C₀ STABILITY OF BOUNDARY ACTIONS AND INEQUIVALENT ANOSOV FLOWS

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ABSTRACT. We give a C₀-structural stability result for the action of the fundamental group of a compact manifold of negative curvature on its boundary at infinity: any nearby action of this group by homeomorphisms of the sphere is semi-conjugate to the standard boundary action. Using similar techniques we prove a global rigidity result for the “slithering actions” of 3-manifold groups that come from skew-Anosov flows. As applications, we construct hyperbolic 3-manifolds that admit arbitrarily many topologically inequivalent Anosov flows, answering a question from Kirby’s problem list, and also give a more conceptual proof of a theorem of the second author on global C₀-rigidity of geometric surface group actions on the circle.

1. Introduction

This paper proves two related rigidity results for group actions on manifolds, with applications to skew-Anosov flows. The first is a general local rigidity result for the boundary action of the fundamental group of a closed negatively curved manifold.

1. Local rigidity of boundary actions. A major historical motivation for the study of rigidity of group actions comes from the classical (Selberg–Calabi–Weil) rigidity of lattices in Lie groups. Perhaps the best known example is Calabi’s original theorem that the fundamental group of a compact, hyperbolic n-manifold M (for n ≥ 3) is locally rigid as a lattice in SO(n, 1), which Mostow extended to a global rigidity result. From a geometric-topological viewpoint, it is natural to consider the action of SO(n, 1) on the boundary sphere of the compactification of hyperbolic n-space (the universal cover of M) and several modern proofs of Mostow rigidity pass through the study of this boundary action. See [10] for a broad introduction to the subject.

In general, if M is a closed n-manifold of (variable) negative curvature, its universal cover \( \tilde{M} \) still admits a natural compactification by a visual boundary sphere, denoted \( \partial_\infty \tilde{M} \), to which the action of \( \pi_1 M \) on \( \tilde{M} \) by deck transformations extends naturally to an action by homeomorphisms. However, \( \partial_\infty \tilde{M} \) typically has no more than a C₀ structure. Few rigidity results are known for actions by homeomorphisms, and few techniques exist, except in the case of actions on 1-manifolds. (Beyond Lie groups, most tools for rigidity of group actions originate in hyperbolic dynamics.) However, motivated by the statement of Calabi–Weil and by the results of [7, 31] in the 1-dimensional setting, we suggest that boundary actions are good candidates for study.

As we will later show, in this C₀ context the best rigidity result one can hope for is structural stability. Recall that an action \( \rho' : \Gamma \to \text{Homeo}(X) \) of a group \( \Gamma \) on a space \( X \) is said to be a topological factor of an action \( \rho : \Gamma \to \text{Homeo}(Y) \) if there is a surjective, continuous map \( h : X \to Y \) (called a semiconjugacy) such that \( h \circ \rho' = \rho \circ h \), and a
group action $\rho : \Gamma \to \text{Homeo}(X)$ is \textit{structurally stable} if any action which is $C^0$-close to it is a factor of it. Our first major result is the following.

**Theorem 1.1** (Structural stability of boundary actions). Let $M$ be a compact, orientable $n$-manifold with negative curvature, and $\rho : \pi_1 M \to \text{Homeo}(S^{n-1})$ the natural boundary action on $\partial \tilde{M}$. There exists a neighborhood of $\rho$ in $\text{Hom}(\pi_1 M, \text{Homeo}(S^{n-1}))$ consisting of representations which are topological factors of $\rho$.

Here and in what follows, $\text{Hom}(\pi_1 M, \text{Homeo}(S^{n-1}))$ is equipped with the standard compact-open topology.

**Sharpness.** As hinted above, one cannot replace “factor of” with “conjugate to” in Theorem 1.1 in fact in Section 4 we show that nearby, non-conjugate topological factors do occur for boundary actions of closed negatively curved manifolds. We give two sample constructions. One comes from Cannon–Thurston maps, special to the case where $M$ is a hyperbolic 3-manifold, and the other is a general “blow-up” type construction, applicable to examples in all dimensions.

2. Global rigidity of slithering actions. In the case where $\dim(M) = 2$, and hence $\partial \infty(M) = S^1$, a stronger global rigidity result for boundary actions of surface groups was proved by the second author in [31] (see also [7], [34]). Using the techniques of Theorem 1.1 we can recover this, and in fact generalize it to the broader context of group actions on $S^1$ arising from slitherings, in the sense of Thurston [38], associated to skew-Anosov flows on 3-manifolds. As we discuss in the next paragraph, these flows are basic examples from hyperbolic dynamics. We show the following.

**Theorem 1.2** (Global rigidity of skew-Anosov slithering actions). Let $F^s$ be the weak stable foliation of a skew-Anosov flow on a closed 3-manifold $M$, and $\rho : \pi_1 M \to \text{Homeo}_+(S^1)$ the associated slithering action. Then the connected component of $\rho$ in $\text{Hom}(\pi_1 M, \text{Homeo}_+(S^1))$ consists of representations “semi-conjugate” to $\rho$ in the sense of [22].

Definitions and properties of skew-Anosov flows and slitherings are recalled in Section 5. We note that the notion of semi-conjugacy of circle maps in the statement above is \textit{not} the same as in the definition of topological factor; unfortunately the terminology “semi-conjugacy” in this sense has also become somewhat standard. We will use the term \textit{weak conjugacy} for this property of actions on the circle to avoid confusion. It has also been referred to as “monotone equivalence” by Calegari.

A consequence of the above theorem is a new, independent proof of the main result of [31] on global $C^0$ rigidity of geometric surface group actions on $S^1$. See Corollary 5.12 below.

3. Inequivalent flows on a common manifold. Anosov (or \textit{uniformly hyperbolic}) flows are important examples of dynamical systems, due to their stability: as originally shown by Anosov, $C^1$-small perturbations of these flows give topologically conjugate systems. Classical examples in dimension 3 include suspension flows of hyperbolic automorphisms of tori, and geodesic flows on the unit tangent bundles of hyperbolic surfaces. The general problem of which manifolds admit Anosov flows, and the classification of such flows, is a fundamental problem in both topology and dynamics.

The first exotic examples of Anosov flows were given by Franks and Williams [19]. They produced non-transitive examples of flows that have separating transverse tori.
Handel and Thurston [25] then gave new transitive examples, and their work furnished the seeds for the definition of a general procedure (namely, the Fried–Goodman Dehn surgery) to produce new flows from old ones, later used to give the first examples of Anosov flows on hyperbolic 3-manifolds.

After existence, the next natural question regarding Anosov flows on a given manifold is that of abundance: how many Anosov flows, up to topological equivalence, does a given manifold support? Results of Ghys [21] and Barbot [3] imply that principal Seifert fibered spaces have unique, up to equivalence, Anosov flows (these are those discussed in Remark 6.1 below). However, the case of graph manifolds, or more generally manifolds with non-trivial JSJ-decompositions, is less rigid and there are indeed examples that exhibit abundance. The first example of a closed 3-manifold admitting at least two distinct Anosov flows was given by Barbot [4], and Beguin-Bonnati-Yu [6] found examples of manifolds admitting \( N \) distinct Anosov flows for arbitrarily large \( N \). All these examples occur on manifolds with non-trivial JSJ-decompositions and have many (incompressible) transverse tori. This leaves open the question of the abundance for hyperbolic manifolds, which appears as Problem 3.53 (C), attributed to Christy, in Kirby’s problem list [30].

Using techniques developed for the proof of Theorem 1.2, we prove the following existence result for flows, thus resolving this problem.

**Theorem 1.3 (Christy’s problem).** For any \( N \in \mathbb{N} \), there exist closed, hyperbolic 3-manifolds that support \( N \) Anosov flows that are distinct up to topological equivalence.

The hyperbolic manifolds in Theorem 1.3 and the flows are described explicitly, using Dehn-filling constructions. In addition to residing on hyperbolic manifolds, our examples contrast with those of Beguin-Bonnati-Yu in that they are skew. They also have the further property of being contact Anosov, meaning that they are Reeb Flows for certain contact structures on these manifolds and are in particular volume preserving.

### 4. Topological stability of geodesic flows.

A major tool in the proof of Theorem 1.1 is a “straightening” result for quasi-geodesic flows. This technique can also be used to recover the topological stability result for Anosov flows of Kato–Morimoto [29, Theorem A] in the special case of the geodesic flow on a compact manifold of negative curvature.

**Theorem 1.4 (Alternative proof of [29], special case).** Let \( M \) be a manifold of negative curvature and \( \Phi_t \) the geodesic flow on \( UTM \). There exist \( \epsilon, R > 0 \) such that, if \( \Psi_t \) is a flow such that each flowline of \( \Psi_t(x) \) remains \( \epsilon \)-close to the flowline \( \Phi_t(x) \) for \( t \in [0, R] \), then there is a continuous function \( p(x, t) \) on \( UTM \times \mathbb{R} \) and surjective map \( h : UTM \to UTM \) such that \( h \circ \Psi_t(x) = \Phi_{p(x,t)} \circ h(x) \).

The idea to prove rigidity through “straightening” quasi-geodesics appears also in Ghys’ work [21]. There, he shows the related result that if \( M \) is a closed \( n \)-manifold admitting a hyperbolic metric, and \( \Phi_t \) an Anosov flow on \( UTM \) that happens to be the geodesic flow associated to some other metric, then \( \Phi_t \) is topologically equivalent to the hyperbolic geodesic flow. However there is an essential difference between our work and his, since Ghys starts with the assumption that flows are Anosov. This simplifies the situation a great deal, allowing one to define a genuine conjugacy (rather than a semi-conjugacy) and also to work within the manifold \( M \). By contrast, without the Anosov
property to start with, we are forced to use more large-scale geometric techniques and define our semi-conjugacy on the level of endpoint maps. Note also that there are many examples of closed manifolds that admit negatively curved metrics, but no hyperbolic metric, in every dimension $> 3$ (see eg. Gromov–Thurston \[24\]), so our setting is somewhat broader in this sense as well.

**Outline.**

- Section 2 covers general background on foliations, suspensions, and large-scale geometry in negative curvature.
- Section 3 is devoted to the proof of Theorem 1.1 followed by Theorem 1.4.
- In Section 4 we construct examples of non-conjugate actions that are $C^0$-close to the boundary actions, illustrating some of the pathological behaviour that can occur despite structural stability.
- In Section 5 we recall the necessary background on skew-Anosov flows and slitherings, prove Theorem 1.2 and derive global rigidity for lifts under finite covers of the boundary action of a surface group.
- Section 6 constructs 3-manifolds with inequivalent skew-Anosov flows, proving Theorem 1.3.

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## 2. Preliminaries

This section contains some general background material on the setting of our work, the results and framework summarized here will be used throughout.

### 2.1. Suspension foliations, flat bundles and holonomy.

For a group $\Gamma$ and manifold $N$, the *representation space* $\text{Hom}(\Gamma, \text{Homeo}_+(N))$ is the space of homomorphisms $\Gamma \to \text{Homeo}_+(N)$ equipped with the compact-open topology. The case of particular interest to us, from a foliations perspective, is when $\Gamma = \pi_1(B)$ is the fundamental group of a closed manifold $B$. In this case one may form the *suspension* of a representation $\rho \in \text{Hom}(\Gamma, \text{Homeo}_+(N))$. The suspension is a foliated $N$-bundle over $B$ with total space given by the quotient

$$E_\rho := (\tilde{B} \times N)/\pi_1(B),$$

where $\pi_1(B)$ acts diagonally by $\rho$ on $N$ and by deck transformations on the universal cover $\tilde{B}$ of $B$. *Horizontal leaves* are subsets of the form $\tilde{B} \times \{p\} \subset N \times \tilde{B}$. The diagonal $\pi_1(B)$-action maps horizontal leaves to horizontal leaves, so the foliation of $\tilde{B} \times N$ by horizontals descends to a foliation on $E_\rho$ transverse to the fibers of the bundle $E_\rho \to B$. In our intended applications, foliations are always co-oriented and representations have image in the group of orientation-preserving homeomorphisms of $N$. Though this is not required for much of the background discussed here, we will take it as a standing assumption from here on.
We will typically use the notation $E_p$ to denote this foliated suspension space, and use other notation (e.g. simply $M$) when we wish to forget the transverse foliation on it.

2.2. Manifolds of negative curvature and boundaries at infinity. We briefly summarize standard results on manifolds of negative curvature that will be used in the sequel. Further background can be found in standard references such as [1, 9].

Let $M$ be a closed Riemannian manifold of negative curvature. Then its universal cover $\tilde{M}$ is a Hadamard manifold of pinched negative curvature. In particular, it is uniquely geodesic and is a $\delta$-hyperbolic space for some $\delta$. Many of the results we use rely primarily on the coarse geometric structure given by $\delta$-hyperbolicity, although we will occasionally make use of the smooth structure.

Any $\delta$-hyperbolic space has a compactification by a “boundary at infinity”. In the case of interest to us, this boundary is topologically a sphere of dimension $\dim(M) - 1$. Points on the boundary correspond to equivalence classes of geodesic rays, where two unit speed geodesics $c_1$ and $c_2 : [0, \infty) \to \tilde{M}$ are equivalent if the distance $d(c_1(t), c_2(t))$ is uniformly bounded. See [9 III.H.3] for a general introduction in the $\delta$-hyperbolic setting, or [1] for the Hadamard manifold setting. One way to specify the topology on $\partial_\infty \tilde{M}$ is as follows: fixing $x \in \tilde{M}$ and a geodesic ray $\alpha$ from $x$ to a point $\xi \in \partial_\infty \tilde{M}$, define neighborhoods $U_r,d(\alpha)$ of $\alpha$ to be the set of (the equivalence classes of) geodesic rays based at $x$ that stay distance at most $d$ from $\alpha$ on a ball of radius $r$ about $x$. Such sets form a basis for the topology. Equivalently, one may take an exhaustion of $\tilde{M}$ by compact sets $K_i$, fix any $d > 0$, and define neighborhoods $U_i(\alpha)$ of a geodesic $\alpha$ to be the sets of geodesic rays that stay within distance $d$ of $\alpha$ on $K_i$. These also form a basis for the topology. Deck transformations of $\tilde{M}$ act by isometries, sending geodesics to geodesics, and this extends to an action by homeomorphisms on the boundary.

Given a unit-speed geodesic ray $\alpha : [0, \infty) \to \tilde{M}$, the Busemann function $B_\alpha : \tilde{M} \to \mathbb{R}$ is defined by

$$B_\alpha(x) = \lim_{t \to \infty} (d(\alpha(t), x) - t).$$

Level sets of $B_\alpha$ are called horospheres. Busemann functions on smooth Hadamard manifolds are always $C^2$ (as was proved in [27]) and the horospheres $B_\alpha$ are perpendicular to geodesics. In our setting of pinched negative curvature and bounded geometry – for us, this comes from the fact that the metric on $\tilde{M}$ is lifted from the compact manifold $M$ – one can show $B_\alpha$ is in fact smooth, although we will not need this higher regularity.

In the case where $M$ is a surface, the boundary at infinity can be used to give a convenient description of the unit tangent bundle of $\tilde{M}$. To any distinct triple of points $(\xi, \eta, \nu)$ in $(\partial_\infty \tilde{M})^3$, one can associate the tangent vector to the (directed) geodesic from $\xi$ to $\eta$ at the unique point $p$ such that the geodesic from $p$ to $\nu$ is orthogonal to the geodesic with endpoints $\xi$ and $\eta$. This assignment defines a homeomorphism between the space of distinct triples in $\partial_\infty \tilde{M}$ and $UT\tilde{M}$.

In general, even for higher dimensional compact manifolds, the action of $\pi_1 M$ on the space of distinct triples of $\partial_\infty \tilde{M}$ is properly discontinuous and cocompact. See [8 Prop 1.13] for a proof (phrased there in terms of the action on the Gromov boundary of a hyperbolic group) and for further discussion. More generally, a group acting on a space such that the induced action on the space of distinct triples is properly discontinuous
and cocompact is said to be a uniform convergence group. The following well-known property applies to any uniform convergence group action, but we state it in the form which will be useful to us in the sequel.

**Proposition 2.1** ([S Prop 3.3]). For each \( x \in \partial_\infty \tilde{\mathcal{M}} \), there exists distinct \( p, q \in \partial_\infty \tilde{\mathcal{M}} \) and a sequence of elements \( \gamma_n \in \pi_1 \mathcal{M} \) such that \( \gamma_n(x) \to p \) and \( \gamma_n(y) \to q \) for all \( y \neq x \).

Points \( x \in \partial_\infty \tilde{\mathcal{M}} \) with the property above are called conical limit points of the action. A proof and further discussion can be found in [S §3].

### 2.3. Geodesic flow.

Associated to the geodesic flow on the unit tangent bundle \( \mathcal{U} \tilde{\mathcal{M}} \) on the universal cover of a manifold of negative curvature are two transverse foliations, each of codimension \( \dim(M) - 1 \). The leaf space of each can be identified with \( \partial_\infty \tilde{\mathcal{M}} \).

The weak stable foliation, denoted \( \mathcal{F}^s \), has leaves \( L^s(\eta) \), for \( \eta \in \partial_\infty \tilde{\mathcal{M}} \), consisting of the union of all geodesics with common forward endpoint \( \eta \). The weak unstable foliation \( \mathcal{F}^u \) consists of leaves \( L^u(\xi) \) formed by geodesics with common negative endpoint \( \xi \).

Both descend to foliations on \( \mathcal{U} \mathcal{M} \).

“Stable” and “unstable” here have a precise dynamical meaning – the geodesic flow in negative curvature is Anosov, and the weak stable (resp. weak unstable) leaves consist of geodesics that converge (resp. diverge) exponentially (see the proof of Lemma 2.5 below), but we do not need any further dynamical framework at the moment, and defer a more detailed discussion to Section [S]. What we will use is that \( \mathcal{F}^s \) and \( \mathcal{F}^u \) are transverse, and also that these foliations can be described naturally in terms of the suspension of the boundary action of \( \pi_1 \mathcal{M} \), as we explain now.

The tangent bundle \( \mathcal{U} \tilde{\mathcal{M}} \) may also be canonically identified with \( \tilde{\mathcal{M}} \times \partial_\infty \tilde{\mathcal{M}} = \tilde{\mathcal{M}} \times S^{n-1} \) via the positive endpoint map which assigns to each unit tangent vector \( v \) the forward endpoint of the oriented geodesic tangent to \( v \). Here the horizontal sets \( \tilde{\mathcal{M}} \times \{p\} \) are the leaves of \( \mathcal{F}^s \). In these coordinates, the natural projection \( \tilde{\mathcal{M}} \times \partial_\infty \tilde{\mathcal{M}} \to \mathcal{U} \tilde{\mathcal{M}} \) is via the diagonal action of \( \pi_1 \mathcal{M} \) by deck transformations on \( \tilde{\mathcal{M}} \) and the boundary action on \( S^{n-1} \); in other words, the suspension foliation of the boundary action gives the weak unstable foliation of geodesic flow. If instead one uses the negative endpoint of oriented geodesics to identify \( \mathcal{U} \tilde{\mathcal{M}} \) with \( \tilde{\mathcal{M}} \times \partial_\infty \tilde{\mathcal{M}} \), the suspension of the boundary action is the weak unstable foliation.

### 2.4. Quasi-geodesics.

Let \( c \geq 0, k \geq 1 \). A curve \( \alpha \) in a metric space \( X \) is a \((c,k)\) quasi-geodesic if

\[
\frac{1}{k} d(\alpha(x), \alpha(y)) - c \leq |x - y| \leq k \, d(\alpha(x), \alpha(y)) + c
\]

holds for all \( x, y \) in the domain of \( \alpha \). Often we will work with unparametrized rectifiable curves in \( X \). Such a curve is quasi-geodesic if its arc length parametrization is. We recall two well-known and useful properties of quasi-geodesics.

**Lemma 2.2** (Local-to-global principle, see [14] Theorem 1.4). Let \( X \) be a \( \delta \)-hyperbolic metric space. For any \( c \geq 0, k \geq 1 \), there exists \( L > 0 \) and \( c', k' \) such that every curve which is a \((c,k)\) quasi-geodesic on each subsegment of length \( L \) is globally a \((c',k')\) quasi-geodesic.

**Lemma 2.3** (Quasi-geodesics are close to geodesics, see [9] III.H.1.7). Let \( X \) be a \( \delta \)-hyperbolic space. There exists a constant \( R = R(\delta, c, k) \) such that if \( \alpha \) is a \((c,k)\)
quasi-geodesic segment in $X$, then the image of $\alpha$ lies in the $R$-neighborhood of the geodesic segment joining its endpoints.

It follows from this latter point that, provided a metric space $X$ is $\delta$-hyperbolic, each (oriented) bi-infinite quasi-geodesic $c$ in $X$ has a unique bi-infinite geodesic at bounded distance. The positive and negative endpoints of $\alpha$ are defined to be the positive and negative endpoints of this geodesic, denoted $e^+(\alpha)$ and $e^-(\alpha)$, respectively. Since quasi-isometries send quasi-geodesics to quasi-geodesics, this means that continuous quasi-isometries of $X$ extend to continuous maps on $\partial_\infty X$. Though this is not essential to what follows, we note that, in particular, when $X = \tilde{M}$, not only deck transformations, but all lifts of homeomorphisms of $M$ to $\tilde{M}$ induce homeomorphisms of $\partial_\infty \tilde{M}$.

A (unparametrized) quasi-geodesic flow of a metric space $X$ is a 1-dimensional foliation whose leaves are quasi-geodesics. The flow is uniform if there exist $k \geq 1, c \geq 0$ such that each leaf is a $(c, k)$ quasi-geodesic. If $\Gamma$ is a group that acts properly discontinuously and cocompactly on a space $X$ and $\mathcal{F}^{QG}$ is a quasi-geodesic foliation such that the action of $\Gamma$ sends leaves to leaves, then local-to-global principle implies that $\mathcal{F}^{QG}$ is automatically uniform.

Using Lemma 2.3 and the definition of the topology on $\partial_\infty X$ described above, one easily attains the following.

**Lemma 2.4.** Let $X$ be $\delta$-hyperbolic and let $\alpha$ be a $(k, c)$ quasi-geodesic ray based at $x_0$. Given a neighborhood $U$ of $e^+(\alpha) \in \partial_\infty X$, and constant $d > 0$, there exists a compact set $K$ such that, if $\beta$ is any $(k, c)$ quasi-geodesic ray that is distance at most $d$ from $\alpha$ on $K$, then $e^+(\beta) \in U$.

The same evidently holds for $e^-$. From this, one may derive the fact that endpoint maps are continuous on the space of $(k, c)$ quasi-geodesics equipped with the compact-open topology, and hence descend to continuous maps on the leaf space of a uniform quasi-geodesic foliation. The following alternative proof of this fact appears essentially in [11]: we include it as it gives another helpful illustration of the behavior of uniform quasi-geodesics in negative curvature.

**Lemma 2.5.** Let $\mathcal{F}$ be an oriented, uniform quasi-geodesic foliation of the universal cover of a compact manifold of negative curvature. Then the endpoint maps, considered as functions on the leaf space of $\mathcal{F}$, are continuous.

**Proof.** Suppose that $\ell_n$ is a sequence of leaves of $\mathcal{F}$ that converge uniformly on compact sets to a leaf $\ell_\infty$. Following the discussion after Lemma 2.3, there exists $D > 0$ (depending on the curvature of $M$ and the quasi-geodesic constants of leaves) such that each $\ell_n$ lies in the $D$-neighborhood of a unique geodesic $\gamma_n$. It follows that the $\gamma_n$ coarsely converge on compact sets: after passing to a subsequence, we may assume that there is a length $n$ segment of $\gamma_n$ which lies in the $3D$-neighborhood of $\gamma_\infty$. Since geodesics in negative curvature have exponential divergence,[1] this implies that $\gamma_n$ lies in a $3De^{-\lambda n}$ neighborhood of $\gamma_\infty$ on a segment of length $n/2$, for some $\lambda > 0$. Thus, the $\gamma_n$ converge and so $e_+(\lambda_n) = e_+(\gamma_n)$ converges to $e_+(\gamma_\infty)$. \hfill $\Box$

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[1] Recall that a divergence function for a metric space $X$ is a function $\Delta : \mathbb{N} \to \mathbb{R}$ such that, for any geodesics $c_1, c_2 : [0, t] \to X$ with $c_1(0) = c_2(0)$, and any $r, R \in \mathbb{N}$; if $R + r \leq t$ and $d_X(c_1(R), c_2(R)) > e(0)$ then any path from $c_1(R + r)$ to $c_2(R + r)$ outside the ball $B(c_1(0), R + r)$ must have length at least $\Delta(r)$. Any $\delta$-hyperbolic space has an exponential divergence function. (See [11] III.H.1.25 for a proof).
This concludes the preliminary material required for the proof of Theorem 1.1. Further material on Anosov flows and an introduction to “slitherings” will be given in Section 5 where it is needed.

3. Proof of Theorem 1.1

3.1. Construction of a well-behaved leafwise immersion. We begin with a general lemma that allows one to construct a well-behaved map between the suspension foliations of nearby representations. Our intended application is where $M$ is negatively curved, $N = S^{n-1}$ and $\rho_0$ is the boundary action.

Lemma 3.1. Let $M$ and $N$ be compact, connected Riemannian manifolds, and $\rho_0 : \pi_1 M \to \text{Homeo}_+(N)$. There exists a neighborhood $U$ of $\rho_0$ in $\text{Hom}(\pi_1 M, \text{Homeo}_+(N))$ and a continuous assignment $\rho \mapsto f_\rho$ from $U$ to the space of $C^0$ maps $\tilde{M} \times N \to \tilde{M} \times N$ with the following properties:

1. $f_{\rho_0}$ is the identity map, and for all $\rho$ the map $f_\rho$ covers the identity $\tilde{M} \to \tilde{M}$.
2. The restriction of $f_\rho$ to each leaf $M \times \{p\}$ is a smooth diffeomorphism onto its image.
3. For each horizontal leaf $L = \tilde{M} \times \{p\}$, the tangent plane to $f_\rho(L)$ at any point is close to the horizontal, with angle to the horizontal at any point varying continuously with $\rho$.
4. The map $f_\rho$ is equivariant with respect to the diagonal actions of $(\pi_1, \rho)$ and $(\pi_1, \rho_0)$ on $\tilde{M} \times N$.

Moreover, if $N$ is 1-dimensional, we may take the maps $f_\rho$ to be homeomorphisms.

Point (1) above is equivalent to the fact that $f_\rho$ is a map of the form $f_\rho(x, y) = (x, h_\rho(x, y))$. In this case, the $\pi_1 M$–equivariance claimed in point (4) is the statement that for all $\gamma \in \pi_1 M$, we have

$$f_\rho(\gamma \cdot x, \rho(\gamma)(y)) := (\gamma \cdot x, h_\rho(\gamma \cdot x, \rho(\gamma)(y))) = (\gamma \cdot x, \rho_0(\gamma) \circ h_\rho(x, y)).$$

Here $\gamma \cdot$ denotes the action of $\pi_1$ by deck transformations. This equivariance means that $f_\rho$ descends to a map $\bar{f}_\rho : E_\rho \to E_{\rho_0}$ between the foliated $N$-bundles over $M$ associated to $\rho$ and to $\rho_0$. The second statement of (1) says that $\bar{f}_\rho$ is a leafwise smooth immersion, and in the case where $\dim(N) = 1$, the construction in the proof will make $\bar{f}_\rho$ a leafwise embedding and a global homeomorphism.

Proof. Let $T$ be a smooth triangulation of $M$. Fix also an enumeration of the vertices of $T$. Let $\tilde{T}$ be the lift of $T$ to $\tilde{M}$. We define $f_\rho$ (or, equivalently, $h_\rho$) iteratively on fibers over the $k$-skeleta of $\tilde{T}$.

Fix a basepoint $x \in \tilde{M}$, we may take this to be a vertex of $\tilde{T}$, and let $D$ be a connected fundamental domain for the deck group action on $\tilde{M}$. Let $N_x$ denote the fiber over $x$, and define $h_\rho(x, \cdot)$ to be the identity map $N_x \to N_x$. Then there is a unique $\pi_1 M$-equivariant extension of $h_\rho$ to a map defined on the fibers over the orbit of $x$. Specifically, for $\gamma \in \pi_1 M$ and $z \in N$, define $h_\rho(\gamma \cdot x, z) := \rho(\gamma) \rho(\gamma)^{-1}(z)$. Repeat this process for each vertex of $\tilde{T}$ lying in the fundamental domain $D$. Note that, if $\rho_0$ is close to $\rho$, then $\rho_0(\gamma) \rho(\gamma)^{-1}$ is close to the identity, for all $\gamma$ in a fixed finite generating set for $\pi_1 M$. 


For the remainder of the construction, fix a smooth, increasing function \( \sigma : [0,1] \to [0,1] \) that is constant 0 in a neighborhood of 0, and 1 in a neighborhood of 1. For each \( k \)-simplex \( c \) of \( T \), fix an identification of \( c \) with the unit ball \( B^k \) in \( \mathbb{R}^k \) and let \( \sigma_c \) be the smooth bump function on \( c \) that (under the identification with \( B^k \)) agrees with \( \sigma \) on each ray from the origin of \( B^k \), taking the value 0 in a neighborhood of \( \partial B^k \) and 1 in a neighborhood of 0.

Inductively, suppose that \( f_\rho \) has been defined on the fibers over the \((k-1)\)-skeleton and suppose further that, for each \( y \in N \) and each \((k-1)\)-simplex \( c \) forming a \((k-1)\)-face in the triangulation, the projection of \( f_\rho(c \times \{y\}) \) to \( N \) is contained in a convex subset of \( N \) small enough so that geodesics between points are unique. (This will hold provided that \( \rho \) is sufficiently close to \( \rho_0 \).) Let \( \pi_N \) denote the natural projection \( \tilde{M} \times N \to N \). We extend \( f_\rho \) to the fibers over a \( k \)-cell \( c \) as follows. Let \( v(c) \) be the minimum vertex of \( c \) (with respect to the enumeration of vertices of \( T \) chosen earlier) and let \( n(c) \) be the image of the projection \( \pi_N(f_\rho(v(c))) \) of \( f_\rho(v(c)) \) to \( N \).

Fix \( x \in \partial c \), let \( \alpha_{x,y} : [0,1] \to N \) denote the geodesic arc with \( \alpha_{x,y}(0) = \pi_N \circ f_\rho(x,y) \) and \( \alpha_{x,y}(1) = n(c) \). Note that we have not parametrized this geodesic by arc-length, but rather we make a continuous choice of parametrizations using the unit interval, such as by taking an arc length parametrization rescaled by the total length of the segment. Now on each radius of \( c \times \{x\} \) (under our chosen identification with \( B^k \), where radii are parametrized by \( t \in [0,1] \)), define \( h_\rho \) to take the value \( \alpha_{x,y}(\sigma(t)) \) at point \( t \).

The point of the preceding paragraph is to define \( h_\rho \) over the \( k \)-skeleton in a way so as to smoothly interpolate between the values it takes over the \((k-1)\)-skeleton. The heavy-handed approach to this taken here is simply for the purpose of giving a construction that is \( \pi_1 M \)-equivariant and depends continuously on \( \rho \). Using \( n(c) \) was one means of accomplishing this, in the case \( N = S^n \) one could just as well take the barycenter of the image of \( \pi_N(f_\rho(\partial c)) \), or any other natural choice of point, instead.

Using this definition, property (1) is clearly satisfied, and equivariance of this map with respect to the diagonal actions of \( \pi_1 M \) is built into the construction since \( \tilde{T} \) and the associated bump functions on cells are lifted from \( M \), and equivariance was built into the map on the 0-skeleton. Continuity in \( \rho \) also follows from the definition, in fact, this map is continuous with respect to leafwise uniform convergence in the \( C^1 \) topology on compact sets.

Since \( f_{\rho_0} \) is the identity map, when \( \rho \) is close to \( \rho_0 \), continuity of the construction means that the leafwise maps will be uniformly close (as \( C^1 \) embeddings) to the identity, and will have tangent distributions close to horizontal, varying continuously with \( \rho \). Since \( M \) is compact and \( f_\rho \) is \( \pi_1 M \)-equivariant, these leafwise maps are \( \text{uniformly} C^1 \) close. Finally, in the case \( \dim(N) = 1 \), the fact that points on \( N \) are locally totally ordered, and that orientation-preserving homeomorphisms of \( N \) are order-preserving, combined with our construction above, ensures that \( f_\rho \) will be bijective. It is then easy to verify that its inverse is also continuous, hence \( f_\rho \) is a homeomorphism. \( \square \)

### 3.2. Quasi-geodesics and endpoint maps. Returning to the situation of interest, we set some notation to be used for remainder of this section. Let \( M \) denote a compact, negatively curved Riemannian \( n \)-manifold, \( \tilde{M} \) its universal cover, and \( \text{UT} \tilde{M} \) the unit tangent bundle of \( \tilde{M} \). As in the statement of Theorem 1.1, \( \rho_0 \) will denote the
standard boundary action of $\pi_1 M$ on $S^{n-1}$. Recall from Section 2.3 that UT$\widetilde{M}$ may be canonically identified with $\widetilde{M} \times S^{n-1}$ where horizontal sets $\widetilde{M} \times \{p\}$ are leaves of the weak stable foliation $F^s$ of the geodesic flow, so projection to UT$\widetilde{M}$ is via the diagonal action of $\pi_1 M$. We will use the notation UT$\widetilde{M}$ rather than $\widetilde{M} \times S^{n-1}$ when we wish to emphasize that UT$\widetilde{M}$ is the suspension foliation of this standard boundary action $\rho_0$.

Let $f_\rho : \widetilde{M} \times S^{n-1} \to$ UT$\widetilde{M} \cong \widetilde{M} \times S^{n-1}$ denote the $\pi_1 M$-equivariant leafwise embedding obtained by applying Lemma 3.1 to a representation $\rho$ close to $\rho_0$ in Hom($\pi_1 M, \text{Homeo}_+(S^{n-1})$). Our next goal is to use this data to produce a quasi-geodesic foliation on $\widetilde{M} \times S^{n-1}$ that is $\rho$-equivariant, so descends to a foliation on the bundle associated to the suspension action $E_\rho$.

**Lemma 3.2.** Provided that $\rho$ is sufficiently close to $\rho_0$ the intersection $f_\rho(L) \cap L_u(\xi)$ of any unstable leaf $L_u(\xi)$ in UT$\widetilde{M}$, and any horizontal leaf $L$ of $\widetilde{M} \times S^{n-1}$ is either empty or a quasi-geodesically embedded bi-infinite line in $\widetilde{M}$ with one endpoint equal to $\xi$.

Note that the intersection may indeed be empty, for instance, when $\rho = \rho_0$, the leaf $L_u(\xi)$ has empty intersection with the stable leaf comprised of geodesics with forward endpoint $\xi$.

**Proof of Lemma 3.2.** Since the tangent space to $f_\rho(L)$ is close to the horizontal in $\widetilde{M} \times S^{n-1}$, if $f_\rho(L)$ and $L_u(\xi)$ intersect, then they intersect transversely, with tangent vector field close to a geodesic path in UT$\widetilde{M}$.

Consider the horosphere foliation $B$ of UT$\widetilde{M}$ whose leaves are level sets of the Busemann function $b$ of a geodesic with forward endpoint $\xi$ (see Section 2.2). For any leaf $L^s$ of the stable foliation of geodesic flow on UT$\widetilde{M}$ (horizontals under our identification UT$\widetilde{M} = \widetilde{M} \times S^{n-1}$) intersecting $L_u(\xi)$, the line $L_u(\xi) \cap L^s$ is perpendicular to leaves of $B$. Since the image of $f_\rho(L)$ is close to horizontal, $L_u(\xi) \cap f_\rho(L)$ meets leaves of $B$ at angle uniformly close to $\pi/2$. It follows that the length of the segment of a leaf of $L_\xi \cap \tilde{F}$ between $b^{-1}(t)$ and $b^{-1}(s)$ is at most $C|t-s|$ for some (uniform) constant $C > 1$; and this still holds after projecting to $\widetilde{M}$. Thus, the image is a quasi-geodesic
with quasi-geodesic constants independent of the choice of $L$. Since the quasi-geodesic enters every horosphere based at $\xi$, it has one endpoint equal to $\xi$. \qed

Thus, after endowing $f_\rho(L)$ with either its induced metric or that pulled back from the canonical projection $UT\tilde{M} \to \tilde{M}$, these intersections of leaves give a quasi-geodesic foliation of $f_\rho(L)$. We make the following orientation convention; when $\rho = \rho_0$, this exactly recovers the oriented geodesics from $UT\tilde{M}$.

**Convention 3.3 (Orientation on leaves).** We orient the lines of the form $f_\rho(L) \cap L^u(\xi)$ so that their negative endpoint is $\xi$.

Since the construction of $f_\rho$ ensures that its restriction to each leaf $L$ is a quasi-isometry, we may pull back the oriented quasi-geodesic foliation on each leaf $f_\rho(L)$ via the restriction of $f_\rho$ to $L$, to get an oriented quasi-geodesic foliation on $L$. Doing this on all leaves gives an oriented, quasi-geodesic foliation on $\tilde{M} \times S^{n-1}$, which we denote by $F^QG_\rho$. Again, $\pi_1M$-equivariance means that the quasi-geodesic constants may be taken to be uniform. See Figure 1 for an illustration.

**Properties of $F^QG_\rho$**. The fact that $f_\rho$ is $\pi_1M$-equivariant and that $F^u$ is a $\pi_1M$-invariant foliation on $UT\tilde{M}$ means that the diagonal action of $\pi_1M$ on $\tilde{M} \times S^{n-1}$ via deck transformations on the first factor and $\rho$ on the second preserves leaves of $F^QG_\rho$. (This can easily be checked directly from the definition.) Furthermore, $F^QG_\rho$ has the property that each quasi-geodesic line $\ell$ in the foliation is contained in a horizontal leaf of $\tilde{M} \times \{p\}$ of $\tilde{M} \times S^{n-1}$. Thus, such a line $\ell$ has a positive and negative endpoint on the boundary sphere $\partial_{\infty}\tilde{M} = S^{n-1}$, giving positive and negative endpoint maps

$$e_\rho^+, e_\rho^- : \tilde{M} \times S^{n-1} \to \partial_{\infty}\tilde{M} = S^{n-1}$$

which assign to a point $x$ in a leaf $\ell$ the positive and negative endpoints $e_\rho^+(x)$ and $e_\rho^-(x)$ of $\ell$, respectively. Note that, since $f_\rho$ covers the identity map on $\tilde{M}$, one may equally well look at the curves $f_\rho(L) \cap L^u(\xi)$ or their pullbacks under $f_\rho$ to determine their endpoints. When $\rho = \rho_0$, the foliation $F^QG_\rho$ is the geodesic foliation of $UT\tilde{M}$, and $e_{\rho_0}$ the usual positive and negative endpoint maps.

Let $e_\rho^\pm$ denote the product map $(e^+, e^-)$ to $S^{n-1} \times S^{n-1}$. The image of this map avoids the diagonal $\Delta$. By definition, this map factors through the projection to the leaf space $\mathcal{L}(F^QG_\rho)$ of $F^QG_\rho$ as summarized in the diagram below.

$$\tilde{M} \times S^{n-1} \xrightarrow{e_\rho^\pm} (S^{n-1} \times S^{n-1}) - \Delta$$

$$\mathcal{L}(F^QG_\rho) \xrightarrow{\tau_\rho}$$

Additionally, since $f_\rho$ is $\pi_1M$-equivariant, a straightforward verification from the definition shows that the same is true of $e_\rho^\pm$, namely

$$e_\rho^\pm(\gamma \cdot x, \rho(\gamma)(y)) = \gamma \cdot e_\rho^\pm(x, y)$$
holds for all \((x,y) \in \tilde{M} \times S^{n-1}\) and \(\gamma \in \pi_1 M\), where the action on the right hand side of the equation is by the standard action of \(\pi_1 M\) on unparametrized geodesics in \(UT\tilde{M}\) – i.e. the diagonal action of the standard boundary action \(\rho_0\).

We now prove various continuity properties.

**Lemma 3.4.** The endpoint maps \(e^\pm_\rho\) and \(\tau_\rho\) are continuous.

**Proof.** Lemma 2.5 implies that the restriction of \(\tau_\rho\) to each leaf \(L\) is continuous. We will use a similar argument to show global continuity. It suffices to show continuity of \(e^\pm_\rho\), since \(\tau_\rho\) is induced map on a quotient space.

Suppose that \(x_n \to x\) is a convergent sequence in \(\tilde{M} \times S^{n-1}\). Let \(L_n\) be the horizontal leaf containing \(x_n\), and \(L\) the leaf containing \(x\). Let \(g_n\) denote the restriction of \(f_\rho\) to \(L_n\), considered as a (based) topological embedding \((\tilde{M}, x_n) \to (UT\tilde{M}, f_\rho(x_n))\). The definition of \(f_\rho\) implies that the maps \(g_n\) converge uniformly on compact sets, i.e. for any \(r > 0, \epsilon > 0\), there exists \(N\) such that for all \(n > N\), the restriction of \(g_n\) to the \(r\)-ball \(B_r(x_n)\) in \(L\) is \(\epsilon\)-close in the \(C^1\)-topology to the restriction of \(g\) to \(B_r(x)\). It follows that, for any fixed leaf \(L^u\) of \(\mathcal{F}^u\), the quasi-geodesic segment \(L^u \cap f_\rho(B_r(x_n))\) lies in some \(C(r)\epsilon\)-neighborhood of \(L^u \cap f_\rho(B_r(x))\), where \(C : (0, \infty) \to (0, \infty)\) is a continuous, increasing function (depending only on the geometry of \(\mathcal{F}^u\)), with \(C(0) = 0\).

Let \(L^u_n\) be the leaf of \(\mathcal{F}^u\) through \(f_\rho(x_n)\). Since \(f_\rho(x_n) \to f_\rho(x)\) these leaves converge on compact sets to the leaf \(L^u_\infty\) through \(f_\rho(x)\). Combined with the above, that shows that, for \(n\) sufficiently large, \(L^u_n \cap f_\rho(B_r(x_n))\) lies in the \(2C(r)\epsilon\)-neighborhood of \(L^u \cap f_\rho(B_r(x))\). Since moreover these are nearly horizontal, there is some \(c' > 1\) (which can be taken as close to 1 as we like, by requiring \(\rho\) close to \(\rho'\), but we only need that this is some fixed constant) such that their projections to \(\tilde{M}\) are quasi-geodesics that \(2C(r)c'\)-fellow-travel each other along segments of length close to \(2r\). Lemma 2.4 now gives the desired continuity.

**Lemma 3.5.** The map \(\rho \mapsto e^\pm_\rho\), defined on a neighborhood of \(\rho_0\) in \(\text{Hom}(\pi_1 M, \text{Homeo}_+(S^{n-1}))\) and with image in the space of continuous maps \(\tilde{M} \times S^{n-1} \to (S^{n-1} \times S^{n-1}) - \Delta\), is continuous with respect to the compact-open topology.

**Proof.** This follows from continuity of \(\rho \mapsto f_\rho\) and the definition of the topology on the end space. Let \(\rho'\) be some fixed representation close to \(\rho_0\), close enough so that \(f_\rho\) and the endpoint maps are defined. The space of continuous maps \(\tilde{M} \times S^{n-1} \to (S^{n-1} \times S^{n-1}) - \Delta\) has the standard compact-open topology, so fix \(K\) compact in \(\tilde{M} \times S^{n-1}\) and an open set \(O\) in \((S^{n-1} \times S^{n-1}) - \Delta\) containing the image of \(K\). Continuity of \(\rho \mapsto f_\rho\) means that, for any \(\epsilon, R > 0\) if \(\rho\) is chosen close enough to \(\rho'\) then quasi-geodesics through points of \(K\) pulled back via \(f_\rho\) will remain \(\epsilon\)-close to quasigeodesics pulled back via \(f_{\rho'}\) on segments of length \(R\). Lemma 2.4 now guarantees that for \(R\) large enough, the endpoints of geodesics through points of \(K\) will remain in \(O\).

**Lemma 3.6.** Any local transversal for the geodesic flow \(\mathcal{F}^{QG}_{\rho_0}\) will be a local transversal for any sufficiently close representation \(\rho\), in particular, the leaf space \(\mathcal{L}(\mathcal{F}^{QG}_{\rho_0})\) is locally homeomorphic to \(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}\). Associating a leaf \(\ell\) to the pair \((e_\rho(\ell), p)\), where \(p \in S^{n-1}\) is the point such that \(\ell\) lies in the horizontal leaf \(\tilde{M} \times \{p\}\), gives a local chart for \(\mathcal{L}(\mathcal{F}^{QG}_{\rho_0})\).
Proof. Continuity of \( \rho \mapsto f_\rho \) and the fact that \( F^Q_{\rho} \) is the pullback of the intersection of (smooth) leaves \( f_\rho(L) \cap L^u \) implies that local transversals for \( F^Q_{\rho_0} \) remain transverse to \( F^Q_{\rho} \) when \( \rho \) is nearby to \( \rho_0 \).

By Lemma 3.2 each leaf \( L^u(\xi) \) of \( F^u \) intersects a leaf \( f_\rho(L) \) along a quasi-geodesic with negative endpoint \( \xi \). Thus, the negative endpoint map gives a parametrization of the leaves of \( F^Q_{\rho} \) which sit inside a fixed leaf \( L \). Continuity of \( f_\rho \) and the negative endpoint map means that these parametrizations vary continuously with the leaf \( L \), giving the desired local chart.

Lemma 3.7. If \( \rho \) is sufficiently close to \( \rho_0 \), then \( \bar{\tau}_\rho \) is surjective.

Proof. Take a \((2n-2)\)-dimensional disc \( D \) in \( \tilde{M} \times S^{n-1} \) which is a local transversal for the geodesic foliation \( F^Q_{\rho} \), and chosen large enough so that the image \( \bar{e}_{\rho_0}(D) \subset S^{n-1} \times S^{n-1} - \Delta \) contains a compact fundamental domain \( K \) for the action of \( \pi_1 M \) on the space of geodesics \( S^{n-1} \times S^{n-1} - \Delta \) in \( \tilde{M} \).

By Lemma 3.6 if \( \rho \) is sufficiently close to \( \rho_0 \), then \( D \) will also be a local transversal for \( F^Q_{\rho} \), and, by continuity of the endpoint map, \( \bar{e}_\rho(D) \) will be \( C^0 \) close to \( \bar{e}_{\rho_0}(D) \) and hence also contain \( K \). Since \( e_\rho^+ \) is \( \pi_1 M \)-equivariant, it follows that the image of \( \bar{e}_\rho \) is a set that is invariant under the action of \( \pi_1 M \) on \( S^{n-1} \times S^{n-1} - \Delta \). We have just shown that it contains a fundamental domain, so the image must be everything. \( \square \)

The following observation will allow us to conclude the proof by arguing that \( e_\rho^+ \) defines a semi-conjugacy.

Proposition 3.8. Under the hypotheses of Lemma 3.7, the restriction of \( e_\rho^+ \) to any horizontal leaf \( L \) of \( \tilde{M} \times S^{n-1} \) is constant.

The broad idea of the proof is to use \( \pi_1 M \)-equivariance and the uniform convergence group property of the boundary action to promote a (hypothetical) leaf where \( e_\rho^+ \) is nonconstant to one where \( e_\rho^- \) is not locally injective, which would contradict Lemma 3.6.

Proof. Suppose for contradiction that \( e_\rho^+ \) is nonconstant on some leaf \( L \), and let \( I \subset L \) be a segment such that \( e_\rho^+(I) \) is a nonconstant path with distinct endpoints in \( S^{n-1} \). We may even take \( I \) to be transverse to \( F^Q_{\rho} \), if desired. Let \( x \) and \( y \) denote the endpoints of \( e_\rho^+(I) \). Since the image of \( (e_\rho^+, e_\rho^-) \) avoids the diagonal, by shrinking \( I \) if needed we may further assume that \( e_\rho^+(I) \) is disjoint from \( e_\rho^-(I) \), in particular \( x \notin e_\rho^-(I) \).

By the uniform convergence group property of the action of \( \pi_1 M \) on its boundary (Proposition 2.1), there exist distinct \( p, q \in S^{n-1} \) and a sequence \( \gamma_n \in \pi_1 M \) such that \( \gamma_n(x) \to p \) and \( \gamma_n(z) \to q \) for all \( z \neq x \). Thus, the image \( \gamma_n \circ e_\rho^+(I) = e_\rho^+ \circ \rho(\gamma_n)(I) \) will contain an arc between some points \( p_n \) and \( q_n \), with \( p_n \to p \) and \( q_n \to q \); while \( \gamma_n \circ e_\rho^-(I) = e_\rho^- \circ \rho(\gamma_n)(I) \) pointwise converges to \( \{q\} \). It will be convenient for us to remain in a compact set of \( S^{n-1} \times S^{n-1} - \Delta \), so fix a small open neighborhood \( N \) of \( \Delta \), and let \( I_n \) denote the connected component of \( \gamma_n \circ e_\rho^+(I) - N \) containing \( p_n \); this is some subinterval of \( \gamma_n \circ e_\rho^+(I) \). After passing to a subsequence, the intervals \( I_n \) converge, in the Hausdorff metric, to a nondegenerate set \( I_\infty \). While \( I_\infty \) may not be an interval, we know at least that its image under \( e_\rho^+ \) is nonconstant.
Consider the sequence of leaves $\rho(\gamma_n)(L)$. Since the leaf space of the horizontal foliation (on $\tilde{M} \times S^{n-1}$) is compact, after passing to a subsequence these converge to some leaf $L_\infty$. Let $D$ be a local transversal for $\mathcal{F}_{qG}^+$, defined in a neighborhood of some quasi-geodesic leaf lying in $L_\infty$ so that for all $n$ sufficiently large, the projection of the segments $I_n \subset \rho(\gamma_n)(L)$ to the leaf space are contained in $D$.

As in Lemma 3.6, we may choose the transversal $D$ so that the restriction of this transversal to each horizontal leaf $L'$ is the parametrization given by the negative endpoint map $e^-_{\rho}$. But then the restriction of $\pi_p$ to $D \cap L_\infty$ has $I_\infty$ in its image, a nontrivial, connected set of the form $\{q\} \times J$, contradicting the fact that the negative endpoint map gives a parametrization of the leaf space on $\mathcal{F}_{qG}^+ \cap L_\infty$.

**Conclusion of proof of Theorem 1.1.** We have just shown that, for representations $\rho$ in some neighborhood of $\rho_0$, the endpoint map $e^+_{\rho}$ is constant on each set $\tilde{M} \times \{p\} \subset \tilde{M} \times S^{n-1}$, so descends to a continuous map $S^{n-1} \rightarrow S^{n-1}$. Lemma 3.7 implies that this map is surjective, and by construction, we have

$$e^+_{\rho} \circ \rho(\gamma)(x) = \rho_0(\gamma) \circ e^+_{\rho}(x)$$

as in Equation (1), so $e^+_{\rho}$ is the desired semi-conjugacy between $\rho$ and the standard boundary action of $\pi_1 M$.

**3.3. Topological stability of geodesic flows.** We conclude this section with a short sketch of how our proof above gives a “soft” geometric proof of topological structural stability of the geodesic flow in negative curvature, as claimed in Theorem 1.4.

**Proof of Theorem 1.4.** Let $M$ be a closed manifold of negative curvature and $\Phi_t$ the geodesic flow on $UTM$. Suppose that $\Psi_t$ is a flow such that the flowlines of the lift $\tilde{\Psi}_t$ to $UT\tilde{M}$ each $\epsilon$ fellow-travel flowlines of $\tilde{\Phi}_t$ on segments of length $R$, as in the statement of Theorem 1.4. The local-to-global principle (Lemma 2.2) implies that there exists $N$ and $c$ such that, if $R \geq N$ and $\epsilon \leq c$, then flowlines of $\Psi_t$ project to quasigeodesics in $\tilde{M}$, and so each flowline of $\tilde{\Psi}_t$ stays within a bounded distance of a unique flowline of $\tilde{\Phi}_t$, and so has well defined endpoints. Lemma 2.4 implies that if $\epsilon$ is sufficiently small, as $R \rightarrow \infty$ these endpoint maps $e^\pm : \tilde{M} \rightarrow \partial_{\infty} \tilde{M} \times \partial_{\infty} \tilde{M}$, sending a point $x$ to the positive and negative endpoints of the flowline $\tilde{\Psi}_t(x)$ converge uniformly on compact sets to the endpoint map for the geodesic flow. By construction $e^\pm$ are $\pi_1(M)$-equivariant, so by the same argument as in Lemma 3.7 we may conclude that $e^\pm$ is surjective onto the complement of the diagonal in $\partial_{\infty} \tilde{M} \times \partial_{\infty} \tilde{M}$, which is of course naturally identified with the flow space of $\tilde{\Psi}_t$. This gives a $\pi_1 M$-equivariant, continuous, surjective map from $\tilde{M}$ to the flow space of $\tilde{\Psi}_t$, which descends to a map defined on the flowspace of $\tilde{\Phi}_t$.

To improve this map on the level of orbit spaces to a topological equivalence of the flows, one may now use the averaging trick in Barbot [2, Theorem 3.4] following Ghys [21, Lemmas 4.3, 4.4]. Specifically, define first a map $h_0$, associating to each point $x \in \tilde{M}$ the closest point to $x$ on the geodesic between $e^+(x)$ and $e^-(x)$. This maps flowlines to flowlines, but may not send a flowline injectively onto its image. Rather, there is simply a continuous function $a : \mathbb{R} \times \tilde{M}$ satisfying $h_0(\tilde{\Phi}_t(x)) = \tilde{\Psi}_{a(t,x)}(h_0(x))$. 


To remedy this, fix $T$ large, and define $A(t) = \frac{1}{T} \int_0^T a(s,x)ds$. One checks that, if $T$ was chosen sufficiently large, the map

$$h(x) := \tilde{\Phi}_{A(t)}(h_0(x))$$

sends each flowline of $\tilde{\Psi}_t$ continuously and injectively onto a flowline of $\tilde{\Phi}_t$, and descends to a continuous map $M \to \tilde{M}$ giving a topological equivalence of the flows. □

4. Examples

In this section we illustrate some of the phenomena that can appear in Theorem 1.1. We give two families of examples of actions that are semi-conjugate, but not conjugate, to the action of the fundamental group of a closed negatively curved manifold on its ideal boundary. The first uses the work of Cannon and Thurston, and is specific to Kleinian groups. The second extends the classical Denjoy Blow-up and applies to any group action of regularity $C^1$.

**Cannon-Thurston Maps.** We briefly summarize the construction of the Cannon–Thurston map (in a special case), following [13]. Let $S$ be a closed, hyperbolic surface, $\phi$ a pseudo-Anosov diffeomorphism, and $M$ a hyperbolic 3-manifold given by the suspension of $\phi$, equipped with the suspension flow $\phi_t$ of the pseudo-Anosov map $\phi$. Lifting flowlines to the universal cover $\tilde{M} = \mathbb{H}^3$ gives a flow $\tilde{\phi}_t$ whose flow space is a topological disk $D$, which may be identified with the universal cover $\tilde{S} \subset \tilde{M}$ of any fiber $S$ of $M$. It is easily verified that flow lines of $\tilde{\phi}_t$ are quasi-geodesics in $\mathbb{H}^3$, so we have continuous endpoint maps $e_{\pm} : D \to \partial_{\infty} \mathbb{H}^3$.

Identifying $D$ with $\tilde{S}$, we have the standard boundary compactification $\hat{D} = \tilde{S} \cup \partial_{\infty} \tilde{S}$. Cannon and Thurston [13] showed the action of $\pi_1 M$ extends to the closed disk $\hat{D}$ in a way that is compatible with the positive and negative endpoint maps. This gives maps

$$\hat{e}_{\pm} : \hat{D} \to \partial_{\infty} \mathbb{H}^3.$$ 

These extensions coincide on the boundary $\partial_{\infty} \tilde{S}$ and are $\pi_1 M$-equivariant. Gluing these together along the boundary, we obtain a $\pi_1 M$-equivariant map

$$h_{CT} = \hat{e}_- \cup \hat{e}_+ : S^2 = \hat{D}_- \cup_{\partial S^1} \hat{D}_+ \to \partial_{\infty} \mathbb{H}^3.$$ 

This gives an induced action $\rho_{CT}$ of $\pi_1 M$ on $S^2$. By equivariance of the construction and by minimality of the action of $\pi_1 M$ on $\partial_{\infty} \mathbb{H}^3$, we conclude that $h_{CT}$ is surjective. Additionally, it follows directly from the construction that preimages of points under $h_{CT}$ are either points, closures of complementary regions of the stable or unstable geodesic lamination of $\varphi$, or closures of geodesics in $\hat{D}$. In particular $h_{CT}$ has contractible point-preimages and hence, by classical work of Moore [35] can be approximated by homeomorphisms. Let $h_n \in \text{Homeo}_+(S^2)$ be a sequence of homeomorphisms such that $h_n \to h_{CT}$ in the compact open topology in the space of continuous maps $S^2 \to S^2$. Then the conjugate actions $h_n \circ \rho_{CT} \circ h_n^{-1}$ converge in the weak sense (element-wise) to the boundary action.

In other words, in any neighborhood of the boundary action, there are conjugates of $\rho_{CT}$. Note that none of these are themselves conjugate to the boundary action, as $\rho_{CT}$ is not minimal – it has an invariant circle. We note also that $\rho_{CT}$ itself (and hence any conjugate of it) is rather flexible: the Alexander trick allows one to produce a
continuous deformation from $\rho_{CT}$ to an action of $\pi_1 M$ on $S^2$ with a global fixed point by continuously shrinking one hemisphere while enlarging the other.

While we have described this construction for fibered hyperbolic 3-manifolds, it applies more broadly: work of Frankel [18] shows that the Cannon–Thurston construction can be modified to give an analogous map on any closed hyperbolic manifold admitting a quasi-geodesic flow.

A “blow-up” example. We describe how to equivariantly blow up an orbit $\Gamma \cdot z$ of a $C^1$ group action on an $n$-sphere to produce an action by homeomorphisms that is semi-conjugate. The semi-conjugating map $h$ will be injective off of the preimage of this orbit, and have the additional property that preimages of points in $\Gamma \cdot z$ are homeomorphic to closed disks. In particular, $h$ may be approximated by homeomorphisms.

While our intended application is boundary actions of manifolds admitting negatively curves metrics, the construction applies quite generally to any $C^1$ action of a countable group on $S^n$ so we work in this broader context. For actions on $S^1$ a similar construction works even for actions by homeomorphisms, and in the $C^1$ case can even be smoothed to a $C^1$ action – this is the classical “Denjoy blow-up”. The construction below could conceivably be generalized to group actions on any manifolds, however ensuring that the space obtained by “blowing up” an orbit is again a manifold requires some care; here we are able to quote Cannon’s description of Sierpinski spaces.

**Proposition 4.1.** Let $\Gamma$ be a countable group and $\rho : \Gamma \to \text{Diff}_+(S^n)$ an action with dense orbit $Z$. Then there exists $\rho' : \Gamma \to \text{Homeo}_+(S^n)$ and a surjective, continuous map $h : S^n \to S^n$, such that the pre-image of each point in $Z$ is a closed disk, that is injective on the complement of $h^{-1}(Z)$, and such that $h \circ \rho = \rho' \circ h$.

While blowing up a finite orbit under a group action is a standard construction, we know of no reference in the literature (beyond that for actions on $S^1$) for this result, so we give a complete proof.

**Proof.** Our strategy is to use an inverse limit construction. For simplicity, we assume that $Z$ is the orbit of a point $z$ with trivial stabilizer, however the construction works more generally using the fact the point stabilizers act naturally on the tangent space at any fixed point. Enumerate $\Gamma = \{\gamma_1, \gamma_2, \ldots \}$, and let $z_n = \gamma_n(z)$. Let $X_0$ denote $S^n$ with the standard round metric. Fix some small $\epsilon_1 > 0$ and let $X_1$ be the space obtained by removing an open $\epsilon_1/2$-ball $D_1$ about $z_1$ from $X_0$, and define $f_1 : X_1 \to X_0$ to be a $C^\infty$ map that is smooth away from the boundary of $D_1$, is identity outside a $\epsilon_1$-ball about (the removed point) $z_1$, and on the $\epsilon_1$-ball, is a contraction that collapses the boundary of $D_1$ to the point $z_1$. We may do this in such a way as to identify $\partial D_1$ with the projectivized tangent space at $z_1$, so that the action of any $C^1$ diffeomorphism $g$ of $S^n$ fixing $z_1$ naturally extends to a homeomorphism $\hat{g}$ of $X_1$ such that $f_1 \circ \hat{g} = g \circ f_1$.

Now inductively, suppose that for all $m \leq k$ we have defined $X_m \subset S^n$ (topologically, a sphere with $m$ holes) and a $C^\infty$ surjective map $f_m : X_m \to X_{m-1}$. Let $F_m$ denote $f_1 \circ \ldots \circ f_{m-1} \circ f_m$. Let $\epsilon_{k+1}$ be half the distance from $F_k^{-1}(z_k)$ to the nearest boundary component of $X_k$, and define $X_{k+1}$ to be $X_k$ with an $\epsilon_{k+1}/2$-disk about $F_k^{-1}(z_{k+1})$ removed, and $f_{k+1} : X_{k+1} \to X_k$ a map that collapses the boundary of the removed disk to the point $F_k^{-1}(z_{k+1})$, with support on a $\epsilon_{k+1}$ disk. Again, we do this blow-up using an identification of the boundary of the disk with the tangent space to the
point $F_{k+1}^{-1}(z_{k+1})$, so that any diffeomorphism of $X_k$ fixing the blown-up point defines a diffeomorphism of $X_{k+1}$. More generally, if $g$ is a diffeomorphism of $S^n$ that preserves the set $\{z_1, \ldots, z_{k+1}\}$, it also defines a homeomorphism of $X_{k+1}$ (via conjugation by $F_{k+1}$) on the invariant set $S^n - \{z_1, \ldots, z_{k+1}\}$ on which $F_{k+1}^{-1}$ is a homeomorphism, and on the inserted boundary disks by the identification of them with the tangent spaces to the points $z_i$.

Let $X$ be the inverse limit of this system, and let $F : X \to X_0 = S^n$ be the natural map. Using the fact that $X_k \subset S^n$ and each map $f_k : X_k \to X_{k-1}$ extends naturally to a map $S^n \to S^n$ with $f_k \to \text{id}$ in the compact-open topology, we may identify $X$ with a subset of $S^n$ by identifying each sequence $(\ldots p_2, p_1, p_0)$ satisfying $p_{k-1} = f_k(p_k)$ with the point $\lim_{k \to \infty} p_k \in S^n$. Note that, if $p_0 \not\in Z$ the sequence is uniquely determined by $p_0$ (since $f_k$ is injective on the complement of $F_{k-1}^{-1}(Z)$), and if $p_0 \in Z$, our construction ensures that $\{p_k\}_{k \in \mathbb{N}}$ is constant for all sufficiently large $k$.

We claim that $X$ is topologically an $(n-1)$-dimensional Sierpinski space, that $F$ gives a homeomorphism between $X - F_{k+1}^{-1}(Z)$ and $S^n - Z$, and that there is an action of $\Gamma$ on $X$ by homeomorphisms such that the restriction to $X - F_{k+1}^{-1}(Z)$ agrees (under this homeomorphic identification) with the original action of $\Gamma$. Given this, collapsing each boundary component of $X$ to a single point gives a sphere $\overline{X}$, and $F$ induces a continuous, surjective map $\overline{X} \to S^n$ that intertwines the two actions, as desired.

To show that $X$ is a Sierpinski space, we use Cannon’s criterion from [12]:

**Theorem 4.2 (Cannon).** Let $X \subset S^n$ be closed, and let $U_i$ denote the connected components of $S^n - X$. Then $X$ is homeomorphic to the (unique up to homeomorphisms) $n-1$ dimensional Sierpinski space if and only if the following hold

1. For each $i$, $S^n - U_i$ is an $n$-cell
2. The closures of the $U_i$ are pairwise disjoint
3. $\bigcup_i U_i$ is dense in $S^n$, and
4. $U_1, U_2, \ldots$ is a null sequence, meaning that $\text{diam}(U_n) \to 0$.

Cannon’s result is for $n \neq 4$, but applies in all dimensions given Quinn’s proof of the Annulus theorem in dimension 4.

As we have already observed, the restriction of $F$ to $X - F_{k+1}^{-1}(Z)$ is injective, which implies that $F : X - F_{k+1}^{-1}(Z) \to S^n - Z$ is a homeomorphism, since it is a continuous bijection induced from the map $F$, and $F$ is a continuous map between compact metric spaces, hence closed. In particular, the third property above is satisfied.

The first condition in Cannon’s theorem is equivalent to the statement that collapsing each set $F_{k+1}^{-1}(z_i)$, for $i > n$, gives a $n$-holed sphere. This collapsing can be seen as induced by the natural map $X \to X_n$ defined by $(\ldots p_2, p_1, p_0) \mapsto p_n$. By a similar argument to the above, this is injective on the complement of $\bigcup_{i>n} F_{k+1}^{-1}(z_i)$, and collapses each $F_{k+1}^{-1}(z_i)$, for $i > n$, to the point $F_{n+1}^{-1}(z_i)$, which are all distinct in $X_n$. The other criteria of theorem [12] are easily verified from the construction of $X$.

Finally, we describe the action of $\Gamma$ on $X$; this comes from our description of the $X_n$ in terms of the tangent space blow-up. For each $i$, each connected component of $F_{k+1}^{-1}(z_i)$ is a circle, identified with the projectivized tangent space of oriented lines $\mathcal{PT}_{z_1}(S^n)$ via projection to $X_i$ and the identification there. This clearly gives an action by bijections of $X$; we show that it is in fact an action by homeomorphisms. For this it suffices to check continuity of the action of each $\gamma \in \Gamma$. Let $x_n \to x_\infty$ be a sequence of
points in $X$. If $x_\infty \notin F^{-1}(Z)$, that $\gamma(x_n)$ converges to $\gamma(x_\infty)$ follows directly from our construction and the definition of the inverse limit. If $x_\infty \in F^{-1}(z_j)$ for some $z_j$, then it suffices to project to $X_j$ and work there. That $x_n$ converges to $x_\infty$ in $X_j$, where $x_\infty$ is a boundary point means precisely that, as $n \to \infty$, the points $F_j(x_n)$ converge to $z_j$ and $\frac{F_j(x_n)-z_j}{||F_j(x_n)-z_j||}$ converges to the tangent direction $v$ represented by $z_\infty$. Continuous differentiability of $\gamma$ at $z_j$ is all that is required to have $\gamma(x_k) \to \gamma(x_\infty)$, this is why we assumed our original action was of class $C^1$. □

5. Global rigidity of slitherings from skew-Anosov foliations

In this section we specialize to actions of fundamental groups of certain 3-manifolds on $S^1$. In this case, the leafwise immersion produced in Lemma 3.1 is actually a homeomorphism, and we will exploit this property to prove a global rather than local rigidity result for both (lifts of) boundary actions and the general case of actions induced by “slitherings” from skew-Anosov flows. We begin by summarizing some standard results and framework needed for the proof.

5.1. Anosov Flows. A flow $\Phi_t$ generated by a vector field $Y$ on a closed 3-manifold $M$ is Anosov if the tangent bundle splits as a sum of (continuous) line bundles that are invariant under the flow $TM = E^{ss} \oplus \langle Y \rangle \oplus E^{uu}$ with the property that for some choice of metric on $M$, there are constants $C, \lambda > 0$ such that

$$||((\phi_t)_* v^s)|| \leq Ce^{-\lambda t}||v^s|| \quad \text{and} \quad ||((\phi_t)_* v^u)|| \geq C^{-1}e^{\lambda t}||v^u||$$

holds for all $t \geq 0$ and all $v^u \in E^{uu}, v^s \in E^{ss}$. By averaging the metric over long time intervals and decreasing $\lambda$, one can assume that $C = 1$. Such a metric is called adapted.

The line fields $E^{uu}, E^{ss}$ are called the strong unstable and strong stable directions of the flow. It is a classical fact that these distributions are uniquely integrable. The foliations to which they are tangent are characterized by the dynamical property that their leaves consist of sets of points that are asymptotic under the flow in forward, respectively backward, time. One also obtains foliations $F^u, F^s$ tangent to the integrable plane fields

$$E^s = E^{ss} \oplus \langle Y \rangle, \quad E^u = E^{uu} \oplus \langle Y \rangle.$$}

These are called the weak stable and unstable foliations $F^s, F^u$ of the flow. In the examples of interest to us, the line fields $E^{ss}$ and $E^{uu}$ will always be orientable, i.e. trivial as line bundles. Thus, for simplicity, we take orientability to be a standing assumption.

The following proposition collects some well-known properties of the weak foliations of an Anosov flow that we will need going forward. The additional $C^1$ structure given by point (1) below will be important in the proof of Theorem 1.2.

**Proposition 5.1.** Let $\Phi_t$ be an Anosov flow on a closed 3-manifold $M$. Then the following hold

1. (Hirsch-Pugh [28]): The weak stable and unstable foliations $F^s$ and $F^u$ are of class $C^1$. 

(2) The leaves of $F^s$ and of $F^u$ have a natural large-scale hyperbolic structure. More precisely, $M$ admits a metric such that the induced metric on weak stable and unstable leaves in $\tilde{M}$ is uniformly bi-Lipschitz equivalent to a metric of constant curvature $-1$. In this metric, the flowlines on each leaf are quasi-geodesics; on a leaf of $F^s$, flowlines share a unique common forward endpoint, and on $F^u$ a common negative endpoint.

For completeness, we give an outline of the proof. The reader may consult [15, Section 5] for more details and general background.

Proof. Item (1) follows from the proof of the Smoothness Theorem part (i) in [28]. Specifically, one applies the graph transform argument there to the quotient bundle $TM/\langle Y \rangle$ upon which the flow acts. This action has two invariant sub-bundles $E^s, E^u$ given by the images of the weak stable and unstable subbundles. Since these are uniformly contracted and expanded, respectively, by $D\Phi_t$, the $C^1$-section Theorem then implies that $E^s$ and $E^u$ are $C^1$. Pulling back to $TM$, one deduces that the subbundles $E^s, E^u$ are of class $C^1$ as well. Since they are invariant under the flow, it follows that they are tangent to $C^1$-foliations.

To show item (2), take a $C^0$-metric on $M$ so that the strong stable/unstable directions and the flow direction are all orthogonal, and the generating vector field has unit length. Without loss of generality we assume that this metric is adapted to the Anosov flow. In general, this metric may only be continuous, but we do not need any higher regularity for the argument. Let $L$ be a leaf of the weak unstable foliation, and $\ell^s$ a strong-stable leaf through some point $p \in L$. Then $\ell^s$ is a section for the restriction of $\Phi_t$ to $L$. Parametrize $\ell^s$ by arc length and call this coordinate $x$. The lift $\tilde{\ell}^s$ gives a section for the induced flow on the universal cover $\tilde{L}$ and hence a global coordinate system $(x,t)$ on $\tilde{L}$ so that the pulled-back metric is of the form

$$
\begin{pmatrix}
  f^2(x,t) & 0 \\
  0 & 1
\end{pmatrix}
$$

In particular, the flow lines are geodesics with respect to this metric. By construction $f(x,0) = 1$ and the Anosov condition gives the bounds

$$
\varepsilon e^{-\lambda t} \leq f(x,t) \leq e^{-\lambda t}.
$$

This implies that the the metric on $\tilde{L}$ is uniformly bi-Lipshitz equivalent to the metric pulled back from $\mathbb{R}^2$ by $(x,t) \mapsto e^{-\lambda t}$, i.e. the constant negative curvature hyperbolic metric on the upper half plane. In the hyperbolic metric, vertical lines are geodesics with the same forward endpoint, and these correspond to flowlines under our bi-Lipshitz identification. The case of the unstable foliation follows mutatis mutandis.

5.2. Slitherings and skew-Anosov flows. We first recall the notion of a slithering, as introduced by Thurston in [38].

**Definition 5.2** (Slithering). Let $M$ be a closed 3-manifold. A slithering of $M$ over $S^1$ is a fibration $s: \tilde{M} \rightarrow S^1$ with 2-dimensional fibers such that deck transformations are bundle automorphisms for $s$, taking fibers to fibers. This means that the foliation of $\tilde{M}$ given by the fibers of $s$ descends to a foliation on $M$. 

Figure 2. The flow space $O$ of a skew-Anosov flow. The two black points are related by the natural map $\mu$ on $O$.

Since deck transformations take fibers to fibers, a slithering $s : \tilde{M} \to S^1$ also induces a natural slithering action $\rho_s : \pi_1 M \to \text{Homeo}_+(S^1)$ on the circle. Following our earlier convention, we continue to assume for simplicity that all foliations are oriented, and this slithering action is by orientation preserving homeomorphisms.

Slitherings generalize both the notion of a fibering over $S^1$ (where $s$ is simply the lift of the bundle projection to $\tilde{M}$), and the notion of a foliated $S^1$-bundle, where $s$ is the projection to the fiber on the induced foliated $S^1$ bundle over $\tilde{M}$. Skew-Anosov flows, which are a generalization of geodesic flows on negatively curved surfaces, provide another important source of examples.

**Example 5.3 (Skew-Anosov flows).** Let $\Phi_t$ be an Anosov flow on a closed 3-manifold $M$, whose stable foliation is oriented and $\mathbb{R}$-covered, meaning that the leaf space on the universal cover is Hausdorff (or equivalently, is homeomorphic to $\mathbb{R}$). Results of Fenley [15] and Barbot [2] show that a flow with this property is either the suspension of an Anosov diffeomorphism of $T^2$ or is skew, meaning that the orbit space of the lift of the flow to $\tilde{M}$ is homeomorphic to the infinite diagonal strip

$$O = \{(x, y) \in \mathbb{R}^2 \mid |x - y| < 1\}$$

in such a way that the preimages of horizontal (respectively, vertical) intervals are the stable (resp. unstable) leaves of the flow, as illustrated in Figure 2.

In this model, each point $o \in O$ can be assigned a point $o_u$ on the upper boundary by following the unstable leaf through $o$, and a point $o_l$ on the lower boundary by following the unstable leaf. Taking the intersection of the stable leaf through $o_u$ and unstable through $o_l$ defines a continuous, fixed point free map $\eta : O \to O$. This map sends stable leaves to unstable leaves and vice versa, so $\tau = \eta^2$ descends to a map on the leaf space $\Lambda^s$ of the weak stable foliation. This map is strictly monotone, and the quotient map

$$\tilde{M} \to \Lambda^s \to \Lambda^s/\tau$$

defines a slithering of $M$. By construction, the foliation associated to this slithering is the weak stable foliation of $\Phi_t$.

The map $\eta$ has many remarkable properties. A concise summary is given in [5, §4]; we will simply state those which are of use to us. First, $\eta$ is a $\pi_1 M$-equivariant
homeomorphism and can be induced from a continuous self-map $\eta_M$ of the underlying manifold $M$ [2, 15]. Barbot [2, Theorem 3.4] showed, using an averaging argument, that this map $\eta_M$ can actually be taken to be a homeomorphism of $M$. (An alternative description of $\eta_M$ map is given in Proposition 7.4 ii) of [38].) Furthermore, if some element of the fundamental group fixes a point $o$ of the leaf space, then it also fixes $\eta^k(o)$ for all $k$, and the corresponding periodic orbits of the flow are freely homotopic. It is not hard to see that the converse is also true: any two periodic orbits of a skew-Anosov flow that are freely homotopic are related by some power of the map $\eta$ on the flow space. We note this fact for later use.

**Proposition 5.4** (see [2, 15]). Let $\alpha, \beta$ be freely homotopic orbits of a skew-Anosov flow with orientable splitting on a closed manifold $M$. Then $\beta = \eta^k_M(\alpha)$ for some integer $k$, where $\eta_M$ is a homeomorphism of $M$ that induces the map $\eta$ on the flow space.

We will also need to use the following result of Barbot on minimality of the slithering action associated to a skew-Anosov flow.

**Proposition 5.5** ([2] Theorem 2.5). Any skew-Anosov flow is transitive and its associated slithering action $\rho_s : \pi_1 M \to \text{Homeo}_+(S^1)$ is minimal.

**Universal circles.** Let $\mathcal{F}^u$ and $\mathcal{F}^s$ denote the lifts to $\tilde{M}$ of the unstable and stable foliations of an Anosov flow. Following Proposition 5.1, the leaves of these foliations have a natural large-scale hyperbolic structure, hence can be compactified by a boundary at infinity. In the case of a skew-Anosov flow, Thurston [38] observed that these foliations are uniform (meaning that leaves remain a bounded distance apart) and hence the leafwise boundaries can be canonically identified, as in the following statement.

**Lemma 5.6** (Lemma 4.1 and Corollary 4.2 of [38]). For each pair of leaves $L$ and $L'$ of $\mathcal{F}^u$, and every infinite geodesic $g$ on $L$, there is a unique geodesic $g'$ on $L'$ at a bounded distance from $g$. This produces a canonical identification of the circles at infinity for all the leaves of $\mathcal{F}^u$, giving one $\pi_1 M$-equivariant “universal” circle.

The same holds with $\mathcal{F}^s$ in the place of $\mathcal{F}^u$.

One way to describe Thurston’s universal circle identification is by considering the intersection of $\mathcal{F}^u$ and $\mathcal{F}^s$. For a fixed leaf $L$ of $\mathcal{F}^u$, the leaves of $\mathcal{F}^s$ intersect $L$ as quasi-geodesics with a common forward endpoint (for the natural choice of orientation), with respect to the large-scale hyperbolic structure. Thus, the boundary of $L$, minus one point, can be identified with a subset of the leaf space of $\mathcal{F}^s$, and there is a natural map defined on subsets of boundaries of any two nearby leaves $L$ and $L'$ of $\mathcal{F}^u$ via leaves of $\mathcal{F}^s$. This gives the following.

**Proposition 5.7** (Prop 7.1 of [38]). Let $S^1_u$ denote the universal circle obtained from Lemma 5.6. There is an identification of $S^1_u$ with $\Lambda^s/\tau$, under which the action of $\pi_1 M$ on $S^1_u$ (i.e. as obtained from the action on the leaf space) agrees up to conjugacy with the slithering action of the foliation.

In other words, $S^1_u$ can be thought of as the space of vertical lines (mod $\tau$) of the orbit space $\mathcal{O}$ depicted in Figure 2.
5.3. **Proof of Theorem 1.2** In this section we use the following notion of semi-conjugacy for circle maps, as defined by Ghys in [22]. Though the terminology “semi-conjugacy” is now widespread, this is not the same as the standard dynamical notion of semi-conjugacy defined in Section 2. To avoid confusion, we will follow [32] and use the term weak conjugacy for Ghys’ definition.

**Definition 5.8.** Let \( \rho_1 \) and \( \rho_2 : \Gamma \to \text{Homeo}_+ (S^1) \) be two actions of a group \( \Gamma \) on the circle \( S^1 = \mathbb{R}/\mathbb{Z} \). These actions are *weakly conjugate* if there is a monotone map \( h : \mathbb{R} \to \mathbb{R} \) commuting with \( x \mapsto x + 1 \), and lifts of each element \( \rho_i(\gamma) \) to \( \text{Homeo}_+(\mathbb{R}) \) satisfying \( h \circ \rho_1(\gamma) = \rho_2(\gamma) \circ h \).

The map \( h \) in the definition above is not required to be continuous or surjective. However, if \( \rho_2 \) is minimal, any weak conjugacy \( h \) between \( \rho_2 \) and any other representation \( \rho_1 \) is necessarily continuous and surjective. Note that, since \( h \) commutes with integer translations, it descends to a map of \( S^1 \). A map of \( S^1 \) so induced is called a *degree one monotone map*. It is easy to verify that the surjective, degree one monotone maps of \( S^1 \) are precisely the orientation-preserving maps of \( S^1 \) which are approximable by homeomorphisms.

We divide the proof of Theorem 1.2 into two propositions, covering first the local then the global result.

**Proposition 5.9** (Local Rigidity). Let \( \rho_1 \) and \( \rho_2 : \Gamma \to \text{Homeo}_+ (S^1) \) be two actions of a group \( \Gamma \) on the circle \( S^1 = \mathbb{R}/\mathbb{Z} \). Then the global result.

**Proposition 5.10** (Global Rigidity). Under the hypotheses above, one can in fact take \( U \) to be the connected component of \( \rho \) in \( \text{Hom}(\pi_1 M, \text{Homeo}_+ (S^1)) \) consisting of representations weakly conjugate to \( \rho \).

The proof of the local version follows roughly the same strategy of the proof of Theorem 1.1 in Section 3. However, here the suspension of the action is one dimension larger than that considered there, forcing us to make use of a carefully chosen section in order to cut down one dimension. The proof of the global result is then a quick consequence of approximability of weak conjugacy maps by homeomorphisms.

**Proof of Proposition 5.9** Let \( \mathcal{F}^s \) be the weak stable foliation of a skew-Anosov flow \( \Phi_t \) on \( M^3 \) with associated slithering action \( \rho_s : \pi_1 M \to \text{Homeo}_+ (S^1) \). Then there exists a neighborhood \( U \) of \( \rho \) in \( \text{Hom}(\pi_1 M, \text{Homeo}_+ (S^1)) \) consisting of representations weakly conjugate to \( \rho \).

**Proof of Proposition 5.10** Let \( \mathcal{F}^s \) be the weak stable foliation of a skew-Anosov flow \( \Phi_t \) on a closed 3-manifold \( M \), and let \( s : \tilde{M} \to S^1 \) be the associated slithering, and \( \rho_s : \pi_1 M \to \text{Homeo}_+ (S^1) \) the slithering action. For clarity, we divide the proof into steps, as indicated by the paragraph headings.

**Setup: a canonical section.** Consider the lift of \( \mathcal{F}^s \) to \( \tilde{M} \). As in Section 3, we abuse notation slightly and let \( \mathcal{F}^s \) also denote the lifted foliation to \( \tilde{M} \), with leaf space \( \Lambda^s \cong \mathbb{R} \). By Proposition 5.1, \( \mathcal{F}^s \) is of class \( C^1 \), giving a \( C^1 \) identification \( \tilde{M} \cong \mathbb{R}^2 \times \Lambda^s \), and an action of \( \pi_1 M \) on \( \Lambda^s \) by \( C^1 \) diffeomorphisms. As explained in our discussion earlier (see Example 5.3), this action commutes with the map \( \tau : \Lambda^s \to \Lambda^s \) used in defining the slithering, and \( \rho_s \) is simply the induced action of \( \pi_1 M \) on the (topological) circle \( \Lambda^s/\tau \). Note, however, that the map \( \tau \) is in general only a homeomorphism and so \( \rho_s \) need not be an action by \( C^1 \) diffeomorphisms. It is for this reason that we work with \( \tau \)-equivariant lifts to \( \Lambda^s \). Fixing notation, let \( \tilde{\rho} \) denote the lift of \( \rho_s \) to an action of \( \pi_1 M \) on \( \Lambda^s \). This is simply the standard action of \( \pi_1 M \) on the leaf space \( \Lambda^s \).
Figure 3. A picture of $\widetilde{\sigma}(\widetilde{M})$ in the familiar case $M = UT\Sigma$ as in [38]. The infinite cylinder is $\widetilde{\sigma}(\widetilde{M}) \cong \widetilde{M} = \mathbb{H}^2 \times \Lambda^s$ i.e. the height of a point in the stack of copies of $\mathbb{H}^2$ corresponds to the positive endpoint of a unit tangent vector based at that point. Horizontal planes are leaves of $\mathcal{F}^s$, one is shown in blue. A leaf of $\widetilde{\sigma}(\mathcal{F}^u)$ is shown in red, the hyperbolic metric on the leaf is that lifted from the projective model of $\mathbb{H}^2$ shown below. In this model, one endpoint at infinity of the unstable leaf is blown up to an interval.

Let $E = \left( \widetilde{M} \times \Lambda^s \right) / \pi_1 M$ denote the suspension foliation of $\hat{\rho}$, and for $p \in \widetilde{M}$, let $\ell(p) \in \Lambda^s$ denote the leaf containing $p$. For $\gamma \in \pi_1 M$, the section $\widetilde{\sigma} : \widetilde{M} \to \widetilde{M} \times \Lambda^s$ given by $p \mapsto (p, \ell(p))$ satisfies the $\hat{\rho}$-equivariance

$$\gamma \cdot p \mapsto (\gamma \cdot p, \hat{\rho}(\gamma)(\ell(p)))$$

so induces to a section $\sigma : M \to E$. Since $\hat{\rho}$ is $C^1$, the section $\widetilde{\sigma}$ (and hence also $\sigma$) are $C^1$ embeddings. Furthermore, $\widetilde{\sigma}$ is transverse to the horizontal foliation on $\widetilde{M} \times \Lambda^s$ since the composition of $\widetilde{\sigma}$ with the projection to $\Lambda^s$ is precisely the quotient map to the leaf space given by $\widetilde{M} \cong \mathbb{H}^2 \times \Lambda^s \to \Lambda^s$, which is a non-singular $C^1$ map. By definition, the leaves of $\widetilde{\sigma}(\mathcal{F}^s)$ are simply the intersection of $\widetilde{\sigma}(M)$ with the leaves of the horizontal foliation of the suspension. In particular, this means that the $C^1$ foliation $\widetilde{\sigma}(\mathcal{F}^s)$ remains transverse to the image of the unstable foliation $\widetilde{\sigma}(\mathcal{F}^u)$.

Nearby actions give nearby foliations. In Lemma 3.1, we showed that there exists a neighborhood $U$ of $\rho_s$ in $\text{Hom}(\pi_1 M, \text{Homeo}_+(S^1))$ such that, for each $\rho' \in U$, the suspension of $\rho'$ can be realized as the holonomy of a foliation $\mathcal{F}_{\rho'}$ on the suspension $E_{\rho_s}$ of $\rho_s$ whose tangent distribution is $C^0$ close to the horizontal distribution defining $E_{\rho_s}$. By taking lifts to the bundle $E$ defined above (which is a fiberwise cover of $E_{\rho_s}$), the same statement holds for the lift of any such perturbation $\rho'$ to the group of $\tau$-equivariant homeomorphisms of $\Lambda^s = \mathbb{R}$. Namely, any nearby representation $\rho'$ admits
a $\tau$-equivariant lift $\tilde{\rho}'$ which can be realized as the holonomy of a near-horizontal $\tau$-equivariant foliation $\mathcal{F}'$ on $E$. Such a foliation will then intersect $\sigma(M)$ to give a foliation transverse to the unstable foliation $\sigma(\mathcal{F}^u)$ on $\sigma(M)$.

Abusing notation, we now let $\mathcal{F}'$ denote the restriction of this foliation to $\sigma(M)$, and we now focus on these two foliations $\sigma(\mathcal{F}^u)$ and $\mathcal{F}'$ on $\sigma(M)$ and their lifts to $\tilde{\sigma}(\tilde{M}) \cong \tilde{M}$, which we will denote by $\tilde{\sigma}(\mathcal{F}^u)$ and $\tilde{\mathcal{F}'}$. The section $\tilde{\sigma}$ also gives a natural identification of the leaf space of $\tilde{\mathcal{F}'}$ with $\Lambda^s$.

**Leafwise quasi-geodesic foliations and the endpoint map.** We now adapt the line of argument carried out in Section 3.2 using “endpoint maps” to define a weak conjugacy between $\rho$ and $\rho'$. By Proposition 5.1(ii), the leaves of $\mathcal{F}^u$ on $\tilde{M}$ are uniformly large-scale hyperbolic and the lines given by $\mathcal{F}^u \cap \mathcal{F}^s$ are quasi-geodesics with uniform constants. Under this quasi-isometry, the leaves of the strong unstable foliation map to horocycles with the same ideal point on the boundary. By construction the intersection $\tilde{\sigma}(\mathcal{F}^u) \cap \tilde{\mathcal{F}'}$ gives a one-dimensional foliation of $\sigma(M)$ and we denote this foliation (suggestively) by $\mathcal{F}^{QG}$. This is indeed a foliation by quasi-geodesic lines on each leaf of $\mathcal{F}^u$, with respect to its large-scale hyperbolic structure; one may see this from the fact that the tangent distribution of $\tilde{\mathcal{F}'}$ is $C^0$ close to the horizontal in $\tilde{M} \times \Lambda^s$, so for any fixed leaf of $\tilde{\sigma}(\mathcal{F}^u)$, its intersection with $\tilde{\mathcal{F}'}$ will form a foliation with tangent distribution close to that of the flowlines defined by $\mathcal{F}^u \cap \mathcal{F}^s$. Thus, for each leaf $L$ of $\mathcal{F}^u$, we have well-defined endpoint maps $e^+_L$ and $e^-_L$ taking leaves of $\mathcal{F}^{QG}$ in $L$ to their positive and negative endpoints on the ideal boundary of $L$.

Recall from Proposition 5.1 that flowlines of $\Phi_1$ on a fixed leaf of $\mathcal{F}^u$ share a common negative endpoint. Using this observation, Lemma 3.2 adapts in a straightforward way, using the strong unstable foliation and dynamics of the flow in place of horocycles for the geodesic flow to show that, for a fixed leaf $L$ of $\mathcal{F}^u$ the map $e^-_L$ is constant.

By Lemma 5.6, the boundaries of leaves of $\mathcal{F}^u$ may be identified in a natural way to give a universal circle $S^1_u$. Thus, we may piece together the maps $e^+_L$ (and $e^-_L$) to obtain globally defined maps $e^+_L$ and $e^-_L$ from the leaf space of $\mathcal{F}^{QG}$ to $S^1_u$. The proof of Lemma 3.4 shows that, for each leaf $L$ of $\mathcal{F}^u$, the map $e^+_L$ is continuous. A straightforward adaptation of this lemma, using only the uniformity of $\mathcal{F}^u$ from Lemma 5.6 shows that the globally defined maps $e^+_L$ and $e^-_L$ are continuous as well.

**Straightening quasi-geodesics to produce the semi-conjugacy.** To conclude the proof, we follow a modified version of the argument from Proposition 3.8. Since leaves of $\mathcal{F}^{QG}$ are transverse to the image of the strong unstable foliation $\tilde{\sigma}(\mathcal{F}^u)$, they can be continuously homotoped via a $\pi_1M$-equivariant homotopy $\tilde{h}_1$ along the one-dimensional leaves of the strong unstable foliation in such a way so that the image of each leaf of $\mathcal{F}^{QG}$ under $\tilde{h}_1$ is the flow line of $\Phi_1$ with the same ideal endpoints in the given leaf.

The time-one map $\tilde{h}_1$ of the homotopy descends to a map $h_1$ from the leaf space of $\mathcal{F}^{QG}$ to the orbit space of the flow $\mathcal{O}$ (which we denote by $\mathcal{O}$), making the following
We claim that for each leaf $L'$ of $\tilde{F}'$, its image $h_1(L')$ agrees with the image in $O$ of some leaf of $F^s$. Equivalently, we need to show that the positive endpoint map is constant on each leaf $L'$ of $\tilde{F}'$. To show this, we will use the picture given by Thurston’s universal circle perspective, as stated in Proposition 5.7. Following this, the negative and positive endpoint maps give local (first and second) coordinates on $O$. Fix any leaf $L'$ of $\tilde{F}'$. Note first that $e^-(h_1(L'))$ is nonconstant, i.e. its image in $O$ does not correspond to a vertical segment in $O$. This is simply because $L'$ intersects at least two distinct leaves of $F^u$. We wish now to show that $h_1(L')$ is horizontal.

Suppose for contradiction that this is not the case. By Proposition 5.5, the skew-Anosov flow $\Phi_t$ is transitive, and so its periodic points are dense. It follows that the image of $L'$ intersects both the unstable leaf and the stable leaf of some periodic orbit.

Let $\gamma \in \pi_1 M$ be the element represented by this periodic orbit, thought of as a closed curve in $M$.

Since $\tilde{h}_1(F^{QG})$ is a $\pi_1 M$-equivariant foliation, $\gamma^n \tilde{h}_1(L')$ are also leaves in the image of $\tilde{h}_1$. See Figure 5 for a schematic picture. Since we are assuming that $\tilde{h}_1(L')$ is not horizontal, then the sequence of leaves $\gamma^n \tilde{h}_1(L')$ approaches (uniformly on compact sets) a vertical segment. The fact that the foliation $\tilde{F}'$ is lifted from a flat $S^1$-bundle structure gives us compactness of the leaf space mod $\tau$, so after passing to a subsequence, in the quotient by $\tau$ the leaves $\gamma^n \tilde{h}_1(L')$ converge to some limit leaf $L_\infty$. By continuity of the map $\tilde{h}_1$, the image of this leaf is vertical, contradicting our earlier observation.

We conclude that leaves of $\tilde{F}'$ map to (subsets of) leaves of $F^s$. We now argue that leaves are sent onto leaves; in other words, the straightening map $h_1$ defines a map from the leaf space of $\tilde{F}'$ to that of $F^s$ on $\tilde{M}$. To see this, consider first a leaf $L'$ of $\tilde{F}'$ whose image contains a periodic orbit representing some $\alpha \in \pi_1 M$. The $\pi_1 M$-equivariance of our construction means that $L'$ is also invariant under the action of $\alpha$, and its image under $h_1$ is a $\alpha$-invariant subset of a stable leaf. We additionally know that, under the

\[
\begin{array}{ccc}
\tilde{\sigma}(\tilde{M}) & \xrightarrow{\tilde{h}_1} & \tilde{M} \\
\downarrow & & \downarrow \\
\tilde{\sigma}(\tilde{M})/F^{QG} & \xrightarrow{h_1} & \mathcal{O}.
\end{array}
\]
Figure 5. The image of a leaf in the flow space $O$ which is non-horizontal is shown in light green. The black point is a periodic orbit corresponding to $\gamma \in \pi_1 \tilde{M}$ and the lines in darker shades of green show the first few iterates under the action of $\gamma$.

negative endpoint map, this subset contains an interval. Thus, it must necessarily be the full leaf. The general case (for leaves not necessarily containing a periodic orbit) now follows from the density of periodic orbits of the flow and continuity.

In summary, we have the following induced $\pi_1 M$-equivariant maps, where the vertical maps denote the maps to the respective leaf spaces:

$$
\begin{align*}
\tilde{\sigma}(\tilde{M}) & \xrightarrow{\tilde{h}_1} \tilde{\mathcal{M}} \\
\tilde{\sigma}(\tilde{M})/\mathcal{F} & \xrightarrow{h_1} \tilde{M}/\mathcal{F}^* = \Lambda^s
\end{align*}
$$

Note that the map $h_1$ is monotone as it preserves the order of leaves in the leaf space, and it is surjective and equivariant with respect to the action of $\tau$. Thus, after quotienting out by the action of $\tau$ on $\tilde{M}$ and $\mathbb{R}$ respectively, we obtain a surjective monotone map $h$ on the circle $S^1 = \mathbb{R}/\tau$ that provides the desired weak conjugacy. $\square$

Proof of Proposition 5.10. Let $\rho$ be a representation as in the statement of Proposition 5.9. We show that the property of being weakly conjugate to $\rho$ is both an open and closed condition in $\text{Hom}(\pi_1 M, \text{Homeo}_+(S^1))$.

Closedness. It follows from work of Ghys [22] and Matsumoto [33] that, for any discrete group $\Gamma$, the closure of a conjugacy class in $\text{Hom}(\Gamma, \text{Homeo}_+(S^1))$ is a weak conjugacy class (called semi-conjugacy rather than weak conjugacy by these authors). This is because weak conjugacy classes can essentially be specified by rotation numbers of elements, rotation number being a continuous function on $\text{Homeo}_+(S^1)$, or by the integer bounded Euler class. A detailed exposition is given in [32, §2].

Openness. This will follow from Proposition 5.9 and approximability of weak conjugacies on $S^1$ by homeomorphisms. (See the remark after Definition 5.8.) Let $U$ be the neighborhood given by Proposition 5.9. Let $\rho'$ be weakly conjugate to $\rho$ via a degree
one monotone map \( h : S^1 \to S^1 \) satisfying \( h \circ \rho' = \rho \circ h \); since \( \rho \) is minimal by Proposition 5.5, \( h \) is continuous. If \( h_0 \in \text{Homeo}_+ (S^1) \) is a sufficiently close \( C^0 \) approximation to \( h \), then \( h_0 \rho' h_0^{-1} \in U \) so admits a neighborhood \( V \) consisting of weakly conjugate representations; and \( h_0^{-1} V h_0 \) is the desired neighborhood of \( \rho' \). \( \square \)

5.4. **Application: global rigidity of geometric representations.** As a first application of Theorem 1.2, we give a new proof of the main result of \[31\]. A second application is discussed in the next section. Both use the following standard construction; further discussion of which can be found in \[31\].

**Fiberwise covers of the geodesic flow.** Let \( \Sigma \) be a closed hyperbolic surface. Then \( \Sigma = \mathbb{H}^2 / \rho_0(\pi_1 \Sigma) \) where \( \rho_0 \) is an embedding as a cocompact Fuchsian group of \( \text{PSL}_2(\mathbb{R}) \).

The action of \( \text{PSL}_2(\mathbb{R}) \) on \( \partial_\infty \mathbb{H}^2 = S^1 \) by Möbius transformations gives a realization of the boundary action of \( \pi_1 \Sigma \). As in Example 5.3, the corresponding suspension foliation of this representation can be naturally identified with the weak stable foliation of the geodesic flow.

Lifts of \( \rho_0 \) to the extension \( \mathbb{Z} / k \mathbb{Z} \to \text{PSL}_2^{(k)}(\mathbb{R}) \to \text{PSL}_2(\mathbb{R}) \) are precisely the holonomy representations of the weak stable foliations of the possible lifts of the geodesic flow to a \( k \)-fold fiberwise cover of \( M \to \text{UT} \Sigma \). Such lifts exist if and only if \( k \) divides the Euler characteristic \( \chi(\Sigma) \); in which case for a genus \( g \) surface there are \( k^{2g} = |\text{Hom}(\pi_1 \Sigma, \mathbb{Z} / k \mathbb{Z})| \) distinct lifts. These lifts can be also be distinguished dynamically: thinking of \( \text{PSL}_2^{(k)}(\mathbb{R}) \subset \text{Homeo}_+(S^1) \), via the natural identification of lifts of Möbius transformations to the \( k \)-fold cover of \( S^1 \), distinct lifts differ by applying a rotation through angles that are multiples of \( 2\pi / k \) to the images of a standard generating set for \( \pi_1 \Sigma \).

**Remark 5.11.** Topologically speaking, the effect of modifying the action of a generator by a rotation is to change the degree of the projection to the fiber of horizontal curves in the suspension of a lift \( \hat{\rho} \) to \( \text{PSL}_2^{(k)}(\mathbb{R}) \). In detail, that some standard generator \( \gamma \) for \( \pi_1 \Sigma \) has image \( \hat{\rho}(\gamma) \in \text{Homeo}_+(S^1) \) with rotation number \( 2\pi n / k \), means precisely that in the suspension of \( \hat{\rho} \), the projection to the \( S^1 \) fiber of the horizontal lift of \( \gamma^n \) to a closed orbit, considered as a map \( S^1 \to S^1 \), has degree \( k \). We will use this perspective again in the proof of Theorem 1.3.

The following is the main result of \[31\] (reproved using a different argument by Matsumoto in \[34\]). By quoting Theorem 1.2 we may give another, shorter independent proof.

**Theorem 5.12** (Mann \[31\]). Let \( \Sigma \) be a surface of genus \( g \geq 2 \), and \( \rho : \pi_1 \Sigma \to \text{PSL}_2(\mathbb{R}) \subset \text{Hom}(\pi_1 \Sigma, \text{Homeo}_+(S^1)) \) an embedding as a cocompact Fuchsian group.

Consider any lift \( \hat{\rho} \) of this action to the \( k \)-fold cover of \( S^1 \):

\[
\begin{array}{ccc}
\text{Homeo}_+^{(k)}(S^1) & \xrightarrow{\hat{\rho}} & \pi_1 \Sigma \\
\downarrow & & \downarrow \rho \\
\pi_1 \Sigma & \xrightarrow{\rho} & \text{Homeo}_+(S^1).
\end{array}
\]

Then the connected component of \( \hat{\rho} \) in \( \text{Hom}(\pi_1 \Sigma, \text{Homeo}_+(S^1)) \) is a single weak conjugacy class.
Proof. Equip \(\Sigma\) with a hyperbolic metric and let \(UT\Sigma\) be its unit tangent bundle. The geodesic flow on \(UT\Sigma\) is skew-Anosov, so determines a slithering action \(\rho_s : \pi_1(UT\Sigma) \to \text{Homeo}_+(S^1)\). We consider lifts \(\hat{\rho}\) of \(\rho = \rho_s\) as per our discussion above. Each lift \(\hat{\rho}\) is the holonomy of the lift of the weak stable foliation of the geodesic flow to a \(k\)-fold fiberwise cover of \(UT\Sigma\). Let \(M\) be such a \(k\)-fold cover, so \(\pi_1M\) sits in a central extension

\[1 \longrightarrow \mathbb{Z} = \langle z \rangle \longrightarrow \pi_1M \longrightarrow \pi_1\Sigma \longrightarrow 1.\]

The lift of geodesic flow to \(M\) is also skew-Anosov, so has a slithering \(\rho_{sk} : \pi_1M \to \text{Homeo}_+(S^1)\). It is easily verified from the definitions that these representations satisfy \(\rho_{sk}(z) = \text{id}\), so descend to representations \(\pi_1\Sigma \to \text{Homeo}_+(S^1)\), which are precisely those appearing in the statement of Theorem 5.12. Theorem 1.2 states that the representation \(\rho_{sk}\) is globally rigid in \(\text{Hom}(\pi_1M, \text{Homeo}_+(S^1))\). Since any small perturbation of a lift \(\hat{\rho} \in \text{Hom}(\pi_1\Sigma, \text{Homeo}_+(S^1))\) can be extended to a deformation of the corresponding slithering representation \(\rho_{sk}\) (one simply declares the central \(\mathbb{Z}\) subgroup to act trivially), this proves global rigidity for the lifts of \(\rho\) as well. \(\square\)

A further consequence of Theorem 1.2 is the following.

**Corollary 5.13.** Let \(M\) be a closed 3-manifold admitting a skew-Anosov flow. Then the component of the space \(\text{Hom}(\pi_1M, \text{Homeo}_+(S^1))\) with trivial Euler class is not connected.

**Proof.** The slithering action \(\rho_s\) corresponds to Thurston’s universal circle action given by compactifying leaves in the universal cover, as described above. The Euler class of this action agrees with the Euler class of \(\mathcal{F}^s\), which is trivial since the tangent bundle to \(\mathcal{F}^s\) admits a nowhere vanishing section determined by the flow. But \(\rho_s\) is not in the same component as the trivial representation by Proposition 5.10, although they have the same Euler class. \(\square\)

6. **Topologically inequivalent Anosov flows on hyperbolic manifolds**

In this section we prove Theorem 1.3 using ideas developed above. Recall that two non-singular flows on a manifold are topologically equivalent if the one-dimensional foliations given by their flow lines are conjugate as foliations. We will consider examples of skew-Anosov flows obtained by lifting geodesic flow to a \(k\)-fold fiberwise cover of the unit tangent bundle of a hyperbolic surface (for large \(k\)), then performing integral Dehn surgery along a closed orbit. (We assume that the reader has some familiarity with hyperbolic Dehn surgery e.g. as described in [37], a brief description of surgery for flows is given below in paragraph [6.1].) Dehn surgery is a necessary component in the construction, as pointed out to us by Barbot in the following remark.

**Remark 6.1** (Lifts to a fiberwise cover give equivalent flows). Recall that the different lifts of geodesic flow on \(UT\Sigma\) to the \(k\)-fold fiberwise cover of \(UT\Sigma\) are determined by cohomology classes in \(H^1(UT\Sigma, \mathbb{Z}_k)\) that pair with the generator to give 1. It is straightforward to check that the group of fiberwise rotations (i.e. smooth gauge transformations of the cover), which can be identified with the group of smooth maps \(\Sigma \to U(1) = S^1\), acts transitively on this affine subspace of cohomology, and hence all covers of the geodesic flow are topologically equivalent. In fact, they are even equivalent by a diffeomorphism rather than homeomorphism of the ambient manifold. These flows are, however, in distinct isotopy classes, having distinct associated slithering actions.
To produce inequivalent flows after performing a Dehn surgery, we choose the orbits and surgery slopes so that the manifolds so obtained have no nontrivial symmetries. Precisely, we desire to perform surgery on a closed orbit $K$ (which we think of as a knot in the $k$-fold fiberwise cover of $M \to UT\Sigma$) with the property that any homeomorphism preserving $K$ up to isotopy is homotopically trivial as a homeomorphism of $M$.

**Asymmetric Knots.** The first ingredient is the following construction of highly asymmetric filling geodesics on surfaces.

**Lemma 6.2.** Let $c_0$ be the arc on the leftmost handle of the surface depicted in Figure 4 and let $\Sigma_g$ be a closed, oriented surface of genus $g \geq 3$. Then there is a simple closed curve $c$ on $\Sigma_g$ so that, for any hyperbolic metric on $\Sigma_g$, the geodesic representative $c_{geo}$ of $c$ has the following properties.

1. There is an embedded one-holed torus $T_0 \subset \Sigma_g$ and an isotopically trivial homeomorphism $T_0 \to T$ taking $T_0 \cap c_{geo}$ to $c_0$.
2. $c$ and $c_{geo}$ are filling, meaning the complimentary regions are disks.
3. The combinatorics (i.e. the self-intersection pattern) of $c_{geo}$ is independent of the choice of hyperbolic structure.
4. If $f$ is a finite order homeomorphism of $\Sigma_g$ so that $f(c_{geo})$ is isotopic to $c_{geo}$, then $f$ is the identity.

For the proof, it will be useful to have the following construction. Let $T$ be a one-holed torus, and $a, b$ curves on $T$ with intersection number 1; we identify these with standard generators of $\pi_1(T)$. Fix a hyperbolic metric on $T$ such that both the boundary curve of $T$ and the curve $a$ are very short in comparison to $b$. Let $c_{m,n}$ denote the geodesic representative of the curve $a^m b^n$. Our choice of metric ensuring that $a$ is a short curve means that, when $m \gg n$, the self-intersection pattern of $c_{m,n}$ is that depicted in Figure 6; i.e. the curve spirals $n$ times in the $b$ direction, followed by $m$ times in the $a$ direction, and its complimentary regions form a “grid” of quadrilaterals on the surface of size $m \times n$. (The assertion regarding the self-intersection pattern of this geodesic will also follow from the discussion below on the disk flow.) Though not needed for the proof here, in our application to the proof of Theorem 1.3, we will wish to choose $m$ and $n$ relatively prime.

**Proof of Lemma 6.2.** We first give an explicit construction, then prove that the claimed properties hold. Fix a pants decomposition $P = \{P_1, \ldots, P_{2g-2}\}$ of $\Sigma$ as in Figure 6 so that each of $P_1, \ldots, P_g$ is a pant decomposition of a one-holed torus subsurface. Fix a hyperbolic metric on $\Sigma_g$ so that for $i = 1, \ldots, g$, the pant $P_i$ is isometric to the torus $T$ from the construction above, where the pants cuffs are the short curves. Choose $m_i \gg n_i$ all distinct pairs, with $n_i \gg m_{i-1}$, and let $c$ be a curve constructed as follows. For $i = 1, \ldots, g$, have $c$ agree with $c_{m_i, n_i}$ on $P_i$. Between $P_i$ and $P_{i+1}$, have $c$ follow a geodesic segment from the end of $c_{m_i, n_i}$ on $P_i$ to the start of $c_{m_{i+1}, n_{i+1}}$, indexing cyclically. Our choice of metric ensures that that this curve $c$ is very close to a geodesic. This ensures that the intersection pattern of $c$ agrees with that of its geodesic representative in this metric. (This fact will also follow from our discussion of disk flow below.) We note for now that no complimentary region of $c$ is a monogon, bigon, or triangle.

Now fix any hyperbolic metric on $\Sigma_g$, and consider the geodesic representative $c_{geo}$ of $c$ in this metric. We claim that it has the same combinatorics as the curve $c$ depicted
in the figure. To see this, we use the disk flow of Hass and Scott defined in [26].

Starting with a curve in general position, this flow may change the combinatorics of
a curve via a) eliminating a monogon or bigon bounding a disk (thus decreasing
the self-intersection number of the curve) or b) moving one edge of a triangle across
the opposite vertex, preserving the self-intersection number. Hass and Scott show [26, Thm
2.1, 2.2] that any curve is homotopic to a representative with minimal self-intersection
number through this process, and that any two distinct representatives of a curve, each
having minimal self-intersection number, are homotopic to each other through moves
of type b) and ambient isotopy of the surface.

Since no complimentary regions of \( c \) are monogons or bigons moves of type a)
are not possible. Thus, \( c \) has minimal self-intersection number (although this was already
ensured by our choice of \( c \) as being close to its geodesic representative). Since no regions
are triangles, Hass and Scott’s theorem implies that \( c_{\text{geo}} \) which, as is well known, also
has minimal self-intersection number, must be attainable from \( c \) by ambient isotopy of
\( \Sigma \). This proves the first three assertions.

Now suppose that \( f \) is a finite order homeomorphism of \( \Sigma \), and \( h \) a homeomorphism
isotopic to identity such that \( hf(c_{\text{geo}}) = c_{\text{geo}} \), setwise. Then \( hf \) induces an automor-
phism of the graph on \( \Sigma \) formed by the image of \( c_{\text{geo}} \). We claim that this graph has no
nontrivial automorphisms. First observe that the “grids” of quadrilateral complimentary
regions on each \( P_i \) are characteristic of this graph and our choices of \( m_i, n_i \) were
all distinct, with \( n_i \gg m_{i-1} \), so each grid must be preserved. Now our specification
that \( g \geq 3 \) ensures that an automorphism preserving each such grid must fix it. Thus,
\( hf \), and hence \( f \), is isotopic to the identity, and since \( f \) was assumed finite order, it is
necessarily the identity.

Our next observation is that the complement of a geodesic in \( \text{UT} \Sigma \), as constructed
above, or the complement of a lift of such to a fiberwise cover, is a hyperbolic 3-
manifold. This will allow us to quote Thurston’s hyperbolization theorem on Dehn
fillings, as well as Mostow rigidity.

**Lemma 6.3** (Calegari/Folklore). Let \( c \subset \Sigma \) be a closed, filling geodesic in a hyperbolic
surface. Then the complement of its image in \( \text{UT} \Sigma \) is irreducible and atoroidal. More
generally, if \( M \rightarrow \text{UT}\Sigma \) is a \( k \)-fold fiberwise cover, and \( K \) a connected component the preimage of \( c \) in \( M \), then \( M - K \) is irreducible and atoroidal.

This is stated and proved in detail for the case where \( M = \text{UT}\Sigma \) in [17, Appendix B], where Foulon and Hasselblatt use it to construct examples of contact Anosov flows on hyperbolic manifolds. However, the proof carries over verbatim when the bundle \( \text{UT}\Sigma \rightarrow \Sigma \) is replaced by any finite fiberwise cover. See also [10] for an alternative exposition.

For the next two lemmas we will use the following set-up. Let \( M \rightarrow \text{UT}\Sigma \) be a \( k \)-fold fiberwise cover of \( \text{UT}\Sigma \) where \( \Sigma \) is a hyperbolic surface of genus \( g \geq 3 \). As in Lemma 6.3 above, let \( K \subset M \) be a connected component of the preimage of a geodesic \( c \) in \( \Sigma \), where \( c \) is chosen as in Lemma 6.2. The first lemma is an easy consequence of the construction of \( c \).

**Lemma 6.4.** With this set-up, if \( h \) is homotopic to a finite order homeomorphism of \( M \), and \( h(K) \) is isotopic to \( K \), then \( h \) is homotopic to the identity.

Note that \( h \) need not be equal to the identity, for instance it may rotate the fibers of \( M \rightarrow \Sigma \).

**Proof.** Consider the action of \( h \) on \( \pi_1 M \). This action preserves the center of \( \pi_1 M \), which is the fundamental group of the fiber, so descends to an action \( \overline{h} \) on \( \pi_1 \Sigma \) modulo inner automorphisms, i.e. \( \overline{h} \in \text{Out}(\pi_1 \Sigma) \). Since \( h \) is finite order, Nielsen realization implies that \( \overline{h} \) can be realized in the isometry group for some hyperbolic structure on \( \Sigma \). Since \( h \) preserves \( K \) up to isotopy, the isometry realizing \( \overline{h} \) preserves \( c \) up to free homotopy, so preserves the geodesic representative of \( c \) in this metric. Since \( c \) was chosen as in Lemma 6.2, this isometry is in fact trivial, so \( \overline{h} \) is a trivial outer automorphism of \( \pi_1 \Sigma \).

Now \( M \) is a \( K(\pi, 1) \) space, so homotopy classes of maps \( M \rightarrow M \) are determined by the action on the fundamental group; in particular homotopy classes of maps that induce the trivial outer automorphism of \( \pi_1 \Sigma \) can be identified with \( \text{Hom}(\pi_1 \Sigma, \mathbb{Z}) \), which is torsion free. Since \( h \) was assumed finite order, it must therefore be homotopic to the identity. \( \square \)

The following is the main technical result of this section.

**Lemma 6.5.** Let \( M_p \) denote the integral Dehn filling of \( M - K \) of slope \( p \). After excluding finitely many slopes, the following hold.

1. The manifold \( M_p \) is hyperbolic and each homeomorphism of \( M_p \) preserves the homotopy class of the core curve of the filling torus.
2. Each homeomorphism \( \phi \in \text{Homeo}_+(M_p) \) that preserves the core of the filling torus determines a homeomorphism \( \phi_M \) of \( M \) that agrees with \( \phi \) away from a tubular neighbourhood of \( K \), preserves the isotopy class of \( K \), and is homotopic to the identity.

**Remark 6.6.** The first point is true (after excluding finitely many slopes, of course) whenever \( K \) is obtained by lifting a filling geodesic on the surface. The fact that \( \phi_M \) in point (2) is homotopic to the identity comes from our choice of \( c \) from Lemma 6.2. We note also that the tubular neighborhood of \( K \) in the second point above may be chosen arbitrarily small.
Proof of Lemma 6.5. By Lemma 6.3 the complement $M - K$ is atoroidal and irreducible, and thus admits a complete hyperbolic metric by geometrisation. Fix this hyperbolic metric, and consider the action of the isometry group $\text{Isom}(M - K)$ on the fundamental group of the cusp, which we identify with $\mathbb{Z} \times \mathbb{Z}$ using generators coming from the meridian and longitude of a tubular neighborhood of $K$. Recall that, by Mostow–Prasad rigidity, $\text{Isom}(M - K)$ is finite. For each of the isometries whose action on $\mathbb{Z} \times \mathbb{Z}$ is not by $\pm I$, record any eigenspace of eigenvalue $\pm 1$. This gives us a collection of finitely many slopes, which we will exclude from the possible slopes of Dehn filling.

Since $M - K$ is hyperbolic, Thurston’s hyperbolisation theorem [37] states that, with finitely many exceptions, the result of Dehn filling $M - K$ is a hyperbolic manifold, in which the core of the filling torus is a closed geodesic of shortest length for this hyperbolic structure. Excluding these finitely many exceptional slopes as well, we claim that $M_p$ will have all the desired properties.

First, suppose that $\phi$ is a homeomorphism of $M_p$. By Mostow rigidity, it is homotopic to an isometry; denote this isometry by $\psi$. Since the core of the filling torus is a geodesic of shortest length, it is preserved by $\psi$, so $\phi$ preserves this curve up to homotopy, proving item (1).

For the second point observe that $\psi$ induces a homeomorphism of the cusped manifold $M - K$ preserving the unoriented isotopy class of a longitude given by the Dehn Surgery slope, which we identify with $K$. Again, by Mostow Rigidity, this homeomorphism is homotopic to an isometry, and by our restriction on the choices of slope $p$, we conclude that the action on the fundamental group of the cusp is by $\pm I$. This means also that $\psi$ can be extended to a homeomorphism, say $\psi_M$, of $M$ by coning off over meridian discs. Moreover, since an isometry of a complete hyperbolic manifold has finite order, we can assume that this extension also has finite order, since the extension over meridian discs preserves this property.

Suppose now that $\phi$ itself has the additional property that it preserves the core of the Dehn filling torus. By the same argument as above, $\phi$ then induces a homeomorphism of $M - K$ preserving the isotopy class of the longitude $K$ and inducing $\pm I$ on the fundamental group of the cusp. Thus, we can extend the action of $\phi$ over meridian discs by “coning off” to give a homeomorphism $\phi_M$ of $M$ preserving the isotopy class of $K$. Since $M$ is a $K(\pi, 1)$ space, the extension of any map over a tubular neighbourhood $N$ of $K$ is well-defined up to homotopy. Moreover, any homotopy of maps on $M - N$ extends to $M$. In particular, if $\psi$ is the isometry homotopic to $\phi$, using the notation as above, then $\phi_M$ is homotopic to $\psi_M$, which is finite order. By Lemma 6.4 we conclude that $\phi_M$ is homotopic to the identity. Finally, since $\phi$ preserves $K$, this homeomorphism $\phi_M$ can be obtained from $\phi$ by undoing the original Dehn surgery in an arbitrarily small neighborhood of $K$. □

6.1. Dehn Surgery and Anosov flows. Given any Anosov flow and a periodic orbit $\gamma$ Goodman [23] and Fried [20] have described how to perform integral Dehn surgery on $\gamma$ in a manner compatible with the flow, giving the following.

Proposition 6.7 (Dehn surgery on Anosov flows [20, 23]). Let $\Phi_t$ be an Anosov flow on a manifold $M$ and let $\gamma$ be a periodic orbit. Then the manifold $M_p(\gamma)$ obtained by integral surgery of slope $p$ admits an Anosov flow that is conjugate to the original flow away from the core of the filling torus.
Goodman’s original construction in [23] produces a smooth Anosov flow. Fried [20] gives an alternative construction which has, a priori, less regularity, but has the property that the dynamics of the flow after surgery are identical to those of the original flow in the complement of the periodic orbit given by the core of the Dehn filling torus. In outline, one simply blows up \( M \) along the normal bundle of the periodic orbit to obtain a manifold homeomorphic to the complement of a small open neighborhood of \( \gamma \) in \( M \), with a torus boundary to which the flow extends in a natural way, having four periodic orbits on the boundary. Choosing a foliation of the torus boundary by circles transverse to the flow, such that each circle leaf intersects each of the periodic orbits in a single point and identifying each circle to a point, one obtains a flow on an integral Dehn-filling of \( M - \gamma \) so that the core of the filling torus (the points obtained by collapsing circles) is a periodic orbit.

While the dynamics under Goodman’s construction are somewhat mysterious, in Fried’s version as described above it is obvious that any Dehn surgery can be undone, on the level of Anosov flows, by an inverse surgery. The drawback of Fried’s construction is that the flows he constructs are not obviously genuinely Anosov, they are only topologically Anosov. It has been largely assumed in the literature that both these surgeries produce topologically equivalent flows, so that in both cases one obtains flows that are Anosov in the usual sense. This has only recently been settled by Mario Shannon [36] for transitive flows, which includes as a special case surgery of skew-Anosov flows (the case of interest to us). This will be crucial in our construction.

The surgery construction as well as some of its properties have been analyzed by Fenley [15]. He shows in particular that surgery on certain Anosov flows produces skew-Anosov examples. We note this for future use.

**Proposition 6.8** (Dehn surgery on skew-Anosov flows [15]). *If the original flow is a cover of the geodesic flow on \( UT\Sigma \), then for \( p > 0 \) the flow on \( M_{-p}(\gamma) \) given by Dehn surgery of slope \( -p \) is skew-Anosov.*

**Constructing inequivalent Anosov flows.** Using the tools above, we now produce examples of hyperbolic 3-manifolds supporting \( N \) topologically inequivalent Anosov flows, proving Theorem 1.3. Recall that this will be done by performing Dehn surgery on fiberwise covers of the unit tangent bundle of a hyperbolic surface.

**Proof of Theorem 1.3.** Let \( \Sigma \) be a hyperbolic surface, with hyperbolic structure defined by a representation \( \rho : \pi_1 \Sigma \to PSL_2(\mathbb{R}) \). Fix some \( k \in \mathbb{N} \) dividing the Euler characteristic of \( \Sigma \), for concreteness one may take \( k = g - 1 \), where \( g \) is the genus of \( \Sigma \). We will give a construction that produces a number of inequivalent skew-Anosov flows via surgery on the \( k \)-fold cover of \( UT\Sigma \), where that number grows linearly in \( k \) (and hence can be taken as large as desired by taking \( g \) large).

Recall from the introduction to Section 5.4 that, for fixed \( k \), the lifts of \( \rho \) to the \( k \)-fold central extension of \( PSL_2(\mathbb{R}) \) are in bijective correspondence with \( \text{Hom}(\pi_1 \Sigma, \mathbb{Z}/k\mathbb{Z}) \), parametrized by the rotation numbers of a standard set of generators of \( \pi_1 \Sigma \). As discussed earlier, these lifts can also be distinguished by understanding the degree of projection to the fiber of horizontal lifts of closed curves from \( \Sigma \) to the suspension \( E_\rho \). Each lift defines an Anosov flow on the \( k \)-fold fiberwise cover of \( UT\Sigma \), (whose weak stable foliation is the suspension \( E_\rho \) of the lift \( \hat{\rho} \) of \( \rho \)) but, as noted in Remark 6.1 these are all topologically equivalent flows. To produce inequivalent flows, we will use
Dehn surgery along the natural lifts of a fixed filling geodesic as constructed in Lemma 6.2.

**Set-up and standing assumptions.** Let \( T \subset \Sigma \) be a one-holed torus and \( c \) a geodesic on \( \Sigma \) as in Lemma 6.2 with \( \alpha_1, \beta_1 \) the standard generators of \( \pi_1(T) \). Complete this to a standard generating set \( \alpha_2, \beta_2, \ldots, \alpha_g, \beta_g \) for \( \pi_1 \Sigma \). We will consider lifts of \( \rho \) that differ only on \( \alpha_1 \) and \( \beta_1 \), agreeing on all other generators.

Identifying the curve \( c \) with an element of \( \pi_1 \Sigma \), fix first a lift \( \hat{\rho} \) of \( \rho \) to \( \text{PSL}_2(\mathbb{R}) \) such that \( \hat{\rho}(c) \) has rotation number 0. Topologically, this corresponds to the fact that the “horizontal lift” of \( c \), meaning the pre-image of the geodesic \( c \) under the covering map \( E_{\hat{\rho}} \to \text{UT} \Sigma \), has \( k \) connected components, each one a periodic orbit of the lift of the geodesic flow to \( E_{\hat{\rho}} \). The following argument shows that there are in fact \( k \) choices for such lifts \( \hat{\rho} \); all of which agree on \( \alpha_i, \beta_i \) for \( i \geq 2 \): Recall that \( c \), as an element of \( \pi_1 \Sigma \), has the form \( w\alpha_1^i \beta_1^m \) where \( w \) is a word in \( \alpha_2, \beta_2, \ldots, \alpha_g, \beta_g \), and \( m \) and \( n \) may be chosen to be relatively prime. Varying the lifts of \( \rho(\alpha_1) \) and \( \rho(\beta_1) \), while preserving the chosen lifts of the other generators amounts to replacing \( \hat{\rho}(\alpha_1) \) with its composition with a rotation by \( \frac{2\pi p}{k} \), and \( \hat{\rho}(\beta_1) \) with its composition by some rotation of the form \( \frac{2\pi n}{k} \). This changes the rotation number of \( \hat{\rho}(c) \) by \( pm + qn \mod k \), which implies what we claimed.

We will further restrict our choice of lifts of \( \rho \) so that the horizontal lifts of \( c \) are all isotopic curves in the \( k \)-fold cover of \( \text{UT} \Sigma \). Following the discussion above, when \( k \) is large (compared to \( n \) and \( m \)), we may choose to vary over a small range of \( p \) and \( q \), so that the holonomies of the lifted representations remain \( C^\infty \) close to each other in \( \text{Hom}(\pi_1 \Sigma, \text{Diff}(S^1)) \). This will give us some number \( C(k) \) of lifts of \( c \) which are sufficiently close to each other to be isotopic, where \( C(k) \) grows linearly in \( k \). Note that the genus of \( \Sigma \) must increase as \( k \) increases (since \( k \) needs to divide \( \chi(\Sigma) \) for a lift to exist), but this does not pose any problem as we are performing modifications only over a fixed torus. Restricting to such lifts of \( \rho \), fix \( p = p(k) \in \mathbb{N} \) large enough so that Proposition 6.8 ensures that Dehn surgery of slope \(-p\) on any connected component of any horizontal lift of \( c \) for any one of this restricted class of lifts gives a skew-Anosov flow. Further restricting \( p \) if needed, we may also ensure that all homeomorphisms of the Dehn-surgered manifold preserve the free homotopy class of the core curve by Lemma 6.5.

The restriction we imposed on our lifts of \( \rho \) ensuring that connected components of lifts of \( c \) are always isotopic means that performing a slope \(-p\) Dehn surgery on any horizontal lift of \( c \) to any of the covers will produce diffeomorphic hyperbolic manifolds. Thus, the remainder of the proof is devoted to showing that the flows produced in this way are inequivalent whenever the lifts of \( \rho \) differ among our \( C(k) \) choices. This means that the Dehn-surgered manifold described above admits \( C(k) \) inequivalent skew-Anosov flows. For this, we need to describe the construction a bit more carefully, setting some more precise notation along the way.

**Proof of inequivalence of flows.** Fixing notation, let \( M \) denote the \( k \)-fold fiberwise cover of \( \text{UT} \Sigma \) and let \( \hat{\rho} \) and \( \hat{\rho}' \) be two lifts of \( \rho \) chosen so as to satisfy the restrictions imposed above. The manifold \( M \) is, topologically, both the suspension \( E_{\hat{\rho}} \) of \( \hat{\rho} \), and the suspension of \( \hat{\rho}' \). Since \( \hat{\rho}' \) is close to \( \hat{\rho} \) (because of our restrictions), we may, as usual, realize \( \hat{\rho}' \) as the holonomy of a foliation on \( M \) that is \( C^1 \) (and in this case actually \( C^\infty \))
close to the horizontal foliation defined by $E_{\hat{\rho}}$. Going forward, we let $E_{\hat{\rho}}$ denote $M$ equipped with this nearby foliation.

Fix a connected component $K$ of the horizontal lift of $c$ to $E_{\hat{\rho}}$, and a connected component $K'$ of the horizontal lift of $c$ to $E_{\rho}$, isotopic to $K$ in $M$. It will be convenient to fix an identification of $K$ and $K'$, so let $g : M \to M$ be an isotopically trivial homeomorphism such that $g(K) = K'$. Then $g\Phi_{t}g^{-1}$ and $\Phi_{t}$ each have $K$ as a periodic orbit. Now perform integral Dehn–Fried–Goodman surgery of slope $-p$ on the knot $K$ to modify the flow $\Phi_{t}$ to a new skew-Anosov flow $\Psi_{t}$ on the Dehn-surgered manifold $M_{-p}$, and separately perform integral Dehn–Fried–Goodman surgery of slope $-p$ on $K$ to modify $g\Psi_{t}g^{-1}$ to obtain a flow $\Psi'_{t}$ on $M_{-p}$. Note that the latter construction is simply the result of performing surgery on the knot $K'$ in $M$, under our identification of $K$ and $K'$ via $g$.

What we are required to show is that $\Psi_{t}$ and $\Psi'_{t}$ are inequivalent. Suppose for contradiction that this is not the case, so there is some homeomorphism $f : M_{-p} \to M_{-p}$ taking flowlines of $\Psi_{t}$ to flowlines of $\Psi'_{t}$. Let $\gamma$ denote the core of the filling torus on $M_{-p}$. By Lemma 6.5, $f(\gamma)$ and $\gamma$ lie in the same free homotopy class, so by Proposition 5.4, there is a homeomorphism $h$ of $M_{-p}$, inducing some power of $\eta$ on the flow space of $\Psi'_{t}$, such that $hf(\gamma) = \gamma$. So we now may as well consider $hf$ as the homeomorphism conjugating the foliations defined by the two flows.

Restricting $hf$ to $M_{-p} - \gamma$ defines a homeomorphism $\phi$ of $M - K$. As in Lemma 6.5, this determines a homeomorphism $\phi_{M}$ of $M$ agreeing with $\phi$ on the complement of a neighborhood of the end, a neighborhood which can be chosen as small as we wish. Choose such a neighborhood small enough so as not to contain any horizontal lift of (powers of) the curves $\alpha_{1}$ or $\beta_{1}$ to either $E_{\hat{\rho}}$ or to (the conjugate by $g$ of) $E_{\rho}$.

By Lemma 6.5 (2), the map $\phi_{M}$ is homotopic to the identity, and, by construction, outside of a small neighborhood of $K$, $\phi_{M}$ maps flowlines of $\Phi_{t}$ to those of $g\Phi'_{t}g^{-1}$. This is where we will derive a contradiction. Let $\hat{\alpha}_{1}$ and $\hat{\beta}_{1}$ denote the horizontal lifts of suitable powers of $\alpha_{1}$ and $\beta_{1}$ to closed orbits of $\Phi_{t}$. Then $\phi_{M}$, being homotopic to the identity, maps these to closed orbits of $g\Phi'_{t}g^{-1}$ that are freely homotopic to $\hat{\alpha}_{1}$ and $\hat{\beta}_{1}$, respectively. In particular, the projection of these orbits to curves on $\Sigma$ are freely homotopic to the corresponding powers $\alpha_{1}$ and $\beta_{1}$ (which must in particular agree). Hence $\phi_{M}(\hat{\alpha}_{1})$ and $\phi_{M}(\hat{\beta}_{1})$ are also the (conjugates under $g$ of) horizontal lifts of suitable powers of $\alpha_{1}$ and $\beta_{1}$ to closed orbits of $\Phi'_{t}$. But, by design, we chose $\hat{\rho'}$ to give these curves different rotation numbers than $\hat{\rho}$. By the discussion in Remark 5.11, this means that their horizontal lifts are not freely homotopic since they wind a different number of times around the fibers over the representative curves on $\Sigma$. This gives the desired contradiction.

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