THE METRIC COMPLETION OF THE RIEMANNIAN SPACE OF KÄHLER METRICS

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Abstract. Let $X$ be a compact Kähler manifold and $\alpha \in H^{1,1}(X, \mathbb{R})$ a Kähler class. We study the metric completion of the space $\mathcal{H}_\alpha$ of Kähler metrics in $\alpha$, when endowed with the Mabuchi $L^2$-metric $d$.

We show that the metric completion $(\overline{\mathcal{H}}_\alpha, d)$ of $(\mathcal{H}_\alpha, d)$ is a CAT(0) space which can be identified with a subset of the class $\mathcal{E}^1(\alpha)$ of positive closed currents with finite energy and contains all currents with higher order finite energy $\mathcal{E}^2(\alpha)$.

We further prove, in the toric setting, that $\overline{\mathcal{H}}_{\alpha, \text{tor}} = \mathcal{E}^2_{\text{tor}}(\alpha)$.

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INTRODUCTION

Let $X$ be a compact Kähler manifold and $\alpha \in H^{1,1}(X, \mathbb{R})$ a Kähler class. The space $\mathcal{H}_\alpha$ of Kähler metrics $\omega$ in $\alpha$ can be seen as an infinite dimensional riemannian manifold whose tangent spaces $T_\omega \mathcal{H}_\alpha$ can all be identified with $C^\infty(X, \mathbb{R})$. Mabuchi has introduced in [Mab87] an $L^2$-metric on $\mathcal{H}_\alpha$, by setting

$$\langle f, g \rangle_\omega := \int_X f g \omega^n V_\alpha,$$

where $n = \dim \mathbb{C} X$ and $V_\alpha = \int_X \omega^n = \alpha^n$ denotes the volume of $\alpha$.

Mabuchi studied the corresponding geometry of $\mathcal{H}_\alpha$, showing in particular that it can formally be seen as a locally symmetric space of non positive curvature. Semmes [Sem92] re-interpreted the geodesic equation as a complex homogeneous equation, while Donaldson [Don99] strongly motivated the search for smooth geodesics through its connection with the uniqueness of constant scalar curvature Kähler metrics.

In a series of remarkable works [Chen00, CC02, CT08, Chen09, CS09] X.X.Chen has studied the metric and geometric properties of the space $\mathcal{H}_\alpha$, showing in particular that it can formally be seen as a locally symmetric space of non positive curvature. Semmes [Sem92] re-interpreted the geodesic equation as a complex homogeneous equation, while Donaldson [Don99] strongly motivated the search for smooth geodesics through its connection with the uniqueness of constant scalar curvature Kähler metrics.

It is a very natural problem to try and understand the metric completion of the metric space $(\mathcal{H}_\alpha, d)$ (see [Cla13] for motivations and similar considerations on the space of all riemannian metrics). This is the main goal of this article.

**Main theorem.** The metric completion $(\overline{\mathcal{H}}_\alpha, d)$ of $(\mathcal{H}_\alpha, d)$ is a Hadamard space which consists of finite energy currents, more precisely

$$\mathcal{E}^2(\alpha) \subset \overline{\mathcal{H}}_\alpha \subset \mathcal{E}^1(\alpha).$$

Moreover weak (finite energy) geodesics are length minimizing.

Recall that a Hadamard space is a complete CAT(0) space, i.e. a complete geodesic space which has non positive curvature in the sense of Alexandrov (see [BH99, Chapter II.1]).

Finite energy currents were introduced in [GZ07], inspired by similar concepts developed by Cegrell in domains of $\mathbb{C}^n$ [Ceg98]. They have become an important tool in constructing singular Kähler-Einstein metrics [EGZ09, BEGZ10]. We refer the reader to section 2 for a precise definition. We just stress here that when the complex dimension is $n = 1$, the class $\mathcal{E}^1(\alpha)$ consists of non negative Radon measures whose potentials have square integrable gradients; we show that the Mabuchi metric dominates the Sobolev norm in this case.

Weak geodesics are generalized solutions of the homogeneous complex Monge-Ampère equation, when the boundary data are no longer smooth points in $\mathcal{H}_\alpha$, but rather finite energy currents in $\overline{\mathcal{H}}_\alpha$. They have played an important role in Berndtsson’s recent generalization of Bando-Mabuchi’s uniqueness result [Ber13]. They also play a crucial role in the variational approach to solving degenerate complex Monge-Ampère equations in finite
energy classes and its application to the Kähler-Einstein problem on singular varieties [BBCZ13, BBEGZ11].

When $X_P$ is a toric manifold associated to a Delzant polytope $P \subset \mathbb{R}^n$, the Legendre transformation allows to linearize the Monge-Ampère operator. We explain in section 4 that all notions (Kähler metrics, finite energy currents, geodesics, etc) have an elementary interpretation on $P$, following previous observations by Guillemin, Guan, Donaldson, Berman-Berndtsson [Gui94, Guan99, Don02, CDG03, BerBer13] (to name a few). Setting $H_{\text{tor}}$ for the set of toric Kähler metrics in $\alpha$ and $\mathcal{E}^2_{\text{tor}}$ for the corresponding set of finite energy toric currents, we then prove:

**Toric theorem.** *The metric completion of $(H_{\text{tor}}, d)$ is $(\mathcal{E}^2_{\text{tor}}, d)$.*

More precisely toric Kähler metrics correspond to certain ”smooth” convex functions on the polytope $P$, geodesics in $H_{\text{tor}}$ correspond to straight lines on $P$, and the Mabuchi distance corresponds to the Lebesgue $L^2$-norm on the latter, while finite energy toric currents in $\mathcal{E}^2$ correspond precisely to convex functions on $P$ that are $L^2$ with respect to Lebesgue measure.

This obviously raises the question whether this equality holds in general, we believe such is the case. When $\alpha$ is a toric Kähler class on a compact toric Kähler manifold $X_P$ our toric result at least insures that the inclusion

$$H_\alpha \subseteq \mathcal{E}^1(\alpha),$$

is strict since toric elements of $\mathcal{E}^1(\alpha) \setminus \mathcal{E}^2(\alpha)$ are at infinite distance from smooth ones.

There are several related questions that we do not address here. It would be for instance interesting to extend the quantization results of Phong-Sturm [PS06], Berndtsson [Ber09], Chen-Sun [CS09] and Song-Zelditch [SZ12], Rubinstein-Zelditch [RZ12] to $H_\alpha$. The reader will find further interesting problems by consulting the surveys [G12, PSS12].

The organization of the paper is as follows. *Section 1* is a recap on the Mabuchi metric and those results of Chen that we shall be using. We refer the interested reader to [G12] for a much broader overview of this exciting field. We introduce in *Section 2* various classes of finite energy currents, compare their natural topologies with the one induced by the Mabuchi distance, and locate the metric completion of $(H_\alpha, d)$.

We study weak geodesics in *Section 3* completing there the proof of our main result, and provide a detailed analysis of the toric setting in *Section 4*.

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*The study of geodesics in $\mathcal{E}^2(\alpha)$ has been independently done by Darvas in [Dar14] by studying envelopes with respect to singularity type. We thank him for communicating his interesting preprint.*

1. **The space of Kähler metrics**

Let $(X, \omega)$ be a compact Kähler manifold of dimension $n$. It follows from the $\partial \overline{\partial}$-lemma that any other Kähler metric on $M$ in the same cohomology
class as \( \omega \) can be written as
\[
\omega_{\varphi} = \omega + d\bar{d}\varphi,
\]
where \( d = \partial + \bar{\partial} \) and \( d^c = \frac{1}{2\pi}(\partial - \bar{\partial}) \). Let \( \mathcal{H} \) be the space of Kähler potentials
\[
\mathcal{H} = \{ \varphi \in C^\infty(X) : \omega_{\varphi} = \omega + d\bar{d}\varphi > 0 \}.
\]
This is a convex open subset of the Fréchet vector space \( C^\infty(X) \), thus itself a Fréchet manifold, which is moreover parallelizable: \( T\mathcal{H} = \mathcal{H} \times C^\infty(X) \). Each tangent space is identified with \( C^\infty(X) \).

As two Kähler potentials define the same metric when (and only when) they differ by an additive constant, we set
\[
\mathcal{H}_\alpha = \mathcal{H}/\mathbb{R}
\]
where \( \mathbb{R} \) acts on \( \mathcal{H} \) by addition. The set \( \mathcal{H}_\alpha \) is therefore the space of Kähler metrics on \( X \) in the cohomology class \( \alpha := \{ \omega \} \in H^{1,1}(X, \mathbb{R}) \).

We briefly review in this section known facts about the riemannian structure of this space, as introduced by Mabuchi in [Mab87] and further investigated by Semmes [Sem92], Donaldson [Don99], Chen [Chen00, CC02]. Introductory references for this material are the lecture notes by Kolev [Kol12] and Boucksom [Bou12].

1.1. The Riemannian structure.

**Definition 1.1.** [Mab87] The Mabuchi metric is the \( L^2 \) Riemannian metric on \( \mathcal{H} \). It is defined by
\[
\langle \psi_1, \psi_2 \rangle_\varphi = \int_X \psi_1 \psi_2 \frac{(\omega + d\bar{d}\varphi)^n}{V_\alpha}
\]
where \( \varphi \in \mathcal{H} \), \( \psi_1, \psi_2 \in C^\infty(X) \) and \( (\omega + d\bar{d}\varphi)^n/V_\alpha \) is the volume element, normalized so that it is a probability measure. Here
\[
V_\alpha := \alpha^n = \int_X \omega^n.
\]

In the sequel we shall also use the notation \( \omega_{\varphi} := \omega + d\bar{d}\varphi \) and
\[
MA(\varphi) := \frac{\omega_{\varphi}^n}{V_\alpha}
\]

Geodesics between two points \( \varphi_0, \varphi_1 \) in \( \mathcal{H} \) are defined as the extremals of the Energy functional
\[
\varphi \mapsto \frac{1}{2} \int_0^1 \int_X (\dot{\varphi}_t)^2 MA(\varphi_1) \, dt.
\]
where \( \varphi = \varphi_t \) is a path in \( \mathcal{H} \) joining \( \varphi_0 \) and \( \varphi_1 \). The geodesic equation is obtained by computing the Euler-Lagrange equation for this Energy functional (with fixed end points). It reduces to
\[
(1) \quad \ddot{\varphi} = \left\| \nabla \varphi \right\|_\varphi^2
\]
where the gradient is relative to the metric \( \omega_{\varphi} \). This identity also writes
\[
\dot{\varphi} MA(\varphi) = \frac{n}{V_\alpha} d\dot{\varphi} \land d^c \dot{\varphi} \land \omega_{\varphi}^{n-1}.
\]
As for Riemannian manifolds of finite dimension, one can find the local expression of the Levi-Civita connection by polarizing the geodesic equation. We define the covariant derivative of the vector field \( \psi_t \) along the path \( \varphi_t \) in \( \mathcal{H} \) by the formula

\[
\frac{D\psi}{Dt} = \partial_t \psi - < \nabla \psi, \nabla \dot{\varphi} >_{\varphi}.
\]

This covariant derivative is symmetric by its very definition, i.e.

\[
\frac{D}{Ds} \partial_t \varphi = \frac{D}{Dt} \partial_s \varphi,
\]

for every family \( \varphi = \varphi_{s,t} \) and it can be checked directly that it preserves the metric,

\[
\frac{d}{dt} < \psi_1, \psi_2 >_{\varphi} = < \frac{D\psi_1}{Dt}, \psi_2 >_{\varphi} + < \psi_1, \frac{D\psi_2}{Dt} >_{\varphi}.
\]

1.2. Dirichlet problem for the complex Monge-Ampère equation.

We are interested in the boundary value problem for the geodesic equation: given \( \varphi_0, \varphi_1 \) two distinct points in \( \mathcal{H} \), can one find a path \( (\varphi(t))_{0 \leq t \leq 1} \) in \( \mathcal{H} \) which is a solution of (1) with end points \( \varphi(0) = \varphi_0 \) and \( \varphi(1) = \varphi_1 \)?

It has been observed by Semmes \([\text{Sem92}]\) that this can be reformulated as a homogeneous complex Monge-Ampère equation.

For each path \( (\varphi_t)_{t \in [0,1]} \) in \( \mathcal{H} \), we set

\[
\varphi(x,t,s) = \varphi_t(x), \quad x \in X, \quad e^{t+is} \in A = [1,e] \times S^1;
\]

i.e. we associate to each path \( (\varphi_t) \) a function \( \varphi \) on the complex manifold \( X \times A \), which is radial in the annulus coordinate: we consider the annulus \( A \) as a Riemann surface with boundary and use the complex coordinate \( z = e^{t+is} \) to parametrize the annulus \( A \). Set \( \omega(x,z) := \omega(x) \).

**Proposition 1.2.** \([\text{Sem92}]\) The path \( \varphi_t \) is a geodesic in \( \mathcal{H} \) if and only if the associated radial function \( \varphi \) on \( X \times A \) is a solution of the homogenous complex Monge-Ampère equation

\[
(\omega + dd^c_x z \cdot \varphi)^{n+1} = 0.
\]

We have stressed the fact that we take here derivatives in all variables \( x,z \).

It is well known that homogeneous complex Monge-Ampère equations in bounded domains of \( \mathbb{C}^n \) admit a unique solution which is at most \( C^{1,1} \) smooth. A fundamental result of Chen \([\text{Chen00}]\) shows the existence (and uniqueness) of almost \( C^{1,1} \)-solutions to the geodesic equation\(^1\).

It was expected that Chen’s regularity result was essentially optimal (although Chen and Tian have proposed in \([\text{CT08}]\) some improvements). This has been confirmed by Lempert, Vivas and Darvas in \([\text{LV13}, \text{DL12}]\).

We will consider in the sequel generalized geodesics with lower regularity, which turn out to be quite useful.

\(^1\)These have bounded Laplacian, hence they are in particular \( C^{1,\alpha} \) for all \( 0 < \alpha < 1 \).
1.3. The Aubin-Mabuchi functional. Each tangent space $T_\varphi \mathcal{H}$ admits the following orthogonal decomposition

$$T_\varphi \mathcal{H} = \{ \psi \in C^\infty(X); \beta_\varphi(\psi) = 0 \} \oplus \mathbb{R},$$

where $\beta = MA$ is the 1-form defined on $\mathcal{H}$ by

$$\beta_\varphi(\psi) = \int_X \psi MA(\varphi).$$

It is a classical observation that the 1-form $\beta$ is closed. Therefore, there exists a unique function $E$ defined on the convex open set $\mathcal{H}$, such that $\beta = dE$ and $E(0) = 0$. It is often called the Aubin-Mabuchi functional and can be expressed (after integration along affine paths) by

$$E(\varphi) = \frac{1}{(n+1)V_\alpha} \sum_{j=0}^{n} \int_X \varphi (\omega + dd^c\varphi)^j \wedge \omega^{n-j}. \quad (2)$$

**Lemma 1.3.** The Aubin-Mabuchi functional $E$ is concave, non-decreasing and satisfies the cocycle condition

$$E(\varphi) - E(\psi) = \frac{1}{(n+1)V_\alpha} \sum_{j=0}^{n} \int_X (\varphi - \psi) (\omega + dd^c\varphi)^j \wedge (\omega + dd^c\psi)^{n-j}$$

It is affine along geodesics in $\mathcal{H}$.

We sketch the proof for the reader’s convenience.

**Proof.** The monotonicity property follows from the definition since the first derivative of $E$ is $dE = \beta = MA \geq 0$, a probability measure. The concavity follows from a direct computation (see e.g. [BEGZ10]), while the cocycle condition follows by differentiating $E(t\varphi + (1-t)\psi)$.

The behavior of $E$ along geodesics is also easily understood:

$$\frac{d^2}{dt^2}E(\varphi_t) = \oint_X \hat{\varphi} MA(\varphi) + \frac{n}{V_\alpha} \oint_X \hat{\varphi} dd^c\hat{\varphi} \wedge \omega^{n-1}$$

$$= \oint_X \left\{ \hat{\varphi} MA(\varphi) - \frac{n}{V_\alpha} d\hat{\varphi} \wedge d\hat{\varphi} \wedge \omega^{n-1} \right\} = 0$$

when $\varphi_t$ is a geodesic. \qed

Since each path $\varphi_t = \varphi + t$ in $\mathcal{H}$ is a geodesic, we obtain in particular

$$E(\varphi + t) = E(\varphi) + t. \quad (3)$$

Given $\varphi \in \mathcal{H}$ there exists a unique $c \in \mathbb{R}$ such that $E(\varphi + c) = 0$. The restriction of the Mabuchi metric to the fiber $E^{-1}(0)$ induces a Riemannian structure on the quotient space $\mathcal{H}_\alpha = \mathcal{H}/\mathbb{R}$ and allows to decompose

$$\mathcal{H} = \mathcal{H}_\alpha \times \mathbb{R}$$

as a product of Riemannian manifolds.
1.4. \( \mathcal{H} \) as a symmetric space. Consider a 2-parameters family \( \varphi(s,t) \in \mathcal{H} \) and a vector field \( \psi(s,t) \in C^\infty(X) \) defined along \( \varphi \). The curvature tensor of the Mabuchi metric on \( \mathcal{H} \) is defined by
\[
R_{\varphi}(\varphi_s,\varphi_t)\psi = (D_s D_t - D_t D_s)\psi,
\]
where \( \varphi_s, \varphi_t \) denote the \( s \) and \( t \) derivatives of \( \varphi \) and
\[
D_t\psi = \psi_t + \Gamma_{\varphi}(\varphi_t,\psi) = \psi_t - \langle \nabla \psi, \nabla \varphi_t \rangle_{\varphi}
\]
is the covariant derivative of \( \psi \).

**Proposition 1.4.** The curvature tensor on \( \mathcal{H} \) can be expressed as
\[
R_{\varphi}(\varphi_s,\varphi_t)\psi = -\{\{\varphi_s, \varphi_t\}, \psi\}
\]
where \( \{\varphi_1, \varphi_2\} \) is the Poisson bracket associated with the symplectic structure \( \omega_{\varphi} \). Furthermore, the covariant derivative \( D_t R(\varphi_s, \varphi_t) \) of the curvature tensor \( R \) vanishes.

In other words, \( \mathcal{H} \) is a **locally symmetric space**: \( \nabla R \equiv 0 \). In finite dimension, such a manifold is characterized by the fact that in the neighbourhood of each point \( x \), the (local) geodesic symmetry of center \( x \): \( \exp_x(X) \mapsto \exp_x(-X) \) is an isometry (see [He01]). Note however that the exponential map is not well defined in our infinite dimensional setting.

1.5. The Mabuchi distance. The length of an arbitrary path \( (\varphi_t)_{t \in [0,1]} \) in \( \mathcal{H} \) is defined in a standard way,
\[
\ell(\varphi) := \int_0^1 |\dot{\varphi}_t| dt = \int_0^1 \sqrt{\int_X \dot{\varphi}_t^2 MA(\varphi_\cdot) dt}.
\]
The distance between two points in \( \mathcal{H} \) is then
\[
d(\varphi_0, \varphi_1) := \inf \{ \ell(\varphi) | \varphi \text{ is a path joining } \varphi_0 \text{ to } \varphi_1 \}.
\]

It is easy to verify that \( d \) defines a semi-distance (i.e. non-negative, symmetric and satisfying the triangle inequality). It is however non trivial, in this infinite dimensional context, to check that \( d \) is non degenerate (see [MM05] for a striking example). This was proved by Chen in [Chen00] (see also [CC02]) who established that

**Theorem 1.5.** [Chen00, CC02]
- \( (\mathcal{H}, d) \) is a metric space of non positive curvature;
- the geodesics \( (\varphi_t) \) are length minimizing;
- any sequence of asymptotically length minimizing paths converge, in the Hausdorff topology, to the unique geodesic.

Moreover for all \( t \in [0,1] \),
\[
d(\varphi_0, \varphi_1) = \sqrt{\int_X \dot{\varphi}_t^2 MA(\varphi_\cdot) dt}.
\]

**Remark 1.6.** A delicate issue here is that it is not clear that the geodesics constructed by Chen lie in \( \mathcal{H} \). This problem has been recently adressed by He [He12] who showed that the whole construction can be extended to
\[
\mathcal{H}_{1,1} := \{ \varphi \in PSH(X,\omega) | \Delta \varphi \in L^\infty(X) \}.
\]
Observe that $d$ induces a distance on $\mathcal{H}_\alpha$ (that we abusively still denote by $d$) compatible with the riemannian splitting $\mathcal{H} = \mathcal{H}_\alpha \times \mathbb{R}$, by setting

$$d(\omega_\varphi, \omega_\psi) := d(\varphi, \psi)$$

whenever the potentials $\varphi, \psi$ of $\omega_\varphi, \omega_\psi$ are normalized by $E(\varphi) = E(\psi) = 0$.

It is rather easy to check that $(\mathcal{H}_\alpha, d)$ is not a complete metric space. Describing the metric completion $(\overline{\mathcal{H}_\alpha}, d)$ is the main goal of this article. We shall always work at the level of potentials, i.e. we will try and understand the metric completion of the space $(\mathcal{H}, d)$.

2. Finite energy classes

We fix $\omega$ a Kähler form representing $\alpha$ and define in this section the set $\mathcal{E}(\alpha)$ (resp. $\mathcal{E}^p(\alpha)$) of positive closed currents $T = \omega + dd^c \varphi$ with full Monge-Ampère mass (resp. finite weighted energy) in $\alpha$, by defining the corresponding class $\mathcal{E}(X, \omega)$ (resp. $\mathcal{E}^p(X, \omega)$) of potentials $\varphi$.

2.1. The space $\mathcal{E}(\alpha)$.

2.1.1. Quasi-plurisubharmonic functions. Recall that a function is quasi-plurisubharmonic if it is locally given as the sum of a smooth and a psh function. In particular quasi-psh functions are upper semi-continuous and $L^1$-integrable. Quasi-psh functions are actually in $L^p$ for all $p \geq 1$, and the induced topologies are all equivalent. A much stronger integrability property actually holds: Skoda’s integrability theorem [Skö72] asserts indeed that $e^{-\varepsilon \varphi} \in L^1(X)$ if $0 < \varepsilon$ is smaller than $2/\nu(\varphi)$, where $\nu(\varphi)$ denotes the maximal logarithmic singularity (Lelong number) of $\varphi$ on $X$.

Quasi-plurisubharmonic functions have gradient in $L^r$ for all $r < 2$, but not in $L^2$ as shown by the local model $\log |z_1|$.

**Definition 2.1.** We let $\text{PSH}(X, \omega)$ denote the set of all $\omega$-plurisubharmonic functions. These are quasi-psh functions $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ such that

$$\omega + dd^c \varphi \geq 0$$

in the weak sense of currents.

The set $\text{PSH}(X, \omega)$ is a closed subset of $L^1(X)$, when endowed with the $L^1$-topology.

2.1.2. Bedford-Taylor theory. Bedford and Taylor have observed in [BT82] that one can define the complex Monge-Ampère operator

$$MA(\varphi) := \frac{1}{V_\alpha}(\omega + dd^c \varphi)^n$$

for all bounded $\omega$-psh function: they showed that whenever $(\varphi_j)$ is a sequence of smooth $\omega$-psh functions locally decreasing to $\varphi$, then the smooth probability measures $MA(\varphi_j)$ converge, in the weak sense of Radon measures, towards a unique probability measure that we denote by $MA(\varphi)$.

At the heart of Bedford-Taylor’s theory lies the following maximum principle: if $u, v$ are bounded $\omega$-plurisubharmonic functions, then

$$(MP) \quad 1_{\{v < u\}}MA(\max(u, v)) = 1_{\{v < u\}}MA(u).$$
This equality is elementary when \( u \) is continuous, as the set \( \{ v < u \} \) is then a Borel open subset of \( X \). When \( u \) is merely bounded, this set is only open in the plurifine topology. Since Monge-Ampère measures of bounded \( q_{ph} \) functions do not charge pluripolar sets (by the so called Chern-Levine-Nirenberg inequalities), and since \( u \) is nevertheless quasi-continuous, this gives a heuristic justification for (MP).

The reader will easily verify that the maximum principle (MP) implies the so called comparison principle:

**Proposition 2.2.** Let \( u, v \) be bounded \( \omega \)-plurisubharmonic functions. Then

\[
\int_{\{ v < u \}} MA(u) \leq \int_{\{ v < u \}} MA(v).
\]

2.1.3. The class \( \mathcal{E}(X, \omega) \). Given \( \varphi \in PSH(X, \omega) \), we consider its canonical approximants

\[
\varphi_j := \max(\varphi, -j) \in PSH(X, \omega) \cap L^\infty(X).
\]

It follows from the Bedford-Taylor theory that the measures \( MA(\varphi_j) \) are well defined probability measures. Since the \( \varphi_j \)'s are decreasing, it is natural to expect that these measures converge (in the weak sense). The following strong monotonicity property holds:

**Lemma 2.3.** The sequence \( \mu_j := \mathbf{1}_{\{ \varphi > -j \}} MA(\varphi_j) \) is an increasing sequence of Borel measures.

The proof is an elementary consequence of (MP) (see [GZ07, p.445]).

Since the \( \mu_j \)'s all have total mass bounded from above by 1 (the total mass of the measure \( MA(\varphi_j) \)), we can consider

\[
\mu_\varphi := \lim_{j \to +\infty} \mu_j,
\]

which is a positive Borel measure on \( X \), with total mass \( \leq 1 \).

**Definition 2.4.** We set

\[
\mathcal{E}(X, \omega) := \{ \varphi \in PSH(X, \omega) \mid \mu_\varphi(X) = 1 \}.
\]

For \( \varphi \in \mathcal{E}(X, \omega) \), we set \( MA(\varphi) := \mu_\varphi \).

The notation is justified by the following important fact: the complex Monge-Ampère operator \( \varphi \mapsto MA(\varphi) \) is well defined on the class \( \mathcal{E}(X, \omega) \), i.e. for every decreasing sequence of bounded (in particular smooth) \( \omega \)-psh functions \( \varphi_j \), the probability measures \( MA(\varphi_j) \) weakly converge towards \( \mu_\varphi \), if \( \varphi \in \mathcal{E}(X, \omega) \).

Every bounded \( \omega \)-psh function clearly belongs to \( \mathcal{E}(X, \omega) \) since in this case \( \{ \varphi > -j \} = X \) for \( j \) large enough, hence

\[
\mu_\varphi \equiv \mu_j = MA(\varphi_j) = MA(\varphi).
\]

The class \( \mathcal{E}(X, \omega) \) also contains many \( \omega \)-psh functions which are unbounded. When \( X \) is a compact Riemann surface \( (n = \dim C X = 1) \), the set \( \mathcal{E}(X, \omega) \) is the set of \( \omega \)-sh functions whose Laplacian does not charge polar sets.
Remark 2.5. If \( \varphi \in PSH(X, \omega) \) is normalized so that \( \varphi \leq -1 \), then \(-(-\varphi)^\varepsilon \) belongs to \( \mathcal{E}(X, \omega) \) whenever \( 0 \leq \varepsilon < 1 \). The functions which belong to the class \( \mathcal{E}(X, \omega) \), although usually unbounded, have relatively mild singularities. In particular they have zero Lelong numbers.

It is shown in [GZ07] that the maximum principle (MP) and the comparison principle continue to hold in the class \( \mathcal{E}(X, \omega) \). The latter can be characterized as the largest class for which the complex Monge-Ampère is well defined and the maximum principle holds. We further note that the domination principle holds:

**Proposition 2.6.** If \( \varphi, \psi \in \mathcal{E}(X, \omega) \) are such that \( \varphi(x) \leq \psi(x) \) for \( MA(\psi) \)-a.e. \( x \), then \( \varphi(x) \leq \psi(x) \) for all \( x \in X \).

Let us stress that the convergence of the canonical approximating measures \( \mu_j = MA(\max(\varphi, -j)) \) towards \( \mu_\varphi \) holds in the (strong) sense of Borel measures, i.e. for all Borel sets \( B \),

\[
\mu_\varphi(B) := \lim_{j \to +\infty} \mu_j(B).
\]

In particular when \( B = P \) is a pluripolar set, we obtain \( \mu_j(P) = 0 \), hence \( \mu_\varphi(P) = 0 \) for all pluripolar sets \( P \).

Conversely, one can show [GZ07, BEGZ10] that a probability measure \( \mu \) equals \( \mu_\varphi \) for some \( \varphi \in \mathcal{E}(X, \omega) \) whenever \( \mu \) does not charge pluripolar sets (one then says that \( \mu \) is non-pluripolar).

It follows from the \( \partial \partial \)-lemma that any positive closed current \( T \in \alpha \) writes \( T = \omega + dd^c \varphi \) for some function \( \varphi \in PSH(X, \omega) \) which is unique up to an additive constant.

**Definition 2.7.** We let \( \mathcal{E}(\alpha) \) denote the set of all positive currents in \( \alpha \), \( T = \omega + dd^c \varphi \), with \( \varphi \in \mathcal{E}(X, \omega) \).

The definition is clearly independent of the choice of the potential \( \varphi \).

2.2. The complete metric spaces \( \mathcal{E}^p(\alpha) \).

2.2.1. Weighted energy classes. Let \( \chi : \mathbb{R}^- \to \mathbb{R}^- \) be an increasing function such that \( \chi(-\infty) = -\infty \). Following [GZ07] we let \( W \) denote the set of all such weights, and set

\[
\mathcal{W}^\pm := \{ \chi \in W \mid \pm \chi \text{ is convex} \}
\]

**Definition 2.8.** We let \( \mathcal{E}_\chi(X, \omega) \) denote the set of \( \omega \)-psh functions with finite \( \chi \)-energy, i.e.

\[
\mathcal{E}_\chi(X, \omega) := \{ \varphi \in \mathcal{E}(X, \omega) / \chi(-|\varphi|) \in L^1(MA(\varphi)) \}.
\]

When \( \chi(t) = -(-t)^p \), \( p > 0 \), we set \( \mathcal{E}^p(X, \omega) = \mathcal{E}_\chi(X, \omega) \). We let

\[
\mathcal{E}_\chi(\alpha) = \{ T = \omega + dd^c \varphi \mid \varphi \in \mathcal{E}_\chi(X, \omega) \}
\]

and \( \mathcal{E}^p(\alpha) \) denote the corresponding sets of finite weighted energy currents.

We list a few important properties of these classes and refer the reader to [GZ07] for the proofs:
• $E(X, \omega) = \cup_{\chi \in W^-} E_{\chi}(X, \omega) = \cup_{\chi \in W^-} E_{\chi}(X, \omega)$;
• $PSH(X, \omega) \cap L^\infty(X) = \cap_{\chi \in W^-} E_{\chi}(X, \omega) = \cap_{\chi \in W^+} E_{\chi}(X, \omega)$;
• when $\chi \in W^+$ any $\varphi \in E_{\chi}(X, \omega)$ is such that $\nabla_{\omega} \varphi \in L^2(\omega^n)$;
• $\varphi \in E^p(X, \omega)$ if and only if for any (resp. one) sequence of bounded $\omega$-functions decreasing to $\varphi$, $\sup_j \int_X (-\varphi_j)^p \cdot MA(\varphi_j) < +\infty$.

**Proposition 2.9.** ([GZ07] Proposition 3.8) Fix $p > 0$. There exists $C_p > 0$ such that for all $0 \geq \varphi_0, \ldots, \varphi_n \in PSH(X, \omega) \cap L^\infty(X)$,

$$0 \leq \int_X (-\varphi_0)^p \omega_{\varphi_1} \wedge \cdots \wedge \omega_{\varphi_n} \leq C_p \max_{0 \leq j \leq n} \left[ \int_X (-\varphi_j)^p \omega_{\varphi_j} \right].$$

In particular the class $E^p(X, \omega)$ is starshaped and convex.

2.2.2. **Strong topology on $E^1(\alpha)$**. The class $E^1(X, \omega)$ plays a central role. Its defining weight $\chi(t) = t$ is the only weight which is both convex and concave,

$$E^1(X, \omega) = \cup_{\chi \in W^-} E_{\chi}(X, \omega) \cap \cup_{\chi \in W^+} E_{\chi}(X, \omega).$$

If we extend the Aubin-Mabuchi functional using its monotonicity property,

$$E(\varphi) := \inf \{ E(\psi) \mid \psi \in PSH(X, \omega) \cap C^\infty(X) \text{ and } \varphi \leq \psi \},$$

then

$$E^1(X, \omega) = \{ \varphi \in PSH(X, \omega) \mid E(\varphi) > -\infty \}.$$

Thus $E^1(X, \omega)$ is the natural frame for the variational approach to studying complex Monge-Ampère operators [BBGZ13]. Set

$$I(\varphi, \psi) = \int_X (\varphi - \psi) \cdot (MA(\psi) - MA(\varphi)).$$

It has been shown in [BBGZ11] that $I$ defines a complete metrizable uniform structure on $E^1(\alpha)$. More precisely we identify $E^1(\alpha)$ with the set

$$E^1_{norm}(X, \omega) = \{ \varphi \in E^1(X, \omega) \mid \sup_X \varphi = 0 \}$$

of normalized potentials. Then

• $I$ is symmetric and positive on $E^1_{norm}(X, \omega)^2 \setminus \{ \text{diagonal} \}$;
• $I$ satisfies a quasi-triangle inequality [BBGZ11] Theorem 1.8;
• $I$ induces a uniform structure which is metrizable [Bourbaki];
• the metric space $(E^1(\alpha), d_1)$ is complete [BBGZ11] Proposition 2.4].

**Definition 2.10.** The strong topology on $E^1(\alpha)$ is the topology defined by $I$. We fix in the sequel a distance $d_1$ which defines this metrizable topology.

The corresponding notion of convergence corresponds to the *convergence in energy* previously introduced in [BBGZ13] (see [BBGZ11] Proposition 2.3]). It is the coarsest refinement of the weak topology such that $E$ becomes continuous. In particular

if $T_j \rightarrow T$ in $(E^1(\alpha), d_1)$, then $T^n_j \rightarrow T^n$

in the weak sense of Radon measures, while the Monge-Ampère operator is usually discontinuous for the weak topology of currents.

\footnote{Up to scaling and translating, two operations which leave the class invariant.}
Example 2.11. When \( \dim \mathbb{C} X = n = 1 \), \( \mathcal{E}^1(X, \omega) = PSH(X, \omega) \cap W^{1,2}(X) \) is the set of \( \omega \)-subharmonic functions with square integrable gradient. The strong topology on \( \mathcal{E}^1(\alpha) \) is the one induced by the Sobolev norm.

When \( X = \mathbb{P}^1 F \) is the Riemann sphere, \( \omega = \omega_{FS} \) is the Fubini-Study Kähler form, and \( \varepsilon_j \nearrow 0^+ \), \( C_j \nearrow +\infty \), the functions

\[
\varphi_j[z] = \varepsilon_j \max(\log \|z\|, \log |z_0| - C_j) - \varepsilon_j \log \|z\| \in \mathcal{E}^1(X, \omega)
\]

converge to zero in \( L^2(X) \) but not in the strong topology if \( \varepsilon_j^2 C_j \geq 1 \) since

\[
I(\varphi_j, 0) = \int_{\mathbb{P}^1} d\varphi_j \wedge d^c \varphi_j \sim \varepsilon_j^2 C_j.
\]

2.2.3. Strong topology on \( \mathcal{E}^2(\alpha) \). We now introduce an analogous strong topology on the class \( \mathcal{E}^2(\alpha) \). For \( \varphi, \psi \in \mathcal{E}^2(X, \omega) \), we set

\[
I_2(\varphi, \psi) := \sqrt{\int_X (\varphi - \psi)^2 \left[ \frac{MA(\varphi) + MA(\psi)}{2} \right]}
\]

This quantity is well-defined (and finite) by \cite{GZ07} Proposition 3.6. It is obviously non-negative and symmetric. It follows from the domination principle (Proposition 2.6) that \( I_2(\varphi, \psi) = 0 \Rightarrow \varphi = \psi \). This can also be seen as a consequence of Propositions 2.15 and 2.17 below.

Definition 2.12. The strong topology on \( \mathcal{E}^2(\alpha) \) is the one induced by \( I_2 \).

It follows from Propositions 2.15 and 2.17 below that the Mabuchi topology (induced by the Mabuchi distance) is stronger than the strong topology on \( \mathcal{E}^1(\alpha) \) and weaker than the strong topology on \( \mathcal{E}^2(\alpha) \). We set

\[
\mathcal{E}^2_{\text{norm}}(X, \omega) = \{ \varphi \in \mathcal{E}^2(X, \omega) \mid \sup_X \varphi = 0 \}
\]

Note that if a sequence \( (\varphi_j) \in \mathcal{E}^2_{\text{norm}}(X, \omega) \) is a Cauchy sequence for \( I_2 \), then it is a Cauchy sequence in \( (\mathcal{E}^1_{\text{norm}}(X, \omega), d_1) \) since

\[
0 \leq I(\varphi, \psi) = \int_X (\varphi - \psi) [MA(\psi) - MA(\varphi)] \leq \sqrt{2} I_2(\varphi, \psi),
\]

as follows from the Cauchy-Schwarz inequality. Since \( (\mathcal{E}^1_{\text{norm}}(X, \omega), d_1) \) is complete, we infer the existence of \( \varphi \in \mathcal{E}^1_{\text{norm}}(X, \omega) \) such that \( d_1(\varphi_j, \varphi) \to 0 \).

Now \( I_2(\varphi_j, 0) \) is bounded and \( MA(\varphi_j) \) weakly converges to \( MA(\varphi) \) (by \cite{BBECZ11} Proposition 1.6]). It follows therefore from Fatou’s and Hartogs’ lemma that \( \varphi \in \mathcal{E}^2_{\text{norm}}(X, \omega) \) and \( I_2(\varphi_j, \varphi) \to 0 \). We have thus proved that \( (\mathcal{E}^2_{\text{norm}}(X, \omega), I_2) \) is ”complete” (with an abuse of terminology, as we haven’t checked that \( I_2 \) induces a uniform structure).

Lemma 2.13. Let \( \varphi, \psi \) be bounded \( \omega \)-psh functions and \( S \) be a positive closed current of bidimension \((1,1)\) on \( X \). If \( \varphi \leq \psi \), then

\[
\int_X (\varphi - \psi)^2 \omega_\psi \wedge S \leq \int_X (\varphi - \psi)^2 \omega_\varphi \wedge S.
\]

In particular for all \( 0 \leq j \leq n \),

\[
V^{-1}_a \int_X (\varphi - \psi)^2 \omega_\psi^j \wedge \omega_\varphi^{n-j} \leq \int_X (\varphi - \psi)^2 MA(\varphi).
\]
Proof. By Stokes theorem,
\[
\int_X (\varphi - \psi)^2 \omega \wedge S - \int_X (\varphi - \psi)^2 \omega \wedge S = 2 \int_X (\psi - \varphi) d(\psi - \varphi) \wedge D(\varphi - \psi) \wedge S
\]
is non-negative if \((\varphi - \psi) \geq 0\).

The second assertion follows by applying the first one inductively. \(\square\)

**Proposition 2.14.** For \(\varphi, \psi \in E^2(X, \omega)\),
\[
I_2(\varphi, \psi)^2 = I_2(\varphi, \max(\varphi, \psi))^2 + I_2(\max(\varphi, \psi), \psi)^2.
\]

**Proof.** Recall that the maximum principle insures that
\[
1_{\{\varphi < \psi\}} MA(\max(\varphi, \psi)) = 1_{\{\varphi < \psi\}} MA(\psi),
\]
while \((\varphi - \max(\varphi, \psi))^2 = 0\) on \((\varphi \geq \psi)\), thus
\[
I_2(\varphi, \max(\varphi, \psi))^2 = \frac{1}{2} \int_{\{\varphi < \psi\}} (\varphi - \psi)^2 [MA(\varphi) + MA(\psi)].
\]

Similarly
\[
I_2(\psi, \max(\varphi, \psi))^2 = \frac{1}{2} \int_{\{\varphi > \psi\}} (\varphi - \psi)^2 [MA(\varphi) + MA(\psi)]
\]
and the result follows since
\[
I_2(\varphi, \psi)^2 = \frac{1}{2} \int_{\{\varphi \neq \psi\}} (\varphi - \psi)^2 [MA(\varphi) + MA(\psi)].
\]
\(\square\)

### 2.3. Comparing Distances

The following lower bound is the first key observation for understanding the metric completion of \((\mathcal{H}, d)\):

**Proposition 2.15.** For all \(\varphi_0, \varphi_1 \in \mathcal{H}\),
\[
\frac{1}{n+1} I(\varphi_0, \varphi_1) + \max \left\{ \int_X (\varphi_1 - \varphi_0) MA(\varphi_0); \int_X (\varphi_0 - \varphi_1) MA(\varphi_1) \right\} \leq d(\varphi_0, \varphi_1).
\]

**Proof.** The cocycle condition (see Lemma 1.3) yields
\[
E(\varphi_0) - E(\varphi_1) - \int_X (\varphi_1 - \varphi_0) MA(\varphi_0) = \frac{1}{(n+1)V_\alpha} \sum_{j=1}^{n-1} \int_X d(\varphi_1 - \varphi_0) \wedge D(\varphi_1 - \varphi_0) \wedge \omega^{n-\ell-1}_{\varphi_0}\]
\[
= \frac{1}{(n+1)V_\alpha} \sum_{\ell=0}^{n-1} (n-\ell) \int_X d(\varphi_1 - \varphi_0) \wedge D(\varphi_1 - \varphi_0) \wedge \omega^{n-\ell-1}_{\varphi_0},
\]
while
\[
I(\varphi_0, \varphi_1) = \int_X (\varphi_0 - \varphi_1) (MA(\varphi_1) - MA(\varphi_0)) = \frac{1}{V_\alpha} \sum_{\ell=0}^{n-1} \int_X d(\varphi_1 - \varphi_0) \wedge D(\varphi_1 - \varphi_0) \wedge \omega^{n-\ell-1}_{\varphi_0},
\]
thus
\[ \frac{1}{n+1} I(\varphi_0, \varphi_1) + \int_X (\varphi_1 - \varphi_0) MA(\varphi_0) \leq E(\varphi_0) - E(\varphi_1). \]

Let \((\varphi_t)_{t \in [0,1]}\) be the geodesic joining \(\varphi_0\) to \(\varphi_1\). Recall from Lemma 1.3 that \(t \mapsto E(\varphi_t)\) is affine. Since \(E\) is a primitive of the (normalized) complex Monge-Ampère, we infer
\[ E(\varphi_1) - E(\varphi_0) = \int_0^1 \int_X \dot{\varphi}_t MA(\varphi_t) dt. \]

By Chen’s theorem the length of this geodesic equals \(d(\varphi_0, \varphi_1)\), while by definition the Riemannian energy \(t \mapsto \int_X (\dot{\varphi}_t)^2 MA(\varphi_t)\) is constant hence equals \(d(\varphi_0, \varphi_1)^2\). Cauchy-Schwarz inequality thus yields
\[ E(\varphi_0) - E(\varphi_1) \leq \sqrt{\int_X (\dot{\varphi}_t)^2 MA(\varphi_t)} = d(\varphi_0, \varphi_1), \]
proving that
\[ \frac{1}{n+1} I(\varphi_0, \varphi_1) + \int_X (\varphi_1 - \varphi_0) MA(\varphi_0) \leq d(\varphi_0, \varphi_1). \]

The desired bound follows by reversing the roles of \(\varphi_0\) and \(\varphi_1\).

**Remark 2.16.** This bound from below contains the non trivial information that \((\mathcal{H}, d)\) is a metric space. Observe indeed that the maximum of \(\int_X (\varphi_1 - \varphi_0) MA(\varphi_0)\) and \(\int_X (\varphi_0 - \varphi_1) MA(\varphi_1)\) is non negative (since the sum equals \(I(\varphi_0, \varphi_1)\) which is non-negative). It follows therefore from Proposition 2.15 that
\[ d(\varphi_0, \varphi_1) = 0 \implies I(\varphi_0, \varphi_1) = 0 \implies \omega_{\varphi_0} = \omega_{\varphi_1}, \]
hence \(\varphi_1 = \varphi_0 + c\) for some \(c \in \mathbb{R}\). The constant \(c\) has to be zero as well since the maximum vanishes.

We also have the following upper-bound:

**Proposition 2.17.** For all \(\varphi_0, \varphi_1 \in \mathcal{H}\),
\[ 0 \leq d(\varphi_0, \varphi_1) \leq 2I_2(\varphi_0, \varphi_1). \]
Moreover if \(\varphi_0 \leq \varphi_1\) then
\[ \sqrt{\int_X (\varphi_1 - \varphi_0)^2 MA(\varphi_1)} \leq d(\varphi_0, \varphi_1) \leq \sqrt{\int_X (\varphi_1 - \varphi_0)^2 MA(\varphi_0)}. \]

**Proof.** We first assume that \(\varphi_0 \leq \varphi_1\). Let \((\varphi_t)\) be the geodesic joining \(\varphi_0\) to \(\varphi_1\). It follows from the maximum principle that \(t \mapsto \varphi_t\) is increasing. As it is convex as well, we infer
\[ 0 \leq \dot{\varphi}_0 \leq \varphi_1 - \varphi_0 \leq \dot{\varphi}_1 \]
hence
\[ \int_X (\varphi_1 - \varphi_0)^2 MA(\varphi_1) \leq \int_X (\dot{\varphi}_1)^2 MA(\varphi_1) = d(\varphi_0, \varphi_1)^2 \]
and similarly for the upper-bound.
We give an alternative proof of the upper bound which does not use Chen’s result. This might be of interest in more singular contexts. We can join $\varphi_0$ to $\varphi_1$ by a straight line $\psi_t = t\varphi_1 + (1-t)\varphi_0$, thus

$$d(\varphi_0, \varphi_1) \leq \ell(\varphi) = \int_0^1 \sqrt{\int_X (\varphi_1 - \varphi_0)^2 MA(\varphi_t) dt}$$

$$\leq \sqrt{\int_0^1 \int_X (\varphi_1 - \varphi_0)^2 MA(\varphi_t) dt},$$

by Cauchy-Schwarz inequality. Now

$$MA(\varphi_t) = V^{-1}_\alpha \sum_{j=0}^n \begin{pmatrix} n \\ j \end{pmatrix} t^j (1-t)^{n-j} \omega_{\varphi_1}^j \wedge \omega_{\varphi_0}^{n-j}$$

and for all $0 \leq j \leq n$,

$$\int_0^1 t^j (1-t)^{n-j} dt = \frac{\begin{pmatrix} n \\ j \end{pmatrix}}{n+1},$$

hence

$$\frac{1}{(n+1)V_\alpha} \sum_{j=0}^n \int_X (\varphi_1 - \varphi_0)^2 \omega_{\varphi_1}^j \wedge \omega_{\varphi_0}^{n-j} \leq \int_X (\varphi_1 - \varphi_0)^2 MA(\varphi_0),$$

as follows from Lemma 2.13, yielding

$$d(\varphi_0, \varphi_1) \leq \sqrt{\int_X (\varphi_1 - \varphi_0)^2 MA(\varphi_0)}.$$

We now treat the general case. It follows from the triangle inequality that

$$d(\varphi_0, \varphi_1) \leq d(\varphi_0, \max(\varphi_0, \varphi_1)) + d(\max(\varphi_0, \varphi_1), \varphi_1) \leq \sqrt{2} \sqrt{\int_X (\varphi_1 - \varphi_0)^2 [MA(\varphi_0) + MA(\varphi_1)]},$$

by using the elementary inequality $\sqrt{a} + \sqrt{b} \leq \sqrt{2(a+b)}$. \hfill \Box

Remark 2.18. The lower bound $\sqrt{\int_X (\varphi_1 - \varphi_0)^2 MA(\varphi_1)} \leq d(\varphi_0, \varphi_1)$ when $\varphi_0 \leq \varphi_1$ is a particular case of the lower bound obtained by Blocki in [Bl12], who showed -generalizing a previous lower bound due to Donaldson [Don99] and Chen [Chen00] - that

$$\sqrt{\int_X (\varphi_1 - \varphi_0)^2 MA(\max(\varphi_0, \varphi_1))} \leq d(\varphi_0, \varphi_1).$$

Indeed since the measure $(\varphi_1 - \varphi_0)^2 MA(\max(\varphi_0, \varphi_1))$ does not charge the set $\{\varphi_0 = \varphi_1\}$, and since $MA(\max(\varphi_0, \varphi_1)) = MA(\varphi_0)$ on the Borel set
\{\varphi_0 > \varphi_1\}, it follows from the maximum principle (MP) that

\[
\int_X (\varphi_1 - \varphi_0)^2 MA(\max(\varphi_0, \varphi_1)) = \int_{(\varphi_0 > \varphi_1)} (\varphi_1 - \varphi_0)^2 MA(\varphi_0) + \int_{(\varphi_0 < \varphi_1)} (\varphi_1 - \varphi_0)^2 MA(\varphi_1)
\geq \max \left\{ \int_{(\varphi_0 > \varphi_1)} (\varphi_1 - \varphi_0)^2 MA(\varphi_0); \int_{(\varphi_0 < \varphi_1)} (\varphi_1 - \varphi_0)^2 MA(\varphi_1) \right\},
\]

which yields the contents of [Blo12, Theorem 1.2].

2.4. Metric completion. Recall that the precompletion of a metric space \((X, d)\) is the set of all Cauchy sequences \(C_X\) of \(X\), together with the semi-distance

\[
\{(x_j), (y_j)\} = \lim_{j \to +\infty} d(x_j, y_j).
\]

The metric completion \((\overline{X}, d)\) of \((X, d)\) is the quotient space \(C_X/\sim\), where

\[
\{x_j\} \sim \{y_j\} \iff \{(x_j), (y_j)\} = 0,
\]
equipped with the induced distance that we still denote by \(d\).

Recall that a path metric space is a metric space for which the distance between any two points coincides with the infimum of the lengths of rectifiable curves joining the two points. By construction the space \((\mathcal{H}, d)\) is a path metric space. For such metric spaces, an alternative description of the metric completion can be obtained as follows: consider \(C'_X\), the set of all rectifiable curves \(\gamma: (0,1] \to X\) equipped with the semi-distance

\[
\langle \gamma, \tilde{\gamma} \rangle := \lim_{t \to 0} d(\gamma(t), \tilde{\gamma}(t)).
\]

The metric completion \((\overline{X}, d)\) is then the quotient space \(C'_X/\sim\) which identifies zero-distance curves \(\gamma, \tilde{\gamma}\).

Both constructions yield a rather abstract view on the metric completion. We are now taking advantage of the fact that \(\mathcal{H}\) leaves inside the complete metric space \((\mathcal{E}^1, d_1)\) to identify \(\overline{\mathcal{H}}\) with a subset of \(\mathcal{E}^1\):

**Theorem 2.19.** The metric completion \((\overline{\mathcal{H}}_\alpha, d)\) of \((\mathcal{H}_\alpha, d)\) is isometric to a subset of \(\mathcal{E}^1(\alpha)\) which contains \(\mathcal{E}^2(\alpha)\),

\[
\mathcal{E}^2(\alpha) \subset \overline{\mathcal{H}}_\alpha \subset \mathcal{E}^1(\alpha).
\]

A similar result holds at the level of potentials,

\[
\mathcal{E}^2(X, \omega) \subset \overline{\mathcal{H}} \subset \mathcal{E}^1(X, \omega).
\]

**Proof.** We first work at the level of normalized potentials, with

\[
\mathcal{E}^i_{\text{norm}}(X, \omega) = \{\varphi \in \mathcal{E}^i(X, \omega) \mid \sup_X \varphi = 0\}
\]

and

\[
\mathcal{H}_{\text{norm}} = \{\varphi \in C^\infty(X, \mathbb{R}) \mid \omega + dd^c \varphi > 0 \text{ and } \sup_X \varphi = 0\}.
\]
Let \((\varphi_j) \in \mathcal{H}^N_{\text{norm}}\) be a Cauchy sequence for the Mabuchi distance. It follows from Proposition 2.15 (applied with \(\varphi_0 = 0\) and \(\varphi_1 = \varphi_j\)) that
\[
\frac{1}{n+1} I(\varphi_j, 0) + \int_X \varphi_j \omega^n \leq d(0, \varphi_j) \leq C
\]
is uniformly bounded from above. Now \(\sup_X \varphi_j = 0\) hence \(\int_X \varphi_j \omega^n\) is bounded as well (by [GZ05, Proposition 2.7]), thus
\[
0 \leq I(\varphi_j, 0) \leq C'.
\]
It follows that \((\varphi_j)\) stays in a weakly compact subset of normalized functions with uniformly bounded energies,
\[
\mathcal{E}^1_C(X, \omega) = \{ \psi \in \mathcal{E}^1(X, \omega) | E(\psi) \geq -C'' \text{ and } \sup_X \psi \leq 0 \},
\]
hence
\[
\mathcal{H}^N_{\text{norm}} \subset \mathcal{E}^1_{\text{norm}}(X, \omega).
\]
It follows moreover from Proposition 2.15 and [BBGZ13, Theorem 3.12] that the injection \(\mathcal{H}^N_{\text{norm}} \hookrightarrow \mathcal{E}^1_{\text{norm}}(X, \omega)\) is continuous.

We now show that \(\mathcal{H}^N_{\text{norm}}\) contains \(\mathcal{E}^2_{\text{norm}}(X, \omega)\). Fix \(\varphi \in \mathcal{E}^2_{\text{norm}}(X, \omega)\) and let \((\varphi_j) \in \mathcal{H}^N_{\text{norm}}\) be a sequence decreasing to \(\varphi\) (the existence of such a sequence follows from Demailly’s regularization result [Dem92]). It follows from Proposition 2.17 that
\[
d(\varphi_{j+p}, \varphi_j)^2 \leq \int_X (\varphi_{j+p} - \varphi_j)^2 MA(\varphi_{j+p}).
\]
Now [GZ07, Lemma 3.5] shows that the latter is bounded from above by
\[
3^n \int_X (\varphi - \varphi_j)^2 MA(\varphi)
\]
which converges to zero as \(j \to +\infty\), as follows from the monotone convergence theorem. Therefore \((\varphi_j)\) is a Cauchy sequence in \((\mathcal{H}^N_{\text{norm}}, d)\). Its limit in \((\mathcal{H}^N_{\text{norm}}, d)\) is \(\varphi\) since
\[
0 \leq d(\varphi, \varphi_j) \leq 2I_2(\varphi_j, \varphi) \to 0
\]
by Proposition 2.17 and [BEGZ10, Theorem 2.17].

We note the following alternative approach of independent interest. One first shows that \(\mathcal{H}^N_{\text{norm}}\) contains all bounded \(\omega\)-psh functions. Given \(\varphi \in \mathcal{E}^2_{\text{norm}}(X, \omega)\) one then considers its “canonical approximants”
\[
\varphi_j = \max(\varphi, -j) \in PSH_{\text{norm}}(X, \omega) \cap L^\infty(X)
\]
which decrease towards \(\varphi \in \mathcal{E}^2(X, \omega)\). It follows from Proposition 2.17 that
\[
d(\varphi_{j+p}, \varphi_j)^2 \leq \int_X (\varphi_{j+p} - \varphi_j)^2 MA(\varphi_{j+p})
\]
\[
= \int_{\varphi \leq -j-p} p^2 MA(\varphi_{j+p}) + \int_{-j-p < \varphi < -j} (\varphi_{j+p} - \varphi_j)^2 MA(\varphi)
\]
\[
= \int_{\varphi \leq -j-p} p^2 MA(\varphi) + \int_{-j-p < \varphi < -j} (\varphi_{j+p} - \varphi_j)^2 MA(\varphi)
\]
\[
\leq \int_{\varphi < -j} \varphi^2 MA(\varphi),
\]
where we have used the maximum principle (MP) together with the fact that since \( \varphi \in \mathcal{E}(X, \omega) \),
\[
\int_{(\varphi \leq -k)} p^2 MA(\varphi_k) = \int_X MA(\varphi_k) - \int_{(\varphi > -k)} MA(\varphi_k) = \int_{(\varphi \leq -k)} MA(\varphi),
\]
as follows again from the maximum principle. We infer that \((\varphi_j)\) is a Cauchy sequence. Its limit point in the metric completion of \((\mathcal{H}, d)\) is again \( \varphi \).

We finally show how to deal with unnormalized potential. If \((\varphi_j) \in \mathcal{H}^N\) is a Cauchy sequence for the Mabuchi distance, we need to show that \(M_j := \sup_X \varphi_j\) is bounded. Set \(\psi_j := \varphi_j - M_j \in \mathcal{H}_{\text{norm}}\). It follows from Proposition 2.15 that there exists \(C > 0\) such that for all \(j \in \mathbb{N}\),
\[
0 \leq I(\psi_j, 0) = I(\varphi_j, 0) \leq C
\]
and
\[
\max \left\{ -M_j + \int_X \psi_j MA(0) ; M_j + \int_X (-\psi_j) MA(\psi_j) \right\} \leq C.
\]
Now \(\int_X \psi_j MA(0)\) is uniformly bounded by [GZ05, Proposition 2.7] while
\[
0 \leq \int_X (-\psi_j) MA(\psi_j) = I(0, \psi_j) - \int_X \psi_j MA(0)
\]
is uniformly bounded as well, thus
\[
\max \{ -M_j + O(1) ; M_j + O(1) \} = O(1),
\]
showing that \(M_j = \sup_X \varphi_j\) is bounded. \(\square\)

**Remark 2.20.** It is natural to wonder, in view of Theorem 4.6, whether \(\overline{H}_\alpha = \mathcal{E}^2(\alpha)\) ? The toric setting is of course very particular, but Theorem 4.6 nevertheless shows that \(\overline{H}_\alpha\) is in general strictly contained in \(\mathcal{E}^1(\alpha)\).

It follows from the continuity properties along decreasing sequences that the lower and upper-bounds obtained in Propositions 2.15 and 2.17 continue to hold for \(\varphi \in \overline{H}\). One can further extend the Mabuchi distance to \(\mathcal{E}^1\) along such sequences, but one should then expect that potentials in \(\mathcal{E}^1 \setminus \mathcal{E}^2\) are at infinite distance from smooth ones.

### 3. Weak geodesics

**3.1. Bounded geodesics.** Homogeneous complex Monge-Ampère equations have been intensively studied since the mid 70’s, after Bedford and Taylor laid down the foundations of pluripotential theory in [BT76, BT82]. An adaptation of the classical Perron envelope technique yields the following:

**Proposition 3.1.** Assume \(\varphi_0, \varphi_1\) are bounded \(\omega\)-psh functions. Then
\[
\Phi(x, z) := \sup \{ \psi(x, z) \mid \psi \in \text{PSH}(X \times A, \omega) \text{ with } \lim_{t \to 0,1} \psi \leq \varphi_{0,1} \}
\]
is the unique bounded \(\omega\)-psh function on \(X \times A\) solution of the Dirichlet problem \(\Phi|_{X \times \partial A} = \varphi_{0,1}\) with
\[
(\omega + dd^c \Phi)^{n+1} = 0 \text{ in } X \times A.
\]
Moreover \(\Phi(x, z) = \Phi(x, t)\) only depends on \(|z|\) and \(\Phi\) is uniformly bounded by \(\|\varphi_1 - \varphi_0\|_{L^\infty(X)}\).
Proof. Recall that $A = [1, e] \times S^1$ is an annulus in $\mathbb{C}$ with complex coordinate $z = e^{t+i\alpha}$. The proof follows from a classical balayage technique, together with a barrier argument: observe that for $A = ||\varphi - \varphi_0||_{L^\infty(X)}$, the function

$$\chi_t(x) = \max\{\varphi_0(x) - A \log |z|, \varphi_1(x) + A(\log |z| - 1)\}$$

belongs to the family and has the right boundary values. Using that $t \mapsto \Phi_t$ is convex (by subharmonicity in $z$), one easily deduces the bound on $|\dot{\Phi}|$.

We refer the reader to [PSS12, Bern13] for more details. □

The function $\Phi$ (or rather the path $\Phi_t \subset PSH(X, \omega) \cap L^\infty(X)$) has been called a bounded geodesic in recent works (see notably [Bern13]). We use the same terminology here (although it might be confusing at first), as we will show that bounded (and later on weak) geodesics are geodesics in the metric sense, i.e. constant speed rectifiable paths which minimize length:

**Proposition 3.2.** Bounded geodesics are length minimizing. More precisely, if $\varphi_0, \varphi_1$ are bounded $\omega$-psh functions and $\Phi(x, z) = \Phi_t(x)$ is the bounded geodesic joining $\varphi_0$ to $\varphi_1$, then for all $t \in [0, 1]$,

$$d(\varphi_0, \varphi_1) = \ell(\Phi) = \sqrt{\int_X (\dot{\Phi}_t)^2 MA(\Phi_t)}.$$

Proof. We prove this in two steps.

Assume first that $\varphi_0 \in \mathcal{H}$, while $\varphi_1 \in PSH(X, \omega) \cap L^\infty(X)$. Let $\varphi^{(j)}_1 \in \mathcal{H}$ be a sequence decreasing to $\varphi_1$. It follows from Proposition 2.17 and [BEGZ10, Theorem 2.17] that $d(\varphi_0, \varphi^{(j)}_1) \to d(\varphi_0, \varphi_1)$.

It follows moreover from the maximum principle and the uniqueness in Proposition 3.1 that $\Phi_{t,j}$ decreases to $\Phi_t$ as $j$ increases to $+\infty$.

We infer now from Chen’s theorem that

$$d\left(\varphi_0, \varphi^{(j)}_1\right)^2 = \ell(\Phi) = \int_X (\dot{\Phi}_{0,j})^2 MA(\Phi_0),$$

where $\Phi_{0,j}$ denotes the geodesic joining $\varphi_0$ to $\varphi^{(j)}_1$.

It follows from the convexity of $t \mapsto \Phi_{t,j}$ that for all $x \in X$,

$$\dot{\Phi}_{0,j}(x) = \frac{\Phi_{t,j}(x) - \Phi_{0,j}(x)}{t}.$$

Letting $j \to +\infty$ and then $t \searrow 0^+$, we infer

$$\limsup_{j \to +\infty} \dot{\Phi}_{0,j}(x) \leq \dot{\Phi}_0(x).$$

Observe conversely that by convexity of $t \mapsto \Phi_t$,

$$\dot{\Phi}_0(x) \leq \frac{\Phi_t(x) - \Phi_0(x)}{t} \leq \frac{\Phi_{t,j}(x) - \Phi_0(x)}{t},$$

where the last inequality follows from the maximum principle, which insures that $\Phi_{t,j}$ decreases to $\Phi_t$. Since $\Phi_0 = \varphi_0 = \Phi_{0,j}$ is fixed, we can let $t$ decrease to zero and obtain that for all $j, x$,

$$\dot{\Phi}_0(x) \leq \dot{\Phi}_{0,j}(x).$$
Therefore $\Phi_{0,j}(x) \to \Phi_0(x)$ as $j \to +\infty$. Since these functions are uniformly bounded by previous Proposition, the dominated convergence theorem insures that

$$d(\varphi_0, \varphi_1)^2 = \lim_{j \to +\infty} d(\varphi_0, \varphi_1^{(j)}) = \lim_{j \to +\infty} \int_X (\dot{\Phi}_{0,j})^2 MA(\Phi_0),$$

$$= \int_X (\dot{\Phi}_0)^2 MA(\Phi_0).$$

We now check that $L^2(\Phi) = \int_X (\dot{\Phi}_0)^2 MA(\Phi_0)$ by showing that for all $t$,

$$\int_X (\dot{\Phi}_t)^2 MA(\Phi_t) = \int_X (\dot{\Phi}_0)^2 MA(\Phi_0).$$

It follows indeed from Stokes theorem that

$$\frac{d}{dt} \int_X (\dot{\Phi}_t)^2 MA(\Phi_t) = 2 \int_X \dot{\Phi}_t \left( \dot{\Phi}_t MA(\Phi_t) - \frac{n}{V_\alpha} d\Phi_t \wedge d^c \Phi_t \wedge \omega^{-1}_0 \right) = 0$$

since $(\dot{\Phi}_t)$ is a bounded geodesic.

We now treat the general case, i.e. $\varphi_0, \varphi_1 \in PSH(X, \omega) \cap L^\infty(X)$. We again approximate $\varphi_1$ by a decreasing sequence of smooth functions $\varphi_1^{(j)} \in \mathcal{H}$. The first step yields the expected formula for the distance $d(\varphi_0, \varphi_1^{(j)})$. The same reasoning allows one to pass again to the limit and conclude. □

Remark 3.3. Let $(\varphi_j)$ be a sequence of bounded $\omega$-psh functions uniformly converging to some $\omega$-psh function $\varphi$. Let $\Phi_t$ denote the geodesic joining $\varphi_j$ to $\varphi$. It follows from Proposition 3.1 that for all $0 \leq t \leq 1$,

$$d(\varphi, \varphi_j) = \sqrt{\int_X (\dot{\Phi}_t)^2 MA(\Phi_t)} \leq ||\varphi - \varphi_j||_{L^\infty(X)}.$$ 

We will see in Example 4.4 that the convergence in the Mabuchi sense is much weaker than the uniform convergence.

3.2. Finite energy geodesics. We now define weak geodesics joining two finite energy endpoints $\varphi_0, \varphi_1 \in \mathcal{E}^1(X, \omega)$. Fix $j \in \mathbb{N}$ and consider $\varphi_0^{(j)}, \varphi_1^{(j)}$ smooth (or bounded) $\omega$-psh functions decreasing to $\varphi_0, \varphi_1$. We let $\varphi_{t,j}$ denote the bounded geodesic joining $\varphi_0^{(j)}$ to $\varphi_1^{(j)}$. It follows from the maximum principle that $j \mapsto \varphi_{t,j}$ is non-increasing. We can thus set

$$\varphi_t := \lim_{j \to +\infty} \varphi_{t,j} \in \mathcal{E}^1(X, \omega).$$

It follows again from the maximum principle that $\varphi_t$ is independent of the choice of the approximants $\varphi_0^{(j)}, \varphi_1^{(j)}$. If we set as previously $\Phi(x, z) := \varphi_t(x)$, with $z = t + is$, then $\Phi$ is a maximal $\omega$-psh function in $X \times \mathbb{R}$, as a decreasing limit of maximal $\omega$-psh functions. It is thus the unique maximal $\omega$-psh function with boundary values $\varphi_0, \varphi_1$. We call it the (unique) weak geodesic joining $\varphi_0$ to $\varphi_1$.

The $\varphi_t$’s form a family of finite energy functions, since $t \mapsto E(\varphi_{t,j})$ is affine hence

$$(1 - t)E(\varphi_0) + tE(\varphi_1) \leq E(\varphi_{t,j})$$

for all $j \in \mathbb{N}$.

We now show that these weak geodesics are minimizing geodesics:
Theorem 3.4. The space $(\mathcal{H}, d)$ is a CAT(0) space. More precisely, given $\varphi_0, \varphi_1 \in \mathcal{H}$, the weak geodesic $\Phi$ joining $\varphi_0$ to $\varphi_1$ is a minimizing geodesic,

$$d(\varphi_0, \varphi_1) = \ell(\Phi) = \sqrt{\int_X (\dot{\Phi}_t)^2 MA(\Phi_t)} \text{ for all } t \in [0, 1].$$

Complete CAT(0) spaces are also called Hadamard spaces. Recall that a CAT(0) space is a geodesic space which has non positive curvature in the sense of Alexandrov. Hadamard spaces enjoy many interesting properties (uniqueness of geodesics, contractibility, convexity properties, see [BH99]).

Proof. The proof that $(\mathcal{H}, d)$ is a geodesic space is very similar to that of Proposition 3.2 and again divided in two steps. Note that $(\mathcal{H}, d)$ is a complete path metric space, being the completion of the path metric space $(\mathcal{H}, d)$. The Hopf-Rinow-Cohn-Vossen theorem (see [BH99, Proposition 1.3.7]) insures that a complete locally compact path metric space is automatically a geodesic space. Here $(\mathcal{H}, d)$ is not locally compact (it is merely locally weakly compact), but we have a natural candidate for the minimizing geodesics.

We assume first that $\varphi_0$ is bounded and approximate $\varphi_1$ by $\varphi_1^{(j)} = \max(\varphi_1, -j)$. Let $\Phi_{t,j}$ denote the corresponding bounded geodesic. The same proof as that of Proposition 3.2 shows that for all $x \in X$,

$$\lim_{j \to +\infty} \dot{\Phi}_{0,j}(x) = \dot{\Phi}_0(x),$$

however there is no domination anymore. Fatou’s lemma nevertheless insures that

$$\int_X (\dot{\Phi}_0)^2 MA(\Phi_0) \leq \liminf_{j \to +\infty} \int_X (\dot{\Phi}_{0,j})^2 MA(\Phi_0) = d(\varphi_0, \varphi_1)^2.$$

Since $t \mapsto \Phi_t$ is a weak geodesic, we observe as previously done that

$$t \mapsto \int_X (\dot{\Phi}_t)^2 MA(\Phi_t) \text{ is constant}$$

thus for all $t \in [0, 1],

$$\ell(\Phi) = \sqrt{\int_X (\dot{\Phi}_0)^2 MA(\Phi_0)} = \sqrt{\int_X (\dot{\Phi}_t)^2 MA(\Phi_t)} \leq d(\varphi_0, \varphi_1),$$

whence equality since $(\mathcal{H}, d)$ is a path metric space.

The second step treats the general case. We approximate $\varphi_0$ by bounded $\omega$-psh functions and apply the first step. The conclusion follows by repeating similar arguments as above.

We finally note that Calabi and Chen proved in [CC02, Theorem 1.1] that $(\mathcal{H}, d)$ satisfies the CN inequality of Bruhat-Tits [BT72]. It follows therefore from [BH99, Exercise 1.9.1.c and Corollary 3.11] that $(\mathcal{H}, d)$ is a CAT(0) space. \(\square\)
4. The toric case

Recall that a compact Kähler toric manifold \((X, \omega, T)\) is an equivariant compactification of the torus \(T = (\mathbb{C}^*)^n\) equipped with a \(T\)-invariant Kähler metric \(\omega\) which writes

\[
\omega = dd^c\psi \quad \text{in} \quad (\mathbb{C}^*)^n,
\]

with \(\psi\) \(T\)-invariant hence \(\psi(z) = F \circ L(z)\) where

\[
L : z \in (\mathbb{C}^*)^n \mapsto (\log |z_1|, \ldots, \log |z_n|) \in \mathbb{R}^n
\]

and \(F : \mathbb{R}^n \to \mathbb{R}\) is strictly convex.

The celebrated Atiyah-Guillemin-Sternberg theorem asserts that the moment map \(\nabla F : \mathbb{R}^n \to \mathbb{R}^n\) sends \(\mathbb{R}^n\) to a bounded convex polytope

\[
P = \{ \ell_i(s) \geq 0, \; 1 \leq i \leq d \} \subset \mathbb{R}^n
\]

where \(d \geq n + 1\) is the number of \((n - 1)\)-dimensional faces of \(P\),

\[
\ell_i(s) = \langle s, u_i \rangle - \lambda_i,
\]

with \(l_i \in \mathbb{R}\) and \(u_i\) is a primitive element of \(\mathbb{Z}^n\), normal to the \(i\)th \((n - 1)\)-dimensional face of \(P\).

Delzant observed in [Del88] that in this case \(P\) is “Delzant”, i.e. there are exactly \(n\) faces of dimension \((n - 1)\) meeting at each vertex, and the corresponding \(u_i\)’s form a \(\mathbb{Z}\)-basis of \(\mathbb{Z}^n\). He conversely showed that there is exactly one (up to symplectomorphism) compact toric Kähler manifold \((X_{\mathcal{P}}, \{\omega_{\mathcal{P}}\}, T)\) associated to a Delzant polytope \(P \subset \mathbb{R}^n\). Here \(\{\omega_{\mathcal{P}}\}\) denotes the cohomology class of the \(T\)-invariant Kähler form \(\omega_{\mathcal{P}}\). Let

\[
G(s) := \sup_{x \in \mathbb{R}^n} \{\langle x, s \rangle - F(x)\}
\]

denote the Legendre transform of \(F\). Observe that \(G = +\infty\) in \(\mathbb{R}^n \setminus P\) and for \(s \in P = \nabla F(\mathbb{R}^n)\),

\[
G(s) = \langle x, s \rangle - F(x) \quad \text{with} \quad \nabla F(x) = s \iff \nabla G(s) = x.
\]

Guillemin observed in [Gui94] that a ”natural” representative of the cohomology class \(\omega_{\mathcal{P}}\) is given by

\[
G(s) = \frac{1}{2} \left\{ \sum_{i=1}^d \ell_i(s) \log \ell_i(s) + \ell_\infty(s) \log \ell_\infty(s) \right\}
\]

where \(\ell_\infty(s) = \sum_{i=1}^d \langle s, u_i \rangle\). We refer the reader to [CDG03] for a neat proof of this beautiful Guillemin formula.

**Example 4.1.** When \(X = \mathbb{C}P^n\) and \(\omega\) is the Fubini-Study Kähler form, then

\[
F(x) = \frac{1}{2} \log \left[ 1 + \sum_{i=1}^n e^{2x_i} \right],
\]

\(P = \nabla F(\mathbb{R}^n)\) is the simplex

\[
P = \left\{ s_i \geq 0, \; 1 \leq i \leq n \text{ and } \sum_{i=1}^n s_i \leq 1 \right\},
\]

thus \(d = n + 1\),

\[
\ell_i(s) = s_i, \; l_i = 0, \; u_i = e_i \quad \text{for} \; 1 \leq i \leq n
\]
and
\[ \ell_{n+1}(s) = 1 - \sum_{i=1}^{n} s_i, \quad \ell_{n+1} = -1, \quad e_{n+1} = - \sum_{j=1}^{n} e_j \]
and \( \ell_{\infty} \equiv 0 \) so that
\[
G(s) = \frac{1}{2} \left\{ \sum_{i=1}^{n} s_i \log s_i + \left( 1 - \sum_{j=1}^{n} s_j \right) \log \left( 1 - \sum_{j=1}^{n} s_j \right) \right\}.
\]

4.1. Toric geodesics. Let \((X, \omega, T)\) be a compact toric manifold. If \(\varphi_0, \varphi_1 \in \mathcal{H}\) are both \(T\)-invariant, it follows from the uniqueness that the geodesic \((\varphi_t)_{0 \leq t \leq 1}\) consists of \(T\)-invariant functions. Let \(F_t\) denote the corresponding potentials in \(\mathbb{R}^n\) so that
\[
F_t \circ L = F \circ L + \varphi_t \text{ in } (\mathbb{C}^n)^n.
\]

Proposition 4.2. [Guan99] The map \((x, t) \mapsto \varphi_t(x)\) is smooth and corresponds to the Legendre transform of an affine path on \(P\).

In other words the Legendre transform \(G_t\) of \(F_t\) is affine in \(t\). The proof is elementary, it suffices to differentiate twice the defining equation for \(G_t\).

In a similar vein we obtain an explicit formula for the Mabuchi distance between \(\varphi_0\) and \(\varphi_1\):

Proposition 4.3.

\[
d(\varphi_0, \varphi_1) = ||G_1 - G_0||_{L^2(P)} = \sqrt{\int_P (G_1 - G_0)^2(s) ds}.
\]

Proof. Recall that
\[
d(\varphi_0, \varphi_1) = \sqrt{\int_X (\varphi_0)^2 MA(\varphi_0)}.
\]

Now \(F_t \circ L = F \circ L + \varphi_t\) has Legendre transform \(G_t = tG_1 + (1-t)G_0\). Thus \(\dot{\varphi}_t = \dot{F}_t \circ L\) with
\[
G_t(s) = \langle x_t, s \rangle - F_t(x_t) \text{ with } s = \nabla F_t(x_t)
\]
hence \(\dot{G}_t(s) = -\dot{F}_t(x)\) and we infer
\[
d(\varphi_0, \varphi_1)^2 = \int_{(\mathbb{C}^*)^n} (\dot{F}_0)^2 MA(F_0 \circ L).
\]

Observe that
\[
\frac{\partial^2(F_0 \circ L)}{\partial z_i \partial z_j} = \frac{1}{s_i s_j} \cdot \frac{\partial^2 F_0}{\partial z_i \partial z_j} \circ L \text{ in } (\mathbb{C}^*)^n
\]
hence
\[
\det \left( \frac{\partial^2(F_0 \circ L)}{\partial z_i \partial z_j} \right) = \frac{1}{\Pi_j |z_j|^2} \cdot MA_{\mathbb{R}}(F_0) \circ L,
\]
where \(MA_{\mathbb{R}}\) denotes the real Monge-Ampère measure (in the Alexandrov sense, see [Gut01]) of the convex function \(F_0\). Thus
\[
\int_{(\mathbb{C}^*)^n} (\dot{F}_0)^2 MA(F_0 \circ L) = \int_{\mathbb{R}^n} (\dot{F}_0)^2 MA_{\mathbb{R}}(F_0).
\]
Now $F_0 = -\dot{G}_0 \circ \nabla F_0$ and $MA_{\mathbb{R}}(F_0) = (\nabla F_0)^* ds$ therefore
$$\int_{\mathbb{R}^n} (F_0)^2 MA_{\mathbb{R}}(F_0) = \int_P (\dot{G}_0)^2(s) ds = \int_P (G_1 - G_0)^2(s) ds.$$ 

\square

**Example 4.4.** Assume $X = \mathbb{CP}^1$ is the Riemann sphere and $\omega$ is the Fubini-Study Kähler form. Let $\varphi_0$ be the toric function associated to the convex potential
$$F_0(x) = \max(x,0)$$ so that $G_0(s) \equiv 0$ on the simplex $P = [0,1]$.

Observe that $\omega_0 = dd^c F_0 \circ L$ is the (normalized) Lebesgue measure on the unit circle $S^1 \subset \mathbb{C}^* \subset \mathbb{CP}^1$.

We consider $\varphi_1 = \varphi_j$ a sequence of toric potentials defined by the convex functions
$$F_j(x) = (1 - \varepsilon_j) F_0(x) + \varepsilon_j \max(x,-C_j),$$
where $\varepsilon_j$ decreases to 0, while $C_j$ increases to $+\infty$. A straightforward computation yields
$$G_j(s) = \max(C_j[\varepsilon_j - s], 0).$$

Therefore
$$d(\varphi_j, \varphi_0) = \frac{C_j \varepsilon_j^{3/2}}{\sqrt{3}}$$

We thus obtain in this case, as $j \to +\infty$,

- $\varphi_j \to \varphi_0$ in $L^1$ iff $\varepsilon_j \to 0$;
- $\varphi_j \to \varphi_0$ in $L^\infty$ iff $\varepsilon_j C_j \to 0$;
- $\varphi_j \to \varphi_0$ in $(\mathcal{E}^1, I_1)$ iff $\varepsilon_j^2 C_j \to 0$;
- $\varphi_j \to \varphi_0$ in $(\mathcal{E}^2, I_2)$ iff $\varepsilon_j^3 C_j^2 \to 0$;
- $\varphi_j \to \varphi_0$ in $(\mathcal{H}, d)$ iff $\varepsilon_j^3 C_j^2 \to 0$.

The convergence in $(\mathcal{E}^1, I_1)$ is here (i.e. in dimension $n = 1$) the convergence in the Sobolev norm $W^{1,2}$. For $\varepsilon_j = 1/j$ and $C_j = j^{3/2}$ we therefore obtain an example of a sequence which converges in the Sobolev sense but not in the Mabuchi metric.

### 4.2. Toric singularities

Let $\varphi \in \mathcal{H}$ be a toric potential. We are going to read off the singular behavior of $\varphi$ from the integrability properties of the Legendre transform of its associated convex potential.

We let $F_\varphi$ and $G_\varphi$ denote the corresponding convex function and its Legendre transform. The function $\varphi$ is bounded (resp. continuous) if and only if so is $F_\varphi - F$ on $\mathbb{R}^n$, since $F_\varphi \circ L = F \circ L + \varphi$, if and only if so is $G_\varphi$ on $P$, as $G$ (Guillemin’s potential) is continuous on $P$.

The same conclusion holds if we take as a reference potential the support function $F_P$ of $P$, defined by
$$F_P(x) := \sup_{s \in P} (s, x).$$

It is the Legendre transform of the function $G_P$ which is identically 0 on $P$ and $+\infty$ in $\mathbb{R}^n \setminus P$. We can similarly understand finite energy classes:
Proposition 4.5.

\[ \varphi \in PSH_{tor}(X, \omega) \cap L^\infty(X) \iff G_\varphi \in L^\infty(P). \]
\[ \varphi \in E^q_{tor}(X, \omega) \iff G_\varphi \in L^q(P). \]

We refer the reader to [BerBer13, Proposition 2.9] for an elegant proof of this result when \( q = 1 \).

**Proof.** We first show that \( \varphi \in E^q_{tor}(X, \omega) \iff G_\varphi \in L^q(P) \). Approximating \( \varphi \) from above by a decreasing sequence of smooth strictly \( \omega \)-psh toric functions, this boils down to show a uniform a priori bound

\[ ||G_\varphi||_{L^q(P)} \leq \left( \int_X |\varphi - \varphi|^q M A(\varphi) \right)^{1/q}. \]

We can assume without loss of generality that \( F \leq F_P \) (since \( \varphi \) is upper semi-continuous hence bounded from above on \( X \) which is compact). Recall that \( \varphi = (F - F_P) \circ L \) in \((\mathbb{C}^*)^n\), where \( F_P \) denotes a reference potential associated to \( \omega \). Changing variables and using the Legendre transform yields

\[ \int_{(\mathbb{C}^*)^n} |\varphi - \varphi|^q M A(\varphi) = \int_{\mathbb{R}^n} |F - F_P|^q M A_{\mathbb{R}}(F) \]
\[ = \int_P |F \circ \nabla G(s) - F_P \circ \nabla G(s)|^q ds, \]

where \( F(x) = \langle x, s \rangle - G(s) \), with \( \nabla G(s) = x \). Therefore

\[ F(\nabla G(s)) = (\nabla G(s), s) - G(s) \]

and

\[ F_P(\nabla G(s)) = F(\nabla G(s)) \]
\[ \geq G(s) - \{ \langle \nabla G(s), s \rangle - F_P \circ \nabla G(s) \} \]
\[ \geq G(s) - G_P(s) = G(s) \geq 0, \]

since \( G_P(s) = \sup_{x \in \mathbb{R}^n} \{ \langle x, s \rangle - F_P(x) \} = 0 \) for \( s \in P \). We infer

\[ ||G_\varphi||_{L^q(P)} \leq \int_P |F_P(\nabla G(s)) - F(\nabla G(s))|^q ds \]
\[ \leq \int_X |\varphi - \varphi|^q M A(\varphi), \]

as claimed.

We now take care of the converse implication. To simplify notations we only treat the case of the complex projective space, using the same notations as in Example 4.1. We want to get an upper bound on \( \int_X |\varphi - \varphi|^q M A(\varphi) \) involving \( \int_P |G - G_0|^q(s) ds \), where

\[ F_0(x) = \frac{1}{2} \log \left[ 1 + \sum_{i=1}^n e^{2x_i} \right] \]

is the convex function associated to the Fubini-Study Kähler form,

\[ P = \left\{ s_i \geq 0, 1 \leq i \leq n \text{ and } \sum_{i=1}^n s_i \leq 1 \right\} \] is the standard simplex.
and

\[ G_0(s) = \frac{1}{2} \left\{ \sum_{i=1}^{n} s_i \log s_i + \left( 1 - \sum_{j=1}^{n} s_j \right) \log \left( 1 - \sum_{j=1}^{n} s_j \right) \right\}. \]

We decompose \((\mathbb{C}^*)^n\) in \(2^n\) pieces, according to whether \(|z_j| \leq 1\) or \(|z_j| \geq 1\).

By symmetry, it suffices to bound

\[ \int_{\mathbb{R}^n} |F_0 - F|^q \mathcal{MA}(F) \]

is bounded from above if and only if so is \(\int_{\mathbb{R}^n} \|F\|^q \mathcal{MA}(F)\). Applying the Legendre transform yields

\[ \int_{\mathbb{R}^n} \|F\|^q \mathcal{MA}(F) = \int_{\nabla F(\mathbb{R}^n)} |G|^q(s) ds \leq \left\| G \right\|^q_{L^q(P)}, \]

as desired.

**Theorem 4.6.** The metric completion of \((\mathcal{H}_{tor}, d)\) is \((\mathcal{E}_{tor}^2, d)\).

**Proof.** We have already observed that the metric completion of \((\mathcal{H}_{tor}, d)\) can be identified to a subset of \(\mathcal{E}_{tor}^1\). It follows from Proposition 4.3 and Proposition 4.5 that the only functions lying at finite distance from smooth toric potentials are those which belong to \(\mathcal{E}_{tor}^2\). \(\square\)

**References**

- [BT76] E. Bedford, B. A. Taylor: The Dirichlet problem for a complex Monge-Ampère equation. Invent. Math. 37 (1976), no. 1, 1-44.
- [BT82] E. Bedford, B. A. Taylor: A new capacity for plurisubharmonic functions. Acta Math. 149 (1982), no. 1-2, 1–40.
- [BerBer13] R. Berman, B. Berndtsson. Real Monge-Ampère equations and Kähler-Ricci solitons on toric log Fano varieties. Annales de la Faculté des Sciences de Toulouse, Vol XXII n°4 (2013), 649-711.
- [BBGZ13] R. Berman, S. Boucksom, V. Guedj, A. Zeriahi: A variational approach to complex Monge-Ampère equations. Publ. Math. I.H.E.S. 117 (2013), 179-245.
- [BBEGZ11] R. Berman, S. Boucksom, P. Eyssidieux, V. Guedj, A. Zeriahi: Kähler-Ricci flow and Ricci iteration on log-Fano varieties. Preprint arXiv:1111.7158.
- [Bern09] B. Berndtsson. Positivity of direct image bundles and convexity on the space of Kähler metrics. J. Differential Geom. 81 (2009), no. 3, 457-482.
- [Bern13] B. Berndtsson. A Brunn-Minkowski type inequality for Fano manifolds and some uniqueness theorems in Kähler geometry. Preprint arXiv:1303.4075.
- [Blo12] Z. Blocki. On geodesics in the space of Kähler metrics. Proceedings of the "Conference in Geometry" dedicated to Shing-Tung Yau (Warsaw, April 2009), in "Advances in Geometric Analysis", ed. S. Janeczko, J. Li, D. Phong, Advanced Lectures in Mathematics 21, pp. 3-20, International Press, 2012.
- [Bou12] S. Boucksom. Monge-Ampère equations on complex manifolds with boundary. Complex Monge-Ampère equations and geodesics in the space of Kähler metrics, 257-282, Lecture Notes in Math., 2038, Springer, Heidelberg, 2012.
- [BEGZ10] S. Boucksom, P. Eyssidieux, V. Guedj, A. Zeriahi: Monge-Ampère equations in big cohomology classes. Acta Math. 205 (2010), 199–262.
- [Bourbaki] N. Bourbaki: Eléments de mathématiques, Topologie générale, Fsc VIII, livre III Chap 9.
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[BH99] M. Bridson, A. Haefliger. Metric spaces of non positive curvature. Grundlehren der Math. Wiss. 319 (1999), 643pp.

[BT72] F. Bruhat, J. Tits. Groupes réductifs sur un corps local. Inst. Hautes Etudes Sci. Publ. Math. No. 41 (1972), 5-251.

[CC02] E. Calabi and X.X. Chen. The space of Kähler metrics. II. J. Differential Geom., 61(2):173–193, 2002.

[CDG03] D. Calderbank, L. David, P. Gauduchon. The Guillemin formula and Kähler metrics on toric symplectic manifolds. J. Symplectic Geom. 1 (2003), no. 4, 767-784.

[Ceg98] U. Cegrell. Pluricomplex energy. Acta Math. 180 (1998), 187-217.

[Chen00] X.X. Chen. The space of Kähler metrics. J. Differential Geom., 56(2):189–234, 2000.

[Chen09] X.X. Chen. The space of Kähler metrics III. On the lower bound of the Calabi energy and geodesic distance. Inventiones Math., 175 (2009), n3, 453-503.

[CS09] X.X. Chen, S. Sun. The space of Kähler metrics V. Kähler quantization. Preprint arXiv:0902.4149.

[CT08] X.X. Chen and G. Tian. Geometry of Kähler metrics and foliations by holomorphic discs. Publ. Math. Inst. Hautes Etudes Sci. No. 107 (2008), 1-107.

[Cl13] B. Clarke. The completion of the manifold of Riemannian metrics. J. Differential Geom. 93 (2013), no. 2, 203268.

[Dar14] T. Darvas. Envelopes, Singularity Types and Geodesics in $\mathcal{E}(X,\omega)$. Preprint (2014).

[DL12] T. Darvas and L. Lempert. Weak geodesics in the space of Kähler metrics. Mathematical Research Letters, 19 (2012), no. 5.

[Don99] S. K. Donaldson. Symmetric spaces, Kähler geometry and Hamiltonian dynamics. In Northern California Symplectic Geometry Seminar, volume 196 of Amer. Math. Soc. Transl. Ser. 2, pages 13–33. Amer. Math. Soc., Providence, RI, 1999.

[Don02] S. K. Donaldson. Scalar curvature and stability of toric varieties. J. Differential Geom. 62 (2002), no. 2, 289-349.

[EGZ09] Eyssidieux, P., Guedj, V., Zeriahi, A. Singular Kähler-Einstein metrics. J. Amer. Math. Soc. 22 (2009), 607-639.

[Guan99] D. Guan. On modified Mabuchi functional and Mabuchi moduli space of Kähler metrics on toric bundles. Math. Res. Lett. 6 (1999), no. 5-6, 547-555.

[G12] Complex Monge-Ampère equations and geodesics in the space of Kähler metrics. Edited by V. Guedj. Lecture Notes in Math. 2038, Springer Heidelberg (2012), 310pp.

[GZ05] V. Guedj, A. Zeriahi: Intrinsic capacities on compact Kähler manifolds. J. Geom. Anal. 15 (2005), no. 4, 607-639.

[GZ07] V. Guedj, A. Zeriahi: The weighted Monge-Ampère energy of quasiplurisubharmonic functions. J. Funct. An. 250 (2007), 442-482.

[Gui94] V. Guillemin Kaehler structures on toric varieties. J. Differential Geom. 40 (1994), no. 2, 285-309.

[Gui01] C. Gutierrez The Monge-Ampère equation. Progress in Nonlinear Differential Equations and their Applications, 44. Birkhäuser Boston, Inc., Boston, MA, 2001. xii+127 pp.

[He12] W. He. On the space of Kähler potentials. Preprint arXiv:1208.1021.

[Hel01] S. Helgason. Differential geometry, Lie groups, and symmetric spaces, volume 34 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2001. Corrected reprint of the 1978 original.

[Kol12] B. Kolev. The Riemannian space of Kähler metrics. Complex Monge-Ampère equations and geodesics in the space of Kähler metrics, 231-255, Lecture Notes in Math., 2038, Springer, Heidelberg, 2012.

[LV13] L. Lempert and L. Vivas. Geodesics in the space of Kähler metrics. Duke Math. J. Volume 162, Number 7 (2013), 1369-1381.
[Mab87] T. Mabuchi. Some symplectic geometry on compact Kähler manifolds. I. Osaka J. Math., 24(2):227–252, 1987.

[MM05] P. W. Michor and D. Mumford. Vanishing geodesic distance on spaces of submanifolds and diffeomorphisms. Doc. Math., 10:217–245 (electronic), 2005.

[PS06] D. H. Phong, J. Sturm. The Monge-Ampère operator and geodesics in the space of Kähler potentials. Invent. Math. 166 (2006), no. 1, 125-149.

[PSS12] D. H. Phong, J. Song, J. Sturm Complex Monge Ampère Equations. Surveys in Differential Geometry, vol. 17 (2012), 327-411

[RZ12] Y. Rubinstein, S. Zelditch. The Cauchy problem for the homogeneous Monge-Ampère equation, I. Toeplitz quantization. J. Differential Geom. 90 (2012), no. 2, 303-327.

[Sem92] S. Semmes. Complex Monge-Ampère and symplectic manifolds. Amer. J. Math., 114(3):495–550, 1992.

[Sko72] H. Skoda: Sous-ensembles analytiques d’ordre fini ou infini dans $\mathbb{C}^n$. Bull. Soc. Math. France 100 (1972), 353–408.

[SZ12] J. Song, S. Zelditch. Test configurations, large deviations and geodesic rays on toric varieties. Adv. Math. 229 (2012), no. 4, 2338-2378.

[Yau78] S. T. Yau. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I. Comm. Pure Appl. Math., 31(3):339–411, 1978.

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