Exact quasi-normal modes for the near horizon Kerr metric

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We study the quasi-normal modes of a massless scalar field in a general sub-extreme Kerr background by exploiting the hidden $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \times SO(3)$ symmetry of the subtracted geometry approximation. This faithfully models the near horizon geometry but locates the black hole in a confining asymptotically conical box analogous to the anti-de-Sitter backgrounds used in string theory. There are just two series of modes, given in terms of hypergeometric functions and spherical harmonics, reminiscent of the left-moving and right-moving degrees in string theory: one is overdamped, the other is underdamped and exhibits rotational splitting. The remarkably simple exact formulae for the complex frequencies would in principle allow the determination of the mass and angular momentum from observations of a black hole. No black hole bomb is possible because the Killing field which co-rotates with the horizon is everywhere timelike outside the black hole.

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INTRODUCTION

Attempts to understand the quantum mechanics of black holes in String/M-theory have led to the construction of a large number of charged, rotating black hole solutions of supergravity theories. These theories admit several generalised Maxwell fields and in addition more than one scalar field. The standard neutral Kerr solutions used to model astrophysical black holes belongs to a family of such metrics\(^1\) which take the form\(^2\):

\[
ds^2 = -\Delta_0^{-1/2}G(dt + A)^2 + \Delta_0^{1/2}\left(\frac{dr^2}{X} + d\theta^2 + \frac{X}{G} \sin^2 \theta d\phi^2\right),
\]

with

\[
X = r^2 - 2mr + a^2,
\]

\[
G = r^2 - 2mr + a^2 \cos^2 \theta.
\]

(0.1)

Remarkably it has been found that for the entire family, the massless scalar wave equation is separable and the solutions expressible in terms of spheroidal functions of $\theta$\(^3\). The general form for $\Delta_0(r)$ and $A$ is complicated but in the astrophysically relevant Kerr case it simplifies:

\[
\Delta_0 = r^4 + 2r^2 a^2 \cos^2 \theta + a^4 \cos^4 \theta,
\]

\[
A = \frac{2mar \sin^2 \theta}{G} d\phi.
\]

(0.3)

and the radial function may be expressed in terms of solutions of a confluent form of Heun’s equation which has two regular singular points and an irregular singular point at infinity.

In \(^4\) it was discovered, in the much more general context of these multi-charged black hole solutions\(^2\) of the so-called STU model, that making the replacement:

\[
\Delta_0 \rightarrow \Delta = (2m)^2 \left(\Pi_c^2 - \Pi_s^2\right) + (2m)^2 \Pi_s^2 - \left((2m)^2 (\Pi_c - \Pi_s)^2 a^2 \cos^2 \theta\right)
\]

(0.4)

in \(^0\) hence reducing the highest power of $r$ in $\Delta_0$ renders the irregular singular point at infinity regular, allowing solutions in terms of hypergeometric functions. Moreover the massless scalar wave equation is now separable in terms of ordinary spherical harmonics, rather than the complicated spheroidal functions needed for the Kerr solution. In addition, while the so-called “subtracted metric” remains non-spherically symmetric, the massless scalar wave equation exhibits a hidden $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \times SO(3)$ symmetry. Furthermore the areas of the outer and inner horizons, their angular velocities $\Omega_\pm$ and their surface gravities $\kappa_\pm$ are unchanged by this replacement, preserving the local geometry and thermodynamic properties of the metric. Thus

\(^1\) We set Newton’s constant and the speed of light equal 1.

\(^2\) In \(^0\), $\Pi_c = \Pi_{i=1}^4 \cosh \delta_i$ and $\Pi_s = \Pi_{i=1}^4 \sinh \delta_i$, where the four boost parameters $\delta_i$ parameterize the four charges $Q_i = 2m \sinh \delta_i \cosh \delta_i$ of the original four-charge rotating solution.
in the special case of the Kerr solution the associated subtracted metric faithfully models the near horizon environment of a general neutral non-extreme Kerr black hole. The aim of this letter is to exploit this fact in order to study analytically its quasi-normal modes in this near horizon approximation.

This enormous simplification was however bought at the cost that the subtracted metric is no longer asymptotically flat but rather asymptotically conical:

\[ ds^2 \approx -\left( \frac{R}{R_0} \right)^6 dt^2 + 16dR^2 + R^2(d\theta^2 + \sin^2 \theta d\phi^2) , \quad (0.5) \]

where \( R = (8m^3r)^{\frac{1}{4}} \), \( R_0 = (2m^3)^{\frac{1}{4}} \) and \(-g_{tt}\) increases as radial distance \( 4R \) to the power six corresponding to a logarithmically increasing gravitational potential. This suggests that the subtracted metric describes a black hole confined in a box in a way which is analogous to the behaviour of a Kerr-anti-De-Sitter black hole. One expects the subtracted geometry to be an excellent approximation to the exact Kerr solution near the horizon but clearly deviates from it considerably at large distances. Except in the purely radially directed case, the asymptotic conical admits no null geodesics which reach infinity and so it is best viewed as a good approximation well inside \( r = 3m \), the radius of the photon sphere in the Schwarzschild case.

The massless scalar wave equation in the Kerr-Anti-De-Sitter background has been the subject of enormous interest of late because of its relevance to the AdS/CFT correspondence. Of particular concern has been the behaviour of quasi-normal modes and stability issues. In the Kerr-Anti-De-Sitter case it has been found that for a suitable range of parameters the system is unstable, behaving like a “black hole bomb”. Of necessity these investigations have been numerical since neither the radial functions nor the spheroidal functions admit simple analytic expressions.

By contrast for the subtracted Kerr metric, the quasi-normal modes may be found exactly using standard properties of spherical harmonics and hypergeometric functions. We find, using the results of [2, 4], that there are two discrete series of solutions, each member being labelled by three integers \( l = 0, 1, \ldots, n = \pm l, \pm(l - 1), \ldots \) and \( N_R, N_L = 0, 1, \ldots \), and having time dependence

\[ e^{-\frac{i}{4m}(l+1)\phi} \quad \text{or} \quad e^{-\frac{i}{4m}(l+1)\phi - \frac{1}{2m}\sqrt{\Delta}(l+1+N_L)\phi} e^{-in\frac{\phi}{2m\sigma}t} . \quad (0.6) \]

Both sets of modes decay exponentially in amplitude: one is overdamped, the other is underdamped and exhibits rotational splitting. The absence of an instability due to super-radiant behaviour may be understood as a consequence of the fact that the Killing vector \( t^+ = \partial_t + \Omega_+ \partial_\phi \), which coincides on the horizon with its null generator, is timelike everywhere outside the horizon. By a result of Hawking and Reall, this is sufficient to preclude super-radiance instabilities.

As expected, the damped oscillatory series of modes exhibit rotational splitting, i.e. dependence on the angular quantum number \( n \), due to gravito-magnetic or frame dragging effects analogous to Zeeman splitting in atomic physics or the free oscillations of the earth. The extraordinarily simple form of allows in principle their use in determining the angular momentum \( a \) and the \( m \) from observational data.

In what follows we shall provide some further details of our results.

THE METRIC

For Kerr solution one has to set in \( \Pi_c = \Pi_{i=1} \cosh \delta_i = 1 \) and \( \Pi_\pm = \Pi_{i=1} \sinh \delta_i = 0 \), i.e. all \( \delta_i = 0 \). The subtracted Kerr metric factor is

\[ \Delta = (2m)^3r - (2m)^2a^2\cos^2 \theta . \quad (0.7) \]

At the inner and outer horizon:

\[ r_\pm = m \pm \sqrt{m^2 - a^2} , \quad (0.8) \]

the angular velocities and surface gravities:

\[ \Omega_\pm = \frac{a}{2m(m \pm \sqrt{m^2 - a^2})} , \quad \kappa_\pm = \frac{\sqrt{m^2 - a^2}}{2m(m \pm \sqrt{m^2 - a^2})} , \quad (0.9) \]

remain the same as for the Kerr black hole. Note that

\[ \frac{\Omega_+}{\kappa_+} = \frac{\Omega_-}{\kappa_-} = \frac{a}{\sqrt{m^2 - a^2}} . \quad (0.10) \]

The subtracted Kerr metric itself can be cast in the following remarkable form:

\[ ds^2 = \sqrt{\Delta} \frac{X}{F^2} (-dt^2 + \frac{F^2 d\theta^2}{X^2}) + \sqrt{\Delta} \frac{F^2 \sin^2 \theta}{\Delta} (d\phi + W dt)^2 , \quad (0.11) \]

where

\[ X = r^2 - 2mr + a^2 , \quad W = -\frac{2m}{F^2} , \quad F = (2m)^3r - (2m)^2a^2 , \quad (0.12) \]

depend only on the radial coordinate \( r \). Note, \( W(r_\pm) = -\Omega_\pm \) and \( F(r_\pm) = \frac{r_\pm - r}{2\kappa_\pm} \).

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3 For a different approach to the massless wave equation in the Kerr background, valid at wave lengths large compared to the horizon scale, see [3].

4 This is a general property of the multi-charged STU black hole solutions, observed already in [2, 6].
KRUSKAL-SZEKERES COORDINATES

We now construct Kruskal-Szekeres type coordinates to cover the outer horizon which allow us to identify suitable boundary conditions there. At infinity the appropriate boundary condition is boundedness of the solution. The construction of Kruskal-Szekeres coordinates is in fact considerably simpler than that used for the Kerr solution [12, 13].

Eddington-Finkelstein null coordinates $u, v$ take the form

$$u = t - r^*,$$  $v = t + r^* \tag{0.13}$$

and satisfy

$$g^{\alpha \beta} \partial_\alpha u \partial_\beta u = 0 = g^{\alpha \beta} \partial_\alpha v \partial_\beta v. \tag{0.14}$$

Using the fact that the Hamilton-Jacobi equation is separable, we find that

$$r^* = \int \frac{F dr}{X}, \tag{0.15}$$

and we define an ingoing angular coordinate

$$\phi_+ = \phi - \Omega_+ t, \tag{0.16}$$

which is constant along the orbits of the co-rotating Killing vector $l^+$

$$l^+ \phi_+ = (\partial_t + \Omega_+ \partial_\phi) \phi_+ = 0. \tag{0.17}$$

Note that the above results are manifest for the metric written as \([11, 11]\).

Kruskal-Szekeres coordinates

$$U = -e^{-\kappa_+ u}, \quad V = e^{\kappa_+ v}, \tag{0.18}$$

imply

$$\frac{2\kappa_+ F dr}{X} = dUU^{-1} + dVV^{-1}, 2\kappa_+ dt = dVV^{-1} - dUU^{-1}. \tag{0.19}$$

In the vicinity of outer horizon, $r \sim r_+$, one obtains $-UV \sim (r - r_+)$, where we have employed $F \sim \frac{r - r_+}{2\kappa_+}$. One can then check explicitly that in coordinates $(U, V, \phi_+, \theta)$, the metric \([11, 11]\) is non-singular and analytic. Moreover, the length-square of the co-rotating Killing vector:

$$g^{\alpha \beta} l^+_\alpha l^+_\beta = -\Delta^2 \left[ X + \frac{a^2 \sin^2 \theta (r_+ - r)(r - r_+)}{r_+^2} \right], \tag{0.20}$$

is manifestly negative for all $r > r_+$.

\[\text{\textsuperscript{5}}\] One can analogously construct Kruskal-Szekeres type coordinates to cover the inner horizon region.

SOLUTIONS OF THE WAVE EQUATION

The separated solutions of the massless scalar wave equation are of the form $e^{-i\omega t} e^{i\nu P_+^+(\theta)} \chi(x)$ where $P^+_l(\theta)$ is an associated Legendre polynomial and \([2, 3]\)

$$x = \frac{r - \frac{1}{2}(r_+ + r_-)}{r_+ - r_-}, \tag{0.21}$$

designed so that the two horizons $r_\pm$ are at $x = \pm \frac{1}{2}$. The overall scale of the black hole is set by $r_+ + r_- = 2m$. The radial wave equation takes the form \([4]\)

$$-\frac{1}{4(1 + \frac{r}{2})} \left( \frac{\omega}{\kappa_+} - \frac{\Omega_+}{\kappa_+} \right)^2 - l(l + 1) \chi(x) = 0. \tag{0.22}$$

Note that in the equation we already employed the symmetry of the ratios \([11, 11]\).

Solutions which are ingoing on the future horizon must be regular at $U = 0$ in Kruskal-Szekeres coordinates and this implies that \([2, 3]\)

$$\chi(x) = (x + \frac{1}{2} - (l + 1))^{\frac{\beta H}{2\pi}} e^{\frac{2nH\left(\omega - n\Omega_+\right)}{2\pi}} \left[ 1 - i \beta H \left( \omega \left( \frac{-n\Omega_+}{2\pi} \right) \right) x + \frac{1}{2} \left( \frac{\omega}{\kappa_+} - \frac{\Omega_+}{\kappa_+} \right) \right], \tag{0.23}$$

where

$$\beta H = \frac{1}{2\pi} \frac{\beta R}{\kappa_+} = \frac{1}{2\pi} \frac{\beta L}{\kappa_+} = \frac{1}{2\pi} \frac{1}{\kappa_+ - \kappa_-.} \tag{0.24}$$

Near the outer horizon $r^* \rightarrow -\infty$, $x \rightarrow \frac{1}{2}$

$$\chi(x) \approx e^{-i(\omega - n\Omega_+)(r^*)} \left( 1 + \ldots \right), \tag{0.25}$$

where the ellipsis denotes a power series in $UV$ which is convergent in a neighbourhood of the future horizon $U = 0$.

At large $x$ \([2, 3]\)

$$\chi(x) \approx (l + 1) \frac{\beta H}{2\pi} e^{i\beta H (\omega - n\Omega_+)} \Gamma(l + 1 - i \frac{2\omega R - 2\beta H \Omega_+}{2\pi}) \left( 1 + \ldots \right), \tag{0.26}$$

and so

$$\chi(x) \approx \left( l + 1 \right)^{-1} \frac{\beta H}{2\pi} e^{i\beta H (\omega - n\Omega_+)} \Gamma(l + 1 - i \frac{2\omega R - 2\beta H \Omega_+}{2\pi}) \left( 1 + \ldots \right). \tag{0.27}$$
In order that $\chi(x)$ be finite at large $x$, we must set
\begin{align}
 i\omega \frac{\beta_L}{2\pi} & = l + 1 + N_L, \\
 or \quad i\omega \frac{\beta_R - 2n\beta H\Omega}{2\pi} & = l + 1 + N_R, \quad (0.28)
\end{align}
where $N_{L,R} = 0, 1, \ldots$ This gives remarkably simple formulae for the frequencies of the quasi-normal modes:
\begin{align}
 \omega & = -\frac{i}{4m}(1 + l + N_L), \\
 or \quad \omega & = -\frac{i\sqrt{m^2 - a^2}}{4m^2} (1 + l + N_R) + \frac{a}{2m^2} n. \quad (0.29)
\end{align}
Both frequencies result in damped modes, with the underdamped branch exhibiting oscillatory behavior and the damping absent in the extremal limit $a \rightarrow m$. The specific asymmetry in the frequencies of the two branches is due to (0.10). The two branches of modes are reminiscent of the left-moving and right-moving modes in string theory, thus potentially relating this subtracted geometry microscopically to a dual two-dimensional conformal field theory description \[4\] and hence the reference to $L$ and $R$ in the preceding formulae.

**CONCLUSION AND FUTURE PROSPECTS**

In this letter we have obtained exact formulae for the quasi-normal modes of the Kerr solution in the subtracted geometry approximation. It would be interesting to compare our results with numerical calculations of the massless wave equation for initial date with support confined to a small neighbourhood of the horizon.

The subtracted geometry is a solution \[6\] of the equations of motion for the Lagrangian of $N=2$ supergravity coupled to three vector super-multiplets, often referred to as the STU model. Its lift to five-dimensions corresponds to $AdS_3 \times S^2$ \[4, 6\]. By taking a scaling limit of Melvin STU black holes in this model, a subtracted geometry with the magnetic field parameter $\beta$ for the Kaluza-Klein gauge field was obtained \[14\]. The lift of this geometry again results in $AdS_3 \times S^2$ where the azimuthal angle is shifted by a coordinate transformation in the circle direction $z$ as $\phi \rightarrow \phi - \beta z$. As a consequence, the wave equation for the massless minimally coupled scalar in this background separates and allows for an explicit analysis of the quasi-normal modes. Details will be presented elsewhere \[15\].

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