A note on two weight commutators of maximal functions on spaces of homogeneous type

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Abstract: We study the two weight quantitative estimates for the commutator of maximal functions and the maximal commutators with respect to the symbol in weighted BMO space on spaces of homogeneous type. These commutators turn out to be controlled by the sparse operators in the setting of space of homogeneous type (developed in [9], originally introduced in [19]). The lower bound of the maximal commutator is also obtained.

Keywords: weighted BMO space; maximal commutator; two weights estimate.
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1 Introduction and statement of main results

In their remarkable result, Coifman–Rochberg–Weiss [7] showed that the commutator of Riesz transforms is bounded on \( L^p(\mathbb{R}^n) \) if and only if the symbol \( b \) is in the BMO space. See also the subsequent result by Janson [16] and Uchiyama [24]. Later, Bloom [4] obtained the two weight version of the commutator of Hilbert transform \( H \) with respect to weighted BMO space. To be more precise, for \( 1 < p < \infty \), let \( \lambda_1, \lambda_2 \) be weights in the Muckenhoupt class \( A_p \) and consider the weight \( \nu = \lambda_1^{1/p} \lambda_2^{-1/p} \). Let \( L^p_\nu(\mathbb{R}) \) denote the space of functions that are \( p \) integrable relative to the measure \( w(x)dx \). Then, by [4], there exist constants \( 0 < c < C < \infty \), depending only on \( p, \lambda_1, \lambda_2 \), such that

\[ c \|b\|_{\text{BMO}_\nu(\mathbb{R})} \leq \| [b, H] : L^p_{\lambda_1}(\mathbb{R}) \to L^p_{\lambda_2}(\mathbb{R}) \| \leq C \|b\|_{\text{BMO}_\nu(\mathbb{R})} \]

in which \([b, H](f)(x) = b(x)H(f)(x) - H(bf)(x)\) denotes the commutator of the Hilbert transform \( H \) and the function \( b \in \text{BMO}_\nu(\mathbb{R}) \), i.e., the Muckenhoupt–Wheeden weighted BMO space (introduced in [22], see also the definition in Section 2.4 below). This result provided a characterization of the boundedness of the commutator \([b, H] : L^p_{\lambda_1}(\mathbb{R}) \to L^p_{\lambda_2}(\mathbb{R})\) in terms of a triple of information \( b, \lambda_1 \) and \( \lambda_2 \). This result was extended very recently to the commutator of Riesz transform in \( \mathbb{R}^n \) by Holmes–Lacey–wick [13] by using different method via representation theorem for the Riesz transforms. Lerner–Ombrosi–Rivera-Ríos also proved it in [19] by using sparse domination and their method was generalised to space of homogeneous type in [9].

In [3, Propositions 4 and 6], Bastero–Milman–Ruiz characterized the class of functions for which the commutator with the Hardy–Littlewood maximal function and the maximal sharp function are bounded on \( L^p \). Later, García-Cuerva et al. [12, Theorem 2.4] proved that the maximal commutator \( C_b \) is bounded from \( L^p_{\lambda_1}(\mathbb{R}^n) \) to \( L^p_{\lambda_2}(\mathbb{R}^n) \), \( 1 < p < \infty \), if and only if \( b \in \text{BMO}_\nu(\mathbb{R}^n) \) with \( \nu = \lambda_1^{1/p} \lambda_2^{-1/p} \), where \( C_b \) is defined by

\[ C_b(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |b(x) - b(y)||f(y)|dy. \]
Hu–Yang [14] also studied the unweighted upper bound of the maximal commutator $C_b(f)$ on spaces of homogeneous type by adapting the approach in [12]. Recently, Agcayazi et al [1] also studied the unweighted version of the maximal commutator $C_b(f)$ on $\mathbb{R}^n$ by using different approach, and this was extended to space of homogeneous type by Fu et al [11].

In this paper, we aim to provide a quantitative estimate for the two weight commutator of maximal functions $[b, \mathcal{M}]$ and the maximal commutator $C_b$ with the symbol $b$ in weighted $\text{BMO}$ space on spaces of homogeneous type. To be more precise, let $(X, d, \mu)$ be a space of homogeneous type in the sense of Coifman and Weiss [8] (see the definition and details in Section 2 below). The Hardy–Littlewood maximal function $Mf(x)$ on $X$ is defined as

$$\mathcal{M}f(x) := \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)| \, d\mu(y),$$

where the supremum is taken over all balls $B \subset X$. The maximal commutator $C_b$ on $X$ with the symbol $b(x)$ is defined by

$$C_b(f)(x) := \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |b(x) - b(y)||f(y)| \, d\mu(y),$$

where the supremum is taken over all balls $B \subset X$.

Our first result is the quantitative estimate of $[b, \mathcal{M}]$.

**Theorem 1.1.** Suppose $1 < p < \infty$, $\lambda_1, \lambda_2 \in A_p$, $\nu := \lambda_1^{\frac{1}{p}} \lambda_2^{-\frac{1}{p}}$. Suppose $b \in \text{BMO}_\nu(X)$. Then there exists a positive constant $C$ such that

$$\|[b, \mathcal{M}] : L^p_{\lambda_1}(X) \to L^p_{\lambda_2}(X)\| \leq C \left( [\lambda_1]_{A_p} [\lambda_2]_{A_p} \right)^{\max\{1, \frac{1}{p'-1}\}} \|b\|_{\text{BMO}_\nu(X)}.$$

Note that $[b, \mathcal{M}]f(x)$ is dominated by $C_b(f)(x)$. To prove the above result, it suffices to show that

**Theorem 1.2.** Suppose $1 < p < \infty$, $\lambda_1, \lambda_2 \in A_p$, $\nu := \lambda_1^{\frac{1}{p}} \lambda_2^{-\frac{1}{p}}$. Suppose $b \in \text{BMO}_\nu(X)$. Then there exists a positive constant $C$ such that

$$\|C_b : L^p_{\lambda_1}(X) \to L^p_{\lambda_2}(X)\| \leq C \left( [\lambda_1]_{A_p} [\lambda_2]_{A_p} \right)^{\max\{1, \frac{1}{p'-1}\}} \|b\|_{\text{BMO}_\nu(X)}.$$

We note that the approach we use is via sparse domination of $C_b$ (see Section 3.1), and hence we obtain a better quantitative estimate with respect to the weights $\lambda_1$ and $\lambda_2$ comparing to the methods used in [12] and [1]. We will explain this in Section 3.2.

We point out that the lower bound of $\|C_b : L^p_{\lambda_1}(X) \to L^p_{\lambda_2}(X)\|$ is also true.

**Theorem 1.3.** Suppose $1 < p < \infty$, $\lambda_1, \lambda_2 \in A_p$, $\nu := \lambda_1^{\frac{1}{p}} \lambda_2^{-\frac{1}{p}}$. Suppose $b \in L^1_{\text{loc}}(X)$ and that $C_b$ is bounded from $L^p_{\lambda_1}(X)$ to $L^p_{\lambda_2}(X)$. Then $b \in \text{BMO}_\nu(X)$, and there exists a positive constant $C$ such that

$$\|b\|_{\text{BMO}_\nu(X)} \leq C \|C_b : L^p_{\lambda_1}(X) \to L^p_{\lambda_2}(X)\|.$$

We will provide the proof in Section 4.

Throughout this paper we assume that $\mu(X) = \infty$ and that $\mu(\{x_0\}) = 0$ for every $x_0 \in X$. Also we denote by $C$ and $\tilde{C}$ positive constants which are independent of the main parameters, but they may vary from line to line. For every $p \in (1, \infty)$, we denote by $p'$ the conjugate of $p$, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. If $f \leq Cg$ or $f \geq Cg$, we then write $f \lesssim g$ or $g \gtrsim f$; and if $f \gtrsim g \lesssim f$, we write $f \approx g$. 

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2 Preliminaries on Spaces of Homogeneous Type

We say that \((X, d, \mu)\) is a space of homogeneous type in the sense of Coifman and Weiss if \(d\) is a quasi-metric on \(X\) and \(\mu\) is a nonzero measure satisfying the doubling condition. A quasi-metric \(d\) on a set \(X\) is a function \(d : X \times X \rightarrow [0, \infty)\) satisfying (i) \(d(x, y) = d(y, x) \geq 0\) for all \(x, y \in X\); (ii) \(d(x, y) = 0\) if and only if \(x = y\); and (iii) the quasi-triangle inequality: there is a constant \(A_0 \in [1, \infty)\) such that for all \(x, y, z \in X\),

\[
d(x, y) \leq A_0[d(x, z) + d(z, y)].
\]

We say that a nonzero measure \(\mu\) satisfies the doubling condition if there is a constant \(C_\mu\) such that for all \(x \in X\) and \(r > 0\),

\[
\mu(B(x, 2r)) \leq C_\mu \mu(B(x, r)) < \infty,
\]

where \(B(x, r)\) is the quasi-metric ball by \(B(x, r) := \{y \in X : d(x, y) < r\}\) for \(x \in X\) and \(r > 0\). We point out that the doubling condition (2.2) implies that there exists a positive constant \(n\) (the upper dimension of \(\mu\)) such that for all \(x \in X\), \(\lambda \geq 1\) and \(r > 0\),

\[
\mu(B(x, \lambda r)) \leq C_\mu \lambda^n \mu(B(x, r)).
\]

A subset \(\Omega \subseteq X\) is open (in the topology induced by \(\rho\)) if for every \(x \in \Omega\) there exists \(\varepsilon > 0\) such that \(B(x, \varepsilon) \subseteq \Omega\). A subset \(F \subseteq X\) is closed if its complement \(X \setminus F\) is open. The usual proof of the fact that \(F \subseteq X\) is closed, if and only if it contains its limit points, carries over to the quasi-metric spaces. However, some open balls \(B(x, r)\) may fail to be open sets, see [15, Sec 2.1].

Constants that depend only on \(A_0\) (the quasi-metric constant) and \(A_1\) (the geometric doubling constant), are referred to as geometric constants.

2.1 A System of Dyadic Cubes

We recall from [15] (see also the previous work by M. Christ [5], as well as Sawyer–Wheeden [23]) the system of dyadic cubes. In a geometrically doubling quasi-metric space \((X, d)\), a countable family

\[
\mathcal{D} = \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k, \quad \mathcal{D}_k = \{Q^k_\alpha : \alpha \in \mathcal{A}_k\},
\]

of Borel sets \(Q^k_\alpha \subseteq X\) is called a system of dyadic cubes with parameters \(\delta \in (0, 1)\) and \(0 < c_1 \leq C_1 < \infty\) if it has the following properties:

\[
X = \bigcup_{\alpha \in \mathcal{A}_k} Q^k_\alpha \quad \text{(disjoint union) for all } k \in \mathbb{Z};
\]

\[
\text{if } \ell \geq k, \text{ then either } Q^\ell_\beta \subseteq Q^k_\alpha \text{ or } Q^k_\alpha \cap Q^\ell_\beta = \emptyset;
\]

\[
\text{for each } (k, \alpha) \text{ and each } \ell \leq k, \text{ there exists a unique } \beta \text{ such that } Q^\ell_\beta \subseteq Q^k_\alpha;
\]

\[
\text{for each } (k, \alpha) \there exists at most } M \text{ (a fixed geometric constant) } \beta \text{ such that } Q^{k+1}_\beta \subseteq Q^k_\alpha, \text{ and } Q^k_\alpha = \bigcup_{Q \in \mathcal{D}_{k+1}, Q \subseteq Q^k_\alpha} Q;
\]
The set \( Q^k \) is called a dyadic cube of generation \( k \) with center point \( x^k \) and side length \( \delta^k \). The interior and closure of \( Q^k \) are denoted by \( \tilde{Q}^k \) and \( Q^k \), respectively.

### 2.2 Adjacent Systems of Dyadic Cubes

In a geometrically doubling quasi-metric space \((X, d)\), a finite collection \( \mathcal{D}^t : t = 1, 2, \ldots, T \) of families \( \mathcal{D}^t \) is called a collection of adjacent systems of dyadic cubes with parameters \( \delta \in (0, 1), 0 < c_1 \leq C_1 < \infty \) and \( 1 < C < \infty \) if it has the following properties: individually, each \( \mathcal{D}^t \) is a system of dyadic cubes with parameters \( \delta \in (0, 1) \) and \( 0 < c_1 \leq C_1 < \infty \); collectively, for each ball \( B(x, r) \subseteq X \) with \( \delta^{k+3} < r \leq \delta^{k+2}, k \in \mathbb{Z} \), there exist \( t \in \{1, 2, \ldots, T\} \) and \( Q \in \mathcal{D}^t \) of generation \( k \) and with center point \( t x^k \) such that \( \rho(x, t x^k) < 2A_0 \delta^k \) and

\[
B(x, r) \subseteq Q \subseteq B(x, Cr).
\]  

We recall from [15] the following construction.

**Theorem 2.1.** Let \((X, d)\) be a geometrically doubling quasi-metric space. Then there exists a collection \( \{\mathcal{D}^t : t = 1, 2, \ldots, T\} \) of adjacent systems of dyadic cubes with parameters \( \delta \in (0, (96A_0^6)^{-1}), c_1 = (12A_0^4)^{-1}, C_1 = 4A_0^2 \) and \( C = 8A_0^3\delta^{-3} \). The center points \( t x^k \) of the cubes \( Q \in \mathcal{D}^k \) have, for each \( t \in \{1, 2, \ldots, T\} \), the two properties

\[
\rho(t x^k, t x^k) \geq (4A_0^2)^{-1} \delta^k (\alpha \neq \beta), \quad \min_{\alpha} \rho(x, t x^k) < 2A_0 \delta^k \quad \text{for all } x \in X.
\]

We recall from [17, Remark 2.8] that the number \( T \) of the adjacent systems of dyadic cubes as in the theorem above satisfies the estimate

\[
T = T(A_0, A_1, \delta) \leq A_0^6(A_0^4/\delta)^{\log_2 A_1}.
\]  

Further, we also recall the following result on the smallness of the boundary.

**Proposition 2.2.** Suppose that \( 144A_0^3 \delta \leq 1 \). Let \( \mu \) be a positive \( \sigma \)-finite measure on \( X \). Then the collection \( \{\mathcal{D}^t : t = 1, 2, \ldots, T\} \) may be chosen to have the additional property that

\[
\mu(\partial Q) = 0 \quad \text{for all } Q \in \bigcup_{t=1}^T \mathcal{D}^t.
\]

### 2.3 Muckenhoupt \( A_p \) Weights

**Definition 2.3.** Let \( \omega(x) \) be a nonnegative locally integrable function on \( X \). For \( 1 < p < \infty \), we say \( \omega \) is an \( A_p \) weight, written \( \omega \in A_p \), if

\[
[w]_{A_p} := \sup_B \left( \frac{1}{f_B} \int_B \left( \frac{1}{w} \right)^{1/(p-1)} \right)^{p-1} < \infty.
\]

Here the suprema are taken over all balls \( B \subseteq X \). The quantity \([w]_{A_p}\) is called the \( A_p \) constant of \( w \). For \( p = 1 \), we say \( w \) is an \( A_1 \) weight, written \( w \in A_1 \), if \( M(w)(x) \leq w(x) \) for \( \mu \)-almost every \( x \in X \), and let \( A_\infty := \cup_{1 \leq p < \infty} A_p \) and we have \([w]_{A_\infty} := \sup_B \left( \int_B w \exp \left( \int_B \log \left( \frac{1}{w} \right) \right) \right) < \infty \).
Next we note that for \( w \in A^p \) the measure \( w(x) d\mu(x) \) is a doubling measure on \( X \). To be more precise, we have that for all \( \lambda > 1 \) and all balls \( B \subset X \),
\[
    w(\lambda B) \leq \lambda^{np}[w]_{A^p} w(B),
\]
where \( n \) is the upper dimension of the measure \( \mu \), as in (2.3).

We also point out that for \( w \in A^\infty \), there exists \( \gamma > 0 \) such that for every ball \( B \),
\[
    \mu\left( \left\{ x \in B : w(x) \geq \gamma \int_B w \right\} \right) \geq \frac{1}{2}\mu(B).
\]
And this implies that for every ball \( B \) and for all \( \delta \in (0, 1) \),
\[
    \fint_B w \leq C \left( \fint_B w^\delta \right)^{1/\delta} ;
\]
see also [20].

2.4 Weighted BMO spaces

Next we recall the definition of the weighted BMO space on space of homogeneous type, while we point out that the Euclidean version was first introduced by Muckenhoupt and Wheeden [22].

**Definition 2.4.** Suppose \( w \in A^\infty \). A function \( b \in L^1_{\text{loc}}(X) \) belongs to the weighted BMO space \( BMO^w(X) \) if
\[
    \|b\|_{BMO^w(X)} := \sup_B \frac{1}{w(B)} \int_B |b(x) - b_B| d\mu(x) < \infty,
\]
where the sup is taken over all quasi-metric balls \( B \subset X \) and
\[
    b_B = \frac{1}{\mu(B)} \int_B f(y) d\mu(y).
\]

2.5 Sparse Operators on Spaces of Homogeneous Type

Let \( \mathcal{D} \) be a system of dyadic cubes on \( X \) as in Section 2.1. We recall the sparse operators on spaces of homogeneous type as studied in [21, 9].

**Definition 2.5.** Given \( 0 < \eta < 1 \), a collection \( S \subset \mathcal{D} \) of dyadic cubes is said to be \( \eta \)-sparse if for every cube \( Q \in \mathcal{D} \),
\[
    \sum_{P \in S : P \subset Q} \mu(P) \leq \frac{1}{\eta} \mu(Q).
\]

Note, that for a collection \( S \subset \mathcal{D} \) of dyadic cubes with the property that for \( 0 < \eta < 1 \) and for every \( Q \in S \), there is a measurable subset \( E_Q \subset Q \) such that \( \mu(E_Q) \geq \eta \mu(Q) \) and the sets \( \{E_Q\}_{Q \in S} \) have only finite overlap, we will have that \( S \) is \( \eta \)-sparse according to Definition 2.5 (following from the standard computation).

We now recall the well-known definition for sparse operator.

**Definition 2.6.** Given \( 0 < \eta < 1 \) and an \( \eta \)-sparse family \( S \subset \mathcal{D} \) of dyadic cubes. The sparse operators \( A_S \) is defined by
\[
    A_S f(x) := \sum_{Q \in S} f_Q \chi_Q(x).
\]
Following the proof of [21, Theorem 3.1], we obtain that
\[ \| A_S f \|_{L^p_w(X)} \leq C_{\eta,n,p} \left[ w \right]_{A_p}^{\max\{1, \frac{1}{p - 1}\}}, \quad 1 < p < \infty. \]

Denote by \( \Omega(b, B) \) the standard mean oscillation
\[ \Omega(b, B) = \frac{1}{\mu(B)} \int_B |b(x) - b_B| d\mu(x). \]

We recall from [9, Lemma 3.5] the following result.

**Lemma 2.7.** Let \( D \) be a dyadic system in \( X \) and let \( S \subset D \) be a \( \gamma \)-sparse family. Assume that \( b \in L^1_{\text{loc}}(X) \). Then there exists a \( \gamma \)\({\frac{\gamma}{\gamma + 1}}\)-sparse family \( \tilde{S} \subset D \) such that \( S \subset \tilde{S} \) and for every cube \( Q \in \tilde{S} \),
\[ |b(x) - b_Q| \leq C \sum_{R \in \tilde{S}, R \subset Q} \Omega(b, R) \chi_R(x) \quad \text{(2.14)} \]
for a.e. \( x \in Q \).

### 3 Upper Bound of the Maximal Commutator \( C_b \)

In this section we provide the proof of Theorem 1.2, which implies Theorem 1.1

#### 3.1 Sparse domination of the maximal commutator \( C_b \)

Given a ball \( B_0 \subset X \), for \( x \in B_0 \) we define a local grand maximal truncated operator \( M_{B_0} \) as follows:
\[ M_{B_0} f(x) := \sup_{B \ni x, B \subset B_0} \text{ess sup}_{\xi \in B} M(f\chi_{4A_0B_0 \setminus 4A_0B})(\xi). \]

Using the idea of [18, Lemma 3.2], we can obtain the following lemma.

**Lemma 3.1.** For \( \mu \)-almost every \( x \in B_0 \),
\[ M(f\chi_{4A_0B_0})(x) \leq C \| M \|_{L^1 \rightarrow L^\infty} |f(x)| + M_{B_0} f(x). \]

**Proof.** Suppose that \( x \in B_0 \), and let \( x \) be a point of approximate continuity of \( M(f\chi_{4A_0B_0}) \) (see, e.g., [10], p. 46). Then for every \( \varepsilon > 0 \), the sets
\[ E_\varepsilon(x) = \{ y \in B(x, s) : |M(f\chi_{4A_0B_0})(y) - M(f\chi_{4A_0B_0})(x)| < \varepsilon \} \]
satisfy \( \lim_{s \to 0} \frac{\mu(E_\varepsilon(x))}{\mu(B(x, s))} = 1 \).

Then for a.e. \( y \in E_\varepsilon(x) \),
\[ M(f\chi_{4A_0B_0})(x) \leq M(f\chi_{4A_0B_0})(y) + \varepsilon \leq M(f\chi_{4A_0B(x,s)})(y) + M_{B_0} f(x) + \varepsilon. \]
Therefore, applying the weak type \((1, 1)\) of \( M \) yields
\[ M(f\chi_{4A_0B_0})(x) \leq \text{ess inf}_{y \in E_\varepsilon(x)} M(f\chi_{4A_0B(x,s)})(y) + M_{B_0} f(x) + \varepsilon \]
Theorem 3.2. Here, \( R \)

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\[
\|M\|_{L^1 \to L^{1,\infty}} \leq C \frac{1}{\mu(E_s(x))} \int_{4A_0 B(x,s)} |f(x)| \, d\mu(x) + M_{B_0} f(x) + \varepsilon.
\]

Assuming additionally that \( x \) is a Lebesgue point of \( f \) and letting subsequently \( s \to 0 \) and \( \varepsilon \to 0 \), we complete the proof of this lemma.

Next, following from [19] (see also [9]) we give the sparse operator \( T_{S,b} \) defined by

\[
T_{S,b}(f)(x) = \sum_{Q \in S} |b(x) - b_Q| f \chi_Q(x).
\]

And we let \( T_{S,b}^* \) denote the adjoint operator to \( T_{S,b} \):

\[
T_{S,b}^*(f)(x) = \sum_{Q \in S} \left( \frac{1}{\mu(Q)} \int_Q |b(y) - b_Q| f(y) \, d\mu(y) \right) \chi_Q(x).
\]

Then we have the following result.

**Theorem 3.2.** For every compactly supported \( f \in L^\infty(X) \), there exists \( T \) dyadic systems \( D^t, t = 1, 2, \ldots, T \) and \( \eta \)-sparse families \( S_t \subset D^t \) such that for a.e. \( x \in X \),

\[
|C_b(f)(x)| \leq C \sum_{t=1}^T \left( T_{S_t,b}(|f|)(x) + T_{S_t,b}^* (|f|)(x) \right).
\]

**Proof.** We recall from Section 2.2, for each ball \( B(x,r) \subset X \) with \( \delta^{k+3} < r \leq \delta^{k+2}, k \in \mathbb{Z} \), there exist \( t \in \{1, 2, \ldots, T\} \) and \( Q \in D^t \) of generation \( k \) and with center point \( t x^k \) such that \( \rho(x, t x^k) < 2A_0 \delta^k \) and \( B(x,r) \subset Q \subset B(x, Cr) \). Here and in what follows, \( A_0 \) denotes the constant in (2.1).

Fix a ball \( B_0 \subset X \), then it is clear that there exist a positive constant \( C, t_0 \in \{1, 2, \ldots, T\} \) and \( Q_0 \in D^{t_0} \) such that \( 4A_0 B_0 \subset Q_0 \subset C(4A_0 B_0) \). We now show that there exists a \( \frac{1}{8C_{A_0, \mu}} \)

sparse family \( F^{t_0} \subset D^{t_0}(B_0) \) such that for a.e. \( x \in B_0 \),

\[
|C_b(f \chi_{4A_0 B_0})(x)| \leq C \sum_{Q \in F^{t_0}} \left( |b(x) - b_{R_Q}| |f|_{4A_0 Q} + |b - b_{R_Q}| f|_{4A_0 Q} \right) \chi_Q(x).
\]

Here, \( R_Q \) is the dyadic cube in \( D^t \) for some \( t \in \{1, 2, \ldots, T\} \) such that \( 4A_0 Q \subset R_Q \subset C(4A_0 Q) \).

It suffices to prove the following recursive claim: there exist disjoint cubes \( P_j \in D^{t_0}(B_0) \) such that \( \sum_j \mu(P_j) \leq \frac{1}{2} \mu(B_0) \) and

\[
|C_b(f \chi_{4A_0 B_0})(x)| \chi_{B_0} \leq C \left( |b(x) - b_{Q_0}| |f|_{4A_0 B_0} + |b - b_{Q_0}| f|_{4A_0 B_0} \right) + \sum_j |C_b(f \chi_{4A_0 P_j})(x)| \chi_{P_j}
\]

a.e. on \( B_0 \). Here we have a \( \frac{1}{2} \)-sparse family since the sets \( E_Q = Q \setminus \bigcup_j P_j \), and then we can appeal to the discussion after Definition 2.5.

Now observe that for arbitrary disjoint cubes \( P_j \in D^{t_0}(B_0) \),

\[
\begin{align*}
|C_b(f \chi_{4A_0 B_0})(x)| \chi_{B_0} \\
= |C_b(f \chi_{4A_0 B_0})(x)| \chi_{B_0 \setminus \bigcup_j P_j} + \sum_j |C_b(f \chi_{4A_0 B_0})(x)| \chi_{P_j}
\end{align*}
\]


\[ |C_b(f \chi_{4A_0B_0})(x)|_{\chi_{B_0} \cup P_j} + \sum_j |C_b(f \chi_{4A_0B_0 \setminus 4A_0P_j})(x)|_{\chi_{P_j}} + \sum_j |C_b(f \chi_{4A_0P_j})(x)|_{\chi_{P_j}}. \]

Hence, in order to prove the recursive claim (3.3), it suffices to show that one can select pairwise disjoint cubes \( P_j \in D^{\mu}(B_0) \) with \( \sum_j \mu(P_j) \leq 2\mu(B_0) \) and such that for a.e. \( x \in B_0, \)

\[ |C_b(f \chi_{4A_0B_0})(x)|_{\chi_{B_0} \cup P_j} + \sum_j |C_b(f \chi_{4A_0B_0 \setminus 4A_0P_j})(x)|_{\chi_{P_j}} \leq C \left( |b(x) - b_{Q_0}| f|_{4A_0B_0} + |(b - b_{Q_0}) f|_{4A_0B_0} \right). \]

To see this, by definition, we obtain that

\[ |C_b(f \chi_{4A_0B_0})(x)|_{\chi_{B_0} \cup P_j} + \sum_j |C_b(f \chi_{4A_0B_0 \setminus 4A_0P_j})(x)|_{\chi_{P_j}} \leq |b(x) - b_{Q_0}| \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)| \chi_{4A_0B_0}(y) d\mu(y) \chi_{B_0 \cup P_j}(x) \]

\[ + |b(x) - b_{Q_0}| \sum_j \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)| \chi_{4A_0B_0 \setminus 4A_0P_j}(y) d\mu(y) \chi_{P_j}(x) \]

\[ + \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |b(y) - b_{Q_0}| |f(y)| \chi_{4A_0B_0}(y) d\mu(y) \chi_{B_0 \cup P_j}(x) \]

\[ + \sum_j \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |b(y) - b_{Q_0}| |f(y)| \chi_{4A_0B_0 \setminus 4A_0P_j}(y) d\mu(y) \chi_{P_j}(x). \]

We now choose \( \alpha \) such that the set \( E := E_1 \cup E_2 \cup E_3 \cup E_4 \), with

\[ E_1 = \{ x \in B_0 : |f(x)| > \alpha |f|_{4A_0B_0} \}, \]

\[ E_2 = \{ x \in B_0 : \mathcal{M}_{B_0} f(x) > \alpha C |f|_{4A_0B_0} \}, \]

\[ E_3 = \{ x \in B_0 : |(b(x) - b_{Q_0}) f(x)| > \alpha |(b - b_{Q_0}) f|_{4A_0B_0} \}, \]

and

\[ E_4 = \{ x \in B_0 : \mathcal{M}_{B_0} ((b - b_{Q_0}) f)(x) > \alpha C |(b - b_{Q_0}) f|_{4A_0B_0} \}, \]

will satisfy

\[ \mu(E) \leq \frac{1}{2^{2n+1}} \mu(B_0). \]

We now apply the Calderón–Zygmund decomposition to the function \( \chi_E \) on \( B_0 \) at the height \( \lambda = \frac{1}{2^{2n+1}} \), where \( n \) is the upper dimension of the measure \( \mu \) as in (2.3), to obtain the pairwise disjoint cubes \( P_j \in D^{\mu}(B_0) \) such that

\[ \chi_E(x) \leq \frac{1}{2^{2n+1}} \quad \text{a.e. } x \notin \cup_j P_j \]
and hence we have that $\mu(E \setminus \cup_j P_j) = 0$. Moreover, we have that
\[
\sum_j \mu(P_j) = \mu\left(\bigcup_j P_j\right) \leq 2^{n+1} \mu(E) \leq \frac{1}{2} \mu(B_0),
\]
and that
\[
\frac{1}{2^{n+1}} \leq \frac{1}{\mu(P_j)} \int_{P_j} \chi_E(x) d\mu(x) = \frac{\mu(P_j \cap E)}{\mu(P_j)} \leq \frac{1}{2},
\]
which implies that
\[
P_j \cap E^c \neq \emptyset.
\]

Therefore, we observe that for each $P_j$, since $P_j \cap E^c \neq \emptyset$, we have that
\[
\mathcal{M}_{B_0}\left((b - b_{Q_0})f\right)(x) \leq \alpha C |(b - b_{Q_0})f|_{A_0B_0}
\]
for some $x \in P_j$, which implies that
\[
\text{ess sup}_{x \in P_j} \sup_{B \ni x \mu(B)} \int_B |b(y) - b_{Q_0}| |f(y)| \chi_{A_0B_0 \setminus A_0P_j}(y) d\mu(y) \leq \alpha C |(b - b_{Q_0})f|_{A_0B_0},
\]
Similarly, we have
\[
\text{ess sup}_{x \in P_j} \sup_{B \ni x \mu(B)} \int_B |f(y)| \chi_{A_0B_0 \setminus A_0P_j}(y) d\mu(y) \leq \alpha C |A_0B_0|.
\]

Also, by Lemma 3.1, we have that
\[
\sup_{B \ni x \mu(B)} \int_B |f(y)| \chi_{A_0B_0}(y) d\mu(y) \leq C |f(x)| + \mathcal{M}_{B_0}f(x).
\]

and
\[
\sup_{B \ni x \mu(B)} \int_B |b(y) - b_{Q_0}| |f(y)| \chi_{A_0B_0}(y) d\mu(y) \leq C |b(x) - b_{Q_0}| |f(x)| + \mathcal{M}_{B_0}\left((b - b_{Q_0})f\right)(x).
\]

Since $\mu(E \setminus \cup_j P_j) = 0$, we have that from the definition of the set $E$, the following estimates
\[
|f(x)| \leq \alpha |f|_{A_0B_0}, \quad |(b(x) - b_{Q_0})f(x)| \leq \alpha |(b - b_{Q_0})f|_{A_0B_0}
\]
hold for $\mu$-almost every $x \in B_0 \setminus \cup_j P_j$, and also
\[
\mathcal{M}_{B_0}f(x) \leq \alpha C |f|_{A_0B_0}, \quad \mathcal{M}_{B_0}\left((b - b_{Q_0})f\right)(x) \leq \alpha C |(b - b_{Q_0})f|_{A_0B_0}
\]
hold for $\mu$-almost every $x \in B_0 \setminus \cup_j P_j$.

Combining these facts with (3.5), we see that (3.4) holds, which further implies that (3.2) holds.

We now consider the partition of the space as follows. Suppose $f$ is supported in a ball $B_0 \subset X$. We have
\[
X = \bigcup_{j=0}^{\infty} 2^j B_0.
\]
Next, we further set $Q_0$ for $x \in B_{0,\mu}$.

Now we let $A_j = 2^j B_0 \setminus 2^{j-1} B_0$ for $j \geq 1$. It is clear that we can choose the balls $\{\tilde{B}_{j,\ell}\}_{\ell=1}^{L_j}$ with radius $2^{j-2} \mu$ to cover $U_j$, satisfying that the center of each the ball $\tilde{B}_{j,\ell}$ is in $U_j \neq \emptyset$ and that $\sup_j L_j \leq C_{A_0,\mu}$, where $C_{A_0,\mu}$ is an absolute constant depending on $A_0$ and $\mu$, only, here $C_\mu$ is the constant as in (2.2).

Moreover, we also have that for each such $\tilde{B}_{j,\ell}$, the enlargement $4A_0 \tilde{B}_{j,\ell}$ covers $B_0$. Also, we note that for each $\tilde{B}_{j,\ell}$, there exist a positive constant $C$, $t_{j,\ell} \in \{1, 2, \ldots, T\}$ and $Q_{j,\ell} \in \mathcal{D}_{j,\ell}$ such that $4A_0 \tilde{B}_{j,\ell} \subseteq Q_{j,\ell} \subseteq C(4A_0 \tilde{B}_{j,\ell})$.

We now apply (3.2) to each $\tilde{B}_{j,\ell}$, then we obtain a $\frac{1}{\gamma}$-sparse family $\tilde{F}_{j,\ell} \subseteq \mathcal{D}_{j,\ell}(\tilde{B}_{j,\ell})$ such that (3.2) holds for a.e. $x \in B_{j,\ell}$.

Now we set $F = \cup_{\ell} \tilde{F}_{j,\ell}$. Note that the balls $\tilde{B}_{j,\ell}$ are overlapping at most $4C_{A_0,\mu}$ times. Then we obtain that $F$ is a $\frac{1}{\gamma C_{A_0,\mu}}$-sparse family and for a.e. $x \in X$, \[ |C_b(f)(x)| \leq C \sum_{Q \in F} \left( |b(x) - b_Q| |f|_{A_0Q} + |(b - b_Q) f|_{A_0Q} \right) \chi_Q(x). \] (3.6)

Since $4A_0 Q \subset Q_R$, and it is clear that $\mu(Q_R) \leq C \mu(4A_0 Q)$, we obtain that $|f|_{A_0Q} \leq C |f|_{R_Q}$. Next, we further set \[ S_t = \{ Q \in D^t : \; Q \in F \}, \quad t \in \{1, 2, \ldots, T\}, \]
and from the fact that $F$ is $\frac{1}{\gamma C_{A_0,\mu}}$-sparse, we can obtain that each family $S_t$ is $\frac{1}{\gamma C_{A_0,\mu}}$-sparse.

Now we let \[ \eta = \frac{1}{8C_{A_0,\mu}}. \]

Then it follows that \[ |C_b(f)(x)| \leq C \sum_{t=1}^T \sum_{R \in S_t} \left( |b(x) - b_R| |f|_R + |(b - b_R) f|_R \right) \chi_R(x), \] (3.7)
finishing the proof.

\[ \square \]

### 3.2 Proof of Theorem 1.2

**Proof.** Let $D$ be a dyadic system in $(X, d, \mu)$ and let $S$ be a sparse family from $D$. By Theorem 3.2, we only need to prove
\[ \|T_{S, b} : L^p_{A_1} (X) \rightarrow L^p_{A_2} (X)\| \leq C \left( [\lambda_1 A_p][\lambda_2 A_p] \right)^{\max\{1, \frac{1}{p'}\}} \|b\|_{\text{BMO}_\nu(X)} \]
and \[ \|T_{S, b}^* : L^p_{A_1} (X) \rightarrow L^p_{A_2} (X)\| \leq C \left( [\lambda_1 A_p][\lambda_2 A_p] \right)^{\max\{1, \frac{1}{p'}\}} \|b\|_{\text{BMO}_\nu(X)}. \]

By duality, we have that
\[ \|T_{S, b} f\|_{L^p_{A_2} (X)} \leq \sup_{g \in [L^p_{A_1} (X)]} \sum_{Q \in S} \left( \int_Q |g(x)\lambda_2(x)||b(x) - b_Q| d\mu(x) \right) |f|_Q. \] (3.8)

Now by Lemma 3.3, there exists a sparse family $\tilde{S} \subset D$ such that $S \subset \tilde{S}$ and for every cube $Q \in \tilde{S}$, for $\mu$-almost every $x \in Q$, \[ |b(x) - b_Q| \leq C \sum_{P \in \tilde{S}, P \subset Q} \Omega(b, P) \chi_P(x). \]
Since $b$ is in $\text{BMO}_\nu(X)$, then we have for $\mu$-almost every $x \in Q$

$$|b(x) - b_Q| \leq C\|b\|_{\text{BMO}_\nu(X)} \sum_{P \in \mathcal{S}, P \subset Q} \frac{\nu(P)}{\mu(P)} \chi_P(x).$$

Then combining this estimate and inequality (3.8), we further have

$$\|T_S b f\|_{L^p_{\mathcal{S}}(X)} \leq C\|b\|_{\text{BMO}_\nu(X)} \sup_{g \in \mathcal{S}} \left( \sum_{Q \in \mathcal{S}} \left( \sum_{P \in \mathcal{S}, P \subset Q} \frac{\nu(P)}{\mu(P)} \int_Q |g(x)\lambda_2(x)| \mathcal{A}_S([g\lambda_2](x)) \nu(x) d\mu(x) \right) |f|_Q \right).$$

Observe that $A_S$ is self-adjoint. Then by Hölder’s inequality, we have

$$\|T_S b f\|_{L^p_{\mathcal{S}}(X)} \leq C\|b\|_{\text{BMO}_\nu(X)} \sup_{g \in \mathcal{S}} \left( \sum_{Q \in \mathcal{S}} \left( \int_Q |g(x)\lambda_2(x)| \mathcal{A}_S([g\lambda_2](x)) \nu(x) d\mu(x) \right) |f|_Q \right).$$

Similarly, we can obtain

$$\|T_{S^*} b f\|_{L^p_{\mathcal{S}}(X)} \leq C\|b\|_{\text{BMO}_\nu(X)} \left( \left[ \lambda_1 \right]_{A_p} \left[ \lambda_2 \right]_{A_p} \right)^{\max\left(1, \frac{1}{p-1}\right)} \|f\|_{L^p_{\mathcal{S}}(X)}.$$
This completes the proof of Theorem 1.2.

We point out that the quantitative estimate with respect to the weights \( \lambda_1 \) and \( \lambda_2 \) that we obtain here is better comparing to the methods used in [12] and [1].

- In [12], for \( \lambda_1^{-1}, \lambda_2^{-1} \in A_1, \nu = \lambda_1 \lambda_2^{-1} \) and \( b \in \text{BMO}_\nu \), they obtained that
  \[
  \|\lambda_2 C_b(f)\|_{L^\infty} \leq C'(\lambda_1, \lambda_2)\|b\|_{\text{BMO}_\nu} \|\lambda_1 f\|_{L^\infty}.
  \]
  Then by extrapolation they obtained that for \( 1 < p < \infty, \lambda_1, \lambda_2 \in A_p, \nu = \lambda_1^{\frac{1}{p}} \lambda_2^{-\frac{1}{p}} \) and \( b \in \text{BMO}_\nu \),
  \[
  \|C_b(f)\|_{L^p_{\nu}} \leq C(b, \lambda_1, \lambda_2)\|f\|_{L^p_{\lambda_1}}.
  \]
  They only showed that \( C(b, \lambda_1, \lambda_2) \) depends on \( b, \lambda_1, \lambda_2 \).

- In [1], to prove the upper bound of \( C_b \) in \( \mathbb{R}^n \), they first proved that (see Corollary 1.11 in [1])
  \[
  C_b(f)(x) \leq C\|b\|_* \mathcal{M}^2(f)(x),
  \]
  where \( \|b\|_* \) is the norm
  \[
  \|b\|_* = \left( \sup_B \frac{1}{|B|} \int_B |b(x) - b_B|^p \, dx \right)^{\frac{1}{p}}.
  \]
  By using John–Nirenberg’s inequality we know that it is equivalent to the BMO norm, that is
  \[
  \|b\|_{\text{BMO}(\mathbb{R}^n)} \leq \|b\|_* \leq C\|b\|_{\text{BMO}(\mathbb{R}^n)}.
  \]
  Using this result directly, one can only get the one weight boundedness for \( C_b \). That is, for \( 1 < p < \infty, \lambda \in A_p \) and \( b \in \text{BMO}(\mathbb{R}^n) \),
  \[
  \|C_b(f)\|_{L^p_{\lambda}} \leq C\|b\|_{\text{BMO}(\mathbb{R}^n)} \lambda^{2\max\{1, \frac{1}{p-1}\}} \|f\|_{L^p_{\lambda}}.
  \]
  If we try to use this approach to obtain the two weight upper bound for \( C_b \) in our setting, we will first need to obtain the quantitative estimate for the John–Nirenberg inequality for weighted BMO. However, this quantitative estimate, together with the quantitative estimate for \( \mathcal{M}^2(f)(x) \), is certainly larger than what we obtained by using sparse operator.

For completeness, we provide the quantitative estimate for the John–Nirenberg inequality for weighted BMO as follows.

We now provide the quantitative version of Bloom’s estimate [4, Lemma 4.3] on spaces of homogeneous type.

**Lemma 3.3.** Suppose \( 1 < p < \infty, \lambda_1, \lambda_2 \in A_p, \nu := \lambda_1^{\frac{1}{p}} \lambda_2^{-\frac{1}{p}} \). Suppose \( b \in \text{BMO}_\nu(X) \). Then there exists \( \varepsilon > 0 \) such that for all \( 1 \leq r < p' + \varepsilon, \)
\[
\frac{1}{\mu(B)} \int_B |b(x) - b_B|^r \lambda_1^{-\frac{r}{p}}(x) \, d\mu(x)
\leq \begin{cases} 
\|b\|_{\text{BMO}_\nu} \lambda_2^{\frac{r}{p'}}(x) \left( \frac{1}{\mu(B)} \int_B \lambda_2^{-\frac{r}{p'}}(x) \, d\mu(x) \right)^{\frac{r}{p'}} & 1 \leq r \leq p' \\
\left[ \lambda_1 A_p \lambda_2 A_p \|b\|_{\text{BMO}_\nu} \left( \frac{1}{\mu(B)} \int_B \lambda_2^{-\frac{r}{p'}}(x) \, d\mu(x) \right)^{\frac{r}{p'}} \right] & p' < r \leq p' + \varepsilon
\end{cases}
\]
for every ball \( B \).
Proof. We will prove result when \( r \leq p' \) using Holder’s Inequality. Without loss of generality we will assume \( r = p' \) to show (3.9). Similar proof works for any \( r < p' \). Fix a ball \( B \) and choose \( s = \frac{1}{p'} \) to use Holder’s Inequality.

\[
\begin{align*}
\frac{1}{\mu(B)} \int_B |b(x) - b_B|^p \lambda_1^{\frac{p'}{r}} d\mu(x) & \\
\leq & \left( \frac{1}{\nu(B)} \int_B |b(x) - b_B| d\mu(x) \right)^{\frac{p}{p'} \nu(B)^p} \left( \int_B \lambda_1^{\lambda_1^{\frac{p'}{r}}(x) d\mu(x)} \right)^{1 - p'} \\
\leq & \|b\|_{BMO} \nu(B)^p \left( \int_B \lambda_1^{\lambda_1^{\frac{p'}{r}}(x) d\mu(x)} \right)^{1 - p'} \\
= & \|b\|_{BMO} \left( \int_B \nu(x) d\mu(x) \right)^{\frac{p}{p'} \nu(B)^p} \left( \int_B \lambda_1^{\lambda_1^{\frac{p'}{r}}(x) d\mu(x)} \right)^{1 - p'} \\
\leq & \|b\|_{BMO} \frac{1}{\mu(B)} \left( \int_B \lambda_1^{\frac{p'}{r}}(x) d\mu(x) \right)^{\frac{p}{p'} \nu(B)^p} \left( \int_B \lambda_1^{\lambda_1^{\frac{p'}{r}}(x) d\mu(x)} \right)^{1 - p'} \\
\leq & \|b\|_{BMO} \left[ \lambda_2 \lambda_1^{\frac{p}{p'}} \right]_{A_p} \left( \frac{1}{\mu(B)} \int_B \lambda_2^{\frac{p}{p'}}(x) d\mu(x) \right)^{\frac{p}{p'}}.
\end{align*}
\]

Here we used Holder’s Inequality in the last step to complete the proof.

Now we will show the proof when \( r > p' \). We choose an index \( r \) for which reverse Holder’s Inequality holds for weights \( \lambda_1^{\frac{p}{r}} \) and \( \lambda_2^{\frac{p}{r}} \) with exponent \( 1 + \delta = \frac{p}{r} \). Fix a ball \( B \) and let \( x \in B \), then

\[
\frac{1}{\mu(B)} \int_B |b(x) - b_B| d\mu(x) \leq \|b\|_{BMO, \nu} \frac{1}{\mu(B)} \int_B \nu(x) d\mu(x).
\]

Hence we have \( \hat{b} \leq \|b\|_{BMO, \nu} \), where

\[
\hat{b} := \sup_B \frac{1}{\mu(B)} \int_B |b(x) - b_B| d\mu(x),
\]

\[
\nu^* := \sup_B \frac{1}{\mu(B)} \int_B \nu(x) d\mu(x).
\]

Following the proof [2, Proposition 2.1] the following Sharp function estimate holds for some constant \( C_{X,b} \) depending on the underlying space \( X \) and the operator \( b \),

\[
\frac{1}{\mu(B)} \int_B |b(x) - b_B|^p \lambda_1^{\lambda_1^{\frac{p}{r}}(x) d\mu(x)} \leq C_{X,b} \lambda_1^{\lambda_1^{\frac{p}{r}}(x) d\mu(x)} \left( \int_B \hat{b}^r \lambda_1^{\lambda_1^{\frac{p}{r}}(x) d\mu(x)} \right) \leq C_{X,b} \lambda_1^{\lambda_1^{\frac{p}{r}}(x) d\mu(x)} \int_B (\nu^*)^r \lambda_1^{\lambda_1^{\frac{p}{r}}(x) d\mu(x)}.
\]
We claim that $\lambda_1^{\frac{r}{p}} \in A_r$ when $\lambda_1 \in A_p$. We will now begin to prove our claim.

\[
\frac{1}{\mu(B)} \int_B \lambda_1(x)^{\frac{r}{p}} \left( \frac{1}{\mu(B)} \int_B \lambda_1^{\frac{r}{p}}(x) d\mu(x) \right)^{\frac{p}{r}} \\
\leq \frac{1}{\mu(B)} \int_B \lambda_1(x)^{\frac{r}{p}} \left( \frac{1}{\mu(B)} \int_B \lambda_1(x) d\mu(x) \right)^{\frac{p}{r}} \\
\leq \left( \frac{1}{\mu(B)} \int_B (\lambda_1^{-1})^{\frac{1}{p-r}}(x) d\mu(x) \right)^{\frac{r}{p}} \left( \frac{1}{\mu(B)} \int_B \lambda_1(x) d\mu(x) \right)^{\frac{p}{r}} \\
\leq \left( \frac{1}{\mu(B)} \int_B (\lambda_1^{-1})^{\frac{1}{p-r}}(x) d\mu(x) \right)^{p-1} \frac{1}{\mu(B)} \int_B \lambda_1(x) d\mu(x) \\
\leq [\lambda_1]_{A_p}^{\frac{r}{p}}.
\]

Last line above follows as we have $\lambda_1 \in A_p$. Hence we have our claim that $\lambda_1^{\frac{r}{p}} \in A_r$. Now following the proof of Muckenhoupt’s Theorem as in [6, Theorem I] we get the following

\[
\frac{1}{\mu(B)} \int_B |b(x) - b_B|^{\lambda_1^{\frac{r}{p}}} d\mu(x) \leq C_{X, b}[\lambda_1]_{A_p}^{\frac{r}{p}}\|b\|_{BMO_v} \int_B (\nu)^{\lambda_1^{-\frac{r}{p}}}(x) d\mu(x) \\
= C_{X, b}[\lambda_1]_{A_p}^{\frac{r}{p}}\|b\|_{BMO_v} \int_B \lambda_2^{\frac{r}{p}} d\mu(x).
\]

Similarly we can show that $\lambda_2^{\frac{r}{p}} \in A_r$ and

\[
\frac{1}{\mu(B)} \int_B \lambda_2^{\frac{r}{p}}(x) d\mu(x) \left( \frac{1}{\mu(B)} \int_B \lambda_2(x) d\mu(x) \right)^{\frac{p}{r}} \leq [\lambda_2]_{A_p}^{\frac{r}{p}},
\]

So we get

\[
\frac{1}{\mu(B)} \int_B \lambda_2^{\frac{r}{p}}(x) d\mu(x) \leq \left[ \lambda_2 \right]_{A_p}^{\frac{r}{p}} \left( \frac{1}{\mu(B)} \int_B \lambda_2(x) d\mu(x) \right)^{\frac{p}{r}}.
\]

Using Cauchy–Schwarz’ inequality, we have

\[
\frac{1}{\mu(B)} \int_B \lambda_2^{\frac{1}{p}}(x) d\mu(x) \left( \frac{1}{\mu(B)} \int_B \lambda_2^{\frac{1}{p}}(x) d\mu(x) \right)^{\frac{p}{r}} \geq 1.
\]

So we have

\[
\frac{1}{\mu(B)} \int_B |b(x) - b_B|^{\lambda_2^{\frac{r}{p}}} d\mu(x) \leq C_{X, b}[\lambda_1]_{A_p}^{\frac{r}{p}}\|b\|_{BMO_v} \int_B \lambda_2^{\frac{r}{p}}(x) d\mu(x) \\
\leq C_{X, b}[\lambda_1]_{A_p}^{\frac{r}{p}}\|b\|_{BMO_v} [\lambda_2]_{A_p}^{\frac{r}{p}} \left( \frac{1}{\mu(B)} \int_B \lambda_2^{\frac{1}{p}}(x) d\mu(x) \right)^{-r} \\
\leq C_{X, b}[\lambda_1]_{A_p}^{\frac{r}{p}}[\lambda_2]_{A_p}^{\frac{r}{p}}\|b\|_{BMO_v} \left( \frac{1}{\mu(B)} \int_B \lambda_2^{\frac{1}{p}}(x) d\mu(x) \right)^{-r}.
\]

The proof of Lemma 3.3 is complete.
4 Lower Bound of the Commutator $C_b$

Proof of Theorem 1.3. Suppose $b \in L^1_{\text{loc}}(X)$ with $\|C_b : L^p_{\lambda_1}(X) \rightarrow L^p_{\lambda_2}(X)\| < \infty$.

For any fixed $B \subset X$, by Hölder’s inequality, we have

$$\frac{1}{\nu(B)} \int_B |b(x) - b_B|d\mu(x) \lesssim \inf_{c} \frac{1}{\nu(B)} \int_B |b(x) - c|d\mu(x)$$

$$\lesssim \inf_{y \in B} \frac{1}{\nu(B)} \int_B |b(x) - b(y)|d\mu(x)$$

$$\lesssim \frac{1}{\nu(B)} \frac{1}{\lambda_2(B)} \int_B \int_B |b(x) - b(y)|d\mu(x)\lambda_2(y)d\mu(y)$$

$$\lesssim \frac{1}{\nu(B)} \frac{1}{\lambda_2(B)} \int_B C_b(\chi_B)(y)\lambda_2(y)d\mu(y)$$

$$\lesssim \frac{1}{\nu(B)} \frac{\mu(B)}{\lambda_2(B)} \left( \int_B |C_b(\chi_B)(y)|^p \lambda_2(y)d\mu(y) \right)^{\frac{1}{p}}$$

$$\lesssim \frac{1}{\nu(B)} \frac{\mu(B)}{\lambda_2(B)} \|C_b(\chi_B)\|_{L^p_{\lambda_2}}$$

$$\lesssim \|C_b : L^p_{\lambda_1}(X) \rightarrow L^p_{\lambda_2}(X)\| \frac{1}{\nu(B)} \frac{\mu(B)}{\lambda_2(B)} \lambda_1(B)^{\frac{1}{p}}. \quad (4.1)$$

Using (2.13) and Hölder’s inequality, we can obtain

$$\frac{1}{\mu(B)} \int_B \lambda_1(x)d\mu(x) \lesssim \left( \frac{1}{\mu(B)} \int_B \lambda_1(x)^{\frac{1}{1+p}} d\mu(x) \right)^{1+p}$$

$$\lesssim \left( \frac{1}{\mu(B)} \int_B \nu(x)^{\frac{1}{1+p}} \lambda_2(x)^{\frac{1}{1+p}} d\mu(x) \right)^{1+p}$$

$$\lesssim \left( \frac{1}{\mu(B)} \int_B \nu(x)d\mu(x) \right)^{p} \left( \frac{1}{\mu(B)} \int_B \lambda_2(x)d\mu(x) \right)$$

$$\lesssim \frac{\nu(B)p\lambda_2(B)}{\mu(B)^{1+p}}.$$

This implies that

$$\frac{1}{\nu(B)} \frac{\mu(B)}{\lambda_2(B)^p} \lambda_1(B)^{\frac{1}{p}} \lesssim 1,$$

and hence, together with (4.1), gives

$$\frac{1}{\nu(B)} \int_B |b(x) - b_B|d\mu(x) \lesssim \|C_b : L^p_{\lambda_1}(X) \rightarrow L^p_{\lambda_2}(X)\|.$$

Therefore, $b \in \text{BMO}_\nu(X)$, and

$$\|b\|_{\text{BMO}_\nu(X)} \leq C\|C_b : L^p_{\lambda_1}(X) \rightarrow L^p_{\lambda_2}(X)\|.$$

The proof of Theorem 1.3 is complete. \qed

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