Gravitational Instability of de Sitter Compactifications

Carlo R. Contaldi (1), Lev Kofman (1), and Marco Peloso (2)

(1) CITA, University of Toronto, Toronto, ON, Canada, M5S 3H8, and
(2) School of Physics and Astronomy, University of Minnesota, Minneapolis, MN 55455, USA.

(Dated: March 27, 2022)

We consider warped compactifications in (4 + d)-dimensional theories, with four dimensional de Sitter dS 4 vacua (with Hubble parameter H) and with a compact internal space. After introducing a gauge-invariant formalism for the generic metric perturbations of these backgrounds, we focus on modes which are scalar with respect to dS 4. The physical eigenmasses of these modes acquire a large universal tachyonic contribution \(-12d/(d+2)H^2\), independently of the stabilization mechanism for the compact space, in addition to the usual KK masses, which instead encode the effects of the stabilization. General arguments, as well as specific examples, lead us to conjecture that, for sufficiently large dS curvature, the compactified geometry becomes gravitationally unstable due to the tachyonic growth of the scalar perturbations. This mean that for any stabilization mechanism the curvature of the dS geometry cannot exceed some critical value. We relate this effect to the anisotropy of the bulk geometry and suggest the end points of the instability. Of relevance for inflationary cosmology, the perturbations of the bulk metric inevitably induce a new modulus field, which describes the conformal fluctuations of the 4 dimensional metric. If this mode is light during inflation, the induced conformal fluctuations will be amplified with a scale free spectrum and with an amplitude which is disentangled from the standard result of slow-roll inflation. The conformal 4d metric fluctuations give rise to a very generic realization of the mechanism of modulated cosmological fluctuations, related to spatial variation of couplings during (p)reheating after inflation.

I. INTRODUCTION

De Sitter or quasi-de Sitter 4d geometries describe the present day acceleration of the Universe as well as the inflationary expansion at very early times. This stimulates significant interest towards the construction of 4 dimensional dS geometry in the context of fundamental string/M theory, which are formulated in 10/11 dimension, with a compact internal space [1, 2, 3]. Similarly, compactification to 4d dS geometry takes place in phenomenological braneworld models, where the inner space is periodic with orbifold branes at the edges [4]. Although most of the activity in this area has been enhanced by very recent progress (both on the observational and theoretical side) the issue of dimensional reduction to the outer cosmological space–time was popular since the 1980s, either in high dimensional supergravity theories or on phenomenological grounds. For example, see the collection of references on KK cosmology given in [3].

The geometries we discuss here have d spatial dimensions wrapped on a compact manifold M, in addition to the standard (3 + 1) space–time. Many of the mentioned examples are covered by the \((3 + d)\) dimensional geometry with the metric

\[
ds^2 = e^{2\tilde{A}(y)} \left( -dt^2 + e^{2H_4}d\vec{x}^2 \right) + e^{-2\tilde{A}(y)} \hat{g}_{ab}(y) dy^a dy^b ,
\]

where the outer space is a dS 4d metric with a Hubble parameter H, while \(\hat{g}_{ab}\) is the metric of the compact inner space with d coordinates \(y^a\). For generality, we also include the warp factor \(A(y)\). In the following we use greek letters to describe the \((3 + 1)\) outer space–time coordinates \((\mu = 0, ..., 3)\) while roman letters span the inner compact space coordinates only. Capitalized roman letters span all coordinates. With the re-definition of the inner space metric \(g_{ab} \equiv e^{-4\tilde{A}(y)}\hat{g}_{ab}\) and the warp factor \(A(y) \equiv e^{\tilde{A}(y)}\), we can re-write (1) in a conformally-factorized form

\[
ds^2 = A(y)^2 \left( ds_4^2 + g_{ab}dy^a dy^b \right) ,
\]

where \(ds_4^2\) is de Sitter metric. This form of the metric is more convenient for developing the formalism of metric perturbations around a warped \(dS_4 \times M\) background, which we will present below.

From the string theory/supergravity perspective, recent studies have concentrated on the compatibility of high \((4 + d)\) dimensional geometries \((d = 6)\) with a 4d de Sitter geometry as the outer space–time, and on the stabilization of the internal space to \(dS_4 \times M\). The bulk geometry, which is usually treated in the supergravity limit, requires a careful study of the \((4 + d)\) dimensional (bulk) Einstein equations. The progress in finding solutions is related to the identification of various possible sources for the bulk stress energy tensor \(T_{\mu\nu}^B\), including supergravity lagrangian fields, branes with fluxes, etc. In phenomenological braneworld models, the \((4 + 1)\) or \((4 + 2)\) dimensional bulk geometry is stabilized, for instance, by means of bulk scalar fields as in the Goldberger-Wise model [4], or more generally through the Casimir effect [7, 8].

Very often, however, the stabilization is studied at the level of a 4 dimensional effective theory, where the inner space geometry emerges in terms of moduli fields. If the effective potential of the moduli has a minimum, the moduli are considered to be stabilized. An example will be the stabilization mechanism with quantum field theory effects which emerge from the properties of \(M\), e.g. the Casimir effect from compact inner dimensions. It is a
reasonable assumption that in principle there shall be a bulk prototype \( T^A_B \) (may be as a complicated functional of the metric and the topology) which generates corresponding terms of the 4d effective potential. The version of string theory dimensional reduction of \( \mathbb{R}^4 \) is realized with the participation of the instanton effects (which are manifest in 4dim superpotential of \( N = 1 \) supergravity). An interesting question is whether it is possible to think about a bulk prototype \( T^A_B \) of these effects.

While it is reasonable to assume that the low energy 4d effective action is Einstein gravity plus moduli fields (or Brans-Diky gravity plus moduli), this picture lacks the full higher dimensional solution of the Einstein equations with proper sources. In particular, the four dimensional description is inadequate for studying the high energy regime (high dS curvature) where the stabilization can break down. For example, in the 5 dimensional braneworld models, the issue of stabilization due to the bulk scalar field can be formulated fully in terms of the stability of the 5 dimensional warped geometry against scalar metric perturbations. From this study one can determine the modes which enter in the exact low energy 4 dimensional description, namely the radion and the other KK modes of the system \( \mathbb{R}^4 \), which cannot be found from an heuristic 4 dimensional effective potential.

We are not concerned here with the details of the stabilization mechanism. Our main goal is to determine the effects of the dS expansion on the stability of these geometries. As we will show, compactifications to a 4dim dS space–time are more difficult to achieve than compactification to a 4dim flat space–time. This is due to a tachyonic contribution to the square of the mass of scalar metric perturbations arising from the de Sitter curvature. In what follows we will discuss the properties of \((4 + d)\) dimensional classical Einstein equations assuming a bulk \( T^A_B \) as a source, but without specifying it. The gravitational instability which we will discuss comes from the gravity sector, so that the exact form of \( T^A_B \) will not be crucial. Of course it will be attractive to check how the effect works for each particular model of \( T^A_B \). However, we will try to argue that the instability effect is generic.

Previous studies of the gravitational sector (i.e. metric perturbations) of the compactification concentrated on the dimensional reduction to 4dim flat space–time \( \mathbb{R}^4 \). The instability effect which we find emerges when the outer space is curved instead of flat, and it is proportional to the curvature. Notice that test scalar fields propagating in \( dS_4 \times M \) do not exhibit tachyonic masses under KK projection \( \mathbb{R}^4 \). Thus, there is a significant difference in the mass spectrum between the test fields and the self-consistent treatment of the metric fluctuations around this background.

The plan of this paper is as follows. In Section II we introduce a generic formalism of metric fluctuations around the \( dS_4 \times M \) background and define scalar, vector and tensor fluctuations according to their transformation properties relative to \( dS_4 \). In Section III we introduce the linearized equations that govern the evolution of scalar perturbations of the metric and show how the tachyonic instability of the scalar modes arises from the gravitational sector. In Sections IV and V we show two explicit example where the instability is manifested. In Section VI we discuss how the instability may be relevant for the generation of the primordial perturbations, providing a very natural and model-independent realization of the idea of modulated perturbations \[25, 27\]. Conclusions and a brief discussion can be found in Section VII.

II. METRIC PERTURBATIONS AROUND THE BULK GEOMETRY

The setting of the problem we are discussing is similar to the routine setup for \((3 + 1)\) dimensional cosmology with the metric \( ds^2 = a(t)^2 (−dt^2 + d\sigma_3^2) \), where \( d\sigma_3^2 \) is the metric of 3d space of constant curvature. Perturbations of this metric can be classified in terms of freely propagating gravitational waves, together with vector and scalar perturbations, supported by the perturbations of the sources (for instance scalar field fluctuations \( δϕ \) in the scalar field cosmology). The classification of cosmological fluctuations is based on the gauge-invariant Bardeen’s variables \[12\] (see also \[13\]).

In the background geometry \( \mathbb{R}^4 \) there are also different modes of metric perturbations, corresponding to different irreducible representations of the background symmetries. Modes living in different representations are not coupled at the linear level. For this reason, it is very useful to extend the classification of \[12, 13\] to this case too.

A. General formalism of \((4 + d)\) dimensional metric fluctuations

Bearing in mind the similarities in the structures of our bulk metric \( \mathbb{R}^4 \) and the \((3 + 1)\) cosmological metric, the generalized perturbed form of the metric \( \mathbb{R}^4 \) can be written as

\[
   ds^2 = A^2(y) \left\{ g_{\alpha\beta} (\delta^\phi_{\alpha\beta} + 2\phi_{,\alpha} dy^\beta + [(1 + 2\psi)g_{\mu\nu} + 2F_{\mu\nu} + 2F_{(\mu,\nu)} + h_{\mu\nu}] dx^\alpha dx^\beta \right\} - 2A_{\alpha\mu} dy^\alpha dx^\mu \right\} \quad (3)
\]

with \( g_{\mu\nu} dx^\mu dx^\nu = ds_4^2 \) and where \( F_{\mu \nu} \) and \( h_{\mu \nu} \) are divergence free and \( h_{\mu \nu} \) is traceless. The cross component

\( A_{\alpha \mu} \) can be further decomposed into individual scalar and
vector contributions as $A_{a\mu} = B_{a;\mu} + S_{a\mu}$. It is simple to check that the variables $\phi^c_i, \psi, E, F_{\mu}, A_{a\mu}, h_{\mu\nu}$ account for all possible physical metric perturbations.

Gauge invariant variables that are counterparts to the Bardeen variables in (3+1) dimensions can be derived in this setup by considering infinitesimal coordinate transformations $x^A \rightarrow \tilde{x}^A = x^A + \xi^A$. In particular we have new scalar variables

$$\Phi_{ab} = \phi_{ab} + A_{ab} \left( B_{c} - E_{c} \right) g_{ab} - \left( B_{(a} - E_{(a}) \right) ; b \right),$$  

$$\Psi = \psi + \frac{A_{ab} \left( B_{c} - E_{c} \right) }{A} \left( B_{c} - E_{c} \right) .$$ (5)

In terms of four dimensional tensor properties in the 4d observable world, we distinguish scalar perturbations of the source term $\delta T_{dS}^{ab}$ at the linear level. The analysis is further simplified (3) is that perturbations of different types are not coupled.

As mentioned, the main advantage of using the split is that perturbations of different types are not coupled at the linear level. The analysis is further simplified if the same decomposition is performed also for the perturbations of the source term $\delta T_{dS}^{A}$. In this work we study metric perturbations of the scalar type. (3)

**B. Scalar perturbations**

For this case we can define a generalized longitudinal gauge analogous to the (3+1) dimensional case. In this gauge the scalars $B_{a}$ and $E$ vanish, and the scalar perturbations $\phi_{ab}$ and $\psi$ coincide with the gauge invariant variables $\Phi_{ab}$ and $\Psi$, respectively. Thus, without lost of generality, the ‘scalar’ part of metric fluctuations around a geometry with $dS_{4} \times M$ symmetry can be simply written as

$$ds^2 = A^2(y) \left\{ g_{ab} (\delta^c_b + 2 \phi^c_a) dy^a dy^b + (1 + 2 \psi) ds^2_{dS} \right\}.$$  

We can obtain a further reduction in the degrees of freedom where the internal space $M$ is maximally symmetric. In this case $\phi^c_a \rightarrow \delta^c_a \phi$, and we obtain

$$ds^2 = A^2(y) \left\{ (1 + 2 \Psi(x,y)) ds^2_{dS} + (1 + 2 \Phi(x,y)) g_{ab} (y) dy^a dy^b \right\} .$$ (7)

Eq. is the the starting point for our treatment of scalar perturbations. It is instructive to compare the simple form of metric fluctuations in the longitudinal gauge around a (4 + d) symmetric $dS_{4} \times M$ background with lower dimensional examples. If we replace $A(y) \rightarrow a(t)$, $ds^2_{dS} \rightarrow ds^2_{dS_2}$, $g_{ab} dy^a dy^b \rightarrow -dt^2$, eq. reduces to the familiar form of scalar metric perturbations around a (3 + 1) dimensional FRW cosmology. Secondly, if $g_{ab} dy^a dy^b = dy^2$ is simply one extra dimensional line element, eq. gives us scalar metric perturbations in the (4 + 1) braneworlds, where $\Phi$ is related to the radion.

**III. LINEARIZED EQUATIONS FOR SCALAR PERTURBATIONS**

First we perform a customary perturbative analysis of the linearized Einstein equations for the scalar metric perturbation. It is remarkable that, for symmetric $M$, the two scalars $\Phi$ and $\Psi$ completely describe all the scalar metric fluctuations around $dS_{4} \times M$. The modes $\Phi$ and $\Psi$ will be coupled to scalar perturbations of $\delta T_{dS}^{A}$, if they are present. Typical example of scalar fluctuations in $\delta T_{dS}^{A}$ are the fluctuations of the bulk scalar field, which is used in the stabilization of $M$.

We will show that $\Phi$ and $\Psi$ are proportional to each other, and decompose $\Psi$ into scalar harmonics (KK modes) on $dS_{4}$. We will then construct the general expression for the lowest eigenvalue (square of the physical mass) of the scalar harmonics, and show that it contains a tachyonic contribution, which signals a possible gravitational instability.

It is very important to stress that $\Phi$ and $\Psi$ exist as dynamical fields even if $\delta T_{dS}^{A}$ is absent, contrary to the case of (3+1) dimensional cosmology. Indeed the bulk geometry $dS \times M$ is anisotropic as a whole. Therefore we may expect the presence of an anisotropic mode of the gravitational instability, even when the fluctuation of the source $\delta T_{dS}^{A}$ are absent. It turns out that this pure gravitational mode is also manifested in the scalar fluctuations $\Phi$ and $\Psi$. In the next section we will put the metric into the context of the generalized anisotropic Kantowsky-Sacks geometries and show that the longitudinal metric fluctuations around correspond to gravitationally unstable modes of the anisotropic metric.

**A. Linearized Einstein Equations**

Consider the linearized Einstein equations for the scalar perturbations $\Phi$ and $\Psi$. The equations for $\delta G_{\mu}^{\nu}$
contain a diagonal part, proportional to $\delta T^\mu_{\nu}$, plus a second term which is nontrivial also for $\mu \neq \nu$. It amounts to

$$
\delta G'^\mu_{\nu} = -A^{-2} \nabla^\mu \nabla_{\nu} [2 \Psi + d \Phi] , \quad \mu \neq \nu ,
$$

where $\nabla_{\mu}$ is the covariant derivative with respect to the non-compact de Sitter space. In the absence of an anisotropic stress tensor in the outer space, $\delta T^\mu_{\nu} = 0$ for $\mu \neq \nu$, (which is typical in cosmology), eq. (8) imposes

$$
\Phi = -\frac{2}{d} \Psi .
$$

\[\text{(9)}\]

This formula replaces the well known relation for $\Phi$ and $\Psi$ scalars in $(3+1)$ dimensional cosmology, $\Phi = -\Psi [13]$, and generalizes the result $\Phi = -2 \Psi$ valid for the $(4+1)$ braneworlds [17].

The linearized Einstein equations for the scalar perturbations $\Psi$ yield

$$
\delta G'^\mu_{\nu} = A^{-2} \delta T^\mu_{\nu} \left[ 6 H^2 + \frac{d + 2}{d} \left( \Box_{d} - \frac{2}{d + 2} R + L_1 \right) \right] \Psi = 8 \pi G \delta T^\mu_{\nu} \times \delta^\nu_{\nu} ,
$$

\[\text{(10)}\]

$$
\delta G'^{a}_{\phantom{a}\alpha} = -A^{-2} \frac{d + 2}{d} [\partial^\mu \partial_a + L_2] \Psi = 8 \pi G \delta T^a_{\mu} ,
$$

\[\text{(11)}\]

$$
\delta G'^{a}_{\phantom{a}\beta} = A^{-2} \frac{d + 2}{d} \left[ -2 \nabla^\mu \nabla_\nu + \frac{4}{d + 2} G^a_{ab} + \delta^a_b \left( \Box_{d} + \frac{12d}{d + 2} H^2 + 2 \Box_{d} \right) + L_3 \right] \Psi = 8 \pi G \delta T^a_{\nu} .
$$

\[\text{(12)}\]

In these equations $\Box_{d} = -\partial^2_t - 3H \partial_t + e^{-2Ht} \partial^2_x$ is the scalar $\nabla^\mu \nabla_\mu$ operator of the $(3+1)$ dimensional de-Sitter space, while $\nabla^a$, $G^a_{ab} = R_a^\alpha - \frac{1}{2} \delta^\alpha_{ab} R$, $\Box_{d}$ are computed on the compact manifold $M$ with the metric $g_{ab}$. $L_1, L_2, L_3$ denote terms proportional to derivatives of $A$. Such terms do not contain second derivatives with respect to $x^\mu$ and can be neglected in the present discussion. For completeness, they are reported in Appendix. Finally, $G$ denotes the higher dimensional Newton constant.

\[\text{B. Tachyonic Contribution to the Eigenmasses}\]

The scalar perturbations [17] as well as the scalar perturbations of the stress-energy tensor $\delta T^A_{B}$, due to the symmetries of the background geometry can be decomposed into the scalar eigenmodes $Q_n(x)$

$$
\Psi (x, y) = \sum_n \tilde{\Psi}_n (y) Q_n (x) ,
$$

$$
\delta T^A_{B} (x, y) = \sum_n \delta T^A_{Bn} (y) Q_n (x) ,
$$

\[\text{(13)}\]

where the sum corresponds to the the KK tower of the 4 dimensional scalar modes. These modes $Q_n$, which are common for all fluctuations, obey the free massive scalar field equations

$$
\left( -\Box_{d} + m_n^2 \right) Q_n = 0 ,
$$

\[\text{(14)}\]

where $m_n^2$ are the separation eigenvalues of the decomposition [13]. Substituting [13] and [14] in [12], we obtain a set of equations for each eigenmode. In practice, we simply replace $\{ \Psi, \delta T^A_{B} \}$ by $\{ \tilde{\Psi}_n, \tilde{\delta T^A_{Bn}} \}$, and $\Box_{d}$ by the corresponding eigenvalue $m_n^2$.

Among [10]-[12], the equation [12] is the dynamical equation while equations [10], [11] are the constraint ones. Moreover, these equations are connected by the Bianchi identity. Thus, it will be enough to work with the two equations [11] and [12] only.

Let us consider the dynamical equation [12]. We make the following observations which is the crucial one for our discussion. We note that the four dimensional operator $\Box_{d}$ only enters in the diagonal components of the Einstein tensor $\delta G^a_{\mu}$ corresponding to the internal space, see eq. [12]. Most importantly, it always appears in the combination $\Box_{d} + \frac{12d}{d + 2} H^2$. Hence, this combination will always generate the contribution

$$
\left( \Box_{d} + \frac{12d}{d + 2} H^2 \right) \Psi \rightarrow \mu_n^2 \tilde{\Psi}_n^2 ,
$$

$$
\mu_n^2 \equiv m_n^2 + \frac{12d}{d + 2} H^2 ,
$$

\[\text{(15)}\]

in the equations for the modes along $M$. This is true for any choice of $M$ and for any underlying stabilization mechanism, since it is due to the symmetry of the background configuration [11]. Thus, the eigenvalue $m_n^2$ from the outer space–time always appears together with $\frac{12d}{d + 2} H^2$.

After the factorization [13], the resulting equations are the equations for the functions $\{ \tilde{\Psi}_n (y), \tilde{\delta T^A_{Bn}} (y) \}$ on $M$. Typically, they can be reduced to a second order elliptic equation. For compact $M$, this corresponds to the eigenvalue problem which determines the combinations
$\mu_n^2$. Once these values have been obtained, the physical masses $m_n^2$ of the perturbations immediately follow from eq. (13).

Further we can use the symmetry of $\mathcal{M}$. If the warping is absent, $A = \text{const}$, on general ground we expect that the solution of the eigenvalue problem on $\mathcal{M}$ will depend on the curvature of $\mathcal{M}$, $\mu_n^2 \sim \mathcal{R}$; we denote it as $\mu_n(\mathcal{R})^2$. On the other hand, for non-vanishing warping operators $L$ may bring in $\mu_n^2$ extra dependence from $A(y)$ in terms of its characteristic scale, which we denote as $\mu_n(A)^2$.

Taking all this together, the eigenvalues of the scalar metric perturbations in $dS$ space–time are expected to have the following structure

$$m_n^2 = -\frac{12 H^2}{1 + 2/d} + \mu_n^2(\mathcal{R}) + \mu_n^2(A) . \quad (16)$$

To support this conjecture, in the next Sections below we will give examples of explicit computations of $m_n^2$.

The most important conclusion is that the curvature of non–compact de Sitter dimensions give a negative (tachyonic) contribution in eq. (16) to the eigenmasses $m_n^2$ of the scalar fluctuations. If de Sitter curvature exceeds a critical value so that $m_n^2$ becomes negative, we encounter gravitational instability of the $dS_4 \times \mathcal{M}$ geometry. This result generalizes a similar finding obtained in [17] in the case of $(4+1)$ dim braneworlds, where $m_n^2 = -4 H^2 + \mu_n^2(\mathcal{A})$. In [17] the interpretation of the tachyonic instability in $m_n^2$ was not given. Here we argue that this is an effect of gravitational instability of anisotropic geometries $dS_4 \times \mathcal{M}$. The present discussion shows that the instability is a generic issue that is not confined solely to the braneworld models studied in [17]. Rather, it is a general property of the geometries of the form $\mathcal{O}$, which includes a very wide class of theories with extra dimensions. In the next Section we discuss how this instability emerges in different contexts.

C. Tachyonic Instability of $dS_4 \times \mathcal{M}$

We now follow what happens if $m_n^2$ is negative. For this we have to turn to equation (14) for the four–dimensional eigenfunction $Q_n(t, \vec{x})$. The four–dimensional massive scalar harmonics $Q_n$ can be further decomposed as

$$Q_n(t, \vec{x}) = \int f_k^{(n)}(t) e^{i\vec{k} \cdot \vec{x}} \, dk.$$  

The temporal mode functions $f_k^{(n)}(t)$ obey the equation

$$\ddot{f} + 3H \dot{f} + (e^{-2Ht} k^2 + m_n^2) f = 0, \quad (17)$$

where dot denotes time derivative, and we have dropped the labels $n$ and $k$ for brevity. We choose the solution of (17) which corresponds to the positive frequency vacuum-like initial conditions in the far past $t \to -\infty$, $f_k(t) \simeq \frac{1}{\sqrt{2k}} e^{\pm i k \eta}$, $\eta = \int dt \, e^{-Ht}$. For the tachyonic mode ($m_n^2 < 0$), the solution to equation (17) with this initial condition is given in terms of Hankel functions $f_k^{(n)}(\eta) = \frac{\sqrt{2}}{\pi} H_n^{(3/2)} (k \eta)$, with the index $\nu = \sqrt{\eta^2 + \frac{|m_n^2|}{H^2}}$. The late-time asymptote of this solution diverges exponentially as $t \to \infty (\eta \to 0)$

$$f_k^{(n)}(t) \propto e^{\lambda_n H t}, \quad (18)$$

with the numerical factor

$$\lambda_n = \left( \sqrt{\frac{9}{4} + \frac{|m_n^2|}{H^2}} - \frac{3}{2} \right). \quad (19)$$

The leading term is associated to the lowest negative mass square $m_n^2 = m_1^2 \sim -f_{\text{ew}} \times H^2$, which defines the instability of metric fluctuations

$$\Psi(t, \vec{x}; y) \propto e^{\lambda_1 H t}. \quad (20)$$

In the following sections we consider examples where $m_n^2$ can be calculated explicitly, and we will provide an interpretations of the tachyonic instability.

IV. EXAMPLE: INSTABILITY OF $(4+1)$ INFLATING BRANEWORLDS

The inclusion of additional fields is typically required to achieve the stabilization of the internal space. However, this complicates significantly the study of the perturbations of the system, so that explicit results have been obtained only for the simplest configurations. The theory of metric perturbations was developed for $(4+1)$ Braneworld models with inflation. Although we do not deal specifically with the braneworlds in this paper these provide a good example for models that are affected by the results discussed in this work and constitute an interesting set-up where calculations have been performed in detail. Here we consider the $(4+1)$ dimensional braneworld with de Sitter branes of codimension one, with and without the presence of a bulk scalar field. The geometry of the system, with the inclusion of scalar perturbations, is again of the form $\mathcal{O}$, with parallel branes as edges of the internal coordinate $y$. The function $A(y)$ is known as warp factor.

A. Inflating Braneworld without stabilization

First let us consider an inflating braneworld without stabilization (i.e. without bulk scalar field). We assume a negative cosmological constant in the AdS bulk. Scalar metric perturbations are given by the form $\mathcal{O}$, and from 14 we have $\Phi = -2 \Psi$. The Einstein equation for the linearized perturbations $\Phi$ has to be supplemented by the junction conditions at the orbifold branes, for details see 4 10 17. Equation (14) gives us

$$\Psi' + \frac{A'}{A} \Psi = 0 , \quad (21)$$

where prime stands for $\partial_y$. This relation shows that the KK tower is absent in this case, and that the only
eigenmode $\tilde{\Psi}_0 \sim 1/A^2$ is present. Equation (12) gives $(\Box + 4H^2) \Psi = 0$. We then see that the perturbation $\Psi$ is described by the free wave equation. In other words, for the branes without stabilization, $\Psi$ is nothing but the gravit-scalar 4 dim projection of the bulk gravitational wave. Second, the square of the mass for this mode is tachyonic

$$m_0^2 = -4H^2,$$  (22)

in exact agreement with our eq. (16). In this case $\mu_0^2(A) = 0$, $\mu_0^2(R) = 0$. The braneworld between two de Sitter branes is unstable. The equation (22) for inflating branes without stabilization was obtained in [19].

B. Inflating Braneworld with Bulk Scalar Stabilization

The addition of a bulk scalar field can, in principle, provide the stabilization of the braneworld configuration. The scalar field $\phi$ can have a potential $V$ in the bulk and potentials $U_{1,2}$ at the two branes. The internal coordinate can be stabilized by suitable choices of these potentials. However, as we will see, the branes can be curved to such a degree that the stabilization mechanism will fail.

Let us consider a stationary configuration, characterized by a fixed interbrane distance, and let us study its stability. Perturbations of the bulk field $\delta \phi$ are coupled with the scalar perturbation of the metric, as we described above. We can decompose them as in eq. (12),

$$\delta \phi (x, y) = \sum_n \delta \phi_n (y) \, Q_n (x).$$  (23)

Next, we can use the well known expression for the bulk stress energy tensor $\delta T^A_{\mu}$ in terms of $\delta \phi$ and the background field $\phi(y)$ (see [10, 17] for details), and substitute them in the linearized Einstein equations. The constraint equation (10) gives a connection between $\Psi_n(y)$ and $\tilde{\phi}_n(y)$

$$(A^2 \tilde{\Psi}_n)' = \frac{A^2}{3} \phi' \tilde{\phi}_n',$$  (24)

Equation (12) yields instead

$$\left( 2\phi'' + \frac{1}{3} \phi'^2 + 4H^2 \right) \tilde{\Psi}_n = \frac{\phi'^2}{3A} \left( \frac{A}{\phi'} \tilde{\phi}_n' \right).$$  (25)

Here, combinations of $A$ and its derivatives are expressed in a more compact manner through $\phi'$, using the background equations.

Because $\Psi$ and $\delta \phi$ are related by the constraint equation (24), in fact there is only one dynamical degree of freedom, which is linear combination of the pair. The mass of the four dimensional modes is most easily obtained from an eigenvalue equation for the wave functions of the modes along $y$.

Combining equations (24) and (25) one obtains [17]

$$\left[ \frac{d^2}{dy^2} + m_n^2 + 4H^2 - \frac{z''}{z} + \frac{2}{3} \phi'^2 \right] \tilde{u}_n = 0,$$

$$\tilde{u}_n \equiv \left( A^{3/2} / \phi' \right) \tilde{\Psi}_n , \quad \tilde{z} \equiv (A \phi')^{-1/2}.$$  (26)

The mass spectrum of the system is determined by the eigenvalue problem, i.e. by the solutions of this equation which satisfies the boundary conditions for the perturbations at the two branes. It can be shown that the lowest eigenmass is [17]

$$m_0^2 = -4H^2 + \mu_0(A),$$  (27)

where $\mu_0(A) \approx \frac{2}{3} \int \frac{dy}{(A \phi')^3}$. From the background Einstein equations we have $\phi'^2 = 6A'' - 3A' - 3H^2$, so that $\mu_0(A)$ is expressed through the warp factor and its derivatives.

As we have argued in the general case (set $d = 1$ in eq. (16)), the eigenmasses $m_n$ enter in the Einstein equations in the combination $m_n^2 + 4H^2$. Hence, the $4H^2$ term results in a negative contribution to the physical masses of the four dimensional modes. The stabilization mechanism has to be “sufficiently strong” to counterbalance this effect, and the configurations for which the right hand side of eq. (24) is negative are subject to tachyonic instability (20).

Since the warp factor and the expansion rate $H$ both enter in $\mu_0(A)$, this term should be considered as a functional of $H$. Hence $H$ affects $m_0^2$ through both terms. However, $\mu_0(A)$ only mildly depends on $H$ and by far, the strongest $H$ dependence is given by the negative $-4H^2$ contribution. Therefore the critical value of dS curvature $H = H_c$ which borders stable and unstable brane configurations can be estimated from $4H^2 = \mu_0(A)$, where $\mu_0(A)$ can be calculated explicitly for concrete braneworld models (for specific examples of the calculation of $\mu_0(A)$, see [3, 10]).

The study of the linear perturbations around the stationary configuration gives information about its stability, but it cannot be used to determine the whole dynamics of the system when the configuration is unstable. The dynamical evolution can be studied numerically with the branecode package, a numerical code designed specifically for this task. It was found numerically (by methods which are completely different from the linear analysis described above) that indeed stationary inflating branes configurations with too high $H$ are unstable. Moreover, the full numerical treatment of the dynamics reveals the end points of the non-linear evolution. It turns out that, depending on the parameters, unstable brane configurations can be re-structured to stable inflating branes configurations with lower $H$ ($H < H_c$). If the parameters of the branemodel do not admit another stationary configurations, the branes are instead colliding. Bulk geometry of the collapsing branes asymptotically reaches the 5 dim Kasner solution $ds_5^2 = ...$
\[-d^2 + t^{2 \nu} \, dy^2 + \sum_{i=1}^3 t^{2 \nu} \, dx_i^2.\] Three of the Kasner indexes \( p_i \) are the same, as dictated by the brane isometries \( 18 \).

V. EXAMPLE: INSTABILITY OF \( dS_4 \times S_d \)

As the next example we consider a \((4 + d)\)-dimensional space–time with the internal dimensions compactified on a \( d \)-sphere of radius \( r_0 \). This is a toy model and has no application to compact extra dimensions. However it serves as a simple geometry with which we can check our conjecture from the GR point of view. We work out in details the case in which the only source term is a cosmological constant \( \Lambda \). The system is rather simple in details the case in which the only source term is a cosmological constant \( \Lambda \). The system is rather simple and its dynamics can be studied exhaustively (for the low dimension case of \( dS_2 \times S_2 \) see \( 20 \)): there is one stationary configuration of the type (7), which, due to the lack of a stabilization mechanism, turns out to be unstable. Our main point is that the instability can be computed precisely with the general approach outlined in Section \( 11 \). Below we will apply this generic formalism to the case of \( dS_4 \times S_d \) with a positive bulk \( \Lambda \).

However, there is a complementary treatment of stability issue of \( dS_4 \times S_d \) based on the investigation of its gravitational evolution within a more general class of anisotropic geometries, the Kantowsky-Sacks metrics, of which \( dS_4 \times S_d \) is a particular solution. We show that the two approaches are equivalent. This provides insight on the nature of the tachyonic instability conjectured for \( dS_4 \times \mathcal{M} \), as the general gravitational instability in the class of anisotropic metrics. The dynamics of the the Kantowsky-Sacks metrics in \((4 + d)\)-dimensions can be analyzed by the qualitative methods of the dynamical systems. The phase diagram can be extended far from the stationary configuration, so we can find the end points of the tachyonic instability.

A. Instability of linear scalar perturbations around \( dS_4 \times S_d \)

We consider the simple situation in which the geometry is factorized \((A = 1\) in eq. \( 11 \)\) and the internal manifold \( \mathcal{M} \) is a sphere of dimension \( d \) and radius \( r_0 \). We assume that the only source term is a cosmological constant \( \Lambda \). This systems admits a stationary solution with static \( S_d \) and de Sitter external space. This configuration is characterized by

\[ R = 3dH^2, \quad H = \sqrt{\frac{2 \Lambda}{3(d + 2)}}. \] (28)

The curvature of the inner space is \( R = d(d-1)/r_0^2 \). The system allows for expanding \( H > 0 \) and contracting \( H < 0 \) phases.

Consider linearized scalar perturbations \( 7 \) around this background in the absence of any dynamical field in the system \( \delta T_\mu^\nu = 0 \). The first Einstein equation \( 10 \) is satisfied identically due to relation \( 28 \). The constraint equation \( 11 \) gives us \( \Psi = \text{const} \), so that only the lightest mode is present; while the tower of KK modes is absent (this is similar to the case of a braneworld with negative bulk cosmological constant without stabilization; also in this case the KK tower would be recovered upon introduction of dynamical fields). The dynamical Einstein equation \( 12 \) gives us

\[ \left( \Box + \frac{12d}{d + 2} H^2 - \frac{4(d - 1)}{d + 2} R \right) \Psi = 0. \] (29)

From here, we immediately have

\[ m_0^2 = \frac{-12H^2}{1 + 2/d} + \mu_0^2(R). \] (30)

where \( \mu_0^2(R) = \frac{2(d - 2)}{d(d + 2)} R \). The term \( \mu_0^2(R) \), proportional to the curvature of the \( d \)-sphere, has comparable magnitude and opposite sign with respect to the tachyonic term. However, the net mass squared of \( \Psi \) is negative for any dimension \( d \),

\[ m_0^2 = -6H. \] (31)

This signals the tachyonic instability of the configuration \( 25 \). Interestingly, the ratio \( m_0/H \) does not depend on \( d \).

Applying eq. \( 20 \) to this example, the linear scalar perturbations are growing with time according to

\[ \Psi(t, \vec{x}; y) \propto e^{m_0 H t} = \exp \left[ \left( \frac{\sqrt{33} - 3}{2} \right) H t \right]. \] (32)

Notice that \( 20 \) is the free wave equation; this means that \( \Phi \) corresponds to the graviscalar projection of the bulk gravitational wave.

B. Gravitational instability of anisotropic geometry \( dS_4 \times S_d \)

The expression \( 32 \) controls the development of the gravitational instability as long as \( |\Phi| \ll 1 \). However, to understand the nature of the instability, we have to study the dynamical evolution beyond the linear regime. To do so, we allow the radius of \( \mathcal{M} \) to be a time dependent function \( r(t) \), and we consider a general FRW evolution for the non-compact space,

\[ ds^2 = -dt^2 + a^2(t) \, dx^2 + r^2(t) \left[ d\theta_1^2 + \sin^2\theta_1 \left( d\theta_2^2 + \ldots \right) \right], \] (33)

where \( \theta_1, \theta_2, \ldots \) are angular coordinates on the \( d \)-sphere. For definiteness, we specify to \( d = 7 \) (the evolution of the system in a lower dimensional context was studied in details e.g. in \( 24 \)). From the \((0,0)\) Einstein equation,
the expansion rate of the non-compact dimensions can be written as a function of $r$ and its time derivative $\dot{r}$,
\[
\frac{\dot{a}}{a} = -21\dot{r} \pm \sqrt{3} \left[ -4\Lambda r^2 + 63\dot{r}^2 - 84 \right]^{1/2} \cdot (34)
\]
We can substitute back this relation into the remaining equations, and obtain a second order differential equation for $r$ alone. As before, we can rescale $\Lambda$ away by redefining $x = \sqrt[3]{8} r$, and $\tau \equiv \sqrt[3]{3} t$. We then obtain the following dynamical system
\[
\frac{dx}{d\tau} = y, \quad \frac{dy}{d\tau} = \frac{4x^2 + 81y^2 - 108 \mp 9\sqrt{3}y [4x^2 + 63y^2 - 84]^{1/2}}{18x}, \tag{35}
\]
whose evolution is summarized in the phase portrait of figure 1.

The set of trajectories lies in a nontrivial manifold, characterized by a forbidden region $(4x^2 + 63y^2 - 84 < 0)$, bounded area of fig. 1, and by two branches which are connected at the boundary of this region but disconnected anywhere else. The two branches are shown in the right and left part of fig. 1 with the right (left) part corresponds to the upper (lower) signs in eqs. (34), (35). Trajectories are shown by continuous lines, while the two dashed lines separate the region for which the external space is expanding ($H > 0$, lower part in both branches) from the (upper) one for which it is contracting ($H < 0$). The trajectories which reach the boundary of the forbidden region in the right branch are identified without discontinuity through opposite points along the boundary into the left branch. The background configuration $dS_4 \times S_7$ studied in the previous Subsection corresponds to the saddle point in the right branch, characterized by a static radius \(\{ x = \sqrt{27}, y = 0 \} \) (in agreement with eq. (28)), and by a de Sitter outer space. The region of the phase portrait around the saddle point $S_+$ is zoomed in the right panel of fig. 1.

The linearized system (35) around this point is
\[
\frac{d}{d\tau} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{4}{9} - \frac{\sqrt{3}}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} . \tag{36}
\]
Next, we have to diagonalize this matrix (36). The diagonal components of it are\( \delta \mathbf{diag} \left( -\frac{\sqrt{3}}{9}, -\frac{\sqrt{3}}{9}, \frac{\sqrt{3}}{9}, \frac{\sqrt{3}}{9} \right) \).

One of the eigenvalue is positive, another is negative, how it shall be for the saddle point. Instability corresponds to the positive eigenvalue. More precisely, metric instability growth with time as exp \((\frac{3\sqrt{3} - 3}{2} \Lambda t)\). Recall that
\[
\sqrt{\Lambda} = \frac{\sqrt{27}}{2} H \text{ in the case of } S_7. \quad \therefore \text{Therefore, the metric instability around saddle point grows as}
\]
\[
r(t) - r_0 \propto \exp \left( \frac{\sqrt{33} - 3}{2} H t \right) . \tag{37}
\]
This is precisely the result (32) for $\Psi$ obtained with the previous calculation. This confirms the equivalency of the two approaches for the study of the instability of the stationary configuration.

The eigenvalues of the matrix (36) are of opposite sign, indicating that the stationary configuration is a saddle point. Its eigenvectors describe the critical directions in the linear regime close to the saddle points. The motion along the unstable line leads to two very different asymptotics along the two opposite directions. Starting with a slightly larger $r$ the system goes toward the decompactification of $M$, with the contribution from the cosmological constant dominating more and more over the curvature of the $d$–sphere. The asymptotic regime is a straight line in the phase portrait, $y/x = \text{constant}$. Substituting this “ansatz” in eq. (36), one finds $y/x \rightarrow 1/\sqrt{45}$, as $x, y \rightarrow \infty$. This solution corresponds to a de Sitter expansion of the internal space. From eq. (36) we see that the non–compact space also approaches the same de Sitter expansion,
\[
\frac{\dot{a}}{a} = \frac{\dot{r}}{r} = \sqrt{\Lambda} . \tag{38}
\]
Hence, the system tends toward $dS_{11}$, with an isotropic expansion of all the coordinates. Notice that the bulk expansion rate (38) is slower than the initial expansion rate of the outer space $H = \frac{\sqrt{2\Lambda}}{27}$. Isotropization of expanding solutions is produces by the bulk cosmological constant.

Alternatively, the instability can lead to the collapse of the internal manifold. In the phase diagram of figure 1 this possibility is described by trajectories which, starting from the stationary point, go from the right to the left branch, and then continue toward $r \to 0$, while the external space expands with increasing expansion rate. The asymptotic evolution is of the Kasner type $ds^2 = -dt^2 + t^{2p_1} dx^2 + t^{2p_4} dy^2$. We have
\[
a \propto (t_0 - t)^{\frac{\sqrt{3} - 1}{10}} , \quad r \propto (t_0 - t)^{\frac{\sqrt{3} + 1}{10}} , \tag{39}
\]
where $t_0$ is the time at which the system becomes singular (one can easily verify that solve the dynamical equations at leading order as $t \to t_0$). Kasner exponents are the same in two groups, $p_1 = p_2 = p_3 = -\frac{\sqrt{3} - 1}{10}$ and $p_4 = \ldots = p_{11} = \frac{3\sqrt{3} + 1}{10}$. The Kasner geometry is a generic collapsing solution [21], already advocated in string cosmology [22] and braneworld cosmology [13]. It is characteristic of a strong gravity regime, where the presence of any source (the cosmological constant, in the case at hand) can be neglected.

VI. MODULATED PERTURBATIONS

In the previous Sections we discussed how the de Sitter expansion can affect the stability of the internal space.
FIG. 1: Phase portrait for the dynamical system (35). Two branches are present, according to different signs in eqs. (34), (35). The branches are connected along the boundary of the oval region symmetrically across the axis passing through the point $P_0$ (as shown by the identification of points $P_1$ and $P_2$ in each branch. The dashed (green) curves show the boundary where the Hubble parameter for the non-compact space changes sign. The stationary configuration discussed in the main text is the saddle point $S_+$. Outgoing trajectories from this point either go to a decompactification of the internal space (toward asymptotic $dS_{11}$) or they change branch and go toward Kasner asymptotic with expanding outer space and collapsing internal manifold. We also show a detailed view of the critical region around the saddle point $S_+$. 

As we discussed, the stability cannot be reached by gravity alone. A point of phenomenological relevance is that the field(s) responsible for the stabilization unavoidably introduce (quantum) fluctuations which perturb the background geometry. Even if these fluctuations are heavy today, so that they have not been yet observed at accelerators, they could have had a mass smaller than the expansion parameter $H$ during inflation. This is particularly true if the the expansionary rate itself gives a negative contribution to their masses, as we discussed above. If this is the case, the fluctuations are amplified to a classical level during the inflationary expansion. These fluctuations can arise independently on slow roll inflaton field. They can be generated even in absence of an inflaton field (if, for example, inflation is due to some metastable form of vacuum energy) since they are “supported” by the bulk dynamical fields responsible for the stabilization of $\mathcal{M}$.

As we will see Fluctuations of the internal manifold will lead to modulated perturbations. The modulated fluctuations were introduced recently as an alternative mechanism of generation of primordial cosmological perturbations [25, 26]. To see this, let us focus on the four dimensional part of the geometry. Induced metric at $dS_4$ is

$$\gamma_{\mu \nu} = A^2(y) (1 + 2 \Psi(x, y)) g_{\mu \nu},$$

where $g_{\mu \nu}$ is the usual four dimensional $dS$ metric. Dimensional reduction leads to appearance of the moduli field $\Psi$ in the four dimensional effective action. Consider only the lightest KK mode, after proper normalization we write

$$S = \int d^4x \sqrt{-g} \left[ R - \frac{1}{2} \Psi_{\mu} \Psi^{\mu} + \frac{1}{2} m_0^2 \Psi^2 - \Psi T^\mu_{\mu} \right].$$

(41)

Such an action may appear if there are $1+3$ dimensional submanifolds which are point-like on $\mathcal{M}$ (in which case $\gamma_{\mu \nu}$ is the induced metric on the submanifolds). More generally, we can also obtain it starting from the geometry and integrating over $\mathcal{M}$ (in which case $A$ and $\Psi$ do not depend on $y$). The first situation is typical of brane models, while the second is more common of general KK theories. In both cases, we have in
mind a more general action than \(\mathcal{L}_0\), which will be relevant for the dynamics of the brane fields or of the KK modes. However, the choice \(\mathcal{L}_0\) is adequate enough for the following discussion.

The action \(\mathcal{L}_0\) differs from a standard theory of inflation due to the presence of interaction term \(\Psi T_{\mu}^\nu\). Perturbations of \(\Psi\) lead to rescaling of masses and couplings during inflation, until the moduli \(\Psi\) is pinned down to its minimum where it has the mass \(m_0^2\). The spatial fluctuations \(\Psi\) of the geometry introduce a small spatial dependency in this redefinition, so that any physical mass scale and couplings will be slightly different in different parts of the universe.

Inhomogeneities of the moduli field \(\Psi\) are generated during inflation if its mass \(m_0\) is lighter than the Hubble parameter. Now we recall the results of the previous section, that \(m_0^2\) contains tachyonic contribution. Therefore, an interesting possibility arises if the net value of \(m_0^2\) is smaller than \(H^2\). In this case during inflation we encounter generation of spatial fluctuations of \(\Psi\). In the late universe long after inflation, when \(H\) significantly drops, we expect \(m_0^2\) to be positive and large relatively to the temperature of the universe, so that \(\Psi\) is stabilized.

Small spatial variation of masses and couplings immediately after inflation, during (p)reheating after inflation, will result in a spatial dependency of the inflaton decay rate \(\Gamma\)

\[
\frac{\Delta \Gamma}{\Gamma} \sim \Psi . \tag{42}
\]

This fluctuation in the decay of the inflaton give rise to the adiabatic mode of the (standard) perturbations: after the reheating epoch the evolution of the perturbations in the various decay products follows the standard picture. The important distinction is that in this case the inflaton assumes a secondary role in defining the nature of the perturbations.

After reheating ends and moduli \(\Phi\) is trapped in its minimum, spatial variations of masses and couplings are erased. However, long wavelength metric perturbations generated during the inflaton decays are preserved as the scalar cosmological perturbations \[25, 26\].

There has been some activity in the computation of the spectrum of CMB anisotropies in brane world models. However, many of these studies do not discuss the general set of metric perturbations in the bulk, but rather focus on the evolution of the standard four dimensional perturbations with the braneworld configuration as a background. On the contrary, the mechanism outlined here exploits the presence of new perturbations, related to the bulk dynamic, in a crucial way (notice that the conformal mode \(\mathcal{L}_0\) is absent on the standard 4 dimensional case).

In addition, the one we have described is a very natural realization of the general idea of modulated perturbations proposed in \[25, 26\], since we provide a definite candidate (generally present in all the configurations \(\mathcal{L}_0\) for the modulus field controlling \(\Gamma\)).

There has been much focus recently on so-called “anomalies” emerging in the observations of LSS and the CMB \[27\] ranging from an intriguing lack of power on the largest scales \[28\], the possibility of “glitches” in the CMB spectrum and an indication of running of the spectral index of the primordial scalar perturbations (for a recent review, see \[29\]), to detections of mild non-Gaussianity and indications of a finite topology for the 3 spatial dimensions. The accuracy of future observations is expected to increase greatly over the next decade and the hope is that if any of the anomalies turn out to be robust results they will help to constrain new physics modifying the dynamics in the early universe.

VII. DISCUSSION

A typical denominator in many extensions of the Standard Model is the presense of extra-dimensions. In the simplest possibilities, the extra space is a compact and static manifold \(\mathcal{M}\). A stabilization mechanism is typically required to explain why \(\mathcal{M}\) should remain static, while the 3 noncompact spatial dimensions are undergoing cosmological expansion. In particular, significant activity has been recently made trying to reconcile this picture with a de Sitter (or quasi de Sitter) geometry for the noncompact coordinates. Indeed, observations are telling us that the expansion of the universe was accelerating at very early times, and it is accelerating also at present.

The present note is focused on the effects of the inflationary expansion on the stability of \(\mathcal{M}\). We have argued that the dS curvature (in other words, a nonvanishing expansion rate \(H\)) has typically the effect to destabilize the internal space, and that any given stabilization mechanism can be effective only up to a certain curvature. We have aimed to discuss this effect in the most general way possible, enlighting the consequences of making the only assumption that the geometry is of the \(dS \times \mathcal{M}_4\) type, with an arbitrary compact space of \(d\) dimensions, and possibly with the presence of a warp factor. Such a set-up is also relevant for string-theory, in the supergravity limit.

To discuss the stability of the system, we have studied the most general set of perturbations of the \(dS_4 \times \mathcal{M}_4\) geometry. It is very convenient to classify the perturbations into irreducible representations of the \(dS_4\) symmetry group. The big advantage of doing so, is that modes belonging to different representations are not coupled at the linear level. The perturbations can be devided into scalar, vector, or tensor modes with respect to the \(dS_4\) isometries. After a general classification, we have focused on the scalar modes, since they are usually the most relevant one for inflationary geometries, and since they encode the effects of the instability we want to discuss. The linearized Einstein equations for the perturbations show that the physical mass squares \(m^2\) of these modes acquire a negative contribution due to the \(dS\) expansion,
\[ m_n^2 (H) = - \frac{12 H^2}{1 + d/2} . \] (43)

This is true for arbitrary \(\mathcal{M}, d\), and for any possible underlying stabilization mechanism.

The presence of this term is a signal for a possible gravitational instability of the system. Indeed, if the whole \(m_n^2\) turns out to be tachyonic, the \(dS \times \mathcal{M}\) is unstable. Clearly, verifying whether (and at which \(H\)) the instability takes place is very model dependent issue which should be verified case by case (namely, for any given source \(T^A\)). Indeed, other contributions to \(m^2\) also depend on \(H\) (although in most cases only indirectly, due to the fact that these terms depend on other background quantities, and that these quantities are related to \(H\) through the background Einstein equations). However, the contribution (43) is generally present, and it has to be taken always into account. Moreover, the study of several different examples, as well as general arguments, lead us to conjecture that the term (43) is the dominant one at large \(H\), and that no stabilization mechanism can be effective up to arbitrarily large \(dS\) curvature.

In this work, we have presented explicit and exact calculations in two specific examples. The first of them is codimension one braneworld configurations with bulk scalar field(s). As shown in [17], in this case it is possible to derive an explicit upper bound \(m^2 \leq -4 H^2 + \mu_0 (A)\) (where \(A\) is the warp factor, see eq. (27) for the mass of the lightest eigenmode). This bound allows to determine at which \(H\) the system becomes unstable. The second example is when \(\mathcal{M}\) is a \(d\)-sphere, and when the only source term is \(\Lambda\) a cosmological constant. This system is well known to be unstable, and we have shown that the instability is precisely due to the tachyonic nature of the scalar modes which we have identified in the general calculation.

There are also general arguments in support of the instability at high \(H\). The first is related to causality. If, as in the standard 4 dimensional case, the Hubble length \(H^{-1}\) has the meaning of a causal horizon, we should expect that any stabilization mechanism cannot be at work when \(H\) becomes much greater than the inverse size of \(\mathcal{M}\) (since different “edges” of \(\mathcal{M}\) would then be causally disconnected). Such a behavior can be found in [21], where the cosmology of the Randall-Sundrum model was studied under the assumption of radion stabilization. It was shown in [22] that radion stabilization cannot be imposed if the physical energy on the hidden brane is greater than \(\sim \text{TeV}^2 M_p^2\). The appearance of this intermediate scale is somewhat surprising, and indeed it was left unexplained in [21]. However, it can be verified that it is precisely at this energy density that \(H^{-1}\) becomes smaller than the distance between the two branes.

A second argument can be inferred from the \(dS_4 \times S_d\) example. We noted that the instability has two possible end-points. One is characterized by a shrinking internal space, with asymptotic Kasner solution (a similar behavior was already noted in [13] in the case of brane collisions). The second one is instead exact \(dS_{4+d}\), with all the compact and noncompact dimensions expanding with the same asymptotic rate. This suggests that the instability can be attributed to the tendency of an inflationary expansion to homogenize and isotropize the whole geometry, in this case by “dragging” also the compact space into a de–Sitter solution. This behaviors, which is a well known feature of inflationary expansions, is encountered also in many Bianchi models.

In the final part of the work, we have also commented on the possible role of the instability in the generation of primordial perturbations. The scalar perturbations appear as a conformal mode from the four dimensional point of view (for example, for an observer living on a 3-brane, or after integrating out the internal space \(\mathcal{M}\)). This mode is a clear signature of the extra dimensions, since it is absent in the standard 4 dimensional case. If, due to the tachyonic contribution, this mode light (\(0 < m < H\) during the de Sitter stage, it will be amplified with a scale invariant spectrum and with an amplitude which is disentangled from the standard result of slow-roll inflation. One can think of a number of ways how the conformal perturbation could be responsible for the adiabatic mode in the later FRW evolution. For instance, consider a mixture of fields conformally and nonconformally coupled to the noncompact geometry. Only the nonconformally coupled fields will be sensitive to the conformal perturbations, so that we expect that the final mode perturbation will be proportional to the amount of these fields in the mixture. In the text we have instead focused on a much simpler possibility, related to the mechanism of modulated perturbations [25, 26]. The conformal factor can be usually rescaled away by a rescaling of the energy scales in the 4 dimensional theory (this is exactly what occurs in the Randall-Sundrum model [21]). Hence, its fluctuations can be interpreted as fluctuations masses of particles and rates of processes in the 4 dimensional theory. In particular, this may be the origin of the fluctuations of the decay rate of the inflaton, which is at the basis of the mechanism of modulated perturbations.

VIII. ACKNOWLEDGMENTS

It is a pleasure to thank Johannes Martin, Dimitry Podolsky, and Lorenzo Sorbo for very useful discussions.
The linearized equations \([\Psi]\) for \([\Psi]\) have been given in the main text in terms of the three operators \(L_{1,2,3}\). These operators are

\[
L_1 = (d + 4) \frac{\hat{\nabla}_a A}{A} \hat{\nabla}_a + 4 \frac{\sqrt{A}}{A} + 2 (d - 1) \frac{\left( \hat{\nabla} A \right)^2}{A^2},
\]
\[
L_2 = 2 \frac{\hat{\nabla}_a A}{A} \hat{\nabla}^a
\]
\[
L_3 = -2 \frac{\hat{\nabla}_a A}{A} \hat{\nabla}_b - 2 \frac{\hat{\nabla}_b A}{A} \hat{\nabla}^a + 8 \frac{\hat{\nabla}_a A \hat{\nabla}_b A}{A^2} - 4 \frac{\hat{\nabla}_a \hat{\nabla}_b A}{A} + \delta_b^a 2 (d + 3) \frac{\hat{\nabla}^c A \hat{\nabla}_c + 4 \frac{\sqrt{A}}{A} + 2 (d - 1) \frac{\left( \hat{\nabla} A \right)^2}{A^2}}{(A1)}
\]

where quantities with tilde are computed from the background metric on \(x\), while quantity with hat from the background metric on \(y\).