MASS-CONSERVING WEAK SOLUTIONS TO THE
COAGULATION AND COLLISIONAL BREAKAGE EQUATION
WITH SINGULAR RATES

Prasanta Kumar Barik*
Tata Institute of Fundamental Research
Centre for Applicable Mathematics
Bangalore-560065, Karnataka, India

Ankik Kumar Giri
Department of Mathematics
Indian Institute of Technology Roorkee
Roorkee-247667, Uttarakhand, India

Rajesh Kumar
Department of Mathematics
Birla Institute of Technology and Science, Pilani
Pilani-333031, Rajasthan, India

(Communicated by Barbara Niethammer)

Abstract. In this article, the existence of mass-conserving solutions is inves-
tigated to the continuous coagulation and collisional breakage equation with
singular collision kernels. Here, the probability distribution function attains
singularity near the origin. The existence result is constructed by using both
conservative and non-conservative truncations to the continuous coagulation
and collisional breakage equation. The proof of the existence result relies on a
classical weak $L^1$ compactness method.

1. Introduction. The coagulation and fragmentation models are a particular class
of partial integro-differential equation in which two particles collide at a particular
instant to form a larger particle by aggregation process or split into more than
two fragments by nonlinear breakage process or a large particle splits into many
particles by linear breakage process. Here, we consider a fully non-linear partial
integro-differential equation i.e., continuous coagulation and collisional breakage
equation (CCBE) where each particle is fully identified by its volume (or size)
v $\in \mathbb{R}_+ := (0, \infty)$. In recent papers [4, 5, 3], we have discussed the existence
and uniqueness of solutions to continuous coagulation and collisional breakage model.
This model has a lot of applications in different field of science, engineering and
technology such as astrology and astrophysics. In this article, we focus on the

2020 Mathematics Subject Classification. Primary: 45K05; Secondary: 34K30.
Key words and phrases. Weak compactness, particle, mass-conserving solution, existence.
* Corresponding author: Prasanta Kumar Barik.
case of existence of mass-conserving solutions to continuous CCBE. The general continuous CCBE reads as \[4, 7, 22, 24, 26\]
\[
\frac{\partial g}{\partial t} = C_B(g) - CB_D(g) + B_B(g),
\]  
(1)
where
\[
C_B(g)(v, t) := \frac{1}{2} \int_0^v E(v - v', v') \varphi(v - v', v') g(v - v', t) g(v', t) dv',
\]
\[
CB_D(g)(v, t) := \int_0^\infty \varphi(v, v') g(v, t) g(v', t) dv',
\]
and
\[
B_B(g)(v, t) := \frac{1}{2} \int_v^\infty \int_0^v P(v|v' - v''; v'') E_1(v' - v'', v'') \times \varphi(v' - v'', v'') g(v'' - v', t) g(v'', t) dv'' dv',
\]
with initial data
\[
g(v, 0) = g^\infty(v) \geq 0 \ a.e.
\]  
(2)
Here the unknown \(g(v, t)\) is a function of volume variable \(v \in \mathbb{R}_+\) and the time variable \(t \geq 0\) and is known as the concentration of particles. The coefficient \(\varphi(v, v')\) is called the collision kernel which is non-negative and symmetric function and describes the rate at which particles of volume \(v\) and \(v'\) interact and produce the larger particles of volume \(v + v'\) with coalescence efficiency \(E(v, v')\) and the breakage efficiency \(E_1(v, v')\). Here, \(E(v, v') + E_1(v, v') = 1\). The probability distribution function \(P(v|v'; v'')\) gives the birth of particles of volume \(v\) with the collision between particles of volumes \(v'\) and \(v''\). In addition, the transfer of volumes between particles \(v'\) and \(v''\) may occur. Furthermore, the probability distribution function \(P\) is assumed to enjoy the following properties
\[
\int_0^{v' + v''} P(v|v'; v'') dv = T_N(v', v'') \quad \forall \quad (v', v'') \in \mathbb{R}_+ \times \mathbb{R}_+,
\]  
(3)
where
\[
\sup_{(v', v'') \in \mathbb{R}_+ \times \mathbb{R}_+} T_N(v', v'') = T_N < \infty, \quad P(v|v'; v'') = 0 \quad \forall \quad v > v' + v'',
\]
and
\[
\int_0^{v' + v''} v P(v|v'; v'') dv = v' + v'' \quad \forall \quad v \in (0, v' + v'').
\]  
(4)
In equation (3), \(T_N(v', v'')\) is the total number of daughter particles obtained due to the collision between particles of volume \(v'\) and \(v''\). We have considered its supremum as \(T_N\) which is a positive constant greater than or equal to 2 and equation (4) shows the conservation of matter in the system during the collisional breakage events. However, the total matter may not conserve during the nonlinear coagulation and nonlinear breakage processes due to the appearance of infinite gel in the system. This happens due to the high aggregation rate in comparison to the fragmentation rate. This physical phenomenon is known as gelation transition and the least time at which this happens is called as gelation time \([12, 20]\).
The total mass of particles in the system for coagulation and collisional breakage equation can be defined as
\[ N_1(t) = N_1(g(t)) := \int_0^\infty v g(v,t) dv, \quad t \geq 0. \]

In particular, if \( t = 0 \), the total mass of particles in the system can be represented by the following notation:
\[ N_1^{in} = N_1(g(0)) := \int_0^\infty v g^{in}(v) dv. \quad (5) \]

In equation (1), the first term \( CB_B \) shows the birth of new particles of volume \( v \) due to the collision between particles of volumes \( v - v' \) and \( v' \) through the aggregation process while the second term \( CB_D \) gives the death of particles of volume \( v \) due to both coagulation and collisional breakage events. The last term \( BB \) represents the formation of particles of volume \( v \) due to the collisional breakage process.

The existence and uniqueness of mass-conserving weak solutions to the continuous coagulation and linear fragmentation equation with both nonsingular and singular kernels have been extensively studied in several articles, see \[2, 1, 6, 8, 14, 16, 21\] and references therein. However, there are only a few articles available in which the nonlinear fragmentation equation is described, see \[9, 10, 11, 15, 19\]. In \[9, 10, 11\], the authors have investigated the scaling solutions to the continuous nonlinear fragmentation equation whereas the analytic solutions for the different cases of kernels are discussed in \[11, 15\]. Later, the existence of weak solutions and the asymptotic behaviour of solutions to the discrete version of the non-linear fragmentation equation are studied in \[19\]. To the best of our knowledge, the fully nonlinear homogeneous continuous coagulation and collisional breakage equation is described by Safronov first time, see \[22\]. Later Wilkins \[26\] gave the geometrical interpretation of this model. Recently, we have studied the existence and uniqueness of classical solutions to the continuous CCBE with a particular class of collision kernel and distribution function in \[4\]. Moreover, we have also discussed the existence of weak solutions to the continuous CCBE with both non-singular and singular collision kernels by using conservative truncation to the original equation, see \[5, 3\], where, in \[3\], the singular collision kernel to the continuous CCBE satisfies
\[ \varphi(v, v') \leq k \frac{(1 + v)^\beta (1 + v')^\beta}{(v + v')^\alpha}, \quad \beta \in [0, 1), \quad \alpha \in (0, 1/2) \quad \text{and} \quad \beta - \alpha \in [0, 1). \]

In addition, a uniqueness result is studied for a special case of collision kernel when \( \beta = 0 \). However, the mass-conservation property of the weak solution is not investigated. Thus, it is clear that there is no uniqueness result available to the continuous CCBE for singular collision kernel which is stated in assumption (A1) in the next section. Hence, there may be some solutions which are either mass-conserving or not conserving the mass. Due to non-availability of the uniqueness result to the continuous CCBE, both conservative and non-conservative approximations are considered in this work. As we know a conservative approximation may always give mass-conserving solution whereas a non-conservative truncation is suitable to study the gelation transition, an obvious question arise is that whether a non-conservative approximation can give a mass-conserving solution or not? The purpose of the present work is to provide a positive answer to this question as well as the existence of mass-conserving weak solution by using a conservative approximation to the original problem. The motivation of the present work is taken from \[6, 1, 13\].
The content of the paper: we describe some assumptions on collision kernel, probability distribution function and coalescence efficiency in Section 2. Furthermore, the main result and some preliminary results for the convex function are stated in Section 3. The proof of the existence of mass-conserving weak solutions is shown by using a weak $L^1$ compactness method in this section.

2. Assumptions, preliminaries and main result. In this section, some assumptions on collision kernel $\varphi$, distribution function $P$, and the coalescence efficiency $E$ are stated.

**Assumptions:**

(A1) Let $\varphi$ be a symmetric and non-negative measurable function on $\mathbb{R}_+ \times \mathbb{R}_+$, and it satisfies $\varphi(v, v') \leq k\frac{1 + v + v'}{(v + v')^2}$ for all $(v, v') \in \mathbb{R}_+ \times \mathbb{R}_+, 0 < \alpha < \frac{1}{2}$ and for some constant $k > 0$,

(A2) there exists a positive constant $\eta(2\alpha) = \frac{(\theta + 2)}{(1 - 2\alpha + \theta)} > 2$ (depending on $\theta$ and $\alpha$) such that

$$\int_0^{v+v''} v^{-2\alpha} P(v|v'; v'')dv \leq \eta(2\alpha)(v' + v'')^{-2\alpha},$$

where $P(v|v'; v'') = (\theta + 2)\frac{v^\alpha}{(v + v'')^{2\alpha}},$ for $-1 < \theta \leq 0$ and $\alpha \& \theta$ are related with the relation $2\alpha - \theta < 1$.

(A3) the symmetric function $E$ satisfies the following condition locally:

$$E(v, v') \geq \frac{\eta(2\alpha) - 2}{\eta(2\alpha) - 1}, \quad \forall \quad (v, v') \in (0, 1) \times (0, 1),$$

and $0 \leq E(v, v') + E_1(v, v') = 1$ for all $(v, v') \in \mathbb{R}_+ \times \mathbb{R}_+$,

(A4) $g^{in} \in L^1_{2\alpha, 1}(\mathbb{R}_+)$, where $L^1_{2\alpha, 1}(\mathbb{R}_+) := L^1(\mathbb{R}_+; (v^{-2\alpha} + v)dv)$.

Next, the main result of this paper is provided.

**Theorem 2.1.** Assume that (A1)–(A4) hold. Let $T \in (0, \infty]$. Then there exists a mass conserving solution $g$ to (1)–(2), that is, $g \in C([0, T); L^1_{-\alpha, 1}(\mathbb{R}_+)) \cap L^\infty(0, T; L^1_{-2\alpha, 1}(\mathbb{R}_+))$ and satisfying the weak formulation

$$\int_0^\infty \{g(v, t) - g^{in}(v)\}h(v)dv = \frac{1}{2} \int_0^t \int_0^\infty \int_0^\infty \tilde{h}(v, v')E(v, v')\varphi(v, v')g(v, s)g(v', s)dv'dvd's$$

$$+ \frac{1}{2} \int_0^t \int_0^\infty \int_0^\infty \Pi_h(v', v'')E_1(v', v'')\varphi(v', v'')g(v', s)g(v'', s)dv''dv'ds,$$

where

$$\tilde{h}(v, v') = h(v + v') - h(v) - h(v')$$

and

$$\Pi_h(v', v'') = \int_0^{v'+v''} h(v)P(v|v'; v'')dv - h(v') - h(v''),$$

for every $t \in (0, T)$ and $h \in L^\infty(\mathbb{R}_+)$. 
In the above theorem, \( C([0, T]^w; L_{1-\alpha,1}^1(\mathbb{R}_+)) \) denotes the space of all weakly continuous functions from \([0, T]\) to \( L_{1-\alpha,1}^1(\mathbb{R}_+)\). In addition, a sequence \((g_n)\) converges to \(g\) in \( C([0, T]^w; L_{1-\alpha,1}^1(\mathbb{R}_+))\) if
\[
\lim_{n \to \infty} \sup_{t \in [0, T]} \int_0^\infty (v^{-\alpha} + v) \{g_n(v, t) - g(v, t)\} h(v) dv = 0,
\]
for every \(h \in L^\infty(\mathbb{R}_+)\).

Next, let us define a particular class of convex functions \( C_{VP,\infty} \). Let \( \Gamma \in C^\infty((0, \infty)) \) be a non-negative and convex function, then \( \Gamma \in C_{VP,\infty} \) if
\[
\Gamma(0) = \Gamma'(0) = 0 \quad \text{and} \quad \Gamma' \text{ is concave}; \quad (7)
\]
\[
\lim_{s \to \infty} \Gamma'(s) = \lim_{s \to \infty} \frac{\Gamma(s)}{s} = \infty. \quad (8)
\]
Since \( g^{in} \in L_{2\alpha,1}^1(\mathbb{R}_+) \), then a refined version of de la Vallée-Poussin theorem, see \([17, \text{Theorem 2.8}]\), ensures that there exists a non-negative function \( \Upsilon_1 \) in \( C_{VP,\infty} \) with
\[
\Upsilon_1 := \int_0^\infty \Gamma_1(v) g^{in}(v) dv < \infty. \quad (9)
\]

Finally, some additional properties of \( C_{VP,\infty} \) which are also required to prove Theorem 2.1 are discussed.

**Lemma 2.2.** Consider \( \Gamma \in C_{VP,\infty} \). Then we have the following results
\[
\Gamma(q_1) \leq q_1 \Gamma'(q_1) \leq 2\Gamma(q_1), \quad (10)
\]
\[
q_1 \Gamma'(q_2) \leq \Gamma(q_1) + \Gamma(q_2), \quad (11)
\]
and
\[
0 \leq \Gamma(q_1 + q_2) - \Gamma(q_1) - \Gamma(q_2) \leq 2 \frac{q_1 \Gamma(q_2) + q_2 \Gamma(q_1)}{q_1 + q_2}, \quad (12)
\]
for all \( q_1, q_2 \in \mathbb{R}_+ \).

**Proof.** This lemma can easily be proved in a similar way as given in \([2, 6, 16]\). \( \Box \)

3. Existence of weak solutions. This section deals with the construction of mass conserving solution for the conservative and non-conservative truncations to (1)–(2). It is expected that a mass conserving solution can be obtained for the conservative approximation under some restricted kernels. Moreover, considering a non-conservative form of coagulation and a conservative approximation of multiple fragmentation is appropriate to study the gelation transition. Therefore, an obvious question arises whether such coupling of mixed approximations will provide a mass-conserving solution or not to (1)–(2)? Interestingly, the answer of this question is yes and we provide the proof to this result in this work.

Let us define here both the conservative and non-conservative approximations to (1)–(2). For a given natural number \( n \in \mathbb{N} \), we set
\[
g_{n}^{in}(v) = g^{in}(v) \mathbb{1}_{(0,n)}(v), \quad (13)
\]
and for \( \tau \in \{0, 1\} \),
\[
\varphi_{n}^{\tau}(v, v') := \varphi(v, v') \mathbb{1}_{(1/n,n)}(v) \mathbb{1}_{(1/n,n)}(v') \{1 - \tau + \tau \mathbb{1}_{(0,n)}(v + v')\}. \quad (14)
\]
Using (13) and (14), the equations (1)–(2) can be rewritten as
\[ \frac{\partial g_n}{\partial t} = C_B^n(g_n) - CB_D^n(g_n) + B_D^n(g_n), \]
with initial data
\[ g_n(v,0) = g_n^{in}(v) \geq 0 \text{ a.e.}, \]
where
\[ C_B^n(g_n)(v,t) := \frac{1}{2} \int_0^v E(v - v',v') \varphi_n^+(v - v',v') g_n(v - v',t) g_n(v,t) dv', \]
\[ B_D^n(g_n)(v,t) := \frac{1}{2} \int_v^n \int_0^{v'} P(v|v' - v'';v'') E_1(v' - v'',v'') \times \varphi_n^+(v' - v'',v'') g_n(v' - v'',t) g_n(v'',t) dv'' dv', \]
and
\[ CB_D^n(g_n)(v,t) := \int_0^{n-v} \varphi_n^+(v,v') g_n(v,t) g_n(v',t) dv'. \]
The additional variable \( \tau \in \{0,1\} \) permits us to handle simultaneously the conservative approximation \((\tau = 1)\) and non-conservative approximation \((\tau = 0)\).

Now, in the following proposition, the result on the positivity of the unique solution is summarized followed by a remark on conservative and non-conservative approximations.

**Proposition 1.** Let \( \tau \in \{0,1\} \) and \( n > 1 \). Then, there exists a unique non-negative solution \( g_n \in C^1([0,\infty);L^1(0,n)) \) to (15)–(16). In addition, it satisfies
\[ \int_0^n v g_n(v,t) dv = \int_0^n v g^{in}_n(v) dv - (1 - \tau) \int_0^t \int_0^n \int_{n-v}^n v E(v,v') \varphi_n^+(v,v') g_n(v,s) g_n(v',s) dv' dv ds, \]
for \( t \geq 0 \).

**Proof.** The proof of the Proposition 1 is similar to [5, Proposition 1]. \(\Box\)

**Remark 1.** It should be mentioned here that the last term on the right-hand side of (17) leads to the case of total mass conservation for \( \tau = 1 \) and therefore known as the conservative approximation. While for \( \tau = 0 \), the truncated form becomes non-conservative as the total mass decreases with respect to time. Moreover, in both these cases, it is clear that
\[ \int_0^n v g_n(v,t) dv \leq \int_0^n v g^{in}_n(v) dv, \text{ for } t > 0. \]

Further, the weak formulation to (15)–(16) for \( n \geq 1 \) and \( h \in L^\infty(\mathbb{R}_+) \) can be obtained as
\[ \int_0^n (g_n(v,t) - g^{in}_n(v)) h(v) dv = \frac{1}{2} \int_0^t \int_0^n \tilde{h}_\tau(v,v') 1_{\{1/n,n\}}(v) 1_{\{1/n,n\}}(v') E(v,v') \varphi(v,v') g_n(v,s) g_n(v',s) dv' dv ds \]
\[ + \frac{1}{2} \int_0^t \int_0^n \int_0^n \Pi_{h,T}(v', v'') \Pi_{(1/n,n)}(v') \Pi_{(1/n,n)}(v'') E_1(v', v'') \]
\[ \times \varphi(v', v'') g_n(v', s) g_n(v'', s) dv'' dv' ds, \]

where
\[ \hat{h}_T(v, v') = h(v + v') \Pi_{(0,n)}(v + v') - \{ h(v) + h(v') \} (1 - \tau + \Pi_{(0,n)}(v + v')) \]
and
\[ \Pi_{h,T}(v', v'') = \Pi_{(0,n)}(v' + v'') \int_0^{v' + v''} h(v) P(v|v'; v'') dv \]
\[ - \{ h(v') + h(v'') \} (1 - \tau + \Pi_{(0,n)}(v' + v'')). \]

Finally, we claim and later prove that the family of solutions \( \{ g_n \}_{n \geq 1} \) is relatively compact in \( C([0,T]; L^{1}_{-\alpha,1}(\mathbb{R}^+)) \). For this purpose, the weak \( L^1 \) compactness method is applied which is also used in the pioneering work of Stewart [23] and Laurencot et al. [18]. To proceed further, the uniform boundedness of the family of solutions \( \{ g_n \}_{n \geq 1} \) in space \( L^{1}_{-2\alpha,1}(\mathbb{R}^+) \) is shown in the next section.

### 3.1. Uniform Bound

**Lemma 3.1.** Let us assume \( (A_1)-(A_4) \) hold. Let \( T > 0 \). Then there exists a positive constant \( B(T) \) depending on \( T \) such that
\[ \int_0^n (v^{-2\alpha} + v) g_n(v, t) dv \leq B(T) \text{ for all } t \in [0,T]. \]

**Proof.** Let \( t \in [0,T] \) and \( n > 1 \). Multiplying the equation (15) by \( v^{-2\alpha} \) and then taking the integration from 0 to \( n \) with respect to \( v \) leads to
\[ \frac{d}{dt} \int_0^n v^{-2\alpha} g_n(v, t) dv = \int_0^n v^{-2\alpha} [C_B^n(g_n)(v, t) - CE_D^n(g_n)(v, t) + E_B^n(g_n)(v, t)] dv. \]

Now, we estimate each integral on the right-hand side of (20), individually. The first term on the right-hand side of (20) can be simplified, by using Fubini’s theorem, the transformation \( v - v' = v_1 \) and \( v'' = v_2 \), and then replacing the variables \( v_1 \) by \( v' \) and \( v_2 \) by \( v \) as
\[ \int_0^n v^{-2\alpha} C_B^n(g_n)(v, t) dv \]
\[ = \frac{1}{2} \int_0^n \int_0^{n-v} (v + v')^{-2\alpha} E(v, v') \varphi^{*}(v, v') g_n(v', t) g_n(v', t) dv' dv. \]

Again, applying the repeated application of Fubini’s theorem to the third term on the right-hand side of (20), using hypothesis \( (A_2) \), the transformation \( v' - v'' = v_2 \) and \( v'' = v_3 \) and finally replacing \( v' \to v \) and \( v'' \to v' \), we get
\[ \int_0^n v^{-2\alpha} E_B^n(g_n)(v, t) dv \]
\[ = \frac{1}{2} \int_0^n \int_0^{n-v} v^{-2\alpha} P(v|v' - v''; v'') E_1(v' - v'', v'') \varphi^{*}(v' - v'', v'') \]
\[ \times g_n(v' - v'', t) g_n(v'', t) dv'' dv' dv \]
\[ = \frac{1}{2} \int_0^n \int_0^{n-v} v^{-2\alpha} P(v|v' - v''; v'') E_1(v' - v'', v'') \varphi^{*}(v' - v'', v'') dv'' dv'. \]
× \int_0^n \int_0^n \int_0^n \int_0^n (v + v')^{-2\alpha} E_1(v, v') \phi_n^\tau(v, v') g_n(v, t) g_n(v', t) d\nu d\nu' dv
dv

\leq \frac{\eta(2\alpha)}{2} \int_0^n \int_0^n (v + v')^{-2\alpha} E_1(v, v') \phi_n^\tau(v, v') g_n(v, t) g_n(v', t) d\nu d\nu' dv
dv.

Using the negativity of the first, third and fourth terms on the right-hand side of (23) which are guaranteed from (A_1), (A_3) and Proposition 1, and then further applying (A_1), we obtain

\frac{d}{dt} \int_0^n v^{-2\alpha} g_n(v, t) dv \leq \int_1^n \int_0^n (v + v')^{-2\alpha} \phi_n^\tau(v, v') g_n(v, t) g_n(v', t) d\nu d\nu' dv
dv

+ \frac{\eta(2\alpha)}{2} \int_1^n \int_0^n v^{-2\alpha} \phi_n^\tau(v, v') g_n(v, t) g_n(v', t) d\nu d\nu' dv

+ \frac{\eta(2\alpha)}{2} \int_0^n \int_0^n v^{-2\alpha} \phi_n^\tau(v, v') g_n(v, t) g_n(v', t) d\nu d\nu' dv

\leq k \left(1 + \frac{\eta(2\alpha)}{2}\right) \int_1^n \int_0^n v^{-2\alpha} \frac{(1 + v + v')}{(v + v')^\alpha} g_n(v, t) g_n(v', t) d\nu d\nu' dv

+ k \frac{\eta(2\alpha)}{2} \int_1^n \int_0^n v^{-2\alpha} \frac{(1 + v + v')}{(v + v')^\alpha} g_n(v, t) g_n(v', t) d\nu d\nu' dv

\leq 2k \left(1 + \frac{\eta(2\alpha)}{2}\right) \int_1^n \int_0^n (v + v') g_n(v, t) g_n(v', t) d\nu d\nu' dv

+ 2k \eta(2\alpha) \int_1^n \int_0^n v^{-2\alpha} v' g_n(v, t) g_n(v', t) d\nu d\nu'.

Finally, by using (5) and (18), the above equation can be further reduced to
\[
\frac{d}{dt} \int_0^n v^{-2\alpha} g_n(v, t) dv \leq 4kN_1^n(1 + \eta(2\alpha)) \int_0^n v^{-2\alpha} g_n(v, t) dv + 2kN_1^n(2 + \eta(2\alpha)).
\]
(24)

Now, an application of Gronwall’s inequality and (A4) to (24) gives
\[
\int_0^n v^{-2\alpha} g_n(v, t) dv \leq B_1(T),
\]
(25)
where
\[
B_1(T) := e^{\alpha T} \|g_0\|_{L^1_{\alpha,n}(\mathbb{R}_+)} + \frac{b}{a} (e^{\alpha T} - 1),
\]
for \(a := 4kN_1^n(1 + \eta(2\alpha))\) and \(b := 2kN_1^n(2 + \eta(2\alpha))\). Hence, using (25) yields
\[
\int_0^n (v + v^{-2\alpha}) g_n(v, t) dv \leq B_1(T) + N_1^n := B(T),
\]
which completes the proof of Lemma 3.1. \(\square\)

In the following lemma, we discuss the behaviour of \(g_n\) for large volume particle \(v\).

**Lemma 3.2.** Let (A1)–(A4) hold. Then for every \(n > 1, T > 0,\)
\[
\sup_{t \in [0, T]} \int_0^n \Gamma_1(v) g_n(v, t) dv \leq G(T),
\]
(26)
and
\[
(1 - \tau) \int_0^T \int_0^n \int_{n-v}^n \Gamma_1(v) \mathbb{I}_{(1/n, n)}(v) \mathbb{I}_{(1/n, n)}(v') \times \varphi(v, v') g_n(v, s) g_n(v', s) dv' dv' ds \leq G(T),
\]
(27)
where the \(\Gamma_1 \in \mathcal{C}_{V_P, \infty}\) satisfies (8) and (9) and \(G(T)\) (depending on \(T\)) is a positive constant.

**Proof.** Choose \(h(v) = \Gamma_1(v) \mathbb{I}_{(0, n)}(v)\), and inserting it into (19) to obtain
\[
\int_0^n \Gamma_1(v) g_n(v, t) dv
= \int_0^n \Gamma_1(v) g_n^{in}(v) dv + \frac{1}{2} \int_0^t \int_0^n \int_{n-v}^n \tilde{\Gamma}_{1, \tau}(v, v') \mathbb{I}_{(1/n, n)}(v) \mathbb{I}_{(1/n, n)}(v') \times E(v, v') \varphi(v, v') g_n(v, s) g_n(v', s) dv' dv' ds
+ \frac{1}{2} \int_0^t \int_0^n \int_0^n \Pi_{\Gamma_{1, \tau}}(v', v'') \mathbb{I}_{(1/n, n)}(v') \mathbb{I}_{(1/n, n)}(v'') E_1(v', v'') \times \varphi(v', v'') g_n(v', s) g_n(v'', s) dv'' dv'' ds,
\]
(28)
where
\[
\tilde{\Gamma}_{1, \tau}(v, v') = \Gamma_1(v + v') \mathbb{I}_{(0, n)}(v + v') - \{\Gamma_1(v) + \Gamma_1(v')\}(1 - \tau + \tau \mathbb{I}_{(0, n)}(v + v'))
\]
and
\[
\Pi_{\Gamma_{1, \tau}}(v', v'') = \mathbb{I}_{(0, n)}(v' + v'') \int_0^{v'+v''} \Gamma_1(v) P(v', v'') dv - \{\Gamma_1(v') + \Gamma_1(v'')\}(1 - \tau + \tau \mathbb{I}_{(0, n)}(v' + v'')).
\]
By using (9) and (13) into (28), we achieve
\[
\int_0^n \Gamma_1(v)g_n(v,t)dv \leq \Upsilon_1 + \frac{1}{2} \int_0^t \{S_1^n(s) + S_2^n(s) + S_3^n(s) + S_4^n(s)\}ds,
\]
where
\[
S_1^n(s) = \int_0^n \int_0^{n-v} \{\Gamma_1(v + v') - \Gamma_1(v) - \Gamma_1(v')\}\Pi_{(1/n,n)}(v)\Pi_{(1/n,n)}(v')E(v, v') \\	imes \varphi(v, v')g_n(v, s)g_n(v', s)dv'dv,
\]
\[
S_2^n(s) = - (1 - \tau) \int_0^n \int_0^{n-v} \{\Gamma_1(v) + \Gamma_1(v')\}\Pi_{(1/n,n)}(v)\Pi_{(1/n,n)}(v')E(v, v') \\	imes \varphi(v, v')g_n(v, s)g_n(v', s)dv'dv,
\]
\[
S_3^n(s) = \int_0^n \int_0^{n-v} \left[ \int_0^{v+v''} \Gamma_1(v)P(v|v'; v'')dv - \{\Gamma_1(v') + \Gamma_1(v'')\} \right] \\	imes \Pi_{(1/n,n)}(v')\Pi_{(1/n,n)}(v'')E(v', v'')\varphi(v', v'')g_n(v', s)g_n(v'', s)dv''dv',
\]
and
\[
S_4^n(s) = - (1 - \tau) \int_0^n \int_0^{n-v} \{\Gamma_1(v') + \Gamma_1(v'')\}\Pi_{(1/n,n)}(v')\Pi_{(1/n,n)}(v'')E(v', v'') \\	imes \varphi(v', v'')g_n(v', s)g_n(v'', s)dv''dv'.
\]
Now, we simplify each term separately. The part \(S_1^n(s)\) is estimated by using (12) and \((A_1)\) as
\[
S_1^n(s) \leq \int_0^n \int_0^{n-v} \{\Gamma_1(v + v') - \Gamma_1(v) - \Gamma_1(v')\}\Pi_{(1/n,n)}(v)\Pi_{(1/n,n)}(v') \\	imes g_n(v, s)g_n(v', s)dv'dv
\leq 2k \int_0^1 \int_0^{1-v} \frac{(1 + v + v')}{(v + v')^\alpha} \times \frac{v\Gamma_1(v') + v'\Gamma_1(v)}{(v + v')}g_n(v, s)g_n(v', s)dv'dv
+ 4k \int_1^\infty \int_0^{1-v} \frac{(1 + v + v')}{(v + v')^\alpha} \times \frac{v\Gamma_1(v') + v'\Gamma_1(v)}{(v + v')}g_n(v, s)g_n(v', s)dv'dv
+ 2k \int_1^\infty \int_0^{1-v} \frac{(1 + v + v')}{(v + v')^\alpha} \times \frac{v\Gamma_1(v') + v'\Gamma_1(v)}{(v + v')}g_n(v, s)g_n(v', s)dv'dv
\leq 12k \int_0^1 \int_0^{1-v} \frac{v\Gamma_1(v')}{(v + v')}g_n(v, s)g_n(v', s)dv'dv
+ 12k \int_1^\infty \int_0^{1-v} \frac{v\Gamma_1(v')}{(v + v')}g_n(v, s)g_n(v', s)dv'dv
+ 4k \int_1^\infty \int_0^{1-v} \frac{v\Gamma_1(v')}{(v + v')}g_n(v, s)g_n(v', s)dv'dv.
\]
Let us first estimate the first term on the right-hand side of (30). By using Lemma 3.1 and the monotonicity of \(\Gamma_1\), it can be simplified as
\[
12k \int_0^1 \int_0^{1-v} \frac{v\Gamma_1(v')}{(v + v')}g_n(v, s)g_n(v', s)dv'dv \leq 12k \Gamma_1(1)B^2(T).
\]
Again considering Lemma 3.1 and the monotonicity of $\Gamma$, the second term on the right-hand side of (30) can be evaluated as

$$12k \int_1^n \int_0^1 \frac{v}{(v+v')^\alpha} \times \frac{v\Gamma_1(v') + v'\Gamma_1(v)}{(v+v')} \, g_n(v,s)g_n(v',s) \, dv' \, dv \leq 12k \Gamma_1(1) \int_1^n \int_0^1 \frac{v\Gamma_1(v)}{(v+v')^\alpha} \, g_n(v,s)g_n(v',s) \, dv' \, dv' \\
+ 12k \int_1^n \int_0^1 \frac{v'}{(v+v')^\alpha} \Gamma_1(v) \, g_n(v,s)g_n(v',s) \, dv' \, dv' \\
\leq 12k \Gamma_1(1)B^2(T) + 12kB(T) \int_0^1 \Gamma_1(v) \, g_n(v,s) \, dv. \tag{32}$$

Finally, the last integral on the right-hand side of (30) is calculated by applying Lemma 3.1 as

$$4k \int_1^n \int_1^n \frac{v\Gamma_1(v') + v'\Gamma_1(v)}{(v+v')^\alpha} \, g_n(v,s)g_n(v',s) \, dv' \, dv \leq 8k \int_1^n \int_1^n \frac{v\Gamma_1(v)}{(v+v')^\alpha} \, g_n(v,s)g_n(v',s) \, dv' \, dv' \\
\leq 8kB(T) \int_0^1 \Gamma_1(v) \, g_n(v,s) \, dv. \tag{33}$$

Inserting (31), (32) and (33) into (30) lead to

$$S^n_3(s) \leq 24k \Gamma_1(1)B^2(T) + 20kB(T) \int_0^1 \Gamma_1(v) \, g_n(v,s) \, dv. \tag{34}$$

Next, to proceed further in estimating the term $S^n_3(s)$, thanks to the monotonicity of the map $v \mapsto \frac{\Gamma_1(v)}{v}$, that follows from (10) and (11), together with the relation (4) which give

$$S^n_3(s) = \int_0^n \int_0^n \left[ \left( \int_0^{v'+v''} \frac{\Gamma_1(v)}{v} \, vP(v|v';v'') \, dv - \Gamma_1(v') - \Gamma_1(v'') \right) \mathbf{1}_{(1/n,n)}(v') \times \mathbf{1}_{(1/n,n)}(v'') \mathbf{E}_1(v',v'') \varphi(v',v'') g_n(v',s)g_n(v'',s) \, dv' \, dv'' \\
\leq \int_0^n \int_0^n \left[ \left( \int_0^{v'+v''} \frac{\Gamma_1(v'+v'')}{(v'+v'')} \, vP(v|v';v'') \, dv - \Gamma_1(v') - \Gamma_1(v'') \right) \mathbf{1}_{(1/n,n)}(v') \times \mathbf{1}_{(1/n,n)}(v'') \mathbf{E}_1(v',v'') \varphi(v',v'') g_n(v',s)g_n(v'',s) \, dv' \, dv'' \\
= \int_0^n \int_0^n \left\{ \Gamma_1(v'+v'') - \Gamma_1(v') - \Gamma_1(v'') \right\} \mathbf{1}_{(1/n,n)}(v') \mathbf{1}_{(1/n,n)}(v'') \mathbf{E}_1(v',v'') \times \varphi(v',v'') g_n(v',s)g_n(v'',s) \, dv' \, dv'' \right].$$

Further, by using (A1) and (12), the above can be simplified as

$$S^n_3(s) \leq 2k \int_0^1 \int_0^1 \frac{(1 + v' + v'')}{(v'+v'')} \times \frac{v\Gamma_1(v'') + v'\Gamma_1(v')}{(v'+v'')} \, g_n(v',s)g_n(v'',s) \, dv' \, dv'' \\
+ 4k \int_1^n \int_0^1 \frac{(1 + v' + v'')}{(v'+v'')} \times \frac{v\Gamma_1(v'') + v'\Gamma_1(v')}{(v'+v'')} \, g_n(v',s)g_n(v'',s) \, dv' \, dv'' \\
+ 2k \int_1^n \int_1^1 \frac{(1 + v' + v'')}{(v'+v'')} \times \frac{v\Gamma_1(v'') + v'\Gamma_1(v')}{(v'+v'')} \, g_n(v',s)g_n(v'',s) \, dv' \, dv''$$
Hence, by applying Lemma 3.1, we get
\[
S_n(s) \leq 24k \Gamma_1(1) B^2(T) + 20k \mathcal{B}(T) \int_0^n \Gamma_1(v) g_n(v, s) dv.
\] (35)

Finally, adding \( S_2^n(s) \) and \( S_1^n(s) \), we have
\[
S_2^n(s) + S_1^n(s) = -2(1 - \tau) \int_0^n \int_{n-v'}^n \Gamma_1(v') \mathbb{1}_{(1/n,n)}(v') \mathbb{1}_{(1/n,n)}(v'') \times \varphi(v, v') g_n(v', s) g_n(v'', s) dv'' dv'.'
\] (36)

Inserting (34), (35) and (36) into (29) gives us
\[
\int_0^n \Gamma_1(v) g_n(v, t) dv + (1 - \tau) \int_0^t \int_0^n \Gamma_1(v) \mathbb{1}_{(1/n,n)}(v) \mathbb{1}_{(1/n,n)}(v') \times \varphi(v, v') g_n(v, s) g_n(v', s) dv' dvds.
\]
\[
\leq \Upsilon_1 + 24k \Gamma_1(1) B^2(T) T + 20k \mathcal{B}(T) \int_0^t \int_0^n \Gamma_1(v) g_n(v, s) dvds.
\]

Then by Gronwall’s inequality, we get
\[
\int_0^n \Gamma_1(v) g_n(v, t) dv + (1 - \tau) \int_0^t \int_0^n \Gamma_1(v) \mathbb{1}_{(1/n,n)}(v) \mathbb{1}_{(1/n,n)}(v') \times \varphi(v, v') g_n(v, s) g_n(v', s) dv' dvds \leq \mathcal{G}(T),
\]
where \( \mathcal{G}(T) = \{ \Upsilon_1 + 24k \Gamma_1(1) B^2(T) T \} e^{20k \mathcal{B}(T) T} \), and it completes the proof of Lemma 3.2.

To proceed further in achieving our main goal, we follow [23] to show the equi-integrability condition for the family of solutions \( \{g_n\}_{n>1} \) in the next subsection.

3.2. Equi-integrability.

**Lemma 3.3.** Assume that the kinetic coefficient \( \varphi \), the probability distribution function \( P \), the coalescence efficiency \( E \), and the initial data satisfy (A1)–(A4), respectively. Let \( \lambda \in (1, n) \) and \( T > 0 \), then for every \( \epsilon > 0 \), there exists a \( \delta \) depending on \( \epsilon \) such that, for every measurable set \( O \subset (0, \lambda) \) with \( |O| < \delta \) and \( t \in [0, T] \),
\[
\int_O v^{-}\alpha g_n(v, t) dv < \epsilon.
\]

**Proof.** For \( t \in [0, T] \), let us first consider \( h_n(v, t) := v^{-}\alpha g_n(v, t) \) and \( n > \lambda \). Next, let us define the following quantity
\[
D(t) := \sup \int_0^\lambda \mathbb{1}_O(v) h_n(v, t) dv,
\] (37)
for every measurable set $O \subset (0, \lambda)$ with $|O| < \delta \in (0, 1)$ and $t \in [0, T]$. Then, by using (15), (37) and the non-negativity of $g_n$, we obtain

\[
\int_0^\lambda \mathcal{I}_O(v)h_n(v,t)dv \leq D(0) + \int_0^t \{\Theta_1(s) + \Theta_2(s)\}ds,
\]

where

\[
\Theta_1(s) := \frac{1}{2} \int_0^\lambda \int_0^v \mathcal{I}_O(v)v^{-\alpha}\varphi_n^+(v-v',v')g_n(v,v',s)g_n(v',s)dv'dv
\]

and

\[
\Theta_2(s) := \frac{1}{2} \int_0^\lambda \int_0^v \int_0^{v'} \mathcal{I}_O(v)v^{-\alpha}P(v|v'-v'';v'')\varphi_n^+(v'-v'',v'')
\]

\[
\times g_n(v'-v'',s)g_n(v'',s)dv''dv'dv.
\]

Now, by employing Fubini’s theorem, the transformation $v-v' = v_1$ and $v' = v_2$ and (A1), $\Theta_1(s)$ can be estimated as

\[
\Theta_1(s) = \frac{1}{2} \int_0^\lambda \int_0^{v_2} \mathcal{I}_O(v_1 + v_2)(v_1 + v_2)^{-\alpha}\varphi_n^+(v_1,v_2)g_n(v_1,v_2,v_2,v_2)dv_1dv_2
\]

\[
\leq \frac{1}{2} k(1 + \lambda) \int_0^\lambda \int_0^{v_2} \mathcal{I}_O(\alpha-\nu',\alpha-\nu'';\beta,\beta)\varphi_n^+(v_2,v_2,v_2,v_3)g_n(v_2,v_3,v_3,v_3)dv_2.
\]

Since \(|(O-v') \cap (0, \lambda-v')| \leq |O| < \delta\) and \((O-v') \cap (0, \lambda-v') \subset (0, \lambda)\), then we infer from the definition in (37), (39) and Lemma 3.1 that

\[
\Theta_1(s) \leq A(\lambda, T)D(s),
\]

where $A(\lambda, T) := \frac{1}{2} k(1 + \lambda)B(T)$. Again, from the repeated application of Fubini’s theorem, and $v_2 - v_3 = v'$ and $v_3 = v''$, $\Theta_2(s)$ can be evaluated as

\[
\Theta_2(s) = \frac{1}{2} \int_0^\lambda \int_0^{v_2} \int_0^{v_2} \mathcal{I}_O(v)v^{-\alpha}P(v|v_2-v_3;v_3)\varphi_n^+(v_2-v_3,v_3)
\]

\[
\times g_n(v_2-v_3,s)g_n(v_3,s)dv_3dv_2
\]

\[
+ \frac{1}{2} \int_0^\lambda \int_0^{v_2} \int_0^{v_2} \mathcal{I}_O(v)v^{-\alpha}P(v|v_2-v_3;v_3)\varphi_n^+(v_2-v_3,v_3)
\]

\[
\times g_n(v_2-v_3,s)g_n(v_3,s)dv_3dv_2
\]

\[
\leq \frac{1}{2} \int_0^\lambda \int_0^{v_2} \int_0^{v_2} \mathcal{I}_O(v)v^{-\alpha}P(v|v';v'')\varphi_n^+(v',v'')g_n(v',v',s)g_n(v'',s)dv'dv'dv''.
\]

Thanks to Hölder’s inequality for $p = \frac{1}{1-\alpha}$ and (A2) for estimating the following integral:

\[
\int_0^{v'+v''} \mathcal{I}_O(v)v^{-\alpha}P(v|v';v'')dv
\]

\[
= \frac{(\theta + 2)}{(v'+v'')^{1+\theta}} \int_0^{v'+v''} \mathcal{I}_O(v)v^{\theta-\alpha}dv
\]

\[
\leq \frac{(\theta + 2)}{(v'+v'')^{1+\theta}} |O|^{1-\alpha} \int_0^{v'+v''} v^{(s-\alpha)}dv
\]

\[
\int_0^{v'+v''} v^{(s-\alpha)}dv
\]

\[
\int_0^{v'+v''} v^{(s-\alpha)}dv
\]

\[
\int_0^{v'+v''} v^{(s-\alpha)}dv
\]

\[
\int_0^{v'+v''} v^{(s-\alpha)}dv
\]

\[
\int_0^{v'+v''} v^{(s-\alpha)}dv
\]

\[
\int_0^{v'+v''} v^{(s-\alpha)}dv
\]
Lemma 3.4. Equi-continuity w.r.t. time in weak sense.

By using \((A_1), (42), \) and Lemma 3.1, we evaluate (41) as

\[
\Theta_2(s) \leq \frac{1}{2} k(\theta + 2) \delta^\alpha \left( \frac{1 - \alpha}{1 + \theta - 2\alpha} \right)^{(1-\alpha)} \int_0^n \int_0^n (v' + v'')^{-2\alpha}(1 + v' + v'')
\times (v' + v'')^{-\alpha} g_n(v', s) g_n(v'', s) dv' dv''
\leq \frac{1}{2} k(\theta + 2) \delta^\alpha \left( \frac{1 - \alpha}{1 + \theta - 2\alpha} \right)^{(1-\alpha)} \int_0^n \int_0^n (1 + v' + v'')
\times v''^{-\alpha} g_n(v', s) g_n(v'', s) dv' dv''
\leq k(\theta + 2) \delta^\alpha \left( \frac{1 - \alpha}{1 + \theta - 2\alpha} \right)^{(1-\alpha)} B(T) \int_0^n (1 + v'')v''^{-\alpha} g_n(v'', s) dv''
\]

\[
= A^\dagger(\alpha, \theta, T) \delta^\alpha,
\]

where

\[
A^\dagger(\alpha, \theta, T) = \frac{k(\theta + 2)}{(1 + \theta - 2\alpha)^{1-\alpha}} (1 - \alpha)^{1-\alpha} B^2(T).
\]

Collecting estimates in (40) and (43), and inserting them into (38), it gives

\[
\int_0^\lambda 1_{\Omega}(v) h_n(v, t) dv \leq D(0) + A(\lambda, T) \int_0^t D(s) ds + A^\dagger(\alpha, \theta, T) \delta^\alpha T.
\]

(44)

Taking the supremum and then finally, using the Gronwall inequality into (44), we obtain

\[
D(t) \leq \left( D(0) + A^\dagger(\alpha, \theta, T) T \delta^\alpha \right) \left( 1 + e^{A(\lambda, T) t} \right).
\]

Observing from (37) that \(D(0) \to 0\) as \(\delta \to 0\). Thus, as \(\delta \to 0\), we obtain

\[
\int_0^\lambda v^{-\alpha} g_n(v, t) dv \to 0.
\]

This proves Lemma 3.3. \(\square\)

3.3. Equi-continuity w.r.t. time in weak sense.

**Lemma 3.4.** Let \(T > 0\) and \(\lambda \in (1, n)\). Assume \((A_1)-(A_4)\) hold. Then for \(0 \leq s \leq t \leq T\) and \(\Delta \in L^\infty(\mathbb{R}_+)\), we have

\[
\left| \int_0^\lambda v^{-\alpha} \Delta(v) [g_n(v, t) - g_n(v, s)] dv \right| \leq \Theta(\lambda, T) \|\Delta\|_{L^\infty(\mathbb{R}_+)} (t - s),
\]

where \(\Theta(\lambda, T)\) is a positive constant depending on \(\lambda\) and \(T\).

**Proof.** Let \(0 \leq s \leq t \leq T\) and \(\Delta \in L^\infty(\mathbb{R}_+)\). Next, we simplify the following integral as

\[
\left| \int_0^\lambda v^{-\alpha} \Delta(v) [g_n(v, t) - g_n(v, s)] dv \right|
\leq \|\Delta\|_{L^\infty(\mathbb{R}_+)} \int_t^0 \int_s^\lambda v^{-\alpha} \left| \frac{\partial g_n}{\partial t}(v, \zeta) \right| dv d\zeta
\]
\[ \leq \| \Delta \|_{L^\infty(\mathbb{R}_+)} \int_s^t \left[ \frac{1}{2} \int_0^\lambda \int_0^v v^{-\alpha} \varphi_n^+(v-v', v') g_n(v-v', \zeta) g_n(v', \zeta) dv' dv \\
+ \frac{1}{2} \int_0^\lambda \int_0^n \int_0^{v'} v^{-\alpha} P(v|v'-v''; v'') \varphi_n^+(v'-v'', v'') \\
\times g_n(v'-v'', \zeta) g_n(v'', \zeta) dv'' dv' dv \right] d\zeta. \] (45)

Now, we estimate each integral on the right-hand side to (45) separately. First, we evaluate the first integral, by using Fubini’s theorem, \( v - v' = v_1 \) & \( v' = v_2 \), \((A_1)\), and Lemma 3.1, as
\[ \frac{1}{2} \int_s^t \int_0^\lambda \int_0^n \int_0^{v'} v^{-\alpha} P(v|v'-v''; v'') \varphi_n^+(v'-v'', v'') \\
\times g_n(v'-v'', \zeta) g_n(v'', \zeta) dv'' dv' dv d\zeta \]
\[ \leq \frac{1}{2} k \int_s^t \int_0^\lambda \int_0^n \int_0^{v'+v''} v^{-\alpha} P(v|v'; v'') \varphi_n^+(v', v'') g_n(v', \zeta) g_n(v'', \zeta) dv'' dv' dv d\zeta \]
\[ \leq \frac{1}{2} k(1+\lambda) \int_s^t \int_0^\lambda \int_0^n v^{-\alpha} v'^{-\alpha} g_n(v, \zeta) g_n(v', \zeta) dv'' dv' dv d\zeta \]
\[ \leq \frac{1}{2} k(1+\lambda) B^2(T)(t-s). \] (46)

The second integral can be estimated, by using the repeated application of Fubini’s theorem, the transformation \( v' - v'' = v_1 \) & \( v'' = v_2 \), \((A_1), (A_2)\) and Lemma 3.1, as
\[ \frac{1}{2} \int_s^t \int_0^\lambda \int_0^n \int_0^{v'} v^{-\alpha} P(v|v'-v''; v') \varphi_n^+(v'-v'', v') \\
\times g_n(v'-v'', \zeta) g_n(v'', \zeta) dv'' dv' dv d\zeta \]
\[ \leq \frac{1}{2} k \frac{(\theta + 2)}{(\theta + 1 - \alpha)} \int_s^t \int_0^\lambda \int_0^n (v' + v'')^{-\alpha} (1 + v' + v'') \\
\left( \frac{1 + v' + v''}{(v' + v'')^{2\alpha}} \right) g_n(v', \zeta) g_n(v'', \zeta) dv'' dv' dv d\zeta \]
\[ \leq \frac{1}{2} k \frac{(\theta + 2)}{(\theta + 1 - \alpha)} \int_s^t \left\{ \int_0^\lambda \int_0^n (v' + v'')^{-\alpha} (1 + v' + v'') \\
\times g_n(v', \zeta) g_n(v'', \zeta) dv'' dv' dv d\zeta \right\} d\zeta \]
\[ \leq \frac{1}{2} k \frac{(\theta + 2)}{(\theta + 1 - \alpha)} \int_s^t \left\{ 3B^2(T) + 3B^2(T) + 3B^2(T) + 4B^2(T) \right\} d\zeta \]
\[ \leq \frac{13}{2} k \frac{(\theta + 2)}{(\theta + 1 - \alpha)} B^2(T)(t-s). \] (47)

Similar to (46), we estimate the last integral, by applying \((A_1)\) and Lemma 3.1, as
\[ \int_s^t \int_0^\lambda \int_0^n v^{-\alpha} \varphi_n^+(v, v') g_n(v, \zeta) g_n(v', \zeta) dv' dv d\zeta \]
\[ \leq k \int_s^t \int_0^\lambda \int_0^n v^{-\alpha} \frac{(1 + \lambda + v')}{(v + v')^{\alpha}} g_n(v, \zeta) g_n(v', \zeta) dv' dv d\zeta \]
\[ \leq k B(T) \int_s^t \int_0^n (1 + \lambda + v') v'^{-\alpha} g_n(v', \zeta) dv' dv d\zeta \leq 2k(1+\lambda) B^2(T)(t-s). \] (48)
Inserting (46), (47), and (48) into (45), we obtain
\[
\left| \int_0^\lambda v^{-\alpha} \Delta(v)[g_n(v, t) - g_n(v, s)] dv \right| \leq \Theta(\lambda, T) \| \Delta \|_{L^\infty(\mathbb{R}_+)} (t - s),
\]
where
\[
\Theta(\lambda, T) = \frac{1}{2} k \left( 3(1 + \lambda) + 13 \frac{(\theta + 2)}{\theta + 1} \right) B^2(T).
\]
This completes the proof of Lemma 3.4. □

3.4. Convergence of integrals. In this section, we complete the proof of Theorem 2.1 by using the above subsections.

Proof of Theorem 2.1: From Lemma 3.1–3.3, and then using the Dunford-Pettis theorem and a variant of the Arzelà-Ascoli theorem, see [25], we conclude that \((g_n)\) is relatively compact in \(C([0, T]^w; L^1_{\alpha}(0, \lambda))\) for each \(T > 0\). Then there exists a subsequence of \((g_n)\) (not relabelled) and a nonnegative function \(g \in C([0, T]^w; L^1_{\alpha}(0, \lambda))\) such that
\[
g_n \to g \quad \text{in} \quad C([0, T]^w; L^1_{\alpha}(0, \lambda))
\]
for each \(T > 0\). We can improve the convergence (49) to
\[
g_n \to g \quad \text{in} \quad C([0, T]^w; L^1_{\alpha,1}(\mathbb{R}_{>0}))
\]
by applying Lemma 3.1, (26) and (8).

Next, we need to show \(g\) is actually a solution to (1)–(2) in the sense of (6). For this, we have to claim that all the truncated integrals in (15) converges weakly to the original integrals in (1), respectively. For this purpose, one can follow [3] and [5] with slight modifications to deal with the convergence of integrals for large and small volume particles. In particular, with the aid of the superlinear growth (8) of \(\Gamma_1\) at infinity, the linear growth of the collision kernel in (A1), (26) and (27), one can handle the behaviour of large size particles whereas the behaviour of small volume particles can be controlled by using (A1), Lemma 3.1 and the weak convergence (50).

Now, using the convergence of integrals and the weak convergence (50) into (15), we have
\[
\int_0^\infty h(v)\{g(v, t) - g^{in}(v)\} dv = \lim_{n \to \infty} \int_0^n h(v)\{g_n(v, t) - g^{in}_n(v)\} dv
\]
\[
= \lim_{n \to \infty} \left\{ \frac{1}{2} \int_0^t \int_0^n \int_0^n \tilde{h}_x(v, v') \mathbb{1}_{(1/n, n)}(v) \mathbb{1}_{(1/n, n)}(v') E(v, v') \times \varphi(v, v') g_n(v, s) g_n(v', s) dv' ds dv \right\}
\]
\[
+ \frac{1}{2} \int_0^t \int_0^n \int_0^n \Pi h,\tau(v', v'') E_1(v', v'') \varphi(v', v'') \mathbb{1}_{(1/n, n)}(v') \times \mathbb{1}_{(1/n, n)}(v'') g_n(v', s) g_n(v'', s) dv'' dv' ds dv
\]
\[
= \frac{1}{2} \int_0^t \int_0^n \int_0^n \tilde{h}(v, v') E(v, v') \varphi(v, v') g(v, s) g(v', s) dv' dv ds
\]
\[
+ \frac{1}{2} \int_0^t \int_0^n \int_0^n \Pi h(v', v'') E_1(v', v'') \varphi(v', v'') g(v', s) g(v'', s) dv'' dv' ds,
\]
for every $h \in L^\infty(\mathbb{R}_+)$. This confirms that $g$ is a weak solution to (1)–(2) in the sense of (6).

Finally for the completeness to the proof of Theorem 2.1, it remains to prove that $g$ is a mass-conserving solution to (1)–(2). On the one hand, for $(\tau = 0)$, it can be easily shown similar to [13, 2] and on the other hand, for conservative case $(\tau = 1)$, we infer from (50) and (17), which completes the proof of Theorem 2.1.

Acknowledgments. A part of this work is supported by the Indo-French Centre for Applied Mathematics (MA/IFCAM/19/58) within the project Collision-induced fragmentation and coagulation: dynamics and numerics. The authors would like to thank the anonymous referees for their careful reading of the manuscript and for the valuable comments and suggestions, which helped us to improve the manuscript significantly.

REFERENCES

[1] P. K. Barik, Existence of mass-conserving weak solutions to the singular coagulation equation with multiple fragmentation, *Evol. Equ. Control Theory*, **9** (2020), 431–446.

[2] P. K. Barik and A. K. Giri, A note on mass-conserving solutions to the coagulation and fragmentation equation by using non-conservative approximation, *Kinet. Relat. Models*, **11** (2018), 1125–1138.

[3] P. K. Barik and A. K. Giri, Existence and uniqueness of weak solutions to the singular kernels coagulation equation with collisional breakage, *arXiv:1806.03911*, (2018).

[4] P. K. Barik and A. K. Giri, Global classical solutions to the continuous coagulation equation with collisional breakage, *Z. Angew. Math. Phys.*, **71** (2020), 1–23.

[5] P. K. Barik and A. K. Giri, Weak solutions to the continuous coagulation model with collisional breakage, *Discrete Contin. Dyn. Syst.*, **40** (2020), 6115–6133.

[6] P. K. Barik, A. K. Giri and P. Laurençot, Mass-conserving solutions to the Smoluchowski coagulation equation with singular kernel, *Proc. Roy. Soc. Edinburgh Sect. A*, **150** (2020), 1805–1825.

[7] P. S. Brown, Structural stability of the coalescence/breakage equations, *J. Atmosph. Sci.*, **52** (1995), 3857–3865.

[8] C. C. Canejo and G. Warnecke, The singular kernel coagulation equation with multifragmentation, *Math. Methods Appl. Sci.*, **38** (2015), 2953–2973.

[9] Z. Cheng and S. Redner, Kinetics of fragmentation, *J. Phys. A. Math. Gen.*, **23** (1990), 1233–1258.

[10] Z. Cheng and S. Redner, Scaling theory of fragmentation, *Phys. Rev. Lett.*, **60** (1988), 2450–2453.

[11] M. H. Ernst and I. Pagonabarraga, The nonlinear fragmentation equation, *J. Phys. A. Math. Theor.*, **40** (2007), F331–F337.

[12] M. Escobedo, P. Laurençot, S. Mischler and B. Perthame, Gelation and mass conservation in coagulation-fragmentation models, *J. Differential Equations*, **195** (2003), 143–174.

[13] F. Filbet and P. Laurençot, Mass-conserving solutions and non-conservative approximation to the Smoluchowski coagulation equation, *Arch. Math.*, **83** (2004), 558–567.

[14] A. K. Giri, P. Laurençot and G. Warnecke, Weak solutions to the continuous coagulation with multiple fragmentation, *Nonlinear Anal.*, **75** (2012), 2199–2208.

[15] M. Kostoglou and A. J. Karabelas, A study of the nonlinear breakage equation: Analytical and asymptotic solutions, *J. Phys. A. Math. Gen.*, **33** (2000), 1221–1232.

[16] P. Laurençot, Mass-conserving solutions to coagulation-fragmentation equations with nonintegrable fragment distribution function, *Quart. Appl. Math.*, **76** (2018), 767–785.

[17] P. Laurençot, Weak compactness techniques and coagulation equations, *Evolutionary Equations with Applications in Natural Sciences*, J. Banasiak & M. Mokhtar-Kharroubi (eds.), Lecture Notes Math., **2126** (2015), 199–253.

[18] P. Laurençot and S. Mischler, From the discrete to the continuous coagulation-fragmentation equations, *Proc. Roy. Soc. Edinburgh Sect. A*, **132** (2002), 1219–1248.

[19] P. Laurençot and D. Wrzosek, The discrete coagulation equations with collisional breakage, *J. Statist. Phys.*, **104** (2001), 193–220.
[20] F. Leyvraz and H. R. Tschudi, Singularities in the kinetics of coagulation processes, *J. Phys. A*, 14 (1981), 3389–3405.

[21] D. J. McLaughlin, W. Lamb and A. C. McBride, An existence and uniqueness result for a coagulation and multiple-fragmentation equation, *SIAM J. Math. Anal.*, 28 (1997), 1173–1190.

[22] V. S. Safronov, *Evolution of the Protoplanetary Cloud and Formation of the Earth and the Planets*, Israel Program for Scientific Translations Ltd. Jerusalem, 1972.

[23] I. W. Stewart, A global existence theorem for the general coagulation-fragmentation equation with unbounded kernels, *Math. Methods Appl. Sci.*, 11 (1989), 627–648.

[24] R. D. Vigil, I. Vermeersch and R. O. Fox, Destructive aggregation: aggregation with collision-induced breakage, *Colloid and Interface Science*, 302 (2006), 149–158.

[25] I. I. Vrabie, *Compactness Methods for Nonlinear Evolutions*, 2nd edition, Pitman Monogr. Surveys Pure Appl. Math., Longman, 1995.

[26] D. Wilkins, A geometrical interpretation of the coagulation equation, *J. Phys. A*, 15 (1982), 1175–1178.

Received March 2020; 1st revision October 2020; final revision January 2021.

E-mail address: prasant.daonly01@gmail.com/pbarik@tifrbng.res.in
E-mail address: ankik.giri@ma.iitr.ac.in
E-mail address: rajesh.kumar@pilani.bits-pilani.ac.in