ON THE EXISTENCE OF NON-TRIVIAL LAMINATIONS IN $\mathbb{C}P^2$

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Abstract. In this article, we show the existence of a non-trivial Riemann surface lamination embedded in $\mathbb{C}P^2$ by using Donaldson’s construction of asymptotically holomorphic submanifolds. Further the lamination that we obtain has property that every leaf is a totally geodesic submanifold of $\mathbb{C}P^2$ with respect to the Fubini-Study metric. This may constitute a step in understanding the conjecture on the existence of minimal exceptional sets in $\mathbb{C}P^2$.

1. Introduction

A Riemann surface lamination is a compact Hausdorff space which is decomposed into disjoint union of Riemann surfaces, called leaves. Riemann surface laminations arise naturally in the context of complex dynamical systems generated by flows of complex vector fields. A Riemann surface lamination is called minimal if all its leaves are dense. It is well known that we can contruct minimal laminations in $\mathbb{C}P^n$ as flows of polynomial vector fields, when $n$ is greater than or equal to 3. However, this construction fails for $\mathbb{C}P^2$ (see [Ghy99]). Motivated by this, Ghys asked the following question in the same article.

**Question 1.1.** Does there exist a minimal lamination that is holomorphically embedded in $\mathbb{C}P^2$ and does not reduce to a compact Riemann surface?

This is related to the question of existence of an exceptional minimal set for polynomial differential equations on $\mathbb{C}^2$, which is a formalized version of Hilbert’s 16th problem. The absence of non-trivial minimal exceptional sets can be viewed as a generalization of the classical Poincaré-Bendixson theorem. In [Ghy99], Ghys notes that the question of existence of non-trivial minimal lamination embedded in $\mathbb{C}P^2$ is stronger than that of an "exceptional minimal set". In fact, there are no known examples of non-trivial Riemann surface laminations embedded in $\mathbb{C}P^2$. Zakeri, in his article [Zak01], states the following broader question whose special case is the earlier question asked by Ghys.

**Question 1.2.** Does there exist a lamination that is holomorphically embedded in $\mathbb{C}P^2$ and not a compact Riemann surface?

We call a Riemann surface lamination, a non-trivial Riemann surface lamination if it is not a compact Riemann surface. In this article, we answer Question 1.2 in affirmative by proving the following theorem.

**Theorem 1.3.** There exists a non-trivial Riemann surface lamination $L$ embedded in $\mathbb{C}P^2$. Moreover, each leaf of $L$ is a totally geodesic submanifold of $\mathbb{C}P^2$ with respect to the Fubini-Study metric.
We construct a non-trivial Riemann surface lamination embedded in $\mathbb{C}P^2$ by taking limits of regions of vanishing sets of asymptotically holomorphic sections constructed by Donaldson [Don96]. Donaldson used asymptotically holomorphic sections to show existence of symplectic submanifolds of any compact symplectic manifold. Further, Proposition 40 from [Don96] says that the currents associated to the symplectic submanifolds converge to the fundamental form $\frac{\omega_{FS}}{2\pi}$ of the Fubini-Study metric, which is supported everywhere on $\mathbb{C}P^2$. Moreover, if there is a lamination in the “limit”, Bezout’s Theorem suggests that there can be atmost one compact leaf in this lamination. Thus, one hopes to get a non-trivial Riemann surface lamination embedded in $\mathbb{C}P^2$.

Continuing the above idea, we elaborate on how we can possibly obtain a holomorphic disk at a given point in $\mathbb{C}P^2$, which is the support of $\omega_{FS}$. Given any point $x$ in $\mathbb{C}P^2$, we can find a sequence of points $x_k$ belonging to these asymptotically holomorphic submanifolds (zero sets of asymptotically holomorphic sections) which converges to $x$. A uniform lower bound on the injectivity radii of these asymptotically holomorphic charts near points $x_k$ will give a smoothly embedded disk at $x$. A version of Montel’s Theorem (see Section 6) for a family of asymptotically holomorphic maps tells us that the limiting disk is holomorphically embedded. This will help us in constructing a chart for the desired lamination near $x$. A uniform lower bound on injectivity radii of these asymptotically holomorphic submanifolds is given by the uniform upper bound on the second derivatives of asymptotically holomorphic sections.

We “continue forward” this disk to obtain a Riemann surface embedded in $\mathbb{C}P^2$. This gives a recipe to construct a Riemann surface $S_x$ passing through arbitrary point $x \in \mathbb{C}P^2$. We, now, choose a point $x_0 \in \mathbb{C}P^2$ and consider $S_{x_0}$. If $S_{x_0}$ is compact, we choose a point $x_1$ in the complement of $S_{x_0}$ and consider $S_{x_1}$. Further, our construction is such that $S_{x_0} \cap S_{x_1} = \emptyset$. We observed (by Bezout’s Theorem) earlier that $S_{x_1}$ can not be compact.

**Remark 1.4.** The construction of non-trivial Riemann surface laminations in $\mathbb{C}P^2$ easily generalizes to the case of $\mathbb{C}P^n$ with $n \geq 3$. The proof is essentially the same as the proof for the case of $\mathbb{C}P^2$. This gives a new method of construction of non-trivial laminations in addition to the known method of considering flows of polynomial vector fields.

**Remark 1.5.** In this article, we make use of Donaldson’s asymptotically holomorphic sections instead of “locally concentrated” holomorphic sections which he uses to give another proof of Kodaira embedding theorem. We do so, because the second derivatives of the latter type of sections diverge to infinity (Corollary 33 in [Don96]), where as we have a uniform upper bound on the second derivatives of the sections of former type.

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2. Preliminaries

2.1. Donaldson’s construction of asymptotically holomorphic submanifolds.

Our argument to show the existence of a non-trivial embedded Riemann surface laminations is based on Donaldson’s construction of asymptotically holomorphic sections. Thus, in view of making our proof of existence theorem more accessible, we will give a brief exposition on Donaldson’s construction of asymptotically holomorphic sections. We employ the notation in [Don96] as far as possible.

To recall the construction here, we start with the fundamental form \( \omega_{FS} \) given by Fubini-Study metric. We consider the tautological line bundle \( \xi \rightarrow \mathbb{C}P^2 \) whose first Chern class is given by \( \frac{\omega_{FS}}{2\pi} \). The \( k \)-th tensor power of \( \xi \) is denoted by \( \xi^\otimes k \).

The construction of asymptotically holomorphic sections proceeds in two stages, namely the construction of locally supported asymptotically holomorphic sections followed by the construction of global section by taking suitable linear combination of locally supported sections.

First, we focus on the construction of locally supported asymptotically holomorphic sections. Let \( z = (z_1, z_2) \) denote a point in \( \mathbb{C}^2 \). Donaldson makes use of Gaussian decay function

\[
f = e^{-\frac{|z|^2}{4}}
\]

as a basic model for the construction of locally supported sections. On \( \mathbb{C}^2 \), we consider the connection given by

\[
A = \frac{1}{4} (z_1 d\bar{z}_1 - \bar{z}_1 dz_1 + z_2 d\bar{z}_2 - \bar{z}_2 dz_2)
\]

then observe that

\[
i dA = \omega_{std}.
\]

We see that

\[
\overline{\partial} A f = \overline{\partial} f + A^{0,1} f = 0
\]

Thus, the Gaussian decay function is holomorphic with respect to the coupled \( \overline{\partial} \)-operator \( \overline{\partial}_A \). Further, Donaldson observes any non-integrable almost complex structure, when scaled sufficiently near a point, becomes close to being integrable. More precisely, the Nijenhius tensor with its higher order derivatives can be made small in size under a suitable scaling transformation.

Let \( J_0 \) denote the standard (integrable) complex structure on \( \mathbb{C}^2 \). We consider the scaling map \( z \mapsto k z \), for \( k > 0 \). We pull back the complex structure \( J_0 \) by this scaling map and denote the pull-back by \( \tilde{J}_k \). The complexified cotangent bundle decomposes into complex linear and anti-linear parts with respect to both complex structures. Let \( \Lambda^{1,0}_{J_0} \) and \( \Lambda^{0,1}_{J_0} \) denote the \( J_0 \)-linear and anti-linear parts respectively. The decomposition induced by \( \tilde{J}_k \) can be tracked by a map \( \Lambda^{1,0}_{J_0} \rightarrow \Lambda^{0,1}_{J_0} \) which is linear at each point. We denote it by \( \mu \).

The derivative of the map \( \mu \) is bounded by a factor of \( k^{-\frac{1}{2}} \), where \( k > 0 \), denotes the scaling factor. One can achieve the effect of scaling, by a factor of \( \sqrt{k} \) near a point in \( \mathbb{C}^2 \) by taking \( k \)-th powers (under tensoring) of the trivial line bundle \( \xi \) over \( \mathbb{C}^2 \), endowed with the connection \( A \), i.e. by considering sections of the bundle \( \xi^\otimes k \) over \( \mathbb{C}^2 \).
The section $f$, which is holomorphic with respect to $J_0$ and connection $A$, becomes approximately holomorphic with respect to $\tilde{J}_k$ and connection $A$. By putting all the above ingredients together, the local section is constructed by using $f$ and multiplying by a suitable cut-off function. We use Darboux charts $\chi : \mathbb{B}_r(0) \to \mathbb{C}^2$ on $\mathbb{CP}^2$ to push-forward these locally supported asymptotically holomorphic functions on $\mathbb{C}^2$. The Darboux charts can be extended to connection preserving bundle maps. Thus, we get locally supported sections of $k$-th tensor power of the tautological line bundle over $\mathbb{CP}^2$. The crucial point here is that the Darboux charts are asymptotic isometries. Hence, all estimates on the derivatives hold true for locally supported sections on $\mathbb{CP}^2$ as well. Denote this locally supported section near point $p$ by $\sigma_p$.

To obtain asymptotically holomorphic submanifolds as zero sets of asymptotically holomorphic sections, Donaldson takes complex linear combinations of locally supported asymptotically holomorphic sections in the following way.

Let $\{B_i\}$ be a finite cover of $\mathbb{CP}^2$ with each $B_i$ being the support of the section $\sigma_{p_i}$. We choose complex numbers $w_i$ with $|w_i| \leq 1$. Let $w$ denote the tuple $(w_i)$. Then, we set

$$s_w := \sum w_i \sigma_{p_i}$$

Often, we will denote the above section by $s_k$ also to emphasize the role of the twisting parameter $k$. The absolute values $|\partial s_k|$ and $|\nabla \partial s_k|$ are of the size $O(k^{-1/2})$. Donaldson uses estimated version of transversality to make sure that $|\partial s_k| > \eta$ for some suitable $\eta > 0$. The estimated transversality is achieved step-by-step in the following way. Firstly, we partition the Darboux charts into $N(D)$ partitions so that any two balls in the same partition are separated by distance $D(> 0)$. The number of Darboux charts required to cover $\mathbb{CP}^2$ grows with respect to parameter $k$ at least as fast as $k^4$. Therefore, it is important to note that the number of partitions $N$ depends only on the separation distance $D$ and it is independent of the parameter $k$. Now, start with the balls belonging to the first partition. We choose coefficients $w_i$ for locally supported sections on the balls in this partition so that some transversality is achieved on the union of balls in this partition. In the next step, we consider balls belonging to the second partition and adjust the coefficients for the local sections supported on these balls to achieve transversality on the new balls. However, we need to make sure that transversality over the balls in the first partition is not completely lost. This is done by controlling the loss in the transversality by carefully choosing new coefficients corresponding to balls in the second partition. The process is repeated till we achieve transversality over the last partition without loosing transversality on the earlier partitions. Theorem 20 and in particular Proposition 23 from [Don96] states that the estimated transversality can be achieved for suitable choice of the separation parameter $D$ and for sufficiently large values of $k$. We reproduce the Proposition 23 from [Don96] below for later use in this article.

**Proposition 2.1.** Let $Q_p(\delta) = \log(\delta^{-1})^p$. There are constants $\rho < 1$ and $p$ such that if section $S_{\eta_{\alpha-1}}$ is $\eta_{\alpha-1}$-transverse over the union of balls belonging to all partitions starting from first till $\alpha - 1$ and if the twisting parameter $k$ and separation distance $D$ satisfy the following conditions

$$1) \ k^{-1/2} < Q_p(\eta_{\alpha-1}) \eta_{\alpha-1},$$
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(2) $\exp(-D^2/5) \leq Q_p(\eta_{\alpha-1})$

then there is another section $S_{\alpha^\alpha}$ such that it is $\eta_{\alpha}$-transverse over union of balls belonging to all partitions $1, 2, \ldots, \alpha - 1, \alpha$, where $\eta_{\alpha} = Q_p(\eta_{\alpha-1})\eta_{\alpha-1}$.

2.2. Riemann surface laminations. A Riemann surface lamination is a compact space $M$ which locally looks like the product of a disk in the complex plane and a metric space. We will make this notion precise. An atlas for a Riemann surface lamination is given by:

- A cover by open sets $U_i$.
- Homeomorphisms $\varphi_i : U_i \to \mathbb{D} \times T_i$ where $\mathbb{D}$ is a disk in $\mathbb{C}$ and $T_i$ is a topological space.
- The transition maps satisfy the following property.

$$\varphi_{ij}(z, t) := \varphi_j \circ \varphi_i^{-1}(z, t) = (\psi_{ij}(z, t), \lambda_{ij}(t)).$$

Two atlases are equivalent if their union is an atlas. A Riemann surface lamination is a compact space $M$ equipped with an equivalence class of atlases. The inverse images $\varphi_i^{-1}(\mathbb{D} \times \{t\})$ are called plaques. Consider the relation $p \sim q$ if $p$ and $q$ lie on the same plaque. This relation is reflexive and symmetric. Consider the transitive closure of this relation and call it $\sim$. Equivalence classes of this equivalence relation are called leaves. A lamination is called minimal if all its leaves are dense.

3. Step 1: Second derivative estimates for $s_k$

In this section, we establish bounds on the second derivative of Donaldson’s asymptotically holomorphic sections.

We note the following facts which are easy to verify

$$\overline{\partial}_A \left( e^{-\frac{|z|^2}{4}} \right) = 0$$

$$\partial_A \left( e^{-\frac{|z|^2}{4}} \right) = (\partial + A^{1,0}) e^{-\frac{|z|^2}{4}}$$

$$= -\frac{1}{2} (\bar{z}_1 dz_1 + \bar{z}_2 dz_2) e^{-\frac{|z|^2}{4}} \quad (3.2)$$

Combining the equations $3.1$ and $3.2$ we get

$$\nabla_A \left( e^{-\frac{|z|^2}{4}} \right) = (d + A) e^{-\frac{|z|^2}{4}} = -\frac{1}{2} (\bar{z}_1 dz_1 + \bar{z}_2 dz_2) e^{-\frac{|z|^2}{4}} \quad (3.3)$$

Therefore, we observe the following estimate on the first covariant derivative.

$$\left| \nabla_A \left( e^{-\frac{|z|^2}{4}} \right) \right| \leq \frac{1}{2} |z| e^{-\frac{|z|^2}{4}} \quad (3.4)$$
Now, we take the second covariant derivative of the term in the rightmost side of Equation 3.3. We get

\[
\nabla_A \left( \nabla_A \left( e^{-\frac{|z|^2}{4}} \right) \right) = \nabla_A - \frac{1}{2} (\overline{z}_1 d z_1 + \overline{z}_2 d z_2) e^{-\frac{|z|^2}{4}}
\]

\[
= \nabla_{A,0}^{1.0} \left( -\frac{1}{2} (\overline{z}_1 d z_1 + \overline{z}_2 d z_2) e^{-\frac{|z|^2}{4}} \right) + \nabla_{A,0}^{0.1} \left( -\frac{1}{2} (\overline{z}_1 d z_1 + \overline{z}_2 d z_2) e^{-\frac{|z|^2}{4}} \right).
\]

The term (1) in the above expression gives 0 after a straightforward computation and the expression in (2) of Equation 3.5 can be simplified to

\[
\frac{1}{2} (d z_1 \wedge d \overline{z}_1 + d z_2 \wedge d \overline{z}_2) e^{-\frac{|z|^2}{4}}
\]

We note that both the terms in Equation 3.5 are bounded in absolute value by \( C e^{-\frac{|z|^2}{4}} \), where \( C \) is a suitable constant.

We now need to transfer the estimates onto the image of any Darboux chart. As observed in [Don96] the Darboux charts \( \chi \circ \delta_k^{-1/2} \) are asymptotic isometries, thus the estimates on the first and second order derivatives will change at most by a factor of \( C k^{-1/2} |z|^3 \) (see Proposition 11 in [Don96]). We recall some notation from the same article by Donaldson. Let \( \sigma_p \) denote a section obtained by push-forward of locally supported asymptotically holomorphic section constructed as above. Let \( d \) denote the distance induced by the Fubini-Study metric. The scaled distance \( k^{1/2} d \) is denoted by \( d_k \). Define the symbol \( e_k(p,q) \) to be \( e^{-d_k^2(p,q)/5} \) if \( d_k(p,q) \leq k^{1/4} \) and is 0 for \( d_k(p,q) > k^{1/4} \). Then, the section \( \sigma_p \), for each \( p \in \mathbb{CP}^2 \), the following estimates hold:

1. \( |\sigma_p(q)| \leq e_k(p,q) \),
2. \( |\nabla \sigma_p(q)| \leq C (1 + d_k(p,q)) e_k(p,q) \),
3. \( |\partial \sigma_p(q)| \leq C k^{-1/2} d_k^2(p,q) e_k(p,q) \),
4. \( |\nabla \sigma_p(q)| \leq C k^{-1/2} (d_k(p,q) + d_k^3(p,q)) e_k(p,q) \).

where \( \nabla \) denotes the Levi-Civita connection on \( \mathbb{CP}^2 \) of the Fubini-Study metric and \( C \) is a constant independent of \( k \).

From the above inequalities it follows that

\[
|\nabla (\nabla \sigma_p)|_q < C (d_k(p,q) + d_k^3(p,q)) e_k(p,q)
\]

Therefore, we see that the second covariant derivative is bounded by some \( m > 0 \) as given below

\[
|\nabla (\nabla \sigma_p)| < m
\]

To get a global estimates on \( s_w \) and its derivatives, we note below the estimates given in Lemma 14 in [Don96] for the later use in this article.
Lemma 3.1. For any choice of coefficients $w = (w_i)$ with $|w_i| \leq 1$, the section $s_w$ satisfies the following estimates everywhere on $\mathbb{CP}^2$

$$|s_w| \leq C,$$
$$|\partial_s w| \leq Ck^{-1/2},$$
$$|\nabla \partial_s w| \leq Ck^{-1/2}.$$ 

The following lemma builds on the Lemma 12 in [Don96]. It plays central role in establishing the fact that lamination we obtain is totally geodesic.

Lemma 3.2. At each point $p \in \mathbb{CP}^2$, the following holds,

- $|(\nabla \nabla s_k)|_p \longrightarrow 0$
- $|d(|\nabla s_k|)|_p \longrightarrow 0$

as $k \rightarrow \infty$. However, the convergence is not uniform.

Proof. In the proof of Lemma 12 in [Don96], Donaldson reduces the argument to that of Euclidean case. Donaldson chooses a cover of $\mathbb{C}^n$ with balls having their centres at the lattice points of some suitable lattice in $\mathbb{C}^n$. Then the sum 3.8 over the chosen lattice is observed to be uniformly bounded. Our argument is the same as in Lemma 12 in [Don96]. We combine it with the observation that for each $a, r > 0$ and $\omega \in \mathbb{C}^2$

$$\sum_{\mu \in \Lambda} |\mu - \omega|^r e^{-a|\mu - \omega|^2}$$

(3.8)

converges to 0 pointwise (but not uniformly) with respect to $\omega$, as $a \rightarrow \infty$. □

4. Step 2: Construction of approximately holomorphic disks around points in $s_k^{-1}(0)$

In this section, we show the existence of embedded disks in the vanishing sets of approximately holomorphic sections $s_k$ such that the radii of these disks are bounded below by some positive constant $r$ which is independent of $k$ and point $p$. Moreover, we show that there is “approximately holomorphic” embeddings of the disk of radius $r$.

Let $s_k$ be a section of the line bundle $\xi^{\otimes k} \to \mathbb{CP}^2$ such that $s_k^{-1}(0)$ is an “approximately” holomorphic submanifold of $\mathbb{CP}^2$. On each affine chart $A_j$ for $\mathbb{CP}^2$, we think of $s_k$ as a function $s_k : \mathbb{C}^2 \to \mathbb{C}$. We further break $s_k$ into real and imaginary parts, $s_k = u_k + iv_k$. Notice that $\frac{\nabla u}{|\nabla u|} = N_1$ and $\frac{\nabla v}{|\nabla v|} = N_2$ are two unit normal vector fields to $s_k^{-1}(0)$. In this section, we will represent the curvature of $s_k^{-1}(0)$ in terms of the the curvature of $\mathbb{CP}^2$ and Weingarten maps. Thus an upper bound on the norm of Weingarten maps would give an upper bound on the curvature of $s_k^{-1}(0)$. We will prove the upper bound on Weingarten maps, by giving bounds on the derivatives of $\frac{\nabla u_k}{|\nabla u_k|}, \frac{\nabla v_k}{|\nabla v_k|}$, in Section 5. The upper bound on curvature will in turn give a lower bound on radii of embedded disks in $s_k^{-1}(0)$ by the celebrated result of Klingenberg.

Recall that $\nabla$ denotes the Levi-Civita connection induced by Fubini-Study metric on $\mathbb{CP}^2$. Let $\nabla$ denote the Levi-Civita connection on $s_k^{-1}(0) = V$ induced by the Fubini-Study metric on $\mathbb{CP}^2$. We define the two Weingarten maps $w_{i,p} : T_p V \times T_p V \to \mathbb{R}$ as
Combining the above expressions, we have,

\[ w_{1,p}(u, v) = \langle \nabla_u N_1 - \langle \nabla_u N_1, N_2 \rangle N_2, v \rangle \]  
\[ w_{2,p}(u, v) = \langle \nabla_u N_2 - \langle \nabla_u N_2, N_1 \rangle N_1, v \rangle. \]  

(4.1)  
(4.2)

For vector fields \( X, Y, Z \) on \( \mathbb{CP}^2 \) (or an affine chart), the curvature tensor is given by

\[ R_T(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \]

As we are interested in computing the curvature of \( V \), we would like to compute values of \( \langle R_T(X, Y, X), Y \rangle \) and \( \langle R_T(X, Y, X), Y \rangle \) when \( X, Y \) are tangent to \( V \). As \( Y \) is orthogonal to \( N_1 \) and \( N_2 \), we will ignore the components of \( R_T(X, Y, X) \) along \( N_1 \) and \( N_2 \).

As \( T_p \mathbb{CP}^2 \) is spanned by \( T_p V, N_1(p) \) and \( N_2(p) \), we can write,

\[ \nabla_Y X = \frac{\nabla Y X}{\Vert \nabla Y X \Vert} (\Vert \nabla Y X \Vert N_1 + \langle \nabla Y X, N_2 \rangle N_2). \]

(4.3)

So,

\[ \nabla_X \nabla_Y X = \nabla_X (\nabla Y X) + \nabla_X (\langle \nabla Y X, N_1 \rangle N_1) + \nabla_X (\langle \nabla Y X, N_2 \rangle N_2). \]

We will compute each term in the right hand side separately. Note that, as \( \langle X, N_i \rangle = 0 \),

\[ \langle \nabla Y X, N_i \rangle = -\langle \nabla Y N_i, X \rangle \]

Thus, we rewrite (4.3) as

\[ \nabla_Y X = \nabla_Y X - \langle \nabla Y N_1, X \rangle N_1 - \langle \nabla Y N_2, X \rangle N_2. \]

(4.4)

Now, we take further covariant derivative and obtain the following

\[ \nabla_X (\nabla Y X) = \nabla_X \nabla Y X - \langle \nabla_X N_1, \nabla Y X \rangle N_1 - \langle \nabla_X N_2, \nabla Y X \rangle N_2 \]

and

\[ \nabla_X (\langle \nabla Y X, N_i \rangle N_i) = -\langle \nabla Y N_i, X \rangle \nabla_X N_i + \text{components along } N_1 \text{ and } N_2. \]

For simplicity, we will call the components along \( N_1 \) and \( N_2 \), normal components.

So, we have,

\[ \nabla_X \nabla_Y X = \nabla_X \nabla Y X - \langle \nabla Y N_1, X \rangle \nabla_X N_1 - \langle \nabla Y N_2, X \rangle \nabla_X N_2 + \text{normal components}. \]

Similarly,

\[ \nabla_Y \nabla_X X = \nabla_Y \nabla X X - \langle \nabla X N_1, X \rangle \nabla_Y N_1 - \langle \nabla X N_2, X \rangle \nabla_Y N_2 + \text{normal components}. \]

And finally,

\[ \nabla_{[X,Y]} X = \nabla_{[X,Y]} X + \text{normal components}. \]

Combining the above expressions, we have,

\[ R_T(X, Y, X) = \nabla_X \nabla_Y X - \langle \nabla Y N_1, X \rangle \nabla_X N_1 - \langle \nabla Y N_2, X \rangle \nabla_X N_2 - \nabla_Y \nabla_X X + \langle \nabla_X N_1, X \rangle \nabla_Y N_1 + \langle \nabla_X N_2, X \rangle \nabla_Y N_2 - \nabla_{[X,Y]} X + \text{normal components} \]

\[ = R_T(X, Y, X) - \langle \nabla Y N_1, X \rangle \nabla_X N_1 - \langle \nabla Y N_2, X \rangle \nabla_X N_2 + \langle \nabla X N_1, X \rangle \nabla_Y N_1 + \langle \nabla X N_2, X \rangle \nabla_Y N_2 + \text{normal components} \]
Now, taking $X, Y$ to be orthonormal vector fields spanning $TV$, we get the curvature of $V$ in $\mathbb{CP}^2$ to be

\[ K^\nabla_V = \left< R_V(X, Y, X), Y \right> \]

\[ = \left< R_\nabla(X, Y, X), Y \right> - \left< \nabla_Y N_1, X \right> \left< \nabla_X N_1, Y \right> - \left< \nabla_Y N_2, X \right> \left< \nabla_X N_2, Y \right> + \left< \nabla_X N_1, X \right> \left< \nabla_Y N_1, Y \right> \]

\[ + \left< \nabla_X N_2, X \right> \left< \nabla_Y N_2, Y \right> \]

\[ = K^\nabla_V + w_1(X, X)w_1(Y, Y) + w_2(X, X)w_2(Y, Y) - w_1(X, Y)w_1(Y, X) - w_2(X, Y)w_2(Y, X). \]

The bounds on the curvature of $\mathbb{CP}^2$ endowed with Fubini-Study metric($1/4 \leq K \leq 1$) and the Weingarten maps (see Lemma 5.2) will give us the following bound on the curvature of $V$

\[ \left| K^\nabla_V \right| \leq 1 + 4\beta^2 \]

where $\beta > 0$ is some constant independent of $k$. Now, we apply Klingenberg’s theorem along with the upper bound on the curvature obtained as above to give a lower bound on the injectivity radius of $V$, say $r$ that depends only on $\beta$. Note that this bound is independent of $k$.

Now, we define what we mean by an approximately holomorphic embedding of a disk.

**Definition 4.1.** Let $\varepsilon > 0$ and $D \subset \mathbb{C}$ be a domain. We say that a smooth embedding $\varphi : D \to \mathbb{C}^2$ is $\varepsilon$-approximately holomorphic if the following holds.

\[ \left| \frac{\partial \varphi}{\partial \bar{z}} \right| < \varepsilon. \]

We will call an $\varepsilon$-approximately holomorphic embedding of a disk $D_r$ as an approximately holomorphic disk in $\mathbb{C}^2$. We construct an approximately holomorphic disk $\varphi : D_r \to V$ as follows. For a point $p \in V$, let $L$ denote a 1-dimensional complex subspace of $T_p\mathbb{CP}^2$ which is close to $T_p V \subset T_p\mathbb{CP}^2$. Such an $L$ exists because $T_p V$ is approximately complex vector subspace of $T_p\mathbb{CP}^2 (= \mathbb{C}^2)$. Let $\pi$ denote the projection of $L$ onto $T_p V$. We note that the antilinear part of $\pi$ satisfies the bound given below

\[ \|\pi^{0,1}\| < Ck^{-1/2}. \]

Let $exp_p$ denote the exponential map $T_p V \to V$. Then consider the map $\varphi = exp_p \circ \pi : D_r \to \mathbb{C}^2$, where $D_r \subset L$ is a disk centred at origin of radius $r$. We observe that the following lemma holds

**Lemma 4.2.** The disk $\varphi$ is $\varepsilon$-approximately holomorphic, where $\varepsilon = C'k^{-1/2}$.

**Proof.** Notice that the derivative of $exp_p$ is bounded by some constant $\tau$. The tangent vector $\frac{\partial}{\partial z}$ to $D_r$ gets mapped to $\frac{\partial \varphi}{\partial \bar{z}}$ whose norm is bounded above by $\tau Ck^{-1/2}$. We take $C' = \tau C$. \[ \square \]

5. **Step 3: Estimates on the Weingarten Maps**

First, we consider the affine charts $A_i$, for $i = 0, 1, 2$, for $\mathbb{CP}^2$. On each chart, the bundle $\xi^{\otimes k}$ is trivial. Thus we can express the section $s_k$ of $\xi^{\otimes k}$ as a function $f : \mathbb{C}^2 \to \mathbb{C}$. Further, we write $f_k = u_k + iv_k$, where $u_k$ and $v_k$ are real valued
functions. We pull-back of the Fubini-Study metric on $A_j$ and denote the pull-back of the connection by $\nabla$. Then, the complex gradient $\nabla f_k$ can be written as $\nabla u_k + i \nabla v_k$.

Note that we have the following estimate as $s_k$ is asymptotically holomorphic

$$|i\nabla u_k - \nabla v_k| < Ck^{-\frac{1}{2}} \quad (5.1)$$

**Lemma 5.1.** There exists $\eta > 0$, independent of $k$, such that $|\nabla u_k| > \eta$ and $|\nabla v_k| > \eta$.

**Proof.** In the proof of estimated transversality (Proposition 23 in [Don96] or see Proposition 2.1), the separation parameter $D$ is chosen in the end to retain some transversality. The separation parameter $D$ was independent of $k$. This means for all sufficiently large $k$, $|\nabla s_k| > c$, for some $c > 0$ independent of $k$. Inequality 5.1 implies that $|\nabla u_k|$ and $|\nabla v_k|$ are asymptotically of same size. Combining it with Donaldson’s estimated transversality, we conclude that both derivatives cannot become arbitrarily small when $k$ is sufficiently large. \hfill \Box

**Lemma 5.2.** Let $N_1^{(k)} = \frac{\nabla u_k}{|\nabla u_k|}$ and $N_2^{(k)} = \frac{\nabla v_k}{|\nabla v_k|}$. Then there is a constant $\beta > 0$ which is independent of $k$ and point $p \in \mathbb{CP}^2$ such that the following inequalities hold

$$|\nabla N_1^{(k)}| \leq \beta, \quad (5.2)$$

$$|\nabla N_2^{(k)}| \leq \beta. \quad (5.3)$$

**Proof.** The proof follows by straightforward computation of the derivative and applying the estimates obtained in the previous lemma. We do the computation for the vector field $N_1^{(k)}$ explicitly. The estimate on $|\nabla N_2^{(k)}|$ follows similarly.

$$\nabla \left( \frac{\nabla u_k}{|\nabla u_k|} \right) = \frac{1}{|\nabla u_k|} \nabla (\nabla u_k) + \nabla u_k \otimes d \left( \frac{1}{|\nabla u_k|} \right) \quad (A)$$

In the term $(A)$, by applying previous lemma we see that

$$1 \leq \frac{1}{|\nabla u_k|} \quad (5.4)$$

Now, observe that

$$|\nabla (\nabla u_k)| \leq |\nabla (\nabla s_k)|$$

and by Inequality 3.2 we have,

$$|\nabla (\nabla u_k)| \leq |\nabla (\nabla s_k)| < M$$

Combining the above we get

$$\left| \frac{1}{|\nabla u_k|} \nabla (\nabla u_k) \right| < \frac{M}{\eta} \quad (5.5)$$

To obtain the estimates in the term $(B)$, we note that

$$|\nabla u_k| \leq |\nabla s_k| < M'$$

and

$$\left| d \left( \frac{1}{|\nabla u_k|} \right) \right| = \left| -\frac{1}{|\nabla u_k|^2} d \left( |\nabla u_k| \right) \right| \leq \frac{C}{\eta^2}$$
The bound on $|d(|\nabla u_k|)|$ follows from $3.3$ as the term $e^{-|\sqrt{k}a|/4}$ survives after differentiating. Combining the above two inequalities we observe that

$$|\nabla u_k \otimes d\left(\frac{1}{|\nabla u_k|}\right)| \leq \frac{M'C}{\eta^2} \quad (5.6)$$

The inequalities $5.5$ and $5.6$ together imply that

$$|\nabla N_1| \leq \frac{M}{\eta} + \frac{M'C}{\eta^2}$$

where all the constants are independent of $k$.

We will now strengthen the above lemma by showing that the derivatives of the normals go to zero pointwise as $k$ goes to infinity.

**Lemma 5.3.** For each point $p \in \mathbb{CP}^2$, as $k \to \infty$, we have the following

$$|\nabla N_1^{(k)}| \to 0$$
$$|\nabla N_2^{(k)}| \to 0.$$  

**Proof.** By Lemma $3.2$ we have, $|\nabla (\nabla u_k)| \to 0$ as $k \to \infty$ pointwise. Recall that from $5.4$ we have $|\nabla u_k|^{-1} < \eta$ for some constant $\eta > 0$ independent of $k$ as given in Lemma $5.1$. Therefore, we see that

$$ \left| \frac{1}{|\nabla u_k|} \nabla (\nabla u_k) \right| \to 0$$

as $k \to \infty$.

Recall that

$$\left| d\left(\frac{1}{|\nabla u_k|}\right)\right| = \left| -\frac{1}{|\nabla u_k|^2} d(|\nabla u_k|) \right|.$$  

Further $|\nabla u_k|^{-2} < \eta^2$ from Lemma $5.1$. Observe that the real part of $d|\nabla s_k|$ gives $d|u_k|$. By Lemma $3.2$ $d|\nabla s_k| \to 0$ as $k \to 0$.

$$\left| \nabla u_k \otimes d\left(\frac{1}{|\nabla u_k|}\right) \right| \to 0$$

as $k \to \infty$ for each point. □

### 6. Step 4: Approximate version of Montel’s theorem

Let $D_r$ be a disk of radius $r$ in $\mathbb{C}$ centred at the origin. Further, let $\varphi_k : D_r \to \mathbb{C}^2$ be approximately holomorphic disks constructed earlier. Let $p_i : \mathbb{C}^2 \to \mathbb{C}$ be the projection map to the $i$-th coordinate, $i = 1, 2$. Note that, $\psi^i_k = p_i \circ \varphi_k : D_r \to \mathbb{C}$ are approximately holomorphic.

**Lemma 6.1.** Some subsequence of $\psi^i_k$ converges to a function $\psi^i : D_r \to \mathbb{C}$.

**Proof.** We will prove this by showing that the family of functions $\psi^i_k$ is uniformly bounded and equi-continuous.
Uniform boundedness:
By Cauchy-Pompeiu formula, given a disk $D \subset D_r$,

$$
\psi^i_k(\zeta) = \frac{1}{2\pi i} \int_{\partial D} \frac{\psi^i_k(z)dz}{z - \zeta} - \frac{1}{\pi} \int_D \frac{\partial \psi^i_k}{\partial \zeta} \frac{dx \wedge dy}{z - \zeta}.
$$

So,

$$
|\psi^i_k(\zeta)| \leq \frac{1}{2\pi} \int_{\partial D} \frac{|\psi^i_k(z)||dz|}{|z - \zeta|} + \frac{1}{\pi} \int_D \frac{|\partial \psi^i_k|}{|z - \zeta|} \frac{dx \wedge dy}{|z - \zeta|}.
$$

By Lemma 4.2 $|\psi^i_k(z)| < M_1$ and $|\frac{\partial \psi^i_k}{\partial \zeta}| < M_2$ for constants $M_1, M_2$ independent of the function and the point $z$. Thus,

$$
|\psi^i_k(\zeta)| \leq \frac{M_1}{2\pi} \int_{\partial D} \frac{|dz|}{|z - \zeta|} + \frac{M_2}{\pi} \int_D \frac{dx \wedge dy}{|z - \zeta|}.
$$

Let $g : \partial D \to \mathbb{R}$ given as $g(z) = |z - \zeta|$. This is clearly continuous and $\partial D$ is compact, so $g$ attains its minimum, say $c$. Thus, if $D$ is a disk of radius $R$,

$$
\int_{\partial D} \frac{|dz|}{|z - \zeta|} \leq \int_{\partial D} \frac{|dz|}{c} = 2\pi R.
$$

On the other hand,

$$
\int_D \frac{dx \wedge dy}{|z - \zeta|} = \int_0^R \int_0^{2\pi} \frac{r \, dr \, d\theta}{r} = \int_0^R \int_0^{2\pi} d\theta \, r = 2\pi R.
$$

The integrability of the integrals gives us uniform boundedness.

Equi-continuity:

$$
|\psi^i_k(\zeta_1) - \psi^i_k(\zeta_2)| \leq \frac{1}{2\pi i} \int_{\partial D} \frac{\psi^i_k(z)dz}{z - \zeta_1} - \frac{1}{2\pi i} \int_{\partial D} \frac{\psi^i_k(z)dz}{z - \zeta_2}
$$

$$
+ \frac{1}{\pi} \int_D \frac{\partial \psi^i_k}{\partial \zeta} \frac{dx \wedge dy}{z - \zeta_2} - \frac{1}{\pi} \int_D \frac{\partial \psi^i_k}{\partial \zeta} \frac{dx \wedge dy}{z - \zeta_1}.
$$

We know from the proof of Montel’s theorem that $\left|\frac{1}{2\pi i} \int_{\partial D} \frac{\psi^i_k(z)dz}{z - \zeta_1} - \frac{1}{2\pi i} \int_{\partial D} \frac{\psi^i_k(z)dz}{z - \zeta_2} \right| < \varepsilon$ if $|\zeta_1 - \zeta_2| < \delta$, where $\delta$ is independent of functions $\psi^i_k$. On the other hand,

$$
\frac{1}{\pi} \int_D \frac{\partial \psi^i_k}{\partial \zeta} \frac{dx \wedge dy}{z - \zeta_2} - \frac{1}{\pi} \int_D \frac{\partial \psi^i_k}{\partial \zeta} \frac{dx \wedge dy}{z - \zeta_1} \leq \frac{1}{\pi} \int_D \frac{\partial \psi^i_k}{\partial \zeta} \frac{\zeta_1 - \zeta_2}{(z - \zeta_1)(z - \zeta_2)}dx \wedge dy
$$

$$
\leq |\zeta_1 - \zeta_2| \frac{1}{\pi} \int_D \frac{\partial \psi^i_k}{\partial \zeta} \frac{1}{(z - \zeta_1)(z - \zeta_2)}dx \wedge dy.
$$

Thus, we have equi-continuity if we have a uniform bound on $\left|\int_D \frac{\partial \psi^i_k}{\partial \zeta}(z - \zeta_1)(z - \zeta_2)dx \wedge dy \right|$. We will first prove $\frac{1}{(z - \zeta_1)(z - \zeta_2)}$ is integrable and later use Stoke’s theorem to obtain a bound. From the estimates on second derivatives, we know that $|\frac{\partial \psi^i_k}{\partial \zeta}| < M_2$ by
Lemma 4.2 So,
\[
\left| \iint_D \frac{1}{(z - \zeta_1)(z - \zeta_2)} \, dx \wedge dy \right| \leq \iint_D \left| \frac{1}{(z - \zeta_1)(z - \zeta_2)} \right| \, dx \wedge dy \\
\leq M_2 \iint_D \left| \frac{1}{(z - \zeta_1)(z - \zeta_2)} \right| \, dx \wedge dy
\]

We will further split the integral on the right hand side of the above inequality as follows.
\[
\iint_D \left| \frac{1}{(z - \zeta_1)(z - \zeta_2)} \right| \, dx \wedge dy = \iint_{B_\rho(\zeta_1)} \left| \frac{1}{(z - \zeta_1)(z - \zeta_2)} \right| \, dx \wedge dy \\
+ \iint_{B_\rho(\zeta_2)} \left| \frac{1}{(z - \zeta_1)(z - \zeta_2)} \right| \, dx \wedge dy \\
+ \iint_{D \setminus (B_\rho(\zeta_1) \cup B_\rho(\zeta_2))} \left| \frac{1}{(z - \zeta_1)(z - \zeta_2)} \right| \, dx \wedge dy
\]

Choose \( \rho \) such that \( 0 < \rho < \frac{d(\zeta_1 - \zeta_2)}{2} \). Then, notice that,
\[
\iint_{D \setminus (B_\rho(\zeta_1) \cup B_\rho(\zeta_2))} \left| \frac{1}{(z - \zeta_1)(z - \zeta_2)} \right| \, dx \wedge dy \leq \frac{1}{\rho^2} \iint_{D \setminus (B_\rho(\zeta_1) \cup B_\rho(\zeta_2))} \, dx \wedge dy \\
\leq \frac{1}{\rho^2} \iint_D \, dx \wedge dy \leq \frac{\text{Area}(D)}{\rho^2}. \quad (6.1)
\]

And,
\[
\iint_{B_\rho(\zeta_1)} \left| \frac{1}{(z - \zeta_1)(z - \zeta_2)} \right| \, dx \wedge dy \leq \iint_{B_\rho(\zeta_1)} \left| \frac{1}{\rho(z - \zeta_1)} \right| \, dx \wedge dy \\
= \frac{1}{\rho} \iint_{B_\rho(\zeta_1)} \left| \frac{1}{(z - \zeta_1)} \right| \, dx \wedge dy \leq 2\pi \quad (6.2)
\]

From (6.1) and (6.2) \( \frac{1}{(z - \zeta_1)(z - \zeta_2)} \) is integrable. Thus, by Stoke’s theorem:
\[
\iint_D \frac{1}{(z - \zeta_1)(z - \zeta_2)} \, dx \wedge dy = \iint_D d \left( \frac{\psi_k^i dz}{(z - \zeta_1)(z - \zeta_2)} \right) = \int_{\partial D} \frac{\psi_k^i dz}{(z - \zeta_1)(z - \zeta_2)}.
\]

Fix \( \zeta_1 \) and choose \( \zeta_2 \) such that \( |\zeta_1 - \zeta_2| < d(\zeta_1, \partial D)/2 \). Thus,
\[
d(\zeta_1, \partial D) \leq d(\zeta_1, \zeta_2) + d(\zeta_2, \partial D) \leq \frac{d(\zeta_1, \partial D)}{2} + d(\zeta_2, \partial D).
\]

So, we have
\[
d(\zeta_2, \partial D) \geq \frac{d(\zeta_1, \partial D)}{2}.
\]

Hence,
\[
\left| \int_{\partial D} \frac{\psi_k^i dz}{(z - \zeta_1)(z - \zeta_2)} \right| \leq \int_{\partial D} \left| \frac{\psi_k^i}{|\zeta_1 - \zeta_2|} \right| dz \leq \frac{4M_2}{d(\zeta_1, \partial D)^2} \int_{\partial D} |dz| = \frac{8\pi RM_2}{d(\zeta_1, \partial D)}.\]
Thus, we get equicontinuity at any point $\zeta_1 \in D$. 

For simplicity of notation, we will denote the subsequence $\psi_{nk}^i$ converging to $\psi^i$ by $\psi_k^i$.

**Lemma 6.2.** The map $\psi^i$ is holomorphic.

**Proof.** From the definition of $\psi^i$, we have,

$$\psi^i(\zeta) = \lim_{k \to \infty} \psi_k^i(\zeta) = \frac{1}{2\pi i} \lim_{k \to \infty} \int_{\partial D} \frac{\psi_k^i(z)dz}{z - \zeta} - \frac{1}{\pi} \lim_{k \to \infty} \int_{D} \frac{\partial \psi_k^i}{\partial \zeta} \frac{dx \wedge dy}{z - \zeta}$$

By the dominated convergence theorem, we can interchange limit and integral. Notice that,

$$\lim_{k \to \infty} \int_{D} \frac{\partial \psi_k^i}{\partial \zeta} \frac{dx \wedge dy}{z - \zeta} = \int_{D} \lim_{k \to \infty} \frac{\partial \psi_k^i}{\partial \zeta} \frac{dx \wedge dy}{z - \zeta}$$

Further, $\frac{\partial \psi_k^i}{\partial \zeta}$ converges to zero as $k$ tends to infinity, by Lemma 4.2. Thus,

$$\psi^i(\zeta) = \frac{1}{2\pi i} \lim_{k \to \infty} \int_{\partial D} \frac{\psi_k^i(z)dz}{z - \zeta} = \frac{1}{2\pi i} \int_{\partial D} \lim_{k \to \infty} \frac{\psi_k^i(z)dz}{z - \zeta} = \frac{1}{2\pi i} \int_{\partial D} \psi^i(z)dz.$$

So, $\psi^i$ satisfies the Cauchy integral formula. Hence it is holomorphic. 

For simplicity of notation, we will denote the subsequence $\psi_{nk}^i$ converging to $\psi^i$ by $\psi_k^i$.

**Lemma 6.2.** The map $\psi^i$ is holomorphic.

**Proof.** From the definition of $\psi^i$, we have,

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By the dominated convergence theorem, we can interchange limit and integral. Notice that,

$$\lim_{k \to \infty} \int_{D} \frac{\partial \psi_k^i}{\partial \zeta} \frac{dx \wedge dy}{z - \zeta} = \int_{D} \lim_{k \to \infty} \frac{\partial \psi_k^i}{\partial \zeta} \frac{dx \wedge dy}{z - \zeta}$$

Further, $\frac{\partial \psi_k^i}{\partial \zeta}$ converges to zero as $k$ tends to infinity, by Lemma 4.2. Thus,

$$\psi^i(\zeta) = \frac{1}{2\pi i} \lim_{k \to \infty} \int_{\partial D} \frac{\psi_k^i(z)dz}{z - \zeta} = \frac{1}{2\pi i} \int_{\partial D} \lim_{k \to \infty} \frac{\psi_k^i(z)dz}{z - \zeta} = \frac{1}{2\pi i} \int_{\partial D} \psi^i(z)dz.$$

So, $\psi^i$ satisfies the Cauchy integral formula. Hence it is holomorphic.

7. Step 5: Construction of a non-trivial lamination

Pick a point $x$ and a sequence of points $x_k \in s_k^{-1}(0)$ such that the sequence converges to $x$. Construct approximately holomorphic disks $\varphi_k$ centred at $x_k$ as in Step 2. We saw in the previous section that $\psi_{nk}^i = p_i \circ \varphi_k$ has a convergent subsequence, say $\psi_{nk}^i$. That is the functions $\varphi_{nk}$ converge to an analytic disk $\varphi : D_r \to \mathbb{CP}^2$ at $x$. It is important to note that the radius $r$ of the disk is independent of the point $x$. For simplicity of notation we will denote $\varphi_{nk}$ by $\varphi_k$.

Fix $0 < \epsilon < r$. Let $x' = \varphi(r - \epsilon)$. The sequence $x'_k = \varphi_k(r - \epsilon)$ converges to $x'$. Construct approximately holomorphic disks (of radius $r$) $\varphi'_k$ centred at $x'_k$ as in Step 2. There exists a further subsequence, say $n_k$, such that $\varphi'_{nk}$ converges to an analytic disc $\varphi' : D_r \to \mathbb{CP}^2$ at $x'$. This disk overlaps the previous disk in an open set. We have thus continued the disk “forward". We construct the maximal surface which is obtained by continuing this way. Let us call this surface $S$. In order to prove that $S$ is an embedded surface, we need the following lemma.

**Lemma 7.1.** Let $\psi : D_r \to \mathbb{CP}^2$ be an analytic disk obtained as above in the limit. Then $\psi(D_r)$ is a totally geodesic submanifold of $\mathbb{CP}^2$ (with respect to the Fubini-Study metric). Therefore, the curvature of $\psi(D_r)$ at any point equals 1.

**Proof.** By Lemma 5.3 and expression (4.4), we conclude that the Levi-Civita connection $\nabla$ and its restriction $\nabla$ to $\psi(D_r)$ are equal. Therefore, $\psi(D_r)$ is a totally geodesic submanifold of $\mathbb{CP}^2$. 

□
Lemma 7.2. Let two analytic disks $\psi : D_r \to \mathbb{CP}^2$ and $\varphi : D_r \to \mathbb{CP}^2$, obtained as limits, intersect nontrivially. Then the disks $\psi(D_r)$ and $\varphi(D_r)$ intersect in an open subset of both or they intersect transversally.

Proof. By Lemma 7.1, the embedded disks $\psi(D_r)$ and $\varphi(D_r)$ are totally geodesic submanifolds of (real) codimension 2. Further, the curvature at point on the disks equals 1. We know that the sectional curvature 1 is attained only at complex subspaces of the tangent space at any point in $\mathbb{CP}^2$.

Now, we recall a general result in this context that given a point $p$ in a Riemannian manifold $(M, g)$ and a subspace $V$ of $T_p M$, if there is a totally geodesic submanifold in a neighborhood of $p$ which passes through $p$ and tangent to $V$ then it must be unique. We apply it our setting to conclude that any given point $p$ and a complex subspace $\xi$ of $T_p \mathbb{CP}^2$, there exists a unique totally geodesic submanifold passing through $p$ and tangent to $\xi$ in a neighborhood of point $p$.

Using the uniqueness result stated above, we conclude that either $\psi(D_r)$ and $\varphi(D_r)$ intersect transversally or they overlap in an open subset of both. □

Lemma 7.3. The surface $S$ is holomorphically embedded in $\mathbb{CP}^2$.

Proof. First we show that $S$, obtained as above, is embedded. Assume the contrary. By Lemma 7.2, this is possible if and only if a disk produced by taking the limit is intersected transversally by a disk formed later by the same process. Note that, at each step we took a further subsequence, so there exist a subsequence, $n_k$, such that both these disks are limits of disks in $s^{-1}_{n_k}(0)$. But, this would imply that $s^{-1}_{n_k}(0)$ would intersect itself for large $k$, which is impossible. □

Theorem 7.4. If $S$ is a surface obtained as above, then $\overline{S}$ is laminated.

Proof. Let $x \in \overline{S}$. Then there exists a sequence $x_n \in S$ such that $x_n$ converges to $x$. Further, there exist disks $\varphi_n : D_r \to \mathbb{CP}^2$ such that $\varphi_n(0) = x_n$ and $\varphi_n(D_r) \subset S$. As before, there exists some subsequence of these $\varphi_n$’s that converge, say $\varphi_{nk}$ converges to $\varphi : D_r \to \mathbb{CP}^2$. This gives us a disk at $x$.

In addition, we observe that the following holds.

Lemma 7.5. The sequence $\varphi_n$ converges to $\varphi$.

Proof. Assume contrary that there exists a subsequence $\varphi_{l_k}$ that converges to $\varphi' : D_r \to \mathbb{CP}^2$. Notice that $\varphi$ and $\varphi'$ has to overlap. Otherwise, the images of the maps $\varphi_{l_k}$ and $\varphi_{nk}$ will intersect for large $k$, which will imply intersection of $S$ with itself, which will contradict Lemma 7.3. □

Further, by the same idea, we can see that,

Lemma 7.6. The disk, $D_x$ at $x$, as obtained above, does not depend on the choice of the sequence $x_n$.

Thus, construct a disk at all points in $\overline{S}$ in this manner. In the following lemma, we prove that this intersection cannot be transverse.

Lemma 7.7. The disk $D_x$ cannot intersect the disk $D_y$ at $y$ transversally.
Proof. Assume the contrary. Let $x_n \in S$ and $y_n \in S$ be points such that $x_n$ converge to $x$ and $y_n$ converge to $y$. Let $D_{x_n}$ be disks at $x_n$ contained in $S$ and $D_{y_n}$ be disks at $y_n$ contained in $S$. The disk $D_x$ intersects $D_y$ transversally implies that $D_{x_n}$ intersects $D_{y_n}$ transversally for all $n$ sufficiently large. This implies that $S$ intersects itself, contradicting Lemma 7.3.

Lemma 7.2 and Lemma 7.7 ensures that all these disks will have to intersect in an open set. So, we have a leaf through every point in $S$. Thus, $S$ is laminated.

Proof of Theorem 1.3 If the surface $S$ given above is not compact, then $S$ is a non-trivial lamination as desired. So, we assume the contrary, that is, $S$ is a compact surface.

Lemma 7.8. If $S$ is compact, there exists a subsequence $s_{n_k}$ such that every point in $S$ is a limit of points in $s_{n_k}^{-1}(0)$.

Proof. As, $S$ is compact, it will be covered by finitely many holomorphic disks. So, the process of continuing the disk forward will stop after finite steps. So, the subsequence, $n_k$ for which the final disk is a limit of disks in $s_{n_k}(0)$, will work.

Rename this sequence as $s_k$. We recall Proposition 40 in [Don96] which states that the limit of currents induced by $W_k$’s has $\mathbb{CP}^2$ as its support. So, there exists a point $y \in S^C$ and a sequence $y_k \in s_k^{-1}(0)$ such that $y_k$ converges to $y$. Construct a maximal surface as before passing through $y$. We call it $R$. If $R$ is compact, then $R$ must intersect $S$ by Bezout’s theorem. This will imply that $s_{n_k}^{-1}(0)$ will intersect itself for sufficiently large $k$. So, $R$ cannot be compact. Therefore, $\overline{R}$ is a non-trivial lamination, by Theorem 7.4.

Now, to show that each leaf of the lamination, obtained as above, is a totally geodesic submanifold, we recall Lemma 5.3 which says that Weingarten maps go to zero pointwise. This implies that the induced connection $\nabla$ on each leaf agrees with the Levi-Civita connection $\nabla$ on $\mathbb{CP}^2$ from relation 4.3.

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