Distributed $Q$-Learning for Dynamically Decoupled Systems

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Abstract—Control of large-scale networked systems often necessitates the availability of complex models for the interactions amongst the agents. However, in many applications building accurate models of these interactions might be prohibitive due to the curse of dimensionality or their inherent complexity. In the meantime, data-guided control methods can circumvent model complexity by directly synthesizing the controller from the observed data. In this paper, we propose a distributed $Q$-learning algorithm to design a feedback mechanism given an underlying graph structure parameterizing the agents’ communication. We assume that the distributed nature of the system arises from a common cost and show that for the particular case of identical dynamically decoupled systems, the learned controller converges to the optimal Linear Quadratic Regulator controller for each subsystem. We provide a convergence analysis and verify the result with an example.

Keywords: Distributed $Q$-learning, data-guided control, linear quadratic regulator, networked control systems

I. INTRODUCTION

Distributed control has undergone an unprecedented growth during the past few years mainly due to the complexity in modeling and analysis of large scale systems. In such scenarios, high-dimensional collective tasks conducted by members of a team are formed from local decisions of each member leading towards the global system-level final decision. Accordingly, the main focus in distributed control design is finding the closest-to-optimal control mechanism for a large-scale system, making use of the structure in information exchange and decision-making. Indeed, such an approach has found broad applications in such areas as robotic swarms [1], structured robust learning [2], and social networks [3].

Decentralized control of large-scale systems is not a new research area. The roots of the field traces back to the socioeconomic literature of 1970’s [4]; an early work in the control literature is [5]. The inspiration of these types of works is that the presupposition of centrality fails to hold due to the lack of either central intelligence or computational capability [6]. This line of work was followed by the pioneering work [7], where stability conditions for multi-channel linear systems were derived. Fast forward a few decades, the stability of networks was studied in [8], where sufficient graph-theoretic conditions were provided for stability of formations comprised of identical vehicles. Graph-based structured controller design was further examined in works such as [9], [10]. The topic was also studied from a spatially distributed control viewpoint [11] or a layered control design approach [12]. However, all of these studies are based on the knowledge of the underlying dynamics; as the system grows in scale, modeling becomes prohibitively difficult and uncertain due to complexities or potential perturbations in high dimensions. This motivates a data-guided approach to evade the difficulties of model-based distributed control.

In this work, we focus on a model-free distributed control design using $Q$-learning [13]. The approach has been used to find the optimal Linear Quadratic Regulator (LQR) feedback controller online for a single system [14]. In that sense, $Q$-learning can be thought of as an adaptive optimal control design method [15]. More recent works on distributed adaptive systems can be found in [16], [17] to name a few. Other related works are [18] that considers network effects within $Q$-functions, and [17] that introduces a decentralized $Q$-learning approach for a general framework but not necessarily on an underlying graph. Moreover, there has emerged several recent works on data-driven control using finite sample opposed to the adaptive control design in the limit [19]–[21].

Our contribution is mainly built upon the work of Bradtke [14], in that we provide a distributed policy iteration to find a collective controller. We assume that the distributed nature of the problem comes from the interconnection of identical dynamically decoupled systems that work together for a common system-level goal in a network. In fact, we assume that the underlying interaction graph is only reflected in the performance index of the corresponding LQR problem. We provide a graph theoretical framework under which each agent synthesizes an estimate of the optimal LQR controller.

The rest of the paper is organized as follows: In II we provide a quick overview of mathematical tools that are used in the paper. In III we introduce the problem setup. In IV the distributed setup and the main algorithm is provided along with its proof of convergence. The section concludes with a discussion on the computational savings due to the adoption of the distributed algorithm. An example is provided in V to validate our theoretical results. Concluding remarks and future directions are discussed in VI.

II. MATHEMATICAL PRELIMINARIES

We denote by $\mathbb{R}$ the set of real numbers. A column vector with $n$ elements is referred to as $v \in \mathbb{R}^n$, where $v_i$ represents the $i$th element in $v$. The matrix $M \in \mathbb{R}^{p \times q}$ contains $p$ rows and $q$ columns with $[M]_{ij}$ denoting the element in the $i$th row and $j$th column of $M$.

For example, computational complexity of order $O(n^3)$ for solving the Algebraic Riccati Equation (ARE) is not desirable for large-scale systems.
row and jth column of $M$. The square matrix $N \in \mathbb{R}^{n \times n}$ is symmetric if $N^\top = N$, where $N^\top$ denotes the transpose of the matrix $N$. The operator $\text{diag}(\cdot)$ makes a square diagonal matrix out of the elements of its argument. The $n \times n$ zero matrix is denoted by $0_n$ and $I_n = \text{diag}(1,1,\ldots,1)$, is the identity matrix. We write $N \succeq 0$ ($\succeq 0$) when $N$ is a positive-(semi)definite matrix, i.e., $x^\top N x > 0$ ($\geq 0$) for all $x \neq 0$. To simplify the vector notation, we use semicolon ($;$) to concatenate column vectors, hence $[v^\top \; w^\top]^\top = [v; w]$. We call the pair $(A,B)$ controllable if and only if the controllability matrix $C = [B \; AB \; \ldots \; A^{n-1}B]$ has full-rank, where $n$ is the size of the system. $A \otimes B \in \mathbb{R}^{p_1 \times q_1 \times p_2 \times q_2}$ refers to the Kronecker product of $A \in \mathbb{R}^{p_1 \times q_1}$ and $B \in \mathbb{R}^{p_2 \times q_2}$, and $I \otimes T$ gives a block diagonal square matrix, with $T$ on each diagonal block. A graph is characterized by $G = (V,E)$ where $V$ is the set of nodes and $E \subseteq V \times V$ denotes the set of edges. An edge exists from node $i$ to $j$ if $(i,j) \in E$; this is also specified by writing $j \in N_i$, where $N_i$ is the set of neighbors of node $i$. Finally, $G$ can be represented by various matrices, in particular, by its graph Laplacian denoted by $L$.

### III. Problem Setup

Herein, we provide the basic formulation and problem setup. First, we introduce the distributed LQR control on a given graph. This is mainly related to [9], where the network contains identical dynamic agents, yet decoupled from other agents’ dynamics. The only coupling between these agents is through a common network-level objective function. Then we introduce the basic setup of $Q$-learning for linear dynamical systems and extend the formulation for the distributed setup. As we shall see later, the distributed nature will be simplified into an additional interaction term in the output of a linear Recursive Least Squares (RLS) algorithm.

#### A. Distributed LQR Problem

Assume that the system contains $N$ agents that form a graph $G$ with each node of the graph indicating a linear time-invariant dynamical system corresponding to that agent as,

$$x_{t+1}^{(i)} = Ax_t^{(i)} + Bu_t^{(i)}, \quad i = 1, 2, \ldots, N,$$

where $x_t^{(i)}$ is the state of agent $i$ at time-step $t$, where $x_t^{(i)} \in \mathbb{R}^{n}$, $u_t^{(i)} \in \mathbb{R}^{m}$, $A \in \mathbb{R}^{n \times n}$, and $B \in \mathbb{R}^{n \times m}$. The assumption that all agents have identical system matrices $A$ and $B$ is relevant in many applications such as formation flight, homogenous mobile robots, and power grids consisting of identical generators. These dynamics can be integrated into a compact form as, $\dot{x}_{t+1} = \bar{A}x_t + \bar{B}u_t$, where $\bar{x} \in \mathbb{R}^{Nn}$ and $\bar{u} \in \mathbb{R}^{Nm}$ are formed by concatenation of all states and inputs into one vector with $\bar{A} = I_N \otimes A \in \mathbb{R}^{Nn \times Nn}$ and $\bar{B} = I_N \otimes B \in \mathbb{R}^{Nn \times Nm}$. The graph structure is reflected in the cost function of the associated LQR problem by the following definition,

$$J = \sum_{i=1}^{N} \left[ \sum_{j=1}^{N} Q_{ij}(x_t^{(i)} - x_t^{(j)})^\top + R_{ii}u_t^{(i)} \right]$$

$$+ \sum_{i=1}^{N} \sum_{j \neq i} (x_t^{(i)} - x_t^{(j)})^\top Q_{ij}(x_t^{(i)} - x_t^{(j)})$$

where the first term indicates the intra-system cost while the second denotes the inter-system coupling. We make the simplifying assumption,

$$Q_i = \tilde{Q}, \quad R_i = \tilde{R}, \quad Q_{ij} = \tilde{Q}_{ij \in N_i} = \begin{cases} 0_n & j \notin N_i \tilde{Q} & j \in N_i \end{cases},$$

where $\tilde{Q} \succeq 0$ and $\tilde{R} \succ 0$. The cost function can also be written in compact form as $J = \tilde{x}_t^\top \tilde{Q} \tilde{x}_t + \tilde{u}_t^\top \tilde{R} \tilde{u}_t$, where $\tilde{Q} = (I + I_n) \otimes \tilde{Q} \succeq 0$ and $\tilde{R} = I_n \otimes \tilde{R} \succ 0$. Solution of the LQR problem in such systems is studied for a particular $\tilde{Q}$ resulting in a structured controller [9]. Suboptimal solutions to the controller design consistent with the graph structure has also been proposed. Nevertheless, in many real-world applications there is no a priori knowledge of the system’s model due to either complexities or model uncertainties [12]. We introduce a model-free approach while considering the optimality criteria for each subsystem. We will show that for an interconnected system with identical dynamically decoupled agents as discussed above, $Q$-learning leads to each subsystem running their respective local LQR optimal controller independent of other agents in the network. This phenomenon is shown to hold asymptotically after each agent collects enough data.

**Remark 1.** We note that the global cost in equation 1 induces a structured way of steering the states of the agents to the origin through an auxiliary consensus term. Our future work will consider further realizations of the global/local cost structure in the LQR setup—that might not be completely aligned with each other.

#### B. Centralized $Q$-Learning

To make the paper self-contained, we refer to some basics of Q-learning and its connections to LQR feedback control design. $Q$-learning describes a methodology where an agent aims to optimize the value of a sum of reward functions from observing the results of its own actions. This value is reformulated by the $Q$-function which is defined for a single agent as,

$$Q(x_t, u_t) = R(x_t, u_t) + \gamma Q(x_{t+1}, u_{t+1}),$$

where $Q(x_t, u_t) = x_t^\top P x_t$ is the state-action $Q$-function, $P$ is the cost-to-go matrix and $R(x_t, u_t) = x_t^\top Q x_t + u_t^\top R u_t$ is the one-step reward with symmetric constant matrices $Q \succeq 0$ and $R \succ 0$. Equation 2 is the simplified form of the well-known Bellman equation for the deterministic case of LQR. Also, the control actions come from a set of optimal policies

\footnote{which is also the solution to the discrete-time ARE in LQR.}
that assume the form of a feedback law \( u_t = -Kx_t \) in the LQR framework. Simplification of (2) results in,

\[
Q(x_t, u_t) = z_t^T H z_t, \tag{3}
\]

where \( z_t = [x_t; u_t] \) and \( H \) is a block matrix defined as,

\[
H = \begin{bmatrix}
H_{11} & H_{12} \\
H_{21} & H_{22}
\end{bmatrix} = \begin{bmatrix}
Q + \gamma A^T PA & \gamma A^T PB \\
\gamma B^T PA & \gamma B^T PB
\end{bmatrix}.
\]

Then the idea is to learn the parameters in \( H \) through observations \( z_t \) and update the estimate of the controller as,

\[
K_{\text{new}} = -H_{22}^{-1} H_{21} = -\gamma(R + \gamma B^T PB)^{-1}(B^T PA),
\]

which can also be obtained by setting \( \partial Q / \partial u_t = 0 \). The adaptive nature of the algorithm is originated from a linear RLS step to learn the parameters of \( H \) in real-time. Hence, we pursue [14] to form a linear parameterization of (3) as,

\[
Q(x_t, u_t) = z_t^T H z_t = \bar{z}_t^T \theta_H, \tag{4}
\]

where \( \bar{z}_t, \theta_H \in \mathbb{R}^{(n+m)(n+m+1)/2} \) are quadratic basis of the elements in \( z_t \) and vector of upper right triangle of symmetric \( H \) in the correct order, respectively. With these definitions,

\[
R(x_t, u_t) = r_t = Q(x_t, u_t) - \gamma Q(x_{t+1}, u_{t+1})
= z_t^T H z_t - \gamma \bar{z}_{t+1}^T H \bar{z}_{t+1} = \phi_t^T \theta_H, \tag{5}
\]

where \( \phi_t = \bar{z}_t - \gamma \bar{z}_{t+1} \). Therefore, assuming that we know \( R(x_t, u_t) \) and \( \phi_t \), RLS can be employed to find an estimate of \( \theta_H \). According to [22], this recursive algorithm converges in the limit if \( \phi_t \) is persistently excited (PE), i.e.,

\[
\alpha I \leq \frac{1}{M} \sum_{i=1}^{M} \phi_{t-i} \phi_{t-i}^T \leq \beta I \quad \forall \ t, M \geq M_0, \tag{6}
\]

for some positive parameters \( M_0, \alpha, \) and \( \beta \). Following the convergence of \( \theta_H \), then \( H \) is obtained using (4).

IV. DISTRIBUTED Q-LEARNING

A. Distributed Q-function

We now switch to a multiagent setup, where several autonomous agents try to minimize their own discounted reward based on a global cost and single-agent control is not applicable since there exist multiple decision-makers. In this section, we extend the Q-learning setup based on the distributed control framework defined in section III-A.

To this end, we assume that each agent enjoys its own Q-function whose reward is a function of the state of the agent as well as the state of its neighbors. For agent \( i \), we define,

\[
Q^{(i)}(x_t^{(i)}, u_t^{(i)}) = R^{(i)}(x_t^{(i)}, u_t^{(i)}) + \gamma Q^{(i)}(x_{t+1}^{(i)}, u_{t+1}^{(i)}) = y_t^{(i)^T} Q^{(i)} y_t^{(i)} + \gamma Q^{(i)}(x_{t+1}^{(i)}, u_{t+1}^{(i)}), \tag{7}
\]

where \( y_t^{(i)} = [x_t^{(i)}; u_t^{(i)}; x_{j_1}^{(i)}; \ldots; x_{j_d}^{(i)}] \), \( d_i \) is the degree of agent \( i \), \( j_{d_i} \in \mathcal{N}_i \) for \( k = 1, \ldots, i, \) and \( Q^{(i)} \) is defined as,

\[
Q^{(i)} = \begin{bmatrix}
(d_i + 1)Q & 0 & \cdots & -Q \\
-\gamma Q & 0 & \cdots & 0 \\
-\gamma Q & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-\gamma Q & 0 & \cdots & 0
\end{bmatrix} \in \mathbb{R}^{(d_i+2)n \times (d_i+2)n}. \tag{8}
\]

The structure of \( Q^{(i)} \) is resulting from equation (1) and implies the new definition of reward function for multiple agents in the system. Note that equations (7) and (8) make two implicit assumptions: (i) there is no control coupling amongst agents and, (ii) each agent has only access to the reward form the coupling between its own state and the states of neighbors. This motivates the existence of zero blocks in (8). Similar to (3), equation (7) can also be re-arranged into,

\[
Q^{(i)}(x_t^{(i)}, u_t^{(i)}) = y_t^{(i)^T} H^{(i)} y_t^{(i)},
\]

where,

\[
H^{(i)} = \begin{bmatrix}
(d_i + 1)Q + \gamma A^T P^{(i)} A & \gamma A^T P^{(i)} B & -\bar{Q} & \ldots & -\bar{Q} \\
-\gamma B^T P^{(i)} A & R + \gamma B^T P^{(i)} B & 0 & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & \ldots \\
-\bar{Q} & 0 & \ldots & 0 & \bar{Q}
\end{bmatrix}
\]

Finally, as in the centralized case, for each agent \( i \) we define \( \phi_t^{(i)} = \bar{z}_t^{(i)} - \gamma \bar{z}_{t+1}^{(i)} \).

B. Main Results

In this section, we introduce the distributed policy iteration algorithm. The analysis in this part is mainly inspired by [14], however, there are fundamental differences as we only assume couplings through a global cost function; as such, the state transition or feedback of each agent only depends on their own history of states and actions. Under these assumptions, we show that this way of coupling in the case of identical systems signifies the interdependency of the agents in the decision-making process.

Algorithm 1 The distributed Q-learning Algorithm

1. Initialize:
2. Random: \( \hat{\theta}_0^{(1)}(0), \ldots, \hat{\theta}_0^{(N)}(0) \)
3. Stabilizable: \( K_0^{(1)}, \ldots, K_0^{(N)} \)
4. \( t = 0, k = 1 \)
5. while convergence:
6. Reset Covariance: \( P_k(0) = P_0 \)
7. For \( j = 1 \) to \( M \):
8. For system \( i = 1, \ldots, N \):
9. Choose \( e_t \) and find \( u_t^{(i)} = K_k^{(i)} x_t^{(i)} + e_t \)
10. Collect \( x_{t+1}^{(i)} \) by applying \( u_t^{(i)} \) to the system
11. Update \( \hat{\theta}_k^{(i)}(j) \) using RLS
12. \( t = t + 1 \)
13. For system \( i = 1, \ldots, N \):
14. Find symmetric \( \hat{H}_k^{(i)} \) corresponding to \( \hat{\theta}_k^{(i)} \)
15. Policy update: \( K_{k+1}^{(i)} = -\hat{H}_k^{(i)} \hat{H}_k^{(i)} \)
16. Initialize parameters \( \hat{\theta}_{k+1}^{(i)}(0) = \hat{\theta}_k^{(i)}(M) \)
17. \( k = k + 1 \)

We briefly explain the steps of the algorithm: \( \hat{\theta}_k^{(i)} \) is the estimate of \( H^{(i)} \) as in (4). In the sequel, \( \hat{\theta}_k^{(i)} \) denotes
the parameters of $H^{(i)}$ obtained using the true system parameters. $K^{(i)}_k$ denotes the controller estimate. The counter $t$ keeps track of the number of collected data while $k$ designates the iteration count on the parameters estimate. Note that these counters are never reset to zero. $P_k(j)$ is the covariance matrix reset to some constant $P_0$ at each iteration to reinitialize the gain. Each RLS estimation interval includes $M$ time-steps. The value of $M$ is dependent on the number of unknown parameters in $\theta_{k}^{(i)}$ and also the desired accuracy. The control signal is PE at each iteration of the RLS and $e_i$ is the excitation component which is assumed to be the same for all agents. After convergence of RLS, the controller for each agent is updated based on (9). The estimation parameters are reinitialized from the final value of the previous iteration such that $\hat{\theta}_{k+1}^{(i)}(0) = \hat{\theta}_{k}(M)$. The reader is referred to Chapter 3 of [22] for exact steps of RLS.

**Theorem 1.** Assume that for all $i = 1, \ldots, N$, the pair $(A, B)$ is a controllable and $K_0^{(i)}$ is stabilizing with a PE signal $\phi_{k}^{(i)}$. Then there exists $M < \infty$ such that Algorithm 1 generates a sequence $\{K_k^{(i)}\}$ with $\lim_{k \to \infty} \|K_k^{(i)} - K^*\| = 0$, where $K^* = \text{LQR}(A, B, Q, R)$.

**Proof:** From [8],

$$r_{k}^{(i)} = y_{k}^{(i)\top} H^{(i)} y_{k}^{(i)} - \gamma y_{k+1}^{(i)\top} H^{(i)} y_{k+1}^{(i)}.$$  

Also from section III-A

$$r_{k}^{(i)} = z_{k}^{(i)\top} Q z_{k}^{(i)} + u_{k}^{(i)\top} R u_{k}^{(i)} + \sum_{j=1}^{N} \left( z_{k}^{(i)} - z_{k}^{(j)\top} \right) Q \left( z_{k}^{(i)} - z_{k}^{(j)\top} \right),$$

and,

$$y_{k}^{(i)\top} H^{(i)} y_{k}^{(i)} = \left[ z_{k}^{(i)\top} u_{k}^{(i)\top} \right] \left[ \begin{array}{ccc} \tilde{Q} + \gamma A^\top P^{(i)} A & \gamma A^\top P^{(i)} B \\ \gamma B^\top P^{(i)} A & \tilde{R} + \gamma B^\top P^{(i)} B \end{array} \right] \left[ \begin{array}{c} z_{k}^{(i)} \\ u_{k}^{(i)} \end{array} \right].$$

Consequently, we obtain,

$$\xi_{k}^{(i)} = z_{k}^{(i)\top} \hat{\theta}_{k}^{(i)}$$  

(10)

where,

$$\xi_{k}^{(i)} = z_{k}^{(i)\top} Q z_{k}^{(i)} + u_{k}^{(i)\top} R u_{k}^{(i)} + \sum_{j=1}^{N} \left( z_{k}^{(i)} - z_{k}^{(j)\top} \right) Q \left( z_{k}^{(i)} - z_{k}^{(j)\top} \right).$$  

Hence the distributed nature of the problem narrows down to a particular distributed form of RLS. We stack the $N$ equations of the form (10) for all agents into vector form as,

$$\begin{bmatrix}
\xi_{1}^{(i)} \\
\vdots \\
\xi_{N}^{(i)}
\end{bmatrix} =
\begin{bmatrix}
z_{1}^{(i)\top} \\
\vdots \\
z_{N}^{(i)\top}
\end{bmatrix}
\begin{bmatrix}
\hat{\theta}_{k}^{(1)} \\
\vdots \\
\hat{\theta}_{k}^{(N)}
\end{bmatrix}.$$  

(12)

Based on the definition of PE in (6), it is straightforward to show that the matrix $Z_t$ is PE if $\xi_{k}^{(i)}$ is PE for all $i$. This results in the convergence of equation (12) to some $\Theta^*$ for large enough $M$. From Theorem 5.1 in [14],

$$\lim_{k \to \infty} \|\hat{\theta}_{k}^{(i)} - \theta_{k}^{(i)}\| = 0, \quad \lim_{k \to \infty} \|\theta_{k}^{(i)} - \theta_{k}^{(i-1)}\| = 0.$$  

(13)

However, the convergence of $\hat{\theta}_{k}^{(i)}$ for all $i$ to some single value is non-trivial due to the interdependency in RLS. We will show that for a connected network of agents,

$$\lim_{t \to \infty} \|x_{t}^{(i)} - x_{t}^{(j)}\| = 0,$$

for any $i$ and $j$. Note that according to (11), if a node is disconnected from the graph it can be individually examined as in the centralized case. Recall that for $\ell = i, j$,

$$x_{t+1}^{(i)} = A x_{t}^{(i)} + B u_{t}^{(i)} = (A - B K_{k}^{(i)}) x_{t}^{(i)} + B e_{t}.$$

As such,

$$x_{t+1}^{(i)} - x_{t}^{(j)} = (A - B K_{k}^{(i)}) x_{t}^{(i)} - x_{t}^{(j)} + B \Delta K_{k} x_{t}^{(j)}$$  

(14)

where $\Delta K_{k} = K_{k}^{(i)} - K_{k}^{(j)}$. Then if we show that $\|\Delta K_{k}\| \to 0$ as $k \to \infty$ we obtain,

$$\|x_{t+1}^{(i)} - x_{t}^{(j)}\| = \|(A - B K_{k}^{(i)}) x_{t}^{(i)} - x_{t}^{(j)}\| \to 0,$$

(15)

given that the policy iteration algorithm leads to a more stabilizing controller $K_{k}^{(i)}$ as $k$ increases [14]. Then,

$$\|K_{k}^{(i)} - K_{k}^{(j)}\| = \|\hat{H}^{(i)}_{k-1} - \hat{H}^{(j)}_{k-1}\| \leq \left|\|\hat{\theta}_{k}^{(i)} - \hat{\theta}_{k}^{(j)}\|\right| \leq \left|\|\theta_{k}^{(i)} - \theta_{k}^{(j)}\|\right|$$

(16)

Since $\hat{H}_{22}$ and $\hat{H}_{21}$ contain only a subset of elements in $\theta$,

$$\|\hat{H}^{(i)}_{k-1} - \hat{H}^{(j)}_{k-1}\| \leq \left|\|\hat{\theta}_{k}^{(i)} - \hat{\theta}_{k}^{(j)}\|\right| \leq \left|\|\theta_{k}^{(i)} - \theta_{k}^{(j)}\|\right|.$$  

(17)

Hence equations (16) and (17) lead to,

$$\|K_{k}^{(i)} - K_{k}^{(j)}\|$$

$$\leq \|\hat{H}^{(i)}_{k-1} - \hat{H}^{(j)}_{k-1}\| \|\hat{\theta}_{k}^{(i)} - \hat{\theta}_{k}^{(j)}\| \|1 + \hat{H}^{(i)}_{k-1} - \hat{H}^{(j)}_{k-1}\|$$

$$\leq \kappa_{0} \|\hat{\theta}_{k}^{(i)} - \hat{\theta}_{k}^{(j)}\|,$$

(18)

where we have used the fact that the estimated parameters are bounded and $\kappa_{0} > 0$ is a constant such that,

$$\|\hat{H}^{(i)}_{k-1} - \hat{H}^{(j)}_{k-1}\| \cdot \|1 + \hat{H}^{(i)}_{k-1} - \hat{H}^{(j)}_{k-1}\| \leq \kappa_{0}.$$  

From Lemma 5.2 in [14],

$$\|\theta_{k}^{(i)} - \theta_{k}^{(j)}\| \leq \epsilon M \|\theta_{k}^{(i)} - \theta_{k}^{(j)}\| + \|\theta_{k}^{(i)} - \hat{\theta}_{k}^{(j)}\|,$$

(19)

Parameter estimation for the multi-output system is an straightforward extension of the scalar case and is discussed in Chapter 3.8 of [22].
which for large enough $M$ results in,
\[
\|\theta_k^{(i)} - \hat{\theta}_k^{(i)}\| + \|\theta_k^{(j)} - \hat{\theta}_k^{(j)}\|
\leq \epsilon_M \left(\|\theta_k^{(i)} - \hat{\theta}_{k-1}^{(i)}\| + \|\theta_k^{(j)} - \hat{\theta}_{k-1}^{(j)}\|ight) + \|\theta_k^{(i)} - \hat{\theta}_k^{(i)}\| + \|\theta_k^{(j)} - \hat{\theta}_k^{(j)}\|. \tag{19}
\]

Using triangle inequality on the left side of this inequality,
\[
\|\hat{\theta}_k^{(i)} - \hat{\theta}_k^{(j)}\| - \|\theta_k^{(i)} - \theta_k^{(j)}\|
\leq \epsilon_M \left(\|\theta_k^{(i)} - \hat{\theta}_{k-1}^{(i)}\| + \|\theta_k^{(j)} - \hat{\theta}_{k-1}^{(j)}\|ight) + \|\theta_k^{(i)} - \hat{\theta}_k^{(i)}\| + \|\theta_k^{(j)} - \hat{\theta}_k^{(j)}\|. \tag{20}
\]

Then, from (19) and (20) and for large $k$,
\[
\|\hat{\theta}_k^{(i)} - \hat{\theta}_k^{(j)}\| \leq \epsilon_M \left(\|\theta_k^{(i)} - \hat{\theta}_{k-1}^{(i)}\| + \|\theta_k^{(j)} - \hat{\theta}_{k-1}^{(j)}\|ight)
\]

Hence using the result in (13),
\[
\|\hat{\theta}_k^{(i)} - \hat{\theta}_k^{(j)}\| \rightarrow 0,
\]
and plugging this into (18),
\[
\|\Delta K_k\| = \|K_k^{(i)} - K_k^{(j)}\| \rightarrow 0, \tag{21}
\]

Hence,
\[
\|x_{i+1}^{(i)} - x_{i+1}^{(j)}\| \rightarrow 0.
\]

This implies that based on (11), for identical systems the algorithm moves towards $N$ decoupled $Q$-learning algorithms for each agent. Thus, although the provided data is from an interconnected system, each controller converges to its optimal value, i.e.,
\[
\lim_{k \rightarrow \infty} \|K_k^{(i)} - K^*\| = 0, \quad \text{for} \quad i = 1, 2, \ldots, N.
\]

**Remark 2.** In Algorithm 1, we have assumed that the exploration signal, $e_i$, is equal for every agent at each time step. This is a valid assumption as long as $Z_i$ in (12) is PE so that RLS is assured to converge. Another option would be to choose the excitation signals $e_i^{(i)}$ and $e_i^{(j)}$ in a way that,
\[
e_i^{(i)} - e_i^{(j)} = -\Delta_k x_i^{(j)}.
\]

Therefore, not only the input to the RLS is PE, the difference cancels out $\Delta_k x_i^{(j)}$ in (14). However, this setup is more challenging to implement, particularly for large-scale systems.

**C. Computational Saving**

The computational saving resulting from using the distributed $Q$-learning algorithm is significant, since for a large system, the design of the LQR controller with the computational complexity of solving ARE of order $O(n^3)$, can be prohibitively expensive. The main computational burden of Algorithm 1 comes from RLS where the complexity of its implementation is $O(\gamma^2)$ with $\gamma$ parameters to learn.

Assuming that the system contains $N$ agents each having $n$ states and $m$ inputs, the computational complexity of the centralized $Q$-learning is obtained by,
\[
O\left(\frac{(Nn + Nm)(Nn + Nm + 1)}{2}\right) \approx O(N^4(n + m)^4),
\]

while for the distributed case the code performs $N$ repetitions of the same RLS leading to the complexity bound,
\[
O\left(N\left(\frac{(n + m)(n + m + 1)}{2}\right)^2\right) \approx O(N(n + m)^4).
\]

Hence the complexity reduction is,
\[
\frac{N^4(n + m)^4 - (n + m)^4}{N^4(n + m)^4} \times 100 = \frac{N^3 - 1}{N^3} \times 100%.
\]

which is substantial for large $N$. Table I compares the computational saving for some values of $N$.

| $N$ | 2   | 3   | 5   | 8   | 100 |
|-----|-----|-----|-----|-----|-----|
| Saving (%) | 87.5 | 96.29 | 99.2 | 99.8 | 99.9 |

**TABLE I: Approximate computational saving of the distributed $Q$-learning compared to the centralized case in [14].**

**V. Example**

In this section, we provide an example to show the efficiency of the distributed $Q$-learning algorithm for a set of identical communicating UAVs. We consider the autonomous flight of a network of six interconnected Unmanned Aerial Vehicles (UAVs) which are set to perform a common task such as geographical data collection or putting out a wildfire. To cover the whole targeted area, these UAVs are programmed to move in parallel and in order for minimal signal transmissions, each UAV only communicates with its closest neighbor in the network as depicted in Figure 1.

![Fig. 1: A group of identical firefighting UAVs maneuvering in parallel aiming to extinguish a blaze (Aerial view of the forest fire - Photo Credit: Alex Punke, Bigstock).](image_url)

The discrete-time dynamics of UAVs is considered by mini-aircraft linear parameters that can be found in [23]. We assume $\bar{Q} = I_3$ and $\bar{R} = I_3$. We will show the results of the distributed policy iteration for $N = 6$, $n = 5$, and $m = 3$ and compare the computational performance with the centralized case. For the distributed algorithm we consider...
$M = 50$ and the exploration signal $e_k$ is generated from a normal distribution. Figure 2 shows the results of simulations regarding the controller error norm.

![Fig. 2: Performance of the distributed Q-learning algorithm.](image)

The plot demonstrates the norm of the error between the LQR optimal controller of each subsystem and the estimate of the algorithm at each iteration $k$.

A comparison between the computational performance of the centralized and distributed methods is also provided in Figure 3. For scaling purposes, $M$ and $e_k$ are re-adjusted for each $N$.

![Fig. 3: Computational performance of distributed and centralized algorithm for different number of interacting agents.](image)

VI. CONCLUSION

In this paper we examined a data-guided approach for the control of large-scale interconnected identical systems with decoupled dynamics; it is assumed that the interconnection is reflected in the cost function for the control design problem. We leveraged a distributed Q-learning as a policy iteration method. In this direction, it is shown that the proposed distributed algorithm converges to each agent’s individual optimal controller, which could have been obtained by running a centralized Q-learning algorithm. The significance of the resulting computational savings are also discussed.

There are a number of directions to pursue as future works. First, the observation in this paper can be further extended to more elaborate cost structure, highlighting the trade-off between local and global optimality in large-scale distributed systems. This can be achieved if other types of interconnections such as dynamics or feedback coupling as well as consensus through the Q-function are adopted for the analysis. Another line of work is to consider other types of data-guided distributed control mechanisms for structures such as layering or systems with switching dynamics.

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