OPTIMAL CURVES OF LOW GENUS OVER FINITE FIELDS

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Abstract. The Hasse-Weil-Serre bound is improved for curves of low genera over finite fields with discriminant in \{-3, -4, -7, -8, -11, -19\} by studying optimal curves.

1. Introduction.

In this paper we give some improvements of the Hasse-Weil-Serre upper and lower bound for the number of rational points on a curve over a finite field for curves of low genus over finite fields of certain discriminants.

Here by a curve over a finite field \(F_q\) we mean an absolutely irreducible nonsingular projective algebraic variety of dimension 1 over \(F_q\) and by the discriminant \(d(F_q)\) of a finite field \(F_q\) we mean the integer \(m^2 - 4q\), where \(m = \lfloor 2\sqrt{q} \rfloor\). Throughout the paper we denote the set \{-3, -4, -7, -8, -11, -19\} by \(D\).

For the number of rational points on a curve \(C\) of genus \(g\) defined over a finite field \(F_q\) we have the well-known bound of Hasse-Weil as improved by J-P. Serre

\[ |\#C(F_q) - q - 1| \leq g[2\sqrt{q}], \]

which now is called the Hasse-Weil-Serre bound.

This bound is interesting in its own right, but also in view of applications in coding theory, as for example for the geometric codes introduced by V. D. Goppa in 1980.

A curve \(C\) of genus \(g\) over a finite field \(F_q\) is called a maximal (resp. minimal) optimal curve if its number of rational points attains the upper (resp. lower) Hasse-Weil-Serre \(q + 1 \pm g[2\sqrt{q}]\).

We study optimal curves of low genera over fields with discriminant \(d(F_q) \in D\). It turns out that such curves are ordinary. This allows us to use the canonical lifts of the Jacobian and the equivalence of categories between the category of ordinary principally polarized abelian varieties over \(F_q\) and the category of certain unimodular irreducible hermitian modules over the ring of integers \(O_K\) of an imaginary quadratic field \(K\) of discriminant equal to \(d(F_q)\). The use of such hermitian modules to study curves over finite fields was initiated by J-P. Serre [12], [13] and continued by K. Lauter [6]. For the discriminants we consider here the class number of \(O_K\) is 1 and there is a classification of hermitian modules. This together with results on defect 1 curves leads to our improvement.
Theorem 1.1. Let $C$ be a curve of genus $g$ over a finite field $\mathbb{F}_q$ of characteristic $p$. Then we have that

$$|\#C(\mathbb{F}_q) - q - 1| \leq g[2\sqrt{q}] - 2,$$

if the conditions on $q$ and $g$ given by a line in the following table hold:

| $d(\mathbb{F}_q)$ | $q$     | $g$        |
|------------------|--------|------------|
| $-3$             | $q \neq 3$ | $3 \leq g \leq 10$ |
| $-4$             | $q \neq 2$ | $3 \leq g \leq 10$ |
| $-7$             | $q \neq 2$ | $4 \leq g \leq 7$ |
| $-8$             | $p \neq 3$ | $3 \leq g \leq 7$ |
| $-11$            | $p \neq 3, \ q < 10^4$ | $g = 4$ |
| $-11$            | $p > 5$   | $g = 5$    |
| $-19$            | $q < 10^4$ | $g = 4$    |
| $-19$            | $q \not\equiv 1 \pmod{5}$ | $g = 5$    |

Our method uses the explicit classification of hermitian lattices by A. Schiemann [9]. We also use some information on generators for the automorphism groups of such lattices of the dimension 4 and 5 over an imaginary quadratic extension $K$ of $\mathbb{Q}$, with discriminant $d(K) = -19$ provided to us by R. Schulze-Pillot [11].

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2. An equivalence of categories.

Let $C$ be a maximal or minimal curve of genus $g$ over $\mathbb{F}_q$. Then by a corollary of the Hasse-Weil-Serre bound (15 Theorem 5) and by the Honda-Tate theorem [17] the Jacobian $\text{Jac}(C)$ is isogenous to $E^g$, where $E$ is a maximal or a minimal elliptic curve, respectively. If the elliptic curve $E$ is ordinary, the Jacobian $\text{Jac}(C)$ is an ordinary principally polarized abelian variety and we can apply an equivalence of categories between the category of ordinary abelian varieties over a finite field and the category of Deligne modules; see [2], [4], or [14] for a more concrete description of this equivalence. The idea is that the canonical lift of our Jacobian is a complex abelian variety with a certain endomorphism ring and in this way we reduce to a classification problem of hermitian modules.

In order to use the classification of unimodular irreducible hermitian modules as given by A. Schiemann in [9], we have to work with maximal orders. According to a result in [24] the isomorphism classes of the abelian varieties in the isogeny class $[E^g]$ correspond one-to-one to the isomorphism classes of $R$-modules that can be embedded as a lattice in the $K$-vector space $K^g$,.
where $R = \text{End}(E)$ and $K = \text{Quot}(R)$. Since our optimal elliptic curve is defined over a finite field $\mathbb{F}_q$ with the discriminant $d \in D$ we have that $R = \mathcal{O}_K$ is the ring of integers of $\mathbb{Q}(\sqrt{d})$, with class number 1 and there is only one isomorphism class of such an $R$-modules. Hence there is only one isomorphism class of abelian varieties in the isogeny class $[E^g]$, so $\text{Jac}(C) \cong E^g$ as abelian varieties over $\mathbb{F}_q$. According to [14] the principal polarizations of $\text{Jac}(C)$ correspond to unimodular hermitian $\mathcal{O}_K$-forms $h : \mathcal{O}_K^g \times \mathcal{O}_K^g \rightarrow \mathcal{O}_K$. Moreover, we have $\text{Aut}_{\mathbb{F}_q}(\text{Jac}(C)) \cong \text{Aut}(\mathcal{O}_K^g, h)$, where $\text{Aut}(\mathcal{O}_K^g, h)$ is the automorphism group of the hermitian module $\mathcal{O}_K^g$ in $(K^g, h)$. This allows us to use the classification of unimodular irreducible hermitian forms with discriminant $d(K)$ and their automorphism groups.

We need conditions on the finite fields $\mathbb{F}_q$ such that there can exist an optimal ordinary elliptic curve over $\mathbb{F}_q$.

**Proposition 2.1.** Let $E$ be an optimal elliptic curve over a finite field $\mathbb{F}_q$ with discriminant $d(\mathbb{F}_q) \in D$. Then $E$ is ordinary if $q \neq 2, 3$.

**Proof.** Let $E$ be an optimal elliptic curve over the finite field $\mathbb{F}_q$ and $d(\mathbb{F}_q) \in D$. If $E$ is supersingular, then $\text{char}(\mathbb{F}_q)$ has to be a divisor of $d(\mathbb{F}_q)$, since $\text{char}(\mathbb{F}_q)$ divides $m$ and hence $\text{char}(\mathbb{F}_q)$ divides $d(\mathbb{F}_q) = m^2 - 4 \cdot q$. Now we explore each discriminant separately.

If $E$ is supersingular and $d(\mathbb{F}_q) = -3$ then $\text{char}(\mathbb{F}_q) = 3$ and $m^2 = 3(4 \cdot 3^{n-1} - 1)$. Then 3 divides $4 \cdot 3^{n-1} - 1$, hence $n = 1$.

If $d(\mathbb{F}_q) = -4$ then the optimal elliptic curve $E$ over $\mathbb{F}_q$ is not supersingular. Otherwise, by $m^2 = 4q - 4$ it follows that $\text{char}(\mathbb{F}_q) = 2$ and $(m/2)^2 = 2^{n-1} - 1$ with $m/2 \in \mathbb{Z}$, hence $(m/2)^2 = 2^{n-1} - 1 \equiv 3 \pmod{4}$, but 3 is not a square in $\mathbb{Z}/4\mathbb{Z}$.

The optimal elliptic curve $E$ is ordinary if the discriminant of the finite field $\mathbb{F}_q$ is in $\{-7, -11, -19\}$. In fact, if $E$ is supersingular then it is defined over $\mathbb{F}_{q^d}$, where $d := d(\mathbb{F}_q)$, and hence $m^2 = |d| \cdot (4(|d|)^{n-1} - 1)$ but this is impossible, since $d$ is prime.

For discriminant $d(\mathbb{F}_q) = -8$ we get a contradiction in a similar way. $\square$

We now have the following corollary.

**Corollary 2.2.** If $C$ is an optimal curve over a finite field $\mathbb{F}_q$ with discriminant $d(\mathbb{F}_q) \in D$ then $\text{Jac}(C)$ is an ordinary abelian variety if $q \neq 2, 3$.

Since our curve embeds in its Jacobian $\text{Jac}(C) \cong E^g$ the projections on the factors $E$ define morphisms $C \rightarrow E$ and we want to calculate the degree of these maps. Here we use the fact that the lattice of the canonical lift is a free $\mathcal{O}_K$-module, since the class number is 1.

**Proposition 2.3.** Let $C$ be an optimal curve over $\mathbb{F}_q$ and fix an isomorphism $\text{Jac}(C) \cong E^g$ such that the theta divisor corresponds to the hermitian form $(h_{ij})$ on $\mathcal{O}_K^g$ on the canonical lift of $\text{Jac}(C)$. Then the degree of the $k$-th projection

$$f_k : C \hookrightarrow \text{Jac}(C) \cong E^g \xrightarrow{pr_k} E$$
equals $\det(h_{ij})_{i,j\neq k}$.

Proof. We denote the abelian variety $E^g$ by $E_1 \times \ldots \times E_g$, where $E_i = E$, and consider the first projection. The degree of the map $f_1$ equals the intersection number $[C] \cdot [E_2 \times \ldots \times E_g]$. The cohomology class $[C]$ of $C$ in an appropriate cohomology theory is $[\Theta^{g-1}/(g-1)!]$. Recall that if $L$ is a line bundle on an abelian variety $A$ of dimension $g$ then by the Riemann-Roch theorem one has $(L^g/g!)^2 = \deg(\varphi_L)$, and $\deg(\varphi_L) = \det(r_{ij})^2$, where the matrix $(r_{ij})$ gives the hermitian form corresponding to the first Chern class of the line bundle $L$. Since the hermitian form $(h_{ij})_{i,j\neq 1}$ corresponds to the line bundle $\Theta\big|_{E_2 \times \ldots \times E_g}$ on the abelian variety $E_2 \times \ldots \times E_g$ the degree of $f_1$ is given by

$$[C] \cdot [E_2 \times \ldots \times E_g] = \frac{1}{(g-1)!}(\Theta|_{E_2 \times \ldots \times E_g})^{g-1} = \det((h_{ij})_{i,j\neq 1}).$$

□

3. Optimal curves of low genus over finite fields with discriminants $-3, -4, -7$ and $-8$.

In this section we prove the non-existence of certain optimal curves using properties of the automorphism groups of abelian varieties over the finite fields $\mathbb{F}_q$ with the discriminant $d(\mathbb{F}_q) \in \{-3, -4, -7, -8\}$.

Throughout this section we denote by $C$ an optimal curve of genus $g$ over $\mathbb{F}_q$ and by $h$ an unimodular irreducible hermitian form which corresponds to the polarization of $\text{Jac}(C)$ under the equivalence of Section 2.

Using Torelli’s theorem (cf. [1]) and an upper bound on the number automorphisms of a curve over a finite field (cf. [3]) we obtain an upper bound on the number of automorphisms of irreducible unimodular hermitian modules $(\mathcal{O}_K^g, h)$.

If a curve $C$ defined over a finite field $\mathbb{F}_q$ is not exceptional (see Subsection 3.1) and $p > g(C) + 1$ then we have $\#\text{Aut}_{\mathbb{F}_q}(C) \leq 84(g(C) - 1)$. Since $\#\text{Aut}_{\mathbb{F}_q}(\text{Jac}(C)) = \#\text{Aut}_{\mathbb{F}_q}(C)$ or $2 \#\text{Aut}_{\mathbb{F}_q}(C)$ depending on whether the curve is hyperelliptic or not, we get

$$\#\text{Aut}(\mathcal{O}_K^g, h) \leq \begin{cases} 84(g(C) - 1) & \text{if } C \text{ is hyperelliptic}, \\ 168(g(C) - 1) & \text{otherwise}. \end{cases}$$

We shall use this bound and other properties of the automorphism group of a unimodular hermitian module to restrict the possibilities for optimal curves of low genus.

To begin with, we treat the case of exceptional curves and their properties.

3.1. Optimal Exceptional Curves. Recall that by an exceptional curve we mean a hyperelliptic curve over the finite field $\mathbb{F}_q$ given by equation of the form $y^2 = x^p - x$, with $p = \text{char} (\mathbb{F}_q)$. An exceptional curve $C$ has genus $g = (p-1)/2$ and $\#\text{Aut}_{\mathbb{F}_q}(C) = 2p(p^2 - 1)$. Optimal exceptional curves over
$\mathbb{F}_q$ of genus $g \leq 10$ can occur only if $\text{char}(\mathbb{F}_q) \in \{7, 11, 13, 17, 19\}$. Now we list the possibilities for $q, g$ and $\#\text{Aut}_{\mathbb{F}_q}(C)$.

| $q$ | $g(C)$ | $\#\text{Aut}_{\mathbb{F}_q}(C)$ |
|-----|--------|--------------------------|
| $7^n$ | 3      | $2^5 \cdot 3 \cdot 7$   |
| $11^n$ | 5      | $2^2 \cdot 3^2 \cdot 5 \cdot 11$ |
| $13^n$ | 6      | $2^4 \cdot 3 \cdot 7 \cdot 13$ |
| $17^n$ | 8      | $2^6 \cdot 3^2 \cdot 17$ |
| $19^n$ | 9      | $2^4 \cdot 3^2 \cdot 5 \cdot 19$ |

The order of the automorphism group $\text{Aut}_{\mathbb{F}_q}(C)$ then equals $\#\text{Aut}(O_K^g, h)$ for some hermitian module $(O_K^g, h)$ and it also has to divide $\#\text{Aut}_{\mathbb{F}_q}(C)$. However from Tables 1,2,3,4 (see here after) for the orders of the automorphism groups of such modules one sees that the order of the automorphism group of any unimodular irreducible module of dimension $g(C)$ does not divide $\#\text{Aut}_{\mathbb{F}_q}(C)$, and hence an exceptional curve cannot be an optimal curve in the case we consider here.

In the remainder of this section we may and shall assume that our optimal curves are not exceptional curves.

3.2. Optimal curves over finite fields with discriminant $-3$. Here we consider optimal curves of genus $g \leq 10$ over finite fields $\mathbb{F}_q$ with $d(\mathbb{F}_q) = -3$. From the classification of unimodular hermitian modules we have the following table of orders of automorphism group of unimodular irreducible hermitian modules over an imaginary quadratic extension $K$ of $\mathbb{Q}$, with discriminant $d(K) = -3$.

| dim | $\#\text{Aut}$ |
|-----|----------------|
| 2 - 5 | —             |
| 6   | $2^9 \cdot 3^7 \cdot 5 \cdot 7$ |
| 7   | —             |
| 8   | $2^{14} \cdot 3^6 \cdot 5^2 \cdot 7$ |
| 9   | $2^8 \cdot 3^{12} \cdot 5 \cdot 7$ |
| 10  | $2^{17} \cdot 3^5 \cdot 5^2 \cdot 7$ |

Table 1

**Proposition 3.1.** There is no optimal curve $C$ of genus $g(C)$ with $2 \leq g(C) \leq 10$ over a finite field $\mathbb{F}_q$ of discriminant $-3$ if $q \neq 3$.

**Proof.** We consider the case of an optimal curve $C$ over $\mathbb{F}_q$. From Table 1 it follows that it cannot have genus 2, 3, 4, 5 and 7, since there is no corresponding irreducible unimodular hermitian module and hence no irreducible polarization.
If \( \text{char}(\mathbb{F}_q) = p \geq 13 \) and \( g(C) = 6, 8, 9 \) or \( 10 \) we would get that 
\[
\#\text{Aut}_{\mathbb{F}_q}(\text{Jac}(C)) > 2 \cdot 84 \cdot (g - 1) \quad \text{and hence } \quad \#\text{Aut}_{\mathbb{F}_q}(C) > 84 \cdot (g - 1)
\]
which contradicts the upper bound mentioned above.

For characteristics \(< 13\) we can apply the Singh bound (cf. [16]), viz.
\[
\#\text{Aut}_{\overline{\mathbb{F}}_q}(C) \leq \frac{4pg^2}{p - 1} \left( \frac{2g}{p - 1} + 1 \right) \left( \frac{4pg^2}{(p - 1)^2} + 1 \right)
\]
For the five characteristics which we have to examine, namely 2, 3, 5, 7 and 11, the Singh bound is smaller than half the number of automorphisms of the unimodular irreducible module, and hence the module cannot correspond to a polarization of a Jacobian. \(\square\)

Combining these results with the non-existence of curves of defect 1, i.e. with \( \#C(\mathbb{F}_q) = q + 1 \pm g[2\sqrt{q}] \mp 1 \) (see [15], page 22), we get the following theorem.

**Theorem 3.2.** If \( C \) is a curve of genus \( g \) with \( 3 \leq g \leq 10 \) over a finite field \( \mathbb{F}_q \) of discriminant \( -3 \) and \( q \not= 3 \) then
\[
|\#C(\mathbb{F}_q) - q - 1| \leq g[2\sqrt{q}] - 2.
\]

### 3.3. Optimal curves over finite fields with discriminant \(-4\).

In this subsection we examine the case of optimal curves of genus \( g \leq 10 \) over a finite field of the discriminant \( d = -4 \). From the classification of unimodular hermitian forms with the discriminant \(-4\) we have the following table of orders of automorphism groups of irreducible hermitian modules over an imaginary quadratic extension \( K \) of \( \mathbb{Q} \) with discriminant \( d(K) = -4 \).

| dim | \#Aut |
|-----|-------|
| 2 - 3 | — |
| 4   | \( 2^{10} \cdot 3^2 \cdot 5 \) |
| 5   | — |
| 6   | \( 2^{15} \cdot 3^2 \cdot 5 \) |
| 7   | \( 2^{11} \cdot 3^4 \cdot 5 \cdot 7 \) |
| 8   | \( 2^{15} \cdot 3^5 \cdot 5^2 \cdot 7, 2^{22} \cdot 3^2 \cdot 5 \cdot 7, 2^{21} \cdot 3^4 \cdot 5^2, 2^{21} \cdot 3^2 \) |
| 9   | \( 2^{16} \cdot 3^3 \cdot 5, 2^{10} \cdot 3^4 \cdot 5^2 \cdot 7, 2^{10} \cdot 3^4 \cdot 5^2 \cdot 7, \) |
| 10  | \( 2^{27} \cdot 3^4 \cdot 5^2 \cdot 7, 2^{25} \cdot 3^4 \cdot 5^2, 2^{21} \cdot 3^5 \cdot 5 \cdot 7, 2^{25} \cdot 3^2, \) |
|     | \( 2^{19} \cdot 3^2 \cdot 5 \cdot 7, 2^{17} \cdot 3^3 \cdot 5, 2^{11} \cdot 3^4 \cdot 5^2 \cdot 7, 2^{23} \cdot 3 \cdot 5, 2^{19} \cdot 3^2, \) |
|     | \( 2^{11} \cdot 3^4 \cdot 5^2, 2^{10} \cdot 3^4 \cdot 5^2 \) |

**Table 2**

**Proposition 3.3.** There is no optimal curve of genus \( g \) with \( 2 \leq g \leq 10 \) over a finite field \( \mathbb{F}_q \) for \( d(\mathbb{F}_q) = -4 \) and \( q \not= 2 \).
**Proof.** Table 2 shows that there is no such optimal curve of genus 2, 3 and 5 since there are no corresponding irreducible unimodular hermitian modules of dimensions 2, 3 and 5, respectively.

Let $C$ be an optimal curve of genus $g \leq 10$ over a finite field $\mathbb{F}_q$ of discriminant $-4$. In case $p \geq 13$ the unimodular hermitian module $(\mathcal{O}_K^g, h)$ corresponding to an optimal curve $C$ of genus $2 \leq g \leq 10$ should have $\#\text{Aut}(\mathcal{O}_K^g, h) \leq 168 \cdot (g - 1)$, but for $g \in \{4, 6, 7, 8, 9, 10\}$ there are no such hermitian modules as Table 2 shows.

It is easy to check that there are no $q = p^n$ with $p \in \{2, 3, 7, 11\}$ and with $d(F_q) = -4$. For example, if $p = 2$ and $n > 1$ then $(m/2)^2 = 2^n - 1 \equiv 3 \pmod{4}$, but 3 is not a square mod 4.

If char($\mathbb{F}_q$) = 5 one sees that number of the automorphisms of the unimodular irreducible hermitian module is greater than twice the Singh bound. □

Since there are no curves with defect 1 (cf. [15]), we get the following theorem.

**Theorem 3.4.** If $C$ is a curve of genus $g$ with $3 \leq g \leq 7$ over a finite field $\mathbb{F}_q$ with discriminant $-4$ and $q \neq 2$ then

$$|\#C(\mathbb{F}_q) - q - 1| \leq g[2\sqrt{q}] - 2.$$  

### 3.4. Optimal curves over finite fields with discriminant $-7$.

As before we begin with producing the table of orders of automorphism groups of the irreducible unimodular hermitian modules over an imaginary quadratic extension $K$ of $\mathbb{Q}$ with discriminant $d(K) = -7$.

| dim | #Aut |
|-----|------|
| 2   | —    |
| 3   | $2^4 \cdot 3 \cdot 7$ |
| 4   | $2^7 \cdot 3^2$ |
| 5   | $2^8 \cdot 3 \cdot 5$ |
| 6   | $2^9 \cdot 3^2 \cdot 7^2, 2^{10} \cdot 3^2 \cdot 5, 2^9 \cdot 3^2 \cdot 5, 2^5 \cdot 3^2 \cdot 5 \cdot 7$ |
| 7   | $2^{10} \cdot 3^4 \cdot 5 \cdot 7, 2^{11} \cdot 3^2 \cdot 5 \cdot 7, 2^{11} \cdot 3^3 \cdot 7 \cdot 2^8 \cdot 3^2 \cdot 5 \cdot 7, 2^{11} \cdot 3^2, 2^{10} \cdot 3^2, 2^6 \cdot 3^3 \cdot 5, 2^{10} \cdot 3, 2^7 \cdot 3 \cdot 5$ |

Table 3

Assume that there is an optimal curve $C$ of genus $g$ with $3 \leq g \leq 7$ over a finite field $\mathbb{F}_q$. By the classification list and Proposition 2.3 we see that there is a projection $f_k : C \to E$ of degree 2 on an optimal elliptic curve $E$, hence there exists an involution $\sigma \in \text{Aut}_{\mathbb{F}_q}(C)$ so that $C/(\langle \sigma \rangle) \cong E$. Let $G$ be a Sylow 2-subgroup of the automorphism group $\text{Aut}_{\mathbb{F}_q}(C)$ containing $\sigma$. The involution $\sigma$ does not belong to the center of the group $G$, since otherwise $\text{Aut}_{\mathbb{F}_q}(C/(\langle \sigma \rangle)) \cong \text{Aut}(E) = \{1, -1\}$ contains the factor group $G/(\langle \sigma \rangle)$ of order greater than $2^3$ as the table shows. Since the center of $G$ is not trivial we can choose an involution $\tau$ in the center of $G$. Then group generated by $\sigma$ and
\( \tau \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) and hence we have the following diagram of coverings

\[
C \quad \begin{array}{c}
\downarrow \quad 2:1 \quad \downarrow \quad 2:1 \quad \downarrow \\
C/\langle \sigma \rangle \cong E \quad C/\langle \sigma \tau \rangle \quad C/\langle \tau \rangle \\
\downarrow \quad 2:1 \quad \downarrow \quad 2:1 \quad \downarrow \\
C/\langle \sigma, \tau \rangle
\end{array}
\]

Relations between idempotents in \( \text{End}(\text{Jac}(C)) \) imply that we have an isogeny

\[
(3.1) \quad \text{Jac}(C) \times (\text{Jac}(C/\langle \sigma, \tau \rangle))^2 \sim E \times \text{Jac}(C/\langle \sigma \tau \rangle) \times \text{Jac}(C/\langle \tau \rangle),
\]

(cf. for example Thm. A of [5]).

**Lemma 3.5.** An optimal curve of genus 3 over the finite field \( \mathbb{F}_q \) is not a hyperelliptic.

**Proof.** Since there exists an irreducible unimodular hermitian module of dimension 3, there exists an irreducible principal polarization on the abelian variety \( E^3 \), where \( E \) is an optimal elliptic curve. Hence by [7] there exists an optimal curve of genus 3 over the finite field \( \mathbb{F}_q \).

If \( C \) is a hyperelliptic optimal curve of genus 3 over \( \mathbb{F}_q \) with hyperelliptic involution \( \iota \) (actually, then \( \iota = \tau \)) then we have

\[
\text{Jac}(C) \sim E \times \text{Jac}(C/\langle \sigma \rangle),
\]

where \( \sigma \) an involution as constructed before. Now the genus of the quotient curve \( C/\langle \sigma \rangle \) is 2, but there is no optimal curve of genus 2, and hence there is no hyperelliptic optimal curve of genus 3. \( \square \)

The key result of this subsection is the following proposition.

**Proposition 3.6.** There is no optimal curve \( C \) of genus \( g \) with \( 2 \leq g \leq 7 \) and \( g \neq 3 \) over a finite field \( \mathbb{F}_q \) of discriminant \(-7\).

**Proof.** If \( C \) is an optimal curve of genus 4, then by Hurwitz’s formula and by the nonexistence of optimal curves of genus 2, the quotient curves \( C/\langle \tau \rangle \) and \( C/\langle \sigma \tau \rangle \) have genus < 2. Therefore, the dimension of the left-hand-side of the equation \((3.1)\) is greater or equal to 4, while the dimension of the right-hand-side is less or equal to 3, which is impossible.

If \( C \) is an optimal curve of genus 5 then according to the choice of \( \tau \) it follows that \( G/\langle \tau \rangle \subseteq \text{Aut}_{\mathbb{F}_q}(C/\langle \tau \rangle) \) and \( \#G/\langle \tau \rangle \geq 2^6 \). By Table 3 it follows that \( g(C/\langle \tau \rangle) \geq 4 \) or \( g(C/\langle \tau \rangle) = 0 \) and by Hurwitz’s formula \( g(C/\langle \tau \rangle) \leq 3 \). Hence the quotient curve \( C/\langle \tau \rangle \) is a rational curve and \( C/\langle \sigma, \tau \rangle \) is also rational. Therefore the dimension of the left-hand-side of the equation \((3.1)\) is 5 and the dimension of the right-hand-side is less or equal to 2, since an optimal curve of genus 3 is not hyperelliptic. This is a contradiction.
If $C$ is an optimal curve of genus 6 then the dimension of the left-hand-side of the equation (3.1) is an even number $\geq 6$, but the dimension of the right-hand-side $\leq 7$ and cannot be equal to 6, since there is no an optimal curve of genus 2.

If $C$ is an optimal curve of genus 7, then the quotient curve $C/\langle \tau \rangle$ is either an optimal curve of genus 3 or a rational curve, since $\#G/\langle \tau \rangle \geq 2^4$. In the first case there is only one choice for the order of a Sylow 2-subgroup $G$, namely $\#G = 2^5$ as Table 3 shows. Then $\#G/\langle \tau \rangle = 2^4$ and since $\#G/\langle \tau \rangle \subset \text{Aut}_{F_q}(C)$ we get that the quotient curve $C/\langle \tau \rangle$ is hyperelliptic by a similar argument, but there is no hyperelliptic optimal curve of genus 3 by Lemma 3.5. In the second case, $\tau$ is the hyperelliptic involution and $\langle \tau, \sigma \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Now (3.1) gives the following splitting of Jacobian (up to isogeny)

$$\text{Jac}(C) \sim \text{Jac}(C/\langle \sigma \rangle) \times \text{Jac}(C/\langle \sigma \tau \rangle).$$

We know that $\text{Jac}(C/\langle \sigma \rangle)$ has the dimension 1. Therefore $C/\langle \sigma \tau \rangle$ has to be an optimal curve of genus 6, but such a curve does not exist as we just showed.

As a consequence of Proposition 3.6 we obtain the following theorem.

**Theorem 3.7.** If $C$ is a curve of genus $g$ with $4 \leq g \leq 7$ over a finite field $\mathbb{F}_q$ of discriminant $-7$ then

$$|\#C(\mathbb{F}_q) - q - 1| \leq g[2\sqrt{q}] - 2.$$

### 3.5. Optimal curves over the finite field $\mathbb{F}_q$ with discriminant $-8$.

We start by giving the table of orders of automorphism groups of the irreducible hermitian modules over an imaginary quadratic extension $K$ of $\mathbb{Q}$ with discriminant $d(K) = -8$.

| dim | $\#\text{Aut}$ |
|-----|----------------|
| 2   | $2^4 \cdot 3$  |
| 3   | $-$            |
| 4   | $2^7 \cdot 3, 2^8 \cdot 3^2, 2^9 \cdot 3^2$ |
| 5   | $2^5 \cdot 3^2 \cdot 5$ |
| 6   | $2^{10} \cdot 3^2 \cdot 5, 2^{12} \cdot 3^3, 2^8 \cdot 3^4 \cdot 5, 2^{13} \cdot 3^4$ |
|     | $2^{10} \cdot 3^2 \cdot 5, 2^{11} \cdot 3^2, 2^{10} \cdot 3, 2^9 \cdot 3^2 \cdot 5, 2^7 \cdot 3^2, 2^{11}, 2^{12} \cdot 3$ |
| 7   | $2^{10} \cdot 3^4 \cdot 5 \cdot 7, 2^{10} \cdot 3^2 \cdot 5, 2^9 \cdot 3^3 \cdot 5, 2^9 \cdot 3^3 \cdot 5, 2^5 \cdot 3^2 \cdot 5 \cdot 7, 2^8 \cdot 3^2, 2^{10} \cdot 3^2, 2^{10} \cdot 3, 2^5 \cdot 3^2 \cdot 5, 2^8 \cdot 3, 2^5 \cdot 3^2, 2^{15} \cdot 3^4$ |

**Table 4**

**Proposition 3.8.** There is no optimal curve of genus $g$ with $3 \leq g \leq 7$ over a finite field $\mathbb{F}_q$ of discriminant $-8$ and $\text{char}(\mathbb{F}_q) \neq 3$.

**Proof.** There is no optimal curve of genus 3 over a finite field $\mathbb{F}_q$ of discriminant $-8$, since does not exist a corresponding hermitian module.
Assume that $C$ is an optimal curve of genus 4 over $\mathbb{F}_q$ of discriminant $-8$ and $\text{char}(\mathbb{F}_q) > 5$. If $C$ is hyperelliptic then by Table 4 we see that $\#\text{Aut}_{\mathbb{F}_q}(C) = \#\text{Aut}_{\mathbb{F}_q}(\text{Jac}(C)) > 84 \cdot (4 - 1)$, a contradiction to the bound we observed. If $C$ is not hyperelliptic then Tabel 4 yields that the automorphism group $\text{Aut}_{\mathbb{F}_q}(C)$ has a subgroup $G$ of order $2^6$. The group $G$ has a normal subgroup $H$ of order 2 and the automorphism group of the quotient curve $C/H$ has a subgroup isomorphic to $G/H$, but this is impossible if $g(C) > 0$, since this order contradicts Table 4. Therefore $C/H$ is a rational curve and $C$ is hyperelliptic.

If $C$ is an optimal curve of genus 5 and $\text{char}(\mathbb{F}_q) \geq 7$, then by Table 4 $\#\text{Aut}_{\mathbb{F}_q}(C) \geq \frac{1}{2} \#\text{Aut}_{\mathbb{F}_q}(\text{Jac}(C)) > 84 \cdot (5 - 1)$, and this contradicts the automorphism bound.

The case of genus 6 is similar to that of genus 5.

Now we consider optimal curves of genus 7. Assume that $C$ is an optimal curve of genus 7 over $\mathbb{F}_q$ of discriminant $-8$. We split the set of orders of the automorphism groups of the irreducible unimodular hermitian modules of the dimension 7 into three subsets, $T_1 = \{2^{10}, 3^4 \cdot 5 \cdot 7, 2^{10} \cdot 3^2 \cdot 5, 2^9 \cdot 3^3 \cdot 5, 2^9 \cdot 3^2 \cdot 5 \cdot 7, 2^5 \cdot 3^4 \cdot 5 \cdot 7 \}$, $T_2 = \{2^8 \cdot 3^3, 2^{10} \cdot 3^2, 2^{10} \cdot 3, 2^{8} \cdot 3 \}$ and $T_3 = \{2^5 \cdot 3^4 \}$.

If $\text{char}(\mathbb{F}_q) \geq 11$ and $\#\text{Aut}_{\mathbb{F}_q}(\text{Jac}(C)) \in T_1$ then we get that $\#\text{Aut}_{\mathbb{F}_q}(C) \geq \frac{1}{2} \#\text{Aut}_{\mathbb{F}_q}(\text{Jac}(C)) > 84(7 - 1)$, and hence there does not exist an optimal curve in this case.

If $\#\text{Aut}_{\mathbb{F}_q}(\text{Jac}(C)) \in T_2$ then it is not hyperelliptic, since otherwise one can get a contradiction as before. The automorphism group $\text{Aut}_{\mathbb{F}_q}(\text{Jac}(C))$ contains a subgroup $G$ of order $2^7$, which has a normal subgroup $H$ of order 2. The automorphism group of the quotient curve $C/H$ then has a subgroup isomorphic to $G/H$. Since there is no optimal curve of genus 3, 4, 5, 6 and $2^6$ does not divide the orders of automorphism groups of optimal curves of genus 1 and 2, it follows that $C/H$ is a rational curve. But this is also impossible, since $C$ is not hyperelliptic.

Let $\#\text{Aut}_{\mathbb{F}_q}(\text{Jac}(C)) \in T_3$ and $G$ be a Sylow 2-subgroup of $\text{Aut}_{\mathbb{F}_q}(C)$. Then similarly as for discriminant $-7$ there are two involutions $\sigma, \tau \in G$ with the same properties. Here we would like to remark that there are two cases: the first hermitian form give us an involution $\sigma$ by Proposition 2.3. In the second case, this does not work, but if we apply an isomorphism given in terms of standard bases as $e_i \to e_i$ if $i \neq 4$ and $e_4 \to e_4 - e_5 + e_6$, then by applying Proposition 2.3 we again have an involution $\sigma$. Therefore there is an isogeny

$$\text{Jac}(C) \times \text{Jac}(C/\langle \sigma, \tau \rangle)^2 \sim \text{Jac}(C/\langle \sigma \rangle) \times \text{Jac}(C/\langle \sigma \tau \rangle) \times \text{Jac}(C/\langle \tau \rangle).$$

The dimension of the left-hand-side is $> 6$, whereas the dimension of the right-hand-side is $< 6$, because one of the factors is an elliptic curve and the other two have genus $\leq 2$.

Finally, note that there is no finite field $\mathbb{F}_q$ of $\text{char}(\mathbb{F}_q) \in \{2, 5, 7\}$ of discriminant $-8$. Indeed, $2(4^n - 1) \equiv 2 \pmod{4}$ is not square in $\pmod{4}$,
similarly, $5^{2n+1} - 2$ is not a square (mod 5) and $7^{2n+1} - 2$ is not a square (mod 3).

□

Theorem 3.9. If $C$ is a curve of genus $g$ with $3 \leq g \leq 7$ over the finite field $\mathbb{F}_q$ of discriminant $-8$ and $\text{char}(\mathbb{F}_q) \neq 3$ then

$$|\#C(\mathbb{F}_q) - q - 1| \leq g[2\sqrt{q}] - 2.$$ 

4. Optimal curves over finite fields with discriminant $-11$.

4.1. Optimal elliptic curves and optimal curves of genus 2. In this subsection we study optimal elliptic curves over $\mathbb{F}_q$, their $\overline{\mathbb{F}}_q$–isomorphism classes and the $\mathbb{F}_q$–isomorphism classes in an isogeny class. In addition, we give the results of concrete calculations for finite fields $\mathbb{F}_q$ with discriminant $-11$ and $q < 10^4$. As we mentioned in Section 1, with that information we can find the number of isomorphism classes of abelian varieties which are candidates for being Jacobians of optimal curves.

Proposition 4.1. Up to isomorphism over a finite field $\mathbb{F}_q$ with discriminant $-11$ there exists exactly one maximal (resp. minimal) optimal elliptic curve $E$ over $\mathbb{F}_q$.

Proof. Let $E$ be an optimal elliptic curve over $\mathbb{F}_q$. The trace of Frobenius of the optimal elliptic curve over $\mathbb{F}_q$ equals $m = \pm[2\sqrt{q}]$ and its Frobenius endomorphism $F$ satisfies the equation $F^2 \pm mF + q = 0$. This fixes the isogeny class of the optimal maximal (resp. minimal) elliptic curve. Then $\text{End}_{\mathbb{F}_q}(E)$ contains a ring $\mathbb{Z}[X]/(X^2 \pm mX + q)$, which is the ring of integers $\mathcal{O}_K$ of $K = \mathbb{Q}(\sqrt{-11})$, hence $\text{End}_{\mathbb{F}_q}(E) = \mathcal{O}_K$. Since the class number of $\mathcal{O}_K$ is 1 the $\mathbb{F}_q$-isogeny class of $E$ contains one $\overline{\mathbb{F}}_q$-isomorphism class and such an $\overline{\mathbb{F}}_q$-isomorphism class contains exactly two $\mathbb{F}_q$-isomorphism classes, since $\text{Aut}_{\overline{\mathbb{F}}_q}(E) = \mathbb{Z}/2\mathbb{Z}$ (see the ”mass formula” in [3]). One of these classes corresponds to a maximal optimal curve, while its twist then gives a minimal elliptic curve.

□

Example 4.2. In the following table we give for various $\mathbb{F}_q$ of discriminant $-11$ an optimal maximal and an optimal minimal elliptic curve. In the table an entry $[a,b]$ means the elliptic curve over $\mathbb{F}_q$ given by an equation $y^2 = x^3 + ax + b$. 
Remark 4.3. The points of order 2 of the elliptic curve consist of two Galois orbits, one of length 1 and one of length 3. Indeed, In view of $-11 = m^2 - 4q$ and $\# E(\mathbb{F}_q) = q + 1 \pm m$ we see that $m$ is odd, hence $\# E(\mathbb{F}_q)$. This implies that there $\# E[2](\mathbb{F}_q) = 1$.

The case of optimal curves of genus 2 was dealt with by J-P. Serre [15]. One finds that over the field $\mathbb{F}_q$ with discriminant $-11$ an optimal curve of genus 2 always exists. J.-P. Serre constructed such curves by gluing two optimal elliptic curves. Here we show how an optimal curve of genus 2 over $\mathbb{F}_q$ can be obtained by gluing two optimal elliptic curves and that there is only one $\mathbb{F}_q$-isomorphism class of such curves. It also follows that up to $\mathbb{F}_q$-isomorphism there are at most two degree 2 maps of an optimal curve of genus 2 to a fixed optimal elliptic curve modulo translation on the elliptic curve.

Proposition 4.4. Up to isomorphism over the field $\mathbb{F}_q$ of discriminant $-11$ there exists exactly one maximal (resp. minimal) optimal curve $C$ of genus 2 over $\mathbb{F}_q$, viz., the fibered product over $\mathbb{P}^1$ of the two maximal (resp. minimal) optimal elliptic curves

$$y^2 = f(x) \quad \text{and} \quad y^2 = f(x)(\alpha x + \beta),$$

where $f(x)$ is polynomial of degree 3.

Proof. By Serre we know that exists at least one maximal (minimal) optimal curve of genus 2 over $\mathbb{F}_q$ with discriminant $-11$. Let $C$ be an optimal curve of genus 2 over $\mathbb{F}_q$. Using the equivalence of categories given in Section 2 the theta divisor of the canonical lift Jacobian $\text{Jac}(C)$ corresponds to a unimodular hermitian module that has the unimodular irreducible hermitian form given by

$$
\begin{pmatrix}
\frac{2}{2} & -\frac{1+\sqrt{-11}}{2} \\
-\frac{1-\sqrt{-11}}{2} & \frac{2}{2}
\end{pmatrix}.
$$
By Proposition 2.3 it follows that $C$ admits a map $\psi$ of degree 2 onto an optimal elliptic curve $E$. This means that the automorphism group of $C$ contains two commuting automorphisms, one $\alpha$ corresponding to $\psi$ and the other one the hyperelliptic involution $i$. It follows that $C$ is (the normalization of) a fibered product over $\mathbb{P}^1 = C/\langle \alpha, i \rangle$ of the two optimal elliptic curves $C/\langle \alpha \rangle$ and $C/\langle i \alpha \rangle$. Now we treat the maximal case; the minimal case follows then by twisting.

Let $C \cong E \times_{\mathbb{P}^1} E'$ be such a fibered product of two maximal elliptic curves. Of course $E'$ is isomorphic to $E$, the curve given in 4.1. In order that the fibered product of $E$ and $E'$ over the $x$-line $\mathbb{P}^1$ is of genus 2, three of their ramification points must coincide. If $E$ is given by an equation of the form $y^2 = f(x)$ then the polynomial $f(x)$ is irreducible over $\mathbb{F}_q$ and it follows that $E'$ can be written as $y^2 = (\alpha x + \beta) f(x)$ for $\alpha, \beta \in \mathbb{F}_q$ with $\alpha \neq 0$. By the result of Proposition 4.1 there exists an automorphism $g \in \text{PGL}(2, \mathbb{F}_q)$ of $\mathbb{P}^1$ that permutes the roots of $f(x)$ and sends $\infty$ to $x = -\beta/\alpha$. Since the roots of $f(x)$ generate a cubic extension of $\mathbb{F}_q$ the element $g$ is necessarily of order 3 or 1, hence there are at most two such equations for the curve $E'$ that can occur. 

\[ \Box \]

Remark 4.5. From the fact that an optimal curve $H$ of genus 2 over $\mathbb{F}_q$ with $d(\mathbb{F}_q) = -11$ is the fibered product of two optimal elliptic curves it follows that $H$ can be given by an equation $z^2 = F(x)$, where $F(x)$ is a polynomial of degree 6.

Remark 4.6. Since the points of order 2 of the elliptic curve consist of two Galois orbits, one of length 1 and one of length 3, it follows that the ramification points of our curve of genus 2 form two Galois orbits of length 3.

Example 4.7. Here we give a list of maximal optimal curves of genus 2 as given Proposition 4.1. In the table we denote a maximal curve of genus 2 given by an equation $z^2 = \alpha x^6 + \beta x^4 + \gamma x^2 + \delta$ by $(\alpha, \beta, \gamma, \delta)$ and a maximal elliptic curve given by an equation $y^2 = (x^3 + ax + b)(cx + d)$ by $[a, b; c, d]$. Minimal optimal curves of genus 2 are obtained by twisting.
4.2. Optimal curves of genus 3, 4 and 5. As J.-P. Serre showed (cf. [14]) there is no optimal curve of genus 3 over a field \( F_q \) with discriminant \(-11\), since there is no irreducible principal polarization of the Jacobian. Therefore we may start with genus 4.

**Lemma 4.8.** If there is an optimal curve \( C \) of genus 4 over \( \mathbb{F}_q \) with discriminant \(-11\), then there exist two commuting involutions \( \sigma, \tau \in \text{Aut}_{\mathbb{F}_q}(C) \) such that \( C/\langle \sigma \rangle \) is an optimal curve of genus 2 over \( \mathbb{F}_q \) and \( C/\langle \tau \rangle \) and \( C/\langle \sigma \tau \rangle \) are optimal elliptic curves over \( \mathbb{F}_q \). Moreover, we can find irreducible polynomials \( h_1(x), h_2(x) \in \mathbb{F}_q[x] \) of degree 3 such that \( C/\langle \sigma \rangle \) is given by an equation \( y^2 = h_1(x) \cdot h_2(x) \) and \( C/\langle \tau \rangle \), \( C/\langle \sigma \tau \rangle \) are given by equations \( y^2 = h_1(x) \) and \( y^2 = h_2(x) \).

**Proof.** Assume that there is an optimal curve \( C \) of genus 4 over \( \mathbb{F}_q \) with discriminant \(-11\). From the classification of unimodular hermitian modules it follows that \( \#\text{Aut}_{\mathbb{F}_q}(\text{Jac}(C)) \in \{2^2 \cdot 3^2, 2^4 \cdot 3 \cdot 5, 2^4 \cdot 3^2\} \). Let \( G \) denote a Sylow 2-subgroup of \( \text{Aut}_{\mathbb{F}_q}(C) \) and \( \sigma \in G \) an involution in the center of \( G \). Since \( G/\langle \sigma \rangle \subseteq \text{Aut}_{\mathbb{F}_q}(C/\langle \sigma \rangle) \) it follows that \( C/\langle \sigma \rangle \) is either a rational curve or an optimal curve of genus 2 since the order \( \#\text{Aut}(C/\langle \sigma \rangle) \) is too large for \( C/\langle \sigma \rangle \) being an elliptic curve. Moreover, Proposition 2.3 together with the classification list show that there is another involution \( \tau \in G \) such that the quotient curve \( C/\langle \tau \rangle \) is an optimal elliptic curve. Therefore \( \sigma, \tau \) generate a group isomorphic to \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \).

If \( \sigma \) is the hyperelliptic involution of \( C \), i.e. \( C/\langle \sigma \rangle \cong \mathbb{P}^1 \), then

\[
\text{Jac}(C) \sim \text{Jac}(C/\langle \tau \rangle) \times \text{Jac}(C/\langle \sigma \tau \rangle).
\]

However, the dimension of the left-hand-side is 4 and of that the right-hand-side is less or equal to 3, so \( \sigma \) cannot be the hyperelliptic involution.

| \( q \) | \( [a, b; c, d] \) | \( (\alpha, \beta, \gamma, \delta) \) |
|---|---|---|
| 23 | \([1, 11; 1, 19]\) | \((1, 12, 3, 10)\) |
| 59 | \([2, 22; 1, 49]\) | \((1, 30, 7, 39)\) |
| 113 | \([5, 44; 1, 24]\) | \((1, 41, 38, 112)\) |
| 383 | \([1, 91; 1, 135]\) | \((1, 361, 290, 356)\) |
| 509 | \([1, 70; 2, 208]\) | \((191, 431, 191, 501)\) |
| 563 | \([2, 363; 1, 189]\) | \((1, 559, 195, 212)\) |
| 1193 | \([5, 11; 1, 1017]\) | \((528, 1074, 1072, 657)\) |
| 1409 | \([6, 221; 3, 1135]\) | \((835, 187, 516, 278)\) |
| 3083 | \([2, 1009; 1, 2569]\) | \((1, 1542, 259, 1880)\) |
| 4973 | \([1, 1748; 2, 2341]\) | \((1865, 987, 4365, 4633)\) |
| 6323 | \([2, 221; 1, 1274]\) | \((1, 2501, 520, 3218)\) |

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Now assume that $C/\langle \sigma \rangle$ is an optimal curve of genus 2. Then there is the isogeny of Jacobians

$$\text{Jac}(C) \times \text{Jac}(C/\langle \sigma, \tau \rangle) \sim \text{Jac}(C/\langle \sigma \rangle) \times \text{Jac}(C/\langle \tau \rangle) \times \text{Jac}(C/\langle \sigma \tau \rangle),$$

again by comparing dimensions and since $g(C/\langle \sigma, \tau \rangle) = 0$ (and there is no double covering of a maximal elliptic curve by a maximal elliptic curve) we have that $C/\langle \sigma \tau \rangle$ is an optimal elliptic curve.

We know by Remark 4.6 that the ramification points of $C/\langle \sigma \rangle$ form two Galois orbits, say $\{p_1, p_2, p_3\}$ and $\{p_4, p_5, p_6\}$. We denote the inverse image of $p_i$ on $C$ by $\{p'_i, p''_i\}$. Then $\sigma(p'_i) = p''_i$ for all $i$, and we have $\tau(p'_i) = p'_i$ and $\tau(p''_{i+3}) = p''_i$ for $i = 1, 2, 3$. The involution $\sigma$ has two ramification points, say $q_1, q_2$ and these are interchanged by $\tau$. We let the image of $q_1$ (and of $q_2$) on $C/\langle \sigma, \tau \rangle$ be equal to $\infty$ and we denote the image of $p_i$ on $C/\langle \sigma, \tau \rangle$ by $P_i$. Then it is clear that the branch points of $C/\langle \tau \rangle \to C/\langle \sigma, \tau \rangle$ are $\{P_1, P_2, P_3, \infty\}$, while those of $C/\langle \sigma \tau \rangle \to C/\langle \sigma, \tau \rangle$ are $\{P_4, P_5, P_6, \infty\}$. This shows that we can choose an equation $y^2 = h_1(x)h_2(x)$ for $C/\langle \sigma \rangle$ and equations for $C/\langle \tau \rangle$ and $C/\langle \sigma \tau \rangle$ as indicated in the statement.

To check for the existence of an optimal curve of genus 4 over $\mathbb{F}_q$ with $d(\mathbb{F}_q) = -11$, it suffices by this lemma to check in view of Proposition 4.2 whether the unique optimal curve of genus 2 over $\mathbb{F}_q$ is of the form $y^2 = h_1(x)h_2(x)$ with $h_1(x), h_2(x)$ as in the lemma. This can easily be checked and we did this on a computer for $q < 10^4$ with $p \neq 3$.

**Corollary 4.9.** There is no optimal curve of genus 4 over a finite field $\mathbb{F}_q$ of discriminant $-11$ if $q < 10^4$ and $\text{char}(\mathbb{F}_q) \neq 3$.

Now we deal with genus 5.

**Theorem 4.10.** There is no optimal curve $C$ of genus 5 over a finite field $\mathbb{F}_q$ of discriminant $-11$ and $\text{char}(\mathbb{F}_q) > 5$.

**Proof.** Assume that $C$ is an optimal curve of genus 5 over $\mathbb{F}_q$ with discriminant $-11$. From the classification list of the unimodular irreducible hermitian modules it follows that $\# \text{Aut}_{\mathbb{F}_q}(\text{Jac}(C)) \in \{2^3 \cdot 3 \cdot 5 \cdot 11, 2^4 \cdot 3 \cdot 5, 2^4 \cdot 3\}$.

If $\text{char}(\mathbb{F}_q) > 5$ then $\# \text{Aut}_{\mathbb{F}_q}(\text{Jac}(C)) \neq 2^3 \cdot 3 \cdot 5 \cdot 11$, since $2^3 \cdot 3 \cdot 5 \cdot 11 \geq 84(5 - 1)$.

If $\# \text{Aut}_{\mathbb{F}_q}(\text{Jac}(C)) \in \{2^4 \cdot 3 \cdot 5, 2^4 \cdot 3\}$ then as for genus 4 there are two involutions $\sigma$ and $\tau$ such that $C/\langle \sigma \rangle$ is an optimal elliptic curve, $\sigma \tau = \tau \sigma$ and $C/\langle \tau \rangle$ is not an optimal elliptic curve. As a result $C$ is the fibered product of $C_1 := C/\langle \sigma \rangle$ and $C_2 := C/\langle \sigma \tau \rangle$ over $C/\langle \sigma, \tau \rangle$ and hence we have the isogeny of Jacobians

$$\text{Jac}(C/\langle \sigma \tau \rangle) \times \text{Jac}(C/\langle \tau \rangle) \times E \sim \text{Jac}(C/\langle \sigma, \tau \rangle)^2 \times \text{Jac}(C).$$

In view of the non-existence of an optimal curve of genus 3, the dimension of the left-hand-side $\leq 5$, but the dimension of the right-hand-side $\geq 5$. The only possibility is that the curves $C/\langle \sigma \tau \rangle$, $C/\langle \tau \rangle$ have genus 2 and that $C/\langle \sigma, \tau \rangle$ has genus 0. But then $C/\langle \sigma \tau \rangle$ and $C/\langle \tau \rangle$ share two of their six
ramification points and this contradicts Remark 4.6, which says that the six ramification points decompose in two Galois orbits of length 3.

5. Optimal curves over finite fields with discriminant $-19$.

5.1. Optimal elliptic curves and optimal curves of genus 2. In this subsection we explore optimal elliptic curves over $\mathbb{F}_q$ and produce the results of calculations for finite fields $\mathbb{F}_q$ with discriminant $-19$ and $q < 10^3$.

Arguments similar to that for discriminant $-11$ give the following results.

**Proposition 5.1.** Up to isomorphism over $\mathbb{F}_q$ with discriminant $-19$ there exists exactly one maximal (resp. minimal) optimal elliptic curve $E$ over $\mathbb{F}_q$.

**Remark 5.2.** The points of order 2 of the elliptic curve consist of two Galois orbits, one of length 1 and one of length 3.

**Proposition 5.3.** Up to isomorphism over the field $\mathbb{F}_q$ with discriminant $-19$ there exists exactly one maximal (resp. minimal) optimal curve $C$ of genus 2 over $\mathbb{F}_q$, viz., the fibered product over $\mathbb{P}^1$ of two maximal (resp. minimal) optimal elliptic curves of the form

$$y^2 = f(x) \quad \text{and} \quad y^2 = f(x)(\alpha x + \beta).$$

**Remark 5.4.** As before (for discriminant $-11$), it follows that the ramification points of our genus 2 curve form two Galois orbits of length 3.

5.2. Optimal Curves of Genus 3 and 4.

**Proposition 5.5.** Up to isomorphism over $\mathbb{F}_q$ with discriminant $-19$ there is an optimal curve $C$ of genus 3 over $\mathbb{F}_q$, namely the double covering of a (maximal or minimal) optimal elliptic curve.

**Proof.** Applying the equivalence of categories as described in Section 2 along with the classification of unimodular hermitian modules, we see that the canonical lift of the Jacobian of an optimal curve of genus 3 has a polarization that corresponds to the unimodular hermitian form

$$
\begin{pmatrix}
2 & 1 & -1 \\
1 & 3 & (-3 + \sqrt{-19})/2 \\
-1 & (-3 - \sqrt{-19})/2 & 3
\end{pmatrix}.
$$

By [7] this is the Jacobian of a curve, hence there exists a (minimal or maximal) optimal curve $C$ of genus 3 over a finite field $\mathbb{F}_q$ and $f_1 : C \to E$ is a double covering onto an optimal elliptic curve by Proposition 2.3.

To finish the proof we remark that there is only one class of such irreducible unimodular hermitian modules, hence by the equivalence of categories and Torelli’s theorem the optimal curve of genus 3 is unique up to isomorphism over $\mathbb{F}_q$. □
In the following we use the explicit description of the automorphism groups of the irreducible unimodular hermitian modules \[ \mathbb{II} \] to prove the following theorem:

**Theorem 5.6.** There is no optimal curve of genus 4 over a finite field \( \mathbb{F}_q \) with discriminant \(-19\) if \( q < 10^4 \).

**Proof.** Let \( C \) be an optimal curve of genus 4 over \( \mathbb{F}_q \). The classification of the unimodular hermitian modules of rank 4 and discriminant \(-19\) tell us that there are 9 irreducible unimodular hermitian modules up to isomorphism. If a unimodular hermitian module corresponds to the Jacobian of a curve \( C \), then its automorphism group is isomorphic to the automorphism group of the polarized abelian variety \( \text{Jac}(C) = (E^g, \Theta) \), hence by the Theorem of Torelli the automorphism group of the hermitian lattice and the automorphism group of the curve \( C \) coincide if the curve is hyperelliptic; in the other case their orders differ by a factor 2.

From the structure of the automorphism group of the unimodular hermitian modules we get that an automorphism group \( \text{Aut}_{\mathbb{F}_q}(C) \) of an optimal curve \( C \) has subgroup isomorphic to either \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) or \( \mathbb{Z}/5\mathbb{Z} \) (cf. the Appendix).

To manage the case when \( \text{Aut}(C) \) has subgroup \( \mathbb{Z}/5\mathbb{Z} \) (this holds for the hermitian unimodular modules in the cases (2) – (4)) we consider the cyclic Galois covering \( C \to C/\mathbb{Z}/5\mathbb{Z} \). Hurwitz’s formula implies that the quotient curve has genus \( g' = 0 \) since \( 6 = 10(g' - 1) + 4t \) with \( t \) the number of branch points, hence \( C/\mathbb{Z}/5\mathbb{Z} \cong \mathbb{P}^1 \). The number of rational points on the curve \( C \) is \( 5s + l \), where \( l \) is the number of rational ramification points.

If \( d(\mathbb{F}_q) = -19 \) then there are only three possibilities for the value \( \#C(\mathbb{F}_q) \pmod{5} \), namely \( \#C(\mathbb{F}_q) \equiv 0, 1 \) or 2 (mod 5).

If \( \#C(\mathbb{F}_q) \equiv 2 \) (mod 5) then two rational points must be ramification points since \( 0 \leq l \leq 4 \). We may assume that the points 0 and \( \infty \) are ramified in the covering, and hence the curve \( C \) can be given by an equation

\[
z^5 = \gamma x^{\nu_1} (x^2 + ax + b)^{\nu_2},
\]
where \( 1 \leq \nu_1, \nu_2 \leq 4 \) and the polynomial \( x^2 + ax + b \) is irreducible in \( \mathbb{F}_q[x] \). Since infinity is ramified, it follows that \( \nu_1 + 2\nu_2 \neq 5 \). If \( \#C(\mathbb{F}_q) \equiv 1 \) (mod 5) then one rational point must be ramification point. One may assume that \( \infty \) is ramified in the covering, and hence the curve \( C \) can be given by an equation

\[
z^5 = \gamma (x^3 + ax + b)^{\nu},
\]
where \( 1 \leq \nu \leq 4 \) and the polynomial \( x^3 + ax + b \) has to be irreducible in \( \mathbb{F}_q[x] \). If \( \#C(\mathbb{F}_q) \equiv 0 \) (mod 5) then there are no rational ramification points. Therefore we may assume that the curve \( C \) is given by equation

\[
z^5 = \gamma (x^2 + ax + b)^{\nu_1} (x^2 + c)^{\nu_1},
\]
where the polynomials \( x^2 + ax + b \) and \( x^2 + c \) are irreducible over \( \mathbb{F}_q \). Since infinity is unramified, it follows that \( 2\nu_1 + 2\nu_2 \equiv 0 \) (mod 5). A computer
search showed that there are no optimal curves of these forms. This finishes the cases when \( \text{Aut}(C) \) contains \( \mathbb{Z}/5\mathbb{Z} \).

For the other cases in view of the Theorem of Torelli we treat the hyperelliptic and the non-hyperelliptic case separately.

If \( C \) is a hyperelliptic curve with the hyperelliptic involution \( \iota \), then by Theorem of Torelli the automorphism group \( \text{Aut}(C) \) equals to \( \text{Aut}(\text{Jac}(C)) \). The automorphism groups of the unimodular hermitian module contain an involution \( \neq \iota \) (cf. the Appendix), and hence there is a non-hyperelliptic involution \( \sigma \) in \( \text{Aut}(C) \). Therefore we can consider the following diagram:

\[
\begin{array}{c}
C \\
\downarrow^{2:1} \\
H_1 := C/\langle \sigma \rangle \\
\downarrow^{2:1} \\
C/\langle \iota \sigma \rangle \\
\downarrow^{2:1} \\
C/\langle \sigma, \iota \sigma \rangle \\
\end{array}
\]

The quotient curve \( C/\langle \sigma, \iota \sigma \rangle \) is a rational line, since it is a quotient of \( C/\iota \cong \mathbb{P}^1 \). In view of the isogeny \( \text{Jac}(C) \sim \text{Jac}(C/\langle \iota \sigma \rangle) \times \text{Jac}(C/\langle \sigma \rangle) \) we can assume that \( H_1 \) and \( H_2 \) are not elliptic curves. From Hurwitz's genus formula and the fact that \( \sigma \) and \( \iota \sigma \) are non-hyperelliptic involutions it follows that genus of the curve \( H_1 \) and \( H_2 \) is 2. By easy Galois theory it follows that \( C \) is the fibered product of the two genus 2 covers of \( \mathbb{P}^1 \). Both \( H_i \) are isomorphic to the optimal curve given in previous subsection. If we represent them as double covers of \( \mathbb{P}^1 \) five of the six ramification points must coincide. But we know that in our case the ramification points form two orbits of three points under the action of Frobenius. Therefore we have three or six common ramification points. This contradicts our assumption.

Finally, we assume that the curve \( C \) is not hyperelliptic, and therefore \( \text{Aut}(C) \cong \text{Aut}(\text{Jac}(C))/\langle -1 \rangle \) by the Theorem of Torelli. In case that the polarization does not corresponds to the modules of the cases (2), (3) or (4) we have two commuting involutions \( \sigma, \tau \in \text{Aut}(\text{Jac}(C), \Theta)/\langle -1 \rangle \), hence \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) in the automorphism group of \( C \). Thus we again obtain a diagram of the coverings:

\[
\begin{array}{c}
C \\
\downarrow^{2:1} \\
C/\langle \sigma \rangle \\
\downarrow^{2:1} \\
C/\langle \sigma \tau \rangle \\
\downarrow^{2:1} \\
C/\langle \sigma, \tau \rangle \\
\end{array}
\]

If any quotient curve is not an optimal elliptic curve then each quotient curves \( C/\langle \sigma \rangle, C/\langle \sigma \tau \rangle \) and \( C/\langle \tau \rangle \) has genus 2. Furthermore, \( C/\langle \sigma, \tau \rangle \) cannot
be \( \mathbb{P}^1 \) since then \( C \) is the fibered product of two optimal genus 2 curves, and this contradicts the splitting of the Jacobian \( \text{Jac}(C) \) as we saw above. So \( C/\langle \sigma, \tau \rangle \) must be an optimal elliptic curve \( E \). Since we saw that up to isomorphism over \( E \) there are only two degree 2 maps of our optimal genus 2 curve to \( E \) two of the three maps \( C/\langle \sigma \rangle \to E \), \( C/\langle \sigma \tau \rangle \to E \) and \( C/\langle \tau \rangle \to E \) coincide; but this contradicts the fact that \( C \) is the fibered product of \( C/\langle \sigma \rangle \) and \( C/\langle \tau \rangle \) over \( E = C/\langle \sigma \tau \rangle \).

If one of the quotient curves is an optimal elliptic curve the \( n \) from the splitting of the Jacobian of curve \( C \) for the Klein 4-group it follows that the second one is also an optimal elliptic curve and the third quotient curve is an optimal curve of genus 2; moreover \( C/\langle \sigma, \tau \rangle \) is isomorphic to \( \mathbb{P}^1 \). As a result we have that an optimal curve of genus 4 is the fibered product of optimal elliptic curves if and only if an optimal curve of genus 2 can be given by an equation \( z^2 = F(x) \) such that \( F(x) \) splits into two irreducible polynomials \( g_1(x), g_2(x) \) of degree 3 and such that the equations \( u^2 = g_1(x), w^2 = g_2(x) \) give optimal elliptic curves. An easy computer check for each \( q \) shows that an optimal curve of genus 2 is not given by such an equation. \( \square \)

5.3. Optimal curves of genus 5.

**Lemma 5.7.** An optimal curve \( C \) of genus 5 over \( \mathbb{F}_q \) with discriminant \(-19\) is not a hyperelliptic curve nor a double cover of a curve of genus \( \geq 3 \).

**Proof.** If \( C \) is a double cover of a genus 3 curve then it is an unramified cover and its number of rational points is even. But \( m = \lfloor 2\sqrt{q} \rfloor \) is odd and hence \( \#C(\mathbb{F}_q) = q + 1 \pm 5 \cdot m \) is odd. The Hurwitz formula excludes the case that it is a double cover of a curve of genus \( \geq 4 \).

If \( C \) is a hyperelliptic curve with the hyperelliptic involution \( \tau \) then from the structure of the automorphism group of the irreducible unimodular hermitian modules it follows that there is an extra involution. Let us denote this involution by \( \sigma \). Then we have the following relation on Jacobians \( E^5 \cong \text{Jac}(C) \sim \text{Jac}(C/\langle \sigma \rangle) \times \text{Jac}(C/\langle -\sigma \rangle) \), where \( E \) is an optimal elliptic curve. Therefore one of the quotient curves is a curve of genus either 4 or 3 and this is impossible. \( \square \)

Now we show that the automorphism group of a maximal curve of genus 5 over \( \mathbb{F}_q \) cannot have the Klein 4-group \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) as subgroup.

**Lemma 5.8.** If \( C \) is a maximal optimal curve of genus 5 over \( \mathbb{F}_q \) with discriminant \(-19\) then \( \text{Aut}(C) \) does not contain the Klein 4-group.

**Proof.** Suppose that \( \sigma \) and \( \tau \) are two commuting involutions of \( C \). Then \( C \) is the fibered product of \( C_1 := C/\langle \sigma \rangle \) and \( C_2 := C/\langle \tau \rangle \) over \( C/\langle \sigma, \tau \rangle \). Put \( C_3 := C/\langle \sigma \tau \rangle \). Then we have the isogeny

\[
\text{Jac}(C_1) \times \text{Jac}(C_2) \times \text{Jac}(C_3) \sim \text{Jac}(C/\langle \sigma, \tau \rangle)^2 \times \text{Jac}(C).
\]

In view of Proposition 5.7 the dimension of the part on the left-hand-side is \( \leq 6 \), but the dimension of the right-hand-side is odd and \( \geq 5 \). The only
possibility is that one of the $C_i$, say $C_1$, has genus 1 and the other two have genus 2 and that $C/\langle \sigma, \tau \rangle$ has genus 0. But then the $C_2$ and $C_3$ share two of their six ramification points and this contradicts Remark 5.5, which implies that the six ramification points decompose in two Galois orbits of length 3.

\[ \square \]

**Theorem 5.9.** There is no optimal curve $C$ of genus 5 over a finite field $F_q$ with $d(F_q) = -19$ and $q \not\equiv 1 \mod 5$

**Proof.** Looking at generators of automorphism groups of unimodular irreducible hermitian modules, we see that $\text{Aut}_{F_q}(C)$ contains either the Klein 4-group, or the cyclic group $\mathbb{Z}/5\mathbb{Z}$ (cf. the Appendix). Since Lemma 5.8 excludes the Klein 4-group we consider a covering $C \to C / (\mathbb{Z}/5\mathbb{Z})$.

The quotient curve $C / (\mathbb{Z}/5\mathbb{Z})$ is an optimal elliptic curve by Hurwitz’s formula; moreover, there are exactly two branch points (here we work over $\overline{\mathbb{F}}_q$). Therefore $\#C(\mathbb{F}_q) \equiv 0 \mod 5$ or $\#C(\mathbb{F}_q) \equiv 2 \mod 5$ and hence $q \equiv 1$ or $3 \mod 5$ since $\#C(\mathbb{F}_q) = q + 1 \pm 5 \cdot m$. On the other hand $q \not\equiv 3 \mod 5$, because otherwise we have for the Legendre symbol $\left( \frac{m^2}{q} \right) = \left( \frac{4q-10}{5} \right) = -1$.

\[ \square \]

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6. Appendix

Here we present the unimodular irreducible hermitian modules of dimension 4 and 5 over an imaginary quadratic extension $K$ of $\mathbb{Q}$ with discriminant $d(K) = -19$ such that their automorphism group does not contain $\mathbb{Z}/5\mathbb{Z}$ as a subgroup. We also show that the Klein 4-group is a subgroup of each automorphism group. The irreducible hermitian modules and elements the automorphism groups were taken from [11].

Throughout we let $w = \frac{1+\sqrt{-19}}{2}$.

6.1. Dimension 4.

$$H_1 = \begin{pmatrix} 3 & -1 & 3 & 2+w \\ -1 & 1+w & 3 & 0 \\ 2-w & 0 & -1 & 3 \end{pmatrix}, \ |\text{Aut}| = 2^3 \cdot 3,$$

$$\alpha_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1+w & -3 & -1 \\ -1 & -2+w & -1-w & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & -1 & 0 & 1+w \\ -1-w & 0 & -1 & -2+w \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\alpha_2^4 = -1, \alpha_2^2 = 1, \ (\alpha_2^2 \cdot \alpha_3) = -\alpha_3 \cdot (\alpha_2^2).$$

$$H_2 = \begin{pmatrix} 2 & 0 & 2 \\ 0 & -1 & 2 \\ -1 & -1+w & 1-w & 4 \end{pmatrix}, \ |\text{Aut}| = 2^3 \cdot 3^2,$$

$$\alpha_3 = \begin{pmatrix} -2 & -1+w & 1-w & -3 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & -1+w & 2 \end{pmatrix}, \alpha_5 = \begin{pmatrix} -1 & -1+w & 1-w & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$(\alpha_3 \cdot \alpha_5)^6 = \alpha_5^2 = 1, \ (\alpha_3 \cdot \alpha_5)^3 \cdot \alpha_5 = \alpha_5 \cdot (\alpha_3 \cdot \alpha_5)^3.$$

$$H_6 = \begin{pmatrix} 2 & -1 & 2 \\ 0 & -1 & 3 \\ 0 & -w & -1+w & 4 \end{pmatrix}, \ |\text{Aut}| = 2^1 \cdot 3,$$

$$\alpha_2 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 2-w & 3-2w & -1-2-w \\ w & w & 0 & 1 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -2+w & -3+2w & 1 & 2+w \\ -w & -w & 0 & -1 \end{pmatrix}.$$

$$\alpha_2^6 = \alpha_2^2 = 1, \ (\alpha_2^2 \cdot \alpha_3^2) = \alpha_2^2 \cdot \alpha_3.$$
\[
\begin{align*}
\alpha_3 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ -1 + w & 0 & 0 & -1 \end{pmatrix}, \\
\alpha_4 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ w & w & -1 & -3 + w \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
\alpha_3^2 = \alpha_4^2 &= 1, \ \alpha_3 \cdot \alpha_4 &= \alpha_4 \cdot \alpha_3.
\end{align*}
\]

\[
H_8 = \begin{pmatrix} 2 & 0 & 2 \\ -w & 0 & 3 \\ 0 & -w & 0 \end{pmatrix}, \ |\text{Aut}| = 2^5,
\]

\[
\alpha_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -w & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \ \alpha_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.
\]

\[
\alpha_2^2 = \alpha_3^2 &= 1, \ \alpha_2 \cdot \alpha_3 = \alpha_3 \cdot \alpha_2.
\]

\[
H_9 = \begin{pmatrix} 2 & 0 & 2 \\ 0 & -1 & 3 \\ -1 & -1 & 1 - w & 3 \end{pmatrix}, \ |\text{Aut}| = 2^4,
\]

\[
\alpha_3 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \ \alpha_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & -1 & 0 & -1 \end{pmatrix}.
\]

\[
\alpha_3^2 = \alpha_4^2 &= 1, \ \alpha_3 \cdot \alpha_4 = \alpha_4 \cdot \alpha_3.
\]

### 6.2. Dimension 5.

\[
H_1 = \begin{pmatrix} 3 & -1 & 3 & 3 \\ -1 & 1 & 3 & 3 \\ 0 & -w & 0 & 3 \\ 1 & -1 + w & -2 + w & -1 & 4 \end{pmatrix}, \ |\text{Aut}| = 2^3,
\]

\[
\alpha_2 = \begin{pmatrix} -1 - w & -2w & 2 & -w & -2 & 4 \\ 0 & 1 & 0 & 0 & 0 \\ w & 2w & -3 + w & 2 & -4 \\ 0 & 0 & 0 & 1 & 0 \\ -w & -w & 2 & -w & -1 & 3 \end{pmatrix},
\]

\[
\alpha_3 = \begin{pmatrix} -w & 1 - 2w & 2 & -w & -2 & 4 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 2 & 5 & 2 + w & 1 - w & -1 + 2w \end{pmatrix}.
\]

\[
\alpha_2^2 = \alpha_3^2 &= 1, \ \alpha_2 \cdot \alpha_3 = \alpha_3 \cdot \alpha_2.
\]

\[
H_6 = \begin{pmatrix} 2 & 1 & 2 \\ -1 & -1 & 3 \\ 1 & 1 & 0 & 3 \\ 0 & 0 & w & -1 \end{pmatrix}, \ |\text{Aut}| = 2^4 \cdot 3,
\]
\[
\alpha_1 = \begin{pmatrix}
1 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1
\end{pmatrix},
\alpha_2 = \begin{pmatrix}
w & w & -2 + 2w & -1 + w & -3 - w \\
w & -1 + w & -2 + 2w & -1 & -3 - w \\
-w & -w & 1 - 2w & 1 + w & 3 + w \\
1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1
\end{pmatrix},
\alpha_3 = \begin{pmatrix}
w & w & -2 + 2w & -1 + w & -3 - w \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
-1 + w & w & -2 + 2w & -w & -3 - w \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

\[
\alpha_2 \cdot \alpha_3 = (\alpha_1 \cdot \alpha_3)^4 = 1, \quad (\alpha_2 \cdot \alpha_3) \cdot (\alpha_1 \cdot \alpha_3)^2 = (\alpha_1 \cdot \alpha_3)^2 \cdot (\alpha_2 \cdot \alpha_3)
\]

\[
H_7 = \begin{pmatrix}
2 & & & & \\
1 & 2 & & & \\
-1 & 0 & 3 & & \\
1 & 0 & 0 & 3 & \\
0 & 0 & 1 - w & -1 & 3
\end{pmatrix}, \quad |\text{Aut}| = 2^4 \cdot 3,
\]

\[
H_8 = \begin{pmatrix}
2 & & & & \\
0 & & 2 & & \\
0 & & -1 & 3 & \\
0 & 1 & -1 - w & 3 & \\
1 - w & 0 & 0 & 0 & 3
\end{pmatrix}, \quad |\text{Aut}| = 2^4 \cdot 3,
\]

\[
H_9 = \begin{pmatrix}
2 & & & & \\
-1 & 2 & & & \\
-1 & 1 & 0 & 3 & \\
-1 & 0 & 1 & 1 - w & 3
\end{pmatrix}, \quad |\text{Aut}| = 2^3 \cdot 3,
\]

\[
\alpha_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & -2 & -2 + w & 1 + w & 0 \\
0 & 1 & -1 - w & -1 & 0 \\
0 & -1 & -2 + w & w & 0 \\
1 - w & 0 & 0 & 0 & -1
\end{pmatrix},
\alpha_3 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & -2 & -2 + w & 1 + w & 0 \\
0 & 1 & 1 - w & -1 - w & 0 \\
0 & -1 & -2 + w & w & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

\[
\alpha_2^2 = -1, \quad \alpha_3^2 = 1, \quad \alpha_2 \cdot \alpha_3 = \alpha_3 \cdot \alpha_2.
\]

\[
H_9 = \begin{pmatrix}
2 & & & & \\
-1 & 2 & & & \\
-1 & 1 & 0 & 3 & \\
-1 & 0 & 1 & 1 - w & 3
\end{pmatrix}, \quad |\text{Aut}| = 2^3 \cdot 3,
\]

\[
\alpha_3 = \begin{pmatrix}
-1 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
-3 & -1 + w & 1 & -1 - 2w & -5 + w \\
0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1
\end{pmatrix},
\alpha_4 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
3 & -w & -1 & 1 + 2w & 5 - w \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

\[
\alpha_2^2 = \alpha_3^2 = 1, \quad \alpha_3 \cdot \alpha_4 = \alpha_4 \cdot \alpha_3.
\]
\[
\alpha_3 = \begin{pmatrix}
0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
1 & -1 & -1 & 0 & 0 \\
-1 & 1 & 0 & -1 & 0 \\
3 - w & -4 & -3 + 2w & 4 + w & 1 \\
\end{pmatrix}, \alpha_4 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
-2 + w & 3 & 3 & -2w & -4 - w & -1 \\
\end{pmatrix}.
\]

\[
\alpha_3^2 = \alpha_3^4 = 1, \alpha_3 \cdot \alpha_4 = \alpha_4 \cdot \alpha_3.
\]

\[
\alpha_2 = \begin{pmatrix}
1 & -3 + 3w & -w & 7 - w & 3 \\
0 & 1 & 0 & 0 & 0 \\
0 & 3 & -1 & -1 - w & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 2 - w & 0 & -3 & -1 \\
\end{pmatrix}, \alpha_3 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & -3 & 1 & 1 + w & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & -2 + w & 0 & 3 & 1 \\
\end{pmatrix}.
\]

\[
\alpha_2^2 = \alpha_3^2 = 1, \alpha_3 \cdot \alpha_2 = \alpha_2 \cdot \alpha_3.
\]

\[
\alpha_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
1 & 0 & -3 & -2 + w & -1 \\
\end{pmatrix}, \alpha_3 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & -2 + w & 1 + w & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 3 & 2 - w & 1 \\
\end{pmatrix}.
\]

\[
\alpha_2^2 = \alpha_3^2 = 1, \alpha_3 \cdot \alpha_2 = \alpha_2 \cdot \alpha_3.
\]

\[
\alpha_2 = \begin{pmatrix}
2 & 1 & 0 & 0 & 0 \\
0 & 3 & 1 & 0 & 0 \\
0 & -1 & -1 - w & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}, \alpha_3 = \begin{pmatrix}
2 & 1 & -w & 3 \\
-1 & -1 & 0 & 3 \\
1 - w & 1 & -1 + w & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

\[
\alpha_2^2 = \alpha_3^2 = 1, \alpha_3 \cdot \alpha_2 = \alpha_2 \cdot \alpha_3.
\]

\[
\alpha_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
-1 + 2w & -2 + w & 1 + w & -1 + w & 2 \\
0 & 0 & 1 & 0 & 0 \\
1 - 2w & 3 - w & -1 - w & 2 - w & -2 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}, \alpha_3 = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
1 - 2w & 2 - w & -1 - w & 1 - w & -2 \\
-2 + 2w & -3 + w & 1 + w & -2 + w & 2 \\
1 - w & 0 & 0 & 0 & -1 \\
\end{pmatrix}.
\]

\[
\alpha_2^2 = \alpha_3^2 = 1, \alpha_3 \cdot \alpha_2 = \alpha_2 \cdot \alpha_3.
\]
\[ \alpha_2 = \begin{pmatrix} -1 & -2 & 2 & 1 + w & -1 + w \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \alpha_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ -1 & -2 & 1 & 1 + w & -1 + w \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \]

\[ \alpha_2^2 = \alpha_4^2 = 1, \quad \alpha_4 \cdot \alpha_2 = \alpha_2 \cdot \alpha_4. \]

\[ H_{14} = \begin{pmatrix} 2 & 3 \\ -1 & 3 \\ 0 & -1 \\ 0 & -1 \\ 1 - w & -1 - w \end{pmatrix}, \quad |\text{Aut}| = 2^4, \]

\[ \alpha_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 -1 & 2 - w & 1 - w & 1 \end{pmatrix}, \quad \alpha_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -w & 0 & 0 & 0 & -1 \end{pmatrix}. \]

\[ \alpha_2^2 = \alpha_4^2 = 1, \quad \alpha_4 \cdot \alpha_2 = \alpha_2 \cdot \alpha_4. \]

\[ H_{15} = \begin{pmatrix} 2 & 0 & 2 \\ 1 & -1 & 3 \\ 0 & 0 & -1 + w & 3 \\ w & w & 1 & 4 \end{pmatrix}, \quad |\text{Aut}| = 2^4, \]

\[ \alpha_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & -1 & 2 - w & 1 - w & 1 \end{pmatrix}, \quad \alpha_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -w & 0 & 0 & 0 & -1 \end{pmatrix}. \]

\[ \alpha_2^2 = \alpha_4^2 = 1, \quad \alpha_4 \cdot \alpha_2 = \alpha_2 \cdot \alpha_4. \]

\[ H_{16} = \begin{pmatrix} 2 & 0 & 2 \\ -1 & 1 & 3 \\ 0 & 0 & -w & 3 \\ 0 & -1 + w & -1 + w & -1 & 4 \end{pmatrix}, \quad |\text{Aut}| = 2^4, \]

\[ \alpha_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -2 + w & 1 & -3 + w & w & -1 \end{pmatrix}, \quad \alpha_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & -1 + w & 0 & 0 & -1 \end{pmatrix}. \]

\[ \alpha_2^2 = \alpha_4^2 = 1, \quad \alpha_4 \cdot \alpha_2 = \alpha_2 \cdot \alpha_4. \]

\[ H_{17} = \begin{pmatrix} 2 & -1 & 3 \\ 0 & -1 & 3 \\ 0 & -1 + w & -1 & 3 \\ 0 & 0 & -1 & 0 & 3 \end{pmatrix}, \quad |\text{Aut}| = 2^4, \]

\[ \alpha_2 = \begin{pmatrix} 4 - w & 8 - 2w & 5 & 5 + 2w & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -w & -2w & 1 - w & 4 - w & 0 \end{pmatrix}, \quad \alpha_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 4 - w & 8 - 2w & 5 & 5 + 2w & 1 \end{pmatrix}. \]

\[ \alpha_2^2 = \alpha_4^2 = 1, \quad \alpha_4 \cdot \alpha_2 = \alpha_2 \cdot \alpha_4. \]

\[ H_{18} = \begin{pmatrix} 2 & 0 & 3 \\ -1 & -1 & 3 \\ 0 & 1 - w & w & 3 \\ 0 & 1 & 1 & 1 & 3 \end{pmatrix}, \quad |\text{Aut}| = 2^4, \]

\[ \alpha_2^2 = \alpha_4^2 = 1, \quad \alpha_4 \cdot \alpha_2 = \alpha_2 \cdot \alpha_4. \]
\[
\begin{align*}
\alpha_3 &= \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix},
\alpha_4 &= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
-1 + w & 2 - w & -2 + 2w & -5 + w & 1 \\
\end{pmatrix}.
\end{align*}
\]
\[\alpha_3^2 = \alpha_4^2 = 1, \quad \alpha_4 \cdot \alpha_3 = \alpha_3 \cdot \alpha_4.\]

\[
H_{19} = \begin{pmatrix}
2 & 0 & 3 \\
0 & 0 & 3 \\
-1 + w & -1 & 3 \\
1 & 0 & 0 & 1 & 3
\end{pmatrix}, \quad |\text{Aut}| = 2^4,
\]

\[
\alpha_3 = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 1 \\
\end{pmatrix},
\alpha_4 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
2 - w & 2 + 3w & 1 & 5 & -4 + 2w \\
1 & 0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 & -1
\end{pmatrix}.
\]
\[\alpha_3^2 = \alpha_4^2 = 1, \quad \alpha_4 \cdot \alpha_3 = \alpha_3 \cdot \alpha_4.\]

\[
H_{20} = \begin{pmatrix}
3 & 1 & 3 \\
0 & -2 + w & 3 \\
-1 & 0 & 0 & 3 \\
-w & -1 & 1 & -1 & 3
\end{pmatrix}, \quad |\text{Aut}| = 2^3,
\]

\[
\alpha_2 = \begin{pmatrix}
-1 & 3 & 1 + w & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 2 + w & 1 - w & 1 & 3 - w \\
0 & -1 - w & 2 - w & 0 & -1
\end{pmatrix},
\alpha_3 = \begin{pmatrix}
1 & -3 & -1 - w & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 + w & -2 + w & 1 & 0
\end{pmatrix}.
\]
\[\alpha_2^2 = \alpha_3^2 = 1, \quad \alpha_3 \cdot \alpha_2 = \alpha_2 \cdot \alpha_3.\]