Scaling, asymptotic scaling and Symanzik improvement. Deconfinement temperature in SU(2) pure gauge theory.

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Abstract

We report on a high statistics simulation of SU(2) pure gauge field theory at finite temperature, using Symanzik action. We determine the critical coupling for the deconfinement phase transition on lattices up to $8 \times 24$, using Finite Size Scaling techniques. We find that the pattern of asymptotic scaling violation is essentially the same as the one observed with conventional, not improved action. On the other hand, the use of effective couplings defined in terms of plaquette expectation values shows a precocious scaling, with respect to an analogous analysis of data obtained by the use of Wilson action, which we interpret as an effect of improvement.

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I. INTRODUCTION

In the last few years the increased computer power available for lattice theorists has allowed a number of studies on relatively large lattices and it has been possible to test accurately scaling and asymptotic scaling.

We have in mind in particular a series of works on $SU(2)$ lattice gauge theory at finite temperature ([1,2] and references therein) which have been able to test for asymptotic scaling on lattices up to $N_t = 16$, showing still considerable scaling violations.

On the other hand, in the same simulations one observes a rather good scaling of ratios of dimensionful quantities, thus supporting the idea that asymptotic scaling is violated by universal terms, which cancel in the ratio, and that some $\beta$ function, other than the perturbative one, exists.

Since the Renormalization Group, parameterized in terms of the “bare” coupling on the lattice, shows large deviations from the asymptotic behavior, we feel that it is very important to test different renormalization schemes, as stressed in a recent work by Lepage et al. [3]. For instance, it has become more and more apparent that effective schemes on one hand improve the asymptotic scaling [4,5], and on the other hand reconcile the perturbative expansion of ultraviolet dominated quantities with the lattice numerical results [3].

In this context, we have decided to test the effect of the use of a different lattice formulation, using an improved Symanzik action in the simulation of 4 dimensional lattice gauge theory. We did not expect a dramatic improvement of asymptotic scaling, even if in a pioneering work [6] some indications of such an improvement were found, but we have considered that by comparison with similar works using Wilson action something about the origin of scaling violations could be understood.

The study of this particular model, $SU(2)$ gauge theory at finite temperature, is justified by its similarity with QCD, as well as by its greater simplicity. Moreover, the particular observable chosen, the temperature of deconfining phase transition, is well defined and can be exactly determined by use of Finite Size Scaling (FSS) techniques, thus allowing us to pursue our main objective, the study of scaling properties.

We can anticipate our results: the pattern of asymptotic scaling violation observed with Symanzik action is essentially the same as the one observed with the Wilson formulation, the only apparent effect is a precocious transition from the strong to the weak coupling regime. This result seems to indicate that a universal lattice $\beta$ function exists.

Analogously the effective coupling approach, formulated in terms of plaquettes expectation values, works well in improving the asymptotic scaling figure and gives results consistent for the two lattice formulations.

In section II we give a short review of the Symanzik approach to the lattice formulation of field theories and we recall physical characteristics of the model under study.

In section III we review the FSS techniques used in this work, in order to give a presentation as self contained as possible.

In section IV we give details about our simulations.

In section V we present the results of our measurements, exploiting the FSS techniques to present various tests of consistency of the results.

We discuss the results in section VI, together with an analysis of the different renormalization schemes.
Appendix A is devoted to technical details on the subtraction of biases and on error estimation. Analogously in Appendix B we discuss our implementation of the Density of States Method (DSM).

II. THE SYMANZIK IMPROVEMENT AND $SU(2)$ LATTICE GAUGE THEORY

1. Generalities

The formulation of a field theory on a lattice is a necessary step if we want to study it in a non perturbative way with a computer simulation. The lattice can be seen as an ultraviolet regulator as any other, introducing in a natural way a short distance cut-off.

We can extract from a computer simulation relations between observables (for example, mass ratios) only for this regularized theory, and obviously we cannot recover completely the continuum limit, so our predictions will be affected by systematic errors (the so-called lattice artifacts).

The cut-off dependence of a theory can be reduced by a clever choice of the regularization scheme: this is a well known technical point which was studied extensively by K. Symanzik [7,8]. He realized that this fact can be used to minimize the consequences of a non-zero lattice spacing $a$ (for an introductory discussion see the pedagogical work of Parisi [9]).

The key observation is that a field theory can be transcribed on a lattice with considerable arbitrariness: the lattice action must only reduce to the continuum one as $a \to 0$. So we can redefine the action by adding an arbitrary combination of irrelevant operators, which vanish in this limit, and this is equivalent to change the regularization scheme. Symanzik has also shown (see for instance [10]) that every lattice regulated theory is perturbatively equivalent to a local effective lagrangian, in a given renormalization scheme, of the type

$$\mathcal{L}_{\text{eff}} = \sum_i c_i^{(0)} O_i^{(0)} + a^2 \sum_i c_i^{(2)} O_i^{(2)} + a^4 \sum_i c_i^{(4)} O_i^{(4)} + \cdots, \quad (1)$$

where $O_i^{(n)}$ are local operators of dimension $d+n$, if $d$ is the dimension of space. It follows that it is possible to build a lattice action in such a way that the corresponding effective lagrangian is

$$\mathcal{L}_{\text{eff},\text{impr}} = \mathcal{L}_{\text{cont}} + a^{2p} \sum_i c_i^{(2p)} O_i^{(2p)} + \cdots, \quad (2)$$

using an appropriate linear combination of the operators $O_i^{(q<2p)}$ transcribed in an arbitrary way on the lattice. The determination of the coefficients for this combination is a non trivial task for an interacting theory: in principle it could be done with a high precision numerical simulation.

We can also compute them perturbatively, if the bare coupling constant is sufficiently small (i.e. near the continuum limit for an asymptotically free theory), using a matching procedure between vertex functions [11], or in the case of gauge theories (for instance)
matching the expectation value of gauge invariant quantities, like the Wilson loops [11] or the inter-quark potential [12]. A exposition of the different improvement strategies can be found in the work of Lüscher et al. [13].

If this program works, we may hope to obtain more accurate predictions for physical quantities with the not extremely big lattices we can use with the current computing resources. The more direct way to verify that lattice artifacts are reduced is to measure an adimensional ratio of physical quantities while making a smaller and smaller. For example using the action in Eq. (2) we expect to find for a mass ratio ($\xi =$ correlation length)

$$ \frac{m_a}{m_b} = K \left( 1 + O \left( \frac{a}{\xi} \right)^{2p+2} \ln \left( \frac{a}{\xi} \right) \right). $$

(3)

A more refined test consists in verifying how well the quantities $M_i$ we can measure on lattice obey the Renormalization Group equation, which is merely a statement of cut-off independence of physical observables (scaling). The lattice artifacts modify this equation adding non-universal (i.e. $M$-dependent) scaling-violating terms, so we have

$$ \left( -a \frac{\partial}{\partial a} + \left[ \bar{\beta}(g) + \bar{\beta}_M(g) \right] \frac{\partial}{\partial g} \right) M_i = \Lambda_M. $$

(4)

Here $\bar{\beta}_M$ is a non analytic contribution that cannot be calculated perturbatively (for instance, in the bidimensional $\sigma$ model it can be evaluated in the large N limit, where one finds $\bar{\beta}_M = O \left( g^{k} \right)$ and $\Lambda_M = O \left[ (a/\xi)^2 \ln (a/\xi) \right]$. If we use the action (2) with $p = 2$, we expect first of all $\Lambda_M = O \left[ (a/\xi)^4 \ln (a/\xi) \right]$, and then a reduced $\bar{\beta}_M$. In the real case we can calculate the improved action at best to the first perturbative order, and we expect contributions $O \left[ g^4 (a/\xi)^2 \ln (a/\xi) \right]$ to $\Lambda_M$, which give a little contribution for an asymptotically free theory if we are sufficiently near the continuum.

2. Scaling vs. asymptotic scaling.

In the general case we know only the first few perturbative terms of the function $\bar{\beta}(g)$, in particular the first two (scheme independent) terms. These are the relevant terms in the continuum limit if Asymptotic Freedom holds, while in an intermediate coupling range higher order contributions can be important. If an observable $M$ follows the renormalization group with the perturbative approximation to $\bar{\beta}(g)$ we say we are in an asymptotic scaling region: if this is not true, it does not imply that the data cannot be trusted, but only that we had to use a better approximation to the exact beta function. We can try to extract this “improved” $\bar{\beta}(g)$ from the simulation itself and then verify the self consistency of this procedure. If we can find a function $\bar{\beta}_{\text{eff}}(g)$ so that the RG evolution holds for different quantities we say we are in the scaling regime.

It is important to note that

- The Symanzik program can improve the scaling, but not necessarily the asymptotic scaling: indeed, the onset of asymptotic scaling is expected in a region where irrelevant operators give negligible contribution.
• The scaling test is more conclusive than the stabilization of mass ratios, as we can imagine that there are some $\mathcal{M}$ independent contributions to $\Lambda_{\mathcal{M}}$ which cancel in Eq. (3).

As it is well known asymptotic scaling in the $SU(2)$ gauge theory with the Wilson action has not yet been found (see for instance [1]). So we must extrapolate the beta function in some physically sensible way in the intermediate coupling region.

3. The model under study

The $SU(2)$ gauge theory at finite temperature has been discussed at length for instance in [14]: we here just recall that the model shares with QCD the presence of a deconfining phase transition, characterized by the breaking of the center $Z(2)$ symmetry [15] and by the appearance of a non zero expectation value for the thermal Wilson loop, or Polyakov line, $P$. It is by now well established that the transition is of the second order and all numerical simulations give results in good agreement with the early ansatz [14] that the model belongs to the same universality class as the 3D Ising model.

In our simulations we have used a “tree improved” action, i.e. we have only corrected in part lattice artifacts in the classical theory.

This can be done by observing that the usual Wilson action is equivalent to the effective lagrangian

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{1}{12} a^2 \partial_{\mu} F_{\mu\nu} \partial_{\mu} F_{\mu\nu} + O(a^4).$$

(5)

We can compensate for the $O(a^2)$ term by adding a suitable irrelevant operator. A possible and widely used choice is

$$S_I = S_W + S_{\text{irr}} = \beta \sum U_{1\times1} + \beta \sum \left( \frac{2}{3} U_{1\times1} - \frac{1}{12} U_{1\times2} \right) = \beta \sum \left( \frac{5}{3} U_{1\times1} - \frac{1}{12} U_{1\times2} \right),$$

(6)

where $U_{n\times m}$ is the $n \times m$ plaquette.

This choice has already been experimented in other works, both in the study of the finite temperature theory [6] and of the quark-antiquark potential [17], using the ichosae-dral approximation to the $SU(2)$ group. To our knowledge the present work is the first where a 4-dimensional gauge theory is simulated with the Symanzik action and the full group.

III. FINITE SIZE SCALING ANALYSIS

The Finite Size Scaling (FSS) technique is by now a widely used tool in the investigation of pseudo-critical properties of statistical systems in finite volume. Its application to the analysis of second order phase transitions permitted a detailed numerical test of theoretical predictions on critical exponents. The effectiveness in the study of $SU(2)$ deconfining phase transition has been already demonstrated in a number of papers, where use of the Wilson’s action was made [12][18].
In this section we collect some formulas needed by our investigation, in order to
give a presentation as self contained as possible of our results. A more detailed general
introduction can be found for instance in [19], while we refer to the original works [1,2]
for the improvements of the method which we have applied to our analysis.

Numerical simulations of statistical systems are limited to finite volumes, characterized by some length scale \( L \). In the vicinity of an (infinite volume) phase transition
the system exhibits a “pseudo-critical” behavior, for instance at a second order phase
transition the susceptibility \( \chi \) shows a peak broadened by the finite volume, while the
correlation length \( \xi \), defined in terms of appropriate pair correlation functions, reaches
the dimensions of the system. For our purposes, the shift in the critical temperature is
the most important effect, which can be easily estimated if one assumes some universality
class for the model under consideration and that the correlation length exhibits a power
behavior in the vicinity of the pseudo-transition.

\[
\xi \propto |t|^{-\nu}
\]  

(7)

where \( t = \frac{T - T_c}{T_c} \) is the reduced temperature ( in units of the infinite volume critical
temperature ). If the finite volume “critical” temperature, \( T_c (L) \), is reached when the
correlation length is of the order of the physical size of the system, it follows that the
expected shift is

\[
|T_c (L) - T_c| \propto T_c L^{-\frac{\nu}{\rho}}.
\]  

(8)

Note that in fitting measurements the critical temperature is a very sensitive parameter,
thus the error in its determination should be reduced as much as possible if one is
interested in determining universal parameters with a good accuracy.

To extract infinite volume limits it is necessary an “ansatz” on the form of the observables.

The main idea is that near a second-order phase transition, as the correlation length \( \xi \) approaches the size \( L \) of the system, the observables \( O \) depend on this two scales only
through the ratio \( r = L/\xi \), apart from a volume dependent prefactor

\[
O_L = L^\omega \bar{Q}_0 (r)
\]  

(9)
or, in terms of the reduced temperature,

\[
O_L = L^\omega \bar{Q}_0 (Lt^{1/\nu}) .
\]  

(10)

If for large \( L \) this observable exhibits a critical behavior, with exponent \(-\rho\), the function \( \bar{Q}_O \) should behave like some power of its argument, and the requirement of
cancellation of volume dependence requires that \( \omega = \rho/\nu \). Hence it results the following
parameterization

\[
O_L = L^{\rho/\nu} \bar{Q}_L (Lt^{1/\nu}) ,
\]  

(11)
or alternatively, and more commonly

\[
O_L = L^{\rho/\nu} Q_L \left( tL^{1/\nu} \right) .
\]  

(12)
The important point is that renormalization group arguments show that all the critical behavior can be derived from the singular part of the free energy density, which is then assumed to have the conventional form

\[ f_s = L^{-d} Q \left( g_T L^{1/\nu}, g_h L^{(\eta+\gamma)/\nu}, g_i L_i^\nu \right) \]  

(13)

where \( g_T, g_h \) are connected to the reduced temperature and the external magnetic field by linear relations, plus corrections

\[ g_t = c_t t + O \left( t^2 \right) \]

\[ g_h = c_h h + O \left( h^2 \right) . \]  

(14)

The additional dependence on irrelevant scaling fields \( g_i \), with negative exponents \( y_i \), cannot be neglected in many practical simulations, and determines corrections to the scaling behavior. Note also that we name \( \eta \) the critical exponent of the “magnetization”, instead of the conventional \( \beta \), to avoid confusion with the coupling in the latticized theory.

Let us come to our system, and recall that we simulate on a lattice of volume \( L^3 \), with \( L = N_\sigma a \), and temporal extent \( N_t \), determining a temperature \( T = 1 / (N_t a) \).

The relation between the spacing \( a \) and the bare coupling \( \beta = 2C_a b_0 \) is unknown, and one of the motivations of our simulation is the study of the validity of the universal two-loop asymptotic formula

\[ a \Lambda = \left( \frac{\beta}{2C_a b_0} \right)^{b_1/2b_0^2} \exp \left( -\frac{\beta}{4C_a b_0} \right) . \]  

(15)

As first noted in [1], it is convenient to rewrite the free energy density in terms of the dimensionless combination \( LT \), that is, of the ratio \( y = N_\sigma / N_T \).

\[ f_s (t, h; N_\sigma; N_T) = y^{-d} Q_f \left( g Ty^\nu; g_h y^{\eta+\gamma}; g_i y_i^\nu \right) . \]  

(16)

Define \( P \) the spatial average of the Polyakov line

\[ L (\vec{x}) = \prod_{t=1}^{N_t} U (\vec{x}, t) \]

\[ P = N_\sigma^{-d} \sum \text{Tr} [L (\vec{x})] \]  

(17)

and introduce a source \( h \) for this quantity in the partition function, through the operator \( hZ (a, N_T) N_\sigma^d P \), where the renormalization factor \( Z \) cancels the divergent self-energy contributions to the Polyakov loop.

Derivatives with respect to the source \( h \) allow to define the physical order parameter and the susceptibilities

\[ ^1\text{In a finite volume system, as tunnelling between inequivalent “vacua” tends to restore the } Z (2) \text{ symmetry, the experimental order parameter is defined as } |P| \]
\[ \langle P \rangle = - \left. \frac{\partial f_s}{\partial h} \right|_{h=0} = y^{-\eta/\nu} Q_P \left( g_t y^{1/2}, g_t y^{\eta/\nu} \right) \]

\[ \chi = \left. \frac{\partial^2 f_s}{\partial h^2} \right|_{h=0} = y^{\gamma/\nu} Q_\chi \left( g_t y^{1/2}, g_t y^{\eta/\nu} \right) \]

where use has been made of the first hyperscaling relation

\[ \gamma/\nu + 2\eta/\nu = d \]

implied by Eq. (18).

A useful quantity in the determination of the critical temperature is then the Binder cumulant \( g_4 \), defined as the fourth derivative of the free energy with respect to “magnetic field”, normalized to the susceptibility

\[ g_4 = \left. \frac{\partial^4 f_s}{\partial h^4} \right|_{h=0} \left( \chi^2 \frac{\langle N_\sigma/N_\tau \rangle^d}{d} \right) . \]

In fact, its expression is a directly scaling function, where explicit dependence on \( N_\sigma \) is canceled by the use of the hyperscaling relation: moreover, the multiplicative renormalization \( Z \) of the Polyakov loop cancels, and therefore the expression for \( g_4 \) can be safely given in terms of expectation values of moments of Polyakov loop

\[ g_4 = \frac{\langle P \rangle^4}{\langle P \rangle^2} - 3.0 . \]

Its directly scaling form can be expressed as

\[ f_s(t) \sim \xi(t)^d \]

which is incorporated in Eq. (16), if for small values of \( x \)

\[ Q_f (x, 0, \ldots) \sim (x)^{\nu d} . \]

Now derivatives in Eq. (18) can be written as

\[ \langle P \rangle = - \langle N_\sigma/N_\tau \rangle^{-d+(\eta+\gamma)/\nu} Q_1 \left( g_t (N_\sigma/N_\tau)^{1/\nu} \right) \]

\[ \chi = \langle N_\sigma/N_\tau \rangle^{-d+2(\eta+\gamma)/\nu} Q_2 \left( g_t (N_\sigma/N_\tau)^{1/\nu} \right) \]

and so, combining the dependence of \( Q_{1,2} \) on \( g_t \) (that is, the same as \( Q_f \), with the requirement that in the infinite volume \( N_\sigma \to \infty \) limit the critical behavior is specified by the exponents \( \eta, \gamma \)

\[ P \sim |t|^\eta \quad \chi \sim |t|^{-\gamma} , \]

it results Eq. (19).
\[ g_4(t; N_\sigma; N_\tau) = Q_g \left( g_t(t, N_\sigma) y^{1/\nu}, g_1 y^{y_1} \right) \]  

(22)

where we have conserved the first irrelevant scaling field. The localization of the critical point is then possible by finding the intersections of \( g_4 \) curves, as functions of the temperature \( t \), at various values of the spatial size \( N_\sigma \).

Taking into account the first irrelevant field one expands the expression given in Eq. (22) in the vicinity of the transition

\[ g_4(t; N_\sigma; N_\tau) \simeq g_{4,0} + g_{4,1} t y^{1/\nu} + g_{4,2} y_{y_1} . \]  

(23)

Hence the intersection of two \( g_4 \) curves at values of the spatial size \( N_\sigma, N'_\sigma \) corresponds to a shifted temperature \[ 2 \]

\[ t = \frac{g_{4,2} [(y')_{y_1} - y_{y_1}]}{g_{4,1} \left[ y^{1/\nu} - (y')^{1/\nu} \right]} \propto \left( \frac{N_\sigma}{N_\tau} \right)^{y_1-1/\nu} \frac{1 - b^{y_1}}{b^{1/\nu} - 1} \]  

(24)

where \( b = N'_\sigma/N_\sigma \).

As \( y_1 < 0 \), this shift in the crossing goes to zero as \( N_\sigma \) grows, while also at fixed \( N_\sigma \) as \( b \) grows the shift goes down, governed by the exponent \(-1/\nu\).

Actually, in the simulation of \( SU(N) \) gauge theories, the connection between the bare coupling \( \beta \) and the lattice spacing, which allows to determine the temperature in units of the reference scale \( \Lambda \), is known only in the asymptotic regime \( g_0 \rightarrow 0 \). As noted in [1], this means that expressing the reduced temperature \( t \) in terms of \( \beta \) through an approximate formula like

\[ t = (\beta - \beta_{c,\infty}) \frac{1}{4C_a b_0} \left[ 1 - \frac{2C_a b_1}{b_0} \beta_{c,\infty}^{-1} \right] \]  

(25)

introduces an error \( O \left( \beta_{c,\infty}^{-1} \right) \) in the determination of \( t \). This shift is of the order of 8% in the relevant coupling regime explored, but one should be aware that this estimate would be \textit{a posteriori} justified by the observation of an asymptotic scaling behavior, which is absent.

In our search for the determination of the critical temperature this shift is immaterial, as long as a relation of the form

\[ t \propto (\beta - \beta_{c,\infty}) \]  

(26)

is valid.

Therefore Eq. (24) translates in the form

\[ \beta_c(N_\sigma, N'_\sigma) = \beta_{c,\infty}(1 - c \varepsilon) \]  

(27)

where

\[ \varepsilon = N'_\sigma y_1^{y_1-1/\nu} \frac{1 - b^{y_1}}{b^{1/\nu} - 1} \quad b = \frac{N'_\sigma}{N_\sigma} \]  

(28)

and the constant \( c \) maintains a dependence on \( N_\tau \) and \( \beta_{c,\infty} \) which is irrelevant in the \( \varepsilon \rightarrow 0 \) extrapolation.
IV. DETAILS OF THE SIMULATION

Most of the work has been done on the APE supercomputer [20]. The model operating at Pisa is the so called “tube” machine, a 128 processor parallel computer with a peak performance of 6 GigaFlops.

All the code has been written in the high-level language [21] proper of this machine (“apese”), and the program runs at about 35% of the peak speed: this result is quite good if one takes into account the complications both in addressing and in memory access due to the use of the Symanzik action. In other words, in our code the floating point performance is slowed by the large number of integer and addressing operations required to load the elements of group in the register-file: this means that an analogous implementation for the SU(3) gauge group should result in a better performance, as the ratio of floating point to integer operations would be increased both in computation and in I/O. A detailed account of the implementation will be given elsewhere.

In the update of lattices with \(N_\tau = 3, 4, ... 8\) we have used an overrelaxed Heat Bath update. An exact Heat Bath algorithm has been implemented, in the modified Kennedy-Pendleton form [22] which results in higher acceptance and is well suited to a parallel machine like APE; a number of complete overrelaxation sweeps ranging from 10 to 16 has been used to decorrelate between subsequent Heat Bath sweeps.

As the Symanzik action couples next-to-nearest neighbors sites, on a lattice \(2 \times N_\sigma\) it results a self coupling of spatial links, and therefore the action is quadratic and prevents the use of an Heat Bath algorithm: hence on these smaller lattices we have used a Metropolis algorithm running on RISC workstations.

In Tabbs. I, II, ..., VI we present the simulation parameters, together with the estimated autocorrelation time of the Polyakov line. The Density of States Method has been used to interpolate between simulated data points. See Appendix B for a discussion of the meaning of the last column, where the “reweighting range” used in the interpolation is listed.

We have performed also a series of runs on large symmetric lattices to measure the expectation values of plaquettes, \(U_{1\times1}, U_{1\times2}\), to be used in determining effective couplings “a la Parisi”. We list in Tab. VII the corresponding run parameters.

V. RESULTS

A. Determination of critical point by FSS techniques

Runs on lattices with \(N_\tau = 2, 3, ... 8\) and up to \(N_\sigma = 24\) have been devoted to explore the transition region and measure accurately the Polyakov line.

1. Binder method

In Figg. 1, 2, ..., 7 we show the plots of the Binder cumulant, Eq. [21] for different values of \(N_\tau\) and \(N_\sigma\).
As the Binder cumulant is not a self-averaging quantity, the resulting bias has been computed, and found to be relevant for the larger lattices, where statistic is relatively poor: details on the evaluation and subtraction of the bias are reported in Appendix A.

The solid lines, as well as the dashed lines which are an estimate of the error, are obtained with our implementation of the Density of States Method [23,24], described in Appendix B.

The determination of the crossing points as well is based on DSM and we report in Tab. VIII the values of intersections $\beta_c(N_\sigma,N_\sigma')$ for the different lattices.

We make use of the extrapolation in Eq. (27) based on the introduction of a single irrelevant field, with exponent $y = -1$, to obtain for lattices $3 - 6$ the infinite volume limit of the critical coupling:

\[
\begin{align*}
N_\tau = 3 & \quad \beta_{c,\infty} = 1.59624(13) & a = 0.23(5) \\
N_\tau = 4 & \quad \beta_{c,\infty} = 1.699(1) & a = 0.20(6) \\
N_\tau = 5 & \quad \beta_{c,\infty} = 1.76948(3) & a = 0.85(10) \\
N_\tau = 6 & \quad \beta_{c,\infty} = 1.8287(11) & a = 3(1).
\end{align*}
\]

For what concerns the lattice with $N_\tau = 2$, we have not enough statistics to distinguish the different intersection points. As this point is in the strong coupling region, an high precision determination is not necessary.

For the $N_\tau = 7,8$ lattices we have only a single crossing, so we can estimate the critical coupling by assuming some value for the $a$ coefficient. By choosing $a \simeq 3 - 5$ for the $N_\tau = 7$ lattice we obtain, having $b = 20/14 = 1.43$, a value for $\varepsilon \simeq 4.2 \times 10^{-4}$, hence $a\varepsilon \simeq 1 - 2 \times 10^{-3}$. Analogously one obtains $a\varepsilon \simeq 3 \times 10^{-3}$ for the $N_\tau = 8$ lattice, so we estimate a systematic error of order $10^{-3}$, giving so

\[
\begin{align*}
N_\tau = 7 & \quad \beta_{c,\infty} = 1.8747 \pm 0.002 \pm 0.002 \\
N_\tau = 8 & \quad \beta_{c,\infty} = 1.920 \pm 0.004 \pm 0.003.
\end{align*}
\]

Using the two loop asymptotic formula, Eq. (17), this values are reexpressed in terms of the $T/\Lambda_I$ ratio in Tab. IX.

2. $\chi$ method

A different way to determine the critical point has been recently proposed by Engels et al. in [18]. The idea is to exploit the specific form of the scaling law for the susceptibility, which can be written as follows

\[
\chi_\nu = N_\sigma^3 \left\langle P^2 \right\rangle,
\]

taking into account that in finite volume the expectation value of Polyakov loops is set to zero by spin flips between degenerate vacua.

Expanding in the vicinity of the transition the expression in Eq. (18) one obtains

\[
\chi_\nu = \left( \frac{N_\sigma}{N_\tau} \right)^{\gamma/\nu} \left\{ c_0 + \left( c_1 + c_2 \left( \frac{N_\sigma}{N_\tau} \right)^{1/\nu} \right) t \frac{N_\sigma}{N_\tau} + c_3 \left( \frac{N_\sigma}{N_\tau} \right)^{y_3} \right\}
\]

\[
\ln \left( \chi_\nu \right)_{t=0} = \ln \left( c_0 \right) + \frac{\gamma}{\nu} \ln \left( \frac{N_\sigma}{N_\tau} \right) + \frac{c_3}{c_0} \left( \frac{N_\sigma}{N_\tau} \right)^{y_3}.
\]
Hence, assumed that the effect of the irrelevant is small \((y_1 \simeq -1)\), at the phase transition the true susceptibility \(\chi_v\), plotted as a function of the spatial size on a doubly logarithmic scale, should exhibit a linear behavior.

Let us apply this method to the \(N_\tau = 3\) lattice. We use the density of states method to determine \(\chi_v\) at various values of \(\beta\), and fit the data to a doubly logarithmic law. The resulting value for the minimum of \(\chi^2\) is at \(\beta_c = 1.5956(4)\) at 95\% c.l., which is in good agreement with the \(g_4\) determination. One obtains also the slope \(\gamma/\nu = 1.91(3)\), which exhibits a 3\% difference from the known value in the 3D Ising model \(\gamma/\nu = 1.965(11)\).

The same procedure has been applied to the other lattices: we find that, when applicable to our data, this method gives essentially the same result as the method of Binder cumulant, so we have used it as a cross check of the validity of the analysis.

### 3. Universal Scaling Behavior

We may test the universal scaling laws, expressed by Eqs. \(22\) and \(18\), by plotting \(g_4(t,y)\), \((y = N_\sigma/N_\tau)\), as a function of the combination \(ty^{1/\nu}\), and the analogous expression for the Polyakov loop, \(N_\tau y^{\eta/\nu} \langle |P| \rangle\): we choose \(\nu = 0.628\), \(\eta/\nu = 0.516\), \(\gamma/\nu = 1.965\) as given by the 3D Ising model. In Fig. 8 one may see that the \(g_4\) values cluster round a straight line, thus supporting the theoretical conclusions. The agreement is not so good for the Polyakov line, Fig. 9, which is sensitive to the self energy contribution.

### B. Critical temperature in the bare scheme

Let us give in Fig. 10 a cumulative plot of the various determinations of the critical temperature. The data on Wilson action are taken from \[1\] and normalized to the point at \(N_\tau = 8\), \((\text{instead of using, for instance, the perturbative } \Lambda \text{ ratio})\).

A few comments are in order:

- the weak coupling regime appears to be reached earlier in the simulation with the Symanzik action,
- in the weak coupling regime the pattern of asymptotic scaling violation appears to be essentially the same, by using Wilson and Symanzik action.

This result is in a clear disagreement with the earlier work by Curci et al. \[3\], where asymptotic scaling was found with the Symanzik action already at \(N_\tau = 5\). We have then analyzed our data with the same method, based on a fit to the critical behavior of the Polyakov line. For \(\beta > \beta_c\), the form
\[
\langle |L| \rangle = A (\beta - \beta_c)^\eta
\]  
(33)
is assumed, while for \(\beta < \beta_c\) the Polyakov line is set to zero: this introduces some arbitrariness in the fitting procedure. In fact, using Eq. \(33\) to determine \(\beta_c\) one is forced to discard data points assumed to be on the “left” of the transition. Moreover, on the “right” side of the transition (smaller lattice spacing) there are stronger renormalization effects, and these introduce a systematic which could be reduced by subtracting perturbative tails.
In Fig. 11 we show the data points, together with the results of a three parameters fit to the critical behavior: the study is done for lattices $3 \times 10, 4 \times 12, 5 \times 16$.

Results of the fit are shown in Tab. X, together with the value of the best reduced $\chi^2$ found. Errors are estimated with the method of $\chi^2$ equisurfaces.

The resulting critical temperatures are also reported in Fig. 10. The high value of the reduced $\chi^2$, especially on the $4 \times 12$ lattice, shows the bad quality of the fit, but for our purposes it is important to point out that the finite volume “critical” temperature determined by this method shows no sign of asymptotic scaling. On the contrary, it is present only a constant shift from the infinite volume limit, and we conclude that the results of [6] were hampered by an insufficient statistic.

1. Perturbative $\Lambda$ ratio

We can exploit the high precision data obtained with the Wilson action by Fingberg et al. [1] to give in Tab. XI the resulting determination of $\Lambda$ ratios, by using the asymptotic formula

$$\frac{\Lambda_I}{\Lambda_L} = \left( \frac{\beta_I}{\beta_L} \right)^{b_1/2b_0} \exp \left( \frac{\beta_L - \beta_I}{4C_0b_0} \right).$$  \hspace{1cm} (34)

These values are to be confronted with the perturbative results from Weisz and Wohlert [25]

$$\frac{\Lambda_I}{\Lambda_L} = 4.13089(1).$$  \hspace{1cm} (35)

showing a discrepancy of the order of 10%.

VI. DISCUSSION OF RESULTS

We can resume the outcome of our simulations as follows: Symanzik and Wilson actions give consistent results in the so called “bare” scheme, that is, the Renormalization Group evolution driven by scheme independent part of the perturbative $\beta$ function does not seem to show scaling of the temperature. The pattern of scaling violation is similar, at least for $N_\tau > 4$.

On the other hand it is well known that adimensional ratios of physical quantities show a good scaling in the coupling range considered in this work, at least in the case of Wilson action, where the ratio $T_c/\sigma^{1/2}$ has been showed to scale over the entire $\beta = 2.30 - 2.74$ range.

In our opinion these two facts, that is, the good scaling and the similar pattern of asymptotic scaling violation in the two actions, are consistent: as anticipated, in the scaling region the improvement has no effect on the asymptotic scaling.

Let us first elaborate on the statement that the two scaling figures are compatible with each other: to do this, we set up a perturbative framework and discuss the effect of the coupling redefinition implied by the use of Symanzik action.
We split \( \beta(g) \) in the universal \( \beta_u = -b_0g^3 - b_1g^5 \) and in the scheme dependent (unknown) part \( \beta_{s.d.} = O(g^7) \), where
\[
\begin{align*}
b_0 &= \frac{1}{(4\pi)^2} \frac{11Ca}{3} \\
b_1 &= \frac{1}{(4\pi)^4} \frac{34C_a^2}{3} .
\end{align*}
\] (36)

The beta function determines the dependence of lattice spacing \( a \) on the \( g \) value:
\[
\ln(a) = \int \frac{dg}{\beta(g)} + C = \int \left( \frac{1}{b_0g^3} - \frac{b_1}{b_0^2g} + \sum_{n=0}^{\infty} c_n g^{2n+1} \right) dg + C
\]
\[
= -\frac{1}{2b_0g^2} - \frac{b_1}{b_0^2} \ln(g) + \sum_{n=0}^{\infty} c_n g^{2n+2}.
\]
\[
a(g) = \frac{1}{\Lambda_L} \left( b_0g^2 \right)^{-\frac{b_1}{2b_0}} \exp \left( -\frac{1}{2b_0g^2} \right) \left( 1 + g^2 \sum_{n=0}^{\infty} d_n g^{2n} \right) .
\] (37)

The coefficients \( d_n \) in Eq. (37) are generated by \( \beta_{s.d.} \); they play the role of asymptotic scaling violating terms. For instance, in terms of the first sub-asymptotic term in \( \beta_{s.d.} \), the \( b_2 \) coefficient, one easily finds
\[
d_0 = \frac{1}{2b_0} \left( \frac{b_2}{b_0} - \left( \frac{b_1}{b_0} \right)^2 \right)
\] (38)

If we redefine the coupling constant
\[
g = g(\bar{g}) = \bar{g} + \alpha_1 \bar{g}^3 + \alpha_2 \bar{g}^5 + \cdots + \alpha_n \bar{g}^{2n+1} + \cdots
\] (39)
we obtain
\[
a(\bar{g}) = \frac{e^{\alpha_1/b_0}}{\Lambda_L} \left( b_0\bar{g}^2 \right)^{-\frac{b_1}{2b_0}} \exp \left( -\frac{1}{2b_0\bar{g}^2} \right) \left( 1 + \bar{g}^2 \sum_{n=0}^{\infty} d_n^* \bar{g}^{2n} \right) .
\] (40)

This is the evolution dictated by a new beta function \( \beta^*(g) \),
\[
\beta^*(\bar{g}) = \beta(g(\bar{g})) \left( \frac{\partial g(\bar{g})}{\partial \bar{g}} \right)^{-1} = \beta_u(\bar{g}) + \beta_{s.d.}(\bar{g}) ,
\] (41)

and the non universal terms are changed, for instance one has
\[
b_2^* = b_2 - 2\alpha_1 b_1 - 3\alpha_1^2 b_0 + 2b_0\alpha_2 .
\] (42)

The prefactor \( \exp(\alpha_1/b_0) \) redefines the scale parameter, consequently
\[
\alpha_1 = -b_0 \ln \left( \frac{\Lambda^*}{\Lambda_L} \right) .
\] (43)

On the other hand the asymptotic scaling violating contribution is changed
\[
d_0^* = d_0 - \alpha_1 \left( \frac{b_1}{b_0^2} + \frac{3\alpha_1}{2b_0} \right) + \frac{\alpha_2}{b_0} .
\] (44)
and one may hope to reduce its effect by a clever (and physically motivated) choice of $\tilde{g}$.

To study if the asymptotic scaling violation can be ascribed to the $d_2$ terms, we have performed an exploration of values of $b_2$ for both actions: in Fig. 12 we show how setting values of $b_2$ for Wilson and Symanzik actions respectively to 0.0018, 0.0011, one is able to obtain a good scaling: the $N_\tau = 2$ term has been dropped for clarity.

On the other hand, we are well aware of the arbitrariness of the procedure, as we have checked that this fit procedure is unstable if also the coefficient $b_3$ is introduced. This is easily understood as values of $b_2$ so large are hard to justify on a perturbative ground: too see this consider the running coupling equation in terms of the characteristic QCD coupling, $\alpha_s/(4\pi)$

$$\frac{d}{da} \left( \frac{\alpha_s}{4\pi} \right) = \frac{22C_a}{3} \left( \frac{\alpha_s}{4\pi} \right)^2 + \frac{68C_a^2}{3} \left( \frac{\alpha_s}{4\pi} \right)^3 + (4\pi)^6 b_2 \left( \frac{\alpha_s}{4\pi} \right)^4 + \ldots$$

$$\simeq \frac{44}{6} \left( \frac{\alpha_s}{4\pi} \right)^2 + \frac{272}{3} \left( \frac{\alpha_s}{4\pi} \right)^3 + 4330 \left( \frac{\alpha_s}{4\pi} \right)^4 + \ldots : \quad (45)$$

for instance at $\beta \simeq 1.9$ one has $\alpha_s/(4\pi) \simeq 0.013$, which means that the last term in Eq. (45) is of the same order of magnitude as the preceding one. So we can merely regard this procedure as the definition of an effective $b_2$, much in the spirit of the effective $\Lambda$ used in [26].

2. Perturbative approach: $\alpha_v$ coupling

Let us consider more closely the poor convergence of the speculative expansion given in Eq. (45): in the preceding subsection we have looked for an appropriate $b_2$ coefficient without any physical motivation. On the other hand, Lepage and Mackenzie [3] point out how a bad choice of the expansion parameter may give rise to a poorer convergence of the perturbation series, and they suggest strategies to design improved couplings, defined in terms of physical quantities.

This is much in the spirit of the old suggestions of Parisi, to define an effective coupling by inverting the perturbative expression for some ultraviolet-dominated quantity, like the single plaquette.

In [3] it is suggested to define an effective coupling $\alpha_v$ in terms of the quark-quark potential, by the relation

$$V(q) = -\frac{C_f (4\pi) \alpha_v (q^2)}{q^2}. \quad (46)$$

This coupling can be perturbatively related to $\alpha_0$, the bare lattice coupling, by the formula

$$\alpha_v (q^2) = \alpha_0 \left( 1 + \alpha_0 \left( (4\pi) b_0 \ln \left( \frac{\pi}{aq} \right)^2 + K \right) \right), \quad (47)$$

where the constant $K$ can be extracted by Kovacs work [27]: she reports

$$V(q) \propto -\frac{1}{q^2} \left( b_0 \ln \left( \frac{q^2}{\pi^2 \Lambda_c^2} \right) + \frac{C_a}{(4\pi)^2 J_L} \right)^{-1}. \quad (48)$$
Numerically one finds
\[ J_L = \begin{cases} -16.954, & \text{for } N = 2 \\ -19.695, & \text{for } N = 3 \end{cases}. \]  
(49)

To translate the result for the Symanzik action it is sufficient to rescale \( \Lambda \) factors and impose equality in the physical result. The defining relation is
\[ b_0 \ln \left( \frac{1}{\Lambda^2} \right) + \frac{C_a}{(4\pi)^2} J_L \equiv b_0 \ln \left( \frac{1}{\Lambda_I^2} \right) + \frac{C_a}{(4\pi)^2} J_I. \]  
(50)

One obtains easily, for \( SU(2) \),
\[ J_I = J_L + \frac{11}{3} \ln \left( \frac{\Lambda_I}{\Lambda_L} \right)^2. \]  
(51)

It is then possible to use the result of Weisz and Wohlert \[25\] on the \( \Lambda \) ratio:
\[ \frac{\Lambda_I}{\Lambda_L} = \exp \left[ \frac{1}{(4\pi) b_0} \left( C_a (0.04329017(1)) - \frac{1}{C_a} (0.04141417(1)) \right) \right] \]
\[ = \begin{cases} 4.13089(1), & \text{for } N = 2 \\ 5.29210(1), & \text{for } N = 3 \end{cases}, \]  
(52)

and it follows
\[ J_I = \begin{cases} -6.54859, & \text{for } N = 2 \\ -7.47609, & \text{for } N = 3 \end{cases}. \]  
(53)

Translating into the expression for \( K \), one easily obtains
\[ K_L = \begin{cases} 2.69831, & \text{for } N = 2 \\ 4.7018, & \text{for } N = 3 \end{cases} \]
\[ K_I = \begin{cases} 1.04224, & \text{for } N = 2 \\ 1.78479, & \text{for } N = 3 \end{cases}. \]  
(54)

We can now use value of \( \alpha_0 \) to set the value \( \alpha_v (\pi/a) \): one has obviously
\[ \alpha_v (\pi/a) = \alpha_0 \left( 1 + \alpha_0 K_{L(I)} \right). \]  
(55)

For a given value \( \beta_c = C_a / (2\pi\alpha_v) \) of the measured critical coupling, the corresponding critical temperature in the \( \alpha_v \) scheme may be obtained by using the relation
\[ \frac{T}{\Lambda_v} = \frac{\pi}{\pi N_f a \Lambda_v}. \]  
(56)

and the two loop scaling formula
\[ \frac{q}{\Lambda_v} = \exp \left[ \frac{1}{(8\pi) b_0} \frac{1}{\alpha_v(q)} - \frac{b_1}{2b_0^2} \ln \left( \frac{1}{4\pi b_0 \alpha_v(q)} \right) \right]. \]  
(57)

Let us report in Tabb. XII and XIII the results of the analysis both for the Wilson and the Symanzik case.

It appears that the use of a perturbative expression for the effective coupling \( \alpha_v \) does not give a better asymptotic scaling. It should be stressed however that this was merely an exercise, as the Lepage scheme is intended to work in conjunction to a non perturbative determination of the coupling \( \alpha_v \), based on the measurement of the heavy quark potential.
3. Non-perturbative approach

The bare coupling itself has no meaning in itself, so it must be eliminated in favor of some physical quantity, connected to it. In the perturbative approach this is a way to hide infinities and obtain a renormalized field theory.

A simple way to implement this program on the lattice is the elimination of the bare coupling in favor of some UV dominated quantity, such as the expectation value of plaquettes. To this end, let us recall the results of the work of Weisz and Wohlert [25] on their perturbative expansion: by defining, for simplicity in the adjoint representation

\[
\ln \left[ \frac{1}{C_A} \langle \text{Tr} U(L, T) \rangle \right] = - \sum_{n=1}^{\infty} \frac{g^{2n}}{(2n)!} w_n(L, T),
\]

the first coefficient is given by

\[
w_1(L, T) = C_f I(L, T)
\]

and the following values are given

\[
I(1, 1) = \left\{ \begin{array}{ll}
\frac{1}{2} & \text{Wilson} \\
0.366262 & \text{Symanzik}
\end{array} \right.
I(1, 2) = \left\{ \begin{array}{ll}
0.862251 & \text{Wilson} \\
0.662624 & \text{Symanzik}
\end{array} \right.
\]

(60)

Coefficients for the Wilson case are known up to fourth $w_1$, while for Symanzik we shall limit ourselves to lowest order.

This allows to define two effective couplings in the Symanzik case, depending on the two expectation values measured. That is, knowing that

\[
\ln \left[ \frac{1}{C_A} \langle \text{Tr} U(L, T) \rangle \right] = - \frac{g^2}{2} C_f I(L, T)
\]

and given $\beta = \frac{2C_a}{g^2}$, we define

\[
\beta_{1(2)} = C_a C_f \frac{I(1, 1(2))}{1 - U_{1x1(1x2)}}.
\]

(62)

Numerically, for the $SU(2)$ case, this corresponds to

\[
\beta_{1(2)} = \frac{0.549393 (0.993936)}{1 - U_{1x1(1x2)}}.
\]

(63)

By using the results of measurements on large symmetric lattices we can obtain via interpolation the values of plaquettes needed to compute the effective coupling: we report the results in Tab. (XIV).

We plot in Fig. 13 the comparison of the different schemes, together with the Wilson data, taken from [1], in the effective scheme deduced from the single plaquette. As usual we normalize to the last point: we drop the point at $N_r = 2$ to show in a larger scale the other points. It is worth noting that
• the two effective schemes used for the Symanzik action agree with each other,

• they both show a good scaling starting at $N_\tau = 4$,

• the same scheme used with Wilson action seems to show, starting at $N_\tau = 5$ the same behavior.

We think that these results can be interpreted as a confirmation of the value of effective schemes, and as an indication that Symanzik action gives an enlargement of the scaling window.

VII. CONCLUSIONS

In this work we have performed an high statistics simulation of the $SU(2)$ pure gauge theory at finite temperature, using the tree-improved Symanzik action and exploring a range of lattices up to $N_\tau = 8$; we have determined, with the help of Finite Size Scaling methods, the critical couplings for the deconfinement transition, with the purpose of studying the scaling properties of the critical temperature and the effect of improvement.

We can resume as follows our findings:

• asymptotic scaling violations are present, and are of the same size as in the case of Wilson action. The pattern of violation is similar and can be interpreted as driven by the same lattice $\beta$ function, modulo scheme redefinitions. This means either that lattice artifacts are small, or they are the same for the two actions, in accordance with previous determinations of the scaling window.

• The use of a non perturbative coupling derived from the plaquette expectation value gives a better scaling figure: this effective coupling was already used in the same context but with the Wilson action, and comparison of the two analyses shows that Symanzik action apparently gives a slightly precocious scaling.

• In our study we are unable to directly test the scaling, so it could be desirable a zero temperature study of some other dimensionful quantity, like masses or the string tension.

From a practical point of view, if we are willing to trust the precocious onset of scaling in the effective scheme, we can say that the gain in volume obtained by use of the improved action compensates the increased complexity in the update routine, thus slightly favouring the use of this technique in simulations. This should be even more true in the case of $SU(3)$ group.

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APPENDIX A: STATISTICAL EVALUATION OF BINDER'S CUMULANT.

In order to extract in a reliable way the quantities we are interested in from a set of $N$ measures we have to correct for the distortions induced by the finite sampling. It is well known that often the “naive” estimator for a statistical quantity is not the best one. The much simpler and well known example is the estimator of variance $\sigma^2$ for a set of $N$ uncorrelated data $x_i$. One may try

$$E_b(\sigma^2) = E(\langle x^2 \rangle) - E(\langle x \rangle)^2 = \left( \frac{1}{N} \sum_{i=1}^{N} x_i^2 \right) - \left( \frac{1}{N} \sum_{i=1}^{N} x_i \right)^2 \quad (A1)$$

but, owing to the fact that $E(\langle x \rangle)$ is a random variable with a finite variance the estimator’s expectation value is

$$\langle E_b(\sigma^2) \rangle = \frac{1}{N} \sum_{i=1}^{N} \langle x^2 \rangle - \frac{1}{N^2} \sum_{i,j=1}^{N} (\delta_{ij} \langle x^2 \rangle + (1-\delta_{ij}) \langle x \rangle^2) = \left( 1 - \frac{1}{N} \right) \sigma^2, \quad (A2)$$

which is “biased” by a finite sample effect $O\left(\frac{1}{\sqrt{N}}\right)$. To evaluate in the best way $\sigma^2$ we must correct for this bias: this can be done using a new redefined estimator

$$E(\sigma^2) = N/(N-1) \, E_b(\sigma^2) \quad (A3)$$

which gives the correct result $\langle E(\sigma^2) \rangle = \sigma^2$.

Now we want to study the Binder’s cumulant $g_4$ of a random variable $L$, where the general cumulant $g_{2n}$ (in the $<L^{2n+1}> = 0$ case which is of interest to us) is defined as

$$g_{2n} = \frac{<L^{2n}>_c}{<L^{2n}>^n_c}, \quad (A4)$$

$$g_4 = \frac{<L^4>_c}{<L^2>_c^2} = \frac{<L^4>}{<L^2>^2} - 3, \quad (A5)$$

$$g_6 = \frac{<L^6> - 15 <L^2> <L^4> + 30 <L^2>^3}{<L^2>^3} \quad (A6)$$

The “natural” estimator for $g_4$ over a set of $N$ measurements is given by

$$\bar{g}_4 = \left( \frac{1}{N} \sum_i L_i^4 \right) \left( \frac{1}{N} \sum_i L_i^2 \right)^{-2} - 3, \quad (A7)$$

but as we will see this is not a self-averaging quantity, in the sense that $\langle \bar{g}_4 \rangle \neq g_4$, and we have to redefine it. To calculate the bias of $g_4$ we introduce the $O(\frac{1}{\sqrt{N}})$ fluctuations $\eta_2$ and $\eta_4$ defined by

$$\eta_2 = \frac{1}{N} \sum_i L_i^2 - <L^2>$$

$$\eta_4 = \frac{1}{N} \sum_i L_i^4 - <L^4> \quad (A8)$$
and we expand Eq. (A7) around $\eta = 0$, to obtain

$$g_4 = g_4 \left[ 1 - \frac{\eta_2}{\langle L^2 \rangle} + \frac{\eta_4}{\langle L^4 \rangle} + 3 \frac{\eta_2^2}{\langle L^2 \rangle^2} - 2 \frac{\eta_2 \eta_4}{\langle L^2 \rangle \langle L^4 \rangle} + o \left( \frac{1}{N} \right) \right]. \quad (A9)$$

Now to evaluate the corrections we take the expectation value, and we get

$$< \bar{g}_4 > = g_4 \left[ 1 + \frac{3}{\langle L^2 \rangle^2} < \eta_2^2 > - \frac{2}{\langle L^2 \rangle \langle L^4 \rangle} < \eta_2 \eta_4 > + o \left( \frac{1}{N} \right) \right] \quad (A10)$$

where

$$< \eta_2 \eta_4 > = \frac{1}{N} \left( < L^6 > - < L^4 > < L^2 > \right) \left( \sum_{\delta=0}^{N-1} \frac{N-\delta}{N} \left[ C(L^2, L^4, \delta) + C(L^4, L^2, \delta) \right] - 1 \right)$$

and

$$< \eta_2^2 > = \frac{1}{N} \left( < L^4 > - < L^2 >^2 \right) \left( \sum_{\delta=0}^{N-1} \frac{N-\delta}{N} \left[ 2C(L^2, L^2, \delta) - 1 \right] \right) \quad (A11)$$

and

$$C(X, Y, \delta) = \frac{<(X_i - < X >)(Y_i+\delta - < Y >)>}{<(X_i - < X >)(Y_i - < Y >)>}. \quad (A12)$$

If $N$ is much bigger than the correlation times of interest then $C(X, Y, \delta) \approx 0$ for $\delta > \bar{\delta}$, with $\bar{\delta} < n$, so we can write approximately

$$< \eta_2 \eta_4 > \approx \frac{1}{N} \left( < L^6 > - < L^4 > < L^2 > \right) \left( \tau_{\text{int}}(L^2, L^4) + \tau_{\text{int}}(L^4, L^2) - 1 \right) \quad (A13)$$

$$< \eta_2^2 > \approx \frac{1}{N} \left( < L^4 > - < L^2 >^2 \right) \left( 2\tau_{\text{int}}(L^2, L^2) - 1 \right) \quad (A14)$$

where we have defined the integrated autocorrelation time

$$\tau_{\text{int}}(X, Y) = \sum_{\delta=0}^{\infty} C(X, Y, \delta). \quad (A15)$$

In order to correct for the $O(\frac{1}{N})$ bias we note that in Eq. (A9) the terms linear in the fluctuations do not contribute to the expectation value. So it is sufficient to define an “improved” estimator $g_4^*$ as

$$g_4^* = g_4 \left[ 1 - \frac{3}{\langle L^2 \rangle^2} \frac{\eta_2^2}{\langle \eta_2 \rangle^2} + 2 \frac{\eta_2 \eta_4}{\langle \eta_2 \rangle \langle \eta_4 \rangle} \right] = g_4 \left[ 1 - \frac{\eta_2}{\langle L^2 \rangle} + \frac{\eta_4}{\langle L^4 \rangle} + o \left( \frac{1}{N} \right) \right]. \quad (A16)$$

To the order we are working we can substitute the quadratic fluctuations terms with the “naive” estimator for their expectation value, and we obtain

$$g_4^* = \frac{1}{N} \sum_{i,j} \frac{L_i L_j}{\sum_{i,j} L_i L_j} \left[ 1 - \frac{3}{N} \left( \frac{1}{N} \sum_{i,j} \frac{L_i^4}{L_i^2} \right) - 1 \right] E \left[ 2\tau_{\text{int}}(L^2, L^2) - 1 \right] + \frac{2}{N} \left( \frac{1}{N} \sum_{i,j} \frac{L_i^6}{L_i^2} \right) - 1 \right] E \left[ \tau_{\text{int}}(L^2, L^2) + \tau_{\text{int}}(L^2, L^4) - 1 \right]. \quad (A17)$$
To evaluate the variance of $g^*_i$ we can write using Eq. (A16)
\[
\sigma^2(g^*_i) = \langle (g^*_i)^2 \rangle - g^2_i = g^2_i \left[ 4 \frac{< \eta^2_2 >}{< L^2 >^2} + \frac{< \eta^4 >}{< L^4 >^2} - 4 \frac{< \eta_2 \eta_4 >}{< L^2 > < L^4 >} \right].
\] (A18)

We can estimate this quantity to the lowest order with
\[
E \left[ \sigma^2(g^*_i) \right] = \left( \frac{\sum_i L^4_i}{\sum_{i,j} L^2_i L^2_j} \right)^2 \left[ 4 \left( \frac{\sum_i L^4_i}{\sum_{i,j} L^2_i L^2_j} - 1 \right) E \left[ 2 \tau_{\text{int}}(L^2, L^2) - 1 \right] + \frac{1}{N} \left( \frac{1}{N} \sum_i L^8_i \right) - 1 \right] E \left[ 2 \tau_{\text{int}}(L^4, L^4) - 1 \right] + \frac{4}{N} \left( \frac{1}{N} \sum_i L^6_i \right) - 1 \right] E \left[ \tau_{\text{int}}(L^2, L^2) + \tau_{\text{int}}(L^2, L^4) - 1 \right].
\] (A19)

**APPENDIX B: DENSITY OF STATES METHOD**

To extract the value of an observable $O$ from a monte carlo simulation we average over the sequence of values $O_i$ generated by the algorithm. This is nothing else than an approximation of the path integral formula
\[
< O > (\beta) = \frac{\int D\psi O[\psi] \exp(-\beta S[\psi])}{\int D\psi \exp(-\beta S[\psi])} \approx \frac{1}{N} \sum_i O_i.
\] (B1)

Using the fact that we know the probability distribution for the fundamental fields we can extrapolate the observable’s value around the $\beta$ value we have used in the simulation. To this effect we note that
\[
< O > (\beta + \Delta \beta) = \frac{\int D\psi O[\psi] \exp(-\Delta \beta S[\psi]) \exp(-\beta S[\psi])}{\int D\psi \exp(-\Delta \beta S[\psi]) \exp(-\beta S[\psi])} = \frac{< O \exp(-\Delta \beta S) > (\beta)}{< \exp(-\Delta \beta S) > (\beta)}
\] (B2)
\[
< O > (\beta + \Delta \beta) \approx \frac{\sum_i O_i \exp(-\Delta \beta S_i)}{\sum_i \exp(-\Delta \beta S_i)},
\] (B3)

so it is sufficient to measure the action $S_i$ correlated with each $O_i$ value, and resum each measure according to Eq. (B3). The “resummed” value will be reliable only for small $\Delta \beta$, because we have a good statistical sampling of action distribution only in a small range around the mean value, so we must do a preliminary determination of the allowed resummation range. During our simulation we have used essentially the same method as Alves et al. in [28]: we plot histograms of the energy distribution, we set a minimum and maximum energy by imposing that at least 2.5% of the energy distribution is present below the minimum and above the maximum, and we translate this energy range in a $\beta$ range by finding the $\beta$ shifts that via resummation give the found energy limits.

If the resumming ranges of different simulations overlap we can “patch” the values coming from different extrapolations to obtain a more accurate estimate.

The determination of an unbiased estimator for the resummed Binder cumulant and its variance can be obtained with the same method as in Appendix [A]. We only note that there are some complications when we combine together different extrapolations, because in this case to cancel the bias we must use an improved estimator for the variances, depending on many-point correlations which can be estimated reliably only with very high statistics.
| $N_x$ | $N_r$ | $\beta$ | $N_{\text{meas}}$ | $\tau_{\text{int}}$ | Rew. range |
|-------|-------|---------|-------------------|-------------------|------------|
| 4     | 2     | 1.340000 | 40960             | 32.8              | 1.28 - 1.40 |
| 4     | 2     | 1.360000 | 40960             | 51.3              | 1.30 - 1.42 |
| 4     | 2     | 1.380000 | 40960             | 105.7             | 1.32 - 1.44 |
| 4     | 2     | 1.400000 | 16384             | 79.4              | 1.34 - 1.46 |
| 8     | 2     | 1.340000 | 16384             | 55.0              | 1.31 - 1.37 |
| 8     | 2     | 1.360000 | 38912             | 154.4             | 1.33 - 1.39 |
| 8     | 2     | 1.380000 | 38912             | 314.9             | 1.35 - 1.41 |
| 8     | 2     | 1.400000 | 38912             | 1264.8            | 1.37 - 1.43 |
| 12    | 2     | 1.360000 | 36864             | 142.9             | 1.34 - 1.37 |
| 12    | 2     | 1.380000 | 45056             | 407.1             | 1.36 - 1.39 |
| 12    | 2     | 1.400000 | 49152             | 3349.7            | 1.39 - 1.41 |

TABLE I. Run parameters

| $N_x$ | $N_r$ | $\beta$ | $N_{\text{meas}}$ | $\tau_{\text{int}}$ | Rew. range |
|-------|-------|---------|-------------------|-------------------|------------|
| 6     | 3     | 1.5600  | 32768             | 3.0               | 1.52 - 1.60 |
| 6     | 3     | 1.5700  | 32768             | 3.4               | 1.53 - 1.61 |
| 6     | 3     | 1.5800  | 32768             | 4.2               | 1.54 - 1.62 |
| 6     | 3     | 1.5900  | 32768             | 4.9               | 1.55 - 1.63 |
| 6     | 3     | 1.6000  | 32768             | 6.7               | 1.56 - 1.64 |
| 6     | 3     | 1.6100  | 32768             | 7.9               | 1.57 - 1.65 |
| 6     | 3     | 1.6200  | 32768             | 10.6              | 1.58 - 1.66 |
| 6     | 3     | 1.6300  | 32768             | 15.3              | 1.59 - 1.67 |
| 10    | 3     | 1.5600  | 30720             | 5.9               | 1.54 - 1.58 |
| 10    | 3     | 1.5700  | 30720             | 8.9               | 1.55 - 1.59 |
| 10    | 3     | 1.5800  | 30720             | 15.9              | 1.56 - 1.60 |
| 10    | 3     | 1.5900  | 30720             | 22.3              | 1.57 - 1.61 |
| 10    | 3     | 1.6000  | 30720             | 53.5              | 1.58 - 1.62 |
| 10    | 3     | 1.6100  | 30720             | 103.6             | 1.59 - 1.63 |
| 10    | 3     | 1.6200  | 30720             | 138.7             | 1.60 - 1.64 |
| 10    | 3     | 1.6300  | 30720             | 327.6             | 1.61 - 1.65 |
| 14    | 3     | 1.5600  | 30336             | 6.5               | 1.551 - 1.569 |
| 14    | 3     | 1.5700  | 65536             | 10.3              | 1.560 - 1.578 |
| 14    | 3     | 1.5800  | 65536             | 17.6              | 1.572 - 1.590 |
| 14    | 3     | 1.5900  | 65536             | 33.6              | 1.579 - 1.599 |
| 14    | 3     | 1.6000  | 65536             | 84.8              | 1.590 - 1.612 |
| 14    | 3     | 1.6100  | 30336             | 206.9             | 1.602 - 1.621 |
| 14    | 3     | 1.6200  | 30336             | 902.2             | 1.612 - 1.629 |
| 14    | 3     | 1.6300  | 30336             | 3.9               | 1.620 - 1.639 |
| 18    | 3     | 1.5800  | 22464             | 12.3              | 1.572 - 1.587 |
| 18    | 3     | 1.5900  | 22464             | 22.1              | 1.584 - 1.597 |
| 18    | 3     | 1.6000  | 22464             | 81.1              | 1.592 - 1.606 |
| 18    | 3     | 1.6100  | 22464             | 563.7             | 1.602 - 1.617 |

TABLE II. Run parameters
| $N_{\sigma}$ | $N_{\tau}$ | $\beta$  | $N_{\text{meas}}$ | $\tau_{\text{int}}$ | Rew. range   |
|------------|-----------|---------|----------------|----------------|-------------|
| 8          | 4         | 1.6500  | 65536          | 3.2            | 1.625 - 1.675 |
| 8          | 4         | 1.6700  | 65536          | 4.8            | 1.646 - 1.694 |
| 8          | 4         | 1.6900  | 65536          | 7.6            | 1.665 - 1.716 |
| 8          | 4         | 1.6950  | 131072         | 8.7            | 1.670 - 1.721 |
| 8          | 4         | 1.7050  | 131072         | 11.0           | 1.680 - 1.731 |
| 8          | 4         | 1.7100  | 196608         | 12.9           | 1.686 - 1.736 |
| 8          | 4         | 1.7200  | 262144         | 18.5           | 1.694 - 1.746 |
| 8          | 4         | 1.7300  | 131072         | 23.0           | 1.704 - 1.757 |
| 12         | 4         | 1.6500  | 4096           | 3.5            | 1.637 - 1.665 |
| 12         | 4         | 1.6700  | 4096           | 6.6            | 1.657 - 1.684 |
| 12         | 4         | 1.6900  | 69632          | 12.5           | 1.676 - 1.704 |
| 12         | 4         | 1.6950  | 131072         | 15.5           | 1.681 - 1.709 |
| 12         | 4         | 1.7000  | 65536          | 22.7           | 1.686 - 1.714 |
| 12         | 4         | 1.7050  | 131072         | 29.1           | 1.691 - 1.719 |
| 12         | 4         | 1.7100  | 135168         | 50.5           | 1.704 - 1.724 |
| 12         | 4         | 1.7200  | 131072         | 84.4           | 1.714 - 1.734 |
| 12         | 4         | 1.7300  | 131072         | 200.7          | 1.717 - 1.744 |
| 12         | 4         | 1.7400  | 131072         | 351.8          | 1.726 - 1.755 |

| 16         | 4         | 1.6900  | 28672          | 16.3           | 1.681 - 1.699 |
| 16         | 4         | 1.6950  | 57344          | 25.4           | 1.686 - 1.704 |
| 16         | 4         | 1.7000  | 57344          | 52.1           | 1.691 - 1.707 |
| 16         | 4         | 1.7050  | 40960          | 55.4           | 1.696 - 1.714 |
| 16         | 4         | 1.7100  | 12288          | 71.2           | 1.701 - 1.719 |

TABLE III. Run parameters (continued)
| $N_\sigma$ | $N_T$ | $\beta$ | $N_{\text{meas}}$ | $\tau_{\text{int}}$ | Rew. range   |
|-------|-------|--------|------------------|----------------|-------------|
| 8     | 5     | 1.7500 | 16384            | 6.7            | 1.727 - 1.772 |
| 8     | 5     | 1.7600 | 40960            | 7.4            | 1.738 - 1.785 |
| 8     | 5     | 1.7650 | 40960            | 8.4            | 1.741 - 1.791 |
| 8     | 5     | 1.7700 | 40960            | 9.0            | 1.745 - 1.796 |
| 8     | 5     | 1.7750 | 40960            | 10.0           | 1.750 - 1.801 |
| 8     | 5     | 1.7800 | 57344            | 11.3           | 1.755 - 1.806 |
| 8     | 5     | 1.7850 | 40960            | 12.0           | 1.759 - 1.811 |
| 8     | 5     | 1.7900 | 40960            | 12.4           | 1.762 - 1.815 |
| 8     | 5     | 1.7950 | 40960            | 13.1           | 1.770 - 1.822 |
| 12    | 5     | 1.7600 | 39936            | 14.0           | 1.746 - 1.774 |
| 12    | 5     | 1.7650 | 39936            | 18.2           | 1.751 - 1.777 |
| 12    | 5     | 1.7700 | 39936            | 22.9           | 1.756 - 1.783 |
| 12    | 5     | 1.7750 | 39936            | 28.5           | 1.761 - 1.789 |
| 12    | 5     | 1.7800 | 23552            | 27.5           | 1.766 - 1.792 |
| 12    | 5     | 1.7850 | 23552            | 43.1           | 1.771 - 1.798 |
| 12    | 5     | 1.7900 | 23552            | 55.8           | 1.776 - 1.804 |
| 12    | 5     | 1.7950 | 23552            | 51.6           | 1.780 - 1.808 |
| 16    | 5     | 1.7600 | 28672            | 24.0           | 1.751 - 1.767 |
| 16    | 5     | 1.7650 | 28672            | 24.2           | 1.756 - 1.774 |
| 16    | 5     | 1.7700 | 28672            | 35.6           | 1.761 - 1.779 |
| 16    | 5     | 1.7750 | 28672            | 46.4           | 1.768 - 1.784 |
| 20    | 5     | 1.7670 | 18432            | 52.4           | 1.761 - 1.773 |
| 20    | 5     | 1.7690 | 18432            | 58.5           | 1.763 - 1.774 |

TABLE IV. Run parameters (continued)
| $N_x$ | $N_T$ | $\beta$  | $N_{\text{meas}}$ | $\tau_{\text{int}}$ | Rew. range     |
|------|------|----------|-------------------|----------------------|---------------|
| 8    | 6    | 1.7750   | 32768             | 3.9                  | 1.754 - 1.799 |
| 8    | 6    | 1.8000   | 32768             | 5.4                  | 1.7775 - 1.824|
| 8    | 6    | 1.8250   | 32768             | 7.5                  | 1.800 - 1.851 |
| 8    | 6    | 1.8750   | 32768             | 11.2                 | 1.850 - 1.902 |
| 12   | 6    | 1.7750   | 49152             | 5.6                  | 1.762 - 1.7875|
| 12   | 6    | 1.8000   | 49152             | 10.1                 | 1.787 - 1.812 |
| 12   | 6    | 1.8125   | 16384             | 14.7                 | 1.799 - 1.826 |
| 12   | 6    | 1.8250   | 49152             | 19.5                 | 1.811 - 1.837 |
| 12   | 6    | 1.8375   | 16384             | 30.6                 | 1.824 - 1.850 |
| 12   | 6    | 1.8750   | 49152             | 87.9                 | 1.861 - 1.889 |
| 16   | 6    | 1.7750   | 28672             | 7.5                  | 1.767 - 1.783 |
| 16   | 6    | 1.8000   | 28672             | 12.0                 | 1.792 - 1.809 |
| 16   | 6    | 1.8125   | 16384             | 21.9                 | 1.805 - 1.820 |
| 16   | 6    | 1.8250   | 28672             | 30.1                 | 1.817 - 1.832 |
| 16   | 6    | 1.8375   | 16384             | 51.2                 | 1.829 - 1.846 |
| 16   | 6    | 1.8750   | 28672             | 466.3                | 1.886 - 1.884 |
| 20   | 6    | 1.8200   | 16384             | 30.3                 | 1.814 - 1.825 |
| 20   | 6    | 1.8250   | 18920             | 55.0                 | 1.819 - 1.831 |
| 20   | 6    | 1.8300   | 16384             | 57.6                 | 1.825 - 1.836 |
| 24   | 6    | 1.8220   | 12288             | 28.5                 | 1.818 - 1.826 |
| 24   | 6    | 1.8250   | 27008             | 62.9                 | 1.821 - 1.829 |
| 24   | 6    | 1.8280   | 27008             | 80.2                 | 1.823 - 1.832 |

TABLE V. Run parameters (continued)
| $N_\sigma$ | $N_\tau$ | $\beta$ | $N_{\text{meas}}$ | $\tau_{\text{int}}$ | Rew. range |
|---|---|---|---|---|---|
| 14 | 7 | 1.8700 | 9216 | 31.3 | 1.861 - 1.880 |
| 14 | 7 | 1.8750 | 11600 | 27.2 | 1.867 - 1.884 |
| 14 | 7 | 1.8800 | 12288 | 23.5 | 1.872 - 1.889 |
| 14 | 7 | 1.8850 | 9216 | 32.0 | 1.878 - 1.892 |
| 20 | 7 | 1.8725 | 15904 | 40.7 | 1.867 - 1.877 |
| 20 | 7 | 1.8750 | 17216 | 47.8 | 1.870 - 1.880 |
| 20 | 7 | 1.8775 | 18400 | 58.3 | 1.871 - 1.884 |
| 20 | 7 | 1.8800 | 14720 | 77.4 | 1.874 - 1.885 |
| 16 | 8 | 1.9000 | 13312 | 19.0 | 1.891 - 1.909 |
| 16 | 8 | 1.9150 | 11536 | 34.0 | 1.907 - 1.922 |
| 16 | 8 | 1.9200 | 13312 | 46.2 | 1.911 - 1.929 |
| 16 | 8 | 1.9250 | 16384 | 39.2 | 1.918 - 1.932 |
| 24 | 8 | 1.9000 | 18432 | 33.1 | 1.896 - 1.904 |
| 24 | 8 | 1.9150 | 29184 | 54.5 | 1.911 - 1.919 |
| 24 | 8 | 1.9180 | 10144 | 27.4 | 1.914 - 1.922 |
| 24 | 8 | 1.9200 | 20736 | 54.9 | 1.916 - 1.924 |
| 24 | 8 | 1.9220 | 10144 | 41.4 | 1.918 - 1.926 |
| 24 | 8 | 1.9250 | 6144 | 54.9 | 1.921 - 1.929 |

**TABLE VI.** Run parameters (continued)

| $N_\sigma$ | $N_\tau$ | $\beta$ | $N_{\text{meas}}$ | $\tau_{\text{int}}$ | $U_{1\times1}$ | $U_{1\times2}$ |
|---|---|---|---|---|---|---|
| 12 | 12 | 1.375000 | 384 | 1.0 | 0.51106(11) | 0.25133(14) |
| 12 | 12 | 1.385000 | 384 | 1.0 | 0.51488(13) | 0.25561(20) |
| 14 | 14 | 1.590000 | 384 | 1.0 | 0.596484(95) | 0.35759(14) |
| 14 | 14 | 1.595000 | 384 | 1.0 | 0.598358(84) | 0.36011(12) |
| 14 | 14 | 1.600000 | 384 | 1.0 | 0.600174(94) | 0.36261(15) |
| 16 | 16 | 1.690000 | 384 | 1.0 | 0.631531(63) | 0.40691(11) |
| 16 | 16 | 1.695000 | 384 | 1.1 | 0.633037(61) | 0.409988(95) |
| 16 | 16 | 1.700000 | 384 | 1.0 | 0.634699(72) | 0.41150(12) |
| 16 | 16 | 1.705000 | 384 | 1.0 | 0.636203(78) | 0.41371(10) |
| 20 | 20 | 1.769000 | 2304 | 1.0 | 0.654686(17) | 0.440752(27) |
| 20 | 20 | 1.770000 | 2304 | 1.0 | 0.654941(13) | 0.441124(22) |
| 20 | 20 | 1.827000 | 2048 | 1.0 | 0.669217(16) | 0.462206(26) |
| 20 | 20 | 1.828500 | 3072 | 1.0 | 0.669557(10) | 0.462711(17) |
| 20 | 20 | 1.830000 | 2048 | 1.0 | 0.669923(13) | 0.463250(22) |
| 20 | 20 | 1.872500 | 1536 | 1.2 | 0.679420(18) | 0.477325(29) |
| 20 | 20 | 1.877500 | 1536 | 1.0 | 0.680491(14) | 0.478897(24) |
| 20 | 20 | 1.915000 | 1536 | 1.0 | 0.688180(16) | 0.490325(27) |
| 20 | 20 | 1.925000 | 1536 | 1.0 | 0.690137(15) | 0.493229(24) |

**TABLE VII.** Run parameters on symmetric lattices, and expectation values of plaquettes
| $N_{\tau}$ | $N_{\sigma}$ | $N_{\sigma'}$ | $\beta_c$ |
|-----------|-------------|-------------|-----------|
| 2         | 4           | 8           | 1.380(4)  |
| 2         | 4           | 12          | 1.380(2)  |
| 2         | 8           | 12          | 1.380(4)  |
| 3         | 6           | 10          | 1.595(1)  |
| 3         | 6           | 14          | 1.5955(8) |
| 3         | 6           | 18          | 1.5958(5) |
| 3         | 10          | 14          | 1.5956(12)|
| 3         | 10          | 18          | 1.5960(7) |
| 3         | 14          | 18          | 1.5963(15)|
| 4         | 8           | 12          | 1.6984(7) |
| 4         | 8           | 16          | 1.6987(5) |
| 4         | 12          | 16          | 1.699(1)  |
| 5         | 8           | 12          | 1.7669(12)|
| 5         | 8           | 16          | 1.7678(8) |
| 5         | 8           | 20          | 1.7682(7) |
| 5         | 12          | 16          | 1.7684(15)|
| 5         | 12          | 20          | 1.769(12) |
| 5         | 16          | 20          | 1.769(3)  |
| 6         | 8           | 12          | 1.812(4)  |
| 6         | 8           | 16          | 1.817(2)  |
| 6         | 8           | 20          | 1.822(2)  |
| 6         | 8           | 24          | 1.824(1)  |
| 6         | 12          | 16          | 1.823(5)  |
| 6         | 12          | 20          | 1.828(2)  |
| 6         | 12          | 24          | 1.825(1)  |
| 6         | 16          | 20          | 1.825(4)  |
| 6         | 16          | 24          | 1.824(3)  |
| 6         | 20          | 24          | 1.824(3)  |
| 7         | 14          | 20          | 1.8747(20)|
| 8         | 16          | 24          | 1.920(4)  |

**TABLE VIII.** Crossings for $g_4$ curves

| $N_{\tau}$ | $\beta_c$      | $T_c/\Lambda_1$ |
|-----------|----------------|-----------------|
| 2         | 1.380(4)       | 8.81(8)         |
| 3         | 1.59624(13)    | 9.889(3)        |
| 4         | 1.699(1)       | 9.526(30)       |
| 5         | 1.76948(3)     | 9.0562(7)       |
| 6         | 1.8287(11)     | 8.73(3)         |
| 7         | 1.8747(30)     | 8.38(6)         |
| 8         | 1.920(5)       | 8.2(1)          |

**TABLE IX.** Critical temperature by two loop asymptotic scaling formula
| $N_{\tau}$ | $\beta_c$   | $T_c/\Lambda$ | $A$   | $\eta$   | $\chi^2$ |
|---------|------------|--------------|-------|----------|---------|
| 3       | 1.588(2)   | 9.70(5)      | 0.76(4) | 0.35(2)  | 5.4     |
| 4       | 1.687(2)   | 9.25(3)      | 0.66(4) | 0.40(2)  | 15.3    |
| 5       | 1.759(3)   | 8.82(5)      | 0.47(2) | 0.38(2)  | 4.0     |

**TABLE X.** Results of the fit

| $N_{\tau}$ | $\beta_I$      | $\beta_L$      | $\Lambda_I/\Lambda_L$ |
|------------|----------------|----------------|------------------------|
| 2          | 1.380(4)       | 1.880(30)      | 3.599(46)              |
| 3          | 1.59624(13)    | 2.1768(30)     | 4.470(35)              |
| 4          | 1.699(1)       | 2.2986(6)      | 4.713(14)              |
| 5          | 1.76948(3)     | 2.3726(45)     | 4.766(56)              |
| 6          | 1.8287(11)     | 2.4265(30)     | 4.709(39)              |
| 8          | 1.920(5)       | 2.5115(40)     | 4.644(77)              |

**TABLE XI.** $\Lambda$ ratios from asymptotic formula

| $N_{\tau}$ | $\alpha_0$ | $\alpha_v$   | $T/\Lambda_L$ | $T/\Lambda_v$ |
|------------|------------|--------------|---------------|---------------|
| 2          | 0.1693(3)  | 0.2466(6)    | 29.7(2)       | 2.27(2)       |
| 3          | 0.1462(2)  | 0.2039(3)    | 41.4(3)       | 2.89(2)       |
| 4          | 0.13848(4) | 0.19022(7)   | 42.1(1)       | 2.848(4)      |
| 5          | 0.1342(2)  | 0.1828(3)    | 40.6(5)       | 2.69(2)       |
| 6          | 0.1312(2)  | 0.1776(3)    | 38.7(3)       | 2.54(2)       |
| 8          | 0.1267(2)  | 0.1700(3)    | 36.0(4)       | 2.32(2)       |
| 16         | 0.1162(4)  | 0.1526(6)    | 32.0(8)       | 1.97(4)       |

**TABLE XII.** Perturbative $\alpha_v$: Wilson case

| $N_{\tau}$ | $\alpha_0$ | $\alpha_v$   | $T/\Lambda_I$ | $T/\Lambda_v$ |
|------------|------------|--------------|---------------|---------------|
| 2          | 0.2306(7)  | 0.286(1)     | 8.81(8)       | 1.49(1)       |
| 3          | 0.19941(2) | 0.24085(3)   | 9.889(3)      | 1.627(6)      |
| 4          | 0.1873(1)  | 0.2239(1)    | 9.526(30)     | 1.549(2)      |
| 5          | 0.179889(3)| 0.213616(4)  | 9.0562(7)     | 1.461(1)      |
| 6          | 0.1741(1)  | 0.2057(1)    | 8.73(3)       | 1.398(2)      |
| 7          | 0.1698(3)  | 0.1998(4)    | 8.38(6)       | 1.34(1)       |
| 8          | 0.1658(4)  | 0.1944(5)    | 8.2(1)        | 1.30(1)       |

**TABLE XIII.** Perturbative $\alpha_v$: Symanzik case
| $\beta$       | $\beta_1$    | $T_1/\Lambda_1$ | $\beta_2$    | $T_2/\Lambda_2$ |
|--------------|--------------|-----------------|--------------|-----------------|
| 1.380(4)     | 1.1280(35)   | 1.832(15)       | 1.331(30)    | 2.953(21)       |
| 1.59624(13)  | 1.36942(16)  | 2.15542(83)     | 1.55480(16)  | 3.3651(13)      |
| 1.699(1)     | 1.5026(13)   | 2.2247(73)      | 1.6876(13)   | 3.485(12)       |
| 1.76948(3)   | 1.59156(3)   | 2.2072(2)       | 1.77784(3)   | 3.4781(3)       |
| 1.8287(11)   | 1.6628(13)   | 2.1876(70)      | 1.850(1)     | 3.462(11)       |
| 1.8747(30)   | 1.7163(34)   | 2.136(18)       | 1.9041(34)   | 3.391(29)       |
| 1.920(5)     | 1.7674(56)   | 2.119(29)       | 1.9557(56)   | 3.371(47)       |

TABLE XIV. Data for the effective schemes
FIG. 1. Binder cumulant: lattices with $N_	au = 2$
FIG. 2. Binder cumulant: lattices with $N_r = 3$
FIG. 3. Binder cumulant: lattices with $N_\tau = 4$
FIG. 4. Binder cumulant: lattices with $N_{\tau} = 5$
FIG. 5. Binder cumulant: lattices with $N_r = 6$
FIG. 6. Binder cumulant: lattices with $N_{\tau} = 7$. 

$g_4$ vs $\beta$ for $7 \times 14$ and $7 \times 20$. The data points with error bars represent the Binder cumulant for different values of $\beta$. The lines connect the data points for visualization.
FIG. 7. Binder cumulant: lattices with $N_T = 8$
FIG. 8. Universal scaling behavior of Binder cumulant.
FIG. 9. Universal scaling behavior of Polyakov loop
FIG. 10. Plot of the $T_c/\Lambda$ ratio by use of the “bare” scheme

- - - - - - - Symanzik: FSS determination
- - - Wilson: FSS determination (Fingberg et al.)
- - - - - - - Symanzik: critical behavior determination
FIG. 11. Critical behavior of Polyakov line
FIG. 12. “Bare” scheme corrected with the first sub-asymptotic coefficient of $\beta$ function
FIG. 13. Comparison of “bare” and effective schemes