Uncertainty Relations in Pre- and Post-Selected Systems

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In this work, we derive Robertson-Heisenberg like uncertainty relation for two incompatible observables in a pre- and post-selected (PPS) system. The newly defined standard deviation and the uncertainty relation in the PPS system have physical meanings which we present here. We demonstrate two unusual properties in the PPS system using our uncertainty relation. First, for commuting observables, the lower bound of the uncertainty relation in the PPS system does not become zero even if the initially prepared state i.e., pre-selection is the eigenstate of both the observables when specific post-selections are considered. This implies that for such case, two commuting observables can disturb each other’s measurement results which is in fully contrast with the Robertson-Heisenberg uncertainty relation. Secondly, unlike the standard quantum system, the PPS system makes it feasible to prepare sharply a quantum state (pre-selection) for non-commuting observables (to be detailed in the main text). Some applications of uncertainty and uncertainty relation in the PPS system are provided: (i) detection of mixedness of an unknown state, (ii) stronger uncertainty relation in the standard quantum system, (iii) “purely quantum uncertainty relation” that is, the uncertainty relation which is not affected (i.e., neither increasing nor decreasing) under the classical mixing of quantum states, (iv) state dependent tighter uncertainty relation in the standard quantum system, and (v) tighter upper bound for the out-of-time-order correlation function.

I. INTRODUCTION

The uncertainty relation, which Heisenberg discovered, is one of the most well-known scientific findings [1, 2]. It asserts that it is impossible to accurately measure the position and the momentum of a particle. In other words, measuring the position of a particle always affects the momentum of that particle and vice versa. Robertson developed the uncertainty relation known as “Robertson-Heisenberg Uncertainty Relation” (RHUR) [3] in the very later years to describe the difficulty of jointly sharp preparation of a quantum state (see Ref. [4] for the notion of sharp preparation) for incompatible observables. This relation not only limits the joint sharp preparation for non-commuting observables but also proved it’s usefulness: to formulate quantum mechanics [5, 6], for entanglement detection [7, 8], for the security analysis of quantum key distribution in quantum cryptography [9], as a fundamental building block for quantum mechanics and quantum gravity [10], etc.

On the one side, we have the standard quantum systems where the RHUR hold while pre- and post-selected (PPS) systems, on the other side, are different kind of quantum mechanical systems that were developed by Aharonov, Bergmann, and Lebowitz (ABL) [11–13] to address the issue of temporal asymmetry in quantum mechanics. Recently, in the references [14, 15], the authors generalized the probabilities of obtaining the measurement results of an observable in a PPS system given by ABL [11].

In the later years, Aharonov, Albert, and Vaidman (AAV) [16] introduced the notion of “weak value” defined as

\[ \langle A_w \rangle_w = \frac{\langle \phi | A | \psi \rangle}{\langle \phi | \psi \rangle}, \]

in a pre- and post-selected system when the observable \( A \) is measured weakly. Here, \( | \psi \rangle \) and \( | \phi \rangle \) are pre- and post-selected states, respectively. Weak values have strange features for being complex and its real part can lie outside the max-min range of the eigenvalues of the operator of interest when the pre- and post-selections are nearly orthogonal.

In order to obtain the real and imaginary parts of the weak value of \( A \) [17, 18], first the system of interest and a pointer (ancilla) is prepared in the product state \( | \psi \rangle \otimes | \xi \rangle \). Then the system-pointer is evolved under the global unitary \( U = \exp(-iHt) \), where \( H = gA \otimes P_x \) is the von Neumann Hamiltonian, \( A \) is the measurement operator of the system, \( P_x \) is the pointer’s momentum observable, ‘\( g \)’ is the coupling coefficient between system and pointer and ‘\( t \)’ is the interaction time. Now after the time evolution of the system-pointer, the system is projected to \( | \phi \rangle \) and as a result, the state of the pointer collapses to the unnormalized state \( | \xi_\phi \rangle \approx | \phi \rangle | \psi \rangle (1 -igt \langle A_w \rangle_w P_x) | \xi \rangle \) in the limit \( g \ll 1 \), i.e., weak interaction. Now, it can be shown that the average position and momentum shifts of the pointer in state \( | \xi_\phi \rangle = \frac{| \xi_\phi \rangle}{\sqrt{\langle \xi_\phi | \xi_\phi \rangle}} \) are \( \langle X \rangle_{\xi_\phi} = g t Re(\langle A_w \rangle_w) \) and \( \langle P_x \rangle_{\xi_\phi} = \frac{gt}{2\pi} Im(\langle A_w \rangle_w) \), respectively with the Gaussian pointer \( | \xi \rangle \approx \left( \frac{1}{\sqrt{2\pi} \sigma} \right)^{1/2} e^{-x^2/4\sigma^2} \), \( \sigma \) is the rms width of the position distribution \( | \langle x | \xi \rangle |^2 \) of the pointer and thus providing the full knowledge of the weak value of \( A \).

Recently, a lot of attention was paid to these aspects [19–40]. The measurements involving weak values are known as “Weak Measurements” or “weak PPS measurements”. Since
it depends on probabilistic post-selection $|\phi\rangle$, a weak value can be thought of as conditional expectation value. Moreover, when the post-selection is same as pre-selection i.e., $|\phi\rangle = |\psi\rangle$, it becomes

$$\langle A \rangle_\psi = \langle \psi | A | \psi \rangle,$$  \hspace{1cm} (2)

the expectation value in the standard quantum system. The PPS systems can therefore be thought of as being more general than the so-called standard quantum systems.

As pre- and post-selected systems are already useful practically as well as fundamentally, then an immediate question can be asked whether there exists any uncertainty relation like the RHUR which can give the limitations on joint sharp preparation of the given pre- and post-selected states when non-commuting observables are measured.

In the present study, we demonstrate the existence of such uncertainty relations in PPS systems, which are expected as PPS systems are more generalised versions of standard ones. We first define the standard deviation of an observable in the PPS system for the given pre- and post-selections with geometrical as well as physical interpretations. After that, we derive our main result of this paper “uncertainty relations in pre- and post-selected systems” using the well known Cauchy-Schwarz inequality.

We provide the following physical applications of our results: (i) detection of the purity of an unknown state of any quantum systems (e.g., qubit, qutrit, two qubit, qutrit-qubit, etc) using two different definitions of the uncertainty of an observable in the PPS system, (ii) stronger uncertainty relation in the standard quantum system (i.e., the uncertainty relation that can not be made trivial or the lower bound can not be made zero for almost all possible choices of initially prepared systems) using the uncertainty relation in the PPS system, (iii) purely quantum uncertainty relations that is, the uncertainty relations which are not affected (i.e., neither increasing nor decreasing) under the classical mixing of quantum states using the uncertainty relations in PPS systems. (iv) state dependent tighter uncertainty relation in the standard system by introducing the idea of post-selection, and finally (v) tighter upper bound for the out-of-time-order correlation function. Moreover, as the RHUR has a plenty of applications, uncertainty relation in the PPS systems can also be applied in quantum optics, information, technologies, etc.

This paper is organized as follows. In Sec. II, we discuss uncertainty relations in standard quantum systems. In Sec. III, we derive our main results of this paper. Application of our results are given in Sec. IV and finally we conclude our work in Sec. V.

II. UNCERTAINTY RELATION IN STANDARD QUANTUM SYSTEM

In this section, we first interpret the standard deviation of an observable in standard quantum systems from geometrical as well as information-theoretic perspective. For establishing the standard deviation in a PPS system, we will introduce a similar interpretation. The RHUR’s well-known interpretation is also provided here.

A. Standard deviation

We consider the system Hilbert space to be $\mathcal{H}$ and let $|\psi\rangle$ be a state vector in $\mathcal{H}$. Due to the probabilistic nature of the measurement outcomes of the observable $A$, the uncertainty in the measurement is defined as the standard deviation:

$$\langle \Delta A \rangle_\psi = \sqrt{\langle \psi | A^2 | \psi \rangle - \langle \psi | A | \psi \rangle^2}. \hspace{1cm} (3)$$

Geometric interpretation.— Standard deviation can be given a geometrical interpretation using the following proposition.

**Proposition 1.** If $|\psi\rangle \in \mathcal{H}$ is an initially prepared state of a standard quantum system and $A \in \mathcal{L}(\mathcal{H})$ is a hermitian operator, then we can decompose $A |\psi\rangle \in \mathcal{H}$ as

$$A |\psi\rangle = \langle A \rangle_\psi |\psi\rangle + \langle \Delta A \rangle_\psi |\psi_A\rangle,$$  \hspace{1cm} (4)

where $|\psi_A\rangle = \frac{1}{\langle \Delta A \rangle_\psi} (A - \langle A \rangle_\psi) |\psi\rangle$, and $\langle A \rangle_\psi = \langle \psi | A | \psi \rangle$.

Eq. (4) is sometimes known as the “Aharonov–Vaidman identity” [41].

**Proof.** Let $A |\psi\rangle$ and $|\psi\rangle$ are two non-orthogonal state vectors. Using Gram-Schmidt orthogonalization process, we find the unnormalized state vector $|\psi_A\rangle \in \mathcal{H}$ orthogonal to $|\psi\rangle$ as

$$|\psi_A\rangle = A |\psi\rangle - \frac{\langle \psi | A | \psi \rangle}{\langle \psi | \psi \rangle} |\psi\rangle = (A-\langle A \rangle_\psi) |\psi\rangle,$$  \hspace{1cm} (5)

where $\langle \psi | \psi \rangle = 1$ and after normalization, Eq. (5) becomes

$$A |\psi\rangle = \langle A \rangle_\psi |\psi\rangle + \langle \Delta A \rangle_\psi |\psi_A\rangle,$$  \hspace{1cm} (6)

where $|\psi_A\rangle = \frac{|\psi_A\rangle}{\sqrt{\langle \psi_A | \psi_A \rangle}}$ and $\sqrt{\langle \psi_A | \psi_A \rangle} = \sqrt{\langle \psi | A^2 | \psi \rangle - \langle \psi | A | \psi \rangle^2} = \langle \Delta A \rangle_\psi$ and $\langle A \rangle_\psi = \langle \psi | A | \psi \rangle$. 

So, Eq. (4) can be interpreted as the unnormalized state vector $A |\psi\rangle$ which has two components and these are $\langle A \rangle_\psi$ along $|\psi\rangle$ and $\langle \Delta A \rangle_\psi$ along $|\psi_A\rangle$. Here we interpret $\langle \Delta A \rangle_\psi$ as disturbance of the state vector due to the measurement of the operator $A$ or as the measurement error (or standard deviation) of that operator when the system is prepared in the state $|\psi\rangle$. For instance, if we set up the system in one of the eigenstates of the observable $A$, then from Eq. (4), it can be seen that the standard deviation of $A$ is zero.

Information-theoretic interpretation.— From an information-theoretic approach, Eq. (3) can be written as

$$\langle \Delta A \rangle_\psi = \sqrt{\sum_{i=1}^{d-1} |\psi_i^+| |A| \psi_i^+|^2}, \hspace{1cm} (7)$$

where $d$ is the dimension of the system and $\{|\psi_i\rangle\}_{i=1}^{d}$ is the eigenbasis of the observable $A$. This equation arises due to the probabilistic nature of the measurement outcomes of the observable $A$.
Geometric definition.— It is well known that when the pre-selection and the post-selection are same, the PPS system becomes the standard quantum system (see introduction). The following proposition generalizes Eq. (4) for the PPS system.

Proposition 2. If a PPS system is in a pre-selected state $|\psi\rangle$ and post-selected state $|\phi\rangle$, then for a hermitian operator $A \in \mathcal{L}(\mathcal{H})$, we can decompose $A|\psi\rangle$ as

$$A|\psi\rangle = \langle \phi | A | \psi \rangle |\phi\rangle + \langle \Delta A \rangle^\circ_{\psi} |\phi^\perp_{A\psi}\rangle,$$

where

$$\langle \Delta A \rangle^\circ_{\psi} = \sqrt{\langle \psi | A^2 | \psi \rangle - |\langle \phi | A | \psi \rangle|^2}$$

and

$$|\phi^\perp_{A\psi}\rangle = \frac{1}{\langle \Delta A \rangle^\circ_{\psi}} (A |\psi\rangle - \langle \phi | A | \psi \rangle |\phi\rangle),$$

a normalized state vector which is orthogonal to $|\phi\rangle$.

Proof. We assume that $A |\psi\rangle$ and $|\phi\rangle$ are two non-orthogonal state vectors. The unnormalized state vector $|\phi^\perp_{A\psi}\rangle \in \mathcal{H}$ which is orthogonal to $|\phi\rangle$ is obtained using Gram-Schmidt orthonormalization process as

$$|\phi^\perp_{A\psi}\rangle = A |\psi\rangle - \frac{\langle \phi | A | \psi \rangle}{\langle \phi | \phi \rangle} |\phi\rangle = A |\psi\rangle - \langle \phi | A | \psi \rangle |\phi\rangle,$$

where $\langle \phi | \phi \rangle = 1$ and after normalization, Eq. (12) becomes

$$A |\psi\rangle = \langle \phi | A | \psi \rangle |\phi\rangle + \langle \Delta A \rangle^\circ_{\psi} |\phi^\perp_{A\psi}\rangle,$$

where

$$|\phi^\perp_{A\psi}\rangle = |\phi^\perp_{A\psi}/\langle \phi^\perp_{A\psi} | \phi^\perp_{A\psi} \rangle|^2$$

and

$$\langle \phi^\perp_{A\psi} | \phi^\perp_{A\psi} \rangle = \sqrt{\langle \psi | A^2 | \psi \rangle - |\langle \phi | A | \psi \rangle|^2} = \langle \Delta A \rangle^\circ_{\psi}.$$

To define the standard deviation of the observable $A$ in the PPS system, we now present an argument which is similar to the one used to describe the standard deviation of an observable in a standard quantum system. So, Eq. (10) can be interpreted geometrically as the unnormalized state vector $A |\psi\rangle$ which has two components $\langle \phi | A | \psi \rangle$ along the post-selection $|\phi\rangle$ and $\langle \Delta A \rangle^\circ_{\psi}$ along $|\phi^\perp_{A\psi}\rangle$. Here we define $\langle \Delta A \rangle^\circ_{\psi}$ as the standard deviation of the observable $A$ when the system is pre-selected in $|\psi\rangle$ and post-selected in $|\phi\rangle$.

The standard deviation $\langle \Delta A \rangle^\circ_{\psi}$ can be realized via the weak value of the observable $A$ as

$$\langle \Delta A \rangle^\circ_{\psi} = \sqrt{\langle \psi | A^2 | \psi \rangle - |\langle A_{\psi} \rangle|^2} |\langle \phi | \psi \rangle|^2$$

$$= \sqrt{\langle \Delta A \rangle^2_{\psi} + \langle A \rangle^2_{\psi} - |\langle A_{\psi} \rangle|^2 |\langle \phi | \psi \rangle|^2},$$

where $\langle A_{\psi} \rangle$ is the weak value of the observable $A$ defined in Eq. (1) and we have used Eq. (3) to derive Eq. (14). $|\langle \phi | \psi \rangle|^2$ is the success probability of the post-selection $|\phi\rangle$. Eq. (13) is no longer a valid expression if pre- and post-selected states are orthogonal to one another because in this situation, weak value is not defined. Then, go back to Eq. (11). It should be noted that Eq. (11) holds true whether the measurement is strong or weak.

Information-theoretic definition.— Another expression of the standard deviation $\langle \Delta A \rangle^\circ_{\psi}$ in the PPS system can be derived by inserting an identity operator $I = |\phi\rangle \langle \phi | + \sum_{i=1}^{d-1} |\phi^i\rangle \langle \phi^i |$, where

$$\{|\phi\rangle, |\phi^1\rangle, |\phi^2\rangle, \ldots, |\phi^{d-1}\rangle\}$$

forms an orthonormal basis,
in the first term of the right hand side of Eq. (13) as

\[ \langle \Delta A \rangle_\psi^2 = \sqrt{\sum_{i=1}^{d-1} (A_{w_i} \phi_i^{d-1})^2} / \langle |\phi_i^d| \psi \rangle^2. \]  

(15)

From an information-theoretic perspective, Eq. (15) may now be understood as follows: non-zero standard deviation in the PPS system arises as a result of the non-zero contributions from the weak values \( \{A_{w_i} \}^{d-1}_{i=1} \) along the orthogonal post-selections \( \{\phi_i^d\}^{d-1}_{i=1} \). Note that two consecutive measurements are taken into account in a PPS system: the operator of interest \( A \) and the projection operator \( \Pi_\phi = \{|\phi\rangle \langle \phi|\} \) which corresponds to the post-selection \( |\phi\rangle \). As a result, it is hard to tell whether or not \( A \) has been measured when the weak value is zero. Because of this, it is crucial to have non-zero weak values which carry the information about the observable \( A \). Null weak values have recently been given a useful interpretation [43]: if a successful post-selection occurs with a null weak value, then the property represented by the observable \( A \) cannot be detected by the weakly coupled quantum pointer. In other words, the pointer state remains unchanged when the weak value is zero (see introduction section). Thus, one should anticipate that the standard deviation in the PPS system should be zero if we obtain null weak values for the post-selections \( \{\phi_i^d\}^{d-1}_{i=1} \), that means the information about the observable \( A \) is not dispersed throughout the post-selections \( \{\phi_i^d\}^{d-1}_{i=1} \).

In addition to the standard deviation’s geometrical and information-theoretical explanations (Eqs. (10) and (15), respectively) in the PPS system, we now study the minimum (zero) and maximum uncertainty (or standard deviation) which provide additional insights to understand the standard deviation.

Zero uncertainty.— The uncertainty \( \langle \Delta A \rangle_\psi^2 \) defined in Eq. (11) in the PPS system is zero if and only if

\[ A |\psi\rangle = \phi_z |A|\psi\rangle \phi_z, \]  

(16)

or \( |\phi_z\rangle \propto A |\psi\rangle \). We have used the notation \( |\phi_z\rangle \) as the post-selection for which uncertainty in PPS system becomes zero. The zero uncertainty in the PPS system can now be realised in the following way: the weak value \( A_{w_i} \phi_i^{d-1} \) becomes non-zero i.e., \( \langle \psi | A^2 |\psi\rangle / \langle \psi | A |\psi\rangle \neq 0 \) when we post-select the system to \( |\phi_z\rangle \), and the weak values for all post-selections \( \{\phi_i^d\}^{d-1}_{i=1} \) orthogonal to \( |\phi_z\rangle \) are zero. As a result, the right side of Eq. (15) is reduced to zero. It should be noted that all post-selections orthogonal to \( |\phi_z\rangle \) are “legitimate post-selections,” meaning that their weak values are clearly specified. Equivalently, we can state that the information about the observable \( A \) is not dispersed along the post-selections \( \{\phi_i^d\}^{d-1}_{i=1} \) as null weak values do not carry informations about the observable \( A \) (according to the above information-theoretic definition). Hence, it is guaranteed that in a particular direction there will be one and only one non-zero weak value of \( A \) in a PPS system if and only if the condition (16) is met.

Usefulness of zero uncertainty state: In this paragraph we provide the following usefulness of the zero uncertainty post-selected state \( |\phi_z\rangle \).

1) In a parameter estimation scenario, where the task is to obtain the precision limit in the estimation of interaction coefficient ‘\( g \)’ in the interaction Hamiltonian \( H = gA \otimes p \) (‘\( p \)’ is the pointer’s momentum variable), Fisher information plays an important role whose maximum value is given by \( F_{\text{max}}(g) = 4\sigma^2 |\langle A |\psi\rangle|^2 \), where \( \sigma \) is the standard deviation of initial distribution of the pointer state and \( |\psi\rangle \) is the initially prepared state of the system [44]. In an arbitrarily post-selected state \( |\phi\rangle \), Fisher information is given by \( F_\phi(g) = 4\sigma^2 |\langle A |A\psi\rangle|^2 \leq F_{\text{max}}(g) \) [44]. Recently, it was shown in the Ref. [45] that the Fisher information can be expressed in terms of quasiprobability distribution of an arbitrary pre-selected state when the system is a PPS system. Although the quasiprobabilities are in general complex and can take negative values as well, the Fisher information is always positive and real. The Fisher information of the pre-selected state with negative quasiprobability distribution can surpass the usual quantum Fisher information (defined in standard quantum system). Violation of such limit implies that the error which occurs in estimating the unknown parameter can be reduced significantly using the Fisher information of the pre-selected state with negative quasiprobability distribution compared to the usual quantum Fisher information. One can immediately see using Eq. (11) that \( F_\phi(g) = F_{\text{max}}(g) \), hence, to achieve the maximum Fisher information \( F_{\text{max}}(g) \) in the PPS system, one must post-select the system in \( |\phi_z\rangle = A |\psi\rangle / \sqrt{|\langle A |A\psi\rangle|^2} \) which corresponds to the zero uncertainty.

2) The post-selection \( |\phi_z\rangle \) alone has the ability to provide the information (e.g., \( \langle \Delta A \rangle_\psi \) and \( \langle A \rangle_\psi \)) about the observable \( A \). Indeed by noting that \( \langle \psi | A^2 |\psi\rangle = p_z (\langle A_{w_i} \phi_i^{d-1} \rangle)^2 \) and \( \langle \psi | A |\psi\rangle = p_z (\langle A_{w_i} \phi_i^{d-1} \rangle)^2 \), we have \( \langle \Delta A \rangle_\psi^2 = (1 - p_z) (\langle A_{w_i} \phi_i^{d-1} \rangle)^2 \), where \( p_z = |\langle \phi_z |A\psi\rangle|^2 \) is the probability of obtaining the post-selection \( |\phi_z\rangle = A |\psi\rangle / \sqrt{|\langle A |A\psi\rangle|^2} \).

Maximum uncertainty.— To achieve the maximum value of \( \langle \Delta A \rangle_\psi^2 \), the weak value \( \langle A_{w_i} \phi_i^{d-1} \rangle \) in Eq. (13) has to be zero i.e., when the post-selection \( |\phi\rangle \) is orthogonal to \( A |\psi\rangle \) and hence max(\( \langle \Delta A \rangle_\psi^2 \)) = \( \sqrt{|\langle A |A\psi\rangle|^2} \). Note that, in a preparation-measurement scenario, maximum measurement error is also found to be \( \sqrt{|\langle A |A\psi\rangle|^2} \) whether the measurement of the observable \( A \) is performed in standard system (see Eq. (3)) or while performing the best estimation the operator \( A \) from the measurement of another hermitian operator [46].

B. Uncertainty relation in PPS system

After defining the standard deviation of an observable in a PPS system, interpreting it geometrically and informationally, and maintaining a parallel comparison and connection with the standard deviation in the standard system, we are now in a position to provide an uncertainty relation in a PPS
system for two incompatible observables. One can formulate many different types of uncertainty relations in PPS systems (for example, [47]), but our interpretation of an uncertainty relation in a PPS system is based on the standard deviation defined in Eq. (11) or (13). Since the weak value of the observable $A$ in the standard deviation Eq. (13) in the PPS system replaces the average value of the same observable $A$ in the standard deviation Eq. (3) in standard quantum system, it is not surprising that the mathematical expression of the uncertainty relation in the PPS system is a modified version of the RHUR (8), where the average values of the incompatible observables $A$ and $B$ in Eq. (8) will be replaced by the weak values of the respective observables when the system is pre-selected in $|\psi\rangle$ and post-selected in $|\phi\rangle$. The explicit form of the uncertainty relation in the PPS system is provided in the following theorem.

**Theorem 1.** Let $A, B \in \mathcal{L}(\mathcal{H})$ be two non-commuting hermitian operators which are measured in the PPS system of our interest with $|\psi\rangle$ and $|\phi\rangle$ being pre- and post-selected states, respectively, then the product of their standard deviations satisfies

$$\langle (\Delta A)^2 \rangle \geq \left[ \frac{1}{2i} \langle [A, B]|\psi\rangle - i\mathcal{M}(W_{AB}) \right]^2,$$

(17)

where $W_{AB} = \langle A|\phi\rangle\langle\phi|B\rangle = \left(\langle A\rangle\phi^*\langle B\rangle\phi^*\right)^2$ (using the definition of the weak value defined in Eq. (1)).

**Proof.** Cauchy-Schwarz inequality for two unnormalized state vectors $|\tilde{\phi}_{A\psi}\rangle$ and $|\tilde{\phi}_{B\psi}\rangle$ in $\mathcal{H}$ becomes

$$\langle \tilde{\phi}_{A\psi}|\tilde{\phi}_{B\psi}\rangle \geq \left| \langle \tilde{\phi}_{A\psi}|\tilde{\phi}_{B\psi}\rangle \right|^2,$$

(18)

Now, as

$$[\mathcal{M}(\tilde{\phi}_{A\psi}|\tilde{\phi}_{B\psi})]^2 = [\mathcal{R}e(\tilde{\phi}_{A\psi}|\tilde{\phi}_{B\psi})]^2 + [\mathcal{I}m(\tilde{\phi}_{A\psi}|\tilde{\phi}_{B\psi})]^2$$

and hence

$$\left| \langle \tilde{\phi}_{A\psi}|\tilde{\phi}_{B\psi}\rangle \right|^2 \geq [\mathcal{M}(\tilde{\phi}_{A\psi}|\tilde{\phi}_{B\psi})]^2,$$

(19)

where $\mathcal{M}(\tilde{\phi}_{A\psi}|\tilde{\phi}_{B\psi}) = \frac{1}{2i} \left( \tilde{\phi}_{A\psi}^\dagger \tilde{\phi}_{B\psi}^\dagger - \tilde{\phi}_{A\psi}^\dagger \tilde{\phi}_{B\psi} \right)$ and $\mathcal{R}e(\tilde{\phi}_{A\psi}|\tilde{\phi}_{B\psi}) = \frac{1}{2} (\tilde{\phi}_{A\psi}^\dagger \tilde{\phi}_{B\psi}^\dagger + \tilde{\phi}_{A\psi}^\dagger \tilde{\phi}_{B\psi})$. Now put $|\tilde{\phi}_{A\psi}\rangle = A|\psi\rangle - |\phi\rangle A|\psi\rangle$ defined in Eq. (12) for operator $A$ and similarly for operator $B$ also, then we have

$$\langle \tilde{\phi}_{A\psi}|\tilde{\phi}_{B\psi}\rangle = \langle A|B\rangle|\psi\rangle - \langle A|\psi\rangle |\phi B\rangle.$$

(20)

Note that $\left| \langle \tilde{\phi}_{A\psi}|\tilde{\phi}_{B\psi}\rangle \right|^2$ is square of the standard deviation of the observable $A$ in the PPS system defined in Eq. (11) and similarly $\left| \langle \tilde{\phi}_{B\psi}|\tilde{\phi}_{A\psi}\rangle \right|^2$ is square of the standard deviation of the observable $B$ in the PPS system. Finally, putting these values and using Eqs. (19) and (20) in Eq. (18), it becomes Eq. (17). \hfill \Box

Eq. (17) is always true for any strong PPS systems [11–13] or weak PPS systems [16]. For weak PPS measurements [16], $W_{AB}$ is expressed in terms of weak values of both the observables. If the pre- and post-selected states are same i.e., $|\phi\rangle = |\psi\rangle$, then one gets back the RHUR (8) as argued before. Eq. (17) with “Schrödinger’s term” becomes

$$\left( \langle \Delta A \rangle \right)^2 \left( \langle \Delta B \rangle \right)^2 \geq \left[ \frac{1}{2i} \langle [A, B] |\psi\rangle - i\mathcal{M}(W_{AB}) \right]^2$$

$$+ \left[ \frac{1}{2} \langle [A, B] |\psi\rangle - i\mathcal{R}e(W_{AB}) \right]^2.$$

(21)

The uncertainty relation (17) can be interpreted in the same way as we did for the RHUR (8). That is, it bounds the sharp preparation of the pair for pre- and post-selections ($|\psi\rangle, |\phi\rangle$) for two non-commuting observables. The lower bound contains an additional term $i\mathcal{M}(W_{AB})$ compared to the RHUR (8). So even if $[A, B] \neq 0$, the bound on the right hand side of Eq. (17) can become zero implying the possibility of both the standard deviations being zero implying further the possibility of sharp preparation of a pair of pre- and post-selected states. Below, we provide the necessary and sufficient condition for such case (see Observation 2). Recently, the authors of the references [48, 49] confirmed that in a PPS system using the ABL- rule [11–13], it is possible to go beyond the standard lower bound in the RHUR for position and momentum observables. Not exactly, but a similar property i.e., achieving arbitrary small lower bound (which depends on the pre- and post-selections) of the product of standard deviations of two non-commuting observables in a PPS system is possible in the relations (17). We now explore two peculiar characteristics of the uncertainty relations (17) and (21) that cannot be observed in standard quantum systems.

**Observation 1.** If the lower bound in an uncertainty relation in any quantum system is non-zero, then we say that two incompatible observables disturb each others’ measurement results. Now consider the following case. If $|\psi\rangle$ is a common eigenstate of both $A$ and $B$, then $\langle \Delta A \rangle = 0$, $\langle \Delta B \rangle = 0$ implying that the measurement of one doesn’t disturbs the outcome of the other. Surprisingly, this property doesn’t hold in the PPS system. Note that, even if $|\psi\rangle$ is a common eigenstate of both $A$ and $B$, the lower bound of the relation (21) doesn’t become zero for specific post-selections which implies $\langle \Delta A \rangle \neq 0$, $\langle \Delta B \rangle \neq 0$. Hence we can say that the measurement of $A$ is invariably disturbed by the measurement of $B$ of vice versa in a PPS system. In the Ref. [50], Vaidman demonstrated the same property in a PPS system using the ABL- rule.

**Observation 2.** With two non-commuting observables in the standard quantum system, sharp preparation of a quantum state is impossible. Or equivalently, for an initially prepared state $|\psi\rangle$, it is impossible to have $\langle \Delta A \rangle = 0$, $\langle \Delta B \rangle = 0$ if $[A, B] \neq 0$. But in the PPS system, we can prepare any quantum state $|\psi\rangle$ which can give $\langle \Delta A \rangle = 0$, $\langle \Delta B \rangle = 0$ for a specific choice of post-selection implying sharp preparation of $|\psi\rangle$ for non-commuting observables $A$ and $B$. It is easy to
show that both the uncertainties $\langle \Delta A \rangle_\psi^\phi$ and $\langle \Delta B \rangle_\psi^\phi$ are zero for the common post-selection $|\phi_2\rangle$ if and only if

$$|\phi_2\rangle \propto A|\psi\rangle, \quad |\phi_2\rangle \propto B|\psi\rangle.$$  

After the normalization, we find the common post-selection condition

$$|\phi_2\rangle = \frac{A|\psi\rangle}{\sqrt{\langle \psi|A^2|\psi\rangle}} = \frac{B|\psi\rangle}{\sqrt{\langle \psi|B^2|\psi\rangle}}, \quad (22)$$

upto some phase factors.

\textit{Example.} Now, consider an example of two non-commuting observables $A = \frac{1}{\sqrt{2}}(I + \sigma_x)$ and $B = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_y)$ with the initially prepared state $|0\rangle$. With these specific choices, it is possible to show that condition (22) is satisfied. Recall that in order to conduct the experiment using weak values, the average values of the observables must not be zero; for this reason, we did not take into account the Pauli observables $\sigma_x$ and $\sigma_y$ with initially prepared state $|0\rangle$. Nonetheless, if one does not adhere to weak values, this example is still true. So, the common post-selection for this case is $|\phi_2\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and hence both the uncertainties $\langle \Delta A \rangle_0^\phi$ and $\langle \Delta B \rangle_0^\phi$ of the non-commuting observables $A$ and $B$, respectively are zero for the given initially prepared state $|0\rangle$ and the conditioned post-selection $|\phi_2\rangle$ in Eq. (22).

In the PPS system, it is now feasible to do the hitherto impossibly difficult task of jointly sharply preparing a quantum state for two non-commuting observables.

The aforementioned Observations 1 and 2 demonstrate that PPS systems are capable of being even stranger than their well-known unusual results e.g., quantum Cheshire Cats [24], measurement of a component of a spin 1/2 particle which can reach 100h [16], etc.

\textit{Comments.}— The characteristics of the uncertainty relations (17) and (21) in PPS systems as compared to the RHUR (8) and Eq. (9) are substantially altered by the post-selections. These uncertainty inequalities (17) and (21) will undoubtedly have applications like the RHUR for quantum foundations, information and technologies. For instances, (i) they can be used for information extraction using commuting observables because the inequalities do not become trivial for particular choices of post-selections, (ii) one can obtain a series of uncertainty inequalities by changing the post-selections and that is advantageous for practical purposes (see stronger uncertainty relations in Sec. IV), (iii) existing applications of uncertainty relations (8) and (9) in standard systems, such as entanglement detection [7], quantum metrology [51, 52], etc., can be revisited using uncertainty relations (17) and (21) in the PPS systems, (iv) PPS system based spin squeezing: spin-squeezed states are a class of states having squeezed spin variance along a certain direction, at the cost of anti-squeezed variance along an orthogonal direction. This is done by using the RHUR (8) in the standard quantum system [53–56]. Such analysis can be reintroduced in the light of PPS systems. As there is no unique definition of spin squeezing in the standard quantum systems, it is, by means of Eq. (17), also possible to define the spin squeezing non uniquely in the PPS system.

A very careful analysis is required to see whether there exists some states in the PPS systems for which $\langle \Delta A \rangle_\psi^\phi = \langle \Delta B \rangle_\psi^\phi$ and inequality (17) is saturated similar to coherent spin states in the standard quantum systems.

1. \textit{Intelligent pre- and post-selected states}

In the standard quantum system, the states for which the equality condition holds in the RHUR (8) are known as intelligent states or minimum-uncertainty states [57–59]. Minimum uncertainty states have been proposed to improve the accuracy of phase measurement in quantum interferometer [60]. Minimum-uncertainty states in the PPS systems can also be defined based on the following condition.

One can find the condition for which the inequality (17) saturates (see Appendix A) is given by

$$A|\psi\rangle - \langle \phi|A|\psi\rangle|\phi\rangle = \pm i \frac{\langle \Delta A \rangle_\psi^\phi}{\langle \Delta B \rangle_\psi^\phi} (B|\psi\rangle - \langle \phi|B|\psi\rangle|\phi\rangle). \quad (23)$$

If the sign of ‘i’ on the RHS of Eq. (23) is taken to be positive (negative) when the observable $A$ appears on the LHS of the Eq. (23), then the sign of ‘i’ on the RHS of Eq. (23) is taken to be negative (positive) when the observable $B$ appears on the LHS of the Eq. (23). So, the pre- and post-selected states which satisfy the condition (23), can be referred as the “intelligent pre- and post-selected states”. For the given pre-selection and observables in Eq. (23), one can find the post-selection which will make the Eq. (17) the most tight i.e., equality.

C. Uncertainty equality in PPS system

Recently, in the reference [61], the authors have shown that there exist variance-based uncertainty equalities from which a series of uncertainty inequalities with hierarchical structure can be obtained. It was shown that stronger uncertainty relation given by Maccone and Pati [62] is a special case of these uncertainty inequalities. Here we show such uncertainty equalities in the PPS systems. We provide interpretation of the uncertainty inequalities derived from the uncertainty equalities. Further, in application section IV, we use uncertainty equalities in PPS systems to obtain stronger uncertainty relations and state dependent tighter uncertainty relations.

\textbf{Theorem 2.} The product of standard deviations of two non-commuting hermitian operators $A, B \in \mathcal{L(H)}$ in a PPS system with pre- and post-selected states $|\psi\rangle$ and $|\phi\rangle$, respectively satisfies

$$\langle \Delta A \rangle_\psi^\phi \langle \Delta B \rangle_\psi^\phi = \pm \left( \frac{1}{2^d} \langle \psi|[A,B]|\psi\rangle - \text{Im}(W_{AB}) \right) \frac{-1 - \frac{1}{2} \sum_{k=1}^{d-1} \left| \langle \psi| \langle A \rangle_\psi^\phi + i \frac{B}{\langle \Delta B \rangle_\psi^\phi} |\phi_k^+\rangle \right|^2}{1 - \frac{1}{2} \sum_{k=1}^{d-1} \left| \langle \psi| \langle A \rangle_\psi^\phi + i \frac{B}{\langle \Delta B \rangle_\psi^\phi} |\phi_k^+\rangle \right|^2},$$  

(24)

where we have assumed that $\langle \Delta A \rangle_\psi^\phi$ and $\langle \Delta B \rangle_\psi^\phi$ are non-zero, and the sign should be considered such that the nu-
merator is always real and positive. Here \( \{ |\phi\rangle, |\phi_k^\pm\rangle_{k=1}^{d-1} \} \) is an complete orthonormal basis in the \( d \)-dimensional Hilbert space.

Proof. Consider an orthonormal complete basis \( \{ |\phi\rangle, |\phi_k^\pm\rangle_{k=1}^{d-1} \} \) in the \( d \)-dimensional Hilbert space \( \mathcal{H} \). Now, define the projection operator \( \Pi = I - |\phi\rangle \langle \phi | \) and the unnormalized state vector \( |\xi^\pm\rangle = \left( \frac{A_{\psi}}{(A_{\psi})^2} \pm i \frac{B_{\psi}}{(B_{\psi})^2} \right) |\psi\rangle \).

Then we have the following identity
\[
\langle \xi^\pm | \Pi | \xi^\pm \rangle = \langle \xi^\pm | \xi^\pm \rangle - \langle \xi^\pm | \phi \rangle \langle \phi | \xi^\pm \rangle
\]
\[
= \left( \frac{|\langle \phi | A \rangle|^2}{(A_{\psi})^2} + \frac{|\langle \phi | B \rangle|^2}{(B_{\psi})^2} \right) \pm \frac{2i m(W_{AB})}{(A_{\psi})^2 - m(W_{AB})^2} + \sum_{k=1}^{d-1} \left| \langle \phi_k^\pm | (A \mp i B)|\phi\rangle \right|^2.
\]

where we have used Eq. (11) and \( W_{AB} = \langle \psi | A |\phi\rangle \langle \phi | B |\psi\rangle \).

Now, we use another expression of \( \Pi = \sum_{k=1}^{d-1} |\phi_k^\pm\rangle \langle \phi_k^\pm | \) to calculate the same identity
\[
\langle \xi^\pm | \Pi | \xi^\pm \rangle = \sum_{k=1}^{d-1} \left| \langle \psi | A |\phi_k^\pm\rangle \pm i \frac{B}{(B_{\psi})^2} \langle \phi_k^\pm | (A \mp i B)|\psi\rangle \right|^2.
\]

So, from the Eqs. (25) and (26), we obtain the uncertainty equality (24) in the PPS system.

Theorem 3. The sum of the variances of two non-commuting hermitian operators \( A, B \in \mathcal{L}(\mathcal{H}) \) in a PPS system with pre- and post-selected states \( |\psi\rangle \) and \( |\phi\rangle \), respectively satisfies
\[
\left( \langle \Delta A \rangle_{\psi}^\phi \right)^2 + \left( \langle \Delta B \rangle_{\psi}^\phi \right)^2 = \pm \left( \frac{i}{2} \left( \langle A | B |\psi\rangle - \langle B | A |\psi\rangle \right) - 2i m(W_{AB}) \right)
\]
\[
+ \sum_{k=1}^{d-1} \left| \langle \phi_k^\pm | (A \mp i B)|\psi\rangle \right|^2.
\]

Here, the ‘±’ sign is taken suitably such that the first term in right side is always positive.

Proof. Consider an orthonormal complete basis \( \{ |\phi\rangle, |\phi_k^\pm\rangle_{k=1}^{d-1} \} \) in the \( d \)-dimensional Hilbert space \( \mathcal{H} \) and hence \( I - |\phi\rangle \langle \phi | = \sum_{k=1}^{d-1} |\phi_k^\pm\rangle \langle \phi_k^\pm | \). By equating the following two
\[
Tr \left( (A \mp i B) | \psi\rangle \langle A \pm i B | (I - |\phi\rangle \langle \phi |) \right),
\]
\[
Tr \left( (A \mp i B) | \psi\rangle \langle A \pm i B | \sum_{k=1}^{d-1} |\phi_k^\pm\rangle \langle \phi_k^\pm | \right),
\]
we have Eq. (27).

An inequality can be obtained by discarding some of the terms in the summation corresponding to ‘\( k \)’ or all the terms except one term in Eq. (24) or Eq. (27). It is also possible to obtain an arbitrarily tight inequality by discarding the minimum valued term inside the summation in the denominator of Eq. (24) for a particular value of ‘\( k \)’. Note that we have to optimize the minimum \( \left| \langle \psi | A |\phi_k^\pm\rangle \pm i \frac{B}{(B_{\psi})^2} \langle \phi_k^\pm | (A \mp i B)|\psi\rangle \right|^2 \) over all possible choice of basis \( \{ |\phi_k^\pm\rangle_{k=1}^{d-1} \} \) in the subspace orthogonal to \( |\phi\rangle \).

In an extreme case, let’s assume that a few post-selected states from \( \{ |\phi_k^\pm\rangle_{k=1}^{d-1} \} \) are not detected by the detector because of certain technical difficulties. Using such imprecise experimental data, one may still be able to obtain an uncertainty relation. In that case, the terms corresponding to the unregistered post-selections in Eq. (24) or Eq. (27) are to be eliminated.

D. Uncertainty relation for mixed pre-selection in PPS system

So far, we have only considered the pre-selected state to be pure in a PPS system. Let us now generalize the definition of the standard deviation and derive the uncertainty relations for mixed pre-selected state in the PPS system. A direct generalization of the standard deviation defined in Eq. (11) is given by
\[
\langle \Delta A \rangle_{\rho}^\phi = \sqrt{Tr(A^2 \rho) - \langle \phi | A \rho A |\phi\rangle}.
\]

See Ref. [63] for the motivation for calling \( \langle \Delta A \rangle_{\rho}^\phi \) as a standard deviation when the pre-selection is a mixed state. If \( \rho = \sum_i p_i |\psi_i\rangle \langle \psi_i | \), where \( \sum_i p_i = 1 \), then the variance is given by
\[
\left( \langle \Delta A \rangle_{\rho}^\phi \right)^2 = \sum_i p_i \left( \langle \Delta A \rangle_{\rho}^\phi \right)^2.
\]

Eq. (29) demonstrates the intriguing fact that, the variance of \( A \) in PPS system i.e., \( \langle Var A \rangle_{\rho}^\phi = \left( \langle \Delta A \rangle_{\rho}^\phi \right)^2 \) respects classical mixing of quantum states. Mathematically, classical mixing of quantum states are represented by a density operator. By taking advantage of this property, one can obtain a purely quantum uncertainty relation when the pre-selection \( \rho \) is a mixed state (see Sec. IV). It may be noted here that in standard quantum systems, the variance \( \langle Var A \rangle = \langle \Delta A \rangle_{\rho}^2 \) increases, in general, under the classical mixing of quantum states.

To realize the standard deviation in PPS system via weak value for mixed pre-selected state, a generalization of the standard deviation \( \langle \Delta A \rangle_{\rho}^\phi \) given in Eq. (13), can be defined as
\[
\langle \Delta A_{w} \rangle_{\rho}^\phi = \sqrt{Tr(A^2 \rho) - \langle \phi \rangle \langle A \phi \rangle},
\]

where \( \langle \Delta A_{w} \rangle_{\rho}^\phi = \langle \phi | A \rho A |\phi \rangle \) is the weak value of the operator \( A \) when the pre- and post-selections are \( \rho \) and \( |\phi\rangle \), respectively.
TABLE I: Comparison of different properties between standard quantum systems and PPS systems.

| Properties                  | Standard quantum systems                                                                 | Pre- and post-selected systems                                                                 |
|-----------------------------|--------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------|
| Standard deviation          | \( \langle \Delta A \rangle _\psi = \left( \langle \psi | A^2 | \psi \rangle - \langle \psi | A | \psi \rangle^2 \right)^{1/2} \) | \( \langle \Delta A \rangle _\psi = \left( \langle \psi | A^2 | \psi \rangle - \langle A_w | \psi \rangle^2 \right)^{1/2} \) |
| Zero standard deviation     | Only if \( |\psi\rangle \) is an eigenstate of \( A \) i.e., \( |\psi\rangle \propto A |\psi\rangle \) | Only if \( |\phi\rangle \propto A |\phi\rangle \)                                                                                           |
| Uncertainty relation        | \( \langle \Delta A \rangle _\psi^2 \langle \Delta B \rangle _\psi^2 \geq \left[ \frac{1}{2\pi} \langle \psi | A, B | \psi \rangle \right]^2 \) | \( \left( \langle \Delta A \rangle _\psi \right)^2 \left( \langle \Delta B \rangle _\psi \right)^2 \geq \left[ \frac{1}{2\pi} \langle \psi | A, B | \psi \rangle - \text{Im} (W_{AB}) \right]^2 \) |
| Joint sharp preparation     | If \( |\psi\rangle \) is the eigenstate of both \( A \) and \( B \)                          | If \( |\phi\rangle = A |\phi\rangle/\sqrt{\langle \psi | A^2 | \psi \rangle} = B |\phi\rangle/\sqrt{\langle \psi | B^2 | \psi \rangle} \) up to some phase factors |

Now, \( \langle \text{Var}_{A_w} \rangle _\rho := (\langle \Delta A \rangle _\rho^2)^\frac{1}{2} \) can be viewed as a variance-like quantity (henceforth called as generalized variance) of \( A \) involving weak value and, it is, in general, different from the variance \( \langle \Delta A \rangle _\rho^2 = (\langle \Delta A \rangle _\rho^2)^2 \), and \( \langle \text{Var}_{A_w} \rangle _\rho \) is always non-decreasing under the classical mixing of quantum states. This property is certified by the following proposition.

**Proposition 3.** The generalized variance \( \langle \text{Var}_{A_w} \rangle _\rho \) is lower bounded by the variance \( \langle \Delta A \rangle _\rho \), that is

\[
\langle \text{Var}_{A_w} \rangle _\rho \geq \sum_i p_i \left( \langle \Delta A \rangle _\psi \right)^2 = \langle \text{Var}_{A} \rangle _\rho, \quad (31)
\]

where equality holds if the pre-selection \( \rho \) is a pure state.

**Proof.** Let \( \rho = \sum_i p_i |\psi_i\rangle \langle \psi_i| \), then using Eq. (30) (after using the definition of the weak value for mixed pre-selection), we have

\[
\langle \text{Var}_{A_w} \rangle _\rho = \left( \langle \Delta A \rangle _\rho^2 \right)^\frac{1}{2} = \sqrt{\text{Tr}(A^2) - \frac{\langle \langle \phi | A \rho | \phi \rangle \rangle^2}{\langle \phi | \rho | \phi \rangle}} = \sum_i p_i \langle \psi_i | A^2 | \psi_i \rangle - \frac{\sum_i \sqrt{p_i} \langle \phi | A | \psi_i \rangle \sqrt{p_i} \langle \psi_i | \phi \rangle^2}{\langle \phi | \rho | \phi \rangle} \geq \sum_i p_i \langle \psi_i | A^2 | \psi_i \rangle - \frac{\sum_i \langle \phi | A | \psi_i \rangle \langle \phi | \rho | \phi \rangle}{\langle \phi | \rho | \phi \rangle} \geq \sum_i p_i \langle \psi_i | A^2 | \psi_i \rangle - \sum_i p_i |\phi | A | \psi_i \rangle^2 = \sum_i p_i \left( \langle \Delta A \rangle _\psi \right)^2 = \langle \text{Var}_{A} \rangle _\rho,
\]

where we have used the Cauchy-Schwarz inequality for the complex numbers in the first inequality and Eq. (29) in the last line. When \( \rho \) is pure, equality holds automatically.

As \( \langle \text{Var}_{A} \rangle _\rho \) does neither increase nor decrease under classical mixing of quantum states, the inequality \( \langle \text{Var}_{A_w} \rangle _\rho \geq \langle \text{Var}_{A} \rangle _\rho \) clearly implies that under classical mixing of quantum states, the generalized variance \( \langle \text{Var}_{A_w} \rangle _\rho \) is always non-decreasing. In fact, one can easily verify that \( \langle \text{Var}_{A_w} \rangle _\rho \) is sum of the quantum uncertainty \( \langle \Delta A \rangle _\rho \) and the classical uncertainty \( C(\rho, A, \phi) := \langle \phi | A \rho | \phi \rangle - \frac{\langle \langle A_w | \psi \rangle^2 \langle \phi | \rho | \phi \rangle}{\langle \phi | \rho | \phi \rangle} \), both of which will be discussed in detail in Sec. IV C.

It is important to note that, in general the equality in Eq. (31) does not imply that the pre-selection \( \rho \) is pure. In Sec. IV A (see below), we show that only in the qubit system, equality of Eq. (31) implies that the pre-selection is a pure state. To make an equality in Eq. (31) in higher dimensional systems, we need to put conditions on the observable and post-selection (see below in Sec. IV A).

The uncertainty relation (17) or (21) can be generalized for mixed pre-selection \( \rho \) also which is given by

\[
\left( \langle \Delta A \rangle _\rho \right)^2 \left( \langle \Delta B \rangle _\rho \right)^2 \geq \left[ \frac{1}{2\pi} \langle \psi | A, B | \psi \rangle - \text{Im}(W_{AB}) \right]^2,
\]

where \( W_{AB} = \langle \phi | B \rho A | \phi \rangle \). See the derivation of Eq. (32) in Appendix B. Eq. (32) holds also when the definition of standard deviation defined in Eq. (30) is considered due the Proposition 3.

See TABLE I for the comparison of different properties between standard quantum systems and PPS systems.

IV. APPLICATIONS

Suitably post-selected systems can offer some essential information regarding quantum systems. Below, we provide a few applications of standard deviations and uncertainty relations in PPS systems.

A. Detection of mixedness of an unknown state

Practically, partial information about a quantum state is often of great help. For example, whether an interaction has taken place with the environment, one must verify the purity
of the system’s state. Quantum state tomography (QST) is the most resource intensive way to verify the purity of a quantum state but here we provide some results that can be used to detect purity of the quantum state using less resources compared to the QST.

We will use the inequality (31) in Proposition 3 to detect the mixedness of an unknown pre-selected state in a PPS system. The proofs of the following Lemmas are given in Appendix C.

Lemma 1. Qubit system: In the case of a two-level quantum system (i.e., a qubit), equality in Eq. (31) holds if and only if the pre-selected state \( \rho \) is pure irrespective of the choice of the observable \( A \) and the post-selected state \( \phi \).

Lemma 2. Qutrit system: If for an observable \( A \) and a complete orthonormal basis \( \{ |\phi_k\rangle \}_{k=1}^3 \) (to be used as post-selected states) of any three-level quantum system (i.e., a qutrit), and the condition \( \langle \phi_1 | A | \phi_2 \rangle = 0 \) also holds good, then equality in Eq. (31) holds good if and only if the pre-selected state \( \rho \) is pure.

Lemma 3. Qubit-qutrit system: Consider any two non orthogonal post-selections \( |\phi_B\rangle \) and \( |\phi'_{B} \rangle \) in the subsystem \( B \). For any observable \( A \), equality of \( \langle \Delta(A \otimes I)_w \rangle_{\rho}^{\phi_{AB}} \) and \( \langle \Delta(A \otimes I)\rangle_{\rho}^{\phi_{AB}} \) and separately of \( \langle \Delta(A \otimes I)_w \rangle_{\rho}^{\phi'_{AB}} \) and \( \langle \Delta(A \otimes I)\rangle_{\rho}^{\phi'_{AB}} \) hold only when the 2 \( \otimes \) 2 pre-selected states \( \rho \) is pure, where \( \langle \phi_{AB} | = \langle \phi_A | \langle \phi_B | \) and \( \langle \phi'_{AB} | = \langle \phi_A | \langle \phi_B' | \). Two non orthogonal post-selections \( |\phi_B\rangle \) and \( |\phi'_{B} \rangle \) in the subsystem \( B \) are required here due to the fact that there exists an unique \( 2 \otimes 2 \) mixed density matrix which satisfies the equality of Eq. (31).

Lemma 4. Qubit-qutrit system: If for an observable \( A \) and any complete orthonormal basis \( \{ |\phi_k\rangle \}_{k=1}^3 \) (to be used as post-selected states) for a qutrit, and the condition \( \langle \phi_1 A | \phi_2 \rangle = 0 \) is considered, then equality of \( \langle \Delta(A \otimes I)_w \rangle_{\rho}^{\phi_{AB}} \) and \( \langle \Delta(A \otimes I)\rangle_{\rho}^{\phi_{AB}} \) and separately of \( \langle \Delta(A \otimes I)_w \rangle_{\rho}^{\phi'_{AB}} \) and \( \langle \Delta(A \otimes I)\rangle_{\rho}^{\phi'_{AB}} \) hold if and only if the 3 \( \otimes \) 2 pre-selected state \( \rho \) is pure.

Extension of this method for higher dimensional systems will require more conditions to be imposed on the observable and post-selections. So it might be difficult to apply our method for higher dimensions. To overcome this difficulties, Eq. (17) or (21) can be used to detect the mixedness of the initially prepared states. Note that, Mal et al. have used the stronger version of the RHUR (9) to do so [64].

### B. Stronger Uncertainty Relation

**Motivation.** If, for example, the initially prepared state of the system is an eigenstate of one of the two incompatible observables \( A \) and \( B \), both the sides of the RHUR (8) becomes trivial (i.e., zero). For certain states, a trivial lower bound is always possible because the right side of the relation (8) contains the average of the commutator of incompatible observables. For such cases, the RHUR (8) does not capture the incompatibility of the non-commuting observables. One can think of adding Schrödinger’s term in the RHUR but still this can be become trivial (e.g., when the prepared states is an eigenstate of either \( A \) or \( B \)). So, none of them are unquestionably appropriate to capture the incompatibility of the non-commuting observables.

For certain states, a trivial lower bound is always possible because the right side of the relation (8) contains the average of the commutator of incompatible observables. For such cases, the RHUR (8) does not capture the incompatibility of the non-commuting observables. One can think of adding Schrödinger’s term in the RHUR but still this can be become trivial (e.g., when the prepared states is an eigenstate of either \( A \) or \( B \)). So, none of them are unquestionably appropriate to capture the incompatibility of the non-commuting observables.
state of the system is an eigenstate of the observable $A$ [66]. It indicates that the first and second inequalities in MPUR for two and arbitrary dimensions, respectively, contain no information about the observable $A$ and are therefore of no practical significance. In other words, we learn nothing new about the quantum system other than the trivial fact that $(\Delta B)_{\psi}^2$ is non-negative. In addition, if the initially prepared state $|\psi\rangle$ is unknown, then $|\psi^+\rangle$ is likewise unknown in the MPUR inequalities and, so is the lower bound of MPUR. The first inequality in MPUR may be useful in a quantum system with Hilbert spaces of more than two dimensions.

Here, we demonstrate that relations (17) and (21) can be used to solve the triviality problem of the RHUR and the problem with MPUR that we have mentioned above, i.e., these uncertainty relations can provide non-trivial information about the observable $A$. Even if the initially prepared state (pre-selection) $|\psi\rangle$ is unknown, the lower bound of our stronger uncertainty relation can be calculated.

Consider the relation (17) which, using Eq. (14), becomes

$$\langle A \rangle_{\psi}^2 + \epsilon_A \langle (\Delta B)_{\psi}^2 + \epsilon_B \rangle \geq \left[ \frac{1}{2i} \langle \psi | [A, B] | \psi \rangle - \Im (W_{AB}) \right]^2,$$

(33)

where $\epsilon_X = \langle X \rangle_{\psi}^2 - |\langle X \rangle_{\psi}^2|^2 \langle \phi | \phi \rangle^2$, with $X = A$ or $B$. Now suppose $|\psi\rangle$ is an eigenstate of $A$, then, the Eq. (33) is nontrivial unless $|\phi\rangle = |\psi\rangle$, as, in the case when $|\phi\rangle \neq |\psi\rangle$, the inequality (33) becomes

$$\epsilon_A \left( \langle (\Delta B)_{\psi}^2 + \epsilon_B \rangle \right) \geq \left[ \Im (W_{AB}) \right]^2.$$

(34)

Notice that, in the both sides of Eq. (34), there is a quantum state $|\phi\rangle$ which can be chosen independently in the standard quantum system. So, it is always possible to choose a suitable $|\phi\rangle$ such that the relation (33) is nontrivial. With “Schrödinger’s term”, the relations (33) and (34) becomes

$$\langle (\Delta A)_{\psi}^2 + \epsilon_A \rangle \langle (\Delta B)_{\psi}^2 + \epsilon_B \rangle \geq \left[ \frac{1}{2i} \langle \psi | [A, B] | \psi \rangle - \Im (W_{AB}) \right]^2,$$

$$+ \left[ \frac{1}{2} \langle \psi | [A, B] | \psi \rangle - \Re (W_{AB}) \right]^2,$$

(35)

$$\epsilon_A \left( \langle (\Delta B)_{\psi}^2 + \epsilon_B \rangle \right) \geq \left[ \Im (W_{AB}) \right]^2,$$

$$+ \left[ \frac{1}{2} \langle \psi | [A, B] | \psi \rangle - \Re (W_{AB}) \right]^2,$$

(36)

respectively. As $\epsilon_A$ and $\epsilon_B$ can also be negative, the left hand side of relation (33) can become lower than the left hand side of relation (8). The same holds true for the right-hand side as well. So, for a fixed $|\psi\rangle$, we always want to have a non-trivial lower bound from the relations (8) and (33) which can be combined in a single uncertainty relation i.e., the stronger uncertainty relation

$$\max \{ L_{RH}, L_{PPS} \} \geq \max \{ R_{RH}, R_{PPS} \},$$

where $L_{RH} = \langle (\Delta A)_{\psi}^2 + \epsilon_A \rangle \langle (\Delta B)_{\psi}^2 + \epsilon_B \rangle$, $L_{PPS} = \left( \langle (\Delta A)_{\psi}^2 + \epsilon_A \rangle \langle (\Delta B)_{\psi}^2 + \epsilon_B \rangle \right)$, $R_{RH} = \left[ \frac{1}{2i} \langle \psi | [A, B] | \psi \rangle \right]^2$ and $R_{PPS} = \left[ \frac{1}{2i} \langle \psi | [A, B] | \psi \rangle - \Im (W_{AB}) \right]^2$.

In FIG. 1, comparison between the relations (8) and, both (33) and (35) is shown. Eqs. (34) and (36) capture the informations about the operator $A$ when the initially prepared state $|\psi\rangle$ is one of the eigenstates of $A$, while MPUR fails to capture such informations which we have already discussed.

Moreover, even if the initial state (i.e., pre-selection) is unknown, the lower bound of the uncertainty relation (33) can be calculated experimentally and in that case we need the average value of the hermitian operator $\frac{1}{2}[A, B]$ and weak values of the operators $A$ and $B$.

Sum uncertainty relation in the PPS system can also be used to obtain stronger uncertainty relation in the standard quantum system. One can easily show that

$$\left( \langle (\Delta A)_{\psi}^2 + \epsilon_A \rangle + \langle (\Delta B)_{\psi}^2 + \epsilon_B \rangle \right) \geq \left( i \langle \psi | [A, B] | \psi \rangle \right),$$

holds in Eq. (27) in Theorem 3 by discarding the summation part which is always a positive number. This inequality remains strong against when $|\psi\rangle$ is one of the eigenstates of $A$ by suitably choosing post-selection $|\phi\rangle$.

C. Purely quantum uncertainty relation

Motivation. In practice, it is not always possible to carry out quantum mechanical tasks with pure states because of interactions with the environment. Because the mixed initial prepared state is a classical mixture of pure quantum states, any task or measurement involves a hybrid of classical and quantum parts. In modern technologies, it is considered that quantum advantage is more effective and superior to classical advantage. Hence, a hybrid of a quantum and classical component may be less advantageous than a quantum component alone. For example, the uncertainty of an observable $A$ in standard quantum system increases in general under classical mixing of quantum states i.e., $\langle (\Delta A)_{\rho}^2 \rangle \geq \sum_{i} p_i \langle (\Delta A)_{\psi_i}^2 \rangle$ (where $\rho = \sum_{i} p_i |\psi_i\rangle \langle \psi_i|)$ and this is disadvantageous in the sense that the knowledge about the observable is more uncertain than only when the average of the pure state uncertainties are considered. The uncertainty that one gets to see due to (classical) mixing of pure states is considered here as ‘classical uncertainty’.

The standard deviation $(\Delta A)_{\rho}$ can be referred as the hybrid of classical and quantum uncertainties and hence the RHUR (8) can be considered as the hybrid uncertainty relation in the standard quantum systems.

Purely quantum uncertainty relation, a crucial component of the quantum world, may be very useful, but in order to obtain it, the classical uncertainty must be eliminated from the
hybrid uncertainty relation. To do this, we first need to determine the purely quantum uncertainty of an observable, which can be done in a number of ways, such as by eliminating the classical component of the hybrid uncertainty or by specifying a purely quantum uncertainty straight away.

Any measure of purely quantum uncertainty should have at least the following intuitive and expected property (below ‘Φ’ represents sometimes quantum observables, sometimes states, etc for different type of quantum mechanical systems. For example, if the system is a PPS system, then ‘Φ’ is the post-selection |ϕ⟩. If the system is a standard quantum system, then the term ‘Φ’ disappears):

(i) Quantum uncertainty should not be affected (neither increasing nor decreasing) by the classical mixing of quantum states i.e.,

\[ Q(\rho, A, \Phi) = \sum_i p_i Q(\psi_i, A, \Phi), \text{ where } \rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|. \]

Here \( Q(\rho, A, \Phi) \) is some measure of purely quantum uncertainty of the observable \( A \) for a given \( \rho \). There might exist some other properties depending upon the nature of the system (e.g., standard systems, PPS systems, etc) but we emphasize that the most important property of a purely quantum uncertainty should be (i).

It is seen that the variance of \( A \) in PPS system i.e., \( (\text{Var}A)^{\rho}_\Phi = (\langle \Delta A \rangle^{\rho}_\Phi)^2 \) is a purely quantum uncertainty which satisfies the property (i). Now the purely quantum mechanical uncertainty relation in this regard is Eq. (32).

As can be seen from Eq. (31) for mixed states, the second definition of the variance of \( A \) i.e., \( (\text{Var}A)^{\rho} = (\langle \Delta A \rangle^{\rho})^2 \) in the PPS system defined in Eq. (30) is a hybrid uncertainty. Hence, the uncertainty in PPS system based on weak value has both classical and quantum parts. When measurement is carried out in the PPS system and weak values are involved, classical uncertainty may be crucial in determining how much classicality (in the form of classical uncertainty) the mixed state \( \rho \) possesses. Mixed states with less classicality should have more quantumness (in the form of quantum uncertainty), and vice versa. To distinguish classical uncertainty from the hybrid uncertainty \( (\text{Var}A)^\rho_\Phi \), we subtract the quantum uncertainty \( \langle \text{Var}A \rangle^\rho_\Phi \) from it i.e.,

\[ C(\rho, A, \phi) = \left( (\langle \Delta A \rangle^\rho_\Phi)^2 - (\langle \Delta A \rangle^\rho)^2 \right). \quad (37) \]

This is one of the good measures of classical uncertainty which should have some intuitive and expected properties:

(i) \( C(\rho, A, \Phi) \geq 0 \) for a quantum state \( \rho \).

(ii) \( C(\rho, A, \Phi) = 0 \) when \( \rho = |\psi\rangle \langle \psi| \) (absence of classical mixing).

(iii) Total classical uncertainty of disjoint systems should be the sum of individual systems’ classical uncertainties:

\[ C(\rho, A_1 \otimes I + I \otimes A_2, \Phi) = C(\rho, A_1 \otimes I, \Phi) + C(\rho, I \otimes A_2, \Phi), \]

when \( \rho = \rho_1 \otimes \rho_2 \).

One can show that all the properties (i)-(iii) of classical uncertainty are satisfied by \( C(\rho, A, \phi) \) defined in Eq. (37). Particularly, property (iii) is satisfied by taking \( \Phi = |\phi_1\rangle |\phi_2\rangle \).

Here, \( |\phi_1\rangle \) and \( |\phi_2\rangle \) are post-selections of the two disjoint systems, respectively.

There are some works by Luo and other authors regarding the purely quantum uncertainty relation. Initial attempt was made by Luo and Zhang [67] to obtain uncertainty relation by using skew information (introduced by Wigner and Yanase [68]) but it was found to be incorrect in general [69]. Later, another attempt was made by Luo himself [70], which is obtained by discarding the classical part from the hybrid uncertainty relation using skew information. But this uncertainty relation can’t be guaranteed to be an intrinsically quantum uncertainty relation (according to property (i)) as the uncertainty they claim to be a quantum uncertainty is a product of skew information (which is a convex function under the mixing of quantum states) and a concave function under the same mixing. After that a series of successful and failed attempts were performed by modifying the works of Luo and other authors [71–74].

Instead, we have given a quantum uncertainty relation although it is based on pre- and post-selections which is different from the standard quantum mechanics but a quantumness can be seen in the relation (32).

D. State dependent tighter uncertainty relations in standard systems

The RHUR (8) or (9) is known not to be the tight one. Some existing tighter bounds are given in [61, 62, 75]. The drawback of these tighter uncertainty relations is that their lower bounds depend on the states perpendicular to the given state of the system. If the given state is unknown, then the lower bound of these uncertainty relations also remain unknown.

Here we show that by the use of arbitrary post-selected state \( |\phi\rangle \), the lower bound of the RHUR based on sum uncertainties can be made arbitrarily tight and even if the given state (i.e., pre-selection here) is unknown, the lower bound of our tighter uncertainty relation can be obtained in experiments.

**Theorem 4.** Let \( \rho \in \mathcal{L}(\mathcal{H}) \) be the density operator of the standard quantum system, then the sum of the standard deviations of two non-commuting observables \( A, B \in \mathcal{L}(\mathcal{H}) \) satisfies

\[ (\Delta A)^2 + (\Delta B)^2 \geq \pm i \text{Tr}([A, B]_\rho) + \langle \phi|C^\pm_+ \rho C^\pm_+ |\phi\rangle, \quad (38) \]

where \( C^+_\pm = A \pm iB - \{A \pm iB\}_\rho I \) and the ‘±’ sign is taken in such a way that the first term in the right hand side is always positive.

**Proof.** Considering Eq. (27) for pre-selection \( |\psi_j\rangle \) and multi-
ply by \( p_j \), and then after summing over \( \{j\} \), we have

\[
\sum_j p_j \left( (\Delta A)_{\psi_j}^\phi \right)^2 + \sum_j p_j \left( (\Delta B)_{\psi_j}^\phi \right)^2 = \pm i \sum_j p_j |\langle A, B|\psi_j\rangle|^2 = \pm 2i \text{Im} \left( \sum_j p_j \langle \phi|A[B|\psi_j\rangle \langle \psi_j|A|\phi\rangle \right)
\]

\[
+ \sum_{k=1}^{d-1} \sum_j p_j (\phi_k^\perp |A \pm iB|\psi_j\rangle \langle \psi_j|(A \mp iB) |\phi_k^\perp \rangle, \quad \text{(39)}
\]

where we have used \( W_{AB} = \langle \phi|B|\psi_j\rangle \langle \psi_j|A|\phi\rangle \). By using Eq. (29) for \( A \) and \( B \) when \( \rho = \sum_j p_j |\psi_j\rangle \langle \psi_j| \), we have

\[
\left( (\Delta A)^\phi_{\rho} \right)^2 + \left( (\Delta B)^\phi_{\rho} \right)^2 = \pm i \text{Tr}([A, B]|\rho\rangle + \sum_{k=1}^{d-1} \sum_j p_j (\phi_k^\perp |A \pm iB|\rho(A \mp iB)|\phi_k^\perp \rangle, \quad \text{(40)}
\]

where \( \mp 2i \text{Im}(\langle \phi|B|\rho(A|\phi\rangle) = \pm i(\langle \phi|B|\rho(A|\phi\rangle - \langle \phi|A|\rho(B|\phi\rangle \rangle - \langle \phi|A|\rho(B|\phi\rangle = \langle \phi|A|\rho(B|\phi\rangle \rangle - \langle \phi|B|\rho(A|\phi\rangle \rangle \)

Defined in Eq. (28) (similarly for \( B \) also) and after subtracting \( \text{Tr}(A|\rho\rangle^2 + \text{Tr}(B|\rho\rangle^2 \) from both sides of Eq. (40) and using \( \langle \phi|\rho\rangle + \sum_{k=1}^{d-1} |\phi_k^\perp \rangle \langle \phi_k^\perp | I, \) we have

\[
(\Delta A)^\phi_{\rho} + (\Delta B)^\phi_{\rho} = \pm i \text{Tr}([A, B]|\rho\rangle + \text{Tr}([A \pm iB]|\rho(A \mp iB)\rangle - \text{Tr}(A|\rho\rangle^2 - \text{Tr}(B|\rho\rangle^2
\]

\[
+ \pm i \text{Tr}([A, B]|\rho\rangle + \text{Tr}([A \pm iB]|\rho(A \mp iB)\rangle - \text{Tr}([A |\rho\rangle (A \mp iB)\rangle
\]

\[
+ \pm i \text{Tr}([A, B]|\rho\rangle + \text{Tr}([\rho(M_{\pm \rho})) - |\text{Tr}(M_{\pm \rho})|^2
\]

\[
= \pm i \text{Tr}([A, B]|\rho\rangle + \text{Tr}([\rho(M_{\pm \rho})) - \text{Tr}(M_{\pm \rho}|\rho\rangle
\]

\[
= \pm i \text{Tr}([A, B]|\rho\rangle + \text{Tr}([M_{\pm \rho}) - \text{Tr}(M_{\pm \rho}|\rho\rangle
\]

\[
\text{Tr}(M_{\pm \rho})|\rho\rangle
\]

\[
\text{Tr}(M_{\pm \rho})|\rho\rangle
\]

where \( M_{\pm} = A \mp iB \). Now let \( C_{\pm} = M_{\pm} - \text{Tr}(M_{\pm}|\rho\rangle \) Then Eq. (41) can be rewritten as

\[
(\Delta A)^\phi_{\rho} + (\Delta B)^\phi_{\rho} = \pm i \text{Tr}([A, B]|\rho\rangle + \langle \phi|C_{\pm}^\phi \rho C_{\pm}^\phi |\phi\rangle
\]

\[
+ \sum_{k=1}^{d-1} \langle \phi_k^\perp |C_{\pm}^\phi \rho C_{\pm}^\phi |\phi_k^\perp \rangle, \quad \text{(41)}
\]

where \( \{|\phi\rangle, |\phi_1^\perp \rangle\} \) is an orthonormal basis in \( \mathcal{H} \). By discarding the summation term which is always a positive number in the above equation, we obtain the inequality (38).

Notice that, the lower bound of Eq. (38) has different non-zero values depending on different choices of the post-selections \( |\phi\rangle \). The inequality (38) becomes an equality when \( |\phi\rangle \propto (A \pm iB - (A \mp iB) |\psi\rangle \), where \( \rho = |\psi\rangle \langle \psi| \) is a pure state. In the references [61, 62, 75], the lower bound of the sum uncertainty relation depends on the state orthogonal to the initial pure state, and if the initial state is a mixed state, then the lower bound can not always be computed at least for the full rank density matrix. The reason is that we cannot find a state which is orthogonal to all the eigenstates of a full rank density matrix. Moreover, if the initial density matrix is unknown then computing the lower bound will be hard. In contrast, Eq. (38) doesn’t have such issues as the first and second terms in right hand side of Eq. (38) are the average values of the hermitian operators \( i|A, B\rangle \langle A, B||\phi \rangle \) and \( (A \pm iB - (A \mp iB) |\phi\rangle \langle \phi| (A \mp iB - (A \mp iB) |\phi\rangle \) in the state \( \rho \), respectively, where \( \langle A \pm iB|\rho\rangle = \langle A \mp iB|\rho\rangle \). All of them can be obtained in experiments even if \( \rho \) is unknown.

E. Tighter upper bound for out-of-time-order correlators

Recently Bong et al. [76] used the RHUR for unitary operators to give upper bound for out-of-time-order correlators (OTOC) which is defined by \( F = \text{Tr}([W_{t}^1 V^1 W_{t} V])|\rho\rangle \), where \( V \) and \( W_{t} \) are fixed and time dependent unitary operators, respectively. The OTOC diagnoses the spread of quantum information by measuring how quickly two commuting operators \( V \) and \( W \) fail to commute, which is quantified by \( \langle [W_{t}^1 V^1]^2 \rangle = 2(1 - \text{Re}[F]) \), where \( |X|^2 = X^\dagger X \). The OTOC has strong connection with chaos and information scrambling [77–79] and also with high energy physics [80–83]. It is known that OTOC’s upper bound is essential for limiting how quickly many-body entanglement can generate [80–82]. The standard upper bound for modulus of the OTOC given by Bong et al. [76] is \( |F| = \cos\left(\theta_{W_{t}} - \theta_{W_{t} V}\right) \), where \( \theta_{W_{t}} = \cos^{-1}\left(\text{Tr}(\rho W_{t} V)^{\dagger}\right) \) and \( \theta_{W_{t} V} = \cos^{-1}\left(\langle W_{t} V | \phi \rangle = \langle \rho W_{t} V | \phi \rangle \right) \). Here, \( \langle . . | \rangle \) defines a vector norm.

**Theorem 5.** Let \( \rho \in \mathcal{L}(\mathcal{H}) \) be the system’s state and \( |\phi\rangle \) be any arbitrary state, then modulus of the OTOC \( F = \text{Tr}([W_{t}^1 V^1 W_{t} V]|\rho\rangle \) for fixed and time dependent unitary operators \( V, W_{t} \in \mathcal{L}(\mathcal{H}) \), respectively is upper bounded by

\[
|F| = |\langle W_{t}^1 V^1 W_{t} V |\rangle \leq \cos\left(\theta_{W_{t}} - \theta_{W_{t} V}\right) \cdot \sqrt{\text{Tr}(\rho W_{t} V)^{\dagger}\phi | \phi \rangle = \cos^{-1}\left(\langle W_{t} V | \phi \rangle = \langle \rho W_{t} V | \phi \rangle \right) \sqrt{\text{Tr}(\rho W_{t} V)^{\dagger}|\phi \rangle \rangle \text{. Here, \( \langle \ . \ | \ \rangle \text{ defines a vector norm.}} \]

**Proof.** For a given mixed state \( \rho \) and arbitrary state \( |\phi\rangle \) which we consider to be pre- and post-selections, respectively, the standard deviation \( \langle \Delta X \rangle_{\rho} \) of any operator \( X \) in the PPS system is defined as \( \langle \Delta X \rangle_{\rho}^2 = \text{Tr}(X X^\dagger |\rho\rangle - \langle \phi|X^\dagger \rho X|\phi\rangle = \text{Tr}(\sqrt{\rho} X_{0}^\phi |\phi|^2 |\phi\rangle = \langle \phi|X_{0}^\phi |\phi\rangle \).
\[ X |\phi\rangle \langle \phi| \text{ and } ||A||_F = \sqrt{Tr(A^\dagger A)} \text{ denotes the Frobenius norm of the operator } A. \text{ When } X \text{ is a hermitian operator,} \  \langle DX \rangle_0^\phi \text{ becomes the standard deviation of } X \text{ defined in Eq. (28). Now consider } X \text{ to be unitary operators } U \text{ and } V. \text{ So, we can derive uncertainty relation for two unitary operators } U \text{ and } V \text{ using the Cauchy-Schwarz inequality for operators with Frobenius norm when the system is in pre- and post-selections } \rho \text{ and } |\phi\rangle, \text{ respectively as}
\]
\[
\langle \Delta U \rangle_\rho^\phi \langle \Delta V \rangle_\rho^\phi \geq \left| Tr \left( (\sqrt{\rho U_0^\dagger})^\dagger (\sqrt{\rho V_0^\dagger}) \right) \right|
= \left| Tr (UV^\dagger) - \langle \phi | U^\dagger \rho V | \phi \rangle \right|, \quad (43)
\]
where \( \langle \Delta U \rangle_\rho^\phi = \sqrt{1 - \langle \phi | U^\dagger \rho U | \phi \rangle} \) and similarly for \( V \) also.

Now, by replacing \( U \rightarrow V^\dagger W_i^\dagger \) and \( V \rightarrow W_i^\dagger V \), (43) becomes
\[
|Tr(W_i^\dagger V_i^\dagger W_i V \rho)|
\leq \left| \langle \phi | (V^\dagger W_i^\dagger)^\dagger W_i V^\dagger | \phi \rangle \right| + \langle \Delta (V^\dagger W_i^\dagger) \rangle_\rho^\phi \langle \Delta (V^\dagger V^\dagger) \rangle_\rho^\phi
\leq \left| \langle \phi | V^\dagger W_i^\dagger | \phi \rangle \right| \left| \langle \phi | W_i V^\dagger | \phi \rangle \right| + \sqrt{1 - \left| \langle \phi | V^\dagger W_i^\dagger | \phi \rangle \right|^2 \left| \langle \phi | W_i V^\dagger | \phi \rangle \right|^2}, \quad (44)
\]
where we have used the Cauchy-Schwarz inequality for vectors and \( \langle \Delta (V^\dagger W_i^\dagger) \rangle_\rho^\phi = \sqrt{1 - \left| \langle \phi | V^\dagger W_i^\dagger | \phi \rangle \right|^2} \) and \( \langle \Delta (W_i V^\dagger) \rangle_\rho^\phi = \sqrt{1 - \left| \langle \phi | W_i V^\dagger | \phi \rangle \right|^2}, \) respectively.

Now, setting \( \left| \sqrt{\rho} (V W_i^\dagger) | \phi \rangle \right| = \cos \theta_{W_i V}^\phi \) and \( \left| \sqrt{\rho} (W_i V^\dagger) | \phi \rangle \right| = \cos \theta_{W_i V}^\phi \) in (44), the inequality (42) is proved.

---

**V. CONCLUSION**

We have defined standard deviation of an observable in a PPS system, interpreted it geometrically as well as informationally from the perspective of weak PPS measurements and subsequently derived the Robertson-Heisenberg like uncertainty relation for two non commuting observables. Such uncertainty relations in PPS system impose limitations on the joint sharp preparation of pre- and post-selected states for two incompatible observables. We provided the necessary and sufficient condition for zero uncertainty of an observable and show its usefulness in achieving optimized Fisher information in quantum metrology. We have derived both product and sum uncertainty equalities from which a series of uncertainty inequalities can be obtained. The generalization of uncertainty relation for mixed pre-selection in PPS system has also been discussed. We have demonstrated that the PPS system can exhibit more bizarre behaviors than the usual ones. For instance, it is possible in PPS system that measurement of two compatible observables can disturb each other’s measurement results i.e., the lower bound in the uncertainty relation can be made non zero by suitably choosing post-selections. A similar property in PPS system was first shown by Vaidman [50].

It is also possible that a quantum state (pre-selection) can be prepared in a PPS system for which both of the standard deviations of incompatible observables are zero although this is not possible in a standard quantum system (see section III B).

The standard deviation and uncertainty relation in the PPS system have been used to provide physical applications. (i) We have used two different definitions of the standard deviations in the PPS system to detect purity of an unknown state. (ii) The uncertainty relation in the PPS system is used to derive the stronger uncertainty relation (i.e., nontrivial for all possible choices of initially prepared states) in the standard quantum system. For two dimensional quantum system, the stronger uncertainty relation by Maccone-Pati [62] fails to provide the informations about the incompatible observables when the system state is an eigenstate of either observables. We have shown that our stronger uncertainty relation overcomes this shortcoming of Maccone-Pati uncertainty relation. (iii) Since the variance in the PPS system remains unaffected (i.e., neither increases nor decreases) by the classical mixing of quantum states, we have concluded that the uncertainty relation in the PPS system is a purely quantum uncertainty relation. In contrast, variance in the standard system increases in general under the classical mixing of quantum states. Following this observation we have provided a measure of classical uncertainty whose less value implies more purely quan-
tum uncertainty. (iv) Tighter sum uncertainty relation in the standard quantum system has been derived where the tightness depends on the post-selection. (v) Uncertainty relation in PPS system for two unitary operators has been used to provide tighter upper bound for out-of-time-order correlators.

Future directions: (i) it will be interesting if the global minimum for sum of uncertainties of non-commuting observables in the PPS system exists because that can be used to detect entanglement by suitably choosing post-selections. Similar to the work by Hofmann and Takeuchi [7]. (ii) Applications and implications of the ideas like ‘zero uncertainty’ and ‘joint sharp preparation of a quantum state for non-commuting observables’ need more attention. (iii) This is a matter of further study if the uncertainty relation (17) in PPS system has applications similar to the RHUR (8), such as quantum metrology, spin squeezing, improving the accuracy of phase measurement in quantum interferometers, etc. (iv) We have derived the condition for the “intelligent pre- and post-selected states” to achieve the minimum bound of the uncertainty relation in the PPS system and intelligent pre- and post-selected states can be exploited to get highly precise phase measurements because many theoretical and experimental efforts have been made in recent years involving the minimum uncertainty states (for which the RHUR saturates) and the spin-squeezing states in the standard quantum systems (see, for example, [52, 56, 60]) for precise phase measurements.

Acknowledgment: We would like to thank Klaus Mølmer and David R. M. Arvidsson-Shukur for bringing the references [14, 15, 48, 49] and [45], respectively to our attention. We acknowledge Debmalya Das for valuable comments on this work.

VI. APPENDICES

Appendix A

Here we derive the condition for which the inequality (17) saturates. In the Cauchy-Schwarz inequality Eq. (18), the remainder and the real term to be vanished for the equality condition of Eq. (17) i.e.,

\[ |\phi_{A\psi}^\perp\rangle - \langle\phi_{B\psi}^\perp|\phi_{A\psi}^\perp\rangle = 0, \]  
\[ \langle\phi_{B\psi}^\perp|\phi_{B\psi}^\perp\rangle = 0. \]

Now take the inner product between \( \langle\phi_{A\psi}^\perp|\phi_{A\psi}^\perp\rangle \) and Eq. (A1), then we have

\[ \langle\phi_{A\psi}^\perp|\phi_{A\psi}^\perp\rangle + \left(\frac{\langle\phi_{B\psi}^\perp|\phi_{A\psi}^\perp\rangle}{\langle\phi_{B\psi}^\perp|\phi_{B\psi}^\perp\rangle}\right)^2 = 0. \]  

Now using \( \langle\phi_{X\psi}^\perp|\phi_{X\psi}^\perp\rangle = \left(\langle\Delta X\rangle^0_{\psi}\right)^2 \) where X={A,B}; the Eq. (A3) becomes

\[ \langle\phi_{A\psi}^\perp|\phi_{A\psi}^\perp\rangle = \pm i \langle\Delta A\rangle^0_{\psi} \langle\Delta B\rangle^0_{\psi}. \]  

Finally, use Eqs. (A4) and (12) in Eq. (A1) to obtain the condition (23).

Appendix B

To show that the uncertainty relation (17) or (21) is also valid for mixed pre-selected state \( \rho \), we consider the following operator

\[ T = A_0^\phi + (\gamma + i\epsilon)B_0^\phi, \]

where \( A_0^\phi = A - A\langle\phi|\langle\phi| \), \( B_0^\phi = B - B\langle\phi|\langle\phi| \) and \( \gamma, \epsilon \) are some real parameters. Now for any operator \( T \), the inequality

\[ Tr(\rho TT^\dagger) \geq 0, \]

holds. Using Eq. (B1), we have

\[ Tr(\rho TT^\dagger) = \left(\langle\Delta A\rangle^0_{\rho}\right)^2 + (\gamma^2 + \epsilon^2) \left(\langle\Delta B\rangle^0_{\rho}\right)^2 + \gamma \left(\langle\{A, B\}\rangle_{\rho} - 2ReW_{AB}\right) - i\epsilon \left(\langle\{A, B\}\rangle_{\rho} - 2iM_{WAB}\right) \geq 0, \]

where \( \langle\Delta A\rangle^0_{\rho} = Tr(\rho A_0^\phi A_0^\phi\dagger) \) is defined in Eq. (28), \( \langle\{A, B\}\rangle_{\rho} = Tr(\rho\{A, B\}) \), \( \{A, B\}_\rho = Tr(\rho\{A, B\}) \) and \( W_{AB} = Tr(\{A, B, A\}) \). Now one finds the quantity \( Tr(\rho TT^\dagger) \) is minimum for \( \gamma = -\frac{\langle\{A, B\}\rangle_{\rho}}{2ReW_{AB}} \) and

\[ \epsilon = \frac{i\langle\{A, B\}\rangle_{\rho} - 2iM_{WAB}}{2\left(\langle\Delta B\rangle^0_{\rho}\right)^2}. \]

Hence, \( min_{\gamma, \epsilon} Tr(\rho TT^\dagger) \geq 0 \) becomes

\[ \left(\langle\Delta A\rangle^0_{\rho}\right)^2 + \left(\langle\Delta B\rangle^0_{\rho}\right)^2 \geq \frac{1}{2\gamma} + \frac{1}{2}\left(\langle\{A, B\}\rangle_{\rho} - 2ReW_{AB}\right)^2. \]  

By discarding the second term which is a positive number in the right hand side of Eq. (B4), the uncertainty relation (32) is achieved.

Appendix C

Here we show the proofs of all the Lemmas to detect mixedness of an unknown state in qubit, qutrit, qubit-qubit and qubit-qutrit systems. Let us recall the mathematical expres-
sion of the statement of the Proposition 3 which is given by
\[ \langle \Delta A \rangle^\phi \rho \geq \langle \Delta A \rangle^\phi . \]  \hfill (C1)

In the following, we will use Eq. (C1) to prove all the Lemmas.

The general form of a mixed state is \( \rho = \sum p_i |\psi_i\rangle \langle \psi_i| \) and the condition for which the equality of Eq. (C1) holds is (see proof of the Proposition 3)
\[ \langle \phi | A | \psi_i \rangle = \lambda \langle \phi | \psi_i \rangle , \]  \hfill (C2)
where \( \lambda \) is some constant which depends on the index of \( |\phi\rangle \) (e.g., for \( |\phi_k\rangle \), it is \( \lambda_k \)).

**The proof of Lemma 1:**

**Proof.** We first assume that each \( |\psi_i\rangle \) is distinct and hence from Eq. (C2), we have a set of equations
\[ \langle \phi | (A - \lambda I) | \psi_i \rangle = 0 , \]  \hfill (C3)
for each \( |\psi_i\rangle \). Denote the unnormalized state vector \( \tilde{\phi}_\lambda^A = (A - \lambda I) |\phi\rangle \). As \( \tilde{\phi}_\lambda^A \) is an unnormalized state vector different from \( |\phi\rangle \) and the \( |\psi_i\rangle \), \( \forall i \) are orthogonal to \( \tilde{\phi}_\lambda^A \), it implies that \( \{ |\psi_i\rangle \}_{i=1} \) are confined in one dimensional Hilbert space. Hence each \( |\psi_i\rangle \) is the same initially prepared state that is \( \rho \) is a pure state in a qubit system.

**The proof of Lemma 2:**

**Proof.** The qubit argument cannot be generalized for the higher dimensional systems. The reason is simply because in three dimensional Hilbert space (for example) all the \( |\psi_i\rangle \) can be confined in a two dimensional subspace of the Hilbert space which is orthogonal to \( \tilde{\phi}_\lambda^A \). To make an "if and only if" condition, we consider the orthogonal basis \( \{ |\phi_k\rangle \}_{k=1}^3 \) as valid post-settings. Here valid post-settings are those post-settings for which weak values are defined.

As there are three post-settings in three dimensional Hilbert space, we have three sets of equations like (C2) for the equality of the inequality (C1)
\[ \{ \langle \phi_i | (A - \lambda_1 I) | \psi_i \rangle = 0 \}_{i=1} , \]  \hfill (C4)
\[ \{ \langle \phi_2 | (A - \lambda_2 I) | \psi_i \rangle = 0 \}_{i=1} , \]  \hfill (C5)
\[ \{ \langle \phi_3 | (A - \lambda_3 I) | \psi_i \rangle = 0 \}_{i=1} . \]  \hfill (C6)

Now, there are three possibilities which is implied by (C4), (C5) and (C6):
(i) The state vectors \( \{ |\phi_k\rangle \}_{k=1}^3 = (A - \lambda_k I) |\psi_i\rangle \) span the whole 3-dimensional Hilbert space \( \mathcal{H} \).
(ii) \( \{ |\phi_k\rangle \}_{k=1}^3 \) span a 2-dimensional Hilbert space \( \mathcal{H} \).
(iii) \( \{ |\phi_k\rangle \}_{k=1}^3 \) span a 1-dimensional Hilbert space \( \mathcal{H} \).

Below, we will show that possibility-(i) is discarded naturally whereas to discard possibility-(iii), we need a condition on observable \( A \) and post-selection \( |\phi\rangle \). Then, possibility-(ii) will automatically indicate that all the \( \{ |\psi_i\rangle \}_{i=1} \) are same i.e., \( \rho \) is pure.

To start with possibility-(i), let’s assume that possibility-(i) is true, then \( \{ |\psi_i\rangle \}_{i=1} \) has to be orthogonal to \( \{ |\phi_k\rangle \}_{k=1}^3 \) according to (C4), (C5) and (C6) implying \( |\psi_i\rangle = 0 \) \( \forall i \) i.e., \( \rho = 0 \). So we discard this possibility.

Possibility-(iii) implies
\[ N_1 (A - \lambda_1 I) |\phi_1\rangle = N_2 (A - \lambda_2 I) |\phi_2\rangle = N_3 (A - \lambda_3 I) |\phi_3\rangle , \]  \hfill (C7)
along z-axis (for example) and hence \( \{ |\psi_i\rangle \}_{i=1} \) span 2-dimensional xy-plane (see Fig. 4). Here \( N_k \) are normalization constants. Now the inner product of (C7) with \( |\phi_1\rangle \), \( |\phi_2\rangle \) and \( |\phi_3\rangle \), respectively gives
\[ N_1 (\langle \phi_1 | A | \phi_1 \rangle - \lambda_1) = N_2 (\langle \phi_1 | A | \phi_2 \rangle - \lambda_2) = N_3 (\langle \phi_1 | A | \phi_3 \rangle - \lambda_3) \]  \hfill (C8)
\[ N_1 (\langle \phi_2 | A | \phi_1 \rangle = N_2 (\langle \phi_2 | A | \phi_2 \rangle - \lambda_2) = N_3 (\langle \phi_2 | A | \phi_3 \rangle , \]  \hfill (C9)
\[ N_1 (\langle \phi_3 | A | \phi_1 \rangle = N_2 (\langle \phi_3 | A | \phi_2 \rangle - \lambda_2) = N_3 (\langle \phi_3 | A | \phi_3 \rangle - \lambda_3) . \]  \hfill (C10)

Now, the particular choice
\[ \langle \phi_1 | A | \phi_2 \rangle = 0 \]  \hfill (C11)
implies that Eq. (C8) and (C9) do not hold if \( \langle \phi_1 | A | \phi_4 \rangle \neq 0 \) and \( \langle \phi_2 | A | \phi_3 \rangle \neq 0 \), respectively. If either of Eq. (C8) and (C9) does not hold then possibility-(iii) is discarded. But, if \( \langle \phi_1 | A | \phi_2 \rangle = 0 \) and \( \langle \phi_2 | A | \phi_3 \rangle = 0 \), then we have to proceed further. Note that, by setting \( \langle \phi_1 | A | \phi_2 \rangle = 0 \) from Eq. (C11), \( \langle \phi_1 | A | \phi_2 \rangle = 0 \) and \( \langle \phi_2 | A | \phi_3 \rangle = 0 \) in Eqs. (C8), (C9) and (C10), we have
\[ \lambda_k = \langle \phi_k | A | \phi_k \rangle \text{ for } k = 1, 2, 3. \]  \hfill (C12)

Now, it is easy to see that with the values of \( \lambda_k \) from Eq. (C12), \( \{ \langle \phi_k | \phi_k^\lambda \rangle = 0 \}_{k=1}^3 \) holds. This implies that \( \{ |\phi_k\rangle \}_{k=1}^3 \) cannot be confined in one dimensional Hilbert space i.e., along a particular axis and in our assumption it is the z-axis. But according to Eq. (C7), \( \{ |\phi_k\rangle \}_{k=1}^3 \) are along the z-axis. Hence it shows the contradiction and we discard the possibility-(iii) when the condition \( \langle \phi_1 | A | \phi_2 \rangle = 0 \) is considered.

Finally the possibility-(iii) implies that \( \{ |\psi_i\rangle \}_{i=1} \) must be spanned in 1-dimensional Hilbert space \( \mathcal{H} \) that is, each \( |\psi_i\rangle \) is the same initially prepared state which is a pure state.

So, we conclude that if for an observable \( A \) and a complete orthonormal basis \( \{ |\phi_k\rangle \}_{k=1}^3 \) (to be used as post-selected states) of any three level quantum system (i.e., a qutrit), the condition \( \langle \phi_1 | A | \phi_2 \rangle = 0 \) is considered, then equality in Eq. (C1) holds good if only when the pre-selected state \( \rho \) is pure.

**The proof of Lemma 3:**

**Proof.** For this bipartite system, we consider the observable and the post-selection to be \( A \otimes I \) and \( |\phi_{AB}\rangle = |\phi_A\rangle |\phi_B\rangle \), respectively. The standard deviations defined in Eq. (28) and
So, let us consider another post-selection $|\phi_B\rangle$ (which is not orthogonal to $\{|\phi_i\rangle\}_{i=1}$) and if we find $\rho^B_{AB}$ to be pure which is equivalent to the equality of $\langle \Delta(A\otimes I)\rangle^\rho_{AB}$ and $\langle \Delta(A\otimes I)\rangle^\rho_{\phi^{AB}}$, then we are sure that the bipartite state $\rho$ is a pure state (due to the virtue of qubit system discussed above), where $|\phi^{AB}\rangle = |\phi_A\rangle|\phi_B\rangle$.

So, here is the conclusion: Consider any two non orthogonal post-selections $|\phi_B\rangle$ and $|\phi'_B\rangle$ in the subsystem B. For any observable $A$, equality of $\langle \Delta(A\otimes I)\rangle^\rho_{\phi^{AB}}$ and $\langle \Delta(A\otimes I)\rangle^\rho_{\phi'^{AB}}$ and separately of $\langle \Delta(A\otimes I)\rangle^\rho_{\phi^{AB}}$ and $\langle \Delta(A\otimes I)\rangle^\rho_{\phi'^{AB}}$ hold only when the $2\otimes 2$ pre-selected state $\rho$ is pure. □

The proof of Lemma 4:

Proof. The treatment above with the condition of the qutrit system, we have the conclusion: if for an observable $A$ and any complete orthonormal basis $\{|\phi^k\rangle\}_{k=1}^3$ (to be used as post-selected states) for a qutrit, the condition $\langle \phi^1_A|\phi^1_A\rangle = 0$ is considered, then equality of $\langle \Delta(A\otimes I)\rangle^\rho_{\phi^{AB}}$ and $\langle \Delta(A\otimes I)\rangle^\rho_{\phi'^{AB}}$ and separately of $\langle \Delta(A\otimes I)\rangle^\rho_{\phi^{AB}}$ and $\langle \Delta(A\otimes I)\rangle^\rho_{\phi'^{AB}}$ hold if and only if the $3\otimes 2$ pre-selected state $\rho$ is pure, where $|\phi^{AB}\rangle = |\phi_A\rangle|\phi_B\rangle$. □

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Let the pre-selection be a mixed state

\[ \rho = \sum_i p_i |\psi_i\rangle \langle \psi_i| \]

Then the generalization of Eq. (10) for the case of \( \rho \) is given by

\[ A\rho A^\dagger = \sum p_i |\phi_i\rangle \langle \psi_i| A = \sum p_i (|\phi_i\rangle \langle \psi_i|)^2 a_i^+ a_i |\phi_i\rangle + (|\Delta A_i\rangle^\dagger a_i^+ a_i |\phi_i\rangle) + \sum p_i (|\phi_i\rangle A |\psi_i\rangle) (\Delta A_i)^\dagger a_i^+ a_i c_i, \]

where we have used Eq. (10) for each \( |\psi_i\rangle \). Now, to see by how much amount the output state deviates from the projection \( I_\phi = |\phi\rangle \langle \phi| \) due to the action of \( A \) on \( \rho \), similar to the case of \( \rho \) being a pure state \( |\psi\rangle \), as evident from Eq. (10), we have to calculate \( Tr[A\rho A(I - I_\phi)] \) and this is given by

\[ (\Delta A_i)_{\phi\psi}^2 = \sum p_i (|\Delta A_i\rangle_{\phi\psi})^2 = \sum p_i (|\psi_i\rangle |A^\dagger| \psi_i\rangle - (|\phi_i\rangle |\psi_i\rangle)^2 = Tr(A^2 \rho) - (|\phi_i\rangle A |\phi_i\rangle). \]

Hence, it is legitimate to call \( (\Delta A_i)_{\phi\psi}^2 \) as a standard deviation of \( A \) in the PPS system.

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