We use the Effective Field Theory (EFT) framework to compute the mass quadrupole moment, the equation of motion, and the power loss of inspiralling compact binaries at the second order in the Post-Newtonian (PN) approximation. We present expressions for the stress-energy pseudo-tensor components of the binary system in higher PN orders. The 2PN correction to the mass quadrupole moment as well as to the acceleration computed in the linearized harmonic gauge presented here are the ingredients needed for the calculation of the next-to-next-to-leading order radiation reaction force, which will be presented elsewhere. While this paper reproduces known results, it supplies the building blocks necessary for future higher order calculations in the EFT methodology.

I. INTRODUCTION

The successful detections of gravitational waves by LIGO and Virgo [1–8] and the consequent advent of Multi-messenger Astronomy [9–11] have expedited the need for precise theoretical descriptions of the dynamics of binary inspirals. While numerical techniques are required for the late stages of inspirals, the early stage admits a perturbative treatment via the PN approximation, which is an expansion in $v^2/c^2$, and can be matched onto numerical results for later stages of the inspiral. Generating higher order PN corrections will allow for more accurate parameter estimations.

In this paper, we will utilize the EFT approach called Non-Relativistic General Relativity (NRGR), proposed in [12] (for reviews see [13–17]), as our calculation framework. To date, most of the results in the non-spinning sector of the EFT formalism have been geared towards the potential sector culminating in the present state of the art 4PN results [18, 19], which agree with results previously derived using other methods [20–23]. In the radiation sector, the EFT results have only been calculated to 1PN [26] as compared to the 3PN results calculated using more traditional GR methods [27]. Therefore, this paper is the next step in the calculation of higher order radiative effects in NRGR. In particular, in a separate paper we will use the results herein to calculate the next-to-next-to leading order radiation reaction force via the generation of an effective action.

The radiation sector of NRGR, the topic of this paper, was first studied in [12]. The effective action that describes radiative effects is determined by the underlying symmetries - reparameterization and diffeomorphism invariances - and is applicable to arbitrary gravitational wave sources in the long wavelength approximation. The Wilson coefficients of the action, the multipole moments, cannot be determined by the symmetries and need to be fixed by a matching procedure. The expression for the effective action to all orders in the multipole expansion and the exact expressions of the multipole moments in terms of the components of the stress-energy tensor were presented in [28]. The NRGR framework provides a systematic way to compute the multipole moments of a binary system by integrating out the modes of the gravitational field that live in the near zone. The stress energy tensor, whose moments are our targeted goal, is determined by calculating the radiation graviton one point function in the presence of the background potentials using Feynman diagrams. The number of Feynman diagrams grows rapidly with PN order.

The goal of this paper is to determine the 2PN correction to the mass quadrupole moment, which comes from various moments of the stress-energy pseudo-tensor. Each such contribution starts at different order in the PN expansion and only a few of these contributions can be derived from known quantities. We also derive the equation of motion of the binary system at 2PN order in the appendix B. Note that this acceleration was calculated previously in the EFT approach in [29], where the authors worked with Kaluza-Klein variables [30] in conjunction with harmonic coordinates. The 2PN acceleration derived here, on the other hand, is written in the linearized (background) harmonic gauge, which leaves a gauge invariant effective action for the radiation field after the potential modes are integrated out, and can be used in combination with previous results obtained in the EFT approach where the linearized harmonic gauge was used. Our results constitute the final missing part necessary for the computation of the next-to-next-to-leading order radiation reaction force as well as for the construction of spinning templates at 2.5 PN for the phase and 3PN for the amplitude. These computations are ongoing and will be reported in a subsequent publication.

---

1 For spinning constituents the relevant multipole moments at 3PN for the flux [24] and 2.5 for the amplitude [25].
This paper is organized as follows. In section II, we provide a summary of NRGR for binary systems of compact bodies with emphasis in the radiation sector, where we explicitly show how the mass quadrupole moment depends on the components of the pseudotensor in different PN orders. The contributions to the quadrupole that come from higher PN order components of the pseudotensor are computed in section III, while the contributions coming from the lower PN order components are obtained in section IV. We use the results obtained in these sections to write down, in section V, the components of the pseudotensor that can be used to compute the multipole moments, which are shown to agree with the literature. The assembly of all contributions constitutes the 2PN correction to the mass quadrupole moment, presented in section VI, in terms of the worldlines of the compact bodies and also in the center of mass frame. In section VII we present our final remarks on the results presented in this paper. Appendix A is intended for readers interested in computing radiation effects in NRGR to higher orders. The necessary ingredients for the computation of the higher PN order components of the pseudotensor are presented therein. In the appendix B we show the result for the acceleration at 2PN order computed in the linearized harmonic gauge, which is necessary to compare the quadrupole moment obtained in this paper with there result in [27], as well as to compute the power loss at the second PN order.

We use the following definitions throughout this paper: \( m = m_1 + m_2 \), \( v \equiv m_1m_2/m^2 \), and \( \mu = mv \). The relative position is defined as \( r \equiv x_1 - x_2 \), while \( v \equiv v_1 - v_2 \) and \( a \equiv a_1 - a_2 \) are the relative velocity and acceleration, respectively. We adopt the mostly minus signature convention for \( \eta^{\mu \nu} \) and Latin indices are contracted with the Euclidean metric. We use \( c = 1 \) units and the Planck mass is defined as \( m_{Pl} \equiv 1/\sqrt{32\pi G} \).

II. EFT SETUP

During the inspiral stage, the physics of a binary system of compact bodies is naturally separated into three length scales: the typical size of the bodies of order of the Schwarzschild radius \( r_s \), the orbital distance between the two bodies given by \( r \), and the wavelength \( \lambda_{GW} \) of the gravitational radiation. As the relative velocity \( v \) of the bodies is small, those three length scales together constitute a hierarchical structure

\[
r_s \ll r \ll \lambda_{GW}.
\]

The first step is to “integrate out” the scale associated with the bodies size.\(^2\) Hence, the binary system can be initially described by the action

\[
S = S_{EH} + S_{GF} + S_{pp},
\]

such that gravity is described by the Einstein-Hilbert (EH) action \( S_{EH} = -2m_{Pl}^2 \int d^4x \sqrt{-g} g^{\mu \nu} R_{\mu \nu} \) with a gauge fixing term \( S_{GF} \), while the massive bodies are described by the point particle action \( S_{pp} = -\sum_a m_a \int d\tau_a \). The index \( a = 1, 2 \) distinguishes the two bodies.

Next, the two different modes of the gravitational field are separated in a diffeomorphism invariant way\(^3\) via

\[
g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu}(x) = \eta_{\mu \nu} + \bar{h}_{\mu \nu}(x) + H_{\mu \nu}(x).
\]

The off-shell potential mode \( H \) obeys \( \partial_\nu H_{\mu \nu} \sim (\frac{v}{r}) H_{\mu \nu} \) and \( \partial_\mu H_{\mu \nu} \sim (\frac{v}{r}) H_{\mu \nu} \); whereas the on-shell radiation mode obeys \( \partial_\nu \bar{h}_{\mu \nu} \sim (\frac{v}{r}) \bar{h}_{\mu \nu} \). Moreover, the radiation field \( \bar{h}_{\mu \nu}(x) \) has to be Taylor expanded around a point inside the source (for instance center of mass of the binary system) at the level of the action in order to achieve a uniform power counting in the parameter \( v^2 \sim \xi \) [32]. With these considerations, the action in (2.2) is then given as an expansion in the fields \( \bar{h}_{\mu \nu}(x) \) and \( H_{\mu \nu}(x) \), each of which scale homogeneously in \( v^2 \).

To describe the dynamics associated to gravitational waves, the potential mode of the gravitational field is integrated leaving an effective action that will depend only on the radiation field and the worldlines. This action will be diffeomorphism invariant if one chooses the linearized harmonic gauge when integrating out the potential field, via the gauge fixing action

\[
S_{GF} = \int d^4x \sqrt{-g} \bar{\Gamma}_\mu \bar{\Gamma}^\mu,
\]

\(^2\) Finite size effects are accounted for by inserting higher-dimensional operators in the effective action, respecting the symmetries of the system.

\(^3\) Double counting subtleties arise at 4PN but can be systematically disentangled [31].
by definition the resulting action takes the form gravitational field. The latter action is used in order to compute the one-graviton emission amplitude. As a result, limit and the action valid below the orbital scale (2.2), which depends on both radiation and potential modes of the electric and the magnetic components of the Weyl tensor.

Gravitational waves propagate. The multipole moments, which constitute the source of radiation, are coupled to the internal dynamics of the radiation emitted. In the center-of-mass frame, the action of the radiation sector is \[ S_{\text{rad}}[\vec{h}, x_a] = -\int dt \sqrt{g_{00}} \left[ m + \frac{1}{2} L_{ij} \omega^{ij} + \sum_{i=2}^{\infty} \frac{1}{i!} I^L \nabla L_{-2} E_{i_{i-1}i} - \frac{2l}{(2l + 1)!} J^L \nabla L_{-2} B_{i_{i-1}i} \right], \tag{2.5} \]

where a multi-index representation \( L = i_1...i_l \) is used. The first two terms generate the Kerr background in which the gravitational waves propagate. The multipole moments, which constitute the source of radiation, are coupled to the electric and the magnetic components of the Weyl tensor.

To determine the moments, one performs a matching between the effective action (2.5) in the long wavelength limit and the action valid below the orbital scale (2.2), which depends on both radiation and potential modes of the gravitational field. The latter action is used in order to compute the one-graviton emission amplitude. As a result, by definition the resulting action takes the form

\[ \Gamma[\vec{h}] = -\frac{1}{2m_{\text{pl}}} \int d^4x T^{\mu\nu} \vec{h}_{\mu\nu}, \tag{2.6} \]

where \( T^{\mu\nu} \) is the stress-energy pseudotensor of the system. Relations from the Ward identity \( \partial_{\mu} T^{\mu\nu} = 0 \) as well as the on-shell equations of motion can be used to bring both actions (2.5) and (2.6) in a comparable form. After that, a general form for the mass quadrupole moment is obtained in terms of the components of the stress-energy pseudotensor and its derivatives,

\[ I_{ij} = \sum_{p=0}^{\infty} \frac{5!!}{(2p)!!(5 + 2p)!!} \left\{ \left( 1 + \frac{2p(3 + p)}{3} \right) \left[ \int d^3x \delta^{2p} T_{00} x^i x^j \right]_{TF} \right. \]
\[ + \left. \left( 1 + \frac{p}{3} \right) \left[ \int d^3x \delta^{2p+1} T_{0l} x^l x^i x^j \right]_{TF} - \frac{4}{3} \left( 1 + \frac{p}{2} \right) \left[ \int d^3x \delta^{2p+2} T_{kl} x^k x^l x^i x^j \right]_{TF} \right. \]
\[ + \left. \frac{1}{6} \left[ \int d^3x \delta^{2p+3} T_{kl} x^k x^l x^i x^j \right]_{TF} \right\}, \tag{2.7} \]

where TF stands for trace-free\(^4\). For the exact expressions for the multipole moments in all orders in the PN expansion, see [28]. The leading-order contribution to the mass quadrupole moment comes from just one term

\[ I_{0}^{ij}_{0PN} = \left[ \int d^3x T_{00}^{0PN} x^i x^j \right]_{TF} = \sum_{a} m_a \left[ x^i_a x^j_a \right]_{TF}, \tag{2.8} \]

while its 1PN correction [26] is given by four different contributions of the components of the stress-energy pseudoten-

\(^4\) More precisely, the multipole moments are symmetric trace-free (STF) quantities, but we are supressing the “S” in the label to avoid redundancy since the general expression for the quadrupole moment is explicitly written as a symmetric tensor already.
mv

If we take into account the zeroth order term of the exponential expanded in the radiation momentum $k$ to be obtained in the EFT formalism.

Introducing the partial Fourier transform of the stress-energy pseudotensor $T^{\mu\nu} (t, x) = \int d^3x T^{\mu\nu} (t, x) e^{-ik \cdot x}$, we consider the long wavelength limit $k \to 0$ to write

$$T^{\mu\nu} (t, k) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \left( \int d^3x T^{\mu\nu} (t, x) x^{i_1} \ldots x^{i_n} \right) k_{i_1} \ldots k_{i_n},$$

where each term in this expansion corresponds to a sum of Feynman diagrams that scale as a definite power of the parameter $v$. This partial Fourier transform is convenient since Feynman graphs are more easily handled in momentum space and, with the pseudotensor written in this way, we can read off the contributions to the mass quadrupole moment (2.10), the ultimate goal of this paper.

### III. Higher Order Stress-Energy Tensors

Introducing the partial Fourier transform of the stress-energy pseudotensor $T^{\mu\nu} (t, x) = \int d^3x T^{\mu\nu} (t, x) e^{-ik \cdot x}$, we consider the long wavelength limit $k \to 0$ to write

$$T^{\mu\nu} (t, k) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \left( \int d^3x T^{\mu\nu} (t, x) x^{i_1} \ldots x^{i_n} \right) k_{i_1} \ldots k_{i_n},$$

where each term in this expansion corresponds to a sum of Feynman diagrams that scale as a definite power of the parameter $v$. This partial Fourier transform is convenient since Feynman graphs are more easily handled in momentum space and, with the pseudotensor written in this way, we can read off the contributions to the mass quadrupole moment (2.10), the ultimate goal of this paper.

#### A. 2PN correction to $T^{00}$

The leading order and the next-to-leading order temporal components of the pseudotensor, obtained in [26] using the EFT techniques summarized in the previous section, are given by

$$T^{00}_{0PN} (t, k) = \sum_a m_a e^{-ik \cdot x_a},$$

$$T^{00}_{1PN} (t, k) = \left[ \sum_a \frac{1}{2} m_a v_a^2 - \sum_{a \neq b} \frac{G m_a m_b}{2\ell} + O (k) + \ldots \right] e^{-ik \cdot x_a}. \quad (3.3)$$

If we take into account the zeroth order term of the exponential expanded in the radiation momentum $k$, we see that the leading order pseudotensor provides the total mass whereas the next-to-leading order represents the Newtonian energy of a dynamical two-body system. These quantities scale as $mv^0$ and $mv^2$, respectively. Hence, to obtain the 2PN correction to the leading order $T^{00}$, we have to calculate all Feynman diagrams that contribute to the one-graviton $h_{00}$ emission and enter at order $v^4$. 

...
FIG. 1: No graviton exchange between the two particles, one external $\bar{h}^{00}$ momentum.

The simplest contribution to the second PN correction for the temporal component of the stress-energy pseudotensor is illustrated in Fig. 1 and comes from the source action term (A7). Comparing this diagram against (2.6), we extract the following contribution to the pseudotensor

$$T_{\text{Fig} 1}^{00} (t, \mathbf{k}) = \sum_a \frac{3}{8} m_a v_a^4 e^{-i \mathbf{k} \cdot \mathbf{x}_a}. \quad (3.4)$$

By expanding the exponential up to the second order in the radiation momentum $\mathbf{k}$, we read off the contribution for the mass quadrupole moment:

$$\int d^3 \mathbf{x} T_{\text{Fig} 1}^{00} [\mathbf{x}^i \mathbf{x}^j]_{TF} = \sum_a \frac{3}{8} m_a v_a^4 \left[ x_a^i x_a^j \right]_{TF}. \quad (3.5)$$

FIG. 2: One-graviton exchange with external $\bar{h}^{00}$ momentum.

The diagrams that contain the exchange of one potential graviton are shown in Fig. 2 and are composed by the couplings between the source action terms (A1-A6) and also the propagator (A15) and its correction (A16). Notice that we need not separate out all of the various terms that arise in the Feynman rules into different orders in the PN expansion as is done in the appendix. We also calculated covariantly vertices, as is done when calculating in the Post-Minkowskian (PM) expansion (see e.g. [33]), and then expand in $v$, as a calculational check. However, for pedagogical purposes we have separated Feynman rules into give orders in the PN expansion. The results from Fig. 2 is given by

$$T_{\text{Fig} 2a}^{00} (t, \mathbf{k}) = \sum_{a \neq b} \frac{5}{2} \frac{G m_a m_b}{r} v_a^2 v_b^2 e^{-i \mathbf{k} \cdot \mathbf{x}_a}, \quad (3.6)$$

$$T_{\text{Fig} 2b}^{00} (t, \mathbf{k}) = \sum_{a \neq b} \frac{3}{2} \frac{G m_a m_b}{r} v_a^2 v_b^2 e^{-i \mathbf{k} \cdot \mathbf{x}_a}, \quad (3.7)$$

$$T_{\text{Fig} 2c}^{00} (t, \mathbf{k}) = - \sum_{a \neq b} \frac{4}{2} \frac{G m_a m_b}{r} v_a \cdot v_b e^{-i \mathbf{k} \cdot \mathbf{x}_a}, \quad (3.8)$$

$$T_{\text{Fig} 2d}^{00} (t, \mathbf{k}) = \sum_{a \neq b} \frac{G m_a m_b}{2r} \left( -v_a^i r_i + v_b^2 - (v_b \cdot n)^2 \right) e^{-i \mathbf{k} \cdot \mathbf{x}_a}. \quad (3.9)$$

Leaving

$$\int d^3 \mathbf{x} T_{\text{Fig} 2a-2d}^{00} [\mathbf{x}^i \mathbf{x}^j]_{TF} = \sum_{a \neq b} \frac{G m_a m_b}{2r} \left[ (5v_a^2 + 4v_b^2 - 8v_a \cdot v_b - a_b \cdot r - (v_b \cdot n)^2) x_a^i x_a^j \right]_{TF}. \quad (3.10)$$
Note the implicit dependence on the indices $a, b$ in the quantities $r = x_a - x_b, \quad r = |r|$ and $n = \frac{r}{r}$ inside the sum.

\[
\frac{1}{q^2 (q + k)^2} = \frac{1}{q^2} - \frac{2 (q \cdot k)}{q^6} + \frac{4 (q \cdot k)^2}{q^8} + ..., \quad (3.11)
\]

In calculating the contributions to the mass quadrupole sourced by the temporal components of the pseudotensor at 2PN, we are allowed to drop terms depending on $k^2$ in the expansion of the denominator, since those terms contribute to the trace part of the mass quadrupole, which is removed in the definition of the STF moment. The results are organized in orders of the radiation momentum, as it is shown below:

\[
T_{\text{Fig3a}}^{00} (t, k) = \sum_{a \neq b} \frac{Gm_a m_b}{4r} e^{-ik \cdot x_a} \left\{ 2 (v_a^2 + a \cdot r - r^2) + 5 v_a \cdot v_b - 5 v_a \cdot n v_b \cdot n \right. \\
+ \frac{1}{2} k^i \left[ \left( v^2 + a \cdot r - r^2 + \frac{5}{2} v_a \cdot v_b - \frac{5}{2} v_a \cdot n v_b \cdot n \right) r^i \\
+ \left( \frac{1}{2} r^i + \frac{5}{2} v_b \cdot r \right) v^i - \left( \frac{2}{2} r^i + \frac{5}{2} v_b \cdot r \right) v^i \\
+ \frac{1}{6} k^i k^j \left[ - (2v^2 + 5 v_a \cdot v_b - 2r^2 - 5v_a \cdot n v_b \cdot n + 2a \cdot r) r^i r^j \\
+ (4v_a \cdot r + v_b \cdot r) v^i r^j - (2v_a \cdot r + 8v_b \cdot r) v^i r^j \\
+ r^2 \left( -4v^i v^j - 7v^j v^i + 2a^i r^j + 4a^j r^i \right) \right] \right\} + O \left( k^3 \right) + ..., \quad (3.12)
\]

\[
T_{\text{Fig3b}}^{00} (t, k) = - \sum_{a \neq b} \frac{Gm_a m_b}{r} e^{-ik \cdot x_a} \left\{ \frac{7}{4} v_b^2 + \frac{3}{4} v_a^2 - \frac{i}{2} k^i v^i v_b \right. \\
+ \frac{1}{2} k^i k^j \left[ \frac{1}{2} v^2 r^i r^j + 2r^i v^i v^j \right] \right\} + O \left( k^3 \right) + ..., \quad (3.13)
\]

\[
T_{\text{Fig3c}}^{00} (t, k) = \sum_{a \neq b} \frac{2Gm_a m_b}{r} e^{-ik \cdot x_a} \left\{ 2v_a \cdot v_b + k^i k^j v^i v^j + O \left( k^3 \right) + ..., \quad (3.14)
\]

\[
T_{\text{Fig3d}}^{00} (t, k) = - \sum_{a \neq b} \frac{2Gm_a m_b}{r} e^{-ik \cdot x_a} \left\{ - \frac{i}{2} k^i k^j \left[ 2r^i a^j_a + 2v_a^i \left( v_b \cdot r + v_a \cdot r \right) \right] \\
- \frac{1}{2} k^i k^j \left[ r^2 \left( v^i_a v^j_a - v^j_b \cdot v^i_b - r^i_a a^j_a \right) - v^i_a r^j_a \left( v_a \cdot r + v_b \cdot r \right) \right] \right\} + O \left( k^3 \right) + ..., \quad (3.15)
\]
\[ T_{Figg3c}^{00}(t, k) = - \sum_{a \neq b} \frac{G m_a m_b}{4r} e^{-i k \cdot x_a} \left\{ 6 \left( -a_b \cdot r + v_b^2 - (v_b \cdot n)^2 \right) \right. \]
\[ - \frac{3i}{2} k^i \left[ \left( a_b \cdot r - v_b^2 + (v_b \cdot n)^2 \right) r^i - 2v_b \cdot r v_b^i + r^2 a_b^i \right] \]
\[ - \frac{1}{2} k^i k^j \left[ \left( -a_b \cdot r + v_b^2 - (v_b \cdot n)^2 \right) r^i r^j + 4v_b \cdot r v_b^i r^j \right] \]
\[ -2r^2 a_b^i r^j + 2r^2 v_b^i v_b^j \right\} + O (k^3) + \ldots \] (3.16)

Together, these quantities provide us with the following contribution,

\[ \int d^3 x T_{Figg3a-3c}^{00} \left[ x^i x^j \right]_{TF} = \sum_{a \neq b} \frac{G m_a m_b}{12r} \left[ \left( -2v_a^2 - 35v_b^2 + 26v_a \cdot v_b - 10v_a \cdot n v_b \cdot n \right) + 3 (v_a \cdot n)^2 + 12 (v_b \cdot n)^2 - 4i^2 + a_a \cdot r + 8a_b \cdot r \right] x_a^i x_b^j \]
\[ + \sum_{a \neq b} \left( v_a^2 + v_a \cdot v_b - 5v_a \cdot n v_b \cdot n + 3 (v_a \cdot n)^2 - 2i^2 + a_a \cdot r \right) x_a^i x_b^j \]
\[ + \left( v_a \cdot r + v_b \cdot r \right) \left( -2v_a^2 x_a^i + 26v_a^2 x_b^i \right) \]
\[ + r^2 \left( 2v_a^i v_a^j - v_a^i v_b^j - 22a_a^i x_b^j - 23a_b^i x_b^j \right) \] \[ \left. \right|_{STF} . \] (3.17)

FIG. 4: Two-potential-graviton exchange with external \( \tilde{h}^{00} \) momentum.

Contributions from Fig. 4 are composed of the source terms (A1), (A4), (A8) and (A9) and yield

\[ T_{Figg4a}^{00}(t, k) = \sum_{a \neq b} \frac{G^2 m_a^2 m_b}{r^2} e^{-i k \cdot x_a} , \] (3.18)
\[ T_{Figg4b}^{00}(t, k) = \sum_{a \neq b} \frac{3G^2 m_a^2 m_b}{2r^2} e^{-i k \cdot x_a} , \] (3.19)
\[ T_{Figg4c}^{00}(t, k) = - \sum_{a \neq b} \frac{3G^2 m_a m_b}{2r^2} e^{-i k \cdot x_a} , \] (3.20)

which gives us

\[ \int d^3 x T_{Figg4a-4c}^{00} \left[ x^i x^j \right]_{TF} = - \sum_{a \neq b} \frac{G^2 m^2 m_b}{2r^2} \left[ x_a^i x_b^j \right]_{TF} . \] (3.21)
These diagrams give:

\[ T_{F_5}^{00}(t, k) = - \sum_{a \neq b} \frac{G^2 m_a m_b^2}{r^2} e^{-i k \cdot x_a}, \]

(3.22)

\[ T_{F_5}^{00}(t, k) = - \sum_{a \neq b} \frac{2G^2 m_a^2 m_b}{r^2} e^{-i k \cdot x_a}, \]

(3.23)

\[ T_{F_5}^{00}(t, k) = - \sum_{a \neq b} \frac{G^2 m_a^2 m_b}{r^2} e^{-i k \cdot x_a} \left( \frac{1}{2} - \frac{7}{2} i k' r^i + \frac{5}{3} k' k^i r^j r^j \right) + O (k^3) + ..., \]

(3.24)

\[ T_{F_5}^{00}(t, k) = \sum_{a \neq b} \frac{G^2 m_a^2 m_b}{r^2} e^{-i k \cdot x_a} \left( 5 - 2 i k' r^i + \frac{2}{3} k' k^i r^j r^j \right) + O (k^3) + ..., \]

(3.25)

\[ T_{F_5}^{00}(t, k) = - \sum_{a \neq b} \frac{G^2 m_a^2 m_b}{r^2} e^{-i k \cdot x_a}. \]

(3.26)

Keeping terms to second order in the radiation momentum we have

\[ \int d^3 x T_{F_5}^{00} \left[ x_i x_j \right]_{TF} = \sum_{a \neq b} \frac{G^2 m_a m_b}{r^2} \left[ \frac{3}{2} (m_a - m_b) x_a^i x_b^j - m_a x_a^i x_b^j + 2m_a x_b^i x_b^j \right]_{TF}. \]

(3.27)

Summing the contributions (3.5), (3.10), (3.17), (3.21) and (3.27), the total contribution of \( T_{2PN}^{00} \) to the mass quadrupole is

\[
\int d^3 x T_{2PN}^{00} \left[ x_i x_j \right]_{TF} = \sum_a \frac{3}{8} m_a v_a \left[ x_a^i x_a^j \right]_{TF} + \sum_{a \neq b} \frac{G m_a m_b}{12r} \left[ \left( 28 v_a^2 - 11 v_b^2 - 22 v_a \cdot v_b - 10 v_a \cdot n v_b \cdot n \right. \right.
\]

\[ + 3 (v_a \cdot n)^2 + 6 (v_b \cdot n)^2 - 4 i^2 + a_a \cdot r + 2a_b \cdot r + 12 \frac{G_m a}{r} + 6 \frac{G_m b}{r} \right] x_a^i x_b^j \]

\[ + \left( v_a^2 + v_a \cdot v_b - 5 v_a \cdot n v_b \cdot n + 3 (v_a \cdot n)^2 - 2 i^2 + a_a \cdot r - 12 \frac{G_m a}{r} \right) x_a^i x_b^j \]

\[ + \left( v_a \cdot r + v_b \cdot r \right) \left( -20 v_a^i x_a^j + 26 v_b^i x_b^j \right) + r^2 \left( 2 v_a^i v_a^j - v_a^i v_b^j - 22 a_a^i x_a^j - 23 a_b^i x_b^j \right) \]_{STF}. \]

(3.28)

### B. 1PN correction to \( T^{0i} \)

The leading order \( T^{0i} \) component was obtained in \([26]\) is

\[ T_{0i}^{0i}(t, k) = \sum_a m_a v_a^i e^{-i k \cdot x_a}. \]

(3.29)

The 1PN correction enter at \( v^3 \) and are shown in Fig. 6.
To extract the $T_{ij}^{0l}$ contributions to the mass quadrupole moment, which is the third term in (2.10), the expansion of the denominator of vertices in Fig. 6c-6d has to be carried out to third order. In addition, $k^2$ terms can not be dropped, since they contribute terms that cannot be included in the trace part of the quadrupole.

![Diagrams](image)

**FIG. 6:** All diagrams that contribute to $T_{ij}^{0l}$. Comparing the diagrams illustrated in Fig.6, which are composed of (A1,A2,A10,A11) together with (A15), (A22) and (A23) we find

$$T_{F1g6a}^{0l}(t, k) = \sum_a \frac{m_a}{2} v_a^l v_a^l e^{-ik \cdot x_a},$$

(3.30)

$$T_{F1g6b}^{0l}(t, k) = \sum_{a \neq b} \frac{Gm_a m_b}{r} v_a^l e^{-ik \cdot x_a},$$

(3.31)

$$T_{F1g6c}^{0l}(t, k) = \sum_{a \neq b} \frac{Gm_a m_b}{4r} e^{-ik \cdot x_a} \left[-2v_a^l + 2ik \cdot (v_a^l r^l - r^l v_a^l) + k^2 v_a^l (r^l r^l v_a^l - v_a^l r^l r^l) + \frac{i}{6} k^2 l^k (2\delta^j il^j v_a^l r^l - r^l r^l v_a^l r^l) + 2v_a^l r^l r^l r^l + O(k^4) + \ldots \right],$$

(3.32)

$$T_{F1g6d}^{0l}(t, k) = \sum_{a \neq b} \frac{Gm_a m_b}{4r} e^{-ik \cdot x_a} \left[v_a^l + v_b^l - \frac{1}{r^2} (v_a + v_b) \cdot rr^l \right.$$

$$- \frac{i}{2} k^l \left(3r^l r^l \delta^l r^l - r^l \left(\delta^l v_a^l + v_b^l + v_b^l + \frac{1}{r^2} (v_a + v_b) \cdot rr^l \right)\right] + \frac{i}{6} k^l j^l \left[-5r^l (v_a^l + v_b^l) \delta^l r^l + (4v_a \cdot r - 5v_b \cdot r) r^l r^l + (v_a^l + 2v_b^l) r^l r^l \right.$$

$$\left. + (v_a^l + v_b^l) + \frac{1}{2} \delta^l r^l r^l + (v_a \cdot r + v_b \cdot r) \left(\frac{1}{2} \delta^l r^l r^l + \frac{1}{r^2} (v_a + v_b) \cdot rr^l \right) \right]$$

$$- \frac{i}{24} k^l j^l k^l [\delta^l (16r^2 v_a^l r^l + 14r^2 v_b^l r^l - 5v_a \cdot rr^l r^l + 7v_b \cdot rr^l r^l) + \delta^l r^l (35k v_a \cdot r - 35k v_b \cdot r - r^l v_a^l + r^l v_b^l + r^l v_b^l - r^l v_a^l) - \delta^l \left(v_a \cdot r + v_b \cdot r \right) r^l r^l \right.$$

$$\left. + r^l r^l r^l \left(v_a^l + v_b^l \right) + (3v_a^l - v_b^l) r^l r^l r^l - \frac{1}{r^2} (v_a + v_b) \cdot rr^l \right] \right \} + O(k^4) + \ldots$$

(3.33)

Expanding the exponentials up to the third order in the radiation momentum, we get

$$\int d^3 \delta T_{1 PN} x^l \left[x^l x^l \right]_{TF} = \sum_a \frac{d}{dt} \left[ \frac{1}{2} m_a v_a^2 v_a \cdot x_a^l x_a^l \right]_{TF}$$

$$+ \sum_{a \neq b} \frac{d}{dt} \left[ \frac{Gm_a m_b}{12r} \left[ (8r^2 - 20r \cdot x_b) v_a^l x_a^l + (20r^2 - 22r \cdot x_b) v_a^l x_b^l \right.$$

$$+ (22v_a \cdot x_a - 30v_b \cdot x_a - 8v_a \cdot x_b + 8v_b \cdot x_b - \frac{2}{r^2} (v_a + v_b) \cdot rr \cdot x_b) x_a^l x_b^l$$

$$\left. + \left(9v_a \cdot x_a - 7v_a \cdot x_b - \frac{1}{r^2} (v_a + v_b) \cdot rr \cdot x_b \right) x_b^l x_a^l \right]_{STF}.$$  

(3.34)


C. 1PN correction to $T^{ii}$

The leading order $T^{ii}$ component obtained in [26] has the form

$$T^{ii}_{0PN}(t, k) = \left( \sum_a m_a v_a^2 - \sum_{a \neq b} \frac{Gm_a m_b}{2r} + O(k) + \ldots \right) e^{-ik \cdot x_a}.$$  \hspace{1cm} (3.35)

Notice that, while $T^{0i}_{0PN}$ in (3.29) is down by $v^1$ relative to $T^{00}_{0PN}$ in (3.2), the leading order spatial component (3.35) is down by $v^2$ compared to $T^{00}_{0PN}$, this fixes the PN hierarchy among the components $T^{00}, T^{0i}$ and $T^{ij}$ of the pseudotensor.

To obtain $T^{ii}_{1PN}$ as well as its contributions to $I^{ij}_{2PN}$ we have to compute all diagrams that enter at $v^4$ with one $\bar{h}^{ii}$ external momentum. To compute the spatial component of the pseudotensor and to extract its contribution to the mass quadrupole moment we have to carry out the expansions up to the second order in the radiation momentum. $k^2$ may be dropped as in section III A.

\begin{align*}
T^{ll}_{Fig7a}(t, k) &= \sum_a m_a v_a^4 e^{-ik \cdot x_a}, \hspace{1cm} (3.36) \\
T^{ll}_{Fig7b}(t, k) &= \sum_{a \neq b} \frac{Gm_a m_b}{r} v_a^2 v_b^2 e^{-ik \cdot x_a}, \hspace{1cm} (3.37) \\
T^{ll}_{Fig7c}(t, k) &= -\sum_{a \neq b} \frac{G^2 m_a m_b m_c}{2r^2} e^{-ik \cdot x_a}. \hspace{1cm} (3.38)
\end{align*}

It is straightforward to extract the contribution for the mass quadrupole moment by expanding the exponentials up to the second order in the radiation momentum,

$$\int d^3x T^{ll}_{Fig7a-7c}[x^i x^j]_{TF} = \sum_a \frac{m_a}{2} v_a^4 [x^i_a x^j_a]_{TF} + \sum_{a \neq b} \frac{Gm_a m_b}{r} \left( v_a^2 - \frac{Gm}{2r} \right) [x^i_a x^j_a]_{TF}. \hspace{1cm} (3.39)$$

\begin{align*}
T^{ll}_{Fig7d}(t, k) &= \sum_{a \neq b} \frac{G^2 m_a m_b m_c}{2r^2} e^{-ik \cdot x_a}, \hspace{1cm} (3.38) \\
T^{ll}_{Fig7e}(t, k) &= -\sum_{a \neq b} \frac{G^2 m_a m_b m_c}{2r^2} e^{-ik \cdot x_a}. \hspace{1cm} (3.38)
\end{align*}

It is straightforward to extract the contribution for the mass quadrupole moment by expanding the exponentials up to the second order in the radiation momentum,
\[ T_{\text{Fig}8a}(t, k) = \sum_{a \neq b} \frac{3G_m m_b}{4r} e^{-ik \cdot x_a} \left\{ 2v^2 + v_a \cdot v_b - 2i^2 - \frac{1}{r^2} v_a \cdot rv_b \cdot r + 2a \cdot r \right. \]

\[ + \frac{i}{2} k^i \left[ \left( 2v^2 + v_a \cdot v_b - 2i^2 - \frac{1}{r^2} v_a \cdot rv_b \cdot r + 2a \cdot r \right) r^i \right. \]

\[ + v_b^i (4v_b \cdot r - 3v_a \cdot r) + v_a^i (3v_b \cdot r - 4v_a \cdot r) - 2r^2 (a_i^a + a_i^b) \]

\[ + \frac{1}{6} k^i k^j \left[ \left( -2v^2 - v_a \cdot v_b - 2a \cdot r + 2i^2 + \frac{1}{r^2} v_a \cdot rv_b \cdot r \right) r^i r^j \right. \]

\[ + (6v_a \cdot r - 8v_b \cdot r) v_b^i r^j + (4v_a \cdot r - 3v_b \cdot r) v_a^i r^j \]

\[ + r^2 \left( -4v_a^i v_b^j - 3v_b^i v_a^j - 4v_a^i v_b^j \right) \left. \right] + O(k^3) + ..., \tag{3.40} \]

\[ T_{\text{Fig}8b}(t, k) = \sum_{a \neq b} \frac{G_m m_b}{r} e^{-ik \cdot x_a} \left\{ \frac{1}{4} \left( v_a^2 + v_b^2 \right) - i v_a^i k^i r^i - k^i k^j \left( r^i v_a^j v_b - \frac{1}{2} v_a^2 r^i r^j \right) \right\} + O(k^3) + ..., \tag{3.41} \]

\[ T_{\text{Fig}8c}(t, k) = \sum_{a \neq b} \frac{G_m m_b}{2r} e^{-ik \cdot x_a} \left\{ -4 v_a \cdot v_b - ik^i (2v_a \cdot v_b r^i + 4v_a \cdot rv_b - 4v_b \cdot rv_a^i) \right. \]

\[ + k^i k^j \left[ 2r^i v_a^j v_b + v_a \cdot v_b r^i r^j - 2v_b \cdot rv_a^i r^j + 2v_a \cdot rv_b^i r^j \right] \left. \right] + O(k^3) + ..., \tag{3.42} \]

\[ T_{\text{Fig}8d}(t, k) = \sum_{a \neq b} \frac{G_m m_b}{r} e^{-ik \cdot x_a} \left\{ -4 \left( v \cdot v_a + a \cdot r - \frac{1}{r^2} v \cdot rv_a \cdot r \right) \right. \]

\[ - 2ik^i \left[ r^i \left( v \cdot v_a + a \cdot r - \frac{j}{r} v_a \cdot r \right) + v_a \cdot r \left( v^i - 2v_a^i \right) \right. \]

\[ + \frac{1}{3} k^i k^j \left. \left[ r^i (v^j v_a^i + a_i^a) \right] + 2r^i r^j (v \cdot v_a + a \cdot r) \right. \]

\[ + r^i v_a^j r^j + 2v_a \cdot r \left( 2v^i r^j - 3v_a^i r^j \right) - 2 \frac{r}{r} v_a \cdot rr^i r^j \left. \right] + O(k^3) + ..., \tag{3.43} \]

\[ T_{\text{Fig}8e}(t, k) = - \sum_{a \neq b} \frac{G_m m_b}{4r} e^{-ik \cdot x_a} \left\{ 2 \left( -a_b \cdot r + v_b^2 - (v_b \cdot n)^2 \right) \right. \]

\[ - \frac{i}{2} k^i \left[ \left( a_b \cdot r - v_b^2 + (v_b \cdot n)^2 \right) r^i - 2v_b \cdot rv_b^i + r^2 a_i^b \right] \]

\[ + \frac{i^2}{6} k^i k^j \left[ \left( -a_b \cdot r + v_b^2 - (v_b \cdot n)^2 \right) r^i r^j + 4v_b \cdot rv_b^i r^j - 2r^2 a_i^j r^j + 2r^2 v_b^i rv_b^j \right] \left. \right] + O(k^3) + ..., \tag{3.44} \]

which provides us with

\[ \int d^3x T_{\text{Fig}8a-8e}^{STF} [x^i x^j] = \sum_{a \neq b} \frac{G_m m_b}{12r} \left\{ \left( 10v_a^2 + 17v_b^2 - 10v_a \cdot v_b \right) \right. \]

\[ + 5 (v_a \cdot n)^2 + 2v_a \cdot nv_b \cdot n - 8 (v_b \cdot n)^2 - 5a_b \cdot r + 8a_b \cdot r \}

\[ + \left( -5v_a^2 + 7v_a \cdot v_b + 5 (v_a \cdot n)^2 - 7v_a \cdot nv_b \cdot n - 5a_b \cdot r \right) x_a^i x_b^j \]

\[ + (4v_a \cdot r - 44v_b \cdot r) v_b^i x_a^j + (14v_a \cdot r - 58v_b \cdot r) v_a^i x_b^j \]

\[ + r^2 \left( 38v_b^i v_a^j - 7v_a^i v_b^j + 14a_i^a x^j_a + 19a_i^b x^j_b \right) \]
FIG. 9: Three-potential-graviton exchange with $\bar{h}^{ij}$ external momentum.

Finally, the diagrams containing a three-potential-graviton exchange shown in Fig. 9 which involve (A1), (A31) and (A34) give

$$T_{\text{Fig}9a}^{ll}(t,k) = \sum_{a \neq b} \frac{G^2 m_a^2 m_b}{r^2} e^{-ik \cdot x_a} \left( -\frac{5}{2} + \frac{7}{2} k^1 r^i - \frac{4}{3} k^1 k^2 r^j \right) + O(k^3) + \ldots$$

(3.46)

$$T_{\text{Fig}9b}^{ll}(t,k) = \sum_{a \neq b} \frac{G^2 m_a^2 m_b}{r^2} e^{-ik \cdot x_a} \left( 1 - 6 k^1 r^i + \frac{7}{3} k^1 k^2 r^j \right) + O(k^3) + \ldots$$

(3.47)

$$T_{\text{Fig}9c}^{ll}(t,k) = \sum_{a \neq b} \frac{7G^2 m_a m_b^2}{2r^2} e^{-ik \cdot x_a},$$

(3.48)

which leads to

$$\int d^3x T_{\text{Fig}9a-9c}^{ll}(t,x)_{TF} = \sum_{a \neq b} \frac{G^2 m_a m_b}{r^2} \left[ \frac{3}{2} m_a x_a^i x_a^j - m_a x_a^i x_b^j \right]_{STF}.$$ 

(3.49)

With this, we now write the total contribution of $T_{1PN}^{ij}$ to the mass quadrupole,

$$\int d^3x T_{1PN}^{ij}(t,x)_{TF} = \sum_a \frac{m_a v_a^4}{2} \left[ x_a^i x_a^j \right]_{TF} + \sum_{a \neq b} \frac{G m_a m_b}{12r} \left( (2v_b^2 - 17v_a^2 - 10v_a \cdot v_b \right. \\
+ 5(v_a \cdot n)^2 + 2v_a \cdot n v_b \cdot n - 8(v_b \cdot n)^2 - 5a_a \cdot r + 8a_b \cdot r + 12 \frac{G m_a}{r} \right) x_a^i x_a^j \\
+ \left( -5v_a^2 + 7v_a \cdot v_b + 5(v_a \cdot n)^2 - 7v_a \cdot n v_b \cdot n - 5a_a \cdot r - 12 \frac{G m_a}{r} \right) x_a^i x_b^j \\
+ (4v_a \cdot r - 44v_b \cdot r) v_a^i x_a^j + (14v_a \cdot r - 58v_b \cdot r) v_a^i x_b^j \\
+ r^2 \left( 38v_a^i v_a^j - 7v_b^i v_b^j + 14a_a^i x_a^j + 19a_a^i x_b^j \right)_{STF}.$$ 

(3.50)

IV. LOWER ORDER STRESS-ENERGY TENSORS

Although $T_{0PN}^{ij}$, $T_{0PN}^{ii}$ and $T_{1PN}^{00}$ have been computed before in [26], to write an expression for the mass quadrupole moment at 2PN order, we need to expand them in the radiation momentum to higher order and terms depending on $k^2$ must be kept.

FIG. 10: Diagrams (a) and (b) contribute to $T_{0PN}^{ij}$ when the external leg is $\bar{h}^{ij}(x)$, while diagrams (a), (b) and (c) contribute to $T_{1PN}^{00}$ when we consider $\bar{h}^{00}(x)$ as the external leg.
To obtain the sixth term of (2.10) we need diagrams in Fig. 10a-b. This gives us an expression for the leading order \( T_{ij}^{00} \), as shown below:

\[
T_{0jPN}^{kl}(t, k) = \sum_a m_a v_a^k v_a^l e^{-ikx_a} + \sum_{a \neq b} \frac{Gm_a m_b}{2r} e^{-ikx_a} \left\{ -\frac{1}{r^2} r^k r^l - \frac{i}{2} k^i k_j + \frac{1}{2} r^k r^l \right\} + \frac{1}{12} k^i k^j \left[ 10 r^2 (\delta^{kl} \delta^{ij} - \delta^{ik} \delta^{jl}) + \delta^{kl} r^i r^j + \delta^{ij} r^k r^l - 2 \delta^{ik} r^j r^l + \frac{1}{r^2} 2 r^i r^j r^k r^l \right] + \frac{i}{24} k^i k^j k^m k^n \left[ r^2 r^m (10 \delta^{ik} \delta^{jl} - 10 \delta^{ij} \delta^{kl}) - \delta^{ij} r^m r^k r^l + r^i r^j \left( 2 \delta^{mk} r^l - \delta^{ml} r^m - \frac{1}{r^2} r^m r^k r^l \right) \right] + \frac{1}{240} \left[ \frac{16}{3} r^4 \delta^{mn} (\delta^{kl} \delta^{ij} - \delta^{ik} \delta^{jl}) + r^2 \delta^{ij} \delta^{mn} r^k r^l - 2 r^2 \delta^{mn} \delta^{ik} r^j r^l \right] + \frac{1}{240} \left[ r^m r^n (34 r^2 \delta^{ik} \delta^{jl} - 33 r^2 \delta^{kl} \delta^{ij} - 3 \delta^{ik} r^j r^l + 6 \delta^{kl} r^i r^j - 3 \delta^{ij} r^k r^l - \frac{2}{r^2} r^i r^j r^k r^l) \right] \} + O(\mathbf{k}^5) + \ldots \quad (4.1)
\]

The first term in the expression above is related to Fig. 10a, which comes from the simple source action term \(- \sum \frac{m_a}{2m_pl} \int dt_a v_a^i v_a^j T_{ij}^{00}(x_a) \). The other terms come from Fig. 10b, which is composed of \( (A1) \) and \( (A17) \) by considering,

\[
F(H_{ij}^{00}, h_{ij}^{00}) = \tilde{h}_{ij} \left[ -\frac{1}{2} q^i q^j - \frac{1}{2} q^i k^j - \frac{1}{2} k^i k^j + \delta^{ij} \left( \frac{1}{4} q^2 + \frac{1}{4} k \cdot q + \frac{1}{2} k^2 \right) \right]. \quad (4.2)
\]

where \( F(H_{ij}^{00}, h_{ij}^{00}) \) is defined in \( (A22) \).

Now, consider the volume term up to the fourth order in the radiation momentum, we extract the contribution

\[
\int d^3x \delta_0^{kl} T_{0jPN} \left[ \left( x^k x^l \right) \right] = \frac{d^2}{dt^2} \left[ \sum_a m_a (v_a \cdot x_a)^2 x_a^i x_a^j \right]_{TF} + \frac{d^2}{dt^2} \left[ \sum_{a \neq b} \frac{Gm_a m_b}{6r} \left[ \left( 27r^2 + x_a^2 - 2x_a \cdot x_b - \frac{2}{r^2} r \cdot x_a r \cdot x_b \right) x_a^i x_a^j \right] \right]_{STF} + \frac{27}{2} r^2 + x_a^2 - \frac{1}{r^2} (r \cdot x_a)^2 \right] x_a^i x_a^j \right) \right]. \quad (4.3)
\]

Taking the trace of (4.1), we get

\[
T_{0jPN}^{kl}(t, k) = \sum_a m_a v_a^k v_a^l e^{-ikx_a} + \sum_{a \neq b} \frac{Gm_a m_b}{2r} e^{-ikx_a} \left\{ -\frac{1}{r^2} r^k r^l - \frac{i}{2} k^i k_j + \frac{1}{4} r^k r^l \right\} + \frac{i}{24} k^i k^j k^m (21r^2 \delta^{ij} r^m + 2 r^i r^j r^m) + \frac{1}{144} k^i k^j k^m k^n (7r^2 \delta^{ij} \delta^{mn} - 42r^2 \delta^{ij} r^m r^n - 3r^i r^j r^m r^n) \right\} + O(\mathbf{k}^5) + \ldots \quad (4.4)
\]

which contributes to the quadrupole in the form below:

\[
\int d^3x \delta_0^{kl} T_{0jPN} \left[ \left( x^k x^l \right) \right] = \frac{d^2}{dt^2} \left[ \sum_a m_a v_a^2 x_a^2 x_a^i x_a^j \right]_{TF} + \frac{d^2}{dt^2} \left[ \sum_{a \neq b} \frac{Gm_a m_b}{12r} \left[ (-104 x_a^2 + 196 x_a \cdot x_b - 98 x_b^2) x_a^i x_a^j - 49 r^2 x_a^i x_a^j \right] \right]_{STF}. \quad (4.5)
\]

To be able to compute the seventh contribution in (2.10), we need an expression for \( T_{1jPN}^{00} \) up to the fourth order in the radiation momentum. We regard the source action term \(- \sum \frac{m_a}{4m_pl} \int dt_a v_a^2 h_{ij}^{00}(x_a) \) and also \( (A15) \), \( (A1) \), \( (A4) \) and \( (A18) \) to solve the diagrams at Fig. 10a-c. With this, we get an expression for \( T_{1jPN}^{00} \) and its contribution to the
mass quadrupole at 2PN, respectively:

\[ T_{1PN}^{00}(t, \mathbf{k}) = \sum_a \frac{1}{2} m_a v_a^2 e^{-i \mathbf{k} \cdot \mathbf{x}_a} + \sum_{a \neq b} \frac{G m_a m_b}{r} e^{-i \mathbf{k} \cdot \mathbf{x}_a} \left[ -\frac{1}{2} \right] \]

\[ -\frac{3}{8} \mathbf{k}^2 r^2 \left( 1 + \frac{i}{2} \mathbf{k}^i r^i - \frac{1}{6} \mathbf{k}^i k^j r^i r^j + \frac{r^2}{36} \mathbf{k}^i \delta^{ij} \right) + O(k^5) + ..., \]  

(4.6)

\[ \int d^3 x_0 \partial_0^2 T_{0PN}^{00} x^2 [x^i x^j]_{TF} = \frac{d^2}{dt^2} \left\{ \sum_a \frac{1}{2} m_a v_a^2 x_a^2 \left[ x_a^i x_a^j \right]_{TF} \right\} \]

\[ + \frac{d^2}{dt^2} \left\{ \sum_{a \neq b} \frac{G m_a m_b}{r} \left[ \frac{7}{4} r^2 \left( 2 x_a^i x_b^i + x_a^i x_b^j \right) - \frac{1}{2} x_a^i x_b^i \right]_{STF} \right\}. \]  

(4.7)

Moreover, considering the expansion up to the fifth and sixth orders in the radiation momentum at (3.29) and (3.2), respectively, in addition to taking time derivatives, we get

\[ \int d^3 x_0 \partial_0^5 T_{0PN}^{00} x^2 [x^i x^j]_{TF} = \frac{d^3}{dt^3} \left[ \sum_a m_a v_a \cdot x_a x_a^2 x_a x_a x_a^2 \right]_{TF} , \]  

(4.8)

\[ \int d^3 x_0 \partial_0^6 T_{0PN}^{00} x^4 [x^i x^j]_{TF} = \frac{d^4}{dt^4} \left[ \sum_a m_a x_a^4 x_a^4 \right]_{TF} . \]  

(4.9)

Before writing the final expression for the 2PN correction to the mass quadrupole moment, we still need to write the contribution of \( T_{1PN}^{ij}(a_{1PN}) \), which is given by the two terms

\[ \left[ \int d^3 x_0 \partial_0^5 T_{0PN}^{00} x^2 [x^i x^j] \right]_{TF} \frac{2PN}{TF} \sum_a m_a a_{1PN} \cdot x_a \left[ x_a^i x_a^j \right]_{TF} , \]  

(4.10)

\[ \left[ \int d^3 x_0 \partial_0^6 T_{0PN}^{00} x^4 [x^i x^j] \right]_{TF} \frac{2PN}{TF} \sum_a 2m_a \left[ x_a^2 a_{1PN}^i x_a^j + a_{1PN}^i \cdot x_a x_a^j \right]_{STF} , \]  

(4.11)

where the 1PN correction to the acceleration, for instance obtained in [34] using the EFT framework, is given by

\[ a_{1PN(1)}^i = \frac{G m_2}{2 r^2} \left\{ \mathbf{n}^i \left[ \frac{2G m}{r} - 3 (\mathbf{v}_1^2 + \mathbf{v}_2^2) + 7 (\mathbf{v}_1 \cdot \mathbf{v}_2) + 3 (\mathbf{v}_1 \cdot \mathbf{n}) (\mathbf{v}_2 \cdot \mathbf{n}) \right] \right. \]

\[ -\mathbf{v}_1^i (\mathbf{v}_2 \cdot \mathbf{n}) - (\mathbf{v}_1 \cdot \mathbf{n}) \mathbf{v}_2^i + \hat{r} \left( 6 \mathbf{v}_1^2 - 7 \mathbf{v}_2 \cdot \mathbf{n} \right) (\mathbf{v}_2 \cdot \mathbf{n}) \]

\[ -6 \mathbf{a}_1^i + 7 \mathbf{a}_2^i + (\mathbf{v}^i - \mathbf{n}^i \hat{r}) (\mathbf{v}_2 \cdot \mathbf{n}) + 2 \mathbf{n}^i (\mathbf{a}_2 \cdot \mathbf{n}) + 3 \mathbf{n}^i (\mathbf{v}_2 \cdot \mathbf{n} \hat{r})) \}

\[ -\frac{1}{2} \mathbf{a}_1^i \mathbf{v}_1^2 - \mathbf{v}_1^i (\mathbf{v}_1 \cdot \mathbf{a}_1) . \]  

(4.12)

V. CONSISTENCY TESTS

Here we check the expressions for the components \( T_{2PN}^{00}, T_{1PN}^{0i} \) and \( T_{1PN}^{ii} \), which were obtained here for the first time in EFT approach with previous results derived using different methods.

The results presented in section IIIA allow us to write down an expression for the temporal component of the
pseudotensor up to 2PN order,

\[ T^{00}(t, \mathbf{k}) = e^{-i\mathbf{k} \cdot \mathbf{x}_a} \left\{ \sum_a m_a \left( 1 + \frac{1}{2} v^2 + \frac{3}{8} v^4 \right) + \sum_{a \neq b} \frac{G m_a m_b}{2r} \left\{ -1 + v^2 + \frac{7}{2} v^2 - \frac{5}{2} v^4 + \frac{5}{2} \mathbf{v}_a \cdot \mathbf{v}_b \right. \right. \]

\[-\frac{5}{2} \mathbf{v}_a \cdot \mathbf{v}_b \cdot \mathbf{n} + 2 (\mathbf{v}_b \cdot \mathbf{n})^2 - r^2 + \mathbf{a} \cdot \mathbf{r} + 2 \mathbf{a}_b \cdot \mathbf{r} + \frac{G}{r} (4m_a - 3m_b) \]

\[ + \frac{1}{2} i \mathbf{k} \left[ \left( v^2 + \frac{5}{2} \mathbf{v}_a \cdot \mathbf{v}_b - r^2 - \frac{5}{2} \mathbf{v}_a \cdot \mathbf{v}_b \cdot \mathbf{n} + \frac{3}{2} (\mathbf{v}_b \cdot \mathbf{n})^2 + \mathbf{a} \cdot \mathbf{r} + \frac{3}{2} \mathbf{a}_b \cdot \mathbf{r} \right) r^2 \right. \]

\[ + \left( 8 \mathbf{v}_a \cdot \mathbf{r} + \frac{11}{2} (\mathbf{v}_b \cdot \mathbf{r} - 2r^2) \right) \mathbf{v}_a + \left( \frac{9}{2} \mathbf{v}_b \cdot \mathbf{r} + \frac{1}{2} r^2 \right) \mathbf{v}_b + r^2 (7a^2_a - 2a^2_b) + \frac{6Gm}{r} \right) \} + O(|\mathbf{k}|^2) + ... \right\}. \tag{5.1} \]

We can use (3.1) to read off different contributions of \( T^{00} \) to the dynamics of the binary system. For instance, at zeroth order in the radiation momentum, we can read off the mechanical energy of the system. It is straightforward to see in (5.1) that the leading order terms in the PN approximation reproduce the total mass of the two-body system, while the next-to-leading order terms provide us with the Newtonian energy. The terms that account for the next-to-next-to-leading order (2PN) correction to this pseudotensor, which were calculated in the section III A of this paper, give us the following contribution to the conserved energy,

\[ E_{1PN} = \int d^3 x T^{00}_{2PN}(x) \]

\[ = \frac{3}{8} \sum_a m_a v^4_a \mathbf{x}_a + \sum_{a \neq b} \frac{G m_a m_b}{4r} \left[ 6v^2_a - 7(v_a \cdot v_b) - (v_a \cdot \mathbf{n}) (v_b \cdot \mathbf{n}) + \frac{Gm}{r} \right]. \tag{5.2} \]

This result is equal to the first correction to the Newtonian energy presented in Eq. (205) of [35] and can also be calculated computing the Hamiltonian function using the Lagrangian obtained by Einstein, Infeld and Hoffman in [36].

Regarding the 2PN terms in Eq. (5.1), we can read off the correction to the center of mass position \( \mathbf{G}_{2PN} = \int d^3 x T^{00}_{2PN}(x) \mathbf{x} \)

\[ = \frac{3}{8} \sum_a m_a v^4_a \mathbf{x}_a + \sum_{a \neq b} \frac{G m_a m_b}{4r} \left[ \left( \frac{19}{2} v_a^2 - 7v_a \cdot v_b - \frac{7}{2} v_b^2 - v_a \cdot \mathbf{n} v_b \cdot \mathbf{n} \right) \right. \]

\[ - \frac{1}{2} (v_a \cdot \mathbf{n})^2 + \frac{1}{2} (v_b \cdot \mathbf{n})^2 - \frac{5Gm_a}{r} + \frac{Gm_b}{r} \mathbf{x}_a - 7(v_a \cdot \mathbf{r} + v_b \cdot \mathbf{r}) \mathbf{v}_a \right\}, \tag{5.3} \]

which agrees with the result presented in Eq. (B2c) of [37], where \( \frac{d\mathbf{G}}{dt} = \mathbf{P} \), the total conserved linear momentum, such that the center of mass frame is defined by \( \mathbf{G} = 0 \). By solving this equation iteratively, using the equations of motion to reduce the second time derivatives of the position, we get the 2PN correction to the center of mass frame,

\[ \delta \mathbf{r}_{2PN} = \frac{\nu \delta m}{m} \left\{ \mathbf{r} \left[ \frac{9}{8} - \frac{3\nu}{2} \right] v^4 + \frac{Gm}{r} \left[ \left( \frac{19}{8} + \frac{3\nu}{2} \right) v^2 \right. \right. \]

\[ + \left( \frac{1}{8} + \frac{3\nu}{4} \right) r^2 + \left( \frac{7}{4} - \frac{\nu}{2} \right) \frac{Gm}{r} \left. \right] - \mathbf{v} \left[ \frac{7}{4} Gmr \right] \right\}, \tag{5.4} \]

which agrees with (B4a), (B4b) and (B5b) of [37].

Let us now consider the results obtained in section IIIIB to write down an expression for \( T^{0l} \) up to 1PN order,

\[ T^{0l}(t, \mathbf{k}) = e^{-i\mathbf{k} \cdot \mathbf{x}_a} \left\{ \sum_a m_a v^4_a \left( 1 + \frac{1}{2} v^2_a \right) + \sum_{a \neq b} \frac{G m_a m_b}{4r} \left[ -3v^l_a + v^l_b - \frac{1}{r^2} (v_a + v_b) \cdot \mathbf{r}^l \right. \right. \]

\[-\frac{i}{2} \mathbf{k} \left[ \left( v^l_a - 16v^l_a r^l + 15r^l a^l - r^l v^l_a + 3r r^l \delta^l i + \frac{1}{r^2} (v_a + v_b) \cdot \mathbf{r}^l r^l \right) \right] + O(|\mathbf{k}|^2) + ... \right\}. \tag{5.5} \]

\[ \]
Taking into account only terms of order zero in the radiation momentum, we obtain the 1PN correction to the linear momentum of the binary system,

\[ P_{1\text{PN}} = \int d^3x T^0_{1\text{PN}}(x) = -\left[ \frac{Gm_1m_2}{2\nu^3} \right] (v_1 + v_2) \cdot r \] \sum_{i=1}^{4} \left[ \frac{m_1}{2} v_1^2 - \frac{Gm_1m_2}{2r} \right] v_i \frac{1}{2} (1 \leftrightarrow 2). \] (5.6)

The result above agrees with Eq. (B1) and Eq. (B2b) of reference [37]. Considering all linear terms in the radiation momentum in (5.5), we are able to obtain the 1PN correction to the angular momentum of the binary system,

\[ L_{1\text{PN}} = -\frac{1}{2} e^{ik} \int d^3x (T^q_{1\text{PN}}) \cdot x^k \frac{1}{2} (v \times v)^i \left[ (1 - 3\nu) v^2 + \frac{Gm}{r} (6 + 2\nu) \right]. \] (5.7)

which agrees with Eq. (2.9b) of reference [38].

Furthermore, considering the result obtained in section III C, we provide an expression for \( T^{ll}(t,k) \) up to 1PN order:

\[ T^{ll}(t,k) = e^{-ik \cdot x_0} \left\{ \sum_a m_a v_a^2 \left( 1 + \frac{1}{2} v_a^2 \right) + \sum_{a \neq b} \frac{Gm_amm_b}{4r} \left\{ -2 - 5v_a^2 + 5v_b^2 - v_a \cdot v_b - 6v^2 \right. \\
- 3v_a \cdot v_b \cdot n + 2 (v_b \cdot n)^2 + 16i v_a \cdot v_n - 10a_a \cdot r - 4a_b \cdot r + \frac{Gm}{r} (-8m_a + 12m_b) \\
\left. + i k \left[ r^i \left( -9v_a^2 + \frac{5}{2} v_b^2 - \frac{1}{2} v_a \cdot v_b - \frac{3}{2} v_a \cdot v_n + \frac{1}{2} (v_b \cdot n)^2 \right) \\
- 3r^i + 8i v_a \cdot n - 5a_a \cdot r - 5a_b \cdot r - \frac{10Gm_a}{r} \right] + \left( 2v_a \cdot r + \frac{5Gm}{2r} v_b \cdot r \right) v_a^i \\
+ \left( -\frac{9}{2} v_a \cdot v + 5v_b \cdot v \right) v_a^i - r^2 \left( 3a_a^i + \frac{5}{2} a_b^i \right) \right\} + O (k^2) + \cdots \}. \] (5.8)

We can use the moment relation

\[ \int d^3x T^{ll} = \frac{1}{2} \int d^2x \int d^3x T^{00}x^2. \] (5.9)

to prove the self-consistency of our results. At leading order in the PN expansion, it is trivial to prove that this relation holds using (5.1) and (5.8), while at next-to-leading order more computation is required. From (5.8) we can read off up to 1PN,

\[ \int d^3x T^{ll} = \sum_a m_a v_a^2 \left( 1 + \frac{1}{2} v_a^2 \right) + \sum_{a \neq b} \frac{Gm_amm_b}{r} \left[ -\frac{1}{2} - \frac{1}{4} v_a \cdot v_b + \frac{3}{2} (v_a \cdot n)^2 - \frac{7}{4} v_a \cdot n + \frac{5Gm_a}{2r} \right]. \] (5.10)

To check if the result above satisfies (5.9), we need a complete expression for \( T^{00}(t,k) \) up to 1PN order and which contains all terms up to the quadratic order in the radiation momentum. In other words, we cannot discard terms proportional to \( k^2 \) as we did in section III A, where we dropped these terms that would not contribute to the trace-free quadrupole moment. Therefore, the expression that we need for \( T^{00}(t,k) \) is the sum of (3.2) with (4.6), which provides us with the following result up to 1PN order:

\[ \frac{1}{2} \int d^2x \int d^3x T^{00}x^2 = \frac{1}{2} \int d^2x \left[ \sum_a m_a \left( 1 + \frac{1}{2} v_a^2 \right) x_a^2 + \sum_{a \neq b} \frac{Gm_amm_b}{r} \left( -\frac{1}{2} x_a^2 + \frac{9}{4} r^2 \right) \right]. \] (5.11)

At this point, it is straightforward to show that, after taking the second order time derivative and imposing the leading and next-to-leading order equations of motion that (5.9) holds, as we expected.
VI. MASS QUADRUPOLE MOMENT AT 2PN ORDER

We are now ready to sum the contributions \((3.28), (3.34), (3.50), (4.3), (4.5), (4.7), (4.8), (4.9)\) and to write down the expression for the 2PN correction to the mass quadrupole moment in a general orbit,

\[
I_{2PN}^{ij} = \sum_a m_a f_{1(a)}^{ij} + \sum_{a \neq b} \frac{Gm_a m_b}{r} f_{2(a,b)}^{ij} + \frac{d}{dt} \left[ \sum_a m_a f_{3(a)}^{ij} + \sum_{a \neq b} \frac{Gm_a m_b}{r} f_{4(a,b)}^{ij} \right] \\
+ \frac{d^2}{dt^2} \left[ \sum_a m_a f_{5(a)}^{ij} + \sum_{a \neq b} \frac{Gm_a m_b}{r} f_{6(a,b)}^{ij} \right] + \frac{d^3}{dt^3} \left[ \sum_a m_a f_{7(a)}^{ij} \right] + \frac{d^4}{dt^4} \left[ \sum_a m_a f_{8(a)}^{ij} \right],
\]

where we have defined the following quantities for convenience:

\[
f_{1(a)}^{ij} = \left[ \frac{7}{8} v_a \cdot x_a x_a^j - \frac{11}{21} x_a^i a_{PN}^i x_a^j - \frac{17}{21} a_{1PN} \cdot x_a x_a^i x_a^j \right]_{STF},
\]

\[
f_{2(a,b)}^{ij} = \frac{1}{12} \left[ (50 v_a^2 - 28 v_b^2 - 32 v_a \cdot v_b - 4 r^2 - 24 v_a \cdot n v_b \cdot n \\
+ 8 (v_a \cdot n)^2 + 14 (v_b \cdot n)^2 - 4 a_a \cdot r + 10 a_b \cdot r + 24 \frac{Gm_a}{r} + 18 \frac{Gm_b}{r} \right] x_a^i x_b^j \\
+ \left( -4 v_a^2 + 8 v_a \cdot v_b - 12 v_a \cdot n v_b \cdot n + 8 (v_a \cdot n)^2 - 2 r^2 - 4 a_a \cdot r - 24 \frac{Gm_1}{r} \right) x_a^i x_b^j \\
+ v_a^i x_b^j (16 v_a \cdot r - 64 v_b \cdot r) + v_a^i x_b^j (40 v_a \cdot r - 32 v_b \cdot r) \\
+ r^2 \left( 40 v_a^i v_a^j - 8 v_a^i v_b^j - 8 a_a^i x_a^j - 4 a_a^i x_b^j \right) \right]_{STF},
\]

\[
f_{3(a)}^{ij} = -\frac{2}{3} v_a^2 v_a \cdot x_a \left[ x_a^i x_a^j \right]_{TF},
\]

\[
f_{4(a,b)}^{ij} = -\frac{1}{9} \left[ (8 r^2 - 20 r \cdot x_a) v_a^i x_a^j + (20 r^2 - 22 r \cdot x_b) v_b^i x_b^j \\
+ (22 v_a \cdot x_a - 30 v_b \cdot x_a - 8 v_a \cdot x_b + 8 v_b \cdot x_b - \frac{2}{r^2} (v_a + v_b) \cdot r r \cdot x_b \right) x_a^i x_b^j \\
+ \left( 9 v_a \cdot x_a - 7 v_a \cdot x_b - \frac{1}{r^2} (v_a + v_b) \cdot r r \cdot x_b \right) x_a^i x_b^j \right]_{STF},
\]

\[
f_{5(a)}^{ij} = \left[ \frac{1}{6} (v_a \cdot x_a)^2 + \frac{19}{84} v_a^2 x_a^2 \right] \left[ x_a^i x_a^j \right]_{TF},
\]

\[
f_{6(a,b)}^{ij} = \left[ \left( \frac{31}{42} x_a^2 - \frac{11}{6} x_a \cdot x_b + \frac{8}{9} x_b^2 - \frac{1}{18} r^2 r \cdot x_a r \cdot x_b \right) x_a^i x_a^j \\
+ \frac{4}{9} r^2 + \frac{1}{36} x_a^2 - \frac{1}{36} r^2 \left( r \cdot x_a \right)^2 \right] x_a^i x_b^j \right]_{STF},
\]

\[
f_{7(a)}^{ij} = -\frac{1}{7} v_a \cdot x_a x_a^2 \left[ x_a^i x_a^j \right]_{TF},
\]

\[
f_{8(a)}^{ij} = \frac{23}{1512} x_a^4 \left[ x_a^i x_a^j \right]_{TF},
\]
With the exception of the accelerations in (6.2) which are of 1PN order, all other accelerations in $I_{2PN}^{ij}$ should be taken as the Newtonian acceleration.

In order to write the 2PN correction of the mass quadrupole moment in the center of mass frame, we must have in mind that the positions of the compact bodies in this frame are given by

\[
x_1 = \frac{m_2}{m} r + \delta r_{1PN} + ..., \tag{6.10}
\]
\[
x_2 = -\frac{m_1}{m} r + \delta r_{1PN} + ..., \tag{6.11}
\]

where $\delta r_{1PN}$ accounts for the 1PN correction to the center of mass frame, which can be obtained following the procedure presented through (5.3) and (5.4) but this time using (4.6). Thus, the corrections to the center of frame necessary to write the 2PN mass quadrupole are

\[
\delta r_{1PN} = \frac{\nu \delta m}{2m} r \left( v^2 - \frac{Gm}{r} \right), 
\]
\[
\delta r_{2PN} = \frac{\nu \delta m}{2m} \left\{ r \left[ \left( \frac{3}{4} - 3\nu \right) v^4 + \frac{Gm}{r} \left( \frac{19}{4} + 3\nu \right) v^2 \right] 
+ \left( -\frac{1}{4} + \frac{3\nu}{2} \right) r^2 + \left( \frac{7}{2} - \nu \right) \frac{Gm}{r} \right\}.
\]

Applying (6.10) and (6.11) to (2.8) and (2.9), we obtain the following contributions at 2PN order:

\[
I_{bPN+2PN}^{ij} = \frac{\nu^2 \delta m^2}{4m} \left( v^4 - 2v^2 \frac{Gm}{r} + \frac{G^2 m^2}{r^2} \right) [r^i r^j]_{TF}, \tag{6.14}
\]
\[
I_{1PN+1PN}^{ij} = \frac{\nu^2 \delta m^2}{21m} \left\{ -29v^4 + \frac{Gm}{r} \left( 41v^2 + \frac{17}{2} r^2 - 12 \frac{Gm}{r} \right) \right\} r^i r^j 
+ \left( 24v^2 - 19 \frac{Gm}{r} \right) r^i \nu^j r^j 
+ \left( -22v^2 + 22 \frac{Gm}{r} \right) r^2 \nu^i v^j \right\}_{STF}. \tag{6.15}
\]

Applying (6.10) and (6.11) to (6.1) after applying (6.10) and (6.11), we finally obtain the expression for the 2PN correction to the mass quadrupole moment in the center of mass frame,

\[
I_{2PN}^{ij} = m\nu \left\{ r^i r^j \left[ \frac{1}{252} (653 - 1906\nu + 337\nu^2) \frac{G^2 m^2}{r^2} + \frac{1}{756} (2021 - 5947\nu - 4883\nu^2) \frac{Gm}{r} v^2 
- \frac{1}{756} (131 - 907\nu + 1273\nu^2) \frac{Gm}{r} r^2 
+ \frac{1}{504} (253 - 1835\nu + 3545\nu^2) \right] v^4 
- 2\nu r^i r^j \left[ \frac{1}{378} (1085 - 4057\nu - 1463\nu^2) \frac{Gm}{r} \right] 
+ \frac{1}{126} (41 - 337\nu + 733\nu^2) v^2 
+ \frac{5}{63} (1 - 5\nu + 5\nu^2) r^2 \right] \right\}_{STF}. \tag{6.16}
\]

We can use the result above to compute, for instance, the 2PN correction to the power loss, whose expression in terms of the multipole moments is given by [28]

\[
P = -\frac{G}{5} \left\{ I_{ij}^{(3)} I_{ij}^{(3)} - \frac{5}{18} I_{ijkl} I_{ijkl}^{(4)} + \frac{5}{9072} I_{ijkl} I_{ijkl}^{(5)} + \frac{16}{9} J_{ij}^{(3)} J_{ij}^{(3)} - \frac{5}{84} J_{ijk}^{(4)} J_{ijk}^{(4)} + ... \right\}. \tag{6.17}
\]

The expressions for these multipole moments below 2PN order are known and can be found for instance in [16]. Considering all terms which contribute to the power loss at 2PN order in the expression above, making use of (6.16)
and the 2PN acceleration (B10) obtained in the appendix B, we get

\begin{align}
P^{2\text{PN}}_{\text{EFT}} &= -\frac{8}{15} \frac{G^3 m^4 \nu^2}{r^4} \left\{ \frac{2}{3} \left( -253 + 1026 \nu - 56 \nu^2 \right) \frac{G^3 m^3}{r^3} + \left[ \frac{1}{756} \left( 245185 + 81828 \nu + 4368 \nu^2 \right) v^2 \\
&\quad - \frac{1}{252} \left( 97247 + 9798 \nu + 5376 \nu^2 \right) \frac{v^2}{r^2} \right] \frac{G^2 m^2}{r^2} + \left[ \frac{1}{21} \left( -4446 + 5237 \nu - 1393 \nu^2 \right) v^4 \\
&\quad + \frac{1}{7} \left( 4987 - 8513 \nu + 2165 \nu^2 \right) v^2 \frac{\nu^2}{r^2} - \frac{1}{63} \left( 33510 - 60971 \nu + 14290 \nu^2 \right) \frac{G m}{r} \\
&\quad + \frac{1}{42} \left( 1692 - 5497 \nu + 4430 \nu^2 \right) v^6 - \frac{1}{14} \left( 1719 - 10278 \nu + 6292 \nu^2 \right) v^4 \frac{\nu^2}{r^2} \\
&\quad + \frac{1}{14} \left( 2018 - 15207 \nu + 7572 \nu^2 \right) v^2 \frac{\nu^2}{r^2} - \frac{1}{42} \left( 2501 - 20234 \nu + 8404 \nu^2 \right) \frac{\nu^2}{r^6} \right\}.
\end{align}

At this point we can see that (6.16) and (6.18) seem to be in disagreement with the results presented at [39] and [40] where the Epstein-Wagoner formalism and multipolar post-Minkowskian approach of Blanchet, Damour, and Iyer (BDI) were used, respectively. For instance, the mass quadrupole moment presented in this paper and ones in the mentioned references differ by a factor of $-\frac{4G^2 m^2}{r^2} \left[ m \nu \mathbf{r} \mathbf{r} \right]_{\text{TQ}}$. The power loss shown above and the energy fluxes at (6.13d) in [39] and at (3.5d) in [40] differ by a global minus sign, as well as by the numerical factors on terms depending on $\frac{G^2 m^2 \nu^2}{r^2}$ and $\frac{G^3 m^3 \nu^2}{r^6}$. The difference in the global sign comes from the relation $P = -\frac{dE}{dt}$, which is actually a matter of convention on how the energy flux is defined. For this reason, we consider instead $|P| = |\frac{dE}{dt}|$ and compare the result for the power loss obtained here against the ones in the literature, and we find the following difference

\begin{equation}
P_{\text{EFT}} - \tilde{P} = \frac{32}{5} \frac{G^5 m^6 \nu^2}{r^6} \left( 4v^2 - 3\nu^2 \right),
\end{equation}

where $\tilde{P}$ is the modulus of the energy flux computed via the Epstein-Wagoner and BDI approaches.

Furthermore, the 2PN acceleration obtained in the appendix B is also different from the one presented in [40], which was computed via the BDI formalism. It turns out that these differences should not be a surprise since the gauge choice adopted here and in other formalisms are not the same: in the BDI and in the Epstein-Wagoner approaches the harmonic gauge is used, while in the EFT approach we use the linearized harmonic gauge (2.4), which depends on the background field metric. The different gauge choices for fixing the gravity action imply different coordinate systems. In fact, the difference between the mass quadrupole moments suggests a coordinate transformation of the form

\begin{equation}
\mathbf{r}_{\text{EFT}} \rightarrow \mathbf{\tilde{r}} - \frac{2G^2 m^2}{r^2} \mathbf{\tilde{r}},
\end{equation}

where $\mathbf{\tilde{r}}$ is the coordinate used in the BDI and Epstein-Wagoner approaches. When this transformation is applied to the power loss (6.18), we can verify that

\begin{equation}
P_{\text{EFT}} (\mathbf{\tilde{r}}) = \tilde{P}.
\end{equation}

An analogous comparison holds for the mass quadrupole moment and the 2PN acceleration, showing the agreement between our results and the literature. It should also be noticed that this coordinate transformation was already brought to attention in [12] when the authors used NRGR to calculate the spacetime metric generated by a point mass at rest.

VII. FINAL REMARKS

In this paper, we provided an independent computation of the 2PN correction to the mass quadrupole moment of a binary system of compact bodies moving in general orbits, using the EFT approach in the linearized harmonic gauge. We calculated high order corrections to the components of the pseudo-stress-energy tensor, which were used to obtain

---

6 If the power is expressed in terms of the gauge invariant frequency of a circular orbit $P = \tilde{P}$. 
the mass quadrupole moment correction as well as the 1PN correction to the conserved energy and to the linear and angular momenta of the system and the 2PN correction to the center of mass frame. We used these quantities to perform tests that confirmed the consistency of our results within the EFT formalism itself and with results presented in the literature computed using different formalisms. Therefore, we not only extracted the contributions of the stress-energy pseudotensor to the 2PN correction to the mass quadrupole, but we provided the expressions for the components of the pseudotensor with higher order corrections that will be useful for future calculations on the dynamics of compact binary system.

We also calculated the 2PN correction to the equation of motion in the linearized harmonic gauge that was used, together with the mass quadrupole moment obtained in this paper, to write down the power loss due to the emission of gravitational waves. We thus compared our results against the literature and we showed that the 2PN correction to the mass quadrupole moment, to the relative acceleration of the two-body system and to the power loss obtained in this paper are in agreement with the results computed via the BDI and in the Epstein-Wagoner formalisms once a coordinate transformation is performed.

Although the 2PN correction to the mass quadrupole and to the equation of motion of compact binary systems obtained here were known in the literature, this derivation establishes the ground work for higher order calculations in the EFT formalism. Finally, these are the final missing ingredients necessary for the analysis of the radiation reaction of the binary system at the next-to-next-to-leading order in the EFT approach, which will be presented in a future paper.

VIII. ACKNOWLEDGEMENTS

A.K.L., N.T.M, and Z.Y. are supported in part by the National Science Foundation under Grant No. PHY-1820760.

Appendix A

In this appendix we show the ingredients used to compute the components of the pseudotensor. We used the package xAct [41] from Mathematica for the extraction of the vertices from the action.

Source terms

The source action terms needed to compute the contributions to $T_{2PN}^{00}$ are given below:

$$S^0 = -\sum_a m_a \int dt_a H^{00}(x_a), \quad (A1)$$

$$S^1 = \sum_a m_a \int dt_a v^a_i H^{00}(x_a), \quad (A2)$$

$$S^2 = -\sum_a m_a \int dt_a \left( \frac{v^a_i}{2} H^{00}(x_a) + v^a_i v^a_j H^{ij}(x_a) \right), \quad (A3)$$

$$S_{h^{00}} = \sum_a \frac{m_a}{4m_{Pl}^2} \int dt_a H^{00}(x_a) \bar{h}^{00}(x_a), \quad (A4)$$

$$S_{h^{0i}} = -\sum_a \frac{m_a}{2m_{Pl}^2} \int dt_a v^a_i \bar{h}^{0i}(x_a), \quad (A5)$$

$$S_{h^{0i}} = \sum_a \frac{m_a}{8m_{Pl}^2} \int dt_a (3v^2_a \bar{H}^{00}(x_a) + 2v^a_i v^a_j \bar{H}^{ij}(x_a)) \bar{h}^{00}(x_a), \quad (A6)$$

$$S_{h^{0i}} = -\sum_a \frac{3m_a}{16m_{Pl}^2} \int dt_a v^a_i \bar{h}^{0i}(x_a), \quad (A7)$$

$$S_{H^2} = \sum_a \frac{m_a}{8m_{Pl}^2} \int dt_a H^{00}(x_a) \bar{h}^{00}(x_a), \quad (A8)$$

$$S_{H^2 \bar{h}^{00}} = -\sum_a \frac{3m_a}{16m_{Pl}^3} \int dt_a H^{00}(x_a) H^{00}(x_a) \bar{h}^{00}(x_a). \quad (A9)$$
In addition, to write down the contributions for $T_{1PN}^{0i}$ we must to consider

\[ S_{h}^{i} = - \sum_{a} \frac{m_{a}}{2mpl} \int dt_{a} \psi_{a}^{i} H^{00}(x_{a}) \tilde{h}_{0i}(x_{a}), \]  

(A10)

\[ S_{h}^{3} = \sum_{a} \frac{m_{a}}{2mpl} \int dt_{a} \psi_{a}^{2} \psi_{a}^{i} \tilde{h}_{0j}(x_{a}), \]  

(A11)

whereas for $T_{1PN}^{ij}$ the following terms are also necessary,

\[ S_{h}^{j} = \sum_{a} \frac{m_{a}}{4mpl} \int dt_{a} \psi_{a}^{i} \psi_{a}^{j} H^{00}(x_{a}) \tilde{h}_{ij}(x_{a}), \]  

(A12)

\[ S_{h}^{ij} = - \sum_{a} \frac{m_{a}}{4mpl} \int dt_{a} \psi_{a}^{i} \psi_{a}^{j} \tilde{h}_{ij}^{l}(x_{a}). \]  

(A13)

Although all the sources terms above are conveniently expressed in position space, effectively we perform the partial Fourier transform\(^{7}\)

\[ H^{\mu\nu}(t, q) = \int d^{3}x H^{\mu\nu}(t, x) e^{-iq\cdot x}, \]  

(A14)

to carry out the Feynman diagrams in momentum space.

**Vertices**

From the EH action expanded in the radiation and potential fields and fixed with the background gauge, we obtain the propagator

\[ \langle H_{\mu\nu}(t, q) H_{\alpha\beta}(t', q') \rangle = -i (2\pi)^{3} P_{\mu\nu\alpha\beta} \delta(t - t') \delta^{3}(q + q') \frac{1}{q^{2}}, \]  

(A15)

as well as its correction

\[ \langle H_{\mu\nu}(t, q) H_{\alpha\beta}(t', q') \rangle_{\text{corr}} = -i (2\pi)^{3} P_{\mu\nu\alpha\beta} \frac{d^2}{dt dt'} \delta(t - t') \delta^{3}(q + q') \frac{1}{q^{2}}. \]  

(A16)

The two-potential-one-radiation vertex regarded inside the momentum integrals of the internal potential momenta coupled to the particles has the form

\[ \int \frac{d^{3}q}{(2\pi)^{3}} \int \frac{d^{3}q'}{(2\pi)^{3}} e^{-iq\cdot x_{1}} e^{-iq'\cdot x_{2}} \langle iS_{h}^{ij} \rangle = - \frac{i}{mpl} \delta(t - t') \int \frac{d^{3}q}{(2\pi)^{3}} e^{-iq\cdot x} \frac{F[k, q, h]}{q^{2}(q + k)^{2}}, \]  

(A17)

for which the different contractions necessary to write down the contributions to $T_{2PN}^{00}$ are

\[ F(H^{00} H^{00}) = \frac{3}{4} \left( q^{2} + k \cdot q \right) - \frac{5}{4} q_{0}^{2} - \frac{5}{4} k_{0} q_{0} - \frac{1}{2} k_{0}^{2}, \]  

(A18)

\[ F(H^{00} H^{00}) = \tilde{h}_{00} \psi_{1}^{k} \frac{-q^{k} \left( q_{0} + \frac{1}{2} k_{0} \right)}{q^{2}(q + k)^{2}}, \]  

(A19)

\[ F(H^{00} H^{00} H^{00}) = \tilde{h}_{00} \psi_{1}^{k} \frac{1}{4} \delta^{kl} \left( q^{2} + 3k \cdot q \right) - \frac{1}{2} k^{k} k^{l}, \]  

(A20)

\[ F(H^{00} H^{00} H^{00} H^{00}) = \tilde{h}_{00} \psi_{2}^{k} \frac{-1}{4} \delta^{kl} \left( q^{2} + k \cdot q \right) + \frac{1}{4} k^{k} k^{l}, \]  

(A21)

\(^{7}\) We consider the partial Fourier transform for the radiation field as well.
On the other hand, to compute the contributions to $T_{1PN}^{\mu}$, the contractions required are

$$F^{(H^{00}H^{00})} [q, k, \bar{h}^{00}] = \bar{h}^{00} \left[ q_0 \left( q^i + \frac{1}{2} k^i \right) + k_0 \left( \frac{1}{2} q^0 + k^0 \right) \right],$$

$$F^{(\nu_{\mu} H^{\nu\kappa} H^{00})} [q, k, \bar{h}^{00}] = \bar{h}^{00} \nu_{\mu} \left[ -\delta^{ik} \left( \frac{1}{2} q^2 + k \cdot q \right) + q^i k^k + \frac{1}{2} k^i k^k \right],$$

whereas for $T_{1PN}^{\mu}$ we need

$$F^{(H^{00}H^{00})} [q, k, \bar{h}^{\mu}] = \bar{h}^{\mu} \left[ \frac{1}{4} q^2 + \frac{1}{2} k \cdot q - \frac{3}{4} (q_0^2 + k_0 q_0 + 2k_0^2) \right],$$

$$F^{(\nu_{\mu} H^{\nu\kappa} H^{00})} [q, k, \bar{h}^{\mu}] = \bar{h}^{\mu} \nu_{\mu} \left[ -k_0 \left( q^k + \frac{1}{2} k^k \right) \right],$$

$$F^{(\nu_{\mu} H^{\nu\kappa} H^{00})} [q, k, \bar{h}^{\mu}] = \bar{h}^{\mu} \nu_{\mu} \left[ k^{kl} \left( -\frac{1}{4} q^2 + \frac{1}{4} k \cdot q \right) - \frac{1}{2} k^k k^k \right],$$

$$F^{(\nu_{\mu} H^{\nu\kappa} H^{00})} [q, k, \bar{h}^{\mu}] = \bar{h}^{\mu} \nu_{\mu} \left[ \frac{1}{4} k^{kl} (q^2 + k \cdot q) - \frac{1}{2} k^k k^l + \frac{1}{4} k^k k^l \right].$$

The three-graviton vertex, in turn, comes naturally in a simple form even not integrated on the internal momenta:

$$\langle H_{\mu}^{00} H_{\nu}^{00} H_{\alpha}^{00} \rangle = -\frac{(2\pi)^3}{4m_{Pl}} \delta(t_2 - t_1) \delta(t_3 - t_1) \delta^3 (q_1 + q_2 + q_3) \left( \frac{q_1^2 + q_2^2 + q_3^2}{q_1^2 q_2^2 q_3^2} \right).$$

The composition of the three-potential-graviton vertex with two-potential-one-radiation-graviton vertex, after integrating in the third momentum, the integrand takes the form

$$F_{[q_1, q_2, k, \bar{h}]} = \frac{k^i (q_1 + k^i)^2 (q_1 + q_2 + k)^2}{q_1^2 q_2^2 (q_1 + k)^2 (q_1 + q_2 + k)^2},$$

in which the numerators for the contractions needed to compute the contributions for $T_{2PN}^{00}$ and $T_{1PN}^{\mu}$ are, respectively,

$$F^{(H^{00}H^{00}H^{00})} [q_1, q_2, k, \bar{h}^{00}] = \frac{1}{4} \bar{h}^{00} \left[ q_1^0 + \frac{5}{2} q_1^2 (q_1 \cdot q_2) + \frac{5}{2} q_2^2 + (q_1 + q_2)^2 (q_1 \cdot k) + \frac{5}{2} q_1^2 (q_2 \cdot k) + 3 (q_1 \cdot k) (q_2 \cdot k) - (q_1 \cdot k)^2 + (q_2 \cdot k)^2 \right],$$

$$F^{(H^{00}H^{00}H^{00})} [q_1, q_2, k, \bar{h}^{\mu}] = -\frac{\bar{h}^{\mu}}{8} \left[ 2 q_1^2 - q_1^2 q_1 \cdot q_2 + 10 q_1^2 q_1 \cdot k + 10 (q_1 \cdot k)^2 - q_1^2 q_2^2 - q_1^2 q_2 \cdot k - 2 (q_1 \cdot k) (q_2 \cdot k) - 2 (q_2 \cdot k)^2 \right].$$

The three-potential-one-radiation-graviton vertex integrated in the internal momenta can be expressed in this way:

$$\prod_{i=1}^{3} \int \frac{d^3 q_i}{(2\pi)^3} e^{i q_i \cdot x_i} \langle i S_{i H^0} \rangle = -\frac{1}{m_{Pl}^2} \delta(t_2 - t_1) \delta(t_3 - t_1) \int \frac{d^3 q_2}{(2\pi)^3} \int \frac{d^3 q_3}{(2\pi)^3} \frac{e^{i (q_2 + q_3) \cdot x_2} F_{[q_2, q_3, k, \bar{h}]}}{q_2^2 q_3^2 (q_2 + q_3 + k)^2},$$

where we have chosen to integrate on $q_1$, for instance coupled to particle 1, and leaving the momenta $q_2$ and $q_3$, both coupled to particle 2, to be integrated in the process of solving the diagrams. For this case, the contractions required to write down the contribution for $T_{2PN}^{00}$ and $T_{1PN}^{\mu}$, respectively, are given by

$$F^{(H^{00}H^{00}H^{00})} [q_2, q_3, k, \bar{h}^{00}] = -\frac{1}{8} \bar{h}^{00} (q_2^2 + q_3^2 + q_2 \cdot q_3 + q_2 \cdot k + q_3 \cdot k),$$

$$F^{(H^{00}H^{00}H^{00})} [q_2, q_3, k, \bar{h}^{\mu}] = -\frac{7}{8} \bar{h}^{\mu} (q_2^2 + q_3^2 + q_2 \cdot q_3 + q_2 \cdot k + q_3 \cdot k).$$
Integrals

To solve integrals in the momentum space, it is helpful to use some general relations that can be obtained by using Feynman parameters [42]. If we consider a spacetime of $d$ dimensions, then for $D = d - 1$ we have

\[
\int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2)^a} \left(\frac{k^2}{4}\right)^{\alpha - \frac{D}{2}} = \frac{1}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma \left( \frac{D}{2} - a \right)}{\Gamma(a)} \left(\frac{r^2}{4}\right)^{\alpha - \frac{D}{2}},
\]

(A35)

\[
\int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2)^a} \left(\frac{k^2}{4}\right)^{\alpha - \frac{D}{2}} = \frac{\Gamma \left( \frac{D}{2} - a - b \right)}{\Gamma(a) \Gamma(b) \Gamma(D - a - b)},
\]

(A36)

\[
\int \frac{d^Dk}{(2\pi)^D} \frac{k^i}{(k^2)^a} \left(\frac{k^2}{4}\right)^{\alpha - \frac{D}{2}} = \frac{\Gamma \left( \frac{D}{2} - a - b \right)}{\Gamma(a) \Gamma(b) \Gamma(D - a - b + 1)},
\]

(A37)

\[
\int \frac{d^Dk}{(2\pi)^D} \frac{k^i k^j}{(k^2)^a} \left(\frac{k^2}{4}\right)^{\alpha - \frac{D}{2}} = \frac{\Gamma \left( \frac{D}{2} - a - b \right)}{\Gamma(a) \Gamma(b) \Gamma(D - a - b + 2)}
\]

\[
\times \left\{ \frac{\eta^{ij}}{2} \Gamma \left( a + b - 1 - \frac{D}{2} \right) \Gamma \left( \frac{D}{2} - a + 1 \right) \Gamma \left( \frac{D}{2} - b + 1 \right) \right. \\
\left. + p^i p^j \Gamma \left( a + b - \frac{D}{2} \right) \Gamma \left( \frac{D}{2} - b \right) \Gamma \left( \frac{D}{2} - a + 2 \right) \right\}.
\]

(A38)

These integrals are especially important to solve diagrams that has a composition of the three-potential-graviton vertex with the two-potential-one-radiation vertex, where an analysis of the integrals in an arbitrary dimension $D$ is required to handle divergences.

Appendix B

In this appendix we present the result for the 2PN acceleration computed via the EFT approach in the linearized harmonic gauge.

To write down the equation of motion of the binary system at 2PN order, we need to obtain the Lagrangian by integrating out the potential modes of the gravitational fields in the action (2.2). Below the diagrams which contribute to the dynamics at 2PN order are presented.

\[ L_{\text{Fig 11}} = \sum_a \frac{1}{16} m_a v_a^6. \]

(B1)

**FIG. 11:** Diagram with no graviton exchange.

The simplest contribution to the 2PN Lagrangian comes from the diagram show in Fig. 11, which gives the following contribution:
FIG. 12: Diagrams with one-graviton exchange.

Next, we have the diagrams with one-graviton exchange illustrated in Fig. 12. Summing those diagrams together yields

$$L_{\text{Fig} 12} = \sum_{a \neq b} \frac{G m_a m_b}{16 r^4} \left\{ 15 r^4 \mathbf{a}_a \cdot \mathbf{a}_b + r^2 \left[ 14 v_a^2 - 20 v_a^2 v_b \cdot v_b + 2 (v_a \cdot v_b)^2 ight. ight.$$

$$+ 3 v_a^2 v_b^2 + 2 v_b^2 a_a \cdot r - a_a \cdot r a_b \cdot r + 28 a_a \cdot v_a v_a \cdot r + 24 a_a \cdot v_a v_b \cdot r \
\left. + 2 (a_b \cdot r - v_b^2) (v_a \cdot r)^2 + 12 (v_a \cdot v_b - v_a^2) v_a \cdot r v_b \cdot r + \frac{3}{r^2} (v_a \cdot r)^2 (v_b \cdot r)^2 \right\}. \quad (B2)$$

FIG. 13: Diagrams with two-graviton exchange.

In Fig. 13 we show all diagrams with two-graviton exchange that enter at the second PN order. The sum of those diagrams is

$$L_{\text{Fig} 13} = \sum_{a \neq b} \frac{G^2 m_a^2 m_b}{4 r^4} \left( 6 r^2 v_a^2 + 7 r^2 v_b^2 - 14 r^2 v_a \cdot v_b + 2 r v_a \cdot r - 2 v_a \cdot r v_b \cdot r \right). \quad (B3)$$

FIG. 14: (a) three-graviton emission from one of the bodies; (b) symmetric three-graviton exchange; (c) composition of a three-graviton vertex with a two-graviton vertex in the source term.
There is also the diagram with a three-graviton source term as well as other two diagrams with combinations of the two-graviton source, as shown in Fig. 14. Their contribution to the Lagrangian is

\[ L_{\text{Fig}14} = -\sum_{a \neq b} \frac{G^3 m_a^2 m_b}{2r^3} (m_a + 3m_b). \]  

(B4)

![Diagram](image1.png)

**FIG. 15:** Diagrams with three-graviton exchange.

The diagrams which contain three-graviton vertices are illustrated in Fig. 15 and give

\[ L_{\text{Fig}15} = \sum_{a \neq b} \frac{G^2 m_a^2 m_b}{2r^4} \left[ r^2 \left( 5v_a^2 - 6v_a \cdot v_b + 2v_b^2 + 2a_b \cdot r \right) - 9 (v_a \cdot r)^2 + 14v_a \cdot rv_b \cdot r - 3 (v_b \cdot r)^2 \right]. \]  

(B5)

![Diagram](image2.png)

**FIG. 16:** Diagrams with four-graviton vertex.

In Fig. 16, we show diagrams with a four-graviton vertex that enter at the 2PN order and, together, yield the result

\[ L_{\text{Fig}16} = \sum_{a \neq b} \frac{G^3 m_a^2 m_b}{r^3}. \]  

(B6)

![Diagram](image3.png)

**FIG. 17:** Diagrams with five propagators.
Lastly, the diagrams with five propagators are shown in Fig. 17 and provide us with the following result:

$$L_{Fig17} = \sum_{a \neq b} \frac{G^3 m_a^2 m_b}{r^3} (m_b - 2m_a).$$  \hspace{1cm} (B7)$$

Summing up all contributions from Fig. 11 to Fig. 17, we write down the Lagrangian at 2PN order in the linearized harmonic gauge:

$$L_{2PN} = \frac{1}{16} m_1 v_1^6 - \frac{G^3 m_1 m_2}{2r^3} (3m_1^2 + m_1 m_2) + \frac{G^2 m_1 m_2}{4r^2} \left[ (16m_1 + 11m_2) v_1^2 - 13m v_1 \cdot v_2 - 4m_2 a_1 \cdot r - \frac{2}{r^2} (8m_1 + 3m_2) (v_1 \cdot r)^2 + \frac{12}{r^2} m v_1 \cdot r v_2 \cdot r \right]$$

$$+ \frac{G m_1 m_2}{8r} \left[ \frac{15}{2} r^2 a_1 \cdot a_2 + 7v_1^2 - 10v_1^2 v_1 \cdot v_2 + (v_1 \cdot v_2)^2 + \frac{3}{2} v_r^2 v_r^2 + a_1 \cdot r v_2^2 - 14a_1 \cdot v_2 v_2 \cdot r + 12a_1 \cdot v_1 v_2 \cdot r - \frac{1}{2} a_1 \cdot r a_2 \cdot r - \frac{1}{r^2} a_1 \cdot r (v_2 \cdot r)^2 \right]$$

$$+ \frac{1}{r^2} \left( 6v_1 \cdot r v_2 \cdot r v_1 \cdot v_2 - (v_1 \cdot r)^2 v_2^2 - 6v_1 \cdot r v_2 \cdot r v_r^2 + \frac{3}{2} v_2^2 (v_1 \cdot r)^2 (v_2 \cdot r)^2 \right) \right]$$

$$+ 1 \leftrightarrow 2.$$  \hspace{1cm} (B8)$$

We use the Lagrangian above to determine the equations of motion of the two-body system at the second PN order. Below we show the acceleration for one of the objects in the binary:

$$a_{2PN}^1 = \frac{1}{8} \frac{G m_2}{r^3} \left\{ \frac{G^2}{r^7} \left[ -2m_2^2 - 20m_1 m_2 + 16m_2^2 \right] + \frac{G}{r} \left[ (18m_1 + 56m_2) v_1^2 - (84m_1 + 128m_2) v_1 \cdot v_2 + (58m_1 + 64m_2) v_2^2 + 30m_1 a_1 \cdot r - 12m_2 a_2 \cdot r \right] + \frac{28}{r^2} \left( m_1 - 4m_2 \right) v_1 \cdot r \left( v_1 \cdot r - 2v_2 \cdot r \right) - \frac{1}{r^2} (56m_1 + 176m_2) (v_2 \cdot r)^2 \right\}$$

$$+ \frac{1}{r^2} \left( 2v_1^4 - 16 (v_1 \cdot v_2)^2 - 16v_2^4 + 32v_1 \cdot v_2 v_2^2 - 2v_1^2 a_2 \cdot r - 2v_2^2 a_2 \cdot r \right.$$ 
$$- \frac{4a_2 \cdot v_2 v_2 \cdot r + (v_2 \cdot r)^2}{r^2} \left( 12v_2^4 - 48v_1 \cdot v_2 + 36v_2^2 \right) - 15 \frac{(v_2 \cdot r)^4}{r^4} \right\}$$

$$+ \frac{1}{4} \frac{G m_2}{r^3} v_1 \left\{ \frac{G}{r} \left[ (48m_2 - 15m_1) v_1 \cdot r + (23m_1 - 40m_2) v_2 \cdot r \right] + v_2 \cdot r \left( 4v_1^2 + 16v_1 \cdot v_2 - 20v_2^2 \right) - 24 \frac{v_1 \cdot r (v_2 \cdot r)^2}{r^2} + 18 \frac{(v_2 \cdot r)^3}{r^2} \right\}$$

$$+ v_1 \cdot r \left( 8v_1^2 - 16v_1 \cdot v_2 + 16v_2^2 - 2a_2 \cdot r \right) + 2r^2 (12a_1 - 7a_2) \cdot v_1 \right\}$$

$$+ 2a_1 \cdot v_1 v_1^2 v_1 + \frac{1}{4} a_1 \left\{ \right. \frac{G^2 m_1 m_2}{r^2} + \frac{G^2 m_2}{r^2} + 12 \frac{G m_2}{r} v_1^2 \left. + v_1^4 \right\}$$

$$+ \frac{1}{4} \frac{G m_2}{r^3} v_2 \left\{ \frac{G}{r} \left[ (31m_1 - 24m_2) v_1 \cdot r + (40m_2 - 9m_1) v_2 \cdot r \right] + v_2 \cdot r \left( -4v_1^2 - 16v_1 \cdot v_2 + 20v_2^2 \right) + 24 \frac{v_1 \cdot r (v_2 \cdot r)^2}{r^2} - 18 \frac{(v_2 \cdot r)^3}{r^2} \right\}$$

$$+ v_1 \cdot r \left( 16v_1 \cdot v_2 - 16v_2^2 \right) - 14r^2 a_2 \cdot v_2 \right\}$$

$$- \frac{7}{4} \frac{G m_2}{r} a_2 \left( \right.$$ 
$$\frac{6G m}{r} + v_1^2 + v_2^2 \right).$$  \hspace{1cm} (B9)$$

All accelerations in the right hand side of the equality above should be regarded as Newtonian accelerations if we want the entire expression to be of definite 2PN order. To write the acceleration in the center of mass frame, we have to consider, in addition to (B9), the reduced contribution from applying the equation of motion inside the (4.12) as well as the PN corrections to the center of mass frame (6.10) and (6.11). Adding these contributions together, we finally obtain the expression for the relative acceleration of the two-body system in the center of mass frame, at the
second PN order, in the linearized harmonic gauge:

\[
\alpha_{2PN} = -\frac{Gm}{8r^3} \left\{ r \left[ (56 + 174\nu) \frac{G^2m^2}{r^2} - (32 + 52\nu - 16\nu^2) \frac{Gm}{r} v^2 + (112 - 200\nu - 16\nu^2) \frac{Gm}{r} i^2 \right] + (24\nu - 32\nu^2) v^4 - (36\nu - 48\nu^2) v^2 i^2 + (15\nu - 45\nu^2) i^4 \right\} + 4r\nu \left[ (-12 + 41\nu + 8\nu^2) \frac{Gm}{r} - (15\nu + 4\nu^2) v^2 + (9\nu + 6\nu^2) i^2 \right].
\]

(B10)

[1] B. P. Abbott et al. Observation of Gravitational Waves from a Binary Black Hole Merger. Phys. Rev. Lett., 116(6):061102, 2016.
[2] B. P. Abbott et al. Astrophysical Implications of the Binary Black-Hole Merger GW150914. Astrophys. J., 818(2):L22, 2016.
[3] B. P. Abbott et al. Binary Black Hole Mergers in the first Advanced LIGO Observing Run. Phys. Rev., X6(4):041015, 2016.
[4] B. P. Abbott et al. Gw170104: Observation of a 50-solar-mass binary black hole coalescence at redshift 0.2. Phys. Rev. Lett., 118:221101, Jun 2017.
[5] B. P. Abbott et al. Gw170814: A three-detector observation of gravitational waves from a binary black hole coalescence. Phys. Rev. Lett., 119:141101, Oct 2017.
[6] B. P. Abbott et al. Gw170817: Observation of gravitational waves from a binary neutron star inspiral. Phys. Rev. Lett., 119:161101, Oct 2017.
[7] B. P. Abbott et al. Gw170608: Observation of a 19 solar-mass binary black hole coalescence. The Astrophysical Journal, 851(2):L35, dec 2017.
[8] B. P. Abbott et al. Gwte-1: A gravitational-wave transient catalog of compact binary mergers observed by ligo and virgo during the first and second observing runs. Phys. Rev. X, 9:031040, Sep 2019.
[9] B. P. Abbott et al. Gravitational waves and gamma-rays from a binary neutron star merger: GW170817 and GRB 170817a. The Astrophysical Journal, 848(2):L13, oct 2017.
[10] B. P. Abbott et al. Multi-messenger observations of a binary neutron star merger. The Astrophysical Journal, 848(2):L12, Oct 2017.
[11] B. P. Abbott et al. Low-latency gravitational-wave alerts for multimessenger astronomy during the second advanced LIGO and virgo observing run. The Astrophysical Journal, 875(2):161, apr 2019.
[12] Walter D. Goldberger and Ira Z. Rothstein. An Effective field theory of gravity for extended objects. Phys. Rev., D73:104029, 2006.
[13] Walter D. Goldberger. Les Houches lectures on effective field theories and gravitational radiation. In Les Houches Summer School - Session 86: Particle Physics and Cosmology: The Fabric of Spacetime Les Houches, France., 2007.
[14] Ira Z. Rothstein. Progress in effective field theory approach to the binary inspiral problem. Gen. Rel. Grav., 46:1726, 2014.
[15] Stefano Foffa and Riccardo Sturani. Effective field theory methods to model compact binaries. Class. Quant. Grav., 31(4):043001, 2014.
[16] Rafael A. Porto. The effective field theoretist’s approach to gravitational dynamics. Phys. Rept., 633:1–104, 2016.
[17] Michele Levi. Effective Field Theories of Post-Newtonian Gravity: A comprehensive review. 2018.
[18] Stefano Foffa and Riccardo Sturani. Conservative dynamics of binary systems to fourth Post-Newtonian order in the EFT approach I: Regularized Lagrangian. Phys. Rev., D100(2):024047, 2019.
[19] Stefano Foffa, Rafael A. Porto, Ira Rothstein, and Riccardo Sturani. Conservative dynamics of binary systems to fourth Post-Newtonian order in the EFT approach II: Renormalized Lagrangian. Phys. Rev., D100(2):024048, 2019.
[20] Donato Bini and Thibault Damour. Analytical determination of the two-body gravitational interaction potential at the fourth post-Newtonian approximation. Phys. Rev., D87(12):121501, 2013.
[21] Thibault Damour, Piotr Jaranowski, and Gerhard Schafer. Nonlocal-in-time action for the fourth post-Newtonian conservative dynamics of two-body systems. Phys. Rev., D89(6):064058, 2014.
[22] Laura Bernard, Luc Blanchet, Alejandro Bohe, Guillaume Faye, and Sylvain Marsat. Fokker action of nonspinning compact binaries at the fourth post-Newtonian approximation. Phys. Rev., D90(3):084037, 2016.
[23] Laura Bernard, Luc Blanchet, Alejandro Bohe, Guillaume Faye, and Sylvain Marsat. Energy and periastron advance of compact binaries on circular orbits at the fourth post-Newtonian approximation. Phys. Rev., D95(4):044026, 2017.
[24] Rafael A. Porto, Andreas Ross, and Ira Z. Rothstein. Spin induced multipole moments for the gravitational wave flux from binary inspirals to third Post-Newtonian order. Phys. Rev., D95(4):044026, 2017.
[25] Rafael A. Porto, Andreas Ross, and Ira Z. Rothstein. Spin induced multipole moments for the gravitational wave amplitude from binary inspirals to 2.5 Post-Newtonian order. JCAP, 1209:028, 2012.
[26] Walter D. Goldberger and Andreas Ross. Gravitational radiative corrections from effective field theory. Phys. Rev., D81:124015, 2010.
[27] Luc Blanchet, Guillaume Faye, Bala R. Iyer, and Benoit Joguet. Gravitational wave inspiral of compact binary systems to 7/2 postNewtonian order. Phys. Rev., D65:061501, 2002. [Erratum: Phys. Rev.D71,129902(2005)].
