Local temperature for dynamical black holes

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Abstract. A local Hawking temperature was recently derived for any future outer trapping horizon in spherical symmetry, using a Hamilton-Jacobi tunneling method, and is given by a dynamical surface gravity as defined geometrically. Descriptions are given of the operational meaning of the temperature, in terms of what observers measure, and its relation to the usual Hawking temperature for static black holes. Implications for the final fate of an evaporating black hole are discussed.

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0. INTRODUCTION

Hawking [1] showed that stationary black holes radiate thermally at a temperature given by their surface gravity. In a quasi-stationary (or adiabatic) approximation, a radiating black hole loses energy and therefore shrinks. The rate accelerates. This raises the question of the final fate of evaporation, including the supposed information paradox.

The fundamental problem is that an evaporating black hole is non-stationary, while the classic derivations of Hawking temperature do not obviously generalize beyond stationary black holes. So the question arises: is there in any sense a Hawking temperature for dynamical black holes?

Traditionally, black holes have generally been defined by event horizons [2, 3], despite their physically unlocatable nature, leading to some confusion that they may be the source of Hawking radiation. Fortunately recent years have seen the development of a local theory of dynamical black holes, based on a refinement of apparent horizons, trapping horizons [4, 5], which have physical properties such as mass and surface gravity, satisfying physically interpretable equations [6, 7]. This theory is practical enough to apply to violent astrophysical processes such as binary black-hole mergers [8], which may be observable in the near future via gravitational-wave detectors.

Contemporaneously, Parikh & Wilczek [9] developed a tunneling method to derive temperature for stationary black holes, making precise the intuitive idea of Hawking radiation in terms of pair production. A Hamilton-Jacobi variant turns out to work even for dynamical black holes [10], yielding a local temperature precisely for future outer trapping horizons, which were previously proposed as a local definition of black holes as part of the above theory [4]. Moreover, the temperature is given by the surface gravity as previously defined for dynamical black holes on geometrical grounds [5].
The article is organized as follows.
1. Geometry of dynamical black holes: trapping horizons, area, mass, surface gravity
2. Hamilton-Jacobi tunneling method: local temperature
3. Operational meaning: redshift and observed temperature
4. Static, asymptotically flat space-times: surface gravity vs. Killing “surface gravity”,
   local temperature vs. Hawking temperature
5. Extremal limits: charged stringy black hole
6. Remarks: evaporation and final fate

General Relativity is assumed throughout, but not the Einstein equation with pre-
where \( \cong \) denotes evaluation on a trapping horizon \( r \cong 2m \), similarly to the usual Killing identity. Then a trapping horizon is outer, degenerate or inner respectively if \( \kappa > 0 \), \( \kappa = 0 \) or \( \kappa < 0 \) \([5]\). Examples of all types are provided by Reissner-Nordström solutions: the future or past trapping horizons are respectively the Killing horizons of the black or white hole, being outer, degenerate or inner as appropriate. In vacuo, \( \kappa = m/r^2 \) \([5]\), therefore reducing to the Newtonian surface gravity in the Newtonian limit, since \( m \) reduces to the Newtonian mass. Thus \( \kappa \) also provides a relativistic definition of the surface gravity of planets and stars.

For an advanced time \( v \), the generalized advanced Eddington-Finkelstein metric

\[
ds^2 = r^2 d\Omega^2 + 2e^\Psi dvdr - e^{2\Psi} Cd v^2
\]

with \((C, \Psi)\) functions of \((r, v)\) and

\[
C = 1 - 2m/r
\]

is valid \([10]\) in untrapped regions, \( C > 0 \), on future marginal surfaces, \( C = 0 \), and in future trapped regions, \( C < 0 \), as appropriate for black holes rather than white holes. Note that \( C \) is an invariant, but \( \Psi \) is not, due to the freedom \( v \rightarrow \tilde{v}(v) \). Also \( K = e^{-\Psi} \partial_r \) and \( \kappa = e^{-\Psi} \partial_r (e^{\Psi} C)/2 \), so

\[
\kappa \cong \partial_r C/2.
\]

2. HAMILTON-JACOBI TUNNELING METHOD

The WKB approximation of the tunneling probability \( \Gamma \) along the classically forbidden trajectory from inside to outside the horizon is

\[
\Gamma \propto \exp(-2\Im \mathcal{I})
\]

in units \( h = 1 \), where \( \Im \mathcal{I} \) is the imaginary part of the action \( \mathcal{I} \) on the classical trajectory. For a massless scalar field \( \phi = \phi_0 \exp(i\mathcal{I}) \) in the eikonal (or geometrical optics) approximation, the amplitude \( \phi_0 \) is slowly varying and the action

\[
\mathcal{I} = \int \omega e^\Psi dv - \int k dr
\]

is rapidly varying, defining angular frequency \( \omega \) and wave number \( k \), where \( e^\Psi \) is included to make \( \omega \) and \( I \) invariant, recalling the freedom \( v \rightarrow \tilde{v}(v) \). Equivalently, \( \omega = K \cdot dI = e^{-\Psi} \partial_r I, k = -\partial_v I \). Then the wave equation \( \nabla^2 \phi = 0 \) yields the Hamilton-Jacobi equation

\[
g^{-1}(\nabla \mathcal{I}, \nabla \mathcal{I}) = 0
\]

which becomes

\[
2\omega k - Ck^2 = 0.
\]

Then \( k = 0 \) yields the ingoing modes, while \( k = 2\omega/C \) yields the outgoing modes. Since \( C \cong 0 \) at a trapping horizon \( r \cong r_0 \), \( \mathcal{I} \) has a pole, evaluated by \( C \approx (r - r_0) \partial_r C \). Thus \( k \approx \omega/\kappa(r - r_0) \) if \( \kappa \not\cong 0 \), yielding

\[
\Im \mathcal{I} \cong \pi \omega/\kappa.
\]
Then the tunneling probability takes a thermal form

\[ \Gamma \propto \exp\left(-\frac{\omega}{T}\right) \]  

with temperature \( T \) given by

\[ T \approx \frac{\kappa}{2\pi}. \]  

For this to be positive, \( \kappa > 0 \), so the trapping horizon is of the outer type. Thus the method has derived a positive temperature if and only if there is a future outer trapping horizon, remarkably confirming the local definition of black hole [4, 5].

Note that this is nothing to do with event horizons. There may be no event horizon in the space-time. If there is, and it does not coincide with a trapping horizon, the above method does not yield a thermal spectrum on it. Generally, the method gives no reason to expect a thermal spectrum everywhere in the space-time, including at infinity, but only on a future outer trapping horizon, and therefore approximately in a neighbourhood.

### 3. OPERATIONAL MEANING

The integral curves of \( K \), outside the horizon, are the worldlines of preferred observers, who would be static observers in the static case. Their velocity vector is \( \hat{\dot{K}} = \frac{\dot{K}}{\sqrt{C}} \). The angular frequency measured by such observers is \( \hat{\omega} = \frac{\hat{K} \cdot dI}{\omega} = \frac{\omega}{\sqrt{C}} \). Such observers therefore measure a thermal spectrum with temperature

\[ \hat{T} \approx T / \sqrt{C} \]  

to leading order near the horizon. The invariant redshift factor \( \sqrt{C} \) is familiar from the Schwarzschild case, where it reflects the acceleration required to keep an observer static [15]. So this is the operational meaning of \( T \): not that someone is measuring \( T \) directly, but that the preferred observers just outside the horizon measure \( T / \sqrt{C} \), which diverges at the horizon. Then \( T \) itself can be interpreted as a redshift-renormalized temperature, finite at the horizon. One might also conjecture that freely falling observers crossing the horizon measure a temperature of the order of \( T \), as predicted for static cases [15].

### 4. STATIC, ASYMPTOTICALLY FLAT SPACE-TIMES

The surface gravity \( \kappa \) coincides with the usual definition of the Killing “surface gravity” \( \kappa_{\infty} \) for standard static black holes such as Schwarzschild and Reissner-Nordström. However, it does not coincide if \( \Psi \neq 0 \), requiring further explanation.

Static metrics can be written as

\[ ds^2 = r^2 d\Omega^2 + C^{-1} dr^2 - C e^{2\Psi} dt^2 \]  

where \((C, \Psi)\) are henceforth functions of \( r \) alone, the notation being consistent with the above. The static Killing vector \( \partial_t \) is

\[ K_{\infty} = e^\Psi K. \]
Then $\kappa_\infty$ is defined by $K_a^\mu \nabla_\mu K_{\infty b} \cong \kappa_\infty K_{\infty b}$, yielding

$$\kappa_\infty \cong e^\Psi \kappa. \quad (18)$$

This discrepancy can be understood as follows. A textbook method derives the gravitational redshift of light along a given ray [3]:

$$\sqrt{-g} (\partial_t, \partial_t) \hat{\omega} = e^\Psi \sqrt{C} \hat{\omega}$$

is constant along the ray. If the space-time is asymptotically flat, with $(t, r)$ being Minkowski coordinates as $r \to \infty$, then $C \to 1$, $\Psi \to 0$ and $\partial_t \to K$. Note that it is precisely here where the generally non-invariant $\Psi$ acquires a specific meaning. Then the angular frequency measured by static observers at infinity is

$$\omega_\infty = e^\Psi \sqrt{C} \hat{\omega} \quad (19)$$

and the corresponding temperature measured by such observers is

$$T_\infty = e^\Psi \sqrt{C} \hat{T} \quad (20)$$

which is the Tolman relation [16]. Thus

$$T_\infty \cong e^\Psi T \quad (21)$$

which indeed corresponds to

$$\kappa_\infty \cong 2\pi T_\infty \quad (22)$$

Hence $e^\Psi$ appears as a relative redshift between the horizon and infinity. The Tolman relation mixes the redshift factors, $\sqrt{C}$ invariant and $e^\Psi$ relative.

So the appropriate local temperature at the horizon is $T$ and generally not $T_\infty$ even in the static case. Likewise, the local surface gravity is $\kappa$ and generally not the textbook definition $\kappa_\infty$. Recall that the physical interpretation of $\kappa_\infty$ is the force at infinity per unit mass required to suspend an object from a massless rope just outside the horizon [3]. This “surface gravity at infinity” would seem to be an oxymoron. It bears no relation to how Newtonian surface gravity is defined, whereas $\kappa$ reduces as above to the Newtonian surface gravity in vacuo.

The relative redshift factor stems from using the Kodama vector $K$ instead of $\partial_t$, since the latter does not exist in dynamic cases. Thus one can deal in a unified way with such situations as an accreting black hole which settles down to a static state, or a static black hole which starts to evaporate.

5. EXTREMAL LIMITS

Lest there still be doubts about the above unorthodox conclusion, a key property of surface gravity is that it should vanish in extremal cases. A good example is provided by charged stringy black holes, which are non-vacuum solutions of Einstein-Maxwell dilaton gravity in the string frame [17,18]:

$$ds^2 = r^2 d\Omega^2 + \frac{dr^2}{(1-a/r)(1-b/r)} - \left( \frac{1-a/r}{1-b/r} \right) dt^2 \quad (23)$$
where $a > b > 0$. The horizon radius is $r \cong a$.

The extremal limit as defined by global structure is $b \to a$. The Killing “surface gravity” $\kappa_\infty \cong 1/2a$ does not vanish in this limit, whereas extremal black holes are expected to be zero-temperature objects. Remarkably, $\kappa \cong (a-b)/2a^2$ vanishes in the extremal limit. This is striking confirmation of the appropriateness of $\kappa$ over $\kappa_\infty$ as a local surface gravity.

**6. REMARKS**

Returning to the main result: dynamical black holes indeed possess a local temperature $T$, with the operational meaning that it determines the redshifted temperature $T/\sqrt{1-2m/r}$ measured by Kodama observers just outside a trapping horizon. The method works precisely for future outer trapping horizons, proposed previously to define black holes on purely geometrical grounds, and $T = \kappa/2\pi$ in terms of the geometrically defined surface gravity $\kappa$. This confirms the quasi-stationary picture of black-hole evaporation in early stages.

Apart from the restriction to spherical symmetry, the derivation is general, exact, simple, independent of model or semi-classical ambiguities, and therefore robust. It yields a clear conclusion on the much debated issue of whether Hawking radiation is a mysterious global effect associated with event horizons, or even the entire space-time, or a local geometrical effect.

The result holds formally for arbitrarily fast evaporation, even in regimes where one normally expects a semi-classical approximation to break down. With this qualification, it strongly suggests that evaporation proceeds until $\kappa \to 0$. While this is reminiscent of quasi-stationary arguments, it has a different meaning, since $\kappa$ is generally not the surface gravity of a static black hole with the same mass and whatever other parameters in a given model.

A common idea is that evaporation results in an extremal remnant [19, 20]. For instance, an outer ($\kappa > 0$) and inner ($\kappa < 0$) trapping horizon might asymptote to the same null hypersurface, effectively forming a degenerate ($\kappa = 0$) trapping horizon. Another idea is that the outer and inner trapping horizons merge smoothly at a single moment of extremality where $\kappa$ vanishes [21]. The results here are consistent with either picture.

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