TOPOLOGICAL HIGHER-RANK GRAPHS AND THE C*-ALGEBRAS OF TOPOLOGICAL 1-GRAPHS

TRENT YEEND

Abstract. We introduce the notion of a topological higher-rank graph, a unified generalization of the higher-rank graph and the topological graph. Using groupoid techniques, we define the Toeplitz and Cuntz-Krieger algebras of topological higher-rank graphs, and show that the C*-algebras defined are coherent with the existing theory.

1. Introduction

We are interested in the generalization of directed-graph C*-algebras, of which, loosely speaking, there are two main themes: either the graphs to which we associate C*-algebras are of higher-rank (for example, \[16, 25, 24, 17, 26, 31, 32, 8, 7\]), or they are given topological structure (for example, \[2, 12, 3, 28, 4, 5, 13, 20, 14\]). Our objective in this article is to present a unified approach to these two strands through the employment of groupoid theory.

This article explores a common approach to the C*-algebras of higher-rank graphs and topological graphs. We begin by introducing topological higher-rank graphs – a unified generalization of higher-rank graphs and topological graphs.

Given a topological higher-rank graph \(\Lambda\), we define a path space \(X_\Lambda\) and a topology on \(X_\Lambda\). There is a natural action of \(\Lambda\) on \(X_\Lambda\), and from it we define a groupoid \(G_\Lambda\) which has \(X_\Lambda\) as its unit space; we call \(G_\Lambda\) the path groupoid of \(\Lambda\).

The topology on \(X_\Lambda\) pulls back to give a topology on \(G_\Lambda\). However, in general the topology may not be locally compact, and the groupoid’s range and source maps may fail to be continuous. Hence we restrict our attention to the class of compactly aligned topological higher-rank graphs – these are the topological analogues of the finitely aligned higher-rank graphs studied in \[20, 24, 8\].

For a compactly aligned topological higher-rank graph \(\Lambda\), \(G_\Lambda\) is a locally compact topological groupoid which is \(r\)-discrete in the sense that the unit space \(G_\Lambda^{(0)}\) is open in \(G_\Lambda\). Furthermore, the range and source maps of \(G_\Lambda\) are local homeomorphisms, so there is a Haar systems of counting measures on \(G_\Lambda\). This allows us to define the full groupoid C*-algebra \(C^*(G_\Lambda)\), which we refer to as the Toeplitz algebra of \(\Lambda\).

Identifying a closed invariant subset \(\partial \Lambda\) of the unit space \(G_\Lambda^{(0)}\), we define the boundary-path groupoid of \(\Lambda\) to be \(G_\Lambda := G_\Lambda|_{\partial \Lambda}\) – a locally compact \(r\)-discrete topological groupoid admitting a system of counting measures – and define the Cuntz-Krieger algebra of \(\Lambda\) to be \(C^*(G_\Lambda)\).

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We finish by showing that when we restrict our attention to topological graphs, we recover the same $C^*$-algebras as those obtained in the existing theory.

Our work builds upon the theories of directed-graph $C^*$-algebras, higher-rank graph $C^*$-algebras and topological-graph $C^*$-algebras, so we mention some of the examples which these classes contain.

The class of directed-graph $C^*$-algebras contains all nonunital, simple, purely infinite, nuclear (SPIN) $C^*$-algebras whose $K_1$ group is torsion-free [33], as well as examples of $C^*$-algebras of quantum spaces [10, 11]. Up to Morita equivalence, graph $C^*$-algebras provide all AF-algebras [6, 34] and unital SPIN $C^*$-algebras with torsion-free $K_1$ [33].

The class of higher-rank graph $C^*$-algebras provides more examples of SPIN $C^*$-algebras, possibly with non-torsion-free $K_1$ [29, 7]. Up to Morita equivalence, the class provides examples of simple $\mathcal{A}T$-algebras with real-rank 0 such as irrational rotation algebras and Bunce Deddens algebras [21]; in particular, these $C^*$-algebras are simple but neither AF nor purely infinite.

Topological graphs generalize directed graphs as well as partially defined local homeomorphisms on locally compact Hausdorff spaces. Katsura shows in [14, 15] that the class of topological-graph $C^*$-algebras contains all AF-algebras and many AH-algebras, purely infinite $C^*$-algebras and stably projectionless $C^*$-algebras. Katsura also shows that from topological graphs one can obtain Exel-Laca algebras (see also [30]), ultragraph algebras, Matsumoto algebras, homeomorphism $C^*$-algebras such as crossed products by partial homeomorphisms, $C^*$-algebras associated with branched coverings, and $C^*$-algebras associated with singly generated dynamical systems.

**Higher-rank graphs.** Higher-rank graphs, or $k$-graphs ($k$ being an element of the natural numbers and the given graph’s rank), were first introduced in [16] as a unified way of approaching the higher-rank Cuntz-Krieger algebras studied by Robertson and Steger [29] and the Cuntz-Krieger algebras of directed graphs. The Toeplitz and Cuntz-Krieger algebras of higher-rank graphs are collectively known as higher-rank graph $C^*$-algebras.

Directed-graph $C^*$-algebras are recovered as the $C^*$-algebras of 1-graphs. However, to pass from a directed graph $E$ to the corresponding 1-graph, we exchange the quadruple $E = (E^0, E^1, r, s)$ for the pair $(E^*, l)$, where $E^*$ is the free category generated by $E$, or the finite-path category of $E$, and $l : E^* \to \mathbb{N}$ is the functor which describes the length of each path.

In general, a $k$-graph $(\Lambda, d)$ comprises a countable category $\Lambda$, where morphisms are referred to as paths and objects as vertices, together with a degree functor $d : \Lambda \to \mathbb{N}^k$ describing the degree, or ‘shape’, of each path. The $k$-graph $(\Lambda, d)$ is subject to one more condition: that of unique factorizations of paths. Just as we may uniquely factorize a path in a directed graph into its constituent edges, given a path $\lambda \in \Lambda$ and any $m, n \in \mathbb{N}^k$ such that $d(\lambda) = m + n$, there are unique paths $\xi, \eta \in \Lambda$ such that $d(\xi) = m$, $d(\eta) = n$ and $\lambda = \xi \eta$.

Given a $k$-graph $(\Lambda, d)$, one may represent it in a $C^*$-algebra by a family of partial isometries $\{S_\lambda : \lambda \in \Lambda\}$ satisfying conditions which encode the structure of the graph and which give rise to a structure theory analogous to that of the original namesake algebras (see [26] Definition 2.5). Hence one may define and study the Toeplitz and Cuntz-Krieger algebras of $(\Lambda, d)$. 
There are a variety of approaches to the theory of higher-rank graph \( C^* \)-algebras. In [16], Kumjian and Pask associate a groupoid to each \( k \)-graph, and define the Cuntz-Krieger algebra of the graph to be the corresponding groupoid \( C^* \)-algebra; this groupoid model is an extension of the groupoid approach to the Cuntz-Krieger algebra of a directed graph used in [13] [18]. Due to groupoid considerations, the \( k \)-graphs studied in [16] are row-finite and have no sources.

In [25], the authors generalize the work of Kumjian and Pask to locally convex row-finite \( k \)-graphs (possibly with sources) by using a direct \( C^* \)-algebraic analysis on the Cuntz-Krieger representations of the \( k \)-graphs. The same authors further generalize the theory to the setting of finitely aligned higher-rank graphs in [26], again using a direct analysis.

In [8] (see also [23]), the theory is brought full-circle with an inverse semigroup and groupoid approach to the Toeplitz and Cuntz-Krieger algebras of finitely aligned higher-rank graphs; the approach recovers the Kumjian-Pask groupoid in the ‘row-finite and no sources’ setting, and is in many respects an extension of the inverse semigroup and groupoid approach to directed-graph \( C^* \)-algebras taken by Paterson in [22].

Product systems of Hilbert bimodules may also be employed to approach the Toeplitz algebras of finitely aligned higher-rank graphs [24], although at present there is a problem in the extending of methods to the theory of Cuntz-Krieger algebras; this intriguing problem is tied up with the need for an appropriate notion of Cuntz-Pimsner covariance of product-system representations for which there is a tractable structure theory.

**Topological graphs.** A topological graph \( E \) is a quadruple \((E^0, E^1, r, s)\), where \( E^0 \) and \( E^1 \) are locally compact Hausdorff spaces, \( r, s : E^1 \to E^0 \) are continuous, and \( s \) is a local homeomorphism. Postponing details until Section 5 we give an outline of a process by which one may associate Toeplitz and Cuntz-Krieger algebras to each topological graph.

Given a topological graph \( E \), Katsura [13] constructs a right-Hilbert \( C_0(E^0) - C_0(E^1) \) bimodule \( C_d(E^1) \), where \( C_d(E^1) \) is a completion of \( C_c(E^1) \). The Toeplitz algebra of \( E \) is then defined to be the universal \( C^* \)-algebra for Toeplitz, or isometric, representations of \( C_d(E^1) \), and the Cuntz-Krieger algebra of \( E \) is defined to be the \( C^* \)-algebra universal for Cuntz-Pimsner covariant, or fully co-isometric, representations of \( C_d(E^1) \). The Toeplitz and Cuntz-Krieger algebras of topological graphs are known collectively as topological-graph \( C^* \)-algebras.

Regarding directed graphs as topological graphs with discrete second countable topologies, one recovers the same Hilbert bimodules and \( C^* \)-algebras as those of the existing theory (see [9] Example 1.2] and [13] Example 1 of Section 2).

The direct \( C^* \)-algebraic analysis used with much success in the discrete setting does not work for topological graphs: the direct analysis involves manipulating generating families of partial isometries and projections, whereas for a general topological graph there may be no such partial isometries or projections present in the representations.

Topological-graph \( C^* \)-algebras may also be approached using groupoid methods [1, 28], [14] Section 10.3]. The groupoid approach was first achieved by Deaconu [2] for topological graphs with compact vertex and edge sets, with \( r \) a homeomorphism, and with \( s \) a surjective local homeomorphisms.
2. Topological higher-rank graphs

Given \( k \in \mathbb{N} \), a topological \( k \)-graph is a pair \((\Lambda, d)\) consisting of a small category \( \Lambda = (\text{Obj}(\Lambda), \text{Mor}(\Lambda), r, s, \circ) \) and a functor \( d : \Lambda \to \mathbb{N}^k \), called the degree map, which satisfy:

1. \( \text{Obj}(\Lambda) \) and \( \text{Mor}(\Lambda) \) are second-countable locally compact Hausdorff spaces;
2. \( r, s : \text{Mor}(\Lambda) \to \text{Obj}(\Lambda) \) are continuous and \( s \) is a local homeomorphism;
3. Composition \( \circ : \Lambda \times_c \Lambda \to \Lambda \) is continuous and open;
4. \( d \) is continuous, where \( \mathbb{N}^k \) has the discrete topology;
5. For all \( \lambda \in \Lambda \) and \( m, n \in \mathbb{N}^k \) such that \( d(\lambda) = m + n \), there exists unique \((\xi, \eta) \in \Lambda \times_c \Lambda \) such that \( \lambda = \xi \eta \), \( d(\xi) = m \) and \( d(\eta) = n \).

We refer to the morphisms of \( \Lambda \) as \emph{paths} and to the objects of \( \Lambda \) as \emph{vertices}. The codomain and domain maps in \( \Lambda \) are called the range and source maps, respectively.

We use the partial ordering of \( \mathbb{N}^k \),
\[
m \leq n \iff m_i \leq n_i \text{ for } i = 1, \ldots, k,
\]
and use the notation \( \lor \) and \( \land \) for the coordinate-wise maximum and minimum.

For \( m \in \mathbb{N}^k \), define \( \Lambda^m \) to be the set \( d^{-1}(\{m\}) \) of paths of degree \( m \). Define \( \Lambda \ast_s \Lambda := \{ (\lambda, \mu) \in \Lambda \times \Lambda : s(\lambda) = s(\mu) \} \), and for \( U, V \subset \Lambda \) define \( U \ast_s V := (U \times V) \cap (\Lambda \ast_s \Lambda) \) and \( UV := \{ \lambda \mu : (\lambda, \mu) \in U \times_s V \} \); in particular, for \( v \in \Lambda^0 \), \( vU := \{ v \} U = \{ \lambda \in U : r(\lambda) = v \} \) and similarly \( Uv := \{ \lambda \in U : s(\lambda) = v \} \). For \( p, q \in \mathbb{N}^k, U \subset \Lambda^p \) and \( V \subset \Lambda^q \), we write
\[
U \lor V := U \Lambda^{(p \lor q) - p} \cap V \Lambda^{(p \lor q) - q}
\]
for the set of \emph{minimal common extensions} of paths from \( U \) and \( V \). For \( \lambda, \mu \in \Lambda \), we write
\[
\Lambda^{\min}(\lambda, \mu) := \{ (\alpha, \beta) : \lambda \alpha = \mu \beta, d(\lambda \alpha) = d(\lambda) \lor d(\mu) \}
\]
for the set of pairs which give minimal common extensions of \( \lambda \) and \( \mu \); that is,
\[
\Lambda^{\min}(\lambda, \mu) = \{ (\alpha, \beta) : \lambda \alpha = \mu \beta \in \{ \lambda \} \lor \{ \mu \} \}.
\]

3. The path groupoid

To describe our path groupoid \( G_\Lambda \), we need some terminology. Let \((\Lambda_1, d_1)\) and \((\Lambda_2, d_2)\) be topological \( k \)-graphs. A graph morphism between \( \Lambda_1 \) and \( \Lambda_2 \) is a continuous functor \( x : \Lambda_1 \to \Lambda_2 \) satisfying \( d_2(x(\lambda)) = d_1(\lambda) \) for all \( \lambda \in \Lambda_1 \).

For \( k \in \mathbb{N} \) and \( m \in (\mathbb{N} \cup \{\infty\})^k \), define the topological \( k \)-graph \((\Omega_{k,m}, d)\) by giving the discrete topologies to the sets
\[
\text{Obj}(\Omega_{k,m}) := \{ p \in \mathbb{N}^k : p \leq m \}
\]
and
\[
\text{Mor}(\Omega_{k,m}) := \{ (p, q) \in \mathbb{N}^k \times \mathbb{N}^k : p \leq q \leq m \},
\]
and setting \( r(p, q) := p, s(p, q) := q, (n, p) \circ (p, q) := (n, q) \) and \( d(p, q) := q - p \).

Let \((\Lambda, d)\) be a topological \( k \)-graph. We define the path space of \( \Lambda \) to be
\[
X_\Lambda := \bigcup_{m \in (\mathbb{N} \cup \{\infty\})^k} \{ x : \Omega_{k,m} \to \Lambda : x \text{ is a graph morphism} \}.
\]
We extend the range and degree maps to \( x : \Omega_{k,m} \to \Lambda \) in \( X_\Lambda \) by setting \( r(x) := x(0) \) and \( d(x) := m \). For \( v \in \Lambda^0 \) we define \( vX_\Lambda := \{ x \in X_\Lambda : r(x) = v \} \).
For \( x \in X_\Lambda, \ m \in \mathbb{N}^k \) with \( m \leq d(x), \) and \( \lambda \in \Lambda \) with \( s(\lambda) = r(x), \) there exist unique graph morphisms \( \lambda x \) and \( \sigma^m x \) in \( X_\Lambda \) satisfying \( d(\lambda x) = d(\lambda) + d(x), \)
\( d(\sigma^m x) = d(x) - m, \)
\[
(\lambda x)(0, p) = \begin{cases} 
\lambda(0, p) & \text{if } p \leq d(\lambda) \\
\lambda x(0, p - d(\lambda)) & \text{if } d(\lambda) \leq p \leq d(\lambda x), 
\end{cases}
\]
and
\[
(\sigma^m x)(0, p) = x(m, m + p) \quad \text{for } p \leq d(\sigma^m x).
\]

For each \( \lambda \in \Lambda \) there is a unique graph morphism \( x_\lambda : \Omega_{k, d(\lambda)} \to \Lambda \) such that \( x_\lambda(0, d(\lambda)) = \lambda; \) in this sense, we may view \( \Lambda \) as a subset of \( X_\Lambda \) and we refer to elements of \( X_\Lambda \) as paths. Indeed, for \( \lambda \in \Lambda \) and \( p, q \in \mathbb{N}^k \) with \( 0 \leq p \leq q \leq d(\lambda) \) we write \( \lambda(0, p), \) \( \lambda(p, q) \) and \( \lambda(q, d(\lambda)) \) for the unique elements of \( \Lambda \) which satisfy \( \lambda = \lambda(0, p)\lambda(p, q)\lambda(q, d(\lambda)), \) \( d(\lambda(0, p)) = p, \) \( d(\lambda(p, q)) = q - p \) and \( d(\lambda(q, d(\lambda))) = d(\lambda) - q.\)

**Definition 3.1.** Let \((\Lambda, d)\) be a topological \( k \)-graph. The path groupoid \( G_\Lambda \) has object set \( \text{Obj}(G_\Lambda) := X_\Lambda, \) morphism set
\[
\text{Mor}(G_\Lambda) := \{ (\lambda x, d(\lambda) - d(\mu), \mu x) \in X_\Lambda \times \mathbb{Z}^k \times X_\Lambda : \\
(\lambda, \mu) \in \Lambda \ast_s \Lambda, \ x \in X_\Lambda \text{ and } s(\lambda) = r(x) \}
\]
\[
= \{ (x, m, y) \in X_\Lambda \times \mathbb{Z}^k \times X_\Lambda : \text{ there exist } p, q \in \mathbb{N}^k \text{ such that } \\
p \leq d(x), \ q \leq d(y), \ p - q = m \text{ and } \sigma^p x = \sigma^q y \},
\]
range and source maps \( r(x, m, y) := x \) and \( s(x, m, y) := y, \) composition
\[
((x, m, y), (y, n, z)) \mapsto (x, m + n, z),
\]
and inversion \( (x, m, y) \mapsto (y, -m, x). \)

To define a topology on \( G_\Lambda, \) we use the following notation. For \( F \subset \Lambda \ast_s \Lambda \) and \( m \in \mathbb{Z}^k, \) define \( Z(F, m) \subset G_\Lambda \) by
\[
Z(F, m) := \{ (\lambda x, d(\lambda) - d(\mu), \mu x) \in G_\Lambda : (\lambda, \mu) \in F, d(\lambda) - d(\mu) = m \}.
\]
For \( U \subset \Lambda, \) define \( Z(U) \subset G_\Lambda^{(0)} \) by
\[
Z(U) := Z(U \ast_s U, 0) \cap Z(\Lambda^0 \ast_s \Lambda^0, 0).
\]

**Proposition 3.2 (\cite{[35] Proposition 3.8}).** Let \((\Lambda, d)\) be a topological \( k \)-graph. The family of sets of the form
\[
Z(U \ast_s V, m) \cap Z(F, m)^c,
\]
where \( m \in \mathbb{Z}^k, U, V \subset \Lambda \) are open and \( F \subset \Lambda \ast_s \Lambda \) is compact, is a basis for a second-countable Hausdorff topology on \( G_\Lambda. \)

Sounds perfect! Not quite, there’s a problem: If \( \Lambda \) is not compactly aligned in the sense that, for compact \( U \subset \Lambda^p \) and \( V \subset \Lambda^q, \) the set \( U \cup V \) is compact, then \( G_\Lambda \) is neither locally compact nor a topological groupoid – the range and source maps in \( G_\Lambda \) fail to be continuous. The property of being compactly aligned is to topological higher-rank graphs what finitely aligned is to discrete higher-rank graphs (see \cite{[23, 29, 3]}, and regardless of the approach to the \( C^* \)-algebras of higher-rank graphs, the property has presented itself as a necessary assumption. Also, it is straightforward to see that every topological 1-graph is compactly aligned.
Theorem 3.3 (FMY Theorem 3.16). Let \((\Lambda, d)\) be a compactly aligned topological \(k\)-graph. Then \(G_\Lambda\) is a locally compact \(r\)-discrete topological groupoid admitting a Haar system consisting of counting measures.

We define the Toeplitz algebra of \(\Lambda\) to be the full groupoid \(C^*\)-algebra \(C^*(G_\Lambda)\).

For a discrete directed graph \(E\), Paterson [22] defines an inverse semigroup \(S_E\) and an action of \(S_E\) on \(X_E\). The topological groupoid \(H_E\) is then defined as the groupoid of germs of the action. Comparison of the topological and groupoid structures reveals that \(H_E\) and \(G_{E^*}\) are isomorphic as topological groupoids.

Given a discrete finitely aligned higher-rank graph \((\Lambda, d)\), the authors of [8] define an \(r\)-discrete groupoid \(G^{[\text{FMY}]}_\Lambda\) and show that its \(C^*\)-algebra is isomorphic to the Toeplitz algebra of \(\Lambda\) [8, Theorem 5.9]. The two groupoids \(G^{[\text{FMY}]}_\Lambda\) and \(G_\Lambda\) are isomorphic, so we see that our Toeplitz algebra \(C^*(G_\Lambda)\) is coherent with the existing theory for higher-rank graphs.

4. The boundary-path groupoid

Let \((\Lambda, d)\) be a topological \(k\)-graph and let \(V \subset \Lambda^0\). A set \(E \subset VA\) is exhaustive for \(V\) if for all \(\lambda \in VA\) there exists \(\mu \in E\) such that \(\Lambda^{\min}(\lambda, \mu) \neq \emptyset\). For \(v \in \Lambda^0\), let \(v\mathcal{C}E(\Lambda)\) denote the set of all compact sets \(E \subset \Lambda\) such that \(r(E)\) is a neighbourhood of \(v\) and \(E\) is exhaustive for \(r(E)\).

A path \(x \in X_\Lambda\) is called a boundary path if for all \(m \in \mathbb{N}_k\) with \(m \leq d(x)\), and for all \(E \in x(m)\mathcal{C}E(\Lambda)\), there exists \(\lambda \in E\) such that \(x(m, m + d(\lambda)) = \lambda\). We write \(\partial \Lambda\) for the set of all boundary paths in \(X_\Lambda\). For \(v \in \Lambda^0\) and \(V \subset \Lambda^0\), we define \(v\partial \Lambda = \{x \in \partial \Lambda : r(x) = v\}\) and \(V(\partial \Lambda) = \{x \in \partial \Lambda : r(x) \in V\}\).

The set of boundary paths of \(\Lambda\) is a nonempty closed invariant subset of \(G^{(0)}_\Lambda\), so we can make the following definition.

Definition 4.1. Let \((\Lambda, d)\) be a compactly aligned topological \(k\)-graph. The boundary-path groupoid \(\mathcal{G}_\Lambda\) is the reduction \(\mathcal{G}_\Lambda := G_\Lambda|_{\partial \Lambda}\), which is a locally compact \(r\)-discrete topological groupoid admitting a Haar system consisting of counting measures. The Cuntz-Krieger algebra of \(\Lambda\) is the full groupoid \(C^*\)-algebra \(C^*(\mathcal{G}_\Lambda)\).

For a discrete finitely aligned higher-rank graph \((\Lambda, d)\), straightforward comparisons with the work in [8] show that the Cuntz-Krieger algebra \(C^*(\Lambda)\) is isomorphic to \(C^*(\mathcal{G}_\Lambda)\).

5. \(C^*\)-algebras of topological 1-graphs

Let \(E\) be a second-countable topological 1-graph as defined in [13, Definition 2.1]; that is, \(E = (E^0, E^1, r, s)\) is a directed graph with \(E^0, E^1\) second-countable locally compact Hausdorff spaces, \(r, s : E^1 \to E^0\) continuous, and \(s\) a local homeomorphism. The free category generated by \(E\), endowed with the relative topology inherited from the union of the product topologies, together with the length functor \(l(e_1 \cdots e_n) := n\), forms a topological 1-graph \((E^*, l)\). Conversely, given a topological 1-graph \((\Lambda, d)\), the quadruple \(E_\Lambda := (\Lambda^0, \Lambda^1, r_{|\Lambda^1}, s_{|\Lambda^1})\) is a second-countable topological graph with \(((E_\Lambda)^*, l) \cong (\Lambda, d)\).

In this section, we will see that the \(C^*\)-algebras, \(C^*(G_\Lambda)\) and \(C^*(\mathcal{G}_\Lambda)\), are isomorphic to the Toeplitz and Cuntz-Krieger algebras, \(T(E_\Lambda)\) and \(O(E_\Lambda)\), of the associated topological graph, as defined in [13].
We note that Muhly and Tomforde \cite{MT} elegantly generalize the theory set forth by Katsura, removing the hypothesis that the source map of the topological graph is a local homeomorphism; in its stead, the authors impose the weaker condition that the source map is open and that there is a family of Radon measures \( \{ \lambda_v \} \) for all \( v \in E^0 \) on \( E^1 \) satisfying:

1. \( \text{supp}(\lambda_v) = s^{-1}(v) \) for all \( v \in E^0 \)
2. \( v \mapsto \int_{E^1} \xi(e) d\lambda_v(e) \) is in \( C_c(E^0) \) for all \( v \in E^0 \).

**Theorem 5.1.** For a topological 1-graph, we have \( \mathcal{T}(E_\Lambda) \cong C^*(G_\Lambda) \).

**Theorem 5.2.** For a topological 1-graph, we have \( \mathcal{O}(E_\Lambda) \cong C^*(G_\Lambda) \).

Henceforth \((\Lambda, d)\) will be a topological 1-graph with associated topological graph \( E_\Lambda = (\Lambda^0, \Lambda^1, r, s) \). We begin by defining the Hilbert bimodule \( C_0(\Lambda^0) C_d(\Lambda^1) C_0(\Lambda^0) \).

For \( \xi \in C(\Lambda^1) \), define \( \langle \xi, \xi \rangle : \Lambda^0 \to [0, \infty) \) by

\[
(\xi, \xi)(v) := \sum_{e \in \Lambda^1 v} |\xi(e)|^2,
\]

and define

\[
C_d(\Lambda^1) := \{ \xi \in C(\Lambda^1) : \langle \xi, \xi \rangle \in C_0(\Lambda^0) \}.
\]

For \( \xi, \eta \in C_d(\Lambda^1) \), define \( \langle \xi, \eta \rangle \in C_0(\Lambda^0) \) by

\[
(\xi, \eta)(v) := \sum_{e \in \Lambda^1 v} \xi(e) \eta(e),
\]

and define left and right actions of \( C_0(\Lambda^0) \) on \( C_d(\Lambda^1) \) by

\[
(f \cdot \xi)(e) = (\phi(f)\xi)(e) := f(r(e))\xi(e) \quad \text{and} \quad (\xi \cdot f)(e) := \xi(e)f(s(e)).
\]

Then \( C_d(\Lambda^1) \) is a right-Hilbert \( C_0(\Lambda^0) \)-module and \( \phi \) is a homomorphism of \( C_0(\Lambda^0) \) into \( \mathcal{L}(C_d(\Lambda^1)) \). Hence \( C_d(\Lambda^1) \) is a right-Hilbert \( C_0(\Lambda^0) \)-bimodule [\cite{R}, Proposition 1.10], and \( C_e(\Lambda^1) \) is dense in \( C_d(\Lambda^1) \) [\cite{R}, Lemma 1.6].

A Toeplitz \( E_\Lambda \)-pair on a \( C^\ast \)-algebra \( B \) is a pair of maps \( \Psi = (\Psi_0, \Psi_1) \) such that \( \Psi_0 : C_0(\Lambda^0) \to B \) is a homomorphism and \( \Psi_1 : C_d(\Lambda^1) \to B \) is a linear map satisfying

1. \( \Psi_1(\xi)^\ast \Psi_1(\eta) = \Psi_0(\langle \xi, \eta \rangle) \) for all \( \xi, \eta \in C_d(\Lambda^1) \), and
2. \( \Psi_0(f)\Psi_1(\xi) = \Psi_1(\phi(f)\xi) \) for all \( f \in C_0(\Lambda^0) \) and \( \xi \in C_d(\Lambda^1) \).

The Toeplitz algebra of \( E_\Lambda \), denoted \( \mathcal{T}(E_\Lambda) \), is the \( C^\ast \)-algebra universal for Toeplitz \( E_\Lambda \)-pairs.

It is straightforward to see that \( \Psi_1 \) is continuous and that \( \Psi_1(\xi)\Psi_0(f) = \Psi_1(\xi \cdot f) \) for all \( \xi \in C_d(\Lambda^1) \) and \( f \in C_0(\Lambda^0) \).

Next we define three open subsets of \( \Lambda^0 \):

\[
\Lambda^0_{\text{sce}} := \Lambda^0 \setminus \overline{r(\Lambda^1)},
\]

\[
\Lambda^0_{\text{fin}} := \{ v \in \Lambda^0 : v \text{ has a neighbourhood } V \text{ such that } VA^1 \text{ is compact} \}
\]

and

\[
\Lambda^0_{\text{rg}} := \Lambda^0_{\text{fin}} \setminus \overline{\Lambda^0_{\text{sce}}}.
\]

Given a Toeplitz \( E_\Lambda \)-pair \( \Psi \) on \( B \), there is a homomorphism \( \Psi^{(1)} \) of \( \mathcal{K}(C_d(\Lambda^1)) \) into \( B \) satisfying

\[
\Psi^{(1)}(\xi \otimes \eta^*) = \Psi_1(\xi)^\ast \Psi_0(\eta),
\]

where \( \xi \otimes \eta^* \in \mathcal{K}(C_d(\Lambda^1)) \) is the ‘rank-one’ operator defined by \( (\xi \otimes \eta^*)(\zeta) = \xi \cdot \langle \eta, \zeta \rangle \) for all \( \zeta \in C_d(\Lambda^1) \).
By [13], Proposition 1.24, the restriction of \( \phi \) to \( C_0(\Lambda_0^0) \) is an injection into \( K(C_0(\Lambda_1^1)) \), so we can make the following definition.

A Cuntz-Krieger \( E_\Lambda \)-pair \( \Psi = (\Psi_0, \Psi_1) \) is a Toeplitz \( E_\Lambda \)-pair which satisfies:

\[
\Psi_0(f) = \Psi_1(\phi(f)) \quad \text{for all } f \in C_0(\Lambda_0^0).
\]

The Cuntz-Krieger algebra of \( E_\Lambda \), denoted \( \mathcal{O}(E_\Lambda) \), is the \( C^* \)-algebra universal for Cuntz-Krieger \( E_\Lambda \)-pairs.

We now construct a Toeplitz \( E_\Lambda \)-pair \( \Psi \) on \( C^*(G_\Lambda) \). For \( f \in C_0(\Lambda^0) \), define \( \psi_0(f) : G_\Lambda^{(0)} \to \mathbb{C} \) by

\[
\psi_0(f)(x,0,x) := f(r(x)) = f(x(0)).
\]

For \( f \in C_0(\Lambda^0) \), we have \( \psi_0(f) \in C_0(G_\Lambda^{(0)}) \), and \( \psi_0 \) is a homomorphism.

Since \( G_\Lambda \) is \( r \)-discrete, it follows that \( C_0(G_\Lambda^{(0)}) \) is a subalgebra of \( C^*(G_\Lambda) \). Letting \( \iota : C_0(G_\Lambda^{(0)}) \to C^*(G_\Lambda) \) be the inclusion homomorphism, we define \( \Psi_0 : C_0(\Lambda^0) \to C^*(G_\Lambda) \) by \( \Psi_0 = \iota \circ \psi_0 \).

For \( \xi \in C_c(\Lambda^1) \), define \( \Psi_1(\xi) : G_\Lambda \to \mathbb{C} \) by

\[
\Psi_1(\xi)(x,m,y) = \delta_{m,1}\delta_{\sigma^1x,y}\xi(x(0,1)).
\]

Then \( \Psi_1 \) is linear, and for \( \xi \in C_c(\Lambda^1) \), we have \( \Psi_1(\xi) \in C_c(G_\Lambda) \) with

\[
(5.1) \quad \text{supp}(\Psi_1(\xi)) \subset Z((\text{supp}(\xi)) \ast_s (s(\text{supp}(\xi))), 1).
\]

**Lemma 5.3.** For \( f \in C_c(\Lambda^0) \) and \( \xi, \eta \in C_c(\Lambda^1) \), we have

(i) \( \Psi_1(\xi)^*\Psi_1(\eta) = \Psi_0(\langle \xi, \eta \rangle) \) and

(ii) \( \Psi_0(f)\Psi_1(\xi) = \Psi_1(\phi(f)\xi) \).

**Proof.** Since \( \Psi_1(\xi), \Psi_1(\eta) \) and \( \Psi_0(f) \) have compact support, we can use convolution on \( C_c(G_\Lambda) \) to calculate products. To begin,

\[
\Psi_1(\xi)^*\Psi_1(\eta)(x,m,y) = \sum_{(x,n,z) \in G_\Lambda} \overline{\Psi_1(\xi)(z,-n,x)}\Psi_1(z,m-n,y).
\]

A summand on the right-hand side may be nonzero only if \( -n = 1, \sigma^1z = x, m-n = 1 \) and \( \sigma^1z = y \), which is precisely when \( m = 0, x = y, n = -1 \) and \( z = ex \) for some \( e \in \Lambda^1(r(x)) \). So the support of \( \Psi_1(\xi)^*\Psi_1(\eta) \) is contained in \( G_\Lambda^{(0)} \), and for \( (x,0,x) \in G_\Lambda^{(0)} \), we have

\[
\Psi_1(\xi)^*\Psi_1(\eta)(x,0,x) = \sum_{e \in \Lambda^1r(x)} \overline{\Psi_1(\xi)(ex,1,x)}\Psi_1(ex,1,x)
\]

\[
= \sum_{e \in \Lambda^1r(x)} (\xi(e)\eta(e))
\]

\[
= \langle \xi, \eta \rangle(r(x))
\]

\[
= \Psi_0(\langle \xi, \eta \rangle)(x,0,x),
\]

giving (i).

For (ii), we have

\[
\Psi_0(f)\Psi_1(\xi)(x,m,y) = \sum_{(x,n,z) \in G_\Lambda} \Psi_0(f)(x,n,z)\Psi_1(\xi)(z,m-n,y).
\]
A summand on the right-hand side may be nonzero only if \( n = 0, x = z, m - n = 1 \) and \( \sigma^2 z = y \). Hence the sum reduces to a single summand, and we have
\[
\Psi_0(f)\Psi_1(\xi)(x, m, y) = \delta_{m,1} \delta_{x,0} \Psi_0(f)(x, 0, x) \Psi_1(x, 1, \sigma^1 x) \\
= \delta_{m,1} \delta_{x,0} f(r(x)) \xi(x(0, 1)) \\
= \delta_{m,1} \delta_{x,0} (\phi(f) \xi)(x(0, 1)) \\
= \Psi_1(\phi(f) \xi)(x, m, y),
\]
as required. \( \square \)

Using Lemma 5.3(i), we see that \( \Psi_1 \) is norm-decreasing, and since \( C_c(\Lambda^1) \) is dense in \( C_d(\Lambda^1) \), it follows that \( \Psi_1 \) extends to a norm-decreasing linear map \( \Psi_1 : C_d(\Lambda^1) \to C^*(G_\Lambda) \). Continuity allows us to extend the properties in Lemma 5.3 to all of \( C_0(\Lambda^0) \) and \( C_d(\Lambda^1) \), giving:

**Proposition 5.4.** For \( f \in C_0(\Lambda^0) \) and \( \xi, \eta \in C_d(\Lambda^1) \),
\[
\begin{align*}
(i) & \quad \Psi_1(\xi^* \Psi_1(\eta) = \Psi_0(\langle \xi, \eta \rangle) \quad \text{and} \\
(ii) & \quad \Psi_0(f) \Psi_1(\xi) = \Psi_1(\phi(f) \xi).
\end{align*}
\]

Therefore \( (\Psi_0, \Psi_1) \) is a Toeplitz \( E_\Lambda \)-pair on \( C^*(G_\Lambda) \), so the universal property of \( T(E_\Lambda) \) gives a homomorphism \( \Psi_0 \times_\tau \Psi_1 : T(E_\Lambda) \to C^*(G_\Lambda) \).

The following notation is handy. For \( m, p, q \in \mathbb{N} \) with \( p \leq q \leq m \), define the continuous map \( \text{Seg}^m_{(p, q)} : \Lambda^m \to \Lambda^{q-p} \) by \( \text{Seg}^m_{(p, q)}(\lambda) := \lambda(p, q) \).

**Proposition 5.5.** \( \Psi_0 \times_\tau \Psi_1 : T(E_\Lambda) \to C^*(G_\Lambda) \) is surjective.

**Proof.** We complete the proof in two steps, first showing that \( C_0(G_\Lambda^{(0)}) \) is in the image of \( \Psi_0 \times_\tau \Psi_1 \), then using this to show \( \Psi_0 \times_\tau \Psi_1 \) is surjective.

Let \( \{ U_j \}_{j \in \mathbb{N}} \) be a basis for \( \Lambda^1 \) comprising relatively compact open sets \( U_j \) such that \( s|_{U_j} \) is a homeomorphism. Define
\[
\begin{align*}
W_1 &= \{ \Psi_0(f) : f \in C_c(\Lambda^0) \}, \\
W_2 &= \{ \Psi_1(\xi) \cdots \Psi_1(\xi_p) \Psi_1(\eta_p)^* \cdots \Psi_1(\eta_1)^* : p \in \mathbb{N}, \\
&\quad \text{and each } \xi_i, \eta_i \in C_0(U_{j_i}) \text{ for some } j_i \in \mathbb{N} \}
\end{align*}
\]
and
\[
W = \text{span}(W_1 \cup W_2).
\]

Then \( W \) is a \( * \)-subalgebra of \( C_0(G_\Lambda^{(0)}) \), and \( W \) separates points in \( G_\Lambda^{(0)} \) and does not vanish identically at any point of \( G_\Lambda^{(0)} \). Therefore, by the Stone-Weierstrass Theorem, \( W \) is uniformly dense in \( C_0(G_\Lambda^{(0)}) \). Since the supremum norm coincides with the \( I \)-norm on elements of \( C_0(G_\Lambda^{(0)}) \), and the \( I \)-norm bounds the \( C^* \)-norm, it follows that \( W \) is dense in \( C_0(G_\Lambda^{(0)}) \) with respect to the \( C^* \)-norm, and \( C_0(G_\Lambda^{(0)}) \) is in the image of \( \Psi_0 \times_\tau \Psi_1 \).

Now fix \( f \in C_c(G_\Lambda) \). Let \( \{ W_j \}_{j=1}^n \) be an open cover of \( \text{supp} \ f \) of the form
\[
W_j = Z(\{(U^j_i \cdots U^j_p) \ast_1 (V^j_i \cdots V^j_p), p_j - q_j)\},
\]
where each \( U^j_i \) and \( V^j_i \) are relatively compact open subsets of \( \Lambda^1 \) such that \( s|_{U^j_i} \) and \( s|_{V^j_i} \) are homeomorphisms. Let \( \{ \varphi_j \}_{j=1}^n \) be a partition of unity subordinate to \( \{ W_j \}_{j=1}^n \); that is, each \( \varphi_j : G_\Lambda \to [0, 1] \) is continuous and \( \text{supp}(\varphi_j) \subset W_j \), and
Fix $j \in \{1, \ldots, n\}$; we will show that $\varphi_j f$ is in the image of $\Psi_0 \times \mathcal{T}$. For convenience, we drop the index $j$, and write

$$W_j = Z((U_1 \cdots U_p) * (V_1 \cdots V_q), p - q).$$

For $x \in G^{(0)}_\Lambda$, it follows from injectivity of $s_{[U_1 \ldots U_p]}$ and $s_{V_1 \cdots V_q}$ that there is at most one $(\lambda, \mu) \in (U_1 \cdots U_p) * (V_1 \cdots V_q)$ such that $s(\lambda) = r(x)$. Hence we may define a function $g : G^{(0)}_\Lambda \to \mathbb{C}$ by

$$g(x) := \begin{cases} 
\varphi_j f(\lambda x, p - q, \mu x) & \text{if there exists } (\lambda, \mu) \text{ such that } (\lambda x, p - q, \mu x) \in W_j \\
0 & \text{otherwise}.
\end{cases}$$

Then $g$ is continuous, and the support of $g$ is compact since it is contained in $Z(s(U_1 \cdots U_p) \cap s(V_1 \cdots V_q))$. Hence $g \in C_c(G^{(0)}_\Lambda)$, and it follows that $g = \Psi_0 \times \mathcal{T} \Psi_1(a)$ for some $a \in \mathcal{T}(E_\Lambda)$.

Define

$$W = \{(\lambda, \mu) \in (U_1 \cdots U_p) * (V_1 \cdots V_q) : (\lambda x, p - q, \mu x) \in \text{supp}(\varphi_j f) \text{ for some } x\},$$

and let $P_1, P_2 : \Lambda \to \Lambda$ be the coordinate projections. Then since $W$ is compact and $P_i$ is continuous, for $i = 1, \ldots, p$, the set $\text{Seg}_{(i-1,i)}^p(P_i(W))$ is a compact subset of $U_i$. By Urysohn’s Lemma, there exists $\xi_i \in C_c(U_i)$ satisfying

$$\xi_i(e) = 1 \quad \text{for all } e \in \text{Seg}_{(i-1,i)}^p(P_i(W)).$$

Similarly, for $i = 1, \ldots, q$, there exists $\eta_i \in C_c(V_i)$ such that

$$\eta_i(e) = 1 \quad \text{for all } e \in \text{Seg}_{(i-1,i)}^q(P_q(W)).$$

Calculations using the convolution product on $C_c(G\Lambda)$ reveal

$$\varphi_j f = \Psi_1(\xi_1) \cdots \Psi_p(\xi_p)(\Psi_0 \times \mathcal{T} \Psi_1(a))\Psi_1(\eta_1)^* \cdots \Psi_1(\eta_p)^*,$$

so $f$ is in the image of $\Psi_0 \times \mathcal{T} \Psi_1$. Since $\Psi_0 \times \mathcal{T} \Psi_1$ has closed range, it follows that $\Psi_0 \times \mathcal{T} \Psi_1$ is surjective. \hfill \square

**Proof of Theorem 5.1.** Fix a faithful nondegenerate representation $\pi$ of $C^*(G_\Lambda)$ on a Hilbert space $H$. Then $(\pi \circ \Psi_0, \pi \circ \Psi_1)$ is a Toeplitz $E_\Lambda$-pair, and the universal property of $\mathcal{T}(E_\Lambda)$ gives

$$(\pi \circ \Psi_0) \times \mathcal{T} (\pi \circ \Psi_1) = \pi \circ (\Psi_0 \times \mathcal{T} \Psi_1).$$

To show $\pi \circ (\Psi_0 \times \mathcal{T} \Psi_1)$ is faithful, we use [3] Theorem 2.1] which says in our setting that $(\pi \circ \Psi_0) \times \mathcal{T} (\pi \circ \Psi_1) : \mathcal{T}(E_\Lambda) \to B(H)$ is faithful if $C_0(\Lambda^0)$ acts faithfully on $(\pi \circ \Psi_0)(C_0(\Lambda^1))H^\perp$; that is, if for nonzero $f \in C_0(\Lambda^0)$, there exists $h \in ((\pi \circ \Psi_1)(C_0(\Lambda^1))H^\perp)$ such that $(\pi \circ \Psi_0)(f)h \neq 0$.

Fix nonzero $f \in C_0(\Lambda^0)$ and $v \in \Lambda^0$ such that $f(v) \neq 0$. Let $V \subset \Lambda^0$ be a relatively compact open neighbourhood of $v$ such that $f(w) \neq 0$ for all $w \in V$, so $\overline{V} \subset \text{supp}(f)$. Then $U := Z(V) \cap Z(\Lambda^1)^c$ is a relatively compact open neighbourhood of $(v, 0, v)$ contained in $\text{supp}(\Psi_0(f))$; choose $g \in C_c(G^{(0)}_\Lambda)$ satisfying $g(v, 0, v) \neq 0$ and $\text{supp}(g) \subset U$. In particular, $(\Psi_0(f))^*(v, 0, v) \neq 0$ and

$$(\Psi_0(f))^*(v, 0, v) \neq 0 \quad \text{and} \quad \text{supp}(g) \subset \{(w, 0, w) \in G^{(0)}_\Lambda : w \in \Lambda^0\}.$$
Since $\Psi_0(fg) \neq 0$ and $\pi$ is faithful, there exists $h' \in H$ such that $\pi(\Psi_0(fg)g)h' \neq 0$. Defining $h := \pi(g)h'$ and using (5.4), we deduce $h \in ((\pi \circ \Psi_1)(C_0(\Lambda^1))H)^\perp$, and we have $(\pi \circ \Psi_0)(f)h = \pi(\Psi_0(fg)g)h' \neq 0$. Thus $C_0(\Lambda^0)$ acts faithfully on $(\pi \circ \Psi_1)(C_0(\Lambda^1))H)^\perp$, and [2, Theorem 2.1] implies $(\pi \circ \Psi_0) \times_\tau (\pi \circ \Psi_1)$ is faithful. Therefore, from [13, Lemmas 1.15 and 1.16], it follows that there exist

\[ \Phi_0((5.8) \Phi_0) \]  

so we must show $(5.4)$ holds for all $(x,m,y)$. It then follows from \[ \text{[13, Lemmas 1.15 and 1.16]} \]  

Proposition 5.5, completes the proof. \[ \square \]

We now construct a Cuntz-Krieger $E_\Lambda$-pair $\Phi$ on $C^*(\mathcal{G}_\Lambda)$. Since $X_\Lambda \setminus \partial \Lambda$ is an open invariant subset of $G_\Lambda$, we can regard $C^*(G_\Lambda|X_\Lambda \setminus \partial \Lambda)$ as an ideal in $C^*(G_\Lambda)$ with quotient $C^*(\mathcal{G}_\Lambda)$. Letting $Q : C^*(G_\Lambda) \to C^*(\mathcal{G}_\Lambda)$ be the quotient homomorphism with kernel $C^*(G_\Lambda|X_\Lambda \setminus \partial \Lambda)$, we define $\Phi_0 := Q \circ \Psi_0$ and $\Phi_1 := Q \circ \Psi_1$.

**Proposition 5.6.** The pair $(\Phi_0, \Phi_1)$ is a Cuntz-Krieger $E_\Lambda$-pair on $C^*(\mathcal{G}_\Lambda)$.

**Proof.** Since $(\Psi_0, \Psi_1)$ is a Toeplitz $E_\Lambda$-pair and $Q$ is a homomorphism, it follows that $(\Phi_0, \Phi_1)$ is a Toeplitz $E_\Lambda$-pair. We must show that

\[ \Phi_0(f) = \Phi^{(1)}(\phi(f)) \quad \text{for all } f \in C_0(\Lambda^0_{\mathcal{B}_\mathcal{G}}). \]

Since $C_c(\Lambda^0_{\mathcal{B}_\mathcal{G}})$ is dense in $C_0(\Lambda^0_{\mathcal{B}_\mathcal{G}})$, it suffices to check \[ \text{[13, Lemmas 1.15 and 1.16]} \] holds for $f \in C_c(\Lambda^0_{\mathcal{B}_\mathcal{G}})$.

Fix $f \in C_c(\Lambda^0_{\mathcal{B}_\mathcal{G}})$. Then $f \circ r \in C_c(\Lambda^1)$ and $\phi(f) = \theta(f \circ r)$, where $\theta : C_0(\Lambda^1) \to \mathcal{L}(C_0(\Lambda^1))$ is the injective homomorphism defined by

\[ (\theta(g)\xi)(e) = g(e)\xi(e). \]

It then follows from \[ \text{[13, Lemmas 1.15 and 1.16]} \] that there exist $l \in \mathbb{N}$ and $\xi_i, \eta_i \in C_c(\Lambda^1)$ for $i = 1, \ldots, l$, such that

\[ f \circ r = \sum_{i=1}^l \xi_i \eta_i, \quad \text{where the product is defined pointwise,} \]

\[ \xi_i(e)\eta_i(e') = 0 \quad \text{for all } i \text{ and } e, e' \in \Lambda^1 \text{ with } s(e) = s(e') \text{ and } e \neq e', \]

and

\[ \phi(f) = \sum_{i=1}^l \xi_i \otimes \eta_i^*. \]

Thus we have

\[ \Phi^{(1)}(\phi(f)) = \sum_{i=1}^l \Phi_1(\xi_i)\Phi_1(\eta_i)^*, \]

so we must show

\[ \Phi_0(f)(x, m, y) = \sum_{i=1}^l \Phi_1(\xi_i)\Phi_1(\eta_i)^*(x, m, y) \]

for all $(x, m, y) \in \mathcal{G}_\Lambda$.

Fixing $(x, m, y) \in \mathcal{G}_\Lambda$, the right-hand side of \[ \text{(5.8)} \] is equal to

\[ \sum_{i=1}^l \sum_{(x, n, z) \in \mathcal{G}_\Lambda} \Phi_1(\xi_i)(x, n, z)\Phi(\eta_i)^*(y, n-m, z). \]
A summand from (5.9) may be nonzero only if \( n = 1, \sigma^1x = z, n - m = 1 \) and \( \sigma^1y = z \); that is, only if \( m = 0, n = 1, \sigma^1x = \sigma^1y \) and, consequently, \( d(x), d(y) \geq 1 \). Thus

\[
\sum_{i=1}^{\ell} \Phi_1(\xi_i) \Phi_1(\eta_i)^*(x, m, y) = \begin{cases} 
\delta_{m,0} \delta_{\sigma^1x,\sigma^1y} \sum_{i=1}^{\ell} \xi_i(x(0,1)) \eta_i(y(0,1)) & \text{if } d(x), d(y) \geq 1 \\
0 & \text{otherwise}
\end{cases}
\]

\[
= \begin{cases} 
\delta_{m,0} \delta_{x,y} \sum_{i=1}^{\ell} \xi_i(x(0,1)) \eta_i(y(0,1)) & \text{if } d(x) \geq 1 \\
0 & \text{otherwise}
\end{cases}
\]

\[
= \begin{cases} 
\delta_{m,0} \delta_{x,y} (f \circ r)(x(0,1)) & \text{if } d(x) \geq 1 \\
0 & \text{otherwise}
\end{cases}
\]

\[
(5.10) = \begin{cases} 
\delta_{m,0} \delta_{x,y} \Phi_0(f)(x, 0, x) & \text{if } d(x) \geq 1 \\
0 & \text{otherwise}
\end{cases}
\]

Hence, if \( d(x) \geq 1 \) then (5.8) follows from (5.10). On the other hand, if \( d(x) = 0 \) then \( x = s(x) \notin \Lambda^0_{rg} \) and since \( \text{supp}(f) \subset \Lambda^0_{rg} \), it follows that \( \Phi_0(f)(x, 0, x) = 0 \). Therefore (5.8) holds for all \( (x, m, y) \in \mathcal{G}_A \), and \( (\Phi_0, \Phi_1) \) is a Cuntz-Krieger \( E_A \)-pair.

Proof of Theorem 5.2. Since \( \Phi \) is a Cuntz-Krieger \( E_A \)-pair, it is also a Toeplitz \( E_A \)-pair, and we have two induced homomorphisms

\[
\Phi_0 \times \tau \Phi_1 : \mathcal{T}(E_A) \rightarrow C^*(\mathcal{G}_A) \quad \text{and} \quad \Phi_0 \circ \Phi_1 : \mathcal{O}(E_A) \rightarrow C^*(\mathcal{G}_A).
\]

By considering the universal Cuntz-Krieger \( E_A \)-pair \( (j_0, j_1) \) on \( \mathcal{O}(E_A) \), the universal property of \( \mathcal{T}(E_A) \) gives a homomorphism \( j_0 \times \tau j_1 : \mathcal{T}(E_A) \rightarrow \mathcal{O}(E_A) \) which satisfies

\[
\Phi_0 \times \tau \Phi_1 = (\Phi_0 \circ \Phi_1) \circ (j_0 \times \tau j_1).
\]

On the other hand, recalling the quotient homomorphism \( Q : C^*(\mathcal{G}_A) \rightarrow C^*(\mathcal{G}_A) \), the universal property of \( \mathcal{T}(E_A) \) gives

\[
\Phi_0 \times \tau \Phi_1 = Q \circ (\Psi_0 \times \tau \Psi_1).
\]

Hence

\[
(\Phi_0 \times \tau \Phi_1) \circ (j_0 \times \tau j_1) = Q \circ (\Psi_0 \times \tau \Phi_1),
\]

and surjectivity of \( \Phi_0 \times \tau \Phi_1 \) follows from surjectivity of \( Q \) and \( \Psi_0 \times \tau \Psi_1 \).

To show that \( \Phi_0 \circ \Phi_1 \) is injective, we use the gauge-invariant uniqueness theorem [24, Proposition 4.5].

The map \( c : \mathcal{G}_A \rightarrow \mathbb{Z} \) defined by \( c(x, m, y) = m \) is a continuous functor, hence by [27, Proposition II.5.1] there is a strongly continuous action \( \beta \) of \( \mathbb{Z} = \mathbb{T} \) on \( C^*(\mathcal{G}_A) \) such that \( \beta_t(g)(x, m, y) = t^m y(x, m, y) \) for all \( t \in \mathbb{T} \) and \( g \in C_c(\mathcal{G}_A) \), and such that \( \beta \) leaves \( C_0(\mathcal{G}_A^{(0)}) \) pointwise fixed. We then have \( \beta_t(\Phi_0(f)) = \Phi_0(f) \) and \( \beta_t(\Phi_1(\xi)) = t^y \Phi_1(\xi) \) for all \( t \in \mathbb{T}, f \in C_0(\mathcal{G}_A^{(0)}) \) and \( \xi \in C_0(\mathcal{G}_A^{(1)}) \).

It is straightforward to see that \( \Phi_0 : C_0(\mathcal{G}_A^{(0)}) \rightarrow C^*(\mathcal{G}_A) \) is injective. Therefore [24, Proposition 4.5] implies \( \Phi_0 \circ \Phi_1 : \mathcal{O}(E_A) \rightarrow C^*(\mathcal{G}_A) \) is injective, and we’re done.

\[\square\]
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