A REMARK ON THE NON-COMPACTNESS OF $W^{2,d}$ IMMERSIONS OF $d$-DIMENSIONAL HYPERSURFACES

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Abstract. We consider the continuous $W^{2,d}$ immersions of $d$-dimensional hypersurfaces in $\mathbb{R}^{d+1}$ with second fundamental forms uniformly bounded in $L^p$. Two results are obtained: first, a family of such immersions is constructed, whose limit fails to be an immersion of a manifold. This addresses the endpoint cases in J. Langer [6] and P. Breuning [1]. Second, under the additional assumption that the Gauss map is slowly oscillating, we prove that any family of such immersions subsequentially converges to a set locally parametrised by Hölder functions.

1. Introduction

In [6] J. Langer proved the following result: denote by $\mathcal{F}(A, E, p)$ the moduli space of immersed surfaces $\psi: M \to \mathbb{R}^3$ with Area($\psi$) $\leq A$, $\|\II\|_{L^p(M)} \leq E$ and $\int_M \psi \, dV = 0$. Here $A, E$ are given finite numbers, $dV$ is the volume/area measure induced by $\psi$, and $p > 2$. Then, any sequence $\{\psi_j\} \subset \mathcal{F}(A, E, p)$ contains a subsequence converging in $C^1$ to an immersed surface, modulo diffeomorphisms of $M$ (written as Diff($M$)). It was motivated by the study of J. Cheeger’s finiteness theorems ([2], also see K. Corlette [5]) and the Willmore energy of surfaces (see e.g., Rivière [7]). In a recent paper [1], P. Breuning generalised the above result to arbitrary dimensions and co-dimensions. More precisely, denote by $\mathcal{F}(V, E, d, n)$ the space of immersions $\psi: M \to \mathbb{R}^n$ where $M$ is a $d$-dimensional closed manifold, $\text{Vol}(M) \leq A$, $\|\II\|_{L^p(M)} \leq E$ and the image $\psi(M)$ contains a fixed point. Let $A, E, d, V$ be as before, let $n > d$ be an arbitrary integer, and let $p > d$. Any sequence $\{\psi_j\} \subset \mathcal{F}(V, E, d, n)$ contains a subsequence converging in $C^1$ to an immersed submanifold, modulo Diff($M$).

The above two compactness theorems on the moduli space of immersions have a crucial assumption: $p > \dim(M) = d$. Indeed, the proofs in [6, 1] utilise the Sobolev–Morrey embedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow C^{0,\alpha}(\mathbb{R}^n)$, where $p > n$ and $\alpha = \alpha(p, n) \in [0, 1[$. It is natural to ask about the endpoint case $p = d$, for which the Sobolev–Morrey embedding fails. In the case $p = d = 2$, J. Langer (p.227, [6]) constructed a counterexample using conformal geometry — the Möbius inversions of the Clifford torus $T_{cl}$ with respect to a sequence of points $x_j \notin T_{cl}$ approaching an outermost point (with distance measured from the centre of the embedded image of $T_{cl}$) on $T_{cl}$ cannot tend to any immersed manifold. Clearly, such counterexamples exist only in $\mathbb{R}^2 \cong \mathbb{C}$.

Our first goal of this paper is to construct a counterexample for the $p = d$ case in arbitrary dimensions. The idea is to construct a family of hypersurfaces that “spiral wildly”, resembling in some sense the motion of vortex sheets in fluid dynamics. This is achieved by letting the Gauss map $n$ (i.e., the outer unit normal vectorfield) increase rapidly from 0 to a large number $N$ as we approach some fixed point $O$, and then decrease rapidly from $N$ to 0 as we leave $O$. 

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To illustrate the geometric picture, we first discuss the toy model in $d = 1$, and then construct a counterexample for general $d$. Instead of using conformal geometric methods, we exploit the scaling invariance of $\|II\|_{L^d(M)}$, which holds in arbitrary dimensions. This is the content of §2.

Our second goal is to establish an affirmative compactness result for the $p = d$ case, with the help of an additional hypothesis: the $BMO$-norm of the Gauss map $n$,

$$\|n\|_{BMO(M)} := \sup_{x \in M, R > 0} \int_{B(x, R)} |n(y) - n_{x,R}| \, dV(y),$$  

(1.1)

is small. Throughout $B(x, R)$ denotes the geodesic ball of radius $R$ centred are $x$ in $M$, $f$ the averaged integral, and $n_{x,R} := \int_{B(x, R)} n \, dV$. This is inspired by the works [8, 9, 10] due to S. Semmes on the chord-arc surfaces with small constant. In §3 we shall use several results in [8, 9, 10] to prove a “partial regularity” result for the weak limit: given any family of immersed hypersurfaces $\mathbb{R}^d$ (equipped with pullback metrics) in the $(d+1)$-dimensional Euclidean space with uniformly $L^d$-bounded second fundamental forms and small $\|n\|_{BMO(M)}$, one may extract a subsequence whose limit can be locally parametrised by Hölder functions.

Finally, we discuss two further questions in §4.

2. A counter-example to the endpoint case $p = d$

Let us first study the toy model $d = 1$. We prove the following simple result:

**Lemma 2.1.** There exist a family of smooth curves $\{M^\epsilon\}$ each homeomorphic to $\mathbb{R}^1$, and a family of immersions $\psi^\epsilon : M^\epsilon \to \mathbb{R}^2$ as planar curves, such that the second fundamental forms $\{\Pi^\epsilon\}$ associated to $\{\psi^\epsilon\}$ are uniformly bounded in $L^1$, but $\psi^\epsilon \circ \sigma^\epsilon$ does not converge in $C^1$-topology to any immersion of $\mathbb{R}$ for arbitrary $\{\sigma^\epsilon\} \subset \text{Diff}(\mathbb{R})$.

**Proof.** Let $J \in C_c^\infty(\mathbb{R})$ be a standard symmetric mollifier; e.g.,

$$J(s) := \Lambda \exp \left\{ \frac{1}{|s^2 - 1|} \right\} \mathbb{1}_{|s| < 1},$$  

(2.1)

where the universal constant $\Lambda > 0$ is chosen such that $\int_{\mathbb{R}} J(s) \, ds = 1$. As usual $J^\epsilon(s) := \epsilon^{-1}J(s/\epsilon)$ for $\epsilon > 0$; then $\|J^\epsilon\|_{L^1(\mathbb{R})} = 1$ for every $\epsilon > 0$. In addition, define the kernel

$$K^\epsilon(x) := J^\epsilon(x + \epsilon) - J^\epsilon(x - \epsilon).$$  

(2.2)

It satisfies $\|K^\epsilon\|_{L^1(\mathbb{R})} = 2$, $K^\epsilon \in C_c^\infty(\mathbb{R})$ and spt$(K^\epsilon) = [-2\epsilon, 2\epsilon]$; in particular, it is smooth at 0.

Now, define an angle function

$$\theta^\epsilon(x) := 10^m \cdot 2\pi \int_{-\infty}^{x} K^\epsilon(s) \, ds,$$  

(2.3)

where $m \in \mathbb{Z}_+$ is to be determined. Then, set the Gauss map $n^\epsilon \in C^\infty(\mathbb{R}; S^1)$:

$$n^\epsilon(x) := \begin{bmatrix} \cos \theta^\epsilon(x) \\ \sin \theta^\epsilon(x) \end{bmatrix} \text{ for each } x \in \mathbb{R}. $$  

(2.4)

The second fundamental form $\Pi^\epsilon$ equals to the negative of the gradient of the Gauss map,

$$|\Pi^\epsilon(x)| = \sqrt{\left( - \sin \theta^\epsilon(x) \left( \theta^\epsilon \right)'(x) \right)^2 + \left( \cos \theta^\epsilon(x) \left( \theta^\epsilon \right)'(x) \right)^2}$$

$$= |(\theta^\epsilon)'(x)| = (2\pi \cdot 10^m) K^\epsilon(x).$$  

(2.5)

Thus, the $L^1$ norm of $\{\Pi^\epsilon\}$ is uniformly bounded by $4\pi \cdot 10^m$. 

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Let \( \psi^{\epsilon} \) be a smooth immersion that realises the Gauss map \( \nu^{\epsilon} \) whose image is the unit circle \( S^1 \) in \( \mathbb{R}^2 \). For each \( \eta > 0 \), we may easily modify \( \psi^{\epsilon} \) to \( \tilde{\psi}^{\epsilon} \) such that \( \| \psi^{\epsilon}(x) \| \) is decreasing on \( ] - \infty, 0] \) and increasing on \( [0, \infty[ \), the image of \( \tilde{\psi}^{\epsilon} \) in \( \mathbb{R}^2 \) is homeomorphic to \( \mathbb{R}^1 \), and that

\[
\| \psi^{\epsilon} - \tilde{\psi}^{\epsilon} \|_{C^{100}(\mathbb{R})} < \eta.
\] (2.6)

Indeed, notice that the image of \( \psi^{\epsilon}[ ] - \infty, 0] \) covers \( S^1 \) for \( 10^m \) times in the positive orientation, and the image of \( \psi^{\epsilon}[0, \infty[ \) covers \( S^1 \) for \( 10^m \) times in the negative orientation. We then choose the perturbed map \( \tilde{\psi}^{\epsilon} \) such that

- As \( x \) goes from \( -\infty \) to \( 0 \), \( \tilde{\psi}^{\epsilon} \) wraps around the origin in a helical trajectory for \( 10^m \) times. Moreover, in each round \( |\tilde{\psi}^{\epsilon}| \) decreases monotonically by \( \sim 10^{-m} \);
- As \( x \) increases from \( 0 \) to \( \infty \), \( \tilde{\psi}^{\epsilon} \) "unwraps" around the origin along a helix for \( 10^m \) times, in each round \( |\tilde{\psi}^{\epsilon}| \) increases monotonically by \( \sim 10^{-m} \);
- For \( x \in ] - \infty, -2\epsilon] \cup [2\epsilon, +\infty[ \), the image of \( \tilde{\psi}^{\epsilon} \) consists of straight line segments ("long flat tails"); hence \( \nu^{\epsilon} \) stays constant on each component of \( ] - \infty, -2\epsilon] \cup [2\epsilon, +\infty[ \);
- Finally, the image \( \tilde{\psi}^{\epsilon}(\mathbb{R}) \) is \( C^\infty \) and homeomorphic to \( \mathbb{R}^1 \).

In view of the above properties, one can take \( m = m(\eta) \in \mathbb{Z}_+ \) sufficiently large to verify (2.6). Let us pick \( \eta = \frac{1}{100} \), so \( m \) is a universal constant fixed once and for all. Without loss of generality, from now on we may assume \( \psi^{\epsilon} = \tilde{\psi}^{\epsilon} \). The point is to ensure that the image of \( \psi^{\epsilon} \) in \( \mathbb{R}^2 \) is free of loops and "concentrates" near the origin \( 0 \in \mathbb{R}^2 \), with Gauss map and second fundamental form arbitrarily close to those constructed in Eqs. (2.4)(2.5).

To conclude the proof, let us define \( M^{\epsilon} \) as the homeomorphic \( \mathbb{R}^1 \) equipped with the pullback metric \( (\psi^{\epsilon})^\# \delta_{ij} \), where \( \delta_{ij} \) is the Euclidean metric on the ambient space \( \mathbb{R}^2 \). It remains to show that the \( C^1 \)-limit (modulo \( \text{Diff}(\mathbb{R}^1) \)) of \( \psi^{\epsilon} \) as \( \epsilon \to 0^+ \) cannot be an immersion. Indeed, note that the topological degree satisfies

\[
\text{deg}(\psi^{\epsilon}[ ] - \infty, 0]) = 10^m, \quad \text{deg}(\psi^{\epsilon}[0, \infty[) = -10^m.
\] (2.7)

These identities are independent of \( \epsilon \). Hence, if \( \tilde{\psi} \) were a limiting immersion, (2.7) would have been preserved. However, \( K_{\epsilon} \xrightarrow{\epsilon} x_0 - \delta_0 = 0 \) as measures, so (2.3)(2.4)(2.5) imply that any pointwise subsequential limit of \( \psi^{\epsilon} \) have zero topological degree. This contradiction completes the proof. \( \square \)

Three remarks are in order:

1. From (2.5) one may infer that

\[
\| II^{\epsilon} \|_{L^\infty(M^\epsilon)} = \frac{2\pi \cdot 10^m \cdot A}{\epsilon \epsilon} + \eta \to \infty \quad \text{as } \epsilon \to 0^+.
\]

2. The construction in Lemma 2.1 can be localised near 0. We can restrict \( M^\epsilon \) to curves of finite \( H^1 \) measure by removing the long tails. This recovers the volume bounds in [6, 1] (§1).

3. We can construct \( \phi^{\epsilon} \) whose limit blows up at a countable discrete set \( \{ x_n \} \) by taking

\[
\tilde{\theta}^{\epsilon}(x) := \sum_{n=1}^{\infty} 2^{-n} \chi_{B(x_n, R_n)}(x) \theta^{\epsilon}(x)
\]
in place of \( \theta^{\epsilon}(x) \), where \( \{ B(x_n, R_n) \} \) are disjoint for all \( n \). Geometrically, the immersed images corresponding to \( \tilde{\theta}^{\epsilon} \) are smooth curves that spiral towards the centres \( x_n \) when \( x < x_n \), and
then spiral away from $x_n$ when $x > x_n$. Near $x_n$ the rate of motion blows up in $L^\infty$ as $\epsilon \to 0^+$; nevertheless, its $L^1$ norm is constant.

Now let us generalise the above construction to $d$-dimensions:

**Theorem 2.2.** Let $d \geq 1$ be an integer. There exist a family of smooth manifolds $\{M^i\}$ each homeomorphic to $\mathbb{R}^d$, and a family of immersions $\psi^i : M^i \to \mathbb{R}^{d+1}$ as smooth hypersurfaces, such that the second fundamental forms $\{H^i\}$ associated to $\{\psi^i\}$ are uniformly bounded in $L^d$, but $\{\psi^i \circ \sigma^i\}$ does not converge in $C^1$-topology to any immersion of $\mathbb{R}^d$ for arbitrary $\{\sigma^i\} \subset \text{Diff}(\mathbb{R}^d)$.

**Proof.** Again the crucial point is to construct the Gauss map $n^i \in C^\infty(\mathbb{R}^d; \mathbb{S}^d)$. We make use of the spherical coordinates on $\mathbb{S}^d$. For $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, one needs to specify the angle functions $\theta^i : \mathbb{R}^d \to \mathbb{S}^d$ for each $i \in \{1, 2, \ldots, d\}$ in the following:

$$n^i(x) = \begin{bmatrix} \cos \theta^i_1(x) \\ \sin \theta^i_1(x) \cos \theta^i_2(x) \\ \sin \theta^i_1(x) \sin \theta^i_2(x) \cos \theta^i_3(x) \\ \vdots \\ \sin \theta^i_1(x) \cdots \sin \theta^i_{d-1}(x) \cos \theta^i_d(x) \\ \sin \theta^i_1(x) \cdots \sin \theta^i_{d-1}(x) \sin \theta^i_d(x) \end{bmatrix}. \tag{2.8}$$

Throughout we view $\mathbb{S}^d = \{z \in \mathbb{R}^{d+1} : |z| = 1\}$ as the round sphere.

Indeed, let us choose

$$\theta^i_j(x) \equiv \Theta^i_j(x_1) := 10^m \cdot 2\pi \int_{-\infty}^{x_i} K_\epsilon(s) \, ds, \tag{2.9}$$

where the kernel $K_\epsilon$ is defined as in (2.2), and $m \in \mathbb{Z}_+$ is a large universal constant fixed later. Each $\theta^i_j$ is a function of $x_1$ only. One can easily compute all the entries in $-H^i = \nabla n^i$, which is a lower-triangular $d \times (d + 1)$ matrix due to the embedding $\mathbb{S}^d \hookrightarrow \mathbb{R}^{d+1}$. The rows $\{r_i\}_{i=1,2,\ldots,d}$ of $\{\nabla n^i\}$ are:

$$r_1 = \left( - (\Theta^i)'(x_1) \sin \Theta^i(x_1), 0, \ldots, 0 \right),$$

$$r_2 = \left( (\Theta^i)'(x_1) \cos \Theta^i(x_1) \cos \Theta^i(x_2), 0, \ldots, 0 \right),$$

$$r_3 = \left( (\Theta^i)'(x_1) \sin \Theta^i(x_1) \sin \Theta^i(x_2) \cos \Theta^i(x_3), \Theta^i(x_2) \sin \Theta^i(x_1) \cos \Theta^i(x_2) \cos \Theta^i(x_3), \right.$$ \[ 
$$- (\Theta^i)'(x_3) \sin \Theta^i(x_1) \sin \Theta^i(x_2) \sin \Theta^i(x_3), 0, \ldots, 0 \bigg)$$

so on and so forth, with the last two being

$$r_{d-1} = \left( (\Theta^i)'(x_1) \cos \Theta^i(x_1) \sin \Theta^i(x_2) \cdots \sin \Theta^i(x_{d-1}) \cos \Theta^i(x_d), *, \ldots, * \right),$$

$$r_d = \left( (\Theta^i)'(x_1) \cos \Theta^i(x_1) \sin \Theta^i(x_2) \cdots \sin \Theta^i(x_{d-1}) \sin \Theta^i(x_d), *, \ldots, * \right),$$

and

$$r_{d-1} = \left( (\Theta^i)'(x_1) \sin \Theta^i(x_1) \sin \Theta^i(x_2) \cdots \cos \Theta^i(x_{d-1}) \cos \Theta^i(x_d), \right.$$ \[ 
$$- (\Theta^i)'(x_d) \sin \Theta^i(x_1) \sin \Theta^i(x_2) \cdots \sin \Theta^i(x_{d-1}) \sin \Theta^i(x_d) \bigg)$$

and

$$r_d = \left( (\Theta^i)'(x_1) \sin \Theta^i(x_1) \sin \Theta^i(x_2) \cdots \sin \Theta^i(x_{d-1}) \sin \Theta^i(x_d), *, \ldots, * \right),$$

$$r_{d-1} = \left( (\Theta^i)'(x_1) \sin \Theta^i(x_1) \sin \Theta^i(x_2) \cdots \cos \Theta^i(x_{d-1}) \sin \Theta^i(x_d), \right.$$ \[ 
$$- (\Theta^i)'(x_d) \sin \Theta^i(x_1) \sin \Theta^i(x_2) \cdots \sin \Theta^i(x_{d-1}) \sin \Theta^i(x_d) \bigg).$$

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A tedious yet straightforward computation yields the Hilbert–Schmidt norm of the above matrix:

\[ \| \Pi \| = \| \nabla f \| = \left\| \left( (\Theta')'(x_1), \cdots, (\Theta')'(x_d) \right) \right\| . \quad (2.10) \]

Thus, in view of (2.9) and Fubini’s theorem, we have

\[ \| \Pi \|_{L^d(\mathbb{R}^d)} = 10^m \cdot 2\pi \left\| \sum_i K_i \otimes \cdots \otimes K_d \right\|_{L^d(\mathbb{R}^d)} = 10^m \cdot 2\pi \| K \|_{L^1(\mathbb{R}^d)} = 10^m \cdot 4\pi . \quad (2.11) \]

Now we shall choose a smooth immersion that (approximately) realises \( n^\epsilon \) precisely as in the \( d = 1 \) case (Lemma 2.1). For the sake of completeness let us sketch the arguments. First, take \( \psi^\epsilon \) whose Gauss map is \( n^\epsilon \) and which takes value in \( S^d \). Then we may modify it — without relabelling and up to an arbitrarily small error, say \( \frac{1}{10^k} \) in the \( C^{100} \)-topology — so that the image of \( \psi^\epsilon \) in \( \mathbb{R}^{d+1} \) is a smooth, homeomorphic copy of \( \mathbb{R}^d \) for each \( \epsilon > 0 \), having flat ends outside \( B(0,2) \), and having \( d \) independent angle functions in the spherical coordinates (i.e., in place of \( \theta^\epsilon \)’s in (2.8)) wrapping around \( 0 \in \mathbb{R}^{d+1} \) for \( 10^m \) times in the positive orientation and unwrapping for \( 10^m \) times in the negative orientation. The second fundamental form of the modified map \( \psi^\epsilon \) satisfies the bound in (2.11), up to an error of \( \pm \frac{1}{10^k} \). Then, define \( M^\epsilon := (\mathbb{R}^d, (\psi^\epsilon)^\# \delta_{ij}) \), where \( \delta_{ij} \) is the Euclidean metric on \( \mathbb{R}^{d+1} \). By a topological degree argument as in (2.7), the limit of \( \psi^\epsilon \) cannot be an immersion up to the action of \( \text{Diff}(\mathbb{R}^n) \). This completes the proof.

Similar to the remarks below the proof of Lemma 2.1, this counterexample can be localised, and an iteration yields a family of immersions of \( \mathbb{R}^d \) that blows up at an infinite discrete set.

**3. Local Hölder Regularity**

In this section we deduce a compactness theorem utilising the works [8, 9, 10] of S. Semmes on the harmonic analysis on chord-arc surfaces with small constants. Consider the moduli space

\[ \mathcal{F}(\delta, d) := \left\{ f \in W^{2,d} \cap C^\infty(M; \mathbb{R}^{d+1}) : f \text{ is an immersion}, M \text{ is an } d\text{-dimensional hypersurface}, \right. \]

\[ \left. M \cup \{ \infty \} \text{ is smooth in } S^{d+1}, \| n \|_{\text{BMO}(M)} \leq \delta, f(M) \text{ contains a fixed point} \right\} . \quad (3.1) \]

We show the following: if the Gauss maps of a family of smooth homeomorphic \( \mathbb{R}^d \) have uniformly small oscillations at all scales, then “a little” regularity persists in the limit. This assumption is natural: if a family of \( W^{2,d} \) immersions of \( d \)-manifolds has uniformly \( L^d \)-bounded second fundamental forms, then their Gauss maps have bounded \( \text{BMO} \)-norms (provided that Poincaré and Sobolev inequalities hold).

For this purpose we need a definition. A set \( \Omega \subset \mathbb{R}^d \) is called a Hölder graph system if it can be locally represented by graphs of \( C^{0,\gamma} \) functions for some \( \gamma \in [0, 1] \). We do not require further geometric information for a Hölder graph system, e.g., whether or not it represents a topological manifold or orbifold. The notion of “graph system” plays an essential role in [6, 1] by J. Langer and P. Breuning.

**Theorem 3.1.** There exists a small constant \( \delta_0 > 0 \) depending only on the dimension \( d \), such that for any \( \delta \in [0, \delta_0] \) and any family of immersions \( \{ \psi^\epsilon \} \subset \mathcal{F}(d, \delta) \), we can find \( \{ \sigma^\epsilon \} \subset \text{Diff} (\mathbb{R}^d) \) such that the limit of \( \psi^\epsilon \circ \sigma^\epsilon \) converges to a Hölder graph system, after passing to subsequences.
It is proved in [9, 10] that for sufficiently small $\delta_0$, $M$ is homeomorphic to $\mathbb{R}^d$ and behaves nicely on small scales— for each $x \in M$ and $R > 0$, $B(x, R) \cap M$ stays close to the hyperplane through $x$ normal to the averaged Gauss map $n_{x,R}$. Indeed, $M$ with small $\|n\|_{\text{BMO}(M)}$ is equivalent to the definition of a chord-arc surface with small constant, defined in [8] as a generalisation of the chord-arc domain for $d = 1$. Although it remains an open question if such $M$ always admits bi-Lipschitz parametrisations by $\mathbb{R}^d$ (cf. T. Toro [11] for a related problem), it is nevertheless known that $M$ has a “bi-Hölder” parametrisation; see Theorem 4.1, [9]. This enables us to prove Theorem 3.1.

**Proof.** Let us first summarise several estimates from [8, 9, 10]. Fix any $t > 0$, e.g. $t = 10^{-5}$. By §3, [9] one can find a new chord-arc surface $M_t$ with the chord-arc constant $\mu$, such that

$$0 \leq \delta \leq \delta_0 \leq C(d)\delta_0 < \mu.$$  

We shall choose $\mu$ later, which is equivalent to the least upper bound for the $\text{BMO}$-norm of the Gauss map; see p.200 [8]. In view of Eq. (3.7) and Lemma 3.8 in [9], $M_t \cap B(x, [2^{-1} + 10^{-10}]t)$ is a Lipschitz graph with constant $\leq C_0 \mu$ for each $x \in M$, provided that $\mu = \mu(t, \delta_0)$ is chosen large enough. Here $C_0 = C(d, \delta_0)$. Under the same condition, $M_t$ can be taken sufficiently close to $M$ (e.g., with distance $\leq 10^{-10}t$ by Lemma 3.8 in [9]). Then, in view of Theorem 4.1 in [9], there exists a homeomorphism $\tau : M \to M_t$ such that

$$\max \left\{ ||\tau||_{C^0, \gamma(B(x, 100t) \cap M)}, ||\tau^{-1}||_{C^0, \gamma(B(x, 100t)) \cap M_t} \right\} \leq C_1 \quad \text{for all } x \in M,$$  

where $C_1 = C(d, \delta_0, t)$ and the Hölder index is given by

$$\gamma = 1 - C_2 d \delta_0$$  

for a dimensional constant $C_2$ (denoted by $k$ in [9]). In fact, putting together Eqs. (1.3)(4.6) and the choice of $p$ on p.178 in [9], Lemma 5.5 in [8] and that $0 \leq \delta \leq \delta_0$, we may explicitly select

$$C_1 = C_3^2 \delta_0 \left\{ \frac{100tC_2 \delta_0}{1 - 2 \cdot 10^4 \delta_0} \right\}. \quad (3.4)$$

Here $C_3 = C_3(d)$ is a dimensional constant. Notice that our estimates (3.4)(3.2) are uniform in $\delta$. We also have to further restrict to $\delta_0 < (C_2 d)^{-1}$ to ensure that $\gamma > 0$ in (3.3).

Now we are ready to give the proof. By considering a compact exhaustion $\{M_k\} \not\subset M$, one may take $M$ to be a bounded domain in $\mathbb{R}^d$. (The argument for non-compact manifolds in the $p > d$ case is more involved, if one needs to check that the limiting object is a manifold; cf. §7 in [1].) Then we can take a $(50t)$-net $N$ of $M$, whose cardinality is

$$|N| \leq C_4 t^{-d}$$

for some geometric constant $C_4 = C(d, \gamma) \equiv C(d, \delta_0)$. In each element of $N$ the hypersurface $M$ is $C^{0, \gamma}$-parametrisable by $M_t$, which is a Lipschitz graph on $(2^{-1} + 10^{-10})$-balls. Using the quantitative estimates in the preceding paragraph, we can refine $N$ to a sub-net $\tilde{N}$ with cardinality $C_5 t^{-d}$, $C_5 = C(d, \delta_0)$ again, such that in each $B \in \tilde{N}$, the set $B \cap M$ is parametrised by a $C^{0, \gamma}$-homeomorphism with the Hölder norm bounded by $C_6 := C_0 \mu \cdot C_1$. Let us choose $\mu = 10C(d)\delta_0$; then $C_6 = C(d, \delta_0, t)$ (where $t > 0$ is fixed from the beginning). Therefore, in view of the Arzelà–Ascoli theorem, i.e., the compactness of $C^{0, \gamma}$ parametrisations, we may complete the proof. □
4. Two Further Questions

Let the moduli space $\mathcal{F}(A,E,p)$ be as in §1. Is the space

$$\mathcal{F}_{\text{isom}}(A,E,p) := \left\{ \psi \in \mathcal{F}(A,E,p) : \psi \text{ is an isometric immersion of a fixed manifold } M \right\}$$

compact in its natural topology? For the end-point case $p = 2 = d$ the answer is affirmative, in contrast to the unconstrained case for $\mathcal{F}(A,E,p)$. The authors of [3] proved this via establishing the weak continuity of the Gauss-Codazzi equations (the PDE system for the isometric immersion), with the help of a div-curl type lemma due to Conti–Dolzmann–Müller in [4]. What about higher dimensions $d \geq 3$ (and co-dimensions greater than 1)? That is, for a family of isometric immersions of some fixed $d$-dimensional manifold with uniformly bounded second fundamental forms in $L^d$, is the subsequential limit an isometric immersion?

Theorem 3.1 leaves open the possibility that the limiting objects of $W^{2,d}$-bounded immersed hypersurfaces may be very irregular (e.g., the nowhere differentiable Weierstrass function is $C^{0,\gamma}$, or other fractals), even if the (somewhat strong) geometrical condition that the Gauss map is slowly oscillating is enforced. Can we find natural geometrical conditions on the moduli space of $d$-dimensional hypersurfaces with uniformly bounded second fundamental forms in $L^d$, which is sufficient to ensure higher regularities for the subsequential limits, e.g., BV or Lipschitz? This is related to the problem of finding good parametrisations of chord-arc surfaces; see the discussions by S. Semmes [9] and T. Toro [11].

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