproduction number $R_0$, which describes the average number of new infections produced by an infected person entering all susceptible populations [48]. It is an important indicator to measure the incidence of infectious diseases.

For model (1), the basic reproduction number $R_0$ is obtained by the next generation matrix approach and found the spectral radius of the matrix [48]. Rewrite the model (1) in the following formula:

$$\frac{dx}{dt} = \Psi(x) - \Phi(x),$$

the quadratic growth rate matrix $\Psi(x)$ and removal rate matrix $\Phi(x)$ are as follows:

$$\Psi(x) = \begin{pmatrix} [(1 - u_1)S_u + (1 - \delta u_1)S_a]\beta_1 I + \beta_2 A \\ 0 \\ 0 \end{pmatrix}, \Phi(x) = \begin{pmatrix} (\mu + \alpha)E \\ -p \alpha E + (\mu + r_1 + d)I \\ -(1 - p) \alpha E + (\mu + r_2)A \end{pmatrix}.$$
The Jacobian matrix at $P_0$ are given by

$$F = \begin{pmatrix}
0 & \beta_1[(1 - u_1)S_u + (1 - \delta u_1)S_0] & \beta_2[(1 - u_1)S_u + (1 - \delta u_1)S_d] \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},$$

$$V = \begin{pmatrix}
(\mu + \alpha) & 0 & 0 \\
-p\alpha & (\mu + r_1 + d) & 0 \\
-(1-p)\alpha & 0 & (\mu + r_2)
\end{pmatrix},$$

where $F$ is a non-negative matrix and $V$ is a non-singular matrix.

Then the basic reproduction number is $R_0 = \rho(FV^{-1})$, where $\rho$ is the spectral radius of the matrix. Matrix $V^{-1}$, inverse of matrix $V$ is given by

$$V^{-1} = \begin{pmatrix}
1 & 0 & 0 \\
(\mu + \alpha) & 0 & 0 \\
(\mu + \alpha)(\mu + r_1 + d) & (\mu + r_1 + d) & 0 \\
(1-\mu)\alpha & 1 & 0 \\
(\mu + \alpha)(\mu + r_2) & 0 & 1
\end{pmatrix}.$$}

Thus, $R_0$ is given as

$$R_0 = \rho(FV^{-1}) = \frac{1 - u_1 + (1 - \delta)u_1 \mu \alpha[\beta_1 p(\mu + r_2) + \beta_2 (1-p)(\mu + r_1 + d)]}{\mu(\mu + \alpha)(\mu + r_2)(\mu + r_1 + d)}.$$}

In the next section, $R_0$ will be used for stability analysis.

2.3 Stability analysis

We can obtain the equilibrium points by making the right-hand side of the equation in model (1) equal to zero. Obviously, the model (1) always has a disease-free equilibrium $P_0 = (\frac{1-\tau}{\tau}A, \frac{1-\mu}{\tau}A, 0, 0, 0, 0)$.

Further, the model (1) has the endemic equilibrium point $P^* = (S_u^*, S_a^*, E^*, I^*, A^*, R^*)$ if and only if $R_0 > 1$. Use $E$ to denote other unknown state variables as follows

$$E^* = \frac{m}{n}I^*,$$

$$A^* = \frac{(1-p)m}{np}I^*,$$

$$R^* = \frac{r_1pn + r_2(1-p)m}{2pn},$$

$$S_u^* = \frac{\mu \alpha(1-\tau)A}{(1-u_1)QI^* + \mu pq},$$

$$S_a^* = \frac{Q(1-u_1)np\mu \alpha(1-\tau)A - qmn[(1-u_1)QI^* + \mu pq]}{[1-(1-u_1)QI^* + \mu pq](\delta u_1 - 1)Q},$$

where $g = \mu + \alpha, m = \mu + r_1 + d, n = \mu + r_2, q = \mu + r_3, Q = [n\beta_1 p\alpha + m\beta_2 (1-p)\alpha], M = r_1 np\alpha + r_2 m(1-p)\alpha, T = \frac{1 - u_1 + (1 - \delta)u_1 \alpha}{\mu}.$ And $I^*$ is the positive solution of quadratic equation

$$c_2 I^{*2} + c_1 I^* + c_0 = 0,$$

where

$$c_2 = (\delta u_1 - 1)(1-u_1)Q^2\mu \alpha [r_3 M - q q n m],$$

$$c_1 = p^2 \alpha^2 q n (\delta u_1 - 1)Q^2 (1-u_1)A (1 - \frac{1}{R_0}) + \frac{c_2 \mu}{1-u_1} Q^2 \mu \alpha + \frac{(\delta - 1)u_1 Q A (1-u_1 + (1 - \delta)u_1 \alpha + r / (\delta u_1 - 1))}{R_0},$$

$$c_0 = p^3 \alpha^3 q^2 \mu^2 g m q (1 - R_0).$$

We discuss the existence of positive roots of the above equations. First prove that $f(I^*) = c_2 I^{*2} + c_1 I^* + c_0$ has at least one positive root. From the positive root, it can be seen that the root should be in the interval $[0, b]$. When $R_0 > 1$, $f(0) = c_0 < 0$, so the required $b$ should satisfy $f(b) > 0$. Because $\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} (c_2 x^{*2} + c_1 x^{*} + c_0) = +\infty$ holds. There is always a point $b$ on $(0, +\infty)$ such that $f(b) > 0$ is satisfied. Therefore, the function values have opposite signs at the endpoints of the interval. Since $I^*$ exists, and $f'(I^*) = 2c_2 I^{*2} + c_1 > 0, I^* \in (0, b)$ holds, so $f(I^*)$ increases monotonically in $(0, b)$. So $f(I^*) = 0$ has only one positive root on $(0, b)$, that is, there is only one positive root on $(0, +\infty)$. Hence, the model (1) has the endemic equilibrium point $P^* = (S_u^*, S_a^*, E^*, I^*, A^*, R^*)$ if and only if $R_0 > 1$.

The globally stability of the equilibrium point is proved in the following.
Theorem 3. The disease free equilibrium of the model (1) is globally asymptotically stable if $R_0 \leq 1$.

Proof. Let the Lyapunov function be defined as:

$$V = E + \frac{[(1 - u_1)\beta_1 S_u + (1 - \delta u_1)\beta_1 S_a]}{m} + \frac{[(1 - u_1)\beta_2 S_u + (1 - \delta u_1)\beta_2 S_a]}{n}$$

The derivative is given as

$$\dot{V} = \dot{E} + \frac{[(1 - u_1)\beta_1 S_u + (1 - \delta u_1)\beta_1 S_a]}{m} + \frac{[(1 - u_1)\beta_2 S_u + (1 - \delta u_1)\beta_2 S_a]}{n}$$

$$= [\beta_1 + \beta_2 A][(1 - u_1)S_u + (1 - \delta u_1)S_a] - gE + \frac{[(1 - u_1)\beta_1 S_u + (1 - \delta u_1)\beta_1 S_a]}{m}$$

$$+ \frac{[(1 - u_1)\beta_2 S_u + (1 - \delta u_1)\beta_2 S_a]}{n}$$

$$\leq [\beta_1 + \beta_2 A]T - gE + \frac{\beta_1 T}{m} (pE - ml) + \frac{\beta_2 T}{n} [(1 - p)\alpha E - ml]$$

$$= -gE + \frac{\beta_1 T p \alpha E}{m} + \frac{\beta_2 T (1 - p) \alpha E}{n}$$

$$= g(R_0 - 1)E$$

From Eq. (4), it shows that $\dot{V} \leq 0$, whenever $R_0 \leq 1$ for all nonnegative state variables and parameters. From the Lyapunov–Lasalle asymptotic stability theorem [49], the disease free equilibrium is globally asymptotically stable when $R_0 \leq 1$. The impose of Theorem 3 is that if the basic reproduction number is less than a unity, then a small number of infected individuals in the general population will not cause a large outbreak of disease. That is to say, the disease will eventually be eliminated.

Theorem 4. If $R_0 > 1$, the unique epidemic equilibrium $P^*$ is globally asymptotically stable.

Proof. Consider the Lyapunov function $L$ defined as:

$$L = (S_u - S_u^* - S_u^* \ln \frac{S_u}{S_u^*}) + (S_a - S_a^* - S_a^* \ln \frac{S_a}{S_a^*}) + (E - E^* - E^* \ln \frac{E}{E^*})$$

$$+ (I - I^* - I^* \ln \frac{I}{I^*}) + (A - A^* - A^* \ln \frac{A}{A^*}) + (R - R^* - R^* \ln \frac{R}{R^*})$$

The time derivative of Lyapunov function is

$$\dot{L} = (1 - \frac{S_u}{S_u^*})\dot{S}_u + (1 - \frac{S_a}{S_a^*})\dot{S}_a + (1 - \frac{E}{E^*})\dot{E} + (1 - \frac{I}{I^*})\dot{I} + (1 - \frac{A}{A^*})\dot{A} + (1 - \frac{R}{R^*})\dot{R}$$

Simplifying the above expression results in

$$\dot{L} = -(1 - \frac{1}{x_1})^2 - \tau \Delta x_2 (1 - \frac{1}{x_2})^2 - (1 - u_1)\beta_1 I^* S_u (x_4 - \frac{x_4}{x_1} + \frac{1}{x_1} - 1)$$

$$-(1 - u_1)\beta_2 A^* S_u (x_5 - \frac{x_5}{x_2} + \frac{1}{x_2} - 1) - (1 - \delta u_1)\beta_1 I^* S_a (x_4 - \frac{x_4}{x_2} + \frac{1}{x_2} - 1)$$

$$-(1 - u_1)\beta_2 A^* S_a (x_5 - \frac{x_5}{x_2} + \frac{1}{x_2} - 1) - r_3 R^* (-x_6 + x_2 + \frac{x_6}{x_2} - 1)$$

$$-(1 - u_1)\beta_1 I^* S_a^* (-x_1 x_4 + x_3 + \frac{x_3}{x_3} - 1) - (1 - u_1)\beta_2 A^* S_a^* (-x_1 x_5 + x_3 + \frac{x_3}{x_3} - 1)$$

where $x_1 = \frac{S_u}{S_u^*}$, $x_2 = \frac{S_a}{S_a^*}$, $x_3 = \frac{E}{E^*}$, $x_4 = \frac{I}{I^*}$, $x_5 = \frac{A}{A^*}$ and $x_6 = \frac{R}{R^*}$.

It follows that $\dot{L} \leq 0$ if $x_1 = 1$, $x_2 = 1$, $x_3 = 1$, $x_4 = 1$, $x_5 = 1$, $x_6 = 1$. Therefore, according to the Lyapunov–Lasalle asymptotic stability theorem [49], the unique epidemic equilibrium $P^*$ is globally asymptotically stable when $R_0 > 1$. 
2.4. Backward bifurcation

In this section, we investigate bifurcation phenomena using a method based on center manifold theory introduced in [50]. The following theorem proves that model (1) exhibits backward bifurcation when \( R_0 = 1 \).

**Theorem 5.** The model (1) exhibits a backward bifurcation at \( \beta_1 = \beta_1^* \) (i.e. at \( R_0 = 1 \)).

**Proof.** Let \( x = (x_1, x_2, x_3, x_4, x_5, x_6)^T = (S_u, S_d, E, I, A, R)^T \), the model (1) can be rewritten as \( \frac{dx}{dt} = f(x) \) with \( f(x) = (f_1(x), f_2(x), f_3(x), f_4(x), f_5(x), f_6(x)) \), that is

\[
\begin{align*}
\frac{dx_1}{dt} &= f_1 = (1 - \tau)A - (1 - u_1)[\beta_1 x_1 x_4 + \beta_2 x_1 x_5] - \mu x_1, \\
\frac{dx_2}{dt} &= f_2 = \tau A - (1 - \delta u_1)[\beta_1 x_2 x_4 + \beta_2 x_2 x_5] - \mu x_2 + r_3 x_6, \\
\frac{dx_3}{dt} &= f_3 = (1 - u_1)[\beta_1 x_3 x_4 + \beta_2 x_3 x_5] + (1 - \delta u_1)[\beta_1 x_3 x_4 + \beta_2 x_3 x_5] - (\mu + \alpha) x_3, \\
\frac{dx_4}{dt} &= f_4 = \rho_1 x_3 - (\mu + r_1 + d) x_4, \\
\frac{dx_5}{dt} &= f_5 = (1 - p) \alpha x_3 - (\mu + r_2) x_5, \\
\frac{dx_6}{dt} &= f_6 = r_1 x_4 + r_2 x_5 - (\mu + r_3) x_6.
\end{align*}
\]

(7)

Let \( \beta_1 \) be the bifurcation parameter and setting \( R_0 = 1 \), we can obtain

\[ \beta_1 = \beta_1^* = \frac{\mu(\alpha + \mu)[(\mu + r_1 + d) + (\mu + r_2)] - [(1 - u_1) + (1 - \delta u_1)1]A\beta_2(1 - p)\alpha(\mu + r_1 + d)}{[(1 - u_1) + (1 - \delta)u_1 \tau]A\rho_1(\mu + r_2)} \]

The Jacobian of the model (7) at the DFE \( P_0 \) with \( \beta_1 = \beta_1^* \) is given by

\[
J(P_0) = \begin{pmatrix}
-\mu & 0 & 0 & -\frac{(1 - u_1)\beta_1(\frac{1 - \tau}{\mu})A}{\mu} & -\frac{(1 - u_1)\beta_2(\frac{1 - \tau}{\mu})A}{\mu} & 0 \\
0 & -\mu & 0 & -\frac{(1 - \delta u_1)\beta_1 \frac{\tau A}{\mu}}{\mu} & -\frac{(1 - \delta u_1)\beta_2 \frac{\tau A}{\mu}}{\mu} & r_3 \\
0 & 0 & -(\mu + \alpha) & 0 & 0 & 0 \\
0 & 0 & -(1 - p) \alpha & 0 & 0 & -(\mu + r_2) \\
0 & 0 & 0 & r_1 & r_2 & -(\mu + r_3)
\end{pmatrix}
\]

The above Jacobian matrix has zero eigenvalues and all other eigenvalues have negative real parts. The right and left eigenvectors associated with zero eigenvalues are given by \( w = [w_1, w_2, w_3, w_4, w_5, w_6]^T \) and \( v = [v_1, v_2, v_3, v_4, v_5, v_6]^T \) respectively, where \( w_1 = \frac{(1 - u_1)Q w_3 + \mu mn}{mn(1 - \tau)A} \), \( w_2 = \frac{q(1 - \delta u_1)Q w_3 + \mu mmn}{mn(1 - \tau)A} \), \( w_3 = w_3 > 0 \), \( w_4 = \frac{(1 - p)\alpha w_3}{(\mu + r_1 + d)} \), \( w_5 = \frac{(1 - p)\alpha w_3}{(\mu + r_2)} \), \( w_6 = \frac{(1 - p)\alpha w_3}{(\mu + r_1 + d)} \), \( v_1 = v_2 = v_6 = 0 \), \( v_3 = v_3 > 0 \), \( v_4 = \beta_1(1 - u_1 + (1 - \delta) u_1 \tau)A v_3 \), \( v_5 = \beta_2(1 - u_1 + (1 - \delta) u_1 \tau)A v_3 \), with \( w \) and \( v \) satisfying \( v \cdot w = 1 \). Based on the theory in [50], we need to compute \( a \) and \( b \) where,

\[
a = \sum_{i,j=1}^{n} v_i w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(P_0, \beta_1^*),
\]

\[
b = \sum_{i,k=1}^{n} v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial \phi}(P_0, \beta_1^*).
\]

Therefore,

\[
a = \frac{v_3 v_3 Q}{mn} \frac{mn(1 - \tau)A(1 - u_1)}{(1 - u_1)w_3 Q + \mu mn} + \frac{q mn \tau A + w_3 r_3 (r_1 p \alpha n + r_2 (1 - p) \alpha m) w_3}{q w_3 Q + \mu mmn/(1 - u_1)} > 0,
\]

\[
b = \frac{[1 - u_1 + (1 - \delta) u_1 \tau]A \rho_1 w_3 v_3}{\mu (\mu + r_1 + d)} > 0.
\]
Table 2

| Parameter Values | Reference | Parameter Values | Reference |
|------------------|-----------|------------------|-----------|
| $\mu$ 3.933 $\times 10^{-5}$ | [2, 51] | $r_3$ 0.41 | Assumed |
| $\beta_1$ 0.476 | Estimated | $\tau$ 0.4 | [52] |
| $\beta_2$ 0.238 | Estimated | $\Lambda$ 9.43 $\times 10^{-5}$ | [2, 51] |
| $\alpha$ 0.19 | [53] | $\varphi$ 0.1 | Assumed |
| $p$ 0.88 | [53, 55] | $\varphi$ 0.1 | Assumed |
| $d$ 0.0013 | [2, 51] | $u_1$ [0, 1] | Simulated |
| $r_1$ 0.2 | [2, 56] | $u_2$ [0, 1] | Simulated |
| $r_2$ 0.1 | [2, 56] | $u_3$ [0, 1] | Simulated |
| $\delta$ 2 | Assumed | $u$ [0, 1] | Simulated |

Fig. 2. The stability of epidemic equilibrium $P^*$ using different initial values when $R_0 > 1$, where the X-axis is $E$, the Y-axis is $I$, and the Z-axis is $A$.

It can be observed that $a > 0$ and $b > 0$ by calculation. So the model (1) undergoes a backward bifurcation when $a > 0$ and $b > 0$ at $R_0 = 1$.

3. Numerical simulation

In this section, we present some numerical simulation results to verify the theoretical analysis results mentioned in the previous section. Our main work is to explore the effectiveness of the model. Cumulative daily reported COVID-19 cases extracted from WHO and Indian Ministry of Health reports using India as the study case [51]. From a biological point of view, the parameter values of all models are non-negative. We use the initial condition $S_a(0) = 0.5964, S_a(0) = 0.3976, E(0) = 0.0000497, I(0) = 0.0003813, A(0) = 0, R(0) = 0.005569$. The other various parameter values are shown in Table 2.

According to the initial value of the model (1) and the various parameter values shown in Table 2, the basic reproduction number $R_0$ is 3.2895, which is greater than 1. It indicates that the endemic equilibrium point $P^*$ is globally asymptotically stable. In the absence of control measures, the COVID-19 has a strong transmission power. In order to illustrate the local stability of the equilibrium point, we choose different initial conditions to simulate the model. Fig. 2 demonstrates the local asymptotic stability of the epidemic equilibrium $P^*$ when the reproduction number $R_0 > 1$.

3.1. Sensitivity analysis of the reproduction number

In this section, we conduct a sensitivity analysis to study the sensitivity of the basic reproduction number $R_0$ due to the uncertainty of the model parameters, and find the relationship between the basic reproduction number and the various parameters related to it. The purpose of this is to find several parameters that have a greater impact on $R_0$ and implement corresponding control measures on them to minimize the spread of the disease.

In calculating the normalized sensitivity index ($\varepsilon_{R_0}^p$) on the $R_0$ for each parameter $p$, we use the following formula in [57]:

$$\varepsilon_{R_0}^p = \frac{\partial R_0}{\partial p} \frac{p}{R_0}.$$
Table 3

| Parameter | Sensitivity index | Parameter | Sensitivity index |
|-----------|-------------------|-----------|-------------------|
| $\Lambda$ | +1.0000           | $p$       | +0.00055          |
| $\beta_1$ | +0.8854           | $d$       | −0.00576          |
| $\beta_2$ | +0.1200           | $r_1$     | −0.8855           |
| $\tau$    | −0.1372           | $r_2$     | −0.1208           |
| $\alpha$  | +0.00002114       | $u_1$     | −0.1373           |

Fig. 3. Global sensitivity plot.

The parameter sensitivity index obtained by applying the above formula is shown in Table 3. It can be seen from Table 3 that in the absence of awareness programs and vaccination, parameters such as $\beta_1$, $\beta_2$ and $p$ are positively correlated with $R_0$. As the values of these parameters increase, the degree of infection of the COVID-19 will become more severe. On the other hand, the parameters $r_1$, $r_2$ and $\tau$ are parameters with negative sensitivity exponents. Sensitivity analysis of the model will play an important role in the fight against the epidemic. The most influential parameter values are $\beta_1$, $r_1$ and $\tau$. Therefore, we propose three control strategies for these parameter values. We denote personal protection measures as $u_1$, vaccinations as $u_2$ and awareness raising programs as $u_3$. In the presence of awareness raising programs and vaccination, other parameter values are the same as those without a control strategy. Then the control parameters $u_1$, $u_2$ and $u_3$ are parameters that have an impact on $R_0$. An increase in the control parameter has an effect on a reduction in $R_0$. Awareness raising programs have a significant effect on $R_0$. Fig. 3 shows the global sensitivity analysis.

4. Optimal control approach

4.1. The formulation of optimal control

Optimal control of infectious diseases is an important and complex issue. In this section, the main task is to study how to minimize the number of infections and costs. We applied an optimal control approach to the model (1) to explore the effects of control on the dynamics of the transmission dynamics of COVID-19.

Based on the sensitivity analysis in the previous section, we extended model (1) to include vaccination and awareness raising programs (such as media campaigns on social platforms, radio, television and other social media). We hypothesized that vaccination of susceptible populations would provide immunity within a short period of time after successful vaccination, thereby effectively preventing infection. Therefore, the time delay is not considered for the time being. We assumed that the rate of successful vaccination of susceptible persons is $u_2$. Furthermore, the awareness raising programs can increase the number of susceptible persons with protective awareness. The rate of awareness raising programs is $u_3$. As mentioned in the first part, $u_1$ represents the implementation rate of protective measures. The control model is
represented by the following system of differential equations:

\[
\begin{align*}
\frac{dS}{dt} & = (1 - \tau)A - (1 - u_1)\beta_1S + \beta_2S_2A - \mu S - \varepsilon u_2 S - \varphi u_3 S, \\
\frac{dS_2}{dt} & = \tau A + \varphi u_3 S_2 - (1 - \delta u_1)\beta_1S_2 + \beta_2S_2A - \mu S_2 + r_3 - \varepsilon u_2 S_2, \\
\frac{dE}{dt} & = (1 - u_1)(\beta_1I + \beta_2A)S - (1 - \delta u_1)(\beta_1I + \beta_2A)S_2 - (\mu + \alpha)E, \\
\frac{dI}{dt} & = p\alpha E - (\mu + r_1 + d)I, \\
\frac{dA}{dt} & = (1 - p)\alpha E - (\mu + r_2)A, \\
\frac{dR}{dt} & = r_1I + r_2A - (\mu + r_3)R + \varepsilon u_2(S_2 + S),
\end{align*}
\]  

(10)

with initial conditions \(S_u(0) = S_{u0} \geq 0, S_0(0) = S_{o0} \geq 0, E(0) = E_0 \geq 0, I(0) = I_0 \geq 0, A(0) = A_0 \geq 0\) and \(R(0) = R_0 \geq 0\).

4.2. Existence and characterization of optimal control

The main goal of optimal control theory applied to disease control is to identify a control set that minimizes the number of exposed, symptomatic and asymptomatic infections within an optimal time interval. However, when considering control measures, the cost of vaccination and implementation of awareness raising programs should also be kept as low as possible. Based on these considerations, we define the following objective function:

\[
J(u_1, u_2, u_3) = \int_0^{t_f} [A_1E + A_2I + A_3A + \frac{B_1}{2}u_1^2 + \frac{B_2}{2}u_2^2 + \frac{B_3}{2}u_3^2]dt
\]

(11)

where \(t_f\) represents the terminal time. The coefficient \(A_1, A_2\) and \(A_3\) are balancing coefficient for the controls. \(B_1, B_2\) and \(B_3\) denote the weight constants on the benefit and cost. \(\frac{B_1}{2}u_1^2, \frac{B_2}{2}u_2^2\) and \(\frac{B_3}{2}u_3^2\) denote costs of disease prevention measures, vaccination and awareness raising programs.

Assume that \(U = \{u = (u_1(t), u_2(t), u_3(t)) : 0 \leq u_i(t) \leq 1, t \in [0, t_f], i = 1, 2, 3\}\) is a measurable control set. The control function \(u_1, u_2\) and \(u_3\) are bounded and Lebesgue measurable function. We choose the quadratic objective functional because the intervention is generally nonlinear, the advantage of which is that the obtained control function is continuous.

We aim to find a set of controls such that

\[
J(u^*_1, u^*_2, u^*_3) = \min_{U} J(u_1(t), u_2(t), u_3(t))
\]

subject to the model (10).

In order to prove the existence of optimal control, we employ the results from Fleming Rishel [58] and Lukes [59]. First, we explain the boundedness of system state variables. We add all the formulas in the model (10) to get \(N(t) \leq N(0)\). From the characteristics of the infectious disease model, it can be known that \(0 \leq S_u(t), S_0(t), E(t), I(t), A(t), R(t) \leq N(t)\). That is, the system state variables are bounded. For the convenience of calculation, let

\[
L(t, \bar{x}, \bar{u}) = A_1E + A_2I + A_3A + \frac{B_1}{2}u_1^2 + \frac{B_2}{2}u_2^2 + \frac{B_3}{2}u_3^2.
\]

When the following conditions are satisfied, the existence of the optimal control solution is guaranteed.

(a) The set of control variables and corresponding state variables is not empty.

(b) The admissible control set is compact and bounded.

(c) The vector function \(f(t, \bar{x}, \bar{u})\) formed by the right side of the system state equation is continuous and can be written in the following form \(f(t, \bar{x}, \bar{u}) = \mu(t, \bar{x}) + v(t, \bar{x}, \bar{u})\).

(d) The integrand \(L(t, \bar{x}, \bar{u})\) of the objective function is a convex function on the control set and there exists constant \(\zeta > 1, \omega_1 > 0\) and \(\omega_2\), such that satisfying \(L(t, \bar{x}, \bar{u}) \geq \omega_1\|u\|^\zeta - \omega_2\).

From the definition of the control set, it can be seen that for each admissible control function, the solution of the system state equation is continuous and bounded. The function on the right side of the model (10) satisfies the Lipschitz condition with respect to the state variables, so the solution of the model (10) exists. Therefore, condition (a) and (b) are satisfied. The right side of the equation of the model (10) is obviously continuous, so condition (c) is satisfied.

To prove that the integrand is a convex function on the control set, just prove that for any \(0 < \theta < 1\), the following formula is satisfied.

\[
L(t, \bar{x}, (1 - \theta)\bar{u} + \theta\bar{u}) \leq (1 - \theta)L(t, \bar{x}, \bar{u}) + \theta L(t, \bar{x}, \bar{u}),
\]
where \( \tilde{u} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)^T, \tilde{u} = (u_1, u_2, u_3)^T \). Let \( Y = A_1 E + A_2 I + A_3 A \). Then there is,

\[
L(t, \tilde{x}, (1 - \theta)\tilde{u} + \theta \tilde{u}) = Y + \frac{B_1}{2} [(1 - \theta)\tilde{u}_1 + \theta \tilde{u}_1] + \frac{B_2}{2} [(1 - \theta)\tilde{u}_2 + \theta \tilde{u}_2]^2 + \frac{B_3}{2} [(1 - \theta)\tilde{u}_3 + \theta \tilde{u}_3]^2
\]

\[
= Y + \frac{B_1}{2} \tilde{u}_1^2 + \frac{B_2}{2} \tilde{u}_2^2 + \frac{B_3}{2} \tilde{u}_3^2 + \frac{B_1}{2} [(\theta^2 - 2\theta)\tilde{u}_1^2 + \theta^2 \tilde{u}_1^2 + 2\theta(1 - \theta)\tilde{u}_1]\]

\[
+ \frac{B_2}{2} [(\theta^2 - 2\theta)\tilde{u}_2^2 + \theta^2 \tilde{u}_2^2 + 2\theta(1 - \theta)\tilde{u}_2]\]

\[
+ \frac{B_3}{2} [(\theta^2 - 2\theta)\tilde{u}_3^2 + \theta^2 \tilde{u}_3^2 + 2\theta(1 - \theta)\tilde{u}_3].
\]

\[\tag{12}\]

\[\text{(12) } L(t, \tilde{x}, (1 - \theta)\tilde{u} + \theta \tilde{u}) = Y + \frac{B_1}{2} \tilde{u}_1^2 + \frac{B_2}{2} \tilde{u}_2^2 + \frac{B_3}{2} \tilde{u}_3^2 + \theta \left( \frac{B_1}{2} (\tilde{u}_1^2 - \tilde{u}_1^2) \right) + \frac{B_2}{2} (\tilde{u}_2^2 - \tilde{u}_2^2) + \frac{B_3}{2} (\tilde{u}_3^2 - \tilde{u}_3^2).\]

\[\tag{13}\]

Subtract the (12) and (13) and we can get

\[
L(t, \tilde{x}, (1 - \theta)\tilde{u} + \theta \tilde{u}) - (1 - \theta)L(t, \tilde{x}, \tilde{u}) + \theta L(t, \tilde{x}, \tilde{u})
\]

\[
= - \frac{B_1}{2} [(\theta - \theta^2)\tilde{u}_1^2 - 2\theta(1 - \theta)\tilde{u}_1\tilde{u}_1 + (\theta - \theta^2)\tilde{u}_1^2]
\]

\[
+ \frac{B_2}{2} [(\theta - \theta^2)\tilde{u}_2^2 - 2\theta(1 - \theta)\tilde{u}_2\tilde{u}_2 + (\theta - \theta^2)\tilde{u}_2^2]
\]

\[
+ \frac{B_3}{2} [(\theta - \theta^2)\tilde{u}_3^2 - 2\theta(1 - \theta)\tilde{u}_3\tilde{u}_3 + (\theta - \theta^2)\tilde{u}_3^2])
\]

\[\tag{14}\]

\[= - \frac{B_1}{2} [\sqrt{\theta(1 - \theta)}(\tilde{u}_1 - \tilde{u}_1)^2] - \frac{B_2}{2} [\sqrt{\theta(1 - \theta)}(\tilde{u}_2 - \tilde{u}_2)]
\]

\[- \frac{B_3}{2} [\sqrt{\theta(1 - \theta)}(\tilde{u}_3 - \tilde{u}_3)^2].\]

Since \( Y = A_1 E + A_2 I + A_3 A \geq 0 \) always holds, we can get

\[
L(t, \tilde{x}, \tilde{u}) = A_1 E + A_2 I + A_3 A + \frac{B_1}{2} u_1^2 + \frac{B_2}{2} u_2^2 + \frac{B_3}{2} u_3^2
\]

\[\geq \min \left\{ \frac{B_1}{2}, \frac{B_2}{2}, \frac{B_3}{2} \right\} (u_1^2(t) + u_2^2(t) + u_3^2(t))
\]

\[\tag{15}\]

\[= \min \left\{ \frac{B_1}{2}, \frac{B_2}{2}, \frac{B_3}{2} \right\} \| \tilde{u} \|_3^2\]

\[\text{Let } \zeta = 2, \omega_1 = \min \left\{ \frac{B_1}{2}, \frac{B_2}{2}, \frac{B_3}{2} \right\}, \omega_2 = 0, \text{ so the condition (d) is proved.}\]

\[\text{Next, we will use the Pontryagin’s Maximum Principle [60] to derive the necessary conditions for the optimal control function, transform the optimal control problem into a Hamiltonian function minimization problem. The Hamiltonian } H \text{ is written as}\]

\[
H = L + \lambda_1 \frac{dS_u}{dt} + \lambda_2 \frac{dS_o}{dt} + \lambda_3 \frac{dE}{dt} + \lambda_4 \frac{dI}{dt} + \lambda_5 \frac{dA}{dt} + \lambda_6 \frac{dR}{dt}
\]

\[
= A_1 E + A_2 I + A_3 A + \frac{B_1}{2} u_1^2 + \frac{B_2}{2} u_2^2 + \frac{B_3}{2} u_3^2
\]

\[+ \lambda_1 [(1 - \tau) A - (1 - u_1)] [\beta_1 S_I + \beta_2 S_o A] - \mu S_o - \epsilon u_2 S_u - \phi u_3 S_u)
\]

\[+ \lambda_2 (\gamma A + \phi u_3 S_u - (1 - \delta u_1) [\beta_1 S_I + \beta_2 S_o A] - \mu S_o + r_1 R + \epsilon u_2 S_u)
\]

\[+ \lambda_3 [(1 - u_1) [\beta_1 I + \beta_2 A] S_u + (1 - \delta u_1) [\beta_1 I + \beta_2 A] S_o - (\mu + \alpha) E]
\]

\[+ \lambda_4 [p a E - (\mu + r_1 + \delta)]
\]

\[+ \lambda_5 [(1 - p) \alpha E - (\mu + r_2) A]
\]

\[+ \lambda_6 [r_1 + r_2 A - (\mu + r_2) R + \epsilon u_2 (S_u + S_o)]
\]

\[\text{where } \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)^T, \lambda_1 \text{ is a comorphic variable associated with } S, S_a, E, I, A, \text{ and } R.\]
Theorem 6. Let $u_1^*, u_2^*$ and $u_3^*$ be the control functions for the control problem given in (10), and $S_u^*, S_a^*, E^*, I^*, A^*$ and $R^*$ be the solutions of state variables. Then there are adjoint variables $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ and $\lambda_6$ that satisfy the following equations

\[
\begin{align*}
\frac{dx_1}{dt} &= (\lambda_1 - \lambda_3)(1 - u_1)(\beta_1 I + \beta_2 A) + \lambda_1 \mu + (\lambda_1 - \lambda_6)\varepsilon u_2 \\
\frac{dx_2}{dt} &= (\lambda_2 - \lambda_3)((1 - \delta u_1)(\beta_1 I + \beta_2 A)) + \lambda_2 \mu + (\lambda_2 - \lambda_6)\varepsilon u_2 \\
\frac{dx_3}{dt} &= \lambda_3(\mu + \alpha) - \lambda_4 \alpha - \lambda_5(1 - p)\alpha - A_1 \\
\frac{dx_4}{dt} &= (\lambda_1 - \lambda_3)(1 - u_1)\beta_1 S_u + (\lambda_2 - \lambda_3)(1 - \delta u_1)\beta_2 S_u + \lambda_4(\mu + r_1 + d) - \lambda_5 r_1 - A_2 \\
\frac{dx_5}{dt} &= (\lambda_1 - \lambda_3)(1 - u_1)\beta_2 S_u + (\lambda_2 - \lambda_3)(1 - \delta u_1)\beta_2 S_u + \lambda_5(\mu + r_2) - \lambda_6 r_2 - A_3 \\
\frac{dx_6}{dt} &= -\lambda_2 r_3 + \lambda_6(\mu + r_3)
\end{align*}
\]

with transversity condition $\lambda_i(t_f) = 0$, $i = 1, 2, 3, 4, 5, 6$. Moreover, the optimal control strategies $u_1^*, u_2^*$ and $u_3^*$ are set as

\[
\begin{align*}
u_1^* &= \max \left\{ 0, \min(1, \frac{(\lambda_3 - \lambda_1)S_u + (\lambda_3 - \lambda_2)\varepsilon S_a}{B_1}) \right\}, \\
u_2^* &= \max \left\{ 0, \min(1, \frac{(\lambda_1 - \lambda_6)\varepsilon S_u + (\lambda_2 - \lambda_6)\varepsilon S_a}{B_2}) \right\}, \\
u_3^* &= \max \left\{ 0, \min(1, \frac{(\lambda_1 - \lambda_2)\varepsilon S_u}{B_3}) \right\}.
\end{align*}
\]

Proof. Using Pontryagin maximum principle, perform derivative operations on Hamiltonian with respect to each state and the following comorphism equations can be obtained.

\[
\frac{d\lambda_i}{dt} = -\frac{\partial H}{\partial x_i},
\]

where $i = 1, 2, 3, 4, 5, 6$ denotes $S_u, S_a, E, I, A, R$ respectively. The following equations are used to find the optimal control functions

\[
\begin{align*}
\frac{\partial H}{\partial u_1} &= B_1 u_1 + (\lambda_1 - \lambda_3)(\beta_1 I + \beta_2 A)S_u + (\lambda_2 - \lambda_3)\delta(\beta_1 I + \beta_2 A)S_a = 0, \\
\frac{\partial H}{\partial u_2} &= B_2 u_2 + (\lambda_6 - \lambda_1)\varepsilon S_u + (\lambda_6 - \lambda_2)\varepsilon S_a = 0, \\
\frac{\partial H}{\partial u_3} &= B_3 u_3 + (\lambda_2 - \lambda_1)\varepsilon S_u = 0.
\end{align*}
\]

Solving for $u_1$, $u_2$ and $u_3$ gives

\[
\begin{align*}
u_1 &= \frac{[(\lambda_3 - \lambda_1)S_u + (\lambda_3 - \lambda_2)\varepsilon S_a](\beta_1 I + \beta_2 A)}{B_1}, \\
u_2 &= \frac{(\lambda_1 - \lambda_6)\varepsilon S_u + (\lambda_2 - \lambda_6)\varepsilon S_a}{B_2}, \\
u_3 &= \frac{(\lambda_1 - \lambda_2)\varepsilon S_u}{B_3}.
\end{align*}
\]

Using the bounds, we can obtain the Eq. (18).

4.3. Optimal control simulation

In this section, the numerical simulation part of the optimal control problem is discussed using the Forward-Backward Sweep method [60] and the results are presented graphically. Model (10) is iterated forward in time, with the accompanying system iterating backward in time. When optimal convergence is reached, the iterations of the forward and backward simulations will stop. The specific algorithm steps are as follows

Step 1. Make an appropriate initial guess $\overline{u_0}$ for the control variable $\overline{u}$ over the interval.

Step 2. Using the initial condition, solve $\overline{X}$ forward in time according to the optimality system (9).
Step 3. Using the current state values for \( \vec{x} \) and \( \vec{u} \) and the transversality condition, solve \( \vec{\lambda} \) backward in time according to the optimality system (9).

Step 4. Update \( \vec{u} \) by entering the values obtained in Step 1 and Step 2 into optimal conditions (20).

Step 5. Check convergence. Determine the value of \( \mid \vec{u}_0^+ (t_0) - \vec{u}_0 (t_0) \mid \). If \( \mid \vec{u}_0^+ (t_0) - \vec{u}_0 (t_0) \mid > 10^{-3} \), substitute \( \frac{1}{2}(\vec{u}_0^+ (t_0) + \vec{u}_0 (t_0)) \) for \( \vec{u}_0 \) and repeat the above three steps. Stop the iteration when the absolute value of the difference between the current and the next two iterations is less than \( 10^{-3} \).

The simulated values of this model were selected to theoretically study the effects of personal protection measures, vaccination and awareness raising programs on mitigating the spread of COVID-19. This is a theoretical analysis. Furthermore, based on the assumptions related to the prevention of COVID-19, both control strategies \( u_2 \) and \( u_3 \) are restricted between 0 and 1, i.e. \( 0 \leq u_i(t) \leq 1, i = 2, 3 \). The control strategy \( u_1 \) is limited between 0.3 and 1, because we believe that even people without protection awareness will passively take protective measures due to some government policies. The proportion of such protection measures will be relatively low initially. It is more efficient for people with protection awareness to take protective measures. We set it as \( \delta u_1 \), and the value of \( \delta \) is set to 2. So the probability of \( S_u \) being infected is \( (1 - u_1)[\beta_1S_u + \beta_2S_u A] \) and the probability of \( S_a \) being infected is \( (1 - \delta u_1)[\beta_1S_u + \beta_2S_u A] \). In the following, we consider four classifications of control strategies for optimal control evaluation.

- Category 1: The combination of personal protection measures \( (u_1) \), vaccination \( (u_2) \) and awareness raising program \( (u_3): u_1 \neq 0, u_2 \neq 0, u_3 \neq 0 \).
- Category 2: The combination of personal protection measures \( (u_1) \) and vaccination \( (u_2) \) and awareness raising program \( (u_3): u_1 \neq 0, u_2 \neq 0 \).
- Category 3: The combination of only vaccination \( (u_2) \): \( u_2 \neq 0 \).
- Category 4: The combination of only personal protection measures \( (u_1) \) and awareness raising program \( (u_3): u_1 \neq 0, u_3 \neq 0 \).

In this way, it can be observed that outbreaks do not occur with optimal control of personal protective measures \( (u_1) \), vaccinations \( (u_2) \) and awareness raising programs \( (u_3) \). Fig. 4 shows all control implementations on the model.

Fig. 4a, b, c, d, e and f represent the dynamics of \( I, A, E, R, S_u \) and \( S_a \) in the case of Category 1. Among them, Fig. 4a, Fig. 4b, Fig. 4c and Fig. 4e show that this category can effectively reduce the number of \( I, A, E \) and \( S_u \). On the contrary, this category greatly increases the proportion of recoverers in the population, as shown in Fig. 4d, the proportion is as high as 0.6. This indicates that more than half of the population has basically achieved immunity. In Fig. 4f, the number of susceptibles with awareness was lower than that of the uncontrolled control group at 165 days, and the number was greater than that of the control group after 165 days. However, in Fig. 4e, the presence of susceptible individuals without awareness can be effectively reduced to zero. Fig. 4h shows the trend of the control variable \( u_1, u_2 \) and \( u_3 \) as a function of time. It can be seen that the control variable \( u_1, u_2 \) and \( u_3 \) maintain a maximum value for the first 155, 154 and 80 days respectively and then decreases monotonically to a minimum value. Fig. 4g is a control effect graph showing the effectiveness of Category 1 for symptomatic and asymptomatic patients, respectively. The results show that Category 1 can reach 100% effectiveness at the end of the simulation period. The functions in the control effect diagram are defined as follows [61]:

\[
E_I = \frac{I(0) - I^*(t)}{I(0)}, E_A = \frac{A(0) - A^*(t)}{A(0)},
\]

where \( I(0) \) and \( A(0) \) are the initial data and \( I^*(t) \) and \( A^*(t) \) are the function relating to the optimal state of control.

Category 2: The combination of personal protection measures \( (u_1) \) and awareness raising program \( (u_3): u_1 \neq 0, u_3 \neq 0 \).

In this category, Fig. 5 shows all control implementations on the model. Fig. 5d shows that with Category 2, the recoverers will be reduced to 0 after ten days and then maintained to 0. This means that if there is a lack of vaccination, there will be no recoverers in the whole society. It can be observed in Fig. 5f that the number of susceptibles with awareness will reach 1 after 50 days. This means that although there are basically all susceptible people in society, they have a good awareness of epidemic prevention and can take corresponding defensive measures. This is in line with our expectations. Similarly, we can notice from Fig. 5g that Category 2 does achieve 100% effectiveness at the end of the simulation period.

Category 3: The use of only vaccination \( (u_2): u_2 \neq 0 \). Fig. 6 illustrates the situation with only \( u_2 \) control on the model (10). Figs. 6a, b, c, d, e and f represent the dynamics of \( I, A, E, R, S_u \) and \( S_a \) in the case of Category 3. This category greatly increases the proportion of recoverers in the population, as shown in Fig. 6d, the proportion is as high as 0.6. In Fig. 6e, the presence of susceptible individuals without awareness can be effectively reduced to zero. Fig. 6f shows that under this control category, the number of susceptible individuals with awareness gradually increases and finally reaches 1. Finally, we can notice from Fig. 6g that Category 3 does achieve 100% at the end of the simulation period. Fig. 6h depicts control \( u_1, u_2 \) and \( u_3 \) which control the simulation time over all.

Category 4: The combination of personal protection measures \( (u_1) \) and vaccination \( (u_2): u_1 \neq 0, u_2 \neq 0 \).

Fig. 7 illustrates the situation with \( u_1 \) and \( u_2 \) control on the model (10). This category greatly increases the proportion of recoverers in the population, as shown in Fig. 7d, the proportion is as high as 0.6. In Fig. 7e, the presence of susceptible
Fig. 4. Simulation results for Category 1. (a), (b), (c), (d), (e) and (f) represent the number of symptomatic infected, asymptomatic infected, exposed, recovered, susceptible without awareness and susceptible with awareness. (g) depicts the control effect. (h) depicts the profiles of optimal controls.

individuals without awareness can be effectively reduced to zero. Fig. 7f shows that under this control category, the number of susceptible individuals with awareness gradually increases and finally reaches 1.
Fig. 5. Simulation results for Category 2. (a), (b), (c), (d), (e) and (f) represent the number of symptomatic infected, asymptomatic infected, exposed, recovered, susceptible without awareness and susceptible with awareness. (g) depicts the control effect. (h) depicts the profiles of optimal controls.

Fig. 8 shows the corresponding dynamics of various groups of people in the model model (10) obtained by simulating the four optimal control categories. However, there are some differences in the control effects of various categories. While many people know that COVID-19 is highly contagious and spreads in a variety of ways. But there are always some people...
Fig. 6. Simulation results for Category 3. (a), (b), (c), (d), (e) and (f) represent the number of symptomatic infected, asymptomatic infected, exposed, recovered, susceptible without awareness and susceptible with awareness. (g) depicts the control effect. (h) depicts the profiles of optimal controls.

who do not take preventive measures seriously and do not actively participate in vaccination. This does not guarantee a very good reduction in the number of the COVID-19 infections. Among them, the implementation of control measure $u_2$ will increase the number of recoverers in society. That is, if there is no vaccination $u_2$, there will be no people in society
who are immune to the virus. They will exist in a susceptible form, which increases the risk of disease. In addition, the implementation of control measures $u_1$ and $u_3$ will increase people's awareness of protection and increase the number
of susceptible people with awareness in the whole society, eventually reaching 1. Our goal is to zero out the number of susceptible people without awareness. All four control categories can effectively reduce the number of infected and susceptible without awareness populations, increase the susceptible with awareness and recovered populations.

Control results suggest that vaccination and awareness raising programs should be implemented at a higher level. In addition, coupled with protective measures taken by individuals with awareness, the level of reduction in transmission rates should be higher and the number of infected people decline faster. In other words, the results of the study show the importance of the awareness raising programs. Because with the addition of the awareness raising programs, the number of susceptibles without awareness will eventually decrease to 0. And the susceptibles existing in society will be the susceptibles with protective awareness. This will reduce the infection rate and thus reduce the risk of disease. The results show that the control measures alone can reduce the number of infected people and the combination of the two measures is more effective than alone. In addition, the combination of the three measures resulted in a greater degree of control, with fewer infections and a higher number of people who eventually recovered.

In general, according to the results of the above analysis, the effective use of the above optimal control will greatly reduce the number of infected people. As far as the pandemic is concerned, as long as we achieve a good level of epidemic prevention and control, the epidemic situation will remain generally stable and people can resume their normal lives.

5. Economic evaluation

In this section, we conduct a cost-effectiveness analysis of the aforementioned COVID-19 control strategies. Through a cost-effectiveness analysis evaluate, we can determine the most cost-effective strategy. Therefore, we use the incremental cost-effectiveness ratio (ICER) [62] to assess costs. The mathematical definitions of this economic assessments are as
Table 4
Comparison between Category 2 and 3.

| Category  | Total infections averted | Total cost | ICER         |
|-----------|--------------------------|------------|--------------|
| Category 3| 56999023.2               | 10604      | 1.860 × 10⁻⁴ |
| Category 2| 57001838.4               | 4001.7     | −2.345       |
| Category 4| 57007652.34              | 14006      | 1.721        |
| Category 1| 57008222.28              | 12005      | −3.511       |

Table 5
Comparison between Category 2 and 4.

| Category  | Total infections averted | Total cost | ICER         |
|-----------|--------------------------|------------|--------------|
| Category 2| 57001838.4               | 4001.7     | 7.020 × 10⁻⁵ |
| Category 4| 57007652.34              | 14006      | 1.721        |
| Category 1| 57008222.28              | 12005      | −3.511       |

Table 6
Comparison between Category 2 and 1.

| Category  | Total infections averted | Total cost | ICER         |
|-----------|--------------------------|------------|--------------|
| Category 2| 57001838.4               | 4001.7     | 7.020 × 10⁻⁵ |
| Category 1| 57008222.28              | 12005      | 1.254        |

follows:

\[ \text{ACER} = \frac{\text{The total cost (TC)}}{\text{The total infection averted (TA)}}. \]

\[ \text{ICER} = \frac{\text{TC(b) } - \text{TC(a)}}{\text{TA(b) } - \text{TA(a)}}. \]

where \( \text{TC(b) } - \text{TC(a)} \) represents the cost difference in strategies \( a \) and \( b \). \( \text{TA(b) } - \text{TA(a)} \) represents the difference in infected averted between strategies \( a \) and \( b \).

The cost function for implementing the control strategy is as follows:

\[ \text{TC} = \int_{0}^{t_f} [(m_1 u_1 + m_3 u_3)(S_u + S_a + E + I + A + R) + m_2 u_2(S_u + S_a + R)] dt. \]

The parameter \( m_1 \) is the per capita unit cost of taking protective measures, \( m_2 \) is the per capita unit cost of vaccination and \( m_3 \) is the per capita unit cost of the awareness raising programs. We assume that \( m_1 = 10, m_2 = 50 \) and \( m_3 = 10 \).

After the results of the model simulation are obtained, the control strategies are ranked in ascending order of effectiveness in averting infection. The calculation result of ICER is as follows:

\[ \text{ICER(2)} = \frac{10604}{56999023.2} = 1.860 \times 10^{-4}, \]

\[ \text{ICER(3)} = \frac{4001.7 - 10604}{57001838.4 - 56999023.2} = -2.345, \]

\[ \text{ICER(4)} = \frac{57007652.34 - 57001838.4}{14006 - 4001.7} = 1.721, \]

\[ \text{ICER(1)} = \frac{12005 - 14006}{57008222.28 - 57007652.34} = -3.511. \]

Tables 4–6 illustrate the rank order of the simulation results in terms of the effectiveness of averting infection and the ICER of control strategies. Comparing ICER(2) and ICER(3) in Table 4, it can be observed that ICER(3) is bigger than ICER(2). This indicates that Category 3 has higher priority. That is, Category 3 is more costly and less effective than Category 2, so we eliminate Category 3 and exclude it from the alternatives. Then recalculate the ICER value of other categories, as shown in Table 5.

It can be seen from Table 5 that the cost of Category 4 is higher and the efficiency is lower, so Category 4 is eliminated. ICER values for the remaining species are recalculated and the results are shown in Table 6. Table 6 shows that the value of ICER(1) is greater than ICER(2). Therefore, Category 1 is dominant, which is more costly than category 2 but less effective. In conclusion, considering the above four types of control strategies, Category 2 (i.e. the only use of vaccination) is the most cost-effective strategy and Category 1 (i.e. a combination of protection measures, vaccination and awareness raising program) is the second most cost-effective strategy.
6. Conclusion

In this paper, the $S_uS_aEIAR$ model containing asymptomatic infections is proposed to study the impact of protective measures, vaccinations, and awareness raising programs on the spread of the virus. Among them, susceptible people are divided into two categories: lack of protection awareness and existence of protection awareness. We carry out qualitative analysis of this mathematical model and study the stability of the model. In order to better control the epidemic and minimize the number of symptomatic and asymptomatic infections, we propose three control methods namely personal protective measures, vaccination and awareness raising programs. By using the optimal control method, the above three control methods are divided into four categories to further compare their control effects. It was found that combining the three methods minimized the number of infected people. And the introduction of awareness improvement program in society will greatly reduce the existence of susceptible individuals without protection awareness. Therefore, programs to raise awareness of protection should be encouraged, such as media campaigns on social platforms, radio, television and other social media. When individuals take precautions, they will reduce the spread of the virus and reduce the rate of contagion. The results of the cost-effectiveness analysis indicated that vaccination is the most cost-effective strategy, and a combination of the three approaches is the second most cost-effective strategy.

Human protection awareness and willingness to take protective measures play an important role in controlling outbreaks. This is confirmed in our research on the importance of raising awareness of personal protection. Our conclusion is that to reduce and eliminate the number of infected people in COVID-19, personal protective measures, vaccinations and awareness raising programs should be strongly supported. This study can be extended in many ways. First, there will be a short delay after vaccination, and the spread of the virus will also be delayed. In future studies, we will use delay with the present problem and to propose adequate control. In addition, according to the randomness of the associations between the various populations, a randomized version of the model could also be a future research direction.

CRediT authorship contribution statement

Yiran Yuan: Conceptualization, Methodology, Software, Writing – original draft. Ning Li: Supervision, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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