Homomorphisms are indeed a good basis for counting:
Three fixed-template dichotomy theorems, for the price of one

Hubie Chen
Departamento LSI
Facultad de Informática
Universidad del País Vasco
San Sebastián, Spain

and
IKERBASQUE, Basque Foundation for Science
Bilbao, Spain

Abstract

Many natural combinatorial quantities can be expressed by counting the number of homomorphisms to a fixed relational structure. For example, the number of 3-colorings of an undirected graph $G$ is equal to the number of homomorphisms from $G$ to the 3-clique. In this setup, the structure receiving the homomorphisms is often referred to as a template; we use the term template function to refer to a function, from structures to natural numbers, that is definable as the number of homomorphisms to a fixed template. There is a literature that studies the complexity of template functions.

The present work is concerned with relating template functions to the problems of counting, with respect to various fixed templates, the number of two particular types of homomorphisms: surjective homomorphisms and what we term condensations. A surjective homomorphism is a homomorphism that maps the universe of the first structure surjectively onto the universe of the second structure; a condensation is a homomorphism that, in addition, maps each relation of the first structure surjectively onto the corresponding relation of the second structure.

In this article, we explain how any problem of counting surjective homomorphisms to a fixed template is polynomial-time equivalent to computing a linear combination of template functions; we also show this for any problem of counting condensations to a fixed template. Via a theorem that characterizes the complexity of computing such a linear combination, we show how a known dichotomy for template functions can be used to infer a dichotomy for counting surjective homomorphisms on fixed templates, and likewise a dichotomy for counting condensations on fixed templates. Our study is strongly inspired by, based on, and can be viewed as a dual of the graph motif framework of Curticapean, Dell, and Marx (STOC 2017); that framework is in turn based on work of Lovász (2012).
1 Preliminaries

When $f : A \rightarrow B$ is a map and $A' \subseteq A$, we use $f(A')$ to denote the set $\{f(a) \mid a \in A'\}$.

1.1 Structures, homomorphisms, and company

A signature is a set of relation symbols; each relation symbol $R$ has an associated arity (a natural number), denoted by $\text{ar}(R)$. A structure $B$ over signature $\sigma$ consists of a universe $B$ which is a set, and an interpretation $B^R \subseteq B^{\text{ar}(R)}$ for each relation symbol $R \in \sigma$. We use $|B|$ to denote the total size of $B$, defined as $|B| + \sum_{R \in \sigma} |B^R|$. We will in general use the symbols $A, B, \ldots$ to denote structures, and the symbols $A, B, \ldots$ to denote their respective universes. In this article, we assume that signatures under discussion are finite, and assume that all structures under discussion are finite; a structure is finite if its universe is finite.

Let $B$ be a structure over signature $\sigma$. When $B' \subseteq B$, we define $B[B']$ as the structure with universe $B'$ and where $B^R[B'] = B^R \cap B'^{\text{ar}(R)}$. We define an induced substructure of $B$ to be a structure of the form $B[B']$, where $B' \subseteq B$. Observe that a structure $A$ has $2^{|A|}$ induced substructures. We define a deduct of $B$ to be a structure obtained from $B$ by removing tuples from relations of $B$, that is, a structure $C$ (over signature $\sigma$) is a deduct of $B$ if $C = B$ and, for each $R \in \sigma$, it holds that $R^C \subseteq R^B$.

Let $A$ and $B$ be structures over the same signature $\sigma$. A homomorphism from $A$ to $B$ is a map $h : A \rightarrow B$ such that for each relation symbol $R \in \sigma$, it holds that $h(R^A) \subseteq R^B$. A surjective homomorphism from $A$ to $B$ is a homomorphism such that $h(A) = B$, that is, such that $h$ is surjective as a mapping from the set $A$ to the set $B$. A condensation from $A$ to $B$ is a surjective homomorphism satisfying the condition that for each relation symbol $R \in \sigma$, it holds that $h(R^A) = R^B$. This condition is sometimes referred to as edge-surjectivity in graph-theoretic contexts. Notions similar to the notion of condensation have been studied in the literature: notably, the term compaction is sometimes used (for example, in [9]) to refer to a homomorphism between graphs that maps the edge relation of the first graph surjectively onto the relation that contains the non-loop edges of the second graph.

Two structures $B, B'$ are homomorphically equivalent if there exists a homomorphism from $B$ to $B'$ and there exists a homomorphism from $B'$ to $B$.

Throughout, we tacitly use the fact that the composition of a homomorphism from $A$ to $B$ and a homomorphism from $B$ to $C$ is a homomorphism from $A$ to $C$.

1.2 Computational problems

We now define the computational problems to be studied. For each structure $B$ over signature $\sigma$:

- Define $\#\text{HOM}(B)$ to be the problem of computing, given a structure $A$ over signature $\sigma$, the number of homomorphisms from $A$ to $B$.
- Define $\#\text{SURJHOM}(B)$ to be the problem of computing, given a structure $A$ over signature $\sigma$, the number of surjective homomorphisms from $A$ to $B$.
- Define $\#\text{CONDENS}(B)$ to be the problem of computing, given a structure $A$ over signature $\sigma$, the number of condensations from $A$ to $B$.

\[\text{We remark that some authors use the term } \text{surjective homomorphism} \text{ to refer to what we refer to as a condensation.}\]
2 Linear combinations of homomorphisms

Our development is strongly inspired by and based on the framework of Curticapean, Dell, and Marx [7], which in turn was based on work of Lovász [12, 11]. It is also informed by the theory developed by the current author with Mengel [3, 4, 5, 6]. In these works, a dual setup is considered, where one fixes the structure $A$ from which homomorphisms originate, and counts the number of homomorphisms that an input structure receives from $A$. Many of our observations and results can be seen to have duals in the cited works.

For each signature $\sigma$, let $\text{STR}[\sigma]$ denote the class of all structures over $\sigma$, and fix $\text{STR}^*[\sigma]$ to be a subclass of $\text{STR}[\sigma]$ that contains exactly one structure from each isomorphism class of structures contained in $\text{STR}[\sigma]$.

For structures $A$, $B$ over the same signature, we use:

- $\text{Hom}(A, B)$ to denote the number of homomorphisms from $A$ to $B$,
- $\text{Surjhom}(A, B)$ to denote the number of surjective homomorphisms from $A$ to $B$,
- $\text{Condens}(A, B)$ to denote the number of condensations from $A$ to $B$,
- $\text{Indsub}(B', B)$ to denote the number of induced substructures of $B$ that are isomorphic to $B'$, and
- $\text{Deducts}(B', B)$ to denote the number of deducts of $B$ that are isomorphic to $B'$.

We use $\text{Hom}(\cdot, B)$ to denote the mapping that sends a structure $A$ to $\text{Hom}(A, B)$, and use $\text{Surjhom}(\cdot, B)$, etc. analogously.

Observe that

$$\text{Hom}(A, B) = \sum_{B' \in \text{STR}^*[\sigma]} \text{Surjhom}(A, B') \cdot \text{Indsub}(B', B).$$

(1)

We briefly justify this as follows. Each homomorphism $h$ from $A$ to $B$ is a surjective homomorphism from $A$ onto an induced substructure of $B$, namely, onto $B[h(A)]$. Let $B' \in \text{STR}^*[\sigma]$ be isomorphic to an induced substructure of $B$, and let us count the number of homomorphisms $h$ from $A$ to $B$ such that $B[h(A)]$ is isomorphic to $B'$. Let $B_1, \ldots, B_k$ be a list of all induced substructures of $B$ that are isomorphic to $B'$. Then, we have $\text{Surjhom}(A, B_1) = \cdots = \text{Surjhom}(A, B_k) = \text{Surjhom}(A, B')$ and $k = \text{Indsub}(B', B)$, so the desired number is $\text{Surjhom}(A, B_1) + \cdots + \text{Surjhom}(A, B_k)$, which is equal to $\text{Surjhom}(A, B') \cdot \text{Indsub}(B', B)$.

Observe that

$$\text{Surjhom}(A, B) = \sum_{B' \in \text{STR}^*[\sigma]} \text{Condens}(A, B') \cdot \text{Deducts}(B', B).$$

(2)

The justification for this equation has the same flavor as that of the previous equation. Each surjective homomorphism $h$ from $A$ to $B$ is a condensation from $A$ to a deduct of $B$; when $B'$ is isomorphic to a deduct of $B$, the product $\text{Condens}(A, B') \cdot \text{Deducts}(B', B)$ is the number of condensations from $A$ to a deduct of $B$ that is isomorphic to $B'$.

It is direct from Equation (1) that

$$\text{Surjhom}(A, B) = \text{Hom}(A, B) - \sum_{B' \in \text{STR}^*[\sigma], |B'| < |B|} \text{Surjhom}(A, B') \cdot \text{Indsub}(B', B).$$

(3)

From this, one can straightforwardly verify by induction on $|B|$ that the function $\text{Surjhom}(\cdot, B)$ can be expressed as a linear combination of functions each having the form $\text{Hom}(\cdot, C)$; moreover, such a linear combination is computable from $B$. We formalize this as follows.
Proposition 2.1 There exists an algorithm that, given as input a structure \( B \) over signature \( \sigma \), outputs a list \((\beta_1, B_1), \ldots, (\beta_k, B_k) \in \mathbb{Q} \times \text{STR}[\sigma]\), where the values \( \beta_i \) are non-zero and the structures \( B_i \) are pairwise non-isomorphic and such that, for all structures \( A \), it holds that

\[
\text{Surjhom}(A, B) = \beta_1 \cdot \text{Hom}(A, B_1) + \cdots + \beta_k \cdot \text{Hom}(A, B_k).
\]

In an analogous fashion, it is direct from Equation 2 that

\[
\text{Condens}(A, B) = \text{Surjhom}(A, B) - \sum_{B' \in \text{STR}^*[\sigma], \|B'\| = \|B\|, \|B'\| < \|B\|} \text{Condens}(A, B') \cdot \text{Deducts}(B', B).
\]

(4)

Remark 2.3 We can write Equation 1 in the following form:

\[
\text{Hom}(A, B) = \sum_{B'} \text{Surjhom}(A, B'),
\]

where the sum is over all induced substructures \( B' \) of \( B \); analogously, we can write Equation 2 in the following form:

\[
\text{Surjhom}(A, B) = \sum_{B'} \text{Condens}(A, B'),
\]

where the sum is over all deducts \( B' \) of \( B \). From these forms, one can use Möbius inversion on posets to express \( \text{Surjhom}(\cdot, B) \) as a linear combination of functions \( \text{Hom}(\cdot, B) \); and likewise to express \( \text{Condens}(\cdot, B) \) as a linear combination of functions \( \text{Surjhom}(\cdot, C) \), which linear combination can then be expressed as a linear combination of functions \( \text{Hom}(\cdot, B) \).

Remark 2.4 Equations 1 and 2 can be conceived of as matrix identities. Let \( \text{Hom}^* \) denote the restriction of \( \text{Hom} \) to pairs in \( \text{STR}^*[\sigma] \times \text{STR}^*[\sigma] \), and view it as an infinite matrix whose indices are such pairs and having entries in \( \mathbb{Q} \); define and view \( \text{Surjhom}^* \), etc. analogously. Then Equation 1 in matrix notation, is expressed by

\[
\text{Hom}^* = \text{Surjhom}^* \cdot \text{Indsub}^*.
\]

Analogously, Equation 2 in matrix notation, is expressed by

\[
\text{Surjhom}^* = \text{Condens}^* \cdot \text{Deducts}^*.
\]

Suppose that, for the indexing, the structures in \( \text{STR}^*[\sigma] \) are ordered in a way that respects total size, that is, whenever \( B \) comes before \( B' \), it holds that \( \|B\| \leq \|B'\| \). Then, the matrices \( \text{Indsub}^* \) and \( \text{Deducts}^* \) are readily seen to be upper triangular and to have all diagonal entries equal to 1; it can be verified that they are invertible.
3 The space of template parameters

We now study the space of linear combinations of functions $\text{Hom}(\cdot, B)$. Fix $\sigma$ to be a signature. Define a \textit{template function} to be a function $f : \text{STR}[\sigma] \to \mathbb{Q}$ such that there exists a structure $B \in \text{STR}[\sigma]$ where, for each $A \in \text{STR}[\sigma]$, it holds that $f(A) = \text{Hom}(A, B)$. Define a \textit{template parameter} to be a function $f : \text{STR}^{*}[\sigma] \to \mathbb{Q}$ that can be expressed as a finite linear combination of template functions. Template parameters naturally form a vector space, and this space is clearly spanned by the template functions. We prove that the template functions $(\text{Hom}(\cdot, B))_{B \in \text{STR}^{*}[\sigma]}$ are linearly independent, and hence form a basis for this vector space.

\textbf{Theorem 3.1} \textit{Let $(\beta_1, B_1), \ldots, (\beta_n, B_n) \in \mathbb{Q} \times \text{STR}^{*}[\sigma]$ be such that the $B_i$ are pairwise distinct. Suppose that, for all structures $A \in \text{STR}[\sigma]$, it holds that $\sum_{i=1}^{n} \beta_i \text{Hom}(A, B_i) = 0$. Then $\beta_1 = \cdots = \beta_n = 0$.}

We first establish a lemma.

\textbf{Lemma 3.2} \textit{Suppose that $B_1, \ldots, B_k \in \text{STR}^{*}[\sigma]$ are pairwise distinct, but all homomorphically equivalent. Then, there exists a structure $A_k$ such that the values $(\text{Hom}(A_k, B_i))_{i=1, \ldots, k}$ are non-zero and pairwise distinct.}

For two structures $A_1, A_2$, we use $A_1 + A_2$ to denote their disjoint union; and, for $N \geq 0$, we use $N A_1$ to denote the $N$-fold disjoint union of $A_1$ with itself. The identity $\text{Hom}(A_1 + A_2, B) = \text{Hom}(A_1, B) \cdot \text{Hom}(A_2, B)$ is known and straightforwardly verified.

\textbf{Proof.} We prove this by induction. In the case that $k = 1$, one can simply take $A_1 = B_1$.

Suppose that $k > 1$. By induction, there exists $A_{k-1}$ such that $(\text{Hom}(A_{k-1}, B_i))_{i=1, \ldots, k-1}$ are non-zero and pairwise distinct. Let us assume for the sake of notation that $\text{Hom}(A_{k-1}, B_1) < \cdots < \text{Hom}(A_{k-1}, B_{k-1})$. Since the structures $B_i$ are homomorphically equivalent, we have $\text{Hom}(A_{k-1}, B_k) > 0$. If $\text{Hom}(A_{k-1}, B_k)$ is distinct from each of the values $(\text{Hom}(A_{k-1}, B_i))_{i=1, \ldots, k-1}$, we are done. Otherwise, there exists a unique index $\ell \in \{1, \ldots, k-1\}$ such that $\text{Hom}(A_{k-1}, B_k) = \text{Hom}(A_{k-1}, B_\ell)$. By Lovasz’s theorem [12], there exists a structure $A'$ such that $\text{Hom}(A', B_k) \neq \text{Hom}(A', B_\ell)$; observe that since $B_k$ and $B_\ell$ are homomorphically equivalent, both of these values are non-zero; indeed, all of the values $(\text{Hom}(A', B_i))_{i=1, \ldots, k}$ are non-zero.

We claim that for all sufficiently large values $M$, the structure $A_k = MA_{k-1} + A'$ has the desired property that the values $(\text{Hom}(A_k, B_i))_{i=1, \ldots, k}$ are non-zero and pairwise distinct. This is indeed straightforward to verify. We have $\text{Hom}(MA_{k-1} + A', B_k) = \text{Hom}(A_{k-1}, B_k)^M \cdot \text{Hom}(A', B_k) > 0$, and since the structures $B_i$ are homomorphically equivalent, we obtain that the values $(\text{Hom}(A_k, B_i))_{i=1, \ldots, k}$ are non-zero. Let us now argue pairwise distinctness. When $j$ is such that $1 < j < i$, for sufficiently large values of $M$, it will hold that $\frac{\text{Hom}(A_k, B_j)}{\text{Hom}(A_k, B_{j+1})} < \frac{\text{Hom}(A_k, B_\ell)}{\text{Hom}(A_k, B_{\ell+1})} = M$, from which it follows that $\text{Hom}(MA_{k-1} + A', B_j) < \text{Hom}(MA_{k-1} + A', B_{j+1})$. In a similar way, one sees that when $j \in \{1, \ldots, k-1\} \setminus \{\ell\}$, for sufficiently large values of $M$, it holds that $\text{Hom}(MA_{k-1} + A', B_j) \neq \text{Hom}(MA_{k-1} + A', B_k)$. Finally, we have for all $M \geq 1$ that $\text{Hom}(MA_{k-1} + A', B_\ell) \neq \text{Hom}(MA_{k-1} + A', B_k)$, as a consequence of $\text{Hom}(A_{k-1}, B_k) = \text{Hom}(A_{k-1}, B_\ell)$ and $\text{Hom}(A', B_k) \neq \text{Hom}(A', B_\ell)$. \hfill \Box

\textbf{Proof.} (Theorem 3.1) We prove this by induction on $n$. It is clear for $n = 1$, so suppose that $n > 1$.

We assume for the sake of notation that $B_1$ is extremal in that for each other structure $B_j$, either $B_1$ is homomorphically equivalent to $B_j$ or does not admit a homomorphism to $B_j$. We assume further that $B_1, \ldots, B_m$ is a list of the structures among $B_1, \ldots, B_n$ that are homomorphically equivalent to $B_1$. 


Applying Lemma \ref{lem:structure_reduction} to \(B_1, \ldots, B_m\), we obtain a structure \(A\) such that the values \((\text{Hom}(A, B_i))_{i=1,\ldots,m}\) are pairwise distinct. Consider the structures \((A_k)_{k=0,\ldots,m-1}\) defined by \(A_k = kA + B_1\). For each \(k \in \{0, \ldots, m-1\}\), we have \(\sum_{i=1}^n \beta_i \text{Hom}(A_k, B_i) = 0\), which implies \(\sum_{i=1}^n \beta_i \text{Hom}(B_1, B_i) \text{Hom}(A, B_i)^k = 0\). Now, form a system of equations by taking this last equation over \(k \in \{0, \ldots, m-1\}\); view it as a system of equations over unknowns \(y_i = \beta_i \text{Hom}(B_1, B_i)\), where \(i\) ranges from 1 to \(m\). The corresponding matrix is a Vandermonde matrix, implying that \(y_1 = \cdots = y_m = 0\). Since the values \((\text{Hom}(B_1, B_i))_{i=1,\ldots,m}\) are all non-zero, we infer that \(\beta_1 = \cdots = \beta_m = 0\). By applying induction, we obtain that \(\beta_{m+1} = \cdots = \beta_n = 0\). \(\square\)

4 The complexity of template parameters

We now study the complexity of computing template parameters, showing in essence that computing a template parameter has the same complexity as being able to compute all of its constituent functions \(\text{Hom}(\cdot, B)\).

**Theorem 4.1** Let \((\beta_1, B_1), \ldots, (\beta_n, B_n) \in \mathbb{Q} \times \text{STR}^*[\sigma]\) be such that the values \(\beta_i\) are non-zero and such that the structures \(B_i\) are pairwise non-isomorphic.

- Let \(f : \text{STR}[\sigma] \to \mathbb{Q}\) be the function defined by \(f(A) = \sum_{i=1}^n \beta_i \cdot \text{Hom}(A, B_i)\).

- Let \(g : \{1, \ldots, n\} \times \text{STR}[\sigma] \to \mathbb{Q}\) be the function defined by \(g(i, A) = \text{Hom}(A, B_i)\).

The functions \(f\) and \(g\) are equivalent under polynomial-time Turing reduction.

For functions \(h, h'\), we use \(h \leq_T^p h'\) to indicate that \(h\) polynomial-time Turing reduces to \(h'\).

**Proof.** It is clear that \(f \leq_T^p g\), so we prove that \(g \leq_T^p f\), by induction on \(n\); the result is clear for \(n = 1\). By rearranging indices if necessary, let us assume that the structures \(B_1, \ldots, B_m\) are as described in the second paragraph of the proof of Theorem \ref{thm:structure_reduction}. Let \(g_1\) be the restriction of \(g\) to \(\{1, \ldots, m\} \times \text{STR}[\sigma]\), and let \(g_2\) be the restriction of \(g\) to \(\{m+1, \ldots, n\} \times \text{STR}[\sigma]\). Let \(f_2 : \text{STR}[\sigma] \to \mathbb{Q}\) be the function defined by \(f_2(A) = \sum_{i=1}^n \beta_i \cdot \text{Hom}(A, B_i)\).

Let us show \(g_1 \leq_T^p f_1\). By applying Lemma \ref{lem:structure_reduction} to \(B_1, \ldots, B_m\), we obtain a structure \(A'\) such that the values \((\text{Hom}(A', B_i))_{i=1,\ldots,m}\) are pairwise distinct. Given a pair \((j, A)\) as input, the reduction constructs the structures \((A_k)_{k=0,\ldots,m-1}\) defined by \(A_k = B_1 + A + kA'\), and then computes the various values \(f(A_k)\). We have, for each \(k\), \(\sum_{i=1}^n \beta_i \text{Hom}(A_k, B_i) = f(A_k)\); from this, we obtain \(\sum_{i=1}^n \beta_i \text{Hom}(B_1, B_i) \text{Hom}(A', B_i)^k = f(A_k)\). Viewing this as a system of equations over unknowns \(y_i = \beta_i \text{Hom}(B_1, B_i)\), the corresponding matrix is Vandermonde. Hence, we may solve for these unknowns \(y_i\), and then from their solution compute the values \(\text{Hom}(A, B_i)\). We then output the desired value \(\text{Hom}(A, B_j)\).

We now argue that \(f_2 \leq_T^p f\). Given a structure \(A\) as input, the reduction first computes \(f(A)\). Since we just showed that \(g_1 \leq_T^p f\), the reduction may also compute the values \(\text{Hom}(A, B_1), \ldots, \text{Hom}(A, B_m)\). By subtracting \(\beta_1 \text{Hom}(A, B_1) + \cdots + \beta_m \text{Hom}(A, B_m)\) from \(f(A)\), the desired value \(f_2(A)\) is computed.

We obtain \(g_2 \leq_T^p f_2\) by induction; it follows that \(g_2 \leq_T^p f\).

As we established that \(g_1 \leq_T^p f\) and \(g_2 \leq_T^p f\), it is immediate that \(g \leq_T^p f\). \(\square\)

5 Complexity results

Previous work established a complexity dichotomy for the family of problems \#HOM(B). Let FP denote the functional version of polynomial time. A criterion was presented that distinguishes the structures \(B\) for
Theorem 5.1 \cite{[8]} Let $B$ be any structure. If $B$ satisfies the $\#\text{HOM}(\cdot)$-tractability condition, then the problem $\#\text{HOM}(B)$ is in $\text{FP}$; otherwise, it is $\text{NP}$-complete under polynomial-time Turing reducibility.

The following was also established.

Theorem 5.2 \cite{[8]} The $\#\text{HOM}(\cdot)$-tractability condition is decidable.

Define the $\#\text{SURJHOM}(\cdot)$-tractability condition to be satisfied by a structure $B$ iff the algorithm of Proposition 2.1 returns a list $(\beta_1, B_1), \ldots, (\beta_k, B_k)$ such that each structure $B_i$ satisfies the $\#\text{HOM}(\cdot)$-tractability condition. (We remark here that all algorithms behaving as described in Proposition 2.1 will output the same list, up to permutation, due to Theorem 5.1.) We obtain the following.

Theorem 5.3 Let $B$ be any structure. If $B$ satisfies the $\#\text{SURJHOM}(\cdot)$-tractability condition, then the problem $\#\text{SURJHOM}(B)$ is in $\text{FP}$; otherwise, it is $\text{NP}$-complete under polynomial-time Turing reducibility. Moreover, the $\#\text{SURJHOM}(\cdot)$-tractability condition is decidable.

Proof. Let $(\beta_1, B_1), \ldots, (\beta_k, B_k)$ be the list obtained by invoking the algorithm of Proposition 2.1 on $B$.

Suppose $B$ satisfies the $\#\text{SURJHOM}(\cdot)$-tractability condition. Let us argue that $\#\text{SURJHOM}(B)$ is in $\text{FP}$. The algorithm is given a structure $A$ as input. By assumption, each $B_i$ satisfies the $\#\text{HOM}(\cdot)$-tractability condition, and so each of the values $\text{Hom}(A, B_i)$ can be computed in polynomial time. The algorithm outputs the sum $\beta_1 \cdot \text{Hom}(A, B_1) + \cdots + \beta_k \cdot \text{Hom}(A, B_k)$.

Suppose that $B$ does not satisfy the $\#\text{SURJHOM}(\cdot)$-tractability condition. There exists an index $\ell$ such that $B_\ell$ does not satisfy the $\#\text{HOM}(\cdot)$-tractability condition, so $\#\text{HOM}(B_\ell)$ is $\text{NP}$-complete by Theorem 5.1. Let $f$ and $g$ be the functions described in the statement of Theorem 4.1. Clearly, $\#\text{HOM}(B_\ell) \leq_p g$. Since $g \leq_p f$ by Theorem 4.1, we obtain that $f$ is $\text{NP}$-complete, as desired.

Decidability of the $\#\text{SURJHOM}(\cdot)$-tractability condition is immediate from its definition and Theorem 5.2. 

Define the $\#\text{CONDENS}(\cdot)$-tractability condition to be satisfied by a structure $B$ iff the algorithm of Proposition 2.2 returns a list $(\beta_1, B_1), \ldots, (\beta_k, B_k)$ such that each structure $B_i$ satisfies the $\#\text{HOM}(\cdot)$-tractability condition. We have the following; the proof is analogous to that of Theorem 5.3.

Theorem 5.4 Let $B$ be any structure. If $B$ satisfies the $\#\text{CONDENS}(\cdot)$-tractability condition, then the problem $\#\text{CONDENS}(B)$ is in $\text{FP}$; otherwise, it is $\text{NP}$-complete under polynomial-time Turing reducibility. Moreover, the $\#\text{CONDENS}(\cdot)$-tractability condition is decidable.

We would like to present further consequences of our theory. From Equation 3 it can be elementarily verified that, for any structure $B$, the expression of $\text{Surjhom}(\cdot, B)$ as a linear combination of functions $\text{Hom}(\cdot, B')$ gives a coefficient of 1 to $\text{Hom}(\cdot, B)$. The same fact holds for $\text{Condens}(\cdot, B)$ in place of $\text{Surjhom}(\cdot, B)$, as can be elementarily seen from Equations 4 and 5 (That $\text{Hom}(\cdot, B)$ receives a coefficient of 1 is these expressions also immediate from Möbius inversion.) We thus obtain the following, via Theorem 4.1.

Corollary 5.5 For each structure $B$, the problem $\#\text{HOM}(B)$ reduces to $\#\text{SURJHOM}(B)$.
Corollary 5.6  For each structure $B$, the problem $\#\text{HOM}(B)$ reduces to $\#\text{CONDENS}(B)$.

In the setting of graphs, results similar to these two corollaries were obtained by Focke, Goldberg, and Zivny \[9\]. We would like to emphasize that here, these two corollaries fall out as very simple consequences of a more general theory.

This work \[9\] presented classifications of undirected graph templates with respect to the problems of counting surjective homomorphisms and of counting compactions.

Let us mention that, for the decision problem of checking existence of a surjective homomorphism, a complexity classification of templates seems to be currently elusive, although there is work in this direction (see for example \[2, 10\] and the references therein).

Acknowledgements. The author is grateful to Radu Curticapean and Holger Dell for discussions about and clear explanations of their joint work \[7\] with Dániel Marx. The author thanks Stefan Mengel for his collaboration on database queries \[3, 4, 5, 6\], in which one can see effects similar to those in the present work. This work was supported by the Spanish Project MINECO COMMAS TIN2013-46181-C2-R, Basque Project GIU15/30, and Basque Grant UFI11/45.

References

[1] Andrei A. Bulatov. The complexity of the counting constraint satisfaction problem. *J. ACM*, 60(5):34, 2013.

[2] Hubie Chen. An algebraic hardness criterion for surjective constraint satisfaction. *Algebra Universalis*, 72(4):393–401, 2014.

[3] Hubie Chen and Stefan Mengel. A trichotomy in the complexity of counting answers to conjunctive queries. *CoRR*, abs/1408.0890, 2014.

[4] Hubie Chen and Stefan Mengel. A trichotomy in the complexity of counting answers to conjunctive queries. In *18th International Conference on Database Theory, ICDT 2015, March 23-27, 2015, Brussels, Belgium*, pages 110–126, 2015.

[5] Hubie Chen and Stefan Mengel. Counting answers to existential positive queries: A complexity classification. In *Proceedings of the 35th ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems*, pages 315–326, 2016.

[6] Hubie Chen and Stefan Mengel. The logic of counting query answers. In *32nd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2017, Reykjavik, Iceland, June 20-23, 2017*, pages 1–12, 2017.

[7] Radu Curticapean, Holger Dell, and Dániel Marx. Homomorphisms are a good basis for counting small subgraphs. In *Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2017, Montreal, QC, Canada, June 19-23, 2017*, pages 210–223, 2017.

[8] Martin E. Dyer and David Richerby. An effective dichotomy for the counting constraint satisfaction problem. *SIAM J. Comput.*, 42(3):1245–1274, 2013.

\[2\] See their Theorem 30 and Theorem 13. We remark that their Theorem 13 concerns compactions, and in their setup, inputs are irreflexive graphs.
[9] Jacob Focke, Leslie Ann Goldberg, and Stanislav Zivny. The complexity of counting surjective homomorphisms and compactions. *CoRR*, abs/1706.08786, 2017.

[10] Benoit Larose, Barnaby Martin, and Daniel Paulusma. Surjective h-colouring over reflexive digraphs, 2017.

[11] László Lovász. *Large Networks and Graph Limits*, volume 60 of *Colloquium Publications*. American Mathematical Society, 2012.

[12] L. Lovsz. Operations with structures. *Acta Mathematica Academiae Scientiarum Hungarica*, 18(3-4):321–328, 1967.