Frame functions in finite-dimensional Quantum Mechanics and its Hamiltonian formulation on complex projective spaces

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Abstract. This work concerns some issues about the interplay of standard and geometric (Hamiltonian) approaches to finite-dimensional quantum mechanics, formulated in the projective space. Our analysis relies upon the notion and the properties of so-called frame functions, introduced by A.M. Gleason to prove his celebrated theorem. In particular, the problem of associating quantum states with positive Liouville densities is tackled from an axiomatic point of view, proving a theorem classifying all possible correspondences. A similar result is established for classical observables representing quantum ones. These correspondences turn out to be encoded in a one-parameter class and, in both cases, the classical objects representing quantum ones result to be frame functions. The requirements of $U(n)$ covariance and (convex) linearity play a central role in the proof of those theorems. A new characterization of classical observables describing quantum observables is presented, together with a geometric description of the $C^*$-algebra structure of the set of quantum observables in terms of classical ones.

1 Introduction

It has been known from a long time \cite{Kib79,AS95,BH01} that quantum mechanics can be formulated as a proper Hamiltonian theory in the complex projective space. This observation, starting from \cite{Kib79} even produced some interesting attempts towards non-linear extensions of quantum mechanics (see the last section of \cite{BH01} for references). Actually, if the Hilbert space of the quantum formulation is infinite-dimensional several technical problems arise, especially related with the notion of infinite dimensional manifold, beyond the obvious fact that the physically relevant observables are unbounded self-adjoint operators. In this paper, we only focus on the $n$-dimensional case, $n < +\infty$, whose interest is due to quantum information theory in particular. Quantum-Hamiltonian formulation relies upon a few ideas. First of all the space of phases is chosen to be the complex projective space $P(H_n)$ constructed out of the Hilbert space $H_n$ of the considered quantum theory. The manifold $P(H_n)$ possesses a natural \textit{almost Kähler structure}. That is a structure made of: (1) a symplectic form $\omega$, accompanied by (2) a Riemannian metric $g$, and (3) an intertwining almost complex structure $j$, transforming $g$ to $\omega$ and \textit{viceversa}. The
symplectic form $\omega$ permits to introduce the standard Hamiltonian formalism. As a second step, quantum observables, i.e. self-adjoint operators $A$, are associated to real-valued functions $f_A : \mathcal{P}(\mathcal{H}_n) \to \mathbb{R}$, thus representing those operators in terms of classical observables. Fixing a Hamiltonian operator $H$ and its classically associated function $H := f_H$ (the classical Hamiltonian), a point $p \in \mathcal{P}(\mathcal{H}_n)$ can be seen to evolve in time, $p = p(t)$, either in accordance with the Schrödinger equation or in accordance with the classical Hamiltonian equations. The above mentioned correspondence of quantum and classical observables is fixed in such a way that the two notions of evolution coincide. It is worth stressing that the existence of such a possibility is far from obvious. Finally, the Riemannian metric $g$ enters the picture in a nice way: Every notion of unitary time evolution corresponds to a notion of evolution along a suitable $g$-Killing flow. In general, it turns out that the unitary quantum symmetries are represented by canonical Hamiltonian symmetries, preserving both the underlying Lie algebra structure and the Riemannian metrical structures of $\mathcal{P}(\mathcal{H}_n)$ (see Theorem 11). There are many other interesting features of this quantum-classical correspondence. However, the agreement seems not to be guaranteed when comparing the two notions of expectation value. Within the classical picture, the expectation value $E_\rho(f)$ of a classical observable $f : \mathcal{P}(\mathcal{H}_n) \to \mathbb{R}$ is defined in terms of integral with respect to a Liouville measure $\rho \, d\mu$. Here, $\mu$ is the Liouville volume form constructed out of $\omega$ (and, for this reason it is invariant under symplectic diffeomorphisms), and the classical state $\rho$ is a positive Liouville density satisfying the Liouville equation. Thus $E_\rho(f) := \int f \rho \, d\mu$. Adopting the quantum framework, instead, an expectation value $\langle A \rangle_\sigma$ is nothing but the trace of the product of the self-adjoint operators $A$, representing the considered observable, and the density matrix $\sigma$, representing the state of the system. Thus $\langle A \rangle_\sigma := tr(\sigma A)$. It is not clear from the literature what is the most general way to define a correspondence from quantum to classical states such that quantum expectation values and classical expectation values coincide: $E_{\rho_\sigma}(f_A) = \langle A \rangle_\sigma$, preserving the requirement $\rho_\sigma \geq 0$. If the last requirement does not hold, $\rho_\sigma$ cannot be interpreted as a classical probability density. This issue is tackled in this paper among others. In particular we classify all possible correspondences from quantum and classical states on the one hand, and quantum and classical observable on the other hand. In both cases, the classical objects representing quantum ones result to be frame functions (see below). These correspondences fulfil a list of natural requirements, $U(n)$ covariance and (convex) linearity in particular. We find that actually, there is room enough in the formalism to obtain positive Liouville densities $\rho_\sigma$, provided one drops another assumption on observables that, however, does not seem so physically cogent. (Theorems 23, 22 and 26). As a byproduct, we also establish another characterization of the small class of functions on $\mathcal{P}(\mathcal{H}_n)$ describing quantum observables (Proposition 25). Eventually, a description of the unital $C^*$-algebra structure of the set of observables will be discussed in terms of the geometric features of $\mathcal{P}(\mathcal{H}_n)$ (Theorem 30). The most important mathematical notion we exploit in our analysis is that of frame function. It was introduced as a technical tool by A. M. Gleason to prove his celebrated theorem [Gle57].
Definition 1  If $H$ is a separable complex Hilbert space, let $\mathbb{S}(H)$ denote the unit sphere centered on the origin. $f : \mathbb{S}(H) \to \mathbb{C}$ is a frame function on $H$ if $W_f \in \mathbb{C}$ exists with:

$$\sum_{\phi \in N} f(\phi) = W_f \quad \text{for every Hilbertian basis } N \text{ of } H. \quad (1)$$

The key step in the proof of Gleason’s theorem is proving that the class of bounded real frame functions coincides to the class of quadratic forms $f(\phi) = \langle \phi | A \phi \rangle$ where $A$ is any self-adjoint trace-class operator on $H$. A frame function on an infinite dimensional Hilbert space is always bounded, whereas in the finite-dimensional case ($\dim(H) \geq 3$), there exist infinitely many unbounded frame functions [Dvu92]. In [MP13], adopting a pure mathematical viewpoint, we proved a proposition concerning sufficient conditions to assure that a frame function on a complex finite-dimensional Hilbert space is representable as a quadratic form without assuming the boundedness requirement a priori. Observing that $\mathbb{S}(H)$ admits a unique positive regular Borel measure $\nu_H$ invariant under the left-action of unitary operators in $H$ and such that $\nu_H(\mathbb{S}(H)) = 1$, we established the following theoretical result we will exploit in this work.

Theorem 2  If $f : \mathbb{S}(H) \to \mathbb{C}$ is a frame function on a finite-dimensional complex Hilbert space $H$, with $\dim(H) \geq 3$ and $f \in L^2(\mathbb{S}(H), d\nu_H)$, then there is a unique linear operator $A : H \to H$ such that: $f(\psi) = \langle \psi | A \psi \rangle \quad \forall \psi \in \mathbb{S}(H)$, where $\langle | \rangle$ is the inner product in $H$. A turns out to be Hermitean if (and only if) $f$ is real.

Though that result is theoretically interesting, nothing was said about the physical meaning of $\nu_H$ and the condition $f \in L^2(\mathbb{S}(H), d\nu_H)$. This paper is also devoted to clarify these issues. As a matter of fact, we will see that functions representing either quantum observables or quantum states must be frame functions for $n > 2$. It happens as a consequence of the (convex) linearity and $U(n)$-covariance of the maps associating classical objects to quantum ones. Moreover $\nu_H$ will be established to be, up a positive factor, just the the Liouville volume form necessary to compare expectation values.

This paper is organized as follows. Sect.2 is devoted to summarize some known aspects of the quantum-classical correspondence, introducing some new ingredients and pointing out the issue of the comparison of expectation values in Sect.2.3. In Sect.3 always sticking to the finite dimensional case, we will present some features of frame functions defined on the projective space, proving some fundamental theorems useful in the rest of the paper (Theorems 17, 20 and 21). In Sect.4 we will exploit the constructed formalism to deeply focus on the interplay of quantum observables/states and classical observables/states, facing the problem of positivity of Liouville densities, also discussing the construction of a suitable $C^*$-algebra structure on the space of frame functions corresponding to the analogous structure of the space of quantum observables. The last section is dedicated to conclusions.
2 Geometric Hamiltonian description of finite-dimensional quantum systems

2.1 Elementary Quantum Mechanics

In the absence of superselection rules, a quantum system is described in a complex Hilbert space $\mathcal{H}$, whose elements determine the pure states of the system in the sense we discuss shortly. With that framework, the self-adjoint elements of the $C^*$-algebra $\mathfrak{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$ describe the observables which only take bounded sets of values. (Unbounded observables can be constructed out of the bounded ones through limit procedures in the strong operator topology.) Two physically relevant two-sided $\ast$-ideals $\mathfrak{B}_1(\mathcal{H}) \subset \mathfrak{B}_2(\mathcal{H})$ of compact operators exist in $\mathfrak{B}(\mathcal{H})$. $\mathfrak{B}_2(\mathcal{H})$ is the space of Hilbert Schmidt operators, $\mathfrak{B}_1(\mathcal{H})$ is the space of trace-class operators. Three distances exist consequently. (1) $d_1(\cdot, \cdot)$ associated with the norm $||A||_1 := \text{tr}(|A|)$ in the Banach space $\mathfrak{B}_1(\mathcal{H})$. (2) $d_2(\cdot, \cdot)$ associated with the norm $||\cdot||_2$ induced by the scalar product on the Hilbert space $\mathfrak{B}_2(\mathcal{H})$, $\langle A|B \rangle_2 = \text{tr}(A^*B)$ for $A, B \in \mathfrak{B}_2(\mathcal{H})$. (3) $d(\cdot, \cdot)$ associated with the standard $C^*$-algebra norm $||\cdot||$ on $\mathfrak{B}(\mathcal{H})$. An elementary computation proves that:

**Proposition 3** If $\mathcal{H}$ is a complex Hilbert space, $||A|| \leq ||A||_2 \leq ||A||_1$ when $A \in \mathfrak{B}_1(\mathcal{H})$.

**Definition 4** If $\mathcal{H}$ is a separable complex Hilbert space with scalar product $\langle \cdot | \cdot \rangle$, $\mathcal{S}(\mathcal{H})$ indicates the convex set of (quantum) states on $\mathcal{H}$. They are the operators $\sigma \in \mathfrak{B}_1(\mathcal{H})$ with $\text{tr}(\sigma) = 1$ which are positive (i.e $\langle \psi | \sigma | \psi \rangle \geq 0$ if $\psi \in \mathcal{H}$). Pure states are the extremal points of $\mathcal{S}(\mathcal{H})$, their set is denoted by $\mathcal{S}_p(\mathcal{H})$. $\sigma \in \mathcal{S}(\mathcal{H})$ is said mixed when $\sigma \not\in \mathcal{S}(\mathcal{H})$.

Pure states are related with the complex projective space on $\mathcal{H}$. The (complex) projective space $\mathcal{P}(\mathcal{H})$ over $\mathcal{H}$ is the quotient $\mathcal{H}/\sim$ deprived of $[0]$ where, for $\psi, \psi' \in \mathcal{H}$, $\psi \sim \psi'$ iff $\psi = \alpha \psi'$, $\alpha \in U(1)$, $\mathcal{S}(\mathcal{H})$ henceforth denotes the unit sphere in $\mathcal{H}$ centred on the null vector. With the topology induced by $\mathcal{H}$, $\mathcal{S}(\mathcal{H})$ is a connected Hausdorff space, compact only if $\dim(\mathcal{H}) < +\infty$. The projection: $\pi : \mathcal{S}(\mathcal{H}) \ni \psi \mapsto [\psi] \in \mathcal{P}(\mathcal{H})$ is surjective, continuous and open when equipping $\mathcal{P}(\mathcal{H})$ with the quotient topology. $\mathcal{P}(\mathcal{H})$ is connected and Hausdorff. States enjoy the following properties, the proofs being well known.

**Proposition 5** If $\mathcal{H}$ is a separable complex Hilbert space, the following facts hold.

1. $\mathcal{S}(\mathcal{H})$ and $\mathcal{S}_p(\mathcal{H})$ are closed in $\mathfrak{B}_1(\mathcal{H})$ and are complete $d_1$-metric spaces.
2. If $\sigma \in \mathcal{S}(\mathcal{H})$, then: $\sigma^2 \leq \sigma$ and $\text{tr}(\sigma^2) \leq 1$, and the following facts are equivalent: (i) $\sigma \in \mathcal{S}_p(\mathcal{H})$; (ii) $\sigma^2 = \sigma$; (iii) $\text{tr}(\sigma^2) = 1$; (iv) $||\sigma|| = 1$; (v) $||\sigma||_2 = 1$, (vi) $\sigma = \psi\langle \psi | \cdot \rangle$ for some $\psi \in \mathcal{S}(\mathcal{H})$.
3. The homeomorphism exists $\mathcal{P}(\mathcal{H}) \ni p \mapsto \psi\langle \psi | \cdot \rangle \in \mathcal{S}_p(\mathcal{H})$ for $\psi \in \mathcal{S}(\mathcal{H})$ with $[\psi] = p$, the topology assumed on $\mathcal{S}_p(\mathcal{H})$ being equivalently induced by $\Vert \Vert$ or $\Vert \Vert_1$ or $\Vert \Vert_2$, since $d_1(p, p') = 2d(p, p') = \sqrt{2}d_2(p, p')$ if $p, p' \in \mathcal{S}_p(\mathcal{H})$.
4. If $\sigma \in \mathcal{S}(\mathcal{H})$, then $\text{sp}(\sigma) \setminus \{0\} \subset \text{sp}_p(\sigma)$ is finite or countable with 0 as uniquely

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1 Actually, separability is not strictly necessary from a mathematical viewpoint at least though it would deserve a physical discussion.
possible limit point. If \( q \in \text{sp}_p(\sigma) \) then \( 0 \leq q \leq 1 \); the associated eigenspace \( H_q \) has finite dimension if \( q \neq 0 \) and the sum of all eigenvalues, taking the geometric multiplicities into account, equals 1. If \( K \) is a Hilbert basis of \( \text{Ker}(\sigma) \) and \( \{\psi_i^{(q)}\}_{i=1,\ldots,\dim(H_q)} \) a Hilbert basis of \( H_q \), then \( K \cup \{\psi_i^{(q)} \mid i = 1, \ldots, \dim(H_q) \}, q \in \text{sp}_p(\sigma) \) is a Hilbert basis of \( H \).

(5) Every \( \sigma \in S(H) \) is a finite or countable convex combination of pure states, referring to the operator strong topology for infinite combinations. The spectral decomposition of \( \sigma \) is an example of such convex decomposition.

From the physical side, for a (bounded) observable \( A = A^* \in \mathfrak{B}(H) \), its spectrum \( \text{sp}(A) \subset \mathbb{R} \) is the set of its possible measured values. Moreover, if \( \sigma \in S(H) \), \( E \subset \text{sp}(A) \) is a Borel set and \( P_E \) is an orthogonal projector of the spectral measure of \( A \), then \( tr(\sigma P_E) \) is the probability of finding the outcome of the measurement of \( A \) in \( E \) when the state is \( \sigma \). Correspondingly, \( tr(\sigma A) \) is the expectation value of \( A \) in the state \( \sigma \). Two observables \( A, B \) are incompatible, i.e., they cannot be measures simultaneously, if and only if \( [A, B] \neq 0 \). Kadison-Wigner symmetries are described by the action of unitary or anti-unitary operators \( U \) on states \( \sigma \) like this: \( \sigma \mapsto U^* \sigma U \). A one-parameter strongly continuous unitary groups \( \{U_s\}_{s \in \mathbb{R}} \) describes continuous symmetries. Its unique self-adjoint operator \( G \) (generally unbounded with domain \( D(G) \)) in the sense of Stone, i.e. \( U_s = e^{-isG} \) for all \( s \in \mathbb{R} \), acquires a particular importance. When \( \{e^{-itH}\}_{t \in \mathbb{R}} \) describes time evolution of the system, \( H \) is the Hamiltonian observable of the system. If \( \sigma(t) := e^{-itH} \sigma e^{itH} \), the Schrödinger equation holds:

\[
\frac{d\sigma(t)}{dt} = -i[H, \sigma(t)].
\]

The derivative refers to the weak operator topology and the commutator is interpreted accordingly in \( D(H) \times D(H) \). If, conversely, one ascribes time evolution to observables, keeping fixed the states, observables evolve as \( A(t) := e^{itH} A e^{-itH} \) (Heisenberg picture).

### 2.2 Finite dimensional case: the geometric Hamiltonian picture

**Remarks 6** \( H_n \) denotes a complex Hilbert space with finite dimension \( n > 1 \) and \( U(n) \) denotes the Lie group of unitary operators on \( H_n \) throughout this paper.

When the dimension of the Hilbert \( H_n \) space is finite, \( S(H_n) \) and \( P(H_n) \) become compact, second countable, topological spaces. However the most interesting differences with respect to the infinite dimensional case concern the space of operators.

**Proposition 7** The following facts hold in \( H_n \).

(1) The topologies of \( || \cdot ||, || \cdot ||_1 \) and \( || \cdot ||_2 \) on \( \mathfrak{B}(H_n) = \mathfrak{B}_2(H_n) = \mathfrak{B}_1(H_n) \) coincide.

(2) \( S(H_n) \) and \( S_p(H_n) \) are compact and, if \( \sigma \in S(H_n) \), the following inequalities hold:

\[ \text{2 Properly speaking, Schrödinger equation arises from (2) when referring to normalized vectors } \psi(t) \text{ describing pure states } |\psi(t)\rangle, \text{ making the simplest choice of the arbitrary phase affecting } \psi(t). \]
$n^{-1/2} \leq ||\sigma||_2 \leq 1$ and $n^{-1} \leq ||\sigma|| \leq 1$. In both cases, the least values of the norms are attained at $\sigma = n^{-1}I$.

(3) Equip the set $T$ of operators $A = A^* \in \mathcal{B}(H_n)$ such that $tr(A) = 1$ with the topology induced by $\mathcal{B}(H_n)$. As a subset of the topological space $T$, $S(H_n)$ fulfils:
\[
\partial S(H_n) = \{ \sigma \in S(H_n) | dim(Ran(\sigma)) < n \}, \quad Int(S(H_n)) = \{ \sigma \in S(H_n) | dim(Ran(\sigma)) = n \}.
\]
In particular: $S_p(H_n) = \{ \sigma \in S(H_n) | dim(Ran(\sigma)) = 1 \} \subset \partial S(H_n)$, and $S_p(H_n) = \partial S(H_n)$ if and only if $n = 2$.

Let us identify $H_n$ with $\mathbb{C}^n$ by choosing a Hilbert basis. With this identification $H_n$ acquires the structure of a real $2n$-dimensional smooth manifold. This structure does not depend on the choice of the basis as one immediately proves. That structure induces analogous structures on the topological spaces $S(H_n)$ and $P(H_n)$. From now on we identify $P(H_n)$ to $S_p(H_n)$ in view of (3) in proposition [5]. As a consequence, we can take advantage of the transitive action of the compact Lie group $U(n)$ on $S_p(H_n) \equiv P(H_n)$:
\[
U(n) \times P(H_n) \ni (U, p) \mapsto \Phi_U(p) := UpU^{-1} \in P(H_n).
\]
A sketch of proof of the following proposition is in the appendix.

**Proposition 8** The following facts hold in the real smooth manifold $H_n$.
(a) $S(H_n)$ is a real $(2n - 1)$-dimensional embedded submanifold of $H_n$.
(b) $P(H_n)$ can be equipped with a real $(2n - 2)$-dimensional smooth manifold structure in a way such that both the continuous projection $\pi : S(H) \ni \psi \mapsto [\psi] \in P(H)$ is a smooth submersion and the transitive action [3] is smooth.

**Remarks 9** Henceforth $iu(n) \subset \mathcal{B}(H_n)$ – where $u(n)$ is the Lie algebra of $U(n)$ – denotes the real space of self-adjoint operators.

The following proposition, whose proof is in the appendix, establishes a useful way to describe the tangent space $T_pP(H_n)$.

**Proposition 10** The tangent vectors $v$ at $p \in P(H_n) \equiv S_p(H_n)$ are all of the elements in $\mathcal{B}(H_n)$ of the form: $v = -i[A_v, p]$, for some $A_v \in iu(n)$. Consequently, $A_1, A_2 \in iu(n)$ define the same vector in $T_pP(H_n)$ iff $[A_1 - A_2, p] = 0$.

As is well known [AS93, CLM83, BH01], $P(H_n)$ has also a structure of a $2n - 2$-dimensional symplectic manifold, where the symplectic form (a closed non-degenerate smooth 2-form) is, for any fixed value of the constant $\kappa > 0$:
\[
\omega_p(u, v) := -i\kappa tr\left(p[A_u, A_v]\right) \quad u, v \in T_pP(H_n).
\]
This definition is well-posed, since, by direct inspection, one sees that the right-hand side of (4) is fixed if adding to $A_u$ or $A_v$ operators commuting with $p$. The constant
\[ \kappa > 0 \] is a natural degree of freedom we do not fix at this stage. It is introduced just for future convenience. As we shall see shortly, \( \kappa \) affects the form of the classical observables associated with the quantum ones [8] and in the literature \( \kappa \) is usually assumed to be either 1 [BH01] or 1/2 [BSS04].

The symplectic structure allows us to take advantage of the usual Hamiltonian machinery, whose relation with quantum mechanics formalism will be examined shortly. Just we recall some general facts (so \( \mathcal{P}(\mathcal{H}_n) \) can be replaced for any symplectic manifold). A diffeomorphism \( F : \mathcal{P}(\mathcal{H}_n) \to \mathcal{P}(\mathcal{H}_n) \) is said to be \textit{symplectic} iff it preserves the symplectic form: \( F_\ast \omega = \omega \). For every smooth \( f : \mathcal{P}(\mathcal{H}_n) \to \mathbb{R} \) one defines the associated \textbf{Hamiltonian (vector) field} \( X_f \) as the unique vector field satisfying \( \omega_p(X_f, \cdot) = df_p \). When \( \mathcal{H} \) is the \textbf{Hamiltonian function} (time-independent for the sake of simplicity) of a physical system described on \( \mathcal{P}(\mathcal{H}_n) \), the integral curves of \( X_{\mathcal{H}} \), the solutions of \textbf{Hamilton equations}:

\[
\frac{dp}{dt} = X_{\mathcal{H}}(p(t)) ,
\]

represent the time evolution of the system. Generally speaking, the evolution along the integral curves of \( X_f \) (which, on \( \mathcal{P}(\mathcal{H}_n) \), are complete since it is compact) defines a one-parameter group of symplectic diffeomorphisms called the \textbf{Hamiltonian flow} generated by the smooth function \( f \). The \textbf{Poisson bracket} of a pair of smooth functions \( f, g : \mathcal{P}(\mathcal{H}_n) \to \mathbb{R} \) is \( \{ f, g \} := \omega(X_f, X_g) \) and the remarkable formula holds \( [X_f, X_g] = X_{\{f,g\}} \), the former commutator being the Lie bracket of vector fields.

Coming back to \( \mathcal{P}(\mathcal{H}_n) \) explicitly, as is known [AS95, BH01], it also admits a positive preferred smooth metric. Up to the factor \( 2\kappa \), it is the so-called \textbf{Fubini-Study metric}\(^3\):

\[
g_p(u, v) = -\kappa \text{tr} (p([A_u, p][A_v, p] + [A_v, p][A_u, p])) , \quad u, v \in T_p \mathcal{P}(\mathcal{H}_n) .
\]

Finally a \( \omega\)-\( g\)-compatible \textbf{almost complex structure} exists [AS95, BH01], explicitly given by the class \( j \) of linear maps [GCM05]:

\[
j_p : T_p \mathcal{P}(\mathcal{H}_n) \ni v \mapsto i[v, p] \in T_p \mathcal{P}(\mathcal{H}_n) , \quad p \in \mathcal{P}(\mathcal{H}_n) .
\]

Indeed \( p \mapsto j_p \) is smooth, fulfils \( j_p j_p = -I \) and \( \omega_p(u, v) = g_p(u, j_p v) \) if \( u, v \in T_p \mathcal{P}(\mathcal{H}_n) \).

(Symmetry of \( g \) and anti-symmetry of \( \omega \) also imply \( \omega(u, j v) = -g(u, v), g(j u, j v) = g(u, v) \) and \( \omega(j u, j v) = \omega(u, v) \).) \( \omega, g, j \) is an \textbf{almost Kähler structure} on \( \mathcal{P}(\mathcal{H}_n) \).

Let us come to the interplay of Hamiltonian and Quantum formalism [AS95, BH01]. It relies upon the idea to associate a quantum observable \( A \in i\mathfrak{U}(n) \) to a classical observable:

\[
f_A : \mathcal{P}(\mathcal{H}_n) \ni p \mapsto \kappa \text{tr}(pA) + c \text{tr} A \in \mathbb{R} .
\]

The constant \( c \in \mathbb{R} \) can be fixed arbitrarily. Once again \( c \) is another natural degree of freedom, allowed since it does not affect the known results we are about stating. The core of

\(^3\)With the suggestive notation \(-i[A, p] = dp, \) for \( A = B, \) the metric [10] assumes the more popular form \( ds^2 = g_p(dp, dp) = 2\kappa \text{tr} (p(dp)^2) \), that is equivalent to [8] through the polarization identity.
the Hamiltonian description of quantum physics is stated in the following theorem proved in the appendix. Other interesting aspects exist, related with, for instance, submanifolds with fixed energy, geometric description of quantum entanglement, theory of integrable systems etc. We omit them for the sake of brevity (see [AS95, BH01, BSS04, GCM05]).

**Theorem 11** Consider a quantum system described on $H^n$. Equip $P(H^n)$ with the triple $(\omega, g, j)$ as before. For every $A \in iu(n)$, define the function $f_A : P(H^n) \to \mathbb{R}$ as in (8). Then the Hamiltonian field associated with $f_A$ reads:

$$X_{f_A}(p) = -i[A, p] \quad \text{for all } p \in P(H^n),$$

and the following facts hold.

(a) $\mathbb{R} \ni t \mapsto p(t) \in S_p(H_n)$ is the evolution of a pure quantum state fulfilling Schrödinger equation (2) with Hamiltonian $H \in iu(n)$ if and only if $\mathbb{R} \ni t \mapsto p(t) \in P(H^n)$ satisfies Hamilton equations (3) with Hamiltonian function $\mathcal{H} := f_H$.

Similarly, Hamiltonian evolution of corresponding quantum observables is equivalent the Heisenberg evolution of classical observables: $f_A(p(t)) = f_{e^{itA}e^{-itH}}(p)$.

(b) If $A, H \in iu(n)$, then:

$$\{f_A, f_H\} = f_{-i[A,H]}.$$  

So in particular $A$ is a quantum constant of motion if and only if $f_A$ is a classical constant of motion when $\mathcal{H} = f_H$ is the Hamiltonian function.

(c) If $U \in U(n)$ the map $\Phi_U : P(H_n) \to P(H_n)$ as in (3), describing the action of the quantum symmetry $U$ on states, is both a symplectic diffeomorphism and an isometry of $P(H_n)$ and thus $X_{f_A}$ is a $g$-Killing fields for every $A \in iu(n)$. Finally the covariance relation holds:

$$f_A(\Phi_U(p)) = f_{U^{-1}AU}(p) \quad \text{for all } A \in iu(n), p \in P(H_n), \text{ and } U \in U(n).$$

**Remarks 12**

(1) Changing the form $ctr(A)$ of the constant term in the right-hand side in (8) only affects the validity of (10) in the thesis of thm 11.

(2) It is possible to prove that, remarkably, a $g$-Killing field is necessarily a Hamiltonian field $X_{f_A}$ for some $A \in iu(n)$ [BH01].

2.3 Matching quantum and classical expectation values

The appearance of the constants $\kappa$ and $c$ in (8), though maybe unusual, does not give rise to any problem in comparing quantum dynamics with Hamiltonian one and in discussing the interplay of classical and quantum symmetries, as done in theorem 11. However one may ask if further degrees of freedom can be found preserving the nice agreement of quantum and classical dynamics. This problem will be tackled shortly, proving that the answer is negative. The said issue is actually related with another problem we go
to introduce. If we take seriously the fact that $A$ and $f_A$ are quantum and classical observables associated to each other, we have to be more precise on how the values obtained by measurements of these observables are related. To focus on this relation we have to compare quantum and classical expectation values, referred to corresponding states. In Hamiltonian mechanics, referring to a statistical state $\rho$ described in terms of a Liouville density, the expectation value is the integral of the product of $\rho$ and the considered observable $f$ with respect to the Liouville (positive Borel) measure $m := \omega \wedge \cdots (n \text{ times}) \cdots \wedge \omega$, where $2n$ is the dimension of the symplectic space:

$$\mathbb{E}_\rho(f) := \int_{\mathcal{P}(H_n)} \rho(p) f(p) \, dm(p).$$

(11)

Here, Liouville (probability) densities are non-negative functions $\rho \in \mathcal{L}^1(\mathcal{P}(H_n), m) \cap \mathcal{L}^2(\mathcal{P}(H_n), m)$ satisfying $||\rho||_{\mathcal{L}^1(m)} = 1$. The classical observables $f$ are supposed to fulfil $f \in \mathcal{L}^2(\mathcal{P}(H_n), m)$ so that (11) makes sense. Liouville densities evolve in time satisfying the celebrated Liouville equation. Sharp states $\rho_{p_0}$ defined by a single point $p_0 \in S(H_n)$ can be thought of as Dirac’s measures $\mu_{p_0}$ on the Borel $\sigma$-algebra on $\mathcal{P}(H_n)$ concentrated on $p_0$. The expectation value of an observable $f$ therefore coincides to its evaluation at $p_0$:

$$\mathbb{E}_{\rho_{p_0}}(f) := \int_{\mathcal{P}(H_n)} f(p) \, d\mu_{p_0}(p) = f(p_0).$$

(12)

In quantum mechanics we have mixed and pure states, respectively resembling statistical and sharp classical states. These states evolve in accordance with Schrödinger equation, i.e., by means of the unitary evolutor associated with the Hamiltonian operator. The expectation value of a quantum observable $A \in iu(n)$ referred to a state $\sigma$ is:

$$\langle A \rangle_\sigma := tr(\sigma A).$$

(13)

Comparing classical and quantum observables, we aspect that a natural requirement that could help fix $\kappa$ and $c$ is a constraint like this:

$$tr(A\sigma) = \int_{\mathcal{P}(H_n)} f_A(p) \rho_\sigma(p) dm(p) \quad \text{for every } A \in iu(n) \text{ and } \sigma \in S(H_n),$$

(14)

$\rho_\sigma$ is a Liouville density associated with $\sigma$ through some unknown procedure. In [Gib92], Gibbons proved that there is a way to associate quantum and classical states such that, if the classical observable $f_A$ with $\kappa = 1$ and $c = 0$ corresponds to $A \in iu(n)$, then (14) holds true. It happens for a measure $\mu$, in place of $m$, related with the Fubini-Study metric ($\mu = n(n+1)\mu_n$, where $\mu_n$ is that defined in proposition 13 below) and for:

$$\rho_\sigma(p) := tr(\sigma p) - \frac{1}{n+1}.$$

(15)

\footnote{Actually, in our case, $\mathcal{L}^1(\mathcal{P}(H_n), m) \subset \mathcal{L}^2(\mathcal{P}(H_n), m)$ since $m(\mathcal{P}(H_n))$ is finite, $\mathcal{P}(H_n)$ being compact.}
With the given definitions one easily sees that the Schrödinger evolution of the quantum states is equivalent to the evolution along the Liouville equation for the associated classical states. Nevertheless, the evident problem is that $\rho \sigma \geq 0$ is not generally true, so $\rho \sigma$ cannot define a probability density. However, as stressed in [Gib92], since (14) is valid, one cannot produce non-physical results (e.g. $f_A \geq 0$ but $\mathbb{E}_\rho(f_A) < 0$) dealing with the few functions of the form $f_A$, varying $A \in iu(n)$. Nevertheless, there is no way to think of $\rho \sigma$ as a classical state when dealing with general classical observables $f : \mathcal{P}(H_n) \to \mathbb{R}$.

In the rest of the paper, we wish to focus on the interplay of quantum and classical description, studying all possible correspondences from quantum states and Liouville densities on $\mathcal{P}(H_n)$ satisfying natural requirements. These requirements in particular, fix a relation between $\kappa$ and $c$. We also establish that both classical observables representing quantum ones and densities representing quantum states must be frame functions. We will also prove that, in the found picture, there is room enough to fix the positivity problem of $\rho \sigma$, preserving the validity of theorem 11.

3 More about frame functions on the projective space

To tackle the issues illustrated in Sect. 2.3, we introduce some new preparatory results about frame functions.

3.1 Frame functions on $\mathcal{P}(H_n)$

First of all we need to restate the definition of frame function and theorem 2 on $\mathcal{P}(H_n)$ rather than on $\mathcal{S}(H_n)$. To this end, we exploit the existence of a suitable measure on $\mathcal{P}(H_n)$ induced by the measure $\nu_n$ defined on $\mathcal{S}(H_n)$ mentioned in theorem 2 and therein denoted by $\nu_4$. As $\mathcal{S}(H_n)$ is homeomorphic to the quotient of compact groups $U(n)/U(n-1)$, it is endowed with a $U(n)$-left-invariant regular positive Borel measure, $\nu_n$, that is uniquely determined by its normalization $\nu_n(\mathcal{S}(H_n)) = 1$ (see [Mac51] and Chapter 4 of [BR00]).

We have the following proposition whose proof is in the appendix.

**Proposition 13** Let $\nu_n : \mathcal{S}(H_n) \to [0, 1]$ denote the unique $U(n)$-left-invariant regular Borel measure with $\nu_n(\mathcal{S}(H_n)) = 1$. There exists a unique positive Borel measure $\mu_n$ over $\mathcal{P}(H_n)$ such that, if $\pi : \mathcal{S}(H_n) \to \mathcal{P}(H_n)$ is the natural projection map, then:

$$f \circ \pi \in L^1(\mathcal{S}(H_n), \nu_n) \quad \text{if} \quad f \in L^1(\mathcal{P}(H_n), \mu_n), \quad \text{and} \quad \int_{\mathcal{P}(H_n)} f d\mu_n = \int_{\mathcal{S}(H_n)} f \circ \pi \, d\nu_n.$$

The measure $\mu_n$ fulfils the following.

(a) Referring to the smooth action (3), $\mu_n$ is the unique $U(n)$-left-invariant regular Borel measure on $\mathcal{P}(H_n)$ with $\mu_n(\mathcal{P}(H_n)) = 1$.

(b) It coincides to the Liouville volume form induced by $\omega$ up to its normalization.

(c) It coincides to the Riemannian measure induced by $g$ up to its normalization.
A frame function as in definition 1 actually determines a function on $P(H_n)$. This is because both the unit vectors $\psi$ and $\alpha \psi$, for $|\alpha| = 1$, can be completed to a Hilbert basis of $H$ by adding the same set of $n - 1$ vectors $\psi_2, \psi_3, \ldots$. Requirement (1) for a frame function $f : \mathbb{S}(H) \rightarrow \mathbb{C}$ therefore implies: $f(\psi) = W_f - \sum_{i \geq 2} f(\psi_i) = f(\alpha \psi)$. We may consequently state an equivalent definition of frame function on $P(H)$. The only problem concerns the analogue of the notion of Hilbert basis stated on $P(H_n)$ instead of $H_n$. We have the following helpful elementary result with $\pi : \mathbb{S}(H_n) \rightarrow P(H_n)$ as before.

**Proposition 14** Let $H$ be a separable complex Hilbert space. $N \subset P(H)$ can be written as $\{\pi(\psi)\}_{\psi \in M}$ for some Hilbertian basis $M \subset H_n$ if and only if both $d_2(p, p') = \sqrt{2}$ for $p, p' \in N$ when $p \neq p'$ and $N$ is maximal with respect to this property.

**Proof.** As $d_2(\psi(\psi|\cdot), \phi(\phi|\cdot)) = \sqrt{|\psi|^2 + |\phi|^2 - 2|\langle \psi, \phi \rangle|^2}$ if $\psi, \phi \in H$, for $\psi, \phi \in \mathbb{S}(H)$, one has $d_2(\psi(\psi|\cdot), \phi(\phi|\cdot)) = \sqrt{2}$ if and only if $\psi \perp \phi$. The proof concludes noticing that the maximality property in the thesis is equivalent to that of a Hilbertian basis. $\square$

We remark the fact that $\sqrt{2} = \max\{d_2(p, p') \mid p, p' \in P(H)\}$ for every separable complex Hilbert space $H$. Moreover, if $\dim(H) = n < +\infty$ the maximality condition is equivalent to say that $N$ contains exactly $n$ elements.

**Definition 15** If $H$ is a separable complex Hilbert space, $N \subset P(H)$ is a basis of $P(H)$ if $d_2(p, p') = \sqrt{2}$ for $p, p' \in N$ with $p \neq p'$ and $N$ is maximal with respect to this property.

We may give a definition of frame function on $P(H_n)$ equivalent to that in definition 1.

**Definition 16** A map $F : P(H_n) \rightarrow \mathbb{C}$ is a frame function if there is $W_F \in \mathbb{C}$ with:

$$\sum_{i=1}^{n} F(x_i) = W_F \quad \text{for every basis } \{x_i\}_{i=1,\ldots,n} \text{ of } P(H_n). \quad (16)$$

Theorem 2 can be now restated referring to the measure $\mu_n$ also completing it by adding some other elementary facts. The non elementary statement is (b).

**Theorem 17** In $H_n$ the following holds.

(a) If $A \in \mathfrak{B}(A)$ then

$$F_A(p) := tr(pA) \quad \text{for } p \in P(H_n). \quad (17)$$

defines a frame function with $W_{F_A} = trA$ which belongs to $\mathcal{L}^2(P(H_n), d\mu_n)$.

(b) If $F : P(H_n) \rightarrow \mathbb{C}$ is a frame function, $n > 2$ and $F \in \mathcal{L}^2(P(H_n), d\mu_n)$, then there is a unique $A \in \mathfrak{B}(H_n)$ such that $F_A = F$.

(c) Defining the subspace, closed if $n > 2$:

$$\mathcal{F}^2(H_n) := \{F : P(H_n) \rightarrow \mathbb{C} \mid F \in \mathcal{L}^2(P(H_n), d\mu_n) \quad \text{and } F \text{ is a frame function}\},$$

11
$M : \mathfrak{B}(H) \ni A \mapsto F_A \in \mathcal{F}^2(H_n)$ is a complex vector space injective homomorphism, surjective if $n > 2$, fulfilling the properties:

(i) $A \geq cI$, for some $c \in \mathbb{R}$, if and only if $F_A(x) \geq c$ for all $x \in \mathcal{P}(H_n)$

(ii) $F_A^* = \overline{F_A}$, where the bar denotes the point-wise complex conjugation. In particular $A = A^*$ if and only if $F_A$ is real.

Proof. The proof of the first part of (a) is trivial. $F_A$ is continuous and thus bounded, since $\mathcal{P}(H_n)$ is compact. Therefore it belongs to $\mathcal{L}^2(\mathcal{P}(H_n), d\mu_n)$ as $\mu_n(\mathcal{P}(H_n)) < +\infty$. Concerning (b), we observe that $f(\psi) := F([\psi])$ is a frame function in the sense of definition\[ due to proposition\[. If $F \in \mathcal{L}^2(\mathcal{P}(H_n), d\mu_n)$, then $f \in \mathcal{L}^2(\mathcal{S}(H_n), d\nu_n)$ in view of the first statement in proposition\[. Thus, whenever $n \geq 3$, we can take advantage of thm\[ obtaining that there is $A \in \mathfrak{B}(H_n)$ with $F_A([\psi]) = f_A(\psi) = \langle \psi|A\psi \rangle = tr(\psi\langle \cdot |A\rangle)$ for all $\psi \in \mathcal{S}(H_n)$, namely $F = F_A$. $A$ is uniquely determined since, as it is simply proved, in complex Hilbert spaces, if $B : H \to H$ is linear, $\langle \psi|B\psi \rangle = 0$ for all $\psi \in \mathcal{S}(H)$ then $B = 0$. The proof of (c) is evident per direct inspection. Closedness of $\mathcal{F}^2(H_n)$ for $n \geq 3$ arises form the fact that $\mathcal{F}^2(H_n)$ is a finite dimensional subspace of a Banach space: The space of quadratic forms on $H_n \times H_n$ for (b).

Remarks 18 The statement (b) is false for $n = 2$. A simple counterexample is the same as for the classical version of Gleason’s theorem. Fix $p_0 \in \mathcal{P}(H_2)$ and consider the map: $F(p) := \frac{1}{2} (1 - (1 - 2tr(p_0p))^3)$ for $p \in \mathcal{P}(H_2)$. Passing to the Bloch representation in $\mathbb{C}^2$, it turns out evident that this is a positive frame function with $W_{\mathcal{F}} = 1$. Next it is simply proved that no $A \in \mathfrak{B}(H_2)$ satisfies $F(p) = tr(Ap)$ for all $p \in \mathcal{P}(H_2)$.

3.2 $U(n)$-covariance and (convex) linearity

Definition 19 If $\mathcal{G} : \mathcal{S}(H_n) \ni \sigma \mapsto g_\sigma$, where the $g_\sigma$ are maps from $\mathcal{S}(H_n)$ to $\mathbb{C}$, we say that $\mathcal{G}$, is $U(n)$-covariant, if

$$g_\sigma(\Phi_U(p)) = g_{U^{-1}U\sigma}(p) \quad \text{for all } U \in U(n), \sigma \in \mathcal{S}(H_n), p \in \mathcal{S}(H_n).$$

There is a nice interplay between $U(n)$-covariance and frame functions we state and prove in the following theorem which will be a key-tool in the next section.

Theorem 20 Assume that $n > 2$ for $H_n$.

(a) If $\mathcal{G} : \mathcal{S}(H_n) \to \mathcal{L}^2(\mathcal{P}(H_n), \mu_n)$ is a convex-linear and $U(n)$-covariant map, then $\mathcal{G}(\mathcal{S}(H_n)) \subset \mathcal{F}^2(H_n)$.

(b) If $\mathcal{G}_1 : \mathfrak{B}(H_n) \to \mathcal{L}^2(\mathcal{P}(H_n), \mu_n)$ is a $\mathbb{C}$-linear map satisfying $\mathcal{G}_1|_{\mathcal{S}(H_n)} = \mathcal{G}$, with $\mathcal{G}$ as in (a), then $\mathcal{G}_1(\mathcal{S}(H_n)) \subset \mathcal{F}^2(H_n)$.

Proof. (a) Suppose that $\sigma = \phi(\phi|\cdot)$ is a given pure state ad suppose that $\{p_i\}_{i=1,2,...,n}$ is a basis of $\mathcal{P}(H_n)$, so that $p_i = \psi_i\langle \psi_i|\cdot \rangle$. With a suitable choice of the arbitrary phase in the definition of the $\psi_i$, there are $n$ operators $U_i$, such that $U_i\phi = \psi_i$ and $U_i = U_i^* = U_i^{-1}$,
(lemma 32 in the appendix). Consequently, taking advantage of the \( U(n) \)-covariance:

\[
g_\sigma(p_i) = g_\sigma(U_i\sigma U_i^*) = g_{U_i^*\sigma U_i}(\sigma) = g_{U_i^*\sigma}(\sigma). \]

Exploiting the convex-linearity:

\[
n^{-1} \sum_i g_\sigma(p_i) = \sum_i n^{-1} g_\sigma(p_i) = n^{-1} g_{\sum_i p_i} = n^{-1} \sum_i p_i = g_{n^{-1}I}(\sigma).
\]

\( U(n) \)-covariance implies:

\[
g_{n^{-1}I}(\sigma) = g_{U^{-1}n^{-1}U}(\sigma) = g_{n^{-1}I}(\Phi_U(\sigma)). \]

Since \( \Phi \) is transitive on \( \mathcal{P}(H_n) \), we conclude that \( n^{-1} \sum_i g_\sigma(p_i) = g_{n^{-1}I}(q) = c \), for every \( q \in \mathcal{P}(H_n) \) and some constant \( c \in \mathbb{R} \). Next consider a mixed \( \sigma \in \mathcal{S}(H_n) \). The found result and convex-linearity of \( \mathcal{G} \), representing \( \sigma \) with its spectral decomposition \( \sigma = \sum_j q_j\sigma_j \) (\( \sigma_j \) being pure), yield:

\[
\sum_i g_\sigma(p_i) = \sum_i \sum_j g_{\sum_j q_j\sigma_j}(p_i) = \sum_i \sum_j q_j g_{\sigma_j}(p_i) = \sum_j q_j \sum_i g_{\sigma_j}(p_i) = \sum_j q_j nc.
\]

As the right most side does not depend on the choice of the basis \( \{p_i\}_{i=1,2,...,n} \), \( g_\sigma \) must be a frame function, that belongs to \( \mathcal{L}^2 \) by hypotheses. (b) of thm 17 implies that \( g_\sigma \in \mathcal{F}^2(\mathbb{H}_2) \).

(b) If \( A \in \mathfrak{B}(H_n) \), decompose it as \( A = \frac{1}{2}(A + A^*) + i\frac{1}{2}(A - A^*) \). Next decompose the self-adjoint operators \( \frac{1}{2}(A + A^*) \) and \( \frac{1}{2}(A - A^*) \) into linear combinations of pure states \( \sigma_k \) exploiting the spectral theorem. Each \( \mathcal{G}_1(\sigma_k) = \mathcal{G}(\sigma_k) \) belongs to the linear space \( \mathcal{F}^2 \).

Linearity of \( \mathcal{G}_1 \) concludes the proof. \( \square \)

3.3 Trace-integral formulas

Frame functions enjoy remarkable properties connecting Hilbert-Schmidt and \( \mathcal{L}^2(\mu_n) \) scalar products. These identities were already discovered in [Gib92] for self-adjoint operators, referring to the measure naturally associated with \( g \), which we proved to be proportional to \( \mu_n \) in proposition 13 above. Here, we establish them directly for \( \mu_n \), using the \( U(n) \) invariance and dealing with generally non self-adjoint operators \( A, B \in \mathfrak{B}(H_n) \).

**Theorem 21** Referring to Theorem 17, if \( F_A \) and \( F_B \) are frame functions respectively constructed out of \( A \) and \( B \) in \( \mathfrak{B}(H_n) \), then:

\[
\int_{\mathcal{P}(H_n)} F_A d\mu_n = \frac{\text{tr}(A)}{n} = \frac{W_F}{n}, \tag{18}
\]

\[
\int_{\mathcal{P}(H_n)} F_A F_B d\mu_n = \frac{1}{n(n+1)} (\text{tr}(A^*B) + \text{tr}(A^*tr(B))) , \tag{19}
\]

which inverts as:

\[
\text{tr}(A^*B) = n(n+1) \int_{\mathcal{P}(H_n)} F_A F_B d\mu_n - n^2 \int_{\mathcal{P}(H_n)} F_A d\mu_n \int_{\mathcal{P}(H_n)} F_B d\mu_n \tag{20}
\]

13
Proof. The second identity in (18) is immediate. (20) arises form (19) and (18) straightforwardly, so we have to prove (18) and (19) only. Actually (18) follows from (19) swapping A and B and taking B = I. Therefore we have to establish (19) to conclude. To this end, we notice that (19) holds true for generic \( A, B \in \mathfrak{B}(\mathcal{H}_n) \) if it is valid for A and B self-adjoint. This result arises decomposing A and B in self-adjoint and anti self-adjoint part and exploiting linearity in various points. Therefore it is enough proving (19) for A and B self-adjoint. Next we observe that, if as before \( iu(n) \) is the real vector space of self-adjoint operators: \( iu(n) \times iu(n) \ni (A,B) \mapsto (n(n + 1))^{-1} (tr(AB) + tr(A)tr(B)) \) is a real scalar product. Similarly, the left-hand side of (19), restricted to the real vector space of real frame functions is a real scalar product. Taking advantage of the polarization identity, we conclude that (19) holds when it does for the corresponding norms on the considered real vector spaces:

\[
\int_{\mathfrak{P}(\mathcal{H}_n)} F^2_A d\mu_n = \frac{1}{n(n+1)} \left( tr(AA) + tr(A)^2 \right) \quad \text{for } A \in iu(n). \tag{21}
\]

Let us establish (21) to conclude. We pass from the integration over \( \mathfrak{P}(\mathcal{H}_n) \) to that over \( \mathbb{S}(\mathcal{H}_n) \) just replacing \( \mu_n \) for \( \nu_n \). If \( \{e_j\}_{j=1,\ldots,n} \) is a Hilbertian basis of \( \mathcal{H}_n \) made of eigenvectors of A such that \( Ae_k = \lambda_k e_k \), we can decompose \( \psi \in \mathbb{S}(\mathcal{H}_n) \) as follows \( \psi = \sum_j \psi_j e_j \) so that:

\[
\int_{\mathbb{S}(\mathcal{H}_n)} F^2_A d\nu_n = \sum_{i=1}^{n} \lambda_i^2 \int_{\mathbb{S}(\mathcal{H}_n)} |\psi_i|^4 d\nu_n + \sum_{i\neq j}^{n} \lambda_i \lambda_j \int_{\mathbb{S}(\mathcal{H}_n)} |\psi_i|^2|\psi_j|^2 d\nu_n.
\]

In view of the \( U(n) \) invariance of \( \nu_n \) and the transitive action of \( U(n) \) on \( \mathbb{S}(\mathcal{H}_n) \), we conclude that: \( \int_{\mathbb{S}(\mathcal{H}_n)} |\psi_i|^4 d\nu_n(\psi) = a \), where a does not depend on \( i \), on the used Hilbertian basis, and on A. If \( \psi, \phi \in \mathbb{S}(\mathcal{H}_n) \) are a pair of vectors satisfying \( \psi \perp \phi \), for every choice of \( i, j = 1, \ldots, n \) with \( i \neq j \), there exist \( U_{i,j} \in U(n) \) such that, both verifies \( U_{i,j} e_i = \psi \) and \( U_{i,j} e_j = \phi \). The invariance of \( \nu_n \) under \( U(n) \) thus proves that, for \( i \neq j \):

\[
\int_{\mathbb{S}(\mathcal{H}_n)} |\psi_i|^2|\psi_j|^2 d\nu_n(\psi) = b \quad \text{where } b \text{ does not depend on } A \text{ on the used Hilbertian basis and on the couple } i, j = 1, \ldots, n \text{ provided } i \neq j.\]

Summing up:

\[
\int_{\mathbb{S}(\mathcal{H}_n)} F^2_A d\nu_n = a \tr(A^2) + b \sum_{i \neq j} \lambda_i \lambda_j = \int_{\mathbb{S}(\mathcal{H}_n)} F^2_A d\nu_n = a \tr(A^2) + b \sum_{i,j} \lambda_i \lambda_j - b \tr(A^2).
\]

That is, redefining \( d := a - b \):

\[
\int_{\mathfrak{P}(\mathcal{H}_n)} F^2_A d\mu_n = d \tr(A^2) + b (\tr(A))^2. \tag{22}
\]

To determine the constants \( d \) and \( b \) we first choose \( A = I \) obtaining: \( 1 = dn + bn^2 \). To grasp another condition, consider the real vector space of self-adjoint operators \( iu(n) \) and complete \( \frac{1}{\sqrt{n}} \) to a Hilbert-Schmidt-orthonormal basis of \( iu(n) \) by adding self-adjoint
operators $T_1, T_2, \ldots, T_{n^2-1}$. Notice that $(I|T_k)_2 = 0$ means $tr(T_k) = 0$. Thus, if $p \in P(H_n)$: $p = \frac{1}{n} + \sum_k p_k T_k$ with $p_k = tr(p T_k) \in \mathbb{R}$. The condition $Tr(p^2) = 1$ ((2) in proposition 5) is equivalent to $\sum_k p_k^2 = 1 - \frac{1}{n}$, so that

$$\sum_{k=1}^{n^2-1} \int_{P(H_n)} F_{T_k}(p)^2 d\mu_n(p) = \int_{P(H_n)} \sum_{k=1}^{n^2-1} p_k^2 d\mu_n(p) = \left(1 - \frac{1}{n}\right) \int_{P(H_n)} d\mu_n(p) = \left(1 - \frac{1}{n}\right).$$

Inserting this result in the left-hand side of (22):

$$\left(1 - \frac{1}{n}\right) = \sum_{i=1}^{n^2-1} d tr(T_i T_i) + b \sum_{i=1}^{n^2-1} (tr(T_i))^2, \quad \text{i.e.} \quad \left(1 - \frac{1}{n}\right) = \sum_{i=1}^{n^2-1} d + b \sum_{i=1}^{n^2-1} (0)^2.$$

Summing up, we have the pair of equations for $b$ and $d$: $1-1/n = d(n^2-1)$ and $1 = dn+bn^2$ with solution $d = b = (n(n+1))^{-1}$ that, inserted in (22), yields (21). \hfill $\Box$

### 4 Observables and states in terms of frame functions

We know that, assuming a quantum observable $A \in \text{iu}(n)$ be associated with a classical one $f_A = F_{\kappa A+c tr(A)}I$ defined in (8), then theorem 11 is true independently from the values of the constants $\kappa > 0$ and $c \in \mathbb{R}$. The first problem we wish to tackle is to study whether there are other possibilities to associate quantum observables to classical observables preserving the validity theorem 11. The second problem we will consider is twofold. On the one hand we want to study if it is possible to associate quantum states $\sigma$ with corresponding classical Liouville densities $\rho_\sigma$ in order to satisfy (14), possibly with $\rho_\sigma \geq 0$. In this juncture, the notion of frame function and the content of Sect.3 will play a crucial rôle. On the other hand we intend to investigate if all these requirements give rise to constraints on the values of $\kappa$ and $c$.

#### 4.1 Observables and states

In the following we focus on two maps respectively associating observables with functions $f_A : P(H_n) \to \mathbb{R}$, the inverse quantization map:

$$\mathcal{O} : \text{iu}(n) \ni A \mapsto f_A,$$

and associating states $\sigma$ with functions $\rho_\sigma : P(H_n) \to \mathbb{R}$:

$$\mathcal{S} : S(H_n) \ni \sigma \mapsto \rho_\sigma.$$

What we intent to do now is fixing $\mathcal{O}$ and $\mathcal{S}$ by requiring some physically natural constraints, most arising from the thesis of theorem 11 concerning $\mathcal{O}$ and from the discussion in Sect. 2.3 regarding $\mathcal{S}$. We assume the almost Kähler structure $(\omega, g, j)$ on $P(H_n)$ as in
Sect. 2.2, with the constant $\kappa > 0$ fixed arbitrarily.

**Requirements on $O$ : $iu(n) \ni A \mapsto f_A$, with $f_A : \text{P}(H_n) \to \mathbb{R}$.**

**(O1)** $O$ is injective.

**(O2)** $O$ is $\mathbb{R}$-linear.

**(O3)** If $H \in iu(n)$, then $f_H$ is $C^1$ so that $X_{f_H}$ can be defined. A curve $p = p(t) \in \text{P}(H_n)$, $t \in (a,b)$, satisfies Hamilton’s equation if and only if it satisfies Schrödinger’s one:

$$\frac{dp}{dt} = X_{f_H}(p(t)) \quad \text{for } t \in (a,b)$$

is equivalent to

$$\frac{dp}{dt} = -i[H, p(t)] \quad \text{for } t \in (a,b).$$

**(O4)** $O$ is $U(n)$-covariant.

**(O5)** If $A \in iu(n)$ then: $\min \text{sp}(A) \leq f_A(p) \leq \max \text{sp}(A)$ for $p \in \text{P}(H_n)$.

The hypothesis (O1) simply says that the map $O$ produces a faithful image of the set of quantum observables in terms of classical ones. Next (O2) establishes that $O$ also preserves the elementary structure of the real vector space enjoyed by the set of quantum observables. The requirement (O3) is a key-requirement, it just concerns the interplay of quantum dynamics and Hamiltonian dynamics that we already know to hold when $f_A$ takes the form $\mathfrak{a}$ in the hypotheses of theorem $\mathfrak{a}$. The requirement (O4), that we know to holds at least when $f_A$ takes the form $\mathfrak{a}$ in the hypotheses of theorem $\mathfrak{a}$ is a natural covariance requirement, since the action of $U(n)$ has both classical and quantum significance. The requirement (O5) focuses on the values of the observables. It is maybe the most elementary possible relation between the values of $A$, the elements of the spectrum, and those of $f_A$, the points in the range. However, there is no unique way to compare a continuous set of values with a discrete one.

It is worth stressing that $f_A$ must be a frame function, $F_{A'}$, in view of (O2) (extended by $\mathbb{C}$-linearity) and (O4) as a straightforward application of (b) in theorem $\mathfrak{a}$. However, it is not so obvious, exploiting (O2) and (O4) only, to determine the form of the operator $A'$ in terms of $A$ itself.

**Requirements on $S : S(H_n) \ni \sigma \mapsto \rho_{\sigma}$ with $\rho_{\sigma} : \text{P}(H_n) \to \mathbb{R}$.**

**(S1)** If $\sigma \in S(H_n)$, then $\rho_{\sigma}(p) \geq 0$ for $p \in \text{P}(H_n)$.

**(S2)** $S$ is convex linear.

**(S3)** With $\mu_n$ as in theorem $\mathfrak{b}$, if $\sigma \in S(H_n)$, then $\rho_{\sigma} \in \mathcal{L}^2(\text{P}(H_n), \mu_n)$ (so that $\rho_{\sigma} \in \mathcal{L}^1(\text{P}(H_n), \mu_n)$) and

$$\int_{\text{P}(H_n)} \rho_{\sigma} d\mu_n = 1.$$

**(S4)** $S$ is $U(n)$-covariant.

**(S5)** If $\sigma \in S(H_n)$ and $A \in iu(n)$ then, assuming $f_A \in \mathcal{L}^2(\text{P}(H_n), \mu_n)$:

$$tr(\sigma A) = \int_{\text{P}(H_n)} \rho_{\sigma} f_A d\mu_n.$$
If we intend to describe quantum expectation values in terms of classical expectation values, the compulsory requirements should be (S1), (S3) and (S5). The hypotheses (S2) focuses on the natural convex structure of the space of the quantum states requiring that it is translated into the analogue structure for the associated classical states. (S4) implies in particular that the Hamiltonian evolution of $\rho_\sigma$ is equivalent to the Schrödinger evolution of $\sigma$. The requirement $f_A \in \mathcal{L}^2(\mathcal{P}(H_n), \mu_n)$ in (S5) is verified if (O3) holds since, in that case, $|f_A|^2$ is continuous and thus bounded on the compact $\mathcal{P}(H_n)$ and $\mu_n(\mathcal{P}(H_n)) < +\infty$. We notice that the map $\mathcal{S}$ it is not required to be injective, so giving rise to a faithful representation of quantum states in terms of classical states. Indeed, we will obtain this result as a consequence of (S2)-(S5).

We are in a position to state two of the main results of this paper. In particular we prove that, for $n > 2$, the densities $\rho_\sigma$ representing quantum states must be frame functions. Beforehand, we establish that even the classical observables $f_A$ are frame functions, just with the form $[\mathcal{S}]$ and that they exhaust the whole space $\mathcal{F}^2(H_n)$.

**Theorem 22** Consider a quantum system described on $H_n$ with $n > 1$. Assume the almost Kähler structure $(\omega, g, j)$ on $\mathcal{P}(H_n)$ as in Sect. 2.2 with the constant $\kappa > 0$ fixed arbitrarily. The following facts hold concerning the map $\mathcal{O} : iu(n) \ni A \mapsto f_A$.

(a) The requirements (O1)-(O4) are valid if and only if both $\mathcal{O}$ has the form $[\mathcal{S}]$ for some constant $c \in \mathbb{R}$ (so that, in particular, theorem 17 holds) and $\kappa + nc \neq 0$.

(b) If the requirements (O1)-(O4) are valid, then $\mathcal{O}$ extends to the whole $\mathcal{B}(H)$ by complex-linearity giving rise to an injective map that, if $n > 2$, satisfies $\mathcal{O}(\mathcal{B}(H_n)) = \mathcal{F}^2(H_n)$.

**Proof.** (a) If $\mathcal{O}$ has the form $[\mathcal{S}]$ then (O2)-(O4) are valid. Let us prove the converse. Assuming (O3), from the definition of Hamiltonian field, it must be $\omega_p(X_{fu}, u_A) = \langle df_u, u_A \rangle$, for every $H \in iu(n)$, $p \in \mathcal{P}(H_n)$ and $u_B = -i[p, B] \in T_p\mathcal{P}(H_n)$. The definition of $\omega$ and some elementary computations permit to re-write the identity above as $\langle df_u, -i[p, B] \rangle = \kappa tr(H(-i[p, B]))$. Consider a smooth curve $q = q(s)$ in $\mathcal{P}(H_n)$ such that $q(s_0) = p$ and $\dot{q}(s_0) = -i[p, B]$. The identity above, taking advantage of the linearity of the trace, entails:

$$\frac{d}{ds} f_H(q(s))|_{s=s_0} = \frac{d}{ds} \kappa tr(Hq(s))|_{s=s_0} = \kappa tr(H \frac{dq}{ds}|_{s=s_0}).$$

Since $s_0$ is arbitrary, we have found that:

$$\frac{d}{ds} f_H(q(s)) = \kappa tr \left( H \frac{dq}{ds} \right).$$

Integrating in $s$ and swapping the integral with the symbol of trace by linearity, we finally obtain $f_H(p) = \kappa tr(Hp) - C_H$, where $p \in \mathcal{P}(H_n)$ is arbitrary. The map $H \mapsto C_H = f_H(p) - \kappa tr(Hp)$ must be linear for (O2). By Riesz’ theorem, referring to the Hilbert-Schmidt (real) scalar product we have that there exists $B \in iu(n)$ such that $C_H = tr(BH)$ for all $H \in iu(n)$. (O4) easily implies that $tr(BUHU^{-1}) = tr(BH)$ for all $U \in U(n)$ and...
H ∈ iu(n). Choosing \( H = \psi \langle \psi | \cdot \rangle \) with \( \psi \in S(H_n) \) and noticing that \( U(n) \) acts transitively on \( S(H_n) \) we conclude that \( \langle \psi | B \psi \rangle = c \) for some constant \( c \in \mathbb{R} \) and all \( \psi \in S(H_n) \). From polarization identity it easily implies that \( B = cI \), so that \( f_A = ktr(pA) + ctr(A) \) for all \( A \in iu(n) \) as requested. Let us prove that \( O \) is injective if and only if (O1) holds. Exploiting \( k \neq 0 \), and dealing with as done above, one easily sees that \( f_A = 0 \) is equivalent to \( A = -ck^{-1}tr(A)I \). Computing the trace of both sides one immediately sees that this equation has \( A = 0 \) as the unique solution if \( 1 + nc/k \neq 0 \), namely \( k + nc \neq 0 \). Conversely, if \( A + ck^{-1}tr(A)I = 0 \) has \( A = 0 \) as unique solution, \( I + ck^{-1}tr(I)I \neq 0 \), namely \( k + nc \neq 0 \).

(b) Assuming (O1)-(O4) and extending \( O \) by complex linearity, exactly as before, \( O \) turns out to be injective. If \( n > 2 \), the elements of \( F^2(H_n) \) are of the form \( F_B \) for every \( B \in \mathfrak{B}(H_n) \) ((b) thm 17). For a fixed \( B \), \( O(A) = F_B \), if \( A := k^{-1}B - c[k(k + nc)]^{-1}tr(B)I \), so that \( O \) is onto \( F^2(H_n) \).

Let us come to the states, proving that \( \rho_\sigma \) is a frame function as well.

**Theorem 23** Consider a quantum system described on \( H_n \) with \( n > 2 \). Assume the almost Kähler structure \((\omega, g, j)\) on \( P(H_n) \) as in Sect.2.2, with the constant \( k > 0 \) fixed arbitrarily and the map \( O : iu(n) \ni A \mapsto f_A \) of the form (8) for some constant \( c \in \mathbb{R} \). The following fact hold.

(a) The requirements (S2)-(S5) are valid if and only if both in the definition (8) of \( O \):

\[
\kappa = k, \quad c = c_k
\]

and the map \( S \) associates states \( \sigma \) with frame functions of the form:

\[
\rho_\sigma(p) := \kappa_k' tr(\sigma p) + c_k'
\]

where

\[
c_k := \frac{1 - k}{n}, \quad \kappa_k' := \frac{n(n + 1)}{k}, \quad c_k' := \frac{k - (n + 1)}{k}.
\]

\( O \) is injective because of the former of the following consequent identities:

\[
k + nc_k = 1 \quad \text{and} \quad \kappa_k' + nc_k' = n.
\]

(b) If (S2)-(S5) are true, also \( S \) is injective.

**Proof.** (a) If (25), (26), (27), (28) are valid, one sees that (S2)-(S5) hold true. In particular, \( O \) is injective because \( k + nc_k = 1 \neq 0 \) and (b) of thm 21 holds. It remains to prove that (S2)-(S5) are valid, then (25), (26), (27), (28) hold. We start (for \( n > 2 \)) by assuming that a map \( S \) verifying (S2)-(S5) and \( O \) of the form (8) with \( k > 0 \) and \( c \in \mathbb{R} \). As the first step we prove that \( \rho_\sigma \) is a frame function, next we shall establish its form.
(S2) and (S4), together with (a) in thm 20 imply that: \( \rho_\sigma(p) = tr(\sigma'p) \) for all \( p \in P(H_n) \), where (i) of (c) of thm 17 entail that \( \sigma' \) is some self-adjoint operator associated with the given \( \sigma \). Using the fact that the total integral of \( \rho_\sigma \) has value 1 from (S3), taking (18) into account, we find \( tr\sigma' = n \). Finally (S5) together the form of \( \mathcal{O} \) require that the following identity holds true for all self-adjoint \( A \in B(H_n) \) and \( \sigma \in S(H_n) \):

\[
tr(\sigma A) = \int_{P(H_n)} tr(\sigma'p) (\kappa tr(\mathcal{A}p) + c trA) d\mu_n(p) .
\]

The right-hand side can be expanded taking (19), (18) and \( tr\sigma' = n \) into account:

\[
tr(\sigma A) = \frac{\kappa}{n(n+1)} tr(\sigma' A) + tr(A) \left( \frac{\kappa}{n+1} + c \right) .
\]

Consequently, for every \( A = A^* \):

\[
tr \left( \left( \sigma - \frac{\kappa}{n(n+1)} \sigma' \right) A \right) = tr(A) \left( \frac{\kappa}{n+1} + c \right) .
\]

Choosing \( A = p \in S_p(H_n) \), arbitrariness of \( p \) easily entails that, for some \( \beta_\sigma \in \mathbb{R} \):

\[
\sigma - \frac{\kappa}{n(n+1)} \sigma' = \beta_\sigma I ,
\]

namely, for some constants \( \kappa' > 0 \) and \( c' \in \mathbb{R} \):

\[
f_\sigma(p) = \kappa' tr(\sigma p) + c' .
\]

Inserting again this expression in (30) he have:

\[
tr(\sigma A) = \int (\kappa' tr(\sigma p) + c') (\kappa tr(\mathcal{A}p) + c trA) d\mu_n(p) .
\]

Finally, using again (18), (19) and \( \mu_n(P(H_n)) = 1 \) we obtain:

\[
\left( 1 - \frac{\kappa \kappa'}{n(n+1)} \right) tr(\sigma A) + tr(A) \left( \frac{\kappa \kappa'}{n(n+1)} + c c' + \frac{c' k}{n} + \frac{c k'}{n} \right) = 0
\]

that has to hold for all \( A \in iu(n) \) and \( \sigma \in S(H_n) \) and for some \( \kappa, \kappa' > 0 \) and \( c, c' \in \mathbb{R} \). Arbitrariness of \( A \) and \( \sigma \) easily lead to the first two requirements of the following triple:

\[
1 - \frac{\kappa \kappa'}{n(n+1)} = 0 , \quad \frac{\kappa \kappa'}{n(n+1)} + c c' + \frac{c' k}{n} + \frac{c k'}{n} = 0 , \quad 1 = \frac{\kappa'}{n} + c' .
\]

the third requirement immediately arises from (S3) using (18). This system can completely be solved parametrizing \( \kappa, \kappa', c \) in terms of \( c' \) with \( c' < 1 \) in order to verify the requirement \( \kappa > 0 \) in the definition of \( f_A \). Finally, parametrizing the solutions in terms of \( \kappa \): \( \kappa, c, k, \kappa' \), we have (27), (28).

(b) If \( \rho_\sigma = \rho_\sigma' \), exploiting \( \kappa', \kappa' \neq 0 \), one has \( tr((\sigma - \sigma')p) = 0 \) for every \( p \in P(H_n) \). Namely \( \langle \psi | (\sigma - \sigma') \psi \rangle = 0 \) for every \( \psi \in H_n \). Polarization leads to \( \sigma - \sigma' = 0 \).
Proof. If \( f \) holds, \( \rho_\sigma \) can be represented by a frame function for expectation values of observables \( f_A \in \mathcal{F}^2(H_n) \) is an immediate consequence of Riesz' theorem, noticing that \( \mathcal{F}^2(P(H)) \) is a closed subspace of \( L^2(P(H_n), \mu_n) \). However this observation nothing says about the nature of \( \rho_\sigma \) when the expectation values are computed for classical observables \( f \) with components in \( \mathcal{F}^2(P(H)) \).

Conversely, assume that (31) holds for a map \( f \). Instead of observables and states, another characterization is the following.

4.2 Characterization of classical observables representing quantum observables

The proved theorem, as a by product, yields a characterization of classical observables representing quantum ones when \( n > 2 \). It is well known (see [BH01]) that these observables \( f \) are exactly those whose Hamiltonian fields \( X_f \) are \( g \)-Killing fields. Focusing instead on the relation of observables and states, another characterization is the following.

Proposition 25 For \( n > 2 \), let \( \mathcal{O} \) and \( \mathcal{S} \) be as in (a) of theorem 23. A map \( f : P(H_n) \rightarrow \mathbb{R} \) in \( \mathcal{L}^2(P(H_n), \mu_n) \) verifies \( f = \mathcal{O}(A) \) for some \( A \in iu(n) \) if and only if there are constants \( a, b \in \mathbb{R} \) with \( a \neq 0 \) and

\[
\int_{P(H_n)} \rho_{p_0}(p)f(p)d\mu_n(p) = af(p_0) + b \quad \text{for every } p_0 \in S_p(H_n). \quad (31)
\]

Proof. If \( f = f_A \) one has immediately:

\[
\int_{P(H_n)} \rho_{p_0}(p)f(p)d\mu_n(p) = tr(p_0 A) = \kappa^{-1}f(p_0) - \kappa^{-1}c_\kappa tr(A) \quad \text{for every } p_0 \in S_p(H_n).
\]

Conversely, assume that (31) holds for a map \( f : P(H_n) \rightarrow \mathbb{R} \) in \( \mathcal{L}^2(P(H_n), \mu_n) \). If \( \{p_i\}_{i=1,...,n} \) is a basis of \( P(H_n) \) one has:

\[
n^{-1}a \left( \sum_i f(p_i) \right) + b = \int_{P(H_n)} \sum_i n^{-1}\rho_{p_i}(p)f(p)d\mu_n(p) = \int_{P(H_n)} \rho_{\sum_i n^{-1}p_i}(p)f(p)d\mu_n(p) = \int_{P(H_n)} \rho_{n^{-1}I}(p)f(p)d\mu_n(p) = \int_{P(H_n)} f(p)d\mu_n(p).
\]

So that

\[
\sum_i f(p_i) = \frac{n}{a} \int_{P(H_n)} f(p)d\mu_n(p) - \frac{nb}{a}
\]

that does not depend on the choice of the basis \( \{p_i\}_{i=1,...,n} \). In view of (b) in thm 17, \( f \) is a real frame function. Due to (d) of thm 23, \( f = \mathcal{O}(A) \) per some \( A \in iu(n) \). \( \Box \)
With the choice, $\kappa = 1$, the proposition above specializes to $a = 1$ and $b = 0$. This gives rise to a suggestive interpretation of the Liouville densities of pure states:

$$\int_{\mathcal{P}(H_n)} \rho_{p_0}(p)f_A(p)d\mu_n(p) = f_A(p_0).$$

If $\kappa = 1$, a map $f: \mathcal{P}(H_n) \to \mathbb{R}$ in $\mathcal{L}^2(\mathcal{P}(H_n), \mu_n)$ can be written as $f = f_A$ for some $A \in iu(n)$ if and only if $f$ “sees” the density $\rho_{p_0}$ of any pure state $p_0$ as a Dirac delta localized at $p_0$ itself.

### 4.3 Bounds on attained values

To conclude our analysis, let us focus on the validity of (S1) and (O5). As stated in the following theorem, they cannot hold simultaneously.

**Theorem 26** For $n > 1$, with $\mathcal{O}$ and $\mathcal{S}$ defined in agreement with (25), (26), (27), (28) for some $\kappa > 0$, the following facts are valid.

(a) (S1) holds if and only if $\kappa \in [n + 1, +\infty)$.

(b) (O5) holds if and only if $\kappa \in (0, 1]$ whereas, in the general case $\kappa > 0$ one has:

$$\min f_A = \min \{\text{sp}(A) + c_{\kappa}(\text{tr}(A) - n \min \text{sp}(A))\}, \quad (32)$$

$$\max f_A = \max \{\text{sp}(A) + c_{\kappa}(\text{tr}(A) - n \max \text{sp}(A))\}, \quad (33)$$

and furthermore, for $A = iu(n)$:

$$||f_A||_{\infty} \leq (1 + 2n|c_{\kappa}|)|A| \quad \text{if } \kappa \in [n + 1, +\infty) , \quad (34)$$

$$||f_A||_{\infty} \leq ||A|| \quad \text{if } \kappa \in (0, 1) , \quad (35)$$

where $\leq$ can be replaced by $=$ if $\kappa = 1$.

**Proof.** (a) if (25), (26), (27), (28) are valid, (S1) holds if and only if $\kappa' > 0$ and $c'_{\kappa} \geq 0$ (notice that $\sigma \geq 0$ for a state by hypotheses and there are states with $\text{tr}(p\sigma) = 0$ for some $p \in \mathcal{P}(H_n)$). From (28), $\kappa' > 0$ and $c'_{\kappa} \geq 0$ are equivalent to $\kappa \in [n + 1, +\infty)$.

(b) We known that, since $A$ is self-adjoint and $H_n$ is finite dimensional, $\min \text{sp}(A) = \min_{|\psi| = 1} \langle \psi | A | \psi \rangle = \min_{p \in \mathcal{P}(H_n)} \text{tr}(pA)$. Therefore $\min f_A = \kappa \min \text{sp}(A) + c_{\kappa} \text{tr}(A)$. Using $\kappa + nc_{\kappa} = 1$ we immediately have (32). The proof of (33) is analogous. Next notice that $\text{tr}(A) - n \min \text{sp}(A) \geq 0$ and $\text{tr}(A) - n \max \text{sp}(A) \leq 0$ so that (32)-(33) imply (O1) if and only if $c_{\kappa} \geq 0$, namely $\kappa \in (0, 1]$. The proof of the remaining estimates easily follows using an analogous procedure, noticing that $\kappa > 0$ and exploiting the inequalities $|\text{tr}(A)| \leq \text{tr}|A| \leq n||A||$ which arises from $||A|| = \max\{|\lambda| \mid \lambda \in \text{sp}(A)\}$ and $\max_{p \in \mathcal{P}(H_n)} |\text{tr}(pA)| = ||A||$. The latter implies the validity of the last statement in (b) out of the fact that $c_{\kappa} = 0$ if $\kappa = 1$. \[\square\]
Remarks 27

(1) For \( n = 2 \), the implication "if" in (a) of theorem \( \text{23} \) holds true.
(2) The choice \( \kappa = n + 1 \) implies \( \kappa' = n, \ c'_n = 0 \), so that: \( \rho_\sigma(p) = nF_\sigma(p) = n\operatorname{tr}(p\sigma) \)
are positive classical densities with the most elementary form allowed by our hypotheses. This form may further be simplified changing the normalization of the measure. Leaving \( \mathcal{O} \) unchanged, one may indeed redefine \( \mu_n \rightarrow \mu'_n := n\mu_n \) and \( \rho_\sigma \rightarrow \rho'_\sigma := n^{-1}\rho_\sigma \) to obtain \( \rho'_\sigma = F_\sigma \) exactly, preserving (S1)-(S5) with \( \rho'_\sigma \) in place of \( \rho_\sigma \), but \( \mu'_n(\mathcal{P}(H_n)) = n \).
(3) Gibbons’ choice corresponds to
(4) The established theorem shows in particular that the pair of requirements (S1) and (O5) cannot hold simultaneously. As long as our goal is describing quantum physics by means of the Hamiltonian formalism, it seems preferable assuming the validity of the former (\( \kappa \in [n + 1, +\infty) \)), dropping the latter. Otherwise \( \rho_\sigma \) would not be represented in terms of a positive Liouville probability density. Sticking to this choice, we can also use \( \rho_\sigma \) to evaluate expectation values for observables that are not of quantum nature, differently form what instead happens if \( \kappa = 1 \). Assuming (S1), the failure of (O5) seems however to remain annoying. Actually, it is not so strong as it could seem at first glance, since as already stressed, there is no unique way to compare a continuous set of reals (the range of \( f_A \)) with a discrete set of real numbers (the spectrum of \( A \)) and the only physically sensible comparison relies upon the identity \( \bigcup \bigcup \) (with \( \mu_n \) in place of \( m \)) that is satisfied. In particular, this identity assures that all elements of \( \text{sp}(A) \) are always obtained as expectation values of \( f_A \) with respect to suitable classical states:\footnote{If \( \kappa = 1 \), the elements \( \text{sp}(A) \) coincides to the singular values of \( f_A \) (i.e. \( df_A(p) = 0 \) iff \( f_A(p) \in \text{sp}(A) \)) as one easily proves.}

4.4 \( C^* \)-algebra of quantum observables in terms of frame functions

In this section we assume to work with \( \mathcal{O} \) of the form \( \text{(8)}, \text{ holding (25), (26), (27), (28)} \). The observables of the systems we are considering are the self-adjoint elements of \( \mathcal{B}(H_n) \). Considering also complex combinations of observables we recover the whole \( C^* \)-algebra \( \mathcal{B}(H_n) \). The map \( \mathcal{O} \), defined with respect to a choice of \( \kappa > 0 \), can be extended by linearity to a map indicated with the same symbol:

\[
\mathcal{O} : \mathcal{B}(H_n) \ni A \mapsto f_A := \kappa F_A + c_\kappa \text{tr}(A) \in \mathcal{F}^2(H_n).
\]

From (d) in theorem \( \text{23} \) this map turns out to be an isomorphism of complex vector spaces with the further property that

\[
\mathcal{O}^{-1}(\mathcal{F}) = (\mathcal{O}^{-1}(f))^* \quad \text{for all} \ f \in \mathcal{F}^2(H_n).
\]
As a consequence, taking advantage from (18), from the explicit expression of \( \kappa > \) in (38) (everything defined for a choice of \( \kappa \) and exploiting \( \kappa \)), we write down two cases explicitly. The case \( L \) working in 28

\[
\text{Proposition 28} \quad \text{If } n > 2 \text{ and } f \in \mathcal{F}^2(\mathbb{H}_n) \text{ is real, referring to the the C*-algebra norm in (38) (everything defined for a choice of } \kappa > 0) \text{ we have:} \\
\| | | f || | = \frac{1}{\kappa} \left\| f - (1 - \kappa) \int_{\mathbb{P}(\mathbb{H}_n)} f d\mu_n \right\|_\infty. \tag{38}
\]

\[
\text{Remarks 29} \quad \text{Even dropping the requirement } f \in \mathcal{F}^2(\mathbb{H}_n) \text{ and assuming, more generally, } f \in C^0(\mathbb{P}(\mathbb{H}_n)), \text{ the right-hand side of (38) defines a norm. The same holds true if working in } L^\infty(C^0(\mathbb{P}(\mathbb{H}_n)), \mu_n) \text{ and interpreting } || | | \text{ as the natural norm referred to the essential supremum computed with respect to } \mu_n. \text{ The proofs are straightforward.}
\]

We write down two cases explicitly. The case \( \kappa = n + 1 \), with \( \mu'_n := n\mu_n \):

\[
\| | | f || | = \frac{1}{n + 1} \left\| f + \int_{\mathbb{P}(\mathbb{H}_n)} f d\mu'_n \right\|_\infty, \quad \text{for every real } f \in \mathcal{F}^2(\mathbb{H}_n),
\]

and the case considered by Gibbons, \( \kappa = 1 \):

\[
\| | | f || | = \| f \|_\infty, \quad \text{for every real } f \in \mathcal{F}^2(\mathbb{H}_n).
\]

Let us finally pass to the product \( \star \), stating a corresponding theorem.
Theorem 30 Let $n > 2$ and $f, g \in \mathcal{F}^2(\mathbf{H}_n)$. If $G_p : T_p^* \mathbf{P}(\mathbf{H}_n) \times T_p^* \mathbf{P}(\mathbf{H}_n) \rightarrow \mathbb{R}$ denotes the scalar product on 1-forms canonically induced by the metric $g$ on $\mathbf{P}(\mathbf{H}_n)$, referring to the $C^*$-algebra product in (36), we have:

$$ f \ast g = \frac{i}{2} \{f, g\} + \frac{1}{2} G(df, dg) + \frac{f g}{\kappa} + \frac{1 - \kappa}{\kappa} \left( \frac{n + 1}{\kappa} \int \! f g \, d\mu_n - f \int \! g \, d\mu_n - g \int \! f \, d\mu_n \right) $$

$$ + \frac{1 - \kappa}{\kappa^2} (\kappa - (n + 1)) \int \! f \, d\mu_n \int \! g \, d\mu_n $$

with, as usual, $\kappa > 0$. In particular, for $\kappa = n + 1$ and defining $\mu'_n := n \mu_n$:

$$ f \ast g = \frac{i}{2} \{f, g\} + \frac{1}{2} G(df, dg) + \frac{1}{n + 1} \left( f g - \int \! f g \, d\mu'_n + f \int \! g \, d\mu'_n + g \int \! f \, d\mu'_n \right) $$

and, for $\kappa = 1$:

$$ f \ast g = \frac{i}{2} \{f, g\} + \frac{1}{2} G(df, dg) + f g . $$

Proof. First of all we notice that, the following identity holds, $g(X_f, X_g) = G(df, dg)$, that immediately follows from $g(X_f, j \cdot) = \omega(X_{\mathbf{H}}, \cdot) = df$, $jj = -I$ and $g(ju, jv) = g(u, v)$. So we replace $G(df, dg)$ for $g(X_f, X_g)$ in the following. Define $f := f_A$ and $g := f_B$. With this choice $f \ast g = f_A \ast f_B = \kappa F_{AB} + c_\kappa \tr(AB)$. Therefore:

$$ f_{AB}(p) = \kappa \tr(pA) + c_\kappa \tr(AB) = \frac{\kappa}{2} \kappa \tr([A, B]) + \frac{\kappa}{2} \tr(p[AB + BA]) + c_\kappa \tr(AB) . $$

A straightforward computation proves that

$$ \tr(p[AB + BA]) = -\tr(p[p, A][p, B]) - \tr(p[p, B][p, A]) + 2 \tr(p) \tr(pB) . $$

Reminding the definition of $\omega$, $\{\cdot, \cdot\}$, $X_\mathbf{H}$ and $g$ presented in Sect.2.2 and putting all together we find:

$$ (f \ast g)(p) = f_{AB}(p) = \frac{i}{2} \{f, g\} + \frac{1}{2} g(X_f, X_g) + \kappa \tr(pA) \tr(pB) + c_\kappa \tr(AB) . $$

From $\kappa + n c_\kappa = 1$ and using (18), one easily finds

$$ \int_{\mathbf{P}(\mathbf{H}_n)} f_A d\mu = \kappa \int_{\mathbf{P}(\mathbf{H}_n)} F_A d\mu + c_\kappa \tr(A) = \frac{1}{n} \tr(A) = \int_{\mathbf{P}(\mathbf{H}_n)} F_A d\mu , $$

and a similar result for $B$ in place of $A$. Using (20) and the definition of $f_A$ ($f_B$) in terms of $F_A$ ($F_B$):

$$ f \ast g = \frac{i}{2} \{f, g\} + \frac{1}{2} g(X_f, X_g) + \frac{1}{\kappa} \left( f - nc_\kappa \int \! f \, d\mu_n \right) \left( g - nc_\kappa \int \! g \, d\mu_n \right) $$

$$ -c_\kappa n^2 \int \! f \, d\mu_n \int \! g \, d\mu_n + \frac{c_\kappa \kappa^2 n(n + 1)}{\kappa^2} \int \! \left( f - nc_\kappa \int \! f \, d\mu_n \right) \left( g - nc_\kappa \int \! g \, d\mu_n \right) d\mu_n . $$

Using again $\kappa + n c_\kappa = 1$ we obtain (39).
Remarks 31

(1) As already noticed in [BH01], for $\kappa = 1$ it turns out that the squared standard deviation $(\Delta A)^2_\psi$ (where $p = \psi(\langle \cdot | \psi \rangle)$ coincides to $\frac{1}{2} G_p(df_A, df_A)$. This allows one to write down a geometrical formulation of Heisenberg inequality. For other choices of $\kappa$ it is still possible, but the expression is more complicated.

(2) A formula similar to (36) for $n = 2$ (stated on the 2-dimensional Bloch sphere) and for the case $\kappa = 1$ is mentioned in [CIMM09].

(3) From (39), the structure of Lie-Jordan Banach algebra [FFIM13, FFMP13] of $F^2(H_n)$ shows up evidently. The Lie commutator is just $\{\cdot, \cdot\}$ whereas the Jordan product reads:

$$f \circ g := \frac{1}{2} G(df, dg) + \frac{fg}{\kappa} + \frac{1 - \kappa}{\kappa} \left( \frac{n + 1}{\kappa} \int_{P(H_n)} fgd\mu_n - \int_{P(H_n)} gfd\mu_n - f \int_{P(H_n)} gdf\mu_n - g \int_{P(H_n)} fdg\mu_n + \frac{1 - \kappa}{\kappa^2} (\kappa - (n + 1)) \int_{P(H_n)} fd\mu_n \int_{P(H_n)} gd\mu_n \right).$$

5 Conclusions and open issues

This paper discussed some issues regarding the interplay of standard and geometric formulation of finite dimensional quantum mechanics, working in the projective space. All the analysis was based upon the properties of the $L^2(\mu_n)$ frame functions, focusing on the rôle of $U(n)$ covariance in particular. The problem of positiveness of Liouville densities associated with quantum states was tackled, establishing that the geometric formalism, in view of the existence of a one-parameter class of natural Kählerian structures on $P(H_n)$, permits to fix several physically safe solutions. A new characterization of classical observables describing quantum observables was discussed, together with a geometric description of the $C^*$-algebra structure (decomposed as a Banach Lie-Jordan algebra) of the set of quantum observables in terms of the Kählerian structure.

It seems worth remarking that the results remain affected by a free parameter, $\kappa > 0$, that could not be fixed out of our physical requirements. A possibility to get rid of this remaining freedom, could arise from the description of compound systems. Compound systems cannot completely be described in classical terms because a problem arises from scratch. In classical physics the phase space of a system composed by two subsystems is the Cartesian product of the space of phases of each subsystem. If one tries to extend the approach of this paper, the phase space of the composed system must be instead the projective space of the tensor product of the Hilbert spaces of the subsystems. That is much larger than the Cartesian product of the projective spaces. The Cartesian product is actually embedded in this larger manifold by means of the well known Segre embedding. A huge literature exists on this topic. It would be interesting to analyse this issue with the help of the technology of frame functions.

Another direction to investigate is, obviously, the infinite dimensional case. Barring evident technical problems to be fixed, a very difficult issue is the generalization of the measure $\mu_n$ on the complex projective space of an infinite dimensional Hilbert space.
A Proof of some results

Proof of proposition 7

Proof. (1) First of all, notice that the three norms $||·||$, $||·||_1$, $||·||_2$ are topologically equivalent since $\mathcal{B}(H_n) = \mathcal{B}_1(H_n) = \mathcal{B}_2(H_n)$ are finite dimensional normed spaces with respect to the corresponding norms.

(2) $S(H_n)$ is compact since it is closed and bounded, with respect to the norm $||·||_1$, in a finite dimensional normed space. Since $S(H_n)$ is compact and the zero operator $0 \not\in S(H_n)$, the continuous functions $S(H_n) \ni σ \mapsto d(0, σ) = ||σ||$, $S(H_n) \ni σ \mapsto d_1(0, σ) = ||σ||_1$, $S(H_n) \ni σ \mapsto d_2(0, σ) = ||σ||_2$ must have strictly positive minima (and maxima). For $d_1$ everything is trivial. Let us pass to consider $d$ and $d_2$. Using the fact that the $n$ eigenvalues $q_k$ of $σ \in S(H_n)$ verify both $q_k ∈ [0, 1]$ and $\sum_{k=1}^n q_k = 1$ one sees that $\sum_{k=1}^n q_k^2 ≥ \frac{n}{2}$, and $1/n$ is indeed the least possible value. All that is equivalent to say that $||\frac{1}{n}I||_2 ≤ ||σ||_2$ where $\frac{1}{n}I$ is an admitted state. Again with the constraints $q_k ∈ [0, 1]$ and $\sum_{k=1}^n q_k = 1$, the maximum of the eigenvalues $q_k = |q_k|$ must be greater than $1/n$, that is equivalent to say $||σ|| ≥ ||\frac{1}{n}I||$. Concerning maxima, $q_k ∈ [0, 1]$ and $\sum_{k=1}^n q_k = 1$ imply, varying $σ \in S(H_n)$: $1 ≥ \sum_{k=1}^n q(σ)_k^2$ and max$\{q(σ)\}_k \in (1, ..., n, σ \in S(H_n)) = 1$ determining the maximum of both $σ \mapsto ||σ||_2$ and $σ \mapsto ||σ||$ since the value 1 of the norms is attained on pure states.

(3). We view $S(H_n)$ a subset of the topological space $T$ of self-adjoint operators $A$ on $H_n$ with $tr(A) = 1$ endowed with the topology induced by $\mathcal{B}(H_n)$.

First of all, notice that $S(H_n) \supset \partial S(H_n)$ because the former is closed with respect to the said topology, so $S(H_n)$ is the disjoint union of $\partial S(H_n)$ and $Int(S(H_n))$. Let $σ$ be an element of $S(H_n)$. First suppose that $dim(Ran(σ)) = n$ we want to show that $σ \in Int(S(H_n))$, that is, there is an open set $O \subset T$ containing $σ$ and such that $σ' \in O$ entails $σ' \in S(H_n)$.

To this end, let us define $m := \min\{⟨ψ|σψ⟩ \mid ||ψ|| = 1\}$. $m$ is real since $σ = σ^*$ and $m > 0$, because: (1) all eigenvalues of $σ$ are strictly positive (since $σ ≥ 0$ and $dim(Ran(σ)) = n$),

(2) $ψ \mapsto ⟨ψ|σψ⟩$ is continuous and (3) the set of vectors $ψ$ with $||ψ|| = 1$ is compact because $dim(H_n) = n < +∞$. Next, if $σ' = σ^* ∈ \mathcal{B}(H_n)$ verifies $||σ - σ'|| < m/2$, one has:

$$m/2 ≤ ||σ - σ'|| = \sup\{|⟨ψ|(σ - σ')ψ⟩| \mid ||ψ|| = 1\}$$

so that: $⟨ψ|σ'ψ⟩ = ⟨ψ|σψ⟩ - ⟨ψ|σψ⟩ + ⟨ψ|σψ⟩ ≥ -\frac{m}{2} + m = \frac{m}{2} > 0$ for $||ψ|| = 1$. Consequently, $σ' ≥ 0$. Summarizing, if $B_{m/2}(σ)$ denotes the open ball in $\mathcal{B}(H_n)$ centred on $σ$ with radius $m/2$, $O := T \cap B_{m/2}(σ)$ is open in $T$ by definition and $σ' \in O$ verifies $σ' = σ^*$, $trσ = 1$ and, as we have proved, $σ' ≥ 0$. In other words, for $σ \in S(H_n)$, $dim(Ran(σ)) = n$ implies $σ \in Int(S(H_n))$.

We pass to the other case for $σ \in S(H_n)$. We suppose that $dim(Ran(σ)) < n$ and we want to show that $σ \in \partial S(H_n)$. $dim(Ran(σ)) < n$ implies $det(σ) = 0$. Thus all eigenvalues are non-negative and one at least vanishes. Let $ψ \in Ker(σ)$. The operators, for $n = 1, 2, \ldots$:

$$σ_n := \left(1 + \frac{1}{n}\right)σ - \frac{1}{n}ψ⟨ψ|::$$
are self-adjoint with \(\text{tr}(\sigma_n) = 1\) so that they stay in \(T\). Furthermore they verify \(\sigma_n \to \sigma\)
for \(n \to +\infty\), but \(\sigma_n \not\in S(H_n)\) because \(\sigma_n\) has the negative eigenvalue \(-\frac{1}{n}\). So \(\sigma \in \partial S(H_n)\).
In particular, since a pure state is a one-dimensional orthogonal projector, it verifies \(\dim(\text{Ran}(\sigma)) = 1 < n\) and thus \(\sigma \in \partial S(H_n)\). If \(n = 2\) this is the only possible case for an element \(\sigma \in \partial S(H_n)\). However, if \(n > 2\), also elements of \(S(H_n)\) with \(\dim(\text{Ran}(\sigma)) \leq n - 1\) belong to \(\partial S(H_n)\). 

Sketch of proof of proposition \[8\]

**Proof.** (a) \(S(H_n)\) is a real \((2n-1)\)-dimensional embedded submanifold of \(H_n\). Let us sketch how it happens. If \((z_01, \ldots, z_{0n}) \in S(H_n)\), there is an open (in \(H_n\)) neighbourhood \(O\) of that point such that for every \((z_1, \ldots, z_n) \in O' := S(H_n) \cap O\) there is a component, say \(z_k = x_k + iy_k\), (the same for all points of \(O\)) such that either \(x_k\) or \(y_k\) can be written as a smooth function of the remaining components \(z_h\), when \((z_1, \ldots, z_n) \in O'\). This procedure define a natural local chart on \(S(H_n)\) with domain \(O'\). Collecting all these charts, that are mutually smoothly compatible, one obtains a smooth differentiable structure on \(S(H_n)\) making it a real \((2n-1)\)-dimensional embedded submanifold of \(H_n\).
(b) Similarly \(P(H_n)\) can be equipped with a real \((2n-2)\)-dimensional smooth manifold structure. Consider \((z_01, \ldots, z_{0n}) \in [\psi_0] \in P(H_n)\). At least one of the components \(z_{0j}\)
cannot vanish, say \(z_{0h}\). By continuity this fact is valid in an open neighbourhood \(V\) of \((z_01, \ldots, z_{0n}) \in S(H_n)\). In that neighbourhood the set of \(n-1\) ratios \(z_j/z_h\) with \(j \neq h\)
determine a point on \(P(H_n)\) biunivocally. These \(n-1\) ratios vary in an open neighborhood \(V' := \pi(V) \subset P(H_n)\) of \([\psi_0]\) when the components \((z_1, \ldots, z_n)\) range in \(V\). Decomposing each of these ratios into real and imaginary part, we obtain a real local chart on \(V' \subset P(H_n)\) with \(2n-2\) real coordinates. Collecting all these local charts, that are mutually smoothly compatible, one obtains a smooth differentiable structure on \(P(H_n)\), making it a \(2n-2\)-dimensional real smooth manifold. With the said structures, the canonical projection \(\pi: S(H_n) \to P(H_n)\) becomes a smooth submersion and the transitive action \[3\] of \(U(n)\) on \(P(H_n)\) turns out to be smooth as one easily proves.

Proof of proposition \[10\]

**Proof.** The action \[3\] is transitive and smooth so, on the one hand \(P(H_n)\) is diffeomorphic to the quotient \(U(n)/G_p\), where \(G_p \subset U(n)\) is the isotropy (closed Lie sub) group of \(p \in P(H_n)\) and on the other hand the projection \(\Pi_p : U(n) \ni U \mapsto UpU^{-1} \in P(H_n)\) is a submersion \[War83\, ONe83\] and thus \(d\Pi_p|_{U=I} : u(n) \to T_p P(H_n)\) is surjective. The thesis is true because \(d\Pi_p(B)|_{U=I} = [B, p]\) for every \(B \in u(n)\) and \(u(n)\) is the real vector space of anti-self adjoint in \(\mathfrak{B}(H)\). 

Proof of theorem \[11\]

**Proof.** Regarding \[3\], consider a smooth curve \(\mathbb{R} \ni t \mapsto p(t) \in P(H_n)\) such that \(\dot{p}(0) = v = -i[B_v, p]\), \[3\] implies:

\[
\langle df_A(p), v \rangle = \kappa \frac{d}{dt}|_{t=0} \text{tr}(p(t)A) = -i \text{tr}([B_v, p], A) = -i \text{tr}(p[A, B]) = \omega_p(-i[A, p], v).
\]

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Since it must also hold $\omega_p(X_{f_A}(p), v) = \langle df_A(p), v \rangle$ we conclude that $\omega_p(X_{f_A}(p) + i[A, p], v) = 0$ for every $v \in T_p\mathcal{P}(H_n)$. As $\omega_p$ is non-degenerate, (9) follows.

(a) In view of (9), Hamilton equation $df_A = X_f p(t)$ is the same as Schrödinger equation $\frac{df_A}{dt} = -i[H, p(t)]$. Now the final statement is obvious form (8) and the cyclic property of the trace:

$$\begin{align*}
f_A(p(t)) &= f_A(e^{-itH}pe^{itH}) = \kappa tr (e^{-itH}pe^{itH}A) + ctr(A) = \kappa tr (pe^{itH}Ae^{-itH}) + ctr(e^{-itH}e^{itH}A) \\
&= \kappa tr (pe^{itH}Ae^{-itH}) + ctr(pe^{itH}Ae^{-itH}) = \int f(u)e^{itH}Ae^{-itH}(p).
\end{align*}$$

(b) The first statement immediately arises using $\{f_A, f_B\} := \omega(X_{f_A}, X_{f_B})$ and (9) noticing that $tr(-i[A, B]) = 0$. Next observe that, since $\mathcal{H}_n$ has finite dimension and so no problems with domains arise, $A$ is a constant of motion with respect to the Hamiltonian $B = H$ iff $[A, H] = 0$. This is equivalent to say $\langle \psi| [A, H]|\psi \rangle = 0$ for all $\psi \in S(H_n)$, that is $tr(p[A, H]) = 0$ for all $p \in \mathcal{P}(H_n)$, that is $f_{-i[A, B]} - tr(-i[A, B]) = 0$. In view of the very identity (10), that is equivalent to say (where $\mathcal{H} = f_H$) $\{f_A, \mathcal{H}\} = 0$, that is eventually equivalent to say that $f_A$ is a constant of motion in Hamiltonian formulation.

(c) The first part of (c) is an evident consequence of the fact that referring to (3): $\Phi_t^{o,\omega} = \omega$ and $\Phi_t^{o, g} = g$ for all $U \in U(n)$. In other words $\omega_p(u, v)$ and $g_p(u, v)$ are invariant if replacing $p$, $A_v$, $A_u$ for $U_{p}U^{-1}$, $U_{A_v}U^{-1}$, $U_{A_u}U^{-1}$ simultaneously as one checks immediately. The last statement is immediately proved by direct inspection. \qed

**Proof of proposition 13**

*Proof.* Henceforth $\mathcal{B}(X)$ denotes the Borel $\sigma$-algebra on the topological space $X$. If $\mu_n$ exists, the requirement $\int_{\mathcal{P}(H_n)} f d\mu_n = \int_{\mathcal{S}(H_n)} f \circ \pi d\nu_n$ entails that, for $f := \chi_E$, it holds:

$$\mu_n(E) = \nu_n(\pi^{-1}(E)) \quad \text{for every } E \in \mathcal{B}(\mathcal{P}(H_n)).$$

(40)

That relation proves that, if $\mu_n$ exists, it is uniquely determined by $\nu_n$. Let us pass to the existence issue. Since $\pi$ is continuous, is Borel-measurable and thus $\pi^{-1}(E) \in \mathcal{B}(\mathcal{S}(H_n))$ if $E \in \mathcal{B}(\mathcal{P}(H_n))$. Since the other requirements are trivially verified, (40) defines, in fact, a positive Borel measure on $\mathcal{P}(H_n)$. That measure fulfills

$$f \circ \pi \in \mathcal{L}^1(\mathcal{S}(H_n), \nu_n) \quad \text{if} \quad f \in \mathcal{L}^1(\mathcal{P}(H_n), \mu_n), \quad \text{and} \quad \int_{\mathcal{P}(H_n)} f d\mu_n = \int_{\mathcal{S}(H_n)} f \circ \pi d\nu_n.$$ 

directly from the definition of integral and $\mu_n(\mathcal{P}(H_n)) = \nu_n(\pi^{-1}(\mathcal{P}(H_n))) = \nu_n(\mathcal{S}(H_n)) = 1$. $\mu_n$ is regular because $\mathcal{P}(H_n)$ is compact it being the image of the compact set $\mathcal{S}(H_n)$ under the continuous map $\pi$ with finite measure [Rud66] and this regularity results also applies the the Liouville measure and the Riemannian one. Concerning the invariance under the action of $PU(n)$, it arises from that of $\mu_n$ under $U(n)$:

$$\mu_n(E) = \nu_n(\pi^{-1}(E)) = \nu_n(U \pi^{-1}(E)) = \nu_n(\pi^{-1}(UEU^{-1})) = \mu_n(UEU^{-1}) \quad \text{if } U \in U(n).$$

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\( P_n(H_n) \) is homeomorphic to the quotient of compact groups \( U(n)/H \) where \( H \) is the isotropy group of any point of \( P_n(H_n) \), since \( H \) is closed in the compact group \( U(n) \), it is compact as well. Thus there is a non-vanishing \( U(n) \) left invariant positive regular Borel measure on \( P_n(H_n) \), uniquely determined by the volume of \( P_n(H_n) \) (Chapter 4 of [BR00]). That measure must thus coincide with \( \mu_n \) up to a strictly positive multiplicative constant.

(b) and (c) the Liouville measure and the Riemannian measure are non-vanishing \( U(n) \) left invariant positive regular Borel measure on \( P_n(H_n) \), because both \( \omega \) and \( g \) are \( U(n) \) invariant. Therefore they have to coincide with \( \mu_n \) up to a strictly positive multiplicative constant.

\( \square \)

**Lemma 32** If \( H \) is a complex Hilbert space and \( \phi, \psi \in S(H) \), then there exists \( U \in \mathfrak{B}(H) \) such that \( U\phi = \alpha \psi \) for some \( \alpha \in \mathbb{C} \), \( |\alpha| = 1 \), and \( U = U^* = U^{-1} \).

**Proof.** If \( \phi \) and \( \psi \) are linearly dependent, choosing \( \alpha \) such that \( \phi = \alpha \psi \), we can define \( U := I \). In the other case, let \( K \) be the closed subspace spanned by \( \phi \) and \( \psi \). It is enough to find \( V : K \to K \) with \( V = V^* = V^{-1} \) and \( V\phi = \alpha \psi \) for some \( \alpha \) with \( |\alpha| = 1 \). If such a \( V \) exists, the wanted \( U \) can be defined ad \( U := V \oplus I \) referring to the orthogonal decomposition \( H = K \oplus K^\perp \). Fixing an orthonormal basis in \( K \) given by \( \phi, \phi_1 \), the problem can be tackled in \( \mathbb{C}^2 \). With the Hilbert-space isomorphism from \( K \) to \( \mathbb{C}^2 \), \( \phi \) corresponds to \( (1,0)^t \) and \( \psi \) with \( (a,b)^t \) where \( |a|^2 + |b|^2 = 1 \). We can choose \( \alpha \) such that \( \alpha \psi \) corresponds to \( (c,d) \) with \( c > 0 \), \( d \in \mathbb{C} \) and \( c^2 + |d|^2 = 1 \). To conclude, we only need to find a complex \( 2 \times 2 \) matrix \( M \) with \( M = \overline{M} = M^{-1} \) and \( M(1,0)^t = (c,d)^t \). The operator \( V \) corresponds to \( M \) through the identification of \( K \) and \( \mathbb{C}^2 \) we have previously introduced. The wanted \( M \) is just the following one:

\[
M := \begin{bmatrix} c & \overline{d} \\ d & -c \end{bmatrix}.
\]

\( \square \)

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**References**

[AS95] A. Ashtekar and T. A. Schilling, *Geometry of quantum mechanics*, AIP Conference Proceedings, **342**, 471-478 (1995).

[BH01] D. C. Brody, L. P. Hughston. *Geometric quantum mechanics*, Journal of Geometry and Physics **38** 19-53 (2001)

[BSS04] A.Benvegnù, N.Sansonetto and M.Spera *Remarks on geometric quantum mechanics*, Journal of Geometry and Physics **51** 229-243 (2004)
A.O. Barut and R. Raczka. *Theory of Group Representations and Applications*. World Scientific, Singapore, 2000

J. F. Carinena, A. Ibort, G. Marmo, G. Morandi. *Geometrical description of algebraic structures: Applications to Quantum Mechanics*, AIP Conf.Proc. **1130** 47-59 (2009)

R. Cirelli, P. Lanzavecchia, and A. Mania. *Normal pure states of the von neumann algebra of bounded operators as kahler manifold* J. Phys. A: Mathematical and General, **16**, 3829 (1983)

A. Dvurečenskij. *Gleason’s theorem and its applications* (Kluwer academic publishers, 1992).

F. Falceto, L. Ferro, A. Ibort and G. Marmo. *Reduction of Lie-Jordan Banach algebras and quantum states* J. Phys. A: Math. Theor. **46** 015201 (2013)

P. Facchi, L. Ferro, G. Marmo and S. Pascazio, *Defining quantumness via the Jordan product*, Preprint arXiv:1309.4635v1

J. Grabowski, M. Kuś and G. Marmo. *Geometry of quantum systems: density states and entanglement*, J. Phys. A: Math. Gen. **38** 10217-10244 (2005)

G.W. Gibbons. *Typical states and density matrices* Journal of Geometry and Physics **8** 147-162 (1992)

A. M. Gleason. *Measures on the closed subspaces of a Hilbert space*, Journal of Mathematics and Mechanics, Vol.6, No.6, 885-893 (1957).

T. W. B. Kibble *Geometrization of Quantum Mechanics*, Commun. Math. Phys. **65**, 189-201 (1979)

V. Moretti and D. Pastorello. *Generalized Complex Spherical Harmonics, Frame Functions, and Gleason Theorem* Ann. Henri Poincaré **14**,1435-1443 (2013).

G. W. Mackey. *Induced representations of locally compact groups I*, The Annals of Mathematics, Second Series, Vol.55, No.1, 101-139 1951.

B. O’Neill , *Semi-Riemannian geometry*, Academic Press, San Diego (1983)

W. Rudin *Real and complex analysis* (McGrow-Hill Book Co. 1986).

F.W. Warner. *Foundations of Differentiable Manifolds and Lie Groups*, Springer, Berlin, 1983