Cluster algebras and semi-invariant rings I. Triple flags

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Abstract
We prove that each semi-invariant ring of the complete triple flag of length \( n \) is an upper cluster algebra associated to an ice hive quiver. We find a rational polyhedral cone \( G_n \) such that the generic cluster character maps its lattice points onto a basis of the upper cluster algebra. As an application, we use the cluster algebra structure to find a special minimal set of generators for these semi-invariant rings when \( n \) is small.

Introduction
The cluster algebra has established its wide connection with many areas in mathematics since its discovery by Fomin and Zelevinsky one and half decade ago. There are two mainstreams of results — one is categorifying cluster algebras, the other is finding cluster structure in natural mathematical objects. The purpose of this project is two-fold. First, we hope to obtain more new examples, and put many old examples in a more uniform framework. Second, we want to use the cluster algebra structure in an essential way to prove new results which seem unreachable by traditional methods.

The earliest examples of cluster algebras are the coordinate rings of the base affine spaces \( \text{SL}_n / U \) and Grassmannians \( \text{Gr}(\mathbb{C}) \) [15]. Later, the latter family was generalized by Scott to all Grassmannians of type \( A \) [33], and by other authors to other types, remarkably [17]. For quite a long time, they served as the main examples of cluster algebras. Afterwards, there are two kinds of generalizations, one is in a Lie-theoretic direction, and the other is in an invariant-theoretic direction. In [1], authors considered double Bruhat cells, and in [17], authors considered unipotent cells and partial flag varieties. Remarkably, Geiss, Leclerc, and Schröer further generalized some of previous examples to the Kac–Moody setting [18]. For the invariant-theoretic direction, Fomin and Pylyavskyy considered in [14] the semi-invariant ring of vectors and covectors in dimension three. It is further generalized to arbitrary dimension by Carde in a work-in-progress.

In this paper, we are going to see the cluster algebra structure in semi-invariant rings of quiver representation spaces. For a fixed dimension vector \( \beta \) of a quiver \( Q \), the space of all \( \beta \)-dimensional representations is

\[
\text{Rep}_\beta(Q) := \bigoplus_{a \in Q_1} \text{Hom}(k^{\beta(t(a))}, k^{\beta(h(a))}).
\]

The product of special linear group

\[
\text{SL}_\beta := \prod_{v \in Q_0} \text{SL}_{\beta(v)}
\]

acts on \( \text{Rep}_\beta(Q) \) by the natural base change. The rings of semi-invariants is by definition

\[
\text{SI}_\beta(Q) := k[\text{Rep}_\beta(Q)]^{\text{SL}_\beta}.
\]
Many previous examples fall into this construction (up to some localization), including Grassmannians, partial flags, unipotent cells, double Bruhat cells, and mixed invariant rings at least in type $A$. So we just saw a tip of an iceberg.

However, it is impractical to consider any quiver with arbitrary dimension vector from the beginning. The first family of quivers we consider here is the triple flag quivers $T_{p,q,r}$. Such a family has established its importance around 2000 when Derksen and Weyman proved the saturation conjecture for the type-$A$ Littlewood–Richardson coefficients $\binom{5}{\mu}$. A triple flag $T_{p,q,r}$ with a dimension vector $\beta$ is called complete if $p = q = r = n$ and $\beta = \beta_n$ is standard as indicated below.

One main result is that

**Theorem 0.1 (Theorem 8.11).** $\text{SI}_{\beta_n}(T_n)$ is the upper cluster algebra $\mathcal{C}(\Delta_n, S_n)$ associated to the seed $(\Delta_n, S_n)$.

Here, $\Delta_n$ is the ice hive quiver to be introduced in Section 6. When $n = 5$, it is displayed in Figure 2. The initial cluster variables together with the coefficients $[16]$ in $S_n$ can be explicitly described in terms of Schofield’s semi-invariants $[30]$. By some rather trivial checking, we can show that the upper cluster algebra $\mathcal{C}(\Delta_n)$ is equal to the cluster algebra $\mathcal{C}(\Delta_n)$ for $n < 6$. However, we do not know if we always have the equality for any $n$.

To show $\mathcal{C}(\Delta_n, S_n) \subseteq \text{SI}_{\beta_n}(T_n)$, we use an idea similar to $[14]$. The hard part is to show the other containment, because we do not even know a finite set of generators of $\text{SI}_{\beta_n}(T_n)$. For this, we consider another well-known model for the (type-A) Littlewood–Richardson coefficients — Knutson–Tao’s hive model $[24]$. In fact, we consider a variation provided by $[26]$ for the sake of exposition. We show that each hive polytope is unimodularly isomorphic to certain polytope $G_n(\sigma)$ inside the rational convex cone $G_n$ spanned by $\mu$-supported $g$-vectors. As a consequence, the lattice points of $G(\sigma)$ count the dimension of $\sigma$-graded piece of $\text{SI}_{\beta_n}(T_n)$. One innovative part of this paper is using geometric invariant theory to explicitly describe the cone $G_n$ in terms of representation theory. The method actually works quite generally (for example, $[10]$).

To finish the proof, we need help from the machinery of quivers with potentials $[7, 8]$. We assign a rigid potential $W_n$ to $\Delta_n$ such that the associated Jacobian algebra $J_n := J(\Delta_n, W_n)$ is finite dimensional. We can view the lattice points of $G_n(\sigma)$ inside $K_0(\text{proj-}J_n)$, the Grothendieck group of the homotopy category of bounded complexes of projective representations of $J_n$. The other containment follows from the fact that the generic character of Plamondon $[27]$ maps the lattice points of $G_n(\sigma)$ bijectively to a linearly independent set in the upper cluster algebra. As a corollary of the proof, we construct a basis of $\mathcal{C}(\Delta_n)$. This can be viewed as an algebraic analog of Fock–Goncharov conjecture for the quotient space $\text{Rep}_{\beta_n}(T_n)/\text{SL}_{\beta_n}$ $[13]$.

**Theorem 0.2 (Corollary 8.12).** The generic character maps the lattice points of $G_n$ bijectively to a basis of $\mathcal{C}(\Delta_n)$.

So far only in few cases, a basis of a (upper) cluster algebra with coefficients is known. To author’s best knowledge, they are bases for cluster algebras of surface type $[25]$, and cluster algebras arising from unipotent cells $[19]$. However, the former only deals with the principal coefficients, the latter only deals with another special set of coefficients. We point out that
Kimura and Qin constructed in [22] a remarkable positive quantum basis in a setting similar to [19].

Another reason why triple flags are important is that many quivers (if not all) can be embedded via quiver exceptional sequences into the (possibly noncomplete) triple flags in the sense of [6, 30]. For example, the quiver semi-invariant ring realizations of all type-A partial flag varieties (including Grassmannians), unipotent cells, and invariants of vector and covectors can be embedded into the triple flags. The results in [12] give an algorithm to find the cluster algebra structure of any such embedding into the triple flag. So finding the above-mentioned cluster structure will be ultimately reduced to the case of complete triple flags. The results here combined with the technique in [12] have been successfully applied to study the Kronecker coefficients [10]. However, the algorithm is not deterministic and does not always succeed (see [12, Example 3.22] for a negative example).

Except for getting new examples, we can use the cluster algebra structure to find generators and relations of semi-invariant rings. This will be done in a more general framework somewhere else. In the present paper, we find a special minimal set of generators of $\text{SI}_{\beta_n}(T_n)$ for each $n < 6$. The author tried to prove the same result for $n = 4$ using a geometric method similar to [3], but the proof is quite long [11]. The method seems hard to be generalized to bigger $n$. Now with the help of the cluster structure, we can finish all proofs in few pages.

This paper is organized as follows. In Section 1, we recall the work of Schofield, Derksen, and Weyman, etc., on the semi-invariant rings of quiver representations. In Section 2, we specialize the general theory to the family of triple flag quivers and make connection with the Littlewood–Richardson coefficients following [5, 6]. In Section 3, we recall the definition of cluster algebras and their upper bounds. We introduce the weight configuration in Definition 3.11, which serves as the first layer of the possible cluster structure.

In Section 4, we recall the mutation of quivers with potentials and their representations following [7, 8]. Since we need to deal with a general coefficient system, we consider something slightly more general, called ice quivers with potentials and their $\mu$-supported representations (Definition 4.8). In Section 5, we recall the cluster character $C$ date back to Caldero–Chapton. The version that we consider is in the setting of [7, 8]. Theorem 5.6 gives a sufficient condition for the image of a set of representations being a linearly independent set in the upper cluster algebra. Specializing this result to the generic character, we get a similar result in [27]. The key difference is that our domain $G(Q,W)$ is a subset of the full lattice $K_0(\text{proj-J})$ due to the difference in the definition of upper cluster algebras.

In Section 6, we introduce the ice hive quiver with potential $(\Delta_{n_1},W_n)$. We give in Theorem 6.8 a complete description of the domain $G(\Delta_{n_1},W_n)$ as lattice points of certain rational polyhedral cone $G_n$. In Section 7, we prove in Theorem 7.3 that there is a unimodular linear transformation mapping the cone $G_n$ onto the Littlewood–Richardson cone.

In Section 8, we prove our main results. We establish the cluster structure of $\text{SI}_{\beta_n}(T_n)$ in Theorem 8.11. One difficult step is to show that our chosen initial seed is algebraically independent (Theorem 8.8). As a corollary, we construct a basis of these (upper) cluster algebras. In Section 9, we use the cluster structure to find a special minimal set of generators of $\text{SI}_{\beta_n}(T_n)$ for $n < 6$.

Notations and conventions. Our vectors are exclusively row vectors. All modules are right modules. For a quiver $Q$, we denote by $Q_0$ the set of vertices and by $Q_1$ the set of arrows. For an arrow $a$, we denote by $t(a)$ and $h(a)$ its tail and head. Arrows are composed from left to right, that is, $ab$ is the path $\cdot \overset{a}{\rightarrow} \cdot \overset{b}{\rightarrow} \cdot$. Throughout the paper, the base field $k$ is algebraically closed of characteristic zero. Unadorned $\text{Hom}$ and $\otimes$ are all over the base field $k$, and the superscript $*$ is the trivial dual. For any representation $M$, $\dim M$ is the dimension vector of $M$. For direct sum of $n$ copies of $M$, we write $nM$ instead of the traditional $M^{\oplus n}$.
1. Semi-invariants of quiver representations

1.1. Schofield’s construction

Let us briefly recall the semi-invariant rings of quiver representations [30]. Let $Q$ be a finite quiver without oriented cycles. We fix an algebraically closed field $k$ of characteristic zero. For a dimension vector $\beta$ of $Q$, let $V$ be a $\beta$-dimensional vector space $\prod_{i \in Q_0} k^{\beta(i)}$. We write $V_i$ for the $i$th component of $V$. The space of all $\beta$-dimensional representations is

$$\text{Rep}_\beta(Q) := \bigoplus_{\alpha \in Q_1} \text{Hom}(V_{t(a)}, V_{h(a)}).$$

The product of general linear group

$$\text{GL}_\beta := \prod_{i \in Q_0} \text{GL}(V_i)$$

acts on $\text{Rep}_\beta(Q)$ by the natural base change. We consider the right action of the subgroup

$$\text{SL}_\beta := \prod_{i \in Q_0} \text{SL}(V_i)$$

on the coordinate ring $k[\text{Rep}_\beta(Q)]$:

$$g(f)(x) = f(gx).$$

We are interested in the rings of semi-invariants

$$\text{SI}_\beta(Q) := k[\text{Rep}_\beta(Q)]^{\text{SL}_\beta} = \{f \in k[\text{Rep}_\beta(Q)] \mid g(f) = f, \forall g \in \text{SL}_\beta\}.$$ 

The ring $\text{SI}_\beta(Q)$ has a weight space decomposition

$$\text{SI}_\beta(Q) = \bigoplus_\sigma \text{SI}_\beta(Q)_\sigma,$$

where $\sigma$ runs through the multiplicative characters of $\text{GL}_\beta$. We refer to such a decomposition the $\sigma$-grading of $\text{SI}_\beta(Q)$. Recall that any character $\sigma : \text{GL}_\beta \to k^\ast$ can be identified with a weight vector $\sigma \in \mathbb{Z}^{Q_0}$

$$(g(i))_{i \in Q_0} \mapsto \prod_{i \in Q_0} (\det g(i))^{\sigma(i)}.$$  \hspace{1cm} (1.1)

Since $Q$ has no oriented cycles, the degree zero component is the field $k$ [23].

Let us understand these multihomogeneous components

$$\text{SI}_\beta(Q)_\sigma := \{f \in k[\text{Rep}_\beta(Q)] \mid g(f) = \sigma(g)f, \forall g \in \text{GL}_\beta\}.$$ 

For any projective presentation $f : P_1 \to P_0$, we view it as an element in the homotopy category $K^b(\text{proj}-Q)$ of bounded complexes of projective representations of $Q$. The weight vector $f$ of $f$ is the corresponding element in the Grothendieck group of $K^b(\text{proj}-Q)$. Concretely, suppose that $P_1 = P(\sigma_1)$ and $P_0 = P(\sigma_0)$ for $\sigma_i \in \mathbb{N}^{Q_0}$, then $f = \sigma_1 - \sigma_0$. Here, we use the notation $P(\sigma)$ for $\bigoplus_{i \in Q_0} \sigma(i)P_i$, where $P_i$ is the indecomposable projective representation corresponding to the vertex $i$. From now on, we will view a weight $\sigma$ as an element in the dual $\text{Hom}_\mathbb{Z}(\mathbb{Z}^{Q_0}, \mathbb{Z})$ via the usual dot product.

We set $\sigma = f$, and assume that $\sigma(\beta) = 0$. We apply the functor $\text{Hom}_Q(-, N)$ to $f$ for $N \in \text{Rep}_\beta(Q)$

$$\text{Hom}_Q(P_0, N) \xrightarrow{\text{Hom}_Q(f, N)} \text{Hom}_Q(P_1, N).$$ \hspace{1cm} (1.2)
Since \( \sigma(\beta) = 0 \), \( \text{Hom}_Q(f, N) \) is a square matrix. Following Schofield \([30]\), we define
\[
s(f, N) := \det \text{Hom}_Q(f, N).
\]
We give a more concrete description for the map \( \text{Hom}_Q(f, N) \).

**Concrete description of \( \text{Hom}_Q(f, N) \):** Recall that a morphism \( P_1 \xrightarrow{f} P_0 \) can be represented by a matrix whose entries are linear combination of paths. Applying \( \text{Hom}_Q(-, N) \) to this morphism is equivalent to that we first transpose the matrix of \( f \), then substitute paths in the matrix by corresponding matrix representations in \( N \).

We set \( s(f)(-) = s(f, -) \) as a function on \( \text{Rep}_\beta(Q) \). It is proved in \([30]\) that \( s(f) \in \text{SI}_\beta(Q)_\sigma \). In fact,

**Theorem 1.1 \([5, 32]\).** The functions \( s(f) \) span \( \text{SI}_\beta(Q)_\sigma \) over the base field \( k \).

It is easy to see that if \( s(f) \neq 0 \), then \( f \) resolves some representation \( M \), say of dimension \( \alpha \)
\[
0 \to P_1 \xrightarrow{f} P_0 \to M \to 0.
\]
From the long exact sequence
\[
\text{Hom}_Q(M, N) \xrightarrow{\text{Hom}_Q(f, N)} \text{Hom}_Q(P_1, N) \to \text{Ext}_Q(M, N),
\]
we see that \( \alpha \) and \( \sigma \) are related by \( \sigma(-) = -\langle \alpha, - \rangle_Q \), where \( \langle -, - \rangle_Q \) is the Euler form of \( Q \). In this case, we call \( \sigma \) the weight vector corresponding to \( \alpha \), and denote it by \( \sigma \alpha \); and conversely we call \( \alpha \) the dimension vector corresponding to \( \sigma \), and denote it by \( \alpha \sigma \). It also follows from (1.3) that \( s(f, N) \neq 0 \) if and only if \( \text{Hom}_Q(M, N) = 0 \), or equivalently \( \text{Ext}_Q(M, N) = 0 \). In this case, we say \( M \) is (left) orthogonal to \( N \), denoted by \( M \perp N \).

We want to point out that the function \( s(f) \) is determined, up to a scalar multiple, by the homotopy equivalence class of \( f \), and thus by the isomorphism class of \( M \). If one hopes to define the function \( s(f) \) uniquely in terms of representations, one can take the canonical resolution \( f_{\text{can}} \) of \( M \). In this case, we denote \( s(M) := s(f_{\text{can}}) \). This is Schofield’s original definition in \([30]\).

### 1.2. Stability

We define the subgroup \( \text{GL}_\beta^\sigma \) to be the kernel of the character map (1.1). The invariant ring of its action is
\[
\text{SI}_\beta(Q)_\sigma := k[\text{Rep}_\beta(Q)]^{\text{GL}_\beta^\sigma} = \bigoplus_{n \geq 0} \text{SI}_\beta(Q)_{n \sigma}.
\]

**Definition 1.2.** A representation \( M \in \text{Rep}_\beta(Q) \) is called \( \sigma \)-semi-stable if there is some nonconstant \( f \in \text{SI}_\beta(Q) \) such that \( f(M) \neq 0 \). It is called stable if the orbit \( \text{GL}_\beta^\sigma.M \) is closed of dimension equal to \( \dim \text{GL}_\beta^\sigma - 1 \).

Based on Hilbert–Mumford criterion, King provided a simple criterion for the stability of a representation.

**Lemma 1.3 \([23, \text{Proposition 3.1}]\).** A representation \( M \) is \( \sigma \)-semi-stable (respectively, \( \sigma \)-stable) if and only if \( \sigma(\dim M) = 0 \) and \( \sigma(\dim L) \geq 0 \) (respectively, \( \sigma(\dim L) > 0 \)) for any nontrivial subrepresentation \( L \) of \( M \).

Let \( \Sigma_\beta(Q) \) be the set of all weights \( \sigma \) such that \( \text{SI}_\beta(Q)_\sigma \) is nonempty. Since \( \text{SI}_\beta(Q)_0 = k \), it spans a pointed cone \( \mathbb{R}^+ \Sigma_\beta(Q) \). The next theorem is an easy consequence of King’s stability
criterion and Theorem 1.1. By a general $\beta$-dimensional representation, we mean in a sufficiently small Zariski open subset (‘sufficient’ here depends on the context). Following [31], we use the notation $\gamma \hookrightarrow \beta$ to mean that a general $\beta$-dimensional representation has a $\gamma$-dimensional subrepresentation.

**Theorem 1.4** [5, Theorem 3]. We have

$$\Sigma_\beta(Q) = \{ \sigma \in \text{Hom}_Z(Z^{Q_{\beta}}, Z) \mid \sigma(\beta) = 0 \text{ and } \sigma(\gamma) \geq 0 \text{ for all } \gamma \hookrightarrow \beta \}.$$  

In particular, $\Sigma_\beta(Q)$ is a saturated semigroup.

Schofield relates the condition $\gamma \hookrightarrow \beta$ to some generic homological condition. Following [31], we introduce the notation $\text{hom}_Q(\alpha, \beta)$ (respectively, $\text{ext}_Q(\alpha, \beta)$) to denote the dimension of the space of homomorphisms (respective, extensions) from a general $\alpha$-dimensional representation to a general $\beta$-dimensional representation of $Q$. Let

$$\text{Rep}_{\gamma \hookrightarrow \beta}(Q) = \{ M \in \text{Rep}_\beta(Q) \mid M \text{ has a } \gamma \text{-dimensional subrepresentation} \}.$$ 

This is a closed subvariety of $\text{Rep}_\beta(Q)$.

**Lemma 1.5** [31, 3]. The codimension of $\text{Rep}_{\gamma \hookrightarrow \beta}(Q)$ is equal to $\text{ext}_Q(\gamma, \beta - \gamma)$.

**Lemma 1.6** [5, Lemma 1]. Suppose that we have an exact sequence of representations of $Q$

$$0 \to L \to M \to N \to 0,$$

with $\langle \text{dim} L, \beta \rangle = \langle \text{dim} N, \beta \rangle = 0$, then as a function on $\text{Rep}_\beta(Q)$, $s(M)$ is, up to a scalar, equal to $s(L)s(N)$.

Recall that a dimension vector $\alpha$ is called a real root if $\alpha$ is a dimension vector of an indecomposable representation and $\langle \alpha, \alpha \rangle_Q = 1$. It is called Schur if $\text{Hom}_Q(M, M) = k$ for $M$ general in $\text{Rep}_\alpha(Q)$. For a real Schur root $\alpha$, $\text{Rep}_{\alpha n}(Q)$ has a dense orbit for any $n \in \mathbb{N}$.

**Lemma 1.7.** If $\alpha$ is a real Schur root, then $\dim\text{SI}_\beta(Q)_{-\langle n\alpha, -\rangle} = 1$ for any $n \in \mathbb{N}$.

**Proof.** As an easy consequence of Theorem 1.1, we have the following reciprocity property

$$\dim\text{SI}_\beta(Q)_{-\langle -\rangle} = \dim\text{SI}_\alpha(Q)_{-\langle -\beta \rangle}.$$  

Since $\text{Rep}_{\alpha n}(Q)$ has a dense orbit, the ring of rational invariants $k(\text{Rep}_{\alpha n}(Q))^{\text{GL}_{\alpha n}}$ is trivial. So if $f, g \in \text{SI}_{\alpha n}(Q)_{-\langle -\beta \rangle}$, then $f/g \in k(\text{Rep}_{\alpha n}(Q))^{\text{GL}_{\alpha n}} = k$. The result follows from the reciprocity property.

A weight is called extremal in $\Sigma_\beta(Q)$ if it lies on an extremal ray of $\mathbb{R}^+\Sigma_\beta(Q)$. An indivisible extremal weight $\sigma$ must be indecomposable in $\Sigma_\beta(Q)$, that is, it cannot be written as $\sigma = \sigma_1 + \sigma_2$ with $\sigma_i \in \Sigma_\beta(Q)$.

**Lemma 1.8.** If $\sigma$ is an indivisible extremal weight, then any semi-invariant function $s$ of weight $\sigma$ is an irreducible polynomial.

**Proof.** Suppose that $\sigma$ spans an extremal ray $r$. Since the cone $\mathbb{R}^+\Sigma_\beta(Q)$ is pointed and $r$ is extremal, there is a set of dimension vectors $\{ \gamma_i \}$ such that $r$ is defined by the hyperplanes $\langle -, \gamma_i \rangle = 0$. Moreover, $\theta(\gamma_i) \geq 0$ for all $\theta \in \Sigma_\beta(Q)$ and all $i$, and $\theta(\gamma_i) > 0$ for some $i$ if $\theta \notin r$. 

We define a total degree $d$ on the semi-invariant ring by setting $d(\sigma) = \sum_i \sigma(\gamma_i) \geq 0$. Note that the total degree 0 component is $k \oplus SI_0^\sigma(Q)$.

Suppose that $s$ factors as $s = s_1s_2$, then both $s_1$ and $s_2$ must be homogeneous of total degree 0. Since $\sigma$ is indivisible, it is clear that one of them has to be a unit. Finally, we remark that being irreducible in the semi-invariant ring is equivalent to being irreducible in the polynomial ring due to \[28, \text{Theorem 3.1}\].

\[\square\]

2. The triple flag quivers

In this section, we specialize the previous discussion to the case of triple flag quivers, and make connection with the Littlewood–Richardson coefficients following \[5, 6\].

Let $T_{p,q,r}$ be the quiver with $p + q + r - 2$ vertices:

\[
\begin{align*}
1^1 & \rightarrow 1^2 \rightarrow \cdots \rightarrow 1^{p-2} \rightarrow 1^{p-1} \\
2^1 & \rightarrow 2^2 \rightarrow \cdots \rightarrow 2^{q-2} \rightarrow 2^{q-1} \rightarrow 3^r \\
3^1 & \rightarrow 3^2 \rightarrow \cdots \rightarrow 3^{r-2} \rightarrow 3^{r-1}
\end{align*}
\]

We use the convention $1^p = 2^q = 3^r$. We know from \[5\] that if we take the standard dimension vector

\[
\beta_n = \begin{pmatrix} 1 & 2 & \cdots & n-1 \\ 1 & 2 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-1 \end{pmatrix}
\]

for $T_n := T_{n,n,n}$, then we can view $\dim SI_{\beta_n}(T_n)_{\sigma}$ as a type-A Littlewood–Richardson coefficient.

\textbf{Theorem 2.1} \[6, 7.1\]. If $\sigma$ is given by

\[
\sigma = \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} \\ b_1 & b_2 & \cdots & b_{n-1} & c_n \\ c_1 & c_2 & \cdots & c_{n-1} \end{pmatrix},
\]

then

\[
\dim SI_{\beta_n}(T_n)_{\sigma} = c^\lambda_{\mu,\nu},
\]

where

\[
\mu = \mu(\sigma) = -(a_1 + \cdots + a_{n-1}, a_2 + \cdots + a_{n-1}, \cdots, a_{n-1}), \quad (2.1)
\]

\[
\nu = \nu(\sigma) = -(b_1 + \cdots + b_{n-1}, b_2 + \cdots + b_{n-1}, \cdots, b_{n-1}), \quad (2.2)
\]

\[
\lambda = \lambda(\sigma) = (c_n, c_n + c_{n-1}, \cdots, c_n + c_{n-1} + \cdots + c_1). \quad (2.3)
\]

Note that for $T_n$ a Schur root either has support on one arm (in which case it corresponds to a positive root of $A_n$), or it is nondecreasing along each arm. A facet of the cone $R_+\Sigma_{\beta_n}(T_n)$ is a codimension one face of $R_+\Sigma_{\beta_n}(T_n)$. A weight in $\Sigma_{\beta_n}(T_n)$ is called extremal if it lies on an extremal ray (dimension one face) of $R_+\Sigma_{\beta_n}(T_n)$.

\textbf{Theorem 2.2} \[6, \text{Theorem 7.8}\]. For every pair $(\gamma, \beta_n)$ with $\beta_n = \gamma + \beta'_n$, $\gamma, \beta'_n$ nondecreasing along arms, and $\gamma \circ \beta'_n = 1$, the inequality $\sigma(\gamma) \geq 0$ defines a facet of $R_+\Sigma_{\beta_n}(T_n)$. All nontrivial facets can be uniquely obtained this way.
Here, $\gamma \circ \beta_n'$ is the number of $\gamma$-dimensional subrepresentations of a general $\beta_n = \gamma + \beta_n'$ dimensional representation. It can also be interpreted as a Littlewood–Richardson coefficient (see [6, Section 7] for detail). Theorem 2.2 has been implemented as an algorithm to compute the cone $\mathbb{R}_+ \Sigma_{\beta_n}(T_n)$. Later we will have a faster algorithm in Section 9.2.

Let $e_1^i, e_2^j, e_3^k$ be the unit vectors supported on the vertex $i, j, k$ respectively, so \{e_n, e_1, e_2, e_3\}_{i=1}^{n-1} form a standard basis of $\mathbb{Z}^{(T_n)}$. We will use the convention that $e_n = e_n$.

It follows from Theorem 1.4 that if $\sigma \in \Sigma_{\beta_n}(T_n)$, then the last coordinate $\sigma(n)$ of $\sigma$ must be positive. The level of $\sigma$ is by definition $\sigma(n)$.

**Lemma 2.3.** Each level-1 weight $\sigma \in \Sigma_{\beta_n}(T_n)$ is extremal. It is of one of the following forms:

(I) $f_{i,j}^{a,b} := e_n - e_i^{a} - e_j^{b}$, $i + j = n, a \neq b$;

(II) $f_{i,j} := e_n - e_i^{1} - e_j^{2} - e_k^{3}$, $i + j + k = n$.

Its corresponding dimension vector $^a\sigma$ is a real Schur root. There are exactly $3(n - 1)$ of first type, and $(^2\sigma)$ of the second type.

**Proof.** Since $\sigma(n)$ must be positive, each level-1 $\sigma \in \Sigma_{\beta_n}(T_n)$ is extremal. The conditions $i + j = n$ and $i + j + k = n$ are clearly necessary. Suppose that $\sigma = e_n - e_i^{a} - e_j^{a} - e_k^{b}$ with $i \leq j$, then $\gamma = \sum_1^n e_i^{a}$ is a dimension vector contradicting Theorem 1.4. So the superscripts of terms in $\sigma$ must be all different. We can also use Lemma 2.5 below to check that the above $f_{i,j}^{a,b}$ and $f_{i,j}$ do lie in $\Sigma_{\beta_n}(T_n)$ because $c_1^{11,1} = 1$. To show $^a\sigma$ is real Schur, we check that $(^a\sigma, ^a\sigma) = 1$, and a general representation of dimension $^a\sigma$ is indecomposable. This is quite obvious. \hfill \Box

**Remark 2.4.** We observe that $3(n - 1)$ is the number of arrows in $T_n$, and $(^3\sigma)$ is exactly the dimension of the GIT quotient of $\text{Rep}_{\beta_n}(T_n)$ for a generic stability. The sum $(^3\sigma) + 3(n - 1)$ is the Krull dimension of $\text{SI}_{\beta_n}(T_n)$. Later we will see that the first type are related to coefficient variables, and the second type are related to (initial) cluster variables.

The author does not like the notation $f_{i,j}^{a,b}$, and wants to treat two types uniformly. So we introduce the convention that $e_0^a$ is the zero vector for $a = 1, 2, 3$, then
\[
\begin{align*}
f_{0,0} &= f_{i,n-1}^{3}, & f_{0,j} &= f_{j,n-j}^{2,3}, & f_{i,j} &= f_{1,j}^{1,2} \text{ (if } i + j = n) \text{.} & (2.4)
\end{align*}
\]

Recall the correspondence (2.1)–(2.3). It is easy to verify that

**Lemma 2.5.** The level-1 weights $f_{i,j}$ are precisely mapped to the partitions $(1^{i+j}, 1^i, 1^j)$.

For any $f_{i,j}$, we can associate a projective presentation
\[
f_{i,j} : P^1_0 \oplus P^2_0 \oplus P^3_0 \overset{(p_1^T, p_2^T, p_3^T)}{\longrightarrow} P_n. \hfill (2.5)
\]

Here, we use the convention that $P_0^a = 0$, and $p_i^a$ is the unique path from $^a i$ to $n$. By Lemma 2.3 and 1.7, up to a scalar multiple the element in $\text{SI}_{\beta_n}(T_n)_{i,j}$ is equal to the Schofield’s semi-invariant function $s(f_{i,j})$. We will see in Section 8 that all functions $s(f_{i,j})$ are algebraically independent over $k$.

**Example 2.6.** Consider the quiver $T_1$ with the standard $\beta_1$. We run Algorithm 9.2, and find 18 extremal rays in $\mathbb{R}_+ \Sigma_{\beta_1}(T_1)$. Besides the 12 level ones, they are
\[
2e_n - e_1^3 - e_2^3 - e_3^3, \quad \text{and} \quad 2e_n - e_2^3 - e_3^3 - e_3^3.
\]
We leave for readers to check that both correspond to real Schur roots. We give a concrete description for some $s(f_{i,j})$. For example,

$$s(f_{01}) = \det \begin{pmatrix} B_1 B_2 B_3 \\ C_3 \end{pmatrix}, \quad s(f_{11}) = \det \begin{pmatrix} A_1 A_2 A_3 \\ B_1 B_2 B_3 \\ C_2 C_3 \end{pmatrix},$$

where $A_i, B_i, C_i$ are generic matrices as shown below.

![Diagram](image)

3. Graded cluster algebras

3.1. Cluster algebras

We follow mostly [14, Section 3]. The combinatorial data defining a cluster algebra is encoded in an ice quiver $\Delta$ with no loops or oriented 2-cycles. The first $p$ vertices of $\Delta$ are designated as mutable; the remaining $q - p$ vertices are called frozen. If we require no arrows between frozen vertices, then such a quiver is uniquely determined by its $B$-matrix $B(\Delta)$. It is a $p \times q$ matrix given by

$$b_{u,v} = |\text{arrows } u \rightarrow v| - |\text{arrows } v \rightarrow u|.$$

**Definition 3.1.** Let $u$ be a mutable vertex of $\Delta$. The quiver mutation $\mu_u$ transforms $\Delta$ into the new quiver $\Delta' = \mu_u(\Delta)$ via a sequence of three steps.

1. For each pair of arrows $v \rightarrow u \rightarrow w$, introduce a new arrow $v \rightarrow w$ (unless both $v$ and $w$ are frozen, in which case do nothing);
2. Reverse the direction of all arrows incident to $u$;
3. Remove all oriented 2-cycles.

The above recipe can be reformulated in terms of $B$-matrix as follows. Let $\phi$ be the $q \times q$ matrix obtained from the identity matrix by replacing the $u$th row by a vector $\phi_u$ where

$$\phi_u(u) = -1, \quad \phi_u(v) = |\text{arrows } v \rightarrow u|.$$

We write $\phi_p$ for the restriction of $\phi$ to its $p \times p$ upper left corner. Then the mutated $B$-matrix $B'$ for $\Delta'$ is related to the original one by

$$B' = \phi_p^T B \phi.$$

We note that $\phi = \phi^{-1}$ so the quiver mutation is an involution.

Quiver mutations can be iterated ad infinitum, using an arbitrary sequence of mutable vertices of an evolving quiver. This combinatorial dynamics drives the algebraic dynamics of seed mutations that we describe next.

**Definition 3.2.** Let $F$ be a field containing $k$. A seed in $F$ is a pair $(\Delta, x)$ consisting of an ice quiver $\Delta$ as above together with a collection $x = \{x_1, x_2, \ldots, x_q\}$, called an extended cluster, consisting of algebraically independent (over $k$) elements of $F$, one for each vertex of $\Delta$. The elements of $x$ associated with the mutable vertices are called cluster variables; they form a cluster. The elements associated with the frozen vertices are called frozen variables, or coefficient variables.
A seed mutation $\mu_u$ at a (mutable) vertex $u$ transforms $(\Delta, x)$ into the seed $(\Delta', x') = \mu_u(\Delta, x)$ defined as follows. The new quiver is $\Delta' = \mu_u(\Delta)$. The new extended cluster is $x' = x \cup \{x'_u\} \setminus \{x_u\}$ where the new cluster variable $x'_u$ replacing $x_u$ is determined by the exchange relation

$$x_u x'_u = \prod_{v \to u} x_v + \prod_{u \to w} x_w.$$  

(3.2)

We note that the mutated seed $(\Delta', x')$ contains the same coefficient variables as the original seed $(\Delta, x)$. It is easy to check that one can recover $(\Delta, x)$ from $(\Delta', x')$ by performing a seed mutation again at $u$. Two seeds $(\Delta, x)$ and $(\Delta', x')$ that can be obtained from each other by a sequence of mutations are called mutation-equivalent, denoted by $(\Delta, x) \sim (\Delta', x')$.

**Definition 3.3** (Cluster algebra). The cluster algebra $C(\Delta, x)$ associated to a seed $(\Delta, x)$ is defined as the subring of $F$ generated by all elements of all extended clusters of the seeds mutation-equivalent to $(\Delta, x)$.

Note that the above construction of $C(\Delta, x)$ depends only, up to a natural isomorphism, on the mutation equivalence class of the initial quiver $\Delta$. So we may drop $x$ and simply write $C(\Delta)$.

**3.2. Upper bounds**

An amazing property of cluster algebras is

**Theorem 3.4** (Laurent Phenomenon $[1, 15]$). Any element of a cluster algebra $C(\Delta, x)$ can be expressed in terms of the extended cluster $x$ as a Laurent polynomial, which is polynomial in coefficient variables.

Since $C(\Delta, x)$ is generated by cluster variables from the seeds mutation equivalent to $(\Delta, x)$, Theorem 3.4 can be rephrased as

$$C(\Delta, x) \subseteq \bigcap_{(\Delta', x') \sim (\Delta, x)} \mathcal{L}_{x'},$$

where $\mathcal{L}_x := k[x_1^{\pm 1}, \ldots, x_p^{\pm 1}, x_{p+1}, \ldots, x_q]$. Note that our definition of $\mathcal{L}_x$ is slightly different from the original one in $[1]$, where the coefficient variables are inverted in $\mathcal{L}_x := k[x_1^{\pm 1}, \ldots, x_p^{\pm 1}, x_{p+1}^{\pm 1}, \ldots, x_q^{\pm 1}]$.

**Definition 3.5** (Upper Cluster Algebra). The upper cluster algebra with seed $(\Delta, x)$ is

$$\mathcal{U}(\Delta, x) := \bigcap_{(\Delta', x') \sim (\Delta, x)} \mathcal{L}_{x'}.$$  

In general, there may be infinitely many seeds mutation equivalent to $(\Delta, x)$. So the above definition is not very useful to test the membership in an upper cluster algebra. However, the following theorem allows us to do only finitely many checking.

**Definition 3.6** $[1]$. Let $x_0 := x$ and $x_u(1 \leq u \leq p)$ be the adjacent cluster obtained from $x$ by applying the mutation at $u$. We define the upper bounds

$$\mathcal{U}(\Delta, x) := \bigcap_{0 \leq u \leq p} \mathcal{L}_{x_u}.$$
Theorem 3.7 [1, Corollary 1.9]. Suppose that \( B(\Delta) \) has full rank, and \( (\Delta, x) \sim (\Delta', x') \), then \( U(\Delta, x) = U(\Delta', x') \). In particular, \( U(\Delta, x) = \overline{C}(\Delta, x) \).

Remark 3.8. This theorem is originally proved for \( U(\Delta, x) \) and \( C(\Delta, x) \) with \( L_x \) defined there. However, if we carefully examine the argument, we find that the result is also valid for our \( L_x \). We only use this theorem in Lemma 5.3. The current version will make the proof of Lemma 5.3 slightly shorter (see Remark 5.4).

Any (upper) cluster algebra, being a subring of a field, is an integral domain (and under our conventions, a \( k \)-algebra). Conversely, given such a domain \( R \), one may be interested in identifying \( R \) as an (upper) cluster algebra. As an ambient field \( F \), we can always use the quotient field \( \text{QF}(R) \). The challenge is to find a seed \( (\Delta, x) \) in \( \text{QF}(R) \) such that \( C(\Delta, x) = R \).

The following lemma is very helpful.

Lemma 3.9 [14, Corollary 3.7]. Let \( R \) be a finitely generated unique factorization domain over \( k \). Let \( (\Delta, x) \) be a seed in the quotient field of \( R \) such that all elements of \( x \) and all elements of clusters adjacent to \( x \) are irreducible elements of \( R \). Then \( R \supseteq C(\Delta, x) \).

Remark 3.10. The original conclusion in [14, Corollary 3.7] is that \( R \supseteq C(\Delta, x) \). However, the proof implies this stronger result (see the comment before the proof of [14, Proposition 3.6]).

3.3. Gradings

Definition 3.11. A weight configuration \( \sigma \) on an ice quiver \( \Delta \) is an assignment for each vertex \( v \) of \( \Delta \) a (weight) vector \( \sigma(v) \) such that for each mutable \( u \), we have that

\[
\sum_{v \to u} \sigma(v) = \sum_{u \to w} \sigma(w).
\]

(3.3)

A mutation \( \mu_u \) at a mutable vertex \( u \) transforms \( \sigma \) into a weight configuration \( \sigma' \) of the mutated quiver \( \mu_u(\Delta) \) defined as

\[
\sigma'(v) = \begin{cases} 
\sum_{u \to w} \sigma(w) - \sigma(u) & \text{if } v = u, \\
\sigma(v) & \text{if } v \neq u.
\end{cases}
\]

(3.4)

By slight abuse of notation, we can view \( \sigma \) as a matrix whose \( v \)th row is the weight vector \( \sigma(v) \). In matrix notation, the condition (3.3) is equivalent to \( B\sigma \) is a zero matrix, and formula (3.4) is \( \sigma' = \phi \sigma \). It follows from (3.1) and induction that for any weight configuration of \( \Delta \), the mutation can be iterated.

Definition 3.12. A weight configuration \( \sigma \) is called full if the null rank of \( B^T \) is equal to the rank of \( \sigma \). The null space of \( B^T \) is called the grading space of the (upper) cluster algebra \( C(\Delta) \).

Given a weight configuration \( \sigma \) of \( \Delta \), we can assign a multidegree (or weight) to the (upper) cluster algebra \( C(\Delta, x) \) by setting \( \deg(x_v) = \sigma(v) \) for \( v = 1, 2, \ldots, q \). Then mutation preserves multihomogeneity. We say that this (upper) cluster algebra is \( \sigma \)-graded, and denoted by \( C(\Delta, x; \sigma) \). We refer to \( (\Delta, x; \sigma) \) a graded seed.
4. Mutations of quivers with potentials

In [7] and [8], the mutation of quivers with potentials is invented to model the cluster algebras. Following [7], we define a potential $W$ on a quiver $\Delta$ as a (possibly infinite) linear combination of oriented cycles in $\Delta$. More precisely, a potential is an element of the trace space $\text{Tr}(\hat{k}\Delta) := \hat{k}\Delta/[\hat{k}\Delta, \hat{k}\Delta]$, where $\hat{k}\Delta$ is the completion of the path algebra $k\Delta$ and $[\hat{k}\Delta, \hat{k}\Delta]$ is the closure of the commutator subspace of $\hat{k}\Delta$. The pair $(\Delta, W)$ is a quiver with potential, or QP for short. For each arrow $a \in \Delta_1$, the cyclic derivative $\partial_a$ on $\hat{k}\Delta$ is defined to be the linear extension of

$$\partial_a(a_1 \cdots a_d) = \sum_{k=1}^d a^*(a_k)a_{k+1} \cdots a_da_1 \cdots a_{k-1}.$$ 

For each potential $W$, its Jacobian ideal $\partial W$ is the (closed two-sided) ideal in $\hat{k}\Delta$ generated by all $\partial_a W$. The Jacobian algebra $J(\Delta, W)$ is $\hat{k}\Delta/\partial W$. A QP is Jacobi-finite if its Jacobian algebra is finite dimensional. If $W$ is polynomial and $J(\Delta, W)$ is finite dimensional, then the completion is unnecessary to define $J(\Delta, W)$. This is the case throughout this paper.

At this stage, we assume that each vertex of $\Delta$ is mutable, but later we will freeze some vertices. The mutation $\mu_u$ of a QP $(\Delta, W)$ at a vertex $u$ is defined as follows. The first step is to define the following new QP $\tilde{\mu}_u(\Delta, W) = (\tilde{\Delta}, \tilde{W})$. We put $\tilde{\Delta}_0 = \Delta_0$ and $\tilde{\Delta}_1$ is the union of three different kinds:

- all arrows of $\Delta$ not incident to $u$,
- a composite arrow $[ab]$ from $t(a)$ to $h(b)$ for each $a, b$ with $h(a) = t(b) = u$,
- an opposite arrow $a^*$ (respectively, $b^*$) for each incoming arrow $a$ (respectively, outgoing arrow $b$) at $u$.

Note that this $\tilde{\Delta}$ is the result of first two steps in Definition 3.1. The new potential on $\tilde{\Delta}$ is given by

$$\tilde{W} := [W] + \sum_{h(a) = t(b) = u} b^*a^*[ab],$$

where $[W]$ is obtained by substituting $[ab]$ for each words $ab$ occurring in $W$. Finally we define $(\Delta', W') = \mu_u(\Delta, W)$ as the reduced part [7, Definition 4.13] of $(\tilde{\Delta}, \tilde{W})$. For this last step, we refer readers to [7, Section 4.5] for details.

Now we start to define the mutation of decorated representations of $J := J(\Delta, W)$.

DEFINITION 4.1. A decorated representation of the Jacobian algebra $J$ is a pair $\mathcal{M} = (M, M^+)$, where $M \in \text{Rep}(J)$, and $M^+$ is a finite-dimensional $k\Delta_0$-module. By abuse of language, we also say that $\mathcal{M}$ is a representation of $(\Delta, W)$.

Consider the resolution of the simple module $S_u$

$$\cdots \to \bigoplus_{h(a) = u} P_{t(a)}^{(\partial(\alpha_{ab})h)} \to \bigoplus_{t(b) = u} P_{h(b)}^{b(b)} \to P_u \to S_u \to 0,$$

and

$$0 \to S_u \to I_u \to \bigoplus_{h(a) = u} I_{t(a)}^{(\partial(\alpha_{ab})h)} \to \bigoplus_{t(b) = u} I_{h(b)} \to \cdots,$$
where $I_u$ is the indecomposable injective representation of $J$ corresponding to a vertex $u$. We thus have the triangle of linear maps with $\beta_u\gamma_u = 0$ and $\gamma_u\alpha_u = 0$.

$$
\begin{array}{c}
\vfill \\
\alpha_u & \rightarrow & M(u) \\
\oplus_{h(a)=u} M(t(a)) & \downarrow & \oplus_{t(b)=u} M(h(b)) \\
\gamma_u & \rightarrow & \beta_u \\
\vfill
\end{array}
$$

We first define a decorated representation $\widetilde{M} = (\widetilde{M}, \widetilde{M}^+)$ of $\tilde{\mu}_u(\Delta, W)$. We set

$$
\widetilde{M}(v) = M(v), \quad \widetilde{M}^+(v) = M^+(v) \quad (v \neq u);
$$

$$
\widetilde{M}(u) = \frac{\text{Ker } \gamma_u}{\text{Im } \beta_u} \oplus \text{Im } \gamma_u \oplus \frac{\text{Ker } \alpha_u}{\text{Im } \gamma_u} \oplus M^+(u), \quad \widetilde{M}^+(u) = \frac{\text{Ker } \beta_u}{\text{Im } \alpha_u}.
$$

We then set $\widetilde{M}(a) = M(a)$ for all arrows not incident to $u$, and $\widetilde{M}([ab]) = M(ab)$. It is defined in [7] a choice of linear maps $\widetilde{M}(a^*), \widetilde{M}(b^*)$ making $\widetilde{M}$ a representation of $(\Delta, W)$. We refer readers to [8, Section 10] for details. Finally, we define $M = \mu_u(M)$ to be the reduced part $[7, \text{Definition 10.4}]$ of $\widetilde{M}$.

Let $Rep(J)$ be the set of decorated representations of $J(\Delta, W)$ up to isomorphism. There is a bijection between two additive category $Rep(J)$ and $K^2(\text{proj-}J)$ mapping any representation $M$ to its minimal presentation in $Rep(J)$, and the simple representation $S^+_u$ of $k\Delta^0$ to $P_u \to 0$. Suppose that $M$ corresponds to a projective presentation $P(\beta_1) \to P(\beta_0)$. We see from the exact sequence (4.2) that

$$
\beta_1(u) = \dim(\text{Ker } \alpha_u / \text{Im } \gamma_u) + \dim M^+(u), \quad \text{and } \beta_0(u) = \dim \text{Coker } \alpha_u.
$$

**Definition 4.2.** The $g$-vector $g(M)$ of a decorated representation $M$ is the weight vector of its image in $K^h(\text{proj-}J)$, that is, $g = \beta_1 - \beta_0$.

**Remark 4.3.** Our $g$-vector is dual to the $g$-vector considered in [8, 16]. It is the negative of the $\delta$-vector considered in [4].

It follows from (4.3) that ([8, (1.13)])

$$
g(u) = \dim(\text{Coker } \gamma_u) - \dim M(u) + \dim M^+(u).
$$

**Definition 4.4.** $M$ is called $g$-coherent if $\min(\beta_1(u), \beta_0(u)) = 0$ for all vertices $u$, or equivalently, $\beta_1 = [g]_+$ and $\beta_0 = [-g]_+$. Here, $[g]_+$ is the vector satisfying $[g]_+(u) = \max(g(u), 0)$.

We also define the mutation $\mu_u$ at the level of $K_0$-groups. For $g \in K_0(\text{proj-}J)$, $g' := \mu_u(g)$ is a vector in $K_0(\text{proj-}J')$ defined by (cf. [8, (1.3)])

$$
g'(v) := \begin{cases} 
-g(u) & \text{if } u = v; \\
g(v) - b_{u,v}[-g(u)]_+ & \text{if } b_{u,v} \geq 0; \\
g(v) - b_{u,v}[g(u)]_+ & \text{if } b_{u,v} < 0.
\end{cases}
$$

This is also an involution. In general, the $g$-vectors of $M$ and $M'$ are related by (5.6). If we compare (4.5) with (5.6) as in the proof of [8, Theorem 1.7], we see that they are equivalent if and only if $\beta_0 = [-g]_+$.

**Lemma 4.5.** The $g$-vectors of $M$ and $M'$ are related by (4.5) if and only if $M$ is $g$-coherent.
An ice quiver with potential, or IQP for short, is a quiver with potential \((\Delta, W)\) with a set \(\Delta_0\) of frozen vertices. In general, we allow arrows between frozen vertices. After some mutations, the ice quiver \(\Delta\) may acquire some oriented 2-cycles, which would make some mutations undefined for the evolving IQP.

**Definition 4.6** ([7]). We say that a potential \(W\) is *nondegenerate* for an ice quiver \(\Delta\) if any finite sequence of mutations can be applied to \((\Delta, W)\) without creating oriented 2-cycles along the way. Such a IQP \((\Delta, W)\) is also called nondegenerate.

It is known ([7], Proposition 7.3) that nondegenerate potential always exists for any 2-acyclic quiver. For example, one can take a generic potential of the quiver. For a nondegenerate IQP \((\Delta, W)\), the ice quiver of the mutated IQP \(\mu_u(\Delta, W)\) is the quiver \(\mu_u(\Delta)\) in Definition 3.1 if we forget the arrows between the frozen vertices.

**Definition 4.7** ([7]). A potential \(W\) is called *rigid* on a quiver \(\Delta\) if every potential on \(\Delta\) is cyclically equivalent to an element in the Jacobian ideal \(\partial W\). Such a QP \((\Delta, W)\) is also called rigid.

It is known ([7], Proposition 8.1, Corollary 6.11] that every rigid QP is 2-acyclic, and the rigidity is preserved under mutations. In particular, any rigid QP is nondegenerate.

It turns out the right category to look at for an IQP is the category of \(\mu\)-supported representations in \(\text{Rep}(J)\).

**Definition 4.8.** A decorated representation \(M\) is called *\(\mu\)-supported* if the supporting vertices of \(M\) are all mutable. We denote by \(\text{Rep}^{\mu}(J)\) the set of all \(g\)-coherent \(\mu\)-supported decorated representations of \(J\). 

### 5. The cluster character

Throughout this section, we assume that \((\Delta, W)\) is a nondegenerate IQP. Let \(x = \{x_1, x_2, \ldots, x_q\}\) be an (extended) cluster. For a vector \(g \in \mathbb{Z}^q\), we write \(x^g\) for the monomial \(x_1^{g_1}x_2^{g_2}\ldots x_q^{g_q}\). For \(u = 1, 2, \ldots, p\), we set \(\hat{y}_u = x^{-b_u}\) where \(b_u\) is the \(u\)th row of the matrix \(B(\Delta)\), and let \(\hat{y} = \{\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_p\}\). The seed mutation of Definition 3.2 induces the \(\hat{y}\)-seed mutation.

We recall the mutation rule for \(\hat{y}\) ([16], (3.8))

\[
\hat{y}_v' = \begin{cases} 
\hat{y}_v^{-1} & \text{if } v = u; \\
\hat{y}_v\hat{y}_u^{[b_{u,v}+1]}(\hat{y}_u + 1)^{b_{v,u}} & \text{if } v \neq u.
\end{cases}
\]  

**Definition 5.1** ([8]). We define the \(F\)-polynomial of a representation \(M\) by

\[
F_M(y) = \sum_e \chi(\text{Gr}^e(M)) y^e,
\]

where \(\text{Gr}^e(M)\) is the variety parameterizing \(e\)-dimensional quotient representations of \(M\), and \(\chi(-)\) denotes the topological Euler-characteristic.

We also define the *cluster character* \(C : \text{Rep}^{\mu}(J) \to \mathbb{Z}(x)\) by

\[
C(M) = x^{g(M)} F_M(\hat{y}) = x^{g(M)} \sum_e \chi(\text{Gr}^e(M)) \hat{y}^e.
\]
Using (4.4), we can reinterpret (5.3) as (cf. [8, Corollary 5.3])

$$C(M) = x^{-d} \sum_e \chi (\text{Gr}^e(M)) x^e,$$

where $d = \dim M$ and $\epsilon(u) = \sum_e \epsilon([b_{v,u}], x(e(v)) + [b_{u,v}]_+ (d(v) - e(v))) - \text{rank } \gamma_u + \dim M^+(u)$.

The key fact we will use is that the vector $\epsilon$ is nonnegative [8, Corollary 5.5]. So the image of $C$ in fact lies in $\mathcal{L}_x$.

Here is the key lemma in [8]:

**Lemma 5.2.** Let $M$ be an arbitrary representation of a nondegenerate QP $(\Delta, W)$, and let $M' = \mu_u(M)$, then

1. the $F$-polynomials of $M$ and $M'$ are related by

$$\left(\hat{y}_u + 1\right)^{-\beta_0(u)} F_M(\hat{y}) = \left(\hat{y}'_u + 1\right)^{-\beta'_0(u)} F_{M'}(\hat{y}'),$$

2. the $g$-vector of $M$ and $M'$ are related by

$$g'(u) = \begin{cases} -g(u) & \text{if } u = v; \\ g(v) + [b_{v,u}]_+ g(u) - b_{v,u} \beta_0(u) & \text{if } u \neq v, \end{cases}$$

and satisfies

$$g(u) = \beta'_0(u) - \beta_0(u).$$

**Lemma 5.3.** The mutations commute with the cluster character $C$. So if $M \in \text{Rep}^g(M(J(\Delta, W))$, then $C(M)$ is an element in the upper cluster algebra $\overline{\mathcal{C}}(\Delta)$. Moreover, if $(\Delta, \mathbf{x})$ is $\sigma$-graded, then $C(M)$ is multihomogeneous of degree $\mathbf{g} \sigma$.

**Proof.** The commuting property $\mu_u(C(M)) = C(\mu_u(M))$ is equivalent to that

$$x^\mathbf{g} F_M(\hat{y}) = (x')^{\mathbf{g}'} F_{M'}(\hat{y}').$$

Comparing with (5.5), we see that it suffices to show that

$$\left(\hat{y}_u + 1\right)^{-\beta_0(u)} x^\mathbf{g} = \left(\hat{y}_u' + 1\right)^{-\beta'_0(u)} (x')^{\mathbf{g}'}.$$

The following equalities are all equivalent to the above.

$$\left(\hat{y}_u + 1\right)^{-\beta_0(u)} x^\mathbf{g} = \hat{y}_u^{-\beta_0(u)} \left(\hat{y}_u + 1\right)^{-\beta'_0(u)} \left(x^{b_{u,v}} + x^{[-b_{u,v}]}\right) x^{-g(u)} x^\mathbf{g} - g(u) x^\mathbf{g} - g(u) e_u \tag{3.2}, \tag{5.1},$$

$$x^{b_{u,v} \beta'_0(u)} x^\mathbf{g} = (x^{b_{u,v}} + 1)^{g(u)} (x^{b_{v,u}} + x^{[-b_{v,u}]} g(u)) x^\mathbf{g} - g(u) e_u \tag{5.7}, \tag{4.5},$$

$$x^{b_{u,\beta_1(u)}} = x^{-[-b_{v,u}]_+ g(u)} x^{b_{u,v} + \beta_1(u) [-b_{v,u}]_+} \tag{5.7}, \tag{4.5}.$$

The last equality can be easily verified using $b_u = [b_{u,v}]_+ - [-b_{u,v}]_+$ and $g = \beta_1 - \beta_0$.

To show $C(M) \in \overline{\mathcal{C}}(\Delta)$, we use an argument similar to that of [27, Theorem 1.3]. Let $x_u$ be the cluster obtained from $x$ by applying the mutation at $u$. By Theorem 3.7, it suffices to show that $C(M) \in \mathcal{L}_{x_u}$ for any mutable vertex $u$. The expression of $C(M)$ with respect to the cluster $x_u$ is given by $\mu_u(C(M))$, which is $C(\mu_u(M)) \in \mathcal{L}_{x_u}$. The last statement about the degree follows directly from (5.3) and (3.4). In certain sense, this says that the $g$-vector governs all gradings.

**Remark 5.4.** We could have used the version of [1, Theorem 3.7], but the proof is slightly longer, as shown below. By the version in [1], it remains to show that $\mu_u(C(M))$ is actually polynomial in coefficient variables for any sequence of mutations $\mu_u$. This is true (initially) for
Again by the commuting property $\mu_u(C(\mathcal{M})) = C(\mu_u(\mathcal{M}))$, it is always polynomial in coefficient variables.

Suppose that $\mathcal{M}$ can be mutated from a positive representation $(0, M^+)$ via a sequence of mutations $\mu_u$. Since $M^+$ is semisimple, it decomposes as a direct sum of simple representations $\bigoplus_{v \in \Delta_0} m_v v_+^v$ with multiplicity $m_v \in \mathbb{N}_0$. Then $C(\mathcal{M}) = \mu_u(C(0, M^+))$ is the cluster monomial $\prod_{v \in \Delta_0} \mu_u(x_v)^{m_v}$. By definition, a cluster monomial is a monomial in elements of any given extended cluster.

Suppose that an element $z \in \mathcal{C}(\Delta)$ can be written as

$$z = x^{g(z)} F(\hat{y}_1, \ldots, \hat{y}_p),$$

(5.8)

where $F$ is a primitive rational polynomial, and $g(z) \in \mathbb{Z}^q$. If we assume that the matrix $B(\Delta)$ has full rank, then the elements $\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_p$ are algebraically independent so that the vector $g(z)$ is uniquely determined [16]. We call the vector $g(z)$ the (extended) $g$-vector of $z$. Note that the $g$-vector of $C(\mathcal{M})$ as in Lemma 5.3 is $g(M)$. Definition implies at once that for two such elements $z_1, z_2$ we have that $g(z_1 z_2) = g(z_1) + g(z_2)$. So the set $G(\Delta)$ of all $g$-vectors in $\mathcal{C}(\Delta)$ forms a sub-semigroup of $\mathbb{Z}^q$.

**Lemma 5.5.** Assume that the matrix $B(\Delta)$ has full rank. Let $Z = \{z_1, z_2, \ldots, z_k\}$ be a subset of $\mathcal{C}(\Delta)$ with well-defined $g$-vectors. If the functions $g(z_i)$ are all distinct, then $Z$ is linearly independent over $k$.

**Proof.** A similar statement is proved in [27] in a slightly different setting. Here, we just adapt his proof to our setting. The full rank condition on $B$ implies that $\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_p$ are algebraically independent, and moreover we can assign a grading on $x_1, x_2, \ldots, x_q$ such that each $\hat{y}_u$ has positive total degree. Then the total degree of the monomial $x^{g_i}$ is minimal among the total degree of all monomials in (5.8). Suppose that $\sum_i a_i z_i = 0$, then we extract the terms of minimal total degree: $\sum_i a_i x^{g(z_i)} = 0$. Since $g(z_i)$ are all distinct, we conclude that $a_i = 0$ for all $i$. \qed

It follows from Lemma 5.3 and 5.5 that

**Theorem 5.6.** Suppose that IQP $(\Delta, W)$ is nondegenerate and $B(\Delta)$ has full rank. Let $\mathcal{R}$ be a set of $g$-coherent $\mu$-supported decorated representations with all distinct $g$-vectors, then $C$ maps $\mathcal{R}$ (bijectively) to a set of linearly independent elements in the upper cluster algebra $\mathcal{C}(\Delta)$.

**Definition 5.7.** If there is some $\mathcal{R}$ as in Theorem 5.6 such that its image under $C$ is a basis of $\mathcal{C}(\Delta)$, then we say that $W$ is a compatible potential for the upper cluster algebra, or in short, $W$ is upper-compatible.

The rest of this section is devoted to find a choice of $\mathcal{R}$ such that $C(\mathcal{R})$ spans a subspace as large as possible. It turns out that for appropriate $g$-vectors, we can take elements of $\mathcal{R}$ generically in some sense.

**Definition 5.8.** To any $g \in \mathbb{Z}^{\Delta_0}$ we associate the reduced presentation space

$$\text{PHom}_J(g) := \text{Hom}_J(P([g]_+), P([-g]_+)).$$

We denote by $\text{Coker}(g)$ the cokernel of a general presentation in $\text{PHom}_J(g)$. 
Reader should be aware that \( \text{Coker}(g) \) is just a notation rather than a specific representation. If we write \( M = \text{Coker}(g) \), this simply means that we take a general presentation in \( \text{PHom}_J(g) \), then let \( M \) to be its cokernel. The following lemma is well-known (see [4]).

**Lemma 5.9.** A general presentation in \( \text{Hom}_J(P_1, P_0) \) of weight \( g \) is homotopy-equivalent to a general presentation in \( \text{PHom}_J(g) \).

**Definition 5.10.** A weight vector \( g \in K_0(\text{proj} - J) \) is called \( \mu \)-supported if \( \text{Coker}(g) \) is \( \mu \)-supported. Let \( G(\Delta, W) \) be the set of all \( \mu \)-supported vectors in \( K_0(\text{proj} - J) \).

A weight vector \( g \in K_0(\text{proj} - J) \) is called positive-free if a general presentation in \( \text{PHom}_J(g) \) does not contain a direct summand of form \( P_1 \to 0 \). Any \( g \in K_0(\text{proj} - J) \) can be decomposed as \( g = g' + g^+ \) with \( g' \) positive-free and \( g^+ \in \mathbb{Z}_{>0}^J \) such that a general presentation in \( \text{PHom}_J(g) \) is a direct sum of a presentation in \( \text{PHom}_J(g') \) and a presentation in \( \text{PHom}_J(g^+) \).

**Lemma 5.11.** A general presentation in \( \text{PHom}_J(g) \) corresponds to a \( g \)-coherent decorated representation.

**Proof.** We decompose \( g \) as \( g = g' + g^+ \). It follows from [4, Theorem 2.3] (see also [27, Corollary 2.8]) that \( \text{Coker}(g') \) is a general representation in some irreducible component \( C \), and thus has its minimal presentation in \( \text{PHom}_J(g') \). The result follows.

It follows from [8, Theorem 7.1] that

**Lemma 5.12.** If \( M \) can be mutated from a positive representation \((0, M^+)\), then it corresponds to a rigid presentation in the sense of [4]. In particular, the presentation is general.

**Definition 5.13** [27]. We define the generic character \( C_W : G(\Delta, W) \to \mathbb{Z}(x) \) by

\[
C_W(g) = x^g \sum_{e} \chi(\text{Gr}^e(\text{Coker}(g))) \hat{y}^e.
\]  

(5.9)

It follows from Theorem 5.6, Lemma 5.11 and 5.12 that

**Corollary 5.14** (cf. [27, Theorem 1.1]). Suppose that IQP \((\Delta, W)\) is nondegenerate and \( B(\Delta) \) has full rank. The generic character \( C_W \) maps \( G(\Delta, W) \) (bijectively) to a set of linearly independent elements in \( \mathbb{Z}(\Delta) \) containing all cluster monomials.

A similar result was first proved by Plamondon in the setting of the (generalized) cluster category. As mentioned in [27, Remark 4.1] the result is also valid in the setting of quivers with potentials. The author wants to thank Plamondon for pointing this out to him. Here, we just gave a simple direct proof. Moreover, the definition of upper cluster algebras in [27] is taken from [1], so instead of \( G(\Delta, W) \) the domain of the generic character is the full lattice \( K_0(\text{proj} - J) \).

**Proposition 5.15.** \( G(\Delta, W) \) is a semigroup, and \( G(\mu_u(\Delta, W)) = \mu_u(G(\Delta, W)) \).

**Proof.** Let \( d \) and \( d' \) be two general presentations in \( \text{PHom}_J(g) \) and \( \text{PHom}_J(g') \). If the cokernels of \( d \) and \( d' \) are not supported on \( v \), then so is \( d \oplus d' \in \text{Hom}_J(P_1, P_0) \). By the semi-continuity of the rank function, this is also true for a general presentation in \( \text{Hom}_J(P_1, P_0) \). Note that the weight of \( d \oplus d' \) is \( g + g' \). Hence, \( g + g' \in G(\Delta, W) \) by Lemma 5.9.
Since $\mu_u$ is an involution, it suffices to show $\mu_u(G(\Delta, W)) \subseteq G(\mu_u(\Delta, W))$. We pick some $g \in G(\Delta, W)$. Let $\mathcal{M}$ be some decorated representation corresponding to a general presentation in $\text{PHom}_J(g)$. By Lemma 5.11, $\mu_u(g(\mathcal{M})) = g(\mu_u(\mathcal{M}))$. $\mathcal{M}$ is $\mu$-supported, then so is $\mu_u(\mathcal{M})$. Hence $\mu_u(g) \in G(\mu_u(\Delta, W))$. □

By Corollary 5.14, we have that $G(\Delta, W) \subseteq G(\Delta)$. It seems difficult to describe both $G(\Delta, W)$ and $G(\Delta)$ in general. The following problems are important.

**Problem 5.16.** Under what conditions, does the upper cluster algebra $\mathcal{C}(\Delta)$ have a basis such that all its elements have distinct $g$-vectors? For which potentials, do we have the equality $G(\Delta) = G(\Delta, W)$?

**Problem 5.17.** Is $G(\Delta, W)$ saturated in $\mathbb{Z}^q$? If yes, is $G(\Delta, W)$ consisted of lattice points in some rational polyhedral cone? We can ask the same question for $G(\Delta)$.

We will see in Section 6 that for ice hive quivers with certain potentials, the answer to both problems are positive.

### 6. The hive quivers

The **hive quiver** $\tilde{\Delta}_n$ of size $n$ is a quiver in the plane with $\binom{n+2}{2}$ vertices arranged in a triangular grid consisting of $n^2$ small triangles formed by arrows. We label the vertices as shown in Figure 1. The **ice hive quiver** $\Delta_n$ is obtained from the hive quiver $\tilde{\Delta}_n$ by forgetting three vertices $\{(0,0), (0,n), (n,0)\}$, then freezing all boundary vertices and deleting all arrows between all boundary vertices as shown in Figure 2. A vertex is called **corner** if it is one of the following six $(0,1), (0,n-1), (1,0), (n-1,0), (1,n-1), (n-1,1)$. We denote by $\delta_n$ the vertex set of $\Delta_n$. The next two lemmas are straightforward.

**Lemma 6.1.** The $B$-matrix of $\Delta_n$ is of full rank.

Recall the definition of the level-1 weight vector $f_{i,j}$ in Lemma 2.3 and (2.4).

**Lemma 6.2.** The assignment $(i,j) \mapsto f_{i,j}$ on the vertices of $\Delta_n$ defines a full weight configuration $\sigma_n$ on $\Delta_n$.
Let $a$ (respectively, $b$, $c$) denote the sum of all northeast (respectively, southeast, west) arrows. We put the potential $W = abc - acb$ on the quiver $\Delta_n$. Then the Jacobian ideal is generated by the elements
\[
e_u(ab - ba), \ e_u(bc - cb), \ e_u(ca - ac) \quad \text{for } u \text{ mutable,} \tag{6.1}
\]
\[
e_v ba, \ ace_v, \ e_v ac, \ cbe_v, \ e_v cb, \ bae_v \quad \text{for } v \text{ noncorner frozen.} \tag{6.2}
\]
We will identify a path $p$ in $\Delta_n$ with a sequence of vertices.

**Definition 6.3.** A straight path in $\Delta_n$ from a vertex $(i, j)$ is any of the following three kinds:
\[
(i, j), (i, j - 1), \ldots, (i, j - k), \quad \text{(NE)}
\]
\[
(i, j), (i + 1, j), \ldots, (i + k, j), \quad \text{(SE)}
\]
\[
(i, j), (i - 1, j + 1), \ldots, (i - k, j + k), \quad \text{(W)}
\]
We define a cone $G_n \subset \mathbb{R}^k_n$ by
\[
\begin{cases}
g(v) \geq 0 & \text{for all frozen vertices } v, \\
\sum_{u \in p} g(u) \geq 0 & \text{for all maximal straight paths } p \text{ from mutable vertices.}
\end{cases}
\]
Here, the notation $\sum_{u \in p} g(u)$ stands for $g(u_1) + g(u_2) + \cdots + g(u_s)$ if $p$ is the path passing $u_1, u_2, \ldots, u_s$.

Let $T_{0,1}$ be the injective (simple) representation $I_{0,1}$. For $j \geq 2$, let $T_{0,j}$ be the unique indecomposable representation supported on the straight path from $(j, 0)$ to $(0, j)$. Note that $\dim T_{0,j}$ is $(1, 1, \ldots, 1)$ on its support.

**Lemma 6.4.** The IQP $(\Delta_n, W_n)$ is rigid and Jacobi-finite. Moreover, we have an (injective) presentation of $T_{0,j}$
\[
0 \to T_{0,j} \to I_{0,j} \xrightarrow{bc} I_{0,j-1} \quad \text{for } 2 \leq j \leq n - 2, \tag{6.3}
\]
\[
0 \to T_{0,n-1} \to I_{0,n-1} \xrightarrow{(bc, ac)} I_{0,n-2} \oplus I_{1,n-1}. \tag{6.4}
\]
Proof. The proof of the first statement is similar to that in [7, Example 8.7]. Because of the relation (6.1) and (6.2), any path not passing a corner vertex $u$ is equivalent to $e_u a^k b^l c^m (0 \leq k, l, m \leq n - 1)$ in the the Jacobian algebra $J(\Delta_n, W_n)$. If a path $p$ passes some corner vertex $v$, then $p$ either starts from $v$ or ends in $v$. If $p$ starts from a vertex in $V_1 := \{(1,0), (0,n-1), (n-1,1)\}$ or ends in a vertex in $V_2 := \{(0,1), (n-1,0), (1,n-1)\}$, then $p$ is a trivial path.

Moreover, for $j = n - 1$ we have that

(1) a dual path in $I_{0,1}$ vanishes under $bc$ if and only if it does not pass $(0,j-1)$;

(2) any dual path to $(0,j)$ not passing $(0,j-1)$ is equivalent to a straight path.

The exact sequences reformulate the above statements.

By symmetry, we also consider the representations $T_{i,0}$ (respectively, $T_{i,n-i}$) supported on the straight path from $(i,n-i)$ to $(i,0)$ (respectively, from $(0,i)$ to $(i,n-i)$). They have the similar presentations.

**Definition 6.5.** A vertex $v$ is called maximal in a representation $M$ if all strict subrepresentations of $M$ are not supported on $v$ and dim $M(v) = 1$.

Note that each above $T_v$ has a maximal vertex.

**Lemma 6.6.** Suppose that a representation $T$ contains a maximal vertex $v$. Let $M = \text{Coker}(g)$, then $\text{Hom}_J(M,T) = 0$ if and only if $g(\text{dim}S) \geq 0$ for all subrepresentations $S$ of $T$.

**Proof.** If $\text{Hom}_J(M,T) = 0$, then $\text{Hom}_J(M,S) = 0$, and thus $g(\text{dim}S) \geq 0$ for all subrepresentations $S$ of $T$. Conversely, suppose that $\text{PHom}_J(g) = \text{Hom}_J(P_1, P_0)$. We add $c = g(\text{dim}T)$ copies of the projective representation $P_v$ to $P_0$ so that a general presentation $P_1 \to P_0 \oplus cP_v \to M' \to 0$ has weight $g' = g + ce_.$. It satisfies that $g'(\text{dim}T) = 0$ and $g'(\text{dim}S) = g(\text{dim}S) \geq 0$ for all subrepresentations $S \subset T$. By King’s criterion (Lemma 1.3), we see that $T$ is $g'$-semi-stable, and thus $\text{Hom}_J(M',T) = 0$. Now a general presentation $P_1 \to P_0 \oplus cP_v$ must have $f$ general in $\text{Hom}_J(P_1, P_0)$. Hence, $\text{Hom}_J(M', T) = 0$ implies $\text{Hom}_J(M, T) = 0$.

**Lemma 6.7.** Let $M = \text{Coker}(g)$, then $\text{Hom}_J(M,T_v) = 0$ for each frozen $v$ if and only if $\text{Hom}_J(M,I_v) = 0$ for each frozen $v$. 
Proof. Since each subrepresentation of $T_v$ is also a subrepresentation of $I_v$, one direction is clear. Conversely, let us assume that $\text{Hom}_J(M, T_v) = 0$ for each frozen $v$. We prove that $\text{Hom}_J(M, I_{0,j}) = 0$ by induction on $k$. For $k = 1$, we have that $T_{0,1} = I_{0,1}$. Now suppose that it is true for $k = j - 1$, that is, $\text{Hom}_J(M, T_{0,j-1}) = 0$. By Lemma 6.4, for $j < n - 1$ $\text{Hom}_J(M, I_{0,j}) = 0$ is equivalent to $\text{Hom}_J(M, T_{0,j}) = 0$. $\text{Hom}_J(M, I_{0,n-1}) = 0$ is equivalent to $\text{Hom}_J(M, T_{0,n-1}) = 0$ and $\text{Hom}_J(M, T_{1,n-1}) = 0$. We are done by symmetry and induction. \(\square\)

Theorem 6.8. The set of lattice points $G_n \cap \mathbb{Z}^\delta_n$ is exactly $G(\Delta_n, W_n)$.

Proof. Due to Lemmas 6.6 and 6.7, it suffices to show that $G_n$ is defined by $g(\dim S) \geq 0$ for all subrepresentations $S$ of $T_v$ and all $T_v$. We notice that these defining conditions are the union of the defining conditions of $G_n$ and $g(\dim T_v) \geq 0$. But the latter conditions for $v \neq (0, 1), (n - 1, 0), (1, n - 1)$ are clearly redundant. \(\square\)

Remark 6.9. For any IQP $(\Delta, W)$, we can define a cone $G_I(\Delta, W) \subset \mathbb{R}^{\Delta_0}$ by $g(\dim S) \geq 0$ for all subrepresentations $S \subset I_v$ and all frozen vertices $v$. It is not hard to see that the cone $G_n = G_I(\Delta_n, W_n)$. It is also clear that $G(\Delta, W)$ is always contained in $G_I(\Delta, W) \cap \mathbb{Z}^{\Delta_0}$. We are curious about the next problem.

Problem 6.10. For what kind of IQP, do we have the equality

$$G(\Delta, W) = G_I(\Delta, W) \cap \mathbb{Z}^{\Delta_0}?$$

For a fixed weight vector $\sigma$, we consider the convex polytope $G_n(\sigma)$ obtained from $G_n$ by adding the condition $g(\sigma) = \sigma$. We will show in the next section that there is a unimodular linear transformation mapping $G_n(\sigma)$ onto a hive polytope of Knutson and Tao.

7. Littlewood–Richardson Triangles

Definition 7.1 [26]. A Littlewood–Richardson triangle of size $n$ is an element $h = \{h(i, j)\} \in \mathbb{R}^{\delta_n}$ that satisfies the following conditions:

$$h(i, j) \geq 0 \quad \text{for } ij \neq 0,$$

$$\sum_{k=0}^{i-j} h(k, j) \geq \sum_{k=0}^{i-j} h(k, j + 1), \quad \text{for } 1 \leq j \leq i \leq n - 1,$$

$$\sum_{k=0}^{j-1} h(i - k, k) \geq \sum_{k=0}^{j} h(i + 1 - k, k), \quad \text{for } 1 \leq j \leq i \leq n - 1.$$

We denote by $LR_n$ the cone of all Littlewood–Richardson triangles in $\mathbb{R}^{\delta_n}$. To each $h = \{h(i, j)\} \in LR_n$ we associate the following numbers:

$$\mu_i = h(i, 0), \quad \text{for } 1 \leq i \leq n - 1,$$

$$\nu_j = \sum_{k=1}^{n-j} h(k, j), \quad \text{for } 1 \leq j \leq n - 1,$$

$$\lambda_i = \sum_{k=0}^{i} h(i - k, k), \quad \text{for } 1 \leq i \leq n.$$
Then it follows from (7.1)–(7.3) that $\lambda, \mu$ and $\nu$ are partitions with $|\lambda| = |\mu| + |\nu|$. We call $(\lambda, \mu, \nu)$ the type of $h$, and denote by $\text{LR}_n(\lambda, \mu, \nu)$ the set of all LR triangles of type $(\lambda, \mu, \nu)$; this is a convex polytope. It is proved in [26] that there is a unimodular linear transformation mapping $\text{LR}_n(\lambda, \mu, \nu)$ onto a hive polytope of Knutson and Tao [24]. In particular,

**Lemma 7.2** [26, Corollary 4.2]. $\text{LR}_n(\lambda, \mu, \nu)$ has $c_{\lambda}^{\mu, \nu}$ integral points.

For any weight $\sigma$, we get a triple of partition $(\lambda(\sigma), \mu(\sigma), \nu(\sigma))$ via (2.1)–(2.3). We denote the polytope $\text{LR}_n(\lambda(\sigma), \mu(\sigma), \nu(\sigma))$ by $\text{LR}_n(\sigma)$. It is clear that $\text{LR}_n(1^i + j^i, 1^i, 1^j)$ has a unique integral point $h_{i,j}$ satisfying

\[
\begin{align*}
    h_{i,j}(k,0) &= h_{i,j}(i,l) = 1 & \text{if } 1 \leq k \leq i, 1 \leq l \leq j, \\
    h_{i,j}(k,l) &= 0 & \text{otherwise.}
\end{align*}
\]

**Theorem 7.3.** There is a unimodular linear isomorphism $\mathbb{R}^{\delta_n} \rightarrow \mathbb{R}^{h_n}$ mapping the polytope $G_n(\sigma)$ onto the polytope $\text{LR}_n(\sigma)$. In particular, $G_n(\sigma)$ has $c_{\lambda(\sigma)}^{\mu(\sigma), \nu(\sigma)}$ integral points.

**Proof.** The definition of the isomorphism is obvious. Let $e_{i,j}$ be the standard basis of $\mathbb{R}^{\delta_n}$, and $f_{i,j}$ be the level-1 weight vector defined in Section 2. By Lemma 2.5 and the remark above, there is a unique integral point $h_{i,j} \in \text{LR}_n(f_{i,j}) \subset \mathbb{R}^{\delta_n}$. Then the assignment $e_{i,j} \mapsto h_{i,j}$ induces a linear map $\varphi : \mathbb{R}^{\delta_n} \rightarrow \mathbb{R}^{h_n}$. We need to show that

1. $\varphi$ is unimodular;
2. $\varphi$ pulls the supporting hyperplanes of $\text{LR}_n(\sigma)$ back to those of $G_n(\sigma)$.

We order the standard basis of $\mathbb{R}^{\delta_n}$ according to the lexicographic order of the subindices, that is, $E = \{e_{0,1}, e_{0,2}, \ldots, e_{1,0}, e_{1,1}, \ldots, e_{n-1,0}, e_{n-1,1}\}$. The matrix of $\varphi$ with respect to $E$ is upper triangular with ones on the main diagonal. So it has determinant one, and thus unimodular.

We write the definition of $\varphi$ in coordinates:

\[
\varphi(g) = h, \text{ where } h(i,j) = \begin{cases} 
    \sum_{l \geq j} g(i,l) & \text{if } j \neq 0, \\
    \sum_{k \geq j, l} g(k,l) & \text{if } j = 0.
\end{cases}
\]

It is easy to verify case by case that (7.1) matches with $\sum_{(i,j) \in P} g(i,j) \geq 0$ for the straight paths $p$ of type (NE) and $g(i,j) \geq 0$ for $j = 0$; (7.2) matches with the type (SE) and $g(i,j) \geq 0$ for $i + j = n$; (7.3) matches with the type (W) and $g(i,j) \geq 0$ for $i = 0$. Finally, by our construction the type of $h$ is $g\sigma_n = \sigma$. \hfill $\square$

8. Cluster structure in $\text{SI}_{\beta_n}(T_n)$

8.1. Algebraic independence

In this subsection, we show that the set of level-1 semi-invariants

\[ S_n := \{ s(f_{i,j}) \in \text{SI}_{\beta_n}(T_n) \mid (i,j) \in \delta_n \} \]

are algebraically independent. The proof is a little technical, so we suggest that readers skip this part for the first-time reading.

We first recall a slice theorem in [30]. Let $Q$ be any finite quiver without oriented cycles. For a representation $M$ of $Q$, the right orthogonal category $M^\perp$ is the abelian subcategory...
\{M \in \text{Rep}(Q) \mid M \perp N\}. A representation \(E\) is called \textit{exceptional} if \(\text{Hom}_Q(E, E) = k\) and \(\text{Ext}_Q(E, E) = 0\), so the dimension vector of \(E\) corresponds to a real Schur root \(\epsilon\).

Schofield showed that the category \(E^\perp\) is equivalent to the category \(\text{Rep}(Q_E)\) for another quiver \(Q_E\) with one vertex less than \(Q\). We compose this equivalence with the embedding \(E^\perp \hookrightarrow \text{Rep}(Q)\), and obtain a functor \(\iota_E : \text{Rep}(Q_E) \hookrightarrow \text{Rep}(Q)\). It induces a linear inclusion of \(K_0(\text{Rep}(Q_E))\) into \(K_0(\text{Rep}(Q))\). If \(E \perp \beta\), which means that \(E\) is right orthogonal to a general representation in \(\text{Rep}_\beta(Q)\), then we denote by \(\beta_\epsilon\) the dimension vector of \(Q_E\) such that \(\beta\) is \(\beta_\epsilon\) under the inclusion.

\textbf{Theorem 8.1} [30, Theorem 3.2]. If \(E \perp \beta\), then \(\text{Rep}_\beta(E^\perp) := \text{Rep}_\beta(Q) \cap E^\perp\) is isomorphic to the homogeneous fibre space \(\text{GL}_{\beta \times \text{GL}_s} \times \text{Rep}_{\beta_\epsilon}(Q_E)\).

It follows ([9, Corollary 6.11], see also [6]) that if \(E \perp \beta\), then we have an isomorphism \(\text{SI}_{\beta_\epsilon}^\tau(\alpha_\epsilon)(Q_E) \cong \text{SI}_{\beta_\epsilon}^\tau(Q)\). So we have an embedding \(\iota : \text{SI}_{\beta_\epsilon}(Q_E) \hookrightarrow \text{SI}_\beta(Q)\).

\textbf{Lemma 8.2.} Let \(s := \iota(E)\) and \(\text{SI}_\beta(Q)_{(s)}\) be the localization of \(\text{SI}_\beta(Q)\) at \(s\), then there is an embedding mapping \(x\) to \(s\)

\[
\text{SI}_{\beta_\epsilon}(Q_E)[x, x^{-1}] \hookrightarrow \text{SI}_\beta(Q)_{(s)}.
\]

In particular, if \(\{s_1, \ldots, s_n\} \subset \text{SI}_{\beta_\epsilon}(Q_E)\) is algebraically independent, then so is \(\{s, \iota(s_1), \ldots, \iota(s_n)\}\).

\textit{Proof.} We define an algebra morphism

\[
\varphi : \text{SI}_{\beta_\epsilon}(Q_E)[x, x^{-1}] \rightarrow \text{SI}_\beta(Q)_{(s)} \quad \text{by} \quad rx^d \mapsto \iota(r)s^d.
\]

To show \(\varphi\) is injective, we suppose that \(\varphi(\sum_{d} rx^d) = \sum_{d} \iota(r_d)s^d = 0\). We define a partial order on the weights such that \(\sigma_1 \geq \sigma_2\) if and only if \(\sigma_1 - \sigma_2 = n(\epsilon\epsilon)\) for some \(n \in \mathbb{N}_0\). Now the leading term of \(\sum_d \iota(r_d)s^d\) has the highest weight because \(\epsilon \notin \epsilon^\perp := \{\alpha \mid \langle \alpha, Q \rangle = 0\}\) and the weight of \(\iota(r_d)\) is in \(\tau(\epsilon^\perp)\). So we see inductively that each \(\iota(r_d)\), and thus each \(r_d\), has to vanish.

\textbf{Remark 8.3.} The above construction together with Lemma 8.2 can be inductively generalized from a exceptional representation to an exceptional sequence \(E\). We refer the readers to [6, Section 2] for details.

We recall that an exceptional sequence of dimension vector \(E := \{e_1, e_2, \ldots, e_n\}\) is a sequence of real Schur roots of \(Q\) such that \(e_i \perp e_j\) for any \(i < j\). It is called quiver if \(\langle e_j, e_i \rangle_Q \leq 0\) for any \(i < j\). It is called \textit{complete} if \(n = |Q_0|\). The quiver of a quiver exceptional sequence \(E\) is by definition the quiver with vertices labeled by \(e_i\) and \(\text{Ext}_Q(e_j, e_i)\) arrows from \(e_j\) to \(e_i\). According to [31, Theorem 4.1], \(\text{Ext}_Q(e_j, e_i) = -\langle e_j, e_i \rangle_Q\). Now we return to the triple flag quivers.

\textbf{Lemma 8.4.}

\[
E := \{\alpha(f_{0,n-1}), \alpha(f_{n-1,0}), \alpha(e_1^3), e_{n-1}^1 + e_{n-1}^2 + e_n, e_{n-1}^3, \ldots, e_n^3, e_{n-2}^2, \ldots, e_1^3, e_{n-2}^1, \ldots, e_2^1\}
\]

is a complete exceptional sequence in \(\text{Rep}(T_n)\) such that the quiver \(T'_{n-1}\) of the quiver exceptional sequence

\[
\{\alpha(e_1^3), e_{n-1}^1 + e_{n-1}^2 + e_n, e_{n-1}^3, \ldots, e_n^3, e_{n-2}^2, \ldots, e_1^3, e_{n-2}^1, \ldots, e_2^1\}
\]
is

\[ e_1 \rightarrow e_2 \rightarrow \cdots \rightarrow e_{n-2} \]

\[ e_1^2 \rightarrow e_2^2 \rightarrow \cdots \rightarrow e_{n-2}^2 \rightarrow e_{n-1}^1 + e_{n-1}^2 + e_n \xrightarrow{\alpha} \alpha(e_1^3) \]

\[ e_1^3 \rightarrow e_2^3 \rightarrow \cdots \rightarrow e_{n-1}^3 \]

**Proof.** It is trivial to check following the definition that \( E \) is a complete exceptional sequence with desired properties. Alternatively, one can perform the *mutation* operators \([29]\) on the standard sequence

\( E_0 := \{ e_n, e_{n-1}, \ldots, e_1, e_{n-1}, \ldots, e_1, e_{n-1}, \ldots, e_1 \} \)

as follows:

1. apply the left mutations to move \( e_1^3 \) to the left, and it becomes \( \alpha(e_1^3) \);
2. apply the left mutations to move \( e_{n-1}^2 \) and \( e_{n-1}^1 \) right next to \( e_n \), and they remain themselves;
3. apply the right mutations 2 times to move \( e_n \) passing \( e_{n-1}^2 \) and \( e_{n-1}^1 \), and it becomes \( e_{n-1}^1 + e_{n-1}^2 + e_n \);
4. apply the left mutations 2 times to move \( e_{n-1}^2 \) and \( e_{n-1}^1 \) passing \( \alpha(e_1^3) \), and they become \( \alpha(f_{0,n-1}) \) and \( \alpha(f_{n-1,0}) \). \( \square \)

To simplify the notation, we label the rightmost vertex of \( T'_{n-1} \) by \( n \), so \( e_n \) is the unit vector supported on \( n \). We also observe that \( T'_{n-1} \) contains \( T_{n-1} \) as a subquiver. So the weight vector \( f_{i,j} \) for \( T_{n-1} \) is naturally a weight vector for \( T'_{n-1} \) if we set its \( n \)th coordinate to be zero.

Let \( F \) be the exceptional sequence \( \{ \alpha(f_{0,n-1}), \alpha(f_{n-1,0}) \} \). By Lemma 8.4 and definition, the quiver \( T'_{n-1} \) is the quiver \((T_n)_{E'}\). So we have an embedding \( \text{Rep}((T_n)_{E'}) \cong \text{Rep}(E) \hookrightarrow \text{Rep}(T_n) \).

Let \( \beta'_{n-1} \) be the following dimension vector of \( T'_{n-1} \).

\[
\begin{align*}
1 & \rightarrow 2 \rightarrow \cdots \rightarrow n-2 \\
1 & \rightarrow 2 \rightarrow n-2 \rightarrow n-1 \rightarrow 1 \\
1 & \rightarrow 2 \rightarrow \cdots \rightarrow n-2
\end{align*}
\]

**Lemma 8.5.** Under the embedding, the dimension vector \( \beta'_{n-1} \) goes to the standard one \( \beta_n \) of \( T_n \). Moreover, \( f_{i,j} \) goes to

\[
\begin{align*}
f_{i,j} & \quad \text{for } i + j = n - 1, ij \neq 0, \\
f_{i,j} + f_{n-1,0} & \quad \text{for } i = 0, \\
f_{i,j} + f_{0,n-1} & \quad \text{for } j = 0, \\
f_{i,j} + f_{0,n-1} + f_{n-1,0} & \quad \text{for } 1 < i + j < n - 1, ij \neq 0; \\
f_i := e_n + e_{n-1} - e_1^1 - e_{n-1}^2 & \quad \text{goes to} \\
f_{i,n-1} & \quad \text{for } 1 \leq i \leq n - 1.
\end{align*}
\]

Here, we use the convention that \( e_{n-1}^1 = e_{n-1}^2 = e_{n-1} \). Moreover, let \( L, N_1, N_2, M_1, M_2, M \) be general representations of
dimension $\alpha(f_{i,j} + f_{n-1,0}), \alpha(f_{i,j} + f_{0,n-1}), \alpha(f_{i,j} + f_{0,n-1} + f_{n-1,0})$ respectively. Then we have exact sequences

$$
0 \to L \to M_1 \to N_1 \to 0 \quad \text{for } i = 0,
0 \to L \to M_2 \to N_2 \to 0 \quad \text{for } j = 0,
0 \to M_1 \to M \to N_2 \to 0 \quad \text{for } 1 < i + j < n - 1, ij \neq 0.
$$

**Proof.** Let us recall the recipe of the linear inclusion $\iota_\epsilon$ of the Grothendieck groups. Let $\beta$ be a dimension vector of $T'_{n-1}$, then $\iota_\epsilon(\beta) = \sum_v \beta(v)e_v$, where $e_v$ is the real Schur root in the exceptional sequence corresponding to the vertex $v$. The recipe for a weight vector $f$ of $T'_{n-1}$ is similar: $\iota_\epsilon(f) = \sum_v (\alpha f)(v)e_v$. Then the first part can be easily verified.

By Lemma 2.3, $L, N_1$ and $N_2$ are general representations. It is easy to check that $\text{hom}_{T_n}(L, N_1) = \text{hom}_{T_n}(N_1, L) = 0$, $\langle L, N_1 \rangle_{T_n} = 0$, and $\langle N_1, L \rangle_{T_n} = -1$, so $\text{ext}_{T_n}(L, N_1) = 0$ and $\text{ext}_{T_n}(N_1, L) = 1$. According to Lemma 1.5, we have the first two exact sequences. The existence of the last exact sequence is proved similarly. \hfill \Box

**Lemma 8.6.** A general representation in $\text{Rep}_{\beta_n}(T_n)$ has a following representative. Its matrices on the first arm are $(I_k \ 0)$; on the second arm are $(0 \ I_k)$, where $I_k$ is a $k \times k$ identity matrix for $k = 1, \ldots, n - 1$ and 0 is a zero column vector.

**Proof.** If we forget the third arm (but keep the central vertex $n$), then we get a quiver of type $A_{2n-1}$ with the dimension vector shown below

$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n - 1 \longrightarrow n \longleftarrow n - 1 \longleftarrow \cdots \longleftarrow 2 \longleftarrow 1$$

This dimension vector decomposes canonically as $\sum_{i=0}^{n-1} 1_i$, where

$$1_i = (0, \ldots, 0, 1, \underbrace{1, 0, \ldots, 0}_n).$$

This means that a general representation of that dimension vector is a direct sum of general representations of dimension $1_i$, which is exactly what we desired. \hfill \Box

Recall the weight vector $f_i = e_n + e_{n-1} - e_1^i - e_{n-i}^2$ in Lemma 8.5. We associate for each $f_i$ a projective presentation $f_i$

$$P_{n-1} \oplus P_n \xrightarrow{(p_1^1, p_{n-1}^2)} P_n \cong P_1^1 \oplus P_{n-i}^2$$

and thus a semi-invariant function $s(f_i) \in \text{SL}_{\beta_n}(T'_{n-1})$. Here $a$ is the rightmost arrow as shown in the figure of Lemma 8.4. From now on, we will write $s_i$ and $s_{i,j}$ for $s(f_i)$ and $s(f_{i,j}).$

**Lemma 8.7.** Assume that elements in $S_{n-1}$ are algebraically independent over $k$. Then $\{s_i \mid 1 \leq i \leq n - 1\} \cup S_{n-1}$ are algebraically independent as well.

**Proof.** We consider a representation $M \in \text{Rep}_{\beta_n}(T'_{n-1})$ whose matrices on the first arm are $(I_k \ 0)$; on the second arm are $(0 \ I_k)$; on the third arm are in general position; and the matrix for the arrow $a$ is generic $(x_1, x_2, \ldots, x_{n-1})^T$. Then the semi-invariant $s_i$ is the determinant of the block matrix

$$
\begin{pmatrix}
I_i & O_{i,n-i-1} & X_i \\
O_{n-i,i-1} & I_{n-i} & O_{n-i,1}
\end{pmatrix}
$$
Recall the weight configuration $\sigma$.

Initial exchanges are algebraically independent. We finish the proof by induction. □

where $I_i$ is the $i \times i$ identity matrix, $O_{i,j}$ is the $i \times j$ zero matrix, and $X_i = (x_1, \ldots, x_i)^T$. It is easy to see that this determinant equals to $(-1)^{n-1} x_i$.

Now suppose that \{ $s_i$ | $1 \leq i \leq n - 1$ \} $\cup S_{n-1}$ are algebraically dependent, that is, there is a nonzero polynomial $p \in k[y_1, y_2, \ldots, y_{\frac{1}{2} \left( n^2 + 3n - 8 \right)}]$ such that

$$p(s_1, \ldots, s_{n-1}, \{s_{i,j}\}_{(i,j) \in \delta_{n-1}}) = 0.$$ 

Its total degree on $y_1, \ldots, y_{n-1}$ must be strictly positive because elements in $S_{n-1}$ are algebraically independent. We evaluate $p$ at the representation $M$, and get

$$p((-1)^{n-1} x_1, \ldots, -x_{n-1}, \{s_{i,j}(M)\}_{(i,j) \in \delta_{n-1}}) = 0.$$ 

Since $x_1, \ldots, x_{n-1}$ are generic variables, we conclude that there is some function $r \in R$ vanishes at $M$, where $R \subset SI_{\beta_{n-1}}(T_{n-1})$ is the subring generated by $S_{n-1}$. By Lemma 8.6, the restriction of $M$ on the subquiver $T_{n-1}$ is a general representation of dimension $\beta_{n-1}$. But a general representation in $\text{Rep}_{\beta_{n-1}}(T_{n-1})$ is stable, so there is no such $r$. We get a contradiction. □

**Theorem 8.8.** The semi-invariant functions in $S_n$ are algebraically independent over the base field $k$.

Proof. We prove by induction on $n$. If $n = 2$, then the statement is trivial. Now suppose that it is true for $n = m$. Applying Lemma 8.2 (see Remark 8.3) to the exceptional sequence $F = \{ \alpha(f_{0,m}), \alpha(f_{m,0}) \}$, we get

$$\text{SI}_{\beta'_m}(T'_m)[x^{\pm 1}, y^{\pm 1}] \hookrightarrow \text{SI}_{\beta_{m+1}}(T_{m+1})_{s_0,m,s_m,0},$$

where the quiver $T'_m$ and dimension vector $\beta'$ are as in Lemma 8.5.

By induction and Lemma 8.7, we have that

$$\{s_{i,j} | (i, j) \in \delta_m\} \cup \{s_i | 1 \leq i \leq m\}$$

are algebraically independent. Then by Lemma 8.2, 8.5, and 1.6

$$\{s_{m,0}, s_{0,m}\} \cup \{s_{i,j}s_{m,0}, s_{i,0}s_{0,m} | 1 \leq i, j \leq m\} \cup$$

$$\{s_{i,j}s_{m,0}s_{0,m} | 1 < i + j < m, ij \neq 0\} \cup \{s_{i,j} | i + j = m, m + 1, ij \neq 0\}$$

is algebraically independent. We note that all elements above, up to some multiple of $s_{m,0}$ and $s_{0,m}$, are elements in $\{s_{i,j} | (i, j) \in \delta_{m+1}\}$. This implies that level-1 semi-invariants in $T_{m+1}$ are algebraically independent. We finish the proof by induction. □

8.2. *Initial exchanges*

Recall the weight configuration $\sigma_n$ for $\Delta_n$. Let $g_{i,j} = f_{i-1,j} + f_{i,j+1} + f_{i,j-1} + f_{i-1,j+1}$. It is not hard to verify that $\alpha'(g_{i,j})$ is an isotropic Schur root of $T_n$, but we do not need this fact. If we mutate $(\Delta_n, \sigma_n)$ at a mutable vertex $(i, j)$, then

$$f'_{i,j} = g_{i,j} - f_{i,j} = 2e_n - e_{i-1}^1 - e_{i+1}^1 - e_{j-1}^2 - e_{j+1}^2 - e_{k-1}^3 - e_{k+1}^3, \quad i, j, k \geq 1, i + j + k = n.$$ 

As before, we use the convention that $e_0^a$ is the zero vector for $a = 1, 2, 3$.

**Lemma 8.9.** For each $(i, j) \in \delta_n$, we have that

(1) $\dim \text{SI}_{\beta_n}(T_n)_{g_{i,j}} = 2$ and $\dim \text{SI}_{\beta_n}(T_n)_{f_{i,j}} = 1$;

(2) the weight $f_{i,j}$ is extremal in $\Sigma_{\beta_n}(T_n)$, so $\text{SI}_{\beta_n}(T_n)_{f_{i,j}}$ is spanned by an irreducible polynomial.
Proof. (1) From the correspondence (2.1)–(2.3), we need to show that \( c_{\mu(\mathfrak{g}_{i,j})}^{\lambda(\mathfrak{g}_{i,j})} = 2 \). It is easy to see that the transposed partitions \( \lambda(\mathfrak{g}_{i,j})^*, \mu(\mathfrak{g}_{i,j})^*, \nu(\mathfrak{g}_{i,j})^* \) is given by

\[
(3, 2, 1) + (i + j - 2)(1, 1, 1), \; (2, 1, 0) + (i - 1)(1, 1, 1), \; (2, 1, 0) + (j - 1)(1, 1, 1).
\]

So everything reduces to show \( c_{\mu, \nu}^{\lambda} = 2 \) for \( (\lambda, \mu, \nu) = ((3, 2, 1), (2, 1), (2, 1)) \). This is clear from the Littlewood–Richards rule. Similarly, we show that \( c_{\mu(\mathfrak{g}_{i,j})}^{\lambda(\mathfrak{g}_{i,j})} = 1 \). The transposed partitions \( \lambda(f_{i,j}'')^*, \mu(f_{i,j}'')^*, \nu(f_{i,j}'')^* \) is given by

\[
(3, 1) + (i + j - 2)(1, 1, 1), \; (2, 0) + (i - 1)(1, 1, 1), \; (2, 0) + (j - 1)(1, 1, 1).
\]

So we only need to observe that \( c_{(3,1)}^{(2,2)} = 1 \).

(2) Suppose that \( f_{i,j}' = f_1 + f_2 \) with \( f_1, f_2 \in \Sigma_{\beta_n}(T_n) \), then \( f_1 \) and \( f_2 \) must be two level-1 weights. But this is clearly impossible. So \( f_{i,j}' \) spans an extremal ray. According to Lemma 1.8, \( \text{SI}_{\beta_n}(T_n)_{f_{i,j}'} \) is spanned by an irreducible polynomial. \( \square \)

For each \( f_{i,j}' \), we associate a Schofield’s semi-invariant function corresponding to the presentation

\[
P_{i-1}^1 \oplus P_{j+1}^2 \oplus P_{k-1}^3 \oplus P_{n+1}^1 \oplus P_{j-1}^2 \oplus P_{k+1}^3 \rightarrow 2P_n.
\]

We keep using the short hand \( s_{i,j} \) and \( s_{i,j}' \) for Schofield’s semi-invariants \( s(f_{i,j}) \) and \( s(f_{i,j}') \). The relations below are the initial exchange relations needed to apply Lemma 3.9.

Lemma 8.10. We have the following relations:

\[
(-1)^ns_{i,j} s_{i,j}' = s_{i-1,j} s_{i,j+1} s_{i+1,j-1} + s_{i+1,j} s_{i,j-1} s_{i-1,j+1}.
\]

Proof. Let

\[
F_0 = (-1)^ns_{i,j} s_{i,j}', \quad F_1 = s_{i-1,j} s_{i,j+1} s_{i+1,j-1}, \quad F_2 = s_{i+1,j} s_{i,j-1} s_{i-1,j+1}.
\]

Since \( \dim \text{SI}_{\beta_n}(T_n)_{\mathfrak{g}_{i,j}} = 2 \), we must have that \( a_0F_0 = a_1F_1 + a_2F_2 \), for some \( a_0, a_1, a_2 \in k \). To find \( a_i \), we consider a special representation \( M \), whose matrices on the first arm are \((I_k 0)\); on the second arm are \((0 I_k)\); on the third arm are \((I_k 0)\) for \( k = 1, \ldots, n - 2 \), where \( I_k \) and \( 0 \) are as in Lemma 8.6. The last matrix on the third arm is

\[
M(a_{n-1}^3)_{rc} = \begin{cases} 1 & \text{if } r-c = i-1, i, i+1 \\ 0 & \text{otherwise.} \end{cases}
\]

Specializing the semi-invariants at \( M \), we find that \( s_{i-1,j}, s_{i+1,j-1}, s_{i+1,j}, s_{i-1,j+1} \) are triangular with all diagonal entries equal to 1. Moreover, elementary linear algebra calculation shows that \( s_{i,j}' = (-1)^ns \)

\[
(s_{i,j}, s_{i,j+1}, s_{i,j-1}) = \begin{cases} (-1)^q(1, 1, 0) & j = 3q \\ (-1)^q(1, 0, 1) & j = 3q + 1 \\ (-1)^q(0, -1, 1) & j = 3q + 2. \end{cases}
\]

Similarly, we consider another representation \( M' \) which is the same as \( M \) except that the last matrix \( M(a_{n-1}^3) \) on the second arm is replaced by \((I_n 0) + (0 I_n)\). We find that the specialization of \( s_{i-1,j}, s_{i+1,j-1}, s_{i+1,j}, s_{i-1,j+1}, s_{i,j}' \) at \( M' \) is the same as at \( M \). But

\[
(s_{i,j}, s_{i,j+1}, s_{i,j-1}) = \begin{cases} (-1)^q(1, 0, 1) & j = 3q \\ (-1)^q(0, -1, 1) & j = 3q + 1 \\ (-1)^q(-1, 0, 0) & j = 3q + 2. \end{cases}
\]

Therefore, we conclude that \( a_0 = a_1 = a_2 \) by solving a linear system. \( \square \)
8.3. Main theorem

**Theorem 8.11.** We assign for each vertex $(i, j)$ of the hive quiver $\Delta_n$ the semi-invariant function $s(f_{ij})$. Then $\text{SI}_{\beta_n}(T_n)$ is the upper cluster algebra $\overline{C}(\Delta_n, S_n)$.

**Proof.** Due to Theorem 8.8, such an assignment does define a cluster algebra $\mathcal{C}(\Delta_n, S_n)$. By [28, Theorem 3.17], any $\text{SI}_{\beta_n}(T_n)$ is a unique factorization domain. Due to Lemma 8.10 and 8.9 (2), we can apply Lemma 3.9 and get the containment $\overline{C}(\Delta_n, S_n) \subseteq \text{SI}_{\beta_n}(T_n)$. By Corollary 5.14, $C_{\mathcal{W}_n}(G_n(\sigma) \cap \mathbb{Z}^n)$ is a linearly independent set in $\text{SI}_{\beta_n}(T_n)$. But the dimension of $\text{SI}_{\beta_n}(T_n)$ is counted by $|G_n(\sigma) \cap \mathbb{Z}^n|$ according to Theorem 2.1 and 7.3. Hence, we have the equality $\overline{C}(\Delta_n, S_n) = \text{SI}_{\beta_n}(T_n)$. □

It follows from the proof that

**Corollary 8.12.** The generic character $C_{\mathcal{W}_n}$ maps the lattice points in $G_n$ (bijectionally) onto a basis of $\overline{C}(\Delta_n)$.

This can be viewed as an algebraic analog of the full Fock–Goncharov conjecture for the quotient space $\text{Rep}_{\beta_n}(T_n)/\text{SL}_{\beta_n}$ [13].

**Proposition 8.13.** Suppose that a cluster algebra $\mathcal{C}(\Delta)$ has only finitely many clusters. If all extremal rays in the g-vector cone $\mathbb{R}_+G(\Delta)$ are generated by the g-vectors of cluster variables, then $G(\Delta)$ is generated by the g-vectors of cluster and coefficient variables over $\mathbb{Z}_+$. 

**Proof.** Let $x = (x_1, x_2, \ldots, x_q)$ be a cluster with g-vectors $g = (g_1, g_2, \ldots, g_q)$. Let $G_{\mathcal{G}}$ be the union of $\mathbb{R}_+g$ for all clusters. It is proved in [4, Proposition 6.1] that for coefficient-free cluster algebras, the finiteness of clusters implies that $G_{\mathcal{G}}$ covers the whole $\mathbb{R}^q$. The extended g-vector of a cluster variable is uniquely determined by its principal part (see the remark after [35, Definition 3.7]). So the same argument as in [4] can show that in general $G_{\mathcal{G}}$ is a polyhedral cone, and thus equal to $\mathbb{R}_+G(\Delta)$. It is known [8, Theorem 1.7] that $g$ form a $\mathbb{Z}$-basis of the lattice $\mathbb{Z}^q$. Since $G_{\mathcal{G}}$ is a union of unimodular cones $\mathbb{R}_+g$, $G(\Delta)$ is generated by the g-vectors of cluster and coefficient variables over $\mathbb{Z}_+$. □

**Corollary 8.14.** For $n < 6$, we have that $\overline{C}(\Delta_n) = \mathcal{C}(\Delta_n)$.

**Proof.** The first nontrivial case is $n = 3$. This case is obvious. For $n = 4, 5$, we use the software Normaliz [2] for computing all extremal rays of $G_n$. We find that there are 18 extremal rays and 45 extremal rays in $G_4$ and $G_5$. Checking with [21], they are all generated by the g-vectors of cluster and coefficient variables. Our claim then follows from Proposition 8.13 and Corollary 8.12. □

We can show that $\overline{C}(\Delta_6) = \mathcal{C}(\Delta_6)$ but the proof is more involved. We do not know if we always have the equality $\overline{C}(\Delta_n) = \mathcal{C}(\Delta_n)$. We remark that for general semi-invariant rings of quiver representations, this equality fails rather frequently [10, 12].

9. Constructing generating sets

9.1. Minimal cluster generating set

Let $Q$ be a finite quiver without oriented cycles as in Section 1. Since $\text{SL}_{\beta}$ is reductive, we know that the semi-invariant ring $\text{SI}_{\beta}(Q)$ is a finitely generated $k$-algebra. Now we assume
that $SI_\beta(Q)$ is a cluster algebra naturally graded by weights of semi-invariants. Although the cluster algebra may contain infinitely many cluster variables, we can always choose a finite minimal set $F$ of generators consisting of cluster variables and coefficient variables. We call such a set a \textit{minimal cluster generating set} of $SI_\beta(Q)$. Recall that all cluster variables and coefficient variables are all multihomogeneous.

Let $F'$ be another minimal set of (not necessary multihomogeneous) generators of $SI_\beta(Q)$. We claim that $F$ and $F'$ have the same cardinality. We know that $SI_\beta(Q)$ is \textit{positively multigraded} $k$-algebra in the sense that the degree zero component $SI_\beta(Q)_0$ is $k$. We can assume that all elements in $F'$ have no constant term, then they belong to the ideal $J$ of $SI_\beta(Q)$ generated by all multihomogeneous elements of nonzero degree. By the (positively multigraded version of) Nakayama lemma, $F'$ modulo $J^2$ forms a basis of $J/J^2$. The same argument can be applied to $F$ as well.

The idea to construct a minimal set of multihomogeneous generators is quite naive. We compute for each weight $\sigma \in \Sigma_\beta(Q)$ the subspace $R_\sigma$ of $SI_\beta(Q)_\sigma$

$$R_\sigma := \sum_{\sigma' \in \Sigma_\beta(Q)} SI_\beta(Q)_{\sigma'} SI_\beta(Q)_{\sigma - \sigma'}.$$ 

Let $B_\sigma$ be a basis of any subspace of $SI_\beta(Q)_\sigma$ complementary to $R_\sigma$. Then $\bigcup_\sigma B_\sigma$ constitutes a minimal set of multihomogeneous generators for $SI_\beta(Q)$. All such minimal sets can be so obtained. In particular, we can choose each element in $B_\sigma$ to be a cluster variable or a coefficient variable.

\textbf{Conjecture 9.1.} If $SI_\beta(Q)$ is a cluster algebra naturally graded by weights of semi-invariants, then it has a unique minimal cluster generating set.

9.2. $\mathbb{R}_+ \Sigma_\beta_n(T_n)$ revisited

Now we come back to the complete triple flags. We will construct a minimal cluster generating set for $SI_\beta_n(T_n)$ when $n < 6$. To carry out our algorithm, we should start with all indivisible extremal weights $\sigma_e$ in $\Sigma_\beta_n(T_n)$. Note that for any such weight $\sigma_e$, $SI_\beta_n(Q)_{\sigma_e}$ is spanned by cluster variables and coefficient variables over $k$.

Recall that we have an algorithm to find extremal rays of $\mathbb{R}_+ \Sigma_\beta_n(T_n)$. Now with Theorem 8.11, we get a faster algorithm as follows.

\textbf{Algorithm 9.2.} We first compute all extremal rays $g_e$ in $G_n$, then all extremal rays in $\mathbb{R}_+ \Sigma_\beta_n(T_n)$ are among $g_e \sigma_n$.

\textbf{Example 9.3.} In contrast to [20, Conjecture 0.19], not all extremal rays of $G_n$ are $g$-vectors of cluster variables. The first counterexample appears when $n = 6$. Let us consider an extremal ray of $G_6$

$$g_0 = (e_{2,0} + e_{0,4} + e_{4,2}) + e_{2,3} + e_{3,1} + e_{1,2} - e_{2,1} - e_{1,3} - e_{3,2}.$$ 

We say a $g$-vector $g$ can be obtained via a sequence of mutations $\mu_{u^1} \cdot \mu_{u^2} \cdot \mu_{u^3}$ from $\Delta$ if $g$ is the $g$-vector of the cluster variable $\mu_{u}(x_{u^1})$ (with respect to the seed $(\Delta, x)$). It is known from [4, 8] that if $g_0$ can be reached via mutations, then the $E$-space $E(f, f) = 0$ for $f = \text{Coker}(g_0)$. But it is not hard to show that $E(f, f) = k$.

The following proposition can greatly expedite our search for minimal cluster generating sets.
Proposition 9.4 [6, Corollary 7.12]. For \( n \leq 7 \), the extremal rays of \( \mathbb{R}^+ \Sigma_{\beta_n}(T_n) \) are generated by weights corresponding to real Schur roots.

As pointed in [6, Example 7.13], \( n = 8 \) is the first example when \( \mathbb{R}^+ \Sigma_{\beta_n}(T_n) \) can contain an extremal weight \( \sigma \) such that \( \sigma \) is an isotropic Schur root. The example given there is

\[
\sigma = 3e_8 - (e_1^2 + e_3^2 + e_4^2 + e_5^2 + e_6^2 + e_8^2).
\]

In this case, \( c_{\mu(\sigma),\nu(\sigma)}^{(\phi)} = 2 \). It turns out that \( \text{SI}_{\beta_8}(T_8) \) is spanned by two cluster variables.

Interested readers can verify using [21] that the two cluster variables have \( g \)-vectors

\[
(e_{0,6} + e_{5,3} + e_{2,0}) + e_{1,3} + e_{3,2} - e_{2,2} - e_{3,3},
\]

which can be obtained respectively by the sequences of mutations from \( \Delta_8 \)

\[(3, 1), (2, 2), (2, 1), (3, 1), (4, 1), (3, 2), (4, 2), (5, 2), (4, 3), (3, 4), (3, 3), (2, 4), (1, 5), (2, 5);\]

\[(2, 5), (2, 4), (3, 4), (3, 3), (4, 3), (4, 2), (4, 1), (3, 2), (3, 1), (2, 2), (2, 1), (1, 2).\]

9.3. When \( n \leq 6 \)

The first nontrivial case appears when \( n = 3 \). It is treated in [34].

Proposition 9.5. The semi-invariant ring \( \text{SI}_{\beta_3}(T_3) \) is a cluster algebra of type \( A_1 \). It can be presented by

\[
k[s_{0,1}, s_{1,0}, s_{0,2}, s_{2,0}, s_{1,2}, s_{2,1}, s_{1,1}, s_{1,1}']/(s_{1,1}s_{1,1}' - s_{0,1}s_{2,0}s_{1,2} - s_{1,0}s_{0,2}s_{2,1}).
\]

Below we will use a well-known fact that the number of cluster variables in a cluster algebra \( C(\Delta) \) is completely determined by the mutable part of \( \Delta \). It is equal to the number of real roots of \( \Delta \) plus \(|\Delta_0|\) if \( \Delta \) is acyclic.

Proposition 9.6. The semi-invariant ring \( \text{SI}_{\beta_4}(T_4) \) is a cluster algebra of type \( A_3 \). It is minimally generated by nine cluster variables and nine coefficient variables in \( C(\Delta_4, S_4) \). They are all of extremal weights in \( \Sigma_{\beta_4}(T_4) \).

Proof. It is enough to observe that there are nine cluster variables for a cluster algebra of type \( A_3 \). Together with the nine coefficient variables, each one has to match one of the 18 extremal rays in the cone \( \mathbb{R}^+ \Sigma_{\beta_4}(T_4) \) (see Example 2.6).

Proposition 9.7. The semi-invariant ring \( \text{SI}_{\beta_5}(T_5) \) is a cluster algebra of type \( D_6 \). It is minimally generated by 30 out of 36 cluster variables and 12 coefficients of \( C(\Delta_5, S_5) \). They are all of extremal weights in \( \Sigma_{\beta_5}(T_5) \). The weights of six cluster variables excluded are, up to symmetry, equal to \( \sigma = 3e_5 - (e_1^1 + e_3^2 + e_2^2 + e_7^2 + e_5^2 + e_1^1) \).

Proof. If we perform a sequence of mutations at \((1,3)\), \((2,1)\), \((1,1)\), and \((1,2)\). then the mutable part of \( \Delta_5 \) transforms into a Dynkin quiver of type \( D_6 \), which has 30 real roots. We use Algorithm 9.2 to find that there are 42 extremal rays in \( \mathbb{R}^+ \Sigma_{\beta_5}(T_5) \). They correspond to the 30 cluster variables and 12 coefficients. The weights of the remaining six cluster variables are, up to symmetry, equal to \( \sigma \). We need to show these six cluster variables are redundant. The weight \( \sigma \) corresponds to the partition \( \lambda = (5, 4, 2), \mu = (4, 2), \nu = (3, 2) \). By Theorem 2.1, the
dimension of $\text{SI}_{\beta_n}(T_n)$ is equal to $c^\lambda_{\mu,\nu}$, which is equal to 2 by the Littlewood–Richardson rule. Consider the products of cluster variables

$$C_{W_5}(e_{1,0} + e_{0,3} + e_{3,2} - e_{1,2})x_{2,1} \quad \text{and} \quad x_{2,3}x_{3,0}x_{0,1}.$$  

The cluster variable with $g$-vector $e_{1,0} + e_{0,3} + e_{3,2} - e_{1,2}$ can be obtained by a sequence of mutations at $\{(1,1),(2,2),(2,1),(1,2)\}$. Both the products have weight $\sigma$ but different $g$-vectors, so they are linearly independent. □

In general, $\text{SI}_{\beta_n}(T_n)$ is not generated by semi-invariants of extremal weights. The following proposition can be proved using the technique of projections [12].

**Proposition 9.8.** The semi-invariant ring $\text{SI}_{\beta_6}(T_6)$ is a cluster algebra of infinite type. It is minimally generated by 103 cluster variables, 15 coefficients of $C(\Delta_6, S_6)$. Six of these cluster variables are not of extremal weights. Their weights up to symmetry are equal to $\sigma = 3e_6 - (e_3^3 + e_2^3 + e_2^3 + e_1^3 + e_0^3)$.

Here, at least we can see that the six cluster variables of nonextremal weights are necessary. We use Algorithm 9.2 to find that there are 112 extremal rays in $\mathbb{R}^{+}\Sigma_{\beta_6}(T_6)$. They corresponds to the 97 cluster variables and 15 coefficients. The weight $\sigma$ corresponds to the partition $\lambda = (6, 5, 2), \mu = (5, 2), \nu = (4, 2)$ with $c^\lambda_{\mu,\nu} = 2$. However, we can only find one decomposition of $\sigma$ into extremal weights,

$$\sigma = (e_6 - e_1^3 - e_3^3) + (e_6 - e_2^3 - e_4^3) + (e_6 - e_2^2 - e_4^3).$$

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