Uncertainty Inequalities for 3D Octonionic-valued Signals Associated with Octonion Offset Linear Canonical Transform

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Abstract. The octonion offset linear canonical transform (O−OLCT) can be defined as a time-shifted and frequency-modulated version of the octonion linear canonical transform (O−LCT), a more general framework of most existing signal processing tools. In this paper, we first define the (O−OLCT) and provide its closed-form representation. Based on this fact, we study some fundamental properties of proposed transform including inversion formula, norm split and energy conservation. The crux of the paper lies in the generalization of several well known uncertainty relations for the (O−OLCT) that include Pitt’s inequality, logarithmic uncertainty inequality, Hausdorff-Young inequality and local uncertainty inequalities.

Keywords: Quaternion offset linear canonical transform (QOLCT); Octonion; Octonion Fourier transform (O−OFT); Octonion offset linear canonical transform (O−OLCT); Uncertainty principle.

2000 Mathematics subject classification: 42B10; 43A32; 94A12; 42A38; 30G30.

1. Introduction

The hyper-complex Fourier transform (FT) is of the interest in the present era. It treats multi-channel signals as an algebraic whole without losing the spectral relations. Presently, many hyper-complex FTs exists in literature which are defined by different approaches, see [1, 2]. The developing interest in hyper-complex FTs including applications in watermarking, color image processing, image filtering, pattern recognition and edge detection [3-8]. Among the various hyper-complex FTs, the most basic ones are the quaternion Fourier transforms (QFTs). QFTs are most widely studied in recent years because of its wide applications in optics and signal processing. QFT [2] is very useful in Cayley-Dickson algebra of order 4 (Quaternions) as it is a substitute to the two-dimensional complex Fourier transform (CFT). Various properties and applications of the QFT were established in [10]-[13]. The generalization of quaternion Fourier transform (QFT) is quaternion linear canonical transform (QLCT), which is more effective signal processing tool than QFT due to its extra parameters, see [14]-[20].

Later, the quaternion linear canonical transform (QLCT) with four parameters has been generalized to a six parameter transform known as quaternion offset linear canonical transform (QOLCT). Due to the time shifting and frequency modulation parameters, the QOLCT has gained more flexibility over classical QLCT. Hence has found wide applications in image and signal processing, see [21]-[23].

On the other hand the Cayley-Dickson algebra of order 8 is known as octonion algebra which deserve special attention in the hyper-complex signal processing. The octonion Fourier transform (O−FT) was proposed by Hahn and Snopek in 2011 [25]. From then O−FT is becoming the hot area of research in modern signal processing community. Some properties and uncertainty relations and applications associated with O−FT have been studied, see [27]-[30]. In 2021 Gao and Li [31] proposed octonion linear canonical transform (OOLCT) as a time-shifted and frequency-modulated version of the octonion Fourier transform (O−FT), a more general framework of most existing signal processing tools. In this paper, we first define the (O−OLCT) and provide its closed-form representation. Based on this fact, we study some fundamental properties of proposed transform including inversion formula, norm split and energy conservation. The crux of the paper lies in the generalization of several well known uncertainty relations for the (O−OLCT) that include Pitt’s inequality, logarithmic uncertainty inequality, Hausdorff-Young inequality and local uncertainty inequalities.

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transform (O–LCT) as a generalization of O–FT by substituting the Fourier kernel with the LCT kernel. They established some vital properties like inversion formula, isometry, Riemann-Lebesgue lemma and proved Heisenberg’s and Donoho-Stark’s uncertainty principles. The generalization of O–FT to other transforms viz linear canonical transform, offset linear canonical transform, Stockwell transform, Quadratic phase transform etc. is still in its infancy.

So motivated and inspired by this, we shall propose the novel octonion offset linear canonical transform (O–OLCT) which have never been proposed up to date, therefore it is worthwhile to rigorously study the Octonion offset linear canonical transform O–OLCT which can be productive for signal processing theory and applications.

The highlights of the paper are pointed out below:

- To introduce a novel integral transform coined as the octonion offset linear canonical transform (O–OLCT).
- To study the fundamental properties of the proposed transform, including the closed-form representation, norm split into quaternions, inversion formula and energy conservation.
- To formulate several classes of uncertainty inequalities, such as the Pitt’s inequality, logarithmic uncertainty inequality, Hausdorff-Young inequality and local uncertainty inequality associated with the octonion offset linear canonical transform (O–OLCT).

The rest of the paper is organized as follows: In Section 2, some general definitions and basic properties of octonions are summarized. The definition and the properties of the O–OLCT are studied in Section 3. In Section 3, we develop a series of uncertainty inequalities such as the Pitt’s inequality, logarithmic uncertainty inequality, Hausdorff-Young inequality and local uncertainty inequality associated with the O–OLCT. Finally, a conclusion is extracted in Section 5.

2. Preliminaries

In this section, we collect some basic facts on the octonion algebra and the offset linear canonical transform (OLCT), which will be needed throughout the paper.

2.1. Octonion algebra.

The octonion algebra denoted by O, [32] is generated by the eighth-order Cayley-Dickson construction. According to His construction, a hypercomplex number $o \in O$ is an ordered pair of quaternions $q_0, q_1 \in \mathbb{H}$

$$o = (q_0, q_1)$$
$$= ((z_0, z_1), (z_2, z_3))$$
$$= q_0 + q_1.e_4$$
$$= (z_0 + z_1.e_2) + (z_2 + z_3.e_2).e_4$$

(2.1)
which has equivalent form

\[ o = s_o + \sum_{i=1}^{7} s_ie_i = s_0 + s_1e_1 + s_2e_2 + s_3e_3 + s_4e_4 + s_5e_5 + s_6e_6 + s_7e_7 \]  

(2.2)

that is \( o \) is a hypercomplex number defined by eight real numbers \( s_i, i = 0, 1, \ldots, 7 \) and seven imaginary units \( e_i \) where \( i = 1, 2, \ldots, 7 \). The octonion algebra is non-commutative and non-associative algebra. The multiplication of imaginary units in the Cayley-Dickson algebra of octonions are presented in Table I or in diagram called Fano scheme, shown in Figure 1.

\begin{table}[h]
\centering
\caption{Multiplication Rules in Octonion Algebra.}
\begin{tabular}{|c||c|c|c|c|c|c|c|c|c|}
\hline
\( \cdot \) & 1 & \( e_1 \) & \( e_2 \) & \( e_3 \) & \( e_4 \) & \( e_5 \) & \( e_6 \) & \( e_7 \) \\
\hline
1 & 1 & \( e_1 \) & \( e_2 \) & \( e_3 \) & \( e_4 \) & \( e_5 \) & \( e_6 \) & \( e_7 \) \\
\hline
\( e_1 \) & \( e_1 \) & -1 & \( e_3 \) & -\( e_2 \) & \( e_5 \) & -\( e_4 \) & -\( e_7 \) & \( e_6 \) \\
\hline
\( e_2 \) & -\( e_2 \) & -1 & \( e_3 \) & \( e_1 \) & -\( e_6 \) & -\( e_4 \) & \( e_7 \) & -\( e_5 \) \\
\hline
\( e_3 \) & \( e_3 \) & -1 & 1 & \( e_6 \) & \( e_7 \) & \( e_4 \) & -\( e_5 \) & -\( e_2 \) \\
\hline
\( e_4 \) & -\( e_4 \) & -1 & -\( e_7 \) & -1 & \( e_5 \) & \( e_6 \) & \( e_2 \) & \( e_3 \) \\
\hline
\( e_5 \) & \( e_5 \) & -\( e_7 \) & -\( e_6 \) & 1 & \( e_4 \) & \( e_3 \) & \( e_1 \) & \( e_2 \) \\
\hline
\( e_6 \) & \( e_6 \) & \( e_7 \) & -\( e_5 \) & -\( e_2 \) & -\( e_3 \) & -1 & -\( e_1 \) & 1 \\
\hline
\( e_7 \) & \( e_7 \) & -\( e_6 \) & \( e_5 \) & \( e_4 \) & -\( e_3 \) & -\( e_2 \) & \( e_1 \) & -1 \\
\hline
\end{tabular}
\end{table}

The conjugate of an octonion is defined as

\[ \overline{o} = s_0 - s_1e_1 - s_2e_2 - s_3e_3 - s_4e_4 - s_5e_5 - s_6e_6 - s_7e_7 \]  

(2.3)

Therefore norm is defined by \( |o| = \sqrt{\overline{o}o} \) and \( |o|^2 = \sum_{i=0}^{7} s_i \). Also \( |o_1o_2| = |o_1||o_2|, \forall o_1, o_2 \in \mathbb{O} \).

From (2.1) it is evident that every \( o \in \mathbb{O} \) can be represented in quaternion form as

\[ o = a + be_4 \]  

(2.4)

where \( a = s_0 + s_1e_1 + s_2e_2 + s_3e_3 \) and \( b = s_4 + s_5e_1 + s_6e_2 + s_7e_3 \) are both quaternions. By direct verification we have following lemma.

**Lemma 2.1.** Let \( a, b \in \mathbb{H} \), then

(1) \( e_4a = \overline{ae_4} \);
(2) \( e_4(\overline{ae_4}) = -\overline{a} \);
(3) \( (ae_4)e_4 = -a \);
(4) \( a(be_4) = (ba)e_4 \);
(5) \( (ae_4)b = (\overline{a}\overline{b})e_4 \);
(6) \( (ae_4)(be_4) = -\overline{ab} \).

It is clear from above Lemma that, for an octonion \( a + be_4, a, b \in \mathbb{H} \), we have

\[ \overline{a + be_4} = \overline{a} - be_4 \]  

(2.5)

and

\[ |a + be_4|^2 = |a|^2 + |b|^2. \]  

(2.6)

An octonion-valued function \( f : \mathbb{R}^3 \rightarrow \mathbb{O} \) has following explicit form

\[ f(x) = f_0 + f_1(x)e_1 + f_2(x)e_2 + f_3(x)e_3 + f_4(x)e_4 + f_5(x)e_5 + f_6(x)e_6 + f_7(x)e_7 \]
\[ = g(x) + h(x)e_4 \]  

(2.7)
where each \( f_i(x) \) is a real valued functions, \( g, h \in \mathbb{H} \) and \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \). For each octonion-valued function \( f(x) \) over \( \mathbb{R}^3 \) and \( 1 \leq p < \infty \), the \( L^p \)-norm of \( f \) is defined by

\[
\|f\|_p^p = \int_{\mathbb{R}^3} |f(x)|^p dx.
\]

And for \( p = \infty \), then the \( L^\infty \)-norm is defined by

\[
\|f\|_\infty = \text{esssup}_{x \in \mathbb{R}^3} |f(x)|.
\]

2.2. Offset linear canonical transform.

The offset linear canonical transform (OLCT)\(^{33}\) of any function \( f : \mathbb{R} \rightarrow \mathbb{O} \) with respect to the matrix parameter \( A = (a, b, c, d, e, \tau, \eta) \) is defined as

\[
\mathcal{O}_A[f(x)]w = \int_{\mathbb{R}} f(x)K_A^x(x, w)dx.
\]

with

\[
K_A(x, w) = \frac{1}{\sqrt{2\pi|b|}} e^{\frac{i}{2}\left[ax^2-2x(w-\tau)-2wdx(\tau-\eta)+d(w^2+\tau^2)-\frac{\pi}{8}\right]}, b \neq 0
\]

where \( e^{-\frac{it}{b}} \) is the polar form of \( \frac{1}{\sqrt{b}} \).

And the Offset linear canonical transform in quaternion\(^{34}\) setting is given as:

Let \( A_s = \begin{bmatrix} a_s & b_s \\ c_s & d_s \end{bmatrix} \eta_s \in \mathbb{R}^{2 \times 2} \) be a matrix parameter satisfying \( \det(A_s) = 1 \), for \( s = 1, 2 \). Then QLCT of signal \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \) is defined by

\[
\mathcal{O}_{i_{A1}}^{A2}[f](w) = \int_{\mathbb{R}^2} f(x)K_{A1}^i(x_1, w_1)K_{A2}^j(x_2, w_2)dx,
\]

where \( w = (w_1, w_2), x = (x_1, x_2) \in \mathbb{R}^2 \), and the kernel signals \( K_{A1}^i(x_1, w_1) \), \( K_{A2}^j(x_2, w_2) \) are respectively given by

\[
K_{A1}^i(x_1, w_1) = \frac{1}{\sqrt{2\pi b_1}} e^{\frac{i}{2b_1}\left[a_1x_1^2-2x_1(w_1-\tau_1)-2w_1(d_1\tau_1-b_1\eta_1)+d_1(w_1^2+\tau_1^2)\right]}, b_1 \neq 0
\]

\[
K_{A2}^j(x_2, w_2) = \frac{1}{\sqrt{2\pi b_2}} e^{\frac{i}{2b_2}\left[a_2x_2^2-2x_2(w_2-\tau_2)-2w_2(d_2\tau_2-b_2\eta_2)+d_2(w_2^2+\tau_2^2)\right]}, b_2 \neq 0
\]

3. Octonionic offset linear canonical transform

In this section we shall formally introduce the notion of proposed transform "The Octonion Offset Linear Canonical Transform (\( \mathbb{O}–\text{OLCT} \)) and study its important properties like closed form representation, inversion formula, split of norm and energy conservation. Prior to establishing the fundamental properties for the proposed transform, we shall revisit the definitions of the octonion Fourier transform (\( \mathbb{O}–\text{FT} \))\(^{35}\) and the octonion linear canonical transform (\( \mathbb{O}–\text{LCT} \))\(^{31}\). Lets begin with definition of \( \mathbb{O}–\text{FT} \).

3.1. Octonion Fourier transform.

Let \( \mu_i, i = 1, 2, \ldots, 7 \) denote the imaginary units in Cayley-Dickson algebra of octonions, then for an octonion-valued function \( f \in L^1(\mathbb{R}^3, \mathbb{O}) \) the one dimensional \( \mathbb{O}–\text{FT} \)\(^{35}\) is given by

\[
\mathcal{F}_{\mu_i}\{f\}(w) = \int_{\mathbb{R}} f(x)e^{-\mu_i 2\pi xw}dx,
\]

(3.1)
with inversion
\[
f(x) = \mathcal{F}_{\mu_1}^{-1}\{\mathcal{F}_{\mu_4}\{f\}\}(x) = \int_{\mathbb{R}} \mathcal{F}_{\mu_4}\{f\}(w)e^{\mu_42\pi xw}dx,
\]
(3.2)
And for octonion valued function \(f \in L^1(\mathbb{R}^3, \mathbb{O}) \cap L^2(\mathbb{R}^3, \mathbb{O})\), the three dimensional \(\mathbb{O}–\)FT \([?, ?]\) is defined as
\[
\mathcal{F}_{\mu_1,\mu_2,\mu_4}\{f\}(w) = \int_{\mathbb{R}^3} f(x)e^{-\mu_12\pi x_1w_1}e^{-\mu_22\pi x_2w_2}e^{-\mu_42\pi x_3w_3}dx,
\]
with inversion
\[
f(x) = \mathcal{F}_{\mu_1,\mu_2,\mu_4}^{-1}\{\mathcal{F}_{\mu_1,\mu_2,\mu_4}\{f\}\}(x) = \int_{\mathbb{R}^3} \mathcal{F}_{\mu_1,\mu_2,\mu_4}\{f\}(w)e^{-\mu_12\pi x_1w_1}e^{-\mu_22\pi x_2w_2}e^{-\mu_42\pi x_3w_3}dw,
\]
(3.4)
where \(w = (w_1, w_2, w_3), \ x = (x_1, x_2, x_3) \in \mathbb{R}^3\).
The multiplication in the above integrals is done from left to right as the octonion is non-associative. And the order of imaginary units in (3.3) is not accidental, see \([36]\). The \(\mathbb{O}–\)FT of 3D octonion-valued signals follows the multiplication rules of Table-I (octonion algebra) because the octonion-valued 3D signals has octonion structure.

### 3.2. Octonion linear canonical transform.
In 2021 Gao, W.B and Li, B.Z \([31]\) introduced linear canonical transform in octonion setting they called it the octonion linear canonical transform \((\mathbb{O}–\text{LCT})\) and defined it as
Let \(f \in L^1(\mathbb{R}^3, \mathbb{O})\), then the one dimensional \(\mathbb{O}–\text{LCT}\) with respect to the uni-modular matrix \(A = (a, b, c, d)\) is given by
\[
\mathcal{L}^A_{\mu_4}\{f(x)\}w = \int_{\mathbb{R}} f(x)K^\mu_4_A(x, w)dx,
\]
(3.5)
where
\[
K^\mu_4_A(x, w) = \frac{1}{\sqrt{2\pi |b|}}e^{\frac{\mu_4}{2b\pi} \left[ax^2-2xw-w^2-\frac{a}{2}\right]}, \ b \neq 0
\]
with inversion
\[
f(x) = \int_{\mathbb{R}} \mathcal{L}^A_{\mu_4}K^{-\mu_4}_A(x, w)dx,
\]
(3.6)
where \(K^{-\mu_4}_A(x, w) = K^\mu_4_{A^{-1}}(w, x)\) and \(A^{-1} = (d, -b, -c, a)\).

And for octonion valued function \(f \in L^1(\mathbb{R}^3, \mathbb{O}) \cap L^2(\mathbb{R}^3, \mathbb{O})\), the three dimensional \(\mathbb{O}–\text{LCT}\) with respect to the matrix parameter \(A_k = (a_k, b_k, c_k, d_k)\), satisfying \(\text{det}(A_k) = , \ k = 1, 2, 3\) is defined as
\[
\mathcal{L}^{A_1, A_2, A_3}_{\mu_1,\mu_2,\mu_4}\{f\}(w) = \int_{\mathbb{R}^3} f(x)K^\mu_1_{A_1}(x_1, w_1)K^\mu_2_{A_2}(x_2, w_2)K^\mu_4_{A_3}(x_3, w_3)dx
\]
(3.7)
where \(x = (x_1, x_2, x_3), w = (w_1, w_2, w_3)\), and \(K^\mu_1_{A_1}(x_1, w_1), K^\mu_2_{A_2}(x_2, w_2)\) and \(K^\mu_4_{A_3}(x_3, w_3)\) are kernel signals given by
\[
K^\mu_1_{A_1}(x_1, w_1) = \frac{1}{\sqrt{2\pi |b_1|}}e^{\frac{\mu_1}{2b_1\pi} \left[a_1x_1^2-2x_1w_1+d_1w_1^2-\frac{a_1}{2}\right]}, \ b_1 \neq 0
\]
\[
K^\mu_2_{A_2}(x_2, w_2) = \frac{1}{\sqrt{2\pi |b_2|}}e^{\frac{\mu_2}{2b_2\pi} \left[a_2x_2^2-2x_2w_2+d_2w_2^2-\frac{a_2}{2}\right]}, \ b_2 \neq 0
\]
\[ K^\mu_4(x_3, w_3) = \frac{1}{\sqrt{2\pi|b|}} e^{\frac{\mu_4}{2\pi} \left[ a_1 x_3^2 - 2 x_3 w_3 + d_3 w_3^2 - \frac{\tau_3}{2} \right]}, \quad b_3 \neq 0. \]

Now we are in a position to define octonion offset linear canonical transform (\(\mathbb{O}-\text{OLCT}\)).

According to the one dimensional octonion Fourier transform (\(\mathbb{O}-\text{FT}\)) and the one dimensional octonion linear canonical transform (\(\mathbb{O}-\text{OLCT}\)), we can obtain the definition of one dimensional octonion offset linear canonical transform (\(\mathbb{O}-\text{OLCT}\)).

**Definition 3.1** (One dimensional \(\mathbb{O}-\text{OLCT}\)). Let \(f \in L^1(\mathbb{R}, \mathbb{O})\), then one dimensional \(\mathbb{O}-\text{OLCT}\) with respect a uni-modular matrix parameter \(A = (a, b, c, d, e, \tau, \eta)\) is defined as follows:

\[ \mathcal{O}^A_{\mu_4} \{ f(x) \} w = \int_{\mathbb{R}} f(x) K^\mu_4(x, w) dx. \]  

where

\[ K^\mu_4(x, w) = \frac{1}{\sqrt{2\pi|b|}} e^{\frac{\mu_4}{2\pi} \left[ a_2 x^2 - 2xw - \frac{\tau_2}{2} \right]}, \quad b \neq 0 \].  

The following lemma gives the relationship of one dimensional \(\mathbb{O}-\text{OLCT}\) and one dimensional \(\mathbb{O}-\text{FT}\) of an octonion-valued signals.

**Lemma 3.1.** The one dimensional \(\mathbb{O}-\text{OLCT}\) of a signal \(f \in L^1(\mathbb{R}, \mathbb{O})\) can be reduced to one dimensional \(\mathbb{O}-\text{FT}\) as

\[ \mathcal{O}^A_{\mu_4} \{ f(x) \} w = \int_{\mathbb{R}} f(x) e^{\frac{\mu_4}{2\pi|x^2 + \frac{\tau_2}{2}|}} \left( \frac{w}{2\pi|b|} \right) e^{\frac{\mu_4}{2\pi \left[ (w^2 + \tau_2) + \frac{\eta_2}{2}(\eta - dr) - \frac{\tau_2}{2} \right]}} dx. \]  

where \(b \neq 0\) and

\[ \mathcal{F}^A_{\mu_4} \{ f(x) \} (w) = \int_{\mathbb{R}} f(x) e^{-\mu_4 2\pi x w} dx \]

represents the one dimensional \(\mathbb{O}-\text{FT}\) of an octonion-valued signal \(f(x)\).

By applying Lemma 3.1 and 3.2, we get the formula for the inversion of one dimensional \(\mathbb{O}-\text{OLCT}\), which is given below as a theorem.

**Theorem 3.1.** Let \(f\) be an octonion-valued signal \(\in L^1(\mathbb{R}, \mathbb{O})\). Then the inversion formula of the one dimensional \(\mathbb{O}-\text{OLCT}\) is

\[ f(x) = \int_{\mathbb{R}} \mathcal{O}^A_{\mu_4} \{ f \}(w) \overline{K^\mu_4(x, w)} dw, \]  

where \(\overline{K^\mu_4(x, w)} = K^{-\mu_4}(x, w)\) and \(b \neq 0\).

On replacing the complex unit \(i\) in the ordinary offset linear canonical transform by the imaginary units in the octonions, the three dimensional octonion offset linear canonical transform (\(\mathbb{O}-\text{OLCT}\)) could be defined as

**Definition 3.2.** (Three dimensional \(\mathbb{O}-\text{OLCT}\))

Let \(A_k = \begin{bmatrix} a_k & b_k & \tau_k \\ c_k & d_k & \eta_k \end{bmatrix}, \) be a matrix parameter such that \(a_k, b_k, c_k, d_k, p_k, q_k \in \mathbb{R}\) and
\(a_k d_k - b_k c_k = 1\), for \(k = 1, 2, 3\). The three dimensional \(\Omega-\text{OLCT}\) of an octonion-valued signal \(f\) over \(\mathbb{R}^3\), is given by

\[
\mathcal{O}_{\mu_1,\mu_2,\mu_4}^{A_1,A_2,A_3\{f\}}(w) = \int_{\mathbb{R}^3} f(x) K_{A_1}^{\mu_1}(x_1,w_1) K_{A_2}^{\mu_2}(x_2,w_2) K_{A_3}^{\mu_4}(x_3,w_3) dx
\]

(3.13)

where \(x = (x_1, x_2, x_3)\), \(w = (w_1, w_2, w_3)\), and \(K_{A_1}^{\mu_1}(x_1,w_1)\), \(K_{A_2}^{\mu_2}(x_2,w_2)\) and \(K_{A_3}^{\mu_4}(x_3,w_3)\) are kernel signals given by

\[
K_{A_1}^{\mu_1}(x_1,w_1) = \frac{1}{\sqrt{2\pi|b_1|}} e^{\frac{\mu_1}{2b_1^2} \left[ a_1 x_1^2 - 2x_1(w_1 - \tau_1) - 2w_1(d_1 \tau_1 - b_1 \eta_1) + d_1(w_1^2 + \tau_1^2) - \frac{\pi}{2} \right]}, \ b \neq 0
\]

(3.14)

\[
K_{A_2}^{\mu_2}(x_2,w_2) = \frac{1}{\sqrt{2\pi|b_2|}} e^{\frac{\mu_2}{2b_2^2} \left[ a_2 x_2^2 - 2x_2(w_2 - \tau_2) - 2w_2(d_2 \tau_2 - b_2 \eta_2) + d_2(w_2^2 + \tau_2^2) - \frac{\pi}{2} \right]}, \ b \neq 0
\]

(3.15)

and

\[
K_{A_3}^{\mu_4}(x_3,w_3) = \frac{1}{\sqrt{2\pi|b_3|}} e^{\frac{\mu_4}{2b_3^2} \left[ a_3 x_3^2 - 2x_3(w_3 - \tau_3) - 2w_3(d_3 \tau_3 - b_3 \eta_3) + d_3(w_3^2 + \tau_3^2) - \frac{\pi}{2} \right]}, \ b \neq 0.
\]

(3.16)

Note that, for \(b_k = 0, k = 1, 2, 3\), the \(\Omega-\text{OLCT}\) boils down to chirp multiplication operator and it is of no particular interest for our objective in this work. Hence for the sake of brevity, we always set \(b_k \neq 0\) in the paper unless stated otherwise.

Also note that, the kernels with imaginary units \(\mu_1, \mu_2, \mu_4\) are octonion-valued and does not reduce to the quaternion cases, thus the present integral transform is more interesting and complicated.

Now we will obtain the closed-form representation of \(\Omega-\text{OLCT}\) defined in (3.13), let us begin by setting

\[
\xi_k = \frac{1}{2b_k} \left[ a_k x_k^2 - 2x_k(w_k - \tau_k) - 2w_k(d_k \tau_k - b_k \eta_k) + d_k(w_k^2 + \tau_k^2) - \frac{\pi}{2} \right], \ k = 1, 2, 3.
\]

(3.17)

Thus,

\[
K_{A_1}^{\mu_1}(x_1,w_1) K_{A_2}^{\mu_2}(x_2,w_2) K_{A_3}^{\mu_4}(x_3,w_3) = \frac{1}{2\pi \sqrt{2\pi|b_1 b_2 b_3|}} e^{\mu_1 \xi_1} e^{\mu_2 \xi_2} e^{\mu_4 \xi_3}
\]

\[
= \frac{1}{2\pi \sqrt{2\pi|b_1 b_2 b_3|}} (c_1 + \mu_1 s_1)(c_2 + \mu_2 s_2)(c_3 + \mu_4 s_3)
\]

\[
= \frac{1}{2\pi \sqrt{2\pi|b_1 b_2 b_3|}} (c_1 c_2 c_3 + s_1 c_2 c_3 \mu_1 + c_1 s_2 c_3 \mu_2 + s_1 s_2 c_3 \mu_3 + c_1 s_2 s_3 \mu_4 + s_1 s_2 s_3 \mu_5 + s_1 s_2 s_3 \mu_7),
\]

(3.18)

where \(c_k = \cos \xi_k\) and \(s_k = \sin \xi_k, \ k = 1, 2, 3\).

Using (3.18) in (3.13) we get closed-form representation of \(\Omega-\text{OLCT}\) given by following lemma.

**Lemma 3.2** (Closed-form representation). The \(\Omega-\text{OLCT}\) of a three dimensional signal \(f : \mathbb{R}^3 \rightarrow \Omega\) has the closed-form representation:

\[
\mathcal{O}_{\mu_1,\mu_2,\mu_4}^{A_1,A_2,A_3\{f\}}(w) = \Phi_0(w) + \Phi_1(w) + \Phi_2(w) + \Phi_3(w) + \Phi_4(w) + \Phi_5(w) + \Phi_6(w) + \Phi_7(w)
\]

(3.19)
where we put the integrals

\[ \Phi_0(w) = \int_{\mathbb{R}^3} f_{ee}(x) \frac{1}{2\pi \sqrt{2\pi |b_1 b_2 b_3|}} c_1 c_2 c_3 dx, \]

and,

\[ \Phi_1(w) = \int_{\mathbb{R}^3} f_{oe}(x) \frac{\mu_1}{2\pi \sqrt{2\pi |b_1 b_2 b_3|}} s_1 c_2 c_3 dx, \]

\[ \Phi_2(w) = \int_{\mathbb{R}^3} f_{oe}(x) \frac{\mu_2}{2\pi \sqrt{2\pi |b_1 b_2 b_3|}} c_1 s_2 c_3 dx, \]

\[ \Phi_3(w) = \int_{\mathbb{R}^3} f_{oe}(x) \frac{\mu_3}{2\pi \sqrt{2\pi |b_1 b_2 b_3|}} s_1 s_2 c_3 dx, \]

\[ \Phi_4(w) = \int_{\mathbb{R}^3} f_{oe}(x) \frac{\mu_4}{2\pi \sqrt{2\pi |b_1 b_2 b_3|}} f_{ce}(x) c_1 c_2 s_3 dx, \]

\[ \Phi_5(w) = \int_{\mathbb{R}^3} f_{oe}(x) \frac{\mu_5}{2\pi \sqrt{2\pi |b_1 b_2 b_3|}} s_1 c_2 s_3 dx, \]

\[ \Phi_6(w) = \int_{\mathbb{R}^3} f_{oe}(x) \frac{\mu_6}{2\pi \sqrt{2\pi |b_1 b_2 b_3|}} c_1 s_2 s_3 dx, \]

\[ \Phi_7(w) = \int_{\mathbb{R}^3} f_{oe}(x) \frac{\mu_7}{2\pi \sqrt{2\pi |b_1 b_2 b_3|}} s_1 s_2 s_3 dx, \]

where the functions \( f_{xyz}, x, y, z \in \{e, o\} \), are eight components of \( f \) of different parity with respect to appropriate variable, for example \( f_{oe}(x) \) is odd with respect to \( x_1 \), even with respect \( x_2 \) and odd with respect to \( x_3 \).

Under suitable conditions, the original octonion-valued signal \( f \) can be reconstructed from \( \Phi-\text{OLCT} \) by its inverse transform.

**Definition 3.3.** The inverse \( \Phi-\text{OLCT} \) of signal \( g : \mathbb{R}^3 \rightarrow \Phi \) is defined by

\[
\{O^A_{\mu_1, \mu_2, \mu_4}\}^{-1} \{g\}(x) = \int_{\mathbb{R}^3} f(w) K_{A_1}^{\mu_4}(x_1, w_1) K_{A_2}^{\mu_2}(x_2, w_2) K_{A_1}^{\mu_1}(x_3, w_3) dw \tag{3.20}
\]

It has the closed-form representation:

\[
\{O^A_{\mu_1, \mu_2, \mu_4}\}^{-1} \{g\}(x) = \Psi_0(w) + \Psi_1(w) + \Psi_2(w) + \Psi_3(w) + \Psi_4(w) + \Psi_5(w) + \Psi_6(w) + \Psi_7(w) \tag{3.21}
\]

where we put the integrals

\[ \Psi_0(w) = \int_{\mathbb{R}^3} f_{ee}(x) \frac{1}{2\pi \sqrt{2\pi |b_1 b_2 b_3|}} c_1 c_2 c_3 dx, \]

and,

\[ \Psi_1(w) = \int_{\mathbb{R}^3} f_{oe}(x) \frac{-\mu_1}{2\pi \sqrt{2\pi |b_1 b_2 b_3|}} s_1 c_2 c_3 dx, \]

\[ \Psi_2(w) = \int_{\mathbb{R}^3} f_{oe}(x) \frac{-\mu_2}{2\pi \sqrt{2\pi |b_1 b_2 b_3|}} c_1 s_2 c_3 dx, \]
\[
\Psi_3(w) = \int_{\mathbb{R}^3} f_{oee}(x) \frac{-\mu_3}{2\pi \sqrt{2\pi |b_1 b_2 b_3|}} s_1 s_2 c_3 dx,
\]
\[
\Psi_4(w) = \int_{\mathbb{R}^3} f_{eoo}(x) \frac{-\mu_4}{2\pi \sqrt{2\pi |b_1 b_2 b_3|}} c_1 s_2 s_3 dx,
\]
\[
\Psi_5(w) = \int_{\mathbb{R}^3} f_{eoo}(x) \frac{-\mu_5}{2\pi \sqrt{2\pi |b_1 b_2 b_3|}} s_1 c_2 s_3 dx,
\]
\[
\Psi_6(w) = \int_{\mathbb{R}^3} f_{eoo}(x) \frac{-\mu_6}{2\pi \sqrt{2\pi |b_1 b_2 b_3|}} c_1 s_2 s_3 dx,
\]
\[
\Psi_7(w) = \int_{\mathbb{R}^3} f_{eoo}(x) \frac{-\mu_7}{2\pi \sqrt{2\pi |b_1 b_2 b_3|}} s_1 s_2 s_3 dx.
\]

The simple way to define inverse of $\mathbb{O}$–$\text{OLCT}$ is to introduce Inversion theorem.

**Theorem 3.2** (Inversion for three dimensional $\mathbb{O}$–$\text{OLCT}$).

Every octonion-valued signal $f : \mathbb{R}^3 \to \mathbb{O}$ can be reconstructed by the formula

\[
f(x) = \{\mathcal{O}^{A_1, A_2, A_3}_{\mu_1, \mu_2, \mu_4}\}^{-1} \{\mathcal{O}^{A_1, A_2, A_3}_{\mu_1, \mu_2, \mu_4} f\}(x) = \int_{\mathbb{R}^3} \mathcal{O}^{A_1, A_2, A_3}_{\mu_1, \mu_2, \mu_4} f(w) K^{-\mu_1}_{A_3}(x_1, w_1) K^{-\mu_2}_{A_2}(x_2, w_2) K^{-\mu_4}_{A_4}(x_3, w_3) dw,
\]

**Proof.** By using the definition of QOLCT\[?, the one dimensional $\mathbb{O}$–$\text{OLCT}$ and three dimensional $\mathbb{O}$–$\text{OLCT}$ the proof of theorem 3.2 follows.

For the clarity of the formulas we denote $x_{l,m,n} = (lx_1, mx_2, nx_3)$, $l, m, n \in \{+, -\}$, i.e. $x_{++} = (x_1, -x_2, x_3)$ and denoting the even and odd part of a function $f(x)$ by $f_e(x)$ and $f_o(x)$ where $f_e = (f(x_{++}) + f(x_{+-}))/2$ which is only even in the third variable $x_3$. Similarly, $f_o = (f(x_{++}) - f(x_{+-}))/2$.

**Lemma 3.3** (Norm split). Let $f : \mathbb{R}^3 \to \mathbb{O}$ beoctonion-valued signal and $\mathcal{O}^{A_1, A_2, A_3}_{\mu_1, \mu_2, \mu_4} f(w)$ be the $\mathbb{O}$–$\text{OLCT}$ of $f$, then

\[
\|\mathcal{O}^{A_1, A_2, A_3}_{\mu_1, \mu_2, \mu_4} f(w)\|^2 = \frac{1}{2\pi |b_3|} \left( \|\mathcal{O}^{A_1, A_2}_{\mu_1, \mu_2} g_e(w)\|^2 + \|\mathcal{O}^{A_1, A_2}_{\mu_1, \mu_2} h_o(w)\|^2 + \|\mathcal{O}^{A_1, A_2}_{\mu_1, \mu_2} h_e(w)\|^2 + \|\mathcal{O}^{A_1, A_2}_{\mu_1, \mu_2} g_o(w)\|^2 \right)
\]

**Proof.** From (2.7) every octonion-valued signal $f$ has explicit form $f = g + h \mu_4$ where $g, h \in \mathbb{H}$. And by the even and odd part we further express the $\mathbb{O}$–$\text{OLCT}$ in (3.13) as
follows,

\[
\mathcal{O}_{\mu_1, \mu_2, \mu_4}^{A_1, A_2, A_3} \{ f \}(w) = \int_{\mathbb{R}^3} g(x) K_{A_1}^{\mu_1}(x_1, w_1) K_{A_2}^{\mu_2}(x_2, w_2) K_{A_3}^{\mu_4}(x_3, w_3) dx \\
+ \int_{\mathbb{R}^3} h(x) K_{A_1}^{-\mu_1}(x_1, w_1) K_{A_2}^{-\mu_2}(x_2, w_2) \mu_4 K_{A_3}^{\mu_4}(x_3, w_3) dx \\
= \frac{1}{\sqrt{2\pi|b_3|}} \int_{\mathbb{R}^3} g_e(x) K_{A_1}^{\mu_1}(x_1, w_1) K_{A_2}^{\mu_2}(x_2, w_2) c_3 dx \\
+ \frac{1}{\sqrt{2\pi|b_3|}} \int_{\mathbb{R}^3} h_o(x) K_{A_1}^{-\mu_1}(x_1, w_1) K_{A_2}^{-\mu_2}(x_2, w_2) s_3 dx \\
+ \frac{1}{\sqrt{2\pi|b_3|}} \int_{\mathbb{R}^3} h_e(x) K_{A_1}^{-\mu_1}(x_1, w_1) K_{A_2}^{-\mu_2}(x_2, w_2) c_3 dx \\
- \frac{1}{\sqrt{2\pi|b_3|}} \int_{\mathbb{R}^3} g_o(x) K_{A_1}^{\mu_1}(x_1, w_1) K_{A_2}^{\mu_2}(x_2, w_2) s_3 dx \mu_4.
\]

(3.23)

With \( c_k = \cos \xi_k \) and \( s_k = \sin \xi_k \) where \( \xi_k \) is given in (3.17). From (3.22) it is clear that \( \mathcal{O} \)-OLCT can be divided into four QOLCTs. Thus the norm of \( \mathcal{O} \)-OLCT splits into four norms of quaternion functions as:

\[
\| \mathcal{O}_{\mu_1, \mu_2, \mu_4}^{A_1, A_2, A_3} \{ f \}(w) \|^2_2 = \frac{1}{2\pi|b_3|} \left\| \int_{\mathbb{R}^3} g_e(x) K_{A_1}^{\mu_1}(x_1, w_1) K_{A_2}^{\mu_2}(x_2, w_2) c_3 dx \right\|^2_2 \\
+ \frac{1}{2\pi|b_3|} \left\| \int_{\mathbb{R}^3} h_o(x) K_{A_1}^{-\mu_1}(x_1, w_1) K_{A_2}^{-\mu_2}(x_2, w_2) s_3 dx \right\|^2_2 \\
+ \frac{1}{2\pi|b_3|} \left\| \int_{\mathbb{R}^3} h_e(x) K_{A_1}^{-\mu_1}(x_1, w_1) K_{A_2}^{-\mu_2}(x_2, w_2) c_3 dx \right\|^2_2 \\
+ \frac{1}{2\pi|b_3|} \left\| \int_{\mathbb{R}^3} g_o(x) K_{A_1}^{\mu_1}(x_1, w_1) K_{A_2}^{\mu_2}(x_2, w_2) s_3 dx \mu_4 \right\|^2_2
\]

(3.24)

where the equality is by the fact that \( f_e \) and \( f_o \) are orthogonal in \( L^2 \) inner product.

\[
\| \mathcal{O}_{\mu_1, \mu_2, \mu_4}^{A_1, A_2, A_3} \{ f \}(w) \|^2_2 = \frac{1}{2\pi|b_3|} (\| \mathcal{O}_{\mu_1, \mu_2}^{A_1, A_2} \{ g_e \}(w) \|^2_2 + \| \mathcal{O}_{\mu_1, \mu_2}^{A_1, A_2} \{ h_o \}(w) \|^2_2 \\
+ \| \mathcal{O}_{\mu_1, \mu_2}^{A_1, A_2} \{ h_e \}(w) \|^2_2 + \| \mathcal{O}_{\mu_1, \mu_2}^{A_1, A_2} \{ g_o \}(w) \|^2_2)
\]

Which completes the proof.

\[\square\]

Again denoting the quaternion form of \( f \) by \( f = g + h\mu_4 \), we have

\[
\| f \|^2_2 = \| g + h\mu_4 \|^2_2 \\
= \| g \|^2 + \| h \|^2 \\
= \| g_e \|^2 + \| g_o \|^2 + \| h_e \|^2 + \| h_o \|^2
\]

(3.24)
Now by the Plancherel theorem for the QOLCT [?], we have

\[ \left\| \int_{\mathbb{R}^3} g_e(x) K_{A_1}^{\mu_1}(x_1, w_1) K_{A_2}^{\mu_2}(x_2, w_2) c_3 \, dx \right\|_2^2 = \| g_e(x) \|_2^2 \]  

(3.25)

Similarly results hold for the remaining three norms.

On applying (3.24) and (3.25) in lemma 3.3 we have proved the following Energy conservation relation for $\mathbb{O}^{-}\text{OLCT}$.

**Theorem 3.3** (Energy conservation).

Let $f : \mathbb{R}^3 \rightarrow \mathbb{O}$ be a continuous and square integrable octonion-valued signal function. Then we have

\[ \| \mathcal{O}_{A_1, A_2, A_3} \{ f \} (w) \|_2^2 = \frac{1}{2\pi |b_3|} \| f \|_2^2 \]  

(3.26)

Now we move towards our main section in which we present some uncertainty principles associated with $\mathbb{O}^{-}\text{OLCT}$.

### 4. Uncertainty principles for $\mathbb{O}^{-}\text{OLCT}$

We know that in signal processing there are different types of uncertainty principles in the QFT, QLCT and QOLCT domains. Recently in [31] authors investigate Heisenberg’s uncertainty principle and Donoho-Stark’s uncertainty principle for $\mathbb{O}^{-}\text{LCT}$. Considering that the $\mathbb{O}^{-}\text{OLCT}$ is a generalized version of the $\mathbb{O}^{-}\text{FT}$ or $\mathbb{O}^{-}\text{LCT}$, it is natural and interesting to study uncertainty principles of a octonion-valued function and its $\mathbb{O}^{-}\text{OLCT}$. So in this section we shall investigate some uncertainty principles for $\mathbb{O}^{-}\text{OLCT}$.

#### 4.1. Pitt’s inequality and the logarithmic uncertainty principle.

Here we prove the sharp Pitt’s inequality for the $\mathbb{O}^{-}\text{OLCT}$ and derive the associated logarithmic uncertainty inequality. The proof of the Pitt’s inequality heavily depends on the QOLCT. So we first present following Pitt’s inequality for QOLCT.

**Lemma 4.1** (Pitt’s inequality for QOLCT). [24]

For $f \in \mathcal{S}(\mathbb{R}^2, \mathbb{H})$, and $0 \leq \alpha < 2$,

\[ \int_{\mathbb{R}^2} \left| \frac{w}{b} \right|^{-\alpha} \left| \mathcal{O}_{A_1, A_2} \{ f(x) \} (w) \right|^2 \, dw \leq \frac{C_{\alpha}}{4\pi^2} \int_{\mathbb{R}^2} |x|^{\alpha} |f(x)|^2 \, dx. \]  

(4.1)

With $C_{\alpha} := \frac{\Gamma(2-\alpha)}{2\pi^2 \Gamma(\frac{2+\alpha}{4})^2}$, and $\Gamma(.)$ is the Gamma function and $\mathcal{S}(\mathbb{R}^2, \mathbb{H})$ denotes the Schwartz space.

**Theorem 4.1** (Pitt’s inequality for the $\mathbb{O}^{-}\text{OLCT}$). For $f \in \mathcal{S}(\mathbb{R}^3, \mathbb{O})$, and $0 \leq \alpha < 3$ and under the assumptions of lemma [4.1] we have

\[ \int_{\mathbb{R}^3} \left| \frac{w}{b} \right|^{-\alpha} \left| \mathcal{O}_{A_1, A_2, A_3} \{ f(x) \} (w) \right|^2 \, dw \leq \frac{C_{\alpha}}{8\pi^3 |b_3|} \int_{\mathbb{R}^3} |x|^{\alpha} |f(x)|^2 \, dx. \]  

(4.2)
Proof. We have split $\mathcal{O}$–OLCT into four QOLCT in lemma [3.3] therefore
\[
\int_{\mathbb{R}^3} \left| \frac{w}{b} \right|^{-\alpha} |\mathcal{O}^{A_1,A_2,A_3}_{\mu_1,\mu_2,\mu_3} \{ f(x) \}(w) |^2 dw = \frac{1}{2\pi |b_3|} \left( \int_{\mathbb{R}^3} \left| \frac{w}{b} \right|^{-\alpha} |\mathcal{O}^{A_1,A_2}_{\mu_1,\mu_2} \{ g_e \}(w) |^2 dw 
+ \int_{\mathbb{R}^3} \left| \frac{w}{b} \right|^{-\alpha} |\mathcal{O}^{A_1,A_2}_{\mu_1,\mu_2} \{ g_o \}(w) |^2 dw 
+ \int_{\mathbb{R}^3} \left| \frac{w}{b} \right|^{-\alpha} |\mathcal{O}^{A_1,A_2}_{\mu_1,\mu_2} \{ h_e \}(w) |^2 dw 
+ \int_{\mathbb{R}^3} \left| \frac{w}{b} \right|^{-\alpha} |\mathcal{O}^{A_1,A_2}_{\mu_1,\mu_2} \{ h_o \}(w) |^2 dw \right) \tag{4.3}
\]

By the Pitt’s inequality for QOLCT (4.1), we have
\[
\int_{\mathbb{R}^3} \left| \frac{w}{b} \right|^{-\alpha} |\mathcal{O}^{i,j}_{A_1,A_2} \{ g_e(x) \}(w) |^2 dw \leq \frac{C_\alpha}{4\pi^2} \int_{\mathbb{R}^3} |x|^\alpha |f(x)|^2 dx. \tag{4.4}
\]
As similar inequalities hold for the remaining three terms. Collecting all and inserting in (4.3), we get
\[
\int_{\mathbb{R}^3} \left| \frac{w}{b} \right|^{-\alpha} |\mathcal{O}^{A_1,A_2,A_3}_{\mu_1,\mu_2,\mu_3} \{ f(x) \}(w) |^2 dw \leq \frac{C_\alpha}{8\pi^2 |b_3|} \left( \int_{\mathbb{R}^3} |x|^\alpha |g_e(x)|^2 dx + \int_{\mathbb{R}^3} |x|^\alpha |g_o(x)|^2 dx 
+ \int_{\mathbb{R}^3} |x|^\alpha |h_e(x)|^2 dx + \int_{\mathbb{R}^3} |x|^\alpha |h_o(x)|^2 dx \right) 
= \frac{C_\alpha}{8\pi^2 |b_3|} \int_{\mathbb{R}^3} |x|^\alpha (|g_e(x)|^2 + |g_o(x)|^2 + |h_e(x)|^2 + |h_o(x)|^2) dx.
\]
where last equality occurs because of [3.24].
Which completes the proof. \qed

Here $C_\alpha$ can’t be smaller any more. It is equal to the ordinary complex and the quaternion cases. Thus, the inequality is sharp. If $\alpha = 0$, it changes to equality, at $\alpha = 0$ differentiating the sharp Pitt’s inequalities led to the following logarithmic uncertainty inequality for the $\mathcal{O}$–OLCT.

**Theorem 4.2** (Logarithmic uncertainty principle for the $\mathcal{O}$–OLCT). Let $f \in \mathcal{S}(\mathbb{R}^3, \mathcal{O})$, then the following inequality is satisfied:
\[
2\pi |b_3| \int_{\mathbb{R}^2} \ln \left| \frac{w}{b} \right| |\mathcal{O}^{A_1,A_2,A_3}_{\mu_1,\mu_2,\mu_3} \{ f \}(w) |^2 dw + \int_{\mathbb{R}^2} \ln |x||f(x)|^2 dx \geq D \int_{\mathbb{R}^2} |f(x)|^2 dx \tag{4.5}
\]
with $D = \ln(2) + \Gamma'(\frac{1}{2})/\Gamma(\frac{1}{2})$.

**Proof.** Following the procedure of theorem 4.11 in [24] we will get desired result.
Alternatively, we can prove Logarithmic uncertainty principle for the $\mathcal{O}$–OLCT from the Logarithmic uncertainty principle for the QOLCT [24].

**Lemma 4.2** (Logarithmic uncertainty principle for the QOLCT). [24]
Let $f \in \mathcal{S}(\mathbb{R}^2, \mathbb{H})$, then
\[
\int_{\mathbb{R}^2} \ln \left| \frac{w}{b} \right| |\mathcal{O}^{A_1,A_2}_{\mu_1,\mu_2} \{ f \}(w) |^2 dw + \int_{\mathbb{R}^2} \ln |x||f(x)|^2 dx \geq D \int_{\mathbb{R}^2} |f(x)|^2 dx \tag{4.6}
\]
with $D = \ln(2) + \Gamma'(\frac{1}{2})/\Gamma(\frac{1}{2})$. 

Proof of theorem 4.2

By lemma 3.3, $\mathbb{O}$–OLCT can be written in split quaternion form as

$$\|O_{\mu_1,\mu_2,\mu_4}^\ast f\|_2^2 = \frac{1}{2\pi|b_3|} \left( \|O_{\mu_1,\mu_2}^\ast g_e\|_2^2 + \|O_{\mu_1,\mu_2}^\ast h_o\|_2^2 + \|O_{\mu_1,\mu_2}^\ast g_o\|_2^2 \right).$$

Therefore

$$\int_{R^2} \ln \left| \frac{w}{b} \right| \|O_{\mu_1,\mu_2,\mu_4}^\ast f\|_2^2 dw = \frac{1}{2\pi|b_3|} \int_{R^2} \ln \left| \frac{w}{b} \right| \left( \|O_{\mu_1,\mu_2}^\ast g_e\|_2^2 + \|O_{\mu_1,\mu_2}^\ast h_o\|_2^2 + \|O_{\mu_1,\mu_2}^\ast g_o\|_2^2 \right) dw.$$

Implies

$$2\pi|b_3| \int_{R^2} \ln \left| \frac{w}{b} \right| \|O_{\mu_1,\mu_2,\mu_4}^\ast f\|_2^2 dw = \int_{R^2} \ln \left| \frac{w}{b} \right| \|O_{\mu_1,\mu_2}^\ast g_e\|_2^2 dw + \int_{R^2} \ln \left| \frac{w}{b} \right| \|O_{\mu_1,\mu_2}^\ast h_o\|_2^2 dw + \int_{R^2} \ln \left| \frac{w}{b} \right| \|O_{\mu_1,\mu_2}^\ast g_o\|_2^2 dw.$$ (4.7)

Also by virtue of (3.24), we can write

$$\int_{R^2} \ln |x| |f(x)|^2 dx = \int_{R^2} \ln |x| |g_e(x)|^2 dx + \int_{R^2} \ln |x| |g_o(x)|^2 dx$$

$$+ \int_{R^2} \ln |x| |h_e(x)|^2 dx + \int_{R^2} \ln |x| |h_o(x)|^2 dx.$$ (4.8)

By logarithmic uncertainty principle for the QOLCT given in lemma 4.2, we get

$$\int_{R^2} \ln \left| \frac{w}{b} \right| \|O_{\mu_1,\mu_2}^\ast g_e\|_2^2 + \int_{R^2} \ln |x| |g_e(x)|^2 dx \geq D \int_{R^2} |g_e(x)|^2 dx.$$ (4.9)

Similarly

$$\int_{R^2} \ln \left| \frac{w}{b} \right| \|O_{\mu_1,\mu_2}^\ast g_e\|_2^2 + \int_{R^2} \ln |x| |g_e(x)|^2 dx \geq D \int_{R^2} |g_e(x)|^2 dx,$$ (4.10)

$$\int_{R^2} \ln \left| \frac{w}{b} \right| \|O_{\mu_1,\mu_2}^\ast g_e\|_2^2 + \int_{R^2} \ln |x| |g_e(x)|^2 dx \geq D \int_{R^2} |h_e(x)|^2 dx.$$ (4.11)

$$\int_{R^2} \ln \left| \frac{w}{b} \right| \|O_{\mu_1,\mu_2}^\ast g_e\|_2^2 + \int_{R^2} \ln |x| |g_e(x)|^2 dx \geq D \int_{R^2} |h_o(x)|^2 dx.$$ (4.12)

Collecting all equations (4.9)–(4.12) and making use of (4.7, 4.8), we obtain the desired result

$$2\pi|b_3| \int_{R^2} \ln \left| \frac{w}{b} \right| \|O_{\mu_1,\mu_2,\mu_4}^\ast f\|_2^2 dw + \int_{R^2} \ln |x| |f(x)|^2 dx \geq D \int_{R^2} |f(x)|^2 dx.$$ (4.13)

Which completes the proof.
4.2. Hausdorff-Young inequality for $\mathcal{O}$–OLCT.
In this subsection we will establish Hausdorff-Young inequality which is very important in signal processing. This inequality will be helpful for researchers in establishing Shannon’s entropy uncertainty relation.

**Lemma 4.3** (Hausdorff-Young inequality for QOLCT). For $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

\[
\|O_{\mu_1, \mu_2}^{A_1, A_2} \{ f \}(w) \|_q \leq (2\pi)^{\frac{1}{q} - \frac{1}{p}} |b_1 b_2|^\frac{1}{q} \|f(x)\|_p.
\]

(4.14)

**Theorem 4.3** (Hausdorff-Young inequality for $\mathcal{O}$–OLCT). For $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

\[
\|O_{\mu_1, \mu_2}^{A_1, A_2, A_3} \{ f \}(w) \|_q \leq (2\pi)^{\frac{1}{q} - \frac{1}{p}} |b_1 b_2|^\frac{1}{q} |b_3|^{-\frac{1}{q}} \|f(x)\|_p.
\]

(4.15)

**Proof.** From (3.22), we obtain

\[
\|O_{\mu_1, \mu_2, \mu_4}^{A_1, A_2, A_3} \{ f \}(w) \|_q = \frac{1}{(2\pi |b_3|)^{\frac{1}{q}}} \left( \|O_{\mu_1, \mu_2}^{A_1, A_2} \{ g_e \}(w) \|_q + \|g_o \|_p \right. \\
+ \left. \|O_{\mu_1, \mu_2}^{A_1, A_2} \{ h_e \}(w) \|_q + \|h_o \|_p \right). \\
\]

Now applying lemma 4.3 to the R.H.S of above equation, we obtain

\[
\|O_{\mu_1, \mu_2, \mu_4}^{A_1, A_2, A_3} \{ f \}(w) \|_q \leq \frac{1}{(2\pi |b_3|)^{\frac{1}{q}}} (2\pi)^{\frac{1}{q} - \frac{1}{p}} |b_1 b_2|^\frac{1}{q} \|g_e(x)\|_p + \|g_o(x)\|_p \\
+ \|h_e(x)\|_p + \|h_o(x)\|_p). \\
= (2\pi)^{\frac{1}{q} - \frac{1}{p}} |b_1 b_2|^\frac{1}{q} |b_3|^{-\frac{1}{q}} \|f(x)\|_p
\]

where last equality follows from (3.24).

Which completes the proof

\[\square\]

4.3. Local uncertainty principle for $\mathcal{O}$–OLCT.

Local uncertainty principle states that if $f$ is highly localized, then the octonion Fourier transform can not be concentrated in a small neighborhood of two or more separated points. We shall establish Local uncertainty principle for $\mathcal{O}$–OLCT in this subsection.

**Lemma 4.4** (Local uncertainty principle for QOLCT). For $0 < \alpha < 1$ and for all $f \in L^2(\mathbb{R}^2, \mathbb{H})$, there is a constant $M_\alpha$ and all measurable set $E \subset \mathbb{R}^3$ that holds

\[
\int_{\partial E} |O_{\mu_1, \mu_2}^{A_1, A_2} \{ f \}(w) |^2 dw \leq M_\alpha |E|^\alpha \| |x|^\alpha f \|^2_2.
\]

(4.16)

(2) If $\alpha > 1$ and for all $f \in L^2(\mathbb{R}^2, \mathbb{H})$, there is a constant $M_\alpha$ and all measurable set $E \subset \mathbb{R}^3$ that holds

\[
\int_{\partial E} |O_{\mu_1, \mu_2}^{A_1, A_2} \{ f \}(w) |^2 dw \leq M_\alpha |b_1 b_2|^\alpha |E|^{\alpha - \frac{1}{\alpha}} \| |f \|^2_2^{\frac{1}{2}} ||x|^\alpha f \|^2_2^{\frac{2}{\alpha}}.
\]

(4.17)

\[
M_\alpha = \begin{cases} 
\frac{(1 + \alpha^2)}{\alpha^{2\alpha}} (2 - 2\alpha)^{\alpha - 2}, & 0 < \alpha < 1, \\
\frac{\pi}{\alpha \Gamma(1/2)} \Gamma \left( \frac{1}{\alpha} \right) \Gamma \left( 1 - \frac{1}{\alpha} \right) (\alpha - 1)^{\alpha} \left( 1 - \frac{1}{\alpha} \right)^{-1}, & \alpha > 1,
\end{cases}
\]

(4.18)
Theorem 4.4 (Local uncertainty principle for $\mathcal{O}$–OLCT).

(1) For $0 < \alpha < 1$ and for all $f \in L^2(\mathbb{R}^3, \mathcal{O})$, there is a constant $M_\alpha$ and all measurable set $E \subset \mathbb{R}^3$ that holds

$$\int_{bE} |\mathcal{O}_{A_1, A_2}^{\mu_1, \mu_2} (f)(w)|^2 dw \leq \frac{1}{2\pi |b|_3} M_\alpha |E|^\alpha ||x|^\alpha f||_2^2.$$  \hspace{1cm} (4.19)

(2) If $\alpha > 1$, and for all $f \in L^2(\mathbb{R}^3, \mathcal{O})$, there is a constant $M_\alpha$ and all measurable set $E \subset \mathbb{R}^3$ that holds

$$\int_{bE} |\mathcal{O}_{A_1, A_2}^{\mu_1, \mu_2} (f)(w)|^2 dw \leq \frac{1}{2\pi |b|_3} M_\alpha |b_1 b_2|^{\alpha - \frac{1}{2}} |E|^\alpha ||f||_2^{2-2\alpha} ||x|^\alpha f||_2^2,$$  \hspace{1cm} (4.20)

$$M_\alpha = \begin{cases} 
\frac{(1 + \alpha^2)}{\alpha^2} (2 - 2\alpha)^{\alpha - 2}, & 0 < \alpha < 1, \\
\frac{\pi}{\alpha \Gamma(1/2)} \Gamma \left(\frac{1}{\alpha}\right) \Gamma \left((1 - \frac{1}{\alpha}) (\alpha - 1)^{\alpha} \right) \left(1 - \frac{1}{\alpha}\right)^{-1}, & \alpha > 1.
\end{cases}$$  \hspace{1cm} (4.21)

Proof. By the splitting of $\mathcal{O}$–OLCT in (3.22), we have

$$\mathcal{O}_{\mu_1, \mu_2, \mu_4}^{A_1, A_2, A_3} (f)(w) = \frac{1}{\sqrt{2\pi |b|_3}} \int_{\mathbb{R}^3} g_c(x) K_{A_1}^{\mu_1}(x_1, w_1) K_{A_2}^{\mu_2}(x_2, w_2) c_3 dx$$

$$+ \frac{1}{\sqrt{2\pi |b|_3}} \int_{\mathbb{R}^3} h_c(x) K_{A_1}^{-\mu_1}(x_1, w_1) K_{A_2}^{-\mu_2}(x_2, w_2) s_3 dx$$

$$+ \left( \frac{1}{\sqrt{2\pi |b|_3}} \int_{\mathbb{R}^3} h_c(x) K_{A_1}^{-\mu_1}(x_1, w_1) K_{A_2}^{-\mu_2}(x_2, w_2) c_3 dx \right) \mu_4,$$

Setting $f_m(x) = g_c(x_{+++}) + h_c(x_{+++}) e_2$ and $f_n(x) = h_c(x_{+++}) - g_c(x_{+++}) e_2$. Then $\mathcal{O}$–OLCT can be written as the combination of two QOLCTs as

$$\|\mathcal{O}_{\mu_1, \mu_2, \mu_4}^{A_1, A_2, A_3} (f)(w)\|^2 = \frac{1}{2\pi |b|_3} \|\mathcal{O}_{\mu_1, \mu_2}^{A_1, A_2} (f_m)(w)\|^2 + \|\mathcal{O}_{\mu_1, \mu_2}^{A_1, A_2} (f_n)(w)\|^2.$$  \hspace{1cm} (4.22)

Now, by lemma 4.4 we have

$$\int_{bE} |\mathcal{O}_{\mu_1, \mu_2, \mu_4}^{A_1, A_2, A_3} (f_j)(w)|^2 dw \leq M_\alpha |E|^\alpha ||x|^\alpha f_j||_2^2, \quad j = m, n$$  \hspace{1cm} (4.23)

Therefore from (4.22), (4.23) and using lemma 2.1 we have

$$\int_{bE} \|\mathcal{O}_{\mu_1, \mu_2, \mu_4}^{A_1, A_2, A_3} (f)(w)\|^2 dw = \frac{1}{2\pi |b|_3} \int_{bE} (\|\mathcal{O}_{\mu_1, \mu_2}^{A_1, A_2} (f_m)(w)\|^2 dw + \|\mathcal{O}_{\mu_1, \mu_2}^{A_1, A_2} (f_n)(w)\|^2 dw)$$

$$\leq \frac{1}{2\pi |b|_3} M_\alpha |E|^\alpha (||x|^\alpha f_m||^2 + ||x|^\alpha f_n||^2)$$

$$= \frac{1}{2\pi |b|_3} M_\alpha |E|^\alpha ||x|^\alpha f||_2^2,$$

which completes the proof.

Similarly we can easily prove (4.20). \hfill \square
5. Conclusions

In this paper, based on the association between the $\mathcal{O}−\text{OLCT}$ the QOLCT via split norm, we have established some basic properties of the proposed transform including the inversion formula and energy conservation. These results are very important for their applications in digital signal and image processing. Finally, the uncertainty inequalities for the $\mathcal{O}−\text{OLCT}$ such as logarithmic uncertainty inequality, Hausdorff-Young inequality and local uncertainty are obtained. In our future works, we will discuss the physical significance and engineering background of this paper. Moreover, we will formulate convolution and correlation theorems for the $\mathcal{O}−\text{OLCT}$.

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