Analysis of static and spherically-symmetric solutions in NDL theory of gravitation

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We investigate a static solution with spherical symmetry of a recently proposed field theory of gravitation. In this so-called NDL theory, matter interacts with gravity in accordance with the Weak Equivalence Principle, while gravitons have a nonlinear self-interaction. It is shown that the predictions of NDL agree with those of General Relativity in the three classic tests. However, there are potential differences in the strong-field limit, which we illustrate by proving that this theory does not allow the existence of static and spherically symmetric black holes.

I. INTRODUCTION

The Equivalence Principle played a fundamental role in the development of General Relativity (GR) and is at the heart of the idea that spacetime is curved [1]. However, a distinction must be made between the Weak Equivalence Principle (WEP) and the Strong Equivalence Principle (SEP). In a nutshell, WEP states that all matter fields (except the gravitational field) interact with gravity in the same way (the so-called “universal coupling”). SEP states, among other things, that all fields (including gravity) interact with gravity in the same way. Let us remark that while the other aspects of SEP (i.e. the absence of preferred-frame and preferred-location effects) have been tested in several instances [3], there is no conclusive evidence about the way that gravity couples to gravity. This is precisely the area explored by a recently presented field theory of gravitation, called NDL by the authors of [2]. This theory is a field-theoretical description of gravity (in the spirit of Feynman [3] and Deser [4]) in which SEP is violated from the beginning: gravity does not couple to itself in the same way it couples to other fields.

Several features of this theory have been studied in [2, 5, 6]. To date, the most important prediction of NDL theory is that gravitational waves do not propagate at the same speed as electromagnetic waves [5], a phenomenon related to the self-coupling of gravity in this theory. In fact, it can be shown that gravitons in NDL theory follow an effective metric, different to the background metric [2], that depends on the abovementioned self-coupling. Indeed, this is a feature of any nonlinear theory: the influence of the effective geometry on nonlinear photons has been extensively analyzed in [2, 3, 4].

To go further with the analysis of the predictions of NDL theory, in this article we shall study some aspects of the static and spherically symmetric solution obtained in [2]. We begin by giving a short summary of NDL theory in Sect.II. The spherically symmetric and static solution is presented in Sect.III. We study in Sect.IV some aspects of gravitation around compact objects. Specifically, we investigate the effective potential felt by particles moving in this solution. Armed with the effective potential, we compare the predictions of NDL with those of GR in the three classic tests. We also investigate if the spherically symmetric and static solution can describe a black hole analog to that of Schwarzschild in GR, and analyze the effective metric felt by gravitons. We close in Sect.V with some comments regarding the nature of the singularities appearing in the solution and prospects for future work.

II. SUMMARY OF NDL THEORY

A detailed presentation of NDL theory was given in [2]. Here we shall list the most important features of the theory, following Ref.[2]:

• The gravitational interaction is represented by a symmetric tensor \( \varphi_{\mu\nu} \) that obeys a nonlinear equation of motion.

• Matter (but not gravity) couples to gravity through the metric \( g_{\mu\nu} = \gamma_{\mu\nu} + \varphi_{\mu\nu} \), where \( \gamma_{\mu\nu} \) is the flat background metric.

• The self-interaction of gravity, given by a nonlinear Lagrangian, breaks the interpretation of gravity as a universal geometric phenomenon. That is, all particles (except gravitons) move following geodesics of \( g_{\mu\nu} \). Gravitons instead move on geodesics of an effective metric, the expression of which we shall give in Sect.V.

It is convenient to define the tensor \( F_{\alpha\beta\mu} \) (called the gravitational field), in terms of \( \varphi_{\mu\nu} \), as follows [2]:

\[
F_{\alpha\beta\mu} = \frac{1}{2} \left( \varphi_{\rho(\alpha;\beta)} + F_{\rho[\alpha} \gamma_{\beta]\mu} \right),
\]

where the covariant derivative is constructed with the background metric \( \gamma_{\mu\nu} \). Indices are raised and lowered.
with that metric also, and
\[ F_\mu = F_{\alpha \mu} \gamma^{\mu \nu}. \]
To construct a nonlinear theory of the gravitational field \( F_{\mu \nu} \) with the correct weak field limit, we assume that the interaction of gravity with itself is described by a functional of \( A - B \), where the scalars \( A \) and \( B \) are given by:
\[ A = F_{\alpha \beta \mu} F^{\alpha \beta \mu}, \quad B = F_{\alpha} F^{\alpha}. \]
From the action
\[ S = \int d^4x \sqrt{-\gamma} L(A - B), \]
where \( \gamma \) is the determinant of the background metric) and using that fact that \( \mathcal{L}_A = -\mathcal{L}_B \) (where \( \mathcal{L}_X \) is the derivative of the Lagrangian with respect to \( X \)), we get the equations of motion
\[ \left( \sqrt{-\gamma} \mathcal{L}_A F^{\lambda}_{(\mu \nu \rho \sigma) \lambda} \right)_{\lambda} = 0. \]  
(1)

In the next section we shall study a particular solution of these equations of motions for a given choice of the Lagrangian.

III. THE STATIC AND SPHERICALLY SYMMETRIC SOLUTION

In what follows, we will restrict our study to a special Lagrangian studied previously in \[10\], and inspired by the Born-Infeld Lagrangian \[10\]:
\[ L(A - B) = \frac{b^2}{\kappa} \left( \sqrt{1 - \frac{A - B}{b^2}} - 1 \right), \]  
(2)
where \( \kappa \) is Einstein’s constant. The parameter \( b \), with dimensions of length \(^{-1}\), is undetermined at this point. Notice that its value should be large enough for the series expansion of Eqn.\([2]\) to be in agreement with the weak-field limit.

We are interested in static and spherically symmetric solutions in a Minkowskian background. Consequently, the only nonzero components of the field \( \varphi_{\mu \nu} \) are
\[ \varphi_{00} \equiv \mu(r), \quad \varphi_{11} \equiv -\nu(r). \]
and the corresponding nonzero components of the tensor \( F_{\alpha \beta \mu} \) are given by
\[ F_{100} = -\frac{\nu}{r}, \]
\[ F_{122} = \frac{1}{2}(\nu r - \mu' r^2), \]
\[ F_{133} = F_{222} \sin^2 \theta. \]
The only nonzero trace component, \( F_1 \), is given by
\[ F_1 = \mu' - \frac{2}{r} \nu. \]
From the equations of motion \([1]\) and the expression of the Lagrangian we get only two nontrivial equations:
\[ 2\nu^3 + b^2 \nu' + b^2 \nu^2 \nu = 0, \]
\[ \mu' r - \nu = 0 \]
The first equation can be easily integrated, and the result is
\[ \nu(r) = \epsilon C \left( 1 - \frac{r_0}{r} \right)^{4/3}, \]  
(3)
where we have defined
\[ r_0^2 = C |b|, \quad \epsilon = \pm 1, \]  
(4)
and \( C \) is an integration constant. From the second equation,
\[ \mu(r) = \int \frac{\nu(r)}{r} dr + \text{const}. \]

Noting that the function \( \nu(r) \) is defined only for \( r \geq r_0 \), we can write using Eqn.\([3]\),
\[ \mu(r) = \epsilon C \int_{r_0}^{r} \frac{dr}{\sqrt{r^2 - r_0^2}} + \text{const}. \]
This integral can be written in terms of an elliptic integral of the first kind \([11]\) using the identity
\[ \int_{\beta}^{\alpha} \frac{dx}{\sqrt{(x^2 + \alpha^2)(x^2 - \beta^2)}} = \frac{1}{\sqrt{\alpha^2 + \beta^2}} F(X, Y), \]
valid for \( u > \beta > 0 \), and
\[ X = \arccos \left( \frac{\beta}{u} \right), \quad Y = \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}. \]
Consequently, \( \mu(r) \) is given by
\[ \mu(r) = \frac{\epsilon C}{\sqrt{2} r_0} F(\arccos(r_0/r), 1/\sqrt{2}) + \text{const}. \]  
(5)
In order to determine the value of the constant, we impose that spacetime be Minkowskian for large \( r \), which means that \( \mu(r) \to 0 \) in that limit. Then,
\[ \mu(r) = \epsilon \sqrt{\frac{|b| C}{2}} \left[ F(\arccos(r_0/r), 1/\sqrt{2}) - F(\pi/2, 1/\sqrt{2}) \right]. \]  
(6)
The value of the constant $C$ and the sign of $\epsilon$ have not been determined up to now: they are dictated by the weak field limit of the solution, which must coincide with that of Schwarzschild. The radial component of the Schwarzschild metric (we are setting $G = c = 1$) is

$$g_{rr}(r) = -1 - \frac{2M}{r},$$

In our case, $g_{rr}(r) = -1 - \nu(r)$, and in the weak field limit we get

$$g_{rr}(r) = -1 - \epsilon \frac{C}{r},$$

and therefore conclude that $\epsilon = +1$ and $C = 2M$.

Thus, the general expression for the static, spherically symmetric and asymptotically flat spacetime in NDL theory previously derived in [1], is given by the metric

$$ds^2 = [1 + \mu(r)] dt^2 - [1 + \nu(r)] dr^2 - r^2 d\Omega^2,$$

with $\mu(r)$ and $\nu(r)$ given by Eqsns. (6) and (7) respectively. In the following sections we shall study some properties of this solution.

IV. GRAVITATIONAL PHYSICS AROUND COMPACT OBJECTS IN NDL THEORY

Here the predictions of NDL theory are compared with those of GR. Using the static and spherically symmetric solution, we will consider first the three classical tests of GR in the framework of NDL theory. Then we shall see if NDL concurs with this prediction. The values

$$C_\odot \approx 3\text{km}, \quad r_+ \approx 7.0 \times 10^7\text{km}, \quad r_- \approx 5.0 \times 10^7\text{km},$$

will be used below, where $r_+$ ($r_-$) is the aphelion (perihelion) of Mercury. With these values, we see that

$$\left( \frac{r_0}{r_\pm} \right)^4 \approx \frac{1}{|b|^2} 10^{-28}.$$

This equation and the discussion following Eqn.(9) ensure that we can keep only the first term in the expansion in Eqn.(8). Notice also that

$$\left( \frac{C_\odot}{r_\pm} \right)^4 \approx 10^{-29},$$

and consequently from the expansions in Eqsns.(10) and (11), the fourth and higher orders terms can be neglected. The effective potential in the case under consideration can thus be approximated by

$$V(r) = k \left( 1 - \frac{C}{r} \right) + \frac{L^2}{r^2} - \frac{CL^2}{r^3} \left( 1 + k \frac{C^2}{L^2} \right) + \frac{C^2}{r^2} (k - E^2).$$

We see that this approximate expression contains new terms not present in the exact expression given by Eqn.(8), even for the massless case. These new terms...
signal potential differences between NDL theory and GR in the weak field regime. Let us try to estimate the magnitude of these terms for the case of Mercury, in which $C^2/L^2 \approx 10^{-7}$, so the term involving this factor can be safely neglected. We now need an estimate for $E^2$ which may be obtained using Eqn. (14) evaluated at $r_+$ and $r_-$ (the values of the radius where $\dot{r} = 0$). The result can be written as a system of two equations in the unknowns $E^2$ and $L^2$. In particular,

$$E^2 = \frac{g_{tt}(r_+)g_{uu}(r_-) (r_+^2 - r_-^2)}{r_+^2 g_{tt}(r_-) - r_-^2 g_{tt}(r_+)}.$$ 

It follows that $E^2 - 1 \approx 10^{-7}$ and so the last term in Eqn. (16) is negligible for $k = 1$. Consequently, the effective potential for NDL theory coincides with that of Schwarzschild. Therefore, NDL agrees with GR in the prediction for the perihelion of Mercury.

2. The deflection of light by the Sun

To derive the equation that governs the deflection of light rays by the Sun, we use the fact that

$$\frac{C}{R} \approx 4.23 \times 10^{-6}. \quad (17)$$

Combining this with the discussion following Eqn. (4), it follows that the effective potential for the deflection of light is given by Eqn. (16) with $k = 0$. That is,

$$V(r) = \frac{L^2}{r^2} \left(1 - \frac{C}{r}\right) - \frac{C^2}{r^2} E^2.$$ 

Consequently, the equation of motion takes the form

$$\dot{r}^2 + \frac{L^2}{r^2} \left(1 - \frac{C}{r}\right) = \left[1 + \left(\frac{C}{r}\right)^2\right] E.$$ 

From Eqn. (14) we have that $(C/r)^2 < 10^{-13}$, which is negligible. Thus it is clear that the equation of motion for photons in NDL coincides with that of Schwarzschild.

3. Time delay of light

Let us sketch the derivation of this effect in NDL theory, adopting the usual procedure for GR (after [13]). Consider the path of a light ray with $\theta = \pi/2$ in the metric given by Eqn. (8):

$$g_{tt}dt^2 + g_{rr}dr^2 - r^2 d\phi^2 = 0.$$ 

This equation can be rewritten as

$$dt^2 = dr^2 \left[r^2 \left(\frac{d\phi}{dr}\right)^2 g_{tt}^{-1} - g_{rr} g_{tt}^{-1}\right].$$ 

Using the expansions given in Eqns. (9) and (10) we get

$$dt^2 = dr^2 \left[r^2 \left(\frac{d\phi}{dr}\right)^2 \left(1 + \frac{C}{r}\right) + 1 + \frac{2C}{r}\right],$$

which coincides with the equation for the GR case (see [13], pp. 204). Consequently, also in this case NDL theory prediction agrees with that of GR.

B. Black holes in NDL

We now investigate the existence of horizons in the metric given by Eqn. (8). The position of the putative horizons is given by the condition $g_{tt}(r_h) = 0$. This implies

$$F(\arccos(r_0/r_h), 1/\sqrt{2}) - F(\pi/2, 1/\sqrt{2}) = -\sqrt{2} \frac{r_0}{C}.$$ 

By standard manipulations (see for instance [11]) we obtain

$$r_h = r_0 \left[\text{cn} \left(F(\pi/2, 1/\sqrt{2}) - \sqrt{2} \frac{r_0}{C}\right)\right]^{-1}, \quad (18)$$

where the function $\text{cn}(x)$ is the cosine-amplitude. This expression provides the value of $r_h$ in terms of $C = 2M$ and the constant $|b|$. Note that $r_h > r_0$ always. Consequently, the region where the geometry is not defined ($r < r_0$) is always inside the surface $r = r_h$.

To determine whether Eqn. (18) defines a horizon or a singular surface, we can compute the components of the Riemann tensor calculated in the tetrad system

$$\omega_\mu^\lambda = \sqrt{1 + \mu(r)}, \quad \omega_\nu^\nu = \sqrt{1 + \nu(r)},$$

$$\omega_\theta^\theta = r, \quad \omega_\phi^\phi = r \sin \theta.$$ 

Here we will give the expression for only one of the components, which will be enough to illustrate the typical behaviour:

$$R_{\theta\phi\theta} = \frac{1}{2r} \frac{\mu'(r)}{(1 + \nu(r)) (1 + \mu(r))}. \quad (19)$$

Clearly this component of the Riemann tensor in the tetrad system depends on the product $(g_{tt}(r)g_{rr}(r))^{-1}$ which diverges at $r = r_h$ because $g_{tt}(r)$ is null at that point. Consequently, we would have a naked singular surface instead of a horizon [13]. We could attempt to cure this divergence by imposing that $g_{rr}(r)$ diverges at the horizon, in the hope of getting a finite and nonzero value for the denominator of Eqn. (13). Note however that $g_{rr}(r)$ only diverges at $r = r_0$, so it would be necessary to impose that $r_h = r_0$ to produce a horizon. To examine the behaviour of the product for $r_h = r_0$, we
expand the metric functions near \( r = r_0 \). It is convenient
to change variables to \( r = r_0 + R \). In terms of this new
variable, the significant terms in the series expansions are
\[
g_{tt}(r_0 + R) \approx 1 - \frac{C}{8(r_0 + R)} \sqrt{\frac{R}{r_0}} + \frac{9C}{8r_0} \arctan \left( \sqrt{\frac{R}{r_0}} \right),
\]
\[
g_{rr}(r_0 + R) \approx -1 - \frac{C}{2(t_0 + R)} \sqrt{\frac{t_0}{R}} \left( 1 + \frac{5R}{4r_0} \right).
\]
The only term that leads to a divergence in the limit
\( R \to 0 \) is the term proportional to \( 1/\sqrt{R} \) appearing in the
second expression. Taking the product \( g_{tt}(r_0 + R)g_{rr}(r_0 + R) \)
in the limit \( R \to 0 \) (and using that \( \arctan(x) \approx x - x^3/3 \)
for small \( x \)), it is clear that the divergence cannot be
eliminated. The analysis of the geometry seen by matter
then implies that static and spherically symmetric black
holes cannot be described by NDL theory: the would-be
horizon \( r = r_h \) is in fact a naked singularity.

C. Effective metric for gravitons

We have shown that there are no black holes for mat-
ter in NDL theory. However, there may be a black hole
configuration for gravitons. As previously mentioned, in
NDL theory gravitons interact nonlinearly with them-
selves. Consequently, the path of these particles is not
governed by the background metric but by an effective
metric given by
\[
\rho_{\mu\nu} = \gamma_{\mu\nu} + \Lambda_{\mu\nu},
\]
with
\[
\Lambda_{\mu\nu} = 2 \frac{\mathcal{L}A_1}{\mathcal{L}A} (F_{\mu}^{\alpha\beta} F_{\nu(\alpha\beta)} - F_{\mu} F_{\nu}),
\]
and \( \mathcal{L}(A-B) \) is an arbitrary Lagrangian. A short calculation
using the Born-Infeld-like Lagrangian from Eqn. (2)
and the metric given by Eqn. (8) shows that
\[
\rho_{00}(r) = 1 - \frac{\nu(r)^2}{b^2 + \nu(r)^2},
\]
with \( \rho_{11}(r) = -\rho_{00}(r) \), while \( \rho_{22}(r) \) and \( \rho_{33}(r) \) are as
in Minkowski spacetime. The result for the compo-
nent of the Riemann tensor given by Eqn. (15)
calculated with this metric indicates that there is a (naked)
singularity seen only by gravitons at \( r = r_0 \). Subse-
quently, the effective metric for gravitons cannot describe
a Schwarzschild black hole.

V. DISCUSSION

Our analysis revealed that NDL agrees with GR for
the three classical tests performed in the weak field limit.
Note however that there are potential disagreements in
situations in which gravity is strong. For instance, for
compact objects like neutron stars, for which typically
\( M/R \approx 10^5 M_{\odot}/R_{\odot} \), the convergence of the series expan-
sions Eqns. (8)-(11) will be much slower, and observable
effects would appear. Another instance in which the two
theories differ in their predictions is in the existence of
black holes. As we have shown, there are no solutions in
NDL theory that describe static and spherically symmet-
ric black holes. In fact, from the analysis in Sect.(III), we
conclude that the two possible cases in the metric seen
by matter lead to naked singularities.

Let us remark that there are two different types of sin-
gularities appearing in the problem, which we will call
geometrical and physical. For \( r_0 \neq r_h \), we encountered
two singularities: one located at \( r = r_h \) and another located
at \( r = r_0 \). We can call the first singularity geo-
metrical because the quantities related to the geometry
(like the effective potential and the Riemann tensor in
tetrads) diverge at \( r = r_h \) but the field \( F_{\alpha\beta\gamma} \) is finite
at that point (see Eqn. (14)). The energy density of the
field is also finite: its energy-momentum tensor is given by
\[
T_{\mu\nu} = -\mathcal{L}(A g) g_{\mu\nu} + 2 [2F_{\mu(\alpha\beta} F_{\nu\gamma)} + F_{\alpha\beta\mu} F_{\mu\nu} - F_{\mu} F_{\nu}]
- F_{(\mu\nu)\alpha\beta} F_{\alpha\beta} - F_{(\mu\nu)\gamma} F_{\gamma}.
\]
From this expression, and using the equations of motion,
we get for the energy density,
\[
T_{00} = -13 \frac{\nu^2}{r^2} + 2 \frac{\nu}{r} (\mu - \nu r) + 2 \frac{\nu^2}{r},
\]
which is finite at \( r = r_h \). On the other hand, at \( r = r_0 \)
the quantities related to the geometry do not diverge,
but there is a divergence of the field and of the energy
density.

The divergences of the geometrical and field quantities
are not compatible for \( r_0 = r_h \) either. We expect that
the divergences of the quantities related to the field
to be physical. The fact that the geometry does not display
the same pattern of divergences as that of the field sug-
gests that some modification should be introduced to the
theory. A possible alternative would be to assume that
the (unobservable) background is not flat spacetime but a
curved geometry (for instance, de Sitter spacetime).
Another assumption that must be checked is the postulated
coupling between matter and gravity (given by the metric
\( g_{\mu\nu} = \gamma_{\mu\nu} + \varphi_{\mu\nu} \)). Work in this direction is currently
under way.

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[14] We are using the index notation \([x,y] = xy - yx\), and \((xy) = xy + yx\).
[15] Note that in Schwarzschild geometry, \(g_{tt}g_{rr} = -1\), and the singularity at the surface \(r = r_h\) is eased out.
[16] Note that there is an additional term in this expression when compared with Eqn. (31) in Ref.[2].