A notion of weak convergence in metric spaces

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Abstract. We discuss some basic properties of polar convergence in metric spaces. This notion has been introduced in [17] as an appropriate mode of convergence which allows to extend to Banach spaces the analysis of the defect of compactness, relative to gauge groups, already established in Hilbert spaces. Polar convergence is closely connected with the notion of ∆-convergence already known from several years (see [12]). Possible existence of a topology which induces polar convergence is also investigated. Some applications of polar convergence follow.

1. Introduction

We discuss, in the setting of metric spaces, some relations between polar convergence introduced by Tintarev-Solimini in [17] and ∆-convergence, introduced by Lim in [12], which, in restriction to Hilbert spaces, both coincide with weak convergence. A sufficient condition for an analog of the Banach-Alaoglu compactness theorem to hold, for ∆-convergence, is existence of asymptotic centers for bounded sequences. Such a property holds true in uniformly convex Banach spaces and in Hadamard spaces (see [11]) and, more generally, in complete metric spaces which are uniformly rotund according to Staples (see [18]). We point out that uniformly rotund Banach spaces are uniformly convex Banach spaces. When a Banach space is uniformly convex and uniformly smooth, ∆-convergence can be characterized as weak convergence of the duals. In this setting weak and ∆-convergence coincide if and only if the space satisfies the classical Opial condition (see [15] Condition (2) and Definition 6.1 below), while, when Opial condition does not hold, ∆-convergence is a relevant mode of convergence for iterations of non-expansive maps. On the other hand, polar convergence is a natural mode of convergence which has been used in [17], instead of weak convergence, in the analysis of the defect of compactness relative to gauge groups allowing the extension of the results, already known in the case of Hilbert spaces, to uniformly convex Banach spaces (where polar convergence and ∆-convergence agree). These results, which will be not surveyed here, have actually motivated our interest in this topic and have led to the introduction of polar convergence in [17]. In analogy to what happens in uniformly convex Banach spaces, where weak and weak-star topology agree and
some local compactness properties (such as Banach-Alaoglu compactness theorem) hold, in complete uniformly rotund metric spaces we prove that $\Delta$-convergence and polar convergence agree and deduce some compactness properties (see Theorem 3.5). We also discuss a version of Brezis-Lieb Lemma [4] where, see Theorem 6.6, the assumption of a.e. convergence is replaced by the assumption of “double weak” convergence (namely polar convergence and usual weak convergence, both are implied by a.e. convergence of a bounded sequence in $L^p$ for $1 < p < +\infty$).

2. $\Delta$-convergence and polar convergence

**Definition 2.1. ($\Delta$-limit)** Let $(E, d)$ be a metric space. A sequence $(x_n)_{n \in \mathbb{N}} \subset E$ is said to $\Delta$-converge to a point $x \in E$ (see [12]), and we shall write $x_n \xrightarrow{\Delta} x$, if

$$\lim_{n} \sup\{d(x_n, x) \leq \limsup_{n} d(x_k, y)$$

for any subsequence $(x_{k_n})_{n \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ and for every $y \in E$.

The following proposition, whose proof is straightforward, gives a characterization of a $\Delta$-limit.

**Proposition 2.2.** Let $(E, d)$ be a metric space. A sequence $(x_n)_{n \in \mathbb{N}} \subset E$ $\Delta$-converges to a point $x \in E$ if and only if

$$\forall y \in E : d(x_n, x) \leq d(x_n, y) + o(1) .$$

**Definition 2.3. (Strong $\Delta$-limit)** Let $(E, d)$ be a metric space. A sequence $(x_n)_{n \in \mathbb{N}} \subset E$ is said to strongly-$\Delta$ converge to a point $x \in E$ (see [12]), and we shall write $x_n \xrightarrow{s-\Delta} x$, if

$$\exists \lim_{n} d(x_n, x) \quad \text{and} \quad \forall y \in E : \lim_{n} d(x_n, x) \leq \liminf_{n} d(x_n, y) .$$

Note that, obviously, if $x_n \xrightarrow{s-\Delta} x$ then $x_n \xrightarrow{\Delta} x$. More characterizations of $\Delta$ and strong-$\Delta$ convergence of a given sequence $(x_n)_{n \in \mathbb{N}}$, which better clarify their meaning, are given by means of the notion of asymptotic centers (denoted by $\text{cen}_{n \rightarrow \infty} x_n$) and asymptotic radius (denoted by $\text{rad}_{n \rightarrow \infty} x_n$) of the sequence $(x_n)_{n \in \mathbb{N}}$ (definitions of asymptotic radius and asymptotic centers can be found for instance in [7] where Edelstein gives a proof of Browder fixed point theorem [5] based on these notions). We emphasize that, while the asymptotic radius always exists and is uniquely determined, asymptotic centers may not exist or may be not uniquely determined. Therefore, the symbol $\text{cen}_{n \rightarrow \infty} x_n$ must be considered in the same way as the limit symbol in a topological space which is not assumed to be Hausdorff. Here we prefer to reformulate the original definition of asymptotic centers and asymptotic radius of a sequence $(x_n)_{n \in \mathbb{N}}$ as minimum points and, respectively, infimum value of the corresponding functional defined by setting for any $y \in E$

$$I(y) = \lim_{n} \sup d(x_n, y).$$

**Remark 2.4.** Let $(E, d)$ be a metric space. A sequence $(x_n)_{n \in \mathbb{N}} \subset E$ $\Delta$-converges to a point $x \in E$ if and only if $x$ is an asymptotic center of every subsequence. On the other hand the strong-$\Delta$ convergence of $(x_n)_{n \in \mathbb{N}}$ to $x$ means that every subsequence has the same asymptotic radius (equal to $I(x)$). This last property allows to prove that any asymptotic center of the whole sequence $(x_n)_{n \in \mathbb{N}}$ is an asymptotic center of every subsequence and therefore that $x$ is a $\Delta$-limit of $(x_n)_{n \in \mathbb{N}}$. 

(Indeed if \((x_k)_n \in \mathbb{N}\) is a subsequence of \((x_n)_n \in \mathbb{N}\), since, by strong-\(\Delta\) convergence, \(\text{rad}_{n \to \infty} x_{k_n} = \text{rad}_{n \to \infty} x_n\) the inequality \(\limsup_n d(x_{k_n}, x) \leq \limsup_n d(x_n, x) = \text{rad}_{n \to \infty} x_n\) forces \(x\) to be an asymptotic center of \((x_{k_n})_{n \in \mathbb{N}}\). Obviously the converse implication is not true.

**Remark 2.5.** In other terms, let \(\Xi\) be the set of bounded sequences of elements in \(E\), for every \(\xi, \zeta \in \Xi\), we shall write \(\xi \leq \zeta\) if the sequence \(\zeta\) is extracted from \(\xi\) after a finite number of terms (note that, in spite of the notation, \(\leq\) is not an ordering, since it is not antisymmetric). The function \(f\) which maps every sequence \(\xi = (x_n)_n \in \Xi\) into \(f(\xi) = -\text{rad}_{n \to \infty} x_n\) is a real “increasing” function (i.e. if \(\xi \leq \zeta\) then \(f(\xi) \leq f(\zeta)\)). We can pass to a coarser relation \(\preceq\), which is a real ordering and makes \(f\) strictly increasing, by setting, for any \(\xi, \zeta \in \Xi\), \(\xi \preceq \zeta\) if \(\xi = \zeta\) or if \(\xi \leq \zeta\) and \(f(\xi) < f(\zeta)\). Under this notation, we can reformulate the second part of the previous remark by stating that \(x_n \overset{s-\Delta}{\to} x\) if and only if \((x_n)_{n \in \mathbb{N}}\) is maximal for \(\preceq\) and \(x\) is an asymptotic center of \((x_n)_{n \in \mathbb{N}}\).

Note that the ordered set \((\Xi, \preceq)\) is countably inductive in the sense specified in [13, Appendix A]. Indeed, if \((\xi_n)_{n \in \mathbb{N}} \subset \Xi\) is increasing with respect to \(\preceq\), after throwing away a finite number of terms from each sequence \(\xi_n\) (in order to make each \(\xi_{n+1}\) a subsequence of \(\xi_n\)) and passing to a diagonal selection, one obtains an upper bound \(\xi\) of the whole sequence.

**Remark 2.6.** Note that a (strong) \(\Delta\)-limit is not necessarily unique. For instance, if \((A_n)_{n \in \mathbb{N}}\) is a decreasing sequence of measurable sets of \(\mathbb{R}\) such that, for any \(n \in \mathbb{N}\), \(A_n \setminus A_{n+1}\) is not negligible, by setting, for any \(n \in \mathbb{N}\), \(x_n = 1_{A_{n+1}} - 1_{A_n \setminus A_{n+1}}\), we get a sequence of bounded functions which (since for any \(n \neq m \in \mathbb{N}\) \(\|x_n - x_m\|_\infty = 2\)) does not admit any subsequence with asymptotic radius strictly smaller than 1. Then, given \(\pi \in \mathbb{N}\), any function \(x\) such that \(\|x\|_\infty \leq 1\) and such that \(x|_{A_\pi} = 0\) (for instance \(x = 1_{\mathbb{R} \setminus A_\pi}\)) satisfies \(\|x_n - x\|_\infty = 1\) for \(n > \pi\).

Therefore \(x\) is an asymptotic center of the sequence \((x_n)_{n \in \mathbb{N}}\) and \(\text{rad}_{n \to \infty} x_n = 1\). Since, as already proved, \((x_n)_{n \in \mathbb{N}}\) does not admit any subsequence with asymptotic radius strictly smaller than 1, it follows that \(\text{rad}_{n \to \infty} x_{k_n} = 1\) for any subsequence \((x_{k_n})_{n \in \mathbb{N}}\). Therefore, by Remark 2.4 \(x\) is a strong-\(\Delta\)-limit of the sequence \((x_n)_{n \in \mathbb{N}}\).

**Definition 2.7.** (Polar limit) Let \((x_n)_{n \in \mathbb{N}}\) be a sequence in a metric space \((E, d)\). One says that \(x \in E\) is polar limit of \((x_n)_{n \in \mathbb{N}}\) if for every \(y \neq x\) there exists \(M(y) \in \mathbb{N}\) such that

\[
d(x_n, x) < d(x_n, y) \quad \text{for all } n \geq M(y).
\]

**Remark 2.8.** Making a comparison between (2.2) and (2.5) it is immediate that polar convergence implies \(\Delta\) convergence. Moreover (2.5) guarantees the uniqueness of polar limit. Therefore, see Remark 2.4 \(\Delta\)-convergence and polar convergence, in general, do not agree.

The following properties for \(\Delta\), strong-\(\Delta\) and polar limits are immediate:

(i) Either of strong-\(\Delta\) convergence and polar convergence implies \(\Delta\)-convergence to the same points.

(ii) Any convergent sequence (in metric) to \(x\) is also polarly convergent and strong-\(\Delta\) convergent to \(x\) (i.e. the usual limit, when it exists, is also both polar and strong-\(\Delta\) limit).
(iii) If \((x_n)_{n \in \mathbb{N}}\) is \(\Delta\) (resp. strong-\(\Delta\), resp. polarly) -convergent to a point \(x\), then any subsequence \((x_{k_n})_{n \in \mathbb{N}}\) of \((x_n)_{n \in \mathbb{N}}\) is \(\Delta\) (resp. strong-\(\Delta\), resp. polarly) -convergent to \(x\).

(iv) \((x_n)_{n \in \mathbb{N}}\) is \(\Delta\) (resp. polarly) -convergent to a point \(x\) if and only if any subsequence \((x_{k_n})_{n \in \mathbb{N}}\) of \((x_n)_{n \in \mathbb{N}}\) admits a subsequence which is \(\Delta\) (resp. polarly) -convergent to \(x\).

(v) If \((x_n)_{n \in \mathbb{N}}\) is \(\Delta\)-convergent to a point \(x\), then it admits a subsequence which is strong-\(\Delta\) convergent to \(x\).

(vi) For any \(x \neq 0\) the sequence \((-1)^n x)_{n \in \mathbb{N}}\) has no \(\Delta\)-limit.

By combining (iv) and (v) it follows that a sequence \((x_n)_{n \in \mathbb{N}}\) \(\Delta\)-converges to a point \(x\) if every subsequence admits a subsequence strong-\(\Delta\) converging to \(x\).

**Proposition 2.9.** Let \((E, d)\) be a metric space, \(x \in E\) and let \((x_n)_{n \in \mathbb{N}}\) be a precompact sequence (or, in particular, a Cauchy sequence) in \(E\). If \(x_n \xrightarrow{\Delta} x\), then \(x_n \rightarrow x\).

**Proof.** Since the sequence \((x_n)_{n \in \mathbb{N}}\) is precompact, in correspondence to any \(\varepsilon > 0\), there exists a finite \(\varepsilon\)-net \(\mathcal{N}_\varepsilon\) of \((x_n)_{n \in \mathbb{N}}\). Then, by (2.2), for large \(n\), we have \(d(x_n, x) \leq \min_{y \in \mathcal{N}_\varepsilon} d(x_n, y) + o(1) < \varepsilon\), i.e., \(d(x_n, x) \rightarrow 0\). \(\square\)

**Definition 2.10.** A metric space \((E, d)\) is \(\Delta\)-complete (or satisfies the \(\Delta\)-completeness property) if every bounded sequence admits an asymptotic center.

An easy maximality argument, see for instance [13] Theorem A.1, allows us to give the following result.

**Theorem 2.11.** Let \((E, d)\) be a \(\Delta\)-complete metric space. Then every bounded sequence in \(E\) has a strong-\(\Delta\) converging subsequence.

**Proof.** Since the ordered set \((\Xi, \preceq)\) introduced in Remark [2.5] is countably inductive and since the function \(f\) which maps every sequence \(\xi = (x_n)_{n \in \mathbb{N}} \in \Xi\) into \(f(\xi) = -\text{rad}_{n \rightarrow \infty} x_n\) is a real strictly increasing function (see Remark [2.5]), by using [13] Theorem A.1 one obtains that every sequence \((x_n)_{n \in \mathbb{N}} \in \Xi\) has a maximal subsequence for \(\preceq\). Since \(E\) is \(\Delta\)-complete, this subsequence has an asymptotic center to which, since it is maximal, strong-\(\Delta\) converges (see Remark [2.5]). \(\square\)

This argument has been employed by Lim in [12] and, in a very close setting, in the proof of [10] Lemma 15.2. Note that the existence of a strictly increasing real valued function \(f\) and the separability of \(\mathbb{R}\) also allow to prove that the countable inductivity leads to inductivity and so one can deduce the existence of a maximal element by Zorn Lemma as in [12] Proposition 1. However, the direct argument in [13] Theorem A.1 looks even simpler than the proof of the inductivity.

**3. Rotund Metric Spaces**

**Definition 3.1.** A metric space \((E, d)\) is a (uniform) SR (“Staples rotund”) metric space (or satisfies (uniformly) property SR) if there exists a function \(\delta : (\mathbb{R}_+)^2 \rightarrow \mathbb{R}_+\) such that for any \(r, \delta > 0\), set \(\delta = \delta(r, \delta)\), for any \(x, y \in E\) with \(d(x, y) \geq \delta\):

\[
\text{rad}(B_{r+\delta}(x) \cap B_{r+\delta}(y)) \leq r - \delta.
\]
(The Chebyshev radius of a set $X$ in a metric space $(E, d)$ is the infimum of the radii of the balls containing $X$. In other words $\text{rad}(X) = \inf_{x \in E} \sup_{y \in X} d(x, y)$. Moreover, the Chebyshev radius and the Chebyshev centers of a set $X \subset E$ can also be defined, analogously to the asymptotic radius and asymptotic centers of a sequence, by replacing the functional $I$ in (2.4) by $I_X : x \mapsto \sup_{y \in X} d(x, y)$).

**Remark 3.2.** If $\delta$ is continuous, one can replace, in the above definition, the occurrences of $r + \delta$ by $r$. Moreover, from Definition 3.1 it is immediate that uniformly convex Banach spaces and Hadamard spaces are uniformly SR metric spaces.

**Remark 3.3.** From (3.1) one easily deduces that uniform SR metric spaces are convex in the sense of Menger [14]: A metric space $(E, d)$ is called convex if for any two different points $x, y \in E$ there is a point $m \in E$ such that $d(x, m) + d(m, y) = d(x, y)$. Every two points in a complete Menger-convex space can be connected by a metric segment.

**Remark 3.4.** Staples rotundity is close to the uniform ball convexity in Foertsch [8] and to the uniform convexity for hyperbolic metric spaces given by Reich and Shafrir [16]. However if, on one hand, the latter properties assume existence of a midpoint map which is not assumed by Staples, on the other one, Staples definition makes an additional assumption that roughly speaking amounts to the continuity of the modulus of convexity with respect to radius of the considered balls. In Banach spaces the above definitions coincide with the notion of uniform convexity. Particular examples of SR metric spaces are given by Hadamard spaces and, more in general, by $\text{CAT}(0)$ spaces.

For complete Staples rotund metric spaces, [18, Theorem 2.5] and [18, Theorem 3.3] state, respectively, uniqueness and existence of the asymptotic center of any bounded sequence (giving actually the $\Delta$-completeness of such spaces, see Definition 2.10). We are going to prove the just mentioned properties as one of the following claims which hold true in complete SR metric spaces.

For complete SR metric spaces we have the following results.

a) $\Delta$-convergence and polar convergence coincide.

b) Every bounded sequence has a unique asymptotic center.

c) The space is $\Delta$-complete. Therefore (the sequential compactness property in) Theorem 2.11 also holds for polar convergence as stated in the following theorem.

**Theorem 3.5.** Let $(E, d)$ be a complete SR metric space. Then every bounded sequence in $E$ has a polarly convergent subsequence.

Actually claim c) is just a restatement of claim b) and therefore it does not need to be proved.

**Lemma 3.6.** Let $(E, d)$ be a SR metric space. Let $(x_n)_{n \in \mathbb{N}} \subset E$ be a bounded sequence and let $x \in E$ be such that (2.3) holds true. Then, for each element $z \in E$, $z \neq x$, there exist positive constants $n_0$ and $c$ depending on $z$ such that

$$d(x_n, x) \leq d(x_n, z) - c \quad \text{for all} \quad n \geq n_0,$$

and so $(x_n)_{n \in \mathbb{N}}$ polarly converges to $x$. 

Proof. If the thesis is false, by (2.2), we can find \( z \neq x \) and a subsequence \((x_{k_n})_{n \in \mathbb{N}}\) such that \( d(x_{k_n}, x) - d(x_{k_n}, z) \rightarrow 0 \). Passing again to a subsequence we can also assume that \( d(x_{k_n}, x) \rightarrow r > 0 \). Set \( \bar{d} = d(x, z) \) and \( \delta = \delta(r, \bar{d}) \). Since, for large \( n \), \( x_{k_n} \in B_{r+\delta}(x) \cap B_{r+\delta}(z) \), we can deduce from (3.1) the existence of \( y \in E \) such that \( d(x_{k_n}, y) < r - \delta \), in contradiction to (2.2).

By taking into account the characterization of \( \Delta \)-limit given by Proposition 2.2, Lemma 3.6 guarantees that if \( x \rightarrow x \) then \( x_n \rightarrow x \). This proves claim a), since, as pointed out in Remark 2.8, polar convergence always implies \( \Delta \)-convergence to the same point.

Lemma 3.7. Let \( (E, d) \) be a SR metric space. Let \( I \) be defined by (2.4) in correspondence to a given sequence \((x_n)_{n \in \mathbb{N}} \subset E \). Then, for any \( \bar{d} > 0 \), there exists \( \varepsilon > 0 \) such that if \( x, y \in E \) satisfy \( \max(I(x), I(y)) < \inf I + \varepsilon \), then \( d(x, y) < \bar{d} \).

Proof. If \( \inf I = 0 \) one can take \( \varepsilon = \frac{\bar{d}}{4} \), otherwise, since \( E \) is a SR metric space (see (3.1)), one can take \( \varepsilon = \delta(\inf I, \bar{d}) \).

Two remarkable and immediate consequences of the above lemma trivially follow.

Corollary 3.8. Let \( (E, d) \) be a SR metric space and let \( I \) be as above. Then

(i) the functional \( I \) admits at most one minimum point;

(ii) any minimizing sequence of the functional \( I \) is a Cauchy sequence.

Corollary 3.9. Let \( (E, d) \) be a SR complete metric space, then any bounded sequence \((x_n)_{n \in \mathbb{N}} \subset E \), admits a unique asymptotic center. Moreover any minimizing sequence for \( I \) converges to the asymptotic center of the sequence \((x_n)_{n \in \mathbb{N}} \).

In other words, Corollary 3.9 states in particular that in a SR complete metric space every bounded sequence \((x_n)_{n \in \mathbb{N}} \) has a unique asymptotic center \( \text{cen}_{n \rightarrow \infty} x_n \), so that claim b) is proved.

4. Polar neighborhoods

In this section we investigate on the existence of a topology for polar convergence.

Let \( (E, d) \) be a metric space, let \( Y \subset E \), \( x \notin Y \), and let

\[
N_Y(x) = \bigcap_{y \in Y} N_y(x) = \{ z \in E \mid d(z, x) < d(z, y), \forall y \in Y \},
\]

where

\[
N_y(x) = \{ z \in E \mid d(z, x) < d(z, y) \}.
\]

In other words, the set \( N_Y(x) \) is the set of all points in \( E \) which are strictly closer to \( x \) than to \( Y \).

Remark 4.1. A trivial restatement of Definition 2.7 is that, if \( x \in E \) and \((x_n)_{n \in \mathbb{N}} \subset E \), then \( x_n \rightarrow x \) if and only if, for any finite set \( Y \neq x \), there exists \( M(Y) \in \mathbb{N} \) such that \( x_n \in N_Y(x) \) for all \( n \geq M(Y) \).

Definition 4.2. Let \( x \in E \). We shall call polar neighborhoods (briefly p-nbd) of the point \( x \) all the subsets \( V \subset E \) containing a set \( N_Y(x) \) given by (4.1), where \( Y \subset E \) is any finite set such that \( x \notin Y \).
Remark 4.3. The polar convergence can be finally tested, as it follows from Remark 4.1 by using polar neighborhoods. Indeed, if $x \in E$ and $(x_n)_{n \in \mathbb{N}} \subset E$, then $x_n \to x$ if and only if $x_n \in V$ for large $n$ for any polar neighborhood $V$ of $x$.

One can easily see that the set of polar neighborhoods of a given point $x$ is a filter of parts of $E$. It is well known (see [3] Prop. 1.2.2) that the union of such filters is a neighborhood base of a (unique) topology $T$ if and only if the following further property is satisfied

$$\forall x \in E, \forall V \text{ p-nbd. of } x, \exists U \text{ p-nbd of } x \text{ s.t. } \forall z \in U : V \text{ is a p-nbd of } z. \tag{4.3}$$

By Definition 4.2 the above condition can be more explicitly stated as

$$\forall x, y \in E, x \neq y, \exists Y \text{ s.t. } \forall z \in N_Y(x) \exists Z \text{ s.t. } N_Z(z) \subset N_Y(x). \tag{4.4}$$

When (4.3) (or equivalently (4.4)) is satisfied we shall call $T$ polar topology induced by $d$ on $E$. Since, for any $x, y \in E$ if $2r = d(x, y)$, $B_r(x) \subset N_y(x)$, polar topology is in general a coarser topology of that usually induced by $d$ by the classical notion of neighborhood. We shall refer to the latter topology as strong topology induced by $d$ when we want to distinguish it from the polar one.

We shall show now that (4.3) is true in some cases and false in others, and even when (4.3) is true, $T$ not always coincides with the strong topology induced by $d$.

Example 4.4. Let $\delta$ be the discrete metric on $E$. Then it is easy to see that $N_y(x) = \{x\}$ for all $x, y \in E, x \neq y$. Therefore the polar topology induced by $\delta$ is the discrete topology as well as the strong topology.

In Hilbert spaces polar topology coincides with weak topology, and thus is different from the strong topology in the infinite-dimensional case.

Example 4.5. Let $E$ be a Hilbert space. Then, for any $x, y \in E, x \neq y$, we have that $z \in N_v(x)$ means $\|z-x\|^2 - \|y\|^2 < 0$, namely $(2z-(x+y)) \cdot (y-x) < 0$. If, with an invertible change of variables, we set $x + y = 2a, y - x = v$, we get that $N_v(x) = \{z \in E \mid (z-a) \cdot v < 0\}$. So $N_v(x)$ gives, for $x, y \in E, x \neq y$, a base of the weak topology.

Combining the previous two examples we can discuss the following case.

Example 4.6. Let $E$ be a finite dimensional vector space and let $d$ be defined as $d(x, y) = |x - y| + \delta(x, y)$ where $\delta$ is as in Example 4.4. Then it is easy to check that the polar convergence induced by $d$ agrees with the natural convergence of vectors while the strong topology induced by $d$ is the discrete topology. In other words the strong topology induced by $d$ is the discrete topology (which, see Example 4.4) coincides with both the strong and the polar topology induced by $\delta$, i.e. by neglecting the contribution of $|x - y|$, while the polar topology induced by $d$ is the natural topology (which, see Example 4.5) coincides with both the strong and the polar topology induced by $|x - y|$, i.e. by neglecting the contribution of $\delta$.

Finally we show an example in which (4.3) is not satisfied and therefore polar topology does not exist.

Example 4.7. Let $E = \mathbb{R}$ and let $D$ be the Dirichlet Function (i.e. $D(x) = 1$ for $x \in \mathbb{Q}$ and $D(x) = 0$ for $x \notin \mathbb{Q}$). Let $d$ be defined as $d(x, y) = |x - y| + (1 + D(x-y))\delta(x, y)$ with $\delta$ as in Example 4.4. It is easy to see that $d$ is a metric (indeed, if $y \notin \{x, z\}$ then $d(x, z) \leq |x-z| + 2 \leq |x-y| + 1 + |y-z| + 1 \leq d(x, y) + d(y, z)$).
We shall prove that (1.4) does not hold. Fix any \( \overline{x}, \overline{y} \in \mathbb{R}, \overline{x} \neq \overline{y}, |\overline{x} - \overline{y}| \leq 1 \) and \( \overline{x} - \overline{y} \notin Q \). Consider \( \mathcal{N}_Y(\overline{x}) \) with an arbitrary finite set \( Y \not= \overline{x} \) and let \( \varepsilon > 0 \) such that \( 2\varepsilon < |\overline{x} - \overline{y}| \) for all \( y \in Y \). Then fix \( \overline{x} \in \mathbb{R} \) such that \( |\overline{x} - \overline{x}| < \varepsilon \) and \( \overline{x} - \overline{x} \notin Q \). This choice implies that

(4.5) \[ \forall y \in Y : \quad 2|\overline{x} - \overline{x}| < |\overline{x} - y| \quad \text{and so} \quad |\overline{x} - \overline{x}| < |\overline{x} - y| \]

by the triangle inequality. Therefore

(4.6) \[ d(\overline{x}, y) = |\overline{x} - x| + 1 < |\overline{x} - y| + 1 \leq d(\overline{x}, y) \quad \forall y \in Y, \]

proving that \( \overline{x} \in \mathcal{N}_Y(\overline{x}) \). In the same way, let \( Z \) be a given finite set such that \( \overline{x} \notin Z \) and fix \( \eta > 0 \) such that \( 2\eta < |\overline{x} - z| \forall z \in Z \). Fix \( x \in \mathbb{R} \) such that \( |\overline{x} - v| < \eta \) and \( \overline{x} - v \in Q \) (therefore \( v - \overline{x} \notin Q \), \( |v - \overline{x}| \notin Q \)). With analogous estimates to (4.3) and (4.4) we see that \( v \in \mathcal{N}_Z(\overline{x}) \). Finally, since \( v - \overline{x} \notin Q, |\overline{x} - v| \leq 1 \) and \( \overline{x} - v \in Q \),

\[ d(v, y) = |v - \overline{y}| + 1 \leq |v - \overline{x}| + |\overline{x} - \overline{y}| + 1 \leq |v - \overline{x}| + 2 = d(v, \overline{x}). \]

Hence for any \( \mathcal{N}_Y(\overline{x}) \) we can find \( \overline{x} \in \mathcal{N}_Y(\overline{x}) \) such that every neighborhood \( \mathcal{N}_Z(\overline{x}) \not= \mathcal{N}_Y(\overline{x}) \) and so (1.4) does not hold.

5. Polar convergence in Banach spaces

In a Hilbert space the polar limit of a sequence is also its weak limit and vice versa, see Example 5.3 (the original argument, although brought up to prove a weaker statement, is due to Opial, see [15] and [17] for more details). The following two examples, suggested by Michael Cwikel, illustrate some differences between polar and weak convergence in a Banach space.

**Example 5.1.** A sequence which is polarly converging need not be bounded, nor to weakly converge. For instance, let \( E \) be the sequence space \( \ell^1 \), and let, for any \( k \in \mathbb{N} \), \( x_k := k(\delta_{k,n})_{n \in \mathbb{N}} \) (we have used the Kronecker delta values, i.e. \( \delta_{k,n} = 1 \) for \( k = n \) and \( \delta_{k,n} = 0 \) otherwise). Since, for each fixed sequence \( \alpha = (\alpha_n)_{n \in \mathbb{N}} \in \ell^1 \), we have \( \lim_{k \to \infty} \|x_k - \alpha\|_{\ell^1} = \|\alpha\|_{\ell^1} \) we get that the sequence \( (x_k)_{k \in \mathbb{N}} \) is polarly convergent to the zero element of \( \ell^1 \).

**Example 5.2.** A weakly convergent sequence need not have any polar limit. For instance let \( E \) be the sequence space \( \ell^\infty \) and let \( x_k := (\delta_{k,n})_{n \in \mathbb{N}} \) for each \( k \in \mathbb{N} \). Clearly the sequence \( (x_k)_{k \in \mathbb{N}} \) converges weakly to the zero of \( \ell^\infty \). However, zero is not a polar limit of the sequence, since for the sequence \( \alpha = (1, 1, 1, \ldots) \) we have \( 1 = d(0, x_k) = d(\alpha, x_k) \) for all \( k \). Let \( \beta := (\beta_n)_{n \in \mathbb{N}} \) be an arbitrary nonzero element of \( \ell^\infty \). Then \( \beta_{n_0} \neq 0 \) for some integer \( n_0 \) and we can define the sequence \( \gamma := (\gamma_n)_{n \in \mathbb{N}} \) by setting \( \gamma_{n_0} = 0 \) and \( \gamma_n = \beta_n \) for all \( n \neq n_0 \). We see that \( d(\beta, x_k) \geq d(\gamma, x_k) \) for all \( k > n_0 \) which shows that \( \beta \) cannot be the polar limit of \( (x_k)_{k \in \mathbb{N}} \).

The following characterization of polar convergence in uniformly convex and uniformly smooth Banach spaces is given in [17]. The duality map \( x \mapsto x^* \) is understood here with normalization \( \|x^*\| = 1 \).

**Theorem 5.3.** Let \( E \) be a uniformly convex and uniformly smooth Banach space. Let \( x \in E \) and let \( (x_n)_{n \in \mathbb{N}} \subset E \) be a bounded sequence such that \( \liminf_n \|x_n - x\| > 0 \). Then \( x_n \to x \) if and only if \( (x_n - x)^* \to 0 \).
From this characterization one can easily conclude that polar and weak convergence coincide in Hilbert spaces and in $\ell^p$-spaces for $1 < p < \infty$. This is not the case for $L^p$-spaces of Euclidean domains: an example of Opial, see [15, Section 5], interpreted in terms of polar convergence, presents bounded sequences in $L^p((0, 2\pi))$, $p \neq 2$, $1 < p < \infty$, whose polar limit and weak limit are different functions.

6. Applications

6.1. Convergence of iterations to fixed points. The requirement [15, Condition (2)] given by Opial and quoted below, plays significant role in the fixed point theory.

**Definition 6.1.** One says that a Banach space $E$ satisfies Opial condition if for every sequence $(x_n)_{n \in \mathbb{N}} \subset E$, which is weakly convergent to a point $x \in E$,

\[(6.1) \quad \liminf_{n} \|x_n - x\| \leq \liminf_{n} \|x_n - y\| \text{ for every } y \in E.\]

It is worth mentioning that van Dulst in [6] proved that a separable uniformly convex Banach space can be provided with an equivalent norm satisfying the Opial condition.

Fixed points of nonexpansive maps (maps with the Lipschitz constant equal to 1) in suitable Banach spaces have been obtained as asymptotic centers of iterative sequences (see [7, Corollary to Theorem 2] for suitable Banach spaces and [18, Claim 2.7] for suitable metric spaces). Also in bounded complete SR metric spaces the existence of fixed points for a nonexpansive map $T$ easily follows, since, for any $x$, the asymptotic center of the iterations sequence $(T^n(x))_{n \in \mathbb{N}}$ is a fixed point of $T$. However, in general, it is not possible to get a fixed point as a polar limit of an iterations sequence. Indeed, Theorem 2.11 only gives the existence of the polar limit of a subsequence and this one, as pointed out in the following example, (even in the case of metric convergence) is not necessarily a fixed point.

**Example 6.2.** Consider the map $T(x) = -x$ on a unit ball in any Banach space. Zero is the unique fixed point of $T$ and the iterations sequence $((-1)^n x)_{n \in \mathbb{N}}$ has zero as asymptotic center. On the other hand, if $x \neq 0$, this sequence has no subsequence that $\Delta$-converges to zero and has two subsequences which converge to $x$ and $-x$, which are not fixed points.

On a bounded complete SR metric space, we can give the following characterization of polarly converging iterations sequences of a nonexpansive map.

**Proposition 6.3.** Let $(E, d)$ be a bounded complete SR metric space. Let $x \in E$ and let $T : E \to E$ be a nonexpansive map. Then, the following properties are equivalent.

(a) The iterations sequence $(T^n(x))_{n \in \mathbb{N}}$ is polarly converging;

(b) The polar limit of any subsequence of $(T^n(x))_{n \in \mathbb{N}}$ (when it exists) is a fixed point of $T$.

**Proof.** We shall only prove that (b) implies (a) (the converse implication being obvious by (iii)). We shall prove, in particular, that if (b) holds true, then the sequence $(T^n(x))_{n \in \mathbb{N}}$ is maximal with respect to the ordering $\preceq$ introduced in Remark 2.5. To this aim let $(T^{k_n}(x))_{n \in \mathbb{N}}$ be a subsequence of $(T^n(x))_{n \in \mathbb{N}}$ and let
$r = \text{rad}_{n \to \infty} T^{k_n}(x)$. By Theorem 2.11 we can replace \((T^{k_n}(x))_{n \in \mathbb{N}}\) by a subsequence (which we still denote by \((T^{k_n}(x))_{n \in \mathbb{N}}\), note that the value of \(r\) does not increase) which is polarly converging to a point \(y\). Then, if \(\varepsilon > 0\) is an arbitrarily fixed real number, there exists \(m \in \mathbb{N}\) such that \(d(T^{k_m}(x), y) < r + \varepsilon\). Since, by assumption (b), \(y\) is a fixed point for \(T\), we get, by induction, \(d(T^n(x), y) < r + \varepsilon\), for any \(n \geq k_m\). (Indeed, if \(n \geq k_m\), we have \(d(T^{n+1}(x), y) = d(T^{n+1}(x), T(y)) \leq d(T^n(x), y) < T^{k_m}(x) < r + \varepsilon\), since \(T\) is nonexpansive). Therefore \(\text{rad}_{n \to \infty} T^n(x) < r + \varepsilon\). By the arbitrariness of \(\varepsilon > 0\) we get \(\text{rad}_{n \to \infty} T^n(x) \leq r = \text{rad}_{n \to \infty} T^{k_n}(x)\).

For spaces where the Opial condition holds, additional conditions, such as asymptotic regularity, are imposed to assure weak convergence of iterative sequences (see [5] and [15] Theorem 2). However, the argument used in [15] suggests that the role of Opial condition consists in deducing weak convergence of iterations from their polar convergence.

**Definition 6.4.** Let \((E, d)\) be a metric space, \(T : E \to E\) and \(x \in E\). We say that the map \(T\) satisfies the PAR-condition (polar asymptotical regularity condition) at \(x\) if, for any \(y \in E\) polar limit of a subsequence \((T^{k_n}(x))_{n \in \mathbb{N}}\) of the iterations sequence \((T^n(x))_{n \in \mathbb{N}}\) and for all \(\varepsilon > 0\), there exists \(m \in \mathbb{N}\) such that \(d(T^{k_m-1}(x), y) < d(T^{k_m}(x), y) + \varepsilon\) for any \(n \geq m\).

**Theorem 6.5.** Let \((E, d)\) be a bounded complete SR metric space. Let \(T : E \to E\) be a nonexpansive map which satisfies the PAR-condition at a point \(x \in E\). Then, the sequence \((T^n(x))_{n \in \mathbb{N}}\) polarly converges to a fixed point.

**Proof.** We shall prove (b) in Proposition 6.3. Let \((T^{k_n}(x))_{n \in \mathbb{N}}\) be a polarly converging subsequence of \((T^n(x))_{n \in \mathbb{N}}\) and let \(y\) be the polar limit of \((T^{k_n}(x))_{n \in \mathbb{N}}\). Since \(T\) is a nonexpansive map which satisfies the PAR-condition in \(E\) we have that \(d(T^{k_n}(x), T(y)) \leq d(T^{k_n-1}(x), y) < d(T^{k_n}(x), y) + \varepsilon\) for any \(\varepsilon > 0\) for large \(n\). Therefore also \(T(y)\) is an asymptotic center of \((T^{k_n}(x))_{n \in \mathbb{N}}\) and, since in SR spaces the asymptotic center is unique, we get \(T(y) = y\). □

### 6.2. Brezis-Lieb Lemma without a.e. convergence

The celebrated Brezis-Lieb Lemma ([4]) is stated for a.e. converging sequences, which need to converge weakly and therefore (by Theorem 5.3) also polarly in \(L^p\). Remarkably, convergence a.e. is not needed when \(p = 2\), which suggests that some version of Brezis-Lieb Lemma may hold for other \(p\) if a.e. convergence in the assumption is replaced by weak and polar convergence. Indeed, in [17] the following result is proved.

**Theorem 6.6.** Let \((\Omega, \mu)\) be a measure space. Assume that \(u_n \rightharpoonup u\) and \(u_n \rightharpoonup u\) in \(L^p(\Omega, \mu)\). If \(p \geq 3\) then

\[
\int_{\Omega} |u_n|^p d\mu \geq \int_{\Omega} |u|^p d\mu + \int_{\Omega} |u_n - u|^p d\mu + o(1).
\]

It is shown in [11] that the condition \(p \geq 3\) cannot be removed, except when \(p = 2\). Moreover, one can see by easy examples with \(p = 4\) (see [17]) that, in general, the equality does not hold in (6.2).

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