A functional model for the Fourier-Plancherel operator truncated on the positive half-axis

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This paper is dedicated to the memory of my colleague Mikhail Solomyak.

Abstract. The truncated Fourier operator $\mathcal{F}_{\mathbb{R}^+}$,

$$(\mathcal{F}_{\mathbb{R}^+} x)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} x(\xi) e^{it\xi} \, d\xi, \quad t \in \mathbb{R}^+,$$

is studied. The operator $\mathcal{F}_{\mathbb{R}^+}$ is considered as an operator acting in the space $L^2(\mathbb{R}^+)$. The functional model for the operator $\mathcal{F}_{\mathbb{R}^+}$ is constructed. This functional model is the multiplication operator on the appropriate $2 \times 2$ matrix function acting in the space $L^2(\mathbb{R}^+) \oplus L^2(\mathbb{R}^+)$. Using this functional model, the spectrum of the operator $\mathcal{F}_{\mathbb{R}^+}$ is found. The resolvent of the operator $\mathcal{F}_{\mathbb{R}^+}$ is estimated near its spectrum.

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Notation:
\begin{itemize}
  \item $\mathbb{R}$ - the set of all real numbers.
  \item $\mathbb{R}^+$ - the set of all positive real numbers.
  \item $\mathbb{C}$ - the set of all complex numbers.
  \item $\mathbb{Z}$ - the set of all integer numbers.
  \item $\mathbb{N} = \{1, 2, 3, \ldots\}$ - the set of all natural numbers.
\end{itemize}

1. The Fourier-Plancherel operator truncated on the positive half-axis.

In this paper we study the truncated Fourier operator $\mathcal{F}_{\mathbb{R}^+}$,

$$(\mathcal{F}_{\mathbb{R}^+} x)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} x(\xi) e^{it\xi} \, d\xi, \quad t \in \mathbb{R}^+. \quad (1.1)$$
The operator $\mathcal{F}_{\mathbb{R}^+}$ is considered as an operator acting in the space $L^2(\mathbb{R}^+)$ of all square summable complex valued functions on $\mathbb{R}^+$ provided with the inner product

$$\langle x, y \rangle_{L^2(\mathbb{R}^+)} = \int_{\mathbb{R}^+} x(t)\overline{y(t)} \, dt.$$  

The operator $\mathcal{F}_{\mathbb{R}^+}^*$ adjoint to the operator $\mathcal{F}_{\mathbb{R}^+}$ with respect to this inner product is

$$(\mathcal{F}_{\mathbb{R}^+}^* x)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} x(\xi) e^{-it\xi} \, d\xi, \quad t \in \mathbb{R}^+. \quad (1.2)$$

The operator $\mathcal{F}_{\mathbb{R}^+}$ is the operator of the form

$$\mathcal{F}_{\mathbb{R}^+} = P_{\mathbb{R}^+} F P_{\mathbb{R}^+}|_{L^2(\mathbb{R}^+)}, \quad (1.3)$$

where $\mathcal{F}$ is the Fourier-Plancherel operator on the whole real axis:

$$(\mathcal{F} x)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x(\xi) e^{it\xi} \, d\xi, \quad t \in \mathbb{R}, \quad (1.4)$$

$\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, and $P_{\mathbb{R}^+}$ is the natural orthogonal projector from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R}^+)$:

$$(P_{\mathbb{R}^+} x)(t) = 1_{\mathbb{R}^+}(t) x(t), \quad x \in L^2(\mathbb{R}), \quad t \in \mathbb{R}^+, \quad (1.5)$$

$1_{\mathbb{R}^+}(t)$ is the indicator function of the set $\mathbb{R}^+$. For any set $E$, its indicator function $1_E$ is

$$1_E(t) = \begin{cases} 1, & \text{if } t \in E, \\ 0, & \text{if } t \not\in E. \end{cases} \quad (1.6)$$

It should be mention that the Fourier operator $\mathcal{F}$ is an unitary operator in $L^2(\mathbb{R})$:

$$\mathcal{F}^* \mathcal{F} = \mathcal{F} \mathcal{F}^* = J_{L^2(\mathbb{R})}, \quad (1.7)$$

$J_{L^2(\mathbb{R})}$ is the identity operator in $L^2(\mathbb{R})$, $\mathcal{F}^*$ is the operator adjoint to the operator $\mathcal{F}$ with respect to the standard inner product in $L^2(\mathbb{R})$.

From (1.3) and (1.7) it follows that the operators $\mathcal{F}_{\mathbb{R}^+}$ and $\mathcal{F}_{\mathbb{R}^+}^*$ are contractive: $\|\mathcal{F}_{\mathbb{R}^+}\| \leq 1$, $\|\mathcal{F}_{\mathbb{R}^+}^*\| \leq 1$. We show later that actually

$$\|\mathcal{F}_{\mathbb{R}^+}\| = 1, \quad \|\mathcal{F}_{\mathbb{R}^+}^*\| = 1. \quad (1.8)$$

Nevertheless, these operators are strictly contractive:

$$\|\mathcal{F}_{\mathbb{R}^+} x\| < \|x\|, \quad \|\mathcal{F}_{\mathbb{R}^+}^* x\| < \|x\|, \quad \forall x \in L^2(\mathbb{R}^+), \quad x \neq 0, \quad (1.9)$$

and their spectral radii $r(\mathcal{F}_{\mathbb{R}^+})$ and $r(\mathcal{F}_{\mathbb{R}^+}^*)$ are less that one:

$$r(\mathcal{F}_{\mathbb{R}^+}) = r(\mathcal{F}_{\mathbb{R}^+}^*) = 1/\sqrt{2}. \quad (1.10)$$

In particular, the operators $\mathcal{F}_{\mathbb{R}^+}$ and $\mathcal{F}_{\mathbb{R}^+}^*$ are contractions of the class $C_{00}$ in the sense of [1]. (See [1] Chapter 2, Section 4.)
In [1], a spectral theory of contractions in a Hilbert space is developed. The starting point of this theory is the representation of the given contractive operator $A$ acting in the Hilbert space $\mathcal{H}$ in the form

$$A = PUP,$$  

(1.11)

where $U$ is an unitary operator acting is some ambient Hilbert space $\mathcal{F}$, $\mathcal{H} \subset \mathcal{F}$, and $P$ is the orthogonal projector from the whole space $\mathcal{F}$ onto its subspace $\mathcal{H}$. In the construction of [1] there is required that not only the equality (1.11) but also the whole series of the equalities

$$A^n = PU^n P, \quad n \in \mathbb{N},$$  

(1.12)

hold. The unitary operator $U$ acting in the ambient Hilbert space $\mathcal{F}$, $\mathcal{H} \subset \mathcal{F}$, is said to be the unitary dilation of the operator $A$, $A : \mathcal{H} \to \mathcal{H}$, if the equalities (1.12) hold. In [1] it is shown that every contractive operator $A$ admits an unitary dilation. Using the unitary dilation, a functional model of the operator $A$ is constructed. This functional model is an operator acting in some Hilbert space of analytic functions. The functional model of the operator $A$ is an operator which is unitary equivalent to $A$. The spectral theory of the original operator $A$ is developed by analyzing its functional model. However the functional model constructed in [1] is not suitable for the spectral analysis of the truncated Fourier-Plancherel operator $\mathcal{F}_{\mathbb{R}^+}$.

The relation (1.3) is of the form (1.11), where $\mathcal{H} = L^2(\mathbb{R}), \mathcal{F} = L^2(\mathbb{R})$, $U = \mathcal{F}, A = \mathcal{F}_{\mathbb{R}^+}, P = P_{\mathbb{R}^+}$ is an orthoprojector from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R}^+)$, (1.5). For these objects the equalities (1.12) do not hold for all $n \in \mathbb{N}$, but only for $n = 1$. So, the operator $\mathcal{F}$ is not an unitary delation of its truncation $\mathcal{F}_{\mathbb{R}^+}$. Nevertheless, we succeeded in constructing such a functional model of the operator $\mathcal{F}_{\mathbb{R}^+}$ which is easily analyzable. Analyzing this model, we can develop the complete spectral theory of the operator $\mathcal{F}_{\mathbb{R}^+}$.

2. The model space.

Definition 2.1.

1. The model space $\mathcal{M}$ is the set of all $2 \times 1$ columns $\varphi = \begin{bmatrix} \varphi_+ \\ \varphi_- \end{bmatrix}$, which entries $\varphi_+$ and $\varphi_-$ are arbitrary complex valued functions from $L^2(\mathbb{R}^+)$.  

2. The space $\mathcal{M}$ is equipped by the natural linear operations.

3. The inner product $\langle \varphi, \psi \rangle_{\mathcal{M}}$ of the columns $\varphi = \begin{bmatrix} \varphi_+ \\ \varphi_- \end{bmatrix}$ and $\psi = \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix}$ belonging to this space is defined as

$$\langle \varphi, \psi \rangle_{\mathcal{M}} = \langle \varphi_+, \psi_+ \rangle_{L^2(\mathbb{R}^+)} + \langle \varphi_-, \psi_- \rangle_{L^2(\mathbb{R}^+)}. $$

(2.1)

In particular, the equality

$$\|\varphi\|_{\mathcal{M}}^2 = \|\varphi_+\|_{L^2(\mathbb{R}^+)}^2 + \|\varphi_-\|_{L^2(\mathbb{R}^+)}^2.$$  

(2.2)
Remark 2.2. The model space \( \mathcal{M} \) is just the orthogonal sum of two copies of the space \( L^2(\mathbb{R}^+) \). The standard notation \( L^2(\mathbb{R}^+) \oplus L^2(\mathbb{R}^+) \) for such orthogonal sum does not reflect that elements of \( \mathcal{M} \) are \( 2 \times 1 \) columns. The notation
\[
\begin{bmatrix}
L^2(\mathbb{R}^+)
\oplus
L^2(\mathbb{R}^+)
\end{bmatrix}
\]
is more logical, but too bulky.

Let us define the linear mapping \( U \) of the space \( L^2(\mathbb{R}) \) into the model space. For \( x \in L^2(\mathbb{R}^+) \), the formal definition is

\[
(Ux)(\mu) = \begin{bmatrix}
(Ux)_+(\mu) \\
(Ux)_-(\mu)
\end{bmatrix}, \quad \mu \in \mathbb{R}^+, \quad (2.3)
\]

where

\[
(Ux)_+(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} x(\xi) \xi^{-1/2} \xi^{+i\mu} d\xi, \quad \mu \in \mathbb{R}^+, \quad (2.4a)
\]

\[
(Ux)_-(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} x(\xi) \xi^{-1/2} \xi^{-i\mu} d\xi, \quad \mu \in \mathbb{R}^+. \quad (2.4b)
\]

Here and in what follows, \( \xi^{\zeta} = e^{\zeta \ln \xi} \), where \( \ln \xi \in \mathbb{R} \) for \( \xi \in \mathbb{R}^+ \).

If \( x \in L^2(\mathbb{R}^+) \), the functions \( x(\xi) \xi^{-1/2} \xi^{\pm i\mu} \), which appear in \( (2.4) \), may be not summable with respect to the Lebesgue measure \( d\xi \) on \( \mathbb{R}^+ \). So the integrals in \( (2.4) \) may not exists as Lebesgue integrals.

Definition 2.3. The set \( \mathcal{D} \) is the set of all functions \( x \in L^2(\mathbb{R}^+) \) which satisfy the condition

\[
\int_{\mathbb{R}^+} |x(\xi)| \xi^{-1/2} d\xi < \infty. \quad (2.5)
\]

Lemma 2.4. If the function \( x \) belongs to \( L^2(\mathbb{R}^+) \) and its support \( \text{supp} x \) is contained strictly inside the positive half-axis \( \mathbb{R}^+ \), then \( x \in \mathcal{D} \).

Proof.

\[
\int_{\mathbb{R}^+} |x(\xi)| \xi^{-1/2} d\xi = \int_{\xi \in \text{supp} x} |x(\xi)| \xi^{-1/2} d\xi \\
\leq \left\{ \int_{\mathbb{R}^+} |x(\xi)|^2 d\xi \right\}^{1/2} \cdot \left\{ \int_{\xi \in \text{supp} x} |\xi|^{-1} d\xi \right\}^{1/2} < \infty.
\]

For \( x \in \mathcal{D} \), the integrals in the right hand sides of \( (2.4a) \) and \( (2.4b) \) exist as Lebesgue integrals for every \( \mu \in \mathbb{R}^+ \). So the functions \( (Ux)_+(\mu) \) and \( (Ux)_-(\mu) \) are well defined for every \( \mu \in \mathbb{R}^+ \).
**Lemma 2.5.** If the function $x$ belongs to $\mathcal{D}$, then both functions $(Ux)_+$ and $(Ux)_-$ belong to $L^2(\mathbb{R}^+)$. Moreover the equality
\[
\|(Ux)_+\|_{L^2(\mathbb{R}^+)}^2 + \|(Ux)_-\|_{L^2(\mathbb{R}^+)}^2 = \|x\|_{\mathbb{R}^+}^2
\] (2.6)
holds.

**Proof.** Changing the variable $\xi = e^\eta$ in (2.4), we obtain
\[
(Ux)_+(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} v(\eta) e^{i\mu\eta} d\eta, \quad \mu \in \mathbb{R}^+, \tag{2.7a}
\]
and
\[
(Ux)_-(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} v(\eta) e^{-i\mu\eta} d\eta, \quad \mu \in \mathbb{R}^+. \tag{2.7b}
\]
where
\[
v(\eta) = e^{\eta/2}x(e^\eta). \tag{2.8}
\]
The equality
\[
\int_{\mathbb{R}} |v(\eta)|^2 d\eta = \int_{\mathbb{R}^+} |x(\xi)|^2 d\xi. \tag{2.9}
\]
holds. Let us define
\[
u(\nu) = \begin{cases} (Ux)_+(\nu), & \text{if } \nu \in \mathbb{R}^+, \\ (Ux)_-(\nu), & \text{if } \nu \in \mathbb{R}^- \end{cases} \tag{2.10}
\]
It is clear that
\[
\int_{\mathbb{R}} |\nu(\nu)|^2 d\nu = \int_{\mathbb{R}^+} |(Ux)_+(\mu)|^2 d\mu + \int_{\mathbb{R}^+} |(Ux)_-(\mu)|^2 d\mu. \tag{2.11}
\]
From (2.7) and (2.10) it follows that
\[
u(\nu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} v(\eta) e^{i\nu\eta} d\eta, \quad \nu \in \mathbb{R}. \tag{2.12}
\]
Thus,
\[
u(\nu) = (\mathcal{F}v)(\nu), \quad \nu \in \mathbb{R}, \tag{2.13}
\]
where $\mathcal{F}$ is the Fourier-Plancherel operator, (1.4). From the Parceval equality
\[
\int_{\mathbb{R}} |\nu(\nu)|^2 d\nu = \int_{\mathbb{R}} |v(\eta)|^2 d\eta \tag{2.14}
\]
and from the equalities (2.9) and (2.11) we derive the equality (2.6). \qed

It is clear that the set $\mathcal{D}$ is a vector subspace (non closed) of the space $L^2(\mathbb{R}^+)$. 

**Lemma 2.6.** The set $\mathcal{D}$ is dense in the space $L^2(\mathbb{R}^+)$. 

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Proof. Given $x \in L^2(\mathbb{R}^+)$, we define
\begin{equation}
  x_n(t) = x(t) \cdot 1_{[1/n, n]}(t), \quad n = 1, 2, \ldots,
\end{equation}
where $1_{[1/n, n]}(t)$ is the indicator function of the interval $[1/n, n]$. (See (1.6).) It is clear that
\begin{equation}
  \|x - x_n\|_{L^2(\mathbb{R}^+)} \to 0 \quad \text{as} \quad n \to \infty.
\end{equation}
Moreover, $x_n \in D$ by Lemma 2.4. □

Definition 2.7. The equality (2.6) means that the operator $U$ which is defined by (2.3)-(2.4) for $f \in D$ maps the subspace $D$ into the model space $\mathfrak{M}$ isometrically. Therefore the operator $U$ can be extended from the subspace $D \subset L^2(\mathbb{R}^+)$ onto its closure $L^2(\mathbb{R}^+)$ by continuity:

If $x \in L^2(\mathbb{R}^+)$, $x_n \in D, (n = 1, 2, \ldots)$, $x = \lim_{n \to \infty} x_n$, then $Ux \overset{\text{def}}{=} \lim_{n \to \infty} Ux_n$.

(2.17)

We preserve the notation $U$ for so extended operator.

From now we consider the operator $U$ which is already extended from $D$ on the whole space $L^2(\mathbb{R}^+)$ according to (2.17). It is clear that the operator $U$ maps the space $L^2(\mathbb{R}^+)$ onto some closed subspace of the model space $\mathfrak{M}$.

Theorem 2.8. The operator $U$ maps the space $L^2(\mathbb{R}^+)$ onto the whole model space $\mathfrak{M}$.

Proof. Let $y = \begin{bmatrix} y_+ \\ y_- \end{bmatrix}$ be an arbitrary element of the space $\mathfrak{M}$. Both functions $y_+$ and $y_-$ belong to $L^2(\mathbb{R}^+)$. We set
\begin{align*}
  u(\nu) &= \begin{cases} 
    y_+(\nu), & \text{if } \nu > 0, \\
    y_-(-\nu), & \text{if } \nu < 0.
  \end{cases}
\end{align*}

(2.18)

It is clear that $u \in L^2(\mathbb{R})$. As a function from $L^2(\mathbb{R})$, the function $u$ is representable in the form
\begin{equation}
  u(\nu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} v(\eta)e^{i\nu\eta} \, d\eta, \quad (\nu \in \mathbb{R}),
\end{equation}

(2.19)

where $v$ is a function from $L^2(\mathbb{R})$.

The formula (2.19) can be interpreted as the pair of formulas
\begin{align*}
  y_+(\mu) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} v(\eta)e^{i\mu\eta} \, d\eta, \quad \mu \in \mathbb{R}^+, \\
  y_-(-\mu) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} v(\eta)e^{-i\mu\eta} \, d\eta, \quad \mu \in \mathbb{R}^+.
\end{align*}

(2.20a, 2.20b)
where \( v \in L^2(\mathbb{R}) \). Changing the variable \( \xi = e^{\eta} \) in (2.20), we reduce (2.20) to the form

\[
y_+(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} x(\xi)\xi^{-1/2}e^{i\mu} d\xi, \quad \mu \in \mathbb{R}^+,
\]

(2.21a)

\[
y_-(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} x(\xi)\xi^{-1/2}e^{-i\mu} d\xi, \quad \mu \in \mathbb{R}^+,
\]

(2.21b)

where

\[
v(\eta) = e^{\eta/2}x(e^{\eta}).
\]

(2.22)

Moreover \( x \in L^2(\mathbb{R}^+) \):

\[
\int_{\mathbb{R}^+} |x(\xi)|^2 d\xi = \int_{\mathbb{R}} |v(\eta)|^2 d\eta.
\]

(2.23)

According to the definition of the operator \( U \), equalities (2.21) mean that

\[
y = Ux.
\]

(2.24)

\[\square\]

Remark 2.9. The function \( v \) in (2.19) may not belong to \( L^1(\mathbb{R}) \). To attach a meaning to the equality (2.19), we use the standard approximating procedure. We choose a sequence \( \{v_n\}_{n=1,2,...} \) such that \( v_n \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \) for every \( n \) and \( \|v_n - v\|_{L^2(\mathbb{R})} \to 0 \) as \( n \to \infty \). The sequence \( \{u_n\}_{n=1,2,...} \), where \( u_n(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} v_n(\xi)e^{it\xi} d\xi \), is well defined and converges to the function \( u \):

\[
\|u_n - u\|_{L^2(\mathbb{R})} \to 0 \quad \text{as} \quad n \to \infty.
\]

Remark 2.10. The transformation (2.4) is nothing but the Mellin transform \( \int_{\mathbb{R}^+} x(t) t^{\xi-1} dt \) restricted on the line \( \text{Re} \, \zeta = \frac{1}{2} : \zeta = \frac{1}{2} + i\mu, \mu \in \mathbb{R} \). The equality (2.6) is the Parceval equality for the Mellin transform. See [2, Section 3.17], Theorem 71. However we would like to emphasize that we consider this Mellin transform not a single function defined for \( \mu \in \mathbb{R} \), but as a pair of functions defined for \( \mu \in \mathbb{R}^+ \).

3. The model of the truncated Fourier-Plancherel operator.

In section 2 we introduced the operator \( U \) which maps the space \( L^2(\mathbb{R}^+) \) onto the model space \( \mathcal{M} \) isometrically. In this section we calculate the operator \( U\mathcal{F}_{\mathbb{R}^+}U^{-1} \), which serves as a model of the operator \( \mathcal{F}_{\mathbb{R}^+} \).

Let \( x \in L^2(\mathbb{R}^+) \) and \( \begin{bmatrix} y_+ \\ y_- \end{bmatrix} = Ux \), that is the equalities (2.4) hold.

We would like to express the pair \( \begin{bmatrix} z_+ \\ z_- \end{bmatrix} = U\mathcal{F}_{\mathbb{R}^+}x \) in terms of the pair \( \begin{bmatrix} y_+ \\ y_- \end{bmatrix} \).
Substituting the function \((F_{\mathcal{R}^+} x)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{R}^+} x(\xi)e^{it\xi} d\xi\) instead the function \(x\) into (2.4), we obtain

\[
z_+(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{R}^+} x(\xi) \left( \frac{1}{\sqrt{2\pi}} \int_{\mathcal{R}^+} e^{it\xi} t^{-1/2} t^{i\mu} dt \right) t^{-1/2} t^{i\mu} d\xi, \quad \mu \in \mathbb{R}^+,
\]

(3.1a)

\[
z_-(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{R}^+} x(\xi) \left( \frac{1}{\sqrt{2\pi}} \int_{\mathcal{R}^+} e^{it\xi} t^{-1/2} t^{-i\mu} dt \right) t^{-1/2} t^{-i\mu} d\xi, \quad \mu \in \mathbb{R}^+,
\]

(3.1b)

Changing the order of integration in (3.1), we come to the equalities

\[
z_+(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{R}^+} x(\xi) \left( \frac{1}{\sqrt{2\pi}} \int_{\mathcal{R}^+} e^{it\xi} t^{-1/2} t^{i\mu} dt \right) d\xi, \quad \mu \in \mathbb{R}^+,
\]

(3.2a)

\[
z_-(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{R}^+} x(\xi) \left( \frac{1}{\sqrt{2\pi}} \int_{\mathcal{R}^+} e^{it\xi} t^{-1/2} t^{-i\mu} dt \right) d\xi, \quad \mu \in \mathbb{R}^+,
\]

(3.2b)

Changing \(t \to t/\xi\) in (3.2), we obtain the pair of equalities

\[
z_+(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{R}^+} x(\xi) \xi^{-1/2} \xi^{-i\mu} \left( \frac{1}{\sqrt{2\pi}} \int_{\mathcal{R}^+} e^{it} t^{-1/2} t^{i\mu} dt \right) d\xi, \quad t \in \mathbb{R}^+,
\]

(3.3a)

\[
z_-(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{R}^+} x(\xi) \xi^{-1/2} \xi^{i\mu} \left( \frac{1}{\sqrt{2\pi}} \int_{\mathcal{R}^+} e^{it} t^{-1/2} t^{-i\mu} dt \right) d\xi, \quad t \in \mathbb{R}^+,
\]

(3.3b)

The inner integrals in (3.3) does not depend on \(\xi\). So we can present the equalities (3.3) in the form

\[
z_+(\mu) = F_{+\mu} y_-(\mu), \quad \mu \in \mathbb{R}^+,
\]

(3.4a)

\[
z_-(\mu) = F_{-\mu} y_+(\mu), \quad \mu \in \mathbb{R}^+,
\]

(3.4b)

where

\[
F_{+\mu} = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{R}^+} e^{it} t^{-1/2} t^{i\mu} dt, \quad \mu \in \mathbb{R}^+,
\]

(3.5a)

\[
F_{-\mu} = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{R}^+} e^{it} t^{-1/2} t^{-i\mu} dt, \quad \mu \in \mathbb{R}^+.
\]

(3.5b)

The functions \(F_{+\mu}\) and \(F_{-\mu}\) can be expressed in term of Euler \(\Gamma\)-functions:

\[
F_{+\mu} = \frac{1}{\sqrt{2\pi}} e^{i\pi/4} e^{-\frac{\pi}{4} \mu} \Gamma\left(\frac{1}{2} + i\mu\right), \quad \mu \in \mathbb{R}^+,
\]

(3.6a)

\[
F_{-\mu} = \frac{1}{\sqrt{2\pi}} e^{i\pi/4} e^{\frac{\pi}{4} \mu} \Gamma\left(\frac{1}{2} - i\mu\right), \quad \mu \in \mathbb{R}^+.
\]

(3.6b)

We will not justify the possibility of changing the order of integration in (3.1). The above reasoning, which leads from (3.1) to (3.4), plays the heuristic role.
Actually we establish the formulas (3.4), where the functions $F_+(-\mu)$ and $F_-(\mu)$ are of the form (3.6), in a different way.

The pair of equalities (3.4) can be presented in the matrix form

$$(U\mathcal{F}_{\mathbb{R}_+}x)(\mu) = F(\mu)(Ux)(\mu), \quad \mu \in \mathbb{R}^+, \quad (3.7)$$

where $F(\mu)$ is a $2 \times 2$ matrix:

$$F(\mu) = \begin{bmatrix} 0 & F_+(-\mu) \\ F_-(\mu) & 0 \end{bmatrix}, \quad \forall \mu \in \mathbb{R}^+. \quad (3.8)$$

**Theorem 3.1.** Let $x$ be an arbitrary function from $L^2(\mathbb{R}^+)$ and $\mathcal{F}_{\mathbb{R}_+}x$ be the truncated Fourier-Plancherel transform of $x$. Then their images $Ux$ and $U(\mathcal{F}_{\mathbb{R}_+}x)$ under the operator $U$ are related by the equalities (3.7), where the entries $F_+(-\mu)$ and $F_-(\mu)$ of the matrix $F(\mu)$ are of the form (3.6).

**Proof.** It is enough to verify the equalities (3.4) only for $x$ of the form $x(t) = e^{at}$, where

$$e^{at}(t) = e^{-at}, \quad t \in \mathbb{R}^+, \quad (3.9)$$

and $a$ is an arbitrary positive number. It is well known that the linear hall of the family of function $\{e^{at}(t)\}_{0 < a < \infty}$ is a dense set in $L^2(\mathbb{R}^+)$. The function $(\mathcal{F}_{\mathbb{R}_+}e^{at})(t)$ can be calculated explicitly:

$$(\mathcal{F}_{\mathbb{R}_+}e^{at})(t) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{a - it}. \quad (3.10)$$

The appropriate elements $\begin{bmatrix} y_+ \\ y_- \end{bmatrix} = Ue^{at}$ and $\begin{bmatrix} z_+ \\ z_- \end{bmatrix} = (U\mathcal{F}_{\mathbb{R}_+})e^{at}$ also can be calculated explicitly. According to the definition (2.3)-(2.4) of the operator $U$,

$$y_+(\mu) = a^{-\frac{1}{2} - i\mu} \Gamma\left(\frac{1}{2} + i\mu\right), \quad \mu \in \mathbb{R}^+, \quad (3.11a)$$

$$y_-(\mu) = a^{-\frac{1}{2} + i\mu} \Gamma\left(\frac{1}{2} - i\mu\right), \quad \mu \in \mathbb{R}^+, \quad (3.11b)$$

and

$$z_+(\mu) = \sqrt{\frac{\pi}{2}} e^{\frac{\pi\mu}{2}} a^{-\frac{1}{2} + i\mu} \frac{e^{-\frac{\pi\mu}{2}}}{\cosh \pi\mu}, \quad \mu \in \mathbb{R}^+, \quad (3.12a)$$

$$z_-(\mu) = \sqrt{\frac{\pi}{2}} e^{\frac{\pi\mu}{2}} a^{-\frac{1}{2} - i\mu} \frac{e^{-\frac{\pi\mu}{2}}}{\cosh \pi\mu}, \quad \mu \in \mathbb{R}^+, \quad (3.12b)$$

The equalities (3.4) follow from (3.11), (3.12), (3.6) and from the identity

$$\Gamma(1/2 + i\mu) \Gamma(1/2 - i\mu) = \frac{\pi}{\cosh \pi\mu}. \quad (3.13)$$

□

If $M = \begin{bmatrix} M_{++} & M_{+-} \\ M_{-+} & M_{--} \end{bmatrix}$ is a $2 \times 2$ matrix with complex entries, then $\|M\|_{C^2 \to C^2}$ is the norm of the matrix $M$ considered as an operator in the space $C^2$, where the space $C^2$ is equipped by the standard Hermitian norm.

---

1 This is a special case of the identity $\Gamma(\zeta) \cdot \Gamma(1 - \zeta) = \frac{\pi}{\sin \pi\zeta}, \quad \zeta \in \mathbb{C} \setminus \mathbb{Z}$. 

---
Definition 3.2. Let \( M(\mu) = \begin{bmatrix} M_{++}(\mu) & M_{+-}(\mu) \\ M_{-+}(\mu) & M_{--}(\mu) \end{bmatrix} \) be a \( 2 \times 2 \) matrix valued function which entries are complex valued functions defined almost everywhere on \( \mu \in \mathbb{R}^+ \). The multiplication operator \( \mathcal{M}_M \) generated by the matrix function \( M \) is defined by the equality

\[
(\mathcal{M}_M y)(\mu) = M(\mu)y(\mu), \quad \forall y = \begin{bmatrix} y_+ \\ y_- \end{bmatrix} \in \mathcal{M}.
\] (3.14)

Lemma 3.3. If the matrix function \( M(\mu) \) is bounded on \( \mathbb{R}^+ \), that is

\[
\text{ess sup}_{\mu \in \mathbb{R}^+} \| M(\mu) \|_{C^2 \rightarrow C^2} < \infty,
\]

then the operator \( \mathcal{M}_M \) is a bounded operator in the space \( \mathcal{M} \) and

\[
\| \mathcal{M}_M \|_{\mathcal{M} \rightarrow \mathcal{M}} = \text{ess sup}_{\mu \in \mathbb{R}^+} \| M(\mu) \|_{C^2 \rightarrow C^2}.
\] (3.15)

The expression in the left hand side of (3.15) means the norm of the multiplication operator \( \mathcal{M}_M \) in the space \( \mathcal{M} \). Concerning the notion \( \text{ess sup} \) see [3, Section 2.11, p.140.]

Remark 3.4. If the matrix function \( M(\mu) \) is continuous on \( \mathbb{R}^+ \), then

\[
\| \mathcal{M}_M \|_{\mathcal{M} \rightarrow \mathcal{M}} = \sup_{\mu \in \mathbb{R}^+} \| M(\mu) \|_{C^2 \rightarrow C^2}.
\] (3.16)

Since \( x \in L^2(\mathbb{R}^+) \) in (3.7) is arbitrary, we can interpret the equality (3.7) as an equality for operators. The following theorem is the core of the present paper.

Theorem 3.5. The truncated Fourier-Plancherel operator \( \mathcal{F}_{\mathbb{R}^+} \) is unitarily equivalent to the multiplication operator \( \mathcal{M}_F \) generated by the matrix function \( F \) of the form (3.8)-(3.6) in the space \( \mathcal{M} \). The equality

\[
\mathcal{F}_{\mathbb{R}^+} = U^{-1} \mathcal{M}_F U
\] (3.17)

holds, where \( U \) is the unitary operator which appeared in Definition 2.7.

Remark 3.6. The multiplication operator \( \mathcal{M}_F \) possesses the same spectral properties that the operator \( \mathcal{F}_{\mathbb{R}^+} \). However to study the operator \( \mathcal{M}_F \) is much easier than to study the operator \( \mathcal{F}_{\mathbb{R}^+} \). We use terminology the model operator for the operator \( \mathcal{M}_F \).

4. The spectrum and the resolvent of the operator \( \mathcal{F}_{\mathbb{R}^+} \).

The unitary equivalence (3.17) allows to reduce the spectral analysis of the operator \( \mathcal{F}_{\mathbb{R}^+} : L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+) \) to the spectral analysis of the operator \( \mathcal{M}_F : \mathcal{M} \rightarrow \mathcal{M} \).

To perform the spectral analysis of the operator \( \mathcal{M}_F \), acting in the infinite dimensional space \( \mathcal{M} \), we have to perform the spectral analysis of the \( 2 \times 2 \) matrix \( F(\mu) \), acting in the two dimensional space \( \mathbb{C}^2 \). The spectral analysis of the matrix \( F(\mu) \) can be done for each \( \mu \in \mathbb{R}^+ \) separately. Then
we can glue the spectrum \(\sigma(M_F)\) of the operator \(M_F\), from the spectra \(\sigma(F(\mu))\) of the matrices \(F(\mu)\), as well as the resolvent of the operator \(M_F\), can be glued from the resolvents of the matrices \(F(\mu)\).

For \(\mu \in [0, \infty)\), let

\[
\zeta(\mu) = e^{i\pi/4} \frac{1}{\sqrt{2} \cosh \pi \mu} \tag{4.1}
\]

and

\[
\zeta_+ (\mu) = \zeta(\mu), \quad \zeta_- (\mu) = -\zeta(\mu). \tag{4.2}
\]

It is clear that \(\zeta(\mu) \neq 0\), so \(\zeta_+(\mu) \neq \zeta_- (\mu)\) for every \(\mu \in [0, \infty)\).

**Lemma 4.1.** For \(\mu \in [0, \infty)\), the spectrum \(\sigma(F(\mu))\) of the matrix \(F(\mu)\), (3.8), is simple, and consists of two different points \(\zeta_+(\mu)\) and \(\zeta_- (\mu)\): (4.2), (4.1):

\[
\sigma(F(\mu)) = \{\zeta_+(\mu), \zeta_- (\mu)\}. \tag{4.3}
\]

**Proof.** Let \(D(z, \mu)\) be the determinant of this matrix:

\[
D(z, \mu) = \det(zI - F(\mu)). \tag{4.4}
\]

According to the structure (3.8) of the matrix \(F(\mu)\),

\[
D(z, \mu) = z^2 - F_{+-}(\mu) \cdot F_{-+}(\mu). \tag{4.5}
\]

The product \(F_{+-}(\mu) \cdot F_{-+}(\mu)\) can be calculated using (3.6) and (3.13):

\[
F_{+-}(\mu) \cdot F_{-+}(\mu) = \frac{i}{2 \cosh \pi \mu}. \tag{4.6}
\]

Thus

\[
D(z, \mu) = z^2 - \frac{i}{2 \cosh \pi \mu}. \tag{4.6}
\]

\(\square\)

**Definition 4.2.** Let \(a\) and \(b\) be points of \(\mathbb{C}\). By definition, the interval \([a, b]\) is the set: \([a, b] = \{(1 - \tau)a + \tau b : \tau \text{ runs over } [0, 1]\}\). The open interval \((a, b)\) as well as half-open intervals are defined analogously.

When \(\mu\) runs over the interval \([0, \infty)\), the points \(\zeta_+(\mu)\) fill the interval

\[
\left(0, \frac{1}{\sqrt{2}} e^{i\pi/4}\right)
\]

and the points \(\zeta_- (\mu)\) fill the interval

\[
\left[-e^{i\pi/4} \frac{1}{\sqrt{2}}, 0\right].
\]

When \(\mu\) increases, the points \(\zeta_+(\mu), \zeta_- (\mu)\) move monotonically: the point \(\zeta_+(\mu)\) moves from \(e^{i\pi/4} \frac{1}{\sqrt{2}}\) to 0, the point \(\zeta_- (\mu)\) moves from \(-e^{i\pi/4} \frac{1}{\sqrt{2}}\) to 0. Thus the mappings \(\mu \to \zeta_+(\mu)\) is a homeomorphism of \([0, \infty)\) onto \(\left(0, e^{i\pi/4} \frac{1}{\sqrt{2}}\right)\) and the mapping \(\mu \to \zeta_- (\mu)\) is a homeomorphism of \([0, \infty)\) onto \(\left[-e^{i\pi/4} \frac{1}{\sqrt{2}}, 0\right]\).

Moreover \(\zeta_+(\infty) = \zeta_- (\infty) = \{0\}\).

So the interval

\[
\left[-e^{i\pi/4} \frac{1}{\sqrt{2}}, e^{i\pi/4} \frac{1}{\sqrt{2}}\right]
\]

is naturally decomposed into the union of three non-intersecting parts

\[
\left[-e^{i\pi/4} \frac{1}{\sqrt{2}}, e^{i\pi/4} \frac{1}{\sqrt{2}}\right] = \left[-e^{i\pi/4} \frac{1}{\sqrt{2}}, 0\right] \cup \{0\} \cup \left(0, e^{i\pi/4} \frac{1}{\sqrt{2}}\right). \tag{4.7}
\]
Theorem 4.3. The spectrum $\sigma(M_F)$ of the model operator $M_F$ is:
\[
\sigma(M_F) = \left[ -e^{i\pi/4} \frac{1}{\sqrt{2}}, e^{i\pi/4} \frac{1}{\sqrt{2}} \right]. \tag{4.8}
\]
In other words, Theorem 4.3 claims that the spectrum $\sigma(M_F)$ of the model operator $M_F$ is represented in the form
\[
\sigma(M_F) = \bigcup_{\mu \in [0, \infty]} \sigma(F(\mu)). \tag{4.9}
\]
Since spectra of unitarily equivalent operators coincide, Theorem 4.3 can be reformulated as

Theorem 4.4. The spectrum $\sigma(F_{\mathbb{R}^+})$ of the truncated Fourier operator $F_{\mathbb{R}^+}$ is:
\[
\sigma(F_{\mathbb{R}^+}) = \left[ -e^{i\pi/4} \frac{1}{\sqrt{2}}, e^{i\pi/4} \frac{1}{\sqrt{2}} \right]. \tag{4.10}
\]

In the present section we prove the description (4.9) of the spectrum $\sigma(M_F)$ of the model operator $M_F$. On the side we obtain some estimates for the resolvents of the matrices $F(\mu)$. These estimates are not quite evident because the matrices $F(\mu)$ are not selfadjoint. In particular, $F(\infty)$ is a Jordan cell.

Lemma 4.5. The norm of an arbitrary $2 \times 2$ matrix $M$,
\[
M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix},
\]
considered as an operator from $\mathbb{C}^2$ to $\mathbb{C}^2$, admits the estimates from above and from below
\[
\frac{1}{2} \text{trace}(M^*M) \leq \|M\|^2 \leq \text{trace}(M^*M). \tag{4.11}
\]
Assuming that $\det M \neq 0$, the norm of the inverse matrix $M^{-1}$ can be estimated as follows:
\[
(det M)^{-2} \text{trace}(M^*M) - \frac{2}{\text{trace}(M^*M)} \leq \|M^{-1}\|^2 \leq |(det M)|^{-2} \text{trace}(M^*M) \tag{4.12}
\]
where
\[
\text{trace} M^*M = |m_{11}|^2 + |m_{12}|^2 + |m_{21}|^2 + |m_{22}|^2. \tag{4.13}
\]

Proof. Let $s_0$ and $s_1$ be singular values of the matrix $M$, that is
\[
0 < s_1 \leq s_0, \tag{4.14}
\]
and the numbers $s_0^2$, $s_1^2$ are eigenvalues of the matrix $M^*M$. Then
\[
\|M\| = s_0, \quad \|M^{-1}\| = s_1^{-1}, \quad \text{trace}(M^*M) = s_0^2 + s_1^2, \quad |det(M)|^2 = \text{det}(M^*M) = s_0^2 \cdot s_1^2.
\]
Therefore the inequality \((4.11)\) takes the form
\[
\frac{1}{2}(s_0^2 + s_1^2) \leq s_0^2 \leq (s_0^2 + s_1^2),
\]
and the inequality \((4.12)\) takes the form
\[
(s_0s_1)^{-2}(s_0^2 + s_1^2) - \frac{2}{s_0^2 + s_1^2} \leq s_1^{-2} \leq (s_0s_1)^{-2}(s_0^2 + s_1^2).
\]
The last inequalities hold for arbitrary numbers \(s_0, s_1\) which satisfy the inequalities \((4.14)\). □

Since the numbers \(\Gamma(1/2 \pm i\mu)\) are complex conjugated, it follows from \((3.13)\) that
\[
|\Gamma(1/2 \pm i\mu)|^2 = \frac{2\pi}{e^{\pi\mu} + e^{-\pi\mu}}, \quad \mu \in \mathbb{R}^+.
\] (4.15)
From \((3.6)\) and \((4.15)\) we calculate the absolute values
\[
|F_{-+}(\mu)| = \frac{1}{\sqrt{1 + e^{-2\pi\mu}}}, \quad \mu \in \mathbb{R}^+.
\] (4.16a)
\[
|F_{+-}(\mu)| = \frac{1}{\sqrt{1 + e^{2\pi\mu}}}, \quad \mu \in \mathbb{R}^+.
\] (4.16b)
We remark that in particular
\[
1/\sqrt{2} \leq |F_{--}(\mu)| < 1, \quad |F_{+-}(\mu)| \leq 1/\sqrt{2}, \quad \mu \in \mathbb{R}^+.
\] (4.17)
If \(\mu\) runs over the interval \([0, \infty)\), then \(|F_{+-}(\mu)|\) increases from \(2^{-1/2}\) to 1 and \(|f_{--}(\mu)|\) decreases from \(2^{-1/2}\) to 0. In particular,
\[
\sup_{\mu \in \mathbb{R}^+} |F_{+-}(\mu)| = \text{ess sup}_{\mu \in \mathbb{R}^+} |F_{-+}(\mu)| = 1.
\] (4.18)
From \((4.16)\) it follows that
\[
|F_{-+}(\mu)|^2 + |F_{+-}(\mu)|^2 = 1.
\] (4.19)
For the matrix \(F(t)\) defined by \((3.8)-(3.6)\) its norm is
\[
\|F(t)\|_{\mathbb{C}^2 \to \mathbb{C}^2} = \frac{1}{\sqrt{1 + e^{-2\pi t}}}, \quad \forall t \in \mathbb{R}^+.
\] (4.20)

**Lemma 4.6.** For every \(\mu \in [0, \infty)\) and every \(z \in \mathbb{C} \setminus \sigma(F(\mu))\) the matrix \((zI - F(\mu))^{-1}\) admits the estimates
\[
|D(z, \mu)|^{-2}(2|z|^2 + 1) - \frac{2}{2|z|^2 + 1} \leq (zI - F(\mu))^{-1} \leq |D(z, \mu)|^{-2}(2|z|^2 + 1),
\] (4.21)
where
\[
D(z, \mu) = \text{det}(zI - F(\mu))
\] (4.22)
and \(\sigma(F(\mu))\) is the spectrum of the matrix \(F(\mu)\).
Proof. We apply the estimate (4.12) to the matrix $M = zI - F(\mu)$. The value \( \text{trace } M^*M \) we calculate according to (4.13):

\[
\text{trace } (zI - F(\mu))^*(zI - F(\mu)) = 2|z|^2 + |F_{++}(\mu)|^2 + |F_{--}(\mu)|^2.
\] (4.23)

Using the equality (4.19), we obtain that

\[
\text{trace } (zI - F(\mu))^*(zI - F(\mu)) = 2|z|^2 + 2|F_+ - (\mu)|^2.
\] (4.24)

\[\square\]

Proof of Theorem 4.3. When \( \mu \) runs over the interval \([0, \infty)\), the complex number \( \frac{i}{2 \cosh \pi \mu} \), which appears in the right hand side of the equality (4.6), fill the interval \((0, \frac{i}{2})\). Therefore

\[
\inf_{\mu \in (0, \infty)} |D(z, \mu)| = \text{dist}(z^2, [0, \frac{i}{2}]),
\] (4.25)

where

\[
\text{dist}(z^2, [0, \frac{i}{2}]) = \min_{\zeta \in [0, \frac{i}{2}]} |z^2 - \zeta|.
\] (4.26)

In particular,

\[
\left( \inf_{\mu \in [0, \infty)} |D(z, \mu)| > 0 \right) \Leftrightarrow \left( z \not\in [0, \frac{i}{2}] \right),
\] (4.27)

or, what is the same,

\[
\left( \inf_{\mu \in [0, \infty)} |D(z, \mu)| > 0 \right) \Leftrightarrow \left( z \not\in \left[ -\frac{1}{\sqrt{2}} e^{i\pi/4}, \frac{1}{\sqrt{2}} e^{i\pi/4} \right] \right).
\] (4.28)

From the inequalities (4.21) it follows that

\[
\frac{2|z|^2 + 1}{\left( \inf_{\mu \in [0, \infty)} |D(z, \mu)| \right)^2} - \frac{2}{2|z|^2 + 1} \leq \sup_{\mu \in [0, \infty)} \| (zI - F(\mu))^{-1} \|_{\mathbb{C}^2 \rightarrow \mathbb{C}^2}^2 \leq \frac{2|z|^2 + 1}{\left( \inf_{\mu \in [0, \infty)} |D(z, \mu)| \right)^2}.
\] (4.28)

From (4.27) and (4.28) it follows that

\[
\left( \sup_{\mu \in [0, \infty)} \| (zI - F(\mu))^{-1} \|_{\mathbb{C}^2 \rightarrow \mathbb{C}^2} < \infty \right) \Leftrightarrow \left( z \not\in \left[ -\frac{1}{\sqrt{2}} e^{i\pi/4}, \frac{1}{\sqrt{2}} e^{i\pi/4} \right] \right).
\] (4.29)

\[\square\]
Fourier-Plancherel operator $\mathcal{F}_{\mathbb{R}^+}$:

$$\frac{2|z|^2 + 1}{(\text{dist}(z^2, [0, i/2]))^2} - \frac{2}{2|z|^2 + 1} \leq \|(z^2 - \mathcal{F}_{\mathbb{R}^+})^{-1}\|_{L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+)} \leq \frac{2|z|^2 + 1}{(\text{dist}(z^2, [0, i/2]))^2}. \quad (4.30)$$

The left of the above equalities can be presented in the form

$$\frac{(2|z|^2 + 1)^{1/2}}{\text{dist}(z^2, [0, i/2])} \sqrt{1 - \frac{(\text{dist}(z^2, [0, i/2]))^2}{(2|z|^2 + 1)^2}} \leq \|(z^2 - \mathcal{F}_{\mathbb{R}^+})^{-1}\|_{L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+)}.$$  

In the previous inequality, the value under the square root is positive because

$$\frac{2\text{dist}^2(z^2, [0, i/2])}{(2|z|^2 + 1)^2} \leq \frac{2|z|^2}{(2|z|^2 + 1)^2} \leq \frac{1}{2}.$$  

Since $(1 - \alpha) \leq \sqrt{1 - \alpha}$ for $0 \leq \alpha \leq 1$, then

$$1 - \frac{2\text{dist}^2(z^2, [0, i/2])}{(2|z|^2 + 1)^2} \leq \sqrt{1 - \frac{2\text{dist}^2(z^2, [0, i/2])}{(2|z|^2 + 1)^2}}.$$  

Thus, the lower estimate for the norm of the resolvent is

$$\frac{(2|z|^2 + 1)^{1/2}}{\text{dist}(z^2, [0, i/2])} - \frac{2\text{dist}(z^2, [0, i/2])}{(2|z|^2 + 1)^{3/2}} \leq \|(z^2 - \mathcal{F}_{\mathbb{R}^+})^{-1}\|_{L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+)} \cdot \quad (4.31a)$$

The upper estimate for the norm of the resolvent is given by the right of the inequalities (4.30):

$$\|(z^2 - \mathcal{F}_{\mathbb{R}^+})^{-1}\|_{L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+)} \leq \frac{(2|z|^2 + 1)^{1/2}}{\text{dist}(z^2, [0, i/2])}. \quad (4.31b)$$

The smaller is the value $\text{dist}(z^2, [0, i/2])$, the closer are the lower estimate (4.31a) and the upper estimate (4.31b).

However, we would like to estimate of the resolvent not in term of the value $\text{dist}(z^2, [0, i/2])$, but in term of the value $\text{dist}(z, \sigma(\mathcal{F}_{\mathbb{R}^+}))$.

**Lemma 4.7.** Let $\zeta$ be a point of the spectrum $\sigma(\mathcal{F}_{\mathbb{R}^+})$:

$$\zeta \in \left[ -\frac{1}{\sqrt{2}} e^{i\pi/4}, \frac{1}{\sqrt{2}} e^{i\pi/4} \right],$$

and the point $z$ lies on the normal to the interval $\left[ -\frac{1}{\sqrt{2}} e^{i\pi/4}, \frac{1}{\sqrt{2}} e^{i\pi/4} \right]$ at the point $\zeta$:

$$z = \zeta \pm |z - \zeta| e^{i3\pi/4}.$$  

Then

$$\text{dist}(z^2, [0, i/2]) = \begin{cases} 2|\zeta| |z - \zeta|, & \text{if } |z - \zeta| \leq |\zeta|, \\ |\zeta|^2 + |z - \zeta|^2 = |z|^2, & \text{if } |z - \zeta| \geq |\zeta|. \end{cases} \quad (4.34)$$
Let \( z = \pm |\zeta|e^{i\pi/4} \). Substituting this expression for \( \zeta \) into (4.33), we obtain
\[
z^2 = \pm 2|\zeta| |z - \zeta| + i(|\zeta|^2 - |z - \zeta|^2).
\]
If \( |z - \zeta| \leq |\zeta| \), then the point \( i(|\zeta|^2 - |z - \zeta|^2) \) lies on the interval \([0, i/2]\).

In this case, \( \text{dist}(z^2, [0, i/2]) = 2|\zeta||z - \zeta| \). If \( |z - \zeta| \geq |\zeta| \), then the point \( i(|\zeta|^2 - |z - \zeta|^2) \) lies on the half-axis \([0, -i\infty)\). In this case,
\[
\text{dist}(z^2, [0, i/2]) = \sqrt{(|\zeta|^2 - |z - \zeta|^2)^2 + 4|\zeta|^2|z - \zeta|^2} = |\zeta|^2 + |z - \zeta|^2 = |z|^2.
\]
Since \( |\zeta|^2 + |z - \zeta|^2 \geq 2|\zeta||z - \zeta| \), in any case, either \( |z - \zeta| \leq |\zeta| \), or \( |z - \zeta| \geq |\zeta| \), the inequality
\[
\text{dist}(z^2, [0, i/2]) \geq 2|\zeta||z - \zeta|.
\]
holds. \(\square\)

**Theorem 4.8.** Let \( \zeta \) be a point of the spectrum \( \sigma(\mathcal{F}_{\mathbb{R}^+}) \) of the operator \( \mathcal{F}_{\mathbb{R}^+} \), and let the point \( z \) lie on the normal to the interval \( \sigma(\mathcal{F}_{\mathbb{R}^+}) \) at the point \( \zeta \). Then

1. The resolvent \((z\mathcal{I} - \mathcal{F}_{\mathbb{R}^+})^{-1}\) admits the estimate from above:
\[
\|(z\mathcal{I} - \mathcal{F}_{\mathbb{R}^+})^{-1}\|_{L^2(\mathbb{R}^+)\to L^2(\mathbb{R}^+)} \leq A(z) \frac{1}{|\zeta|} \cdot \frac{1}{|z - \zeta|},
\]
where
\[
A(z) = \frac{(2|z|^2 + 1)^{1/2}}{2}.
\]
2. If moreover the condition \( |z - \zeta| \leq |\zeta| \) is satisfied, then the resolvent \((z\mathcal{I} - \mathcal{F}_{\mathbb{R}^+})^{-1}\) also admits the estimate from below:
\[
A(z) \frac{1}{|\zeta|} \cdot \frac{1}{|z - \zeta|} - B(z)|\zeta||z - \zeta| \leq \|(z\mathcal{I} - \mathcal{F}_{\mathbb{R}^+})^{-1}\|_{L^2(\mathbb{R}^+)\to L^2(\mathbb{R}^+)}
\]
where \( A(z) \) is the same that in (4.37) and
\[
B(z) = \frac{4}{(2|z|^2 + 1)^{3/2}}.
\]
3. For \( \zeta = 0 \), then the resolvent \((z\mathcal{I} - \mathcal{F}_{\mathbb{R}^+})^{-1}\) admits the estimates
\[
2A(z) \frac{1}{|z|^2} - B(z) \leq \|(z\mathcal{I} - \mathcal{F}_{\mathbb{R}^+})^{-1}\|_{L^2(\mathbb{R}^+)\to L^2(\mathbb{R}^+)} \leq 2A(z) \frac{1}{|z|^2},
\]
where \( A(z) \) and \( B(z) \) are the same that in (4.37), (4.39), and \( z \) is an arbitrary point of the normal.

In particular, if \( \zeta \neq 0 \), and \( z \) tends to \( \zeta \) along the normal to the interval \( \sigma(\mathcal{F}_{\mathbb{R}^+}) \), then
\[
\|(z\mathcal{I} - \mathcal{F}_{\mathbb{R}^+})^{-1}\|_{L^2(\mathbb{R}^+)\to L^2(\mathbb{R}^+)} = \frac{A(\zeta)}{|\zeta|} \frac{1}{|z - \zeta|} + O(1).
\]
If $\zeta = 0$ and $z$ tends to $\zeta$ along the normal to the interval $\sigma(\mathcal{F}_R^+)$, then
\[
\left\| (zJ - \mathcal{F}_R^+)^{-1} \right\|_{L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)} = |z|^{-2} + O(1),
\]
where $O(1)$ is a value which remains bounded as $z$ tends to $\zeta$.

**Proof.** The proof is based on the estimates (4.31) for the resolvent and on Lemma 4.7. Combining the inequality (4.35) with the estimate (4.31b), we obtain the estimate (4.36), which holds for all $z$ lying on the normal to the interval $\sigma(\mathcal{F}_R^+)$ at the point $\zeta$. If moreover $z$ is close enough to $z$, namely the condition $|z-\zeta| \leq |\zeta|$ is satisfied, then the equality holds in (4.35). Combining the equality (4.35) with the estimate (4.31a), we obtain the estimate (4.38).

The asymptotic relation (4.41) is a consequence of the inequalities (4.38) and (4.40) since $|\Delta(z) - \Delta(\zeta)| = O(1)$ as $z$ tends to $\zeta$.

The asymptotic relation (4.42) is a consequence of the inequalities (4.31) and the equality $\text{dist}(z^2, [0, i/2]) = |z|^2$ which holds for all $z$ lying on the normal to the interval $\sigma(\mathcal{M}_F)$ at the point $\zeta = 0$. (See (4.34) for $\zeta = 0$.) □

**Corollary 4.9.** The operator $\mathcal{F}_R^+$ is not similar to any normal operator.

If the operator $\mathcal{F}_R^+$ were similar to some normal operator $\mathcal{N}$, then the resolvent of the operator $\mathcal{F}_R^+$ would admit the estimate
\[
\left\| (zJ - \mathcal{F}_R^+)^{-1} \right\|_{L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)} \leq C \left( \text{dist}(z, \sigma(\mathcal{F}_R^+)) \right)^{-1},
\]
where $C < \infty$ is a value which does not depend on $z$. However, this estimate is not compatible with the asymptotic relation (4.42).

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