Normalization of modes in an open universe

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We discuss the appropriate normalization of modes required to generate a homogeneous random field in an open Friedmann-Robertson-Walker universe. We consider scalar random fields and certain tensor random fields that can be obtained by covariantly differentiating a scalar. Modes of interest fall into three categories: the familiar sub-curvature modes, the more recently discussed super-curvature modes, and a set of discrete modes with positive eigenvalues which can be used to generate homogeneous tensor random fields even though the underlying scalar field is not homogeneous. A particular example of the last case which has been discussed in the literature is the bubble wall fluctuation in open inflationary universes.

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I. INTRODUCTION

Recently, considerable attention has been directed to models of structure formation in an open universe [1]. The description of perturbations in an open universe is more subtle than in the spatially flat case, because one has to make a distinction between the concept of a function and that of a random field. The latter can be thought of as an ensemble of functions with a probability assigned to each one. In cosmology we are interested in some finite region around us, possibly much bigger than the Hubble distance. A given perturbation about the homogeneous background, say the density perturbation, is some function of space-time in this region, but the specific form of the function is not thought to be very important. Rather, we make the hypothesis that it is a typical realization of a random field. It is assumed that the field is homogeneous with respect to the translations and rotations of the coordinate system. It is usually also assumed to be Gaussian, which means that there exists an expansion into mode functions with independent Gaussian probability distributions for the coefficients. From now on we usually take the homogeneity and Gaussianity as read, referring simply to ‘a random field’.

Let us first recall the situation for a scalar field, such as the density perturbation. In order to generate a random field (and also to decouple the time dependence of the modes in the linear approximation) the mode functions will be eigenfunctions of the Laplacian in comoving coordinates. In a flat universe, the most general square-integrable function may be expanded in terms of the eigenfunctions with \( k^2 > 0 \) (as usual \(-k^2\) is the eigenvalue of the Laplacian). In that case these same functions also give the most general scalar random field. In an open universe the situation is different. In units of the curvature scale, the most general square-integrable function may be expanded using only the \( k^2 > 1 \) modes (sub-curvature modes). In particular a perturbation defined in any finite region can be so expanded, which is all we need for any cosmological application. But now the situation for a random field is different; to obtain the most general scalar random field one needs, in addition, the \( 0 < k^2 \leq 1 \) modes (super-curvature modes). This fact has been known to mathematicians for half a century [2,3], though it has only recently been brought to the attention of the astrophysics community [4]. One of the objects of the present paper is to derive, in a simple way, the normalization of the mode functions which will lead to a homogeneous random field (with a natural choice for the variances of the coefficients).

In cosmology we might also be interested in vector and (second rank or higher) tensor functions, and the corresponding random fields. In general such objects have to be expanded in terms of different mode functions, which we shall not consider. But an important special case arises when the vector or tensor is the spatial derivative of a scalar. The central purpose of this paper is to treat this case, and focus on a fact concerning the open universe which does not seem to have been discussed in the mathematics literature. Namely, that a homogeneous vector or tensor random field can be constructed by differentiating an inhomogeneous scalar random field. We
conjuncture that the most general random field derived from a scalar can be expanded in terms of the continuum modes plus some new mode functions with \( k^2 = 1 - n^2 \) (discrete modes) where \( n \) is an integer. We support our conjecture by displaying an explicit normalization of the mode functions for the cases \( k^2 = 0 \) and \( k^2 = -3 \) which is suitable for generating a tensor random field. Examples of both cases exist in the recent literature. A discrete mode with \( k^2 = 0 \) is generated by fluctuations a massless scalar field in de Sitter space-time and the effect on the microwave background of such a mode was evaluated in Ref. [3]. A discrete \( k^2 = -3 \) mode can be generated by fluctuations of the bubble wall in single-bubble models of open universe inflation.

To be more specific, we write down the mode expansion for a scalar random field in terms of eigenfunctions of the spatial Laplacian, \( \nabla^2 Z_{klm} = -k^2 Z_{klm} \):

\[
f(r, \theta, \phi) = \int dk \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{klm} Z_{klm}(r, \theta, \phi) ,
\]

where the \( f_{klm} \) are members of an ensemble. The possible values of \( k \) will be discussed in what follows. We will consider both a continuous spectrum of modes and discrete values of \( k \). We work in a spherical coordinate system with line element

\[
ds^2 = dr^2 + \sinh^2 r(d\theta^2 + \sin^2 \theta d\phi^2),
\]
corresponding to the homogeneous spatial hypersurfaces in an open Friedmann–Robertson–Walker universe. The normalized eigenfunctions can be written as

\[
Z_{klm}(x) = \Pi_{kl}(r) Y_{lm}(\theta, \phi),
\]

where \( Y_{lm}(\theta, \phi) \) are the usual spherical harmonics on the two-sphere and \( \Pi_{kl}(r) \) are eigenfunctions of the operator

\[
\frac{1}{\sinh^2 r} \frac{d}{dr} \left( \sinh^2 r \frac{d}{dr} \right) + \frac{l(l+1)}{\sinh^2 r}.
\]

The normalization of the \( \Pi_{kl}(r) \) is the subject of the present paper.

In a Gaussian random field, if one uses the complex form of the spherical harmonics the magnitudes \( |f_{klm}| \) of each coefficient have independent Gaussian probability distributions for \( m \geq 0 \). The reality condition

\[
f_{kl,-m} \Pi_{kl} = (-1)^m (f_{klm} \Pi_{kl})^*,
\]

which follows from \( Y_{l,-m}(\theta, \phi) = (-1)^m Y_{lm}^*(\theta, \phi) \), fixes the phase of the \( m = 0 \) modes, while the other modes have uniformly distributed random phases subject to the above equation. In what follows we will find it more convenient to use the real form of the \( Y_{lm}(\theta, \phi) \); the coefficients then have fixed phases, \( \arg(f_{klm}) = -\arg(\Pi_{kl}) \), and the magnitudes of \( f_{klm} \) are independent random variables for all \( m \) from \(-l\) to \(+l\).

The variance of the distribution is defined by

\[
\langle f_{klm}^* f_{k'l'm'} \rangle = A_k \mathcal{P}(k) \delta(k-k') \delta_{l'l'} \delta_{m'm}.
\]

where the brackets denote an ensemble average. As we shall discuss in the next subsection, the variance is taken to be independent of \( l \) and \( m \) to allow us to construct a homogeneous random field. Several different conventions exist in the literature concerning the definition of the power spectrum \( \mathcal{P}(k) \), corresponding to different choices of the prefactor \( A_k \). In what follows we shall not need to define a particular separation. Since the field is Gaussian, it is completely defined by the two-point correlation function. For the continuous case this is defined by

\[
\langle f^*(x_1) f(x_2) \rangle = \int dk A_k \mathcal{P}(k) \sum_{lm} Z_{klm}^*(x_1) Z_{klm}(x_2),
\]

and in the discrete case the integral over \( k \) is replaced by a sum. Here and throughout we use the abbreviated notation

\[
\sum \rightarrow \sum_{l=-m}^{m} \sum_{m=-l}^{l}.
\]

II. UNITARITY AND HOMOGENEITY

The above procedure generates a scalar Gaussian random field. The field is said to be homogeneous if the correlation function depends only on the geodesic distance between the two points. By covariantly differentiating such a field, one can obtain a tensor random field which is likewise homogeneous if its correlation function is unaffected by a coordinate change (except of course for the transformation that the change induces in the components of the tensor). It is obvious that differentiating a homogeneous scalar random field always gives a homogeneous tensor random field, but we shall demonstrate that in some circumstances one can also obtain a homogeneous tensor field by differentiating an inhomogeneous scalar field.

The central purpose of this paper is to ask what restriction is placed on the normalization of the modes by the requirement that the random field be homogeneous, and to discuss which values of \( k \) are compatible with this requirement. The homogeneity requirement, that the correlation function depends only on the geodesic distance between the points, is equivalent to the requirement that under a shift in the origin or orientation of the coordinate system the Gaussianity is preserved and the correlation function is unaltered. We begin by showing that for a scalar field this is the case if and only if such a coordinate change corresponds to a unitary transformation of
The basis functions. Since a coordinate change does not affect the eigenvalues of the Laplacian, it will be useful to consider a field constructed from modes with a single value of \( k \)

\[
f(r, \theta, \phi) = \sum_{lm} f_{klm} Z_{klm}(r, \theta, \phi).
\]

The joint probability distribution for the coefficients is

\[
\text{probability} = N \exp \left( -\frac{\sum_{lm} |f_{klm}|^2}{2 A_k P(k)} \right),
\]

where the normalization factor \( N \) is actually infinitesimal because the sum is infinite).

Under a change of coordinates, the mode functions undergo a linear transformation

\[
Z_{klm}(r, \theta, \phi) = \sum_{l'm'} U_{l'm'}^k Z_{k'l'm'}(r', \theta', \phi'),
\]

and the mode expansion in Eq. (3) becomes

\[
f(r, \theta, \phi) = \sum_{lm} f_{klm} \sum_{l'm'} U_{l'm'}^k Z_{k'l'm'}(r', \theta', \phi') = \sum_{l'm'} f'_{kl'm'} Z_{k'l'm'}(r', \theta', \phi'),
\]

where

\[
f'_{kl'm'} = \sum_{lm} U_{l'm'}^k f_{klm}.
\]

The form of the joint probability distribution, Eq. (10), with respect to the transformed coefficients is clearly unaltered if and only if

\[
\sum_{l'm'} |f'_{kl'm'}|^2 = \sum_{lm} |f_{klm}|^2,
\]

which requires the matrix \( U \) to be unitary:

\[
\sum_{lm} (U_{l'm'}^{k'})^* U_{lm}^k = \delta_{l'l'} \delta_{m'm'}.
\]

Although we have shown this only for modes with a single value of \( k \), the correspondence expression for a continuous spectrum of modes will also be unaffected, since the transformation does not act on \( k \). One can show that the correlation function defined by Eq. (7), or the spectrum in Eq. (8), are also unaffected for a unitary transformation between modes. Thus the field is homogeneous if and only if \( U \) is unitary.

The unitarity requirement allows one to change the normalization of the modes by an arbitrary \( k \)-dependent real factor, and a completely arbitrary phase. What is important is the dependence of the magnitude of the normalization factor on \( l \) and \( m \). The normalization of the spherical harmonics \( Y_{lm}(\theta, \phi) \) ensures that they transform unitarily under an arbitrary rotation (and hence the distribution defined in Eq. (3) is isotropic), but we have to ensure that the complete basis functions \( Z_{klm} \) transform unitarily under both rotations and shifts of origin.

Note that a rotation about a fixed origin leaves \( k \) and \( l \) fixed but mixes the different \( m \)-multipoles, while a shift along the \( \theta = 0 \) axis leaves \( k \) and \( m \) fixed but mixes different \( l \) multipoles. Because an arbitrary shift and rotation of the origin can be decomposed into a rotation, followed by a shift along \( \theta = 0 \), followed by another rotation, the unitarity of the spherical harmonics under rotations fixes the \( m \)-dependence of the normalization. We now seek the correct \( l \)-dependence of the normalization of the radial functions \( \Pi_{kl}(r) \) which ensures homogeneity under a shift of the origin.

### III. Homogeneous Scalar Random Fields

#### Sub-curvature modes

For modes with \( k^2 > 1 \), we can split \( \Pi_{kl}(r) \) as

\[
\Pi_{kl}(r) = N_{kl} \tilde{\Pi}_{kl}(r),
\]

where \( N_{kl} \) is a normalization factor, to be determined, and \( \tilde{\Pi}_{kl}(r) \) are the unnormalized functions.\(^\dagger\)

\[
\tilde{\Pi}_{kl}(r) = q^{-2} (\sinh r)^l \left( \frac{-1}{\sinh r} \frac{d}{dr} \right)^{l+1} \cos(qr),
\]

where \( q^2 = k^2 - 1 \). For these modes, \( q = \pm \sqrt{k^2 - 1} \) is real and the eigenfunctions are exponentially decreasing beyond the curvature scale [which equals unity for the line element in Eq. (2)]. It is possible to choose the normalization factor \( N_{kl} \) to give an orthogonality relation between the modes (not necessarily orthonormality) of the form

\[
\int dV Z_{klm}(x) Z_{k'l'm'}(x) = B_k \delta(|q| - |q'|) \delta_{ll'} \delta_{mm'},
\]

where \( B_k \) is a finite real function of \( k \), independent of \( l \) and \( m \). This relation ensures unitarity of the matrix \( U \) for a coordinate shift.\(^\ddagger\) as we show in Appendix A. In particular, the usual normalization factor is taken to be \( 1 \).

\(^\dagger\)The invariance of the correlation function was shown in Ref. 1. As in that reference we treat infinite sums as finite, which should be valid if the infinite sum Eq. (5) is uniformly convergent. That has been demonstrated in Ref. 3 for \( k^2 > 0 \), but we have not investigated the question for \( k^2 \leq 0 \).
corresponding to \( B_k = 1 \) in Eq. (15).

Note that the radial functions \( \Pi_{kl}(r) \) and the normalization Eq. (15) are real, and one can always use the real form of the spherical harmonics. With that convention, \( U \) is real and unitarity of the transformation, given in Eq. (13), implies that the matrix is also orthogonal:

\[
\sum_{lm} U^k_{lm} U^\dagger_{lm} = \delta_{kk'} \delta_{mm'}.
\]

**IV. POSITIVE-EIGENVALUE MODES**

Modes with positive eigenvalues not only extend beyond the curvature scale, but actually diverge as \( r \to \infty \). However we have seen that the super-curvature modes with \( 0 < k^2 < 1 \) can form a homogeneous random field despite being non-square integrable, so one might ask whether a similar analytic continuation from the normalized sub-curvature modes might also give a homogeneous random field for \( k^2 < 0 \).

For \( k^2 < 0 \), the unnormalized radial functions given by Eq. (22) remain real; however the monopole normalization in Eq. (14) is purely imaginary, while the dipole acquires an extra factor of \( i \) and becomes real. Thus the normalized mode functions no longer have a unique phase. This means that the transformation matrices \( U \) can no longer be purely real, unless the modes of differing phase do not mix with one another under a change of coordinates. Thus Eq. (15), although it still holds, no longer guarantees unitarity, Eq. (15).

For instance, for \(-3 < k^2 < 0\), the \( l \geq 1 \) multipoles are all real but the monopole is purely imaginary. One might hope to build a homogeneous random field from only the higher multipoles, but the orthogonality relation, Eq. (24), involves a sum over all the multipoles. Moreover a distribution with no monopole in one coordinate system will in general acquire a monopole term after a shift of the origin, which implies inhomogeneity.

More generally, for \( 1 - (n+1)^2 < k^2 < 1 - n^2 \) (where \( n \geq 1 \) is an integer), the normalization given by Eq. (24)

\[
N_{kl} = -\sqrt{\frac{2}{\pi}} |q|^2 \prod_{s=0}^l (s^2 - |q|^2)^{-1/2},
\]

which is purely imaginary for all \( l \), the transformation matrix \( U \) will also be an analytic continuation. We know that Eq. (15) holds in the sub-curvature regime, and in fact it also holds in the super-curvature regime. At first sight it appears that we cannot appeal to the uniqueness of analytic continuation to demonstrate this, because the left hand side of Eq. (15) is not holomorphic. However, if real mode functions are used the unitarity relation becomes the orthogonality relation Eq. (24), which is preserved under the analytic continuation into the super-curvature regime. The transformation matrices \( U \) remain real in this regime because all the \( N_{kl} \) in Eq. (24) are purely imaginary, and so Eq. (24) implies unitarity and homogeneity.
has a phase \((l - 1)\pi/2\) for \(l < n\), while for all \(l \geq n\) it has
the same phase, \((n - 1)\pi/2\). A shift of the origin mixes all
the multipoles and any attempt to construct a random
scalar field only from modes with the same phase fails to
respect homogeneity.

**Homogeneous tensor fields when \(k^2 = 1 - n^2\)**

An interesting situation arises for the discrete set of
modes when \(k^2 = 1 - n^2\) with \(n \geq 1\) an integer. These
modes can be obtained from the discrete closed uni-
verse modes with \(k^2 = n^2 - 1\) by analytically continuing
both the radial coordinate \(r \rightarrow ir\) and the wavenumber
\(k \rightarrow ik\) \(^\dagger\).

Note that Eqs. (22) and (23) give finite expressions
for the first \(n\) multipoles, while for the higher multipoles
\(N_{kl}\Pi_{kl}(r) = \mathcal{O}(\epsilon)\), where \(\epsilon^2 = n^2 - 1 + k^2\). An alternative
is to define

\[
\tilde{N}_{kl} \equiv \lim_{\epsilon \to 0} \epsilon N_{kl}, \tag{24}
\]

\[
\Pi_{kl} \equiv \lim_{\epsilon \to 0} \frac{\Pi_{kl}}{\epsilon^2}, \tag{25}
\]

which gives finite expressions for the \(l \geq n\) multipoles,
while with this normalization the lower multipoles di-
 verge as \(1/\epsilon\). From Eqs. (17) and (25), we have, for \(l \geq n\),

\[
\Pi_{kl} = \frac{1}{2n^2}(\sinh r)^l \left(\frac{-1}{\sinh r} \frac{d}{dr}\right)^{l+1} (r \sinh nr). \tag{26}
\]

Rescaling all the mode functions independently of \(l\) and
\(m\) does not change the transformation matrix \(U\).

We show in Appendix B that the matrix \(U\) becomes
block diagonal in the limit \(\epsilon \to 0\), so that the orthogo-
nality relation, Eq. (23), holds separately for transforma-
tions between the \(l < n\) multipoles and for transforma-
tions between the \(l \geq n\) multipoles.

In a closed universe it is well known that the \(l < n\)
multipoles form a closed unitary group under coordinate
transformations. In an open universe, while the \(l < n\)
modes still form a closed group, the mode functions have
alternating phases and so the transformation matrix
is not purely real. Thus the orthogonality relation no longer
implies unitary, and so one cannot form a homogeneous
random scalar field.

By contrast, the \(l \geq n\) multipoles all have the same
phase in an open universe (whereas in a closed universe
they have alternating phases). Hence the sub-matrix of
\(U\) connecting the higher multipoles is real, and the trans-
formation is unitary. One might think that it is possible
to construct a homogeneous scalar random field from the
\(l \geq n\) multipoles alone. However this is not possible,
because the lower multipoles can be regenerated by a co-
deordinate transformation. This is despite the fact that
\(U\) becomes block diagonal, as the contribution from the
diverging low multipoles given in Eqs. (24) and (25) re-
 mains finite in the limit \(k^2 \to 1 - n^2\) even though the
matrix elements approach zero.

However if we act on the scalar field with an operator
which kills the lower multipoles, we obtain a homoge-
nous tensor random field even though the underlying
scalar field is inhomogeneous. We now discuss two phys-
ical cases in which this does in fact occur.

\[k^2 = 0\] modes

As \(k^2 \to 0\), the normalization factor for the monopole
in Eq. (23) gives \(N_{00} = i\sqrt{2/\pi}\) while the higher mul-
tipoles diverge as \(N_{0l} \propto i/k\). At the same time, the
unnormalized monopole \(\Pi_{00} \to 1\), while the mode func-
tions given by Eq. (22) vanish for \(l \geq 1\) as \(\Pi_{kl} \propto k^2\). One

\[
\tilde{N}_{0l} \equiv \lim_{k \to 0} k N_{kl}, \tag{27}
\]

\[
\Pi_{0l} = \lim_{k \to 0} \frac{\Pi_{kl}}{k^2}. \tag{28}
\]

which leaves these modes finite (although the monopole
becomes infinite with this normalization). Such a mode
appears if one considers the quantum fluctuations of a
massless scalar field in de Sitter space-time using an open
universe coordinate system \([8]\). The anisotropy of the
microwave background sky due to curvature perturbations
with \(k^2 = 0\) in an open universe has also been discussed
recently \([4]\).

The monopole \(N_{00} \Pi_{00}(\theta, \phi)\) is a constant when
\(k^2 = 0\). As a result any tensor field constructed by covar-
iant differentiation of the scalar field will be homogeneous.
The simplest example is the vector field

\[
V^i \equiv \nabla^i f. \tag{29}
\]

Note that, in the notation of Ref. [12], \(V^i\) is indistinguish-
able from an intrinsically ‘vector’ quantity when \(k^2 = 0\),
as it is solenoidal, \(\nabla_i V^i = 0\).

\[k^2 = -3\] modes

Another interesting case of the \(k^2 = 1 - n^2\) modes
discussed above is \(n = 2\). In the limit \(k^2 \to -3\), the
monopole normalization is \(N_{k0} = 2i\sqrt{2/\pi}\) and that of
the dipole is \(N_{k1} = 2i\sqrt{2/3\pi}\), while the normalizations
of the higher multipoles diverge as \(N_{kl} \propto 1/\epsilon\), where
\(\epsilon^2 = 3 + k^2\). On the other hand, the radial functions
behave as \(\Pi_{k0} \to \cosh r, \Pi_{k1} \to -\sinh r, \Pi_{kl} \propto \epsilon^2\)

\(^\dagger\)Note there is a typographical error in Eq. (100) of Ref. [4].
for \( l \geq 2 \). We can use the normalization given in Eqs. (24) and (25) to render the \( l \geq 2 \) multipoles finite.

To construct a homogeneous tensor random field in this case we need to kill both the monopole and the dipole. The simplest operator which does this is

\[
T_{ij} = \nabla_i \nabla_j - \frac{1}{3} \gamma_{ij} \nabla^2,
\]

where \( \gamma_{ij} \) is the spatial metric. The explicit form of the tensor components for the metric given by Eq. (2) are given in Appendix C. Acting on a scalar, this gives a traceless and (for \( k^2 = -3 \)) transverse second-rank tensor. Because of the transversality, there is no distinction between these ‘scalar’ metric perturbations and intrinsically ‘tensor’ gravitational waves, in the notation of Ref. [4]. A physical example of this case is the metric perturbation associated with quantum fluctuations of the bubble wall in open inflation models [8].

The lowest non-vanishing modes are the quadrupole and octopole whose radial dependence is given from Eq. (26) as

\[
\Pi_{k2} = -\frac{1}{8} \left( 2 \cosh r - \frac{3 \cosh r}{\sinh^2 r} + \frac{3r}{\sinh^3 r} \right),
\]

\[
\Pi_{k3} = -\frac{1}{8} \left( 2 \sinh r - \frac{5}{\sinh r} - \frac{15}{\sinh^3 r} + \frac{15r \cosh r}{\sinh^3 r} \right).
\]

Note that these higher multipoles of the scalar field diverge as \( \Pi_{kl} \sim e^r \) as \( r \to \infty \). The action of the tensor operator renders some components of \( T_{ij} \) finite at infinity, such as \( T_{rr}, T_{r\theta} \) and \( T_{r\phi} \) which are of order \( e^{-r} \), but the remaining components of \( T_{ij} \) still diverge as \( e^r \). Nonetheless, due to the form of the metric inverse [\( \gamma^{ij} = \text{diag}(1,1/\sin^2 r, 1/\sin^2 \theta \sin^2 r) \)] this is sufficient to leave scalar invariants finite at infinity, e.g., \( T_{ij} T^{ij} \sim e^{-2r} \).

V. CONCLUSIONS

To summarize, the normalization defined by Eq. (19) for sub-curvature modes can be used to generate a homogeneous field because of the orthogonality relation in Eq. (18). We have shown that it remains valid in the super-curvature regime, by virtue of the fact that the radial mode functions for a given eigenvalue \( k^2 \) have the same phase.

These normalized eigenfunctions can be written, for both sub- and super-curvature modes, as

\[
\Pi_{kl}(r) = \left( \frac{\Gamma(l + 1 + i|q|)\Gamma(l + 1 - i|q|)}{\Gamma(|q|)\Gamma(-i|q|)} \right)^{1/2} \times \frac{P_{|q|-1/2}^{-l-1/2}(\cosh r)}{\sqrt{\sinh r}},
\]

where \( q^2 = k^2 - 1 \). For sub-curvature modes, these functions are real, while for super-curvature modes they are purely imaginary.

If we were to multiply the mode functions by an \( l \)-dependent phase in the sub-curvature regime, they would still be suitably normalized there but the continuation of the phase factor to the super-curvature regime would in general spoil the normalization of the super-curvature modes. An example of this would be to replace the first factor in Eq. (19) by \( \Gamma(l + 1 + iq)/\Gamma(iq) \). Both normalizations are equivalent in the sub-curvature regime, but the latter is not suitable for analytic continuation to the super-curvature regime, where it gives an incorrect normalization and also fails to be symmetric under \( q \leftrightarrow -q \). Although Lyth and Woszczyna [4] and Hamazaki et al. [9] both quoted this latter form for the sub-curvature modes, they did not use it directly for analytic continuation and in fact both these papers obtained the satisfactory super-curvature mode normalization given by Eq. (33) above.

For \( k^2 \leq 0 \) there is no unique phase, and therefore one cannot use the analytically continued mode functions to construct a homogeneous scalar random field. However, in the specific case \( k^2 = 1 - n^2 \), for integer \( n \), the transformation between multipoles with \( l \geq n \) is unitary, which allows a homogeneous tensor random field to be constructed by acting on the scalar with a covariant differential operator, provided that the operator kills the lower multipoles. For \( k^2 = 0 \) we have noted that any differential operator does this, since the monopole is spatially constant. For \( k^2 = -3 \) we have seen that the traceless symmetric second-rank tensor does this. A physical example of the latter is the metric perturbation generated by bubble wall fluctuations in open inflation models. The normalized eigenfunctions for the higher multipoles, given in Eqs. (24) and (25), can be written as

\[
\Pi_{nl}(r) = \left( \frac{\Gamma(l + 1 + n)\Gamma(l + 1 - n)}{2} \right)^{1/2} \times \frac{P_{|n|-1/2}^{-l-1/2}(\cosh r)}{\sqrt{\sinh r}}.
\]

We recover the flat-space limit by taking \( r \to 0 \) while keeping \( |q|r \) fixed. Only the sub-curvature modes survive in this limit, where they tend to the usual spherical Bessel functions [4]. The super-curvature modes, which have \( |q| < 1 \), are not present in this limit, and the \( l \geq n \) multipoles of the discrete modes with \( k^2 = 1 - n^2 \) can be discarded in this limit since \( n \to \infty \).

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**APPENDIX A: MODE ORTHOGONALITY AND TRANSFORMATION UNITARITY**

Here we prove that the orthogonality of the sub-curvature modes implies that the transformation induced by shifts in origin and orientation of coordinates is unitary. An alternative proof was given in Ref. [4]. This means that a random field generated from them is homogeneous.

Substituting the transformation

$$Z_{klm}(r, \theta, \phi) = \sum_{l'm'} U_{mlm'}^k Z_{klm'}(r', \theta', \phi'),$$  \hspace{1cm} (A1)

into the orthogonality relation

$$\int dV Z^*_{klm}(x) Z_{kl'm'}(x) = B_k \delta(|q| - |q'|) \delta_{ll'} \delta_{mm'},$$  \hspace{1cm} (A2)

gives

$$\int dV Z^*_{klm}(x) Z_{kl'm'}(x) = \int dV \sum_{l'm'} (U^k_{mlm'})^* Z^*_{klm'}(x') \times \sum_{l'm'} U_{mlm'}^k Z_{klm'm'}(x')$$
$$= \sum_{l'm'} (U^k_{mlm'})^* U_{mlm'}^k B_k \delta(|q| - |q'|),$$ \hspace{1cm} (A3)

where the last equality uses the orthogonality relation, Eq. (A2), for the $x'$ coordinates. Thus we have

$$\sum_{l'm'} (U^k_{mlm'})^* U_{mlm'}^k = \delta_{ll'} \delta_{mm'},$$ \hspace{1cm} (A4)

and hence the transformation is unitary.

Note that orthogonality also gives us an expression for the matrix $U_{mlm'}^k$, namely

$$\int dV Z^*_{klm'}(x') Z_{klm}(x) = \int dV' Z^*_{klm'}(x') \sum_{l'm'} U^k_{mlm'} Z_{klm'm'}(x')$$
$$= U^k_{mlm'} B_k \delta(|q| - |q'|).$$ \hspace{1cm} (A5)

**APPENDIX B: BLOCK DIAGONALITY OF $U$ FOR $k^2 = 1 - n^2$**

In this Appendix we shall demonstrate that the matrix $U$ becomes block diagonal in the limit $k^2 \to 1 - n^2$ for integer $n \geq 1$. We split the transformation matrix $U$ into block matrix form

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$  \hspace{1cm} (B1)

where $A_{mlm'}^k \equiv U_{mlm'}^k$, for $l, l' < n$ and so on. Then the orthogonality condition $UU^T = I$ gives

$$AA^T + BB^T = I,$$  \hspace{1cm} (B2)
$$AC^T + BD^T = 0,$$  \hspace{1cm} (B3)
$$CA^T + DB^T = 0,$$  \hspace{1cm} (B4)
$$CC^T + DD^T = I,$$  \hspace{1cm} (B5)

where $I$ is the identity matrix.

We adopt the normalization of the mode functions given in Eqs. (B4) and (B5), which leaves the $l \geq n$ multipoles finite but leads to the $l < n$ multipoles diverging as $1/\epsilon$ where $k^2 = 1 - n^2 + \epsilon^2$. Consider a function composed solely of modes with $k^2 = 1 - n^2 + \epsilon^2$

$$f_k(x) = \sum_{lm} f_{klm} \bar{\Delta}_{kl}(r) Y_{lm}(\theta, \phi).$$ \hspace{1cm} (B6)

If this is to remain bounded at finite $r$ as $\epsilon \to 0$, we must have $f_{klm} = O(\epsilon)$ for $l < n$ and $f_{klm}$ finite for $l \geq n$. Under a coordinate transformation

$$f_k(x) = \sum_{l'm'} f'_{klm'} \bar{\Delta}_{kl}(r') Y_{l'm'}(\theta', \phi'),$$ \hspace{1cm} (B7)

where $f'_{klm'}$ is given by Eq. (B3), and we must likewise have $f'_{klm'} = O(\epsilon)$ for $l' < n$.

Thus the sub-matrix $C = O(\epsilon)$. In the limit $\epsilon \to 0$, Eq. (B4) becomes $BD^T = 0$ and Eq. (B5) becomes $DB^T = 0$. Because $D$ is orthogonal this implies $B = 0$, and hence the matrix $U$ is block diagonal. Finally, we note from Eq. (B2) that $AA^T = I$.

**APPENDIX C: $k^2 = -3$ TENSOR MODES**

In this Appendix we give the actual components of the (symmetric) operator $T_{ij}$ given in Eq. (C9) for the metric of Eq. (C3),

$$T_{rr} = \frac{\partial^2}{\partial r^2} - \frac{1}{3} \nabla^2,$$  \hspace{1cm} (C1)
$$T_{r\theta} = \frac{\partial^2}{\partial \theta^2} + \sinh r \left( \cosh r \frac{\partial}{\partial r} - \frac{1}{3} \sinh r \nabla^2 \right),$$  \hspace{1cm} (C2)
$$T_{r\phi} = \frac{\partial^2}{\partial \phi^2} + \sin \theta \cos \theta \frac{\partial}{\partial \theta}$$
\[ T_{r\theta} = \left( \frac{\partial}{\partial r} - \frac{\cosh r}{\sinh r} \right) \frac{\partial}{\partial \theta}, \] (C4)

\[ T_{r\phi} = \left( \frac{\partial}{\partial r} - \frac{\cosh r}{\sinh r} \right) \frac{\partial}{\partial \phi}, \] (C5)

\[ T_{\theta\phi} = \left( \frac{\partial}{\partial \theta} - \frac{\cos \theta}{\sin \theta} \right) \frac{\partial}{\partial \phi}. \] (C6)

The unnormalized \( k^2 = -3 \) monopole and dipole modes are given, using Eq. (22), by

\[ \tilde{\Pi}_{k0}(r) Y_{00}(\theta, \phi) = \sqrt{\frac{1}{4\pi}} \cosh r, \] (C7)

\[ \tilde{\Pi}_{k1}(r) Y_{10}(\theta, \phi) = -\sqrt{\frac{3}{4\pi}} \sinh r \cos \theta, \] (C8)

\[ \tilde{\Pi}_{k1}(r) Y_{11}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \sinh r \sin \theta \cos \phi, \] (C9)

\[ \tilde{\Pi}_{k1}(r) Y_{1(-1)}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \sinh r \sin \theta \sin \phi. \] (C10)

It is then straightforward to verify that each component of \( T_{ij} \) vanishes everywhere when applied to the monopole and dipole modes. Thus the action of the operator in Eq. (30) on the \( k^2 = -3 \) modes can indeed form a homogeneous tensor field from the \( l \geq 2 \) multipoles of the scalar field whose normalized radial functions are given by Eqs. (24) and (25).