A comparison of FQHE quasi electron trial wave functions on the sphere

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We study Haldane’s and Jain’s proposals for the quasiparticle wave function on the sphere. The expectation values of the energy and the pair angular momenta distribution are calculated at filling factor \( \frac{1}{q} \) and compared with the data of an exact numerical diagonalization for up to 10 electrons with Coulomb and truncated quasipotential interaction.

I. INTRODUCTION

The dynamics of interacting planar electrons in the lowest Landau level (LLL) of a strong magnetic field shows some interesting and not yet fully understood features at filling factors \( \nu < 1 \) experimentally observed as the Fractional Quantum Hall Effect (FQHE).

Following Haldane’s proposal we will study the physics of the FQHE in a spherical geometry. This gives a clear meaning to the concept of filling factor since the one particle LLL Hilbert space is finite dimensional. For a sphere of radius \( R \) and filled with \( \nu \) flux quanta this dimension is \( 2\nu + 1 \) and the filling factor is given by \( \nu = \frac{N}{2\nu} \) for \( N \) electrons.

Trial wave functions play an important rôle in our attempts to understand the FQHE. The prime example is the Laughlin wave function, which describes very accurately the ground state at filling factors \( \nu = \frac{1}{q} \) with odd integer \( q \).

There are two main attempts to understand other filling factors, the hierarchy model by Haldane and Halperin and the composite fermion (CF) model by Jain. Both describe the reaction of Laughlin’s ground state to the addition or removal of flux quanta by the creation of quasiholes or quasiparticles and give analytic expressions for the quasiparticle wavefunctions.

After some remarks concerning the structure of these wavefunctions on the sphere this paper gives a comparison of them with the results of exact numerical diagonalizations including up to 10 electrons. Besides energy expectation values and overlaps we discuss the distribution of pair angular momenta as a more demanding test. Jain’s proposal gives better results but contrary to the Laughlin wavefunction it is not exact even for a electron interaction including only one quasipotential.

II. WAVE FUNCTIONS ON THE SPHERE

Using the Wu-Yang gauge for the field of a magnetic monopole in the center of the sphere (radius \( R \), \( e < 0 \), \( 0 \leq \theta < \pi \))

\[
A_{\theta} = \frac{hS}{eR} \left( 1 - \cos \theta \right) \sin \theta, \tag{2.1}
\]
a basis for the \( 2\nu + 1 \) dimensional one particle LLL Hilbert space is given by

\[
\psi_{m}(z) = \left( \frac{2S}{S-m} \right)^{\frac{1}{2}} z^{S-m}, \quad m = 1, -S, -S + 1, \ldots S. \tag{2.2}
\]

Here \( z = \tan \frac{\theta}{2} e^{-i\phi} \) is the complex stereographic coordinate.

These wave functions are orthonormal

\[
\frac{2S+1}{\pi} \int \frac{d^{2}z}{(1+|z|^{2})^{2S+2}} \psi_{m}^{\ast}(z) \psi_{n}(z) = \delta_{n,m} \tag{2.3}
\]

and related to Haldane’s basis by a phase (gauge) factor \( e^{iS\phi} \).

In this gauge, the Laughlin function has up to a normalization factor \( \Pi_{i}^{N} (S_{-m})^{1/2} \) the usual shape \( \Psi_{L} = \Pi_{i}^{N} \langle z_{i} \rangle^{q} \). In the following we will always work with unnormalized wave functions.

The conserved angular momentum \( \vec{l} = \vec{r} \times (-i\hbar \vec{\nabla} + e\vec{A} / r) \) acts in this basis by

\[
l_{z} = S - z \frac{\partial}{\partial z}, \quad l_{r} = \frac{\partial}{\partial z}, \quad l_{\perp} = 2Sz - z^{2} \frac{\partial}{\partial z}. \tag{2.4}
\]

The total angular momentum of all electrons \( \vec{L} = \sum \vec{l}_{i} \) commutes with the two particle interaction giving a decomposition of the spectrum into degenerate \( SU(2) \) multiplets. So is the Laughlin wave function a \( |\vec{L} = 0 \rangle \) state with an homogeneous charge distribution and the quasiparticle appears as a \( |\vec{L} = N/2 \rangle \) multiplet.

The first proposal for such a quasiparticle multiplet on the sphere is due to Haldane in the framework of the hierarchy model and reads in our coordinates

\[
\Psi^{H}(z_{0} ; z_{1}, \ldots, z_{N}) \mid_{m=N/2} = \sum_{m=-N/2}^{N} z_{0}^{q_{m}+m} \Psi_{m}^{H}(z_{1}, \ldots, z_{N}) \]

\[
= \prod_{i=1}^{N} \left( q(N-1) - z_{i} \frac{\partial}{\partial z_{i}} + z_{0} \frac{\partial}{\partial z_{1}} \right) \Psi_{L}(z_{1}, \ldots, z_{N}). \tag{2.5}
\]
One easily checks using eq. (2.4) that the $\Psi^H_m$ constitute a $SU(2)$ multiplet of spin $N/2$.

In the CF model, on the other side, a quasiparticle at $\nu = 1/3$ is described as a state of $N$ composite fermions in a reduced magnetic flux $2S' = 2(S - N + 1)$. Since there is only room for $N - 1$ particles in this LLL, one composite particle has to be in the second Landau level.

\[
\chi_n(z_1, \ldots, z_N) = \frac{z_1^n \cdots z_N^n}{1 \cdots 1} \quad z_1 \cdots z_N \quad \cdots \quad z_{N-2} \cdots z_{N-2} \quad (2.6)
\]

Now one has to make an electron wavefunction out of this composite fermion wave function according to

\[
\Psi_n(z_1, \ldots, z_N) = \mathcal{P} \prod_{i<j}(z_i - z_j)^2 \chi_n \quad (2.7)
\]

where $\mathcal{P}$ is the projector to the LLL. This gives for $n = N/2 - 1 - m = -1, 0, \ldots, N - 2$ only $N$ out of $N + 1$ members of an $SU(2)$ multiplet. Setting $n = N - 1$ in eq. (2.7) gives a wrong answer for the last member, but it can of course be obtained by acting with $L_-$ on $\Psi_n^{N-2}$.

Another projection scheme would be

\[
\Psi_n(z_1, \ldots, z_N) = \prod_{i<j}(z_i - z_j) \mathcal{P} \prod_{i<j}(z_i - z_j)^2 \chi_n \quad (2.8)
\]

Remarkably, both schemes give identical wave functions since the difference between eq. (2.7) and eq. (2.8) includes the factor

\[
\sum_{i}^{N} z_i^n \prod_{j \neq i}(z_j - z_i) = \begin{vmatrix}
z_1^n & \cdots & z_N^n \\
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & 1
\end{vmatrix} = 0 \quad (2.9)
\]

vanishing for $n = 0, \ldots, N - 2$ as well as a factor $n + 1$ vanishing for $n = -1$.

### III. Numerical Calculations

The $(2S+1)$-dimensional $N$ electron LLL Hilbert space has a basis of Slater determinants $\Psi[m_1, \ldots, m_N](z_i) = \det[z_i^{S-m_i}]$. Since $L_z$ is diagonal with eigenvalue $M = \sum m_i$, it is possible to work in a Hilbert space $\mathcal{H}_M$ with fixed $M$. The possible values for $M$ lie between $-M_{\max}$ and $+M_{\max}$ with $M_{\max} = (2S + 1 - N)/2$.

A generating function for its dimension can be found by considering the grand partition function for a system having $2S + 1$ fermionic energy levels with energy proportional to $m$. A Laplace transform projects out the $N$ particle contribution

\[
\sum_M \dim(H_M)x^{M + M_{\max}} = \int_0^{2\pi} e^{iN\alpha} \prod_{j=1}^{2S+1} (1 + e^{-i\pi x/2}) \frac{d\alpha}{2\pi} = \prod_{j=1}^{N} \frac{1 - xe^{2S+2-j}}{1 - xe^{j}} \quad (3.1)
\]

As an illustration, the right hand side evaluates for $N = 4$ electrons at $\nu = 1/3, 2S = 9 M_{\max} = 12$ to

\[
x^{24} + x^{23} + 2x^{22} + 3x^{21} + 5x^{20} + 6x^{19} + 9x^{18} + 10x^{17} + 13x^{16} + 14x^{15} + 16x^{14} + 16x^{13} + 18x^{12} + 16x^{11} + 16x^{10} + 14x^{9} + 13x^{8} + 10x^{7} + 9x^{6} + 6x^{5} + 5x^{4} + 3x^{3} + 2x^{2} + x + 1 \quad (3.2)
\]

and we can read off from the middle term $x^{12}$ that $\mathcal{H}_0$ has dimension 18.

A rotational invariant two particle interaction in this Slater basis can be expressed as

\[
H = \frac{1}{2} \sum_{m_1, m_2, m_3, m_4} V_{m_1 m_2 m_3 m_4} a_{m_1}^{\dagger} a_{m_2}^{\dagger} a_{m_3} a_{m_4}. \quad (3.3)
\]

where the $a_{m}^{\dagger}$ and $a_{m}$ create or annihilate an electron in the state $z^{S-m}$.

With the help of Clebsch–Gordan coefficients this is written as sum of contributions from different pair angular momentum states

\[
V_{m_1 m_2 m_3 m_4} = \sum_{M} \sum_{J=0}^{2S} \left( \begin{array}{c}
S \\
1 \frac{m_1}{2} \\
1 \frac{m_2}{2}
\end{array} \right) M \times \left( \begin{array}{c}
S \\
\frac{m_3}{2} \\
\frac{m_4}{2}
\end{array} \right) V_{2S-J} \quad (3.4)
\]

Due to Fermi statistics only odd quasiparticle coefficients $V_{1,i}, i = 1, 3, \ldots$ contribute to eq. (3.3).

For the case of Coulomb interaction proportional to $(\text{chord distance})^{-1}$ the $V_i$ are calculated by Fano et al.

We consider also the truncated quasipotential model, where two particles repel each other only if they are in a state of maximal relative angular momentum, i.e. only $V_1$ is nonvanishing.

The lowest eigenvalue and the corresponding eigenvector of the Hamiltonian eq. (3.3) were calculated by an iterative Lanczos procedure. In order to handle the very large matrices (of e. g., dimension 165 821$^2$ for $N = 10$) a sophisticated algorithm for indexing and storing was used.

To calculate the energy (and other) expectation values of the trial wave functions they are expanded in Slater states in order to have them in the same form as the exact eigenvectors.
IV. RESULTS

Fig. 1 shows the finite size dependence of the energies for $V_1$ interaction. The data are fitted by a quadratic polynomial in $N^{-1}$. Obviously, Jain’s wavefunction works better. In the $N \to \infty$ limit its energy is about 7% too high. This has to be compared with an 28% error in the hierarchy model. The results are qualitatively similar for Coulomb interaction and are confirmed by a calculation of the overlaps of the different wave functions, shown for $N = 10$ electrons in Table I. The superiority of the CF quasiparticle wave function in the reproduction of finite size calculations has also been found in the disk geometry.

To give an optical impression of a quasiparticle, Fig. 2 shows the charge distribution of the $L_z = \frac{N}{2}$ member of the quasiparticle multiplet. In the sense of eq. (2.5) this can be interpreted as a quasiparticle sitting at the south pole $z \to \infty$. As already observed in the disk geometry, the quasiparticle excess charge is concentrated on a ring of roughly the size of a magnetic length ($R/\sqrt{S}$ in our units). Higher quasipotentials introduce more inhomogeneities outside the excitation as the case of Coulomb interaction (dashed line) shows.

As emphasized by Gros and MacDonald, a crucial rôle in the dynamics of the FQHE is played by the distribution of pair angular momenta. The Laughlin state is the only state in the $\nu = 1/3$ Hilbert space with vanishing contribution for the highest possible pair angular momentum $2S - 1$. For higher filling factors (i.e., in the presence of quasiparticles) no such state exists in the Hilbert space. But due to the $V_1$ interaction, the one quasiparticle ground state is still a state with very small probability of finding an electron pair with angular momentum $2S - 1$. These probabilities are shown for a $N = 10$ electron system in Fig. 3 normalized to sum up to the number of pairs $N(N-1)/2 = 45$. One sees clearly, how the interaction nearly empties the pair states coupling to $V_1$ resulting in a very high probability for pairs in the next-to-highest angular momentum state. This picture is not very much changed in the Coulomb case, an astonishing result facing the fact that the $V_3$ interaction should suppress the amplitude of the next-to-highest angular momentum state. The smallness of this suppression confirms the point of view that the $V_1$ hard core model includes the essential physics of the FQHE and higher quasipotentials just make small qualitative changes. It seems that even in the hard core model the amplitudes for higher angular momentum pairs have already nearly reached their maximum for this Hilbert space. It would be nice to have a theoretical insight in the occurrence of these filling factor dependent maximal and minimal probabilities of finding electron pairs in states of some relative angular momentum.

Remarkably, both the hierarchy and the CF proposal reproduce very well the exact distributions. This supports the conclusion that both of them capture in their analytic expressions some of the essential physics of the FQHE. The ground state energies show, however, a much better behaviour for the CF wavefunction especially in the large $N$ extrapolation.

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**FIG. 1.** Quasiparticle energy for $V_1$ interaction vs. $N^{-1}$.

**FIG. 2.** Charge density distribution $\rho(\theta)$ of 10 electrons for one quasiparticle at the south pole normalized to $\int \rho(\theta) \, d\cos \theta = 10|e|$.  

**FIG. 3.** Pair angular momenta distribution.

**TABLE I.** Quasiparticle wave function overlaps for $N = 10$.

|                | exact ($V_1$) | Jain     | Haldane  |
|----------------|--------------|----------|----------|
| exact (Coulomb)| 0.987584     | 0.985416 | 0.972998 |
| exact ($V_1$)  |              | 0.988918 | 0.968469 |
| Jain           |              | 0.993149 |          |