On the Manhattan pinball problem

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Abstract

We consider the periodic Manhattan lattice with alternating orientations going north-south and east-west. Place obstructions on vertices independently with probability $0 < p < 1$. A particle is moving on the edges with unit speed following the orientation of the lattice and it will turn only when encountering an obstruction. The problem is that for which value of $p$ is the trajectory of the particle closed almost surely. We prove this for $p > \frac{1}{2} - \varepsilon$ with some $\varepsilon > 0$.

1 Introduction

We consider the Manhattan pinball problem which was previously proposed and analysed in [BOC03]. See also [Spe12] and [Car10]. There are equivalent ways to state the problem formally and we state it by using bond percolation. Consider the $\mathbb{Z}^2$ lattice embedded into the plane $\mathbb{R}^2$. Denote the tilted lattice by

$$\tilde{\mathbb{Z}}^2 = \left\{ \left( x + \frac{1}{2}, y + \frac{1}{2} \right) : x, y \text{ are integers and } x - y \text{ is even} \right\},$$

and for any $a, b \in \tilde{\mathbb{Z}}^2$, $a$ and $b$ are connected by an edge if and only if $|a - b| = \sqrt{2}$. Here, $|\cdot|$ is the Euclidean distance on $\mathbb{R}^2$. See Figure 1 for an illustration. Given $0 < p < 1$, we consider the Bernoulli bond percolation on $\tilde{\mathbb{Z}}^2$. We declare each edge of $\mathbb{Z}^2$ to be closed with probability $p$ and open with probability $1 - p$, independent of all other edges. We use $\omega$ to denote a configuration of open and closed edges. We place a two-sided plane mirror on each closed edge. Suppose a ray of light starts from the origin and initially moves towards east. When the light reaches an open edge, it passes through without deflection. When the light reaches a closed edge, it is deflected through a right angle by the mirror on the edge. Let $L(\omega)$ denote the trajectory of the light.

Our main result is the following

**Theorem 1.1.** Let $\mathcal{E}_n$ denote the event that the diameter of $L(\omega)$ is at most $n$. Then there exists $\varepsilon_0 > 0$ such that following holds. For each $p > \frac{1}{2} - \varepsilon_0$, there are $\alpha, c > 0$ such that for $n > \alpha$, we have

$$\Pr_p[\mathcal{E}_n] \geq 1 - \exp(-cn). \quad (1)$$

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The black dots in the picture illustrate the lattice $\tilde{\mathbb{Z}}^2$. The dashed lines are edges of $\tilde{\mathbb{Z}}^2$. The arrows indicate the orientation of the Manhattan lattice.

The left figure illustrates the configuration $G_0$ where solid black edges are closed and dashed edges are open. In the right hand side, after placing mirrors on closed edges of $G_0$, the light comes from the west of the red edge will travel in the blue path before leaving.

Theorem 1.1 implies that almost surely, the trajectory of the light is finite when $p > \frac{1}{2} - \varepsilon_0$. This result is previously known to hold for $p > \frac{1}{2}$ by using bond percolation, see e.g. \[BOC03\], \[Spe12\]. It is suggested by physicists that the same happens for any $p \in (0, 1)$, see e.g. \[BOC03\], \[Car10\].

2 Enhancement

The proof of Theorem 1.1 uses an enhancement procedure which we define now. Denote by $G_0$ the finite configuration illustrated in the left hand side of Figure 2 and its caption. Given any configuration $\omega$, we enhance it by the following procedure. For any translated copy of $G_0$ which appears in $\omega$, we close the open edge which coincides with the red edge (illustrated in Figure 3). We denote the
Figure 3: The picture contains a doubly-infinite closed path if and only if the enhancement is activated so that the red edge is turned to be closed. This means the enhancement is essential.

configuration after this procedure by \( \tilde{\omega} \).

The procedure above is a special case of the general enhancement (see e.g. [Gri99 Section 3.3]). Intuitively, we have added closed edges to \( \omega \) in a systematic way and we expect that the resulting configuration \( \tilde{\omega} \) is more “percolative” than \( \omega \). This is the content of [Gri99 Theorem (3.16)]. For [Gri99 Theorem (3.16)] to hold, the enhancement is required to be essential (a concept defined in [Gri99 Page 64]) which is true for our case (see Figure 3 and its caption).

Remark 2.1. There are other choices of the enhancement \( G_0 \) for our proof of Theorem 1.1. We will see in Section 3 that, our choice of \( G_0 \) makes the proof easier because of an important feature contained in Lemma 2.2 below.

The following lemma can be directly checked from Figure 3.

Lemma 2.2. Suppose \( \omega \) is a configuration and \( G_0, \overrightarrow{G_0} \) are two translated copies of \( G_0 \) that appear in \( \omega \). Then any open edge in \( G_0 \) does not coincide with the red edge in \( G_0 \).

Denote

\[
Q_n = \left\{ (x, y) \in \mathbb{Z}^2 : |x + y - 1| \leq n \quad \text{and} \quad |x - y| \leq n \right\}
\]

to be the tilted box centered at \( \left( \frac{1}{2}, \frac{1}{2} \right) \). For any edge \( e \) of \( \mathbb{Z}^2 \), we say \( e \) is inside \( Q_n \) if \( e \) connects two vertices in \( Q_n \), otherwise we say it is outside \( Q_n \). The following proposition follows from the proof of [Gri99 Theorem (3.16)] (more precisely, the argument near [Gri99 Equation (3.8)]) and the fact that the enhancement we constructed is essential.

**Proposition 2.3** (Theorem (3.16) in [Gri99]). There exists \( \varepsilon_1 > 0 \) such that, for each \( p > \frac{1}{2} - \varepsilon_1 \),

\[
P_p[\{ \omega : \tilde{\omega} \in A_n \}] \geq P_{\frac{1}{2} + \varepsilon_1}[\{ \omega \in A_n \}]
\]

(2)
for large enough \( n \). Here, \( A_n \) denotes the event that there exists a path of closed edges joining \((\frac{1}{2}, \frac{1}{2})\) to some vertex in \( \mathbb{Z}^2 \setminus Q_n \).

In fact, by directly adapting the proof of [Gri99, Theorem (3.16)], we can substitute \( A_n \) in Proposition 2.3 by \( A'_n \) which denotes the crossing event in a \( 4n \times n \) tilted rectangle. Thus we have

**Proposition 2.4.** There exists \( \varepsilon_1 > 0 \) such that, for each \( p > \frac{1}{2} - \varepsilon_1 \),

\[
P_p\left(\left\{ \omega : \tilde{\omega} \in A'_n \right\}\right) \geq P_{\frac{1}{2} + \varepsilon_1}\left(\left\{ \omega \in A'_n \right\}\right)
\]

for large enough \( n \). Here, \( A'_n \) denotes the event that there is a path of closed edges in the tilted rectangle

\[
T_n = \left\{ (x, y) \in \mathbb{Z}^2 : 1 \leq x + y - 1 \leq n, |x - y| \leq 2n \right\}
\]

joining some vertex on northwest side \( \{(x, y) \in T_n : x - y = -2n\} \) to some vertex on southeast side \( \{(x, y) \in T_n : x - y = 2n\} \). See Figure 4 for an illustration.

The following well-known lemma states that in the supercritical phase (i.e. \( p > \frac{1}{2} \)), the crossing event \( A'_n \) happens with high probability.

**Lemma 2.5.** For any \( p > \frac{1}{2} \), there is constant \( c > 0 \) such that

\[
P_p\left(\left\{ \omega \in A'_n \right\}\right) \geq 1 - \exp(-cn)
\]

for large enough \( n \).

**Proof.** Follow the argument prior to [Gri99, Lemma (11.22)].

Together with Proposition 2.4 and Lemma 2.5 we have

**Proposition 2.6.** There exists \( \varepsilon_1 > 0 \) such that, for each \( p > \frac{1}{2} - \varepsilon_1 \),

\[
P_p\left(\left\{ \omega : \tilde{\omega} \in A''_n \right\}\right) \geq 1 - \exp(-c_p n).
\]

for a constant \( c_p > 0 \) and large enough \( n \). Here, \( A''_n \) denotes the event that \( Q_n \) lies in the interior of a circuit which consists of closed edges in \( Q_{2n} \).
Figure 5: The four green lines are four closed paths that cross tilted rectangles $T^{(i)}(i = 1, 2, 3, 4)$. If such paths exist, then $Q_n$ (the yellow region) lies in the interior of a closed circuit.

Proof. Consider the four tilted rectangles

1. $T^{(1)} = \{(x, y) \in \mathbb{Z}^2 : n + 1 \leq x + y - 1 \leq 2n, |x - y| \leq 2n\},$
2. $T^{(2)} = \{(x, y) \in \mathbb{Z}^2 : -2n \leq x + y - 1 \leq -n - 1, |x - y| \leq 2n\},$
3. $T^{(3)} = \{(x, y) \in \mathbb{Z}^2 : n + 1 \leq x - y \leq 2n, |x + y - 1| \leq 2n\},$
4. $T^{(4)} = \{(x, y) \in \mathbb{Z}^2 : -2n \leq x - y \leq -n - 1, |x + y - 1| \leq 2n\}.$

See Figure 5 for an illustration. If these four tilted rectangles are crossed in the ‘long direction’ by closed paths as indicated in Figure 5, then $Q_n$ lies in the interior of a closed circuit. By symmetry, each of these four crossing events has the same probability as $A_n'$. Thus our proposition follows from Proposition 2.4 and Lemma 2.5.

3 Proof of Theorem 1.1

Proof of Theorem 1.1. By Proposition 2.6 it suffices to prove $A''_n \subseteq E_{2n+10}$ for $n > 100$. Thus we suppose $\omega \in A''_n$. Let configuration $\omega_0$ be obtained by $\omega_0 = \omega$ inside $Q_{100}$ and $\omega_0 = \tilde{\omega}$ outside $Q_{100}$. Recall that $L(\omega_0)$ is the trajectory of the light under the configuration $\omega_0$. By definition of $A''_n$ and $n > 100$, the closed circuit gives rise to an enclosure of mirrors surrounding the origin and thus traps the light. Hence, $L(\omega_0)$ is contained inside $Q_{2n}$.

Note that, for each open edge $e$ in $\omega$, $e$ will be closed in the configuration $\omega_0$ if and only if

1. $e$ is outside $Q_{100},$
2. $e$ is at the position of the red edge in a translated copy of $G_0$. 

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Figure 6: $L(\omega)$ is obtained from $L(\omega_0)$ (the pink path) by attaching translated copies of the blue path (or its inverse). If $L(\omega_0)$ is inside $Q_{2n}$, then $L(\omega)$ is inside $Q_{2n+10}$.

By Lemma 2.2, if $G_0$ is a translated copy of $G_0$ which appears in $\omega$, then only the red edge will be closed under the enhancement and other edges in $G_0$ will remain unchanged. We denote the set of edges (outside $Q_{100}$) which are closed under the enhancement by $E$. We now start from $\omega_0$ and open edges in $E$ to reach the initial configuration $\omega$ and we keep track of the trajectory. Only those edges in $E$ which intersects with $L(\omega_0)$ will affect the trajectory by adding translated copies of the blue path (or its inverse, see Figure 2) to $L(\omega_0)$. Here, we used the fact that the edges in $E$ are outside $Q_{100}$ and thus if the light comes to some $e \in E$, it must come from west or south (otherwise, $L(\omega_0)$ is a translated copy of the blue loop and is outside $Q_{90}$ which contradicts with the fact that $L(\omega_0)$ starts from the origin). Hence, $L(\omega)$ is “trapped” inside $Q_{2n+10}$ (see Figure 6) and our theorem follows.

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References

[BOC03] E. Beamond, A. Owczarek, and J. Cardy. Quantum and classical localization and the Manhattan lattice. *J. Phys. A*, 36(41):10251, 2003.

[Car10] J. Cardy. Quantum network models and classical localization problems. *Int. J. Mod. Phys. B*, 24(12n13):1989–2014, 2010.

[Gri99] G. Grimmett. *Percolation*, volume 321. Springer Berlin Heidelberg, 2nd edition, 1999.

[Spe12] T. Spencer. Duality, statistical mechanics and random matrices. In *Current Developments in Mathematics*, pages 229–260. International Press, Somerville, 2012.