ON THE CONVERGENCE OF PARTIAL SUMS WITH RESPECT TO VILENKIN SYSTEM ON THE MARTINGALE HARDY SPACES

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Abstract. In this paper we derive characterizations of boundedness of the subsequences of partial sums with respect to Vilenkin system on the martingale Hardy spaces when $0 < p < 1$. Moreover, we find necessary and sufficient conditions for the modulus of continuity of $f \in H_p$ martingales, which provide convergence of subsequences of partial sums on the martingale Hardy spaces. It is also proved that these results are the best possible in a special sense. As applications, both some well-known and new results are pointed out.

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1. Introduction

The definitions and notations used in this introduction can be found in our next Section. It is well-known that (for details see e.g. [13])

$$\|S_n f\|_p \leq c_p \|f\|_p,$$

where $S_n f$ is $n$-th partial sum with respect to bounded Vilenkin system.

Moreover, it can be proved also a more stronger result (for details see e.g. [11]):

$$\|S^* f\|_p \leq c_p \|f\|_p,$$

where

$$S^* f := \sup_{n \in \mathbb{N}} |S_n f|.$$

Lukomsckii [13] proved two-side estimate for Legesgue constants $L_n$ with respect to Vilenkin sistem. By using this result we easily obtain that $S_{n_k} f$ convergence to $f$ in $L_1$-norm, for every integrable function $f$, if and only if

$$\sup_{k \in \mathbb{N}} L_{n_k} \leq c < \infty.$$

Point-wise and uniform convergence and some approximation properties of partial sums in $L_1$ norm were studied by a lot of authors. We mentioned,
for instance, the paper of Goginava [9], Goginava and Sahaki [10] and Avdipsahić and Memić [2]. Fine [4] obtained sufficient conditions for the uniform convergence which are in complete analogy with the Dini-Lipschits conditions. Guličev [12] estimated the rate of uniform convergence of a Walsh-Fourier series by using Lebesgue constants and modulus of continuity. Uniform convergence of subsequence of partial sums with respect to Walsh system was investigate also in [8]. This problem for Vilenkin group \(G_m\) was considered by Blahota [3], Fridli [5] and Gát [7].

It is known (for details see e.g. [18]) that Vilenkin system does not form basis in the space \(L_1(G_m)\). Moreover, there is a function in the martingale Hardy space \(H_1(G_m)\), such that the partial sums of \(f\) are not bounded in \(L_1(G_m)\)-norm, but subsequence \(S_{M_n}\) of partial sums are bounded from the martingale Hardy space \(H_p(G_m)\) to the Lebesgue space \(L_p(G_m)\), for all \(p > 0\).

In [21] it was proved that if \(0 < p \leq 1\) and \(\{\alpha_k : k \in \mathbb{N}\}\) be a increasing secuence of nonnegative integers such that

(1) \[\sup_{k \in \mathbb{N}} \rho(\alpha_k) < \infty,\]

where \(\rho(n) = |n| - \langle n \rangle\)

and

\[\langle n \rangle := \min\{j \in \mathbb{N} : n_j \neq 0\}, \hspace{1cm} |n| := \max\{j \in \mathbb{N} : n_j \neq 0\},\]

for \(n = \sum_{k=0}^{\infty} n_j M_j, \hspace{1cm} n_j \in Z_{m_j} \hspace{1cm} (j \in \mathbb{N})\), then the restricted maximal operator

\[\tilde{S}^{*}\triangle f := \sup_{k \in \mathbb{N}} |S_{\alpha_k}f|\]

is bounded from the Hardy space \(H_p\) to the Lebesgue space \(L_p\). Moreover, if \(0 < p < 1\) and \(\{\alpha_k : k \in \mathbb{N}\}\) be a increasing secuence of nonnegative integers satisfying the condition

(2) \[\sup_{k \in \mathbb{N}} \rho(\alpha_k) = \infty,\]

then there exists a martingale \(f \in H_p\) such that

\[\sup_{k \in \mathbb{N}} \|S_{\alpha_k}f\|_{L_p,\infty} = \infty.\]

It immediately follows that for any \(p > 0\) and \(f \in H_p\), the following restricted maximal operator

\[\tilde{S}^{*}_{\#} f := \sup_{n \in \mathbb{N}} |S_{M_n}f| ,\]

where \(M_0 := 1, \hspace{1cm} M_{k+1} := \prod_{i=0}^{k} m_i\) and \(m := (m_0, m_1, ...)\) be a sequence of positive integers not less than 2, which generates a Vilenkin system, is bounded.
from the Hardy space $H_p$ to the space $L_p$:

\[ \left\| \sum_{k=1}^{n} S_k f \right\|_p \leq \left\| f \right\|_{H_p}, \quad f \in H_p. \]

For the Vilenkin system Simon [15] proved that there is an absolute constant $c_p$, depending only on $p$, such that

\[ \sum_{k=1}^{\infty} \frac{\| S_k f \|_p}{k^{2-p}} \leq c_p \left\| f \right\|_{H_p}, \]

for all $f \in H_p(G_m)$, where $0 < p < 1$. In [17] we proved that sequence $\{1/k^{2-p} : k \in \mathbb{N}\}$ can not be improved.

Analogical theorem for $p = 1$ with respect to the unbounded Vilenkin systems was proved in Gát [6].

In [18] we proved that if $0 < p < 1$, $f \in H_p(G_m)$ and

\[ \omega \left( \frac{1}{M_n}, f \right)_{H_p(G_m)} = o \left( \frac{1}{M_n^{1/p-1}} \right), \quad \text{as } n \to \infty, \]

then

\[ \left\| S_{n_k} f - f \right\|_{H_p} \to 0, \quad \text{as } k \to \infty. \]

Moreover, for every $p \in (0, 1)$ there exists martingale $f \in H_p(G_m)$, for which

\[ \omega \left( \frac{1}{M_n}, f \right)_{H_p(G_m)} = O \left( \frac{1}{M_n^{1/p-1}} \right), \quad \text{as } n \to \infty \]

and

\[ \left\| S_k f - f \right\|_{L_p(G_m)} \to 0, \quad \text{as } k \to \infty. \]

In [20] we investigated some $(H_p, p)$, $(H_p, L_p)$ and $(H_p, L_{p, \infty})$ type inequalities of subsequences of partial sums with respect to Walsh-Fourier series for $0 < p \leq 1$.

In this paper we derive characterizations of boundedness (or even the ratio of divergence of the norm) of the subsequences of partial sums with respect to Vilenkin system on the martingale Hardy spaces when $0 < p < 1$. Moreover, we find necessary and sufficient conditions for the modulus of continuity of $f \in H_p$, which provide convergence of subsequences of partial sums on the martingale Hardy spaces. It is also proved that these results are the best possible in a special sense. As applications, both some well-known and new results are pointed out.

This paper is organized as follows: In order not to disturb our discussions later on some definitions and notations are presented in Section 2. For the proofs of the main results we need some auxiliary Lemmas, some of them are new and of independent interest. These results are also presented in Section 2. The formulations and detailed proofs of our main results and some of its consequences can be found in Sections 3 and 4.
2. Preliminaries

Let \( \mathbb{N}_+ \) denote the set of the positive integers, \( \mathbb{N} := \mathbb{N}_+ \cup \{0\} \).

Let \( m := (m_0, m_1, \ldots) \) denote a sequence of the positive integers not less than 2. Denote by \( \mathbb{Z}_{m_k} := \{0, 1, \ldots, m_k - 1\} \) the additive group of integers modulo \( m_k \). Define the group \( G_m \) as the complete direct product of the group \( \mathbb{Z}_{m_j} \) with the product of the discrete topologies of \( \mathbb{Z}_{m_j} \) s. The direct product \( \mu \) of the measures

\[
\mu_k (\{j\}) := 1/m_k \ (j \in \mathbb{Z}_{m_k})
\]

is the Haar measure on \( G_m \) with \( \mu (G_m) = 1 \).

If the sequence \( m := (m_0, m_1, \ldots) \) is bounded than \( G_m \) is called a bounded Vilenkin group, else it is called an unbounded one.

The elements of \( G_m \) are represented by sequences \( x := (x_0, x_1, \ldots, x_j, \ldots) \ (x_k \in \mathbb{Z}_{m_k}) \).

It is easy to give a base for the neighborhood of \( G_m \)

\[
I_0 (x) := G_m, \quad I_n (x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\} \ (x \in G_m, \ n \in \mathbb{N}).
\]

Denote \( I_n := I_n (0) \) for \( n \in \mathbb{N} \) and \( T_n := G_m \setminus I_n \). It is evident that

\[
(6) \quad T_N = \bigcup_{s=0}^{N-1} I_s \setminus I_{s+1}.
\]

If we define the so-called generalized number system based on \( m \) in the following way \( M_0 := 1, M_{k+1} := m_k M_k \ (k \in \mathbb{N}) \) then every \( n \in \mathbb{N} \) can be uniquely expressed as \( n = \sum_{k=0}^{\infty} n_j M_j \), where \( n_j \in \mathbb{Z}_{m_j} \ (j \in \mathbb{N}) \) and only a finite number of \( n_j \)'s differ from zero.

For all \( n \in \mathbb{N} \) let us define

\[
\langle n \rangle := \min\{j \in \mathbb{N} : n_j \neq 0\}, \quad |n| := \max\{j \in \mathbb{N} : n_j \neq 0\}, \quad \rho (n) = |n| - \langle n \rangle.
\]

For the natural number \( n = \sum_{j=1}^{\infty} n_j M_j \), we define functions \( v \) and \( v^* \) by

\[
v (n) = \sum_{j=1}^{\infty} |\delta_j + 1 - \delta_j| + \delta_0, \quad v^* (n) = \sum_{j=1}^{\infty} \delta_j^*,
\]

where

\[
\delta_j = \text{sign} n_j = \text{sign} (\oplus n_j), \quad \delta_j^* = |\oplus n_j - 1| \delta_j
\]

and \( \oplus \) is the inverse operation for \( a_k \oplus b_k := (a_k + b_k) \mod m_k \).

The norms (or quasi-norms) of the spaces \( L_p (G_m) \) and \( L_{p,\infty} (G_m) \ (0 < p < \infty) \) are respectively defined by

\[
\|f\|^p_p := \int_{G_m} |f|^p \, d\mu, \quad \|f\|_{L_{p,\infty}} := \sup_{\lambda > 0} \lambda \mu (f > \lambda)^{1/p}.
\]
Next, we introduce on \(G_m\) an orthonormal system which is called the Vilenkin system.

First define the complex valued function \(r_k(x) : G_m \rightarrow \mathbb{C}\), the generalized Rademacher functions as

\[
r_k(x) := \exp \left(\frac{2\pi ixk}{m_k}\right) \quad (i^2 = -1, \ x \in G_m, \ k \in \mathbb{N}).
\]

Now define the Vilenkin system \(\psi := (\psi_n : n \in \mathbb{N})\) on \(G_m\) as:

\[
\psi_n(x) := \prod_{k=0}^{\infty} r_{nk}^{n_k}(x) \quad (n \in \mathbb{N}).
\]

Specifically, we call this system the Walsh-Paley one if \(m \equiv 2\), that is \(m_k = 2\) for all \(k \in \mathbb{N}\).

The Vilenkin system is orthonormal and complete in \(L_2(G_m)\) (for details see e.g. [1, 22]).

If \(f \in L_1(G_m)\) we can establish the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels with respect to the Vilenkin system \(\psi\) in the usual manner:

\[
\hat{f}(k) := \int_{G_m} f \overline{\psi}_k d\mu, \quad (k \in \mathbb{N})
\]

\[
S_n f := \sum_{k=0}^{n-1} \hat{f}(k) \psi_k, \quad (k \in \mathbb{N})
\]

\[
D_n := \sum_{k=0}^{n-1} \psi_k, \quad (n \in \mathbb{N}_+).
\]

Recall that (for details see e.g. [1])

\[
D_{Mn}(x) = \begin{cases} M_n, & x \in I_n \\ 0, & x \notin I_n \end{cases}
\]

and

\[
D_n = \psi_n \left(\sum_{j=0}^{\infty} D_{M_j} \sum_{u=m_j-n_j}^{m_j-1} r_j^u\right).
\]

Moreover, (for details see Tephnadze [16] and [19]) if \(n \in \mathbb{N}\) and \(x \in I_s \setminus I_{s+1}, \ 0 \leq s \leq N - 1\), then the following apper and bellow etimates are true:

\[
|D_n(x)| = \left|D_{n-M_{|n|}}(x)\right| \geq M_{\langle n \rangle}, \quad |n| \neq \langle n \rangle
\]

and

\[
\int_{I_n} |D_n(x-t)| d\mu(t) \leq \frac{cM_s}{M_N}.
\]

The \(n\)-th Lebesgue constant is defined in the following way:

\[
L_n := \|D_n\|_1.
\]
It is known that (see Lukomskii [13]) that for every $n = \sum_{i=1}^{\infty} n_i M_i$, the following two-side estimate is true:

$$\frac{1}{4\lambda^v(n)} + \frac{1}{8\lambda^v(n)} \leq L_n \leq \frac{3}{2\lambda^v(n)} + 4\lambda^v(n) - 1,$$

where $\lambda := \sup_{n \in \mathbb{N}} m_n$.

The $\sigma$-algebra generated by the intervals $\{I_n(x) : x \in G_m\}$ will be denoted by $F_n (n \in \mathbb{N})$. Denote by $f = (f_n, n \in \mathbb{N})$ a martingale with respect to $F_n (n \in \mathbb{N})$. (for details see e.g. Weisz [23]).

The maximal function of a martingale $f$ is defined by

$$f^* = \sup_{n \in \mathbb{N}} |f(n)|.$$

In case $f \in L_1(G_m)$, the maximal functions are also be given by

$$f^*(x) = \sup_{n \in \mathbb{N}} \frac{1}{|I_n(x)|} \left| \int_{I_n(x)} f(u) d\mu(u) \right|.$$

For $0 < p < \infty$ the Hardy martingale spaces $H_p(G_m)$ consist of all martingales, for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

Let $X = X(G_m)$ denote either the space $L_1(G_m)$, or the space of continuous functions $C(G_m)$. The corresponding norm is denoted by $\|\cdot\|_X$. The modulus of continuity, when $X = C(G_m)$ and the integrated modulus of continuity, where $X = L_1(G_m)$ are defined by

$$\omega \left( \frac{1}{M_n}, f \right)_X = \sup_{h \in I_n} \|f(\cdot + h) - f(\cdot)\|_X.$$

The concept of modulus of continuity in $H_p(G_m) (0 < p \leq 1)$ can be defined in following way

$$\omega \left( \frac{1}{M_n}, f \right)_{H_p(G_m)} := \|f - S_{M_n}f\|_{H_p(G_m)}.$$

If $f \in L_1(G_m)$, then it is easy to show that the sequence $(S_{M_n}f : n \in \mathbb{N})$ is a martingale.

If $f = (f_n, n \in \mathbb{N})$ is a martingale then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}(i) := \lim_{k \to \infty} \int_{G_m} f_k(x) \psi_i(x) d\mu(x).$$

The Vilenkin-Fourier coefficients of $f \in L_1(G_m)$ are the same as the martingale $(S_{M_n}f : n \in \mathbb{N})$ obtained from $f$.

A bounded measurable function $a$ is p-atom, if there exist an interval $I$, such that

$$\int_I a d\mu = 0, \quad \|a\|_{\infty} \leq \mu(I)^{-1/p}, \quad \text{supp}(a) \subset I.$$
The martingale Hardy spaces $H_p(G_m)$ for $0 < p \leq 1$ have atomic characterizations (for details see e.g. Weisz [23] and [24]):

**Lemma 1.** A martingale $f = (f_n, n \in \mathbb{N})$ is in $H_p(0 < p \leq 1)$ if and only if there exist a sequence $(a_k, k \in \mathbb{N})$ of $p$-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that, for every $n \in \mathbb{N}$,

$$\sum_{k=0}^{\infty} \mu_k S_m a_k = f_n, \quad \text{a.e.,}$$

where

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$

Moreover,

$$\|f\|_{H_p} \sim \inf \left( \sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p},$$

where the infimum is taken over all decomposition of $f$ of the form (12).

By using atomic decomposition of $f \in H_p$ martingales, we can bring counterexample, which play a central role to prove sharpness of our main results and it will be used several times in the paper (for details see e.g Tephnadze [21], Section 1.7., Example 1.48):

**Lemma 2.** Let $0 < p \leq 1$, $\lambda = \sup_{n \in \mathbb{N}} m_n$, $\{\lambda_k : k \in \mathbb{N}\}$ be a sequence of real numbers, such that

$$\sum_{k=0}^{\infty} |\lambda_k|^p \leq c_p < \infty.$$

and $\{a_k : k \in \mathbb{N}\}$ be a sequence of $p$-atoms, defined by

$$a_k := \frac{M_{|\alpha_k|}^{1/p-1}}{\lambda} \left(D_{M_{|\alpha_k|}+1} - D_{M_{|\alpha_k|}}\right),$$

where $|\alpha_k| := \max \{j \in \mathbb{N} : (\alpha_k)_j \neq 0\}$ and $(\alpha_k)_j$ denotes the $j$-th binary coefficient of $\alpha_k \in \mathbb{N}$. Then $f = (f_n : n \in \mathbb{N})$, where

$$f_n := \sum_{\{k : |\alpha_k| < n\}} \lambda_k a_k$$

is a martingale, $f \in H_p$ for all $0 < p \leq 1$ and

$$\hat{f}(j) = \begin{cases} \frac{\lambda_k M_{|\alpha_k|}^{1/p-1}}{\lambda}, & j \in \{M_{|\alpha_k|}, ..., M_{|\alpha_k|+1} - 1\}, \quad k \in \mathbb{N}, \\ 0, & j \notin \bigcup_{k=1}^{\infty} \{M_{|\alpha_k|}, ..., M_{|\alpha_k|+1} - 1\}. \end{cases}$$
Let $M_{|\alpha|} \leq j < M_{|\alpha|} + 1$, $l \in \mathbb{N}$. Then

\begin{align*}
S_j f &= S_{M_{|\alpha|}} + \frac{\lambda_l M_{|\alpha|}^{1/p-1} \psi_{M_{|\alpha|}} D_j - M_{|\alpha|}}{\lambda} \\
&= \sum_{\eta=0}^{l-1} \frac{\lambda_{\eta} M_{|\alpha|}^{1/p-1}}{\lambda} \left( D_{M_{|\alpha|}+1} - D_{M_{|\alpha|}} \right) \\
&\quad + \frac{\lambda_l M_{|\alpha|}^{1/p-1} \psi_{M_{|\alpha|}} D_j - M_{|\alpha|}}{\lambda}.
\end{align*}

Moreover,

\begin{equation}
\omega \left( \frac{1}{M_n}, f \right)_{H_p(G_m)} = O \left( \sum_{\{k: |\alpha_k| \geq n\}} |\lambda_k|^p \right)^{1/p}, \text{ as } n \to \infty,
\end{equation}

There exists an intimate connection between the $H_p$ and $L_p$ norms of partial sums (for details see e.g. Tephnadze [21], Section 1.7., Example 1.45):

Lemma 3. Let $M_k \leq n < M_{k+1}$ and $S_n f$ be the $n$-th partial sum with respect to Vilenkin system, where $f \in H_p$ for some $0 < p \leq 1$. Then for every $n \in \mathbb{N}$ we have the following estimate:

\begin{align*}
\|S_n f\|_p &\leq \|S_n f\|_{H_p} \\
&\leq \left\| \sup_{0 \leq l \leq k} |S_{M_l} f| \right\|_p + \|S_n f\|_p \\
&\leq \left\| S_{\#f}^* \right\|_p + \|S_n f\|_p.
\end{align*}

3. CONVERGENCE OF SUBSEQUENCES OF PARTIAL SUMS ON THE MARTINGALE HARDY SPACES

Our first main result reads:

**Theorem 1.** a) Let $0 < p < 1$ and $f \in H_p$. Then there exists an absolute constant $c_p$, depending only on $p$, such that

$$
\|S_n f\|_{H_p} \leq \frac{c_p M_{|\alpha|}^{1/p-1}}{M_{(n)}^{1/p-1}} \|f\|_{H_p}.
$$

b) Let $0 < p < 1$ and $\{n_k : k \in \mathbb{N}\}$ be an increasing sequence of non-negative integers such that condition (12) is satisfied and $\{\Phi_n : n \in \mathbb{N}\}$ be any non-decreasing sequence, satisfying the condition

\begin{equation}
\lim_{k \to \infty} \frac{M_{|\alpha|}^{1/p-1}}{M_{(n_k)}^{1/p-1} \Phi_{n_k}} = \infty.
\end{equation}
Then there exists a martingale $f \in H_p$, such that

$$\sup_{k \in \mathbb{N}} \left\| \frac{S_{n_k} f}{\Phi_{n_k}} \right\|_{L^p, \infty} = \infty.$$ 

Proof. Suppose that

$$\left\| \frac{M_{(n)}^{1/p-1} S_n f}{M_{[n]}^{1/p-1}} \right\|_{H_p} \leq c_p \| f \|_{H_p}. \tag{18}$$

According to Lemma 3 and estimates (3) and (18) we get that

$$\left\| \frac{M_{(n)}^{1/p-1} S_n f}{M_{[n]}^{1/p-1}} \right\|_{H_p} \leq \left\| S^*_f \right\|_p + \left\| \frac{M_{(n)}^{1/p-1} S_n f}{M_{[n]}^{1/p-1}} \right\|_p \leq c_p \| f \|_{H_p}. \tag{19}$$

By using Lemma 1 and (19) the proof of part a) of Theorem 1 will be complete, if we show that

$$\int_G \left| \frac{M_{(n)}^{1/p-1} S_n a}{M_{[n]}^{1/p-1}} \right| d\mu \leq c_p < \infty, \tag{20}$$

for every $p$-atom $a$, with support $I$ and $\mu (I) = M_N^{-1}$.

We may assume that this arbitrary $p$-atom $a$ has support $I = I_N$. It is easy to see that $S_n a = 0$, when $M_N \geq n$. Therefore, we can suppose that $M_N < n$. According to $\| a \|_{\infty} \leq M_{[n]}^{1/p}$ we can write that

$$\left| \frac{M_{(n)}^{1/p-1} S_n a (x)}{M_{[n]}^{1/p-1}} \right| \leq \frac{M_{(n)}^{1/p-1} \| a \|_{\infty}}{M_{[n]}^{1/p-1}} \int_{I_N} |D_n (x - t)| d\mu (t) \leq \frac{M_{(n)}^{1/p-1} M_{[n]}^{1/p}}{M_{[n]}^{1/p-1}} \int_{I_N} |D_n (x - t)| d\mu (t). \tag{21}$$

Let $x \in I_N$. Since $x - t \in I_N$, for $t \in I_N$ and

$$v (n) + v^* (n) \leq c (|n| - \langle n \rangle) = c \rho (n)$$
if we apply (11) we get that

\[
\left| \frac{M_{(n)}^{1/p-1} S_n a (x)}{M_{|n|}^{1/p-1}} \right| \leq \frac{M_{(n)}^{1/p-1} M_N^{1/p}}{M_{|n|}^{1/p-1}} \int_{I_N} |D_n (t)| \, d\mu (t)
\]

\[
\leq \frac{M_{(n)}^{1/p-1} M_N^{1/p} (v (n) + v^* (n))}{M_{|n|}^{1/p-1}}
\]

\[
\leq \frac{c M_{(n)}^{1/p-1} M_N^{1/p} (|n| - \langle n \rangle)}{M_{|n|}^{1/p-1}}
\]

\[
\leq \frac{c M_N^{1/p} \rho (n)}{2 \rho (n) (1/p - 1)}
\]

and

\[
\int_{I_N} \left| \frac{M_{(n)}^{1/p-1} S_n a (x)}{M_{|n|}^{1/p-1}} \right|^p \, d\mu (x) \leq \frac{\rho^p (n)}{2 \rho (n) (1/p - 1)} < c_p < \infty.
\]

Let \( x \in I_s \setminus I_{s+1}, \) \( 0 \leq s \leq N - 1 < \langle n \rangle \) or \( 0 \leq s \leq \langle n \rangle \leq N - 1. \) Then \( x - t \in I_s \setminus I_{s+1}, \) for \( t \in I_N. \) By combining (17) and (18) we get that

\[
D_n (x - t) = 0
\]

and

\[
\left| \frac{M_{(n)}^{1/p-1} S_n a}{M_{|n|}^{1/p-1}} \right| = 0.
\]

Let \( x \in I_s \setminus I_{s+1}, \) \( 0 \leq \langle n \rangle < s \leq N - 1 \) or \( 0 \leq \langle n \rangle < s \leq N - 1. \) Then \( x - t \in I_s \setminus I_{s+1}, \) for \( t \in I_N. \) By applying (10) we get that

\[
\left| \frac{M_{(n)}^{1/p-1} S_n a (x)}{M_{|n|}^{1/p-1}} \right| \leq \frac{c_p M_{(n)}^{1/p-1} M_N^{1/p}}{M_{|n|}^{1/p-1} M_N}
\]

\[
= c_p M_{(n)}^{1/p-1} M_s.
\]
By combining (6), (24) and (25) we have that

\begin{equation}
\int_{I_N} \left| \frac{M_{(n)}^{1/p-1} S_n a}{M_{[n]}^{1/p-1}} \right|^p d\mu
\end{equation}

\begin{align*}
&= \sum_{s=0}^{N-1} \int_{I_s \setminus I_{s+1}} \left| \frac{M_{(n)}^{1/p-1} S_n a}{M_{[n]}^{1/p-1}} \right|^p d\mu \\
&\leq c_p \sum_{s=(n)}^{N-1} \int_{I_s \setminus I_{s+1}} \left| M_{(n)}^{1/p-1} M_s \right|^p d\mu \\
&= c_p \sum_{s=(n)}^{N-1} c_p M_{(n)}^{1-p} M_s^{1-p} \leq c_p < \infty.
\end{align*}

Hence, the proof of part a) is complete.

Under condition (17), there exists a sequence \( \{\alpha_k : k \in \mathbb{N}\} \subset \{n_k : k \in \mathbb{N}\} \), such that

\begin{equation}
\sum_{\eta=0}^{\infty} \frac{M_{(\eta)}^{(1-p)/2} \Phi_{\eta}^{p/2}}{M_{[\eta]}^{(1-p)/2}} < \infty.
\end{equation}

We note that such increasing sequence \( \{\alpha_k : k \in \mathbb{N}\} \) which satisfies condition (27) can be constructed.

Let \( f = (f_n, n \in \mathbb{N}) \) be a martingale from the Lemma where

\begin{equation}
\lambda_k = \frac{M_{(\alpha_k)}^{(1/p-1)/2} \Phi_{\alpha_k}^{1/2}}{M_{[\alpha_k]}^{(1/p-1)/2}}.
\end{equation}

In view of (28) we conclude that (13) is satisfied and by using Lemma we obtain that \( f \in H_p \).

By now using (15) with \( \lambda_k \) defined by (28) we get that

\begin{align*}
\frac{S_{\alpha_k} f}{\Phi_{\alpha_k}}
&= \frac{1}{\Phi_{\alpha_k}} \sum_{\eta=0}^{k-1} M_{[\alpha_\eta]}^{(1/p-1)/2} M_{(\alpha_\eta)}^{(1/p-1)/2} \Phi_{\alpha_\eta}^{1/2} \left( M_{[\alpha_\eta]+1} - D_{M_{[\alpha_\eta]}} \right) M_{(\alpha_k)}^{(1/p-1)/2} M_{[\alpha_k]}^{(1/p-1)/2} D_{\alpha_k} - M_{[\alpha_k]} \Phi_{\alpha_k}^{1/2}
&= I + II.
\end{align*}
According to (27) we can write that
\[
\|I\|_{L_p,\infty}^p \leq \frac{1}{\Phi_{\alpha_k}} \sum_{q=0}^{\infty} M_{(\alpha_q)}^{(1-p)/2} \Phi_{\alpha_q}^{p/2} \left( M_{(\alpha_q)}^{(1/p-1)} \left( M_{(\alpha_q)+1} - D M_{(\alpha_q)} \right) \right)_{L_p,\infty}^p \leq \frac{1}{\Phi_{\alpha_k}} \sum_{q=0}^{\infty} M_{(\alpha_q)}^{(1-p)/2} \Phi_{\alpha_q}^{p/2} \left( M_{(\alpha_q)}^{(1/p-1)} \left( M_{(\alpha_q)+1} - D M_{(\alpha_q)} \right) \right)_{L_p,\infty}^p \leq c \frac{\Phi_{\alpha_k}^{1/2}}{\Phi_{\alpha_k}^{1/2}} \leq c < \infty.
\]

Let \( x \in I_{(\alpha_k)} \setminus I_{(\alpha_k)+1} \). If we apply (9) we conclude that
\[
|II| = \frac{M_{(\alpha_k)}^{(1/p-1)/2} M_{(\alpha_k)}^{(1/p-1)/2} D_{\alpha_k} - M_{(\alpha_k)}}{\Phi_{\alpha_k}^{1/2}} \geq \frac{M_{(\alpha_k)}^{(1/p-1)/2} M_{(\alpha_k)}^{(1/p+1)/2}}{\Phi_{\alpha_k}^{1/2}}.
\]

By combining (30) and (31) for the sufficiently large \( k \), we can write that
\[
\left\| \frac{S_{\alpha_k} f}{\Phi_{\alpha_k}} \right\|_{L_p,\infty}^p \geq |II|_{L_p,\infty}^p - |I|_{L_p,\infty}^p \geq \frac{1}{2} |II|_{L_p,\infty}^p \geq \frac{c M_{(\alpha_k)}^{(1-p)/2} M_{(\alpha_k)}^{(1+p)/2}}{\Phi_{\alpha_k}^{p/2}} \mu \left\{ x \in G_m : |II| \geq \frac{c M_{(\alpha_k)}^{(1/p-1)/2} M_{(\alpha_k)}^{(1/p+1)/2}}{\Phi_{\alpha_k}^{1/2}} \right\} \geq \frac{c M_{(\alpha_k)}^{(1-p)/2} M_{(\alpha_k)}^{(1+p)/2}}{\Phi_{\alpha_k}^{p/2}} \mu \{ I_{(\alpha_k)} \setminus I_{(\alpha_k)+1} \} \geq \frac{c M_{(\alpha_k)}^{(1-p)/2}}{M_{(\alpha_k)}^{(1-p)/2} \Phi_{\alpha_k}^{p/2}} \to \infty, \text{ as } k \to \infty.
\]

Thus, also part b) is proved so the proof is complete. \( \square \)

Next, we present equivalent characterizations of boundedness of the sub-
sequences of partial sums with respect to the Vilenkin system of \( f \in H_p \) martingales in terms of measurable properties of a Dirichlet kernel:

**Corollary 1.** a) Let \( 0 < p < 1 \) and \( f \in H_p \). Then there exists an absolute constant \( c_p \), depending only on \( p \), such that
\[
\|S_n f\|_{H_p} \leq c_p (n \mu \{ \text{supp} (D_n) \})^{1/p-1} \|f\|_{H_p}.
\]
b) Let $0 < p < 1$ and $\{n_k : k \in \mathbb{N}\}$ be an increasing sequence of nonnegative integers such that

\[(32) \sup_{k \in \mathbb{N}} n_k \mu \{\text{supp} \ D_{n_k}\} = \infty\]

and $\{\Phi_n : n \in \mathbb{N}\}$ be any nondecreasing sequence, satisfying the condition

\[
\lim_{k \to \infty} \left( \frac{n_k \mu \{\text{supp} \ D_{n_k}\}}{\Phi_{n_k}} \right)^{1/p - 1} = \infty.
\]

Then there exists a martingale $f \in H_p$, such that

\[
\sup_{k \in \mathbb{N}} \left\| \frac{S_{n_k} f}{\Phi_{n_k}} \right\|_{L_p, \infty} = \infty.
\]

Remark 1. Corollary [1] shows that when $0 < p < 1$ the main reason of divergence of partial sums of Vilenkin-Fourier series is unboundedness of Fourier coefficients, but in the case when measure of supp of $n_k$-th Dirichlet kernels tends to zero, then the divergence rate drops and in the case when it is maximally small

\[
\mu(\text{supp}D_{n_k}) = O\left( \frac{1}{M_{|n_k|}} \right), \text{ as } k \to \infty, \quad (M_{|n_k|} < n_k \leq M_{|n_k| + 1})
\]

we have convergence.

Proof. By combining (7) and (8) we can write that

\[I_{(n)} \setminus I_{(n) + 1} \subset \text{supp}D_n \subset I_{(n)}\]

and

\[
\frac{1}{2M_{(n)}} \leq \mu(\text{supp}D_n) \leq \frac{1}{M_{(n)}}
\]

Since $M_{|n|} \leq n < M_{|n| + 1}$ we immediately get that

\[
\frac{M_{|n|}}{2M_{(n)}} \leq n \mu(\text{supp}D_n) \leq \lambda M_{|n|} \frac{M_{|n|}}{M_{(n)}},
\]

where $\lambda = \sup_{n \in \mathbb{N}} m_n$.

It follows that

\[
\frac{M_{|n|}^{1/p - 1}}{2M_{(n)}^{1/p - 1}} \leq (n \mu(\text{supp}D_n))^{1/p - 1} \leq \frac{\lambda^{1/p - 1}M_{|n|}^{1/p - 1}}{M_{(n)}^{1/p - 1}}.
\]

The proof follows by using these estimates in Theorem [1].

A number of special cases of our result are of particular interest and give both well-known and new information. We just give the following examples of such Corollaries (see Corollaries [2][3]):
Corollary 2. Let $0 < p < 1$, $f \in H_p$ and $\{n_k : k \in \mathbb{N}\}$ be an increasing sequence of nonnegative integers. Then

$$\|S_{n_k}f\|_{H_p} \leq c_p \|f\|_{H_p}$$

if and only if condition (1) is satisfied.

Proof. It is easy to show that

$$2^{\rho(n_k)} \leq \frac{M_{|n_k|}}{M_{(n_k)}} \leq \lambda^{\rho(n_k)},$$

where $\lambda = \sup_{n \in \mathbb{N}} m_n$. It follows that

$$\sup_{k \in \mathbb{N}} \frac{M_{1/p-1}}{M^{1/p-1}_{(n_k)}} < \infty$$

if and only if (1) holds, so the proof follows by using Theorem 1. □

Corollary 3. Let $n \in \mathbb{N}$ and $0 < p < 1$. Then there exists a martingale $f \in H_p$, such that

$$(33) \quad \sup_{n \in \mathbb{N}} \|S_{M_n+1}f\|_{L_p, \infty} = \infty.$$

Proof. To prove Corollary 3 we only have to calculate that

$$(34) \quad |M_n + 1| = n, \quad \langle M_n + 1 \rangle = 0$$

and

$$(35) \quad \rho(M_n + 1) = n.$$

By using Corollary 2 we obtain that there exists a martingale $f \in H_p$ $(0 < p < 1)$, such that (33) holds.

The proof is complete. □

Corollary 4. Let $n \in \mathbb{N}$, $0 < p \leq 1$ and $f \in H_p$. Then

$$(36) \quad \|S_{M_n+M_{n-1}}f\|_{H_p} \leq c_p \|f\|_{H_p}.$$

Proof. Analogously to (34) and (35) we can write that

$$|M_n + M_{n-1}| = n, \quad \langle M_n + M_{n-1} \rangle = n - 1$$

and

$$\rho(M_n + M_{n-1}) = 1.$$

By using Corollary 2 we immediately get inequality (37), for all $0 < p \leq 1$.

The proof is complete. □

Corollary 5. Let $n \in \mathbb{N}$, $0 < p \leq 1$ and $f \in H_p$. Then

$$(37) \quad \|S_{M_n}f\|_{H_p} \leq c_p \|f\|_{H_p}.$$
Proof. Analogously to (34) and (35) we can write that 

\[ |M_n| = n, \quad \langle M_n \rangle = n \quad \text{and} \quad \rho(M_n) = 0. \]

By using Corollary 2 we immediately get inequality (37), for all \(0 < p \leq 1\).

The proof is complete. \(\square\)

4. NECESSARY AND SUFFICIENT CONDITION FOR THE CONVERGENCE OF PARTIAL SUMS IN TERMS OF MODULUS OF CONTINUITY

The main result of this section reads:

**Theorem 2.** Let \(0 < p < 1\), \(f \in H_p\) and \(M_k < n \leq M_{k+1}\). Then there exists an absolute constant \(c_p\), depending only on \(p\), such that

\[ \|S_n f - f\|_{H_p} \leq \frac{c_p M_1^{1/p-1}}{M_{(n)}^{1/p-1}} \omega \left( \frac{1}{M_k}, f \right)_{H_p(G_m)}, \quad (0 < p < 1). \]

Moreover, if \(\{n_k : k \in \mathbb{N}\}\) be increasing sequence of nonnegative integers such that

\[ \omega \left( \frac{1}{M_{|n_k|}}, f \right)_{H_p(G_m)} = o \left( \frac{M_1^{1/p-1}}{M_{(n_k)}^{1/p-1}} \right), \quad \text{as} \quad k \to \infty, \]

then

\[ \|S_{n_k} f - f\|_{H_p} \to 0, \quad \text{as} \quad k \to \infty. \]

b) Let \(\{n_k : k \in \mathbb{N}\}\) be an increasing sequence of nonnegative integers such that condition (2) is satisfied. Then there exists a martingale \(f \in H_p\) and a subsequence \(\{\alpha_k : k \in \mathbb{N}\} \subset \{n_k : k \in \mathbb{N}\}\), for which

\[ \omega \left( \frac{1}{M_{|\alpha_k|}}, f \right)_{H_p(G_m)} = O \left( \frac{M_1^{1/p-1}}{M_{(\alpha_k)}^{1/p-1}} \right), \quad \text{as} \quad k \to \infty \]

and

\[ \limsup_{k \to \infty} \|S_{\alpha_k} f - f\|_{L_p, \infty} > c > 0, \quad \text{as} \quad k \to \infty. \]

**Proof.** Let \(0 < p < 1\). Then, by using Theorem 1 we immediately get that

\[
\begin{align*}
\|S_n f - f\|_{H_p}^p & \leq \|S_n f - S_{M_k} f\|_{H_p}^p + \|S_{M_k} f - f\|_{H_p}^p \\
& = \|S_n (S_{M_k} f - f)\|_{H_p}^p + \|S_{M_k} f - f\|_{H_p}^p \\
& \leq \left( \frac{c_p M_1^{1/p-1}}{M_{(n)}^{1/p-1}} + 1 \right) \omega_{H_p}^p \left( \frac{1}{M_k}, f \right).
\end{align*}
\]
and

\[ \|S_nf - f\|_{H^p} \leq \frac{c_p M_1^{1/p-1}}{M_1^{(n)}} \omega \left( \frac{1}{M_k}, f \right)_{H^p(G_m)}. \]

According to condition (38) if we apply estimate (42) we immediately get that (39) holds.

For the proof of part b) we first note that under the conditions of part b) of Theorem 2, there exists \( \{\alpha_k : k \in \mathbb{N}\} \subset \{n_k : k \in \mathbb{N}\} \), such that

\[ M_{|\alpha_k|} \uparrow \infty, \quad \text{as} \quad k \rightarrow \infty \]

\[ \frac{M_{2(1/p-1)}}{M_{(\alpha_k)}} \leq \frac{M_{(\alpha_k+1)}}{M_{|\alpha_k|}}. \]

Let \( f = (f_n, n \in \mathbb{N}) \) be a martingale from the Lemma 2, where

\[ \lambda_k = \frac{\lambda M_{1/p-1}}{M_{|\alpha_k|}}. \]

If we apply (43) and (44) in the case when \( \lambda_k \) are defined by (45) we conclude that (13) is satisfied and by using Lemma 2 we obtain that \( f \in H^p \).

By using (45) with \( \lambda_k \) defined by (45) we get that

\[ \omega \left( \frac{1}{M_{|\alpha_k|}}, f \right)_{H^p(G_m)} \leq \sum_{i=k}^{\infty} \frac{M_{1/p-1}}{M_{|\alpha_k|}} = O \left( \frac{M_{1/p-1}}{M_{|\alpha_k|}} \right), \quad \text{as} \quad k \rightarrow \infty. \]

By applying (45) with \( \lambda_k \) defined by (45) we get that

\[ S_{\alpha_k}f = S_{M_{|\alpha_k|}} + M_{(\alpha_k)}^{1/p-1} \psi M_{|\alpha_k|} D_{J_{M_{|\alpha_k|}}}. \]

In view of (9) we can conclude that

\[ \left| D_{\alpha_k-M_{|\alpha_k|}} \right| \geq M_{(\alpha_k)}, \quad \text{for} \quad I_{(\alpha_k)} \setminus I_{(\alpha_k)+1} \]

and

\[ M_{(\alpha_k)} \mu \left\{ x \in G_m : \left| D_{\alpha_k-M_{|\alpha_k|}} \right| \geq M_{(\alpha_k)} \right\} \]

\[ M_{(\alpha_k)} \mu \left\{ I_{(\alpha_k)} \setminus I_{(\alpha_k)+1} \right\} \geq M_{(\alpha_k)}^{1-p}. \]

If we combine Corollary 5 and (47), for the sufficiently large \( k \), we can write that

\[ \|S_{\alpha_k}f - f\|_{L^p, \infty} \geq M_{(\alpha_k)}^{1/p-1} \|D_{\alpha_k}\|_{L^p, \infty} - \|S_{M_{|\alpha_k|}} f - f\|_{L^p, \infty} \geq \frac{M_{(\alpha_k)}^{1/p-1} \|D_{\alpha_k}\|_{L^p, \infty}}{2} \geq c, \quad \text{as} \quad k \rightarrow \infty. \]

The proof is complete.
Next, we present simple consequence of Theorem 2 which was proved in Tephnadze [18]:

**Corollary 6.** a) Let $0 < p < 1$, $f \in H_p$ and

$$
\omega \left( \frac{1}{M_n}, f \right)_{H_p(G_m)} = o \left( \frac{1}{M_n^{1/p-1}} \right), \quad \text{as } n \to \infty.
$$

Then

$$
\|S_k f - f\|_{H_p} \to 0, \quad \text{as } k \to \infty.
$$

b) For every $0 < p < 1$ there exists martingale $f \in H_p$, for which

$$
\omega \left( \frac{1}{M_n}, f \right)_{H_p(G_m)} = O \left( \frac{1}{M_n^{1/p-1}} \right), \quad \text{as } n \to \infty
$$

and

$$
\|S_k f - f\|_{L_p, \infty} \to 0, \quad \text{as } k \to \infty.
$$

Finally, we present equivalent conditions for the modulus of continuity in terms of measurable properties of a Dirichlet kernel, which provide boundedness of the subsequences of partial sums with respect to the Vilenkin system of $f \in H_p$ martingales:

**Corollary 7.** Let $0 < p < 1$, $f \in H_p$ and $M_k < n \leq M_{k+1}$. Then there exists an absolute constant $c_p$, depending only on $p$, such that

$$
\|S_n f - f\|_{H_p} \leq c_p \left( n \mu(\text{supp} D_n) \right)^{1/p-1} \omega_{H_p} \left( \frac{1}{M_k}, f \right), \quad (0 < p < 1)
$$

Moreover, if $\{n_k : k \in \mathbb{N}\}$ be a sequence of nonnegative integers such that

$$
\omega \left( \frac{1}{M_{|n_k|}}, f \right)_{H_p(G_m)} = o \left( \frac{1}{(n_k \mu(\text{supp} D_{n_k}))^{1/p-1}} \right), \quad \text{as } k \to \infty,
$$

then (39) holds.

b) Let $\{n_k : k \in \mathbb{N}\}$ be an increasing sequence of nonnegative integers such that condition (2) is satisfied. Then there exists a martingale $f \in H_p$ and a subsequence $\{\alpha_k : k \in \mathbb{N}\} \subset \{n_k : k \in \mathbb{N}\}$, for which

$$
\omega \left( \frac{1}{M_{|\alpha_k|}}, f \right)_{H_p(G_m)} = O \left( \frac{1}{(\alpha_k \mu(\text{supp} D_{\alpha_k}))^{1/p-1}} \right), \quad \text{as } k \to \infty
$$

and

$$
\limsup_{k \to \infty} \|S_{\alpha_k} f - f\|_{L_p, \infty} > c > 0, \quad \text{as } k \to \infty.
$$
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