New modelling technique for aperiodic-sampling linear systems

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Abstract

A general input-output modelling technique for aperiodic-sampling linear systems has been developed. The procedure describes the dynamics of the system and includes the sequence of sampling periods among the variables to be handled. Some restrictive conditions on the sampling sequence are imposed in order to guarantee the validity of the model. The particularization to the periodic case represents an alternative to the classic methods of discretization of continuous systems without using the Z-transform. This kind of representation can be used largely for identification and control purposes.

1 Introduction

Aperiodic sampling is a very interesting technique for improving the solution of several problems in control and identification. In particular, aperiodic sampling systems have been found useful in signal adaptation and compensation [4, 2] and in optimal transmission of measuring errors in problems involving the solutions of systems of linear equations, such as observability [1, 9], controllability [8] and identifiability [7, 8].

In all these cases it is essential to find a general model that

(a) describes the dynamics of the system;

(b) is adapted to real experimentation conditions (external representation against internal-state representation);

(c) includes the sequence of sampling periods among the variables to be handled.

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A well-known input-output formulation is used successfully for linear systems sampled periodically. The scalar output of the plant at an arbitrary instant is described by means of input and output samples taken at previous instants. It seems natural to consider the aperiodic case in the same way, except that the finite difference equation coefficients, depending on the fixed sampling period, would have to be replaced by multivariable functions, depending on the aperiodic sampling sequence. Such a formulation would satisfy the above conditions.

There must be restrictions on the aperiodic sampling sequence in order that the model be valid, whereas in the periodic case any sampling interval is valid.

The systems considered are characterized by their impulse response (weighting function), which describes the output of the plant through a convolution expression. Consequently, the properties and results obtained are always given in terms of that function.

The paper is organized as follows. In §1 several basic assumptions and general considerations, which will be assumed through the work, are given. Section 2 is devoted to a modelling technique for aperiodic-sampling linear systems. This also includes the action of a sampler and zeroth-order hold preceding the plant. Section 3 contains some restrictive conditions on the sampling sequence. The problem of the selection of the aperiodic sequence is strictly considered in geometric terms and the results found in the literature are improved. In §4 a particularization to the periodic case has been made. The proposed technique represents an alternative to the classical methods of discretization of continuous systems, in order to obtain the discrete-model coefficients, without using the Z-transform. Finally conclusions in §5 end the paper.

## 2 Basic assumptions

This discussion is restricted to:

(i) linear time-invariant single-input/single-output differential systems of finite order \( n \);

(ii) systems whose transfer function \( G(S) \) is a strictly proper rational function, and whose impulse response \( h(t) \) is therefore a particular solution of

\[
    h^{(n)}(t) + a_1 h^{(n-1)}(t) + \ldots + a_n h(t) = 0, \quad t \geq 0
\]

an \( n \)th-order homogeneous linear differential equation with constant coefficients \( (a_i \in \mathbb{R}) \).

\( h(t) \) can be then be written as

\[
    h(t) = \sum_{i=1}^{n} C_i \varphi_j(t), \quad t \geq 0
\]

where \( C_i \in \mathbb{C} \) are constant coefficients and \( \varphi_i; \mathbb{R} \to \mathbb{C} \quad (i = 1, \ldots, n) \) is the fundamental system of solutions of eqn. [1].
We conclude this preliminary section with the following statement. Let \((G_i)\) be a family of vector functions
\[
G_i : \mathbb{R}^n \to \mathbb{R}^n \quad (i = 0, 1, \ldots, n)
\]
\(C^\infty(\mathbb{R}^n, \mathbb{R}^n)\) being the set of infinitely differentiable functions on \(\mathbb{R}^n\). It then follows that if there exist

(a) an integer \(r \leq n\) such that the elements \((G_0(z), \ldots, G_r(z))\) are linearly independent for all \(z \in \mathbb{R}^n\), and

(b) an integer \(k > r\) such that \(G_k(z)\) depends linearly on \((G_0(z), \ldots, G_r(z))\), then there are functions
\[
f_0, f_1, \ldots, f_n \in C^\infty(\mathbb{R}^n, \mathbb{R})
\]
such that
\[
\sum_{i=0}^{n} f_i(z) G_i(z) = 0 \quad \forall z \in \mathbb{R}^n
\]

3 Modelling technique

3.1 Generalities and key ideas of the methodology

The class of linear time-invariant SISO systems is characterized by the impulse response, which describes for zero initial conditions the output of the plant through a convolution expression
\[
y_k = \sum_{i=0}^{k} h(t_k - t_i) u_i
\]
where \(y_k\) is the output of the plant at time \(t_k\), \(u_i\) is the impulse input at time \(t_i\), and \(h(t)\) is the impulse response of the plant. According to (1), the impulse response can also be written in matrix form by means of the equivalent linear system.
\[
\dot{X}(t) = AX(t)
\]
where
\[
X(t) = [h(t), \dot{h}(t), \ldots, h^{(n-1)}(t)]'
\]
the symbol ‘ denotes the transpose and \(A\) is an \(n \times n\) bottom-companion matrix.
\[
A = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_1
\end{bmatrix}
\]
The solution of the linear system
\[ \dot{X}(t) = \exp(At)X_0 \quad (X_0 = X(0)) \] (8)
is related to the impulse response through the expression
\[ h(t) = c \exp(At)X_0 \] (9)
with
\[ c = [1, 0, \ldots, 0] \] (10)

Note that the components of the vector \( X_0 \)
\[ X_0 = [h_1, h_2, \ldots, h_n]' \] (11)
correspond to the first \( n \) Markov parameters
\[ h_{i+1} = \frac{d^i h(t)}{dt^i} \bigg|_{t=0} \quad (i = 0, 1, \ldots, n-1) \] (12)

It should also be noted that the triad \((A, X_0, c)\) leads us naturally to the observability canonical realization from the scalar impulse response.

From these matrices \( A, X_0, c \) we are going to define a family of vectors functions
\[ (G_0, G_1, \ldots, G_n) \]
\[ G_i : \mathbb{R}^n \to \mathbb{R}^n \quad (i = 0, 1, \ldots, n) \]
given by
\[ G_i(z_1, \ldots, z_n) = \exp(A(z_1 + \ldots + z_i))X_0 \quad (i = 1, \ldots, n) \] (13)
\[ G_0(z_1, \ldots, z_n) = X_0 \] (14)
with
\[ z = [z_1, z_2, \ldots, z_n]' \in \mathbb{R}^n \]
The functions \( G_i \) and the impulse response are related by means of the expression
\[ h(z_1 + \ldots + z_i) = c G_i(z) \quad z_1 + \ldots + z_i \geq 0 \] (15)

Let us now consider the statement given in §1 for the kind of function defined above.

From an analytical viewpoint, the functions \( G_i \) belong to \( C^\infty(\mathbb{R}^n, \mathbb{R}^n) \), as compositions of \( C^\infty \) functions.

Let \( I^n \subset \mathbb{R}^n \) be an open subset of \( \mathbb{R}^n \) such that the vectors \((G_0(z), \ldots, G_{n-1}(z))\) are linearly independent for all \( z \in I^n \). In this case, it is easy to see that for the new domain \( I^n \) the conditions \((a)\) and \((b)\) in the previous statement
hold. In fact, condition (a) holds by definition of the subset $I^n$, and condition (b) holds by the dimensionality of the vector $G_i$. Hence there will be functions

$$f_i(z) \in C^\infty(I^n, \mathbb{R}) \quad (i = 0, \ldots, n)$$

such that

$$\sum_{i=0}^{n} f_{n-i}(z) G_i(z) = 0 \quad \forall z \in I^n \quad (16)$$

Rewriting (16) in matrix form, with

$$G_i(z) = [G_{i1}(z) \ldots G_{in}(z)]' \quad (i = 0, \ldots, n - 1)$$

we get

$$\begin{bmatrix} G_{10}(z) & \ldots & G_{1,n-1}(z) \\ \vdots & \ddots & \vdots \\ G_{n0}(z) & \ldots & G_{n,n-1}(z) \end{bmatrix} \begin{bmatrix} f_n(z) \\ \vdots \\ f_1(z) \end{bmatrix} = \begin{bmatrix} G_{1n}(z) \\ \vdots \\ G_{nn}(z) \end{bmatrix} (-f_0(z)) \quad (17)$$

($f_0(z) = -1$ for simplicity) and the functions $(f_1(z), \ldots, f_n(z))$ can be obtained by solving a compatible system of linear equations.

The general form of the functions $f_i(z)$ is

$$f_i(z) = \frac{\text{Det}[G_0(z) \ldots G_{n-1}(z)]}{\text{Det}[G_0(z) \ldots G_{n-1}(z)]} \quad (18)$$

where the numerator is the determinant obtained from the matrix

$$[G_0(z), \ldots, G_{n-1}(z)]$$

by replacing the $i$th column by the column vector $G_n(z)$.

At this point, we identify the components of $z$ with the elements of the sampling period sequence

$$\begin{cases} z_n = t_k - t_{k-1} = T_k \\ z_{n-1} = t_{k-1} - t_{k-2} = T_{k-1} \\ \vdots \\ z_1 = t_{k-n+1} - t_{k-n} = T_{k-n+1} \end{cases} \quad (19)$$

We multiply both sides of (16) by $c \exp(Az^*)$, with $z^*$ successively taking the values

$$z^* = - \sum_{i=1}^{i} z_l \quad (i = 1, \ldots, n) \quad (20)$$
and we get in each case

\[
\begin{cases}
  c \exp(A(z_{i+1} + \ldots + z_n))X_0 = c \sum_{l=1}^{n-i} f_l(z) \exp(A(z_{i+1} + \ldots + z_{n-l}))X_0 + \\
  c \sum_{l=0}^{i-1} f_{n-l}(z) \exp(-A(z_{l+1} + \ldots + z_i))X_0 \\
(\text{for } i = 1, \ldots, n)
\end{cases}
\]  

(21)

We define

\[
g_{n-i}(z) = c \sum_{l=0}^{i-1} f_{n-l}(z) \exp(A(z_{l+1} + \ldots + z_{n-l}))X_0 \quad (i = 1, \ldots, n)\]  

(22)

From (22) the functions \(g_{n-i}(z)\) are of class \(C^\infty\) as compositions of \(C^\infty\) functions.

It can be checked that the functions \(g_{n-i}(z)\) correspond to the first component of linear combinations of vectors \(G_i\) with negative argument.

If we write

\[
\begin{cases}
  f^k_i = f_i(z) \\
g^k_j = g_j(z)
\end{cases}
\]

at time \(t_k\), we can condense the preceding expressions into two sets of equations involving the functions \(f^k_i, g^k_j\) and \(h(t)\):

\[
\sum_{i=0}^{n} f^k_i h(t_{k-i} - t_j) + g^k_{k-j} = 0 \quad (j = k, k-1, \ldots, k-n+1) \]  

(23)

\[
\sum_{i=0}^{n} f^k_i h(t_{k-i} - t_j) = 0 \quad (k \geq n; j = k-n, \ldots, 0)\]  

(24)

Multiplying every equation by \(u_j\) \((j = k, k-1, \ldots, 1, 0)\) (impulse inputs at the sampling instants) respectively and summing, we get

\[
f^k_1 \sum_{l=0}^{k-1} u_l h(t_{k-l} - t_l) + \ldots + f^k_n \sum_{l=0}^{k-n} u_l h(t_{k-n} - t_l) + \sum_{j=0}^{n-1} g^k_j u_{k-j} = \sum_{l=0}^{k} u_l h(t_k - t_l)\]  

(25)

and, according to (4), the preceding expression becomes

\[y_k = \sum_{i=1}^{n} f^k_i y_{k-i} + \sum_{j=0}^{n-1} g^k_j u_{k-j}\]  

(26)

which is called the input-output model for linear time-invariant aperiodic-sampling systems.
3.2 Simplified form of the functions \( f_i, g_j \)

Companion matrices are an important example of what are called cyclic (or non-derogatory) matrices, which have only one (normalized) eigenvector associated with each distinct eigenvalue. This means that

(i) the Jordan canonical form is clearly simplified (there is only one Jordan block for each distinct eigenvalue);

(ii) the similarity transformation of the given matrix \( A \) to the Jordan canonical form can be obtained in a standard way.

Indeed,

\[
A = BJB^{-1}
\]

where \( J \) is the Jordan canonical form of the matrix \( A \), and \( B \) is an invertible matrix of a well-known general form.

In this way, (16) becomes

\[
\sum_{i=0}^{n} f_{n-i}(z) \exp(J\alpha_i)Y_0 = 0
\]

with

\[
\alpha_i = z_1 + \ldots + z_i \quad (i = 1, \ldots, n)
\]

\[
\alpha_0 = 0
\]

\[
Y_0 = B^{-1}X_0
\]

Factorizing \( \text{det}\{\exp(J\alpha_0)Y_0, \ldots, (J\alpha_{n-1})Y_0\} \) by means of Laplace’s expansion by minors and cancelling common factors in the numerator and denominator of (18), the functions \( f_i^k \) can be written as

\[
f_i^k = \frac{\Delta_i}{\Delta} \quad (i = 1, \ldots, n)
\]

where

\[
\begin{align*}
\Delta &= |\varphi_l(\alpha_j)| \\
\Delta_i &= |\varphi_l(\alpha_j)| \\
(l = 1, \ldots, n)
\end{align*}
\]

\[
(j = 0, 1, \ldots, n - 1)
\]

\[
(j = 0, \ldots, n, \ldots, n - 1)
\]

The linear independence of the vectors \( G_i \) implies the non-nullity of the determinant \( \Delta \). We can check that the functions \( f_i \) depend exclusively on the poles of the transfer function and on the sampling sequence. The result could be expected and agrees for the periodic case with the direct correspondence

\[
\lambda_i \rightarrow \exp(\lambda_i T_0)
\]

between the poles of the continuous and pulse transfer function.
The functions $g^k_j$ are given by

$$ g^k_{k-j} = - \sum_{i=0}^{n} f^k_i h(t_{k-i} - t_j) \quad (j = k, k-1, \ldots, k-n+1) \quad (34) $$

and depend on the poles and zeros of the transfer function, since they are obtained as a linear combination of the impulse response for specific arguments.

### 3.3 General formulation for systems with zeroth-order hold

If the sampler at the input is followed by a zeroth-order hold the input-output model is modified in the following way:

$$ y_k = \sum_{i=0}^{k} h(t_k - t_i) x_i \quad (35) $$

with

$$ x_i = u_i - u_{i-1} \quad (36) $$

$$ h(t) = L^{-1} \left[ \frac{G(s)}{s} \right] \quad (37) $$

As $x_i$ represents the difference between two consecutive values of the input signal, the expressions (23) and (24) become

$$ \begin{align*}
\sum_{i=0}^{n} f^k_i [h(t_{k-i} - t_j) - h(t_{k-i} - t_{j+1})] + g^k_{k-j} &= 0 \quad (j = k, k-1, \ldots, k-n) \\
\sum_{i=0}^{n} f^k_i [h(t_{k-i} - t_j) - h(t_{k-i} - t_{j+1})] &= 0 \quad (j = k - n - 1, \ldots, 0) \\
& \quad (k > n)
\end{align*} \quad (38) $$

Briefly, the presence of a zeroth-order hold implies that

(a) the functions $f_i$ are the same as before;

(b) the functions $g_j$ are modified: the number functions $g_j$ is increased by one (so that now $0 \leq j \leq n$), and their general expression is similar to (34) but with

$$ h(t_{k-i} - t_j) \rightarrow h(t_{k-i} - t_j) - h(t_{k-i} - t_{j+1}) $$

for $h(.)$ defined in (37).

The results obtained were to be expected, since the functions $f_i$ depend only on the poles of the transfer function, which reflect the internal coupling in the system and its autonomous behaviour. However, the functions $g_j$ reflect the internal plant coupling to the input signal, which has been affected by the presence of the zeroth-order hold.
3.4 Main results

We recall briefly the main results we have obtained.

(a) We have shown that the functions \( f_i, g_j \) are infinitely differentiable.

(b) We have obtained a general and systematic formulation for every function \( f_i, g_j \) in contrast with the procedure found in [6], where all these functions are computed globally.

(c) We have developed a procedure to impose restrictive conditions on the sampling sequence in order to guarantee the linear independence of the vectors \( (G_0(z), G_1(z), \ldots, G_{n-1}(z)) \).

In the next section we shall determine the set of vectors \( I^n \subset \mathbb{R}^n \) whose elements \( z \) satisfy the above condition.

4 Choice of the sampling-period sequence. A geometric interpretation

The problem of the choice of the sampling sequence has been treated in [9] analytically. Making use of the concept of a Chebyshev system, some intervals of the real line are selected in which the sampling instants can be chosen freely. In this way, the non-nullity of the determinant \( \Delta \) is guaranteed, and consequently the difference equation for the aperiodic case can be obtained (via (26), (32) and (34)).

In the present paper the same problem is considered geometrically. The choice of sampling-period sequence is directly related to the properties of certain vectors in the space \( \mathbb{R}^n \). In this way, the intervals found by Troch [9] are largely increased for low-order models and some general considerations for higher-order models are made.

4.1 Second-order model \((n = 2)\)

We are going to consider a second-order model with a pair of complex eigenvalues

\[ a + jb \in \mathbb{C} \quad (b > 0) \]  

(39)

The problem depends on an adequate choice of the sampling periods in such a way that the linear independence of the vectors

\[ [Y_0, Y_1] = [exp(J\alpha_0)z_0, exp(J\alpha_1)z_0] \]  

(40)

will be preserved, where

\[ \alpha_0 = 0 \]  

(41)

\[ \alpha_1 = t_{k-1} - t_{k-2} = T_{k-1} \]  

(42)

\( J_r \) is the real canonical form of the matrix \( A \):
\[ J_r = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad (b > 0) \]  

and \( T \) is the (invertible) matrix governing the change of basis:

\[ z_0 = T^{-1} X_0 \]  

As we are in \( \mathbb{R}^2 \), the geometric interpretation is very simple. The generic operator \( \exp(J_r \alpha) \) applied to the vector \( z_0 \) can be viewed as follows. It is a counterclockwise rotation through \( b \alpha \) radians, followed by a stretching (or shrinking) of the length of \( z_0 \) by a factor \( \exp(a \alpha) \). 3

From this interpretation \( Y_0 \) and \( Y_1 \), will be linearly independent if and only if

\[ b \alpha_1 = b T_{k-1} \neq \hat{\pi} \]  

where \( \hat{\pi} \) denotes an integral multiple of \( \pi \). Otherwise the vectors will be colinear.

Comparing these results to those of Troch \( 9 \) we have the following. According to \( 9 \), given \( t_{k-2}, t_{k-1} \) can be fixed so that

\[ t_{k-1} \in (t_{k-2}, t_{k-2} + \frac{\pi}{b}) \]  

According to \( 45 \), given \( t_{k-2}, t_{k-1} \) can be fixed so that

\[ t_{k-1} \in I = (t_{k-2}, \infty) - (t_{k-2} + \frac{\pi}{b}) \]  

Only point values will be rejected.

It can also be checked that such a formulation is concerned with differences between sampling instants but not with the actual position of such instants on the real axis. This agrees perfectly with the kind of time-invariant systems we are dealing with.

If the condition \( 45 \) is violated, the sampling process resonates with the system dynamics and the formulation then obtained no longer affords a faithful representation of the system.

We can also see that, for the second-order model, \( 45 \) is a necessary and sufficient condition to guarantee the linear independence of the vectors \( Y_i \) whereas the condition given by Troch is only sufficient.

4.2 Third-order model \((n = 3)\)

We are going to consider a third-order model with a real pole and a complex pair

\[ \lambda \in \mathbb{R}, \quad a + jb \in \mathbb{C} \quad (b > 0) \]  

The point is to choose the sampling instants in the right way; that is, such that the vectors
\[ [Y_0, Y_1, Y_2] = [\exp(J_\alpha z_0), \exp(J_\alpha z_0), \exp(J_\alpha z_0)] \] (49)

are linearly independent, where

\[
\alpha_0 = 0
\] (50)

\[
\alpha_1 = t_{k-2} - t_{k-3} = T_{k-2}
\] (51)

\[
\alpha_2 = t_{k-1} - t_{k-3} = T_{k-1} + T_{k-2}
\] (52)

\[
J_\tau = \begin{bmatrix}
  a & -b & 0 \\
  b & a & 0 \\
 0 & 0 & \lambda
\end{bmatrix}
\] (b > 0) (53)

with \( z_0, T \) defined as before.

The problem is treated in three-dimensional space, and the geometric structure can be described as follows. The generic vector \( Y(\alpha) \) is written as

\[
Y(\alpha) = [\exp(a\alpha \cos(b\alpha)), \exp(a\alpha \sin(b\alpha)), \exp(\lambda\alpha)]'
\] (54)

It is therefore, a spiral on the surface of revolution

\[
z = (x^2 + y^2)^{\lambda/2a}
\] (55)

whose form is determined by the real part of the system eigenvalues.

The vectors \( Y_0, Y_1, Y_2 \) have their origin at the point \((0,0,0)\) and their ends at the points \( Y(\alpha_0), Y(\alpha_1), Y(\alpha_2) \) of the parametric curve.

Heuristically we can imagine the same vectors as before \((n = 2)\) pointing upwards from the \((X,Y)\) plane as they have a third component on the \(Z\) axis.

From this geometric interpretation we can study some interesting cases of linear dependence.

(a) If the sampling-period sequence is chosen in such a way that

\[
b\alpha_i = \pi \quad (i = 0, 1, 2)
\] (56)

according to Fig. 1, the vectors \( Y_0, Y_1, Y_2 \) will be coplanar (in a plane containing the \(Z\) axis) and therefore they will be linearly dependent since every vector can be written as a linear combination of the other two.

In physical terms, this can be regarded as a resonance of the sampling sequence with a pair of complex eigenvalues. This situation can be generalized to higher-order models if the equation \((56)\) holds for the imaginary part of any pair of complex eigenvalues.

(b) If the real eigenvalue and the real part of the complex pair are equal then the revolution surface is a cone and there will be linear dependence if at least two vectors are on the same generator (see Fig. 2). Analytically,

\[
b(\alpha_j - \alpha_i) = 2\pi
\] (57)
for any $i, j$ such that $0 \leq i < j \leq 2$.

Physically, this situation can be regarded as a resonance of all the eigenvalues with the time interval between two sampling instants, not necessarily consecutive.

This situation can be generalized to higher-order models if the eigenvalues are such that

(i) the real parts are the same, and
(ii) all the imaginary parts $b_j$ ($j = 1, \ldots, q$) satisfy (57).

(c) This is the general case where there is linear dependence because the vectors $Y_i$ are contained in an arbitrary plane, which obliquely intersects the surface of revolution passing through the origin.

The intersection of the oblique plane and the surface of revolution is a closed curve $\Gamma$. Therefore the spiral will intersect such a curve in only two points for each rotation of $2\pi$ radians. From this interpretation, different geometric structures can be obtained, depending on the parameter values. In fact, we have the following possibilities:

$$\begin{align*}
\lambda > 0 & \quad \left\{ \begin{array}{c}
a > 0 \\
a < 0 
\end{array} \right. \\
\lambda < 0 & \quad \left\{ \begin{array}{c}
a > 0 \\
a < 0 
\end{array} \right.
\end{align*}$$

Now $\lambda > 0 (\lambda < 0)$ implies that the spiral goes up (down) as far as $\alpha \to \infty$; and $a > 0 (a < 0)$ implies that the surface of revolution expands (shrinks) when we move up the $Z$ axis.

**Case 1: $\lambda > 0, a > 0$**

From examination of $P_0$ and $P_1$ the normalized projections of $Y_0$ and $Y_1$ on the $(X, Y)$ plane in Fig. 4, it is clear that if $Y_2$ is contained in the oblique plane then its normalized projection on the $(X, Y)$ plane is situated on the $arcP_0P_1 = b\alpha_1$ (here $P_0$ and $P_1$ denote the ends of the vectors $Y_0$ and $Y_1$ respectively). Therefore a sufficient condition that guarantees the linear independence of the three vectors is that the rotation $b\alpha_2$ is not part of such an arc.

In this way, $P_0P_1$ and the corresponding multiples of $2\pi$ are the forbidden intervals, and consequently $P_1P_0$ and the corresponding multiples of $2\pi$ are the allowed ones.

It has to be pointed out that inside the forbidden intervals there would only be two intersection points of the spiral with the curve $\Gamma$ for each rotation through angle $2\pi$. However, we reject the whole interval since we intend to find sampling intervals in which the linear independence of the vectors $Y_i$ is automatically guaranteed without analytical computations.

The most favourable case corresponds to a rotation $P_0P_1$ as short as possible, because this would increase the length of the allowed interval. Reciprocally, the most disfavourable case corresponds to a rotation $P_0P_1$, with angle close to $\pi$. 

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For values of $\alpha$ such that
\[ |Y(\alpha)| > |Y(\alpha^*)| \] (58)

where $Y(\alpha^*)$ is the vector of maximum modulus with its origin at the point $(0,0,0)$ and its end on the curve $\Gamma$, the choice of the third vector is completely arbitrary, since $Y_2$ can never be on the oblique plane. Therefore the number of forbidden intervals is finite and depends on the previous relation.

For the case $\lambda > 0$, $a < 0$ the allowed intervals are the same as before.

Case 2: $\lambda < 0$, $a < 0$

We proceed in the same way as before. Figure 5 shows the projections of $Y_0, Y_1$ and the curve $\Gamma$ on the $(X,Y)$ plane. The diameter $R_1 R_0$ is parallel to the chord joining the end of the normalized projections $P_0$ and $P_1$. For each rotation through angle $2\pi$, the spiral will intersect the curve $\Gamma$ in only one point at $P_1 R_1$, and the same will happen at $R_0 P_0$. Therefore a sufficient condition to guarantee the linear independence of the three vectors in that the rotation $b_0 \alpha_2$ is not part of such an arc. Consequently, $P_0 P_1$ and $R_1 R_0$ and the corresponding multiples of $2\pi$ radians would be the allowed intervals.

It should be pointed out that the arc $P_1 R_1$ is allowed only for the first rotation. The most favourable case corresponds to a rotation $P_0 P_1 = b_0 \alpha_1$, with angle as close to $\pi$ as possible. Conversely, the most unfavourable case corresponds to a rotation with angle as small as possible.

In this situation the number of forbidden intervals is not finite, since there are always two intersection points for each rotation of angle $2\pi$.

For the case $\lambda < 0$, $a > 0$ the allowed intervals are the same as before.

Comparing these results with those of [9], we can see that for the latter the allowed interval, in rotational form, can be expressed as
\[ b_0 \alpha_2 < \pi \] (59)

which, according to the geometric interpretation, is clearly a shorter interval. In both cases the conditions are sufficient.

The only interesting cases for second- and third-order models are those considered in §§ 4.1 and 4.2.

For systems without complex eigenvalues, the linear independence of the vectors $Y_i$ is automatically guaranteed for any arbitrary choice of the sampling instants [9].

4.3 General considerations for higher-order models

From the fourth-order upwards, the situation becomes more and more complex because we do not have the geometrical insight given by the plane or three-dimensional space. However, from the basic structures developed for $\mathbb{R}^2$ and $\mathbb{R}^3$, we can make some general considerations that simplify the choice of sampling instants in higher-order models.
In order to clarify these ideas, we are going to consider a 4th-order model 
\( (n = 4) \) with one pair of complex eigenvalues and one pair of real eigenvalues:

\[
\lambda_1, \lambda_2 \in \mathbb{R}, \quad a + jb \in \mathbb{C}
\] (60)

The parametric curve is

\[
Y(\alpha) = [\exp(aa) \cos(b\alpha), \exp(aa) \sin(b\alpha), \exp(\lambda_1 \alpha), \exp(\lambda_2 \alpha)]'
\] (61)

which can be decomposed into

\[
c_1(\alpha) = [\exp(aa) \cos(b\alpha), \exp(aa) \sin(b\alpha), \exp(\lambda_1 \alpha)]
\] (62)

\[
c_2(\alpha) = [\exp(aa) \cos(b\alpha), \exp(aa) \sin(b\alpha), \exp(\lambda_2 \alpha)]
\] (63)

and we can study the evolution of each curve separately in the same way as
before.

The normalized projections of the vectors

\[
(c_1(\alpha_i)), (c_2(\alpha_i)) \quad (i = 0, \ldots, 3)
\]
on the \((X, Y)\) plane will coincide, since the only element varying is the third
component and not the angle of rotation.

Take \((\alpha_0, \alpha_1, \alpha_2, \alpha_3)\) such that each subset of three elements in either of the
sets \((c_1(\alpha_i)) (c_2(\alpha_i))\) is linearly independent. Then we have that each subset of
three elements in \(Y_i \ (i = 0, \ldots, 3)\) is linearly independent. Therefore it suffices
to check the linear dependence of any one vector with respect to the other three.

In analytical terms, this means that

\[
M_1 c_1(\alpha_j) = M_2 c_2(\alpha_j)
\] (64)

where \(M_1\) and \(M_2\) are \(3 \times 3\) matrices of general form

\[
\begin{align*}
M_1 &= (c_1(\alpha_i)) \\
M_2 &= (c_2(\alpha_i))
\end{align*}
\] (i = 0, \ldots, 3), \(i \neq j\) (65)

If (64) does not hold then the linear independence of the vectors \(Y_i \ (i = 0, \ldots, 3)\), obtained through the manipulation of 3-dimensional vectors, is guar-
anteed. Therefore we have been able to reduce the problem dimension by one.

The method can be generalized to arbitrary order \(n\), although the problem
becomes more and more complicated as the order of the model is increased.

If the multiplicity of the poles is greater than 1, the third component of the
vectors \(c_i(\alpha)\) is a polynomial expression with trigonometric functions, which will
give a more complicated curve.

If there is more than one pair of complex eigenvalues, that with the greatest
imaginary part (maximum eigenfrequency) will be the main pair; since it
determines the rotation with greater angle.

When the dimension of the model is very large, it is more convenient to use
the shorter intervals found in the literature or to reduce the order of the model
and then apply the techniques developed to the reduced model.
5 Particularization to the periodic case

The aperiodic model developed in previous sections is also valid for the periodic case. In this situation, the general expressions and the geometric interpretation can be simplified.

5.1 Periodic model

Once the sampling period $T_0$ has been fixed, the multivariable functions $f_i, g_j$ are reduced to constant coefficients $a_i, b_j$.

It can be demonstrated by induction that for the one-variable case the coefficients $a_i$ can be written as

$$a_i = (-1)^{i-1} \sum \phi_{j_1} \cdots \phi_{j_i}$$

where the sum is taken for $1 \leq jp \leq n$ and $1 \leq p \leq i$ ($i = 1, \ldots, n$)

$$\phi_{ji}(T_0) = \exp(\lambda_i T_0)$$

($\lambda_i$ is an eigenvalue of the continuous system). If the multiplicity of $\lambda_i$ is $m_i > 1$ then there are $m_i$ functions $\phi_i$ which are all equal. The $a_i$ correspond to coefficients with opposite sign of the polynomial $A(z)$ in the $Z$-transfer function

$$G(z) = \frac{B(z)}{A(z)}$$

Indeed,

$$A(z) = 1 + a'_1 z^{-1} + \ldots + a'_n z^{-n}$$

with

$$a'_i = -a_i$$

It is well known that between the coefficients and the roots of a polynomial there is a variational relation like that given by (66).

According to (34), the coefficients $b_j$ are

$$b_{k-j} = -\sum_{i=0}^{n} a_i h(t_{k-i} - t_j) \quad (j = k, k-1, \ldots, k-n+1)$$

with $a_0 = -1, k \geq n$. Equations (66) and (71) represent an alternative approach to that presented by the $Z$-transform in order to compute discrete-model coefficients from the weighting function.

The formulation corresponding to the case with zeroth-order hold can be also transcribed to the periodic version by means of (38).
### 5.2 Influence of sampling period on the discrete model parameters

To discuss the effect of sampling period on the absolute values of parameters we give the following example. We consider the continuous transfer function

$$G(s) = \frac{K}{(1 + T_1 s)(1 + T_2 s)(1 + T_3 s)}$$

(72)

with zeroth-order hold. The external representation is

$$y_k = \sum_{i=1}^{3} a_i y_{k-i} + \sum_{j=0}^{3} b_i^j u_{k-j}$$

(73)

where $a_i$ and $b_j$ can be obtained according to (66) and (71) with

$$\phi_i(T_0) = \exp\left(-\frac{1}{T_i} T_0\right) \quad (i = 1, 2, 3)$$

(74)

For the following values of the continuous model parameters

$$K = 1, \ T_1 = 10s, \ T_2 = 7.5s, \ T_3 = 5s$$

the values of the discrete-model coefficients are shown in Table (1) for different sampling periods $T_0$.

These results agree with those obtained by Isermann [5]. Because of the sign of $T_i$ and the general form of $a_i$, $b_j$, the magnitudes of the $a_i$ parameters decrease (in absolute value) and those of $b_j$ increase with increasing sampling period $T_0$.

If $T_i$ had the opposite sign the parameter behaviour would be entirely different.

For a small sampling period $T_0 = 1s$

$$b_i \ll |a_i|, \ \Sigma b_i \ll |a_i|$$

---

**Table 1: Values of the parameters for different sampling periods**

| $T_0$ | 2     | 4     | 6     | 8     | 10    | 12    |
|-------|-------|-------|-------|-------|-------|-------|
| $b_1$ | 0.0026| 0.0186| 0.0510| 0.0989| 0.1586| 0.2260|
| $b_2$ | 0.0092| 0.0486| 0.1086| 0.1718| 0.2257| 0.2643|
| $b_3$ | 0.0018| 0.0078| 0.0139| 0.0174| 0.0181| 0.0167|
| $a_1$ | 2.2549| 1.7063| 1.2993| 0.9953| 0.7668| 0.5938|
| $a_2$ | -1.689 | -0.958 | -0.547 | -0.314 | -0.182 | -0.106 |
| $a_3$ | 0.4203| 0.1767| 0.0742| 0.0312| 0.0131| 0.0055|

$\Sigma b_i = 1 + \Sigma a_i$ 0.0139 0.0750 0.1736 0.2882 0.4025 0.5071
This is why small errors in the estimated parameters can have a significant influence on the input-output behaviour of the model. Indeed, $\Sigma b_i$ depends on the 4th or 5th place of $b_i$ after the decimal point.

If the sampling period is chosen too small, ill-conditioned matrices result, which leads to numerical problems. On the other hand, if the sampling period is chosen too large then the dynamical behaviour is described inexactly. For $T_0 = 10$ s the model is practically reduced to second order because

$$a_3 \ll 1 + \Sigma |a_i|, \quad b_3 \ll \Sigma b_i$$

and for even greater sampling periods we get a first-order model.

A proper choice of sampling interval in most cases is not critical, because the range between too-small and too-large values is relatively wide.

5.3 Geometric particularization

The geometric interpretation of §3 can be particularized to the periodic case. We are going to consider again the third order model because it is the most significant. For the cases previously presented, we get the following results.

(a) The vectors $Y_i$ are coplanar if and only if

$$bT_0 = \pi$$

(b) This is included in the previous case.

(c) This situation never occurs, since different rotations of the vectors $Y_i$ would imply different sampling periods, which is not possible in the periodic case.

We can see that the transition from the aperiodic case to the periodic case is a simple particularization. In the opposite sense, the procedure is much more complicated, because there are many aperiodic situations that have no equivalent in the periodic version.

In analytical terms, the coefficients $a_i$, and consequently the $b_j$, are perfectly defined for all sampling periods $T_0$, according to (66) and (71). We recall that the $Z$-transform imposes no constraints on the choice of $T_0$. Conversely, in the aperiodic formulation there are some sampling sequences that are forbidden for the model developed. This confirms the complexity of the aperiodic case compared with the periodic one.

5.4 Coefficients of the discrete model for systems with dead time

We are going to consider the same system with zeroth-order hold as before, but also with dead time $T_d$. The convolution expression that reflects this situation is
y_k = \sum_{i=0}^{k} h(t_k - t_i - T_d) x_i \quad (76)

with \( x_i \) and \( h(t) \) defined as before. For

\[(p - 1)T_0 < T_d \leq pT_0 \quad (77)\]

with \( p \in \mathbb{Z} \), the general input-output model can be written as

\[y_k = \sum_{i=1}^{n} a_i y_{k-i} + \sum_{j=0}^{n} b_j u_{k-j-p} \quad (78)\]

In order to determine the coefficients \( a_i \) and \( b_j \), the procedure is similar to that previously developed. Indeed, now we have the expressions

\[\sum_{i=0}^{n} a_i [h(t_{k-i} - t_j - T_d) - h(t_{k-i} - t_{j+1} - T_d)] + b_{k-j} = 0 \quad (79)\]

with \((j = k - p, k - p - 1, \ldots, k - p - n)\).

\[\sum_{i=0}^{n} a_i [h(t_{k-i} - t_j - T_d) - h(t_{k-i} - t_{j+1} - T_d)] = 0 \quad (j = k - p - n - 1, \ldots, 0) \quad (80)\]

and we see that for \( j = k, k - 1, \ldots, k - p + 1\)

\[h(t_k - t_j - T_d) = 0 \quad (81)\]

There are two different cases.

(i) \( T_d = pT_0 \), which implies that the coefficients \( a_i \) are not modified by the dead time; the coefficients \( b_j \) are not modified either, except that these coefficients multiply the impulse inputs \( u_{k-j-p} \) \((j = 0, 1, \ldots, n)\), rather than the inputs \( u_{k-j} \) \((j = 0, 1, \ldots, n)\) for the case without dead time.

(ii) \((p-1)T_0 < T_d < pT_0 \), which implies that the coefficients \( a_i \) are not modified by the dead time; the coefficients \( b_j \) are now functions of \( T_d \) whose general expression is given by \((79)\) and which multiply the inputs \( u_{k-j-p} \) \((j = 0, 1, \ldots, n)\).

6 Conclusions

A general aperiodic model for linear time-invariant SISO systems has been developed. The model also covers more general cases such as systems with zeroth-order hold and dead time. The formulation considered stresses the importance of the sampling period against other system parameters. In this way, such systems have an additional element for analysis and manipulation. The periodic-sampling case appears as a simple particularization of the general procedure.
The results obtained are simplified and the use of tables of Z-transforms are avoided. For every sampled system, in a periodic or aperiodic way, there will always be sampling-period sequences more or less adequate according to the general characteristics of the process under study. In the aperiodic case, there will also be some restrictive conditions on these sequences, although it has also been possible to give strategies for some special cases.

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