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Pot and Ladle: Seat Allocation and Seat Bias under the Jefferson-D’Hondt Method

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Abstract

We propose a simple and new formula for estimating national seat shares in multidistrict elections employing the Jefferson-D’Hondt method for seat allocation solely on the basis of the national vote shares and fixed parameters of the electoral system. The proposed formula clarifies the relationship between seat bias and both the number of parties and the number of districts. We discuss how our formula differs from the simple generalization of the single-district asymptotic seat bias formulae and what assumptions must hold for our method to give exact results. We further demonstrate that despite minor violations of those assumptions, the formula provides a good estimate of actual seat allocations for all EU countries that have a national party system and employ the Jefferson-D’Hondt seat allocation method in multi-district parliamentary elections, i.e., Croatia, the Czech Republic, Finland, Luxembourg, the Netherlands, Poland, Portugal, and Spain. Moreover, we show that the formula constitutes a generalization and extension of the McGhee-Stephanopoulos efficiency gap test for political gerrymandering. Finally, we present a number of additional applications of the formula for the evaluation of political strategies.

Keywords: Jefferson-D’Hondt method, seats-votes relationship, seat bias, proportionality

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The Jefferson-D’Hondt method is one of the most popular ways for allocating parliamentary seats to party lists in proportional representation electoral systems (Colomer, 2004; Bormann & Golder, 2013; Carey, 2017). It was originally devised in 1792 by Thomas Jefferson to apportion seats in the U.S. House of Representatives among the states (Jefferson, 1792), and later it was proposed by a Belgian mathematician and lawyer Victor D’Hondt (D’Hondt, 1882; 1885) for use in parliamentary elections. It is used to allocate all parliamentary seats in, inter alia, Albania, Argentina, Aruba, Belgium, Cape Verde, Chile, the Czech Republic, East Timor, Fiji, Finland, Greenland, Israel, Luxembourg, Macedonia, the Netherlands, Paraguay, Peru, Poland, Portugal, São Tome and Príncipe, Serbia, Spain, Suriname, Switzerland, and Turkey, as well as nearly all the seats in Croatia and Montenegro. It is also employed, in combination with other methods, in Austria, Denmark, the Dominican Republic, Faroe Islands, Iceland, and Japan, and had been historically used in, among others, Bulgaria, France, Germany, Italy, Moldova, Norway, and Sweden.

It is well known that the Jefferson-D’Hondt method is biased in favor of larger parties (see e.g., Humphreys, 1911; Huntington, 1921; 1928; 1931; Morse, Von Neumann & Eisenhart, 1948; Rae, 1967; Taagepera & Laakso, 1980; Carstairs, 1980; Woodall, 1986; Taagepera & Shugart, 1989; Lijphart, 1990; Gallagher, 1991; Oyama & Ichimori 1995; Benoit, 2000; Balinski & Young, 2001: 72-74; Marshall, Olkin & Pukelsheim, 2002; van Eck et al., 2005; Pukelsheim, 2014). The magnitude of such bias has been estimated by Sainte-Laguë (1910), Pólya (1918a, 1918b, 1919a, 1919b, 1919c), Schuster et al. (2003), Schwingenschlögl & Drton (2004), Drton & Schwingenschlögl (2005), Schwingenschlögl (2008), Pukelsheim (2014), and Janson (2014). However, though earlier research focused on a single-district scenario, the majority of countries employing the Jefferson-D’Hondt method allocate seats within each of their multiple electoral districts separately, with the most notable exceptions being Israel and the Netherlands. In those countries, the political effects of the advantage provided by the Jefferson-D’Hondt method to larger parties can only be assessed on the national scale, when looking at the composition of the legislature as a whole.

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1 The Jefferson-D’Hondt method is also known as the Hagenbach-Bischoff method (Szpiro, 2010: 204), the method of greatest divisors (Huntington, 1921; 1928; 1931), the method of highest averages (Carstairs, 1980: 17-19), and the method of rejected fractions (Chafee, 1929). In Israel the method is called the Bader-Ofer system after two members of the Knesset who proposed it in 1975: Yohanan Bader, an eminent alumnus of the authors’ Alma Mater, and Avraham Ofer.

2 In fact, the method was also rediscovered by several authors in various contexts between 1860 and 1874, see Mora (2013: 6) for details. Dančičín (2013b) discusses the evolution of D’Hondt’s ideas on the subject of proportional representation and the origins of his method.

3 The Jefferson-D’Hondt method is also employed to allocate European Parliamentary seats in a majority of the EU member states (Poptcheva 2016), as well as at the regional and local level in a number of countries.

4 Pukelsheim (2017: 133) noted that seat biases scale with the number of districts, but stopped short of providing an explicit summation formula. Of course, he has been primarily concerned with an expected seat bias of the k-th largest party, rather than a seat bias of a party whose vote share is fixed. Unless one assumes that the distribution of party vote shares on the probability simplex is so highly concentrated that the order of parties is the same in every district (an assumption that fails to match empirical data), a simple summation of the expected biases over districts will not produce a nationwide seat bias.
Drawing on earlier works by Janson (2014) and Pukelsheim (2014), we develop here a new formula for estimating seat allocation that depends solely on nationwide electoral results and the fixed parameters of the electoral system. We show that the number of seats for the \(i\)\textsuperscript{th} party (\(s_i\)) is given by the seat allocation formula:

\[
 s_i = p_i \cdot s + p_i \cdot \frac{cn}{2} - \frac{c}{2},
\]

where \(p_i\) is the renormalized share of votes cast for that party, \(s\) – the total number of seats, \(c\) – the number of electoral districts, and \(n\) – the renormalized number of parties, i.e., the number of “relevant” parties determined in accordance with the procedure described in Sec. 3.3. This formula is exact if the five underlying assumptions discussed in Sec. 1.2 are fully satisfied, and it provides an approximation if they are only approximately satisfied. Note also that this formula is not a simple consequence of the asymptotic results valid for a single district, but requires different assumptions and justifications.

Metaphorically, the formula can be thought of in terms of a potluck:\(^5\) each party provides a contribution of equal value to the common bounty pot (averaging to half a seat per district, that is \(c/2\)). But when the pot is later divided among the parties, the size of the bounty is proportional to the ladle, namely its (renormalized) share of the overall number of votes (\(p_i\)).\(^6\) Accordingly, small parties are disadvantaged, since they contribute more than they get back from the pot, while large parties receive a bonus. What the formula does make clear, however, is that the size of that bonus depends not only on the size of the ladle, but also on the size of the bounty pot (\(cn/2\)). This is, in fact, the basic mechanism the formula reveals: the party bonus created by the D’Hondt system is a function of both the number of districts and of the number of parties.

While for the political practitioners the number of seats is the key magnitude of interest, scholars have traditionally thought of electoral systems in terms of seat shares (\(q_i := s_i/s\)), which are, \textit{inter alia}, more suitable for international comparisons. In that case, it can be easier to substitute for the number of seats and the number of districts a single variable, the mean district magnitude (\(m := s/c\)) and to express the formula in the following manner:

\[
 q_i = (1 + \frac{n}{2m})p_i - \frac{1}{2m},
\]

underscoring the linear (affine) dependence between the seat shares and the vote shares.

Hence, the difference between the seat share and the vote share is the seat bias for the \(i\)\textsuperscript{th} party (\(\Delta_i := q_i - p_i\)) given by the following formula:

\[
 \Delta_i = \frac{n}{2m} \cdot \left( p_i - \frac{1}{n} \right).
\]

\(^5\) Or \textit{Jacob’s join}, for those of our readers who are from Lancashire, see Crosby (2000).
\(^6\) See Janson (2014: Remark 3.9) for a similar heuristic for \(c = 1\).
Since the sign of the bias is determined by its last factor, the difference between the party’s vote share and the mean vote share, it is evident that whether a party stands to gain or lose from the D’Hondt formula depends on its vote share being above or below the mean. It is likewise evident that the system is neutral (i.e., $\Delta_i = 0$) only towards those parties for which $p_i = 1/n$. Such parties receive exactly the number of seats corresponding to their vote shares.

In addition to explaining the magnitude of the nationwide bias for the winning parties, the formulae have a purely practical application. The Jefferson-D’Hondt method requires that district-level results should be known, and as all divisor methods can be sensitive to small variations in vote shares, those results have to be known exactly. By contrast, our formula provides a surprisingly good prediction of the nationwide seat allocation (with accuracy within 1.5% of the national seat total for more than 94.0% of parties for the analyzed data from eight countries) while requiring only aggregate party vote shares to be known. Hence, it can be used to accurately model seat allocation on the basis of opinion polls, exit polls, and preliminary election results, when aggregate vote shares are usually all that is known.\(^7\)

The model can be also applied to detect gerrymandering in electoral systems employing Jefferson-d’Hondt method. In fact, it generalizes and extends the McGhee-Stephanopoulos efficiency gap test (McGhee, 2014; Stephanopoulos & McGhee, 2015), currently one of the most prominent methods for detecting gerrymandering in two-party FPTP systems, in three aspects: it allows for relaxation of a restrictive assumption that the turnout is equal across all districts and it permits the test to be applied to multiparty systems, as well as to systems with multimember districts.

The proposed model has several additional potential applications. It can be useful for calculating political strategies (e.g., for estimating consolidation benefits or secession losses). It can also provide a simple method for estimating the effects of changes in electoral system parameters (particularly the number of seats and the number of districts) on particular parties, as well as on general systemic incentives.

In Sec. 1 of this paper we discuss the basic features and assumptions of the proposed formula, demonstrating how it fits into earlier literature on the subject and where it introduces new findings. In Sec. 2, we analyze actual data from eight European countries to demonstrate that the formula provides a reasonably accurate estimate of actual seat allocation results and is quite robust against minor violations of the assumptions. In Sec. 3 we focus on the connection between the mathematical basis of the Jefferson-D’Hondt seat allocation method and our formula, explaining how the latter can be derived from the former. In particular, we discuss at which stages the assumptions formulated in Sec. 1 are employed. Finally, in Sec. 0 we explore the political consequences of the seat bias formula. Moreover, we examine possible applications of the formula to single-member systems, as well as the generalization of the McGhee-Stephanopoulos efficiency gap test mentioned above. In addition, in Appendix A we quantify and analyze the sources of approximation error in the formula that arises when the

\(^7\) For earlier attempts to estimate seat allocations on the basis of nationwide polls, see, e.g., Pavia & García-Cárceles (2016) and Udina & Delicado (2005). It should be noted, however, that those prior works approach the problem from an entirely different perspective, attempting to fit a statistical model to data (a task heavily reliant on overt and latent distributional assumptions), while we seek to derive a theoretical model from the Jefferson-D’Hondt method itself and use the empirical data only for test purposes.
assumptions given in Sec. 1.2 are partially violated, and in Appendix B we discuss in more depth the two technical assumptions and demonstrate why they seem to be less restrictive than the ones made in prior works.

All parts of the article are designed to be sufficiently independent of each other to permit the reader to skip immediately to the one they are interested in.

1. Basic features and assumptions

1.1. Seat bias under the Jefferson-D’Hondt method

The Jefferson method’s bias in favor of larger parties had already been recognized in the context of congressional apportionment discussions long before Victor D’Hondt introduced (or rather reinvented) it as a method for allocating seats among parties in the context of proportional representation systems (Balinski & Young, 2001: 23-24; Pukelsheim, 2017: 323-324), but Sainte-Laguë (1910) was the first to quantify this effect, finding its expected value to equal \( \ln 2 - 1/2 \) seats under the assumption that the ratio of the smaller-party vote count to larger-party vote count is uniformly distributed over \((0,1)\). Pólya (1918a, 1918b, 1919a, 1919b, 1919c), assuming instead a uniform distribution of party vote shares over the probability simplex, employed geometric approach to calculate expected seat biases for three-party elections. This line of research has been continued by Schuster et al. (2003), Schwingenschlögl & Drton (2004), Drton & Schwingenschlögl (2005), and Schwingenschlögl (2008), who provide expected seat biases for the \( k \)-th largest party in an \( n \)-party election.\(^8\)

However, one is frequently interested in estimating expected seat bias for a specific party (characterized by a given vote share) rather than for an average \( k \)-th largest party. Moreover, the latter problem necessarily involves difficulties related to the choice of appropriate distributional assumptions.

Analytic formulae expressing single-district seat bias as a function of a party’s vote share have been proposed by Bochsler (2010), Janson (2014), and Pukelsheim (2014). \textit{Prima facie}, they appear identical to each other and match our seat bias formula for \( c = 1 \). However, despite those similarities, they address different problems and employ different assumptions. Under the assumption that the votes shares follow an arbitrary absolutely continuous distribution over the probability simplex, Pukelsheim (2014: Sec. 6.10) has proven that the rounding residuals, i.e., the quantities that are discarded when dividing a party’s number of votes by the Jefferson-D’Hondt divisor and rounding down to full seats, converge in distribution to a vector of \( n \) variables drawn from a uniform distribution and stochastically independent of each other and of the party vote shares as the number of seats \( s \) approaches infinity.\(^9\) Hence he has deduced (Pukelsheim, 2014: Sec. 7.3), that the seat biases approaches those given by (0.3) (with \( c = 1 \)) in this case. Janson (2014: Theorem 3.4) has shown that for

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\(^8\) It should be noted that their results for expected seat bias for the \( k \)-th largest party (obtained under the assumption that vote share vectors are drawn from the uniform distribution on the unit simplex) match exactly the results produced by our seat bias formula when \( p_i \) equals the expected vote share of the \( k \)-th largest party.

\(^9\) The proof is a more general case of an earlier proof by Tukey (1938), who established that the rounding residuals of a scalar variable converge in distribution to a uniform distribution on \((0,1)\). See also Heinrich, Pukelsheim & Schwingenschlögl (2004).
any choice of votes shares (under the mild assumption that they are linearly independent over rationals), the mean seat biases for the number of seats being randomly drawn from the uniform distribution on \{1, \ldots, S\} also approach those given by (0.3) (again with \(c = 1\)) as \(S\) tends to infinity. Bochsler (2010: p. 621) has obtained the single-district mean seat bias formula by assuming that the rounding residuals are independent of that party’s vote share and always have an expected value of 1/2.

While all three cited works significantly advanced our knowledge of seat biases, they share a common limitation. The single-district formulae described by their authors are only correct in an asymptotic sense – as the district magnitude approaches infinity. Moreover, numerical simulations demonstrate that the rate of convergence in some of the theorems on which the authors rely is sufficiently slow as to render them of limited usefulness in those real-life electoral systems where the district magnitude is typically on the order of 3 to 30 seats (and, as we discuss in more detail in Sec. 0, it is exactly there that the magnitude of seat biases is most significant). It is primarily in the area described that we seek to advance prior knowledge by demonstrating that when seat allocations and seat biases are summed over multiple districts, restrictive assumptions about the rounding residuals on which Bochsler, Janson, and Pukelsheim rely can be exchanged for more liberal ones, dealing only with inter- and intra-district averages (which converge to 1/2 far more rapidly).

While our formulae are superficially similar to the those obtained by Janson, Pukelsheim, and Bochsler, they are not merely generalizations of the latter three to the multi-district case. Indeed, the difference is one of kind. First, our formulae are deterministic rather than probabilistic: if the five assumptions described in the following section are exactly satisfied, they yield the exact rather than just the expected values of the seat allocation, seat share, and seat bias. If the assumptions do not hold, the formulae are not guaranteed to work at all, but in Sec. 2 we demonstrate that in a number of real-life elections they still provide a reasonably good approximation of the empirical results. Second, our formulae rely on rather different assumptions. Janson and Pukelsheim both effectively assume that there is only one district, but its magnitude approaches infinity,\(^{10}\) while we do not require the district magnitude even to be large, though our assumption A5 involves averaging over districts and thereby relies on the law of large numbers, working better when the number of districts is substantial.

1.2. Assumptions

The five assumptions discussed in this subsection (the first three of which are essentially political and the latter two – technical) are sufficient for the seat allocation and seat bias formulae to produce exact results. Note that in this case it can be shown that the right hand side of formula (0.1) yields an integer number of seats. Otherwise, the formulae provide only an approximation.

Technically, we define the set of relevant parties as the set of those parties that exceed both the relevant statutory thresholds (if any) and the mean natural threshold, defined in Sec. 3.3

\(^{10}\) Technically, Janson treats the district magnitude as randomly drawn from a discrete uniform distribution on \((0, S)\), with \(S\) approaching infinity. But in such case the expected district magnitude also approaches infinity, while the probability of the district magnitude being equal or smaller than some \(x > 0\) (i.e., the value of its cumulative distribution function) approaches 0 for every \(x\).
below. The renormalized number of parties $n$ that appears in the seat allocation and seat bias formulae is the cardinality of that set. Moreover, by “the rounding residuals” we mean the quantities that are discarded when dividing each relevant party’s vote share by the Jefferson-D'Hondt divisor and rounding down, see Sec. 3.1.

The five assumptions are as follows:

A1 parties that are too small to be relevant in the above sense are allocated no seats in every electoral district;

A2 renormalized party vote shares\[^{11}\] are not correlated with district size measured by the sum of votes cast for relevant parties;

A3 renormalized party vote shares are not correlated with district magnitude measured by the total number of seats;

A4 in each district the rounding residuals average to 1/2 over all relevant parties;

A5 for each relevant party the rounding residuals average to 1/2 over all districts.

Assumption A1 is potentially most troublesome, in the sense that its violations cannot be easily remedied. Such violations can occur in two instances.

1) First, if the electoral support of some parties is highly concentrated in a number of districts small enough that the nationwide vote share is insufficient for those parties to be included in the set of relevant parties. The most extreme example of the former case occurs where one or more parties are regional in character (for instance because they represent a national or ethnic minority) and register party lists only in a single region, but win a non-zero number of seats therein. Such parties are a permanent fixture of political environments in such countries as Spain, Finland, or Croatia. The solution to this problem consists of employing a regional correction, which envisages estimating regional party votes shares on the basis of nationwide results and then applying the formula to each region separately, see Sec. 1.3.

2) Second, if the variances in district magnitude translate into such variances in natural thresholds that parties too small to be qualify for seat allocation in an average-sized district nevertheless gain seats in the larger ones. For instance, in Spain the average district magnitude is ca. 6.73 seats, but (as of the last election) 36 seats have been allocated in the Madrid district and 31 seats – in the Barcelona district. Accordingly, outside the capital the best threshold estimate (see Taagepera, 2002) is at ca. 10.48% of the votes, while in Madrid it is more than five times smaller – just ca. 2.02%. For a recent discussion of this effect and its political consequences, see Barceló and Muraoka (2018).

Assumptions A2 and A3 are fairly intuitive and we would expect them to be approximately satisfied in most real-life elections, except in two cases (both of which apply only to assumption A2): if one party’s vote share is highly correlated with turnout at the level of

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\[^{11}\] By renormalized vote share of the $i$-th party in the $k$-th district ($p_i^k$), we mean here the ratio of that party’s vote count ($v_i^k$) and the sum of votes cast for all relevant parties ($v^k$).
electoral district or with aggregate support for the unrepresented parties. A major violation of both A2 and A3 would likely be indicative of some form of gerrymandering (or at least a non-intentional partisan bias). Smaller violations occur in some countries, such as Spain, for the parties whose support is highly concentrated in the urban areas, which are also the largest electoral districts both in terms of size and magnitude.

Assumptions A4 and A5 are probably the least intuitive ones. At first glance they resemble the restrictive assumptions of the single-district formulae and, in fact, if \( c = 1 \) assumption A5 is equivalent to the assumption underlying Bochsler (2010). However, averaging the rounding residuals over parties (in the case of A4) or over districts (in the case of A5) leads, both empirically and theoretically, at least for distributions that are reasonably good approximations of real-life vote share distributions, to quantities close to 1/2. Assumption A4 is equivalent to the assumption that the Jefferson-D'Hondt divisor equals the Gfeller-Joachim quota, which has been shown by Happacher & Pukelsheim (1996) to be asymptotically optimal in the sense of generating the smallest discrepancy from the desired number of seats, see Sec. 3.1. We reserve more detailed discussion of assumption A5 for Appendix B. Note that an empirical analysis of those two assumptions shows that A4 is a much smaller source of error than A5, which is the single largest source of error in the entire formula, see Appendix A.

1.3. Regional correction

As noted in connection with assumption A1, regional parties pose a significant problem in applying the seat allocation formula, as all of their votes are concentrated in a small number of districts. Accordingly, their nationwide vote shares are a poor approximation of their actual level of support in those districts, leading to such errors as eliminating them from the set of relevant parties due to being too small to cross the natural threshold, thereby misallocating all their seats to non-regional parties. If exact regional vote shares are known, for instance, because separate polls are held for regions with strong local parties, as is usually the case in Spain, this can be entirely avoided by applying the seat allocation formula for each region independently and then summing over all the regions. However, even if no such data are available, seat allocations can still be estimated on the basis of the nationwide distribution of party votes, provided that it is already known before the election which parties contest which districts and that certain additional assumptions discussed below hold.

Let a party that contests less than a majority of all districts be a “regional party,” and let other parties be “national parties.” Let a set of districts contested by a given regional party be a “region.” For technical reasons, we will treat the set of districts with no regional parties as another region. Using those terms, we can express the assumptions for the regional correction as follows:

- **R1** there is no partial overlap between any regions;
- **R2** the voters-to-seats ratio does not vary between regions (i.e., there is no interregional malapportionment);
- **R3** relative size of the national parties (i.e., the ratios of their respective vote
shares) does not vary between regions.

Of those, assumptions $\textbf{R1}$ and $\textbf{R2}$ sound natural (although $\textbf{R1}$ can be quite restrictive – for instance, it prevents the regional correction from being applied in Belgium, as the Flemish and Walloon regions overlap in Brussels). $\textbf{R3}$ is less obvious, but the effects of its violations tend to partially cancel each other when nationwide seat allocations are considered.

The main premise of the regional correction is the fact that if the votes of a regional party $i$ are entirely concentrated in a single region $r$ (per $\textbf{R1}$) then we can express its regional vote share as simply

$$P_i^r := P_i \frac{v}{vr} = P_i \frac{s}{sr}, \quad (1.1)$$

where $P_i$ is the $i$-th party nationwide vote share and $P_i^r$ is its regional vote share (both are non-renormalized). Of course, for regional vote shares to sum up to 1, vote shares of the national parties need to be rescaled (per $\textbf{R3}$) to

$$P_j^r := P_j \frac{1 - \sum_{l \in R} P_l^r}{1 - \sum_{l \in R} P_l}, \quad (1.2)$$

where $R$ is the set of all regional parties and $R^r$ is the set of regional parties running in region $r$. In the region with no regional parties, this will simplify to

$$P_j^0 := \frac{P_j}{1 - \sum_{l \in R} P_l}. \quad (1.3)$$

With those approximations of regional vote shares, as well as exact data on the number of seats and districts within each region, the seat allocation formula can be applied for each region without any further modifications.

2. Empirical test

As noted above, the Jefferson-D’Hondt method is used in numerous electoral systems around the world. Due to data availability and length constraints, we restrict ourselves to a limited subset of cases that meet the following criteria:

1) Post-1945 national lower house elections…
2) … in the EU member states…
3) … with national (rather than fragmented) party systems…
4) … and electoral systems based on multi-member districts (average district magnitude greater than 1)…
5) … where the Jefferson-D’Hondt method is used to allocate all seats…
6) … and continues to apply as of October 2018.

There are seven countries that fully satisfy those criteria: the Czech Republic, Finland, Luxembourg, the Netherlands, Poland, Portugal, and Spain. We also include Croatia, although it does not fully satisfy criterion (4): it uses the FPTP method to allocate seats in special districts set aside for ethnic minorities. However, the number of those minority seats is
relatively small in comparison to the size of the Croatian parliament (6 out of ca. 150) and elections for those seats are held at different dates, so we shall simply omit them from our calculations. We do not include Belgium, given its use of multietier elections in the Brussels region, but also given its absence of a nationwide party system (our formula would have to be applied separately for Flanders and Wallonia). For Finland, we omit elections prior to 2003 due to the fact that the available data sets do not include information about the use of apparentments.

For elections before 2007, we have obtained our data from the Global Election Database (Brancati, 2007). For subsequent elections, we have used the Constituency-Level Elections Archive (Kollman et al., 2017) and the web sites of the respective national electoral authorities. Table 1 sets forth the general electoral system parameters of the countries discussed above:

**Table 1. General parameters of electoral systems of the test country data set.**

| Country         | Earliest election included | Number of elections | Number of seats (s) | Number of districts (c) | Number of relevant parties (n) |
|-----------------|---------------------------|---------------------|---------------------|-------------------------|-------------------------------|
| Croatia         | 2000                      | 6                   | 143-146             | 11                      | 4-8                           |
| Czech Republic  | 2002                      | 5                   | 200                 | 14                      | 4-9                           |
| Finland         | 2003                      | 4                   | 200                 | 13-15                   | 7-8                           |
| Luxembourg      | 1945                      | 16                  | 26-64               | 2-4                     | 4-8                           |
| Netherlands     | 1948                      | 21                  | 100-150             | 1                       | 7-14                          |
| Poland          | 2005                      | 4                   | 460                 | 41                      | 4-6                           |
| Portugal        | 1975                      | 15                  | 230-263             | 22-24                   | 4-6                           |
| Spain           | 1977                      | 13                  | 350                 | 52                      | 3-6                           |

In five of those eight countries there have been at least a few regional parties that managed to win parliamentary seats, such as the Convergence and Union (CiU), the Republican Left of Catalonia (ERC), the Basque Nationalist Party (EAJ / PNV), and many others in Spain, the Swedish People’s Party (SFP / RKP) in Finland, the Croatian Democratic Alliance of Slavonia and Baranja (HDSSB) and the Istrian Democratic Party (IDS) in Croatia, the German Minority (MN) in Poland, the Independent Democratic Association of Macau in Portugal in 1975, or the Party of Independents of the East in Luxembourg in 1945. Accordingly, in those cases we employ the regional correction.

To test the accuracy of the seat allocation formula (if it is found accurate, the accuracy of the seat bias formula necessarily follows), we have compared actual seat allocation with the results yielded by formula (0.1), with regional correction if necessary. For further comparison, we have also included the results obtained by the “naïve proportional” seat allocation, i.e., by allocating to \(i\)-th party exactly \(p_i s\) seats, without accounting for the rounding effects, the district magnitude variations, and the bias in favor of larger parties, as well as results obtained by employing the modified cube law for proportional elections (Taagepera, 1986)\(^{12}\). Table 2 sets forth the results for the most recent elections in the eight test countries.

\(^{12}\) Under Taagepera’s modified cube law for proportional elections, the expected number of seats of \(i\)-th party equals \(s_i = s p_i^x / (p_i^{x} + (N - 1)^{1-x})^{1/m}\), where \(x := (\log v / \log s)^{1/m}\) and \(N\) is the effective number of parties, i.e., \(N := (\sum_{i=1}^{n} p_i^x)^{-1} \).
Table 2. Comparison of predicted and actual seat allocations in most recent parliamentary elections in eight European countries (with regional corrections when necessary).

| Country       | s   | c   | n | party\(^{14}\) | \(p_i\)   | \(s_j\) | formula allocation | naïve proportionality | modified cube law |
|---------------|-----|-----|---|----------------|-----------|--------|-------------------|-----------------------|-------------------|
| Croatia 2016  | 143 | 11  | 5 | HDZ            | 38.95%    | 61     | 61.8 ± 0.8        | 55.7 ± 5.3            | 57.2 ± 3.8        |
|               |     |     |   | NK             | 35.64%    | 54     | 56.1 ± 2.1        | 51.0 ± 3.0            | 51.9 ± 2.1        |
|               |     |     |   | MOST           | 10.48%    | 13     | 12.6 ± 0.4        | 15.0 ± 2.0            | 13.4 ± 0.4        |
|               |     |     |   | ZIVI ZID       | 6.56%     | 8      | 5.8 ± 2.2         | 9.4 ± 1.4             | 8.0 ± 0.0         |
|               |     |     |   | BM 365         | 4.31%     | 2      | 2.0 ± 0.0         | 6.2 ± 4.2             | 5.0 ± 3.0         |
|               |     |     |   | NL-Zeljko      | 0.31%     | 1      | 0.0 ± 1.0         | 0.4 ± 0.6             | 0.2 ± 0.8         |
|               |     |     |   | IDS            | 2.42%     | 3      | 3.0 ± 0.0         | 3.5 ± 0.5             | 3.0 ± 0.0         |
|               |     |     |   | HDSSB          | 1.32%     | 1      | 1.6 ± 0.6         | 1.9 ± 0.9             | 1.9 ± 0.9         |
| Czech Republic 2017 | 200 | 14  | 9 | ANO            | 31.62%    | 78     | 76.2 ± 1.8        | 63.2 ± 14.8           | 66.2 ± 11.8       |
|               |     |     |   | ODS            | 12.08%    | 25     | 24.8 ± 0.2        | 24.2 ± 0.8            | 23.6 ± 1.4        |
|               |     |     |   | Pirati         | 11.52%    | 22     | 23.3 ± 1.3        | 23.0 ± 1.0            | 22.4 ± 0.4        |
|               |     |     |   | SPD            | 11.35%    | 22     | 22.9 ± 0.9        | 22.7 ± 0.7            | 22.1 ± 0.1        |
|               |     |     |   | KSCM           | 8.29%     | 15     | 14.8 ± 0.2        | 16.6 ± 1.6            | 15.7 ± 0.7        |
|               |     |     |   | CSSD           | 7.76%     | 15     | 13.4 ± 1.6        | 15.5 ± 0.5            | 14.6 ± 0.4        |
|               |     |     |   | KDU-CSL        | 6.19%     | 10     | 9.3 ± 0.7         | 12.4 ± 2.4            | 11.4 ± 1.4        |
|               |     |     |   | TOP 09         | 5.67%     | 7      | 7.9 ± 0.9         | 11.3 ± 4.3            | 10.4 ± 3.4        |
|               |     |     |   | STAN           | 5.53%     | 6      | 7.5 ± 1.5         | 11.1 ± 5.1            | 10.1 ± 4.1        |
| Finland 2015  | 200 | 13  | 8 | Center         | 21.44%    | 49     | 47.4 ± 1.6        | 42.9 ± 6.1            | 44.3 ± 4.7        |
|               |     |     |   | NCP            | 18.49%    | 37     | 40.0 ± 3.0        | 37.0 ± 0.0            | 37.8 ± 0.8        |
|               |     |     |   | True Finns     | 17.93%    | 38     | 38.6 ± 0.6        | 35.9 ± 2.1            | 36.5 ± 1.5        |
|               |     |     |   | SDP            | 16.78%    | 34     | 35.7 ± 1.7        | 33.6 ± 0.4            | 34.0 ± 0.0        |
|               |     |     |   | Greens         | 8.67%     | 15     | 15.3 ± 0.3        | 17.3 ± 2.3            | 16.7 ± 1.7        |
|               |     |     |   | Left           | 7.25%     | 12     | 11.7 ± 0.3        | 14.5 ± 2.5            | 13.7 ± 1.7        |
|               |     |     |   | Ch. Dem.       | 3.60%     | 5      | 2.5 ± 2.5         | 7.2 ± 2.2             | 6.4 ± 1.4         |
|               |     |     |   | Aland          | 0.92%     | 1      | 1.0 ± 0.0         | 1.8 ± 0.8             | 1.5 ± 0.5         |
|               |     |     |   | Swedes         | 4.91%     | 9      | 8.8 ± 0.2         | 9.8 ± 0.8             | 9.4 ± 0.4         |
| Luxembourg 2013 | 60  | 4   | 7 | CSV            | 35.15%    | 23     | 24.0 ± 1.0        | 20.9 ± 2.1            | 21.7 ± 1.3        |
|               |     |     |   | LSAP           | 19.83%    | 13     | 12.7 ± 0.3        | 12.6 ± 0.4            | 12.5 ± 0.5        |
|               |     |     |   | DP             | 19.70%    | 13     | 12.6 ± 0.4        | 11.3 ± 1.7            | 11.1 ± 1.9        |
|               |     |     |   | Dei Grenz      | 10.65%    | 6      | 5.9 ± 0.1         | 6.3 ± 0.3             | 5.8 ± 0.2         |
|               |     |     |   | ADR            | 7.00%     | 3      | 3.2 ± 0.2         | 4.1 ± 1.1             | 3.7 ± 0.7         |
|               |     |     |   | Dei Lenk       | 4.61%     | 2      | 1.4 ± 0.6         | 3.1 ± 1.1             | 2.7 ± 0.7         |
|               |     |     |   | Piratepart.    | 3.05%     | 0      | 0.3 ± 0.3         | 1.8 ± 1.8             | 1.5 ± 1.5         |
| Netherlands 2017 | 150 | 1   | 13 | VVD            | 21.62%    | 33     | 33.3 ± 0.3        | 32.4 ± 0.6            | 32.6 ± 0.4        |
|               |     |     |   | PVV            | 13.26%    | 20     | 20.3 ± 0.3        | 19.9 ± 0.1            | 19.9 ± 0.1        |
|               |     |     |   | CDA            | 12.57%    | 19     | 19.2 ± 0.2        | 18.9 ± 0.1            | 18.9 ± 0.1        |
|               |     |     |   | D66            | 12.42%    | 19     | 18.9 ± 0.1        | 18.6 ± 0.4            | 18.6 ± 0.4        |
|               |     |     |   | GrLinks        | 9.27%     | 14     | 14.0 ± 0.0        | 13.9 ± 0.1            | 13.9 ± 0.1        |
|               |     |     |   | SP             | 9.23%     | 14     | 13.9 ± 0.1        | 13.8 ± 0.2            | 13.8 ± 0.2        |
|               |     |     |   | PvdA           | 5.79%     | 9      | 8.6 ± 0.4         | 8.7 ± 0.3             | 8.6 ± 0.4         |
|               |     |     |   | CU             | 3.44%     | 5      | 4.9 ± 0.1         | 5.2 ± 0.2             | 5.1 ± 0.1         |
|               |     |     |   | PvdD           | 3.24%     | 5      | 4.6 ± 0.4         | 4.9 ± 0.1             | 4.8 ± 0.2         |

\(^{13}\) Without regional parties.

\(^{14}\) Regional parties are in shaded rows.
It is immediately apparent that in all countries bar the Netherlands the proposed formula produces a much better approximation of the final result than the “naïve proportionality” approach (in the Netherlands both formulae produce very small errors). Indeed, only in 10 cases out of 66 did our formula’s margin of error exceed 1% of the national seat total. This is despite quite significant deviations from assumptions A2 and A3 (see Appendix A). In most cases (46 out of 66) it also provides a better approximation than the modified cube law, and in a further 6 cases both approximations are equally good (within a rounding error).

Eight elections and sixty-six parties is still a rather small sample to establish a claim of empirical validity. For this reason, we have repeated the test for all available elections from our eight test countries. To keep the length of this article within the bounds of reasonableness, we do not present the full results for each party. Instead, for each country we have computed two measures of error: the Loosemore-Hanby index (Loosemore & Hanby, 1971):

\[
LH := \frac{1}{2} \sum_{i=1}^{n} |s_i^{act} - s_i|;
\]

the Bhattacharyya angle between the actual and estimated seat vectors (Bhattacharyya, 1943):

\[
BA := \arccos \left( \sum_{i=1}^{n} \left( \frac{s_i^{act}}{s} \cdot \frac{s_i}{s} \right)^{1/2} \right),
\]

where \(s_i^{act}\) is the number of seats awarded to the \(i\)-th party under the actual allocation. Table 3 sets forth the values of the error measure for successive elections.
Table 3. Aggregate errors for post-1945 parliamentary elections in eight European countries.

| country          | year | s   | c   | n   | aggregate error | mod. cube law error
|------------------|------|-----|-----|-----|-----------------|---------------------|
|                  |      |     |     |     | BA  | LH  | LH/s | LH  | LH/s |
| Croatia          | 2000 | 146 | 11  | 4   | 0.025 | 1.55 | 1.1%  | 2.62 | 1.8%  |
| Croatia          | 2003 | 144 | 11  | 7   | 0.034 | 2.32 | 1.6%  | 10.81 | 7.5%  |
| Croatia          | 2007 | 145 | 11  | 6   | 0.052 | 2.42 | 1.7%  | 12.30 | 8.5%  |
| Croatia          | 2011 | 143 | 11  | 6   | 0.174 | 4.39 | 3.1%  | 7.50  | 5.2%  |
| Croatia          | 2015 | 143 | 11  | 5   | 0.106 | 2.60 | 1.8%  | 8.02  | 5.6%  |
| Croatia          | 2016 | 143 | 11  | 5   | 0.094 | 3.62 | 2.5%  | 5.56  | 3.9%  |
| Czech Republic   | 2002 | 200 | 14  | 4   | 0.011 | 1.99 | 1.0%  | 1.53  | 0.8%  |
| Czech Republic   | 2006 | 200 | 14  | 5   | 0.041 | 3.14 | 1.6%  | 7.80  | 3.9%  |
| Czech Republic   | 2010 | 200 | 14  | 5   | 0.010 | 1.81 | 0.9%  | 2.95  | 1.5%  |
| Czech Republic   | 2013 | 200 | 14  | 7   | 0.030 | 4.92 | 2.5%  | 4.97  | 2.5%  |
| Czech Republic   | 2017 | 200 | 14  | 9   | 0.032 | 4.58 | 2.3%  | 11.87 | 5.9%  |
| Finland          | 2003 | 200 | 15  | 6   | 0.102 | 3.61 | 1.8%  | 7.73  | 3.9%  |
| Finland          | 2007 | 200 | 15  | 7   | 0.080 | 5.70 | 2.9%  | 7.84  | 3.9%  |
| Finland          | 2011 | 200 | 15  | 7   | 0.086 | 7.04 | 3.5%  | 8.21  | 4.1%  |
| Finland          | 2015 | 200 | 13  | 7   | 0.087 | 5.54 | 2.8%  | 6.35  | 3.2%  |
| Luxembourg       | 1945 | 51  | 4   | 4   | 0.041 | 1.06 | 2.1%  | 3.84  | 7.5%  |
| Luxembourg       | 1948 | 26  | 2   | 4   | 0.056 | 1.23 | 4.7%  | 1.30  | 5.0%  |
| Luxembourg       | 1951 | 26  | 2   | 3   | 0.016 | 0.38 | 1.5%  | 1.09  | 4.2%  |
| Luxembourg       | 1954 | 52  | 4   | 4   | 0.032 | 1.18 | 2.3%  | 3.36  | 6.5%  |
| Luxembourg       | 1959 | 52  | 4   | 4   | 0.036 | 1.39 | 2.7%  | 2.91  | 5.6%  |
| Luxembourg       | 1964 | 56  | 4   | 5   | 0.016 | 0.77 | 1.4%  | 3.32  | 5.9%  |
| Luxembourg       | 1968 | 56  | 4   | 4   | 0.035 | 1.50 | 2.7%  | 2.74  | 4.9%  |
| Luxembourg       | 1974 | 59  | 4   | 5   | 0.031 | 1.10 | 1.9%  | 1.71  | 2.9%  |
| Luxembourg       | 1979 | 59  | 4   | 6   | 0.043 | 1.67 | 2.8%  | 4.02  | 6.8%  |
| Luxembourg       | 1984 | 64  | 4   | 5   | 0.055 | 1.99 | 3.1%  | 3.47  | 5.4%  |
| Luxembourg       | 1989 | 60  | 4   | 7   | 0.092 | 2.25 | 3.7%  | 3.99  | 6.7%  |
| Luxembourg       | 1994 | 60  | 4   | 5   | 0.016 | 0.76 | 1.3%  | 1.78  | 3.0%  |
| Luxembourg       | 1999 | 60  | 4   | 6   | 0.059 | 2.32 | 3.9%  | 2.97  | 5.0%  |
| Luxembourg       | 2004 | 60  | 4   | 5   | 0.024 | 1.07 | 1.8%  | 2.26  | 3.8%  |
| Luxembourg       | 2009 | 60  | 4   | 6   | 0.049 | 0.96 | 1.6%  | 2.70  | 4.5%  |
| Luxembourg       | 2013 | 60  | 4   | 7   | 0.074 | 1.46 | 2.4%  | 3.34  | 5.6%  |
| Netherlands      | 1948 | 100 | 1   | 8   | 0.014 | 1.00 | 1.0%  | 1.31  | 1.3%  |
| Netherlands      | 1952 | 100 | 1   | 8   | 0.015 | 0.73 | 0.7%  | 1.44  | 1.4%  |
| Netherlands      | 1956 | 150 | 1   | 7   | 0.006 | 0.63 | 0.4%  | 0.90  | 0.6%  |
| Netherlands      | 1959 | 150 | 1   | 8   | 0.015 | 1.26 | 0.8%  | 2.31  | 1.5%  |
| Netherlands      | 1963 | 150 | 1   | 10  | 0.018 | 1.22 | 0.8%  | 1.78  | 1.2%  |
| Netherlands      | 1967 | 150 | 1   | 11  | 0.014 | 1.35 | 0.9%  | 2.25  | 1.5%  |
| Netherlands      | 1971 | 150 | 1   | 14  | 0.021 | 1.75 | 1.2%  | 3.35  | 2.2%  |
| Netherlands      | 1972 | 150 | 1   | 14  | 0.018 | 1.60 | 1.1%  | 2.55  | 1.7%  |
| Netherlands      | 1977 | 150 | 1   | 11  | 0.020 | 0.73 | 0.5%  | 2.80  | 1.9%  |
| Netherlands      | 1981 | 150 | 1   | 10  | 0.025 | 1.33 | 0.9%  | 1.52  | 1.0%  |

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15 Without regional parties.
16 The Bhattacharyya angle is not applicable to the modified cube law results, as the modified cube law does not guarantee that seat estimates for all parties will sum up to national seat totals, which means that seat shares can lie outside the probability simplex.
This much larger sample of elections demonstrates that the seat allocation formula does indeed work as expected and is robust against violations of its assumptions. Only in 4 out of...
84 elections have more than 4% seats been misallocated. The robustness of the formula undoubtedly requires further investigation, which is, however, beyond the scope of this article. Nevertheless, our preliminary study of the subject suggests that under typical conditions encountered in real-life elections (unless the system is gerrymandered or otherwise deliberately skewed in favor of some type of parties), the errors introduced at different stages of approximation tend to largely cancel each other out, thereby making the overall error much smaller than initially expected. See Appendix A for details.

Finally, we have created two density plots illustrating the distribution of absolute party errors (absolute differences between actual and predicted seat allocations) and relative party errors (absolute party errors divided by national seat totals), see Fig. 1 and 2. The plots illustrate that for more than 93.3% of the cases (each party’s electoral run being a separate case) the absolute error is within the $[-3, 3]$ interval, and for more 94.0% of the cases the relative error is within the $[-1.5\%, 1.5\%]$ interval.

3. Mathematical underpinnings

3.1. The Jefferson-D’Hondt method

There are in common use several formulations of the Jefferson-D’Hondt seat allocation method. The original method proposed by Victor D’Hondt (1882) closely tracked a 1792 proposal by Thomas Jefferson for apportioning seats among the states in the U.S. House of

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17 Three of those cases are the Spanish elections of 1977, 1979, and 2015, where there were significant correlations between the vote shares of the left-wing parties (PSOE and PCE in 1977 and 1979, Podemos in 2015) and district size and magnitude, and the last one is the Luxembourg election of 1948, which was held in only two districts (so all parties exhibited a perfect positive or negative correlation between vote share and district size and magnitude).

18 At least part of that overall error is systematic in nature and correlated with the number of parties. It can be shown that as the number of dimensions of a simplex $n$ increases to infinity, the expected taxicab distance between two random points drawn from that simplex with a distribution supported evenly on the whole simplex (which is equivalent up to a constant to the expected value of the normalized Loosemore-Hanby index) increases asymptotically to ca. 1.
Representatives (Jefferson, 1792) (though it is unclear whether D’Hondt knew of Jefferson’s work on the subject\textsuperscript{19}). It called for finding such a divisor $D$ that if each party (or state) were to be allocated as many seats $s_i$ as its number of votes (or population) $v_i$ divided by $D$, rounding down to the nearest integer, i.e., if $s_i = \lfloor v_i / D \rfloor$, no seats would remain unallocated, i.e., $\sum_{i=1}^{n} s_i = s$\textsuperscript{20}. It is easy to demonstrate that – unless an electoral tie occurs – such divisors always exist\textsuperscript{21} and they will always yield the same distribution of seats.

There remains the question of finding an appropriate divisor. As noted by Pukelsheim (2014, 2017), all solutions in common use are similar in form. First, we pick a starting value $D \in (0, \infty)$ (a divisor initialization) and let the seat vector $s' := (\lfloor v_1 / D \rfloor, ..., \lfloor v_n / D \rfloor)$. Then, as long as the allocated seats do not sum up to $s$, we adjust the divisor using the jump-and-step algorithm. If $\sum_{i=1}^{n} s'_i < s$, we choose any $D \in \left( Q_{(n-1)}, Q_{(n)} \right]$, and if $\sum_{i=1}^{n} s'_i > s$, we choose any $D \in \left( q_{(1)}, q_{(2)} \right]$, where $q_i := v_i / s'_i$ is the votes-to-seats ratio for the $i$-th party, $Q_i := v_i / (s'_i + 1)$ is the votes-to-seats-plus-one ratio for the $i$-th party, and $X_{(k)}$ denotes the $k$-th smallest element of the set $\{ X_1, ..., X_n \}$. Among the possible initializations one can consider the simple quota, $v/s$ (D’Hondt, 1882; cf. Hare, 1859), the Hagenbach-Bischoff quota $[v/(s + 1)] + 1$ (Hagenbach-Bischoff, 1888, 1905; cf. Droop, 1881), and the Gfeller-Joachim-Pukelsheim quota, $v/(s + n/2)$ (Gfeller, 1890; Joachim, 1917).\textsuperscript{22} The latter quota has been established by Happacher & Pukelsheim (1996, 2000) to have the unique property of being asymptotically unbiased as the number of seats approaches infinity.\textsuperscript{23}

An alternative formulation of the D’Hondt method, first introduced by D’Hondt himself in 1885, and by far the most popular among legislators and political scientists,\textsuperscript{24} is an

\textsuperscript{19} James (1897: 36) has probably been the first to notice that the Jefferson method is equivalent to the D’Hondt method, but it appears that this finding has escaped the attention of the subsequent generations of scholars. As far as we are aware, Balinski & Young (1975: 703) have been the first modern authors to credit Jefferson with the original authorship of the method. It should be noted that the Jefferson method was enacted into law (Act of Apr. 14, 1792, c. 23, 1 Stat. 253) and has remained in use for apportioning representatives among the states until 1842. For an extensive discussion of the history of congressional apportionment methods, see Biles (2017).

\textsuperscript{20} Under the original Jefferson plan, the divisor has been fixed, while the number of seats has been allowed to vary (Balinski & Young, 1978). In such a case, of course, the jump-and-step algorithm is omitted, but all the subsequent steps, the final results, and the mathematical properties of the method are otherwise identical.

\textsuperscript{21} Electoral ties are very rare empirically (no such tie occurred in any of the elections analyzed in Sec. 2; for a more general discussion of the frequency of ties, see Mulligan & Hunter, 2001) and negligible theoretically (as the set of points on the probability simplex of vote shares for which such ties occur has a Lebesgue measure zero). In addition, a tie-resolution rule will always be arbitrary from the standpoint of the Jefferson-D’Hondt method, and an analysis of such a rule’s application will therefore provide no useful insight into the subject under discussion. For those reasons, we do not concern ourselves with ties in the following analysis.

\textsuperscript{22} For an in-depth discussion of the origin and attribution of the most popular electoral quotas, see Dančišin (2013a).

\textsuperscript{23} A number of scholars have analyzed the distribution of discrepancies arising under the proposed quotas. Happacher (2001) has provided an analytic formula for a probability mass function of the discrepancy under the assumption that vote shares are drawn from a uniform distribution on the unit simplex. Janson (2013: Theorem 7.5) has established that as the number of seats approaches infinity, the discrepancy distribution approaches the Euler-Frobenius distribution. Finally, Heinrich, Pukelsheim & Wachtel (2017) find that the discrepancy distribution can be approximated by the distribution obtained by applying standard rounding to a sum of uniformly distributed random variables.

\textsuperscript{24} For instance, all EU countries employing the D’Hondt method for legislative elections – except Luxembourg – include this algorithmic formulation in their electoral legislation. It can be noted that this formulation of the d’Hondt method closely resembles an earlier proposal by Burnitz & Varrentrapp (1863), who called for a modified version of the Borda count, with each elector ranking no more than $s$ candidates, ranks being
algorithmic one. Let \( s \) be the number of seats to be allocated within a given district and \( v_i \) be the number of votes cast for the \( i \)-th party (\( i = 1, \ldots, n \)) in that district. We define a \( k \)-th quotient for the \( i \)-th party as \( v_i / k \) for \( k = 1, 2, \ldots \). Let \( q_s \) be the \( s \)-th highest quotient overall, i.e., across all parties. The number of seats \( s_i \) allocated to the \( i \)-th party is then defined as the number of the smallest quotient for the \( i \)-th party larger than or equal to \( q_s \).

It is well known that the two methods are equivalent, i.e., they always generate an identical allocation of seats. For an early proof, see Equer (1911). To check it quickly, assume \( s_i \) (\( i = 1, \ldots, n \)) is the number of seats awarded to the \( i \)-th party under the “algorithmic” method. Clearly, \( s_i \) be the number of votes cast for the \( i \)-th party (\( i = 1, \ldots, n \)). Let \( D \in (q_{s+1}, q_s] \). Observe that we have then \( v_i / s_i \geq q_s \geq D > q_{s+1} \geq v_i / (s_i + 1) \) and, in consequence, \( s_i = [v_i / D] \) for every \( i = 1, \ldots, n \), as desired.

Unfortunately, the divisor formula still does not permit a party’s seat allocation to be estimated without full knowledge of vote shares of the other parties. Put \( M := v / D \), where \( D \) is any divisor in the D’Hondt method. Then \( s \leq M < s + n \). It should be noted that \( M \) need not be an integer. Let \( i = 1, \ldots, n \). We have \( s_i = [v_i / D] = [v_i M / v] = [p_i M] \), where \( p_i := v_i / v \) is the fraction of total votes (the “vote share”) cast for the \( i \)-th party. Let us denote the rounding residual of \( p_i M \) as

\[
   r_i := p_i M - [p_i M] = p_i M - s_i. \tag{3.1}
\]

Summing this over \( i = 1, \ldots, n \) we get \( \sum_{i=1}^n r_i = M - s \). Now we use the assumption \( A4 \), which states that \( \sum_{i=1}^n r_i = n/2 \) or, equivalently, \( M = n/2 + s \). Substituting this equality in (3.1) we obtain

\[
   s_i = p_i(s + n/2) - r_i. \tag{3.2}
\]

Hence under \( A4 \) the number of seats for the \( i \)-th party in a single district can be expressed as above.

### 3.2. Aggregation over multiple districts

Let \( v_i^k \), \( s_i^k \), and \( r_i^k \) be, respectively, the number of votes received, the number of seats awarded, and the fractional expression defined by (3.1) for the \( i \)-th party (\( i = 1, \ldots, n \)) in the \( k \)-th electoral district (\( k = 1, \ldots, c \)).\(^{27}\) Let \( v_i := \sum_{k=1}^c v_i^k \) and \( s_i := \sum_{k=1}^c s_i^k \) be, respectively, the nationwide vote and seat numbers for the \( i \)-th party, and let \( v^k := \sum_{i=1}^n v_i^k \) and \( s^k := \sum_{i=1}^n s_i^k \) be, respectively, the district vote and seat totals for the \( k \)-th district. Finally, let \( v := \sum_{i=1}^n v_i = \sum_{k=1}^c v^k \) and \( s := \sum_{i=1}^n s_i = \sum_{k=1}^c s^k \) be, respectively, nationwide vote and

---

\(^{25}\) If \( q_{s+1} = q_s \), the procedure described herein will allocate more than \( s \) seats. This situation is equivalent to an electoral tie.

\(^{26}\) In fact, the algorithmic formulation of the D’Hondt method is equivalent to the jump-and-step algorithm with initialization \( D = 0 \) (which is always the least optimal choice of initialization, as for each possible vote share vector it requires exactly \( s \) iterations to arrive at a correct divisor).

\(^{27}\) For districts uncontested by the \( i \)-th party we assume that \( v_i^k = 0 \).
seat totals. In this case, using assumption A5, which states that $\sum_{k=1}^{c} r_i^k = c/2$, we get by aggregating single-district formulae (3.2) applied to all $c$ districts

$$s_i = \sum_{k=1}^{c} \left( \frac{v_i^k}{v^k} \left( s^k + \frac{n}{2} \right) - r_i^k \right) =$$

$$\sum_{k=1}^{c} \frac{v_i^k}{v^k} s^k + \sum_{k=1}^{c} \frac{v_i^k n}{v^k} - \frac{1}{c} \sum_{k=1}^{c} r_i^k =$$

$$\sum_{k=1}^{c} \frac{v_i^k}{v^k} s^k + \sum_{k=1}^{c} \frac{v_i^k n}{v^k} - \frac{c}{2}.$$  

(3.3)

Let $i = 1, \ldots, n$. Note that

$$p_i s + p_i \frac{c n}{2} - \frac{c}{2} =$$

$$\sum_{k=1}^{c} \frac{v_i^k}{v^k} \sum_{k=1}^{c} v_i^k s^k + \sum_{k=1}^{c} \frac{v_i^k n}{v^k} + \sum_{k=1}^{c} \frac{v_i^k \cdot n}{2} - \frac{c}{2}.$$  

(3.4)

Comparing (3.3) and (3.4) we deduce that (0.1) holds if and only if

$$\sum_{k=1}^{c} \frac{v_i^k}{v^k} s^k + \sum_{k=1}^{c} \frac{v_i^k n}{v^k} = \frac{1}{c} \sum_{k=1}^{c} v_i^k \cdot n$$

$$\sum_{k=1}^{c} \frac{v_i^k}{v^k} s^k + \sum_{k=1}^{c} \frac{v_i^k \cdot n}{v^k} = \frac{c}{2}.$$  

(3.5)

This equality is satisfied if

$$\sum_{k=1}^{c} \frac{v_i^k}{v^k} s^k = \frac{1}{c} \sum_{k=1}^{c} v_i^k$$  

(3.6)

and

$$\sum_{k=1}^{c} \frac{v_i^k \cdot n}{v^k} = \frac{1}{c} \sum_{k=1}^{c} v_i^k \cdot n$$  

(3.7)

(Or if neither (3.6) nor (3.7) holds, but the differences cancel each other out.)

Now, note first that (3.7) is equivalent to

$$\frac{1}{c} \sum_{k=1}^{c} p_i^k = \frac{1}{c} \sum_{k=1}^{c} v_i^k \cdot p_i^k$$

(3.8)

— which is true if the vote shares of the $i$-th party ($p_i^k$) are not correlated with the district size ($v^k$) (assumption A2).
The left-hand side of (3.6) is
\[ \sum_{k=1}^{c} \frac{v_i^k}{v^k} s^k = \sum_{k=1}^{c} p_i^k s^k. \] (3.9)

On the other hand, applying (3.8) we can transform the right-hand side of (3.6) in the following way:
\[ \frac{\sum_{k=1}^{c} v_i^k}{\sum_{k=1}^{c} v^k} \sum_{k=1}^{c} s^k = \frac{\sum_{k=1}^{c} v_i^k}{\sum_{k=1}^{c} p_i^k v^k} \sum_{k=1}^{c} p_i^k \cdot \left( \sum_{k=1}^{c} s^k \right) = \frac{1}{c} \sum_{k=1}^{c} p_i^k \sum_{k=1}^{c} s^k. \] (3.10)

From (3.9) and (3.10) we deduce that (3.6) is equivalent to
\[ \frac{1}{c} \sum_{k=1}^{c} p_i^k s^k = \frac{1}{c} \sum_{k=1}^{c} p_i^k \cdot \frac{1}{c} \sum_{k=1}^{c} s^k \] (3.11)
— which is true if the vote shares of the \( i \)-th party \( (p_i^k) \) are not correlated with the district magnitude \( (s^k) \) (assumption A3).

Summing up, (0.1) follows from the assumptions, and so we arrive at our main result:

**Proposition.** If assumptions A1-A5 hold, then for every \( i = 1, \ldots, n \) we have
\[ s_i = p_i \cdot s + p_i \cdot \frac{cn}{2} - \frac{c}{2}, \]
or equivalently
\[ q_i = p_i + \frac{p_i - 1/n}{2m/n} = \left( 1 + \frac{n}{2m} \right) p_i - \frac{1}{2m}, \]
where \( q_i \) is the seats share for \( i \)-th party and \( m \) is the mean number of seats per district.

### 3.3. Natural threshold

As noted in Sec. 1.2 above, the formula can only be applied to relevant parties, i.e., such parties that exceed both the applicable statutory threshold, if any, and the mean natural threshold. The former are easy to account for: since parties that fall below them are ignored in the seat allocation process, we simply eliminate them from the electoral results data set before starting any processing. Natural thresholds call for a more sophisticated treatment, since they depend in each case on the actual distribution of party vote shares. Yet we cannot ignore them, since not only applying formula (0.1) for the sub-threshold parties can yield negative seat numbers (which are obviously incorrect), but including them in the number of relevant parties would inflate \( n \) and thereby distort results for the supra-threshold parties as well.

To avoid this kind of error, we use an iterative algorithm for determining the number of relevant parties and for identifying the supra-threshold parties. The basic strategy is as follows: first, we sort \( N \) parties degressively according to their original (non-renormalized) vote share \((P_1 > \cdots > P_N)\). Then we start with only one party, the largest one, in the model, and continue to add parties, according to the sort order, until we encounter the first party with a negative seat number. At that point, we eliminate such a party and end the algorithm.
From (0.1) we deduce that \( n \) has to fulfill condition
\[
\frac{p_n}{2m + n} > \frac{1}{2m + n}, \tag{3.12}
\]
where \( m := s/c \) is the mean district magnitude and \( p_n \) is the renormalized support for the \( n \)-th party, with only the first \( n \) parties taken into account. This is equivalent to
\[
n > \frac{\sum_{i=1}^{n} p_i}{p_n} - 2m \tag{3.13}
\]
and so to
\[
2m > \sum_{i=1}^{n} \frac{p_i - p_n}{p_n} \tag{3.14}
\]
For our algorithm to be correct, the above condition should hold for a given \( n \), if it holds for \( n - 1 \). To show that this is indeed the case, it is enough to observe that the right-hand side of (3.14) increases in \( n \), since
\[
\sum_{i=1}^{n+1} \frac{p_i - p_{n+1}}{p_{n+1}} - \sum_{i=1}^{n} \frac{p_i - p_n}{p_n} = \sum_{i=1}^{n} \frac{p_i(p_n - p_{n+1})}{p_{n+1}p_n} \geq 0, \tag{3.15}
\]
which is true, as parties are sorted degressively by the number of votes. In consequence, \( n \) is the largest natural number such that
\[
\frac{1}{2m + n} < \frac{p_n}{\sum_{i=1}^{n} p_i} \tag{3.16}
\]
holds, and so
\[
t := \frac{1}{2m + n} \tag{3.17}
\]
is our estimate of the natural threshold. It can be used to express (0.2) in yet another form,
\[
q_i = (1 + \frac{n}{2m})(p_i - t), \tag{3.18}
\]
which demonstrates that the seat shares depend linearly on the over-threshold vote shares.

We can further note that if
\[
\frac{p_n}{\sum_{i=1}^{n} p_i} > \frac{1}{2m}, \tag{3.19}
\]
then the inequality (3.12) will always be satisfied, meaning that we can easily start the algorithm with the first party that does not satisfy equation (3.19).

Note that for the single-district case \( (m = s) \) this result is in accord with the earlier works on the subject by D’Hondt (1883), Rokkan (1968), Rae, Hanby & Loosemore (1971), Lijphart & Gibberd (1977), and Palomares & Ramírez (2003), who estimated the threshold of exclusion to be \( 1/(s + 1) \) and the threshold of inclusion to be \( 1/(N + s - 1) \), where \( N \) is the overall number of parties. Namely, it is easy to show that our formula gives at least 1/2 seat for a party fulfilling \( P > 1/(s + 1) \geq 1/(2s + n) \), and at most \( (s + 1)/(s + N - 1)/2 \leq 1/2 \) for a party satisfying \( P < 1/(N + s - 1) \), though the latter is not necessarily excluded from our algorithm as it cannot be ruled out that \( t < 1/(N + s - 1) \).
Given discreteness of the number of parties \( n \) appearing in formula (0.1), one could \textit{prima facie} expect that as a new party crosses the natural threshold or as one of the existing parties falls below the threshold, the number of seats for every other party would change discontinuously. This would obviously constitute a significant obstacle to applying formula (0.1) to estimate seat allocations under circumstances where \( P_1, \ldots, P_n \) are known only approximately (for instance, obtained from opinion polls) and some parties are in the vicinity of the natural threshold. Fortunately, as the next paragraph demonstrates, this is not the case.

Imagine that we know that \( n \) largest parties are relevant in the sense of the algorithm described above and that we have one more (smaller) party to allocate some seat \( s \). Let parties numbered from 1 to \( n + 1 \) be sorted degressively, i.e., \( P_1 \geq \cdots \geq P_n \geq P_{n+1} \). Put \( P := P_{n+1} \).

We would like to show that \( s_1, \ldots, s_n, s_{n+1} \) depend continuously on \( P \in [0, P_n] \). In fact, the only doubtful case is \( P = P' \), where \( P' \) fulfills

\[
P' = \frac{t'}{1-t'} \sum_{j=1}^{n} P_j,
\]

where

\[
t' = \frac{1}{2m} \frac{1}{n+1}.
\]

For \( P < P' \) we have

\[
s_i = s_i(P) = \frac{1}{2m} \frac{P_i}{P + \sum_{j=1}^{n} P_j} - \frac{1}{2m}, \quad i = 1, \ldots, n.
\]

On the other hand, for \( P > P' \) and \( i = 1, \ldots, n, n + 1 \) we obtain

\[
s_i = s_i(P) = \frac{1}{2m} \frac{P_i}{P + \sum_{j=1}^{n} P_j} - \frac{1}{2m} - \frac{1}{2m} \sum_{j=1}^{n} P_j - \frac{1}{2m}.
\]

Now, for \( P \to P', P > P' \) we have for \( i = 1, \ldots, n \)

\[
s_i(P) = \frac{1}{2m} \frac{P_i}{P + \sum_{j=1}^{n} P_j} - \frac{1}{2m} \to \frac{1}{2m} \frac{P_i}{P' + \sum_{j=1}^{n} P_j} - \frac{1}{2m} = \frac{1-t'}{2m} \frac{P_i}{\sum_{j=1}^{n} P_j} - \frac{1}{2m} = \frac{1}{2m} \frac{P_i}{\sum_{j=1}^{n} P_j} - \frac{1}{2m},
\]

and for \( i = n + 1 \)

\[
s_n(P) = \frac{1}{2m} \frac{P_n}{P + \sum_{j=1}^{n} P_j} - \frac{1}{2m} \to \frac{1}{2m} \frac{P_n}{P' + \sum_{j=1}^{n} P_j} - \frac{1}{2m} = 0,
\]

as desired.

3.4. \textit{Artificial threshold}

Unlike the natural thresholds, the artificial thresholds (if greater than the former) give rise to discontinuities in the seats-votes curves of all parties (not just those that are immediately affected by it)\textsuperscript{28}. Accordingly, if the electoral results are not known exactly, but only predicted

\textsuperscript{28} Among the seven countries for which we test our formula in Sec. 2, four – Croatia, the Czech Republic, the Netherlands and Poland – employ some form of a national artificial threshold. In the Netherlands it equals the
on the basis of polls or other models, and confidence intervals for the results overlap the artificial threshold, an additional source of potential error is introduced. The discontinuities in question, however, can be easily quantified. Let us initially consider a simple case where \( \tau > t \) is the statutory threshold, \( n - 1 \) parties are certain to cross it and only the \( n \)-th party is uncertain. Note that it is not necessary for the \( n \)-th party to be the smallest one – due to different classes of thresholds for different types of electoral actors, it is possible for a party to fail to cross its threshold, while some other smaller parties are certain to cross theirs. Then the \( i \)-th party (with \( p_i > t, i = 1, \ldots, n - 1 \)) increase of seat share arising from the \( n \)-th party failure to cross the threshold (and conversely – its decrease of seat share arising from the \( n \)-th party success in doing so) equals:

\[
T_i := \lim_{p_n \to \tau^{-}} s_i - \lim_{p_n \to \tau^{+}} s_i = \frac{p_i}{1 - \tau} \left(1 + \frac{n - 1}{2m}\right) - \frac{1}{2m} - p_i \left(1 + \frac{n}{2m}\right) + \frac{1}{2m} = \frac{p_i}{2m} \left(\frac{2m + n - 1}{1 - \tau} - 2m - n\right) = \frac{p_i}{2m} \left(\tau(2m + n) - 1\right) = \frac{p_i}{2m} \left(\tau/t - 1\right),
\]

where \( t \) is given by (3.17). Note that for \( \tau = t \) (i.e., the artificial threshold equaling the natural threshold), \( T_i = 0 \), and the discontinuity disappears. Formula (3.24) can be easily extended to the case of \( k \) out of \( n \) parties being uncertain to cross the threshold:

\[
T_i(k) = \frac{p_i}{2m} \left(\tau/t - 1\right) \quad (1/k - 1/k - \tau).
\]

A party’s relative seat gain from the others’ failure to cross the threshold can be expressed as:

\[
T_i(k) = \frac{s_i}{\tau - t} \approx \frac{\tau - t}{(1/k - \tau)(1 - t/p_i)} \geq k \frac{\tau - t}{1 - t/p_i} \geq k(\tau - t),
\]

Note that

\[29\] For instance, in Poland in 2015 the United Left coalition failed to cross the coalition threshold of 8% with the vote share of 7.55%, while the Polish People’s Party qualified for seats with a smaller vote share of 5.13%, as the threshold for individual parties was only 5%.

Hare quota (1/150), while in Croatia, the Czech Republic, and Poland – 5%. However, in the latter two countries there are also higher thresholds for coalitions of parties (8% in Poland, and 10%, 15% or even 20% – depending on the number of the coalition partners – in the Czech Republic). The Turkish threshold of 10% seems to provide currently the upper bound for the artificial thresholds applicable to individual parties.
\[
\frac{T_i(k)}{s_i} = \frac{\tau - t}{(1/k - \tau)(1 - t/p_i)} \leq \frac{\tau}{1/k - \tau} \tag{3.28}
\]

Accordingly, for \( k = 1 \) a party’s relative seat gain from a single competitor’s failure to cross the threshold satisfies \( \tau - t \leq T_i/s_i \leq \tau/(1 - \tau) \).
4. Political consequences

4.1. Political determinants of the seat bias

The political implications of the seat bias formula become immediately clear when the latter is stated in a form (0.3) which expresses the seat bias of the $i$-th party as the product of three parameters: the inverse of the mean district magnitude, the number of relevant parties, and the deviation of the $i$-th party vote share from the mean vote share. Of those, the mean district magnitude is the most stable one, as it is not only completely exogenous to the election results, but is virtually always fixed by statute. In most of the eight countries discussed in Sec. 2, the mean district magnitude varies between 11 and 16. The exceptions are Spain (with a mean district magnitude of 6.73) and the Netherlands (with a single 150-seat district).

The number of parties is a more variable parameter, though in institutionalized party systems it rarely varies drastically and can be predicted with reasonable accuracy from the exogenous parameters, such as the size of the legislature (see Shugart & Taagepera, 2017), permitting it to be still considered “systemic” in nature. As expected, it correlates with the mean district magnitude, with Spain (three nationally relevant parties in 2000 and 2004) and the Netherlands (fourteen relevant parties in 1971 and 1972) again being the extreme cases. Most countries under consideration tend to average at five-six relevant parties.

Finally, the last parameter influencing the seat bias is party size (normalized vote share). While variability of this parameter is related to the number of parties (Taagepera, 2007), the relationship is nontrivial. Similarly sized parties occur in party systems with very different numbers of relevant parties. Yet the party size is ultimately the key determinant of the size of the seat bonus, distinguishing the winners and losers of the Jefferson-D’Hondt method.

Some of the relationships revealed by the seat bias formula are well known to students of electoral systems. For instance, there is nothing new about a finding of negative effects of small districts on small parties, as the same follows from the celebrated micromega rule (“the large prefer the small and the small prefer the large,” see Colomer, 2004) and has already been well established by Rae (1967), Taagepera & Laakso (1980), Taagepera (1986), and Taagepera & Shugart (1989). Nevertheless, note that under the Jefferson-D’Hondt method this effect is magnified, as the negative seat bias arises entirely apart from the exclusionary effect of the small district magnitude documented by Lijphart & Gibberd (1977). Also observe that where very large districts are involved – as in the Netherlands – the bias becomes negligible.

The effect of the second parameter – the number of parties – on the magnitude of the seat bias appears to have escaped the attention of many electoral scholars. It demonstrates, however, an

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30 In three out of the eight countries for which we tested the formula (the Czech Republic, Poland, and Spain) the mean district magnitude remained constant since the introduction of the D’Hondt formula. In four of the others its standard deviation of the mean district magnitude across all elections did not exceed 1. Only in the Netherlands a major change in district magnitude occurred due to a 50% house size increase in 1956.

31 Even in those countries where apportionment of representatives among districts is made without legislative involvement, the legislature – or the national constitution – fixes the number of seats to be apportioned and the number of districts, which together determine mean district magnitude. Pilet et al. (2016) consider each change in the latter two parameters to be a major change in the electoral rules.

32 Baldini & Pappalardo (2009: 67-69) note that Spain has one of the lowest district magnitudes in Europe, which makes its electoral system one of the least proportional PR systems.
important self-correcting aspect of the electoral systems based on the Jefferson-D’Hondt method: as the number of parties increases, so does the bias of the largest party, at least partially alleviating difficulties in government formation caused by legislative fragmentation.

To illustrate the importance of the two systemic parameters, on Figs. 5 and 6 we plot the seat biases for two hypothetical parties with a normalized vote share of, respectively, 40% and 10% – as it varies depending on the number of parties (from 2 to 9) and the district magnitude.

![Fig. 5. Seat bias of a party with a normalized vote share of 40% as a function of mean district magnitude.](image)

![Fig. 6. Seat bias of a party with a normalized vote share of 10% as a function of district magnitude.](image)
There remains the relationship between the vote share and the seat bias. The first glance at the seat bias formula (0.3) reveals that it is affine and increasing. Its slope $n/(2m)$ and intercept $-1/(2m)$ depend on the two electoral parameters: the mean district magnitude ($m$) and the number of parties ($n$). To illustrate that, on Fig. 7 we plot the bias as a function of the vote share as the number of parties is either four or eight and as the mean district magnitude is either three (the smallest district magnitude for which speaking of proportional seat allocation makes sense) or sixteen (the upper boundary of the mean district magnitude for all multi-district countries analyzed in Sec. 2). The points representing the votes share $((m + 1)/(2m + n))$ and the seat bias $((n - 2)/(4m + 2n))$ required to secure a majority of seats are likewise indicated, see also Fig. 7.

![Fig. 7. Seat bias ($\Delta$) as a function of vote share ($p$) for specified electoral parameters. The dotted line $\Delta = 1/2 - p$ represents the size of the bias required to secure a majority of seats with the given vote share.](image)

4.2. Applications for the evaluation of political strategies

As already noted, the seat bias formula can be applied to evaluate the mathematical effects of a number of electoral strategies, such as party divisions and mergers or manipulation of the district magnitude. Let us consider a few effects of the seat bias formula in this area.

First. It is well known that as the D’Hondt system favors large parties, it provides an inducement for coalition formation, cf., Bochsler (2010). The seat bias formula (0.3) facilitates easy assessment of the theoretical integration bonus, i.e., the difference between the estimated bias $\Delta_{ij}$ of the coalition of parties $i$ and $j$ and the sum of the estimated biases of
the individual parties \((\Delta_i + \Delta_j)\) under the assumption that the vote shares of all other relevant parties remain constant:

\[
\Delta_{i,j} - (\Delta_i + \Delta_j) = \frac{1 - (p_i + p_j)}{2m},
\]

(4.1)

cf. Janson (2014: Theorem 8.1) for analogous asymptotic formula for \(c = 1\) and \(s \to \infty\). Note that the integration bonus does not depend on the number of third parties, but only on their total vote share, and that it is negatively related to the sum of the coalition parties’ vote shares.

Of course, it should be noted that the seat bias formula is by itself insufficient to forecast the exact effects of coalition formation, as it is necessary to remember that such strategies carry a risk of alienating each party’s fringe electorate, as well as some probability of attracting additional voters due to the bandwagon effect (Kaminski, 2001), but at least the formula provides an initial estimate of the integration bonus and its derivative with respect to vote share changes. Moreover, while the coalition, taken together, will nearly always benefit and never lose from the merger, it does not necessarily follow that each member thereof will have adequate incentive to join. That will depend on how the seats are allocated within the coalition, see Leutgäb & Pukehlseim (2009), Janson (2014), and Karpov (2015).

In addition, formula (4.1) can be easily transformed to model the reverse case: a breakup of a party or a coalition (for such applications, see Kaminski, 2018). Again, note that as long as all successor parties are above the natural and statutory thresholds, and the set of relevant third parties remains constant, the sum of the disintegrated coalition’s seat shares remains invariant with respect to the distribution of the successor parties’ vote shares.

Second. Political strategies can also include electoral engineering. In proportional systems, such engineering usually takes the form of changes to the seat allocation method or to the number of electoral districts (Kaminski, 2002). In the latter case, the seat bias formula can be applied to estimate the effects of the change, again, modulo secondary effects, such as induced coalition formation among the opposition or changes in the distribution of vote shares. In particular, it can be easily deduced from (0.1) that the number \(\delta_i^1\) of districts that need to be added for the \(i\)-th party \((i = 1, \ldots, n)\) to gain just one seat equals, depending on its vote share \(p_i\) and the number of relevant parties \(n\):

\[
\delta_i^1 = \left\lceil \frac{2}{p_i n - 1} \right\rceil.
\]

(4.2)
Fig. 8. The number of districts $\delta_i^1$ that need to be added (to some initial number of districts $c$) for the $i$-th party to gain a single seat, depending on its vote share $p_i$ and the number of relevant parties $n$.

Somewhat counterintuitively, $\delta_i^1$ does not depend on the initial number of districts $c$. Moreover, it follows from (4.2) that $\delta_i^1$ has singularity at $1/n$, which is not surprising, since there is no bias for mean-sized parties, so no matter how many districts are added, they will gain no seats as long as the number of relevant parties remains unchanged. Of course, in practice the interval where no change in seats is possible is wider, as the number of districts that can be added is bounded by the total number of seats.

We shall complete the discussion of the political consequences of the seat bias formula with one final issue. While all parties seek to maximize their vote shares, most elections – and especially those in parliamentary systems – are still primarily about winning the legislative majority. The seat share formula (0.2) can be transformed to yield, for any combination of parameters $m$ (the mean district magnitude) and $n$ (the number of relevant parties), the minimum vote share $p_{\text{Maj}}$ that translates to a half of the total number of seats:

$$p_{\text{Maj}} = \frac{1 + 1/m}{2 + n/m}.$$  

(4.3)
This is illustrated by a contour plot.

![Contour Plot](image)

**Fig. 9.** The minimum share of votes necessary for a party to obtain a half of the total number of seats as a function of the number of relevant parties and the mean district magnitude.

### 4.3. Efficiency gap and application of the formula to single-member systems

Let us consider for a while the simplest case of electoral system to which the seat allocation and seat bias formulas can be applied: a two-party system\(^3\) \((n = 2)\) with single-seat districts where seats are allocated by the first-past-the-post rule (which can be thought of as a limiting case of Jefferson-D’Hondt), i.e., with \(m = 1\), that is, \(c = s\). In this case, the seat allocation formula reduces to

\[
 s_t = s \left(2p_t - \frac{1}{2}\right),
\]

and the natural threshold becomes equal to \(1/(2m + n) = 1/4\). A number of assumptions can also be simplified: \(A2\) is equivalent to the requirement that the effective vote shares of both parties are independent of the effective district size (which is the number of votes cast for

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\(^3\) By a two-party system we mean a system in which there are exactly two relevant parties.
the two largest parties), and A3 becomes trivial, as district magnitude is constant. Moreover, one can easily demonstrate that A4 is always satisfied: let $p_1^k$ be the larger party’s vote share, and $p_2^k = 1 - p_1^k$ be the smaller party’s share in any district $k = 1, \ldots, c$. It follows that $p_1^k > 1/2$ and $p_2^k < 1/2$, so $D^k = v^k/2$ (which is equal to the Gfeller-Joachim-Pukelsheim quota, see Sec. 3.1) is a proper divisor. Then the rounding residuals (see Sec. 3.1) are

$$r_1^k = p_1^k \frac{v^k}{D^k} - \left[ p_1 \frac{v^k}{D^k} \right] = 2p_1^k - \left[ 2p_1^k \right] = 2 \left( p_1^k - \frac{1}{2} \right)$$

(4.5)

and

$$r_2^k = p_2^k \frac{v^k}{D^k} = 2p_2^k = 2(1 - p_1^k).$$

(4.6)

Hence, we get

$$r_1^k + r_2^k = 2p_1^k - 1 + 2 - 2p_2^k = 1 = n/2,$$

(4.7)

as desired. Finally, A5 can be stated in different terms. Let us express the sum of rounding residuals in terms of (4.5) and (4.6):

$$\sum_{k=1}^{c} r_i^k = 2 \sum_{k \in W_i} (p_i^k - \frac{1}{2}) + 2 \sum_{k \in L_i} p_i^k,$$

(4.8)

where $W_i$ is the set of districts where the $i$-th party ($i = 1, 2$) wins the election (i.e., $p_i^k > 1/2$), and $L_i$ is the set of districts where it loses one (i.e., $p_i^k < 1/2$). Note that (4.5) and (4.6) are equal up to a constant factor of 2 to the winning and losing party’s share of wasted votes, as defined by McGhee (2014), and accordingly the right-hand side of (4.8) is equal twice the $i$-th party’s share of wasted votes. In consequence, if our assumption A5 is satisfied, then

$$\sum_{k=1}^{c} r_i^k = \sum_{k=1}^{c} r_2^k,$$

and so both parties waste the same number of votes. It also follows that since by (4.7), the sum of the rounding residuals of all parties over all districts equals the number of districts $c$, the wasted vote shares can only be equal if for both parties, $i = 1, 2$, $\sum_{k=1}^{c} r_i^k = 1/2$. Accordingly, A5 is equivalent to the assumption underlying the McGhee–Stephanopoulos efficiency gap test (Stephanopoulos & McGhee, 2015) that has been proposed as a method for detecting partisan gerrymandering and obtained a great deal of recent attention in the field (see, e.g., Bernstein & Duchin, 2017; Chambers, Miller & Sobel, 2017; Plenner Cover, 2018; Tapp, 2018; Veomett, 2018). Moreover our formula applied to single-member districts is equivalent to their seats-votes formula. In fact, our assumptions constitute a generalization of the efficiency gap test, since instead of a quite restrictive assumption made by McGhee and Stephanopoulos that the effective district size is constant, we only assume that it is independent of party vote shares.

34 Note that the absence of malapportionment is neither necessary nor sufficient for A2 to be satisfied. Even if districts are perfectly equal in terms of the number of residents, but there are turnout differences or differences in the number of votes cast for third parties, a violation of assumption A2 can still occur. Cf. Brookes (1960).
There have been several attempts to extend the efficiency gap test to multi-party elections (McGhee, 2017; Stephanopoulos & McGhee, 2018; Tapp, 2018). It seems, however, that none of them have been deemed satisfactory. The main issue arises from the difficulties of providing a natural definition of wasted votes when multiple parties are competing. McGhee (2017, p. 429) suggested to retain the two-party definition, i.e., to put $v_i^k$ for the losing parties and $\max(v_i^k - 1/2, 0)$ for the winning parties, observing that no result below $1/2$ guarantees success, as the probability of winning depends on the distribution of the competitors’ votes. This definition is based on an implicit assumption that all such distributions, including those where the entire competition vote concentrates on a single candidate, are realistic. Especially for party-based elections this appears unlikely (unless some parties do not contest all seats).

We propose a natural extension of the definition of wasted votes for fully-contested (see below) elections, enabling our formula to be expressed in the language of equality of the number of wasted votes. First, we note that for such equality to be satisfied, each relevant party needs to obtain certain minimum share of votes. We call that minimum share the threshold of relevancy ($t$), where $t \in (0,1)$. We shall later show that this threshold must coincide with the natural threshold defined in Sec. 3.3. Second, we consider an election to be fully contested if each party $i = 1, \ldots, n$ obtains at least $tv^k$ votes in every district $k = 1, \ldots, c$, where $v^k$ is the total number of votes cast for relevant parties in the $k$-th district. Third, we follow McGhee’s intuition that the wasted votes should be defined for the winner as the surplus votes over the number of votes ensuring the victory and for the losers as all the votes casted. Let $W \in (0,1)$ be the threshold above which the victory is ensured. It is clear now, that it should be defined not as half of all the votes, but only as half of the $1 - nt$ redistributable votes plus $t$, i.e.,

$$W := (1 - nt)/2 + t.$$ (4.9)

Formally, we define the number of wasted votes for the $i$-th ($i = 1, \ldots, n$) party in the $k$-th ($k = 1, \ldots, c$) district as

$$w_i^k := \begin{cases} v_i^k - Wv^k, & \text{where } s_i^k = 1 \\ v_i^k, & \text{where } s_i^k = 0 \\ v_i^k - Wv^k s_i^k. & \end{cases}$$ (4.10)

To avoid the number of wasted votes being negative, we assume that the winning party’s vote share equals or exceeds $W$ in each district.

Moreover, we shall assume that district size (the total number of votes cast for relevant parties) is not correlated with electoral victory for any party, i.e.,

$$\frac{1}{c} \sum_{k=1}^{c} v^k s_i^k = \frac{1}{c} \sum_{k=1}^{c} v^k \cdot \frac{1}{c} \sum_{k=1}^{c} s_i^k$$

or equivalently

$$35$$ For instance, it is easy to observe that if one party has zero votes (and necessarily zero wasted votes), the efficiency gap criterion can never be satisfied.

\[\sim 30 \sim\]
\[
\sum_{k=1}^{c} v^k s_i^k = v s_i / c
\]  
(4.11)

for each \( i = 1, \ldots, n \) (note that, as in the two-party case, this is a relaxed version of the McGhee-Stephanopoulos assumption that district size is constant).

Within each district, the number of wasted votes sums up to

\[
\sum_{i=1}^{n} w_i^k = \sum_{i=1}^{n} (v_i^k - W v^k s_i^k) = \sum_{i=1}^{n} v_i^k - W v^k \sum_{i=1}^{n} s_i^k = (1 - W) v^k.
\]  
(4.12)

Thus, the nationwide sum of wasted votes equals

\[
\sum_{k=1}^{c} (1 - W) v^k = (1 - W) v.
\]  
(4.13)

If each of the \( n \) parties is to have equal number of wasted votes (per the efficiency gap criterion), it follows that

\[
\sum_{k=1}^{c} w_i^k = (1 - W) \frac{v}{n}
\]  
(4.14)

must hold for each \( i = 1, \ldots, n \). Applying (4.11) and (4.12) we get

\[
\sum_{k=1}^{c} w_i^k = \sum_{k=1}^{c} (v_i^k - W v^k s_i^k) = v_i - W v s_i / c = v(p_i - W q_i).
\]  
(4.15)

Now (4.14) and (4.15) imply

\[
(1 - W) / n = p_i - W q_i.
\]  
(4.16)

Hence \( p_i > (1 - W) / n \). Accordingly, we obtain the formula for the threshold of relevancy

\[
t := (1 - W) / n.
\]  
(4.17)

Now, if we substitute (4.17) into (4.9), we obtain \( W = W/2 + t \), and so \( t = W/2 \). Accordingly,

\[
W = \frac{1 - nW/2}{2} + \frac{2}{2 + n} = \frac{1}{1 + n/2},
\]  
(4.18)

and

\[
t = \frac{1}{2 + n}.
\]  
(4.19)
Note that $W$ is our optimal divisor defined in Sec. 3.1 and $t$ is our natural threshold introduced in Sec. 3.3. Accordingly, the definitions of relevancy given in Sec. 3.3 and in this section coincide. Now substituting (4.18) into (4.15) we have

$$v \left( p_i - \frac{2}{2 + n} q_i \right) = \frac{v}{n} \left( 1 - \frac{2}{2 + n} \right),$$

(4.20)

This gives

$$q_i = \left( 1 + n/2 \right) p_i - 1/2,$$

(4.21)

which is exactly our formula (0.2) for single-seat ($m = 1$) districts. Conversely, assuming (4.21) we shall get (4.22), so both approaches coincide.

Accordingly, it follows that under the definition of wasted votes given by (4.10) and the assumptions stated above, viz.

1. that all districts are fully contested, i.e., $p_i^k > t$ for each relevant party $i = 1, \ldots, n$ and for every district $k = 1, \ldots, n$,
2. that wasted votes are always nonnegative, i.e., that the winner in every district obtains a vote share equal to or in excess of $W$,
3. that district size is not correlated with electoral victory for any party,

the only values of $W$ and $t$ that lead to the efficiency gap criterion being capable of satisfaction are those that result from our formula, i.e., $W = 2/(2 + n)$ and $t = 1/(2 + n)$.

4.4. Application of power law to modeling the Jefferson-D’Hondt seat allocation method

Scholars have usually considered the relationship between seat ratios and vote ratios in FPTP electoral systems in terms of the power law. Cube law – the original form of the power law – has been first proposed in 1909 (Hearings before the Royal Commission on Systems of Election 1909: 77-86) by James Parker Smith (who attributed it to a British mathematician P. A. MacMahon36).

**Proposition (“the cube law”).** Let us pick any of the two parties and denote its vote share as $p$ and its seat share as $q$. It follows that the other party’s vote share equals $1 - p$ and its seat share equals $1 - q$. It claimed that in a two-party system with FPTP the seat ratio would be a cube of the vote ratio, i.e. that

$$\frac{q}{1 - q} = \left( \frac{p}{1 - p} \right)^3.$$

(4.23)

Since the first seminal work by Kendall and Stuart (1950), both the theoretical underpinnings and empirical correctness of the cube law have been extensively studied; see, inter alia, works by Butler (1952), March (1957), Coleman (1964), Qualter (1968), Theil (1970), Taagepera

36 On MacMahon’s role in the development of the cube law, see Garcia (2006).
(1973), Quandt (1974), Gudgin and Taylor (1979), Stanton (1980), Gilliland (1985), Taagepera and Shugart (1989), Blais and Massicotte (1996), Maloney, Pearson and Pickering (2003), Dolez and Laurent (2005) and others. Taagepera (1973), Tufte (1973) and Linehan and Schrod (1977) appear to have been the first to propose a more general power law, where the exponent in (4.23) – instead of being fixed at 3 – becomes an exogenous parameter permitted to vary within $[1, \infty)$, i.e.

$$\frac{q}{1-q} = \left(\frac{p}{1-p}\right)^\beta.$$  \hspace{1cm} (4.24)

Attempts to fit such more general models led to findings of exponents within the $(2, 2.5)$ interval for the United States (Tufte, 1973, Gudgin and Taylor, 1979: 29, Schrod, 1981, Gryski, Reed and Elliott, 1990), New Zealand (Tufte, 1973), South Africa (Schrod, 1981) and even Great Britain (Laakso, 1979, Blau, 2004). Theoretical arguments in favor of square (rather than cube) law have been made by Sankoff and Mellos (1972), who relied on game theoretic approach, and Maloney, Pearson and Pickering (2003), who argued that Taagepera’s (1973) original assumptions on fractal character of electoral geography should produce “$\sqrt{3}$-law” rather than cube law.

A seat ratio under our formula can be obtained from (4.4):

$$\frac{q}{1-q} = \frac{p - 1/4}{3/4 - p}.$$ \hspace{1cm} (4.25)

The relationship between this ratio and the votes ratio $p/(1 - p)$ can also be described by a power law, but with the exponent itself being a function of $p$:

$$\beta(p) = \left(\ln \frac{q}{1-q}\right)/\left(\ln \frac{p}{1-p}\right) = \left(\ln \frac{4p - 1}{3 - 4p}\right)/\left(\ln \frac{p}{1-p}\right).$$ \hspace{1cm} (4.26)

We plot that function on Fig. 10 below.
We note that, except in extreme cases when the stronger party wins more than two thirds of the vote, the exponent falls exactly within the interval indicated by “square-plus law” findings cited above (and as the votes ratio tends to 1, the exponent tends to 2). Assuming that the vote share of the stronger party is uniformly distributed on \((1/2, 3/4)\), the expected value of \(\beta\) equals approximately 2.38.

Of course, the above model can be generalized to any district magnitude, as long as there are still only two parties. For arbitrary \(m \in [1, \infty)\) the seats ratio formula is

\[
\frac{q}{1-q} = \frac{p - \frac{1}{2m+2}}{1-p - \frac{1}{2m+2}},
\]

and the function

\[
\beta(p) = \frac{\ln(p(m+1) - 1/2) - \ln(m - p(m+1) + 1/2)}{\ln \frac{p}{1-p}}
\]

for \(m = 1,2,3,4,12\) is plotted on Fig. 11 below.
Two properties of this function are of particular interest: its limit as \( p \) approaches \( 1/2 \) and the expected value throughout its natural domain. The former can be calculated by applying de L'Hôpital's rule:

\[
\lim_{p \to 1/2} \beta(p) = \lim_{p \to 1/2} \frac{\partial}{\partial p} \ln \frac{p(m + 1) - 1/2}{m - p(m + 1) + 1/2} = 1 + \frac{1}{m},
\]

while expected \( \beta \) equals

\[
E(\beta) = \int_{1/2}^{2m+1} \frac{1}{2m+2} \ln \frac{p(m + 1) - 1/2}{m - p(m + 1) + 1/2} f(p) \, dp,
\]

where \( f \) is the probability density of \( p \).

It is interesting to see how the limiting exponent (4.29) and the expected exponent (4.30) compare with the leading model of seats-votes relationship in multi-seat districts – Rein Taagepera’s generalized cube law for proportional elections (Taagepera 1986). Taagepera proposed a fixed power law exponent (i.e., one invariant with respect to \( p \)) equal to \( 3^{1/m} \). The relationship between the three for realistic district magnitudes (\( m \in \{1, \ldots, 30\} \)) is plotted as Fig. 12 below.
Of special note is the rapid convergence of Taagepera’s generalized cube law exponent \(3^{1/m}\) and our limiting exponent \((1 + 1/m)\). This is however easily explained when one notes that for large values of \(s\)

\[
1 + \frac{1}{m} = \left(1 + \frac{1}{m}\right)^{m} \approx e^{1/m},
\]

which differs from Taagepera’s formula only as to base. While empirical test of the generalized \(e\) law versus the generalized cube law is beyond the scope of this article, we submit that at least on theoretical grounds there are sound arguments to expect \(e\) law to hold for multiple-seat districts.

**Conclusions**

As noted above, the seat allocation and seat bias formulae presented in this article have a number of both practical and theoretical applications. First, the seat allocation formula facilitates easy translation of vote shares into seat numbers, which constitute a natural complement of opinion and exit poll results. In such cases, aggregate national vote shares are usually all that is known. Their disaggregation into district-level results can only be done with complex and volatile election demographic models. In addition, such models are especially unreliable for new parties, whose territorial support patterns cannot be inferred from earlier elections. Our proposed formula provides a simpler alternative that relies only on aggregate
results and on the numbers of seats and districts, which are known before the election, and yet still provides a high degree of accuracy.

Second, the seat allocation and seat bias formulae enable researchers and practitioners alike to simulate counterfactual election results without relying on restrictive assumptions with regard to the territorial distribution of party votes. This makes it a useful tool for evaluating political strategies and what-if scenarios and for assessing the effects of electoral engineering. The formula can be used to quantify for each party the expected costs and benefits of electoral reforms that involve changes in the mean district magnitude to a greater degree of precision than with earlier approaches. Such predictions were presented by one of the authors of this article during the 2017 public debate on changing the local and regional electoral system in Poland, contributing to the governing majority’s decision to withdraw the controversial proposals to shrink the electoral districts.

Thirdly, the article explains how and why the seat bias under the Jefferson-D’Hondt method depends on both the mean district magnitude and the number of parties. While the former relationship – captured by the micromega rule – is well known to students of electoral systems, the latter has been somewhat underappreciated beyond the field of purely theoretical studies of election bias. We demonstrate how, and under what conditions, those two strands of electoral system research can be combined into a complete picture of the conditions determining the magnitude of the seat bias.

Finally, our formula, generalizing the McGhee-Stephanopoulos efficiency gap approach, provides a consistent normative criterion for lack of ‘skewness’ in the Jefferson-D’Hondt variant of proportional voting system. An election the results of which deviate significantly from the formula must be in a sense ‘skewed,’ either as a result of some unnatural correlations possibly, though not necessarily, caused by malapportionment or gerrymandering, or due to some random numerical artefacts of the system.
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References

Baldini, Gianfranco and Pappalardo, Adriano (2009) Elections, Electoral Systems and Volatile Voters. Basingstoke: Palgrave Macmillan.

Balinski, Michel L. and Young, H. Peyton (1975) ‘The Quota Method of Apportionment’, American Mathematical Monthly, 82 (7), pp. 701–730, doi: 10.2307/2318729.

Balinski, Michel L. and Young, H. Peyton (1978) ‘The Jefferson Method of Apportionment’, SIAM Review, 20 (3), pp. 278–284, doi: 10.1137/1020040.

Balinski, Michel L. and Young, H. Peyton (2001) Fair Representation: Meeting the Ideal of One Man, One Vote. Washington, DC: Brookings Institution Press.

Barceló, Joan and Muraoka, Taishi (2018) ‘The Effect of Variance in District Magnitude on Party System Inflation’, Electoral Studies, doi: 10.1016/j.electstud.2018.04.016.

Benoit, Kenneth (2000) ‘Which Electoral Formula Is the Most Proportional? A New Look with New Evidence’, Political Analysis, 8 (4), pp. 381–388, doi: 10.1093/oxfordjournals.pan.a029822.

Bernstein, Mira and Duchin, Moon (2017) ‘A Formula Goes to Court: Partisan Gerrymandering and the Efficiency Gap’, AMS Notices, 64 (9), pp. 1020–1024, doi: 10.1090/noti1573.

Bhattacharyya, Anil K. (1943) ‘On a Measure of Divergence between Two Statistical Populations Defined by Their Probability Distributions’, Bulletin of the Calcutta Mathematical Society, 35, pp. 99–109.

Biles, Charles M. (2017) The History of Congressional Apportionment. Arcata, CA: Humboldt State University Press.

Blais, André and Massicotte, Louis (1996) ‘Mixed Electoral Systems: An Overview’, Representation, 33 (4), pp. 115-118, doi: 10.1080/00344899608522970.

Blau, Adrian (2004), ‘A Quadruple Whammy For First-Past-the-Post’, Electoral Studies, 23 (3), pp. 431–453, doi: 10.1016/S0261-3794(03)00030-1.
Bochsler, Daniel (2008) ‘Who Gains from Apparentments under D’Hondt?’, Electoral Studies, 29 (4), pp. 617–627, doi: 10.1016/j.electstud.2010.06.001.

Bormann, Nils-Christian and Golder, Matt (2013) ‘Democratic Electoral Systems Around the World, 1946–2011’, Electoral Studies, 32 (2), pp. 360-369, doi: 10.1016/j.electstud.2013.01.005.

Brancati, Dawn (2007) Global Elections Database. New York: Global Elections Database, http://www.globalelectionsdatabase.com/.

Burnitz, Gustav and Varrentrapp, Georg (1863) Methode, bei jeder Art von Wahlen sowohl der Mehrheit als den Minderheiten die ihrer Stärke entsprechende Zahl von Vertretern zu sichern. Frankfurt am Main: I. D. Sauerländer. Trans. by F. Weitenkampf in Forney, Matthias N. (Ed.) (1894) Political Reform by the Representation of Minorities. New York, pp. 159-174.

Butler, David E. (1952) The British General Election of 1951. London: Macmillan.

Carey, John M. (2017), ‘Electoral System Design in New Democracies’. In Herron, Erik, Pekkanen, Robert and Shugar, Matthew (Eds.), Oxford Handbook of Electoral Systems. New York: Oxford UP, pp. 85-111, doi: 10.1093/oxfordhb/9780190258658.001.0001.

Carstairs, Andrew M. (1980) A Short History of Electoral Systems in Western Europe. London: George Allen & Unwin.

Chafee, Zechariah, Jr., (1929) ‘Congressional Reapportionment’, Harvard Law Review, 42 (8), pp. 1015–1047, doi: 10.2307/1331072.

Chambers, Christopher P., Miller, Alan D., and Sobel, Joel (2017) ‘Flaws in the Efficiency Gap’, Journal of Law & Politics, 33 (1), pp. 1–33.

Coleman, James S. (1964) Introduction to Mathematical Sociology. New York: Free Press.

Colomer, Josep M. (2004) The Handbook of Electoral System Choice. London: Palgrave Macmillan.

Crosby, Alan (2000) The Lancashire Dictionary of Dialect, Tradition and Folklore. Skipton: Dalesman.

Curiel, Imma (2014) ‘A Multifaceted Analysis of the Electoral System of the Republic of Suriname’, Operations Research and Decisions, 4, pp. 29–49, doi: 10.5277/ord140403.

Dančišin, Vladimir (2013a) ‘Notes on the Misnomers Associated with Electoral Quotas’, European Electoral Studies, 9 (2), pp. 160–165.

Dančišin, Vladimir (2013b) ‘Hľadanie volebného deliteľa Victorom D’Hondtom’, European Electoral Studies, 10 (1), pp. 63–70.

D’Hondt, Victor (1882) Système pratique et raisonné de représentation proportionnelle. Bruxelles: Librairie C. Muquardt, doi: 10.3931/e-rara-39876.

D’Hondt, Victor (1883) ‘Formule du minimum dans la représentation proportionnelle. Moyen facile de trouver le diviseur’, Représentation proportionnelle. Revue mensuelle, 2, pp. 117–128, 129–130.

~ 39 ~
D’Hondt, Victor (1885) *Exposé du système pratique de représentation proportionnelle. Adopté par le Comité de l’Association Réformiste Belge*. Gand: Eug. Vanderhaeghen.

Dolez, Bernard and Laurent, Annie (2005) ‘The Seat–Vote Equation in French Legislative Elections (1978–2002)’, *French Politics*, 3 (2), pp. 124-141, doi: 10.1057/palgrave.fp.8200077.

Droop, Henry R. (1881) ‘On Methods of Electing Representatives’, *Journal of the Statistical Society of London*, 44 (2), pp. 141–202, doi: 10.2307/2339223.

Droton, Mathias and Schwingenschlögl, Udo (2005) ‘Asymptotic Seat Bias Formulae’, *Metrika*, 62 (1), pp. 23–31, doi: 10.1007/s001840400352.

Equer, Maurice (1911) ‘Relation entre la méthode d'Hondt et la proportionnalité’, *La Grande Revue, Deuxième série*, 31 (10 Jan.), pp. 130–137.

Gallagher, Michael (1991) ‘Proportionality, Disproportionality and Electoral Systems’, *Electoral Studies*, 10 (1), pp. 33–51, doi: 10.1016/0261-3794(91)90004-c.

Gfeller, Jules (1890) ‘Du transfert des suffrages et de la répartition des sièges complémentaires’, *Répresentation proportionnelle – Revue mensuelle*, 9, pp. 120–131.

Gilliland, Dennis C. (1985) ‘On the Macmahon Cube Law For Election Results,’ *Journal of Applied Statistics*, 12 (2), pp. 127-135, doi: 10.1080/02664768500000018.

Gryski, Gerard S., Reed, Bryce, and Elliott, Euel (1990) ‘The Votes-Seats Relationship in State Legislative Elections’, *American Politics Quarterly*, 18 (2), pp. 141-157, doi: 10.1177/1532673X8001800202.

Gudgin, Graham and Taylor, Peter J. (1979) *Seats, Votes and the Spatial Organization of Elections*. London: Pion.

Hagenbach-Bischoff, Eduard (1888) *Die Frage der Einführung einer Proportionalvertretung statt des absoluten Mehres*. Basel: H. Georg.

Hagenbach-Bischoff, Eduard (1905) *Die Verteilungsrechnung beim Basler Gesetz nach dem Grundsatz der Verhältniswahl*. Basel: Berichthaus.

Happacher, Max and Pukelsheim, Friedrich (1996) ‘Rounding Probabilities: Unbiased Multipliers’, *Statistics & Decisions*, 14 (4), pp. 373–382, doi: 10.1524/strm.1996.14.4.373.

Happacher, Max and Pukelsheim, Friedrich (2000) ‘Rounding Probabilities: Maximum Probability and Minimum Complexity Multipliers’, *Journal of Statistical Planning and Inference*, 85 (1–2), pp. 145–158, doi: 10.1016/S0378-3758(99)00077-4.

Happacher, Max (2001) ‘The Discrepancy Distribution of Stationary Multiplier Rules for Rounding Probabilities’, *Metrica* 53 (2), pp 171–181, doi: 10.1007/s001840000099.

Hare, Thomas (1859) *A Treatise on the Election of Representatives, Parliamentary and Municipal*. London: Longman & al.

Hearings before the Royal Commission on Systems of Election (1909), Minutes of Evidence, Cd 5352. London: HMSO.
Heinrich, Lothar, Pukelsheim, Friedrich, and Schwingenschlögl, Udo (2004), ‘Sainte-Laguë’s Chi-Square Divergence for the Rounding of Probabilities and Its Convergence to a Stable Law’, *Statistics & Decisions*, 22 (1), pp. 43–60, doi: 10.1524/stnd.22.1.43.32717.

Heinrich, Lothar, Pukelsheim, Friedrich, and Wachte, Vitali (2017), ‘The Variance of the Discrepancy Distribution of Rounding Procedures, and Sums of Uniform Random Variables’, *Metrika*, 80 (3), pp. 363–375, doi: 10.1007/s00184-017-0609-0.

Humphreys, John H. (1911) *Proportional Representation: A Study in Methods of Election*. London: Methuen & Co.

Huntington, Edward V. (1921) ‘The Mathematical Theory of the Apportionment of Representatives’, *Proceedings of the National Academy of Sciences*, 7 (4), pp. 123–127.

Huntington, Edward V. (1928) ‘The Apportionment of Representatives in Congress’, *Transactions of the American Mathematical Society*, 30 (1), pp. 85–110.

Huntington, Edward V. (1931) ‘Methods of Apportionment in Congress’, *American Political Science Review*, 25 (4), pp. 961–965, doi: 10.2307/1946616.

James, Edmund J. (1897) ‘The First Apportionment of Federal Representatives in the United States’, *Annals of the American Academy of Political and Social Science*, 9 (1), pp. 1–41.

Janson, Svante (2013) ‘Euler-Frobenius Numbers and Rounding’, arXiv:1305.3512 [math.PR].

Janson, Svante (2014) ‘Asymptotic Bias of Some Election Methods’, *Annals of Operations Research*, 215 (1), pp. 89–136, doi: 10.1007/s10479-012-1141-2.

Jefferson, Thomas (1792) ‘Opinion on Apportionment Bill’, *The Papers of Thomas Jefferson, Digital Edition*, ed. Oberg, Barbara and Looney, J. Jefferson, Charlottesville: University of Virginia Press, Rotunda, 2008.

Joachim, Václav (1917) ‘K otázce poměrného zastoupení’, *Správní obzor*, 9 (8), pp. 289–298.

Kaminski, Marek M. (2001) ‘Coalitional Stability of Multi-Party Systems: Evidence from Poland’, *American Journal of Political Science*, 45 (2), pp. 294-312, doi: 10.2307/2669342.

Kaminski, Marek M. (2002) ‘Do Parties Benefit from Electoral Manipulation? Electoral Laws and Heresthetics in Poland’, *Journal of Theoretical Politics*, 14 (3), pp. 325-359, doi: 10.1177/095169280201400303.

Kaminski, Marek M. (2018) ‘Spoiler Effects in Proportional Representation Systems: Evidence from Eight Polish Parliamentary Elections, 1991–2015’, *Public Choice*, 176 (3-4), pp. 441–460, doi: 10.1007/s11127-018-0565-x.

Karpov, Alexander (2015) ‘Alliance Incentives under the D’Hondt Method’, *Mathematical Social Sciences*, 74 (C), pp. 1-7, doi: 10.1016/j.mathsocsci.2014.12.001.

Kendall, M. G. and Stuart, A. (1950) ‘The Law of the Cubic Proportion in Election Results’, *British Journal of Sociology*, 1 (3), pp. 183-196, doi: 10.2307/588113.
Kollman, Ken, Hicken, Allen, Caramani, Daniele, Backer, David, and Lublin, David (2017) ‘Constituency-Level Elections Archive’ Ann Arbor: Center for Political Studies, University of Michigan, MI, http://www.electiondataarchive.org/.

Laakso, Markku (1980) ‘Should a Two-and-a-Half Law Replace the Cube Law in British Elections?’, British Journal of Political Science, 9 (3), pp. 355–362, doi: 10.1017/S0007123400001824.

Leutgäb, Peter and Pukelsheim, Friedrich (2009) ‘List Apparentements in Local Elections – A Lottery’. In Holler, Manfred J. and Nurmi, Hannu (Eds.), Power, Voting, and Voting Power: 30 Years After. Berlin: Springer, pp. 489–500, doi: 10.1007/978-3-642-35929-3_7.

Lijphart, Arend (1990) ‘The Political Consequences of Electoral Laws, 1945-85’, American Political Science Review, 84 (2), pp. 481–496, doi: 10.2307/1963530.

Lijphart, Arend and Gibberd, Robert W. (1977) ‘Thresholds and Payoffs in List Systems of Proportional Representation’, European Journal of Political Research, 5 (3), pp. 219–244, doi: 10.1111/j.1475-6765.1977.tb01289.x.

Linehan, William J., and Schrod, Philip A. (1978) ‘A New Test of the Cube Law,’ Political Methodology, 4 (4): 353–367.

Loosemore, John, and Hanby, Victor J. (1971) ‘The Theoretical Limits of Maximum Distortion: Some Analytic Expressions for Electoral Systems’ British Journal of Political Science, 1 (4), pp. 467–477, doi: 10.1017/S000712340000925X.

Maloney, John, Pearson, Bernard, and Pickering, Andrew (2003) ‘Behind the Cube Rule: Implications of, and Evidence Against a Fractal Electoral Geography’, Environment and Planning A, 35 (8), pp. 1405-1414, doi: 10.1068/a35184.

March, James G. (1957) ‘Party Legislative Elections as a Function of Election Results’, Public Opinion Quarterly, 21 (4), pp. 521-542, doi: 10.1086/266748.

Marshall, Albert W., Olkin, Ingram, and Pukelsheim, Friedrich (2002) ‘A Majorization Comparison of Apportionment Methods in Proportional Representation’, Social Choice and Welfare, 19 (4), pp. 885–900, doi: 10.1007/s003550200164.

McGhee, Eric (2014) ‘Measuring Partisan Bias in Single-Member District Electoral Systems’, Legislative Studies Quarterly, 39 (1), pp. 55–85, doi: 10.1111/lsq.12033.

McGhee, Eric (2017) ‘Measuring Efficiency in Redistricting’, Election Law Journal, 16 (4), pp. 417–442, doi: 10.1089/elj.2017.0453.

Mora, Xavier (2013) ‘La regla de Jefferson – D’Hondt i les seves alternatives’, Departament de Matemàtiques, Universitat Autònoma de Barcelona, MATERials MATemàtics No. 4, http://mat.uab.cat/matmat/PDFv2013/v2013n04.pdf.

Morse, Marston, von Neumann, John, and Eisenhart, Luther P. (1948) Report to the President of the National Academy of Sciences.
Mulligan, Casey B. and Hunter, Charles G. (2001) ‘The Empirical Frequency of a Pivotal Vote’, National Bureau of Economic Research, Working Paper No. 8590, http://www.nber.org/papers/w8590.pdf.

Oyama, Tatsuo and Ichimori, Tetsuo (1995) ‘On the Unbiasedness of the Parametric Divisor Method for the Apportionment Problem’, Journal of the Operations Research Society of Japan, 38 (3), pp. 301–321, doi: 10.15807/jorsj.38.301.

Palomares, Antonio and Ramírez Gonzáles, Victoriano (2003), ‘Thresholds of the Divisor Methods’, Numerical Algorithms, 34 (2), pp. 405–415, doi: 10.1023/B:NUMA.0000005353.82970.ce.

Pavia, Jose and García-Cárceles, Belén (2016) ‘Estimating Representatives from Election Poll Proportions: The Spanish Case’, Statistica Applicata – Italian Journal of Applied Statistics, 25 (3), pp. 325-340.

Pilet, Jean-Benoit, Renwick, Alan, Núñez, Lidia, Reimink, Elwin, and Simón, Pablo (2016) Electoral System Change in Europe (ESCE): Database of Electoral Systems Codebook, http://www.electoralsystemchanges.eu/Files/media/MEDIA_767/F ILE/Codebook_Database_of_Electoral_Systems_15-02-2016.pdf.

Plener Cover, Benjamin (2018) ‘Quantifying Partisan Gerrymandering: An Evaluation of the Efficiency Gap Proposal’, Stanford Law Review, 70 (4), pp. 1131–1233.

Pólya, György (1918a) ‘Sur la représentation proportionnelle en matière électorale’, L’Enseignement Mathématique, 20, pp. 355–379.

Pólya, György (1918b) ‘Über die Verteilungssysteme der Proportionalwahl’, Zeitschrift für schweizerische Statistik und Volkswirtschaft, 54, pp. 363–387.

Pólya, György (1919a) ‘Proportionalwahl und Wahrscheinlichkeitsrechnung’, Zeitschrift für die gesamte Staatswissenschaft, 74, pp. 297–322.

Pólya, György (1919b) ‘Über Sitzverteilung bei Proportionalwahlverfahren’, Schweizerisches Zentralblatt für Staats- und Gemeinde-Verwaltung, 20, pp. 1–5.

Pólya, György (1919c) ‘Über die Systeme der Sitzverteilung bei Proportionalwahl’, Wissen und Leben—Schweizerische Halbmonatsschrift, 12, pp. 259–268, 307–312.

Poptcheva, Eva-Maria (2016) Understanding the D’Hondt Method. Allocation of Parliamentary Seats and Leadership Positions. European Parliamentary Research Service Briefing PE 580.901, http://www.europarl.europa.eu/RegData/etudes/BRIE/2016/580901/EPRS_BRI(2016)580901_EN.pdf.

Pukelsheim, Friedrich (2014) Proportional Representation: Apportionment Methods and Their Applications. Heidelberg: Springer Verlag.

Pukelsheim, Friedrich (2017) Proportional Representation: Apportionment Methods and Their Applications. Second Edition. Heidelberg: Springer Verlag.
Qualter, Terence H. (1968) ‘Seats and Votes: An Application of the Cube Law to the Canadian Electoral System’, Canadian Journal of Political Science, 1 (3), pp. 336-344, doi: 10.1017/S0008423900036817.

Quandt, Richard (1974) ‘A Stochastic Model of Elections in Two-Party Systems’, Journal of the American Statistical Association, 69 (346), pp. 315-324, doi: 10.1080/01621459.1974.10482946.

Rae, Douglas W. (1967) The Political Consequences of Electoral Laws. New Haven, CT: Yale University Press.

Rae, Douglas W., Hanby, Victor J., and Loosemore, John (1971) ‘Thresholds of Representation and Thresholds of Exclusion. An Analytic Note on Electoral Systems’, Comparative Political Studies, 3 (4), pp. 479–488, doi: 10.1177/001041407100300406.

Rokkan, Stein (1968) ‘Elections: Electoral Systems’, International Encyclopaedia of the Social Sciences, ed. Sills, David L. New York: Crowell-Collier-Macmillan, vol. 5, pp. 6–21.

Sainte-Laguë, André (1910) ‘La représentation proportionnelle et la méthode des moindres carrés’, Annales scientifiques de l'École Normale Supérieure, Sér. 3, 27, pp. 529–542, doi: 10.24033/asens.627.

Sankoff, David, and Mellos, Koulla (1972) ‘The Swing Ratio and Game Theory’, American Political Science Review, 66 (2), pp. 551-554. doi: 10.2307/1957798.

Schrodt, Philip A. (1981 ) ‘A Statistical Study of the Cube Law in Five Electoral Systems’, Political Methodology, 7 (2), pp. 31-53.

Schuster, Karsten, Pukelsheim, Friedrich, Drton, Mathias, and Draper, Norman R. (2003) ‘Seat Biases of Apportionment Methods for Proportional Representation’, Electoral Studies, 22 (4), pp. 651–676, doi: 10.1016/S0261-3794(02)00027-6.

Schwingenschlögl, Udo and Drton, Mathias (2004) ‘Seat Allocation Distributions and Seat Biases of Stationary Apportionment Methods for Proportional Representation’, Metrika, 60 (2), pp. 191–202, doi: 10.1007/s001840400347.

Schwingenschlögl, Udo (2008) ‘Seat Biases of Apportionment Methods under General Distributional Assumptions’, Applied Mathematics Letters, 21 (1), pp. 1–3, doi: 10.1016/ j.aml.2007.02.005.

Shugart, Matthew S. and Taagepera, Rein (2017) Votes from Seats: Logical Models of Electoral Systems. Cambridge, UK: Cambridge University Press.

Stanton, Ralph G. (1980) ‘A Result of MacMahon on Electoral Predictions’, Annals of Discrete Mathematics, 8, pp. 163-167, doi: 10.1016/S0167-5060(08)70866-5.

Stephanopoulos, Nicholas O., and McGhee, Eric M. (2015) ‘Partisan Gerrymandering and the Efficiency Gap’, University of Chicago Law Review, 82 (2), pp. 831–900.

Stephanopoulos, Nicholas O., and McGhee, Eric M. (2018) ‘The Measure of a Metric: The Debate over Quantifying Partisan Gerrymandering’, Stanford Law Review, 70 (5), pp. 1503–1568.
Szpiro, George G. (2010) *Numbers Rule: The Vexing Mathematics of Democracy, from Plato to the Present*. Princeton, NJ: Princeton University Press.

Taagepera, Rein (1973) ‘Seats and Votes: A Generalization of the Cube Law of Elections’, *Social Science Research*, 2 (3), pp. 257-275. doi: 10.1016/0049-089X(73)90003-3.

Taagepera, Rein (1986) ‘Reformulating the Cube Law for Proportional Representation Elections’, *American Political Science Review*, 80 (2), pp. 489-504, doi: 10.2307/1958270.

Taagepera, Rein (2002) ‘Nationwide Threshold of Representation’, *Electoral Studies*, 21 (3), pp. 383-401, doi: 10.1016/S0261-3794(00)00045-7.

Taagepera, Rein (2007) *Predicting Party Sizes: The Logic of Simple Electoral Systems*. Oxford: Oxford University Press.

Taagepera, Rein and Laakso, Markku (1980) ‘Proportionality Profiles of West European Electoral Systems’, *European Journal of Political Research*, 8 (4), pp. 423–446, doi: 10.1111/j.1475-6765.1980.tb00582.x.

Taagepera, Rein and Shugart, Matthew S. (1989) *Seats and Votes: The Effects and Determinants of Electoral Systems*. New Haven, CT: Yale University Press.

Tapp, Kristopher (2018) ‘Measuring Political Gerrymandering’, arXiv: 1801.02541 [physics.soc-ph].

Theil, Henri (1970) ‘The Cube Law Revisited’, *Journal of the American Statistical Association*, 65 (331), pp. 1213-1219. doi: 10.1080/01621459.1970.10481156.

Tufte, Edward R. (1973) ‘The Relationship between Seats and Votes in Two-Party Systems’, *American Political Science Review*, 67 (2), pp. 540-554. doi: 10.2307/1958782.

Tukey, John Wilder (1938) ‘On the Distribution of the Fractional Part of a Statistical Variable’, *Recueil Mathématique*, 46 (3), pp. 561–562.

Udina, Frederic and Delicado, Pedro (2005) ‘Estimating Parliamentary Composition through Electoral Polls’, *Journal of the Royal Statistical Society. Series A (Statistics in Society)*, 168 (2), pp. 387-399.

van Eck, Liezl, Visagie, Stephanus E., and de Kock, Hendrik C. (2005) ‘Fairness of Seat Allocation Methods in Proportional Representation’, *ORiON*, 21 (2), pp. 93–110, doi: 10.5784/21-2-22.

Veomett, Elen (2018) ‘The Efficiency Gap, Voter Turnout, and the Efficiency Principle’, arXiv: 1801.05301 [physics.soc-ph].

Woodall, Douglas R. (1987) ‘How Proportional is Proportional Representation?’, *The Mathematical Intelligencer*, 8 (4), pp. 36–46, doi: 10.1007/BF03026117.
Appendix A. The relationship between errors and assumptions

As noted in Sec. 1.2 and explained in detail in Sec. 3.2, the seat allocation and seat bias formulae produce exact results if assumptions $A_1$ to $A_5$ hold exactly. Any deviation from those assumptions introduces errors. The key to the formula’s robustness, however, lies in the fact that those errors tend to cancel each other out.

There are, broadly speaking, three principal sources of error in the seat allocation formula. First, there is an error resulting from violations of assumption $A_1$, i.e., allocation of some seats to parties that fall below the mean natural threshold. Because we eliminate those parties from the set of relevant parties, those seats get misallocated even on the district level under formula (3.1) (which is otherwise exactly correct in all cases). The number of such misallocated seats is a natural measure of this kind of error (we will denote it as $\varepsilon_1$).

Second, there is an error arising from the summation procedure described in Sec. 3.2. Analyzing this procedure, we see that for the $i$-th party ($i = 1, \ldots, n$) the error can be written as the sum of three components:

$$
\varepsilon_{2a}^i := p_i s - \sum_{k=1}^{c} \frac{v_i^k s^k}{v_k^{}} = \sum_{k=1}^{c} \left( p_i m - \frac{v_i^k}{v_k^{} s^k} \right),
$$

$$
\varepsilon_{2b}^i := p_i \frac{cn}{2} - \sum_{k=1}^{c} \frac{v_i^k}{v_k^{} n} \sum_{j=1}^{n} r_j^k = \sum_{k=1}^{c} \left( p_i n - \frac{v_i^k}{v_k^{} n} \sum_{j=1}^{n} r_j^k \right),
$$

and

$$
\varepsilon_{2c}^i := \frac{c}{2} - \sum_{k=1}^{c} r_i^k = \sum_{k=1}^{c} \left( \frac{1}{2} - r_i^k \right).
$$

It follows from Sec. 3.2 that $\varepsilon_{2a}^i$ accounts for violations of combined assumptions $A_2$ and $A_3$, $\varepsilon_{2b}^i$ – for those of combined assumptions $A_2$ and $A_4$, and $\varepsilon_{2c}^i$ – for those of assumption $A_5$.

Finally, for countries where the regional correction is employed there exists a third source of potential error, involving violations of one or more of the three assumptions underlying that correction ($R_1$, $R_2$, and $R_3$). For national parties such an error equals

$$
\varepsilon_{3N}^i := \sum_{k=1}^{r} \left( s^k + \frac{c^k (|N| + |R^k|)}{2} \right) \left( p_i \frac{1 - \sum_{j \in R^k} P_j^k}{1 - \sum_{j \in R} P_j} - \frac{v_i^k}{v_k^{} s^k} \right),
$$

where $r$ is the number of regions, $N$ – the set of all national parties, $R$ – the set of all regional parties, and $R^k$ – the set of regional parties running in the $k$–th region, while for regional parties running in region $k$ the error introduced by the regional correction equals simply

~ 46 ~
\[ \varepsilon_{3R}^i := P_i \left( s^k + \frac{c^k(|N| + |R^k|)}{2} \left( \frac{s}{s^k} - \frac{P}{pk} \right) \right). \] (5.5)

Note that the above enumeration of error sources is exhaustive, i.e., those sources account for all actual errors in the formula. In other words, comparing – for every case under consideration – the sum of \( \varepsilon_1, \varepsilon_{2a}, \varepsilon_{2b} \) and \( \varepsilon_{2c} \) over all regions and of \( \varepsilon_3 \) – the regional error – with the deviation of the estimated seat allocation from the actual seat allocation, the difference between the two is zero.

For each kind of error we plot a kernel density estimate (a histogram for \( \varepsilon_1 \), which is discrete). We normalize all error measures by dividing them by national seat totals.

Fig. 13. Histogram of errors (\( \varepsilon_1 \)) arising from the violations of A1.
Fig. 14. Density of errors arising from the violations of A2 and A3 ($\varepsilon_{2a}^1$ – upper left), of A2 and A4 ($\varepsilon_{2b}^1$ – upper right), and A5 ($\varepsilon_{2c}^1$ – lower left), and of the assumptions underlying the regional correction ($\varepsilon_3$ – lower right).

Figs. 13 and 14 illustrate the fact that assumption $A5$ (that the rounding residuals average $1/2$ for each relevant party) is responsible for the largest number of errors in its application. The errors generated by violations of the other four assumptions are surprisingly few in number.

We are also interested in the interaction between the five sources of errors, and particularly in correlations between them. They are described by the following correlation matrix:

|      | $\varepsilon_1$ | $\varepsilon_{2a}$ | $\varepsilon_{2b}$ | $\varepsilon_{2c}$ | $\varepsilon_3$ |
|------|------------------|--------------------|---------------------|--------------------|----------------|
| $\varepsilon_1$ | 1.0000 | 0.0002 | 0.0832 | 0.0253 | 0.0504 |
| $\varepsilon_{2a}$ | 0.0002 | 1.0000 | 0.6717 | -0.1410 | 0.4561 |
| $\varepsilon_{2b}$ | 0.0832 | 0.6717 | 1.0000 | -0.1666 | 0.5316 |
| $\varepsilon_{2c}$ | 0.0253 | -0.1410 | -0.1666 | 1.0000 | -0.1017 |
| $\varepsilon_3$ | 0.0504 | 0.4561 | 0.5316 | -0.1017 | 1.0000 |
It appears that error $\varepsilon_1$ is very weakly correlated with others. On the other hand, errors $\varepsilon_{2a}$, $\varepsilon_{2b}$, and $\varepsilon_3$ are strongly positively correlated and reinforce each other, but each of them is also negatively correlated with the largest source of error – $\varepsilon_{2c}$ – so when summed up, they tend to cancel one another.

Appendix B. Discussion of assumption A5

Assumption A5 may prima facie appear arbitrary and restrictive. It can be shown, however, that under reasonable distributional assumptions we can expect it to be well-satisfied. A5 can be expressed as

$$\frac{1}{c} \sum_{k=1}^{c} r_{ik}^k = \frac{1}{2},$$

for any party $i = 1, \ldots, n$. Assuming that both the vote shares of the $i$-th party in given districts and those districts’ magnitudes are independent and identically distributed (for $k = 1, \ldots, c$) random variables, one can deduce from the law of large numbers that the equality (6.1) is at least approximately fulfilled, providing that the expected value of $r_{ik}^k$ equals 1/2.

Let us assume that the $i$-th party’s vote share $p_i$ has an absolutely continuous distribution on $[0,1]$ with the density $f_i$ and the cumulative distribution function $F_i$. Moreover, let $g$ be the probability mass function of the distribution of district magnitudes. Let us assume that for every party $i = 1, \ldots, n$:

- $f_i$ is smooth (of class $C^3$) on $[0,1]$ (A6),
- $f_i(0) = f_i([M]/M) = 0$ (A7),
- $F_i([M]/M) = 1$ (A8).

It should be noted that those assumptions are only mildly restrictive. A6 can be freely assumed, since a smooth density function can always be fitted to a finite set of empirical data points. A7 and A8 follow from the fact that in real-life elections extreme vote shares arise only infrequently – in the eight countries analyzed in Sec. 2 such an occurrence has happened only in 9 out of 6358 cases and, in fact, 5 of them has occurred in uncontested districts.

For given $k = 1, \ldots, c$ put $\mu := s_k$ and $M := M_k = \mu + n/2$. By A4 the conditional distribution of $r_{ik}^k$ given the district magnitude being equal to $\mu = 1, 2, \ldots$ is given by a density function

$$\frac{1}{M} \sum_{l=0}^{[M]} f_i \left( \frac{l + x}{M} \right),$$

for $x \in [0,1]$. Accordingly, the unconditional density of $r_{ik}^k$ is given by

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40 Four of those cases occurred in the Aland Islands (where all four elections represented in our data set have been uncontested), two in Croatian expatriate district (elections of 2011 and 2015), two in Portuguese expatriate district (elections of 1979 and 1980), and one in Mozambique (which, as a Portuguese colony, elected one deputy in the Portuguese election of 1975). Note also that all districts involved have been rather small in terms of the district magnitude: five of them have been single-member districts and none was larger than three seats.
\[ \sum_{\mu=1}^{\infty} g(\mu) \frac{1}{M} \sum_{l=0}^{\lfloor M \rfloor} f_i \left( \frac{l + x}{M} \right), \]  

(6.3)

for \( x \in [0,1] \), and its expected value equals

\[ E(r_i^k) = \int_0^1 \sum_{\mu=1}^{\infty} \frac{x}{M} \sum_{l=0}^{\lfloor M \rfloor} g(\mu) f_i \left( \frac{l + x}{M} \right) \, dx. \]  

(6.4)

Let us substitute \( y := (l + x)/M \) and exchange integration and summation to obtain

\[ E(r_i^k) = \sum_{\mu=1}^{\infty} \sum_{l=0}^{\lfloor M \rfloor} \int_{\frac{l}{M}}^{\frac{l+1}{M}} My \, g(\mu) f_i(y) M \, dy = \]

\[ \sum_{\mu=1}^{\infty} \sum_{l=0}^{\lfloor M \rfloor} \int_{\frac{l}{M}}^{\frac{l+1}{M}} My \, g(\mu) f_i(y) \, dy = \]

\[ E(M)E(p_i) - \sum_{\mu=1}^{\infty} \sum_{l=0}^{\lfloor M \rfloor} l g(\mu) \left( F_i \left( \frac{l+1}{M} \right) - F_i \left( \frac{l}{M} \right) \right) = \]

\[ E(M)E(p_i) - \sum_{\mu=1}^{\infty} g(\mu) \left( [M] F_i \left( \frac{[M]+1}{M} \right) - \sum_{l=0}^{\lfloor M \rfloor} F_i \left( \frac{l}{M} \right) \right) = \]

\[ E(M)E(p_i) - E([M]) + \sum_{\mu=1}^{\infty} g(\mu) \sum_{l=0}^{\lfloor M \rfloor} F_i \left( \frac{l}{M} \right). \]  

(6.5)

Put \( \Phi_i(l) := F_i(l/M) \) for \( 0 \leq l \leq M \). From the Euler-Maclaurin summation formula it follows that for \( \Phi_i \) smooth enough (of class \( C^{\rho+2} \)) we have

\[ \sum_{l=1}^{\lfloor M \rfloor} \Phi_i(l) = \int_0^M \Phi_i(x) \, dx + \frac{\Phi_i([M]) + \Phi_i(0)}{2} \]

\[ + \sum_{j=1}^{\rho/2} \frac{B_{2j}}{(2j)!} \left( \Phi_i^{(2j-1)}([M]) - \Phi_i^{(2j-1)}(0) \right) + R_\rho, \]  

(6.6)

where \( \rho \) is an even natural number, \( B_{2j} \) is the \((2j)\)-th Bernoulli number (for \( j = 1, \ldots, \rho/2 \)), \( \Phi_i^{(K)} \) is the \( K \)-th derivative of \( \Phi_i \), and

\[ \sim 50 \sim \]
\[ |R_\rho| \leq \frac{2\zeta(\rho)}{(2\pi)^\rho} \int_0^{[M]} \left| \Phi_i^{(\rho)}(x) \right| \, dx, \]  

(6.7)

where \( \zeta \) is the Riemann zeta function. Let us note that

\[ \int_0^{[M]} \Phi_i(x) \, dx = M \int_0^{[M]/M} F_i(y) \, dy = M \int_0^{1} F_i(y) \, dy - M \int_{[M]/M}^{1} F_i(y) \, dy, \]  

(6.8)

\[ \Phi([M]) + \Phi(0) = \frac{F_i([M]/M)}{2}, \]  

(6.9)

\[ \Phi^{(K)}_i(x) = M^{-K} F_i^{(K)}(x/M), \]  

(6.10)

\[ |R_\rho| \leq \frac{2\zeta(\rho)}{(2\pi)^\rho} \int_0^{[M]} \left| \Phi_i^{(\rho)}(x) \right| \, dx = \frac{2M\zeta(\rho)}{(2\pi M)^\rho} \int_0^{[M]/M} \left| F_i^{(\rho)}(y) \right| \, dy. \]  

(6.11)

Combining (6.6) with (6.8)-(6.11), we get for \( \rho = 2 \)

\[ \sum_{i=1}^{[M]} \Phi_i(l) = M \int_0^{1} F_i(y) \, dy - M \int_{[M]/M}^{1} F_i(y) \, dy + \frac{F_i([M]/M)}{2} + \frac{f_i \left( \frac{[M]}{M} \right) - f_i(0)}{12M} + R_2, \]  

(6.12)

and

\[ |R_2| \leq \frac{2M\zeta(2)}{(2\pi M)^2} \int_0^{[M]/M} \left| F_i^{(2)}(y) \right| \, dy = \frac{1}{12M} \int_0^{[M]/M} \left| f_i'(y) \right| \, dy. \]  

(6.13)

From A7 we get

\[ f_i([M]/M) - f_i(0) = 0. \]  

(6.14)

Note also that if \( n \) is even, and so \( [M] = M \), then

\[ M \int_{[M]/M}^{1} F_i(y) \, dy = 0, \]  

(6.15)

and accordingly

\[ F_i([M]/M)/2 - M \int_{[M]/M}^{1} F_i(y) \, dy = F_i(1)/2 = 1/2, \]  

(6.16)

while if \( n \) is odd, and so \( [M] = M - 1/2 \), then by A8 we obtain

\[ M \int_{[M]/M}^{1} F_i(y) \, dy = F_i([M]/M)/2, \]  

(6.17)
and accordingly
\[
\frac{F_i(\lfloor M \rfloor / M)}{2} - M \int_{\lfloor M \rfloor / M}^{1} F_i(y) dy = 0. \tag{6.18}
\]

By incorporating the foregoing results into (6.5) we obtain for even values of \(n\):
\[
E(r_i^k) = E(M)E(p_i) - E(\lfloor M \rfloor) + \sum_{\mu=1}^{\infty} \frac{1}{2} g(\mu) + \sum_{\mu=1}^{\infty} g(\mu)M \int_0^{1} F_i(y) dy + \sum_{\mu=1}^{\infty} g(\mu)R_2 =
\]
\[
E(M)E(p_i) - E(M) + E(M)(1 - E(p_i)) + \frac{1}{2} + \sum_{\mu=1}^{\infty} g(\mu)R_2 =
\]
\[
\frac{1}{2} + \sum_{\mu=1}^{\infty} g(\mu)R_2, \tag{6.19}
\]

and for odd values of \(n\):
\[
E(r_i^k) = E(M)E(p_i) - E(\lfloor M \rfloor) + \sum_{\mu=1}^{\infty} g(\mu)M \int_0^{1} F_i(y) dy + \sum_{\mu=1}^{\infty} g(\mu)R_2 =
\]
\[
E(M)E(p_i) - E(M - 1/2) - E(M)(1 - E(p_i)) + \sum_{\mu=1}^{\infty} g(\mu)R_2 =
\]
\[
\frac{1}{2} + \sum_{\mu=1}^{\infty} g(\mu)R_2. \tag{6.20}
\]

Note that the right-hand sides of (6.19) and (6.20) are identical, so the parity of \(n\) has no effect on the expected value of \(r_i^k\). Finally, we get
\[
|E(r_i^k) - \frac{1}{2}| \leq \sum_{\mu=1}^{\infty} g(\mu)R_2 = E\left(\frac{1}{12M} \int_0^{M} f_i'(y) dy\right) = \frac{E(M^{-1})}{12} \int_0^{1} f_i'(y) dy, \tag{6.21}
\]

where \(\int_0^{1} f_i'(y) dy\) is simply the total variation of \(f_i\). For unimodal distributions, it equals \(2f_i(\varphi_i)\), where \(\varphi_i\) is the mode of \(f_i\).

Hence, if the number of relevant parties or the district magnitudes are large enough, then the expected reciprocal of the unbiased multiplier \(M = \mu + n/2\) is small, and so \(E(r_i^k)\) are approximately equal to \(1/2\) for \(i = 1, \ldots, n\) and \(k = 1, \ldots, c\). When the number of districts \(c\) is large, it further follows from the law of large numbers that means \((1/c) \sum_{k=1}^{c} r_i^k (i = 1, \ldots, n)\) are close to the expected values, thereby approximately satisfying assumption \(A5\).