Schwarzschild black hole surrounded by quintessential matter field as an accelerator for spinning particles

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We study the collision of two massive particles with non-zero intrinsic spin moving in the equatorial plane in the background of a Schwarzschild black hole surrounded by quintessential matter field (SBHQ). For the quintessential matter equation of state (EOS) parameter, we assume three different values. It is shown that for collisions outside the event horizon, but very close to it, the centre-of-mass energy (ECM) can grow without bound if exactly one of the colliding particles is what we call near-critical, i.e., if its constants of motion are fine tuned such that the time component of its four-momentum becomes very small at the horizon. In all other cases, ECM only diverges behind the horizon if we respect the Møller limit on the spin of the particles. We also discuss radial turning points and constraints resulting from the requirement of subluminal motion of the spinning particles.

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I. INTRODUCTION

The first simplest black hole (BH) solution of Einstein’s field equations was obtained by Schwarzschild in 1916 [1] immediately after the discovery of general relativity (GR) by Einstein. The BH solution found by Schwarzschild is the simplest in the sense that it has only one observable parameter (i.e., mass). Black Holes (BHs) are one of the most interesting topics of research in GR and in alternative theories of gravity (ATG) for researchers. It took almost a century to confirm that these mysterious objects do exist in our universe and recently the LIGO and VIRGO collaborations have detected the first ever gravitational waves signals from BH merger [2]. More recently, the first ever direct image of a BH observed by the Event Horizon Telescope (EHT) suggests to us to strongly believe in the presence of BHs in our universe [3].

The appearance of BHs is not only limited to GR or ATG like string theory [4], but they have also played a crucial role in understanding the cosmology. There are two major classes of cosmological models for dark energy. One of them is the cosmological constant Λ [5] having an equation of state (EOS) parameter $\epsilon = -1$. But in this model, there is a problem known as the fine tuning problem which is yet to be resolved [6]. The other class of cosmological model mainly depends on a dynamical scalar field such as, but not confined to, quintessence [7], chameleon fields [8], K-essence [9], tachyons [10], phantom [11] and dilatons [12]. In these models, the main difference is the EOS parameter $\epsilon$ which varies from -1 to -1/3 for quintessence like models and lesser than -1 for phantom like models. The complete study of various dark energy models is presented in [13]. As we know that the existence of dark energy having negative pressure is not well studied in the context of BHs. Hence, we try to understand the consequences of BHs in this paper surrounded by the quintessence like models. In particular, we aim to focus on some aspects like particle acceleration in the background of a static BH solution surrounded by quintessence like matter obtained by Kiselev in [14]. The geodesic motion and geodesic deviation around this BH spacetime is investigated in detail in [15].

A rotating BH under some rare conditions can act as a particle accelerator for two spinless particles which start from rest at infinity and collide near the event horizon of a rotating BH (Kerr BH) pointed out by Bañados, Silk, and West (BSW) [16]. They showed that the collisional energy (i.e., center-of-mass (CM) energies) of these spinless particles will be infinitely high if the BH is rotating in addition to the condition that one of the particle must have attained a critical value (a very fine-tuned value) of the angular momentum. They also mentioned that if the BH is non-rotating (i.e., Schwarzschild), it is not possible to obtain an infinite amount of CM energy. After this pioneering work by BSW [16], a number of studies have been performed on the particle acceleration by all sorts of BHs in GR [17–56] and in different ATG models [57–98]. These studies conclude in their individual works that the con-

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conditions obtained by BSW to get infinite amount of high CM energy are universal and these results were also generalized by Harada in [99]. It is worth noting here that the conditions mentioned by the BSW such as the BH must be rotating and one of the colliding particles should have a critical angular momentum are very rare to observe in nature. In turn, the BSW process is a very rare event to observe in nature which needs careful attention in diverse context.

The BSW mechanism is so far mainly studied for spinless test particles (i.e. particles that follow geodesics) only. However, in general a particle moving in the vicinity of a BH is an extended object having self interaction such as the case of a spinning particle. It has been shown by Matisson, Papapetrou and Dixon (MPD) [100–102] that the trajectory followed by a spinning particle is non-geodesic due to the coupling between the spin of the particle and curvature of the spacetime around a massive central object like a BH.

Recently (2016), it was shown by Armaza et al. [103] that it is still possible to obtain an infinite amount of CM energy for the Schwarzschild BH if one considers the collision of the spinning particles instead of a collision of spinless particles. The study of BHs as a particle accelerator for spinning particles is further extended to the case of charged spinning BHs in [104], where they have shown that it is possible to obtain infinitely high CM energy outside the event horizon of non-extremal Reissner-Nordstrom (RN) BH. They also showed that the area belonging to the infinitely high CM energy in spin and total orbital angular momentum $(s,l)$ plane of the spinning particles is very sensitive to the BH charge as it decreases as the charge of the black hole increases. They further showed that for a non-extremal Kerr BH case, we can also obtain infinitely high CM away from the event horizon and infinitely high CM energy area in $(s,l)$ plane increases with an increase in the spin of the BH. In the same work, they also discussed the CM energy of the colliding spinning particles in the background of Kerr-Newman BH as well. They finally concluded that the spin parameter $(a)$ and the charge $(Q)$ of the BH affect the CM energy of the colliding particles in a completely opposite way. Recently, the universality of BSW mechanism for spinning particle, for a class of stationary axisymmetric BH is also discussed in [105]. However, the study of BHs as particle accelerators for spinning particles has been performed for only a few BH models. Hence, it is very interesting to explore more about BH particle acceleration processes in the context of spinning particles. Based on such standpoints, in this work, we plan to investigate the particle acceleration process for two spinning particles colliding outside the event horizon of the non-extremal Schwarzschild BH which is surrounded by the quintessence like matter and it will be mentioned as SBHQ henceforth [15]. We have observed that the CM energy of the colliding particles might be infinitely high for the collisions of the spinning particles but the collisions must take place inside the cosmological horizon of the SBHQ. The CM energy in our case is found to be very sensitive to the value of normalization constant $(\lambda)$ and the EOS parameter $(\epsilon)$ which, for quintessential matter, varies from $-1$ to $-1/3$.

Our paper is organized as follows. In Sec. II, we will present a brief overview of the equations of motion for spinning particles in Einstein’s theory of general relativity (GR). In Sec. III, we discuss about the spacetime geometry around the SBHQ and talk about its event and cosmological horizons. In this section, we have also obtained the expressions for the four-momentum of a spinning particle by following the algorithm discussed in Ref. [106–108]. In Sec. IV, we obtained the expression for the CM energy of the colliding spinning particles in the vicinity of SBHQs and showed that CM energy expression obtained for SBHQ reduced to SBH case [103] when normalization constant $\lambda$ vanishes and energy $e$ per unit mass become unity. In this section, we also discussed about the possible scenarios where arbitrarily high $E_{CM}$ can be obtained. Section V is devoted to the study of the effective potential $(V_{eff})$ and radial turning points for the trajectories of the spinning particles. We have divided this section into two parts: in the first part, we found the expression for $V_{eff}$, as it helps to characterize the path of the spinning particle moving in the background of SBHQ. In second part, we classified the spinning particle according to [109] into three sub-classes: usual particle, critical particle and near-critical particle, respectively. Further, we also classify the trajectories of a spinning particle depending on its behavior (i.e., usual, critical and near-critical). In Sec. VI, we study the superluminal constraint and the conditions to avoid the superluminal region for the spinning particles. Finally, Sec. VII is devoted to the summary and conclusions drawn from the results obtained and to future prospects.

Throughout our work in this paper, we set the fundamental constants to unity (i.e., $G = 1$), the signature of spacetime as $(−, +, +, +)$, Greek indices (i.e., $\alpha, \beta, \ldots$) run from 0 to 3 and Latin indices runs from 1 to 3 unless otherwise stated. Also, in the following sections, we chose the spin $s$ per unit mass of the colliding particles within the Møller limit (i.e., $r_p > s$) [110, 111], where $r_p$ is the size of the spinning particle. It is important to note that size of the spinning particle is very less than the size of the BH (i.e. $r_p \ll r_{(1)}$), therefore we have $s \ll M$ [112].

II. EQUATIONS OF MOTION OF SPINNING PARTICLES IN CURVED SPACETIME

The study of the chargeless spinning particles in GR started with the pioneering work of MPD [100–102] on spinning tops in curved spacetime. In their formulation, they showed that the trajectories followed by the chargeless spinning tops were not in accordance with the equivalence principle i.e. the above massive particles follow the non-geodesic paths. Further, Hojman [106, 113] extensively studied and extended the formulation by MPD. In this section, we will present a brief overview of the equations of motion developed by Hojman with the help of Lagrangian formulation. The aforesaid equations of the motion read as

$$\frac{dx^\alpha}{d\tau} = u^\alpha, \quad (1)$$

$$\frac{DP^\alpha}{D\tau} = -\frac{1}{2} R^\alpha_{\beta \gamma \delta} u^\beta S^\gamma_{\delta}, \quad (2)$$
\[ DS^{\alpha\beta} \frac{D\tau}{D\tau} = S^{\alpha\gamma} \sigma_{\gamma}^\beta - \sigma^{\alpha\gamma} S_{\gamma}^\beta = P^\alpha u^\beta - P^\beta u^\alpha, \]  

where \( \tau, u^\alpha, P^\alpha, S^{\alpha\beta} \) and \( \sigma^{\alpha\beta} \) are an affine parameter, the 4-velocity, the 4-momentum vector, the spin tensor, and the antisymmetric angular velocity tensor, respectively. The antisymmetric angular velocity tensor is in turn defined as

\[ \sigma^{\alpha\beta} \equiv \eta^{(\gamma\delta)} \epsilon^\alpha_\gamma \epsilon^\alpha_\delta \frac{D\epsilon^\beta_\alpha}{D\tau} = -\sigma^{\beta\alpha}. \]  

Here, \( \epsilon^\alpha_\gamma \) is an orthonormal tetrad which is used to define the orientation of the top, \( D\epsilon^\beta_\alpha / D\tau \) is the usual covariant derivative of the orthonormal tetrad and \( \eta^{(\gamma\delta)} \equiv \text{diag}(-1, 1, 1, 1) = \eta^{(\gamma\delta)} \).

As the Eqs. (1)-(3) does not form a closed set of equations (i.e., they are insufficient to determine the complete trajectory of spinning particles in a curved spacetime) and hence, spin supplementary conditions are needed. For simplicity purposes, we choose the Tulczyjew spin supplementary condition (TSSC) \( S^{\alpha\beta} P_\beta = 0 \) which conserves the dynamical mass of the spinning particle and choose a particular frame of the spinning particles for which only 3-components of \( S^{\alpha\beta} \) are non-vanishing (i.e., \( S^{\alpha\theta} = 0 \) [103]).

Additionally, the 4-momentum \( P^\alpha \) is not parallel to the four velocity \( u^\alpha \) for the case of a spinning particle and a relation between \( P^\alpha \) and \( u^\alpha \) is essential and can be written as [114]

\[ u^\alpha = \frac{\kappa}{m} \left[ P^\alpha + \frac{2S^{\alpha\gamma} \lambda R^\gamma_{\beta\gamma\epsilon}}{4m^2 + R_{\mu
u\lambda \kappa} S^{\mu\nu} / S^{\kappa\lambda}} \right]. \]  

Here, \( \kappa \) is a normalization constant. It is worth mentioning here that the above condition on the spin tensor comes naturally from the theory if one suitably chooses the correct Lagrangian (for detailed analysis see [115]).

We now define the conserved quantities [115] related to the spinning top and these are the mass \( m \) of the spinning top

\[ m^2 = -P^\alpha P_\alpha, \]  

and its spin \( S \),

\[ S^2 = \frac{1}{2} S^{\alpha\beta} S_{\alpha\beta}. \]  

In addition to the above-mentioned conserved quantities, we have an extra conserved quantity \( D_\xi \) defined as below,

\[ D_\xi \equiv P^\alpha \xi_\alpha - \frac{1}{2} S^{\alpha\beta} \xi_{\alpha\beta}, \]  

which is independent of the choice of the background metric as shown in [102]. Here, \( \xi_\alpha \) is a Killing vector associated with the spacetime metric. The motion of the tops in the background of SBHQ is presented in the next section.

**III. SPINNING PARTICLES IN SBHQ BACKGROUND**

The metric for SBHQ in the Schwarzschild coordinate system \((r, \theta, \phi)\) reads as

\[ ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega_2^2, \]  

where

\[ f(r) = \left( 1 - \frac{2M}{r} - \frac{\lambda}{r^{3\epsilon+1}} \right), \]  

\[ d\Omega_2^2 = d\theta^2 + \sin^2 \theta d\phi^2. \]  

Here, \( \lambda \) is a normalization constant whose physical interpretation depends on the specific EOS parameter value \( \epsilon \). The behaviour of \( f(r) \) is shown in Fig. 1 for different combinations of \( \lambda \) and \( \epsilon \).

In order to analyze the properties of SBHQ, we study the structure of horizon which has a two-sphere topology (except in the case \( \epsilon = -1/3 \) and \( 0 < \lambda < 1 \) which has the topology of a two-sphere but a deficit solid angle [116]-[117]) and is calculated by the equation \( \sigma^r = 0 \) of the above metric. Now, using Eqs. (9), (10) and the above definition, the horizon satisfies the following condition

\[ \Delta_0 = r^{3\epsilon+1} - 2Mr^{3\epsilon} - \lambda = 0. \]  

From Eq. (12), we find that the horizon of SBHQ depends upon two extra parameters, i.e. \( \lambda \) and \( \epsilon \) respectively, besides the usual mass \( M \) of a static spherical BH as in general relativity (i.e. SBH). We consider in this work three different choices of the EOS parameter, namely \( \epsilon = -1/3, -2/3, -1 \).

For these choices, we now analyse the possible horizons of the spacetime:

- When \( \epsilon = -1/3 \) and \( 0 < \lambda < 1 \), the Eq. (12) becomes linear in \( r \) and has only one root at \( r = r_{0(1)} = 2M/(1 - \lambda) \), known as event horizon.

- For \( \epsilon = -2/3 \), the Eq. (12) becomes quadratic in \( r \) and has two roots \( r_{0(1)} \) and \( r_{0(2)} \), known as event and cosmological horizons, located at

\[ r = r_{0(1,2)} = \frac{1 \pm \sqrt{1 - 8M\lambda}}{2\lambda}. \]  

It is clear from above equation that for \( \lambda = 1/8M \) both horizons coincide at the position \( r = 4M \).

- For \( \epsilon = -1 \), Eq. (12) becomes a depressed cubic equation in \( r \) whose discriminant and roots are as follows:

\[ \Delta = \frac{1}{27\lambda^3} (-1 - 27M^2 \lambda), \]  

\[ \tilde{r}_1 = Y_1 + Y_2, \]  

\[ \tilde{r}_{2,3} = Y_1 Y_2 \pm \frac{i\sqrt{3}}{2} (Y_1 - Y_2), \]  

where

\[ Y_{1,2} = \sqrt{\frac{-M}{\lambda}} \pm \sqrt{\Delta}. \]  

Depending on the values of \( \lambda \) we have following three sub-cases:

i) If \( \lambda = 1/27M^2 \implies \Delta = 0 \), then all roots are real, and at least two are equal (i.e. \( \tilde{r}_1 < 0, \tilde{r}_2 \equiv r_{0(2)} = 3M \) and \( \tilde{r}_3 \equiv r_{0(1)} = 3M \)). This means both the event \( r_{0(1)} \) and the cosmological \( r_{0(2)} \) horizons coincide.

...


\[ S^{\alpha\beta}P_\beta = 0, \]  

the components \( S^{t^\phi} \) and \( S^{r^\phi} \) come out as

\[ S^{r^\phi} = s P^{r^\phi}, \quad S^{t^\phi} = \frac{s P^t}{r f(r)} \]  

and \( S^{r^\phi} = \frac{s f(r) P^t}{r} \). (22)

It is worth to note here that \( s = \pm S/m \) is the spin per unit mass; the \( \pm \) signs are related to (anti) parallel spin of the particle with respect to the total angular momentum, respectively. The component of spin perpendicular to the equatorial plane may then reads as

\[ S_z = r S^{r^\phi} = s \left( \frac{2 e r - j s f(r)'}{2 r - s^2 f(r)' r} \right). \]  

Further, all the non-zero components of the 4-momentum vector \( P^\alpha \) calculated with the help of Eqs. (18), (19), (20), (21), (22), and (23) as follows,

\[ P^t = m \left( \frac{r^{3e+1}}{s} \right) K, \]  

\[ P^\phi = m \left( \frac{2}{r} \right) L, \]  

\[ (P^r)^2 = m^2 \left[ \frac{K^2 - f(r) (1 + 4L^2)}{r} \right]. \] (26)

where

\[ K = \frac{2 e r - j s f(r)'}{2 r - s^2 f(r)' r}, \]  

\[ L = \frac{j - es}{2 r - s^2 f(r)' r}. \] (27)

Here, \( e = E/m \) is energy per unit mass and \( j = J/m \) is the total angular momentum per unit mass. Hereafter, we normalize \( m \) to unity for simplicity.

Finally, one can write the expression for \( \dot{\phi} \) and \( \dot{r} \) as follows:

\[ \dot{\phi} = \frac{u^\phi}{u^t} = \frac{[2 r - r s^2 f(r)'] P^\phi}{[2 r - s^2 f(r)'] P^t}, \]  

\[ \dot{r} = \frac{u^r}{u^t} = \frac{P^r}{P^t}. \] (29)

It is worth mentioning here that the parameter corresponding to the proper time (\( \tau \)) has to be fixed in order to obtain the velocity components \( u^r, u^\phi, \) and \( u^t \). However, for the above discussed relativistic invariants, one does not need to make any such specific choices.

## IV. CENTRE-OF-MASS ENERGY OF THE SPINNING PARTICLES

Let us consider two spinning massive particles \( (m_1, m_2) \) colliding near to the horizon of the BH. The centre-of-mass energy \( (E_{CM}) \) of these two particles can with the help of the formula derived as in [16] be written as

\[ E_{CM}^2 = -g_{\alpha\beta} (P^\alpha + P^\beta) \left( P^{\alpha}_1 + P^{\alpha}_2 \right), \]  

\[ = m_1^2 + m_2^2 - 2 g_{\alpha\beta} P^\alpha_1 P^\beta_2. \] (31)
In [119], it is shown that the spinless particles with equal mass must have maximum $E_{CM}$ in comparison to the particles with unequal masses. This $E_{CM}$ increases as the BH spin increases and diverges for the extremal rotating BH under specific conditions on the angular momentum of one of the particles. Hence, to have the maximum collisional energy, it is assumed that both the spinning particles have the same mass (i.e., $m_1 = m_2 = m$) and for simplicity we consider $m = 1$. Therefore, the Eq. (31) with these assumptions in the equatorial plane becomes

$$\frac{E_{CM}^2}{r^2} = 2 \left( 1 - g_{tt}P_1^t P_2^t + g_{rr}P_1^r P_2^r + g_{\theta \varphi} P_1^\theta P_2^\varphi \right),$$

(Eq. 32)

which after substituting the values of $P^t$, $P^r$ and $P^\varphi$ from Eqs. (24), (25) and (26) respectively, reduces to

$$\frac{E_{CM}^2}{r^2} = \frac{2}{\Delta_0 C_1 C_2} \left[ r^{3r+2} D_1 D_2 + \Delta_0 \left[ C_1 C_2 - 4r^{6r+4} (j_1 - e_1 s_1)(j_2 - e_2 s_2) \right] - \sqrt{r^{3r+1} D_1^2 - \Delta_0 \left[ C_1^2 + 4r^{6r+4} (j_1 - e_1 s_1)^2 \right]} \right]$$

\(\frac{E_{CM}^2}{r^2} = \frac{2}{\Delta_0 C_1 C_2} \left[ r^{3r+2} D_1 D_2 + \Delta_0 \left[ C_1 C_2 - 4r^{6r+4} (j_1 - e_1 s_1)(j_2 - e_2 s_2) \right] - \sqrt{r^{3r+1} D_1^2 - \Delta_0 \left[ C_1^2 + 4r^{6r+4} (j_1 - e_1 s_1)^2 \right]} \right],

(Eq. 33)

where

$$C_{1,2} = 2r^{3r+2} - s_{1,2}^2 \left[ 2Mr^{3r} + \lambda(3s + 1) \right],$$

$$D_{1,2} = 2r^{3r+2} e_{1,2} - j_{1,2} s_{1,2} \left[ 2Mr^{3r} + \lambda(3s + 1) \right].$$

(Eq. 34)

One can easily verify from Eq. (33) that $E_{CM}$ could possibly diverge not only for $\Delta_0 = 0$ but also for $C_{1,2} = 0$. In case one substitutes $\lambda = 0$ and $e = 1$ in Eq. (33), the expression

\[\Delta = r - 2M \quad \text{and} \quad \Delta_{1,2} = r^3 - M s_{1,2}^2.\]

(Eq. 35)

for $E_{CM}$ reduces to

\[E_{CM}^2 = \frac{2}{\Delta_1 \Delta_2} \left[ r (r^3 - M j_1 s_1)(r^3 - M j_2 s_2) + \Delta_1 \Delta_2 - r^4 (j_1 - s_1)(j_2 - s_2) \right] - \sqrt{r (r^3 - M j_1 s_1)^2 - \Delta_1 \Delta_2 + r^4 (j_1 - s_1)^2} \sqrt{r (r^3 - M j_2 s_2)^2 - \Delta_1 \Delta_2 + r^4 (j_2 - s_2)^2}.

\[\Delta = r - 2M \quad \text{and} \quad \Delta_{1,2} = r^3 - M s_{1,2}^2. \quad \text{Eq. (35) matches with} \quad E_{CM} \quad \text{of two spinning test particles colliding near the Schwarzschild BH [103].}

For Eq. (33), the case when $\Delta_0 = 0$ is not of much interest because both numerator and denominator vanish at the horizon, and the energy in this limit becomes finite. It can be generally shown from (32) that in the limit $\Delta_0 = 0$ or, equiv-
The radial velocity $u^r$ is proportional to the radial component of the conjugate momenta $P^r$ and, therefore, we can determine the radial turning points from $P^r = 0$. We rewrite $P^r$ in the form of an effective potential,

$$
\left(\frac{P^r}{m}\right)^2 = A \left[1 - \frac{s^2 f(r)}{2r}\right]^{-2} (e - V_{\text{eff}(+)}(r))(e - V_{\text{eff}(-)}(r)),
$$

$$
V_{\text{eff}(\pm)}(r) = B \pm C^{1/2} \lambda \frac{1}{A},
$$

where

$$
A = 1 - \frac{f(r)s^2}{r^2},
$$

$$
B = \frac{js}{r^2} \left(\frac{f(r) r - f(r)}{2}\right),
$$

$$
C = f(r) \left(1 - \frac{s^2 f(r)}{2r}\right)^2 \left[1 + \frac{j^2}{r^2} - \frac{f(r)s^2}{r^2}\right].
$$

One needs to restrict the values of $r$ such that $e > V_{\text{eff}(+)}(r)$ or $e < V_{\text{eff}(-)}(r)$ whenever $A > 0$, in order to have $u^r$ to be real for the motion of the spinning particle. We can easily check for the cases $e = -1, \frac{1}{3}, -\frac{2}{3}$ that $A$ has some minimum outside the event horizon. For $e = -1$ we find a minimum at $r = 3M$ with $A = 1 - s^2 \left(\frac{1}{27M^2} + \lambda\right)$ implying that $A$ is positive between event and cosmological horizon for $s \ll M$. Analogously, for $e = -\frac{2}{3}$ the minimum $A = 1 - s^2 \left(\frac{1}{27M^2}\right)$ is at $r = \frac{4M}{3\lambda}$, so again $A$ is positive for $s \ll M$. Finally, for $e = -\frac{2}{3}$ we have a minimum between event and cosmological horizon at $r = \left(1 - \sqrt{1 - 6M\lambda}\right)/\lambda$, and $A$ is always positive for $s \ll M$.

In the original paper by BSW [16] it is assumed that the colliding particles start from rest at infinity. In our case this is however not generally possible due to the presence of a cosmological horizon. Let us discuss the cases $e = -1/3, -2/3, -1$ separately: (i) If $e = -1/3$, we only have an event horizon as explained in section III. We can therefore assume that the particle starts from rest at infinity. In this case, the energy of the particle is given by $e = 1 - \lambda$. (ii) For $e = -2/3$, we have an event and a cosmological horizon if we choose $\lambda < 1/(8M)$, see section III. Therefore, it does not make sense to consider a particle starting from infinity. Instead, we could choose to let the particle start from rest from the static radius, see e.g. [120, 121], representing an equilibrium between gravitational attraction and cosmological expansion. A particle with $P^r = 0, P^\phi = 0$ can sit at radius $r = \sqrt{2/\lambda}$ with energy $e^2 = 1 - 2\sqrt{2/\lambda}$. (iii) For $e = -1$ we may again choose the static radius as starting point, giving $r = \lambda^{-\frac{2}{3}}$ and $e^2 = 1 - 3\lambda^{-\frac{2}{3}}$.

In Fig. 2, the behavior of the positive component of the effective potential $V_{\text{eff}(+)}$ is shown as a function of $r$ for two different values of the spin $(s)$ and several different $j$. We plotted here $V_{\text{eff}(+)}$ only for those values of the particle spin $s$ and the total angular momentum $j$ for which the particle starting from rest from infinity or the static radius, as respectively explained

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$\lambda$  & 0.00001 & 0.0001 & 0.001 & 0.01 \\
\hline
$s$ & $r_d$ & $r_d$ & $r_d$ & $r_d$ \\
\hline
0.2 & 0.341995 & 0.341991 & 0.341991 & 0.341949 \\
0.4 & 0.42883 & 0.428855 & 0.426594 & 0.42594 \\
0.6 & 0.711378 & 0.711293 & 0.710527 & 0.710527 \\
0.8 & 0.861772 & 0.861590 & 0.859943 & 0.859943 \\
0.99 & 0.993319 & 0.993289 & 0.992998 & 0.990098 \\
\hline
\end{tabular}
\end{table}

\section{V. EFFECTIVE POTENTIAL AND RADIAL TURNING POINTS}

The study of the effective potential and the radial turning points are very important as this help us to characterize the different trajectories of the spinning particles moving around the BHs.
above, will fall into the SBHQ and does not meet the turning point first. It is shown in Fig. 2 that the maximum value of $V_{\text{eff}(+)}$ decreases with increase in $j$ for each value of $s$ (i.e., $s = 0.2$ and $s = 0.99$) corresponding to $\epsilon = -1/3, -2/3$ and $-1,$ respectively.

We showed the behavior of the positive component of $P^r$ with $r$ in Fig. 3 for different combinations of particle spin $s$, total angular momentum $j$, $\lambda$ and $\epsilon$ as it will help in visualizing for which combinations of these parameters the spinning particle will reach the event horizon $r_0(1)$ first before meeting the turning point. In the figure, we fixed the normalization parameter $\lambda = 0.00001$ and increase the EOS parameter for quintessential matter $\epsilon$ from top to bottom in each column. It is easy to conclude from the Fig. 3 (see first column) that all the spinning particles fall into the SBHQ if they obey $s \ll M$, as implied by the Möller limit, for $j = 0$. In the second and third columns the value of $s$ is fixed to 0.2 and 0.99, respectively. It is found from the second and third columns that for each $\epsilon$ value the range of this total angular momentum $j$ increases with increase in $s$. However, the radial distance for which $P^r = 0$ decreases as $s$ increases.

**B. Classification of the spinning particles and their trajectories**

Let us return now to particles that might produce arbitrarily high center of mass energies. According to equation Eq. (36) this may happen for collisions near the horizon if $\mathcal{K}$ of at least one of the colliding particles becomes very small.

From now onwards we denote the event horizon $r_0(1) \equiv r_0$ until and otherwise stated. We start by classifying the spinning particles into three different classes: We call a particle critical if $\mathcal{K}|_{r=r_0} = 0$, near-critical if $\mathcal{K}|_{r=r_0} = \mathcal{O}(\sqrt{r_c - r_0})$ with the point of collision $r_c$, and all other particles usual.

Let us start with critical particles. The condition $\mathcal{K}|_{r=r_0} = 0$ implies

$$e = \frac{jsf'(r_0)}{2r_0}. \quad (39)$$

Then, the expression for $\mathcal{K}$ near the event horizon (in the first approximation) reads

$$\mathcal{K} \approx \frac{3js}{r_0} \left[ \frac{2Mr_0}{2r_0^4 + s^2(2Mr_0^3 + (3e + 1)\lambda)} \right] (r - r_0). \quad (40)$$

Thus, the second term in Eq. (26) becomes larger than $\mathcal{K}^2$ close to the horizon, where the collision should take place. Hence $(P^r)^2$ is negative there which in turn means that the spinning particle cannot reach the event horizon and meets the turning point first.

For a near critical particle, to have $\mathcal{K}|_{r=r_0} = \mathcal{O}(\sqrt{r_c - r_0})$, we may for instance choose the energy as

$$e = \frac{jsf'(r_0)}{2r_0} + a\sqrt{r_c - r_0}(2r_0 - s^2 f'(r_0)), \quad (41)$$

where $a$ is some positive constant. At the point of collision $r_c$ we then find

$$\mathcal{K}|_{r=r_c} = \mathcal{K}|_{r=r_0} + \mathcal{K}'|_{r=r_0} (r_c - r_0) + \ldots = a\sqrt{r_c - r_0} + \mathcal{O}(r_c - r_0). \quad (42)$$

Now consider the case that one particle, say particle 1, is usual and the other particle is near-critical. To calculate the center of mass energy for this case we write $f = (r - r_0)f$ and derive from (32),

$$\frac{1}{2}E_{\text{CM}} = 1 - 4L_1L_2 + \frac{1}{f} \left( K_1K_2 - \sqrt{R_1R_2} \right), \quad (43)$$

where $R = \mathcal{K}^2 - f(1 + 4L^2)$. If we evaluate all quantities at $r = r_c$ we find
FIG. 3. The variation of $P^n$ with $r$ for a Schwarzschild BH surrounded by quintessential matter. Left column: shows different combinations of spin $s$, keeping $j = 0$. Middle column: different combinations of $j$, keeping $s = 0.2$. Right column: different combinations of $j$, keeping $s = 0.99$. In each of the rows the EOS parameter is fixed to $\epsilon = -1/3, -2/3$ and $-1$, respectively, for the corresponding value of normalization constant (i.e., $\lambda = 0.00001$). Here, for the corresponding value of parameter $\epsilon (-1/3, -2/3$ and $-1)$, the value of particle energy per unit mass $\epsilon$ is 0.9999, 0.995518 and 0.967144. The vertical (blue) solid line represents the location of event horizon ($M=1$) (Color online).

\[
\frac{1}{2} E_{\text{CM}} = 1 - 4\mathcal{L}_1 \mathcal{L}_2 + \frac{K_1 a_2}{f\sqrt{r_c - r_0}} + O(\sqrt{r_c - r_0}) \\
\sqrt{\left[K_1^2 - (r_c - r_0)\bar{f}(1 + 4\mathcal{L}_1^2)\right] \left[a_2^2 + O(\sqrt{r_c - r_0}) - \bar{f}(1 + 4\mathcal{L}_2^2)\right] \sqrt{r_c - r_0}} \\
\frac{1}{\bar{f}(r_c - r_0)} \\
= 1 - 4\mathcal{L}_1 \mathcal{L}_2 + \frac{K_1 \left[a_2 - \sqrt{a_2^2 - \bar{f}(1 + 4\mathcal{L}_2^2)}\right]}{f\sqrt{r_c - r_0}} + O(1).
\] (45)

We see that this expression is only valid if $a_2^2 - \bar{f}(1 + 4\mathcal{L}_2^2) > 0$. In the limit $s = 0$ this condition can be fulfilled, and by continuity it should also hold for small $s$. If the point of collision $r_c$ now approaches the horizon $r_0$ the center of mass energy (45) can grow without bound.

If both particles are near-critical, we can calculate the center of mass energy analogously.
\[ \frac{1}{2} E_{\text{CM}} = 1 - 4 \mathcal{L}_1 \mathcal{L}_2 + \frac{a_1 a_2}{f} - \sqrt{\left[a_1^2 - f(1 + 4 \mathcal{L}_1^2)\right]} \left[a_2^2 - f(1 + 4 \mathcal{L}_2^2)\right] \frac{1}{f} + \mathcal{O}(r_c - r_0), \] (46)

which will remain finite for \( r_c \rightarrow r_0 \). Finally, if both particles are usual, we can directly see that the diverging parts will cancel and the center of mass energy remains finite, too.

VI. AVOIDANCE OF SUPERLUMINAL REGION

It is shown in [103, 115, 122–124] that the four-momentum satisfies the relation \( P^\alpha P_\alpha = -1 \) and hence is a conserved quantity, in contrast to the four-velocity \( u^\alpha \), which is not a conserved quantity for the spinning test particles moving in the curved background. Therefore, the \( P^\alpha \) vector remains timelike throughout the motion of spinning particle around the BH, whereas the \( u^\alpha \) vector might change from the subluminal (timelike) to superluminal (spacelike) region depending upon the invariant relation \( u^\alpha u_\alpha < 0 \) or \( u^\alpha u_\alpha > 0 \), respectively. As the four-velocity \( u^\alpha \) of two colliding spinning test particles will not always lie in the subluminal region, it becomes important to examine closely the behavior of the square of the four-velocity in the region where \( E_{\text{CM}} \) diverges. The square of the four velocity thus reads as

\[ U^2 = \frac{u^\alpha u_\alpha}{(u^t)^2} = g_{tt} + 2g_{tr} \left( \frac{u^\alpha}{u^t} \right)^2 + g_{\phi \phi} \left( \frac{u^\phi}{u^t} \right)^2. \] (47)

Using Eqs. (24), (25), (26), (28) and (29) in Eq. (47) leads to:

\[ U^2 = -f(r)^2 \left( \frac{2r - s^2 f(r)^{1/3}}{2r - j s f(r)^{1/3}} \right)^2 (1 - \Sigma), \] (48)

\[ \Sigma = \frac{(2j - es s^2)^2 (\eta_-)}{(2r - s^2 f(r)^{1/3})^4}, \] (49)

where \( \eta_\pm = f(r)^{1/3} \pm rf(r)^{2/3} \).

The \( E_{\text{CM}} \) calculated in Eq. (33) diverges when either \( C_1 = 0 \) or \( C_2 = 0 \) as mentioned in section IV. We already showed there that the point where \( C_i = 0 \) always lies behind the horizon and, therefore, is of little importance for our analysis. We note here that, in addition, the condition \( C_i = 0 \) leads to a transition of \( U^2 \) (i.e. Eq. (48)) of the colliding spinning particle from the subluminal region (physical) to the superluminal region (unphysical) as seen in the Fig 4. We have also concluded earlier from Eq. (36) that the center of mass energy remains finite when the collision takes place at the event horizon. Hence, in this work we are more interested in finding location outside the event horizon where the square of the four-velocity lies in subluminal region. This leads to the condition \( \Sigma < 1 \) according to Eq. (48).

We may rewrite the expression for \( \Sigma \) in Eq. (49) as

\[ \Sigma = 4 \mathcal{L}^2 (G - 1), \] (50)

\[ G := \left( \frac{2r - r s f(r)^{1/3}}{2r - s^2 f(r)^{1/3}} \right)^2. \] (51)

We first notice that both \( G \) and \( \mathcal{L}^2 \) are monotonically decreasing functions, and that \( \Sigma > 0 \), for \( \epsilon = -\frac{1}{3} \) and \( \epsilon = -1 \). For \( \epsilon = -\frac{2}{3} \) this only holds in the vicinity of the horizon. We therefore find that \( \Sigma < 1 \) holds in the vicinity of the horizon \( r_0 \) if

\[ \Sigma_{r_0} < 1. \] (52)

In order to have an arbitrarily high collisional \( E_{\text{CM}} \) outside the event horizon, one of the colliding particles must be the usual particle (i.e. a particle for which \( \mathcal{K}_{r_0} \neq 0 \)) and the other must be a near-critical one as shown in the previous section.

For the usual particle with \( s = 0 \), the condition (52) is satisfied automatically. If \( s \neq 0 \) we can always choose \((j - es)^2\rangle\) such that (52) holds, for instance, one could choose \( j = es \). For near-critical particles, we fixed an energy \( e \) in (41). To achieve the inequality (52), we could then for instance choose \( j = es \) again. Near-critical particles we can also explicitly solve \( \Sigma = 1 \) for \( j \), using the energy \( e \) from Eq. (41). We find

\[ j = as \sqrt{r_c - r_0} \pm \frac{r_0}{s} N_e \] (53)

with

\[ N_{-1} = \frac{Ms^2 - r_0^3 (\lambda s^2 - 1)}{\sqrt{3M (Ms^2 + 2r_0^3 (\lambda s^2 + 1))}}, \] (54)

\[ N_{-1/3} = \frac{Ms^2 - r_0^3}{\sqrt{3M (Ms^2 + 2r_0^3)}}, \] (55)

\[ N_{-2/3} = \frac{2Ms^2 - r_0^3 (\lambda s^2 - 2r_0)}{\sqrt{(6M - 2Mr_0^3) (2Ms^2 + r_0^3 (\lambda s^2 + 4r_0))}}. \] (56)

Hence, we may conclude that the collision of a near-critical particle with a usual particle can produce arbitrarily high center-of-mass energy \( E_{\text{CM}} \) if we fine tune the parameters. For instance, we could choose a usual particle starting from rest from infinity or the static radius, respectively, with vanishing total angular momentum \( j = es \), and a near-critical particle with energy as given in (41) and vanishing total angular momentum starting from a radius close to the event horizon (but outside of it).
VII. SUMMARY, CONCLUSION AND FUTURE PROSPECTS

The equations of motion for spinning test particles are summarized and the useful quantities for the spinning particles like conserved mass and conserved spin have been noted down. This in addition to the relation between particle 4-particles like conserved mass and conserved spin have been summarized and the useful quantities for the spinning particles are given in Eqs. (24), (25), (26), (28) and (29). These expressions are used in later sections to study the collision of spinning particles in the background of the SBHQ.

- Explicit form of 4-momentum and 4-velocity for spinning particles that move in a curved spacetime of SBHQ are given in Eqs. (24), (25), (26), (28) and (29). These expressions are used in later sections to study the collision of spinning particles in the background of the SBHQ.

- The behavior of $P^r$ as a function of $r$ plotted in Fig. 3, is used to analyze and distinguish between different trajectories of the spinning particles. It is found from the Fig. 3 that within the Møller limit, all the spinning particles reach the event horizon of SBHQ for $j = 0$ first before meeting the turning point. It is worth noting from Fig. 3, that for the value of $\epsilon = -1/2$, $-2/3$ and $-1$, the maximum allowed range of total angular momentum $j$ for which a spinning particle fall inside the event horizon of SBHQ without meeting the turning point first, increases with increase in the value of particle’s spin $s$. However, it is also observed that the radius
at which $P^r$ becomes zero decreases with increase in $s$ for respective $\epsilon$.

- We showed that the center of mass energy remains in general finite in the limit that the collision takes place at the horizon. Moreover, potential additional points of divergence given by $C_{1,2} = 0$, see Eq. (33), are shown to always lie behind the event horizon. It is easily concluded from the expression for $C_{1,2}$ in Eq. (34) that the divergence radius $r_d$ for the EOS parameter $\epsilon = -1/3$ is independent of normalization constant $\lambda$ and it increases as the spin $s$ of the particle increases. The numerical value of $r_d$ for different spin $s$ is shown in Table II. Contrary to the case $\epsilon = -1/3$, the divergence radius $r_d$ for the cases $\epsilon = -2/3$ and $\epsilon = -1$ depends on both $\lambda$ and $s$. From Tables III and IV, it is easily concluded that $r_d$ decreases as $\lambda$ increases for the fixed value particle’s spin. On the other hand, if $\lambda$ is fixed and spin increases then the value of $r_d$ increases.

- Characterization of the spinning particles into three different classes: usual, critical and near critical is also done in Sec. V B, similarly as shown in [109]. Later, different combinations of these colliding spinning particles types (say usual-usual, critical-usual, usual-near critical and near critical-near critical) are studied in order to see for which case infinitely high $E_{\text{CM}}$ scenario can be achieved. It is interestingly found that for SBHQ, there is one possible combination of the spinning particles (i.e., usual-near critical) for which infinitely high $E_{\text{CM}}$ can be obtained.

- It is observed from the variation $U^2$ as a function of $r$ for different values of the EOS parameter $\epsilon$ as in Fig. 4, that the 4-velocity squared of the spinning particle is always timelike outside the event horizon if spin of the particle is sufficiently small (say $s \leq 0.6$ for $\lambda = 0.00001$ and $j = 4$). It is also found, if the spin is small enough (say $s = 0.2$), $U^2$ always falls in a subluminal region outside the event horizon $r_{0(1)}$ of SBHQ for all values of $j$. Unlike the case when $s = 0.99$ and $\lambda = 0.00001$, where $U^2$ is not always fall in subluminal region for all values of $j$ beyond event horizon $r_0$ of SBHQ.

In future works, we intend to study the particle acceleration process and chaos of spinning particles moving in the vicinity of rotating BHs surrounded by quintessential matter fields.

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[1] K. Schwarzschild, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) 1916, 189 (1916), physics/9905030.
[2] B. P. Abbott et al. (LIGO Scientific, Virgo), Phys. Rev. Lett. 116, 061102 (2016), 1602.03837.
[3] K. Akiyama et al. (Event Horizon Telescope), Astrophys. J. 875, L1 (2019), 1906.11238.
[4] J. M. Maldacena, Ph.D. thesis, Princeton U. (1996), hep-th/9607235, URL: http://wwwlib.umi.com/dissertations/fullcit?p9627605.
[5] T. Padmanabhan, Phys. Rept. 380, 235 (2003), hep-th/0212290.
[6] S. Weinberg, Rev. Mod. Phys. 61, 1 (1989), [S69(1988)].
[7] S. M. Carroll, Phys. Rev. Lett. 81, 3067 (1998), astro-ph/9806099.
[8] J. Khoury and A. Weltman, Phys. Rev. Lett. 93, 171104 (2004), astro-ph/0309300.
[9] C. Armendariz-Picon, V. F. Mukhanov, and P. J. Steinhardt, Phys. Rev. Lett. 85, 4438 (2000), astro-ph/0004134.
[10] T. Padmanabhan, Phys. Rev. D66, 021301 (2002), hep-th/0204150.
[11] R. R. Caldwell, Phys. Lett. B545, 23 (2002), astro-ph/9908168.
[12] M. Gasperini, F. Piazza, and G. Veneziano, Phys. Rev. D65, 023508 (2002), gr-qc/0108016.
[13] E. J. Copeland, M. Sami, and S. Tsujikawa, Int. J. Mod. Phys. D15, 1753 (2006), hep-th/0603057.
[14] V. V. Kiselev, Class. Quant. Grav. 20, 1187 (2003), gr-qc/0210040.
[15] R. Uniyal, N. Chandrachani Devi, H. Nandan, and K. D. Purohit, Gen. Rel. Grav. 47, 16 (2015), 1406.3931.
[16] M. Banados, J. Silk, and S. M. West, Phys. Rev. Lett. 103, 111102 (2009), 0909.0169.
[17] I. Jacobson and T. P. Sotiriou, Phys. Rev. D79, 064034 (2009), 0906.2262.
[18] T. Jacobson and T. P. Sotiriou, Phys. Rev. Lett. 104, 021101 (2010), 0911.3363.
[19] A. A. Grib and Yu. V. Pavlov, Astropart. Phys. 34, 581 (2011), 1001.0756.
[20] K. Lake, Phys. Rev. Lett. 104, 211102 (2010), [Erratum: Phys. Rev. Lett.104,259903(2010)], 1001.5463.
[21] S.-W. Wei, Y.-X. Liu, H. Guo, and C.-E. Fu, Phys. Rev. D82, 103005 (2010), 1006.1056.
[22] A. A. Grib and Y. V. Pavlov (2010), 1007.3222.
[23] O. B. Zaslavskii, JETP Lett. 92, 571 (2010), [Pisma Zh. Eksp.
