Auslander-Reiten Quivers and the Coxeter complex

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Abstract: Let $Q$ be a quiver of type $ADE$. We construct the corresponding Auslander-Reiten quiver as a topological complex inside the Coxeter complex associated with the underlying Dynkin diagram. We use the notion of chamber weights coming from the theory of the canonical basis of quantized enveloping algebras, and show this set has a special linearity property in our setting. Finally, we consider $A_n$ case, and describe how Auslander-Reiten quivers correspond to particular wiring diagrams.

Introduction

Let $D$ be a Dynkin diagram of type $ADE$, and $Q$ a quiver obtained by orienting its edges. Associated with $Q$ is a finite dimensional algebra over $\mathbb{C}$, the path algebra $\mathbb{C}Q$. The category of representations of the quiver $Q$ over $\mathbb{C}$ identifies with the category $\text{mod} \mathbb{C}Q$ of finitely generated $\mathbb{C}Q$ modules. Gabriel’s theorem states that isomorphism classes of indecomposables (“indecomposables”) of this category are indexed by the positive roots of the root system $\Phi$ associated with $D$. Furthermore only quivers of $ADE$ types have finitely many indecomposables.

The Auslander-Reiten quiver $\Gamma_Q$ codifies the structure of the category $\text{mod} \mathbb{C}Q$. Vertices are the indecomposables, arrows are irreducible morphisms between them. Our aim is to construct $\Gamma_Q$ as a topological complex $\tilde{\Gamma}_Q$, while retaining the spirit of Gabriel’s theorem. We use the Coxeter complex $\Sigma$, that is the simplicial complex determined by hyperplanes perpendicular to the positive roots, in order to get a unified labelling of both vertices and arrows of $\tilde{\Gamma}_Q$.

The vertices of $\tilde{\Gamma}_Q$ are one dimensional rays inside $\Sigma$, that is weights lying in Weyl group orbits of the fundamental weights. To be more precise, $Q$ defines a bilinear form $R$ on the Euclidean space $E$ generated by $\Phi$ which corresponds to the homological form at the level of mod $\mathbb{C}Q$. Every column of the matrix of $R$ in the basis of simple roots defines a weight $-\rho_i$. The vertices of $\tilde{\Gamma}_Q$ obtain then out of the $\rho_i$’s by the same linear combinations which construct $\Phi^+$ out of the simple roots $\alpha_1, \ldots, \alpha_n$.

$Q$ defines a particular reduced expression $\tilde{w}_0$ of the longest word of $W$, up to commutation of simple reflections, referred to as an adapted reduced expression. $\tilde{w}_0$ induces a total order $R_{\tilde{w}_0}$ of $\Phi^+$, and the main result of [2] is that the pair $\tilde{w}_0, R_{\tilde{w}_0}$

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encodes the structure of $\Gamma_Q$ (Theorem 1.1). Conversely, one can recover $\Gamma_Q$ out of $\tilde{w}_0$ (Proposition 1.2).

Passing from the simple roots to fundamental weights, one associates with $\tilde{w}_0$ the set of chamber weights $\mathcal{C}_{\tilde{w}_0}$ (Definition 2.1). This set was introduced in \cite{4} in the study of total positivity questions in real semi-simple groups. We show (Corollary 2.5) $\mathcal{C}_{\tilde{w}_0}$ is the set of chamber weights $\mathcal{C}$ defined above using $R$. Actually one has linearity of the $1:1$ correspondence $\varphi_{\tilde{w}_0} : R_{\tilde{w}_0} \rightarrow \mathcal{C}_{\tilde{w}_0}$ (Theorem 2.4), which turns out to be a restriction to $\Phi^+$ of a linear application $-\varphi_R$ defined by $R$. We conjecture linearity is a characterization of reduced expressions $\tilde{w}_0$ adapted to quivers.

Two dimensional faces of $\Sigma$ (planar cones) are uniquely determined by the two rays to which they are adjacent. They may therefore be seen as edges between vertices. This allows us to use the set of vertices $\mathcal{C}_{\tilde{w}_0}$ in order to associate with any reduced expression $\tilde{w}_0$ a quiver $\Gamma_{\tilde{w}_0}$ (Definition 3.5). This quiver depends on $\tilde{w}_0$ only up to commutation of simple reflections. When $\tilde{w}_0$ is adapted to a quiver $Q$, $\Gamma_{\tilde{w}_0}$ depends only on $Q$, so we can note it by $\tilde{\Gamma}_Q$. We show this quiver is isomorphic to $\Gamma_Q$ (Theorem 3.7). We use a direct combinatorial approach, thus isomorphism is proved independently of linearity. Note however Theorem 2.4 defines naturally the isomorphism involved. The $\tilde{\Gamma}_Q$ being special cases of the $\Gamma_{\tilde{w}_0}$, we call the later abstract Auslander-Reiten quivers.

The last section describes as an example $A_n$ type. This case is characterized by the fact all fundamental representations are minuscule. We can therefore index one-dimensional rays in $\Sigma$ by Young columns, and arrows of $\Gamma_{\tilde{w}_0}$ become simply couples of column tableaus characterized by inclusion of indices (Lemma 4.1). This fact, combined with the existence of action of a Coxeter element in $\tilde{\Gamma}_Q$ (Proposition 2.7) similar to the well-known one on $\Gamma_Q$, allows us to give a combinatorial description (Proposition 4.3) of $\tilde{\Gamma}_Q$. On the theoretical level, use of Young columns makes the link between $\Gamma_{\tilde{w}_0}$ and the wiring diagram $\mathcal{WD}(\tilde{w}_0)$ of $\tilde{w}_0$, used in \cite{3} as a key tool in the study of Lusztig’s parameterizations of the canonical basis (consult the more recent \cite{5} for use of chamber weights).

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1 Auslander-Reiten quivers

We recall in this section basic facts about Auslander-Reiten quivers, and proceed to study their link with reduced words of the longest word of the Weyl group.

Let us fix $\mathbb{C}$ as the ground field $k$. Consider a $ADE$ Dynkin diagram $D$, of rank $n$, and index its vertices as in \cite{14}, page 53. For sake of simplicity we shall identify the set of simple roots $\{\alpha_1, \ldots, \alpha_n\}$ with $I = \{1, \ldots, n\}$.

We note by $\Phi$ the root system associated with $D$, $E$ the real vector space in which it lies, $(\ ,\ )$ the scalar product defined on $E$ by the Cartan matrix, and by $\alpha_i^\vee$ the coroots. Let $W$ be the Weyl group, $\{s_1, \ldots, s_n\}$ the set of simple reflections, $l()$ the length function, and $w_0$ the unique longest element of $W$. $w_0$ induces an involution on
I (paragraphs (XI) of [8], Planches I, IV, V, VI, VII), which we shall note by $i \mapsto i^\ast$.

We obtain a quiver $Q$ by orienting the edges of $D$. $Q$ is given therefore as a couple $(Q_0, Q_1)$, with $Q_0 (= I)$ a set of vertices, and $Q_1$ a set of arrows. In this work $Q$ will always denote a fixed quiver whose underlying Dynkin diagram is of type $ADE$. Let $\text{mod } \mathbb{C}Q$ denote the category of finite type representations of $Q$ over the complex numbers $\mathbb{C}$.

77.1. An object in this category is given by $M = (V, \varphi)$, where with every $i \in I$ we associate a finite dimensional vector space $V_i$, and with every arrow $i \rightarrow j$ in $Q_1$ a linear mapping $\varphi_{i,j} : V_i \rightarrow V_j$. One obtains the simple objects by associating a one dimensional vector space to a vertex $i$, and 0 spaces to the others. They are therefore in 1:1 correspondence with the simple roots.

We shall note by $\text{Ind } Q$ the set of isomorphism classes $[M]$ of indecomposable representations in $\text{mod } \mathbb{C}Q$. The dimension vector of a representation $M$ is the linear combination $\dim M = \sum_{i \in I} a_i \alpha_i$ with $a_i = \dim V_i$.

**Gabriel’s Theorem**  Let $Q$ be a quiver of type $ADE$. Then the mapping $[M] \mapsto \dim M$ establishes a one to one correspondence between $\text{Ind } Q$ and $\Phi^+$. Furthermore, the quivers of type $ADE$ are the only ones for which $\text{Ind } Q$ is a finite set.

The category $\text{mod } \mathbb{C}Q$ is hereditary. As observed by Ringel [18] (see also [19]), the homological form $\langle M, N \rangle := \dim_{\mathbb{C}} \text{Hom}(M, N) - \dim_{\mathbb{C}} \text{Ext}^1(M, N)$ depends only on the dimension vectors. If we note by $[\beta]$ the isomorphism class of the indecomposable corresponding to $\beta \in \Phi^+$, then $\langle [\alpha], [\beta] \rangle = R(\alpha, \beta)$ where $R$ is the bilinear form on $E$ defined by

$$r_{i,j} := R(\alpha_i, \alpha_j) = \begin{cases} 1 & \text{if } i = j \\ -1 & \text{if } i \rightarrow j \in Q_1 \\ 0 & \text{otherwise} \end{cases}$$

We shall refer to $R$ as the Ringel form of $Q$.

The structure of $\text{mod } \mathbb{C}Q$ is codified by the Auslander-Reiten quiver $\Gamma_Q$. Vertices are $\text{Ind } Q$, arrows are irreducible morphisms, namely morphisms that cannot be written as a non-trivial composition of two morphisms ([1] page 166). $\Gamma_Q$ is endowed with translation $\tau$ ([1] page 225), which stratifies it into levels $\Gamma_Q^i$ indexed by $I$. Such a level is of the form

$$(P_i(Q) =) \tau^{N_i-1}I_i(Q), \tau^{N_i-2}I_i(Q), \ldots, \tau I_i(Q), I_i(Q)$$

with $P_i(Q)$ the projective cover of the simple module corresponding to $i \in I$, and $I_i(Q)$ the injective envelope of the simple module corresponding to $i^\ast \in I$.

A detailed construction of $\Gamma_Q$ our case is given by [11] section 6.5. With $D$ one associates an infinite quiver $\mathbb{Z}D$ [11] fig 13. page 49, whose vertices are $I \times \mathbb{Z}$. Arrows can exist between vertices only if their first coordinates are linked in $D$. The vertices of $D$ are ordered in a natural way such that for $i, j$ linked in $D$, the corresponding arrows
in \( \mathbb{Z}D \) are given by the configuration

\[
\begin{array}{ccc}
  (j-1) & (j,0) & (j,1) \\
  (i,-1) & (i,0) & (i,1) \\
  \vdots & \vdots & \vdots
\end{array}
\]

Thus \( \mathbb{Z}D \) is an “infinite mesh” deduced out of \( D \). One defines on it the Nakajima involution \( \nu \) \([1]\) page 48. The opposite quiver \( Q^{op} \) to \( Q \) may be realized inside \( \mathbb{Z}D \) by a slice \( S_{\text{proj}} \) starting at \((1,0)\). Its image by \( \nu \) gives a slice \( S_{\text{inj}} \) which has the same arrows as \( Q \). The Auslander-Reiten quiver then identifies with the subgraph of \( \mathbb{Z}D \) delimited by \( S_{\text{proj}} \) to the left, and \( S_{\text{inj}} \) to the right. Vertices of \( S_{\text{proj}} \) (resp. \( S_{\text{inj}} \)) are the projectives in \( \text{mod } \mathbb{C}Q \) (resp. injectives). Each translation level \( \Gamma^i_Q \) corresponds to the vertices in its realization inside \( \mathbb{Z}D \) having \( i \) as first coordinate. One may deduce recursively the dimension vectors of the vertices of \( \Gamma_Q \) out of the slice \( S_{\text{proj}} \) by use of additivity \([1]\) page 50) (for examples, see \([2]\) Examples 2.7, 2.8).

**Example 1.1**: Consider the \( A_n \) type quiver \( Q^{(d)} : \bullet \to 2 \to \ldots \to n \), which is distinguished by the regularity of its structure. We reproduce the figure in \([19]\) page 85, \( \Gamma_{Q^{(d)}} \) is given by

\[
\begin{array}{ccc}
M_{1,n+1} & M_{2,n+1} & M_{1,n} \\
M_{3,n+1} & M_{2,n} & M_{1,n-1} \\
\vdots & \vdots & \vdots \\
M_{n-1,n+1} & M_{n,n+1} & M_{n-1,n} \\
M_{n,n+1} & M_{n-1,n} & M_{n,n-1} \\
\end{array}
\]

where \( M_{i+1,j+1} := [\alpha_i + \alpha_i+1 + \ldots + \alpha_j] \). We have \( S_{\text{proj}} = \{M_{1,n+1}, M_{2,n+1}, \ldots M_{n,n+1}\} \), \( (M_{n,n+1} \) corresponding to \((0,1) \in \mathbb{Z}D\), and \( S_{\text{inj}} = \{M_{1,n+1}, M_{1,n}, \ldots M_{1,3}, M_{1,2}\} \). For \( i \in I \), the translation level \( \Gamma^i_{Q^{(d)}} \) is \( i \)th horizontal level from the top (i.e. containing \([M_{i,n+1}]\)).

Given \( w \in W \), we shall use the notation \( \bar{w} = s_{i_1} \ldots s_{i_N} \) for a reduced expression of \( w \), the \( \sim \) indicating we are referring not to \( w \) but to its reduced expression. Recall one has for the longest element \( l(w_0) = N \) where \( N = \text{card} \Phi^+ \). A fixed reduced expression \( \bar{w}_0 = s_{i_1} s_{i_2} \ldots s_{i_N} \) induces a total ordering of the positive roots \( R_{\bar{w}_0} = (\beta_1, \beta_2, \ldots \beta_N) \) by

\[
\beta_j = s_{i_1} s_{i_2} \ldots s_{i_{j-1}}(\alpha_{i_j}).
\]

\( R_{\bar{w}_0} \) obtains as a linear refinement of convex partial order on \( \Phi^+ \). Convex (or normal) means a partial order \( \preceq \) verifying conditions \([3]\):

\[
\forall \alpha, \beta \in \Phi^+, \text{ such that } \gamma = \alpha + \beta \in \Phi^+: \text{ either } \alpha \preceq \gamma \preceq \beta \text{ or } \beta \preceq \gamma \preceq \alpha.
\]
Conversely, a convex order $\preceq$, determines $\bar{w}_0$, up to commutation of simple reflections, that is up to ”2-moves” replacing $s_is_{i+1}$ by $s_{i+1}s_i$, $i_j$ and $i_{j+1}$ not linked in $D$. If we consider two reduced expressions of $w_0$ as equivalent if one may be obtained from the other by a sequence of 2-moves, then the equivalence classes we obtain are known as commutation classes \[2\] 1.4. We have a 1 : 1 correspondence between them and convex orderings. All constructions in the sequel will actually depend only on the commutation class (or equivalently a convex ordering).

A quiver $Q$ defines a convex ordering $\preceq_Q$ \[9\]. One may use the Ringel form, putting $\beta \preceq \gamma$ if $\langle \beta, \gamma \rangle > 0$, and taking the transitive closure of these relations. The same partial order may be obtained from $\Gamma_Q$, one has $\alpha \preceq \beta$ iff there is a path in $\Gamma_Q$ from $[\alpha]$ to $[\beta]$. Equivalence of the two constructions follows form the fact arrows in $\Gamma_Q$ are couples $[\alpha], [\beta]$ such that $R(\alpha, \beta) > 0$ and there is no $\gamma$ such that $R(\alpha, \gamma) > 0$ and $R(\gamma, \beta) > 0$. We shall say $\preceq$ is the convex ordering adapted to $Q$, and use the same reference to reduced expressions in the corresponding commutation class.

We say, following \[3\] a vertex $k$ is a sink of $Q$ if there are only entering arrows into it. Call it a source if there are only exiting arrows out of it. If $k$ is a sink of $Q$ define $S_kQ$ as the quiver obtained by reversing the direction of the arrows with $k$ as end, thus turning $k$ into a source. A reduced expression $\bar{w}_0 = s_is_{i+1}\ldots s_{iN}$ is adapted to $Q$ if and only if $i_1$ is a sink of $Q$, $i_2$ a sink of $S_iQ$, $i_3$ is a sink of $S_{i_2}S_iQ$ and so on.

**Example 1.2 :** Consider $Q = Q^{(d)} : \bullet \rightarrow \bullet \rightarrow \bullet$ of type $A_3$. 1 is a source, 3 is a sink, and $Q' = S_3Q$ is then $$Q' : \bullet \rightarrow \bullet \rightarrow \bullet$$

We leave it to the reader to verify $\bar{w}_0 = s_is_{i+1}\ldots s_{iN}$ is adapted to $Q^{(d)}$ while $\bar{w}_0 = s_is_{i+1}s_{i+2}s_{i+3}$ is adapted to $Q'$. For arbitrary $n$, one may verify $\bar{w}_0 = s_{n}s_{n-1}\ldots s_{1}s_{n}s_{n-1}\ldots s_{2}\ldots s_{n}s_{n-1}s_n$ is adapted to the distinguished orientation $Q_d$. Given $Q$, let us renumber its vertices in a compatible way with its arrows. Namely we renumber $\{\alpha_1, \ldots, \alpha_n\}$ as $\{\beta_1, \ldots, \beta_n\}$ such that $\beta_j \rightarrow \beta_k$ in $Q$ implies $j > k$. One then associates to $Q$ (\[3\] §1.2) a Coxeter element $c = s_{\beta_n}\ldots s_{\beta_2}s_{\beta_1}$. There might be several compatible indexations, yet by \[3\] Lemma 1.2, $c$ is independent of a particular choice. For instance $c = s_is_{is_1}s_1$ is the Coxeter element associated to the quiver $Q^{(d)}$ in the example above, and $c = s_is_{is_2}(=s_{is_1}s_1)$ the Coxeter element to the quiver $Q'$. Observe, as in the proof of \[3\] Theorem 1.2, $s_{\beta_n}\ldots s_{\beta_2}s_{\beta_1}$ is a sequence adapted to $Q$. It is well known that translation $\tau$ of $\Gamma_Q$ becomes action of the Coxeter element $c$ at the level of dimension vectors.

If $k$ is a sink of $Q$, and $Q' := S_kQ$ then the categories $\text{mod} \mathbb{C}Q$ and $\text{mod} \mathbb{C}Q'$ are closely linked by the BGP-reflection functor \[3\] 77.4 corresponding to $s_k$. Bedard \[2\] combines use of these functors together with the description of $\Gamma_Q$ inside $\mathbb{Z}D$ in order to show how to obtain from $\Gamma_Q$ all $\bar{w}_0$ adapted to $Q$.

If for $i \in I$, the occurrences of $s_i$ in $\bar{w}_0$ are at positions $j_1, j_2, \ldots, j_r$ define $\mathcal{R}_{\bar{w}_0}^{i_j}$ as the ordered $r$-tuple $(\gamma_1, \gamma_2, \ldots, \gamma_r)$ with $\gamma_k = s_{i_1}s_{i_2}\ldots s_{i_{j_{k-1}}} (\alpha_{j_k})$ Theorem 2.17 of \[2\] may
Theorem 1.1 Let $Q$ be a quiver of type $ADE$.

a) Associate with every vertex $[\beta]$ of $\Gamma_Q$ the translation level $i_\beta$ it belongs to. Read $\text{Ind}_Q$ sequentially, in a manner compatible with the arrows (i.e. $[\alpha] \rightarrow [\beta]$ implies $[\alpha]$ appears before $[\beta]$). Replacing vertices $[\beta]$ by the corresponding $i_\beta$ one produces an $N$-tuple $(i_1, i_2, \ldots, i_N)$ giving a reduced expression adapted to $Q$. Furthermore, all $\tilde{w}_0$ adapted to $Q$ are obtained this way.

b) Fix $\tilde{w}_0$ as in a). For all $i \in I$ taking dimension vectors establishes a 1:1 correspondence (as ordered sets) between $\Gamma_i^Q$ and $R_{\tilde{w}_0}^i$.

Part b) tells us in particular each $[\alpha] \in \Gamma_i^Q$ has the form $\alpha = s_{i_1} s_{i_2} \ldots s_{i_{j-1}} (\alpha_{i_j})$ with $i_j = i$. We shall note by $N_i(Q)$ the cardinality of $\Gamma_i^Q$ (or of $R_{\tilde{w}_0}^i$).

Example 1.3: Consider the quiver $Q := \bullet \rightarrow \bullet \leftarrow \bullet$ of type $A_3$. $\Gamma_Q$ is

```
[\alpha_1 + \alpha_2] 
[\alpha_2]  \rightarrow  [\alpha_1 + \alpha_2 + \alpha_3] 
[\alpha_2 + \alpha_3]  \rightarrow  [\alpha_3] 
```

Replacing vertices by their translation levels one obtains

```
1 \rightarrow 2 \leftarrow 1 
2 \leftarrow 3 \rightarrow 2 
1 \rightarrow 3 \leftarrow 1 
```

There are 4 ways of reading the vertices in a compatible way with the arrows $(2, 1, 3, 2, 1, 3)$, $(2, 1, 3, 2, 3, 1)$, $(2, 3, 1, 2, 1, 3)$, $(2, 3, 1, 2, 3, 1)$. The 4 corresponding reduced expressions form the commutation class determined by $Q$.

Theorem 1.1 allows us also to describe the opposite direction, recovering $\Gamma_Q$ from a $\tilde{w}_0$ adapted to $Q$. Citing the referee, this result is well known to the experts, yet unpublished so we provide its proof.

Proposition 1.2 Let $\tilde{w}_0$ be a reduced expression adapted to a quiver $Q$ of type $ADE$. Consider a couple of roots $\alpha = s_{i_1} s_{i_2} \ldots s_{i_{j-1}} (\alpha_{i_j})$, $\beta = s_{i_1} s_{i_2} \ldots s_{i_{k-1}} (\alpha_{i_k})$ in $R_{\tilde{w}_0}$. $[\alpha] \rightarrow [\beta]$ is an arrow in $\Gamma_Q$ iff

i) $i_j$ and $i_k$ are linked in the Dynkin diagram $D$,

ii) $j < k$,

iii) For $j < p < k$, $i_p \neq i_j$. 


Proof: Since $\Gamma_Q$ lies inside $\mathbb{Z}D$, observe there are arrows only between translation levels whose indices are linked in $D$. We need therefore only to consider the case of $i_j$ and $i_k$ linked in $D$. Put $a := i_j$, $b := i_k$. As $\tilde{w}_0$ is adapted to $Q$, by Theorem 1.1 it obtains by reading translation levels of vertices of $\Gamma_Q$ in a compatible manner with its arrows. Now the mesh structure of $\mathbb{Z}D$ implies an arrow from a vertex $[\alpha]$ on $\Gamma^\alpha_Q$ to a vertex $[\beta]$ on $\Gamma^\beta_Q$ is immediately followed by an arrow from $[\beta]$ to the vertex $[\gamma]$ following $[\alpha]$ on the translation level $\Gamma^\alpha_Q$, and vice-versa. In order to comply with this configuration, if we delete from $\tilde{w}_0$ all simple reflection $s_{i_l}$ with $l \neq a, b$ we have to obtain a word either of the form $s_as bs s_b \ldots$ or of the form $s_b s_a s_b s_a \ldots$. By ii) and iii), our criterion gives then the same structure of arrows as the one of $\Gamma_Q$ inside $\mathbb{Z}D$.

**Definition 1.3** Let $\tilde{w} = s_{i_1} \ldots s_{i_m}$ be a reduced expression of $w \in W$. We shall say $\tilde{w}$ is alternating if for every couple $j, k$ of indices linked in $D$, the word one obtains by erasing from $\tilde{w}$ all $s_{i_r}$, $i_r \neq j, k$ is either of the form $s_is_is_j\ldots$ or of the form $s_js_is_j\ldots$.

**Lemma 1.4** Any reduced expression $\tilde{w}_0$ adapted to a quiver $Q$ is alternating.

The proof is included in that of Proposition 1.2.

**Example 1.4:** Consider $\tilde{w}_0 = s_3s_2s_1s_3s_2s_3$. For the couple $\{1, 2\}$, we get by erasing occurrences of $s_3$, the word $s_2 s_1 s_2$, and for the couple $\{2, 3\}$ erasing occurrences of $s_1$ gives $s_3 s_2 s_3 s_2 s_3$.

## 2 The linearity phenomenon

We import in this section the notion of chamber weights $C_{\tilde{w}_0}$ from quantum groups and show in case $\tilde{w}_0$ is adapted to $Q$, the correspondence $\varphi_{\tilde{w}_0} : R_{\tilde{w}_0} \longrightarrow C_{\tilde{w}_0}$ is linear and given by the Ringel form.

Let us consider orbits of fundamental weights $\omega_i$ instead of positive roots.

**Definition 2.1** Let $W$ be a Weyl group of finite type, and let us fix a reduced expression $\tilde{w}_0$. The set of chamber weights associated with $\tilde{w}_0$ is the $N$-tuple $C_{\tilde{w}_0} := (\mu_1, \mu_2, \ldots, \mu_N)$ where $\mu_j = s_{i_1}s_{i_2} \ldots s_{i_j}(\omega_{i_j})$.

Chamber weights were introduced in [3] as a key tool in the study of total positivity in semi-simple groups. Our convention is in opposite order to that in [3]. All elements of $C_{\tilde{w}_0}$ are distinct ([3] 2.9). By its construction, $C_{\tilde{w}_0}$ seen as a set depends only on the commutation class of $\tilde{w}_0$. Let us observe the cardinality of $\mathcal{F} = \bigcup_{i \in I} W \omega_i$ is greater than $N$, so unlike the positive roots case, $\tilde{w}_0$ actually “cuts out” a subset inside $\mathcal{F}$.

For any $\tilde{w}_0$, one has a natural $1 : 1$ correspondence $\varphi_{\tilde{w}_0}$ between $R_{\tilde{w}_0} = (\beta_1, \ldots, \beta_N)$ and $C_{\tilde{w}_0} = (\mu_1, \ldots, \mu_N)$ defined by $\varphi_{\tilde{w}_0}(\beta_k) = \mu_k$, $k = 1, \ldots, N$. Observe this bijection depends only on the commutation class, as interchange of two commuting reflections $s_{i_j}, s_{i_{j+1}}$ results in an interchange between $\beta_j, \beta_{j+1}$ in $R_{\tilde{w}_0}$, and $\mu_j, \mu_{j+1}$ in $C_{\tilde{w}_0}$.

If we consider columns of the matrix $(r_{i,j})$ of the Ringel form $R$ as coordinate vectors in the basis of the $\omega_i$’s, we obtain the following weights :

$$\forall i \in I : \rho_i := \sum_{\substack{l \in I \\mid \varnothing \cap \{i\} \neq \varnothing}} r_{l,i} \omega_i.$$

We note by $\varphi_R$ the linear mapping from $E$ to $E$ defined by $\varphi_R(\alpha_i) = \rho_i$ for $i$ in $I$. 

7
Lemma 2.2 Suppose \( k \) is a sink of \( Q \) and \( Q' := S_k Q \). Let \( R \) and \( R' \) be the corresponding Ringel forms. Then

\[
s_k \varphi_{R'} = \varphi_{R} s_k
\]

Proof: The lemma is a direct consequence of the behavior of Ringel forms under BGP reflection functors. Let \( \{ \rho_i \}_{i \in I}, \{ \rho'_i \}_{i \in I} \) be the respective weights defined by \( R \) and \( R' \). Note by \( D_k \) the set of \( j \neq k \) linked to \( k \) in \( D \).

The coefficient of \( \omega_k \) in \( \rho_j \) verifies \([\rho_j : \omega_k] = 0 \) for \( j \neq k \), while \( \rho_k = \omega_k - \sum_{j \in D_k} \omega_j \).

In comparison, we have for \( Q' \):

\[
\rho'_j = \begin{cases} 
\omega_k & j = k, \\
\rho_j - \omega_k & j \in D_k, \\
\rho_j & \text{otherwise.}
\end{cases}
\]

Let us show directly \( \forall j \in I : \varphi_{R'}(\alpha_j) = s_k \varphi_R s_k(\alpha_j) \) (we give details for convenience of the reader).

Case 1: \( j = k \).

\[
s_k \varphi_R s_k(\alpha_k) = s_k(-\rho_k) = s_k(-\omega_k + \sum_{j \in D_k} \omega_j) = -\omega_k + \alpha_k + \sum_{j \in D_k} \omega_j.
\]

Now \( \alpha_k = 2\omega_k - \sum_{j \in D_k} \omega_j \) so we get \( s_k(-\rho_k) = \omega_k = \rho'_k = \varphi_{R'}(\alpha_k) \).

Case 2: \( j \in D_k \).

\[
s_k \varphi s_k(\alpha_j) = s_k \varphi_R(\alpha_j + \alpha_k) = s_k(\rho_j + \rho_k).
\]

\( s_k(\rho_j) = \rho_j \) since \([\rho_j : \omega_k] = 0 \). We have just seen \( s_k(\rho_k) = -\omega_k \). We get therefore \( s_k \varphi_{R} s_k(\alpha_j) = \rho_j - \omega_k = \rho'_j = \varphi_{R'}(\alpha_j) \).

Case 3: \( j \neq k \) and not in \( D_k \).

Immediate, \( \rho_j = \rho'_j \) and \( s_k \) stabilizes both \( \alpha_j \) and \( \rho_j \).

Corollary 2.3 Assume the expression \( s_{i_1} s_{i_2} \ldots s_{i_{k-1}} \) is adapted to \( Q \). Note for \( j = 1 \ldots k-1 : Q_j := S_{i_{j-1}} s_{i_{j-2}} \ldots S_{i_1} Q \) (\( Q_1 := Q \)) and \( R_j \) the Ringel form corresponding to \( Q_j \). Then

\[
\varphi_{R_1 s_{i_1} s_{i_2} \ldots s_{i_{k-1}}} = s_{i_1} s_{i_2} \ldots s_{i_{k-1}} \varphi_{R_k}.
\]

Proof: For any \( j = 1, \ldots k-1, i_j \) is a sink of \( Q_j \) so that \( \varphi_{R_j} s_{i_j} = s_{i_j} \varphi_{R_{j+1}} \).

Theorem 2.4 (Linearity) Let \( \tilde{w}_0 \) a reduced expression adapted to \( Q \) of type ADE. Then the mapping \( \varphi_{\tilde{w}_0} \) is linear and given by the Ringel form \( R \):

\[
\forall 1 \leq k \leq N, \mu_k = -\varphi_R(\beta_k)
\]

Corollary 2.5 The \( -\rho_i \)'s, \( i \in I \) are elements of \( C_{\tilde{w}_0} \), and \( C_{\tilde{w}_0} \) is obtained from them by the same linear combinations which construct \( \Phi^+ \) out of the simple roots \( \{ \alpha_1, \ldots, \alpha_n \} \).
Proof of Theorem 2.4: Consider an arbitrary \( k, 1 \leq k \leq N \). By hypothesis, \( \tilde{w}_0 = s_{i_1}s_{i_2} \ldots s_{i_N} \) is adapted to \( Q \) so may apply Corollary 2.3 and its conventions (in particular \( R_1 = R \)).

\[
\varphi_R(\beta_k) = \varphi_{R_1} s_{i_1} s_{i_2} \ldots s_{i_{k-1}} (\alpha_{i_k}) \\
= s_{i_1} s_{i_2} \ldots s_{i_{k-1}} \varphi_R(s_{i_k}(-\alpha_{i_k})) \\
= s_{i_1} s_{i_2} \ldots s_{i_{k-1}} s_{i_k} \varphi_{R_{k+1}}(-\alpha_{i_k}) \\
= -s_{i_1} s_{i_2} \ldots s_{i_{k-1}} s_{i_k} \varphi_{R_{k+1}}(\alpha_{i_k})
\]

Observe \( i_k \) is a source of \( Q_{k+1} \). We have seen while proving Lemma 2.2, in that case \( \varphi_{R_{k+1}}(\alpha_{i_k}) = \omega_{i_k} \), so finally

\[
\varphi_R(\beta_k) = -s_{i_1} s_{i_2} \ldots s_{i_k} (\omega_{i_k}) = -\mu_k. \quad \square
\]

Examples 2.1 Consider \( A_3 \) type.

1) We have seen \( \tilde{w}_0 = s_2s_1s_3s_2s_3s_1 \) is adapted to a quiver. \( R_{\tilde{w}_0} \) is given by

\[
\beta_1 = \alpha_2, \quad \beta_3 = \alpha_2 + \alpha_3, \quad \beta_5 = \alpha_1, \\
\beta_2 = \alpha_1 + \alpha_2, \quad \beta_4 = \alpha_1 + \alpha_2 + \alpha_3, \quad \beta_6 = \alpha_3,
\]

and \( C_{\tilde{w}_0} \) by

\[
\mu_1 = \omega_1 - \omega_2 + \omega_3, \quad \mu_3 = \omega_1 - \omega_2, \quad \mu_5 = -\omega_1, \\
\mu_2 = -\omega_2 + \omega_3, \quad \mu_4 = -\omega_2, \quad \mu_6 = -\omega_3.
\]

The matrix of the Ringel form is

\[
\begin{bmatrix}
1 & -1 & 0 \\
0 & 1 & 0 \\
0 & -1 & 1
\end{bmatrix}
\]

\( \mu_5, \mu_1, \mu_6 \) are the respective images by \( \varphi_{\tilde{w}_0} \) of the simple roots, and indeed

\(-\rho_1 = -\omega_1 = \mu_5, -\rho_2 = \omega_1 - \omega_2 + \omega_3 = -\mu_1, -\rho_3 = -\omega_3 = \mu_6. \)

Finally, observe one has \( \mu_2 = \varphi_{\tilde{w}_0}(\alpha_1 + \alpha_2) \), and as expected \(-\rho_1 - \rho_2 = -\omega_2 + \omega_3. \)

Similarly, \( \mu_3 = -\rho_2 - \rho_3 \)

and \( \mu_4 = -\rho_1 - \rho_2 - \rho_3. \)

2) The reduced expression \( \tilde{w}_0 = s_2s_1s_2s_3s_2s_1 \) is not adapted to a quiver. \( R_{\tilde{w}_0} = (\beta'_1, \ldots, \beta'_6) \) with

\[
\beta'_1 = \alpha_2, \quad \beta'_3 = \alpha_1, \quad \beta'_5 = \alpha_2 + \alpha_3, \\
\beta'_2 = \alpha_1 + \alpha_2, \quad \beta'_4 = \alpha_1 + \alpha_2 + \alpha_3, \quad \beta'_6 = \alpha_3,
\]

while \( C_{\tilde{w}_0} = (\mu'_1, \ldots, \mu'_6) \) with

\[
\mu'_1 = \omega_1 - \omega_2 + \omega_3, \quad \mu'_3 = -\omega_1 + \omega_3, \quad \mu'_5 = -\omega_2, \\
\mu'_2 = -\omega_2 + \omega_3, \quad \mu'_4 = -\omega_1, \quad \mu'_6 = -\omega_3.
\]

In this case, linearity fails. Observe \( \mu'_2 = \varphi_{\tilde{w}_0}(\alpha_1 + \alpha_2) = -\omega_2 + \omega_3 \) while \( \mu'_3 = \varphi_{\tilde{w}_0}(\alpha_1) \), \( \mu'_1 = \varphi_{\tilde{w}_0}(\alpha_2) \) verify \( \mu'_3 + \mu'_1 = -\omega_2 + 2\omega_3. \)

The set \( \mathcal{A} \) of adapted convex orderings (commutation classes) is a small subset of \( \mathcal{O} \) of all convex orderings, we provide the table below for small ranks:


|     | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $D_4$ | $D_5$ |
|-----|-------|-------|-------|-------|-------|-------|
| card $\mathcal{A}$ | 4     | 8     | 16    | 32    | 8     | 16    |
| card $\mathcal{O}$  | 8     | 62    | 908   | 24968 | 182   | 13198 |

**Conjecture** \( \tilde{w}_0 \) is adapted to a quiver iff the mapping \( \varphi_{\tilde{w}_0} \) is linear.

Computer testing shows linearity fails for all convex orderings in $\mathcal{O}\setminus\mathcal{A}$ for the Dynkin types in the table above. It is not clear to us at the moment how linearity should be modified in order to describe \( \varphi_{\tilde{w}_0} \) for arbitrary commutation classes.

The set $\mathcal{C}_{\tilde{w}_0}$ may be stratified into levels the same way $\mathcal{R}_{\tilde{w}_0}$ is, namely the \( i \)th level $\mathcal{C}_{\tilde{w}_0}^i$ is the set of weights $\mu_k = s_{i_1} \ldots s_{i_k}(\omega_{i_k})$ with $i_k$ equal to $i$.

**Lemma 2.6** For all $i \in I$, the last element of $\mathcal{C}_{\tilde{w}_0}^i$ is $-\omega_i^\ast$.

**Proof:** Consider the last appearance $s_{i_k} = s_i$ in $\tilde{w}_0$. All following reflections $s_{j_m}$ with $m > k$ stabilize $\omega_i$. Hence $s_{j_1}s_{j_2} \ldots s_{j_k}\omega_i = w_0\omega_i = -\omega_i^\ast$. \( \square \)

**Proposition 2.7** Let $\tilde{w}_0$ be adapted to $Q$, and $c$ the Coxeter element associated to $Q$. Then for all $i \in I$:

\[
\mathcal{C}_{\tilde{w}_0}^i = (c^{N_i-1}(-\omega_i^\ast), c^{N_i-2}(-\omega_i^\ast), \ldots, c(-\omega_i^\ast), -\omega_i^\ast)
\]

where $N_i = N_i(Q)$.

**Proof:** Recall $c = s_{j_1}s_{j_2} \ldots s_{j_n}$ with $j_1$ a sink of $Q_1 := Q$, $j_2$ a sink of $Q_2 := S_1Q$ and so on. As $(j_1, j_2, \ldots, j_n)$ is a permutation of \{1, \ldots, $n$\}, every arrow in $Q$ is inverted twice in the sequence $S_{j_n} \ldots S_{j_1}Q$ so have $Q = Q_{n+1}$. Using Corollary 2.3 and $R_{n+1} = R_1 = R$, we get $c\varphi_R = \varphi_Rc$. By Theorem 2.4 we can replace $\varphi_R$ by $\varphi_{\tilde{w}_0}$ so that :

\[
(*) \ c\varphi_{\tilde{w}_0} = \varphi_{\tilde{w}_0}c.
\]

We already know by Auslander-Reiten theory the levels of $\mathcal{R}_{\tilde{w}_0}$ are given by the action of $c$. It remains to apply $(*)$ to get the statement of the proposition. \( \square \)

**Remark:** If $\tilde{w}_0$ is not adapted to a quiver $Q$, the levels $\mathcal{C}_{\tilde{w}_0}^i$ may not be given through the action of a Coxeter element. We do not have for the moment a Weyl group model for the levels in general case.

## 3 The Coxeter complex construction

In this section we consider elements of $\mathcal{C}_{\tilde{w}_0}$ as one dimensional rays in the Coxeter complex $\Sigma$. The structure of $\Sigma$ allows us to construct arrows, and associate a quiver $\Gamma_{\tilde{w}_0}$ with $\tilde{w}_0$. When $\tilde{w}_0$ is adapted to a quiver $Q$, we show we recover $\Gamma_Q$.

Recall the geometry of the root system $\Phi$ is captured by the Coxeter complex $\Sigma$ (see [8] chapters 1-3, [7] chapter V). Geometrically $\Sigma$ is the simplicial complex induced by the hyperplanes orthogonal to the positive roots. Maximal elements are the open chambers. Recall a parabolic subgroup of $W$ is a subgroup $W^J = \langle s_j \rangle_{j \in J}$ with $J$ a subset of $I$. $\Sigma$ can be constructed out of $W$ as the complex of all left classes of all parabolic subgroups partially ordered by inverse inclusion.

It would be more convenient for us to work in $\Sigma^{op}$, the inversely ordered complex, as the partial order becomes plain inclusion. Note $W^{(J)} := \langle s_j \rangle_{j \in I \setminus J}$. The maximal
elements (facets) of $\Sigma^{op}$ are left classes of maximal parabolics $W^{(i)}$ (equivalently, one dimensional rays). Two facets $wW^{(i)}$, $w'W^{(j)}$ of $\Sigma^{op}$ are adjacent iff their intersection is non-empty, which in that case, it is a left class of $W^{(i,j)}$.

Let us recall now basic facts about parabolic subgroups, one may consult for details [13] Chapter 1 or [12] Chapters 1 and 2 (Geck and Pfeiffer work with right classes, but all the arguments can be easily adapted to left classes).

Fix $J \subset I$. $W^J$ is a itself a Coxeter group, and as such is endowed with its proper length function $l^J$. 

Proposition 3.1 ([12] 1.2.10) Consider $w \in W^J$. Then $l^J(w) = l(w)$. Moreover, if $\tilde{w} = s_{i_1} \ldots s_{i_m}$ is any reduced expression of $w$ inside $W$, then $\forall k$, $i_k$ is in $J$.

Proposition 3.2 ([12] 2.1.1) Define $X^J = \{ w \in W \mid l(ws) > l(w) \forall x \in J \}$.

a) Every left class $wW^J$ contains a unique element $u \in X_J$ and $u$ is the unique element of minimal length inside $wW^J$.

b) For any element $y \in W^J$, if we write $y = ux$ with $x \in W^J$, then $l(y) = l(u) + l(x)$.

Consider $J \subset K \subset I$, and note by $X^K_J$ the set of minimal representatives of the left classes of $W^K$ in $W^K$.

Lemma 3.3 ([12] 2.2.5) $X_J = X_K X^K_J$.

Finally, $W^{(i)}$ is the stabilizer of $\omega_i$, thus a weight $\mu = w\omega_i$ may be identified with the left class $wW^{(i)}$. If $u$ is the minimal length element inside $wW^{(i)}$ we shall say $u\omega_i$ is the minimal presentation of $\mu$.

Lemma 3.4 ([4] 2.7) Let $\mu = w\omega_i$, and $\tilde{w} = s_{i_1} s_{i_2} \ldots s_{i_m}$ a reduced expression of $w$. The minimal representation of $\mu$ is obtained by erasing from $\tilde{w}$ all factors $i_k$ for which $\langle \omega_i, \beta_k \rangle = 0$, where $\beta_k = s_{i_m} s_{i_{m-1}} \ldots s_{i_k+1}(\alpha_{i_k})$.

Chamber weights can be seen as facets of $\Sigma^{op}$ by identification above. We shall feel free to consider them as appropriate either as weights or left classes.

Definition 3.5

Let $\tilde{w}_0$ be a reduced expression of $w_0$ in $W$ a Weyl group of finite type. The abstract Auslander-Reiten quiver $\Gamma_{\tilde{w}_0}$ associated with $\tilde{w}_0$ is defined as follows

- The set of vertices is the totally ordered set $C_{\tilde{w}_0} = (\mu_1, \ldots, \mu_N)$, seen as lying inside $\Sigma^{op}$.
- There is an arrow from $\mu_j = w^j W^{(i_j)}$ to $\mu_k = w^k W^{(i_k)}$ iff
  i) $i_j$ and $i_k$ are linked in Dynkin diagram;
  ii) $w^j W^{(i_j)}$ and $w^k W^{(i_k)}$ are adjacent in $\Sigma$;
  iii) $j < k$. 

11
We are forced to introduce \( iii \) as intersection of left classes is non-oriented. Observe also that by considering only "Dynkin couples" in \( i \), \( \Gamma_{\tilde{\mu}} \) actually depends only on the commutation class \([\tilde{\mu}]\) of \( \tilde{\mu} \). Definition 3.5 provides therefore a unique quiver \( \Gamma_Q \) associated to the reduced expressions adapted to a quiver \( Q \). Our aim is to show it is isomorphic to \( \Gamma_Q \). We shall follow a combinatorial approach, as suggested by the referee.

**Example 3.1:** Consider the quiver \( Q : \bullet \to \bullet \to \bullet \) of type \( A_3 \). An adapted expression to \( Q \) is \( \tilde{w}_0 = s_1 s_2 \ldots s_6 \) with \( \underline{r} = (2, 1, 3, 2, 3, 1) \). \( \Gamma_{\tilde{\mu}_0} \) is

\[
\begin{array}{ccc}
& w_1 W(2) & \\
\downarrow & & \downarrow \\
w_2 W(1) & \rightarrow & w_6 W(1) \\
& w_3 W(3) & \\
\downarrow & & \downarrow \\
w_4 W(2) & \rightarrow & w_5 W(3)
\end{array}
\]

where \( w_k = s_{i_1} \ldots s_{i_k}, k = 1, \ldots 6 \).

Note for example

\[
w_1 W(2) = \{s_2, s_2 s_1, s_2 s_3, s_2 s_1 s_3\} \\
w_2 W(1) = \{s_2 s_1, s_2 s_1 s_2, s_2 s_1 s_3, s_2 s_1 s_2 s_3, s_2 s_1 s_3 s_2, s_2 s_1 s_2 s_3 s_2\}
\]

are adjacent since \( w_1 W(2) \cap w_2 W(1) = \{s_2 s_1, s_2 s_1 s_3\} (= s_2 s_1 W(1, 2)) \), while \( w_1 W(2) \) and \( w_6 W(1) \) are not since \( w_1 W(2) \cap w_6 W(1) = \emptyset \). We leave it to the reader to verify that there is an arrow between vertices on adjacent levels (=Dynkin couples) exactly when intersection of left classes is non empty.

Let \( \tilde{\mu} = s_{i_1} s_{i_2} \ldots s_{i_m} \) a reduced expression of an arbitrary element \( w \in W, w \neq e \). We can define a partial set of chamber weights \( C_{\tilde{\mu}} = (\mu_1, \ldots, \mu_n) \) with \( \mu_j = s_{i_1} s_{i_2} \ldots s_{i_{j-1}} (\mu_{i_j}) \), and a partial quiver \( \Gamma_{\tilde{\mu}} \) as above.

**Proposition 3.6** Assume \( \tilde{w} \) is alternating. Then combinatorial the description of Proposition 1.2 is valid for \( \Gamma_{\tilde{\mu}} \). There is an arrow \( \mu_j \to \mu_k \) if and only if

i) \( \mu_j \) and \( \mu_k \) are linked in the Dynkin diagram \( D \);

ii) \( j < k \);

iii) For \( j < p < k \), \( i_p \neq i_j \).

**Proof:** We proceed by induction on \( l(w) \). If \( l(w) = 1 \) there is nothing to prove. Suppose the proposition is valid for an alternating reduced expression \( \tilde{\mu} = s_{i_1} \ldots s_{i_m} \), and consider an alternating \( \tilde{\mu}' = s_r \tilde{\mu} \) with \( l(\tilde{\mu}') = l(\tilde{\mu}) + 1 \). Note \( C_{\tilde{\mu}'} = (\mu'_1, \ldots, \mu'_{m+1}) \).

For any \( x, y \in W, x W(i) \cap y W(j) \neq \emptyset \iff s_r x W(i) \cap s_r y W(j) \neq \emptyset \). Using induction hypothesis for \( \tilde{\mu} \), the combinatorial description is valid for arrows \( \mu'_j \to \mu'_k \) inside \( \Gamma_{\tilde{\mu}'} \) with \( j \geq 2 \). It remains to study arrows out of \( \mu'_1 \), that is intersections \( s_r W(r) \cap s_r s_{i_1} \ldots s_{i_k} W(k) \). By the above, this amounts to consider intersections \( W(r) \cap s_{i_1} \ldots s_{i_k} W(k) \). An arrow \( \mu'_1 \to \mu'_{k+1} \) exists in \( \Gamma_{\tilde{\mu}'} \) iff such an intersection is non-empty. We shall consider \( 1 \leq k \leq m \) with \( i_k \) linked to \( r \) in \( D \) and put \( w_k = s_{i_1} \ldots s_{i_k} \).
Let us show first all arrows $\mu'_1 \longrightarrow \mu'_{k+1}$ given by the criterion appear in $\Gamma_{\tilde{w}}$. If $k$ verifies for all $1 < p < k$, $i_p \neq r$, as also $i_k \neq r$, we have $w_k \in W^{(r)}$. Hence $W^{(r)} \cap w_k W^{(i_k)} \neq \emptyset$ as expected.

Conversely, let us show there are no more arrows $\mu'_1 \longrightarrow \mu'_{k+1}$ in $\Gamma_{\tilde{w}}$ than those given by the criterion. We suppose therefore there is $p$, $1 < p < k$ such that $i_p = r$. We can take $p$ maximal with that property. $\tilde{w}$ is alternating by hypothesis, hence $i_{p+1}, i_{p+2} \ldots i_{k-1}$ are different from $r$ and $i_k$. By definition of the simple reflections we have the following coefficients ($\alpha_{i_p} = \alpha_r$):

\[
\begin{cases}
[s_{i_{k-1}} s_{i_{k-2}} \ldots s_{i_{p+1}} (\alpha_{i_p}) : \alpha_r] = 1 \\
[s_{i_{k-1}} s_{i_{k-2}} \ldots s_{i_{p+1}} (\alpha_{i_p}) : \alpha_{i_k}] = 0
\end{cases}
\]

Now $s_{i_k} (\alpha_{i_p}) = \alpha_r + \alpha_{i_k}$, so we obtain from (*) :

\[
[s_{i_k} s_{i_{k-1}} \ldots s_{i_{p+1}} (\alpha_{i_p}) : \alpha_{i_k}] > 0
\]

and therefore $(\omega_{i_k}, s_{i_k} s_{i_{k-1}} \ldots s_{i_{p+1}} (\alpha_{i_p}^\vee)) \neq 0$. By Lemma 3.4, $s_{i_p} = s_r$ appears in the reduced expression $\tilde{u}$ for minimal representative $u$ of $w_k W^{i_k}$ obtained by erasing procedure out of $\tilde{w}_k$.

Consider a left class of $W^{(r,i_k)}$ inside $w_k W^{(i_k)}$. By Lemma 3.3, its minimal representative is of the form $y = u x$ with $x$ a minimal representative of a class of $W^{(r,i_k)}$ inside $W^{(i_k)}$. Take any reduced expression $\tilde{x}$ of $x$. $\tilde{y} = \tilde{u} \tilde{x}$ is a reduced expression of $y$, in which $s_r$ appears. By Proposition 3.1 this means $y \notin W^{(r)}$, hence $y W^{(r,i_k)}$ is not included in $W^{(r)}$. We see no left class of $W^{(r,i_k)}$ can be contained in both $W^{(r)}$ and $w_k W^{(i_k)}$, so there is no arrow in $\Gamma_{\tilde{w}}$ from $\mu'_1$ towards $\mu'_{k+1}$.

**Remark** : Experimental data suggests the statement of the proposition is actually true in general, without the assumption that $\tilde{w}$ is alternating (see example 3.2 below). For this reason we insist that the third condition in the combinatorial characterization be $i_p \neq i_j$ while $i_p \neq i_j$, $i_k$ would have been sufficient for the adapted case.

**Theorem 3.7** (Isomorphism)

Let $Q$ be a quiver of type $ADE$, and $R$ its Ringel form. Then $\varphi_R$ induces isomorphism of oriented graphs of $\Gamma_Q$ onto $\tilde{\Gamma}_Q$.

**Proof** : Let $\tilde{w}_0$ be a reduced expression adapted to $Q$ Combining Theorems 1.1 and 2.4, we get a correspondence between vertices of $\Gamma_Q$ and $\tilde{\Gamma}_Q$ given by $\varphi_R$. By Lemma 1.4, $\tilde{w}_0$ is alternating, hence applying Propositions 1.2 and 3.6, $\varphi_R$ extends to an isomorphism of oriented graphs.

Theorem 3.7 justifies the use of the term Abstract Auslander-Reiten quiver for $\Gamma_{\tilde{w}_0}$. There is a finite dimensional algebra, behind the commutation class corresponding to $Q$, namely $CQ$. Definition 3.5 arises the question of associating a finite dimensional algebra with any commutation class. One aim would be controlling the corresponding parameterizations of the canonical basis (see [17] for some adapted cases). Another application of finite dimensional algebra techniques in general reduced expressions setting, might be solving the problem of realizability of the set of commutation classes with braid moves (see [2]) as a 1-skeleton of a polytopal complex. This problem is also known as realizability of second Bruhat order [21]; we point for instance to [10] as
representing the current trend of research concerning this problem. We observe for the moment, even the number of commutation classes in $A_n$ case is unknown.

Example 3.2: We have seen linearity fails for non-adapted cases. The same applies for the “mesh” structure coming from $ZD$. Take $\bar{w}_0 = s_2s_1s_2s_3s_2s_1$. One may check $s_2s_1W(1) \cap s_2s_1s_2s_3W(2) = s_2s_1s_3s_2W^{(1,2)}$. $\Gamma_{\bar{w}_0}$ is given by

$$\begin{align*}
&\xrightarrow{w_2W(1)} & \xrightarrow{w_6W(1)} \\
&\xrightarrow{w_1W(2)} & \xrightarrow{w_3W(2)} & \xrightarrow{w_5W(2)} \\
& & & \xrightarrow{w_4W(3)}
\end{align*}$$

We see there are two arrows from $s_2s_1W(1)$ towards the second level. Observe also $\bar{w}_0$ is no longer alternating, erasing $s_3$ we get $s_2s_1s_2s_2s_1$. The combinatorial description of Proposition 3.6 still applies.

4 The $A_n$ case: wiring diagrams

We describe in this section $A_n$ case. We use Young tableaus (columns) to describe $\bar{\Gamma}_Q$. On the theoretical level, we describe briefly the correspondence of abstract Auslander-Reiten quivers with wiring diagrams used in $[3]$.

Linearity allows us to compute easily the vertices of $\Gamma_{\bar{w}_0}$, however, we are unable for the moment to provide an generalized adjacency criteria inside $\Sigma^{op}$, which would give in a straight-forward manner the arrows. There are in the literature case by case combinatorial descriptions of $\Sigma$ according to Dynkin type, we restrict ourselves here to the simplest case: $A_n$.

All fundamental representations $E(\omega_i)$ of a simple complex lie algebra of type $A_n$ are minuscule, so that $W\omega_i$ is the set of weights of $E(\omega_i)$. Furthermore as $E(\omega_i)$ is isomorphic to $\bigwedge^i E(\omega_1)$, it has a basis of weight vectors indexed by all Young columns of size $i$, filled in a strictly increasing manner with indices in $\{1, \ldots, n+1\}$. We get therefore an identification of $W\omega_i$ (equivalently the set of left classes of $W^{(i)}$) with multi-indices $J = (j_1, j_2 \ldots j_i)$, $1 \leq j_1 < j_2 \ldots < j_i \leq n$. Conversely, given a strictly increasing multi-index $J$ of cardinality $i$ (that is a Young column of size $i$), one may recover its weight as the sum of weights of its boxes

$$wt(J) = \sum_{k=1}^{i} wt([j_k]),$$

with $wt([j_k]) = \omega_k - \omega_{k-1}$, as such a box corresponds to a weight vector inside $E(\omega_1)$ (by convention $\omega_0 = \omega_{n+1} = 0$).

In $A_n$ case, $W$ is the symmetric group $S_{n+1}$, so that simple reflections are the transpositions $(i \ i + 1)$, and action on weight orbits becomes usual $S_{n+1}$ action on sets of indices. Let us note for two integers $a < b$ the interval $(a, a+1, \ldots, b)$ by $[a, b]$. 
Proposition 4.1 Fix \( i \) between 1 and \( n \), and consider two left cosets \( wW^{(i)}, w'W^{(i+1)} \), with \( J, J' \) corresponding multi-indices. Then \( wW^{(i)} \) and \( w'W^{(i+1)} \) are adjacent in \( \Sigma_{op} \) if and only if \( J \subset J' \).

Proof: The closure of the dominant chamber is a fundamental domain for the action of \( W \). Furthermore pairs \( (W^{(i)}, w'W^{(i+1)}) \) correspond to left classes of \( W^{(i+1)} \) inside \( W^{(i)} \) which form a single \( W^{(i)} \) orbit. Hence, the adjacent pairs \( (wW^{(i)}, w'W^{(i+1)}) \) in \( \Sigma_{op} \) form one orbit under diagonal action, that of \( (W^{(i)}, W^{(i+1)}) \).

The multi-indices corresponding to \( W^{(i)} \) and \( W^{(i+1)} \) are \([1, i]\) and \([1, i+1]\). As Weyl group action preserves inclusion of multi-indices we get from the discussion above that \( wW^{(i)} \), \( w'W^{(i+1)} \) adjacent implies \( J \subset J' \).

For the converse we need to show any couple \((J, J')\) with \( J \subset J' \) can be obtained by diagonal action from \(([1, i], [1, i+1])\). Since all multi-indices of cardinality \( i \) form a single Weyl group orbit, it is enough to consider couples of the form \(([1, i], J')\). Since \([1, i] \subset J' \), we must have \( J' = [1, i] \cup \{k\} \) with \( k \geq i + 1 \). If \( k > i + 1 \), then \( s_{i+1}s_{i+2}\ldots s_{k-1}J' = [1, i + 1] \) while \( s_{i+1}s_{i+2}\ldots s_{k-1} \) stabilizes \([1, i]\). Thus

\[
([1, i], J') = s_{k-1}s_{k-2}\ldots s_{i+1}([1, i], [1, i+1])
\]

\(\square\)

Let us now describe \( \tilde{\Gamma}_Q \) in terms of column tableaus. It seems somewhat simpler to use the classical method of Coxeter translation, instead of linearity.

As \( W \cong S_{n+1} \), the Coxeter element \( c \) defined by \( Q \) is a \( n + 1 \) cycle. Let us fix a cycle writing \([c] = (a_1 a_2 \ldots a_{n+1})\) by imposing \( a_{n+1} = n + 1 \). In order to work inside \([c]\), we shall use \( \overline{m} \) for \( m \mod n + 1 \). Finally let \([c]_{\overline{m}, i}\) be the multi-index obtained by ordering the "interval" \( \{a_{\overline{m}}, a_{\overline{m}+1}, \ldots, a_{\overline{m+1}+1}\} \).

Lemma 4.2 Define for \( i \in I \), \( \overline{m}_i \) recursively by \( \overline{m}_1 = n + 1 \) and for \( i \geq 2 \)

\[
\overline{m}_i = \begin{cases} 
\overline{m}_{i-1} & \text{if } i^* \rightarrow i^{*+1} \text{ in } Q \\
\overline{m}_{i-1} - 1 & \text{if } i^* \leftarrow i^{*+1} \text{ in } Q 
\end{cases}
\]

Then \([c]_{\overline{m}_i, i}\) is the multi-index \((i^* + 1, i^* + 2, \ldots, n + 1)\).

Proof: We proceed by induction on \( n \). For case \( n = 2 \) there is nothing to prove. Consider \( Q \) of type \( A_n \), and let \( Q' \) be its sub-quiver with vertices 1 to \( n - 1 \), with \( c' \) the corresponding Coxeter element, and \( m' \) defined for \( c' \) as above.

Case 1: \( Q \) is obtained from \( Q' \) by adding \( n \rightarrow n+1 \). \( n \) is a sink, hence \( c = s_n c = (n + 1)c' \). If \( c' = (b \ldots an) \) then \( c = (b \ldots an + 1n) \). We have therefore \( \overline{m}_2 = \overline{m}_1 \).

Case 2: \( Q \) is obtained from \( Q' \) by adding \( n \leftarrow n+1 \). \( n \) is a source, hence \( c = c's_n = c'(n + 1n) \). If \( c' = (b \ldots an) \) then \( c = (b \ldots an + 1n) \). We get \( \overline{m}_2 = \overline{m}_1 - 1 \).

In both cases, the cycle writing of \( c \) is obtained from that of \( c' \) by replacing \( n \) by either \( n + 1 \) or \( n + 1n \). The validity of the Lemma for \( c' \) in cases \( i = 2 \ldots n - 1 \) implies its validity for \( c \) in cases \( i = 3 \ldots n \). Also, the recursive formula for \( \overline{m}_i \) for \( i \leq 3 \) is the same as that for \( \overline{m}_i - 1 \) \( \square \)

We see \( c \) has a "segment property": for any \( j \), indices \( j, j + 1, \ldots n + 1 \) form a "connected" band (modulo \( n + 1 \)) in the cycle writing of \( c \). This property leads to the description of levels of \( \tilde{\Gamma}_Q \). Let use note by \( J_j^{(i)}(Q) \) the multi-index of the \( j^\text{th} \) vertex from the right (starting with 0) on the \( i^\text{th} \) level of \( \tilde{\Gamma}_Q \).
Proposition 4.3 Consider $Q$ and fix $i \in I$, then
\begin{itemize}
  \item[(a)] (Corollary 2.20 (a)) The cardinality $N_i(Q)$ of the $i$th level of $\bar{\Gamma}_Q$ is $(n + 1 + a_i - b_i)/2$ where $a_i$ (respectively $b_i$) is the number of arrows in the unoriented path from $i$ to $i^*$ that are directed towards $i$ (respectively $i^*$).
  \item[(b)] For $j = 0, 1, \ldots, N_i(Q) - 1$ one has
    
    \[ J_j^{(i)}(Q) = [c]^{m_{i-j,i}} \]

Proof of (b) : Fix $i \in I$. By Lemma 2.6, the first weight from the right on the $i$th level of $\bar{\Gamma}_Q$ is $-\omega_i$. It corresponds to the multi-index $(i^* + 1, i^* + 2, \ldots n + 1)$. By the Lemma above, the proposition is true for $j = 0$.

For $1 \leq j \leq N_i(Q) - 1$, we use Proposition 2.7 :

\[ J_j^{(i)}(Q) = c^{-j} J_0^{(i)}(Q) = c^{-j}[c] = [c]^{m_{i-j,i}} \]

We see getting $\bar{\Gamma}_Q$ out of $Q$ requires two operations : computing the cycle writing of the Coxeter element associated to $Q$, and the cardinalities $N_i(Q)$. Both involve working directly with $Q$. Once vertices are known, getting the arrows is immediate due to Lemma 4.1.

Example 4.1 : Consider the quiver $Q : 1 \rightarrow 2 \leftarrow 3$ of type $A_3$. $1^* = 3$ so that $a_1 = 1, b_1 = 1$ hence $N_1(Q) = 2$. We obtain in a similar manner $N_2(Q) = 2, N_3(Q) = 2$. The Coxeter element $c$ is the cycle $(2134)$. The levels of $\bar{\Gamma}_Q$ are given respectively by $\bar{\Gamma}_Q^1 : (\cdot 3\cdot), (\cdot 4\cdot); \bar{\Gamma}_Q^2 : (\cdot 3\cdot), (\cdot 3\cdot 4\cdot); \bar{\Gamma}_Q^3 : (\cdot 3\cdot 4\cdot), (2\cdot 3\cdot 4\cdot)$. Considering inclusion between sets of indices we get the arrows :

Reduced expressions $\bar{w}_0$ of the longest word in $W$ play a central role in the theory of the canonical basis $[13, 10]$ of the positive part $U_q(n^+)$ of the quantized enveloping algebra $U_q(g)$. As this base is difficult to compute, Lusztig’s model consists in approximating it by a basis composed of PBW monomials. Every convex ordering of $\Phi^+$ (or commutation class) gives rise to a particular PBW base, which provides an indexation of the canonical basis. The control over these parameterizations involves a geometrical interpretation of $\bar{w}_0$, the wiring diagram $\mathcal{WD}(\bar{w}_0)$. We refer to [3] §2.3 for the
details, and reduce the presentation here to example 4.2. Our conventions are “opposite” (twisted by *) compared to those of [3].

Given \( \tilde{w}_0 = s_is_is_2 \ldots s_{iv} \) in \( W \cong S_{n+1} \), the wiring diagram \( WD(\tilde{w}_0) \) is an arrangement of \( n+1 \) pseudo-lines of the plane \( \mathbb{R}^2 \), going from \(-\infty\) to \(+\infty\). The pseudo-lines cross pairwise in points \( x_1, \ldots, x_N \), situated in levels (top to bottom) in accordance with \( \tilde{w}_0 \), namely the crossing \( x_k \) is on the level \( i_k \). We get a braid reversing the order of the lines, and which partitions \( \mathbb{R}^2 \) into zones (or chambers). One numbers the pseudo-lines from 1 to \( n+1 \) as in the example (at \( x \) coordinate \(-\infty : \) from top to bottom). This gives a labelling of each zone \( Z \) by the set \( J(Z) \) of indices of lines passing above it. Each zone has a distinct label. Observe [3] geometrically \( WD(\tilde{w}_0) \) depends only on the commutation class of \( \tilde{w}_0 \), and there is a 1 : 1 bijection between the set of commutation classes of \( A_k \) and possible wiring diagrams with \( n+1 \) strands.

**Example 4.2**: Take \( \tilde{w}_0 = s_2s_1s_3s_2s_3s_1 \) (type \( A_3 \)). \( WD(\tilde{w}_0) \) is:

```
  4 3 2 1
level 1
  
  3 4 1 2
level 2
  
  1 3 4
level 3
```

Consider the set of zones bounded to the left by a crossing point. These are in bijection with crossing points \( x_k \). Let us note therefore by \( Z_k \) the zone whose leftmost point is \( x_k \) in \( WD(\tilde{w}_0) \), and associate to it the level of \( x_k \).

**Lemma 4.4**

a) The label \( J(Z_k) \) is \( w_k : (1, 2, \ldots, i_k) \) where \( w_k = s_is_is_2 \ldots s_{i_k} \).

b) Two chambers \( Z_j \) on the \( i \)th level and \( Z_k \) on the \( (i+1) \)th level are adjacent if and only if a pseudo-line \( L_m \) passes between them, that is if and only if \( J(Z_k) = J(Z_j) \cup \{m\} \).

**Proof**: Pass a vertical line \( p_k \) through each crossing \( x_k \). This divides the plane into vertical bands \( \mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_N \). \( \mathcal{V}_0 \) lies between \(-\infty\) and \( p_1 \), for \( k = 1, \ldots, N-1 \), \( \mathcal{V}_k \) lies between \( p_k \) and \( p_{k+1} \), and \( \mathcal{V}_N \) lies between \( p_N \) and \(+\infty\).

Note by \( J_k \) the ordered set of indices of pseudo-lines passing through \( \mathcal{V}_k \), read top to bottom. Initially \( J_0 = (1, 2, \ldots, n) \). If we note \( i = i_k \), the crossing \( x_k \) consists in interchanging the lines in positions \( i \) and \( i + 1 \). Suppose \( J_{k-1} = (w_{k-1}(1), w_{k-1}(2), \ldots, w_{k-1}(i), w_{k-1}(i + 1), \ldots, w_{k-1}(n)) \), then

\[
J_k = (w_{k-1}(1), w_{k-1}(2), \ldots, w_{k-1}(i + 1), w_{k-1}(i), \ldots, w_{k-1}(n)) \\
= (w_{k-1}s_is_{i+1}(1), w_{k-1}s_is_{i+1}(2), \ldots, w_{k-1}s_is_{i+1}(i), w_{k-1}s_is_{i+1}(i + 1), \ldots, w_{k-1}s_is_{i+1}(n)) \\
= (w_k(1), w_k(2), \ldots, w_k(n)).
\]
Proceeding by induction we get for \( k = 1 \ldots N \), \( J_k = (w_k(1), w_k(2), \ldots, w_k(n)) \). Part a) follows then by observing for a zone \( Z_k \), the label \( J(Z_k) \) consists in the first \( i_k \) indices of \( J_k \).

Part b) follows directly from our definition of labelling of the zones in \( WD(\bar{w}_0) \).

We may consider labels of zones \( Z_k \) as Young columns. Lemma 4.4 a) shows this establishes a bijection with \( C_{\bar{w}_0} \) (thus the term chamber weights used in [4]). Comparing Lemma 4.1 with Lemma 4.4 b), we see arrows in \( \Gamma_{\bar{w}_0} \) correspond to adjacencies of zones in \( WD(\bar{w}_0) \). We invite the reader to compare Examples 4.1 and 4.2. Observe the non-left bounded zones may be considered as trivial, as they appear in any wiring diagram, and always have the same label. We therefore have a natural \( 1:1 \) correspondence between the set of abstract Auslander-Reiten quivers and that of wiring diagrams.

Remark : Wiring diagrams were generalized to \( B_n, C_n \) cases in [4]. No such construction exists in \( D_n \) or \( E_n \) cases.

In \( A_n \) case, we may summarize our results in the following figure:

\[
\begin{array}{ccc}
\text{Abstract A.R. quivers} & \Gamma_{\bar{w}_0} & 1:1 \\
\downarrow & & \downarrow \\
\text{A.R. quivers} & \Gamma_Q & \text{Wiring diagrams} \\
& & \text{WD}(\bar{w}_0)
\end{array}
\]

We obtain another instance of the link between theories of finite dimensional algebras and of the canonical basis of \( U_q(\mathfrak{g}) \).

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