The $1/D$ expansion in general relativity critically revisited

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Abstract

The large dimension ($D$) limit of general relativity has been used in problems involving black holes as an analytical approximation tool. Further it has been proposed that both linear and nonlinear problems involving black holes can be systematically studied in a $1/D$ expansion. As an example, certain quasinormal modes of higher-dimensional Schwarzschild black holes with $\omega \sim \mathcal{O}(1)$ were studied in the large $D$ limit by assuming an expansion for the mode function as a series in $1/D$. In this paper, we critically revisit this linear perturbation problem and compute quasinormal modes without this assumption. We show that for $\omega \sim \mathcal{O}(1)$, the general solutions to the perturbation equations cannot be written in a series expansion in $1/D$. This result has ramifications for recent proposals for an effective membrane approach to tackle nonlinear problems in black hole physics in a $1/D$ expansion. We also discuss modes with $\omega \sim \mathcal{O}(D)$ and the difficulties in computing them beyond leading order.

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I Introduction

Recent studies of black holes/branes have employed the large dimension $D$ limit to address classical stability issues \cite{1}, \cite{2}, \cite{3}, \cite{4}, \cite{5}, \cite{7}. Specifically, the large $D$ limit has been used to study the equations for linearized perturbations of black holes, analyze stability and to compute quasinormal modes. Motivated by this, an effective membrane approach has also been initiated in \cite{4}, \cite{6} to study large $D$ black holes systematically in a $1/D$ expansion. The aim is to eventually tackle the nonlinear regime in problems involving black holes (such as mergers) by using the large $D$ limit. The most pertinent questions then are (i) In a process such as a merger, what would a hypothetical gravitational wave detector see in a large $D$ spacetime? (ii) What does this imply for such processes in four dimensions - i.e., can one make qualitative predictions in four dimensions based on what one has obtained in the large $D$ limit?

While the nonlinear regime is difficult to study even in the large $D$ limit, one can certainly address problems in linearized perturbation theory for large $D$ Schwarzschild-Tangherlini black holes. In particular, quasinormal modes, which characterize the response of a final black hole after a nonlinear process such as a merger, can be studied in the large $D$ limit. The hope is that the large $D$ limit would allow an analytical computation of these modes, and a systematic procedure of incorporating $1/D$ corrections might yield an analytical approximation to quasinormal modes of black holes of finite dimension $D$. To this end, Emparan and collaborators \cite{2} assumed a $1/D$ expansion for the quasinormal mode functions and the quasinormal mode frequencies, and computed a set of modes with frequency $\omega \sim \mathcal{O}(1)$. In fact, two distinct sets of modes were identified in the large $D$ limit - one set with frequency $\omega \sim \mathcal{O}(D)$ and the other set with frequency $\omega \sim \mathcal{O}(1)$, the mode functions for the latter decaying away from the horizon.

In our paper, we evaluate the vector and scalar quasinormal modes of Schwarzschild-Tangherlini black holes in the large $D$ limit without any assumption about a $1/D$ expansion for either the quasinormal mode functions or frequencies. In section II, we first find the leading order result for vector quasinormal modes with frequency $\omega \sim \mathcal{O}(D)$ and discuss the obstructions to going beyond the leading order in $D$. We then compute modes with frequency $\omega \sim \mathcal{O}(1)$. We also find that the perturbation equation for $\omega \sim \mathcal{O}(1)$ is a hypergeometric equation which reduces to a degenerate case in the $D \to \infty$ limit and its general solution cannot be obtained from solutions to the degen-
erate case in a $1/D$ expansion. In section III, we analyze the perturbation equation for scalar quasinormal modes with frequencies $\omega \sim \mathcal{O}(1)$. This is a Heun equation in which two of the singular points merge in the $D \to \infty$ limit. A careful analysis must be done of nearby singular points in a Heun equation. Our analysis yields the result that there are no scalar modes with $\omega \sim \mathcal{O}(1)$ that are ingoing at the horizon and decay away from it. Our result is in contrast to previous numerical [10] and analytical [2] approaches that claim the presence of such modes. However, these approaches do not seem to have dealt with the issue of nearby singularities that merge in the $D \to \infty$ limit which must be handled carefully. In particular, it seems unlikely that solutions to the perturbation equation can be obtained in a naive $1/D$ expansion around the solutions to the equation in the $D \to \infty$ limit without dealing with the issue of merging singularities. In section IV, we discuss the leading order scalar quasinormal modes with $\omega \sim \mathcal{O}(D)$.

For a large $D$ analysis to be significant in four dimensions, there must be a way to systematically incorporate corrections to the mode functions or frequencies beyond the answer obtained in the large $D$ limit. The fact that there are obstructions to going beyond leading order or that solutions to perturbation equations are not likely to admit a $1/D$ expansion for the simplest set of problems in linear perturbation theory is of concern for effective membrane approaches [4], [6] that seek to address nonlinear problems involving black holes in a $1/D$ expansion. A summary of results and a discussion of these points is presented in section V. Appendices A and B contain some of the computational details of sections II and III.

II Vector quasinormal modes

We now consider perturbations of $D$ dimensional Schwarzschild-Tangherlini black holes. The black hole metric is

$$g_{AB}dx^A dx^B = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2d\Omega_n^2; \quad (II.1)$$

where $f(r) = \left(1 - \frac{b^{D-3}}{r^{D-3}}\right)$ and $r \geq b$. The event horizon is located at $r = b$.

$d\Omega_n^2 = \gamma_{ij}d\tilde{y}^id\tilde{y}^j$ is the metric of a $n$-dimensional sphere of unit radius and $n = D - 2$.

We consider perturbations of the metric (II.1), with the perturbed metric $\bar{g}_{AB} = g_{AB} + h_{AB}$, in linearized perturbation theory. We assume the mode
decomposition $h_{AB} = \tilde{h}_{AB} e^{i\omega t}$ where $\tilde{h}_{AB}$ is time-independent. The standard procedure consists of classifying perturbations $\tilde{h}_{AB}$ as tensor, scalar and vector depending on whether they are proportional to the tensor, scalar or vector spherical harmonics (on the $n$-sphere) respectively. Using appropriately defined master variables in each class of perturbations, Ishibashi and Kodama [8] obtained decoupled equations for each class in a simple Schrodinger form.

Denoting the master variable corresponding to vector perturbations as $\Psi_V$, the equation governing vector perturbations is

$$\frac{d^2}{dr_*^2} \Psi_V + (\omega^2 - V_V) \Psi_V = 0. \quad \text{(II.2)}$$

Here $dr_* = \frac{dr}{f(r)}$ and

$$V_V = \frac{(D-3)^2 f(r)}{4r^2} \left\{ \left(1 + \frac{2\ell}{D-3}\right)^2 - \frac{1}{(D-3)^2} - 3 \left(1 + \frac{1}{D-3}\right)^2 \left(\frac{b}{r}\right)^{D-3} \right\}. \quad \text{(II.3)}$$

For future reference, we note the various distinct regions that appear in this perturbation problem in the large $D$ limit. The function $f(r)$ increases steeply from zero in $b < r < b + \frac{b}{D}$ and is almost constant outside this region. We therefore define as the ‘far’ region, $r >> b + \frac{b}{D}$ where the metric is almost flat. The potential $V_V$ has its maximum at $r \sim b + c \ln D$ where $c$ is a $D$-independent constant. Also, following Emparan and collaborators, [2] we can define a near zone by $(r - b) \ll b$. Thus, the so defined near region and the far region have an overlap for $\frac{b}{D} < (r - b) < b$ and the maximum of the potential lies in this region. The overlap region is significant only for large $D$. We also note here that when $b < r << b + \frac{b}{D}$, the $t - r$ part of the black hole metric can be approximately written in Rindler form. So this is what would be traditionally called ‘near-horizon’. In this paper, we retain the terminology in [2] for our definition of the near region $(r - b) \ll b$ which, for large $D$ is much bigger than the traditional near-horizon region. We keep in mind that this definition of the near and far regions with an overlap is nothing more than a convenient analytical approximation tool for large $D$.

We change to coordinate $R = \left(\frac{r}{b}\right)^{D-3}$. From (II.2), the equation for
\(\Psi_V(R)\) is:

\[
R(1 - R)\Psi''_V(R) + \left[-R + \frac{R - 1}{D - 3}\right] \Psi'_V(R) + \left[\frac{1}{4} \left(1 + \frac{2\ell}{D - 3}\right)^2 - \frac{1}{4(D - 3)^2}\right] \Psi_V
- \frac{3}{4R} \left(1 + \frac{1}{D - 3}\right)^2 - \frac{\omega^2 r^2}{(D - 3)^2} - \frac{\omega^2 r^2}{(D - 3)^2(R - 1)}
\Psi_V = 0.
\]

(II.4)

### II.1 Near zone analysis

This equation cannot be solved exactly for the entire range of \(1 \leq R < \infty\) due to the fact that we need to write \(r\) as a function of \(R\) in (II.4). Let us now focus on the near zone, defined by \((r - b) \ll b\). In this region,

\[
\ln R = (D - 3) \ln \left(\frac{r}{b}\right) = (D - 3) \ln \left(1 + \frac{r - b}{b}\right)
\sim (D - 3) \left(\frac{r - b}{b}\right).
\]

(II.5)

Thus, in the entire near region, inverting this relation, we have

\[
r = b \left(1 + \frac{\ln R}{D - 3}\right).
\]

(II.6)

We can therefore approximate \(r\) by \(b\) in (II.4) as the error \((r - b) \ll b\). We note that this approximation does not require us to take the large \(D\) limit explicitly and is valid in the entire near region. This region is expressed in the \(R\) coordinate as \(R \ll e^{D-3}\). We will depart from the previous computations of Emparan and collaborators [2] in that we will not assume that function \(\Psi_V\) and the quasinormal mode frequency can be expanded as a series in \(1/D\). Rather, in this section, we will derive the approximate equation valid in the entire near region without resorting to a \(1/D\) expansion for \(\Psi_V\) or \(\omega\). Let us now write \(\Psi_V = R^\alpha (R - 1)^\beta \chi\) where

\[
\alpha = \frac{3}{2} \left(1 + \frac{1}{D - 3}\right); \quad \beta = \frac{i\omega b}{D - 3}.
\]

(II.7)
The equation for $\chi$ reduces in the near region to a hypergeometric differential equation, using which we can write in this region;

$$
\Psi_V = R^\alpha(R - 1)^\beta \left[ c_1 F(a, b, a + b - c + 1; 1 - R) + c_2(1 - R) c^{-a - b} F(c - a, c - b, c - a - b + 1; 1 - R) \right];
$$

$$
a = \alpha + \beta - \frac{1}{2(D - 3)} + \sqrt{\omega_\ell^2 - \frac{\omega^2 b^2}{(D - 3)^2}};
$$

$$
\tilde{b} = \alpha + \beta - \frac{1}{2(D - 3)} - \sqrt{\omega_\ell^2 - \frac{\omega^2 b^2}{(D - 3)^2}};
$$

$$
c = 2\alpha - \frac{1}{D - 3}. \quad (\text{II.8})
$$

Here, $\omega_\ell = (\frac{1}{2} + \frac{\ell}{D - 3})$.

We now aim to compute quasinormal modes. As has been observed [2], there are two kinds of distinct quasinormal modes in the large $D$ limit. A set of modes termed non-decoupled modes are such that the mode frequency $\omega \sim O(D)$. There is another distinct set of modes that can appear due to the confining shape of the potential $V_V$ in the large $D$ limit. These modes are ingoing at the horizon and decay far from the horizon and are termed decoupled modes. They have frequency $\omega \sim O(1)$ and they are significant for $\tilde{b} < r < b + \frac{b}{D - 3}$, the region where the $t - r$ part of the metric can be approximately written in Rindler form. Their possible significance lies in processes that are localized in this region very close to the horizon.

Let us first consider the problem of computing the non-decoupled modes. As has been emphasized many times in the literature, analytically finding modes that are purely outgoing at infinity or purely ingoing at the horizon is fraught with difficulties. Quasinormal modes have complex frequencies with a sign of the imaginary part such that they decay in time. As a function of the radial coordinate, the solution to the perturbation equation (II.2) which is outgoing at infinity is non-normalizable, whereas the ingoing solution at infinity is exponentially decaying as $r \to \infty$. Further, we have an irregular singularity at infinity, and therefore, we only have asymptotic expansions for the outgoing and ingoing solutions. Any contamination of the outgoing solution by an ingoing part will not change the asymptotic expansions, which are typically in powers of $(\frac{1}{r})$ whereas the ingoing piece decays exponentially in $r$ and therefore faster than any power of $r$. However, such a contamination will
of course lead to erroneous values for the quasinormal frequencies. Nollert and Schmidt [9] characterized quasinormal modes as poles of the Laplace transform of the Green’s function for the problem - their procedure eliminates many of the computational difficulties. Given the quasinormal mode frequency $\omega$, formally replace $\omega = -i\Omega$ in the equation (II.2). Denote by $f_+(\Omega, r_*)$ the unique solution to (II.2) that is bounded at infinity, and of the form $e^{-\Omega r_*}$ as $r_* \to \infty$ for positive $\Omega$. Similarly let $f_-$ denote the bounded solution at the horizon that is of the form $e^{\Omega r_*}$ as $r_* \to -\infty$. It is easy to see that the inverse transformation from $\Omega$ to $\omega$ takes these boundary conditions to the quasinormal mode ones. The investigation of the time evolution of an initially bounded perturbation of the black hole spacetime is done by taking its Laplace transform, and then evaluating the transform using the Green’s function. The (Laplace transform of the ) Green’s function can be obtained in a standard way from $f_+$ and $f_-$:

$$G(\Omega, x, x') = \frac{1}{W(\Omega)} f_-(\Omega, x') f_+(\Omega, x), (x' < x);$$

$$= \frac{1}{W(\Omega)} f_-(\Omega, x) f_+(\Omega, x'), (x' > x).$$ (II.9)

This Green’s function has poles in the complex $\Omega$ plane at the zeroes of the Wronskian of $f_-$ and $f_+$, i.e., for those complex $\Omega$ for which $f_-$ and $f_+$ are linearly dependent. The procedure of taking an inverse Laplace transform by integrating over $\Omega$ to get the perturbation as a function of time involves an integral over an appropriate contour in the complex $\Omega$ plane. The major contribution to the integral comes from the poles of the Green’s function which are precisely the quasinormal modes, and possible branch cuts which contribute to late-time power law behaviour. This method makes the significance of quasinormal modes clear in the time evolution of bounded perturbations. As we will show in our problem, it also makes evaluation of these modes free from errors, as one is computing the zeroes of the Wronskian of two functions bounded at either end (horizon or infinity).

Replacing $\omega = -i\Omega$ in the solution (II.8) which is valid in the near region $r - b \ll b$, the bounded solution at the horizon has $c_2 = 0$. We designate this $f_-$. The strategy is to work in the large $D$ limit, evaluate $f_+$ in the far region $r - b >> \frac{b}{D-3}$ and evaluate the Wronskian of the two solutions in the overlap region $\frac{b}{D-3} << r - b << b$. To this end, we evaluate $f_-$ in the overlap region. This is obtained from (II.8) by setting $c_2 = 0$ and then taking $R$ large. We
\[ f_- \sim R^{a+b} \left[ \frac{\Gamma(a+b+1-c)\Gamma(b-a)}{\Gamma(b+1-c)} R^{-a} F \left( a, a-c+1, a-b+1; \frac{1}{R} \right) \right. \]
\[ \left. + \frac{\Gamma(a+b+1-c)\Gamma(a-b)}{\Gamma(a+1-c)} R^{-b} F \left( b, b-c+1, a-b+1; \frac{1}{R} \right) \right] \] \tag{II.10}

Simplifying, we get the leading large \( R \) behaviour
\[ f_- \sim \left[ \frac{\Gamma(a+b+1-c)\Gamma(b-a)}{\Gamma(b+1-c)} R^{-1+\frac{1}{2(D-3)}} \sqrt{\frac{\omega^2 + \Omega^2}{(D-3)^2}} \right. \]
\[ \left. + \frac{\Gamma(a+b+1-c)\Gamma(a-b)}{\Gamma(a+1-c)} R^{-1+\frac{1}{2(D-3)}} \sqrt{\frac{\omega^2 + \Omega^2}{(D-3)^2}} \right]. \] \tag{II.11}

### II.2 Far region analysis

The far region is defined as the region for which \( r - b \gg b/(D-3) \). In this limit, the ratio \( (b/r)^{D-3} \sim C e^{-(D-3)\ln r} \) is a small quantity for both large \( D \) and large \( r \). Hence we can approximate \( f(r) \approx 1 \) and \( dr_* = dr \). The far region equation becomes,
\[ -\frac{d^2 \Psi_V}{dr^2} + \frac{(D-3)^2}{4r^2} \left[ \left( 1 + \frac{2\ell}{D-3} \right)^2 - \frac{1}{(D-3)^2} \right] \Psi_V = -\Omega^2 \Psi_V \] \tag{II.12}

Solutions of this equation are the modified Bessel functions of order \( \nu = \frac{D-3}{2} \left( 1 + \frac{2\ell}{(D-3)} \right) \).
\[ \Psi_V = (\Omega r)^{1/2} [d_1 I_\nu(\Omega r) + d_2 K_\nu(\Omega r)] \] \tag{II.13}

For large \( D \) it is more convenient to use a new coordinate \( z = \frac{\Omega}{\nu} r \). In the terms of \( z \), we can use the uniform asymptotic expansion of modified Bessel functions for large order and argument. The asymptotic behaviour of the solutions is \( I_\nu(\nu z) \sim e^{\nu z} \) and \( K_\nu(\nu z) \sim e^{-\nu z} \). Demanding the solution to be bounded at infinity, we set the coefficient of the growing solution \( I_\nu(\nu z) \) to zero. The expansion of \( K_\nu(\nu z) \) is given by
\[ K_\nu(\nu z) = \sqrt{\frac{\pi}{2\nu (1+z^2)^{1/4}}} \left[ 1 + \sum_{m=1}^{\infty} (-1)^m \frac{U_m(\nu)}{\nu^m} \right] \] \tag{II.14}
where

\[ \eta = \sqrt{1 + z^2} + \ln \left[ \frac{z}{1 + \sqrt{1 + z^2}} \right]; \]
\[ \tilde{t} = \frac{1}{\sqrt{1 + z^2}}. \]  

(II.15)

and \( U_m(\tilde{t}) \) are polynomials in \( \tilde{t} \). This solution can be extended to the overlap region of the near and far regions to get the leading order solution for large \( D \). The large \( D \) limit leads, as we recall, to this distinct overlap region \( \frac{b}{D-3} \ll r - b \ll b \). To find the solution in the overlap region, we write \( r \) in terms of \( R \) by using (II.6) which is valid in the entire near region, and therefore, in particular, in the overlap region.

The leading order solution in the large \( D \) limit in the overlap region, denoted by \( f_+ \) is (see Appendix A for details):

\[ f_+ = \tilde{C} R^{\frac{1}{2(D-3)}} \sqrt{\frac{\omega^2 + \Omega^2 b^2}{(D-3)^2}} \]  

(II.16)

We note that \( f_+ \) is the solution that decays as \( r \to \infty \) and it is impossible for its asymptotic expansion to be contaminated by the other (linearly independent) growing solution. This is the distinct advantage of Nollert and Schmidt’s method. We also observe that the exponent contains a term of order \( 1/D \), namely \( \frac{1}{2(D-3)} \). This also matches exactly with one of the linearly independent solutions coming from the near region equation, where we have not resorted to a \( 1/D \) expansion.

### II.3 Non-decoupled quasinormal modes

We have now obtained the form of both the solutions \( f_- \), bounded at the horizon, and \( f_+ \), bounded at \( \infty \) in the overlap region, given by (II.11) and (II.16) respectively. We look for complex values of \( \Omega \) when their Wronskian is zero, i.e., they are linearly dependent in the overlap region. By inspection, it is clear that one way that this can happen is that the coefficient of the term increasing in \( R \) in (II.11) must go to zero. The coefficient is \( \frac{\Gamma(a+b+1-c)\Gamma(a-b)}{\Gamma(a)\Gamma(a-c+1)} \), which can only go to zero at the poles of the Gamma functions in the denominator, provided the numerator remains finite. These cases can be checked, and this possibility ruled out (the Gamma function in the numerator also has a pole in this case). The only other way for \( f_- \) and \( f_+ \) to be linearly dependent
is the limiting case \( \sqrt{\omega_{\ell}^2 + \frac{\Omega^2 b^2}{(D-3)^2}} \to 0 \), which happens for \( \Omega b \to i(D-3)\omega_{\ell} \).

This corresponds precisely to the quasinormal mode expected on general grounds (see, for example, [2]), \( \omega = (D-3)\omega_{\ell} \). It is important to note that this is true only in a limiting sense. This observation assumes importance if we try to go beyond leading order in \( D \) in the far region. Let \( \omega = \hat{D}\omega \). For example, we can conjecture that the quasinormal modes can be expanded order by order in \( (D-3)^k \) as \( (D-3)^k[\hat{\omega}_0 + \hat{\omega}_1/(D-3)^k + \hat{\omega}_2/(D-3)^{2k} + .....] \) as has been done in [2] for decoupled modes, and try to obtain the corrections to \( \omega_0 b = (D-3)\omega_{\ell} \). To obtain \( \omega_1 \), one would have to analytically continue to \( \Omega \), go beyond leading order in \( D \) in the asymptotic expansion for the modified Bessel function (III.14) and do the same for \( f_- \) in the near region. The hope is that the zeroes of the Wronskian beyond leading order in \( D \) would yield \( \omega_1 \) upon replacing \( \omega_0 b = (D-3)\omega_{\ell} \) at leading order. However, in the near region, there is an obstruction to going beyond leading order in \( D \) as it entails having to deal with coefficient functions in the near region differential equation of the form \( \ln R \). The near region equation is also a degenerate case for the hypergeometric equation when the value of the leading order quasinormal mode \( \omega_0 \) is explicitly put in the equation. This is true for any \( k \).

Further, from the far region, for a range of \( k \), the terms in the asymptotic expansion for the modified Bessel function become ill-defined in the overlap region upon plugging in the leading order value for \( \omega_0 \) that we have obtained in the large \( D \) limit (see Appendix A ). A direct approach which uses the asymptotic expansion for the Hankel function corresponding to the outgoing solution at \( \infty \) also suffers from the same problem in the overlap region for the same range of \( k \). The necessary condition for the asymptotic expansion of either the modified Bessel or Hankel function to converge is \( k \leq 2/3 \). (see Appendix A for details).

**Note on past work on non-decoupled modes:** The method followed in [2] for computing quasinormal modes suffers from a more serious problem. There, the asymptotic expansion for the Hankel function is used at infinity to identify the outgoing solution and obtain it in the overlap region. This is then matched with the solution from the near region to obtain quasinormal modes. However, one is then faced with the problem of contamination of the asymptotic expansion for the Hankel function from the ingoing (exponentially decaying) solution which was mentioned in a previous subsection. This does not affect the leading order quasinormal mode computed in [2] as the
leading order mode is real, and for real frequencies, the outgoing boundary condition is unique. However, the quasinormal mode does have an imaginary part which is subleading in $D$. No conclusion can be drawn about its $D$ dependence or its value from such methods. There is thus a serious obstruction to going beyond the leading order result at large $D$. Using a different and promising approach to obtain answers beyond leading order in $D$, Emparan and Tanabe [3] have approximated the potential $V_V$ in (II.2) by a simpler expression. The function $f(r)$ increases steeply from zero in $b < r < b + \frac{b}{D}$ and is almost constant outside this region. The potential $V_V$ has its maximum at $r \sim b + c \ln \frac{D}{L}$ where $c$ is a $D$-independent constant. The simplification consists of identifying the maximum of the potential with the horizon, assuming it rises to its maximum there from zero and then decays as an inverse square of $r_*$. This effectively ignores the variation of the potential over the entire region $0 < (r - b) < b/D$ and the fact that the maximum actually occurs for $(r - b) \sim O((\ln D)/D))$. The horizon is where the ingoing boundary condition is put (and $r_* \rightarrow -\infty$), and is distinct from the maximum. The large $D$ limit must be taken carefully so that the fine features of the potential are not lost, as the answer may depend very sensitively on such details. Emparan and Tanabe obtain $(Im \hat{\omega}) \sim O(D^{-2/3})$ whereas other naive approximations to the potential as also WKB methods yield $(Im \hat{\omega}) \sim O(D^{-1/2})$. While Emparan and Tanabe are critical of the WKB approximation in this set-up, a cross-check of their approximation would clarify how sensitively the answer depends on the approximation. In our view, in the spirit of Poschl-Teller-type approximations to the black hole potential in four dimensions, the approximate potential should be an exponential one for $b < r < b + \frac{b}{D}$, an inverse square potential beyond the maximum, with a two-step matching at $b + b/D$ and the maximum. This would yield a cross-check on Emparan and Tanabe’s computation. Assuming that this computation were to give an approximate answer for the quasinormal mode beyond leading order, it is not obtained in a $1/D$ expansion for the quasinormal mode or mode function (and the correction beyond leading order, i.e., $(Im \hat{\omega}) \sim O(D^{-2/3}))$. Interestingly, as mentioned before, assumptions about any $1/D^k, k \geq 2/3$ expansion for the quasinormal mode lead to a breakdown of the asymptotic expansion for the outgoing solution in the far region, when taken to the overlap region.

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The decoupled modes $\omega \sim \mathcal{O}(1)$ are ingoing at the horizon and decaying far from the horizon for large $R$. We take $\ell \sim \mathcal{O}(1)$ as well. From (II.11), we observe that the solution that is ingoing at the horizon has a piece growing as $R^{\frac{1}{2}}$ in the overlap region at leading order in $D$. The coefficient of this piece must vanish for the decoupled quasinormal mode. The two possibilities are $a = -m$ or $a - c + 1 = -m$ where $m$ is a non-negative integer. There is no decoupled mode corresponding to the first possibility. But there is a mode for $a - c + 1 = 0$.

We have

$$-\frac{1}{2} - \frac{1}{(D - 3)} + \frac{\Omega b}{(D - 3)} + \frac{1}{2} \sqrt{\left(1 + \frac{2\ell}{(D - 3)}\right)^2 + \frac{4\Omega^2 b^2}{(D - 3)^2}} = -m \quad (\text{II.17})$$

For $\Omega, \ell \sim \mathcal{O}(1)$, the square root can be approximated by a series. Keeping the terms upto $\mathcal{O}((D - 3)^{-2})$, we get

$$-\frac{1}{2} - \frac{1}{(D - 3)} + \frac{\Omega b}{(D - 3)} + \frac{1}{2} \left[1 + \frac{2\ell}{(D - 3)} + \frac{2\Omega^2 b^2}{(D - 3)^2}\right] = -m \quad (\text{II.18})$$

This equation has a solution only for $m = 0$. This is the decoupled mode with frequency upto $\left(\frac{1}{(D-3)}\right)$

$$\omega = i \left[(\ell - 1) + \frac{1}{(D-3)}(\ell - 1)^2\right]. \quad (\text{II.19})$$

This answer agrees with past numerical work [10].

**Note on past work on decoupled modes:** Emparan and collaborators [2] used the assumption of an analytic expansion of the mode function and quasinormal mode frequency to address the problem of obtaining decoupled modes. Taking a large $D$ limit of the hypergeometric equation for the mode function, we find that the parameters in the leading order are such that we have a degenerate case of the hypergeometric equation in this limit. The general solution (II.8) to the complete equation beyond leading order cannot be analytically expanded about the leading order answer of the degenerate equation (which has a logarithmic singularity) in powers of $1/D$. The problem lies with one of the two linearly independent solutions which is outgoing at the horizon. We note that if one assumed a $1/D$ expansion for
the mode function and attempted to compute quasinormal modes order by order viewing the terms from the previous order as source terms, to evaluate the particular solution requires both linearly independent solutions to the homogeneous differential equation, which is degenerate. We would like our solution to be ingoing at the horizon. However, upon choosing the ingoing solution, the problem recurs in the far limit (II.11). One of the Gamma functions in (II.11) cannot be naively expanded about the leading order expression in powers of $1/D$. The reason is that the leading order expression is a pole of the Gamma function (at leading order, $b - a = -1$). This is a manifestation of the fact that the hypergeometric equation reduces to a degenerate case at leading order, and this issue cannot in fact be ignored.

III Scalar quasinormal modes: decoupled modes

We investigate the equations governing scalar quasinormal modes next. As shown in [8], these can be reduced to one Schrödinger-type equation

$$\frac{d^2}{dr^2} \Psi_S + (\omega^2 - V_S)\Psi_S = 0.$$  \hspace{1cm} (III.20)

The form of $V_S$ is given in [8] and it is not possible to solve (III.20) exactly for all $r$. We will therefore resort to a near and far region analysis of this equation. We work with the $R$ coordinate instead of $r^*$. Then the equation obeyed by $\Psi_S$ in the near region is of the form

$$\frac{d^2}{dR^2} \Psi_S + \left[ \frac{1}{R-1} - \frac{1}{(D-3)R} \right] \frac{d}{dR} \Psi_S$$

$$+ \left[ \frac{\omega^2 b^2}{(D-3)^2(R-1)^2} - \frac{(c_3 R^3 + c_4 R^2 + c_5 R + c_6)}{4 R^2 (R-1) (p R + q)^2} \right] \Psi_S = 0.$$ \hspace{1cm} (III.21)

Here, $c_3, c_4, c_5, c_6, p, q$ are constants that depend on the angular momentum mode $\ell$ and dimension $D$. We have not resorted to any assumptions about the mode functions being expandable as a series in $1/D$. Neither have we done so for $\omega$.

$$p = \frac{2(-1 + \ell)}{(D-3)} + \frac{2(-1 + \ell^2)}{(D-3)^2}$$ \hspace{1cm} (III.22)

$$q = 1 + \frac{3}{(D-3)} + \frac{2}{(D-3)^2}$$ \hspace{1cm} (III.23)
Constants $c_3, c_4, c_5, c_6$ are given in Appendix B.

The equation (III.21) has four regular singular points at $R = -q/p, 0, 1, \infty$ and can therefore be rewritten as a Heun differential equation. However, the solutions to this equation are harder to analyze and in particular, it is difficult to look for solutions satisfying specified boundary conditions at the horizon and infinity. The reason is that unlike the hypergeometric equation, whose solutions around the singular points are connected by linear transformations, a similar result, termed the ‘connection problem’ for the Heun equation is as yet, unsolved except in specific cases. A useful reference for the Heun equation is [13]. We will employ different strategies to obtain the non-decoupled modes and the decoupled modes.

In this section, we only consider the decoupled modes. To get the $R$ equation in the form of a Heun equation we define

$$\Psi_S = R^\alpha(R - 1)^\beta(pR + q)^\gamma \chi$$

The equation for $\chi$ is

$$\frac{d^2 \chi}{dR^2} + \left[\frac{2\alpha + d_1}{R} + \frac{1 + 2\beta}{R - 1} + \frac{2\gamma}{R + \frac{q}{p}}\right] \frac{d\chi}{dR} + \frac{ABR - C}{R(R - 1)(R + \frac{q}{p})} \chi = 0$$

Here,

$$\alpha = \frac{1}{2} + \frac{1}{2(D - 3)} \quad \beta = \frac{i\omega b}{(D - 3)} \quad \gamma = 2 \quad d_1 = -\frac{1}{(D - 3)}$$

The constants $A, B, C$ are

$$A + B = 2(\alpha + \beta + \gamma) + d_1$$
$$AB = (\alpha + \gamma)(1 + 2\beta) + 2\alpha\gamma + (\gamma + \beta)d_1 + \frac{(c_5 + c_6)}{4pq} - \frac{c_6}{2q^2} - \frac{(c_3 + c_4 + c_5 + c_6)}{4p(p + q)}$$
$$C = -\frac{\alpha(1 + 2\beta)q}{p} + 2\alpha\gamma - \frac{\beta d_1 q}{p} + \gamma d_1 + \frac{c_5 + c_6}{4pq} - \frac{c_6}{2q^2}$$

3 The confluent Heun equation appears in the study of the Regge-Wheeler equation in four dimensions, see for instance [11] and references therein.

4 For some results for the connection problem for adjacent singularities, see [12]. The singular points of interest in our problem, the horizon and $\infty$ are not adjacent; nor can they be made so by a transformation of $R$ that results in another Heun equation. Hence we cannot use these results.
At leading order in $D$, the various constants in the equation can be evaluated, and $p \to 0$ in this limit ($q \to 1$). This results from setting $\ell \sim \mathcal{O}(1)$ for decoupled modes. The singular points in the Heun equation are at $R = -q/p$, 0, 1, $\infty$, and in this limit, the singular point at $-q/p$ approaches the point at infinity.\[5\] We now examine this limit carefully. We will employ a rescaling that is often used to obtain the confluent Heun equation from the Heun equation. We define a new coordinate $\bar{R}$ by $R = -\frac{q}{p} \bar{R}$. We also define the constant $\bar{C}$ by $C = -\frac{q}{p} \bar{C}$. We first perform the rescaling and then take the limit $p \to 0$ and $q \to 1$ (the values at leading order in $D$). This yields the equation

$$\frac{d^2 \chi}{d\bar{R}^2} + \left[ \frac{2\alpha + d_1 + 1 + 2\beta}{\bar{R}} + \frac{2\gamma}{\bar{R} - 1} \right]\frac{d\chi}{d\bar{R}} + \frac{AB\bar{R} - \bar{C}}{\bar{R}^2(\bar{R} - 1)} \chi = 0. \quad (III.30)$$

However, we also need to evaluate all the constants in this equation in the large $D$ limit. We find that in this limit, $\bar{C} = 0$ in (III.30). The equation (III.25), upon taking the limit $D \to \infty$ reduces to a hypergeometric equation

$$\frac{d^2 \chi}{d\bar{R}^2} + \left[ \frac{2\alpha + d_1 + 1 + 2\beta}{\bar{R}} + \frac{2\gamma}{\bar{R} - 1} \right]\frac{d\chi}{d\bar{R}} + \frac{AB}{\bar{R}(\bar{R} - 1)} \chi = 0. \quad (III.31)$$

It must be noted that all the constants in (III.31) take their leading order in $D$ values, and do not contain the quasinormal mode frequency $\omega$ which is not of $\mathcal{O}(D)$. Thus this equation cannot be used to obtain the decoupled mode frequency $\omega$ as this equation is valid only at leading order. We mention this as in our problem, the singular points are ‘nearby’ and merge only in the $D \to \infty$ limit. To obtain the mode frequencies, one must analyze the Heun equation (III.25) somewhat carefully.

For any finite $D$, in the Heun equation (III.25), the singular point at $R = -q/p$ has not actually merged with the singular point at infinity. Rather, they are ‘nearby’ and merge in the large $D$ limit. To compute decoupled modes with $\omega \sim \mathcal{O}(1)$, if they are present, we need to take into account sub-leading corrections in $D$ in the various constants in (III.25). A delicate method due to Lay and Slavyanov \[14\] is employed to deal with nearby singularities of the Heun equation. We re-write the equation in terms of $z = \frac{1}{R}$. In the new coordinates, the black hole horizon lies at $z = 1$ and infinity is mapped to $z = 0$.\[5\] For non-decoupled modes for which $\ell \sim \mathcal{O}(D)$, we do not have $p \to 0$.\[15\]
The singularity at $z = -\frac{p}{q}$ tends to zero in the decoupled mode case. To get the equation (III.25) in canonical Heun form in terms of $z$, we define $\chi = z^\delta \Phi$. Then, for an appropriate value of $\delta$, $\Phi$ obeys a Heun equation

$$\frac{d^2 \Phi}{dz^2} + \left[ \frac{2\delta + 1 - 2(\alpha + \beta + \gamma) - d_1}{z} + \frac{1 + 2\beta}{z - 1} + \frac{2\gamma}{z + \frac{p}{q}} \right] \frac{d\Phi}{dz} + \frac{(MNz - L)\Phi}{z(z-1)\left(z + \frac{p}{q}\right)} = 0$$

(III.32)

In terms of the constants in the original equation (III.25),

$$\delta = \frac{1}{2} \left[ 2(\alpha + \beta + \gamma) + d_1 + \sqrt{(2(\alpha + \beta + \gamma) + d_1)^2 - 4AB} \right]$$  \hspace{1cm} \text{(III.33)}$$

$$L = -\frac{p}{q}C + \delta \left( 2\gamma - \frac{p(1 + 2\beta)}{q} \right) - AB \left( 1 - \frac{p}{q} \right)$$

$$M = \delta; \hspace{0.5cm} N = \delta + 1 - (2\alpha + d_1)$$

(III.34)

(III.35)

We are interested in finding those quasinormal mode frequencies for which the mode function is ingoing at the horizon and bounded at $\infty$ (since we are working with the near region equation, we are interested in solutions that decay for large $R$). We let $\Omega = i\omega$ and map an ingoing solution at the horizon to a bounded solution. Then, we are interested in finding $\Omega$ for which the solution to (III.32) is bounded both at $z = 0$ and $z = 1$. The complication is that $\Omega$ is sub-leading in $D$, but in the large $D$ limit, the singular point at $z = -\frac{p}{q} \to 0$ and merges with the singular point at $z = 0$. Lay and Slavyanov [14] split the interval $0 \leq z \leq 1$ into two overlapping regions, one of which contains the interval $0 \leq z \leq \frac{p}{q}$. In each region, the solution to the Heun equation is written as a hypergeometric series and the Wronskian of the bounded solution at $z = 0$ is computed with the bounded solution at $z = 1$, the computation being carried out in the overlap region. The zeroes of the Wronskian give the condition for a solution that is bounded at both ends. This has been evaluated at leading order by Lay and Slavyanov. This translates into a condition for the constant $L$ in (III.32) at leading order in the limit when the two nearby singularities merge. Substituting values of all the constants, we find that, to leading order $L = 6$. Following Lay and Slavyanov, for the desired quasinormal mode to exist, the necessary condition is that for some non-negative integer $n$,

$$L = n(1 + 2\beta - M - N - n)$$  \hspace{1cm} \text{(III.36)}$$

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For our problem, there is no non-negative integer that satisfies this condition as \( L = 6 \) implies \( n = -2 \) or \( n = -3 \). If this necessary condition had been satisfied, we would have had to go beyond leading order to find the value of the mode. We note that to go beyond leading order, Lay and Slavyanov’s method needs to be modified to take into account the fact that the constants in the Heun equation depend on \( D \). The fact that this condition is not satisfied even at leading order implies that there is no such mode and we do not have to employ such procedures. We note that merger of singularities is distinct from confluence (which yielded the hypergeometric equation valid as \( D \to \infty \)).

**Note on previous work on decoupled scalar quasinormal modes:** Emparan and collaborators claim that there is such a scalar quasinormal mode with a non-zero real part in past work [2]. However, their computation uses a hypergeometric equation, while in our computations, we find that such an equation only emerges in the leading order limit \( D \to \infty \) upon doing a rescaling of coordinate, and cannot be used to compute these decoupled modes. In [2], the issue of merger of singular points is not dealt with. Previous numerical work by Dias, Hartnett and Santos [10] has found quasinormal modes ingoing at the horizon and outgoing at \( \infty \), which are order \( \mathcal{O}(1) \). The equation they use is in a coordinate \( Z \) where \( Z = 0 \) corresponds to the radial coordinate \( r \to \infty \) and \( Z = 1 \) corresponds to the horizon. Both these are singular points of the perturbation equation. However, in the complex plane, there is another singular point at a radius \( \bar{R} < 1 \) from \( Z = 0 \) such that a series expansion of various functions in the differential equation around \( Z = 0 \) will in general, only have a radius of convergence \( \bar{R} \). Further in the large \( D \) limit, \( \bar{R} \to 0 \), so this singular point merges with the singular point at \( Z = 0 \). If a method approximates functions by a sum of polynomials, this is tantamount to a series expansion for the function. So one has to be careful with numerical methods that seek an approximate solution in \( 0 \leq Z \leq 1 \) if the methods require analyticity of various functions in the interval \( 0 \leq |Z| \leq 1 \) in the complex plane. We do not know how the authors of this numerical work have dealt with this problem. We have focussed on this delicate issue of merging singular points. The most that we can do is to study the far limit of the near region equation, and look for an ingoing solution that decays far from the horizon. We find analytically that there is no such solution upon doing a careful analysis of the nearby singular points. The only other computation we can do is to obtain non decoupled modes \( \omega \sim \mathcal{O}(D) \), which we do in the next section.
IV Scalar Quasinormal Modes - non-decoupled modes

The Heun equation is hard to work with for scalar non-decoupled modes. The reason is that the far limit of the ingoing near region solution cannot be obtained as in the case of solutions to the hypergeometric equation (this is the connection problem). Our strategy is to work with a set of three coupled equations which describe the scalar perturbations. As shown in [8], these can be reduced to the Heun equation after the use of identities, but we will not do so as we would like to circumvent the problem of analyzing a Heun equation. We refer to our past work on linearized scalar perturbation equations for black strings in (D+1) dimensions (an extra dimension added to the D-dimensional Schwarzschild-Tangherlini metric) which were obtained in ([5]). As shown in ([4]), these equations can be simplified to three coupled equations for three perturbation variables. We refer the reader to ([5]) for a detailed derivation of these equations. Now, if we consider perturbations independent of the extra dimension along the black string, we recover the scalar perturbations of the Schwarzschild-Tangherlini black hole, and our equations can be used to analyze the non-decoupled scalar quasinormal modes. One of the equations decouples in the near region limit, and since its solutions can be written in terms of hypergeometric functions, it is easy to take the solution to the overlap region (as we did in the case of vector perturbations). The coupled equations are for perturbation variables ̂ψ(r, t), ̂φ(r, t), ̂η(r, t) which are defined in ([5]) in terms of the metric perturbations.

Writing, for example, ̂ψ(r, t) = ψ(r)e\(i\omega t\), the system of coupled equations describing Schwarzschild-Tangherlini (scalar) perturbations is:

\[
\begin{align*}
-\frac{d^2\psi}{dr^2} &+ \left[\frac{(D-2)^3 - 2(D-2)^2 + 8(D-2) - 8}{4(D-2)r^2}\right] + \frac{f'^2}{4f^2} \\
- \frac{(D-2)^2 + 2(D-2) - 4}{2(D-2)} \frac{f'}{fr} - \frac{f''}{2f} - \frac{2(D-3)}{(D-2)r^2f} + \frac{k^2}{fr^2} - \frac{\omega^2}{f^2} \psi = \\
\left[\frac{4}{f} - \frac{2f'r}{f^2}\right] (i\omega)\eta + \left[\frac{2(D-3)}{(D-3)f} + \frac{2}{(D-2)} - \frac{D}{(D-2)} \frac{rf'}{f} - \frac{r^2f''}{f} + \frac{f'^2}{2f^2}\right] \phi
\end{align*}
\]

(IV.37)
\[- \frac{d^2 \phi}{dr^2} + \left[ \frac{(D - 2)^3 - 2(D - 2)^2 + 8(D - 2) - 8}{4(D - 2)r^2} \right] + \frac{f'^2}{4f^2} \]

\[- \frac{(D - 2)^2 + 2(D - 2) - 4}{2(D - 2)} \frac{f'}{fr} - \frac{f''}{2fr} - \frac{2(D - 3)}{(D - 2)r^2f} + \frac{k^2}{fr^2} - \frac{\omega^2}{f^2} \right] \phi = \]

\[\frac{2f'}{fr^2} \eta(i\omega) + \left[ \frac{2(D - 3)}{(D - 2)r^4f} - \frac{2(D - 3)}{(D - 2)r^3f} - \frac{2 - (D - 2)}{(D - 2)r^3f} - \frac{f''}{r^2} + \frac{f'^2}{2f^2r^2} \right] \psi \]

(IV.38)

As in the previous cases, we solve these equations in the near and far region separately. In the near region, we rewrite the equations (IV.38) in terms of \( R \) variable using (II.6). As we are interested in the nondecoupled modes, we consider \( \Omega, \ell \sim \mathcal{O}(D) \) where \( \Omega = i\omega \). For simplicity, let us denote \( \hat{\Omega} = \frac{\Omega}{(D - 3)} \) and \( \hat{k}^2 = \frac{\ell(D - 3)}{(D - 3)^2} \). Taking the large \( D \) limit of the equations, we see that these equations only decouple at leading order. We define new variables

\[ H = \psi + \phi b^2 \]

and

\[ G = \psi - \phi b^2. \]

(IV.40)

The solution to this equation is written in terms of hypergeometric functions. Choosing the bounded solution at the horizon, we find

\[ H = C_1 R(R - 1) \frac{\hat{k}^2}{2} F(p, q, 1 + 2\hat{\Omega}b, 1 - R) \]

(IV.42)

where,

\[ p = \frac{1}{2} \left[ 3 + 2\hat{\Omega}b - \sqrt{1 + 4\hat{k}^2 + 4\hat{\Omega}^2b^2} \right] \quad q = \frac{1}{2} \left[ 3 + 2\hat{\Omega}b + \sqrt{1 + 4\hat{k}^2 + 4\hat{\Omega}^2b^2} \right] \]

(IV.43)
We can extend this solution to the overlap region using the standard formulae for hypergeometric functions and taking $R \to \infty$.

\[
H = C_1 \left[ \frac{\Gamma(p + q - 2)\Gamma(q - p)}{\Gamma(q)\Gamma(q - 2)} \right] R^{\frac{\sqrt{1 + 4\hat{\Omega}^2} b^2 + 4k^2}{2}} + \frac{\Gamma(p + q - 2)\Gamma(p - q)}{\Gamma(p)\Gamma(p - 2)} R^{-\frac{\sqrt{1 + 4\hat{\Omega}^2} b^2 + 4k^2}{2}} \right]
\]

(IV.44)

The equations for $G$ and $\eta$ in the near region are:

\[
\frac{d^2 G}{dR^2} + \frac{1}{R} \frac{dG}{dR} - \left[ \frac{1}{4R^2} + \frac{3}{4R^2(R - 1)^2} + \frac{1}{R^2(R - 1)} + \frac{\hat{k}^2}{R(R - 1)} + \frac{\hat{\Omega}^2 b^2}{(R - 1)^2} \right] G = \frac{4\hat{\Omega}}{R(R - 1)^2} \eta b^2
\]

(IV.45)

\[
\frac{d^2 \eta}{dR^2} + \frac{1}{R} \frac{d\eta}{dR} - \left[ \frac{1}{4R^2} + \frac{3}{4R^2(R - 1)^2} + \frac{1}{R^2(R - 1)} + \frac{\hat{k}^2}{R(R - 1)} + \frac{\hat{\Omega}^2 b^2}{(R - 1)^2} \right] \eta = \frac{\hat{\Omega}}{R(R - 1)^2} G
\]

(IV.46)

We cannot solve this system of equations, but using the arguments in [51] it can be shown that the solution to $G$ and $\eta$ in the overlap region will be of the form

\[
G = \eta = a_1 R^{\frac{\sqrt{1 + 4\hat{\Omega}^2} b^2 + 4k^2}{2}} + a_2 R^{-\frac{\sqrt{1 + 4\hat{\Omega}^2} b^2 + 4k^2}{2}}
\]

(IV.47)

The constants $a_1$ and $a_2$ are both non-zero.

Now let us look at the far region. In this region we have $r \gg b$. Hence we can neglect terms with $f'(r)$ and $f''(r)$ in equations (IV.37)–(IV.39) compared to the other $r^{-2}$ terms in the potential. As in the near region, we assume $\Omega^2, k^2$ to be of order $D^2$ and keep only the leading $D$ terms. We also write $(D - 3) \sim D$ etc. at leading order. Approximating $f(r) \approx 1$, we get the far region equations

\[
- \frac{d^2 \psi}{dr^2} + \left[ \frac{D^2}{4r^2} + \frac{k^2}{r^2} + \Omega^2 \right] \psi = 2\phi + 4\Omega \eta
\]

(IV.48)

\[
- \frac{d^2 \phi}{dr^2} + \left[ \frac{D^2}{4r^2} + \frac{k^2}{r^2} + \Omega^2 \right] \phi = 0
\]

(IV.49)

\[
- \frac{d^2 \eta}{dr^2} + \left[ \frac{D^2}{4r^2} + \frac{k^2}{r^2} + \Omega^2 \right] \eta = -2\Omega \phi
\]

(IV.50)
The equation for $\phi$ now decouples. The solution for $\phi$ is given in terms of modified Bessel functions. For $\nu = \sqrt{\frac{D^2+1}{4} + k^2}$,

$$\phi = D_1 \sqrt{r} I_\nu(\Omega r) + D_2 \sqrt{r} K_\nu(\Omega r). \quad (IV.51)$$

Boundedness as $r \to \infty$ dictates $D_1 = 0$. Extending this solution to the overlap region by changing $r$ to $R$, we get

$$\phi = D_0 R^{-\sqrt{\frac{1+4k^2+4\Omega^2}{2}}} \quad (IV.52)$$

Using this solution as source term, we can find the solution to $\psi$ and $\eta$ from equations (IV.48) and (IV.50). The leading order solution for both turns out to be the same as $\phi$. Taking the combination $(\psi + \phi b^2)$, we calculate zeros of its Wronskian with (IV.44). The Wronskian only becomes zero in a limiting sense as

$$\omega b \to D \left( \frac{1}{2} + \frac{\ell}{D} \right) \quad (IV.53)$$

We will obtain the same condition by solving the equations for $G$ and $\eta$.

For decoupled modes, these equations are difficult to work with since the quasinormal mode is sub-leading in $D$ and at that order, all the equations (IV.37)–(IV.39) are coupled.

V Summary and Discussion

In this paper, we have revisited the problem of linear perturbations of $D$-dimensional Schwarzschild-Tangherlini black holes in the large $D$ limit. Following [2], we can define a near zone and a far region with an overlap region which is significant for large $D$. The perturbation equations cannot be solved exactly. They are therefore analyzed in the near zone and far region, and the solutions matched in the overlap region in the large $D$ limit. The interest is in obtaining the quasinormal modes in the large $D$ limit. In previous work, attempts have been made to set up an expansion of the quasinormal mode frequency and mode function in powers of $1/D$ in the case when $\omega \sim O(1)$. The motivation was to systematically obtain the mode frequency for any finite $D$ as corrections to the answer in the large $D$ limit.

We have addressed this problem without any assumptions that the mode function or frequency can be expanded as a series in $1/D$. Rather, we find
several serious issues with that assumption. Our results and observations are:

(i) For the vector quasinormal modes $\omega \sim \mathcal{O}(1)$ (decoupled modes), the perturbation equation reduces to a degenerate case of the hypergeometric equation at leading order in $D$ and its general solution cannot be obtained from that of the leading order equation as a series in $1/D$. We compute these modes without an assumption of a series expansion.

(ii) For the vector quasinormal modes $\omega \sim \mathcal{O}(D)$ (non-decoupled modes), previous approaches to match an ingoing solution from the near region with an outgoing solution from the far region suffer from problems. In particular, the asymptotic expansion for the Hankel function used for the outgoing solution can be contaminated by the ingoing one. We use a method of Nollert and Schmidt that circumvents such problems. There is an obstruction to going beyond the leading order in $D$ both in the near and far regions. The obstruction in the near region is, for us, a computational one. However, if one assumes a series expansion for the frequency in powers of $(1/D^\alpha)$, then the near region equation becomes a degenerate case of the hypergeometric equation. In the far region, we find that upon plugging the leading order quasinormal mode, the asymptotic expansion fails to converge for $\alpha > \frac{2}{3}$. For $\alpha = \frac{2}{3}$, the asymptotic expansion cannot be consistently truncated as an approximation. Thus, it is not possible to compute the corrections to the leading order answer systematically in powers of $1/D^\alpha$ for $\alpha \geq \frac{2}{3}$ in the far region.

(iii) The scalar perturbation equation can be written in the form of a Heun differential equation. This is hard to analyze analytically. We first study decoupled modes for which $\omega \sim \mathcal{O}(1)$. A complication is that in the large $D$ limit, two of the singular points of the associated Heun equation are ‘nearby’ and merge in the $D \to \infty$ limit. The issue of merging singularities in the Heun equation needs to be handled carefully. Quasinormal mode functions, if they exist, cannot therefore be naively expanded in a series in $1/D$ without careful consideration of the merging singularities. We do a careful analysis of this issue by using a result of Lay and Slavyanov [14] on merging singularities of the Heun equation. We find that there are no modes with $\omega \sim \mathcal{O}(1)$ whose mode functions decay away from the horizon. Since this is the expected behaviour of the decoupled scalar quasinormal modes, we are led to the conclusion that there are no such modes. Our conclusion differs from previous groups [10], [2] who have claimed such modes numerically or analytically as they do not seem to have analyzed the issue of merging singularities.
points in the large $D$ limit.

(iv) For nondecoupled scalar modes, an analysis of the Heun equation is difficult. We instead use an equivalent set of three coupled perturbation equations we had derived in previous work in the context of a study of black brane stability [5]. The equations partly decouple in the near and far regions in the large $D$ limit, allowing us to discuss the leading order nondecoupled scalar quasinormal modes in this $D$ limit.

We now comment on the proposed $1/D$ expansion as a means to study nonlinear processes involving black holes in an effective membrane approach [2], [4], [6]. We find that general solutions for decoupled quasinormal modes (which are localized near the horizon) do not admit an expansion as a series in $1/D$. This is among the simplest problems involving black holes, as it involves linear perturbation theory. This is of concern for approaches seeking to address nonlinear problems in black holes in a $1/D$ expansion. In the Introduction, we had mentioned the two key questions that are pertinent for a study of black holes in the large $D$ limit. We now discuss their answers.

**Question 1:** In a nonlinear process involving black holes, such as a merger, what would a hypothetical gravitational wave detector see in a large $D$ space-time?

While we have not analyzed nonlinear processes in this paper, clearly after the merger, the detector would detect quasinormal modes. Such a detector, *placed anywhere* would not detect gravitational wave oscillations from the decoupled modes, which, when they are present, are pure imaginary. Moreover, they are expected to be localized in a region very close to the horizon, where the metric is approximately Rindler. Further we have not found scalar modes of this type. Their possible significance is therefore unclear.

**Question 2:** What can such studies of a large $D$ black hole spacetime tell us about four dimensions?

In four dimensions, we do not have two sets of modes like the decoupled and non-decoupled modes in a large $D$ spacetime. The fact that in a large $D$ spacetime, the quasinormal modes cannot be systematically studied in a $1/D$ expansion (as we find) or that there are obstructions to going beyond leading order in the case of non-decoupled modes is worrisome. It implies that we cannot recover answers to questions in linear perturbation theory for finite $D$ systematically as order by order corrections to the leading order answer. Therefore, we do not think that any new knowledge can be gained for four dimensional black holes from this large $D$ approach.
References

[1] V Asnin, D Gorbonos, S Hadar, B Kol, M Levi, U Miyamoto, Class Quant Grav 24, 4915(2007).

[2] R Emparan, R Suzuki, K Tanabe, JHEP 07, 113(2014).

[3] R Emparan, R Tanabe, Phys Rev D89, 064028(2014).

[4] R Emparan, R Suzuki, K Tanabe, JHEP 06, 009(2013); R Emparan, R Suzuki, K Tanabe, JHEP 06, 106(2014); R Emparan, R Suzuki, K Tanabe, JHEP 04, 085(2015); R Emparan, T Shiromizu, R Suzuki, K Tanabe, T Tanaka, arXiv:1504.06489.

[5] A Sadhu, V Suneeta, Phys Rev D93, 124002(2016).

[6] S Bhattacharyya, A De, S Minwalla, M Ravi and A Saha JHEP 1604 (2016) 076; Y Dandekar, S Mazumdar, S Bhattacharyya, A De, S Minwalla and A Saha JHEP 1612 (2016) 113; Y Dandekar, S Mazumdar, S Minwalla, A Saha JHEP 1612 (2016) 140; S Bhattacharyya, A Mandal, M Mandlik, U Mehta, S Minwalla, U Sharma and S Thakur JHEP 1705 (2017) 098; S Bhattacharyya, P Biswas, B Chakrabarty, Y Dandekar and A Dinda arXiv:1704.06076; Y Dandekar, S Kundu, S Mazumdar, S Minwalla, A Mishra and A Saha arXiv:1712.09400.

[7] B Chen, Z Fan, P Li and W Ye JHEP 1601 (2016) 085; B Chen and P Li arXiv:1607.04713; B Chen, P Li and Z Wang JHEP 1704 (2017) 167; B Chen and P Li JHEP 1705 (2017) 025; B Chen, P Li and C Zhang JHEP 1710 (2017) 123.

[8] A Ishibashi, H Kodama, O Seto Phys Rev D 62, 064022; A Ishibashi and H Kodama Prog. Theor. Phys. Supplement (2011) 189 165-209.

[9] H Nollert, B Schmidt Phys Rev D 45, 2617.

[10] O J C Dias, G S Hartnett, J E Santos Class Quant Grav 31:245011 (2014).

[11] P P Fiziev, J Phys Conf Ser 66:012016 (2007).

[12] R Schafke, D Schmidt, page 306, Lecture Notes in Mathematics, vol. 810, Edited by R Martini, Springer Verlag, Berlin (1980).
VI Appendix A: Expansion of Modified Bessel Functions

In this section, we describe the procedure to extend the far region solution \( f_+ = (\Omega r)^{1/2}K_\nu(\Omega r) \) to the overlap region in terms of \( R \). Here \( \nu = \frac{(D-3)}{2} \left( 1 + \frac{2\ell}{D-3} \right) \). In the case \( \Omega \sim \mathcal{O}(D) \), both the amplitude and the order of the modified Bessel function \( K_\nu(\Omega r) \) is large. Hence \( f_+ \) is described using the uniform asymptotic expansions for modified Bessel functions. To keep the notation simple, we define a new coordinate \( z = \Omega r / \nu = \beta r \).

The large order and large argument expansion of the modified Bessel function is,

\[
K_\nu(\nu z) = \sqrt{\frac{\pi}{2\nu}} \frac{e^{-\nu \eta}}{(1 + z^2)^{1/4}} \left[ 1 + \sum_{m=1}^{\infty} (-1)^m \frac{U_m(\tilde{t})}{\nu^m} \right]
\]  

(VI.54)

where

\[
\eta = \sqrt{1 + z^2} + \ln \left[ \frac{z}{1 + \sqrt{1 + z^2}} \right] \quad \tilde{t} = \frac{1}{\sqrt{1 + z^2}}
\]  

(VI.55)

and \( U_m(\tilde{t}) \) are polynomials in \( \tilde{t} \).

\[
U_0(\tilde{t}) = 1, \quad U_1(\tilde{t}) = \frac{1}{24} (3\tilde{t}^3 - 5\tilde{t}^2), \quad U_2(\tilde{t}) = \frac{1}{1152} (81\tilde{t}^6 - 462\tilde{t}^4 + 385\tilde{t}^2)
\]  

(VI.56)

In the large \( D \) limit, we can truncate the asymptotic expansion in (VI.54). In the large \( D \) limit, we can truncate the asymptotic expansion in (VI.54). We retain terms to order \( \mathcal{O}(1/D) \) and substitute the expression for \( \eta \) to get

\[
K_\nu(\nu z) \approx \sqrt{\frac{\pi}{2\nu}} \frac{(1 + \sqrt{1 + z^2})^{1/2} e^{-\nu \sqrt{1 + z^2}}}{z^\nu(1 + z^2)^{1/4}} \left[ 1 - \frac{1}{24\nu} \left( \frac{3}{(1 + z^2)^{3/2}} - \frac{5}{(1 + z^2)^{5/4}} \right) \right]
\]  

(VI.57)
In order to express $K_{\nu}(\nu z)$ in terms of $R$, we recall $r = bR^{(D-3)}$ and in the overlap region, this can be approximated as

$$r = b \left[ 1 + \frac{\ln R}{(D-3)} + \frac{(\ln R)^2}{2(D-3)^2} \right].$$

Note that this expansion is valid only for $\frac{\ln R}{(D-3)} \ll \frac{1}{2}$ which is true in the overlap region.

We will now look at each term in (VI.57) individually. The term $z^\nu$ is directly proportional to $r^{(D-3)}$. Using $r^{(D-3)} = b^{(D-3)}R$ we get

$$\frac{1}{z^\nu} = \frac{1}{(\beta r)^\nu} = (\beta b)^{-\nu} R^{-\nu(D-3)}.$$ (VI.58)

Similarly the prefactor $(\Omega r)^{1/2}$ in $f_+$ is written as $\sqrt{\nu z} = (\nu b)^{1/2}R^{1/2(D-3)}$. We are interested in finding $f_+$ to the leading order in $D$. Hence we keep terms up to $\frac{\ln R}{(D-3)}$ in expansion of $r$. The next term in (VI.57) becomes

$$\left[1 + \sqrt{1 + z^2}\right]^\nu = \exp \left[ \nu \ln \left\{ 1 + \left[ 1 + \beta^2 b^2 \left( 1 + 2 \frac{\ln R}{(D-3)} \right) \right]^{1/2} \right\} \right]$$

$$= \left(1 + \sqrt{1 + \beta^2 b^2}\right)^\nu \exp \left[ \frac{\nu \beta^2 b^2}{\sqrt{1 + \beta^2 b^2} \left( 1 + \sqrt{1 + \beta^2 b^2} \right)} \frac{\ln R}{(D-3)} \right].$$ (VI.59)

Similarly substituting for $z$ we get

$$\exp \left[ -\nu \sqrt{1 + z^2} \right] = (1 + \beta^2 b^2)^{-\nu/2} \exp \left[ -\nu \beta^2 b^2 \frac{\ln R}{\sqrt{1 + \beta^2 b^2} (D-3)} \right].$$ (VI.60)

The remaining terms in (VI.57) can be written in terms of various powers of $(1 + z^2)^{-1}$. In the overlap region, this can be written as

$$\frac{1}{(1 + z^2)} = (1 + \beta^2 b^2)^{-1} \left[ 1 + \frac{\beta^2 b^2}{1 + \beta^2 b^2} \frac{2 \ln R}{(D-3)} \right].$$ (VI.61)

Comparing the coefficients of all the $\ln R$ terms in the above expressions, we see that the contribution from the term $(1 + z^2)^{-1/4}$ and the series in
(VI.57) is smaller by an order $D$ than the other terms. For the leading order solution in $D$, we neglect these subleading terms. Substituting all the expressions in $R$ back in (VI.57), we get the following expression for $f_+$ (we have absorbed all the constants in one constant $D_0$):

\[
f_+ = D_0 R^{\frac{3}{2(D-3)}} - \frac{\nu \sqrt{1+\beta^2 b^2}}{(D-3)} = D_0 R^{\frac{3}{2(D-3)}} - \frac{\nu}{2} \sqrt{(1+\beta^2 b^2)^2 + 4\nu^2 b^2/(D-3)}
\] (VI.62)

In order to calculate the quasinormal modes, we compute the zeroes of the Wronskian of $f_+$ with $f_-$ (obtained from the near region computation). Recall

\[
f_- = c_1 R^{\frac{3}{2(D-3)}} - \frac{\nu \sqrt{1+\beta^2 b^2}}{(D-3)} + c_2 R^{\frac{3}{2(D-3)}} + \frac{\nu \sqrt{1+\beta^2 b^2}}{(D-3)}
\] (VI.63)

The constants $c_1$ and $c_2$ depend on $\Omega$ and $\ell$. We could in principle extend solution (VI.62) to further orders in $D$ by keeping higher order terms in the $r$ expansion. But we cannot compute the next order corrections to $f_-$ in the overlap region from the near region. Calculating the zeroes of the Wronskian of the far solution beyond leading order with only the leading order solution from the near region is inconsistent and leads to contradictions as one might expect. We therefore cannot obtain the value of $\Omega$ beyond the leading order.

**Comments on use of Hankel functions**

In previous work [2], the authors have used Hankel functions as outgoing solutions for the computation of quasinormal modes. We have already commented on the problems involved in identifying a pure outgoing solution from an asymptotic expansion for the Hankel function. We now discuss other issues involved with the method pursued in [2]. The idea in [2] was to match the outgoing solution in the far region with the ingoing solution from the near region. The matching is done in the overlap region and the quasinormal mode frequency and mode function are expanded in powers of $1/D$. Here, we discuss the issues involved in assuming such an expansion for the quasinormal mode frequency. First, we change from $\Omega$ back to $i\omega$. Then the solutions of far region equation (II.12) are given by the standard Bessel functions. To compare with previous results, we choose the solution as $r \to \infty$ to be

\[
f_{+n} = (i\omega r)^{\frac{1}{2}} H_{\nu}^{(1)}(\omega r)
\] (VI.64)
As in the case of \( K_\nu(\Omega r) \), we resort to uniform asymptotic expansions as the order and argument of \( f_{+n} \) are large. For \( z\nu = \omega r \), \( H^{(1)}_\nu(\nu z) \) becomes,

\[
H^{(1)}_\nu(z) \sim 2e^{\pm \pi i/3} \left( \frac{4\zeta}{1 - z^2} \right)^{\frac{1}{4}} \left( \frac{\text{Ai} e^{\pm 2\pi i/3 \nu^2 \zeta}}{\nu^{\frac{5}{3}}} \right) \sum_{k=0}^{\infty} \frac{A_k(\zeta)}{\nu^{2k}} \\
+ \frac{e^{\pm 2\pi i/3} \text{Ai}' e^{\pm 2\pi i/3 \nu^2 \zeta}}{\nu^{\frac{5}{3}}} \sum_{k=0}^{\infty} \frac{B_k(\zeta)}{\nu^{2k}},
\]

(VI.65)

\[
\frac{2}{3}(-\zeta)^{\frac{1}{2}} = \sqrt{z^2 - 1} - \text{arcsec} \, z = \sqrt{z^2 - 1} - i \ln \left( \frac{1 + \sqrt{1 - z^2}}{z} \right).
\]

(VI.66)

Here

\[
A_k(\zeta) = \sum_{j=0}^{2k} \left( \frac{3}{2} \right)^j v_j \zeta^{-3j/2} U_{2k-j} \left( (1 - z^2)^{-\frac{1}{2}} \right)
\]

(VI.67)

\[
B_k(\zeta) = -\zeta^{-\frac{1}{2}} \sum_{j=0}^{2k+1} \left( \frac{3}{2} \right)^j u_j \zeta^{-3j/2} U_{2k-j+1} \left( (1 - z^2)^{-\frac{1}{2}} \right)
\]

(VI.68)

\( U_i \) are the polynomials (VI.56) in \((\sqrt{1 - z^2})^{-1}\) of degree 3k and \( u_i, v_i \) are constants. We substitute the asymptotic expansions for the Airy functions \( \text{Ai} e^{\pm 2\pi i/3 \nu^2 \zeta} \) and the sums \( A_k \) and \( B_k \) and keep terms until order \( O \left( \frac{1}{D} \right) \).

Denoting \( \frac{2}{3}(-\zeta)^{\frac{1}{2}} = x \), we get

\[
H^{(1)}_\nu(\nu z) = \frac{2\sqrt{2}e^{-i\pi/3}}{\sqrt{\nu \nu(1 - z^2)^{1/4}}} \left[ \cos \left( \nu x - \frac{\pi}{4} \right) + \sin \left( \nu x - \frac{\pi}{4} \right) \left( \frac{1}{i \nu x - \frac{\pi}{4}} \right) \right]
\]

(VI.69)

We observe that this expansion fails to select a purely outgoing solution even at the leading order. For large \( z \), \( i\nu x \sim i\omega r \). We can write

\[
2\sqrt{2} \cos \left( \nu x - \frac{\pi}{4} \right) = \left( \frac{1 + i}{i} \right) e^{i\nu x} + \left( \frac{i - 1}{i} \right) e^{-i\nu x}
\]

In order to compare coefficients with the near region solution, we extend the solution \( f_{+n} \) to the overlap region as done for the modified Bessel functions. We again substitute \( r \) in terms of \( R \). This is achieved using the same
procedure as that for $K_\nu(\nu z)$. In fact, the final expressions for both the near and far region solutions in overlap region are same as (VI.63) with $\beta^2$ replaced by $-\omega^2/\nu^2$.

$$f_+ = d_1 R^{\frac{\mu}{\nu z}} + d_2 R^{\frac{\mu}{\nu z}} + (\nu^2 b^2)$$  \text{(VI.70)}

Matching of the two solutions gives

$$\left(1 - \frac{\omega^2 b^2}{\nu^2}\right) = 0.$$  \text{(VI.71)}

After substituting $\nu$, we see that this equation gives us the leading order non-decoupled mode frequency. Even with the Hankel functions, we cannot find the value of $\omega$ beyond leading order due to the fact that in the near region, it is computationally difficult to go beyond leading order. But if we assume an expansion for $\omega = \omega_0 + \frac{\omega_1}{(D-3)\nu} + \ldots$, we obtain a bound on $\alpha$ by requiring the asymptotic expansion for $f_+$ to converge in the large $D$ limit. To see this, write the asymptotic expansion in the overlap region using

$$1 - z^2 = \left[1 - \frac{\omega_0^2 b^2}{\nu^2} - \frac{2\omega_1\omega_0 b^2}{\nu^2 (D-3)^\alpha} - \frac{\omega_0^2 b^2}{\nu^2 (D-3)} \right]$$

$$= \left(1 - \frac{\omega_0^2 b^2}{\nu^2}\right) \left[1 - \frac{2\omega_1\omega_0 b^2}{\nu^2 (1 - \frac{\omega_0^2 b^2}{\nu^2}) (D-3)^\alpha} - \frac{2\omega_0^2 b^2}{\nu^2 (1 - \frac{\omega_0^2 b^2}{\nu^2}) (D-3)} \ln R \right].$$  \text{(VI.72)}

It can be seen that this expansion only changes $\omega$ in the leading order solutions to $\omega_0$. In this case the $\omega$ in the expressions (VI.70) and (VI.71) are replaced by $\omega_0$.

To see the effect of leading order matching on the subleading terms of (VI.69), let us express the polynomials $U_k$ in the coordinate $R$. $U_k$ are polynomials in the variable $(\sqrt{1 - z^2})^{-1}$. Using (VI.72) we get (at leading order in the large $D$ limit)

$$\frac{1}{\sqrt{1 - z^2}} = \left(1 - \frac{\omega_0^2 b^2}{\nu^2}\right)^{-\frac{1}{2}} \left[1 + \frac{\omega_1\omega_0 b^2}{\nu^2 (1 - \frac{\omega_0^2 b^2}{\nu^2}) (D-3)^\alpha} + \frac{\omega_0^2 b^2}{\nu^2 (1 - \frac{\omega_0^2 b^2}{\nu^2}) (D-3)} \ln R \right].$$  \text{(VI.73)}

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For the $\omega_0$ obtained in (VI.71), $1 - \frac{\omega_0^2 b^2}{\nu^2} = 0$. We therefore have to evaluate (VI.73) carefully beyond leading order.

\[
\frac{1}{\sqrt{1 - \frac{1}{z^2}}} = \left[ -\frac{2\omega_1 \omega_0 b^2}{\nu^2 (D-3)^{\alpha}} - \frac{\omega_0 b^2}{\nu^2} \frac{2 \ln R}{(D-3)} \right]^{-\frac{1}{2}}
\]

\[
= (D-3)^{-\frac{\alpha}{2}} \left( \frac{-2\hat{\omega}_1 \hat{\omega}_0 b^2}{(1 + 2\hat{\ell})^2} \right)^{-\frac{1}{2}} \left[ 1 + \frac{\hat{\omega}_0}{\hat{\omega}_1} \frac{\ln R}{\omega_1 (D-3)^{1-\alpha}} \right]\]

(VI.74)

The hatted quantities denote $\hat{\omega} = \frac{\omega}{(D-3)}$ and $\hat{\ell} = \frac{\ell}{(D-3)}$. As we are in the overlap region, $1 \ll \ln R \ll (D-3)^2$. Let us now look at the subleading term in (VI.69). The $\sin\left(\nu x - \frac{\pi}{4}\right)$ again can be written in terms of (VI.70). The leading contribution from the series term after substituting for $U_1$ is,

\[
\frac{1}{\nu} \left\{ U_1 \left( \frac{1}{\sqrt{z^2 - 1}} \right) - \frac{5}{72x} \right\} =
\]

\[
(D-3)^{-\frac{\alpha}{2}} \left( \frac{1 + 2\hat{\ell}}{(D-3)} \right) \left( \frac{-2\hat{\omega}_1 \hat{\omega}_0 b^2}{(1 + 2\hat{\ell})^2} \right)^{-\frac{1}{2}} \frac{1}{8} \left[ 1 + \frac{\hat{\omega}_0}{\hat{\omega}_1} \frac{\ln R}{\omega_1 (D-3)^{1-\alpha}} \right]\]

\[
= (D-3)^{-\frac{3\alpha}{2}} \left( \frac{1 + 2\hat{\ell}}{(D-3)} \right) \frac{5}{24} \left( \frac{-2\hat{\omega}_1 \hat{\omega}_0 b^2}{(1 + 2\hat{\ell})^2} \right)^{-\frac{1}{2}} \left[ 1 + \frac{\hat{\omega}_0}{\hat{\omega}_1} \frac{\ln R}{\omega_1 (D-3)^{1-\alpha}} \right]\]

\[
= (D-3)^{-\frac{3\alpha}{2}} \left( \frac{1 + 2\hat{\ell}}{(D-3)} \right) \frac{5}{24} \left( \frac{-2\hat{\omega}_1 \hat{\omega}_0 b^2}{(1 + 2\hat{\ell})^2} \right)^{-\frac{1}{2}} \left[ 1 + \frac{\hat{\omega}_0}{\hat{\omega}_1} \frac{\ln R}{\omega_1 (D-3)^{1-\alpha}} \right] - \frac{5}{72x^2y}
\]

(VI.75)

Consider the $D$ dependence of the second term. The coefficient of the second term on (VI.75) is proportional to $(D-3)^{3\alpha - 2}$. For $\alpha > \frac{2}{3}$, the exponent $\frac{3\alpha - 2}{2}$ becomes positive. We can further see that from the expressions for (VI.67) and (VI.68), for each $k$ the series terms in (VI.65), $\sum_{k=0}^{\infty} \frac{A_k(\xi)}{\nu^k}$ and $\sum_{k=0}^{\infty} \frac{B_k(\xi)}{\nu^k}$, the leading term will be $\approx (D-3)^{(3\alpha - 2)k}$. For $\alpha > \frac{2}{3}$, $(3\alpha - 2)k$ is positive and both the series diverge for large $(D-3)$. The asymptotic expansion of the solution $f_{+,n}$ thus breaks down for $\alpha > \frac{2}{3}$. For $\alpha = \frac{2}{3}$, the leading order term for each $k$ is $O(1)$. Hence we cannot truncate the asymptotic expansion for large $D$. We therefore need $\alpha < \frac{2}{3}$ in our expansion for the quasinormal mode frequency. Interestingly, $\alpha = \frac{2}{3}$ corresponds to the correction $\omega_1 \sim O(D^{1/3})$. Using a different approach, it is claimed in [3] that the correction is of this order. If correct, such a result would imply that the correction can never be recovered by the direct method of matching.
the ingoing and outgoing solutions and likely ω would not admit a series expansion in $1/(D-3)^a$.

VII Appendix B: Constants in the scalar perturbation equation

Here, we give the values of various constants in the equation (III.21). We denote $(D-3) = d$.

$$c_3 = \frac{4(-1 + \ell)^2}{d^2} + \frac{8(1 + \ell - 5\ell^2 + 3\ell^3)}{d^3} + \frac{4(10\ell - 7\ell^2 - 16\ell^3 + 13\ell^4)}{d^4} + \frac{8(-1 + 3\ell + 5\ell^2 - 9\ell^3 - 4\ell^4 + 6\ell^5)}{d^5} + \frac{4(-1 + 6\ell^2 - 9\ell^4 + 4\ell^6)}{d^6}$$

(VII.76)

$$c_4 = -\frac{12(-1 + \ell)}{d} - \frac{12(-3 + \ell + 2\ell^2)}{d^2} - \frac{12(-4 + 3\ell - \ell^2 + 2\ell^3)}{d^3} - \frac{12(-4 + 7\ell - 4\ell^3 + \ell^4)}{d^4} - \frac{12(3 - 4\ell - 7\ell^2 + 6\ell^3 + 2\ell^4)}{d^5} + \frac{12(1 - 4\ell^2 + 3\ell^4)}{d^6}$$

(VII.77)

$$c_5 = 1 + \frac{2(-3 + 4\ell)}{d} + \frac{4(-11 + 7\ell + 2\ell^2)}{d^2} + \frac{2(-45 + 20\ell + 14\ell^2)}{d^3} + \frac{-89 + 44\ell + 40\ell^2}{d^4} + \frac{4(-12 + 6\ell + 11\ell^2)}{d^5} + \frac{12(-1 + 2\ell^2)}{d^6}$$

(VII.78)

$$c_6 = 1 + \frac{8}{d} + \frac{26}{d^2} + \frac{44}{d^3} + \frac{41}{d^4} + \frac{20}{d^5} + \frac{4}{d^6}$$

(VII.79)