Remarks on the extended Brauer quotient

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Abstract

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1. Introduction

Let $G$ be a group, and let $A$ be a $G$-algebra over a complete discrete valuation ring $\mathcal{O}$ with residue field $k$ of characteristic $p > 0$. The well-known Brauer quotient $A(P)$ with respect to a $p$-subgroup $P$ of $G$ (introduced by M. Broué and L. Puig, see [8, §11]) is an $N_G(P)$-algebra. If moreover, $A$ is $G$-interior (that is, $A$ is endowed with a unitary algebra homomorphism $\mathcal{O}G \to A$), then $A(P)$ becomes a $C_G(P)$-interior $N_G(P)$-algebra. This means that one may construct, as in [5, Chapter 9], the $N_G(P)/C_G(P)$-graded $N_G(P)$-interior algebra $A(P) \otimes_{C_G(P)} N_G(P)$, so $A(P)$ is extended by automorphisms of $P$ given by conjugation with elements of $G$.

L. Puig and Y. Zhou [6] extended $A(P)$ by all automorphisms of $P$, obtaining the so-called extended Brauer quotient $\tilde{N}_A^{\text{Aut}(P)}(P)$ as an $N_G(P)$-interior $k$-algebra. The interiority assumption is necessary, because the main feature used is the $\mathcal{O}(P \times P)$-module structure of $A$. This construction was further generalized by T. Coconet and C.-C. Todea [3] to the case of $H$-interior $G$-algebras, where $H$ is a normal subgroup of $G$.

Our aim here is to unify and generalize these constructions, by introducing an extended Brauer quotient of a group graded algebra. The main ingredients of our construction are
a $\tilde{G}$-graded algebra $A$, a group homomorphism $P \to \tilde{G}$ (which induces a $\tilde{G}$-grading on the group algebra $\mathcal{O}P$), and a homomorphism $\mathcal{O}P \to A$ of $\tilde{G}$-graded algebras.

In Section 2 below we recall the Puig and Zhou definition of $N_{A}^{\text{Aut}(P)}(P)$, pointing out its $\text{Aut}(P)$-graded algebra structure. Our alternative construction in Section 3 is based on the easy observation that if $A$ is a $\tilde{G}$-graded $P$-algebra with identity component $B$ such that the action of $P$ on $\tilde{G}$ is trivial, then the Brauer quotient $A(P)$ inherits the $\tilde{G}$-grading such that the identity component of $A(P)$ is $B(P)$. Here we apply the classical Brauer quotient to the $\text{Aut}(P)$-graded algebra $A = A \otimes _{\mathcal{O}} \text{Aut}(P)$, and we get that $A(P)$ is isomorphic to $N_{A}^{\text{Aut}(P)}(P)$ as $\text{Aut}(P)$-graded algebras. In Section 4 we construct the extended Brauer quotient of a $\tilde{G}$-graded $P$-interior algebra $A$ as mentioned above, this time with $P$ acting nontrivially on $\tilde{G}$. We also discuss the exact relationship to the construction from [3]. Section 5 investigates the extended Brauer quotient of tensor products of $P$-interior algebras, in Section 6 we give an application towards correspondences for covering blocks.

Our general notations and assumptions are standard, and closely follow [8], [5] and [4].

2. The extended Brauer quotient

2.1. The construction of Puig and Zhou

We begin with a $p$-group $P$ and a $P$-interior algebra $A$. Let $\varphi \in \text{Aut}(P)$, and as in [6], we consider the $\varphi$-twisted diagonal

$$\Delta_{\varphi}(P) = \{(u, \varphi(u)) \mid u \in P\}.$$ 

Then the set of $\Delta_{\varphi}(P)$-fixed elements, is the following $\mathcal{O}$-submodule of $A$:

$$A^{\Delta_{\varphi}(P)} = \{a \in A \mid ua = a\varphi(u) \text{ for any } u \in P\}.$$ 

Further, we consider $Q < P$ and denote by $A_{\Delta_{\varphi}(Q)}^{\Delta_{\varphi}(P)}$ the $\mathcal{O}$-module consisting of elements of the form

$$\text{Tr}_{\Delta_{\varphi}(Q)}^{\Delta_{\varphi}(P)}(c) = \sum_{u \in [P/Q]} u^{-1} c\varphi(u),$$

where $c \in A^{\Delta_{\varphi}(Q)}$. At last, we denote by $A(\Delta_{\varphi}(P))$ the quotient

$$A(\Delta_{\varphi}(P)) = A^{\Delta_{\varphi}(P)}/ \sum_{Q < P} A_{\Delta_{\varphi}(Q)}^{\Delta_{\varphi}(P)}.$$
and we obtain the usual Brauer homomorphism
\[ \text{Br}_{\Delta \phi}(P) : A^{\Delta \phi(P)} \to A(\Delta \phi(P)). \]

If \( K \) is a subgroup of \( \text{Aut}(P) \), it is easily checked that the external direct sum \( \bigoplus_{\phi \in K} A^{\Delta \phi(P)} \) is an algebra, while its subset \( \bigoplus_{\phi \in K} \sum_{Q \prec P} A^{\Delta \phi(Q)} \) is a two-sided ideal, hence we have the following definition.

**Definition 2.1** ([6]). The *extended Brauer quotient* associated to the \( P \)-interior algebra \( A \) and the subgroup \( K \) of \( \text{Aut}(P) \) is the external direct sum
\[
\tilde{N}^K_A(P) := \bigoplus_{\phi \in K} A^{\Delta \phi(P)} / \bigoplus_{\phi \in K} \sum_{Q \prec P} A^{\Delta \phi(Q)} \simeq \bigoplus_{\phi \in K} A(\Delta \phi(P)).
\]

**Remark 2.2.** Note that in this case, one deduces easily from the details given in [6, Section 3] and [7, Section 3] that \( \tilde{N}^K_A(P) \) is a \( K \)-graded algebra, and the map \( \text{Br}_P := \bigoplus_{\phi \in K} \text{Br}_{\Delta \phi(P)} \) is a homomorphism of \( K \)-graded algebras. This fact will become even more transparent in the next section.

### 2.2. The case of \( G \)-interior algebras

In addition to the situation of subsection 2.1, we assume the \( A \) is a \( G \)-interior algebra, where \( G \) is a (not necessarily finite) group, and \( P \) is a \( p \)-subgroup of \( G \). Conjugation induces the group homomorphisms
\[ N_G(P) \to \text{Aut}(P) \quad \text{and} \quad N_G(P)/C_G(P) \to \text{Aut}(P), \quad (1) \]
and for the subgroup \( K \) in \( \text{Aut}(P) \), \( N^K_G(P) \) denotes the inverse image of \( K \) in \( N_G(P) \). If \( x \in N_G(P) \), we use denote by \( \phi_x \) the automorphism of \( P \) given by \( \phi_x(u) = u^x = x^{-1}ux \) for all \( u \in P \).

In this setting, we obtain some additional properties of the extended Brauer quotient (the details are left to the reader).

**Proposition 2.3.** With the above notation, the following statements hold:

1) \( \tilde{N}^K_A(P) \) is a \( K \)-graded \( N^K_G(P) \)-interior algebra;
2) If \( K = N_G(P)/C_G(P) \), then we have the isomorphism
\[ \tilde{N}^K_A(P) \simeq A(P) \otimes_{kC_G(P)} kN_G(P) \]
of \( N_G(P)/C_G(P) \)-graded \( N_G(P) \)-interior algebras.
Proof. 1) We only need to notice that any \( x \in \mathbb{N}_K(P) \) verifies \( u^{-1}x\varphi_x(u) = x \).

2) We define the \( N_G(P)/C_G(P) \)-graded map

\[
A(P) \otimes_{kC_G(P)} kN_G(P) \to \tilde{N}_A^K(P), \quad \tilde{a} \otimes x \mapsto \tilde{a}x,
\]

whose restriction to the identity component is an isomorphism. \( \square \)

Remark 2.4. Note that if \( K = N_G(P)/C_G(P) \), then \( \tilde{N}_A^K(P) \) is just the group algebra \( kN_G(P) \) considered with the obvious \( K \)-grading. Moreover, the construction of \( \tilde{N}_A^K(P) \) is clearly functorial in \( A \), so the \( N_G(P) \)-interior algebra structure of \( \tilde{N}_A^K(P) \) comes from applying the construction to the algebra map \( \mathcal{O}G \to A \).

3. An alternative construction

3.1. The \( \mathcal{O}P \)-interior algebra \( A \) admits an obvious \( (\mathcal{O}P, \mathcal{O}P) \)-bimodule structure. Consider the group algebra \( \mathcal{O}[P \rtimes \text{Aut}(P)] \), of the semidirect product \( P \rtimes \text{Aut}(P) \). This algebra is also a left \( \mathcal{O}P \)-module, hence it makes sense to consider the \( \text{Aut}(P) \)-graded \( (\mathcal{O}P, \mathcal{O}P) \)-bimodule

\[
\tilde{A} := A \otimes_{\mathcal{O}} \mathcal{O}(P \rtimes \text{Aut}(P)).
\]

We may also use the isomorphism

\[
\tilde{A} \simeq A \otimes_{\mathcal{O}} \mathcal{O} \text{ Aut}(P)
\]

of \( \mathcal{O} \)-modules, which becomes an isomorphism of \( (\mathcal{O}P, \mathcal{O}P) \)-bimodules, by defining the bimodule structure of \( A \otimes_{\mathcal{O}} \mathcal{O} \text{ Aut}(P) \) as follows:

\[
u(a \otimes \phi)v = u \cdot a \cdot \phi(v) \otimes \phi,
\]

for \( u, v \in P \) and \( \phi \in \text{Aut}(P) \). Then we regard \( A \otimes_{\mathcal{O}} \mathcal{O} \text{ Aut}(P) \) as an \( \text{Aut}(P) \)-graded \( P \)-algebra with \( P \)-action given by

\[
(a \otimes \phi)^u = u^{-1} \cdot a \cdot \phi(u) \otimes \phi,
\]

With the notations of Sections 2 and 3 we have:

Theorem 3.2. There is an isomorphism

\[
\tilde{A}(P) \simeq \tilde{N}^{\text{Aut}(P)}_A(P)
\]

of \( \text{Aut}(P) \)-graded algebras, where \( \tilde{A}(P) \) is the usual Brauer quotient of \( \tilde{A} \).
Proof. As the $p$-group $P$ is a normal subgroup of $P \rtimes \text{Aut}(P)$, we get the decomposition
\[
\tilde{A}(P) = \bigoplus_{\varphi \in \text{Aut}(P)} (A \otimes (1, \varphi))(P).
\]
If $a \otimes (1, \varphi) \in (A \otimes (1, \varphi))^P$, then
\[
u^{-1} \cdot a \otimes (1, \varphi) \cdot u = u^{-1} \cdot a \otimes (1, \varphi)(u, 1) = u^{-1} a \varphi(u) \otimes (1, \varphi) = a \otimes (1, \varphi).
\]
Then $a \in A^{\Delta \varphi(P)}$, and consequently
\[(A \otimes (1, \varphi))(P) \rightarrow \tilde{N}^\varphi_A(P), \quad a \otimes (1, \varphi) \mapsto \tilde{a},
\]
is a well-defined map of $\mathfrak{O}$-modules for every $\varphi \in \text{Aut}(P)$. We extend this map to a $\text{Aut}(P)$-graded map between these two modules and we notice that, with all the above identifications, it is actually an isomorphism of algebras.

Remark 3.3. We often use subgroups of $P$, and we obviously have the isomorphism
\[(A \otimes \mathfrak{O}_Q \mathfrak{O}[Q \rtimes K])(Q) \simeq \tilde{N}^K_A(Q)
\]
of $K$-graded algebras, for any subgroups $Q \leq P$ and $K \leq \text{Aut}(Q)$.

4. The extended Brauer quotient of a group graded algebra

In this paragraph we set $\tilde{G} := G/H$, where $H$ is a normal subgroup of the finite group $G$, $P$ is a $p$-subgroup of $G$, and let
\[A := B \otimes_{\mathfrak{O}H} \mathfrak{O}G
\]
for some $H$-interior $G$-algebra $B$, so $A$ is the $G$-interior $\tilde{G}$-graded algebra induced from $B$.

The following lemma says that we restrict ourselves, without loss, to a certain subgroup of $\text{Aut}(P)$.

Lemma 4.1. Let $\varphi \in \text{Aut}(P)$, and let $O(\tilde{x})$ be the orbit of $\tilde{x} \in \tilde{G}$ under the action of $\Delta \varphi(P)$ on $\tilde{G}$. If $| O(\tilde{x}) | \neq 1$ then $(\bigoplus_{z \in O(\tilde{x})} B \otimes z)(\Delta \varphi(P)) = 0$.

Proof. Consider the element $a = \sum b_{z_i} \otimes z_i$ such that $u^{-1} a \varphi(u) = a$. Since the elements $z_i$ are all representatives of the classes of an orbit, we can choose them such that for any $u \in P$ we obtain $u^{-1} z_i \varphi(u) = z_{i'}$. It follows that $b_{z_i}^u = b_{z_{i'}}$, and then there is one element, say $b_z$, such that $b_{z_i}^u = b_z$ for any $i$ and any $u \in P$. Hence
\[a = \text{Tr}_{\Delta \varphi(Q)}(b_z \otimes z),
\]
where $Q$ is the stabilizer of $b_z \otimes z$ in $P$. \qed
4.2. The above lemma gives the motivation to introduce two subgroups of \( \text{Aut}(P) \), because it implies that \((A \otimes \phi)(P) = 0\) for \( \phi \in \text{Aut}(P) \) not satisfying \( \phi(u) = \bar{u}g \) in \( \bar{G} \), for some \( g \in G \). So let

\[
\text{Aut}_\bar{G}(P) = \{ \phi \in \text{Aut}(P) \mid \phi(u) = \bar{u}g \text{ for some } \bar{g} \in \bar{G} \text{ and for any } u \in P \}
\]

and

\[
\text{Aut}_1(P) = \{ \phi \in \text{Aut}(P) \mid \phi(u) = \bar{u} \text{ for any } u \in P \}.
\]

Denote also

\[
K := \text{Aut}_\bar{G}(P), \quad K_1 := \text{Aut}_1(P),
\]

and let \( \bar{A} := A \otimes \sigma_P \mathcal{O}[P \rtimes K] \) as in Section 3.

Finally, let \( N^K_G(\bar{P}) \) denote the subgroup of \( N_G(\bar{P}) \) whose elements define an element of \( K \) and let \( U \) be the inverse image of \( N^K_G(\bar{P}) \) in \( G \). Also let \( U' \) be the inverse image of \( C_G(\bar{P}) \) in \( G \). Observe that \( N^K_G(P) = N_G(P) \) and \( N^K_1(P) = N_G(P) \cap U' \).

**Lemma 4.3.** The group \( \text{Aut}_1(P) \) is a normal subgroup of \( \text{Aut}_\bar{G}(P) \), hence \( U' \) is normal in \( U \). Furthermore, we have the isomorphisms

\[
\text{Aut}_\bar{G}(P)/\text{Aut}_1(P) \simeq N^K_G(\bar{P})/C_G(\bar{P}) \simeq U/U'.
\]

**Proof.** If \( \phi_1 \in \text{Aut}_1(P) \) then \( \phi(u) = \bar{u} \) for all \( u \in P \). Hence, if \( \phi \in \text{Aut}_\bar{G}(P) \) with \( \phi(u) = \bar{u}g \), we get

\[
(\phi^{-1} \circ \phi_1 \circ \phi)(u) = (\phi_1 \circ \phi)(u)g^{-1} = \phi(u)g^{-1} = \bar{u}.
\]

Further if \( x \in \bar{g} \) then \( \phi(u) = \bar{u}g = \bar{u}x \) and then \( g^{-1}x \in C_G(\bar{P}) \). With all of the above, the map

\[
\text{Aut}_\bar{G}(P) \ni \phi \mapsto \bar{g} \in N_G(\bar{P})
\]

gives the first isomorphism. The second isomorphism is obvious. \( \square \)

We will denote by \( \bar{\phi} \) the image of \( \phi \) in the quotient group \( \text{Aut}_\bar{G}(P)/\text{Aut}_1(P) \).

**Theorem 4.4.** The algebra \( \bar{N}^\text{Aut}(P)_A \), as constructed in [2.1] is the \( U/U' \)-graded \( N_G(P) \)-interior algebra \( \bar{N}^K_P \) with identity component the \( N^K_1(P) \)-interior algebra

\[
\bar{N}^K_1(P) = \bigoplus_{g' \in U/H} \bar{N}^K_{B \otimes g'}(P),
\]

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and for any $\tilde{g} \in U/U'$ (corresponding to $\tilde{\phi}$), the $\tilde{g}$-component is

$$\tilde{N}^K_A(P)_{\tilde{g}} = \bigoplus_{\varphi \in \tilde{\phi}, \tilde{z} \in \tilde{g}} (B \otimes z)(\Delta_{\varphi}(P)),$$

where $\varphi \in \tilde{\phi}$ satisfies $\overline{\varphi(u)} = \overline{u^g}$, for any $u \in P$.

**Proof.** By Lemma 4.1, we obtain the following decomposition of the extended Brauer quotient

$$\tilde{N}^{Aut(P)}_A(P) = \left( \bigoplus_{\phi \in K_1} \tilde{N}^\phi_A(P) \right) \oplus \tilde{N}^{K\setminus K_1}_A(P)$$

$$= \left( \bigoplus_{\tilde{g} \in U'/H} \tilde{N}^{K_1}_{B \otimes \tilde{g}}(P) \right)$$

$$\oplus \bigoplus_{\phi \in K/K_1} \left( \bigoplus_{\varphi \in \tilde{\phi}, \tilde{z} \in \tilde{g}} (B \otimes z)(\Delta_{\varphi}(P)) \right),$$

where in the second sum $\tilde{\phi}$ corresponds to $\tilde{g}$.

We see that, for any $\tilde{g}$ and any $\tilde{z} \in \tilde{g}$,

$$B \otimes z = B \otimes z\cdot x^{-1} \cdot x = (B \otimes z\cdot x^{-1}) \cdot (B \otimes x),$$

for any $x \in U'$. Then

$$\tilde{N}^K_A(P)_{\tilde{g}} \cdot \tilde{N}^{K_1}_A(P) = \tilde{N}^{K_1}_A(P) \cdot \tilde{N}^K_A(P)_{\tilde{g}} = \tilde{N}^K_A(P)_{\tilde{g}}.$$

The fact that this algebra is $N_G(P)$-interior is immediate since for any $x \in N_G(P)$ the element $1 \otimes x$ is $\Delta_{\varphi}(P)$-invariant.

**Remark 4.5.** 1) The fact that in the above theorem every $\tilde{g}$-component of $\tilde{N}^K_A(P)$ is a direct sum suggests that this algebra actually has a finer grading than stated. Indeed, it is not difficult to see that $\tilde{N}^{K_1}_A(P)$ is graded by the group

$$\tilde{K}_1 := \{ (\phi, \tilde{g}) \mid \phi \in K_1, \tilde{g} \in U'/H \text{ such that } \overline{\phi(u)} = \overline{u^g} \},$$

and in general, $\tilde{N}^K_A(P)$ is graded by the group

$$\tilde{K} := \{ (\phi, \tilde{g}) \mid \phi \in K, \tilde{g} \in U/H \text{ such that } \overline{\phi(u)} = \overline{u^g} \}.$$
2) Applying the construction to the group algebra $\mathcal{O}G$ yields $\tilde{N}_{\mathcal{O}G}^K(P) = kN_G(P)$. The map
\[ N_G(P) \to \tilde{K}, \quad g \mapsto (\phi_g, \tilde{g}), \]
where $\phi_g(u) = gu^{-1}$, is a group homomorphism with kernel $C_H(P)$. The $\tilde{G}$-graded algebra map $\mathcal{O}G \to A$ induces by functoriality the $\tilde{K}$-graded algebra map
\[ kN_G(P) \to \tilde{N}_{A}^K(P). \]

3) Observe finally that the construction of $\tilde{N}_{A}^K(P)$ does not require the $G$-interiority of $A$. We only need a $\tilde{G}$-graded algebra $A$, a group homomorphism $P \to \tilde{G}$ inducing a $\tilde{G}$-grading on the group algebra $\mathcal{O}P$, and a homomorphism $\mathcal{O}P \to A$ of $\tilde{G}$-graded algebras.

4.6. Next, we establish the connection between $\tilde{N}_{A}^K(P)$ $\simeq \tilde{A}(P)$ and the extended Brauer quotient $\tilde{N}_{B}^{K_1}(P)$ of the $H$-interior $G$-algebra $B$, introduced in [3]. Recall that $\tilde{N}_{B}^{K_1}(P)$ is a $N_H^{K_1}(P)$-interior $N_G(P)$-algebra constructed formally as in Section 1 above. One can easily see from the definition in [3, Section 2] that $\tilde{N}_{B}^{K_1}(P)$ is actually a $K_1$-graded $N_G^{K_1}(P)$-interior $N_G(P)$-algebra.

Let $Q := P \cap H$. Then, as in Section 3, let
\[ \tilde{B} := B \otimes_{\mathcal{O}Q} \mathcal{O}(Q \rtimes K_1) \simeq B \otimes_{\mathcal{O}} \mathcal{O}K_1. \]

**Proposition 4.7.** The $\mathcal{O}$-module $\tilde{B}$ is a $\mathcal{O} \Delta(P \times P)$-module via
\[
(b \otimes (1, \varphi))(u, v) = b^u \otimes v^{-1} \cdot (1, \varphi) \cdot u = b^u \otimes (u^{-1} \varphi(u), \varphi) = b^u u^{-1} \varphi(u) \otimes (1, \varphi),
\]
for any $u \in P, b \in B$ and $\varphi \in K_1$. Furthermore, we have the isomorphism
\[ \tilde{B}(\Delta(P \times P)) \simeq \tilde{N}_{B}^{K_1}(P) \]

of $K_1$-graded $N_G^{K_1}(P)$-interior $N_G(P)$-algebras with identity component $B(P)$.

**Proof.** It is clear that for any $\varphi \in K_1$ we have $\varphi(u) \in Q$ for any $u \in Q$, hence $K_1$ acts on $Q$ and $\tilde{B}$ is a well-defined $\Delta(P \times P)$-module and we have
\[ \tilde{B}(\Delta(P \times P)) = \bigoplus_{\varphi \in K_1} (B \otimes (1, \varphi))(\Delta(P \times P)). \]

For any $\varphi \in K_1$ the map
\[
(B \otimes (1, \varphi))(\Delta(P \times P)) \ni b \otimes (1, \varphi) \mapsto \tilde{b} \in \tilde{N}_{B}^{\varphi}(P)
\]
is an isomorphism of $k$-vector spaces. The direct sum of these maps is the required algebra isomorphism. \qed
Remark 4.8. 1) According to Theorem 3.2 and Theorem 4.4, we have the decompositions
\[
\tilde{\mathcal{A}}(P) \simeq \tilde{\mathcal{N}}_1^K(P) = \tilde{\mathcal{N}}_B^{K_1}(P) \oplus \tilde{\mathcal{N}}_A^{K\setminus K_1}(P)
\]
\[
= \tilde{\mathcal{N}}_B^{K_1}(P) \oplus \left( \bigoplus_{\bar{x} \in U'/H, \, x \notin H} \tilde{\mathcal{N}}_B^K(P) \right) \oplus \tilde{\mathcal{N}}_A^{K\setminus K_1}(P).
\]

The above statements show that the \(N_1^K(P)\)-interior algebra \(\tilde{\mathcal{B}}(P)\) can be identified with a unitary subalgebra of \(\tilde{\mathcal{A}}(P)\), and even of \(\tilde{\mathcal{N}}_A^K(P)\), such that the \(N_1^G(P)\)-action and the \(K_1\)-grading are preserved. For the particular case of the \(H\)-interior \(G\)-invariant group algebra \(B = \mathcal{O}_H\), the component \(\tilde{\mathcal{N}}_B^{K_1}(P)\) is the \(N_1^G(P)\)-algebra studied in [2, Section 5].

2) The Brauer quotient \(B(P)\) of \(B\) is a \(C_H(P)\)-interior \(N_1^G(P)\)-algebra. The argument of Proposition 2.3 implies that the induced algebra
\[
B(P) \otimes_{kC_H(P)} kN_1^G(P)
\]
is isomorphic to a \(\tilde{\mathcal{K}}\)-graded subalgebra of \(\tilde{\mathcal{N}}_A^K(P)\), while
\[
B(P) \otimes_{kC_H(P)} kC_G(P)
\]
is isomorphic to a \(\tilde{\mathcal{K}}_1\)-graded subalgebra of \(\tilde{\mathcal{N}}_B^K(P)\).

5. Tensor products of algebras

Recall that if \(A\) and \(A'\) are two \(G\)-graded algebras, then the diagonal subalgebra of the \(G \times G\)-graded algebra \(A \otimes A'\) is the \(G\)-graded subalgebra
\[
\Delta(A \otimes A') := \bigoplus_{g \in G} (A_g \otimes A'_g).
\]
The following result is an extension of [6, Proposition 3.9]

Theorem 5.1. Assume that \(A\) and \(A'\) are two \(G\)-interior algebras such that \(A'\) has a \(P \times P\)-invariant \(\mathcal{O}\)-basis, and let \(K\) be a subgroup of \(\text{Aut}(P)\).

1) There is an isomorphism
\[
\tilde{\mathcal{N}}_A^K(P) \otimes_{\mathcal{O}P} \tilde{\mathcal{N}}_A^{K'}(P) \simeq \Delta(\tilde{\mathcal{N}}_A^K(P) \otimes_k \tilde{\mathcal{N}}_A^{K'}(P))
\]
of \(K\)-graded \(N_1^K(P)\)-interior algebras.
2) Assume in addition that, as \( K \)-graded \( N_G^K(P) \)-interior algebras,

\[
\tilde{N}_A^K(P) \simeq A(P) \otimes_k kK.
\]

Then

\[
\bar{N}_{A \otimes \sigma A'}^K(P) \simeq A(P) \otimes_k \bar{N}_{A'}^K(P)
\]

as \( K \)-graded \( N_G(P) \)-interior algebras.

Proof. 1) We consider the \( K \times K \)-graded \( N_G^K(P) \)-interior algebra

\[
\tilde{N}_A^K(P) \otimes_k \tilde{N}_{A'}^K(P) = \bigoplus_{\phi, \psi \in K} \tilde{N}_A^\phi(P) \otimes_k \tilde{N}_{A'}^\psi(P),
\]

whose diagonal subalgebra

\[
\Delta(\tilde{N}_A^K(P) \otimes_k \tilde{N}_{A'}^K(P)) = \bigoplus_{\phi \in K} \tilde{N}_A^\phi(P) \otimes_k \tilde{N}_{A'}^\phi(P)
\]

is an \( N_G^K(P) \)-interior \( K \)-graded algebra. Due to the inclusion

\[
A^\Delta_{\phi}(P) \otimes_{\sigma} (A')^\Delta_{\phi}(P) \subseteq (A \otimes_{\sigma} A')^\Delta_{\phi}(P),
\]

we obtain an \( \sigma \)-module map

\[
A^\Delta_{\phi}(P) \otimes_{\sigma} (A')^\Delta_{\phi}(P) \to \tilde{N}_A^\phi(P)
\]

sending \( a \otimes a' \) to \( \overline{a \otimes a'} \). If \( c \in A^\Delta_{\phi}(Q) \) and \( c' \in (A')^\Delta_{\phi}(R) \), for some subgroups \( Q \) and \( R \) of \( P \) then

\[
\text{Tr}_{\Delta_{\phi}(Q)}(c) \otimes \text{Tr}_{\Delta_{\phi}(R)}(c') = \text{Tr}_{\Delta_{\phi}(Q)}(c \otimes \text{Tr}_{\Delta_{\phi}(R)}(c')) = \text{Tr}_{\Delta_{\phi}(Q)}(\text{Tr}_{\Delta_{\phi}(Q)}(c) \otimes c') \\
\subseteq (A \otimes_{\sigma} A')^\Delta_{\phi}(P) \cap (A \otimes_{\sigma} A')^\Delta_{\phi}(Q).
\]

This determines an \( \sigma \)-module homomorphism

\[
\tilde{N}_A^\phi(P) \otimes \tilde{N}_{A'}^\phi(P) \to \tilde{N}_{A \otimes \sigma A'}^\phi(P), \quad \bar{a} \otimes \bar{a}' \mapsto \overline{a \otimes a'}
\]
for every \( \phi \in K \). The direct sum of all these homomorphisms is a \( K \)-graded algebra homomorphism between \( \Delta(\tilde{N}_A^K(P) \otimes_k \tilde{N}_{A'}^K(P)) \) and \( \tilde{N}_A^K(P) \otimes_{\tilde{G}} \tilde{N}_{A'}^K(P) \), which is in fact an isomorphism of interior \( N_G^K(P) \)-algebras since by our assumptions we have

\[
(A \otimes_{\tilde{G}} A')(P) \simeq A(P) \otimes_k A'(P).
\]

2) By the additional assumption we obtain

\[
\Delta(\tilde{N}_A^K(P) \otimes_k \tilde{N}_{A'}^K(P)) = \bigoplus_{\phi \in K} (A(P) \otimes_k k \phi) \otimes \tilde{N}_{A'}^\phi(P).
\]

We define the \( k \)-linear map

\[
A(P) \otimes_k \tilde{N}_{A'}^\phi(P) \to (A(P) \otimes_k k \phi) \otimes \tilde{N}_{A'}^\phi(P), \quad \tilde{a} \otimes \tilde{a}' \mapsto (\tilde{a} \otimes \phi) \otimes \tilde{a}',
\]

for every \( \phi \in K \). The sum of these maps determine the required isomorphism of \( K \)-graded interior \( N_G(P) \)-algebras between \( A(P) \otimes_k \tilde{N}_A^K(P) \) and \( \Delta(\tilde{N}_A^K(P) \otimes_k \tilde{N}_{A'}^K(P)) \).

**Remark 5.2.** 1) Statement 2) of the previous theorem applies in the situation of [6, Proposition 3.8]. More precisely, let

\[
A = \text{End}_{\tilde{G}}(N)
\]

for an indecomposable endopermutation \( \tilde{G}P \)-module \( N \), such that \( A(P) \neq 0 \). Let \( Q \leq P \), and let \( \delta \) be the unique local point of \( Q \) on \( A \). Let \( K := F_{\bar{A}}(Q_\delta) \). Then [6, Proposition 3.8] says that there is an isomorphism

\[
\tilde{N}_A^K(Q) \simeq A(Q) \otimes_k kK
\]

of \( N_G^K(Q) \)-interior \( K \)-graded algebras.

2) Assume in addition that \( A' \) is \( \tilde{G} \)-graded \( G \)-interior as in Section 4, and has a \( P \times P \)-invariant \( \tilde{G} \)-basis consisting of \( \bar{G} \)-homogeneous elements. Then, by Remark 4.5, the isomorphism in Theorem 5.1 2) is in fact an isomorphism of \( \tilde{K} \)-graded \( N_G(P) \)-interior algebras.

**6. A correspondence for covering points**

In this section we establish a correspondence between covering points in the case of a \( G \)-interior algebra that has a stable basis.
6.1. We keep the notations of Section 4 and we assume that the $G$-interior $\tilde{G}$-graded algebra $A := B \otimes_{\mathcal{O}H} \mathcal{O}G$ has a $G \times G$-stable $\mathcal{O}$-basis consisting of $\tilde{G}$-homogeneous elements. Further, we assume that $A$ is projective regarded as a left and as a right $\mathcal{O}G$-module. By these assumptions, $B$ is an $H$-interior permutation $G$-algebra, and it is projective both as a left and a right $\mathcal{O}H$-module.

6.2. We fix a normal subgroup $N$ of $G$ that contains $P$ and a point $\beta$ of $N$ on $B$ with defect group $P$. Then our assumptions and [3, Theorem 3.1] imply that $\tilde{\beta} := \text{Br}_P(\beta)$ is a point of $N_N(P)$ on $\tilde{B}(P)$ with defect group $P$.

We adopt here the definition of covering points from [1]. We say that the point $\alpha$ of $G$ on $A$ covers $\beta$ if $\alpha$ has defect group $P$ and for any $i \in \alpha$ there is an idempotent $j_1 \in A^N$ that lies in the conjugacy class of $\beta$ and there is a primitive idempotent $f \in A^N$ belonging to a point with defect group $P$ such that $j_1 f = f j_1 = f$ and $i f = f i = f$.

Clearly in this case we consider a particular setting in which a defect group of the points that are covered coincides with a defect group of the points that cover the given ones.

Now we can state our result.

**Theorem 6.3.** The Brauer homomorphism

$$\text{Br}_P : A^P \rightarrow A(P)$$

determines a bijective correspondence between the points of $A^G$ with defect group $P$ that cover $\beta$ and the points of $\tilde{A}(P)^{N_G(P)}$ with defect group $P$ that cover $\tilde{\beta}$.

**Proof.** Theorem 3.2 and [6, Proposition 3.3] already provide a bijection between the points of $G$ on $A$ and the points of $N_G(P)$ on $\tilde{A}(P)$ with the same defect group $P$. Even more, this bijection coincides with the bijection determined by the epimorphism given by the restriction of the Brauer homomorphisms

$$\text{Br}_P : A^G_P \rightarrow A(P)^{N_G(P)}_P.$$

Since $N$ is normal in $G$, hence $N_N(P)$ is also normal in $N_G(P)$, the fact that this bijection restricts to a bijection between the points that cover $\beta$ and $\tilde{\beta}$ is an easy verification given by [1, Theorem 3.5].

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