Derived noncommutative schemes, geometric realizations, and finite dimensional algebras

D. O. Orlov

Abstract. The main purpose of this paper is to describe various phenomena and certain constructions arising in the process of studying derived noncommutative schemes. Derived noncommutative schemes are defined as differential graded categories of special type. Different properties of both noncommutative schemes and morphisms between them are reviewed and discussed. In addition, the concept of a geometric realization for a derived noncommutative scheme is introduced and problems of existence and construction of such realizations are discussed. Also studied are the construction of gluings of noncommutative schemes via morphisms, along with certain new phenomena such as phantoms, quasi-phantoms, and Krull–Schmidt partners which arise in the world of noncommutative schemes and which enable us to find new noncommutative schemes. The last sections consider noncommutative schemes connected with basic finite-dimensional algebras. It is proved that such noncommutative schemes have special geometric realizations under which the algebra goes to a vector bundle on a smooth projective scheme. Such realizations are constructed in two steps, the first of which is the well-known construction of Auslander, while the second step is connected with the new concept of a well-formed quasi-hereditary algebra, for which there are very special geometric realizations sending standard modules to line bundles.

Bibliography: 50 titles.

Keywords: differential graded categories, triangulated categories, derived noncommutative schemes, finite-dimensional algebras, geometric realizations.

Contents

Introduction 866
1. Preliminaries on triangulated and differential graded categories 870
   1.1. Triangulated categories, generators, and semi-orthogonal decompositions 870
   1.2. Differential graded categories 873
   1.3. Differential graded functors and quasi-functors 875

This work was supported by the Russian Science Foundation under grant no. 14-50-00005.

AMS 2010 Mathematics Subject Classification. Primary 14A22, 14F05, 16E45, 18E30.

© 2018 Russian Academy of Sciences (DoM), London Mathematical Society, Turpion Ltd.
Noncommutative algebraic geometry is based on the fact that, as in commutative geometry, affine schemes are directly related to rings or algebras over a field. And in spite of the fact that, in contrast to the commutative case, for a noncommutative algebra $A$ we do not have any construction of the topological space $\text{Spec } A$, we can nevertheless freely speak about the category of quasi-coherent sheaves on an affine noncommutative scheme, having in view the category $\text{Mod–} A$ of (right) modules over $A$. This observation is a serious reason to shift the focus from varieties (or schemes) and go directly to the categories of sheaves on these varieties (schemes), making these categories the main object of investigation. This is natural in many ways also because the theory of sheaves is one of the most powerful methods for studying algebraic varieties. In addition, it should be noted that quasi-coherent sheaves do not depend on the choice of topology on schemes, being sheaves in all natural topologies, and they best reflect the algebraic structure of schemes.

To the next (already quite non-trivial) question of how to glue together noncommutative affine schemes, there are several different approaches. However, the most fruitful approach here is again from a point of view connected with the category of sheaves, but with some natural generalizations dictated by homological algebra. This approach consists in working in fact with the derived category of quasi-coherent sheaves and with the category of perfect complexes on a noncommutative scheme when we talk about such a noncommutative scheme. It is natural to begin a more detailed consideration by returning to commutative algebraic geometry.
Let us consider a scheme $X$ over a field $k$ and impose some finiteness conditions on it. We will assume that $X$ is quasi-compact and quasi-separated, that is, it has a finite covering by affine schemes whose intersections have the same property. With any such scheme we associate the unbounded derived category of complexes of $\mathcal{O}_X$-modules with quasi-coherent cohomology $D_{\text{Qcoh}}(X)$. In [31] and [12] it was shown that this category has enough compact objects, and the triangulated subcategory of compact objects actually coincides with the category $\mathcal{P}erf-X$ of perfect complexes. It should be recalled that a complex is said to be perfect if it is locally quasi-isomorphic to a bounded complex of locally free sheaves of finite type. In addition, it was also shown in [31] and [12] that the category $\mathcal{P}erf-X$ of perfect complexes can be generated by a single object, called a classical generator. This means that the minimal full triangulated subcategory of $\mathcal{P}erf-X$ containing this object and closed under taking direct summands coincides with the whole category $\mathcal{P}erf-X$. Such an object will also be a compact generator for the category $D_{\text{Qcoh}}(X)$. We remark that [31] dealt with separated schemes, for which the category $D_{\text{Qcoh}}(X)$ is equivalent to the usual unbounded derived category of quasi-coherent sheaves $D(\text{Qcoh}X)$, while in [12] these statements were proved in the general case of a quasi-compact and quasi-separated scheme.

The existence of such a generator $E \in \mathcal{P}erf-X$ gives us an opportunity to look at the derived category $D_{\text{Qcoh}}(X)$, as well as the triangulated category $\mathcal{P}erf-X$, from a different angle. The results of [21] and [22] allow us to assert that in this situation the category $D_{\text{Qcoh}}(X)$ is equivalent to the unbounded derived category of differential graded (DG) modules $D(\mathcal{R})$ over some differential graded (DG) algebra $\mathcal{R}$, and the triangulated category of perfect complexes $\mathcal{P}erf-X$ is equivalent to the category of perfect DG modules $\mathcal{P}erf-\mathcal{R}$. The differential graded algebra $\mathcal{R}$ depends directly on the choice of the generator $E \in \mathcal{P}erf-X$ and is obtained as the DG algebra of endomorphisms $\text{End}(E)$ of the given generator, although not as an object of the category $\mathcal{P}erf-X$, but rather as its lift to a differential graded category $\mathcal{P}erf-X$ which is a natural enhancement of the category $\mathcal{P}erf-X$. In particular, this means that $\mathcal{P}erf-X$ is equivalent to the homotopy category $\mathcal{H}^0(\mathcal{P}erf-X)$. A differential graded (DG) category $\mathcal{A}$ is a category whose morphisms have the structure of complexes of $k$-vector spaces. Passing from the complexes to their zero cohomology spaces, we obtain a $k$-linear category $\mathcal{H}^0(\mathcal{A})$ with the same objects, which is called the homotopy category for the DG category $\mathcal{A}$. If there is an equivalence $\epsilon: \mathcal{H}^0(\mathcal{A}) \sim \Rightarrow \mathcal{T}$, then $(\mathcal{A}, \epsilon)$ is called a DG enhancement for the category $\mathcal{T}$.

Triangulated categories usually have natural enhancements arising in the process of constructing these categories. In our case, the triangulated category $D_{\text{Qcoh}}(X)$ has several natural enhancements: the DG category of $h$-injective complexes, the DG quotient of all complexes by acyclic complexes, the DG quotient of $h$-flat complexes by acyclic $h$-flat complexes, and so on. All these enhancements are also quasi-equivalent to each other in a natural way, and when working with them, we can choose any convenient model from the class of quasi-equivalent DG categories. A DG enhancement of the category $D_{\text{Qcoh}}(X)$ induces a DG enhancement of the triangulated subcategory $\mathcal{P}erf-X$ of perfect complexes, which will be denoted by $\mathcal{P}erf-X$. 
Combined together, the results in [31], [12], [21], and [22] mentioned above tell us that the DG category $\text{Perf} - X$ is quasi-equivalent to a category of the form $\text{Perf} - R$, where $R$ is the DG algebra of endomorphisms of some generator $E \in \text{Perf} - X$. We can also note that in the case of a quasi-compact and quasi-separated scheme, the DG algebra $R$ is cohomologically bounded, that is, it has only a finite number of non-trivial cohomology spaces. Thus, the following definition of a derived noncommutative scheme over the field $k$ arises (see Definition 2.1). By a derived noncommutative scheme $\mathcal{X}$ we shall mean a $k$-linear DG category of the form $\text{Perf} - R$, where $R$ is a cohomologically bounded DG algebra over $k$. It is natural to call the derived category $D(R)$ the derived category of quasi-coherent sheaves on this noncommutative scheme, while the triangulated category $\text{Perf} - R$ will be called the category of perfect complexes on $\mathcal{X}$. It was mentioned above that there is an equivalence of triangulated categories
\[ \mathcal{H}^0(\text{Perf} - R) \cong \text{Perf} - R. \]

Note also that, after considering DG categories of the form $\text{Perf} - R$, we not only were able to glue together noncommutative schemes from affine pieces, but in fact obtained derived noncommutative affine schemes as well by passing from algebras to DG algebras. Thus, this definition allows us to talk not only about noncommutative schemes, but also about derived noncommutative schemes.

Many important properties of usual schemes extend to derived noncommutative schemes. In particular, we can talk about smoothness, regularity, and properness for noncommutative schemes. We can also define morphisms between schemes as quasi-functors between DG categories. Despite the fact that the usual morphisms between schemes are also morphisms between them in the noncommutative sense, it should be noted that there are many more morphisms in the noncommutative world, and they form a category (and even a DG category). This implies that they can be added together, and one can talk about maps between morphisms. Many natural concepts from the usual commutative algebraic geometry also generalize to noncommutative schemes: the concepts of compactification, resolution of singularities, and Serre functor are defined.

In this paper we discuss some properties of derived noncommutative schemes and draw various analogies with the commutative case. However, one of the important concepts, to which a large part of our article is devoted, is a gluing together of noncommutative schemes, an operation existing in the noncommutative world and having no analogue in the commutative world. Another important concept is the geometric realization of derived noncommutative schemes. It arises naturally for two reasons. First, for each abstract algebraic structure it is always useful and interesting to find some geometric representations. On the other hand, many noncommutative schemes come to us from the usual geometry with a given geometric realization.

The most natural and often encountered but highly non-trivial examples of geometric realizations are connected with admissible subcategories $\mathcal{N} \subset \text{Perf} - X$ of categories of perfect complexes on smooth projective schemes $X$. In this case we obtain a noncommutative scheme as a DG category $\mathcal{N} \subset \text{Perf} - X$. It is easy to see that such a DG category can be realized in the form $\text{Perf} - R$, and moreover, the noncommutative scheme $\mathcal{N}$ itself is smooth and proper. The initial embedding
$N \subset \text{Perf} - X$ is a particular but most interesting case of a geometric realization of smooth and proper noncommutative schemes (see Definition 2.17). Such realizations are said to be pure.

As mentioned above, for any two DG categories $\mathcal{A}$ and $\mathcal{B}$ and a $\mathcal{B}^\circ - \mathcal{A}$-bimodule $T$ we can define a new DG category $\mathcal{C} = \mathcal{A} \oplus \mathcal{T} \mathcal{B}$, which is called the gluing of these two DG categories via the bimodule $T$. This construction allows us to introduce gluing of derived noncommutative schemes $\mathcal{X}$ and $\mathcal{Y}$ by imposing some boundedness condition on the gluing bimodule $T$. Noncommutative schemes of the form $\mathcal{X} \oplus \mathcal{T} \mathcal{Y}$ inherit many properties of the schemes $\mathcal{X}$ and $\mathcal{Y}$ under appropriate conditions on $T$. For example, the smoothness and properness of $\mathcal{X}$ and $\mathcal{Y}$ imply the smoothness and properness of the gluing $\mathcal{X} \oplus \mathcal{T} \mathcal{Y}$ if the bimodule $T$ is perfect.

In [40] we studied the problem of geometric realizations of noncommutative schemes that are obtained by gluing smooth and proper noncommutative schemes $\mathcal{X}$ and $\mathcal{Y}$. In particular, it was proved that if noncommutative schemes $\mathcal{X}$ and $\mathcal{Y}$ arise as admissible subcategories in categories of perfect complexes on smooth projective schemes, then their gluing $\mathcal{X} \oplus \mathcal{T} \mathcal{Y}$ via a perfect bimodule $T$ can be realized in the same way. This problem is discussed in detail in §§3.2–3.4.

Derived noncommutative schemes may be quite different from commutative schemes in general even when they are smooth and proper. In §3.5 phenomena such as quasi-phantoms and phantoms are discussed. Without going into details, we can say that phantoms are smooth and proper noncommutative schemes $\mathcal{X}$ for which $K$-theory $K_*(\mathcal{X})$ is completely trivial. Moreover, we also assume that a phantom noncommutative scheme has a geometric realization in the form of an admissible subcategory in the category of perfect complexes on a smooth projective scheme. In [19] it was proved that phantoms exist, and a procedure for constructing them was described that involves the product of surfaces of general type with $p_g = q = 0$. In [5] one of the phantoms was constructed as an admissible subcategory in the category of perfect complexes on the Barlow surface. In §3.6 we discuss the so-called Krull–Schmidt partners, which were introduced in [42]. These are smooth and proper noncommutative schemes $\mathcal{X}$ and $\mathcal{X}'$ for which there exists a smooth proper noncommutative scheme $\mathcal{Y}$ with the condition that some gluings $\mathcal{X} \oplus \mathcal{T} \mathcal{Y}$ and $\mathcal{X}' \oplus \mathcal{T} \mathcal{Y}$ are isomorphic to each other. In particular, we give a new procedure for constructing smooth and proper noncommutative schemes that are Krull–Schmidt partners for the usual schemes and have the same additive invariants.

The last two sections are devoted to the study of geometric realizations for finite-dimensional algebras. Any finite-dimensional algebra $\Lambda$ gives a noncommutative scheme $\mathcal{V} = \text{Perf} - \Lambda$ that is proper. In [40] it was proved that for any such noncommutative scheme $\mathcal{V}$ if the semisimple part $\Lambda = \Lambda / r$ is separable over the field $k$, then one can find a geometric realization $\text{Perf} - \Lambda \to \text{Perf} - X$ for which the scheme $X$ is smooth and projective. An explicit construction of such a geometric realization was also given there. On the other hand, of special interest in this case are the geometric realizations for which the image of the algebra $\Lambda$ is not an arbitrary perfect complex, but some vector bundle on a scheme $X$. This problem can be reformulated as follows: for an arbitrary finite-dimensional algebra $\Lambda$ find
and construct a smooth projective scheme $X$ and a vector bundle $E$ on it such that

$$\text{End}_X(E) \cong \Lambda \quad \text{and} \quad \text{Ext}^j_X(E,E) = 0 \quad \text{for all} \ j > 0.$$  

In the last section such a construction is proposed for an arbitrary basic algebra, that is, for an algebra $\Lambda$ whose semisimple part $\overline{\Lambda} = \Lambda/\tau$ is an $m$-fold product $k \times \cdots \times k$ of the base field. If the field $k$ is algebraically closed, then this result implies a positive answer for any finite-dimensional algebra.

The construction of a smooth projective scheme $X$ and a vector bundle $E$ takes place in two steps and uses the theory of quasi-hereditary algebras. In §4 the new concept of well-formed quasi-hereditary algebra is introduced, and for such algebras we construct very special geometric realizations that send standard modules to line bundles. This construction is a generalization of the construction for quiver algebras described in the paper [41]. Applying this new procedure to the algebra $\Gamma$ that is obtained from a basic finite-dimensional algebra $\Lambda$ by the Auslander construction, we obtain a geometric realization for $\Lambda$ such that $\Lambda$ goes to a vector bundle $E$ on a smooth projective scheme $X$. We also note that the scheme $X$ is a tower of projective bundles, and the rank of the bundle $E$ is exactly equal to the dimension of the algebra $\Lambda$.

The author is very grateful to Anton Fonarev, Alexander Kuznetsov, and Amnon Neeman for useful discussions and valuable comments.

1. Preliminaries on triangulated and differential graded categories

1.1. Triangulated categories, generators, and semi-orthogonal decompositions. Let $\mathcal{T}$ be a triangulated category. We say that a set $S \subset \mathcal{T}$ of objects classically generates the triangulated category $\mathcal{T}$ if the smallest full triangulated subcategory of $\mathcal{T}$ containing $S$ and closed under taking direct summands coincides with the whole of $\mathcal{T}$. If the set $S$ consists of a single object $E \in \mathcal{T}$, then $E$ is called a classical generator for $\mathcal{T}$.

A classical generator will be called strong if it generates the whole triangulated category $\mathcal{T}$ in a finite number of steps (see [12], for instance). To define it precisely, we introduce a multiplication on the set of strictly full subcategories. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two full subcategories of $\mathcal{T}$. Denote by $\mathcal{I}_1 \ast \mathcal{I}_2$ the full subcategory of $\mathcal{T}$ consisting of all objects $X$ such that there is a distinguished triangle $X_1 \to X \to X_2$ with $X_i \in \mathcal{I}_i$. For any subcategory $\mathcal{I} \subset \mathcal{T}$ denote by $\langle \mathcal{I} \rangle$ the smallest full subcategory of $\mathcal{T}$ containing $\mathcal{I}$ and closed under taking direct summands coincides with the whole of $\mathcal{T}$. If the set $S$ consists of a single object $E \in \mathcal{T}$, then $E$ is called a classical generator for $\mathcal{T}$.

A classical generator will be called strong if it generates the whole triangulated category $\mathcal{T}$ in a finite number of steps (see [12], for instance). To define it precisely, we introduce a multiplication on the set of strictly full subcategories. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two full subcategories of $\mathcal{T}$. Denote by $\mathcal{I}_1 \ast \mathcal{I}_2$ the full subcategory of $\mathcal{T}$ consisting of all objects $X$ such that there is a distinguished triangle $X_1 \to X \to X_2$ with $X_i \in \mathcal{I}_i$. For any subcategory $\mathcal{I} \subset \mathcal{T}$ denote by $\langle \mathcal{I} \rangle$ the smallest full subcategory of $\mathcal{T}$ containing $\mathcal{I}$ and closed under taking direct summands, direct summands, and shifts. Now we can define a new multiplication on the set of strictly full subcategories closed under finite direct sums. Put $\mathcal{I}_1 \circ \mathcal{I}_2 = \langle \mathcal{I}_1 \ast \mathcal{I}_2 \rangle$ and define by induction $\langle \mathcal{I} \rangle_k = \langle \mathcal{I} \rangle_{k-1} \circ \langle \mathcal{I} \rangle_1$, where $\langle \mathcal{I} \rangle_1 = \langle \mathcal{I} \rangle$. If $\mathcal{I}$ consists of a single object $E$, then we denote $\langle \mathcal{I} \rangle$ by $\langle E \rangle_1$ and put $\langle E \rangle_k = \langle E \rangle_{k-1} \circ \langle E \rangle_1$ by induction.

Definition 1.1. An object $E$ will be called a strong generator if $\langle E \rangle_n = \mathcal{T}$ for some $n \in \mathbb{N}$.

Note that $E$ is a classical generator if and only if $\bigcup_{k \in \mathbb{Z}} \langle E \rangle_k = \mathcal{T}$. It is evident that if a triangulated category $\mathcal{T}$ has a strong generator, then any classical generator of $\mathcal{T}$ is strong too, that is, the existence of a strong generator is a property of a triangulated category (see [39]).
**Definition 1.2.** A triangulated category $\mathcal{T}$ will be called *regular* if it has a strong generator.

Following [43], we introduce the notion of dimension for a regular triangulated category $\mathcal{T}$ as the smallest integer $d \geq 0$ such that there exists an object $E \in \mathcal{T}$ for which $\langle E \rangle_{d+1} = \mathcal{T}$.

Now we recall the notion of a compact object. An object $E$ of a triangulated category $\mathcal{T}$ is said to be *compact* (in $\mathcal{T}$) if the functor $\text{Hom}_{\mathcal{T}}(E, -)$ commutes with arbitrary direct sums (coproducts) existing in $\mathcal{T}$, that is, for each family of objects $\{X_i\} \subset \mathcal{T}$ such that $\bigoplus_i X_i$ exists, the canonical map $\bigoplus_i \text{Hom}(E, X_i) \to \text{Hom}(E, \bigoplus_i X_i)$ is an isomorphism. Compact objects form a full triangulated subcategory of $\mathcal{T}$ which is usually denoted by $\mathcal{T}^c \subset \mathcal{T}$.

Let $\mathcal{T}$ be a triangulated category that admits arbitrary (small) direct sums. A full triangulated subcategory $\mathcal{L} \subset \mathcal{T}$ which is closed under taking any direct sums is called a *localizing subcategory*. This means that the inclusion functor preserves direct sums. Note that $\mathcal{L}$ is also closed under taking direct summands (see [31]).

A set $S \subset \mathcal{T}^c$ is called a set of *compact generators* if the smallest localizing subcategory containing $S$ coincides with the whole of $\mathcal{T}$. This property is equivalent to the following: for an object $X \in \mathcal{T}$, we have $X \cong 0$ if $\text{Hom}(Y, X[n]) = 0$ for all $Y \in S$ and all $n \in \mathbb{Z}$.

Let $\mathcal{T}$ be a triangulated category with small Hom-sets, that is, Hom between any two objects should be a set. Assume that $\mathcal{T}$ admits arbitrary direct sums and let $\mathcal{L} \subset \mathcal{T}$ be a localizing triangulated subcategory. We can consider the Verdier quotient $\mathcal{T}/\mathcal{L}$ with the natural localization map $\pi: \mathcal{T} \to \mathcal{T}/\mathcal{L}$. It is known that the category $\mathcal{T}/\mathcal{L}$ also has arbitrary direct sums and, moreover, the functor $\pi$ preserves direct sums (see [32], 3.2.11).

Note, however, that Hom-sets in $\mathcal{T}/\mathcal{L}$ need not be small. Assume nevertheless that the Verdier quotient $\mathcal{T}/\mathcal{L}$ is a category with small Hom-sets. If the triangulated category $\mathcal{T}$ has a set of compact generators, then the Brown representability theorem holds for $\mathcal{T}$ and the quotient functor $\pi: \mathcal{T} \to \mathcal{T}/\mathcal{L}$ has a right adjoint $\mu: \mathcal{T}/\mathcal{L} \to \mathcal{T}$ (see [32], 8.4.5). This adjoint is called the *Bousfield localization* functor.

Let $j: \mathcal{N} \hookrightarrow \mathcal{T}$ be a full embedding of triangulated categories. The subcategory $\mathcal{N}$ is said to be *right admissible* (left admissible) if there is a right (respectively, left) adjoint $q: \mathcal{T} \to \mathcal{N}$ to the embedding functor $j: \mathcal{N} \hookrightarrow \mathcal{T}$. The subcategory $\mathcal{N}$ is said to be *admissible* if it is both right and left admissible.

The *right orthogonal* (left orthogonal) to the subcategory $\mathcal{N} \subset \mathcal{T}$ is defined to be the full subcategory $\mathcal{N}^\perp \subset \mathcal{T}$ (respectively, $^\perp \mathcal{N}$) consisting of all objects $M$ such that $\text{Hom}(N, M) = 0$ (respectively, $\text{Hom}(M, N) = 0$) for any $N \in \mathcal{N}$. It is clear that the subcategories $\mathcal{N}^\perp$ and $^\perp \mathcal{N}$ are triangulated subcategories.

**Definition 1.3.** A *semi-orthogonal decomposition* of a triangulated category $\mathcal{T}$ is a sequence of full triangulated subcategories $\mathcal{N}_1, \ldots, \mathcal{N}_n$ of $\mathcal{T}$ such that there is an increasing filtration $0 = \mathcal{T}_0 \subset \mathcal{T}_1 \subset \cdots \subset \mathcal{T}_n = \mathcal{T}$ by left admissible subcategories for which the left orthogonals $^\perp \mathcal{T}_{r-1}$ in $\mathcal{T}_r$ coincide with $\mathcal{N}_r$ for all $1 \leq r \leq n$. We then write $\mathcal{T} = \langle \mathcal{N}_1, \ldots, \mathcal{N}_n \rangle$. 
In some cases one can hope that $T$ has a semi-orthogonal decomposition $T = \langle N_1, \ldots, N_n \rangle$ in which each $N_r$ is as simple as possible, that is, each of them is equivalent to the bounded derived category of finite-dimensional vector spaces.

From now on we will assume that $T$ is a $k$-linear triangulated category, where $k$ is an arbitrary base field.

**Definition 1.4.** An object $E$ of a $k$-linear triangulated category $T$ is said to be **exceptional** if
\[
\text{Hom}(E, E[m]) = 0 \quad \text{for all } m \neq 0, \quad \text{and} \quad \text{Hom}(E, E) \cong k.
\]

An **exceptional collection** in $T$ is a sequence $\sigma = (E_1, \ldots, E_n)$ of exceptional objects satisfying the semi-orthogonality condition
\[
\text{Hom}(E_i, E_j[m]) = 0 \quad \text{for all } m \text{ if } i > j.
\]

If a triangulated category $T$ has an exceptional collection $\sigma = (E_1, \ldots, E_n)$ that generates the whole of $T$, then the collection is said to be **full**. In this case $T$ has a semi-orthogonal decomposition with $N_r = \langle E_r \rangle$. Since $E_r$ is exceptional, each of these categories is equivalent to the bounded derived category of finite-dimensional $k$-vector spaces. In this case we write $T = \langle E_1, \ldots, E_n \rangle$.

We now recall the notion of a proper triangulated category.

**Definition 1.5.** We say that a $k$-linear triangulated category $T$ is **proper** if the vector space $\bigoplus_{m \in \mathbb{Z}} \text{Hom}(X, Y[m])$ is finite-dimensional for any pair of objects $X, Y \in T$.

Proper and regular triangulated categories have good properties. In particular, they are saturated and admissible if they are idempotent complete. Recall that $T$ is said to be idempotent complete if the kernels of all projections $p: X \to X$, $p^2 = p$, exist as objects of $T$. The following theorem is due to Bondal and Van den Bergh.

**Theorem 1.6** ([12], Theorem 1.3). Let $T$ be a regular and proper triangulated category that is idempotent complete (Karoubian). Then every cohomological functor from $T^\circ$ to the category of finite-dimensional vector spaces is representable, that is, it has the form $h^Y = \text{Hom}(-, Y)$.

In [8] and [12] a triangulated category having such a representability property is said to be **right saturated**. It is proved in [8], §2.6, that if a right saturated triangulated category $T$ is a full subcategory of a proper triangulated category, then it is right admissible there. By Theorem 1.6 a regular and proper idempotent complete triangulated category is right saturated. Since the opposite category is also regular and proper, it is left saturated as well. Thus, we obtain the following proposition.

**Proposition 1.7.** Let $N \subset T$ be a full triangulated subcategory of a proper triangulated category $T$. Assume that $N$ is regular and idempotent complete. Then $N$ is admissible in $T$.

We now recall the definition of a Serre functor (see [8], [10], [11]). Let $T$ be a proper $k$-linear triangulated category. A $k$-linear autoequivalence $S: T \to T$ is called a **Serre functor** if there exists an isomorphism of bifunctors
\[
\text{Hom}_T(Y, SX) \xrightarrow{\sim} \text{Hom}_T(X, Y)^*;
\]
where $V^*$ is the dual vector space for a vector space $V$. If such a functor exists, then it is exact and unique up to a natural isomorphism (see [8]). It was proved in [8] that any saturated triangulated category $T$ has a Serre functor. Taking into account Theorem 1.6, we obtain the following proposition.

**Proposition 1.8.** Let $T$ be a regular and proper $k$-linear triangulated category that is idempotent complete. Then it has a Serre functor.

### 1.2. Differential graded categories

In this subsection we recall some facts on differential graded (DG) categories. The main references are [21], [22], [16], and [27]. Let $k$ be a field and let $C$ be a $k$-linear differential graded (DG) category. This means that the morphism spaces $\text{Hom}_{C}(X, Y)$ are complexes of $k$-vector spaces (DG $k$-modules), and for any $X, Y, Z \in \text{Ob} C$ the composition $\text{Hom}(Y, Z) \otimes \text{Hom}(X, Y) \to \text{Hom}(X, Z)$ is a morphism of DG $k$-modules.

For any DG category $C$ we denote by $\mathcal{H}^0(C)$ its homotopy category. The homotopy category $\mathcal{H}^0(C)$ has the same objects as $C$, but its morphisms are defined by taking the 0th cohomology $H^0(\text{Hom}_{C}(X, Y))$ of the complex $\text{Hom}_{C}(X, Y)$.

As usual, a DG functor $F: \mathcal{A} \to \mathcal{B}$ is given by a map $F: \text{Ob}(\mathcal{A}) \to \text{Ob}(\mathcal{B})$ and by morphisms of DG $k$-modules

$$F_{X, Y}: \text{Hom}_{\mathcal{A}}(X, Y) \to \text{Hom}_{\mathcal{B}}(FX, FY), \quad X, Y \in \text{Ob}(\mathcal{A}),$$

which are compatible with composition and the units.

A DG functor $F: \mathcal{A} \to \mathcal{B}$ is called a quasi-equivalence if $F_{X, Y}$ is a quasi-isomorphism for all pairs of objects $X, Y$ of $\mathcal{A}$, and the induced functor $H^0(F): \mathcal{H}^0(\mathcal{A}) \to \mathcal{H}^0(\mathcal{B})$ is an equivalence. DG categories $\mathcal{A}$ and $\mathcal{B}$ are said to be quasi-equivalent if there exist DG categories $\mathcal{C}_1, \ldots, \mathcal{C}_n$ and a chain of quasi-equivalences $\mathcal{A} \leftarrow \mathcal{C}_1 \to \cdots \to \mathcal{C}_n \rightarrow \mathcal{B}$. Actually, for any DG category $\mathcal{A}$ we can find a (quasi-equivalent) cofibrant replacement $\mathcal{A}_{\text{cot}} \to \mathcal{A}$ such that any chain of quasi-equivalences between $\mathcal{A}$ and $\mathcal{B}$ can be realized by a simple roof $\mathcal{A} \leftarrow \mathcal{A}_{\text{cot}} \rightarrow \mathcal{B}$ (see [22] and [48]).

Let $\mathcal{A}$ be a small $k$-linear DG category. A (right) DG $\mathcal{A}$-module is a DG functor $M: \mathcal{A}^\circ \to \mathcal{Mod}_{\text{-}k}$, where $\mathcal{Mod}_{\text{-}k}$ is the DG category of complexes of $k$-vector spaces, and $\mathcal{A}^\circ$ is the opposite DG category. We denote by $\mathcal{Mod}_{\text{-}\mathcal{A}}$ the DG category of right DG $\mathcal{A}$-modules, and by $\mathcal{A}_{\text{c}}$ the full DG subcategory of $\mathcal{Mod}_{\text{-}\mathcal{A}}$ consisting of all acyclic DG modules, that is, DG modules $M$ for which the complexes of vector spaces $M(X)$ have trivial cohomology for all $X \in \mathcal{A}$. The homotopy category $\mathcal{H}^0(\mathcal{Mod}_{\text{-}\mathcal{A}})$ has a natural triangulated category structure, and $\mathcal{H}^0(\mathcal{A}_{\text{c}})$ forms a localizing triangulated subcategory of it.

**Definition 1.9.** The derived category $\mathcal{D}(\mathcal{A})$ (of DG $\mathcal{A}$-modules) is defined as the Verdier quotient

$$\mathcal{D}(\mathcal{A}) := \mathcal{H}^0(\mathcal{Mod}_{\text{-}\mathcal{A}})/\mathcal{H}^0(\mathcal{A}_{\text{c}}).$$

Any object $Y \in \mathcal{A}$ defines a representable right DG module

$$\mathcal{h}_{\mathcal{A}}^Y(-) := \text{Hom}_{\mathcal{A}}(-, Y).$$

This gives the Yoneda DG functor $\mathcal{h}^\bullet: \mathcal{A} \to \mathcal{Mod}_{\text{-}\mathcal{A}}$, which is a full embedding. A DG module is said to be free if it is isomorphic to a direct sum of DG modules
of the form $h^Y[m]$. A DG module $P$ is said to be semifree if it has a filtration $0 = \Phi_0 \subset \Phi_1 \subset \cdots = P$ with free quotients $\Phi_{i+1}/\Phi_i$. The full DG subcategory of semifree DG modules is denoted by $\mathcal{IF}_{\text{fg}}\mathcal{A}$.

It is also natural to consider the category of $h$-projective DG modules. A DG $\mathcal{A}$-module $P$ is said to be $h$-projective (homotopically projective) if

$$\text{Hom}_{\mathcal{H}^0(\Mod-\mathcal{A})}(P, N) = 0$$

for every acyclic DG module $N$. (Dually, we can define $h$-injective DG modules.) Let $\mathcal{P}(\mathcal{A}) \subset \Mod-\mathcal{A}$ denote the full DG subcategory of $h$-projective objects. It can be easily checked that a semifree DG-module is $h$-projective, and the natural embedding $\mathcal{IF}_{\text{fg}}\mathcal{A} \hookrightarrow \mathcal{P}(\mathcal{A})$ is a quasi-equivalence. Moreover, for any DG $\mathcal{A}$-module $M$ there is a quasi-isomorphism $pM \rightarrow M$ such that $pM$ is a semifree DG $\mathcal{A}$-module (see [21], [20], and [16]). Hence, the canonical DG functors $\mathcal{IF}_{\text{fg}}\mathcal{A} \hookrightarrow \mathcal{P}(\mathcal{A}) \hookrightarrow \Mod-\mathcal{A}$ induce equivalences $\mathcal{H}^0(\mathcal{IF}_{\text{fg}}\mathcal{A}) \sim \mathcal{H}^0(\mathcal{P}(\mathcal{A})) \sim \mathcal{D}(\mathcal{A})$ of triangulated categories. Dually, it can be shown that for any DG $\mathcal{A}$-module $M$ there is a quasi-isomorphism $M \rightarrow iM$ such that $iM$ is $h$-injective (see [21]).

We denote by $\mathcal{IF}_{\text{fg}}\mathcal{A} \subset \mathcal{IF}\mathcal{A}$ the full DG subcategory of finitely generated semifree DG modules, that is, $\Phi_n = P$ for some $n$ and $\Phi_{i+1}/\Phi_i$ is a finite direct sum of DG modules of the form $h^Y[m]$ for any $i$.

**Definition 1.10.** The DG category of perfect DG modules $\text{Perf-}\mathcal{A}$ is the full DG subcategory of $\mathcal{IF}\mathcal{A}$ consisting of all DG modules that are isomorphic to direct summands of objects from $\mathcal{IF}_{\text{fg}}\mathcal{A}$ in the homotopy category $\mathcal{H}^0(\mathcal{IF}\mathcal{A})$.

Denote by $\text{Perf-}\mathcal{A}$ the homotopy category $\mathcal{H}^0(\text{Perf-}\mathcal{A})$. It is triangulated and is equivalent to the subcategory of compact objects $\mathcal{D}(\mathcal{A})^c \subset \mathcal{D}(\mathcal{A})$ (see [31] and [22]).

With any DG category $\mathcal{A}$ we can associate a DG category $\mathcal{A}^{\text{pre-tr}}$ called the pretriangulated hull and a canonical fully faithful DG functor $\mathcal{A} \hookrightarrow \mathcal{A}^{\text{pre-tr}}$ (see [9] and [22]). The idea of the definition of $\mathcal{A}^{\text{pre-tr}}$ is to add to $\mathcal{A}$ all shifts, all cones, cones of morphisms between cones, and so on. There is a canonical fully faithful DG functor (the Yoneda embedding) $\mathcal{A}^{\text{pre-tr}} \rightarrow \Mod-\mathcal{A}$, and under this embedding $\mathcal{A}^{\text{pre-tr}}$ is equivalent to the DG category of finitely generated semifree DG modules $\mathcal{IF}_{\text{fg}}\mathcal{A}$. If $\mathcal{A}$ is small, then the pretriangulated hull $\mathcal{A}^{\text{pre-tr}}$ is also small, and in a certain sense it is just a small version for the essentially small DG category $\mathcal{IF}_{\text{fg}}\mathcal{A}$.

A DG category $\mathcal{A}$ is said to be pretriangulated if the canonical DG functor $\mathcal{A} \rightarrow \mathcal{A}^{\text{pre-tr}}$ is a quasi-equivalence. This property is equivalent to requiring that the homotopy category $\mathcal{H}^0(\mathcal{A})$ is triangulated as a subcategory of $\mathcal{H}^0(\Mod-\mathcal{A})$. The DG category $\mathcal{A}^{\text{pre-tr}}$ is always pretriangulated, so $\mathcal{H}^0(\mathcal{A}^{\text{pre-tr}})$ is a triangulated category.

If $\mathcal{A}$ is pretriangulated and $\mathcal{H}^0(\mathcal{A})$ is idempotent complete, then the Yoneda functor $\mathcal{h}^\bullet: \mathcal{A} \rightarrow \text{Perf-}\mathcal{A}$ is a quasi-equivalence, and hence the induced exact functor $h: \mathcal{H}^0(\mathcal{A}) \rightarrow \text{Perf-}\mathcal{A}$ is an equivalence of triangulated categories.

**Definition 1.11.** Let $\mathcal{T}$ be a triangulated category. An enhancement of $\mathcal{T}$ is a pair $(\mathcal{A}, \varepsilon)$, where $\mathcal{A}$ is a pretriangulated DG category and $\varepsilon: \mathcal{H}^0(\mathcal{A}) \sim \mathcal{T}$ is an equivalence of triangulated categories.
Thus, the DG category $\mathcal{I}\mathcal{F} - \mathcal{A}$ of semifree DG modules is an enhancement of the derived category $\mathcal{D}(\mathcal{A})$, while the DG category $\text{Perf} - \mathcal{A}$ of perfect DG modules is an enhancement of the triangulated category $\text{Perf} - \mathcal{A}$.

1.3. Differential graded functors and quasi-functors. Let $F: \mathcal{A} \to \mathcal{B}$ be a DG functor between small DG categories. It induces the restriction DG functor $F_*: \text{Mod} - \mathcal{B} \to \text{Mod} - \mathcal{A}$ which sends a DG $\mathcal{B}$-module $N$ to $N \cdot F$.

The restriction functor $F_*$ has left and right adjoint functors $F^*$ and $F^!$ defined as follows:

$$F^* M(Y) = M \otimes_\mathcal{A} F_* h_Y \quad \text{and} \quad F^! M(Y) = \text{Hom}_{\text{Mod} - \mathcal{A}}(F_* h_Y, M),$$

where $Y \in \mathcal{B}$, $M \in \text{Mod} - \mathcal{A}$, and $h_Y(-) := \text{Hom}_{\mathcal{B}}(-, Y)$ is a right DG $\mathcal{B}$-module, while $h_Y(-) := \text{Hom}_{\mathcal{B}}(Y, -)$ is a left DG $\mathcal{B}$-module. The DG functor $F^*$ is called the induction functor and is an extension of $F$ to the category of DG modules, that is, there is an isomorphism of DG functors $F^* \cdot h^*_\mathcal{A} \cong h^*_\mathcal{B} \cdot F$.

The DG functor $F_*$ preserves acyclic DG modules and induces a derived functor $F_*: \mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{A})$. By adjunctions, the DG functors $F^*$ and $F^!$ preserve $h$-projective and $h$-injective DG modules, respectively. The existence of $h$-projective and $h$-injective resolutions allows us to define derived functors $L F^*$ and $R F^!$ from $\mathcal{D}(\mathcal{A})$ to $\mathcal{D}(\mathcal{B})$. For example, the derived functor $L F^* : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ is the induced homotopy functor $H^0(F^*)$ for the extension DG functor $F^*: \mathcal{I}\mathcal{F} - \mathcal{A} \to \mathcal{I}\mathcal{F} - \mathcal{B}$.

More generally, let $T$ be an $\mathcal{A}$-$\mathcal{B}$-bimodule, that is (by definition), a right DG-module over $\mathcal{A}^{\text{op}} \otimes \mathcal{B}$. For each DG $\mathcal{A}$-module $M$ we obtain a DG $\mathcal{B}$-module $M \otimes_\mathcal{A} T$. The DG functor

$$(\_ \otimes_\mathcal{A} T): \text{Mod} - \mathcal{A} \to \text{Mod} - \mathcal{B}$$

admits a right adjoint $\text{Hom}_{\mathcal{B}}(T, \_ \otimes_\mathcal{A} T)$. These functors do not respect quasi-isomorphisms in general, but by applying them to $h$-projective ($h$-injective) DG modules we obtain an adjoint pair of derived functors $(\_ \otimes_\mathcal{A} T)$ and $R \text{Hom}_{\mathcal{B}}(T, \_)$ between the derived categories $\mathcal{D}(\mathcal{A})$ and $\mathcal{D}(\mathcal{B})$ (see [21] and [22]).

It is evident that the categories $\mathcal{D}(\mathcal{A})$ and $\text{Perf} - \mathcal{A}$ are invariant under quasi-equivalences of DG categories. Moreover, if a DG functor

$F: \mathcal{A} \to \mathcal{B}$

is a quasi-equivalence, then the functors

$$F^*: \text{Perf} - \mathcal{A} \to \text{Perf} - \mathcal{B} \quad \text{and} \quad F^*: \mathcal{I}\mathcal{F} - \mathcal{A} \to \mathcal{I}\mathcal{F} - \mathcal{B}$$

are also quasi-equivalences.

Furthermore, we have the following proposition that is essentially equivalent to Lemma 4.2 in [21] (see also [27], Proposition 1.15).

**Proposition 1.12** (see [21]). Let $F: \mathcal{A} \to \mathcal{B}$ be a full embedding of DG categories and let $F^*: \text{Perf} - \mathcal{A} \to \text{Perf} - \mathcal{B}$ ($F^*: \mathcal{I}\mathcal{F} - \mathcal{A} \to \mathcal{I}\mathcal{F} - \mathcal{B}$) be the extension DG functor. Then the derived functor $L F^*: \text{Perf} - \mathcal{A} \to \text{Perf} - \mathcal{B}$ (respectively, $L F^*: \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$) is fully faithful.
If, in addition, the category $\text{Perf}-B$ is classically generated by $\text{Ob }A$, then $LF^*$ is an equivalence, and the DG functor $F^*: \text{Perf-}A \to \text{Perf-}B$ (respectively, $F^*: \mathcal{I}F-\mathcal{A} \to \mathcal{I}F-B$) is a quasi-equivalence.

Remark 1.13. Applying this proposition to the case where $B = \text{Perf-}A$, we obtain quasi-equivalences $B = \text{Perf-}A \sim \text{Perf-}B$ and $\mathcal{I}F-\mathcal{A} \sim \mathcal{I}F-B$ that induce an equivalence between the derived categories $D(A)$ and $D(\text{Perf-}A)$.

On the other hand, we can also consider the restriction DG functor

$$F_*: \text{Mod-B} \to \text{Mod-A}$$

and the induced derived functor $F_*: D(B) \to D(A)$. The functor $F_*$ is right adjoint to the derived functor $LF^*: D(A) \to D(B)$. The composition of the DG functor $F_*$ with the Yoneda DG functor $h^*$ gives a DG functor $B \to \text{Mod-A}$ and a homotopy functor $H^0(B) \to D(A)$. As a result we obtain the following proposition, a proof of which can be found in [27].

**Proposition 1.14.** Let $F: A \hookrightarrow B$ be a full embedding of DG categories. Assume that $\text{Ob }A$ forms a set of compact generators of $D(B)$. Then the derived functor $F_*: D(B) \to D(A)$ is an equivalence. If, in addition, $B$ is pretriangulated and the homotopy category $H^0(B)$ is idempotent complete, then the derived functor $F_*$ induces an equivalence between $H^0(B)$ and the triangulated category $\text{Perf-}A$, and the DG categories $B$ and $\text{Perf-}A$ are quasi-equivalent.

Let $A$ and $B$ be two small DG categories. Since we consider DG categories up to quasi-equivalence, it is natural to consider morphisms from $A$ to $B$ as roofs $A \sim A_{\text{cof}} \to B$, where $A \sim A_{\text{cof}}$ is a cofibrant replacement (see [22], for example). These sets of morphisms are much better described in term of quasi-functors.

**Definition 1.15.** A DG $A-B$-bimodule $T$ is called a quasi-functor from $A$ to $B$ if the tensor functor $(-) \otimes_A^L T: D(A) \to D(B)$ takes every representable DG $A$-module to an object which is isomorphic in $D(B)$ to a representable $B$-module.

In other words, a quasi-functor is represented by a DG functor $A \to \text{Mod-B}$ whose essential image consists of quasi-representable DG $B$-modules, where ‘quasi-representable’ means ‘quasi-isomorphic to a representable DG module’. Since the category of quasi-representable DG $B$-modules is equivalent to $H^0(B)$, a quasi-functor $T$ defines a functor $H^0(T): H^0(A) \to H^0(B)$. By the same reasons the restriction of the tensor functor $(-) \otimes_A^L T$ to the category of perfect modules $\text{Perf-}A$ induces a functor $\text{Perf-}A \to \text{Perf-}B$. In the particular case when $A$ is just a DG algebra, the quasi-functor $T$ considered as a DG $B$-module is quasi-isomorphic to a representable DG $B$-module, and hence it is a perfect $B$-module.

Denote by $\text{Rep}(A, B)$ the full subcategory of the derived category $D(A^{\text{op}} \otimes B)$ of $A-B$-bimodules that consists of all quasi-functors.

It is known that the morphisms from $A$ to $B$ in the localization of the category of all small DG $k$-linear categories with respect to quasi-equivalences are in a natural bijection with the isomorphism classes of $\text{Rep}(A, B)$ (see [50]). Due to this theorem, any morphism of the form $A \sim A_{\text{cof}} \to B$ will also be called a quasi-functor.
Let $F: \mathcal{A} \to \mathcal{B}$ be a quasi-functor. It can be realized as a roof
$\xymatrix{ \mathcal{A} \ar[r]^{a} & \mathcal{A}_{\text{cof}} \ar[r]^{F'} \ar[l]_{a} & \mathcal{B},}$
where $a$ and $F'$ are DG functors, and $a$ is a quasi-equivalence. The quasi-functor $F$
induces functors

$$LF^* = LF'^* \cdot a_*: \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B}) \quad \text{and} \quad RF_* := Ra'^* \cdot F'^*: \mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{A}). \quad (2)$$

Since $a$ is a quasi-equivalence, the functor $a_*: \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{A}_{\text{cof}})$ is an equivalence, and $Ra'^*$ is quasi-inverse to $a_*$. Hence, the right adjoint functor $Ra'^*$ is isomorphic to the left adjoint $La^*$, and $RF_* \cong La^* \cdot F'^*$. Thus, we conclude that the functor $RF_*$ has a right adjoint functor

$$RF'^* = RF^* \cdot a_*: \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$$

and thus commutes with all direct sums.

The inverse image functor $LF^*: \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ induces the functor

$$LF^*: \text{Perf}_{-}\mathcal{A} \to \text{Perf}_{-}\mathcal{B}$$

between subcategories of perfect modules, while the functors $RF_*$ and $RF'^*$ do not necessarily send perfect modules to perfect modules.

Now if we consider the quasi-functor $F$ as a DG $\mathcal{A}-\mathcal{B}$-bimodule $T$, then there are isomorphisms of functors

$$LF^* \cong (-) \otimes_{\mathcal{A}} T: \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B}) \quad \text{and} \quad RF_* \cong R\text{Hom}_{\mathcal{B}}(T, -): \mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{A}).$$

Actually, quasi-functors from $\mathcal{A}$ to $\mathcal{B}$ form a DG category $\text{Rep}(\mathcal{A}, \mathcal{B})$ that can be defined as the full DG subcategory of the DG category of semifree DG $\mathcal{A}-\mathcal{B}$-bimodules $\mathcal{A} \mathcal{B} \cdot (\mathcal{A}^\circ \otimes \mathcal{B})$ that consists of all the objects of $\text{Rep}(\mathcal{A}, \mathcal{B})$. It follows from the definition that $\mathcal{H}^0(\text{Rep}(\mathcal{A}, \mathcal{B})) \cong \text{Rep}(\mathcal{A}, \mathcal{B})$. The DG category of all quasi-functors $\text{Rep}(\mathcal{A}, \mathcal{B})$ can also be described as $R\text{Hom}(\mathcal{A}, \mathcal{B})$, where $R\text{Hom}$ is an internal Hom-functor in the localization of the category of all small DG $k$-linear categories with respect to quasi-equivalences (see [50]).

2. Derived noncommutative schemes, their properties, and geometric realizations

2.1. Derived noncommutative schemes and their properties. Let $X$ be a quasi-compact and quasi-separated scheme over a field $k$. Let $\text{Qcoh} X$ denote the Abelian category of quasi-coherent sheaves on $X$. With any such scheme one can associate the derived category of $\mathcal{O}_X$-modules with quasi-coherent cohomology $\mathcal{D}_{\text{Qcoh}}(X)$. It admits arbitrary direct sums. It is also known and proved in [31] and [12] that the subcategory of compact objects in this category coincides with the subcategory of perfect complexes $\text{Perf}_{-}X$. Recall that a complex of $\mathcal{O}_X$-modules on a scheme is said to be perfect if it is locally quasi-isomorphic to a bounded complex of locally free sheaves of finite type.

In [31] and [12] it was proved that the category $\text{Perf}_{-}X$ admits a classical generator. Let us take such a generator $\mathcal{R}$ as an object of the DG category $\text{Perf}_{-}X$. Denote by $\mathcal{R}$ its DG algebra of endomorphisms, that is, $\mathcal{R} = \text{Hom}(\mathcal{R}, \mathcal{R})$. Proposition 1.12 implies that the DG category $\text{Perf}_{-}X$ is quasi-equivalent to $\text{Perf}_{-}\mathcal{R}$. In this case, we also obtain as a corollary an equivalence between the derived categories $\mathcal{D}(\mathcal{R}) \sim \mathcal{D}_{\text{Qcoh}}(X)$ and the triangulated categories $\text{Perf}_{-}\mathcal{R} \sim \text{Perf}_{-}X$.  

\[
\text{Derived noncommutative schemes, their properties, and geometric realizations}
\]

2.1. Derived noncommutative schemes and their properties. Let $X$ be a quasi-compact and quasi-separated scheme over a field $k$. Let $\text{Qcoh} X$ denote the Abelian category of quasi-coherent sheaves on $X$. With any such scheme one can associate the derived category of $\mathcal{O}_X$-modules with quasi-coherent cohomology $\mathcal{D}_{\text{Qcoh}}(X)$. It admits arbitrary direct sums. It is also known and proved in [31] and [12] that the subcategory of compact objects in this category coincides with the subcategory of perfect complexes $\text{Perf}_{-}X$. Recall that a complex of $\mathcal{O}_X$-modules on a scheme is said to be perfect if it is locally quasi-isomorphic to a bounded complex of locally free sheaves of finite type.

In [31] and [12] it was proved that the category $\text{Perf}_{-}X$ admits a classical generator. Let us take such a generator $\mathcal{R}$ as an object of the DG category $\text{Perf}_{-}X$. Denote by $\mathcal{R}$ its DG algebra of endomorphisms, that is, $\mathcal{R} = \text{Hom}(\mathcal{R}, \mathcal{R})$. Proposition 1.12 implies that the DG category $\text{Perf}_{-}X$ is quasi-equivalent to $\text{Perf}_{-}\mathcal{R}$. In this case, we also obtain as a corollary an equivalence between the derived categories $\mathcal{D}(\mathcal{R}) \sim \mathcal{D}_{\text{Qcoh}}(X)$ and the triangulated categories $\text{Perf}_{-}\mathcal{R} \sim \text{Perf}_{-}X$.  

\[
\text{Derived noncommutative schemes, their properties, and geometric realizations}
\]
Since $R$ is a perfect complex, the DG algebra $R$ has only finitely many non-trivial cohomology groups. This fact allows us to propose the following definition of a derived noncommutative scheme over $k$.

**Definition 2.1.** A derived noncommutative scheme $X$ over a field $k$ is a $k$-linear DG category of the form $\text{Perf}-R$, where $R$ is a cohomologically bounded DG algebra over $k$. The derived category $D(R)$ is called the derived category of quasi-coherent sheaves on this noncommutative scheme, while the triangulated category $\text{Perf}-R$ is called the category of perfect complexes on it.

Henceforth, for brevity we will sometimes omit the adjective ‘derived’ and refer to such an object just as a ‘noncommutative scheme’.

For any noncommutative scheme $X$ we have the opposite noncommutative scheme $X^{\circ}$ that is the DG category $\text{Perf}-R^{\circ}$, where $R^{\circ}$ is the opposite DG algebra. We can also define the tensor product $X \otimes_k Y$ of noncommutative schemes $X = \text{Perf}-R$ and $Y = \text{Perf}-I$ as the derived noncommutative scheme $\text{Perf}-(R \otimes_k I)$.

Now we consider some natural properties of noncommutative schemes.

**Definition 2.2.** A noncommutative scheme $X = \text{Perf}-R$ will be said to be proper if the triangulated category $\text{Perf}-R$ is proper, that is, the $k$-vector spaces $\bigoplus_{p \in \mathbb{Z}} H^p(\text{Hom}_{\text{Perf}-R}(M, N))$ are finite-dimensional for any two perfect DG modules $M, N \in \text{Perf}-R$.

This property can be described in terms of the DG algebra $R$. It can be checked that the noncommutative scheme $\text{Perf}-R$ is proper if and only if the cohomology algebra $\bigoplus_{p \in \mathbb{Z}} H^p(R)$ is finite-dimensional. It can be shown that Definition 2.2 is consistent with the usual concept of a proper scheme.

**Proposition 2.3** ([40], Proposition 3.30). Let $X$ be a separated scheme of finite type over a field $k$. Then $X$ is proper if and only if the category of perfect complexes $\text{Perf}-X$ is proper.

Another fundamental property of the usual commutative schemes that can be extended to noncommutative schemes is regularity.

**Definition 2.4.** A noncommutative scheme $X = \text{Perf}-R$ is said to be regular if the triangulated category $\text{Perf}-R$ is regular, that is, has a strong generator.

It was proved in [33] that for a quasi-compact and separated scheme $X$ the triangulated category $\text{Perf}-X$ is regular if and only if $X$ can be covered by open affine subschemes $\text{Spec}(R_i)$, where each $R_i$ has finite global dimension. There is also a short proof of this fact for a separated Noetherian scheme over $k$ of finite Krull dimension, with Noetherian square $X \times_k X$ (see [40], Theorem 3.27).

Regularity of a scheme is closely related to another important property that is called smoothness. However, smoothness depends on the base field $k$. A small $k$-linear DG category $\mathcal{A}$ is said to be $k$-smooth if it is perfect as a DG bimodule, that is, as a DG module over $\mathcal{A}^\circ \otimes_k \mathcal{A}$ (see [23]). Thus, we obtain a definition of smoothness for noncommutative schemes.

**Definition 2.5.** A noncommutative scheme $X = \text{Perf}-R$ is said to be $k$-smooth if the DG category $\text{Perf}-R$ is $k$-smooth, that is, it is perfect as a DG bimodule.
Smoothness is invariant under Morita equivalence [28]. This means that if $\mathcal{D}(\mathcal{A})$ and $\mathcal{D}(\mathcal{B})$ are equivalent through a functor of the form $(-) \otimes_{\mathcal{A}} \mathcal{T}$, where $\mathcal{T}$ is an $\mathcal{A}$-$\mathcal{B}$-bimodule, then $\mathcal{A}$ is smooth if and only if $\mathcal{B}$ is smooth. Thus, the DG category $\text{Perf} - \mathcal{B}$ is smooth if and only if $\mathcal{B}$ is smooth. It is proved in [26] that a smooth DG category $\mathcal{A}$ is regular. Thus, a smooth noncommutative scheme is also regular.

A usual commutative scheme $X$ over a field $k$ is said to be smooth if it is of finite type and the scheme $X = X \otimes_k \overline{k}$ is regular, where $\overline{k}$ is the algebraic closure of $k$. It is proved in [26] that a separated scheme $X$ of finite type is smooth if and only if the DG category $\text{Perf} - X$ is smooth (see also [40] and [29]). Thus, we have defined and can talk about smooth, regular, and proper noncommutative schemes.

For any two derived noncommutative schemes $X$ and $Y$ we can consider the tensor product $X \otimes_k Y$. If both $X$ and $Y$ are proper, then $X \otimes_k Y$ is also proper. It can also be shown that the tensor product $X \otimes_k Y$ is smooth when $X$ and $Y$ are smooth. However, the tensor product of regular schemes is not necessarily regular even for usual commutative schemes.

2.2. Morphisms of derived noncommutative schemes. Let $\mathcal{X} = \text{Perf} - \mathcal{R}$ and $\mathcal{Y} = \text{Perf} - \mathcal{S}$ be two derived noncommutative schemes over an arbitrary field $k$.

**Definition 2.6.** A morphism $f: \mathcal{X} \to \mathcal{Y}$ of noncommutative schemes is a quasi-functor $F: \text{Perf} - \mathcal{S} \to \text{Perf} - \mathcal{R}$.

With any usual morphism $f: X \to Y$ of commutative schemes $X$ and $Y$ one can associate the inverse image functor $f^*: \text{Perf} - Y \to \text{Perf} - X$. Therefore, any morphism between commutative schemes induces a morphism between the corresponding noncommutative schemes. Meanwhile, in the derived noncommutative world we have a lot of additional morphisms between commutative schemes, because there are many other quasi-functors between DG categories of perfect complexes.

Let $\mathcal{X} = \text{Perf} - \mathcal{R}$ and $\mathcal{Y} = \text{Perf} - \mathcal{S}$ be two noncommutative schemes and let $f: \mathcal{X} \to \mathcal{Y}$ be a morphism, that is, a quasi-functor $F: \text{Perf} - \mathcal{S} \to \text{Perf} - \mathcal{R}$. Any such morphism induces derived functors

$$LF^*: \mathcal{D}(\mathcal{S}) \longrightarrow \mathcal{D}(\mathcal{R}) \quad \text{and} \quad RF_*: \mathcal{D}(\mathcal{R}) \longrightarrow \mathcal{D}(\mathcal{S})$$

that are defined for any quasi-functor $F$ in (2) and will be called the inverse image and the direct image functors, respectively. We also have the functor $f^! := RF^!$ that is right adjoint to $RF_*$. The inverse image functor $LF^*$ sends perfect modules to perfect modules, and its restriction to $\text{Perf} - \mathcal{S}$ is isomorphic to the homotopy functor $\mathcal{H}^0(F): \text{Perf} - \mathcal{S} \to \text{Perf} - \mathcal{R}$. The direct image functor $RF_*: \mathcal{D}(\mathcal{R}) \to \mathcal{D}(\mathcal{S})$ commutes with arbitrary direct sums.

The most important morphisms for us are those for which the inverse image functor $LF^*$ is fully faithful.

**Definition 2.7.** A morphism $f: \mathcal{X} \to \mathcal{Y}$ of noncommutative schemes is called an $\text{ff}$-morphism (fully faithful morphism) if the inverse image functor $LF^*$ is fully faithful.
Note that the functor $L\mathfrak{f}^*: \mathcal{D}(\mathcal{I}) \to \mathcal{D}(\mathcal{R})$ is fully faithful if and only if its restriction $\mathfrak{f}^*: \text{Perf} - \mathcal{I} \to \text{Perf} - \mathcal{R}$ is fully faithful (see Proposition 1.12 and Remark 1.13, for example). Furthermore, for any ff-morphism $\mathfrak{f}: \mathcal{X} \to \mathcal{Y}$ the functor $\mathfrak{f}^!: \mathcal{D}(\mathcal{I}) \to \mathcal{D}(\mathcal{R})$ is also fully faithful, because the composition functor $\mathfrak{R}\mathfrak{f}_* \mathfrak{f}^!$ is right adjoint to $\mathfrak{R}\mathfrak{f}_* \mathfrak{f}^* \cong \text{id}$.

It is easy to see that by the projection formula a morphism $\mathfrak{f}: X \to Y$ between usual commutative schemes is an ff-morphism if and only if the direct image $\mathfrak{R}\mathfrak{f}_* \mathcal{O}_X$ is isomorphic to $\mathcal{O}_Y$. Only in this case is the inverse image functor $\mathfrak{L}\mathfrak{f}^*: \mathcal{D}_{\text{Qcoh}}(Y) \to \mathcal{D}_{\text{Qcoh}}(X)$ fully faithful.

Another class of morphisms that can be extended to derived noncommutative schemes is the class of so-called perfect proper morphisms.

**Definition 2.8.** A morphism $\mathfrak{f}: \mathcal{X} \to \mathcal{Y}$ of derived noncommutative schemes is called a pp-morphism (perfect proper morphism) if the direct image functor $\mathfrak{R}\mathfrak{f}_*$ sends perfect modules to perfect modules.

This also means that the inverse image functor $\mathfrak{L}\mathfrak{f}^*$ as a functor from $\text{Perf} - \mathcal{I}$ to $\text{Perf} - \mathcal{R}$ has a right adjoint $\mathfrak{R}\mathfrak{f}_*: \text{Perf} - \mathcal{R} \to \text{Perf} - \mathcal{I}$. As a consequence, the right adjoint to the quasi-functor $\mathcal{F}$ corresponding to $\mathfrak{f}$ induces a quasi-functor $\mathcal{G}: \text{Perf} - \mathcal{R} \to \text{Perf} - \mathcal{I}$ for which $\mathcal{H}^0(\mathcal{G}) \cong \mathfrak{R}\mathfrak{f}_*$. Thus, in this case we obtain a morphism $\mathfrak{g}: \mathcal{Y} \to \mathcal{X}$ that can be called a ‘right adjoint’ morphism to $\mathfrak{f}$. If a morphism $\mathfrak{f}$ is both an ff-morphism and a pp-morphism, then we obtain an isomorphism of morphisms $\mathfrak{f} \cdot \mathfrak{g} \cong \text{id}_Y$.

Perfect morphisms of schemes were defined in [4], III, as pseudocoherent morphisms of locally finite Tor-dimension. For a locally Noetherian scheme $Y$ a pseudocoherent morphism $\mathfrak{f}: X \to Y$ is the same as a morphism locally of finite type, and in this case if $\mathfrak{f}$ is perfect and proper, then the direct image functor $\mathfrak{R}\mathfrak{f}_*$ sends perfect complexes on $X$ to perfect complexes on $Y$ (see [4], III, and [49], 2.5.5).

We can also define other types of morphisms. For example, we can say that a morphism $\mathfrak{f}: \mathcal{X} \to \mathcal{Y}$ is quasi-affine if the set of images of perfect modules $\text{Perf} - \mathcal{I}$ under the inverse image functor $\mathfrak{L}\mathfrak{f}^*$ classically generates the category $\text{Perf} - \mathcal{R}$. By Proposition 1.12 we see that a quasi-affine ff-morphism $\mathfrak{f}: \mathcal{X} \to \mathcal{Y}$ induces an equivalence $\mathfrak{L}\mathfrak{f}^*: \text{Perf} - \mathcal{I} \to \text{Perf} - \mathcal{R}$ and a quasi-equivalence between the DG categories $\text{Perf} - \mathcal{I}$ and $\text{Perf} - \mathcal{R}$. Thus, we conclude that the noncommutative schemes $\mathcal{X}$ and $\mathcal{Y}$ are isomorphic in this case.

In the commutative situation smooth projective varieties $X$ and $Y$ that have equivalent triangulated categories $\text{Perf} - X$ and $\text{Perf} - Y$ are called Fourier–Mukai partners. Since any equivalence is represented by an object on the product (see [36] and [27]), the DG categories $\text{Perf} - X$ and $\text{Perf} - Y$ are quasi-equivalent, and Fourier–Mukai partners determine the same derived noncommutative schemes. The most famous example is due to Mukai [30] and is given by an Abelian variety $A$ and the dual Abelian variety $\hat{A}$ (see also [37] and [38]).

One can introduce the **triangulated category of cohomologically bounded pseudocoherent complexes** on $\mathcal{X}$ and denote it by $\mathcal{D}^b(\text{coh} \mathcal{X})$, because it is equivalent to a bounded derived category of coherent sheaves on a Noetherian scheme in the commutative case. First we consider the full triangulated subcategory $\mathcal{D}^b(\mathcal{R}) \subset \mathcal{D}(\mathcal{R})$ consisting of all DG $\mathcal{R}$-modules $\mathcal{N}$ that are cohomologically bounded. Now we say that an object $\mathcal{M} \in \mathcal{D}^b(\mathcal{R})$ belongs to the full subcategory $\mathcal{D}^b(\text{coh} \mathcal{X}) \subset \mathcal{D}^b(\mathcal{R})$ if
for any sufficiently large $N \in \mathbb{N}$ there exist a perfect DG module $P \in \text{Perf}-R$ and a morphism $P \to M$ such that the induced maps $H^k(P) \to H^k(M)$ are isomorphisms for all $k \geq -N$ (see [34], for example). Although the definition of the cohomology $H^k(M) = \text{Hom}_{D(R)}(R, M)$ depends on $R$, the definition of a pseudocoherent module does not depend on the choice of a classical generator for $\text{Perf}-R$. Obviously, there is a natural inclusion of triangulated categories $\text{Perf}-R \subseteq D^b(\text{coh} \mathcal{X})$.

Any morphism $f: \mathcal{X} \to \mathcal{Y}$ is a quasi-functor $F: \text{Perf}-\mathcal{I} \to \text{Perf}-\mathcal{R}$ and is given by a DG $\mathcal{I}$-$\mathcal{R}$-bimodule $T$. Since $\mathcal{R}$ is cohomologically bounded and $\mathcal{I}$ is perfect as a DG $\mathcal{R}$-module, the direct image functor $Rf_* = R\text{Hom}_{\mathcal{R}}(T, -)$ sends $D^b(\mathcal{R})$ to $D^b(\mathcal{I})$ for any morphism $f$. Now we can say that a morphism $f: \mathcal{X} \to \mathcal{Y}$ is of finite Tor-dimension if the inverse image functor $Lf^*: D(\mathcal{I}) \to D(\mathcal{R})$ sends $D^b(\mathcal{I})$ to $D^b(\mathcal{R})$, and we say that $f$ is proper if the direct image functor $Rf_*$ sends $D^b(\text{coh} \mathcal{X})$ to $D^b(\text{coh} \mathcal{Y})$.

**Definition 2.9.** A morphism of derived noncommutative schemes $f: \mathcal{X} \to \mathcal{Y}$ is called an immersion if the direct image functor $Rf_*: D(\mathcal{R}) \to D(\mathcal{I})$ is fully faithful.

In this case the derived category $D(\mathcal{R})$ can be obtained as a Bousfield localization (and colocalization) of $D(\mathcal{I})$ (see, for example, [32], Chap. 9). An example of such a morphism in commutative algebraic geometry is given by a usual open immersion $j: U \hookrightarrow Y$.

**Definition 2.10.** Let $\mathcal{X}$ be a noncommutative scheme. A compactification of $\mathcal{X}$ is a morphism $f: \mathcal{X} \to \mathcal{X}$ such that $f$ is an immersion and the noncommutative scheme $\mathcal{X}$ is proper.

Another important notion is resolution of singularities, or desingularization.

**Definition 2.11.** For a derived noncommutative scheme $\mathcal{X}$, a regular (smooth) desingularization of $\mathcal{X}$ is a morphism $f: \mathcal{X} \to \mathcal{X}$ such that $f$ is an ff-morphism, and the noncommutative scheme $\mathcal{X}$ is regular (respectively, smooth).

We should note that a usual resolution of singularities $f: \mathcal{X} \to \mathcal{X}$ of a commutative scheme is not necessarily a desingularization in the noncommutative case, because we require that the inverse image functor $Lf^*$ be fully faithful. This condition is fulfilled only if $Rf_*O_X \cong O_X$, that is, if $X$ has rational singularities. However, it follows from Theorem 1.4 in [25] that a separated scheme of finite type $X$ over a field of characteristic 0 has a smooth desingularization as in Definition 2.11. By the construction of this resolution, there is a quasi-functor $G: \text{Perf}-X \to D$, where $D$ is a gluing of DG categories of perfect complexes on smooth proper schemes, and the induced homotopy functor $G: \text{Perf}-Y \to D$ is fully faithful.

**2.3. Categories of morphisms and Serre functors.** It was discussed above that quasi-functors form a DG category. Therefore, morphisms between derived noncommutative schemes $\mathcal{X}$ and $\mathcal{Y}$ also form a DG category, which will be denoted by $\text{Mor}(\mathcal{X}, \mathcal{Y})$. For any derived noncommutative schemes $\mathcal{X} = \text{Perf}-\mathcal{R}$ and $\mathcal{Y} = \text{Perf}-\mathcal{I}$ we have quasi-equivalences of DG categories:

$$\text{Mor}(\mathcal{X}, \mathcal{Y}) := \text{Rep}(\text{Perf}-\mathcal{I}, \text{Perf}-\mathcal{R})$$

$$\cong \text{Rep}(\mathcal{I}, \text{Perf}-\mathcal{R}) \subset \mathcal{I}\mathcal{F}-(\mathcal{I}^o \otimes_k \mathcal{R}).$$
The DG category $\mathcal{M}or(\mathcal{X}, \mathcal{Y})$ is pretriangulated, and there is a triangulated category of morphisms

$$\mathcal{M}or(\mathcal{X}, \mathcal{Y}) = \mathcal{H}^0(\mathcal{M}or(\mathcal{X}, \mathcal{Y})).$$

In particular, we can add any two morphisms and consider morphisms between morphisms. It is evident that the DG category $\mathcal{M}or(\mathcal{X}, \text{pt})$ of morphisms from $\mathcal{X}$ to the point pt is quasi-equivalent to $\text{Perf-}\mathcal{R}$. Besides, the DG category $\mathcal{M}or(\text{pt}, \mathcal{X})$ consists of DG $\mathcal{R}$-modules $\mathcal{N}$ that are perfect as complexes of $k$-vector spaces, that is, $\dim_k \bigoplus_i H^i(\mathcal{N}) < \infty$.

By Theorem 1.6, if a triangulated category $\mathcal{T}$ is regular, proper, and idempotent complete, then any exact functor from $\mathcal{T}$ to $\text{Perf-}k$ is representable. The proof of Theorem 1.6 (see [12]) works for DG categories without any changes and, moreover, the assertion for DG categories can be deduced from the theorem.

**Proposition 2.12** ([40], Theorem 3.18). Let $\mathcal{A}$ be a small DG category that is regular and proper. Then a DG module $\mathcal{M}$ is perfect if and only if $\dim \bigoplus_i H^i(\mathcal{M}(X)) < \infty$ for all $X \in \mathcal{A}$.

In particular, we obtain the following corollary.

**Corollary 2.13.** Let $\mathcal{Y}$ be a derived noncommutative scheme that is regular and proper. Then there is a quasi-equivalence $\mathcal{M}or(\text{pt}, \mathcal{Y}) \cong \mathcal{Y}^\circ$.

If the noncommutative scheme $\mathcal{Y} = \text{Perf-}\mathcal{I}$ is proper, then the DG category $\mathcal{M}or(\mathcal{X}, \mathcal{Y})$ contains $\text{Perf-}(\mathcal{I}^\circ \otimes_k \mathcal{R})$ as a DG subcategory, because the perfect DG $\mathcal{I}-\mathcal{R}$-bimodule $\mathcal{I} \otimes_k \mathcal{R}$, which classically generates the category $\text{Perf-}(\mathcal{I}^\circ \otimes_k \mathcal{R})$, is also perfect as a DG $\mathcal{R}$-module.

Besides, if a noncommutative scheme $\mathcal{Y}$ is smooth, then we have the reverse inclusion $\mathcal{M}or(\mathcal{X}, \mathcal{Y}) \subseteq \text{Perf-}(\mathcal{I}^\circ \otimes_k \mathcal{R})$. Indeed, for a smooth $\mathcal{Y}$ the DG $\mathcal{I}-\mathcal{I}$-bimodule $\mathcal{I}$ is perfect. The category of perfect $\mathcal{I}-\mathcal{I}$-bimodules is classically generated by the bimodule $\mathcal{I} \otimes_k \mathcal{I}$. Thus, any DG $\mathcal{I}-\mathcal{R}$-bimodule $\mathcal{T} \cong \mathcal{I} \otimes_\mathcal{I} \mathcal{T}$ belongs to the subcategory generated by the DG $\mathcal{I}-\mathcal{R}$-bimodule

$$(\mathcal{I} \otimes_k \mathcal{I}) \otimes_\mathcal{I} \mathcal{T} \cong \mathcal{I} \otimes_k \mathcal{T}.$$ 

Suppose that $\mathcal{T}$ is perfect as a DG $\mathcal{R}$-module; then $\mathcal{I} \otimes_k \mathcal{T}$ is perfect as a DG $\mathcal{I}-\mathcal{R}$-bimodule, and hence $\mathcal{T}$ is also perfect as a DG $\mathcal{I}-\mathcal{R}$-bimodule. Thus, we get the inclusion $\mathcal{M}or(\mathcal{X}, \mathcal{Y}) \subseteq \text{Perf-}(\mathcal{I}^\circ \otimes_k \mathcal{R})$. Finally, we obtain the following proposition.

**Proposition 2.14.** Let $\mathcal{Y}$ be a derived noncommutative scheme that is smooth and proper. Then there is a natural quasi-equivalence

$$\mathcal{M}or(\mathcal{X}, \mathcal{Y}) := \text{Perf-}(\mathcal{I}^\circ \otimes_k \mathcal{R}) \cong \mathcal{Y}^\circ \otimes_k \mathcal{X}$$

for any derived noncommutative scheme $\mathcal{X}$.

At the same time there is the following theorem of Toën.
Consider the DG

\[ M \triangleright \omega \]

\textit{category of perfect DG modules over a usual regular projective scheme for any equivalence.}

Then there is a canonical quasi-equivalence

\[ \text{Mor}(X, Y) \cong \text{Rep}(\text{Perf} - Y, \text{Perf} - X) \cong \text{Perf} - (Y \times_k X). \]

In particular, the DG category \( \text{Perf} - (Y \times_k X) \) is quasi-equivalent to the DG category of perfect DG modules over \( (\text{Perf} - Y)^\circ \otimes_k (\text{Perf} - X) \).

The quasi-equivalence between the DG category \( \text{Perf} - (Y \times_k X) \) and the DG category of perfect DG modules over \( (\text{Perf} - Y)^\circ \otimes_k (\text{Perf} - X) \) can be described explicitly. Consider the DG functors

\[ \text{pr}_1^* : \text{Perf} - Y \rightarrow \text{Perf} - (Y \times_k X) \quad \text{and} \quad \text{pr}_2^* : \text{Perf} - X \rightarrow \text{Perf} - (Y \times_k X) \]

induced by the projections \( \text{pr}_1 : Y \times_k X \rightarrow Y \) and \( \text{pr}_2 : Y \times_k X \rightarrow X \). For any perfect complex \( E \) on the product \( Y \times_k X \) we can define a bimodule \( T_E \) by the rule

\[ T_E(N, M) \cong \text{Hom}_{\text{Perf} - (Y \times_k X)}(\text{pr}_2^* M, \text{pr}_1^* N \otimes E), \]

where \( M \in \text{Perf} - X \) and \( N \in \text{Perf} - Y \). This is exactly the required quasi-equivalence.

Let \( \mathcal{X} = \text{Perf} - \mathcal{R} \) be a derived noncommutative scheme. If it is proper and regular, then by Proposition 1.8 the triangulated category \( \text{Perf} - \mathcal{R} \) has a Serre functor. Recall that an autoequivalence \( S_{\mathcal{X}} \) is a Serre functor if it induces bifunctorial isomorphisms

\[ \text{Hom}_{\text{Perf} - \mathcal{R}}(N, S_{\mathcal{X}}(M)) \cong \text{Hom}_{\text{Perf} - \mathcal{R}}(M, N)^* \]

for any \( M, N \in \text{Perf} - \mathcal{R} \) (see [8], [10], [11], and also [45] for the DG case). For a usual regular projective scheme \( X \) the Serre functor is isomorphic to \( (-) \otimes \omega_X[n] \), where \( \omega_X \) is the canonical sheaf and \( n \) is the dimension of \( X \).

For any right DG \( \mathcal{R} \)-module \( M \) we can define the left DG \( \mathcal{R} \)-modules

\[ M^* := \text{Hom}_k(M, k) \quad \text{and} \quad M^\vee := \text{Hom}_{\mathcal{R}}(M, \mathcal{R}). \]

Consider the DG \( \mathcal{R} \)-bimodule \( \mathcal{R}^* := \text{Hom}_k(\mathcal{R}, k) \) and the derived functor \( (-) \otimes_{\mathcal{R}} \mathcal{R}^* \) from \( D(\mathcal{R}) \) to itself. If \( P \) is a perfect DG \( \mathcal{R} \)-module, then \( (P^\vee)^\vee \cong P \) and there is an isomorphism \( P \otimes_{\mathcal{R}} \mathcal{R}^* \cong (P^\vee)^* \), because

\[ P \otimes_{\mathcal{R}} \mathcal{R}^* \cong \text{Hom}_{\mathcal{R}}(\mathcal{R}, P \otimes_{\mathcal{R}} \mathcal{R}^*) \cong \text{Hom}_{\mathcal{R}}(P^\vee, \text{Hom}_k(\mathcal{R}, k)) \]

\[ \cong \text{Hom}_k(P^\vee, k) \cong (P^\vee)^*. \]

Let us consider the functor \( (-) \otimes_{\mathcal{R}} \mathcal{R}^* \) as a functor from \( \text{Perf} - \mathcal{R} \) to \( D(\mathcal{R}) \). We have the sequence of isomorphisms

\[ \text{Hom}_{\mathcal{X}}(P, M)^* \cong \text{Hom}_k(M \otimes_{\mathcal{R}} P^\vee, k) \cong \text{Hom}_{\mathcal{X}}(M, \text{Hom}_k(P^\vee, k)) \]

\[ \cong \text{Hom}_{\mathcal{X}}(M, P \otimes_{\mathcal{R}} \mathcal{R}^*), \]
where the notation $\text{Hom}_X$ is used for the space of morphisms in the triangulated category $\text{Perf} - \mathcal{R}$. These isomorphisms show us that the functor $(-) \otimes_{\mathcal{R}} \mathcal{R}^* : \text{Perf} - \mathcal{R} \to \mathcal{D}(\mathcal{R})$ induces the Serre duality. If it sends $\text{Perf} - \mathcal{R}$ to itself, then we obtain a Serre functor on the category $\text{Perf} - \mathcal{R}$. This happens when $\mathcal{R}^*$ is a perfect (right) DG $\mathcal{R}$-module.

Furthermore, a perfect DG module $P \in \text{Perf} - \mathcal{R}$ defines a morphism $f : X \to \text{pt}$. By the definition of this morphism, we have $Lf^*k \cong P$. Consider the functor $f^! : \mathcal{D}(k) \to \mathcal{D}(\mathcal{R})$ and apply it to $k$. The sequence of isomorphisms

$$
\text{Hom}_X(M, f^!k) \cong \text{Hom}_k(Rf_*M, k) \cong \text{Hom}_k(k, Rf_*M)^*
$$

shows us that the object $f^!k$ is isomorphic to $P \otimes_{\mathcal{R}} \mathcal{R}^*$. When the category $\text{Perf} - \mathcal{R}$ has a Serre functor $S_X$, we get that $S_X(P) \cong S_X(Lf^*k) \cong f^!k$. For example, this holds when $X$ is regular and proper.

Consider a morphism $f : X \to Y$ of noncommutative schemes $X = \text{Perf} - \mathcal{I}$ and $Y = \text{Perf} - \mathcal{J}$ and assume that the category $\text{Perf} - \mathcal{I}$ has a Serre functor $S_Y$. Then we have the sequence of isomorphisms

$$
\text{Hom}_X(Lf^*N, M)^* \cong \text{Hom}_Y(N, Rf_*M)^*
$$

$$
\cong \text{Hom}_Y(Rf_*M, S_Y N) \cong \text{Hom}_X(M, f^!S_Y N),
$$

where $N \in \text{Perf} - \mathcal{I}$ and $M \in \text{Perf} - \mathcal{R}$. This shows that there is a relation between the Serre functor on $Y$ and a Serre functor on $X$ if it exists. Therefore, we obtain the following proposition.

**Proposition 2.16.** Let $f : X \to Y$ be a morphism of noncommutative schemes $X = \text{Perf} - \mathcal{I}$ and $Y = \text{Perf} - \mathcal{J}$ which possess Serre functors $S_X$ and $S_Y$, respectively. Then for any $N \in \text{Perf} - \mathcal{I}$ there is an isomorphism

$$
S_X(Lf^*N) \cong f^!(S_Y N)
$$

in the triangulated category $\text{Perf} - \mathcal{R}$.

A Serre functor is an intrinsic invariant of a triangulated category, which in some cases allows one to talk about a certain notion of dimension for noncommutative schemes and about various other concepts. For example, a proper noncommutative scheme $X$ will be called a weak Calabi–Yau variety of dimension $m$ if it has a Serre functor $S_X$ that is isomorphic to the shift functor $[m]$.

### 2.4. Geometric realizations of derived noncommutative schemes.

The most interesting derived noncommutative schemes appear as full DG subcategories of the DG categories $\text{Perf} - Z$, where $Z$ is some usual commutative scheme. Moreover, any such realization carries an important geometric meaning and gives us a new way to look at these derived noncommutative schemes.

**Definition 2.17.** A geometric realization of a derived noncommutative scheme $X = \text{Perf} - \mathcal{R}$ is a usual commutative scheme $Z$ and a localizing subcategory $\mathcal{L} \subseteq \mathcal{D}_{\text{Qcoh}}(Z)$ such that a natural enhancement $\mathcal{L}$ of it is quasi-equivalent to $\text{Perf} - \mathcal{R}$.
Thus, the derived category $\mathcal{D}(\mathcal{R})$ is equivalent to the localizing subcategory $\mathcal{L}$ and there is a fully faithful functor $\mathcal{D}(\mathcal{R}) \to \mathcal{D}_{Qcoh}(Z)$ whose image coincides with the subcategory $\mathcal{L}$. Moreover, since this inclusion functor preserves direct sums and the category $\mathcal{D}(\mathcal{R})$ is compactly generated, there exists a right adjoint to the inclusion functor as a consequence of Brown’s representability theorem (see [32], 9.1.19). This implies that the category $\mathcal{L} \cong \mathcal{D}(\mathcal{R})$ can be realized as a Verdier quotient (localization) of the category $\mathcal{D}_{Qcoh}(Z)$.

The most important class of geometric realizations is given by the ff-morphisms $f: Z \to \mathcal{X}$. In this case the inverse image functor $Lf^*: \mathcal{D}(\mathcal{R}) \to \mathcal{D}_{Qcoh}(Z)$ gives a geometric realization for $\mathcal{X}$. The functor $Lf^*$ gives a full embedding $\text{Perf}-\mathcal{R} \hookrightarrow \text{Perf}-Z$. This is exactly the case when the localizing subcategory $\mathcal{L}$ is compactly generated by some perfect complexes on $Z$ and, hence, there is an inclusion of subcategories of compact objects $\mathcal{L}^c \cong \text{Perf}-\mathcal{R} \subset \text{Perf}-Z$.

Conversely, if we have an inclusion $\mathcal{L}^c \cong \text{Perf}-\mathcal{R} \subset \text{Perf}-Z$, then it gives an ff-morphism $f: Z \to \mathcal{X}$. In this case the inverse image functor $Lf^*: \mathcal{D}(\mathcal{R}) \to \mathcal{D}_{Qcoh}(Z)$ gives a geometric realization for $\mathcal{X}$, while the direct image functor $Rf_*$ realizes $\mathcal{D}(\mathcal{R})$ as a localization of the category $\mathcal{D}_{Qcoh}(Z)$. Moreover, it is a so-called Bousfield localization, because there exists a right adjoint functor $f^!: \mathcal{D}(\mathcal{R}) \to \mathcal{D}_{Qcoh}(Z)$. Such geometric realizations will be called plain. If, in addition, the morphism $f$ is a pp-morphism, that is, the inclusion $\text{Perf}-\mathcal{R} \subset \text{Perf}-Z$ has a right adjoint, then such a geometric realization will be called perfectly plain.

**Example 2.18.** Let $\pi: \widetilde{X} \to X$ be a proper birational morphism that is a resolution of singularities of $X$. Assume that $R\pi_*\mathcal{O}_{\widetilde{X}} \cong \mathcal{O}_X$. Then $\pi$ is an ff-morphism and the inverse image functor $L\pi^*$ gives a geometric realization for $X$. This realization is plain, but it is not perfectly plain when $X$ is singular.

Another class of geometric realizations is given by immersive morphisms $j: \mathcal{X} \to Z$ (see Definition 2.9). In this situation the direct image functor $Rj_*: \mathcal{D}(\mathcal{R}) \to \mathcal{D}_{Qcoh}(Z)$ is fully faithful, while the inverse image functor $Lj^*: \mathcal{D}_{Qcoh}(Z) \to \mathcal{D}(\mathcal{R})$ is a Bousfield localization. Such geometric realizations will be called immersive.

Suppose now that an immersive morphism $j: \mathcal{X} \to Z$ is a pp-morphism, that is, $Rj_*$ is fully faithful and sends perfect modules to perfect modules. Thus, there is a quasi-functor $G: \text{Perf}-\mathcal{R} \to \text{Perf}-Z$ such that $\mathcal{H}^0(G) \cong Rj_*$. The quasi-functor $G$ itself gives an ff-morphism $g: Z \to \mathcal{X}$. We have an isomorphism $g \cdot j \cong id_{\mathcal{X}}$. In this case the geometric realization is connected with a pair of adjoint morphisms $(j, g)$, where $Rj_* \cong Lg^*$ is fully faithful. It will be called perfectly immersive.

If, in addition, the morphism $g$ is a pp-morphism, then the subcategory $\text{Perf}-\mathcal{R} \subset \text{Perf}-Z$ is admissible from both sides. In this case the geometric realization is both perfectly plain and perfectly immersive, and it will be called pure.

**Example 2.19.** Let $j: U \to X$ be an open immersion. Then the functor $Rj_*$ is fully faithful and gives a geometric realization for $U$ in $X$. This realization is immersive, but it is not perfectly immersive in general.

Many interesting examples of noncommutative schemes appear naturally as admissible subcategories $\mathcal{N} \subset \text{Perf}-X$ for some smooth projective scheme $X$. 
More precisely, for any such subcategory we can consider a DG enhancement \( N \subset \text{Perf} - X \) of it. This is a DG category that has a generator and hence can be realized as \( \text{Perf} - \mathcal{E} \) for some cohomologically bounded DG algebra \( \mathcal{E} \). Indeed, since \( N \) is admissible in \( \text{Perf} - X \), the inclusion functor has right and left adjoint projections. Thus, a projection of a generator in \( \text{Perf} - X \) to \( N \) gives a classical generator for \( N \). Moreover, the noncommutative scheme \( N \) is proper, being a full subcategory of the proper category \( \text{Perf} - X \). Furthermore, it is regular, because a projection of a strong generator gives a strong generator in \( N \). It can also be shown that the noncommutative scheme \( N \) is smooth as an admissible subcategory of the smooth category \( \text{Perf} - X \) (see Proposition 3.10). Note that, by construction, the derived noncommutative scheme \( N \subset \text{Perf} - X \) comes with a geometric realization which, moreover, is pure because \( N \) is admissible.

**Example 2.20.** Let \( X \) be a proper scheme such that \( H^0(X, \mathcal{O}_X) \cong k \) and \( H^i(X, \mathcal{O}_X) = 0 \) for \( i > 0 \). Then the structure sheaf \( \mathcal{O}_X \) is exceptional. The subcategory \( \langle \mathcal{O}_X \rangle \subset \text{Perf} - X \) is admissible and gives a pure geometric realization for the point \( \text{pt} \). Consider the left and right orthogonals \( N \cong \langle \mathcal{O}_X \rangle \) and \( M \cong \langle \mathcal{O}_X \rangle \perp \), respectively. The right orthogonal \( M \) is left admissible in \( \text{Perf} - X \). It defines a derived noncommutative scheme \( M \subset \text{Perf} - X \) together with a perfectly immersive geometric realization. In addition, the left orthogonal \( N \) is right admissible in \( \text{Perf} - X \). It defines a derived noncommutative scheme \( N \subset \text{Perf} - X \) together with a perfectly plain geometric realization. When \( X \) is smooth, the geometric realizations of \( M \) and \( N \) are pure as well. However, for a singular scheme \( X \) these realizations are not necessarily pure.

All other geometric realizations, which are neither plain nor immersive, will be called *mixed*. We can obtain different mixed realizations as compositions of functors of the form \( Lf^* \) and \( Rj_* \) for ff-morphisms \( f \) and immersive morphisms \( j \), where the target category of the last functor is \( \mathcal{D}_{\text{Qcoh}}(Z) \) for some commutative scheme \( Z \). Any such composition of functors preserves all direct sums and has a right adjoint functor that is a composition of functors of the form \( Rf_* \) and \( j! \).

### 3. Gluing of derived noncommutative schemes and geometric realizations

#### 3.1. Gluing of differential graded categories.**

Let \( \mathcal{A} \) and \( \mathcal{B} \) be two small DG categories, and let \( T \) be a DG \( \mathcal{B} - \mathcal{A} \)-bimodule, that is, a DG \( \mathcal{B}^\circ \otimes \mathcal{A} \)-module. We construct the so-called lower triangular DG category corresponding to the data \((\mathcal{A}, \mathcal{B}; T)\).

**Definition 3.1.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be two small DG categories and let \( T \) be a DG \( \mathcal{B} - \mathcal{A} \)-bimodule. The *lower triangular* DG category \( \mathcal{C} = \mathcal{A} \downarrow T \mathcal{B} \) is defined as follows:

1. \( \text{Ob}(\mathcal{C}) = \text{Ob}(\mathcal{A}) \sqcup \text{Ob}(\mathcal{B}) \),
2. \( \text{Hom}_\mathcal{C}(X, Y) = \begin{cases} 
\text{Hom}_\mathcal{A}(X, Y) & \text{when } X, Y \in \mathcal{A}, \\
\text{Hom}_\mathcal{B}(X, Y) & \text{when } X, Y \in \mathcal{B}, \\
T(Y, X) & \text{when } X \in \mathcal{A}, Y \in \mathcal{B}, \\
0 & \text{when } X \in \mathcal{B}, Y \in \mathcal{A}, 
\end{cases} \)
with the composition law coming from the DG categories $\mathcal{A}$ and $\mathcal{B}$ and the bimodule structure on $T$.

The lower triangular DG category $\mathcal{C} = \mathcal{A} \downarrow_{\mathcal{T}} \mathcal{B}$ is not necessarily pretriangulated even if the components $\mathcal{A}$ and $\mathcal{B}$ are pretriangulated. To make this operation well-defined on the class of pretriangulated categories, we introduce the following notion of gluing pretriangulated categories together (see [48], [25], [18], and [40]).

**Definition 3.2.** Let $\mathcal{A}$ and $\mathcal{B}$ be two small pretriangulated DG categories, and let $T$ be a DG $\mathcal{B}$-$\mathcal{A}$-bimodule. The gluing $\mathcal{A} \oplus_{\mathcal{T}} \mathcal{B}$ of DG categories $\mathcal{A}$ and $\mathcal{B}$ via $T$ is defined to be the pretriangulated hull of $\mathcal{A} \downarrow_{\mathcal{T}} \mathcal{B}$, that is,

$$\mathcal{A} \oplus_{\mathcal{T}} \mathcal{B} = (\mathcal{A} \downarrow_{\mathcal{T}} \mathcal{B})^{\text{pre-tr}}.$$

The natural fully faithful DG inclusions

$$a: \mathcal{A} \hookrightarrow \mathcal{A} \downarrow_{\mathcal{T}} \mathcal{B} \quad \text{and} \quad b: \mathcal{B} \hookrightarrow \mathcal{A} \downarrow_{\mathcal{T}} \mathcal{B}$$

induce the fully faithful extension DG functors $a^*: \mathcal{A} \rightarrow \mathcal{A} \oplus_{\mathcal{T}} \mathcal{B}$ and $b^*: \mathcal{B} \rightarrow \mathcal{A} \oplus_{\mathcal{T}} \mathcal{B}$. These quasi-functors induce exact functors

$$a^*: \mathcal{H}^0(\mathcal{A}) \longrightarrow \mathcal{H}^0(\mathcal{A} \oplus_{\mathcal{T}} \mathcal{B}) \quad \text{and} \quad b^*: \mathcal{H}^0(\mathcal{B}) \longrightarrow \mathcal{H}^0(\mathcal{A} \oplus_{\mathcal{T}} \mathcal{B})$$

between triangulated categories, which are also fully faithful. The following proposition is almost obvious.

**Proposition 3.3** ([40], Proposition 3.7). Let the DG category $\mathcal{E}$ be a gluing $\mathcal{A} \oplus_{\mathcal{T}} \mathcal{B}$. Then the DG functors $a^*: \mathcal{A} \rightarrow \mathcal{E}$ and $b^*: \mathcal{B} \rightarrow \mathcal{E}$ induce a semi-orthogonal decomposition for the triangulated category $\mathcal{H}^0(\mathcal{E})$ of the form $\mathcal{H}^0(\mathcal{E}) = \langle \mathcal{H}^0(\mathcal{A}), \mathcal{H}^0(\mathcal{B}) \rangle$.

Furthermore, we can show that any enhancement of a triangulated category with a semi-orthogonal decomposition can be obtained as a gluing of enhancements of the summands.

**Proposition 3.4** ([40], Proposition 3.8). Let $\mathcal{E}$ be a pretriangulated DG category. Assume a semi-orthogonal decomposition $\mathcal{H}^0(\mathcal{E}) = \langle \mathcal{A}, \mathcal{B} \rangle$. Then the DG category $\mathcal{E}$ is quasi-equivalent to a gluing $\mathcal{A} \oplus_{\mathcal{T}} \mathcal{B}$, where $\mathcal{A}, \mathcal{B} \subset \mathcal{E}$ are full DG subcategories with the same objects as $\mathcal{A}$ and $\mathcal{B}$, respectively, and the DG $\mathcal{B}$-$\mathcal{A}$-bimodule is given by the rule

$$T(Y, X) = \text{Hom}_{\mathcal{E}}(X, Y), \quad \text{with } X \in \mathcal{A} \text{ and } Y \in \mathcal{B}. \quad (5)$$

Actually, we can show much more. The following proposition is also not very difficult to prove.

**Proposition 3.5** ([40], Proposition 3.11). Let

$$a: \mathcal{A} \rightarrow \mathcal{A}' \quad \text{and} \quad b: \mathcal{B} \rightarrow \mathcal{B}'$$
be quasi-functors between small DG categories. Let $T$ and $T'$ be DG modules over $B \otimes A$ and $B' \otimes A'$, respectively. Suppose that there is a map $\phi : T \to R(b \otimes a)_* T'$ in $D(B \otimes A)$. Then there are quasi-functors

$$a \phi b : \mathcal{A} \otimes_T B \to \mathcal{A}' \otimes_{T'} B'$$

$$a \oplus b : \mathcal{A} \oplus_T B \to \mathcal{A}' \oplus_{T'} B'.$$

Moreover, assume that $\phi$ is a quasi-isomorphism and the homotopy functors $a : \mathcal{H}^0(\mathcal{A}) \to \mathcal{H}^0(\mathcal{A}')$ and $b : \mathcal{H}^0(\mathcal{B}) \to \mathcal{H}^0(\mathcal{B}')$ are fully faithful. Then the induced functors

$$a \phi b : \mathcal{H}^0(\mathcal{A} \otimes_T B) \to \mathcal{H}^0(\mathcal{A}' \otimes_{T'} B')$$

$$a \phi b : \mathcal{H}^0(\mathcal{A} \oplus_T B) \to \mathcal{H}^0(\mathcal{A}' \oplus_{T'} B')$$

are also fully faithful. If $a$ and $b$ are quasi-equivalences, then both $a \phi b$ and $a \phi b$ are also quasi-equivalences.

It is easy to see that the restriction functor $b_* : Mod-(\mathcal{A} \otimes_T B) \to Mod-B$ sends semifree DG modules to semifree DG modules, as well as finitely generated semifree DG modules to finitely generated semifree DG modules. Thus, we obtain a DG functor $\mathcal{I} \mathcal{F}_{fg}(\mathcal{A} \otimes_T B) \to \mathcal{I} \mathcal{F}_{fg}-B$. By assumption $B$ is pretriangulated, and we know that a pretriangulated hull is quasi-equivalent to the DG category of finitely generated semifree DG modules. Therefore, we obtain a quasi-functor $b_* : \mathcal{A} \oplus_T B \to \mathcal{B}$ that is right adjoint to $b^*$.

3.2. Gluing of derived noncommutative schemes. The construction above lets us define gluing of derived noncommutative schemes. Let $X = \text{Perf}-R$ and $Y = \text{Perf}-\mathcal{I}$ be two derived noncommutative schemes, and let $T$ be a DG $Y \leftarrow X$-bimodule. By the construction above we can define the DG category

$$Z := X \oplus_T Y := \text{Perf}-R \oplus_T \text{Perf}-\mathcal{I},$$

which will be called the gluing of $X$ and $Y$ via (or with respect to) $T$. Since by Proposition 1.12 and Remark 1.13 there is a quasi-equivalence between semifree DG $Y \leftarrow X$-bimodules and semifree DG $\mathcal{I}-R$-bimodules, the DG category $Z$ is quasi-equivalent to $\text{Perf}-(R \otimes_T \mathcal{I})$. (We use the same letter $T$ for a DG $Y \leftarrow X$-bimodule and for its restriction as a DG $\mathcal{I}-R$-bimodule.) The DG category $Z$ is a derived noncommutative scheme according to our definition if and only if the DG algebra $R \otimes_T \mathcal{I}$ is cohomologically bounded, that is, the DG bimodule $T$ belongs to the bounded derived category $D^b(\mathcal{I} \leftarrow R)$ of $\mathcal{I}-R$-bimodules. When $T \cong 0$, the gluing will be denoted by $X \oplus_T Y$, and it is the biproduct of $X$ and $Y$ in the ‘world’ of derived noncommutative schemes. The natural inclusions

$$\text{Perf}-R \leftarrow \text{Perf}-(R \otimes_T \mathcal{I}) \quad \text{and} \quad \text{Perf}-\mathcal{I} \leftarrow \text{Perf}-(R \otimes_T \mathcal{I})$$

define morphisms $p_X : Z \to X$ and $p_Y : Z \to Y$ that will be called projections. Both $p_X$ and $p_Y$ are ff-morphisms of noncommutative schemes, and moreover, the
projection \( p_\mathcal{Y} \) is a pp-morphism, because the direct image functor \( R p_\mathcal{Y} \) sends perfect objects to perfect objects. This also implies that we have a right adjoint morphism \( r_\mathcal{Y} : \mathcal{Y} \to \mathcal{X} \), which is called a right section and for which there is an isomorphism \( L r^*_\mathcal{Y} \cong R p_\mathcal{Y} \). The composition \( p_\mathcal{Y} \cdot r_\mathcal{Y} \) is isomorphic to the identity \( \text{id}_\mathcal{Y} \). By symmetry, the natural inclusion \( \text{Perf}(-R) \hookrightarrow \text{Perf}(R \mathcal{X}) \) has a left adjoint functor. It defines a morphism \( l_\mathcal{X} : \mathcal{X} \to \mathcal{Y} \), which is called a left section, and the composition \( p_\mathcal{X} \cdot l_\mathcal{X} \) is isomorphic to the identity \( \text{id}_\mathcal{X} \), while the compositions \( p_\mathcal{X} \cdot r_\mathcal{Y} \) and \( p_\mathcal{X} \cdot l_\mathcal{X} \) are equal to 0-morphisms. Thus, the gluing \( \mathcal{Z} = \mathcal{X} \oplus \mathcal{Y} \) comes with the set of morphisms \( (l_\mathcal{X}, p_\mathcal{X}, p_\mathcal{Y}, r_\mathcal{Y}) \) which form a diagram of morphisms of derived noncommutative schemes

\[
\begin{array}{ccc}
\mathcal{X} & \xleftarrow{p_\mathcal{X}} & \mathcal{Z} \xrightarrow{r_\mathcal{Y}} \mathcal{Y} \\
\downarrow{l_\mathcal{X}} & & \downarrow{p_\mathcal{Y}} \\
\end{array}
\tag{6}
\]

with the following properties:

(a) \( l_\mathcal{X} \) is a left adjoint section for the projection \( p_\mathcal{X} \);
(b) \( r_\mathcal{Y} \) is a right adjoint section for the projection \( r_\mathcal{Y} \);
(c) \( p_\mathcal{X} \cdot l_\mathcal{X} \cong \text{id}_\mathcal{X} \) and \( p_\mathcal{Y} \cdot r_\mathcal{Y} \cong \text{id}_\mathcal{Y} \);
(d) \( p_\mathcal{X} \cdot r_\mathcal{Y} = 0 \), so that by adjointness \( p_\mathcal{Y} \cdot l_\mathcal{X} = 0 \), and moreover, the kernel of the functor \( L r^*_\mathcal{Y} \) is essentially the image of the functor \( L p^*_\mathcal{X} \).

In particular, we have a semi-orthogonal decomposition \( (\text{Perf}(-R), \text{Perf}(-\mathcal{X})) \) for the triangulated category of perfect objects of the noncommutative scheme \( \mathcal{X} \), and by Proposition 3.4 it characterizes the noncommutative scheme \( \mathcal{X} \) as a gluing of \( \mathcal{X} \) and \( \mathcal{Y} \).

Consider two morphisms of derived noncommutative schemes, \( f : \mathcal{X}' \to \mathcal{X} \) and \( g : \mathcal{Y}' \to \mathcal{Y} \). They induce a morphism \( (g \circ f) : \mathcal{Y}' \circ \mathcal{X}' \to \mathcal{Y} \circ \mathcal{X}' \). Let \( T \) be a DG \( \mathcal{Y}-\mathcal{X}' \)-bimodule and let \( T' \) be a DG \( \mathcal{Y}'-\mathcal{X}' \)-bimodule. Consider the respective gluings \( \mathcal{X} \oplus \mathcal{Y} \) and \( \mathcal{X}' \oplus \mathcal{Y}' \). By Proposition 3.5, with any map \( \phi : T \to R(g \circ f)_* T' \) in the derived category \( \mathcal{D}(\mathcal{X} \circ \mathcal{T}) \) we can associate a morphism \( (f \circ g) : \mathcal{X}' \circ \mathcal{Y}' \to \mathcal{X} \circ \mathcal{Y} \) between the gluings. When \( \mathcal{X}' \cong \mathcal{X} \) and \( \mathcal{Y}' \cong \mathcal{Y} \), then for any map \( \phi : T \to T' \) in \( \mathcal{D}(\mathcal{X} \circ \mathcal{T}) \) there exists a morphism \( \mathcal{X} \oplus \mathcal{Y} \to \mathcal{X} \oplus \mathcal{Y} \), which is \( (\text{id} \circ \text{id}) \). As a special case, we obtain morphisms

\[
\begin{align*}
\mathcal{X} \oplus \mathcal{Y} & \to \mathcal{X} \oplus \mathcal{Y} \\
\mathcal{X} \oplus \mathcal{Y} & \to \mathcal{X} \oplus \mathcal{Y},
\end{align*}
\]

the composition of which is the identity morphism of \( \mathcal{X} \oplus \mathcal{Y} \).

A particular case of gluing of two noncommutative schemes involves a morphism \( f : \mathcal{X} \to \mathcal{Y} \). Any such morphism is a quasi-functor \( F : \text{Perf}(-\mathcal{X}) \to \text{Perf}(-\mathcal{Y}) \) that is represented by a DG \( \mathcal{Y}-\mathcal{X} \)-bimodule \( T \). The gluing \( \mathcal{X} \oplus \mathcal{Y} \) with respect to this DG bimodule \( T \) will be also denoted by \( \mathcal{X} \oplus \mathcal{Y} \). Let \( f' : \mathcal{X} \to \mathcal{Y} \) be another morphism of the noncommutative schemes that is represented by a DG \( \mathcal{Y}-\mathcal{X} \)-bimodule \( T' \). A map \( \phi : f \to f' \) in the triangulated category \( \mathcal{Mor}(\mathcal{X}, \mathcal{Y}) \) is a map between DG bimodules \( \phi : T \to T' \) in the derived category \( \mathcal{D}(\mathcal{X} \circ \mathcal{T}) \). As above, with any such map \( \phi : f \to f' \) we can associate a morphism \( \mathcal{X} \oplus \mathcal{Y} \to \mathcal{X} \oplus \mathcal{Y} \) of the gluings.
Let us describe morphisms between a noncommutative scheme \( \mathcal{V} \) and a gluing \( \mathcal{X} \# \mathcal{Y} \) of two other noncommutative schemes (see, for example, [25], § 7.2, for details).

**Proposition 3.6.** Let \( f: \mathcal{X} \rightarrow \mathcal{Y} \) be a morphism of derived noncommutative schemes and let \( \mathcal{Z} = \mathcal{X} \# \mathcal{Y} \) be the gluing. For any noncommutative scheme \( \mathcal{V} \) a morphism \( v: \mathcal{Z} \rightarrow \mathcal{V} \) is a triple \((g, h; \phi)\), where \( g: \mathcal{X} \rightarrow \mathcal{V} \) and \( h: \mathcal{Y} \rightarrow \mathcal{V} \) are morphisms of noncommutative schemes, and \( \phi: h \cdot f \rightarrow g \) is a map between morphisms in the category \( \text{Mor}(\mathcal{X}, \mathcal{V}) \).

Similarly, a morphism \( v': \mathcal{V}' \rightarrow \mathcal{Z} \) is a triple \((g', h'; \psi)\), where \( g': \mathcal{V} \rightarrow \mathcal{X} \) and \( h': \mathcal{Y} \rightarrow \mathcal{Z} \) are morphisms of noncommutative schemes and \( \psi: f \cdot g' \rightarrow h' \) is a map between morphisms in the category \( \text{Mor}(\mathcal{V}', \mathcal{X}) \).

A composition \( v \cdot v': \mathcal{V}' \rightarrow \mathcal{V} \) of any two morphisms \( v': \mathcal{V}' \rightarrow \mathcal{Z} \) and \( v: \mathcal{Z} \rightarrow \mathcal{V} \) is a morphism that can be obtained as a cone of the map

\[
h \cdot f \cdot g' \xrightarrow{\tilde{\phi} + \tilde{\psi}} g \cdot g' \oplus h \cdot h'
\]

in the triangulated category \( \text{Mor}(\mathcal{V}', \mathcal{V}) \), where

\[
\tilde{\phi}: h \cdot f \cdot g' \longrightarrow g \cdot g' \quad \text{and} \quad \tilde{\psi}: h \cdot f \cdot g' \longrightarrow h \cdot h'
\]

are maps induced by \( \phi: h \cdot f \rightarrow g \) and \( \psi: f \cdot g' \rightarrow h' \), respectively.

Under the description above, the projections \( p\mathcal{X} \) and \( p\mathcal{Y} \) are equal to \((id\mathcal{X}, 0; 0)\) and \((f, id\mathcal{Y}; id)\), respectively, while the sections \( l\mathcal{X} \) and \( r\mathcal{Y} \) coincide with the morphisms \((id\mathcal{X}, 0; 0)\) and \((0, id\mathcal{Y}; 0)\), respectively. Furthermore, in this situation there is a right adjoint morphism \( r\mathcal{X} : \mathcal{X} \rightarrow \mathcal{Z} \) for which \( Lr^*\mathcal{X} \cong Rp\mathcal{X}^* \). It is given by the triple \((id\mathcal{X}, f; id)\). There is also another projection \( \overline{p}\mathcal{Y} \) which is determined by the triple \((0, id\mathcal{Y}; 0)\). For this morphism we have isomorphisms of functors \( Lp^*\mathcal{Y} \cong Rr\mathcal{Y}^* \cong p\mathcal{Y}^* \). Finally, we obtain the ‘recollement’ for derived noncommutative schemes (see [3])

\[
\begin{array}{ccc}
\mathcal{X} & \xleftarrow{p\mathcal{X}} & \mathcal{Z} \\
\xrightarrow{l\mathcal{X}} & & \xrightarrow{r\mathcal{Y}} \\
\mathcal{Y} & \xrightarrow{p\mathcal{Y}} & \\
\end{array}
\]

with the following properties:

(a) \( l\mathcal{X} \) and \( r\mathcal{X} \) are left and right adjoint sections for the projection \( p\mathcal{X} \);
(b) \( p\mathcal{Y} \) and \( \overline{p}\mathcal{Y} \) are left and right adjoint projections for the section \( r\mathcal{Y} \);
(c) \( p\mathcal{X} \cdot l\mathcal{X} \cong id\mathcal{X} \cong p\mathcal{X} \cdot r\mathcal{X} \) and \( p\mathcal{Y} \cdot r\mathcal{Y} \cong id\mathcal{Y} \cong \overline{p}\mathcal{Y} \cdot r\mathcal{Y} \);
(d) \( p\mathcal{X} \cdot r\mathcal{X} = 0 \), so that by adjointness \( p\mathcal{Y} \cdot 1\mathcal{X} = 0 = \overline{p}\mathcal{Y} \cdot r\mathcal{X} \), and moreover, the kernel of the functor \( Lr^*\mathcal{Y} \) is essentially the image of the functor \( Lp^*\mathcal{X} \).

In addition, we have an isomorphism \( p\mathcal{Y} \cdot r\mathcal{X} \cong f \), while \( \overline{p}\mathcal{Y} \cdot l\mathcal{X} \cong f[1] \).

Consider the direct image functor \( Rf_*: D(\mathcal{R}) \rightarrow D(\mathcal{I}) \). It is isomorphic to \( R\text{Hom}_\mathcal{R}(T, -) \), where \( T \) is the corresponding quasi-functor. Since \( T \) is perfect as a DG \( \mathcal{R} \)-module, this functor commutes with direct sums. Therefore, there is a DG \( \mathcal{R}-\mathcal{I} \)-bimodule \( U \) such that the functor \( Rf_* \) is represented as \((-) \otimes_{\mathcal{R}} U \). In fact, the DG \( \mathcal{R}-\mathcal{I} \)-bimodule \( U \) is isomorphic to \( \text{Hom}_{\mathcal{R}}(T, \mathcal{R}) \). We can consider another
gluing $\mathcal{X}' = \mathcal{Y} \cup \mathcal{X}'$. However, this gluing is quasi-equivalent to $\mathcal{X} = \mathcal{X}' \cup \mathcal{Y}$ and determines the same noncommutative scheme, which we also denote by $\mathcal{Y} \cup \mathcal{X}'$. In fact, to obtain the decomposition $\mathcal{Y} \cup \mathcal{X}'$ for our noncommutative scheme $\mathcal{Z}$, we have to consider the diagram (6) with the set of morphisms $(r_{\mathcal{Y}}, l_{\mathcal{Y}}; p_{\mathcal{Y}}, r_{\mathcal{X}})$ instead of the set $(l_{\mathcal{X}}, p_{\mathcal{X}}; p_{\mathcal{Y}}, r_{\mathcal{Y}})$ for $\mathcal{Z} = \mathcal{X} \cup \mathcal{Y}$. Thus, the noncommutative schemes $\mathcal{X} \cup \mathcal{Y}$ and $\mathcal{Y} \cup \mathcal{X}$ are isomorphic despite the constructions and the decompositions being different.

Note that if the morphism $f: \mathcal{X} \rightarrow \mathcal{Y}$ is a pp-morphism and $g: \mathcal{Y} \rightarrow \mathcal{X}$ is the right adjoint morphism, then by construction $\mathcal{Y} \cup \mathcal{X} \cong \mathcal{Y} \cup \mathcal{X} \cong \mathcal{Y} \cup \mathcal{X}$.

The next example comes from the usual commutative geometry.

Example 3.7. Let $i: Z \hookrightarrow Y$ be a closed immersion of a smooth proper scheme $Z$ into the smooth proper scheme $Y$, with $Z$ of codimension $2$. Denote by $\tilde{Y}$ the blowup of $Y$ along the closed subscheme $Z$. A blow up formula from [35] gives a semi-orthogonal decomposition of the category $\text{Perf} - \tilde{Y}$ in the form $\langle \text{Perf} - Y, \text{Perf} - Z \rangle$. Moreover, by Proposition 3.4 the DG category $\text{Perf} - \tilde{Y}$ is quasi-equivalent to the gluing $(\text{Perf} - Y) \cup (\text{Perf} - Z)$, where $U$ is a DG bimodule of the form

$$
U(P, Q) \cong \text{Hom}_{\text{Perf} - X}(i^* Q, P) \quad \text{with} \quad P \in \text{Perf} - Z \quad \text{and} \quad Q \in \text{Perf} - Y,
$$

and $i^*$ is the inverse image quasi-functor from $\text{Perf} - Y$ to $\text{Perf} - Z$. The gluing $(\text{Perf} - Y) \cup (\text{Perf} - Z)$ is actually the gluing $(\text{Perf} - Y) \cup (\text{Perf} - Z)$ along the morphism $i: Z \hookrightarrow Y$.

3.3. Properties of gluings. Now we discuss some properties of derived noncommutative schemes that are obtained by gluing. First of all, it is easy to see when a gluing $\mathcal{X} \cup \mathcal{Y}$ is proper and regular (see [40], Propositions 3.20 and 3.22).

Proposition 3.8. Let $\mathcal{X} = \text{Perf} - \mathcal{B}$ and $\mathcal{Y} = \text{Perf} - \mathcal{I}$ be two derived noncommutative schemes over $k$, and let $T$ be a DG $\mathcal{Y} \cdot \mathcal{X}$-bimodule. Then the following conditions are equivalent:

1) the gluing $\mathcal{X} \cup \mathcal{Y}$ is proper;

2) $\mathcal{X}$ and $\mathcal{Y}$ are proper, and

$$
\dim_k \bigoplus_i H^i(T(Q, P)) < \infty \quad \text{for all} \quad P \in \text{Perf} - \mathcal{B} \quad \text{and} \quad Q \in \text{Perf} - \mathcal{I}.
$$

Note that it is sufficient to check the property of $T$ mentioned above in the case where $P = \mathcal{B}$ and $Q = \mathcal{I}$, that is, $\dim_k \bigoplus_i H^i(T) < \infty$ for $T$ as a DG $\mathcal{I} \cdot \mathcal{B}$-bimodule.
Proposition 3.9. Let $\mathcal{X} = \text{Perf} - \mathcal{R}$ and $\mathcal{Y} = \text{Perf} - \mathcal{I}$ be two derived noncommutative schemes over $k$ and let $T$ be a DG $\mathcal{Y} - \mathcal{X}$-bimodule. Then the following conditions are equivalent:

1) the gluing $\mathcal{X} \oplus_{T} \mathcal{Y}$ is regular;
2) $\mathcal{X}$ and $\mathcal{Y}$ are regular.

Here we see that the property of being regular does not depend on the DG bimodule $T$.

On the other hand, the property of being smooth does depend on $T$. Since $\mathcal{R} \oplus_{T} \mathcal{I}$ and $\mathcal{X} \oplus_{T} \mathcal{Y}$ are Morita equivalent, $\mathcal{R} \oplus_{T} \mathcal{I}$ is smooth if and only if $\mathcal{X} \oplus_{T} \mathcal{Y}$ is smooth. Further, we can compare the smoothness of a gluing with the smoothness of the summands. We obtain the following.

Proposition 3.10 ([28], 3.24). Let $\mathcal{X} = \text{Perf} - \mathcal{R}$ and $\mathcal{Y} = \text{Perf} - \mathcal{S}$ be two derived noncommutative schemes over $k$ and let $T$ be a DG $\mathcal{Y} - \mathcal{X}$-bimodule. Then the following conditions are equivalent:

1) the gluing $\mathcal{X} \oplus_{T} \mathcal{Y}$ is smooth;
2) $\mathcal{X}$ and $\mathcal{Y}$ are smooth and $T$ is a perfect DG $\mathcal{Y} - \mathcal{X}$-bimodule.

Consider the point $pt = \text{Perf} - k$ and a noncommutative scheme $\mathcal{X}$ that is the gluing $pt \oplus_{V} pt$, where $V$ is a $k$-vector space. If the vector space $V$ is infinite-dimensional, then the noncommutative scheme $\mathcal{X}$ is not smooth despite being regular. Of course, in this case $\mathcal{X}$ is also not proper.

3.4. Geometric realizations of gluings. Let $X$ and $Y$ be two usual smooth irreducible projective schemes over a field $k$. Let $E \in \text{Perf} - (X \times_{k} Y)$ be a perfect complex on the product $X \times_{k} Y$. Note that in the case of projective schemes any perfect complex is globally (not only locally) quasi-isomorphic to a strictly perfect complex, that is, a bounded complex of locally free sheaves of finite type (see [49], 2.3.1, for example).

Consider the DG category obtained as the gluing $(\text{Perf} - X) \oplus_{E} (\text{Perf} - Y)$. It is a derived noncommutative scheme which will be denoted by $\mathcal{Z} := X \oplus_{E} Y$. Taking Theorem 2.15 and Propositions 3.8 and 3.10 into account, we can deduce that the noncommutative scheme $\mathcal{Z}$ is smooth and proper. The derived noncommutative scheme $\mathcal{Z}$ is not commutative in general. However, it is natural to ask about the existence of a geometric realization for such derived noncommutative schemes. The following theorem is proved in [40].

Theorem 3.11 ([40], Theorem 4.11). Let $X$ and $Y$ be smooth irreducible projective schemes over a field $k$, and let $E$ be a perfect complex on the product $X \times_{k} Y$. Let $\mathcal{Z} = X \oplus_{E} Y$ be the derived noncommutative scheme that is the gluing of $\mathcal{X}$ and $\mathcal{Y}$ via $E$. Then there exist a smooth projective scheme $V$ and an ff-morphism $f: V \rightarrow \mathcal{Z}$ giving a pure geometric realization for the noncommutative scheme $\mathcal{Z}$.

A proof of this theorem can be found in [40]. It is constructive, and it is useful to take into account that the category $\text{Perf} - V$ in Theorem 3.11 has a semi-orthogonal
decomposition of the form
\[ \text{Perf} - V = \langle N_1, \ldots, N_k \rangle \]
such that each \( N_i \) is equivalent to one of four categories: namely, \( \text{Perf} - k \), \( \text{Perf} - X \), \( \text{Perf} - Y \), and \( \text{Perf} - (X \times_k Y) \).

Now we can extend this result to the case of derived noncommutative schemes. Let \( X_i, i = 1, \ldots, n \), be smooth and projective schemes, and let \( N_i \subset \text{Perf} - X_i \), \( i = 1, \ldots, n \), be full pretriangulated DG subcategories. Assume that the homotopy triangulated categories \( N_i \approx \mathcal{H}^0(N_i) \) are admissible in \( \text{Perf} - X_i \). By Propositions 3.8 and 3.10 these conditions imply that the derived noncommutative schemes \( N_i \) are proper and smooth. Moreover, they come with pure geometric realizations.

**Theorem 3.12** ([40], Theorem 4.15). Let the DG categories \( N_i, i = 1, \ldots, n \), and the smooth projective schemes \( X_i, i = 1, \ldots, n \), be as above. Let \( \mathcal{X} = \text{Perf} - \mathcal{B} \) be a proper derived noncommutative scheme with full embeddings of the DG categories \( N_i \subset \text{Perf} - \mathcal{B} \) such that \( \text{Perf} - \mathcal{B} \) has a semi-orthogonal decomposition of the form \( \langle N_1, N_2, \ldots, N_n \rangle \), where \( N_i = \mathcal{H}^0(N_i) \). Then there exist a smooth projective scheme \( X \) and an ff-morphism \( f: X \to \mathcal{X} \) which give a pure geometric realization for the noncommutative scheme \( \mathcal{X} \).

Note that in this case the derived noncommutative scheme \( \mathcal{X} \) is also smooth. Indeed, it is a gluing of smooth proper noncommutative schemes \( N_i \) with respect to DG bimodules that are DG functors from \( N_i \otimes N_i^\circ \) to \( \text{Perf} - k \). By Proposition 2.12 all such DG bimodules are perfect, because the \( N_i \) are smooth and proper. Theorem 3.12 implies that the world of all smooth proper geometric noncommutative schemes is closed under gluing via perfect bimodules.

These theorems have useful applications. Using results in [25], we get that for any usual proper scheme \( Y \) over a field of characteristic 0 there is a full embedding of \( \text{Perf} - Y \) into \( \text{Perf} - V \), where \( V \) is smooth and projective.

**Corollary 3.13** ([40], Corollary 4.16). Let \( Y \) be a proper scheme over a field of characteristic 0. Then there exist a smooth projective scheme \( X \) and a quasi-functor \( F: \text{Perf} - Y \to \text{Perf} - X \) such that the induced functor \( F: \text{Perf} - Y \to \text{Perf} - X \) is fully faithful, that is, \( Y \) has a plain geometric realization that is a smooth desingularization \( f: X \to Y \).

When a proper derived noncommutative scheme \( \mathcal{X} = \text{Perf} - \mathcal{B} \) has a full exceptional collection, there is another and more useful procedure for constructing a smooth projective geometric realization. Any such derived noncommutative scheme \( \mathcal{X} \) is smooth and could be obtained by a procedure of sequential gluing of copies of the point \( \text{pt} \). In this case one can find a usual smooth projective scheme \( X \) and an exceptional collection \( \sigma = (L_1, \ldots, L_n) \) of line bundles on \( X \) such that the DG subcategory \( \mathcal{N} \subset \text{Perf} - X \) generated by \( \sigma \) is quasi-equivalent to \( \text{Perf} - \mathcal{B} \). Moreover, by construction the scheme \( X \) is rational and has a full exceptional collection.

**Theorem 3.14** ([40], Theorem 5.8). Let \( \mathcal{X} = \text{Perf} - \mathcal{B} \) be a proper derived noncommutative scheme over \( k \) such that the homotopy category \( \text{Perf} - \mathcal{B} \) has a full exceptional collection \( \text{Perf} - \mathcal{B} = \langle E_1, \ldots, E_n \rangle \). Then there exist a smooth projective
scheme $X$ and an exceptional collection $\sigma = (L_1, \ldots, L_n)$ of line bundles on $X$ such that the DG subcategory of $\text{Perf} - X$ generated by $\sigma$ is quasi-equivalent to $\text{Perf} - \mathcal{R}$. Moreover, $X$ can be chosen in such way that it is a tower of projective bundles and thus has a full exceptional collection.

The scheme $X$ has a full exceptional collection as a tower of projective bundles (see [35]). Furthermore, it follows from the construction that a full exceptional collection on $X$ can be chosen in such a way that it contains the collection $\sigma = (L_1, \ldots, L_n)$ as a subcollection.

In the proof of this theorem a quasi-functor from the DG category $\text{Perf} - \mathcal{R}$ to the DG category $\text{Perf} - X$ was constructed that sends the exceptional objects $E_i$ to shifts of the line bundles $L_i[r_i]$ for some integers $r_i$. In other words, we have an ff-morphism $\mathcal{f} : X \to \mathcal{R}$ that gives a pure geometric realization for the noncommutative scheme $\mathcal{R}$, and $L\mathcal{f}^* E_i \cong L_i[r_i]$. Of course, we cannot expect in general that the $E_i$ will simply go to line bundles without shifts. On the other hand, in the case of strong exceptional collections it is natural to seek geometric realizations as collections of vector bundles (without shifts) on smooth projective varieties. It can be shown that in general we cannot realize a strong exceptional collection as a collection of unshifted line bundles, but trying to present it in terms of vector bundles seems quite reasonable.

**Theorem 3.15** ([41], Corollary 2.7). Let $\mathcal{R} = \text{Perf} - \mathcal{R}$ be a proper derived noncommutative scheme such that the category $\text{Perf} - \mathcal{R}$ has a full strong exceptional collection $\text{Perf} - \mathcal{R} = \langle E_1, \ldots, E_n \rangle$. Then there exist a smooth projective scheme $X$ and an ff-morphism $\mathcal{f} : X \to \mathcal{R}$ such that the functor $L\mathcal{f}^*$ sends the exceptional objects $E_i$ to vector bundles $E_i$ on $X$.

There is a special class of derived noncommutative schemes connected with finite-dimensional algebras. Let $\Lambda$ be a finite-dimensional algebra over a base field $k$. Consider the derived noncommutative scheme $\mathcal{V} = \text{Perf} - \Lambda$. This noncommutative scheme is proper for any such $\Lambda$. It is regular if and only if the algebra $\Lambda$ has finite global dimension. Denote by $\mathfrak{r}$ the (Jacobson) radical of $\Lambda$. We know that $\mathfrak{r}^n = 0$ for some $n$. Let $S$ be the quotient algebra $\Lambda/\mathfrak{r}$. It is semisimple and has only a finite number of simple non-isomorphic modules.

Recall that a semisimple algebra $S$ over a field $k$ is said to be separable over $k$ if it is projective as an $S$-$S$-bimodule. It is well known that a semisimple algebra $S$ is separable if it is a direct sum of simple algebras whose centers are separable extensions of the field $k$. This also means that the noncommutative scheme $\text{Perf} - S$ is smooth. Moreover, the noncommutative scheme $\mathcal{V} = \text{Perf} - \Lambda$ is smooth over $k$ if $\Lambda$ has finite global dimension and $S = \Lambda/\mathfrak{r}$ is separable (see [43], for example).

**Theorem 3.16** ([40], Theorem 5.3). Let $\Lambda$ be a finite-dimensional algebra over $k$. Assume that the semisimple algebra $S = \Lambda/\mathfrak{r}$ is $k$-separable. Then there exist a smooth projective scheme $X$ and a perfect complex $E \in \text{Perf} - X$ such that $\text{End}(E) \cong \Lambda$ and $\text{Hom}(E, E[l]) = 0$ for all $l \neq 0$.

**Corollary 3.17** ([40], Theorem 5.4). Let $\mathcal{V} = \text{Perf} - \Lambda$ be a derived noncommutative scheme with $\Lambda$ a finite-dimensional algebra over $k$ such that $\Lambda/\mathfrak{r}$ is $k$-separable. Then there exist a smooth projective scheme $X$ and an ff-morphism $\mathcal{f} : X \to \mathcal{V}$ which
give a plain geometric realization for \( \mathcal{V} \). Moreover, if \( \Lambda \) has finite global dimension, then this realization is pure.

Note that over a perfect field all semisimple algebras are separable. Thus, if \( \mathbf{k} \) is perfect, then these results can be applied to any finite-dimensional \( k \)-algebra.

Consider a smooth and proper noncommutative scheme \( \mathcal{X} = \text{Perf} - \mathcal{A} \) such that the category \( \text{Perf} - \mathcal{A} \) has a full strong exceptional collection \( \sigma = \langle E_1, \ldots, E_n \rangle \). The object \( E = \bigoplus_{i=1}^n E_i \) is a generator of \( \text{Perf} - \mathcal{A} \), and the DG category \( \text{Perf} - \mathcal{B} \) is quasi-equivalent to the DG category \( \text{Perf} - \Lambda \), where \( \Lambda = \text{End}(\bigoplus_{i=1}^n E_i) \) is the algebra of endomorphisms of the collection \( \sigma \). It is evident that the algebra \( \Lambda \) is a quiver algebra on \( n \) directed vertices. Recall that \( \Lambda \) is called a quiver algebra on \( n \) directed vertices, if \( \Lambda \cong \mathbf{k}Q/I \), where \( Q \) is a quiver for which \( Q_0 = \{1, \ldots, n\} \) is the ordered set of \( n \) elements, and for any arrow \( a \in Q_1 \) the source \( s(a) \in Q_0 \) is less than the target \( t(a) \in Q_0 \), while \( I \) is a two-sided ideal of the path algebra \( \mathbf{k}Q \) generated by a subspace of \( \mathbf{k}Q \) spanned by linear combinations of paths of length at least 2 having a common source and a common target (see [41], for example).

On the other hand, any quiver algebra \( \Lambda \) on \( n \) directed vertices has finite global dimension, and moreover, the category \( \text{Perf} - \Lambda \) has a strong full exceptional collection consisting of the indecomposable projective modules \( P_i \) for \( i = 1, \ldots, n \). The algebra \( \Lambda \) is exactly the algebra of endomorphisms of this full strong exceptional collection. Thus, Theorem 3.15 implies that for any quiver algebra \( \Lambda \) on \( n \) directed vertices there exist a smooth projective scheme \( X \) and a vector bundle \( \mathcal{E} \) on \( X \) such that \( \text{End}_X(\mathcal{E}) = \Lambda \) and \( \text{Ext}^p_X(\mathcal{E}, \mathcal{E}) = 0 \) for all \( p \neq 0 \). Moreover, they can be chosen so that the rank of \( \mathcal{E} \) is equal to the dimension of \( \Lambda \) (see [41], Corollary 2.8).

### 3.5. Quasi-phantoms and phantoms.

Let \( \mathcal{T} \) be a triangulated category and let \( \mathcal{T} = \langle \mathcal{N}_1, \ldots, \mathcal{N}_n \rangle \) be a semi-orthogonal decomposition. Any such decomposition induces the following decomposition for the Grothendieck group:

\[
K_0(\mathcal{T}) \cong K_0(\mathcal{N}_1) \oplus K_0(\mathcal{N}_2) \oplus \cdots \oplus K_0(\mathcal{N}_n).
\]

In the particular case of a full exceptional collection we obtain an isomorphism of the Grothendieck group \( K_0(\mathcal{T}) \) with a free Abelian group \( \mathbb{Z}^n \).

For any small DG category \( \mathcal{A} \) we can define the K-theory spectrum \( K(\mathcal{A}) \) by applying Waldhausen’s construction to a certain category with cofibrations and weak equivalences that can be obtained from the DG category \( \text{Perf} - \mathcal{A} \) (see [17], [44], [22], [48]). More precisely, the objects of this category are perfect DG modules, the cofibrations are the morphisms of degree zero that admit retractions as morphisms of graded modules, and the weak equivalences are the quasi-isomorphisms. This construction is invariant under quasi-equivalences between \( \text{Perf} - \mathcal{A} \) and \( \text{Perf} - \mathcal{B} \). Thus, with any derived noncommutative scheme \( \mathcal{X} = \text{Perf} - \mathcal{B} \) we can associate a K-theory spectrum \( K(\mathcal{X}) := K(\mathcal{B}) = K(\text{Perf} - \mathcal{B}) \). K-theory gives us an additive invariant for derived noncommutative schemes in the sense that for any gluing \( \mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2 \) there is an isomorphism \( K(\mathcal{X}) \cong K(\mathcal{X}_1) \oplus K(\mathcal{X}_2) \).
Other natural additive invariants are given by Hochschild and cyclic homology. Hochschild homology $HH_*(X)$ can be defined by

$$HH_*(X) \cong H^{-*}(R \overset{L}{\otimes}_{R} R),$$

and for any gluing $\mathcal{X} = \mathcal{X} \oplus \mathcal{Y}$ there is an isomorphism

$$HH_*(\mathcal{X}) \cong HH_*(\mathcal{X}) \oplus HH_*(\mathcal{Y}).$$

Recall that for a smooth projective scheme $X$ over a field of characteristic $0$ there is an isomorphism $HH_i(X) = \bigoplus_p H^{p+i}(X, \Omega^p_X)$ (see [46]), which allows one to describe the Hochschild homology in terms of the usual cohomology of the scheme $X$.

**Definition 3.18.** A smooth and proper derived noncommutative scheme $\mathcal{X}$ will be called a **quasi-phantom** if $HH_*(\mathcal{X}) = 0$ and $K_0(\mathcal{X})$ is a finite Abelian group. It will be called a **phantom** if, in addition, $K_0(\mathcal{X}) = 0$.

This definition is a result of the successive appearance and study of such objects, but more natural and more important is the following definition of a universal phantom.

**Definition 3.19.** We say that a phantom $\mathcal{X}$ is a **universal phantom** if $\mathcal{X} \otimes_k \mathcal{Y}$ is also a phantom for any smooth and proper noncommutative scheme $\mathcal{Y}$.

It can be shown that it is sufficient to verify this property for $\mathcal{Y} = \mathcal{X}$. Moreover, it is also known that any universal phantom $\mathcal{X}$ has a trivial $K$-motive, and hence its $K$-theory $K(\mathcal{X})$ vanishes as well (see [19]).

Different examples of geometric quasi-phantoms were constructed as semi-orthogonal complements to exceptional collections of maximal length on some smooth projective surfaces of general type with $q = p_g = 0$ for which Bloch’s conjecture holds, that is, the Chow group $\text{CH}^2(S)$ is isomorphic to $\mathbb{Z}$. In greater detail, let $S$ be such a surface. In this case the Grothendieck group $K_0(S)$ is isomorphic to $\mathbb{Z} \oplus \text{Pic}(S) \oplus \mathbb{Z} \cong \mathbb{Z}^{r+2} \oplus \text{Pic}(S)_{\text{tors}}$, where $r$ is the rank of the Picard lattice $\text{Pic}(S)/\text{Pic}(S)_{\text{tors}}$. Since the Picard group $\text{Pic}(S)$ of this surface is isomorphic to $H^2(S(\mathbb{C}), \mathbb{Z})$, we get that $\text{Pic}(S)_{\text{tors}}$ is finite and the equalities $r+2 = b_2+2 = e$ hold, where $b_2$ is the second Betti number and $e$ is the topological Euler characteristic of $S$. Assume that the triangulated category $\mathcal{P}erf-S$ has an exceptional collection $(E_1, \ldots, E_e)$ of the maximal possible length $e$. In this case there is a semi-orthogonal decomposition of the form

$$\mathcal{P}erf-S = \langle E_1, \ldots, E_e, \mathcal{N} \rangle,$$

where $\mathcal{N}$ is the left orthogonal to the subcategory $\mathcal{T} = \langle E_1, \ldots, E_e \rangle$ generated by the exceptional collection. We have $K_0(\mathcal{T}) \cong \mathbb{Z}^e$, and hence $K_0(\mathcal{N}) \cong \text{Pic}(S)_{\text{tors}}$.

Consider now the DG category $\mathcal{P}erf-S$ and its DG subcategory $\mathcal{N} \subset \mathcal{P}erf-S$ that has the same objects as $\mathcal{N} \subset \mathcal{P}erf-S$. The DG category $\mathcal{N}$ is a derived noncommutative scheme, which is smooth and proper because the subcategory $\mathcal{N}$ is admissible. We have already mentioned that $K_0(\mathcal{N}) \cong \text{Pic}(S)_{\text{tors}}$. Moreover, it
is evident that the Hochschild homology $\text{HH}_*(\mathcal{N})$ is trivial. Thus, the noncommutative scheme $\mathcal{N}$ is a quasi-phantom coming with a pure geometric realization $\mathcal{N} \subset \text{Perf}^{-S}$.

By now a lot of different examples of quasi-phantoms have been constructed as we described above. The first example was constructed in [6] for the classical Godeaux surface $S$ that is the $\mathbb{Z}/5\mathbb{Z}$-quotient of the Fermat quintic in $\mathbb{P}^3$. In this case $e = 11$, and the Grothendieck group $K_0(\mathcal{N})$ is isomorphic to the cyclic group $\mathbb{Z}/5\mathbb{Z}$.

The next examples were Burniat surfaces with $e = 6$, for which exceptional collections of maximal length were constructed in [1]. In this case we have a four-dimensional family of such surfaces, and we obtain a four-dimensional family of quasi-phantoms $\mathcal{N}$ with $K_0(\mathcal{N}) = (\mathbb{Z}/2\mathbb{Z})^6$. (It was proved in [24] that the second Hochschild cohomology of the quasi-phantoms $\mathcal{N}$ coincides with the second Hochschild cohomology of the corresponding Burniat surfaces $S$.)

Suppose that we have two different quasi-phantoms $\mathcal{N}$ and $\mathcal{N}'$. It is natural to consider their tensor product $\mathcal{N} \otimes_k \mathcal{N}'$. If the orders of the Grothendieck groups $K_0(\mathcal{N})$ and $K_0(\mathcal{N}')$ are coprime, then we can hope that the Grothendieck group of $\mathcal{N} \otimes_k \mathcal{N}'$ will be trivial. In the case of surfaces this can be proved.

**Theorem 3.20** [19]. Let $S$ and $S'$ be smooth projective surfaces over $\mathbb{C}$ with $q = p_g = 0$ for which Bloch's conjecture for 0-cycles holds. Assume that the categories $\text{Perf}^{-S}$ and $\text{Perf}^{-S'}$ have exceptional collections of maximal lengths $e(S)$ and $e(S')$, respectively. Let $\mathcal{N} \subset \text{Perf}^{-S}$ and $\mathcal{N}' \subset \text{Perf}^{-S'}$ be the left orthogonals to these exceptional collections. If the orders of $\text{Pic}(S)_{\text{tors}}$ and $\text{Pic}(S')_{\text{tors}}$ are coprime, then the noncommutative scheme $\mathcal{N} \otimes_k \mathcal{N}' \subset \text{Perf}^-(S \times_k S')$ is a universal phantom.

This theorem also tells us that the noncommutative scheme $\mathcal{N} \otimes_k \mathcal{N}'$ has a trivial K-motive, that is, it is in the kernel of the natural map from the world of smooth and proper derived noncommutative schemes to the world of K-motives (they are called noncommutative motives now), and in particular, it has trivial K-theory; that is, $K_i(\mathcal{N} \otimes_k \mathcal{N}') = 0$ for all $i$ (see [19]). It is known that K-theory is a universal additive invariant (see [47] and [48]), and hence all additive invariants of universal phantoms vanish.

**Corollary 3.21** [19]. Let $S$ be a Burniat surface with $e = 6$ and let $S'$ be the classical Godeaux surface over $\mathbb{C}$. Let $\mathcal{N} \subset \text{Perf}^{-S}$ and $\mathcal{N}' \subset \text{Perf}^{-S'}$ be quasi-phantoms that are the left orthogonals to exceptional collections of maximal lengths. Then the derived noncommutative scheme $\mathcal{N} \otimes_k \mathcal{N}' \subset \text{Perf}^-(S \times_k S')$ is a universal phantom and $K_i(\mathcal{N} \otimes_k \mathcal{N}') = 0$ for all $i \in \mathbb{Z}$.

Another type of a geometric phantom was constructed in [5] as a semi-orthogonal complement to an exceptional collection of maximal length on the determinantal Barlow surface. Since the Barlow surface is simply connected, it does not have torsion in the Picard group. In this case any quasi-phantom constructed as described above is actually a phantom, because the Grothendieck group is trivial. The results in [19] applied to a phantom coming from the Barlow surface give us that this phantom is universal.
3.6. Krull–Schmidt partners. Let \( \mathcal{X} = \text{Perf-} \mathcal{R} \) and \( \mathcal{Y} = \text{Perf-} \mathcal{I} \) be two derived noncommutative schemes, and let \( f: \mathcal{X} \to \mathcal{Y} \) be a morphism that is represented by a DG \( \mathcal{Y}-\mathcal{X} \)-bimodule \( T \) as a quasi-functor
\[
F: \text{Perf-} \mathcal{I} \longrightarrow \text{Perf-} \mathcal{R}.
\]
Consider the gluing \( \mathcal{X} = \mathcal{X} \sqcup \mathcal{Y} \), which is by definition the gluing
\[
\mathcal{X} \sqcup \mathcal{Y} = \text{Perf-} \binom{\mathcal{R}}{\mathcal{T}} \mathcal{I}
\]
of \( \mathcal{X} \) and \( \mathcal{Y} \) via \( T \).

Any morphism \( c: \mathcal{X} \to \mathcal{Y} \) induces a morphism \( \zeta: \mathcal{X} \to \mathcal{Y} \) that is the composition of \( c \) and the natural projection \( p_\mathcal{X}: \mathcal{X} \to \mathcal{X} \). By Proposition 3.6, \( \zeta \) is given by the triple \((c, 0; 0)\). If \( c \) is represented by a DG \( \mathcal{Y}-\mathcal{X} \)-bimodule \( P \), then \( \zeta \) is connected with the DG \( \mathcal{Y}-\mathcal{X} \)-bimodule \( P = (P, 0) \) which coincides with \( P \) on the subcategory \( \text{Perf-} \mathcal{R} \subset \text{Perf-} \binom{\mathcal{R}}{\mathcal{T}} \mathcal{I} \) and is equal to 0 on the subcategory \( \text{Perf-} \mathcal{I} \subset \text{Perf-} \binom{\mathcal{R}}{\mathcal{T}} \mathcal{I} \).

Let \( g: \mathcal{X} \to \mathcal{Y} \) be another morphism that is represented by a DG \( \mathcal{Y}-\mathcal{X} \)-bimodule \( U \). Suppose that there is a map \( \phi: f \to g \) between the morphisms. By Proposition 3.6, the map \( \phi: f \to g \) induces a morphism \( \tilde{g}_\phi: \mathcal{X} \to \mathcal{Y} \) that is given by the triple \((g, \text{id}\mathcal{Y}; \phi)\). The morphism \( \tilde{g}_\phi \) is represented by a DG \( \mathcal{Y}-\mathcal{X} \)-bimodule \( \tilde{U}_\phi := (U, \mathcal{I}) \) coinciding with \( U \) on the subcategory \( \text{Perf-} \mathcal{R} \subset \text{Perf-} \binom{\mathcal{R}}{\mathcal{T}} \mathcal{I} \), and \( \tilde{U}_\phi(-, \cdot) = \text{Hom}_\mathcal{Y}(\cdot, -) \) on the subcategory \( \text{Perf-} \mathcal{I} \subset \text{Perf-} \binom{\mathcal{R}}{\mathcal{T}} \mathcal{I} \). The map \( \phi: T \to U \) enables us to define a natural \( \mathcal{X} \)-module structure on \( \tilde{U}_\phi = (U, \mathcal{I}) \).

There is an isomorphism \( \tilde{f}_{\text{id}} \cong p_\mathcal{Y} \). Moreover, it is easy to see that, by Proposition 3.6, the composition of \( \tilde{g}_\phi: \mathcal{X} \to \mathcal{Y} \) with the right section \( r_\mathcal{X}: \mathcal{X} \to \mathcal{X} \) is exactly the morphism \( g: \mathcal{X} \to \mathcal{Y} \).

We denote by \( c \) the cone of the map \( \phi: f \to g \). The map \( \phi \) induces a map \( \tilde{\phi}: p_\mathcal{Y} \to \tilde{g}_\phi \) whose cone is isomorphic to the morphism \( \zeta = cp_\mathcal{X} \). Thus, we have two exact triangles of morphisms
\[
(\text{Tr1}) \quad c[-1] \longrightarrow f \overset{\phi}{\longrightarrow} g \longrightarrow c \quad \text{and} \quad (\text{Tr2}) \quad \zeta[-1] \longrightarrow p_\mathcal{Y} \overset{\tilde{\phi}}{\longrightarrow} \tilde{g}_\phi \longrightarrow \zeta
\]
in the triangulated categories \( \text{Mor}(\mathcal{X}, \mathcal{Y}) \) and \( \text{Mor}(\mathcal{X}, \mathcal{Y}) \). Moreover, the triangle (Tr1) can be obtained from Tr2 by applying the section \( r_\mathcal{X} \), that is, we have \( (\text{Tr1}) = (\text{Tr2}) \cdot r_\mathcal{X} \).

Let us describe a case when the constructed morphism \( \tilde{g}_\phi: \mathcal{X} \to \mathcal{Y} \) is an ff-morphism, that is, it induces a fully faithful embedding \( L\tilde{g}_\phi^*: \text{Perf-} \mathcal{I} \to \text{Perf-} \binom{\mathcal{R}}{\mathcal{T}} \mathcal{I} \). By construction, the projection \( \tilde{f}_{\text{id}} \cong p_\mathcal{Y} \) is such a morphism.

**Theorem 3.22.** Let \( \mathcal{X} = \text{Perf-} \mathcal{R} \) and \( \mathcal{Y} = \text{Perf-} \mathcal{I} \) be noncommutative schemes, and let \( \phi: f \to g \) be a map between morphisms from \( \mathcal{X} \) to \( \mathcal{Y} \). Let \( c = \text{Cone}(\phi) \) be the cone of \( \phi \). Let the morphisms \( g \) and \( c \) be represented by DG \( \mathcal{Y}-\mathcal{X} \)-bimodules \( U \) and \( P \), respectively. Then the following properties are equivalent:

1. \( \tilde{g}_\phi: \mathcal{X} = \mathcal{X} \sqcup \mathcal{Y} \to \mathcal{Y} \) is an ff-morphism, that is, \( L\tilde{g}_\phi^* \) is fully faithful;
2) \( \text{Hom}_\mathcal{X}(Lc^*\mathcal{I}, L\tilde{g}_\phi^*\mathcal{I}[m]) \cong \text{Hom}_\mathcal{X}(P, \tilde{U}_\phi[m]) = 0 \) for all \( m \in \mathbb{Z} \);
3) \( \text{Hom}_\mathcal{X}(Lc^*\mathcal{I}, Lg^*\mathcal{I}[m]) \cong \text{Hom}_\mathcal{X}(P, U[m]) = 0 \) for all \( m \in \mathbb{Z} \).

**Proof.** The triangle (Tr2) of morphisms from \( \mathcal{X} \) to \( \mathcal{Y} \) induces an exact triangle

\[
Lc^*\mathcal{I}[-1] \longrightarrow Lp^*\mathcal{I} \longrightarrow L\tilde{g}_\phi^*\mathcal{I} \longrightarrow Lc^*\mathcal{I}
\]  
(8)

in the triangulated category \( \text{Perf}(\mathcal{X}, \mathcal{I}) \). Since \( \tilde{g}_\phi \cdot r_\mathcal{X} = g \) and \( \xi = c \cdot p_\mathcal{X} \), the relation \( Lr^* \cong Rp_{\mathcal{X}*} \) gives us the sequence of isomorphisms

\[
\text{Hom}_\mathcal{X}(Lc^*\mathcal{I}, L\tilde{g}_\phi^*\mathcal{I}[m]) \cong \text{Hom}_\mathcal{X}(Lp^*\mathcal{I}, Lc^*\mathcal{I})
\]

\[
\cong \text{Hom}_\mathcal{X}(Lc^*\mathcal{I}, Rp^*\mathcal{I}, L\tilde{g}_\phi^*\mathcal{I}[m])
\]

\[
\cong \text{Hom}_\mathcal{X}(Lc^*\mathcal{I}, Lg^*\mathcal{I}[m]).
\]

Thus, condition 2) is equivalent to condition 3).

Consider now the map \( \tilde{\phi} : p_\mathcal{Y} \to \mathcal{G}_\phi \) between morphism from \( \mathcal{X} \) to \( \mathcal{Y} \). It induces a natural transformation \( L^*p_\mathcal{Y} \to L^*\mathcal{G}_\phi \) between inverse image functors. In particular, for any pair of objects \( M, N \in \text{Perf}(\mathcal{X}) \) and a morphism \( u : M \to N \) there is a commutative diagram

\[
L^*p_\mathcal{Y}(M) \xrightarrow{L^*p_\mathcal{Y}(u)} L^*p_\mathcal{Y}(N)
\]

\[
L^*\mathcal{G}_\phi(M) \xrightarrow{L^*\mathcal{G}_\phi(u)} L^*\mathcal{G}_\phi(N)
\]

Putting \( M = \mathcal{I} \) and \( N = \mathcal{I}[m] \), we obtain the commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_\mathcal{Y}(\mathcal{I}, \mathcal{I}[m]) & \xrightarrow{L\tilde{g}_\phi^*} & \text{Hom}_\mathcal{X}(L\tilde{g}_\phi^*\mathcal{I}, L\tilde{g}_\phi^*\mathcal{I}[m]) \\
\downarrow \text{Lp}_\mathcal{Y}^* & & \downarrow h^*(\tilde{\phi}_{\mathcal{Y}[m]}) \\
\text{Hom}_\mathcal{X}(Lp_\mathcal{Y}^*\mathcal{I}, Lp_\mathcal{Y}^*\mathcal{I}[m]) & \xrightarrow{h^*(\tilde{\phi}_\mathcal{Y})} & \text{Hom}_\mathcal{X}(Lp_\mathcal{Y}^*\mathcal{I}, L\tilde{g}_\phi^*\mathcal{I}[m])
\end{array}
\]  
(9)

The left vertical arrow is an isomorphism because the functor \( Lp_\mathcal{Y}^* \) is fully faithful. The bottom horizontal arrow is also an isomorphism for all \( m \in \mathbb{Z} \). Indeed, let us apply the functor \( \text{Hom}_\mathcal{X}(Lp_\mathcal{Y}^*\mathcal{I}, -) \) to the exact triangle (8). Taking into account the semi-orthogonal decomposition for perfect complexes on \( \mathcal{X} \), we obtain \( \text{Hom}_\mathcal{X}(Lp_\mathcal{Y}^*\mathcal{I}, Lc^*\mathcal{I}) = 0 \) because \( Lc^*\mathcal{I} \cong Lp_\mathcal{Y}^* \cdot Lc^*\mathcal{I} \). This implies that the bottom arrow is an isomorphism.

1) \( \Leftrightarrow \) 2). Now, the top horizontal arrow in the diagram (9) is an isomorphism for all \( m \in \mathbb{Z} \) if and only if the right vertical arrow is an isomorphism for all \( m \in \mathbb{Z} \). But this is equivalent to the property \( \text{Hom}_\mathcal{X}(Lc^*\mathcal{I}, L\tilde{g}_\phi^*\mathcal{I}[m]) = 0 \) for all \( m \in \mathbb{Z} \), because the right vertical arrow \( h^*(\tilde{\phi}_\mathcal{Y}) \) is a part of the long exact sequence obtained by applying the functor \( \text{Hom}_\mathcal{X}(-, L\tilde{g}_\phi^*\mathcal{I}) \) to the exact triangle (8). Thus, if the functor \( L\tilde{g}_\phi^* \) is fully faithful, then the top horizontal arrow is an isomorphism, and this implies condition 2).
Conversely, if 2) holds for all $m \in \mathbb{Z}$, then the right vertical arrow in (9) is an isomorphism for all $m \in \mathbb{Z}$. Hence, the top horizontal arrow is an isomorphism for all $m \in \mathbb{Z}$. The object $\mathcal{I}$ is a classical generator for the category $\mathbf{Perf}\mathcal{I}$. Thus, by Proposition 1.12 the functor $L\tilde{g}_\phi^*$ is fully faithful and $\tilde{g}_\phi^*: \mathcal{L} = \mathcal{X} \oplus \mathcal{Y} \to \mathcal{Y}$ is an ff-morphism. □

Let $\mathcal{X} = \mathbf{Perf}\mathcal{R}$ and $\mathcal{Y} = \mathbf{Perf}\mathcal{I}$ be derived noncommutative schemes, and let $\phi: f \to g$ be a map between morphisms from $\mathcal{X}$ to $\mathcal{Y}$. Suppose, as in Theorem 3.22, that the morphism $\tilde{g}_\phi: \mathcal{L} = \mathcal{X} \oplus \mathcal{Y} \to \mathcal{Y}$ is an ff-morphism.

Assume also that the morphism $\tilde{g}_\phi$ is a pp-morphism and hence has a right section $r_\mathcal{Y}: \mathcal{Y} \to \mathcal{X}$. This implies that the embedding functor $L\tilde{g}_\phi^*$ realizes the category $\mathbf{Perf}\mathcal{I}$ as a right admissible subcategory of $\mathbf{Perf}(\mathcal{R} \downarrow \mathcal{I})$. Thus, we obtain another semi-orthogonal decomposition of the form $\mathbf{Perf}(\mathcal{R} \downarrow \mathcal{I}) = \langle \mathbf{Perf}\mathcal{I}^-, \mathbf{Perf}\mathcal{I} \rangle$. A natural enhancement of the subcategory $\mathbf{Perf}\mathcal{I}^-$ gives a derived noncommutative scheme $\mathcal{X}' = \mathbf{Perf}\mathcal{R}'$. By Proposition 3.4 the decomposition above says that the noncommutative scheme $\mathcal{X}$ can also be represented as a gluing $\mathcal{X}' \oplus \mathcal{Y}$, where $\mathcal{T}'$ is a DG $\mathcal{Y}$-$\mathcal{X}'$-bimodule. The morphism $\tilde{g}_\phi^*$ is a new projection $p_\mathcal{Y}'$. The composition $p_\mathcal{Y}' \cdot r_\mathcal{Y}$ is the identity by adjointness, but the composition $p_\mathcal{Y} \cdot r_\mathcal{Y}$ is also isomorphic to the identity by construction. This implies that the adjoint composition $\tilde{p}_\mathcal{Y} \cdot r_\mathcal{Y}'$ is also isomorphic to the identity.

In the case when $\mathcal{X}$ and $\mathcal{Y}$ are regular and proper, all projections have right and left adjoint sections by Proposition 1.7. The noncommutative schemes $\mathcal{X}$ and $\mathcal{X}'$ are also regular and proper by Propositions 3.8 and 3.9.

**Definition 3.23.** Let $\mathcal{X} = \mathbf{Perf}\mathcal{R}$ and $\mathcal{X}' = \mathbf{Perf}\mathcal{R}'$ be two smooth and proper derived noncommutative schemes. Suppose that there exist another smooth and proper noncommutative scheme $\mathcal{Y} = \mathbf{Perf}\mathcal{I}$, morphisms $f: \mathcal{X} \to \mathcal{Y}$ and $f': \mathcal{X}' \to \mathcal{Y}$, and isomorphisms $v$ and $w$ between the gluings

$$
\mathcal{L} = \mathcal{X} \oplus \mathcal{Y} \overset{v}{\leftarrow} \mathcal{L}' = \mathcal{X}' \oplus \mathcal{Y} \overset{w}{\leftarrow}
$$

such that the morphisms $p_\mathcal{Y}' \cdot v \cdot r_\mathcal{Y}$ and $p_\mathcal{Y} \cdot w \cdot r_\mathcal{Y}'$ are isomorphisms of $\mathcal{Y}$. In this case we call $\mathcal{X}$ and $\mathcal{X}'$ Krull–Schmidt partners.

Since the composition $p_\mathcal{Y}' \cdot v \cdot r_\mathcal{Y}$ is an isomorphism, the K-theories of $\mathcal{X}$ and $\mathcal{X}'$ are isomorphic. Moreover, their K-motives are also isomorphic.

Let $X$ be a smooth projective scheme and let $\mathcal{R}$ be a DG algebra such that $\mathbf{Perf}X$ is quasi-equivalent to $\mathbf{Perf}\mathcal{R}$. Let $P_s \in \mathbf{Perf}X$, $s = 1, 2$, be two perfect complexes such that their supports $\text{supp} P_1$ and $\text{supp} P_2 \subset X$ do not meet. Put $T = P_1 \oplus P_2$ and consider the gluing $\mathcal{L} = X \oplus \text{pt}$. Take $U = P_2$ and $P = P_1$.

Since condition 3) in Theorem 3.22 holds for $U = P_2$ and $P = P_1$, the morphism $\mathcal{L} \to \text{pt}$ given by the object $U$ is an ff-morphism. Therefore, we obtain another semi-orthogonal decomposition $\langle \mathbf{Perf}\mathcal{R}', \mathbf{Perf}k \rangle$ for the category $\mathbf{Perf}(\mathcal{R} \downarrow k)$.

Thus, we get a Krull–Schmidt partner $\mathcal{X}' := \mathbf{Perf}\mathcal{R}'$ for the usual commutative scheme $X$. In general this Krull–Schmidt partner is not isomorphic to $X$. 
For example, let $X$ be a smooth projective curve of genus $g$, and let $P_1$ and $P_2$ be torsion coherent sheaves of lengths $l_s = \text{length } P_s$, $s = 1, 2$, such that $\text{supp } P_1 \cap \text{supp } P_2 = \emptyset$. It can be easily checked that the Krull–Schmidt partner $\mathcal{X}'$ is not isomorphic to $X$. Indeed, the integral bilinear form

$$\chi(E, F) = \sum_m (-1)^m \dim \text{Hom}(E, F[m])$$

on $K_0(X)$ goes through $\mathbb{Z}^2 = H^{\text{ev}}(X, \mathbb{Z})$. In this case the forms $\chi$ for $X$ and for $\mathcal{X}'$ are respectively equal to

$$\chi_X = \begin{pmatrix} 1 - g & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \chi_{\mathcal{X}'} = \chi := \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{where } t = 1 - g - l_1 l_2. \quad (10)$$

The integral bilinear forms $\chi_t$ are not equivalent for different $t$. Hence, the categories $\text{Perf} - X$ and $\text{Perf} - \mathcal{X}'$ are not equivalent. Moreover, the categories $\text{Perf} - \mathcal{X}'$ for different $t$ are not equivalent to each other. Thus, for any smooth projective curve $X$ we obtain infinitely many different Krull–Schmidt partners. For the same $t$ we have many Krull–Schmidt partners that depend on the torsion sheaves $P_1$ and $P_2$. It is reasonable to expect that these Krull–Schmidt partners have non-trivial moduli spaces. The case of $X = \mathbb{P}^1$ and two points $P_1 = P_2$, $s = 1, 2$, was discussed in [41], §3.1.

4. Finite-dimensional algebras, quasi-hereditary algebras, and gluings

4.1. Finite-dimensional algebras. In this section we consider derived noncommutative schemes connected with finite-dimensional algebras.

Let $k$ be a field, and let $\Gamma$ be a finite-dimensional $k$-algebra with Jacobson radical $\mathfrak{R}$. The quotient algebra $\Gamma = \Gamma/\mathfrak{R}$ is semisimple. We will assume that $\Gamma$ is basic, that is, the algebra $\Gamma$ is isomorphic to $k \times \cdots \times k$ ($n$ times), where $k$ is the base field. Denote by $\{e_1, \ldots, e_n\}$ a complete sequence of primitive orthogonal idempotents of $\Gamma$, so that $\sum_{i=1}^n e_i = 1$. Let $\Pi_i = e_i \Gamma$ for $1 \leq i \leq n$ be the corresponding indecomposable projective right $\Gamma$-modules, and let $\Sigma_i = e_i \Gamma/e_i \mathfrak{R}$ be the simple right $\Gamma$-modules. The quotient algebra $\Gamma = \Gamma/\mathfrak{R}$ is isomorphic to $\bigoplus_{i=1}^n \Sigma_i$ as a $\Gamma$-module. Since $\Gamma$ is basic, all the $\Sigma_i$ are one-dimensional as $k$-vector spaces.

Denote by $\text{Mod–}\Gamma$ the category of right $\Gamma$-modules. The full subcategory of finitely generated right $\Gamma$-modules will be denoted by $\text{mod–}\Gamma$. Any algebra $\Gamma$ can be regarded as a DG algebra. The derived category $D(\Gamma)$ of all DG modules over this DG algebra is nothing but the unbounded derived category $D(\text{Mod–}\Gamma)$, and the DG category $\mathcal{I} \mathcal{F} – \Gamma$ of semifree DG modules is an enhancement of this triangulated category. The triangulated category $\text{Perf–}\Gamma$ of perfect DG modules (from now on, perfect complexes) consists of all bounded complexes of finitely generated projective modules. The DG category $\text{Perf–}\Gamma$ is a natural enhancement of $\text{Perf–}\Gamma$, and it defines a derived noncommutative scheme $\mathcal{W} = \text{Perf–}\Gamma$ (see Definition 2.1). Since the algebra $\Gamma$ is finite-dimensional, the noncommutative scheme $\mathcal{W}$ is proper.

We can also consider the bounded derived category $D^b(\text{mod–}\Gamma)$ of finitely generated $\Gamma$-modules. It contains the triangulated category $\text{Perf–}\Gamma$ as a full triangulated subcategory. Moreover, the subcategory of perfect complexes
\( \mathcal{P}ef - \Gamma \subseteq D^b(\text{mod} - \Gamma) \) is equivalent to the whole bounded derived category \( D^b(\text{mod} - \Gamma) \) if and only if the algebra \( \Gamma \) is of finite global dimension. In this case the derived noncommutative scheme \( \mathcal{W} = \mathcal{P}ef - \Gamma \) is regular (see [43]).

4.2. Quasi-hereditary algebras. Let \( N_1, \ldots, N_k \) be finitely generated right \( \Gamma \)-modules. We denote by \( \text{Filt}(N_1, \ldots, N_k) \) the full subcategory of the Abelian category \( \text{mod} - \Gamma \) consisting of all modules \( M \) that admit a finite filtration \( 0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_s = M \) such that each quotient \( M_p/M_{p-1} \) is isomorphic to an object of the form \( N_j \).

Most of our constructions will depend on a linear order on the sequence of idempotents. This ordering will be denoted by \( \overline{e} = (e_1, e_2, \ldots, e_n) \). The idempotents \( e_i + e_{i+1} + \cdots + e_n \) for \( 1 \leq i \leq n \) will be denoted by \( \varepsilon_i \), and we set \( \varepsilon_{n+1} = 0 \).

**Definition 4.1.** For each \( 1 \leq i \leq n \) define the standard module \( \Delta_i \) as the largest quotient of the projective module \( \Pi_i \) having no simple composition factors \( \Sigma_j \) with \( j > i \).

In other words, we have the following definition of the standard modules \( \Delta_i = e_i \Gamma/e_i \varepsilon_{i+1} \Gamma \), and this definition depends on the ordering \( \overline{e} \).

We denote by \( \Theta_i \) the kernels of the natural epimorphisms \( \Pi_i \rightarrow \Delta_i \) and by \( \Xi_i \) the kernels of the natural surjections \( \Delta_i \rightarrow \Sigma_i \). Thus, for all \( 1 \leq i \leq n \) there are short exact sequences
\[
0 \rightarrow \Theta_i \rightarrow \Pi_i \rightarrow \Delta_i \rightarrow 0 \quad (11)
\]
and
\[
0 \rightarrow \Xi_i \rightarrow \Delta_i \rightarrow \Sigma_i \rightarrow 0. \quad (12)
\]

**Definition 4.2** [13]. The algebra \( \Gamma \) is said to be quasi-hereditary (with respect to the ordering \( \overline{e} \)) if the following conditions hold:

1) the modules \( \Xi_i \) belong to \( \text{Filt}(\Sigma_1, \ldots, \Sigma_{i-1}) \) for all \( 1 \leq i \leq n \);
2) the modules \( \Theta_i \) belong to \( \text{Filt}(\Delta_{i+1}, \ldots, \Delta_n) \) for all \( 1 \leq i \leq n \).

Condition 1) in Definition 4.2 means that all the modules \( \Delta_i \) are Schurian, that is, \( \text{End}_\Gamma(\Delta_i) \) are division rings for all \( 1 \leq i \leq n \). In our case, when \( \Gamma \) is a basic algebra, condition 1) implies that \( \text{End}_\Gamma(\Delta_i) \cong k \) for all \( i \). In particular, there is an isomorphism \( \Delta_1 \cong \Sigma_1 \), while, by definition, \( \Delta_n \cong \Pi_n \). Property 2) also implies that for every \( 1 \leq i \leq n \) the projective module \( \Pi_i \) belongs to the subcategory \( \text{Filt}(\Delta_i, \ldots, \Delta_n) \). The following proposition is well known.

**Proposition 4.3.** Let \( \Gamma \) be a basic quasi-hereditary algebra. Then \( \Gamma \) has finite global dimension, and the sequence of standard modules \( (\Delta_1, \ldots, \Delta_n) \) forms a full exceptional collection in the triangulated category \( \mathcal{P}ef - \Gamma \).

**Proof.** Descending induction and the short exact sequences (11) give us that all the \( \Delta_i \) belong to \( \mathcal{P}ef - \Gamma \), but ascending induction and the short exact sequences (12) show us that all the simple modules \( \Sigma_i \) also belong to \( \mathcal{P}ef - \Gamma \). This implies that \( \Gamma \) has finite global dimension and \( \mathcal{P}ef - \Gamma \cong D^b(\text{mod} - \Gamma) \).

Since \( \Delta_i \) belongs to \( \text{Filt}(\Sigma_1, \ldots, \Sigma_i) \), we have \( \text{Hom}(\Pi_j, \Delta_i) = 0 \) when \( j > i \). Applying this to \( \Delta_n = \Pi_n \), we get that \( \text{Ext}^k(\Delta_n, \Delta_i) = 0 \) for all \( k \geq 0 \) and \( i < n \). Using descending induction on \( j \), we can now show that \( \text{Ext}^k(\Delta_j, \Delta_i) = 0 \) for all \( k \geq 0 \) and \( i < j \). Indeed, we know that \( \text{Ext}^k(\Pi_j, \Delta_i) = 0 \) for all \( k \geq 0 \) and \( i < j \).
Moreover, by the induction hypothesis \( \text{Ext}^k(\Theta_j, \Delta_i) = 0 \) for all \( k \geq 0 \) and \( i < j \), because \( \Theta_i \) belongs to \( \text{Filt}(\Delta_{i+1}, \ldots, \Delta_n) \). Now the short exact sequence (11) for \( \Delta_j \) implies vanishing of all the Ext groups from \( \Delta_j \) to \( \Delta_i \) when \( j > i \).

Since \( \Theta_i \) belongs to \( \text{Filt}(\Delta_{i+1}, \ldots, \Delta_n) \), we have \( \text{Ext}^k(\Theta_i, \Delta_i) = 0 \) for all \( k \geq 0 \). Hence for any \( i \)

\[
\text{Ext}^k(\Delta_i, \Delta_i) \cong \text{Ext}^k(\Pi_i, \Delta_i) \cong \text{Ext}^k(\Pi_i, \Sigma_i).
\]

This implies that \( \text{Ext}^k(\Delta_i, \Delta_i) = 0 \) when \( k > 0 \), and \( \text{End}(\Delta_i) \cong k \). Thus, the sequence of standard modules \( (\Delta_1, \ldots, \Delta_n) \) forms a full exceptional collection in the triangulated category \( \text{Perf} - \Gamma \cong \mathcal{D}^b(\text{mod} - \Gamma) \). \( \square \)

The proposition above implies that the derived noncommutative scheme \( \mathcal{W} = \text{Perf} - \Gamma \) is smooth and proper and can be obtained as a gluing of \( n \) copies of the point \( pt \). For any \( 1 \leq i \leq n \) we denote by \( \mathcal{T}_i \subseteq \text{Perf} - \Gamma \) the full admissible subcategory \( \mathcal{T}_i = (\Delta_1, \ldots, \Delta_i) \) that is generated by the exceptional subcollection \( (\Delta_1, \ldots, \Delta_i) \). Let \( \mathcal{U}_{i+1} = \perp \mathcal{T}_i \) be the left orthogonal, which is generated by the exceptional subcollection \( (\Delta_{i+1}, \ldots, \Delta_n) \), that is, \( \mathcal{U}_{i+1} = (\Delta_{i+1}, \ldots, \Delta_n) \), and this subcategory is also admissible. The following lemma is evident.

**Lemma 4.4.** For every \( 1 \leq i \leq n \) the subcategories \( \mathcal{T}_i, \mathcal{U}_i \subseteq \text{Perf} - \Gamma \) have the following properties:

1) \( \mathcal{T}_i \) contains the simple modules \( \{\Sigma_1, \ldots, \Sigma_i\} \) and is generated by this set of objects;

2) \( \mathcal{U}_i \) contains the projective modules \( \{\Pi_i, \ldots, \Pi_n\} \) and is generated by this set of objects.

Let \( \mathcal{T}_i \) and \( \mathcal{U}_i \) be full DG subcategories of the DG category of perfect complexes \( \text{Perf} - \Gamma \) with the same objects as \( \mathcal{T}_i \) and \( \mathcal{U}_i \), respectively, that is, \( \mathcal{T}_i \) and \( \mathcal{U}_i \) are the induced DG enhancements for \( \mathcal{T}_i \) and \( \mathcal{U}_i \).

There are two recursive constructions of quasi-hereditary algebras described in the literature. We will use the construction based on extensions of centralizers, described in [14]. Actually, when we have an algebra \( \Gamma \), we can consider the sequence of algebras \( \Gamma_k \cong \varepsilon_k \Gamma \varepsilon_k \), where \( 1 \leq k \leq n \). We get that \( \Gamma_n \) is isomorphic to the field \( k \), while the algebra \( \Gamma_1 \) is isomorphic to \( \Gamma \). The algebras \( \Gamma_k \) are the endomorphism algebras \( \Gamma_k \cong \text{End}_\Gamma(\bigoplus_{i=k}^n \Pi_i) \).

**Proposition 4.5.** Let \( (\Gamma, \varepsilon) \) be a basic quasi-hereditary algebra with indecomposable projective modules \( \Pi_1, \ldots, \Pi_n \). Let \( \Gamma_k \) be the endomorphism algebra \( \text{End}_\Gamma(\bigoplus_{i=k}^n \Pi_i) \).

Then the following properties hold.

1) For any \( 1 \leq k \leq n \) the algebra \( \Gamma_k \) is basic and quasi-hereditary.

2) For any \( 1 \leq k \leq n \) the DG category \( \text{Perf} - \Gamma_k \) is quasi-equivalent to the DG subcategory \( \mathcal{U}_k \subseteq \text{Perf} - \Gamma \). Moreover, under this quasi-equivalence the indecomposable projective \( \Gamma_k \)-modules go to indecomposable projective \( \Gamma \)-modules, and standard \( \Gamma_k \)-modules go to standard \( \Gamma \)-modules.

3) For any \( 1 \leq k < n \) the DG category of perfect complexes \( \text{Perf} - \Gamma_k \) is quasi-equivalent to the gluing \( \bigoplus_{i=k+1}^n \text{Perf} - \Gamma_{k+1} \) via the left DG \( \Gamma_{k+1} \)-module

\[
\mathcal{T}_k = \text{Hom}_{\text{Perf} - \Gamma}(\Delta_k, \bigoplus_{i=k+1}^n \Pi_i).
\]
Proof. By Lemma 4.4 the projective module $\bigoplus_{i=k}^n \Pi_i$ belongs to $\mathcal{U}_k$ and generates it. Moreover, the DG algebra of endomorphisms of this object in the DG category $\mathcal{U}_k$ is quasi-isomorphic to the algebra $\Gamma_k$. Hence, by Proposition 1.14 there is a quasi-equivalence between $\mathcal{U}_k$ and $\text{Perf} - \Gamma_k$. This quasi-equivalence is actually given by the DG functor $\text{Hom}_\Gamma(\bigoplus_{i=k}^n \Pi_i, -)$. Under this DG functor the projective modules $\Pi_j, j \geq k$, go to the projective $\Gamma_k$-modules $\text{Hom}_\Gamma(\bigoplus_{i=k}^n \Pi_i, \Pi_j)$. Note that the projective modules $\Pi_j, j \geq k$, belong to $\mathcal{U}_k$.

The simple modules $\Sigma_j, j \geq k$, go to simple $\Gamma_k$-modules, while the simple modules $\Sigma_j, j < k$, go to 0. However, the simple modules $\Sigma_j, j \geq k$, do not necessarily belong to $\mathcal{U}_k$, and hence simple $\Gamma_k$-modules do not go to simple $\Gamma$-modules. Besides, the standard modules $\Delta_j = e_j \Gamma/e_j \Gamma e_{j+1} \Gamma$ also go to standard $\Gamma_k$-modules when $j \geq k$, and go to 0 when $j < k$. Moreover, the standard modules $\Delta_j, j \geq k$, belong to $\mathcal{U}_k$, and therefore standard $\Gamma_k$-modules correspond to the standard $\Gamma$-modules $\Delta_j$ for $j \geq k$. We also have that exact sequences (11) and (12) for $i \geq k$ go to the same exact sequences in $\text{mod} - \Gamma_k$, and conditions 1) and 2) of Definition 4.2 hold. Thus, the algebra $\Gamma_k$ is also quasi-hereditary. By Proposition 3.4, the semi-orthogonal decomposition $\mathcal{U}_k = \langle \Delta_k, \mathcal{U}_{k+1} \rangle$ implies property 3). □

Any path algebra of a directed quiver with relations is quasi-hereditary in two different ways. First, we can take all the simple modules as standard modules. In this case the category $\text{Filt}(\Delta_1, \ldots, \Delta_n)$ coincides with the whole Abelian category $\text{mod} - \Gamma$. The second way is to take the indecomposable projective modules as standard modules. In this case the subcategory $\text{Filt}(\Delta_1, \ldots, \Delta_n)$ contains only the finitely generated projective modules.

4.3. Well-formed quasi-hereditary algebras. Let $\Gamma$ be a basic quasi-hereditary algebra, and let $\langle \Delta_1, \ldots, \Delta_n \rangle$ be the complete sequence of standard $\Gamma$-modules that forms a full exceptional collection in the triangulated category of perfect complexes $\text{Perf} - \Gamma$.

Definition 4.6. An algebra $\Gamma$ is well-formed if for every $1 \leq i \leq n$ there exist a right $\Gamma$-module $\Psi_i \in \text{Filt}(\Delta_{i+1}, \ldots, \Delta_n)$ and a morphism $\pi_i: \Pi_i \rightarrow \Psi_i$ such that the canonical morphism of the functor

$$\text{Hom}(\Psi_i, -) \rightarrow \text{Hom}(\Pi_i, -)$$

is an isomorphism on the subcategory $\mathcal{U}_{i+1} \subset \text{Perf} - \Gamma$.

Remark 4.7. In other words, this property means that the complex $\Pi_i \xrightarrow{\pi_i} \Psi_i$ belongs to the left orthogonal $\perp \mathcal{U}_{i+1}$ in the category $\mathcal{U}_i$, and the corresponding projection of $\Pi_i$ on $\mathcal{U}_{i+1}$ is a module in the subcategory $\text{Filt}(\Delta_{i+1}, \ldots, \Delta_n)$. We remark that the right orthogonal $\mathcal{U}_{i+1}^\perp$ in $\mathcal{U}_i$ is exactly the subcategory generated by the exceptional object $\Delta_i$.

Note that $\Psi_n = 0$. Moreover, for $n - 1$ the module $\Psi_{n-1}$ also exists for any quasi-hereditary algebra $\Gamma$. It is isomorphic to $\Pi_{n-1}^m$, where $m$ is the dimension of the space of homomorphisms $\text{Hom}_\Gamma(\Pi_{n-1}, \Pi_n)$. However, already for $i = n - 2$ the existence of the module $\Psi_{n-2} \in \text{Filt}(\Delta_{n-1}, \Delta_n)$ with the property described in Definition 4.6 is an additional restrictive condition.
Recall that any exceptional collection in a proper triangulated category has the right and left dual exceptional collections (see [7], for example). First, let us describe the left dual to the collection \((\Delta_1, \ldots, \Delta_n)\). Denote by \(I_i\) the injective envelope of the simple module \(\Sigma_i\) for all \(1 \leq i \leq n\), and define the costandard modules \(\nabla_i\) to be the maximal submodules of \(I_i\) having no composition factors \(\Sigma_j\) with \(j > i\). It is evident that \(\nabla_1 \cong \Delta_1 \cong \Sigma_1\). Moreover, it is not difficult to check that the collection \((\nabla_n, \ldots, \nabla_1)\) is a full exceptional collection in the triangulated category \(\text{Perf} - \Gamma\) and is left dual to the collection \((\Delta_1, \ldots, \Delta_n)\). The last property means that the following conditions hold:

\[
\text{Ext}^l(\Delta_i, \nabla_j) \cong \begin{cases} k, & \text{when } i = j \text{ and } l = 0, \\ 0, & \text{otherwise.} \end{cases}
\]

In this case we get that for every \(1 \leq i \leq n\) the admissible subcategory \(T_i = \langle \Delta_1, \ldots, \Delta_i \rangle\) coincides with the subcategory \(\langle \nabla_i, \ldots, \nabla_1 \rangle\), and the object \(\nabla_{i+1}\) generates the right orthogonal to the subcategory \(T_i\) in the category \(T_{i+1}\).

Now we consider a full exceptional collection \((K_n, \ldots, K_1)\) of objects in \(\text{Perf} - \Gamma\) that is right dual to the collection \((\Delta_1, \ldots, \Delta_n)\). Thus, we have

\[
\text{Ext}^l(K_i, \Delta_j) \cong \begin{cases} k, & \text{when } i = j \text{ and } l = 0, \\ 0, & \text{otherwise.} \end{cases}
\]

In particular, there are isomorphisms \(K_n \cong \Delta_n \cong \Pi_n\). It follows directly from the definition that the objects \(K_i\) are isomorphic to the complexes \(\{\Pi_i \xrightarrow{\pi_i} \Psi_i\}\). Hence, the well-formedness property of a quasi-hereditary algebra can also be regarded as a property of the right dual exceptional collection \((K_n, \ldots, K_1)\). Denote by \(\psi_i : \Theta_i \rightarrow \Psi_i\) the composition of \(\pi_i\) and the natural inclusion of \(\Theta_i\) in \(\Pi_i\), and let \(\Upsilon_i\) be the cone of \(\psi_i\). For any \(1 \leq i \leq n\) we have the following commutative diagram of exact triangles in the triangulated category \(\text{Perf} - \Gamma\):

\[
\begin{array}{ccc}
K_i & \longrightarrow & K_i \\
\downarrow & & \downarrow \\
\Theta_i & \longrightarrow & \Pi_i \\
\downarrow & & \downarrow \\
\Theta_i & \longrightarrow & \Psi_i \\
\downarrow & & \downarrow \\
\Theta_i & \longrightarrow & \Upsilon_i
\end{array}
\]

and the algebra \(\Gamma\) is well-formed if the object \(\Psi_i\) not only belongs to \(\mathcal{U}_{i+1}\) but is also in \(\text{Filt}(\Delta_{i+1}, \ldots, \Delta_n)\) for any \(1 \leq i \leq n\). We also have that the objects \(\Theta_i\), \(\Psi_i\), and \(\Upsilon_i\) belong to \(\mathcal{U}_{i+1}\), while \(\Delta_i\) and \(K_i\) generate the right and left orthogonals \(\mathcal{U}_{i+1}^r\) and \(\mathcal{U}_{i+1}^l\) in \(\mathcal{U}_i\), respectively.

\textbf{Remark 4.8.} Let us consider the algebra \(\Gamma = kQ/I\) of a directed quiver with relations \((Q, I)\) for which the vertex set \(Q_0 = \{1, \ldots, n\}\) is the ordered set of \(n\) elements and \(s(a) > t(a)\) for any arrow \(a \in Q_1\), where \(s, t : Q_1 \rightarrow Q_0\) are the maps associating to each arrow its source and target. In this case the algebra \(\Gamma\) is quasi-hereditary with respect to the ordering \(\bar{e} = (e_1, e_2, \ldots, e_n)\). Moreover, for this ordering the
standard modules $\Delta_i$ are isomorphic to the simple modules $\Sigma_i$ for all $1 \leq i \leq n$. The quasi-hereditary algebra $(\Gamma, \bar{e})$ is well-formed in this case because the projective module $\Pi_i$ belongs to the subcategory $\mathcal{U}_{i+1} \subseteq \mathcal{T}$ for each $i$. Thus, for all $1 \leq i \leq n$ we have $\Psi_i = 0$.

Remark 4.9. Note that if we take the opposite ordering of idempotents, then the algebra $\Gamma$ is also quasi-hereditary, but in this case the standard modules are the indecomposable projective modules, and the algebra is not necessarily well-formed. The simplest example is a quiver $Q = (\bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet)$ with three vertices, two arrows $a$ and $b$, and the single relation $ba = 0$.

5. Geometric realizations of finite-dimensional algebras

5.1. Geometric realizations of well-formed quasi-hereditary algebras. In this section we discuss some geometric realizations for finite-dimensional algebras. We consider the case when an algebra is basic. By Theorem 3.16, any such algebra has a plain geometric realization. However, Theorem 3.16 only tells us that for any such algebra $\Lambda$ there is a perfect complex $E$ on a smooth projective scheme $X$ for which $\text{End}(E) \cong \Lambda$ and $\text{Hom}(E, E[l]) = 0$ when $l \neq 0$. A reasonable question about finite-dimensional algebras here is to find a geometric realization such that the perfect complex $E$ is a vector bundle $E$ on $X$. We know that this question has a positive answer for any quiver algebra $\Lambda$ by Theorem 3.15. Moreover, in this case we can find $X$ and a vector bundle $E$ such that the rank of $E$ is equal to the dimension of $\Lambda$ (see [41], Corollary 2.8).

First we consider the case of a quasi-hereditary algebra, and we try to find a geometric realization such that all the standard modules $\Delta_i$ go to line bundles on $X$.

Definition 5.1. Let $\Gamma$ be a basic quasi-hereditary algebra over $k$, and let $X$ be a smooth projective scheme. Let $G: \text{Perf} - \Gamma \to \text{Perf} - X$ be a quasi-functor. We say that $G$ satisfies property $(V)$ if the following conditions hold:

(V.1) the functor $G = \mathcal{H}^0(G): \text{Perf} - \Gamma \to \text{Perf} - X$ is fully faithful, that is, $G$ gives a plain geometric realization of $\text{Perf} - \Gamma$;

(V.2) the standard modules $\Delta_i$, $1 \leq i \leq n$, go to line bundles $L_i$ on $X$ under $G$;

(V.3) there is a line bundle $N$ on $X$ such that $N \in \mathcal{U} \subseteq \text{Perf} - X$, the line bundles $N \otimes L_i^{-1}$ are generated by global sections, and $H^j(X, N \otimes L_i^{-1}) = 0$ when $j > 1$ for all $i = 1, \ldots, n$.

Since any module $M \in \text{Filt}(\Delta_1, \ldots, \Delta_n)$ has a filtration with successive quotients being standard modules, condition (V.2) in Definition 5.1 implies that any such $\Gamma$-module goes to a vector bundle under the functor $G$. Moreover, by the same reasoning, the vector bundle $G(M)$ has a filtration with successive quotients isomorphic to the line bundles $L_i$. Now it is not difficult to check that condition (V.3) implies the following condition:

(V.3') there is a line bundle $N$ on $X$ such that $N \in \mathcal{U}$ and the vector bundles $N \otimes G(M)^\vee$ are generated by global sections for all $M \in \text{Filt}(\Delta_1, \ldots, \Delta_n)$ and have no higher cohomology.

Note that the quasi-functor $G: \text{Perf} - \Gamma \to \text{Perf} - X$ gives an ff-morphism $g: X \to \mathcal{W}$, where $\mathcal{W} = \text{Perf} - \Gamma$ is the derived noncommutative scheme connected
with the algebra $\Gamma$. Property (V.2) tells us that the standard modules $\Delta_i \in \text{Perf} - \Gamma$ go to the line bundles $\mathcal{L}_i$ on $X$ under the inverse image functor $Lg^*$.

For any quasi-hereditary algebra $\Gamma$ the projective modules $\Pi_i$ belong to the subcategory $\text{Filt}(\Delta_1, \ldots, \Delta_n)$, and hence they go to vector bundles under the functor $G$. Denote by $\mathcal{P}_i$ these vector bundles $G(\Pi_i)$ for all $i = 1, \ldots, n$. Since $G$ is fully faithful, the vector bundles $\mathcal{E}_k = \bigoplus_{i=k}^n \mathcal{P}_i$ possess the following properties:

$$\text{Hom}_X(\mathcal{E}_k, \mathcal{E}_k) \cong \Gamma_k \quad \text{and} \quad \text{Ext}_X^l(\mathcal{E}_k, \mathcal{E}_k) = 0 \quad \text{for all } l \neq 0.$$ 

Thus, if the quasi-functor $G: \text{Perf} - \Gamma \rightarrow \text{Perf} - X$ has property (V), then the algebra $\Gamma$ goes to the vector bundle $\mathcal{E}_1$ under such a geometric realization.

The following proposition gives us the inductive step of a general construction.

**Proposition 5.2.** Let $(\Gamma, \mathcal{E})$ be a basic well-formed quasi-hereditary algebra with indecomposable projective modules $\Pi_1, \ldots, \Pi_n$. Let $\Gamma_l$ for $l = 1, \ldots, n$ be the endomorphism algebras $\text{End}_\Gamma(\bigoplus_{i=l}^n \Pi_i)$. Suppose that there exist a smooth projective scheme $X_{k+1}$ and a quasi-functor $G_{k+1}: \text{Perf} - \Gamma_{k+1} \rightarrow \text{Perf} - X_{k+1}$ that satisfies property (V). Then there exist a smooth projective scheme $X_k$ and a quasi-functor $G_k: \text{Perf} - \Gamma_k \rightarrow \text{Perf} - X_k$

that also satisfies property (V).

Moreover, the scheme $X_k \cong \mathbb{P}(\mathcal{F}_k)$ is a projective vector bundle over $X_{k+1}$, and the restriction of $G_k$ to the subcategory $\text{Perf} - \Gamma_{k+1} \subset \text{Perf} - \Gamma_k$ is isomorphic to $Lp^* \cdot G_{k+1}$, where $p: \mathbb{P}(\mathcal{F}_k) \rightarrow X_{k+1}$ is the natural projection.

**Proof.** Consider the quasi-functor

$$G_{k+1}: \text{Perf} - \Gamma_{k+1} \rightarrow \text{Perf} - X_{k+1}.$$ 

By Proposition 4.5, we can identify the DG category $\text{Perf} - \Gamma_{k+1}$ with the DG subcategory $\mathcal{U}_{k+1} \subset \text{Perf} - \Gamma$ that is generated by the standard modules $\Delta_{k+1}, \ldots, \Delta_n$. Consider the quasi-functor $G_{k+1}$ as a quasi-functor from $\mathcal{U}_{k+1}$ to $\text{Perf} - X_{k+1}$.

Since $G_{k+1}$ satisfies property (V), the standard modules $\Delta_{k+1}, \ldots, \Delta_n$ go to line bundles $\mathcal{L}_{k+1}, \ldots, \mathcal{L}_n$, respectively. The DG subcategory of $\text{Perf} - X_{k+1}$ that is generated by the line bundles $\mathcal{L}_{k+1}, \ldots, \mathcal{L}_n$ is quasi-equivalent to $\mathcal{U}_{k+1}$, and the quasi-functor $G_{k+1}$ establishes this quasi-equivalence.

Any module $M \in \text{Filt}(\Delta_{k+1}, \ldots, \Delta_n)$ goes to a vector bundle under the functor $G_{k+1} = \mathcal{H}^0(G_{k+1})$. Now take the projective module $\Pi_k$ and consider the short exact sequence

$$0 \longrightarrow \Theta_k \longrightarrow \Pi_k \longrightarrow \Delta_k \longrightarrow 0. \quad (14)$$

By the definition of a quasi-hereditary algebra, the module $\Theta_k$ belongs to $\text{Filt}(\Delta_{k+1}, \ldots, \Delta_n)$. The algebra $\Gamma$ is well-formed, and hence there is a module $\Psi_k \in \text{Filt}(\Delta_{k+1}, \ldots, \Delta_n)$ such that the complex $K_k = \{ \Pi_k \xrightarrow{\pi_k} \Psi_k \}$ belongs to $\mathcal{U}_{k+1}$. As in the diagram (13), we denote by $\psi_k: \Theta_k \rightarrow \Psi_k$ the composition of $\pi_k$ with the natural inclusion of $\Theta_k$ in $\Pi_k$. By Proposition 4.5, the DG category $\mathcal{U}_k \subset \text{Perf} - \Gamma$ is quasi-equivalent to a gluing $pt \bigoplus_{T_k} \mathcal{U}_{k+1}$, where the left DG
$\mathcal{U}_{k+1}$-module $T_k$ is $\text{Hom}_{\mathcal{U}_k}(\Delta_k, Q)$ with $Q \in \mathcal{U}_{k+1}$. Using (14) as a resolution for the standard module $\Delta_k$, we obtain the following quasi-isomorphism of left DG $\mathcal{U}_{k+1}$-modules:

$$T_k = \text{Hom}_{\mathcal{U}_k}(\Delta_k, Q) \cong \text{Hom}_{\mathcal{U}_{k+1}}(Y_k, Q), \quad Q \in \mathcal{U}_{k+1},$$

where $Y_k \in \mathcal{U}_{k+1}$ is the complex $\{\Theta_k \xrightarrow{\psi_k} \Psi_k\}$ concentrated at degrees $-1$ and $0$.

Consider the vector bundle $G_{k+1}(\Theta_k)$. The morphism $\psi_k$ induces a map $G_{k+1}(\psi_k)$ between the vector bundles $G_{k+1}(\Theta_k)$ and $G_{k+1}(\Psi_k)$ and also a map $G_{k+1}(\Psi_k)^{\vee} \rightarrow G_{k+1}(\Theta_k)^{\vee}$ between the dual vector bundles. By assumption (V.3) and its consequence (V.3') there is a surjection $(N^{-1})^{\oplus m} \rightarrow G_{k+1}(\Theta_k)^{\vee}$ for some $m \in \mathbb{N}$.

Consider the induced map

$$(N^{-1})^{\oplus m} \oplus G_{k+1}(\Psi_k)^{\vee} \rightarrow G_{k+1}(\Theta_k)^{\vee},$$

which is also a surjection. Denote by $F$ the vector bundle on $X$ that is dual to the kernel of this surjection. We obtain the following exact sequence of vector bundles on $X_{k+1}$:

$$0 \rightarrow G_{k+1}(\Theta_k) \rightarrow N^{\oplus m} \oplus G_{k+1}(\Psi_k) \rightarrow F \rightarrow 0. \quad (15)$$

Since the line bundle $N$ belongs to the orthogonal $\perp G_{k+1}(\mathcal{U}_{k+1})$, we obtain the quasi-isomorphism of left DG $\mathcal{U}_{k+1}$-modules

$$\text{Hom}_{\text{perf} - X_{k+1}}(F, G_{k+1}(Q)) \cong \text{Hom}_{\mathcal{U}_{k+1}}(Y_k, Q) \cong \text{Hom}_{\mathcal{U}_k}(\Delta_k, Q) \quad \text{with} \quad Q \in \mathcal{U}_{k+1}.$$

Taking $m$ sufficiently large, we can assume that the rank of $F$ is greater than 2. Let us consider the projective bundle $p: \mathbb{P}(F) \rightarrow X_{k+1}$ and denote it by $X_k$. There are natural exact sequences on $X_k$ of the form

$$0 \rightarrow \Omega_{X_k/X_{k+1}}(1) \rightarrow p^*F^{\vee} \rightarrow \mathcal{O}_{X_k}(1) \rightarrow 0$$

and

$$0 \rightarrow \mathcal{O}_{X_k}(-1) \rightarrow p^*F \rightarrow \mathcal{T}_{X_k/X_{k+1}}(-1) \rightarrow 0,$$

where $\mathcal{O}_{X_k}(-1)$ is the relative tautological line bundle, and $\mathcal{T}_{X_k/X_{k+1}}$ and $\Omega_{X_k/X_{k+1}}$ are the relative tangent and the relative cotangent bundles, respectively. We have

$$\mathbf{R}p_*\mathcal{O}_{X_k}(1) \cong F^{\vee} \quad \text{and} \quad \mathbf{R}p_*\mathcal{O}_{X_k}(-1) = 0.$$

Denote by $\tilde{L}_i$ the pullback line bundles $p^*L_i$ for $i = k+1, \ldots, n$ and consider the DG subcategory $\mathcal{U}_{k+1} \subset \text{perf} - X_k$ that is generated by these line bundles. Since the functor $Lp^*$ is fully faithful, the DG category $\mathcal{U}_{k+1}$ is quasi-equivalent to $\mathcal{U}_{k+1}$. Put

$$\tilde{L}_k = \mathcal{O}_{X_k}(-1)$$

and consider the DG subcategory $\mathcal{V} \subset \text{perf} - X_k$ generated by the line bundles $\tilde{L}_k, \ldots, \tilde{L}_n$. Since $\mathbf{R}p_*\mathcal{O}_{X_k}(-1) = 0$, the line bundles $\tilde{L}_k, \ldots, \tilde{L}_n$ form a full.
exceptional collection in the homotopy category \( \mathcal{V} = \mathcal{H}^0(\mathcal{V}) \), and the category \( \mathcal{V} \subset \text{Perf} - X_k \) is admissible.

By Proposition 3.4, the DG category \( \mathcal{V} \) is quasi-equivalent to a gluing \( \text{pt} \oplus \widetilde{\mathbb{U}}_{k+1} \) (where \( \text{pt} = \text{Perf} - k \)) with respect to a left DG \( \mathbb{U}_{k+1} \)-module \( S = \text{Hom}_\mathcal{V}(\mathcal{L}_k, -) \). There is the following sequence of quasi-isomorphisms of left DG \( \mathbb{U}_{k+1} \)-modules:

\[
S = \text{Hom}_\mathcal{V}(\mathcal{L}_k, Lp^*G_{k+1}(Q)) \cong \text{Hom}_{\text{Perf} - X_k}(\mathcal{O}_{X_k}, Lp^*G_{k+1}(Q) \otimes \mathcal{O}_{X_k}(1)) \\
\cong \text{Hom}_{\text{Perf} - X_{k+1}}(\mathcal{O}_{X_{k+1}}, G_{k+1}(Q) \otimes \mathcal{F}^\vee) \cong \text{Hom}_{\text{Perf} - X_{k+1}}(\mathcal{F}, G_{k+1}(Q)) \\
\cong \text{Hom}_{\mathbb{U}_k}(\Delta_k, Q) \cong T_k.
\]

Thus, by Proposition 3.5 we get that the DG category \( \mathcal{V} \cong \text{pt} \oplus \widetilde{\mathbb{U}}_{k+1} \) is quasi-equivalent to the DG category \( \mathbb{U}_k \cong \text{pt} \oplus \mathbb{U}_{k+1} \). By Proposition 4.5 these DG categories are also quasi-equivalent to \( \text{Perf} - \Gamma_k \). Therefore, we obtain a quasi-functor \( G_k : \text{Perf} - \Gamma_k \rightarrow \text{Perf} - X_k \) that establishes a quasi-equivalence between \( \text{Perf} - \Gamma_k \) and \( \mathcal{V} \). By construction, conditions (V.1) and (V.2) hold for the quasi-functor \( G_k \).

Finally, we have to show that condition (V.3) also holds for an appropriate line bundle \( \mathcal{N} \) on \( X_k \). Choosing \( \mathcal{N} \) as a line bundle of the form \( \mathcal{O}_{X_k}(1) \otimes p^*\mathcal{R}^{\otimes s} \), where \( \mathcal{R} \) is an ample line bundle on \( X_{k+1} \) and \( s \) is sufficiently large, we can guarantee that (V.3) will hold. Indeed, since the rank of \( \mathcal{F} \) is greater than 2, the line bundle \( \mathcal{N} \) belongs to \( +G_k(\text{Perf} - \Gamma_k) \). Moreover, for \( k < i < n \) we have isomorphisms

\[
H^j(X_k, L\mathcal{L}_i^{-1} \otimes \mathcal{N}) \cong H^j(X_{k+1}, L\mathcal{L}_i^{-1} \otimes \mathcal{F}^\vee \otimes \mathcal{R}^{\otimes s}).
\]

Additionally, for the line bundle \( \mathcal{N} \) we also have

\[
H^j(X_k, L\mathcal{N}^{-1} \otimes \mathcal{N}) \cong H^j(X_{k+1}, S^2(\mathcal{F}^\vee) \otimes \mathcal{R}^{\otimes s}).
\]

Taking a sufficiently large \( s \), we obtain the vanishing of all the cohomology for \( j > 0 \), by the Serre vanishing theorem, and we can guarantee that all these bundles are generated by global sections on \( X_k \). Since the natural maps

\[
p^*\mathcal{F}^\vee \longrightarrow \mathcal{O}_{X_k}(1) \quad \text{and} \quad p^*S^2(\mathcal{F}^\vee) \longrightarrow \mathcal{O}_{X_k}(2)
\]

are surjective, condition (V.3) also holds for the quasi-functor \( G_k \). \( \square \)

We can also give a precise construction of the vector bundles on \( X_k \) that are the images of the projective modules under the functor \( G_k \). Denote by \( \mathcal{P}_{k+1}, \ldots, \mathcal{P}_n \) the vector bundles on \( X_{k+1} \) that are the images of the projective modules \( \Pi_{k+1}, \ldots, \Pi_n \in \text{Filt}(\Delta_{k+1}, \ldots, \Delta_n) \) under the functor \( G_{k+1} \). Under the functor \( G_k \) these projective modules go to the vector bundles \( \mathcal{P}_i = p^*\mathcal{P}_i \). Now we construct the vector bundle \( \mathcal{P}_k \) which is the image of \( \Pi_k \). Consider the sequence of isomorphisms

\[
\text{Ext}^1_{X_k}(\mathcal{O}_{X_k}(-1), p^*G_{k+1}(\Theta_k)) \cong H^1(X_k, p^*G_{k+1}(\Theta_k) \otimes \mathcal{O}_{X_k}(1)) \\
\cong H^1(X_{k+1}, G_{k+1}(\Theta_k) \otimes \mathcal{F}^\vee) \\
\cong \text{Ext}^1_{X_{k+1}}(\mathcal{F}, G_{k+1}(\Theta_k)).
\]
The element $e \in \text{Ext}^1_{X_{k+1}} (\mathcal{F}, G_{k+1} (\Theta_k))$, which determines the short exact sequence (15), gives some element

$$e' \in \text{Ext}^1_{X_k} (\mathcal{O}(-1), p^* G_{k+1} (\Theta_k)).$$

The element $e'$ induces the extension

$$0 \longrightarrow p^* G_{k+1} (\Theta_k) \longrightarrow \mathcal{P}_k \longrightarrow \mathcal{O}_{X_k} (-1) \longrightarrow 0,$$  

(16)

which can be considered as the definition of the vector bundle $\mathcal{P}_k$. Finally, the algebra $\Gamma_k$ itself goes to the vector bundle $\mathcal{E}_k = \bigoplus_{i=k}^n \mathcal{P}_i$ under the functor $G_k$.

Proposition 5.2 as an induction step implies the following theorem.

**Theorem 5.3.** Let $(\Gamma, \mathcal{E})$ be a basic well-formed quasi-hereditary algebra. Then there exist a smooth projective scheme $X$ and a quasi-functor $\mathcal{G}: \text{Perf} - \Gamma \rightarrow \text{Perf} - X$ such that the following conditions hold:

1) the induced homotopy functor

$$G = \mathcal{H}^0 (\mathcal{G}): \text{Perf} - \Gamma \longrightarrow \text{Perf} - X$$

is fully faithful;

2) the standard modules $\Delta_i$ go to line bundles $\mathcal{L}_i$ on $X$ under $\mathcal{G}$;

3) the scheme $X$ is a tower of projective bundles and has a full exceptional collection.

**Proof.** The proof proceeds by induction on $n$. The base of induction is $n = 1$ and $\Gamma = \mathbf{k}$. In this case $X = \mathbb{P}^1$, the quasi-functor $\mathcal{G}$ sends $\Gamma$ to $\mathcal{O}_{\mathbb{P}^1}$, and $\mathcal{N} = \mathcal{O}(1)$. The inductive step is Proposition 5.2. By construction, the scheme $X$ is a tower of projective bundles and hence has a full exceptional collection. \(\square\)

**Remark 5.4.** Note that the well-formedness property of the quasi-hereditary algebra $\Gamma$ is essential in the theorem. Indeed, consider the algebra of the quiver

$$Q = ( \begin{array}{c} \bullet \\ \downarrow a \end{array} \begin{array}{c} \bullet \\ \downarrow b \end{array} \begin{array}{c} \bullet \\ \end{array} )$$

with the relation $ba = 0$ as in Remark 4.9. This algebra is quasi-hereditary but is not well-formed in the case when the standard modules are indecomposable projective modules. It is easy to see that we cannot find a geometric realization for which the standard modules go to line bundles. Indeed, any non-zero morphism of line bundles on a smooth irreducible projective scheme is an isomorphism at the generic point, and this contradicts the fact that $ba = 0$.

### 5.2. Auslander construction and geometric realizations of finite-dimensional algebras.

Now we will discuss geometric realizations for an arbitrary basic finite-dimensional algebra.

Let $\Lambda$ be a basic finite-dimensional algebra over a field $\mathbf{k}$. Denote by $\mathfrak{r}$ the Jacobson radical of $\Lambda$. We know that $\mathfrak{r}^N = 0$ for some $N$. The index of nilpotency of $\Lambda$ is defined as the smallest integer $N$ such that $\mathfrak{r}^N = 0$. The following amazing result was proved by Auslander.
Theorem 5.5 [2]. Let $\Lambda$ be a finite-dimensional algebra with index of nilpotency $N$. Then the finite-dimensional algebra $\tilde{\Gamma} = \text{End}_\Lambda(\bigoplus_{p=1}^{N} \Lambda/\tau^p)$ has the following properties:

1) $\text{gl.dim} \tilde{\Gamma} \leq N + 1$;
2) there is a finite projective $\tilde{\Gamma}$-module $\Pi$ such that $\text{End}_{\tilde{\Gamma}}(\Pi) \cong \Lambda$.

The algebra $\tilde{\Gamma}$ is usually not basic, even if the algebra $\Lambda$ is. Indeed, since $\Lambda$ is basic, each non-projective indecomposable summand occurs in a direct decomposition of $\bigoplus_{p=1}^{N} \Lambda/\tau^p$ with multiplicity 1, whereas each indecomposable projective $\Lambda$-module of Loewy length $l$ occurs with multiplicity $N - l + 1$. If we delete the repeated copies of the indecomposable projective summands of $\bigoplus_{p=1}^{N} \Lambda/\tau^p$, then we obtain a module whose endomorphism algebra is basic and Morita equivalent to $\tilde{\Gamma}$. We describe this more precisely.

Let us denote by $\{f_1, \ldots, f_m\}$ a complete sequence of primitive orthogonal idempotents of the basic algebra $\Lambda$, so that

$$\sum_{j=1}^{m} f_j = 1.$$ 

Let $P_j = f_j \Lambda$ for $1 \leq j \leq m$ be the corresponding indecomposable projective $\Lambda$-modules, and let $S_j = P_j / P_j \tau$ be the simple right $\Lambda$-modules. The quotient algebra $\overline{\Lambda} = \Lambda/\tau$ is semisimple, and it is isomorphic to $\bigoplus_{j=1}^{m} S_j$ as a $\Lambda$-module. Moreover, since $\Lambda$ is basic, the algebra $\overline{\Lambda}$ is isomorphic to $k \times \cdots \times k$ ($m$ times).

We will choose a linear order on the set of idempotents $\{f_1, \ldots, f_m\}$ such that

$$L(P_k) \geq L(P_l), \quad \text{when } k \leq l.$$ 

Here $L(P_j)$ denotes the Loewy length of the projective module $P_j$, that is, the minimal positive integer $i$ such that $P_j \tau^i = 0$. Consider the finite set $T$ consisting of pairs $(j, l)$ with $1 \leq j \leq m$ and $1 \leq l \leq L(P_j)$. The cardinality of $T$ is equal to

$$n = \sum_{j=1}^{m} L(P_j).$$ 

With any element $t = (j, l)$ of $T$ we associate the $\Lambda$-module

$$M_t \cong P_j / P_j \tau^l.$$ 

We introduce a linear order on $T$ by the following rule:

$$t_1 = (j_1, l_1) < t_2 = (j_2, l_2) \iff \begin{cases} l_1 > l_2, \\
\text{or} \quad l_1 = l_2 \text{ and } j_1 < j_2. \end{cases}$$

In particular, we have isomorphisms

$$M_1 \cong P_1, \quad \text{while} \quad M_n \cong S_m.$$
Consider the $\Lambda$-module $M = \bigoplus_{t \in T} M_t$ and denote by $\Gamma$ the endomorphism algebra $\text{End}_\Lambda(M)$. Since all the modules $M_t$ are indecomposable and non-isomorphic to each other, the algebra $\Gamma$ is basic. Its complete set of primitive orthogonal idempotents $\{e_1, \ldots, e_n\}$ is in bijection with the set $T$ and has the linear order introduced above. We fix this linear order and denote it by $\overline{\mathfrak{e}}$. The $\Lambda$-module $M$ gives a standard functor

$$\text{Hom}_\Lambda(M, -): \text{Mod}-\Lambda \longrightarrow \text{Mod}-\Gamma$$

between Abelian categories of modules, and it is left exact. For a $\Lambda$-module $N$ we denote by $\widehat{N}$ the $\Gamma$-module $\text{Hom}_\Lambda(M, N)$ for brevity. Since the $\Lambda$-module $M$ is a generator in the Abelian category $\text{Mod}-\Lambda$, for any two $\Lambda$-modules $N_1$ and $N_2$ the canonical map

$$\text{Hom}_\Lambda(N_1, N_2) \xrightarrow{\sim} \text{Hom}_\Gamma(\widehat{N}_1, \widehat{N}_2) \quad (17)$$

is an isomorphism. By construction, the indecomposable projective $\Gamma$-module $\Pi_t = e_t \Gamma$ is isomorphic to $\widehat{M}_t$. The submodules $M_t r \subset M_t$ induce the submodules $\Theta_t = \widehat{M}_t r \subset \Pi_t$, and we denote the quotient $\Gamma$-modules $\Pi_t/\Theta_t$ by $\Delta_t$.

**Proposition 5.6.** Let $\Lambda$ be a basic finite-dimensional algebra over a field $k$. Then the algebra $(\Gamma, \overline{\mathfrak{e}})$ constructed above is a basic well-formed quasi-hereditary $k$-algebra for which the $\Delta_t = \widehat{M}_t/\widehat{M}_t r$ with $t \in T$ are the standard modules.

**Proof.** We already know that $\Gamma$ is basic. It is proved in [15] that the algebra $(\Gamma, \overline{\mathfrak{e}})$ is quasi-hereditary. However, let us show this.

First we calculate $\text{Hom}_\Gamma(\Pi_{t'}, \Delta_t)$ for any $t', t \in T$. For each $t = (j, l) \in T$ we have $M_t r \subset M_t$ induce the submodules $\Theta_t = \widehat{M}_t r \subset \Pi_t$, and we denote the quotient $\Gamma$-modules $\Pi_t/\Theta_t$ by $\Delta_t$.

Since $S_j = M_s$ for $s = (j, 1) \in T$, we get that $\widehat{S}_j \cong \Pi_s \cong \Delta_s$ is both a projective and a standard module for any $1 \leq j \leq m$. The isomorphisms

$$\text{Hom}_\Gamma(\Pi_{t'}, \widehat{S}_j) \cong \text{Hom}_\Lambda(M_{t'}, S_j) \cong \text{Hom}_\Lambda(P_{t'}, S_j)$$

tell us that if $\text{Hom}_\Gamma(\Pi_{t'}, \widehat{S}_j) \neq 0$, then it is one-dimensional and $t' = (j, l')$ with the same $j$. Since for any $t = (j, l)$ there is an inclusion $\Delta_t \subseteq \widehat{S}_j$, we conclude that if $\text{Hom}_\Gamma(\Pi_{t'}, \Delta_t) \neq 0$, then $t' = (j, l')$ with the same $j$. Furthermore, on the one hand a non-trivial morphism $\Pi_{t'} = \widehat{M}_{t'} \to \Delta_t \subset \widehat{S}_j$ is induced by a non-trivial morphism $M_{t'} \to S_j$. On the other hand, it can be lifted to a map $\Pi_{t'} \to \Pi_t = \widehat{M}_t$ because $\Pi_{t'}$ is projective. Therefore, the map $M_{t'} \to S_j$ should go through a map to $M_t$. Hence, in this case $l' \geq l$. Thus, for any $t = (j, l)$ and $t' = (j', l')$ we obtain

$$\text{Hom}_\Gamma(\Pi_{t'}, \Delta_t) = \begin{cases} k & \text{if } j' = j \text{ and } l' \geq l, \\ 0 & \text{otherwise.} \end{cases} \quad (18)$$
In particular, we obtain
\[
\dim_k \Delta_t = L(P_j) - l + 1 \quad \text{for } t = (j, l).
\]

Moreover, the sequence
\[
0 \subset \Delta_{(j,L(P_j))} \subset \cdots \subset \Delta_{(j,1)} = \tilde{S}_j \quad \text{with } \Delta_{(j,l)}/\Delta_{(j,l+1)} \cong \Sigma_{(j,l)}, \quad (19)
\]
where the \( \Sigma_{(j,l)} \) are the corresponding simple \( \Gamma \)-modules, gives a composition series for \( \tilde{S}_j \cong \Pi_s \cong \Delta_s \), where \( s = (j, 1) \). This implies that any simple quotient of the standard module \( \Delta_t \) is only \( \Sigma_t \), and any submodule of the standard module \( \Delta_t \) with \( t = (j, l) \) is isomorphic to a standard module \( \Delta_{t'} \), where \( t' = (j, l') \) with \( l' \geq l \).

As a consequence, we also get that for any \( t = (j, l) \) the kernel \( \Xi_t \) of the canonical morphism from \( \Delta_t \) to the corresponding simple module \( \Sigma_t \) is isomorphic to the standard module \( \Delta_{t'} \) with \( t' = (j, l + 1) \). Hence, taking the composition series (19) into account, we see that \( \Xi_t \cong \Delta_{t'} \) belongs to the subcategory \( \text{Filt}(\Sigma_1, \ldots, \Sigma_{t-1}) \).

Thus, condition 1) in Definition 4.2 holds.

To prove that the algebra \( \Gamma \) is quasi-hereditary, we have to show that the modules \( \Theta_t \) belong to the subcategories \( \text{Filt}(\Delta_{t+1}, \ldots, \Delta_n) \). Let us prove that any \( \Gamma \)-module of the form \( \tilde{N} \) belongs to the subcategory \( \text{Filt}(\Delta_1, \ldots, \Delta_n) \). We proceed by induction on the length of a \( \Lambda \)-module \( N \). The base of induction is the simple modules \( S_j \), and we already know that the \( \tilde{S}_j \) themselves are standard. Consider an exact sequence
\[
0 \longrightarrow N' \longrightarrow N \longrightarrow S_k \longrightarrow 0
\]
coming from a composition series for \( N \). By the induction hypothesis the \( \Gamma \)-module \( \tilde{N}' \) belongs to \( \text{Filt}(\Delta_1, \ldots, \Delta_n) \). On the other hand, the quotient module \( \tilde{N}/\tilde{N}' \) is standard as a non-zero submodule of \( \tilde{S}_k \). Thus, \( \tilde{N} \) also belongs to \( \text{Filt}(\Delta_1, \ldots, \Delta_n) \).

Consider the \( \Gamma \)-module \( \Theta_t = \widetilde{M_t \tau} \) for \( t = (j, l) \). We would like to show that \( \Theta_t \) belongs not only to the subcategory \( \text{Filt}(\Delta_1, \ldots, \Delta_n) \) but also to the subcategory \( \text{Filt}(\Delta_{t+1}, \ldots, \Delta_n) \). It is enough to check that any morphism from \( \Pi_{t'} \) to \( \Theta_t = \widetilde{M_t \tau} \) can be factorized through \( \Pi_s \) with \( s > t \). The Loewy length of the \( \Lambda \)-modules \( M_t \tau = P_j \tau/P_j \tau^{l} \) is equal to \( l-1 \). Hence, any morphism from a projective module \( P_k \) to \( M_t \tau \) goes through the module \( P_k/P_k \tau^{l-1} \). This implies that
\[
\Theta_t \in \text{Filt}(\Delta_{t+1}, \ldots, \Delta_n).
\]

Thus, condition 2) in Definition 4.2 also holds, and the algebra \( (\Gamma, \bar{e}) \) is thus quasi-hereditary with \( \Delta_t, t \in T \), as the standard modules.

Finally, we check that \( \Gamma \) is well-formed. Consider the projective \( \Gamma \)-module \( \Pi_t \) with \( t = (j, l) \). The canonical morphism \( P_j/P_j \tau^{l} \rightarrow P_j/P_j \tau^{l-1} \) induces a canonical map \( \pi_t: \Pi_t \rightarrow \Pi_{t'} \), where \( t' = (j, l - 1) \). The calculations above show that the natural map \( \text{Hom}(\Pi_{t'}, \Delta_s) \rightarrow \text{Hom}(\Pi_t, \Delta_s) \) is an isomorphism for any \( \Delta_s \) when \( s > t \). Since \( \Delta_{t+1}, \ldots, \Delta_n \) generate the subcategory \( \mathcal{U}_{t+1} \subset \text{Perf} - \Gamma \), the natural map
\[
\text{Hom}(\Pi_{t'}, -) \rightarrow \text{Hom}(\Pi_t, -)
\]
is an isomorphism on the whole subcategory \( \mathcal{U}_{t+1} \subset \text{Perf} - \Gamma \). Thus, \( \Psi_t \cong \Pi_{t'} \), and the algebra \( (\Gamma, \bar{e}) \) is well-formed. \( \square \)
The algebra $\Gamma = \text{End}_\Lambda(M)$ has finite global dimension, and the bounded derived category of finite $\Gamma$-modules $\text{D}^b(\text{mod} \ - \Gamma)$ is equivalent to the triangulated category of perfect complexes $\text{Perf} \ - \Gamma$. Consider the $\Gamma$-module

$$\Pi = \text{Hom}_\Lambda(M, \Lambda) = \hat{\Lambda}.$$

It is projective and actually isomorphic to $\bigoplus_{j=1}^{m} \Pi_{s_j}$, where $s_j = (j, l_j)$ and $l_j = L(P_j)$ is the Loewy length of $P_j$. Moreover, we have $\text{End}_\Gamma(\Pi) \cong \Lambda$. Thus, the projective $\Gamma$-module $\Pi$ is a $\Lambda$-$\Gamma$-bimodule, and it gives us two functors

$$(-) \otimes^L \Pi: \text{D}(\text{Mod} \ - \Lambda) \longrightarrow \text{D}(\text{Mod} \ - \Gamma)$$

and

$$\text{RHom}_\Gamma(\Pi, -): \text{D}(\text{Mod} \ - \Gamma) \longrightarrow \text{D}(\text{Mod} \ - \Lambda)$$

that are adjoint. The first functor is fully faithful and sends perfect complexes to perfect complexes, while the second functor is a quotient. They induce the following functors:

$$(-) \otimes^L \Pi: \text{Perf} \ - \Lambda \longrightarrow \text{Perf} \ - \Gamma$$

and

$$\text{RHom}_\Gamma(\Pi, -): \text{Perf} \ - \Gamma \longrightarrow \text{D}^b(\text{mod} \ - \Lambda).$$

Thus, the $\Lambda$-$\Gamma$-bimodule $\Pi$ determines a quasi-functor

$$\text{R}: \text{Perf} \ - \Lambda \longrightarrow \text{Perf} \ - \Gamma,$$

that is, we have an ff-morphism of noncommutative schemes

$$r: \mathcal{W} \longrightarrow \mathcal{V}, \quad \text{where } \mathcal{V} = \text{Perf} \ - \Lambda \text{ and } \mathcal{W} = \text{Perf} \ - \Gamma.$$

The morphism $r$ is the simplest but very important example of a smooth resolution of singularities of the noncommutative scheme $\mathcal{V}$ (see Definition 2.11).

**Theorem 5.7.** Let $\Lambda$ be a basic finite-dimensional algebra over a field $k$. Then there exist a smooth projective scheme $X$ and a quasi-functor $\mathcal{F}: \text{Perf} \ - \Lambda \rightarrow \text{Perf} \ - X$ such that the following conditions hold:

1) the induced homotopy functor

$$F = \mathcal{H}^0(\mathcal{F}): \text{Perf} \ - \Lambda \longrightarrow \text{Perf} \ - X$$

is fully faithful;

2) the indecomposable projective modules $P_i$ go to vector bundles $\mathcal{P}_i$ on $X$ under $F$, and the rank of $\mathcal{P}_i$ is equal to $\dim_k P_i$;

3) the scheme $X$ is a tower of projective bundles and has a full exceptional collection.

**Proof.** The quasi-functor $\mathcal{F}$ can be defined as the composition of the quasi-functor

$$\text{R}: \text{Perf} \ - \Lambda \longrightarrow \text{Perf} \ - \Gamma.$$
defined above and the quasi-functor

\[ G : \mathcal{P}erf - \Gamma \longrightarrow \mathcal{P}erf - X \]

constructed in Theorem 5.3. Thus, conditions 1) and 3) hold. Moreover, the indecomposable projective \( \Lambda \)-modules \( P_i \) go to projective \( \Gamma \)-modules under the quasi-functor \( R \), and after that they go under the action of \( G \) to vector bundles which will be denoted by \( P_i \).

Consider the \( \Gamma \)-module \( \Pi \) that, as a \( \Lambda - \Gamma \)-bimodule, defines the quasi-functor \( R \).

It is the direct sum \( \bigoplus_{j=1}^{m} \Pi_{s_j} \), where \( s_j = (j, l_j) \) and \( l_j = L(P_j) \) is the Loewy length of \( P_j \). We have isomorphisms

\[ R(\Lambda) \cong \Pi, \quad \text{and} \quad R(P_i) \cong \Pi_{s_i}. \]

The property (18) implies that \( \text{Hom}_\Gamma(\Pi, \Delta_t) \cong k \) for any \( t \in T \). Since the standard modules go to line bundles under the quasi-functor \( G \), for any \( \Gamma \)-module \( N \in \text{Filt}(\Delta_1, \ldots, \Delta_n) \) the rank of the vector bundle \( G(N) \) is equal to \( \dim_k \text{Hom}_\Gamma(\Pi, N) \). In particular, we obtain the equalities

\[ \text{rk} F(P_i) = \text{rk} G(\Pi_{s_i}) = \dim_k \text{Hom}_\Gamma(\Pi, \Pi_{s_i}) = \dim_k \text{Hom}_\Lambda(\Lambda, P_i) = \dim_k P_i. \]

Thus, condition 2) also holds. \( \Box \)

In the case when the base field \( k \) is algebraically closed, any finite-dimensional algebra is Morita equivalent to a basic finite-dimensional algebra. This means that any finite-dimensional algebra \( A \) is isomorphic to \( \text{End}_\Lambda(\bigoplus_{i=1}^{m} P_i^{\otimes k_i}) \), where \( \Lambda \) is basic and \( P_1, \ldots, P_m \) is the complete set of indecomposable projective \( \Lambda \)-modules. Therefore, we have the following corollary.

**Corollary 5.8.** Let \( k = \overline{k} \) be an algebraically closed field. Then for any finite-dimensional \( k \)-algebra \( A \) there exist a smooth projective scheme \( X \) and a vector bundle \( E \) on \( X \) such that

\[ \text{End}_X(E) \cong A \quad \text{and} \quad \text{Ext}^l_X(E, E) = 0 \quad \text{for all} \ l > 0. \]

Moreover, such an \( X \) can be constructed as a tower of projective bundles, and \( \text{rk} E \leq \dim_k A \).

**Remark 5.9.** There are many reasons to believe that the assertion of Theorem 5.7 also holds for an arbitrary algebra with one refinement: the variety \( X \) can no longer be chosen to be a tower of projective bundles but must be an appropriate twisted form of such a variety, that is, it becomes isomorphic to a tower of projective bundles after an extension of the base field.

**Bibliography**

[1] V. Alexeev and D. Orlov, “Derived categories of Burniat surfaces and exceptional collections”, *Math. Ann.* **357**:2 (2013), 743–759.

[2] M. Auslander, *Representation dimension of Artin algebras*, Queen Mary College Mathematical Notes, Queen Mary College, London 1971, 179 pp.; republished in *Selected works of Maurice Auslander*, Part 1, Amer. Math. Soc., Providence, RI 1999, pp. 505–574.
[3] A. A. Beilinson, J. Bernstein, and P. Deligne, “Faisceaux pervers”, *Analysis and topology on singular spaces*, vol. I (Luminy 1981), Astérisque, vol. 100, Soc. Math. France, Paris 1982, pp. 5–171.

[4] P. Berthelot, A. Grothendieck, and L. Illusie (eds.), *Théorie des intersections et théorème de Riemann–Roch*, Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6), Lecture Notes in Math., vol. 225, Springer-Verlag, Berlin–New York 1971, xii+700 pp.

[5] Ch. Böhning, H.-Ch. Graf von Bothmer, L. Katzarkov, and P. Sosna, “Determinantal Barlow surfaces and phantom categories”, *J. Eur. Math. Soc. (JEMS)* 17:7 (2015), 1569–1592.

[6] Ch. Böhning, H.-Ch. Graf von Bothmer, and P. Sosna, “On the derived category of the classical Godeaux surface”, *Adv. Math.* 243 (2013), 203–231.

[7] А.И. Бондал, “Представления ассоциативных алгебр и когерентные пучки”, *Изв. АН СССР. Сер. матем.* 53:1 (1989), 25–44; English transl., A.I. Bondal, “Representation of associative algebras and coherent sheaves”, *Math. USSR-Izv.* 34:1 (1990), 23–42.

[8] А.И. Бондал, М.М. Капранов, “Представимые функторы, функторы Серра и перестройки”, *Изв. АН СССР. Сер. матем.* 53:6 (1989), 1183–1205; English transl., A.I. Bondal and M.M. Kapranov, “Representable functors, Serre functors, and mutations”, *Math. USSR-Izv.* 35:3 (1990), 519–541.

[9] А.И. Бондал, М.М. Капранов, “Оснащенные триангулированные категории”, *Матем. сб.* 181:5 (1990), 669–683; English transl., A.I. Bondal and M.M. Kapranov, “Enhanced triangulated categories”, *Math. USSR-Sb.* 70:1 (1991), 93–107.

[10] A. Bondal and D. Orlov, *Semiorthogonal decomposition for algebraic varieties*, Preprint MPIM 95/15, 56 pp., https://www.mpim-bonn.mpg.de/node/263; 1995, 55 pp., arXiv:alg-geom/9506012.

[11] A. Bondal and D. Orlov, “Reconstruction of a variety from the derived category and groups of autoequivalences”, *Compositio Math.* 125:3 (2001), 327–344.

[12] A. Bondal and M. Van den Bergh, “Generators and representability of functors in commutative and noncommutative geometry”, *Mosc. Math. J.* 3:1 (2003), 1–36.

[13] E. Cline, B. Parshall, and L. Scott, “Finite dimensional algebras and highest weight categories”, *J. Reine Angew. Math.* 391 (1988), 85–99.

[14] V. Dlab and C. M. Ringel, “Quasi-hereditary algebras”, *Illinois J. Math.* 33:2 (1989), 280–291.

[15] V. Dlab and C. M. Ringel, “Every semiprimary ring is the endomorphism ring of a projective module over a quasi-hereditary ring”, *Proc. Amer. Math. Soc.* 107:1 (1989), 1–5.

[16] V. Drinfeld, “DG quotients of DG categories”, *J. Algebra* 272:2 (2004), 643–691.

[17] D. Dugger and B. Shipley, “K-theory and derived equivalences”, *Duke Math. J.* 124:3 (2004), 587–617.

[18] A. I. Efimov, *Homotopy finiteness of some DG categories from algebraic geometry*, 2017 (v1 – 2013), 65 pp., arXiv:1308.0135.

[19] S. Gorchinskiy and D. Orlov, “Geometric phantom categories”, *Publ. Math. Inst. Hautes Études Sci.* 117 (2013), 329–349.

[20] V. Hinich, “Homological algebra of homotopy algebras”, *Comm. Algebra* 25:10 (1997), 3291–3323.

[21] B. Keller, “Deriving DG categories”, *Ann. Sci. École Norm. Sup.* (4) 27:1 (1994), 63–102.
[22] B. Keller, “On differential graded categories”, *International Congress of Mathematicians*, vol. II, Eur. Math. Soc., Zürich 2006, pp. 151–190.

[23] M. Kontsevich and Y. Soibelman, “Notes on $A_\infty$-algebras, $A_\infty$-categories and non-commutative geometry”, *Homological mirror symmetry*, Lecture Notes in Phys., vol. 757, Springer, Berlin 2009, pp. 153–219.

[24] A. Kuznetsov, “Height of exceptional collections and Hochschild cohomology of quasiphantom categories”, *J. Reine Angew. Math.* **708** (2015), 213–243.

[25] A. Kuznetsov and V. A. Lunts, “Categorical resolutions of irrational singularities”, *Int. Math. Res. Not. IMRN* **2015**:13 (2015), 4536–4625.

[26] A. Kuznetsov, “Height of exceptional collections and Hochschild cohomology of quasiphantom categories”, *J. Reine Angew. Math.* **708** (2015), 213–243.

[27] A. Kuznetsov and V.A. Lunts, “Categorical resolutions of singularities”, *J. Algebra* **323**:10 (2010), 2977–3003.

[28] V.A. Lunts, “Categorical resolution of singularities”, *J. Algebra* **446** (2016), 203–274.

[29] V.A. Lunts, “Categorical resolution of singularities”, *J. Algebra* **323**:10 (2010), 2977–3003.

[30] V.A. Lunts and D.O. Orlov, “Uniqueness of enhancement for triangulated categories”, *J. Amer. Math. Soc.* **23**:3 (2010), 853–908.

[31] A. Neeman, “The Grothendieck duality theorem via Bousfield’s techniques and Brown representability”, *J. Amer. Math. Soc.* **9**:1 (1996), 205–236.

[32] A. Neeman, “Triangulated categories”, *Ann. of Math. Studies*, vol. 148, Princeton Univ. Press, Princeton, NJ 2001, viii+449 pp.

[33] A. Neeman, “Strong generators in $D^b_{coh}(X)$ and applications”, *J. Algebra* **446**(2016), 203–274.

[34] A. Neeman, “Triangulated categories with a single compact generator and a Brown representability theorem”, 2018, 59 pp., arXiv:1804.02240.

[35] Д.О. Орлов, “Проективные расслоения, моноидальные преобразования и производные категории когерентных пучков”, *Изв. РАН. Сер. матем.*, **56**:4 (1992), 852–862; English transl., D. O. Orlov, “Projective bundles, monoidal transformations, and derived categories of coherent sheaves”, *Russian Acad. Sci. Izv. Math.* **41**:1 (1993), 133–141.

[36] D. Orlov, “Equivalences of derived categories and K3 surfaces”, *J. Math. Sci. (N. Y.)* **84**:5 (1997), 1361–1381.

[37] Д. О. Орлов, “Производные категории когерентных пучков на абелевых многообразиях и эквивалентности между ними”, *Изв. РАН. Сер. матем.*, **66**:3 (2002), 131–158; English transl., D. O. Orlov, “Derived categories of coherent sheaves on Abelian varieties and equivalences between them”, *Izv. Math.* **66**:3 (2002), 569–594.

[38] Д. О. Орлов, “Производные категории когерентных пучков и эквивалентности между ними”, *УМН* **58**:3(351) (2003), 89–172; English transl., D. O. Orlov, “Derived categories of coherent sheaves and equivalences between them”, *Russian Math. Surveys* **58**:3 (2003), 511–591.

[39] D. Orlov, “Remarks on generators and dimensions of triangulated categories”, *Mosc. Math. J.* **9**:1 (2009), 153–159.

[40] D. Orlov, “Smooth and proper noncommutative schemes and gluing of DG categories”, *Adv. Math.* **302** (2016), 59–105.

[41] Д. О. Орлов, “Geometric realizations of ringed algebras”, *Современные проблемы математики, механики и математической физики*, Сборник статей, Тр. МИАН, **290**, МАИК “Наука/Интерпериодика”, М. 2015, с. 80–94;
Dmitri O. Orlov

Steklov Mathematical Institute
of Russian Academy of Sciences, Moscow

E-mail: orlov@mi-ras.ru

Received 20/JUL/18
Translated by THE AUTHOR

English transl., D. O. Orlov, “Geometric realizations of quiver algebras”, Proc. Steklov Inst. Math. 290:1 (2015), 70–83.

[42] Д. О. Орлов, “Склейка категорий и партнеры Крулля–Шмидта”, УМН 71:3(429) (2016), 203–204; English transl., D. O. Orlov, “Gluing of categories and Krull–Schmidt partners”, Russian Math. Surveys 71:3 (2016), 594–596.

[43] R. Rouquier, “Dimensions of triangulated categories”, J. K-Theory 1:2 (2008), 193–256.

[44] M. Schlichting, “Negative K-theory of derived categories”, Math. Z. 253:1 (2006), 97–134.

[45] D. Shklyarov, On Serre duality for compact homologically smooth DG algebras, 2007, 13 pp., arXiv: math/0702590.

[46] R. G. Swan, “Hochschild cohomology of quasiprojective schemes”, J. Pure Appl. Algebra 110:1 (1996), 57–80.

[47] G. Tabuada, “Invariants additifs de dg-catégories”, Int. Math. Res. Not. 2005:53 (2005), 3309–3339.

[48] G. Tabuada, Théorie homotopique des DG-catégories, Ph. D. thesis, Univ. Paris Diderot – Paris 7, Paris 2007, 178 pp., arXiv: 0710.4303.

[49] R. W. Thomason and T. Trobaugh, “Higher algebraic K-theory of schemes and of derived categories”, The Grothendieck Festschrift, vol. III, Progr. Math., vol. 88, Birkhäuser Boston, Boston, MA 1990, pp. 247–435.

[50] B. Toën, “The homotopy theory of dg-categories and derived Morita theory”, Invent. Math. 167:3 (2007), 615–667.