Gauge equivalence classes of flat connections in the Aharonov–Bohm effect

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In this paper we present a simplified derivation of the fact that the moduli space of flat connections in the abelian Aharonov-Bohm effect is isomorphic to the circle. The length of this circle is the electric charge.

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1. Introduction

In a recent paper, Aguilar and Socolovsky 1 have studied geometrical and topological aspects of the abelian Aharonov-Bohm (A-B) 2 effect. This is a gauge invariant non local quantum effect that geometrically can be thought to be induced by a non trivial flat connection on a product bundle over a space with a non trivial topology. In particular it was determined that the principal bundle relevant for the A-B effect is the product bundle $\xi_{A-B} : U(1) \rightarrow \mathbb{R}^2 \times U(1) \rightarrow \mathbb{R}^2$ (for scalar particles, the wave function is a section of the associated vector bundle $\xi_C : C \rightarrow \mathbb{R}^2 \times C \rightarrow \mathbb{R}^2$), where $\mathbb{R}^2 \times U(1)$, the total space of $\xi_{A-B}$, is homeomorphic to an open solid 2-torus minus a circle. The moduli space of flat connections, that is, the set of gauge equivalence classes of flat connections $\mathcal{M}_0 = C_0/G$, where $C_0$ is the set of flat connections and $G$ is the gauge group of the bundle, was shown to be isomorphic to the circle $S^1$; finally, the holonomy groups of these connections in terms of $\rho \in [0, 1)$ were shown to be either the cyclic groups $H(\rho) = \mathbb{Z}_q$, for $\rho = p/q \in Q$, or the integers $\mathbb{Z}$, the infinite cyclic group, for $\rho \notin Q$.

These geometrical properties of the A-B effect are independent of the ideal case considered here, namely that of an infinitesimally thin solenoid carrying the magnetic flux $\Phi$. In a real situation, the base space of the bundle is the plane minus a disk, which is topologically equivalent to $\mathbb{R}^2$.

The result about $\mathcal{M}_0$ in Ref. 1, was obtained as a corollary of a general result which is valid for product bundles over any manifold $M$. This result shows that the A-B effect is caused by the non trivial topology of $M$. Here we find $\mathcal{M}_0$ i.e., the moduli space for the case $M = \mathbb{R}^2$ in a simpler way. The result coincides with the previous derivation, but the explicit inclusion of the coupling constant i.e. the electric charge $e$, leads to the result that the length of the circle is $|e|$. In section 2 we describe the gauge group $G$, and in section 3 we rederive $\mathcal{M}_0$. Section 4 is a remark on the relation of this length with other fundamental lengths.

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2. The gauge group

Since the A-B bundle $\xi_{A-B} : U(1) \rightarrow R^{2*} \times U(1) \xrightarrow{\pi} R^{2*}$ is trivial, then its gauge group $\mathcal{G}$ (see e.g. Refs. 1 or 3) is given by $\mathcal{G} = C^\infty(M,G)$ where $G$ is the fiber and $M$ is the base space; in our case, $G = U(1)$ and $M = R^{2*}$, then

$$\mathcal{G} = C^\infty(R^{2*}, U(1)).$$  \hspace{1cm} (1)

Since differentiable functions are continuous, then $\mathcal{G} \subset C^0(R^{2*}, U(1))$, and therefore the elements of $\mathcal{G}$ fall into different homotopy classes:

$$[R^{2*}, U(1)] = \{\text{homotopy classes of maps } R^{2*} \rightarrow U(1)\} \cong [S^1, S^1] \cong \pi_1(S^1) \cong Z.$$ \hspace{1cm} (2)

So, if $f \in \mathcal{G}$, then there exists a unique $n \in Z$ such that $f$ is homotopic to $f_n$ ($f \sim f_n$), where $f_n(re^{i\phi}) = e^{in\phi}$, and $\phi \in [0, 2\pi)$, in other words, $f$ is an element of the homotopy class of $f_n$ ($f \in [f_n]$). This means there exists a differentiable map $h : R^{2*} \times [0,1] \rightarrow U(1)$, such that $h((re^{i\phi}), 0) = f(re^{i\phi})$ and $h((re^{i\phi}), 1) = e^{in\phi}$. In fact, in Ref. 1 it is shown that the group of smooth homotopy classes of smooth maps from $R^{2*}$ to $U(1)$ is isomorphic to $Z$.

3. Flat connections

By Ref. 1, the space of flat connections on $\xi_{A-B}$ is given by the set

$$\mathcal{C}_0 = \{A \in \Omega^1(R^{2*}; u(1)), \ dA = 0\}$$ \hspace{1cm} (3)

where $u(1) = iR$ is the Lie algebra of $U(1)$ and $d$ is the exterior derivative operator on $R^{2*}$. And the action of $\mathcal{G}$ on $\mathcal{C}_0$ is given by $A \cdot f = A + f^{-1}df$, where $f^{-1}(x, y) = f(x, y)^{-1} \in U(1)$.

We shall prove the following result.

Theorem.

There is a bijection between $\mathcal{M}_0 = \mathcal{C}_0 / \mathcal{G} = \{\text{gauge equivalence classes of flat connections on } \xi_{A-B}\}$ and $S^1$, with $length(S^1) = |e|$.

Proof. The 1-form in $R^{2*}$ which induces the A-B effect is given by

$$a_0 = \frac{\Phi_0 }{2\pi^2} \frac{xdy - ydx}{x^2 + y^2}$$ \hspace{1cm} (4)

where $\Phi_0$ is the magnetic flux associated with the charge $|e| : \frac{\Phi_0 }{2\pi^2} = \frac{hc}{|e|}$, and is such that for an arbitrary flux $\Phi$ in the solenoid, $\Phi = \lambda \Phi_0$, with $\lambda \in R$; it is useful to express $\Phi_0$ in terms of the fine structure constant $\alpha$ and in the natural system of units: $\frac{\alpha^2}{e^2hc} = \alpha$, so $|e| = \sqrt{4\pi\alpha}$ and then $\frac{\Phi_0 }{2\pi^2} = \frac{\Phi_0 }{2\pi^2} \cong \sqrt{\frac{137}{4\pi}}$. So, $a_0 = \frac{1}{\sqrt{4\pi\alpha} \sqrt{\frac{137}{4\pi}}} \frac{xdy - ydx}{x^2 + y^2}$ and therefore

$$A_0 = ia_0 = \frac{i}{\sqrt{4\pi\alpha}} \frac{xdy - ydx}{x^2 + y^2} \in \mathcal{C}_0.$$ \hspace{1cm} (5)

Though closed, $A_0$ is not exact since only locally, i.e. for $\phi \in (0, 2\pi)$, $\frac{xdy - ydx}{x^2 + y^2} = d\phi$.

In particular, $A_0$ generates the De Rahm cohomology (with coefficients in $iR$) of $R^{2*}$ in dimension 1

$$H^1_{DR}(R^{2*}; iR) \cong H^1_{DR}(S^1; iR) = \{\lambda [A_0]_{DR}\}_{\lambda \in R} \cong R,$$ \hspace{1cm} (6)
where \([A_0]_{DR} = \{A_0 + d\beta, \beta \in \Omega^0(R^2; iR)\}\). Though \(\beta\) does not generate the most general gauge transformation of \(A_0\), it gives, however, the gauge transformation defined by the composite \(\exp \circ \beta : R^{2*} \to U(1)\),

In general, a gauge transformed of \(A_0\) is of the form \(A'_0 = A_0 + f^{-1}df\) with \(f \in \mathcal{G}\). Therefore, the gauge class of \(A_0\) is

\[
[A_0] = \{A_0 + f^{-1}df\}_{f \in \mathcal{G}}.
\] (7)

In order to calculate the quotient \(C_0/\mathcal{G}\), consider the homomorphism \(\exp : C^\infty(R^{2*}, iR) \to \mathcal{G} = C^\infty(R^{2*}, U(1))\), given by \(\exp(\beta) = \exp \circ \beta\).

It is easy to see that \(C_0/\mathcal{G} \cong (C_0/\text{Im}(-)\mathcal{G})/\mathcal{G}\), where the action of \(\text{Im}(\exp\#)\) on \(C_0\) is the action as a subgroup of \(\mathcal{G}\), i.e., \(A \cdot \exp(\beta) = A + \exp(\beta^{-1}d\exp(\beta))\). Since \(\exp(\beta)^{-1}d\exp(\beta) = (\exp \circ \beta)^{-1}(\exp \circ \beta)d\beta\), then \(A \cdot \exp(\beta) = A + d\beta\). Therefore \(C_0/\text{Im}(\exp\#) = H^1(R^{2*}; iR) = \{\lambda[A_0]_{DR}\}_{\lambda \in R}\). The restriction imposed by the action of the full group \(\mathcal{G}\) on the parameter \(\lambda\) is obtained as follows.

Let \((\lambda + \sigma)A_0 \in [\lambda A_0]\), then there exists \(f \in \mathcal{G}\) (\(f\) depends on \(\sigma\)) such that \((\lambda + \sigma)A_0 = \lambda A_0 + f^{-1}df\) and therefore \(f^{-1}df = \sigma A_0\) i.e.

\[
\frac{1}{f} \frac{\partial}{\partial x} f = \frac{\partial}{\partial x} \ln f(x, y) = -\frac{i\sigma}{\sqrt{4\pi\alpha}} \frac{y}{x^2 + y^2},
\] (8)

\[
\frac{1}{f} \frac{\partial}{\partial y} f = \frac{\partial}{\partial y} \ln f(x, y) = \frac{\sigma}{\sqrt{4\pi\alpha}} \frac{x}{x^2 + y^2}.
\] (9)

Since \(\int \frac{dx}{x^2 + y^2} = \frac{i}{2\pi} \ln(x + iy) + \text{const.}\), \(\int \frac{dy}{x^2 + y^2} = \frac{i}{2\pi} \ln(y + iy) + \text{const.}\), and \(\frac{dy}{x + iy} = -\frac{x + iy}{x - iy}\), we obtain

\[
\ln f(x, y) = \frac{\sigma}{\sqrt{16\pi\alpha}} \ln\left(\frac{x + iy}{x - iy}\right) + c_1 = -\frac{\sigma}{\sqrt{16\pi\alpha}} \ln\left(\frac{y + ix}{y - ix}\right) + c_2;
\] (10)

where \(c_1\) and \(c_2\) are constants; if \(z = x + iy\) then \(\tilde{z} = x - iy\) and \(\frac{z}{\tilde{z}} = e^{2i\arg(z)} = e^{2i\phi}\), then

\[
\ln f(x, y) = \frac{\sigma i\phi}{\sqrt{4\pi\alpha}} + c_1 = \frac{\sigma}{\sqrt{16\pi\alpha}} (2i\phi + i\pi) + c_2 = \frac{i\sigma}{\sqrt{4\pi\alpha}} + \frac{i\pi\sigma}{\sqrt{16\pi\alpha}} + c_2;
\] (11)

let \(\sigma = n\sqrt{4\pi\alpha} (= n|\epsilon|)\) with \(n \in Z\), then \(\ln f(x, y) = \ln f + c_1 = \ln f + \frac{i\pi n}{4} + c_2\) i.e. \(f(x, y) = f_n(r e^{i\phi}) = K_1 e^{i\phi} = K_2 e^{i\frac{\pi n}{4}} e^{i\phi}\). Choosing \(K_2 = e^{-i\frac{\pi n}{4}}\) implies \(K_1 = 1\), and we have the solutions

\[
f_n(r e^{i\phi}) = e^{i\phi},
\] (12)

with \(f_n(e^{i0}) = f_n(e^{i2\pi}) = 1\). In particular, for \(n = 1\), we obtain

\[
[\lambda A_0] = [\lambda + \sqrt{4\pi\alpha} A_0],
\] (13)

which, as far as the classification of equivalence classes of connections and the calculation of holonomy groups is concerned, restricts the possible values of \(\lambda\) to an interval of length \(\sqrt{4\pi\alpha}\) which, without loss of generality, can be chosen to be \([0, \sqrt{4\pi\alpha}] \cong [0, \sqrt{\frac{3}{2}}]\) with \(\sqrt{4\pi\alpha}\) identified with 0, which corresponds to the trivial connection i.e. the electromagnetic vacuum. Then, one obtains

\[
\left(\begin{array}{c}
\text{gauge equivalence classes} \\
\text{of flat connections on } \xi_{A-B}
\end{array}\right) \cong \frac{\{[\lambda A_0]\}_{\lambda \in [0, \sqrt{4\pi\alpha}]}}{0 \sim \sqrt{4\pi\alpha}} \cong [0, \sqrt{4\pi\alpha}] \cong S^1.
\] (14)
4. Final remark

In terms of the electric charge, \( \frac{0,|e|}{0,|e|} \cong S^1 \). The “small” value of \( \alpha (\alpha \cong 1_{137.04}) \) reduces the pure geometrical upper limit 1 of the interval of \( \lambda^1 \), since \( \sqrt{4\pi\alpha} \cong 3028 < 1 \); then \( \text{length}(S^1) = |e| \) (approximately \( 5.5 \times 10^{-9} (\text{erg} \times \text{cm})^{1/2} \) in the c.g.s. system of units). It is interesting that this “length” can be related to the Kaluza-Klein \( 4 \) length \( l_{KK} \) for 5 dimensional gravity with the 5th dimension compactified in a circle giving the Maxwell field, and to the Planck length \( l_P = \sqrt{G_N} \), through \( l_{KK} = \frac{2\pi|e|}{\sqrt{G_N}} \).

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