Sphere-like Solutions in Surface Functional Theory and Dirac’s Membrane Model

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Abstract

A surface functional theory for $p$-dimensional extended objects, the $p$-branes, was proposed in previous papers. The field equations for toroidal $p$-branes was exactly solved in $d = p + 2$ dimensions, yielding equally spaced mass-squared spectrum with massless states. In this paper, we obtain the asymptotic distribution of mass spectrum in the point-particle limit of the theory with sphere-like membranes ($p = 2$) in $d = 4$ dimensions. Similarity between this spectrum and that obtained in the Dirac’s membrane model of electron is discussed.
1. Introduction

The revival of interest and the excitement aroused by string theory has stimulated interest in the theory of other higher dimensional extended objects, generally known as $p$-branes [1]. Green-Schwartz-type actions had been constructed for $p$-branes, which are closely related to the four classical superstring theories [2]. The most interesting Green-Schwartz-type super $p$-brane theories is the 11D supermembrane theory, since it could have maximally extended supergravity as its low energy approximation. It is also realized that membrane theory has very interesting algebraic structure: it has an area-preserving algebra which contains the Virasoro algebra as a subalgebra [3]. This area-preserving algebra is also closely related to a new field in mathematics, namely, the quantum group [4]. In view of the beautiful and rich structures inherent in $p$-brane theory, a better understanding of the theory is desirable: it might provide new insight to string theory and related mathematics.

On the other hand, it is tremendously difficult to further develop $p$-brane theory as unified theories. Firstly, it is very difficult to quantize $p$-branes, because the $p$-brane actions are inextricably nonlinear. Most quantization schemes are only semiclassical. Secondly, one has to settle the question of the existence of massless states and the discreteness of the mass spectrum in $p$-brane theory. It has been argue that the 11D supermembrane theory is not likely to contain a massless state, and that its spectrum is continuous [5].

In [6], a surface functional theory of geometric $p$-branes was proposed, which keeps the fundamental reparametrization invariance manifest. There both strings (regarded as 1-branes) and general $p$-branes were given a unified treatment. The $p$-brane fields are considered as multicomponent surface functionals. Dirac-type field equations were obtained for the surface functionals. Generalized Dirac algebras were introduce in order to linearize the Dirac-Nambu-Goto actions of $p$-branes. In this way quantization of $p$-branes could be carried out in a completely quantum-mechanical manner. The field equations were solved exactly for toroidal $p$-branes in $d = p + 2$ dimensions and $X_0 = \tau$ gauge, yielding an equally spaced mass-squared (string-like) spectrum. The ground states are always massless. It was later shown that the above results are the manifestation of $N = p + 1$ hidden supersymmetries in the theory [7].
Of course, solutions with toroidal $p$-branes represent only a very small fraction of all the possible solutions that our theory admits. It is, however, very difficult to find exact solutions with other topological configurations.

In this paper, we shall attempt to find the asymptotic distribution of mass spectrum in the point-particle limit of our theory with sphere-like membranes in four dimensions. The surface functional theory is first reviewed in Sect. 2. The mass spectrum of the theory with sphere-like membranes is then obtained in Sect. 3. In Sect. 4, we discuss similarity between our mass spectrum and that obtained in Dirac’s membrane model [8], initially proposed to explain the existence of muon. Sect. 5 concludes the paper.

2. Surface Functional Theory for $p$-branes

We first briefly review the essence of the surface functional theory of $p$-branes as proposed in Ref. [6].

Consider a $p$-brane moving in a $d$-dimensional flat Minkowski space-time. The classical Dirac-Nambu-Goto action for $p$-brane is given by the volume of the world volume swept out by the extended object in the course of its evolution from some initial to some final configuration:

$$S = -\frac{1}{\kappa} \int d^{p+1}\xi \sqrt{(-1)^p} \det \hat{h}_{\alpha\beta} \hat{h}_{\alpha\beta}(\xi) = \partial_\alpha X^\mu \partial_\beta X_\mu,$$

(2.1)

where $X^\mu(\xi) (\mu = 0, \ldots, d-1)$ and $\xi^\alpha = (\tau, \sigma_1, \ldots, \sigma_p) (\alpha = 0, \ldots, p)$ are space-time and world volume coordinates, respectively. $\kappa$ is a proportional constant the inverse of which could be thought of as the tension of the $p$-brane. Space-time metric is taken to be $\eta^{\mu\nu} = \text{diag}(1,-1,\ldots,-1)$. One could then derive a set of primary constraints from this action. It is found that the Poisson-bracket algebra of these constraints closes only weakly, except in the case for string ($p = 1$). Furthermore, when the theory is quantized, the commutator algebra of the constraints would give rise to operator anomalies.

These difficulties are avoided in the surface functional theory proposed in [6]. In this approach, the basic entities are taken to be $p$-dimensional surfaces, or $p$-surfaces.
Fields are functions of $p$-surfaces, and hence are called $p$-surface functionals. These surface functionals depend only on the location and topology of the $p$-surfaces, but not on their parametrization. In this way, the important symmetry inherent in $p$-brane dynamics, namely the reparametrization invariance, is maintained manifestly in the theory. The equations of motion for the surface functionals are required to reproduce the Dirac-Nambu-Goto $p$-brane dynamics in the $X_0 = \tau$ gauge. This connection is made in accordance with Dirac’s treatment of spin-$\frac{1}{2}$ particles. This leads to the following covariant equations of motion for multi-component fields $\Psi$ of space-time $p$-surfaces:

$$
\int d^p\sigma \left[ \Gamma^\mu(\sigma)p_\mu(\sigma) - \frac{\sqrt{(-1)^p}}{\kappa} \hat{h} \Lambda(\sigma) \right] \Psi[X^\mu(\sigma)] = 0 ,
$$

(2.2)

$$
(\partial_k X^\mu) p_\mu \Psi = 0 ,
$$

$k = 1, \ldots, p$, $\sigma \equiv \{\sigma_1, \ldots, \sigma_p\}$.

Here $p_\mu = i \frac{\delta}{\delta x^\mu(\sigma)}$, and $\hat{h} = \det \hat{h}_{jk}$ ($j, k = 1, \ldots, p$). $\Gamma^\mu(\sigma)$ and $\Lambda(\sigma)$ are generalised Dirac matrices. If we define a totally anti-symmetric tangent tensor

$$
t_{\mu_1 \cdots \mu_p} \equiv \frac{1}{\sqrt{(-1)^p}} \frac{\partial \left( X_{\mu_1}, \ldots, X_{\mu_p} \right)}{\partial (\sigma_1, \ldots, \sigma_p)} ,
$$

(2.3)

with $t^2 = (-1)^p p!$, then $\Gamma^\mu(\sigma)$ and $\Lambda(\sigma)$ are given by

$$
\Gamma^\mu(\sigma) = a_1 \Gamma^\mu + \frac{a_2}{p!} \Gamma_A^{\mu_1 \cdots \nu_p} t_{\nu_1 \cdots \nu_p} + \frac{a_3}{p!} \Gamma_M^{\mu_1 \cdots \nu_p} t_{\nu_1 \cdots \nu_p}
$$

$$
\Lambda(\sigma) = 1.
$$

(2.4)

[Actually $\Lambda(\sigma)$ can contain arbitrary powers of $t_{\nu_1 \cdots \nu_p}$, we set $\Lambda(\sigma) = 1$ for simplicity]. Here $a_1$, $a_2$ and $a_3$ are arbitrary constants. $\Gamma^\mu$ are the usual Dirac matrices. $\Gamma_A^{\mu_1 \cdots \mu_{p+1}}$ are totally anti-symmetric in their indices, while $\Gamma_M^{\mu_1 \cdots \mu_{p+1}}$ are only anti-symmetric in the last $p$ indices and are taken to be traceless, in view of the second equation in (2.2). Algebra satisfied by the $\Gamma_M$-piece is generally very complicated, and equation which involves the $\Gamma_M$-piece is also difficult to solve in general. In this paper, we shall consider only the $\Gamma^\mu$ and the totally anti-symmetric $\Gamma_A^{\mu_1 \cdots \mu_{p+1}}$ piece. Full equations containing the $\Gamma_M$ piece for membranes and strings are given in [6].
Eq. (2.2) can now be written as

\[
\left[ a_1 \Gamma^\mu P_\mu + \frac{a_2}{p!} \Gamma^{\mu_1 \cdots \mu_{p+1}}_A P_{\mu_1 \cdots \mu_{p+1}} - \frac{V_p}{\kappa} \right] \Psi = 0 ,
\]

where

\[
V_p = \int d^p \sigma \sqrt{(-1)^p \hbar} ,
\]

\[
P_\mu = \int d^p \sigma p_\mu ,
\]

\[
P_{\mu_1 \cdots \mu_{p+1}} = \frac{1}{p+1} \int d^p \sigma \left[ \sum_\rho \sign(\rho) t_{\mu_1 \cdots \mu_p \mu_{p+1}} \right] .
\]

Here \( \rho \) represents cyclic permutation of the indices \( \mu_1, \ldots, \mu_{p+1} \). The two constants \( a_1 \) and \( a_2 \) satisfy

\[
a_1^2 + (-1)^p a_2^2 = 1 .
\]

(2.6)

The \( \Gamma \)'s matrices obey the following anti-commutation relations:

\[
\{ \Gamma^\mu, \Gamma^\nu \} = 2 \eta^{\mu\nu} ,
\]

(2.7)

\[
\{ \Gamma^{\mu_1 \cdots \mu_{p+1}}_A, \Gamma^{\nu_1 \cdots \nu_{p+1}}_A \} = 2 \eta^{\mu_1 \cdots \mu_{p+1}, \nu_1 \cdots \nu_{p+1}} ,
\]

(2.8)

All others = 0 .

The generalized \( \eta \)-tensor is defined by

\[
\eta^{\mu_1 \cdots \mu_{p+1}, \nu_1 \cdots \nu_{p+1}} = \begin{vmatrix} \eta^{\mu_1 \nu_1} & \eta^{\mu_1 \nu_2} & \cdots & \eta^{\mu_1 \nu_{p+1}} \\ \eta^{\mu_2 \nu_1} & \eta^{\mu_2 \nu_2} & \cdots & \eta^{\mu_2 \nu_{p+1}} \\ \vdots & \vdots & \ddots & \vdots \\ \eta^{\mu_{p+1} \nu_1} & \eta^{\mu_{p+1} \nu_2} & \cdots & \eta^{\mu_{p+1} \nu_{p+1}} \end{vmatrix} .
\]

(2.9)

The algebras (2.7) and (2.8) are Clifford algebras of dimensions \( d \) and \( \frac{d(d-1) \cdots (d-p)}{(p+1)!} \) respectively. As such, the field functional \( \Psi \) transforms non-trivially under Lorentz transformations.

The equation of motion can be greatly simplified in \( d = p + 2 \) dimensions and \( X_0 = \tau \) gauge. Under these conditions, the \( p \)-surfaces become hypersurfaces in \( (p+1) \)-dimensional space. Eq.(2.5) now reduces to

\[
\left[ a_1 \Gamma^\mu P_\mu + ia_2 A^0 Q - \frac{V_p}{\kappa} \right] \Psi = 0 .
\]

(2.10)
The dual matrices
\[
\Lambda^\mu \equiv \frac{1}{(p + 1)!} \epsilon^{\mu \mu_1 \ldots \mu_{p+1}} \Gamma_{\mu_1 \ldots \mu_{p+1}}
\]  
(2.11)
satisfy
\[
\{\Lambda^\mu, \Lambda^\nu\} = (-1)^{p+1} 2 \delta^{\mu \nu},
\]
(2.12)
which is, up to a sign, just the usual Dirac algebra. Only \(\Lambda^0\) contributes in (2.10). The operator \(Q\) is given by
\[
Q = \int d^p \sigma \ N(\sigma) \cdot \frac{\delta}{\delta X(\sigma)}. 
\]
(2.13)
\(N(\sigma)\) is the unit normal vector at each point of the \(p\)-surface in \(E^{p+1}\), and is related to the tangent tensor by
\[
N_j = \frac{1}{p!} \epsilon_{i_1 \ldots i_p j} t_{i_1 \ldots i_p}, \quad j, i_k = 1, 2, \ldots, p + 1.
\]
(2.14)
The operator \(Q\) is in fact the generator of small deformation of \(p\)-surface along the normal direction at each of its points, and is therefore related to the differential-geometric properties of \(p\)-dimensional hypersurfaces, as will be shown below.

Let \(h = \det h_{ij} = \det (\partial_i X \cdot \partial_j X)\) and \(H_r\) be the \(r\)-th mean curvatures of the hypersurface. Then we can define the following \(\sigma\)-reparametrization invariant coordinates
\[
h_\ell = \int d^p \sigma \sqrt{h} \ H_\ell, \quad \ell = 0, 1, \ldots, p,
\]
\[
y = \frac{1}{h_0} \int d^p \sigma \sqrt{h} \ X,
\]
\[
\bar{\lambda}_r = \binom{p}{r} \int d^p \sigma \sqrt{h} \ (X - y) \ H_r,
\]
and
\[
\tilde{\zeta}_r = \binom{p}{r} \int d^p \sigma \sqrt{h} \ N \ H_r, \quad r = 1, \ldots, p.
\]
(2.15)
We note here that \(h_0 = V_p\), and \(h_p\) is the total Gauss curvature. \(y\) is the center-of-mass coordinates of the \(p\)-surface. In terms of the coordinates \(\{\tau, h_\ell, y, \bar{\lambda}_r, \tilde{\zeta}_r\}\), we have
\[
P^\mu = \left( i \frac{\partial}{\partial \tau}, -i \frac{\partial}{\partial y} \right),
\]
(2.16)
and

\[
Q = - \left[ ph_1 \frac{\partial}{\partial h_0} + (p - 1)h_2 \frac{\partial}{\partial h_1} + \cdots + h_p \frac{\partial}{\partial h_{p-1}} \right] + \frac{\tilde{\lambda}_1}{h_0} \cdot \frac{\partial}{\partial y} + \left[ -2\tilde{\lambda}_2 + \tilde{\zeta}_1 + p \frac{\tilde{\lambda}_1}{h_0} h_1 \right] \cdot \frac{\partial}{\partial \tilde{\lambda}_1} + \left[ -3\tilde{\lambda}_3 + \tilde{\zeta}_2 + \left( \frac{p}{2} \right) \frac{\tilde{\lambda}_1}{h_0} h_2 \right] \cdot \frac{\partial}{\partial \tilde{\lambda}_2} - 2\tilde{\zeta}_2 \cdot \frac{\partial}{\partial \tilde{\zeta}_1} + \cdots + \left[ - (r + 1)\tilde{\lambda}_{r+1} + \tilde{\zeta}_r + \left( \frac{p}{r} \right) \frac{\tilde{\lambda}_1}{h_0} h_r \right] \cdot \frac{\partial}{\partial \tilde{\lambda}_r} - r\tilde{\zeta}_r \cdot \frac{\partial}{\partial \tilde{\zeta}_{r-1}} + \cdots + \left[ \tilde{\zeta}_p + \frac{\tilde{\lambda}_1}{h_0} h_p \right] \cdot \frac{\partial}{\partial \tilde{\lambda}_p} - p\tilde{\zeta}_p \cdot \frac{\partial}{\partial \tilde{\zeta}_{p-1}}. \tag{2.17}
\]

It can be easily checked that the total Gauss curvature, \( h_p \), and the coordinate \( \zeta_p \) are constants of motion.

3. Sphere-like Solutions in the Surface functional Theory for Membrane

We now proceed to solve the field equation (2.10) in this section. In general, the field equation (2.10) together with equations (2.16) and (2.17) is still difficult to solve. In [6] the field equation is exactly solved in \( p + 2 \) dimensions for toroidal \( p \)-branes for which \( h_2 = h_3 = \ldots = h_p = 0 \), and \( h_1 \neq 0 \), yielding the string-like mass-squared spectrum with massless states. But for \( p \)-branes of other topological configurations, the field equations are difficult to solve. Here we shall consider the asymptotic solutions for sphere-like membranes in \( d = 4 \) with constant \( h_2 > 0 \). Note that \( h_2 \) is a constant of motion, since \([Q, h_2] = 0\).

We shall look for positive energy solutions which have the following form

\[
\Psi = e^{-ip \cdot y} U(p, y, h_2, \tilde{\lambda}_r, \tilde{\zeta}_r) \Phi(h_0, h_1, h_2). \tag{3.1}
\]

with \( y_0 \equiv \tau \). We require that \( U \) satisfies

\[
Q(e^{-ip \cdot y} U) = 0 \tag{3.2}
\]
If this is so, then one can factor out the factor \((e^{-ip y}U)\) from the field equation. It is not too difficult to find the required form of \(U\) that solves eq.(3.2). Let us note that in the case we are interested in, the operator \(Q\) has the following form,

\[
Q = -2h_1 \frac{\partial}{\partial h_0} - h_2 \frac{\partial}{\partial h_1} + \frac{\vec{\lambda}_1}{h_0} \cdot \frac{\partial}{\partial \vec{y}} \\
+ \left[-2\vec{\lambda}_2 + \vec{\zeta}_1 + \frac{2\vec{\lambda}_1}{h_0} h_1\right] \cdot \frac{\partial}{\partial \vec{\lambda}_1} \\
+ \left[\vec{\zeta}_2 + \frac{\vec{\lambda}_1}{h_0} h_2\right] \cdot \frac{\partial}{\partial \vec{\lambda}_2} - 2\vec{\zeta}_2 \cdot \frac{\partial}{\partial \vec{\zeta}_1}.
\]

The required \(U\) is found to be:

\[
U = \exp\left\{ -\frac{i}{h_2} \vec{\rho} \cdot [\vec{\lambda}_2 + \vec{\zeta}_1/2] \right\}.
\]

With (2.16), (3.1), (3.2) and (3.3), the field equation (2.10) reads

\[
\left[a_1 \Gamma^\mu p_\mu + i a_2 \Lambda^0 Q - \frac{h_0}{\kappa}\right] \Phi(h_0, h_1, h_2) = 0,
\]

where \(p_\mu\) is the energy-momentum vector, and the operator \(Q\) is now given by

\[
Q = -2h_1 \frac{\partial}{\partial h_0} - h_2 \frac{\partial}{\partial h_1}.
\]

To solve eq.(3.5), we take the explicit representations of the \(\Gamma^\mu\) and \(\Lambda^\mu\) matrices as follows:

\[
\Gamma^0 = \sigma_1, \Gamma^1 = i\sigma_2, \Gamma^2 = i\sigma_3 \sigma_1, \Gamma^3 = i\sigma_3 \sigma_2, \ldots \\
\bar{\Gamma} = \sigma_3 \sigma_3, \quad \Lambda^\mu = i\bar{\Gamma} \otimes \Gamma^\mu.
\]

Here the constants \(a_1\) and \(a_2\) are taken to be real (and positive, without loss of generality) so as to ensure that the mass squared, \(p^2 = m^2\), is bounded from below. The wavefunction \(\Phi\) has 16 components, which can be decomposed into four 4-component fields \(\Phi_{ab}\). The indices \(a\) and \(b\) \((= + \text{ or } -)\) denote eigenvalues of \(\bar{\Gamma}\) and \(I \otimes \bar{\Gamma}\) where \(I\) is a \(4 \times 4\) matrix of unity. After taking the square of eq.(3.5) and rearranging the components, we obtain

\[
a_1^2 m^2 (\Phi_{++} \pm \Phi_{+-}) = \left[ -a_2^2 Q^2 + \frac{h_0^2}{\kappa^2} \pm \frac{2h_1}{\kappa} a_2 \right] (\Phi_{++} \pm \Phi_{+-}),
\]

\[
a_1^2 m^2 (\Phi_{-+} \pm \Phi_{--}) = \left[ -a_2^2 Q^2 + \frac{h_0^2}{\kappa^2} \pm \frac{2h_1}{\kappa} a_2 \right] (\Phi_{-+} \pm \Phi_{--}).
\]
This equation is still difficult to solve. Let us then consider only the high membrane
tension limit (also called the “point particle limit”) in which \( \kappa \to 0 \). In this case we can
neglect terms in the first powers of \( 1/\kappa \) in eq.(3.8). The original eigenvalue problem now
reduces to a scalar problem:

\[
a_1^2 m^2 \Phi = \left[ -a_2^2 Q^2 + \frac{h_0^2}{\kappa^2} \right] \Phi ,
\]

with the boundary condition \( \Phi \to 0 \) as \( h_0, h_1 \to \infty \). With the expression of \( Q \) in (3.6),
eq(3.9) becomes:

\[
\left( 4h_1^2 \frac{\partial^2}{\partial h_0^2} + 4h_1 h_2 \frac{\partial^2}{\partial h_0 \partial h_1} + h_2^2 \frac{\partial^2}{\partial h_1^2} + 2h_2 \frac{\partial}{\partial h_0} \frac{1}{a_2^2 \kappa^2} h_0^2 + \frac{a_1^2 m^2}{a_2^2} \right) \Phi = 0.
\]

If we denote the coefficients of the first three derivative operators in (3.10) by \( c_1, 2c_2 \) and
\( c_3 \) respectively, then the discriminant of (3.10) [9] is given by

\[
\Delta \equiv c_2^2 - c_1 c_3 \\
= (2h_1 h_2)^2 - (4h_1^2)(h_2^2) \\
= 0.
\]

This indicates that eq.(3.10) is of the parabolic type. That means solutions of the differ-
ential equation can be classified by family of curves along which the basic variables satisfy
the equation

\[
\frac{dh_1}{dh_0} = \frac{c_2}{c_1} = \frac{h_2}{2h_1} ,
\]

or \( h_1^2 - h_0 h_2 = \text{constant} \). Such curves are called the characteristics of the differential
equation. Hence of the two variables \( h_0 \) and \( h_1 \) only one is independent. We can therefore
transform the partial differential equation (3.10) into an ordinary one (the canonical form)
by the following change of variables:

\[
\xi(h_0, h_1) = h_0 ,
\]

\[
\eta(h_0, h_1) = h_1^2 - h_0 h_2 .
\]

In terms of these new variables, eq.(3.10) becomes

\[
4(\eta + h_2 \xi) \frac{d^2 \Phi}{d\xi^2} + 2h_2 \frac{d\Phi}{d\xi} + \left( \frac{a_1^2 m^2}{a_2^2} - \frac{1}{a_2^2 \kappa^2} \xi^2 \right) \Phi = 0 .
\]
Note that $\eta$ is a constant of motion as $[Q, \eta] = 0$. Hence the evolution of the system is along the characteristic curve corresponding to fixed $\eta$.

We now solve (3.14) for sphere-like membranes. For a regular sphere of radius $r$, one has $h_0 = 4\pi r^2$, $h_1 = 4\pi r$, and $h_2 = 4\pi$ from the Gauss-Bonnet theorem, and hence $\eta = 0$. We consider membrane configurations corresponding to $\eta = 0$ as sphere-like in our theory. Restricting ourselves to sphere-like membranes, eq.(3.14) reads

$$4\xi \frac{d^2 \Phi}{d\xi^2} + 2 \frac{d\Phi}{d\xi} + \left( \frac{a_1^2 m^2}{a_2^2 h_2} - \frac{1}{a_2^2 \kappa^2 h_2} \xi^2 \right) \Phi = 0 .$$

(3.15)

To solve for the eigenvalues of (3.15), we make a further change of variable. Let $x \equiv (a_2 \kappa h_2^{1/2})^{-\frac{1}{3}} \xi^{\frac{1}{2}}$. Then (3.15) is changed into a Schrödinger-type equation:

$$\frac{d^2 \Phi}{dx^2} + (\lambda^2 - x^4) \Phi = 0 ,$$

(3.16)

with boundary condition $\Phi \rightarrow 0$ as $x \rightarrow \infty$. Here the eigenvalues $\lambda$ is defined as

$$\lambda \equiv \frac{a_1}{a_2^{2/3}} \left( \frac{\kappa}{h_2} \right)^{\frac{1}{3}} m .$$

(3.17)

The asymptotic distribution of the eigenvalues of eq.(3.16) has been well studied in the literature [10]. Applying the WKBJ method, one gets

$$\lambda_n = \left[ \frac{(n + \frac{1}{2})\pi}{2 \int_0^1 \sqrt{1 - t^4} dt} \right]^{\frac{1}{2}} .$$

(3.18)

for large positive integers $n$. The integral in (3.18) can be expressed in terms of the $\Gamma$ functions:

$$\int_0^1 \sqrt{1 - t^4} dt = \frac{\Gamma(\frac{7}{4})\Gamma(\frac{3}{2})}{4\Gamma(\frac{5}{4})} .$$

(3.19)

From (3.17) and (3.18) one gets the mass spectrum of our theory:

$$m = \left[ \frac{a_2^2 h_2}{4 \kappa a_1^3} \right]^{\frac{1}{3}} \left[ \frac{(n + \frac{1}{2})\pi}{\int_0^1 \sqrt{1 - t^4} dt} \right]^{\frac{1}{2}} .$$

(3.20)
Hence for sphere-like membranes in our theory, the mass varies as \( m \sim n^{\frac{2}{3}} \) for sufficiently large \( n \). For comparison, let us recall that for toroidal membranes, one has the string-like spectrum \( m \sim n^{\frac{1}{2}} \) [6].

The spectrum (3.20) turns out to be very similar to that obtained by Dirac in a model of electron proposed in 1962 [8]. We shall now turn to discussing the connection between the two spectra.

4. Dirac’s Membrane Model

In an attempt to explain the existence of muon, which has properties so similar to the electron except the mass, Dirac proposed a membrane model of the electron [8]. There he considered the classical electron as a charged conducting surface, with a surface tension to prevent it from flying apart under the repulsive forces of the charge. The action of the system thus consists of two parts: the Maxwell term outside the surface, and a surface tension term on the surface. The second part is indeed the action of membrane. Such an electron has an equilibrium state with spherical symmetry (radius \( \rho \)), and if disturbed its shape and size oscillate. Dirac deduced the equation of motion of the extended object from an action principle and a Hamiltonian formulation. From the classical equation of motion he showed that the equilibrium radius \( \rho = a \) of the electron is given by

\[
a^3 = \frac{e^2}{4\omega},
\]

where Dirac’s membrane tension \( \omega \) is related to our \( \kappa \) by \( \omega \equiv \frac{4\pi \kappa}{\kappa} \). To connect \( a \) with the mass \( m_e \) of the electron, he considered the total energy of an electron instantaneously at rest. The total energy \( E \) consist of two parts: the electrostatic energy of the Coulomb field, \( e^2/2\rho \), and a surface tension energy proportional to \( \rho^2 \):

\[
E = \frac{e^2}{2\rho} + \beta\rho^2.
\]

As the minimum value of \( E \) must occur when \( \rho = a \), one obtained

\[
\beta = \frac{e^2}{4a^3}.
\]
This shows that $\beta = \omega$, and that the surface energy in the equilibrium state is half the electrostatic energy. Hence the total energy is $E = 3e^2/4a$. But since $E = m_e(c = 1)$, one gets

$$a = \frac{3e^2}{4m_e}.$$  \hfill (4.4)

To find the energy or mass spectrum of the excited states, Dirac proceeded to formulate a quantum theory of the model, hoping that the first excited state would give the mass of the muon. For this purpose Dirac had to base his formulation on his then newly developed method of constraint quantisation \cite{11}. For spherically symmetric motion, he showed that the only independent dynamical variables are the radius $\rho$ of the electron and its conjugate momentum $P$. The corresponding Hamiltonian of the symmetric system was then found to be

$$H = (P^2 + \omega^2 \rho^4)^{\frac{1}{2}} + e^2/2\rho.$$  \hfill (4.5)

Note that (4.5) reduces to (4.2) when $P = 0$. The Schrödinger equation with this Hamiltonian is difficult to solve, and Dirac obtained the mass spectrum by means of the Bohr-Sommerfeld quantisation rule:

$$2\pi n = 2\int_{\min}^{\max} Pd\rho.$$  \hfill (4.6)

From (4.5) we have

$$P^2 = \left( E - \frac{e^2}{2\rho} \right)^2 - \omega^2 \rho^4.$$  \hfill (4.7)

Let us define $x = \rho/a$ and $l = 4aE/e^2$. Then eq.(4.6) becomes

$$\frac{4n\pi}{e^2} = \int \left( \frac{l^2}{x^4} - x^4 \right)^{\frac{1}{2}} dx.$$  \hfill (4.8)

For large values of $E$, eq.(4.8) is approximately given by

$$\frac{4n\pi}{e^2} = \int_{0}^{l^\frac{4}{3}} (l^2 - x^4)^{\frac{1}{2}} dx$$

$$= l^\frac{3}{2} \int_{0}^{1} \sqrt{1 - t^4} dt.$$  \hfill (4.9)
Eq.(4.9), together with eq.(4.4) and the definition of $l$, lead to the mass spectrum in Dirac membrane theory:

$$m = \frac{m_e}{3} \left( \frac{4}{e^2} \right)^{\frac{2}{3}} \left[ \frac{n\pi}{\int_0^1 \sqrt{1-t^4}dt} \right]^{\frac{2}{3}}.$$  \hspace{1cm} (4.10)

To Dirac’s disappointment, the mass of the first excited state ($n = 1$) turns out to be $m = 53m_e$, which is only about a quarter of the observed muon mass ($m_\mu = 210m_e$).

Comparing eq.(4.10) and (3.20), we see that the mass spectra in both theories have the same asymptotic behaviour for large mass, i.e., $m \sim n^{\frac{2}{3}}$. In fact, Dirac’s mass spectrum could be obtained from the spectrum in our theory by an appropriate choice of the membrane tension and the values of $a_1$ and $a_2$. To see this, let us assume that the membrane tension is given by Dirac’s expression (4.1) together with eq.(4.4), then we get

$$\frac{1}{\kappa} = \frac{\omega}{4\pi} = \frac{4m_e^3}{3^3e^4\pi}.$$  \hspace{1cm} (4.11)

Substituting this expression into (3.20), and using the fact that $h_2 = 4\pi$, we obtain

$$m = \frac{m_e}{3} \left( \frac{4}{e^2} \right)^{\frac{2}{3}} \left[ \frac{a_2^2}{4a_1^3} \right]^{\frac{1}{3}} \left[ \frac{(n + \frac{1}{2})\pi}{\int_0^1 \sqrt{1-t^4}dt} \right]^{\frac{2}{3}}.$$  \hspace{1cm} (4.12)

Hence (4.10) and (4.12) will be identical (except for the irrelevant difference in $n$ and $n + 1/2$ for large $n$), provided that

$$\frac{a_2^2}{4a_1^3} = \frac{1 - a_1^2}{4a_1^3} = 1.$$  \hspace{1cm} (4.13)

Here we have used the relation $a_1^2 + a_2^2 = 1$ (eq.(2.6)). Eq.(4.13) is solved by $a_1 = 0.557$ and $a_2 = 0.831$. It is also of interest to note that if the mass of the first excited state ($n = 1$) in our theory were to correspond to the muon mass, then we must have $a_1 = 0.206$ and $a_2 = 0.979$. However, the power law $m \sim n^{\frac{2}{3}}$ does not allow one to account for both the observed values of the masses of the muon and the $\tau$-meson.
5. Summary

We have obtained the asymptotic distribution of mass spectrum in the point-particle limit of the surface functional theory for sphere-like membranes in four dimensions. By a specific choice of the membrane tension and the parameters in our theory, we reproduce the mass spectrum in Dirac’s membrane theory of electron.

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