Thermodynamics of Fractal Universe

Ahmad Sheykhi\textsuperscript{1,2}\textsuperscript{*}, Zeinab Teimoori \textsuperscript{3} and Bin Wang \textsuperscript{4}\textsuperscript{†}

\textsuperscript{1} Physics Department and Biruni Observatory, College of Sciences, Shiraz University, Shiraz 71454, Iran
\textsuperscript{2} Research Institute for Astronomy and Astrophysics of Maragha (RIAAM), P.O. Box 55134-441, Maragha, Iran
\textsuperscript{3} Department of Physics, Shahid Bahonar University, P.O. Box 76175, Kerman, Iran
\textsuperscript{4} INPAC and Department of Physics, Shanghai Jiao Tong University, Shanghai 200240, China

We investigate the thermodynamical properties of the apparent horizon in a fractal universe. We find that one can always rewrite the Friedmann equation of the fractal universe in the form of the entropy balance relation \( \delta Q = T_h \delta S_h \), where \( \delta Q \) and \( T_h \) are the energy flux and Unruh temperature seen by an accelerated observer just inside the apparent horizon. We find that the entropy \( S_h \) consists two terms, the first one which obeys the usual area law and the second part which is the entropy production term due to nonequilibrium thermodynamics of fractal universe. This shows that in a fractal universe, a treatment with nonequilibrium thermodynamics of spacetime may be needed. We also study the generalized second law of thermodynamics in the framework of fractal universe. When the temperature of the apparent horizon and the matter fields inside the horizon are equal, i.e. \( T = T_h \), the generalized second law of thermodynamics can be fulfilled provided the deceleration and the state parameters ranges either as \(-1 \leq q < 0, -1 \leq \omega < -1/3\) or as \( q < -1, \omega < -1\) which are consistent with recent observations. We also find that for \( T_h = bT \), with \( b < 1\), the GSL of thermodynamics can be secured in a fractal universe by suitably choosing the fractal parameter \( \beta \).

I. INTRODUCTION

Nowadays, it is a general belief that there is a deep connection between thermodynamics and gravity. The story started with the discovery of black holes thermodynamics in 1970's by Hawking and Bekenstein \textsuperscript{[1-4]}. According to their discovery, a black hole can be regarded as a thermodynamic system, with temperature and entropy proportional to its surface gravity and horizon area, respectively. After that, people were speculating that maybe there is a direct connection between thermodynamics and Einstein equation, a hyperbolic second order partial differential equation for the spacetime metric. In 1995, Jacobson \textsuperscript{2} was indeed able to derive the Einstein equation from the requirement that the Clausius relation \( \delta Q = T \delta S \) holds for all local acceleration horizons through each spacetime point, where \( \delta S \) is one-quarter the horizon area change in Planck units and \( \delta Q \) and \( T \) are the energy flux across the horizon and the Unruh temperature seen by an accelerating observer just inside the horizon. Jacobson’s derivation of the Einstein field equation from thermodynamics opened a new window for understanding the thermodynamic nature of gravity. After Jacobson, a lot of works have been done to disclose the profound connection between gravity and thermodynamics. It was shown that the gravitational field equations in a wide range of theories, can be rewritten in the form of the first law of thermodynamics and vice versa \textsuperscript{[6-14]}. The studies were also generalized to the cosmological setup, where it was shown that the differential form of the Friedmann equation in the Friedmann-Robertson-Walker (FRW) universe can be transformed to the first law of thermodynamics on the apparent horizon \textsuperscript{[15-26]}. On the other side, the second law of black hole mechanics expresses that the total area of the event horizon of any collection of classical black holes can never decrease, even if they collide and swallow each other. This is remarkably similar to the second law of thermodynamics where the area is playing the role of entropy. Note that the second law of black hole thermodynamics can be violated if one take into account the quantum effect, such as the Hawking radiation. To overcome this difficulty, Bekenstein \textsuperscript{[2, 4]} introduced the so-called total entropy \( S_{\text{tot}} \) which is defined as

\[ S_{\text{tot}} = S_h + S_m, \]  

where \( S_h \) and \( S_m \) are, respectively, the black hole entropy and the entropy of the surrounding matter. According to Bekenstein’s argument, in general, the total entropy should be a non decreasing function. This statement is known as the generalized second law (GSL) of thermodynamics,

\[ \Delta S_{\text{tot}} \geq 0. \]  

Besides, if thermodynamical interpretation of gravity near the apparent horizon is a generic feature, one needs to verify whether the results may hold not only for more general spacetimes but also for the other principles of thermodynamics, especially for the GSL of thermodynamics. The GSL of thermodynamics is a universal principle governing the evolution of the universe. It was argued that in the accelerating universe the GSL is valid provided the boundary of the universe is chosen the apparent horizon \textsuperscript{[27-31]}. In this paper, we would like to extend the study to the fractal universe. Fractal cosmology was recently pro-
posed by Calcagni 32, 33 for a power-counting renormalizable field theory living in a fractal spacetime. It is interesting to see whether the Friedmann equation of a fractal universe can be written in the form of the first law of thermodynamics. As we will see, in a fractal universe, the Friedmann equation can be transformed to Clausius relation, but a treatment with nonequilibrium thermodynamics of spacetime is needed.

In the next section we review the basic equations in the framework of fractal cosmology. In section III, we show that the Friedmann equation of a fractal universe can be written in the form of the fundamental relation δQ = T_h d S_h, where δQ and T_h are, respectively, the energy flux and Unruh temperature seen by an accelerated observer just inside the apparent horizon. In section IV, we check the validity of the GSL of thermodynamics for a fractal cosmology. The last section is devoted to some concluding remarks.

II. FRACTAL UNIVERSE

The total action of Einstein gravity in a fractal spacetime is given by 32, 33

\[ S = S_G + S_m, \]  

(3)

where the gravitational part of the action is given by

\[ S_G = \frac{1}{16\pi G} \int d^x g(R - 2\Lambda - \omega \partial_{\mu} v \partial^{\mu} v), \]  

(4)

and the matter part of the action is

\[ S_m = \int d^x g(x) \sqrt{-g} L_m. \]  

(5)

Here g is the determinant of the dimensionless metric g_{\mu\nu}. Λ and R are, respectively, the cosmological constant and Ricci scalar. v is the fractional function and ω is the fractal parameter. The standard measure d^4x replaced with a Lebesgue-Stieltjes measure dg(x). The derivation of the Einstein equations goes almost like in scalar-tensor models. Taking the variation of the action 33 with respect to the FRW metric g_{\mu\nu}, one can obtain the Friedmann equations in a fractal universe as 33

\[ H^2 + \frac{k}{a^2} + \frac{\dot{H}}{H} - \omega \frac{v^2}{6} = \frac{8\pi G}{3} \rho + \frac{\Lambda}{3}, \]  

(6)

\[ \dot{H} + H^2 - H \frac{\dot{v}}{v} - \omega \frac{v}{3} \frac{\Box v}{v} = \frac{8\pi G}{6} (\rho + 3p) + \frac{\Lambda}{3}, \]  

(7)

where \( H = \dot{a}/a \) is the Hubble parameter, ρ and p are the total energy density and pressure of the ideal fluid composing the universe, respectively. The curvature constant k = 0, 1, -1 corresponding to a flat, closed and open universe, respectively. The continuity equation in a fractal universe takes the form 33

\[ \dot{\rho} + \left( 3H + \frac{\dot{v}}{v} \right) (\rho + p) = 0. \]  

(8)

It is clear that for v = 1, the standard Friedmann equations are recovered. We further assume that only the time direction is fractal, while spatial slices have usual geometry. Indeed, in the framework of fractal cosmology, classically fractals can be timelike \([v = v(t)]\) or even spacelike \([v = v(x)]\) (see Ref. 33 for details). These two cases lead to different classical physics, but at quantum level all configurations should be taken into account, so there is no quantum analogue of space or timelike fractals. In this paper we take a timelike fractal. Thus, those parameters that depend on time change and those parts that related to x remain fixed.

Assuming a timelike fractal profile \( v = t^{-\beta} \) 32, where \( \beta = 4(1 - \alpha) \) is the fractal dimension, the Friedmann equations (6) and (7) in the absence of the cosmological constant can be written as

\[ H^2 + \frac{k}{a^2} - \frac{\beta}{t} H - \frac{\omega \beta^2}{6(2\beta+1)} = \frac{8\pi G}{3} \rho, \]  

(9)

\[ \dot{H} + H^2 - \frac{\beta}{2t} H - \frac{\beta(\beta + 1)}{2t^2} + \frac{\omega \beta^2}{3(2\beta+1)} = -\frac{8\pi G}{6} (\rho + 3p), \]  

(10)

while, the continuity equation (8) takes the form

\[ \dot{\rho} + \left( 3H - \frac{\beta}{t} \right) (\rho + p) = 0. \]  

(11)

From the definition of the fractional integral 33, 34, we know that α ranges as 0 < α ≤ 1. Thus for α = 1, we obtain β = 0 which physically means that the universe does not have any fractal structure and one can recovers the well-known Friedmann equations in standard cosmology. As one can see from Friedmann equations (6) and (10), we have no limit \( t \to 0 \) for a timelike fractal profile, since in this case the Friedmann equations diverge unless \( \beta = 0 \). This implies that at the early stages of the universe, we could not have the timelike fractal structure.

In the remaining part of this paper we show that the differential form of the Friedmann equation (6) can be written in the form of the fundamental relation δQ = T_h d S_h, where S_h is the entropy associated with the apparent horizon. We also investigate the validity of the GSL of thermodynamics for the fractal universe surrounded by the apparent horizon.

III. FIRST LAW OF THERMODYNAMICS IN FRACTAL COSMOLOGY

For a homogeneous and isotropic FRW universe the line elements can be written

\[ ds^2 = h_{\mu\nu} dx^\mu dx^\nu + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \]  

(12)

where \( r = a(t)r, \) \( x^0 = t, x^1 = r, \) and \( h_{\mu\nu} = \text{diag} (-1, a^2/(1 - kr^2)) \) is the two dimensional metric. The
dynamical apparent horizon, a marginally trapped surface with vanishing expansion, is determined by the relation $h^\mu\nu \partial_\mu \tilde{r} \partial_\nu \tilde{r} = 0$. Straightforward calculation gives the apparent horizon radius for the FRW universe as
\begin{equation}
\tilde{r}_A = \frac{1}{\sqrt{H^2 + k/a^2}}.
\end{equation}
The associated temperature $T$ with the apparent horizon is given by
\begin{equation}
T_h = \frac{1}{2\pi \tilde{r}_A},
\end{equation}
where $A = 4\pi \tilde{r}_A^3$ is the apparent horizon area and we have assumed the apparent horizon radius is fixed. We shall assume the matter source in the fractal universe has a perfect fluid form with stress-energy tensor
\begin{equation}
T_{\mu \nu} = (\rho + p) u_\mu u_\nu + pg_{\mu \nu},
\end{equation}
where $\rho$ and $p$ are the energy density and pressure, respectively. Assuming the total energy inside the apparent horizon is given by $E = \rho V$, where $V = \frac{4}{3} \pi \tilde{r}_A^3$ is the volume of the 3-sphere of radius $\tilde{r}_A$. Taking differential form of energy, we can obtain the energy flux $dE = V dp$. Here we have assumed the volume enveloped by the apparent horizon is fixed during the infinitesimal internal of time $dt$. Thus $-dE$ is actually just the heat flux $\delta Q$ in crossing the apparent horizon within an infinitesimal internal of time $dt$, it is not the change in the matter energy inside the apparent horizon due to the volume change, so there is no term of volume change. Hence we can write
\begin{equation}
dE = V \dot{p} dt.
\end{equation}

Differentiating Eq. (13) with respect to the cosmic time $t$, we obtain
\begin{equation}
\dot{\tilde{r}}_A = -H \left( \frac{H}{a^2} \right) \tilde{r}_A^3.
\end{equation}
Substituting Eq. (20) into (19) we can rewrite it as
\begin{equation}
\frac{1}{G} \left[ d\tilde{r}_A - \beta \frac{H}{2t^2} \frac{\dot{H}}{2t} \tilde{r}_A^3 dt - \frac{\omega}{6} \frac{\beta^2 (\beta + 1)}{t^{2\beta + 3}} \tilde{r}_A^3 dt \right] = 4\pi H (\rho + p) \tilde{r}_A^3 dt
\end{equation}
Combining Eq. (17) with (21) and using the fact that $\delta Q = -dE$ is just the energy flux crossing through the apparent horizon, we have
\begin{equation}
\delta Q = \frac{1}{G} \left[ d\tilde{r}_A - \beta \frac{H}{2t^2} \frac{\dot{H}}{2t} \tilde{r}_A^3 dt - \frac{\omega}{6} \frac{\beta^2 (\beta + 1)}{t^{2\beta + 3}} \tilde{r}_A^3 dt \right].
\end{equation}

Using definition (12) for the temperature, one can see that Eq. (22) is just the entropy balance relation,
\begin{equation}
\delta Q = T_h dS_h,
\end{equation}
provided we define
\begin{equation}
dS_h = \frac{2\pi \tilde{r}_A d\tilde{r}_A}{G} + \frac{2\pi \beta}{G} \left[ \frac{\dot{H}}{2t - \frac{H}{2t^2}} - \frac{\omega}{6} \frac{\beta (\beta + 1)}{t^{2\beta + 3}} \right] \tilde{r}_A^3 dt.
\end{equation}
Integrating, we find
\begin{equation}
S_h = \frac{A}{4G} + \frac{2\pi \beta}{G} \int \left[ \frac{\dot{H}}{2t - \frac{H}{2t^2}} - \frac{\omega}{6} \frac{\beta (\beta + 1)}{t^{2\beta + 3}} \right] \tilde{r}_A^3 dt.
\end{equation}
As one can see, in a fractal universe, the entropy $S_h$ associated with the apparent horizon consists two parts, the first one obeys the usual area law and the second part is the entropy term developed internally in the system as a result of being out of equilibrium \cite{019, 021}. The entropy production rate vanishes for standard cosmology where $\beta = 0$. It is worth mentioning that even in standard cosmology one can still have non-equilibrium thermodynamics depending on the assumptions \cite{021}.

IV. GSL OF THERMODYNAMICS IN FRACTAL UNIVERSE

In this section we investigate the validity of the GSL (19) of thermodynamics in a region enclosed by the apparent
horizon in the framework of the fractal universe. Let us put \( k = 0 \) for simplicity, so we have \( \dot{r}_A = 1/H \), \( \dot{H} = -\dot{r}_A/r_A^2 \). The total entropy associated with the apparent horizon, \( S_h \), can be written

\[
dS_h = \frac{2\pi}{G} \left[ \frac{\dot{r}_A d\dot{r}_A}{r_A^2} - \frac{\beta}{2\pi} \frac{\dot{r}_A^2 d\dot{r}_A}{r_A^3} dt - \frac{\omega \beta^2 (\beta + 1)}{6 \pi t^{2\beta+3}} \left( t^{2\beta+1} \right) r_A^4 dt \right].
\]

(27)

Dividing Eq. (27) by \( dt \), we arrive at

\[
\dot{S}_h = \frac{2\pi}{G} \left[ \frac{\dot{r}_A^2}{r_A^2} - \frac{\beta^2}{2\pi} \frac{\dot{r}_A^3}{r_A^3} - \frac{\omega \beta^2 (\beta + 1)}{6 \pi t^{2\beta+3}} \left( t^{2\beta+1} \right) r_A^4 \right].
\]

(28)

Eq. (28), can be written as

\[
\dot{S}_h = \frac{2\pi}{G} \dot{r}_A \left( 1 - \frac{\beta}{2\pi} \frac{\dot{r}_A}{r_A} - \frac{\beta^2}{2\pi} \frac{\dot{r}_A^2}{r_A^2} - \frac{\omega \beta^2 (\beta + 1)}{6 \pi t^{2\beta+3}} \left( t^{2\beta+1} \right) r_A^4 \right).
\]

(29)

The Friedmann equation (11), for a flat universe, becomes

\[
\frac{1}{r_A^2} - \frac{1}{t} \frac{1}{r_A} - \frac{\omega \beta^2}{6 \pi t^{2\beta+3}} = \frac{8\pi G}{3} \rho.
\]

(30)

Differentiating the above equation with respect to the cosmic time and using the continuity equation (11), after some simplification, we get

\[
\dot{\dot{r}}_A = \left( 1 - \frac{\beta}{2\pi} \frac{\dot{r}_A}{r_A} \right)^{-1} \frac{\dot{r}_A^3}{r_A^2} \left[ 4\pi G H (\rho + p) - \frac{4\pi G \beta}{3} (\rho + p) \right].
\]

(31)

Solving this equation for \( \dot{r}_A \), we find

\[
\dot{r}_A = \left( 1 - \frac{\beta}{2\pi} \frac{\dot{r}_A}{r_A} \right)^{-1} \frac{r_A^3}{\dot{r}_A} \left[ 4\pi G H (\rho + p) - \frac{4\pi G \beta}{3} (\rho + p) \right] - \frac{4\pi G \beta}{3} (\rho + p) + \frac{\beta}{2\pi} \frac{1}{r_A^2} \left[ \frac{\omega \beta^2 (\beta + 1)}{6 \pi t^{2\beta+3}} \left( t^{2\beta+1} \right) r_A^4 \right].
\]

(32)

Substituting \( \dot{r}_A \) from Eq. (32) into (29), we obtain

\[
\dot{S}_h = \frac{2\pi}{G} H^{-1} \left[ 4\pi G H (\rho + p) - \frac{4\pi G \beta}{3} (\rho + p) \right],
\]

(33)

which can also be rewritten in the following form

\[
\dot{S}_h = 8\pi^2 H^{-3} (\rho + p) \left( 1 - \frac{\beta}{3Ht} \right),
\]

(34)

where we have used \( \dot{r}_A = H^{-1} \) for the flat universe. Let us discuss the two cases in which \( \dot{\dot{S}}_h \geq 0 \). In the first case we assume the dominant energy condition valid, \( \rho + p \geq 0 \), therefore \( \dot{S}_h \geq 0 \), provided \( \beta \leq 3Ht \). However, in an accelerating universe the dominant energy condition may violate, \( \rho + p < 0 \). In this case \( \dot{S}_h \geq 0 \) provided \( \beta \geq 3Ht \). However, as we will see below, the GSL can be still fulfilled in an accelerating fractal universe.

For latter convenience we also calculate \( T_h \dot{S}_h \),

\[
T_h \dot{S}_h = 4\pi H^{-2} (\rho + p) \left( 1 - \frac{\beta}{3Ht} \right).
\]

(35)

Next, we study the GSL of thermodynamics, namely the time evolution of the total entropy including the entropy \( S_h \) associated with the apparent horizon together with the matter field entropy \( S_m \) inside the apparent horizon. The entropy of the universe inside the horizon can be related to its energy and pressure in the horizon by the Gibbs equation [36]

\[
T dS_m = d(\rho V) + pdV = V d\rho + (\rho + p)dV.
\]

(36)

We assume the temperature of the perfect fluid inside the apparent horizon scales as the temperature of the horizon, which for flat universe is \( T_h = H/(2\pi) \). In general, if the temperature of the horizon differs much from that the fluid, then the energy would spontaneously flow between the horizon and the fluid, something at variance with FRW universe [34, 37]. Thus we suppose that the temperature \( T_h \) associated with the apparent horizon is \( T_h = bT_h \) [28], where \( b \) is a real proportional constant. Since at the present time the horizon temperature is lower than that of the CMB by many orders of magnitude, we will not consider the case \( b > 1 \). We limit ourselves to the assumption of the local equilibrium hypothesis, that the energy would not spontaneously flow between the horizon and the fluid. Indeed, this will certainly be the situation at late times, that is when the universe fluids and the horizon will have interacted for a long time, it is ambiguous if it will be the case at early or intermediate times [38]. However, in order to avoid nonequilibrium thermodynamical calculations, which would lead to lack of mathematical simplicity and generality, the assumption of equilibrium, although restricting, has widely accepted for studying the GSL in the literature [38]. Thus, we follow this assumption and we notice that our results are valid only at the late stages of the universe evolution where the universe fluids and the horizon will interact for a long time.

Let us first consider the case where \( b = 1 \). Physically, this means that we have assumed during the infinitesimal internal time \( dt \), the temperature of the perfect fluid is equal to the temperature \( T_h \) associated with the apparent horizon. Therefore, from Gibbs equation, after using the continuity equation (11), we get

\[
T \dot{S}_m = -4\pi H^{-2} (\rho + p) \left( 1 - \frac{\beta}{3Ht} \right) + 4\pi H^{-2} (\rho + p) \dot{r}_A.
\]

(37)

To check the GSL of thermodynamics, we have to examine the evolution of the total entropy \( S_h + S_m \). Adding
the case the horizon temperature is lower than that of the state parameter crosses the phantom line, the GSL is again fulfilled, but in this case the equation of accelerated expansion. For an accelerating universe we have $q < 1$, the GSL can be fulfilled if $q < -b^{-1}$. In an accelerating universe $q < 0$, and hence the condition for keeping the GSL in a fractal universe reduced to $-b^{-1} < q < 0$.

On the other hand, if equation of state parameter crosses the phantom line, which some observational data support it, then we have $w < -1$. In this case, the GSL holds provided

$$\beta(1-b) > 3Ht(1+ bq).$$  

For $b < 1$, the GSL can be fulfilled if

$$\beta > 3Ht \left( \frac{1+bq}{1-b} \right).$$

We conclude that the requirement of the GSL of thermodynamics in a fractal universe leads to constraint on the fractal dimension parameter $\beta$. It is worth noting that our local equilibrium hypothesis, which we used for checking the GSL, only valid at the late time. Thus, the result obtained in this section have no $t \to 0$ limit.

Finally, it is instructive to discuss the sign of the entropy and temperature, when the universe lies in the phantom phase ($\omega < -1$). A lot of works have been done in the literature, showing that in the absence of chemical potential in phantom regime, the temperature must be negative, while the energy density and the entropy should be positive. In [43] it was found that the phantom temperature is positive and its entropy negative. Finally, in [44] it was shown that with an arbitrary chemical potential the density and entropy are always positive, while the temperature of a phantom universe ($w < -1$) is negative, and that of quintessence universe ($w > -1$) is positive. The temperature negativity can only be interpreted in the quantum framework [43]. In [39, 40] it was found that the phantom temperature is positive and its entropy negative. Finally, in [44, 45] it was argued that one can describe the phantom universe either with negative temperature and positive entropy, or with negative entropy and positive temperature. Since the horizon temperature is always positive, it is deduced that the universe temperature will be positive even if it lies in the phantom phase. Thus we should have a negative universe entropy in this case. Although the total entropy is always positive. Because negative entropy of the universe ingredients is overcome by the positive horizon entropy.

\section{V. CONCLUDING REMARKS}

In summary, in this paper we studied thermodynamics of the apparent horizon in the framework of fractal cosmology. We found that the Friedmann equation of a fractal universe can be transformed to the fundamental relation $\delta Q = T_h dS_h$ on the apparent horizon, where $\delta Q$ and $T_h$ are the energy flux and Unruh temperature.
seen by an accelerated observer just inside the apparent horizon. We showed that the entropy $S_h$ consists two terms, the first one which obeys the usual area law and the second part which is the entropy production term and appears due to the nonequilibrium thermodynamics of the fractal universe. This indicates that in a fractal universe, a treatment with nonequilibrium thermodynamics of spacetime maybe needed. We also investigated the time evolution of the total entropy including the entropy $S_m$ inside the apparent horizon. We assumed the temperature of the apparent horizon is proportional to the matter fields temperature inside the horizon, i.e. $T_h = bT$. We studied several cases including $b < 1$ and $b = 1$, in which the GSL of thermodynamics can be secured in a fractal universe. Interestingly enough, we found that for an accelerating fractal universe the GSL can be preserved, at least for the late times where the local equilibrium hypothesis holds. The fulfillment of the GSL of thermodynamics in an accelerating fractal universe leads also to constraints on the fractal dimension parameter $\beta$.

Acknowledgments

We thank the anonymous referee for constructive comments which helped us to improve the paper significantly. This work has been supported financially by Research Institute for Astronomy and Astrophysics of Maragha (RI-AAM) and also by Shiraz University Research Council.

[1] J. M. Bardeen, B. Carter and S.W. Hawking, Commun. Math. Phys. 31, 161 (1973).
[2] J. D. Bekenstein, Phys. Rev. D 7, 2333 (1973).
[3] S. W. Hawking, Nature 248, 30 (1974).
[4] J. D. Bekenstein, Phys. Rev. D 9, 12 (1974).
[5] T. Jacobson, Phys. Rev. Lett. 75, 1260 (1995).
[6] C. Eling, R. Guedens, and T. Jacobson, Phys. Rev. Lett. 96, 121301 (2006).
[7] M. Akbar and R. G. Cai, Phys. Lett. B 635, 7 (2006).
[8] M. Akbar and R. G. Cai, Phys. Lett. B 648, 243 (2007).
[9] T. Padmanabhan, Class. Quantum Grav 19, 5387 (2002).
[10] T. Padmanabhan, Phys. Rep. 406, 49 (2005).
[11] T. Padmanabhan, Int. J. Mod. Phys. D 15, 1659 (2006).
[12] A. Paranjape, S. Sarkar and T. Padmanabhan, Phys. Rev. D 74, 104015 (2006).
[13] D. Kothawala, S. Sarkar and T. Padmanabhan, Phys. Lett. B 652, 338 (2007).
[14] T. Padmanabhan and A. Paranjape, Phys. Rev. D 75, 064004 (2007).
[15] M. Akbar and R. G. Cai, Phys. Rev. D 75, 084003 (2007).
[16] R. G. Cai and L. M. Cao, Phys. Rev. D 75, 064008 (2007).
[17] R. G. Cai and S. P. Kim, JHEP 0502, 050 (2005).
[18] A. V. Frolov and L. Kohfler, J. Cosmol. Astropart. Phys. 0305, 009 (2003).
[19] U. K. Danielsson, Phys. Rev. D 71, 023516 (2005).
[20] B. Wang, E. Abdalla and R. K. Su, Phys. Lett. B 503, 394 (2001).
[21] B. Wang, E. Abdalla and R. K. Su, Mod. Phys. Lett. A 17, 23 (2002).
[22] R. G. Cai and Y. S. Myung, Phys. Rev. D 67, 124021 (2003).
[23] R. G. Cai and L. M. Cao, Nucl. Phys. B 785, 135 (2007).
[24] A. Sheykhi, B. Wang and R. G. Cai, Nucl. Phys. B 779, 1 (2007).
[25] A. Sheykhi, B. Wang and R. G. Cai, Phys. Rev. D 76, 023515 (2007).
[26] A. Sheykhi, J. Cosmol. Astropart. Phys. 05, 019 (2009); A. Sheykhi, Eur. Phys. J. C 69, 265 (2010).
[27] B. Wang, Y. Gong, E. Abdalla, Phys. Rev. D 74, 083520 (2006).
[28] J. Zhou, B. Wang, Y. Gong, E. Abdalla, Phys. Lett. B 652, 86 (2007).
[29] A. Sheykhi, B. Wang, Phys. Lett. B 678, 434 (2009).
[30] A. Sheykhi, Class. Quantum Grav 27, 025007 (2010).
[31] A. Sheykhi, B. Wang, Mod. Phys. Lett. A 25(14), 1199 (2010).
[32] G. Calcagni, Phys. Rev. Lett. 104, 251301 (2010).
[33] G. Calcagni, JHEP 03, 120 (2010).
[34] R. Hilfer (Ed.), Applications of Fractional Calculus in Physics (Word Scientific Publishing, New Jersey, 2000).
[35] S. R. de Groot and P. Mazur, Non-Equilibrium Thermodynamics (North-Holland, Amsterdam, 1962).
[36] G. Izquierdo and D. Pavon, Phys. Lett. B 633, 426 (2006).
[37] M. Jamil, E. N. Saridakis, M. R. Setare, Phys. Rev. D 81, 023007, (2010).
[38] M. Jamil, E. N. Saridakis and M. R. Setare, J. Cosmol. Astropart. Phys. 1011, 032, (2010).
[39] Y. S. Myung, Phys. Lett. B 671, 216 (2009).
[40] P. F. Gonzalez-Diaz and C. L. Siguenza, Nucl. Phys. B 697, 363 (2004).
[41] J. A. S. Lima and S. H. Pereira, Phys. Rev. D 78, 083504 (2008).
[42] S. H. Pereira and J. A. S. Lima, Phys. Lett. B 669, 266 (2008).
[43] E. N. Saridakis, P. F. Gonzalez-Diaz and C. L. Siguenza, Class. Quantum Grav 26, 165003 (2009).
[44] I. H. Brevik, S. Nojiri, S. D. Odintsov and L. Vanzo, Phys. Rev. D 70, 043520 (2004).
[45] S. Nojiri and S. D. Odintsov, Phys. Rev. D 70, 103522 (2004).