Distributed Zero-Order Optimization under Adversarial Noise

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Abstract

We study the problem of distributed zero-order optimization for a class of strongly convex functions. They are formed by the average of local objectives, associated to different nodes in a prescribed network. We propose a distributed zero-order projected gradient descent algorithm to solve the problem. Exchange of information within the network is permitted only between neighbouring nodes. An important feature of our procedure is that it can query only function values, subject to a general noise model, that does not require zero mean or independent errors. We derive upper bounds for the average cumulative regret and optimization error of the algorithm which highlight the role played by a network connectivity parameter, the number of variables, the noise level, the strong convexity parameter, and smoothness properties of the local objectives. The bounds indicate some key improvements of our method over the state-of-the-art, both in the distributed and standard zero-order optimization settings. We also comment on lower bounds and observe that the dependency over certain function parameters in the bound is nearly optimal.

1 Introduction

We study the problem of distributed optimization where each node (or agent) has an objective function $f_i : \mathbb{R}^d \rightarrow \mathbb{R}$ and exchange of information is limited between neighbouring agents within a prescribed network of connections. The goal is to minimize the average of these objectives on a closed bounded convex set $\Theta \subset \mathbb{R}^d$.

$$\min_{x \in \Theta} f(x) \quad \text{where} \quad f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x).$$

Distributed optimization has been widely studied in the literature, we refer to Tsitsiklis et al. [1986], Nedic and Ozdaglar [2009], Nedic et al. [2010], Boyd et al. [2011], Duchi et al. [2012], Jakovetic et al. [2014], Lobel et al. [2011], Xia et al. [2013], Shi et al. [2014], Jakovetic [2019], Scaman et al. [2019], Pu et al. [2021] and references therein. This problem has broad applications such as multi-agent target seeking [Liu et al. 2017], distributed learning [Kraska et al. 2013], and wireless networks [Park et al. 2020], among others.

We address problem (1) from the perspective of zero-order distributed optimization. That is we assume that only function values can be queried by the algorithm, subject to measurement noise.

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During the optimization procedure, each agent maintains a local copy of the variables which are sequentially updated based on local and neighboring functions’ queries. We wish to devise such optimization procedures which are efficient in bounding the average optimization error and cumulative regret in terms of the functions’ properties and network topology.

**Contributions** Our principal contribution is a distributed zero-order optimization algorithm, introduced in Section 2, which we show to achieve tight rates of convergence under certain assumptions on the objective functions, outlined in Section 5. Specifically, we consider that the local objectives \( f_i \) are \( \beta \)-Hölder and the average objective \( f \) is \( \alpha \)-strongly convex. The algorithm relies on a novel zero-order gradient estimator, presented in Section 4. Although conceptually very simple, this estimator, when employed within our algorithm, allows us to obtain an \( O(d^2) \) computational gain as well as improved error rates than previous state-of-the-art zero-order optimization procedures [Akhavan et al., 2020, Bach and Perchet, 2016], in the special case of standard (undistributed) setting. Another key advantage of our approach is due to the general noise model presented in Section 5 under which function values are queried. The noise variables do not need to be zero mean or independently sampled, and thus they include “adversarial” noise. In Section 6 we derive the rates of convergence for the cumulative regret and the optimization error of the proposed algorithm, and in Section 7 we consider the special case of 2-smooth functions. The rates highlight the dependency with respect to the number of variables \( d \), the number of function queries \( T \), the spectral gap of the network matrix \( 1 - \rho \), and the parameters \( n, \alpha \) and \( \beta \). The bounds enjoy a better dependency on \( 1 - \rho \) than previous bounds on zero-order distributed optimization [Qu and Li, 2018], Yu et al. [2019], Tang et al. [2019]. We also compare our bounds to related lower bounds in [Akhavan et al., 2020] for undistributed setting, observing that our rates are optimal either with respect to \( T \) and \( \alpha \), or with respect to \( T \) and \( d \).

**Previous Work** We briefly comment on previous related work and defer to Section 8 for a more in depth discussion and comparison. For both deterministic and stochastic scenarios of problem (1), a large body of literature is devoted to first-order gradient based methods with a consensus scheme (see the papers cited above and references therein). On the other hand, the study of zero-order methods was started only recently [Qu and Li, 2018, Sahu et al., 2018a,b, Hajinezhad et al., 2019, Yu et al., 2019, Tang et al., 2019]. The works [Qu and Li, 2018], Yu et al. [2019], Tang et al. [2019] are dealing with zero-order distributed methods in noise-free settings while the noisy setting is developed in [Hajinezhad et al., 2019, Sahu et al., 2018a]. Namely, Hajinezhad et al. [2019] considers 2-point zero-order methods with stochastic queries for non-convex optimization but assume that the noise is the same for both queries, which makes the problem analogous to noise-free scenario in terms of optimization rates. Papers [Sahu et al., 2018a,b] study zero-order distributed optimization for strongly convex and \( \beta \)-smooth functions \( f_i \) with \( \beta \in \{2, 3\} \), they derive bounds on the optimization error, without providing closed form expressions.

**Notation** Throughout we denote by \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \) be the standard inner product and Euclidean norm on \( \mathbb{R}^d \), respectively, and by \( \| \cdot \|_1 \) the spectral norm of a matrix. The notation \( I \) is used for the \( n \)-dimensional identity matrix and \( e_j \) for the vector in \( \mathbb{R}^n \) with all entries equal to 1. We denote by \( e_j \) the \( j \)-th canonical basis vector in \( \mathbb{R}^d \). For any set \( A \), the number of elements in \( A \) is denoted by \( |A| \).

For \( x \in \mathbb{R} \), the value \( \lfloor x \rfloor \) is the maximal integer less than \( x \). For every closed convex set \( \Theta \subset \mathbb{R}^d \) and \( x \in \mathbb{R}^d \) we denote by \( \text{Proj}_\Theta(x) = \arg\min\{\|z - x\| : z \in \Theta\} \) the Euclidean projection of \( x \) onto \( \Theta \). We denote by \( \text{diam}(\Theta) \) the Euclidean diameter of \( \Theta \). Finally we let \( U[-1, 1] \) be the uniform distribution on \( [-1, 1] \).

2 The Problem

Let \( n \) be the number of agents and let \( G = (V, E) \) be an undirected graph, where \( V = \{1, \ldots, n\} \) is the set of nodes and \( E \subseteq V \times V \) is the set of edges. The adjacency matrix of \( G \) is the symmetric matrix \( (A_{ij})_{i,j=1}^n \) defined as \( A_{ij} = 1 \), if \((i, j) \in E\) and zero otherwise. We consider the following sequential learning framework, where each agent \( i \) gets values of function \( f_i \) corrupted by noise and shares information with other agents. At step \( t \), agent \( i \) acts as follows:

- makes queries and gets noisy values of \( f_i \),
- provides a local output \( u^i(t) \) based on these queries and on the past information,
- broadcasts \( u^i(t) \) to neighboring agents,
Algorithm 1 Distributed Zero-Order Gradient

Input  Communication matrix \((W_{ij})_{i,j=1}^n\), step sizes \((\eta_t > 0)_{t=1}^{T_0-1}\)
Initialization  Choose initial vectors \(x^1(1) = \cdots = x^n(1) \in \mathbb{R}^d\)
For \(t = 1, \ldots, T_0 - 1\)
    For \(i = 1, \ldots, n\)
        1. Build an estimate \(g^i(t)\) of the gradient \(\nabla f_i(x^i(t))\) using noisy evaluations of \(f_i\)
        2. Update \(x^i(t+1) = \sum_{k=1}^n W_{ik} \text{Proj}_f(x^k(t) - \eta_t g^k(t))\)
    End
End
Output  Approximate minimizer \(\hat{x}(T_0) = \frac{1}{n} \sum_{i=1}^n x^i(T_0)\) of the average objective \(\hat{f} = \frac{1}{n} \sum_{i=1}^n f_i\)

• updates its local variable using information from other agents as follows:

\[x^i(t+1) = \sum_{j=1}^n W_{ij} w^j(t),\]

where \(W = (W_{ij})_{i,j=1}^n\) is a given matrix called the consensus matrix.

Below we use the following condition on the consensus matrix.

**Assumption A.** Matrix \(W\) is symmetric, doubly stochastic, and \(\rho := \|W - n^{-1} \mathbb{I} \mathbb{T}\|_\infty < 1\).

Matrix \(W\) accounts for the connectivity properties of the network. If \(W_{ij} = 0\) the agents \(i\) and \(j\) are not connected (do not exchange information). Often \(W\) is defined as a doubly stochastic matrix function of the adjacency matrix \(A\) of the graph. One popular example is as follows:

\[W_{ij} = \begin{cases} \frac{A_{ij}}{\max\{d(i),d(j)\}} & \text{if } i \neq j, \\ 1 - \frac{\sum_{k:k \neq i} A_{ik}}{\gamma \max\{d(i),d(k)\}} & \text{if } i = j, \end{cases}\]

where \(d(i) = \sum_{j=1}^n A_{ij}\) is the degree of node \(i\) and \(\gamma > 0\) is a constant. Then, clearly, \(W = (W_{ij})\) is a symmetric and doubly stochastic matrix, and \(W_{ij} = 0\) if agents \(i\) and \(j\) are not connected. Moreover, we have \(\rho < 1 - c/n^2\) for a constant \(c > 0\) (see Qu and Li [2018], Olshevsky [2014]). Values of spectral gaps \(\rho\) for some other \(W\) reflecting different network topologies can be found in Duchi et al. [2012]. Typically, \(\rho < 1 - a_n\), where \(a_n = \Omega(n^{-1})\) or \(a_n = \Omega(n^{-2})\). Parameter \(\rho\) can be viewed as a measure of difference between the distributed problem and a standard optimization problem. If the graph of communication is a complete graph a natural choice is \(W = n^{-1} \mathbb{I} \mathbb{T}\), and then \(\rho = 0\). For more examples of consensus matrices \(W\), see Olshevsky and Tsitsiklis [2009], Duchi et al. [2012] and references therein.

The local outputs \(w^i\) can be defined in different ways. Our approach is outlined in Algorithm 1. At Step 1, an estimate of the gradient of the local objective \(f_i\) at \(x^i(t)\) is constructed. This involves a randomized procedure that we describe and justify in Section 4. The local output \(w^i\) is defined as an update of the projected gradient algorithm with such an estimated gradient. At Step 2 of the algorithm, each agent computes the next point by a local consensus gradient descent step, which uses local and neighbor information. Step 2 of the algorithm is known as gossip method, see e.g., Boyd et al. [2004], which was initially introduced as an approach for the networks with the imposed connection between the nodes changing by time. We also refer to Sayin et al. [2017] for similar algorithms in the context of distributed stochastic first-order gradient methods.

3  **Assumptions on Local Objectives**

In this section, we give some definitions and introduce our assumptions on the local objective functions \(f_1, \ldots, f_n\).

**Definition 1.** Denote by \(\mathcal{F}_\beta(L)\) the set of all functions \(f : \mathbb{R}^d \rightarrow \mathbb{R}\) that are \(\ell = \lfloor \beta \rfloor\) times differentiable and satisfy, for all \(x, z \in \mathbb{R}^d\) the Hölder-type condition

\[
|f(z) - \sum_{0 \leq |m| \leq \ell} \frac{1}{m!} D^m f(x)(z-x)^m| \leq L \|z-x\|^\beta,
\]

where \(m! = 1 \cdot 2 \cdot \cdots \cdot m\) is the factorial of \(m\).
The gradient estimator crucially requires a kernel function $K$ (Appendix A.3 in Bach and Perchet [2016]) for a derivation. Algorithm 2 is an estimate of $\nabla F$.

Algorithm 2: Gradient Estimator with 2d Queries

**Input** Function $F : \mathbb{R}^d \to \mathbb{R}$ and point $x \in \mathbb{R}^d$

**Requires** Kernel $K : [-1, 1] \to \mathbb{R}$, parameter $h > 0$

**Initialization** Generate random $r$ from uniform distribution on $[-1, 1]$

For $j = 1, \ldots, d$

1. Obtain noisy values $y_j = F(x + hre_j) + \xi_j$ and $y_j' = F(x - hre_j) + \xi_j'$
2. Compute $g_j = \frac{1}{2n}(y_j - y_j')K(r)$

**End**

**Output** $g = (g_j)_{j=1}^d \in \mathbb{R}^d$ estimator of $\nabla F(x)$

where $L > 0$, the sum is over the multi-index $m = (m_1, \ldots, m_d) \in \mathbb{N}^d$, we used the notation $m! = m_1! \cdots m_d!$, $|m| = m_1 + \cdots + m_d$, and we defined, for every $\nu = (\nu_1, \ldots, \nu_d) \in \mathbb{R}^d$,

$$D^m f(x)\nu^m = \frac{\partial^{m_1|f(x)}\partial^{m_d}}{\partial x_1^{m_1} \cdots \partial x_d^{m_d}} \nu_1^{m_1} \cdots \nu_d^{m_d}.$$  

Elements of the class $\mathcal{F}_\beta(L)$ are referred to as $\beta$-Hölder functions.

**Definition 2.** Function $f : \mathbb{R}^d \to \mathbb{R}$ is called 2-smooth if there exists $L > 0$ such that, for every $(x, x') \in \mathbb{R}^d \times \mathbb{R}^d$, it holds that

$$\|\nabla f(x) - \nabla f(x')\| \leq L \|x - x'\|.$$  

**Definition 3.** Let $\alpha > 0$. Function $f : \mathbb{R}^d \to \mathbb{R}$ is called $\alpha$-strongly convex if $f$ is differentiable on $\mathbb{R}^d$ and

$$f(x) - f(x') \geq \langle \nabla f(x'), x - x' \rangle + \frac{\alpha}{2} \|x - x'\|^2, \forall x, x' \in \mathbb{R}^d.$$  

**Assumption B.** Functions $f_1, \ldots, f_n$: (i) belong to the class $\mathcal{F}_\beta(L)$, for some $\beta \geq 2$, and (ii) are 2-smooth.

In Section 6 we will analyse the convergence properties of Algorithm 1 when the objective function $f$ in 1 is $\alpha$-strongly convex. We stress that we do not need the functions $f_1, \ldots, f_n$, to be as well $\alpha$-strongly convex. It is enough to make such an assumption on the compound function $f$, while the local functions $f_i$ only need to satisfy the smoothness conditions stated in Assumption B above.

## 4 Gradient Estimator

In this section, we detail our choice of gradient estimators $g^t(\ell)$ used at Step 1 of Algorithm 1. We consider Algorithm 2. For any function $F : \mathbb{R}^d \to \mathbb{R}$ and any point $x$, the vector $g$ returned by Algorithm 2 is an estimate of $\nabla F(x)$ based on noisy observations of $F$ at randomized points. The estimator is computed for every node $i$ at each step $\ell$, thus giving the vectors $g = g^t(\ell)$ in Algorithm 1. The gradient estimator crucially requires a kernel function $K : [-1, 1] \to \mathbb{R}$ that allows us to take advantage of possible higher order smoothness properties of $f$. Specifically, in what follows we assume that

$$\int uK(u)du = 1, \int u^jK(u)du = 0, j = 0, 2, 3, \ldots, \ell, \text{ and } \kappa_\beta \equiv \int |u|^\beta|K(u)|du < \infty, \quad (3)$$

for given $\beta \geq 2$ and $\ell = |\beta|$. In Polyak and Tsybakov [1990] such kernels can be constructed as weighted sums of Legendre polynomials, in which case $\kappa_\beta \leq 2\sqrt{2\beta}$ with $\beta \geq 1$; see also Appendix A.3 in Bach and Perchet [2016] for a derivation.

The gradient estimator in Algorithm 2 differs from the standard 2d-point Kiefer-Wolfowitz type estimator in that it uses multiplication by a random variable $K(r)$ with a well-chosen kernel $K$. On the other hand, it is also different from the previous kernel-based estimators in zero-order optimization literature Polyak and Tsybakov [1990], Bach and Perchet [2016], Akhavan et al. [2020] in that it needs $2d$ function queries per step, whereas those estimators require only one or two queries; see, in particular, Algorithm 1 in Akhavan et al. [2020] for a comparison. At first sight, this seems a big drawback of the estimator proposed here, however we will show below that thanks to this estimator
we achieve both a more efficient optimization procedure and better rate of convergences for the optimization error.

When the estimator in Algorithm 2 is used at the $t$-th outer step of Algorithm 1, it should be intended as a random variable that depends on the randomization used during the current estimation at the given node, as well as on the randomness of the past iterations, inducing the $\sigma$-algebra $F_t$ (see Section 5 for the definition). Bounds for the bias of this estimator conditional on the past and for its second moment play an important role below, in our analysis of the convergence rates. These bounds are presented in the next two lemmas, whose proofs are presented in Appendix B. We state them in the simpler setting of Algorithm 2, with no reference to the filtration $(F_t)_{t \in \mathbb{N}}$.

**Lemma 1.** Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a function in $\mathcal{F}_\beta(L)$, $\beta \geq 2$, and let the random variables $\xi_1, \ldots, \xi_d$ and $\xi_1', \ldots, \xi_d'$ be independent of $r$ and satisfy $E[|\xi_j|] < \infty$, $E[|\xi_j'|] < \infty$, for $j = 1, \ldots, d$. Let the kernel satisfy conditions (3). If the gradient estimator $g$ of $f$ given by Algorithm 2 is used in the proofs.

\[ \|E[g] - \nabla f(x)\| \leq L\kappa_\beta \sqrt{d}h^{\beta-1}. \]

It is straightforward to see that the bound of Lemma 1 holds when the estimators are build recursively during the execution of Algorithm 1 and the expectation is taken conditionally on $F_t$. This will be used in the proofs.

**Lemma 2.** Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be 2-smooth and let $\max_{x \in \Theta} \|\nabla f(x)\| \leq G$, $\kappa \equiv \int K^2(u)du < \infty$. Let the random variables $\xi_1, \ldots, \xi_d$ and $\xi_1', \ldots, \xi_d'$ be independent of $r$ and satisfy $E[\xi_j^2] \leq \sigma^2$, $E[(\xi_j')^2] \leq \sigma^2$ for $j = 1, \ldots, d$. If $g$ is defined by Algorithm 2, where $x$ is a random variable with values in $\Theta$ independent of $r$ and depending on $\xi_1, \ldots, \xi_d$ and $\xi_1', \ldots, \xi_d'$ in an arbitrary way, then

\[ E\|g\|^2 \leq \frac{3d\kappa}{2} \left( \frac{\sigma^2}{h^2} + \frac{3L^2}{4}h^2 \right) + 9G^2\kappa. \]

## 5 Noise Model

Algorithm 2 is called to compute estimators of gradients of the local functions $f_i, i = 1, \ldots, n$, at each iteration $t$ of Algorithm 1. Thus, we assume that agent $i$ at iteration $t$ generates a uniform random variable $r_i(t) \sim U[-1, 1]$ and gets $2d$ noisy observations, defined, for $j = 1, \ldots, d$

\[ y_{i,j}(t) = f(x^i(t) + h_tr_i(t)e_j) + \xi_{i,j}(t) \]
\[ y'_{i,j}(t) = f(x^i(t) + h_tr_i(t)e_j) + \xi'_{i,j}(t) \]

where the parameters $h_t > 0$ will be specified later.

In what follows, we denote by $F_t$ the $\sigma$-algebra generated by the random variables $x^i(t)$, for $i = 1, \ldots, n$. In order to meet the conditions of Lemmas 1 and 2 for each $(i, t)$, we impose the following assumption on the collection of random variables $(r_i(t), \xi_{i,1}(t), \xi'_{i,1}(t))$.

**Assumption C.** For all integers $t$ and $i \in \{1, \ldots, n\}$ the following properties hold.

(i) The random variables $r_i(t) \sim U[-1, 1]$ are independent of $\xi_{i,1}(t), \ldots, \xi_{i,d}(t)$, $\xi'_{i,1}(t), \ldots, \xi'_{i,d}(t)$ and from the $\sigma$-algebra $F_t$,

(ii) $E[(\xi_{i,j}(t))^2] \leq \sigma^2$, $E[(\xi'_{i,j}(t))^2] \leq \sigma^2$ for $j = 1, \ldots, d$, and some $\sigma \geq 0$.

Assumption C is very mild. Indeed, its part (i) occurs as a matter of course since it is unnatural to assume dependence between the random environment noise and artificial random variables $r_i(t)$ generated by the agents. We state (i) only for the purpose of formal rigor. Remarkably, we do not assume the noises $\xi_{i,j}(t)$ and $\xi'_{i,j}(t)$ to have zero mean. What is more, these variables can be deterministic and no independence between them for different $i, j, t$ is required, so we consider an adversarial environment. Having such a relaxed assumption on the noise is possible because of the multiplication by the zero-mean variable $K(r)$ in Algorithm 2. This and the fact that all components of the vectors are treated separately allows the proofs go through without the zero-mean assumption and under arbitrary dependence between the noises.
6 Main Results

In this section, we provide upper bounds on the performance of the proposed algorithms. Recall
that \( T_0 \) is the number of outer iterations in Algorithm 1. Let \( T \) be the total number of times that we observed noisy values of each \( f_i \). At each iteration of Algorithm 2 we make \( 2d \) queries. Thus, to keep the total budget equal to \( T \) we need to make \( T_0 = T/(2d) \) steps of Algorithm 1 (assuming that \( T/(2d) \) is an integer). We compare our results to lower bounds for any algorithm with the total budget of \( T \) queries.

For given \( \beta \geq 2 \), we choose the tuning parameters \( \eta_t \) and \( h_t \) in Algorithms 1 and 2 as

\[
\eta_t = \frac{2}{\alpha t} \quad \text{and} \quad h_t = t^{-\frac{1}{2\beta}}.
\]  

(4)

Inspection of the proofs in Appendix C shows that these values of \( \eta_t \) and \( h_t \) lead to the best rates minimizing the bounds. As one can expect, there are two contributions to the bounds, one representing the usual stochastic optimization error, while the second one accounts for the distributed character of the problem. This second contribution to the bounds is driven by the following quantity that we call the mean discrepancy: \( \Delta(t) \equiv n^{-1} \sum_{i=1}^{n} E[\|x^i(t) - \bar{x}(t)\|^2] \). It plays an important role in our argument and may be of interest by itself, cf. [14]. The next lemma gives a control of the mean discrepancy.

Lemma 3. Let Assumptions A, B, and C hold. Let \( \Theta \) be a convex compact subset of \( \mathbb{R}^d \). Assume that \( \text{diam}(\Theta) \leq K \) and \( \max_{x \in \Theta} \|Vf(x)\| \leq G \). If the updates \( x^i(t), \bar{x}(t) \) are defined by Algorithm 1 in which the gradient estimators for \( i \)-th agent are defined by Algorithm 2 with \( F = f_i, i = 1, \ldots, n \), and parameters (4) then

\[
\Delta(t) \leq A \left( \frac{\rho}{1 - \rho} \right)^2 \frac{d}{\alpha^2} t^{-\frac{2\beta-1}{\beta}},
\]  

(5)

where \( A \) is a constant independent of \( t, d, \alpha, n, \rho \). The explicit value of \( A \) can be found in the proof.

Proof Sketch. Let \( V(t) = \sum_{i=1}^{n} \|x^i(t) - \bar{x}(t)\|^2 \), and \( z^i(t) = \text{Proj}_\Theta(x^i(t) - \eta_t g^i(t)) - (x^i(t) - \eta_t g^i(t)) \). The first step is to show that, due to the definition of the algorithm and Assumptions A on matrix \( W \), we have

\[
V(t+1) \leq \rho^2 \sum_{i=1}^{n} \|x^i(t) - \bar{x}(t) - \eta_t (g^i(t) - \bar{g}(t)) + z^i(t) - \bar{z}(t)\|^2,
\]  

(6)

where \( \bar{g}(t) \) and \( \bar{z}(t) \) denote the averages of \( g^i(t) \)’s and \( z^i(t) \)’s over the agents \( i \). From (5), by using the fact that \( \|z^i(t)\| \leq \eta_t \|g^i(t)\| \), applying Lemma 1 conditionally on \( \mathcal{F}_t \), taking expectations and then applying Lemma 2, we deduce the recursion

\[
\Delta(t+1) \leq \rho \Delta(t) + A_1 \frac{\rho^2}{1 - \rho} \cdot \frac{d}{\alpha^2} t^{-\frac{2\beta-1}{\beta}},
\]

where \( A_1 > 0 \) is a constant. The initialization of Algorithm 1 is chosen so that \( \Delta(1) = 0 \). It follows that \( \Delta(t) \) is bounded by a discrete convolution that can be carefully evaluated leading to (5).

Using Lemma 3 we obtain the following theorem.

Theorem 4. Let \( f \) be an \( \alpha \)-strongly convex function and let the assumptions of Lemma 2 be satisfied. Then for any \( x \in \Theta \) the cumulative regret satisfies

\[
\sum_{t=1}^{T_0} E[f(\bar{x}(t)) - f(x)] \leq \frac{d}{\alpha} T_0^{\frac{1}{\beta}} \left( B_1 + B_2 \rho^2 \right) + \frac{B_3}{\alpha(1 - \rho)} \log(T_0) + 1,
\]

where the positive constants \( B_1, B_2, B_3 \) are independent of \( T_0, d, \alpha, n, \rho \). The explicit values of these constants can be found in the proof. Furthermore, if \( x^* \) is the minimizer of \( f \) over \( \Theta \) the optimization error of the averaged estimator \( \hat{x}(T_0) = \frac{1}{T_0} \sum_{t=1}^{T_0} \bar{x}(t) \) satisfies

\[
E[f(\hat{x}(T_0)) - f(x^*)] \leq \frac{d}{\alpha} T_0^{\frac{1}{\beta}} \left( B_1 + B_2 \rho^2 \right) + \frac{B_3}{\alpha(1 - \rho)} \left( \log(T_0) + \frac{1}{T_0} \right).
\]  

(7)
Proof sketch. Note first that, due to the definition of Algorithm 1 and to the properties of matrix \( W \) we have \( \bar{x}(t+1) = \bar{x}(t) - \eta_t \bar{g}(t) + \bar{z}(t) \). This resembles the usual recursion of the gradient algorithm with an additional term \( \bar{z}(t) = n^{-1} \sum_{i=1}^{n} z^i(t) \), where \( \|z^i(t)\| \leq \eta_t \|g^i(t)\| \). Using this bound and \( \alpha \)-strong convexity of \( f \), analyzing the recursion in the standard way and taking conditional expectations we obtain that, for any \( x \in \Theta \),

\[
\begin{align*}
    f(\bar{x}(t)) - f(x) & \leq \frac{1}{2\eta_t} E[a_t - a_{t+1}|\mathcal{F}_t] - \alpha a_t^2 + \frac{2\eta_t}{n} \sum_{i=1}^{n} E[\|g^i(t)\|^2 |\mathcal{F}_t] \\
    & + \left[ E[\bar{g}(t)|\mathcal{F}_t] - \nabla f(\bar{x}(t)) \right] \|\bar{x}(t) - x\| + \frac{1}{\eta_t} E[\langle \bar{z}(t), \bar{x}(t) - x \rangle |\mathcal{F}_t],
\end{align*}
\]

where \( a_t = \|\bar{x}(t) - x\|^2 \). Here, the term \( \text{Bias}2 \) is entirely induced by the distributed nature of the problem. Using the properties of Euclidean projection and some algebra, it can be bounded as

\[
\text{Bias}2 \leq \frac{3\eta_t}{2(1 - \rho)n} \sum_{i=1}^{n} E[\|g^i(t)\|^2 |\mathcal{F}_t] + \frac{1 - \rho}{2n\eta_t} \sum_{i=1}^{n} \|x^i(t) - \bar{x}(t)\|^2.
\]

On the other hand, \( \text{Bias}1 \) accumulates two contributions, the first due to the gradient approximation (cf. Lemma 1) and the second due to the distributed nature of the problem:

\[
\begin{align*}
    \text{Bias}1 & \leq \kappa_2 L \sqrt{dh_t} \beta_{t-1} \|\bar{x}(t) - x\| + \bar{L} \sum_{i=1}^{n} \|x^i(t) - \bar{x}(t)\| \|\bar{x}(t) - x\| \\
    & \leq \left( \frac{\kappa_2 L^2}{\alpha}dh_t \beta_{t-1} + \frac{\alpha a_t^2}{4} \right) + \left( \bar{L}t \alpha(1 - \rho) \sum_{i=1}^{n} \|x^i(t) - \bar{x}\|^2 + \bar{L}K^2 \frac{1}{4t\alpha(1 - \rho)} \right).
\end{align*}
\]

Next, we combine inequalities (8) - (10), take expectations of both sides of the resulting inequality, and use Lemmas 2 and 3 to bound the second moments \( E[\|g^i(t)\|^2] \) and the mean discrepancy. The final result is obtained by summing up from \( t = 1 \) to \( t = T_0 \) and recalling that \( \eta_t = \frac{\beta}{\alpha t}, h_t = t^{-\frac{1}{2\beta}} \).

Due to \( \alpha \)-strong convexity of \( f \), Theorem 4 immediately implies a bound on the estimation error \( E[\|\hat{x}(T_0) - x^*\|^2] \). The bound is of the order of the right-hand side of (7) divided by \( \alpha \). Furthermore, we get the following result about local estimators, which follows from a slight modification of Lemma 4 and Theorem 4.

**Corollary 5.** Let Assumptions A, B, and C hold. Let \( \Theta \) be a convex compact subset of \( \mathbb{R}^d \). Assume that \( \text{diam}(\Theta) \leq K \) and \( \max_{x \in \Theta} \|\nabla f(x)\| \leq G \). If the updates \( x^i(t) \) are defined by Algorithm 2 in which the gradient estimators for \( i \)-th agent are defined by Algorithm 2 with \( F = f_{x^i}, i = 1, \ldots, n \), and parameters \( \eta_t = \frac{4}{\alpha(t+1)}, h_t = t^{-\frac{1}{2\beta}} \), then the local average estimator \( \hat{x}^i(T_0) = \frac{1}{T_0} \sum_{t=1}^{T_0} x^i(t) \) satisfies

\[
E[\|\hat{x}^i(T_0) - x^*\|^2] \leq C \min \left\{ 1, \frac{d}{\alpha^2(1 - \rho) T_0^{-\frac{1}{2\beta}}} \left( 1 + \frac{\eta T_0^2}{(1 - \rho)} \right) \right\}, \quad i = 1, \ldots, n,
\]

where \( C > 0 \) is a positive constant independent of \( T_0, d, \alpha, n, \rho \).

We now state a corollary of Theorem 4 for an algorithm with total budget of \( T \) queries. Assume that \( T_0 = T/(2d) \) is an integer. As our algorithm makes \( 2d \) queries per step the estimator \( \hat{x}(T/(2d)) \) uses the total budget of \( T \) queries. Combining Theorem 4 with the trivial bound \( E[f(\hat{x}(T/(2d)) - f(x^*))] \leq G K \) we get the following result.

**Corollary 6.** Let \( T \geq 2d \) and let the assumptions of Theorem 4 be satisfied. Then we have

\[
E[f(\hat{x}(T/(2d)) - f(x^*))] \leq C \min \left\{ 1, \frac{d^{2-1/\beta}}{\alpha(1 - \rho) T^{-\frac{1}{2\beta}}} \right\},
\]

where \( C > 0 \) is a positive constant independent of \( T, d, \alpha, n, \rho \).

We now state several important implications of our results.
While Theorem 4 covers a more difficult noisy setting, our bound displays a better dependency of $(1 - \rho)^{-1}$ in the "connectivity" parameter $\rho$. Since most common values of $1 - \rho$ are of the order $n^{-2}$ (or $n^{-1}$), this represents a substantial gain.

Remark 2. The case $n = 1$, $\rho = 0$ corresponds to usual (undistributed) zero-order stochastic optimization. Then Corollary 6 gives a bound of order $\min (1, \frac{d^2}{\alpha T}\cdot\frac{\beta}{2})$. This improves upon the bound $\min (1, \frac{d^2}{\alpha T}\cdot\frac{\beta}{2})$ obtained under the same assumptions in Akhavan et al. [2020]. Still our bound does not match the minmax lower bound established in Akhavan et al. [2020] and equal to

$$\min \left( \max (\alpha, T^{-1/2+1/\beta}), \frac{d}{\sqrt{T}}\cdot \frac{T^{-\beta+1}}{\alpha} \right).$$

For $\alpha \approx 1$ the lower bound $[11]$ scales as $\min (1, \frac{d^2}{\alpha T}\cdot\frac{\beta}{2})$. It has the same behavior in the interesting regime of $\alpha$ not too small ($\alpha \geq T^{-1/2+1/\beta}$) and $T \geq d$. Note, however, that the lower bound $[11]$ is obtained for the setting with i.i.d. noise, while our upper bound is valid under adversarial noise. Therefore, it may seem rather surprising that the ratio is only $d^{2-1/\beta}$.

Remark 3. With the same budget of queries $T$, the 2d-point method in Algorithm 3 is computationally simpler than the methods with one or two queries per step Polyak and Tsypkin [1990], Bach and Perchet [2016], Akhavan et al. [2020] previously suggested for the same setting. For example, the method in Bach and Perchet [2016], Akhavan et al. [2020] prescribes, at each step $t = 1, \ldots, T$, to generate a random variable uniformly distributed on the unit sphere in $\mathbb{R}^d$. This requires of order $dT$ calls of one-dimensional random variable generator. Overall, in $T$ steps, the number of calls is of order $dT$. For our method with the same budget $T$, we make of order $T_0 = T/(2d)$ steps and at each step we need to call the generator only once in order to get $r \sim U[-1,1]$. Thus, with the same budget of queries, Algorithm 2 needs $\sim 1/d^2$ less calls of random variable generator than the gradient estimator in Bach and Perchet [2016], Akhavan et al. [2020].

Finally, we notice that in Appendix E we present numerical comparisons between our algorithm and that in Akhavan et al. [2020]. These results confirm our theoretical findings: our method converges faster and the advantage is more pronounced as $d$ increases.

7 Improved Bounds for $\beta = 2$

In this section we provide improved upper bounds for the case $\beta = 2$ in Corollary 6, where we relax the dependency over $d$, from $d^{-3/2}$ to $d$.

Following the literature on undistributed zero-order optimization, we use a standard 2-point method with elements of the analysis developed in [Flaxman et al. 2005], [Agarwal et al. 2010], [Duchi et al. 2015], [Shamir 2013], 2017, Akhavan et al. 2020 among others. Specifically, we define

$$g^i(t) = \frac{d}{2h_t}(y_i(t) - y'_i(t))\xi_i(t)$$

where $y_i(t) = f_i(x_i(t) + h_i\xi_i(t)) + \xi_i(t)$, $y'_i(t) = f_i(x_i(t) - h_i\xi_i(t)) + \xi'_i(t)$, with the random variables $\xi_i(t), 1 \leq i \leq n, 1 \leq t \leq T$, that are i.i.d. uniformly distributed on the unit Euclidean sphere in $\mathbb{R}^d$. We make the following assumption on the noise analogous to Assumption C.

Assumption D. For all integers $t$ and all $i \in \{1, \ldots, n\}$ the following properties hold.

(i) The random variables $\xi_i(t)$ are independent of $\xi_i(t), \xi'_i(t)$ and from the $\sigma$-algebra $\mathcal{F}_t$,

(ii) $E[(\xi_i(t))^2] \leq \sigma^2, E[(\xi'_i(t))^2] \leq \sigma^2$ for some $\sigma > 0$.

Theorem 7. Let $f$ be an $\alpha$-strongly convex function. Let Assumptions A, B, and D hold with $\beta = 2$. Let $\Theta$ be a convex compact subset of $\mathbb{R}^d$, and assume that $\text{diam}(\Theta) \leq K$. Assume that $\max_{x \in \Theta} \|\nabla f_i(x)\| \leq G, \forall 1 \leq i \leq n$. Let the updates $x^i(t), \bar{x}(t)$ be defined by Algorithm 7 in

\[\text{The recent work Novitskii and Gaskov [2021] obtains the same improvement, using the gradient estimator of Akhavan et al. [2020]. However as we notice below that estimator is less computationally appealing.}\]
which the gradient estimator for $i$-th agent is defined by (12), and $\eta_t = \frac{1}{\alpha t}$, $h_t = \left(\frac{3d^2 \sigma^2}{2L\alpha t + 9L^2 d^2}\right)^{1/4}$. Then for the estimator $\hat{x}(T) = \frac{1}{T-T/2} \sum_{t=T/2+1}^{T} \tilde{x}(t)$ we have

$$\mathbb{E}[f(\hat{x}(T)) - f(x^*)] \leq \frac{B}{1 - \rho} \left( \frac{d}{\alpha T} + \frac{d^2}{\alpha T} \right),$$

where $B > 0$ is a constant independent of $T, d, \alpha, n, \rho$.

The main idea of the proof is to use surrogate functions $\tilde{f}_i(x)$, for $1 \leq i \leq n$, defined for every $x \in \mathbb{R}^d$, as $\tilde{f}_i(x) = \mathbb{E} f_i(x + h_i \tilde{z})$, where the expectation with respect to the random vector $\tilde{z}$ is uniformly distributed on the unit ball $B_d = \{u \in \mathbb{R}^d : \|u\| \leq 1\}$. A result, which can be traced back to Nemirovsky and Yudin [1983], implies the fact that $g'(t)$ is an unbiased estimator of the gradient of the surrogate function $\tilde{f}_i$ at $x^*(t)$. Thus, we can consider Algorithm [1] as a gradient descent for the surrogate function. Then replacing $f_i$ and $f$ by the surrogate functions with the cost of the order $h_i^2$, we can recover the initial problem. This method does not work for $\beta > 2$ since the error of approximation by surrogate function becomes of bigger order than the optimal rate $T^{-\frac{1}{2\beta - 1}}$. The results that we implement as tools for this section are given in Appendix D.

Combining Theorem 7 with the obvious bound $\mathbb{E}[f(\hat{x}(T)) - f(x^*)] \leq GC$ we obtain

$$\mathbb{E}[f(\hat{x}(T)) - f(x^*)] \leq \frac{B'}{1 - \rho} \min \left(1, \frac{d}{\alpha T}\right),$$

where $B' > 0$ is a constant independent of $T, d, \alpha, n, \rho$. By comparing this upper bound with the minmax lower bound (11) for $\beta = 2$, one can note that (13) is optimal with respect to the parameters $T$ and $d$ when $\alpha \approx 1$.

8 Discussion

We expand our discussion on previous related work, comparing our results to the state-of-the-art distributed and undistributed zero-order optimization settings, and highlight few key open problems.

Comparison to Zero-Order Distributed Settings. Distributed optimization with noisy functions’ queries was considered in detail in Sahu et al. [2018b,a], where the setting differs from ours in some key aspects: the updates are obtained not as in Step 2 of Algorithm 1 but rather via decentralized techniques, matrix $W$ is random, the noise is zero-mean random rather than adversarial, and 2-point gradient estimator is used. Papers Sahu et al. [2018b,a] provide, for $\beta = 2$ and $\beta = 3$, bounds on $\mathbb{E}[\|x^*(T) - x^*\|^2]$ of the order at least $\frac{d^2}{\alpha^2 \rho^2} T^{-1/2}$ and $\frac{d^3}{\alpha^2 \rho^2} T^{-2/3}$, respectively, as functions of $n, \rho$ and $T$. Their bounds contain uncontrolled terms of the form $\mathbb{E}[\|x^*(k_0) - x^*\|^2]$ for some large enough $k_0 = k_0(n, \alpha, d)$ leaving unclear the resulting rate. Paper Hajinezhad et al. [2019] considers 2-point methods with stochastic queries but assume that the noise is the same for both queries and deal with non-convex optimization. Noisy-free zero-order distributed optimization is studied by Qu and Li [2018], Yu et al. [2019], Tang et al. [2019]. From these, Tang et al. [2019] is the closest to our work as it builds on the updates as at Step 2 of Algorithm 1 (though without projections). The bounds obtained therein are of the order $(1 - \rho)^{-2}$ considered as functions of $\rho$, although they hold for the larger class of gradient dominant functions. As noted in Remark 1, the bound of Theorem 4 scales only as $(1 - \rho)^{-1}$ and this bound holds true, in particular, for noisy-free setting, which is its special case corresponding to $\sigma = 0$. Since most common values of $1 - \rho$ are of the order $n^{-2}$ (or $n^{-1}$), this represents a substantial gain. Moreover, Theorem 4 covers a difficult noise setting as we deal with adversarial noise. It is also worthwhile to note that the first-order distributed optimization exhibits much better dependency on $\rho$ since bounds that scale as $(1 - \rho)^{-1/2}$ can be achieved Duchi et al. [2013], Scaman et al. [2019].

Computational and Statistical Advantage of the Proposed Gradient Estimator As we highlighted in Section 6 the gradient estimator in Algorithm 1 requires $2d$ function queries. At first sight this seems problematic when the dimension $d$ is high, as they need at least $T = 2d$ queries. However, the lower bounds in Shamir [2013], Akhavan et al. [2020] reported in (11) above indicate that no estimator can achieve nontrivial convergence rate for zero-order optimization when $T \lesssim d^{9/2}$. Thus,
having the total budget of $T \gg d$ queries is a necessary condition for success of any zero-order stochastic optimization method. Algorithms with one or two queries per step can, of course, be realized for $T \lesssim d$ but in this case they do not enjoy any nontrivial error behavior. Moreover, by Remark 3, with the same total budget of queries $T$, the gradient estimator from Algorithm 2 is computationally more efficient\footnote{One may object that the computation bottleneck in zero-order optimization is in function evaluation; however such costs are external to the optimization procedure, for example they may be performed by black-box software running on external machines or devices. Thus such costs should not be taken into account in evaluating the procedure itself. In this sense our computational speedup is important for high dimensional settings.} than the estimators in Polyak and Tsybakov [1990], Bach and Perchet [2016], Akhavan et al. [2020], since with the same budget of queries, it needs $1/d^2$ less calls of random variable generator than it would be for the gradient estimator in [Bach and Perchet 2016, Akhavan et al. 2020]. At the same time, as detailed in Remark 2, the proposed gradient estimator yields a better rates on the optimization error. We conclude that the proposed zero-order optimization procedure provides both a computational and statistical improvement over the state-of-the-art methods in [Akhavan et al. 2020].

**Limitations and Future Work** A main problem, which remains open, is to study whether the dependency of $(1 - \rho)^{-1}$ in the upper bounds in Corollary 6 and Theorem 7 is minimax optimal. Moreover, in the standard (undistributed) setting it remains an open problem to design a zero-order optimization procedure that meets the minimax lower bound\footnote{One may object that the computation bottleneck in zero-order optimization is in function evaluation; however such costs are external to the optimization procedure, for example they may be performed by black-box software running on external machines or devices. Thus such costs should not be taken into account in evaluating the procedure itself. In this sense our computational speedup is important for high dimensional settings.} with respect all problem parameters ($T, d, \beta$ and $\alpha$). Further directions of research include the analysis of disturbed zero-order algorithms for larger classes of functions, such as $\alpha$-gradient dominant ones, as well as extension of our results to stochastic updates or asynchronous activation schemes.

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A Auxiliary Lemma

Lemma 8. Let $W$ be a matrix satisfying Assumption A and let $x^i = \sum_{j=1}^n W_{i,j} u^j$ for $i = 1, \ldots, n$, where $u^1, \ldots, u^n$ are some vectors in $\mathbb{R}^d$. Set $\bar{x} = n^{-1} \sum_{i=1}^n x^i$, $\bar{u} = n^{-1} \sum_{i=1}^n u^i$. Then

$$\sum_{i=1}^n \|x^i - \bar{x}\|^2 \leq \rho^2 \sum_{i=1}^n \|u^i - \bar{u}\|^2.$$  

Proof. Introduce the matrices $X^\top = (x^1, \ldots, x^n) \in \mathbb{R}^{d \times n}$, $U^\top = (u^1, \ldots, u^n) \in \mathbb{R}^{d \times n}$ and the centering matrix $H = I - \frac{1}{n} \mathbb{1} \mathbb{1}^\top \in \mathbb{R}^{n \times n}$. Notice that $\sum_{i=1}^n \|x^i - \bar{x}\|^2 = \text{Tr}(\Sigma)$, where $\text{Tr}(\Sigma)$ is the trace of the matrix $\Sigma = \sum_{i=1}^n (x^i - \bar{x})(x^i - \bar{x})^\top = \sum_{i=1}^n x^i(x^i)^\top - \bar{x}\bar{x}^\top = X^\top H X$.

It is not hard to check that $\text{Tr}(\Sigma) = \text{Tr}(U^\top W H W U)$. Moreover, as $W$ is symmetric and $W \mathbb{1} = \mathbb{1}$ we have $HW = W - \frac{1}{n} \mathbb{1} \mathbb{1}^\top := W = WH$. Thus, $WHW = WH^2 W = H W^2 H$ and

$$\text{Tr}(\Sigma) = \text{Tr}(U^\top H W^2 H U) \leq \|W^2\|, \quad \text{Tr}(U^\top H^2 U) \leq \rho^2 \text{Tr}(U^\top H U) = \rho^2 \sum_{i=1}^n \|u^i - \bar{u}\|^2.$$  

\hfill \Box

B Proofs for Section 4

Lemma 1. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a function in $\mathcal{F}_\beta(L)$, $\beta \geq 2$, and let the random variables $\xi_1, \ldots, \xi_d$ and $\xi'_1, \ldots, \xi'_d$ be independent of $r$ and satisfy $E[|\xi_j|] < \infty$, $E[|\xi'_j|] < \infty$, for $j = 1, \ldots, d$. Let the kernel satisfy conditions (3). If the gradient estimator $g$ of $f$ given by Algorithm 2 then, for all $x \in \mathbb{R}^d$,

$$\|E[g] - \nabla f(x)\| \leq L \kappa_\beta \sqrt{dh^{\beta-1}}.$$  

Proof. By Taylor expansion we have

$$\frac{f(x+h\xi_j)-f(x-h\xi_j)}{2h} = \frac{\partial f(x)}{\partial x_j} + \frac{1}{h} \sum_{2 \leq m \leq \ell, m \text{ odd}} \frac{(rh)^m}{m!} \frac{\partial^m f(x)}{\partial x_j^m} + \frac{R(h\xi_j) - R(-h\xi_j)}{2h},$$  

where $|R(\pm h\xi_j)| \leq L|\xi_j|^\beta = L|r|^\beta h^\beta$. Using (3) it follows that

$$\left|E[g_j] - \frac{\partial f(x)}{\partial x_j} \right| = \left|E \left[ \frac{f(x+h\xi_j) - f(x-h\xi_j)}{2h} K(r) - \frac{\partial f(x)}{\partial x_j} \right] \right| \leq L \kappa_\beta h^{\beta-1},$$  

which implies the result.  

\hfill \Box

Lemma 2. Let $f : \mathbb{R}^d \to \mathbb{R}$ be 2-smooth and let $\max_{x \in \Theta} \|\nabla f(x)\| \leq G$, $\kappa \equiv \int K^2(u) du < \infty$. Let the random variables $\xi_1, \ldots, \xi_d$ and $\xi'_1, \ldots, \xi'_d$ be independent of $r$ and let $\xi_j$, $\xi'_j$ be independent of $r$ and $E[\xi_j^2] \leq \sigma^2$, $E[(\xi'_j)^2] \leq \sigma^2$ for $j = 1, \ldots, d$. If $g$ is defined by Algorithm 2 where $x$ is a random variable with values in $\Theta$ independent of $r$ and depending on $\xi_1, \ldots, \xi_d$ and $\xi'_1, \ldots, \xi'_d$ in an arbitrary way, then

$$E\|g\|^2 \leq \frac{3dk}{2} \left( \frac{\sigma^2}{h^2} + \frac{3L^2}{4} h^2 \right) + 9G^2 \kappa.$$  

Proof. Fix $j = 1, \ldots, d$. Using the inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ and the independence between $r$ and $(\xi_j, \xi'_j)$ we have

$$E[g_j^2] = \frac{1}{4h^2} E \left[ (f(x+h\xi_j) - f(x-h\xi_j) + \xi - \xi'_j)^2 K^2(r) \right] \leq \frac{3}{4h^2} E \left[ \left( (f(x+h\xi_j) - f(x-h\xi_j))^2 + 2\sigma^2 \right) K^2(r) \right].$$  

13
The same calculations as in the proof of Lemma 2.4 in Akhavan et al. [2020] yield
\[
(f(x + hre_j) - f(x - hre_j))^2 \leq 3 \left( \frac{L^2}{2} ||hre_j||^4 + 4(\nabla f(x), hre_j)^2 \right),
\]
Finally, we combine this inequality with (14) to obtain
\[
\mathbb{E}[g_t^2] \leq \frac{3}{2} \kappa \left( \frac{\sigma^2}{\eta^2} + \frac{3L^2}{4} \eta^2 \right) + 9\kappa \mathbb{E}[\langle \nabla f(x), e_i \rangle^2],
\]
which immediately implies the lemma. \(\square\)

C Proofs for Section 6

Recall the notation \(\Delta(t) = n^{-1} \sum_{i=1}^{n} \mathbb{E}[||x^i(t) - \bar{x}(t)||^2], \bar{g}(t) = \frac{1}{n} \sum_{i=1}^{n} g^i(t),\) and \(z^i(t) = \text{Proj}_{\Theta}(x^i(t) - \eta_t g^i(t)) - (x^i(t) - \eta_t g^i(t)).\) We also set \(\bar{z}(t) = \frac{1}{n} \sum_{i=1}^{n} z^i(t).\)

Lemma 3. Let Assumptions \(A, B,\) and \(C\) hold. Let \(\Theta\) be a convex compact subset of \(\mathbb{R}^d.\) Assume that \(\text{diam}(\Theta) \leq \mathcal{K}\) and \(\max_{x \in \Theta} ||\nabla f(x)|| \leq G.\) If the updates \(x^i(t), \bar{x}(t)\) are defined by Algorithm 1, in which the gradient estimators for \(i\)-th agent are defined by Algorithm 2, with \(F = f_i, \ i = 1, \ldots, n,\) and parameters \(\beta, \) then
\[
\Delta(t) \leq \mathcal{A} \left( \rho \frac{\rho}{1 - \rho} \right)^2 \alpha^2 t^{-\frac{2d-1}{2}}\frac{d}{\alpha^2 t^{-\frac{2d-1}{2}}},
\]
where \(\mathcal{A}\) is a constant independent of \(t, d, \alpha, n, \rho.\) The explicit value of \(\mathcal{A}\) can be found in the proof.

Proof. Set \(V(t) = \sum_{i=1}^{n} ||x^i(t) - \bar{x}(t)||^2.\) The definition of Algorithm 1 and Lemma 8 imply:
\[
V(t + 1) \leq \rho^2 \sum_{i=1}^{n} ||x^i(t) - \bar{x}(t) - \eta_t g^i(t) + z^i(t) - \bar{z}(t)||^2.
\]
The result is immediate if \(\rho = 0.\) Therefore, in rest of the proof we assume that \(\rho > 0.\) We have
\[
V(t + 1) \leq \rho^2 \sum_{i=1}^{n} \left[ V(t) + \eta_t^2 ||g^i(t) - \bar{g}(t)||^2 + ||z^i(t) - \bar{z}(t)||^2 - 2\eta_t \langle x^i(t) - \bar{x}(t), g^i(t) - \bar{g}(t) \rangle \right. \\
- 2\eta_t \langle g^i(t) - \bar{g}(t), z^i(t) - \bar{z}(t) \rangle \\
+ \left. 2 \langle x^i(t) - \bar{x}(t), z^i(t) - \bar{z}(t) \rangle \right].
\]
For any \(z \in \mathbb{R}^d,\) we have \(\sum_{i=1}^{n} ||g^i(t) - \bar{g}(t)||^2 \leq \sum_{i=1}^{n} ||g^i(t) - z||^2,\) so that
\[
\eta_t^2 \sum_{i=1}^{n} \mathbb{E} \left[ ||g^i(t) - \bar{g}(t)||^2 | \mathcal{F}_t \right] \leq \eta_t^2 \sum_{i=1}^{n} \mathbb{E} \left[ ||g^i(t)||^2 | \mathcal{F}_t \right].
\]
Next, from the definition of the projection,
\[
||z^i(t)|| = \left| \left| \text{Proj}_{\Theta}(x^i(t) - \eta_t g^i(t)) - (x^i(t) - \eta_t g^i(t)) \right| \right|
\leq \left| \left| x^i(t) - (x^i(t) - \eta_t g^i(t)) \right| \right| = \eta_t ||g^i(t)||.
\]
Therefore, for the term containing \(||z^i(t) - \bar{z}(t)||^2\) in (15) we obtain
\[
\sum_{i=1}^{n} \mathbb{E}[||z^i(t) - \bar{z}(t)||^2 | \mathcal{F}_t] \leq \sum_{i=1}^{n} \mathbb{E}[||z^i(t)||^2 | \mathcal{F}_t] \leq \eta_t^2 \sum_{i=1}^{n} \mathbb{E}[||g^i(t)||^2 | \mathcal{F}_t].
\]
where $\Delta(1) = 0$ is a value to be chosen later. For the expression in (17), we have

$$-2\eta_t \sum_{i=1}^{n} E \left[ \left( g'(t) - \bar{g}(t), z'(t) - \bar{z}(t) \right) | \mathcal{F}_t \right] \leq 2 \eta_t^2 \sum_{i=1}^{n} E \left[ \| g'(t) \|^2 | \mathcal{F}_t \right] \leq -2 \eta_t \sum_{i=1}^{n} E \left[ \| g'(t) \|^2 | \mathcal{F}_t \right].$$

Similarly, for the expression in (18), using the Cauchy–Schwarz inequality we get

$$2 \sum_{i=1}^{n} E \left[ \left( x'(t) - \bar{x}(t), z'(t) - \bar{z}(t) \right) | \mathcal{F}_t \right] \leq 2 \sum_{i=1}^{n} E \left[ \| x'(t) - \bar{x}(t) \| \| z'(t) - \bar{z}(t) \| | \mathcal{F}_t \right] \leq \lambda V(t) + \frac{1}{\lambda} \sum_{i=1}^{n} E \left[ \| z'(t) - \bar{z}(t) \|^2 | \mathcal{F}_t \right] \leq \lambda V(t) + \frac{\eta_t^2}{\lambda} \sum_{i=1}^{n} E \left[ \| g'(t) \|^2 | \mathcal{F}_t \right].$$

Combining the above inequalities yields

$$E[V(t+1)|\mathcal{F}_t] \leq \rho^2 (1 + 2\lambda) V(t) + \rho^2 \left( 4 + \frac{2}{\lambda} \right) \eta_t^2 \sum_{i=1}^{n} E \left[ \| g'(t) \|^2 | \mathcal{F}_t \right].$$

Taking expectations in (20) and applying Lemma 2, we obtain

$$\Delta(t+1) \leq \rho \Delta(t) + A_1 \rho^2 \frac{d}{\alpha^2} t^{-\frac{2\beta-1}{\alpha}} \leq \rho \Delta(t),$$

where $A_1 = \frac{144\kappa G^2}{d} + 18\kappa L^2 + 24\kappa\sigma^2$. Due to the recursion in (21) we have, for any $t \geq 3$,

$$\Delta(t+1) \leq \rho \Delta(t) + A_1 \rho^2 \frac{d}{\alpha^2} \sum_{s=1}^{t} s^{-\frac{2\beta-1}{\alpha}} \rho^{t-s} \leq A_1 \rho^2 \frac{d}{\alpha^2} \left( \frac{1}{\frac{1}{2}} \sum_{s=1}^{\frac{1}{2}} s^{-\frac{2\beta-1}{\alpha}} + \sum_{k=t-\frac{1}{2}}^{t-1} \rho^k + \frac{1}{\frac{1}{2}} \sum_{s=t+\frac{1}{2}}^{t} s^{-\frac{2\beta-1}{\alpha}} \sum_{k=0}^{t-\frac{1}{2}} \rho^k \right),$$

where $\Delta(1) = 0$ by the choice of initial values and the last inequality uses the fact that if the function $\phi_1(\cdot)$ is monotone decreasing and $\phi_2(\cdot)$ is monotone increasing then

$$\frac{1}{S} \sum_{s=1}^{S} \phi_1(s) \phi_2(s) \leq \left( \frac{1}{S} \sum_{s=1}^{S} \phi_1(s) \right) \left( \frac{1}{S} \sum_{s=1}^{S} \phi_2(s) \right),$$

see, e.g., [Devroye et al. 1996] Theorem A.19. The sums in (22) satisfy

$$\sum_{s=\frac{1}{2}}^{\frac{1}{2}} s^{-\frac{2\beta-1}{\alpha}} \leq 1 + \int_{1}^{\infty} s^{-\frac{2\beta-1}{\alpha}} = \frac{2\beta-1}{\beta-1}, \quad \sum_{s=\frac{1}{2}}^{t} s^{-\frac{2\beta-1}{\alpha}} \leq t \left( \frac{t}{2} \right)^{-\frac{2\beta-1}{\alpha}} = \frac{2}{\beta-1} t^{-\frac{\beta-1}{\alpha}}.$$
\[
\sum_{k=0}^{t - \lceil \frac{k}{2} \rceil - 1} \rho^k \leq \frac{1}{1 - \rho}, \quad \sum_{k=t - \lceil \frac{k}{2} \rceil}^{t - 1} \rho^k \leq \sum_{k=\lceil \frac{k}{2} \rceil}^{t - 1} \rho^k \leq t \rho^{1/2} \leq \frac{8}{\log(1/\rho) t},
\]
where the last inequality follows from the fact that \(\rho^k \leq \frac{1}{(\log(1/\rho))^k} \) for any positive integer \(k\).

Plugging the above inequalities in (22) gives
\[
\Delta(t + 1) \leq A_1 \frac{\rho^2}{1 - \rho} \frac{d}{\alpha^2} \left( \frac{24}{\log(1/\rho) t^2} \frac{2\beta - 1}{\beta - 1} + 3 \left( 2 \frac{\beta - 1}{\beta - 3} \right) t \frac{1}{1 - \rho} \right)
\leq A_2 \frac{\rho^2}{(1 - \rho)^2} \frac{d}{\alpha^2} t ^ {\frac{2\beta - 1}{\beta - 3}},
\]
where \(A_2 = \left( 2 \frac{2\beta - 1}{\beta - 1} + 3 \left( 2 \frac{\beta - 1}{\beta - 3} \right) \right) A_1\). Therefore, setting \(A := 2A_2\) we conclude that, for \(t \geq 3\),
\[
\Delta(t) \leq A \frac{\rho^2}{(1 - \rho)^2} \frac{d}{\alpha^2} t ^ {\frac{2\beta - 1}{\beta - 3}}.
\]
For \(t \in \{1, 2\}\) the bound of the lemma holds trivially since \(\bar{x}\) and all \(x^i\) belong to the compact \(\Theta\).

\[\square\]

**Theorem 4.** Let \(f\) be an \(\alpha\)-strongly convex function and let the assumptions of Lemma 3 be satisfied. Then for any \(x \in \Theta\) the cumulative regret satisfies
\[
\sum_{t=1}^{T_0} \mathbb{E} \left[ |f(\bar{x}(t)) - f(x)| \right] \leq \frac{d}{\alpha} T_0 \frac{\alpha}{\alpha - 1} \left( B_1 + \frac{B_2 \rho^2}{1 - \rho} \right) + \frac{B_3}{\alpha (1 - \rho)} \left( \log(T_0) + 1 \right),
\]
where the positive constants \(B_i\) are independent of \(T_0, d, \alpha, n, \rho\). The explicit values of these constants can be found in the proof. Furthermore, if \(x^*\) is the minimizer of \(f\) over \(\Theta\) the optimization error of the averaged estimator \(\bar{x}(T_0) = \frac{1}{T_0} \sum_{t=1}^{T_0} \bar{x}(t)\) satisfies
\[
\mathbb{E} \left[ |f(\bar{x}(T_0)) - f(x^*)| \right] \leq \frac{d}{\alpha} T_0 ^ {\frac{\alpha - 1}{\alpha}} \left( B_1 + \frac{B_2 \rho^2}{1 - \rho} \right) + \frac{B_3}{\alpha (1 - \rho)} \left( \log(T_0) + \frac{1}{T_0} \right). \tag{7}
\]

**Proof.** From the definition of Algorithm 1 and 10 we obtain
\[
\|ar{x}(t + 1) - x\|^2 = \|ar{x}(t) - x\|^2 + \|ar{z}(t)\|^2 + \eta_t^2 \|ar{g}(t)\|^2 - 2\eta_t \langle \bar{g}(t), \bar{x}(t) - x \rangle + 2 \langle \bar{z}(t), \bar{x}(t) - x \rangle - 2 \eta_t \langle \bar{z}(t), \bar{g}(t) \rangle \leq \|ar{x}(t) - x\|^2 - 2\eta_t \|ar{g}(t)\|^2 + 2 \langle \bar{z}(t), \bar{x}(t) - x \rangle + \frac{4 \eta_t^2}{n} \sum_{i=1}^{n} \|g^i(t)\|^2.
\]
It follows that
\[
\langle \bar{g}(t), \bar{x}(t) - x \rangle \leq \frac{\|ar{x}(t) - x\|^2 - \|ar{x}(t + 1) - x\|^2}{2 \eta_t} + \frac{1}{\eta_t} \|ar{z}(t), \bar{x}(t) - x \rangle + \frac{2 \eta_t}{n} \sum_{i=1}^{n} \|g^i(t)\|^2.
\]
The strong convexity assumption implies
\[
f(\bar{x}(t)) - f(x) \leq \langle \nabla f(\bar{x}(t)), \bar{x}(t) - x \rangle - \frac{\alpha}{2} \|ar{x}(t) - x\|^2.
\]
Combining the last two displays and taking conditional expectations from both sides we get
\[
\mathbb{E} \left[ |f(\bar{x}(t)) - f(x)| \right] \leq \left( \mathbb{E} \left[ \|ar{g}(t)\|^2 \right] - \nabla f(\bar{x}(t)) \right) \|ar{x}(t) - x\| + \frac{1}{2 \eta_t} \mathbb{E} \left[ |a_t - a_{t+1}| \right] + \frac{2 \eta_t}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \|g^i(t)\|^2 \right] - \frac{\alpha}{2} \mathbb{E} \left[ \|ar{z}(t), \bar{x}(t) - x\| \right]. \tag{23}
\]

16
where \( a_t = \| \bar{x}(t) - x \|^2 \).

The first term in right hand side of (23) is bounded as follows

\[
\left\| \mathbb{E}[\bar{g}(t)|\mathcal{F}_t] - \nabla f(\bar{x}(t)) \right\| \| \bar{x}(t) - x \| \leq \left[ \left\| \mathbb{E}[\bar{g}(t)|\mathcal{F}_t] - \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(x^i(t)) \right\| \right.
\]
\[
+ \left. \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(x^i(t)) - \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(\bar{x}(t)) \right\| \| \bar{x}(t) - x \|
\]
\[
\leq \kappa_\beta L \sqrt{dh^\beta_t \| \bar{x}(t) - x \| + \frac{\tilde{L}}{n} \sum_{i=1}^{n} \| x^i(t) - \bar{x}(t) \| \| \bar{x}(t) - x \| ,
\]

(24)

where the last inequality is due to Lemma [1] and Assumption [2 ii]. We now decouple the terms in (24) using the fact that \( ab \leq \frac{a^2}{\alpha} + \frac{b^2}{\beta}, \forall a, b \geq 0, \nu > 0 \). Thus, we obtain

\[
\kappa_\beta L \sqrt{dh^\beta_t \| \bar{x}(t) - x \|} \leq \frac{(\kappa_\beta L)^2}{\alpha} \frac{\nu}{dh^\beta_t} \frac{\alpha}{4} \| \bar{x}(t) - x \|^2
\]

(25)

and

\[
\tilde{L} \sum_{i=1}^{n} \| x^i(t) - \bar{x}(t) \| \| \bar{x}(t) - x \| \leq \frac{\tilde{L} \alpha (1 - \rho)}{n} \sum_{i=1}^{n} \| x^i(t) - \bar{x} \|^2 + \frac{\tilde{L} K^2}{4 \alpha (1 - \rho)}.
\]

(26)

Combining (25) and (26) with (24) gives

\[
\left\| \mathbb{E}[\bar{g}(t)|\mathcal{F}_t] - \nabla f(\bar{x}(t)) \right\| \| \bar{x}(t) - x \| \leq \frac{(\kappa_\beta L)^2}{\alpha} \frac{\nu}{dh^\beta_t} \frac{\alpha}{4} \| \bar{x}(t) - x \|^2 +
\]
\[
+ \frac{\tilde{L} \alpha (1 - \rho)}{n} \sum_{i=1}^{n} \| x^i(t) - \bar{x}(t) \|^2 + \frac{\tilde{L} K^2}{4 \alpha (1 - \rho)}.
\]

(27)

Next, we have

\[
\frac{1}{n} \left\langle x(t), \bar{x}(t) - x \right\rangle = \frac{1}{n \eta t} \sum_{i=1}^{n} \left\langle x^i(t), \bar{x}(t) - x \right\rangle
\]
\[
\leq \frac{1}{n \eta t} \sum_{i=1}^{n} \left\langle x^i(t), \bar{x}(t) - (x^i(t) - \eta t g^i(t)) \right\rangle + \left\langle z^i(t), (x^i(t) - \eta t g^i(t)) - x \right\rangle.
\]

(28)

Since \( \text{Proj}_\Theta(\cdot) \) is the Euclidean projection on the convex set \( \Theta \), for any \( w \in \mathbb{R}^d, x \in \Theta \) we have

\[
\langle \text{Proj}_\Theta(w) - w, \text{Proj}_\Theta(w) - x \rangle \leq 0,
\]

which implies

\[
\langle \text{Proj}_\Theta(w) - w, w - x \rangle = - \| \text{Proj}_\Theta(w) - w \|^2 + \langle \text{Proj}_\Theta(w) - w, \text{Proj}_\Theta(w) - x \rangle \leq 0.
\]

Therefore,

\[
\left\langle z^i(t), x^i(t) - \eta t g^i(t) - x \right\rangle = \langle \text{Proj}_\Theta(x^i(t) - \eta t g^i(t)) - (x^i(t) - \eta t g^i(t)), x^i(t) - \eta t g^i(t) - x \rangle \leq 0.
\]

Applying this inequality in (28) and using (19) we find

\[
\frac{1}{n \eta t} \left\langle x(t), \bar{x}(t) - x \right\rangle \leq \frac{1}{n \eta t} \sum_{i=1}^{n} \left\langle z^i(t), (\bar{x}(t) - x^i(t)) + \eta t g^i(t) \right\rangle
\]
\[
\leq \frac{1}{n \eta t} \sum_{i=1}^{n} \| z^i(t) \| \| x^i(t) - \bar{x}(t) \| + \frac{1}{n} \sum_{i=1}^{n} \| z^i(t) \| \| g^i(t) \|
\]
\[
\leq \frac{1}{2 n \eta t} \sum_{i=1}^{n} \left[ \frac{\eta t^2}{1 - \rho} \| g^i(t) \|^2 + (1 - \rho) \| x^i(t) - \bar{x}(t) \|^2 \right] + \frac{\eta t}{n} \sum_{i=1}^{n} \| g^i(t) \|^2
\]
\[
\leq \frac{3 \eta t}{2 (1 - \rho) n} \sum_{i=1}^{n} \| g^i(t) \|^2 + \frac{1 - \rho}{2 n \eta t} \sum_{i=1}^{n} \| x^i(t) - \bar{x}(t) \|^2 .
\]

(29)
Inserting (29) and (27) in (23) and using the fact that \( \eta_t = \frac{2t}{\alpha t} \) we get

\[
\mathbb{E}[f(\bar{x}(t)) - f(x)|\mathcal{F}_t] \leq \frac{1}{2\eta_t} \mathbb{E}[a_t - a_{t+1}|\mathcal{F}_t] - \frac{\alpha}{4} a_t + \frac{(1 + 4\bar{L})\alpha (1 - \rho)}{4} \sum_{i=1}^{n} \|x^i(t)\|^2 + \frac{7\eta_t}{2(1 - \rho)n} \sum_{i=1}^{n} \mathbb{E}[\|g_i^i(t)\|^2 |\mathcal{F}_t] + \frac{(\kappa_\beta L)^2}{\alpha} d h_t^{2(\beta - 1)} + \frac{\bar{L}K^2}{4t\alpha(1 - \rho)}.
\]

where the last inequality follows from. Taking the expectations, setting \( r_t := \mathbb{E}[a_t] \) and applying Lemma 2 we get

\[
\mathbb{E}[f(\bar{x}(t)) - f(x)] \leq \frac{r_t - r_{t+1}}{2\eta_t} - \frac{\alpha}{4} r_t + \frac{(1 + 4\bar{L})\alpha (1 - \rho)}{4} \Delta(t) + \frac{7}{\alpha (1 - \rho)t} \left( 9\kappa G^2 + 4\left( \frac{9\kappa^2 L^2}{8} + \frac{3\kappa_\beta^2}{2} \right) \right) + \frac{(\kappa_\beta L)^2}{\alpha} d h_t^{2(\beta - 1)} + \frac{\bar{L}K^2}{4t\alpha(1 - \rho)}.
\]

Notice that since \( \eta_t = \frac{2t}{\alpha t} \) we have

\[
\sum_{i=1}^{T_0} \left( \frac{r_t - r_{t+1}}{2\eta_t} - \frac{\alpha}{4} r_t \right) \leq 0.
\]

Thus, recalling that \( h_t = t^{-\frac{1}{\beta}} \) and summing over \( t \) we get

\[
\sum_{i=1}^{T_0} \mathbb{E}[f(\bar{x}(t)) - f(x)] \leq \frac{(1 + 4\bar{L})\alpha (1 - \rho)}{4} \sum_{i=1}^{T_0} t \Delta(t) + B_1 \frac{d}{\alpha} T_0^{\frac{1}{\beta}} + \frac{\bar{L}K^2}{4\alpha(1 - \rho)} \left( \log(T_0) + 1 \right),
\]

where \( B_1 = 7\beta \left( \frac{9\kappa G^2}{d} + \left( \frac{9\kappa L^2}{8} + \frac{3\kappa_\beta^2}{2} \right) \right) + \beta(\kappa_\beta L)^2 \). Finally, using Lemma 3 we obtain

\[
\sum_{i=1}^{T_0} \mathbb{E}[f(\bar{x}(t)) - f(x)] \leq B_1 d T_0^{\frac{1}{\beta}} + B_2 \frac{\rho^2}{1 - \rho} d T_0^{\frac{1}{\beta}} + \frac{B_3}{\alpha(1 - \rho)} \left( \log(T_0) + 1 \right),
\]

where \( B_2 = \frac{\beta(1 + 4\bar{L})}{4} A \), and \( B_3 = \bar{L}K^2 \). This proves the first bound of the theorem. The second bound follows immediately by the convexity of \( f \).

**Corollary 5.** Let Assumptions A, B, and C hold. Let \( \Theta \) be a convex compact subset of \( \mathbb{R}^d \). Assume that \( \text{diam}(\Theta) \leq \bar{K} \) and \( \max_{x \in \Theta} \| \nabla f(x) \| \leq G \). If the updates \( x^i(t) \) are defined by Algorithm 7 in which the gradient estimators for \( i \)-th agent are defined by Algorithm 2 with \( F = f_i \), \( i = 1, \ldots, n \), and parameters \( \eta_t = \frac{4}{\alpha(t+1)} \), \( h_t = t^{-\frac{1}{\beta}} \) then the local average estimator \( \bar{x}(T_0) = \frac{1}{n} \sum_{i=1}^{n} x^i(T_0) \) satisfies

\[
\mathbb{E}[\|\bar{x}(T_0) - x^*\|^2] \leq C \min \left\{ \frac{1}{\alpha^2(1 - \rho)} \left( 1 + \frac{n\rho^2}{(1 - \rho)T_0} \right) \right\}, \quad i = 1, \ldots, n,
\]

where \( C > 0 \) is a positive constant independent of \( T_0, d, \alpha, n, \rho \).

**Proof.** In contrast to the previous proofs, now we have \( \eta_t = \frac{4}{\alpha(t+1)} \) rather than \( \eta_t = \frac{2t}{\alpha t} \).

1°. Inspection of the proof of Lemma 3 immediately yields that Lemma 3 remains valid with \( \eta_t = \frac{4}{\alpha(t+1)} \) instead of \( \eta_t = \frac{2t}{\alpha t} \), up to a change in constant \( A \). Thus,

\[
\Delta(t) \leq \bar{A} \left( \frac{\rho}{1 - \rho} \right)^2 \frac{d}{\alpha^2} t^{-\frac{2\beta - 1}{\beta}}, \tag{31}
\]

\[
\mathbb{E}[\|\bar{x}(t) - \bar{x}(t)\|^2] \leq \bar{A} n \left( \frac{\rho}{1 - \rho} \right)^2 \frac{d}{\alpha^2} t^{-\frac{2\beta - 1}{\beta}}, \quad i = 1, \ldots, n, \tag{32}
\]
We first restate the following three lemmas from Akhavan et al. [2020].

\[ \bar{A} > 0 \] is a constant independent of \( t, d, \alpha, n, \rho. \)

2. Next, we show that, up to changes in constants \( B_1, \) the bound (7) of Theorem 4 remains valid with \( \eta_t = \frac{4}{\alpha(t+1)} \) instead of \( \eta_t = \frac{2}{\alpha t} \) if we replace \( \hat{x}(T_0) \) in (7) by the estimator

\[
\hat{x}_*(T_0) := \frac{2}{T_0(T_0 + 1)} \sum_{t=1}^{T_0} t \bar{x}(t).
\]

Indeed, repeating the proof of Theorem 4 until (30), multiplying both sides of (30) by \( t, \) summing up from \( t = 1 \) to \( T_0 \) and using the fact that

\[
\sum_{t=1}^{T_0} \left( \frac{t(r_t - r_{t+1})}{2\eta_t} - \frac{\alpha r_t}{4} \right) \leq 0 \quad \text{if} \quad \eta_t = \frac{4}{\alpha(t+1)},
\]

we find that, for all \( x \in \Theta, \)

\[
\sum_{t=1}^{T_0} t \mathbb{E} [f(\bar{x}(t)) - f(x)] \leq \frac{(1 + 4\bar{L}\alpha)(1 - \rho)}{4} \sum_{t=1}^{T_0} t^2 \Delta(t) + \bar{B}_1 \frac{d}{\alpha} T_0^{\frac{1}{2}} + \frac{\bar{L}K^2}{4\alpha(1 - \rho)},
\]

where \( \bar{B}_1 \) is a positive constant independent of \( T_0, d, \alpha, n, \rho. \) Using (31) we get, for all \( x \in \Theta, \)

\[
\frac{2}{T_0(T_0 + 1)} \sum_{t=1}^{T_0} t \mathbb{E} [f(\bar{x}(t)) - f(x)] \leq \bar{B}_2 \frac{d}{\alpha(1 - \rho)} T_0^{-\frac{1}{2}},
\]

where \( \bar{B}_2 \) is a positive constant independent of \( T_0, d, \alpha, n, \rho. \) In view of the convexity of \( f, \) it follows that

\[
\mathbb{E} [f(\hat{x}_*(T_0)) - f(x^*)] \leq \bar{B}_2 \frac{d}{\alpha(1 - \rho)} T_0^{-\frac{1}{2}}.
\]

As \( f \) is strongly convex we also have

\[
\mathbb{E} [\|\hat{x}_*(T_0) - x^*\|^2] \leq 2\bar{B}_2 \frac{d}{\alpha^2(1 - \rho)} T_0^{-\frac{1}{2}}.
\]

On the other hand, convexity of function \( \| \cdot \|^2 \) implies that

\[
\|\hat{x}(T_0) - \hat{x}_*(T_0)\|^2 = \left\| \frac{2}{T_0(T_0 + 1)} \sum_{t=1}^{T_0} t (x(t) - \bar{x}(t)) \right\|^2
\]

\[
\leq \frac{2}{T_0(T_0 + 1)} \sum_{t=1}^{T_0} t \|x(t) - \bar{x}(t)\|^2.
\]

Combining (32) and (34) we obtain

\[
\mathbb{E} [\|\hat{x}(T_0) - \hat{x}_*(T_0)\|^2] \leq \bar{C}(\frac{\rho}{1 - \rho})^2 \frac{d}{\alpha^2} T_0^{-\frac{2a-1}{a}},
\]

where \( \bar{C} > 0 \) is a constant independent of \( T_0, d, \alpha, n, \rho. \) The desired result now follows from (33), (35) and the fact that \( \|\hat{x}(T_0) - x^*\| \) is trivially bounded by the diameter of \( \Theta. \)

\[ \square \]

D Proofs for Section 7

We first restate the following three lemmas from Akhavan et al. [2020].

**Lemma 9.** Let for \( \beta = 2, \) Assumptions B and D hold. Let \( \bar{g}(t) \) be the average of gradient estimators for \( n \) agents defined each by (12), and \( \bar{g} \) be the gradient estimator for one agent defined by (12). If \( \max_{x \in \Theta} \|\nabla f_i(x)\| \leq G, \) for \( 1 \leq i \leq n, \) then

\[
\mathbb{E}[\|\bar{g}(t)\|^2] \leq 9\kappa \left( G^2 d + \frac{L^2 \bar{d}^2}{2} + \frac{3\kappa d^2 \sigma^2}{2h^2} \right).
\]
Introduce the notation
\[ \hat{f}_i(x) = \mathbb{E} f(x + h_i \zeta), \quad \forall x \in \mathbb{R}^d, \]
and
\[ \hat{f}_i(x) = \mathbb{E} f_i(x + h_i \zeta), \quad \forall x \in \mathbb{R}^d. \]

**Lemma 10.** Suppose \( f_i \) is differentiable. For the conditional expectation given \( \mathcal{F}_t \), we have
\[ \mathbb{E}[g^i(t)|\mathcal{F}_t] = \nabla \hat{f}_i^i(t(x)). \]

**Lemma 11.** If \( f \) is \( \alpha \)-strongly convex then \( \hat{f} \) is \( \alpha \)-strongly convex. If \( f \in \mathcal{F}_2(L) \), for any \( x \in \mathbb{R}^d \) and \( h_i > 0 \), we have
\[ |\hat{f}_i(x) - f(x)| \leq L h_i^2, \]
and
\[ |\mathbb{E} f(x + h_i \zeta) - f(x)| \leq L h_i^2. \]

**Lemma 12.** Let Assumptions \([A][B]\) and \([D]\) hold with \( \beta = 2 \). Let \( \Theta \) be a convex compact subset of \( \mathbb{R}^d \), and assume that \( \text{diam}(\Theta) \leq k \). Assume that \( \max_{x \in \Theta} \|\nabla f_i(x)\| \leq G \), for \( 1 \leq i \leq n \). Let the updates \( x^i(t), \tilde{x}(t) \) be defined by Algorithm \([I]\) in which the gradient estimator for \( i \)-th agent is defined by \([12]\), and \( \eta_t = \frac{1}{\alpha t}, h_t = \left(\frac{3d^2 \sigma^2}{2L\alpha t + 9L^2 \sigma^2} \right)^{1/4} \). Then
\[ \Delta(t) \leq \left(\frac{\rho}{1-\rho}\right)^2 \left( A_1 \frac{d}{\alpha^{3/2}} t^{-3/2} + A_2 \frac{d^2}{\alpha^2} t^{-2} \right), \]
where \( A_1 \) and \( A_2 \) are positive constants independent of \( T, d, \alpha, n, \rho \).

**Proof.** Similarly to Lemma\(^3\) we obtain
\[ \mathbb{E}[V(t+1)|\mathcal{F}_t] \leq \rho^2 (1 + 2\lambda) V(t) + \rho^2 (4 + \frac{2}{\lambda}) \eta_t^2 \sum_{i=1}^n \mathbb{E}[\|g^i(t)\|^2|\mathcal{F}_t]. \]
Choosing \( \lambda \) as above and using Lemma\(^3\) we get
\[ \mathbb{E}[V(t+1)|\mathcal{F}_t] \leq \rho V(t) + \frac{4\rho^2}{1-\rho} \eta_t^2 \left( 9(G^2d + \frac{L^2d^2h_i^2}{2}) + \frac{3d^2 \sigma^2}{2L\alpha t + 9L^2 \sigma^2} \right). \]
Taking here the expectations and setting \( \eta_t = \frac{1}{\alpha t} \) and \( h_t = \left(\frac{3d^2 \sigma^2}{2L\alpha t + 9L^2 \sigma^2} \right)^{1/4} \) yields
\[ \Delta(t+1) \leq \rho \Delta(t) + \frac{\rho^2}{1-\rho} \left( A_3 \frac{d}{\alpha^{3/2} t^{3/2}} + A_4' \frac{d^2}{\alpha^2 t^2} \right) \]
with \( A_3 = 2\sqrt{6}L\sigma \), and \( A_4' = 12\sqrt{3}L\sigma + \frac{36\sigma^2}{d} \). On the other hand, by recursion we have
\[ \Delta(t+1) \leq \rho^t \Delta(1) + \frac{\rho^2}{1-\rho} \frac{d}{\alpha^{3/2} t^{3/2}} \left( A_3' \sum_{s=1}^t s^{-3} \rho^{t-s} + A_4' \frac{d^2}{\alpha^2 t^2} + \sum_{s=1}^t s^{-2} \rho^{t-s} \right). \]
Here \( \Delta(1) = 0 \) due to the initialization. The sums on right hand side can be estimated by using an argument, which is quite analogous to what was done in the proof of Lemma\(^3\) after equation \([22]\), leading to the result of the lemma. \( \square \)

**Lemma 13.** Let the assumptions of Lemma\(^12\) hold and let \( f \) be an \( \alpha \)-strongly convex function. Then
\[ \mathbb{E}[\|\tilde{x}(t) - x^*\|^2] \leq \frac{C}{1-\rho} \left( \frac{d}{t^{1/2} \alpha^{3/2}} + \frac{d^2}{t \alpha^2} \right), \]
where \( C > 0 \) is a constant independent of \( T, d, \alpha, n, \rho \).
Proof. First note that due to the strong convexity assumption we have
\[ \| \tilde{x}(t) - x^* \|^2 \leq \frac{G^2}{\alpha^2}. \]
Therefore, for \( t = 1 \) the result holds. For \( t \geq 2 \), by the definition of the algorithm we have
\[ \| \tilde{x}(t+1) - x^* \|^2 \leq \| \tilde{x}(t) - x^* \|^2 + \eta_t^2 \| \bar{g}(t) \|^2 + \| \tilde{z}(t) \|^2 - 2\eta_t (\bar{g}(t), \tilde{x}(t)) - 2\eta_t (\bar{z}(t), \tilde{x}(t) - x^*) + 2(\tilde{x}(t) - x^*, \tilde{z}(t)). \]

Taking conditional expectations we get
\[ \mathbb{E}[a_{t+1}|\mathcal{F}_t] \leq a_t + \frac{2\eta_t^2}{n} \sum_{i=1}^{n} \mathbb{E}[\| g^i(t) \|^2 |\mathcal{F}_t] - 2\eta_t \mathbb{E}[(\bar{g}(t), \tilde{z}(t))|\mathcal{F}_t] - 2\eta_t \mathbb{E}[(\bar{g}(t), \tilde{x}(t) - x^*)|\mathcal{F}_t] + 2\mathbb{E}[(\tilde{x}(t) - x^*, \tilde{z}(t))|\mathcal{F}_t], \]
where we used the fact that \( \| z^i(t) \| \leq \eta_t \| g^i(t) \| \) for \( 1 \leq i \leq n \).

For the term \(-2\eta_t \mathbb{E}[(\bar{g}(t), \tilde{x}(t) - x^*)|\mathcal{F}_t] \) in (36), we have
\[ -2\eta_t \mathbb{E}[(\bar{g}(t), \tilde{x}(t) - x^*)|\mathcal{F}_t] \leq -\frac{2\eta_t}{n} \sum_{i=1}^{n} \left( \mathbb{E}[\langle g^i(t), \nabla f^i_t(x^i(t)) \rangle, \tilde{x}(t) - x^*] + \langle \nabla f^i_t(\tilde{x}(t)), \tilde{x}(t) - x^* \rangle \right) \]

For the term in (38), by Lemma 10 we have
\[ -\frac{2\eta_t}{n} \sum_{i=1}^{n} \mathbb{E}[(\bar{g}(t), \nabla f^i_t(x^i(t)), \tilde{x}(t) - x^*)|\mathcal{F}_t] = 0. \]

For the term in (39), decoupling yields
\[ -\frac{2\eta_t}{n} \sum_{i=1}^{n} \langle \nabla f^i_t(\tilde{x}(t) - x^*), \tilde{x}(t) - x^* \rangle \leq \frac{\eta_t \alpha}{n} (1 - \rho)V(t) + \frac{\bar{L}^2 \eta_t}{\alpha (1 - \rho)} a_t. \]
Next, we use the strong convexity (cf. Lemma 11) to handle (40):
\[ -2\eta_t \langle \nabla f^i_t(\tilde{x}(t)), \tilde{x}(t) - x^* \rangle \leq -2\eta_t \alpha a_t. \]
Finally, for the term containing \( 2(\tilde{x}(t) - x^*, \tilde{z}(t)) \) in (37) we obtain similarly to (29) that
\[ 2\mathbb{E}[(\tilde{x}(t) - x^*, \tilde{z}(t))|\mathcal{F}_t] \leq \frac{3\eta_t^2}{(1 - \rho)n} \sum_{i=1}^{n} \mathbb{E}[\| g^i(t) \|^2 |\mathcal{F}_t] + \frac{1}{n} V(t). \]
Combining the above inequalities yields
\[ \mathbb{E}[a_{t+1}|\mathcal{F}_t] \leq (1 - 2\eta_t \alpha) a_t + \frac{2\eta_t^2}{n} \sum_{i=1}^{n} \mathbb{E}[\| \bar{g}(t) \|^2 |\mathcal{F}_t] - 2\eta_t \mathbb{E}[(\bar{g}(t), \tilde{z}(t))|\mathcal{F}_t] + \frac{\eta_t \bar{L}^2 K^2}{\alpha (1 - \rho)} + \frac{\eta_t \alpha + 1}{n} (1 - \rho)V(t) + \frac{3\eta_t^2}{(1 - \rho)n} \sum_{i=1}^{n} \mathbb{E}[\| g^i(t) \|^2 |\mathcal{F}_t]. \]
Now, recalling that \( \eta_t = \frac{1}{\sqrt{\alpha t}}, h_t = \left( \frac{3d^2 \sigma^2}{2L^2 \alpha t + 9L_\alpha} \right)^{1/4} \), taking the expectations and applying Lemma 9 we find
\[ r_{t+1} \leq \left( 1 - \frac{2}{t} \right) r_t + 2(1 - \rho) \Delta(t) + \frac{C}{(1 - \rho)} \left( \frac{d}{t^{3/2} \alpha^{3/2}} + \frac{\sigma^2}{t^{2} \alpha^2} \right), \]
where \( r_t = \mathbb{E}[a_t] \), and \( C > 0 \) is a constant independent of \( T, d, \alpha, n, \rho \). Using Lemma 12 to bound \( \Delta(t) \) in (41) we get
\[ r_{t+1} \leq \left( 1 - \frac{2}{t} \right) r_t + \frac{C'}{(1 - \rho)} \left( \frac{d}{t^{3/2} \alpha^{3/2}} + \frac{\sigma^2}{t^{2} \alpha^2} \right), \]
where \( C' > 0 \) is a constant independent of \( T, d, \alpha, n, \rho \). The desired result follows from this recursion by applying [Akhavan et al., 2020, Lemma D.1].
Theorem 7. Let $f$ be an $\alpha$-strongly convex function. Let Assumptions $A$, $B$, and $D$ hold with $\beta = 2$. Let $\Theta$ be a convex compact subset of $\mathbb{R}^d$, and assume that $\text{diam}(\Theta) \leq K$. Assume that $\max_{x \in \Theta} \|\nabla f_i(x)\| \leq G$, for $1 \leq i \leq n$. Let the updates $x^i(t), \bar{x}(t)$ be defined by Algorithm (7) in which the gradient estimator for $i$-th agent is defined by (12), and $\eta_t = \frac{1}{\alpha t}$, $h_t = \left( \frac{3\beta^2 \sigma^2}{2L \alpha t + 9L \alpha^2} \right)^{1/4}$.

Then for the estimator $\bar{x}(T) = \frac{1}{T-\lceil T/2 \rceil} \sum_{t=\lceil T/2 \rceil+1}^T \bar{x}(t)$ we have

$$
\mathbb{E}[f(\bar{x}(T)) - f(x^*)] \leq \frac{B}{1 - \rho} \left( \frac{d}{\sqrt{\alpha T}} + \frac{d^2}{\alpha T} \right),
$$

where $B > 0$ is a constant independent of $T, d, \alpha, n, \rho$.

Proof. Fix $x \in \Theta$. Due to the $\alpha$-strong convexity of $\hat{f}_i$, we have

$$
\hat{f}_i(\bar{x}(t)) - \hat{f}_i(x^*) \leq \langle \nabla \hat{f}_i(\bar{x}(t)), \bar{x}(t) - x^* \rangle - \frac{\alpha}{2} \|\bar{x}(t) - x^*\|^2.
$$

Thus, by Lemma (11) we get

$$
f(\bar{x}(t)) - f(x^*) \leq 2\bar{L}h_t^2 + \langle \nabla \hat{f}_i(\bar{x}(t)), \bar{x}(t) - x^* \rangle - \frac{\alpha}{2} \|\bar{x}(t) - x^*\|^2.
$$

Let $a_t = \|\bar{x}(t) - x^*\|^2$. Taking conditional expectations and applying Lemma (10) we obtain

$$
\mathbb{E}[f(\bar{x}(t)) - f(x^*)|\mathcal{F}_t] \leq 2\bar{L}h_t^2 + \frac{1}{n} \sum_{i=1}^n \langle \nabla \hat{f}_i(\bar{x}(t)) - \nabla \hat{f}_i(x^i(t)), \bar{x}(t) - x^* \rangle - \frac{\alpha}{2} a_t + \mathbb{E}[(\langle \eta_t \rangle, \bar{x}(t) - x^*)|\mathcal{F}_t] - \frac{\alpha}{2} a_t + \mathbb{E}[a_{t+1}|\mathcal{F}_t] + \frac{\eta_t}{n} \sum_{i=1}^n \mathbb{E}[\|g^i(t)\|^2|\mathcal{F}_t],
$$

where the last inequality uses the definition of the algorithm. Now, by decoupling we find

$$
\frac{1}{n} \sum_{i=1}^n \langle \nabla \hat{f}_i(\bar{x}(t)) - \nabla \hat{f}_i(x^i(t)), \bar{x}(t) - x^* \rangle \leq \frac{t\alpha}{2n} (1 - \rho) V(t) + \frac{L^2}{2(1 - \rho) t\alpha} K^2,
$$

while similarly to (29) we also have

$$
\frac{1}{\eta_t} \mathbb{E}[\langle \eta_t \rangle, \bar{x}(t) - x^*|\mathcal{F}_t] \leq \frac{1 - \rho}{1 - \rho} \frac{3\eta_t}{2n} \sum_{i=1}^n \mathbb{E}[\|g^i(t)\|^2|\mathcal{F}_t] + (1 - \rho) \frac{1}{2n\eta_t} V(t).
$$

Combining the above inequalities and applying Lemma (9) yields

$$
\mathbb{E}[f(\bar{x}(t)) - f(x^*)|\mathcal{F}_t] \leq \left( \frac{1}{\eta_t} + t\alpha \right) \frac{1 - \rho}{2n} V(t) + \frac{L^2}{2(1 - \rho) t\alpha} K^2 - \frac{\alpha}{2} a_t + \frac{a_t - \mathbb{E}[a_{t+1}|\mathcal{F}_t]}{2\eta_t} + 2\bar{L}h_t^2 + \left( 2 + \frac{3}{2(1 - \rho)} \right) \frac{\eta_t}{n} \sum_{i=1}^n \mathbb{E}[\|g^i(t)\|^2|\mathcal{F}_t].
$$

Let $r_t = \mathbb{E}[a_t]$. Using the fact that $\eta_t = \frac{1}{\alpha t}$, $h_t = \left( \frac{3\beta^2 \sigma^2}{2L \alpha t + 9L \alpha^2} \right)^{1/4}$, taking the expectations in (45) and applying Lemma (9) we find

$$
\mathbb{E}[f(\bar{x}(t)) - f(x^*)] \leq t\alpha \left( \frac{r_t - r_{t+1}}{2} \right) - \frac{\alpha}{2} r_t + (1 - \rho) \alpha t \Delta(t) + \frac{C_3}{1 - \rho} \left( \frac{d}{\sqrt{\alpha t}} + \frac{d^2}{\alpha t} \right),
$$

where $\Delta(t) = \mathbb{E}[\|\nabla f_i(x^i(t))\|^2|\mathcal{F}_t]$. 

22
We wish to minimize the following function \( f \) where \( \beta \) is a randomly generated sparse positive definite matrix in \( \mathbb{R}^{d \times d} \) with smallest eigenvalue equal to 1, and \( \psi(x) = \int_{-\infty}^{x} \int_{-\infty}^{z} \phi(t) dt dz \), with

\[
\phi(x) = \begin{cases} \\
0 & \text{if } x < -a \\
\frac{-3}{a^2} x + 2 & \text{if } -a \leq x < -\frac{a}{2} \\
\frac{-2}{a^2} x & \text{if } -\frac{a}{2} \leq x < \frac{a}{2} \\
\frac{2}{a^2} x - 2 & \text{if } \frac{a}{2} \leq x < a \\
0 & \text{if } a < x, 
\end{cases}
\]

where \( a > 0 \). A direct computation gives that

\[
\psi(x) = \begin{cases} \\
\frac{1}{3} x^3 + ax^2 + ax + \frac{a^3}{3} & \text{if } x < -a \\
-\frac{1}{3} x^3 + \frac{a}{2} x + \frac{a^2}{4} & \text{if } -a \leq x < -\frac{a}{2} \\
\frac{x^3}{3} - ax^2 + ax + \frac{a^2}{6} & \text{if } -\frac{a}{2} \leq x < \frac{a}{2} \\
\frac{a^3}{2} & \text{if } \frac{a}{2} \leq x < a \\
0 & \text{if } a < x.
\end{cases}
\]

Let \( \Theta = \{ x \in \mathbb{R}^d : \|x\| \leq 1, \text{ and } x_i \leq 0, \text{ for } 1 \leq i \leq d \} \). Since for any \( x \in \Theta, \phi(x) \geq 0 \), then \( \psi \) is convex on \( \Theta \), which implies \( \alpha \)-strong convexity of \( f \) on \( \Theta \). Also, the second derivative of \( Lh^3 \psi(h^{-1} x) \) is Lipschitz continuous with Lipschitz constant equal to \( \frac{2L}{a} \). Therefore \( f \) is \( \beta \)-Hölder with \( \beta = 3 \). We choose the kernel function, \( K : [-1, 1] \to \mathbb{R} \), such that \( K(x) = \frac{15}{\pi} x (5 - 7x^2) \). For each iteration \( t \), we fix \( h_t = t^{-\frac{4}{5}} \), and \( \eta_t = \frac{2}{a^2} \). Function evaluations at a fixed point \( x \in \mathbb{R}^d \) are obtained in the form \( f(x) + \zeta \) where \( \zeta \) is a random variable uniformly distributed in \([-5, 5]\).

In this implementation we assign \( \alpha = 2, h = 10^{-3}, L = 10^{7.5}, a = 10 \). We also let \( A = B + I \), where \( B \) is a randomly generated sparse positive definite matrix in \( \mathbb{R}^{d \times d} \) and \( I \) is the \( d \)-dimensional identity matrix. For the initialization, we generate a \( d \)-dimensional Gaussian random variable and project it on \( \Theta \).
Figure 1: Optimization error vs. number of function evaluations for the 2-Point Estimator in Akhavan et al. [2020] and our method, run on function (46) for different number of variables ($d = 25, 50, 100, 150$ clockwise from top-left).

The design of $f$ in (46) is inspired by the function that has been used in the proof of the lower bound in Akhavan et al. [2020]. It is a quadratic function plus the perturbation $L h^3 \sum_{i=1}^{d} \psi(h^{-1} x_i)$, which adds difficulty to estimation of the minimizer. We have chosen this worst case function to provide a comparison between two algorithms in a long run and growing dimension. In Figure 1 we display the average optimization error of the method proposed in this paper and that of the 2-Point estimator from Akhavan et al. [2020] versus the total number of function evaluations, for different dimensions $d$. This result is averaged over 40 trials, corresponding to different random initialization, noisy function evaluations and randomization in the optimization procedures. We would like to emphasize that both methods are considered with the same budget of function evaluations, which means that the number of iterations for the two algorithms differ. Thus, if $T$ is the total number of function evaluations, the 2-point estimator makes $T/2$ iterations, while the proposed method makes only $T/(2d)$ iterations.