THE COMPLEX ORTHOGONAL GELFAND-ZEITLIN SYSTEM

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Abstract. In this paper, we use the theory of algebraic groups to prove a number of new and fundamental results about the orthogonal Gelfand-Zeitlin system. We show that the moment map (orthogonal Kostant-Wallach map) is surjective and simplify criteria of Kostant and Wallach for an element to be strongly regular. We further prove the integrability of the orthogonal Gelfand-Zeitlin system on regular adjoint orbits and describe the generic flows of the integrable system. We also study the nilfibre of the moment map and show that in contrast to the general linear case it contains no strongly regular elements. This extends results of Kostant, Wallach, and Colarusso from the general linear case to the orthogonal case.

1. INTRODUCTION

Kostant and Wallach studied the Gelfand-Zeitlin system of functions for the Lie algebra of complex $n$ by $n$ matrices in two fundamental papers [KW06a, KW06b], and found a number of new features that did not appear in the Gelfand-Zeitlin system of functions on Hermitian matrices [GS83]. The Gelfand-Zeitlin functions generate a Poisson commutative family of functions and in [KW06a], Kostant and Wallach show they generate an integral system on each regular conjugacy class of matrices. To show this, they introduced and studied the set of strongly regular matrices, which is the set where the Gelfand-Zeitlin functions have linearly independent differentials. There is also a Gelfand-Zeitlin system on the complex orthogonal Lie algebra $\mathfrak{so}(n, \mathbb{C})$. Many fundamental and incisive linear algebra constructions used to study matrices in [KW06a, Col11] do not carry over to the orthogonal case, and as a consequence, much less is known about the complex orthogonal Gelfand-Zeitlin system. The purpose of this paper and its sequel is to remedy this situation. We establish the complete integrability of the orthogonal Gelfand-Zeitlin system on regular adjoint orbits in $\mathfrak{so}(n, \mathbb{C})$ and extend a number of basic results on the strongly regular set to $\mathfrak{so}(n, \mathbb{C})$, which were established in the general linear case in [KW06a, Col11, CE12, CE15]. In particular, we describe the generic leaves of the foliation given by the integrable system as well as aspects of the geometry of the nilfibre of the moment map of the system. The key idea is to use results from the theory of algebraic group actions due to Knop, Panyushev, and Luna to extend the general linear results to
Theorem 1.2. We then use this result, together with Theorem 1.1, to prove one of our main results. In Proposition 4.14, we prove that elements of \( g \) also define \( x \) to a regular adjoint orbit in \( 4.19 \). In the final section, we study the nilfibre \( \Phi^{-1}(0) \). We begin by studying the partial nilfibre \( \Phi^{-1}_{n}(0) \). We show that its irreducible components can be described in terms of closed \( G_{n-1} = SO(n-1, \mathbb{C}) \)-orbits on the flag variety \( B \) of \( g \) (Theorem 5.11). We accomplish this by using the Luna slice theorem to describe the generic fibres of \( \Phi_{n} \).

Theorem 1.1. (see Theorem 3.6) The morphism \( \Phi \) is surjective, and every fibre contains a regular element of \( g \).

An element \( x \in g \) is called strongly regular if the set \( \{df(x) : f \in J_{GZ} \} \) is linearly independent. In the case of \( gl(n, \mathbb{C}) \), Kostant and Wallach consider the sequence of subalgebras \( g_{1} \subset g_{2} \subset \cdots \subset g_{n} = gl(n, \mathbb{C}) \) by letting \( g_{i} \cong gl(i, \mathbb{C}) \) be the upper left \( i \) by \( i \) corner, and given \( x \in gl(n, \mathbb{C}) \), define \( x_{i} \in g_{i} \) by orthogonal projection. They define Gelfand-Zeitlin functions and the strongly regular set as above, and prove that \( x \in gl(n, \mathbb{C}) \) is strongly regular if and only if (i) \( x_{i} \) is regular in \( g_{i} \) for each \( i \) and (ii) the centralizers \( Z_{g_{i}}(x_{i}) \cap Z_{g_{i+1}}(x_{i+1}) = 0 \) for \( i = 1, \ldots, n-1 \). We use results on spherical varieties due to Panyushev [Pan90] to prove that we may omit condition (i), and extend this result to the case of \( so(n, \mathbb{C}) \) (Proposition 4.8). Here, given \( x \in g = so(n, \mathbb{C}) \), we also define \( x_{i} \in g_{i} \cong so(i, \mathbb{C}) \) by orthogonal projection. Using this simplified criterion for strong regularity, we can construct a large collection of strongly regular elements. In particular, we consider the set \( g_{\Theta} \) in \( so(n, \mathbb{C}) \) given by the property that the (suitably defined) spectra of \( x_{i} \) and \( x_{i+1} \) do not intersect for \( i = 2, \ldots, n-1 \) (see Notation 2.10 and Equation 4.12). In Proposition 4.14 we prove that elements of \( g_{\Theta} \) are strongly regular. We then use this result, together with Theorem 1.1, to prove one of our main results.

Theorem 1.2. (see Theorem 4.17) The restriction of the Gelfand-Zeitlin functions \( J_{GZ} \) to a regular adjoint orbit in \( g \) forms a completely integrable system on the orbit.

We further show that for \( x \in g_{\Theta} \), the fibre \( \Phi^{-1}(\Phi(x)) \) has a free action by an abelian linear algebraic group, and thereby extend a result of Kostant and Wallach and the first author [KW06a, Col11] from the general linear case to the orthogonal case (Theorem 4.19). In the final section, we study the nilfibre \( \Phi^{-1}(0) \). We begin by studying the partial nilfibre \( \Phi^{-1}_{n}(0) \). We show that its irreducible components can be described in terms of closed \( G_{n-1} = SO(n-1, \mathbb{C}) \)-orbits on the flag variety \( B \) of \( g \) (Theorem 5.11). We accomplish this by using the Luna slice theorem to describe the generic fibres of \( \Phi_{n} \).
and then degenerate a generic fibre to $\Phi_{n}^{-1}(0)$ using Knop’s flatness result. Using another interlacing argument and well-known facts about the closed $G_{n-1}$-orbits on $B$, we prove:

**Proposition 1.3. (see Proposition [5.14])** The nilfibre of $\Phi$ contains no strongly regular elements.

This stands in contrast to the case of $\mathfrak{gl}(n, \mathbb{C})$ studied extensively in [CE12]. As a consequence of Proposition 1.3 we prove that there is no analogue of the Hessenberg matrices, which play a fundamental role in [KW06a] (Corollary 5.19). In the sequel, we plan to further develop these methods to completely understand the strongly regular set, and describe in full detail each partial KW fiber $\Phi_{n}^{-1}(\Phi_{n}(x))$.

This paper is organized as follows. In Section 2, we introduce notation and results for later use. In Section 3 we identify the partial KW map $\Phi_{n}$ as an invariant theory quotient and show that $\Phi$ is surjective. In Section 4 we study the strongly regular set, and prove that $\mathfrak{g}_{\Theta}$ consists of strongly regular elements. We further prove complete integrability of regular orbits, and study the KW fibers for elements in $\mathfrak{g}_{\Theta}$. In Section 5 we study the nilfibres of $\Phi_{n}$ and $\Phi$ and prove that the nilfibre of $\Phi$ contains no strongly regular elements. In the body of the paper, all Lie algebras are complex, as are all algebraic groups. In particular, we will write $\mathfrak{so}(n)$ and $SO(n)$ to denote $\mathfrak{so}(n, \mathbb{C})$ and $SO(n, \mathbb{C})$, and similarly with $\mathfrak{gl}(n)$.

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2. Preliminaries

In this section, we recall basic facts about the orthogonal Lie algebras (Section 2.1). After introducing some notation, we review basic properties of the orthogonal Gelfand-Zeitlin systems (hereafter referred to as GZ systems) and its moment map (orthogonal Kostant-Wallach map) and summarize known results (Section 2.5). We begin by describing the realization of $\mathfrak{so}(n)$ that we will use throughout the paper.

2.1. Realization of Orthogonal Lie algebras. We give explicit descriptions of standard Cartan subalgebras and corresponding root systems of $\mathfrak{so}(n)$. Our exposition follows Chapters 1 and 2 of [GW98].

Let $\beta$ be the non-degenerate, symmetric bilinear form on $\mathbb{C}^{n}$ given by

$$\beta(x, y) = x^{T} S_{n} y,$$

(2.1)
where \( x, y \) are \( n \times 1 \) column vectors and \( S_n \) is the \( n \times n \) matrix:

\[
S_n = \begin{bmatrix}
0 & \ldots & \ldots & 0 & 1 \\
\vdots & & & 1 & 0 \\
\vdots & & & \ddots & \\
0 & 1 & \ldots & 0 & \\
1 & 0 & \ldots & \ldots & 0
\end{bmatrix}
\]

with ones down the skew diagonal and zeroes elsewhere. The special orthogonal group, \( SO(n) \), consists of the \( g \in SL(n) \) such that \( \beta(gx, gy) = \beta(x, y) \) for all \( x, y \in \mathbb{C}^n \). Its Lie algebra, \( \mathfrak{so}(n) \), consists of the \( Z \in \mathfrak{sl}(n) \) such that \( \beta(Zx, y) = -\beta(x, Zy) \) for all \( x, y \in \mathbb{C}^n \).

We consider the cases where \( n \) is odd and even separately. Throughout, we denote the standard basis of \( \mathbb{C}^n \) by \( \{e_1, \ldots, e_n\} \).

### 2.1.1. Realization of \( \mathfrak{so}(2l) \)

Let \( \mathfrak{g} = \mathfrak{so}(2l) \) be of type \( D \). The subalgebra of diagonal matrices \( \mathfrak{h} := \{\text{diag}[a_1, \ldots, a_l, -a_l, \ldots, -a_1], a_i \in \mathbb{C}\} \) is a Cartan subalgebra of \( \mathfrak{g} \). We refer to \( \mathfrak{h} \) as the standard Cartan subalgebra. Let \( \epsilon_i \in \mathfrak{h}^\ast \) be the linear functional \( \epsilon_i(\text{diag}[a_1, \ldots, a_l, -a_l, \ldots, -a_1]) = a_i \), and let \( \Phi(\mathfrak{g}, \mathfrak{h}) \) be the roots of \( \mathfrak{g} \) with respect to \( \mathfrak{h} \). Then \( \Phi(\mathfrak{g}, \mathfrak{h}) = \{\epsilon_i - \epsilon_j, \pm(\epsilon_i + \epsilon_j) : 1 \leq i \neq j \leq l\} \), and we take as our standard positive roots the set \( \Phi^+(\mathfrak{g}, \mathfrak{h}) := \{\epsilon_i - \epsilon_j, \epsilon_i + \epsilon_j : 1 \leq i < j \leq l\} \) with corresponding simple roots \( \Pi := \{\alpha_1, \ldots, \alpha_{l-1}, \alpha_l\} \) where \( \alpha_i = \epsilon_i - \epsilon_{i+1} \) for \( i = 1, \ldots, l-1 \), and \( \alpha_l = \epsilon_{l-1} + \epsilon_l \). The Borel subalgebra \( \mathfrak{b}_+ := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{h})} \mathfrak{g}_\alpha \) is easily seen to be the set of upper triangular matrices in \( \mathfrak{g} \).

For the purposes of computations with \( \mathfrak{so}(2l) \), it is convenient to relabel part of the standard basis of \( \mathbb{C}^{2l} \) as \( e_{-j} := e_{2l+1-j} \) for \( j = 1, \ldots, l \).

### 2.1.2. Realization of \( \mathfrak{so}(2l+1) \)

Let \( \mathfrak{g} = \mathfrak{so}(2l + 1) \) be of type \( B \). The subalgebra of diagonal matrices \( \mathfrak{h} := \{\text{diag}[a_1, \ldots, a_l, 0, -a_l, \ldots, -a_1], a_i \in \mathbb{C}\} \) is a Cartan subalgebra of \( \mathfrak{g} \). We again refer to \( \mathfrak{h} \) as the standard Cartan subalgebra. Let \( \epsilon_i \in \mathfrak{h}^\ast \) be the linear functional \( \epsilon_i(\text{diag}[a_1, \ldots, a_l, 0, -a_l, \ldots, -a_1]) = a_i \). In this case, the roots are \( \Phi(\mathfrak{g}, \mathfrak{h}) = \{\epsilon_i - \epsilon_j, \pm(\epsilon_i + \epsilon_j) : 1 \leq i \neq j \leq l\} \cup \{\pm\epsilon_k : 1 \leq k \leq l\} \). We take as our standard positive roots the set \( \Phi^+(\mathfrak{g}, \mathfrak{h}) := \{\epsilon_i - \epsilon_j, \epsilon_i + \epsilon_j : 1 \leq i < j \leq l\} \cup \{\epsilon_k : 1 \leq k \leq l\} \) with corresponding simple roots \( \Pi := \{\alpha_1, \ldots, \alpha_{l-1}, \alpha_l\} \) where \( \alpha_i = \epsilon_i - \epsilon_{i+1}, i = 1, \ldots, l-1, \alpha_l = \epsilon_l \). The Borel subalgebra \( \mathfrak{b}_+ := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{h})} \mathfrak{g}_\alpha \) is easily seen to be the set of upper triangular matrices in \( \mathfrak{g} \).

We relabel part of the standard basis of \( \mathbb{C}^{2l+1} \) by letting \( e_{-j} := e_{2l+2-j} \) for \( j = 1, \ldots, l \) and \( e_0 := e_{l+1} \).
2.2. Split Rank 1 symmetric subalgebras. For later use, recall the realization of \( \mathfrak{so}(n - 1) \) as a symmetric subalgebra of \( \mathfrak{so}(n) \). For \( \mathfrak{g} = \mathfrak{so}(2l + 1) \), let \( t \) be an element of the Cartan subgroup with Lie algebra \( \mathfrak{h} \) with the property that \( \text{Ad}(t)|_{\mathfrak{g}_{\alpha_i}} = \text{id} \) for \( i = 1, \ldots, l - 1 \) and \( \text{Ad}(t)|_{\mathfrak{g}_{\alpha_l}} = -\text{id} \). Consider the involution \( \theta_{2l+1} := \text{Ad}(t) \). Then \( \mathfrak{k} = \mathfrak{so}(2l) = \mathfrak{g}^{\theta_{2l+1}} \) (see [Kna02], p. 700). Note that \( \mathfrak{h} \subset \mathfrak{k} \). In the case \( \mathfrak{g} = \mathfrak{so}(2l) \), \( \mathfrak{k} = \mathfrak{so}(2l - 1) = \mathfrak{g}^{\theta_{2l}} \), where \( \theta_{2l} \) is the involution induced by the diagram automorphism interchanging the simple roots \( \alpha_{l-1} \) and \( \alpha_l \) relative to a fixed choice of simple root vectors (see [Kna02], p. 703). Note that in this case, \( \theta_{2l}(\epsilon_i) = -\epsilon_i \) and \( \theta_{2l}(\epsilon_i) = \epsilon_i \) for \( i = 1, \ldots, l - 1 \). We will omit the subscripts \( 2l + 1 \) and \( 2l \) from \( \theta \) when \( \mathfrak{g} \) is understood.

We also denote the corresponding involution of \( G = SO(n) \) by \( \theta \). The fixed subgroup \( G^\theta = S(O(n - 1) \times O(1)) \) is disconnected. We let \( K := (G^\theta)^0 \) be the identity component of \( G^\theta \). Then \( K = SO(n - 1) \), and \( \text{Lie}(K) = \mathfrak{k} = \mathfrak{g}^\theta \).

In both cases, \( \theta \) preserves the standard Cartan subalgebra \( \mathfrak{h} \), and hence acts on the roots \( \Phi(\mathfrak{g}, \mathfrak{h}) \). A root \( \alpha \) is called real if \( \theta(\alpha) = -\alpha \), imaginary if \( \theta(\alpha) = \alpha \), and complex if \( \theta(\alpha) \neq \pm\alpha \). If \( \alpha \) is imaginary, then \( \alpha \) is called compact if \( \theta|_{\mathfrak{g}_{\alpha}} = \text{id} \) and noncompact if \( \theta|_{\mathfrak{g}_{\alpha}} = -\text{id} \). If \( \alpha \) is complex, then \( \alpha \) is called complex \( \theta \)-stable if \( \theta(\alpha) \) is positive, and otherwise is called complex \( \theta \)-unstable.

Example 2.1. Let \( \mathfrak{g} = \mathfrak{so}(2l + 1) \) and \( \mathfrak{k} = \mathfrak{so}(2l) \), and let \( \theta = \text{Ad}(t) \) be as above. The roots \( \{\pm(\epsilon_i - \epsilon_j), \pm(\epsilon_i + \epsilon_j), 1 \leq i < j \leq l\} \) are compact imaginary, and the roots \( \{\pm\epsilon_i, i = 1, \ldots, l\} \) are non-compact imaginary. As noted above \( \mathfrak{h} \subset \mathfrak{k} \), so that \( \mathfrak{h} \) is a Cartan subalgebra of \( \mathfrak{k} \), and it is easy to see that the set of roots \( \{\alpha_1, \alpha_2, \ldots, \alpha_{l-1}, \epsilon_{l-1} + \epsilon_l\} \) may be identified with the standard set of simple roots \( \Pi \) of \( \mathfrak{so}(2l) \) given in Section 2.1.1.

Now let \( \mathfrak{g} = \mathfrak{so}(2l) \) and \( \mathfrak{k} = \mathfrak{so}(2l - 1) \) and \( \theta = \theta_{2l} \) be as above. Then the simple roots \( \alpha_{l-1} = \epsilon_{l-1} - \epsilon_l \) and \( \alpha_l = \epsilon_{l-1} + \epsilon_l \) are complex \( \theta \)-stable with \( \theta(\alpha_{l-1}) = \alpha_l \). Note that we can choose \( \theta \) to be the involution which acts on the basis of \( C^{2l} \) as \( \theta(\epsilon_l) = -\epsilon_l \) and \( \theta(\epsilon_j) = \epsilon_j \) for \( j \neq l \). Therefore, the roots \( \{\pm(\epsilon_i + \epsilon_j), \pm(\epsilon_i - \epsilon_j), 1 \leq i < j \leq l - 1\} \) are compact imaginary, whereas the roots \( \{\pm(\epsilon_i + \epsilon_l), \pm(\epsilon_i - \epsilon_l), 1 \leq i < l \leq l - 1\} \) are complex \( \theta \)-stable with \( \theta(\epsilon_i \pm \epsilon_l) = \epsilon_i \pm \epsilon_l \). The \( \theta \)-stable subspace \( \mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{\theta(\alpha)} \) decomposes as \( \mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{\theta(\alpha)} = ((\mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{\theta(\alpha)}) \cap \mathfrak{k}) \oplus ((\mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{\theta(\alpha)}) \cap \mathfrak{g}^{-\theta}) \).

In this case, \( \mathfrak{h} \cap \mathfrak{k} \) can be identified with the standard diagonal Cartan subalgebra of \( \mathfrak{k} \). Under this identification, the set \( \{\alpha_1, \ldots, \alpha_{l-2}, \frac{1}{2}(\alpha_{l-1} + \theta(\alpha_{l-1}))\} \) is identified with the standard simple roots \( \Pi \) of \( \mathfrak{so}(2l - 1) \) given in Section 2.1.2.

2.3. Notation. We lay out some of the notation that we will use throughout the paper. Recall that all Lie algebras and algebraic groups are assumed to be complex.

Notation 2.2. (1) We let \( \mathfrak{g} = \mathfrak{so}(n) \) unless otherwise specified. We let \( \mathfrak{k} = \mathfrak{so}(n - 1) \) be the symmetric subalgebra given in Section 2.2. We denote the corresponding connected algebraic subgroup by \( K \). It will also be convenient at times to denote the Lie algebras \( \mathfrak{so}(i) \) by \( \mathfrak{g}_i \), and the algebraic group \( SO(i) \) by \( G_i \) for \( i = 2, \ldots, n \). In particular, \( \mathfrak{g}_{n-1} = \mathfrak{k} \).
(2) We let \( r_i \) be the rank of \( g_i \).

(3) Let \( \langle \langle x, y \rangle \rangle = \text{Tr}(xy) \), for \( x, y \in g \) be the (nondegenerate) trace form on \( g \).

(4) In the Cartan decomposition, \( g = \mathfrak{k} + \mathfrak{p} \) is direct, where \( \mathfrak{p} = g^\theta \). For \( x \in g \), we let \( x_{\mathfrak{k}} \) and \( x_{\mathfrak{p}} \) be the \( \mathfrak{k} \) and \( \mathfrak{p} \) components. We recall that \( \mathfrak{p} = \mathfrak{k} \perp \), where for \( U \subset g \), \( U \perp \) is the perpendicular subspace relative to the trace form.

(5) Let \( G \) be an algebraic group with Lie algebra \( g \). Suppose \( c \subset g \) is an algebraic subalgebra, and \( S \subset g \) is a subset. We denote by \( z_c(S) \) the centralizer of \( S \) in \( c \), i.e., \( z_c(S) := \{ Y \in c : [Y, s] = 0 \text{ for all } s \in S \} \). If \( C \subset G \) is an algebraic subgroup, then we denote the centralizer of \( S \) in \( C \) by \( Z_C(S) = \{ g \in C : gsg^{-1} = s \text{ for all } s \in S \} \). Of course \( \text{Lie}(Z_C(S)) = z_c(S) \) if \( C \) is connected with Lie algebra \( c \).

(6) For any complex variety \( X \), we denote by \( \mathbb{C}[X] \) its ring of regular functions. If \( X \) is a \( G \)-variety, we let \( \mathbb{C}[X]^G \) be the \( G \)-invariant regular functions. We call an element \( x \in X \) regular (or \( G \)-regular if \( G \) is ambiguous) if \( x \) lies in the open dense set consisting of \( G \)-orbits of maximal dimension, and let \( X_{reg} \) be the regular elements. If \( G \) is a reductive algebraic group acting on its Lie algebra \( g \) via the adjoint action, we let \( \mathbb{C}[g]^G \) be the ring of adjoint invariant polynomial functions on \( g \), and recall that \( x \in g \) is regular if and only if \( z_g(x) \) has dimension equal to the rank of \( g \).

(7) We let \( B = B_{\mathfrak{r}} \) denote the flag variety of a Lie algebra \( \mathfrak{r} \), which we identify with the Borel subalgebras of \( \mathfrak{r} \).

**Definition 2.3.** We shall call a Borel subalgebra \( b \in B \) standard if \( h \subset b \), where \( h \subset g \) is the standard Cartan subalgebra of diagonal matrices of \( g \).

2.4. **Numerical identities.** We record several numerical identities which are used in the remainder of the paper. Let \( g = \mathfrak{so}(n) \) and \( \mathfrak{f} = \mathfrak{so}(n-1) \).

\[
\sum_{i=2}^{n-1} r_i = \frac{1}{2}(\dim g - r_n). \tag{2.3}
\]

\[
\dim(B_g) + \dim(B_{\mathfrak{f}}) = \dim(g) - r_n - r_{n-1}. \tag{2.4}
\]

\[
\dim(g) - r_n - r_{n-1} = \dim \mathfrak{f}. \tag{2.5}
\]

These assertions are routine and are left to the reader. We note that they are also true for \( \mathfrak{gl}(n) \), provided the sum in the first equation goes from 1 to \( n-1 \).

2.5. **The Gelfand-Zeitlin Integrable Systems.** The complex general linear GZ system was first introduced by Kostant and Wallach in [KW06a], and the orthogonal GZ system was introduced by the first author in [Col09]. We briefly recall the construction of the orthogonal GZ systems here, especially since we are using a different realization of the
orthogonal Lie algebra than was used in [Col09]. To construct the GZ system, let \( g_i \cong \mathfrak{so}(i) \) be defined by downward induction, by taking \( g_n = g \), and letting \( g_i = g_{i+1}^{\theta_{i+1}} \), where \( \theta_{i+1} \) is the involution from Section 2.2. Thus, we have a chain of Lie subalgebras
\[
(2.6) \quad g_2 \subset g_3 \subset \cdots \subset g_i \subset \cdots \subset g_n.
\]
The Lie algebra \( g \) has nondegenerate, invariant, symmetric bilinear form \( \langle \langle x, y \rangle \rangle \) given by the trace form from Notation 2.2 (3). The form is nondegenerate on each \( g_i \), so we may identify \( g_i \) with its dual, and regard it as a Poisson variety using the Lie-Poisson structure. For \( i = 2, \ldots, n \), the inclusion \( g_i \subset g \) then dualizes to give us a map: \( \pi_i : g \rightarrow g_i \), which projects an element \( x \in g \) to its projection \( x_i \) in \( g_i \) off of \( g_i^\perp \). Define functions \( \psi_{i,j} \) on \( g_i \) for \( i = 2, \ldots, n \) as follows.
\[
(2.7) \quad \psi_{i,j}(y) := \text{Tr}(y^{j}), \quad j = 1, \ldots, r_i.
\]
If \( i \) is odd, then \( \psi_{i,j}(y) := \text{Tr}(y^{j}), \quad y \in g_i, \quad j = 1, \ldots, r_i. \)
If \( i \) is even, then \( \psi_{i,j}(y) := \text{Tr}(y^{j}), \quad y \in g_i, \quad j = 1, \ldots, r_i - 1 \), and \( \psi_{i,r_j}(y) := \text{Pfaff}(y) \), where \( \text{Pfaff}(y) \) denotes the Pfaffian of \( y \).

Then it is well-known that for each \( i \), \( \mathbb{C}[g_i]^{G_i} \) is a polynomial algebra with free generators \( \psi_{i,j}, j = 1, \ldots, r_i \), and \( \mathbb{C}[g_i]^{G_i} \) Poisson commutes with \( \mathbb{C}[g_i] \). Since \( g_i \subset g \) is an inclusion of Lie algebras, the transpose \( \pi_i : g \rightarrow g_i \) is easily seen to be a map of Poisson varieties. For \( i = 2, \ldots, n \) and \( j = 1, \ldots, r_i \), let \( f_{i,j} = \pi_i^* \psi_{i,j} \). We define the Gelfand-Zeitlin functions:
\[
(2.8) \quad J_{GZ} = \{ f_{i,j} : i = 2, \ldots, n, \ j = 1, \ldots, r_i \},
\]
and let \( J(g) \) be the subalgebra of \( \mathbb{C}[g] \) generated by the functions in \( J_{GZ} \).

**Remark 2.4.** The Gelfand-Zeitlin functions \( J_{GZ} \) and the associated subalgebra \( J(g) \) were first considered for the \( n \times n \) complex general linear Lie algebra \( g = \mathfrak{gl}(n) \) by Kostant and Wallach in [KW06a]. In that setting, there is also a chain of subalgebras as in (2.6), where \( g_i = \mathfrak{gl}(i) \subset g \) is identified with the top lefthand \( i \times i \) corner of \( g \). See [KW06a] or [CE14] for more details.

The following routine proposition may be proved by the same method as for the general linear case in Proposition 2.5 of [CE14].

**Proposition 2.5.** The algebra \( J(g) \) is a Poisson commutative subalgebra of \( \mathbb{C}[g] \).

Our main goals in this paper are to show that the restriction of the functions \( J_{GZ} \) to any regular \( \text{Ad}(G) \)-orbit in \( g \) forms a completely integrable system and to understand the generic leaves of the foliation given by the integrable system. For both of these issues, we need to study the moment map for the system, which we call the (orthogonal) Kostant-Wallach map or KW map for short. This is the morphism \( \Phi : g \rightarrow \mathbb{C}^{r_2} \times \cdots \times \mathbb{C}^{r_n} \) given by:
\[
(2.9) \quad \Phi : g \rightarrow \mathbb{C}^{r_2} \times \cdots \times \mathbb{C}^{r_n}; \quad \Phi(x) := (f_{2,1}(x), \ldots, f_{i,1}(x), \ldots, f_{i,r_i}(x), \ldots, f_{n,1}(x), \ldots, f_{n,r_n}(x)).
\]
Notation 2.6. For $x \in \mathfrak{g}$, let $\sigma(x)$ denote the spectrum of $x$. If $\mathfrak{g}$ is type $B$, then zero occurs as an eigenvalue of $x$ with multiplicity at least one. In this case, we only consider $0 \in \sigma(x)$, if it occurs as an eigenvalue of $x$ with multiplicity strictly greater than one.

Remark 2.7. We observe that if $y \in \Phi^{-1}(\Phi(x))$, then $\sigma(x_i) = \sigma(y_i)$ for all $i = 2, \ldots, n$. This follows from the well-known fact that the values of $x_i$ on the basic adjoint invariants $\psi_{i,1}, \ldots, \psi_{i,r_i}$ in (2.7) determine the characteristic polynomial of $x_i$.

Elements of $x \in \mathfrak{g}$ which lie in a regular level set of $\Phi$ play a very important role in the study of the GZ systems.

Definition-Notation 2.8. An element $x \in \mathfrak{g}$ is said to be strongly regular if the differentials

\begin{equation}
\{df(x) : f \in J_{GZ}\} \subset T^*_x(\mathfrak{g})
\end{equation}

are linearly independent elements of the cotangent space $T^*_x(\mathfrak{g})$. We denote the set of strongly regular elements as $\mathfrak{g}_{\text{sreg}}$ and note that $\mathfrak{g}_{\text{sreg}} \subset \mathfrak{g}$ is Zariski open. For $x \in \mathfrak{g}$, we set $\Phi^{-1}(\Phi(x))_{\text{sreg}} := \Phi^{-1}(\Phi(x)) \cap \mathfrak{g}_{\text{sreg}}$.

The term strongly regular is suggested by the following classical result of Kostant. Let $\mathfrak{g}$ be a complex reductive Lie algebra with adjoint action, and let $\phi_1, \ldots, \phi_r$ be generators of the polynomial algebra $\mathbb{C}[\mathfrak{g}]^G$. Then (Theorem 9, [Kos63])

\begin{equation}
x \in \mathfrak{g}_{\text{reg}} \text{ if and only if } d\phi_1(x) \wedge \cdots \wedge d\phi_r(x) \neq 0.
\end{equation}

Hence, by Equation (2.5), we see that

\begin{equation}
\text{If } x \in \mathfrak{g}_{\text{sreg}}, \text{ then } x_i \in \mathfrak{g}_i \text{ is regular for all } i = 2, \ldots, n.
\end{equation}

We briefly recall the notion of an integrable system from symplectic geometry. Recall that if $M$ is a complex manifold then the holomorphic functions $\{F_1, \ldots, F_r\}$ are said to be independent on $m$ if the open subset

\[ U = \{m \in M : dF_1(m) \wedge \cdots \wedge dF_r(m) \neq 0\} \]

is dense in $M$.

Definition 2.9. Let $(M, \omega)$ be a complex symplectic manifold of dimension $2r$. An integrable system on $M$ is a collection of $r$ independent, holomorphic functions $\{F_1, \ldots, F_r\}$ which Poisson commute with respect to the natural Poisson bracket on the space of holomorphic functions defined by the symplectic form $\omega$.

We recall that for any $x \in \mathfrak{g}$, the adjoint orbit through $x$, $\text{Ad}(G) \cdot x$ is a symplectic manifold with Kostant-Kirillov-Souriau symplectic structure. The connection between the complete integrability of the GZ system on regular adjoint orbits and strongly regular elements is given by the following proposition.
Proposition 2.10. Let \( x \in g_{\text{reg}} \), and let \( \text{Ad}(G) \cdot x \) be the adjoint orbit of \( G \) through \( x \). Then the restriction of the GZ functions \( J_{\text{GZ}} \) in (2.8) form a completely integrable system on \( \text{Ad}(G) \cdot x \) if and only if
\[
(2.13) \quad \text{Ad}(G) \cdot x \cap g_{\text{sreg}} \neq \emptyset.
\]

Proof. Consider the GZ functions \( J_{\text{GZ}} \) and let
\[
T = \{ f_{i,j} \text{ for } i = 2, \ldots, n-1, j = 1, \ldots, r_i \},
\]
and let \( S = \{ f|_{\text{Ad}(G)\cdot x} : f \in T \} \). Since the functions \( f_{n,j}, j = 1, \ldots, r_n \), restrict to constant functions on \( \text{Ad}(G) \cdot x \), we do not include them in \( T \). Let \( x \in g_{\text{reg}} \) and let \( z \in \text{Ad}(G) \cdot x \cap g_{\text{sreg}} \neq \emptyset \). Let \( \hat{R} \) be the span of the differentials \( \{ df(z) : f \in T \} \subset T_z^* (g) \), and let \( R \) be the restriction of \( \hat{R} \) to \( T_z (\text{Ad}(G) \cdot x) \). If \( \dim(R) < |S| \), there is a non-zero linear functional \( \lambda \in \hat{R} \cap T_z (\text{Ad}(G) \cdot x)^\perp \). But this contradicts the fact that \( z \in g_{\text{sreg}} \), since the trace form identifies \( T_z (\text{Ad}(G) \cdot z)^\perp \) with the span of \( df_{n,j}(z), j = 1, \ldots, r_n \), and hence \( \dim(R) = |S| \). The complete integrability of \( S \) on \( \text{Ad}(G) \cdot x \) now follows from standard assertions, Proposition 2.5, and Equation (2.3). We leave the converse assertion to the reader.

Q.E.D.

For \( g = \mathfrak{gl}(n) \), Kostant and Wallach introduced the GZ integrable system using a chain of subalgebras analogous to the one we used in (2.6) (Remark 2.4). The definitions of the Kostant-Wallach map and strong regularity are the same. Abusing notation, we will also refer to Kostant-Wallach map for \( \mathfrak{gl}(n) \) as \( \Phi \). In the general linear case, Kostant and Wallach show that Equation (2.13) is satisfied for every \( x \in \mathfrak{gl}(n)_{\text{reg}} \) (see Theorem 3.36 of [KW06a]). Their proof makes use of the fact that in this case \( \Phi \) possesses a natural cross-section given by the so-called upper Hessenberg matrices. These are matrices of the form:

\[
(2.14) \quad Hess := \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n-1} & a_{1n} \\
    1 & a_{22} & \cdots & a_{2n-1} & a_{2n} \\
    0 & 1 & \cdots & a_{3n-1} & a_{3n} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & \cdots & 1 & a_{nn}
\end{bmatrix},
\]

with \( a_{ij} \in \mathbb{C} \). In Theorem 2.3 of [KW06a], the authors prove that the restriction of \( \Phi \) to \( Hess \) is an isomorphism of varieties from which it follows that
\[
(2.15) \quad \Phi^{-1}(\Phi(x))_{\text{sreg}} \neq \emptyset \text{ for any } x \in \mathfrak{gl}(n),
\]
and (2.13) follows easily. It is not clear in the orthogonal case whether \( \Phi \) is even surjective nor that every non-empty fibre contains strongly regular elements.
3. The Partial Kostant-Wallach Map and Surjectivity of $Φ$

Rather than considering all of the Lie algebras in the chain in (2.6) simultaneously and working directly with the KW map, it is easier to consider one step in the chain at a time: $g_{n-1} \subset g$. Accordingly, we define the **partial Kostant-Wallach map** to be

$$Φ_n : g \to C^{r_n-1} \oplus C^{r_n},$$

(3.1)

$$Φ_n(x) = (f_{n-1,1}(x), \ldots, f_{n-1,r_{n-1}}(x), f_{n,1}(x), \ldots, f_{n,r_n}(x)).$$

The map $Φ_n$ has the advantage that it is a geometric invariant theory quotient (GIT quotient) as well as having other remarkable properties. To see these properties of $Φ_n$, we will need the theory of spherical pairs.

### 3.1. Spherical pairs and their coisotropy representations.

In this section, we consider pairs $(M, H)$ where $M$ is a reductive algebraic group and $H \subset M$ is an algebraic subgroup. Let $m$ and $h$ be the Lie algebras of $M$ and $H$ respectively.

**Definition 3.1.**

1. The pair $(M, H)$ is called spherical if $H$ acts on the flag variety $B = B_m$ with finitely many orbits.
2. The pair $(M, H)$ is called reductive if $H \subset M$ is a reductive algebraic subgroup.

**Definition 3.2.** For a reductive pair $(M, H)$ we say the branching law from $M$ to $H$ is multiplicity free if for all irreducible rational representations $V$ and $U$ of $M$ and $H$ respectively, $\dim(\text{Hom}_H(U, V)) \leq 1$.

For a pair $(M_1, R)$, we define a new pair $(M, H)$ by taking $M = \tilde{M}_1 := M_1 \times R$ and taking $H = R_\Delta = \{(g, g) : g \in R\} \subset \tilde{M}_1$. Let $\tilde{m}_1 = m_1 \oplus r$ and $r_\Delta$ be the corresponding Lie algebras. The following result is well-known.

**Proposition 3.3.**

1. Let $(M_1, R)$ be a reductive pair. The pair $(\tilde{M}_1, R_\Delta)$ is spherical if and only if the branching rule from $M_1$ to $R$ is multiplicity free.
2. For the pair $(M_1, R) = (SO(n), SO(n-1))$, the pair $(\tilde{M}_1, R_\Delta) = (SO(n) \times SO(n-1), SO(n-1)_\Delta)$ is spherical.

**Proof.** The first statement follows by Theorem B of [Bru97], together with the easy observation that a Borel subgroup $B_R$ of $R$ has finitely many orbits on the flag variety $B_m$, of $m_i$ if and only if $R_\Delta$ has finitely many orbits on $B_{m_i} \times B_r$. The second statement follows from the first statement and well-known branching laws (see [Joh01]).

Q.E.D.

For a reductive spherical pair $(M, H)$, let $\langle \cdot, \cdot \rangle$ be a non-degenerate symmetric $M$-invariant bilinear form on $m$, and let $h^\perp$ be the annihilator of $h$ with respect to $\langle \cdot, \cdot \rangle$. Then the adjoint action of $M$ on $m$ restricts to an action of $H$ on $h^\perp$, which is referred to in the literature as the coisotropy representation of $H$ (see [Pan90]). Then it is well-known that $\mathbb{C}[h^\perp]^H$ is a polynomial algebra (Kor 7.2 of [Kno90] or Corollary 5 of [Pan90]).
3.2. Surjectivity of $\Phi_n$ and $\Phi$. We now return to our convention with $G = SO(n)$ and $K = SO(n-1)$, and similarly with Lie algebras. We consider the bilinear form on $\tilde{g} = g + \mathfrak{k}$ given by taking the trace form on each factor, and note that by an easy calculation,

$$\mathfrak{t}_\Delta^\perp = \{(x, -x) : x \in g, x \in \mathfrak{k}\} \cong g$$

as a $K \cong K_\Delta$-representation.

**Proposition 3.4.**

1. $\mathbb{C}[\mathfrak{g}]^K = \mathbb{C}[\mathfrak{g}]^G \otimes \mathbb{C}[\mathfrak{k}]^K$.
2. $\Phi_n$ coincides with the invariant theory quotient morphism $g \to g//K$. In particular, $\Phi_n$ is surjective.
3. The morphism $\Phi_n$ is flat. In particular, its fibres are equidimensional varieties of dimension $\dim g - r_n - r_{n-1}$.

**Proof.** Recall the well-known fact that the fixed point algebra $U(\mathfrak{g})^K$ of $K$ in the enveloping algebra $U(\mathfrak{g})$ is commutative [Joh01]. Hence, $U(\mathfrak{g})^K$ coincides with its centre, $Z(U(\mathfrak{g})^K)$. In Theorem 10.1 [Kno94], Knop shows that $Z(U(\mathfrak{g})^K) \cong U(\mathfrak{g})^G \otimes \mathbb{C} U(\mathfrak{g})^K$. The first assertion now follows by taking the associated graded algebra with respect to the usual filtration of $U(\mathfrak{g})$. By the first assertion, $\Phi_n$ coincides with the invariant theory quotient $g \to g//K$, which gives the second assertion. By Equation (3.2), we identify $\mathfrak{t}_\Delta^\perp \cong g$ $K$-equivariantly. Then the flatness of $\Phi_n$ follows by Korollar 7.2 [Kno90], which gives a criterion for flatness of invariant theory quotients in the setting of spherical homogeneous spaces (see also [Pan90]).

Q.E.D.

**Notation 3.5.** For ease of notation, we denote the nilfibre $\Phi_n^{-1}(0,0), (0,0) \in \mathbb{C}^{r_n-1} \times \mathbb{C}^{r_n}$ of $\Phi_n$ by $\Phi_n^{-1}(0)$.

Using the flatness of the partial KW map $\Phi_n$, we can now show that the orthogonal KW map $\Phi$ is surjective.

**Theorem 3.6.** Let $\Phi : g \to \mathbb{C}^{r_2} \times \cdots \times \mathbb{C}^{r_n}$ be the the Kostant-Wallach map. The morphism $\Phi$ is surjective and every fibre of $\Phi$ contains a regular element of $g$.

To prove Theorem 3.6, we need some preparation. Consider the nonempty Zariski open set:

$$U_{r,\mathfrak{t}} := \{x \in \mathfrak{g}_{\text{reg}} : x \in \mathfrak{t}_{\text{reg}}\}.$$ 

**Lemma 3.7.** Let $\Phi_n : g \to \mathbb{C}^{r_n-1} \times \mathbb{C}^{r_n}$ be the partial KW map. The restriction $\Phi_n|_{U_{r,\mathfrak{t}}} : U_{r,\mathfrak{t}} \to \mathbb{C}^{r_n-1} \times \mathbb{C}^{r_n}$ is surjective.

**Proof.** Since $\Phi_n$ is a flat morphism, it is an open morphism by Exercise III.9.1 [Har77]. Thus, $\Phi_n(U_{r,\mathfrak{t}}) \subseteq \mathbb{C}^{r_n-1} \times \mathbb{C}^{r_n}$ is Zariski open. We suppose that $\Phi_n(U_{r,\mathfrak{t}}) \neq \mathbb{C}^{r_n-1} \times \mathbb{C}^{r_n}$. Then $C := \mathbb{C}^{r_n-1} \times \mathbb{C}^{r_n} \setminus \Phi_n(U_{r,\mathfrak{t}})$ is a non-empty, closed subset of $\mathbb{C}^{r_n-1} \times \mathbb{C}^{r_n}$. Since $\Phi_n$ is...
surjective, it follows that $\Phi_n^{-1}(C)$ is a non-empty, closed subset of $\mathfrak{g}$. Now it follows from definitions that

$$x \in \Phi_n^{-1}(C) \text{ if and only if } \Phi_n^{-1}(\Phi_n(x)) \cap U_{r,\xi} = \emptyset.$$  

Since the functions defining $\Phi_n$ are homogeneous (Equation (3.1)), scalar multiplication by $\lambda \in \mathbb{C}^\times$ induces an isomorphism: $\Phi_n^{-1}(\Phi_n(x)) \to \Phi_n^{-1}(\Phi_n(\lambda x))$. Note that $U_{r,\xi}$ is also preserved by scalar multiplication by elements of $\mathbb{C}^\times$. Thus, (3.4) implies that $\Phi_n^{-1}(C)$ is preserved by scalar multiplication by elements of $\mathbb{C}^\times$. Since $\Phi_n^{-1}(C)$ is closed, it follows that $0 \in \Phi_n^{-1}(C)$, whence

$$\Phi_n^{-1}(0) \cap U_{r,\xi} = \emptyset.$$  

But we claim that $\Phi_n^{-1}(0) \cap U_{r,\xi} \neq \emptyset$, contradicting the initial assumption that $\Phi_n(U_{r,\xi}) \neq \mathbb{C}^{n-1} \times \mathbb{C}^n$. 

For this, let $b_+ \subset \mathfrak{g}$ be the Borel subalgebra of upper triangular matrices in $\mathfrak{g}$, and let $n_+ = [b_+, b_+]$. We consider the cases where $\mathfrak{g}$ is type $B$ and $D$ separately. If $\mathfrak{g} = \mathfrak{so}(2l + 1)$ is type $B$, we let $e := e_{a_1} + \cdots + e_{a_l} + e_{i_{l-1} + i_l}$, where $e_{a_i} \in \mathfrak{g}_{a_i}$, $i = 1, \ldots, l$ and $e_{i_{l-1} + i_l} \in \mathfrak{g}_{i_{l-1} + i_l}$ are nonzero root vectors. Then by Kostant’s criterion for an element to be regular nilpotent (Theorem 5.3 of [Kos59]), $e \in n_+$ is regular nilpotent and $\pi_{\mathfrak{t}}(e) = e_{a_1} + \cdots + e_{a_{l-1}} + e_{i_{l-1} + i_l}$ by Example 2.1. By Kostant’s criterion, the element $\pi_{\mathfrak{t}}(e)$ is a regular nilpotent element of $\mathfrak{t}$. Thus, $e \in \Phi_n^{-1}(0) \cap U_{r,\xi}$. 

Now let $\mathfrak{g} = \mathfrak{so}(2l)$ be type $D$. Let $e = e_{a_1} + \cdots + e_{a_{l-2}} + \frac{1}{2}(e_{a_{l-1}} + \theta(e_{a_{l-1}}))$. By Example 2.1 we have $\theta(e_{a_{l-1}}) = a_l$, so $e$ is regular nilpotent by Kostant’s criterion. Example 2.1 also implies that $e \in \mathfrak{t}$, and that in terms of the simple roots for $\mathfrak{t} = \mathfrak{so}(2l - 1)$, we have $e = e_{a_1} + \cdots + e_{a_{l-2}} + e_{a_{l-1}}$. Therefore, again by Kostant’s criterion, $e$ is also regular nilpotent when viewed as an element of $\mathfrak{t}$. Thus, $e \in \Phi_n^{-1}(0) \cap U_{r,\xi}$.

We conclude that $\Phi_n^{-1}(0) \cap U_{r,\xi} \neq \emptyset$, contradicting (3.5) and the initial assumption that $\Phi_n|_{U_{r,\xi}}$ is not surjective.

Q.E.D.

**Proof of Theorem 3.6.** The proof proceeds by induction on $n$. The case $n = 3$, follows from the fact that $\mathfrak{so}(3) \cong \mathfrak{sl}(2)$, and the theorem is known to hold for $\mathfrak{sl}(2)$. We now assume that the result is true for $\mathfrak{t}$. Let $(c_2, \ldots, c_{n-1}, c_n) \in \mathbb{C}^r \times \cdots \times \mathbb{C}^{r_{n-2}} \times \mathbb{C}^{r_n}$ be arbitrary. Consider $\Phi^{-1}(c_2, \ldots, c_{n-1}, c_n)$. Let $\Phi_{\mathfrak{t}} : \mathfrak{t} \to \mathbb{C}^r \times \cdots \times \mathbb{C}^{r_{n-2}}$ be the KW map for $\mathfrak{t}$. By the inductive hypothesis, there exists an element $y \in \Phi_{\mathfrak{t}}^{-1}(c_2, \ldots, c_{n-1})$ with $y \in \mathfrak{t}_{\text{reg}}$. By Lemma 3.7, there exists $x \in \Phi_n^{-1}(c_{n-1}, c_n)$ with $x \in \mathfrak{g}_{\text{reg}}$ and $x_{\mathfrak{t}} \in \mathfrak{t}_{\text{reg}}$. It follows that $x_{\mathfrak{t}}, y \in \chi_{\mathfrak{t}}^{-1}(c_{n-1}) \cap \mathfrak{t}_{\text{reg}}$, where $\chi_{\mathfrak{t}} : \mathfrak{t} \to \mathfrak{t}/K$ is the adjoint quotient, so that $y = \text{Ad}(k) \cdot x_{\mathfrak{t}}$ for some $k \in K$ by Theorem 3 of [Kos63] - [Kos63]. Let $z := \text{Ad}(k) \cdot x$. Then $z \in \Phi_n^{-1}(c_{n-1}, c_n)$ is regular, and $z_{\mathfrak{t}} = \text{Ad}(k) \cdot x_{\mathfrak{t}} = y$. It follows from definitions that $z \in \Phi^{-1}(c_2, \ldots, c_{n-1}, c_n)$. This completes the proof of the theorem.
4. Spherical Pairs and strongly regular elements

In this section, we use the geometry of spherical varieties to study the set of strongly regular elements of $\mathfrak{g}_{\text{sreg}}$ of $\mathfrak{g}$ introduced in Definition-Notation 2.8. We simplify the criterion of Kostant and Wallach for an element $x \in \mathfrak{g}$ to be strongly regular and use our new characterization of strongly regular elements to construct a new set of strongly regular elements $\mathfrak{g}_s$ for $\mathfrak{g}$. As a consequence, we obtain the complete integrability of the GZ system on regular adjoint orbits on $\mathfrak{g}$ and determine the structure of the KW fibres $\Phi^{-1}(\Phi(x))$ for $x \in \mathfrak{g}_s$. We show that each such fibre has a free action of an abelian algebraic group thereby obtaining “angle coordinates” for the GZ system on these fibres.

4.1. Further results on the coisotropy representation of a spherical pair. Recall the reductive spherical pair $(M, H)$ from Definition 3.1. We consider reductive spherical pairs satisfying:

\[(4.1) \quad \dim \mathcal{B} = \dim \mathfrak{h}^\perp - \dim \mathfrak{h}^\perp//H.\]

**Remark 4.1.** If we let $G = GL(n)$, and $K = GL(n - 1)$, where $K$ is embedded in $G$ in the top lefthand corner (Remark 2.4), then the pair $(\tilde{G}, K_\Delta)$ is spherical by Part (1) of Proposition 3.3. Further, this pair satisfies Equation (4.1). In Theorem 2.3, [CE15] we gave a different proof of Knop’s flatness result (Korollar 7.2, [Kno90]) for the spherical pair $(GL(n) \times GL(n - 1), GL(n - 1)_\Delta)$ using a variant of the Steinberg variety along with a dimension estimate obtained from (4.1). Our proof can be generalized to any reductive spherical pair $(M, H)$ satisfying Equation (4.1). See the appendix of [CE] for details.

We now analyze further the meaning of the condition in Equation (4.1) for the coisotropy representation. We study the set of $H$-regular elements $\mathfrak{h}_{\text{reg}}^\perp$ in $\mathfrak{h}^\perp$ consisting of $H$-orbits of maximal dimension, and recall the regular set $\mathfrak{m}_{\text{reg}}$ of $\mathfrak{m}$ (Notation 2.2 (6)).

**Theorem 4.2.** Let $(M, H)$ be a reductive spherical pair. Then the following conditions are equivalent.

1. Equation (4.1) holds.
2. We have $\mathfrak{h}_{\text{reg}}^\perp \subset \mathfrak{m}_{\text{reg}}$.

**Proof.** By Theorems 3 and 6 and Equation (15) of [Pan90], there is an open dense set $U$ of $\mathfrak{h}_{\text{reg}}^\perp$ such that for $y \in U$,

\[(4.2) \quad \text{codim}_{\mathfrak{h}^\perp}(\text{Ad}(H) \cdot y) = \dim(\mathfrak{h}^\perp//H),\]

and

\[(4.3) \quad \dim(\text{Ad}(M) \cdot y) = 2 \dim(\text{Ad}(H) \cdot y).\]
We consider \( x \in \mathfrak{h}_{\text{reg}}^\perp \). By (1.2), we have \( \text{codim}_{\mathfrak{h}^\perp} (\text{Ad}(H) \cdot x) = \dim(\mathfrak{h}^\perp / H) \). Assuming (1), we see that \( \dim(\text{Ad}(H) \cdot x) = \dim B \).

By Proposition (1) of [Pan90], \( 2 \dim(\text{Ad}(H) \cdot x) \leq \dim(\text{Ad}(M) \cdot x) \) so that \( \dim(\text{Ad}(M) \cdot x) \geq 2 \dim(B) \), and thus \( x \in \mathfrak{m}_{\text{reg}} \), which proves one direction of the assertion.

Conversely, for any element \( x \) in the open set \( U \subset \mathfrak{h}_{\text{reg}}^\perp \), then \( x \in \mathfrak{m}_{\text{reg}} \) by assumption. Hence by (4.3), \( \dim(\text{Ad}(H) \cdot x) = \frac{1}{2} \dim(\text{Ad}(M) \cdot x) = \dim(B) \). The assertion of (1) now follows from Equation (4.2).

Q.E.D.

Remark 4.3. If \( \theta \) is an involution of \( \mathfrak{m} \) with fixed subalgebra \( \mathfrak{h} = \mathfrak{m}^\theta \), the involution is called quasi-split if \( \mathfrak{m} \) has a Borel subalgebra \( \mathfrak{b} \) such that \( \mathfrak{b} \cap \theta(\mathfrak{b}) \) is a Cartan subalgebra of \( \mathfrak{m} \). In Proposition 4.4 of [CE], we show that for \( M \) and \( H \) the corresponding connected groups, \((M, H)\) satisfies Equation \((4.1)\) if and only if \( \theta \) is quasi-split.

4.2. A new criterion for strong regularity. We now apply Theorem 4.2 to study the \( K \)-action on \( \mathfrak{g} \) in the case where \((K, \mathfrak{g}) = (SO(n - 1), \mathfrak{so}(n))\) and prove an analogue of Kostant’s Theorem \((2.11)\) for the action of \( K \) on \( \mathfrak{g} \) by restricting the adjoint action of \( G \) to \( K \) (Theorem 4.5). For this, we consider the reductive spherical pair \((\tilde{G}, K_\Delta)\) with \( \tilde{G} = G \times K \). Recall that we identify \( \mathfrak{k}^\perp \cong \mathfrak{g} \) as a \( K \)-module (Equation \((3.2)\)).

Lemma 4.4. Consider the reductive spherical pair \((\tilde{G}, K_\Delta)\).

1. Equation \((4.1)\) holds.
2. \( (\mathfrak{t}^\perp_\Delta)_{\text{reg}} \cong \{ x \in \mathfrak{g} : \dim(\text{Ad}(K) \cdot x) = \dim K \} \).
3. \( (\mathfrak{t}^\perp_\Delta)_{\text{reg}} \cong \{ x \in \mathfrak{g} : \mathfrak{z}_K(x) \cap \mathfrak{z}_G(x) = 0 \} \).

Proof. Equation \((4.1)\) is equivalent to the identity
\[
\dim(B_\mathfrak{g}) + \dim(B_\mathfrak{t}) = \dim(g) - r_n - r_{n-1} = \dim(g) - \dim(g / K),
\]
which follows by Equation \((2.3)\) and Proposition \((3.3)\) (2). To prove the second assertion, let \( x \in (\mathfrak{t}^\perp_\Delta)_{\text{reg}} \). By (1), we can apply Equation \((4.4)\) to conclude that \( \dim(\text{Ad}(K) \cdot x) = \dim(B_\mathfrak{g}) + \dim(B_\mathfrak{t}) \). The assertion now follows from \((4.6)\) and Equation \((2.5)\). The second assertion implies that \( x \in (\mathfrak{t}^\perp_\Delta)_{\text{reg}} \) if and only if \( \mathfrak{z}_K(x) = 0 \). The third assertion follows since \( \mathfrak{z}_K(x) = \mathfrak{z}_K(x) \cap \mathfrak{z}_G(x) \).

Q.E.D.

We now describe the regular elements of the coisotropy representation of the reductive spherical pair \((\tilde{G}, K_\Delta)\), which establishes an analogue of Kostant’s theorem. Recall that
an element $x \in \mathfrak{g}_{\text{reg}}$ if $\dim \mathfrak{j}_\mathfrak{g}(x) = \text{rank}(\mathfrak{g})$. If we identify $T^*_x(\mathfrak{g})$ with $\mathfrak{g}$ using the non-degenerate form on $\mathfrak{g}$, then Kostant’s basic result in (2.11) implies that

$$\text{(4.7)} \quad \text{span}\{df_{n,i}(x) : i = 1, \ldots, r_n\} = \mathfrak{j}_\mathfrak{g}(x).$$

Recall also that $\mathbb{C}[\mathfrak{g}]^K = \mathbb{C}[f_{n-1,1}, \ldots, f_{n-1,r_n-1}, f_{n,1}, \ldots, f_{n,r_n}]$ (see Proposition 3.4). Let

$$\omega_{\mathfrak{g}/K} := df_{n-1,1} \wedge \cdots \wedge df_{n-1,r_n-1} \wedge df_{n,1} \wedge \cdots \wedge df_{n,r_n} \in \Omega^{r_n-1+r_n}(\mathfrak{g}).$$

**Theorem 4.5.** $x \in (\mathfrak{t}_\Delta)_{\text{reg}}$ if and only if $\omega_{\mathfrak{g}/K}(x) \neq 0$, and if so, then $x \in \mathfrak{g}_{\text{reg}}$ and $x_\mathfrak{k} \in \mathfrak{k}_{\text{reg}}$.

**Proof.** We first suppose that $\omega_{\mathfrak{g}/K}(x) \neq 0$. By Equation (2.11), it follows that $x \in \mathfrak{g}_{\text{reg}}$ and $x_\mathfrak{k} \in \mathfrak{k}_{\text{reg}}$. Equation (4.7) then implies that $\mathfrak{j}_\mathfrak{k}(x_\mathfrak{k}) \cap \mathfrak{j}_\mathfrak{g}(x) = 0$, so $x \in (\mathfrak{t}_\Delta)_{\text{reg}}$ by Equation (4.5).

Conversely, suppose $x \in (\mathfrak{t}_\Delta)_{\text{reg}}$. Then by Theorem 4.2, Equation (3.2), and part (1) of Lemma 4.3, $(x,-x_\mathfrak{k}) \in \mathfrak{p}_{\text{reg}}$. Thus, both $x \in \mathfrak{g}_{\text{reg}}$ and $x_\mathfrak{k} \in \mathfrak{k}_{\text{reg}}$. Hence by Equation (2.11),

$$\text{(4.8)} \quad df_{n-1,1}(x_\mathfrak{k}) \wedge \cdots \wedge df_{n-1,r_n-1}(x_\mathfrak{k}) \neq 0 \quad \text{and} \quad df_{n,1}(x) \wedge \cdots \wedge df_{n,r_n}(x) \neq 0.$$ 

Since $x \in (\mathfrak{t}_\Delta)_{\text{reg}}$, $\mathfrak{j}_\mathfrak{k}(x_\mathfrak{k}) \cap \mathfrak{j}_\mathfrak{g}(x) = 0$ by Equation (4.5). It now follows from (4.8) and (4.7) that $\omega_{\mathfrak{g}/K}(x) \neq 0$.

Q.E.D.

Theorem 4.5 can be obtained as a special case of a more general result proven by Knop, \[Kno86\]. We include our proof here because of its simplicity. Theorem 4.5 has an immediate corollary which is of interest in linear algebra.

**Corollary 4.6.** Let $x \in \mathfrak{g}$ and suppose that $\mathfrak{j}_\mathfrak{k}(x_\mathfrak{k}) \cap \mathfrak{j}_\mathfrak{g}(x) = 0$. Then $x \in \mathfrak{g}_{\text{reg}}$ and $x_\mathfrak{k} \in \mathfrak{k}_{\text{reg}}$.

**Proof.** This follows by Equation (4.5) and Theorem 4.5.

Q.E.D.

We observed in the proof of Lemma 4.4 that for $x \in \mathfrak{g}$, $\mathfrak{j}_\mathfrak{k}(x) = \mathfrak{j}_\mathfrak{k}(x_\mathfrak{k}) \cap \mathfrak{j}_\mathfrak{g}(x)$. Elements of $(\mathfrak{t}_\Delta)_{\text{reg}}$ can be used to inductively construct strongly regular elements of $\mathfrak{g}$, so we give them a special name.

**Definition 4.7.** An element $x \in \mathfrak{g}$ such that $\mathfrak{j}_\mathfrak{k}(x) = 0$ is said to be $n$-strongly regular. We denote the set of $n$-strongly regular elements by $\mathfrak{g}_{n_{\text{reg}}}$.

We can use Corollary 4.6 to simplify the characterization of strongly regular elements originally given by Kostant and Wallach for the general linear case in Theorem 2.14 of \[KW06a\] and generalized by the first author to the orthogonal case in Proposition 2.11 of \[Col09\].
Proposition 4.8. An element $x \in \mathfrak{g}$ is strongly regular if and only if
\[ \mathfrak{z}(x_i) \cap \mathfrak{z}(x_{i+1}) = 0 \text{ for } i = 2, \ldots, n - 1. \]

Proof. By Proposition 2.11 of [Col09], an element $x \in \mathfrak{g}$ is strongly regular if and only if the following two conditions hold:

1. $x_i \in \mathfrak{g}_i$ are regular for all $i = 2, \ldots, n$.
2. $\mathfrak{z}(x_i) \cap \mathfrak{z}(x_{i+1}) = 0$ for $i = 2, \ldots, n - 1$.

It follows from Corollary 4.6 that if $x_{i+1} \in \mathfrak{g}_{i+1}$ satisfies $\mathfrak{z}(x_i) \cap \mathfrak{z}(x_{i+1}) = 0$, then $x_{i+1} \in \mathfrak{g}_{i+1}$ is regular.

Q.E.D.

Remark 4.9. We note that arguments given above also apply to the general linear case. Indeed, we observed in Remark 4.1 that if we let $G = GL(n)$, $K = GL(n-1)$, then the pair $(G, K_\Delta)$ is spherical and satisfies Equation (1.1). Further, Lemma 4.4 also holds for this spherical pair, and we obtain Theorem 4.5 by the same proof. Corollary 4.6 follows, and we obtain a simplification of Kostant and Wallach’s characterization of strongly regular elements in Theorem 2.14 of [KW06a].

In [CE15], we defined the set of $n$-strongly regular elements for $\mathfrak{g} = gl(n)$ to be the set of elements $x \in \mathfrak{g}$ for which $\omega_{\mathfrak{g}_i/\mathfrak{k}}(x) \neq 0$. It follows from Theorem 4.5 and Equation (1.5) that our definition in Definition 4.7 is consistent with the previous one and $\mathfrak{g}_{\text{nsreg}} \simeq (\mathfrak{t}_\Delta^\perp)_{\text{reg}}$.

The following technical result will be useful in the next section in our description of the geometry of generic fibres of the KW map (see Theorem 4.18).

Lemma 4.10. For $x \in \mathfrak{g}_{\text{nsreg}}$, the group $Z_K(x) = Z_K(x_\mathfrak{t}) \cap Z_G(x) = \{e\}$ is the trivial group.

Proof. Since $x \in \mathfrak{g}_{\text{nsreg}}$, $\text{Lie}(Z_K(x)) = \mathfrak{z}(x) = 0$, so that $Z_K(x)$ is a finite group. Decompose $x = x_\mathfrak{t} + x_\mathfrak{p}$ with respect to the Cartan decomposition, so that $Z_K(x) = Z_K(x_\mathfrak{t}) \cap Z_K(x_\mathfrak{p})$. Consider the Jordan decomposition of $x_\mathfrak{t}$ in $\mathfrak{t}$, $x_\mathfrak{t} = s + n$, with $s$ semisimple and $n$ nilpotent. Consider the Levi subgroup $L := Z_K(s)$ of $K$, let $\mathfrak{l} = \text{Lie}(L)$, and let $Z$ be the centre of $L$. We claim that

\[ Z_K(x) = Z_{x_\mathfrak{p}} := \{ z \in Z : \text{Ad}(z) \cdot x_\mathfrak{p} = x_\mathfrak{p} \}. \]

Indeed, if $z \in Z_{x_\mathfrak{p}}$, then since $s, n \in \mathfrak{l}$ and $z \in Z$, it follows that $z \in Z_K(x_\mathfrak{t})$, and hence $z \in Z_K(x_\mathfrak{t}) \cap Z_K(x_\mathfrak{p}) = Z_K(x)$. Conversely, by standard properties of the Jordan decomposition, $Z_K(x_\mathfrak{t}) = Z_K(s) \cap Z_K(n) = Z_L(n)$. Since $x \in \mathfrak{g}_{\text{nsreg}}$, $x_\mathfrak{t} \in \mathfrak{t}_{\text{reg}}$ by Corollary 4.6, so $n$ is regular nilpotent in $\mathfrak{l}$, and hence $Z_L(n) = Z \cdot U$, where $U$ is a unipotent subgroup of $L$. It follows that the finite subgroup $Z_K(x)$ of $Z_L(n)$ is in $Z$, and hence that $Z_K(x) \subset Z_{x_\mathfrak{p}}$, and this establishes (1.9). For later use, we let $Z^0$ denote the identity component of $Z$. 


From the classification of Levi subgroups of $K = SO(n - 1)$, it follows that up to $K$-conjugacy,

$$L = GL(s_1) \times \cdots \times GL(s_d) \times SO(r),$$

where $r \equiv n - 1 \pmod{2}$, and $r \neq 2$, or

$$L = GL(m_1) \times \cdots \times GL(m_d).$$

If $n - 1$ is even and $L$ is as in (4.10), then the centre $Z$ of $L$ is $GL(1)^d \times < \epsilon >$, where $\epsilon$ is the negative of the identity in $SO(r)$. Otherwise, $Z = Z^0 = GL(1)^d$. Recall also that the $K$-module $p \cong V$, where $V$ is the standard representation of $K$. It follows that the $Z^0$-weights of $p$ consist of $\Gamma = \{ \pm \mu_i : i = 1, \ldots, d \}$, where $\mu_1, \ldots, \mu_d$ is a basis of the character group $X^*(Z^0)$ of $Z^0$, along with the trivial character when $L$ is given by (4.10).

Let $x_p = \bigoplus_{\lambda \in \Gamma} x_{\lambda}$ be the decomposition of $x_p$ into $Z^0$-weight vectors, and let $\Gamma_0$ consist of the $\mu_i$ such $x_{\mu_i}$ or $x_{-\mu_i}$ is nonzero. Let $Z^0_{\Gamma_0} := Z^0 \cap Z_{x_p}$ and note that

$$Z^0_{\Gamma_0} = \{ z \in Z^0 : \mu_i(z) = 1 \text{ for all } \mu_i \in \Gamma_0 \}.$$

If $|\Gamma_0| < d$, then $\dim(Z^0_{\Gamma_0}) \geq 1$, which contradicts the finiteness of $Z_K(x)$ (4.9). Hence $|\Gamma_0| = d$, so that $\Gamma_0$ is a basis of $X^*(Z^0)$, and it follows that $Z^0_{\Gamma_0} = \{ e \}$. When $Z = Z^0$, we have

$$\{ e \} = Z^0_{\Gamma_0} = Z_{x_p} = Z_K(x)$$

by (4.9), and the lemma is proven. If, on the other hand, $Z \neq Z^0$, then by our remarks above, $n - 1$ must be even and $L$ is given by (4.10). In this case, we can decompose $p = \bigoplus_{i=1}^d p_{x_{\mu_i}} \oplus p_0$, where $p_{\mu_i}$ is the standard representation of $GL(s_i)$, $p_{-\mu_i}$ is its contragradient, and $p_0$ is the standard representation of $SO(r)$. Thus, $\epsilon$ acts as the negative of the identity on $p_0$ while $Z^0$ acts trivially on $p_0$, and $\epsilon$ acts trivially on $\bigoplus_{i=1}^d p_{\pm \mu_i}$. If $g \in Z_K(x) \setminus \{ e \}$, then since $Z^0_{\Gamma_0} = \{ e \}$, we would have $g = z \cdot \epsilon$ with $z \in Z^0$. If this were the case, then $x_{p_0} = 0$. It follows that if $x_{\ell} = \bigoplus_{i=1}^d x_i \oplus y$, with $x_i \in \mathfrak{gl}(s_i)$ and $y \in \mathfrak{so}(r)$, is the decomposition of $x_{\ell}$ in $l$, then $y \in \mathfrak{t}(x_{\ell}) \cap \mathfrak{g}(x) = 0$, so $y = 0$. But then $x_{\ell}$ is not regular, which is a contradiction. Thus, $Z_K(x) = \{ e \}$.

Q.E.D.

4.3. Generic Elements for GZ integrable systems. We use Proposition 4.8 to construct a new set $\mathfrak{g}_0 \subset \mathfrak{g}$ of strongly regular elements. We then use the set $\mathfrak{g}_0$ to show that the GZ system is completely integrable on regular adjoint orbits of $\mathfrak{g}$.

We begin by studying the relation between the spectra of $x$ and $x_{\ell}$. For $x \in \mathfrak{g}$, recall that $\sigma(x)$ denotes the spectrum of $x$ (see Notation 2.6). We show the Zariski open subset:

$$\mathfrak{g}(0) = \{ x \in \mathfrak{g} : \sigma(x_{\ell}) \cap \sigma(x) = \emptyset \}.$$

consists of $n$-strongly regular elements in the sense of Definition 4.7. We require a result from linear algebra.
Lemma 4.11. Let $V = V_1 \oplus V_2$ be a direct sum decomposition of a finite dimensional complex vector space. Let $X, Y \in \text{End}(V)$ with $Y \neq 0$. Suppose that $Y(V_1) \subset V_1$, $Y(V_2) = 0$, and $[X, Y] = 0$. Define $X_1 := \pi_{V_1} \circ X|_{V_1} \in \text{End}(V_1)$ where $\pi_{V_1}$ is projection off $V_2$. Let $V(\lambda)$ be the generalized eigenspace of $X$ of eigenvalue $\lambda$ and let $V_1(\lambda)$ be the generalized eigenspace of $X_1$ of eigenvalue $\lambda$. Then $Y(V(\lambda)) \subset V_1(\lambda)$.

Proof. Let $v \in V(\lambda)$. Then there is $j \in \mathbb{Z}^{\geq 0}$ such that $(X - \lambda \text{Id}_V)^j v = 0$ where $\text{Id}_V$ is the identity operator on $V$. For $k \geq 0$, let $v_k = (X - \lambda \text{Id}_V)^k Y v$. Since $[X, Y] = 0$, then $v_k \in \text{Im}(Y) \subset V_1$. We show that $v_k = (X_1 - \lambda \text{Id}_{V_1})^k Y v$ by induction on $k$. The case $k = 0$ is clear, and note that if $u \in V_1$ and $(X - \lambda \text{Id}_V) u \in V_1$, then $(X - \lambda \text{Id}_V) u = (X_1 - \lambda \text{Id}_{V_1}) u$. The inductive step follows easily from this observation since each $v_k \in V_1$. By the choice of $j$,

$$(X_1 - \lambda \text{Id}_{V_1})^j Y v = (X - \lambda \text{Id}_V)^j Y v = Y(X - \lambda \text{Id}_V)^j v = 0,$$

and this establishes the lemma.

Q.E.D.

Theorem 4.12. Let $x \in \mathfrak{g}(0)$. Then $x \in \mathfrak{g}_{\text{nsreg}}$.

Proof. Suppose $x \not\in \mathfrak{g}_{\text{nsreg}}$ so that $\mathfrak{z}_\mathfrak{k}(x) \neq 0$. We show that $\sigma(x) \cap \sigma(x_\mathfrak{k}) \neq \emptyset$ by considering the types $B, D$ separately. Suppose that $\mathfrak{g} = \mathfrak{so}(2l)$. We apply Lemma 4.11 to the vector space $V = \mathbb{C}^{2l} = V_1 \oplus V_2$, where $V_1 = \text{span}\{e_{\pm 1}, \ldots, e_{\pm(l-1)}, e_l + e_{-l}\}$, and $V_2 = \text{span}\{e_l - e_{-l}\}$ (see Section 2.1.1). We take $X = x$ and $Y$ any nonzero element of $\mathfrak{z}_\mathfrak{k}(x)$, so that $[X, Y] = 0$. The reader can check that the involution $\theta$ acts on $e_{\pm i}$ via $\theta(e_{\pm i}) = e_{\pm i}$ for $i \neq l$ and $\theta(e_l) = e_{-l}$ (see Example 2.1). Since $\theta(Z \cdot \theta(v)) = \theta(Z) \cdot v$ for any $Z \in \mathfrak{g}$ and $v \in V$, it follows that $\mathfrak{k}(V_1) \subset V_1$ and $\mathfrak{p}(V_1) \subset V_2$. Hence, $Y(V_1) \subset V_1$, $x_\mathfrak{k}(V_1) \subset V_1$ and $x_\mathfrak{p}(V_1) \subset V_2$. Thus, $x_\mathfrak{k}$ is the element $X_1$ from Lemma 4.11. Let $V = \bigoplus_{\mu \in \sigma(x)} V(\mu)$ be the decomposition of $V$ into generalized eigenspaces of $x$. Since $Y \neq 0$, there exists $\mu \in \sigma(x)$ such that $Y(V(\mu)) \neq 0$, so by Lemma 4.11, $V_1(\mu) \neq 0$. We show by contradiction that if $Y(V(0)) \neq 0$, then $\dim(V_1(0)) > 1$. Indeed, otherwise, $V_1(0) = \mathbb{C} Y v$ for some nonzero $v \in V(0)$. Recall the nondegenerate bilinear form $\beta(\cdot, \cdot)$ on $\mathbb{C}^{2l}$ given in Equation (2.2) and note that $\beta|_{V_1 \times V_1}$ is the bilinear form on $V_1 \cong \mathbb{C}^{2l-1}$ defining $K = SO(2l - 1)$ (see Section 2.2). Since $\beta(v_1, v_2) = 0$ unless $v_1 \in V(\mu_1)$ and $v_2 \in V(-\mu_1)$, we may assume that $\beta(Y v, Y v) = 1$. Since $Y \in \mathfrak{z}_\mathfrak{k}(x) \subset \mathfrak{z}_\mathfrak{k}(x_\mathfrak{k})$, $Y$ stabilizes the subspace $V_1(0)$. Since $Y \in \mathfrak{k}$, $\beta(Y^2 v, Y v) = -\beta(Y v, Y^2 v) = -\beta(Y^2 v, Y v)$, and $\beta(Y^2 v, Y v) = 0$. Then $\dim(V_1(0)) = 1$ implies that $Y^2 v = 0$, so that $1 = \beta(Y v, Y v) = -\beta(v, Y^2 v) = 0$. This contradiction establishes the claim. Hence there is $\mu \in \mathbb{C}$ such that $V(\mu)$ and $V_1(\mu)$ are nonzero, and if $\mu = 0$, $\dim(V_1(0)) > 1$. It follows that $\mu \in \sigma(x) \cap \sigma(x_\mathfrak{k})$ (Notation 2.6), so $x \not\in \mathfrak{g}(0)$, completing the proof for type $D$.

If $\mathfrak{g} = \mathfrak{so}(2l + 1)$, we let $V = \mathbb{C}^{2l+1}$ and $V_1 = \text{span}\{e_{\pm 1}, \ldots, e_{\pm l}\}$, $V_2 = \text{span}\{e_0\}$ (see Section 2.1.2). As above, we take $Y$ to be any nonzero element of $\mathfrak{z}_\mathfrak{k}(x)$, and verify that $x$ and $Y$ satisfy the hypotheses of Lemma 4.11 and $x_\mathfrak{k}$ is the element $X_1$ from the lemma.
Let $\beta(\cdot, \cdot)$ be the bilinear form from Equation (2.1). Then by Lemma 4.11 there is a generalized eigenspace $V(\mu)$ such that $Y(V(\mu))$ is a nonzero subspace of $V_1(\mu)$. We claim that if $\dim(V(0)) = 1$, then $Y(V(0)) = 0$. Indeed, if $\dim(V(0)) = 1$, then $V(0)$ is spanned by a nonzero vector $v$ and as above $Y(V(0)) \subseteq V(0)$. Then $\beta(Yv, v) = -\beta(v, Yv) = -\beta(Yv, v)$, so that $Yv = 0$ since $\beta$ is nondegenerate on $V(0)$. Hence, there is $\mu$ such that $Y(V(\mu))$ is a nonzero subspace of $V_1(\mu)$ and either $\mu \neq 0$ or $\mu = 0$ and $\dim(V(0)) \geq 2$. As above, we conclude that $x \not\in \mathfrak{g}(0)$.

Q.E.D.

Let $c = (c_{r_{n-1}}, c_{r_n}) \in \mathbb{C}^{r_{n-1}} \times \mathbb{C}^{r_n}$ and write $c_{r_i} = (c_{i,1}, \ldots, c_{i,r_i}) \in \mathbb{C}^{r_i}$ for $i = n - 1, n$. Let $I_{n,c}$ be the ideal of $\mathbb{C}[\mathfrak{g}]$ generated by the functions $f_{i,j} - c_{i,j}$ for $i = n - 1, n$ and $j = 1, \ldots, r_i$, and note that the zero set of $I_{n,c}$ is $\Phi_i^{-1}(c)$.

**Corollary 4.13.** Let $x \in \mathfrak{g}(0)$, and let $c = \Phi_i^{-1}(x)$.

1. The ideal $I_{n,c}$ is the ideal of functions vanishing on $\Phi_i^{-1}(c)$, and the variety $\Phi_i^{-1}(c)$ is smooth.

2. The fibre $\Phi_i^{-1}(c)$ is a single closed $K$-orbit.

**Proof.** As in Remark 2.4, the fibre $\Phi_i^{-1}(c) \subset \mathfrak{g}(0)$. By Theorem 4.12 every element of the fibre $\Phi_i^{-1}(c)$ is $n$-strongly regular. Hence Theorem 4.5 implies that the differentials $\{df_{i,j}(x) : i = n - 1, n; j = 1, \ldots, r_i\}$ are independent for all $x \in \Phi_i^{-1}(c)$. By Theorem 18.15 (a) of [Eis95], the ideal $I_{n,c}$ is radical, so $I_{n,c}$ is the ideal of $\Phi_i^{-1}(c)$. The smoothness of $\Phi_i^{-1}(c)$ now follows since the differentials of the generators of $I_{n,c}$ are independent at every point of $\Phi_i^{-1}(c)$. For the second assertion, note first that

$$\dim(K) = \dim(\mathfrak{g}) - \dim(\mathfrak{g}/\mathfrak{k}) = \dim(\Phi_i^{-1}(c)),$$

where the first equality follows from Equations (4.6) and (2.4), and the second equality follows from Proposition 3.4 (3). By Lemma 4.4 $\dim(\text{Ad}(K) \cdot x) = \dim(K)$ for all $x \in \Phi_i^{-1}(c)$. By Proposition 3.4 (2), each fibre $\Phi_i^{-1}(c)$ has a unique closed $K$-orbit, which implies the assertion.

Q.E.D.

Consider the Zariski open subvariety of $\mathfrak{g}$:

$$\mathfrak{g}_{\Theta} := \{x \in \mathfrak{g} : \sigma(x_i) \cap \sigma(x_{i+1}) = \emptyset \text{ for } i = 2, \ldots, n - 1\}. \tag{4.12}$$

**Proposition 4.14.** The elements of $\mathfrak{g}_{\Theta}$ are strongly regular.

**Proof.** If $x \in \mathfrak{g}_{\Theta}$, then $x_i \in \mathfrak{g}_{i}(0)$ for $i = 3, \ldots, n$. By Theorem 4.12 $\mathfrak{g}_{i-1}(x_{i-1}) \cap \mathfrak{g}_{i}(x_i) = 0$. The result now follows from Proposition 4.8.

Q.E.D.

**Corollary 4.15.** Let $x \in \mathfrak{g}_{\Theta}$. Then $\Phi^{-1}(\Phi(x)) = \Phi^{-1}(\Phi(x))_{\text{sreg}}$. 
Proof. By Remark 2.7, \( \Phi^{-1}(\Phi(x)) \subset g_\Theta \) for \( x \in g_\Theta \). The result then follows from Proposition 4.14.

Q.E.D.

Remark 4.16. Let \( g = gl(n) \). In Theorem 5.15 of [Col11], the first author proved that \( g_\Theta \subset g_{sreg} \) for the analogously defined set \( g_\Theta \). The methods of this section also prove that \( g_\Theta \subset g_{sreg} \), and our proof is significantly simpler than the proof in [Col11]. To prove Theorem 4.12 for \( gl(n) \) we simply apply Lemma 4.11 with \( g = gl(n) \), \( k = gl(n-1) \), \( V = \mathbb{C}^n \), with \( V_1 = \text{span}\{e_1, \ldots, e_{n-1}\} \) and \( V_2 = \text{span}\{e_n\} \). Proposition 4.14 follows, since the analogue of Proposition 4.8 also holds in this case as we observed in Remark 4.9. Thus, we can construct the strongly regular elements \( g_\Theta \) of \( g \) in both orthogonal and general linear cases using the same framework.

We can now prove one of our main results.

Theorem 4.17. The restriction of the GZ functions \( J_{GZ} \) to a regular adjoint orbit in \( g \) forms a completely integrable system on the orbit.

Proof. Let \( x \in g_{reg} \) and let \( \text{Ad}(G) \cdot x \) be the adjoint orbit containing \( x \). By Proposition 2.10, it suffices to show that Equation (2.13) holds. Let \( \chi : g \to g//G \) be the adjoint quotient. Since the Kostant-Wallach map \( \Phi \) is surjective (Theorem 3.6), there exists \( y \in g_\Theta \) such that \( \chi(y) = \chi(x) \). It follows from Proposition 4.14 that \( y \in g_{sreg} \), whence \( y \in g_{reg} \) by (2.12). Therefore \( y \in \chi^{-1}(\chi(x)) \cap g_{reg} = \text{Ad}(G) \cdot x \), and \( \text{Ad}(G) \cdot x \cap g_{sreg} \neq \emptyset \).

Q.E.D.

We end this section by describing the KW fibre \( \Phi^{-1}(\Phi(x)) \) for \( x \in g_\Theta \). In particular, we generalize Corollary 5.18 of [Col11] to the orthogonal setting using Proposition 4.14 and Corollary 4.13. Our argument below can also be used to give an easier proof of Corollary 5.18 of [Col11] in the general linear case. In the proof of the following theorem, we use repeatedly the easy fact that the projection \( \pi_i : g \to g_i \), \( \pi_i(x) = x_i \), is \( \text{Ad}(G_i) \)-equivariant.

Theorem 4.18. Let \( x \in g_\Theta \), and let \( Z_{G_i}(x_i) \) be the centralizer of \( x_i \in g_i \) in \( G_i \) viewed as a subgroup of \( G \). Then the morphism:

\[
\Psi : \prod_{i=2}^{n-1} Z_{G_i}(x_i) \to \Phi^{-1}(\Phi(x)) \text{ given by } \Psi(z_2, \ldots, z_{n-1}) = \text{Ad}(z_2) \cdots \text{Ad}(z_{n-1}) \cdot x,
\]

\( z_i \in Z_{G_i}(x_i) \), is an isomorphism of non-singular algebraic varieties.

Proof. We first note that the image of \( \Psi \) is contained in the fibre \( \Phi^{-1}(\Phi(x)) \). Indeed, observe that \( \text{Ad}(z_{n-1}) \cdot x \in \Phi^{-1}(\Phi(x)) \), since \( (\text{Ad}(z_{n-1}) \cdot x)_i = x_i \) for all \( i \leq n-1 \). In fact, for any \( i = 2, \ldots, n-1 \) we have:

\[
(\text{Ad}(z_i \cdots z_{n-1}) \cdot x)_i = x_i.
\]
We prove Equation (4.14) by downward induction on $i$, with the base case $i = n - 1$ following from our discussion above. Suppose for any $j$ with $i < j \leq n - 1$, we have $(\text{Ad}(z_j \ldots z_{n-1}) \cdot x)_j = x_j$. Now
\[(\text{Ad}(z_i \ldots z_{n-1}) \cdot x)_i = [(\text{Ad}(z_i \ldots z_{n-1}) \cdot x)_{i+1}]_i.\]

Since $z_i \in G_i \subset G_i+1$, we have
\[(\text{Ad}(z_i \ldots z_{n-1}) \cdot x)_{i+1} = \text{Ad}(z_i) \cdot (\text{Ad}(z_{i+1} \ldots z_{n-1}) \cdot x)_{i+1} = \text{Ad}(z_i) \cdot x_{i+1}\]
by the induction hypothesis. But then it follows from (4.15) that
\[(\text{Ad}(z_i \ldots z_{n-1}) \cdot x)_i = (\text{Ad}(z_i) \cdot x_{i+1})_i = \text{Ad}(z_i) \cdot x_i = x_i,
\]
since $z_i \in Z_{G_i}(x_i)$, yielding (4.14).

Using Equation (4.14), we now show that for any $i = 2, \ldots, n$,
\[(4.16) \quad \chi_i((\text{Ad}(z_2 \ldots z_{n-1}) \cdot x)_i) = \chi_i(x_i),\]
where $\chi_i : g_i \to g_i//G_i$ is the adjoint quotient for $g_i$, and thus $\text{Ad}(z_2 \ldots z_{n-1}) \cdot x \in \Phi^{-1}(\Phi(x))$ by the definition of the KW map in (2.9). We first note that (4.16) is easily seen to be true for $i = n$, since $z_2 \ldots z_{n-1} \in G_{n-1} \subset G$. For $i < n$,
\[(4.17) \quad (\text{Ad}(z_2 \ldots z_{i-1}z_i \ldots z_{n-1}) \cdot x)_i = \text{Ad}(z_2 \ldots z_{i-1}) \cdot (\text{Ad}(z_i \ldots z_{n-1}) \cdot x)_i = \text{Ad}(z_2 \ldots z_{i-1}) \cdot x_i\]
by Equation (4.14). Since $z_2 \ldots z_{i-1} \in G_{i-1} \subset G_i$, Equation (4.17) implies
\[\chi_i((\text{Ad}(z_2 \ldots z_{n-1}) \cdot x)_i) = \chi_i(\text{Ad}(z_2 \ldots z_{i-1}) \cdot x_i) = \chi_i(x_i),\]
yielding (4.16) in this case.

We now claim $\Psi$ is surjective. Suppose $y \in \Phi^{-1}(\Phi(x))$. Let $\Phi(x) = (c_2, \ldots, c_n) \in \mathbb{C}^{r_2} \times \cdots \times \mathbb{C}^{r_n}$. Since $\chi_2$ is bijective, we see that $x_2 = y_2$. Further, $y_3, x_3 \in \Phi_3^{-1}(c_2, c_3)$, where $\Phi_3$ is the partial KW map for $g_3$. Since $x \in g_0$, it follows from definitions that $\Phi_3^{-1}(c_2, c_3) \subset g_3(0)$. By part (2) of Corollary 4.13, $x_3$ are $G_2 = Z_{G_2}(x_2)$-conjugate. Let $z_2 \in Z_{G_2}(x_2)$ be such that $\text{Ad}(z_2) \cdot y_3 = x_3$. Let $y' := \text{Ad}(z_2) \cdot y$, so that $y'_3 = x_3$. Now observe that $y'_4, x_4 \in \Phi_4^{-1}(c_3, c_4) \subset g_4(0)$. Again, by part (2) of Corollary 4.13, $y'_4, x_4$ are $G_3$-conjugate. But since $y'_3 = x_3$, we see that $y'_4, x_4$ are $Z_{G_3}(x_3)$-conjugate. Let $z_3 \in Z_{G_3}(x_3)$ be such that $\text{Ad}(z_3) \cdot y'_4 = x_4$. Let $y'' := \text{Ad}(z_3) \cdot y$. Then we claim $y''_4 = x_4$. Indeed,

\[y''_4 = (\text{Ad}(z_3z_2) \cdot y)_4 = (\text{Ad}(z_3) \cdot y')_4 = \text{Ad}(z_3) \cdot y'_4 = x_4.\]

Continuing, in this fashion we can find $z_2, \ldots, z_{n-1} \in Z_{G_2}(x_2), \ldots, Z_{G_{n-1}}(x_{n-1})$ respectively such that $x = \text{Ad}(z_{n-1} \ldots z_2) \cdot y$. It follows immediately that $\Psi(z_2^1, \ldots, z_{n-1}^1) = y$.

We now show that $\Psi$ is injective. Our main tool will be Lemma 4.10. Suppose we have $z_i, \tilde{z}_i \in Z_{G_i}(x_i)$, for $i = 2, \ldots, n-1$ such that
\[(4.18) \quad \text{Ad}(z_2 \ldots z_{n-1}) \cdot x = \text{Ad}(\tilde{z}_2 \ldots \tilde{z}_{n-1}) \cdot x.\]

Then $(\text{Ad}(z_2 \ldots z_{n-1}) \cdot x)_3 = (\text{Ad}(\tilde{z}_2 \ldots \tilde{z}_{n-1}) \cdot x)_3$, which is equivalent to $\text{Ad}(z_2) \cdot x_3 = \text{Ad}(\tilde{z}_2) \cdot x_3$ by (4.14). It follows that $z_2^{-1}\tilde{z}_2 \in Z_{G_2}(x_2) \cap Z_{G_3}(x_3)$. Since $x \in g_0$, $x \in g_{sreg}$ by Proposition 4.14 so that $x_3 \in (g_3)_{3-sreg}$ by Proposition 4.8. Lemma 4.10 now implies
that \( z_2 = \tilde{z}_2 \). By induction, we may assume that \( z_3 = \tilde{z}_3, z_4 = \tilde{z}_4, \ldots, z_{i-1} = \tilde{z}_{i-1} \), so that Equation (4.18) becomes

\[
\text{Ad}(z_1 \ldots z_{n-1}) \cdot x = \text{Ad}(\tilde{z}_1 \ldots \tilde{z}_{n-1}) \cdot x.
\]

Using (4.17) again, we have \( \text{Ad}(z_i) \cdot x_{i+1} = \text{Ad}(\tilde{z}_i) \cdot x_{i+1} \), yielding \( z_i^{-1} \tilde{z}_i \in Z_{G_i}(x_i) \cap Z_{G_{i+1}}(x_{i+1}) \). Lemma 4.10 again gives that \( z_i = \tilde{z}_i \). By induction \( \Psi \) is injective.

Finally, we show that \( \Phi^{-1}(\Phi(x)) \) is a nonsingular variety. Let \( \Phi(x) = (c_{i,j})_{i=2,\ldots,n} \in \mathbb{C}^{r^2} \times \cdots \times \mathbb{C}^{r^2} \). Let \( I_c \subset \mathbb{C}[\mathfrak{g}] \) be the ideal generated by the functions \( g_{i,j} = f_{i,j} - c_{i,j} \) for \( i = 2, \ldots, n \) and \( j = 1, \ldots, r_i \). By Corollary 4.15 \( \Phi^{-1}(\Phi(x)) \subset \mathfrak{g}_{sreg}^c \). It follows from the definition of strong regularity in Notation-Definition 2.8 that the differentials of the functions \( g_{i,j} \) are independent at any point \( y \in \Phi^{-1}(\Phi(x)) \). Again using Theorem 18.15 of [Lis94], we see that \( y \) is a smooth point of \( \Phi^{-1}(\Phi(x)) \). Thus, \( \Psi : Z_{G_i}(x_2) \times \cdots \times Z_{G_{n-1}}(x_{n-1}) \to \Phi^{-1}(\Phi(x)) \) is a bijective morphism of non-singular varieties. It follows from Zariski’s main theorem that \( \Psi \) is a isomorphism.

Q.E.D.

### 4.4. Generic orbits of the GZ group \( A \) on \( \mathfrak{g} \)

The GZ integrable system in Equation (2.8) integrates to a global, holomorphic action of \( A := \mathbb{C}^d \) on \( \mathfrak{g} \) where

\[
d = \frac{\dim \mathfrak{g} - r_n}{2}
\]

is half the dimension of a regular Ad(\( G \))-orbit on \( \mathfrak{g} \). To see this, one considers the Lie algebra of Hamiltonian GZ vector fields on \( \mathfrak{g} \):

\[
\mathfrak{a}_{GZ} = \text{span}\{\xi_f : f \in J_{GZ}\}.
\]

For \( f \in J_{GZ} \), the Hamiltonian vector field \( \xi_f \) is complete and integrates to a global action of \( \mathbb{C} \) on \( \mathfrak{g} \) (see Theorem 2.4. [Col09]). Since the functions \( J_{GZ} \) Poisson commute, the Lie algebra \( \mathfrak{a}_{GZ} \) is abelian, and therefore the global flows of the vector fields \( \xi_f \) simultaneously integrate to give a holomorphic action of \( A \) on \( \mathfrak{g} \) (see Section 2.3. [Col09] for details). We refer to the group \( A \) as the Gelfand-Zeitlin group (or GZ group) (see [Col11], [CE14]). Since the Lie algebra \( \mathfrak{a}_{GZ} \) is commutative, the GZ functions \( f_{i,j} \in J_{GZ}, i = 2, \ldots, n, j = 1, \ldots, r_i \) are invariant under one another’s Hamiltonian flows, whence the action of \( A \) preserves the fibres of the KW map \( \Phi \). In fact, Theorem 2.14 of [Col09] implies that the irreducible components of the strongly regular fibre \( \Phi^{-1}(\Phi(x))_{sreg} \) are precisely the orbits of \( A \) on the fibre \( \Phi^{-1}(\Phi(x)) \). Theorem 4.18 can be now be used to describe the action of the GZ group \( A \) on the fibres \( \Phi^{-1}(\Phi(x)) \) for \( x \in \mathfrak{g}_\Theta \).

**Theorem 4.19.** Let \( x \in \mathfrak{g}_\Theta \), and let \( Z_{G_i}(x_i)^0 \) be the identity component of the group \( Z_{G_i}(x_i) \). Let

\[
\mathcal{Z} := \prod_{i=2}^{n-1} Z_{G_i}(x_i)^0.
\]
Then the $A$-orbits on $\Phi^{-1}(\Phi(x))$ coincide with orbits of a free, algebraic action of the connected, abelian algebraic group $Z$ on $\Phi^{-1}(\Phi(x))$. In particular, the Lie algebra of $GZ$ vector fields in (4.19) is algebraically integrable on the fibre $\Phi^{-1}(\Phi(x))$.

Proof. By Theorem 4.18 the morphism

\[
\Psi : \prod_{i=2}^{n-1} Z_{G_i}(x_i) \to \Phi^{-1}(\Phi(x)) \text{ given by } \Psi(z_2, \ldots, z_n) = \text{Ad}(z_2) \cdots \text{Ad}(z_{n-1}) \cdot x.
\]

is an isomorphism of varieties (cf. Equation (4.13)). Therefore, we can use $\Psi$ to define an action of $Z$ on $\Phi^{-1}(\Phi(x))$ via

\[
(4.20) \quad (z_2, \ldots, z_n) \cdot y := \Psi((z_2, \ldots, z_n) \circ \Psi^{-1}(y)),
\]

where $\circ$ in the above equation represents multiplication in the group $\prod_{i=2}^{n-1} Z_{G_i}(x_i)$. The action of $Z$ in (4.20) is clearly algebraic and free and its orbits are the irreducible components of $\Phi^{-1}(\Phi(x))$, since $\Psi$ is an isomorphism of varieties. Since $\Phi^{-1}(\Phi(x)) = \Phi^{-1}(\Phi(x))_{sreg}$ by Corollary 4.15, the orbits of $Z$ are then precisely the orbits of the $GZ$ group $A$ on $\Phi^{-1}(\Phi(x))$ by our remarks above.

Q.E.D.

Using Theorem 4.18, we can count the number of irreducible components (and hence the number of $A$-orbits) in $\Phi^{-1}(\Phi(x))$ for $x \in \mathfrak{g}_\Theta$.

Corollary 4.20. Let $\mathfrak{g} = \mathfrak{so}(n)$ with $n > 3$, and let $x \in \mathfrak{g}_\Theta$. Let $m$ be the number of indices $i$, $i = 4, \ldots, n-1$, satisfying the following two conditions:

1. $i$ is even.
2. zero occurs as an eigenvalue of $x_i$ with multiplicity at least 4.

Then $m \in \{0, \ldots, r_{n-1} - 1\}$ and the number of irreducible components of $\Phi^{-1}(\Phi(x))$ is $2^m$.

Further, for any $m \in \{0, \ldots, r_{n-1} - 1\}$, there exists an $x \in \mathfrak{g}_\Theta$ with the property that $\Phi^{-1}(\Phi(x))$ contains $2^m$ irreducible components.

Proof. It follows immediately from the isomorphism in Theorem 4.18 that the number of irreducible components of $\Phi^{-1}(\Phi(x))$ for $x \in \mathfrak{g}_\Theta$ coincides with the number of irreducible (i.e. connected) components of the product of algebraic groups $Z_{G_i}(x_2) \times \cdots \times Z_{G_{n-1}}(x_{n-1})$. We compute the number of components of $Z_{G_i}(x_i)$ by considering the Jordan decomposition of $x_i \in \mathfrak{g}_i$, $x_i = s_i + n_i$ with $s_i$ semisimple and $n_i$ nilpotent. Then $Z_{G_i}(x_i) = Z_{L_i}(n_i)$, where $L_i := Z_{G_i}(s_i)$ is a Levi subgroup of $G_i$. Since $x \in \mathfrak{g}_\Theta \subset \mathfrak{g}_{sreg}$ by Proposition 4.14, $x_i$ is regular for all $i = 2, \ldots, n$ by (2.12), so that $n_i \in L_i = \text{Lie}(L_i)$ is regular nilpotent. Thus, $Z_{L_i}(n_i) \cong Z_i \times U_i$, where $Z_i$ is the centre of $L_i$ and $U_i$ is a unipotent subgroup of $L_i$. Suppose $i > 2$ is even. As we observed in the proof of Lemma 4.10 up to $G_i$-conjugacy the Levi subgroup $L_i$ is either a product of subgroups $\text{GL}(m_j)$ in which case $Z_i$ is connected, or $L$ is a product of $\text{GL}(s_j)$ and precisely one factor of the form $\text{SO}(2k)$ with $k > 1$, in


which case $Z_i$ has exactly two components (see Equations (4.10) and (4.11)). The latter case occurs if and only if zero occurs as an eigenvalue of $s_i$ of multiplicity at least 4. On the other hand, if $i$ is odd, a similar argument shows that $Z_{G_i}(x_i)$ is always a connected, algebraic group. Lastly, if $i = 2$, the group $Z_{G_2}(x_2) = G_2$ is connected.

The upper bound on the value of $m$ follows from an easy calculation that there are precisely $r_n - 1 - 1$ subalgebras of type $D$ in the chain $g_3 \subset \cdots \subset g_{n-1}$. To prove the second statement of the theorem, suppose we are given $m \in \{0, \ldots, r_n - 1\}$. Then we can find $m$ subalgebras $g_{i_1} \subset g_{i_2} \subset \cdots \subset g_{i_m}$ of type $D$ in the chain $g_3 \subset \cdots \subset g_{n-1}$. It follows from the surjectivity of the KW map (Theorem 3.6) that we can find $x \in g_0$ with $x_{i_j}$ regular nilpotent in $g_{i_j}$ for all $j = 1, \ldots, m$. It follows that $\Phi^{-1}(\Phi(x))$ contains $2^m$ irreducible components from the first statement of the theorem.

Q.E.D.

**Remark 4.21.** Theorems 4.18 and 4.19 improve on results from [Col09] and [Col11]. In [Col09], the first author proved Theorem 4.18 for the elements $x$ of $g$ such that $x_i$ is regular semisimple for $i = 2, \ldots, n$ and $x_i$ and $x_{i+1}$ have disjoint spectra for $i = 2, \ldots, n - 1$. In this case, the fibres $\Phi^{-1}(\Phi(x))$ are isomorphic to a product of tori. Our result here is more general and has a simpler proof. In [Col11], the first author proved the analogue of Theorem 4.19 for $g = gl(n)$. Our proof here carries over to $gl(n)$ with minor modifications, and is simpler than the proof in [Col11].

5. The orthogonal KW nilfibre and strongly regular elements

In this section, we show that $\Phi^{-1}(0)_{sreg} = \emptyset$ (see Proposition 5.14). This stands in stark contrast to the situation in the general linear setting where the rich structure of $\Phi^{-1}(0)_{sreg}$ is studied extensively in [CE12]. As with the generic fibres of $\Phi$ studied in the previous sections, our study of $\Phi^{-1}(0)$ begins with studying the nilfibre of the partial KW map $\Phi_n^{-1}(0)$.

5.1. Structure of Partial KW nilfibre. Our goal in this section is to describe the structure of the nilfibre $\Phi_n^{-1}(0)$ of the partial KW map (see Notation 3.3). This will be achieved by degenerating a generic fibre $\Phi_n^{-1}(\Phi_n(x))$ to $\Phi_n^{-1}(0)$ together with the Luna slice theorem.

We recall the basic ingredients of Luna’s slice theorem. Let $M$ be a complex reductive algebraic group acting linearly on a finite dimensional vector space $V$. Let $M \cdot v$ be a closed $M$-orbit in $V$ and note that the stabilizer $M_v$ is reductive. Hence, we may find a representation $\mathfrak{s}$ of $M_v$ such that $V \cong T_v(M \cdot v) \oplus \mathfrak{s}$ as $M_v$-modules. The representation $\mathfrak{s}$ is called the slice representation at $v$. Let $q : \mathfrak{s} \to \mathfrak{s}/M_v$ be the GIT quotient for the slice representation, and let $\mathfrak{N}_{\mathfrak{s}} := q^{-1}(0) = \{ y \in \mathfrak{s} : 0 \in M \cdot y \}$ be the nullcone of the slice representation. Consider the GIT quotient $\pi : V \to V//M$ for the $M$-action on $V$. 
The Luna slice theorem asserts that the $M$-equivariant morphism
\[ M \times_{M_\theta} (v + \mathfrak{n}_\mathfrak{g}) \xrightarrow{\sim} \pi^{-1}(\pi(v)) \] given by \((g, v + n) \mapsto g \cdot (v + n)\)

is an isomorphism, where \(g \in M\) and \(n \in \mathfrak{n}_\mathfrak{g}\) (Theorem 6.6 of [VP89], [Lun73]). In particular, the \(M\)-orbits on \(\pi^{-1}(\pi(v))\) are in one-to-one correspondence with the \(M_\theta\)-orbits in \(\mathfrak{n}_\mathfrak{g}\).

To apply this construction to the \(K = SO(n - 1)\)-action on \(\mathfrak{g} = \mathfrak{so}(n)\), we recall some facts about the involution \(\theta\) in Section 2.2. Suppose that \(\mathfrak{b} \in \mathcal{B}\) is \(\theta\)-stable, i.e., \(\theta(\mathfrak{b}) = \mathfrak{b}\), and let \(\mathfrak{t} \subset \mathfrak{b}\) be a \(\theta\)-stable Cartan subalgebra of \(\mathfrak{g}\). Then the corresponding Borel subgroup \(B \subset G\) with \(\text{Lie}(B) = \mathfrak{b}\) is also \(\theta\)-stable and contains the \(\theta\)-stable Cartan subgroup \(T \subset G\) with \(\text{Lie}(T) = \mathfrak{t}\). It follows from Lemma 5.1 of [Ric82] that \(B \cap K\) is a Borel subgroup of \(K\) and \(T \cap K\) is a Cartan subgroup of \(K\). Hence, \(\mathfrak{b} \cap \mathfrak{t}\) is a Borel subalgebra of \(\mathfrak{t}\) with nilradical \(\mathfrak{n} \cap \mathfrak{t} = [\mathfrak{b} \cap \mathfrak{t}, \mathfrak{b} \cap \mathfrak{t}]\), and \(\mathfrak{b} \cap \mathfrak{t}\) contains the Cartan subalgebra \(\mathfrak{t} \cap \mathfrak{t}\) of \(\mathfrak{t}\).

Recall the notion of a standard Borel subalgebra in Definition 2.3. We will make use of the following notation throughout this section.

**Notation 5.1.** Let \(\mathcal{B}^\theta \subset \mathcal{B}\) denote the set of \(\theta\)-stable Borel subalgebras of \(\mathfrak{g}\). We denote by \(\mathcal{B}^\theta_{\text{std}} \subset \mathcal{B}^\theta\) the subset of standard \(\theta\)-stable Borel subalgebras of \(\mathfrak{g}\). Similarly, we let
\[ \mathcal{N}^\theta := \{ \mathfrak{n} = [\mathfrak{b}, \mathfrak{b}] \text{ with } \mathfrak{b} \in \mathcal{B}^\theta\}, \]

and
\[ \mathcal{N}^\theta_{\text{std}} := \{ \mathfrak{n} = [\mathfrak{b}, \mathfrak{b}] \text{ with } \mathfrak{b} \in \mathcal{B}^\theta_{\text{std}}\} \]

be the collections of the \(\theta\)-stable nilradicals and \(\theta\)-stable, standard nilradicals respectively. Note that the standard Borel subalgebra of upper triangular matrices \(\mathfrak{b}_+\) in \(\mathfrak{g}\) belongs to the set \(\mathcal{B}^\theta_{\text{std}}\).

Recall that \(\mathfrak{h} \subset \mathfrak{g}\) denotes the standard Cartan subalgebra of diagonal matrices and that \(\mathfrak{h}\) is preserved by \(\theta\). Since \(\mathfrak{b}_+ \in \mathcal{B}^\theta_{\text{std}}\), it follows that \(\mathfrak{h} \cap \mathfrak{t}\) is a Cartan subalgebra of \(\mathfrak{t}\) with corresponding Cartan subgroup \(H \cap K\) of \(K\). Let \((\mathfrak{h} \cap \mathfrak{t})_{\text{reg}}\) denote the regular semisimple elements of \(\mathfrak{t}\) in \(\mathfrak{h} \cap \mathfrak{t}\). Then for \(x \in (\mathfrak{h} \cap \mathfrak{t})_{\text{reg}}\), the orbit \(\text{Ad}(K) \cdot x\) is closed in \(\mathfrak{t}\), and thus in \(\mathfrak{g}\), and the isotropy group of \(x\) is the Cartan subgroup \(H \cap K\) of \(K\). Let \(\Phi_\mathfrak{t}\) be the \(\mathfrak{h} \cap \mathfrak{t}\)-roots in \(\mathfrak{t}\). Then we may identify \(T_x(\text{Ad}(K) \cdot x) = [\mathfrak{t}, x] = \oplus_{\alpha \in \Phi_\mathfrak{t}} x_\alpha\). Let \(\mathfrak{s} = \mathfrak{h} + \mathfrak{g}^{-\theta}\), so that \(\mathfrak{s}\) is a \(H \cap K\)-slice to \(\text{Ad}(K) \cdot x\) in \(\mathfrak{g}\). Note that the \(H \cap K\)-nullcone \(\mathfrak{n}_\mathfrak{g} = \mathfrak{n}_\mathfrak{g}^{-\theta}\), since \(H \cap K\) acts trivially on \(\mathfrak{h}\).

We compute \(\mathfrak{n}_\mathfrak{g}^{-\theta}\) by considering type \(B\) and type \(D\) separately. Let \(\mathfrak{g} = \mathfrak{so}(2l + 1)\) and recall from Example 2.1 that \(H \cap K = H\) and \(\mathfrak{g}^{-\theta} = \oplus_{i=1}^l \mathfrak{a}_{\pm \epsilon_i}\). An element of \(H \cap K\) is \(h = \text{diag}(h_1, \ldots, h_l, 1, h_l^{-1}, \ldots, h_1^{-1})\) with \(h_i \in \mathbb{C}^\times\), and note that if \(e_{\pm \epsilon_i}\) a root vector of \(\mathfrak{g}_{\pm \epsilon_i}\), then \(\text{Ad}(h) \cdot e_{\pm \epsilon_i} = h_{\pm 1} e_{\pm \epsilon_i}\). Hence, if \(x = \sum_{i=1}^l \lambda_i e_{\epsilon_i} + \mu_i e_{-\epsilon_i}\) with \(\lambda_i, \mu_i \in \mathbb{C}\), then
\[ \text{Ad}(h) \cdot x = \sum_{i=1}^l \lambda_i h_{\epsilon_i} + \mu_i h_{\epsilon_i}^{-1} e_{-\epsilon_i}. \]
It follows that
\[ 0 \in \overline{\text{Ad}(H)} \cdot x \text{ if and only if } \mu_i \lambda_i = 0 \text{ for all } i = 1, \ldots, l. \]

Now let \( \mathfrak{g} = \mathfrak{so}(2l) \), and note that since \( \theta(\epsilon_l) = -\epsilon_l \) by Example 2.1,
\[ \mathfrak{g}^{-\theta} = \mathfrak{h}^{-\theta} \oplus \bigoplus_{i=1}^{l-1} (\mathfrak{g}_{\epsilon_i - \epsilon_l} \oplus \mathfrak{g}_{\epsilon_i + \epsilon_l})^{-\theta} \oplus \bigoplus_{i=1}^{l-1} (\mathfrak{g}_{-(\epsilon_i - \epsilon_l)} \oplus \mathfrak{g}_{-(\epsilon_i + \epsilon_l)})^{-\theta}. \]

Let \( f_{\pm i} \) be a root vector of \( \mathfrak{g}_{\pm(\epsilon_i - \epsilon_l)} \) for \( i = 1, \ldots, l - 1 \), so that \( \mathfrak{g}^{-\theta} = \mathfrak{h}^{-\theta} + \sum_{i=1}^{l-1} \mathcal{C}(f_{\pm i} - \theta(f_{\pm i})) \). Elements of \( H \cap K \) are of the form \( h = \text{diag}[h_1, \ldots, h_{l-1}, 1, h_{l-1}^{-1}, \ldots, h_1^{-1}] \subset H \) with \( h_i \in \mathbb{C}^\times \). Then \( \text{Ad}(h) \cdot (f_{\pm i} - \theta(f_{\pm i})) = h_i^{\pm 1}(f_{\pm i} - \theta(f_{\pm i})) \). Thus, if
\[ x = x_{\mathfrak{h}^{-\theta}} + \sum_{i=1}^{l-1} \lambda_i (f_i - \theta(f_i)) + \mu_i (f_{-i} - \theta(f_{-i})), \]
with \( x_{\mathfrak{h}^{-\theta}} \in \mathfrak{h}^{-\theta} \), then
\[ \text{Ad}(h) \cdot x = x_{\mathfrak{h}^{-\theta}} + \sum_{i=1}^{l-1} h_i \lambda_i (f_i - \theta(f_i)) + h_i^{-1} \mu_i (f_{-i} - \theta(f_{-i})), \]
since \( H \cap K \) acts trivially on \( \mathfrak{h}^{-\theta} \). It follows that
\[ 0 \in \overline{\text{Ad}(H \cap K)} \cdot x \text{ if and only if } \mu_i \lambda_i = 0 \text{ for all } i = 1, \ldots, l - 1, \text{ and } x_{\mathfrak{h}^{-\theta}} = 0. \]

To describe the irreducible components of \( \mathfrak{N}_{\mathfrak{g}^{-\theta}} \), introduce symbols \( U \) and \( L \) for upper and lower. For a \( r_{n-1} \)-tuple of symbols \( (i_1, \ldots, i_{r_{n-1}}) \) where \( i_j = U \) or \( i_j = L \) for \( j = 1, \ldots, r_{n-1} \), define
\[ C_{i_1, \ldots, i_{r_{n-1}}} := \{ x \in \mathfrak{g}^{-\theta} : \mu_j = 0 \text{ if } i_j = U, \lambda_j = 0 \text{ if } i_j = L, \text{ and } x_{\mathfrak{h}^{-\theta}} = 0 \text{ if } \mathfrak{g} = \mathfrak{so}(2l) \}. \]

We have proved:

**Lemma 5.2.** The irreducible components of \( \mathfrak{N}_{\mathfrak{g}^{-\theta}} \) are the \( 2^{|r_{n-1}|} \) varieties \( C_{i_1, \ldots, i_{r_{n-1}}} \cong \mathbb{C}^{r_{n-1}} \). They are indexed by the choices of \( r_{n-1} \)-tuples \( (i_1, \ldots, i_{r_{n-1}}) \), with \( i_j = U \) or \( i_j = L \) for \( j = 1, \ldots, r_{n-1} \).

We now give a Lie-theoretic description of the components \( C_{i_1, \ldots, i_{r_{n-1}}} \).

**Lemma 5.3.** Let \( \mathfrak{b} \in \mathcal{B}^\theta \) with nilradical \( \mathfrak{n} \). Then \( \dim \mathfrak{n}^{-\theta} = r_{n-1} = \text{rank}(\mathfrak{k}) \).

**Proof.** Since \( \mathfrak{b} \in \mathcal{B}^\theta \), \( \mathfrak{n} \in \mathcal{N}^\theta \), and therefore
\[ \dim \mathfrak{n} = \dim \mathfrak{n}^{-\theta} + \dim(\mathfrak{n} \cap \mathfrak{k}). \]

By our discussion above, \( \mathfrak{n} \cap \mathfrak{k} \) is the nilradical of the Borel subalgebra \( \mathfrak{b} \cap \mathfrak{k} \) of \( \mathfrak{k} \), so by Equation (2.3),
\[ \dim \mathfrak{n} = \frac{1}{2}(\dim \mathfrak{g} - r_n) = \sum_{i=2}^{n-1} r_i, \text{ and } \dim(\mathfrak{n} \cap \mathfrak{k}) = \sum_{i=2}^{n-2} r_i. \]
Thus, Equation (5.4) becomes

$$\dim n^{-\theta} = \dim n - \dim n \cap \mathfrak{k} = \sum_{i=2}^{n-1} r_i - \sum_{i=2}^{n-2} r_i = r_{n-1}.$$ 

Q.E.D.

**Proposition 5.4.** Let $\mathfrak{M}_{\mathfrak{g},-\theta}$ be the nullcone for the $H \cap K$-action on $\mathfrak{g}^{\theta}$, and let $C_{i_1,\ldots,i_{r_n-1}}$ be an irreducible component of $\mathfrak{M}_{\mathfrak{g},-\theta}$. Then there exists $n \in N_{\text{std}}^\theta$ such that $C_{i_1,\ldots,i_{r_n-1}} = n^{-\theta}$.

Conversely, if $n \in N_{\text{std}}^\theta$, then $n^{-\theta}$ is an irreducible component of $\mathfrak{M}_{\mathfrak{g},-\theta}$ and therefore $n^{-\theta} = C_{i_1,\ldots,i_{r_n-1}}$ for some choice of indices $i_j = U, L$ as in (5.3).

**Proof.** We prove the proposition when $\mathfrak{g} = \mathfrak{so}(2l)$ is type $D$. The case where $\mathfrak{g}$ is type $B$ is similar and left to the reader. Suppose $C_{i_1,\ldots,i_{r_n-1}} \subset \mathfrak{M}_{\mathfrak{g},-\theta}$ is an irreducible component. Then there exists an $x \in (\mathfrak{h} \cap \mathfrak{k})_{\text{reg}}$ with $x \in \mathfrak{h}_{\text{reg}}$ such that $\text{ad}(x)$ acts on $C_{i_1,\ldots,i_{r_n-1}}$ with positive weights. Indeed, using Equation (5.2), we let $x$ be the element $\text{diag}[x_1, \ldots, x_{l-1}, 0, 0, -x_{l-1}, \ldots, -x_1]$ with $x_j \in \mathbb{Z}$ for $j = 1, \ldots, l - 1$, and satisfying $x_j > 0$ if $i_j = U$, $x_j < 0$ if $i_j = L$, and $x_j \neq \pm x_i$ for $j \neq i$, and note that $x$ has the desired properties. For $k \in \mathbb{Z}^+$, let $\mathfrak{g}(k)$ be the $k$-eigenspace for the action of $\text{ad}(x)$. Since $x \in \mathfrak{k}$, the Lie subalgebra $\mathfrak{b} := \mathfrak{h} \oplus \bigoplus_{k \in \mathbb{Z}^+} \mathfrak{g}(k)$ is $\theta$-stable. Further, since $x$ is regular semisimple in $\mathfrak{g}$, $\mathfrak{b}$ is a Borel subalgebra, with nilradical $\mathfrak{n} = \bigoplus_{k \in \mathbb{Z}^+} \mathfrak{g}(k)$ (Lemma 3.84 of [CM93]). By construction $C_{i_1,\ldots,i_{r_n-1}} \subset \mathfrak{n}^{-\theta}$, and $n \in N_{\text{std}}^\theta$. By Lemmas 5.2 and 5.3, $\dim C_{i_1,\ldots,i_{r_n-1}} = \dim n^{-\theta} = r_{n-1}$, so that $C_{i_1,\ldots,i_{r_n-1}} = n^{-\theta}$.

For the converse, let $n \in N_{\text{std}}^\theta$ with $n = [b, b], b \in B_{\text{std}}^\theta$. Then the Cartan subgroup $H \cap K$ of the Borel subgroup $B \cap K$ acts on $n$ and on $n^{-\theta}$ with positive weights. Hence, $n^{-\theta} \subset \mathfrak{M}_{\mathfrak{g},-\theta}$. Further, $n^{-\theta}$ is a closed, irreducible subvariety of $\mathfrak{M}_{\mathfrak{g},-\theta}$ of dimension $\dim n^{-\theta} = r_{n-1}$ by Lemma 5.3. It follows that $n^{-\theta}$ is an irreducible component of $\mathfrak{M}_{\mathfrak{g},-\theta}$ and hence $n^{-\theta}$ is of the form $C_{i_1,\ldots,i_{r_n-1}}$ by Lemma 5.2.

Q.E.D.

Given Proposition 5.4, we define an equivalence relation on $N_{\text{std}}^\theta$ by:

(5.5) $n \equiv n' \iff n^{-\theta} = (n')^{-\theta}$.

Let $\mathfrak{S}^{-\theta} := N_{\text{std}}^\theta/\equiv$ be the set of equivalence classes of $N_{\text{std}}^\theta$ modulo the relation in (5.5). We shall denote the equivalence class of $n \in N_{\text{std}}^\theta$ by $[n] \in \mathfrak{S}^{-\theta}$.

**Corollary 5.5.** The cardinality of the set of equivalence classes $\mathfrak{S}^{-\theta}$ is $2^{r_{n-1}}$.

**Proof.** The result follows by combining Lemma 5.2 with Proposition 5.4.

Q.E.D.
We now use the Luna slice theorem and Proposition 5.4 to completely describe the fibres \( \Phi_n^{-1}(\Phi_n(x)) \) for \( x \in (\mathfrak{h} \cap \mathfrak{t})_{\text{reg}} \).

**Theorem 5.6.** Let \( x \in (\mathfrak{h} \cap \mathfrak{t})_{\text{reg}} \). The irreducible component decomposition of the fibre \( \Phi_n^{-1}(\Phi_n(x)) \) is

\[
\Phi_n^{-1}(\Phi_n(x)) = \bigcup_{[n] \in S^{-\theta}} \text{Ad}(K) \cdot (x + n^{-\theta}),
\]

Further, each irreducible component \( \text{Ad}(K) \cdot (x + n^{-\theta}) \) is smooth, and there are exactly \( 2^{n-1} \) irreducible components in \( \Phi_n^{-1}(\Phi_n(x)) \).

**Proof.** By Part (2) of Proposition 3.4, the partial KW map \( \Phi_n \) is a GIT quotient for \( K \)-action on \( \mathfrak{g} \). Therefore Equation (5.1) implies that we have a \( K \)-equivariant isomorphism

\[
K \times_{H \cap K} (x + \mathfrak{n}_{\mathfrak{g}^{-\theta}}) \cong \Phi_n^{-1}(\Phi_n(x))
\]

given by \( (k, x + n) \mapsto \text{Ad}(k) \cdot (x + n) \), where \( k \in K \) and \( n \in \mathfrak{n}_{\mathfrak{g}^{-\theta}} \). Proposition 5.4 implies that the irreducible component decomposition of the fibre bundle \( K \times_{H \cap K} (x + \mathfrak{n}_{\mathfrak{g}^{-\theta}}) \) is:

\[
K \times_{H \cap K} (x + \mathfrak{n}_{\mathfrak{g}^{-\theta}}) = \bigcup_{[n] \in S^{-\theta}} K \times_{H \cap K} (x + n^{-\theta}).
\]

Equation (5.6) now follows from (5.7) and (5.8). The smoothness of the varieties \( \text{Ad}(K) \cdot (x + n^{-\theta}) \) with \( n \in \mathfrak{n}_{\mathfrak{g}^{-\theta}} \) follows from (5.7) and the fact that the fibre bundles \( K \times_{H \cap K} (x + n^{-\theta}) \) are smooth. The final statement of the theorem follows from (5.7), (5.8), and Corollary 5.5.

Q.E.D.

**Lemma 5.7.** Let \( x \in (\mathfrak{h} \cap \mathfrak{t})_{\text{reg}} \) and \( n \in \mathfrak{N}^0_{\text{std}} \). Then

\[
\text{Ad}(K) \cdot (x + n^{-\theta}) = \text{Ad}(K) \cdot (x + n).
\]

**Proof.** Let \( b \in B^0_{\text{std}} \) with \( n = [b, b] \). Since \( b \in B^0_{\text{std}} \), \( b \cap \mathfrak{t} \) is a Borel subalgebra of \( \mathfrak{t} \) with nilradical \( \mathfrak{n} \cap \mathfrak{t} \) and corresponding Borel subgroup \( B \cap K \) of \( K \). Since \( x \in (\mathfrak{h} \cap \mathfrak{t})_{\text{reg}} \), then \( \text{Ad}(B \cap K) \cdot x = x + n \in \mathfrak{t} \) by Lemma 3.1.44 of [CG97]. Hence, since \( n^{-\theta} \) is \( B \cap K \)-stable, \( \text{Ad}(K) \cdot (x + n) = \text{Ad}(K) \cdot (x + n \cap \mathfrak{t} + n^{-\theta}) = \text{Ad}(K) \cdot (\text{Ad}(B \cap K) \cdot x + n^{-\theta}) = \text{Ad}(K) \cdot (x + n^{-\theta}) \).

Q.E.D.

Combining Lemma 5.7 with Equations (5.6) and (5.9), we can decompose the fibre \( \Phi_n^{-1}(\Phi_n(x)) \), \( x \in (\mathfrak{h} \cap \mathfrak{t})_{\text{reg}} \) into irreducible subvarieties as:

\[
\Phi_n^{-1}(\Phi_n(x)) = \bigcup_{[n] \in S^{-\theta}} \text{Ad}(K) \cdot (x + n).
\]

The following result on closed \( K \)-orbits in \( \mathcal{B} = B_{\mathfrak{g}} \) is well-known (figure 4.3 of [Col85]). The reader can find a proof in Section 2.7 of [CE].
Proposition 5.8. (1) Let $\mathfrak{g} = \mathfrak{so}(2l + 1)$, let $\mathfrak{b}_+$ be the upper triangular matrices in $\mathfrak{g}$, let $\mathfrak{b}_- := \text{Ad}(s_{\alpha_1}) \cdot \mathfrak{b}_+$, where $s_{\alpha_1}$ is a representative of the simple reflection $s_{\alpha_1}$, and let $Q_{\mathfrak{b}_+}$ and $Q_{\mathfrak{b}_-}$ be the $K$-orbits on $\mathcal{B}$ containing $\mathfrak{b}_+$ and $\mathfrak{b}_-$ respectively. Then the flag variety $\mathcal{B}$ has two closed $K$-orbits which are $Q_{\mathfrak{b}_+}$ and $Q_{\mathfrak{b}_-}$.

(2) Let $\mathfrak{g} = \mathfrak{so}(2l)$, and let $\mathfrak{b}_+$ be the set of upper triangular matrices in $\mathfrak{g}$. Then $Q_{\mathfrak{b}_+}$ is the only closed $K$-orbit.

Remark 5.9. It is well-known that $\mathfrak{b} \in \mathcal{B}^0$ if and only if the $K$-orbit $Q_\mathfrak{b}$ of $\mathfrak{b}$ in $\mathcal{B}$ is closed in $\mathcal{B}$ (see Proposition 4.12 of [CE14]).

Lemma 5.10. Let $\mathfrak{n}, \mathfrak{n'} \in \mathcal{N}^0_{\text{std}}$, and let $\mathfrak{b}, \mathfrak{b}' \in \mathcal{B}^0_{\text{std}}$, with $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ and $\mathfrak{n'} = [\mathfrak{b}', \mathfrak{b}]$.

(1) Then $\text{Ad}(K) \cdot \mathfrak{n} = \text{Ad}(K) \cdot \mathfrak{n'}$ if and only if $Q_\mathfrak{b} = Q_\mathfrak{b'}$.

(2) If $Q_\mathfrak{b} \neq Q_\mathfrak{b'}$, then $[\mathfrak{n}] \neq [\mathfrak{n'}]$ in $\mathcal{G}^{-\theta}$.

Proof. The sufficiency of (1) is clear. For the necessity, suppose that $\text{Ad}(K) \cdot \mathfrak{n} = \text{Ad}(K) \cdot \mathfrak{n'}$, and let $e \in \mathfrak{n}$ be principal nilpotent. Then $e = \text{Ad}(k) \cdot e'$ for some $k \in K$, and $e' \in \mathfrak{n}'$ principal nilpotent. Thus, $e \in \mathfrak{b} \cap \text{Ad}(k) \cdot \mathfrak{b'}$. But since $e$ is principal nilpotent, it is contained in a unique Borel subalgebra (Proposition 3.2.14 of [CG97]), forcing $\mathfrak{b} = \text{Ad}(k) \cdot \mathfrak{b'}$. Thus, $Q_\mathfrak{b} = Q_\mathfrak{b'}$.

To prove (2), suppose that $[\mathfrak{n}] = [\mathfrak{n'}] \in \mathcal{G}^{-\theta}$. Then by Lemma 5.7 we have $\text{Ad}(K) \cdot (x + \mathfrak{n}) = \text{Ad}(K) \cdot (x + \mathfrak{n'})$ for any $x \in (\mathfrak{h} \cap \mathfrak{k})_{\text{reg}}$. In particular, $\text{Ad}(K) \cdot (\lambda x + \mathfrak{n}) = \text{Ad}(K) \cdot (\lambda x + \mathfrak{n'})$ for any $\lambda \in \mathbb{C}^\times$. Taking the limit as $\lambda \to 0$, we obtain $\text{Ad}(K) \cdot \mathfrak{n} = \text{Ad}(K) \cdot \mathfrak{n'}$, and the assertion in (2) now follows from (1).

Q.E.D.

The following theorem determines the nilfibre.

Theorem 5.11. Let $\Phi_\mathfrak{n} : \mathfrak{g} \to \mathbb{C}^{r_{\mathfrak{n}-1}} \times \mathbb{C}^{r_{\mathfrak{n}}}$ be the partial KW map (Equation (3.7)).

If $\mathfrak{g} = \mathfrak{so}(2l)$, then $\Phi_\mathfrak{n}^{-1}(0) = \text{Ad}(K) \cdot \mathfrak{n}_+$ is irreducible, where $\mathfrak{n}_+ = [\mathfrak{b}_+, \mathfrak{b}_+]$.

If $\mathfrak{g} = \mathfrak{so}(2l + 1)$, then $\Phi_\mathfrak{n}^{-1}(0) = \text{Ad}(K) \cdot \mathfrak{n}_+ \cup \text{Ad}(K) \cdot \mathfrak{n}_-$ has two irreducible components, where $\mathfrak{n}_\pm = [\mathfrak{b}_\pm, \mathfrak{b}_\pm]$.

Proof. We first prove that $\Phi_\mathfrak{n}^{-1}(0) = \bigcup_{[\mathfrak{n}] \in \mathcal{G}^{-\theta}} \text{Ad}(K) \cdot (\lambda x + \mathfrak{n})$. Let $x \in (\mathfrak{h} \cap \mathfrak{k})_{\text{reg}}$, and $\lambda \in \mathbb{C}^\times$. Then by Equation (5.10),

$$
\Phi_\mathfrak{n}^{-1}(\Phi_\mathfrak{n}(\lambda x)) = \bigcup_{[\mathfrak{n}] \in \mathcal{G}^{-\theta}} \text{Ad}(K) \cdot (\lambda x + \mathfrak{n}).
$$

Now consider $\lim_{\lambda \to 0} \Phi_\mathfrak{n}^{-1}(\Phi_\mathfrak{n}(\lambda x))$. Since $\Phi_\mathfrak{n}$ is flat by Part (3) of Proposition 3.31, it follows from Theorem VIII.4.1 of [Gro03] that $\lim_{\lambda \to 0} \Phi_\mathfrak{n}^{-1}(\Phi_\mathfrak{n}(\lambda x)) = \Phi_\mathfrak{n}^{-1}\left(\lim_{\lambda \to 0} \Phi_\mathfrak{n}(\lambda x)\right)$. Therefore,

$$
\lim_{\lambda \to 0} \Phi_\mathfrak{n}^{-1}(\Phi_\mathfrak{n}(\lambda x)) = \Phi_\mathfrak{n}^{-1}\left(\lim_{\lambda \to 0} \Phi_\mathfrak{n}(\lambda x)\right) = \Phi_\mathfrak{n}^{-1}(\Phi_\mathfrak{n}(0)) = \Phi_\mathfrak{n}^{-1}(0).
$$

Q.E.D.
On the other hand,

\[ \lim_{\lambda \to 0} \bigcup_{[n] \in \mathcal{S}^{-\theta}} \text{Ad}(K) \cdot (\lambda x + n) = \bigcup_{[n] \in \mathcal{S}^{-\theta}} \text{Ad}(K) \cdot (\lim_{\lambda \to 0} \lambda x + n) = \bigcup_{[n] \in \mathcal{S}^{-\theta}} \text{Ad}(K) \cdot n. \]

By (5.12) and (5.13), we obtain

\[ \Phi_n^{-1}(0) = \bigcup_{[n] \in \mathcal{S}^{-\theta}} \text{Ad}(K) \cdot n, \]

Since each \( n \in \mathcal{N}_{\text{std}}^{\theta} \) is stabilized by a Borel subgroup \( B \cap K \) of \( K \), the set \( \text{Ad}(K) \cdot n \) is closed by Lemma 39.2.1 of [TY05]. Since each \( \text{Ad}(K) \cdot n \) is evidently irreducible, the distinct \( \text{Ad}(K) \cdot n \) form the irreducible components of \( \Phi_n^{-1}(0) \). The theorem now follows from Remark 5.9, Lemma 5.10, and the classification of the closed \( K \)-orbits on \( B \) given in Proposition 5.8.

Q.E.D.

Remark 5.12. For \( \mathfrak{g} = \mathfrak{gl}(n) \) an analogous description of the nilfibre is proven in Proposition 3.10 of [CE15]. The reasoning used here gives a simpler determination of the nilfibre than in [CE15].

Remark 5.13. In upcoming work, we use the Luna slice theorem and Theorem 5.11 to describe arbitrary fibres of the partial KW map in terms of closed \( K \)-orbits on \( B \) and closed \( K \)-orbits on certain partial flag varieties. We determine the \( x \in \mathfrak{g} \) such that \( \Phi^{-1}(\Phi(x))_{\text{sreg}} \neq \emptyset \) and develop geometric descriptions of \( \Phi^{-1}(\Phi(x))_{\text{sreg}} \) using the theory of \( K \)-orbits on \( B \). This is in the same spirit as the main results of [CE12].

5.2. The orthogonal KW nilfibre. We now use Theorem 5.11 to show that \( \Phi^{-1}(0)_{\text{sreg}} \) is empty.

Proposition 5.14. Let \( \mathfrak{g} = \mathfrak{so}(n) \), with \( n > 3 \), and let \( \Phi : \mathfrak{g} \to \mathbb{C}^{r_2} \times \cdots \times \mathbb{C}^{r_n} \) be the KW map. Then \( \Phi^{-1}(0)_{\text{sreg}} = \emptyset \).

To show Proposition 5.14, we first observe that the nilfibre of the partial KW map \( \Phi^{-1}(0) \) has no \( n \)-strongly regular elements. The key observation is the following proposition, which can be viewed as an extension of Proposition 3.8 in [CE12].

Proposition 5.15. Let \( n > 3 \), let \( \mathfrak{g} = \mathfrak{so}(n) \), and let \( K = \text{SO}(n - 1) \). Let \( Q \subset B \) be a closed \( K \)-orbit, and let \( b \in Q \), with nilradical \( n \). Then

\[ \mathfrak{z}_\mathfrak{g}(n \cap \mathfrak{t}) \cap \mathfrak{z}_\mathfrak{g}(n) \neq 0. \]

Proof. By \( K \)-equivariance, it suffices to show (5.15) for a representative \( b \) of the closed \( K \)-orbit \( Q \). By Proposition 5.8 we can assume that the standard diagonal Cartan subalgebra \( \mathfrak{h} \) is in \( b \). Let \( \phi \in \Phi^+(\mathfrak{g}, \mathfrak{h}) \) be the highest root of \( b \). We claim for \( n > 4 \) that \( \phi \) is compact imaginary. It then follows that the root space

\[ \mathfrak{g}_\phi \subset \mathfrak{z}_\mathfrak{g}(n \cap \mathfrak{t}) \cap \mathfrak{z}_\mathfrak{g}(n), \]
and the result follows. For the claim, suppose first that \( g = \mathfrak{so}(2l) \). By Part (2) of Proposition 5.8, we can assume that \( b = b_+ \). The highest root is then \( \epsilon_1 + \epsilon_2 \), which is compact imaginary for \( l > 2 \) (Example 2.1). If \( g = \mathfrak{so}(2l + 1) \), then by Part (1) of Proposition 5.8, we can assume that \( b = b_+ \) or \( b = b_- = \text{Ad}(s_{\alpha_i}) \cdot b_+ \). For both \( b_+ \) and \( b_- \), the highest root is \( \epsilon_1 + \epsilon_2 \), which is compact imaginary (Example 2.1).

If \( g = \mathfrak{so}(4) \), then \( \phi = \epsilon_1 + \epsilon_2 \) is complex \( \theta \)-stable. Since \( n \) is abelian in this case, and \( n \) is \( \theta \)-stable by Remark 5.9, (\( g_{\phi} \oplus g_{\theta(\phi)} \)) \( \cap \mathfrak{z}(n) \cap \mathfrak{j}(n) \).

Q.E.D.

**Corollary 5.16.** Let \( n > 3 \), and let \( \Phi_n : g \rightarrow \mathbb{C}^{r_{n-1}} \oplus \mathbb{C}^{r_n} \) be the partial Kostant-Wallach map. Then \( \Phi_n^{-1}(0) \) contains no \( n \)-strongly regular elements.

**Proof.** Suppose \( x \in \Phi_n^{-1}(0) \), so by Theorem 5.11 \( x \) is contained in \( n \), the nilradical of a Borel subalgebra \( b \) with \( Q_b \subset B \) closed. By Proposition 5.15, there is a nonzero element \( y \) of \( J_{\mathfrak{t}}(n \cap \mathfrak{t}) \cap \mathfrak{j}(n) \). By Remark 5.9, \( x \in n \cap \mathfrak{t} \), so that \( y \in J_{\mathfrak{t}}(n) \cap \mathfrak{j}(n) \), and therefore \( x \) is not \( n \)-strongly regular.

Q.E.D.

**Remark 5.17.** The assertion of the corollary is false for \( n = 3 \). In this case, \( \mathfrak{so}(3) \cong \mathfrak{sl}(2) \) and \( \mathfrak{t} = \mathfrak{h} \subset \mathfrak{sl}(2) \), where \( \mathfrak{h} \) is the standard Cartan subalgebra of \( \mathfrak{sl}(2) \). Further, the KW map for \( \mathfrak{so}(3) \cong \mathfrak{sl}(2) \) coincides with the partial KW map for \( \mathfrak{so}(3) \cong \mathfrak{sl}(2) \). In this case, it follows by Proposition 3.11 from [CE15] that each irreducible component of \( \Phi_n^{-1}(0) \) contains strongly regular elements.

**Proof of Proposition 5.14** It follows from Proposition 4.8 that \( x \in \mathfrak{g}_{\text{reg}} \) if and only if \( x_i \in (\mathfrak{g}_i)_{\text{reg}} \) for all \( i \). Thus, if \( x \in \Phi_{-1}(0)_{\text{reg}} \), then \( x_i \in \Phi_{i-1}(0)_{\text{reg}} \), for all \( i = 3, \ldots, n \) where \( \Phi_i : \mathfrak{g}_i \rightarrow \mathbb{C}^{r_{i-1}} \times \mathbb{C}^{r_i} \) is the partial KW map for \( \mathfrak{g}_i \). But Corollary 5.16 implies that \( \Phi_i^{-1}(0)_{\text{reg}} = \emptyset \) for all \( i = 4, \ldots, n \), and therefore \( \Phi_{-1}(0)_{\text{reg}} = \emptyset \).

Q.E.D.

Let \( I_{\text{GZ}} \) be the ideal of \( \mathbb{C}[\mathfrak{g}] \) generated by the GZ functions \( J_{\text{GZ}} \) in Equation (2.8). We can use Proposition 5.14 to determine when the ideal \( I_{\text{GZ}} \) is radical.

**Corollary 5.18.** Let \( g = \mathfrak{so}(n) \). Then the ideal \( I_{\text{GZ}} \) is radical if and only if \( n = 3 \).

**Proof.** By Theorem 18.15(a) of [Eis95], the ideal \( I_{\text{GZ}} \) is radical if and only if the set of differentials \( \{ df(x) : f \in J_{\text{GZ}} \} \) is linearly independent on an open, dense subset of each irreducible component of \( \Phi^{-1}(0) \). It follows from Definition-Notation 2.8 that \( I_{\text{GZ}} \) is radical if and only if each irreducible component of \( \Phi^{-1}(0) \) contains strongly regular elements. But it follows from Proposition 5.14 and the case of \( \mathfrak{so}(3) \) in Remark 5.17 that each irreducible component of \( \Phi^{-1}(0) \) contains strongly regular elements if and only if \( n = 3 \).
Using Proposition 5.14, we can see that there is no orthogonal analogue of the Hessenberg matrices, which play an important role for \( \mathfrak{gl}(n) \) (Equation (2.14)).

**Corollary 5.19.** Let \( \mathfrak{g} = \mathfrak{so}(n) \) with \( n > 3 \). There is no subvariety \( \mathfrak{X} \subset \mathfrak{g} \) such that the restriction of the KW map to \( \mathfrak{X} \) is an isomorphism:

\[
\Phi : \mathfrak{X} \rightarrow \mathbb{C}^{r_2} \times \cdots \times \mathbb{C}^{r_n}.
\]

**Proof.** The existence of such a subvariety \( \mathfrak{X} \) would imply that every fibre of the KW map contained strongly regular elements, contradicting Proposition 5.14.

Q.E.D.

**Remark 5.20.** In upcoming work, we show that the KW map \( \Phi \) is flat in both the general linear and orthogonal cases, so that \( \Phi^{-1}(0) \) is an equidimensional variety of dimension

\[
\dim \Phi^{-1}(0) = \dim \mathfrak{g} - \sum_{i=2}^{n} r_i = \frac{\dim \mathfrak{g} - r_n}{2} = \dim n
\]

using Equation (2.3). This was previously known only for \( \mathfrak{gl}(n) \) by work of Ovsienko [Ovs03, FO05]. We use this result to study the category of Gelfand-Zeitlin modules for the enveloping algebra \( U(\mathfrak{g}) \) of \( \mathfrak{g} \) studied by Futorny, Ovsienko, and others [DFO94, FO14]. This enables us to extend known results for Gelfand-Zeitlin modules for \( \mathfrak{gl}(n) \) to the orthogonal case.

Q.E.D.

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