Statistics of two-particle dispersion in two-dimensional turbulence

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(March 30, 2022)

We investigate Lagrangian relative dispersion in direct numerical simulation of two-dimensional inverse cascade turbulence. The analysis is performed by using both standard fixed time statistics and an exit time approach. Our results are in good agreement with the original Richardson’s description in terms of diffusion equation. The deviations are only of the quantitative nature. These deviations, and the observed strong sensitivity to the finite size effects, are associated with the long-range-correlated nature of the particles’ relative motion. The correlation, or persistence, parameter is measured by means of a Lagrangian “turning point” statistics.

I. INTRODUCTION

Understanding the statistics of particle pairs dispersion in turbulent velocity fields is of great interest for both theoretical and practical implications. At variance with single particle dispersion which depends mainly on the large scale, energy containing eddies, pair dispersion is driven (at least at intermediate times) by velocity fluctuations at scales comparable with the pair separation. Since these small scale fluctuations have universal characteristics, independent from the details of the large scale flow, relative dispersion in fully developed turbulence is expected to show universal behavior. From an applicative point of view, a deep comprehension of relative dispersion mechanisms is of fundamental importance for a correct modelization of small scale diffusion and mixing properties.

Since the pioneering work by Richardson, many efforts have been done to confirm experimentally or numerically his law. Nevertheless, the main obstacle to a deep investigation of relative dispersion in turbulence remains the lack of sufficient statistics due to technical difficulties in laboratory experiments and to the moderate inertial range reached in direct numerical simulations.

In this Paper we present a detailed investigation of the statistics of relative dispersion from extensive direct numerical simulations of particle pairs in two-dimensional Navier-Stokes turbulence. We will see that the main ingredient of the original Richardson description, i.e. Richardson diffusion equation, is sufficient for rough description of relative dispersion in this flow. Nevertheless, our simulations show that two-particle statistics is rather sensible to finite size effects. This asks for a different analysis based on doubling time statistics which has recently introduced for the analysis of relative dispersion.

Comparison of numerical results with ones based on the Richardson’s equation show that the last delivers a qualitatively good description of the doubling-time-distributions. The quantitative deviations found are attributed to the fact that the dispersion process is not purely diffusive and is influenced by ballistic (persistent) motion.

The article is organized as follows: In Section II we discuss the Richardson’s approach to the two-particle dispersion, in Section III the fixed-scale properties of dispersion process (such as doubling-time statistics) are considered. The numerical approach and the results of simulations are discussed in Section IV. Section V is devoted to conclusions. The mathematical details of calculations of doubling-time statistics for the Richardson’s case are given in Appendices A and B.

II. STATISTICS OF RELATIVE DISPERSION

Relative dispersion in turbulence is often phenomenologically described in terms of a diffusion equation for the probability density function of pair separation $p(r,t)$.
\[
\frac{\partial p(r, t)}{\partial t} = \frac{\partial}{\partial r_j} \left( K(r, t) \frac{\partial p(r, t)}{\partial r_j} \right) \tag{1}
\]

with a space and time dependent diffusion coefficient \( K(r, t) \). The original Richardson proposal obtained from experimental data in the atmosphere corresponds to \( K(r, t) = K(r) = k_0 \varepsilon r^{4/3} \), where \( \varepsilon \) has the dimension of energy dissipation (see below) and \( k_0 \) is an adimensional constant. In the \( d \)-dimensional isotropic case, this diffusion equation takes the form

\[
\frac{\partial p(r, t)}{\partial t} = \frac{1}{d-1} \frac{\partial}{\partial r} r^{d-1} K(r) \frac{\partial p(r, t)}{\partial r} \tag{2}
\]

Its solution leads to the well known non-Gaussian distribution

\[
p(r, t) = \frac{A}{(k_0 t)^3 \varepsilon} \exp \left( - \frac{9r^{2/3}}{4k_0 \varepsilon^{1/3} t} \right) \tag{3}
\]

where \( A \) is a normalizing factor. The growth of pair separation is described in terms of a single exponent as

\[
R^{2n}(t) \equiv \langle r^{2n}(t) \rangle = C_{2n} \varepsilon^n t^{3n}. \tag{4}
\]

The so called Richardson constant in (4) is \( g = C_2 = \frac{1280}{27} k_0^3 \).

The Richardson's conjecture was formulated based on the scaling nature of the diffusion coefficient and in analogy with diffusion. No information about the nature of the pdf was available by that time. Any chose of the form \( K(r, t) \approx r^{4/3-\alpha} \langle \delta v^2(t) \rangle^{\alpha/2} \) (i.e. \( K(r, t) \approx r^{4/3-\alpha} t^{3\alpha/2} \)) would give the same scaling law \( R^2 \propto t^3 \) but with different pdf’s (see [1,2,12,13]).

The possibility to describe the dispersion process by means of a diffusion equation is based on essentially two important physical assumptions which can be verified a posteriori. The first one is that the dispersion process is self-similar in time, which is probably true in non intermittent velocity field [1], the second one is that the velocity field is short correlated in time [14]. In the limit of velocity field \( \delta \)-correlated in time the diffusion equation (1) becomes exact [15,16]. As we proceed to show, the Richardson’s conjecture (2), which is exact under small values of the persistence parameter of the flow, vide infra and [13,17], still delivers a qualitatively good approximation for realistic 2d turbulent flows, whose persistence parameter of the order of 1.

Richardson scaling in turbulence is a consequence of Kolmogorov scaling for the velocity differences [2]. Under Kolmogorov scaling, the mean square relative velocity and the correlation time in the inertial range are given by

\[
\langle \delta v(r)^2 \rangle = v_0^2 \left( \frac{r}{r_0} \right)^{2/3} \approx \varepsilon^{2/3} r^{2/3} \tag{5}
\]

and

\[
\tau(r) = r_0 \left( \frac{r}{r_0} \right)^{2/3} \approx \varepsilon^{-1/3} r^{2/3}, \tag{6}
\]

where \( r_0, \tau_0 \) and \( v_0 \) are some (large scale) characteristic length, time and velocity scale and \( \varepsilon \approx \varepsilon_0^2/\tau_0 \) is the energy flux in the inertial range. The value of the dimensionless combination \( Ps = \frac{\varepsilon t_0}{\tau_0} \) remains however unspecified by scaling considerations. It is referred to as a persistence parameter of the flow and plays a central role in describing single particle diffusion and pair separation [13,17].

The persistence parameter gives the ratio of the velocity correlation time to the Lagrangian characteristic time. In order to see how \( Ps \) influences Lagrangian dispersion, let us consider the following simple model, which has been used as a basis for building a stochastic model of turbulent dispersion [14]. We take that the magnitude of the separation velocity is a function of \( r \) only so that \( \delta v(r) = v_0 (r/r_0)^{1/3} \). The temporal changes of the flow can be accounted for by letting the particle change its velocity direction from time to time, while keeping the velocity’s magnitude constant. The probability that the relative velocity of particle separation changes its direction during time interval \( dt \) can be assumed to be \( dp = dt/\tau(r) \). The growth of the magnitude of the interparticle separation \( \delta v(r) \) in \( dt \) is \( dr \simeq \delta v(r) dt \), thus the probability to change the direction of velocity within \( dr \) is, using (4)

\[
dp = p(r) dr = \frac{dr}{\delta v(r) \tau(r)} = \frac{r_0}{v_0 \tau_0} \frac{dr}{r} = \frac{1}{Ps} \frac{dr}{r}. \tag{7}
\]

The distribution of the position of turning points in the separation follows from [6,17]. The conditional probability density to find a next turning point at \( r_2 \) provided a previous one was at \( r_1 < r_2 \) is given by

\[\text{2}\]
\[
\Psi(r_2|r_1) = \frac{1}{P_s r_1} \left( \frac{r_2}{r_1} \right)^{-1/P_s-1}.
\] (8)

Note that the dependence of \( \Psi(r_2|r_1) \) only on the relative positions of the turning points, i.e. on \( r_2/r_1 \), is a clear consequence of scaling.

The tail of \( \Psi(r_2|r_1) \) decides about the existence of the second moment of this distribution, i.e. on the fact whether the corresponding motion is short- or long-range correlated in space. Depending on the persistence parameter \( P_s \), the dispersion can be either diffusive (\( P_s < 1 \)) or ballistic (\( P_s > 1 \)) in nature. Note that the power-law tail of the distributions make the problem extremely sensitive to the finite-size effects, especially for large \( P_s \), when the weights of ballistic events (Lévy-walks) is considerable.

We note that the value of \( P_s \) is not a free parameter, but is fixed for a given physical situation. The scaling nature of turbulence supposes that this parameters is a constant, depending only on general properties of the flow, e.g. on its 2d or 3d nature (in this last case also the overall geometry of the flow can be of importance). On the other hand, since the nature of dispersion process depends crucially on the value of \( P_s \), the only way for getting quantitative information about the dispersion is through direct numerical simulations. The strong finite-size effects in relative dispersion simulations require the introduction of quantities which are less sensitive to finite resolution.

### III. EXIT TIME STATISTICS

All the results discussed in the previous Section can be observed only in high Reynolds number flows in which the inertial range, where the scaling laws (3) hold, is sufficiently wide. The need for large Reynolds is particularly severe in the case of Richardson dispersion, as a consequence of the long tails in (3). Moreover, the observation of time scaling laws as (4) requires sufficiently long times in order to forget the initial separation \( r(0) \).

For these reasons, the observation of Richardson scaling (i.e. (3) or (4)) is very difficult in direct numerical simulations where the Reynolds number is limited by the resolution. The same kind of limitations arise in laboratory experiments, as a consequence of the necessity to follow the Lagrangian trajectories which limits again the Reynolds number (3).

To partially overcome these difficulties, an alternative approach based on doubling time (or exit time) statistics has been recently proposed (3). Given a set of thresholds \( R_n = \rho^n R(0) \) within the inertial range, one computes the “doubling time” \( T_p(R_n) \) defined as the time it takes for the particle pair separation to grow from threshold \( R_n \) to the next one \( R_{n+1} \). Averages are then performed over many dispersion experiments, i.e., particle pairs, to get the mean doubling time \( \langle T_p(R) \rangle \). The outstanding advantage of this kind of averaging at fixed scale separation, as opposite to a fixed time, is that it removes crossover effects since all sampled particle pairs belong to the same scales.

The problem of doubling time statistics is a first-passage problem for the corresponding transport process. For the Richardson case, in 2d it is given by the solution of the Richardson’s diffusion equation, Eqs. (2), with initial condition \( p(r,0) = \delta(r - R/\rho)/2\pi \) and absorbing boundary at \( r = R \) (so that \( p(R,t) = 0 \)). The pdf of doubling time can be obtained as the time derivative of the probability that the particle is still within the threshold

\[
p_D(t) = -\frac{d}{dt} \int_{|r|<R} p(r,t) dr.
\] (9)

Using (2) one obtains

\[
p_D(t) = -2\pi \varepsilon^{1/3} k_0 R^{4/3} \left. \frac{\partial p(r,t)}{\partial r} \right|_{r=R}.
\] (10)

The solution using the eigenfunction decomposition is given in Appendix A and shows that the long-time asymptotic of \( p_D(t) \) is exponential:

\[
p_D(t) \simeq \exp(-\kappa k_0 \varepsilon^{1/3} R^{-2/3} t)
\] (11)

where \( \kappa \approx 2.93 \) is a number factor. This exponential nature of the tail of \( p_D(t) \)-distribution will be confirmed by direct simulations in Section IV.

Note that the combination \( \varepsilon^{-1/3} R^{2/3} \) has a dimension of time and is proportional to the average doubling time \( \langle T_p(R) \rangle \). This time can be obtained by a simple argument reported in the Appendix B. In the 2-dimensional case one obtains
\[ \langle T_p(R) \rangle = \frac{3}{4} \rho^{2/3} - 1 \frac{R^{2/3}}{k_0} \]

Prediction (12) contains the parameter \( k_0 \) which is, as shown in Section II, is equivalent to the Richardson constant \( g \). As a consequence, the computation of average doubling time can be used for an alternative (and more robust, as we will see) estimation of the Richardson constant. It is convenient to rewrite the doubling time pdf (11) in terms of the average doubling time \( \langle T(R) \rangle \). Making use of (12), one obtains in 2d

\[ p_D(t) \simeq \exp(-0.252 t / \langle T(R) \rangle) \]

which is a parameterless, universal function.

IV. DIRECT NUMERICAL SIMULATIONS

Pair dispersion statistics has been investigated by high-resolution direct numerical simulations of the inverse energy cascade in two-dimensional turbulence [19]. There are several reasons for considering 2D turbulence. First of all, the dimensionality of the problem makes feasible direct numerical simulations at high Reynolds numbers. Moreover, the observed absence of intermittency [20] makes the 2D inverse energy cascade an ideal framework for the study of Richardson scaling in Kolmogorov turbulence.

The 2D Navier-Stokes equation for the vorticity \( \omega = \nabla \times \mathbf{v} = -\Delta \psi \) is:

\[ \partial_t \omega + J(\omega, \psi) = \nu \Delta \omega - \alpha \omega + \phi, \]

where \( \psi \) is the stream function and \( J \) denotes the Jacobian. The friction linear term \( -\alpha \omega \) extracts energy from the system to avoid Bose-Einstein condensation at the gravest modes [21]. The forcing is active only on a typical small scale \( l_f \) and is \( \delta \)-correlated in time to ensure the control of the energy injection rate. The viscous term has the role of removing enstrophy at scales smaller than \( l_f \) and, as customary, it is numerically more convenient to substitute it by a hyperviscous term (of order eight in our simulations). Numerical integration of (14) is performed by a standard pseudospectral method on a doubly periodic square domain at resolution \( N = 1024 \) or \( N = 2048 \). All the results presented are obtained in conditions of stationary turbulence.

In Figure 2 we plot the energy spectrum, which displays Kolmogorov scaling \( E(k) \propto C k^{-5/3} \) over about two decades with Kolmogorov constant \( C \simeq 6.0 \). In the inset we plot the third order structure function \( S_3(r) = \langle \delta v(r)^3 \rangle \) compensated with the theoretical prediction \( S_3(r) = 3/2 r^3 \). The observation of the plateau confirms the existence of an inverse energy cascade and indicates the extension of the inertial range. Previous numerical investigation has shown that velocity differences statistics in the inverse cascade is not affected by intermittency corrections [20]. In this case we may expect the Richardson statistics to be self-similar with Richardson scaling [9].

Lagrangian statistics is obtained by integrating the trajectories of 64000 particle pairs in the turbulent velocity field, initially uniformly distributed with constant separation \( R(0) \).

The Lagrangian data reported below are in dimensionless units in which separations are rescaled with the box size \( L = 2\pi \) and time with the large scale time \( T_0 = (L^2/\varepsilon)^{1/3} \).

A. Relative dispersion analysis

In Figure 2 we plot the relative dispersion \( R^2(t) \) for two different initial separation, \( R(0) = \delta x / 2 \) and \( R(0) = \delta x \) (where \( \delta x = 2\pi/N \) is the grid mesh). The Richardson \( t^3 \) law [4] is observed in a limited time interval, especially for the larger \( R(0) \) run. It is remarkable that the relative separation law displays such a strong dependence on the initial conditions even in our high resolution runs.

This dependence makes the determination of the Richardson constant particularly difficult. In Figure 2 we also show the compensated plot \( R^2(t)/t^3 \) which, in the adimensional units, should directly give the constant \( g \). It is clear that a precise determination of \( g \) is impossible; even the Richardson scaling [3], when looked in a compensated plot, is rather poor. Figure 2 suggests that starting with an intermediate initial separation would give a wider scaling range. Of course, one would like to avoid this “fine tuning”, which is probably impossible to implement in the case of experimental data.

The probability distribution function of pair separations is plotted in Figure 2 for the \( R(0) = \delta x / 2 \) run. At short time \( t = 0.015 \), in the beginning of the \( t^3 \) range in Figure 2, we found that the Richardson pdf [6] fits pretty well our data, although some deviations can be detected. Of course, at time comparable with the integral time \( t = 0.77 \),
particle separations are of the order of the integral scale and we observe Gaussian distribution. The crossover between these two regimes is extremely broad: deviations from Richardson pdf are clearly seen already for the times well within the Richardson’s $t^3$ range. To observe better this transition, in Figure 3 we plot, in log-log plot, the right tail of $-\ln(p(r,t)/p(0,t))$. We see that the tail slope can be fitted with an exponent which change continuously in time, from $2/3$ to the Gaussian value 2 (see the inset of Figure 3). Thus self similarity, if it exists, is reduced to the very short time at the beginning of dispersion. Moreover, the scaling region is strongly affected by the choice of initial separation, as shown in Figure 3.

B. Doubling time data

From the same Lagrangian trajectories discussed in the previous Section, we have computed the doubling time statistics. The average doubling time as function of the scale is plotted in Figure 5. The line represent the dimensional prediction $\langle T(R) \rangle \approx R^{2/3}$. In the inset we plot the quantity $\frac{20}{9} \frac{(r^{2/3} - 1)^3}{r^2} \frac{r^2}{\epsilon(T)}$, which, from (1) and (2) gives the value of the Richardson constant $g \approx 3.8$, which is compatible with the estimation from Figure 3. It is interesting to compare our result with previous estimations of $g$. The only experimental estimation of $g$ for 2D inverse cascade [1] gives a value about 7 times smaller (but the Reynolds number in the experiment is even smaller than in present simulations). Other estimations of $g$ are based on kinematic simulations with synthetic flows. In all these cases [1,22] the reported values are even smaller. On the other hand, our numerical finding is not very far from the prediction of turbulence closure theory [3,23].

From Figure 3, we observe that at very small separations $R \approx 10^{-3}$, the doubling time has a tendency to a constant value $\langle T(R) \rangle \approx 0.0016$. On these scales we are below the forcing scale (see Figure 3), and the velocity field can be assumed smooth. As a consequence of Lagrangian chaos we expect on these scales an exponential amplification of separations $\frac{23}{24} R$ at a rate given by the Lagrangian Lyapunov exponent $\lambda$. The latter can be obtained as $\lambda = \lim_{R \to 0} \ln R/\langle T(R) \rangle$, and gives $\lambda \approx 100$ (in adimensional units). The Lagrangian Lyapunov exponent $\lambda$ is a small scale quantity (i.e. depends on the Reynolds number of the simulation), and thus has to be compared with a small scale characteristic time. One can estimate the smallest characteristic time $\tau_{\min}$ by the minimum value of $(k^3 E(k))^{-1}$. We obtain $\lambda \approx 0.23 r_{\min}^{-1}$.

The comparison of Figures 3 and 5 shows the advantages of fixed scale statistics with respect to fixed time statistics. In particular, the dependence of $R(t)$ and the Richardson constant on the initial separation are absent in fixed scale statistics $\langle T(R) \rangle$.

In Figure 3, we plot the doubling time pdf $p_D(T)$ compensated with the mean value $\langle T(R) \rangle$ at different scales in the inertial range $0.003 \leq R \leq 0.046$. First, we obtain a very nice collapse of the different curves, indicating that the process is really self-similar. Second, we observe the exponential tail predicted in Section III with a fitted coefficient 0.3 which is indeed not far from the theoretical prediction (13) based on the Richardson’s picture. The difference between the predicted and measured values of the prefactors is not large, but perceptible: it shows that the Richardson’s equation gives a correct qualitative description of the dispersion process, but is not exact. The reasons for deviations from the diffusive picture proposed by Richardson are the long-range correlations in the particles’ motion, as seen from the analysis of the turning points of their relative trajectories.

C. Turning points statistics and the persistence parameter

A possible explanation for the deviations of our high resolution numerical data from the Richardson’s picture is the not too small value of the persistence parameter. As discussed in Section III, at large values of $Ps$ the contribution of ballistic events may lead to non-Richardson distributions and moreover makes the dispersion strongly sensible to finite-size effects, cutting the longer trajectories.

We have computed the persistence parameter making use of (8). We have recorded, for each pair, the set of turning points $r_i$ at which the pair’s relative velocity changes sign. From the set of $r_i$ we have then computed the pdf of the ratio $r_{i+1}/r_i$, accumulating for all the $i$ and all the pairs. The result, plotted in Figure 4, gives $Ps \approx 0.87$. The requirement that both $r_1$ and $r_2$ are in the inertial range, strongly limits the statistics on turning points and the numerical result is affected by rather large uncertainty. Nevertheless, it is remarkable that the power law tail in the conditional probability density $\Psi(r_2|r_1)$ is well observed in our numerical simulations. This justifies, a posteriori, the use of models based on $\Psi(r_2|r_1)$ for describing relative dispersion [4]. The numerical value of the effective persistence parameter $Ps \approx 0.87$ is not so small, and can explain the observed deviations from Richardson pdf (which are however less pronounced in a 2d flow than in a theoretical one-dimensional model [17]): The transport in a 2d turbulent flow is neither purely diffusive nor ballistic.
In order to be more confident on the numerical value of $Ps$ obtained through the turning-points statistics, let us show that it agrees with a simple estimates based on the values of the Kolmogorov’s and the Richardson’s constants. According to the Kolmogorov’s scaling, the mean squared relative velocity of the pair is given by

$$\langle \delta v^2(r) \rangle = C_2 \varepsilon^{2/3} r^{2/3},$$

with $C_2 \approx 12.9$ \cite{21}. If the particles separate ballistically with the rms velocity

$$\delta v(r) = C_2^{1/2} \varepsilon^{1/3} r^{1/3}$$

the distance between them should grow as

$$R_{\text{max}}^2 = \left( \frac{2}{3} \right)^3 C_2^{3/2} \varepsilon t^3.$$  

On the other hand, due to the unsteadiness of the separation velocity, the distance between the particles grows slower, namely as $R^2 = g t^2$, so that the factor

$$\xi^2 = \frac{R^2}{R_{\text{max}}^2} = \left( \frac{3}{2} \right)^3 \frac{g}{C_2^{3/2}}$$

serves as a measure of this unsteadiness and $\xi^2$ is connected with the value of the persistence parameter. In our case, $\xi^2 \approx 0.28$. Within the stochastic model of Ref. \cite{17} this corresponds to a value of $Ps$ between 1.1 and 1.2, in reasonable agreement with the direct measurement from the turning-point statistics, and again corroborates the stochastic approach.

We also note a possibility to “tune” the $Ps$ value by performing simulations in which the Lagrangian trajectories are integrated according to $\dot{x} = \lambda v(x, t)$. By changing the value of parameter $\lambda$ one effectively changes $v_0$ and thus $Ps$. In the extreme case $\lambda \to 0$ the trajectories resemble those in a time $\delta$-correlated velocity field. In the opposite limit, $\lambda \gg 1$ we have dispersion on a quenched field. Of course, it is only for the standard value $\lambda = 1$ that Lagrangian trajectories move consistently with velocity field (i.e. for $\nu = \alpha = \phi = 0$ \cite{14} conserves vorticity along the Lagrangian trajectories). For other values of $\lambda$ such simulations suffer the typical problem of advection in synthetic field (i.e. wrong reproduction of the sweeping effect, see \cite{14} for a discussion). Simulations for several values of $\lambda$ show that existence of the power-law tails of $\Psi(r_2|r_1)$ is a robust effect, as supposed by the model of Refs. \cite{14,17}, and that $Ps$ grows with $\lambda$. As an example, in Figure \cite{3} we plot the probability density $\Psi(r_2|r_1)$ obtained from a simulation with $\lambda = 0.5$. All the Eulerian parameters are the same of Figure \cite{2}. We again observe a clear power law tail but now with $Ps \approx 0.58$.

V. CONCLUSIONS

In this paper we have investigated the Lagrangian relative dispersion in direct numerical simulation of two-dimensional turbulence. The inverse energy cascade of two-dimensional turbulence displays Kolmogorov scaling without intermittency and it is thus the natural framework for investigating possible deviations from the classical Richardson picture. Moreover, the large enough values of the accessible Reynolds numbers make it possible to obtain the quantitative results not inferior to those of laboratory experiments.

The analysis of the numerical data was performed by using both standard statistics at fixed time and exit time statistics at fixed scale. The latter is shown to be more robust to finite Reynolds effects. An application of exit time statistics is developed for measuring the Richardson constant. The value found is somewhat larger than one obtained experimentally and in kinematic simulations, and thus nearer to the predictions of the closure theories.

The results of numerical simulation of statistics of the interparticle distances are in good agreement with the original Richardson’s description in terms of diffusion equation. The large deviations (with respect to Richardson theory) observed in the pdf of separation at fixed times are mostly related to crossover effects due to finite Reynolds numbers and disappear when looking at exit time statistics. The deviations found here are only of the quantitative nature. Thus, the Richardson’s equation gives a good basis for qualitative description of the dispersion in turbulent flows.

The deviations found in the fixed-scale statistics and also the strong sensitivity to the finite size effects can be associated with the long-range-correlated nature of the particles’ relative motion in turbulent flow. Paying attention to the turning points of the relative trajectories allows for estimating the effective persistence parameter of the motion.
which is found to be of the order of unity. Thus, the motion shows a strong ballistic component and is not purely diffusive. On the other hand, the correlations are not too strong to fully destroy the Richardson’s picture.

We note that the methodology of analysis proposed here based on the fixed-scale statistics and on the analysis of the relative trajectories can be also applied to the analysis of laboratory experiments. It would be extremely interesting to see, to what extent the laboratory flows reproduce the results reported here.

ACKNOWLEDGMENTS

We gratefully acknowledge the support of the DFG through SFB428, of the Fonds der Chemischen Industrie and of INFM (PRA-TURBO). We thank M. Cencini and J. Klafter for useful discussions. Numerical simulations were partially performed at CINECA within the INFM project “Fully developed two-dimensional turbulence”.

APPENDIX A: THE PDF OF DOUBLING TIMES

Let us discuss the probability \( p_D(t) \) of the time when the a pair of particles initially at distance \( R/\rho \) separates up to the distance \( R \), and obtain its asymptotic decay of this probability for \( t \) large.

Changing to a variable \( \xi = (k_0^2)^{-1/3} - 1/2 \rho^{1/3} \) reduces (2) to a radial part of a spherically symmetric diffusion equation in 3d dimensions with constant diffusion coefficient. In 2d one has

\[
\frac{\partial p}{\partial t} = \frac{1}{9\xi^5} \frac{\partial}{\partial \xi} \xi^5 \frac{\partial p}{\partial \xi}
\]

with the initial condition \( p(\xi, 0) = \delta(\xi_{\min} - \xi) \) with \( \xi_{\min} = (k_0^2)^{-1/3} - 1/2(R/\rho)^{1/3} \) and with the boundary condition \( p(\xi_{\max}, t) = 0 \), with \( \xi_{\max} = (k_0^2)^{-1/3} - 1/2R^{1/3} \).

The solution of a boundary-value problem for (A1) can be obtained by means of eigenfunction decomposition. Assuming the variable separation we get the solution in the form \( p(\xi, t) = \sum e^{-\lambda_i^2 t} \psi_i(\xi) \), where \( \psi_i(\xi) \) is an eigenfunction of the equation

\[
\frac{1}{9\xi^5} \frac{\partial}{\partial \xi} \xi^5 \frac{\partial \psi_i}{\partial \xi} = -\lambda_i^2 \psi_i
\]

satisfying the boundary condition \( \psi(\xi_{\max}) = 0 \). The corresponding solution which is nonsingular in zero is \( \psi_i = \xi^{-2} J_2(3\lambda_i \xi) \) (\( J_2 \) is the Bessel function [25]). The fact that \( \psi \) vanishes at \( \xi_{\max} \) gives \( 3\lambda_i \xi_{\max} = j_{2,i} \), where \( j_{2,i} \) is the \( i \)-the real zero of \( J_2(x) \). For example, the smallest eigenvalue is \( \lambda_1^2 = j_{2,1}^2 / 9\xi_{\max}^2 \approx 2.93k_0^{1/3}R^{-2/3} \). Thus, the long-time asymptotic of the doubling-time distribution is \( \exp(-2.93k_0^{1/3}R^{-2/3}t) \).

APPENDIX B: AVERAGE DOUBLING TIME

The mean doubling time can be obtained from a stationary solution of the Richardson diffusion equation. Imagine that one particle per unit time is introduced at \( r = R/\rho \) and there are respectively a reflecting and absorbing boundaries at \( r = 0 \) and \( r = R \). The stationary solution of (3) in 2d with the appropriate boundary conditions and continuity are \( R/\rho \) is

\[
p(r) = \begin{cases} 
C \left[ \rho^{4/3} - 1 \right] & \text{for } 0 < r < R/\rho \\
C \left[ (r)^{-4/3} - 1 \right] & \text{for } R/\rho < r < R
\end{cases}
\]

(B1)

The number of particle in \( r < R \) is

\[
N = \int_{|r|<R} p(r) \, dr = 2\pi \int_0^R rp(r) \, dr
\]

(B2)

where \( S_d \) is a surface of a unit sphere in \( d \) dimensions. By using (B1) one obtains

\[
N = 2\pi C(1 - \rho^{-2/3})R^d.
\]

(B3)
The current at $r = R$, i.e. the number of particle exiting from the boundary $R$ per unit time, is given, as in (10), as

$$J = -2\pi \frac{2}{3} \varepsilon^{1/3} k_0 R^{7/3} \frac{\partial \rho(r)}{\partial r} \bigg|_{r=R} = \frac{8\pi}{3} C \varepsilon^{1/3} k_0 R^{4/3}$$  \hspace{1cm} (B4)

The mean doubling time is the average time spent by a particle at $r < R$. It is given by the ratio $N/J$ and thus

$$\langle T_\rho(R) \rangle = \frac{3}{4} \frac{\rho^{2/3} - 1}{\epsilon^{1/3} k_0 \rho^{2/3}} R^{2/3}$$  \hspace{1cm} (B5)

which is Eq.(12).

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FIG. 1. Energy spectrum of the inverse cascade simulations $E(k)$. The dashed line has the Kolmogorov slope $-5/3$. In the inset it is shown the compensated third order longitudinal structure function $S_3(r)/(\varepsilon r)$. The dashed line is the prediction $3/2$.

FIG. 2. Relative dispersion $R^2(t)$ with $R(0) = \delta x/2$ (+) and $R(0) = \delta x$ (×). In the inset the compensated plot $R^2(t)/(\varepsilon t^3)$. 
FIG. 3. Probability distribution function of relative separations at times $t = 0.031$ (•) and $t = 0.77$ (×) rescaled with $R(t) = \langle r^2(t) \rangle^{1/2}$. The continuous line is the Richardson prediction [3], the dashed line is the Gaussian distribution.

FIG. 4. Right tail of $-\log(p(r,t)/p(0,t))$ at times $t = 0.015$ (+), $t = 0.041$ (×), $t = 0.067$ (•) and $t = 0.77$ (□) in log-log plot. The two lines represent the Richardson slope $2/3$ and the Gaussian slope $2$. The inset shows the exponent of the right tail of the pdf as a function of time.
FIG. 5. Mean doubling time $\langle T(R) \rangle$ as function of the separation $R$. The ratio is $r = 1.2$ and the average is obtained over about $5 \times 10^5$ events. The line represents the dimensional scaling $R^{2/3}$. In the inset the compensated plot gives the value of Richardson constant $g \simeq 3.8$, as explained in the text.

FIG. 6. Pdf of doubling times for distances $R = 0.003$ (+), $R = 0.075$ (×), $R = 0.02$ (*) and $R = 0.046$ (◻). The dashed line is the exponential $\exp(-0.3 T/\langle T \rangle)$.
FIG. 7. Probability density function of turning point ratio $\Psi(r_2/r_1)$. The exponent of the power law (dashed line) gives the value $Ps \simeq 0.87$.

FIG. 8. The same of Figure 7 but for $\lambda = 0.5$. The persistence parameter is now $Ps \simeq 0.58$. 