STU Black Holes as Four Qubit Systems

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Abstract

In this paper we describe the structure of extremal stationary spherically symmetric black hole solutions in the STU model of $D = 4$, $N = 2$ supergravity in terms of four-qubit systems. Our analysis extends the results of previous investigations based on three qubits. The basic idea facilitating this four-qubit interpretation is the fact that stationary solutions in $D = 4$ supergravity can be described by dimensional reduction along the time direction. In this $D = 3$ picture the global symmetry group $SL(2,\mathbb{R})^3$ of the model is extended by the Ehlers $SL(2,\mathbb{R})$ accounting for the fourth qubit. We introduce a four qubit state depending on the charges (electric, magnetic and NUT) the moduli and the warp factor. We relate the entanglement properties of this state to different classes of black hole solutions in the STU model. In the terminology of four qubit entanglement extremal black hole solutions correspond to nilpotent, and nonextremal ones to semisimple states. In arriving at this entanglement based scenario the role of the four algebraically independent four qubit $SL(2,\mathbb{C})$ invariants is emphasized.

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1. INTRODUCTION

Recently striking multiple relations have been discovered between two seemingly unrelated fields: Quantum Information Theory (QIT) and the physics of black hole solutions in String Theory. Although the physical basis for this black hole qubit correspondence (or black hole analogy) is still to be clarified, it has repeatedly proved to be useful for obtaining additional insight into one of the two fields by exploiting methods and techniques of the other. The main correspondence found is between the macroscopic entropy formulas obtained for certain black hole solutions in supergravity theories and multiqubit and qutrit entanglement measures used in Quantum Information Theory. The basic reason for this correspondence is the occurrence of similar groups of symmetry in these very different contexts. On the stringy black hole side the groups in question are the global symmetry groups of \(D=4\) classical supergravities, and on the QIT one the groups of local transformations for entangled subsystems not changing their multipartite entanglement. As far as physics is concerned an attempt has been made to understand these mathematical coincidences in terms of wrapped brane configurations giving rise to qubits.

Apart from understanding black hole entropy in quantum information theoretic terms the desire for an entanglement based understanding for issues of dynamics also arose. In particular in the special case of the STU model it has been realized that it is possible to rephrase the attractor mechanism as a distillation procedure of entangled ”states” of very special kind on the event horizon. Such ”states” for \(D=4\) extremal static spherical symmetric solutions are arising from more general ones of the form:

\[
|\psi(\tau)\rangle \equiv (\mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V})(S_3(\tau) \otimes S_2(\tau) \otimes S_1(\tau))|\gamma\rangle, \quad \tau \equiv \frac{1}{r}.
\]

Here

\[
\mathcal{V} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix}, \quad S_j \equiv \frac{1}{\sqrt{y_j(\tau)}} \begin{pmatrix} y_j(\tau) & 0 \\ -x_j(\tau) & 1 \end{pmatrix}, \quad j = 1, 2, 3
\]

\[
|\gamma\rangle = \sum_{a_3,a_2,a_1=0,1} \gamma_{a_3a_2a_1}|a_3a_2a_1\rangle |a_3a_2a_1\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2
\]

\[
\begin{pmatrix} \gamma_{000}, \gamma_{001}, \gamma_{010}, \gamma_{100} \\ \gamma_{111}, \gamma_{110}, \gamma_{101}, \gamma_{011} \end{pmatrix} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} p^0, & p^1, & p^2, & p^3 \\ -q_0, & q_1, & q_2, & q_3 \end{pmatrix}
\]

where \(r\) is the radial distance from the event horizon, \(z_j(\tau) = x_j(\tau) - iy_j(\tau), j = 1, 2, 3\) are the scalar fields, \(q_I\), and \(p^I, I = 0, 1, 2, 3\) are the electric and magnetic charges occurring.
in the STU model\(^1\). As we see these quantities are organized into a complex three-qubit state. This instructive notation clearly expresses the triality symmetry of the STU model\(^1\). Moreover, the classical symmetry group of the model (i.e. \(SL(2, \mathbb{R})^\times 3\)) is manifested in this formalism by the fact that apart from the unitary matrices \(V\), \(|\psi\rangle\) is lying on the \(SL(2, \mathbb{R})^\times 3\) orbit of the "charge-state" \(|\gamma\rangle\). The unitaries \(V^\otimes 3\) provide an embedding of the \(SL(2, \mathbb{R})^\times 3\) symmetry group of this \(N = 2\) supergravity model into \(GL(2, \mathbb{C})^\times 3\).

The state of Eq.(1) has a number of remarkable properties\(^3, 7, 16\).

1. The three-tangle\(^17\) \(\tau_{123}\), the unique triality and \(SL(2, \mathbb{C})^\times 3\) invariant 3-qubit entanglement measure based on Cayley’s hyperdeterminant\(^18, 19\), for \(|\psi\rangle\) is related to the macroscopic black hole entropy in the STU model as

\[
S = \pi \sqrt{\tau_{123}(|\psi\rangle)} = \pi \sqrt{\tau_{123}(|\gamma\rangle)}. \tag{5}
\]

2. The norm of \(|\psi\rangle\) with respect to the usual scalar product in \(\mathbb{C}^8\) with complex conjugation in the first factor is the Black Hole Potential\(^11\) \(V_{BH}\).

3. The flat covariant derivatives with respect to the Kähler connection are acting on \(|\psi\rangle\) as bit flip errors on the qubits.

4. For BPS-solutions and for non-BPS solutions with vanishing central charge\(^8\) \(|\psi(\infty)\rangle\) is a \(GHZ\)-state\(^20\). For non-BPS solutions with non-vanishing central charge the corresponding states are graph-states known from QIT\(^21\). In this respect moduli stabilization is related to a distillation procedure of states with special entanglement properties at the event horizon.

5. On the horizon bit flip errors on \(|\psi\rangle\) are supressed for BPS solutions and for non-BPS ones they are not. The non-BPS solutions can be characterized by the number and types of bit-flip errors.

6. After solving the equations of motion one obtains the attractor flow \(z_j(\tau)\) in moduli space. There is a flow \(|\psi(\tau)\rangle\) associated to this one. For the non-BPS seed solution\(^22\) it is possible to study how the distillation procedure unfolds itself\(^16\) with the following result. In the asymptotically flat region we are starting with a \(|\psi(0)\rangle\) having 7 nonequal nonvanishing amplitudes and finally at the horizon we get a graph state \(|\psi(\infty)\rangle\) with merely 4 nonvanishing ones with equal magnitudes.

7. The magnitude of the nonvanishing amplitudes of such "attractor states" is proportional to the black hole entropy. The relative phases of the amplitudes reflect the structure of the fake superpotential.
8. If we are starting with the very special values for the moduli corresponding to flat directions this uniform structure at the horizon deteriorates, with the interpretation of errors of more general types acting on the qubits of the relevant attractor states.

In addition to these interesting results based on three-qubit states there are ones which strongly hint at the possibility that for a complete understanding of STU black holes we have to embed our three-qubit states into four-qubit ones. In particular one can generalize Eq. (1) by also including the warp factor $U(\tau)$ occurring in the static, spherically symmetric ansatz for the 4D space-time metric

$$ds^2 = -e^{2U(\tau)} dt^2 + e^{-2U(\tau)} d\mathbf{x}^2$$ (6)

into a new state $|\chi\rangle$ defined as

$$|\chi(\tau)\rangle = e^{U(\tau)}|\psi(\tau)\rangle.$$ (7)

For the non-BPS seed solution it has been shown that the 7 nonvanishing $\tau$ dependent amplitudes of this state depending on the charges, the moduli and the warp factor satisfy a system of first order differential equations. This finding conforms with recent work done within the framework of the first order formalism for non-BPS solutions based on the so called fake superpotential.

Moreover, within the realm of the more general class of stationary solutions it is well-known that the warp factor taken together with the NUT potential $\sigma$ forms another $SL(2,\mathbb{R})$ doublet, a doublet with respect to the Ehlers group. Hence it is natural to suspect that for stationary solutions objects like $|\chi(\tau)\rangle$ are really four-qubit states in disguised form with the Ehlers group acting on a hidden extra qubit. The properties of these hypothetical 4-qubit states should account for the first order formalism hiding behind the integrability of the non-BPS flow equations.

Recent investigations clearly demonstrated that this should indeed be the case. The key observation is that stationary solutions in $D = 4$ supergravity can be elegantly described by dimensional reduction along the time direction. In this picture stationary solutions can be identified as solutions to a $D = 3$ non-linear sigma model with target space being a symmetric space $G/H$ with $H$ non-compact. The property that is of basic significance for us is that the group $G$ in this case extends the global symmetry group $G_4$ of $D = 4$ supergravity, by also incorporating the Ehlers $SL(2,\mathbb{R})$. In our specific case the
$N = 2$ STU model can be regarded as a consistent truncation of maximal $N = 8$, $D = 4$ supergravity with $G_4 = E_{7(7)}$, truncating to $SL(2, \mathbb{R})^*$. Timelike reduction in the general case then yields the coset $E_{8(8)}/SO^*(16)$, or in the case of the STU truncation the one $\mathcal{M}_3 = SO(4,4)/SL(2,\mathbb{R})^*$.

We then expect the four copies of $SL(2,\mathbb{R})$s giving rise to the group of local operations acting on the four qubits. Here the fourth qubit which accounts for the Ehlers group will then play a special role.

The manifold $\mathcal{M}_3$ is the target space of the aforementioned sigma model. It has been proved\textsuperscript{31} that for such symmetric target spaces stationary spherically symmetric black hole solutions can be obtained as geodesic curves on this pseudo Riemannian target space. Such geodesic curves are classified in terms of the Noether charges of the solutions. In the case of the STU model the coset representative $\mathcal{P}$ of our target space $\mathcal{M}_3$ and the related Noether charge can be written in the form reminiscent of a 4-qubit state\textsuperscript{27,31,32}. Moreover the line element on $\mathcal{M}_3$ can be written in the form\textsuperscript{27} $\text{Tr}(\mathcal{P}^2)$ which turns out to be just the quadratic 4-qubit invariant, one of the four algebraically independent invariants characterizing 4-qubit systems\textsuperscript{37}. It has also been observed\textsuperscript{31,32} that the entanglement properties of such 4-qubit-like states seem to be related to the fact whether the extremal solution in question is BPS, non-BPS or non-BPS with vanishing central charge. Based on this finding the authors of this paper\textsuperscript{31} mention that there might be a connection with issues concerning the black hole qubit correspondence, though in this field the $D = 3$ reformulation has never been used. (See Eqs. (5.51)-(5.52) of that paper.)

The aim of the present paper is to show, that using the $D = 3$ picture such 4-qubit interpretation indeed emerges naturally. Moreover, after establishing the desired connection we see that in this framework many aspects of the usual three-qubit interpretation can be understood in a nice and unified way.

The organization of this paper is as follows. In Section II. we present the background material on the STU model and the basics of the $D = 3$ picture emerging after reduction along the time direction. In Section III. in a four-qubit notation we reconsider the usual Iwasawa parametrization of the physical patch of the pseudo-Riemannian manifold $\mathcal{M}_3$. This formalism is exploited in Section IV. where we describe the line element on $\mathcal{M}_3$ as the canonical quadratic 4-qubit $SL(2,\mathbb{C})^*$ invariant. Here after a sequence of 4-qubit transformations (Hadamard gates, phase gates, and permutations) a very convenient realization for the “vierbein” $\mathcal{P}$ is obtained. These transformations correspond to a special choice of basis in $T_{\mathcal{C}}\mathcal{M}_3$.
similar to the ones used in Ref.\textsuperscript{27} rendering the ”quaternionic vierbein” covariantly constant with respect to the spin connection. In Section V. we discuss the structure of conserved charges in an entanglement based framework. Here we see how our remarkable three-qubit state of Eq. (1) originates from the geometric data on $\mathcal{M}_3$. As an important generalization we write down a generalization of Eq. (1) for stationary solutions when the NUT charge is not zero. Section VI. is devoted to an analysis of the static, spherically symmetric solutions. Our treatment is based on the algebraically independent 4-qubit $SL(2,\mathbb{C})^\times 4$ invariants. It is shown that in the language of QIT extremal solutions correspond to nilpotent, and nonextremal ones to semisimple 4-qubit states. Nilpotent states are the ones for which all of the four algebraically independent invariants vanish. This picture is dual to the usual characterization in terms of nilpotent orbits. Next in this entanglement based approach a study of the usual BPS and non-BPS solutions with vanishing central charge, and the non-BPS seed solution is given. These investigations culminate in establishing an explicit connection between the results of Ref.\textsuperscript{31} and some standard ones on four-qubit entangled systems in QIT. Finally we present our conclusions and comments in Section VII. In an Appendix for the convenience of the reader we also included some background material concerning four-qubit systems.

II. THE STU MODEL

In the following we consider ungauged $N = 2$ supergravity in $d = 4$ coupled to $n$ vector multiplets. The $n = 3$ case corresponds to the $STU$ model. The bosonic part of the action (without hypermultiplets) is

$$
S = \frac{1}{16\pi} \int d^4 x \sqrt{|g|} \left\{ -\frac{R}{2} + G_{ij} \partial_\mu z^i \partial^\nu \bar{z}^j g^{\mu\nu} + (\text{Im} N_{I,I'}F^I \cdot F^{I'} + \text{Re} N_{I,I'}F^I \cdot F^{I'}) \right\}
$$

(8)

Here $F^I$, and $*F^I$, $I = 0, 1, 2 \ldots n$ are two-forms associated to the field strengths $F^I_{\mu\nu}$ of $n + 1$ $U(1)$ gauge-fields and their duals.

The $z^i$, $i = 1, 2 \ldots n$ are complex scalar (moduli) fields that can be regarded as local coordinates on a projective special Kähler manifold. This manifold for the STU model is
\[ SL(2, \mathbb{R})/U(1) \times 3. \] In the following we will denote the three complex scalar fields as
\[ z_j \equiv x_j - iy_j, \quad j = 1, 2, 3, \quad y_j > 0. \] (9)

With these definitions the metric and the connection on the scalar manifold are
\[ G_{\bar{j}j} = \frac{\delta_{\bar{j}j}}{(2y_j)^2}, \quad \Gamma^j_{jj} = \frac{-i}{y_j}. \] (10)

The metric above can be derived from the Kähler potential
\[ K = -\log(8y_1y_2y_3) \] (11)
as \[ G_{\bar{j}j} = \partial_{\bar{j}}\partial_j K. \] For the STU model the scalar dependent vector couplings \( \text{Re} \mathcal{N}_{IJ} \) and \( \text{Im} \mathcal{N}_{IJ} \) take the following form
\[ \nu_{IJ} \equiv \text{Re} \mathcal{N}_{IJ} = \begin{pmatrix}
2x_1x_2x_3 & -x_2x_3 & -x_1x_3 & -x_1x_2 \\
-x_2x_3 & 0 & x_3 & x_2 \\
-x_1x_3 & x_3 & 0 & x_1 \\
-x_1x_2 & x_2 & x_1 & 0
\end{pmatrix}, \] (12)

\[ \mu_{IJ} \equiv \text{Im} \mathcal{N}_{IJ} = -y_1y_2y_3 \begin{pmatrix}
1 + \left( \frac{x_1}{y_1} \right)^2 + \left( \frac{x_2}{y_2} \right)^2 + \left( \frac{x_3}{y_3} \right)^2 - \frac{x_1}{y_1} - \frac{x_2}{y_2} - \frac{x_3}{y_3} & & & \\
-\frac{x_1}{y_1} & 1/y_1 & 0 & 0 \\
-\frac{x_2}{y_2} & 0 & 1/y_2 & 0 \\
-\frac{x_3}{y_3} & 0 & 0 & 1/y_3
\end{pmatrix}, \] (13)

\[ \mu^{IJ} \equiv (\mu^{-1})_{IJ} = -\frac{1}{y_1y_2y_3} \begin{pmatrix}
1 & x_1 & x_2 & x_3 \\
x_1 & |z_1|^2 & x_1x_2 & x_1x_3 \\
x_2 & 1 & |z_2|^2 & x_2x_3 \\
x_3 & x_1x_2 & x_2x_3 & |z_3|^2
\end{pmatrix}. \] (14)

We note that these vector couplings can be derived from the holomorphic prepotential
\[ F(X) = \frac{X^1X^2X^3}{X^0}, \quad X^I = (X^0, X^0z^a), \] (15)
via the standard procedure characterizing special Kähler geometry\(^{28}\).

Our aim is to describe stationary solutions of the Euler-Lagrange equations arising from the Lagrangian of the STU model in a four-qubit entanglement based language. It is well-known that the most general ansatz for stationary solutions in four dimensions is
\[ ds^2 = -e^{2U}(dt + \omega)^2 + e^{-2U}h_{ab}dx^a dx^b, \] (16)
\[ \mathcal{F}^I = dA^I = d(\xi^I(dt + \omega) + A^I), \tag{17} \]

where \(a, b = 1, 2, 3\) correspond to the spacial directions. The quantities \(U, \xi^I, A_a, \omega_a\) and \(h_{ab}\) are regarded as 3D fields, i.e. the ansatz above corresponds to dimensional reduction to \(D = 3\) along the timelike direction. In achieving this we have chosen the gauge such that the Lie-derivative of \(A^I\) with respect to the timelike Killing vector vanishes, and have chosen coordinates such that the isometry corresponding to this Killing vector is just a (time) translation. In this case the quantities in Eqs. (16-17) are merely depending on \(x^a\), \(a = 1, 2, 3\). The ansatz for the gauge fields \(A^I\) reflects its decomposition to terms parallel (\(\xi^I\)), and orthogonal (\(A^I\)) components with respect to the timelike Killing vector \(35\).

After performing the dimensional reduction to \(D = 3\) our starting Lagrangian of Eq. (8) takes the following form \(33, 35\)

\[ \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3, \tag{18} \]

where

\[ \mathcal{L}_1 = -\frac{1}{2} \sqrt{h} R[h] + dU \wedge *dU + \frac{1}{4} e^{-4U}(d\sigma + \tilde{\xi}_I d\xi^I - \xi^I d\tilde{\xi}_I) \wedge *(d\sigma + \tilde{\xi}_J d\xi^J - \xi^J d\tilde{\xi}_J) \], \tag{19} \]

\[ \mathcal{L}_2 = G^{ij} dz_i \wedge *dz^j, \tag{20} \]

\[ \mathcal{L}_3 = \frac{1}{2} e^{-2U} \mu_{IJ} d\xi^I \wedge *d\xi^J + \frac{1}{2} e^{-2U} \mu^{IJ} (d\tilde{\xi}_I - \nu_{IK} d\xi^K) \wedge *(d\tilde{\xi}_J - \nu_{JL} d\xi^L). \tag{21} \]

Here the new (axionic) scalars \(\sigma\) and \(\tilde{\xi}_I\) are coming from dualizing \(\omega\) and \(A^I\) by \(35\)

\[ d\tilde{\xi}_I \equiv \nu_{IJ} d\xi^J - e^{2U} \mu_{IJ} \ast (dA^I + \xi^I d\omega) \tag{22} \]

\[ d\sigma \equiv e^{4U} \ast d\omega + \xi^I d\tilde{\xi}_I - \tilde{\xi}_I d\xi^I. \tag{23} \]

Note also that here the exterior derivative is understood on the (generally curved) spatial slice with local coordinates \(x^j, j = 1, 2, 3\).

The dimensionally reduced Lagrangian \(\mathcal{L}\) can be written in the nice form of 3D gravity coupled to a nonlinear sigma model defined on the spatial slice with target manifold \(27\)

\( \mathcal{M}_3 = SO(4, 4)/SL(2, \mathbb{R}) \times 4\) with the Lagrangian

\[ \mathcal{L} = -\frac{1}{2} \sqrt{h} R[h] + g_{mn} \partial_a \Phi^m \partial^a \Phi^n \tag{24} \]

where \(\Phi^m, m = 1, 2, \ldots 16\) refers to the scalar fields: \(U, \sigma, \xi^I, \tilde{\xi}_I, z^j, \tilde{z}^j\) with \(I = 0, 1, 2, 3\) and \(j = 1, 2, 3\). Here the line element on \(\mathcal{M}_3\) defines \(g_{mn}\) as \(ds^2_{\mathcal{M}_3} = g_{mn} \Phi^m \Phi^n\) with the explicit
In this paper we are only discussing the special case of stationary, weakly extremal solutions i.e. solutions when the spacial slices are flat \(^{27,33}\). Single centered black holes with spherical symmetry are of this type. In this case the dynamics of the moduli \(\Phi^m\) are decoupled from the 3D gravity and the metric ansatz can be chosen to be the form

\[
\frac{1}{4}ds^2_{M_3} = G_{ij}(z, \bar{z})dz^i dz^\overline{j} + dU^2 + \frac{1}{4}e^{-4U}(d\sigma + \bar{\xi}_I d\xi^I - \xi^I d\bar{\xi}_I)^2 + \frac{1}{2}e^{-2U}[\mu_{IJ}d\xi^I d\xi^J + \mu^{IJ}(d\xi_I - \nu_{IK} d\xi^K)(d\bar{\xi}_J - \nu_{JL} d\bar{\xi}^L)].
\] (25)

with the warp factor depending merely on \(r\). Now the equations of motion are equivalent to light-like geodesic motion on \(M_3\) with the affine parameter \(\tau = \frac{1}{r}\). Since \(M_3\) is a symmetric space there is a number of conserved Noether charges associated with this geodesic motion. The most important ones are the electric and magnetic charges \(p^I\) and \(q_I\) and the NUT charge \(k^{27,28,31}\). Static solutions are characterized by the vanishing of the NUT charge i.e. \(k = 0\). In this case the dynamics is described by the Lagrangian of a fiducial particle in a "black-hole potential" \(V_{BH}\)

\[
\mathcal{L}(U(\tau), z^i(\tau), \bar{z}^i(\tau)) = \left(\frac{dU}{d\tau}\right)^2 + G_{ij}(z, \bar{z}) \frac{dz^i}{d\tau} \frac{d\bar{z}^j}{d\tau} + e^{2U} V_{BH}(z, \bar{z}, p, q),
\] (27)

with the constraint

\[
\left(\frac{dU}{d\tau}\right)^2 + G_{ij}(z, \bar{z}) \frac{dz^i}{d\tau} \frac{d\bar{z}^j}{d\tau} - e^{2U} V_{BH}(z, \bar{z}, p, q) = 0.
\] (28)

Here the black hole potential \(V_{BH}\) is depending on the moduli as well on the charges. Its explicit form is given by

\[
V_{BH} = \frac{1}{2} (p^I q_I) \begin{pmatrix}
(\mu + \nu^{-1} \nu)_{IJ} & -(\nu \mu^{-1})^J_I \\
-(\mu^{-1} \nu)^J_I & (\mu^{-1})^J_I
\end{pmatrix}
\begin{pmatrix}
p^J \\
q_J
\end{pmatrix}.
\] (29)

An alternative expression for \(V_{BH}\) can be given in terms of the central charge of \(N = 2\) supergravity, i.e. the charge of the graviphoton.

\[
V_{BH} = Z \bar{Z} + G^{ij}(D_i Z)(\overline{D_j Z})
\] (30)

where for the STU model...
\[ Z = e^{K/2}W = e^{K/2}(q_0 + z_1q_1 + z_2q_2 + z_3q_3 + z_1z_2z_3p^0 - z_2z_3p^1 - z_1z_3p^2 - z_1z_2p^3), \]

and \( D_a \) is the Kähler covariant derivative

\[ D_i Z = (\partial_i + \frac{1}{2} \partial_i K) Z, \]

and \( W \) is the superpotential.

Extremization of the effective Lagrangian Eq.(27) with respect to the warp factor and the scalar fields yields the Euler-Lagrange equations

\[ \ddot{U} = e^{2U} V_{BH}, \quad \ddot{z}^i + \Gamma^i_{jk} \dot{z}^j \dot{z}^k = e^{2U} \partial^i V_{BH}. \]

In these equations the dots denote derivatives with respect to \( \tau = \frac{1}{r} \). These radial evolution equations taken together with the constraint Eq.(28) determine the structure of static, spherically symmetric, extremal black hole solutions in the STU model. For the more general stationary case with nonvanishing NUT charge the motion along \( \xi^I, \tilde{\xi}^I \) and \( \sigma \) does not separate from the one on \( U \) and \( z^i \). In this case we obtain the generalization of Eqs.(33).

Since for our four-qubit picture we will not consider solutions of such kind we will not give the corresponding equations here.

As we have seen from this section the radial evolution associated to stationary spherical symmetric black hole solutions of the \( D = 4 \) STU model can be described as geodesic motion in the moduli space \( \mathcal{M}_3 \) of a dimensionally reduced \( D = 3 \) theory. The key issue of this reduction relevant to this paper is the enlargement of the \( D = 4 \) symmetry group from \( SL(2, \mathbb{R}) \times SO(2, 2) \) to the \( D = 3 \) one \( SO(4, 4) \) containing \( SL(2, \mathbb{R}) \times 4 \) as a subgroup. This result paves the way for the possibility to reinterpret our STU black holes as four-qubit systems.

### III. THE Iwasawa Parametrization and Four Qubits

Our starting point is the Iwasawa parametrization of the coset \( \mathcal{M}_3 = SO(4, 4)/SO(2, 2) \times SO(2, 2) \simeq SO(4, 4)/SL(2, \mathbb{R}) \otimes 4 \) as used in the paper of Bossard et.al.\(^{27}\) For this parametrization the 16 dimensional coset is (locally) coordinatized by the fields \( x_j, y_j, \phi \equiv 2U, \sigma \), and the potentials \( \xi^I \) and \( \tilde{\xi}^I \) quantities featuring the Lagrangian \( \mathcal{L} \) of Eq.(24).
In order to avoid using disturbing factors of $\sqrt{2}$ we rescale the potentials and define new quantities $\zeta^I, \tilde{\zeta}_I$ as

$$\zeta^I \equiv \sqrt{2}\xi^I, \quad \tilde{\zeta}_I = \sqrt{2}\tilde{\xi}_I. \quad (34)$$

In terms of these quantities the coset representative is

$$V \equiv e^{-\frac{1}{2}\phi H_0} \left( \prod_{j=1}^3 e^{-\frac{1}{2}\log y_j H_j e^{-x_j E_j}} \right) e^{-\zeta^I E_{qI} - \tilde{\zeta}_I E_{pI} e^{-\sigma E_0}}. \quad (35)$$

Here the four copies of $SL(2, \mathbb{R})$ generators $H_\alpha, E_\alpha, F_\alpha, \alpha = 0, 1, 2, 3$ satisfy the commutation relations

$$[E_\alpha, F_\alpha] = H_\alpha, \quad [H_\alpha, E_\alpha] = 2E_\alpha, \quad [H_\alpha, F_\alpha] = -2F_\alpha, \quad (36)$$

and the 16 generators of $so(4, 4)$ not belonging to the $sl(2) \oplus sl(2) \oplus sl(2) \oplus sl(2)$ algebra are denoted by the symbols $E_{pI}, E_{qI}, F_{pI}, F_{qI}, I = 0, 1, 2, 3$. This decomposition of generators answers the split

$$so(4, 4) = [sl(2, \mathbb{R})]^4 \oplus (2, 2, 2, 2) = h \oplus m, \quad (37)$$

which we would like to explicitly describe. (For an explicit connection between our conventions described below, and the one as given by Bossard et al. we refer the reader to the Appendix.)

The Lie-algebra $so(4, 4)$ adapted to our 4-qubit description will be regarded as the set of $8 \times 8$ matrices $D$ satisfying

$$DG + GD^T = 0 \quad (38)$$

where

$$G = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}, \quad g = \varepsilon \otimes \varepsilon, \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (39)$$

An element of $so(4, 4)$ will be parametrized as

$$\mathcal{D}(s_3, s_2, s_1, s_0; D) = \begin{pmatrix} s_3 \otimes I_2 + I_3 \otimes s_2 & Dg \\ -D^T g & s_1 \otimes I_0 + I_1 \otimes s_0 \end{pmatrix}. \quad (40)$$

Here the $m$-type generators are labelled by a real $4 \times 4$ matrix

$$D = \begin{pmatrix} D_{0000} & D_{0001} & D_{0010} & D_{0011} \\ D_{0100} & D_{0101} & D_{0110} & D_{0111} \\ D_{1000} & D_{1001} & D_{1010} & D_{1011} \\ D_{1100} & D_{1101} & D_{1110} & D_{1111} \end{pmatrix}. \quad (41)$$
which is expressed in terms of the amplitudes of a 4-qubit state with index structure

\[ D_{i_3 i_2 i_1 i_0}, \quad i_3, i_2, i_1, i_0 = 0, 1. \] (42)

Notice that for convenience we have labelled the qubits from the right to the left. Moreover, the first qubit will be regarded as special explaining the somewhat unusual label: \( i_0 \).

The \( h \) type generators are featuring the \( 2 \times 2 \) matrices \( s_\alpha \) of the form

\[ s_\alpha \equiv \begin{pmatrix} h_\alpha & e_\alpha \\ f_\alpha & -h_\alpha \end{pmatrix}, \quad \alpha = 0, 1, 2, 3. \] (43)

These matrices are expanded in terms of the ones

\[ H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \] (44)

satisfying the relations of Eq. (36).

The labels of the \( 2 \times 2 \) matrices appearing in Eq. (40) are referring to the qubits they act on. This action is induced by commutators of the form \([h, m] \subset m\). More precisely after commuting the block off-diagonal \( m \) part with the block-diagonal \( h \) one using

\[ s \varepsilon + \varepsilon s^T = 0, \quad s \in sl(2) \] (45)

we get the action

\[ (s_3 \otimes I_2 + I_3 \otimes s_2)D + D(s_1^T \otimes I_0 + I_1 \otimes s_0^T), \] (46)

which is the first order term in the \( SL(2, \mathbb{R}) \times 4 \) group action

\[ D \mapsto (S_3 \otimes S_2)D(S_1 \otimes S_0)^T, \quad S_\alpha \in SL(2, \mathbb{R}), \quad \alpha = 0, 1, 2, 3. \] (47)

Clearly this action in 4-qubit notation reads as

\[ D_{i_3 i_2 i_1 i_0} \mapsto \sum_{i_0' \neq i_0} (S_3)_{i_3 i_3'} (S_2)_{i_2 i_2'} (S_1)_{i_1 i_1'} (S_0)_{i_0 i_0'} D_{i_3 i_3' i_2 i_2' i_1 i_1' i_0 i_0'}, \] (48)

or in the notation used in Quantum Information Theory

\[ \lvert D \rangle \mapsto (S_3 \otimes S_2 \otimes S_1 \otimes S_0) \lvert D \rangle, \quad \lvert D \rangle = \sum_{i_3 i_2 i_1 i_0 = 0, 1} D_{i_3 i_3' i_2 i_2' i_1 i_1' i_0} \lvert i_3 i_2 i_1 i_0 \rangle. \] (49)

We remark that for the convenience of the reader in the Appendix we included more details on the correspondence between the structure of the group \( SO(4, 4) \) and 4-qubit entanglement.
Now returning to our coset representative of Eq. (33), we introduce the new coordinates

\[ x_0 \equiv \sigma, \quad y_0 \equiv e^{\phi} = e^{2U}. \]  

Using our 4-qubit realization in these coordinates we have

\[ \prod_{\alpha=0}^{3} e^{-\frac{1}{2} \log y_\alpha H_{\alpha}} e^{-x_\alpha E_{\alpha}} = \begin{pmatrix} M_3 \otimes M_2 & 0 \\ 0 & M_1 \otimes M_0 \end{pmatrix}, \]

where

\[ M_\alpha \equiv \frac{1}{\sqrt{y_\alpha}} \begin{pmatrix} 1 -x_\alpha \\ 0 \end{pmatrix}. \]

As a next step we introduce the 4 \times 4 matrix and its associated 4-qubit state

\[ \zeta \equiv \begin{pmatrix} \zeta_{0000} & \zeta_{0001} & \zeta_{0010} & \zeta_{0011} \\ \zeta_{0100} & \zeta_{0101} & \zeta_{0110} & \zeta_{0111} \\ \zeta_{1000} & \zeta_{1001} & \zeta_{1010} & \zeta_{1011} \\ \zeta_{1100} & \zeta_{1101} & \zeta_{1110} & \zeta_{1111} \end{pmatrix} = \begin{pmatrix} -\tilde{\zeta}_0 & 0 & \tilde{\zeta}_1 & 0 \\ \tilde{\zeta}_2 & 0 & \zeta^2 & 0 \\ \tilde{\zeta}_3 & 0 & \zeta^2 & 0 \\ \zeta^1 & 0 & \zeta^0 & 0 \end{pmatrix}, \]

Using this we write

\[ \zeta^I E_{qI} + \tilde{\zeta}^I E_{pI} = \begin{pmatrix} 0 & \zeta g \\ -\zeta^T g & 0 \end{pmatrix}. \]

Using the special form of the matrix \( \zeta \) we have the property \( \zeta g \zeta^T g = 0 \) hence a straightforward calculation shows that

\[ e^{-\zeta^I E_{qI} - \tilde{\zeta}^I E_{pI}} = \begin{pmatrix} 1 & -\zeta g \\ \zeta^T g & 1 + \frac{1}{2} \Delta \end{pmatrix}, \]

where \( 1 \equiv I \otimes I \) and

\[ \Delta = -\zeta^T g \zeta g = \begin{pmatrix} \zeta^{(0)} & \zeta^{(0)} & \zeta^{(0)} & \zeta^{(1)} \\ \zeta^{(0)} & \zeta^{(1)} & \zeta^{(1)} & \zeta^{(1)} \end{pmatrix} \varepsilon \otimes E. \]

Here the 4-component vectors \( \zeta^{(0)} \) and \( \zeta^{(1)} \) are just the first and third columns of the matrix \( \zeta \) of Eq. (53), and the \( \cdot \) product is defined by Eq. (185) of the Appendix.

Due to the special structure of \( \zeta \) we also have the property

\[ e^{x_0 E_0} e^{-\zeta^I E_{qI} - \tilde{\zeta}^I E_{pI}} e^{-x_0 E_0} = e^{-\zeta^I E_{qI} - \tilde{\zeta}^I E_{pI}}, \]
resulting in our final form for the coset representative in the Iwasawa gauge

$$V = \begin{pmatrix} M_3 \otimes M_2 & 0 \\ 0 & M_1 \otimes M_0 \end{pmatrix} \begin{pmatrix} 1 & -\zeta g \\ \zeta^T g & 1 + \frac{1}{2} \Delta \end{pmatrix}. \quad (58)$$

We close this section with some important comments. From the particular form of our coset representative in the Iwasawa gauge, also reflected in our choice of the matrix $\zeta$ of Eq.(53), we see that the role of the first qubit labelled by $i_0$ is special. The corresponding $SL(2, \mathbb{R})$ action refers to the Ehlers-group. However, our choice of $\zeta$ also gives special status to the second qubit labelled by $i_1$. This is also reflected in the structure of the matrix $\Delta$ of Eq.(56). The 8 components of $\zeta$ can be regarded as the ones arising from an embedding of a three-qubit state sitting inside a four-qubit one having merely 8 nonvanishing amplitudes. The grouping of these amplitudes of this three-qubit state into two four-vectors $\zeta^{(0)}$ and $\zeta^{(1)}$ is based on the special role we have also attached to the second qubit. However, we would have chosen any of the remaining two qubits to play this role. This would have resulted in another 4 plus 4 split for the 8 nonzero components of $\zeta$. This freedom for different arrangements is related to the triality of $so(4,4)$ connected to the permutation symmetry inherent in the embedded three-qubit system. For more details on this point we refer the reader to the Appendix.

IV. THE LINE ELEMENT ON $\mathcal{M}_3$ AS A FOUR-QUBIT INVARIANT.

The line element on $\mathcal{M}_3$ is given by the formula

$$ds^2 = \text{Tr}(\mathcal{P})^2 \quad (59)$$

where

$$\mathcal{P} \equiv \frac{1}{2}(dVV^{-1} + \eta(dVV^{-1})^T \eta) \quad (60)$$

and the involution compatible with our conventions is

$$\eta = \begin{pmatrix} I \otimes I & 0 \\ 0 & -I \otimes I \end{pmatrix}. \quad (61)$$
Using the explicit form for $V$ as given by Eq. (58) a straightforward calculation gives the result for $\mathcal{P}$

$$\mathcal{P} = \frac{1}{2} \begin{pmatrix} \Sigma_3 \otimes I_2 + I_3 \otimes \Sigma_2 & -g \Psi - \Psi g \\ g \Psi^T + \Psi^T g & \Sigma_1 \otimes I_0 + I_1 \otimes \Sigma_0 \end{pmatrix}$$  \hspace{1cm} (62)$$

where

$$\Sigma_j = \frac{1}{y_j} \begin{pmatrix} -dy_j & -dx_j \\ -dx_j & dy_j \end{pmatrix}, \quad j = 1, 2, 3$$  \hspace{1cm} (63)$$

and

$$\Sigma_0 = \frac{1}{y_0} \begin{pmatrix} -dy_0 & -dx_0 + w \\ -dx_0 + w & dy_0 \end{pmatrix}, \quad w = \frac{1}{2} (\zeta^I d\zeta_I - \tilde{\zeta}_I d\zeta^I).$$  \hspace{1cm} (64)$$

The important part we have not discussed yet is the $4 \times 4$ matrix

$$\Psi \equiv (M_3 \otimes M_2)d\zeta (M_1 \otimes M_0)^T,$$  \hspace{1cm} (65)$$

which by virtue of Eqs. (47-49) can be written as a differential form on the symplectic torus determined by the Wilson lines based on a four-qubit state

$$|\Psi\rangle = (M_3 \otimes M_2 \otimes M_1 \otimes M_0)|d\zeta\rangle.$$  \hspace{1cm} (66)$$

Recalling our conventions of Eqs. (50), (52), (53) we expect that $|\Psi\rangle$ is depending on the warp factor, the NUT potential, the moduli, and the Wilson lines $d\zeta^I$ and $d\tilde{\zeta}_I$. The four-qubit state $|d\zeta\rangle$ depending only on the Wilson lines clearly determines the entanglement type, since $|\Psi\rangle$ is lying on the $SL(2, \mathbb{R})^4$ orbit of this state. However, due to the special role of our first qubit $|\Psi\rangle$ is of special kind. Like in Eq. (53) its nonzero amplitudes when displayed in a $4 \times 4$ array are located in the first and the third columns. An important consequence of this is that the NUT potential is not appearing in the explicit form of $|\Psi\rangle$.

We can get a four-qubit state $|\Phi\rangle$ of a more general type after reinterpreting the term $g \Psi g + \Psi = (g \Psi + \Psi g)g$ found in the upper right block of Eq. (62) as a superposition

$$|\Phi\rangle = (\varepsilon \otimes \varepsilon \otimes \varepsilon \otimes \varepsilon)|\Psi\rangle + |\Psi\rangle.$$  \hspace{1cm} (67)$$

The explicit form of this state is

$$|\Phi\rangle = (M_3 \otimes M_2 \otimes M_1 \otimes M_0)|d\zeta\rangle + (M_3 \otimes M_2 \otimes M_1 \otimes M_0)^{T-1}|d\tilde{\zeta}\rangle$$  \hspace{1cm} (68)$$

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where $|\tilde{\zeta}\rangle = (\varepsilon \otimes \varepsilon \otimes \varepsilon \otimes \varepsilon)|\zeta\rangle$. We see that $|\tilde{\zeta}\rangle$ is transforming with respect to the contragredient action. Using Eq. (53) the explicit form of the transformation $|\zeta\rangle \mapsto |\tilde{\zeta}\rangle$ is

$$
\begin{pmatrix}
-\tilde{\zeta}_0 & 0 & \tilde{\zeta}_1 & 0 \\
\tilde{\zeta}_2 & 0 & \tilde{\zeta}^3 & 0 \\
\tilde{\zeta}_3 & 0 & \tilde{\zeta}^2 & 0 \\
\zeta^1 & 0 & \zeta^0 & 0
\end{pmatrix} \mapsto \begin{pmatrix}
0 & -\zeta^0 & 0 & \zeta^1 \\
0 & \zeta^2 & 0 & -\tilde{\zeta}_3 \\
0 & \zeta^3 & 0 & -\tilde{\zeta}_2 \\
0 & -\tilde{\zeta}_1 & 0 & -\tilde{\zeta}_0
\end{pmatrix}.
$$

(69)

i.e. fields with a tilde are transformed into the corresponding ones without a tilde up to some crucial signs ($\tilde{\zeta}_I \mapsto \zeta_I$ and $\zeta_I \mapsto -\tilde{\zeta}_I$), and their locations are shifted by one column.

Using these results the final form of $\mathcal{P}$ is

$$
\mathcal{P} = \frac{1}{2} \begin{pmatrix}
\Sigma_3 \otimes I_2 + I_3 \otimes \Sigma_2 & -\Phi g \\
\Phi^T g & \Sigma_1 \otimes I_0 + I_1 \otimes \Sigma_0
\end{pmatrix}.
$$

(70)

Using Eqs. (59-62) we obtain for the line element the following form

$$
ds_{M3}^2 = \sum_{j=1}^3 \frac{dx_j^2 + dy_j^2}{y_j^2} + \frac{(dx_0 - w)^2 + dy_0^2}{y_0^2} - ||\Psi||^2,
$$

(71)

where

$$
||\Psi||^2 \equiv \langle \Psi | \Psi \rangle, \quad |\Psi\rangle = (M_3 \otimes M_2 \otimes M_1 \otimes M_0)|d\zeta\rangle.
$$

(72)

Alternatively we can consider Eq. (70) featuring $\Phi = g\Psi g + \Psi$ which is the $4 \times 4$ version of the state $|\Phi\rangle$. Then by virtue of the special structure of the matrix $\Psi$ (which is similar to the one of Eq. (53)) satisfying $\Psi g \Psi^T g = 0$ one gets

$$
\langle \Psi | \Psi \rangle = \text{Tr}(\Psi^T \Psi) = \frac{1}{2} \text{Tr}(\Phi g \Phi^T g) = \frac{1}{2} \varepsilon^{i_3 i'_3} \varepsilon^{i_2 i'_2} \varepsilon^{i_1 i'_1} \varepsilon^{i_0 i'_0} \Phi_{i_3 i_2 i_1 i_0} \Phi_{i'_3 i'_2 i'_1 i'_0}.
$$

(73)

We see that the term $||\Psi||^2$ occurring in the expression of the line element has the immediate interpretation as the norm of a four-qubit state. However, again due to the special structure of $|d\zeta\rangle$ which determines the orbit type of $|\Psi\rangle$ it is natural to give a three-qubit reinterpretation as follows. Define

$$
|\psi\rangle \equiv (M_3 \otimes M_2 \otimes M_1 \otimes I)|d\zeta\rangle, \quad |\Psi\rangle = (I \otimes I \otimes I \otimes M_0)|\psi\rangle.
$$

(74)

Then we have

$$
||\Psi||^2 = \frac{1}{y_0}||\psi||^2 = e^{-2U}||\psi||^2,
$$

(75)
where by virtue of
\[ |dζ⟩ = \sum_{i_3i_2i_1=0,1} dζ_{i_3i_2i_1} |i_3i_2i_1⟩ \] (76)

\( ||ψ||^2 \) can be regarded as the norm squared of a three-qubit state. Let us now recall Eqs. (12)-(13) and (14). One can check that
\[ -e^{-2U}(dζ^I \ dζ_I^*) \begin{pmatrix} (μ + νμ^{-1}ν)_{IJ} - (νμ^{-1})_{I}^{J} \\ -μ \end{pmatrix} \begin{pmatrix} dζ^J \\ dζ_J \end{pmatrix} = e^{-2U}||ψ||^2 = ||Ψ||^2, \] (77)
i.e. we get back to the usual notation used in the supergravity literature. Notice that unlike its usual form the new version as a norm squared is not explicitly \( SL(2, \mathbb{R})^{×3} \subset Sp(8, \mathbb{R}) \) invariant. However, by virtue of Eq. (73) we have another interpretation for this term, which clearly displays its \( SL(2, \mathbb{R})^{×3} \) invariance. (The expression is actually the canonical quadratic \( SL(2, \mathbb{R})^{×4} \) four-qubit invariant, however, the Ehlers \( SL(2, \mathbb{R}) \) transformations of the form \( I ⊗ I ⊗ I ⊗ S \) are not preserving the special form of \( |Φ⟩ \).)

Proceeding further let us define the 8 × 8 unitary matrix
\[ U = \begin{pmatrix} U ⊗ U & 0 \\ 0 & U ⊗ U \end{pmatrix}, \quad U = HP = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix}, \] (78)
where \( H \) and \( P \) are the Hadamard (discrete Fourier transform) and phase gates known from Quantum Information Theory
\[ H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}. \] (79)

We notice that for \( α = 0, 1, 2, 3 \)
\[ UΣ_α U^† = \frac{i}{y_α} \begin{pmatrix} 0 & dz_α \\ -dz_α & 0 \end{pmatrix} \quad dz_j = dx_j - i dy_j, \quad dz_0 = (dx_0 - w) - i dy_0, \] (80)
with \( j = 1, 2, 3 \). After introducing the right invariant one-forms \( e_α = \frac{1}{y_j} dz_α \) on the cosets \([SL(2, \mathbb{R})/SO(2)]_α\) we can define
\[ \hat{e}_α ≡ UΣ_α U^† = \begin{pmatrix} 0 & z_α \\ e_α & 0 \end{pmatrix}. \] (81)

Using the unitary matrix of Eq. (78) we can transform \( \mathcal{P} \) of Eq. (62) to the form
\[ \hat{\mathcal{P}} = U \mathcal{P} U^† = \frac{1}{2} \begin{pmatrix} \hat{e}_3 ⊗ I_2 + I_3 ⊗ \hat{e}_2 & g\hat{Ψ} + \Psi g \\ -g\hat{Ψ}^T - \hat{Ψ} \hat{g} & \hat{e}_1 ⊗ I_0 + I_1 ⊗ \hat{e}_0 \end{pmatrix}. \] (82)
Here we have introduced $\hat{\Psi} = (U \otimes U)\Psi(U \otimes U)^T$ answering the new 4-qubit state

$$ |\hat{\Psi}\rangle = (H \otimes H \otimes H \otimes H)(P \otimes P \otimes P \otimes P)(M_3 \otimes M_2 \otimes M_1 \otimes M_0)|d\zeta\rangle. \quad (83) $$

Notice that this new 4-qubit state is now on the $GL(2,\mathbb{C})^{\times 4}$ orbit of the one $|d\zeta\rangle$ due to the presence of the matrices $U \in U(2)$. Moreover, $|\hat{\Psi}\rangle$ can also be regarded as the discrete Fourier transform of the one $(PM_3 \otimes PM_2 \otimes PM_1 \otimes PM_0)|d\zeta\rangle$ incorporating the important phase factors $e^{i\pi \frac{d}{2}}$ via the phase gates.

In order to gain some insight into the structure of $\hat{\Phi}$ we define the $4 \times 4$ matrix

$$ \hat{\Phi} = g\bar{\Psi}g + \hat{\Psi}, \quad (84) $$
corresponding to the complex four-qubit state

$$ |\hat{\Phi}\rangle = (\epsilon \otimes \epsilon \otimes \epsilon \otimes \epsilon)|\bar{\Psi}\rangle + |\hat{\Psi}\rangle. \quad (85) $$

Notice that though this state is now complex it is again of special form since it satisfies the reality condition

$$ \overline{|\hat{\Phi}\rangle} = (\epsilon \otimes \epsilon \otimes \epsilon \otimes \epsilon)|\bar{\Phi}\rangle. \quad (86) $$

In order to understand the structure of $|\hat{\Phi}\rangle$ we write its component state $|\bar{\Psi}\rangle$ in a three-qubit-like notation

$$ |\bar{\Psi}\rangle = (I \otimes I \otimes I \otimes UM_0)|\psi\rangle, \quad |\hat{\psi}\rangle = (UM_3 \otimes UM_2 \otimes UM_1 \otimes I)|d\zeta\rangle \quad (87) $$

where again $\hat{\psi}_{i_3i_2i_10} \neq 0$ but $\hat{\psi}_{i_3i_2i_11} = 0$ thanks to the structure similar to that of $\zeta_{i_3i_2i_10}$. Introducing the shorthand

$$ \hat{\psi}_{i_3i_2i_1} \equiv \hat{\psi}_{i_3i_2i_10} \quad (88) $$
in $4 \times 4$ notation we get

$$ \hat{\Psi} = \frac{i}{\sqrt{2g_0}} \begin{pmatrix}
\hat{\psi}_{000} & \hat{\psi}_{000} & \hat{\psi}_{001} & \hat{\psi}_{001} \\
\hat{\psi}_{010} & \hat{\psi}_{010} & \hat{\psi}_{011} & \hat{\psi}_{011} \\
\hat{\psi}_{100} & \hat{\psi}_{100} & \hat{\psi}_{101} & \hat{\psi}_{101} \\
\hat{\psi}_{110} & \hat{\psi}_{110} & \hat{\psi}_{111} & \hat{\psi}_{111}
\end{pmatrix} \quad (89) $$
i.e. the first and the last two columns are the same. Now using the special structure of the matrix $U \otimes U \otimes U$ one can verify that the following reality conditions hold

$$ \hat{\psi}_{111} = -\overline{\hat{\psi}_{000}}, \quad \hat{\psi}_{001} = -\overline{\hat{\psi}_{110}}, \quad \hat{\psi}_{010} = -\overline{\hat{\psi}_{101}}, \quad \hat{\psi}_{100} = -\overline{\hat{\psi}_{011}}. \quad (90) $$
As a result of these considerations the matrix $\hat{\Phi}$ takes the following form

$$
\hat{\Phi} = \begin{pmatrix}
\mathcal{E}_0 & 0 & 0 & \mathcal{E}_1 \\
0 & \mathcal{E}_2 & \mathcal{E}_3 & 0 \\
0 & \mathcal{E}_3 & \mathcal{E}_2 & 0 \\
\mathcal{E}_1 & 0 & 0 & \mathcal{E}_0
\end{pmatrix},
$$

with

$$
\mathcal{E}_0 = 2\hat{\Psi}_{1110} = i\sqrt{\frac{2}{y_0}}\hat{\psi}_{111}, \quad \mathcal{E}_1 = 2\hat{\Psi}_{1100} = i\sqrt{\frac{2}{y_0}}\hat{\psi}_{110},
$$

$$
\mathcal{E}_2 = 2\hat{\Psi}_{1010} = i\sqrt{\frac{2}{y_0}}\hat{\psi}_{101}, \quad \mathcal{E}_3 = 2\hat{\Psi}_{0110} = i\sqrt{\frac{2}{y_0}}\hat{\psi}_{011}.
$$

After using this result in the expression for $\hat{\mathbb{P}}$ of Eq.(82) we arrive at the explicit form

$$
\hat{\mathbb{P}} = \frac{1}{2} \begin{pmatrix}
0 & e_2 & e_3 & 0 & \overline{\mathcal{E}_1} & 0 & 0 & \overline{\mathcal{E}_0} \\
\overline{e}_2 & 0 & 0 & e_3 & 0 & -\mathcal{E}_3 & -\overline{\mathcal{E}_2} & 0 \\
\overline{e}_3 & 0 & 0 & e_2 & 0 & -\mathcal{E}_2 & -\overline{\mathcal{E}_3} & 0 \\
0 & \overline{\mathcal{E}_3} & \overline{e}_2 & 0 & \mathcal{E}_0 & 0 & 0 & \mathcal{E}_1 \\
-\mathcal{E}_1 & 0 & 0 & -\overline{\mathcal{E}_0} & 0 & e_0 & e_1 & 0 \\
0 & \overline{\mathcal{E}_2} & \overline{e}_3 & 0 & \overline{\mathcal{E}_1} & 0 & 0 & e_0 \\
-\mathcal{E}_0 & 0 & 0 & -\overline{\mathcal{E}_1} & 0 & \overline{\mathcal{E}_2} & \overline{\mathcal{E}_3} & 0
\end{pmatrix}.
$$

The line element in terms of these complex quantities is the familiar one of Eq.(71)

$$
ds^2_{\mathcal{M}_3} = \sum_{\alpha=0}^3 (\overline{e}_\alpha e_\alpha - \overline{\mathcal{E}_\alpha} \mathcal{E}_\alpha) = \sum_{\alpha=0}^3 \frac{d\overline{z}_\alpha z_\alpha}{y_\alpha^2} - \frac{1}{y_0} ||\hat{\Psi}||^2 = \sum_{\alpha=0}^3 \frac{d\overline{z}_\alpha z_\alpha}{y_\alpha^2} - ||\hat{\Psi}||^2.
$$

Here $||\hat{\Psi}||^2 = \langle \hat{\Psi} | \hat{\Psi} \rangle$ is the usual scalar product on $\mathbb{C}^4$ with complex conjugation in the first factor.

It is important to realize that our quantities $\mathcal{E}_\alpha$ can be written in the familiar form

$$
\mathcal{E}_0 = \sqrt{2e^{i\Theta} - U} X^I (\bar{N}_{IJ} d\zeta^J - d\bar{\zeta}_I), \quad \mathcal{E}_j = 2i\sqrt{2y_0} e^{-i\Theta} f_j (\bar{N}_{IJ} d\zeta^J - d\bar{\zeta}_I)
$$

in terms of the quantities known from special Kähler geometry. Here

$$
f_j^I = e^{i\Theta} D_1 X^I = e^{i\Theta} (\partial_1 + (\partial_1 K)) X^I = e^{i\Theta} \frac{1}{z_1 - z_1} \begin{pmatrix} 1 \\
\overline{z}_1 \\
z_2 \\
z_3
\end{pmatrix}, \quad \text{e.t.c.}
$$

(97)
with \( X^I = (1, z_1, z_2, z_3)^T \), \( K = -\log(y_1 y_2 y_3) \) and \( \mathcal{N}_{IJ} \) is defined by Eqs. (12)(13). By virtue of Eqs. (92)(93) these quantities are nicely compressed into the four-qubit state \( |\tilde{\Psi}\rangle \) of Eq. (83).

The special structure of \( \hat{P} \) of Eq. (94) reveals yet another way for obtaining a four-qubit state. Indeed, \( \hat{P} \) contains precisely 16 nonzero quantities which can be organized to form the amplitudes of this new state. In order to motivate our construction of this new state let us consider the space of \( 8 \times 8 \) matrices of the following form

\[
\hat{R} = \frac{1}{2} \begin{pmatrix} A & C \\ -gC^T g & B \end{pmatrix}, \quad C = \begin{pmatrix} 0 & b & a & 0 \\ -\gamma & 0 & 0 & c \\ -\delta & 0 & 0 & d \\ 0 & -\beta & 0 & 0 \end{pmatrix},
\]

\[B = \begin{pmatrix} -\gamma - \delta & 0 & 0 & 0 \\ 0 & -\alpha + \beta & 0 & 0 \\ 0 & 0 & \alpha - \beta & 0 \\ 0 & 0 & 0 & \gamma + \delta \end{pmatrix}, \quad A = \begin{pmatrix} -\alpha - \beta & 0 & 0 & 0 \\ 0 & -\gamma + \delta & 0 & 0 \\ 0 & 0 & \gamma - \delta & 0 \\ 0 & 0 & 0 & \alpha + \beta \end{pmatrix},\]

where \( \alpha, \beta, \gamma, \delta \in \mathbb{R} \), and \( a, b, c, d \in \mathbb{C} \). Clearly this 12 real parameter family is complementary to the 16 real parameter one characterizing \( \hat{P} \). In both cases the off-diagonal blocks are related as \( C \mapsto -gC^T g \). In the case of \( \hat{P} \) the off-diagonal block satisfies the reality condition \( \Phi = g(\Phi)g \), and for \( C \) this condition is \( \overline{C} = -gCg \). Both of the matrices \( \hat{P} \) and \( \hat{R} \) are satisfying Eq. (38) hence they are elements of the Lie algebra of \( SO(4, \mathbb{R}) \simeq SO(8, \mathbb{C}) \).

Let us now label the rows and columns of these \( 8 \times 8 \) matrices as \( 0, 1, 2, 3, 4, 5, 6, 7 \) or in binary notation \( 000, 001, 010, 011, 100, 101, 110, 111 \). Now we employ the following permutation to the rows and columns

\[
(0, 1, 2, 3, 4, 5, 6, 7) \mapsto (7, 1, 2, 4, 3, 5, 6, 0),
\]

\[
(000, 001, 010, 011, 100, 101, 110, 111) \mapsto (111, 001, 010, 100, 011, 101, 110, 000).
\]

The binary notation is instructive since it clearly shows that after applying the permutation we get two 4 element blocks labelled by numbers containing an even number of zeros for the first block and an odd number of zeros in the second. (Another mnemonic: the numbers \( 1, 2, 4 \) are the quadratic residues modulo 7 and the ones \( 3, 5, 6 \) are the quadratic nonresidues.) Now it is easy to check that our fundamental matrix \( G \) of Eq. (39) is invariant under this permutation.
Applying this permutation to the matrix $R$ yields the one

$$
R' = \frac{1}{2} \begin{pmatrix}
\gamma + \delta & \bar{c} & \bar{d} & 0 & 0 & 0 & 0 & 0 \\
c & -\gamma + \delta & 0 & \bar{d} & 0 & 0 & 0 & 0 \\
d & 0 & \gamma - \delta & \bar{c} & 0 & 0 & 0 & 0 \\
0 & d & c & -\gamma - \delta & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha + \beta & \bar{c} & \bar{b} & 0 \\
0 & 0 & 0 & 0 & a & -\alpha + \beta & 0 & \bar{b} \\
0 & 0 & 0 & 0 & b & 0 & \alpha - \beta & \bar{a} \\
0 & 0 & 0 & 0 & b & a & -\alpha - \beta & \end{pmatrix}
$$

(102)

This matrix contains the two $4 \times 4$ blocks in its block diagonal part

$$
\frac{1}{2}(d \sigma \otimes I + I \otimes c \sigma), \quad d = \begin{pmatrix} d_1 \\ d_2 \\ \delta \end{pmatrix}, \quad c = \begin{pmatrix} c_1 \\ c_2 \\ \gamma \end{pmatrix}, \quad d = d_1 + id_2, \quad c = c_1 + ic_2,
$$

(103)

$$
\frac{1}{2}(b \sigma \otimes I + I \otimes a \sigma), \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \beta \end{pmatrix}, \quad a = \begin{pmatrix} a_1 \\ a_2 \\ \alpha \end{pmatrix}, \quad b = b_1 + ib_2, \quad a = a_1 + ia_2.
$$

(104)

The same permutation acting on $P$ results in the new form

$$
P'_s = \frac{1}{2} \begin{pmatrix} 0 & \Lambda g \\ -\Lambda^T g & 0 \end{pmatrix},
$$

(105)

$$
\Lambda = \begin{pmatrix}
\Lambda_{0000} & \Lambda_{0001} & \Lambda_{0010} & \Lambda_{0011} \\
\Lambda_{0100} & \Lambda_{0101} & \Lambda_{0110} & \Lambda_{0111} \\
\Lambda_{1000} & \Lambda_{1001} & \Lambda_{1010} & \Lambda_{1011} \\
\Lambda_{1100} & \Lambda_{1101} & \Lambda_{1110} & \Lambda_{1111} \\
\end{pmatrix} = \begin{pmatrix}
-\epsilon_0 & -\epsilon_0 & -\epsilon_1 & -\bar{\epsilon}_1 \\
e_2 & \bar{\epsilon}_2 & \epsilon_3 & \bar{\epsilon}_3 \\
e_3 & \bar{\epsilon}_3 & \epsilon_2 & \bar{\epsilon}_2 \\
-\epsilon_1 & -\bar{\epsilon}_1 & -\epsilon_0 & -\bar{\epsilon}_0 \end{pmatrix}.
$$

(106)

Now we define a new four-qubit state

$$
|\Lambda\rangle = \sum_{a_3,a_2,a_1,a_0=0,1} \Lambda_{a_3a_2a_1a_0} |a_3a_2a_1a_0\rangle.
$$

(107)

Looking at the structure of our matrix $R'$ it is clear that it defines an infinitesimal $SU(2)^\otimes 4$ action on our state based on the $4 \times 4$ complex matrix $\Lambda$ related to the decomposition

21
so(8, C) = [sl(2, C)]^4 \oplus (2, 2, 2, 2),

and the embedding of su(2) in sl(2, C).

It is important to realize that after the transformation

\[ C \mapsto C' \equiv \begin{pmatrix} 0 & -b & -a & 0 \\ \overline{d} & 0 & 0 & -c \\ \overline{c} & 0 & 0 & -d \\ 0 & \overline{a} & \overline{b} & 0 \end{pmatrix}, \]  

with \( C' \) having the property \( C' = gC'g \) and also the one of Eq.(101) the matrix replacing Eq.(102) will contain the diagonal blocks

\[ \frac{1}{2}(d\tau \otimes I + I \otimes c\tau), \quad \frac{1}{2}(b\tau \otimes I + I \otimes a\tau). \]  

Here the matrices

\[ \tau_1 \equiv \frac{i}{2}\sigma_2, \quad \tau_2 \equiv -\frac{i}{2}\sigma_1, \quad \tau_3 \equiv \frac{1}{2}\sigma_3, \]  

satisfy the commutation relations of an su(1, 1) subalgebra of sl(2, C).

Recall that one of our qubits (i.e. the first one labelled by \( a_0 \)) is still special. According to Eq.(106) transformations of the form \( I \otimes I \otimes I \otimes S, \quad S = e^{i\alpha_0} \in SL(2, C) \) acting on this qubit relate the first column with the second, and the third with the fourth. Hence these transformations relate \( E_0 \) to \( e_0 \) and the \( E_j \) to the \( e_j \) with the same index \( j \).

Notice that the \( SU(2) \) subgroup of this \( SL(2, C) \) is just the R-symmetry group arising from the restricted holonomy group, of the quaternionic-Kähler space which is the analytically continued version of our para-quaternionic \( M_3 \). This holonomy implies\(^{27,36,40}\) that the complexified tangent bundle of that space splits locally as \( W \otimes V \) where \( W \) and \( V \) are vector bundles of dimension 8 and 2. In our case a similar split exists where the former space corresponds to the three-qubit part of our \( \Lambda \) labelled by the indices \( a_3, a_2, a_1 \) and the latter to the special qubit labelled by \( a_0 \). Indeed our transformations of Eqs.(78) and (101) correspond to a change of basis in \( T_C M_3 \) similar to the usual one rendering the ”quaternionic vierbein” covariantly constant with respect to the spin connection\(^{27}\).

Eqs.(105-107) are of central importance for our considerations of the following sections.
They define a complex four-qubit state satisfying the reality condition

$$|\Lambda\rangle = (\sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1)|\Lambda\rangle, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

(112)

where $\sigma_1$ is the bit flip gate of Quantum Information Theory. It is straightforward to check that the subgroup of transformations of the group $SL(2, \mathbb{C})^4$ leaving invariant this reality condition is $SU(1, 1)^4$ i.e. precisely those transformations as described by Eqs. (109)-(111).

Hence in the notation used in quantum information the admissible transformations are of the form

$$|\Lambda\rangle \mapsto (S \otimes S_2 \otimes S_1 \otimes S_0)|\Lambda\rangle, \quad S_3, S_2, S_1, S_0 \in SU(1, 1).$$

(113)

In what follows our basic concern will be a study of quantities invariant under the larger group of transformations i.e. $SL(2, \mathbb{C})^4$. Such invariants are clearly also $SU(1, 1)^4$ ones.

It is known that the number of such algebraically independent invariants is four. We have a quadratic, two quartic, and one sextic invariant. The structure and geometry of such invariants has been investigated. Here in closing this section we just observe that the quadratic four-qubit invariant for our state $|\Lambda\rangle$ is precisely the line element $ds^2_{M_3}$ i.e.

$$ds^2_{M_3} = -\frac{1}{2}e^{a_3 a_1^*} e^{a_2 a_1^*} e^{a_1 a_1^*} e^{a_0 a_0^*} \Lambda_{a_3 a_2 a_1 a_0} \Lambda_{a_3' a_2' a_1' a_0'} = \sum_{\alpha=0}^{3} (\bar{e}_\alpha e_\alpha - \bar{E}_\alpha E_\alpha).$$

(114)

This formula first appeared in the paper of Bossard et.al. Here we have also clarified its intimate connection to four-qubit systems. We also remark that the quadratic invariant is also a permutation invariant. However, from the physical point of view the special role we have attached to the first qubit obviously breaks this permutation invariance.

V. CONSERVED CHARGES

The 3D duality group acts isometrically on our $M_3$ by right multiplication and yields a conserved Noether charge

$$Q = V^{-1}P V = \begin{pmatrix} Q_{11} & -gQ_{12} \\ gQ_{12}^T & Q_{22} \end{pmatrix}.$$ 

(115)

The explicit expression of $Q$ is given by

$$2Q = \begin{pmatrix} 1 & \zeta g \\ -\zeta^T g & 1 + \frac{1}{2}\Delta \end{pmatrix} \begin{pmatrix} \rho_3 \otimes I_2 + I_3 \otimes \rho_2 & -gd\dot{\zeta} - d\zeta g \\ gd\dot{\zeta}^T + d\zeta^T g & \rho_1 \otimes I_0 + I_1 \otimes \rho_0 \end{pmatrix} \begin{pmatrix} 1 & -\zeta g \\ \zeta^T g & 1 + \frac{1}{2}\Delta \end{pmatrix}.$$ 

(116)
where
\[
\rho_\alpha \varepsilon \equiv \text{Re} \left( \frac{dz_\alpha}{y_\alpha^2} \begin{pmatrix} \overline{z}_\alpha \\ 1 \end{pmatrix} \right), \quad z_\alpha = x_\alpha - iy_\alpha, \quad \alpha = 0, 1, 2, 3 \tag{117}
\]
and
\[
d\hat{\zeta} \equiv (N_3 \otimes N_2) d\zeta (N_1 \otimes N_0), \quad N_\alpha \equiv M_\alpha^T M_\alpha, \quad \alpha = 0, 1, 2, 3 \tag{118}
\]
where for \(dz_j, j = 1, 2, 3\) and \(dz_0\) we used the definitions of Eq.(80).

Now we are interested in the conserved electric and magnetic charges coming from the first and third column of \(Q_{12}\). (This part has the same structure as the matrix \(\zeta\) of Eq.(53)). Since the matrix \(\zeta g\) has vanishing first and third column, any \(4 \times 4\) matrix multiplied by \(\zeta g\) from the right also has this property. Hence terms having this structure will not contribute to the relevant part of \(Q_{12}\). Using \(\Delta = -\zeta^T g \zeta g\) the relevant part of \(Q_{12}\) is
\[
2[Q_{12}]_{\text{relevant}} = \frac{1}{y_0} (N_3 \otimes N_2) d\zeta (N_1 \otimes I) + \frac{dx_0 - w}{y_0^2} (\varepsilon \otimes \varepsilon) \zeta (\varepsilon \otimes I). \tag{119}
\]

On the other hand let us look at the conserved quantity
\[
k \equiv \frac{1}{2} \text{Tr}(I_1 \otimes E_0) Q_{22} = \frac{(dx_0 - w)}{2y_0^2}. \tag{120}
\]
Since the line element is related to the Lagrangian and the Lagrangian according to Eq.(71) contains a term \(((\dot{x}_0 - w)^2 + y_0^2) / 4y_0^2\) hence \(p_{x_0} = p_\sigma = \frac{\partial L}{\partial \dot{x}_0} = k\) i.e. our quantity is just the NUT charge. These considerations show that the relevant part \(\Gamma\) of \(Q_{12}\) in four-qubit notation is
\[
|\Gamma\rangle = \frac{1}{2y_0} (N_3 \otimes N_2 \otimes N_1 \otimes I) |d\zeta\rangle - p_{x_0} (\varepsilon \otimes \varepsilon \otimes \varepsilon \otimes I) |\zeta\rangle. \tag{121}
\]
Eq.(121) comprises 8 conserved quantities represented as the 8 nonzero amplitudes of a four-qubit state. Note, that our formula also contains the NUT charge. In a three-qubit-like notation we can alternatively write this as
\[
|\Gamma\rangle = \frac{1}{2} e^{-2U} (N \otimes I) |d\zeta\rangle - p_\sigma (\varepsilon \otimes I) |\zeta\rangle, \tag{122}
\]
with
\[
N \equiv N_3 \otimes N_2 \otimes N_1, \quad \varepsilon \equiv \varepsilon \otimes \varepsilon \otimes \varepsilon. \tag{123}
\]

Let us see how this set of conserved quantities is related to the momenta \(p_{\zeta I} = \frac{\partial L}{\partial \zeta I}\) and \(p_{\zeta I} = \frac{\partial L}{\partial \zeta I}\). A calculation based on the Lagrangian related to the line element Eq.(71) shows
that these quantities can be also organized into a state

\[ |p_\zeta\rangle \equiv -\frac{1}{2}e^{-2U(N \otimes I)|d\zeta\rangle} + \frac{p_\sigma}{2}(\epsilon \otimes I)|\zeta\rangle. \quad (124) \]

Hence

\[ |\Gamma\rangle = -\frac{p_\sigma}{2}(\epsilon \otimes I)|\zeta\rangle - |p_\zeta\rangle. \quad (125) \]

Let us now introduce the new quantity

\[ |P_\zeta\rangle \equiv |p_\zeta\rangle - \frac{p_\sigma}{2}(\epsilon \otimes I)|\zeta\rangle. \quad (126) \]

Notice that after writing out the 8 amplitudes explicitly we get

\[ P_{\zeta I} = p_{\zeta I} - \frac{p_\sigma}{2}\tilde{\zeta}_{I}, \quad P_{\tilde{\zeta} I} = p_{\tilde{\zeta} I} + \frac{p_\sigma}{2}\zeta_{I}, \quad (127) \]

in accordance with Eq. (4.14) of Bossard et.al.\(^{27}\) (The \(\sigma\) used by them is different by a factor of 2). Now one can verify that the Hamiltonian is

\[ H = \sum_{\alpha=0}^{3} y_{\alpha}^2(p_{x_{\alpha}}^2 + p_{y_{\alpha}}^2) - y_{0}\langle P_\zeta|N^{-1} \otimes I|P_\zeta\rangle. \quad (128) \]

From Eqs. (125,126) we have

\[ |\hat{\Gamma}\rangle \equiv |\Gamma\rangle + p_\sigma(\epsilon \otimes I)|\zeta\rangle = -|P_\zeta\rangle \]

arriving at an alternative expression as

\[ H = \sum_{\alpha=0}^{3} y_{\alpha}^2(p_{x_{\alpha}}^2 + p_{y_{\alpha}}^2) - e^{2U}\langle \hat{\Gamma}|N^{-1} \otimes I|\hat{\Gamma}\rangle \quad (130) \]

in a three-qubit-like notation.

Let us now parametrize \(\Gamma\) in terms of the electric and magnetic charges as

\[
\Gamma = \frac{1}{\sqrt{2}} \begin{pmatrix}
    p^0 & 0 & -p^1 & 0 \\
    -p^2 & 0 & q_3 & 0 \\
    -p^3 & 0 & q_2 & 0 \\
    q_1 & 0 & q_0 & 0
\end{pmatrix}, \quad (131)
\]

where the rows and columns of this matrix are related to the four-qubit labels as in Eq.(53) defining the four-qubit state \(|\Gamma\rangle\). This parametrization in the conventional language amounts to

\[
\frac{1}{\sqrt{2}}p^I = \frac{1}{2}p_\sigma \tilde{\zeta}^I + p_{\tilde{\zeta} I}, \quad \frac{1}{\sqrt{2}}q^I = \frac{1}{2}p_\sigma \zeta_I - p_{\zeta I}. \quad (132)
\]
Notice that the origin of the factors of $\sqrt{2}$s appearing in Eqs.(131-132) can be traced back to the fact\textsuperscript{27,31} that the electric and magnetic charges should be proportional to the generators $\sqrt{2}E_{p^i}$ and $\sqrt{2}E_{q^j}$. For vanishing NUT charge $p_\sigma = 0$ we get
\[ e^{2U}V_{BH} = e^{2U} \langle \Gamma | \mathcal{N}^{-1} \otimes I | \Gamma \rangle \] (133)
which can be checked to yield the usual expression for the $V_{BH}$ black hole potential.

Let us now rewrite our expression of Eq.(66) for $|\Psi\rangle$ in terms of our conserved quantities. First by using Eqs.(122) and (129) we express $|d\zeta\rangle$ in terms of the charges as
\[ |d\zeta\rangle = 2e^{2U}(\mathcal{N}^{-1} \otimes I)|\hat{\Gamma}\rangle, \] (134)
to arrive at the expression
\[ |\Psi\rangle = 2e^{2U}(M_T^3 \otimes M_T^2 \otimes M_T^1 \otimes I)^{-1}|\hat{\Gamma}\rangle. \] (135)
We can further transform this to obtain $|\tilde{\Psi}\rangle = (UUU|\Psi\rangle$ of Eq.(83)
\[ |\tilde{\Psi}\rangle = 2e^{2U}(I \otimes I \otimes I \otimes UM_0)(V \otimes V \otimes V \otimes I)(S_3 \otimes S_2 \otimes S_1 \otimes I)|\hat{\gamma}\rangle, \] (136)
where we have used that
\[ UM_T^{-1} = VS_3. \] (137)
Here the matrices $V$ and $S_j$, $j = 1, 2, 3$ are the ones of Eq.(2) discussed in the Introduction and
\[ |\hat{\gamma}\rangle = (\sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes I)|\hat{\Gamma}\rangle. \] (138)
Clearly for $k = 0$ i.e. vanishing NUT charge $|\gamma\rangle$ is just the phase-flipped version of $|\Gamma\rangle$ of Eq.(131). This $|\gamma\rangle$ reinterpreted as a three-qubit state is just the charge state mentioned in Eq.(1). Now notice that we have
\[ UM_0 = \frac{i}{\sqrt{2}}e^{-U} \begin{pmatrix} 1 & * \\ 1 & * \end{pmatrix}, \] (139)
where the terms in the second column are not needed since for the four-qubit state $|\hat{\Gamma}\rangle$ we have as usual $\hat{\Gamma}_{j_3j_2j_1} = 0$, a structure that dates back to the similar one of $|\zeta\rangle$ and $|\Gamma\rangle$. Now obviously $\tilde{\Psi}_{j_3j_2j_1} = \tilde{\Psi}_{j_3j_2j_10}$ and either of them can be reinterpreted as the ones of a
three-qubit state. (See also Eq. (89) in this respect.) For this three-qubit projection we have

$$|\hat{\Psi}\rangle_3 \equiv i\sqrt{2}|\hat{\chi}\rangle = i\sqrt{2}e^{iU}(V \otimes V \otimes V)(S_3 \otimes S_2 \otimes S_1)|\hat{\gamma}\rangle,$$

(140)

where by virtue of Eqs. (138) and (129)

$$|\hat{\gamma}\rangle = |\gamma\rangle + p_\sigma(\sigma_1 \otimes \sigma_1 \otimes \sigma_1)|\zeta\rangle.$$

(141)

Eqs. (140) and (141) clearly show how our state $|\chi\rangle$ of Eq. (7) as a special case of $|\hat{\chi}\rangle$ is embedded in a four-qubit state $|\hat{\Psi}\rangle$. Moreover, in achieving this we managed to present a generalization also valid in the case of nonvanishing NUT charge. The important new property to be noted here is that unlike $|\gamma\rangle$ which is constant the one $|\hat{\gamma}\rangle$ is depending on $\tau = \frac{1}{r}$ via the Wilson lines $\zeta^I$ and $\tilde{\zeta}_I$. Notice that we also have $||\hat{\Psi}||^2 = 4e^{2U}V_{BH} = 4||\hat{\chi}||^2_3$, i.e. the Black Hole potential is just the norm of a three-qubit state$^{7,16}$.

For later use for static spherically symmetric solutions let us write out explicitly the quantities $E_\alpha$ of Eq.(92)-(93) in terms of the three-qubit state $|\chi\rangle$ of Eqs.(7) and (1)

$$E_0 = i\sqrt{8}\chi_{111}, \quad E_1 = i\sqrt{8}\chi_{110}, \quad E_2 = i\sqrt{8}\chi_{101}, \quad E_3 = i\sqrt{8}\chi_{011},$$

(142)

$$\overline{E}_0 = i\sqrt{8}\chi_{000}, \quad \overline{E}_1 = i\sqrt{8}\chi_{001}, \quad \overline{E}_2 = i\sqrt{8}\chi_{010}, \quad \overline{E}_3 = i\sqrt{8}\chi_{100}.$$

(143)

For the more general stationary solutions we have to use $|\hat{\chi}\rangle$ as given by Eq.(140).

In closing this section let us also calculate the conserved quantity

$$m \equiv \frac{1}{4}\text{Tr}(I_1 \otimes H_0)Q_{22}.$$  

(144)

Writing out explicitly $Q_{22}$ using Eq.(116) we notice that many terms end with the matrix $\zeta g$. Using the cyclic property of the trace these terms in some cases result in ones begining with $\zeta g(I_1 \otimes H_0)\zeta^T g$ which is vanishing. Employing Eq.(132) and the definition $p_{y_0} = \frac{\dot{y}_0}{2y_0}$ the result of these considerations will be just two nonvanishing terms yielding the final result

$$m = \frac{1}{2}\langle\Gamma|\zeta\rangle + x_0p_{x_0} + y_0p_{y_0} = -\frac{1}{2}(\zeta^T \dot{p}_\zeta^I + \tilde{\zeta}_I \dot{p}_{\tilde{\zeta}_I}) + \sigma p_\sigma + \dot{U}.$$ 

(145)

which is the ADM mass of the black hole$^{28-30}$. 

27
VI. BLACK HOLE SOLUTIONS AS ENTANGLED SYSTEMS

A. BPS solutions

Let us consider our four-qubit state $|\Lambda\rangle$ of Eqs. (105)-(107). In this section we would like to investigate issues of separability for this state. In particular in this subsection we will be interested in the sufficient and necessary condition for the separability of the first qubit, i.e. the one which is labelled by $a_0$ in Eq. (107). From our previous considerations it is clear that this qubit is the one of special status, i.e. it is the one transforming as a doublet under the $R$-symmetry.

In QIT terms separability of this qubit from the rest is equivalent to the condition that the (unnormalized) $2 \times 2$ reduced density matrix $\varrho_1 \equiv \text{Tr}_1 |\Lambda\rangle\langle \Lambda|$ represents a pure state. This density matrix is of the form

$$\varrho_1 = \begin{pmatrix} \langle \Lambda_0 | \Lambda_0 \rangle & \langle \Lambda_0 | \Lambda_1 \rangle \\ \langle \Lambda_1 | \Lambda_0 \rangle & \langle \Lambda_1 | \Lambda_1 \rangle \end{pmatrix}, \quad \langle \Lambda_0 | \Lambda_0' \rangle = \sum_{a_3,a_2,a_1=0,1} \Lambda_{a_3a_2a_10} \Lambda_{a_3a_2a_10}'.$$

This is a pure state i.e. a projector if and only if $\text{Det}\varrho_1 = 0$. Equivalently this condition is satisfied iff the two 8 component vectors $\Lambda_{a_3a_2a_10}$ and $\Lambda_{a_3a_2a_11}$ are proportional, i.e. $\Lambda_{a_3a_2a_10} = \lambda \Lambda_{a_3a_2a_11}$. By virtue of the reality condition of Eq. (112) we also have the constraint $|\lambda| = 1$. Using the definitions in Eq. (106) this means that

$$\mathcal{E}_0 = \lambda \varepsilon_0, \quad \mathcal{E}_j = \lambda \varepsilon_j, \quad |\lambda| = 1.$$

The first consequence of these considerations is that for the state $\Lambda$ the quadratic invariant $I_1$ of the Appendix (see Eq. (189)) which is related to $d\mathbf{s}^2$ of Eq. (114) is vanishing. Moreover, since the first column of the matrix $\Lambda$ of Eq. (106) is proportional to the second and the third one is proportional to the fourth, the invariant $I_4$ which is according to Eq. (191) just the determinant of $\Lambda$ is also vanishing. A straightforward calculation based on Eqs. (190) and (193) shows that the remaining two algebraically independent invariants $I_3$ and $I_2$ are also vanishing.

Now the $4 \times 4$ matrix $\Omega \equiv \Lambda^T g \Lambda g$ satisfies Eq. (197) of the Appendix, so it follows that $\Omega^4 = 0$. This implies that the matrix $\mathcal{R}_\Lambda$ of Eq. (202) is nilpotent. According to the terminology of Ref. 44 states with the property that their associated $8 \times 8$ matrix $\mathcal{R}_\Lambda$ is nilpotent are called nilpotent states. Hence our separable state is a (trivial) example of

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a nilpotent 4-qubit state. Moreover, since $R_\Lambda$ is just $2P'$ of Eq. (105) and this matrix is unitarily related to the matrix $Q$ of Eq. (113) of conserved charges, it follows that $Q$ is also nilpotent.

In order to link these considerations to the usual static, extremal spherically symmetric BPS black hole solutions we choose $\lambda$ as

$$\lambda = -i \sqrt{\frac{Z}{Z}}.$$  \hspace{1cm} (148)

Note that for static solutions the NUT charge is zero, hence $x_0 = 0$ and $e_0 = -\frac{\partial U}{\partial \phi} = d\phi$, i.e. $\overline{e_0} = e_0$.

Now in the language of supergravity the above discussed condition on separability is just the usual one on the existence of Killing spinors\cite{27,30} expressed in terms of the quaternionic vierbein

$$\Lambda_{a_3 a_2 a_1 a_0} e^{a_0} = 0, \quad e^{a_0} = \begin{pmatrix} 1 \\ \lambda \end{pmatrix}, \hspace{1cm} (149)$$

and Eqs. (117), (96) and (31), give rise to the attractor flow equations\cite{15,27}\cite{15,27}

$$\dot{U} = -e^U |Z|, \quad \dot{z}^j = -2e^U G^{j\overline{k}} \partial_{\overline{k}} |Z|.$$  \hspace{1cm} (150)

As it is well-known these first order equations imply that the corresponding second order equations of Eq. (33) hold. Moreover, by virtue of the vanishing of the invariant $I_1$ the constraint of Eq. (28) is also satisfied hence the solution is extremal.

From this analysis we have learnt that the condition of separability for the first qubit for the 4-qubit state $|\Lambda\rangle$ taken together with the special choice of Eq. (148) yields the first order attractor flow equations. Moreover, we have seen that in this case $|\Lambda\rangle$ is a nilpotent state. This property of $|\Lambda\rangle$ is related to the well-known nilpotency of the Noether charge\cite{27,28,31} $Q$. Notice however, that our approach does not directly yield the order of nilpotency of $Q$ for BPS solutions which is three\cite{27}.

**B. Non-BPS solutions with vanishing central charge**

Let us discuss the separability properties of $|\Lambda\rangle$ associated with the remaining qubits not playing any distinguished role. Here we chose to consider separability of the 4th qubit. An argument similar to the one as given in the previous subsection shows that the sufficient and
necessary condition of separability for this qubit is that the first row is proportional to the third and the second is proportional to the fourth. Due to the reality condition we again have $|\lambda| = 1$ and we get

\[
\mathcal{E}_0 = -\lambda e_3, \quad \mathcal{E}_1 = -\lambda e_2, \quad \mathcal{E}_2 = -\lambda e_2, \quad \mathcal{E}_3 = -\lambda e_0.
\] (151)

Using the definitions of Eq.(96) these conditions take the explicit form

\[
\frac{\dot{z}_0}{y_0} = \overline{\lambda} e^U Z_3, \quad \frac{\dot{z}_1}{y_1} = \lambda e^U Z_2, \quad \frac{\dot{z}_2}{y_2} = \lambda e^U Z_1, \quad \frac{\dot{z}_3}{y_3} = -\overline{\lambda} e^U Z,
\] (152)

where $Z_j \equiv -2iy_j D_j Z$ with $D_j$ as given by Eq.(32). Now for static solutions we again have no twist potential i.e. $x_0 = 0$ hence by choosing

\[
\lambda = -i \sqrt{\frac{Z_3}{Z_3}}
\] (153)

we get

\[
\dot{U} = -e^U |Z_3|, \quad \dot{z}^j = -2e^U G^j_k \partial_k |Z_3|.
\] (154)

These expressions show that demanding separability for the fourth qubit taken together with the choice of Eq.(153) yields the first order equations characterizing attractors with vanishing central charge\textsuperscript{25}.

Clearly similar considerations apply for issues of separability for the second and third qubits. The result will be similar sets of equations with $|Z_3|$ replaced by $|Z_1|$ and $|Z_2|$. This amounts to taking different forms for the fake superpotential\textsuperscript{27}.

Note that the value for the four-qubit invariant $I_1$ is related to the extremality parameter. Unlike the other three algebraically independent invariants, this is also a permutation invariant. Of course the value of $I_1$ is zero for both BPS and non-BPS solutions with vanishing central charge, expressing the fact that our solutions are extremal. Moreover, for all of our non-BPS solutions some rows or columns of the $4 \times 4$ matrix are proportional, hence the invariant $I_4$ is zero as well. Calculations show that the remaining invariants $I_2$ and $I_3$ also give zero, hence our considerations on the nilpotency of $|\Lambda\rangle$ familiar from the previous subsection still apply.

In closing this subsection we note that the conditions for separability can be written in the familiar form\textsuperscript{27} of Eq.(149) with the label of $e^{a_0}$ is $a_0$ for BPS, $a_j, j = 1, 2, 3$ for non-BPS solutions with vanishing central charge. Of course $\lambda$ should be modified accordingly.
C. Non-BPS seed solutions

From the previous subsections it is obvious that the condition of extremality related to the vanishing of the invariant $I_1$ can be satisfied in a number of different ways. Explicitly the relevant equation to be satisfied is

$$\sum_{\alpha=0}^{3} \overline{\mathcal{E}}_{\alpha} \mathcal{E}_{\alpha} = \sum_{\alpha=0}^{3} \overline{\mathcal{E}}_{\alpha} e_{\alpha}. \quad (155)$$

For static solutions we have already remarked that $\overline{e}_0 = e_0$, hence for BPS solutions Eqs. (147)-(148) can be written in the form $\mathcal{E}_{\alpha} = \lambda \overline{\mathcal{E}}_{\alpha}$, i.e. $\mathcal{E}_{\alpha}$ is related to $\overline{\mathcal{E}}_{\alpha}$ via a special element of $U(4)$ containing merely phase factors $\lambda$ in its diagonal. In the case of non-BPS solutions with vanishing central charge Eqs. (151)-(153) of the previous subsection can be written in a similar way in terms of another element of $U(4)$

$$\begin{pmatrix} \mathcal{E}_0 \\ \mathcal{E}_1 \\ \mathcal{E}_2 \\ \mathcal{E}_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -\lambda \\ 0 & 0 & -\overline{\lambda} & 0 \\ 0 & -\overline{\lambda} & 0 & 0 \\ -\lambda & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix}. \quad (156)$$

Similarly the basic equations of the remaining two cases of the previous subsection can be expressed in terms of similar unitaries. These unitaries are just permutation matrices combined with phase factors and their conjugates. As we have shown this structure is related to the separability of one of the qubits of the state $|\Lambda\rangle$. In simple terms this means that some of the rows or columns of the $4 \times 4$ matrix $\Lambda$ corresponding to $|\Lambda\rangle$ are proportional to each other.

In order to obtain states $|\Lambda\rangle$ which are entangled and at the same time give rise to static spherically symmetric non-BPS black hole solutions with non-vanishing central charge we have to experiment with elements of $U(4)$ of more general type.

Let us consider the following choice

$$\begin{pmatrix} \mathcal{E}_0 \\ \mathcal{E}_1 \\ \mathcal{E}_2 \\ \mathcal{E}_3 \end{pmatrix} = -\frac{i}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix}. \quad (157)$$

Due to the unitarity of the relevant matrix the condition of extremality is satisfied, moreover obviously none of the qubits can be separated from the rest. However, apart from satisfying
Eq. [155] or equivalently Eq. [28] we still have to satisfy the equations of motion i.e. Eq. [33]. In the following we show that the choice of Eq. [157] indeed gives rise to a solution of the latter equations namely the non-BPS seed solution\(^\text{22}\). Clearly apart from characterizing the seed solution in a nice and compact way Eq. [157] also serves as a mnemonic for the structure of the corresponding entangled 4-qubit state \(|\Lambda\rangle\) of Eqs. [105]-[107].

In order to reveal the structure of the seed solution for special non-BPS charge configurations we recall Eqs. [142]-[143] and (7) and employ a discrete Fourier (Hadamard) transformation to \(|\chi\rangle\) as

\[
|\tilde{\chi}(\tau)\rangle = (H \otimes H \otimes H)|\chi(\tau)\rangle, \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix},
\]  

(158)

The amplitudes of this state are

\[
\sqrt{2y_1 y_2 y_3} \chi_{000} = -i e^U y_1 y_2 y_3 p_0^0,
\]

(159)

\[
\sqrt{2y_1 y_2 y_3} \chi_{110} = i e^U y_1 (x_2 x_3 p_0^0 - x_2 p^3 - x_3 p^2 + q_1)
\]

(160)

\[
\sqrt{2y_1 y_2 y_3} \chi_{101} = i e^U y_2 (x_1 x_3 p_0^0 - x_1 p^3 - x_3 p^1 + q_2),
\]

(161)

\[
\sqrt{2y_1 y_2 y_3} \chi_{011} = i e^U y_3 (x_1 x_2 p_0^0 - x_1 p^2 - x_2 p^1 + q_3),
\]

(162)

\[
\sqrt{2y_1 y_2 y_3} \chi_{111} = e^U (x_1 x_2 x_3 p_0^0 - x_2 x_3 p^1 - x_1 x_3 p^2 - x_1 x_2 p^3 + x_1 q_1 + x_2 q_2 + x_3 q_3 + q_0),
\]

(163)

\[
\sqrt{2y_1 y_2 y_3} \chi_{010} = e^U y_2 y_3 (p^1 - x_1 p^0),
\]

(164)

\[
\sqrt{2y_1 y_2 y_3} \chi_{100} = e^U y_1 y_3 (p^2 - x_2 p^0),
\]

(165)

\[
\sqrt{2y_1 y_2 y_3} \chi_{001} = e^U y_1 y_2 (p^3 - x_3 p^0).
\]

(166)

Now one can check that Eq. (157) can be expressed in terms of these quantities as

\[
\tilde{\chi}_{000} = \frac{i \dot{x}_0}{2 y_0}, \quad \tilde{\chi}_{110} = \frac{i \dot{x}_1}{2 y_1}, \quad \tilde{\chi}_{101} = \frac{i \dot{x}_2}{2 y_2}, \quad \tilde{\chi}_{011} = \frac{i \dot{x}_3}{2 y_3},
\]

(167)

\[
\tilde{\chi}_{111} = \frac{1}{4} \left( \frac{\dot{y}_0}{y_0} - \frac{\dot{y}_1}{y_1} - \frac{\dot{y}_2}{y_2} - \frac{\dot{y}_3}{y_3} \right), \quad \tilde{\chi}_{001} = \frac{1}{4} \left( \dot{y}_0 \frac{y_0}{y_0} + \frac{\dot{y}_1}{y_1} - \frac{\dot{y}_2}{y_2} - \frac{\dot{y}_3}{y_3} \right),
\]

(168)

\[
\tilde{\chi}_{010} = \frac{1}{4} \left( \frac{\dot{y}_0}{y_0} - \frac{\dot{y}_1}{y_1} + \frac{\dot{y}_2}{y_2} - \frac{\dot{y}_3}{y_3} \right), \quad \tilde{\chi}_{100} = \frac{1}{4} \left( \dot{y}_0 \frac{y_0}{y_0} - \frac{\dot{y}_1}{y_1} + \frac{\dot{y}_2}{y_2} + \frac{\dot{y}_3}{y_3} \right).
\]

(169)

For static solutions we have vanishing NUT charge i.e. \(x_0 = 0\) hence the first of these equations reads as \(\tilde{\chi}_{000} = 0\) which by virtue of Eq. (159) means that \(p^0 = 0\). Hence our
candidate for a non-BPS solution should have only seven nonvanishing Fourier amplitudes and no $D6$ brane charges (in the type IIA duality frame).

Let us now introduce the notation

\[ y_0 = e^{\phi_0}, \quad y_j = e^{\phi_j}, \quad \beta \equiv U - \frac{1}{2}(\phi_1 + \phi_2 + \phi_3), \quad \alpha_j \equiv U + \frac{1}{2}\phi_j, \quad (170) \]

with and $j = 1, 2, 3$ (recall also that according to Eq.\(\frac{50}{50}\) now $\phi \equiv \phi_0 = 2U$.) Now our equations take the form

\[ \tilde{\chi}_{111} = \frac{1}{2}\dot{\beta}, \quad \tilde{\chi}_{110} = \frac{i}{2}e^{-\phi_1}\dot{x}_1, \quad \tilde{\chi}_{101} = \frac{i}{2}e^{-\phi_2}\dot{x}_2, \quad \tilde{\chi}_{011} = \frac{i}{2}e^{-\phi_3}\dot{x}_3, \quad (171) \]

\[ \tilde{\chi}_{001} = \frac{1}{2}(\dot{\alpha}_1 - \dot{\alpha}_2 - \dot{\alpha}_3), \quad \tilde{\chi}_{010} = \frac{1}{2}(\dot{\alpha}_2 - \dot{\alpha}_3 - \dot{\alpha}_1), \quad \tilde{\chi}_{100} = \frac{1}{2}(\dot{\alpha}_3 - \dot{\alpha}_1 - \dot{\alpha}_2). \quad (172) \]

Now using Eqs.\(\frac{160}{160}\)-\(\frac{166}{166}\) with the further charge constraints $q_j = 0$, $q_0 < 0$, and $p^1, p^2, p^3 > 0$ one can see that the equations are precisely the ones found in the Appendix of the paper of Gimon et.al.\(\frac{22}{22}\) characterizing the seed solutions for the $D0-D4$ system.

We remark in closing that one can verify by an explicit calculation that all of the four algebraically independent four-qubit invariants $I_k, k = 1, 2, 3, 4$ are vanishing. This means that the corresponding matrix $Q$ of conserved charges is nilpotent. Hence in the terminology of four-qubit entanglement we obtained the result: the relevant state $|\Lambda\rangle$, is a nilpotent state. However, unlike in the previous cases now neither of the qubits can be separated from the rest, hence $|\Lambda\rangle$ is also an entangled state.

Notice however, that neither the order of nilpotency nor the particular entanglement type follows from our simple considerations. It would be interesting to extend our analysis and identify the particular entanglement class to which $|\Lambda\rangle$ belongs case by case. It is important to realize in this respect, that in our simplified considerations we have merely used complex four qubit states and the corresponding $SL(2, \mathbb{C})^\times 4$ invariants. However, we must recall that our state $|\Lambda\rangle$ also have to satisfy the reality condition of Eq. \(\frac{112}{112}\). The result of the implementation of this constraint is that in the black hole context we have to classify orbits under the group $SU(1, 1)^\times 4$ which is merely a subgroup of the full group of admissible local operations\(\frac{42}{42}\). Hence a full entanglement based understanding of black hole solutions in the STU model should rely on the classification of entanglement types of real four qubit states defined by Eq. \(\frac{112}{112}\). This classification should be founded on a study of $SU(1, 1)^\times 4$ invariants. Clearly this reformulation would relate the known classification\(\frac{27}{27}\) of satric black
hole solutions in the STU model in terms of nilpotent orbits to a similar one based on entanglement classes of the relevant real four qubit states.

VII. CONCLUSIONS

In this paper we managed to understand the structure of extremal stationary spherically symmetric black hole solutions in the STU model of $D = 4$, $N = 2$ supergravity in terms of four-qubit systems. Our analysis extended the results obtained in our previous papers based on three qubit systems$^{3,7,16}$. The basic idea facilitating this 4-qubit description was the fact that stationary solutions in $D = 4$ supergravity can be elegantly described by dimensional reduction along the time direction$^{25}$. In this picture stationary solutions can be identified as solutions to a $D = 3$ non-linear sigma model with target space being a symmetric space $G/H$ with $H$ non-compact. The group $G$ extends the global symmetry group $G_4$ of $D = 4$ supergravity, by also incorporating the Ehlers $SL(2, \mathbb{R})$. In our specific case the $N = 2$ STU model can be regarded as a consistent truncation of maximal $N = 8$, $D = 4$ supergravity with $G_4 = E_7(7)$, truncating to $G_4 = SL(2, \mathbb{R})^\times 3$. Timelike reduction then yields the coset $E_8(8)/SO^*(16)$, or in the case of the STU truncation the one $\mathcal{M}_3 = SO(4,4)/SL(2,\mathbb{R})^\times 4$. We have shown that the four copies of $SL(2, \mathbb{R})$s occurring in this coset can be reinterpreted as the group of local operations acting on four qubits subject to special reality constraints. Here the fourth qubit which accounts for the Ehlers group played a special role.

The central object of our considerations was the complex 4-qubit state $|\Lambda\rangle$ of Eqs. (106) and (107), also satisfying the reality condition Eq.(112). The amplitudes of this state of odd parity contain the right invariant one-forms $e_\alpha$, $\alpha = 0, 1, 2, 3$ defined by Eqs.(80)-(81). On the other hand the 8 amplitudes of even parity are just the 8 amplitudes of the 3-qubit state well-known from previous studies concerning the black hole qubit correspondence. According to Eqs.(93) and (96) these amplitudes are related to well-known quantities of special geometry. We have shown that the state $|\Lambda\rangle$ is connected to the line element on $\mathcal{M}_3$ via Eqs. (59), (105) and (114). We also realized that this expression for the line element is minus the quadratic 4-qubit $SL(2, \mathbb{C})$ invariant $I_1$ of Eq.(189). After expressing the 8 amplitudes of the embedded 3-qubit state in terms of the conserved electric, magnetic and NUT charges as in Eqs.(140), (141) this invariant also has the physical interpretation as the BPS parameter$^{28}$. (For nonrotating solutions this parameter is just the extremality
We clarified the relationship between the warp factor, moduli and charge dependent 3-qubit state of Eq. (11), and (17) used in previous studies and the 4-qubit one |Λ⟩. Our considerations enabled a formal generalization for this state (see Eq. (141)) also valid for nonvanishing NUT charge. Notice that for general stationary solutions the entanglement type of this state (i.e. the value of the three-tangle) is also depending on the Wilson lines ζ and ˜ζ. This is in sharp contrast to the static case where the entanglement type is merely depending on the conserved electric and magnetic charges.

Note that one of the qubits of the state |Λ⟩ was special. We have seen that the special status of this qubit is related to the R-symmetry group arising from the restricted holonomy group of the para quaternionic Kähler space M3. We realized that our special set of transformations, based on Hadamard and phase gates and permutations, resulting in the explicit form for |Λ⟩ correspond to the basis transformations similar to the ones rendering the quaternionic vierbein covariantly constant with respect to the spin connection.

The separability properties of this special qubit are related to the solution being BPS or non-BPS. We demonstrated within our formalism the observation of Bergshoeff et al. that static, extremal BPS and non-BPS-solutions with vanishing central charge correspond to states for which one of the qubits is separable from the rest. On the other hand using the non-BPS seed solution for nonvanishing central charge we have shown that |Λ⟩ in this case is entangled. We revealed a connection between the classification of nilpotent states within the realm of quantum information theory and the similar classification of nilpotent orbits. The details of this connection should be explored further.

It is amusing to see that nonextremal solutions should correspond to states which are semisimple. Since nilpotent states are rather exceptional among the 4-qubit ones, semisimple states are the ones that represent genuine 4-qubit entanglement. According to our Appendix for such states at least one of the algebraically independent invariants (namely I1 related to the extremality parameter) are non-vanishing. Such states with special entanglement properties should correspond to nonextremal solutions of special kind.

Notice also in this respect that in the paper of Chemissany et al. dealing with the full integration of black hole solutions in symmetric supergravity solutions the authors notice that in their Lax pair approach exactly non-extremal solutions are easier to describe than extremal ones. Such solutions correspond to diagonalizable initial conditions in terms of the
Lax matrix. On the other hand they note that the initial conditions summarized in nondiagonalizable Lax matrices correspond to extremal BPS and non-BPS solutions, however such Lax matrices represent a subset of measure zero within the space of Lax matrices. Within the context of the STU-model clearly semisimple states should correspond to diagonalizable Lax matrices, and nilpotent states to nondiagonalizable ones. Hence there should be a correspondence between giving the Lax matrix at some initial time and specifying the entanglement properties of the corresponding entangled 4-qubit state.

Notice in particular the highly symmetrical nature of the genuine entangled 4-qubit class (see Eq. (201) of the Appendix and the structure of its invariants.) This state is the 4-qubit analogue of the famous GHZ state familiar from 3-qubit entanglement. Notice that for choosing \( a, b, c, d \in \mathbb{R} \) this state automatically satisfies the reality condition of Eq. (112). Acting on this state with \( SU(1,1)^{\times 4} \) transformations preserving this reality condition and also the values of the algebraically independent invariants results in a state \( |\Lambda\rangle \) containing 16 real parameters. Using this parametrization and the black hole qubit correspondence it would be amusing to find a corresponding highly symmetrical non-extremal solution.

Recall also the classification of black hole solutions in the STU model in terms of three qubit entanglement classes as given by Kallosh and Linde\(^2\). In this paper the authors noticed a similarity between the classification of complex three qubit states and the corresponding classification of small and large black holes in the STU model related to real three qubit ones. In the light of our results we might substantially generalize this interesting result. Indeed, by embedding the usual three-qubit picture into the four qubit one as described here, we also have the possibility to include such notions as BPS and non-BPS, extremal and non-extremal solutions into an entanglement based picture. As we have seen the extremality parameter is related to the quadratic four qubit invariant. Extremal black holes are characterized by the vanishing of this quantity. Though the remaining four qubit invariants are all vanishing for the known extremal solutions, but such solutions are still distinguished by their entanglement properties. For BPS and non-BPS solutions with vanishing central charge one of the qubits is separable from the rest, and for the \( Z \neq 0 \) case none of the qubits is separable. Since the states describing such solutions are real (i.e. they are satisfying the reality condition of Eq. (112)) in order to classify their orbit structure we also have to include some additional \( SU(1,1)^{\times 4} \) invariants. We know that the classes in question for extremal solutions are just the nilpotent orbits classified in the paper of Bossard et. al\(^{27} \), hence these classes might
be distinguished by additional $SU(1,1)^\times 4$ invariants whose physical meaning is still to be clarified. On the other hand the three qubit part of our four qubit states is classified by the value of Cayley’s hyperdeterminant i.e. the three-tangle$^{17}$. For small black holes this invariant is vanishing and for large ones it is nonzero and its value is proportional to the black hole entropy.

Finally notice that we have deliberately emphasized the possibility for reformulating the well-known results of the STU model in a suggestive permutation invariant language (see e.g. Eqs. (111)-(112)). Though instructive, this language is deceptive due to the special role we have attached to our first qubit via the use of the Ehlers $SL(2,\mathbb{R})$. However, since the quadratic invariant of Eq.(114) related to the line element and the extremality parameter is a permutation invariant quantity one might speculate whether there is a further possibility for embedding the STU model into an even greater picture where permutation symmetry is manifest. In this respect an exciting possibility is to find the physical relevance (if any) of the permutation invariant quantity of Eq.(200) i.e. the four qubit generalization of Cayley’s hyperdeterminant (the ”four-tangle”).

VIII. APPENDIX

A four qubit state can be written in the form

$$|\Psi\rangle = \sum_{i_3i_2i_1i_0=0,1} \Psi_{i_3i_2i_1i_0}|i_3i_2i_1i_0\rangle, \quad |i_3i_2i_1i_0\rangle \equiv|i_3\rangle\otimes|i_2\rangle\otimes|i_1\rangle\otimes|i_0\rangle \in V_3\otimes V_2\otimes V_1\otimes V_0, \quad (173)$$

where $V_{3,2,1,0} \equiv \mathbb{C}^2$. Let the subgroup of stochastic local operations and classical communication$^{42}$ representing admissible fourpartite protocols be $SL(2,\mathbb{C})^\times 4$ acting on $|\Psi\rangle$ as

$$|\Psi\rangle \mapsto (S_3 \otimes S_2 \otimes S_1 \otimes S_0)|\Psi\rangle, \quad S_\alpha \in SL(2,\mathbb{C}), \quad \alpha = 0, 1, 2, 3. \quad (174)$$

Our aim in this appendix is to give a unified description of four-qubit states taken together with their SLOCC transformations and their associated invariants. As we will see states and transformations taken together can be described in a unified manner using the group $SO(4,4,\mathbb{C})$. 
Let us introduce the $2 \times 2$ matrices
\[
E_{00} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{01} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{10} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_{11} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Then we arrange the 16 complex amplitudes appearing in $\Psi_{i_3 i_2 i_1 i_0}$ in a $4 \times 4$ matrix in three different ways
\[
D_1(\Psi) = \sum_{i_3 i_2 i_1 i_0 = 0, 1} \Psi_{i_3 i_2 i_1 i_0} E_{i_3 i_1} \otimes E_{i_2 i_0} = \begin{pmatrix} \Psi_{0000} & \Psi_{0001} & \Psi_{0010} & \Psi_{0011} \\ \Psi_{0100} & \Psi_{0101} & \Psi_{0110} & \Psi_{0111} \\ \Psi_{1000} & \Psi_{1001} & \Psi_{1010} & \Psi_{1011} \\ \Psi_{1100} & \Psi_{1101} & \Psi_{1110} & \Psi_{1111} \end{pmatrix},
\]

\[
D_2(\Psi) = \sum_{i_3 i_2 i_1 i_0 = 0, 1} \Psi_{i_3 i_2 i_1 i_0} E_{i_3 i_2} \otimes E_{i_1 i_0} = \begin{pmatrix} \Psi_{0000} & \Psi_{0001} & \Psi_{0010} & \Psi_{0011} \\ \Psi_{0010} & \Psi_{0011} & \Psi_{0100} & \Psi_{0101} \\ \Psi_{1000} & \Psi_{1001} & \Psi_{1100} & \Psi_{1101} \\ \Psi_{1010} & \Psi_{1011} & \Psi_{1110} & \Psi_{1111} \end{pmatrix} = \begin{pmatrix} X & Y \\ W & Z \end{pmatrix},
\]

\[
D_3(\Psi) = \sum_{i_3, i_2, i_1, i_0 = 0, 1} \Psi_{i_3 i_2 i_1 i_0} E_{i_3 i_2} \otimes E_{i_1 i_0} = \begin{pmatrix} X & W \\ Y & Z \end{pmatrix}
\]

where the $2 \times 2$ matrices $X, Y, W, Z$ are introduced merely to illustrate the block structure of the relevant matrices. The first matrix is obtained by arranging the components of $X, Y, W, Z$ as the first, second, third and fourth rows. Notice that the arrangement $D_1$ of Eq.(176) is one of Eq.(11). Clearly changing $D_1$ to $D_2$ or to $D_3$ corresponds to the two generators of the permutation group $S_3$ acting on the last three qubits. Applying such permutations to the qubits corresponds to a similar permutation of the entries $s_3, s_2, s_1$ of Eq.(40) resulting in the matrices $\mathcal{D}(s_3, s_1, s_2, s_0; D_2)$, and $\mathcal{D}(s_2, s_3, s_1, s_0; D_3)$. These permutations give rise to alternative forms for the matrix exponentials of Eq.(55-56) with special roles attached to the second and the third qubit respectively.

Our matrix $\mathcal{D} \equiv \mathcal{D}(s_3, s_2, s_1, s_0; D_1)$ of Eq.(40) in the parametrization used in Bossard et.al. takes the following form.

38
\[
\begin{pmatrix}
H_3 + H_2 & E_2 & E_3 & 0 & -F_{q_1} & -E_{p_1} & -F_{q_0} & -E_{p_0} \\
F_2 & H_3 - H_2 & 0 & E_3 & F_{p_3} & -E_{q_3} & F_{q_2} & E_{p_2} \\
F_3 & 0 & H_2 - H_3 & E_2 & F_{p_2} & -E_{q_2} & F_{q_3} & E_{p_3} \\
0 & F_3 & F_2 & -H_3 - H_2 & F_{p_1} & -E_{q_0} & -F_{p_0} & E_{q_1} \\
-E_{q_3} & E_{p_3} & E_{p_2} & E_{p_0} & H_1 + H_0 & E_0 & E_1 & 0 \\
-F_{p_3} & -E_{q_3} & -F_{q_2} & -F_{q_0} & F_0 & H_1 - H_0 & 0 & E_1 \\
-E_{q_0} & E_{q_2} & E_{q_3} & -E_{p_3} & F_1 & 0 & H_0 - H_1 & E_0 \\
-F_{p_0} & F_{p_2} & F_{p_3} & F_{q_1} & 0 & F_1 & F_0 & -H_1 - H_0
\end{pmatrix}
\]

(179)

It can be checked that the matrix \( SDS^T \) where

\[
S = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0
\end{pmatrix}
\]

(180)

is just the one used in Eq.(4.6) of that paper. This matrix also relates our matrix \( G \) of
Eq.(39) to the usual \( SO(4,4) \) invariant one

\[
\eta = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}, \quad 1 = I \otimes I.
\]

(181)

Relating the upper right block of Eq.(179) to the \( 4 \times 4 \) matrix \( D_g \) of Eq.(40) shows that in
this parametrization

\[
D = D_1 = \begin{pmatrix}
-E_{p_0} & F_{q_0} & E_{p_1} & -F_{q_1} \\
E_{p_2} & -F_{q_2} & E_{q_3} & F_{p_3} \\
E_{p_3} & -F_{q_3} & E_{q_2} & F_{p_2} \\
E_{q_1} & F_{p_1} & E_{q_0} & F_{p_0}
\end{pmatrix}
\]

(182)

which justifies our parametrization of \( \zeta^I E_{q_I} + \bar{\zeta}_I E_{p_I} \) used in Eq.(54).
Let us discuss now the structure of four-qubit $SL(2, \mathbb{C})^4$ invariants. The number of algebraically independent four-qubit invariants is four. We have one quadratic, two quartic, and one sextic invariant. In our recent paper\textsuperscript{41} we investigated the structure of these invariants in the special frame where two of our qubits played a distinguished role. Clearly this is the case in the black hole context, since one of the special qubits is associated with the Ehlers-group and the choice of the other is just a matter of convention related to the special choice $D_1$, $D_2$ or $D_3$ of Eqs.\textsuperscript{176,177,178}.

To an arbitrary state

$$|\Lambda\rangle = \sum_{i_3i_2i_1i_0=0,1} \Lambda_{i_3i_2i_1i_0} |i_3i_2i_1i_0\rangle,$$  \hspace{1cm} (183)

we can associate the $4 \times 4$ matrix

$$\Lambda \equiv \begin{pmatrix} \Lambda_{0000} & \Lambda_{0001} & \Lambda_{0010} & \Lambda_{0011} \\ \Lambda_{0100} & \Lambda_{0101} & \Lambda_{0110} & \Lambda_{0111} \\ \Lambda_{1000} & \Lambda_{1001} & \Lambda_{1010} & \Lambda_{1011} \\ \Lambda_{1100} & \Lambda_{1101} & \Lambda_{1110} & \Lambda_{1111} \end{pmatrix} \equiv \begin{pmatrix} A^1 & B^1 & C^1 & D^1 \\ A^2 & B^2 & C^2 & D^2 \\ A^3 & B^3 & C^3 & D^3 \\ A^4 & B^4 & C^4 & D^4 \end{pmatrix},$$  \hspace{1cm} (184)

or four four-vectors. The splitting of the amplitudes of $|\Lambda\rangle$ into four four-vectors reflects our special choice for the distinguished qubits compatible with our conventions. Now we introduce on the vector space $\mathbb{C}^4 \simeq \mathbb{C}^2 \times \mathbb{C}^2$ corresponding to the third and fourth qubit a symmetric bilinear form $g : \mathbb{C}^4 \times \mathbb{C}^4 \rightarrow \mathbb{C}$ with matrix representation: $g \equiv \varepsilon \otimes \varepsilon$. This means that we have an $SL(2, \mathbb{C})^2$ invariant quantity with the explicit form

$$g(A, B) \equiv g_{\alpha\beta} A^\alpha B^\beta = A_\alpha B^\alpha = A \cdot B = A^1 B^4 - A^2 B^3 - A^3 B^2 + A^4 B^1.$$  \hspace{1cm} (185)

We can also introduce a dual four-qubit state

$$|\lambda\rangle = \sum_{i_3i_2i_1i_0=0,1} \lambda_{i_3i_2i_1i_0} |i_3i_2i_1i_0\rangle,$$  \hspace{1cm} (186)

with the associated matrix

$$\lambda \equiv \begin{pmatrix} \lambda_{0000} & \lambda_{0001} & \lambda_{0010} & \lambda_{0011} \\ \lambda_{0100} & \lambda_{0101} & \lambda_{0110} & \lambda_{0111} \\ \lambda_{1000} & \lambda_{1001} & \lambda_{1010} & \lambda_{1011} \\ \lambda_{1100} & \lambda_{1101} & \lambda_{1110} & \lambda_{1111} \end{pmatrix} \equiv \begin{pmatrix} a^1 & b^1 & c^1 & d^1 \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \\ a^4 & b^4 & c^4 & d^4 \end{pmatrix},$$  \hspace{1cm} (187)

where

$$a^\alpha = \epsilon^{\alpha\beta\gamma\delta} B_\beta C_\gamma D_\delta, \hspace{0.5cm} b^\beta = \epsilon^{\alpha\beta\gamma\delta} A_\alpha C_\gamma D_\delta \hspace{0.5cm} c^\gamma = \epsilon^{\alpha\beta\gamma\delta} A_\alpha B_\beta D_\delta \hspace{0.5cm} d^\delta = \epsilon^{\alpha\beta\gamma\delta} A_\alpha B_\beta C_\gamma.$$  \hspace{1cm} (188)
Here $\varepsilon^{1234} = +1$, and indices are lowered by the matrix of $g$. Notice that the amplitudes of the dual four-qubit state are *cubic* in the original ones. Such dual states were first introduced in Ref.41, and were later defined differently in the three-qubit context by Borsten et.al.10. These dual states have also made their debut to the physics of black holes admitting Freudenthal or Jordan duals10,12.

Using these definitions we define the quadratic and sextic invariants as

$$I_1 \equiv \frac{1}{2}(A \cdot D - B \cdot C), \quad I_3 \equiv \frac{1}{2}(a \cdot d - b \cdot c).$$

(189)

(The labelling convention and normalization for our invariants will be clarified below.) This form of the sextic invariant is deceptively simple. Its explicit form in terms of the dot product of Eq.(185) is

$$2I_3 = \text{Det} \begin{pmatrix} A \cdot A & A \cdot B & A \cdot D \\ A \cdot C & B \cdot C & C \cdot D \\ A \cdot D & B \cdot D & D \cdot D \end{pmatrix} - \text{Det} \begin{pmatrix} A \cdot B & B \cdot B & B \cdot C \\ A \cdot C & B \cdot C & C \cdot C \\ A \cdot D & B \cdot D & C \cdot D \end{pmatrix}.$$  

(190)

We also recall that the explicit form of $I_1$ is hiding its permutation invariance. A permutation invariant form is the one we used in Eq.(114), i.e. we have $ds^2 = -I_1(|\Lambda\rangle)$ where $|\Lambda\rangle$ is the special state of Eq.(106). Moreover, though the expression of $I_3$ of Eq.(189) is similar to the one of $I_1$ the invariant $I_3$ is *not* invariant under the permutation of the qubits.

Now we turn to the structure of quartic invariants. We have two of them and the simplest is the obvious expression

$$I_4 \equiv \text{Det} \Lambda$$

(191)

i.e. the determinant of the $4 \times 4$ matrix of Eq.(184). In order to present the definition of our last invariant we define separable bivectors of the form

$$\Pi_{\mu\nu\alpha\beta} \equiv \Lambda_{\mu\alpha}\Lambda_{\nu\beta} - \Lambda_{\mu\beta}\Lambda_{\nu\alpha}, \quad \alpha, \beta, \mu, \nu = 1, 2, 3, 4.$$  

(192)

Here our labelling convention $\Lambda_{\mu\alpha}$ indicates that $\alpha = 1, 2, 3, 4$ identifies the four-vector in question (i.e. $A, B, C$ or $D$ of Eq.(184)), and the label $\mu = 1, 2, 3, 4$ refers to the component of the particular vector. Now our last invariant is the quartic combination

$$I_2 = \frac{1}{6} \Pi_{\mu\nu\alpha\beta} \Pi^{\mu\nu\alpha\beta}. $$

(193)
For the explicit form of this invariant we introduce the product of two separable bivectors as

$$(A \wedge B) \cdot (C \wedge D) \equiv 2(A \cdot C)(B \cdot D) - 2(A \cdot D)(B \cdot C).$$  \hspace{1cm} (194)

Then the explicit form is

$$I_2 = \frac{1}{6} \left[ (A \wedge B) \cdot (C \wedge D) + (A \wedge C) \cdot (B \wedge D) - \frac{1}{2}(A \wedge D)^2 - \frac{1}{2}(B \wedge C)^2 \right].$$  \hspace{1cm} (195)

Let us now present the reason for considering these particular combinations for the set of algebraically independent $SL(2, \mathbb{C})^4$ invariants. Let us consider the matrix

$$\Omega \equiv \Lambda^T g \Lambda.$$  \hspace{1cm} (196)

Then the characteristic polynomial of this $4 \times 4$ matrix is

$$\text{Det}(1t - \Omega) = t^4 - 4I_1 t^3 + 6I_2 t^2 - 4I_3 t + I_4^2.$$  \hspace{1cm} (197)

Clearly we have

$$I_1 = \frac{1}{4} \text{Tr} \Omega, \quad I_2 = \frac{1}{12} [(\text{Tr} \Omega)^2 - \text{Tr} \Omega^2],$$  \hspace{1cm} (198)

$$I_3 = \frac{1}{24} [(\text{Tr} \Omega)^3 - 3 \text{Tr} \Omega \text{Tr} \Omega^2 + 2 \text{Tr} \Omega^3], \quad (I_4)^2 = \text{Det} \Omega.$$  \hspace{1cm} (199)

This form of writing our invariants is related to the fact that there is a $1 - 1$ correspondence between the $SL(2, \mathbb{C})^4$ orbits of four-qubit states and the $SO(4, \mathbb{C}) \times SO(4, \mathbb{C})$ ones of $4 \times 4$ matrices.

The polynomial of Eq.(197) first appeared in Ref.\textsuperscript{41} its role as a characteristic polynomial has been emphasized in Ref.\textsuperscript{44}. The discriminant of this fourth order polynomial is the hyperdeterminant $D_4$ of the $2 \times 2 \times 2 \times 2$ hypercube $\Lambda_{i_1i_2i_3i_4}$. It is a polynomial of degree 24 in the 16 amplitudes and has 2894276 terms\textsuperscript{46}. It can be shown\textsuperscript{37,41} that $D_4$ can be expressed in terms of our fundamental invariants in the form

$$256D_4 = S^3 - 27T^2$$  \hspace{1cm} (200)

where

$$S = (I_4^2 - I_2^2) + 4(I_2^2 - I_1 I_3), \quad T = (I_4^2 - I_2^2)(I_1^2 - I_2) + (I_3 - I_1 I_2)^2.$$  \hspace{1cm} (201)

In closing this appendix we briefly discuss some results on the full classification of entanglement classes for four qubits\textsuperscript{44,47}. By entanglement classes we mean orbits under
$SL(2, \mathbb{C})^4 \cdot \text{Sym}_4$ where $\text{Sym}_4$ is the symmetric group on four symbols. The basic result states that four qubits can be entangled in nine different ways\textsuperscript{44,47}. It is to be contrasted with the two entanglement classes\textsuperscript{42} obtained for three qubits.

Let us consider the matrix

$$\mathcal{R}_\Lambda \equiv \begin{pmatrix} 0 & \Lambda g \\ -\Lambda^T g & 0 \end{pmatrix}.$$  \hfill (202)

If $\Lambda$ is the special matrix of Eq.(106) used in the black hole context $\mathcal{R}_\Lambda$ is just $2\mathcal{P}'$ of Eq.(105). If the matrix $\mathcal{R}_\Lambda$ is diagonalizable under the action

$$\mathcal{R}_\Lambda \mapsto S \mathcal{R}_\Lambda S^{-1}, \quad S = \begin{pmatrix} S_3 \otimes S_2 & 0 \\ 0 & S_1 \otimes S_0 \end{pmatrix}, \quad S_\alpha \in SL(2, \mathbb{C})$$  \hfill (203)

we say that the corresponding four-qubit state $|\Lambda\rangle$ is \textit{semisimple}. If $\mathcal{R}_\Lambda$ is \textit{nilpotent} then we call the corresponding state $|\Lambda\rangle$ \textit{nilpotent too}. It is known that a nilpotent orbit is \textit{conical} i.e. if $|\Lambda\rangle$ is an element of the orbit then $t|\Lambda\rangle$ is also an element for all nonzero complex numbers $t$. Hence a nilpotent orbit is also a $GL(2, \mathbb{C})^\otimes 4$ orbit. (Recall that our $\mathcal{P}'$ is in the $GL(2, \mathbb{C})^\otimes 4$ orbit of the original $\mathcal{P}$ of Eq.(62).) It is clear that for nilpotent states all of our algebraically independent invariants are zero. These are the states we associated to extremal black hole solutions of BPS and non-BPS type.

A generic semisimple state of four qubits can always be transformed to the form\textsuperscript{47}

$$|G_{abcd}\rangle = \frac{a + d}{2}(|0000\rangle + |1111\rangle) + \frac{a - d}{2}(|0011\rangle + |1100\rangle) + \frac{b + c}{2}(|0101\rangle + |1010\rangle) + \frac{b - c}{2}(|0110\rangle + |1001\rangle),$$  \hfill (204)

where $a, b, c, d$ are complex numbers. This class corresponds to the so called GHZ class found in the three-qubit case\textsuperscript{42}. For this state the reduced density matrices obtained by tracing out all but one of the qubits are proportional to the identity. This is the state with maximal four-partite entanglement.

Another interesting property of this state is that it does not contain true three-partite entanglement. A straightforward calculation shows that the values of our invariants $(I_1, I_2, I_3, I_4)$ occurring for the state $|G_{abcd}\rangle$ representing the generic class are

$$I_1 = \frac{1}{4}[a^2 + b^2 + c^2 + d^2], \quad I_2 = \frac{1}{6}[(ab)^2 + (ac)^2 + (ad)^2 + (bc)^2 + (bd)^2 + (cd)^2].$$  \hfill (205)
\[ I_4 = \frac{1}{4}[(abc)^2 + (abd)^2 + (acd)^2 + (bcd)^2], \quad I_3 = abcd, \]  

hence the values of the invariants \((4I_1, 6I_2, 4I_3, I_4^2)\) are given in terms of the elementary symmetric polynomials in the variables \((x_1, x_2, x_3, x_4) = (a^2, b^2, c^2, d^2)\). On the generic class \(|G_{abcd}|\) the value of \(D_4\) can be expressed as\(^{37,41}\)

\[ D_4 = \frac{1}{256} \prod_{i<j} (x_i - x_j)^2 = \frac{1}{256} V(a^2, b^2, c^2, d^2)^2, \]  

where \((x_1, x_2, x_3, x_4) \equiv (a^2, b^2, c^2, d^2)\) and \(V\) is the Vandermonde determinant.

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