Form Factors for Quasi-particles
in $c = 1$ Conformal Field Theory

R.A.J. van Elburg and K. Schoutens

Van der Waals-Zeeman Institute and Institute for Theoretical Physics
University of Amsterdam, Valckenierstraat 65
1018 XE Amsterdam, The Netherlands

The non-Fermi liquid physics at the edge of fractional quantum Hall systems is described by specific chiral Conformal Field Theories with central charge $c = 1$. The charged quasi-particles in these theories have fractional charge and obey a form of fractional statistics. In this paper we study form factors, which are matrix elements of physical (conformal) operators, evaluated in a quasi-particle basis that is organized according to the rules of fractional exclusion statistics. Using the systematics of Jack polynomials, we derive selection rules for a special class of form factors. We argue that finite temperature Green’s functions can be evaluated via systematic form factor expansions, using form factors such as those computed in this paper and thermodynamic distribution functions for fractional exclusion statistics. We present a specific case study where we demonstrate that the form factor expansion shows a rapid convergence.
1 Introduction and summary

Shortly after the first observation of the fractional quantum Hall (fqH) effect [1], R.B. Laughlin proposed that this phenomenon has its origin in a new state of matter, which is formed by electrons but nevertheless admits excitations of fractional charge [2]. The experimental evidence for the existence of fractionally charged excitations includes the results of shot-noise measurements for charge transport through fqH samples [3].

The unusual properties of the fqH quasi-particles do not stop at their fractional charge; in addition, the (bulk and edge) quasi-particles exhibit various forms of fractional quantum statistics [4, 5, 6, 7].

In a previous paper [8] we initiated a program that aims at computing various transport properties of fqH systems in a formalism that makes direct reference to the (fractionally) charged quasi-particles. In particular, we presented a quasi-particle basis of the edge Conformal Field Theory (CFT) for the $\nu = \frac{1}{p}$ principal Laughlin states. We demonstrated that the CFT quasi-particles satisfy a form of fractional statistics that is closely related to Haldane’s notion of ‘fractional exclusion statistics’ [6].

The CFT for the $\nu = 1/p$ fqH edges can be identified with a continuum ($N \to \infty$) limit of the Calogero-Sutherland (CS) model for quantum mechanics with inverse square exchange. The natural quasi-particles for the fqH system correspond to eigenstates of the CS Hamiltonian. In the work of many groups, the eigenstates of Hamiltonians of the CS-type have been understood in terms of Jack symmetric polynomials. In particular, S. Iso [9] presented two alternative Jack polynomial bases for the continuum CS theory. In section 3.4 below, we discuss the precise correspondence between these CS bases and the fqH basis we presented in [8].

It has been recognized by many groups that quantum many-body systems with inverse square exchange come close to being ‘ideal gases of fractional statistics particles’. Supporting this claim are the observations that (i) equilibrium thermodynamic quantities can be evaluated with the help of 1-particle distribution functions for fractional statistics and, (ii) the zero-temperature correlation functions of simple operators (such as electron and hole operators) are mediated by intermediate states with a minimal number of propagating quasi-particles [see section 4].

When turning to finite temperature, one quickly discovers that the second ‘free gas’ property (ii) no longer holds. There are important many-body effects which can not be ignored when computing correlation functions of physical operators at finite temperature. This implies that great care needs to be taken when setting up arguments that link the $T$-dependence of physical observables ($I-V$ and shot-noise characteristics in particular) to the fractional statistics of the fundamental charge carriers [10].

In this paper we turn to the problem of extending the formalism based on
quasi-particles with fractional statistics to finite temperatures. We argue that specific finite-$T$ correlation functions can be written in a so-called form factor expansion. Such expansions start with a term that refers to a minimal number of quasi-particles; to this leading term successive corrections are added that refer to more and more quasi-particles participating in physical process described by the correlation function. To test the validity of the proposed expansion, we explicitly evaluate the form factor expansion for a specific finite-$T$ Green’s function, collecting all terms that refer to one or two quasi-particles. The numerical results show a rather rapid convergence to the (known) exact expression.

The form factor expansion that we propose is similar in spirit to expressions that were proposed in [11] (see also [12]) in the context of integrable quantum field theories whose structure is set by a factorizable scattering matrix. Despite this similarity, the two approaches are rather different: in our approach we nowhere rely on scattering data or on the associated form factor axioms, but instead perform explicit computations in a CFT that is regularized by the finite size $L$ of the spatial direction.

This paper is organized as follows.

In section 2 we review results obtained in our earlier paper [8]. We present a basis for the edge theories for the $\nu = \frac{1}{p}$ fqH state, employing edge electrons and edge quasi-holes as the fundamental charged quasi-particles. We describe the associated 1-particle thermodynamic distribution functions and discuss the fundamental duality between the two types of quasi-particles.

In section 3 we discuss the continuum CS models and the associated Jack polynomial bases as first given by Iso. We give an explicit 1-1 connection between the states in the CS basis and the states in the fqH basis.

In section 4 we turn to form factors. In 4.1 we discuss the ones relevant for $T = 0$, while in 4.2 we present simple examples of form factors that contribute at non-zero $T$. The expressions that we obtain make clear that the simple picture of ‘an ideal gas’ breaks down at finite temperature. A rather general set of selection rules is presented in 4.3, while in section 4.4 we briefly discuss the extent to which our explicit results can be understood from an axiomatic approach based on a factorized S-matrix.

In section 5 we compute 1 and 2-particle diagonal form factors for the ‘edge electron counting operator’. These form factors are then used to evaluate the leading terms in a form factor expansion for the finite-$T$ Green’s function that determines the $I-V$ characteristics of edge tunneling processes.

In section 6 we present our conclusions. In the appendices we specify algebraic properties of the charged edge operators for $\nu = \frac{1}{2}$, we summarize some relevant results on Jack polynomials, and we provide a proof of the form factor selection rules presented in section 4.3.
2 Quasi-particle basis of fqH edge theories

2.1 fqH edge theories as CFT

It is well known that many (though not all!) aspects of the low energy dynamics of fqH systems can be captured by an effective edge theory. These edge theories are so-called chiral Luttinger Liquids or chiral CFTs. For the specific case of a principal Laughlin state at filling $\nu = 1/p$, the CFT describing a single edge (in isolation) is a specific $c = 1$ CFT.

The standard interpretation of this effective theory is in terms of a chiral boson, which is identified with a quantized density wave (magnetoplasmon) along the edge of the fqH sample. By exploiting the relation between the chiral anomaly and the quantized Hall conductance, one finds that the chiral boson field is compactified on a circle of radius $R^2 = p$ \[13]. This construction directly leads to a space of states (the so-called chiral Hilbert space) with partition function

$$Z^{1/p}(q) = \sum_{Q=-\infty}^{\infty} q^{Q^2/p},$$

(2.1)

with $(q)_\infty = \prod_{l=1}^{\infty} (1 - q^l)$ and $q = e^{-\beta 2\pi \frac{1}{\rho_0}}$. [The 1-particle energies are spaced by $l^{2\pi \frac{1}{L \rho_0}}$ with $l$ integer and $\rho_0$ the density of states per unit length, $\rho_0 = (\hbar v_F)^{-1}$.]

In eq. (2.1), the parameter $Q$ is the electric $U(1)$ charge in units of $\frac{e}{p}$. The Hilbert space is obtained as a collection of charge sectors $Q$. Within each sector, there is a leading state of minimal energy $Q^2/2p$; all other states in that sector are reached via the application of (neutral) Fourier modes of the density operator. Together, these modes form a $U(1)$ Kac-Moody algebra, and the factor $\frac{1}{(q)_\infty}$ is the well-known character of a highest weight module of this affine algebra.

In our earlier paper [8], we proposed that the CFT for the $\nu = \frac{1}{p}$ fqH edge can be interpreted in terms of a set of fundamental charged quasi-particles. We have worked out a formulation in terms of edge electrons (of charge $-e$) and edge quasi-holes (of charge $+\frac{e}{p}$). Our main motivation has been that, using this novel formulation, one learns how to understand some of the unusual and spectacular phenomenology of the fqH systems as manifestations of unusual properties (fractional charge and fractional statistics in particular) of their fundamental quasi-particles.

2.2 The fqH quasi-particle basis

In [8], we demonstrated how the collection of states (2.1) can be understood as a collection of multi-particle states, the fundamental (quasi-)particles being the edge electron and the edge quasi-hole, of charge $Q = -p$ and $Q = 1$, respectively.
The edge electron and quasi-holes are described by the conformal, primary fields
\[ J^{(-p)}(z) = \sum_t J^{(-p)}_{-t} z^{-\frac{2}{p}} , \quad \phi^+(z) = \sum_s \phi^+_{-s} z^{-\frac{2}{p}} . \] (2.2)

Clearly, one can employ the Fourier modes \( J^{(-p)}_{-t} \) and \( \phi^+_{-s} \) as ‘creation operators’ for the corresponding quasi-particles. [The reason why we put quotation marks here will soon become clear.] In [8] we identified a collection of multi-\( J \), multi-\( \phi \) states which together form a basis for the chiral Hilbert space. It is given by the states (from now on we omit the charge superindex on the operators \( J \), \( \phi \))
\[ J_{-(2M-1)p+Q-m_M} \cdots J_{2p+Q-m_1} \phi_{-(2N-1)p+Q-n_N} \cdots \phi_{2p+Q-n_1} |Q \rangle \]
with \( m_M \geq m_{M-1} \geq \ldots \geq m_1 \geq 0 \), \( n_N \geq n_{N-1} \geq \ldots \geq n_1 \geq 0 \),\n\[ n_1 > 0 \quad \text{if} \quad Q < 0 , \] (2.3)
where \(|Q \rangle\) denotes the lowest energy state of charge \( Q \), with \( Q \) taking the values \(-(p-1), -(p-2), \ldots, -1, 0\).

The associated character identity is
\[ Z^{1/p}(q) = \sum_{Q=-(p-1)}^{0} q^{Q^2/2p} Z_{Q}^{\text{qh}}(q) Z_{Q}^{\text{e}}(q) , \] (2.4)

where the factor \( q^{Q^2/2p} \) takes into account the energy of the initial states and we denoted by \( Z_{Q}^{\text{qh}}, Z_{Q}^{\text{e}} \) the partition functions for quasi-holes and electrons in the sector with vacuum charge \( Q \). They are naturally written as
\[ Z_{Q}^{\text{qh}} = \sum_{N=0}^{\infty} q^{\frac{1}{2p}(N^2 + 2QN) + \delta_{Q,0}N} (q)_N , \quad Z_{Q}^{\text{e}} = \sum_{M=0}^{\infty} q^{\frac{M^2 - QM}{2p}} (q)_M , \] (2.5)
with \((q)_L = \prod_{l=1}^{L}(1-q^l)\). The identity (2.4) can be rigorously established by employing partition counting theorems that are available in the mathematical literature (see [4] for a discussion).

An important special case is \( p = 1 \), where the quasi-particle basis (2.3) is the standard multi-particle basis in a theory of free, charged but spin-less fermions.

2.3 Fractional statistics and duality

2.3.1 Fractional exclusion statistics

In a 1991 paper [3], F.D.M. Haldane proposed the notion ‘fractional exclusion statistics’, as a tool for the analysis of strongly correlated many-body systems.
The central assumption that is made concerns the way a many-body spectrum is built by filling available one-particle states. In words, it is assumed that the act of filling a one-particle state effectively reduces the dimension of the space of remaining one-particle states by an amount $g$. The choices $g = 1$, $g = 0$ correspond to fermions and bosons, respectively. The thermodynamics for general ‘$g$-ons’, and in particular the appropriate generalization of the Fermi-Dirac distribution function, have been obtained in [15]. The so-called IOW equations

$$\bar{n}_g(\epsilon) = \frac{1}{[w(\epsilon) + g]} , \quad \text{with} \quad [w(\epsilon)]^g[1 + w(\epsilon)]^{1-g} = e^{\beta(\epsilon - \mu)} \quad (2.6)$$

provide an implicit expression for the 1-particle distribution function $\bar{n}_g(\epsilon)$ for $g$-ons at temperature $T$ and chemical potential $\mu$. The solutions $\bar{n}_g(\epsilon)$ have limiting value $\bar{n}_g^{\max} = \frac{1}{g}$ for $\epsilon \to -\infty$.

### 2.3.2 Spectral shift statistics and the fqH effect

In [8] we analyzed the exclusion statistics behind the states (2.3) that form a basis for the chiral Hilbert space for a $\nu = \frac{1}{p}$ fqH edge. This analysis employed a technique, first proposed in [16], based on recursion relations satisfied by truncated chiral partition functions. The remarkable conclusions are that

- the ‘microscopic’ state-filling rules differ from those formulated by Haldane, but
- the state counting, and thereby the 1-particle thermodynamic distribution functions, agree with those associated to fractional exclusion statistics, i.e. the distribution functions are solutions of the IOW equations.

The precise statement is that the edge electrons are described by the IOW distribution with $g = p$, while the edge quasi-hole states are thermally occupied according to the distribution with $\tilde{g} = 1/p$. It is important to remark that there is no mutual statistics between the two types of excitations.

For later reference, we list the explicit expressions for the distribution functions for the case $p = 2$

$$\bar{n}_2(\epsilon) = \frac{1}{2} \left( 1 - \frac{1}{\sqrt{1 + 4e^{-\beta(\epsilon - \mu)}}} \right) , \quad \bar{n}_{\frac{1}{2}}(\epsilon) = \frac{2}{\sqrt{1 + 4e^{2\beta(\epsilon - \mu)}}} . \quad (2.7)$$

### 2.3.3 Duality

The distribution functions for fractional exclusion statistics with parameters $g$ and $\tilde{g} = 1/g$ satisfy the following duality relation [17]

$$g \bar{n}_g(\epsilon) = 1 - \tilde{g} \bar{n}_{\tilde{g}}(-\tilde{g}\epsilon) . \quad (2.8)$$
The interpretation of this result is that the $\tilde{g}$ quanta with positive energy act as holes in the ground state distribution of negative energy $g$-quanta.

Translating back to the $\nu = \frac{1}{p}$ fqH edge, we observe a fundamental duality between edge electrons and edge quasi-holes, in agreement with the physical interpretation of strong and weak backscattering limits of edge-to-edge tunneling processes [13]. Under the duality, the removal of a single edge electron corresponds to the creation of a total of $p$ quasi-holes.

This duality further implies that, when setting up a quasi-particle description for fqH edges, we can opt for (i) either quasi-holes or edge electrons, with energies over the full range $-\infty < \epsilon < \infty$ ('particle picture'), or (ii) a combination of both types of quasi-particles, each having positive energies only ('excitation picture'). The option (ii) is the one realized in the fqH basis of section 2.2. In section 3 we shall discuss the CS bases proposed by Iso, which in a sense uses the option (i).

### 2.4 Equilibrium quantities

#### 2.4.1 Specific heat

The specific heat of a conformal field theory is well-known to be proportional to the central charge $c_{\text{CFT}}$

$$
\frac{C(T)}{L} = \gamma \rho_0 k_B^2 T , \quad \gamma = \frac{\pi}{6} c_{\text{CFT}}, \quad (2.9)
$$

where $\rho_0 = (\hbar v_F)^{-1}$ is the density of states per unit length. In [8] we demonstrated that the fqH quasi-particle basis specified in (2.3) leads to (with $g = p$, $\tilde{g} = \frac{1}{p}$)

$$
\gamma = \gamma_{g,+} + \gamma_{\tilde{g},+} \quad (2.10)
$$

with

$$
\gamma_{g,+} = \partial_\beta \int_0^\infty d\epsilon \epsilon \bar{n}_g(\epsilon) , \quad \gamma_{\tilde{g},+} = \partial_\beta \int_0^\infty d\epsilon \epsilon \bar{n}_{\tilde{g}}(\epsilon) . \quad (2.11)
$$

One finds that, while the individual contributions $\gamma_{g,+}, \gamma_{\tilde{g},+}$ depend on $g$, their sum is equal to $\frac{\pi}{6}$ for all $g$, in agreement with $c_{\text{CFT}} = 1$. For $g = 2$, $\tilde{g} = \frac{1}{2}$ one has

$$
\gamma_{2,+} = \frac{\pi}{6} \frac{2}{5} , \quad \gamma_{\frac{1}{2},+} = \frac{\pi}{6} \frac{3}{5} . \quad (2.12)
$$

#### 2.4.2 Hall conductance

By a simple back-of-the-envelope argument, the Hall conductance is related to the edge capacitance, i.e. to the charge $\Delta Q$ that is accumulated on a given edge
in response to an applied voltage $V$. One quickly derives the following expression for the Hall conductance

$$G/\left[\frac{e^2}{h}\right] = \frac{1}{eV} \left[ - \int_0^\infty d\epsilon \tilde{n}_p(\epsilon + eV) + \frac{1}{p} \int_0^\infty d\epsilon \tilde{n}_1(\epsilon - \frac{e}{p}V) \right].$$

Using the properties of the distribution functions, one shows that this expression is independent of the temperature, and gives $G = \frac{1}{p} \frac{q^2}{h}$ for the $\nu = \frac{1}{p}$ edge. For $T = 0$ eq. (2.13) reduces to

$$G = \tilde{n}_{\text{max}} \frac{q^2}{h},$$

with $q$ the charge and $g$ the statistics parameter of the quasi-particles that are pulled into the edge by the applied voltage. Depending on the sign of $V$ these are the edge electrons ($q = -e$, $\tilde{n}_{\text{max}} = \frac{1}{p}$) or the quasi-holes ($q = \frac{e}{p}$, $\tilde{n}_{\text{max}} = p$); both give the canonical value of the Hall conductance.

### 3 CS models and fractional statistics

#### 3.1 Inverse square exchange in the CFT setting

While the states specified in (2.3) form a complete basis of the chiral Hilbert space, they are not mutually orthogonal, and as such they do not form a proper starting point for further analysis. In principle one may go through a orthogonalization procedure to arrive at a proper canonical quasi-particle basis. We shall here reach this goal in a more efficient way, by exploiting the close connection with so-called Calogero-Sutherland (CS) models of many-body quantum mechanics.

The CS model describes the (non-relativistic) quantum mechanics of $N$ particles on a circle, with 2-body interaction that is proportional to the inverse square of the chord distance between the particles. In [9], S. Iso demonstrated that the limit $N \to \infty$ of a CS model with interaction strength $p(p - 1)$ can be identified with the $c = 1$ CFT of a chiral boson compactified on a circle with radius $R^2 = p$. Iso also specified a collective hamiltonian $H_{CS}$, acting in the CFT Hilbert space, whose eigenstates precisely correspond to the multi-particle states of the underlying CS model.

It turns out that the eigenstates of $H_{CS}$ are in 1-1 correspondence with the states specified in eq. (2.3): by adding subleading terms to the expressions in (2.3), one arrives at a (orthogonal and complete) set of eigenstates of $H_{CS}$. Comparing with Iso’s formulation, one finds that the ‘superfermions’ of [9] correspond to what we call edge electrons and the ‘anyons’ of [9] are the edge quasi-holes of the fqH system. Nevertheless, the ‘CS basis’ specified by Iso differs from the
'fqH basis' of this paper through the way in which the quasi-particle content of a given state is specified. In subsection 3.4 below we shall spell out the precise correspondence between the two formulations.

The spectrum of the CS models has been analyzed with the help of so-called Jack symmetric polynomials, which provide explicit wave functions and eigenstates for the CS hamiltonian. With the help of 'Jack technology', important conjectures \[19\] about zero-temperature correlation functions of models with inverse square exchange have been proven \[20, 21\]. In the present paper, where we are interested in finite-temperature correlation functions, we shall apply the 'Jack technology' to obtain a set of selection rules on form factors that are relevant for computations at finite temperature. We shall complement these considerations with explicit computations of form factors involving states with up to two quasi-particles.

### 3.2 The hamiltonian $H_{CS}$ and the fqH basis

To specify the operator $H_{CS}$, we employ a free chiral scalar field $\varphi(z)$. In terms of this scalar field, the charged fields $J$ and $\phi$ take the form of so-called vertex operators,

$$J(z) = e^{-i\sqrt{p}\varphi(z)}, \quad \phi(z) = e^{i\sqrt{p}\varphi(z)}.$$  \hspace{1cm} (3.1)

The operator $Q = \oint (i\sqrt{p}\partial \varphi)$ measures the electric charge in units $e/p$. Following \[3\], we define

$$H_{CS} = \frac{p-1}{p} \sum_{l=0}^{\infty} \left( l+1 \right)(i\sqrt{p}\partial \varphi)_{l-1}(i\sqrt{p}\partial \varphi)_{l+1} + \frac{1}{3p} \left[ (i\sqrt{p}\partial \varphi)^3 \right]_0,$$  \hspace{1cm} (3.2)

where $\partial \varphi(z) = \sum_l (\partial \varphi)z^{-l-1}$ and where the second term on the r.h.s. denotes the zero-mode of the normal ordered product of three factors $(i\sqrt{p}\partial \varphi)(z)$. As a first result, one finds the following action of $H_{CS}$ on states containing a single quasi-particle of charge $Q = +1$ or $Q = -p$

$$H_{CS} \phi_{-\frac{1}{2p}-n}|0\rangle = h_\phi(n) \phi_{-\frac{1}{2p}-n}|0\rangle, \quad h_\phi(n) = \left[ \frac{1}{3p} + pn(n + \frac{1}{p}) \right],$$

$$H_{CS} J_{-\frac{p}{2}-m}|0\rangle = h_J(m) J_{-\frac{p}{2}-m}|0\rangle, \quad h_J(m) = \left[ -\frac{p^2}{3} - m(p + m) \right].$$  \hspace{1cm} (3.3)

We would like to stress that the fact that both $J_s$ and $\phi_t$ diagonalize $H_{CS}$ is quite non-trivial. If one evaluates $H_{CS}$ on any vertex operator $\phi^{(Q)}(z)$ (of charge $Q_{\frac{2}{p}}$), one typically runs into the field product $(T\phi^{(Q)})(z)$, where $T(z) = -\frac{1}{2}(\partial \varphi)^2(z)$ is the
stress-energy of the scalar field \( \varphi \). Only for \( Q = 1 \) and \( Q = -p \) do such terms cancel and do we find that the quasi-particle states are eigenstates of \( H_{CS} \).

We can now continue and construct eigenstates of \( H_{CS} \) which contain several \( \phi \) or \( J \)-quanta. What one then finds is that the simple product states specified in (2.3) are not \( H_{CS} \) eigenstates, but that they rather act as head states that need to be supplemented by a tail of subleading terms. For the \( H_{CS} \) eigenstate headed by the multi-particle state (2.3) (with unit coefficient), we shall use the notation

\[
|\{m_j; n_i\}\rangle_Q^Q
\]

so that

\[
H_{CS}|\{m_j; n_i\}\rangle_Q^Q = 
\left[ \frac{Q^3}{3p} + \sum_{j=1}^{M} h_J((j-1)p - Q + m_j) + \sum_{i=1}^{N} h_{\varphi}\left(\frac{1}{p}(Q + i - 1) + n_i\right) \right]|\{m_j; n_i\}\rangle_Q^Q. \tag{3.5}
\]

The states (3.4), with the \( m_j \), \( n_i \) and \( Q \) as specified in and below (2.3), form a complete and orthogonal basis for the chiral Hilbert space.

For the sake of illustration, we give explicit results for a few simple states of the fqH basis. The 1-particle states over the \( Q = 0 \) vacuum are given by

\[
|\{m_1\}\rangle^0 = J_{-\frac{p}{2}-m_1}|0\rangle, \quad |\{n_1\}\rangle^0 = \phi_{-\frac{1}{2p}-n_1}|0\rangle. \tag{3.6}
\]

The norms of these states can explicitly be evaluated by exploiting the (generalized) commutation relations satisfied by the modes \( J_s \) and \( \phi_t \). One finds

\[
0\langle\{m_1\}|\{m_1\}\rangle^0 = C_{m_1}^{(-p)} \, , \quad 0\langle\{n_1\}|\{n_1\}\rangle^0 = C_{n_1}^{(-\frac{1}{2})} \, , \tag{3.7}
\]

where the \( C_{k}^{(\alpha)} \) are the expansion coefficients of \( (1 - x)^{\alpha} = \sum_{k \geq 0} C_{k}^{(\alpha)} x^k \).

In our discussion below we shall often restrict ourselves to the vacuum sector \( Q = 0 \), and omit the explicit sector label \( Q \) on the fqH states (3.4).

For later use, we present the explicit form of the fqH states \(|\{m_2, m_1\}\rangle\) and \(|\{m_1; n_1\}\rangle\) at \( p = 2 \)

\[
|\{m_2, m_1\}\rangle = |m_2, m_1\rangle + \frac{2}{m_2 - m_1 + 3} \sum_{l>0} |m_2 + l, m_1 - l\rangle, \tag{3.8}
\]

\[
|\{m_1; n_1\}\rangle = |m_1; n_1\rangle - \frac{1}{(m_1 + 2n_1 + 1)} \sum_{l>0} |m_1 + l; n_1 - l\rangle. \tag{3.9}
\]
with
\[ |m_2, m_1\rangle = J_{-3-m_2}J_{-m_1}|0\rangle, \quad |m_1; n_1\rangle = J_{-m_1}J_{\frac{-1}{2}-n_1}|0\rangle. \] (3.10)

By explicit evaluation, we obtain the following norms for these states (again at \( p = 2 \))
\[
N_{\{m_2, m_1\}} = \langle \{m_2, m_1\} | \{m_2, m_1\} \rangle = \frac{m_2 - m_1 + 1}{m_2 - m_1 + 3}(m_2 + 3)(m_1 + 1),
\]
\[
N_{\{m_1; n_1\}} = \langle \{m_1; n_1\} | \{m_1; n_1\} \rangle = \frac{m_1 + 2n_1 + 2}{m_1 + 2n_1 + 1}(m_1 + 1)C_{n_1}^{\left(-\frac{1}{2}\right)}. \] (3.11)

### 3.3 Jack polynomials and the CS bases

In the previous section, we specified a complete set of eigenstates of the hamiltonian \( H_{CS} \) in terms of charged quasi-particles \( J \) and \( \phi \). It is clear that these same eigenstates can be obtained by applying an appropriate polynomial in the modes \( a_n = (\partial \varphi)_n \) of the auxiliary scalar field to a vacuum state \( |q\rangle \). It turns out that the polynomials that are needed are so-called Jack polynomials. In appendix B we briefly discuss some of their relevant properties.

Following Iso [9], we may specify a basis of CS eigenstates as follows
\[
|\{\mu\}_J, q\rangle = J_{\frac{1}{p}}^{\frac{1}{p}}(\{\sqrt{pa_n}\})|q\rangle = J_{\frac{1}{p}}^{\frac{1}{p}}|q\rangle,
\] (3.12)

with the \( U(1) \) charge \( q \) running over all integers, and \( \{\mu\} \) running over all Young tableaus. The norms of these states are given by
\[
\langle \{\mu\}_J, q | \{\mu\}_J, q \rangle = j_{\mu'}^{\frac{1}{p}}. \] (3.13)

An alternative ‘dual’ basis, consists of the states
\[
|\{\nu\}_\phi, q\rangle = J_{\frac{p}{p}}^{p}(\{\frac{a_n}{\sqrt{p}}\})|q\rangle = J_{\frac{p}{p}}^{p}|q\rangle,
\] (3.14)

with \( q \) integer and \( \{\nu\} \) running over all Young tableaus, with norms given by
\[
\langle \{\nu\}_\phi, q | \{\nu\}_\phi, q \rangle = j_{\nu'}^{p}. \] (3.15)
3.4 Correspondence between fqH and CS bases

Knowing that both the fqH quasi-particle basis and the CS Jack polynomial basis (3.12) are complete bases of orthogonal eigenstates of the operator $H_{CS}$, one quickly concludes that there is a 1-1 identification between these two bases. In this section we explicitly describe this 1-1 correspondence.

We start by observing that the Jack state

$$\left\{ \mu \right\} J_{\{\nu\}} = J_{\{\nu\}} \phi_{\{\nu\} M} \left| q \right>$$

(3.16)

can be written as the sum of a leading state

$$J_{-m_{1}+q+\frac{3}{2}} \cdots J_{-m_{2}+q+\frac{2}{2}+\frac{1}{p}} \left| q + \tilde{M} p \right>$$

(3.17)

and a ‘tail’ of sub-leading corrections, where ‘subleading’ refers to the triangular form of $H_{CS}$ on states of the form (3.17). Here we identified $m_{j} = \mu_{M+1-j} \geq 1$, with $\tilde{M} = l(\{\mu\})$.

Similarly we identify

$$\left\{ \nu \right\} \phi_{\{\nu\} + N} = J_{\{\nu\}} \phi_{\{\nu\} + N} \left| q \right>$$

(3.18)

with the $H_{CS}$ eigenstate headed by

$$\phi_{-n_{1}+\frac{3}{2}+\frac{3}{2p}} \cdots \phi_{-n_{i}+\frac{3}{2}+\frac{3}{2p}} \left| q - \tilde{N} \right>$$

(3.19)

with $n_{i} = \nu_{N+1-i} \geq 1$ and $\tilde{N} = l(\{\nu\})$.

Note that, as they stand, the expressions (3.17) and (3.19) are, in general, not of the form (2.3) that defines a member of the fqH basis.

Using the above, we find the following identifications for fqH states which contain only one type of mode operator

$$\left\{ n_{i} \right\}^{Q} = \left\{ \nu \right\} \phi_{\{\nu\} + N}$$

$$\left\{ m_{j} \right\}^{Q} = \left\{ \mu \right\} J_{\{\nu\}} ,$$

(3.20)

with $n_{i} = n_{N-i+1} \geq 1$ for $\tilde{N} - i \geq 0$ and similarly $\mu_{j} = m_{M-j+1} \geq 1$ for $\tilde{M} - j \geq 0$. Note that only the $n_{i}$ that are non-zero become a part of the tableau $\{\nu\}$; the $\phi$-modes with $n_{i} = 0$ change the charge of the vacuum without exciting any of the $(\partial \phi)_{n}$ modes. [A similar remark applies to the $J$-modes with $m_{j} = 0$.]

The duality eq. (B.3) on the Jack polynomials leads to the following duality relation for the Jack operators,

$$J_{\{\nu\}}^{p} = (-1)^{|\lambda|} j_{\{\lambda\}}^{p} J_{\{\lambda\}}^{\frac{1}{p}} .$$

(3.21)

This relation enables us rewrite the states $\left\{ n_{i} \right\}^{Q}$ to either the $\left\{ \nu \right\} J, Q + N$ or the $\left\{ \nu \right\} \phi, Q + N$ form. We can for example rewrite

$$\left\{ n_{i} \right\}^{Q} = \left\{ \nu \right\} \phi, Q + N$$

$$= (-1)^{|\nu|} j_{\{\nu\}}^{p} \left\{ \nu \right\} J, Q + N .$$

(3.22)
The last state is equivalent to the state which has
\[ J_{-\nu'_1 + Q + N + \frac{p}{2}} J_{-\nu'_2 + Q + N + \frac{p}{2}} \cdots J_{-\nu'_N + Q + N + \frac{2nN - 1}{2}} |Q + N + nNp\rangle \] (3.23)
as its leading state (assuming \( n_1 > 0 \)).

One may explicitly check that the eigenvalues of both the Virasoro zero-mode \( L_0 \) and and the CS hamiltonian \( H_{CS} \) are invariant under the duality transformation that identifies the state headed by (3.19) to the state headed by (3.23).

If there are no \( J \)-operators present in thefqH head state we can use the duality to transform the \( \phi \)-operators in the head state into \( J \)-operators and we achieve our goal of identitying the fqH state with a member of the CS \( J \)-basis.

If the fqH state has both \( J \) and \( \phi \)-operators present, we can still map the \( \phi \)-operators to a dual set of \( J \)-operators. Starting from the state \(|\{m_j\}, \{n_i\}\rangle_Q\) we see that, upon using the identity (3.22), the \( J \)-modes associated to the \( m_j \) ‘see’ their vacuum shifted from \(|Q\rangle\) to \(|Q + N\rangle\). Since the vacuum charge leads to a shift in the values of the mode-indices (see (3.17)), this means that in the CS basis, the corresponding \( J \)-modes will be labeled by \( m_i + N \) instead of \( m_i \). It is important to remark that this shift does not affect the contribution of the \( J \) modes to the eigenvalue of \( H_{CS} \) on the state. This is because the eigenvalues of \( H_{CS} \), as specified in eq. (3.5), depend directly on the full mode-indices in the head state and not just on the labels \( m_j \), as can be seen by comparing eq.(3.5) with (2.3).

We can now give the full mapping from a fqH basis state to a state in the CS basis
\[ |\{m_j; n_i\}\rangle_Q = |\{\sigma\}_{J}, Q + N - pM\rangle \]
with \( \{\sigma\} = (\{m\} + N^M) \cup \{\nu'\} \), (3.24)
where the sum of the partitions is \( \{\mu\} + N^M = (\mu_1 + N, \mu_2 + N, \ldots, \mu_M + N) \), and the cup product \( \{\lambda\} \cup \{\rho\} \) denotes the partition obtained from sorting the parts \( (\lambda_1, \ldots, \lambda_S, \rho_1, \ldots, \rho_R) \) in descending order.

The mapping from the CS basis back to the fqH basis is slightly more complicated. We start from a CS state \(|\{\sigma\}_{J}, q\rangle\), with \( S = l(\{\sigma\}) \), to which we associate a multi-\( J \) state as in (3.17). We note that the quantity \( \sigma_j - q - pj \) decreases with increasing label \( j \) which allows us to fix \( j \) such that
\[ \sigma_j - q - pj \geq 0 \ , \quad \sigma_{j+1} - q - p(j + 1) < 0 \] (3.25)
The condition on \( \sigma_j \) guarantees that the associated \( J \)-mode is an allowed mode on a fqH vacuum \(|Q\rangle\) with \(- (p - 1) \leq Q \leq 0 \). We now consider the state that remains when all \( J \)-modes left from and at the position \( j \) are removed. With the use of duality this state can be rewritten,
\[ |\{\nu'\}_{J}, q + pj\rangle \propto |\{\nu\}_{\phi}, q + pj\rangle \] (3.26)
with

\[ \nu'_i = \sigma_{i+j} \text{ for } i = 1 \ldots S - j. \] (3.27)

The \( \phi \)-operators in the leading state of \(|\{\nu\}_\phi, q + p j\rangle\) act on the vacuum with charge \( \tilde{Q} = q + p j - \sigma_{j+1} \) vacuum. The second condition in (3.25) guarantees that \( \tilde{Q} \geq -(p - 1) \). If one has \( \tilde{Q} \leq 0 \), one identifies \( Q = \tilde{Q} \) as the vacuum charge of the fqH basis state. If, however, \( \tilde{Q} \) is larger than zero the state \(|\tilde{Q}\rangle\) was created from \(|0\rangle\) using \( \phi \)-operators with the highest allowed mode index. From this argument we obtain that the state \(|\{\sigma\}_j, q\rangle\) from the CS basis can be rewritten as \(|\{m_i; n_i\}\rangle^Q\) with the following rules for selecting \( Q, M \) and \( N \) and the mode indices \( \{m_i\} \) and \( \{n_i\}\)

\[ M = j, \quad N = \max(q + p j, \sigma_{j+1}), \quad Q = q + p M - N \] (3.28)

\[ m_i = \sigma_{j+1-i} - N \text{ for } i = 1 \ldots j \]

\[ n_i = \nu_{\sigma_{j+1-i}} \text{ for } i = 1 \ldots \sigma_{j+1} \]

\[ = 0 \text{ for } i = \sigma_{j+1} + 1 \ldots N, \] (3.29)

with \( j \) and \( \{\nu\} \) as specified in (3.25) and (3.27).

### 3.5 Norms for the fqH basis

Of importance for later calculations are the norms of the states \(|\{m_j; n_i\}\rangle^Q\) of the fqH basis. These norms can be evaluated by using the fqH-CS correspondence,
together with the norms in the CS basis, as specified in (3.13) and (3.15). The
general result is
\[ N_{\{m_j; n_i\}} = \left| \langle \{m_j; n_i\} \rangle \right|^2 = (j_{\{m\}}^p)^2 (j_{\{m\}}^q)^2 , \]
where \{\nu\} and \{\mu\} are the tableaus corresponding to \{n_i\} and \{m_j\}, respectively,
and the tableau \{\sigma\} is specified in (3.24). The norm can be factorized into
the norms of the \(J\) and \(\phi\) parts separately times an extra factor associated with the
added partition \(N^M\)
\[ N_{\{m_j; n_i\}} = j_{\{m\}}^p j_{\{m\}}^q \prod_{(i,j) \in M^N} \frac{(\mu_j - i + M) + p(\nu_i - j + N + 1)}{(\mu_j - i + 1 + M) + p(\nu_i - j + N)} . \]
The expressions (3.31) are special cases of this general formula.

In the case where \(m_j, n_i \gg 1\) the expressions simplify and one finds the
following factorized form
\[ N_{\{m_j; n_i\}} \approx \prod_{j=1}^M m_j^{p-1} \prod_{i=1}^N n_i^{\frac{1}{2}-1} . \]

4 Form factors

4.1 Vacuum form factors

We start by considering the simplest non-vanishing form factors of the basic
electron operators \(J(z)\), \(J^\dagger(z)\) against the multi-particle states in the fqH basis
\[ \langle 0 | J^\dagger_{\{m\}} | \{n_p, \ldots, n_2, n_1\} \rangle_N = [N_{\{n_p, \ldots, n_2, n_1\}}]^{\frac{1}{2}} f_{J^\dagger}(n_p, \ldots, n_1) \delta_{m, n_p+\ldots+n_1} , \]
\[ \langle 0 | J_{\{m\}} | \{m_1\} \rangle_N = [N_{\{m_1\}}]^{\frac{1}{2}} f_{J}(m_1) \delta_{m, m_1} , \]
where the subscript \(N\) indicates that the state has been properly normalized.

One immediately finds
\[ f_{J}(m_1) = 1 . \]

We briefly explain the exact evaluation of the form factor \(f_J(n_p, \ldots, n_1)\) as
declared in (4.1). Let us consider the special case \(p = 2\) first. In that case the
operator \(J(z)\) has conformal dimension 1 and may be identified with one of the
currents of the affine Kac-Moody algebra \(\widehat{su}(2)_1\) (see appendix A). By exploiting
the OPE
\[ \phi(w_1)\phi(w_2) = (w_1 - w_2)^{\frac{1}{2}} \left[ J^\dagger(w_2) + O(w_1 - w_2) \right] \]
one obtains
\[ J^\dagger(w_2) = \oint_{C_{w_2}} \frac{dw_1}{2\pi i} (w_1 - w_2)^{-\frac{1}{2}} \phi(w_1)\phi(w_2) . \]
Using the expansion formula (B.7) we obtain
\[ J^\dagger(w_2)|0\rangle = \sum_{n_2,n_1} P^\frac{j}{m}_{\{n_2,n_1\}}(w_2,w_2)|\{n_2,n_1\}\phi, q = 2 \]  
(4.5)

and it follows that
\[ \langle 0|J_{1+m}|\{n_2,n_1\}\rangle_N = [N_{\{n_2,n_1\}}]^\frac{j}{m} P^\frac{j}{m}_{\{n_2,n_1\}}(1,1) \delta_{m,n_2+n_1}. \]  
(4.6)

For general \( p \) one obtains a similar result in terms of Jack polynomials with label \( \frac{j}{p} \). Using explicit expressions [22, 21] for \( P^\frac{j}{p}_{\{\ldots,1\}} \), we obtain
\[ f_J(n_p,\ldots,n_1) = \prod_{i=0}^{p-1} \Gamma(1 - \frac{i}{p}) \prod_{i<j} \Gamma(n_j - n_i + \frac{j-i+1}{p}) / \Gamma(n_i - \frac{i}{p}). \]  
(4.7)

In the large volume limit, where all \( n_i \gg 1 \), this becomes
\[ f_J(n_p,\ldots,n_1) = \frac{[\Gamma(\frac{j}{p})]^p}{\prod_{i=0}^{p-1} \Gamma(1 - \frac{i}{p})} \prod_{i<j} (n_j - n_i)^{\frac{1}{p}}. \]  
(4.8)

The form (4.8) of the form factor can be viewed as a limit in (chiral) CFT of a result on correlation functions for the ‘classical’ model of quantum mechanics with inverse square exchange. This result was conjectured by Haldane [19] and later proven in [20, 21].

An important insight is that there are no other non-vanishing form factors of the state \( J^\dagger_{-\frac{1}{2} - m}|0\rangle \) with the elements of the fqH basis. In other words, the spectral weight of this state is completely accounted for by states having precisely the minimal number of \( p \) quasi-holes. On the basis of this observation, it has been proposed that the \( T = 0 \) particle system underlying the fqH basis be viewed as an ‘ideal gas of fractional statistics particles’. In formula, this completeness is expressed by the following identity for the \( T = 0 \) 2-point function of the edge electron operator \( J_{-t} \)
\[ \frac{1}{m+1} \langle 0|J_{m+1}J^\dagger_{m-1}|0\rangle = \frac{1}{m+1} \sum_{n_2 \geq n_1, n_2 + n_1 = m} \langle 0|J_{m+1}|\{n_2,n_1\}\rangle_N \langle \{n_2,n_1\}|J^\dagger_{m-1}|0\rangle \]
\[ = \frac{1}{m+1} \sum_{n_2 \geq n_1, n_2 + n_1 = m} \frac{(n_2 - n_1)}{(n_2n_1)^{\frac{1}{2}}} = \int_0^{\frac{1}{m+1}} dx \frac{1 - 2x}{x(1-x)} = 1, \]  
(4.9)
where \( x = \frac{m_1}{m} \), and we inserted asymptotic expressions valid for \( m, n_2, n_1 \gg 1 \). This result is in agreement with a direct computation using algebraic properties of the \( J \) modes.

We remark here that, as we shall see in the next sections, the structure of more general form factors, shows ‘many body effects’ and is not easily reconciled with a notion of an ideal gas of fractional statistics particles.

4.2 More general form factors

We now consider more general form factors for the edge electron ‘annihilation operator’ \( J^\dagger(z) \). The simplest form factor with a 2-particle in-state is

\[
N\langle \{m'_1\}|J^\dagger_{\frac{m'}{2+m}}|\{m_2, m_1\}\rangle_N = \left[ \frac{N_{\{m_2,m_1\}}}{N_{m'_1}} \right]^{\frac{1}{2}} f_{J,J}(m'_1, m_2, m_1) \delta_{m,m_2+m_1-m'_1} .
\]  

Using the expansion formula (B.8) we derive the following general result

\[
f_{J,J}(m'_1, m_2, m_1) = \sum_{s=0}^{p} C_s^{(p)} p_{m'_1-m_1-s}^{(p)} \delta_{m'_1=m_{m_1}} .
\]  

(4.11)

where

\[
P_{\{m_2,m_1\}}^{(p)}(z_1, z_2) = z_1^{m_2} z_2^{m_1} \sum_{l=0}^{m_2-m_1} p_l^{(p)} \left( \frac{z_2}{z_1} \right)^l .
\]  

(4.12)

Specializing to \( p = 1, 2, 3 \), we have the following explicit results

\begin{align*}
\text{p = 1} & : \quad f_{J,J}(m'_1, m_2, m_1) = \delta_{m'_1=m_1} - \delta_{m'_1=m_2+1} \\
\text{p = 2} & : \quad f_{J,J}(m'_1, m_2, m_1) = \\
& \quad \delta_{m'_1=m_1} + \delta_{m'_1=m_2+2} - \frac{2}{m_2-m_1+1} \Theta(m_1 < m'_1 < m_2+2) \\
\text{p = 3} & : \quad f_{J,J}(m'_1, m_2, m_1) = \\
& \quad \delta_{m'_1=m_1} - \delta_{m'_1=m_2+3} - \frac{6(m_2+m_1-2m'_1)}{(m_2-m_1+1)(m_2-m_1+2)} \Theta(m_1 < m'_1 < m_2+3) .
\end{align*}

(4.13)

We also consider the case where the in-state contains one \( \phi \) and one \( J \) quantum

\[
N\langle \{n'_1\}|J^\dagger_{\frac{n'}{2+m}}|\{m_1; n_1\}\rangle_N = \left[ \frac{N_{\{m_1;n_1\}}}{N_{n'_1}} \right]^{\frac{1}{2}} f_{\phi,J}(n'_1, m_1, n_1) \delta_{m,m_1+n_1-n'_1} .
\]  

(4.14)
Using the expansion formula

$$: J(z) \phi(w) : |0\rangle = \sum_{m_1;n_1} r_{m_1,n_1}(z,w)|\{m_1;n_1\}\rangle$$  \hspace{1cm} (4.15)$$

with

$$r_{m,n}(z,w) = z^{m+1}w^n - \frac{m+1+pn}{m+p(n+1)}z^{m}w^{n+1},$$  \hspace{1cm} (4.16)$$

we derive

$$f_{\phi|J\phi}(n'_1, m_1, n_1) = \left[ \delta_{n'_1=n_1} - \frac{p-1}{m_1 + p(n_1+1)} \Theta(n'_1 > n_1) \right].$$ \hspace{1cm} (4.17)$$

While the results for $p = 1$ are a direct consequence of the Wick theorem, the expressions for $p \neq 1$ show that the ‘ideal gas interpretation’ is no longer applicable for general $p$: both form factors (4.10) and (4.14) can be non-vanishing when the energy of the electron annihilation operator $J^\dagger$ does not match the incoming electron energies $m_2$ or $m_1$, and where the energy difference is transferred to a second ‘spectator particle’. Furthermore, $f_{J|J,J}$ and $f_{\phi|J\phi}$ are not the only non-vanishing form factors of $J^\dagger$ with two incoming particles. For example, there are non-vanishing overlaps between a state created by applying $J^\dagger$ on the a 2-electron state and states containing more quasi-particles than just a single electron. The additional quasi-particles can be visualized as (neutral) density waves or excitons, which are composed of a single electron and $p$ quasi-holes.

In the next sub-section, we explore the selection rules that determine in a more general setting the possible out-states for which the form factor of $J^\dagger$ with a given in-state is non-vanishing.

### 4.3 A form factor selection rule

In this section we put a bound on the possible out-state that arise upon acting with an electron creation or annihilation operator on a given in-state. We perform this analysis in the CS basis, where the systematics of Jack polynomials come to our help. Using the mapping of section 3.4, the results can be translated to the fqH basis.

We focus on the form factors

$$\langle \{\mu\}_J, q|J_{-m_1+q+p-\frac{m}{2}}|\{\nu\}_J, q+p\rangle,$$ \hspace{1cm} (4.18)$$

with (unnormalized) Jack states as in (3.12).

The power of Jack Polynomial technology in analyzing this form factor comes from the fact a product of vertex-operators can be written as a sum over products of a ‘coordinates’ Jack polynomials $P_{\{\lambda\}}^P(\{z_i\})$ and ‘bosonic modes’ Jack.
operators $J_{\{\lambda\}}^\frac{1}{2}$. Both the coordinate and the bosonic mode Jack polynomials can be manipulated using the results for Jack polynomials in mathematical literature \cite{22, 23}. In Appendix B, we used this to rearrange the part of the vertex operators that survives after applying them to the vacuum. Here we apply the same method which enables us to analyze the action of a single mode operator $J_{-m+q+p-\frac{p}{2}}$ on a state created by a Jack operator. We note that for $m \geq 0$ this mode operator creates an additional edge electron.

A product of $N + 1$ edge electron operators acting on a vacuum $|\tilde{q}\rangle$ can be expanded in Jack polynomials and Jack operators,

$$J(w) \prod_{i=1}^{N} J(z_i)|\tilde{q}\rangle =$$

$$w^{-\tilde{q}} \prod_{i=1}^{N} (w - z_i)^p \prod_{i<j} (z_i - z_j)^p \prod_{i=1}^{N} z_i^{-\tilde{q}}$$

$$\times \left( \sum_{\{\lambda\}} (-1)^{|\lambda|} P_{i\{\lambda\}}^p (w) J_{\frac{1}{2}} \left( \sum_{\{\nu\}} (-1)^{|\nu|} P_{i\{\nu\}}^p (\{z_i\}) J_{\frac{1}{2}} \right) \right) \left( \sum_{\{\nu\}} (-1)^{|\nu|} P_{i\{\nu\}}^p (\{z_i\}) J_{\frac{1}{2}} \right) |\tilde{q} - pN - p\rangle. \tag{4.19}$$

The state $|\{\nu\}_J, \tilde{q} - Np\rangle$ with $N = l(\{\nu\})$ can be extracted from $\prod_{i=1}^{N} J(z_i)|\tilde{q}\rangle$ by applying the operator

$$O_{i\{\nu\}, \tilde{q}} (\{z_i\}) =$$

$$(P_{i\{\nu\}, l(\{\nu\})})^{-1} \left( \prod_{i=1}^{l(\{\nu\})} \oint \frac{dz_i}{2\pi i z_i^{-\tilde{q}+1}} \right) P_{\{\nu\}}^p (\{z_i^{-1}\}) \Delta^p (\{z_i^{-1}\}) , \tag{4.20}$$

where we made use of the inner product (3.6) on coordinate dependent Jack polynomials. The norm $P_{\{\nu\}, l(\{\nu\})}^p$ of the Jack polynomials will drop out of the final result. We can write

$$J_{-m+q+p-\frac{p}{2}} |\{\nu\}_J, \tilde{q} - p l(\{\nu\})\rangle =$$

$$M_{m-q-p}(w) J(w) O_{\{\nu\}, \tilde{q}} (\{z_i\}) \prod_{i=1}^{l(\{\nu\})} J(z_i)|\tilde{q}\rangle \tag{4.21}$$

where

$$M_m(w) = \oint \frac{dw}{2\pi i w^{m+1}}. \tag{4.22}$$
From the definition it is clear that $O_{\nu,\tilde{q}}^{p}(\{z_i\})$ commutes with $J(w)$, which allows us to interchange the order in the expression above and to use the expansion \[ \text{(4.19)} \] of $J(w)\prod_{i=1}^{N} J(z_i)|\tilde{q}\rangle$ in terms of Jack polynomials. By taking the inner product of the resulting expression with $\langle\{\mu\},q|\{\nu\}J,\tilde{q}\rangle$, where $q = \tilde{q} - (N + 1)p$, we obtain an expression for the form factor \[ \text{(4.18)}. \]

To decide for which choices of $\{\mu\}, \{\nu\}$ and $m$ the form factor can be non-vanishing we proceed as follows. We first rewrite the product

$$
\prod_{i=1}^{N} (w - z_i)^p = \sum_{\{n_i\}} C_{\{n_i\}}^{(p)} w^{pN} \prod_{i=1}^{N} \left( \frac{z_i}{w} \right)^{n_i}
$$

(4.23)

where $n_i = 0, 1, \ldots, p$. We insert this into the expansion given in \[ \text{(4.19)}, \]

and we use that $P_{\{\lambda\}}^{(p)}(w)$ is zero when $\{\lambda\}$ is not of the form $\{\lambda\} = (\lambda_1, 0, 0, \ldots)$. The contour integration contained in $M_{m-p-q}(w)$ selects the following value for $\lambda_1$

$$
\lambda_1 = |n| + m .
$$

(4.24)

Varying $|n| = \sum_i n_i$ over all allowed values $|n| = 0, \ldots, pN$, we find that $\lambda_1$ has to satisfy the inequalities

$$
m \leq \lambda_1 \leq m + pN .
$$

(4.25)

We write $\langle\{\mu\},J|\{\lambda\},J\{\rho\},J\rangle$ for the inner product $\langle\{\mu\},J,q|J^{\hat{\lambda}}_{\{\lambda\}},J^{\hat{\rho}}_{\{\rho\}}|q\rangle$. Using a result by Stanley (Proposition 5.3 from \[ \text{[22]} \]) we learn that this inner product is non-zero if and only if $\rho \subseteq \mu$ and $\mu/\rho$ is a horizontal $\lambda_1$-strip. The relation $\rho \subseteq \mu$ indicates that for all $i$ we have $\rho_i \leq \mu_i$. The skew tableau $\mu/\rho$ is the tableau containing all boxes which are in the tableau $\mu$ but not in the tableau $\rho$. If every column of the skew tableau contains at most one box it is called a horizontal $\lambda_1$-strip and if furthermore the total number of boxes in it is $\lambda_1$ it is called a horizontal $\lambda_1$-strip.

A particular consequence is that $\lambda_1$ satisfies

$$
0 \leq \lambda_1 \leq \mu_1 .
$$

(4.26)

Combining this inequality with \[ \text{(4.25)}, \]

we conclude that

$$
\max(0, m) \leq \lambda_1 \leq \mu_1 ,
$$

(4.27)

which implies that

$$
\mu_1 < m \Rightarrow J_{m-q-p+\frac{p}{2}}^{\hat{\mu}}(\{\mu\},q) = 0 ,
$$

(4.28)

in agreement with explicit step-functions in the form factors \[ \text{(4.13)} \] and \[ \text{(4.17)}. \]
Figure 4.1: $l$ arms and $p$ legs.

The form factor is now written as

$$
\langle q, \{\mu\}_J | J_{-m+q+p \frac{q}{2}} | \{\nu\}_J, q + p \rangle =
$$

$$
(p^p_{\{\nu\},(\{\nu\})})^{-1} \sum_{\{n\},\{\lambda_1\},\{\rho\}} \delta_{|n|+m,\lambda_1} C_{\{n\}}^{(p)} \langle \{\mu\}_J | \{\lambda_1\}_J \{\rho\}_J \rangle
$$

$$
\times \left( \prod_{i=1}^N M_0(z_i) \right) m_{\{n\}}(\{z_i\}) \Delta^p(\{z_i^{-1}\}) \Delta^p(\{z_i\}) P^p_{\{\rho\}}(\{z_i\}) P^p_{\{\nu\}}(\{z_i^{-1}\})
$$

where the summations extend over $i,j = 1,\ldots,N$ and $\{n\} = (n_1,\ldots,n_N)$ with $n_i \in 0,1,\ldots,p$.

The last part of this expression is the inner product on products of Jack polynomials with a finite number of arguments $\{z_i\}$. In Appendix C we discuss restrictions on the tableaus $\{n\}$, $\{\nu\}$ and $\{\rho\}$ that follow from imposing that this final inner product be non-zero. Combining all ingredients, one arrives at the following

**Form factor selection rule**

The form factor

$$
\langle \{\mu\}_J, q | J_{-m+q+p \frac{q}{2}} | \{\nu\}_J, q + p \rangle
$$

can only be non-zero if $\{\mu\}, \{\nu\}, m$ satisfy the following conditions

a. $|\nu| + m = |\mu|$
b. 1. \( \nu_j \geq \mu_{j+1} \) for all \( j \)

2. \( \nu_i \leq p \) for \( i > l(\{\mu\}) \)

c. \( m + \sum_{i \geq l(\{\mu\})} \nu_i \leq \mu_1 \)

d. \( \sum_{i=1}^j \nu_i \leq \sum_{i=1}^j (\mu_i + p) \).

These conditions imply that the tableau \( \{\nu\} \) should have at most \( p \) legs and \( l(\{\mu\}) \) arms, see fig. [4.3].

We refer to Appendix C for a complete proof of this result.

We remark that the above selection rule can be viewed as a generalization of a selection rule that was used by Lesage, Pasquier and Serban [21] for the evaluation of the zero-temperature density-density correlation function in the Calogero-Sutherland model. These authors found that the (neutral) density operator \( \rho \) when acting on the vacuum creates an ‘exiton’ with \( p \) quasi-holes and a single electron, corresponding to a Young tableau with \( p \) legs and a single arm. In the processes described by the form factor discussed in this section a similar structure is found. Starting from a multi-\( J \) state described by a tableau \( \{\mu\} \), the operator labeled by \( m \) annihilates one of the \( J \)-quanta. If there is a mismatch between the modes of the operator and of the quantum that is annihilated, the remaining momentum is carried away by a density fluctuation, which roughly speaking corresponds to one extra arm and the \( p \) legs that can be present in the tableau \( \{\nu\} \). If one starts from a state which, in the fqH basis, has a number of \( J \)-quanta and \( N \) quasi-holes, with maximal mode \( n_N \), one finds that upon annihilating a \( J \) mode up to \( n_N + 1 \) exitons can be created.

### 4.4 Relation with S-matrix approach

In this section, we consider the structure of the quasi-particle form factors from the point of view of an associated \( S \)-matrix structure.

Via the TBA procedure, the distribution functions for fractional exclusion statistics are linked to an \( S \)-matrix with the following dependence on particle rapidities \( \theta = \theta_2 - \theta_1 \)

\[
S_{ab}(\theta) = \exp[2\pi i(\delta_{ab} - G_{ab})\Theta(\theta)].
\]  

(4.30)

Although the quasi-particle states that we have considered are part of the discrete spectrum of a finite size system, it is natural to identify the quasi-particle states with a set of asymptotic particle states in a scattering theory with 2-body \( S \)-matrix of this type, with diagonal statistics matrix \( G_{11} = p, G_{22} = \frac{1}{p} \).

Via the well-known form factor axioms, this identification leads to specific properties of the form factors. In particular, we expect factors

\[
(\epsilon_i - \epsilon_j)^p, \quad (\tilde{\epsilon}_i - \tilde{\epsilon}_j)^{\frac{1}{p}},
\]

(4.31)
in form factors with particles $J(\epsilon_i)$ and $\phi(\tilde{\epsilon}_j)$ in the in-state, and annihilation poles between particles in the in- and out-states.

The explicit result (4.8) for the vacuum form factor $f_J$ has the expected zero’s $(n_i - n_j)^p$. For the more general form factors discussed in section 4.2 the structure is less clear. We observe however that, upon heuristically replacing

$$\delta_{m_2,m_1} \to \frac{1}{(\epsilon_2 - \epsilon_1)} , \quad \Theta(m_2 - m_1) \to \log(\epsilon_2 - \epsilon_1)$$

we have (for $p = 1, 2, 3$)

$$[\partial_{\epsilon'_1}]^{p-1} [f_{J|J}(\epsilon'_1, \epsilon_2, \epsilon_1)] \to \frac{(\epsilon_2 - \epsilon_1)^p}{(\epsilon'_1 - \epsilon_2)^p (\epsilon'_1 - \epsilon_2)^p} .$$

It will be most interesting to investigate whether the asymptotic limit of the form factors considered and computed in this paper can be obtained by means of an axiomatic approach.

5 Form factor expansion at finite temperature

5.1 General remarks

In a system of non-interacting electrons, transport properties such as $I$-$V$ and noise characteristics are obtained by computing the relevant amplitudes for transmission and reflection of single particles, and then performing a statistical average using a one-particle Fermi-Dirac distribution function. An important goal, that we had in mind when setting up the quasi-particle formulation of fqH edges, is to arrive at a similar description of transport processes in these interacting, highly non-Fermi liquid, systems.

As a first attempt in this direction, one may try to simply replace free electron amplitudes by corresponding amplitudes for fqH quasi-particles, and simultaneously replace the Fermi-Dirac distribution by an appropriate distribution function for fractional statistics. While, as we shall argue, this idea is essentially correct, we stress that a correct implementation is subtle and involves the important concept of a so-called form factor expansion.

In this section, we shall focus on the following finite temperature Green’s functions in the CFT for the $\nu = \frac{1}{p}$ fqH edge

$$h(\epsilon) = \langle \psi^{\dagger}_{\nu=\frac{1}{p}}(\epsilon_{\nu=\frac{1}{p}}(\epsilon)) \rangle_T , \quad H(\epsilon) = \langle \psi_{\nu=\frac{1}{p}}(\epsilon_{\nu=\frac{1}{p}}(\epsilon)) \rangle_T ,$$

where the operators $\psi^{\dagger}_{\nu=\frac{1}{p}}(\epsilon)$ and $\psi_{\nu=\frac{1}{p}}(\epsilon)$ are the continuum limits of the edge electron operators $J_s$ and $J^\dagger_s$ considered in this paper.
In the next subsection, we recall how this Green’s function is used for the computation of the $I-V$ characteristics for the tunneling of electrons into a $\nu = \frac{1}{3}$ $\text{fqH}$ edge. After that, we give the general form of the form factor expansions for finite temperature correlation functions. We shall then zoom in on the case $m = 2$, and explain how the finite-$T$ Green’s function $h(\epsilon)$ can be approximated in a form factor expansion.

### 5.2 Kinetic equation for electron tunneling

As explained in [24, 8], the Green’s functions (5.1) can be used to computed the finite temperature $I-V$ characteristics for the tunneling of electrons from a Fermi Liquid (FL) reservoir into a $\nu = \frac{1}{3}$ $\text{fqH}$ edge. Starting from the tunneling hamiltonian

$$H_{\text{int}} \propto t \int d\epsilon \left[ \psi^\dagger_{\text{FL}}(\epsilon) \psi_{\nu=\frac{1}{3}}(\epsilon) + \text{h.c.} \right],$$

one can show that, in lowest order perturbation theory, the current-voltage characteristics are given by

$$I(V,T) \propto e t^2 \int_{-\infty}^{\infty} d\epsilon \left[ f(\epsilon - eV)H(\epsilon) - F(\epsilon - eV)h(\epsilon) \right],$$

where $f(\epsilon)$ and $F(\epsilon)$ are the standard Fermi-Dirac distributions for particles and holes. Using the conformal mapping from a plane to a cylinder, or employing an imaginary time approach, one finds the following exact expression for the case $\nu = \frac{1}{3}$

$$H(\epsilon) = \frac{\epsilon^2 + \frac{\pi^2}{\beta^2}}{e^{-\beta \epsilon} + 1}, \quad h(\epsilon) = \frac{\epsilon^2 + \frac{\pi^2}{\beta^2}}{1 + e^{\beta \epsilon}}.$$

They lead to $I-V$ characteristics

$$I(V,T) \propto e t^2 \beta^{-3} \left( \frac{\beta eV}{2\pi} + \left( \frac{\beta eV}{2\pi} \right)^3 \right),$$

in agreement with the result obtained in different approaches [23, 20]. The $I-V$ characteristics (5.5) show cross-over from a linear (thermal) regime into a power-law behavior at high voltages and thus presents a clear fingerprint of the Luttinger liquid features of the $\text{fqH}$ edge. The experimental results of [27] are in agreement with these predictions. (See [20] for a further theoretical analysis of the data.)
5.3 Form factor expansion

As a proto-type study for a form factor expansion based on CFT quasi-particles, we now analyze the Green's function $h(\epsilon)$, for $p = 2$ in that spirit. Obviously, an exact result is easily obtained

$$h(\epsilon) = \frac{\epsilon}{e^{\beta \epsilon} - 1}.$$ (5.6)

The Bose-Einstein denominator in this expression has its origin in the fact that the operators $J, J^\dagger$ satisfy bosonic commutation relations. In the spirit of the quasi-particle formulation of this paper, we wish to treat the $J, J^\dagger$-quanta as quasi-particles with exclusion statistics $g = 2$, and see if we can recover the Green’s function $h(\epsilon)$ in such an approach.

The Green’s function $h(\epsilon)$ can be viewed as a one-point function for the operator $N_\psi(\epsilon) = \psi^\dagger_{\nu=1} \psi_{\nu=p}(\epsilon)$. In the formulation on the finite system of size $L$, this operator is represented as $N_J(m) = a J_{-m} J_{m}^\dagger$, with $\epsilon = a m$, with $a = \frac{2\pi}{L \rho_0}$ the energy level spacing in the finite size system. This one-point function is formally expressed as

$$\sum_{\Psi \in \mathcal{H}} \langle \Psi | N_J(m) | \Psi \rangle \sum_{\Psi \in \mathcal{H}} \langle \Psi | \Psi \rangle.$$ (5.7)

The sum runs over a basis the full Hilbert space of the edge CFT, and we can opt for the fqH quasi-particle basis discussed in this paper. The idea is now that the matrix elements $\langle \Psi | N_J(m) | \Psi \rangle$ are dominated by processes where only a few of the quasi-particles that are present in a concrete basis state $|\{m_i; n_j\}\rangle$ participate (we restrict our attention to states in the $Q = 0$ sector of the fqH basis).

For the case at hand, the lowest contributions comes from 1-particle states $|\{m_1\}\rangle$, for which one computes the form factor

$$D^{(1,0)}(m; m_1) = \langle \{m_1\} | J_{-m} J_{+1+m}^\dagger | \{m_1\} \rangle_N = (m + 1) \delta_{m, m_1} + 2 \left( 1 - \frac{m + 1}{m_1 + 1} \right) \Theta(m < m_1).$$ (5.8)

The expected presence of an edge electron of energy $m_1$ is given by the distribution function $\tilde{n}_2(\epsilon_1 = a m_1)$. This leads to the following contribution to the Green’s function

$$h^{(1,0)}(\epsilon) = a \sum_{m_1} D^{(1,0)}(m, m_1) \tilde{n}_2(a m_1).$$ (5.9)

If we now consider the form factor of $N_J(m)$ against a two-electron state, we find (see next subsection) that it is not simply the sum of two 1-particle contributions.
The left-over part is what we call the irreducible 2-electron form factor
\[ D^{(2,0)}(m; m_1, m_2) = N\langle\{m_1, m_2\}|J_{-3-m}J_{+3+m}\rangle_{N} \]
\[ - N\langle\{m_1\}|J_{-3-m}J_{+3+m}\rangle_{N} - N\langle\{m_2\}|J_{-3-m}J_{+3+m}\rangle_{N} .\] (5.10)

It leads to an additional contribution \( h^{(2,0)}(m) \) to the Green’s function
\[ h^{(2,0)}(\epsilon) = a\sum_{m_1, m_2} D^{(2,0)}(m; m_1, m_2)\bar{n}_2(\alpha m_1)\bar{n}_2(\alpha m_2) . \] (5.11)

Similarly, we define
\[ D^{(1,1)}(m; m_1, n_1) = \]
\[ N\langle\{n_1, m_1\}|J_{-1-m}J_{+1+m}\rangle_{N} - N\langle\{m_1\}|J_{-1-m}J_{+1+m}\rangle_{N} \] (5.12)
and
\[ h^{(1,1)}(\epsilon) = a\sum_{m_1, n_1} D^{(1,1)}(m; m_1, n_1)\bar{n}_2(\alpha m_1)\bar{n}_{1\frac{1}{2}}(\alpha n_1) . \] (5.13)

Continuing in this manner, we build up the following expansion
\[ h = \sum_{M, N} h^{(M,N)}(\epsilon) , \]
\[ h^{(M,N)}(\epsilon) = a\sum_{\{m_i; n_j\}} D^{(M,N)}(m; \{m_i; n_j\}) \prod_i \bar{n}_2(\alpha m_i) \prod_j \bar{n}_{1\frac{1}{2}}(\alpha n_j) . \] (5.14)

We remark that an expansion of precisely this type has been proposed by LeClair and Mussardo [11], see also [12]. This work was done in the context of integrable qft’s, that are fully characterized by a factorized \( S \)-matrix. In such a context, the irreducible form factors are constrained by the form factor axioms, and the distribution functions have their origin in a TBA procedure. Although clearly in the same spirit, the analysis that we present here is very different at the technical level. We obtain the relevant form factor by explicit computation in a theory that is regularized by the finite size of the fqH edge, and we have identified the relevant distribution functions by analyzing the state counting of the (discrete) spectrum of the finite-size system. We thus do not rely on an underlying (massless) \( S \)-matrix point of view. Despite these differences, it seems clear that the two approaches are closely related: in subsection 4.5 we briefly indicated that our form factor have symmetry properties that are expected on the basis of a ‘purely statistical \( S \)-matrix’. We leave this interesting issue for further study.

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5.4 Irreducible form factors

To evaluate explicitly the leading terms in the form factor expansion (5.14) for $h(\epsilon)$, we need to evaluate the relevant irreducible form factors. While it is clear that these form factors have very special mathematical properties, we here compute them by a simple brute force computation, relying on the explicit form of the two-particle states (3.8) and (3.9), and on the algebraic properties of the operators $J$, $J^\dagger$ and $\phi$ (see appendix A).

5.4.1 Two electrons

For the irreducible two electron form factor we find

$$D^{(2,0)}(m; m_2, m_1) =$$

$$\delta_{m-m_2} - \frac{2(m_2 + 3)}{m_2 - m_1 + 3} + \delta_{m-m_1+2} - \frac{2(m_1 + 1)}{m_2 - m_1 + 1}$$

$$+ \frac{1}{(m_2 - m_1 + 3)} \frac{1}{(m_2 - m_1 + 1)} \frac{1}{(m_1 + 1)(m_2 + 3)}$$

$$\times [\Theta(m < m_1 - 2) P(m; m_1, m_2)$$

$$+ \Theta(m < m_2 < m + m_1) Q(m; m_1, m_2)$$

$$+ \Theta(m < m_2) R(m; m_1, m_2)] ,$$

(5.15)

with

$$P(m; m_1, m_2) =$$

$$(m_2 - m_1 + 3)(m_1 - m - 2)(2m_1 - m_2 - 3)$$

$$+ (m_1 - m - 2)(m_1 - m - 3)(-3m_2 + \frac{5}{3}m_1 + \frac{1}{3}m - \frac{26}{3})$$

$$+ (m + 3)[-2(m_2 - m_1 + 3)(2m_1 - m - 1)$$

$$-2m_1(m_1 + 1) + (m + 3)(m_2 + m_1 - m + 1)]$$

$$Q(m; m_1, m_2) =$$

$$(m_1 - m_2 + m + 1)[(m_2 - m_1 + 3)^2 + 2(m_2 - m_1 + 3)(m_1 - m_2 + m)$$

$$+ \frac{2}{3}(m_1 - m_2 + m)(m_1 - m_2 + m - 1)]$$

$$R(m; m_1, m_2) =$$

$$(m_2 - m)(m_1 + 1)(m_2 - m_1 + 3) + \frac{1}{3}m_1(m_1 + 1)(m_1 + 3m_2 - 3m + 2) .$$

(5.16)
The polynomials $P, Q$ and $R$ enjoy special properties, which include

$$(P + Q + R)(m; m_1, m_2) = -\frac{1}{3}(m_1 - m_2 - 1)(m_1 - m_2 - 2)(m_1 - m_2 - 3). \quad (5.17)$$

### 5.4.2 One electron and one quasi-hole

The irreducible form factor with one electron and one hole is found to be

$$D^{(1,1)}(m; m_1, n_1) = \delta_{m_1, m_1} \frac{m_1 + 1}{m_1 + 2n_1 + 1} + \Theta(m < m_1) \frac{1}{C_{n_1}^{(-1)}(m_1 + 2n_1 + 2)(m_1 + 2n_1 + 1)(m_1 + 1)} \times \left[ C_{n_1 - m_1 + m}^{(-\frac{1}{2})} S(m; m_1, n_1) + C_{n_1}^{(-\frac{1}{2})} T(m; m_1, n_1) \right], \quad (5.18)$$

with

$$S(m; m_1, n_1) = (m_1 + 2n_1 + 1)^2 + (m + n_1 - m_1)(\frac{8}{3} - 4(m_1 + 2n_1 + 2)) + \frac{4}{3}(m + n_1 - m_1)^2$$

$$T(m; m_1, n_1) = 2(m_1 - m)((m_1 + 2n_1 + 1)^2 - 1) + 2(2n_1 + 1)(m_1 - m - 1) + 2(\frac{2}{3}n_1 + 1)(2n_1 + 1). \quad (5.19)$$

### 5.5 Evaluating the series

With the information collected in the previous subsections, we can evaluate the 1-particle and 2-particle contributions $h^{(1,0)}$, $h^{(2,0)}$ and $h^{(1,1)}$ to the Green's function $h(\epsilon)$.

The expressions (5.9), (5.11) and (5.13) for $h^{(2,0)}$ and $h^{(1,1)}$ are discrete sums, which we wish to study in the limit $a \to 0$. In this limit, one may view the expressions as Riemann sums and evaluate them using continuous integrals; however, one needs to be careful because the integrands as they stand have singularities, and the sums are not term-by-term convergent. One may check however that by carefully redistributing some of the terms, one obtains convergent sums that can be approximated by the corresponding continuous integrals. Proceeding in this manner, and using a numerical integrator, we obtained the results plotted in figure 5.1.
We observe that the form factor series converge in the following sense: while the 1-particle terms agree with the exact result for \( \epsilon \) greater than about \( 3k_B T \), the approximation including 2-particle terms reaches the exact curve at \( \epsilon \) around \( 2k_B T \). For energies \( \epsilon \ll k_B T \), the thermal factors do not efficiently suppress many particle contributions, and the convergence of the form factor expansion is expected to be slow.

We remark that the asymptotic behavior for \( \epsilon \gg k_B T \) of the 2-particle terms is
\[
h^{(2,0)}(\epsilon) \sim c_2 e^{-\beta \epsilon}, \quad h^{(1,1)}(\epsilon) \sim c_1 e^{-\beta \epsilon}
\]
with
\[
c_2 = -2 \int_0^{\infty} d\epsilon_1 \bar{n}_2(\epsilon_1), \quad c_1 = \int_0^{\infty} d\epsilon_1 \bar{n}_1^2(\epsilon_1).
\]

Remarkably, the duality relation (2.8) leads to the relation
\[
c_2 = -c_1
\]
meaning that the Boltzmann tails of the 2-particle terms precisely cancel. This ‘conspiracy’ was needed as, numerically, it is seen that the deviation between the exact curve \( h(\epsilon) \) and the 1-particle term \( h^{(1,0)}(\epsilon) \) is far smaller than the individual Boltzmann tails of \( h^{(2,0)} \) and \( h^{(1,1)} \).

6 Conclusions

Summarizing the results collected in this paper, we have made some first steps on the way to realizing a computational scheme where the \( T \)-dependence of physical observables in a fqH system (charge transport properties in particular) is computed with direct reference to fractional statistics of the fundamental quasi-particles. We expect that on the basis of the formalism presented here, meaningful claims about the observability of the fractional statistics of CFT edge quasi-particles can be formulated. We leave this most interesting aspect for further study.

We remark the the continuum (CFT) limit of the CS model provides an ideal testing ground for form factor expansions for finite temperature correlation functions, such as discussed in section 5.3 and in the literature [11, 12]. This is because on the one hand the theory is explicitly regularized by the finite extent of the spatial direction and, on the other, the finite temperature Green’s functions are known from standard CFT considerations.

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A Algebraic properties of $\nu = \frac{1}{2}$ edge operators

The charged edge operators $J = J^-, J^+ = J^+$ in the edge theory at $\nu = \frac{1}{2}$ are part of a $SU(2)_1$ affine symmetry algebra. Together with the charge density $Q = i\sqrt{p}\partial\varphi$ they satisfy the commutation relations

$$[J_{m_2}^+, J_{m_1}^-] = m_1\delta_{m_2+m_1} + Q_{m_2+m_1},$$
$$[Q_{m_2}, J_{m_1}^+] = \pm 2J_{m_2+m_1}^+, \quad [Q_{m_2}, Q_{m_1}] = 2m_1\delta_{m_2+m_1}. \quad (A.1)$$

The fractionally charged edge quasi-particles $\phi^\pm$ transform in the spin-$\frac{1}{2}$ representation of the $SU(2)_1$ symmetry

$$[J_m^+, \phi_s^\pm] = 0, \quad [J_m^-, \phi_s^\pm] = \pm\phi_{m+s}^\pm, \quad [Q_m, \phi_s^\pm] = \pm\phi_{m+s}^\pm. \quad (A.2)$$

Among themselves, the modes of $\phi^\pm$ satisfy so-called generalized commutation relations, which have been studied in the context of the spinon formulation of the $SU(2)_1$ CFT [28, 29].

B Jack polynomials and Jack operators

In this appendix we briefly introduce the Jack polynomials that are used in sections 3 and 4 of this paper. We essentially follow the conventions of Iso [1], but we introduce different notations for the coordinate dependent Jack polynomials $P_{\beta}^\mu (\{z_i\})$ and the bosonic mode Jack operators $J_{\beta}^\mu (\{\frac{a-n}{\sqrt{\beta}}\})$.

We start by specifying an inner product on the ring of symmetric polynomials,

$$\langle p_{\{\lambda\}} | p_{\{\mu\}} \rangle_\beta = \delta_{\{\lambda\},\{\mu\}}\beta^{-l(\{\lambda\})} z_{\lambda} \quad (B.1)$$

where $p_{\{\lambda\}} (z_i) = \prod_{j=1}^{l(\{\lambda\})} p_{\lambda_j} (\{x_i\})$ with $p_{\lambda_j} (\{x_i\}) = \sum x_i^{\lambda_j}$ is the power sum set by a Young tableau $\{\lambda\} = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, $z_{\{\lambda\}}$ is $\prod_{i\geq 1} i^{l_i} l_i!$ with $l_i$ the number of entries in $\{\lambda\}$ which satisfy $\lambda_i = j$ and $\beta$ a rational number.

The coordinate Jack polynomials $P_{\{\lambda\}}^\beta (\{z_i\})$ are symmetric functions in the coordinates $\{z_i\}$ labeled by a Young tableau $\{\lambda\}$ and a rational number $\beta$. They are defined by the following properties

orthogonality:

$$\langle P_{\{\lambda\}}^\beta (\{z_i\}) | P_{\{\nu\}}^\beta (\{z_i\}) \rangle_\beta = \delta_{\{\lambda\},\{\nu\}} J_{\{\nu\}}^\beta$$

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triangularity:
\[ P^\beta_{\{\lambda\}}(\{z_i\}) = \sum_{\mu} v_{\lambda,\mu}(\beta)m_{\{\mu\}} \text{ where } v_{\lambda,\mu}(\beta) = 0 \text{ unless } \mu \leq \lambda \]
normalization:
\[ \text{the coefficient } v_{\lambda,\lambda} = 1. \]

In this definition, \( m_{\{\lambda\}}(\{z_i\}) \) are the monomial symmetric functions \( \sum_\sigma \prod_i z_{\lambda_i}^{\sigma_i} \) where \( \sum_\sigma \) denotes the sum over all permutations of the indices \( i \). The partial ordering \( \leq \) on partitions is the so-called dominance ordering on partitions of equal weight (\( |\lambda| = |\mu| \)): \( \lambda \leq \mu \iff \sum_i i^j \lambda_i \leq \sum_i i^j \mu_j \) for all \( j \). The function \( j^\beta_{\{\nu\}} \) in the inner product can be shown to be given by \[ j^\beta_{\{\nu\}} = \prod_{(i,j) \in \{\nu\}} \frac{\beta(\nu'_j - i) + \nu_i - j + 1}{\beta(\nu'_j - i + 1) + \nu_i - j}. \] (B.2)

In a notation where Jack polynomials are written as functions of power sums \( p_n \), they satisfy a duality between \( \beta = p \) and \( \beta = \frac{1}{p} \).

\[ P^p_{\{\lambda\}} \left( \left\{ \frac{p_n}{p} \right\} \right) = (-1)^{|\lambda|} j^p_{\{\lambda\}} P^{\frac{1}{p}}_{\{\lambda'\}} (-\{p_n\}) , \] (B.3)

where \( \{\lambda'\} \) is the Young tableau dual to \( \{\lambda\} \). It follows that
\[ j^p_{\{\nu\}} j^{\frac{1}{p}}_{\{\nu'\}} = 1. \] (B.4)

The following elementary property of the Jack polynomials
\[ \prod_{i,j} (1 - x_i y_j) = \sum_{\{\lambda\}} (-1)^{|\lambda|} P^\beta_{\{\lambda\}} (\{x_i\}) P^{\frac{1}{\beta}}_{\{\lambda'\}} (\{y_j\}) \] (B.5)
can be used to rewrite expressions involving vertex-operators.

For integer \( \beta \) an alternative inner product [23] on the Jack polynomials \( P^\beta_{\{\mu\}} \) depending on only a finite set of coordinates \( \{z_i\} = \{z_1, ..., z_n\} \) is given by
\[ \langle\langle P^\beta_{\{\nu\}} (\{z_i\}) | P^\beta_{\{\mu\}} (\{z_i\}) \rangle\rangle = \left( \prod_{i=1}^n \int \frac{dz_i}{2\pi i z_i} \right) \Delta^\beta(\{z_i^{-1}\}) \Delta^\beta(\{z_i\}) P^\beta_{\{\nu\}} (\{z_i^{-1}\}) P^\beta_{\{\mu\}} (\{z_i\}) , \] (B.6)

where \( \Delta^\beta(\{x_i\}) = \prod_{i<j} (x_i - x_j)^\beta \) denotes a generalized Vandermonde determinant. Although it is also possible to define this inner product for fractional \( \beta \) [23], we will use it in this form for integer \( \beta \). The Jack polynomials are orthogonal w.r.t. this alternative inner product.
For the product of \( N \) quasi-hole vertex operators \( \phi(z_i) \), the following expression can be derived

\[
\phi(z_1) \ldots \phi(z_N)|q\rangle = 
\Delta^\frac{1}{\sqrt{p}}(\{z_i\}) \sum_{\lambda} (-1)^{|\lambda|} P_{\{\lambda\}}^p (\{z_j\}) \ J_{\{\lambda\}}^p \left( \left\{ \frac{a_n}{\sqrt{p}} \right\} \right) \prod_{j=1}^N z_j^\frac{\sqrt{p}}{q} |q + N\rangle, \tag{B.7}
\]

where we wrote \( J_{\{\lambda\}}^p \left( \left\{ \frac{a_n}{\sqrt{p}} \right\} \right) \) for a Jack polynomials in which power sums \( p_n \) are replaced by bosonic modes, writing \( a_n = (\partial \varphi)_n \). [We refer to such expressions as Jack operators.] Similarly,

\[
J(z_1) \ldots J(z_N)|q\rangle = 
\Delta^p(\{z_i\}) \sum_{\lambda} (-1)^{|\lambda|} P_{\{\lambda\}}^p (\{z_j\}) \ J_{\{\lambda\}}^p (\{\sqrt{p}a_n\}) \prod_{j=1}^N z_j^{-q}|q - Np\rangle. \tag{B.8}
\]

For brevity, we sometimes drop the explicit reference to the bosonic modes and write

\[
J_{\{\lambda\}}^\frac{1}{\sqrt{p}} \equiv J_{\{\lambda\}}^\frac{1}{\sqrt{p}} (\{\sqrt{p}a_n\}), \quad J_{\{\lambda\}}^p \equiv J_{\{\lambda\}}^p (\left\{ \frac{1}{\sqrt{p}} a_n \right\}). \tag{B.9}
\]

C  Proof of selection rules

We present a proof of the form factor selection rules of section 4.3.

(a.) This is a consequence of energy conservation and can be found from the product of three delta functions,

\[
\delta_{|n|+m, \lambda_1} \delta_{n+|\rho|, |\mu|} \delta_{|n|+|\rho|, |\nu|} \tag{C.1}
\]

present implicitly in eq. (1.29).

(b.) Proposition 2.4 in \cite{22} states that it is possible to rewrite any Jack polynomial as a linear combination of products of Jack polynomials labeled with horizontal strips \( J_{\{\lambda\}}^p = \prod_i P_{\{\lambda_i\}}^p \),

\[
P_{\{\lambda\}}^p = \sum_{\{\sigma\} \geq \{\lambda\}} \tilde{q}(\{\lambda\}, \{\sigma\}) J_{\{\sigma\}}^p. \tag{C.2}
\]

An important difference between this expansion and the expansion of a Jack polynomial in monomial symmetric functions is that the sum runs over tableaus \( \{\sigma\} \) satisfying \( \{\sigma\} \geq \{\lambda\} \) instead of \( \{\sigma\} \leq \{\lambda\} \). From repeated application of proposition 5.3 in \cite{22} (see also section 4.3) to a product of two Jack polynomials, where one is expanded using the expansion above, it follows that the tableau
labeling the non-expanded Jack polynomial is contained in every tableau labeling a Jack polynomial appearing in the product. Exchanging the roles we see that also the tableau labeling the other Jack polynomial is contained in these tableaus.

Combining this knowledge with the fact that by triangularity the monomial symmetric functions $m_{\{n\}}$ can be expanded in Jack polynomials,

$$m_{\{n\}} = \sum_{\{\tau\} \leq \{n\}} \tilde{v}_{\{n\}\{\tau\}} P_{\{\tau\}}, \quad (C.3)$$

and applying this to the coordinate inner product in eq. (1.29) we find that $\{\rho\}$ is contained in $\{\nu\}$. The operator inner product shows that $\{\mu\}$ differs at most a horizontal strip from $\{\rho\}$ and thus $\{\rho\}$ contains $\{\tilde{\mu}\} = (\mu_2, \ldots, \mu_M)$. We can conclude that $\{\nu\}$ contains $\{\tilde{\mu}\}$ and we have obtained (b.1.).

We can extract extra information from examining this construction once more, under the addition of a horizontal strip the length of a column can grow with one box. If we now multiply the Jack polynomials $P_{\{\text{tau}\}}^p$ appearing in the expansion of the monomial symmetric function $m_{\{n\}}$ with the $J_{\{\sigma\}}^p$ appearing in the expansion of the Jack polynomial $P_{\{\rho\}}^p$ we find that the maximal difference in column length between $\{\nu\}$ and $\{\tau\}$ is $l(\{\rho\})$ from which we can conclude that the only columns in $\{\nu\}$ which have a length exceeding $l(\{\mu\}) \geq l(\{\rho\})$ are those columns for which the corresponding tableau labeling the monomial symmetric function has a column of non-zero length. Because only monomial symmetric functions with at most $p$ non-zero columns appear we obtain (b.2.).

(c.) This is a simple consequence of (a.) and (b.1.)

(d.) By definition the Jack polynomials can be expanded in monomial symmetric functions, so we have

$$P_{\{\rho\}}^p = \sum_{\{\sigma\} \leq \{\rho\}} v_{\{\rho\}\{\sigma\}} m_{\{\sigma\}}. \quad (C.4)$$

If we now use this expansion $P_{\{\rho\}}^p$ and then multiply the resulting $m_{\{\sigma\}}$ in the expansion by $m_{\{n\}}$, then the products in the expansion will be linear combinations of $m_{\{\sigma'\}}$ satisfying $\{\sigma'\} \leq \{\sigma\} + \{n\},$

$$m_{\{n\}} P_{\{\rho\}}^p = \sum_{\{\sigma\} \leq \{\rho\}} v_{\{\rho\}\{\sigma\}} m_{\{n\}} m_{\{\sigma\}} = \sum_{\{\sigma'\} \leq \{\rho\} + \{n\}} u_{\{n\}\{\rho\}\{\sigma'\}} m_{\{\sigma'\}}. \quad (C.5)$$

Expanding the $m_{\{\sigma'\}}$ as we did in the proof of (b.) the sum over products of monomial symmetric functions can be rewritten in terms of Jack polynomials again,

$$m_{\{n\}} P_{\{\rho\}}^p = \sum_{\{\sigma\} \leq \{\rho\} + \{n\}} w_{\{n\}\{\rho\}\{\sigma\}} P_{\{\sigma\}}^p, \quad (C.6)$$
and we find that \( \{ \nu \} \) is smaller in the sense of dominance order than a tableau \( \{ \sigma \} \) of the form \( \{ \rho \} + \{ n \} \). Since \( n_i \leq p \) and \( \rho_i \leq \mu_i \), the result (d.) follows.

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Figure 5.1: One-particle Green’s function $h(\epsilon)$ for $p = 2$ as a function of energy. The solid curve is the exact result; the data points are the numerical results for: $h^{(1,0)}$ (diamonds), $h^{(2,0)}$ (circles) and $h^{(1,1)}$ (plusses). The sum of all contributions with up to two particles is represented by squares.