Continuous time mean–variance–utility portfolio problem and its equilibrium strategy

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ABSTRACT
In this paper, we propose a new class of optimization problems, which maximize the terminal wealth and accumulated consumption utility subject to a mean–variance criterion controlling the final risk of the portfolio. The multiple-objective optimization problem is firstly transformed into a single-objective one by introducing the concept of overall ‘happiness’ of an investor defined as the aggregation of the terminal wealth under the mean–variance criterion and the expected accumulated utility, and then solved under a game-theoretic framework. We have managed to maintain analytical tractability; the closed-form solutions found for a set of special utility functions enable us to discuss some interesting optimal investment strategies that have not been revealed in the literature before.

1. Introduction
The optimal portfolio selection problem is essentially to achieve a balance between uncertain returns and risks, for which the mean–variance methodology has become one of the most important tools, ever since Markowitz’s pioneering work [1] on a static investment model. This approach conveys a nice and elegant idea, maximizing the expected return at a given level of risk or minimizing the risk at a given level of the expected return, and has been applied to finance practice (see [2–15]).

Although Markowitz’s work is theoretically very appealing, it only provides results based on a single-period (static) model, in which the investors can only make a decision at the very beginning, while they are not allowed to make any adjustments before the investment period ends. In fact, dynamic mean–variance optimization is not a trivial task, as Bellman’s dynamic programming principle cannot be directly applied to these kinds of path-dependent optimization
problems due to the nonlinearity of the variance operator. In this sense, such kind of problems is usually referred to as time-inconsistent problems and the corresponding solutions are called time-inconsistent solutions.

A natural approach to solve time-inconsistent problems is to optimize the objective with a fixed initial point under the mean–variance criterion, and the derived solution is regarded as an optimal pre-commitment solution. For instance, with the market being complete and continuous timewise, various results have been presented for the variance-minimizing policy using the martingale methods, given that the expected terminal wealth is equal to a certain level (see [16–19]). In contrast, Cochrane [20] derived the optimal investment policy that minimizes the ‘long-term’ variance of the portfolio return subject to the constraint that the long-term mean of the portfolio return equals a pre-specified target level under an incomplete market. This approach has also been applied in futures trading strategies by Duffie and Richardson [21] through setting a mean–variance objective at the initial date. They obtained a pre-commitment solution, which also solves the optimal problem with a quadratic objective for some specific parameters. A similar approach developed for the continuous-time complete-market settings has also been widely discussed (see [22–25]).

However, Basak and Chabakauri [26] challenged the pre-commitment assumption [25], and assumed investors are sophisticated in the sense that they will maximize their mean–variance objective over time considering all future updates, instead of finding an optimal solution at a fixed given time moment. Following this, Kryger and Steffensen [27] worked under the Black–Scholes framework without the pre-commitment assumption and showed that the optimal strategy derived for a mean-standard deviation investor is to take no risk at all. The latest contribution to the relevant literature was presented in [28], in which the mean–variance optimization problem was considered under a game-theoretic framework [29], and the optimal strategies were derived in the context of the sub-game perfect Nash equilibrium.

Although all the work mentioned above is very appealing, consumption was usually not considered in mean–variance optimization problems, which is not consistent with what is happening in practice, as the decisions made for consumption would naturally have an impact on the optimality of the investment strategy. Thus, Christiansen and Steffensen [30] as well as Kronborg and Steffensen [31] started to incorporate some consumption choices into the mean–variance problem, investigating the optimal investment-consumption problem together with the mean–variance criterion. Unfortunately, the optimal consumption strategies derived under their particular model assumptions and the formulations of the problem have led to a rather absurd conclusion that an investor could suddenly be required to switch his/her consumption strategy from consuming as much as possible to as little as possible, in order to achieve the ‘optimal’ objective set at the beginning of the investment period. Their results indicate that directly incorporating the consumption choices into the mean–variance problem leads
to a fundamental flaw in the state-of-the-art model frame of the portfolio selection problem. This has prompted us to propose a new model frame that would not only eliminate this rather strange behaviour but also maintain mathematical tractability so that a more rational investment behaviour can be discussed economically for some simple utility functions through the obtained analytical closed-form solutions.

A new class of continuous-time portfolio selection problems is proposed in this paper, which combine the maximization of the terminal wealth under the mean–variance criterion and the maximization of the accumulated consumption utility together. Instead of adding consumption back to the terminal wealth in the objective value function of the optimization problem as previously presented in the literature [30,31], we introduce the concept of overall ‘happiness’ of an investor, which is measured by the aggregation of the terminal wealth under the mean–variance criterion and the expected accumulated utility, using a consumption preference parameter. Amazingly, the newly formulated optimization problem preserves the analytical tractability under a continuous-time game-theoretic framework, and the analytical optimal continuous investment and consumption strategies derived in the sense of equilibrium have an intuitive economic explanation.

The rest of this paper is organized as follows. Section 2 reviews the classical mean–variance problem and proposes the new portfolio selection problem. In Section 3, we analytically derive the optimal strategies based on the definition of the equilibrium strategy. Explicit solutions to the optimal strategies are then presented in Section 4. Numerical examples and discussions are provided in Section 5, followed by some concluding remarks given in the last section.

2. The portfolio selection problem

In this section, we first briefly review the classical continuous-time mean–variance portfolio selection models, after which our new model is established by introducing consumption into a classical model.

2.1. The classical mean–variance portfolio problem

We now assume that we work under the standard Black–Scholes market, where an investor has access to a risk-free bank account $M$ and a risky asset $S$ whose dynamics can be specified as

$$
\begin{align*}
\text{d}M(t) &= rM(t) \, \text{d}t, \quad M(0) = 1, \\
\text{d}S(t) &= \mu S(t) \, \text{d}t + \sigma S(t) \, \text{d}B(t), \quad S(0) = s_0 > 0.
\end{align*}
$$

(1)

Here, $r > 0$, $\mu$ and $\sigma$ are constants, and it is assumed that $\mu > r$. The process $B(t)$ is a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the filtration $\{\sigma ; 0 \leq s \leq t\}, \forall t \in [0, T]$. 

We also assume that the investor in this market needs to make investment decisions on a finite time horizon \([0, T]\), and he/she allocates a proportion \(\pi(t)\) and \(1 - \pi(t)\) of his/her wealth into the stock and bank account, respectively, at time \(t\). Let \(X^\pi(t)\) be the wealth of the investor at time \(t\) following the investment strategy \(\pi(\cdot)\) with an initial wealth of \(x_0\) at time 0, in which case we must have \(X^\pi(t) = \pi(t)S(t) + (1 - \pi(t))M(t)\). Therefore, the dynamic of the investor’s wealth follows:

\[
\begin{align*}
\text{d}X^\pi(t) &= [(r + \pi(t)(\mu - r))X^\pi(t)] \text{d}t + \pi(t)\sigma X^\pi(t) \text{d}B(t), \quad t \in [0, T), \\
X(0) &= x_0 > 0.
\end{align*}
\]

(2)

If \(L^2_F(0, T; R)\) denotes the set of all \(R\)-valued, measurable stochastic processes \(f(t)\) adapted to \(\{F_t\}_{t \geq 0}\) such that \(E[\int_0^T f^2(t) \text{d}t] < \infty\), the classical continuous-time mean–variance portfolio optimization problem is stated below.

**Definition 2.1 ([25]):** A portfolio strategy \(\pi(\cdot)\) is admissible if \(\pi(\cdot) \in L^2_F(0, T; R)\).

**Definition 2.2 ([25]):** The continuous-time mean–variance portfolio optimization problem is a multi-objective optimization problem, which is defined as

\[
\begin{align*}
\min_{\pi(\cdot)} & \quad (V_1(\pi(\cdot)), V_2(\pi(\cdot))) \equiv (-E(X(T)), \text{Var}(X(T))), \\
\text{s.t.} & \quad \pi(\cdot) \in L^2_F(0, T; R), \\
& \quad (X(\cdot), \pi(\cdot)) \text{ satisfy Equation (2),}
\end{align*}
\]

(3)

where \(E(X(T))\) and \(\text{Var}(X(T))\) are the expectation and variance of \(X(T)\), respectively.

Optimization problem (3) can be transformed into a single-objective optimization problem by introducing a weight parameter \(\gamma\) such that the new objective becomes the weighted average of two original objectives using the mild convexity condition [25]

\[
\begin{align*}
\min_{\pi(\cdot)} & \quad V_1(\pi(\cdot)) + \frac{\gamma}{2} V_2(\pi(\cdot)) \equiv -E(X(T)) + \frac{\gamma}{2} \text{Var}(X(T)), \\
\text{s.t.} & \quad \pi(\cdot) \in L^2_F(0, T; R), \\
& \quad (X(\cdot), \pi(\cdot)) \text{ satisfy Equation (2),}
\end{align*}
\]

(4)

where the weight parameter satisfies \(\frac{\gamma}{2} > 0\). In fact, this particular optimization problem (4) has been extensively studied in the past 20 years with various theoretical results, numerical algorithms, and applications being available in the literature. Interested readers are referred to [19,25,26] and the references therein for more details.
2.2. The mean–variance–utility consumption and investment problem

It should be pointed out that classical mean–variance optimization problem (3) or transformed optimization problem (4) has a fundamental flaw that the wealth of the investor does not take into account the investor’s income and consumption. One main possible reason is that the incorporation of the consumption choices in the classical mean–variance problem will destroy the tractability of the original problem. However, this is apparently not appropriate as it is not consistent with real situations, and thus, we consider a natural extension of the mean–variance problem. Assume that the investor possesses a continuous deterministic income rate \( l(t) \) and chooses a non-negative consumption rate \( c(t) \). Under these assumptions, the dynamic of the investor’s wealth can be derived as

\[
\begin{align*}
\frac{dX^{c,\pi}(t)}{dt} &= [(r + \pi(t)(\mu - r))X^{c,\pi}(t) + l(t) - c(t)] dt \\
&+ \pi(t)\sigma X^{c,\pi}(t) dB(t), \quad t \in [0, T), \quad X(0) = x_0 > 0.
\end{align*}
\]

(5)

Obviously, after incorporating the consumption into the mean–variance problem, the investor is also seeking his/her maximum utility through the consumption choices, apart from the mean–variance type objective as specified in optimization problem (4). In other words, the investor again faces a dual-objective optimization problem; he/she wants to achieve the maximum accumulated utility over a consumption choice, while at the same time minimizing the investment risk by considering a mean–variance objective over terminal wealth \( X(T) \). In this case, we need a measurement for the utility obtained through consumption. With \( \rho \) representing a constant discounting rate and \( U(\cdot) \) denoting a utility function, the accumulated utility of the investor through his/her continuous consumption on the time interval \([t, T]\) can be defined as

\[
V^{c,\pi}_3(t, x) = E\left[\int_t^T e^{-\rho(s-t)}U(c(s)) \, ds\right],
\]

(6)

where \( E(\cdot) \) denotes taking the expectation, and the utility function \( U(\cdot) \) is continuous, concave and increasing with \( U(0) = 0 \). It is this new optimal portfolio selection problem, named as the ‘mean–variance–utility consumption and investment optimization problem’, that is presented below.

**Definition 2.3:** A new mean–variance–utility consumption and investment optimization problem can be formulated as

\[
\max \left[ V^{c,\pi}_1(t, x), V^{c,\pi}_2(t, x), V^{c,\pi}_3(t, x) \right] = \left[ E(X(T)), -\text{Var}(X(T)), E\left(\int_t^T e^{-\rho(s-t)}U(c(s)) \, ds\right) \right].
\]
Similarly to what have been presented in the previous subsection, optimization problem (7) can also be converted into a single-objective optimization problem

\[
\max_{c(\cdot), \pi(\cdot)} \ E(X(T)) - \frac{\gamma}{2} \text{Var}(X(T)) + \beta E \left( \int_t^T e^{-\rho(s-t)} U(c(s)) \, ds \right)
\]

s.t. \( c(\cdot), \pi(\cdot) \in L^2_F(0, T; R), \)

\( (X(\cdot), c(\cdot), \pi(\cdot)) \) satisfy Equation (5), \( \quad (7) \)

where \( \beta \) is a positive constant. It should be noted that problem (8) degenerates to classical mean–variance portfolio selection model (4) when \( \beta \) approaches zero.

Clearly, the parameter \( \beta \) can be treated as a trade-off between acquiring more terminal wealth in the mean–variance sense and achieving more accumulated utility through consumption; the larger the value of \( \beta \) is, the more inclined the investor is to consume to maximize his/her accumulated utility. In this sense, \( \beta \) is actually a consumption preference parameter. It should also be noted that the introduced parameter \( \beta \) can be regarded as a conversion operator that converts the utility units to the wealth units.

It should be remarked here that various authors tried to incorporate consumption into the optimization problem, and one of the most popular approaches is to use the mean and variance framework to approximate other expected utility functions in the theoretical modelling as well as practical applications of the portfolio selection problems [32–34]. However, these are only approximations and will inevitably induce biases. It should also be particularly emphasized that although Christiansen and Steffensen [30] as well as Kronborg and Steffensen [31] have already tried to incorporate consumption into the mean–variance framework, the problem they discussed is essentially different from our new problem (8). This is because they directly added the accumulated consumption to the terminal wealth to formulate an ‘adjusted’ terminal wealth and considered the mean–variance optimization of the adjusted wealth, while we have distinguishably introduced a parameter \( \beta \) so that the mean–variance of the terminal wealth and the accumulated consumption utility can be added together to form a new objective. In this way, the new objective can be economically interpreted as the overall 'happiness' of an investor towards his/her investment return as well as the undertaken risk level during the time period \([0, T]\).

The investor aims at achieving the maximum overall happiness through the combination of maximizing the terminal wealth with the mean–variance criteria and the consumption utility, leading to a new class of the mean–variance–utility optimization problems. What our new model suggests is that an investor should not proportionally consume as suggested by Merton's classic framework and she/he should also consider the balance of the total wealth management under
Markowitz’s mean–variance criterion. The fundamental reason is because now he/she still wants to minimize his/her total investment risk at the end of the investment period, while maximizing his/her expected return and accumulated consumption utility. Of course, it is interesting to explore whether the introduction of the consumption utility would affect the previous structure of the investment strategy reported in [26,28,31]. The new challenge here is to solve new optimal portfolio selection problem (8), which will be discussed in the next section.

3. The optimal portfolio selection strategy

Having successfully established a new optimal portfolio selection problem in (8), a natural question is whether or not there exists a solution to the optimal portfolio selection strategy, and how it can be derived if it does exist. This is a challenging problem because we are not able to find time-consistent solutions in the sense that the condition for the Bellman Optimality Principle no longer holds, given that the law of the iterated expectations does not apply for a given strategy. Therefore, in this section, we attempt to seek an optimal solution to problem (8) in the sense of time inconsistency.

Similar to [28,29], a natural way for an investor to deal with any time-inconsistent problem is to solve the problem by setting $t = 0$, and the investor will follow the resulting optimal strategy during the finite time horizon. This is the so-called pre-commitment control, i.e. the investor pre-commits at a fixed time moment. However, one of the main drawbacks of the optimal solution with pre-commitment is that it will not be optimal for the control problem at any time $t > 0$.

In fact, most investors in practice would assign the same weights to all time instances, implying that they are looking for an optimal strategy that is optimal from the point of view at any time $t$ during the considered time horizon instead of time 0. Therefore, instead of seeking a pre-commitment solution, our problem is considered under a game-theoretic framework without pre-commitment, which was introduced in [28,29] and developed by Kronborg and Steffensen [31] as well as Kryger et al. [35].

To solve optimization problem (8), we consider a more general optimization problem with a discount factor as follows:

$$\max_{c(\cdot), \pi(\cdot)} \quad E(e^{-\delta(T-t)}X(T)) - \frac{\gamma}{2} \text{Var}(e^{-\delta(T-t)}X(T))$$

$$+ \beta E \left( \int_t^T e^{-\rho(s-t)} U(c(s)) \, ds \right) \quad (9)$$

$$\text{s.t.} \quad \left\{ \begin{array}{l}
    c(\cdot), \pi(\cdot) \in L^2_T(0, T; R), \\
    (X(\cdot), c(\cdot), \pi(\cdot)) \text{ satisfy Equation (5),}
  \end{array} \right. $$
where $\delta$ is a discount rate, $\gamma$ is the risk-aversion coefficient, and $\beta$ is a positive parameter introduced to represent the trade-off between the terminal wealth and consumption utility. Obviously, optimization problem (9) degenerates to the original one (8) when the investor requires a discount rate of 0.

The equilibrium strategy under the continuous-time game theoretic equilibrium for problem (9) can be defined below.

**Definition 3.1:** Consider a strategy $(c^*, \pi^*)$ and a fixed point $(c, \pi)$. For a fixed number $h > 0$ and an initial point $(t, x)$, we define the strategy $(\tilde{c}_h(s), \tilde{\pi}_h(s))$ as

$$
(\tilde{c}_h(s), \tilde{\pi}_h(s)) = \begin{cases} (c, \pi), & \text{for } t \leq s < t + h, \\
(c^*(s), \pi^*(s)), & \text{for } t + h \leq s < T.
\end{cases}
$$

(10)

If

$$
\lim_{h \to 0} \inf \frac{1}{h} \left( f(c^*, \pi^*, t, x, y^c, \pi^c, z^c, \pi^c, w^c, \pi^c) - f(\tilde{c}_h, \tilde{\pi}_h, t, x, y^{\tilde{c}_h, \tilde{\pi}_h}, z^{\tilde{c}_h, \tilde{\pi}_h}, w^{\tilde{c}_h, \tilde{\pi}_h}) \right) \geq 0
$$

(11)

for all $(c, \pi) \in \mathbb{R}_+ \times \mathbb{R}$, where $f$ is a general objective function and

$$
y^{c, \pi} := y^{c, \pi}(t, x) = E \left[ e^{-\delta(T-t)} X^{c, \pi}(T) \bigg| X(t) = x \right],
$$

$$
z^{c, \pi} := z^{c, \pi}(t, x) = E \left[ \left( e^{-\delta(T-t)} X^{c, \pi}(T) \right)^2 \bigg| X(t) = x \right],
$$

$$
w^{c, \pi} := w^{c, \pi}(t, x) = E \left[ \int_t^T e^{-\rho(T-t)} U(c(s)) \bigg| X(t) = x \right],
$$

(12)

then $(c^*, \pi^*)$ is called an equilibrium strategy.

The equilibrium strategy under the Nash Equilibrium Criteria defined in Definition 3.1 is a time-inconsistent solution to control problem (8), which is essentially different from time-consistent solutions discussed in the context of optimization [26]. If we denote $(c^*, \pi^*)$ as the equilibrium strategy satisfying Definition 3.1, and let $V$ be the corresponding value function with the equilibrium strategy, we can obtain

$$
V(t, x) = f^{c, \pi}(t, x, y^{c, \pi}, z^{c, \pi}, w^{c, \pi}).
$$

(13)

Clearly, our problem is to search for the corresponding optimal strategies and the optimal value function $f : [0, T] \times \mathbb{R}^4 \to \mathbb{R}$ as a $C^{1,2,2,2}$ function of the form

$$
f^{c, \pi}(t, x, y^{c, \pi}, z^{c, \pi}, w^{c, \pi}) = y - \frac{\gamma}{2} (z - y^2) + \beta w, \quad (c, \pi) \in \mathcal{A},
$$

(14)

where $\mathcal{A}$ is the class of admissible strategies to be defined below. As pointed in [35], the investor continuously deviates from this strategy and thus does not
actually achieve any of the determined supremums. Instead, the investor con--centrates on determining the equilibrium control law, as introduced in [27–29]. The desired investment strategy is determined so that it maximizes the present objective at any time moment $t$, under the restriction that the future strategy is assumed to be given. In other words, the strategy is determined through the backward recursion, and thus, this recursively optimal solution under the equilibrium control law is also regarded as the optimal control (see [27,35]).

Before we are able to present the optimal solution, some preliminaries need to be outlined. In particular, we establish an extension of the Hamilton–Jacob–Bellman (HJB) equation which will be discussed in detail later in Lemma 3.2 for the characterization of the optimal value function and the corresponding optimal strategy, so that the stochastic problem can be transformed into a system of deterministic differential equations and a deterministic point-wise minimization problem.

Let $A$ be the set of admissible strategies that contains all strategies $(c, \pi)$ satisfying the following two assumptions: (i) there exist solutions to partial differential equations (15)–(17); (ii) the stochastic integrals in (A3), (A8), (A11) and (A20) are martingales. In this way, the equilibrium strategy we solve below is restricted in set $A$. Now, we can prove the following two lemmas.

**Lemma 3.1:** If there exist three functions $Y = Y(t,x), Z = Z(t,x)$ and $W = W(t,x)$ such that

\[
\begin{align*}
Y_t(t,x) &= -[(r + \pi(\mu - r))x + l - c]Y_x(t,x) \\
&\quad - \frac{1}{2}\pi^2\sigma^2 x^2 Y_{xx}(t,x) + \delta Y(t,x), \\
Y(T,x) &= x,
\end{align*}
\]

(15)

\[
\begin{align*}
Z_t(t,x) &= -[(r + \pi(\mu - r))x + l - c]Z_x(t,x) \\
&\quad - \frac{1}{2}\pi^2\sigma^2 x^2 Z_{xx}(t,x) + 2\delta Z(t,x), \\
Z(T,x) &= x^2,
\end{align*}
\]

(16)

and

\[
\begin{align*}
W_t(t,x) &= -[(r + \pi(\mu - r))x + l - c]W_x(t,x) \\
&\quad - \frac{1}{2}\pi^2\sigma^2 x^2 W_{xx}(t,x) - e^{-\rho t} U(c), \\
W(T,x) &= 0,
\end{align*}
\]

(17)

where $(c, \pi)$ is an arbitrary admissible strategy, then

\[
Y(t,x) = y^{c,\pi}(t,x), \quad Z(t,x) = z^{c,\pi}(t,x), \quad W(t,x) = w^{c,\pi}(t,x),
\]

(18)

where $y^{c,\pi}, z^{c,\pi}$ and $w^{c,\pi}$ are given by (12).
**Proof:** See Appendix 1 for the proof.

**Lemma 3.2:** If there exists a function $F = F(t, x)$ such that

\[
F_t = \inf_{c, \pi \in A} \left\{ - [r + \pi (\mu - r)]x + l - c(F_x - Q) - \frac{1}{2} \pi^2 \sigma^2 x^2 (F_{xx} - K) + J \right\},
\]

where $Q = f^c, \pi^*$, 

\[
K = f^{c, \pi^*}_{xx} + f^{c, \pi^*}_{yy} (F^{(1)}_x)^2 + f^{c, \pi^*}_{yy} (F^{(2)}_x)^2 + f^{c, \pi^*}_{ww} (F^{(3)}_x)^2 
+ 2 f^{c, \pi^*}_{xy} F^{(1)}_x + 2 f^{c, \pi^*}_{yz} F^{(2)}_x 
+ 2 f^{c, \pi^*}_{zw} F^{(3)}_x + 2 f^{c, \pi^*}_{yw} F^{(1)}_x F^{(2)}_x + 2 f^{c, \pi^*}_{ww} F^{(3)}_x F^{(3)}_x
\]

and

\[
J = \int f^{c, \pi^*}_t + f^{c, \pi^*}_y \delta F^{(1)} + 2 f^{c, \pi^*}_z \delta F^{(2)} - f^{c, \pi^*}_w e^{-\rho t} U(c(t)).
\]

with

\[
F^{(1)} = y^{c, \pi^*} (t, x), \quad F^{(2)} = z^{c, \pi^*} (t, x), \quad F^{(3)} = w^{c, \pi^*} (t, x),
\]

then

\[
F(t, x) = V(t, x),
\]

where $V$ is the optimal value function defined by (13).

**Proof:** See Appendix 2 for the proof.

**Remark 3.1:** The representation corresponds to pseudo-Bellman equation (19), originally presented in [27] and applied in Theorem 2.1 of [31], calls for optimization across strategies, whereas the whole point of the dynamic programming is to appeal only to optimization across vectors. Therefore, the optimal solution solved by this approach belongs to the subspace of $A$. Similar to Björk and Murgoci [29], the optimality of our obtained strategy can also be confirmed in the sense of equilibrium.

## 4. The optimal consumption and investment

In this section, we present the optimal solutions to optimal portfolio selection problem (8) based on the results derived in the previous section, and some detailed discussions are provided to illustrate the behaviour of the optimal strategies.
A candidate strategy for optimal value function (19) can be derived by simply differentiating (19) with respect to $c$ and $\pi$, respectively. By noticing that the terms containing $c$ and $\pi$ are independent of each other, we obtain

$$\frac{\partial}{\partial c} \left( c(F_x - Q) - f_w e^{-\rho t} U'(c) \right) = 0$$

(22)

and

$$\frac{\partial}{\partial \pi} \left( -\pi (\mu - r)x(F_x - Q) - \frac{1}{2} \pi^2 \sigma^2 x^2 (F_{xx} - K) \right) = 0.$$  

(23)

A further simplification then yields

$$\begin{cases} 
  c^* = [U']^{-1} \left( \frac{F_x - Q}{f_w} e^{-\rho t} \right), \\
  \pi^* = -\frac{\beta - r}{x\sigma^2} \frac{F_x - Q}{F_{xx} - K},
\end{cases}$$

(24)

where $[f]^{-1}(\cdot)$ is the inverse function of $f$, and stars denote that they are the optimal strategies.

Substituting the corresponding objective form

$$f(t, x, y, z, w) = y - \gamma(z - y^2) + \beta w$$

(25)

into (20) and (21) gives

$$Q = 0, \quad K = \gamma (F_x^{(1)})^2, \quad J = \delta F^{(1)} - \gamma \delta \left( F^{(2)} - (F^{(1)})^2 \right) - \beta e^{-\rho t} U(c).$$

(26)

Given the linear structure of dynamics (15), (16) and (17), as well as of the boundary conditions, $F$, $F^{(1)}$ and $F^{(3)}$ can be written in the following form:

$$F(t, x) = A(t)x + B(t), \quad F^{(1)}(t, x) = a(t)x + b(t), \quad F^{(3)}(t, x) = p(t)x + q(t),$$

(27)

which then naturally leads to $F^2$ being written in the form

$$F^{(2)}(t, x) = \frac{2}{\gamma} \left[ a(t)x + b(t) + \beta \left[ p(t)x + q(t) \right] \right. \\
- \left. [A(t)x + B(t)] + [a(t)x + b(t)]^2 \right].$$

(28)

This is a quite generally adopted approach in the literature [28,29,31], and the correctness of the assumption is usually verified at the end of the derivation that the found solution is indeed the required form. Now, the substitution of (27) and (28)
into (26) results in
\[ Q = 0, \quad K = \gamma (a(t))^2, \]
and
\[ J = \delta [a(t)x + b(t)] - 2\delta [a(t)x + b(t) + \beta [p(t)x + q(t)]] ~ \]
\[ - [A(t)x + B(t))] - \beta e^{-\rho t}U(c(t)), \]
with which optimal strategy (24) becomes
\[
\begin{align*}
  c^* &= \left[U'\right]^{-1} \left(\beta^{-1} e^{-\rho t} A(t)\right), \\
  \pi^* &= \frac{1}{\gamma} \frac{\mu - r}{x a^2} A(t).
\end{align*}
\]
If we further substitute (29), (30) and (31) into (15)–(17) and (19) with the corresponding terminal conditions, it is straightforward to show that
\[
\begin{align*}
  A_t x + B_t &= -r A - \frac{1}{2\gamma} \frac{(\mu - r)^2 A^2}{\sigma^2 a^2} - l A + c^* A - \beta e^{-\rho t} U(c^*) + \delta(ax + b) \\
  &- 2\delta(ax + b - Ax - B + \beta(p x + q)), \\
  a_t x + b_t &= -r A - \frac{1}{2\gamma} \frac{(\mu - r)^2 A^2}{\sigma^2 a^2} - l a + c^* A + \delta(ax + b), \\
  p_t x + q_t &= -r p - \frac{1}{\gamma} \frac{(\mu - r)^2 A p}{\sigma^2 a^2} - l p + c^* p - e^{-\rho t} U(c^*(t)),
\end{align*}
\]
with the terminal conditions \( A(T) = a(T) = 1 \) and \( B(T) = b(T) = p(T) = q(T) = 0. \)

After some further simplifications, we can obtain \( A(t) = a(t) = e^{(r-\delta)(T-t)}, \)
\( p(t) = 0, q(t) = \int_t^T e^{-\rho s} U(c^*(s)) \, ds, \)
and
\[
\begin{align*}
  b(t) &= e^{\delta t} \int_t^T \left[ \frac{1}{\gamma} \frac{(\mu - r)^2}{ \sigma^2 a^2} + l(s)e^{(r-\delta)(T-s)} - c^*(s)e^{(r-\delta)(T-s)} \right] e^{-\delta s} \, ds, \\
  B(t) &= e^{2\delta t} \int_t^T \left[ \frac{1}{2\gamma} \frac{(\mu - r)^2}{ \sigma^2 a^2} + l(s)e^{(r-\delta)(T-s)} - c^*(s)e^{(r-\delta)(T-s)} \right] e^{-2\delta s} \, ds.
\end{align*}
\]
Therefore, the solution to the mean–variance–utility problem can be presented in the following proposition.

**Proposition 4.1:** Suppose that the utility function \( U(\cdot) \) is continuous, concave and increasing. The optimal consumption and investment strategy for problem (9) are,
respectively, given by

\[ c^* = [U']^{-1} \left( \beta^{-1} e^{(r(T-t)-\delta T)} \right) \]  
(34)

and

\[ \pi^* = \frac{1}{\gamma} \frac{\mu - r}{x \sigma^2} e^{-(r-\delta)(T-t)}. \]  
(35)

In particular, when the discount rate \( \delta \) is 0, we can obtain

\[ c^* = [U']^{-1} \left( \beta^{-1} e^{r(T-t)} \right) \]  
(36)

and

\[ \pi^* = \frac{1}{\gamma} \frac{\mu - r}{x \sigma^2} e^{-r(T-t)}, \]  
(37)

which are exactly the desired optimal strategy of mean–variance–utility problem (8). This shows that under the mean–variance–utility criterion, the optimal consumption rate of the investor is independent of the current wealth, being an increasing function of time \( t \), while the optimal investment rate is reversely proportional to the current wealth.

Remark 4.1: The new optimal portfolio selection problem subject to a minimized risk at the end of an investment period has led to at least two very interesting features that clearly distinguish themselves from those of Merton’s classic framework [36]:

(i) The optimal consumption strategy derived under the current mean–variance–utility framework is independent of the wealth, as suggested by Equation (36). In Merton’s classic framework, optimal consumption depends on one’s current total accumulated wealth. This of course makes sense economically as one would probably feel that he/she can afford to consume more when his/her total wealth is larger. However, in our new problem, the newly introduced risk control at the end of the investment period has magically balanced out such a dependence; our solution (36) suggests that one’s optimal consumption should be independent of the current wealth and be an increasing function of \( t \). But, his/her consumption is no longer directly proportional to the total wealth, as there is not much time left at the end of an investment period to control the total investment risk while optimizing his/her return. When there is no need to worry about the investment risk at all, his/her investment behaviour would naturally be different as suggested by Merton’s original framework. It should also be noted that the final wealth under our framework can be negative with a positive probability. This is because of the finite slope of the utility function at \( x = 0 \), and a non-negative wealth can only be achieved if this is explicitly imposed as
a side constraint. However, with this additional constraint being imposed, one is not able to obtain explicit results as what we have presented here. In fact, Korn and Trautmann [37] were the first to solve a continuous-time mean–variance problem under the additional constraint of a non-negative final wealth, and interested readers are referred to [38,39] for further details on this.

(ii) On the other hand, the optimal investment strategy in our problem (37) shows that the optimal investment strategy \( \pi^* \) obtained in this paper is dependent on the current wealth, which is consistent with the reality. The optimal investment rate is in fact inversely related to the current wealth value, since investors have to manage the risk of the current wealth under the mean–variance criterion, in which case the investment rate will be slowed down when the wealth value increases. If we further rewrite (37) as

\[
\pi^* x = \frac{1}{y} \frac{\mu - r}{\sigma^2} e^{-r(T-t)}.
\]

it is not difficult to find that the dollar amount invested in the risky asset at time \( t \) is independent of the current wealth \( x \), which agrees well with the previous relevant literature [26,28,31]. The most astonishing part, however, is that all the optimal investment ends with a same form, as long as a mean variance is built into a model. Specifically, the optimal investment found by Basak and Chabakauri [26], who solve the dynamic mean–variance portfolio problem and derive its time-consistent solution using dynamic programming, and Björk et al. [28], who place the mean–variance problem within a game-theoretic framework and obtain the subgame perfect Nash equilibrium strategies with time inconsistency, and even Kronborg and Steffensen [31], who take into account the consumption term in the mean–variance framework and consider the optimal investment strategy, all share a common formula, Equation (37), no matter where the mean variance is placed at. This suggests that the mean–variance term added to minimize the investment risk only exerts the influence on the investment strategy itself, while the consumption choice is not strictly restricted. This makes economical sense as risks associated with investing in a risky asset are a different type of risk from those associated with consumption, which has a direct impact on the total wealth available to be invested at each point of the investment horizon.

It should be noted that as the consumption is deterministic, the investor can actually just adjust the initial capital by the present value of future (deterministic) income and future (deterministic) consumption and then invest the remaining capital according to Basak and Chabakauri [26]. Since the Basak and Chabakauri strategy (amount invested) is independent of wealth, the consumption/investment combination actually has a simple and reasonable interpretation, which is stated below.
**Remark 4.2:** The optimal investment strategy obtained under our mean– variance–utility framework is exactly the same as that derived in [30,31], with the optimal amount of money being independent of wealth. A possible explanation is that the intermediary consumption is independent of the wealth and involves no risk, which indicates that the structure of the solution to the remaining mean–variance problem remains the same, independent of this consumption term. In particular, one has to firstly finance the deterministic optimal consumption, and the rest of the capital is invested according to a mean–variance problem without consumption. As the capital is invested independently of the size of the wealth, financing the optimal consumption does not play a role there. Therefore, the presence of the consumption utility could not yield the fundamental change in the optimal investment strategy, and the form of the investment strategy remains the same as that under the mean–variance criterion. We also note that the optimal consumption strategies under the two frameworks are completely different, as the one in [30,31] is discrete, taking either the maximal or minimal allowed value, while ours is continuous. Moreover, from an economic point of view, their results are actually not reasonable as it is usually not possible for a normal investor to make sudden changes in his/her consumption strategy from consuming the maximal to the minimal allowed value.

Having successfully derived the optimal consumption and investment strategy, it is not difficult to formulate the optimal value function of problem (9) as

$$V(t, x) = e^{(r - \delta)(T - t)} x + B(t),$$  \hspace{1cm} (39)

where $B(t)$ is specified in (33). Obviously, the optimal value function at the terminal point is constructed with the accumulated amount of the wealth $x$ at time $t$ and the additional amount resulted from the continuous consumption and investment strategy. With

$$V_x(t, x) = e^{(r - \delta)(T - t)} > 0,$$  \hspace{1cm} (40)

the optimal value increases with wealth, which financially matches with one’s intuition. The sensitivity of the optimal value function with respect to the time is affected by two aspects, i.e. the wealth, and the consumption and investment strategy.

The optimal strategy derived in (34) and (35) can also give rise to the conditional expected value and conditional second moment of the discounted optimal terminal wealth, yielding

$$E\left[e^{-\delta(T-t)}X^{e^{\pi}}(T) \mid X(t) = x\right] = e^{(r - \delta)(T - t)} x + b(t),$$  \hspace{1cm} (41)

and

$$E\left[\left(e^{-\delta(T-t)}X^{e^{\pi}}(T)\right)^2 \mid X(t) = x\right] = \frac{2}{\gamma} \left\{b(t) - B(t) + \beta \left[p(t)x + q(t)\right]\right\},$$  \hspace{1cm} (42)
respectively. One can also similarly compute the conditional expectation of the discounted accumulated utility of the consumption as

\[ E \left[ \int_t^T e^{-\rho(T-t)} U(c^*(s)) \left| X(t) = x \right. \right] = \int_t^T e^{-\rho s} U(c^*(s)) \, ds. \]  

(43)

To further investigate the properties of the optimal consumption strategy as well as the corresponding optimal value function, we now provide three examples with the specific utility functions.

**Proposition 4.2:** With some particular choices for the utility functions in problem (7), the corresponding consumption strategies can be specified according to Proposition 4.1.

(i) With a logarithmic utility function \( U(c) = \log(c) \), optimal consumption strategy (34) can be simplified to the form

\[ c^* = \beta e^{-r(T-t)} \]  

(44)

Then,

\[ c^* = \beta e^{-r(T-t)} \]  

(45)

is the optimal solution for mean–variance–utility problem (8).

(ii) With a power utility function \( U(c) = c^{\theta}/\theta \), where \( \theta < 1 \) and \( \theta \neq 0 \), optimal consumption strategy (34) becomes

\[ c^* = \left( \beta^{-1} e^{r(T-t) - \delta T} \right)^{\frac{1}{\theta-1}}. \]  

(46)

Then,

\[ c^* = \left( \beta^{-1} e^{r(T-t)} \right)^{\frac{1}{\theta-1}} \]  

(47)

is the optimal solution for mean–variance–utility problem (8).

(iii) With an exponential utility function \( U(c) = -e^{-\eta c}/\eta \) with \( \eta > 0 \), optimal consumption strategy (34) can be explicitly obtained as

\[ c^* = \frac{1}{\eta} \left[ \ln \beta - r(T-t) + \delta T \right]. \]  

(48)

Then,

\[ c^* = \frac{1}{\eta} \left[ \ln \beta - r(T-t) \right] \]  

(49)

is the optimal solution for mean–variance–utility problem (8).

With different optimal consumption strategies being derived corresponding to different utility functions, it is of interest to investigate the effect of the newly
introduced parameter, $\beta$, on the optimal objective value function $V(t, x)$ defined in (13). Let’s adopt the logarithmic utility function to mean–variance–utility problem (8) (setting $\delta = 0$ in (33) and (39)) as an example for illustration. Substituting (45) into (39) yields

$$\frac{dV}{d\beta} = M(t) + M(t) \log \beta$$

(50)

and

$$\frac{d^2V}{d\beta^2} = \frac{1}{\beta} M(t),$$

(51)

where

$$M(t) = \int_t^T e^{-\rho s}[-r(T - s) + \rho T] ds.$$  

From the expression of the first-order derivative, one can easily observe that the changes of the value function with respect to $\beta$ are dependent on both the parameter values and some other time-dependent functions, which implies that the sensitivity of the value function over $\beta$ will be adjusted over time. In addition, it is not difficult to find that $V_{\beta \beta} > 0$ when $r < \rho$, which suggests that the sensitivity of the value function towards $\beta$ is a monotonic increasing function of $\beta$ in the case where the expected return on the saved money is less than the expected return on the consumption utility. This is also reasonable, as $\beta$ denotes the consumption preference, and when $\beta$ is large, any tiny changes in $\beta$ value would result in a large change in the consumption strategy, leading to a significant impact on the value function.

Apart from the optimal value function, one may also be interested to see how the optimal strategies behave with respect to different parameter values, the details of which are provided in the next section. Before we finish here, it should be remarked that we may not always be able to derive the specific form of the optimal consumption strategy, since it depends on whether we can find the inverse of the derivative of the chosen utility function. However, even when the analytical inversion of the selected utility function is not available, e.g. when some mixed utility functions are adopted, it is still very straightforward to implement (34) in some numerical software like Matlab to compute.

One may wonder what happens if $\beta$ goes to infinity. Mathematically, this limit process will lead to an ill-posed problem, as far as the optimization is concerned. In fact, the infinite $\beta$ value will lead to abnormal (infinite) consumption, which can only be allowed if the income of investors is also abnormal (infinite). If we assume that investors only have limited initial wealth and normal income, then in order to maintain the balance of income and expenditure of investors, $\beta$ should be constrained to a reasonable but not infinite range. Financially, such a limit has actually freed the investor from the ‘hassle’ of trying to optimize his/her portfolio in the sense that he/she could consume without restraints which is actually not
reasonable as the investor would normally keep a balanced budget. If $\beta$ goes to infinity, the optimal consumption will also approach infinity. This is because $\beta$ going to infinity means that the investor does not care about the final wealth and the mean–variance concern for the terminal wealth becomes redundant, in which case the investor will consume as much as possible.

5. Numerical examples

In this section, the properties of the optimal consumption strategies under three common utility functions discussed in Proposition 4.2 are investigated. Numerical examples and detailed discussions are provided by setting $\mu$ and $r$ as 0.05 and 0.01, respectively. All experiments are conducted using Matlab software. The optimal strategy in this paper is applicable to general utility functions, as long as the basic definition of the utility functions is satisfied: the more satisfied a person is with consumption, the better, or in other words, the first derivative of the utility function is greater than zero; with the increased consumption, the speed of satisfaction decreases, and the second derivative of the utility function is less than zero. Once a specific utility function is selected, the optimal investment strategy will be determined. Now, we choose some simple and representative utility functions in economics to illustrate.

First of all, depicted in Figure 1 is the optimal consumption strategy with different $\beta$ values when the investor chooses a logarithmic utility function. One can easily observe that the investor tends to consume more when the $\beta$ value is higher. This is indeed reasonable as an increase in $\beta$ places a higher weight on the accumulated utility when calculating the value function, and this corresponds to the case where the investor prefers to increase consumption to achieve a higher utility than managing the wealth under the mean–variance framework. It is also interesting to find that the investor would like to raise the level of consumption when the end of the pre-determined investment period is approached under our mean–variance–utility framework. This may appear to be strange at a first glance, but this could also be understood from an economic point of view. At the early stage, a rational investor tends to be conservative in terms of consumption given that there may be plenty of uncertainty with maximizing the terminal wealth being part of his/her long-term goal in achieving maximum 'happiness', and thus managing his/her terminal wealth through investment has higher priority over consumption. However, when time passes by and the investor has accumulated a certain amount of wealth, he/she would gain more confidence in consuming more to achieve more 'happiness'. This is indeed consistent with our theoretical findings for the optimal value function.

Figures 2 and 3 display how the optimal consumption strategy varies when the utility function is in the form of a power and an exponential function, respectively. What can be observed first from both figures is a similar pattern as shown in Figure 1 that the optimal consumption strategy is still a monotonic increasing
function of $\beta$, as a higher value of $\beta$ still implies that more consumption is preferred. Another phenomenon that should be noted is that the investor is willing to consume more when $\theta$ ($\eta$) takes smaller values. The main explanation for this is that $\theta$ ($\eta$) indicates the degree of risk aversion, and a larger $\theta$ ($\eta$) value implies avoiding excessive consumptions.

It is also interesting to show the difference between the optimal consumption strategy derived under our framework and that obtained in [30], as there are two different approaches used to incorporate the consumption into the mean–variance problem. In particular, the problem proposed in [30] does not admit an explicit and analytical solution, and it was to be numerically solved.
Figure 3. Optimal consumption under the exponential utility function. The fixed parameter $\eta$ used in the left subfigure and $\beta$ used in the right subfigure are set to be 3 and 10, respectively.

Figure 4. Comparisons of the optimal consumption strategies.

with the fixed-point method, while our optimal consumption strategy is completely closed-form, which facilitates its practical applications. For comparison purposes, we respectively choose the power utility function with $\theta = 0.5$ and $\beta = 1$ and the exponential utility function with $\eta = 1$ and $\beta = 10$ in our model. As displayed in Figure 4, the optimal consumption strategy derived in [30] is not continuous, and there exists a sudden drop from the maximal to minimal allowed consumption rate for any investor using their framework. This is by no means reasonable, since an investor would never consider making substantial changes in his/her consumption in normal situations. In contrast, our optimal consumption strategy turns out to be continuous as shown in Equations (47) and (49), being a monotonic increasing function of the time, which is more reasonable for the same reason stated above. In fact, one can easily verify the monotonicity of the optimal consumption strategy with respect to $t$ according to the analytical formulae (47) and (49).
6. Concluding remarks

In this paper, we introduce the concept of overall ‘happiness’ of an investor, with which the terminal wealth under the mean–variance criterion and accumulated consumption utility can be directly added together using a consumption preference parameter, to formulate a new class of investment–consumption optimization problems as described in (8). As reported in Proposition 4.1, derived optimal consumption strategy (36) for our mean–variance–utility portfolio problem (8) is continuous and increases over time, which is consistent with financial intuition that a normal investor would prefer investment over consumption when it is far away from the end of the period to achieve more happiness, while he/she would gradually increase the level of consumption when the wealth starts to be accumulated.

Notes

1. Although the two wealth dynamics (2) and (5) are different, the optimal solution to the mean–variance problem (4) with (2) and that to the mean–variance problem (4) with (5) are the same. In particular, by setting \( \rho = 0 \) in Proposition 3.1 of [31], one can easily guarantee the optimal strategy, and the results in [31] also show that the deterministic cash flow does not bring any possible adjustment to the investment strategy under the mean–variance criterion.

2. These stochastic integrals are indeed local martingales. However, we actually adopted the idea in [27] to make the assumption that these are martingales to build and study the solution set of all admissible strategies.

3. Note that the utility function \( U(\cdot) \) is a monotonically increasing and concave function, implying that its derivative and the inverse of the derivative are both decreasing functions.

4. As pointed out in [31], this seems to be economically unreasonable for a multi-period model, and a possible way to resolve this issue is to make the risk aversion be time- and wealth-dependent. We refer interested readers to [31] for a more detailed discussion.

5. The codes for numerical experiments presented in this paper are provided in the appendix of arXiv version (arXiv:2005.06782).

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Appendices

Appendix 1. The proof of Lemma 3.2

Proof: Define

\[ \bar{Y}(t, x) = e^{-\delta t} Y(t, x). \] (A1)
Similarly, if we denote
\[
\tilde{Z}(t,x) = e^{-\delta t} Z(t,x),
\]
we can obtain
\[
\begin{aligned}
\tilde{Z}_t(t,x) &= -[(r + \pi(\mu - r))x + l - c] \tilde{Z}_x(t,x) - \frac{1}{2} \pi^2 \sigma^2 x^2 \tilde{Z}_{xx}(t,x), \\
\tilde{Z}(T,x) &= e^{-\delta T} x^2.
\end{aligned}
\]
Again, applying Itô’s lemma leads to

\[
\tilde{Z}(t, X^{c,\pi} (t)) = - \int_t^T d\tilde{Z}(s, X^{c,\pi} (s)) + \tilde{Z}(T, X^{c,\pi} (T))
\]

\[
- \int_t^T \left( \tilde{Z}_s(s, X^{c,\pi} (s)) + \left[ (r + \pi(s)(\mu - r)) X^{c,\pi} (s) + l(s) - c(s) \right] \tilde{Z}_x(s, X^{c,\pi} (s)) 
+ \frac{1}{2} \pi^2(s) \sigma^2(X^{c,\pi} (s))^2 \tilde{Z}_{xx}(s, X^{c,\pi} (s)) \right) ds
\]

\[
- \int_t^T \pi(s) \sigma X^{c,\pi} (s) \tilde{Z}_x(s, X^{c,\pi} (s)) dB(s) + \tilde{Z}(T, X^{c,\pi} (T))
\]

\[
= \left( e^{-\delta T} X^{c,\pi} (T) \right)^2 - \int_t^T \pi(s) \sigma X^{c,\pi} (s) \tilde{Z}_x(s, X^{c,\pi} (s)) dB(s).
\]  

(A8)

Taking the expectation on both sides of (A8) conditional upon \( X(t) = x \), it is straightforward that

\[
\tilde{Z}(t, X^{c,\pi} (t)) = E \left[ \left( e^{-\delta T} X^{c,\pi} (T) \right)^2 \ \bigg| \ X(t) = x \right],
\]

(A9)

and thus,

\[
Z(t, x) = e^{2\delta t} \tilde{Z}(t, x) = z^{c,\pi} (t, x).
\]  

(A10)

Finally, following a similar fashion, \( W(t, X^{c,\pi} (t)) \) satisfying (17) can be founded as

\[
W(t, X^{c,\pi} (t)) = - \int_t^T dW(s, X^{c,\pi} (s)) + W(T, X^{c,\pi} (T))
\]

\[
= - \int_t^T \left( W_s(s, X^{c,\pi} (s)) + \left[ (r + \pi(s)(\mu - r)) X^{c,\pi} (s) + l(s) - c(s) \right] W_x(s, X^{c,\pi} (s)) 
+ \frac{1}{2} \pi^2(s) \sigma^2(X^{c,\pi} (s))^2 W_{xx}(s, X^{c,\pi} (s)) \right) ds
\]

\[
- \int_t^T \pi(s) \sigma X^{c,\pi} (s) W_x(s, X^{c,\pi} (s)) dB(s) + W(T, X^{c,\pi} (T))
\]

\[
= \int_t^T e^{-\delta s} U(c(s)) ds - \int_t^T \pi(s) \sigma X^{c,\pi} (s) \tilde{W}_x(s, X^{c,\pi} (s)) dB(s).
\]  

(A11)

Taking the conditional expectation on (A11) yields the desired result. This has completed the proof.

\[\blacksquare\]

**Appendix 2. The proof of Lemma 3.2**

**Proof:** The proof process is divided into three steps. The first step is to derive an expression for

\[
f^{c,\pi} (t, X^{c,\pi} (t), y^{c,\pi} (t, X^{c,\pi} (t)), z^{c,\pi} (t, X^{c,\pi} (t)), w^{c,\pi} (t, X^{c,\pi} (t))).
\]  

(A12)
Using Itô’s lemma, we have

\[
\begin{align*}
    f^c_{\cdot t}(t, X^c_{\cdot t}(t), y^c_{\cdot t}(t), X^c_{\cdot t}(t)), z^c_{\cdot t}(t, X^c_{\cdot t}(t)), w^c_{\cdot t}(t, X^c_{\cdot t}(t)) \\
    &= -\int_t^T df^c_{\cdot s}(t, X^c_{\cdot s}(t), y^c_{\cdot s}(t, X^c_{\cdot s}(t)), z^c_{\cdot s}(t, X^c_{\cdot s}(t)), w^c_{\cdot s}(t, X^c_{\cdot s}(t)) \\
    &+ f^c_{\cdot T}(T, X^c_{\cdot T}(T), y^c_{\cdot T}(T, X^c_{\cdot T}(T)), z^c_{\cdot T}(T, X^c_{\cdot T}(T)), w^c_{\cdot T}(t, X^c_{\cdot T}(T))) \\
    &= -\int_t^T \left( (f^c_z + f^c_y Z_x + f^c_w W_x) ds \\
    &+ (f^c_z + f^c_y Z_x + f^c_w W_x) d\mu + \frac{1}{2} \pi^2(s) \sigma^2(X_{\cdot s}(s))^2 \left[ f^c_{\cdot y} Y_{xx} + f^c_{\cdot z} Z_{xx} + f^c_{\cdot w} W_{xx} \\
    &+ f^c_{\cdot y}(Y_x)^2 + f^c_{\cdot z}(Z_x)^2 + f^c_{\cdot w}(W_x)^2 \\
    &+ 2 f^c_{\cdot y} Y_x + 2 f^c_{\cdot z} Z_x + 2 f^c_{\cdot w} W_x + 2 f^c_{\cdot y} Y_x Z_x + 2 f^c_{\cdot y} Y_x W_x + 2 f^c_{\cdot w} Z_x W_x \right] \right) ds \\
    &+ f^c_{\cdot T}(T, X^c_{\cdot T}(T), Y^c_{\cdot T}(T, X^c_{\cdot T}(T)), Z^c_{\cdot T}(T, X^c_{\cdot T}(T)), W^c_{\cdot T}(T, X^c_{\cdot T}(T)). \quad (A13)
\end{align*}
\]

Using (15), (16) and (17), we further have

\[
\begin{align*}
    f^c_{\cdot t}(t, X^c_{\cdot t}(t), y^c_{\cdot t}(t, X^c_{\cdot t}(t)), z^c_{\cdot t}(t, X^c_{\cdot t}(t)), w^c_{\cdot t}(t, X^c_{\cdot t}(t)) \\
    &= -\int_t^T \left( f^c_z ds + f^c_y \left( -[(r + \pi(\mu - r)) X_{\cdot s}(s)] + l(s) - c\right) Y_x \\
    &- \frac{1}{2} \pi^2(s) \sigma^2(X_{\cdot s}(s))^2 Y_{xx} + (r + \pi(\mu - r)) X_{\cdot s}(s)] + l(s) - c\right) Z_x - \frac{1}{2} \pi^2(s) \sigma^2(X_{\cdot s}(s))^2 Z_{xx} + 2 \delta Z \\
    &+ f^c_w \left( -[(r + \pi(\mu - r)) X_{\cdot s}(s)] + l(s) - c\right) W_x - \frac{1}{2} \pi^2(s) \sigma^2(X_{\cdot s}(s))^2 W_{xx} - e^{-\rho s} U(c(s)) \\
    &+ (f^c_{\cdot y} + f^c_{\cdot z} Z_x + f^c_{\cdot w} W_x) \left( [(r + \pi(\mu - r)) X_{\cdot s}(s)] + l(s) - c\right) ds \\
    &+ \pi(s) \sigma^c_{\cdot t}(s) dB(s) \\
    &+ \frac{1}{2} \pi^2(s) \sigma^2(X_{\cdot s}(s))^2 \left[ f^c_{\cdot y} Y_{xx} + f^c_{\cdot z} Z_{xx} + f^c_{\cdot w} W_{xx} \right] \left( [(r + \pi(\mu - r)) X_{\cdot s}(s)] + l(s) - c\right) ds \\
    &+ f^c_{\cdot T}(T, X^c_{\cdot T}(T), Y^c_{\cdot T}(T, X^c_{\cdot T}(T)), Z^c_{\cdot T}(T, X^c_{\cdot T}(T)), W^c_{\cdot T}(T, X^c_{\cdot T}(T)). \quad (A14)
\end{align*}
\]

Therefore,

\[
\begin{align*}
    f^c_{\cdot t}(t, X^c_{\cdot t}(t), y^c_{\cdot t}(t, X^c_{\cdot t}(t)), z^c_{\cdot t}(t, X^c_{\cdot t}(t)), w^c_{\cdot t}(t, X^c_{\cdot t}(t)) \\
    &= -\int_t^T \left( f^c_z ds + f^c_y \delta Y + 2 f^c_z \delta Z - f^c_w e^{-\rho s} K(c(s)) ds \\
    &+ f^c_x [ (r + \pi(t)(\mu - r)) X_{\cdot s}(s) + l(s) - c(s)] ds
\end{align*}
\]
\begin{align}
+ \pi(s)\sigma X^{c,\pi}(s)(f_x^{c,\pi} + f_y^{c,\pi} Y_x + f_z^{c,\pi} Z_x + f_w^{c,\pi} W_x) \, dB(s)
+ \frac{1}{2} \pi^2(s)^2 (X^{c,\pi}(s))^2 \left[ f_{xx}^{c,\pi} + f_{yy}^{c,\pi} (Y_x)^2 + f_{zz}^{c,\pi} (Z_x)^2 + f_{ww}^{c,\pi} (W_x)^2 \right]
+ 2 f_{xy}^{c,\pi} Y_x + 2 f_{xz}^{c,\pi} Z_x + 2 f_{yw}^{c,\pi} Y_x W_x + 2 f_{zw}^{c,\pi} Z_x W_x \right] \, ds
+ f^{c,\pi} (T, X^{c,\pi}(T), Y^{c,\pi} (T, X^{c,\pi} (T)), Z^{c,\pi} (T, X^{c,\pi} (T)), W^{c,\pi} (T, X^{c,\pi} (T)). \quad (A15)
\end{align}

For an arbitrary admissible strategy \((c, \pi)\), we further define
\begin{align}
\tilde{K} &= f_{xx}^{c,\pi} + f_{yy}^{c,\pi} (Y_x)^2 + f_{zz}^{c,\pi} (Z_x)^2 + f_{ww}^{c,\pi} (W_x)^2 + 2 f_{xy}^{c,\pi} Y_x + 2 f_{xz}^{c,\pi} Z_x
+ 2 f_{yw}^{c,\pi} Y_x W_x + 2 f_{zw}^{c,\pi} Z_x W_x
+ f^{c,\pi} (T, X^{c,\pi}(T), Y^{c,\pi} (T, X^{c,\pi} (T)), Z^{c,\pi} (T, X^{c,\pi} (T)), W^{c,\pi} (T, X^{c,\pi} (T)). \quad (A16)
\end{align}

and
\begin{align}
\tilde{J} &= f^{c,\pi} + f_y^{c,\pi} \delta Y + f_w^{c,\pi} \delta Z - f_w^{c,\pi} e^{-\rho t} U(c(t)). \quad (A17)
\end{align}

This leads to
\begin{align}
f^{c,\pi} (t, X^{c,\pi}(t), Y^{c,\pi} (t, X^{c,\pi} (t)), Z^{c,\pi} (t, X^{c,\pi} (t)), W^{c,\pi} (t, X^{c,\pi} (t))
&= - \int_t^T \{ \tilde{J}(s) + f_x^{c,\pi} [(r + \pi(t) (\mu - r)) X^{c,\pi}(s) + l(s) - c(s)] \, ds
+ \pi(s)\sigma X^{c,\pi}(s)(f_x^{c,\pi} + f_y^{c,\pi} Y_x + f_z^{c,\pi} Z_x + f_w^{c,\pi} W_x) \, dB(s)
+ \frac{1}{2} \pi^2(s)^2 (X^{c,\pi}(s))^2 \tilde{K}(s) \, ds
+ f^{c,\pi} (T, X^{c,\pi}(T), Y^{c,\pi} (T, X^{c,\pi} (T)), Z^{c,\pi} (T, X^{c,\pi} (T)), W^{c,\pi} (T, X^{c,\pi} (T)). \quad (A18)
\end{align}

With the utilization of Itô’s lemma, one can easily derive
\begin{align}
F(t, X^{c,\pi}(t)) &= - \int_t^T \{ F_x \, ds + F_x \, dX^{c,\pi}(s) + \frac{1}{2} F_{xx}(\pi(s))^2 (X^{c,\pi}(s))^2 \, ds \}
+ F(T, X^{c,\pi}(T)). \quad (A19)
\end{align}

Since \(F\) solves pseudo-HJB equation (19), we can obtain that for any arbitrary strategy \((c, \pi)\), we have
\begin{align}
F_t \leq -(r + \pi(s)(\mu - r)) X + k - c(F_x - Q) - \frac{1}{2} \pi^2(s) \sigma^2 (F_{xx} - K) + J.
\end{align}

Setting \(x = X^{c,\pi}(s)\) in (5) and using terminal conditions (15), (16) and (17) directly lead to
\begin{align}
F(t, X^{c,\pi}(t)) \geq - \int_t^T \{ [(r + \pi(s)(\mu - r)) X^{c,\pi}(s) + l(s) - c(s)] (F_x - Q)
- \frac{1}{2} \pi^2(s)^2 (X^{c,\pi}(s))^2 (K(s) + F_{xx}) + J(s) \} \, ds
+ F_x \{ [(r + \pi(s)(\mu - r)) X^{c,\pi}(s) + l(s) - c(s)] \, ds + \pi(s) \sigma X^{c,\pi}(s) \, dB(s)\}
\end{align}
are satisfied. If we assume that the strategy

The last step is to check whether the Nash equilibrium criteria specified in Definition 3.1

Taking the expectation on both sides of the above equality conditional upon

where the second equality follows from (A18), after taking the expectation of the first equality conditional upon $X(t) = x$. We can thus arrive at

$$\begin{align*}
f^{c, \pi}(t, x, y^{c, \pi}(t, x), z^{c, \pi}(t, x), w^{c, \pi}(t, x)) \\
\leq F(t, x) + \int_t^T \left\{ [(r + \pi(t)(\mu - r))X^{c, \pi}(s) + l(s) - c(s)] \right. \\
\times (f^{e, \pi*}(s) - f^{c, \pi}(s)) + J(s) - \tilde{J}(s) \\
\left. + \frac{1}{2} \pi^2(s) \sigma^2(X^{c, \pi}(s))^2 (K(s) - \tilde{K}(s)) \right\} ds. & \tag{A21}
\end{align*}$$

The last step is to check whether the Nash equilibrium criteria specified in Definition 3.1 are satisfied. If we assume that the strategy $(c^*, \pi^*)$ satisfies the infimum in (19), it follows from (18) that

$$\begin{align*}
F^{(1)}(t, x) &= y^{c, \pi*}(t, x), \quad F^{(2)}(t, x) = z^{c, \pi*}(t, x), \quad F^{(3)}(t, x) = w^{c, \pi*}(t, x). & \tag{A22}
\end{align*}$$

As (A20) holds for any admissible strategy $(c, \pi)$, it also applies for the specific strategy $(c^*, \pi^*)$, i.e.

$$F_t = -[(r + \pi(\mu - r))x + l - c](F_x - Q) - \frac{1}{2} \pi^2 \sigma^2 x^2 (F_{xx} - K) + J,$$

leading to

$$\begin{align*}
F(t, X^{c, \pi*}(t)) \\
= \int_t^T \left\{ (f^{e, \pi*}(s) + f^{c, \pi*}(s) + f^{e, \pi*}(s) + f^{c, \pi*}(s)) \right. \\
\left. \frac{\pi^2(s) \sigma^2(X^{c, \pi}(s))^2 (K(s) - \tilde{K}(s)) \right\} ds. & \tag{A23}
\end{align*}$$

Taking the expectation on both sides of the above equality conditional upon $X(t) = x$ yields

$$\begin{align*}
F(t, x) = \frac{1}{2} F_x (\pi(s))^2 \sigma^2(X^{c, \pi}(s))^2 ds \\
+ f^{c, \pi*}(T, X^{c, \pi}(T), Y^{c, \pi}(T, X^{c, \pi}(T)), Z^{c, \pi}(T, X^{c, \pi}(T)), W^{c, \pi}(T, X^{c, \pi}(T))) \\
= -\int_t^T \left\{ [(r + \pi(s)(\mu - r))X^{c, \pi}(s) + l(s) - c(s)] \right. \\
\times (f^{e, \pi*}(s) - f^{c, \pi}(s)) + J(s) - \tilde{J}(s) \\
\left. + \frac{1}{2} \pi^2(s) \sigma^2(X^{c, \pi}(s))^2 (K(s) - \tilde{K}(s)) \right\} ds.
\end{align*}$$

(A24)
If we consider the strategy \((\tilde{c}_h, \tilde{\pi}_h)\) defined in (10), Equations (A21) and (A23) yield

\[
\begin{align*}
&\lim_{h \to 0} \inf_\frac{1}{h} \left\{ \int_t^{t+h} \left[ (r + \pi(s)(\mu - r))X^{c^\pi}(s) + l(s) - c(s) \right] \left( f_{X}^{c^\pi}(s) - f_{X}^{c^\ast, \pi^\ast}(s) \right) ds \\
&\quad + \int_t^{t+h} \left( J_h(s) - J(s) + \frac{1}{2} \sigma^2(\pi(s))^2(X^{c^\pi}(s))^2(\tilde{K}_h(s) - K(s)) ds \right) \right\} \\
&= \lim_{h \to 0} \inf_\frac{1}{h} \left\{ \int_t^{t+h} \left[ (r + \pi(t)(\mu - r))X^{c^\pi}(t) + l(t) - c(t) \right] \left( f_{X}^{c^\ast}(t) - f_{X}^{c^\ast, \pi^\ast}(t) + J_{0}(t) - J(t) \\
&\quad + \frac{1}{2} \sigma^2(\pi(t))^2(X^{c^\ast, \pi^\ast}(t))^2(\tilde{K}_0(t) - K(t)) \right) \right\} \\
&= 0,
\end{align*}
\]

which implies that \(F(t, x) = V(t, x)\) and \((c^\ast, \pi^\ast)\) is the desired optimal strategy. \(\blacksquare\)