COLLISION OF A SOLID BODY WITH ITS CONTAINER
IN A 3D COMPRESSIBLE VISCOUS FLUID

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ABSTRACT. We consider a bounded domain $\Omega \subset \mathbb{R}^3$ and a rigid body $S(t) \subset \Omega$ moving inside a viscous compressible Newtonian fluid. We exploit the roughness of the body to show that the solid collides its container in finite time. We investigate the case when the boundary of the body is of $C^{1,\alpha}$-regularity and show that collision can happen for some suitable range of $\alpha$.

Keywords. Fluid-structure interaction, Compressible Navier–Stokes, Collision.
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1. INTRODUCTION

We consider the compressible Navier-Stokes equations of a barotropic fluid in a bounded domain $\Omega \subset \mathbb{R}^3$ and a rigid body $S(t)$ with center of mass at $G(t)$ moving inside the fluid, where the fluid domain is $\mathcal{F}(t) = \Omega \setminus S(t)$. The equations of motion are given by

$$
\begin{align*}
\begin{cases}
\partial_t \rho_f + \text{div}(\rho_f u_f) &= 0 &\text{in } \mathcal{F}(t), \\
\partial_t (\rho_f u_f) + \text{div}(\rho_f u_f \otimes u_f) - \text{div} S(u_f) + \nabla p &= \rho_f f &\text{in } \mathcal{F}(t), \\
u_f &= G(t) + \omega(t) \times (x - G(t)) &\text{on } \partial S(t), \\
u_f &= 0 &\text{on } \partial \Omega, \\
& m \ddot{G} = -\int_{\partial S} S(u_f) - pI \, n \, dS + \int_{\mathcal{S}} \rho_s f \, dS, \\
& \frac{d}{dt}(\omega) = -\int_{\partial S} (x - G) \times (S(u_f) - pI) \, n \, dS + \int_{\mathcal{S}} (x - G) \times \rho_s f \, dS, \\
& \rho_f(0) = \rho_0, \quad \rho_f u_f(0) = q_0, \quad G(0) = G_0, \quad \dot{G}(0) = V_0, \quad \omega(0) = \omega_0 &\text{in } \mathcal{F}(0).
\end{cases}
\end{align*}
$$

(1.1)

Here, $u_f$ is the fluid’s velocity, $\rho_f$ and $\rho_s$ are the fluid’s and solid’s density, respectively. In the above, $G(t)$ and $\omega(t)$ are the translational and rotational velocities of the rigid body, respectively.

The pressure $p$ of the fluid is given by

$$
p = p(\rho_f) = (\rho_f)^\gamma \quad \text{for some } \gamma > \frac{3}{2}.
$$

(1.2)

The stress tensor satisfies Newton’s rheological law

$$
S(u) = 2\mu \mathbb{D}(u) + \lambda \text{div } u,
$$

where $\mathbb{D}(u)$ is the symmetric part of the velocity gradient tensor.

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where $\mu > 0$, $2\mu + 3\lambda \geq 0$, and $\mathcal{D}(u) = \frac{1}{2}(\nabla u + \nabla^T u)$ is the symmetric part of the gradient of $u$. Further, we assume that $\rho_s > 0$ is constant. The mass and moment of inertia of the rigid body are given by

$$m = \rho_s |S(0)|, \quad \mathcal{J}(t) = \int_{S(t)} \rho_s \left( |x - G|^2 I - (x - G) \otimes (x - G) \right) \, dx.$$  

The centre of mass is defined as

$$G(t) = \frac{1}{m} \int_{S(t)} \rho_s x \, dx.$$  

Note especially that we have for any $\omega \in \mathbb{R}^3$

$$\mathcal{J} \omega \cdot \omega = \int_{S(t)} \rho_s \left( |x - G|^2 |\omega|^2 - |(x - G) \cdot \omega|^2 \right) \, dx = \int_{S(t)} \rho_s |\omega \times (x - G)|^2 \, dx \geq 0.$$  

We will also assume that the solid’s mass is independent of time, that is, $m = \rho_s |S(t)|$ for any $t \geq 0$.

Understanding the dynamics of a solid body immersed in a fluid is a very active area of research. During the last years, there are many interesting studies focused on the collision/no collision results of a moving body with the boundary of the domain. However, all these available results (see [8, 9, 10, 13, 15, 16, 17] and the references therein) are for the incompressible fluid-rigid body interaction system only. The aim of the present paper is therefore to deal with viscous compressible fluids.

The answer to the question whether or not collision occurs mainly depends on the “physical roughness” of the system’s boundary, which can be further split into two issues: the shape of the falling body, and the boundary conditions imposed on the solid’s and container’s wall. This can also be translated into the properties of the velocity gradient. A simple argument reveals that the velocity gradient must become singular (unbounded) at the contact point since otherwise the streamlines would be well defined, in particular, they could never meet each other.

Experimentally, the influence of the particle roughness on the collision problem had been studied in [19] and the influence of boundary conditions at the fluid–solid interfaces had been investigated in [20]. Mathematically, the roughness-induced effect on the collision process in two space dimensions is analyzed in [8]. They considered a vertical motion of a $C^{1, \alpha}$ rigid body falling over a horizontal flat surface, and proved that collision happens in finite time if and only if $\alpha < 1/2$. The authors in [9] investigated the relation between the roughness and the drag force in more detail. They studied the evolution of a three dimensional rough solid falling towards a rough wall, introducing the roughness by special shapes of the solid’s and container’s boundary, respectively, or by considering Navier slip conditions. The works [15, 17] deal with the free fall of a rigid sphere over a wall with no-slip conditions at the solid’s and container’s boundary in two and three spatial dimensions, respectively. In this setting, no collision between the sphere and the wall happens. The effect of Navier slip boundary conditions was analyzed in [10]. They established that under certain assumptions on the slip coefficients the rigid ball touches the boundary of the domain in finite time. In addition, the authors in [22] showed that collision appears under the prescription of the motion of a given ball in the case of slip boundary conditions.

Let us also mention the paper [26], where the author constructed a collision example of a rigid body with the boundary of the physical 2D domain resulting from the action of a very singular driving force. On the other hand, the same author [27] showed that collisions, if any, must occur with zero relative translational velocity as soon as the boundaries of the rigid objects are smooth and the gradient of the underlying velocity field is square-integrable - a hypothesis satisfied by any Newtonian fluid flow of finite
energy.

Concerning non-Newtonian incompressible fluids, the existence of global-in-time solutions of the motion of several rigid bodies was proven in [7], where, in accordance with [27], collisions of two or more rigid objects do not appear in finite time unless they were present initially.

In this paper we want to discuss the collision phenomenon of a moving rigid body with the boundary of its container when the container is filled with a viscous compressible fluid. Precisely, we want to construct an example in which the rigid body collides with the boundary of the domain filled by the compressible Newtonian fluid. We want to exploit the roughness of the body to achieve this collision result, thus considering the case when the boundary of the body is of $C^{1,\alpha}$-regularity and searching for a suitable range for the exponent $\alpha$ such that the collision can happen.

1.1. Weak formulation and Main result. Let us introduce the notion of weak solutions to problem (1.1). First, we extend $\rho$ and $u$ to the whole of $\mathbb{R}^3$ via

\[
\rho = \begin{cases} 
\rho_f & \text{in } F(t), \\
\rho_s & \text{in } S(t), \\
0 & \text{in } \mathbb{R}^3 \setminus \Omega,
\end{cases}
\]

\[u = \begin{cases} 
\mathbf{u}_f(t) & \text{in } F(t), \\
\mathbf{G}(t) + \omega(t) \times (x - \mathbf{G}(t)) & \text{in } S(t), \\
0 & \text{in } \mathbb{R}^3 \setminus \Omega.
\end{cases}
\]

Then we consider the following notion of weak solutions:

\textbf{Definition 1.1.} A triplet $(\rho, \mathbf{u}, \mathbf{G})$ is a renormalized finite energy weak solution to (1.1) if:

- The solution belongs to the regularity class
  \[\rho \geq 0, \quad \rho \in L^\infty(0,T;L^\gamma(\Omega)) \cap C([0,T];L^1(\Omega)), \quad \mathbf{u} \in L^2(0,T;W_0^{1,2}(\Omega)),\]
  \[\mathbf{G} \in W^{1,\infty}(0,T), \quad \mathbf{u} = \dot{\mathbf{G}}(t) + \omega(t) \times (x - \mathbf{G}(t)) \text{ in } S(t);\]

- The weak form of the continuity and renormalized continuity equation hold:
  \[
  \int_0^T \int_{\mathbb{R}^3} [\rho \partial_t \phi + (\rho \mathbf{u}) \cdot \nabla \phi] \, dx \, dt = 0,
  \int_0^T \int_{\mathbb{R}^3} \left[ b(\rho) \partial_t \phi + (b(\rho) \mathbf{u}) \cdot \nabla \phi + (b(\rho) - b'(\rho)) \rho \div \mathbf{u} \phi \right] \, dx \, dt = 0,
  \]
  for any $\phi \in C^\infty_c((0,T) \times \mathbb{R}^3)$ and for any $b \in C^1(\mathbb{R})$ such that $b'(z) = 0$ for $z$ large enough;

- The weak form of the momentum equation holds for a.e. $t \in [0,T]$:
  \[
  \int_0^t \int_{\Omega} \left[ (\rho \mathbf{u}) \cdot \partial_t \phi + (\rho \mathbf{u} \otimes \mathbf{u}) : \nabla \phi + p(\rho) \div \mathbf{u} - S(\mathbf{u}) : \nabla \phi + \rho \mathbf{f} \cdot \phi \right] \, dx \, dt
  = \int_{\Omega} \left[ \rho(t) \mathbf{u}(t) \cdot \phi(t) - \mathbf{q}_0 \cdot \phi(0) \right] \, dx
  \]
  for any $\phi \in C^\infty_c([0,T] \times \Omega)$ with $\phi(t,y) = \ell_\phi(t) + \omega_\phi(t) \times (y - \mathbf{G}(t))$ in a neighborhood of $S(t);

- The following energy inequality holds for a.e. $t \in [0,T]$:
  \[
  \int_{\Omega} \left( \frac{1}{2} \rho(t,x)|\mathbf{u}(t,x)|^2 + \frac{\rho_\gamma(t,x)}{\gamma - 1} \right) \, dx + \int_0^t \int_{\Omega} (2\mu|\nabla \mathbf{u}|^2 + \lambda|\div \mathbf{u}|^2) \, (\tau, x) \, dx \, d\tau
  \leq \int_{\Omega} \left( \frac{1}{2} |\mathbf{q}_0(x)|^2 + \frac{\rho_\gamma(x)}{\gamma - 1} \right) \, dx + \int_0^t \int_{\Omega} \rho \mathbf{f} \cdot \mathbf{u} \, dx,
  \]

where we extended $\rho, u$ as in (1.3), and $q_0 = \rho(0)u(0)$.

Let us remark that the regularity classes of $\rho$ and $u$ immediately imply

$$(1.7) \quad \rho u \in L^\infty(0, T; L^{2\gamma+1}(\Omega)).$$

Before stating precisely the well-posedness result of the system (1.1), we want to mention previous mathematical results related to the existence theory of solutions to this system. Regarding the evolution of a system of a rigid body in a compressible fluid with Dirichlet boundary conditions, the existence of a weak solution up to collision is proved in [4]. In [5] this result was generalized to allow also for collisions. Recently, existence of weak solutions for Navier slip boundary conditions at the interface as well as the boundary of the domain has been established in [21]. The existence of strong solutions was studied in [1, 12, 14, 23].

Let us now state the existence result of the system (1.1) which can be established by following [5, Theorem 4.1]:

**Theorem 1.2.** Let $\Omega$ and $S_0$ be two bounded domains of $\mathbb{R}^3$. Let $p$ be defined through (1.2), and $f = \nabla F$ with $F = -gx_3$, where $g > 0$ is the acceleration due to gravity. Assume that the initial data satisfy

$$(1.8) \quad \rho_0 \in L^2(\Omega), \quad \rho_0 \geq 0 \text{ a.e. in } \Omega,$$

$$(1.9) \quad q_0 \in L^{2\gamma+1}(\Omega), \quad q_0 \mathbf{1}_{\{\rho_0=0\}} = 0 \text{ a.e. in } \Omega, \quad \frac{|q_0|^2}{\rho_0} \mathbf{1}_{\{\rho_0>0\}} \in L^1(\Omega),$$

$$(1.10) \quad u_0 = V_0 + \omega_0 \times (x - G_0) \text{ on } S_0 \text{ with } V_0, \omega_0, G_0 \in \mathbb{R}^3.$$

Then the system (1.1) admits a weak solution in the sense of Definition 1.1.

In this paper, we study whether the solid body can collide with the boundary $\partial \Omega$ in finite time or not, that is, whether the maximal existence time $T_*$ of a weak solution is finite or not. We consider a $C^{1,\alpha}$ solid moving vertically over a flat horizontal surface under the influence of gravity. More precisely, we make the following assumptions (see Figure 1.1 for the main notations):

1. The source term is provided by the gravitational force $f = -ge_3$ and $g > 0$.
2. The solid moves along the vertical axis $\{x_1 = x_2 = 0\}$.
3. The only possible collision point is at $x = 0$, and the solid’s motion is a vertical translation.
4. Near $r = 0$, $\partial \Omega$ is flat and horizontal, where $r = \sqrt{x_1^2 + x_2^2}$.
5. Near $r = 0$, the lower part of $\partial S(t)$ is given by $x_3 = h(t) + r^{1+\alpha}$, $r \leq 2r_0$ for some small enough $r_0 > 0$.
6. The collision just happens near the flat boundary of $\Omega$:

$$\inf_{t > 0} \text{dist} \left( S(t), \partial \Omega \setminus [-r_0, r_0]^2 \times \{0\} \right) \geq d_0 > 0.$$  

Let us also assume that the position of the solid is characterized by its height $h(t)$, in the sense that $G(t) = G(0) + (h(t) - h(0))e_3$, and $S(t) = S(0) + (h(t) - h(0))e_3$.

Our main result regarding collision now reads as follows:

1 The mathematical analysis of systems describing the motion of a rigid body in a viscous incompressible fluid is nowadays well developed. The proof of existence of weak solutions until a first collision can be found in several papers, see [2, 3, 11, 18, 25]. Later, the possibility of collisions in the case of weak solutions was included, see [6, 24].
Theorem 1.3. Let $0 < \alpha \leq 1$ and $\Omega, S \subset \mathbb{R}^3$ be bounded domains of class $C^{1,\alpha}$. Let $(\rho, u, G)$ be a renormalized finite energy weak solution of the compressible Navier-Stokes equations (1.1) in the sense of Definition 1.1 satisfying the assumptions (1)–(6). If the solid’s mass is large enough, and its initial vertical and rotational velocity are small enough, then the solid touches $\partial \Omega$ in finite time provided

$$\gamma > 3 \quad \text{and} \quad \alpha < \frac{\gamma - 3}{3\gamma + 1}.$$ 

Remark 1.4. The terms “large enough” and “small enough” should be interpreted in such a way that inequality (3.13) is satisfied.

Remark 1.5. The above constraint on $\alpha$ seems to be optimal in the sense that $\alpha = \frac{1}{3}$ is the critical value for the incompressible case, which would (loosely speaking) correspond to $\gamma = \infty$ (see [9] for details). As a matter of fact, the outcome of Theorem 1.3 remains true for $\alpha < \frac{1}{3}$ as long as $\rho \in L^\infty(\Omega)$ uniformly in time.

Organization of the paper. In Section 2, we give a priori bounds on the velocity and the density needed for the sequel. Finally, Section 3 is devoted to the construction of an appropriate test function and to the proof of Theorem 1.3.

2. A PRIORI BOUNDS AND ENERGY ESTIMATES

From the energy inequality (1.5) we obtain a priori bounds on the density and velocity.
Proposition 2.1. Let \( \gamma > 1 \) and \((\rho, u, G)\) be a weak solution to (1.1) on \( (0, T) \times \Omega \) with external force \( f = \nabla F \in L^{2,\infty}(\Omega) \), where \( F \in L^{2,\infty}(\Omega) \) is a time-independent potential. Then,

\[
\sup_{t \in (0, T)} \int_{F(t)} \left( \frac{1}{2} \rho |u|^2 + \frac{\rho^{\gamma}}{2(\gamma - 1)} \right) \, dx + \sup_{t \in (0, T)} \int_{\partial \Omega} \frac{1}{2} \rho |u|^2 \, dx
\]

\[
+ \int_{0}^{T} \int_{F(t)} (2\mu |\nabla u|^2 + \lambda |\text{div} u|^2) \, dx \, dt
\]

\[
\leq \int_{F(0)} \left( \frac{1}{2} \frac{|q|^2}{\rho_0} + \frac{\rho_0^{\gamma}}{\gamma - 1} \right) \, dx + \int_{0}^{T} \int_{\Omega} \rho f \cdot u \, dx \, dt.
\]

Remark 2.2. We remark that \( f = \nabla F \in L^{2,\infty}(\Omega) \), together with the bound (1.7), imply that the force term \( \int_{\Omega} \rho f \cdot u \, dx \) is well-defined.

Proof of Proposition 2.1. By the energy inequality (1.5), we have

\[
\int_{\Omega} \left( \frac{1}{2} \rho |u|^2 + \frac{\rho^{\gamma}}{\gamma - 1} \right) \, dx + \int_{0}^{T} \int_{F(t)} (2\mu |\nabla u|^2 + \lambda |\text{div} u|^2) \, dx \, dt
\]

\[
\leq \int_{\Omega} \left( \frac{1}{2} \frac{|q|^2}{\rho_0} + \frac{\rho_0^{\gamma}}{\gamma - 1} \right) \, dx + \int_{0}^{T} \int_{\Omega} \rho f \cdot u \, dx \, dt.
\]

We split the pressure term in the first integral to get

\[
\int_{\Omega} \frac{\rho^{\gamma}}{\gamma - 1} \, dx - \int_{F} \frac{\rho^{\gamma}}{\gamma - 1} \, dx = \int_{\partial \Omega} \frac{\rho^{\gamma}}{\gamma - 1} \, dx + \frac{\rho_0^{\gamma}}{\gamma - 1} |\partial \Omega| = \frac{\rho_0^{\gamma - 1}}{\gamma - 1} m,
\]

where we used that \( \rho |\partial \Omega| = \rho_s \) is constant, and the mass of \( \partial \Omega \) is independent of time. Obviously, the same holds if we replace \( \rho \) by \( \rho_0 \). Thus, the energy inequality turns into

(2.2)

\[
\int_{F(t)} \left( \frac{1}{2} \rho |u|^2 + \frac{\rho^{\gamma}}{\gamma - 1} \right) \, dx + \int_{0}^{T} \int_{F(t)} \frac{1}{2} \rho |u|^2 \, dx + \int_{0}^{T} \int_{\Omega} (2\mu |\nabla u|^2 + \lambda |\text{div} u|^2) \, dx \, dt
\]

\[
\leq \int_{F(0)} \left( \frac{1}{2} \frac{|q|^2}{\rho_0} + \frac{\rho_0^{\gamma}}{\gamma - 1} \right) \, dx + \int_{0}^{T} \int_{\Omega} \frac{1}{2} \frac{|q|^2}{\rho_0} \, dx + \int_{0}^{T} \int_{\Omega} \rho f \cdot u \, dx \, dt.
\]

Next we note that the continuity equation is satisfied in the whole of \( \Omega \) due to the specific form of \( u \) on \( \partial \Omega \). Since \( f = \nabla F \) and \( F \in L^{2,\infty}(\Omega) \) is time-independent, we obtain

\[
\int_{0}^{T} \int_{\Omega} \rho u \cdot f \, dx \, dt = \int_{0}^{T} \int_{\Omega} \rho u \cdot \nabla F \, dx \, dt = -\int_{0}^{T} \int_{\Omega} \text{div}(uF) \, dx \, dt
\]

\[
= \int_{0}^{T} \int_{\Omega} \partial_t \rho F \, dx \, dt = \int_{\Omega} \rho(T) F \, dx - \rho_0 F \, dx
\]

\[
= \int_{F(T)} \rho(T) F \, dx - \int_{F(0)} \rho_0 F \, dx,
\]

where in the last equality we used again that \( \rho |\partial \Omega| = \rho_s \) is constant. Thus, we estimate the force term as

\[
\left| \int_{0}^{T} \int_{\Omega} \rho u \cdot f \, dx \, dt \right| \leq 2 \sup_{t \in (0, T)} \|\rho F\|_{L^{2}(F(t))} \leq 2 \|\rho\|_{L^{2}(0, T; L^{2}(F(\cdot)))} \|F\|_{L^{2,\infty}(\Omega)}
\]
\[
\leq \frac{1}{2(\gamma - 1)} \| \rho \|_{L^\infty(O; L^\gamma(\mathcal{F}))} + C(\gamma) \| F \|_{L^{\gamma/\tau}(\Omega)}.
\]

The last line is coming from the following form of Young’s inequality:

\[
ab \leq \varepsilon a^p + \left( \frac{p \varepsilon}{q} \right)^{1 - q} b^q \quad \forall (a, b, \varepsilon) \in (0, \infty)^3, \quad \forall p, q \in (1, \infty), \quad \frac{1}{p} + \frac{1}{q} = 1.
\]

Hence, we can absorb the first term on the right-hand site by the left-hand site of (2.2). Further, the definitions of \( \rho \) and \( u \) in (1.3), together with \( q_0 = \rho(0) u(0) \), yield

\[
\int_{S(0)} \frac{|q_0|^2}{2\rho_0} \, dx = \int_{S(0)} \frac{1}{2} \rho_s |V_0 + \omega_0 \times (x - G_0)|^2 \, dx
\]

\[
= \frac{1}{2} \rho_s \int_{S(0)} |V_0|^2 + 2V_0 \cdot (\omega_0 \times (x - G_0)) + |\omega_0 \times (x - G_0)|^2 \, dx
\]

\[
= \frac{1}{2} m |V_0|^2 + 2\|\omega_0 \cdot \omega_0 + \int_{S(0)} \rho_s V_0 \cdot (\omega_0 \times (x - G_0)) \, dx.
\]

Together with (2.2) and the fact that \( G_0 = \frac{1}{m} \int_{S(0)} \rho_s x \, dx \), we conclude the desired estimate (2.1). \( \square \)

**Remark 2.3.** In the case of gravitational force, we have \( f = -g \varepsilon_3 = -g \varepsilon_3 \), where \( g > 0 \) is the acceleration due to gravity. Thus, \( F = -g x_3 \) and the energy inequality (2.1) becomes

\[
\begin{align*}
&\sup_{t \in (0, T)} \int_{\mathcal{F}(t)} \left( \frac{1}{2} \rho |u|^2 + \frac{\rho^\gamma}{2(\gamma - 1)} \right) \, dx + \sup_{t \in (0, T)} \int_{S(t)} \frac{1}{2} \rho |u|^2 \, dx \\
&\quad + \int_0^T \int_{\mathcal{F}(t)} \left( 2 \mu \| D(u) \|^2 + \lambda |\text{div} u|^2 \right) \, dx \, dt \\
&\leq \int_{\mathcal{F}(0)} \left( \frac{1}{2} \frac{|q_0|^2}{\rho_0} + \frac{\rho_0^\gamma}{\gamma - 1} \right) \, dx + \frac{m}{2} |V_0|^2 + \frac{1}{2} \| \omega_0 \cdot \omega_0 + C(\gamma) \| g x_3 \|_{L^{\gamma/\tau}(\Omega)}
\end{align*}
\]

(2.3)

where we used that \( x_3 \leq \text{diam} \Omega \) and \( |\Omega| \leq (\text{diam} \Omega)^3 \).

If we define the initial energy by

\[
E_0 = \int_{\mathcal{F}(0)} \left( \frac{1}{2} \frac{|q_0|^2}{\rho_0} + \frac{\rho_0^\gamma}{\gamma - 1} \right) \, dx + \frac{m}{2} |V_0|^2 + \frac{1}{2} \| \omega_0 \cdot \omega_0,
\]

we obtain from (2.3) the following energy estimate:

\[
\begin{align*}
&\sup_{t \in (0, T)} \int_{\mathcal{F}(t)} \left( \frac{1}{2} \rho |u|^2 + \frac{\rho^\gamma}{2(\gamma - 1)} \right) \, dx + \sup_{t \in (0, T)} \int_{S(t)} \frac{1}{2} \rho |u|^2 \, dx \\
&\quad + \int_0^T \int_{\Omega} \left( 2 \mu \| D(u) \|^2 + \lambda |\text{div} u|^2 \right) \, dx \, dt \\
&\leq E_0 + L(g, \gamma, \Omega),
\end{align*}
\]

(2.6)

where

\[
L(g, \gamma, \Omega) = C(\gamma) g \| g x_3 \|_{L^{\gamma/\tau}(\Omega)} \|_{\gamma/\tau+3},
\]

(2.8)
In particular, this together with Korn’s inequality yields
\begin{equation}
\|\rho|u|^2\|_{L^\infty(0,T;L^1(\Omega))} + \|\rho|^2\|_{L^\infty(0,T;L^1(\mathcal{F}(\cdot)))} + \|\nabla u\|^2_{L^2(0,T;L^2(\Omega))} \leq C(\Omega, \mu, \gamma)(E_0 + L).
\end{equation}

Note carefully that we do not have such an estimate for $\rho|_S = \rho_s$. Therefore, we have to distinguish between the fluid and solid part in the estimates just follow in Section 3.2.

3. Construction of the test function

Assume that $(\rho, u, G)$ is a weak solution of (1.1) satisfying the assumptions (1)–(6) in the time interval $(0, T_\ast)$ before collision. From now on we denote $\mathcal{S}_h = S_h(t) = S(0) + (h(t) - h(0))e_3$ and $\mathcal{F}_h = \mathcal{F}_h(t) = \Omega \setminus \mathcal{S}_h(t)$.

Collision can occur if and only if $\lim_{t \to T_\ast} h(t) = 0$. Note further that $\text{dist}(\mathcal{S}_h(t), \partial \Omega) = \min\{h(t), d_0\}$ by assumptions (2) and (6).

3.1. Test function. In this section, we will use the notation $a \lesssim b$ whenever there is a constant $C > 0$ which is independent of $a$, $b$, $h$, and $T$ such that $a \leq Cb$. Further, we will make use of cylindrical coordinates $(r, \theta, x_3)$ with the standard basis $(e_r, e_\theta, e_3)$. As in the papers \cite{8, 9, 10}, we construct a test function $w_h$ associated with the solid particle $\mathcal{S}_h$ frozen at distance $h$. This function will be defined for $h \in (0, h_M)$ with $h_M = \sup_{t \in [0, T_\ast]} h(t)$. Note that when $h \to 0$, a cusp arises in $\mathcal{F}_h$, which is contained in the domain
\begin{equation}
\Omega_{h,r_0} = \{x \in \mathcal{F}_h : 0 \leq r < r_0, \ 0 \leq x_3 \leq h + r^{1+\alpha}, \ r^2 = x_1^2 + x_2^2\}.
\end{equation}

For the sequel, we fix $h$ as a positive constant and define $\psi(r) := h + r^{1+\alpha}$. Note that the common boundary $\partial \Omega_{h,r_0} \cap \partial \mathcal{S}_h$ is precisely given by the set $\{0 \leq r \leq r_0, \ x_3 = \psi(r)\}$.

We use the same function as in \cite{9}: define smooth functions $\chi, \eta$ satisfying
\begin{align}
\chi &= 1 \text{ on } (0, r_0)^3, \quad \chi = 0 \text{ on } \Omega \setminus (0, 2r_0)^3, \\
\eta &= \frac{r}{2} \text{ near } \mathcal{S}_h, \quad \eta = 0 \text{ near } \partial \Omega.
\end{align}

Let us set
\begin{equation}
\phi_h(r, x_3) = (1 - \chi(r, x_3))\eta(r, x_3 - h + h(0)) + \chi(r, x_3)r^2\frac{\Phi(x_3)}{\psi(r)}, \quad \Phi(t) = t^2(3 - 2t).
\end{equation}

Let $w_h = \nabla \times (\phi_h e_\theta)$, which we can write as
\begin{equation}
w_h = -\partial_3 \phi_h e_r + \frac{1}{r} \partial_r (r \phi_h) e_3.
\end{equation}

Observe that the function $w_h$ satisfies
\begin{align*}
w_h|_{\partial \mathcal{S}_h} &= e_3, \quad w_h|_{\partial \Omega} = 0, \quad \text{div } w_h = 0.
\end{align*}

Indeed, the divergence-free condition is obvious from the definition of $w_h$. Further, since $x_3 = \psi(r)$ and $\chi = 1$ on $\partial \mathcal{S}_h$, $\Phi(1) = 1$, and $\Phi'(1) = 0$, we have
\begin{align*}
w_h|_{\partial \mathcal{S}_h} &= \frac{1}{r} \partial_r \left( r^2 \frac{\Phi}{2} \right)(1)e_3 = \Phi(1)e_3 + \frac{r}{2} \Phi'(1) \partial_r \psi = e_3.
\end{align*}

Moreover, since $\chi = \eta = 0$ near $\partial \Omega$, we have $w_h|_{\partial \Omega} = 0$.

We summarize further properties in the following Lemma:
Lemma 3.1. $w_h \in C_c^\infty(\Omega)$ and
\begin{equation}
\|\partial_h w_h\|_{L^\infty(\Omega \setminus \Omega_{h, r_0})} + \|w_h\|_{W^{1, \infty}(\Omega \setminus \Omega_{h, r_0})} \lesssim 1.
\end{equation}
Moreover,
\begin{align*}
\|w_h\|_{L^p(\Omega_{h, r_0})} \lesssim 1 \text{ for any } p < 1 + \frac{3}{\alpha},
\|\partial_h w_h\|_{L^p(\Omega_{h, r_0})} + \|\nabla w_h\|_{L^p(\Omega_{h, r_0})} \lesssim 1 \text{ for any } p < \frac{3 + \alpha}{1 + 2\alpha}.
\end{align*}

**Proof.** We know from the definition of $w_h$ in (3.5) that $w_h \in C_c^\infty(\Omega)$. Moreover, $w_h$ is bounded outside a bounded region, so the first inequality (3.6) is obvious.

Due to the property (3.2) of $\chi$, the function $\partial_h$ (see (3.4)) in $\Omega_{h, r_0}$ (see (3.1)) becomes
\begin{equation}
\phi_h(r, x_3) = \frac{r}{2} \Phi \left( \frac{x_3}{\psi(r)} \right) \quad \text{in } \Omega_{h, r_0}.
\end{equation}

By definition (3.5) of $w_h$, we have
\begin{equation}
w_h = -\frac{r}{2} \Phi' \left( \frac{x_3}{\psi} \right) \frac{1}{\psi} e_r + \Phi \left( \frac{x_3}{\psi} \right) e_3 - \frac{r}{2} \Phi' \left( \frac{x_3}{\psi} \right) \frac{r x_3 \partial_r \psi}{\psi^2} e_3 \quad \text{in } \Omega_{h, r_0}.
\end{equation}

Further, $x_3 \leq \psi$ in $\Omega_{h, r_0}$. Hence,
\begin{equation}
|\Phi| + |\Phi'| + |\Phi''| \lesssim 1,
\end{equation}
leading to
\begin{equation}
|w_h| \lesssim 1 + \frac{r}{\psi}(1 + \partial_r \psi).
\end{equation}

Due to legibility, we will not write the argument of $\Phi$ in the sequel. Similarly, we obtain
\begin{align*}
|\partial_r w_h| &\lesssim \Phi' \frac{1}{\psi} + \frac{r}{2} \Phi'' \frac{x_3 \partial_r \psi}{\psi^2} - \frac{r}{2} \Phi' \frac{\partial_r \psi}{\psi^2} \\
&\quad + \Phi' \frac{x_3 \partial_r \psi}{\psi^2} + \frac{r}{2} \Phi'' \left( \frac{x_3 \partial_r \psi}{\psi^2} \right)^2 + \frac{r}{2} \Phi' \frac{x_3 \partial_r \psi}{\psi^2} + r \Phi' \frac{x_3 (\partial_r \psi)^2}{\psi^3},
\end{align*}
\begin{align*}
|\partial_3 w_h| &\lesssim \frac{r}{2} \Phi' \frac{x_3}{\psi^3} + \frac{r}{2} \Phi'' \frac{1}{\psi^2} + \frac{r}{2} \Phi'' \frac{x_3 \partial_r \psi}{\psi^2} + \frac{r}{2} \Phi' \frac{\partial_r \psi}{\psi^2},
\end{align*}
\begin{align*}
|\partial_h w_h| &\lesssim \frac{r}{2} \Phi'' \frac{x_3}{\psi^3} + \frac{r}{2} \Phi' \frac{1}{\psi^2} + \Phi' \frac{x_3}{\psi^2} + \frac{r}{2} \Phi' \frac{x_3 \partial_r \psi}{\psi^2} + \Phi' \frac{x_3 \partial_r \psi}{\psi^2}.
\end{align*}

Using again $x_3 \leq \psi$ and the bounds (3.7), we have
\begin{align*}
|\nabla w_h| &\lesssim |\partial_r w_h| + |\partial_3 w_h| + \frac{|w_h \cdot e_r|}{r} \approx \frac{1}{\psi} + \frac{r}{\psi^2} + \frac{r \partial_r \psi}{\psi^2} + \partial_r \psi + \frac{r (\partial_r \psi)^2}{\psi^2} + \frac{r \partial_r^2 \psi}{\psi},
\end{align*}
\begin{align*}
|\partial_h w_h| &\lesssim \frac{1}{\psi} + \frac{r}{\psi^2} + \frac{r \partial_r \psi}{\psi^2}.
\end{align*}

Note that these bounds hold independently of the specific form of $\psi$. In our setting, $\psi(r) = h + r^{1+\alpha}$. Thus, the proof of the remaining estimates on $w_h$, $\nabla w_h$ and $\partial_h w_h$ are based on the following result, which can be proven analogously to [15, Lemma 13]: we have
\begin{equation}
\int_0^{r_0} \frac{r^p}{(h + r^{1+\alpha})^q} \, dr \lesssim 1 \quad \forall (\alpha, p, q) \in (0, \infty)^3 \quad \text{satisfying } \quad p + 1 > q(1 + \alpha).
\end{equation}
Using the estimate (3.8), we get
\[
\int_{\Omega_{h,r_0}} |\nabla w_h|^p \, dx \lesssim 1 + \int_0^{r_0} \int_0^r \frac{\psi r^{p+1}}{\psi^p} + \frac{r(p+1)\psi^{p+1}}{\psi^p} \, dx_3 \, dr \\
\lesssim 1 + \int_0^{r_0} \frac{\psi r^{p+1}}{\psi^p} + \frac{r(p+1)\psi^{p+1}}{\psi^p} \, dr \lesssim 1 \\
\iff p + 2 > (p - 1)(1 + \alpha) \quad \text{and} \quad p(1 + \alpha) + 2 > (p - 1)(1 + \alpha) \\
\iff \alpha(p - 1) < 3.
\]

Using the estimates \( r\partial_r \psi \lesssim \psi \) and \( r\partial_r^2 \psi \lesssim \partial_r \psi \), we have
\[
|\nabla w_h| \lesssim |\partial_r w_h| + |\partial_3 w_h| + \left| \frac{w_{h,r}}{r} \right| \lesssim \frac{1}{\psi} + \frac{r}{\psi^2} + \frac{\partial_r \psi}{\psi},
\]
\[
|\partial_{h} w_h| \lesssim \frac{1}{\psi} + \frac{r}{\psi^2}.
\]

In particular, it is enough to get estimates for \( \nabla w_h \), since the most restrictive term is \( r/\psi^2 \). Hence, we obtain
\[
\int_{\Omega_{h,r_0}} |\nabla w_h|^p \, dx \lesssim \int_0^{r_0} \int_0^r \frac{\psi r^{p+1}}{\psi^p} + \frac{r(p+1)\psi^{p+1}}{\psi^p} \, dx_3 \, dr \\
\lesssim \int_0^{r_0} \frac{\psi r^{p+1}}{\psi^p} + \frac{r(p+1)\psi^{p+1}}{\psi^p} \, dr \lesssim 1 \\
\iff 2 > (p - 1)(1 + \alpha) \quad \text{and} \quad p + 2 > (2p - 1)(1 + \alpha) \quad \text{and} \quad \alpha p + 2 > (p - 1)(1 + \alpha) \\
\iff p < \frac{3 + \alpha}{1 + 2\alpha}.
\]

\[\square\]

**Remark 3.2.** Let us remark that the specific form of the function \( \Phi \) is due to the following observation (see [9, Section 3.1]): searching for a minimizer of the energy functional \( \int_{\mathcal{F}_h} |\nabla u|^2 \, dx \) in the class
\[
\{ u \in W^{1,2}_{\text{loc}}(\mathcal{F}_h) : u = \nabla \times (\phi e_3) = -\partial_3 \phi e_r + \frac{1}{r} \partial_r (r \phi) e_3 \},
\]
where \( \phi \) satisfies the boundary conditions
\[
\partial_3 \phi |_{\partial S_h} = \partial_3 \phi |_{\partial \Omega} = 0, \quad \partial_r (r \phi) |_{\partial S_h} = r, \quad \phi |_{\partial \Omega} = 0,
\]
and anticipating that most of the energy comes from the \( x_3 \)-derivative as \( h \to 0 \), one ends up with a relaxed problem of searching a minimizer to
\[
\mathcal{E}_h = \int_{\{0 < r < r_0, \ 0 < x_3 < \psi(r)\}} |\partial_3 u_r|^2 \, dx = \int_{\{0 < r < r_0, \ 0 < x_3 < \psi(r)\}} |\partial_3^2 \phi|^2 \, dx
\]
in the class
\[
\{ u \in W^{1,2}(\{0 < r < r_0, \ 0 < x_3 < \psi(r)\}) : u = -\partial_3 \phi e_r + \frac{1}{r} \partial_r (r \phi) e_3 \},
\]
supplemented with boundary conditions
\[
\partial_3 \phi(r, \psi(r)) = 0, \quad \phi(r, \psi(r)) = \frac{r}{2},
\]
\[
\partial_3 \phi(r, 0) = 0, \quad \phi(r, 0) = 0.
\]
According to the Euler–Lagrange equation $\partial^2_{xx}\phi = 0$, one easily finds that the unique minimizer of $\mathcal{E}_h$ is given by

$$\phi_{\text{min}}(r, x_3) = \frac{r}{2} \Phi \left(\frac{x_3}{\psi(r)}\right), \quad \Phi(t) = t^2(3-2t), \quad t \in [0, 1].$$

### 3.2. Estimates near the collision – Proof of Theorem 1.3

Let $0 < T < T_\ast$ and let $\zeta \in C^1_c([0, T))$ with $0 \leq \zeta \leq 1$, $\zeta' \leq 0$, and $\zeta = 1$ near $t = 0$. (For instance, the properly extended function $\zeta_k(t) = \zeta(kt - (k - 1)T)$ for some $k \geq 1$ and $\zeta(t) = \exp[T^{-2} - (T^2 - t^2)^{-1}]$ will do.) We take $\zeta(t)w_{h(t)}$ as test function in the weak formulation of the momentum equation (1.4) with the source term $f = -ge_3$, $g > 0$.

Recalling $\text{div}w_h = 0$ and $\partial_t w_h(t) = \dot{h}(t)\partial_h w_{h(t)}$, we have the identity

$$\int_0^T \zeta \int_\Omega \rho \mathbf{u} \otimes \mathbf{u} : \mathbb{D}(w_h) \, dx \, dt + \int_0^T \zeta' \int_\Omega \rho \mathbf{u} \cdot w_h \, dx \, dt$$

$$+ \int_0^T \zeta \dot{h} \int_\Omega \rho \mathbf{u} \cdot \partial_h w_h \, dx \, dt - \int_0^T \zeta \int_\Omega \mathbf{S}(\mathbf{u}) : \mathbb{D}(w_h) \, dx \, dt$$

$$= \int_0^T \zeta \int_\Omega \rho g e_3 \cdot w_h \, dx \, dt - \int_\Omega \mathbf{q}_0 \cdot w_h \, dx$$

$$= \int_0^T \zeta \int_{S_h} \rho g e_3 \cdot w_h \, dx \, dt + \int_0^T \zeta \int_{\mathcal{F}_h} \rho g e_3 \cdot w_h \, dx \, dt - \int_\Omega \mathbf{q}_0 \cdot w_h \, dx.$$

Observe that we have $w_h = e_3$ on $S_h$, so for a sequence $\zeta_k \to 1$ in $L^1([0, T))$,

$$\int_0^T \zeta_k \int_{S_h} \rho g e_3 \cdot w_h \, dx \, dt = \int_0^T \zeta_k \int_{S_h} \rho s g \to mgT.$$

In particular, for a proper choice of $\zeta$, it follows that

$$\frac{1}{2} mgT \leq \int_0^T \zeta \int_\Omega \rho \mathbf{u} \otimes \mathbf{u} : \mathbb{D}(w_h) \, dx \, dt + \int_0^T \zeta' \int_\Omega \rho \mathbf{u} \cdot w_h \, dx \, dt + \int_0^T \zeta \dot{h} \int_\Omega \rho \mathbf{u} \cdot \partial_h w_h \, dx \, dt$$

$$- \int_0^T \zeta \int_\Omega \mathbf{S}(\mathbf{u}) : \mathbb{D}(w_h) \, dx \, dt - \int_0^T \zeta \int_{\mathcal{F}_h} \rho g e_3 \cdot w_h \, dx \, dt + \int_\Omega \mathbf{q}_0 \cdot w_h \, dx = \sum_{j=1}^{6} I_j.$$

We will estimate each $I_j$ separately, and set our focus on the explicit dependence on $T$ and $m$. For the latter purpose, we split each density dependent integral into its fluid and solid part $I_j^f$ and $I_j^s$, respectively. To get a lean notation, we will use the symbol $a \lesssim b$ whenever there is a constant $C > 0$ such that $a \leq Cb$, where $C$ does not depend on $a$, $b$, $E_0$, $L$, and $T$.

- For $I_j^f$, we have by $\zeta' \leq 0$, $\zeta(T) = 0$, and $\zeta(0) = 1$

$$|I_j^f| \leq -\int_0^T \zeta' \int_{\mathcal{F}_h} \rho |\mathbf{u}| \||w_h| \, dx \, dt = -\int_0^T \zeta' \int_{\mathcal{F}_h} \sqrt{\rho} \sqrt{\rho} |\mathbf{u}| \||w_h| \, dx \, dt$$

$$\leq -\int_0^T \zeta' \||\sqrt{\rho}||L_2(\mathcal{F}_h)||\sqrt{\rho}||L_2(\mathcal{F}_h)||w_h||_{L_2(\mathcal{F}_h)} \, dt$$

$$\leq \||\rho||_{L_\infty(0, T; L_2(\mathcal{F}_h))}||\rho||_{L_\infty(0, T; L_1(\Omega))} \||w_h||_{L_\infty(0, T; L_2(\mathcal{F}_h))} \zeta(0) \lesssim (E_0 + L)^{\frac{1}{2}} + \frac{1}{2},$$
where we have used the estimate (2.9) and Lemma 3.1 under the condition
\[ \frac{2\gamma}{\gamma - 1} < 1 + \frac{3}{\alpha} \iff \alpha < \frac{3\gamma - 3}{\gamma + 1}. \]

- For \( I_2^f \), notice that \( \mathbf{w}_h|_{S_h} = \mathbf{e}_3 \), \( \rho|_{S_h} = \rho_s \), and \( \mathbf{u}|_{S_h} = \mathbf{h}\mathbf{e}_3 + \omega(t) \times (x - \mathbf{h}\mathbf{e}_3) \). Further, as before, \( \frac{1}{m} \int_{S_h} \rho_s x \, dx = \mathbf{G}_h = \mathbf{h}\mathbf{e}_3 = \frac{1}{m} \int_{S_h} \rho_s \mathbf{h}\mathbf{e}_3 \, dx \), and
\[ \int S_h \frac{1}{2} \rho |\mathbf{u}|^2 \, dx = \int S_h \frac{1}{2} \rho_s (|\mathbf{h}\mathbf{e}_3|^2 + |\omega \times (x - \mathbf{h}\mathbf{e}_3)|)^2 \, dx = \frac{1}{2} m |\mathbf{h}|^2 + \frac{1}{2} \omega \cdot \omega. \]
Hence, we infer from the energy inequality (2.3) that
\[ \sup_{t \in (0,T)} |\mathbf{h}| = (\sup_{t \in (0,T)} |\mathbf{h}|^2)^{\frac{1}{2}} \leq \sqrt{\frac{2}{m} (E_0 + L)^{\frac{1}{2}}}. \]
Thus,
\[ |I_2^f| = \left| \int_0^T \zeta' \int_{S_h} \rho_s \mathbf{e}_3 \cdot (\mathbf{h}\mathbf{e}_3 + \omega \times (x - \mathbf{h}\mathbf{e}_3)) \, dx \, dt \right| \]
\[ = \left| \int_0^T \zeta' \mathbf{h} \mathbf{m} \, dt \right| \leq m \sup_{t \in (0,T)} |\mathbf{h}| \lesssim \sqrt{m} (E_0 + L)^{\frac{1}{2}}. \]

- For \( I_3 \), observe that \( I_3 \) is 0 due to \( \partial_h \mathbf{w}_h|_{S_h} = \partial_h \mathbf{e}_3 = 0 \). Thus,
\[ |I_3| = |I_3^f| \leq \int_0^T \zeta |\mathbf{h}(t)| \| \rho \|_{L^\infty(0,T;L^\gamma(f(\cdot)))} \| \rho |\mathbf{u}|^2 \|_{L^\infty(0,T;L^1(\Omega))} \| \partial_h \mathbf{w}_h \|_{L^{\frac{2\gamma}{3\gamma - 3}}(f(\cdot))} \, dt \]
\[ \lesssim (E_0 + L)^{\frac{\gamma - \frac{1}{2}}{\gamma - 1}} \| \mathbf{h} \|_{L^\infty(0,T)} \| \zeta \|_{L^1(0,T)} \lesssim \sqrt{\frac{1}{m} (E_0 + L)^{\frac{1}{2}}} T^{\frac{1}{2}}, \]
where we have used the estimates (2.9), (3.11), and Lemma 3.1 under the condition
\[ \frac{2\gamma}{\gamma - 1} < 1 + \frac{3}{2\alpha} \iff \alpha < \frac{\gamma - 3}{\gamma + 1}. \]

- Regarding \( I_4 \), by using the fact that \( \text{div} \mathbf{w}_h = 0 \), we have
\[ \text{S(} \mathbf{u} \text{)} : \mathbb{D} (\mathbf{w}_h) = 2\mu \mathbb{D} (\mathbf{u}) : \mathbb{D} (\mathbf{w}_h) + \lambda \text{div} \mathbf{u} \mathbb{I} : \mathbb{D} (\mathbf{w}_h) = 2\mu \mathbb{D} (\mathbf{u}) : \mathbb{D} (\mathbf{w}_h) + \lambda \text{div} \mathbf{u} \text{div} \mathbf{w}_h \]
\[ = 2\mu \mathbb{D} (\mathbf{u}) : \mathbb{D} (\mathbf{w}_h). \]
Hence, using the bounds on \( \mathbb{D} (\mathbf{u}) \) already obtained in (2.5), we calculate
\[ |I_4| \lesssim \int_0^T \zeta \| \mathbb{D} (\mathbf{u}) \|_{L^2(\Omega)} \| \nabla \mathbf{w}_h \|_{L^2(\Omega)} \, dt \leq \| \zeta \|_{L^2(0,T)} \| \mathbb{D} (\mathbf{u}) \|_{L^2((0,T) \times \Omega)} \| \nabla \mathbf{w}_h \|_{L^\infty(0,T;L^2(\Omega))} \]
\[ \lesssim (E_0 + L)^{\frac{1}{2}} T^{\frac{1}{2}}, \]
where we have used Lemma 3.1 under the condition
\[ 2 < \frac{3 + \alpha}{1 + 2\alpha} \iff \alpha < \frac{1}{3}. \]

- For \( I_5 = I_5^f \),
\[ |I_5| \leq g \int_0^T \zeta \| \rho \|_{L^\gamma(f_\rho)} \| \mathbf{w}_h \|_{L^{\frac{2\gamma}{3\gamma - 3}}(\Omega)} \leq g \| \zeta \|_{L^1(0,T)} \| \rho \|_{L^\infty(0,T;L^\gamma(f(\cdot)))} \| \mathbf{w}_h \|_{L^\infty(0,T;L^{\frac{2\gamma}{3\gamma - 3}}(\Omega))} \]
\[ \leq g(E_0 + L)^{\frac{1}{2}} T, \]

by using Lemma 3.1 under the condition
\[ \frac{\gamma}{\gamma - 1} < 1 + \frac{3}{\alpha} \iff \alpha < 3 - \frac{3}{\gamma}. \]

- Similar to \( I_2^f \), we have for \( I_6^f \) the estimate
\[ |I_6^f| \leq \|q_0\|_{L^\infty(0,T;L^{\frac{2\gamma}{2\gamma-3}}(\Omega))} w_h \|_{L^\infty(0,T;L^{\frac{2\gamma}{2\gamma-3}}(\Omega))} \lesssim \left\| \frac{q_0}{\rho_0} \right\|_{L^1(\Omega)} \rho_0 \|_{L^\gamma(\Omega)} \lesssim (E_0 + L)^{\frac{1}{2}} + \frac{1}{\gamma}. \]

- For \( I_6^g \), where \( w_h = e_3 \) and \( q_0 = \rho(0)u(0) = \rho_s(h e_3 + \omega \times (x - h e_3)) \), we have similarly to \( I_2^g \) that
\[ |I_6^g| = \left| \int_{S(0)} q_0 \cdot e_3 \, dx \right| = \left| \int_{S(0)} \rho_s \dot{h} \, dx \right| \leq m \| h \|_{L^\infty(0,T)} \lesssim \sqrt{m}(E_0 + L)^{\frac{1}{2}}. \]

- Let us turn to \( I_1 \). First, by Sobolev embedding and (2.9),
\[ \|u\|_{L^2((0,T),L^6(\Omega))} \lesssim \|\nabla u\|_{L^2((0,T),L^2(\Omega))} \lesssim E_0 + L. \]

Next, due to \( w_h|_{S_h} = e_3 \), we see that \( I_1^f = 0 \) since \( \mathbb{D}(w_h) = 0 \) there. Hence, we calculate
\[ |I_1| = |I_1^f| \lesssim \int_0^T \zeta \| \rho \|_{L^\gamma(\mathcal{F}_h)} \|u\|_{L^6(\Omega)} \|\nabla w_h\|_{L^2(\Omega)} \]
\[ \lesssim \| \rho \|_{L^\infty(0,T;L^\gamma(\mathcal{F}_h))} \|\nabla w_h\|_{L^\infty(0,T;L^{\frac{2\gamma}{2\gamma-3}}(\Omega))} \int_0^T \zeta \|\nabla u\|_{L^2(\Omega)} \]
\[ \lesssim (E_0 + L)^{\frac{1}{2}} \| \zeta \|_{L^\infty(0,T)} \|\nabla u\|_{L^2((0,T) \times \Omega)} \lesssim (E_0 + L)^{\frac{1}{2}} + 1. \]

by using the estimate (2.9) and Lemma 3.1 under the condition
\[ \frac{3\gamma}{2\gamma - 3} < \frac{3 + \alpha}{1 + 2\alpha} \iff \alpha < \frac{3\gamma - 9}{4\gamma + 3}. \]

Note further that for any \( \gamma \geq 3 \),
\[ \frac{\gamma - 3}{3\gamma + 1} \leq \min \left\{ \frac{1}{3}, \frac{3\gamma - 3}{\gamma + 1}, \frac{3 - 3\gamma - 9}{\gamma}, \frac{3\gamma - 9}{4\gamma + 3} \right\}, \]

and that all estimates are independent of the choice of \( \zeta \). Hence, we can take a sequence \( \zeta_k \to 1 \) in \( L^\infty((0,T)) \) without changing the bounds obtained. In turn, collecting all estimates above, we finally arise at
\[ \frac{1}{2} m g T \leq C_0 \left( 1 + \sqrt{m} + \sqrt{m}^{-1} \right) \left( (E_0 + L)^{\frac{1}{2}} + (E_0 + L)^{\frac{1}{2}} + (E_0 + L)^{1 + \frac{1}{2}} \right) \]
\[ + g(E_0 + L)^{\frac{1}{4}} + (E_0 + L)^{1 + \frac{1}{4}} \right) \left( 1 + T^{\frac{1}{2}} + T \right), \]

which after dividing by \( \frac{1}{2} m \) and using Young’s inequality on several terms, leads to
\[ g T \leq C_0 (m^{-1} + m^{-\frac{1}{2}} + m^{-\frac{3}{4}}) \left( 1 + (E_0 + L)^{1 + \frac{1}{2}} + g(E_0 + L)^{\frac{1}{2}} \right) (1 + T), \]
where $C_0$ only depends on $\gamma, \alpha, \mu$, the bounds on $w_h$ obtained in Lemma 3.1, and the Sobolev and Korn constant of $\Omega$, provided
\[
\gamma > 3 \quad \text{and} \quad \alpha < \frac{\gamma - 3}{3\gamma + 1}.
\]

Recalling the definitions of $E_0$ from (2.4) and $L$ from (2.8) as
\[
E_0 = \int_{\mathcal{F}(0)} \left( \frac{1}{2} |\mathbf{q}_0|^2 + \frac{\rho_0}{\gamma - 1} \right) \, dx + \frac{m}{2} |V_0|^2 + \frac{1}{2} \mathbb{J}_0 \omega_0 \cdot \omega_0,
\]
\[
\mathbb{J}_0 = \int_{\mathcal{S}(0)} \rho_0 \left( |x - \mathbf{G}_0|^2 I - (x - \mathbf{G}_0) \otimes (x - \mathbf{G}_0) \right) \, dx,
\]
\[
L = C(\gamma) g^{\frac{2}{\gamma - 1}} (\text{diam } \Omega)^{\frac{2}{\gamma - 1} + 3},
\]

we see that collision can occur only if the solid’s mass in (3.12) is large enough, meaning in fact it’s density is very high. Since $E_0$ depends on solid’s mass and, through $\mathbb{J}_0$, on solid’s density, we require the solid initially to have low vertical and rotational speed. More precisely, choosing $V_0$ and $\omega_0$ such that $|V_0|, |\omega_0| = O(m^{-\frac{1}{2}})$, and choosing $m$ high enough such that
\[
C_0 (m^{-1} + m^{-\frac{1}{4}} + m^{-\frac{3}{4}}) \left( 1 + (E_0 + L)^{1+\frac{4}{\gamma}} + g(E_0 + L)^{\frac{2}{\gamma}} \right) < g,
\]
the solid touches the boundary of $\Omega$ in finite time, ending the proof of Theorem 1.3.

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