Why are the rational and hyperbolic Ruijsenaars–Schneider hierarchies governed by the same $R$–operators as the Calogero–Moser ones?

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Abstract. We demonstrate that in a certain gauge the Lax matrices of the rational and hyperbolic Ruijsenaars–Schneider models have a quadratic $r$-matrix Poisson bracket which is an exact quadratization of the linear $r$–matrix Poisson bracket of the Calogero–Moser models. This phenomenon is explained by a geometric derivation of Lax equations for arbitrary flows of both hierarchies, which turn out to be governed by the same dynamical $R$–operator.
1 Introduction

In the recent years the interest in the Calogero–Moser type of models [1]–[4] is considerably revitalized. One of the directions of this recent development was connected with the notion of the dynamical $r$–matrices and their interpretation in terms of Hamiltonian reduction [5]–[14]. Very recently, this line of research touched also the so–called Ruijsenaars–Schneider models [13], [14], which may be seen as relativistic generalizations of the Calogero–Moser ones [3], [4].

In the present paper the same subject as in [14] is handled, namely, the rational and hyperbolic Ruijsenaars–Schneider models. However, the results are somewhat different, and, as we hope, somewhat more beautiful. The difference is due to another gauge we choose for the Lax matrix of the Ruijsenaars–Schneider models. A striking circumstance comes out when using our gauge, namely that the both classes of models are governed by one and the same dynamical $R$–operator. This seems to pass unnoticed in the existing literature and is explained in Sect.4. The computations presented there are hardly new, at least for the non–relativistic case, but we could not find in the literature the main message following from these computations, namely that they give Lax representations for an arbitrary flow of the corresponding hierarchies, and hence are perfectly suited for guessing (not proving!) the correct $r$–matrix ansatz. Sect.3 contains the main result of the paper, namely a quadratic $r$–matrix Poisson structure for the rational and hyperbolic Ruijsenaars–Schneider models. We compare our results with the previously known ones in the Sect.5. Besides, for the convenience of a general reader we give a short review of relevant notions from the $r$–matrix theory in the Sect.2. Sect.6 is devoted to some problems arising from our results.

2 General framework

In this section we recall some basic notions about integrable hierarchies and their $r$–matrix theory. Our formulations result from the observations on the large ”experimental material” collected in the last decades of research in this area.

Let $\mathcal{P}$ be a Poisson manifold; in fact we consider here only the simplest case of a symplectic space $\mathbb{R}^{2N}\{x, p\}$ with canonically conjugated coordinates.
Let \( H(x, p) \) be a completely integrable Hamiltonian, i.e. suppose that the Hamiltonian system
\[
\dot{x} = \{x, H\}, \quad \dot{p} = \{p, H\}
\] (2.2)
possesses \( N \) functionally independent integrals in involution. Then (2.2) usually (probably always) admits a Lax representation, i.e. there exist two maps \( L : \mathcal{P} \mapsto g \) and \( M : \mathcal{P} \mapsto g \) into some Lie algebra \( g \) such that (2.2) is equivalent to
\[
\dot{L} = [M, L].
\] (2.3)
In the cases we are dealing with in this paper, \( g = gl(N) \), and it carries some additional structures. In particular, it is an associative algebra with respect to the usual matrix multiplication, and admits a non-degenerate scalar product \( \langle U, V \rangle = \text{tr}(UV) \). In other cases \( g \) could be a more complicated algebra, for example a loop algebra; then one would speak about Lax representation with a spectral parameter.

An important observation is that usually
\[
H(x, p) = \varphi(L),
\]
where \( \varphi(L) \) is an \( Ad \)-invariant function on \( g \). This is related to the fact that integrable systems appear not separately, but are organized in hierarchies. Namely, to every invariant function \( \varphi(L) \) there corresponds a Lax equation of the form (2.3). Moreover, there often holds the following relation:
\[
M = R(\nabla \varphi(L)),
\] (2.4)
where \( R : g \mapsto g \) is a linear operator, depending, may be, on some of the coordinates on \( \mathcal{P} \) (it is called then dynamical). We shall call \( R \) an \( R \)-operator governing the corresponding hierarchy. (Recall that the gradient \( \nabla \varphi(L) \) of a smooth function \( \varphi \) on an algebra \( g \) with a scalar product \( \langle U, V \rangle \) is defined by the relation
\[
\langle \nabla \varphi(L), U \rangle = \frac{d}{d\epsilon} \varphi(L + \epsilon U) \bigg|_{\epsilon=0}, \quad \forall U \in g,
\]
and for an $Ad$–invariant $\varphi$ one has $[\nabla \varphi(L), L] = 0$).

Sometimes the relation (2.4) has actually another form:

$$M = R(L \nabla \varphi(L)).$$

(2.5)

Of course, in the general setting, where $R$ in (2.4) can be dynamic, the equation (2.3) can be seen as $M = R(\nabla \varphi(L))$ with an operator $R(\cdot) = R(L\cdot)$. However, if, as it is often the case in applications, an operator $R$ in (2.3) has some special properties (e.g. is independent on some or all dynamical variables), then it is advantageous to consider this particular case on its own rights. We shall call $R$ in (2.3) also an $R$–operator governing the corresponding hierarchy, keeping in mind difference between (2.4) and (2.3).

An $r$–matrix theory \[15\], \[16\] provides a sort of explanation of relations (2.4), (2.3). Namely, the formula (2.4) is usually a consequence of a more fundamental fact, namely that $L = L(x, p)$ form a Poisson submanifold in $g$ equipped with a so called linear $r$–matrix bracket. This is expressed in the formula

$$\{L \otimes L\} = [I \otimes L, r] - [L \otimes I, r^*].$$

(2.6)

The $N^2 \times N^2$ matrix $r$ corresponding to the operator $R$ is defined as

$$r = \sum_{i,j,k,m=1}^{N} r_{ij,km} E_{ij} \otimes E_{km},$$

where

$$r_{ij,km} = \langle R(E_{ji}), E_{mk} \rangle = \text{coeff. by } E_{km} \text{ in } R(E_{ji}).$$

(2.7)

and $r^*$ corresponds in the same way to the operator $R^*$, so that

$$r^* = \sum_{i,j,k,m=1}^{N} r_{km,ij} E_{ij} \otimes E_{km}.$$ 

In case of constant (non–dynamical) $R$–operators a sufficient condition for (2.6) to define a Poisson bracket is given by the so called modified Yang–Baxter equation \[15\]. In case of dynamical $R$–operators the corresponding theory is less developed; the most general known sufficient conditions are given in \[13\]. The lax representation (2.3) with $M$ given in (2.4) is a consequence of of (2.6) for an arbitrary $Ad$–invariant function $\varphi(L)$,
Analogously, the formula (2.3) is usually a consequence of a more fundamental fact that $L = L(x,p)$ form a Poisson submanifold in $g$ equipped with a so called quadratic $r$-matrix bracket. The most general quadratic bracket is given by

$$\{L \otimes L\} = (L \otimes L)a_1 - a_2(L \otimes L) + (I \otimes L)s_1(L \otimes I) - (L \otimes I)s_2(I \otimes L). \quad (2.8)$$

Such general quadratic $r$-matrix structures were discovered several times independently [17], [18], [19]; see [19] for an application to the Toda and relativistic Toda hierarchy.

In (2.8) the matrices $a_1, a_2, s_1, s_2$ correspond in a canonical way to some linear (in principle, dynamical) operators $A_1, A_2, S_1, S_2$ and satisfy conditions

$$a_1^* = -a_1, \quad a_2^* = -a_2, \quad s_1^* = s_1, \quad (2.9)$$

and

$$a_1 + s_1 = a_2 + s_2 = r. \quad (2.10)$$

The first of these conditions assures the skew–symmetry of the Poisson bracket (2.8), and the second one guarantees that the Hamiltonian flows with invariant Hamiltonian functions $\varphi(L)$ have the Lax form (2.3) with the $M$–matrix (2.3). If (2.10) is satisfied, we call the bracket (2.8) a quadratization of the bracket (2.6). In the case of constant operators a sufficient condition for (2.8) with (2.9), (2.10) to be a Poisson bracket is validity of the modified Yang–Baxter equation for three operators $R, A_1, A_2$; nothing is known for dynamical case.

It should be remarked that, while in the linear case the correspondence between operator $R$ in (2.4) and matrix $r$ in (2.6) is rather unambiguous, the situation is quite different in the quadratic case. There exist in principle infinitely many possibilities of choice of $a_1, a_2, s_1, s_2$ in (2.8), satisfying (2.4), (2.10) and resulting in the same operator $R$ in (2.3). All such quadratizations are parametrized by one skew–symmetric matrix, for example, by $a_1$, because $a_2 = r - r^* - a_1, \quad s_1 = r - a_1, \quad s_2 = r^* + a_1$. Hence finding a quadratic $r$–matrix Poisson structure for a given Lax matrix $L$ is a non–trivial entertainment even if the $R$–operator governing the whole hierarchy is known. Guessing a correct quadratization remains more or less a matter of art. The present paper is devoted to finding such quadratic structure for rational and hyperbolic Ruijsenaars–Schneider models.
3 Ruijsenaars–Schneider models and their $r$–matrix formulation

The hyperbolic relativistic Ruijsenaars–Schneider (RS) hierarchy is described in terms of the Lax matrix

$$L = L_{RS}(x, p) = \sum_{k,j=1}^{N} \frac{\sinh(\gamma)}{\sinh(x_k - x_j + \gamma)} b_j E_{kj}.$$  \hspace{1cm} (3.1)

Here $\gamma$ is a parameter of the model, usually supposed to be pure imaginary. We use an abbreviation

$$b_k = \exp(p_k) \prod_{j \neq k} \left(1 - \frac{\sinh^2(\gamma)}{\sinh^2(x_k - x_j)}\right)^{1/2}, \hspace{1cm} (3.2)$$

so that in the variables $(x, b)$ the canonical Poisson brackets (2.1) take the form

$$\{x_k, x_j\} = 0, \quad \{x_k, b_j\} = b_k \delta_{kj}, \quad \{b_k, b_j\} = \pi_{kj} b_k b_j \hspace{1cm} (3.3)$$

with

$$\pi_{kj} = \coth(x_j - x_k + \gamma) - \coth(x_k - x_j + \gamma) + 2(1 - \delta_{kj}) \coth(x_k - x_j). \hspace{1cm} (3.4)$$

The Hamiltonian of the RS model proper (i.e. of the simplest member of the hierarchy) is given by

$$H(x, p) = \sum_{k=1}^{N} b_k = \text{tr} L(x, p).$$

This model admits a so called non–relativistic limit, achieved by rescaling $p \mapsto \beta p$, $\gamma \mapsto \beta \gamma$ and subsequent sending $\beta \to 0$. In this limit we have

$$L_{RS} = I + \beta L_{CM} + O(\beta^2),$$

where the Lax matrix of the rational Calogero–Moser (CM) hierarchy is introduced:

$$L = L_{CM}(x, p) = \sum_{k=1}^{N} p_k E_{kk} + \sum_{k \neq j} \frac{\gamma}{\sinh(x_k - x_j)} E_{kj}. \hspace{1cm} (3.5)$$

The Hamiltonian of the CM model proper is given by

$$H(x, p) = \frac{1}{2} \sum_{k=1}^{N} p_k^2 + \frac{1}{2} \sum_{k \neq j} \frac{\gamma^2}{\sinh^2(x_k - x_j)} = \frac{1}{2} \text{tr} L^2(x, p).$$
A simple, but important particular case of these models constitute the rational ones, which are obtained by rescaling $x \mapsto \mu x$, $\gamma \mapsto \mu \gamma$ and sending $\mu \to 0$. In this limit one gets the Lax matrix of the RS hierarchy:

$$L = L_{RS}(x, p) = \sum_{k,j=1}^{N} \frac{\gamma}{x_k - x_j + \gamma} b_j E_{kj}, \quad (3.6)$$

where

$$b_k = \exp(p_k) \prod_{j \neq k} \left(1 - \frac{\gamma^2}{(x_k - x_j)^2}\right)^{1/2}. \quad (3.7)$$

The canonical Poisson brackets in terms of $(x, b)$ are still given by (3.3) with

$$\pi_{kj} = \frac{1}{x_j - x_k + \gamma} - \frac{1}{x_k - x_j + \gamma} + \frac{2(1 - \delta_{kj})}{x_k - x_j}. \quad (3.8)$$

As a Lax matrix of the CM hierarchy one gets

$$L = L_{CM}(x, p) = \sum_{k=1}^{N} p_k E_{kk} + \sum_{k \neq j}^{N} \frac{\gamma}{x_k - x_j} E_{kj}. \quad (3.9)$$

Now we are in a position to formulate the main result of this paper.

**Theorem.** For the Lax matrices of the relativistic models (3.6), (3.1) there holds a quadratic $r$–matrix ansatz (2.8) with the matrices

$$a_1 = a + w, \quad s_1 = s - w,$$
$$a_2 = a + s - s^* - w, \quad s_2 = s^* + w,$$

where in the rational case

$$a = \sum_{k \neq j} \frac{1}{x_k - x_j} E_{jk} \otimes E_{kj}, \quad (3.10)$$
$$s = - \sum_{k \neq j} \frac{1}{x_k - x_j} E_{jk} \otimes E_{kk}, \quad (3.11)$$
$$w = \sum_{k \neq j} \frac{1}{x_k - x_j} E_{kk} \otimes E_{jj}. \quad (3.12)$$
and in the hyperbolic case

\[ a = \sum_{k \neq j} \coth(x_k - x_j) E_{jk} \otimes E_{kj}, \]  

\[ s = -\sum_{k \neq j} \frac{1}{\sinh(x_k - x_j)} E_{jk} \otimes E_{kk}, \]  

\[ w = \sum_{k \neq j} \coth(x_k - x_j) E_{kk} \otimes E_{jj}. \]  

Let us stress three remarkable properties of the found $r$–matrix structure.

- All matrices $a, s, w$ are dynamical, but depend only on the coordinates $x_k$, not on the momenta $p_k$.

- All matrices $a, s, w$ do not depend on the parameter $\gamma$ of the model.

- The most important and striking fact is that the structure found is an exact quadratization of a linear $r$–matrix bracket with

\[ r = a + s, \]

which is just the $r$–matrix of the non–relativistic CM model found by Avan–Talon in [5]. So, the rational and hyperbolic RS hierarchies turn out to be governed by the same $R$–operators as the rational and hyperbolic CM hierarchies, respectively.

The proof of the Theorem above is a matter of rather direct computations, and is therefore omitted. The next section is devoted to a geometric derivation of the $R$–operators for the RS and CM hierarchies. This derivation is independent on the our Theorem, and provides some explanation of the third property mentioned above.

### 4 Derivation of $R$–operator for CM and RS hierarchies

An important tool to investigate the hyperbolic CM and RS models is, along with the Lax matrices, an auxiliary diagonal matrix

\[ X = \text{diag}(\exp(x_1), \ldots, \exp(x_N)) = \sum_{k=1}^{N} \exp(x_k) E_{kk}. \]  

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The fundamental commutation relation, describing the structure of the Lax matrix $L$ in terms of a given diagonal matrix $X$, reads for the CM model:

$$XLX^{-1} - X^{-1}LX = 2\gamma \sum_{k\neq j} E_{kj} = 2\gamma (ee^T - I),$$

(4.2)

and for the RS model:

$$e^{\gamma}XLX^{-1} - e^{-\gamma}X^{-1}LX = 2\sinh(\gamma)eb^T.$$  

(4.3)

Here $I$ stands for the $N \times N$ unity matrix, $e = (1, \ldots, 1)^T$, and $b = (b_1, \ldots, b_N)^T$.

We shall derive the $R$–operators of these models from the results about their explicit solution obtainable from [1], [3]. These results may be formulated as follows. Let $\varphi(L)$ be an $Ad$–invariant function on $gl(N)$, and take its value on one of the Lax matrices $L_{CM}$, $L_{RS}$ as a Hamiltonian function for the corresponding model. Then the quantities $\exp(2x_k(t))$ are just the eigenvalues of the matrix

$$X_0^2 \exp(2tf(L_0)),$$

where

$$f(L) = \nabla \varphi(L) \quad \text{for the non–relativistic CM case},$$

(4.4)

$$f(L) = L\nabla \varphi(L) = \nabla \varphi(L)L \quad \text{for the relativistic RS case}.$$  

(4.5)

From this statement the Lax equations for the corresponding flows can be derived. Indeed, define the evolution of the matrices $X, L$ by the equations

$$X^2 = X^2(t) = VX_0^2 \exp(2tf(L_0))V^{-1},$$

(4.6)

$$L = L(t) = VL_0V^{-1}.$$  

(4.7)

Let us explain, in which sense these equations define an evolution. The matrix $X^2 = X^2(t)$ consists of eigenvalues of the matrix $X_0^2 \exp(2tf(L_0))$, and the matrix $V = V(t)$ is a diagonalizing one. (It is easy to see that the matrix $X_0^2 \exp(2tf(L_0))$ is similar to a self–adjoint one, if $\gamma$ is pure imaginary). If we fix the ordering of $x_k$ (for example, $x_1 < \ldots < x_N$), then the only freedom in the definition of $V$ is a left multiplication by a diagonal matrix. We fix now $V$ by the condition

$$VX_0e = Xe,$$

(4.8)
and show that this assures that the corresponding requirements (4.2) and (4.3) are satisfied for all \( t \), provided they were satisfied for \( t = 0 \).

Indeed, we have for the CM case:

\[
XLX^{-1} - X^{-1}LX = X^{-1}VX_0 \left( X_0L_0X_0^{-1} - X_0^{-1}L_0X_0 \right) X_0^{-1}V^{-1}X
\]

\[
= 2\gamma X^{-1}VX_0(\varepsilon\varepsilon^T - I)X_0^{-1}V^{-1}X = 2\gamma \left( X^{-1}VX_0\varepsilon\varepsilon^TX_0^{-1}V^{-1}X - I \right).
\]

Since the diagonal entries of the matrix on the left-hand side vanish, we see that (4.8) implies \( \varepsilon^TX_0^{-1}V^{-1}X = \varepsilon^T \), which proves the commutation relation (4.2) for all \( t \).

Analogously, for the RS case we have:

\[
e^{\gamma}XLX^{-1} - e^{-\gamma}X^{-1}LX
\]

\[
= X^{-1}VX_0 \left( e^{\gamma}X_0L_0X_0^{-1} - e^{-\gamma}X_0^{-1}L_0X_0 \right) X_0^{-1}V^{-1}X
\]

\[
= 2\sinh(\gamma)X^{-1}VX_0e_0^Tb_0^TX_0^{-1}V^{-1}X.
\]

so that denoting \( b_0^TX_0^{-1}V^{-1}X = b^T \), we see that (4.8) implies the validity of the commutation relation (4.3) for all \( t \).

From this point the derivation of the evolution equations for \( L, X \) is identical for both the non-relativistic and the relativistic cases. Differentiating (4.7), (4.6), we get:

\[
\dot{L} = [M, L],
\]

(4.9)

(so that the evolution of \( L \) is governed by a Lax equation),

\[
2X\dot{X} = [M, X^2] + 2X^2f(L),
\]

(4.10)

where

\[
M = \dot{V}V^{-1}.
\]

(4.11)

In order to find the matrix \( M \) explicitly, consider first the off-diagonal part of (4.10), which implies:

\[
M_{kj} = \frac{\exp(x_k - x_j)}{\sinh(x_k - x_j)}f(L)_{kj} = (1 + \coth(x_k - x_j))f(L)_{kj}, \quad k \neq j.
\]

(4.12)

The normalizing condition following from (4.8), (4.11) reads:

\[
MXe = \dot{X}e,
\]

(4.13)
hence
\[ M_{kk} = \dot{x}_k - \sum_{j \neq k} M_{kj} \exp(x_j - x_k). \]

An expression for \( \dot{x}_k \) can be read off the diagonal part of (4.10):
\[ \dot{x}_k = f(L)_{kk}. \]
Substituting this in the previous formula and using (4.12), we get:
\[ M_{kk} = f(L)_{kk} - \sum_{j \neq k} \frac{f(L)_{kj}}{\sinh(x_k - x_j)}. \]

Finally, notice that we can redefine \( M - f(L) \) as a new \( M \) (this does not influence the equations of motion described by the Lax pair), which results in more convenient expressions
\[ M_{kj} = \coth(x_k - x_j) f(L)_{kj}, \quad k \neq j, \]
\[ M_{kk} = -\sum_{j \neq k} \frac{f(L)_{kj}}{\sinh(x_k - x_j)}. \]

Analogous results for the rational case may be obtained by the limiting process, or derived in parallel. An auxiliary diagonal matrix is then given by
\[ X = \text{diag}(x_1, \ldots, x_N) = \sum_{k=1}^{N} x_k E_{kk}. \]

Given the diagonal matrix \( X \), the structure of the Lax matrix \( L \) is completely described by the following fundamental commutation relations: for the CM case
\[ XL - LX = \gamma \sum_{k \neq j} E_{kj} = \gamma(\epsilon\epsilon^T - I), \]
and for the RS case
\[ XL - LX + \gamma L = \gamma e b^T. \]

The results about an explicit solution of these models [1], [3] read: let \( \varphi(L) \) be an \( \text{Ad} \)–invariant function on \( gl(N) \), and take its value on one of the Lax matrices (4.9), (4.6) as a Hamiltonian function of the corresponding model. Then the quantities \( x_k(t) \) are just the eigenvalues of the matrix
\[ X_0 + tf(L_0), \]
where (4.4), (4.5) still hold. Defining the evolution of the matrices $X$, $L$ by the equations
\begin{align}
X &= X(t) = V(X_0 + tf(L_0))V^{-1}, \\
L &= L(t) = VL_0V^{-1},
\end{align}
(4.20)
(4.21)
we see that in order to assure that the commutation relations (4.18) and (4.19) are preserved in the dynamics, one has to normalize the matrix $V$ by the condition
\begin{equation}
Ve = e,
\end{equation}
(4.22)

Now the calculations are again identical for both the CM and the RS cases. Differentiating (4.21), (4.20), we get:
\begin{equation}
\dot{L} = [M, L],
\end{equation}
(4.23)
(so that the evolution of $L$ is governed by a \emph{Lax equation}),
\begin{equation}
\dot{X} = [M, X] + f(L),
\end{equation}
(4.24)
where
\begin{equation}
M = \dot{V}V^{-1}.
\end{equation}
(4.25)
Differentiating (4.22) gives:
\begin{equation}
Me = 0,
\end{equation}
(4.26)

Now we can find the matrix $M$ explicitly. From (4.24) we immediately obtain the off–diagonal entries of the matrix $M$:
\begin{equation}
M_{kj} = \frac{f(L)_{kj}}{x_k - x_j}, \quad k \neq j.
\end{equation}
(4.27)
The normalizing condition (4.26) implies:
\begin{equation}
M_{kk} = -\sum_{j \neq k} M_{kj} = -\sum_{j \neq k} \frac{f(L)_{kj}}{x_k - x_j}.
\end{equation}
(4.28)

According to (4.4), (4.5), we see that the following statement is proved.

\textbf{Proposition.} \textit{The general flows of the (rational or hyperbolic) CM and RS hierarchies with Ad–invariant Hamiltonians $\varphi(L)$ have Lax representations (2.3) with the $M$–matrices given by (2.4), (2.5), respectively. The $R$–operator is one and the same for the CM and RS cases and is given by:}
\begin{equation}
R = A + S,
\end{equation}
(4.29)
where $A$ is a skew-symmetric operator on $\mathfrak{gl}(N)$, and $S$ is a non-skew-symmetric one, whose image consists of diagonal matrices. For the rational models

$$
A(E_{kj}) = \frac{1 - \delta_{kj}}{x_k - x_j} E_{kj},
$$

$$
S(E_{kj}) = -\frac{1 - \delta_{kj}}{x_k - x_j} E_{kk},
$$

and for the hyperbolic models

$$
A(E_{kj}) = (1 - \delta_{kj}) \coth(x_k - x_j) E_{kj},
$$

$$
S(E_{kj}) = -\frac{1 - \delta_{kj}}{\sinh(x_k - x_j)} E_{kk}.
$$

Obviously, operators $A$, $S$ canonically correspond to the matrices $a$, $s$ from (3.10), (3.11) and (3.13), (3.14), respectively.

## 5 Comparison with the previous results

As pointed out in the Introduction, the $r$–matrices for the RS models were previously discussed by Babelon–Bernard in [13] and by Avan–Rollet in [14]. In both papers another gauge for the Lax matrix is chosen, namely a self-adjoint one:

$$
L = \sum_{k,j=1}^{N} \frac{\sinh(\gamma)}{\sinh(x_k - x_j + \gamma)} (b_k b_j)^{1/2} E_{kj}.
$$

Let us reformulate our results for this gauge. Calculations analogous to those presented in Sect.4 show that the $M$–matrix for the flow of RS hierarchy with a Hamiltonian function $\varphi(L)$ is given in the gauge (5.1) by

$$
M_{kj} = \coth(x_k - x_j) f(L)_{kj}, \quad j \neq k,
$$

$$
M_{kk} = -\frac{1}{2} \sum_{j \neq k} \left( \frac{b_j}{b_k} \right)^{1/2} \frac{f(L)_{kj} - f(L)_{jk}}{\sinh(x_k - x_j)},
$$

where $f(L) = L \nabla \varphi(L)$, as in (4.3). Accordingly, the matrices $a_1$, $a_2$, $s_1$, $s_2$ in a quadratic Poisson bracket (2.8) for the matrix (5.1) depend with necessity not only on $x_k$ but also on the momenta $p_k$. After direct but
tedious calculations one can get a gauged version of the bracket from our
Theorem in the form \((2.8)\) with

\[
a_1 = a + \frac{1}{2}(u_1 - u_1^*) + v, \quad s_1 = \frac{1}{2}(u_2 + u_2^*) - v,
\]

\[
a_2 = a + \frac{1}{2}(u_2 - u_2^*) - v, \quad s_2 = \frac{1}{2}(u_1 + u_2^*) + v,
\]

(5.3)

where

\[
\begin{align*}
 u_1 &= \sum u_{kj} E_{kj} \otimes E_{kk}, \\
 u_2 &= -\sum u_{kj} E_{jk} \otimes E_{kk}, \\
 v &= \sum v_{kj} E_{kk} \otimes E_{jj},
\end{align*}
\]

the coefficients of these matrices being given by

\[
v_{kj} = \frac{1}{4} \left( \coth(x_k - x_j + \gamma) - \coth(x_j - x_k + \gamma) + 2(1 - \delta_{jk}) \coth(x_k - x_j) \right)
\]

and

\[
u_{kj} = \left( \frac{b_j}{b_k} \right)^{1/2} \frac{1 - \delta_{jk}}{\sinh(x_k - x_j)}.
\]

Let us stress that these matrices, first, depend on momenta \(p_k\), second, de-
pend on the parameter \(\gamma\), and third, do not reproduce the \(R\)–operator govern-
ing the CM hierarchy. This demonstrates how crucially depend the pro perties
of \(r\)–matrices on the choice of gauge for Lax matrix.

### 5.1 On the Avan–Rollet \(r\)–matrix

Avan–Rollet found a linear \(r\)–matrix structure of the type \((2.6)\) for the Lax
matrix \((5.1)\). They looked for an \(r\)–matrix with entries being linear combina-
tions of elements \(L_{kj}\) of the Lax matrix \((5.1)\) with coefficients depending only
on coordinates \(x_k\), but not on the momenta \(p_k\). According to the remarks
above, such linear combinations can not be cast in the form

\[
r = (L \otimes I)r_1 + r_2(L \otimes I),
\]

(5.4)

necessary for putting linear \(r\)–matrix structure \((2.6)\) into quadratic form
\((2.8)\). Indeed, in order to have \((5.4)\) one is forced to admit coefficients de-
pending on momenta.
Actually, after some manipulations based on the explicit expressions of $L_{kj}$, the Avan–Rollet $r$–matrix can be demonstrated to be equal to (5.4) with

$$r_1 = \frac{1}{2}(a + u_1 + v), \quad r_2 = \frac{1}{2}(a + u_2 - v),$$

which gives back the expressions (5.3). Hence our results and those by Avan–Rollet are in a sort of "hidden" equivalence.

5.2 On the Babelon–Bernard $r$–matrix

Babelon–Bernard consider the hyperbolic RS system with a specific value of parameter $\gamma = i\pi/2$ (which, by the way, precludes the possibility of both the non–relativistic and the rational reductions). They use the gauge (5.1) and obtain a quadratic Poisson bracket with $a_1 = a_2 = \frac{1}{2}a$, $s_1 = s_2 = \frac{1}{2}a^\tau$, where $\tau$ denotes the transposition in the first factor of tensor products. In particular, all $r$–matrix objects depend only on $x_k$’s. This seems to be in contradiction with the remark after (5.2). However, a closer look at (5.2) solves this paradox. For $\gamma = i\pi/2$ the Lax matrix $L$ (5.1) is symmetric, which forces diagonal entries $M_{kk}$ of the matrix $M$ to vanish. It means that for this particular case the $R$–operator is equal symply to $A$, or, taking into account the symmetry of $L$, to the $\frac{1}{2}(A + A \circ T)$, where $T$ stands for the transposition operator. The structure found by Babelon–Bernard is the simplest possible quadratization of a twisted linear $r$–matrix bracket with $r = \frac{1}{2}(a + a^\tau)$. So, it essentially takes care of specific properties of the model for $\gamma = i\pi/2$ and is not covered by our general construction.

6 Conclusions

The results of this paper surely constitute only one link in a long chain of (already achieved and still hypothetical) results concerning the RS models. The whole work made for their non–relativistic counterparts should be repeated, or, better, generalized.

- After this paper was submitted for publication, two apparently different (spectral parameter dependent) quadratic $r$–matrix structures were found for the elliptic RS hierarchy [22], [23]. The structure found in the
second of these preprints serves as a direct generalization of the present results: it turns out that the elliptic RS hierarchy is also governed by the same $R$–operator as the elliptic CM one.

- The spin Ruijsenaars–Schneider models, introduced recently in [20], should be also put into the $r$–matrix framework, as it was done for the spin Calogero–Moser systems in [8].

- Another important point is a derivation of our Theorem in the framework of Hamiltonian reduction; to this end the results in [11] should be used and further developed.

- It would be rather important to investigate the dynamical analogs of the modified Yang–Baxter equation assuring the Poisson bracket properties of the ansatz (2.8). The investigations of these objects were started in [8] for the linear ansatz (2.6) and are expected to unveil new interesting structures in the case of quadratic brackets.

Further, our Theorem being established, situation with the Calogero–Moser type models becomes perfectly analogous to the situation with the Toda–like ones. Namely, the transition from the non–relativistic to the relativistic systems corresponds to the transition from a linear $r$–matrix Poisson structure to its quadratisation. (See [21], [19] for the Toda case). A deeper understanding of this phenomenon is highly desirable.

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