INRADIUS COLLAPSED MANIFOLDS

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Abstract. In this paper, we study collapsed manifolds with boundary, where we assume a lower sectional curvature bound, two side bounds on the second fundamental forms of boundaries and upper diameter bound. Our main concern is the case when inradii of manifolds converge to zero. This is a typical case of collapsing manifolds with boundary. We determine the limit spaces of inradius collapsed manifolds as Alexandrov spaces with curvature uniformly bounded below. When the limit space has co-dimension one, we completely determined the topology of inradius collapsed manifold in terms of singular $I$-bundles. General inradius collapse to almost regular spaces are also characterized. In the general case of unbounded diameters, we prove that the number of boundary components of inradius collapsed manifolds is at most two, where the disconnected boundary happens if and only if the manifold has a topological product structure.

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1. Introduction

In the present paper, we are concerned with collapsing phenomena of Riemannian manifolds with boundary under a lower sectional curvature bound. The study of collapse of closed manifolds has a long history. In the case of two side bounds on sectional curvatures, a deep general theory was established in [7]. Then for the case of lower sectional curvature bound, in [35], [9], [15], the structure of the first Betti numbers and the fundamental groups with their topological rigidity were determined through a fibration theorem. Later on, those results were partly extended to the case of a lower Ricci curvature bound in [5], [6], [8], [16]. Especially the general manifold structure results of lower dimensional collapsed manifolds under a lower sectional curvature bound were established in [30], [37], [31].

In those results, it is crucial to study Alexandrov spaces with curvature bounded below which appear as the Gromov-Hausdorff limit spaces. In particular, Perelman’s topological stability theorem has played significant roles. In connection with the study of Alexandrov spaces, the collapsing phenomena of three-dimensional closed Alexandrov spaces with curvature bounded below has been classified in a recent work [19].

For collapsing Riemannian manifolds with boundary, there is a pioneering work by J. Wong [33], [34] on this subject after the investigation in the non-collapsing and bounded curvature case due to [17], [2]. In the study of convergence and collapsing Riemannian manifolds with boundary, it is obvious that the main problem is to control the boundary behavior in a geometric way. It is in [33] that a nice extension procedure over the boundary was first carried out to study collapsed manifolds with boundary under a lower sectional curvature bound. The study of collapse of three-dimensional Alexandrov spaces with boundary is now undergoing in the work [20], where all the details of collapses will be made clear.

In the present paper, partly motivated by [20], we develop and extend results in [34] to a great extent. Let $\mathcal{M}(n, \kappa, \lambda, d)$ denote the set of all isometry classes of $n$-dimensional compact Riemannian manifolds $M$ with boundary whose sectional curvature, second fundamental form and diameter satisfy

$$K_M \geq \kappa, \ |\Pi_{\partial M}| \leq \lambda, \text{diam}(M) \leq d.$$
Every Riemannian manifold in $\mathcal{M}(n,\kappa,\lambda,d)$ can be glued with a warped cylinder along their boundaries in such a way that the resulting space becomes an Alexandrov space with curvature bounded below having $C^0$-Riemannian structure and that its boundary is totally geodesic ([33]). Investigating such a cylindrical extension, Wong proved that $\mathcal{M}(n,\kappa,\lambda,d)$ is precompact with respect to the Gromov-Hausdorff distance. He also proved that if $\mathcal{M}(n,\kappa,\lambda,d,v)$ denote the set of all elements $M \in \mathcal{M}(n,\kappa,\lambda,d)$ having volume $\text{vol}(M) \geq v > 0$, then it contains only finitely many homeomorphism types.

Under the situation above, the main problem we are concerned in this paper is as follows:

**Problem 1.1.** Let $M_i$ be a sequence in $\mathcal{M}(n,\kappa,\lambda,d)$ converging to a length space $N$ with respect to the Gromov-Hausdorff distance.

1. Characterize the structure of $N$;
2. Find geometric and topological relations between $M_i$ and $N$ for large enough $i$.

The *inradius* of $M$ is defined as the largest radius of metric ball contained in the interior of $M$:

$$\text{inrad}(M) := \sup_{x \in M} d(x, \partial M).$$

In the present paper, we first consider the case of $\text{inrad}(M_i)$ converging to zero. We prove in Corollary 3.11 that if inrad($M_i$) converges to zero, then $M_i$ actually dimension collapses in the sense that any limit space $N$ has dimension

$$\dim N \leq n - 1.$$

Therefore in this case, we say that $M_i$ inradius collapses. The inradius collapse is a typical case of collapsing of manifolds with boundary. Actually in the forthcoming paper [38], we show that if a sequence $M_i$ in $\mathcal{M}(n,\kappa,\lambda,d)$ converges to a topological closed manifold or a closed Alexandrov space, then $M_i$ inradius collapses.

The main results in this paper are stated as follows. The first one is about the limit spaces of inradius collapse.

**Theorem 1.2.** Let $M_i \in \mathcal{M}(n,\kappa,\lambda,d)$ inradius collapses to a length space $N$ with respect to the Gromov-Hausdorff distance. Then $N$ is an Alexandrov space with curvature $\geq c(\kappa,\lambda)$, where $c(\kappa,\lambda)$ is a constant depending only on $\kappa$ and $\lambda$.

It should be noted that $M_i$ are not Alexandrov spaces unless $\Pi_{\partial M_i} \geq 0$, and that the constant $c(\kappa,\lambda)$ really depend on both $\kappa$ and $\lambda$. Moreover, if one assume only $\Pi_{\partial M_i} \geq -\lambda^2$ or $\Pi_{\partial M_i} \leq \lambda^2$ instead of $|\Pi_{\partial M_i}| \leq \lambda^2$, there is a counterexamples to Theorem 1.2 (see Examples 3.14, 3.15 and 3.16).
Let $\mathcal{M}(n, \kappa, \lambda)$ denote the set of all isometry classes of $n$-dimensional complete Riemannian manifolds $M$ satisfying

$$K_M \geq \kappa, \ |\Pi_{\partial M}| \leq \lambda.$$ 

This family is also precompact with respect to the pointed Gromov-Hausdorff convergence. Theorem 1.2 actually holds true for the limit of manifolds in $\mathcal{M}(n, \kappa, \lambda)$ with respect to the pointed Gromov-Hausdorff convergence (see Theorem 6.3).

Next we discuss the topological structure of inradius collapsed manifolds. First consider the case of inradius collapse of codimension one. We can give a complete characterization of codimension one inradius collapsed manifolds as follows. Let $D^2_+$ be the upper half disk on $xy$-plane, and $J := D^2_+ \cap \{y = 0\}$.

**Theorem 1.3.** Let $M_i \in \mathcal{M}(n, \kappa, \lambda, d)$ inradius collapse to an $(n - 1)$-dimensional Alexandrov space $N$. Then there is a singular $I$-fiber bundle:

$$I \rightarrow M_i \xrightarrow{\pi} N$$

whose singular locus coincides with $\partial N$, and $M_i$ is a gluing of $I$-bundle $N \times I$ over $N$ and $D^2_+$-bundle $\partial N \times D^2_+$ over $\partial N$,

$$M_i = N \times I \cup \partial N \times D^2_+,$$

where the gluing is done via $\partial N \times I = \partial N \times J$, and $\times$ denotes either the product or a twisted product.

In particular $M_i$ has the same homotopy type as $N$.

Next, we consider inradius collapse to almost regular spaces. We say that an Alexandrov space $N$ is almost regular if any point of $N$ has the space of directions whose volume is close to $\text{vol} S^{\dim N - 1}$, where $S^m$ denotes the unit $m$-sphere.

**Theorem 1.4.** Let a sequence $M_i$ in $\mathcal{M}(n, \kappa, \lambda, d)$ inradius collapse to an Alexandrov space $N$, and suppose that the limit of $\partial M_i$ is almost regular and

$$\text{vol}(\Sigma_x(N)) > \frac{1}{2} \text{vol} S^{m-1}$$

for all $x \in N$. Then the topology of $M_i$ can be classified into the following two types:

(a) There exists a locally trivial fiber bundle

$$F_i \times I \rightarrow M_i \rightarrow N,$$

where $F_i$ is a closed almost nonnegatively curved manifold in a generalized sense as in [35];

(b) There exists a locally trivial fiber bundle

$$\text{Cap}_i \rightarrow M_i \rightarrow N,$$
where $\text{Cap}_i$ (resp. $\partial \text{Cap}_i$) is an almost nonnegatively curved manifold with boundary (resp. a closed almost nonnegatively curved connected manifold) in a generalized sense as in [35].

In general, the almost regularity of $N$ implies that of the limit of $\partial M_i$ (Proposition 4.30). However the converse if not true (see Example 3.18).

It should also be pointed out that several fibration theorem were obtained in [34] in some cases, where the nonnegativity of the second fundamental form $\Pi_{\partial M_i} \geq 0$, or the upper bound $K_{M_i} \leq \kappa^2$ and the lower bound for the injectivity radius $\text{inj}(M_i) \geq i_0 > 0$ were assumed.

Next we discuss the number of boundary components of inradius collapsed manifolds, where we do not assume the diameter bound.

**Theorem 1.5.** There exists a positive number $\epsilon = \epsilon_n(\kappa, \lambda)$ such that if $M$ in $\mathcal{M}(n, \kappa, \lambda)$ satisfies $\text{inrad}(M) < \epsilon$, then

1. the number $k$ of connected components of $\partial M$ is at most two;
2. if $k = 2$, then $M$ is diffeomorphic to $W \times [0, 1]$, where $W$ is a component of $\partial M$.

Theorem 1.5 (1) was stated in [34, Theorem 5]. However it seems to the authors that the argument there is unclear (see Remark 6.1). Theorem 1.5 may be considered as a generalization of a result in Gromov [11] and Alexander and Bishop [1], where an $I$-bundle structure was found for an inradius collapsed manifold under the two-sides bound on sectional curvature. It should be pointed out that the constants $\epsilon(\kappa, \lambda)$ in [11] and [1] are explicit and independent of $n$ while our constant $\epsilon_n(\kappa, \lambda)$ is neither. This is because our argument is by contradiction.

The organization and the outline of the proofs are as follows.

In section 2 we first recall basic notions and facts on the Gromov-Hausdorff convergence and Alexandrov spaces with curvature bounded below. Then we focus on Wong’s extension procedure of a Riemannian manifold with boundary by gluing a warped cylinder along their boundaries. By [18], the result of the gluing is a $C^{1,\alpha}$-manifold with $C^0$-Riemannian metric, and becomes an Alexandrov spaces with curvature uniformly bounded below. This construction is quite effective and used in an essential way in the present paper.

In section 3 we describe limit spaces of glued Riemannian manifolds with boundary. The limit spaces also have gluing structure. In this section we focus on the estimate of multiplicities of gluing, the intrinsic metric structure of the limit space and a general description of the limit spaces of extensions.

In Section 4 we determine the metric structure of limit spaces. First we study the spaces of directions of the limit space at gluing points, and prove that the gluing map preserves the length of curves. This implies that the gluing in the limit space is done metrically in a natural
manner, and yields significant structure results (see Theorem 4.32) on the limits including Theorem 1.2.

Those structure results are applied in Section 5 to obtain the fiber structures of inradius collapsed manifolds. Theorems 1.3 and 1.4 are proved there. To prove Theorem 1.3, we need to analyze the singularities of the singular $I$-fiber bundle in details. To prove Theorem 1.4, we apply an equivariant fibration-capping theorem in [37].

To prove Theorem 1.5, we consider the case of unbounded diameter in Section 6. Applying the results in Section 4, we obtain basically three types on local connectedness of the boundary of an inradius collapsed complete manifold, according to the types of the local limit spaces. After such local observation, Theorem 1.5 follows from a monodromy argument.

Our approach can be applied to the general case of non inradius collapse of Riemannian manifolds with boundary. As a continuation of the present paper, in [38], we obtain the structure of limit spaces, stabilities of topological types and volumes, in the general framework of non inradius collapse/convergence, and get an obstruction to the general collapse.

2. Preliminaries

In order to make the presented paper more accessible, we fix some basic definition, notations and conventions.

- $\tau(\delta)$ is a function which depends on $\delta$ such that $\lim_{\delta \to 0} \tau(\delta) = 0$.
- For topological spaces $X$ and $Y$, $X \approx Y$ means $X$ is homeomorphic to $Y$.
- The distance between two points $x, y$ in a metric space is denoted by $d(x, y)$, $|x, y|$ or $|xy|$.
- For a point $x$ and a subset $A$ of a metric space $X$, $B(x, r) = B^X(x, r)$ and $B(A, r) = B^X(A, r)$ denote open $r$-balls in $X$ around $x$ and $A$ respectively.
- For a metric space $(X, d)$, and $r > 0$, the rescaled metric space $(X, rd)$ is denoted by $rX$.
- The Euclidean cone $K(\Sigma)$ over a metric space $(\Sigma, \rho)$ is $\Sigma \times [0, \infty)$ equipped with the metric $d$ defined as
\[ d((x_1, t_1), (x_2, t_2)) = (t_1^2 + t_2^2 - 2t_1t_2\cos(\min\{\rho(x_1, x_2), \pi\}))^{1/2}, \]
for any two points $(x_1, t_1), (x_2, t_2) \in \Sigma \times [0, \infty)$.
- For a subspace $M$ of a metric space $(M, d_M)$, $M^{\text{ext}}$ denotes $(M, d_M)$, which is called the exterior metric of $M$.
- The metric $d$ of a connected metric space $(X, d)$ induces a length metric $d_{\text{int}}$ of $X$ defined as the infimum of the length of all curves joining given two points. We denote by $X^{\text{int}}$ the new metric space $(X, d_{\text{int}})$.
- The length of a curve $\gamma$ is denoted by $L(\gamma)$.
2.1. The Gromov-Hausdorff convergence. A (not necessarily continuous) map \( f : X \rightarrow Y \) between two metric spaces \( X \) and \( Y \) is called an \( \varepsilon \)-approximation if it satisfies

1. \( |d(x, y) - d(f(x), f(y))| < \varepsilon \), for all \( x, y \in Y \),
2. \( f(X) \) is \( \varepsilon \)-dense in \( Y \), i.e., \( B(f(X), \varepsilon) = Y \).

The Gromov-Hausdorff distance \( d_{GH}(X, Y) \) is defined as the infimum of those \( \varepsilon \) such that there are \( \varepsilon \)-approximations \( f : X \rightarrow Y \) and \( g : Y \rightarrow X \).

A map \( f : (X, x) \rightarrow (Y, y) \) between two pointed metric spaces is called a pointed \( \varepsilon \)-approximation if it satisfies

1. \( f(x) = y \),
2. \( |d(x, y) - d(f(x), f(y))| < \varepsilon \), for all \( x, y \in B^X(x, 1/\varepsilon) \),
3. \( f(B^X(x, 1/\varepsilon)) \) is \( \varepsilon \)-dense in \( B^Y(y, 1/\varepsilon) \).

The pointed Gromov-Hausdorff distance \( d_{pGH}((X, x), (Y, y)) \) is defined as the infimum of those \( \varepsilon \) such that there are pointed \( \varepsilon \)-approximations \( f : (X, x) \rightarrow (Y, y) \) and \( g : (Y, y) \rightarrow (X, x) \).

Consider a pair \((X, \Lambda)\) of a metric space \( X \) and a group \( \Lambda \) of isometries of \( X \). For such pairs \((X, \Lambda), (Y, \Gamma)\), a triple \((f, \varphi, \psi)\) of maps \( f : X \rightarrow Y \), \( \varphi : \Lambda \rightarrow \Gamma \) and \( \psi : \Gamma \rightarrow \Lambda \) is called an equivariant \( \varepsilon \)-approximation from \((X, \Lambda)\) to \((Y, \Gamma)\) if the following holds

1. \( f \) is an \( \varepsilon \)-approximation,
2. if \( \lambda \in \Lambda, x \in X \), then \( d(f(\lambda x), (\varphi \lambda)(fx)) < \varepsilon \),
3. if \( \gamma \in \Gamma, x \in X \), then \( d(f(\psi(\gamma)x), \gamma(fx)) < \varepsilon \).

The equivariant Gromov-Hausdorff distance \( d_{eGH}((X, \Lambda), (Y, \Gamma)) \) is defined as the infimum of those \( \varepsilon \) such that there are \( \varepsilon \)-approximations from \((X, \Lambda)\) to \((Y, \Gamma)\) and from \((Y, \Gamma)\) to \((X, \Lambda)\).

2.2. Alexandrov spaces. Let \( X \) be a geodesic metric space, where every two points of \( X \) can be joined by a shortest geodesic. For a fixed real number \( \kappa \) and a geodesic triangle \( \Delta pqr \) in \( X \) with vertices \( p, q \) and \( r \), denote by \( \Delta pqr \) a comparison triangle in the complete simply connected model surface \( M^2_\kappa \) with constant curvature \( \kappa \). This means that \( \Delta pqr \) has the same side lengths as the corresponding ones in \( \Delta pqr \).

Here we suppose that the perimeter of \( \Delta pqr \) is less than \( 2\pi/\sqrt{\kappa} \) if \( \kappa > 0 \).

The metric space \( X \) is called an Alexandrov space with curvature \( \geq \kappa \), sometimes Alexandrov space for short if we do not emphasis the lower curvature bound, if each point of \( X \) has a neighborhood \( U \) satisfying the following: For any geodesic triangle in \( U \) with vertices \( p, q \) and \( r \) and for any point \( x \) on the segment \( qr \), we have \( |px| \geq |\hat{x}r| \), where \( \hat{x} \) is the point on \( \hat{q}r \) corresponding to \( x \). From now on we assume that an Alexandrov space is always finite dimensional.

For an Alexandrov space \( X \) with curvature bounded below by \( \kappa \), let \( \alpha : [0, s_0] \rightarrow X \) and \( \beta : [0, t_0] \rightarrow X \) be two geodesics starting from a point \( x \). The angle between \( \alpha \) and \( \beta \) is defined by \( \angle(\alpha, \beta) = \)
\[ \lim_{s,t \to 0} \tilde{\alpha}(s) x \beta(t) \], where \( \tilde{\alpha}(s) x \beta(t) \) denotes the angle of a comparison triangle \( \tilde{\Delta}(s) x \beta(t) \) at the point \( \tilde{x} \). Two geodesics \( \alpha, \beta \) from \( x \in X \) is called equivalent if \( \angle(\alpha, \beta) = 0 \). We denoted by \( \Sigma'_x(X) \) the set of equivalent classes of geodesics emanating from \( x \). The space of directions at \( x \), denoted by \( \Sigma_x = \Sigma_x(X) \), is the completion of \( \Sigma'_x(X) \) with the angle metric. A direction of minimal geodesic from \( p \) to \( x \) is also denoted by \( \uparrow_p \). Let \( X \) be \( n \)-dimensional. Then \( \Sigma_x \) is an \((n-1)\)-dimensional compact Alexandrov space with curvature \( \geq 1 \).

A point \( x \in X \) is called regular if \( \Sigma_x \) is isometric to \( S^{n-1} \). Otherwise we call \( x \) a singular point. We denote by \( X^{\text{reg}} \) (resp. \( X^{\text{sing}} \)) the set of all regular points (resp. singular points) of \( X \).

The tangent cone at \( x \in X \), denoted by \( T_x(X) \), is the Euclidean cone \( K(\Sigma_x) \) over \( \Sigma_x \). It is known that \( T_x(M) = \lim_{r \to 0} (\frac{1}{r} M, x) \).

For a closed subset \( A \) of \( X \) and \( p \in A \), the space of directions \( \Sigma_p(A) \) of \( A \) at \( p \) is defined as the set of all \( \xi \in \Sigma_x(X) \) which can be written as the limit of directions from \( p \) to points \( p_i \) in \( A \) with \( |p, p_i| \to 0 \):

\[ \xi = \lim_{i \to \infty} \uparrow_{p_i}^p. \]

For \( x, y \in X \setminus A \), consider a comparison triangle on \( M^2_x \) having the side-lengths \( (|A, x|, |x, y|, |y, A|) \) whenever it exists. Then \( \angle x \) denotes the angle of this comparison triangle at the vertex corresponding to \( x \).

For \( x, y, z \in X \), we denote by \( \angle xyz \) (resp. \( \angle x\bar{y}z \)) the angle between the geodesics \( yx \) and \( yz \) at \( x \) (resp. the geodesics \( \bar{y}x \) and \( \bar{y}z \) at \( x \) in the comparison triangle \( \Delta xyz = \Delta x\bar{y}z \)).

Let \( X \) be an \( n \)-dimensional Alexandrov space with curvature bounded below by \( \kappa \). For \( \delta > 0 \), a system of \( n \) pairs of points, \( \{a_i, b_i\}_{i=1}^n \) is called an \((n, \delta)\)-strainer at \( x \in X \) if it satisfies

\[
\tilde{\alpha}(a_i x b_i) > \pi - \delta, \quad \tilde{\alpha}(a_i x a_j) > \pi/2 - \delta, \\
\tilde{\alpha}(b_i x b_j) > \pi/2 - \delta, \quad \tilde{\alpha}(a_i x b_j) > \pi/2 - \delta,
\]

for every \( 1 \leq i \neq j \leq n \). If \( x \in X \) has a \((n, \delta)\)-strainer, then we say \( x \) is \((n, \delta)\)-strained. In this case, we call \( x \) \( \delta \)-regular. We call \( X \) almost regular if every point of \( X \) is \( \delta_n \)-regular for some \( \delta_n < 1/100n \). It is known that a small neighborhood of any almost regular point is almost isometric to an open subset in \( \mathbb{R}^n \).

Inductively on the dimension, the boundary \( \partial X \) is defined as the set of points \( x \in X \) such that \( \Sigma_x \) has non-empty boundary \( \partial \Sigma_x \). We denote by \( D(X) \) the double of \( X \), which is also an Alexandrov space with curvature \( \geq \kappa \) (see [23]). By definition, \( D(X) = X \amalg \partial X \), where tow copies of \( X \) are glued along their boundaries.

A boundary point \( x \in \partial X \) is called \( \delta \)-regular if \( x \) is \( \delta \)-regular in \( D(X) \). We say that \( X \) is almost regular with almost regular boundary if every point of \( X \) is \( \delta \)-regular in \( D(X) \) for \( \delta < 1/100n \).
In Section 5.1, we need the following result on the dimension of the interior singular point sets. We set \( \text{int} X := X \setminus \partial X \).

**Theorem 2.1** ([4], cf. [21]).

\[
\dim_H (X^\text{sing} \cap \text{int} X) \leq n - 2, \quad \dim_H (\partial X)^\text{sing} \leq n - 2,
\]

where \((\partial X)^\text{sing} = D(X)^\text{sing} \cap \partial X\).

**Theorem 2.2** ([23], cf. [14]). If a sequence \( X_i \) of \( n \)-dimensional compact Alexandrov spaces with curvature \( \geq \kappa \) Gromov-Hausdorff converges to an \( n \)-dimensional compact Alexandrov space \( X \), then \( X_i \) is homeomorphic to \( X \) for large enough \( i \).

A subset \( E \) of an Alexandrov space \( X \) is called extremal ([25]) if every distance function \( f = \text{dist}_q, \ q \in M \setminus E \) has the property that if \( f|_E \) has a local minimum at \( p \in E \), then \( df_p(\xi) \leq 0 \) for every \( \xi \in \Sigma_p(X) \). Extremal subsets possess quite important properties.

**Theorem 2.3** ([25]). Let \( E \) be an extremal subset of \( X \).

1. For every \( p \in E \), \( \Sigma_p(E) \) is an extremal subset of \( \Sigma_p(X) \);
2. \( E \) is totally quasigeodesic in the sense that any nearby two points of \( E \) can be joined by a quasigeodesic (see [26]).
3. \( E \) has a topological stratification.

**Theorem 2.3(1), (2)** implies the following

**Corollary 2.4.** For an extremal subset \( E \) of \( X \) and \( p \in E \), \( \dim \Sigma_p(E) \leq \dim E - 1 \).

Suppose that a compact group \( G \) acts on \( X \) as isometries. Then the quotient space \( X/G \) is an Alexandrov space ([4]). Let \( F \) denote the set of \( G \)-fixed points.

**Proposition 2.5** ([25] Prop 5.10). \( \pi(F) \) is an extremal subset of \( X/G \), where \( \pi : X \to X/G \) is the projection.

Boundaries of Alexandrov spaces are typical examples of extremal subsets.

**Proposition 2.6** ([37] Prop 5.10). The boundary \( \partial X \) of any finite dimensional Alexandrov space \( X \) has a collar neighborhood.

An \( n \)-dimensional Alexandrov space is called smoothable if it is a Gromov-Hausdorff limit of \( n \)-dimensional closed Riemannian manifolds with a uniform lower sectional curvature bounds.

**Theorem 2.7** ([13]). Let \( X \) be a smoothable Alexandrov space. Then for any \( p \in X \), every iterated space of directions

\[
\Sigma_{\xi_k}(\Sigma_{\xi_{k-1}}(\cdots (\Sigma_{\xi_1}(\Sigma_p(X)) \cdots)),
\]

is homeomorphic to a sphere, where

\[
\xi_1 \in \Sigma_p(X), \ \xi_2 \in \Sigma_{\xi_1}(X), \ldots, \ \xi_k \in \Sigma_{\xi_{k-1}}(\cdots (\Sigma_{\xi_1}(\Sigma_p(X)) \cdots))
\]
2.3. Manifolds with boundary and gluing. In this section, we consider a Riemannian manifold $M$ with boundary in $\mathcal{M}(n, \kappa, \lambda, d)$. First, we recall some fundamental properties of $\partial M$, which were derived by Wong [33]. We also recall Wong’s cylindrical extension procedure based on Kosovskii’s Gluing theorem [18].

Let $M$ be a Riemannian manifold with boundary, and $\partial M^a$ denote a boundary component of $\partial M$. $(\partial M^a)^{\text{int}}$ means $\partial M^a$ with intrinsic length metric.

The following is an immediate consequence of the Gauss equation.

**Proposition 2.8.** For every $M \in \mathcal{M}(n, \kappa, \lambda)$, $\partial M$ has a uniform lower sectional curvature bound: $K_{\partial M} \geq K$, where $K = K(\kappa, \lambda)$.

**Proposition 2.9** ([33]). Let $M \in \mathcal{M}(n, \kappa, \lambda, d)$.

1. There exists a constant $D = D(n, \kappa, \lambda, d)$ such that any boundary component $\partial M^a$ has intrinsic diameter bound $\text{diam}((\partial M^a)^{\text{int}}) \leq D$;

2. $\partial M$ has at most $J$ components, where $J = J(n, \kappa, \lambda, d)$;

It follows from Proposition 2.9 that every boundary component of $M \in \mathcal{M}(n, \kappa, \lambda, d)$ is an Alexandrov space with curvature $\geq \kappa$ and diameter $\leq D$, where $K = K(\kappa, \lambda), D = D(n, \kappa, \lambda, d)$.

In general, a Riemannian manifold with boundary is not necessarily an Alexandrov space. Wong ([33]) carried out a gluing of warped cylinders and $M$ along their boundaries in such a way that the resulting manifold becomes an Alexandrov space having totally geodesic boundary.

This is based on Kosovskii’s gluing theorem:

**Theorem 2.10** ([18]). Let $M_0$ and $M_1$ be Riemannian manifolds with boundaries $\Gamma_0$ and $\Gamma_1$ respectively with sectional curvature $K_{M_i} \geq \kappa$ for $i = 0, 1$. Assume that there exists an isometry $\phi : \Gamma_0 \rightarrow \Gamma_1$, and let $M$ denote the space with length metric obtained by gluing $M_0$ and $M_1$ along their boundaries via $\phi$. Let $L_i, i = 0, 1$, be the second fundamental form of $\Gamma := \Gamma_0 \cong \phi \Gamma_1 \subset M$ with respect to the normal inward to $M_1$. Then $M$ is an Alexandrov space with curvature $\geq \kappa$ if and only if the sum $L := L_1 + L_2$ is positive semidefinite.

**Remark 2.11.** Actually, for every $\delta > 0$, a smooth Riemannian metric $g_\delta$ on $M$ is constructed in [18] in such a way that the sectional curvature of $g_\delta$ is greater than $\kappa(\delta)$ with $\lim_{\delta \rightarrow 0} \kappa(\delta) = \kappa$ and that $(M, g_\delta)$ Gromov-Hausdorff converges to $M$ as $\delta \rightarrow 0$.

Now let us recall the extension construction in [33].

Suppose $M$ is an $n$-dimensional complete Riemannian manifold with $K_M \geq \kappa, \lambda^- \leq \Pi_{\partial M} \leq \lambda^+$. 
Let $\lambda := \min\{0, \lambda^-\}$. Then for arbitrarily $t_0 > 0$ and $0 < \varepsilon_0 < 1$ there exists a monotone non-increasing function $\phi : [0, t_0] \to \mathbb{R}^+$ satisfying
\[
\phi''(t) + K\phi(t) \leq 0, \quad \phi(0) = 1, \quad \phi(t_0) = \varepsilon_0,
\]
for some constant $K = K(\lambda, \varepsilon_0, t_0)$. Now consider the warped product metric on $\partial M \times [0, t_0]$ defined by
\[
g(x, t) = dt^2 + \phi^2(t)g_{\partial M}(x)
\]
where $g_{\partial M}$ is the Riemannian metric of $\partial M$ induced from that of $M$. We denote by $\partial M \times \phi [0, t_0]$ the warped product. It follows from the construction that
\[
\begin{align*}
&\cdot II_{\partial M \times \{0\}} \geq |\min\{0, \lambda^-\}|, \\
&\cdot II_{\partial M \times \{t_0\}} \equiv 0,
\end{align*}
\]
the sectional curvature of $\partial M \times \phi [0, t_0]$ is greater than a constant $c(\kappa, \lambda^+, \varepsilon_0, t_0)$
\[
\cdot \text{the second fundamental form of } \partial M \times \{t\} \text{ is given by}
\]
\[
II_{\partial M \times \{t\}}(V, W) = \frac{\phi'(t)}{\phi(t)}g(V, W),
\]
for vector fields $V, W$ on $\partial M \times \{t\}$.
Clearly, $\partial M \times \{0\}$ in $\partial M \times \phi [0, t_0]$ is canonically isometric to $\partial M$. Thus we can glue $M$ and $\partial M \times \phi [0, t_0]$ along $\partial M$ and $\partial M \times \{0\}$. The resulting space
\[
\tilde{M} := M \amalg_{\partial M} (\partial M \times \phi [0, t_0])
\]
carries the structure of differentiable manifold of class $C^{1, \alpha}$ with $C^0$-Riemannian metric \cite{18}. Obviously $M$ is diffeomorphic to $\tilde{M}$.

**Proposition 2.12** \cite{33}. For $M \in \mathcal{M}(n, \kappa, \lambda)$, we have
\begin{enumerate}
\item $\tilde{M}$ is an Alexandrov space with curvature $\geq \tilde{\kappa}$, where $\tilde{\kappa} = \tilde{\kappa}(\kappa, \lambda)$;
\item the exterior metric $M_{\text{ext}}$ is $L$-bi-Lipschitz homeomorphic to $M$ for the uniform constant $L = 1/\varepsilon_0$;
\item $\text{diam}(\tilde{M}) \leq \text{diam}(M) + 2t_0$.
\end{enumerate}

The notion of warped product also works for metric spaces.

Let $X$ and $Y$ be metric spaces, and $\phi : Y \to \mathbb{R}_+$ a positive continuous function. Then the warped product $X \times_\phi Y$ is defined as follows (see \cite{32}). For a curve $\gamma = (\sigma, \nu) : [a, b] \to X \times Y$, the length of $\gamma$ is defined as
\[
L_\phi(\gamma) = \sup_{|\Delta| \to 0} \sum_{i=1}^k \sqrt{\phi^2(\nu(s_i))|\sigma(t_{i-1}) - \sigma(t_i)|^2 + |\nu(t_{i-1}) - \nu(t_i)|^2},
\]
where $\Delta : a = t_0 < t_1 < \cdots < t_k = b$ and $s_i$ is any element of $[t_{i-1}, t_i]$. The warped product $X \times_{\phi} Y$ is defined as the topological space $X \times Y$ equipped with the length metric induced from $L_{\phi}$.

**Proposition 2.13** ([32], Proposition B.2.6). Let $X_i$ be a convergent sequence of length spaces. If $Y$ is a compact length space, we have
\[
\lim_{i \to \infty} GH(X_i \times_{\phi} Y) = (\lim_{i \to \infty} GH X_i) \times_{\phi} Y.
\]
whenever the limits exist.

### 3. Descriptions of limit spaces and examples

Under the notations in section 2.3, throughout this section unless otherwise stated, we assume $M_i \in \mathcal{M}(n, \kappa, \lambda, d)$ Gromov-Hausdorff converges to a compact length space $N$, where $\text{inrad}(M_i) \to 0$. Let $\hat{M}_i$ converge to a compact Alexandrov space $Y$, and $M_i^{\text{ext}}$ converge to a closed subset $X$ of $Y$ under the convergence $\hat{M}_i \to Y$.

Here we fix some notations used later on.

- $C_{M_i}$ denotes $\partial M_i \times_{\phi} [0, t_0]$;
- $C_{M_i, t}$ denotes the subspace $\partial M_i \times_{\phi} \{t\}$ in $C_{M_i}$;
- For $C_{M_i} \subset \hat{M}_i$, $C_{M_i}^{\text{ext}}$ denotes $(C_{M_i}, d_{\hat{M}_i})$.

In this section, we first investigate the relation between the limit $C$ (resp. $C_0$) of $C_{M_i}$ (resp of $\partial M_i$) and $Y$ (resp. $X$), and discuss the intrinsic structure of $X$ and prove that $X^{\text{int}}$ is isometric to $N$ (Proposition 3.8). Then we describe the metric structure of $Y$ (Proposition 3.9).

#### 3.1. Descriptions of $X$ and $Y$

Under the notation presented in the beginning of this section, in view of Proposition 2.9 and (2.1), passing to a subsequence, we may assume that $C_{M_i}$ converges to some compact Alexandrov space $C$ with curvature $\geq K = K(\kappa, \lambda)$. Here $C_{M_i}$ is not necessarily connected, and therefore the convergence $C_{M_i} \to C$ should be understood componentwisely. It follows from Proposition 2.13 that
\[
C = C_0 \times_{\phi} [0, t_0], \quad C_0 = \lim_{i \to \infty} (\partial M_i)^{\text{int}},
\]
where $(\partial M_i)^{\text{int}}$ denotes $\partial M_i$ endowed with length metric induced by its original metric. For simplicity we denote
\[
C_0 := C_0 \times \{0\}, \quad C_t := C_0 \times \{t\} \subset C,
\]
Since the identity map $\iota_t : C_{M_i} \to C_{M_i}^{\text{ext}}$ is 1-Lipschitz, we can define a surjective 1-Lipschitz map $\eta : C \to Y$ in the limits. More precisely, define $\eta : C \to Y$ by
\[
\eta = \lim_{i \to \infty} g_i \circ \iota_i \circ f_i,
\]
where $f_i : C \to C_{M_i}$, $g_i : \hat{M}_i \to Y$ are component-wise $\varepsilon_i$-approximations with $\lim \varepsilon_i = 0$. 
From now on, we consider

$$\eta_0 := \eta|_{C_0 \times \{0\}} : C_0 \to X,$$

which is also a surjective 1-Lipschitz map with respect to the exterior metrics of $C_0$ and $X$, and hence with respect to the interior metrics, too.

The following two lemmas are obvious.

**Lemma 3.1.** The map $\eta : C \setminus C_0 \to Y \setminus X$ is a bijective local isometry.

**Lemma 3.2.** For $(p, t) \in C \setminus C_0$, we have $|\eta(p, t), X| = t$.

We now study the multiplicities of the gluing map $\eta_0$.

**Lemma 3.3.** For every $x \in X$, we have the following:

1. $\#\eta_0^{-1}(x) \leq 2$;
2. Suppose $\#\eta_0^{-1}(x) = 2$ for some $x \in X$, and take $p_k \in C_0$, $k = 1, 2$, with $\eta_0(p_k) = x$. Then $\Sigma_x(Y)$ is isometric to a spherical suspension with the two vertices $\{\xi_1, \xi_2\}$, where

$$\xi_k := \frac{1}{2}\eta(p_k, t_0).$$

**Proof.** Suppose that $\#\eta_0^{-1}(x) \geq 2$ and take $p_1, p_2 \in \eta_0^{-1}(x)$, and let $y_i := \eta(p_i, t)$, $i = 1, 2$, for some $t > 0$. We show that $|y_1, y_2| = 2t$ or equivalently,

$$\tilde{y}_1py_2 = \pi,$$

if $t < \phi(t_0)|p_1p_2|_{C_0}^\text{int}/2$, which yields $\#\eta_0^{-1}(x) = 2$ and the conclusions (1) and (2).

Let $\gamma : [0, \ell] \to Y$ be a minimal geodesic in $Y$ joining $y_1$ and $y_2$. If $\gamma$ meets $X$, we certainly have $|y_1, y_2| = 2t$. Suppose that $\gamma$ does not meet $X$. Then $\tilde{\gamma} = \eta^{-1}(\gamma)$ is well-defined and is a minimal geodesic joining $(p_1, t)$ and $(p_2, t)$. Write $\tilde{\gamma}$ as $\tilde{\gamma}(s) = (\sigma(s), \nu(s)) \in C_0 \times_{\phi} [0, t_0]$. Then we have

$$L(\gamma) = L(\tilde{\gamma}) = \int_0^\ell \sqrt{\phi^2(\nu(s))|\dot{\sigma}(s)|^2 + |\dot{\nu}(s)|^2} \, ds.$$

$$\geq \int_0^\ell \phi(t_0)|\dot{\sigma}(s)| \, dt \geq \phi(t_0)|p_1, p_2|_{C_0}^\text{int}.$$

Thus we have $|y_1, y_2| = L(\gamma) \geq \phi(t_0)|p_1, p_2|_{C_0}^\text{int}$. On the other hand, the triangle inequality shows that $|y_1, y_2| \leq 2t < \phi(t_0)|p_1, p_2|_{C_0}^\text{int}$. This is a contradiction, and therefore $\gamma$ meets $X$ and $|y_1, y_2| = 2t$.  

Next we construct a good approximation map $\tilde{M}_i \to Y$, which helps us to grasp a whole picture on the several convergences.

Let $\psi_i : \partial M_i = C_{M_i,0} \to C_0$ be an $\varepsilon_i$-approximation with $\lim_{i \to \infty} \varepsilon_i = 0$.  

Lemma 3.4. The map $\Psi_i : C_{M_i} \to C$ defined by
$$\Psi_i(p, t) = (\psi_i(p), t)$$
is an $\epsilon_i$-approximation with $\lim_{i \to \infty} \epsilon_i = 0$. Actually, for any approximation map $\Psi : C_{M_i} \to C$ there is a $\psi_i : \partial M_i = C_{M_i, 0} \to C_0$ such that
$$|\Psi_i(p, t), \Psi(p, t)| < \epsilon_i$$for all $\Psi_i = (\psi_i, \text{id})$.

Proof. This follows from Proposition 2.13.

Recall that $\eta : C \setminus C_0 \to Y \setminus X$ is a locally isometric bijection. In particular for every $y = (p, t_0) \in C_{t_0} \subset Y$, there is a unique minimal geodesic $\gamma_y : [0, t_0] \to Y$ between $X$ and $y$ such that $\gamma_y(0) \in X$, $\gamma(t_0) = y$. Actually $\gamma_y$ is defined as $\gamma_y(t) = \eta(p, t)$. Define $g_i^* : C_{M_i}^{\text{ext}} \to Y$ by

$$(3.3) \quad g_i^*(p, t) = \eta \circ \Psi_i \circ i_i^{-1}(p, t) = \eta(\psi(p), t).$$

Proposition 3.5. The map $g_i^* : C_{M_i}^{\text{ext}} \to Y$ defined above provides an $\epsilon_i$-approximation.

Let $g_i : C_{M_i} \to Y$ be any $\epsilon_i$-approximation such that $g_i = g_i^*$ on $C_{M_i, t_0}$, namely $g_i(p, t_0) = g_i^*(p, t_0)$.

For the proof of Proposition 3.5, it suffices to show the following.

Lemma 3.6. $|g_i(p, t), g_i^*(p, t)| < \epsilon_i$ for all $(p, t) \in C_{M_i}^{\text{ext}}$.

Proof. We have to show that
$$\lim_{i \to \infty} \sup_{(p, t) \in C_{M_i}} |g_i(p, t), g_i^*(p, t)| = 0.$$Suppose the contrary. Then there is a subsequence $\{i_j\} \subset \{i\}$ and $(p_j, t_j) \in C_{M_j}$ such that

$$(3.4) \quad |g_i(p_j, t_j), g_i^*(p_j, t_j)| \geq c > 0,$$
for some constant $c$ independent of $j$. Passing to a subsequence, we may assume that $(\psi_j(p_j), t_j)$ converges to $(p_\infty, t_\infty) \in C$. Let $\gamma_j(t) = (p_j, t)$, $0 \leq t \leq t_0$, which is a minimal geodesic in $C_{M_j}$ between $\partial M_j$ and $C_{M_j, t_0}$. Now $g_j^* \circ \gamma_j(t) = \eta(\psi(p_j), t)$ converges to a minimal geodesic $\gamma_\infty(t) = \eta(p_\infty, t)$ realizing the distance between $X$ and $(p_\infty, t_0) \in C_{t_0} \subset Y$. Since $g_j$ is $\epsilon_j$-approximation, any limit of $g_j \circ \gamma_j$, say $\gamma_\infty$, must also be minimal geodesic between $X$ and $(p_\infty, t_0)$. From the uniqueness of such geodesic, we have $\gamma_\infty(t) = \gamma_\infty(t)$, which contradicts (3.4).

Next, we determine the intrinsic structure of $X$, and prove Proposition 3.8 below, which will be crucial in our start for the description of $Y$ in terms of $N$ (see Proposition 3.9).

Recall that $X \subset Y$ is the limit of $M_i^{\text{ext}}$ under the convergence $\hat{M}_i \to Y$. By Proposition 2.12, the identity $\iota_i : M_i \to M_i^{\text{ext}}$ is a $L$-bi-Lipschitz homeomorphism. Therefore we have that

Lemma 3.7. For a subsequence, $\iota_i : M_i \to M_i^{\text{ext}}$ converges to an $L$-bi-Lipschitz homeomorphism $\iota_\infty : N \to X$. 

Proposition 3.8. $X^{\text{int}}$ is isometric to $N$.

Proof. Passing to a subsequence if necessary, we may assume that the $L$-Lipschitz map $\iota_i : M^\text{ext}_i \to M_i$, where $L = 1/\epsilon_0$, converges to a surjective map $h : X \to N$ satisfying

$$|x, y|_X \leq |h(x), h(y)|_N \leq L|x, y|_X,$$

for every $x, y \in X$. Let $\sigma : [0, d] \to N$ be a minimal geodesic joining $h(x)$ and $h(y)$. Then we have

$$|h(x), h(y)|_N = L(\sigma) \geq L(h^{-1}(\sigma)) \geq |x, y|_{X^{\text{int}}}.$$

Next we show the reverse inequality. Let $\gamma : [0, \ell] \to X$ be a minimal geodesic in $X^{\text{int}}$ joining $x$ to $y$. For any $\epsilon > 0$, take a subdivision $\Delta$ of $\gamma : x = x_0 < x_1 < \cdots < x_n < \cdots x_k = y$ such that denoting by $\gamma_\Delta$ the broken geodesic consisting of minimal geodesic joining $x_{\alpha - 1}$ and $x_\alpha$ in $Y$ for $1 \leq \alpha \leq k$, we have

1. $|L(\gamma_\Delta) - |x, y|_{X^{\text{int}}}| < \epsilon$;
2. $\max_t |\gamma_\Delta(t), X| < \epsilon$.

Take $p_\alpha^i \in M_i$ converging to $x_\alpha$ under the convergence $\hat{M}_i \to Y$, and denote by $\gamma_\Delta^i$ a broken geodesic consisting of minimal geodesic joining $p_{\alpha - 1}^i$ and $p_\alpha^i$ in $\hat{M}_i$ for $1 \leq \alpha \leq k$. Note that for large enough $i$

1. $|L(\gamma_\Delta) - L(\gamma_\Delta^i)| < \epsilon$;
2. $\max_t |\gamma_\Delta^i(t), M_i| < \epsilon$.

Let $\sigma_i := \pi_i \circ \gamma_\Delta^i$, where $\pi_i : \hat{M}_i \to M_i$ is the canonical projection defined by $\pi_i(p, t) = p$. From the warped product metric construction, we have $L(\gamma_\Delta^i) \geq \phi(\epsilon)L(\sigma_i)$ for large $i$. It follows that

$$|x, y|_{X^{\text{int}}} \geq L(\gamma_\Delta) - \epsilon > L(\gamma_\Delta^i) - 2\epsilon \geq \phi(\epsilon)L(\sigma_i) - 2\epsilon \geq \phi(\epsilon)|p_i, q_i|_{M_i} - 2\epsilon,$$

where $p_i \to x$ and $q_i \to y$ under $\hat{M}_i \to Y$. Letting $|\Delta| \to 0$ and $i \to \infty$, we conclude that $|x, y|_{X^{\text{int}}} \geq |h(x), h(y)|_N$. This completes the proof. \hfill \Box

Let $X^{\text{int}} \cup_{\eta_0} C_0 \times_{\phi} [0, t_0]$ denote the length space obtained by the result of gluing of the two length spaces $X^{\text{int}}$ and $C_0 \times_{\phi} [0, t_0]$ by the map $\eta_0 : C_0 \times 0 \to X^{\text{int}}$.

Proposition 3.9. $Y$ is isometric to the length space $X^{\text{int}} \cup_{\eta_0} C_0 \times_{\phi} [0, t_0]$.

Proof. Let $Z := X^{\text{int}} \cup_{\eta_0} C_0 \times_{\phi} [0, t_0]$, and $\Phi : Y \to Z$ the canonical map. Note that $\Phi$ is bijective. For every $y_0, y_1 \in Y$, let $\gamma : [0, \ell] \to Y$ be a minimal geodesic joining $y_0$ and $y_1$. Decompose $\gamma$ into the two parts:

$$\gamma = \gamma_{\gamma X} \cup \gamma_{X},$$
where $\gamma_{Y\setminus X} = \gamma \cap (Y \setminus X)$ and $\gamma_X = \gamma \cap X$. Let $\gamma_{Y\setminus X} = \bigcup_{\alpha} \gamma_{\alpha}$ be the at most countable union consisting of open arc components of $\gamma_{Y\setminus X}$. For any $\epsilon > 0$, take $\gamma_{\alpha}$ of length $\leq \epsilon$ such that the endpoints $z_\alpha$ and $w_\alpha$ of $\gamma_{\alpha}$ are contained in $X$ if such a $\gamma_{\alpha}$ exists. Take $p_\i, q_\i \in M_\i$ such that $p_\i \to z_\alpha$, $q_\i \to w_\alpha$ under the convergence $M_\i \to Y$. For a minimal geodesic $\gamma_\i$ joining $p_\i$ and $q_\i$ in $M_\i$, let $\sigma_\i := \pi_\i(\gamma_\i)$, where $\pi_\i : M_\i \to M_\i$ is the projection. Note that $\max |\gamma_\i(t), M_\i| < \epsilon$. Using the warped metric structure, we have $L(\gamma_\i) \geq \phi(\epsilon)L(\sigma_\i)$, which implies

$$|z_\alpha, w_\alpha|_Y \geq |p_\i, q_\i|_{\tilde{M}_\i} - \epsilon \geq \phi(\epsilon)|p_\i, q_\i|_{\tilde{M}_\i} - \epsilon,$$

where $\lim \epsilon_\i = 0$. Letting $i \to \infty$, we have $|z_\alpha, w_\alpha|_Y \geq \phi(\epsilon)|z_\alpha, w_\alpha|_{X^{\text{int}}}$. Now we replace $\gamma_{\alpha}$ by a minimal geodesic joining $z_\alpha$ and $w_\alpha$ in $X^{\text{int}}$. Repeating this procedure at most countably many times if necessarily, we construct a Lipschitz curve $\tilde{\gamma}$ joining $y_0$ to $y_1$ such that in the decomposition $\tilde{\gamma} = \tilde{\gamma}_{Y\setminus X} \cup \tilde{\gamma}_X$, $\tilde{\gamma}_{Y\setminus X}$ (resp. $\tilde{\gamma}_X$) consists of finitely many $Y$-minimal geodesics each of whose length $\geq \epsilon$ (resp. finitely many $X$-minimal geodesic) and that

$$|y_0, y_1|_Y = L(\gamma) \geq \phi(\epsilon)L(\tilde{\gamma}) \geq \phi(\epsilon)|\Phi(y_0), \Phi(y_1)|_Z.$$

Letting $\epsilon \to 0$, we conclude that $|y_0, y_1|_Y \geq |\Phi(y_0), \Phi(y_1)|_Z$.

Next taking a $Z$-minimal geodesic joining $\Phi(y_0)$ and $\Phi(y_1)$ and replacing it by a Lipschitz curve in a similar way, we obtain the reverse inequality $|y_0, y_1|_Y \leq |\Phi(y_0), \Phi(y_1)|_Z$. This completes the proof. □

**Remark 3.10.** Both Propositions 3.8 and 3.9 hold true for pointed Gromov-Hausdorff limits of inradius collapsed manifolds (see Section 6). Moreover, in the above proofs, we do not need the assumption of inradius collapse. Therefore Propositions 3.8 and 3.9 also hold for Gromov-Hausdorff limits of non-inradius collapsed manifolds.

**Corollary 3.11.** If $M_\i \in \mathcal{M}(n, \kappa, \lambda, d)$ inradius collapses to $N$, then it actually collapses to $N$. Namely we have

1. $\dim M_\i > \dim N$;
2. $\lim \text{vol}(M_\i) = 0$.

**Proof.** (1) From Lemma 3.7 and Proposition 3.9 we have

$$\dim M_\i = \dim \tilde{M}_\i \geq \dim Y \geq \dim X + 1 = \dim N + 1.$$

(2) We proceed by contradiction. Suppose $\text{vol}(M_\i) > v_\i > 0$ for some constant $v_\i$ independent of $\i$. By Proposition 2.12 there is a uniform bound $V$ with $\text{vol}(\partial M_\i) \leq V$. Choose any $v_0 \in (0,1)$ and $t_0 \in (0, v_0/2V)$, and perform the extension procedure with warping function as in Subsection 2.3. Then $C_{M_\i}$ has volume

$$\text{vol}(C_{M_\i}) < Vt_0 < \frac{v_0}{2}.$$
Passing to a subsequence, we may assume that $\tilde{M}_i$ converges to $Y$. Since $\text{vol}(\tilde{M}_i) \geq v_0$, we have $\dim Y = n$. It follows from the volume convergence, 

$$\text{vol}(Y) = \lim \text{vol}(\tilde{M}_i) \geq v_0.$$ 

However 

$$\text{vol}(Y) = \text{vol}(Y \setminus X) + \text{vol}(X) = \text{vol}(C_0 \times_{\phi} [0, t_0]) < V_0 t_0 \leq v_0/2,$$

which is a contradiction. □

Remark 3.12. Wong proved $\dim M_i > \dim N$ in [34, Lemma 1] under the condition that $N$ is an absolute Poincaré duality space. In [38], we shall show that if $N$ is a closed topological manifold or a closed Alexandrov space, then $M_i$ inradius collapses. Hence Corollary 3.11 give another version of Wong’s result. It should also be noted that the conclusion of Corollary 3.11 holds for limit spaces of inradius collapsed manifolds with respect to the pointed Gromov-Hausdorff topology (see Corollary 6.2).

Definition 3.13. In view of Lemma 3.3 and Proposition 3.9, we make an identification $N = X^{\text{int}}$ and set for $k = 1, 2,$ 

$$N_k = Xk := \{x \in X | \#\eta_0^{-1}(x) = k\},$$ 

$$C_0^k := \{p \in C_0 | \eta_0(p) \in X_k\}.$$

3.2. Examples. We exhibit some examples of collapse of manifolds with boundary. All the examples except Example 3.20 are inradius collapses.

Example 3.14. Let $S^{n-1}(r) := \{x \in \mathbb{R}^n | \sum_{i=1}^n (x_i)^2 = r^2\}$. For $\epsilon > 0$, define $\tilde{M}_\epsilon$ as the closed domain in $\mathbb{R}^n$ bounded by $S^{n-1}(r + \epsilon)$ and $S^{n-1}(r)$. Then $K_{\tilde{M}_\epsilon} \equiv 0$ and $|\Pi_{BM}| \leq 1/r$, and $M_\epsilon$ inradius collapses to $N := S^{n-1}(r)$, where the limit space is an Alexandrov space with curvature $\geq r^{-2}$. Note that $N_2 = N$, and that the limit $Y$ of $M_\epsilon$ is isometric to the form 

$$Y = (S^{n-1}(r) \amalg S^{n-1}(r)) \times_{\phi} [0, t_0]/(f(x), 0) \sim (x, 0),$$

where $f : S^{n-1}(r) \amalg S^{n-1}(r) \to S^{n-1}(r) \amalg S^{n-1}(r)$ is the canonical involution. Equivalently $Y$ is isometric to the warped product 

$$S^{n-1}(r) \times_{\tilde{\phi}} [-t_0, t_0],$$

where $\tilde{\phi}(t) = \phi(|t|)$.

This example shows that the lower Alexandrov curvature bound of the limit in Theorem 1.2 really depends on the bound $\lambda \geq |\Pi_{BM}|$.

Example 3.15 ([34]). Let $N \subset \mathbb{R}^2 \times 0 \subset \mathbb{R}^3$ be a non-convex domain with smooth boundary, and let $M'_\epsilon$ denote the closure of $\epsilon$-neighborhood of $N$ in $\mathbb{R}^3$. After a slight smoothing of $M'_\epsilon$, we obtain a flat Riemannian
manifold $M_\epsilon$ with boundary such that $\Pi_{\partial M_\epsilon} \geq -\lambda$ for some $\lambda > 0$ independent of $\epsilon$. Note that $M_\epsilon$ inradius collapses to $N$, where $N$ has no lower Alexandrov curvature bound.

This example shows that Theorem 1.2 does not hold if one drops the upper bound $\lambda \geq \Pi_{\partial M}$.

**Example 3.16.** Let $N \subset \mathbb{R}^2$ be the union of the unit circle $\{(x, y) | x^2 + y^2 = 1\}$ and the segment $\{(x, y) | x = 0, -1 \leq y \leq 1\}$. Let $M_\epsilon$ be the intersection of the closed $\epsilon$-neighborhood of $N$ in $\mathbb{R}^2$ and the unit disk $\{(x, y) | x^2 + y^2 \leq 1\}$. After slight smoothing of $M_\epsilon$, it is a compact surface with $K_{M_\epsilon} \equiv 0, \Pi_{\partial M_\epsilon} \leq \lambda^2$ for some $\lambda$. However $\inf \Pi_{\partial M_\epsilon} \to -\infty$ as $\epsilon \to 0$, and $M_\epsilon$ inradius collapses to $N$, which is not an Alexandrov space with curvature bounded below.

This example shows that Theorem 1.2 does not hold if one drops the lower bound $-\lambda^2 \leq \Pi_{\partial M}$.

**Example 3.17.** Let $\pi : P \to N$ be a Riemannian double covering between closed Riemannian manifolds with the deck transformation $\varphi : P \to P$. Define $\Phi : P \times [-\epsilon, \epsilon] \to P \times [-\epsilon, \epsilon]$ by

$$\Phi(x, t) = (\varphi(x), -t),$$

and consider $M_\epsilon := P \times [-\epsilon, \epsilon]/\Phi$, which is a twisted $I$-bundle over $N$. Note that $M_\epsilon \in \mathcal{M}(n, \kappa, 0, d)$ for some $\kappa$ and $d$, and that $M_\epsilon$ inradius collapses to $N$ as $\epsilon \to 0$. In this case, we have $N_2 = N$. Note that the limit $Y$ of $\tilde{M}_\epsilon$ is isometric to the form

$$Y = P \times \phi [0, t_0]/(\varphi(x), 0) \sim (x, 0),$$

or equivalently $Y$ is doubly covered by the warped product

$$P \times \phi [-t_0, t_0].$$

**Example 3.18.** Let $N$ be a convex domain in $\mathbb{R}^{n-1} \times 0 \subset \mathbb{R}^{n+1}$ with smooth boundary. Let $M'_\epsilon$ denote the intersection of the boundary of $\epsilon$-neighborhood of $N$ in $\mathbb{R}^{n+1}$ with the upper half space $H_u = \{(x_1, \ldots, x_{n+1}) | x_{n+1} \geq 0\}$. After a slight smoothing of $M'_\epsilon$, we obtain a nonnegatively curved Riemannian manifold $M_\epsilon$ with totally geodesic boundary. Note that $M_\epsilon$ inradius collapses to $N$ as $\epsilon \to 0$. Note also that $(\partial M_\epsilon)^{\text{int}}$, a smooth approximation of the boundary of $\epsilon$-neighborhood of $N$ in $\mathbb{R}^n$, converges to the double $D(N)$ of $N$. It follows that $N_1 = \partial N$ and $N_2 = N \setminus \partial N$, and that the limit $Y$ of $\tilde{M}_\epsilon$ is isometric to the form

$$Y = D(N) \times \phi [0, t_0]/(r(x), 0) \sim (x, 0),$$

where $r : D(N) \to D(N)$ denotes the canonical reflection of $D(N)$.

Next let us consider more general examples. The following ones come from Example 1.2 in [35], where general examples of collapse of closed manifolds were given.
Example 3.19. Let ̂\(\pi: M \to N\) be a fiber bundle over a closed manifold \(N\) with fiber \(F\) having non-empty boundary and with the structure group \(G\) such that

1. \(G\) is a compact Lie group;
2. \(F\) has a \(G\)-invariant metric \(g_F\) of nonnegative curvature which smoothly extends to the double \(D(F)\);

Fix a bi-invariant metric \(b\) on \(G\) and a metric \(h\) on \(N\). Let \(\pi: P \to N\) be the principal \(G\)-bundle associated with ̂\(\pi: M \to N\). Define \(G\)-invariant metric \(g_\epsilon\) on \(P\) by
\[
g_\epsilon(u, v) = h(d\pi(u), d\pi(v)) + \epsilon^2 b(\omega(u), \omega(v)),
\]
where \(\omega\) is a \(G\)-connection on \(P\). Define a metric ̃\(g_\epsilon\) on \(P \times D(F)\) as
\[
\text{̃}g_\epsilon = g_\epsilon + \epsilon^2 g_F.
\]
For the \(G\)-action on \(P \times D(F)\) defined by \((p, f) \cdot g = (pg, g^{-1}f)\), \(\text{̃}g_\epsilon\) is \(G\)-invariant and invariant under the action of reflection of \(D(F)\). Therefore it induces a metric \(g_{D(M),\epsilon}\) on \(D(M) = P \times D(F)/G\). Since \(g_{D(M),\epsilon}\) is invariant under the action of reflection of \(D(M)\), it induces a metric \(g_{M,\epsilon}\) on \(M\) with totally geodesic boundary such that \((M, g_{M,\epsilon})\) inradius collapse to \((N, h)\) under a lower sectional curvature bound.

Example 3.20. Let \(M\) be a compact manifold with boundary, and suppose that a compact Lie group of positive dimension effectively acts on \(M\) which extends to the action on \(D(M)\). Suppose that \(D(M)\) has \(G\)-invariant and reflection-invariant smooth metric \(g\). As in Example 1.2 of [35], one can construct a metric \(g_{D(M),\epsilon}\) on \(D(M)\) which collapses to \((D(M), g_{D(M),\epsilon})/G\) under a lower curvature bound. It follows that the metric \((M, g_{M,\epsilon})\) induced by \(g_{D(M),\epsilon}\) also collapses to \((M, g_{M,\epsilon})/G\) under a lower curvature bound. Note that \((M, g_{M,\epsilon})\) has totally geodesic boundary.

4. Metric structure of limit spaces

Let \(X \subset Y\) and \(N\) be as in Section 3. The main purpose of this section is to show that \(Y\) and \(N\) are actually isometric to \(C/\eta_0\) and \(C_0/\eta_0\) respectively. To study how this gluing is made, we first analyze the tangent cones of \(C\), \(C_0\), \(Y\) and \(X\) at gluing points, and their relations via the the differential \(d\eta_0\) of the gluing map \(\eta_0\). It turns out that the identification map \(\eta_0\) preserves length of curves. Finally, we see that \(N\) is isometric to a quotient of \(C_0^{\text{int}}\) by an isometric \(\mathbb{Z}_2\) action. (see Proposition 4.30), which implies Theorems 1.2 and 4.32.

4.1. Preliminary argument. In this subsection, we study geodesic behavior in \(C\) and the property of a rescaling limit of the map \(\eta: C \to Y\). These will be useful in the next subsection to investigate geodesic behavior in \(Y\)
Let $\tilde{\pi} : C \to C_0$ and $\pi : Y \to X$ be the projections. To be precise, $\pi(y) := \eta_0 \circ \tilde{\pi}(\eta^{-1}(y))$, which are surjective Lipschitz maps. For every $p \in C_0$, let $\tilde{\gamma}_+(t) = (p, t)$ and $\gamma_+(t) = \eta(\tilde{\gamma}(t)), t \in [0, t_0]$. We call $\tilde{\gamma}_+$ (resp. $\gamma_+$) a perpendicular to $C_0$ (resp. to $X$) at $p$ (resp. at $\eta_0(p)$). The map $\tilde{\pi}$ and $\pi$ are the projections along perpendiculars. Note that $\eta : C \setminus C_0 \to Y \setminus X$ is a locally isometric bijective map. Therefore $C \setminus C_0$ and $Y \setminus X$ are isometric to each other with respect to the their length metrics.

For simplicity, we use the notation

$$C_t := \{x \in C \mid d(C_0, x) = t\}, \quad C_t^Y := \{y \in Y \mid d(X, y) = t\}$$

for every $t \in (0, t_0]$. We also denote by

$$\tilde{\pi}_t : C \to C_t, \quad \pi_t : Y \setminus X \to C_t$$

the canonical projections along perpendiculars. Recall that

$$X_1 = \{x \in X \mid \#\eta_0^{-1}(x) = 1\}, \quad X_2 = \{x \in X \mid \#\eta_0^{-1}(x) = 2\},$$

$$C_t^k = \{p \in C_0 \mid \eta_0(p) \in X_k\}, \quad k = 1, 2.$$

First we investigate the behavior of geodesics in $C$. To do this we make use of the Gromov-Hausdorff convergence $C_{M_i} \to C$.

Recall that for every $t \in [0, t_0]$, we set

$$C_{M_i, t} = \{x \in C_{M_i} \mid d(x, \partial M_i) = t\}.$$  

We also use $t$ to denote the distance functions on $C$ and $C_{M_i}$ from $C_0$ and $\partial M_i$, respectively.

Let $\gamma : [0, t] \to C$ be a unit speed geodesic, and $\xi = \frac{\partial}{\partial t}$, the unit vector fields on $C$. Take a geodesic $\gamma_i$ in $C_{M_i}$ such $\gamma_i \to \gamma$. We denote by $\Pi_i^T$ the second fundamental form of $C_{M_i, t}$:

$$\Pi_i^T(V, W) = -\langle \nabla_V \xi_i, W \rangle, \quad V, W \in T(C_{M_i, t}),$$

where $\xi_i = \frac{\partial}{\partial t}$ is the unit vector fields on $C_{M_i}$. Consider the function $\rho_i(s) = t(\gamma_i(s)) = |\gamma_i(s)|, \partial M_i|$. We have

$$\rho_i'(s) = \langle \xi_i(\gamma_i(s)), \gamma_i(s) \rangle$$

$$\rho_i''(s) = \langle \nabla_{\gamma_i^T} \xi_i, \gamma_i^T \rangle = -\Pi_i(\gamma_i^T, \gamma_i^T)$$

$$= \frac{\phi'(\rho_i(s))}{\phi(\rho_i(s))} |\gamma_i^T(s)|^2,$$

where $\gamma_i^T(s)$ is the component of $\gamma_i(s)$ tangent to $C_{M_i, \rho_i(s)}$. Note that $0 \geq \rho_i''(s) \geq -c$ for some uniform constant $c > 0$. In particular, we have

**Lemma 4.1.** $\rho_0$ and $\rho$ are concave functions.

**Lemma 4.2.** For every $t \in [0, t_0]$, and $p_1, p_2 \in C_t$, we have

$$\frac{|p_1, p_2|_{C_t}^m}{|p_1, p_2|_C} - 1 < O(|p_1, p_2|_C^2).$$
Proof. Let $\gamma : [0, \ell] \to C$ be a unit speed minimal geodesic joining $p_1$ to $p_2$. Take a unit speed minimal geodesic $\gamma_i : [0, \ell_i] \to C_{M_i}$ such that $\gamma_i \to \gamma$ under the Gromov-Hausdorff convergence $C_{M_i} \to C$. We may assume that $\rho_i(\gamma_i(0)) = \rho_i(\gamma_i(\ell_i)) = t$. Putting 
\[ \rho_i(s) = \rho_i(\gamma_i(s)) = |\gamma_i(s), \partial M_i|, \]
$\rho_i(s)$ takes a maximum $t_i = \rho_i(u_i) > t$ at some $u_i \in (0, \ell)$. By the mean value theorem, we obtain
\[ \frac{\rho_i(u_i)}{u_i} = \rho_i'(v_i), \quad \frac{\ell_i - \rho_i(u_i)}{\ell_i - u_i} = \rho_i'(v'_i), \quad \frac{\rho_i'(v'_i) - \rho_i'(v_i)}{v'_i - v_i} = \rho''(w_i), \]
for some $0 < v_i < u_i < v'_i < \ell$ and $v_i < w_i < v'_i$. Adding the first two equalities, we get
\begin{equation}
(4.5) \quad \rho_i(u_i) - t \leq \frac{(\ell_i - u_i)u_i}{\ell_i}(v'_i - v_i)(-\rho''(w_i)) \leq c|\gamma_i(0), \gamma_i(\ell_i)|^2.
\end{equation}
Setting $t^* := \max_{[0, \ell]} \rho$ and letting $i \to \infty$, we have
\begin{equation}
(4.6) \quad t^* - t = \max_{[0, \ell]} \rho - t \leq c|p_1, p_2|^2,
\end{equation}
and hence
\begin{equation}
(4.7) \quad \left| \frac{\phi(t)}{\phi(t^*)} - 1 \right| \leq c|p_1, p_2|^2.
\end{equation}
Let $\pi_t : C \to C_t$ be the canonical projection. Since $\pi_t$ has Lipschitz constant $\frac{\phi(t)}{\phi(t^*)}$ on the domain bounded by $C_t$ and $C_{t^*}$, it follows from
\begin{equation}
(4.8) \quad |p_1, p_2|_{C_t^{int}} \leq L(\pi_t \circ \gamma) \leq \frac{\phi(t)}{\phi(t^*)}|p_1, p_2| < (1 + O(|p_1, p_2|^2))|p_1, p_2|.
\end{equation}
This completes the proof. \qed

Lemma 4.3. For every $p_1, p_2 \in C_t$ and unit speed minimal geodesic $\gamma : [0, \ell] \to C$ joining $p_1$ to $p_2$, we have 
\[ \rho'(0) \leq C|p_1, p_2|, \]
where $\rho'(0) = \gamma'(s), C_t|$. 

Proof. Let $\rho(s)$ takes the maximum at $s = s_0$. Using the mean value theorem, we obtain $-\rho'(0)/s_0 = \inf \rho'' \geq -c$, from which the conclusion is immediate. \qed

Next we discuss a rescaling limit of the map $\eta : C \to Y$. Fix $p \in C_0$ and $x = \eta_0(p) \in X$, and let $t_i$ be an arbitrary sequence of positive numbers with $\lim t_i = 0$. Passing to a subsequence, we may assume that 
\[ \eta_i = \eta : \left( \frac{1}{t_i} C_t, p \right) \to \left( \frac{1}{t_i} Y, x \right) \]
converges to a 1-Lipschitz map
\[ \eta_\infty : (T_p(C), o_p) \to (T_x(Y), o_x) \]
between the tangent cones of the Alexandrov spaces. We may also assume that \( \left( \frac{1}{t_i}X, x \right) \) converges to a closed subset \( (T^*_x(X), o_x) \) of \( (T_x(Y), o_x) \) under the convergence \( \left( \frac{1}{t_i}Y, x \right) \to (T_xY, o_x) \).

**Sublemma 4.4.** \( \eta_\infty : T_p(C) \setminus T_p(C_0) \to T_xY \setminus T^*_x(X) \) is a bijective local isometry.

**Proof.** Let \( \tilde{\rho} = | \cdot, C_0 |, \rho = | \cdot, X | \). Under the \( 1/t_i \)-rescaling, we may assume that \( \tilde{\rho}, \rho \) converges to the maps
\[ \tilde{\rho}_\infty = | \cdot, T_p(C_0) |, \quad \rho_\infty = | \cdot, T^*_x(X) |, \]
respectively satisfying \( \tilde{\rho}_\infty = \rho_\infty \circ \eta_\infty \). For any \( \tilde{w} \in T_p(C) \setminus T_p(C_0) \), let \( \epsilon = \tilde{\rho}_\infty(\tilde{w}) \) and \( w = \eta_\infty(\tilde{w}) \). Since \( \rho_\infty(w) = \epsilon \), it is easily checked that \( \eta_\infty : B(\tilde{w}, \epsilon/2) \to B(w, \epsilon/2) \) is an isometry.

Next let us show that \( \eta_\infty : T_p(C) \setminus T_p(C_0) \to T_xY \setminus T^*_x(X) \) is bijective. Suppose that \( w := \eta_\infty(\tilde{w}_1) = \eta_\infty(\tilde{w}_2) \) for \( \tilde{w}_j \in T_p(C) \setminus T_p(C_0) \). Take \( q^*_1, q^*_2 \in C \) such that \( q^*_j \) converge to \( \tilde{w}_j \) under the \( 1/t_i \)-rescaling. Let \( \eta(q^*_j) = y^*_j \). Since \( y^*_j \) converges to the same point \( w \), any minimal geodesic joining \( y^*_1 \) and \( y^*_2 \) does not meet \( X \). This implies that \( |q^*_1, q^*_2| = |y^*_1, y^*_2| \). However this must imply that \( \tilde{w}_1 = \tilde{w}_2 \). Hence \( \eta_\infty \) is injective on \( T_p(C) \setminus T_p(C_0) \). It is easy to see that \( \eta_\infty : T_p(C) \setminus T_p(C_0) \to T_xY \setminus T^*_x(X) \) is surjective, and hence the proof is omitted. \( \square \)

**4.2. Spaces of directions and differential of \( \eta_0 \).** In this subsection, we study the the spaces of directions of \( C, C_0, Y \) and \( X \) at the points where the gluing is done, and the relation between them. We also study the differential of the gluing map \( \eta_0 \) at those points.

**Lemma 4.5.** For every \( p \in C_0 \), let \( \tilde{\gamma}^+(t) = (p, t) \) and \( \gamma^+(t) = \eta(\tilde{\gamma}(t)) \). Then

1. \( \Sigma_p(C) \) is isometric to the half-spherical suspension \( \{ \tilde{\gamma}^+(0) \} \star \Sigma_p(C_0) \);
2. for every \( s \in (0, t_0) \), \( \Sigma_{(p,s)}(C) \) and \( \Sigma_{(p,s)}(\eta(C)) \) are isometric to the spherical suspensions \( \{ \pm \tilde{\gamma}^+(s) \} \star \Sigma_p(C_0) \) and \( \{ \pm \gamma^+(s) \} \star \Sigma_p(C_0) \) respectively.

**Proof.** From the suspension structure \( C = C_0 \times_\phi [0, t_0] \), obviously we have \( T_p(C) = T_p(C_0) \times [0, \infty) \), which implies the conclusion (1). Since both \( \tilde{\gamma}^+ \) and \( \gamma^+ \) are geodesic, the splitting theorem shows (2). \( \square \)

**Lemma 4.6.** For every \( x \in X \) and \( \xi \in T_x(Y) \setminus K(\Sigma_x(X)) \) which is not a perpendicular direction, assume that there is a geodesic \( \gamma : [0, \ell] \to Y \) with \( \gamma'(0) = \xi \), and let
\[ \tilde{\gamma} = \eta^{-1} \gamma, \quad \tilde{\sigma} = \tilde{\pi} \circ \tilde{\gamma}, \quad \sigma = \pi \circ \gamma, \quad p := \tilde{\gamma}(0). \]
Let \( \tilde{\gamma}_+ \) be the perpendicular to \( C_0 \) at \( p \), and set \( \gamma_+ = \eta(\tilde{\gamma}_+) \). Put
\[
\tilde{\xi} = \tilde{\gamma}'(0), \quad \tilde{\xi}_+ = \tilde{\gamma}'(0), \quad \tilde{v} = \tilde{\sigma}'(0), \quad \xi_+ = \gamma'_+(0).
\]
Then
\begin{enumerate}
\item \( \sigma \) defines a unique vector \( v = \sigma'(0) \in K(\Sigma_x(X)) \) and we have
\[
\angle(\xi, \xi) = \angle(\tilde{\xi}, \tilde{\xi}), \quad \angle(\xi, v) = \angle(\tilde{\xi}, \tilde{v})
\]
\[
\angle(\xi, \xi) + \angle(\xi, v) = \angle(\tilde{\xi}, \tilde{v}) = \pi/2;
\]
\item there is a unique limit \( \eta_\infty : T_p(C) \rightarrow T_x(Y) \) of \( \eta_t = \eta : (\frac{1}{t}C, p) \rightarrow (\frac{1}{t}X, x) \) as \( t \rightarrow 0 \), and we have
\[
\eta_\infty(\tilde{v}) = v, \quad |\tilde{v}| = |v|.
\end{enumerate}

Proof. Let \( \zeta \in \Sigma_x(X) \) be a direction defined by the curve \( \sigma \). By definition, this means that \( \zeta = \lim_{t \rightarrow \infty} \frac{1}{t}C(1, \sigma(t), \gamma(t)) \) for a sequence \( t \rightarrow 0 \). Since \( \gamma \) is minimal, so is \( \tilde{\gamma} \). Note that \( \tilde{v} \) is uniquely determined since \( \tilde{\sigma} \) is a shortest curve. From Lemma 4.5 we have
\[
\angle(\tilde{\xi}, \tilde{\xi}) \angle(\tilde{\xi}, \tilde{v}) = \angle(\tilde{\xi}_+, \tilde{v}) = \pi/2.
\]

Now we show (4.9). Consider the \( 1/t \)-rescaling limits,
\[
(T_x^\ast(Y), o_x) = \lim_{t \rightarrow \infty} \left( \frac{1}{t}Y, x \right), \quad (T_p(C), o_p) = \lim_{t \rightarrow \infty} \left( \frac{1}{t}C, p \right).
\]
Let \( \gamma, \gamma_\infty \) (resp. \( \tilde{\gamma}, \tilde{\gamma}_\infty \)) be the perpendicular to \( X \) at \( \sigma(t) \) (resp. to \( C_0 \) at \( \tilde{\sigma}(t) \)). Passing to a subsequence, we may assume that the quadruplet \( (\gamma_+, \gamma, \sigma, \gamma_\infty) \) converges to \( (\gamma_+, \gamma_\infty, \sigma_\infty, \gamma_\infty) \) under the convergence \( \left( \frac{1}{t}Y, x \right) \rightarrow (T_x(Y), o_x) \). For instance, this explicitly means that the Lipschitz curve \( \frac{1}{t}C \sigma(t, t) \) converges to a Lipschitz curve \( \sigma_\infty(t) \) in \( T_x(Y) \). Thus \( \gamma_+ \) and \( \gamma_\infty \) are perpendicular to \( T_x^\ast(X) \) at \( o_x \) and \( \sigma_\infty(1) \) and \( \gamma_\infty \) is the geodesic from \( o_x \) with \( \gamma_\infty(1) = \xi \). Here we assume that \( \left( \frac{1}{t}X, x \right) \) converges to a closed subset \( (T_x^\ast(X), o_x) \) of \( T_x(Y), o_x) \).

Similarly passing to a subsequence, we may assume that the quadruplet \( (\tilde{\gamma}_+, \tilde{\gamma}, \tilde{\sigma}, \tilde{\gamma}_\infty) \) converges to \( (\tilde{\gamma}_+, \tilde{\gamma}_\infty, \tilde{\sigma}_\infty, \tilde{\gamma}_\infty) \) under the convergence \( \left( \frac{1}{t}C, p \right) \rightarrow (T_p(C), o_p) \). Thus \( \tilde{\gamma}_+ \) and \( \tilde{\gamma}_\infty \) are perpendicular to \( T_p(C_0) \) at \( o_p \) and \( \tilde{\sigma}_\infty(1) \) and \( \tilde{\gamma}_\infty \) is the geodesic from \( o_p \) with \( \tilde{\gamma}_\infty(1) = \tilde{\xi} \).

We set
\[
\rho(t) = |C_0, \tilde{\gamma}(t)| = |X, \gamma(t)|.
\]
Notice that
\begin{enumerate}
\item \( |\xi, \sigma_\infty(1)| = |\tilde{\xi}, \tilde{\sigma}_\infty(1)| = \rho'(0) \);
\item \( \rho'(0) = |\xi| \sin \angle(\tilde{\xi}, \tilde{v}) \).
\end{enumerate}
Let $\tilde{\lambda}$ be a minimal geodesic joining $\tilde{\xi} = \gamma_\infty(1)$ to the geodesic $\gamma_{+\infty}$.
Let $\eta_\infty : T_p(C) \to T_x(Y)$ be any limit of $\eta_{i} = \eta : (\frac{1}{t_i} C, p) \to (\frac{1}{t_i} Y, x)$.
Since $\eta_\infty$ is 1-Lipschitz, we have
\[
|\tilde{\xi}| \sin \angle(\tilde{\xi}_+, \tilde{\xi}) = L(\lambda) = L(\eta_\infty \circ \tilde{\lambda}) \geq |\xi| \sin \angle(\xi_+, \xi),
\]
and hence
\[
\angle(\xi_+, \xi) \leq \angle(\tilde{\xi}_+, \tilde{\xi}). \tag{4.11}
\]
Next we show that
\[
\angle(\xi, \zeta) = \angle(\tilde{\xi}, \tilde{\sigma}). \tag{4.12}
\]
Put for simplicity
\[
\tilde{\theta} := \angle(\tilde{\xi}, \tilde{v}), \quad \tilde{\theta}_i := \angle(\tilde{\gamma}(t_i), T_{\tilde{\gamma}(t_i)} C_{\rho(t_i)}), \\
\theta := \angle(\xi, \zeta), \quad \theta_i := \angle(\gamma(t_i), T_{\gamma(t_i)} Y_{\rho(t_i)}).
\]
From the warping product structure of $C$, we easily have
\[
\lim_{i \to \infty} \tilde{\theta}_i = \tilde{\theta}.
\]
On the other hand, under the convergence $\left( \frac{1}{t_i} C, p \right) \to (T_p(C), o_p)$ (resp. under the convergence $\left( \frac{1}{t_i} Y, x \right) \to (T_x(Y), o_x)$), we may assume that $C_{st_i}$ converges to some space, denoted by $C_{s\infty}$. (resp. $C^{Y}_{s\infty}$ converges to some space $C^{Y}_{s\infty}$). Then we have
\[
(1) \quad \tilde{\theta} = \angle(\tilde{\gamma}(0), \tilde{\sigma}(0)); \\
(2) \quad \tilde{\theta} = \lim \theta_i = \angle(\gamma_{1\infty}(1), T_{\gamma_{1\infty}(1)} (C_{\rho(0)\infty})).
\]
Since $\angle(\gamma_{1\infty}(1), \gamma'_{1\infty}(0)) = \pi / 2 - \tilde{\theta}$, we have
\[
\angle(o_p \tilde{\gamma}(1) \tilde{\sigma}(1)) = \pi / 2 - \tilde{\theta}. \tag{4.13}
\]
On the other hand, since $\eta : C \setminus C_0 \to C^{Y} \setminus X$ is a local isometry, we have
\[
\tilde{\theta}_i = \theta_i.
\]
From the lower semi-continuity, of angles, we have
\[
\lim \theta_i = \angle(\gamma'_{1\infty}(1), T_{\gamma_{1\infty}(1)} (C_{\rho(0)\infty})).
\]
It follows from the spherical suspension structure of $\Sigma_{\gamma_{1\infty}(1)} T_p(C)$ that
\[
\angle(\gamma'_{1\infty}(1), \gamma'_{1\infty}(0)) = \pi / 2 - \angle(\gamma'_{1\infty}(1), T_{\gamma_{1\infty}(1)} (C_{\rho(0)\infty})))
\]
and hence
\[
\angle(o_p \xi \sigma_{\infty}(1)) = \pi / 2 - \tilde{\theta}. \tag{4.14}
\]
By (4.13) and (4.14), the two Euclidean triangles $\triangle o_p \xi \sigma_{\infty}(1)$ and $\triangle o_p \tilde{\xi} \tilde{\sigma}_{\infty}(1)$ are congruent to each other, and we conclude that $\angle(\xi, \zeta) = \angle(\tilde{\xi}, \tilde{v})$ as required.
The first variation formula immediately implies $\angle(\xi_+, \zeta) \geq \pi/2$. It follows \[4.11\] and \[4.12\] that
\[
\frac{\pi}{2} \leq \angle(\xi_+, \zeta) \leq \angle(\xi_+, \xi) + \angle(\xi, \zeta)
\leq \angle(\tilde{\xi}_+, \tilde{\xi}) + \angle(\tilde{\xi}, \tilde{v}) = \frac{\pi}{2}.
\]
(4.15)
Thus we conclude that
\[
\angle(\xi_+, \xi) + \angle(\xi, \zeta) = \angle(\xi_+, \zeta) = \frac{\pi}{2},
\]
which shows the uniqueness of $\zeta$. Namely $\sigma$ determines a unique direction at $x$. Note that
\[
(4.16) \quad v := \sigma'(0) = \lim_{i \to \infty} \frac{\sigma(t_i)}{t_i}, \quad \tilde{\sigma} = |o_x, \sigma(1)|, \quad |\tilde{v}| = |o_x, \sigma(1)| = |\tilde{v}|.
\]
Since $\eta_\infty(\tilde{v}) = v$, this shows that $\eta_\infty$ does not depend on the choice of $t_i \to 0$. This completes the proof. □

Remark 4.7. The argument in the proof of Lemma 4.6 also shows that
\[
(\tilde{\xi}) \cos \hat{\theta} = |o_p, \tilde{\sigma}_\infty(1)| = |o_x, \sigma(1)|
\leq L(\sigma_\infty|_{[0,1]}) \leq L(\tilde{\sigma}_\infty|_{[0,1]}) = \cos \hat{\theta},
\]
which implies that
\[
(4.17) \quad \sigma_\infty \text{ is minimizing in the direction } \sigma'(0).
\]

Corollary 4.8. For every $x \in X$ and $\xi \in \Sigma_x(Y) \setminus \Sigma_x(X)$ which is not a perpendicular direction, there is a unique perpendicular direction $\xi_+ \in \Sigma_x(Y)$ to $X$ at $x$ and a unique $v \in \Sigma_x(X)$ such that
\[
(4.18) \quad \angle(\xi_+, \xi) + \angle(\xi, v) = \angle(\xi_+, v) = \frac{\pi}{2};
\]
Proof. This immediately follows from Lemma 4.6 and a limit argument. □

By Lemma 4.6, for every geodesic $\gamma$ in $Y$ starting from $x \in X$ such that $\gamma'(0) \in \Sigma_x(Y) \setminus \Sigma_x(X)$, the Lipschitz curve $\sigma = \pi(\gamma)$ determines a unique direction $[\sigma] \in \Sigma_x(X)$. In general, we call such a direction $[\sigma]$ an intrinsic direction if $\sigma$ is a Lipschitz curve in $X$ starting from $x$ and having a unique direction $[\sigma] = \sigma'(0)$ in the sense that for any sequence $t_i \to 0$, $\tilde{\sigma}^{(t_i)}$ converges to $[\sigma]$.

The next lemma shows that every direction in $\Sigma_x(X)$ can be approximated by intrinsic directions.

Lemma 4.9. For every $v \in \Sigma_x(X)$, we have the following:

1. For any perpendicular direction $\xi_+ \in \Sigma_x(Y)$, we have
   \[
   \angle(\xi_+, v) = \frac{\pi}{2}.
   \]
2. There are intrinsic directions $[\sigma_i] \in \Sigma_x(X)$ satisfying
   \[
   \lim [\sigma_i] \in \Sigma_x(X) \text{ satisfying } [\sigma_i] = v.
   \]
Lemma 4.6 (2) as the limit of present case, we have only geodesic \( \gamma \) \( o \) \( o \) where \( \lim \) a point \( \gamma \) \( o \) \( o \) considered the points \( \gamma \) \( o \) \( o \). An argument similar to that in Lemma 4.6 implies that Proof. For every \( v \in \Sigma_x(X) \) take a sequence \( y_i \in X \) with \( y_i \to x \) and \( v_i := t^{y_i}_x \to v \). Let \( \mu_i : [0, t_i] \to Y \) be a minimal geodesic from \( x \) to \( y_i \). Let \( \gamma_+ \) be a perpendicular to \( X \) at \( x \) with \( \gamma'_+(0) = \xi_+ \). Let \( \lambda_i \) be a minimal geodesic joining \( \gamma_+(t_i) \) to \( y_i \). Considering perpendiculars to \( X \) through the points of \( \lambda_i \) and taking the limit, we obtain a perpendicular \( \gamma_{y_i} \) to \( X \) at \( y_i \). Let \( \gamma_i : [0, t_i] \to Y \) be a minimal geodesic from \( x \) to \( \gamma_{y_i}(s_i) \), and set \( \sigma_i(t) := \pi(\gamma_i(t)) \), \( \tilde{\gamma}_i = \eta^{-1}(\gamma_i) \), \( \tilde{\sigma}_i = \tilde{\pi}(\tilde{\gamma}_i) \).

By Lemma 4.6 \( \sigma_i \) defines a unique direction \( \tilde{\nu}_i \in \Sigma_x(X) \) such that

\[
\angle(\xi_+ + \xi_i) + \angle(\xi_i, \tilde{\nu}_i) = \angle(\xi_+, \tilde{\nu}_i) = \pi/2,
\]

where \( \xi_i = \gamma'_i(0) \). Note that \( y_i = \sigma_i(t_i) \).

We now use an argument similar to that of Lemma 4.6 Consider the convergence

\[
\left( \frac{1}{t_i} Y, x \right) \to (T_x(Y), o_x), \quad \left( \frac{1}{t_i} C, p \right) \to (T_p(C), o_p).
\]

Passing to a subsequence, we may assume that \( \xi_i \) converge to some \( \xi \in \Sigma_x(Y) \subset T_x(Y) \). We may also assume that

(a) \( \gamma_i(t_i, s) \) and \( \sigma_i(t_i, s) \) converge to geodesic \( \gamma_{\infty}(s) \) and a Lipschitz curve \( \sigma_{\infty}(s) \) respectively;

(b) \( \tilde{\gamma}_i(t_i, s) \) and \( \tilde{\sigma}_i(t_i, s) \) converge to geodesics \( \tilde{\gamma}_{\infty}(s) \) and \( \tilde{\sigma}_{\infty}(s) \) in \( T_p(C) \) respectively.

Let \( \eta_{\infty} : (T_p(C), o_p) \to (T_x(Y), o_x) \) be the 1-Lipschitz map defined in Lemma 4.6 (2) as the limit of

\[
\eta_i = \eta : \left( \frac{1}{t_i} C, p \right) \to \left( \frac{1}{t_i} Y, x \right).
\]

Note that \( \eta_{\infty}(\tilde{\sigma}_{\infty}(s)) = \sigma_{\infty}(s) \). Consider the geodesic triangles

\[
\Delta_{o_x} := \Delta_{o_x \gamma_{\infty}(1)} \sigma_{\infty}(1) \subset T_x(Y), \quad \Delta_{o_p} := \Delta_{o_p \tilde{\gamma}_{\infty}(1)} \tilde{\sigma}_{\infty}(1) \subset T_p(C).
\]

An argument similar to that in Lemma 4.6 implies that

\[
\angle o_x \gamma_{\infty}(1) \sigma_{\infty}(1) = \angle o_p \tilde{\gamma}_{\infty}(1) \tilde{\sigma}_{\infty}(1),
\]

It should be remarked that in the case of Lemma 4.6 the geodesic \( \gamma : [0, \ell] \to Y \) in the direction \( \xi \) was given in the beginning, and we considered the points \( \gamma(t_i) \) with \( t_i \to 0 \). On the other hand, in the present case, we have only geodesic \( \gamma_i : [0, t_i] \to Y \). Therefore we take a point \( z_i \in Y \) instead, in such a way that

\[
\tilde{\gamma} x \gamma_i(t_i) z_i > \pi - \alpha, \quad |\gamma_i(t_i), z_i| = t_i,
\]

where \( \lim \alpha = 0 \). Then with almost parallel argument, we obtain (1.20) and that \( \Delta_{o_x} \) and \( \Delta_{o_p} \) are congruent to each other as Euclidean
flat triangles. In particular we conclude that
\begin{equation}
|o_x, \sigma_\infty(1)| = |o_p, \bar{\sigma}_\infty(1)|.
\end{equation}
Since \( L(\sigma_\infty) \leq L(\bar{\sigma}_\infty) \), this implies that \( \sigma_\infty \) is a minimal geodesic in the direction \( v \). As in Lemma 4.6, (4.20) also implies that
\begin{equation}
\angle(\xi_+, \xi) + \angle(\xi, v) = \angle(\xi_+, v) = \pi/2,
\end{equation}
where \( \xi = \lim \xi_i \). This proves (1). It follows from (4.19), (4.22) that
\begin{equation}
v = \lim  \hat{v}_i.
\end{equation}
which shows (2).

Furthermore, it follows from
\[
\begin{cases}
\frac{L(\bar{\sigma}_i)}{t_i} \geq \frac{L(\sigma_i)}{t_i} \geq \frac{L(\mu_i)}{t_i}, \\
\lim \frac{L(\bar{\sigma}_i)}{t_i} = |o_p, \bar{\sigma}_\infty(1)|, \quad \lim \frac{L(\mu_i)}{t_i} = |o_x, \sigma_\infty(1)|,
\end{cases}
\]
that
\begin{equation}
\lim \frac{L(\sigma_i)}{t_i} = L(\sigma_\infty).
\end{equation}

Proposition 4.10. \( \Sigma_x(X) \) coincides with the closure of \( \Sigma_x^0(X) \) in \( \Sigma_x(Y) \).

The space of direction \( \Sigma_x(X) \) was originally defined in an extrinsic way (see Subsection 2.2). Proposition 4.10 shows that it coincides with the one defined in an intrinsic way.

For \( x \in X_1 \) (resp. \( x \in X_2 \)), let \( \xi_+ \in \Sigma_x(Y) \) (resp. \( \xi_\pm \in \Sigma_x(Y) \)) be the unique (resp. the two) direction (resp directions) of the perpendicular (resp. perpendiculars) to \( X \) at \( x \).

Corollary 4.11. For every \( x \in X \), we have the following:

1. If \( x \in X_1 \), then
   \[ \Sigma_x(X) = \{ v \in \Sigma_x(Y) \mid \angle(\xi_+, v) = \pi/2 \} \].

2. If \( x \in X_2 \), then \( \Sigma_x(Y) \) is isometric to the spherical suspension
   \[ \{ \xi_\pm \} \ast \Sigma_x(X) \]
In either case, \( \Sigma_x(X) \) is an Alexandrov space with curvature \( \geq 1 \) of dimension equal to \( \dim Y - 2 \).

Proof. (1) is a direct consequence of Corollary 4.8 and Lemma 4.9. (2) is a direct consequence of Lemma 3.3, Corollary 4.8 and Lemma 4.9. In an Alexandrov space \( \Sigma \) with curvature \( \geq 1 \), for any \( \xi \in \Sigma \), the set \( \{ v \in \Sigma \mid |v, \xi| \geq \pi/2 \} \) is convex, which implies the last conclusion. \( \Box \)
Our next purpose is to show the following.

**Proposition 4.12.** Under the convergence \( \lim_{\delta \to 0}(\frac{1}{\delta}Y, x) = (T_x(Y), o_x) \), \((\frac{1}{\delta}X, x)\) converges to the Euclidean cone \( K(\Sigma_x(X)), o_x \) as \( \delta \to 0 \).

We set
\[ T_x(X) = K(\Sigma_x(X)) \]
and call it the tangent cone of \( X \) at \( x \).

For the proof of Proposition 4.12, we need three lemmas.

**Lemma 4.13.** For every minimal geodesic \( \gamma : [0, \ell] \to Y \) joining any \( x \in X \) and \( y \in Y \), the curve \( \sigma(t) := \pi(\gamma(t)) \) has a unique direction at \( t = 0 \), and hence defines an intrinsic direction \([\sigma] \in \Sigma_x(X)\).

**Proof.** By Lemma 4.6, it suffices to consider only the case \( \xi := \gamma'(0) \in \Sigma_x(X) \). Suppose that for sequences \( s_j \to 0 \) and \( t_j \to 0 \) we have limits:
\[ \eta := \lim_{j \to \infty} \tau_x^{\sigma(s_j)}, \quad \zeta := \lim_{j \to \infty} \tau_x^{\sigma(t_j)}. \]
Take \( \xi_j \in \Sigma_x(Y) \setminus \Sigma_x(X) \) such that \( \xi_j \to \xi \) and geodesic \( \gamma_i \) in the direction \( \xi_j \) is defined. Set \( \sigma_j = \pi \circ \gamma_i \). By Lemma 4.6, \( \sigma_j \) defines an intrinsic direction \([\sigma_j] \in \Sigma_x(X)\), and passing to a subsequence we may assume that
\[ \angle(\xi_j, \xi_j) + \angle(\xi_j, [\sigma_i]) = \angle(\xi, [\sigma_i]) = \pi/2, \]
for some perpendicular direction \( \xi_j \) at \( x \). It follows that \( \angle(\xi_j, [\sigma_i]) \to 0 \) and \([\sigma_i] \to \xi \). From \( \xi_j \to \xi \), we have
\[ |\gamma_i(t), \gamma(t)| < o_i t, \]
where \( \lim o_i = 0 \). Since \( \pi \) is 1-Lipschitz, it follows that
\[ |\sigma_i(t), \sigma(t)| < o_i t, \]
which implies that
\[ |\sigma_i(s_j), \sigma(s_j)| < o_i s_j, \quad |\sigma_i(t_j), \sigma(t_j)| < o_i t_j. \]
Passing to a subsequence, we may assume that there are limits:
\[ \alpha = \lim_{j \to \infty} \frac{|x, \sigma(s_j)|}{s_j}, \quad \alpha_i = \lim_{j \to \infty} \frac{|x, \sigma_i(s_j)|}{s_j}, \]
\[ \beta = \lim_{j \to \infty} \frac{|x, \sigma(t_j)|}{t_j}, \quad \beta_i = \lim_{j \to \infty} \frac{|x, \sigma_i(t_j)|}{t_j}. \]
(4.26) implies \( \alpha_i \to \alpha \) and \( \beta_i \to \beta \). On the other hand, since \( \xi \in \Sigma_x(X) \),
(4.16) shows \( \alpha_j \to 1 \) and \( \beta_j \to 1 \). Thus we have \( \alpha = \beta = 1 \). Since (4.26)
implies
\[ |\alpha_i[\sigma_i], \alpha \eta| \leq o_i, \quad |\beta_i[\sigma_i], \beta \zeta| \leq o_i, \]
we conclude that
\[ [|\sigma_i, \eta| \leq o_i, \quad [|\sigma_i, \zeta| \leq o_i, \]
and hence the uniqueness \( \eta = \zeta = \xi \). This completes the proof. □
Lemma 4.14. For every \( x, y \in X \) and every minimal geodesic \( \mu : [0, \ell] \to Y \) joining them, let \( \sigma = \pi \circ \mu \) and set \( \rho(t) = |\mu(t), X| \). Then we have

(1) \( \max \rho \leq O(|x, y|^2) \);

(2) \( \angle(\mu'(0), [\sigma]) \leq O(|x, y|) \);

(3) \( \left| \frac{L(\sigma)}{L(\mu)} - 1 \right| < O(|x, y|^2) \).

Proof. (1) Let \( \rho(s^*) = \max \rho \) and take \( 0 \leq a < b \leq \ell \) such that \( s^* \in (a, b) \), \( \rho > 0 \) on \( (a, b) \) and \( \rho(a) = \rho(b) = 0 \). Then \( \tilde{\mu} = \eta^{-1}(\mu_{|[a,b]}) \) and \( \tilde{\sigma} = \tilde{\pi}(\tilde{\mu}) \) are well defined. By (4.13), we have

\[
\rho(s^*) = |\tilde{\mu}(s^*), C_0| \leq O(|\tilde{\mu}(a), \tilde{\mu}(b)|^2) \leq O(|\mu(a), \mu(b)|^2) \leq O(|x, y|^2).
\]

(2) We may assume \( \mu'(0) \neq [\sigma] \). Take the smallest \( s_1 \in (0, \ell] \) satisfying \( \mu(s_1) \in X \). Note that \( \tilde{\mu} = \eta^{-1}(\mu_{|[0,s_1]}) \) is well defined and \( |x, \mu(s_1)| = |\tilde{\mu}(0), \tilde{\mu}(s_1)| \). For \( \rho(s) = |\mu(s), X| = |\tilde{\mu}(s), C_0|, 0 \leq s \leq s_1 \), Lemmas 4.6 and 4.9 imply

\[
\rho'(0) = \sin \angle(\mu'(0), [\sigma]) \leq C|\tilde{\mu}(0), \tilde{\mu}(s_1)| = C|x, \mu(s_1)| \leq C|x, y|.
\]

(3) Take at most countable disjoint open intervals \((a_i, b_i)\) of \([0, \ell]\) such that

- \( \mu(a_i), \mu(b_i) \in X \) and \( \mu((a_i, b_i)) \subset Y \setminus X \);
- \( \mu(s) \in X \) for all \( s \in J := [0, \ell] \setminus \cup(a_i, b_i) \).

Set

\[
\mu_i = \mu_{|[a_i,b_i]}, \quad \sigma_i = \sigma_{|[a_i,b_i]}, \quad \tilde{\mu}_i = \eta^{-1}(\mu_i), \quad \tilde{\sigma}_i = \eta^{-1}(\sigma_i).
\]

Let \( L(J) \) denote the measure of \( J \). Since \( L(\tilde{\mu}_i) = L(\mu_i), L(\tilde{\sigma}_i) \geq L(\sigma_i) \) and \( \mu(J) = \sigma(J) \), Lemma 4.2 implies that

\[
L(\sigma) = \sum L(\sigma_i) + L(J) \leq \sum L(\tilde{\sigma}_i) + L(J) \leq \sum (1 + O(|\tilde{\sigma}_i(a_i), \tilde{\sigma}_i(b_i)|^2)L(\tilde{\mu}_i) + L(J) \leq \sum (1 + O(|\mu(a_i), \mu(b_i)|^2)L(\mu_i) + L(J) \leq (1 + O(|x, y|^2)L(\mu).
\]

\]

Lemma 4.15. For any \( \nu = [\sigma] \in \Sigma^0_x(X) \), if we consider the arc-length parameter of \( \sigma \), \( \sigma(\delta t) \) converges to the geodesic ray \( \sigma_\infty(t) \) in \( T_x(Y) \) from the origin \( \alpha_x \) in the direction \( \nu \) as \( \delta \to 0 \) under the convergence \((\frac{\delta}{\delta} Y, x) \to T_x(Y), o_x \).
Proof. Since $\sigma$ determines the unique direction $v$ we have for any $0 < R_1 < R_2$,
\[ \lim_{\delta \to 0} \pi^{\delta R_1} = \lim_{\delta \to 0} \pi^{\delta R_2}. \]
This implies that the image $\sigma_\infty([0, \infty))$ coincides with the ray in the direction $v$. \hfill \square

Proof of Proposition \[4,12\]. We have to show that the Gromov-Hausdorff distance between $\left( \frac{1}{4}B(x, \delta R; X), x \right)$ and $(B(o_x, R; K(\Sigma_x(X))), o_x)$ converges to zero as $\delta \to 0$ for any fixed $R > 0$. For every small $\epsilon > 0$ take a $\epsilon$-dense subset $\{[\sigma_i]\}_{i=1}^I$ of $\Sigma_x(X)$, and put $K := [R/\epsilon] + 1$. Taking small enough $\delta$, we may assume that $\sigma_i$ are defined on $[0, \delta R]$ and that $\sigma_i$ can be written as $\sigma_i = \pi(\gamma_i)$, where $\gamma_i$ is a minimal geodesic in $Y$ joining $x$ to $\sigma_i(\delta R)$. Let us consider the following sets:

\[ N_\infty := \left\{ \frac{kR}{K}\sigma | 1 \leq i \leq I, 0 \leq k \leq K \right\} \subset B(o_x, R; K(\Sigma_x(X))), \]
\[ N_\epsilon := \left\{ \sigma_i \left( \frac{\delta kR}{K} \right) | 1 \leq i \leq I, 0 \leq k \leq K \right\} \subset \frac{1}{\delta}B(x, \delta R; X) \]

First we show that both $N_\infty$ and $N_\epsilon$ are $(\tau(R|\epsilon) + \tau(R|\delta))$-dense. For simplicity, set

\[ v_{i,k} = \frac{kR}{K}\sigma, \quad x_{i,k} = \sigma_i \left( \frac{\delta kR}{K} \right). \]

For every $y \in B(x, \delta R; X)$, let $\gamma : [0, \ell] \to Y$ be a minimal geodesic joining $x$ to $y$, and let $\sigma := \pi(\gamma)$. Choose $i$ and $k$ with $\angle([\sigma], [\sigma_i]) < \epsilon$ and $|\ell - kR/K| < R/K < \epsilon$. Since Lemma \[4,14\] implies that

\[ \angle(\gamma(0), \gamma_i(0)) \leq \angle(\gamma(0), [\sigma]) + \angle([\sigma], [\sigma_i]) + \angle([\sigma_i], \gamma_i(0)) \leq \epsilon + 2\tau(\delta). \]

we obtain

\[ |\gamma(\ell), \gamma_i(\ell)| \leq C\ell(\epsilon + 2\tau(\delta)). \]

It follows from Lemma \[4,14\] that

\[ |y, x_{i,k}| \leq |\gamma(\ell), \gamma_i(\ell)| + |\gamma_i(\ell), \sigma_i(\ell)| + |\sigma_i(\ell), x_{i,k}| \]
\[ \leq C\ell(\epsilon + 2\tau(\delta)) + (\delta R)^2 + C\delta R/K \]
\[ \leq \delta(\tau(R|\epsilon) + \tau(R|\delta)). \]

Thus $N_\epsilon$ is $(\tau(R|\epsilon) + \tau(R|\delta))$-dense in $\frac{1}{4}B(x, \delta R)$.

For every $v \in B(o_x, R; K(\Sigma_x(X)))$, take $i$ and $k$ satisfying $\angle(v, [\sigma_i]) < \epsilon$ and $|\|v\| - kR/K| < R/K < \epsilon$. Then we have

\[ |v, v_{i,k}| \leq R/K + \epsilon R = \tau(R|\epsilon). \]

Hence $N_\infty$ is $\tau(R|\epsilon)$-dense in $B(o_x, R; K(\Sigma_x(X)))$.

Finally define $f : N_\infty \to N_\epsilon$ by $f(v_{i,k}) = x_{i,k}$. For simplicity put

\[ w_{i,k} = \frac{kR}{K}\gamma_i(0), \quad y_{i,k} = \gamma_i \left( \frac{\delta kR}{K} \right). \]
By Lemma 4.14, we then have
\[
||x_{i,k}, x_{j,\ell}| - |y_{i,k}, y_{j,\ell}| < 2C(\delta R)^2
\]
\[
||y_{i,k}, y_{j,\ell}| - |w_{i,k}, w_{j,\ell}| < \tau(R|\delta)
\]
\[
||w_{i,k}, w_{j,\ell}| - |v_{i,k}, v_{j,\ell}| < \tau(R|\delta),
\]
which implies that \( f \) is \( \tau(R|\delta) \)-approximation. In this way, we conclude that \( d_{GH}((\frac{1}{3}B(x, \delta R), x), (B(o_x, R; T_x(X)))) < \tau(R|\epsilon) + \tau(R|\delta) \). This completes the proof of Proposition 4.12.

Lemma 4.16. Fix any \( x \in X \) and take \( p \in C_0 \) with \( \eta_0(p) = x \). Then for every \( y \in X \), there is a point \( q \in \eta_0^{-1}(y) \) such that
\[
\left| \frac{|x, y|_Y}{|p, q|_C} - 1 \right| < \tau_x(|x, y|),
\]
where \( \tau_x(t) \) is a function depending on \( x \) with \( \lim_{t \to 0} \tau_x(t) = 0 \).

Proof. Suppose the lemma does not hold. Then since \( \eta_0 \) is 1-Lipschitz, we have a sequence \( y_i \in X \) with \( \lim y_i = x \) such that for every \( q_i \in \eta_0^{-1}(y_i) \),
\[
\frac{|x, y_i|_Y}{|p, q_i|_C} < 1 - \epsilon,
\]
for some \( \epsilon > 0 \) independent of \( i \).

We proceed as in the proof of Lemma 4.9. Let \( \mu_i : [0, s_i] \to Y \) be a minimal geodesic from \( x \) to \( y_i \), and take a perpendicular \( \gamma_{y_i} \) to \( X \) at \( y_i \). Let \( \gamma_i : [0, t_i] \to Y \) be a minimal geodesic from \( x \) to \( \gamma_{y_i}(s_i) \), and set
\[
\sigma_i(t) := \pi(\gamma_i(t)), \quad \tilde{\sigma}_i = \eta^{-1}(\sigma_i).
\]
Let \( q_i := \tilde{\sigma}(t_i) \). Under the convergences
\[
\left( \frac{1}{t_i}y, x \right) \to (T_x(Y), o_x), \quad \left( \frac{1}{t_i}C, p \right) \to (T_p(C), o_p)
\]
passing to a subsequence if necessarily, we may assume that the triplet \( (\mu_i(t_i, s), \gamma_i(t_i, s), \sigma_i(t_i, s)) \) (resp. the pair \( (\tilde{\gamma}_i(t_i, s), \tilde{\sigma}_i(t_i, s)) \)) converges to a triplet \( (\mu_\infty(s), \gamma_\infty(s), \sigma_\infty(s)) \) (resp. a double \( (\tilde{\gamma}_\infty(s), \tilde{\sigma}_\infty(s)) \)). From (4.21), we see that
\[
\lim_{t_i \to \infty} \frac{|x, y_i|_Y}{|p, q_i|_C} = \lim_{t_i \to \infty} \frac{|o_x, \sigma_\infty(1)|}{|o_p, \tilde{\sigma}_\infty(1)|} = 1,
\]
which yields a contradiction to the hypothesis (4.28):
\[
\lim_{i \to \infty} \frac{|x, y_i|_Y}{|p, q_i|_C} = 1.
\]
\( \square \)
For any \( p \in C_0 \), by Lemma 4.10 as \( t \to 0 \), \( \eta_0 : (\frac{1}{t}C_0, p) \to (\frac{1}{t}X, x) \) converges to a 1-Lipschitz map \( (d\eta_0)_p : T_p(C_0) \to T_x(X) \), which is called the differential of \( \eta_0 \) at \( p \).

Lemma 4.11 immediately implies the following.

**Proposition 4.17.** For every \( p \in C_0 \), the differential \( d\eta_0 : T_p(C_0) \to T_x(X) \) satisfies

\[
|d\eta_0(\tilde{v})| = |\tilde{v}|.
\]

for every \( \tilde{v} \in T_p(C_0) \). In particular, \( \eta_0 : C_0 \to X \) preserves the length of Lipschitz curves in \( C_0 \).

By Proposition 4.17, \( d\eta_0 \) provides a surjective 1-Lipschitz map \( d\eta_0 : \Sigma_p(C_0) \to \Sigma_x(X) \).

**Remark 4.18.** By Lemma 4.11, \( x \in X_2 \) is a regular point of \( Y \) if and only if the tangent cone \( T_x(X) \) is isometric to \( \mathbb{R}^{m-1} \), where \( m = \dim Y \). From this reason, in that case we call \( x \) a regular point of \( X \), and set \( X^{\text{reg}} := X \cap Y^{\text{reg}} \). Later we show that every \( x \in X_1 \) is a singular point of \( X \) unless \( X = X_1 \) (see Corollary 4.35).

**Proposition 4.19.** For every \( p \in C_0^* \), we have

1. the differential \( d\eta_0 : T_p(C) \to T_x^+(Y) \) which preserves the half suspension structures of both \( \Sigma_p(C) = \{ \xi_+ \} * \Sigma_p(C_0) \) and \( \Sigma_x^+(Y) := \{ \xi_+ \} * \Sigma_x(X) \), where \( T_x^+(Y) = T_x(X) \times \mathbb{R}_+^* \);
2. \( p \in C_0^{\text{reg}} \) if and only if \( x \in X^{\text{reg}} \). In this case, \( (d\eta_0)_p : T_p(C_0) \to T_x(X) \) is a linear isometry.

**Proof.** (1) For every \( \tilde{v}_1, \tilde{v}_2 \in \Sigma_p(C_0) \), put \( v_i := d\eta_0(\tilde{v}_i) \). We show that \( \angle(\tilde{v}_1, \tilde{v}_2) = \angle(v_1, v_2) \). Let \( \xi_i \) (resp. \( \xi_+ \)) be the midpoint of the geodesic joining \( \xi_+ \) to \( \tilde{v}_i \) (resp. \( \xi_+ \) to \( \tilde{v}_i \)). Note that \( d\eta(\xi_i) = \xi_i \). We may assume that there are geodesics \( \tilde{\gamma}_i(t) \) with \( \tilde{\gamma}_i(0) = \xi_i \), and set \( \gamma_i(t) := \eta(\tilde{\gamma}_i(t)) \). Since \( T_x(Y) = T_x(X) \times \mathbb{R}_+^* \), any minimal geodesic joining \( \tilde{\gamma}_1(t) \) and \( \tilde{\gamma}_2(t) \) does not meet \( X \) for any small \( t > 0 \). It follows from the fact that \( \eta : C \setminus C_0 \to Y \setminus X \) is locally isometric that

\[
|\tilde{\gamma}_1(t), \tilde{\gamma}_2(t)| = |\gamma_1(t), \gamma_2(t)|,
\]

which implies that \( \angle(\tilde{\xi}_1, \tilde{\xi}_2) = \angle(\xi_1, \xi_2) \). From the suspension structures, we conclude that \( \angle(\tilde{v}_1, \tilde{v}_2) = \angle(v_1, v_2) \).

(2) is an immediate consequence of (1). \( \square \)

### 4.3. Gluing maps

Using the results of the last subsection, we study the metric properties of the gluing map.

From Lemma 3.3, we can define a map \( f : C_0 \to C_0 \) as follows: For an arbitrary point \( p \in C_0 \), let \( f(p) := q \) if \( \{ p, q \} = \eta_0^{-1}(\eta_0(p)) \), where \( q \) may be equal to \( p \) if \( \eta_0(p) \in X_1 \). Note that \( f \) is an involutive map, i.e., \( f^2 = \text{id} \). Moreover
Lemma 4.20. \( f : C_0 \to C_0 \) is a homeomorphism.

Proof. Since \( f \) is involutive, it suffices to prove that \( f \) is continuous. For a sequence \( p_i \) converging to a point \( p \) in \( C_0 \), we show that \( f(p_i) \to f(p) \). Set \( x = \eta_0(p), x_i = \eta_0(p_i) \).

Case 1) \( x \in X_1 \). In this case, \( f(p) = p \). If \( x_i \in X_1 \), then \( f(p_i) = p_i \), and we have nothing to do. Suppose \( x_i \in X_2 \). Let \( \gamma_i^\pm \) be the two perpendiculars to \( X \) at \( x_i \). Letting \( s_i = |x,x_i| \), consider minimal geodesics \( \gamma_i^\pm \) joining \( x \) to \( \gamma_i^\pm(s_i) \). If we set \( \hat{\gamma}_i^\pm := \hat{p} \circ \gamma_i^\pm \), where \( \hat{\gamma}_i^\pm = \eta^{-1}(\gamma_i^\pm) \), then \( \hat{\gamma}_i^\pm \) are minimal geodesics joining \( p \) to \( p_i \) and \( f(p) \) to \( f(p_i) \) respectively. Lemma 4.16 then shows that

\[
|x,x_i| - 1 < \tau_x(\gamma_i,\gamma_i^\pm(s_i)) \quad \text{and} \quad \tau_x(\gamma_i,\gamma_i^\pm(s_i)) - 1 < \tau_x(\gamma_i,\gamma_i^\pm(s_i))
\]

which implies \( f(p_i) \to f(p) \).

Case 2) \( x \in X_2 \). In this case, \( f(p) \neq p \). Let \( \gamma_i^\pm \) be the two perpendiculars to \( X \) at \( x \). By Lemma 4.11, we have

\[
\angle \gamma_i^\pm(s_0)x_i\gamma_i^\pm(s_0) \geq \hat{\gamma}_i^\pm(s_0)x_i\hat{\gamma}_i^\pm(s_0) > \pi - \tau(s_0),
\]

If \( x_i \in X_1 \), then both \( \angle(x^+_i,\gamma_i^\pm(s_0)) \) and \( \angle(x^-_i,\gamma_i^\pm(s_0)) \) become small, yielding a contradiction to (4.30). Thus we have \( x_i \in X_2 \). Then in a way similar to Case 1), we have the formula (4.29), which implies \( f(p_i) \to f(p) \).

Corollary 4.21. \( \eta_0|_{C_0^2} : C_0^2 \to X_2 \) is a double covering space and \( X_2 \) is open in \( X \).

Proof. For \( x \in X_2 \) set \( \eta_0^{-1}(x) = \{p_1,p_2\} \), and take an open neighborhood \( D_1 \) of \( p_1 \) in \( C_0 \) such that \( D_1 \cap f(D_1) \) is empty. We set \( D_2 = f(D_1) \). We show that \( E := \eta_0(D_1) \) is open in \( X \). Suppose that \( E \) is not open, and take \( y \in E \) for which there are \( y_i \in X \setminus E \) converging to \( y \). Let \( \{q_1,q_2\} := \eta_0^{-1}(y) \) with \( q_k \in D_k, k = 1,2 \). Applying Lemma 4.16 to \( y_i \to y \) and \( q_k \in \eta_0^{-1}(y) \), we have \( q_{k,i} \in \eta_0^{-1}(y_i) \) such that

\[
\frac{|q_k,q_{k,i}|}{|y,y_i|} - 1 < \tau_y(|y,y_i|), \quad k = 1,2.
\]

This implies that \( q_{k,i} \in D_k \) and \( y_i \in E \) for large \( i \). Since this is a contradiction, \( E \) is open. Similarly one can show that each restriction \( \eta_0|_{D_k} : D_k \to E \) is an open map, and hence is a homeomorphism.

Corollary 4.22. \( Y \) and \( X \) are homeomorphic to the quotient spaces \( C_0 \times_{\phi} [0,t_0]/f \) and \( C_0/f \) respectively, where \( (x,0) \) and \( (f(x),0) \) are identified for every \( x \in C_0 \).

Corollary 4.23. If the inradius of \( M_i \in M(n,k,\lambda,d) \) converges to zero, then the number of components of \( \partial M_i \) is at most two for large enough \( i \).
Proof. Since $f$ is an involutive homeomorphism, $f$ gives a transposition of two components of $C_0$. The conclusion is immediate from the connectedness of $X$. □

Remark 4.24. In Theorem 1.5, we remove the diameter bound to get the diameter free result.

Lemma 4.25. $\eta_0|_{C_0^2} : (C_0^2)^{\text{int}} \to X_2^{\text{int}}$ is a local isometry.

Proof. Since $\eta_0|_{C_0^2} : C_0^2 \to X_2$ is a covering by Corollary 4.21, we can find relatively compact open subsets $D$ and $E$ of $C_0^2$ and $X_2$ respectively such that $\eta_0 : D \to E$ is a homeomorphism. We must show that $\eta_0 : D \to E$ is an isometry with respect to the interior distances of $C_0$ and $X$ respectively. Since $\eta_0$ is 1-Lipschitz, it suffices to show that $g := \eta_0^{-1} : E \to D$ is 1-Lipschitz. We may assume that $D$ is small enough so as to satisfy that for every $x, y \in E$, there is a minimal geodesic $\gamma : [0, 1] \to X_2$ joining $x$ to $y$. We do not know if $g \circ \gamma$ is a Lipschitz curve yet. However by Proposition 4.17, $g \circ \gamma$ has the speed $v_{g \circ \gamma}(t)$ (see [3])

$$v_{g \circ \gamma}(t) = \lim_{\epsilon \to 0} \frac{|g \circ \gamma(t), g \circ \gamma(t + \epsilon)|}{|\epsilon|},$$

which is equal to the speed $v_{\gamma}(t)$ of $\gamma$, and therefore

$$|x, y| = L(\gamma) = \int_0^1 v_{g \circ \gamma}(t) dt = L(g \circ \gamma) \geq |g(x), g(y)|.$$

This completes the proof. □

Lemma 4.26. If $X_1$ has non-empty interior in $X$, then $X = X_1$ and $\eta_0 : (C_0)^{\text{int}} \to X^{\text{int}}$ is an isometry.

Proof. If the interior $U$ of $X_1$ is non-empty, then $V := \eta_0^{-1}(U) \subset C_0$ is open in $C_0$. From the non-branching property of geodesics in Alexandrov spaces, we have $V = C_0$ and $X = X_1$. An argument similar to the proof of Lemma 4.25 shows that $\eta_0 : (C_0)^{\text{int}} \to X^{\text{int}}$ is an isometry. □

Proposition 4.27. $f : (C_0)^{\text{int}} \to (C_0)^{\text{int}}$ is an isometry.

Proof. For $x \in X_2$ with $\eta_0^{-1}(x) = \{q_1, q_2\}$, by lemma 4.25, we can take disjoint open sets $q_i \in D_i$, $i = 1, 2$, and $E$ such that $\eta_0^\circ_{D_i} : D_i \to E$ are isometry. Thus $f|_{D_i} = (\eta_0^\circ_{D_i})^{-1} \circ \eta_0 : D_1 \to D_2$ is an isometry with respect to the interior distances. Note that $f$ is identity on $C_0^\circ$, and by Lemma 4.25, $f : (C_0^\circ)^{\text{int}} \to (C_0^\circ)^{\text{int}}$ is a locally isometry. For every $p_1, p_2 \in C_0$ we show that $|f(p_1), f(p_2)| = |p_1, p_2|$. This is obvious if $p_1, p_2 \in C_0^\circ$. Let $\gamma : [0, 1] \to C_0$ be a minimal geodesic joining $p_1$ to $p_2$. If $p_1, p_2 \in C_0^\circ$, applying Lemma 4.25, we may assume that $\gamma$ meets $C_0^\circ$. Let $t_0 \in (0, 1)$ be the smallest parameter with $\gamma(t_0) \in C_0^\circ$. By Lemma 4.25, we have $|f(p_1), f(\gamma(t_0))| = |p_1, \gamma(t_0)|$. Therefore the non-branching property of geodesics in Alexandrov space implies that
\( \gamma \cap C^1_0 \) consists of only the single point \( \gamma(t_0) \), and therefore we also have 
\[ |f(p_2), f(\gamma(t_0))| = |p_2, \gamma(t_0)|. \]
It follows that
\[
|f(p_1), f(p_2)| \leq |f(p_1), f(\gamma(t_0))| + |f(\gamma(t_0)), f(p_2)| \\
\leq |p_1, \gamma(t_0)| + |\gamma(t_0), p_2| = |p_1, p_2|.
\]
Repeating this, we also have \( |p_1, p_2| \leq |f(p_1), f(p_2)| \), and \( |f(p_1), f(p_2)| = |p_1, p_2| \). The case of \( p_1 \in C^1_0 \) and \( p_2 \in C^2_0 \) is similar, and hence is omitted. This completes the proof. \( \square \)

### 4.4. Structure theorems
In this subsection, making use of the results on gluing maps in the last subsection, we obtain structure results for limit spaces.

We begin with

**Lemma 4.28.** \( X_2 \) is convex in \( X \).

**Proof.** Suppose this is not the case. Then we have a minimal geodesic \( \gamma : [0, 1] \to X \) joining points \( x, y \in X_2 \) such that \( \gamma \) is not entirely contained in \( X_2 \). Let \( t_1 \) be the first parameter with \( \gamma(t_1) \in X_1 \). Set \( z := \gamma(t_1) \). By Lemma 4.25 for any \( p \in \eta_0^{-1}(x) \), there exists a unique geodesic \( \tilde{\gamma} : [0, t_1] \to C_0 \) such that \( \tilde{\gamma}(0) = p \) and \( \eta_0 \circ \tilde{\gamma}(t) = \gamma(t) \), for every \( t \in [0, t_1] \). Put \( \tilde{z} := \tilde{\gamma}(t_1) \in C_0^1 \), and take \( \tilde{v} \in \Sigma_2(C_0) \) such that \( (d\eta_0)_z(\tilde{v}) = \frac{d}{dt} \gamma(t_0) \in \Sigma_2(X) \).

Let \( \tilde{\gamma}_1 : [0, t_1] \to C_0 \) and \( \gamma_1 : [0, t_1] \to X \) be the reversed geodesic to \( \tilde{\gamma} \) and \( \gamma : [0, t_1] \to X \) be the reversed geodesic to \( \gamma \) and \( \gamma_1(t) = \tilde{\gamma}(t_0 - t), \gamma_1(t) = \gamma(t_1 - t) \), and set \( \tilde{\gamma}_2(t) := f(\tilde{\gamma}_1(t)) \). Since \( (d\eta_0)_z \) preserves norm and is 1-Lipschitz, we have
\[
\angle(\tilde{v}, \tilde{\gamma}_i'(0)) \geq \angle \left( \frac{d}{dt} \gamma(t_1), \frac{d}{dt} \gamma_1(0) \right) = \pi,
\]
for \( i = 1, 2 \). Since \( \tilde{\gamma}_1'(0) \neq \tilde{\gamma}_2'(0) \), this is impossible in the Alexandrov space \( C_0 \). \( \square \)

**Lemma 4.29.** For every \( x, y \in X \), let \( \gamma : [0, 1] \to X \) be a minimal geodesic joining \( x \) to \( y \), and let \( p \in C_0 \) be such that \( \eta_0(p) = x \). Then there exists a unique minimal geodesic \( \tilde{\gamma} : [0, 1] \to C_0 \) starting from \( p \) such that \( \eta_0 \circ \tilde{\gamma} = \gamma \).

In particular, if \( X_1 \) is not empty, then \( C_0 \) is connected.

**Proof.** From Lemmas 4.28 and the discussion there using non-branching property of geodesics in Alexandrov spaces, we have only the following possibilities:
- (1) \( \gamma \) is included in \( X_1 \) or \( X_2 \);
- (2) only one end point of \( \gamma \) is contained in \( X_1 \) and the other part of \( \gamma \) is included in \( X_2 \).

The conclusion follows immediately from Lemmas 4.25 and 4.26. \( \square \)

**Proposition 4.30.** \( N = X^{\text{int}} \) is isometric to \( C_0^{\text{int}}/f \).
Proof. In the case of \( X = X_1 \) or \( X = X_2 \), the conclusion follows from Lemma \( \ref{lemma4.26} \) or Proposition \( \ref{prop4.25} \) respectively. Next assume that both \( X_1 \) and \( X_2 \) are non-empty. We set \( Z := C_0^\mathrm{int}/f \), which is an Alexandrov space, and decompose \( Z \) as

\[
Z = Z_1 \cup Z_2, \quad Z_i := C_i^0/f, \quad i = 1, 2.
\]

For every \([p] \in Z_1\), \( \Sigma_{[p]}(Z) \) is isometric to \( \Sigma_{[p]}(C_0)/f_* \), where \( f_* : \Sigma_p(C_0) \to \Sigma_p(C_0) \) is an isometry induced by \( f \). Since \( X_1 \) is a proper subset of \( X \), \( f_* \) defines a non-trivial isometric \( Z_2 \)-action on \( \Sigma_{[p]}(C_0) \). Thus \([p] \) is a single point of \( Z \): \([p] \in Z^\mathrm{sing} \), and therefore \( Z_1 \subset Z^\mathrm{sing} \).

Conversely since \( X_2 \) is convex in \( X \) by Lemma \( \ref{lemma4.28} \), \( F_0^{-1} \) defines a 1-Lipschitz map \( G_1 : (X_2)^\text{ext} \to Z_2 \) which extends to a 1-Lipschitz map \( G : X \to Z \) satisfying \( G \circ F = 1_Z \). Therefore \( X \) must be isometric to \( Z \). □

Proof of Theorem \( \ref{thm1.2} \). By Proposition \( \ref{prop4.27} \) \( f : C_0^\mathrm{int} \to C_0^\mathrm{int} \) is an involutive isometry. By Propositions \( \ref{prop3.8} \) and \( \ref{prop4.30} \), \( N \) is isometric to \( C_0^\mathrm{int}/f \). Since \( C_0^\mathrm{int} \) is an Alexandrov space with curvature \( \geq c(\kappa, \lambda) \), so is \( N \). □

In view of Proposition \( \ref{prop2.5} \), Proposition \( \ref{prop4.30} \) immediately implies

**Corollary 4.31.** \( X_1 \) is an extremal subset of \( X^\mathrm{int} \).

**Theorem 4.32.** Let \( M_i \in \mathcal{M}(n, \kappa, \lambda, d) \) inradius collapse to a compact length space \( N \). Let \( M_i \) Gromov-Hausdorff converge to \( Y \), and \( M_i^\mathrm{ext} \) converge to \( X \subset Y \) under the convergence \( \tilde{M}_i \to Y \). Then

1. \( X^\mathrm{int} \) is isometric to \( N \);
2. \( Y \) is isometric to \( C_0^\mathrm{int} \times_{\tilde{\phi}} [0, t_0]/(f(x), 0) \sim (x, 0) \), or equivalently, isometric to the following quotient by an isometric involution \( \tilde{f} = (f, -\text{id}) \):

\[
C_0^\mathrm{int} \times_{\tilde{\phi}} [-t_0, t_0]/\tilde{f},
\]

where \( \tilde{\phi}(t) = \phi(|t|) \).

In particular, \( Y \) is a singular \( I \)-bundle over \( N \), where singular fibers occur exactly on \( X_1 \) unless \( X = X_1 \).

Compare Examples \( \ref{example3.14} \), \( \ref{example3.17} \) and \( \ref{example3.18} \).

**Proof of Theorem 4.32**. (1) is just Proposition \( \ref{prop3.8} \). (2) follows immediately from Propositions \( \ref{prop3.9} \) and \( \ref{prop4.30} \). □

**Remark 4.33.** Theorem \( \ref{thm4.32} \) can be generalized to the unbounded diameter case (see Section \( \ref{section6} \).
Proposition 4.34. If \( x \in X_1 \), then \( \Sigma_x(X) \) is isometric to the quotient space \( \Sigma_p(C_0)/f_* \), and \( \Sigma_x(Y) \) is isometric to the quotient space \( \Sigma_p(C)/f_* \), where \( f_* : \Sigma_p(C_0) \to \Sigma_p(C_0) \) is an isometry induced by \( f \).

Proof. Take an \( f \)-invariant neighborhood \( U_p \) of \( p \) in \( C_0 \), where \( \eta_0(p) = x \). It is easy to check that \( V_x := \eta_0(U_p) \) is a neighborhood of \( x \) isometric to \( U_p/f \). The conclusion of (2) follows immediately. \( \square \)

Corollary 4.35. Let \( \dim N = m \). Suppose that both \( X_1 \) and \( X_2 \) is non-empty. Then every element \( x \in X_1 \) satisfies that

\[
\text{vol} \Sigma_x(X) \leq \frac{1}{2} \text{vol} S^{m-1}.
\]

In particular, \( \dim(X_1 \cap \partial X) \leq m - 1 \) and \( \dim(X_1 \cap \text{int} X) \leq m - 2 \).

Proof. For \( x \in X_1 \), take \( p \in C_0 \) with \( \eta_0(p) = x \). Note that \( C_0 \) is connected by Lemma 4.29. If \( f_* : \Sigma_p(C_0) \to \Sigma_p(C_0) \) is the identity, then the non-branching property of geodesics in Alexandrov spaces implies that \( f \) is the identity on \( C_0 \). Therefore \( f_* \) must be non-trivial on \( \Sigma_p(C_0) \). The conclusion follows since

\[
\text{vol} \Sigma_x(X) = (1/2)\text{vol} \Sigma_p(C_0) \leq (1/2)\text{vol} S^{m-1}.
\]

By Corollary 4.35 if every \( x \in X \) satisfies that

\[
\text{vol} \Sigma_x(X) > (1/2)\text{vol} S^{m-1},
\]

then \( X = X_1 \) or \( X = X_2 \).

Next let us consider such a case. If \( X = X_1 \), then by Lemma 4.26 \( \eta_0 \) is an isometry. If \( X = X_2 \), then by Lemma 4.25 \( \eta_0 \) is a locally isometric double covering. Therefore it is straightforward to see the following.

Corollary 4.36. If \( X = X_1 \) or \( X_2 \), then \( Y \) can be classified by \( N \) as follows.

1. if \( X = X_1 \), then \( Y \) is isometric to \( N \times \phi[0,t_0]\).
2. if \( X = X_2 \), then either \( Y \) is isometric to the gluing
   \[
   N \times \hat{\phi}[-t_0,t_0],
   \]
   with length metric, or else, \( Y \) is a nontrivial \( I \)-bundle over \( N \), and is doubly covered by
   \[
   C_0^{\text{int}} \times \hat{\phi}[-t_0,t_0],
   \]
   where \( \hat{\phi}(t) = \phi(|t|) \).

Compare Examples 3.14 and 3.17.

From now, we write for simplicity as \( C_0 := C_0^{\text{int}} \).
5. Inradius collapsed manifolds with bounded diameters

In this section, we investigate the structure of inradius collapsed manifolds \( M_i \) applying the structure results for limit spaces in Section \[ \text{[1]} \]. First, we study the case of inradius collapse of codimension one to determine the manifold structure. To carry out this, some additional considerations on the limit spaces are needed to determine the singularities of singular \( I \)-fibered spaces. In the second part of this section, we study inradius collapse to almost regular spaces.

5.1. Inradius collapse of codimension one. We consider \( M_i \in \mathcal{M}(n, \kappa, \lambda, d) \) inradius collapse to an \((n-1)\)-dimensional Alexandrov space \( N \). Then by Theorem \[ \text{[2.2]} \] \( M_i \) is homeomorphic to \( Y \), and by Theorem \[ \text{[4.32]} \] we have

\[
Y = C_0 \times \rho [\{-t_0, t_0\}] / \tilde{f}, \quad N = C_0 / f,
\]

where \( \tilde{f} = (f, -\text{id}) \) is an isometric involution. The singular locus of the singular \( I \)-bundle structure on \( Y \) defined by the above form coincides with \( C_0^1 \) unless \( X \neq X_1 \). Later in Lemma \[ \text{[5.3]} \] we show that \( \eta_0(C_0^1) = \partial N \).

Assuming that \( N \) has non-empty boundary, we begin with construction of singularity models of singular \( I \)-fibered spaces around each boundary component of the limit space \( N \).

By Proposition \[ \text{[2.6]} \] each component \( \partial_\alpha N \) of \( \partial N \) has a collar neighborhood \( V_\alpha \). Let \( \varphi : V_\alpha \to \partial_\alpha N \times [0, 1) \) be a homeomorphism. Let \( \pi : Y \to N \) be the projection. Then \( I \)-fiber structure on \( \pi^{-1} \varphi^{-1}\left(\{p\} \times [0, 1)\right) \) is isomorphic to the form

\[
R_{t_0} := [0, 1) \times [-t_0, t_0] / (0, y) \sim (0, -y),
\]

with the projection \( \pi : R_{t_0} \to [0, 1) \) induced by \( (x, y) \to x \). Therefore \( \pi^{-1}(V_\alpha) \) is an \( R_{t_0} \)-bundle over \( \partial_\alpha N \).

Now we define two singularity model for the singular \( I \)-bundle \( \pi^{-1}(V_\alpha) \): one is the case when \( \pi^{-1}(V_\alpha) \) is a trivial \( R_{t_0} \)-bundle over \( \partial_\alpha N \), and the other one is the case of non-trivial \( R_{t_0} \)-bundle.

**Definition 5.1.** (1). First, set

\[
\mathcal{U}_1(\partial_\alpha N) := \partial_\alpha N \times R_{t_0},
\]

and define \( \pi : \mathcal{U}_1(\partial_\alpha N) \to \partial_\alpha N \times [0, 1) \) by \( \pi(p, x, y) = (p, x) \) for \( (p, x, y) \in \partial_\alpha N \times R_{t_0} \). This gives \( \mathcal{U}_1(\partial_\alpha N) \) the structure of a singular \( I \)-bundle over \( \partial_\alpha N \times [0, 1) \) whose singular locus is \( \partial_\alpha N \times 0 \). We call this the product singular \( I \)-bundle model around \( \partial_\alpha N \).

(2). For the second model, suppose that \( \partial_\alpha N \) admits a double covering space \( \rho : P_\alpha \to \partial_\alpha N \) with the deck transformation \( \varphi \). Let

\[
\mathcal{U}_2(\partial_\alpha N) := (P_\alpha \times R_{t_0}) / \Phi,
\]

where \( \Phi \) is the isometric involution on \( P_\alpha \times R_{t_0} \) defined by \( \Phi = (\varphi, g) \), where \( g : R_{t_0} \to R_{t_0} \) is the reflection induced from \( (x, y) \to (x, -y) \).
Define \( \pi : \mathcal{U}_2(\partial \alpha N) \to \partial \alpha N \times [0,1) \) by \( \pi([p,x,y]) = (\rho(p), x) \) for \((p,x,y) \in P_\alpha \times R_\epsilon \). This gives \( \mathcal{U}_2(\partial \alpha N) \) the structure of a singular \( I \)-bundle over \( \partial \alpha N \times [0,1) \) whose singular locus is \( \partial \alpha N \times 0 \). The second model is a twisted one, and is doubly covered by the first model \( \mathcal{U}_1(P_\alpha) = P_\alpha \times R_\epsilon \). We call this the twisted singular \( I \)-bundle model around \( \partial \alpha N \).

**Example 5.2.** Let us consider the codimension one inradius collapse in Example 3.18. Recall that the limit space \( Y \) of \( \tilde{M}_\epsilon \) is isometric to the form

\[
Y = D(E) \times _\delta [-t_0,t_0]/(x,t) \sim (r(x),-t),
\]

where \( r : D(E) \to D(E) \) denotes the canonical reflection of \( D(E) \). If \( \pi : Y \to E \) denotes the projection, then \( \pi^{-1}(V) \) is isomorphic to the product singular \( I \)-bundle model around \( \partial E \), where \( V \) is any collar neighborhood of \( \partial E \).

**Example 5.3.** Let \( Q_\epsilon \) denote the space obtained from the disjoint union of two copies of the completion \( \tilde{R}_\epsilon \) of \( R_\epsilon \) glued along each segment \( 1 \times [-\epsilon,\epsilon] \) of the boundaries:

\[
Q_\epsilon = \tilde{R}_\epsilon \coprod_{[-\epsilon,\epsilon]} \tilde{R}_\epsilon.
\]

Let \( r : Q_\epsilon \to Q_\epsilon \) be the reflection induced from \((x,y) \to (x,-y)\). Let \( M_\epsilon = (S^1(1) \times Q_\epsilon)/(z,p) \sim (-z,r(p)) \). As \( \epsilon \to 0 \), \( M_\epsilon \) inradius collapses to \( S^1(1/2) \times [0,2] \). Let \( \pi_\epsilon : M_\epsilon \to S^1(1/2) \times [0,2] \) be the projection induced by \([z,(x,y)] \to (z,x)\). Then both \( \pi_\epsilon^{-1}(S^1(1/2) \times [0,1]) \) and \( \pi_\epsilon^{-1}(S^1(1/2) \times (1,2]) \) are solid Klein bottle and their \( I \)-fiber structures are isomorphic to the twisted singular \( I \)-bundle model around respective boundary of \( S^1(1/2) \times [0,2] \).

The following is a detailed version of Theorem 1.3. Recall that \( D^2_+ \) is the upper half disk on \( xy \)-plane, and \( J := D^2_+ \cap \{y = 0\} \).

**Theorem 5.4.** Let \( M_\epsilon \in \mathcal{M}(n,\kappa,\lambda,d) \) inradius collapse to an \( (n - 1) \)-dimensional Alexandrov space \( N \). Then there is a singular \( I \)-fiber bundle:

\[
I \to M_\epsilon \xrightarrow{\pi} N
\]

whose singular locus coincides with \( \partial N \), and \( M_\epsilon \) is a gluing of \( I \)-bundle \( N \tilde{\times} I \) over \( N \) and \( D^2_+\)-bundle \( \partial N \tilde{\times} D^2_+ \) over \( \partial N \),

\[
M_\epsilon = N \tilde{\times} I \cup \partial N \tilde{\times} D^2_+,
\]

where the gluing is done via \( \partial N \tilde{\times} I = \partial N \tilde{\times} J \), and \( \tilde{\times} \) denotes either the product or a twisted product. In particular \( M_\epsilon \) has the same homotopy type as \( N \).

More precisely,

(1) If \( N \) has no boundary, then \( M_\epsilon \) is homeomorphic to a product \( N \times I \) or a twisted product \( N \times \tilde{I} \).
If $N$ has non-empty boundary, each component $\partial_\alpha N$ of $\partial N$ has a neighborhood $V$ such that $\pi^{-1}(V)$ is isomorphic to either the product or the twisted singular $I$-fiber bundle around $\partial_\alpha N$;

(3) If $\pi^{-1}(V)$ is isomorphic to the product singular $I$-fiber bundle for some component $\partial_\alpha N$, then $M_i$ is homeomorphic to $D(N) \times [-1, 1]/(x, t) \sim (r(x), -t)$, where $r$ is the canonical reflection of the double $D(N)$.

Recall that

$$Y = C_0 \times_{\tilde{\phi}} [-t_0, t_0]/\tilde{f},$$

where $\tilde{f} = (f, -\text{id})$, $C_0$ and $Y$ are the noncollapsing limit of $(\partial M_i)^{\text{int}}$ and $M_i$ respectively. Therefore both $C_0$ and $Y \setminus C_{t_0}$ are smoothable spaces in the sense of [13]. See also Remark 2.11.

Let $F \subset C_0$ denote the fixed point set of the isometry $f : C_0 \to C_0$. By Proposition 2.5 and Theorem 2.3, $\eta_0(F)$ is an extremal subset of $N$ and it has a topological stratification.

**Lemma 5.5.** $\eta_0(F)$ coincides with $\partial N$ if $f$ is not the identity.

We postpone the proof of Lemma 5.5 for a moment.

**Proof of Theorem 5.4** (1) By Lemma 5.5 if $N$ has no boundary, $F$ is empty, and therefore either $N = N_1$ or $N = N_2$. If $N = N_1$, then $C_0 = N$ and $Y$ is homeomorphic to $N \times I$. If $N = N_2$, then $N = C_0/f$ has no boundary, and $Y$ is homeomorphic to either $N \times I$ or $C_0 \times [-1, 1]/(x, t) \sim (f(x), -t)$ which is a twisted $I$ bundle over $N$.

(2) Suppose $N$ has non-empty boundary. Note that $N_1 = \eta_0(F)$.

By Proposition 2.6 each component $\partial_\alpha N$ of $\partial N$ has a collar neighborhood $V_\alpha$. Let $\varphi : V_\alpha \to \partial_\alpha N \times [0, 1)$ be a homeomorphism. Let $\pi : Y \to N$ be the projection. By the $I$-fiber structure of $Y$, $\pi^{-1}(\varphi^{-1}(x \times [0, 1)))$ is canonically homeomorphic to $R_{t_0}$. In particular $\pi^{-1}(V_\alpha)$ is an $R_{t_0}$-bundle over $\partial_\alpha N$. If this bundle is trivial, $\pi^{-1}(V_\alpha)$ is isomorphic to the product singular $I$-bundle structure $\mathcal{U}_1(\partial_\alpha N) = \partial_\alpha N \times R_{t_0}$.

Suppose that this bundle is nontrivial, and let $P_\alpha$ be the boundary of $\pi^{-1}(\varphi^{-1}(\partial_\alpha N \times \{1/2\}))$, which is a double covering of $\partial N_\alpha$. Let $\Phi = (\varphi, g)$, and $\rho : P_\alpha \to \partial_\alpha N$ the projection.

**Lemma 5.6.** $\pi^{-1}(V_\alpha)$ is isomorphic to the twisted singular $I$-bundle structure $\mathcal{U}_2(\partial_\alpha N) = (P_\alpha \times R_{t_0})/\Phi$.

**Proof.** Note that

$$\mathcal{U}_2(\partial_\alpha N) := (P_\alpha \times R_{t_0})/(p, x, y) \sim (\varphi(p), x, -y),$$

$$\pi^{-1}(V_\alpha) = \pi^{-1}(\varphi^{-1}(\partial_\alpha N \times [0, 1)).$$

We define a map $\Psi : \mathcal{U}_2(\partial_\alpha N) \to \pi^{-1}(V_\alpha)$ as follows: Note that for each $(p, x) \in P_\alpha \times [0, 1)$, $\{p, \varphi(p)\}$ can be identified with with the
boundary of the $I$-fiber $I_{\rho(p),x} := \pi^{-1}\varphi^{-1}(\rho(p) \times \{x\})$. Define $\Psi(p,x,y)$, $-t_0 \leq y \leq t_0$, be the arc on the fiber $I_{\rho(p),x}$ from $p$ to $\varphi(p)$. It is easy to see that $\Psi : U_2(\partial_\alpha N) \to \pi^{-1}(V_\alpha)$ gives an isomorphism between $I$ fibred spaces. 

(3) Put $\text{int}N := N \setminus \partial N$ for simplicity.

**Assertion 5.7.** There is an isometric imbedding $g : N \to C_0$ such that $\eta_0 \circ g = 1_N$.

**Proof.** Set $F_\alpha := \eta_0^{-1}(\partial_\alpha N)$. From the assumption, we may assume that $F_\alpha$ is two-sided in the sense that the complement of $F_\alpha$ in some connected neighborhood of it is disconnected. Thus there is a connected neighborhood $V_\alpha$ of $\partial_\alpha N$ in $\text{int}N$ for which there is an isometric imbedding $g_\alpha : V_\alpha \to C_0 \setminus F$ such that $\eta_0 \circ g_\alpha = 1_{V_\alpha}$.

Let $W$ be the maximal connected open subset of $\text{int}N$ for which there is an isometric imbedding $g_0 : W \to C_0 \setminus F$ such that $\eta_0 \circ g_0 = 1_W$ and $g_0(W) \supset g_\alpha(V_\alpha)$. We only have to show that $W = \text{int}N$. Otherwise, there is a point $x \in \partial W \cap \text{int}N$. Take a connected neighborhood $W_x$ of $x$ in $\text{int}N$ such that $\eta_0^{-1}(W_x)$ is a disjoint union of open sets $U_1$ and $U_2$ such that $\eta_0 : U_i \to W_x$ is an isometry for $i = 1, 2$. Obviously one of $U_i$, say $U_1$, meets $g_0(W)$ and the other does not. We extend $g_0$ to $g_1 : W \cup W_x \to C_0 \setminus F$ by requiring $g_1|_{W_x} = \eta_0^{-1} : W_x \to U_1$. Since $g_1$ is an isometric imbedding, this is a contradiction to the maximality of $W$.

Thus we have an isometric imbedding $g_0 : \text{int}N \to C_0 \setminus F$. Since $\text{int}N$ is convex and $\eta_0$ is 1-Lipschitz, $g_0$ preserves the distance. It follows that $g_0$ extends to an isometric imbedding $g : N \to C_0$ which preserves distance. 

**Assertion 5.7** shows that every component of $F$ is two-sided. It follows that $C_0 = D(N)$, and that $f$ is the reflection of the double $D(N)$. This completes the proof of Theorem 5.4.

**Proof of Lemma 5.5.** Obviously $\partial N \subset \eta_0(F)$. Suppose that $\eta_0(F) \cap (\text{int}N)$ is not empty.

**Sublemma 5.8.** $\dim(\eta_0(F) \cap \text{int}N) \leq m - 2$, where $m := \dim N$.

**Proof.** If $\dim(\eta_0(F) \cap \text{int}N) = m - 1$, then the top-dimensional strata $S$ of $\eta_0(F) \cap \text{int}N$ is a topological $(m-1)$-manifold, and therefore it meets the $m$-dimensional strata of $N$ because $N^{\text{sing}} \cap \text{int}N$ has codimension $\geq 2$ (Theorem 2.1). Take $p \in \eta_0^{-1}(S)$. It is now easy to see that $f$ is the reflection with respect to $\eta_0^{-1}(S)$ in a small neighborhood of $p$. It follows that $S$ is a subset of $\partial N$, contradiction to the hypothesis.

Take a point $x = \eta_0(p) \in \eta_0(F) \cap \text{int}N$, and consider the directional derivatives $f_* : \Sigma_\alpha(C_0) \to \Sigma_\alpha(C_0)$ of $f$ at $p$ which is again an isometric involution with fixed point set $F_* := \Sigma_\alpha(F)$. 

By Corollary 4.3 and Sublemma 5.8, \( \dim F_* \leq m - 3 \) while \( \dim \Sigma_p(C_0) = m - 1 \). Repeating this we have a finite sequence of directional derivatives of \( f, f_*, \ldots \), each of which is an isometric involution:

\[
f_{*k} : \Sigma_{*k}(C_0) \to \Sigma_{*k}(C_0),
\]

where \( \Sigma_{*k}(C_0) \) denotes a \( k \) iterated space of directions,

\[
\Sigma_{*k}(C_0) = \Sigma_{*k-1}(\cdots (\Sigma_1(\Sigma_p(C_0))) \cdots),
\]

and \( \xi_i \) is taken from the fixed point set of the iterated directional derivatives:

\[
\xi_1 \in \Sigma_p(F), \xi_2 \in \Sigma_{*1}(F), \ldots, \xi_k \in \Sigma_{*k-1}(F_{*(k-1)}),
\]

and \( F_{*i} \) denotes the fixed point set of \( f_{*i} : \Sigma_{*i}(C_0) \to \Sigma_{*i}(C_0) \) which coincides with \( F_{*i} = \Sigma_{*i-1}(F_{*(i-1)}) \).

Note that the iterated space of directions \( \Sigma_{*k}(C_0) \) has dimension \( m - k \), and the iterated fixed point set \( F_{*k} \subset \Sigma_{*k}(C_0) \) has dimension \( \leq m - k - 2 \). It follows that for some \( k \leq m - 2 \), \( F_{*k} \) becomes a finite set. It follows that for any \( \xi_{k+1} \in F_{*k} \),

\[
f_{*(k+1)} : \Sigma_{*k+1}(\Sigma_{*k}(C_0)) \to \Sigma_{*k+1}(\Sigma_{*k}(C_0))
\]

has no fixed points. Put

\[
D := C_0 \times \varphi [-t_0, t_0],
\]

and let \( \tilde{f} \) be an isometric involution on \( D \) defined by \( \tilde{f} = (f, -\text{id}) \).

From Theorem 4.32, \( Y = D/\tilde{f} \).

Let \( x = \eta_0(p), p = (p, 0), \xi_i \in \Sigma_{*i-1}(F_{*(i-1)}), 1 \leq i \leq k + 1, \) be as above. Note that

\[
\Sigma_x(Y) = \Sigma_p(D)/f_{*k}, \Sigma_x(X) = \Sigma_p(C_0)/f_*.
\]

Let \( \zeta_1 \in \Sigma_x(\eta_0(F)) \subset \Sigma_x(X) \subset \Sigma_x(Y) \) be the element corresponding to \( \xi_1 \in \Sigma_p(F) \subset \Sigma_p(C_0) \subset \Sigma_p(D) \). Note that

\[
\Sigma_p(D) = \{\xi_+\} * \Sigma_p(C_0)
\]

and \( \tilde{f}_i = (f, -\text{id}) \) interchanges \( \xi_+ \) and \( \xi_- \) and preserves \( \Sigma_p(C_0) \). Next consider

\[
\Sigma_{\zeta_1}(\Sigma_x(Y)) = \Sigma_{\zeta_1}(\Sigma_p(D))/\tilde{f}_{*\ast},
\]

where \( \tilde{f}_{*\ast} \) denotes the directional derivatives of \( f_* \) at \( \zeta_1 \). Note that \( \Sigma_{\zeta_1}(\Sigma_p(D)) \) is still isometric to \( \{\xi_\ast\} * \Sigma_{\xi_1}(\Sigma_p(C_0)) \) and \( \tilde{f}_{*\ast} = (f_{*\ast}, -\text{id}) \) interchanges \( \xi_+ \) and \( \xi_- \) and preserves \( \Sigma_{\zeta_1}(\Sigma_p(C_0)) \). Similarly and finally we consider

\[
(5.31) \quad \Sigma_{\zeta_{k+1}}(\Sigma_{*k}(Y)) = \Sigma_{\zeta_{k+1}}(\Sigma_{*k}(D))/\tilde{f}_{*k+1},
\]

where \( \zeta_{k+1} \in \Sigma_{*k}(Y) \) is the element corresponding to \( \xi_{k+1} \in \Sigma_{*k}(D) \), and \( \tilde{f}_{*k+1} = (f_{*k+1}, -\text{id}) \) freely acts on \( \Sigma_{\zeta_{k+1}}(\Sigma_{*k}(D)) \). Recall that

\[
\ell := \dim \Sigma_{\zeta_{k+1}}(\Sigma_{*k}(D)) = m - k \geq 2.
\]
Note that the iterated spaces of directions of the smoothable spaces $Y \setminus \partial C_t$ must be all homeomorphic to spheres (Theorem 2.7). However (5.31) shows that $\Sigma_{k+1}(\Sigma_{\ast k}(Y))$ is homeomorphic to a quotient $S^\ell/\mathbb{Z}_2$ for $\ell \geq 2$ by a free $\mathbb{Z}_2$-action, which is a contradiction. This completes the proof of Lemma 5.5.

□

5.2. Inradius collapse to almost regular spaces. Next we consider the case where $M_i$ inradius collapses to almost regular Alexandrov space $N$. The idea of using an equivariant fibration-capping theorem in [37] was inspired by a recent work [20].

First we recall this theorem. Let $X$ be a $k$-dimensional complete Alexandrov space with curvature $\geq \kappa$ possibly non-empty boundary. We denote by $D(X)$ the double of $X$, which is also an Alexandrov space with curvature $\geq \kappa$ (see [23]). By definition, $D(X) = X \cup X^*$ glued along their boundaries, where $X^*$ is another copy of $X$.

A $(k, \delta)$-strainer $\{(a_i, b_i)\}$ of $D(X)$ at $p \in X$ is called admissible if $a_i \in X, b_j \in X$ for every $1 \leq i \leq k, 1 \leq j \leq k - 1$ (clearly, $b_k \in X^*$ if $p \in \partial X$ for instance). Let $R_D^\delta(X)$ denote the set of points of $X$ at which there are admissible $(k, \delta)$-strainers. It has the structure of a Lipschitz $k$-manifold with boundary. Note that every point of $R_D^\delta(X) \cap \partial X$ has a small neighborhood in $X$ almost isometric to an open subset of the half space $\mathbb{R}^k_+$ for small $\delta$.

If $Y$ is a closed domain of $R_D^\delta(X)$, then the $\delta_D$-strain radius of $Y$ is defined as the infimum of positive numbers $\ell$ such that there exists an admissible $(k, \delta)$-strainer of length $\geq \ell$ at every point in $Y$, denoted by $\delta_D\text{-str.rad}(Y)$.

For a small $\nu > 0$, we put

$$Y_\nu := \{x \in Y \mid d(\partial X, x) \geq \nu\}.$$ 

We use the following special notations:

$$\partial_0 Y_\nu := Y_\nu \cap \{d_{\partial X} = \nu\}, \quad \text{int}_0 Y_\nu := Y_\nu - \partial_0 Y_\nu.$$ 

Let $M^n$ be another $n$-dimensional complete Alexandrov space with curvature $\geq \kappa$ having no boundary. Let $R_\delta(M)$ denote the set of all $(n, \delta)$-strained points of $M$.

A surjective map $f : M \to X$ is called an $\epsilon$-almost Lipschitz submersion if

1. it is an $\epsilon$-approximation;
2. for every $p, q \in M$

$$\left| \frac{d(f(p), f(q))}{d(p, q)} - \sin \theta_{p,q} \right| < \epsilon,$$

where $\theta_{p,q}$ denotes the infimum of $\angle qpx$ when $x$ runs over $f^{-1}(f(p))$.

Now let a Lie group $G$ act on $M^n$ and $X$ as isometries. Let

$$d_{e, GH}((M, G), (X, G))$$
denote the equivariant Gromov-Hausdorff distance as defined in Section 2.1. We need to assume the following on the existence of slice for $G$-orbits:

**Assumption 5.9.** For each $p \in X$, there is a slice $L_p$ at $p$. Namely $U_p := GL_p$ provides a $G$-invariant tubular neighborhood of $Gp$ which is $G$-isomorphic to $G \times_{G_p} L_p$.

Obviously Assumption 5.9 is automatically satisfied if $G$ is discrete. By [12], Assumption 5.9 also holds true if $G$ is compact.

**Theorem 5.10 (Equivariant Fibration-Capping Theorem([34], Thm 18.9)).** Let $X$ and $G$ be as above such that $X/G$ is compact. Given $k$ and $\mu > 0$ there exist positive numbers $\delta = \delta_k(\mu)$ and $\nu = \nu_{X,G}(\mu)$ satisfying the following: Suppose $X = R^D(\delta, \nu)$ and $d_{eGH}((M,G), (X,G)) < \epsilon$ for some $\epsilon \leq \epsilon_{X,G}(\mu)$. Then there exists a $G$-invariant decomposition

$$M = M_{\text{int}} \cup M_{\text{cap}}$$

of $M$ into two closed domains glued along their boundaries, and a $G$-equivariant Lipschitz map $f : M \to X_{\nu}$ such that

1. $M_{\text{int}}$ is the closure of $f^{-1}(\text{int}_0 X_{\nu})$, and $M_{\text{cap}} = f^{-1}(\partial_0 X_{\nu})$;
2. the restrictions $f|_{M_{\text{int}}} : M_{\text{int}} \to X_{\nu}$ and $f|_{M_{\text{cap}}} : M_{\text{cap}} \to \partial_0 X_{\nu}$ are
   a. locally trivial fiber bundles;
   b. $\tau(\delta, \nu, \epsilon/\nu)$-Lipschitz submersions.

Here, $\tau(\epsilon_1, \ldots, \epsilon_k)$ denotes a function depending on a priori constants and $\epsilon_i$ satisfying

$$\lim_{\epsilon_i \to 0} \tau(\epsilon_1, \ldots, \epsilon_k) = 0.$$ 

**Remark 5.11.** If $X$ has no boundary, then $X_{\nu}$ is replaced by $X$, $M_{\text{cap}} = \emptyset$ and $M = N$ in the statement above.

We go back to the situation of Theorem 1.4. Assume that $M_i$ inradius collapses to an almost regular Alexandrov space $N$. Let us consider the double and the partial double of $	ilde{M}_i$ and $Y$ respectively

$$D(\tilde{M}_i) := \tilde{M}_i \sqcup_{\partial \tilde{M}_i} \tilde{M}_i, \quad W := Y \sqcup_{C_{t_0}} Y,$$

where two copies of $Y$ are glued along $C_{t_0}$. From Perelman’s result [23], both $D(\tilde{M}_i)$ and $W$ are Alexandrov space. Note that both $D(\tilde{M}_i)$ and $W$ admit canonical isometric $\mathbb{Z}_2$ actions by the reflections.

The proof of the following lemma is standard, and hence omitted.

**Lemma 5.12.** $(D(\tilde{M}_i), \mathbb{Z}_2)$ converges to $(W, \mathbb{Z}_2)$ with respect to the equivariant Gromov-Hausdorff convergence.

**Proof of Theorem 1.4.** By Lemma 5.12 for any $\epsilon > 0$, if $i$ is large, $d_{eGH}((D(\tilde{M}_i), \mathbb{Z}_2), (W, \mathbb{Z}_2)) < \epsilon$. 

By Theorem 5.32, $Y$ is almost regular possibly with almost regular boundary. Hence, $W = R^2(W)$ and $\delta_{\text{str.rad}}(W) > \mu$ for some $\mu > 0$. Thus by Theorem 5.10 and its remark, there exists a $\mathbb{Z}_2$-equivariant capping fibration

$$\tilde{f}_i : D(\tilde{M}_i) \to W_\nu,$$

where

$$W_\nu = \{ x \in W \mid d(x, \partial W) \geq \nu \}.$$

Notice that $W_\nu$ is homeomorphic to $W$ because of the form of $Y$. Obviously, $\tilde{f}_i$ induces a map $f_i : \tilde{M}_i \to Y$. By the remark after Corollary 4.35, $\eta_0 : C_0 \to X$. By Theorem 5.10, there exists a $\mathbb{Z}_2$-invariant decomposition

$$(5.32) \quad D(\tilde{M}_i) = (D(\tilde{M}_i))_{\text{int}} \cup (D(\tilde{M}_i))_{\text{cap}},$$

do $D(\tilde{M}_i)$ into two closed domains glued along their boundaries such that

1. $(D(\tilde{M}_i))_{\text{int}}$ is the closure of $\tilde{f}_i^{-1}(\text{int}_0 W_\nu)$, and $(D(\tilde{M}_i))_{\text{cap}} = f_i^{-1}(\partial_0 W_\nu)$;
2. $\tilde{f}_i|_{(D(\tilde{M}_i))_{\text{int}}} : (D(\tilde{M}_i))_{\text{int}} \to W_\nu$, $\tilde{f}_i|_{(D(\tilde{M}_i))_{\text{cap}}} : (D(\tilde{M}_i))_{\text{cap}} \to \partial_0 W_\nu$ are locally trivial fiber bundles,

where

$$\partial_0 W_\nu := \{ x \in W \mid d(x, \partial W) = \nu \}, \quad \text{int}_0 W_\nu := W_\nu \setminus \partial_0 W_\nu.$$

Since (5.32) is $\mathbb{Z}_2$-invariant, it induces a decomposition

$$\tilde{M}_i = (\tilde{M}_i)_{\text{int}} \cup (\tilde{M}_i)_{\text{cap}}.$$

Since $\tilde{f}_i$ is $\mathbb{Z}_2$-equivariant, these fibrations induce fibrations

$$F_i \longrightarrow (\tilde{M}_i)_{\text{int}} \longrightarrow Y_\nu,$$

$$\text{Cap}_i \longrightarrow (\tilde{M}_i)_{\text{cap}} \longrightarrow \partial_0 Y_\nu.$$

From construction, $\partial\text{Cap}_i$ is homeomorphic to $F_i$. Note that every cylindrical geodesic in the warped cylinder $C_i \subset \tilde{M}_i$ is almost perpendicular to the fibers (35, 36). This implies that $(\tilde{M}_i)_{\text{int}}$ is homeomorphic to $\partial(\tilde{M}_i)_{\text{int}} \times [0, 1]$, and therefore $M_i$ and hence $M_i$ is homeomorphic to $(\tilde{M}_i)_{\text{cap}}$. Noting $\partial_0 Y_\nu$ is homeomorphic to $N$, we obtain a fiber
bundle
\[ \text{Cap}_i \rightarrow M_i \rightarrow N. \]
This completes the proof. \(\square\)

6. The case of unbounded diameters

In this section we provide the proof of Theorem 1.5. In view of Corollary 4.23, for the proof of Theorem 1.5(1), it suffices to consider inradius collapsed manifolds with unbounded diameters.

Remark 6.1. Theorem 1.5 (1) was stated in [34], Theorem 5, where the following argument was employed: If \( k \geq 3 \) and if \( p \in M \) is the furthest point from \( \partial M \), then \( B(p, r), r = \text{inrad}(M) \), touches \( \partial M \) at least three points. However it seems to the authors that this is unclear.

6.1. Description of pointed inradius collapse. In the case of unbounded diameter, we do not know the uniform boundedness of the numbers of boundary components of inradius collapsed manifolds yet. This forces us to reconsider the descriptions of limit spaces in Section 3.

Let \( \mathcal{M}(n, \kappa, \lambda) \) (resp. \( \mathcal{M}(n, \kappa, \lambda)_{\text{pt}} \)) denote the set of all isometry classes of \( n \)-dimensional complete Riemannian manifolds \( M \) (resp. pointed complete Riemannian manifolds \( (M, p) \) with \( p \in \partial M \)) satisfying
\[ K_M \geq \kappa, \ |\Pi_{\partial M}| \leq \lambda. \]
We carry out the extension procedure for \( M \) to obtain \( \tilde{M} \). Let \( \tilde{\mathcal{M}}(n, \kappa, \lambda)_{\text{pt}} \) denote the set of all \( (\tilde{M}, M, p) \) with \( M \in \mathcal{M}(n, \kappa, \lambda) \) and \( p \in \partial M \). We denote by
\[ \partial_0 \tilde{\mathcal{M}}(n, \kappa, \lambda)_{\text{pt}} \]
the set of all pointed Gromov-Hausdorff limit spaces \( (Y, X, x) \) of sequences \( (\tilde{M}_i, M_i, p_i) \) in \( \tilde{\mathcal{M}}(n, \kappa, \lambda)_{\text{pt}} \) with
\[ \text{inrad}(M_i) \rightarrow 0. \]
From now on, \( (\tilde{M}_i, M_i, p_i) \) and \( (\tilde{M}, M, p) \) are always assumed to be elements in \( \tilde{\mathcal{M}}(n, \kappa, \lambda)_{\text{pt}} \).

Now suppose that a sequence \( (M_i, p_i) \in \mathcal{M}(n, \kappa, \lambda)_{\text{pt}} \) converges to a complete length space \( (N, q) \) with \( \text{inrad}(M_i) \rightarrow 0 \), while \( (\tilde{M}_i, M_i, p_i) \) converges to \( (Y, X, x) \). In a way similar to Proposition 3.8 we see that \( X^{\text{int}} \) is isometric to \( N \).

Next we describe \( Y \) as
\[ Y = X \bigcup_{\eta} C_0 \times_{\eta} [0, t_0], \]
as in the bounded diameter case. In the bounded diameter case, the number of components of \( \partial M_i \) is uniformly bounded, and \( C_0 \) is the component wise Gromov-Hausdorff limit of \( \partial M_i \). In the case of unbounded diameter, we do not know the boundedness of the number of
components of $\partial M_i$ yet. This is why we need a bit careful consideration to define $C_0$, which will be carried out in the following.

Let 
\[ C_{t_0}^Y := \{ y \in Y \mid \|X, y\| = t_0 \}. \]

We begin with the decomposition of $C_{t_0}^Y$ into the connected components:
\[ C_{t_0}^Y = \prod_{\alpha \in A} C_{t_0}^\alpha. \]

Let
\[ C_0^\alpha := \frac{1}{\phi(t_0)} C_{t_0}^\alpha, \quad C_0^\alpha = C_0^\alpha \times [0, t_0], \]

and
\[ C := \prod_{\alpha \in A} C_0^\alpha, \quad C_0 = \prod_{\alpha \in A} C_0^\alpha \times \{0\}. \]

Note that each component of $C$ and $C_0^{\text{int}}$ is an Alexandrov space with curvature $\geq c(\kappa, \lambda)$.

Each $p \in C_0$ can be identified with the element of $C_{t_0}^Y$ which we write as $\eta(p, t_0)$, and there is a unique perpendicular $\gamma^\eta(p, t_0)(t), 0 \leq t \leq t_0$, to $X$ satisfying $\gamma^\eta(p, t_0)(t_0) = \eta(p, t_0)$. We then define the surjective 1-Lipschitz map $\eta : C \to Y$ as
\[ \eta(p, t) = \gamma^\eta(p, t_0)(t). \]

Obviously $\eta : C \setminus C_0 \to Y \setminus X$ is bijective locally isometric map.

Let $\varphi_i : B^M(p_i, 1/\delta_i) \to B^Y(x, 1/\delta_i)$ be a $\delta_i$-approximation, with $\lim \delta_i = 0$. Note that for each component $C_{t_0}^\alpha$ of $C_{t_0}^Y$ and any fixed point $y_\alpha \in C_{t_0}^\alpha$, there is a component, say $\partial^\alpha M_i$, of $\partial M_i$ for which we have a point $q_i^\alpha \in \partial^\alpha M_i$ satisfying $|\varphi_i(q_i^\alpha), y_\alpha| < \delta_i$. For a distinct component $C_{t_0}^\beta$ and $y_\beta \in C_{t_0}^\beta$, we also have a component $\partial^\beta M_i$ of $\partial M_i$ for which there is a point $q_i^\beta \in \partial^\beta M_i$ satisfying $|\varphi_i(q_i^\beta), y_\beta| < \delta_i$. Since $|C_{t_0}^\alpha, C_{t_0}^\beta| \geq 2t_0$, it is easily checked that $\partial^\alpha M_i$ and $\partial^\beta M_i$ are distinct components, and hence $|\partial^\alpha M_i, \partial^\beta M_i| \geq 2t_0$. Thus we have that
\[ \lim_{i \to \infty} (\partial^\alpha \tilde{M}_i, q_i^\alpha) = (C_{t_0}^\alpha, y_\alpha), \quad \lim_{i \to \infty} (\partial^\beta \tilde{M}_i, q_i^\beta) = (C_{t_0}^\beta, y_\beta), \]

under the convergence $(\tilde{M}_i, p_i) \to (Y, x)$. In particular, the component $\partial^\alpha \tilde{M}_i$ is uniquely determined by $C_{t_0}^\alpha$.

Now for the map
\[ \eta_0 := \eta|_{C_0} : C_0 \to X, \]
from an argument similar to the bounded diameter case, we see that
\[ \# \eta_0^{-1}(x) \leq 2, \]

for every $x \in X$. Thus we can define the involution $f : C_0 \to C_0$ as in Section 4. Note that all the results in Subsections 4.1, 4.2 and 4.3...
still holds true for the present situation of noncompact \(C_0\), because the arguments there are local. In particular, we see that (6.34) the number of components of \(C_0\) is at most two.

Thus we see that all the results in Section 4 holds true except Corollary 4.23, and therefore we conclude that

\[
Y = C_0^{\text{int}} \times \tilde{\phi} \left(-t_0, t_0\right)/\tilde{f},
\]

where \(\tilde{\phi}(t) = \phi(|t|)\), and \(N = X^{\text{int}}\) is isometric to \(C_0^{\text{int}}/f\).

Now we immediately have the following:

**Corollary 6.2.** If \((M_i, p_i) \in \mathcal{M}(n, \kappa, \lambda)\) inradius collapse to \((N, q)\) with respect to the pointed Gromov-Hausdorff convergence, then we have \(\dim M_i > \dim N\).

**Theorem 6.3.** Let a sequence of pointed complete Riemannian manifolds \((M_i, p_i)\) in \(\mathcal{M}(n, \kappa, \lambda)\) inradius collapse to a pointed length space \((N, q)\) with respect to the pointed Gromov-Hausdorff convergence. Then \(N\) is an Alexandrov space with curvature \(\geq c(\kappa, \lambda)\), where \(c(\kappa, \lambda)\) is a constant depending only on \(\kappa\) and \(\lambda\).

To have Corollary 4.23 in the case when \(Y\) is noncompact is a main purpose of the rest of this section.

**Remark 6.4.** Note that (6.34) only shows that there are at most two components of \(\partial M_i\) meeting a bounded region from the reference point \(p_i\). Thus (6.34) does not immediately imply that the number of components of \(\partial M_i\) is at most two. This is because the convergence \((M_i, p_i) \to (Y, p)\) is only under the pointed Gromov-Hausdorff topology. Namely, there is still a possibility that some component of \(\partial M_i\) disappear to infinity under that convergence.

To overcome the difficulty stated in Remark 6.4, we investigate the local connectedness of \(\partial M_i\) in more detail. To carry out this, it is helpful to consider a special pointed Gromov-Hausdorff approximations similar to (3.3), which is verified below.

Let \(\iota_{0M} : \left((\partial M)^{\text{int}}, d_{\partial M^{\text{int}}}\right) \to \left((\partial M)^{\text{ext}}, d_{\partial M^{\text{ext}}}\right)\) be the canonical map, where \((\partial M)^{\text{ext}}\) is equipped with the exterior metric in \(M\). Let \(\omega_M : M \to \partial M\) be a nearest point map. It should be noted that although \(\omega_M\) is not continuous in general, it will not affect the argument below (compare Proposition 3.5).

For \((M, M, p)\) and \((Y, X, x) \in \partial_0\mathcal{M}(n, \kappa, \lambda)_{\text{pt}}\) with

\[
Y = X \bigcup_{\eta_0} C_0^{\text{int}} \times \phi \left[0, t_0\right],
\]

as described above, set

\[
\partial M^{\text{int}}(p, 1/\delta) := (\partial M \cap B^\delta(p, 1/\delta), d_{\partial M^{\text{int}}}),
\]

\[
C_0^{\text{int}}(p_0, 1/\delta) := (C_0 \cap B^C(p_0, 1/\delta, d_{C_0^{\text{int}}}),
\]

where \(\tilde{\phi}(t) = \phi(|t|)\), and \(N = X^{\text{int}}\) is isometric to \(C_0^{\text{int}}/f\).
where \( p_0 \in \eta_0^{-1}(x) \). Note that if \( q, q' \in \partial M^\text{int}(p, 1/\delta) \) belong to distinct components of \( \partial M \), then the distance between them in \( \partial M^\text{int}(p, 1/\delta) \) is infinity: \( d_{\partial M^\text{int}}(q, q') = \infty \). Similarly, we also consider

\[
\partial \tilde{M}^\text{int}(p, 1/\delta) := (\partial \tilde{M} \cap B^\tilde{M}(p, 1/\delta), d_{\partial \tilde{M}}),
\]

\[
C_{t_0}(p_0, 1/\delta) := (C_{t_0} \cap B^C(p_0, 1/\delta), d_{C_{t_0}}).
\]

**Sublemma 6.5.** The number of connected components of \( \partial M \) which intersect \( \partial M^\text{int}(p_i, 1/\delta_i) \) is at most two.

**Proof.** Suppose that there are three points \( q^\alpha_i \in \partial M^\text{int}(p_i, 1/\delta_i) \) which belong to three distinct components \( \partial^\alpha_i M_i, 1 \leq \alpha \leq 3 \). Let \( \hat{q}^\alpha_i \in \partial M(p_i, 1/\delta) \) be the image of \( q^\alpha_i \) under the projection to \( \partial \tilde{M}_i \) along perpendiculars, which belongs to the component \( \partial^\alpha_i \tilde{M}_i \) corresponding to \( \partial^\alpha_i M_i \). Since \( C_0 \) as well as \( C_{t_0} \) has at most two components, we may assume that \( \phi_i(q^1_i) \) and \( \phi_i(q^2_i) \) are in the same component of \( C_0 \). It turns out that \( \phi_i(\hat{q}^1_i) \) and \( \phi_i(\hat{q}^2_i) \) are in the same component of \( C_{t_0} \), which contradicts (6.33). \( \square \)

**Definition 6.6.** For \((\tilde{M}, M, p)\) and \((Y, X, x) \in \partial_0 \mathcal{M}(n, \kappa, \lambda)_{pt}\) with

\[
Y = X \bigcup_{\eta_0} C_0 \times_\phi [0, t_0],
\]

we define the pointed Gromov-Hausdorff distance

\[
d^*_\text{pGH}((\tilde{M}, M, p), (Y, X, x)) \tag{6.35}
\]

as the infimum of those \( \delta > 0 \) such that

1. there exists a component-wise \( \delta \)-approximation \( \psi : (\partial M)^\text{int}(p, 1/\delta) \to C_0^\text{int}(x, 1/\delta) \);
2. the map \( \varphi : B^M(\tilde{M}, p, 1/\delta) \to B^X^\text{ext}(x, 1/\delta) \) defined by

\[
\varphi = \eta_0 \circ \psi \circ t^{-1}_{\partial M} \circ \omega_M
\]

is a \( \delta \)-approximation:

\[
\begin{array}{ccc}
M^\text{ext} & \xrightarrow{\omega_M} & \partial M^\text{ext} \xrightarrow{t^{-1}_M} \partial M^\text{int} \\
\varphi \downarrow & & \psi \\
X^\text{ext} & \xrightarrow{\eta_0} & C^\text{int}_{0}
\end{array}
\]

3. the map \( \Phi : B^\tilde{M}(\tilde{M}, p, 1/\delta) \to B^Y(x, 1/\delta) \) defined by

\[
\Phi(q) = \begin{cases} 
\varphi(q), & \text{if } q \in M \cap B^\tilde{M}(p, 1/\delta) \\
(\varphi(q_0), t), & \text{if } q = (q_0, t) \in \partial M \times_\phi [0, t_0] \cap B^\tilde{M}(p, 1/\delta).
\end{cases}
\]

is a \( \delta \)-approximation.

This definition is justified by the following lemma.
Lemma 6.7. Let $(\tilde{M}_i, M_i, p_i) \in \tilde{\mathcal{MM}}(n, \kappa, \lambda)_{pt}$ converge to $(Y, X, x)$ in $\partial_0 \tilde{\mathcal{MM}}(n, \kappa, \lambda)_{pt}$ for the usual pointed Gromov-Hausdorff topology. Then there exists a component-wise $\delta_i$-approximation
\[
\psi_i : (\partial M_i)^{\text{int}}(p_i, 1/\delta_i) \to C_0^{\text{int}}(x, 1/\delta_i)
\]
with $\lim \delta_i = 0$ such that the maps
\[
\varphi_i : B_{M_i}^{\text{ext}}(p_i, 1/\delta_i) \to B_{X}^{\text{ext}}(x, 1/\delta_i), \Phi_i : B_{\tilde{M}_i}^{\tilde{M}_i}(p_i, 1/\delta_i) \to B_Y(x, 1/\delta_i)
\]
defined as in Definition 6.6 via $\psi_i$ are $\delta_i'$-approximations with $\lim \delta_i' = 0$.

Proof. Let $\lambda_i : B_{\tilde{M}_i}^{\tilde{M}_i}(p_i, 1/\epsilon_i) \to B_Y(x, 1/\epsilon_i)$ be an $\epsilon_i$-approximation with $\lim \epsilon_i = 0$. We may assume that when it is restricted to the boundary, it provides a component-wise $\epsilon_i$-approximation $\lambda_i : B_{\tilde{M}_i}^{\tilde{M}_i}(p_i, 1/\epsilon_i) \cap \partial M_i \to B_Y(x, 1/\epsilon_i) \cap C_0^{\text{师}}$. Since $\partial M_i$ and $C_0^{\text{师}}$ are convex and $1/\phi(t_0)$-homothetic to $(\partial M_i)^{\text{int}}$ and $C_0$ respectively, $\lambda_i$ gives a component-wise $\epsilon_i/\phi(t_0)$-approximation
\[
\psi_i : (\partial M_i)^{\text{int}}(p_i, 1/(\phi(t_0)\epsilon_i)) \to C_0^{\text{师}}(p_0, 1/\phi(t_0)\epsilon_i)).
\]
Let $\delta_i := \phi(t_0)\epsilon_i$, and define $\varphi_i$ and $\Phi_i$ as in Definition 6.6. In a way similar to Proposition 3.5, one can easily show that the restriction
\[
\Phi_i : B_{\tilde{M}_i}^{\tilde{M}_i}(p_i, 1/\delta_i) \setminus M_i \to B_Y(x, 1/\delta_i) \setminus X
\]
is a $\delta_i'$-approximation with $\lim \delta_i' = 0$. In particular this implies that
\[
\varphi_i : B_{M_i}^{\text{ext}}(p_i, 1/\delta_i) \cap \partial M_i \to B_{X}^{\text{ext}}(x, 1/\delta_i)
\]
is also a $\delta_i'$-approximation. Let $\nu_i := \text{inrad}(M_i)$. Since $B_{M_i}^{\text{ext}}(p_i, 1/\delta_i) \cap \partial M_i$ is $\nu_i$-dense in $B_{M_i}^{\text{ext}}(p_i, 1/\delta_i) \cap \partial M_i$, $\varphi_i$ is certainly a $\delta_i''$-approximation with $\lim \delta_i'' = 0$. Since $\Phi_i$ is a natural extension of $\varphi_i$, $\Phi_i$ is also a $\delta_i''$-approximation. \hfill $\Box$

Lemma 6.8. For each $\delta > 0$ there exists a positive number $\epsilon = \epsilon(\delta)$ such that if $(M, p)$ in $\mathcal{MM}(n, \kappa, \lambda)_{pt}$ satisfies $\text{inrad}(M) < \epsilon$, then
\[
d^*_\text{pGH}((\tilde{M}, M, p), (Y, X, x)) < \delta,
\]
for some $(Y, X, x)$ contained in $\partial_0 \tilde{\mathcal{MM}}(n, \kappa, \lambda)_{pt}$.

Proof. Lemma 6.8 follows from Lemma 6.7 and the precompactness of $\tilde{\mathcal{MM}}(n, \kappa, \lambda)_{pt}$ combined with a contradiction argument. \hfill $\Box$

If $(Y, X, x) \in \partial_0 \tilde{\mathcal{MM}}(n, \kappa, \lambda)_{pt}$ satisfies the conclusion of Lemma 6.8 for $(M, p) \in \mathcal{MM}(n, \kappa, \lambda)_{pt}$, we call it a $\delta$-limit of $(\tilde{M}, M, p)$, which is also denoted by $\mathcal{Y}(M, p)$ for simplicity:
\[
\mathcal{Y}(M, p) = (Y, X, x).\]
6.2. **Local/global connectedness of boundary.** In this subsection, using Lemma 6.8 we first investigate the local behavior of connectedness of boundary of a inradius collapsed manifold, and provide the proof of Theorem 1.5.

**Definition 6.9.** Let \((Y, X, x) \in \partial_0 \mathcal{M}(n, \kappa, \lambda)\) and \(y \in X\). We call \(y\) a single point (resp a double point) if \(#\eta_0^{-1}(y) = 1\) (resp. \(#\eta_0^{-1}(y) = 2\)). We say that \((Y, X, x)\) is single (resp double) if every element of \(X\) is single (resp double). If \((Y, X, x)\) neither single nor double, it is called **mixed**. We also say that \((Y, X, x)\) is single (resp. double) in scale \(R\) if every element of \(X \cap B^X(x, R)\) is single (resp. double). If \((Y, X, x)\) is neither single nor double in scale \(R\), it is called mixed in scale \(R\).

From now on, to prove Theorem 1.5, we analyze the local structure of \(\partial M\) about the connectedness when \(\text{inrad}(M) < \epsilon\). By Lemma 6.8 for any \(p \in M\), there exists a \(\delta\)-limit \(\mathcal{Y}(M, p) = (Y, X, x)\) together with

1. a \(\delta\)-approximation \(\psi : (\partial M)^\text{int}(p, R) \rightarrow C_0^\text{int}(p, R)\);
2. a \(\delta\)-approximation \(\varphi := \eta_0 \circ \psi \circ \iota_{\partial M}^{-1} \circ \omega_M : B^M_{\text{ext}}(p, R) \rightarrow B^{X_{\text{ext}}}(x, R)\).

Note that for every \(p_1, p_2 \in B^M_{\text{ext}}(p, R)\),

\[
|\varphi(p_1), \varphi(p_2)|_{X_{\text{ext}}} \leq L|\varphi(p_1), \varphi(p_2)|_{X_{\text{ext}}}
\]

(6.36)

\[
\leq L(|p_1, p_2|_{M_{\text{ext}}} + \delta)
\]

\[
\leq L(|p_1, p_2|_{M_{\text{int}}} + \delta)
\]

Those approximation maps are effectively used in the proofs of the following lemma.

**Lemma 6.10.** For any \(R > 0\) there exists \(\delta_0 > 0\) satisfying the following: For every \(0 < \delta \leq \delta_0\), let \(\epsilon = \epsilon(\delta) > 0\) be as in Lemma 6.8. For each \(M \in \mathcal{M}(n, \kappa, \lambda)\) with \(\text{inrad}(M) < \epsilon\) and for each \(p \in \partial M\), we have the following: Let \(\mathcal{Y}(M, p)\) any \(\delta\)-limit of \((M, p)\).

1. If \(\mathcal{Y}(M, p)\) is single in scale \(R\), then every \(p_1, p_2 \in \partial M \cap B^M(p, R)\) can be joined by a curve in \(\partial M\) of length \(\leq L|p_1, p_2|_{M} + (L + 1)\delta\).
2. If \(\mathcal{Y}(M, p)\) is double in scale \(R\), then there exists a point \(p' \in \partial M\) with \(|p, p'|_{M} < \delta\) such that every \(q \in \partial M \cap B^M(p, R)\) can be joined to \(p\) or \(p'\) by a curve in \(\partial M\) of length \(\leq L|p, q|_{M} + (L + 2)\delta\);
3. If \(\mathcal{Y}(M, p)\) is mixed in scale \(R\), then there exists a point \(p_0 \in \partial M \cap B^M(p, R)\) such that every point \(q \in \partial M \cap B^M(p, R)\) can be joined to \(p_0\) by a curve in \(\partial M\) of length \(L|p_0, q|_{M} + (L + 2)\delta\).

**Proof.** Let \((Y, X, x) := \mathcal{Y}(M, p)\), and \(\psi, \varphi\) be approximation maps as above.

1. Put \(x_i := \varphi(p_i) \in X, i = 1, 2\). Take \(\tilde{x}_i \in C_0\) such that \(\eta_0(\tilde{x}_i) = x_i\). Lemma 1.29 shows \(|\tilde{x}_1, \tilde{x}_2| = |x_1, x_2|\). Since \(\psi\) is a \(\delta\)-approximation
and \( \psi(p_i) = \tilde{x}_i \), it follows from (6.36) that
\[
|p_1, p_2|_{\partial M^{\text{int}}} < |\tilde{x}_1, \tilde{x}_2|_{C_{\text{int}}^\circ} + \delta = |x_1, x_2|_{X_{\text{int}}} + \delta < L|p_1, p_2|_M + (L + 1)\delta.
\]

(2) Set \( x := \varphi(p), y := \varphi(q) \). Since \((Y, X, x)\) is double in scale \( R \), we can put \( \{\tilde{x}_1, \tilde{x}_2\} := \eta_0^{-1}(x) \) and \( \{\tilde{y}_1, \tilde{y}_2\} := \eta_0^{-1}(y) \). Let \( \gamma : [0, 1] \to X \) be a minimal geodesic joining \( x \) to \( y \). From Lemma 4.29 there are lifts \( \tilde{\gamma}_0 : [0, 1] \to C_0 \) of \( \gamma \) starting from \( \tilde{x}_i \), where we may assume \( \tilde{\gamma}_0(1) = \tilde{y}_1 \) and \( \tilde{x}_1 = \psi(p) \). If \( \psi(q) = \tilde{y}_1 \), then
\[
|p, q|_{\partial M^{\text{int}}} < |\tilde{x}_1, \tilde{y}_1|_{C_{\text{int}}^\circ} + \delta = |x, y|_{X_{\text{int}}} + \delta < L|p, q|_M + (L + 1)\delta.
\]

If \( \psi(q) = \tilde{y}_2 \), then take a point \( p' \) with \( |\psi(p'), \tilde{x}_2| < \delta \). Then similarly we have \( |p', q|_{\partial M^{\text{int}}} < L|p, q|_M + (L + 2)\delta \).

(3) Let \( x_0 \in X \) be a single point with \( |x, x_0| \leq R \), and take \( \tilde{x}_0 \in C_0 \) and \( p_0 \in \partial M \) such that \( \eta_0(\tilde{x}_0) = x_0 \) and \( |\psi(p_0), \tilde{x}_0| < \delta \). Let \( \gamma : [0, 1] \to X \) be a minimal geodesic from \( x_0 \) to \( \varphi(q) \). Since \( \tilde{x}_0 \in C_0^1 \), there is a unique minimal geodesic \( \tilde{\gamma} : [0, 1] \to C_0 \) from \( \tilde{x}_0 \) to \( \tilde{\psi}(q) \) with \( \eta_0 \circ \tilde{\gamma} = \gamma \). We then have
\[
|p_0, q|_{\partial M^{\text{int}}} < |\tilde{x}_0, \tilde{\psi}(q)|_{C_{\text{int}}^\circ} + \delta = |x_0, \varphi(q)|_{X_{\text{int}}} + \delta < L|p_0, q|_M + (L + 2)\delta.
\]

\[ \square \]

From now on, for a fixed \( R > 0 \), let \( \delta = \delta_0(n, \kappa, \lambda, R) > 0 \) and \( \epsilon = \epsilon(\delta) > 0 \) be as determined in Lemma 6.10.

**Lemma 6.11.** If \( M \in \mathcal{M}(n, \kappa, \lambda) \) has inradius \( \text{inrad}(M) < \epsilon \) and disconnected boundary, then every \( \delta \)-limit \( \mathcal{Y}(M, p) \) is double in scale \( R \) for every \( p \in \partial M \).

**Proof.** Suppose that some \( \delta \)-limit \( \mathcal{Y}(M, p) = (Y, X, x) \) is single or mixed in scale \( R \). First note that by Lemma 6.10 (1), (3), every points \( q_1, q_2 \) in \( \partial M \cap B_M^\delta(p, R) \) can be joined by a curve in \( \partial M \). Take a point \( p_\alpha \in \partial M \) contained in a component different from the component containing \( p \). Let \( c : [0, \ell] \to M \) be a unit speed minimal geodesic in \( M \) from \( p \) to \( p_\alpha \). For each \( k \) with \( 1 \leq k \leq [2\ell/R] \), take \( p_k \in \partial M \) with \( |p_k, c(kR/2)|_M < \epsilon \).

Note that
\[
B_M^\delta(p_k, R) \cap B_M^\delta(p_{k+1}, R) \cap \partial M \neq \emptyset
\]
for each \( 1 \leq k \leq [2\ell/R] - 1 \). By applying Lemmas 6.10 to \( p_k \) together with a standard monodromy argument, we see that \( p_\alpha \) can be joined to \( p \) by a curve in \( \partial M \), which is a contradiction. \( \square \)

**Lemma 6.12.** Suppose that \( M \in \mathcal{M}(n, \kappa, \lambda) \) has inradius \( \text{inrad}(M) < \epsilon \) and disconnected boundary. For any \( p \in \partial M \), let \((Y, X, x)\) be any
\( \delta \)-limit for \((M, p)\). For any \( y \in X \), take distinct points \( y_1 \neq y_2 \in C^Y_{t_0} \) such that \( |y, y| = t_0 \). Then we have \( |y_1, y_2| = 2t_0 \).

Remark 6.13. In Lemma 6.12, we need the assumption on the disconnectedness of \( \partial M \). Namely, for some \((Y, X, x)\) which is double, the conclusion of Lemma 6.12 does not hold. For instance, take the Möbius band

\[ Y = S^1_x \tilde{x}_\tau[-t_0, t_0]. \]

If the length \( \ell \) of \( X = S^1_x \) is smaller than \( t_0 \), then \( |y_1, y_2| < 2t_0 \) for every \( y \in X \) and \( y_i \in C^Y_{t_0} \) with \( |y_i, X| = t_0, i = 1, 2 \).

Proof of Lemma 6.12. First note that \((Y, X, x)\) is double in scale \( R \). Suppose \( |y_1, y_2| < 2t_0 \), and take a minimal geodesic \( \gamma \) joining them in \( Y \). Then \( \gamma \) does not meet \( X \), and therefore we can project \( \gamma \) to \( C^Y_{t_0} \). The obtained curve \( \pi_{t_0}(\gamma) \) joins \( y_1 \) and \( y_2 \) in \( C^Y_{t_0} \). Thus the two elements \( \tilde{y}_1, \tilde{y}_2 \) of \( \eta^{-1}_0(y) \) can be joined in \( C_0 \). Take \( q_1, q_2 \in \partial M \) such that \( |\psi(q_k), \tilde{y}_k| < \delta \) for \( k = 1, 2 \). Lemma 6.10(2) shows that every \( p' \in \partial M \cap B^M(p, R) \) can be joined to \( q_1 \) or \( q_2 \) by a curve in \( \partial M \).

By a monodromy argument as in Lemma 6.11, we can conclude that every \( q \in \partial M \) can be joined to \( q_1 \) or \( q_2 \) by a curve in \( \partial M \), which is a contradiction. \( \square \)

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. We assume that \( \text{inrad}(M) < \epsilon \).

1. Suppose that \( \partial M \) is disconnected. By Lemma 6.11, every \( \delta \)-limit \( \mathcal{Y}(M, p) \) is double in scale \( R \) for every \( p \in M \). Take \( p_\alpha \) and \( p_\beta \) from distinct components of \( \partial M \). For every \( p \in \partial M \), let \( c : [0, \ell] \to M \) be a unit speed curve in \( M \) from \( p_{\alpha} \) to \( p_{\beta} \) through \( p \). For each \( k \) with \( 1 \leq k \leq 2[\ell/R] \), take \( p_k \in \partial M \) with \( |p_k, c(kR/2)|_M < \epsilon \). By applying Lemma 6.10(2) to each \( p_k \) together with a standard monodromy argument as in Lemma 6.11, we see that \( p \) can be joined to \( p_\alpha \) or \( p_\beta \) by a curve in \( \partial M \). Therefore we conclude that the number of boundary components of \( M \) is at least two.

2. Suppose that \( \partial M \) has two components. By Lemma 6.11, any \( \delta \)-limit \( \mathcal{Y}(M, p) = (Y, X, x) \) is double in scale \( R \) for every \( p \in \partial M \). Therefore for any \( y \in X \), there are distinct \( y_1 \neq y_2 \in C^Y_{t_0} \) with \( |y_k, y| = t_0, k = 1, 2 \). Lemma 6.12 shows that

\[ |y_1, y_2| = 2t_0. \]

Let \( W \) be a component of \( \partial \hat{M} \), and consider the distance function \( d_W \) from \( W \). The above observation shows that for any \( p \in M \), there exists a point \( q \in \partial M \) such that

\[ d_W(pq) > \pi - \tau(\delta). \]

That is, \( d_W \) is \( \pi/2 - \tau(\delta) \)-regular on a neighborhood of \( M \) in \( \hat{M} \). This makes it possible to define locally defined gradient-like vector fields
for $d_W$ on neighborhoods of the points of $M$. Then by gluing those local gradient-like vector fields, we get a globally defined gradient-like vector field $V$ on $M$ whose support is contained in a neighborhood of $M$. It is now straightforward to obtain a diffeomorphism between $M$ and $W \times [0, 1]$ by means of integral curves of $V$. 

\textbf{Theorem 6.14 (Gromov\cite{11}, Alexander-Bishop\cite{1}).} There exists a positive number $\epsilon = \epsilon(n, \kappa, \lambda)$ such that if $M \in \mathcal{M}(n, \kappa, \lambda)$ has the two side bounds on sectional curvature $|K_M| \leq \kappa^2$ in addition and if the inradius $\text{inrad}(M) < \epsilon$, then either $M$ or its double cover is diffeomorphic to a product $W \times [0, 1]$, where $W$ is a closed manifold.

The following example shows that Theorem 6.14 does not hold in the connected boundary case if one drops the upper sectional curvature bound $K_M \leq \kappa^2$. Namely there are some $M \in \mathcal{M}(n, \kappa, \lambda, d)$ with connected boundary and with small inradius that are not finitely covered by any topological product of the form $W \times [0, 1]$, where $W$ is a closed manifold.

\textbf{Example 6.15.} Let $N$ be a compact surface of genus one with connected boundary, and consider a Riemannian metric on $N$ such that $\partial N$ has a cylindrical neighborhood $U_\epsilon$. Namely there is an isometric embedding $f : S_\ell^1 \times [0, \epsilon) \rightarrow U_\epsilon$ such that $f(S_\ell^1 \times 0) = \partial N$, where $\ell = L(\partial N)$. Consider a segment $I = \{(x, 0, 0) \mid 0 \leq x \leq 2\epsilon\}$ in the $xyz$-space $\mathbb{R}^3$, and let $D_\epsilon$ denote the intersection of the boundary of $\epsilon$-neighborhood of $I$ with $\{x \leq \epsilon, z \leq 0\}$. Let

\begin{align*}
J_\epsilon &= D_\epsilon \cap \{x = \epsilon\}, \\
K_\epsilon &= D_\epsilon \cap \{x \leq 0, z = 0\}, \\
L_\epsilon &= D_\epsilon \cap \{z = -\epsilon\}, \\
E_\epsilon &= D_\epsilon \cap \{0 \leq x \leq \epsilon\}.
\end{align*}

Note that $J_\epsilon$ and $K_\epsilon$ (resp. $L_\epsilon$) are segments of length $\pi \epsilon$ (resp. length $\epsilon$). Since there is an isometry $\varphi : \tilde{U}_\epsilon \times [-\pi \epsilon/2, \pi \epsilon/2] \rightarrow S_\ell^1 \times E_\epsilon$, we have an obvious gluing to obtain three-dimensional Riemannian manifold $M_\epsilon$ with totally geodesic boundary:

$$M_\epsilon = N \times [-\pi \epsilon/2, \pi \epsilon/2] \amalg S_\ell^1 \times D_\epsilon.$$ 

Note that after slight smoothing of $M_\epsilon$, we may assume that $M_\epsilon \in \mathcal{M}(3, \kappa, 0, d)$ for some $\kappa, d$ and it inradius collapses to $N$ as $\epsilon \rightarrow 0$. Note that $M_\epsilon$ is homeomorphic to $N \times I \cup \partial N \times \bar{D}^2$ as in Theorem 5.3.

Note that any finite cover $\hat{M}_\epsilon$ of $M_\epsilon$ is not homeomorphic to $W \times [0, 1]$, for any closed surface. Otherwise a finite cover $\hat{M}_\epsilon$ of $M_\epsilon$ is homeomorphic to $W \times [0, 1]$ as above. Since $M_\epsilon$ has the same homotopy type as $N$, $\pi_1(M_\epsilon)$ is a free group generated by two elements. It turns out that $\pi_1(M_\epsilon) = \pi_1(W)$ is a free group, which is a contradiction.
7. Remark on locally convex manifolds

In the argument so far, the assumption $|\Pi_{\partial M}| \leq \lambda^2$ was used to have lower sectional curvature bound $K_{\partial M} \geq c(\kappa, \lambda)$. It is a challenging problem to study the case where only lower bound $\Pi_{\partial M} \geq -\kappa$ is assumed.

In the case of locally convex boundary in the sense that $\Pi_{\partial M} \geq 0$, the Gauss equation implies $K_{\partial M} \geq \kappa$ as long as $K_M \geq \kappa$. Therefore taking $\phi(t) = 1$ as the warping function, we can extend $M$ to $\tilde{M} = M \cup \partial M \times [0, t_0]$, and proceed by the same argument as in the previous sections, to obtain the results corresponding to Theorems 1.3, 1.4 and 1.5.

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