Fully localised three-dimensional gravity-capillary solitary waves on water of infinite depth

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Abstract

Fully localised solitary waves are travelling-wave solutions of the three-dimensional gravity-capillary water wave problem which decay to zero in every horizontal spatial direction. Their existence for water of finite depth has recently been established, and in this article we present an existence theory for water of infinite depth. The governing equations are reduced to a perturbation of the two-dimensional nonlinear Schrödinger equation, which admits a family of localised solutions. Two of these solutions are symmetric in both horizontal directions and an application of a suitable version of the implicit-function theorem shows that they persist under perturbations.

1 Introduction

Three-dimensional gravity-capillary water waves on the surface of a body of water of infinite depth are described by the Euler equations in a domain bounded above by a free surface \( \{ y = \eta(x, z, t) \} \), where the function \( \eta \) depends upon the two horizontal spatial directions \( x, z \) and time \( t \). In terms of an Eulerian velocity potential \( \varphi \) and in dimensionless coordinates, the mathematical problem is to solve Laplace’s equation

\[
\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0, \quad -\infty < y < \eta,
\]

with boundary conditions

\[
\varphi_y \rightarrow 0, \quad y \rightarrow -\infty, \quad \eta_t = \varphi_y - \eta_x \varphi_x - \eta_z \varphi_z, \quad y = \eta,
\]

and

\[
\varphi_t = -\frac{1}{2} (\varphi^2_x + \varphi^2_y + \varphi^2_z) - \eta + \left[ \frac{\eta_x}{\sqrt{1 + \eta_x^2 + \eta_y^2}} \right]_x + \left[ \frac{\eta_z}{\sqrt{1 + \eta_z^2 + \eta_y^2}} \right]_z, \quad y = \eta.
\]

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In this article we consider fully localised solitary waves, that is nontrivial travelling-wave solutions to (1.1)–(1.4) of the form
\[ \eta(x, z, t) = \eta(x - ct, z), \varphi(x, y, z, t) = \varphi(x - ct, y, z) \]
(so that the waves move with unchanging shape and constant speed c from left to right) with \( \eta(x - ct, z) \to 0 \) as \( |(x - ct, z)| \to \infty \) (so that the waves decay in every horizontal direction).

**Theorem 1.1.** Suppose that \( c^2 = 2(1 - \varepsilon^2) \). For each sufficiently small value of \( \varepsilon > 0 \) there exist two solitary-wave solutions of (1.1)–(1.4) for which \( \eta \in H^3(\mathbb{R}^2) \) is symmetric in \( x \) and \( z \) and given by
\[ \eta(x, z) = \pm \varepsilon \zeta_0(x, z) \cos x + o(\varepsilon) \]
uniformly over \( (x, z) \in \mathbb{R}^2 \), where \( \zeta_0 \) is the unique symmetric, positive (real) solution of the two-dimensional nonlinear Schrödinger equation
\[ -\frac{1}{2} \zeta_{xx} - \zeta_{zz} + \zeta = \frac{11}{16} |\zeta|^2 \zeta = 0. \] (1.5)

This result confirms the prediction made on the basis of model equations (see below) and numerical computations by Parau, Vanden-Broeck & Cooker [11] (see Figure 1 for sketches of typical free surfaces in their simulations). Qualitative properties of (two- and three-dimensional) solitary waves on deep water have been discussed by Wheeler [15].

![Figure 1: Sketch of a symmetric fully localised solitary wave of elevation (left) and depression (right); the arrow shows the direction of wave propagation.](image)

We proceed by formulating the water-wave problem (1.1)–(1.4) in terms of the variables \( \eta \) and \( \Phi = \varphi|_{y=\eta} \) (see Zakharov [16] and Craig & Sulem [5]). The Zakharov-Craig-Sulem formulation of the water-wave problem is
\[ \eta_t - G(\eta) \Phi = 0, \]
\[ \Phi_t + \eta + \frac{1}{2} \Phi_x^2 + \frac{1}{2} \Phi_z^2 - \frac{(G(\eta) \Phi + \eta_x \Phi_x + \eta_z \Phi_z)^2}{2(1 + \eta_x^2 + \eta_z^2)} - \left[ \frac{\eta_x}{\sqrt{1 + \eta_x^2 + \eta_z^2}} \right]_x - \left[ \frac{\eta_z}{\sqrt{1 + \eta_x^2 + \eta_z^2}} \right]_z = 0, \]
where \( G(\eta) \Phi = \varphi_y - \eta_x \varphi_x - \eta_z \varphi_z \mid_{y=\eta} \) and \( \varphi \) is the (unique) solution of the boundary-value problem
\[ \varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0, \quad y < \eta, \]
\[ \varphi_y \to 0, \quad y \to -\infty, \]
\[ \varphi = \Phi, \quad y = \eta. \]
Travelling waves are solutions of the form \( \eta(x, z, t) = \eta(x - ct, z) \), \( \Phi(x, z, t) = \Phi(x - ct, z) \); they satisfy

\[
-c\eta_x - G(\eta)\Phi = 0, \tag{1.6}
\]

\[
-c\Phi_x + \eta + \frac{1}{2}\Phi_x^2 + \frac{1}{2}\Phi_z^2
- \frac{(G(\eta)\Phi + \eta_x\Phi_x + \eta_z\Phi_z)^2}{2(1 + \eta_x^2 + \eta_z^2)}
- \left[ \frac{\eta_x}{\sqrt{1 + \eta_x^2 + \eta_z^2}} \right]_x
- \left[ \frac{\eta_z}{\sqrt{1 + \eta_x^2 + \eta_z^2}} \right]_z
= 0. \tag{1.7}
\]

Equations (1.6), (1.7) can be reduced to a single equation for \( \eta \). Using (1.6), one finds that \( \Phi = -cG(\eta)^{-1}\eta_x \), and inserting this formula into (1.7) yields the equation

\[
K'(\eta) - c^2L'(\eta) = 0, \tag{1.8}
\]

where

\[
K'(\eta) = \eta - \left[ \frac{\eta_x}{\sqrt{1 + \eta_x^2 + \eta_z^2}} \right]_x
- \left[ \frac{\eta_z}{\sqrt{1 + \eta_x^2 + \eta_z^2}} \right]_z, \tag{1.9}
\]

\[
L'(\eta) = -\frac{1}{2}(K(\eta)\eta)^2 - \frac{1}{2}(L(\eta)\eta)^2
+ \frac{(\eta_x - \eta_xK(\eta)\eta - \eta_zL(\eta)\eta)^2}{2(1 + \eta_x^2 + \eta_z^2)} + K(\eta)\eta \tag{1.10}
\]

and

\[
K(\eta)\xi = -(G(\eta)^{-1}\eta_x)_x, \quad L(\eta)\xi = -(G(\eta)^{-1}\eta_x)_z.
\]

Note the equivalent definitions

\[
K(\eta)\xi = -(\varphi|_{y=\eta})_x, \quad L(\eta)\xi = -(\varphi|_{y=\eta})_z, \tag{1.11}
\]

where \( \varphi \) is the solution of the boundary-value problem

\[
\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0, \quad y < \eta, \tag{1.12}
\]

\[
\varphi_y \to 0, \quad y \to -\infty, \tag{1.13}
\]

\[
\varphi_y - \eta_x\varphi_x - \eta_z\varphi_z = \xi_x, \quad y = \eta \tag{1.14}
\]

(which is unique up to an additive constant); the operators \( K \) and \( L \) are studied in Section 2 below. Although this fact is not used in the present paper, let us note that (1.8) is in fact the Euler-Lagrange equation for the functional

\[
\mathcal{J}(\eta) := K(\eta) - c^2L(\eta),
\]

where

\[
K(\eta) = \int_{\mathbb{R}^2} \left( \frac{1}{2}\eta^2 + \beta\sqrt{1 + \eta_x^2 + \eta_z^2} - \beta \right) \, dx \, dz,
\]

\[
L(\eta) = \frac{1}{2} \int_{\mathbb{R}^2} \eta K(\eta)\eta \, dx \, dz;
\]

the functions \( K' \) and \( L' \) are the \( L^2(\mathbb{R}^2) \)-gradients of respectively \( K \) and \( L \) (see Buffoni et al. [2,3]). Finally, observe that equation (1.8) is invariant under the reflections \( \eta(x, z) \mapsto \eta(-x, z) \).
and $\eta(x, z) \mapsto \eta(x, -z)$; a solution which is invariant under these transformations is termed symmetric.

It is instructive to review the formal derivation of the nonlinear Schrödinger equation for travelling waves (see Ablowitz & Segur [1, §2.2]), beginning with sinusoidal wave trains. The linearised version of (1.8) admits a solution of the form

$$\eta(x, z, t) = A \cos k_1(x - ct)$$

whenever $c > 0$ and $k_1 \geq 0$ satisfy the linear dispersion relation

$$c^2 = k_1 + \frac{1}{k_1}$$

(see Figure 2); note that the function $s \mapsto c(k_1)$, $k_1 \geq 0$ has a unique global minimum $c_{\text{min}} = \sqrt{2}$ at $k_1 = 1$. Bifurcations of nonlinear solitary waves are expected whenever the linear group and phase speeds are equal, so that $c'(k_1) = 0$ (see Dias & Kharif [6, §3]). We therefore expect the existence of small-amplitude solitary waves with speed near $\sqrt{2}$; the waves bifurcate from a linear sinusoidal wave train with unit wavenumber. Substituting $c^2 = 2(1 - \varepsilon^2)$ and the Ansatz

$$\eta(x, z) = \frac{1}{2}\varepsilon(A_1(X, Z)e^{ix} + A_1(X, Z)e^{-ix}) + \varepsilon^2 A_0(X, Z) + \frac{1}{2}\varepsilon^2(A_2(X, Z)e^{2ix} + A_2(X, Z)e^{-2ix}) + \cdots ,$$

where $X = \varepsilon x$, $Z = \varepsilon z$, into equation (1.8), one finds that $A_1$ satisfies the stationary nonlinear Schrödinger equation (1.5). This equation has a unique symmetric, positive (real) solution $\zeta_0 \in S(\mathbb{R}^2)$ which is characterised as the ground state of the functional $\tilde{J} : H^1(\mathbb{R}^2) \to \mathbb{C}$ with

$$\tilde{J}(\zeta) = \int_{\mathbb{R}^2} \left( \frac{1}{4} |\zeta_x|^2 + \frac{1}{2} |\zeta_z|^2 + \frac{1}{2} |\zeta|^2 - \frac{11}{64} |\zeta|^4 \right) \, dx \, dz$$

(see Sulem & Sulem [13, §4.2] and the references therein).

The above Ansatz suggests that the Fourier transform of a fully localised solitary wave is concentrated near the points $(1, 0)$ and $(-1, 0)$. We therefore decompose $\eta$ into the sum of functions $\eta_1$ and $\eta_2$ whose Fourier transforms $\hat{\eta}_1$ and $\hat{\eta}_2$ are supported in the region $B =$
The support of \( \hat{\eta}_1 \) is contained in the set \( B = B_\delta(1, 0) \cup B_\delta(-1, 0) \) (with \( \delta \in (0, \frac{1}{5}) \)) and its complement (see Figure 3). (The Fourier transform \( \hat{u} = \mathcal{F}[u] \) is defined by the formula

\[
\hat{u}(k) = \frac{1}{2\pi} \int_{\mathbb{R}^2} u(x, z)e^{-i(k_1 x + k_3 z)} \, dx \, dz, \quad k = (k_1, k_3),
\]

and we use the notation \( m(D) \) with \( D = (-i\partial_x, -i\partial_z) \) for the Fourier multiplier-operator with symbol \( m \), so that \( m(D)u = \mathcal{F}^{-1}[m \hat{u}] \); in particular \( \eta_1 = \chi(D)\eta_1, \eta_2 = (1 - \chi(D))\eta_1 \), where \( \chi \) is the characteristic function of the set \( B \).) Writing \( c_2^2 = 2(1 - \varepsilon^2) \) and decomposing (1.8) into

\[
\chi(D)(K'(\eta_1 + \eta_2) - 2(1 - \varepsilon^2)L'(\eta_1 + \eta_2)) = 0,
\]

\[
(1 - \chi(D))(K'(\eta_1 + \eta_2) - 2(1 - \varepsilon^2)L'(\eta_1 + \eta_2)) = 0,
\]

one finds that the second equation can be solved for \( \eta_2 \) as a function of \( \eta_1 \) for sufficiently small values of \( \varepsilon \); substituting \( \eta_2 = \eta_2(\eta_1) \) into the first yields the reduced equation

\[
\chi(D)(K'(\eta_1 + \eta_2(\eta_1)) - 2(1 - \varepsilon^2)L'(\eta_1 + \eta_2(\eta_1))) = 0
\]

for \( \eta_1 \). Finally, the scaling

\[
\eta_1(x, z) = \frac{1}{2}\varepsilon\zeta(X, Z)e^{i \xi} + \frac{1}{2}\varepsilon\zeta(X, Z)e^{-i \xi}
\]

transforms the reduced equation into a perturbation of the equation

\[
\varepsilon^{-2}g(e + \varepsilon D)\zeta + 2f(e + \varepsilon D)\zeta - \frac{11}{8}|\zeta|^2\zeta = 0,
\]

where \( e = (1, 0) \) and

\[
g(s) = 1 + |s|^2 - 2f(s), \quad f(s) = \frac{s^2}{|s|}
\]

(see Sections 3 and 4; the reduced equation is stated precisely in equation (4.2)). Equation (1.16) is termed a full-dispersion version of the stationary nonlinear Schrödinger equation (1.5) since it retains the linear part of the original equation (1.8); noting that

\[
\varepsilon^{-2}g(e + \varepsilon k) + 2f(e + \varepsilon k) = 2 + k_1^2 + 2k_3^2 + O(\varepsilon),
\]

we obtain the fully reduced model equation in the formal limit \( \varepsilon = 0 \) (see Obrecht & Saut [10] for a discussion of related full-dispersion model equations for three-dimensional water waves).
The existence theory is completed in Section 5, where we exploit the fact that the reduction procedure preserves the invariance of equation (1.8) under \( \eta(x, z) \mapsto -\eta(x, z) \) and \( \eta(x, z) \mapsto \eta(x, -z) \), so that equation (1.16) is invariant under the reflections \( \zeta(x, z) \mapsto -\zeta(x, z) \) and \( \zeta(x, z) \mapsto \zeta(x, -z) \). We demonstrate that the reduced equation for \( \zeta \) has two symmetric solutions \( \zeta_{\pm} \) which satisfy \( \zeta_{\pm} \to \pm \zeta_0 \) in \( H^1(\mathbb{R}^2) \) as \( \epsilon \to 0 \). The key step is a nondegeneracy result for the solution \( \zeta_0 \) of (1.5) (see Weinstein [14], Kwong [9] and Chang et al. [4]) in a symmetric setting which allows one to apply a suitable version of the implicit-function theorem. A similar method was recently used by Stefanov & Wright [12] to establish the existence of solitary-wave solutions to the Whitham equation (a full-dispersion Korteweg-de Vries equation).

The scaling (1.15) implies that our waves have small amplitude but finite energy. When splitting our basic function space \( \mathcal{X} = H^3(\mathbb{R}^2) \) into two parts \( \mathcal{X}_1 = \chi(D)\mathcal{X}, \mathcal{X}_2 = (1 - \chi(D))\mathcal{X} \) for \( \eta_1 \) and \( \eta_2 \), we respect this scaling by equipping \( \mathcal{X}_1 \) with the scaled norm \( \| \cdot \| \) defined by

\[
\| \eta_1 \|^2 := \int_{\mathbb{R}^2} (1 + \epsilon^{-2}((|k_1| - 1)^2 + k_3^2))|\hat{\eta}_1(k)|^2 \, dk_1 \, dk_3 = \| \zeta_0 \|^2_{H^1(\mathbb{R}^2)}
\]

and taking \( \zeta \) in a ball \( B_R(0) \subseteq H^1(\mathbb{R}^2) \) which is large enough to contain \( \zeta_0 \); solving the equation for \( \eta_2 \) yields the estimate

\[
\| \eta_2(\eta_1) \|_{H^3(\mathbb{R}^2)} \lesssim \epsilon^\theta \| \eta_1 \|^2,
\]

where \( \theta \) is a fixed number in the interval \((0, 1)\). Equation (1.17) shows that our waves have finite \( H^3(\mathbb{R}^2) \)-norm, while the estimates

\[
\| \eta_1 \|_{\infty} \lesssim \epsilon^\theta \| \eta_1 \|, \quad \| \eta_2 \|_{\infty} \lesssim \| \eta_2 \|_{H^3(\mathbb{R}^2)},
\]

shows that they have small amplitude.

Our result complements recent existence theories for fully localised gravity-capillary solitary waves on water of finite depth (Groves & Sun [7] and Buffoni et al. [2, 3]), which also confirm predictions made by model equations, namely the KP-I equation for ‘strong’ and Davey-Stewartson equation for ‘weak’ surface tension (see Ablowitz & Segur [1]). In particular, Buffoni et al. [3] present a variational counterpart of the theory in the present paper by reducing a classical variational principle for fully localised solitary waves to a locally equivalent variational principle featuring a perturbation of the functional associated with the Davey-Stewartson equation. A nontrivial critical point of the reduced functional is found by showing that an appropriate direct method for the Davey-Stewartson functional (minimisation over its natural constraint set) is robust under perturbation. This variational method is also applicable here, allowing one to reduce the functional \( \mathcal{J} \) to a perturbation of the functional \( \tilde{\mathcal{J}} \). The present method however has the advantages of being more explicit and yielding two distinct families of fully localised solitary waves.
2 Analyticity

In this section we show that the operators $K$, $L$ given by (1.11) and hence $K'$ and $L'$ given by (1.9), (1.10) are analytic at the origin in suitable function spaces (see Corollaries 2.2 and 2.3 below).

The boundary-value problem (1.12)–(1.14) is handled using the change of variable

$$y' = y - \eta(x, z), \quad u(x, y', z) = \varphi(x, y, z),$$

which maps $\Sigma_\eta = \{(x, y, z) : x, z \in \mathbb{R}, -\infty < y < \eta(x, z)\}$ to the lower half-space $\Sigma = \mathbb{R} \times (-\infty, 0) \times \mathbb{R}$. Dropping the primes, one finds that (1.12)–(1.14) are transformed into

$$u_{xx} + u_{yy} + u_{zz} = \partial_x F_1(\eta, u) + \partial_y F_2(\eta, u) + \partial_z F_3(\eta, u), \quad y < 0,$$

$$u_y \to 0, \quad y \to -\infty,$$

$$u_y = F_2(\eta, u) + \xi_x, \quad y = 0,$$

where

$$F_1(\eta, u) = \eta_x u_y, \quad F_2(\eta, u) = \eta_x u_x + \eta_z u_z - (\eta_x^2 + \eta_z^2) u_y, \quad F_3(\eta, u) = \eta_z u_y$$

and $K(\eta) \xi = -u_x|_{y=0}$, $L(\eta) \xi = -u_z|_{y=0}$. We study this boundary-value problem in the space

$$Z = \{ \eta \in S'(\mathbb{R}^2) : \| \|_Z := \| \hat{\eta}_1 \|_{L^1(\mathbb{R}^2)} + \| \hat{\eta}_2 \|_3 < \infty \}$$

for $\eta$ and $H^3_\nu(\Sigma)$ for $u$, in which $H^\nu_{n+1}(\Sigma)$, $n \in \mathbb{N}$, is the completion of

$$S(\Sigma, \mathbb{R}) = \{ u \in C^\infty(\Sigma) : |(x, z)|^m \| \partial_x^{\alpha_1} \partial_y^{\alpha_2} \partial_z^{\alpha_3} u \| \text{ is bounded for all } m, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{N}_0 \}$$

with respect to the norm

$$\| u \|_{n+1, s}^2 := \| u_x \|_n^2 + \| u_y \|_n^2 + \| u_z \|_n^2$$

and $\| \cdot \|_s$ denotes the usual norm for the standard Sobolev space $H^s(\mathbb{R}^2)$ or $H^s(\Sigma)$.

**Lemma 2.1.** For each $\xi \in H^{5/2}(\mathbb{R}^2)$ and sufficiently small $\eta \in Z$ the boundary-value problem (2.1)–(2.3) admits a unique solution $u \in H^2_\nu(\Sigma)$. Furthermore, the mapping $\eta \mapsto (\xi \mapsto u)$ defines a function $Z \to L(H^{5/2}(\mathbb{R}^2), H^3_\nu(\Sigma))$ which is analytic at the origin.

**Proof.** First note that for each $F_1$, $F_2$, $F_3 \in H^2(\Sigma)$ and $\xi \in H^{5/2}(\mathbb{R}^2)$ the boundary-value problem

$$u_{xx} + u_{yy} + u_{zz} = \partial_x F_1 + \partial_y F_2 + \partial_z F_3, \quad y < 0,$$

$$u_y \to 0, \quad y \to -\infty,$$

$$u_y = F_2(\eta, u) + \xi_x, \quad y = 0,$$

admits a unique solution $u = S(F_1, F_2, F_3, \xi)$ in $H^3_\nu(\Sigma)$ whose gradient is obtained from the explicit formula

$$S(F_1, F_2, F_3, \xi) = \mathcal{F}^{-1} \left[ \int_{-\infty}^0 \left( \frac{-ik_1}{2|k|} \hat{F}_1 - \frac{ik_3}{2|k|} \hat{F}_3 + \frac{1}{2} \text{sgn}(y - \tilde{y}) \hat{F}_2 \right) e^{-|k||y-\tilde{y}|} d\tilde{y} \right. $$

$$+ \left. \int_{-\infty}^0 \left( \frac{-ik_1}{2|k|} \hat{F}_1 - \frac{ik_3}{2|k|} \hat{F}_3 + \frac{1}{2} \hat{F}_2 \right) e^{|k|(y+\tilde{y})} d\tilde{y} + \frac{ik_1}{|k|} \hat{\xi} e^{k|y|} \right]$$
(with a slight abuse of notation), so that
\[ \|S(F_1, F_2, F_3, \xi)\|_{3,*} \lesssim \|F_1\|_2 + \|F_2\|_2 + \|F_3\|_2 + \|\xi\|_{5/2}. \]

Define
\[ T : H^3_*(\Sigma) \times \mathcal{Z} \times H^{5/2}(\mathbb{R}^2) \rightarrow H^3_*(\Sigma) \]
by
\[ T(u, \eta, \xi) = u - S(F_1(\eta, u), F_2(\eta, u), F_3(\eta, u), \xi) \]
and note that the solutions of (2.1)–(2.3) are precisely the zeros of \( T(\cdot, \eta, \xi) \). Using the estimates
\[
\|\eta_j^2 w\|_2 \lesssim \|\eta_j^2|_{\infty} \|w\|_2 + \|\eta_j^3 \| \|w\|_2 \\
\lesssim (\|\tilde{\eta}_1\|_{L^1(\mathbb{R}^2)} + \|\eta_j\|_{\infty}) \|w\|_2 \\
= \|\eta\|_{\mathcal{Z}} \|w\|_2, \quad j = 1, 2,
\]
and similarly
\[
\|\eta_j^2 w\|_2 \lesssim \|\eta\|_{\mathcal{Z}} \|w\|_2, \quad j = 1, 2
\]
(where we have used the fact that \( \tilde{\eta}_1 \) has compact support), we find that the mappings \( H^3_*(\Sigma) \times \mathcal{Z} \rightarrow H^2(\Sigma) \) given by \((\eta, u) \mapsto F_j(\eta, u), j = 1, 2, 3\), are analytic at the origin; it follows that \( T \) is also analytic at the origin. Furthermore \( T(0, 0, 0) = 0 \) and \( d_1 T[0, 0, 0] = I \) is an isomorphism. By the analytic implicit-function theorem there exist open neighbourhoods \( V_1 \) and \( V_2 \) of the origin in \( \mathcal{Z} \) and \( H^{5/2}(\mathbb{R}^2) \) and an analytic function \( v : V_1 \times V_2 \rightarrow H^3_*(\Sigma) \) such that
\[ T(v(\eta, \xi), \eta, \xi) = 0. \]
Since \( v \) is linear in \( \xi \) one can take \( V_2 \) to be the entire space \( H^{5/2}(\mathbb{R}^2) \).

**Corollary 2.2.** The mappings \( K(\cdot), L(\cdot) : \mathcal{Z} \rightarrow \mathcal{L}(H^{5/2}(\mathbb{R}^2), H^{3/2}(\mathbb{R}^2)) \) are analytic at the origin.

In view of Corollary 2.2 we choose \( M \) sufficiently small and study the equation
\[ K'(\eta) - 2(1 - \varepsilon^2) L'(\eta) = 0 \tag{2.4} \]
in the set
\[ U = \{ \eta \in H^3(\mathbb{R}^2) : \|\eta\|_{\mathcal{Z}} < M \}, \]
noting that \( H^3(\mathbb{R}^2) \) is continuously embedded in \( \mathcal{Z} \) and that \( U \) is an open neighbourhood of the origin in \( H^3(\mathbb{R}^2) \); we proceed accordingly by decomposing \( \mathcal{X} = H^3(\mathbb{R}^2) \) into the direct sum of \( \mathcal{X}_1 = \chi(D) H^3(\mathbb{R}^2) \) and \( \mathcal{X}_2 = (1 - \chi(D)) H^3(\mathbb{R}^2) \).

**Corollary 2.3.** The formulae (1.9), (1.10) define functions \( U \rightarrow H^1(\mathbb{R}^2) \) which are analytic at the origin and satisfy \( K'(0) = L'(0) = 0 \).
Proof. The result for $K'$ follows from (1.9), Corollary 2.2 and the fact that $H^{3/2}(\mathbb{R}^2)$ is an algebra. The result for $L'$ follows from (1.10) and the observation that $\eta \mapsto \eta_x(1 + \eta_x^2 + \eta_z^2)^{-1/2}$ and $\eta \mapsto \eta_z(1 + \eta_x^2 + \eta_z^2)^{-1/2}$ define functions $U \to H^2(\mathbb{R}^2)$ which are analytic at the origin since $H^2(\mathbb{R}^2)$ is an algebra.

In keeping with Lemma 2.1 and Corollary 2.2 we write

$$u(\eta, \xi) = \sum_{j=0}^{\infty} u^j(\eta, \xi),$$

where $u^j$ is homogeneous of degree $j$ in $\eta$ and linear in $\xi$, and

$$K(\eta) = \sum_{j=0}^{\infty} K_j(\eta), \quad L(\eta) = \sum_{j=0}^{\infty} L_j(\eta), \quad K'(\eta) := \sum_{j=1}^{\infty} K'_j(\eta), \quad L'(\eta) := \sum_{j=1}^{\infty} L'_j(\eta).$$

where $K_j(\eta)$, $L_j(\eta)$ and $K'_j(\eta)$, $L'_j(\eta)$ are homogeneous of degree $j$ in $\eta$. A straightforward calculation shows that

$$u^0(\xi) = \mathcal{F}^{-1}\left[\frac{ik_1}{|k|} e^{ik\hat{y}\hat{\xi}}\right]$$

and hence that $K_0$ and $L_0$ are Fourier-multiplier operators, namely

$$K_0 \hat{\xi} = \mathcal{F}^{-1}\left[\frac{k_1^2}{|k|} \hat{\xi}\right], \quad L_0 \hat{\xi} = \mathcal{F}^{-1}\left[\frac{k_1k_3}{|k|} \hat{\xi}\right]$$

(we have omitted the argument $\eta$ on the left-hand sides of these equations).

The following lemma gives expressions for the first few terms in the Maclaurin expansions of $K'(\eta)$ and $L'(\eta)$; it is proved by expanding (1.9), (1.10) and examining the boundary-value problems for $u^1(\eta, \eta)$ and $u^2(\eta, \eta)$ to derive the formulæ

$$K_1(\eta)\eta = -(\eta_{xx})_x - K_0(\eta K_0 \eta) - L_0(\eta L_0 \eta), \quad \frac{2}{3}\left((\eta_x^2 + \eta_z^2)\eta_x\right)_x + \frac{1}{2}(\eta_x^2 + \eta_z^2)\eta_x)$$

with a similar formula for $K_2(\eta_1)\eta_1$ (see Buffoni et al. [2] pp. 1032–1033] for details in a similar setting; the restriction to $\eta_1$ is necessary to allow the use of higher-order derivatives in these expressions).

Lemma 2.4.

(i) The identities

$$K'_1(\eta) = \eta - \eta_{xx} - \eta_{zz},$$

$$K'_2(\eta) = 0,$$

$$K'_3(\eta) = \frac{1}{2}((\eta_x^2 + \eta_z^2)\eta_x) + \frac{1}{2}(\eta_x^2 + \eta_z^2)\eta_x)$$

hold for each $\eta \in H^3(\mathbb{R}^2)$. 

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(ii) The identities

\[ L'_1(\eta) = K_0 \eta, \]
\[ L'_2(\eta) = \frac{1}{2} (n_x^2 - (K_0 \eta)^2) - (L_0 \eta)^2 - 2(\eta_x \eta) - 2K_0(\eta K_0 \eta) - 2L_0(\eta L_0 \eta) \]

hold for each \( \eta \in H^3(\mathbb{R}^2) \).

(iii) The identity

\[ L'_3(\eta_1) = K_0 \eta_1 K_0(\eta_1 K_0 \eta_1) + K_0 \eta_1 L_0(\eta_1 L_0 \eta_1) + L_0 \eta_1 L_0(\eta_1 K_0 \eta_1) + L_0 \eta_1 M_0(\eta_1 L_0 \eta_1) + K_0(\eta_1 K_0 \eta_1) + K_0(\eta_1 L_0(\eta_1 L_0 \eta_1)) + L_0(\eta_1 L_0(\eta_1 K_0 \eta_1)) + L_0(\eta_1 L_0(\eta_1 \eta_1) L_0) + \frac{1}{2} K_0(\eta_2 \eta_1) + \frac{1}{2} (\eta_1^2 K_0 \eta_1) + \frac{1}{2} \eta_1(\eta_2 \eta_2) + \frac{1}{2} L_0(\eta_2 \eta_2) + \frac{1}{2} (\eta_1^2 L_0 \eta_1) \]

where

\[ M_0(\xi) = \mathcal{F}^{-1} \left[ \frac{k_3^2}{|k|^3} \right], \]

holds for each \( \eta_1 \in \mathcal{X}_1 \) and more generally for any function \( \eta_1 \) whose Fourier transform has compact support.

Finally, we present some useful estimates for their cubic and higher-order parts of \( K'_1(\eta) \) and \( L'(\eta) \). The results for \( L'(\eta) \) are established by substituting

\[ K(\eta) = \sum_{j=0}^{2} K_j(\eta) + K_c(\eta), \]
\[ L(\eta) = \sum_{j=0}^{2} L_j(\eta) + L_c(\eta) \]

into (1.10) and estimating the resulting formulae for \( L'_c(\eta) \) and \( L'_c(\eta) \) using the rules

\[ \|K_j(\eta)\|_{3/2} \lesssim \|\eta\|_{5/2} \|\eta\|_{5/2}, \]
\[ \|K_c(\eta)\|_{3/2} \lesssim \|\eta\|_{3/2} \|\eta\|_{3/2} \]

(with corresponding estimates for \( L_j(\eta) \), \( L_c(\eta) \) and derivatives). Since this method yields only

\[ \|(K_1(\eta)\eta)^2\|_1, \quad (L_1(\eta)\eta)^2\|_1 \lesssim \|\eta\|_{5/2} \|\eta\|_{5/2} \]

we do not include the fourth-order terms \( -\frac{1}{2}(K_1(\eta)\eta)^2, -\frac{1}{2}(L_1(\eta)\eta)^2 \) in \( L'_c(\eta) \) and treat them separately later (see in particular Proposition 4.3).

**Lemma 2.5.**

(i) The quantities

\[ K'_c(\eta) := \sum_{j=3}^{\infty} K'_j(\eta), \quad L'_c(\eta) := \sum_{j=3}^{\infty} L'_j(\eta) \]

satisfy the estimates

\[ \|K'_c(\eta)\|_1 \lesssim \|\eta\|_{5/2} \|\eta\|_{5/2}, \quad \|dK'_c(\eta)\|_1 \lesssim \|\eta\|_{5/2} \|\eta\|_{5/2} \]
\[ \|L'_c(\eta)\|_1 \lesssim \|\eta\|_{5/2} \|\eta\|_{5/2}, \quad \|dL'_c(\eta)\|_1 \lesssim \|\eta\|_{5/2} \|\eta\|_{5/2} \]

for each \( \eta \in U \) and \( v \in H^2(\mathbb{R}) \).
(ii) The quantities

$$K'_i(\eta) := \sum_{j=4}^{\infty} K'_j(\eta), \quad L'_i(\eta) := \sum_{j=4}^{\infty} L'_j(\eta) + \frac{1}{2}(K_1(\eta)\eta)^2 + \frac{1}{2}(L_1(\eta)\eta)^2$$

satisfy the estimates

$$\|K'_i(\eta)\|_1 \lesssim \|\eta\|_2^4 \|\eta\|_3, \quad ||dK'_i[\eta](v)||_1 \lesssim \|\eta\|_2 \|v\|_3 + \|\eta\|_2^3 \|\eta\|_3 \|v\|_z$$

$$\|L'_i(\eta)\|_1 \lesssim \|\eta\|_2^3 \|\eta\|_3, \quad ||dL'_i[\eta](v)||_1 \lesssim \|\eta\|_2 \|v\|_3 + \|\eta\|_2^2 \|\eta\|_3 \|v\|_z$$

for each \(\eta \in U\) and \(v \in H^2(\mathbb{R})\).

3 Reduction

Observe that \(\eta \in U\) satisfies (2.4) if and only if

$$\eta_1 - \eta_{1xx} - \eta_{1zz} - 2K_0\eta_1 + 2\varepsilon^2 K_0\eta_1 + \chi(D)N(\eta_1 + \eta_2) = 0, \quad (3.1)$$

$$\eta_2 - \eta_{2xx} - \eta_{2zz} - 2K_0\eta_2 + 2\varepsilon^2 K_0\eta_2 + (1 - \chi(D))N(\eta_1 + \eta_2) = 0, \quad (3.2)$$

in which

$$N(\eta) = K'_c(\eta) - 2(1 - \varepsilon^2)(L'_2(\eta) + L'_e(\eta)).$$

The nonlinear term in (3.1) is at leading order cubic in \(\eta_1\) because \(\chi(D)L'_2(\eta_1)\) vanishes; we therefore write it as

$$\eta_1 - \eta_{1xx} - \eta_{1zz} - 2K_0\eta_1 + 2\varepsilon^2 K_0\eta_1 + \chi(D)\left[N(\eta_1 + \eta_2) + 2(1 - \varepsilon^2)L'_2(\eta_1)\right] = 0 \quad (3.3)$$

and make the corresponding adjustment to (3.2), that is ‘replacing’ its nonlinearity with

$$(1 - \chi(D))\left[N(\eta_1 + \eta_2) + 2(1 - \varepsilon^2)L'_2(\eta_1)\right],$$

by writing

$$\eta_2 = F(\eta_1) + \eta_3, \quad F(\eta_1) := 2(1 - \varepsilon^2)\mathcal{F}^{-1}\left[\frac{1 - \chi(k)}{g(k)}\mathcal{F}[L'_2(\eta_1)]\right]$$

(with the requirement that \(\eta_1 + F(\eta_1) + \eta_3 \in U\)). Equation (3.2) may thus be cast in the form

$$\eta_3 = -\mathcal{F}^{-1}\left[\frac{1 - \chi(k)}{g(k)}\mathcal{F}\left[2(1 - \varepsilon^2)L'_2(\eta_1) + N(\eta_1 + F(\eta_1) + \eta_3) + 2\varepsilon^2 K_0(F(\eta_1) + \eta_3)\right]\right], \quad (3.4)$$

where

$$g(k) = 1 + |k|^2 - 2\frac{k^2}{|k|} \geq 0$$

with equality if and only if \(k = \pm(1, 0)\).
**Proposition 3.1.** The mapping

\[ f \mapsto F^{-1} \left[ \frac{1 - \chi(k)}{g(k)} f \right] \]

defines a bounded linear operator \( H^1(\mathbb{R}^2) \to H^3(\mathbb{R}^2) \).

We proceed by solving (3.4) for \( \eta_3 \) as a function of \( \eta_1 \) using the following following fixed-point theorem, which is a straightforward extension of a standard result in nonlinear analysis.

**Theorem 3.2.** Let \( \mathcal{X}_1, \mathcal{X}_2 \) be Banach spaces, \( X_1, X_2 \) be closed, convex sets in, respectively, \( \mathcal{X}_1, \mathcal{X}_2 \) containing the origin and \( \mathcal{G}: X_1 \times X_2 \to \mathcal{X}_2 \) be a smooth function. Suppose that there exists a continuous function \( r: X_1 \to [0, \infty) \) such that

\[ \|G(x_1, 0)\| \leq \frac{1}{2} r, \quad \|d_2 G[x_1, x_2]\| \leq \frac{1}{3} \]

for each \( x_2 \in B_r(0) \subseteq X_2 \) and each \( x_1 \in X_1 \).

Under these hypotheses there exists for each \( x_1 \in X_1 \) a unique solution \( x_2 = x_2(x_1) \) of the fixed-point equation \( x_2 = G(x_1, x_2) \) satisfying \( x_2(x_1) \in B_r(0) \). Moreover, \( x_2(x_1) \) is a smooth function of \( x_1 \in X_1 \) and in particular satisfies the estimate

\[ \|dx_2[x_1]\| \leq 2\|d_1 G[x_1, x_2(x_1)]\|. \]

We apply Theorem 3.2 to equation (3.4) with \( \mathcal{X}_1 = \chi(D) H^3(\mathbb{R}^2), \mathcal{X}_2 = (1 - \chi(D)) H^3(\mathbb{R}^2) \), equipping \( \mathcal{X}_1 \) with the scaled norm

\[ \|\eta\| := \left( \int_{\mathbb{R}^2} (1 + \varepsilon^{-2}((|k_1| - 1)^2 + k_3^2))|\hat{\eta}(k)|^2 \, dk_1 \, dk_3 \right)^{1/2} \]

and \( \mathcal{X}_2 \) with the usual norm for \( H^3(\mathbb{R}^2) \), and taking

\[ X_1 = \{ \eta_1 \in \mathcal{X}_1 : \|\eta_1\| \leq R_1 \}, \quad X_3 = \{ \eta_3 \in \mathcal{X}_2 : \|\eta_3\| \leq R_3 \}; \]

the function \( \mathcal{G} \) is given by the right-hand side of (3.4). (Here we write \( X_3 \) rather than \( X_2 \) for notational clarity.) The calculation

\[
\int_{\mathbb{R}^2} |\hat{\eta}_1(k)| \, dk_1 \, dk_3 = \int_{\mathbb{R}^2} \frac{1}{(1 + \varepsilon^{-2}((|k_1| - 1)^2 + k_3^2))^{1/2}} |\hat{\eta}_1(k)| \, dk_1 \, dk_3 \\
\leq 2\|\eta\| \left( \int_{B(1, 0)} \frac{1}{1 + \varepsilon^{-2}((k_1 - 1)^2 + k_3^2)} \, dk_1 \, dk_3 \right)^{1/2} \\
= 2\sqrt{\pi} \varepsilon (\log(1 + \delta^2 \varepsilon^{-2}))^{1/2} \|\eta\| 
\]

shows that

\[ \|\hat{\eta}_1\|_{L^1(\mathbb{R}^2)} \leq \varepsilon^\theta \|\eta_1\|, \quad \eta_1 \in \mathcal{X}_1, \quad \eta_1 \in \mathcal{X}_1, \quad (3.5) \]

for each fixed \( \theta \in (0, 1) \). We can therefore guarantee that \( \|\hat{\eta}_1\|_{L^1(\mathbb{R}^2)} < M/2 \) for all \( \eta_1 \in X_1 \) for an arbitrarily large value of \( R_1 \); the value of \( R_3 \) is then constrained by the requirement that \( \|F(\eta_1) + \eta_3\|_3 < M/2 \) for all \( \eta_1 \in X_1 \) and \( \eta_3 \in X_3 \), so that \( \eta_1 + F(\eta_1) + \eta_3 \in U \) (Corollary 3.4 below asserts that \( \|F(\eta_1)\|_3 = O(\varepsilon^\theta) \) uniformly over \( \eta_1 \in X_1 \).
We proceed by systematically estimating each term appearing in the equation for $G$, using the inequalities
\[ \|\eta\|_{\infty} \lesssim \|\eta\|_{L}, \quad \|\eta\|_{Z} \lesssim \varepsilon\|\eta_{1}\| + \|\eta_{3}\|_{3}, \quad \|\eta\|_{3} \lesssim \|\eta_{1}\| + \|\eta_{3}\|_{3} \]
and making extensive use of the fact that the support of $\hat{\eta}$ is contained in the fixed bounded set $B$, so that for example
\[ \|\eta_{1}\|_{n} \lesssim \|\eta_{1}\|_{0}, \quad \|\eta_{1}\|_{n,\infty} \lesssim \varepsilon\|\eta_{1}\| \]
for each $n \in \mathbb{N}_{0}$.

In order to estimate $F(\eta)$ we write $\mathcal{L}_{2}(\eta) = m(\{\eta\}^{2})$, where
\[ m(u, v) = \frac{1}{2}(u_{x}v_{x} - (K_{0}u)(K_{0}v) - (L_{0}u)(L_{0}v)) + \frac{1}{2}(-u_{x}v + u_{x}v_{x})_{x} - K_{0}(uK_{0}v + vK_{0}u) - L_{0}(uL_{0}v + vL_{0}u) \]  
(3.6)
(see Lemma 2.4(ii)), and note that
\[ d\mathcal{L}_{2}[\eta](v) = 2m(\eta, v). \]

**Proposition 3.3.** The estimate
\[ \|m(u, v)\|_{1} \lesssim \|u\|_{Z}\|v\|_{3} \]
holds for each $u, v \in H^{3}(\mathbb{R}^{2})$.

**Corollary 3.4.** The estimates
\[ \|F(\eta_{1})\|_{3} \lesssim \varepsilon\|\eta_{1}\|^{2}, \quad \|dF[\eta_{1}]\|_{\mathcal{L}(X_{1}, X_{2})} \lesssim \varepsilon\|\eta_{1}\| \]
hold for each $\eta_{1} \in X_{1}$.

**Remark 3.5.** Noting that
\[ K_{0}F(\eta_{1}) = 2(1 - \varepsilon^{2})F^{-1}\left[ \frac{1 - \chi(k)}{g(k)} \frac{k^{2}}{|k|} F(\mathcal{L}_{2}[\eta_{1}]) \right] \]
and that $F(\mathcal{L}_{2}(\eta_{1}))$ has compact support, one finds that $K_{0}F(\eta_{1})$ satisfies the same estimates as $F(\eta_{1})$.

**Lemma 3.6.** The quantity
\[ \mathcal{N}_{1}(\eta_{1}, \eta_{3}) = \mathcal{L}_{2}(\eta_{1} + F(\eta_{1}) + \eta_{3}) - \mathcal{L}_{2}(\eta_{1}) \]
satisfies the estimates
\[ (i) \quad \|\mathcal{N}_{1}(\eta_{1}, \eta_{3})\|_{1} \lesssim \varepsilon^{2}\|\eta_{1}\|^{2} + \varepsilon\|\eta_{1}\|^{2}\|\eta_{3}\|_{3} + \varepsilon\|\eta_{1}\|\|\eta_{3}\|_{3} + \|\eta_{3}\|_{3}^{3}, \]
\[ (ii) \quad \|d_{1}\mathcal{N}_{1}[\eta_{1}, \eta_{3}]\|_{\mathcal{L}(X_{1}, H^{1}(\mathbb{R}^{2}))} \lesssim \varepsilon^{2}\|\eta_{1}\|^{2} + \varepsilon\|\eta_{1}\|\|\eta_{3}\|_{3} + \varepsilon\|\eta_{3}\|_{3}, \]
\[ (iii) \quad \|d_{2}\mathcal{N}_{1}[\eta_{1}, \eta_{3}]\|_{\mathcal{L}(X_{2}, H^{1}(\mathbb{R}^{2}))} \lesssim \varepsilon\|\eta_{1}\| + \|\eta_{3}\|_{3} \]
for each $\eta_{1} \in X_{1}$ and $\eta_{3} \in X_{3}$.
Proof. We estimate

$$\mathcal{N}_1(\eta_1, \eta_3) = 2m(\eta_1, F(\eta_1) + \eta_3) + m(F(\eta_1) + \eta_3, F(\eta_1) + \eta_3)$$

by combining Proposition 3.3 with Corollary 3.4 using the chain rule. \hfill \square

**Lemma 3.7.** The quantity

$$\mathcal{N}_2(\eta_1, \eta_3) = K'_c(\eta_1 + F(\eta_1) + \eta_3) - 2(1 - \varepsilon^2)\mathcal{L}'_c(\eta_1 + F(\eta_1) + \eta_3),$$

satisfies the estimates

1. $$\|\mathcal{N}_2(\eta_1, \eta_3)\|_1 \lesssim (\varepsilon^\theta\|\eta_1\| + \|\eta_3\|)^2(\|\eta_1\| + \|\eta_3\|),$$
2. $$\|d_1\mathcal{N}_2[\eta_1, \eta_3]\|_{L(X_1, H^1(\mathbb{R}^2))} \lesssim (\varepsilon^\theta\|\eta_1\| + \|\eta_3\|)^2,$$
3. $$\|d_2\mathcal{N}_2[\eta_1, \eta_3]\|_{L(X_2, H^1(\mathbb{R}^2))} \lesssim (\varepsilon^\theta\|\eta_1\| + \|\eta_3\|)(\|\eta_1\| + \|\eta_3\|) + \varepsilon^2$$

for each $$\eta_1 \in X_1$$ and $$\eta_3 \in X_3$$.

**Proof.** We compute the derivatives of $$\mathcal{N}_2$$ using the chain rule and estimate these expressions using the linearity of the derivative, Lemma 2.5(i) and Corollary 3.4. \hfill \square

Altogether we have established the following estimates for $$G$$ and its derivatives (see Proposition 3.1, Remark 3.3 and Lemmata 3.6, 3.7).

**Lemma 3.8.** The function $$G : X_1 \times X_3 \to X_2$$ satisfies the estimates

1. $$\|G(\eta_1, \eta_3)\|_3 \lesssim (\varepsilon^\theta\|\eta_1\| + \|\eta_3\|)^2(1 + \|\eta_1\| + \|\eta_3\|) + \varepsilon^2\|\eta_3\|_3,$$
2. $$\|d_1G[\eta_1, \eta_3]\|_{L(X_1, X_2)} \lesssim (\varepsilon^\theta\|\eta_1\| + \|\eta_3\|)(\varepsilon^\theta + \varepsilon^\theta\|\eta_1\| + \|\eta_3\|),$$
3. $$\|d_2G[\eta_1, \eta_3]\|_{L(X_2)} \lesssim (\varepsilon^\theta\|\eta_1\| + \|\eta_3\|)(1 + \|\eta_1\| + \|\eta_3\|) + \varepsilon^2$$

for each $$\eta_1 \in X_1$$ and $$\eta_3 \in X_3$$.

**Theorem 3.9.** Equation (3.4) has a unique solution $$\eta_3 \in X_3$$ which depends smoothly upon $$\eta_1 \in X_1$$ and satisfies the estimates

$$\|\eta_3(\eta_1)\|_3 \lesssim \varepsilon^2\|\eta_1\|_3^2, \quad \|d\eta_3[\eta_1]\|_{L(X_1, X_2)} \lesssim \varepsilon^2\|\eta_1\|.$$ 

**Proof.** Choosing $$R_3$$ and $$\varepsilon$$ sufficiently small and setting $$r(\eta_1) = \sigma\varepsilon^2\|\eta_1\|_3^2$$ for a sufficiently large value of $$\sigma > 0$$, one finds that

$$\|G(\eta_1, 0)\|_3 \lesssim \frac{1}{2}r(\eta_1), \quad \|d_2G[\eta_1, 0]\|_{L(X_2)} \lesssim \varepsilon^\theta$$

for $$\eta_1 \in X_1$$ and $$\eta_3 \in \overline{B}_{r(\eta_1)}(0) \subset X_3$$ (Lemma 3.8(i), (iii)). Theorem 3.2 asserts that equation (3.4) has a unique solution $$\eta_3$$ in $$\overline{B}_{r(\eta_1)}(0) \subset X_3$$ which depends smoothly upon $$\eta_1 \in X_1$$, and the estimate for its derivative follows from Lemma 3.8(ii). \hfill \square

Substituting $$\eta_2 = \eta_1 + F(\eta_1) + \eta_3(\eta_1)$$ into (3.3) yields the reduced equation

$$\eta_1 - \eta_{1XX} - \eta_{1zz} - 2K_0\eta_1$$

$$+ 2\varepsilon^2K\omega_1 + \chi(D)\left(\mathcal{N}(\eta_1 + F(\eta_1) + \eta_3(\eta_2)) + 2(1 - \varepsilon^2)\mathcal{L}'_c(\eta_1)\right) = 0$$

(3.7)

for $$\eta_1 \in X_1$$. Observe that this equation is invariant under the reflections $$\eta_1(x, z) \mapsto \eta_1(-x, z)$$ and $$\eta_1(x, z) \mapsto \eta_1(x, -z)$$; a familiar argument shows that they are inherited from the corresponding invariance of (3.3), (3.4) under $$\eta_1(x, z) \mapsto \eta_1(-x, z)$$, $$\eta_3(x, z) \mapsto \eta_3(-x, z)$$ and $$\eta_1(x, z) \mapsto \eta_1(x, -z)$$, $$\eta_3(x, z) \mapsto \eta_3(x, -z)$$ when applying Theorem 3.2.
4 Derivation of the reduced equation

In this section we compute the leading-order terms in the reduced equation (3.7). To this end we write
\[ \eta_1 = \eta_1^+ + \eta_1^- , \]
where \( \eta_1^+ = \chi^+(D)\eta_1, \eta_1^- = \chi^-(D)\eta_1 \) and \( \chi^+, \chi^- \) are the characteristic functions of respectively \( B_\delta(1,0) \) and \( B_\delta(-1,0) \), so that \( \eta_1^+ \) satisfies the equation
\[ \eta_1^+ - \eta_1^{+xx} - \eta_1^{+zz} - 2K_0\eta_1^+ \]
\[ + 2\varepsilon^2 K_0\eta_1 + \chi^+(D) \left( \mathcal{N}(\eta_1 + F(\eta_1) + \eta_3(\eta_2)) + 2(1-\varepsilon^2)\mathcal{L}_2(\eta_1) \right) = 0 \]
(4.1)
(and \( \eta_1^- \) satisfies its complex conjugate). It is also convenient to introduce some additional notation.

Definition 4.1.

(i) The symbol \( \mathcal{Q}(\varepsilon^\gamma \|\eta_1\|_r^r) \) denotes a smooth function \( N : X_1 \to H^1(\mathbb{R}^2) \) which satisfies the estimates
\[ \|N(\eta_1)\|_1 \lesssim \varepsilon^\gamma \|\eta_1\|_r^r, \quad \|dN[\eta_1]\|_{L(\mathcal{L}(X_1, H^1(\mathbb{R}^2)))} \lesssim \varepsilon^\gamma \|\eta_1\|_r^{r-1} \]
for each \( \eta_1 \in X_1 \) (where \( \gamma \geq 0, r \geq 1 \)). Furthermore
\[ \mathcal{Q}_0(\varepsilon^\gamma \|\eta_1\|_r^r) := \chi_0(D)\mathcal{Q}(\varepsilon^\gamma \|\eta_1\|_r^r), \quad \mathcal{Q}_+(\varepsilon^\gamma \|\eta_1\|_r^r) := \chi^+(D)\mathcal{Q}(\varepsilon^\gamma \|\eta_1\|_r^r), \]
where \( \chi_0 \) and \( \chi^+ \) are the characteristic functions of the sets \( B_\delta(0,0) \) and \( B_\delta(1,0) \).

(ii) The symbol \( \mathcal{Q}_s^s(\|u\|_n^1) \) denotes \( \chi_0(\varepsilon D)N(u) \), where \( N \) is a smooth function \( B_\delta(0) \subseteq \chi_0(\varepsilon D)H^1(\mathbb{R}^2) \to H^n(\mathbb{R}^2) \) or \( B_\delta(0) \subseteq H^1(\mathbb{R}^2) \to H^n(\mathbb{R}^2) \) which satisfies the estimates
\[ \|N(u)\|_n \lesssim \|u\|_1^r, \quad \|dN[u]\|_{L(\mathcal{L}(H^1(\mathbb{R}^2), H^n(\mathbb{R}^2)))} \lesssim \|u\|_1^{r-1} \]
for each \( u \in B_\delta(0) \) (with \( r \geq 1, n \geq 0 \)).

We begin with a result which shows how a Fourier-multiplier operator \( m(D) \) may be approximated by \( m(\omega,0) \) when acting upon a function whose Fourier transform is supported near the point \( (\omega,0) \). Its proof is given by Buffoni et al. [3, Lemma 11] (in a slightly different context).

Lemma 4.2. The estimates

(i) \( \partial_x \eta_1^\pm = \pm i\eta_1^\pm + \mathcal{Q}(\varepsilon \|\eta_1\|) \),
(ii) \( \partial_y \eta_1^\pm = -\eta_1^\pm + \mathcal{Q}(\varepsilon \|\eta_1\|) \),
(iii) \( \partial_z \eta_1^\pm = \mathcal{Q}(\varepsilon \|\eta_1\|) \),
(iv) \( K_0 \eta_1^\pm = \eta_1^\pm + \mathcal{Q}(\varepsilon \|\eta_1\|) \),
(v) \( L_0 \eta_1^\pm = \mathcal{Q}(\varepsilon \|\eta_1\|) \).
(vi) \( K_0((\eta_1^\pm)^2) = 2(\eta_1^\pm)^2 + O(\varepsilon^{1+\theta}\|\eta_1\|^2) \),

(vii) \( L_0((\eta_1^\pm)^2) = O(\varepsilon^{1+\theta}\|\eta_1\|^2) \),

(viii) \( K_0(\eta_1^+ \eta_1^-) = O(\varepsilon^{1+\theta}\|\eta_1\|^2) \),

(ix) \( L_0(\eta_1^+ \eta_1^-) = O(\varepsilon^{1+\theta}\|\eta_1\|^2) \),

(x) \( \mathcal{F}^{-1}[g(k)^{-1}\mathcal{F}[((\eta_1^\pm)^2)] = (\eta_1^\pm)^2 + O(\varepsilon^{1+\theta}\|\eta_1\|^2) \),

(xi) \( K_0(\eta_1^- (\eta_1^+)^2) = \eta_1^-(\eta_1^+)^2 + O(\varepsilon^{1+2\theta}\|\eta_1\|^3) \).

hold for each \( \eta_1 \in X_1 \).

We proceed by approximating each term in the nonlinearity on the right-hand side of (4.1) according to the rules given in Lemma 4.2.

**Proposition 4.3.** The estimate

\[
F(\eta_1) = -2 \left( (\eta_1^+)^2 + (\eta_1^-)^2 \right) + F_t(\eta_1), \quad F_t(\eta_1) = O(\varepsilon^{1+\theta}\|\eta_1\|^2)
\]

holds for each \( \eta_1 \in X_1 \).

**Proof.** Using the expansions given in Lemma 4.2, we find that

\[
\mathcal{L}_t(\eta_1) = m(\eta_1, \eta_1) = - \left( (\eta_1^+)^2 + (\eta_1^-)^2 \right) + O(\varepsilon^{1+\theta}\|\eta_1\|^2).
\]

It follows that

\[
\mathcal{F}^{-1} \left[ \frac{1 - \chi(k)}{g(k)} \mathcal{F}[\mathcal{L}_t(\eta_1)] \right] = - \left( (\eta_1^+)^2 + (\eta_1^-)^2 \right) + O(\varepsilon^{1+\theta}\|\eta_1\|^2)
\]

because of Lemma 4.2(x) and the fact that

\[
\mathcal{F}^{-1} \left[ \frac{1 - \chi(k)}{g(k)} \mathcal{F}[O(\varepsilon^{1+\theta}\|\eta_1\|^2)] \right] = O(\varepsilon^{1+\theta}\|\eta_1\|^2)
\]

(because \((1 - \chi(k))g(k)^{-1}\) is bounded). We conclude that

\[
F(\eta_1) = 2(1 - \varepsilon^2) \mathcal{F}^{-1} \left[ \frac{1 - \chi(k)}{g(k)} \mathcal{F}[\mathcal{L}_t(\eta_1)] \right] = -2 \left( (\eta_1^+)^2 + (\eta_1^-)^2 \right) + O(\varepsilon^{1+\theta}\|\eta_1\|^2). \quad \Box
\]

**Remark 4.4.** The remainder term \( F_t(\eta_1) \) in the formula for \( F(\eta_1) \) given in Proposition 4.3 satisfies

\[
\| F_t(\eta_1) \|_n \lesssim \varepsilon^{1+\theta}\|\eta_1\|^2, \quad \| d F_t(\eta_1) \|_{\mathcal{L}(X_1, H^{\alpha}(\mathbb{R}^2))} \lesssim \varepsilon^{1+\theta}\|\eta_1\|
\]

for all \( n \in \mathbb{N}_0 \) since its Fourier transform is supported in the region \( B + B \).

**Proposition 4.5.** The estimate

\[
\chi^+(D)\mathcal{N}_t(\eta_1, \eta_3(\eta_1)) = 4\chi^+(D)(\eta_1^-)(\eta_1^+)^2 + O_+(\varepsilon^3\|\eta_1\|^3)
\]

holds for each \( \eta_1 \in X_1 \).
Proof. Observe that
\[
\chi^+(D)\mathcal{N}_1(\eta_1, \eta_3(\eta_1)) = \chi^+(D)(2m(\eta_1, F(\eta_1) + \eta_3) + m(F(\eta_1) + \eta_3, F(\eta_1) + \eta_3))
\]
\[
= 2\chi^+(D)m(\eta_1, F(\eta_1)) + Q(\varepsilon^{39}\|\eta_1\|^3),
\]
in which we have used the calculations
\[
m(\eta_1, \eta_3) = Q(\varepsilon^{39}\|\eta_1\|^3), \quad m(F(\eta_1), \eta_3) = Q(\varepsilon^{39}\|\eta_1\|^4)
\]
(see Proposition 3.3, Corollary 3.4 and Theorem 3.9) and
\[
m(F(\eta_1), F(\eta_1)) = Q(\varepsilon^{39}\|\eta_1\|^4)
\]
(because of (3.6) and Proposition 4.3). Observing that
\[
\eta \in F(\eta_1) + \eta_3(\eta_1)
\]
and it follows from (3.6) and Lemma 4.2 that
\[
m(\eta_1^-, (\eta_1^+)^2) = -\eta_1^- (\eta_1^+)^2 + O(\varepsilon^{39}\|\eta_1\|^3).
\]
\[
\square
\]
Proposition 4.6. The estimates
\[
\chi^+(D)\mathcal{K}_3'(\eta_1 + F(\eta_1) + \eta_3(\eta_1)) = -\frac{3}{2}\chi^+(D)(\eta_1^- (\eta_1^+)^2) + O_+(\varepsilon^{39}\|\eta_1\|^3),
\]
\[
\chi^+(D)\mathcal{L}_3'(\eta_1 + F(\eta_1) + \eta_3(\eta_1)) = -2\chi^+(D)(\eta_1^- (\eta_1^+)^2) + O_+(\varepsilon^{39}\|\eta_1\|^3)
\]
hold for each \(\eta_1 \in X_1\).

Proof. Using the estimates for \(F(\eta_1)\) and \(\eta_3(\eta_1)\) given in Corollary 3.4 and Theorem 3.9, we find that
\[
\mathcal{K}_3'(\eta_1 + F(\eta_1) + \eta_3(\eta_1)) = \mathcal{K}_3'(\eta_1) + O(\varepsilon^{39}\|\eta_1\|^4)
\]
and
\[
\chi^+(D)\mathcal{K}_3'(\eta_1) = -\frac{3}{2}\chi^+(D)(\eta_1^- (\eta_1^+)^2) + O_+(\varepsilon^{39}\|\eta_1\|^3)
\]
(because of equation (2.7)). It similarly follows from the formula
\[
\mathcal{L}_3'(\eta_1) = -K_0\eta K_1(\eta)\eta - L_0\eta L_1(\eta)\eta - \eta^2 K_0\eta - \eta^2 L_0\eta + K_2(\eta)\eta
\]
and the fact that \(K_2(\eta) = m_2(\eta, \eta)\), where \(m_2\) is a bounded, symmetric bilinear mapping \(\mathcal{Z} \times \mathcal{Z} \rightarrow L(H^{5/2}(\mathbb{R}^2), H^{3/2}(\mathbb{R}^2))\), that
\[
\mathcal{L}_3'(\eta_1 + F(\eta_1) + \eta_3(\eta_1)) = \mathcal{L}_3'(\eta_1 + \eta_1) + O(\varepsilon^{39}\|\eta_1\|^4);
\]
using Lemma 2.4(iii) twice yields
\[
\mathcal{L}_3'(F(\eta_1) + \eta_1) = \mathcal{L}_3'(\eta_1) + O(\varepsilon^{39}\|\eta_1\|^3)
\]
and
\[
\chi^+(D)\mathcal{L}_3'(\eta_1) = -2\chi^+(D)(\eta_1^- (\eta_1^+)^2) + O_+(\varepsilon^{39}\|\eta_1\|^3).
\]
\[
\square
\]
**Proposition 4.7.** The estimates
\[
K'_3(\eta_1 + F(\eta_1) + \eta_3(\eta_1)) = O(\varepsilon^{10} \| \eta_1 \|^5), \\
L'_3(\eta_1 + F(\eta_1) + \eta_3(\eta_1)) = O(\varepsilon^{10} \| \eta_1 \|^4)
\]
hold for each $\eta_1 \in X_1$.

**Proof.** This result follows from Proposition 2.3 (ii), Corollary 3.4 and Theorem 3.9. 

**Proposition 4.8.** The estimates
\[
-\frac{1}{2} \chi^+(D)(K(\eta_1 + F(\eta_1) + \eta_3(\eta_1))(\eta_1 + F(\eta_1) + \eta_3(\eta_1)))^2 = O_+(\varepsilon^{30} \| \eta_1 \|^4), \\
-\frac{1}{2} \chi^+(D)(L(\eta_1 + F(\eta_1) + \eta_3(\eta_1))(\eta_1 + F(\eta_1) + \eta_3(\eta_1)))^2 = O_+(\varepsilon^{30} \| \eta_1 \|^4)
\]
hold for each $\eta_1 \in X_1$.

**Proof.** Using Corollary 3.4 and Theorem 3.9 we find that
\[
-\frac{1}{2}(K(\eta_1 + F(\eta_1) + \eta_3(\eta_1))(\eta_1 + F(\eta_1) + \eta_3(\eta_1)))^2 = -\frac{1}{2}(K(\eta_1)\eta_1)^2 + O(\varepsilon^{30} \| \eta_1 \|^4),
\]
and furthermore
\[
-\frac{1}{2} \chi^+(D)(K(\eta_1)\eta_1)^2 = -\frac{1}{2} \chi^+(D)((\eta_1\eta_{1x})_x + K_0(\eta_1 K_0 \eta_1) + L_0(\eta_1 L_0 \eta_1))^2 = 0
\]
because of equation (2.5). The second estimate is derived in the same fashion (with equation (2.6)).

**Corollary 4.9.** The estimate
\[
\chi^+(D)\mathcal{N}_2(\eta_1, \eta_3(\eta_1)) = 5 \frac{1}{2} \chi^+(D)(\eta_1^-(\eta_1^+)^2) + O_+(\varepsilon^{30} \| \eta_1 \|^3)
\]
holds for each $\eta_1 \in X_1$.

We conclude that the reduced equation for $\eta_1$ is
\[
\eta_1^+ - \eta_{1xx} - \eta_{1zz} - 2K_0 \eta_1^+ + 2\varepsilon^2 K_0 \eta_1^+ - \frac{11}{2} \chi^+(D)(|\eta_1^+|^2 \eta_1^+) + O_+(\varepsilon^{30} \| \eta_1 \|^3) = 0,
\]
which can be further simplified to
\[
\eta_1^+ - \eta_{1xx} - \eta_{1zz} - 2K_0 \eta_1^+ + 2\varepsilon^2 \eta_1^+ - \frac{11}{2} \chi^+(D)(|\eta_1^+|^2 \eta_1^+) + O_+(\varepsilon^{30} \| \eta_1 \|) = 0
\]
by an application of Lemma 4.2 (iv). Finally, we introduce the nonlinear Schrödinger scaling
\[
\eta_1^+(x, z) = \frac{1}{2} \varepsilon \zeta(x, \varepsilon z) e^{ix},
\]
so that $\zeta \in B_R(0) \subseteq \chi_0(\varepsilon D) H^1(\mathbb{R}^2)$ solves the perturbed full-dispersion nonlinear Schrödinger equation
\[
\varepsilon^{-2} g(e + \varepsilon D) \zeta + 2 \zeta - \frac{11}{8} \chi_0(\varepsilon D)(|\zeta|^2 \zeta) + \varepsilon^{30-2} O_6(\| \zeta \|_1) = 0,
\]
where $R = R_1/\sqrt{2}$ and $e = (1, 0)$ (note that $\| \eta_1 \|^2 = \| \zeta \|^2$ and the change of variables from $(x, z)$ to $\varepsilon(x, z)$ introduces an additional factor of $\varepsilon$ in the remainder term). The invariance of the reduced equation under $\eta_1(x, z) \mapsto \eta_1(-x, -z)$ and $\eta_1(x, z) \mapsto \eta_1(x, -z)$ is inherited by (4.2), which is invariant under the reflections $\zeta(x, z) \mapsto -\zeta(x, z)$ and $\zeta(x, z) \mapsto \zeta(x, -z)$.

**Remark 4.10.** In the formal limit $\varepsilon = 0$ equation (4.2) reduces to the nonlinear Schrödinger equation
\[
-\frac{1}{2} \zeta_{xx} - \zeta_{zz} + \zeta - \frac{11}{16} |\zeta|^2 \zeta = 0.
\]

---

The text primarily discusses the estimates and properties of certain equations and their solutions, particularly focusing on the invariance and reduction of equations under specific transformations. The proof of these propositions relies on previous results and the application of specific lemmas and theorems. The introduction of nonlinear Schrödinger scaling is used to simplify the equations and to highlight the invariance properties of the solutions.
5 Solution of the reduced equation

In this section we complete our existence theory by proving the following theorem.

Theorem 5.1. For each sufficiently small value of $\varepsilon > 0$ equation (4.2) has two small-amplitude solutions $\zeta^\pm_\varepsilon$ in $\chi_0(\varepsilon D)H^1(\mathbb{R}^2)$ which satisfy $\zeta^\pm_\varepsilon(x, z) = \zeta^\pm_\varepsilon(-x, z)$, $\zeta^\pm_\varepsilon(x, z) = \zeta^\pm_\varepsilon(x, -z)$ and $\|\zeta^\pm_\varepsilon - (\pm \zeta_0)\|_1 \lesssim \varepsilon^{1/2}$, where $\zeta_0 \in S(\mathbb{R}^2)$ is the unique symmetric, positive (real) solution of the nonlinear Schrödinger equation (4.3).

The first step is a result which allows us to ‘replace’ the nonlocal operator in equation (4.2) with a differential operator.

Proposition 5.2. The inequality

$$\left| \frac{\varepsilon^2}{2\varepsilon^2 + g(e + \varepsilon k)} - \frac{1}{2 + k_1^2 + 2k_3^2} \right| \lesssim \frac{\varepsilon |k|^3}{(1 + |k|^2)^2}$$

holds uniformly over $|k| < \delta/\varepsilon$.

Proof. Clearly

$$\left| \frac{\varepsilon^2}{2\varepsilon^2 + g(e + \varepsilon k)} - \frac{1}{2 + k_1^2 + 2k_3^2} \right| = \frac{|g(e + \varepsilon k) - \varepsilon^2(k_1^2 + 2k_3^2)|}{(2\varepsilon^2 + g(e + \varepsilon k))(2 + k_1^2 + 2k_3^2)},$$

while

$$g(e + s) - s_1^2 - 2s_2^2 \lesssim |s|^3, \quad |s| \leq \delta$$

and

$$g(e + s) \gtrsim |s|^2, \quad s \in \mathbb{R}^2.$$ 

It follows that

$$\left| \frac{\varepsilon^2}{2\varepsilon^2 + g(e + \varepsilon k)} - \frac{1}{2 + k_1^2 + 2k_3^2} \right| \lesssim \frac{\varepsilon |k|^3}{(1 + |k|^2)^2}, \quad |k| < \delta/\varepsilon. \quad \square$$

Using this proposition, one can write equation (4.2) as

$$\zeta + F_\varepsilon(\zeta) = 0,$$ (5.1)

where

$$F_\varepsilon(\zeta) = -\frac{11}{16} (1 - \frac{1}{2} \partial_x^2 - \partial_z^2)^{-1} \chi_0(\varepsilon D)\bigl(|\zeta|^2 \zeta\bigr) + \varepsilon^{1/2} Q_1^\varepsilon(\|\zeta\|_1)$$

and we have chosen the concrete value $\theta = 5/6$, so that $\varepsilon^{3\theta - 2} = \varepsilon^{1/2}$. It is convenient to replace equation (5.1) with

$$\zeta + \tilde{F}_\varepsilon(\zeta) = 0,$$ (5.2)

where $\tilde{F}_\varepsilon(\zeta) = F_\varepsilon(\chi_0(\varepsilon D)\zeta)$ and study it in the fixed space $H^1(\mathbb{R}^2)$ (the solution sets of (5.1) and (5.2) evidently coincide). Equation (5.2) is solved using the following version of the implicit-function theorem.
Theorem 5.3. Let $\mathcal{X}$ be a Banach space, $X_0$ and $\Lambda_0$ be open neighbourhoods of respectively $x^*$ in $\mathcal{X}$ and the origin in $\mathbb{R}^n$ and $G : X_0 \times \Lambda_0 \to \mathcal{X}$ be a function which is differentiable with respect to $x \in X_0$ for each $\lambda \in \Lambda_0$. Suppose that $G(x^*, 0) = 0$, $d_1G[x^*, 0] : \mathcal{X} \to \mathcal{X}$ is an isomorphism,

$$\lim_{x \to x^*} \|d_1G[x, 0] - d_1G[x^*, 0]\|_{\mathcal{L}(\mathcal{X})} = 0$$

and

$$\lim_{\lambda \to 0} \|G(x, \lambda) - G(x, 0)\|_{\mathcal{X}} = 0, \quad \lim_{\lambda \to 0} \|d_1G[x, \lambda] - d_1G[x, 0]\|_{\mathcal{L}(\mathcal{X})} = 0$$

uniformly over $x \in X_0$.

There exist open neighbourhoods $X$ of $x^*$ in $\mathcal{X}$ and $\Lambda$ of 0 in $\mathbb{R}^n$ (with $X \subseteq X_0$, $\Lambda \subseteq \Lambda_0$) and a uniquely determined mapping $h : \Lambda \to X$ with the properties that

(i) $h$ is continuous at the origin (with $h(0) = x^*$),

(ii) $G(h(\lambda), \lambda) = 0$ for all $\lambda \in \Lambda$,

(iii) $x = h(\lambda)$ whenever $(x, \lambda) \in X \times \Lambda$ satisfies $G(x, \lambda) = 0$.

Furthermore, the existence of $\alpha > 0$ such that $\|G(x, \lambda) - G(x, 0)\|_{\mathcal{X}} \lesssim |\lambda|^\alpha$ for all $\lambda \in \Lambda_0$ and $x \in X_0$ implies that $\|h(\lambda) - h(0)\|_{\mathcal{X}} \lesssim |\lambda|^\alpha$ for all $\lambda \in \Lambda$.

We establish Theorem 5.1 by applying Theorem 5.3 with $\alpha > 0$ such that $\|G(x, \lambda) - G(x, 0)\|_{\mathcal{X}} \lesssim |\lambda|^\alpha$ for all $\lambda \in \Lambda_0$ and $x \in X_0$ implies that $\|h(\lambda) - h(0)\|_{\mathcal{X}} \lesssim |\lambda|^\alpha$ for all $\lambda \in \Lambda$.

We establish Theorem 5.1 by applying Theorem 5.3 with

$$\mathcal{X} = H^1_c(\mathbb{R}^2, \mathbb{C}) = \{ \zeta \in H^1(\mathbb{R}^2, \mathbb{C}) : \zeta(x, z) = \overline{\zeta(-x, z)}, \zeta(x, z) = \zeta(x, -z) \},$$

$X = B_R(0)$, where $R$ is chosen large enough that $\zeta_0 \in X$, $\Lambda_0 = (-\varepsilon_0, \varepsilon_0)$ for a sufficiently small value of $\varepsilon_0$ and

$$G(\zeta, \varepsilon) := \zeta + \tilde{F}_{|\varepsilon|}(\zeta)$$

(here $\varepsilon$ is replaced by $|\varepsilon|$ so that $G(\zeta, \varepsilon)$ is defined for $\varepsilon$ in a full neighbourhood of the origin in $\mathbb{R}$).

Observe that

$$G(\zeta, \varepsilon) - G(\zeta, 0)$$

$$= -\frac{11}{16} \left(1 - \frac{1}{2} \partial_{z}^2 - \partial_{\bar{z}}^2\right)^{-1} \left(\chi_0(|\varepsilon|D) \left|\chi_0(|\varepsilon|D)\zeta_0(|\varepsilon|D)\zeta\right| - |\zeta|^2\zeta + |\varepsilon|^{1/2}G_1[|\zeta||_{\mathcal{H}}] \right)$$

$$= -\frac{11}{16} \left(1 - \frac{1}{2} \partial_{z}^2 - \partial_{\bar{z}}^2\right)^{-1} \left(\chi_0(|\varepsilon|D) \left|\chi_0(|\varepsilon|D)\zeta_0(|\varepsilon|D)\zeta - |\zeta|^2\zeta\right| - |\varepsilon|^{1/2}G_1[|\zeta||_{\mathcal{H}}] \right)$$

$$+ \zeta_0(|\varepsilon|D)\zeta(\chi_0(|\varepsilon|D) - I)\zeta$$

$$+ \left(\chi_0(|\varepsilon|D) - I\right)|\zeta|^2\zeta + |\varepsilon|^{1/2}G_1[|\zeta||_{\mathcal{H}}].$$

Noting that

$$\|\chi_0(|\varepsilon|D) - I\|_{\mathcal{L}(H^1(\mathbb{R}^2, \mathbb{C}), H^{1/2}(\mathbb{R}^2, \mathbb{C}))} \lesssim |\varepsilon|^{1/2}$$
because
\[ \|\chi_0(\epsilon|D)u - u\|_{1/2}^2 = \int_{|k|>\frac{\delta}{|\tau|}} (1 + |k|^2)^{1/2} |\tilde{u}|^2 \mathrm{d}k \]
\[ \leq \sup_{|k|>\frac{\delta}{|\tau|}} (1 + |k|^2)^{-1/2} \int_{|k|>\frac{\delta}{|\tau|}} (1 + |k|^2) |\tilde{u}|^2 \mathrm{d}k \]
\[ \leq \frac{1}{(1 + \frac{\delta^2}{|\tau|^2})^{1/2}} \|u\|_1^2, \]
and similarly
\[ \|\chi_0(\epsilon|D) - I\|_{\mathcal{L}(H^{1/2}(\mathbb{R}^2, \mathbb{C}), L^2(\mathbb{R}^2, \mathbb{C}))} \lesssim |\epsilon|^{1/2}, \]
and that pointwise multiplication defines bounded trilinear mappings \( H^1(\mathbb{R}^2, \mathbb{C})^3 \to H^{1/2}(\mathbb{R}^2, \mathbb{C}) \) and \( H^1(\mathbb{R}^2, \mathbb{C})^2 \times H^{1/2}(\mathbb{R}^2, \mathbb{C}) \to L^2(\mathbb{R}^2, \mathbb{C}) \) (see Hörmander [8 Theorem 8.3.1]), we find that
\[ \|G(\zeta, \epsilon) - G(\zeta, 0)\|_1 \lesssim |\epsilon|^{1/2} \]
uniformly over \( \zeta \in B_R(0) \). Here we have also used the estimate \( \|\chi_0(\epsilon|D)u\|_s \leq \|u\|_s \) for all \( u \in H^s(\mathbb{R}^2, \mathbb{C}) \) and and the fact that \( (1 - \frac{1}{2}\partial_x^2 - \partial_z^2)^{-1} \) maps \( L^2(\mathbb{R}^2, \mathbb{C}) \) continuously into \( H^1(\mathbb{R}^2, \mathbb{C}) \). A similar calculation shows that
\[ \|d_1 G[\zeta, \epsilon] - d_1 G[\zeta, 0]\|_{\mathcal{L}(H^1(\mathbb{R}^2, \mathbb{C}))} \lesssim |\epsilon|^{1/2} \]
uniformly over \( \zeta \in B_R(0) \).

Furthermore the equation
\[ G(\zeta, 0) = \zeta - \frac{11}{16} (1 - \frac{1}{2}\partial_x^2 - \partial_z^2)^{-1} |\zeta|^2 \zeta = 0 \quad (5.3) \]
has a unique symmetric, positive (real) solution \( \zeta_0 \in \mathcal{S}(\mathbb{R}^2, \mathbb{C}) \) (see Sulem & Sulem [13, §4.2] and the references therein). The fact that \( d_1 G[\pm \zeta_0, 0] \) is an isomorphism is conveniently established by using real coordinates. Define \( \zeta_1 = \text{Re} \zeta \) and \( \zeta_2 = \text{Im} \zeta \), so that
\[ d_1 G[\pm \zeta_0, 0](\zeta_1 + i\zeta_2) = G_1(\zeta_1) + iG_2(\zeta_2), \]
where \( G_1 : H_0^1(\mathbb{R}^2, \mathbb{R}) \to H_0^1(\mathbb{R}^2, \mathbb{R}) \) and \( G_2 : H_0^1(\mathbb{R}^2, \mathbb{R}) \to H_0^1(\mathbb{R}^2, \mathbb{R}) \) are defined by
\[ G_1(\zeta_1) = \zeta_1 - \frac{33}{16} (1 - \frac{1}{2}\partial_x^2 - \partial_z^2)^{-1} \zeta_0^2 \zeta_1, \quad G_2(\zeta_2) = \zeta_2 - \frac{11}{16} (1 - \frac{1}{2}\partial_x^2 - \partial_z^2)^{-1} \zeta_0^2 \zeta_2 \]
with
\[ H_0^n(\mathbb{R}^2, \mathbb{R}) = \{ \zeta_1 \in H^n(\mathbb{R}^2, \mathbb{R}) : \zeta_1(x, z) = \zeta_1(-x, z), \zeta_1(x, z) = \zeta_1(x, -z) \}, \]
\[ H_0^n(\mathbb{R}^2, \mathbb{R}) = \{ \zeta_2 \in H^n(\mathbb{R}^2, \mathbb{R}) : \zeta_2(x, z) = -\zeta_2(-x, z), \zeta_2(x, z) = \zeta_1(x, -z) \} \]
for \( n \in \mathbb{N}_0 \). The formulae
\[ \zeta_1 \mapsto \frac{33}{16} (1 - \frac{1}{2}\partial_x^2 - \partial_z^2)^{-1} \zeta_0^2 \zeta_1, \quad \zeta_2 \mapsto \frac{11}{16} (1 - \frac{1}{2}\partial_x^2 - \partial_z^2)^{-1} \zeta_0^2 \zeta_2 \]
define compact operators $H^1(\mathbb{R}^2, \mathbb{R}) \to H^1(\mathbb{R}^2, \mathbb{R})$, $H^1_c(\mathbb{R}^2, \mathbb{R}) \to H^1(\mathbb{R}^2, \mathbb{R})$ and $H^1(\mathbb{R}^2, \mathbb{R}) \to H^1(\mathbb{R}^2, \mathbb{R})$, so that $G_1$, $G_2$ are Fredholm operators with index 0. Writing

$$T_1 \zeta_1 = \zeta_1 - \frac{1}{2} \zeta_{1xx} - \zeta_{1zz} - \frac{33}{16} \zeta_0^2 \zeta_1, \quad T_2 \zeta_2 = \zeta_2 - \frac{1}{2} \zeta_{2xx} - \zeta_{2zz} - \frac{11}{10} \zeta_0^2 \zeta_2,$$

we find that the kernels of $G_1$ and $G_2$ coincide with respectively the kernels of the linear operators

$$T_1 : H^2(\mathbb{R}^2, \mathbb{R}) \subseteq L^2(\mathbb{R}^2, \mathbb{R}) \to L^2(\mathbb{R}^2, \mathbb{R}) \quad \text{and} \quad T_2 : H^2(\mathbb{R}^2, \mathbb{R}) \subseteq L^2(\mathbb{R}^2, \mathbb{R}) \to L^2(\mathbb{R}^2, \mathbb{R}).$$

It is however known that the kernels of $T_1$, $T_2 : H^2(\mathbb{R}^2, \mathbb{R}) \subseteq L^2(\mathbb{R}^2, \mathbb{R}) \to L^2(\mathbb{R}^2, \mathbb{R})$ are respectively $\langle \zeta_{0x}, \zeta_{0z} \rangle$ and $\langle \zeta_0 \rangle$ (see Chang et al. [4]). The kernels of $G_1$, $G_2$ are therefore trivial, so that $G_1$, $G_2$ and hence $d_1G[\pm \zeta_0, 0]$ are isomorphisms.

It remains to confirm that tracing back the changes of variable

$$\eta = \eta_1 + F(\eta_1) + \eta_3(\eta_1), \quad \eta_1 = \eta_1^+ + \eta_1^-, \quad \eta_1^+(x, z) = \frac{1}{2} \epsilon \zeta^+_e(\epsilon x, \epsilon z)e^{ix}$$

leads to the estimate

$$\eta(x, z) = \pm \epsilon \zeta_0(\epsilon x, \epsilon z) \cos x + o(\epsilon)$$

uniformly over $(x, z) \in \mathbb{R}^2$. The key is to show that

$$\|\zeta^+_e - \zeta_0\|_\infty \preccurlyeq \epsilon^\Delta$$

for any $\Delta \in (0, 1/2)$; here we choose the concrete value $\Delta = 1/4$. This result follows from the calculation

$$\|\zeta^+_e - \zeta_0\|_\infty \preccurlyeq \|\zeta^+_e - \zeta_0\|_{L^2(|k|<\delta/\epsilon)}^{5/4}$$

(because the support of $\hat{\zeta}_e$ lies in $\mathcal{B}_{\delta/\epsilon}(0)$) and

$$\|\zeta^+_e - \zeta_0\|_{L^2(|k|<\delta/\epsilon)}^{5/4} \preccurlyeq \epsilon^{-1/4}\|\zeta^+_e - \zeta_0\|_{L^2(|k|<\delta/\epsilon)}^{1/2} \preccurlyeq \epsilon^{-1/4}\|\zeta^+_e - \zeta_0\|_{L^2(|k|>\delta/\epsilon)}^{1/2} \preccurlyeq \epsilon^{-1/4}\|\zeta^+_e - \zeta_0\|_{L^2(|k|>\delta/\epsilon)}^{1/4} \preccurlyeq \epsilon^{1/4},$$

(because $\hat{\zeta}_0 \in \mathcal{S}(\mathbb{R}^2)$, so that in particular $|\hat{\zeta}_0(|k|)|^2 \preccurlyeq (1 + |k|^2)^{-11/4}$. It follows that

$$\eta_1^+(x, z) = \epsilon \zeta_0(\epsilon x, \epsilon z)e^{ix} + \frac{1}{2} \epsilon (\zeta^+_e - \zeta_0)(\epsilon x, \epsilon z)e^{ix}$$

$$= \epsilon \zeta_0(\epsilon x, \epsilon z)e^{ix} + O(\epsilon^{5/4}),$$

uniformly in $(x, z)$. (These estimates remain valid when $\zeta^+_e$ and $\zeta_0$ are replaced by respectively $\zeta^-_e$ and $-\zeta_0$.)
Furthermore
\[ \| \eta_3(\eta_1) \|_\infty \lesssim \| \eta_3(\eta_1) \|_3 \lesssim \varepsilon^{10/6} \| \eta_1 \|_2^2 \lesssim \varepsilon^{10/6} \]
by Theorem 3.9 (recall that we have chosen \( \theta = 5/6 \)), while
\[ \| F(\eta_1) \|_\infty = O(\varepsilon^{11/6}) \]
because
\[ F(\eta_1) = -2 \left( (\eta_1^+)^2 + (\eta_1^-)^2 \right) + F_\varepsilon(\eta), \]
where
\[ \| F_\varepsilon(\eta_1) \|_\infty \lesssim \| F_\varepsilon(\eta_1) \|_3 \lesssim \| F_\varepsilon(\eta_1) \|_1 \lesssim \varepsilon^{11/6} \| \eta_1 \|_2^2 \lesssim \varepsilon^{11/6} \]
(see Proposition 4.3; the second estimate follows by the fact that the support of \( F[F_\varepsilon(\eta_1)] \) is bounded independently of \( \varepsilon \)).

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