Numerical Results For The 2D Random Bond 3-state Potts Model

Marco Picco
LPTHE
Université Pierre et Marie Curie, PARIS VI
Université Denis Diderot, PARIS VII
Boîte 126, Tour 16, 1er étage
4 place Jussieu
F-75252 Paris CEDEX 05, FRANCE
picco@lpthe.jussieu.fr

ABSTRACT

We present results of a numerical simulation of the 3-state Potts model with random bond, in two dimension. In particular, we measure the critical exponent associated to the magnetization and the specific heat. We also compare these exponents with recent analytical computations.

*Laboratoire associé No. 280 au CNRS*
These last years, many studies have been devoted to the problem of the effect of randomness on 2-dimensional statistical models. The effect of randomness is supposed to be directly related to the critical exponent of the specific heat, $\alpha$, according to the well-known Harris criterion \[1\]. If $\alpha$ is positive then the disorder will be relevant, i.e. under the effect of the disorder, the model will reach a new critical behavior at a new critical point. Otherwise, if $\alpha$ is negative, disorder is irrelevant, the critical behavior will not change. The Harris criterion does not give any information on the behavior of models with $\alpha = 0$ (marginal case), and the most studied case, the 2D Ising model, falls in this category. Due to its apparent simplicity, this model is the most studied example of a disordered model with many analytical results \[2, 3, 4, 5, 6, 7\] as well as numerical ones \[8, 9, 10, 11\].

In the present work, we will focus on the 3-state Potts model with disorder. The pure 3-state Potts model having a positive $\alpha$ ($\alpha = 1/9$), it is expected that a new critical behavior will be obtained. The main purpose of the present work is to study numerically this new behavior. A certain number of analytical studies of this model predict new critical exponents \[12, 13, 14, 15\]. In these papers, the 3-state Potts model is considered like a generalization of the Ising model by shifting a regularization parameter (namely the central charge of the corresponding conformal field theory describing the behavior of the pure model at the critical point.) Computations are performed by using a perturbed conformal field theory with a near marginal operator associated to the disorder. Predictions for critical exponents of the 3-state Potts model in presence of disorder are obtained by employing perturbative expansion with the machinery of the renormalisation group. In particular, the exponent associated to the spin-spin correlation function is given by

$$< \sigma(0)\sigma(R) > \simeq R^{-2\Delta'_{\sigma}},$$

with

$$2\Delta'_{\sigma} = 2\Delta_{\sigma} - \frac{27}{16} \frac{\Gamma^2\left(-\frac{2}{3}\right)\Gamma^2\left(\frac{1}{3}\right)}{\Gamma^2\left(-\frac{1}{3}\right)\Gamma^2\left(-\frac{1}{6}\right)} \epsilon^3 + O(\epsilon^4),$$

where $\epsilon$ is the perturbation parameter. It measures the deviation from the Ising model, see \[13, 14\] for details.) The critical exponent associated to the energy-energy correlation

\[\text{continued...}\]
function is given by

\[ <\epsilon(0)\epsilon(R)> \simeq R^{-2\Delta_\epsilon'} \]  

with

\[ 2\Delta_\epsilon' = 2\Delta_\epsilon - 3\epsilon + \frac{9}{4}\epsilon^2 + O(\epsilon^3). \]  

Here, \( \epsilon \) takes the value

\[ \epsilon = -\frac{2}{15} \]  

for the 3-state Potts model. Plugging this value into the previous expressions, we obtain

\[ \Delta_\sigma' = \frac{2}{15} + 0.00132 + O(\epsilon^4) = 0.13465 + O(\epsilon^4). \]  

It will be more convenient to express \( \Delta_\sigma' \) in function of \( \Delta_\sigma = \frac{2}{15} \). Then

\[ \Delta_\sigma' = \Delta_\sigma(1 + \delta) \quad ; \quad \delta = 0.0099 \simeq 0.01. \]  

For \( \Delta_\epsilon' \), we have

\[ \Delta_\epsilon' = 1.02 + O(\epsilon^3) \]  

which can be compared to the value for the pure 3-state Potts model

\[ \Delta_\epsilon = 0.8. \]  

It is clear at this stage that the deviation of \( \Delta_\sigma \) due to the disorder will be very difficult to measure, while the one of \( \Delta_\epsilon \) should be observed easily.

\( \Delta_\sigma' \) and \( \Delta_\epsilon' \) do also appear in the magnetization and in the specific heat. Using some standard finite size scaling arguments \[16\], we have at the critical point

\[ M(L) \simeq L^{-\Delta_\sigma'} \]  

\[ C(L) \simeq L^{2-2\Delta_\epsilon'}. \]  

These results were in fact obtained by using perturbative computations around the replica symmetry solution. It can also be argued that it is necessary to break the replica symmetry, in a fashion similar to the Parisi solution for the mean field spin glass \[17\]. If such a solution is employed, then the modification appears only at the third order in
\( \epsilon \) for the energy-energy correlation function, while there is no modification for the spin-spin correlation function (at least up to fourth order in \( \epsilon \)). The new exponent for the energy-energy correlation function is given by \[15\]

\[
\Delta''_\epsilon = \Delta_\epsilon - \frac{3}{2} \epsilon + O(\epsilon^3) = 1 + O(\epsilon^3). \tag{12}
\]

Thus we have a certain number of exponents that we can compare to the numerical results that will be presented below.

The Hamiltonian of the simulated model is given by

\[
H = - \sum_{\langle i,j \rangle} J_{ij} \delta_{\sigma_i,\sigma_j} \tag{13}
\]

where the coupling constant between nearest neighbor spins takes the value

\[
J_{ij} = pJ_0 + (1 - p)J_1. \tag{14}
\]

Without any lost of generality, we can consider the case where \( p = \frac{1}{2} \). Then the model is self-dual and thus the critical temperature is exactly known. It is given by the solution of the equation

\[
\frac{1 - e^{-\beta J_0}}{1 + (q - 1)e^{-\beta J_0}} = e^{-\beta J_1}. \tag{15}
\]

In this Letter, we will only consider the case with a strong disorder, i.e. \( J_0 = 1, J_1 = \frac{1}{10} \). The reason for such a choice is that there exists a cross-over length \( l_J \), depending on the strength of the disorder. It is only for distance larger than \( l_J \) that we expect to measure the critical behavior of the model with disorder. Extrapolating the results of the 2d random bond Ising model, we expect that with the present disorder, the cross-over length will take the value \( l_J \simeq 2 - 5 \). We will come back later to the manifestation of this cross-over length. Thus by running on large lattice, up to 1000\(^2\) for results presented here, we are quite sure to measure the critical behavior of the model with disorder.

Monte Carlo data were obtained by using the well known Wolff cluster algorithm \[18\] as well as the Swendsen-Wang algorithm \[19\]. Details on the algorithms and the simulation parameters will be presented in a subsequent paper. Measurements were performed on a square lattice with helical boundary conditions. Due to the very strong disorder
that we consider, we needed to have huge statistics over the number of configurations of disorder. Simulations were performed for lattice with size ranging from $L = 2$ to $L = 1000$. For the magnetization, the number of configurations of disorder were 100000 for $L = 2 - 50$, then 40000 for $L = 100$ to 4000 for $L = 1000$. For each of these configuration of disorder, measurements were taken over $t_1 = 1000$ configurations. For the specific heat, the number of configurations need to be larger, namely $t_2 = 20000$. 10000 configurations of disorder were simulated for $L = 2 - 50$, then from 1000 for $L = 100$ to 100 for $L = 1000$. Finally, statistical errors were computed by taking the mean value over the configurations of disorder. The values of $t_1$ and $t_2$ were determined in order that the thermal fluctuations are smaller than the one coming from the distribution of disorder.

The first result that we present is the Log-Log plot of the magnetization versus the lattice size $L$ at the critical point (see Fig. 1). This plot exhibits a perfect scaling behavior with a very small deviation between $L = 2$ and $L = 5$. It is the manifestation of the cross-over length and justifies our previous assumption on its value. This is even more obvious on the second figure. There we plot $M(L) L^{\Delta_\sigma}$ vs. $\ln(L)$. We see a jump in the cross-over region for $2 \leq L \leq 5$. Outside this region we observe a rather good plateau.

We now compute $\Delta'_\sigma$ inside the plateau region with the parametrization of Eq. 7:

$$\Delta'_\sigma = 0.1337 \pm 0.0007 = \Delta_\sigma (1 + \delta) \quad ; \quad \delta = 0.003 \pm 0.005 . \quad (16)$$

This is to be compared with the analytical value

$$\delta = 0.01 . \quad (17)$$

At this level of precision, it is too difficult to draw any conclusion on the validity of the computed value vs. the measured one. The only obvious fact is that $\Delta'_\sigma$ is very close to $\Delta_\sigma$. With the statistics that we present here, it is not possible to differentiate these two exponents.

For the specific heat, the situation is different and we see clearly the influence of disorder. For the pure 3-state Potts model

$$C(L) \simeq L^{2 - 2\Delta_\epsilon} \simeq L^{0.4} , \quad (18)$$
while for the model with disorder we obtain a negative exponent (see Fig. 3). The only possible parametrization of the curve is the one with a cusp, i.e.

\[ C(L) = A + BL^{-x}. \]  (19)

With such a parametrization, we obtain

\[ x = 0.13 \pm 0.04 \]  (20)

Using the relation \( x = 2\Delta'_\varepsilon - 2 \) gives

\[ \Delta'_\varepsilon = 1.065 \pm 0.02. \]  (21)

This result should be compared with the analytical values

\[ \Delta'_\varepsilon = 1.02 + O(\varepsilon^3) \quad \text{and} \quad \Delta''_\varepsilon = 1 + O(\varepsilon^3). \]  (22)

In conclusion, we have measured the critical exponent of the magnetization of the 3 state Potts model in the presence of disorder. We have shown that its value does not differ significantly from the pure case value. Comparisons with analytical computations of this exponent \cite{13, 14} are beyond the level of precision of the present simulation. For the critical exponent associated to the specific heat (or the energy-energy correlation function), the numerical value obtained in Eq. is very close to the analytical one \cite{12, 13, 14, 15}. The result given by the numerical simulation seems to agree better with the analytical value obtained with a replica symmetry ansatz \( (\Delta'_\varepsilon = 1.02 + O(\varepsilon^3)) \) than the one given by the replica symmetry breaking ansatz \( \Delta''_\varepsilon = 1 + O(\varepsilon^3) \).

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**Note**

As this work was being completed I became aware of a related work by J.-K. Kim which deals with a similar problem. In particular, Kim also measured the exponent of the magnetisation (the parameter \( \eta \) of the magnetic susceptibility in fact) for the case with weak disorder and finds it unchanged from the case without disorder.
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Figure Captation

Fig. 1  \( \ln \text{Mag}(L) \) vs. \( \ln L \) for 3-state Potts model with disorder 1/10

Fig. 2  \( \text{Mag}(L) \times L^{2/15} \) vs. \( \ln L \) for 3-state Potts model with disorder 1/10

Fig. 3  \( C(L) \) vs. \( L \) for 3-state Potts model with disorder 1/10, with a best fit.
Figure 1. ln(Mag(L)) vs. ln(L), 3-state Potts model with disorder 1/10.
Figure 2. Magetisation for 3-state Potts model with disorder 1/10.
Cs(L) vs. L, 3-state Potts model with disorder 1/10

Figure 3.

Cs(L) = 0.598481 - 0.523881*(L**(-0.132875))