Exceptional Lie algebras, SU(3) and Jordan pairs: part 2. Zorn-type representations

Alessio Marrani\textsuperscript{1} and Piero Truini\textsuperscript{2}

\textsuperscript{1} Instituut voor Theoretische Fysica, KU Leuven, Celestijnenlaan 200D, B-3001 Leuven, Belgium
\textsuperscript{2} Dipartimento di Fisica, Università degli Studi, via Dodecaneso 33, I-16146 Genova, Italy

E-mail: alessio.marrani@fys.kuleuven.be and truini@ge.infn.it

Received 24 April 2014, revised 26 May 2014
Accepted for publication 27 May 2014
Published 17 June 2014

Abstract
A representation of the exceptional Lie algebras reflecting a simple unifying view, based on realizations in terms of Zorn-type matrices, is presented. The role of the underlying Jordan pair and Jordan algebra content is crucial in the development of the structure. Each algebra contains three Jordan pairs sharing the same Lie algebra of automorphisms and the same external $su(3)$ symmetry. The applications in physics are outlined.

Keywords: exceptional Lie algebras, Jordan algebras, Zorn matrices
PACS numbers: 02.20.Sv, 11.30.-j, 02.20.Qs

1. Introduction
Groups in physics have at least a threefold valence. First, they represent symmetries that, by definition, introduce elegance in all the equations which are manifestly symmetry invariant. Moreover, symmetries also arise as fundamental principles in constructing new theories, like, for example, gauge symmetries for the standard model (SM) of particle physics, conformal symmetry for string theory, or general covariance for the Einstein theory of relativity. Finally, symmetries—hence groups—play a key role in solving the equations of motion.

A particular class is represented by (semi)-simple Lie groups and algebras, which find application in a large number of mathematical and physical fields. All finite-dimensional complex Lie algebras have been classified by Killing, whose proofs have been made rigorous by Cartan, who has also extended the classification to non-compact real forms. This classification has led to the discovery, beyond the famous classical series, of five exceptional algebras together with the corresponding real forms: $g_2$, $f_4$, $e_6$, $e_7$ and $e_8$.

Exceptional Lie groups and algebras appear naturally as gauge symmetry groups of field theories which are low-energy limits of string models [1].
Various non-compact real forms of exceptional algebras occur in supergravity theories in different dimensions as $U$-duality\(^3\). The related symmetric spaces are relevant by themselves for general relativity, because they are Einstein spaces [4]. In supergravity, some of these cosets, namely those pertaining to the non-compact real forms, are interpreted as scalar fields of the associated non-linear sigma model (see e.g. [5, 6], and also [7] for a review and list of references). Moreover, they can represent the charge orbits of electromagnetic fluxes of black holes when the Attractor Mechanism [8] is studied ([9]; for a comprehensive review, see e.g. [10]), and they also appear as the moduli spaces [11] for extremal black hole attractors; this approach has been recently extended to all kinds of branes in supergravity [12]. Fascinating group theoretical structures arise clearly in the description of the Attractor Mechanism for black holes in the Maxwell–Einstein supergravity, such as the so-called magic exceptional $\mathcal{N} = 2$ supergravity [13] in four dimensions, which is related to the minimally non-compact real $\mathfrak{e}_7(-25)$ form [14] of $\mathfrak{e}_7$.

The smallest exceptional Lie algebra, $\mathfrak{g}_2$, occurs for instance in the deconfinement phase transitions [15], in random matrix models [16], and in matrix models related to $D$-brane physics [17]; it also finds application to Monte Carlo analysis [18].

$\mathfrak{f}_4$ enters the construction of integrable models on exceptional Lie groups and of the corresponding coset manifolds. Of particular interest, from the mathematical point of view, is the coset manifold $\mathbb{CP}^2 = \mathfrak{f}_4/\text{Spin}(9)$, the octonionic projective plane (see e.g. [19], and references therein). Furthermore, the split real form $\mathfrak{f}_4(-4)$ has been recently proposed as the global symmetry of an exotic ten-dimensional theory in the context of gauge/gravity correspondence and ‘magic pyramids’ in [20].

Starting from the pioneering work of Gürsey [21, 22] on grand unified theories (GUTs), exceptional Lie algebras have been related to the study of the SM, and to the attempts to go beyond it: for example, the discovery of neutrino oscillations, the fine tuning of the mixing matrices, the hierarchy problem, the difficulty in including gravity, and so on. The renormalization flow of the coupling constants suggests the unification of gauge interactions at energies of the order of $10^{15}$ GeV, which can be improved and fine tuned by supersymmetry. In this framework the gauge group $G$ of GUT is expected to be simple, to contain the SM gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$ and also to predict the correct spectra after spontaneous symmetry breaking. The particular structure of the neutrino mixing matrix has led to the proposal of $G$ given by the semi-direct product between the exceptional group $E_6$ and the discrete group $S_4$ [23]. For some mathematical studies on various real forms of $\mathfrak{e}_6$, see e.g. [24], and references therein.

Recently, $\mathfrak{e}_7$ and ‘groups of type $E_7$’ [25] have appeared in several indirectly related contexts. They have been investigated in relation to minimal coupling of vectors and scalars in cosmology and supergravity [26]. They have been considered as gauge and global symmetries in the so-called Freudenthal gauge theory [27]. Another application is in the context of entanglement in quantum information theory; this is actually related to its application to black holes via the black hole/qubit correspondence (see [28] for reviews and list of references). For various studies on the split real form of $\mathfrak{e}_7$ and its application to maximal supergravity, see e.g. [29–32].

The largest finite-dimensional exceptional Lie algebra, namely $\mathfrak{e}_8$, appears in supergravity [33] in its maximally non-compact (split) real form, whereas the compact real form appears in heterotic string theory [34]. Rather surprisingly, in recent times the popular press has been dealing with $\mathfrak{e}_8$ more than once. Firstly, the computation of the Kazhdan–Lusztig–Vogan

---

\(^3\) Here $U$-duality is referred to as the ‘continuous’ symmetries of [2]. Their discrete versions are the $U$-duality non-perturbative string theory symmetries introduced in [3].
polynomials [35] involved the split real form of $\mathfrak{e}_8$. Then, attempts at formulating a `theory of everything’ were considered in [36], but they were proved to be unsuccessful (cfr. e.g. [37]). More interestingly, the compact real form of $\mathfrak{e}_8$ appears in the context of the cobalt niobate (CoNb$_2$O$_6$) experiment, making this the first actual experiment to detect a phenomenon that could be modeled using $\mathfrak{e}_8$ [38].

It should also be recalled that alternative approaches to quantum gravity, such as loop quantum gravity, [39] have also led towards the exceptional algebras, and $\mathfrak{e}_8$ in particular (see e.g. [40]).

It is worth mentioning that the adjoint of $\mathfrak{e}_8$ is its smallest fundamental representation; this sets $\mathfrak{e}_8$ on a different footing with respect to all other Lie algebras for unifying theories, which all exhibit a fundamental representation of lower dimension than the adjoint—of dimension 7, 26, 27, 56, in particular, for the exceptional algebras $\mathfrak{g}_2$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$ respectively. In the framework of a unified physical theory, therefore, only an $\mathfrak{e}_8$-based model has matter particles, intermediate bosons, Higgs(es) etc all in the same (adjoint, 248-dimensional) representation.

There is a wide consensus in both mathematics and physics on the appeal of the largest exceptional Lie algebra $\mathfrak{e}_8$, considered by many beautiful in spite of its complexity (for an explicit realization of its octic invariant, see [41]).

The present paper is the continuation of a previous one [42], in which the (finite-dimensional) exceptional Lie algebras were studied from a unifying point of view represented by the diagram in figure 1.

Figure 1 shows the projection of the roots of the exceptional Lie algebras on a complex $\mathfrak{su}(3) = \mathfrak{a}_2$ plane, recognizable by the dots forming the external hexagon, and it exhibits the Jordan pair content of each exceptional Lie algebra. There are three Jordan pairs $(\mathfrak{J}_n^3, \mathfrak{J}_n^3)$, each of which lies on an axis symmetrically with respect to the center of the diagram. Each pair doubles a simple Jordan algebra of rank 3, $\mathfrak{J}_n^3$, with involution—the conjugate representation $\mathfrak{J}_n^3$, which is the algebra of $3 \times 3$ Hermitian matrices over $\mathbb{H}$, where $\mathbb{H} = \mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{O}$, $\mathbb{C}$ for $n = 1, 2, 4, 8$ respectively, stands for real, complex, quaternion, octonion algebras, the four composition algebras according to Hurwitz’s Theorem—see e.g. [43]. Exceptional Lie algebras $\mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$ are obtained for $n = 1, 2, 4, 8$, respectively. $\mathfrak{g}_2$ can be also represented in the same way, with the Jordan algebra reduced to a single element; this corresponds to setting $n = -2/3$; in table 1 below. The Jordan algebras $\mathfrak{J}_n^3$ (and their conjugate $\mathfrak{J}_n^3$) globally behave like a 3 (and a 3) dimensional representation of the outer $\mathfrak{a}_2$. The algebra denoted by $\mathfrak{g}_0^3$ in the center (plus the Cartan generator associated with the axis along which the pair...
lies) is the algebra of the automorphism group of the Jordan Pair; namely, \( g^n \) is the reduced structure group of the corresponding Jordan algebra \( J^n \): \( g^n = \text{str}_o(J^n) \). Notice that \( J^n \) fits into a \((3n + 3)\)-dimensional irreducible representation of \( g^n \) itself.

The base field considered throughout the present paper is \( \mathbb{C} \). Therefore, all parameters in the whole paper are complex numbers. For instance, \( J^1 \) is a real Jordan algebra over the complex numbers, which means that the Hermitian conjugation is the transposed of matrices: \( J^1 \) is an algebra of symmetric complex matrices.

The reason for choosing complex numbers as base field lies in the fact that we are dealing with the root diagrams of the Lie algebras, therefore we need an algebraically closed field. The various real compact and non-compact forms of the exceptional Lie algebras follow as a consequence, using some more or less laborious tricks, whose treatment we leave to a future study, and they do not affect the essential structure.

The real, complex, quaternion, octonion attributes corresponding to setting \( n = 1, 2, 4, 8 \) in \( J^n \) refer to algebras—over the complex field—whose role is that of a book-keeping device: they are used in order to make the language easier and more compact. In this sense, they fall naturally into the Lie structure.

While the algebras \( \mathbb{R} \) and \( \mathbb{C} \) are commutative, the algebras \( \mathbb{Q} \) and \( \mathbb{C} \) are non-commutative (octonions—Cayley numbers—\( \mathbb{C} \) are also non-associative), but they all are alternative, a fundamental property without which our whole construction would fall apart\(^4\). They, however, have nothing to do with the base field—the complex field \( \mathbb{C} \)—of the corresponding Lie algebras. On the other hand, it is true the opposite: having complex alternative algebras allows to have nilpotents, which are as useful as \( J^+ \) and \( J^- \) are in the algebra of spin, or as creation and annihilation operators are in the description of the quantum harmonic oscillator or in quantum field theory.

By varying \( n \), figure 1 depicts the following decomposition, [42]:

\[
L^n = a_2 \oplus \text{str}_o(J^n) \oplus 3 \times J^n \oplus 3^* \times J^n,
\]

with the corresponding compact cases given in table 1.

The sequence \( L^n \) is usually named ‘exceptional sequence’ (or ‘exceptional series’; see e.g. [44], and references therein). This can be either interpreted as a sequence of Lie algebras over the complex numbers \( \mathbb{C} \), as we will consider throughout the present investigation, or as a sequence of corresponding compact real forms.

It is here worth pointing out that, by considering suitable non-compact, real forms, one obtains the \( n \)-parametrized sequence of \( U \)-duality Lie algebras \( L^n \) in \( D = 3 \) (Lorentzian) space-time dimensions\(^5\) [45]:

\[
L^n = \text{sl}(3, \mathbb{R}) \oplus \text{str}_o(J^n) \oplus 3 \times J^n \oplus 3^* \times J^n.
\]

Note that the reduced structure Lie algebra \( \text{str}_o(J^n) \), which, as stated above, is a suitable non-compact real form of \( g^n \), is nothing but the \( D = 5 \) \( U \)-duality Lie algebra. Also,

---

4 Non-alternative extensions beyond \( \mathbb{C} \), such as sedenions and trigintaduonions (cfr. e.g. [66]) would require a different approach.

5 Jordan pairs of semi-simple Euclidean–Jordan algebras of rank 3 in supergravity theories (among which the case of \( \text{so}(8), n = 0 \)) has been presented in [45].
\( L^n = \text{qconf}(J^n_3) \) \hspace{1cm} (1.3)

is the quasi-conformal Lie algebra of \( J^n_3 \) \cite{46, 47}, i.e. the \( U \)-duality Lie algebra in \( D = 3 \) (see e.g. \cite{48} and \cite{10} for an introduction to the application of Jordan algebras and their symmetries in supergravity\textsuperscript{6}, and lists of references). Suitable real, non-compact forms of all exceptional Lie algebras can thus be characterized as quasi-conformal algebras\textsuperscript{7} of Euclidean simple Jordan algebras of rank 3.

At group level, the algebraic decompositions (1.1) and (1.2) are Cartan decompositions respectively pertaining to the following maximal non-symmetric embeddings:

\[
\text{QConf}_f(J^n_3) \supset \text{SU}(3) \times \text{Str}_{0,\text{c}}(J^n_3); \tag{1.4}
\]

\[
\text{QConf}(J^n_3) \supset \text{SL}(3, \mathbb{R}) \times \text{Str}_0(J^n_3). \tag{1.5}
\]

As mentioned above, the non-semi-simple part of the rhs of (1.1) and (1.2) is given by a triplet of Jordan pairs.

Finally, we recall that in \cite{45}, by exploiting the Jordan pair structure of \( U \)-duality Lie algebras in \( D = 3 \) and the relation to the super-Ehlers symmetry in \( D = 5 \) \cite{49}, the massless multiplet structure of the spectrum of a broad class of \( D = 5 \) supergravity theories was investigated.

In general, many properties of Lie algebras and groups can be already inferred from abstract theoretical considerations; however, for most applications, it is useful to have explicit concrete realizations in terms of matrices\textsuperscript{8}.

In this paper we develop the results of \cite{42} and fully exploit Jordan pairs and the corresponding unifying view depicted in figure 1. We introduce Zorn-type matrix realizations of all exceptional finite-dimensional Lie algebras, which make the Jordan pair structure manifest and are written in the form of a \( 2 \times 2 \) matrix, endowed with a quite peculiar matrix product accounting for the complexity and non-associativity of the underlying structure. As a consequence of (1.3), this corresponds to the explicit construction of Zorn-type matrix realizations of the compact form of quasi-conformal algebras of simple Jordan algebras of rank 3; we point out that in the present paper we will deal with Lie algebras over \( \mathbb{C} \), leaving the analysis of real forms to future investigation.

The paper is organized as follows.

In section 2 we briefly review the concept of a Jordan pair. Most of the section can be found also in \cite{42} and is repeated here for completeness.

For the same reason, as well as for introducing some notation, we present in section 3 a summary on the octonion algebra and its representation through the Zorn matrices, on which we base the development of our representations. The key idea which we exploit here is that the octonions’ non-associativity can be cast into a properly defined product of \( 2 \times 2 \) complex matrices.

With this in mind, we are able to define, formally using \( 2 \times 2 \) matrices, a representation of \( g_2 \) in section 4, \( f_4 \) in section 5 (where we also make a comparison with Tits’ construction), \( e_6 \) in section 6, \( e_7 \) in section 7. In section 8 we prove the Jacobi identity for all these algebras.

\textsuperscript{6} In these theories, the \( U \)-duality Lie algebra in \( D = 4 \) (Lorentzian) space-time dimensions is given by the conformal Lie algebra \( \text{conf}(J^n_4) = \text{aut}(3(J^n_4)) \), where \( 3(J^n_4) \) denotes the Freudenthal triple system constructed over \( J^n_4 \).

\textsuperscript{7} The case \( n = -1 \) is trivial, and it corresponds to ‘pure’ \( N = 2 \) supergravity in four-dimensional Lorentzian space-time; therefore, it does not admit an uplift to five dimensions, and it will henceforth not be considered. Moreover, \( \text{su}(2) \) might be considered as the \( n = -4/3 \) element of the sequence in table below (1.1), as well. However, this is a limit case of the ‘exceptional’ sequence reported in table 1, not pertaining to Jordan pairs nor to supergravity in \( D = 3 \) dimensions, and thus we will disregard it.

\textsuperscript{8} Explicit realizations of exceptional groups have been obtained e.g. in \cite{50}. Our results, however, displays a much more manageable form, with manifest \( a_2 \) covariance, as a consequence of the full exploitation of the underlying Jordan pair structure.
In the case of $\mathfrak{e}_8$, section 9, a new difficulty occurs due to non-associativity. Not only the octonions are non-associative, but so is the underlying standard matrix product of the Jordan algebra elements. This forces a new definition of matrix elements and of their product, which still allows us to formally describe the representation of $\mathfrak{e}_8$ through $2 \times 2$ matrices. The proof of the Jacobi identity for this case heavily relies on the Jordan Pair axioms, and it is presented in section 10.

The paper ends with some proposals of future developments of the present work.

2. Jordan pairs

In this section we review the concept of a Jordan Pair, [51] (see also [43] for an enlightening overview).

Jordan Algebras have traveled a long journey, since their appearance in the 30’s [52]. The modern formulation [53] involves a quadratic map $U_{xy}$ (like $xyx$ for associative algebras) instead of the original symmetric product $x \cdot y = \frac{1}{2}(xy + yx)$. The quadratic map and its linearization $V_{xyz} = (U_{xz} - U_x - U_z)y$ (like $xyz + zyx$ in the associative case) reveal the mathematical structure of Jordan Algebras much more clearly, through the notion of inverse, inner ideal, generic norm, etc. The axioms are:

$$ U_1 = \text{Id,} \quad U_x V_{yx} = V_{xy} U_x, \quad U_{1,y} = U_1 U_x U_x. \quad (2.1) $$

The quadratic formulation led to the concept of Jordan Triple systems [54], an example of which is a pair of modules represented by rectangular matrices. There is no way of multiplying two matrices $x$ and $y$, say $n \times m$ and $m \times n$ respectively, by means of a bilinear product. But one can do it using a product like $xyx$, quadratic in $x$ and linear in $y$. Notice that, like in the case of rectangular matrices, there needs not be a unity in these structures. The axioms are in this case:

$$ U_x V_{yx} = V_{xy} U_x, \quad U_{1,y} = U_1 U_x U_x. \quad (2.2) $$

Finally, a Jordan Pair is defined just as a pair of modules $(V^+, V^-)$ acting on each other (but not on themselves) like a Jordan Triple:

$$ U_{e,x} V_{e,\sigma} = V_{e,\sigma} U_{e,a}, $$

$$ U_{e,x} V_{e,-\sigma} = V_{e,-\sigma} U_{e,a}, $$

$$ U_{e,x} V_{e,\sigma} = U_{e,a} U_{x,\sigma} U_{e,a} \quad (2.3) $$

where $\sigma = \pm$ and $x^\sigma \in V^{+\sigma}$, $y^- \in V^{-\sigma}$.

Jordan pairs are strongly related to the Tits–Kantor–Koecher construction of Lie Algebras $\mathfrak{L}$ [55–57] (see also the interesting relation to Hopf algebras, [58]):

$$ \mathfrak{L} = J \oplus \text{str}(J) \oplus \bar{J} \quad (2.4) $$

where $J$ is a Jordan algebra and $\text{str}(J) = L(J) \oplus \text{Der}(J)$ is the structure algebra of $J$ [43]; $L(x)$ is the left multiplication in $J$: $L(x)y = xy$ and $\text{Der}(J) = [L(J), L(J)]$ is the algebra of derivations of $J$ (the algebra of the automorphism group of $J$) [59, 60].

In the case of complex exceptional Lie algebras, this construction applies to $\mathfrak{e}_7$, with $J = J_3(\mathbb{C})$, the 27-dimensional exceptional Jordan algebra of $3 \times 3$ Hermitian matrices over the complex octonions, and $\text{str}(J) = \mathfrak{e}_6 \oplus \mathbb{C}$ - $\mathbb{C}$ denoting the complex field. The algebra $\mathfrak{e}_6$ is called the reduced structure algebra of $J$, $\text{str}_0(J)$, namely the structure algebra with the generator corresponding to the multiplication by a complex number taken away: $\mathfrak{e}_6 = L(J_0) \oplus \text{Der}(J)$, with $J_0$ denoting the traceless elements of $J$. 

A Marrani and P Truini
As we introduced in section 1, Octonions

the complex field (for earlier studies, see e.g. [67]).

Another fundamental product is the sharp product #, [43]. It is the linearization of
\[ x^8 := x^3 - t(x)x - \frac{1}{2} t(t(x^2) - t(x)^2)I, \]
with \( t(x) \) denoting the trace of \( x \in J^n \), in terms of which
we can write the fundamental cubic identity for \( J^n \), \( n = 1, 2, 4, 8 \):
\[ x^2 \cdot (x \cdot z) - x \cdot (x^2 \cdot z) = 0. \]  (2.5)

We shall make use of the following identities, which can be derived from the Jordan Pair
axioms, [51]:
\[ \{x, y, z\} := V_{x,y}z := t(x, y)z + t(z, y)x - (x#z)#y \]
\[ = 2[(x \cdot y)z + (y \cdot z)x - (z \cdot x)y]. \]  (2.7)

Notice that the last equality of (2.7) is not trivial at all. \( V_{x,y}z \) is the linearization of the quadratic map \( U_{x,y} \). The equation (2.3.15) at page 484 of [43] shows that:
\[ U_{x,y} = t(x, y)x - x^8#y = 2(x \cdot y)x - x^2\cdot y. \]  (2.8)

We shall make use of the following identities, which can be derived from the Jordan Pair
axioms, [51]:
\[ [V_{x,y}, V_{z,w}] = V_{V_{x,y}z,w} - V_{V_{z,w}x,y} \]  (2.9)

and, for \( D = (D_+, D_-) \) a derivation of the Jordan Pair \( V \) and \( \beta(x, y) = (V_{x,y}, -V_{y,x}) \),
\[ [D, \beta(x, y)] = \beta(D_+(x), y) + \beta(x, D_-(y)). \]  (2.10)

3. Octonions

As we introduced in section 1, \( \mathbb{O} \) stands for the algebra of the octonions (Cayley numbers) over
the complex field \( \mathbb{C} \) whose multiplication rule goes according to the Fano diagram in figure 2
(for earlier studies, see e.g. [67]).

If \( a \in \mathbb{O} \) we write \( a = a_0 + \sum_{k=1}^{7} a_ku_k \), where \( a_k \in \mathbb{C} \) for \( k = 1, \ldots, 7 \) and \( u_k \) for
\( k = 1, \ldots, 7 \) denote the octonion imaginary units. We denote by \( i \) the imaginary unit in \( \mathbb{C} \).

Thence, we introduce two idempotent elements:
\[ \rho^\pm = \frac{1}{2}(1 \pm ir) \]
and six nilpotent elements:
\[ \varepsilon_k^\pm = \rho^\pm u_k, \quad k = 1, 2, 3. \]

One can readily check that:
\[
(\rho^\pm)^2 = \rho^\pm, \quad \rho^\pm \rho^\mp = 0
\]
\[
\rho^\pm \varepsilon_k^\pm = \varepsilon_k^\pm \rho^\mp = \varepsilon_k^\mp
\]
\[
\rho^\mp \varepsilon_k^\pm = \varepsilon_k^\pm \rho^\mp = 0
\]
\[
(\varepsilon_k^\pm)^2 = 0, \quad k = 1, 2, 3
\]
\[
\varepsilon_k^\pm \varepsilon_{k+1}^\pm = -\varepsilon_{k+1}^\pm \varepsilon_k^\pm = \varepsilon_{k+2}^\mp \quad \text{(indices modulo 3)}
\]
\[
\varepsilon_j^\pm \varepsilon_k^\mp = 0 \quad j \neq k
\]
\[
\varepsilon_k^\pm \varepsilon_k^\mp = -\rho^\pm, \quad k = 1, 2, 3.
\]

It is known that octonions can be represented by Zorn matrices, [61]. If \( a \in \mathbb{C} \), \( A^\pm \in \mathbb{C}^3 \) is a vector with complex components \( a_k^\pm, \quad k = 1, 2, 3 \) (and we use the standard summation convention over repeated indices throughout), then we have the identification:
\[
a = a_0^+ \rho^+ + a_0^- \rho^- + a_k^+ \varepsilon_k^+ + a_k^- \varepsilon_k^- \leftrightarrow \begin{bmatrix} a_0^+ & A^+ \\ A^- & a_0^- \end{bmatrix};
\]
therefore, through equation (3.2), the product of \( a, b \in \mathbb{C} \) corresponds to:
\[
\begin{bmatrix} \alpha^+ & A^+ \\ A^- & \beta^- \end{bmatrix} = \begin{bmatrix} \alpha^+ \beta^+ + A^+ \cdot B^- & \alpha^+ B^+ + \beta^- A^+ + A^- \land B^- \\ \alpha^- B^+ + \beta^+ A^- + A^+ \land B^+ & \alpha^- \beta^- + A^- \cdot B^+ \end{bmatrix},
\]
where \( A^\pm \cdot B^\mp = -\alpha_k^\pm \beta_k^\mp \) and \( A \land B \) is the standard vector product of \( A \) and \( B \).

4. \( g_2 \) action on Zorn matrices

In this section, we derive the matrix representation of \( g_2 \), and its action on Zorn matrices. Let \( a, b, c \in \mathbb{C} \). Then the derivations of the octonions, [59, 62], can be written as \( D_{a,b} \):
\[
D_{a,b,c} = \frac{1}{2}([a, b], c) = (a, b, c) \quad \text{where} \quad (a, b, c) = (ab)c - a(bc).
\]

We choose the following \( g_2 \) generators, for \( k = 1, 2, 3 \) (mod 3):
\[
d_k^\pm = \mp D_{\varepsilon_k^+, \varepsilon_k^+} = \mp L_{\varepsilon_k^+, \varepsilon_k^+}
\]
\[
H_1 = \frac{\sqrt{2}}{2} (D_{\varepsilon_1^+, \varepsilon_1^+} - D_{\varepsilon_2^+, \varepsilon_2^+}) = \frac{\sqrt{2}}{2} (L_{\varepsilon_1^+} L_{\varepsilon_1^+} - L_{\varepsilon_2^+} L_{\varepsilon_2^+})
\]
\[
H_2 = \frac{\sqrt{6}}{6} (D_{\varepsilon_1^+, \varepsilon_1^+} + D_{\varepsilon_2^+, \varepsilon_2^+} - 2D_{\varepsilon_3^+, \varepsilon_3^+}) = \frac{\sqrt{6}}{6} (L_{\varepsilon_1^+} L_{\varepsilon_1^+} + L_{\varepsilon_2^+} L_{\varepsilon_2^+} - 2L_{\varepsilon_3^+} L_{\varepsilon_3^+})
\]
\[
g_k^\pm = 3 D_{\rho^+, \varepsilon_k^+} = L_{\varepsilon_k^+} - R_{\varepsilon_k^+} = 3 L_{\rho^+} L_{\varepsilon_k^+}.
\]
We notice that \( D_\rho \sigma = 0, D_\rho \tau = -D_\rho \sigma \), and that \( D_\sigma \tau = D_\sigma \tau + D_\tau \sigma \), hence the 14 generators introduced above span all the derivations of \( \mathfrak{e} \).

The action of these generators on \( a \in \mathfrak{e}, a = \alpha_0^+ \rho^+ + \alpha_0^- \rho^- + \alpha_k^+ \varepsilon_k^+ + \alpha_k^- \varepsilon_k^- \) is:

\[
d_k^\pm : a \rightarrow \pm (\alpha_k^+ \varepsilon_k^+ + \alpha_k^- \varepsilon_k^-) - \alpha_k^\mp \varepsilon_k^{\mp}
\]

\[
H_1 : a \rightarrow \frac{\sqrt{2}}{2} (\alpha_1^+ \varepsilon_1^+ - \alpha_1^- \varepsilon_1^-) + \alpha_2^+ \varepsilon_2^+ - \alpha_2^- \varepsilon_2^-
\]

\[
H_2 : a \rightarrow \frac{\sqrt{6}}{6} (\alpha_3^+ \varepsilon_3^+ - 2 \alpha_1^+ \varepsilon_1^+ - \alpha_1^- \varepsilon_1^- - \alpha_2^+ \varepsilon_2^+ + 2 \alpha_2^- \varepsilon_2^-)
\]

\[
g_k^\pm : a \rightarrow -\alpha_k^+ \varepsilon_k^+ + \alpha_k^- \varepsilon_k^-
\]

One can thus readily check that \( [H_1, H_2] = 0 \) and that the \( g_k^\pm \)'s are eigenvectors of \( (H_1, H_2) \), with respect to the Lie product, with eigenvalues \( (\pm \frac{\sqrt{2}}{2}, \pm \frac{\sqrt{6}}{6}) \) and \( (0, \pm \frac{\sqrt{12}}{12}) \); the same with for \( d_k^\pm \)'s with eigenvalues \( (\pm \frac{\sqrt{2}}{2}, \pm \frac{\sqrt{6}}{6}) \) and \( (\pm \sqrt{12}, 0) \), namely:

\[
[H_1, H_2] = 0
\]

\[
[H_1, g_k^\pm] = \pm \frac{\sqrt{6}}{3} g_k^\pm \quad [H_2, g_k^\pm] = \pm \frac{\sqrt{6}}{3} g_k^\pm \quad [H_1, d_k^\pm] = \pm \frac{\sqrt{12}}{12} d_k^\pm \quad [H_2, d_k^\pm] = \pm \frac{\sqrt{12}}{12} d_k^\pm
\]

Therefore, we have found out that the \( d_k^\pm \) generators correspond to the external \( a_2 \) in the root diagram of \( g_2 \), whereas the \( g_k^\pm \) generators correspond to the internal hexagon \( (3 \text{ and } 3) \) of \( a_2 \).

The remaining non-vanishing commutation relations are:

\[
[d_k^\pm, d_{k+1}^\pm] = \pm d_{k+2}^\pm
\]

\[
[d_k^+, d_{k-1}^-] = -\frac{1}{2} (\sqrt{2} H_1 - \sqrt{2} H_2) \quad [d_k^-, d_{k+1}^+] = -\frac{1}{2} (\sqrt{2} H_1 + \sqrt{2} H_2) \quad [d_k^+, d_{k+1}^-] = \sqrt{2} H_1
\]

We now introduce the following complex algebra of \( 4 \times 4 \) Zorn-type matrices:

\[
\left[ \begin{array}{cc}
a & A^+ \\
A^- & t(a)
\end{array} \right]
\]

(4.1)

where \( a \) is a \( 3 \times 3 \) complex matrix, \( A^+, A^- \in \mathbb{C}^3 \), viewed as column and row vectors respectively and \( t(a) \) denotes the trace of \( a \).

The product of two such matrices is defined by:

\[
\left[ \begin{array}{cc}
a & A^+ \\
A^- & t(a)
\end{array} \right] \left[ \begin{array}{cc}
b & B^+ \\
B^- & t(b)
\end{array} \right] = \left[ \begin{array}{cc}
ab + A^+ \circ B^- & aB^+ + A^- \wedge B^- \\
A^- b + A^+ \wedge B^+ & t(a)t(b) + t(B^+ \circ A^-)
\end{array} \right]
\]

(4.2)

where

\[
A^+ \circ B^- = t(A^+ B^-)I - t(I)A^+ B^-\]

(4.3)

(with standard matrix products of row and column vectors and with \( I \) denoting the \( 3 \times 3 \) identity matrix); \( A \wedge B \) is the standard vector product of \( A \) and \( B \). Notice that \( t(X^+ \circ Y^-) = 0 \), hence we have an algebra.
In particular, we get a sub-algebra by imposing the \(a\) matrices to be traceless. We use this algebra in order to define the following adjoint representation \(\varrho\) of the Lie algebra \(g_2\):

\[
\begin{bmatrix}
a \\
A^+ \\
A^- \\
0
\end{bmatrix}
\]

(4.4)

where \(a \in a_2, A^+, A^- \in \mathbb{C}^3\), viewed as column and row vector respectively.

Indeed, the commutator of two such matrices, using (4.2), can be computed to read:

\[
\begin{bmatrix}
a & A^+ \\
A^- & 0
\end{bmatrix}
\begin{bmatrix}
b & B^+ \\
B^- & 0
\end{bmatrix} = \begin{bmatrix}
[a, b] + A^+ \circ B^- - B^+ \circ A^- & aB^+ - bA^+ + 2A^- \wedge B^- \\
A^- b - B^- a + 2A^+ \wedge B^+ & 0
\end{bmatrix}.
\]

(4.5)

and therefore one is led to the following identifications of the \(g_2\) generators shown above:

\[
\varrho(d_{\pm}) = E_{k\pm1} \quad (\text{mod } 3), \quad k = 1, 2, 3
\]

\[
\varrho(\sqrt{2}H_1) = E_{11} - E_{22} \quad \varrho(\sqrt{6}H_2) = E_{11} + E_{22} - 2E_{33}
\]

(4.6)

where \(E_{ij}\) denotes the matrix with all zero elements except a 1 in the \([ij]\) position: \(E_{ij} = \delta_{i,j}\). Let us define \(e^k_j\) as the standard basis vectors of \(\mathbb{C}^3\) (\(e^k_j\) denote their transpose).

On the other hand, a direct calculation shows that:

\[
\varrho([X, Y]) = [\varrho(X), \varrho(Y)] \quad X, Y \in g_2
\]

(4.7)

thus proving that \(\varrho\) (which is obviously linear) is indeed a representation.

It is useful to extend this correspondence to the roots of \(g_2\), obtaining the pictorial view of the diagram in figure 3.

For future use, we here explicitly associate a matrix in the form (4.4) to the derivation \(D_{c,d}\) for \(c, d \in \mathfrak{e}\). Let us define \(e_{jk} := D_{c,d}\) (notice the switch of indices). A straightforward calculation shows that, for \(c, d \in \mathfrak{e}, c = \alpha_0^c \rho^+ + \upsilon_k^c \epsilon^+_k, d = \rho_0^d \rho^+ + \upsilon_k^d \epsilon^+_k\), it holds that:

\[
D_{c,d} = \frac{1}{2}((\alpha_0^c - \alpha_0^d) \upsilon^+_w - (\beta_0^+ - \beta_0^+) \upsilon^+_w - (\upsilon^+ \wedge \upsilon^+)_{\bar{c}})g^+_k + (\upsilon^+_w - \upsilon^+_w) e_{jk}.
\]

(4.8)
Notice that \( \varrho(e_{ij}) = E_{ij} \) for \( i \neq j = 1, 2, 3 \), whereas

\[
\varrho(e_{11}) = \varrho\left(\frac{\sqrt{2}}{2} H_1 + \frac{\sqrt{6}}{6} H_2 \right) = \frac{1}{3} (2E_{11} - E_{22} - E_{33})
\]

\[
\varrho(e_{22}) = \varrho\left(-\frac{\sqrt{2}}{2} H_1 + \frac{\sqrt{6}}{6} H_2 \right) = \frac{1}{3} (-E_{11} - E_{22} + 2E_{33})
\]

\[
\varrho(e_{33}) = \varrho\left(-\frac{\sqrt{6}}{3} H_2 \right) = \frac{1}{3} (-E_{11} - E_{22} + 2E_{33}).
\]

We thus obtain from (4.8):

\[
\varrho(D_{c,d}) = \begin{pmatrix}
D_{11} & D_{12} \\
D_{12} & 0
\end{pmatrix}, \quad \text{where}
\]

\[
D_{11} = -\frac{1}{3} (v_i^- w_i^+ - v_i^+ w_i^-) I + (v_i^- w_i^+ - v_i^+ w_i^-) E_{ij}
\]

\[
D_{12} = \frac{1}{3} ((\alpha_0^+ - \alpha_0^-) w^- + (\beta_0^+ - \beta_0^-) v^+ + (v^- \wedge w^-))
\]

\[
D_{21} = \frac{1}{3} ((\alpha_0^+ - \alpha_0^-) w^- - (\beta_0^+ - \beta_0^-) v^+ - (v^- \wedge w^+)).
\]

We introduce the following action of \( \varrho(\mathfrak{g}_2) \) on the octonions represented by Zorn matrices:

\[
\begin{bmatrix}
A^+ & 0 \\
A^- & 0
\end{bmatrix}
\begin{bmatrix}
\alpha_0^+ & v^+ \\
v^- & \alpha_0^-
\end{bmatrix} =
\begin{bmatrix}
-A^- + A^+ & av^+ + (\alpha_0^- - \alpha_0^+) A^- - A^+ \wedge v^- \\
v^- A - (\alpha_0^- - \alpha_0^+) A^- A^+ \wedge v^+
\end{bmatrix}.
\]

We see that \( \varrho(\mathfrak{g}_2) \) acts non-trivially on traceless octonions, hence we can write \( \alpha_0^+ = -\alpha_0^- \) to get a ‘matrix-like’ expression of the seven-dimensional (fundamental) representation 7 of \( \mathfrak{g}_2 \). A direct calculation confirms that the action (4.10) corresponds to the action of the \( \mathfrak{g}_2 \) generators on the octonions shown above.

It can also be shown that the action (4.10) is indeed a derivation of the octonions, confirming that \( \text{Der}(\mathbb{O}) = \mathfrak{g}_2 \). The only ingredients needed for the proof are identities from elementary three-dimensional geometry, like \( (A \wedge B) \cdot C = (C \wedge A) \cdot B \) or \( (A \wedge B) \wedge C = (A \cdot B) C - (B \cdot C) A \), plus the following identity for a \( 3 \times 3 \) traceless matrix \( a \):

\[
\sum_j a_{ij} \varepsilon_{jkl} + a_{kj} \varepsilon_{ijk} + a_{ij} \varepsilon_{ijk} = 0.
\]

### 5. \( n = 1 \): Matrix representation of \( \mathfrak{f}_4 \)

We introduce in this section the representation \( \varrho \) of \( \mathfrak{f}_4 \) in the form of a matrix. For \( \mathbf{f} \in \mathfrak{f}_4 \):

\[
\varrho(\mathbf{f}) = \begin{pmatrix}
a \otimes I + I \otimes a_1 & x^+ \\
-I \otimes a_1 & x^- \end{pmatrix},
\]

where \( a \in \mathfrak{a}_2^{(1)}, a_1 \in \mathfrak{a}_2^{(2)} \) (the superscripts being merely used to distinguish the two copies of \( \mathfrak{a}_2 \)) \( a_1^T \) is the transpose of \( a_1 \), \( I \) is the \( 3 \times 3 \) identity matrix, \( x^+ \in \mathbb{C}^3 \otimes J^1_3 \), \( x^- \in \mathbb{C}^3 \otimes J^1_3 \):

\[
x^+ := \begin{pmatrix}
x_1^+ \\
x_2^+ \\
x_3^+
\end{pmatrix}, \quad x^- := (x_1^-, x_2^-, x_3^-), \quad x_i^+ \in J^1_3, x_i^- \in J^1_3, \quad i = 1, 2, 3.
\]
The commutator is set to be:

$$\left[ (a \otimes I + I \otimes a_1) \begin{pmatrix} x^+ & y^+ \\ -x^- & -y^- \end{pmatrix}, (b \otimes I + I \otimes b_1) \begin{pmatrix} y^+ & y^- \\ -x^- & x^- \end{pmatrix} \right] := \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

(5.2)

where:

$$C_{11} = [a, b] \otimes I + I \otimes [a_1, b_1] + x^+ \circ y^- - y^+ \circ x^-$$

$$C_{12} = (a \otimes I)y^+ - (b \otimes I)x^+ + (I \otimes a_1)y^+ + y^+ (I \otimes a_1^T)$$

$$- (I \otimes b_1)x^+ - x^- (I \otimes b_1^T) + x^- \times y^-$$

$$C_{21} = -y^- (a \otimes I) + x^- (b \otimes I) - (I \otimes a_1^T)y^- - y^- (I \otimes a_1)$$

$$+ (I \otimes b_1^T)x^- + x^- (I \otimes b_1) + x^+ \times y^+$$

$$C_{22} = I \otimes [a_1^T, b_1^T] + x^- \bullet y^- - y^- \bullet x^+$$

with the following definitions:

$$x^+ \circ y^- := \left( \frac{i}{2}t(x^+_j, y^-_j)I - t(x^+_j, y^-_j)E_{ij} \right) \otimes I + I \otimes \left( \frac{i}{2}t(x^+_j, y^-_j)I - x^+_jy^-_j \right)$$

$$x^- \bullet y^+ := I \otimes \left( \frac{i}{2}t(x^+_j, y^-_j)I - x^+_jy^-_j \right)$$

$$\left( x^\pm \times y^\pm \right)_j := \epsilon_{ijk}x^\pm_k y^\pm_j + x^\pm_k y^\pm_j - y^\pm_k t(x^\pm_j) - x^\pm_k t(y^\pm_j) - t(x^\pm_j, y^\pm)$$

Notice that:

1. \( x \in J_1^1 \) is a symmetric complex matrix;
2. writing \( x^+ \circ y^- := c \otimes I + I \otimes c_1 \) we have that both \( c \) and \( c_1 \) are traceless hence \( c, c_1 \in \mathfrak{a}_2 \), and indeed they have 8 (complex) parameters, and \( y^- \bullet x^+ = I \otimes c_1^T \);
3. terms like \((I \otimes a_1)y^+ + y^+ (I \otimes a_1^T)\) are in \( \mathbb{C}^3 \otimes J_1^1 \), namely they are matrix valued vectors with symmetric matrix elements;
4. the sharp product \# of \( J_1^1 \) matrices appearing in \( x^\pm \times y^\pm \) is the fundamental product in the theory of Jordan Algebras, introduced in section 2.

In order to prove that \( g \) is a representation of the Lie algebra \( \mathfrak{f}_4 \) we make a comparison with Tits’ construction of the fourth row of the magic square, [63, 64]. If \( J_0 \) denotes the traceless elements of \( J \), \( \mathcal{C}_0 \) the traceless octonions (the trace being defined by \( t(a) := a + \bar{a} \in \mathbb{C} \), for \( a \in \mathcal{C} \) where the bar denotes the octonion conjugation—that does not affect the field \( \mathbb{C} \) ) it holds that:

$$\mathfrak{f}_4 = \text{Der} (\mathcal{C}) \oplus (\mathcal{C}_0 \otimes J_0) \oplus \text{Der} (J)$$

(5.5)

with commutation rules, for \( D \in \text{Der} (\mathcal{C}) = \mathfrak{g}_2 \), \( c, d \in \mathcal{C}_0, x, y \in J_0, E \in \text{Der} (J) \), given by:

$$[\text{Der} (\mathcal{C}), \text{Der} (\mathcal{C})] = \text{Der} (\mathcal{C})$$

$$[\text{Der} (J), \text{Der} (J)] = \text{Der} (J)$$

$$[\text{Der} (\mathcal{C}), \text{Der} (J)] = 0$$

(5.6)

$$[D, c \otimes x] = D(c) \otimes x$$

$$[E, c \otimes x] = c \otimes E(x)$$

$$[c \otimes x, d \otimes y] = t(xy)D_{c,d} + 2(c \ast d) \otimes (x \ast y) + \frac{i}{2}t(cd)[x, y]$$

where \( \mathcal{C}_0 \ni c \ast d = cd - \frac{i}{2}t(cd), J_0 \ni x \ast y = \frac{1}{2}(xy + yx) - \frac{i}{2}t(xy)I \), \( J = J_1^1 \).
The derivations of $J$ are inner: $Der(J) = [L(J), L(J)]$ where $L$ stands for the left (or right) multiplication with respect to the Jordan product: $L_{a}y = \frac{1}{2}(xy + yx)$. In the case under consideration, the product $x, y \rightarrow xy$ is associative and $[L_{x}, L_{y}]z = \frac{1}{2}[[x, y], z]$. Since $[x, y]$ is antisymmetric, then $Der(J) = so(3)_{c} \equiv a_{1}$.

We can thus put forward the following correspondence:

\[
\varrho(D) = \begin{pmatrix}
\frac{1}{2}tr(x_{+}^{\top}) \otimes I & \frac{1}{2}tr(x_{+}^{\top}) \otimes I \\
0 & 0
\end{pmatrix}
\]

\[
\varrho(E) = \begin{pmatrix}
I \otimes a_{1} & 0 \\
0 & I \otimes a_{1}
\end{pmatrix}
\]

\[
\varrho(\epsilon_{+}^{\top} \otimes J_{0}) = \begin{pmatrix}
0 & x_{k}^{+} - \frac{1}{2}tr(x_{k}^{+}) \otimes I \\
0 & 0
\end{pmatrix}
\]

\[
\varrho(\epsilon_{-}^{\top} \otimes J_{0}) = \begin{pmatrix}
x_{k}^{-} - \frac{1}{2}tr(x_{k}^{-}) \otimes I & 0 \\
0 & 0
\end{pmatrix}
\]

\[
\varrho((\rho^{+} - \rho^{-}) \otimes J_{0}) = \begin{pmatrix}
I \otimes a_{1}^{\delta} & 0 \\
0 & -I \otimes a_{1}^{\delta}
\end{pmatrix}
\]

where $a_{1}^{\delta}$ and $a_{1}^{\epsilon}$ are the antisymmetric and symmetric parts of $a_{1}$, and

\[
tr(x^{\top}) := \begin{pmatrix}
t(x_{1}^{+}) \\
t(x_{2}^{+}) \\
t(x_{3}^{+})
\end{pmatrix}, \quad tr(x^{-}) = (t(x_{1}^{-}), t(x_{2}^{-}), t(x_{3}^{-}))
\]

with $x_{k}^{\pm}$ denoting a matrix-valued vector whose $k$th component is the only non-vanishing one.

### 5.1. Comparison with Tits' construction

It is here worth commenting that there is some apparent difference between the way we write Tits' construction, (5.6), and the way it is written in the mathematical literature; see for instance [65], page 93.

Firstly, we have the operators acting from the left, contrary to the action from the right often used by mathematicians. This implies that the third and fourth commutators in (5.6) are written in the reverse order. Moreover, the last commutator of (5.6) is instead written in [65] (using the superscript $\top$ in order to distinguish it from ours) as follows:

\[
[c \otimes x, d \otimes y]^{\top} = \frac{1}{12}[t(xy)D_{c,d}^{\top} + ((c * d) \otimes (x \ast y))^{\top} + \frac{1}{4}t(cd)[L_{x}, L_{y}]].
\]  

Furthermore, we observe that we have defined the derivation $D_{c,d}^{\top} = \frac{1}{4}D_{c,d}^{\top}$ (up to a sign due to left versus right action). Because of this and the fact that $[L_{x}, L_{y}]z = \frac{1}{2}[[x, y], z]$, we have 4 times the first and third terms in (5.8), and 2 times the middle one. However, these factors can be reabsorbed by changing $\varrho(c \times x) \rightarrow \frac{1}{2}\varrho(c \times x)$, thus proving the equivalence of the two ways of writing all the commutation relations.

### 5.2. $\varrho$ as a representation of $f_{4}$

By exploiting the correspondence (5.7), we now prove in six steps that the commutators (5.3) satisfy (5.6), thus proving the following.

**Theorem.** $\varrho$ realizes the adjoint representation of $f_{4}$.
Proof. (1) \([\text{Der}(\mathcal{C}), \text{Der}(\mathcal{C})] = \text{Der}(\mathcal{C})\).

In order to prove this first step, let us denote by \(A^\pm\) and \(B^\pm\) the \(3\) vectors \(\frac{1}{2}tr(x^\pm)\) and \(\frac{1}{2}tr(y^\pm)\) respectively. Then, we have to compute:

\[
\begin{pmatrix}
  (a \otimes I) & A^+ \otimes I \\
  A^- \otimes I & 0
\end{pmatrix}
\begin{pmatrix}
  b \otimes I & B^+ \otimes I \\
  B^- \otimes I & 0
\end{pmatrix} :=
\begin{pmatrix}
  C_{11} & C_{12} \\
  C_{21} & C_{22}
\end{pmatrix}.
\]  

(5.9)

Let us calculate some terms separately:

\(A^+ \otimes I \circ B^- \otimes I = (A^+_i B^-_j I - 3A^+_i B^-_j E_{ij}) \otimes I + I \otimes \left( \frac{1}{2} A^+_i B^-_j t(I) I - A^+_i B^-_j I \right)\);

the first bracket on the right-hand side is \(A^+ \circ B^-,\) as defined in (4.3), whereas the second one vanishes

\((A^+ \otimes I) \times (B^+ \otimes I)) = \epsilon_{ijk} A^+_i B^+_j \left( 2J - 2t(I) I - (t(I) - t(I)\t(\t(I)\t)) I \right) = 2(A^+ \wedge B^+) I;

similarly with \(A^-\) and \(B^-\), hence

\((A^\pm \otimes I) \times (B^\pm \otimes I) = 2(A^\pm \wedge B^\pm \otimes I)\).

Therefore, we obtain

\[
\begin{align*}
C_{11} &= ([a, b] + A^+ \circ B^- - B^- \circ A^-) \otimes I \\
C_{12} &= (aB^+ - bA^+ + 2A^- \wedge B^-) \otimes I \\
C_{21} &= (-B^- a + A^+ b + 2A^+ \wedge B^-) \otimes I \\
C_{22} &= 0,
\end{align*}
\]

(5.10)

which make the commutaton relations (5.9) correspond to those of \(g_{\mathbb{S}}\) introduced in (4.5).

(2), (3) \([\text{Der}(\mathcal{J}), \text{Der}(\mathcal{J})] = \text{Der}(\mathcal{J}), [\text{Der}(\mathcal{C}), \text{Der}(\mathcal{J})] = 0\).

One can prove this in a straightforward way, e.g. by explicit computation.

(4) \([D, c \otimes x] = D(c) \otimes x\).

In order to prove this, let us write \(c = \alpha (\rho^+ - \rho^-) + v^+_k \epsilon^+_k \in \mathcal{C}_0\) (summed over \(\pm\) and \(k\)), and let us consider \(V^\pm \in \mathcal{C}\) with components \(v^\pm_i\). Then, we have to calculate:

\[
\begin{pmatrix}
  (a \otimes I) & A^+ \otimes I \\
  A^- \otimes I & 0
\end{pmatrix}
\begin{pmatrix}
  (aI \otimes x) & (V^+ \otimes x) \\
  (V^- \otimes x) & -(aI \otimes x)
\end{pmatrix} :=
\begin{pmatrix}
  C_{11} & C_{12} \\
  C_{21} & C_{22}
\end{pmatrix}.
\]

(5.11)

Let us calculate some terms separately:

\(A^+ \otimes I \circ V^- \otimes x = \left( \frac{1}{2} A^+_i v^-_i t(x) I - A^+_i v^-_i t(x) E_{ij} \right) \otimes I + I \otimes \left( \frac{1}{2} A^+_i v^-_i t(x) I - A^+_i v^-_i x \right) = -I \otimes A^+_i v^-_i x,\)

since \(t(x) = 0\) by hypothesis.

\((A^+ \otimes I) \times (V^+ \otimes x)) = \epsilon_{ijk} A^+_i v^+_j (2x - xt(I) I) = -(A^+ \wedge V^+) x,\)

once again because \(t(x) = 0\). Similarly with \(A^-\) and \(V^-\), hence

\((A^\pm \otimes I) \times (V^\pm \otimes x) = -(A^\pm \wedge V^\pm \otimes x)\).

Consequently, we obtain

\[
\begin{align*}
C_{11} &= I \otimes (-v^-_i A^+_i + A^-_i v^+_i x) \\
C_{12} &= (aV^+ - 2\alpha A^+ - A^- \wedge B^-) \otimes x \\
C_{21} &= (-V^- a + 2\alpha A^- - A^+ \wedge B^+) \otimes x \\
C_{22} &= I \otimes (v^-_i A^+_i - A^-_i v^+_i x),
\end{align*}
\]

(5.12)

which is the \(g_{\mathbb{S}}\) action on \(c \in \mathcal{C}_0\) introduced in (4.10) tensored with \(x\).

(5) \([E, a \otimes x] = a \otimes E(x)\) (this can be proved by explicit computation).

(6) \([c \otimes x, d \otimes y] = t(xy) D_{c,d} + 2(c \ast d) \otimes (x \ast y) + \frac{1}{2} t(c d)[x, y].\)
Let us use notations analogous to the ones in the proof of (4). Then, we have to compute:
\[
\begin{pmatrix}
\alpha I \otimes x & V^+ \otimes x \\
V^- \otimes x & -\alpha I \otimes x
\end{pmatrix} + \begin{pmatrix}
\beta I \otimes y & W^+ \otimes y \\
W^- \otimes y & -\beta I \otimes y
\end{pmatrix} = \begin{pmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{pmatrix}.
\] (5.13)

Let us calculate some terms separately:
\[
(V^+ \otimes x) \otimes (W^- \otimes y) = \left( \frac{1}{2}v^+_i w^-_j t(xy)I - v^+_i w^-_j t(xy)E_{ij} \right) \otimes I + I \otimes \left( \frac{1}{2}v^-_i w^+_j t(xy)I - v^-_i w^+_j t(xy)E_{ij} \right)
\]
\[
= \left( \frac{1}{2}(v^+_i w^-_j - w^+_i v^-_j) \mu(xy)I - (v^+_i w^-_j - w^+_i v^-_j) \mu(xy)E_{ij} \right) \otimes I + I \otimes \left( (\alpha \beta - \frac{1}{2}(v^-_i w^+_j + v^+_i w^-_j)) [x, y] + (v^-_i w^+_j - w^-_i v^+_j) \left( \frac{1}{2}(xy + yx) - \frac{1}{4}t(xy)I \right) \right)
\]
\[
C_{11} = \left( \frac{1}{2}(v^+_i w^-_j - w^+_i v^-_j) \mu(xy)I - (v^+_i w^-_j - w^+_i v^-_j) \mu(xy)E_{ij} \right) \otimes I + I \otimes \left( (\alpha \beta - \frac{1}{2}(v^-_i w^+_j + v^+_i w^-_j)) [x, y] + (v^-_i w^+_j - w^-_i v^+_j) \left( \frac{1}{2}(xy + yx) - \frac{1}{4}t(xy)I \right) \right)
\]
\[
C_{12} = (\alpha W^+ - \beta V^+) \otimes (xy + yx) + (V^- \otimes W^-) \otimes (xy + yx - t(xy)I)
\]
\[
= 2(\alpha W^+ - \beta V^+ + V^- \otimes W^-) \otimes \left( \frac{1}{2}(xy + yx) - \frac{1}{4}t(xy)I \right)
\]
\[
+ 2(\alpha W^+ - \beta V^+ - V^- \otimes W^-) \otimes \frac{1}{4}t(xy)I
\]
with similar results for $C_{21}$ and $C_{22}$.

Finally, for $c, d \in \mathbb{C}$, $c = \alpha (\rho^+ - \rho^-) + v^+_k e^+_k$ and $d = \beta (\rho^+ - \rho^-) + w^-_k e^+_k$ (summed over $\pm$ and $k$), $V^\pm, W^\pm \in \mathbb{C}^3$ with components $v^\pm_k, w^\pm_k$, it can be computed that:
\[
c \ast d = \frac{1}{2}(v^-_k w^+_k - v^+_k w^-_k) (\rho^+ - \rho^-) + (\pm \alpha w^\pm_k \mp \beta v^\pm_k + (V^\pm \otimes W^\mp) k) e^\pm_k
\]
\[
\frac{1}{2}t(cd) = \alpha \beta - \frac{1}{2}(v^-_k w^+_k - v^+_k w^-_k) (\rho^+ - \rho^-) + (\pm \alpha w^\pm_k \mp \beta v^\pm_k + (V^\pm \otimes W^\mp) k) e^\pm_k.
\] (5.14)

From (5.14) and (4.9) we obtain indeed the proof of (6).

This completes the proof that $\varphi$ (5.1) is the adjoint representation of $f_4$. $\square$

Notice that (5.1) reproduces the well known branching rule of the adjoint of $f_4$ with respect to its maximal and non-symmetric subalgebra $a^{(1)}_2 \oplus a^{(2)}_2$:
\[
52 = (8, 1) + (1, 8) + (3, 6) + (\bar{3}, 6).
\] (5.15)

It is here worth anticipating that in section 8 we prove the Jacobi identity in the more general case of $\mathfrak{g}$, which includes in an obvious manner this case of $f_4$. The validity of the Jacobi identity, together with the fact that the representation $\varphi$ fulfils the root diagram of $f_4$ (the proof is straightforward, and it can also be considered as a particular case of the proof given at the end of section 6 for $\mathfrak{e}_8$) proves in an alternative way that $\varphi$ is indeed a representation of $f_4$.

6. $n = 2$: Matrix representation of $\mathfrak{e}_6$

We present in this section the representation $\varphi$ of $\mathfrak{e}_6$ in the form of a matrix. We have to complexify the Jordan structure with respect to $f_4$. We introduce the imaginary unit $\mathbf{i}$—leaving $\mathbf{i}$ as the imaginary unit of the base field. In particular, $\mathbf{J}_2$ is Hermitian with respect
to the \( u_1 \)-conjugation, and we are going to denote such an Hermitian conjugation with the symbol \( \dagger \) throughout.

In a similar fashion to (5.1), for \( \mathfrak{f} \in \mathfrak{e}_6 \), we thus write:

\[
\varrho(f) = \begin{pmatrix} a \otimes I + I \otimes a_1 & x^+ \\ x^- & -I \otimes a_1^T \end{pmatrix}
\]  

(6.1)

where \( a \in a_3^1, a_1 \in a_3^{(1)} \oplus u_1 a_3^{(2)}, a_1^\dagger \) is the Hermitian conjugate of \( a_1 \) (with respect to \( u_1 \)), \( I \) is the \( 3 \times 3 \) identity matrix, \( x^+ \in \mathbb{C}^3 \otimes j_3, x^- \in \mathbb{C}^3 \otimes j_3^* \):

\[
x^+ = \begin{pmatrix} x^+_1 \\ x^+_2 \\ x^+_3 \end{pmatrix}, \quad x^- = (x^-_1, x^-_2, x^-_3), \quad x^+_i \in j_3, \quad x^-_i \in j_3^*, \quad i = 1, 2, 3.
\]

The commutator of two such matrices is the same as for \( \mathfrak{f}_4 \), with \( \dagger \) instead of \( T \) (cfr. (5.2)):

\[
\left[ \begin{pmatrix} (a \otimes I + I \otimes a_1) & x^+ \\ x^- & -I \otimes a_1^T \end{pmatrix}, \begin{pmatrix} (b \otimes I + I \otimes b_1) & y^+ \\ y^- & -I \otimes b_1^T \end{pmatrix} \right] := \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}
\]  

(6.2)

where:

\[
C_{11} = [a, b] \otimes I + I \otimes [a_1, b_1] + x^+ \circ y^- - y^+ \circ x^-
\]

\[
C_{12} = (a \otimes I) y^+ - (b \otimes I) x^+ + (I \otimes a_1) y^+ + y^+ (I \otimes a_1^T) - (I \otimes b_1) x^+ - x^+ (I \otimes b_1^T) + x^- \times y^-
\]

(6.3)

\[
C_{21} = -y^- (a \otimes I) + x^- (b \otimes I) - (I \otimes a_1^T) y^- - y^- (I \otimes a_1) + (I \otimes b_1^T) x^- + x^- (I \otimes b_1) + x^+ \times y^+
\]

\[
C_{22} = I \otimes [a_1, b_1^T] + x^- \bullet y^+ - y^- \bullet x^+
\]

with products defined as in (5.4).

Notice that:

1. \( x \in j_3^* \) is a Hermitian matrix (with respect to \( u_1 \)) over the complex field (with imaginary unit \( i \));
2. by writing \( a_1 \in a_3^{(1)} \oplus u_1 a_3^{(2)} \) we state that \( a_1 \) is the sum of a traceless skew-Hermitian matrix and a traceless hermitian matrix (namely a matrix in \( j_3^0 \), with \( J = j_3^3 \)), hence \( a_1 \in \mathfrak{sl}(3, \mathbb{C}) \) is a generic \( 3 \times 3 \) traceless matrix over \( \mathbb{C} \otimes \mathbb{C} \);
3. writing \( x^+ \circ y^- := c \otimes I + I \otimes c_1 \) we have that both \( c \) and \( c_1 \) are traceless, \( c \in a_2 \) and \( c_1 \in \mathfrak{sl}(3, \mathbb{C}) \), and \( y^- \bullet x^+ = I \otimes c_1^T \); if \( x, y \in j_3^3 \), then \( C \ni t(x, y) = t(xy) \), and \( c_1 \) has indeed 16 (complex) parameters. It is here worth anticipating that this will not be the case for \( j_3^3 \) and \( j_3^3 \), as we shall stress in the next sections on \( \mathfrak{e}_7 \) and \( \mathfrak{e}_8 \);
4. terms like \( (I \otimes a_1) y^+ + y^+ (I \otimes a_1^T) \) are in \( \mathbb{C}^3 \otimes j_3^* \), namely they are matrix valued vectors with Hermitian matrix elements;
5. the correspondence between matrix elements in (6.1) and Tits’ construction is similar to the one shown in (5.7) and is omitted here;
6. the Jacobi identity can be demonstrated as a particular case of the proof for \( \mathfrak{e}_7 \), shown in section 8. The validity of the Jacobi identity, together with the fact that the representation \( \varrho \) fulfills the root diagram of \( \mathfrak{e}_6 \), as we show next, prove that \( \varrho \) (6.1) is the adjoint representation of \( \mathfrak{e}_6 \).
As regards the counting of parameters, we refer to our comment in the introduction about the use of $\mathbb{C}$ as base field.

We end this section with the correspondence between the roots of $e_6$ and the matrix elements in (6.1).

The roots of $e_6$ can be written in terms of an orthonormal basis $\{ k_i \mid i = 1, \ldots, 6 \}$ as, $[42]$:

$$e_6 \text{ 72 roots}$$

$$\pm k_i \pm k_j \quad i \neq j = 1, \ldots, 5 \quad 4 \times \binom{5}{2} = 40$$

$$\frac{1}{2}(\pm k_1 \pm k_2 \pm k_3 \pm k_4 \pm k_5 \pm \sqrt{3} k_6)^* \quad 2^5 = 32$$

* [odd number of + signs].

We refer to figure 4 and write the roots associated with the highest weight $J^2_3$ as:

$$\begin{align*}
-k_3 & \pm k_4, \quad -k_1 \pm k_5, \quad k_2 + k_3 \\
\frac{1}{2}(-k_1 + k_2 + k_3 + k_4 - k_5 - \sqrt{3} k_6) & \\
\frac{1}{2}(-k_1 + k_2 + k_3 - k_4 + k_5 + \sqrt{3} k_6) & \\
\frac{1}{2}(-k_1 + k_2 - k_3 - k_4 - k_5 + \sqrt{3} k_6) & \quad (6.4)
\end{align*}$$

The other $J^2_3$’s correspond to a cyclic permutation of $k_1, k_2, k_3$, and each $J^2_3$ in a Jordan pair $(J^1_3, J^2_3)$ corresponds to the roots of a $J^1_3$ with opposite signs.

The subalgebra $g^2_2 \cong a_2 \oplus a_2$ has roots:

$$\begin{align*}
\pm (k_4 + k_5) \\
\pm \frac{1}{2}(k_1 + k_2 + k_3 - k_4 - k_5 - \sqrt{3} k_6) & \quad (6.5)
\end{align*}$$

and

$$\begin{align*}
\pm (k_4 - k_5) \\
\pm \frac{1}{2}(k_1 + k_2 + k_3 - k_4 + k_5 + \sqrt{3} k_6) & \\
\pm \frac{1}{2}(k_1 + k_2 + k_3 + k_4 - k_5 + \sqrt{3} k_6) & \quad (6.6)
\end{align*}$$

Furthermore, the roots of $a^1_2$ relate in the standard way to the matrix elements of $a \otimes I$ in (6.1). The roots of each Jordan Pair project on the plane of $a^1_2$ according to figure 4, as it can be easily checked. Therefore, we are only left with the roots corresponding to $\mathfrak{sl}(3, \mathbb{C})$ and to
The algebras \(a_2^{(1)}\) and \(a_2^{(2)}\) are related to Tits' construction. Now, we twist them in the following way: we denote by \(\rho_{\pm} := i(1 \pm i u_1)\) and introduce \(a_2^{\pm} = \rho_i^\pm \text{sl}(3, C)\). Then, it follows that \((\rho_+)^2 = \rho_+\) and \(\rho_+ \rho^- = 0\). If \(a \in \text{sl}(3, C)\) then \(a = a^+ + a^-\), \(a^\pm = \rho^\pm a\) and, if we write \(a = a_r + u_1 a_i\) (where \(a_r\) and \(a_i\) are the self-conjugate parts of \(a\) with respect to \(u_1\)), one can easily check that \(a^\pm = (a_r \mp i a_i)\rho^\pm\). Moreover, for \(a, b \in \text{sl}(3, C)\) then \([a, b] = [a^+, a^-] + [b^+, b^-] = [a^+, b^+] + [a^-, b^-]\). Therefore \(a_2^+\) and \(a_2^-\) are both isomorphic to \(a_2\) and \(\text{sl}(3, C) \cong a_2^+ \oplus a_2^-\).

We now write \(x \in J_2 = a_i E_{ii} + a_{i+1} E_{i,i+1} + a_{i+1,j+1} E_{i+1,i,j+1}\), where the indices run over \(1, 2, 3\) \(\text{mod}(3), \alpha \in C\) and \(a_{\alpha} \in C \otimes C\). Obviously \(a_i = a_i(\rho^+ + \rho^-)\) and \(ai_j = a_{ij}(\rho^+ + \rho^-)\). The matrix \(x\) is therefore in the linear span of the nine generators

\[
X_i = E_{ii}, \quad X_{i+1,i} := \rho_{\pm} E_{i,i+1} + \rho^\mp E_{i+1,i}, \quad (6.7)
\]

We fix the Cartan subalgebra of \(a_2^+ \oplus a_2^-\) in the obvious way, by introducing the Cartan generators

\[
H_{1,2} := \rho_{\pm} H_{1,2}, \quad H_1 = \frac{\sqrt{2}}{3} (E_{11} - E_{22}), \quad H_2 = \frac{\sqrt{2}}{3} (E_{11} + E_{22} - 2E_{33}). \quad (6.8)
\]

We let \(H_{1,2}^+\) and \(H_{1,2}^-\) correspond to the axes along the directions of the unit vectors \(\pm (k_1 + k_2 + k_3)\) and \(H_1^+, H_2^+\) to \(\pm (k_4 + k_5)\) respectively. Consequently, we are all set to establish the correspondence between the roots and the generators of the highest weight \(J_2^+\), by exploiting the commutation rule (6.2). This is shown in table 2.

We thus reproduce the well known branching rule of the adjoint of \(e_8\) with respect to its maximal and non-symmetric subalgebra \(a_2^+ \oplus a_2^- \oplus a_2^+ \oplus a_2^-\):

\[
\{ 78 \} = \{ 8, 1, 1 \} + \{ 1, 8, 1 \} + \{ 1, 1, 8 \} + \{ 3, 3, 3 \} = \{ 3, 3, 3 \}, \quad (6.9)
\]

with the exact correspondence of each single root with a matrix elements of (6.1).

It is intriguing to remark the quantum information meaning of the maximal non-symmetric embedding of \(a_2^+ \oplus a_2^+ \oplus a_2^-\) into \(e_8\) has been investigated in [68], within the context of the so-called “black hole—qubit correspondence” [28].

7. \(n = 4\): Matrix representation of \(e_7\)

In the present section, we briefly mention how the results of the previous sections can be extended to the case of \(e_7\). Nothing different really occurs, as of course the Jordan algebras

| Table 2. Roots and \(a_2^+ \oplus a_2^-\) weights of the highest weight \(J_2^+\). |
|----------------|----------------|----------------|
| Root          | Generator \(a_2^+\) weights | \(a_2^-\) weights |
| \(-k_1 + kodel\) | \(X_1\) | \(\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}\) | \(\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}\) |
| \(\frac{1}{2}(-k_1 + kodel)\) | \(X_{i1}\) | 0, \(-\frac{\sqrt{2}}{3}\) | 0, \(-\frac{\sqrt{2}}{3}\) |
| \(-k_1 + kodel\) | \(X_{i2}\) | \(-\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}\) | \(-\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}\) |
| \(\frac{1}{2}(-k_1 + kodel)\) | \(X_{i1}\) | \(\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}\) | 0, \(-\frac{\sqrt{2}}{3}\) |
| \(kodel\) | \(X_{i3}\) | 0, \(-\frac{\sqrt{2}}{3}\) | \(-\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}\) |
| \(\frac{1}{2}(-k_1 + kodel)\) | \(X_{i2}\) | \(-\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}\) | \(-\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}\) |
involved are of the type \( J_3^3 \), whose elements associate with respect to the standard product of matrices.

For \( f \in e_7 \), we write:

\[
\varrho(f) = \begin{pmatrix}
  a \otimes I + I \otimes a_1 & x \\
  z & -I \otimes a_1
\end{pmatrix}
\]

(7.1)

where \( a \in a_1, a_1 \in a_3, a_1^* \) is the Hermitian conjugate of \( a_1 \) (with respect to the quaternion units), \( I \) is the \( 3 \times 3 \) identity matrix, \( x \in C^3 \otimes J_3^3 \), \( z \in C^3 \otimes \bar{J}_3^3 \):

\[
x = \begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix} \quad z = (z_1, z_2, z_3), \quad x_1 \in J_3^4, z_i \in \bar{J}_3^4, \quad i = 1, 2, 3.
\]

The commutator of two such matrices is formally the same as for \( e_6 \) (cfr. (6.2)).

A few remarks are in order:

1. since \( a_3 \simeq \mathfrak{sl}(3, Q) \) (cfr. e.g. [19, 69, 70]), then \( a_1 \in a_3 \) can be written as the sum of a skew-Hermitian matrix and a traceless Hermitian matrix in \( J_0 \), with \( J = J_3^3 \); it is worth noting that \( \mathfrak{sl}(3, Q) \) has 35 parameters, only one less than \( gl(3, Q) \) since the trace that is taken away from \( gl(3, Q) \) is in \( C \), not in \( C \otimes Q \);

2. writing \( x^+ \otimes y^+ := c \otimes I + I \otimes c_1 \), we have that both \( c \) and \( c_1 \) are traceless, \( c \in a_2 \) and \( c_1 \in \mathfrak{sl}(3, Q) \) (and indeed this latter has 35 complex parameters), and \( y^+ \otimes x^+ = I \otimes c_1^* \); according to the previous point, the trace that we take away with the term \( I \otimes \frac{1}{2} t(x^+, y^+) I \) in (5.4) is in \( C \) and \( t(x, y) \neq t(yx) \) in general, due to non-commutativity;

3. terms like \( (I \otimes a_1) y^+ + y^+ (I \otimes a_1^*) \) are in \( C^3 \otimes J_3^3 \), namely they are matrix-valued vectors with Hermitian matrix elements;

4. the correspondence between matrix elements in (7.1) and Tits’ construction is similar to the one shown in (5.7) (and commented in section 6), and it is omitted here;

5. the Jacobi identity is demonstrated in section 8;

6. the adjoint action in \( e_7 \) implicitly provides us with the action of \( e_6 \) on the fundamental representations \( 27 \) and \( \bar{27} \), since \( e_7 \simeq e_6 \oplus C \oplus (J_3^3, \bar{J}_3^3) \).

This last point deserves to be commented a little further, since it allows us to write the action of \( e_7 \) by means of matrices that associate with respect to the standard matrix product instead of non-associative matrices of \( J_3^3 \). In a way, we are nothing but doubling the procedure already implemented for \( e_6 \) in section 4, where we have realized the octonions within a Zorn-type matrix, which was the basic structure for building up our representations. Here, we have to branch \( J_3^3 \) into associative matrices, and still recover non-associativity through a non-standard matrix product.

As a first step, we consider the \( e_6 \) subalgebra. We select an imaginary unit in \( Q \), say \( u_1 \), and restrict \( J_3^3 \) to \( J_3^3 \) accordingly. Then, we pick two \( a_3 \)’s inside \( a_3 \) by setting \( a_3^\pm = \rho^\pm \mathfrak{sl}(3, C) \subset \mathfrak{sl}(3, Q) \), and \( \rho^\pm = \frac{1}{2}(1 \pm iu_1) \). We thus get the following \( e_6 \) subalgebra of matrices:

\[
\begin{pmatrix}
  a \otimes I + I \otimes (\rho^+ a_1^+ \oplus \rho^- a_1^-) & x \\
  z & -I \otimes (\rho^- a_1^T \oplus \rho^+ a_1^-)
\end{pmatrix}
\]

(7.2)

where \( a_1^\pm \in a_3 \) and the vectors \( x, z \) have components \( x_i \in J_3^3, z_i \in \bar{J}_3^3 (i = 1, 2, 3) \).

We now introduce the nilpotent elements \( \varepsilon^\pm := \rho^\pm u_2 \), so that a generic quaternion can be written as \( Q \equiv q = q_0^\pm \rho^\pm + q_0^\pm \varepsilon^\pm \). The Jordan pair \((27, \bar{27})\) reads then:

\[
\begin{pmatrix}
  (\varepsilon^+ \eta^+ + \varepsilon^- \eta^-) & \varepsilon^+ \xi^+ + \varepsilon^- \xi^- \\
  \varepsilon^+ \xi^+ + \varepsilon^- \xi^- & I \otimes (\varepsilon^+ \eta^T + \varepsilon^- \eta^-)
\end{pmatrix}
\]

(7.3)
where $\eta^\pm \in \mathfrak{gl}(3)$ are complex $3 \times 3$ matrices, and $\zeta^+, \zeta^-, \xi^+, \xi^- \in \mathfrak{b}_1$ are skew-symmetric complex matrix-valued vectors.

As a convention, we associate the 27 with all the ‘+’ signs in (7.3), and thus the $\overline{27}$ with the ‘−’ signs.

The only parameter left with respect to an element of $\mathfrak{e}_7$ is the sum of the diagonal elements of type $\lambda \mathbf{u}_1 = \lambda(\rho^+ - \rho^-)$, ($\lambda \in \mathbb{C}$), which is associated to the generator $C$ in the decomposition of $\mathfrak{e}_7$ (see point 6 above).

The action of $\mathfrak{e}_8$ on its 27 is:

$$\begin{bmatrix} a \otimes I + I \otimes \rho^+ a_1^+ & x \\ -I \otimes \rho^+ a_1^+ & z \end{bmatrix} \cdot \begin{bmatrix} I \otimes \varepsilon^+ \eta & \varepsilon^+ \xi \\ \varepsilon^+ \xi & I \otimes \varepsilon^+ \eta^T \end{bmatrix} := \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$ (7.4)

where, for $x_i, z_i \in J_3^1$, $x_i = x_{i+}, \rho^+ + x_{i-}, \rho^-$, $z_i = z_{i+}, \rho^+ + z_{i-}, \rho^-$. $C_{11} = \varepsilon^+ (I \otimes (a_1^+ \eta - a_1^- \eta)) - (x_i \xi_i - \xi_z z_i)$.

$C_{12} = \varepsilon^+ (a \otimes I) \xi + (I \otimes a_1^+) \zeta + (I \otimes a_1^+) + x_i (I \otimes \eta^T) - (I \otimes \eta)x_i + z \times \xi)$

$C_{21} = \varepsilon^+ (- \xi (a \otimes I) - (I \otimes a_1^+) \zeta - \xi (I \otimes a_1^-) + (I \otimes \eta^T) x + z (I \otimes \eta) + x \times \xi)

C_{22} = \varepsilon^+ (I \otimes (-a_1^T \eta^T + \eta^T a_1^+)) - (z_i \xi_i - \xi_z z_i)$.

Notice that if $x \in J_3^1$, $x = x_+ \rho^+ + x_-, \rho^-$, then $x = x_+ \rho^+ + x_- \rho^-$ shows that $x_+ = x_-$. Therefore $x \cdot (\varepsilon^+ \xi) = (x, \zeta + \xi x) \varepsilon^+ \xi$ where $(x, \zeta + \xi x)$ is skew-symmetric. In particular, $t(x, \zeta) = 0$. It also holds that $(x_i, \xi_i - \xi_z z_i) = (z_i, \xi_i - \xi_z z_i)$, thus showing that $C_{22} = -C_{11}$.

It can also be shown that $C_{12}$ and $C_{21}$ are the product of $\varepsilon^+$ with a skew-symmetric complex matrix.

Analogous calculation can be performed for the $\overline{27}$.

The action of the $C$ generator $\lambda(\rho^+ - \rho^-)$ on the 27 and on the $\overline{27}$ is just a multiplication by $2\lambda$ on the 27 and by $-2\lambda$ on the $\overline{27}$.

We thus reproduce the well-known branching rule of the adjoint of $\mathfrak{e}_7$ with respect to its maximal and non-symmetric subalgebra $\frak{a}_2 \oplus \frak{a}_8$:

$$133 = (8, 1) + (1, 35) + (3, 1\overline{5}) + (\overline{3}, 15).$$ (7.6)

### 8. Jacobi identity for $\frak{f}_4, \frak{e}_6, \frak{e}_7$

An equivalent way of proving that $\varrho$ (given by (5.1),(6.1),(7.1)) is a representation of $\frak{f}_4, \frak{e}_6, \frak{e}_7$ respectively, is to directly prove the Jacobi identity for $\varrho$, and check that one gets the root diagram of the corresponding Lie algebra.

We consider the most general setting of $\mathfrak{e}_7$, which involves the Jordan algebra $\mathfrak{J}_3^1$ with non-commutative, but associative matrix elements. The $\varrho(\mathfrak{f}_4)$ and $\varrho(\mathfrak{e}_6)$ cases are obviously included as particular instances.

Recalling (7.1), we thus write:

$$\varrho(\mathfrak{f}_4) = \begin{bmatrix} a \otimes I + I \otimes a_1 & A^+ \\ A^- & -I \otimes a_1^+ \end{bmatrix}$$ (8.1)

where $a \in \mathfrak{a}_2, a_1 \in \mathfrak{a}_8 \simeq \mathfrak{su}(3, \mathbb{Q})$ and $A^+, A^-$ are three-vectors with elements in $\mathfrak{J}_3^1$. Similarly, one can define $\varrho(\mathfrak{f}_2)$ and $\varrho(\mathfrak{f}_3)$, by respectively replacing $a \to b$ and $a \to c$ in (8.1), and:

$$[[\varrho(\mathfrak{f}_1), \varrho(\mathfrak{f}_2)], \varrho(\mathfrak{f}_3)] + \text{cyclic permutations} := \begin{bmatrix} \mathfrak{3}_{11} & \mathfrak{3}_{12} \\ \mathfrak{3}_{21} & \mathfrak{3}_{22} \end{bmatrix}.$$ (8.2)
In order for the Jacobi identity to hold for the matrix realization (8.1) of the adjoint of \( e_7 \), we have to prove that \( \mathcal{J}_{11} = \mathcal{J}_{12} = \mathcal{J}_{21} = \mathcal{J}_{22} = 0 \).

After some algebra, one computes:

\[
\mathcal{J}_{11} = [(a, b), c] \otimes I + I \otimes [(a_1, b_1), c_1] + (A^+ \otimes B^- - B^+ \otimes A^-)(c \otimes I + I \otimes c_1) \\
\quad - (c \otimes I + I \otimes c_1)(A^+ \otimes B^- - B^+ \otimes A^-) \\
\quad + (a \otimes I)B^+ - (b \otimes I)A^+ + (I \otimes a_1)B^+ + B^+(I \otimes a_1) \\
\quad - (I \otimes b_1)A^+ - A^+(I \otimes b_1) + A^- \otimes B^- \otimes C^- \\
\quad - C^+ \otimes (-B^- (a \otimes I) + A^- (b \otimes I) - (I \otimes a_1)B^+ - B^- (I \otimes a_1) \\
\quad + (I \otimes b_1)A^+ + A^- (I \otimes b_1) + A^+ \otimes B^+) + \text{cyclic permutations.}
\]

The first two terms of (8) vanish upon cyclic permutations because of the Jacobi identity in \( a_2 \) and \( a_5 \). Let us consider then the terms in the rhs of (8) containing \( A^+, B^-, c \); by denoting by \( a_k, b_k \in J^1_3 \), \( k = 1, 2, 3 \) the components of \( A^+ \) and \( B^- \), respectively, one computes that:

\[
[A^+ \otimes B^- , c \otimes I] + ((c \otimes I)A^+) \otimes B^- - A^+ \otimes (B^- (c \otimes I)) \\
= \left[ \left( \frac{1}{2}t(a_1, b_1)I - t(a_1, b_1)E_{ij} \right) \otimes I, c \otimes I \right] + \left[ I \otimes \left( \frac{1}{2}t(a_1, b_1)I - a_1b_1 \right), c \otimes I \right] \\
\quad + \left( \frac{1}{2}t(c_1a_2, b_1)I - t(c_1a_2, b_1)E_{ij} \right) \otimes I + I \otimes \left( \frac{1}{2}t(c_1a_2, b_1)I - c_1a_2b_1 \right) \\
\quad - \left( \frac{1}{2}t(a_1, b_1c_2)I - t(a_1, b_1c_2)E_{ij} \right) \otimes I + I \otimes \left( \frac{1}{2}t(a_1, b_1c_2)I - a_1b_1c_2 \right) \\
\quad = \left( -t(a_1, b_1)E_{ij}c + t(a_1, b_1)cE_{ij} - t(c_1a_2, b_1)E_{ij} + t(a_1, b_1)E_{ij} \right) \otimes I \\
\quad = \left( -t(a_1, b_1)c_2 + t(a_1, b_1)cE_{ij} - t(a_1, b_1)c_2 + t(a_1, b_1)cE_{ij} \right) \otimes I \\
\quad = 0.
\]

Next, we consider the terms in the rhs of (8) containing \( A^+, B^-, c_1 \). They read:

\[
[A^+ \otimes B^- , I \otimes c_1] + ((I \otimes c_1)A^+ + A^+(I \otimes c_1^+)) \otimes B^- - A^+ \otimes ((I \otimes c_1^+)B^- + B^- (I \otimes c_1)) \\
= I \otimes (c_1a_2b_1 - a_2b_1c_1) + \left( \frac{1}{2}t(c_1a_2, c_1) + (c_1, e_3)I \right) - t(c_1a_2 + a_2c_1, b_1)E_{ij} \otimes I \\
\quad + I \otimes \left( \frac{1}{2}t(c_1a_2 + a_2c_1, b_1I - (c_1a_2) - a_2c_1 \right) \\
\quad - \left( \frac{1}{2}t(a_1, c_1^+b_1 + b_1c_1)I - t(a_1, c_1^+b_1 + b_1c_1)E_{ij} \right) \otimes I \\
\quad - I \otimes \left( \frac{1}{2}t(a_1, c_1^+b_1 + b_1c_1)I - a_1(c_1^+b_1 + b_1c_1) \right) \\
\quad = I \otimes \left[ \frac{1}{2} \left( t(c_1a_2 + a_2c_1, b_1) - t(a_1, c_1^+b_1 + b_1c_1)I \\
\quad - (c_1a_2 + a_2c_1, b_1) - t(a_1, c_1^+b_1 + b_1c_1) \right) \right].
\]

In order to prove that the rhs of (8) is zero, we write \( \mathfrak{sl}(3, \mathbb{Q}) \supseteq c_1 = h + s \), where \( h \in J^1_3 \) is Hermitian, and \( s \) skew-Hermitian (with respect to quaternion conjugation). Note that the action \( x \mapsto sx + xs^\dagger = sx - xs \) is a derivation in \( J^3_3 \). Therefore, by exploiting the identities [43, 65]:

\[
t(x, y \cdot z) = t(z, x \cdot y) \\
t(Dx, y) + t(x, Dy) = 0 \quad \text{where } D \text{ is a derivation in } J^4_3
\]

one proves that the terms under consideration in the rhs of (8) sum up to zero.

Finally, we consider terms in the rhs of (8) which contain structures like \( (A^- \times B^-) \otimes C^- \); they read:
\[(A^- \times B^-) \circ C^- + (B^- \times C^-) \circ A^- + (C^- \times A^-) \circ B^-\]
\[= \epsilon_{ijk} \left( \frac{1}{2} t(a_i \# b_k, c_j) I - t(a_i \# b_k, c_j) E_{ij} \right) \otimes I \]
\[+ I \otimes \epsilon_{ijk} \left( \frac{1}{2} t(a_i \# b_k, c_j) I - t(a_i \# b_k, c_j) \right) + \text{cyclic permutations} \]
\[:= M^{(1)} \otimes I + I \otimes M^{(2)}. \tag{8.4}\]

In order to show that \(M^{(1)} = M^{(2)} = 0\), we observe, after [43], that \(t(a \# b, c)\) is symmetric in \(a, b, c\). Let us consider \(M^{(1)}\) first. For \(i \neq j\), then either \(j = \ell\) or \(j = k\). The coefficient of \(E_{ij}\) is therefore:
\[\epsilon_{ijk} (t(a_i \# b_k, c_j) - t(a_i \# b_k, c_j) + t(b_j \# c_k, a_j) - t(b_j \# c_k, a_j)) + t(c_k \# a_j, b_j) = 0.\]

For \(i = j\), by summing over \(i, \ell, k\) and using the notation \(\tau_{\ell ki} := t(a_i \# b_k, c_j) + t(b_j \# c_k, a_i) + t(c_k \# a_j, b_j)\), one can easily check that:
\[\epsilon_{ijk} \tau_{\ell ki} = \epsilon_{2ik} \tau_{i2i} = \epsilon_{3ik} \tau_{i3i} := \omega.\]

Thus:
\[\epsilon_{ijk} \tau_{\ell ki} \left( \frac{1}{2} I - E_{ii} \right) = \omega (I - E_{11} - E_{22} - E_{33}) = 0.\]

This proves that \(M^{(1)} = 0\). For what concerns \(M^{(2)}\), we observe that \(\frac{1}{2} t(x \# y, z) I - (x \# y) z + [\text{cyclic permutations}]\) is linear and symmetric in \(x, y, z\). It is indeed the polarization of (2.6), hence it is zero, implying that \(M^{(2)} = 0\). We stress that it is crucial to have associativity with respect to the standard matrix product of elements in \(J^3_1\), in order to apply the polarization statement; we do need in particular \(x^2 x = x^2 = x^2 \cdot x\), which does indeed hold in the associative case.

Analogous calculations for the other terms in the rhs of (8) involving \([B^+, A^-, c], [B^+, A^-, c_1], [B^+, A^+, C^+]\) plus their cyclic permutations prove that \(\tilde{J}_{12} = 0\).

Next, we proceed to consider \(\tilde{J}_{12}\) which, after some algebra, can be computed to read:
\[\tilde{J}_{12} = ([a, b] \otimes I + I \otimes [a_i, b_i]) + A^+ \circ B^- - B^+ \circ A^-) C^-\]
\[-((a \otimes I) B^- - (b \otimes I) A^+) + (I \otimes a_i) B^+ + B^+ (I \otimes a_i)\]
\[-(I \otimes b_i) A^+ - A^+ (I \otimes b_i) + A^- \times B^- (I \otimes c_i)\]
\[-(c \otimes I + I \otimes c_i) ((a \otimes I) B^- - (b \otimes I) A^+) + (I \otimes a_i) B^+ + B^+ (I \otimes a_i)\]
\[-(I \otimes b_i) A^+ - A^+ (I \otimes b_i) + A^- \times B^-\]
\[-C^- (I \otimes [a_i, b_i]) + A^- \circ B^- - B^+ \circ A^-\]
\[+(-B^- (a \otimes I) A^- (b \otimes I) - (I \otimes a_i) B^- - B^- (I \otimes a_i)\]
\[+(I \otimes b_i) A^+ - A^+ (I \otimes b_i) + A^+ \times B^-) \times C^- + \text{cyclic permutations}.\]

Many terms cancel out trivially, and one remains with terms of the following three types:
\[(1) \quad (I \otimes c_i) (A^- \times B^-) + (A^- \times B^-) (I \otimes c_i)\]
\[+ ((I \otimes c_i) A^-) + (A^- (I \otimes c_i)) \times B^- - (I \otimes c_i) B^- + (B^- (I \otimes c_i)) \times A^-\]
\[(2) \quad - (c \otimes I) (A^- \times B^-) - (A^- (c \otimes I)) \times B^- + (B^- (c \otimes I)) \times A^-\]
\[(3) \quad (A^- + B^-) \times C^- + (B^+ \otimes C^-) A^+ - (A^+ \otimes C^-) B^+ + A^+ (C^- \bullet B^+) - B^+ (C^- \bullet A^+)\]
where we remark that the first two terms show the action of the \(a_3\) and \(a_{12}\) subalgebras as derivations.

Let us analyze each of the terms (1)–(3) separately.
(1) Writing this term explicitly, one obtains:
\[ \epsilon_{ijk} (a_j \# b_k + (a_j \# b_k) c_i - \epsilon_{ijk} (c_i a_j + a_i c_j) \# b_k - \epsilon_{ijk} (c_i b_k + b_i c_j) \# a_j) \]
\[ = \epsilon_{ijk} [c_i (a_j \# b_k) + (a_j \# b_k) c_i + (c_i a_j + a_i c_j) \# b_k + (c_i b_k + b_i c_j) \# a_j]. \]

In order to show that the expression in brackets is identically zero, we write \( \mathfrak{sl}(3, \mathbb{Q}) \ni c_1 = h + s \), namely as the sum of a traceless Hermitian matrix \( h \) and of a skew-Hermitian matrix \( s \). Since the expression under consideration is linear in \( c_1 \), we can consider the two contributions of \( h \) and \( s \) separately. The contribution of \( h \) reads:
\[
4((a \cdot b) \cdot h + (a \cdot h) \cdot b + (b \cdot h) \cdot a) - 4(t(a) b \cdot h + (b \cdot h) a + (a \cdot h) t h) + (a(b) - (a) t(b)) h \]
\[ - 2(t(a) b h + (a \cdot h) t b) - (a \cdot h) t b h + (a) t(b) + (a, D(b)) = 0. \]

(2) We can write this expression as:
\[ -(\epsilon_{ijk} c_{ij} + \epsilon_{i\ell k} c_{j\ell} - \epsilon_{i\ell j} c_{k\ell}) (a_j \# b_k). \]

For \( \ell = k \) or \( \ell = j \), or \( k = j \), the first round bracket trivially vanishes. For \( \ell \neq j \neq k \neq \ell \), it can be written as \( \epsilon_{ijk} c_{ij} + \epsilon_{i\ell k} c_{j\ell} + \epsilon_{i\ell j} c_{k\ell} = 0 \), since \( t(e) = 0 \).

(3) Explicit calculation shows that the \( \ell \)th component of this term reads:
\[
(a \cdot b) \cdot h c_{\ell} - (a \cdot b) h c_{\ell} + t(b, c) a_{\ell} + t(a, c) b_{\ell} - t(a, c) a_{\ell} - a_{\ell} c_{\ell} b_{\ell} - b_{\ell} c_{\ell} a_{\ell} \]
\[ = -[c_{\ell}, c_{\ell}, b_{\ell}] + [a_{\ell}, c_{\ell}, b_{\ell}] - a_{\ell} c_{\ell} b_{\ell} - b_{\ell} c_{\ell} a_{\ell} + a_{\ell} c_{\ell} b_{\ell} + b_{\ell} c_{\ell} a_{\ell} = 0, \]

where the triple product \([x, y, z] := V_{x, y, z}\) has been introduced in section 2 and, in the associative case we are considering here: \([x, y, z] = xyz + zyx\), thus implying that also the term 3) vanishes.

This ends the proof of the fact that \( \mathfrak{g}_{12} = 0 \).

Analogous calculations show that also \( \mathfrak{g}_{11} = \mathfrak{g}_{22} = 0 \), thus proving the Jacobi identity for the matrix realization \( \mathfrak{g} \) (7.1) of the adjoint of \( \mathfrak{e}_7 \), implying the Jacobi identity for the matrix realizations \( \mathfrak{g} \) (5.1) and (6.1) of the adjoint of \( \mathfrak{f}_4 \) and \( \mathfrak{e}_6 \), respectively.

9. \( n = 8 \): Matrix representation of \( \mathfrak{e}_8 \)

Finally, we consider the case of \( \mathfrak{e}_8 \), the largest finite-dimensional exceptional Lie algebra.

We use the notation \( L_x := x \cdot z \) and, for \( x \in \mathbb{C}^3 \otimes \mathfrak{f}_4^3 \) with components \( (x_1, x_2, x_3) \), \( L_x \in \mathbb{C}^3 \otimes \mathfrak{f}_4^3 \) denotes the corresponding operator-valued vector with components \( (L_{x_1}, L_{x_2}, L_{x_3}) \).

We can write an element \( a_i \) of \( \mathfrak{e}_8 \) as \( a_i = L_x + \sum [L_{x_i}, L_{x_i}] \) where \( x, x_i \in \mathfrak{f}_4^3 \) (\( i = 1, 2, 3 \)) and \( t(x) = 0, [60, 65] \). The adjoint is defined by \( a_i^\dagger := L_x - [L_{x_i}, L_{x_i}] \). Notice that the operators
It should be stressed that the products occurring in (9) is a derivation in the Jordan Pair \((J^3_3, J^3_3)\), and it is here useful to recall the relationship between the structure group of a Jordan algebra \(J\) and \(\text{Aut}(V)\). In our case, for \(g = 1 + \epsilon (L_x + F)\), at first order in \(\epsilon\) we get (namely, in the tangent space of the corresponding group manifold) \(U_{\frac{g}{g(1)}}^{-1} g = 1 + \epsilon (-L_x + F) + O(\epsilon^2)\).

Next, we introduce a product \(\ast\), such that \(L_x \ast L_y := L_{x y} + [L_x, L_y]\), \(F \ast L_x := 2FL_x\), \(L_x \ast F := 2L_x F\) for \(x, y \in J^3_3\), including each component \(x\) of \(X \in C^3 \otimes J^3_3\) and \(y\) of \(Y \in C^3 \otimes J^3_3\). By denoting with \([\ : \ :]\) the commutator with respect to the \(\ast\) product, we also require that \(L_x \ast L_y = L_{x y} + 2L_{x y} = 2L_{xy}\) and \(L_x \ast F = 2L_x F\). Moreover, one can readily check that \(L_x \ast L_y = L_{x y} + 2L_{x y} = 2L_{xy}\) and \(L_x \ast F = 2L_x F\).

After some algebra, the commutator of two matrices like (9.1) can be computed to read:

\[
\left[ (a \otimes I d + I \otimes a_1 \begin{pmatrix} | & | \\ L_x & L_y \end{pmatrix}, b \otimes I d + I \otimes b_1 \begin{pmatrix} | & | \\ L_x & L_y \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \right]
\]

where:

\[
\begin{align*}
C_{11} &= [a, b] \otimes I d + 2I \otimes [a_1, b_1] + L_{x y} \ast L_{x - y} - L_{y - x} \ast L_{x - y} \\
C_{12} &= (a \otimes I d) L_{x - y} - (b \otimes I d) L_{x y} + 2L_{(I \otimes a_0) x - y} - 2L_{(I \otimes b_0) x - y} + L_{x - x} \ast L_{y - y} \\
C_{21} &= -L_{x y} (a \otimes I d) + L_{x - y} (b \otimes I d) - 2L_{(I \otimes a_0) x - y} + 2L_{(I \otimes b_0) x - y} + L_{x - y} \ast L_{y - y} \\
C_{22} &= 2I \otimes [a_1, b_1] + L_{x - y} \ast L_{x - y} - L_{y - x} \ast L_{x - y}
\end{align*}
\]

It should be stressed that the products occurring in (9.3) do differ from those of (5.4); namely, they are defined as follows:

\[
L_{x - y} \ast L_{x - y} := \left( \begin{pmatrix} \frac{1}{2} (x_+^+, y_+^-) I - t(x_+^+, y_+^-) E_{ij} \end{pmatrix} \otimes I d + I \otimes \left( \frac{1}{2} t(x_+^+, y_+^-) I - L_{y^- - y^-} \right) \right)
\]

\[
L_{x - y} \ast L_{x - y} := \left( \begin{pmatrix} \frac{1}{2} t(x_+^+, y_+^-) I - L_{y^- - y^-} \right) \right)
\]

\[
L_{x - y} \ast L_{x - y} := L_{x - y} \otimes L_{x - y} = L_{(x_+^+, y_+^-) \otimes (x_+^+, y_+^-)}
\]

From the properties of the triple product of Jordan algebras (discussed in section 2), it holds that \(L_{x_+^+, y_-^-} + [L_{x_+^+, y_-^-}] = \frac{1}{2} V_{x_+^+, y_-^-} \in \mathfrak{e}_6 \otimes \mathbb{C}\), see (2.7). Moreover, one can readily check

\[
\text{It should be stressed here that the matrix products } x \circ y, x \cdot y \in \mathfrak{e} \text{ defined in (9.4), never appeared (to the best of our present knowledge) in the literature, and are an original result of the present investigation.}
\]

\[J. \text{ Phys. A: Math. Theor. 47 (2014) 265202}\]

A Marrani and P Truini
that \([a_i^1, b_l^1] = \{a_1, b_1\}\), \((a \otimes \text{Id})L_b = L_{(a)\text{Id}}b\) and \(L_y \bullet L_x = I \otimes ((x^+_y, y^+_y)I - [y^+_x, x^+_x])\); this result implies that we are actually considering an algebra.

In the next section we are going to prove that Jacobi’s identity holds for the algebra of Zorn-type matrices \((9,1)\), with Lie product given by \((9.2) – (9.4)\). On the other hand, once Jacobi’s identity is proven, the fact that the Lie algebra so represented is \(e_8\) is made obvious by a comparison with the root diagram in figure 1, for \(n = 8\); in this case, we have:

1) an \(e_0^1 = e_6\), commuting with \(a_2^1\);
2) As in general, the three Jordan Pairs which globally transform as a \((3, 3)\) of \(a_2^1\); in this case, each of them transforms as a \((27, \overline{27})\) of \(e_6\).

As a consequence, we reproduce the well known branching rule of the adjoint of \(e_8\) with respect to its maximal and non-symmetric subalgebra \(a_2^1 \oplus e_6\):

\[
248 = (8, 1) + (1, 78) + (3, 27) + (3, \overline{27}).
\]

(9.5)

10. Jacobi identity for \(e_8\)

We use the same notation as in section 8, and write \((9.1)\) in a slight different way, namely, for for \(f_1 \in e_8\):

\[
\varrho(f_1) = \left(\begin{array}{cc}
aa & A^+ \\
A^- & -I \otimes a_1^1
da & A^+
\end{array}\right);
\]

(10.1)

where \(a \in a_2^1, a_1 \in e_6\) and \(A^+, A^-\) three vectors with elements in \(J_3^2, J_3^3\). Similarly, one can define \(\varrho(f_2)\) and \(\varrho(f_3)\) by respectively replacing \(a \rightarrow b\) and \(a \rightarrow c\) in \((10.1)\). Let us then write:

\[
[[\varrho(f_1), \varrho(f_2)], \varrho(f_3)] + \text{cyclic permutations} := \begin{pmatrix} \overline{3}_{11} & \overline{3}_{12} \\ \overline{3}_{21} & \overline{3}_{22} \end{pmatrix}.
\]

(10.2)

In order for the Jacobi identity to hold for the matrix realization \((10.1)\) of the adjoint of \(e_8\), we have to prove that \(3_{11} = 3_{12} = 3_{21} = 3_{22} = 0\).

After some algebra, we compute:

\[
\overline{3}_{11} = [(a, b), c] \otimes \text{Id} + 4(a \otimes [a_1, b_1], c_1) + (A^+ \circ B^- - B^+ \circ A^-)(c \otimes \text{Id} + I \otimes c_1)
\]

\[
- (c \otimes \text{Id} + I \otimes c_1)(A^+ \circ B^- - B^+ \circ A^-)
\]

\[
+ ((a \otimes \text{Id})B^+ - (b \otimes \text{Id})A^+ + (I \otimes a_1)B^+ + B^+(I \otimes a_1^1)
\]

\[
- (I \otimes b_1)A^+ - A^- (I \otimes b_1^1) + A^- \times B^-) \circ C^-
\]

\[
-C^+ \circ (-B^-(a \otimes \text{Id}) + A^- (b \otimes \text{Id}) - (I \otimes a_1^1)B^- - B^- (I \otimes a_1)
\]

\[
+ (I \otimes b_1^1)A^- + A^- (I \otimes b_1) - B^- \circ a_1)
\]

\[
+ \text{cyclic permutations}.
\]

The first two terms in the rhs of \((10)\) vanish upon cyclic permutations, because of the Jacobi identity in \(a_2^1\) and \(e_6\). The terms containing \(A^+, B^-, c\) can be proved to vanish, by the very same arguments used in section 8.

Next, we consider the terms containing \(A^+, B^-, c_1\). By denoting with \(a_k, b_k \in J_k^3\)

\((k = 1, 2, 3)\), the components of \(A^+\) and \(B^-\) respectively, and using the shorthand notation: \(E(x, y) := L_{x,y} + [L_{x}, L_{y}] = \frac{1}{2}V_{x,y}\), for \(x, y \in J_3^3\), one can compute that:

\[
\overline{2}_{11} = \overline{2}_{11} + (I \otimes c_1)(A^+ + A^+(I \otimes c_1^1)) \circ B^- - A^+ \circ ((I \otimes c_1^1)B^- + B^- (I \otimes c_1))
\]

\[
= 2I \otimes [c_1, E(a_i, b_j)] + 2 \left(\frac{1}{4}t(c_1(a_i), b_j) - t(c_1(a_i), b_j)E_{ij}\right) \otimes I +
\]

\[
+ 2I \otimes \left(\frac{1}{4}t(c_1(a_i), b_j)Id - E(c_1(a_i), b_j)\right)
\]

\[
= 0.
\]
This is easily shown by writing \( J. \phantom{a} \text{Phys. A: Math. Theor.} \)

\[
-2 \left( \frac{1}{\hbar} \{ t(a_i, c^j_i(b_j)) \} - t(a_i, c^j_i(b_j)) E_{ij} \right) \otimes \text{Id}
\]

\[
-2 I \otimes \left( \frac{1}{\hbar} \{ t(a_i, c^j_i(b_j)) \} \text{Id} - E(a_i, c^j_i(b_j)) \right)
\]

\[
= \left( \frac{1}{\hbar} \{ t(c_1(a_i), b_j) - t(a_i, c^j_i(b_j)) \} \right) - 2 \left( t(c_1(a_i), b_j) - t(a_i, c^j_i(b_j)) \right) E_{ij} \otimes \text{Id}
\]

\[
+ 2 I \otimes \left( \{ c_1, E(a_i, b_j) \} - E(c_1(a_i), b_j) + E(a_i, c^j_i(b_j)) \right)
\]

In order to prove that (10) sums up to zero, we start and observe that \( t(c_1(a), b) = t(a, c^j_i(b)) \); this is easily shown by writing \( c_1 = L_x + F \) (hence \( c_1^+ = L_x - F \)) and noticing that \( t(L_x, a, b) = t(x \cdot a, b) = t(a, x \cdot b) = t(a, L_x(b)) \) and \( t(Fx, y) + t(x, Fy) = 0 \), being \( F \) a derivation in \( \mathfrak{g} \). Moreover, \( [c_1, E(a, b)] = E(c_1(a), b) - E(a, c^j_i(b)) \), by (2.10). This indeed implies that (10) vanishes.

Finally, we consider terms in the rhs of (10) which contain structures like \( (A^- \times B^-) \cdot C^- \); they read:

\[
(A^- \times B^-) \cdot C^- + (B^- \times C^-) \cdot A^- + (C^- \times A^-) \cdot B^-
\]

\[
= \epsilon_{ijk} \left( \frac{1}{\hbar} \{ t(a_i \# b_k, c_j) \} - t(a_i \# b_k, c_j) E_{ij} \right) \otimes \text{Id}
\]

\[
+ I \otimes \epsilon_{ijk} \left( \frac{1}{\hbar} \{ t(a_i \# b_k, c_j) \} - E(a_i \# b_k, c_j) \right) + \text{cyclic permutations}
\]

\[
: = M^{(1)} \otimes I + I \otimes M^{(2)}.
\]

(10.3)

\( M^{(1)} = 0 \), by the same argument used in section 8. Let us here show that \( M^{(2)} = 0 \). In order to do this, we write \( (a_j, b_k, c_i) := \frac{1}{\hbar} t(a_j \# b_k, c_i) \). Thence:

\[
M^{(2)} = \epsilon_{ijk} (L_{(a_j, b_k, e_i)} - [L_{a_j \# b_k}, L_{e_i}]) + \text{cyclic permutations}.
\]

(10.4)

For each fixed \( i,j,k \), it holds that

\[
\epsilon_{ijk}(a_j, b_k, c_i) + \epsilon_{ijk}(b_k, c_i, a_j) + \epsilon_{ijk}(c_i, a_j, b_k) = \epsilon_{ijk}((a_j, b_k, c_i) + (b_k, c_i, a_j) + (c_i, a_j, b_k)).
\]

(10.5)

Since \( (x, y, z) \) is symmetric in \( x, y \) and linear in \( x, y, z \), the above expression is linear and symmetric in \( a_j, b_k, c_i \), thus it is the polarization of \( (x, x, x) = 2(\frac{1}{\hbar} t(x^i, x) I - x^i \cdot x) = 0 \), by (2.6). Similarly, for \([L_x, \# b_y, L_z] + \text{cyclic permutations}, we get the polarization of [L_x, L_z], which is zero by the Jordan identity (2.5), namely:

\[
[L_x, L_z] z = x^i \cdot (x \cdot z) - x \cdot (x^i \cdot z) = x^i \cdot (x \cdot z) - x \cdot (x^i \cdot z) = 0 \forall z \in J.
\]

(10.6)

Analogous calculations for terms in the rhs of (10) which contain structures like \([B^+, A^-, c_i], [B^+, A^+, c_i], [B^+, A^+, C^+] + \text{plus their cyclic permutations}\) prove that \( \tilde{J}_{12} = 0 \).

Next, we proceed to consider \( \tilde{J}_{12} \) which, after some algebra, can be computed to read:

\[
\tilde{J}_{12} = ([a, b] \otimes I + 2 I \otimes [a_1, b_1] + A^+ \otimes B^- + B^+ \otimes A^-) C^+
\]

\[
- ((a \otimes I) B^+ - (b \otimes I) A^+) + (I \otimes a_1) B^+ + B^+ (I \otimes a_1)
\]

\[
- (I \otimes b_1) A^- + A^- (I \otimes b_1^+) + A^- \otimes B^+ (I \otimes c^j_i)
\]

\[
- (c \otimes I + I \otimes c_1)((a \otimes I) B^+ - (b \otimes I) A^+) + (I \otimes a_1) B^+ + B^+ (I \otimes a_1)
\]

\[
- (I \otimes b_1) A^- + A^- (I \otimes b_1^+) + A^- \otimes B^-
\]

\[
- C^+ (I \otimes [a_1, b_1]) + A^- \cdot B^- + B^- \cdot A^+
\]

\[
+ (-B^- (a \otimes I) + A^- (b \otimes I) - (I \otimes a_1) B^- - B^- (I \otimes a_1)
\]

\[
+ (I \otimes b_1^+) A^- + A^- (I \otimes b_1) + A^- \times B^+ + C^- + \text{cyclic permutations}.
\]
By noticing that \([a_1, b_1^+] = -[a_1, b_1]^\dagger\) and that, as already noticed, \((a \otimes \text{Id})L_b = L_{(a \otimes \text{Id})b}\), as already noticed, one finds that many terms cancel out trivially, and only terms of the following three types remain:

1. \[-2L_{(a \otimes \text{Id})b}^−_{\otimes \text{Id}}a−b− = L_{(a \otimes \text{Id})b}^−_{\otimes \text{Id}}b−a− + L_{(a \otimes \text{Id})b}^−_{\otimes \text{Id}}a−b−\]

2. \[-(c \otimes \text{Id})L_a−_b− = L_{(c \otimes \text{Id})a−_b−} = L_{(c \otimes \text{Id})b−_a−} + L_{(c \otimes \text{Id})a−_b−}\]

3. \[L_{(a^+ \otimes b^+)}(c \otimes \text{Id}) + (c^+ \otimes b^+)(a \otimes \text{Id}) - (c \otimes \text{Id})(b^+ \otimes \text{Id}) - (c^+ \otimes b^+)(a \otimes \text{Id}) = L^+_{(c \otimes \text{Id})a^+_b^+} + L^+_{(c \otimes \text{Id})a^+_b^+}\]

Terms like (1) and (2) can be shown to vanish using similar arguments to those of section 8.

The \(i\)th component of terms like (3) can be written as (omitting the +, − superscripts):

\[L_{(a \otimes b)_{\otimes c}} = L_{(a \otimes b)_{\otimes c}} + (b_1, c_1)L_{a_1} + (a_1, c_1)L_{b_1} - (a_1, c_1)L_{b_1}\]

which vanishes because of (2.7).

This ends the proof of the fact that \(\mathcal{J}_{12} = 0\).

Analogous calculations show that also \(\mathcal{J}_{21} = \mathcal{J}_{22} = 0\), thus proving the Jacobi identity for the matrix realization (10.1) (or, equivalently (9.1)) of the adjoint of \(e_8\).

11. Future developments

There are several topics that we are planning to develop in the future.

One is the extension of the Zorn-type representations to the Lie algebra of the semi-direct product group \(E_{7,8}\), through a representation of the sextonions [71, 72] and of the algebra of their derivations.

A second interesting venue of developments is the characterization of all real forms of these representations of the exceptional Lie algebras, as well as the treatment of split forms of Hurwitz’s algebras \(\mathbb{C}, \mathbb{Q}, \mathbb{E}\), with a particular attention to the coset spaces related to the scalar manifolds in supergravity. This would yield a Zorn-like realization of (some of) the maximal non-symmetric embeddings considered in [45], and proved in a broader framework in [49].

Moreover, it would be interesting to consider Jordan pairs for semi-simple Jordan algebras of rank 3 of relevance for supergravity theories, along the lines of the treatment given in [45].

We plan then to proceed to the study of the representations of quantum exceptional groups—in particular quantum \(e_8\)—and of integrable models built on them. We aim at a new perspective of elementary particle physics at the early stages of the Universe based on the idea that interactions, defined in a purely algebraic way, are the fundamental objects of the theory, whereas space-time, hence gravity, are derived structures.

Acknowledgments

The work of AM is supported in part by the FWO—Vlaanderen, project no. G.0651.11, and in part by the Interuniversity Attraction Poles Programme initiated by the Belgian Science Policy (P7/37). The work of PT is supported in part by the Istituto Nazionale di Fisica Nucleare grant In. Spec. GE 41.

References

[1] Ramond P 2002 Exceptional Groups and Physics Plenary Conference Groupe 24, Paris arXiv:hep-th/0301050v1
[24] Dobrev V K 2009 Invariant differential operators for non-compact Lie groups: the \( E_{6(-14)} \) case
Proc. SFN Ser. A: Conferences, A1 ed B Dragovich and Z Rakic (Belgrade: Institute of Physics) pp 95–124 arXiv:0812.2655 [math-ph]

[25] Brown R B 1969 Groups of type \( E_7 \) J. Reine Angew. Math. 236 79

[26] Ferrara S and Kallosh R 2011 Creation of matter in the universe and groups of type \( E_7 \) J. High Energy Phys. JHEP12(2011)096

Ferrara S, Kallosh R and Marrani A 2012 Degeneration of groups of type \( E_7 \) and minimal coupling in supergravity J. High Energy Phys. JHEP06(2012)074

[27] Marrani A, Qiu C-X, Shih S-Y D, Tagliaferro A and Zumino B 2013 Freudenthal gauge theory J. High Energy Phys. JHEP03(2013)132

[28] Borsten L, Duff M J, Marrani A and Rubens W 2011 On the black-hole/qubit correspondence Eur. Phys. J. Plus 126 37

Borsten L, Duff M J and Lévay P 2012 The black-hole/qubit correspondence: an up-to-date review Class. Quantum Grav. 29 224008

[29] Kallosh R and Soroush M 2008 Explicit action of \( E_7 \) on \( N=8 \) supergravity fields

Nucl. Phys. B 801 25

[30] Kallosh R and Kugo T 2009 The footprint of \( E_7 \) in amplitudes of \( N=8 \) supergravity

J. High Energy Phys. JHEP01(2009)072

[31] Bianchi M and Ferrara S 2008 Enriques and octonionic magic supergravity models

J. High Energy Phys. JHEP02(2008)054

[32] Marcus N and Schwarz J H 1983 Three-dimensional supergravity theories

Nucl. Phys. B 228 145

[33] Gross D J, Harvey J A, Martinec E and Rohm R 1985 Heterotic string

Phys. Rev. Lett. 54 502

[34] Rovelli C 2004

Quantum Gravity (Cambridge: Cambridge University Press)

[35] Lisi A G, Smolin L and Speziale S 2010 Unification of gravity, gauge fields and Higgs bosons

J. Phys. A: Math. Theor. 43 444501

[36] Cacciatori S L, Cerchiai B L, Vedova A D, Ortenzi G and Scotti A 2005 Euler angles for \( G_2 \)

J. Math. Phys. 46 083512

Cacciatori S L 2005 A simple parametrization for \( G_2 \)

J. Math. Phys. 46 083520

Bernardoni F, Cacciatori S L, Cerchiai B L and Scotti A 2008 Mapping the geometry of the \( F_4 \) group

Adv. Theor. Math. Phys. 12 889

Bernardoni F, Cacciatori S L, Cerchiai B L and Scotti A 2008 Mapping the geometry of the \( E_6 \) group

J. Math. Phys. 49 012107
Cacciatori S L and Cerchiai B L 2010 Exceptional groups, symmetric spaces and applications
Group Theory: Classes, Representations, Connections, and Applications ed C W Danellis (New York: Nova Science Publishers) p 177
Cacciatori S L, Piazza F D and Scotti A 2010 $E_7$ groups from octonionic magic square arXiv:1007.4758 [math-ph]
Cacciatori S L, Piazza F D and Scotti A 2012 A simple $E_8$ construction arXiv:1207.3623 [math-ph]
[51] Loos O 1975 Jordan Pairs (Lecture Notes in Mathematics vol 460) (Berlin: Springer)
[52] Jordan P, von Neumann J and Wigner E 1934 On an algebraic generalization of the quantum mechanical formalism Ann. Math. 35 29
[53] Jacobson N 1966 Structure theory for a class of Jordan algebras Proc. Natl Acad. Sci. USA 55 243
[54] Meyberg K 1970 Jordan-triplesysteeme und die Koecher-konstruktion von Lie algebren Math. Z. 115 58
[55] Tits J 1962 Une classe d’algèbres de Lie en relation avec les algèbres de Jordan Nederl. Akad. Wetensch. Proc. Ser. A 65 Indagationes Math. 24 530
[56] Kantor I L 1964 Classification of irreducible transitive differential groups Dokl. Akademii Nauk SSSR 158 1271
[57] Koecher M 1967 Imbedding of Jordan algebras into Lie algebras: I Am. J. Math. 89 787
[58] Faulkner J R 2000 Jordan pairs and Hopf algebras J. Algebra 232 152
[59] Schafer R D 1966 An Introduction to Non Associative Algebras (New York: Academic)
[60] Schafer R D 1949 Inner derivations of non associative algebras Bull. Am. Math. Soc. 55 769
[61] Zorn M 1933 Alternativkörper und quadratische systeme Abh. Math. Sem. Univ. Hamburg 9 395
[62] Loos O, Petersson H P and Racine M L 2008 Inner derivations of alternative algebras over commutative rings Algebra Number Theory 2 927
[63] Tits J 1966 Algèbres alternatives, algèbres de Jordan et algèbres de Lie exceptionnelles, I Construction Nederl. Akad. Wetensch. Proc. Ser. A 69 223
[64] Freudenthal H 1959 Beziehungen der $E_7$ und $E_8$ zur Oktavenebene V-iX Proc. K. Ned. Akad. Wet. A 62 447
[65] Jacobson N 1971 Exceptional Lie Algebras (Lecture Notes in Pure and Applied Mathematics vol 1) (New York: Dekker)
[66] Imaeda K and Imaeda M 2000 Sedenions: algebra and analysis Appl. Math. Comp. 115 77
[67] Moreno G 1997 The zero divisors of the Cayley–Dickson algebras over the real numbers Soc. Mat. Mex. Bol. Tercera Ser. 4 arXiv:q-alg/9710013
[68] Dundarer R, Gürsey F and Tze C 1984 Generalized vector products, duality and octonionic identities in $\mathbb{O} = 8$ geometry J. Math. Phys. 25 1496
[69] Duff M J and Ferrara S 2007 $E_6$ and the bipartite entanglement of three qutrits Phys. Rev. D 76 124023
[70] Kugo T and Townsend P 1983 Supersymmetry and the division algebras Nucl. Phys. Sect. B 221 357
[71] Westbury B W 2006 Sextonions and the magic square J. London Math. Soc. 73 455
[72] Landsberg J M and Manivel L 2006 The sextonions and $E_{7\frac{1}{2}}$ Adv. Math. 201 143