ON THE HÖRMANDER MULTIPLIER THEOREM AND MODULATION SPACES

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Abstract. It is known that the Sobolev space $L^2_s$ with $s > n/2$ appeared in the Hörmander multiplier theorem can be replaced by the Besov space $B^2_{n/2}$. On the other hand, the Besov space $B^2_{n/2}$ is continuously embedded in the modulation space $M^2_0$. In this paper, we consider the problem whether we can replace $B^2_{n/2}$ by $M^2_0$.

1. Introduction

Sjöstrand [18] proved the $L^2$-boundedness of pseudo-differential operators with symbols in the modulation space $M^\infty_1(\mathbb{R}^n)$ which contains the Hörmander class $S^{0,0}_0$. Since then, modulation spaces have been recognized as a useful tool for pseudo-differential operators. See Benyi-Gröchenig-Okoudjou-Rogers [3], Cordero-Nicola-Rodino [5], Gröchenig-Heil [10] and Toft [23] for further developments. The purpose of this paper is to apply modulation spaces to (singular) Fourier multipliers.

We recall some known results on the boundedness of Fourier multipliers on $L^p(\mathbb{R}^n)$. The Mihlin multiplier theorem says that if $m \in C^{[n/2] + 1}(\mathbb{R}^n \setminus \{0\})$ satisfies

\begin{equation}
|\partial^\alpha m(\xi)| \leq C_{\alpha} |\xi|^{-|\alpha|} \quad \text{for all } |\alpha| \leq [n/2] + 1
\end{equation}

then $m(D)$ is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ (see [6, Corollary 8.11]), where $[n/2]$ stands for the largest integer $\leq n/2$. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\psi \geq c > 0$ on $\{2^{1/2} \leq |\xi| \leq 2^{1/2}\}$ and supp $\psi \subset \{2^{-1} \leq |\xi| \leq 2\}$. For $m \in \mathcal{S}'(\mathbb{R}^n)$, we set

\begin{equation}
m_j(\xi) = \psi(\xi) m(2^j \xi).
\end{equation}

The Hörmander multiplier theorem [11] states that if $m \in \mathcal{S}'(\mathbb{R}^n)$ satisfies

\begin{equation}
\sup_{j \in \mathbb{Z}} \|m_j\|_{L^2_s} < \infty \quad \text{with } s > n/2
\end{equation}

then $m(D)$ is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ (see also [6, Theorem 8.10]), where $L^2_s(\mathbb{R}^n)$ is the Sobolev space. We note that (1.3) is weaker than (1.1). By using the Besov space $B^{2,1}_{n/2}(\mathbb{R}^n)$ instead of the Sobolev space $L^2_s(\mathbb{R}^n)$ in (1.3), Seeger [17] proved that if $m \in \mathcal{S}'(\mathbb{R}^n)$ satisfies

\begin{equation}
\sup_{j \in \mathbb{Z}} \|m_j\|_{B^{2,1}_{n/2}} < \infty
\end{equation}

then $m(D)$ is bounded from the Hardy space $H^1(\mathbb{R}^n)$ to the Lorentz space $L^{1,2}(\mathbb{R}^n)$ (see [16, 17] for the definition of $L^{1,2}$). Then, by interpolation and duality, (1.4) implies the boundedness of $m(D)$ on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$. It should be
pointed out that the $L^p$-boundedness of $m(D)$ satisfying (1.4) follows from a slight modification of Stein’s approach in [19, Chapter 4, Section 3]. Since
\[ L^2(R^n) = B^{2,2}_s(R^n) \hookrightarrow B^{2,1}_{n/2}(R^n) \quad \text{if } s > n/2, \]
we see that (1.4) is weaker than (1.3).

It is known that the Besov space $B^{2,1}_{n/2}(R^n)$ is continuously embedded in the modulation space $M^{2,1}_{s}(R^n)$, and this embedding yields the problem
\[ \text{“Can we replace } B^{2,1}_{n/2}(R^n) \text{ in (1.4) by } M^{2,1}_{0}(R^n)?” \]

At least, we have

**Theorem 1.1.** Let $s > 0$. If $m \in S'(R^n)$ satisfies
\[ (1.5) \quad \sup_{j \in Z} \| m_j \|_{M^{2,1}} < \infty, \]
then $m(D)$ is bounded on the Hardy space $H^1(R^n)$, where $m_j$ is defined by (1.2).

We note that, if $m$ satisfies (1.5) with $s \geq 0$, then $m(D)$ is bounded on $L^2(R^n)$ (see the proof of Theorem 1.1). Then, by interpolation and duality, (1.5) with $s > 0$ implies the boundedness of $m(D)$ on $L^p(R^n)$ for all $1 < p < \infty$. Hence, Theorem 1.1 covers the Hörmander multiplier theorem, since
\[ L^2(R^n) = M^{2,2}_{s}(R^n) \hookrightarrow M^{2,1}_{s'}(R^n) \quad \text{if } s' < s - n/2. \]

Let us compare (1.4) and (1.5). Toft [23, Theorem 3.1] proved the embeddings
\[ B^{2,1}_{n/2}(R^n) \hookrightarrow M^{2,1}_{s}(R^n) \hookrightarrow B^{2,1}_0(R^n), \]
and the optimality was proved by Sugimoto-Tomita [22, Theorem 1.2] (see also [26]). More precisely, if $B^{2,1}_{s}(R^n) \hookrightarrow M^{2,1}_{s}(R^n)$ then $s \geq n/2$, and if $M^{2,1}_{s}(R^n) \hookrightarrow B^{2,1}_{s'}(R^n)$ then $s' \leq 0$. Then, since $\|f\|_{B^{2,1}} \asymp \| (I - \Delta)^{s/2} f \|_{B^{2,1}_{0}}$ and $\|f\|_{M^{2,1}} \asymp \| (I - \Delta)^{s/2} f \|_{M^{2,1}_{s}}$, we see that $B^{2,1}_{n/2}(R^n) \hookrightarrow M^{2,1}_{s}(R^n)$ if and only if $s \leq 0$, and $M^{2,1}_{s}(R^n) \hookrightarrow B^{2,1}_{n/2}(R^n)$ if and only if $s \geq n/2$. Therefore, $B^{2,1}_{n/2}(R^n)$ and $M^{2,1}_{s}(R^n)$ have no inclusion relation with each others if $0 < s < n/2$:

We also mention the relation between Theorem 1.1 and Baernstain-Sawyer [1] (see also Carbery [4] and Seeger [15] for some related results). Since
\[ (1.6) \quad C^{-1} \| \widehat{m_j} \|_{K^{1,1}_s} \leq \| m_j \|_{M^{2,1}_s} \leq C \| \widehat{m_j} \|_{K^{1,1}_s} \]
(see Section 8), where $K^{1,1}_s$ is the Herz space, we have by Theorem 1.1

**Corollary 1.2.** Let $s > 0$. If $m \in S'(R^n)$ satisfies
\[ \sup_{j \in Z} \| \widehat{m_j} \|_{K^{1,1}_s} < \infty, \]
then $m(D)$ is bounded on the Hardy space $H^1(R^n)$, where $m_j$ is defined by (1.2).
We remark that Corollary 1.2 is a special case of [11, Theorem 3]. As another corollary of Theorem 1.4 we have by the norm equivalence
\[
C_p^{-1} \| m_j \|_{M_p^{-1}} \leq \| m_j \|_{M_p^{2,1}} \leq C_p \| m_j \|_{M_p^{-1}}.
\]
**Corollary 1.3.** Let \( 1 \leq p \leq \infty \) and \( s > 0 \). If \( m \in \mathcal{S}'(\mathbb{R}^n) \) satisfies
\[
\sup_{j \in \mathbb{Z}} \| m_j \|_{M_p^{2,1}} < \infty,
\]
then \( m(D) \) is bounded on the Hardy space \( H^1(\mathbb{R}^n) \), where \( m_j \) is defined by (1.2).

However, in the critical case \( s = 0 \), we have the following negative answer:

**Proposition 1.4.** Let \( 1 < p < \infty \) and \( p \neq 2 \). Then there exists a Fourier multiplier \( m \in \mathcal{S}'(\mathbb{R}^n) \) such that \( \sup_{j \in \mathbb{Z}} \| m_j \|_{M_p^{2,1}} < \infty \), but \( m(D) \) is not bounded on \( L^p(\mathbb{R}^n) \).

The proofs of Theorem 1.1, (1.6), (1.7) and Proposition 1.4 will be given in Section 3.

2. Preliminaries

Let \( \mathcal{S}(\mathbb{R}^n) \) and \( \mathcal{S}'(\mathbb{R}^n) \) be the Schwartz spaces of all rapidly decreasing smooth functions and tempered distributions, respectively. We define the Fourier transform \( \mathcal{F} \) and the inverse Fourier transform \( \mathcal{F}^{-1} \) of \( f \in \mathcal{S}(\mathbb{R}^n) \) by
\[
\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) \, dx \quad \text{and} \quad \mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\xi \cdot x} f(\xi) \, d\xi.
\]
For \( m \in \mathcal{S}'(\mathbb{R}^n) \), the Fourier multiplier operator \( m(D) \) is defined by \( m(D)f = \mathcal{F}^{-1}[m \hat{f}] \) for \( f \in \mathcal{S}(\mathbb{R}^n) \). The notation \( A \asymp B \) stands for \( C^{-1}A \leq B \leq CA \) for some positive constant \( C \) independent of \( A \) and \( B \).

We introduce Besov and modulation spaces, and suppose that \( 1 \leq p, q \leq \infty \) and \( s \in \mathbb{R} \). Let \( \psi \in \mathcal{S}(\mathbb{R}^n) \) be such that \( \psi \geq c > 0 \) on \( \{ 2^{-1/2} \leq |\xi| \leq 2^{1/2} \} \),
\[
\text{supp } \psi \subset \{ 1/2 \leq |\xi| \leq 2 \} \quad \text{and} \quad \sum_{j \in \mathbb{Z}} \psi(2^{-j}\xi) = 1 \quad \text{for all } \xi \neq 0.
\]
We set
\[
\psi_0(\xi) = 1 - \sum_{j=1}^{\infty} \psi(2^{-j}\xi) \quad \text{and} \quad \psi_j(\xi) = \psi(2^{-j}\xi) \quad \text{if } j \geq 1.
\]
Then the Besov space \( B_{p,q}^s(\mathbb{R}^n) \) consists of all \( f \in \mathcal{S}'(\mathbb{R}^n) \) such that
\[
\| f \|_{B_{p,q}^s} = \left( \sum_{j=0}^{\infty} 2^{jsq} \| \psi_j(D)f \|_{L^p}^q \right)^{1/q} < \infty
\]
(with obvious modification in the case \( q = \infty \)). We refer to Triebel [25] and the references therein for more details on Besov spaces. Let \( \varphi \in \mathcal{S}(\mathbb{R}^n) \) be such that
\[
\text{supp } \varphi \subset [-1,1]^n \quad \text{and} \quad \sum_{k \in \mathbb{Z}^n} \varphi(\xi - k) = 1 \quad \text{for all } \xi \in \mathbb{R}^n.
\]
Then the modulation space \( M_{p,q}^s(\mathbb{R}^n) \) consists of all \( f \in \mathcal{S}'(\mathbb{R}^n) \) such that
\[
\| f \|_{M_{p,q}^s} = \left( \sum_{k \in \mathbb{Z}^n} (1 + |k|)^s \| \varphi(D-k)f \|_{L^p}^q \right)^{1/q} < \infty
\]
(with obvious modification in the case \( q = \infty \)). We remark that

\[
(2.4) \quad \| f \|_{M^{p,q}_r} \coloneqq \left\{ \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |V_g f(x, \xi)|^p dx \right)^{q/p} (1 + |\xi|^2)^{sq/2} d\xi \right\}^{1/q},
\]

where \( V_g f \) is the short-time Fourier transform of \( f \in \mathcal{S}'(\mathbb{R}^n) \) with respect to \( g \in \mathcal{S}(\mathbb{R}^n) \backslash \{0\} \) defined by

\[
V_g f(x, \xi) = \int_{\mathbb{R}^n} f(t) \overline{g(t-x)} e^{-ix\cdot t} \, dt \quad \text{for } x, \xi \in \mathbb{R}^n
\]

(see, for example, \([23]\)). The definition of \( M^{p,q}_r(\mathbb{R}^n) \) is independent of the choice of the window function \( g \in \mathcal{S}(\mathbb{R}^n) \backslash \{0\} \), that is, different window functions yield equivalent norms ([9, Proposition 11.3.2]). It is also well known that \( M^{2,2}_r(\mathbb{R}^n) \) is independent of the choice of the window function \( g \in \mathcal{S}(\mathbb{R}^n) \backslash \{0\} \), that is, different window functions yield equivalent norms ([9, Proposition 11.3.1]), where \( L^2(\mathbb{R}^n) \) is the Sobolev space defined by the norm \( \| f \|_{L^2} = \| (I - \Delta)^{s/2} f \|_{L^2} \) and \( (I - \Delta)^{s/2} f = \mathcal{F}^{-1}[(1 + |\xi|^2)^{s/2} \hat{f}] \). We refer to Feichtinger [7] and Gröchenig [9] for more details on modulation spaces (see also Bényi-Grafakos-Gröchenig-Okoudjou [2], Feichtinger-Narimani [8] for Fourier multipliers on modulation spaces).

We next introduce the Hardy and Herz spaces. Let \( \eta \in \mathcal{S}(\mathbb{R}^n) \) be such that \( \int_{\mathbb{R}^n} \eta(x) \, dx = 1 \). Then the Hardy space \( H^1(\mathbb{R}^n) \) consists of all \( f \in L^1(\mathbb{R}^n) \) such that

\[
\| f \|_{H^1} = \int_{\mathbb{R}^n} \sup_{t>0} |\eta_t \ast f(x)| \, dx < \infty,
\]

where \( \eta_t(x) = t^{-n} \eta(t^{-1}x) \). It is well known that

\[
(2.5) \quad \| f \|_{H^1} \simeq \| f \|_{L^1} + \sum_{j=1}^n \| R_j f \|_{L^1},
\]

where \( R_j \) is the Riesz transform defined by

\[
R_j f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot \xi} \left( -i \frac{\xi_j}{|\xi|} \right) \hat{f}(\xi) \, d\xi.
\]

The Herz space \( K^{p,q}_r(\mathbb{R}^n) \) consists of all \( f \in L^1_{loc}(\mathbb{R}^n) \) such that

\[
\| f \|_{K^{p,q}_r} = \left( \sum_{j=0}^\infty 2^{jq} \| \psi_j f \|_{L^p}^q \right)^{1/q} < \infty,
\]

where \( \{ \psi_j \}_{j=0}^\infty \) is as in \((2.2)\). See Baernstein-Sawyer [1] and Stein [20] for more details on Hardy and Herz spaces.

3. Proof

Before proving Theorem 1.1 we note that \( M^{2,1}_0 \hookrightarrow \mathcal{F}L^1 \). In fact, by Schwarz’s inequality and Plancherel’s theorem,

\[
\| \hat{f} \|_{L^1} \leq \sum_{k \in \mathbb{Z}^n} \| \varphi(\cdot - k) \hat{f} \|_{L^1} \leq C \sum_{k \in \mathbb{Z}^n} \| \varphi(\cdot - k) \hat{f} \|_{L^2} \leq C \sum_{k \in \mathbb{Z}^n} \| \varphi(D - k) f \|_{L^2} = C \| f \|_{M^{2,1}_0},
\]

(3.1) where \( \{ \varphi(\cdot - k) \}_{k \in \mathbb{Z}^n} \) is as in \((2.3)\).
Proof of Theorem 1.1. Let \( \psi \) be as in (2.1) and \( \sup_{j \in \mathbb{Z}} \|m_j\|_{M^{2,1}_s} < \infty \), where \( s > 0 \) and \( m_j(\xi) = \psi(\xi) m(2^j \xi) \). Since
\[
\|m_j\|_{L^\infty} \leq C \|\hat{m}_j\|_{L^1} \leq C \|m_j\|_{M^{2,1}_0} \leq C \|m_j\|_{M^{2,1}_s},
\]
and \( \psi(\xi) \geq c > 0 \) on \( \{2^{-1/2} \leq |\xi| \leq 2^{1/2}\} \), we see that \( m \in L^\infty \). This implies that \( m(D) \) is bounded on \( L^2 \). Then, by the Calderón-Zygmund theory (see, for example, [6 Corollary 6.3], [20 Chapter 3, Theorem 3]), if \( K = F^{-1} m \in L^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \) and
\[
(3.3) \quad \sup_{y \neq 0} \int_{|x| > 2|y|} |K(x-y) - K(x)| \, dx < \infty,
\]
then \( m(D) \) is bounded from \( H^1 \) to \( L^1 \).

We only consider (3.3) (see Remark 3.1 for the proof of \( K \in L^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \)). By (2.1), we have
\[
m(\xi) = \sum_{j \in \mathbb{Z}} \psi(2^{-j} \xi) m(\xi) = \sum_{j \in \mathbb{Z}} m_j(2^{-j} \xi),
\]
and consequently \( K(x) = \sum_{j \in \mathbb{Z}} 2^{in} K_j(2^j x) \), where \( K_j = F^{-1} m_j \). Then,
\[
\int_{|x| > 2|y|} |K(x-y) - K(x)| \, dx \leq \sum_{j \in \mathbb{Z}} \int_{|x| > 2|y|} |2^{jn} K_j(2^j(x-y)) - 2^{jn} K_j(2^j x)| \, dx.
\]
Note that \( \text{supp } m_j \subset \{-2 \leq |\xi| \leq 2\} \) for all \( j \in \mathbb{Z} \). Since
\[
(3.4) \quad \|K_j\|_{L^1} \leq C \|m_j\|_{M^{2,1}_s} \quad \text{and} \quad \|\nabla K_j\|_{L^1} \leq C \|K_j\|_{L^1} \leq C \|m_j\|_{M^{2,1}_0},
\]
(see (2.3) for the left hand inequality, and [25 Theorem 1.4.1 (ii)] for the right hand one), we have by Taylor's formula
\[
(3.5) \quad \int_{|x| > 2|y|} |2^{2jn} K_j(2^j(x-y)) - 2^{jn} K_j(2^j x)| \, dx
\]
\[
= \int_{|x| > 2|y|} \left| 2^{jn} \sum_{\ell=1}^n (-2^j y_\ell) \int_0^1 (\partial_\ell K_j)(2^j(x - ty)) \, dt \right| \, dx
\]
\[
\leq C 2^j |y| \|\nabla K_j\|_{L^1} \leq C 2^j |y| \|m_j\|_{M^{2,1}_s},
\]
where \( y = (y_1, \ldots, y_n) \). On the other hand,
\[
(3.6) \quad \int_{|x| > R} |K_j(x)| \, dx \leq CR^{-s} \|m_j\|_{M^{2,1}_s} \quad \text{for all } j \in \mathbb{Z} \text{ and } R > 0.
\]
In fact, since \( \text{supp } \varphi(-k) \subset k + [-1, 1]^n \subset \{|x-k| \leq \sqrt{n}\} \) (see (2.30)), we have
\[
\int_{|x| > R} |K_j(x)| \, dx \leq \sum_{k \in \mathbb{Z}^n} \int_{|x| > R} |\varphi(-x-k) K_j(x)| \, dx
\]
\[
\leq \sum_{|k| > R/2} \int_{\mathbb{R}^n} |\varphi(-x-k) F^{-1} m_j(x)| \, dx \leq \sum_{|k| > R/2} \left( \int_{\mathbb{R}^n} |\varphi(x-k) \hat{m}_j(x)|^2 \, dx \right)^{1/2}
\]
\[
= (2\pi)^{n/2} \sum_{|k| > R/2} (1 + |k|)^{-s} (1 + |k|)^s \|\varphi(D-k)m_j\|_{L^2} \leq CR^{-s} \|m_j\|_{M^{2,1}_s},
\]
for all \( R > 2\sqrt{n} \), and
\[
\int_{|x| > R} |K_j(x)| \, dx \leq (1 + R)^{-s} (1 + R)^s \|K_j\|_{L^1} \leq CR^{-s} \|m_j\|_{M^{2,1}_s}
\]
for all $0 < R \leq 2\sqrt{n}$, where we have used (3.4). Then, (3.6) gives
\begin{equation}
\int_{|x| > 2|y|} |2^{jn} K_j(2^j (x - y)) - 2^{jn} K_j(2^j x)| \, dx
\end{equation}
for all $y \neq 0$. Hence, it follows from (3.5) and (3.7) that
\begin{equation}
\sum_{j \in \mathbb{Z}} \int_{|x| > 2|y|} |2^{jn} K_j(2^j (x - y)) - 2^{jn} K_j(2^j x)| \, dx
\end{equation}
for all $y \neq 0$. Therefore, $m(D)$ is bounded from $H^1$ to $L^1$. This implies the boundedness of $m(D)$ on $H^1$. In fact, by (2.5),
\begin{equation}
\|m(D)f\|_{H^1} \leq C \left( \|m(D)f\|_{L^1} + \sum_{j=1}^{n} \|R_j(m(D)f)\|_{L^1} \right)
\end{equation}
where we have used the fact that $R_j$ is bounded on $H^1$. The proof is complete. □

Remark 3.1. It is not difficult to prove that, if $m$ satisfies (1.3) with $s > 0$, then $K = F^{-1}m \in L^1_{loc}(\mathbb{R}^n \setminus \{0\})$. Let $m_j$ and $K_j$ be as in the proof of Theorem 1.1 and recall that $K(x) = \sum_{j \in \mathbb{Z}} 2^{jn} K_j(2^j x)$. Since $sup \, m_j \subset \{1/2 \leq |x| \leq 2\}$ for all $j \in \mathbb{Z}$, it follows from (2.2) that $|K_j|_{L^\infty} = \|F^{-1}m_j|_{L^\infty} \leq C \|m_j\|_{L^\infty} \leq C \|m_j\|_{M^{2,1}_s}$ for all $j \in \mathbb{Z}$. Hence, for any $0 < R_1 < R_2 < \infty$,
\begin{equation}
\sum_{j=-\infty}^{0} \int_{R_1 \leq |x| \leq R_2} |2^{jn} K_j(2^j x)| \, dx \leq C_{R_1, R_2} \sum_{j=-\infty}^{0} 2^{jn} \|K_j\|_{L^\infty}
\end{equation}
\begin{equation}
\leq C_{R_1, R_2} \sum_{j=-\infty}^{0} 2^{jn} \|m_j\|_{M^{2,1}_s} \leq C_{R_1, R_2} \sup_{j \leq 0} \|m_j\|_{M^{2,1}_s}.
\end{equation}
On the other hand, by (3.3),
\begin{equation}
\sum_{j=1}^{\infty} \int_{R_1 \leq |x| \leq R_2} |2^{jn} K_j(2^j x)| \, dx \leq \sum_{j=1}^{\infty} \int_{|x| \geq 2^j R_1} |K_j(x)| \, dx
\end{equation}
\begin{equation}
\leq \sum_{j=1}^{\infty} C(2^j R_1)^{-s} \|m_j\|_{M^{2,1}_s} \leq C_{R_1} \sup_{j \geq 1} \|m_j\|_{M^{2,1}_s}.\end{equation}
Therefore, we see that
\begin{equation}
\int_{R_1 \leq |x| \leq R_2} |K(x)| \, dx \leq \sum_{j \in \mathbb{Z}} \int_{R_1 \leq |x| \leq R_2} |2^{jn} K(2^j x)| \, dx \leq C_{R_1, R_2} \sup_{j \in \mathbb{Z}} \|m_j\|_{M^{2,1}_s},
\end{equation}
that is, \( K \in L^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \).

To prove (1.6) and (1.7), we use the following fact [14, Remark 4.2] (see also [13, Lemma 1] for the case \( s = 0 \)), and give the proof for reader’s convenience.

**Proposition 3.2.** Let \( 1 \leq p, q \leq \infty \), \( s \in \mathbb{R} \), and let \( \Omega \) be a compact subset of \( \mathbb{R}^n \). Then there exists a constant \( C_\Omega > 0 \) such that

\[
C_\Omega^{-1} \|(I - \Delta)^{s/2} f\|_{\mathcal{F}L^q} \leq \|f\|_{M^p,q} \leq C_\Omega \|(I - \Delta)^{s/2} f\|_{\mathcal{F}L^q}
\]

for all \( f \in \mathcal{S}'(\mathbb{R}^n) \) with \( \text{supp} f \subset \Omega \), where \( \|f\|_{\mathcal{F}L^q} = \|\hat{f}\|_{L^q} \).

**Proof.** Our proof is based on one of [13, Lemma 1]. Let \( \Omega \subset \{|x| \leq R\} \), and let \( f \in \mathcal{S}'(\mathbb{R}^n) \) and \( g \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\} \) be such that \( \text{supp} f \subset \{|x| \leq R\} \), \( \text{supp} g \subset \{|x| \leq 4R\} \) and \( g = 1 \) on \( \{|x| \leq 2R\} \). Then \( \text{supp} V_g f(\cdot, \xi) \subset \{x : |x| \leq 5R\} \) for all \( \xi \in \mathbb{R}^n \). By Plancherel’s theorem,

\[
|V_g f(x, \xi)| = \frac{1}{(2\pi)^n} \left| \int_{\mathbb{R}^n} \hat{f}(t) \overline{\hat{g}(t - \xi)} e^{ix\cdot t} dt \right| \leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left| \hat{f}(t) \overline{\hat{g}(t - \xi)} \right| dt
\]

for all \( x \in \mathbb{R}^n \). Hence, by (2.4),

\[
\|f\|_{M^p,q} \leq C \left[ \int_{\mathbb{R}^n} \left( \int_{|t| \leq 5R} \left| \hat{f}(t) \overline{\hat{g}(t - \xi)} \right| dt \right)^p dx \right]^{1/p} \left( 1 + |\xi|^2 s^{q/2} \right)^{1/q} \leq CR^{n/p} \left[ \int_{\mathbb{R}^n} \left( \int_{|t| \leq 5R} (1 + |t|^2 s^{q/2}) \left| \hat{f}(t) \right| (1 + |t - \xi|^2)^{q/2} \left| \hat{g}(t - \xi) \right| dt \right)^q dx \right]^{1/q} \leq C_R \|(I - \Delta)^{s/2} g\|_{\mathcal{F}L^q} \|(I - \Delta)^{s/2} f\|_{\mathcal{F}L^q},
\]

where \( C > 0 \) is independent of \( f \).

On the other hand, since \( g = 1 \) on \( \{|x| \leq 2R\} \), we see that

\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} f(t) e^{-ix\cdot t} dt = \int_{|t| \leq R} f(t) \overline{\hat{g}(t - x)} e^{-ix\cdot t} dt = V_g f(x, \xi)
\]

for all \( |x| \leq R \). This gives

\[
\left| \hat{f}(\xi) \right| \leq CR^{-n/p} \left( \int_{|x| \leq R} \left| V_g f(x, \xi) \right|^p dx \right)^{1/p} \leq CR^{-n/p} \|V_g f(\cdot, \xi)\|_{L^p}
\]

for all \( \xi \in \mathbb{R}^n \). Therefore,

\[
\|(I - \Delta)^{s/2} f\|_{\mathcal{F}L^q} = \left( \int_{\mathbb{R}^n} \left| (1 + |\xi|^2)^{s/2} \hat{f}(\xi) \right|^q d\xi \right)^{1/q} \leq C_R \left\{ \int_{\mathbb{R}^n} \left( (1 + |\xi|^2)^{s/2} |V_g f(\cdot, \xi)| \right)^q d\xi \right\}^{1/q} \leq C_R\|f\|_{M^p,q},
\]

where \( C > 0 \) is independent of \( f \). The proof is complete. \(\square\)

We are now ready to prove (1.6), (1.7) and Proposition 1.3.

**Proofs of (1.6) and (1.7).** Let \( m_j \) be defined by (1.2). Note that \( \text{supp} m_j \subset \{2^{-1} \leq |\xi| \leq 2\} \) for all \( j \in \mathbb{Z} \). Then, by Proposition 3.2

\[
(3.8) \quad C^{-1} \|(I - \Delta)^{s/2} m_j\|_{\mathcal{F}L^1} \leq \|m_j\|_{M^2,1} \leq C \|(I - \Delta)^{s/2} m_j\|_{\mathcal{F}L^1},
\]

as required.
for all $j \in \mathbb{Z}$. On the other hand, since $(1 + |x|^2)^{1/2} \leq 2^j$ on $\text{supp} \psi_\ell$ for all $\ell \geq 0$ (see (2.1) and (2.2)), we have

$$
\|(I - \Delta)^{s/2} m_j\|_{B^s_{p,1} L^1} \leq \sum_{\ell=0}^{\infty} \int_{\mathbb{R}^n} |\psi_\ell(x) (1 + |x|^2)^{s/2} \widehat{m_j}(x)| \, dx
$$

(3.9)

Hence, combining (3.8) and (3.9), we have (1.7).

Let $\mathbf{1} \leq p \leq \infty$. Then, by Proposition 3.2

$$
C_p^{-1} \|(I - \Delta)^{s/2} m_j\|_{B^s_{p,1} L^1} \leq \|m_j\|_{M^{s,1}_p} \leq C_p \|(I - \Delta)^{s/2} m_j\|_{B^s_{p,1} L^1}
$$

(3.10)

for all $j \in \mathbb{Z}$. Therefore, combining (3.8) and (3.10), we have (1.7). \hfill \Box

Proof of Proposition 1.4 The following counterexample (Triebel [25, Proposition 2.6.4]) is known:

$$
\begin{cases}
\text{m}(D) & \text{is bounded on } B^{s,q}_p(\mathbb{R}^n) \text{ for all } 1 \leq p, q \leq \infty \text{ and } s \in \mathbb{R} \\
\text{m}(D) & \text{is not bounded on } L^p(\mathbb{R}^n) \text{ for any } p \neq 2
\end{cases}
$$

(see also Littman-McCarthy-Riviere [12] and Stein-Zygmund [21]). Let $m$ be as in (3.11), and we prove that $m$ satisfies $\sup_{j \in \mathbb{Z}} \|m_j\|_{M^{s,1}_p} < \infty$, where $m_j$ is defined by (1.2). We remark that $\text{m}(D)$ is bounded on $B^{1,q}_s(\mathbb{R}^n)$ for some $1 \leq q \leq \infty$ and $s \in R$ if and only if $\mathcal{F}^{-1} m \in B^{1,\infty}_0(\mathbb{R}^n)$ (see [25, Theorem 2.6.3]). Then, the boundedness of $\text{m}(D)$ on $B^{1,q}_s$ implies $\mathcal{F}^{-1} m \in B^{1,\infty}_0$. Hence, since $B^{1,\infty}_0 \hookrightarrow B^{1,\infty}_0$, we see that

$$
\begin{align*}
\sup_{j \in \mathbb{Z}} \|\mathcal{F}^{-1} m_j\|_{L^1} &= \sup_{j \in \mathbb{Z}} \|2^{-jn} \mathcal{F}^{-1}[\psi(2^{-j}) \cdot m](2^{-j})\|_{L^1} \\
&= \sup_{j \in \mathbb{Z}} \|\mathcal{F}^{-1}[\psi(2^{-j}) \cdot m]\|_{L^1} = \sup_{j \in \mathbb{Z}} \|\psi(2^{-j} D)(\mathcal{F}^{-1} m)\|_{L^1} \\
&= \|\mathcal{F}^{-1} m\|_{B^{1,\infty}_0} \leq C \|\mathcal{F}^{-1} m\|_{B^{1,\infty}_0} < \infty.
\end{align*}
$$

(3.12)

On the other hand, since $\text{supp} m_j \subset \{2^{-1} \leq |\xi| \leq 2\}$ for all $j \in \mathbb{Z}$, we have by Proposition 3.2

$$
C^{-1} \|\mathcal{F}^{-1} m_j\|_{L^1} \leq \|m_j\|_{M^{s,1}_p} \leq C \|\mathcal{F}^{-1} m_j\|_{L^1} \text{ for all } j \in \mathbb{Z}.
$$

(3.13)

Therefore, combining (3.12) and (3.13), we see that $\sup_{j \in \mathbb{Z}} \|m_j\|_{M^{s,1}_p} < \infty$, but $\text{m}(D)$ is not bounded on $L^p(\mathbb{R}^n)$ for any $p \neq 2$. The proof is complete. \hfill \Box

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