On the Geometry of Spaces of Oriented Geodesics

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Abstract: Let $M$ be either a simply connected pseudo-Riemannian space of constant curvature or a rank one Riemannian symmetric space other than $OH^2$, and consider the space $L(M)$ of oriented geodesics of $M$. The space $L(M)$ is a smooth homogeneous manifold and in this paper we describe all invariant symplectic structures, (para)complex structures, pseudo-Riemannian metrics and (para)Kähler structure on $L(M)$.

Keywords: space of geodesics, rank one symmetric spaces, homogeneous Riemannian manifolds, pseudo-Riemannian metrics, symplectic structures, complex structures, Kähler structures, flag manifolds, Cayley projective plane.

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1 Introduction

1.1 Background

The geometry of the set of straight lines of projective space $P^3$ and Euclidean space $E^3$ is a classical subject of investigations of such 19th century geometers as Grassmann, Plücker, F. Klein and Study. They studied natural correspondences between submanifolds (i.e. points, curves and surfaces) in $E^3$ and submanifolds of $L(E^3)$, the space of oriented lines of $E^3$. For example, a point $p \in E^3$ defines a surface in $L(E^3)$ which consists of the oriented lines through $p$, a curve $C \subset E^3$ defines three curves in $L(E^3)$ associated with the Frenet frame of $C$, and a surface $S \subset E^3$ defines a surface in $L(E^3)$ by its oriented normal lines. Conversely, distinguished (from the point of view of intrinsic geometry of $L(E^3)$) submanifolds determine special families of oriented lines in $E^3$.

In [14], Study identified the space $L(E^3)$ of oriented lines in $E^3$ with the 2-sphere (the Study sphere) over the dual numbers and defined and studied a notion of distance.
between oriented lines. For a nice modern exposition and generalization of these results with applications to computational geometry, computer graphics and visualization, see [9].

A natural complex structure in the space $L(E^3)$ of oriented lines has been considered by Hitchin, who used it for a description of monopoles [7]. In [5], two of the authors defined a neutral Kähler structure in the space $L(E^3)$ and gave its geometric description. Recently, it has been used in the solution of a long-standing conjecture of Carathéodory [6].

In general, however, the space $L(M)$ of oriented geodesics of a (complete) Riemannian manifold is not a smooth manifold and has very bad topology. But $L(M)$ is a smooth manifold if $(M, g)$ is either an Hadamard manifold (i.e. complete simply connected Riemannian manifold of non-positive curvature) or a manifold with closed geodesics of the same length (aufwiederseen manifolds). Symplectic and Riemannian structures in the space of geodesics $L(M)$ of aufwiederseen manifolds are discussed by Besse [1], see also Reznikov [10] [11]. The symplectic form on the space of geodesics of a Hadamard manifold is described in [2].

Geometric structures in the space of oriented geodesics of hyperbolic 3-space are studied in [3], while Salvai has addressed the existence and uniqueness of pseudo-Riemannian metrics in the spaces $L(E^n)$ and $L(H^n)$ [12] [13]. Note that both spaces are homogeneous manifolds of the corresponding isometry group. Salvai proved that $L(E^n)$ admits a pseudo-Riemannian metric invariant under a transitive subgroup $G$ of the isometry group $I(E^n) = E(n) = SO(n) \cdot E^n$ only for $n = 3, 7$ and gave an explicit description of the corresponding metric.

### 1.2 Main results

The aim of the paper is to describe the natural geometric structures in the space $L(M)$ of oriented geodesics of $M$, where $M$ is either a simply connected pseudo-Riemannian space of constant curvature, or a rank one Riemannian symmetric space other than $OH^2$. In these cases the space of geodesics $L(M)$ is a smooth homogeneous manifold and we use Lie groups and Lie algebras to describe all invariant symplectic structures, (para)complex structures, pseudo-Riemannian metrics and (para)Kähler structure on $L(M)$.

More specifically, let $S^{p,q} = \{x \in E^{p+1,q}, x^2 = 1\}$, where $E^{p+1,q}$ is $\mathbb{R}^{p+q+1}$ endowed with the flat metric of signature $(p + 1, q)$. Here, for $p = 0$ we assume in addition that $x^0 > 0$ so that $S^{p,q}$ is connected. The induced metric on $S^{p,q}$ has signature $(p, q)$ and is
of constant curvature 1.

Let \( L^+(S^{p,q}) \) (respectively \( L^-(S^{p,q}) \)) be the set of spacelike (respectively, timelike) geodesics in \( S^{p,q} \) and similarly for \( L^\pm(E^{p+1,q}) \). We prove:

**Main Theorem 1** For the flat pseudo-Euclidean spaces \( E^{p+1,q} \)

i) The space \( L^-(E^{p+1,q}) = E(p+1,q)/SO(p,q) \cdot \mathbb{R}^+ \) is a symplectic symmetric space with an invariant Grassmann structure defined by a decomposition \( \mathfrak{m} = W \otimes \mathbb{R}^2 \).

Moreover, if \( n = p+1+q = 3 \), it has an invariant Kähler structure \((g,J)\) of neutral signature \((2,2)\). In addition, \( L^+(E^{p+1,q}) = L^-(E^{q,p+1}) \).

while for the non-flat constant curvature manifolds \( S^{p,q} \)

ii) Suppose that \( p + q > 3 \). Then there exists a unique (up to scaling) invariant symplectic structure \( \omega \) and a unique (up to a sign) invariant complex structure \( I^+ = J \) on \( L^+(S^{p,q}) \) and a unique (up to sign) invariant para-complex structure \( K = I^- \) on \( L^-(S^{p,q}) \). There exists unique (up to scaling) invariant pseudo-Riemannian metric \( g^\varepsilon = \omega \circ I^\varepsilon \) on \( L^\varepsilon(S^{p,q}) \) which is Kähler of signature \((2(p-1),2q)\) for \( \varepsilon = + \) and para-Kähler (of neutral signature) for \( \varepsilon = - \).

iii) Suppose that \( p + q = 3 \). Then there are 2-linearly independent invariant (parallel and closed) 2-forms \( \omega, \omega' \) on \( L^\varepsilon(S^{p,q}) \) with values \( \omega_{\mathfrak{m}} = \omega_H \otimes g_V, \omega'_{\mathfrak{m}} = g_H \otimes \omega_V \).

Any invariant metric has the form \( h = \lambda g + \mu g' \) where \( g' \) is a neutral metric with value \( g'_{\mathfrak{m}} = \omega_H \otimes \omega_V \). Any metric \( h \) is Kähler for \( \varepsilon = 1 \) (respectively, para-Kähler for \( \varepsilon = - \)) with respect to the complex (respectively, para-complex) structure \( I^\varepsilon = I^\varepsilon_H \otimes 1 \) with the Kähler form \( h \circ I^\pm = \lambda \omega + \mu \omega' \). Moreover, the endomorphism \( I' = 1 \otimes I_V \) of \( \mathfrak{m} \) defines an invariant parallel h-skew-symmetric complex structure of \( L^\varepsilon(S^{p,q}) \) if \( \varepsilon = 1, (p-1,q) = (2,0) \) or \( (0,2) \) or \( \varepsilon = - \) and \( (p,q-1) = (2,0) \) or \( (0,2) \) and skew-symmetric parallel para-complex structure if \( \varepsilon = +, (p-1,q) = (1,1) \) or \( \varepsilon = -, (p,q-1) = (1,1) \). The Kähler or para-Kähler structure \((h,I')\) has the Kähler form \( h \circ I' = \lambda \omega + \mu \omega' \).

Consider now the rank one Riemannian symmetric spaces of non-constant curvature.

That is, \( M = G/K \) is one of the projective spaces

\[
\mathbb{C}P^n = SU_{n+1}/U_n, \quad \mathbb{H}P^n = Sp_{n+1}/Sp_1 \cdot Sp_n, \quad \mathbb{O}P^2 = F_4/Spin_9
\]

or one of the dual hyperbolic spaces

\[
\mathbb{C}H^n = SU_{1,n}/U_n, \quad \mathbb{H}H^n = Sp_{1,n}/Sp_1 \cdot Sp_n, \quad \mathbb{O}H^2 = F_4^{\text{non-comp}}/Spin_9.
\]
For the projective spaces we prove:

**Main Theorem 2**

i) The space $L(\mathbb{C}P^n) = SU_{n+1}/T^2 \cdot SU_{n-1}$ has a one–parameter family $\omega^t = \omega_1 + t\omega_0$ of invariant symplectic forms (up to scaling) and four invariant almost complex structures, up to sign, two of them being integrable. All (almost) complex structures $J$ are compatible with $\omega^t$ i.e. they define an (almost) Kähler or pseudo-Kähler metric $g = \omega^t \circ J$.

ii) The spaces $L(\mathbb{H}P^n) = Sp_{n+1}/T^1 \cdot Sp_1 \cdot Sp_{n-1}$ and $L(\mathbb{O}P^2) = F_4/T^1 \cdot Spin_7$ has a unique (up to scaling) invariant symplectic forms $\omega$ and unique (up to sign) invariant complex structure $J$. The pair $(\omega, J)$ defines a unique (up to scaling) invariant Kähler metric $g = \omega \circ J$.

iii) None of the above spaces have any invariant almost para-complex structures.

iv) The canonical symplectic structure on $L(\mathbb{C}P^n)$ is identified with $\omega_1$.

while for the hyperbolic spaces (other than $\mathbb{O}H^2$) we show

**Main Theorem 3**

i) The space $L(\mathbb{C}H^n) = SU_{1,n}/T^2 \cdot SU_{n-1}$ has a one–parameter family of invariant symplectic structures $\omega^t = \omega_1 + t\omega_0$, and two (up to sign) invariant almost para-complex structures $K^\pm$, one of them being integrable; and both are consistent with $\omega^t$ i.e. $(K^\pm, \omega^t)$ defines a para-Kähler metric $g = \omega \circ K^\pm$.

ii) The space $L(\mathbb{H}H^n) = Sp_{1,n}/T^1 \cdot Sp_1 \cdot Sp_{n-1}$ admits a unique (up to scaling) invariant symplectic form $\omega$ and two (up to sign) invariant almost para-complex structures, one of which is integrable. Both are compatible with $\omega$.

iii) None of the above spaces have any invariant almost complex structures.

iv) The canonical symplectic structure on the space geodesics $L(\mathbb{C}H^n)$ is $\omega_1$.

In Table 1 we summarize the results of Main Theorems 1, 2 and 3.
Table 1: Invariant Geometric Structures

| Structure | Symplectic | Complex | Para-Complex | Kähler | ParaKähler |
|-----------|------------|---------|--------------|--------|------------|
| $L^\pm(E^{p+1,q})$ | Symmetric | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| | | $\mathbb{R}$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| $L^+(S^{p,q})$ | $\mathbb{R}$ | $1$ | $0$ | $0$ | $1$ | $0$ | $\emptyset$ |
| | | $1$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| | | | | | | | $\emptyset$ |
| $L^-(S^{p,q})$ | $\mathbb{R}$ | $1$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| | | | | | | | $\emptyset$ |
| $L(\mathbb{CP}^n)$ | $\mathbb{R}$ | $2$ | $2$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| $L(\mathbb{HP}^n)$ | $1$ | $1$ | $\emptyset$ | $\emptyset$ | $1$ | $\emptyset$ | $\emptyset$ |
| $L(\mathbb{OP}^2)$ | $1$ | $1$ | $\emptyset$ | $\emptyset$ | $1$ | $\emptyset$ | $\emptyset$ |
| $L(\mathbb{CH}^n)$ | $\mathbb{R}$ | $1$ | $1$ | $\emptyset$ | $\emptyset$ | $1$ | $1$ |
| $L(\mathbb{HH}^n)$ | $1$ | $0$ | $\emptyset$ | $\emptyset$ | $1$ | $1$ | $\emptyset$ | $\emptyset$ | $1$ | $1$ |

1.3 Outline of paper

This paper is organised as follows. In the following section we consider the set of oriented geodesics $L = L(M)$ of a general complete Riemannian manifold $M$. Though $L$ may not be a manifold, we can still define differential geometric objects on $L$ in terms of $G_1$-invariant objects on $SM$, where $SM$ is the unit sphere bundle and $G_1 = \{ \text{exp} \Gamma \}$ is the geodesic flow. For example, the algebra of smooth functions $\mathcal{F}(L)$ on $L$ is defined as the algebra of $G_1$-invariant functions on $SM$ (that is first integrals of the geodesic flow). We sketch this approach in section 2 and define the canonical symplectic form on $L$. In the case when $L$ is a manifold this coincides with the standard symplectic form defined in [1].

Section 3 considers the oriented geodesics of pseudo-Riemannian manifolds of constant curvature, taking the flat and non-flat cases separately. Main Theorem 1 follows from Theorems 1 and 2 of sections 3.1 and 3.2, respectively.

In section 4 we turn to rank one Riemannian symmetric spaces with non-constant curvature, dealing separately with the complex projective and hyperbolic spaces, the
quaternionic projective and hyperbolic spaces and the Cayley plane. In particular, Main
Theorem 2 follows from Theorem 3 of section 4.1, Theorem 5 of section 4.2 and Theorem 7
of section 4.3, while Main Theorem 3 follows from Theorem 4 of section 4.1 and Theorem
6 of section 4.2.

2 The Space of Oriented Geodesics on a Riemannian Manifold

2.1 Tangent and unit sphere bundle

Let $M$ be an $n$-dimensional manifold and $\pi : TM \to M$ be the tangent bundle of $M$. Local coordinates $(x^i)$ on $M$ give rise to local coordinates $(x^i, y^i)$ on $TM$ via $v_x = y^i \partial/\partial x^i$. Then $T_{(x,y)} TM \ni (x, y, \dot{x}, \dot{y})$. The vertical subspace $V_{(x,y)} TM$ is given by $(x, y, 0, \dot{y})$ and we denote by $N$ the canonical endomorphism

$$N : T_{(x,y)} TM \to V_{(x,y)} TM \to 0 : (x, y, \dot{x}, \dot{y}) \to (x, y, 0, \dot{x}) \to 0,$$

with $N^2 = 0$. Note that

$$\text{Ker } N = \text{Im } N = T^v(TM) = \{(x, y, 0, \dot{y})\},$$

is the vertical subbundle of $T(TM)$. The restriction of the projection $\pi_* : T(TM) \to TM$
onto the vertical subspace is an isomorphism

$$\pi_* : T^v_{(x,y)}(TM) \to T_{x} M, \ (x, y, \dot{x}, \dot{y}) \mapsto (x, \dot{x}).$$

Assume now that $(M, g)$ is Riemannian and $SM$ is the unit sphere bundle:

$$SM = \{(x, y) \in TM, g_{ij}(x)y^iy^j = 1\}.$$ 

The metric $g$ induces an isomorphism of the tangent bundle onto cotangent bundle

$$T^* M = \{\alpha = (x, p), \alpha = p_i dx^i\}$$
given by

$$(x^i, y^i) \to (x^i, p_i = g_{ij} y^j).$$

The pull-back of the canonical one-form $\alpha = p_i dx^i$ and symplectic form $\omega = d\alpha$ of $T^* M$
are given by

$$\alpha^g = g^* \alpha = g_{ij} y^i dx^j, \ \omega^g = g \circ \omega = g \circ d\alpha = d(g_{ij} y^j) \wedge dx^i.$$ 

The tangent bundle of the unit sphere bundle has the canonical decomposition $T_{(x,y)} SM =
V_{(x,y)} SM + H^g_{(x,y)} SM$ into vertical space and horizontal subspaces.
Lemma 1 The sphere bundle of the Riemannian manifold \((M, g)\) admits the canonical contact structure

\[
\theta = \alpha^g|SM, \quad \alpha^g = g^*\alpha = g_{ij} y^i dx^j,
\]

where the associated Reeb vector field is the geodesic vector field given by

\[
\Gamma = y^i \partial/\partial x^i - \Gamma^i_{jk} y^j y^k \partial/\partial y^i.
\]

The horizontal lift of \(\partial/\partial x^i = \partial_i\) into \(SM\) is given by

\[
\nabla_i = \partial_i - \Gamma^k_{ij}(x)y^j \partial/\partial y^k, (x, y) \in SM.
\]

In particular, \(\Gamma\) preserves \(\theta\) and \(d\theta\).

Proof. The last claims are verified as follows. A vector field along \(x(t)\) given by \(X(t) = y^i(x(t)) \partial_i\) is parallel if

\[
\dot{X}^i + \Gamma^i_{jk}(x(t)) \dot{x}^j \dot{X}^j = 0.
\]

Then \((x(t), X(t))\) is the horizontal lift of the curve \(x(t)\). The horizontal space is spanned by the tangent vectors of such lifts, namely

\[
H_{(x,y)} = \text{span}\{(\dot{x}^i, -\Gamma^i_{jk} \dot{x}^j y^k)\}.
\]

The vector field \(\Gamma = y^i \partial/\partial x^i - \Gamma^i_{jk} y^j y^k \partial/\partial y^i\) on \(TM\) is tangent to \(SM\) since

\[
\nabla_k (g_{ij} y^j y^j) = g_{ij,k} - g_{im}\Gamma^m_{kl} - g_{jm}\Gamma^m_{ki} = g_{ij,k} - \Gamma_{i,j}^k - \Gamma_{j,i}^k = 0.
\]

Moreover, \(\Gamma\) is a geodesic vector field since its integral curves satisfy the geodesic equation. Therefore we have the decomposition into vertical and horizontal parts

\[
T_{(x,y)}SM = V_{(x,y)}SM + H_{(x,y)}SM = \{v^i \partial/\partial y^i : g_{ij} v^j v^j = 0\} + \{u^k \nabla_k\}.
\]

We compute

\[
d\theta = g_{ij,k} y^j dx^i \wedge dx^k + g_{ij} dy^j \wedge dx^i.
\]

One easily checks that \(\theta(\Gamma) = 1\) and \(d\theta(\Gamma) = 0\). Clearly the form \(d(g^*\alpha) = g^*d\alpha = g^*\omega\) is non-degenerate on \(TM\). Therefore its restriction to \(SM\) has one-dimensional kernel spanned by \(\Gamma\) and so \(\Gamma\) is the Reeb vector field of the contact form \(\theta\) and it preserves \(\theta\) and \(d\theta\). □

We associate with the vector field \(X = X^i \partial_i\) the function \(f_X\) on \(SM\) given by

\[
f_X(x, y) = g_{ij} X^i y^j.
\]
Lemma 2 The covariant derivative $\nabla_i X$ corresponds to the Lie derivative of $f_X$ in direction of the vector field $\nabla_i$:

$$f_{\nabla_i X} = \nabla_i f_X.$$ 

Proof. Applying the vector field $\nabla_i$ to the function $f_X$ on $SM$ we get

$$\nabla_i f_X = (g_{kl} X^k)_i y^l - g_{kl} X^k \Gamma^l_{ij} y^j,$$

$$= g_{kl} X^k y^l + (g_{kj,i} - \Gamma_{k,ij}) X^k y^j,$$

$$= g_{kl} X^k y^l - \frac{1}{2} (g_{kj,i} + g_{ki,j} - g_{ij,k}) X^k y^j,$$

$$= g_{ka} (X^a + \Gamma^a_{li} X^l) y^k,$$

$$= f_{\nabla_i X}.$$

\qed

Denote the tangent bundle without the zero section by $T'M$. Then $T'M = SM \times \mathbb{R}^+$ with coordinate $r$ on the second factor. We denote $E = r \partial/r = y^i \partial/\partial y^i$ the Euler vector field. Note that the symplectic form $\omega^g$ is homogeneous of degree one: $E \cdot \omega^g = \omega^g$ and $E \cdot (\omega^g)^{-1} = -\omega^g$. Denote the homogeneous functions of degree $k$ by $\mathcal{F}_k(T'M)$ and the extension of the function $f \in \mathcal{F}(SM)$ to $\mathcal{F}_k$ by $f(k)$. Furthermore, denote the Poisson structure by $\{f, g\} = \omega^{-1}_g(df, dg)$. Since $\omega^{-1}_g$ has degree $-1$, we have

$$\{\mathcal{F}_k, \mathcal{F}_l\} \subset \mathcal{F}_{k+l-1}.$$

We identify $\mathcal{F}(SM)$ with $\mathcal{F}_1(T'M)$, $f \mapsto \tilde{f} = f_1 = f \otimes \mathbb{R}$ and define the Legendrian bracket in the space $\mathcal{F}(SM)$ by

$$\{f, h\} := \{\tilde{f}, \tilde{h}\}_{SM}.$$

\qed

2.2 Smooth structure in the space of geodesics $L(M)$

We first consider the topology of $L(M)$.

Let $(M, g)$ be a smooth complete Riemannian $n$-dimensional manifold. By geodesic we mean an oriented maximally extended geodesic on $M$ and such a geodesic $\gamma$ has a natural parameterization by arc-length $\gamma = \gamma(s)$ defined up to a shift $s \to s + C$. We denote by $\Gamma$ the canonical geodesic vector field on the unit sphere bundle $SM$.

The maximal integral curves of $\Gamma$ through $(x, v) \in SM$ have the form $(\gamma(s), \gamma'(s)) = (\exp_x(sv), d/ds \exp_x(sv))$, where $\gamma(s) = \exp_x(sv)$ is the maximal geodesic defined by
$(x, v) \in SM$. The set $L(M)$ is identified with the set of orbits of the flow generated by $\Gamma$, i.e. the maximal integral curves of $\Gamma$. Denote by $\pi : SM \to L(M) = SM/\Gamma$ the natural projection, equip $L(M)$ with the weakest topology such that $\pi$ is continuous, and call the resulting topological space $L(M)$ the space of geodesics of $M$. In general, $L(M)$ is not Hausdorff, but if $M$ is complete and compact, then $L(M)$ is compact.

**Lemma 3** The projection $\pi : SM \to L(M)$ is an open map if $(M, g)$ is a complete Riemannian manifold.

**Proof.** We have to prove that if $U \subset SM$ is an open set then $\pi(U)$ is open, i.e. $V := \pi^{-1}(\pi(U)) \subset SM$ is open. It is clear since $V = \cup_{t \in \mathbb{R}} \varphi_t U$ where $\varphi_t = \exp(t\Gamma)$ is the 1-parameter group of transformations which is generated by $\Gamma$.  

To define a smooth structure in $L(M)$ we consider:

**Definition 1** i) A function $f$ on an open subset $U \subset L(M)$ is called smooth if its pull back $\pi^* f$ is a smooth function on $\pi^{-1}U$. We identify the algebra $\mathcal{F}(U)$ of smooth functions on $U$ with the algebra $\mathcal{F}(\pi^{-1}U)$ of $\Gamma$-invariant functions on $\pi^{-1}U$. In particular, $\mathcal{F}(L(M)) = \mathcal{F}(SM)^\Gamma$. We denote by $\mathcal{F}_\gamma(L(M))$ the germ of smooth functions at $\gamma$.

ii) A tangent vector $v$ of $L(M)$ at $\gamma$ is a derivation $v : \mathcal{F}_\gamma(L(M)) \to \mathbb{R}$, i.e. $v : f \mapsto v \cdot f$ such that $v \cdot (f h) = h(\gamma)v \cdot f + (\gamma)h \cdot v \cdot (h)$. They give rise to a vector space denoted by $T_\gamma L(M)$.

iii) A vector field on $U \subset L$ is a derivation of the algebra $\mathcal{F}(U)$. We identify the Lie algebra $\mathfrak{X}(U)$ of vector fields with the Lie algebra $\mathfrak{X}(\pi^{-1}U)^\Gamma$ of $\Gamma$-invariant vector fields on $\partial^{-1}U \subset SM$.

iv) A $k$-form $\omega \in \Omega^k(U)$ is a $\mathcal{F}(U)$-polylinear skew-symmetric map

$$\omega : \mathfrak{X}(U) \times \ldots \times \mathfrak{X}(U) \to \mathfrak{X}(U).$$

The standard definition of the exterior differential holds on $\Omega^k$. Note in particular that if the manifold $M$ has a dense geodesic, then $C^\infty(L) = \mathbb{R}$ and there are no non-trivial vectors and vector fields on $L$. But if we restrict $M$ to a sufficiently small neighborhoods $M'$ of a point, the algebra $\mathcal{F}(M')$ will be non-trivial and we get a non-trivial Lie algebra of vector fields.

### 2.3 Canonical symplectic structure on $L(M)$

We now give another, more general, definition of tangent vectors in $T_\gamma L(M)$ in terms of Jacobi fields along $\gamma$. 

10
**Definition 2** A Jacobi tangent vector \( v \in T_\gamma L(M) \) at a point \( \gamma \in L(M) \) is a Jacobi vector field \( Y \) along \( \gamma \) which is normal to \( \gamma \). The space \( T^J_\gamma(L(M)) := \text{Jac}^{\perp}_\gamma \) of such vector fields is called the Jacobi tangent vector space of \( L(M) \) at \( \gamma \).

Note that \( \dim T^J_\gamma(L(M)) = 2n - 1 \). It is useful to give a relation between the tangent vector space \( T_\gamma L(M) \) and the Jacobi tangent vector space.

For a tangent vector \( Y \in T_m M \) we denote by \( Y_{v}^{x,y} \in V_{x,y}TM \) the vertical lift and by \( Y_h^{x,y} \in H_{x,y}TM \) the horizontal lift. We denote by \( \gamma^S = (\gamma, \dot{\gamma}) \) the natural lift of a geodesic \( \gamma \) to \( SM \).

**Lemma 4** The horizontal lift \( Y^h \) of a Jacobi field \( Y \in \text{Jac}^{\perp}_\gamma = T^J_\gamma L(M) \) is a horizontal \( \Gamma \)-invariant vector field along \( \gamma^S \subset SM \). It defines a tangent vector \( \hat{Y} : \mathcal{F}(L(M)) \to \mathbb{R}, f \mapsto Y^h(f) \).

The map \( Y \mapsto \hat{Y} \) is a homomorphism of \( T^J_\gamma(L(M)) \) into \( T_\gamma L(M) \).

Note that Proposition 1.90 in [1] is incorrect: not every Jacobi field is the transverse field of a geodesic variation, a counterexample being the vertical constant vector field along the minimal geodesic of \( x_1 + x_2^2 - x_3^2 = 1 \).

We now define the canonical symplectic 2-form \( \omega \) on \( L(M) \).

**Lemma 5** The 2-form \( \omega = d\theta \) on \( SM \) is \( \Gamma \)-horizontal (\( \iota_\Gamma \omega = 0 \)) and \( \Gamma \)-invariant.

*Proof.* We have \( \theta(\Gamma) = 1 \) and \( \iota_\Gamma \omega = 0 \), therefore \( \Gamma \cdot \omega = (d\iota_\Gamma + \iota_\Gamma d)\omega = 0 \) and \( \Gamma \in \ker \omega \). Therefore \( d\theta \) pushes down to a closed 2-form \( \omega = \omega_L \) on \( L(M) = SM/\Gamma \).

Now we describe a Poisson structure on \( L(M) \), namely

**Lemma 6** \( \mathcal{F}(L(M)) = \mathcal{F}(SM)^\Gamma \) is a subalgebra of the Lie algebra \( (\mathcal{F}(SM), \{,\}) \).

*Proof.* It is known that the Hamiltonian field preserves the symplectic form. If \( \Gamma \cdot f = 0 \), then \( \Gamma \cdot \tilde{f} = 0 \) since \( \Gamma \cdot r = 0 \). We then have \( \Gamma \cdot \{f, h\} = \Gamma \cdot \{\tilde{f}, \tilde{h}\} = \Gamma \cdot \omega^{-1}_\gamma(df, dh)|SM = 0 \). We conclude that \( L(M) \) has a canonical Poisson structure. □

One can check that if the form \( \omega_L \) is non-degenerate, then the Poisson structure on \( L(M) \) is associated with the symplectic structure \( \omega_L \), i.e. \( \{f, h\} = \omega^{-1}_L(df, dh), f, h \in \mathcal{F}(L) = \mathcal{F}_1(SM)^\Gamma \) if \( \Gamma \cdot f = \Gamma \cdot h = 0 \), where \( \omega^{-1}_L(df, dh) := \omega^{-1}(d\tilde{f}, d\tilde{h}) \).
3 Pseudo-Riemannian Spaces of Constant Curvature

3.1 Spaces of zero curvature

Let \( E = E^{p+1,q} \) be a pseudo-Euclidean vector space of signature \((p + 1, q)\) with basis \((e^+_0, ..., e^+_p, e^-_1, ..., e^-_q)\). The scalar product is given by

\[
g(X, Y) = < x, y > = \sum_{i=0}^{p} x^i y^i_+ - \sum_{i=1}^{q} x^i y^-_i.
\]

**Definition 3** A vector \( v \in E^{p+1,q} \) is said to be timelike (respectively spacelike, null) if its norm is negative (positive, vanishes). A straight line is said to be timelike (respectively spacelike, null) if its tangent vector has that type.

The space of oriented timelike (respectively spacelike) geodesics of \( E^{p+1,q} \) is denoted by \( L^-(E^{p+1,q}) \) (respectively \( L^+(E^{p+1,q}) \)). Note that by changing the sign of the metric \( L^+(E^{p+1,q}) = L^-(E^{q,p+1}) \).

We denote the unit pseudosphere by \( S^{p,q} = \{ x \in E^{p+1,q}, x^2 = 1 \} \). Here, for \( p = 0 \) we assume in addition that \( x^0 > 0 \) so that \( S^{p,q} \) is connected. The induced metric has signature \((p, q)\) and constant curvature 1.

We denote by \( SO(E) = SO^0(p+1, q) \) the connected pseudo-orthogonal group which preserves the scalar product and by \( SO(E)_e = SO^0(p, q) \) the connected subgroup which preserves the vector \( e = e^+_0 \). The group \( SO^0(p, q) \) acts transitively on \( S^{p,q} \) and we can identify \( S^{p,q} \) with the quotient \( SO^0(p + 1, q)/SO^0(p, q) \). Note that the tangent space \( T_eS^{p,q} \) has the orthonormal basis \((e^+_1, ..., e^+_p, e^-_1, ..., e^-_q)\)

Any non-null oriented straight line in \( E \) can be canonically written in the form

\[ \ell_{e,v}(t) = \{ v + te \}, \]

where \( e \) is the unit tangent vector (s.t. \( e^2 = \pm 1 \)) and \( v \) is a vector orthogonal to \( e \). So we can identify the space \( L^-(E) \) of timelike lines in \( E \) with the tangent bundle \( TS^{p,q} \). The group \( E(p + 1, q) = SO^0(p + 1, q) \cdot E^{p+1,q} \) of pseudo-Euclidean motions acts in the space \( L^-(E) = TS^{p,q} \) of timelike lines \( (e, v) := l_{e,v} \) by

\[
T_a(e, v) = (e, v + a_{e\perp}), \quad A(e, v) = (Ae, Av), \quad a \in E, \quad A \in SO^0(p + 1, q),
\]

where \( a_{e\perp} = a - < a, e > e \). The group \( E(p + 1, q) \) also naturally acts on \( S^{p+1,q} \) with the kernel of effectivity \( E^{p+1,q} \).
Proposition 1 The isometry group $E(p + 1, q)$ acts transitively on the space $L^-(E)$ of straight lines with stabilizer $SO(n - 1) \cdot R^+$ and this action commutes with the projection $\pi : L^-(E) \to S^{p,q}, (e,v) \to e$.

The proof follows from equation (3.1).

Proposition 2 A necessary condition that a subgroup $G \subset E(p + 1, q)$ acts transitively on the space $L^-(E^{p+1,q})$ of timelike geodesics is that its linear part $L_G = G/G \cap E$ acts transitively on $S^{p,q}$. If the group $G$ contains the group $T_E$ of parallel translations this condition is also sufficient.

Proof. The first claim follows from the previous Proposition. Assume now that $G$ contains $E$ and $L_G$ acts transitively on $S^{p,q}$. Let $\ell = \ell_{e,v}, \ell' = \ell'_{e',v'}$ be two lines. Using a transformation from $G$ we transform $\ell_{e',v'}$ into a line $\ell'' = \ell_{e,v''}$ with the tangent vector $e$ and then using parallel translation we transforms $\ell''$ into $\ell$.

Corollary 1 Let $E = E^n$ be the Euclidean vector space. Then any connected subgroup $G$ of the group $E(n) = SO(E) \cdot E$ of Euclidean motions has the form $G = L_G \cdot E$ where $L_G \subset SO(E)$ is a connected orthogonal group which acts transitively on $S^{n-1}$, that is $L_G$ is one of the groups

$$SO(n), U(n/2), SU(n/2), Sp(1) \cdot Sp(n/4), Sp(n/4),$$

$$G_2, (n = 7), Spin(7), (n = 8), Spin(9), (n = 16).$$

We identify the space $L^-(E^{p+1,q})$ with the homogeneous space

$$G/H = E(p + 1, q)/SO(p, q) \cdot R^*$$

where $H = SO(p, q) \cdot R^+$ is the stabilizer of the line $\ell_0 = \ell_{e_0,0}$. Let $E = \mathbb{R}e_0 + W$ be the orthogonal decomposition. We write the corresponding reductive decomposition of the homogeneous space $G/H$ as

$$\mathfrak{g} = \mathfrak{e}(p + 1, q) = \mathfrak{h} + \mathfrak{m} = (\mathfrak{so}(W)) + \mathbb{R}e_0 + (U + W),$$

where $\mathfrak{so}(E) = \mathfrak{so}(W) + U$ is the reductive decomposition of $\mathfrak{so}(E)$. In matrix notation, elements from the algebra of Euclidean isometries $\mathfrak{e}(p + 1, q) \subset \mathfrak{gl}(n + 1)$ can be written as
\[(A, \lambda e_0, u, w) = \begin{pmatrix} A & u & w \\ -u^t & 0 & \lambda \\ 0 & 0 & 0 \end{pmatrix}, \lambda \in \mathbb{R}, u, w \in \mathbb{R}^{p+q}.\]

The adjoint action of \((A, \lambda e_0)\) on \(\mathfrak{m} = U \oplus W = \{(u, w)\}\) is given by

\[
\text{ad}_{(A, \lambda)}(u, w) = (Au, Aw + \lambda u),
\]

and the bracket of two elements of \(= U \oplus W\) is given by

\[
[(u, w), (u', w')] = -(u \wedge u', (u \cdot w' - u' \cdot w)e_0) \in \mathfrak{h},
\]

where \(u \wedge u'\) is an element from \(\mathfrak{so}(W)\) and dot means the standard scalar product of vectors from \(\mathbb{R}^{n-1} = U = W\). Note that \(L^-(E^{p+1,q}) = G/H\) is a symmetric manifold since \(\mathfrak{g} = \mathfrak{h} + \mathfrak{m}\) is a symmetric decomposition.

The isomorphism \(\text{ad}_{e_0} : U \rightarrow W\) allows one to identify \(U\) with \(W\) and the tangent space \(\mathfrak{m} = U \oplus W\) with a tensor product \(\mathfrak{m} = W \otimes \mathbb{R}^2\), where \(U = W \otimes f_1, W = W \otimes f_2\) and \(f_1, f_2\) is the standard basis of \(\mathbb{R}^2\). The isotropy representation \(\text{ad}_{\mathfrak{h}}\) preserves the Grassmann structures \(\mathfrak{m} = W \otimes \mathbb{R}^2\) and \(\mathfrak{so}(W)\) act on the first factor \(W\) and \(e_0\) acts on the second factor \(\mathbb{R}^2\) by the matrix

\[
\lambda \text{ad}_{e_0} = \begin{pmatrix} 0 & 0 \\ -\lambda & 0 \end{pmatrix}.
\]

Moreover, the isotropy action \(\text{ad}_{\mathfrak{h}} | \mathfrak{m}\) preserves the metric \(g^W = g|W\) of signature \((p, q)\) in \(W\) and the symplectic structure \(\omega_0 = f_1 \wedge f_2\) in \(\mathbb{R}^2\). The tensor product \(\omega^\mathfrak{m} = g^W \otimes \omega_0\) defines a non-degenerate \(\mathfrak{h}\)-invariant 2-form in \(\mathfrak{m}\) which is extended to an invariant symplectic form \(\omega\) in \(L^-(E^{p+1,q})\). The form \(\omega\) is closed since it is invariant and the manifold \(L^-(E^{p+1,q})\) is a symmetric space. In the case of dimension \(n = 3\) the action \(\text{ad}_{\mathfrak{so}(W)}\) on \(W\) also preserves a 2-form \(\omega^W\) (which is the volume form of \(W\)). Hence we get an invariant metric \(g^\mathfrak{m} = \omega^W \otimes \omega_0\) on \(\mathfrak{m}\) which extends to an invariant pseudo-Riemannian metric \(g\) of signature \((2, 2)\) on \(L^-(E^{p+1,q})\). The quotient \(J = g^{-1} \circ \omega\) is an invariant (hence, integrable) complex structure and the pair \((g, J)\) is an invariant Kähler structure. Summarizing, we get (cf. [12]):

**Theorem 1** The space \(L^-(E^{p+1,q}) = E(p + 1, q)/SO(p, q) \cdot \mathbb{R}^+\) is a symplectic symmetric space with an invariant Grassmann structure defined by a decomposition \(\mathfrak{m} = W \otimes \mathbb{R}^2\). Moreover, if \(n = p + 1 + q = 3\), it has an invariant Kähler structure \((g, J)\) of neutral signature \((2, 2)\).
3.2 Spaces of constant non-zero curvature

We now describe the space of oriented timelike and spacelike geodesics of the pseudo-Riemannian space $S^{p,q}$ of constant curvature 1. Any such geodesic through $e \in S^{p,q}$ in direction of a unit vector $e_1^\pm$ with $(e_1^\pm)^2 = \pm 1$ is given by

$$\gamma^+ = \gamma^+_{e_1^+} = \cos(s)e + \sin(s)e_1^+, \quad \gamma^- = \gamma^-_{e_1^-} = \cosh(s)e + \sinh(s)e_1^-.$$  

The subgroup of the stability group $SO^0(p,q)$ preserving the spacelike geodesic $\gamma^+$ is $SO(p-1, q)$ and the timelike geodesic $\gamma^-$ is $SO(p, q - 1)$. The one-parameter subgroup $SO(2)$ generated by the element $e \wedge e_1^\pm$ preserves $\gamma^+$ and the one-parameter subgroup $SO(1,1)$ generated by $e \wedge e_1^-$ preserves $\gamma^-$. Since the group $SO^0(p+1, q)$ acts transitively on the space $L^+(S^{p,q})$ of spacelike geodesics and on the space $L^-(S^{p,q})$ of timelike geodesics, we can represent these spaces as

$$L^+(S^{p,q}) = SO^0(p+1, q)/SO(2) \cdot SO(p-1, q),$$

$$L^-(S^{p,q}) = SO^0(p+1, q)/SO^0(1, 1) \cdot SO(p, q - 1).$$

To get the reductive decomposition associated with these spaces, fix the orthogonal decomposition

$$E^{p+1,q} = \mathbb{R}e \oplus \mathbb{R}e_1^\pm \oplus V^\pm,$$

where $V^+$ is the vector space of signature $(p-1,q)$ with basis $(e_2^+, \ldots, e_p^+, e_1^-, \ldots, e_q^-)$ and $V^-$ is the vector space of signature $(p, q - 1)$ with basis $(e_1^+, \ldots, e_p^+, e_2, \ldots, e_q^-)$. Using the metric, we identify the Lie algebra $\mathfrak{so}(p+1, q)$ of $SO^0(p+1, q)$ with the space of bivectors $\Lambda^2(E^{p+1,q})$. Then the reductive decomposition associated with the unit sphere bundles

$$S^\pm(S^{p,q}) = \{e_1^\pm \in T_eS^{p,q}, \langle e_1^\pm, e_1^\pm \rangle = \pm 1\}$$

is given by

$$\mathfrak{so}(p+1, q) = \Lambda^2(V^\pm) \oplus (e \wedge V^\pm \oplus e_1^\pm \wedge V^\pm) \oplus \mathbb{R}(e \wedge e_1^\pm).$$

The bivector $e \wedge e_1^+$ (resp., $e \wedge e_1^-$) is invariant under the stability subgroup $SO(p-1,q)$ (resp., $SO(p,q-1)$) and defines an invariant vector field $\Gamma$ on $S^\pm(S^{p,q})$, which is the geodesic field. It is the velocity field of the right action of the subgroup $SO^\pm(2) = SO(2), SO(1,1)$ of $SO(p+1,q)$. The space of geodesics is the quotient

$$L^+(S^{p,q}) = SO(p+1,q)/SO(p-1,q) \cdot SO^\pm(2), \quad L^-(S^{p,q}) = SO(p+1,q)/SO(p,q-1) \cdot SO(1,1).$$


The corresponding reductive decomposition may be written as
\[ \mathfrak{so}(p + 1, q) = \mathfrak{h}^\pm \oplus \mathfrak{m}^\pm = \mathbb{R}(e \wedge e_1^\pm) \oplus \Lambda^2(V^\pm) \oplus (e \wedge V^\pm \oplus e_1^\pm \wedge V^\pm). \]

We identify \( \mathfrak{m}^\pm \) with the tangent space \( T_{\gamma^\pm}(L^\pm S^{p,q}) \). There is also a natural identification with the tensor product \( \mathfrak{m}^\pm = H \otimes V^\pm \), where \( H = \text{span}(e, e_1^\pm) \simeq \mathbb{R}^2 \) is the 2-dimensional oriented pseudo-Euclidean vector space. Then the action of the isotropy subalgebra \( \mathfrak{h}^\pm \) takes the form:

\[
ad_{e \wedge e_1^\pm} : e \otimes x \mapsto (e \wedge e_1^\pm)e \otimes x = -e_1^\pm \otimes x,
\]

\[
ad_{e \wedge e_1^\pm} : e_1^\pm \otimes x \mapsto (e \wedge e_1^\pm)e_1^\pm \otimes x = \pm e \otimes x,
\]

\[
ad_{a \wedge b} : e' \otimes x \mapsto (e' \otimes (a \wedge b))x = e' \otimes < b, x > a - < a, x > b,
\]

for all \( a, b, x \in V^\pm \) and \( e' \in \mathbb{R}^2 \).

Note that \( L^\pm(S^{p,q}) \) is identified with the Grassmanian \( Gr_2^\pm(\mathbb{R}^{p+1,q}) \) of two-planes of signature \((2,0)\) or \((1,1)\) and the decomposition
\[ T_{\gamma^\pm}(L^\pm S^{p,q}) = H \otimes V^\pm, \]

defines an invariant Grassmann structure in \( L^\pm(S^{p,q}) \).

Denote by \( H^\varepsilon \) two-dimensional vector space with a scalar product \( g_H = g_H^\varepsilon \) of signature \((2,0)\) for \( \varepsilon = + \) and \((1,1)\) for \( \varepsilon = - \), and by \( I_H = I_H^\varepsilon \) the \( \text{SO}(H^\varepsilon) \)-invariant endomorphism of \( H^\varepsilon \) with \( I_H^2 = -\varepsilon I_1 \) and by \( \omega_H = \omega_{H^\varepsilon} = g_\varepsilon \circ I_H \) the invariant volume form. Let \((V, g_V)\) be a pseudo-Euclidean vector space of dimension \( m \). If \( m = 2 \), we denote by \( I_V \) the \( \text{SO}(V) \)-invariant endomorphism with \( I_V^2 = -1 \) for signature \((2,0)\) or \((0,2)\) and with \( I_V^2 = 1 \) for the signature \((1,1)\). Denote also by \( \omega_V = g_V \circ I_V \) the volume form of \( V \).

**Lemma 7**

i) Any \( \text{SO}(H^\varepsilon) \times \text{SO}(V) \)-invariant endomorphism of the space \( W^\varepsilon = H^\varepsilon \otimes V \) has the form \( A = 1 \otimes A + I_H^\varepsilon \otimes B \) where \( A, B \in \mathfrak{gl}(V)^{\text{SO}(V)} \) are invariant endomorphisms of \( V \).

ii) Any invariant endomorphism \( I \) of \( W^\varepsilon \) different from 1 with \( I^2 = \pm 1 \) is given (up to a sigh) by \( I_\varepsilon = I_H \otimes 1 \) if \( m > 2 \) and by \( I_\varepsilon, I'_\varepsilon := 1 \otimes I_V, I''_\varepsilon = I_H \otimes I_V \) if \( m = 2 \).

iii) Any invariant metric on \( W^\varepsilon \) is proportional to \( g := g_H \otimes g_V \) if \( m > 2 \) and is a linear combination of the metric \( g' := \omega_H \otimes \omega_V \) otherwise.
iv) The space of invariant 2-forms has the basis \( \omega = \omega_H \otimes g_V \) if \( m > 2 \) and \( \omega, \omega' := g_H \otimes \omega_V \) if \( m = 2 \).

v) The endomorphisms \( I, I' \) are skew-symmetric with respect to any invariant metric \( h \) on \( W^{\varepsilon} \), hence define a Hermitian or para-Hermitian structure, and the endomorphism \( I'' \) is symmetric with respect to any invariant metric \( h \).

Note that the tensor product of two complex or two para-complex structures is a para-complex structure and the tensor product of a complex and a para-complex structures is a complex structure.

Proof. To prove part i), it is sufficient to write the endomorphism \( A \) in block matrix form with respect to the decomposition \( W = h_1 \otimes V + h_2 \otimes V \), where \( h_1, h_2 \) is an orthonormal basis of \( H^+ \) or isotropic basis of \( H^- \) and write the conditions that it is \( SO(H) \times SO(V) \)-invariant. Since the only invariant endomorphism of \( V \) is a scalar if \( m > 2 \) and is a linear combination of \( 1, I_V \) if \( m = 2 \), part ii) follows from part i). Parts iii) and iv) follow from the fact that the space of symmetric bilinear forms \[ S^2(H \otimes V) = S^2(V) + \Lambda^2 H \otimes \Lambda^2 V, \]
and that the space of 2-forms \[ \Lambda^2(H \otimes V) = \Lambda^2(V) + S^2(H) \otimes \Lambda^2 V. \] Now part v) follows from parts iii) and iv). \( \square \)

Since the spaces
\[ L^+(S^{p,q}) = SO^0(p + 1, q)/SO(2) \cdot SO^0(p - 1, q) \]
\[ L^-(S^{p,q}) = SO^0(p + 1, q)/SO^0(1, 1) \cdot SO^0(p, q - 1) \]
of spacelike and timelike geodesics are symmetric spaces, any Hermitian pair \((h, I)\) which consists of invariant pseudo-Euclidean metric on \( m^\varepsilon \) and skew-symmetric invariant complex or para-complex structure \( I \) (such that \( I^2 = -1 \) or \( I^2 = 1 \)) defines an invariant Kähler or para-Kähler structure on \( L^{\pm}S^{p,q} \).

We get the following theorem.

**Theorem 2**

i) Let \( L^+(S^{p,q}) \), (respectively \( L^-(S^{p,q}) \)) be the space of spacelike (respectively timelike) geodesics in \( S^{p,q} \) and \( p + q > 3 \). Then there exists a unique (up to scaling) invariant symplectic structure \( \omega \) and a unique (up to sign) invariant complex structure \( I^+ = J \) on \( L^+(S^{p,q}) \) (respectively para-complex structure \( K = I^- \) on \( L^-(S^{p,q}) \)). There exists unique (up to a scaling) invariant pseudo-Riemannian metric \( g^\varepsilon = \omega \circ I^\varepsilon \) on \( L^\varepsilon(S^{p,q}) \) which is Kähler of signature \((2(p - 1), 2q)\) for \( \varepsilon = + \) and para-Kähler (of neutral signature) for \( \varepsilon = - \).
ii) If $p + q = 3$, then there are two linearly independent invariant (parallel and closed) 2-forms $\omega, \omega'$ on $L^\varepsilon(S^{p,q})$ with values $\omega^m = \omega_H \otimes g_V$, $\omega'^m = g_H \otimes \omega_V$. Any invariant metric has the form $h = \lambda g + \mu g'$ where $g'$ is a neutral metric with value $g'^m = \omega_H \otimes \omega_V$. Any metric $h$ is Kähler for $\varepsilon = 1$ (respectively, para-Kähler for $\varepsilon = -1$) with respect to the complex (respectively, para-complex) structure $I^\varepsilon = I_H \otimes 1$ with the Kähler form $h \circ I^\pm = \lambda \omega + \mu \omega'$. Moreover, the endomorphism $I' = 1 \otimes I_V$ of $m$ defines an invariant parallel $h$-skew-symmetric complex structure of $L^\varepsilon(S^{p,q})$ if $\varepsilon = 1, (p - 1, q) = (2, 0)$ or $(0, 2)$ or $\varepsilon = -1$ and $(p, q - 1) = (2, 0)$ or $(0, 2)$ and skew-symmetric parallel para-complex structure if $\varepsilon = +, (p - 1, q) = (1, 1)$ or $\varepsilon = -, (p, q - 1) = (1, 1)$. The Kähler or para-Kähler structure $(h, I')$ has the Kähler form $h \circ I' = \lambda \omega + \mu \omega'$.

One can easily check that the form $\omega$ is the canonical symplectic form of the space of geodesics $L^\varepsilon(S^{p,q})$. 
4 Rank One Symmetric Spaces of Non-Constant Curvature

In this section we discuss the invariant geometric structures on the space $L(M)$ of oriented geodesics of a rank one Riemannian symmetric space of non-constant curvature $M = G/K$, that is for the projective spaces

$$\mathbb{C}P^n = SU_{n+1}/U_n, \quad \mathbb{H}P^n = Sp_{n+1}/Sp_1 \cdot Sp_n, \quad \mathbb{O}P^2 = F_4/Spin_9,$$

and the dual hyperbolic spaces

$$\mathbb{C}H^n = SU_{1,n}/U_n, \quad \mathbb{H}H^n = Sp_{1,n}/Sp_1 \cdot Sp_n, \quad \mathbb{O}H^2 = F_4^{non\text{-}comp}/Spin_9.$$

In all of these cases, the space of geodesics is a homogeneous manifold $L(M) = G/H$ where the stability subgroup $H$ is the same for the compact and dual non-compact case, and is given by

$$H = T^2 \cdot SU_{n-1}, \; T^1 \cdot Sp_1 \cdot Sp_{n-1}, T^1 \cdot Spin_7.$$

Moreover, in the case of a classical Lie group $G$, the space $L(M)$ is the adjoint orbit $L(M) = \text{Ad}_GI^\varepsilon$ of the element $I^\varepsilon = h_1^\varepsilon = \text{diag}(I_2^\varepsilon, 0, 0, ..., 0)$, where

$$I_2^\varepsilon = \begin{pmatrix} 0 & -\varepsilon \\ 1 & 0 \end{pmatrix},$$

and $\varepsilon = 1$ in the compact case and $\varepsilon = -1$ otherwise. Main Theorems 2 and 3 describe all invariant structures (symplectic structures, complex and para-complex, Kähler and para-Kähler structures) on the space of geodesics $L(M)$.

We prove these in three stages: first for the complex and quaternionic projective spaces, then for their hyperbolic counterparts and finally for the Cayley projective plane.

4.1 Complex projective and hyperbolic spaces

We now describe the space of real geodesics in complex projective space

$$M^1 = \mathbb{C}P^n = SU_{n+1}/U_n$$

and in complex hyperbolic space

$$M^{-1} = \mathbb{C}H^n = SU_{1,n}/U_n.$$
We set $g^1 = \mathfrak{su}_{n+1}$ and $g^{-1} = \mathfrak{su}_{1,n}$. We then choose the associated reductive decompositions $g^\epsilon = j(u_n) + p^\epsilon$, $\epsilon = \pm 1$, where
\[
j(u_n) = \{ \begin{pmatrix} -\text{tr} A & 0 \\ 0 & A \end{pmatrix} \mid A \in u_n \}, \quad p_\epsilon = \{ \begin{pmatrix} 0 & -\epsilon X^* \\ X & 0 \end{pmatrix} \mid X \in \mathbb{C}^n \},
\]
and where $X$ is a column vector and $X^*$ denotes the Hermitian conjugate. We identify $p^\epsilon$ with the tangent space $T_oM^\epsilon$, $o = eU_n$. We next describe the stability subalgebra $\mathfrak{h}^\epsilon$ of the geodesic $\gamma = \exp(th_1^1)(o)$, where the element $h_1^\epsilon$ is represented by the matrix
\[
h_1^\epsilon = I^\epsilon = \begin{pmatrix} I_2^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad I_2^2 = \begin{pmatrix} 0 & -\epsilon \\ 1 & 0 \end{pmatrix}.
\]
We have that $\mathfrak{h}^\epsilon = Zg(h_1^1) = \mathbb{R}h_1^\epsilon + Zu_n(h_1^1)$, where $Zu_n(h_1^1)$ is the centraliser of $h_1^\epsilon$ in $u_n$. We now describe the reductive decomposition $g^\epsilon = \mathfrak{h}^\epsilon + l^\epsilon$. We have
\[
\mathfrak{h}^\epsilon = \{ A = \begin{pmatrix} i\alpha & -\epsilon\beta & 0 \\ \beta & i\alpha & 0 \\ 0 & 0 & A_{n-1} \end{pmatrix} \mid \alpha, \beta \in \mathbb{R}, \quad A_{n-1} \in u_{n-1}, \text{tr} A_{n-1} + 2i\alpha = 0 \},
\]
and the complimentary subspace is
\[
l^\epsilon = \{ X = (x_1, x_2, X_1, X_2) = \begin{pmatrix} ix_1 & \varepsilon ix_2 & -\epsilon X_1^* \\ ix_2 & -ix_1 & -X_2^* \\ X_1 & X_2 & 0 \end{pmatrix} \mid x_1, x_2 \in \mathbb{R}, \quad X_1, X_2 \in \mathbb{C}^{n-1} \}.
\]
We may write
\[
\mathfrak{h}^\epsilon = \mathbb{R}h_0 + \mathbb{R}h_1^\epsilon + \mathfrak{su}_{n-1},
\]
where
\[
h_0 = \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & \frac{2}{n-1}I_{n-1} \end{pmatrix}, \quad h_1^\epsilon = I^\epsilon = \begin{pmatrix} 0 & -\varepsilon & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]
Similarly,
\[
l^\epsilon = V_0^\epsilon + V_+^\epsilon + V_-^\epsilon,
\]
where
\[
V_0^\epsilon = \{ (x_1, x_2, 0, 0) = \mathbb{R}E_1 + \mathbb{R}E_2^\epsilon. \quad V_\pm^\epsilon = \{ X_\pm = (0, 0, X, \pm X), X \in \mathbb{C}^{n-1} \},
\]
\[
E_1 = (1, 0, 0, 0) = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2^\epsilon = (0, 1, 0, 0) = \begin{pmatrix} 0 & \varepsilon i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]
We denote the canonical Hermitian form in the space of vector columns $\mathbb{C}^{n-1}$ by $\eta(X, Y) = X^*Y$. Then
\[
g(X, Y) = \text{Re} \eta(X, Y) = \frac{1}{2}(X^*Y + Y^*X),
\]
\[
\rho(X, Y) = \text{Im} \eta(X, Y) = \frac{1}{2i}(X^*Y - Y^*X).
\]
For any $X \in \mathbb{C}^{n-1}$ we set $X_\pm = (0, 0, X, \pm X)$.

**Lemma 8** We have the following commutator relations:

\[
[E_1, E_2^\varepsilon] = 2h_1^\varepsilon, \tag{4.1}
\]
\[
[E_1, (0, 0, X_1, X_2)] = (0, 0, -iX_1, iX_2), \tag{4.2}
\]
\[
[E_2^\varepsilon, (0, 0, X_1, X_2)] = (0, 0, -iX_2, -\varepsilon iX_1). \tag{4.3}
\]

The isotropy action of $h^\varepsilon$ on $l^\varepsilon$ is given by
\[
ad_{h_0}(x_1, x_2, X_1, X_2) = (0, 0, -iX_1, -iX_2),
\]
\[
ad_{h_1^\varepsilon}(x_1, x_2, X_1, X_2) = (-2\varepsilon x_2, 2x_1, -X_2, \varepsilon X_1),
\]
\[
ad_{A_n^{-1}}(x_1, x_2, X_1, X_2) = (0, 0, A_n^{-1}X_1, A_n^{-1}X_2).
\]
Moreover
\[
[X_+^1, Y_+^1] = 2\rho(X, Y)(-h_0 \mp E_2^1) \mod \mathfrak{su}_{n-1}, \tag{4.4}
\]
\[
[X_-^1, Y_-^1] = 2\rho(X, Y)(E_1 \pm E_2^{-1}) \mod \mathfrak{su}_{n-1}, \tag{4.5}
\]
\[
[X_+^{-1}, Y_-^{-1}] = -2\rho(X, Y)E_1 + 2g(X, Y)h_1 \mod \mathfrak{su}_{n-1}, \tag{4.6}
\]
\[
[X_-^{-1}, Y_+^{-1}] = 2\rho(X, Y)h_0 + 2g(X, Y)h_{-1} \mod \mathfrak{su}_{n-1}, \tag{4.7}
\]
for all $X_\pm^1, Y_\pm^1 \in V_\pm^\varepsilon$.

**Proposition 3** The $h$-module $l^\varepsilon$ has the following decomposition into irreducible components.

- **a)** For $\varepsilon = 1$

\[
l_+^1 = V_0^1 + V_+^1 + V_-^1
\]
\[
ad_{h_0} : 0 & -i\text{Id} & -i\text{Id}
\]
\[
ad_{h_1^1} : 2J_0 & i\text{Id} & -i\text{Id}
\]
\[
A_{n-1} : 0 & A_{n-1} & A_{n-1}
\]
b) For \( \varepsilon = -1 \)

\[
\begin{align*}
T^{-1} &= \mathbb{R}E_+ + \mathbb{R}E_- + V_+^{-1} + V_-^{-1} \\
\text{ad}_{h_0} : &= 0 0 -i\text{Id} -i\text{Id} \\
\text{ad}_{h_1} : &= 2 -2 \text{Id} -\text{Id} \\
A_{n-1} : &= 0 0 A_{n-1} A_{n-1},
\end{align*}
\]

where

\[
E_\pm = (1, \pm 1, 0, 0) = E_1 \pm E_2^{-1}.
\]

With this notation in case b), the commutation relations read as follows:

\[
\begin{align*}
[E_\pm, V_\pm] &= 0, \quad [E_+, E_-] = -4h_1^{-1}, \quad (4.8) \\
[E_+, X_-] &= -2iX_+, \quad [E_-, X_+] = -2iX_- \quad (4.9) \\
[X_\pm, Y_\pm] &= 2\rho(X,Y)E_\pm, \quad [X_+, Y_-] = 2\rho(X,Y)h_0 + 2g(X,Y)h_1 \mod \mathfrak{su}_{n-1}, \quad (4.10)
\end{align*}
\]

for \( X_\pm \in V_\pm^{-1} \). Recall that any invariant 2-form on \( L(M) \) is generated by an \( \text{ad}_{\mathfrak{h}} \)-invariant two form \( \omega \) on the tangent space \( \mathfrak{t} = T_0L(M) \). Any such form may be represented as \( \omega = d(B \circ h) \), where \( h \in Z(\mathfrak{h}) \) is a central element and \( B \) is the Killing form.

**Theorem 3**

i) The only invariant almost complex structures on \( L(\mathbb{C}P^n) \) are defined by

\[
J_{\varepsilon_0\varepsilon_1\varepsilon_2} = \varepsilon_0\varepsilon_1\varepsilon_2 \text{ad}_{h_1}|_{V_0} \oplus \varepsilon_1\text{ad}_{h_1}|_{V_+} \oplus \varepsilon_2\text{ad}_{h_1}|_{V_-} = \varepsilon_0J^{V_0} \oplus \varepsilon_1J^{V_+} \oplus \varepsilon_2J^{V_-},
\]

where \( \varepsilon_k = \pm 1 \). The integrable ones among these are (up to a sign) \( J_\pm = (\pm J^{V_0}) \oplus J^{V_-} \oplus J^{V_+} \).

ii) Any \( SU(n+1) \)-invariant closed 2-form on \( L(\mathbb{C}P^n) \) is a linear combination \( \omega = \lambda_0\omega_0 + \lambda_1\omega_1 \) of the invariant differential forms defined by \( \omega_0 = d(B \circ h_0)|_{\mathfrak{t}^1} \) and \( \omega_1 = d(B \circ h_1)|_{\mathfrak{t}^1} \) where \( B \) is the Killing form.

Moreover

\[
\ker \omega_0 = V_0, \quad \omega_0(X_\pm, Y_\pm) = 2\rho(X,Y), \quad \omega_0(V_\pm^1, V_\mp^1) = 0,
\]

\[
\omega_1(E_1, E_2) = -2, \quad \omega_1(V_\pm, V_\mp) = 0, \quad \omega_1(X_+, Y_-) = -\omega_1(Y_-, X_+) = 2g(X,Y).
\]

Here \( X_\pm = (0,0,X,\pm X), Y_\pm = (0,0,Y,\pm Y) \in \mathfrak{t}^1 \) and

\[
g(X,Y) = \text{Re} \eta(X,Y), \omega(X,Y) = \text{Im} \eta(X,Y)
\]

22
are the real and imaginary parts of the standard Hermitian form \( \eta(X, Y) = X^* Y \) on \( \mathbb{C}^{n-1} \).

iii) The canonical symplectic structure on the space oriented geodesics is \( \omega_1 \).

**Corollary 2** Up to scaling, any invariant symplectic form on \( L(\mathbb{C}P^n) \) may written as

\[
\omega^t = \omega_1 + t\omega_0, \quad t \in \mathbb{R}
\]

It is compatible with any invariant complex structure \( J_\pm \). That is, the pair \( (\omega^t, \pm J_\pm) \) is a Kähler structure on \( L \).

Proof of the Theorem: The description in part i) follows directly from the previous Proposition. For integrability, we calculate the Niejenhuis bracket

\[
N_J(X, Y) = [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y],
\]

for \( X, Y \in \mathfrak{t} = \mathbb{R}E_1 + \mathbb{R}E_2^\varepsilon + V_+ + V_- \). For example, for \( X_+ \in V_+, Y_- \in V_- \), \( J = J_{\varepsilon 01} \) we calculate:

\[
N_J(X_+, Y_-) = [iX_+ \varepsilon Y_-] - J[X_+ \varepsilon Y_-] - J[iX_+, Y_-] - [X_+, Y_-]
= (-\varepsilon - 1)[X_+, Y_-] + (\varepsilon - 1)iJ[X_+, Y_-].
\]

Hence \( N_J(X_+, Y_-) = 0 \) iff \( J = J_{\varepsilon 01} \) i.e. \( \varepsilon = 1 \). Similarly we calculate \( N_J(X_+, E_1) \):

\[
N_J(X_+, E_1) = [iX_+ \varepsilon E_1] - J[X_+ \varepsilon E_1] - J[iX_+, E_1] - [X_+, E_1]
= (i - J)[X_+, E_2] + (iJ - 1)[X_+, E_1]
= (-i + J)iX_+(iJ - 1)X_+
\]

This always vanishes. The proof of part ii) follows from the fact that the center \( Z(\mathfrak{g}^1) \) has basis \( h_0, h_1 \). To verify part iii) we consider \( h_1^1 \in \mathfrak{p}^1 = T_0\mathbb{C}P^n \). The stability subalgebra of this element is \( \mathfrak{h}_{h_1^1} = \mathbb{R}h_0 + \mathfrak{su}_{n-1} \subset \mathfrak{h}^1 \). The sphere bundle \( S\mathbb{C}P^n \) is identified with \( SU_{n+1}/T^1 \cdot SU_n \) with the reductive decomposition \( \mathfrak{su}_{n+1} = (\mathbb{R}h_0 + \mathfrak{su}_{n-1}) + (\mathbb{R}h_1^1 + \mathfrak{l}^1) \). The geodesic vector field \( \Gamma \) on \( S\mathbb{C}P^n \) is the invariant vector field generated by the element \( h_1^1 \) and the contact form \( \theta \) is the invariant form associated the one-form \( \theta_0 = cB \circ h_1^1 \). This shows that the canonical form coincides with \( \omega^1 \) (up to scaling).
Theorem 4  

i) There is no invariant almost complex structure on the space $L(CH^n)$.

There exist two (up to sign) almost para-complex structures $K^\pm$ with $(\pm 1)$-eigenspace decompositions given by

\[ K^+: l^{-1} = l_+ + l_- = (\mathbb{R}E_+ + V_+) + (\mathbb{R}E_- + V_-), \]
\[ K^-: l^{-1} = l_+ + l_- = (\mathbb{R}E_+ + V_+) + (\mathbb{R}E_- + V_+). \]

Only $K^+$ is integrable.

ii) Any closed invariant two-form is a linear combination of the form defined by

\[ \omega_0 = d(B \circ h_0)|l^{-1}, \quad \omega_1 = d(B \circ h_{-1})|l^{-1}, \]
where $B$ is the Killing form.

Moreover, we have

\[ \ker \omega_0 = V_0, \quad \omega_0(V_+, V_+) = 0, \quad \omega_0(X_+, Y_-) = 2\rho(X, Y), \]
\[ \omega_1(E_+, E_-) = 4, \quad \omega_1(X_+, Y_-) = -\omega_1(Y_-, X_+) = 2g(X, Y), \]

where $X_+ = (0, 0, X, X), \quad Y_- = (0, 0, Y, -Y)$.

Proof: An invariant almost complex structure on $L(CH^n)$ preserves the one-dimensional $ad_{h^{-1}}$-eigenspaces $\mathbb{R}E_\pm$, which is impossible. Since $l_\pm = \mathbb{R}E_\pm + V_\pm$ are subalgebras, the endomorphisms $K$ of $l^{-1}$ with $K|l_\pm = \pm \text{Id}$ define an invariant para-complex structure. It is unique up to sign since $[V_\pm, V_\pm] = \mathbb{R}E_\pm$. This proves i). The first claim of ii) follows from the remark that $h_0, h_{-1}$ form a basis of the center $Z(\mathfrak{h}^{-1})$. The explicit formulas for $\omega_0, \omega_1$ follow from equations (4.8), (4.9) and (4.10). \n
Corollary 3  Any invariant symplectic form on $LCH^n$ may be written as

\[ \omega^t = \omega_1 + t\omega_0. \]

They are compatible with the para-complex structures $K^\pm$, i.e. $(\omega^t, K^\pm)$ is a para-Kähler structure and, in particular, $g^t = \omega^t \circ K^\pm$ is a para-Kähler metric.
4.2 Quaternionic projective and hyperbolic spaces

Consider now the spaces $M^+ = \mathbb{HP}^n = Sp_{n+1}/Sp_1 \cdot Sp_n$ and $M^- = \mathbb{HH}^n = Sp_{1,n}/Sp_1 \cdot Sp_n$. The reductive decomposition $\mathfrak{g}^c = (sp_1 + sp_n) + \mathfrak{p}^c$ associated to the homogeneous space $M^c$ may be written as

$$sp_1 + sp_n = \left\{ \begin{pmatrix} a & 0 \\ 0 & A_n \end{pmatrix} \mid a \in \text{Im} \mathbb{H} = sp_1, A_n \in sp_n \right\}, \quad \mathfrak{p}^c = \left\{ \begin{pmatrix} 0 & -\varepsilon X^* \\ X & 0 \end{pmatrix} \mid X \in \mathbb{H}^n \right\}.$$

We next describe the stability subalgebra $\mathfrak{h}^c$ of the geodesic $\gamma = \exp(th_1')(o)$, $o = e(Sp_1 \cdot Sp_n) \in M^c$, which is the orbit of the one-parameter group $\exp(th_1')$, where $h_1^c = \text{diag}(I_2^c, 0)$. We have $\mathfrak{h} = Z_{\mathfrak{g}}(h_1^c) = \mathbb{R}h_1^c + Z_{(sp_1+sp_n)}(h_1^c)$, where $Z_{(sp_1+sp_n)}(h_1^c)$ is the centraliser of $h_1^c$.

The reductive decomposition associated to $L(M)^c$ may be written as

$$\mathfrak{h}^c = \{ A = \begin{pmatrix} a & -\varepsilon \alpha & 0 \\ \alpha & a & 0 \\ 0 & 0 & A_{n-1} \end{pmatrix} \mid a \in \mathbb{R}, A_{n-1} \in sp_{n-1} \},$$

where $h_0 = \text{diag}(1, 1, 0)$, $h_1^c = \text{diag}(I_2^c, 0)$, and the complimentary subspace is

$$\mathfrak{l}^c = \{ X = (x_1, x_2, X_1, X_2) = \begin{pmatrix} x_1 & \varepsilon x_2 & -\varepsilon X_1^* \\ x_2 & -x_1 & -X_2^* \\ X_1 & X_2 & 0 \end{pmatrix} \mid x_1, x_2 \in sp_1, X_1, X_2 \in \mathbb{H}^{n-1} \}.$$

We set

$$E_1 = \text{diag}(1, -1, 0), \quad E_2^c = h_1^c.$$

Then

$$\mathfrak{l}^c = sp_1 E_1 + sp_1 E_2^c + \{ (0, 0, X_1, X_2), X_i \in \mathbb{H}^{n-1} \}.$$

The isotropy action of $\mathfrak{h}^c$ on $\mathfrak{l}^c$ is given by

$$\text{ad}_{ah_0}(x_1, x_2, X_1, X_2) = ([a, x_1], [a, x_2], -X_1 a, -X_2 a),$$

$$\text{ad}_{h_1^c}(x_1, x_2, X_1, X_2) = (-2\varepsilon x_2, 2x_1, -X_2, \varepsilon X_1),$$

$$\text{ad}_{A_{n-1}}(x_1, x_2, X_1, X_2) = (0, 0, A_{n-1} X_1, A_{n-1} X_2).$$

As in the complex case, we introduce the canonical Hermitian form $\eta(X, Y) = X^* Y$ on $\mathbb{H}^{n-1}$. We have the following commutator relations:

$$[x_1 E_1, x_2 E_2^c] = -(x_1 x_2 + x_2 x_1) h_1^c = -2Re(x_1 x_2) h_1^c,$$

which completes the description.
Theorem 5  

The commutator relations imply the following Theorem in a similar way as in the complex submodule:

For \( \epsilon = 1 \) we have the following decomposition of the \( \mathfrak{h}^1 \)-module \( \mathfrak{t}^1 \) into two irreducible submodules: \( \mathfrak{t}^1 = V_0 + V_1, V_0 = \text{Im} \mathbb{H} + \text{Im} \mathbb{H} = \mathbb{R}^6, V_1 = \mathbb{H}^{n-1} + \mathbb{H}^{n-1} \). Define an \( \mathfrak{h}^1 \)-invariant complex structure \( J^V_0, J^V_1 \) by

\[
J^V_0 = 1/2 \text{ad} h_1 | V_0 : (x_1, x_2) \rightarrow (-x_2, x_1) \quad J^V_1 = \text{ad} h_1 | V_1 : (X_1, X_2) \rightarrow (-X_2, X_1).
\]

The commutator relations imply the following Theorem in a similar way as in the complex case.

**Theorem 5**

i) There exist (modulo sign) two invariant almost complex structures on \( L(\mathbb{H}P^n) \) defined by \( J_\pm = J^V_0 \oplus \pm J^V_1 \). Only one of them, namely \( J_+ \), is integrable.

ii) Up to scaling, there exists a unique invariant symplectic form \( \omega \) on \( L(\mathbb{H}P^n) \) defined by \( \omega = d(h^\epsilon_1)^* \). More precisely we have

\[
\omega((x_1, x_2, X_1, X_2), (y_1, y_2, Y_1, Y_2)) = \\
+2 \text{Re}(y_1 x_2) + 2 \text{Re}(y_1 x_2) + \text{Re}(X_1, Y_2) - \text{Re}(Y_1, X_2).
\]

iii) The symplectic structure \( \omega \) consistent with the complex structure \( J_+ \) and gives rise to a \( \text{Kähler} \) structure.

For \( \epsilon = -1 \) we have the decomposition into irreducible \( \text{ad} \mathfrak{h}^{-1} \)-submodules

\[ \mathfrak{t}^{-1} = V_2 + V_{-2} + V_1 + V_{-1} \]

such that \( \text{ad} \mathfrak{h}_{-1} | V_k = k \text{Id}, V_2 \simeq V_{-2} \simeq \text{Im} \mathbb{H} \simeq \mathfrak{sp}_1 \) and \( V_1 \simeq V_{-1} \simeq \text{Im} \mathbb{H}^{n-1} \) with the standard action of \( \mathfrak{h}^{-1} = \mathfrak{sp}_1 + \mathfrak{sp}_{n-1} \). We finally define two \( \text{ad} \mathfrak{h}^{-} \)-invariant para-complex structures \( K_\pm \) on \( \mathfrak{h}_{-1} \) by

\[
K_+|_{V_1+V_2} = 1, \quad K_+|_{V_{-1}+V_{-2}} = -1, \quad K_-|_{V_1} = 1, \\
K_-|_{V_2} = -1, \quad K_-|_{V_{-2}} = -1.
\]

We then have the following
**Theorem 6** On $L(\mathbb{H}H^n)$ there exist no invariant almost complex structures and two (up to sign) unique invariant almost para-complex structures $K_\pm$, with $K_+$ being integrable and $K_-$ being non-integrable.

### 4.3 Cayley projective plane

Let $M = \mathbb{O}P^2 = F_4/Spin_9$ be the octonian projective plane and $\mathfrak{g} = \mathfrak{f}_4 = spin_9 + \mathfrak{p}$ the associated reductive decomposition. The isotropy group $Spin_9$ acts on the 16-dimensional tangent space $\mathfrak{p} = T_oM$ by the spinor representation with 15-dimensional spheres as orbits. Let $\gamma = \exp(t h_1))o, h_1 \in \mathfrak{p}$ be the geodesic through the point $o = e(Spin_9) \in M$.

The stability subgroup of $\gamma$ is $H = SO_2 \cdot Spin_7$ and the stability subalgebra $\mathfrak{h} = \mathbb{R}h_1 + Z_{spin}(h_1) = \mathbb{R}h_1 + \mathfrak{spin}_7$. We identify the space of geodesics in $M = \mathbb{O}P^2$ with $L(M) = F_4/\mathbb{SO}_2 \cdot Spin_7$.

Following [4] we choose the root system $R = \{\pm \varepsilon_i, \pm \varepsilon_i \pm \varepsilon_j, 1/2(\pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)\}$ of the complex Lie algebra $\mathfrak{f}_4$ with respect to a Cartan subalgebra $\mathfrak{a}$ and a system of simple roots as follows

$$\alpha_1 = 1/2(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4), \alpha_2 = \varepsilon_4, \alpha_3 = \varepsilon_3 - \varepsilon_4, \alpha_4 = \varepsilon_4 - \varepsilon_3.$$  

Here, $\varepsilon_i, \ i = 1, ..., 4$ is an orthonormal basis of the real space $\mathfrak{a}_R = B^{-1} \circ \text{span}_\mathbb{R} R$. We may assume that $\frac{1}{2}d = -i h_1 \in \mathfrak{a}_\mathbb{R}$ is the vector dual to the fundamental weight $\pi_1 = \varepsilon_1$.

Then the adjoint operator $ad_d$ defines a gradation

$$\mathfrak{f}_4 = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2,$$

where $\mathfrak{g}_0 = Z_{\mathfrak{h}}(h_1) = C h_1 + \mathfrak{spin}^C_7$, and $\mathfrak{spin}^C_7$ has the root system given by

$$\{ \pm \varepsilon_i, \pm \varepsilon_i \pm \varepsilon_j, i, j = 2, 3, 4\}.$$  

The space $\mathfrak{g}_{\pm 1}$ is spanned by the root vectors with roots $1/2(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$ and $\mathfrak{g}_{\pm 2}$ is spanned by the root vectors with roots $\pm \varepsilon_i, \pm (\varepsilon_i \pm \varepsilon_i), i = 2, 3, 4$. Let $\tau$ be the standard compact involution of $\mathfrak{f}_4$ such that $\mathfrak{f}_4^\tau$ is the compact real form of $\mathfrak{f}_4$. Then the reductive decomposition associated with the space of geodesics can be written as

$$\mathfrak{f}_4^\tau = \mathfrak{h} + \mathfrak{l} = \mathfrak{g}_0^\tau + (\mathfrak{g}_{-1} + \mathfrak{g}_1)^\tau + (\mathfrak{g}_{-2} + \mathfrak{g}_2)^\tau.$$  

The decomposition

$$\mathfrak{f}_4^C = \mathfrak{l}^{00} + \mathfrak{l}^{01} = (\mathfrak{g}_1 + \mathfrak{g}_2) + (\mathfrak{g}_{-1} + \mathfrak{g}_{-2})$$

defines a unique (up to sign) invariant complex structure $J$ on the space of geodesics defined by $J|_{\mathfrak{l}^{00}} = i \text{Id}, J|_{\mathfrak{l}^{01}} = -i \text{Id}$. The 2-form $\omega = d(B \circ h_1)$ associated with
the central element \( h_1 \in Z(\mathfrak{h}) = \mathbb{R}h \) defines a unique (up to scaling) symplectic form compatible with \( J \), where \( B \) is the Killing form. We get

**Theorem 7** The space \( L(\mathbb{Q}P^2) = F_4/\text{SO}_2 \cdot \text{Spin}_7 \) admits a unique (up to a sign) invariant complex structure \( J \), unique (up to a scaling) invariant symplectic structure \( \omega = dB \circ h_1, h_1 = \in Z(\mathfrak{h}) \) and a unique invariant Kähler structure \((\omega, J)\).

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