A basis of algebraic de Rham cohomology of complete intersections over a characteristic zero field

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\textbf{ABSTRACT}

Let $k$ be a field of characteristic 0. Let $X$ be a smooth complete intersection over $k$ of dimension $n - k$ in the projective space $\mathbb{P}^n_k$ for given positive integers $n$ and $k$. When $k = \mathbb{C}$, Terasoma and Konno provided an explicit representative (in terms of differential forms) of a basis for the primitive middle-dimensional algebraic de Rham cohomology $H_{dR, \text{prim}}^{n-k}(X; \mathbb{C})$. Later Dimca constructed another explicit representative of a basis of $H_{dR, \text{prim}}^{n-k}(X; \mathbb{C})$. Moreover, he proved that his representative gives the same cohomology class as the previous representative of Terasoma and Konno. The goal of this article is to examine the above two different approaches without assuming that $k = \mathbb{C}$ and provide a similar comparison result for any field $k$. Dimca’s argument depends heavily on the condition $k = \mathbb{C}$ and our idea is to find appropriate Cech-de Rham complexes and spectral sequences corresponding to those two approaches, which work without restrictions on $k$.

\textbf{ARTICLE HISTORY}

Received 16 April 2021
Revised 19 July 2021
Communicated by Daniel Erman

\textbf{KEYWORDS}

De Rham cohomology; Gysin map; projective smooth complete intersections; residues

\textbf{2020 MATHEMATICS SUBJECT CLASSIFICATION}

14M10; 14F40 (primary)

\section{1. Introduction}

\subsection{1.1. The main result}

Let $k$ be a field of characteristic zero. Let $n$ and $k$ be positive integers such that $n \geq k \geq 1$. Let $X_G$ be a smooth complete intersection variety over $\mathbb{Q}$ of dimension $n - k$ embedded in the projective space $\mathbb{P}^n$. We use $x = [x_0 : x_1 : \ldots : x_n]$ as a projective coordinate of the projective $n$-space $\mathbb{P}^n$ and let $G_1(x), \ldots, G_k(x)$ be defining homogeneous polynomials in $k[x]$ such that $\text{deg}(G_i) = d_i$ for $i = 1, \ldots, k$.

The main object of study is the primitive middle-dimensional algebraic de Rham cohomology group $H_{dR, \text{prim}}^{n-k}(X_G; k)$. For this, we introduce new variables $y_1, \ldots, y_k$ corresponding to $G_1, \ldots, G_k$. Let $N = n + k + 1$ and

$$A := k[y_1, \ldots, y_k, x_0, \ldots, x_n] = k[q_{\ell}]_{i=1,\ldots,N}$$

where $q_1 = y_1, \ldots, q_k = y_k$ and $q_{k+1} = x_0, \ldots, q_N = x_n$. Then consider the Dwork potential

$$S(q) := \sum_{\ell=1}^k y_{\ell} \cdot G_{\ell}(x).$$
When \( k = \mathbb{C} \), Konno [8] described \( H^n_{dR, \text{prim}}(X_G; \mathbb{C}) \) in terms of the Jacobian ideal of \( S \): he constructed an isomorphism

\[
A_{cG} / A_{cG} \cap \text{Jac}(S) \simeq H^n_{dR, \text{prim}}(X_G; \mathbb{C}),
\]

where we refer to (2.5) for detailed notations.

When \( k = \mathbb{C} \), Dimca found another isomorphism (see Section 2.3 for our review) from \( (A / \text{Jac}(S))_{cG} \) to \( H^n_{dR, \text{prim}}(X_G; \mathbb{C}) \) in [2], which is more close to the spirit of Griffiths [3] than Konno by showing that his map (described in a different way) is, in fact, same as the (Terasoma’s and) Konno’s map via a new use of relative Bochner–Martinelli and Andreotti–Norguet integral formulas for the residue map \( \text{Res}_G : H^{n+k-1}_{dR}(\mathbb{P}^n \setminus X_G; \mathbb{C}) \cong H^n_{dR, \text{prim}}(X_G; \mathbb{C}) \). The goal here is to prove a version of Dimca’s result when \( \mathbb{C} \) is replaced by any field \( k \) of characteristic zero.

More precisely, Dimca’s result [2, Proposition 10] (see also Proposition 2.3) can be stated as the commutativity of the (diamond) diagram below when \( k = \mathbb{C} \):

\[
\begin{array}{ccc}
A_{cG} / A_{cG} \cap \text{Jac}(S) & \xrightarrow{d_{cG}} & H^n_{dR}(\mathbb{P}^n \setminus D_G; k) \\
\downarrow k_{cG} & \downarrow & \downarrow \\
H^n_{dR}(\mathbb{P}(\mathcal{E}) \setminus X_{\mathcal{E}}; k) & \xrightarrow{s^*} & H^n_{dR}(\mathbb{P}^n \setminus X_G; k) \\
\end{array}
\]

\[
\begin{array}{ccc}
& & \xrightarrow{\text{Res}_G} \\
& & \downarrow \\
\downarrow & & \downarrow \\
& & \\
& & \\
\end{array}
\]

\[
\begin{array}{ccc}
H^{n+k-1}_{dR}(\mathbb{P}^n \setminus X_G; k) & \xrightarrow{d_{cG}} & H^n_{dR, \text{prim}}(X_G; k) \\
\end{array}
\]

where \( s \) is any section to the natural projection map \( pr_1 : \mathbb{P}(\mathcal{E}) \setminus X_{\mathcal{E}} \to \mathbb{P}^n \setminus X_{cG} \) and we refer to Definitions 3.3, 3.7 and 3.9 for Terasoma–Konno’s map \( k_{cG} \), Dimca’s map \( d_{cG} \) and the natural epimorphism \( \delta_{cG} \). Note that \( k_{cG} \) and \( s^* \) are isomorphisms but neither \( d_{cG} \) nor \( \delta_{cG} \) is an isomorphism unless \( k = 1 \). Our main Theorem 3.11 states that the above diagram commutes (up to a precise numerical factor). Dimca’s argument depends heavily on the condition \( k = \mathbb{C} \), since Dimca’s proof relies on an explicit description (integral formulas) of \( \text{Res}_G \). On the other hand, the residue (isomorphism) map \( \text{Res}_G : H^{n+k-1}_{dR}(\mathbb{P}^n \setminus X_G; k) \to H^n_{dR, \text{prim}}(X_G; k) \) over a general \( k \) is defined to be the connecting homomorphism of the distinguished triangle (3.3) and we do not have an integral formula. So, we need a new idea to prove a comparison result for a general field \( k \) : we will find appropriate Cech-de Rham complexes and spectral sequences, which work without restrictions on \( k \), corresponding to Dimca’s and Terasoma-Konno’s maps. Our argument does not use \( \text{Res}_G \) : it involves direct computations in the level of Cech-de Rham cochains and has more algebraic nature using “partition of unity”.

1.2. An application

We briefly indicate an application of our result to the deformation formula (in the form of the Feynman path integral) for the period integrals of smooth projective complete intersection varieties over \( k = \mathbb{Q} \).

We fix a homology class \([\gamma]\) \( \in H_{n-k}(X_G(\mathbb{C}), \mathbb{Q})_0 = \ker(H_{n-k}(X_G(\mathbb{C}), \mathbb{Q}) \to H_{n-k}(\mathbb{P}^n(\mathbb{C}), \mathbb{Q})) \).

Then, we consider the following period integral (which compares the \( \mathbb{Q} \)-rational structures of the de Rham cohomology and the singular cohomology) from the de Rham cohomology to \( \mathbb{C} \):

\[1\text{More precisely, the isomorphism is due to Griffiths in the hypersurface case [3], Terasoma in the equal degree complete intersection case [11], and Konno in the general case [8].} \]
\[ C^G_T : H^{n-k}_{dR,\text{prim}}(X_G; \mathbb{Q}) \rightarrow \mathbb{C}, \quad [\sigma] \mapsto \int_T \sigma, \]  

(1.3)

where \( \sigma \) is a representative of a cohomology class \([\sigma] \in H^{n-k}_{dR,\text{prim}}(X_G; \mathbb{Q})\). By using the isomorphism of \( \mathbb{Q} \)-vector spaces (the lower maps in the commutative diagram in Section 1.1)

\[ \text{Res}_G \circ s^* \circ k_G : A_{cG}/A_{cG} \cap \text{Jac}(S) \rightarrow H^{n-k}_{dR,\text{prim}}(X_G; \mathbb{Q}), \]

we interpret the above period integral as a linear map

\[ C^G_T := C^G_T \circ J_G : A \rightarrow \mathbb{C} \]  

(1.4)

from the polynomial ring \( A \) to \( \mathbb{C} \), where \( J_G \) is the composition of the projection map from \( A \rightarrow A_{cG} \), the natural quotient map \( A_{cG} \rightarrow A_{cG}/A_{cG} \cap \text{Jac}(S) \), and \( \text{Res}_G \circ s^* \circ k_G \).

First, the main Theorem 3.11 enables us to interpret the period integral \( C^G_T \) as a form of the Feynman path integral\(^2\) by the Laplace transform (2.9): the description of the upper map \( d_G \) (see (2.9) and (2.10)) of the diagram in Section 1.1 and its commutativity (Theorem 3.11) essentially say that

\[ C^G_T(f) = \int_{\mathbb{A}^n} f(\mathbf{z}) e^{S(\mathbf{z})} d\mu \]  

(1.5)

for “some measure” \( \mu \) on \( \mathbb{A}^N \).

Second, this interpretation has a benefit of getting a simple deformation formula. For the explanation of the deformation formula, let \( X_U \subset \mathbb{P}^n \) be a smooth projective complete intersection over \( \mathbb{Q} \) which is deformed from \( X = X_G \) by homogeneous polynomials \( H = (H_1(\mathbf{x}), ..., H_k(\mathbf{x})) \), that is, \( U = (U_1(\mathbf{x}), ..., U_k(\mathbf{x})) \) with \( U_i(\mathbf{x}) = G_i(\mathbf{x}) + H_i(\mathbf{x}) \) for each \( i = 1, ..., k \) are the defining equations for \( X_U \). Note that smooth projective complete intersections with fixed degrees \( d_1, ..., d_k \) have same topological types and their singular (vanishing) homologies and singular (primitive) cohomologies are isomorphic \([7, \text{Corollary A.2}]\). To make the context more clear, we make an assumption that our homology class \([\gamma] \) is supported in \( X_G(\mathbb{C}) \cap X_U(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C}) \). In this case, we use \([7, \text{Proposition A.1}]\) to get a particular identification

\[ H_\bullet(X_G(\mathbb{C}), \mathbb{Z}) \xrightarrow{\sim} H_\bullet(X_U(\mathbb{C}), \mathbb{Z}) \quad \text{such that} \quad [\gamma] \mapsto [\gamma]. \]  

(1.6)

Using this identification, we can directly compare \( C^G_T \) and \( C^U_T \). Let \( \Gamma = \sum_{i=1}^k \gamma_i H_i(\mathbf{x}) \). One can verify the following simple deformation formula: for any homogeneous polynomial \( u \in A \), the power series\(^3\)

\[ C^G_T(f \cdot e^\Gamma) := \sum_{n=0}^{\infty} \frac{C^G_T(f \cdot \Gamma^n)}{n!} \]

converges to the polynomial realization \( C^U_T(f) \) of the period integral of \( X_U \). The key idea is to notice that \( C^U_T(f) = C^G_T(f) + H(f) = \int_{\mathbb{A}^n} f(\mathbf{z}) e^S d\mu \). We refer to \([7, \text{Theorem 1.1}]\) for its proof when \( k = \mathbb{C} \) (the deformation theory of the period integrals when \( k = \mathbb{C} \) had been studied in \([7]\)); the \( k = \mathbb{Q} \) case proof can be done exactly same way, once we have a commutative diagram over \( \mathbb{Q} \) (Theorem 3.11).

\( ^2\)See \([9, \text{Section 1.4}]\) for physical explanation how to understand \( S \) as an action functional of a 0-dimensional quantum field theory.

\( ^3\)Here, \( e^\Gamma = 1 + \Gamma + \frac{\Gamma^2}{2} + \cdots \) and, for \( x \in A \), we think of \( C^G_T(x \cdot e^\Gamma) = C^G_T(x) + C^G_T(x \cdot \Gamma) + C^G_T(x \cdot \frac{\Gamma^2}{2}) + \cdots \) as a formal expression.
1.3. The plan of the paper

The paper consists of two sections. Section 2 is a summary of Dimca’s result over \( \mathbb{C} \) in terms of the language which is suitable for generalization to arbitrary field of characteristic zero. We set up the basic notations in Section 2.1. In Section 1.2, we review Terasoma–Konno’s approach to cohomology of complete intersections. In Section 2.3, we explain Dimca’s approach and his comparison result.

Section 3 is the main part of the paper. In Section 3.1, we examine the Gysin exact sequence for cohomologies with coefficient \( k \). Section 3.2 (respectively, 3.3) is devoted to study of Terasoma–Konno’s approach (respectively, Dimca’s approach) over \( k \). Finally, in Section 3.4, we prove the main comparison theorem.

2. Dimca’s comparison result over \( \mathbb{C} \)

2.1. A brief review over complex numbers \( \mathbb{C} \)

We review Terasoma’s and Konno’s approaches to explicit descriptions of \( H_{dR, \text{prim}}^{n-k}(X_{\mathbb{C}}; \mathbb{C}) \). Their ideas are same, which are based on the Gysin exact sequence and “the Cayley trick”. Terasoma’s result covers the case when \( d_1 = \cdots = d_k \) and Konno’s result works generally.

Let us briefly explain the Gysin exact sequence and “the Cayley trick”. There is a long exact sequence, called the Gysin exact sequence:

\[
\cdots \rightarrow H_{dR}^{n+k-1}(\mathbb{P}^n; \mathbb{C}) \rightarrow H_{dR}^{n+k-1}(\mathbb{P}^n \setminus X_{\mathbb{C}}; \mathbb{C}) \xrightarrow{\text{Res}_{\mathbb{C}}} H_{dR}^{n-k}(X_{\mathbb{C}}; \mathbb{C}) \xrightarrow{\text{Gys}} H_{dR}^{n+k}(\mathbb{P}^n; \mathbb{C}) \rightarrow \cdots, \tag{2.1}
\]

where \( \text{Res}_{\mathbb{C}} \) is the residue map (see p. 96 of [2]) and \( \text{Gys} \) is the Gysin map (the cup product with \( k \)th wedge product of the fundamental Kähler 2-form on \( X_{\mathbb{C}} \)). This sequence gives rise to an isomorphism

\[
\text{Res}_{\mathbb{C}} : H_{dR}^{n+k-1}(\mathbb{P}^n \setminus X_{\mathbb{C}}; \mathbb{C}) \xrightarrow{\sim} H_{dR, \text{prim}}^{n-k}(X_{\mathbb{C}}; \mathbb{C}), \tag{2.2}
\]

where the primitive cohomology \( H_{dR, \text{prim}}^{n-k}(X_{\mathbb{C}}; \mathbb{C}) \) is defined to be the kernel of \( \text{Gys} \). The Cayley trick is about translating a computation of the cohomology of the complement of a complete intersection into a computation of the cohomology of the complement of a hypersurface in a bigger space. Let \( \mathcal{E} = \mathcal{O}_\mathbb{P}(d_1) \oplus \cdots \oplus \mathcal{O}_\mathbb{P}(d_k) \) be the locally free sheaf of \( \mathcal{O}_\mathbb{P} \)-modules with rank \( k \). Let \( \mathbb{P}(\mathcal{E}) \) be the projective bundle associated to \( \mathcal{E} \) with fiber \( \mathbb{P}^{k-1} \) over \( \mathbb{P}^n \). Then \( \mathbb{P}(\mathcal{E}) \) is the smooth projective toric variety with Picard group isomorphic to \( \mathbb{Z}^2 \) whose (toric) homogeneous coordinate ring\(^4\) is given by

\[
A := A_{\mathbb{P}(\mathcal{E})} := \mathbb{C}[y_1, \ldots, y_k, x_0, \ldots, x_n], \tag{2.3}
\]

where \( y_1, \ldots, y_k \) are new variables corresponding to \( G_1, \ldots, G_k \). This is the polynomial ring introduced in (1.1). There are two additive gradings \( ch \) and \( wt \), called the charge and the weight, corresponding to the Picard group \( \mathbb{Z}^2 \):

\[
\begin{align*}
ch(y_i) &= -d_i, & \text{for } i = 1, \ldots, k, \\
ch(x_j) &= 1, & \text{for } j = 0, \ldots, n, \\
wt(y_i) &= 1, & \text{for } i = 1, \ldots, k, \\
wt(x_j) &= 0, & \text{for } j = 0, \ldots, n.
\end{align*}
\]

We have the weight and the charge decomposition of \( A \) such that

\[
A = \bigoplus_{\lambda \in \mathbb{Z}^2} \bigoplus_{w \geq 0} A_{\lambda, (w)},
\]

\(^4\)This was already introduced in (1.1)
where sub-indices $\lambda$ and $(w)$ means the charge and the weight, respectively. Then

$$S(y, x) := \sum_{j=1}^{k} y_j G_j(x) \in A_{\theta(1)}$$

defines a hypersurface $X_\theta$ in $P(E)$. The natural projection map $pr_1 : P(E) \to P^n$ induces a morphism $pr_1 : P(E) \setminus X_\theta \to P^n \setminus X_{\overline{\theta}}$ which can be checked to be a homotopy equivalence (the fibers are affine spaces). Hence, there exists an isomorphism

$$H^{n+k-1}_{dR}(P(E) \setminus X_\theta; \mathbb{C}) \xrightarrow{\pi^*} H^{n+k-1}_{dR}(P^n \setminus X_{\overline{\theta}}; \mathbb{C}),$$

where $s$ is a section to $pr_1$. The cohomology group $H^{n+k-1}_{dR}(P(E) \setminus X_\theta; \mathbb{C})$ of a hypersurface complement in $P(E)$ can be described explicitly in terms of the de Rham cohomology of $P(E)$ with poles along $X_\theta$. Terasoma and Konno proved\(^5\) that there is an isomorphism

$$\Phi_{S} : A_{c_{\mathbb{G}}} / A_{c_{\mathbb{G}}} \cap Jac(S) \xrightarrow{\sim} H^{n+k-1}_{dR}(P(E) \setminus X_\theta; \mathbb{C}),$$

where $Jac(S)$ is the Jacobian ideal of $S(y, x)$, and

$$c_{\mathbb{G}} := \sum_{i=1}^{k} d_i - (n + 1).$$

Here, the sub-index $c_{\mathbb{G}}$ means the submodule in which the charge is $c_{\mathbb{G}}$ and $Jac(S)$ is the sum of the images of the endomorphisms $\frac{\partial S}{\partial y_i}, \frac{\partial S}{\partial x_j}$ of $A (i = 1, \ldots, k; j = 0, \ldots, n)$.

### 2.2. An explicit basis due to Terasoma and Konno

To review the explicit basis of Terasoma and Konno, we need to examine a map from $(A/Jac(S))_{c_{\mathbb{G}}}$ to $H^{n+k-1}_{dR}(P(E) \setminus X_\theta; \mathbb{C})$.

Let

$$\Omega_x = \sum_{i=0}^{n} (-1)^i x_i (dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n), \quad \Omega_y = \sum_{i=1}^{k} (-1)^j y_i (dy_1 \wedge \cdots \wedge \widehat{dy_i} \wedge \cdots \wedge dy_n).$$

(2.6)

For a multi-index $i = (i_1, \ldots, i_k)$ with $|i| = i_1 + \cdots + i_k$, consider the rational differential form (given in [2, p. 90] and originally studied by [11] and [8])

$$\alpha(S, y^i u(x)) := (-1)^{n(k-1) + |i|} (k + |i| - 1)! \frac{y^i u(x)}{S(q)^{k+|i|}} \Omega_x \Omega_y \in \Omega^{n-2}(P(E) \setminus X_\theta),$$

(2.7)

where $y^i = y_1^{i_1} \cdots y_k^{i_k}$ and $y^i u(x) \in A_{c_{\mathbb{G}}}$. Define $k_{\mathbb{G}} : A_{c_{\mathbb{G}}} / A_{c_{\mathbb{G}}} \cap Jac(S) \to H^{n+k-1}_{dR}(P(E) \setminus X_\theta; \mathbb{C})$ as follows:

$$k_{\mathbb{G}} \left( \left[ y^i u(x) \right] \right) = \left[ \alpha(S, y^i u(x)) \right]$$

(2.8)

for $y^i u(x) \in A_{c_{\mathbb{G}}}$ (linearly extended to every element of $A_{c_{\mathbb{G}}}$). Here, $[\cdot]$ is the cohomology class. Konno [8] proved that this map $k_{\mathbb{G}}$ is an isomorphism of finite dimensional $\mathbb{C}$-vector spaces.

---

\(^5\)More precisely, Terasoma [11] provided such an isomorphism in the case $d_1 = d_2 = \cdots = d_k$ and Konno [8] extended the result of Terasoma to the general case when $d_j$'s are not equal. Also see [3] for the pioneering work of Griffiths in the case $k = 1$, hypersurface case.
Definition 2.1 (Terasoma’s and Konno’s approaches). We call the map \( k_G \) the Terasoma/Konno map.

2.3. An explicit basis due to Dimca and a comparison

Here, we review Dimca’s approach and his comparison result with Terasoma’s and Konno’s approaches.

We consider the natural epimorphism which appeared in [2, p. 95],
\[
\delta_G : H^*_{dr}(\mathbb{P}^n \setminus D_G; \mathbb{C}) \rightarrow H^{n+k-1}_{dr}(\mathbb{P}^n \setminus X_G; \mathbb{C}),
\]
where \( D_G \) is the divisor defined by \( G_1(x) \cdots G_k(x) = 0 \).

Dimca considered the following rational differential form [2, p. 93]
\[
\beta(S, y^i u(x)) := (-1)^{|i|+1} \frac{i_1! \cdots i_k! \cdot u(x)}{G^{i_1+1}_1(x) \cdots G^{i_k+1}_k(x)} \Omega_x \in \Omega^n(\mathbb{P}^n \setminus D_G)
\]
for \( y^i u(x) \in A_{e_G} \). This explicit representation has a benefit of being written as the Laplace transform
\[
-\int_0^\infty \cdots \int_0^\infty y^i u(x) e^{s(x)} \, dy_1 \cdots dy_k = \frac{(-1)^{|i|+1} i_1! \cdots i_k! u(x)}{G^{i_1+1}_1 \cdots G^{i_k+1}_k} \tag{2.9}
\]
for \( y^i u(x) \in A_{e_G} \). Define \( d_G : A_{e_G} / A_{e_G} \cap \text{Jac}(S) \rightarrow H^*_{dr}(\mathbb{P}^n \setminus D_G; \mathbb{C}) \) as follows:
\[
d_G \left( \left[ y^i u(x) \right] \right) = \left[ \beta(S, y^i u(x)) \right], \quad y^i u(x) \in A_{e_G} \cdot \tag{2.10}
\]

Definition 2.2 (Dimca’s approach). We call the map \( d_G \) the Dimca map.

In order to describe Dimca’s comparison result, we need to know an explicit description of a section \( s \) of the second map \( s^* \) in (2.4). For this, we review the toric quotient construction of the projective bundle \( \mathbb{P}(E) \). Let \( G = \mathbb{C}^* \times \mathbb{C}^* \). Let \( U_G = (\mathbb{C}^{n+1} \setminus \{0\}) \times (\mathbb{C}^1 \setminus \{0\}) \), consider the following \( G \)-action on \( U_G \):
\[
(u, v)(x, y) = (ux_0, ..., ux_n, u^{-d_1} vy_1, ..., u^{-d_k} vy_k).
\]

The hypersurface \( X_S \) in \( U_G \) is \( G \)-invariant. Lemma 17, [2], says that the geometric quotient \( U_G / G \) is naturally identified to the projective bundle \( \mathbb{P}(E) \). Let \( d = \text{lcm}(d_1, ..., d_k) \) and define the positive integers \( e_i = d/d_i \). According to page 98, [2], one has a well-defined section
\[
s_G(x) = (x, G^{e_1}_1(x) G^{e_2}_1(x) \cdots, G^{e_k}_1(x) G^{e_k}_1(x)) \tag{2.11}
\]
to the projection map \( pr_1 : \mathbb{P}(E) \setminus X_S \rightarrow \mathbb{P}^n \setminus X_G \).

The following is the main result of Dimca [2, Proposition 10] which compares his approach to Terasoma–Konno’s approach.

Proposition 2.3 (Proposition 10, [2]). We have the following equality
\[
\delta_G \left( \left[ \beta(S, y^i u(x)) \right] \right) = s_G^*( \left[ \alpha(S, y^i u(x)) \right] )
\]
for \( [y^i u(x)] \in A_{e_G} / A_{e_G} \cap \text{Jac}(S) \). In other words, we have

\footnote{This fact is crucially used to develop the deformation theory of period integrals of \( X_G \) in [7].}
\[
\delta_G \circ d_G = s_G^* \circ k_G \quad \text{(or} \quad pr_1^* \circ \delta_G \circ d_G = k_G). 
\]

### 3. Dimca’s comparison over any field \( k \) of characteristic 0

In this section, we will work with a field \( k \) of characteristic 0 and homogeneous polynomials \( G_1(x), \ldots, G_k(x) \in \mathbb{K}[x] \) defining a smooth complete intersection \( X_G \) in \( \mathbb{P}^n \) over \( k \). In order to simplify the notations, we will use the notations as the \( \mathbb{C} \)-case for \( A, \text{Res}_G, k_G, d_G, \) and \( \delta_G \).

Throughout this section, we will use the affine open covering
\[
U := \{ D_+(G_\lambda) \}_{\lambda=1,\ldots,k} \tag{3.1}
\]
of \( \mathbb{P}^n \setminus X_G \) where \( D_+(G_\lambda) \) denotes the nonvanishing locus of \( G_\lambda \) in \( \mathbb{P}^n \) (see [6, Proposition 2.5] for the notation). Denote \( D_G \subseteq \mathbb{P}^n \) the divisor cut out by \( G_1 \cdots G_k \).

Then
\[
D_+(G_1) \cap \cdots \cap D_+(G_k) = D_+(G_1 \cdots G_k) = \mathbb{P}^n \setminus D_G. \tag{3.2}
\]

#### 3.1. The Gysin exact sequence

Here, we show that the Gysin exact sequence (2.1) holds when \( \mathbb{C} \) is replaced by \( k \).

Given an \( \mathcal{O}_{\mathbb{P}^n} \)-module \( F \), denote \( H_{X_G}^\bullet (F) \subseteq F \) the subsheaf of sections supported on \( X_G \). Since the assignment \( F \mapsto H_{X_G}^\bullet (F) \) defines a left exact functor, there are derived functors \( H^p_{X_G}(F) := R^pH_{X_G}^\bullet (F) \). Since \( X_G \subseteq \mathbb{P}^n \) is a smooth complete intersection, we can describe \( H^p_{X_G}(F) \) using the affine open covering \( U = \{ D_+(G_\lambda) \}_{\lambda=1,\ldots,k} \) of (3.1). Let
\[
F_{\lambda_0 \cdots \lambda_p} := (D_+(G_{\lambda_0} \cdots G_{\lambda_p}) \hookrightarrow \mathbb{P}^n)_*(F|_{D_+(G_{\lambda_0} \cdots G_{\lambda_p})})
\]
where \( \{ \lambda_0, \ldots, \lambda_p \} \subset \{ 1, \ldots, k \} \). Then there is the associated Čech complex of \( \mathcal{O}_{\mathbb{P}^n} \)-modules:
\[
\tilde{\mathcal{C}}^\bullet (U, F) : 0 \longrightarrow F \longrightarrow \prod_{\lambda_0} F_{\lambda_0} \longrightarrow \prod_{\lambda_0 < \lambda_1} F_{\lambda_0,\lambda_1} \longrightarrow \cdots \longrightarrow F_{1 \cdots k} \longrightarrow 0
\]
where \( F \) is put in cohomological degree 0.

**Lemma 3.1.** If \( F \) is a quasicoherent \( \mathcal{O}_{\mathbb{P}^n} \)-module, then the following hold.

1. \( \tilde{\mathcal{C}}^\bullet (U, F) \) represents \( R\mathcal{H}^\bullet_{X_G}(F) \) in the derived category of \( \mathcal{O}_{\mathbb{P}^n} \)-modules whose cohomology sheaves are supported on \( X_G \). In other words,
   \[
   H^p_{X_G}(F) \cong H^p(\tilde{\mathcal{C}}^\bullet (U, F))
   \]
   holds for every \( p \in \mathbb{Z} \).
2. There is a canonical map
   \[
   F \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{X_G} \longrightarrow H^k_{X_G}(F) \quad s \longmapsto \frac{s}{G_1 \cdots G_k}
   \]
3. If, furthermore, \( F \) is flat over \( \mathcal{O}_{\mathbb{P}^n} \), then \( H^p_{X_G}(F) = 0 \) for \( p \neq k \).

**Proof.** For (1), see [10, Tag 0G7M]. For (2), see [10, Tag 0G7Q]. For (3), see [10, Tag 0G7P].

\(^7\)For example, \( A = \mathbb{K}[y,x], \text{Res}_G : H^k_{an}(\mathbb{P}^n, k) \rightarrow H^{2k+1}_{an}(X_G, k) \).
On the other hand, for the closed embedding $i : X_G \hookrightarrow \mathbb{P}^n$ and the open inclusion $j : (\mathbb{P}^n \setminus X_G) \hookrightarrow \mathbb{P}^n$, there is a distinguished triangle (see [10, Tag 0G72])

\[ i_* \mathcal{R}H_{X_G}(\Omega^*_{p^n/k}) \longrightarrow \Omega^*_{p^n/k} \longrightarrow \mathcal{R}j_* \Omega^*_{(\mathbb{P}^n \setminus X_G)/k} \longrightarrow i_* \mathcal{R}H_{X_G}(\Omega^*_{p^n/k})[1]. \]

Using Lemma 3.1, we may choose a representative of $\mathcal{R}H_{X_G}(\Omega^*_{p^n/k})$. Using Cartan–Eilenberg resolutions, we get a spectral sequence (see [10, Tag 015J]):

\[ E_1^{p,q} = H^p_{X_G}(\Omega^q_{p^n/k}) \Rightarrow H^{p+q}_{X_G}(\Omega^*_{p^n/k}). \]

This spectral sequence degenerates by Lemma 3.1:

\[ E_1^{p,q} = \begin{cases} H^k_{X_G}(\Omega^q_{p^n/k}) & \text{if } p = k, \\ 0 & \text{otherwise}, \end{cases} \]

which implies

\[ \mathcal{R}H_{X_G}(\Omega^*_{p^n/k}) \cong H^k_{X_G}(\Omega^*_{p^n/k})[-k]. \]

To compute the right hand side, consider the conormal exact sequence (see [6, Proposition 8.12]):

\[ 0 \longrightarrow C_{X_G/p^n} \longrightarrow \Omega^1_{p^n/k} \otimes \mathcal{O}_{p^n} \mathcal{O}_{X_G} \longrightarrow \Omega^1_{X_G/k} \longrightarrow 0 \]

where $C_{X_G/p^n}$ is the conormal module of the embedding $X_G \hookrightarrow \mathbb{P}^n$; the exactness at $C_{X_G/p^n}$ follows from the smoothness of $X_G$ over $\mathbb{P}^n$. Hence, the composition

\[ \Omega^*_{p^n/k} \longrightarrow \Omega^*_{p^n/k} \otimes \mathcal{O}_{X_G} \rightarrow \Omega^*_{X_G/k} \]

is surjective (the first map is obviously surjective). In this setting, we may define the Gysin map as follows:

\[ \gamma^q_G : \Omega^q_{X_G/k} \longrightarrow H^k_{X_G}(\Omega^{q+k}_{p^n/k}) \quad \omega \longmapsto \frac{\tilde{\omega}|_{X_G}}{G_1 \cdots G_k} \wedge dG_1 \wedge \cdots \wedge dG_k. \]

Here, we may take a lift $\tilde{\omega}$ for a given local section $\omega$ over affine open subsets, since $\Omega^q_{p^n/k} \rightarrow \Omega^q_{X_G/k}$ still remains surjective over affine open subsets. Since any two lifts of $\omega$ differ by sections of the form

\[ \sum_{i=1}^k G_i \alpha_i + \sum_{i=1}^k dG_i \wedge \beta_i, \]

which maps to zero by construction, $\gamma^q_G$ are well-defined. Moreover, $\gamma^q_G$ gather to define a cochain map (see [10, Tag 0G87]) so we get a map

\[ \gamma_G : \Omega^*_{X_G/k} \longrightarrow H^k_{X_G}(\Omega^*_{p^n/k})[k] \sim \mathcal{R}H_{X_G}(\Omega^*_{p^n/k})[2k] \]

This map is a quasi-isomorphism (see [5, II, Lemma (3.1)] and its proof). Therefore, the distinguished triangle (3.3) is isomorphic to
By taking the derived global section $R\Gamma (P^n, -)$, the associated long exact sequence gives us

$$\cdots \rightarrow H^0_{dR}(\mathbb{P}^n; k) \rightarrow H^0_{dR}(\mathbb{P}^n \setminus X_G; k) \rightarrow H^0_{dR}(-2k+1)(X_G; k) \rightarrow H^1_{dR}(\mathbb{P}^n; k) \rightarrow \cdots$$  \hspace{1cm} (3.4)

**Definition 3.2.** We define the residue map $\text{Res}_G : H^0_{dR}(\mathbb{P}^n \setminus X_G; k) \rightarrow H^0_{dR}(-2k+1)(X_G; k)$ to be the connecting homomorphism in (3.4).

### 3.2. Terasoma–Konno’s approach over $k$

**Definition 3.3** (Terasoma-Konno’s map over $k$). We define $k_G$ exactly same as (2.8), which works for any field of characteristic 0.

Then the map $k_G$ is an $k$-linear isomorphism:

$$k_G : A_{cG} / A_{cG} \cap \text{Jac}(S) \sim H^0_{dR}((\mathbb{P}(E)) \setminus X_S; k).$$  \hspace{1cm} (3.5)

This is because one can find a $\mathbb{Q}$-basis $\mu$ of $\mathbb{Q}[q]_{cG}/\mathbb{Q}[q]_{cG} \cap \text{Jac}(S)$ and a $\mathbb{Q}$-basis $\nu$ of $H^0_{dR}(\mathbb{P}(E) \setminus X_S; \mathbb{Q})$ such that $\mu$ (respectively, $\nu$) extends to a $\mathbb{C}$-basis of $\mathbb{C}[q]_{cG}/\mathbb{C}[q]_{cG} \cap \text{Jac}(S)$ (respectively, $H^0_{dR}(\mathbb{P}(E) \setminus X_S; \mathbb{C})$). Since $k$ is a $\mathbb{Q}$-algebra, we can extend these bases to $k$-bases to obtain the isomorphism in (3.5).

Here, we provide a detailed structure of the map $k_G$. For any commutative $k$-algebra $C$, let us consider the de Rham complex $(X_C, d)$ and the twisted de Rham complex $(X_C, d + dS \wedge)$ for any element $S \in C$. We define the charge and the weight on the de Rham complex by:

$$\begin{align*}
\text{charge} & : ch(q_i) = ch(dq_i), \quad i = 1, 2, \ldots, N, \\
\text{weight} & : wt(q_i) = wt(dq_i), \quad i = 1, 2, \ldots, N.
\end{align*}$$

We have the weight and the charge decomposition of $\Omega$ such that

$$\Omega = \bigoplus_{0 \leq j \leq N} \bigoplus_{i \in \mathbb{Z}} \bigoplus_{w \geq 0} \Omega^j_{i, (w)},$$

where $B := k[q, S^{-1}]_{0, (0)}$. Then $\mathbb{P}(E) \setminus X_S$ is a smooth affine variety whose coordinate ring is given by $B$.

**Definition 3.4.** We define a sequence of maps as follows (here $N = n + k + 1$):

We define a map $\varphi_S$ as the composition of the following maps:

$$A_{cG} \xrightarrow{\mu} \left( \Omega_A^N \right)_0 \xrightarrow{\rho} \left( \Omega_A^{N-1} \right)_{0, (0)} \xrightarrow{\theta_{wt \circ \theta_{dR}}} \Omega^{N-2}_B \xrightarrow{\text{quotient}} \Omega^{N-2}_B / d(\Omega^{N-3}_B) = H^{N-2}_{dR}(\mathbb{P}(E) \setminus X_S; k)$$

where

(i) For $q^u := q_1^{u_1} \cdots q_N^{u_N} \in A_{cG}$,

$$\mu(q^u) = -q^u dq_1 \wedge \cdots \wedge dq_N.$$
(ii) For \( \bar{y} = y_0^\infty \cdots y_n^\infty \), \( \bar{y} := y_1^m \cdots y_k^m \) with \( |\bar{y}| = v_1 + \cdots + v_k \),
\[
\rho(\bar{y} \bar{y}^r d\bar{q}_1 \wedge \cdots \wedge d\bar{q}_N) = (-1)^{|\bar{y}|+|r|-1}(|\bar{y}|+k-1)! \frac{\bar{y} \bar{y}^r}{S(\bar{y})^{k+|\bar{y}|}} dq_1 \wedge \cdots \wedge dq_N.
\]

(iii) Let \( \theta_{ch} \) be the contraction operator with the vector field \( \sum_{i=1}^N ch(q_i) \frac{\partial}{\partial q_i} \), and \( \theta_{wt} \) be the contraction operator with the vector field \( \sum_{i=1}^N wt(q_i) \frac{\partial}{\partial q_i} \).

Consider the rational differential form same as (2.7)
\[
\hat{\omega}(S, y^l u(\bar{x})) := (-1)^{n(k-1)+|\bar{y}|}(k+|\bar{y}|-1)! \frac{y^l u(\bar{x})}{S(q)^{k+|\bar{y}|}} \Omega_y \Omega_y \in \Omega^{N-2}_p,
\]
where \( y^l u(\bar{x}) \in A_{\bar{y}}^p \). Then a simple computation confirms that
\[
\varphi_S(y^l u(\bar{x})) = \left[ \hat{\omega}(S, y^l u(\bar{x})) \right],
\]
where \( [\cdot] \) means the cohomology class.

**Proposition 3.5.** The kernel of the map \( \varphi_S \) is \( K_G = \Gamma_G \cap A_{\bar{y}} \) where
\[
K_G := \bigoplus_{i=1}^N \left( \frac{\partial}{\partial q_i} + \frac{\partial S(q)}{\partial q_i} \right) A.
\]

By this proposition, \( \varphi_S \) induces a map \( \bar{\varphi}_S : A_{\bar{y}} / A_{\bar{y}} \cap K_G \to H^{n+k-1}_{dR}(\mathbb{P}(E) \setminus X_S; \mathbb{k}) \).

**Proposition 3.6.** Let \( \{ u_x + \text{Jac}(S) \} \) be a basis of \( A_{\bar{y}} / A_{\bar{y}} \cap \text{Jac}(S) \). The assignment \( u_x + \text{Jac}(S) \mapsto u_x + K_G \) provides us an isomorphism, denoted by \( \bar{q}_S \)
\[
q_S : A_{\bar{y}} / A_{\bar{y}} \cap \text{Jac}(S) \to A_{\bar{y}} / A_{\bar{y}} \cap K_G.
\]

We define \( J_G : A_{\bar{y}} \to H^{n+k-1}_{dR, \text{prim}}(X_G; \mathbb{k}) \) by the composition of the following maps:
\[
J_G : A_{\bar{y}} \xrightarrow{\phi_S} H^{n+k-1}_{dR}(\mathbb{P}(E) \setminus X_S; \mathbb{k}) \xrightarrow{s} H^{n+k-1}_{dR}(\mathbb{P}(E) \setminus X_S; \mathbb{k}) \xrightarrow{\text{Res}_G} H^{n+k-1}_{dR, \text{prim}}(X_G; \mathbb{k}),
\]
where we choose a section \( s \) to the projection \( pr_1 \). Denote the induced maps on \( A_{\bar{y}} / A_{\bar{y}} \) by \( \bar{\varphi}_S \) and \( J_G \). Therefore, we have an isomorphism
\[
A_{\bar{y}} / A_{\bar{y}} \cap \text{Jac}(S) \xrightarrow{q_S} A_{\bar{y}} / A_{\bar{y}} \cap K_G \xrightarrow{J_G} = \text{Res}_G \circ s \circ \bar{\varphi}_S H^{n+k-1}_{dR, \text{prim}}(X_G; \mathbb{k}).
\]

The Terasoma/Konno map \( k_G \) over \( \mathbb{k} \) can be understood as \( \bar{\varphi}_S \circ q_S : A_{\bar{y}} / A_{\bar{y}} \cap \text{Jac}(S) \to H^{n+k-1}_{dR}(\mathbb{P}(E) \setminus X_S; \mathbb{k}) \).

### 3.3. Dimca’s map over \( \mathbb{k} \) and construction of \( \delta_G \)

**Definition 3.7** (Dimca’s map over \( \mathbb{k} \)) We define \( d_G \) exactly same as (2.10), which works for any field of characteristic 0.

Now, we study how to construct \( \delta_G \) over \( \mathbb{k} \). The affine open covering \( \mathcal{U} = \{ D_+(G_i) \}_{i=1, \ldots, k} \) gives Čech-de Rham (double) complex \( \mathcal{C}^* (\mathcal{U}, \Omega(\mathbb{P}(E)) / \mathbb{k}^*) \) and a spectral sequence
\[ E_2^{p,q} = \tilde{H}^q \left( \Omega, H^p(\Omega^*_{(p^n \setminus X_G) / k}) \right) \Rightarrow H^{p+q}_{dR}(p^n \setminus X_G; k), \] (3.10)

where \( H^p(\Omega^*_{(p^n \setminus X_G) / k}) \) is the \( p \)th cohomology sheaf of \( \Omega^*_{(p^n \setminus X_G) / k} \).

**Remark 3.8.** Since \( \Omega^*_{(p^n \setminus X_G) / k} \) is quasicoherent, its cohomology sheaves are quasicoherent. Since each \( D_+(G_{j_0} \cdots G_{j_q}) \) are affine,

\[ H^r \left( D_+(G_{j_0} \cdots G_{j_q}), H^p(\Omega^*_{(p^n \setminus X_G) / k}) \right) = 0 \]

for \( r > 0 \) so

\[ \tilde{H}^q \left( \Omega, H^p(\Omega^*_{(p^n \setminus X_G) / k}) \right) \cong H^q \left( p^n \setminus X_G, H^p(\Omega^*_{(p^n \setminus X_G) / k}) \right). \]

Hence, the spectral sequence (3.10) represents the one coming from the Cartan–Eilenberg resolution and the derived global section \( R\Gamma(p^n \setminus X_G, -) \).

By (3.2), we have

\[ C_{n-1}(\Omega^*_{(p^n \setminus X_G) / k}) = \Gamma \left( p^n \setminus D_G, \Omega^*_{(p^n \setminus X_G) / k} \right), \]

so the zeroth page of the spectral sequence (3.10) is given as follows:

\[ \Gamma \left( p^n \setminus D_G, \Omega^*_{(p^n \setminus X_G) / k} \right) \rightarrow \Gamma \left( p^n \setminus D_G, \Omega^*_{(p^n \setminus X_G) / k} \right) \rightarrow \cdots \rightarrow \Gamma \left( p^n \setminus D_G, \Omega^*_{(p^n \setminus X_G) / k} \right) \]

Since \( p^n \setminus D_G = D_+(G_1 \cdots G_k) \) is affine, the top row computes the algebraic de Rham cohomology of \( p^n \setminus D_G \) over \( k \) so the next page with respect to the horizontal differential is the following:

\[ H^0_{dR}(p^n \setminus D_G; k) \rightarrow H^1_{dR}(p^n \setminus D_G; k) \rightarrow \cdots \rightarrow H^n_{dR}(p^n \setminus D_G; k) \]

Now the convergence of the spectral sequence (3.10) gives an exact sequence.
Definition 3.9. Define $\delta_G$ to be the surjection in the exact sequence (3.11).

3.4. Computation of $\delta_G$ and a comparison result

Since $E = O \oplus \cdots \oplus O_{p^k}$ is trivialized over $D_+(G)$ using $G^{-1}_k$, there is a commutative diagram

\[
\begin{array}{cccc}
(P^n \setminus D_G) \times_k \mathbb{A}^{k-1} & \longrightarrow & D_+(G) \times_k \mathbb{A}^{k-1} & \longrightarrow & P(E) \setminus X_S \\
\downarrow & & \downarrow & & \downarrow \\
P^n \setminus D_G & \longrightarrow & D_+(G) & \longrightarrow & P^n \setminus X_G
\end{array}
\]

where the squares are Cartesian. Let

\[U \times_k \mathbb{A}^{k-1} = \{D_+(G) \times_k \mathbb{A}^{k-1}\}_{i=1,\ldots,k}\]

be the affine open covering of $P(E) \setminus X_S$ corresponding to (3.1). Then

\[
\tilde{C}^d\left(U \times_k \mathbb{A}^{k-1}, H^p_{P}(\Omega^*_P)\right) \cong \tilde{C}^d\left(U_0, H^p_{\mathbb{R}}(\Omega^*_P)\right)
\]

for every $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ by the Künneth formula and $H^*_d(A^{k-1}) \cong \mathbb{k}[0]$. By the naturality of the spectral sequence associated to the Čech-de Rham complexes, we get a commutative square

\[
\begin{array}{ccc}
H^d_{dR}(P^n \setminus D_G) \times_k \mathbb{A}^{k-1} \setminus \mathbb{k} & \longrightarrow & H^{n+k-1}_{dR}(P(E) \setminus X_S; \mathbb{k}) \\
\uparrow & & \uparrow \\
\delta_D & & \delta_G \\
H^d_{dR}(P^n \setminus D_G; \mathbb{k}) & \longrightarrow & H^{n+k-1}_{dR}(P^n \setminus X_G; \mathbb{k})
\end{array}
\]

(3.12)

where the column isomorphism come from the projection $pr_1: P(E) \setminus X_S \to P^n \setminus X_G$ and the top row $\delta_D$ is defined from an exact sequence analogous to (3.11).

Now we will represent $\delta_D$ and $\tilde{\delta}_G$ analogously as in [2, Section 3] or [4, pp. 651–654], but using only polynomial differential forms. For this, we choose a map of affine schemes

\[
\sigma_G : P^n \setminus D_G \longrightarrow (P^n \setminus D_G) \times_k \mathbb{A}^{k-1}
\]

(3.13)

given by the ring homomorphism

\[
[\bar{x}, y_1, \frac{1}{G_1} \cdots \frac{1}{G_k}, \frac{1}{S}]_{0,(0)} \longrightarrow [\bar{x}, \frac{1}{G_1} \cdots \frac{1}{G_k}]_0, \quad (x, y_1, \cdots, y_k) \mapsto \left(\bar{x}, \frac{1}{G_1}, \cdots, \frac{1}{G_k}\right)
\]

which is a section to the projection $(P^n \setminus D_G) \times_k \mathbb{A}^{k-1} \to P^n \setminus D_G$ (recall that sub-indices 0 and (0) mean the charge 0 and weight 0 submodule, respectively). Using the presentation
\[(\mathbb{P}^n \setminus D_{\bar{G}}) \times_k \mathbb{A}^{k-1} \cong \text{Spec} \left( k \left[ x, \frac{1}{G_1 \cdots G_k} \right] \left[ \frac{y_1 G_1}{S}, \ldots, \frac{y_k G_k}{S} \right] \right), \]

we see that the above map can be written as

\[
\mathbb{P}^n \setminus D_{\bar{G}} \longrightarrow (\mathbb{P}^n \setminus D_{\bar{G}}) \times_k \mathbb{A}^{k-1} \quad x \longmapsto \left( x, \frac{1}{k}, \ldots, \frac{1}{k} \right).
\]

This suggests a “partition of unity”

\[
\left\{ \frac{S_\lambda}{S} \mid S_\lambda := y_\lambda G_\lambda, \quad \lambda = 1, \ldots, k \right\}.
\]

Here, the quotation mark means that this “partition of unity” works only for the differential forms with poles along the divisor \( D_{\bar{G}} \times_k \mathbb{A}^{k-1} \) of order at most 1. Any differential \( n \)-form on \( \mathbb{P}^n \setminus D_{\bar{G}} \) can be written as

\[
\frac{1}{G_1^{i_1+1} \cdots G_k^{i_k+1}} \xi, \quad \xi \in \Gamma \left( (\mathbb{P}^n \setminus X_{\bar{G}}), \Omega_+^{\mathbb{P}^n \setminus D_{\bar{G}}}/[k^n] \right),
\]

such that \( \text{ch}(\xi) = \text{ch}(G_1^{i_1+1} \cdots G_k^{i_k+1}). \) But this can be lifted via \( \sigma_{\bar{G}} \) to a differential form \( \tilde{\xi} \) with poles along the divisor \( D_{\bar{G}} \times_k \mathbb{A}^{k-1} \) of order at most 1:

\[
\sigma_{\bar{G}}(\xi) = \frac{1}{G_1^{i_1+1} \cdots G_k^{i_k+1}} \tilde{\xi}, \quad \tilde{\xi} := \frac{1}{G_1 \cdots G_k} \frac{k^{i_1} y_1^{i_1}}{S^{i_1}} \xi \in \Gamma \left( (\mathbb{P}^n \setminus D_{\bar{G}}) \times_k \mathbb{A}^{k-1}, \Omega_+^{\mathbb{P}^n \setminus D_{\bar{G}}}/[k^n] \right),
\]

(3.14)

where \( i = (i_1, \ldots, i_k) \) denotes a multi-index.

For \( \omega \in \check{\mathcal{C}}^q \left( \mathbb{L} \times_k \mathbb{A}^{k-1}, \Omega_+^{\mathbb{P}(\mathcal{E}) \setminus X_\mathbb{S}}/\mathbb{L} \right) \) with poles along the divisor \( D_{\bar{G}} \times_k \mathbb{A}^{k-1} \) of order at most 1, we write \( \omega \) in components as (cf. [1, Definition 8.2]):

\[
\omega = (\omega_{i_0 \cdots i_q}) \in \prod_{1 \leq i_0 < \cdots < i_q \leq k} \Gamma \left( D_+ (G_{i_0} \cdots G_{i_q}), \mathcal{O}_+^{\mathbb{P}(\mathcal{E}) \setminus X_\mathbb{S}}/\mathbb{L} \right)
\]

and we define

\[
(\tau \omega)_{i_0 \cdots i_q} := \frac{1}{k^{i_0+1} \cdots k^{i_q+1}} S_{i_0} \omega_{i_0 \cdots i_q} \in \Gamma \left( D_+ (G_{i_0} \cdots G_{i_q}), \mathcal{O}_+^{\mathbb{P}(\mathcal{E}) \setminus X_\mathbb{S}}/\mathbb{L} \right).
\]

Note that terms with repeated indexes are regarded as zero. Hence, we get an element

\[
\tau \omega \in \check{\mathcal{C}}^{q-1} \left( \mathbb{L} \times_k \mathbb{A}^{k-1}, \Omega_+^{\mathbb{P}(\mathcal{E}) \setminus X_\mathbb{S}}/\mathbb{L} \right).
\]

Let

\[
d^+ : \check{\mathcal{C}}^q \left( \mathbb{L} \times_k \mathbb{A}^{k-1}, \Omega_+^{\mathbb{P}(\mathcal{E}) \setminus X_\mathbb{S}}/\mathbb{L} \right) \longrightarrow \check{\mathcal{C}}^{q+1} \left( \mathbb{L} \times_k \mathbb{A}^{k-1}, \Omega_+^{\mathbb{P}(\mathcal{E}) \setminus X_\mathbb{S}}/\mathbb{L} \right)
\]

be the vertical differential coming from the Čech differential; let

\[
d_{\rightarrow} : \check{\mathcal{C}}^q \left( \mathbb{L} \times_k \mathbb{A}^{k-1}, \Omega_+^{\mathbb{P}(\mathcal{E}) \setminus X_\mathbb{S}}/\mathbb{L} \right) \longrightarrow \check{\mathcal{C}}^{q+1} \left( \mathbb{L} \times_k \mathbb{A}^{k-1}, \Omega_+^{\mathbb{P}(\mathcal{E}) \setminus X_\mathbb{S}}/\mathbb{L} \right)
\]

be the horizontal differential coming from the de Rham differential.
Lemma 3.10. For

\[ \omega = \omega_{1\ldots k} \in \tilde{C}^{k-1}(\mathbb{U} \times_k \mathbb{A}^{k-1}, \Omega^\alpha_{(\mathbb{P}(E) \setminus X_k)/k}) \]

such that \( d_- \omega = 0 \) (\( d_1 \omega = 0 \) is automatic),

\[ (-d_- \tau)^{k-1} \omega \in \tilde{C}^0(\mathbb{U} \times_k \mathbb{A}^{k-1}, \Omega^\nu_{(\mathbb{P}(E) \setminus X_k)/k}) \]

represents the cohomology class \( \delta_D([\omega]) \).

**Proof.** For \( \omega \in C_\mathbb{Q}(\mathbb{U} \times_k \mathbb{A}^{k-1}, \Omega^\nu_{(\mathbb{P}(E) \setminus X_k)/k}) \) with poles along the divisor \( D_\mathbb{Q} \times_k \mathbb{A}^{k-1} \) of order at most 1, we compute

\[ (d_1 \tau \omega)_{\lambda_0 \ldots \lambda_q} = \sum_{j=0}^{q} (-1)^j (\tau \omega)_{\lambda_0 \ldots \lambda_j \ldots \lambda_q} = \sum_{j=0}^{q} \sum_{k=1}^{k} (-1)^j \frac{S_j}{S_x} \omega_{\lambda_0 \ldots \lambda_j \ldots \lambda_q}. \]

This implies that

\[ (\tau d_1 \omega)_{\lambda_0 \ldots \lambda_q} = \sum_{k=1}^{k} \frac{S_j}{S_x} (d_1 \omega)_{\lambda_0 \ldots \lambda_j \ldots \lambda_q} \]

\[ = \sum_{k=1}^{k} \left( \frac{S_j}{S_x} \omega_{\lambda_0 \ldots \lambda_q} + \sum_{j=0}^{q} (-1)^j \frac{S_j}{S_x} \omega_{\lambda_0 \ldots \lambda_j \ldots \lambda_q} \right) \]

\[ = \omega_{\lambda_0 \ldots \lambda_q} - \sum_{j=0}^{q} \sum_{k=1}^{k} (-1)^j \frac{S_j}{S_x} \omega_{\lambda_0 \ldots \lambda_j \ldots \lambda_q} \]

\[ = \omega_{\lambda_0 \ldots \lambda_q} - (d_1 \tau \omega)_{\lambda_0 \ldots \lambda_q} \]

that is, \( (d_1 + \tau d_1) \omega = \omega \) (cf. [1, Proposition 8.5]).

Now assume that \( d_1 \omega = 0 \) and \( d_- \omega = 0 \). In this case, \( \omega_1 := -d_- \tau \omega \) satisfies

\[ d_- \omega_1 = -d_1 \tau \omega = 0, \quad d_1 \omega_1 = -d_1 d_- \tau \omega = d_- d_1 \tau \omega = d_- (\omega - \tau d_1 \omega) = 0. \]

Hence, we may apply \(-d_- \tau \) to \( \omega_1 \) again. Moreover,

\[ (d_1 + d_- \tau) \omega = d_1 \tau \omega - \omega_1 = \omega - \tau d_1 \omega - \omega_1 = \omega - \omega_1, \]

so \( \omega \) and \( \omega_1 = -d_- \tau \omega \) define the same class in the cohomology of the total complex. In particular, for \( \omega = \omega_{1\ldots k} \in \tilde{C}^{k-1}(\mathbb{U} \times_k \mathbb{A}^{k-1}, \Omega^\alpha_{(\mathbb{P}(E) \setminus X_k)/k}) \) such that \( d_- \omega = 0 \) (\( d_1 \omega = 0 \) is automatic),

\[ (-d_- \tau)^{k-1} \omega \in \tilde{C}^0(\mathbb{U} \times_k \mathbb{A}^{k-1}, \Omega^\nu_{(\mathbb{P}(E) \setminus X_k)/k}) \]

represents the cohomology class \( \delta_D([\omega]) \) by the definition of \( \delta_D \).

Recall the maps \( k_\mathbb{Q} : A_{\mathbb{Q}} / A_{\mathbb{Q}} \cap \text{Jac}(S) \to H^{n+k-1}_{\mathbb{Q}}(\mathbb{P}(E) \setminus X; \mathbb{Q}) \) and \( d_\mathbb{Q} : A_{\mathbb{Q}} / A_{\mathbb{Q}} \cap \text{Jac}(S) \to H^{n+k}_{\mathbb{Q}}(\mathbb{P}(E) \setminus D_\mathbb{Q}; \mathbb{Q}) \) from (2.8) and (2.10):

\[ k_\mathbb{Q} \left( \left[ y^lu(x) \right] \right) = \left[ 1 \right] \left( \left[ S_y y^lu(x) \right] \right) = \left[ (-1)^{n+k} y^lu(x) \right], \]

\[ d_\mathbb{Q} \left( \left[ y^lu(x) \right] \right) = \left[ 2 \right] \left( \left[ S_y y^lu(x) \right] \right) = \left[ (-1)^{n+k} y^lu(x) \right]. \]
Theorem 3.11. We have

\[
(pr_1^* \circ \delta_G) \left[ \beta \left( S, y^i u(x) \right) \right] = \frac{i_1! \cdots i_k!(k-1)!k!}{(-1)^{k+1}(i_1 + k - 1)!} \left[ \alpha \left( S, y^i u(x) \right) \right]
\]

for \( y^i u(x) \in A_{c_G} \). In other words, we have \( pr_1^* \circ \delta_G \circ d_G = \frac{i_1! \cdots i_k!(k-1)!k!}{(-1)^{k+1}(i_1 + k - 1)!} \cdot k_G \).

Proof. By the commutativity \( pr_1^* \circ \delta_G \circ (pr_1^*)^{-1} = \delta_D \) of the diagram (3.12) and the fact \( (pr_1^*)^{-1} = \sigma_{c_G}^* \), it suffices to construct a differential form \( \omega(S, y^i u(x)) \in \Gamma((\mathbb{P}^n \setminus D_G) \times_k \mathbb{A}^{k-1}, \Omega^n_{\mathbb{P}(\mathbb{P}^n), X_0/k}) \) which satisfies

\[
\sigma_{c_G}^* \left( \left[ \omega(S, y^i u(x)) \right] \right) = \frac{(-1)^k(i_1 + k - 1)!}{i_1! \cdots i_k!(k-1)!k!} \left[ \beta(S, y^i u(x)) \right], \ \delta_D \left( \left[ \omega(S, y^i u(x)) \right] \right) = \left[ \alpha(S, y^i u(x)) \right].
\]

(3.15)

We claim that the following differential form

\[
\omega(S, y^i u(x)) := \frac{(-1)^{i_1 + k - 1}(i_1 + k - 1)!}{(k-1)!} \frac{y^i u(x)}{S^{i_1} \cdots S^{i_k}} \Omega_x \in \Gamma((\mathbb{P}^n \setminus D_G) \times_k \mathbb{A}^{k-1}, \Omega^n_{\mathbb{P}(\mathbb{P}^n), X_0/k})
\]
serves our purpose. The first equality in (3.15) clearly holds from the definition (3.14) of \( \sigma_{c_G}^* \). For the second equality, based on Lemma 3.10, we compute \( (-d_-\tau)^{k-1} \omega \) on \( D_+(G_1) \times_k \mathbb{A}^{k-1} \) : its component is

\[
(-d_-\tau)^{k-1} \omega \big|_{x_0} = (-1)^{k-1} \left( \sum_{\mu=1}^k d\left( \frac{S_\mu}{S} \right) \right) \wedge \cdots \wedge \left( \sum_{\mu=1}^k d\left( \frac{S_{k-1}}{S} \right) \right) \wedge \omega_{\mu_1 \cdots \mu_{k-1}}
\]

\[
= (-1)^{k-1}(k-1)!d\left( \frac{S_1}{S} \right) \wedge \cdots \wedge d\left( \frac{S_k}{S} \right) \wedge \cdots \wedge d\left( \frac{S_k}{S} \right) \wedge \omega_{1 \cdots k}.
\]

From the computation

\[
d\left( \frac{S_1}{S} \right) \wedge \cdots \wedge d\left( \frac{S_k}{S} \right) \wedge \cdots \wedge d\left( \frac{S_k}{S} \right)
\]

\[
= dS_1 \wedge \cdots \wedge d\tilde{S}_\lambda \wedge \cdots \wedge d\tilde{S}_\lambda \wedge \cdots \wedge dS_k
\]

\[
= \left( S_1 + \cdots + \tilde{S}_\lambda \wedge \cdots + \tilde{S}_\lambda \right) dS_1 \wedge \cdots \wedge d\tilde{S}_\lambda \wedge \cdots \wedge dS_k
\]

\[
+ \sum_{\mu \neq \lambda} (-1)^{i_\mu} S_\mu dS_1 \wedge \cdots \wedge \tilde{S}_\mu \wedge \cdots \wedge dS_k
\]

\[
= (-1)^k \left( \sum_{\mu=1}^k (-1)^{i_\mu} S_\mu dS_1 \wedge \cdots \wedge \tilde{S}_\mu \wedge \cdots \wedge dS_k \right),
\]

we obtain
\[
\left( -d_{\tau} \right)^{k-1} \omega_x \equiv (-1)^{\frac{1}{2}} \left( -1 \right)^{k-1} (k-1)! \sum_{i=1}^{k} (-1) S_i \sum_{j=1}^{k} (-1)^{\frac{1}{2}} \tilde{S}_j \sum_{m=1}^{k} dS_m \wedge \omega_{1-k}.
\]

Since the local sections agree up to \((-1)^{\frac{1}{2}}\), which is the index of \(D_+ (G_x)\), they glue to define a global section of \(\Omega^{n+k-1}_{(\mathcal{P}(\mathcal{E}) \setminus X_0)/k}\)

\[
\delta(\omega) = (-1)^{k-1} (k-1)! \sum_{i=1}^{k} (-1)^{\frac{1}{2}} S_i \sum_{j=1}^{k} \tilde{S}_j \sum_{m=1}^{k} dS_m \wedge \omega,
\]

that is, if we extend the Čech complex as usual:

\[
d_{\tau} : \Gamma \left( \mathcal{P}(\mathcal{E}) \setminus X_0, \Omega^{n+k-1}_{(\mathcal{P}(\mathcal{E}) \setminus X_0)/k} \right) \rightarrow \tilde{\mathcal{C}}^0 \left( \mathfrak{U} \times_k \mathbb{A}^{k-1}, \Omega^{n+k-1}_{(\mathcal{P}(\mathcal{E}) \setminus X_0)/k} \right) \quad \psi \mapsto \left( -1 \right)^{\frac{1}{2}} \psi_{x_{1,...,k}},
\]

then \(d_{\tau} \delta(\omega) = \left( -d_{\tau} \right)^{k-1} \omega\). By the sheaf property, \(d_{\tau} d_{\tau} \delta(\omega) = -d_{\tau} d_{\tau} \delta(\omega) = 0\) implies that \(\delta(\omega)\) is also a de Rham cocycle. Thus, \(\delta_D ([\omega]) = [\delta(\omega)]\).

Note that

\[
\delta(\Omega_x) = (-1)^{k-1} (k-1)! \sum_{i=1}^{k} (-1)^{\frac{1}{2}} S_i \sum_{j=1}^{k} \tilde{S}_j \sum_{m=1}^{k} dS_m \wedge \Omega_x
\]

\equiv (-1)^{(k-1)(n-1)} (k-1)! \frac{G_1 \cdots G_k}{\mathfrak{S}_k} \Omega_x \wedge \Omega_y \mod dx_0 \wedge \cdots \wedge dx_n
\]

together with the vanishing of \([dx_0 \wedge \cdots \wedge dx_n]\) in the cohomology shows that

\[
[\delta(\Omega_x)] = \left[ (-1)^{(k-1)(n-1)} (k-1)! \frac{G_1 \cdots G_k}{\mathfrak{S}_k} \Omega_x \wedge \Omega_y \right].
\]

Therefore, we get

\[
\delta \left( \omega(S, y^I u(x)) \right) \equiv (-1)^{\frac{1}{2} + n(k-1)} \left( \frac{1}{2} + n(k-1) \right)! \frac{y^I u(x)}{\mathfrak{S}_{\left[\frac{1}{2} + n(k-1) \right]}} \Omega_x \wedge \Omega_y \mod dx_0 \wedge \cdots \wedge dx_n,
\]

which proves the second equality in (3.15). \(\square\)

**Acknowledgements**

Jeehoon Park thanks KIAS (Korea Institute for Advanced Study), where the part of work was done, for its hospitality. The authors thank the anonymous referee for useful comments to improve the article.

**Funding**

The work of Jeehoon Park was supported by the National Research Foundation of South Korea (NRF-2018R1A4A1023590 and NRF-2021R1A2C1006696). The work of Jeehoon Park was also supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No.2020R1A5A1016126).
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