Explicit justification stit logic: a completeness result

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Abstract. We consider the explicit fragment of the basic justification stit logic introduced in [9]. We define a Hilbert-style axiomatic system for this logic and show that this system is strongly complete relative to the intended semantics.

stit logic, justification logic, completeness, compactness

1 Introduction

Basic justification stit (or jstit, for short) logic was introduced in [9] as an environment for analysis of doxastic actions related to proving activity within a somewhat idealized community of agents. This logic combines expressive means of stit logic by N. Belnap et al. [4] with those of justification logic by S. Artemov et al. [2]. The two latter logics provide for the pure agency side and the pure proof ontology side of the proving activity, respectively, so that it is assumed that doing something is in effect seeing to it that something is the case, and that every actual proof can be understood as a realization of some proof polynomial from justification logic. The only missing element in this picture is then the link between the two components, i.e. how agents can see to it that a proof is realized. Such a realization may come in different forms, researchers may, for example, exchange emails or put the proofs they have found on a common whiteboard. In basic jstit logic this rather common situation is idealized in that only public proving activity of agents is taken into account. In other words, taking up the whiteboard metaphor, the agents in question can only participate in proving activity by putting their proofs on the common whiteboard for everyone to see, and not by sending one another private messages or scribbling in their private notebooks.

In order to represent the proving activity of agents under the above-described set of assumptions, basic jstit logic features a set of four new modalities which were called in [9] proving modalities. Proving modalities capture four different modes in which one can speak about proving activity of an agent. The idea is that one gets a right classification of such modes if one intersects the distinction between agentive and factual (aka moment-determinate) events developed in stit logic with the distinction between
explicit and implicit modes of knowledge which is central to justification logic. The first distinction, when applied to proofs, corresponds to a well-known philosophical discussion of proofs-as-objects vs proofs-as-acts. One refers to a proof-as-act when one says that agent $j$ proves some proposition $A$, but one refers to a proof-as-object when saying that $A$ was proved. While doing that, one can either simply say that $A$ was proved, or add that $A$ was proved by some proof $t$; and the difference between these two modes of speaking is exactly the difference between implicit and explicit reference to proofs. The resulting classification of proving modalities looks then as follows:

|                | Agentive | Moment-determinate |
|----------------|----------|-------------------|
| **Explicit**   | $j$ proves $A$ by $t$ | $A$ has been proven by $t$ |
| $Prove(j, t, A)$ | $Proven(t, A)$ |
| **Implicit**   | $j$ proves $A$ | $A$ has been proven |
| $Prove(j, A)$  | $Proven(A)$ |

In [9] the semantics of these modalities was presented and informally motivated in some detail. However, axiomatizing basic jsit logic proved to be an uphill task. The first partial success was achieved in [7] where an axiomatization of implicit fragment of basic jsit logic (obtained by omitting the two explicit proving modalities of the above table) was presented and shown to be complete. In the present paper we focus on this omitted set of proving modalities and look into what happens when one omits the implicit proving modalities from the basic jsit logic and keeps the explicit ones. The resulting system, which will be called here the explicit jsit logic, complements, in a sense, the implicit jsit logic to the full set of proving modalities. In this paper we will axiomatize this logic and thus complement the main result of [7] with a similar result for the explicit proving modalities.

The layout of the rest of the paper is then as follows. In Section 2 we define the language and the semantics of the logic at hand. We then connect explicit jsit logic to JA-STIT, the stit logic of justification announcements. JA-STIT was the subject of some of our earlier papers (see [8]) and is a proper extension of explicit jsit logic in terms of expressive power. The proof of the main result of the current paper displays some very close parallels to the completeness proof for JA-STIT given in [8], to the point that we can re-use a dozen of technical lemmas proven in [8] without altering one letter in their respective proofs. We also give some facts about expressivity of explicit jsit logic, namely, we mention the failure finite model properties and show that this logic, just as JA-STIT, can tell the difference between the general class of its intended models and the subclass of models based on discrete time.

The strongly complete axiomatization for explicit jsit logic is then presented in Section 3. We immediately show this system to be sound w.r.t. the semantics introduced in Section 2 and we end the section with a proof for a number of theorems of the system and a note on an alternative axiomatization of the same set of theorems.

Section 4 then contains the bulk of technical work necessary for the completeness theorem. It gives a stepwise construction and adequacy check for all the numerous components of the canonical model and ends with a proof of a truth lemma. This section displays the highest degree of dependency on lemmas proved in [8] and we give a table connecting the lemmas of this section to the results of [8]. Section 5 then wraps up, giving a concise proof of the completeness result, drawing some conclusions and drafting directions for future work.
Completeness for explicit jstit logic

In what follows we will be assuming, due to space limitations, a basic acquaintance with both stit logic and justification logic. We recommend to peruse [5, Ch. 2] for a quick introduction to the basics of stit logic, and [1] for the same w.r.t. justification logic.

2 Basic definitions and notation

2.1 Language

We fix some preliminaries. First we choose a finite set $A_g$ disjoint from all the other sets to be defined below. Individual agents from this set will be denoted by letters $i$ and $j$. Then we fix countably infinite sets $P Var$ of proof variables (denoted by $x, y, z$) and $P Const$ of proof constants (denoted by $c, d$). When needed, subscripts and superscripts will be used with the above notations or any other notations to be introduced in this paper. Set $Pol$ of proof polynomials is then defined by the following BNF:

\[ t ::= x \mid c \mid s + t \mid s \times t \mid !t, \]

with $x \in P Var$, $c \in P Const$, and $s, t$ ranging over elements of $Pol$. In the above definition $+$ stands for the sum of proofs, $\times$ denotes application of its left argument to the right one, and $!$ denotes the so-called proof-checker, so that $!t$ checks the correctness of proof $t$.

In order to define the set $Form^{Ag}$ of formulas we fix a countably infinite set $Var$ of propositional variables to be denoted by letters $p, q$. Formulas themselves will be denoted by letters $A, B, C, D$, and the definition of $Form^{Ag}$ is supplied by the following BNF:

\[ A ::= p \mid A \land B \mid \neg A \mid [j]A \mid \Box A \mid t: A \mid KA \mid Prove(j, t, A) \mid Proven(t, A), \]

with $p \in Var$, $j \in A_g$ and $t \in Pol$.

It is clear from the above definition of $Form^{Ag}$ that we are considering a version of modal propositional language. As for the informal interpretations of modalities, $[j]A$ is the so-called cstit action modality and $\Box$ is the historical necessity modality, both modalities are borrowed from stit logic. The next two modalities, $KA$ and $t:A$, come from justification logic and the latter is interpreted as “$t$ proves $A$”, whereas the former is the strong epistemic modality “$A$ is known”.

We assume $\Diamond$ and $\langle j \rangle$ as notations for the dual modalities of $\Box$ and $[j]$, respectively. As usual, $\omega$ will denote the set of natural numbers including 0, ordered in the natural way.

2.2 Semantics

For the language at hand, we assume the following semantics. A justification stit (jstit, for short) model for a given agent community $A_g$ is a structure of the form:

\[ \mathcal{M} = \langle Tree, \preceq, Choice, Act, R, R_e, \mathcal{E}, V \rangle \]

such that:
1. Tree is a non-empty set. Elements of Tree are called moments.

2. \( \preceq \) is a partial order on Tree for which a temporal interpretation is assumed. We will also freely use notations like \( \succeq, \prec, \) and \( \triangleright \) to denote the inverse relation and the irreflexive companions.\footnote{A more common notation \( \leq \) is not convenient for us since we also widely use \( \leq \) in this paper to denote the natural order relation between elements of \( \omega \).}

3. Hist is a set of maximal chains in Tree w.r.t. \( \preceq \). Since Hist is completely determined by Tree and \( \preceq \), it is not included into the structure of model as a separate component. Elements of Hist are called histories. The set of histories containing a given moment \( m \) will be denoted \( H_m \). The following set:

\[ MH(\mathcal{M}) = \{(m, h) \mid m \in Tree, h \in H_m\}, \]

called the set of moment-history pairs, will be used to evaluate formulas of the above language.

4. Choice is a function mapping \( Tree \times Ag \) into \( 2^{Hist} \) in such a way that for any given \( j \in Ag \) and \( m \in Tree \) we have as Choice\(_m\)(\( m, j \)) (to be denoted as Choice\(_m^m\) below) a partition of \( H_m \). For a given \( h \in H_m \) we will denote by Choice\(_j^m\)(\( h \)) the element of partition Choice\(_j^m\) containing \( h \).

5. Act is a function mapping \( MH(\mathcal{M}) \) into \( 2^{Pol} \).

6. \( R \) and \( R_e \) are two pre-order on Tree giving two versions of epistemic accessibility relation. They are assumed to be connected by inclusion \( R \subseteq R_e \).

7. \( E \) is a function mapping \( Tree \times Pol \) into \( 2^{Form^Ag} \) called admissible evidence function.

8. \( V \) is an evaluation function, mapping the set \( Var \) into \( 2^{MH(\mathcal{M})} \).

A structure of the above described type is admitted as a jstit model iff it satisfies the following list of additional constraints. In order to facilitate their exposition, we introduce a couple of useful notations first. For a given \( m \in Tree \) and any given \( h, g \in H_m \) we stipulate that:

\[ Act_m := \bigcap_{h \in H_m} Act(m, h); \quad Act_{(m, h, j)} := \bigcap_{g \in Choice_j^m(h)} Act(m, g); \]

and:

\[ h \approx_m g \iff (\exists m' \triangleright m)(h, g \in H_{m'}). \]

Whenever we have \( h \approx_m g \), we say that \( h \) and \( g \) are undivided at \( m \).

We then demand satisfaction of the following constraints by the jstit models:

1. Historical connection:

\[ (\forall m, m_1 \in Tree)(\exists m_2 \in Tree)(m_2 \leq m \land m_2 \leq m_1). \]
2. No backward branching:

\[(\forall m, m_1, m_2 \in \text{Tree})(m_1 \leq m \land m_2 \leq m) \Rightarrow (m_1 \leq m_2 \lor m_2 \leq m_1).\]

3. No choice between undivided histories:

\[(\forall m \in \text{Tree})(\forall h, h' \in H_m)(h \approx_m h' \Rightarrow \text{Choice}^n_j(h) = \text{Choice}^n_j(h'))\]

for every \(j \in Ag\).

4. Independence of agents:

\[(\forall m \in \text{Tree})(\forall f : Ag \rightarrow 2^{H_m})(\forall j \in Ag)(f(j) \in \text{Choice}^n_j) \Rightarrow \bigcap_{j \in Ag} f(j) \neq \emptyset).\]

5. Monotonicity of evidence:

\[(\forall t \in \text{Pol})(\forall m, m' \in \text{Tree})(R_e(m, m') \Rightarrow \mathcal{E}(m, t) \subseteq \mathcal{E}(m', t)).\]

6. Evidence closure properties. For arbitrary \(m \in \text{Tree}, s, t \in \text{Pol}\) and \(A, B \in \text{Form}^Ag\) it is assumed that:

(a) \(A \rightarrow B \in \mathcal{E}(m, s) \land A \in \mathcal{E}(m, t) \Rightarrow B \in \mathcal{E}(m, s \times t)\);

(b) \(\mathcal{E}(m, s) \cup \mathcal{E}(m, t) \subseteq \mathcal{E}(m, s + t)\).

(c) \(A \in \mathcal{E}(m, t) \Rightarrow t : A \in \mathcal{E}(m, !t)\);

7. Expansion of presented proofs:

\[(\forall m, m' \in \text{Tree})(m' \prec m \Rightarrow \forall h \in H_m(\text{Act}(m', h) \subseteq \text{Act}(m, h))).\]

8. No new proofs guaranteed:

\[(\forall m \in \text{Tree})(\text{Act}_m \subseteq \bigcup_{m' < m, h \in H_m} \text{Act}(m', h))).\]

9. Presenting a new proof makes histories divide:

\[(\forall m \in \text{Tree})(\forall h, h' \in H_m)(h \approx_m h' \Rightarrow (\text{Act}(m, h) = \text{Act}(m, h'))).\]

10. Future always matters:

\[\triangle U \subseteq R.\]

11. Presented proofs are epistemically transparent:

\[(\forall m, m' \in \text{Tree})(R_e(m, m') \Rightarrow (\text{Act}_m \subseteq \text{Act}_{m'})).\]
We offer some intuitive explanation for the above-defined notion of jstit model. Due to space limitations, we only explain the intuitions behind jstit models very briefly, and we urge the reader to consult [9, Section 3] for a more comprehensive explanations, whenever needed.

The components like Tree, $\leq$, Choice and V are inherited from stit logic, whereas $R, R_e$ and $E$ come from justification logic. The only new component is Act which represents the above-mentioned common pool of proofs demonstrated to the community at any given moment under a given history. When interpreting Act, we invoke the classical stit distinction between dynamic (agentive) and static (moment-determinate) entities, assuming that the presence of a given proof polynomial $t$ on the community whiteboard only becomes an accomplished fact at $m$ when $t$ is present in $\text{Act}(m,h)$ for every $h \in H_m$. On the other hand, if $t$ is in $\text{Act}(m,h)$ only for some $h \in H_m$ this means that $t$ is rather in a dynamic state of being presented, rather than being present, to the community.

The numbered list of semantical constraints above then just builds on these intuitions. Constraints 1–4 are borrowed from stit logic, constraints 5 and 6 are inherited from justification logic. Constraint 7 just says that nothing gets erased from the whiteboard, constraint 8 says a new proof cannot spring into existence as a static (i.e. moment-determinate) feature of the environment out of nothing, but rather has to come as a result (or a by-product) of a previous activity. Constraint 9 is just a corollary to constraint 3 in the richer environment of jstit models, constraint 10 says that the possible future of the given moment is always epistemically relevant in this moment, and constraint 11 says that the community knows everything that has firmly made its way onto the whiteboard.

Right away we establish some elementary facts about jstit models to be used in what follows:

**Lemma 1.** Let $\mathcal{M} = \langle \text{Tree}, \leq, \text{Choice}, \text{Act}, R, R_e, E, V \rangle$ be a jstit model. Then:

1. $\forall m \in \text{Tree} \forall h \in H_m (\{m_1 \in \text{Tree} \mid m_1 \leq m\} \subseteq h)$;
2. $\forall m \in \text{Tree} \forall h, g \in H_m (h \neq g \Rightarrow (\exists m'' \triangleright m)(h \in H_{m''}))$;
3. $\forall m, m' \in \text{Tree} (m \leq m' \Rightarrow H_{m'} \subseteq H_m)$.

**Proof.** (Part 1). We clearly have $m \in h$. Consider an arbitrary $m_1 \leq m$. Then $h \cup \{m_1\}$ must be an $\leq$-chain. Indeed, if $m' \in h$ then either $m \leq m'$ or $m' \leq m$. In the former case we get $m_1 \leq m'$ by transitivity of $\leq$, in the latter case we get $m_1 \leq m' \vee m' \leq m_1$ by the absence of backward branching. But since $h$ is a maximal chain, this means that we must have $m_1 \in h$.

(Part 2). To obtain a contradiction, assume that $h, g \in H_m$ are different, but we have:

$$\forall m'' \triangleright m (h \notin H_{m'}).$$

(1)

Given that every two elements of $h$ must be $\leq$-comparable, this means that $h \subseteq \{m_1 \in \text{Tree} \mid m_1 \leq m\}$ and, by Part 1, that $h = \{m_1 \in \text{Tree} \mid m_1 \leq m\}$. Note that Part 1 also entails that $g \supseteq \{m_1 \in \text{Tree} \mid m_1 \leq m\}$, so that in this case we must have $g \supseteq h$. We can have neither $g \supset h$, since this would violate the maximality of $h$, nor $g = h$, since this is in contradiction with our assumption. Therefore, (1) must be false.

(Part 3). Immediately by the absence of backward branching. \qed
Completeness for explicit jstit logic

For the members of $\text{Form}^A$, we will assume the following inductively defined satisfaction relation. For every jstit model $\mathcal{M} = \langle \text{Tree}, \leq, \text{Choice}, \text{Act}, R, R_e, E, V \rangle$ and for every $(m, h) \in MH(\mathcal{M})$ we stipulate that:

\[ \mathcal{M}, m, h \models p \iff (m, h) \in V(p); \]
\[ \mathcal{M}, m, h \models [j]A \iff (\forall h' \in \text{Choice}^j(h))(\mathcal{M}, m, h' \models A); \]
\[ \mathcal{M}, m, h \models \Box A \iff (\forall h' \in H_m)(\mathcal{M}, m, h' \models A); \]
\[ \mathcal{M}, m, h \models KA \iff \forall m'\forall h'(R(m, m') \& h' \in H_m) \Rightarrow \mathcal{M}, m', h' \models A; \]
\[ \mathcal{M}, m, h \models t: A \iff A \in E(m, t) \& (\forall m' \in \text{Tree})(\forall h' \in H_m')(R_e(m, m') \Rightarrow \mathcal{M}, m', h' \models A); \]
\[ \mathcal{M}, m, h \models \text{Prove}(j, t, A) \iff t \in \text{Act}(m, h, j) \& \mathcal{M}, m, h \models t: A \& t \notin \text{Act}_m; \]
\[ \mathcal{M}, m, h \models \text{Proven}(t, A) \iff t \in \text{Act}_m \& \mathcal{M}, m, h \models t: A. \]

In the above clauses we assume that $p \in \text{Var}$; we also assume standard clauses for Boolean connectives.

Explicit jstit logic is closely connected to JA-STIT, the stit logic of justification announcement. In JA-STIT the two proving modalities of explicit jstit logic are replaced with a single modality $E_t$ for $t \in \text{Pol}$, with the following semantics:

\[ \mathcal{M}, m, h \models E_t \iff t \in \text{Act}(m, h). \]

Since JA-STIT is interpreted over the same class of models as basic jstit logic, it turns out that one can retrieve the explicit proving modalities in JA-STIT using the following definitions:

\[ \text{Prove}(j, t, A) = _\text{af} [j]E_t \land \Diamond \neg E_t \land t: A; \quad \text{Proven}(t, A) = _\text{af} \Box E_t \land t: A. \]

On the other hand, it is easy to show that $E_t$ cannot be defined in explicit jstit logic, so that JA-STIT is its proper extension. Despite the difference in expressive powers, it was possible to borrow many constructions and lemmas for the main result of this paper directly from the completeness proof for JA-STIT without any modifications at all.

We observe that even though all elements of $\text{Form}^A$ are interpreted over moment-history pairs, for some of them their evaluations are obviously independent from the history component:

**Lemma 2.** For every agent community $A$, every $A \in \text{Form}^A$ and every $t \in \text{Pol}$, all of the formulas $\Box A$, $KA, t: A$ and $\text{Proven}(t, A)$ are moment-determinate, that is to say, if $\alpha \in \{ \Box A, KA, t: A, \text{Proven}(A) \}$, then for an arbitrary jstit model $\mathcal{M} = \langle \text{Tree}, \leq, \text{Choice}, \text{Act}, R, R_e, E, V \rangle$, arbitrary $m \in \text{Tree}$ and $h' \in H_m$ we have:

\[ \mathcal{M}, m, h \models \alpha \iff \mathcal{M}, m, h' \models \alpha. \]

Also, Boolean combinations of these formulas are moment-determinate.

**Proof.** For $\alpha \in \{ \Box A, KA, t: A \}$ it suffices to note that their respective satisfaction conditions at a given $(m, h) \in MH(\mathcal{M})$ in a given $\mathcal{M}$ have no free occurrences of $h$. When we turn, further, to the corresponding condition for $\text{Proven}(t, A)$, the only free occurrence of $h$ will be within the context $\mathcal{M}, m, h \models t: A$ which was shown to be moment-determinate.

Of course, Boolean combinations of moment-determinate formulas must be moment-determinate, too. \qed
It follows from Lemma 2 that we might as well omit the histories when discussing satisfaction of such formulas and write $\mathcal{M}, m \models KA$ instead of $\mathcal{M}, m, h \models KA$, etc.

One can in principle simplify the above semantics by introducing the additional constraint that $R_e \subseteq R$. This leads to a collapse of the two epistemic accessibility relation into one. Therefore, we will call jstit models satisfying $R_e \subseteq R$ unirelational jstit models. It is known that such a simplification in the context of pure justification logic does not affect the set of theorems (see, e.g. [6] and [2, Comment 6.5]), and we have shown in [8] that this is also the case for JA-STIT. The main result of this paper will show that also in this respect the explicit jstit logic follows the suit. In fact, the canonical model to be constructed in our completeness proof below is unirelational. In view of this, we offer some comments as to the simplifications of semantics available in the unirelational setting.

We observe that one can equivalently define a unirelational jstit model as a structure $\mathcal{M} = \langle \text{Tree}, \subseteq, \text{Choice}, \text{Act}, R, E, V \rangle$ satisfying all the constraints for the jstit models, except that in the numbered constraints one substitutes $R$ for $R_e$. Also, in the context of unirelational jstit models, it is possible to simplify the satisfaction clause for $t:A$ as follows:

$$\mathcal{M}, m, h \models t:A \iff A \in E(m, t) \& \mathcal{M}, m, h \models KA.$$

### 2.3 Concluding remarks

Before we start with the task of axiomatizing the explicit jstit logic, we briefly mention some facts about its expressive powers which are relevant to our chosen format of completeness proof. Firstly, it is worth noting that under the presented semantics some satisfiable formulas cannot be satisfied over finite models, or even over infinite models where all histories are finite. The argument for this is the same as in implicit fragment of basic jstit logic, for which this claim was proved in [7] using $K(\Box p \land \Box \neg p)$ as an example of a formula which is satisfiable over jstit models but not over jstit models with finite histories. This already rules out some methods of proving completeness like filtration method.

Second, we mention that, just like JA-STIT, explicit jstit logic has enough expressive power to tell the difference between the general class of jstit models and its subclass of jstit models based on discrete time. To be more precise, we define that a jstit model $\mathcal{M}$ is based on discrete time if every chain in $\text{Hist} (\mathcal{M})$ is isomorphic to an initial segment of $\omega$, the set of natural numbers. Then it can be shown that:

**Proposition 1.** Let $Ag$ be an agent community. The subset of $\text{Form}^{Ag}$-validities over the class of (unirelational) jstit models for $Ag$ is a proper subset of the set of $\text{Form}^{Ag}$-validities over the class of (unirelational) jstit models for the same community based on discrete time.

**Proof.** We fix an arbitrary agent community $Ag$. We clearly have the subset relation. As for the properness part, consider the following formula:

$$A := K(\neg\text{Proven}(x, p) \lor \text{Proven}(y, q)) \rightarrow (\neg\text{Prove}(j, x, p) \lor$$

$$\lor (y:q \rightarrow (\text{Proven}(y, q) \lor \text{Prove}(j, y, q)))),$$

with $x, y \in PVar$, $p \in Var$, and $j \in Ag$. We show that $A$ is not valid over the class of all unirelational jstit models (hence not valid over the class of all jstit models either).
Consider the following unirelational model $\mathcal{M} = \langle Tree, \preceq, Choice, Act, R, E, V \rangle$ for the community $Ag$:

- $Tree = \{a, -1\} \cup \{r \in \mathbb{R} \mid 0 \leq r < 1\}$;
- $\preceq = \{(0, a), (-1, a), (a, a)\} \cup \{(r, r') \mid r, r' \in \mathbb{R} \cap Tree, r \leq r'\}$;
- $R = \preceq$;
- $\mathcal{E}(m, t) = Form^Ag$, for all $m \in Tree$ and $t \in Pol$.
- $V(p) = V(q) = MH(\mathcal{M})$, $V(p') = \emptyset$ for all $p' \in Var \setminus \{p, q\}$.

It is straightforward to see that the above-defined components of $\mathcal{M}$ satisfy all the constraints imposed on normal jstit models except possibly those involving $Choice$ and $Act$. Note that, among other things, we will have, under the above settings, that:

$$\mathcal{M}, m \models x:p \wedge y:q$$

(2)

for all $m \in Tree$. Before we go on and define the remaining components, let us pause a bit and reflect on the structure of histories in the model $\mathcal{M}$ that is being defined. We only have two histories in it, one is $h_1 = \{-1, 0, a\}$ and the other is $h_2 = \{-1\} \cup \{r \in \mathbb{R} \mid 0 \leq r < 1\}$. So we define:

$$Choice_i^m = \begin{cases} 
H_m, & \text{if } i \neq j \text{ or } m \neq 0; \\
\{h_1, \{h_2\}\}, & \text{if } i = j \text{ and } m = 0.
\end{cases}$$

$$Act(m, h) = \begin{cases} 
\{x, y\}, & \text{if } m \in \mathbb{R} \text{ and } m > 0; \\
\{x\}, & \text{if } m = 0 \text{ and } h = h_2; \\
\emptyset, & \text{otherwise}.
\end{cases}$$

Again, most of the constraints on jstit models are now easily seen to be satisfied. The no new proofs guaranteed constraint is perhaps less straightforward, so we consider it in some detail. We have, on the one hand, $Act_m = \emptyset$, whenever $m \in \{-1, 0, a\}$ so neither of these moments can falsify the constraint. The only remaining option is that $m \in \{r \in \mathbb{R} \mid 0 < r < 1\}$, say $m = r$. But then the only history passing through $r$ is $h_2$ and we have, on the other hand, $\frac{a}{\mathcal{M}} \in Tree$, $\frac{a}{\mathcal{M}} < r$, and $Act(\frac{a}{\mathcal{M}}, h_2) = Act(r, h_2) = Act_r$ so that the no new proofs guaranteed constraint is again verified.

Now, consider $0 \in Tree$. The set of its epistemic alternatives is $Tree \setminus \{-1\}$. We have all of the following: $Choice_i^0(h_2) = \{h_2\}$, $x \in Act(0, h_2)$, $H_0 = \{h_1, h_2\}$, $x \notin Act(0, h_1)$, and $y \notin Act(0, h_2)$. In virtue of all this and by (2), we obtain that:

$$\mathcal{M}, 0, h_2 \models Prove(j, x, p) \wedge y:q \wedge \neg Prove(j, y, q) \wedge \neg Proven(y, q).$$

(3)

On the other hand, if $m \in Tree$ and $h \in H_m$, then either $m \in \{r \in \mathbb{R} \mid 0 < r < 1\}$ or $m$ is outside of this set. In the former case we have $H_m = \{h_2\}$, whence $Act_m = \{x, y\}$ so that, by (2), we get:

$$\mathcal{M}, m, h \models Proven(y, q).$$

\footnote{It is also easy to see that $\mathcal{M}$ is CS-normal for any possible constant specification $CS$ as defined in the next section. Therefore, Proposition I persists when one restricts attention to a class of jstit models normal relative to a constant specification.}
for all \( h \in H_m \). In the latter case we have \( \text{Act}_m = \emptyset \), which means that we must also have:

\[
\mathcal{M}, m, h \models \neg \text{Proven}(x, p),
\]

for all \( h \in H_m \). So, in any case, the formula \( \neg \text{Proven}(x, p) \lor \text{Proven}(y, q) \) gets to be satisfied throughout all of the moment-history pairs in \( \mathcal{M} \), which further means that:

\[
\mathcal{M}, 0, h_2 \models K(\neg \text{Prove}(j, x, p) \lor \text{Prove}(j, y, q))
\]

is satisfied. Adding the latter with (3), we see that (0, \( h_2 \)) falsifies \( A \) in \( \mathcal{M} \).

On the other hand, \( A \) is valid in the class of jstit models based on discrete time (hence also over unirelational jstit models based on discrete time). In order to show this, we will assume its invalidity and obtain a contradiction. Indeed, let \( \mathcal{M} = \langle \text{Tree}, \preceq, \text{Choice}, \text{Act}, R, R_e, E, V \rangle \) be a jstit model based on discrete time such that \( \mathcal{M}, m, h \not\models A \). Then we will have both

\[
\mathcal{M}, m, h \models K(\neg \text{Proven}(x, p) \lor \text{Proven}(y, q)),
\]

and

\[
\mathcal{M}, m, h \models \text{Prove}(j, x, p) \land y : q \land \neg \text{Prove}(j, y, q) \land \neg \text{Proven}(y, q).
\]

Note that the failure of \( \text{Proven}(y, q) \) combined with satisfaction of \( y : q \) shows that we cannot have \( y \in \text{Act}_m \). On the other hand, the failure of \( \text{Prove}(j, y, q) \) at \( (m, h) \) leaves us with the following options:

\[
(\mathcal{M}, m, h \not\models y : q) \lor y \in \text{Act}_m \lor (\exists g \in \text{Choice}_j^m(h))(y \not\in \text{Act}(m, g)).
\]

Thus we know that for some \( h' \in \text{Choice}_j^m(h) \) we will have \( y \not\in \text{Act}(m, h') \). Also, note that for any such \( h' \) we will have \( \text{Choice}_j^m(h') = \text{Choice}_j^m(h) \). Adding this up with the satisfaction of \( \text{Prove}(j, x, p) \) at \( (m, h) \), we get that one can choose an \( h' \in \text{Choice}_j^m(h) \) in such a way that the following holds:

\[
y \not\in \text{Act}(m, h') \land (\mathcal{M}, m, h' \models \text{Prove}(j, x, p) \land y : q).
\]

By Lemma 2 and (4), we also know that for such \( h' \) we will have:

\[
\mathcal{M}, m, h' \models K(\neg \text{Proven}(x, p) \lor \text{Proven}(y, q)),
\]

Next, we observe that since \( (m, h') \) satisfies \( \text{Prove}(j, x, p) \), we know that \( x \in \text{Act}(m, h') \) and also that there is a \( g \in H_m \) such that \( x \notin \text{Act}(m, g) \). This shows that we must have \( h' \neq g \), and, by Lemma 2, this means \( m \) must have \( < \)-successors along \( h' \). Since \( \mathcal{M} \) is based on discrete time, consider embedding \( f \) of \( h' \) into an initial segment of \( \omega \). Suppose that \( f(m) = n \). We have established that \( m \) is not the \( \leq \)-last moment along \( h' \), so there must be an \( m' \in h' \) such that \( f(m') = n + 1 \). By the embedding properties of \( f \), this means that \( m < m' \) and for no \( m'' \in \text{Tree} \) it is true that \( m < m'' < m' \). By the future always matters constraint, we know that \( R(m, m') \), therefore, by (7) we must have:

\[
\mathcal{M}, m', h' \models \neg \text{Proven}(x, p) \lor \text{Proven}(y, q).
\]

On the other hand, let \( g \in H_m \) be arbitrary. Then, by Lemma 3, \( g \in H_m \), and, moreover, \( g \equiv_m h' \). Therefore, by the presenting a new proof makes histories divide
constraint, we must have \( \text{Act}(m, g) = \text{Act}(m, h') \). By \( \theta \) we know that \( x \in \text{Act}(m, h') \), which means that also \( x \in \text{Act}(m, g) \). Since \( g \in H_{m'} \) was chosen arbitrarily, the latter means that \( x \in \bigcap_{g \in H_{m'}} (\text{Act}(m, g)) \), and, by the expansion of presented proofs constraint, \( x \in \text{Act}_{m'} \). Further, it follows from \( \theta \) that:

\[
\mathcal{M}, m, h' \models x:p.
\] (9)

Given that \( R \subseteq R_e \), we must have \( R_e(m, m') \), whence by the monotonicity of evidence and the pre-order properties of \( R_e \) we further obtain that:

\[
\mathcal{M}, m', h' \models x:p.
\] (10)

Since we know that \( x \in \text{Act}_{m'} \), \( \theta \) immediately leads to:

\[
\mathcal{M}, m', h' \models \text{Proven}(x, p).
\] (11)

Whence, in view of \( \Theta \), it follows that

\[
\mathcal{M}, m', h' \models \text{Proven}(y, q).
\] (12)

The latter means that \( y \in \text{Act}_{m'} \), and by the no new proofs guaranteed constraint, it follows that for some \( g \in H_{m'} \) and some \( m'' \in g \) such that \( m'' \preceq m' \), we must have \( y \in \text{Act}(m'', g) \). Now, if \( m'' \preceq m' \), then \( m'' \preceq m \), since \( m' \) was chosen as the immediate \( \preceq \)-successor of \( m \) along \( h' \). The latter means, by the expansion of presented proofs, that \( y \in \text{Act}(m, g) \). Since, as we have shown above, \( g \approx_m h' \), this means, by the presenting a new proof makes histories divide constraint, that \( \text{Act}(m, g) = \text{Act}(m, h') \) and, further, that \( y \in \text{Act}(m, h') \). The latter is in obvious contradiction with \( \Theta \).

The obtained contradiction shows that \( A \) is valid over the class of jstit models based on discrete time, so that it must also be valid over its unirelational subclass. □

3 Axiomatic system and soundness

Throughout this section, and the next one, we are going to let \( Aq \) serve as a constant denoting arbitrary but fixed agent community. We consider the Hilbert-style axiomatic
system Π with the following set of axiomatic schemes:

A full set of axioms for classical propositional logic (A0)

S5 axioms for □ and [j] for every j ∈ Ag (A1)

□A → [j]A for every j ∈ Ag (A2)

(◊[j1]A1 ∧ ... ∧ ◊[jn]An) → ◊([j1]A1 ∧ ... ∧ [jn]An) (A3)

(s:(A → B) → (t:A → (s × t):B)) (A4)

t:A → (t;(t:A) ∧ KA) (A5)

(s: A ∨ t:A) → (s + t):A (A6)

S4 axioms for K (A7)

KA → ◇KA (A8)

Prove(j, t, A) → (∼Proven(t, A) ∧ [j]Prove(j, t, A) ∧ ◇Prove(j, t, A) ∧ □Prove(j, t, A) ∧ t:A) (B9)

(Prove(j, t, A) ∧ t:B) → Prove(j, t, B) (B10)

Proven(t, A) → (KProven(t, A) ∧ t:A) (B11)

(Proven(t, A) ∧ t:B) → Proven(t, B) (B12)

¬Prove(j, t, A) → (j)( ∩ i∈Ag ¬Prove(i, t, A)) (B13)

The assumption is that in (A3) j1, ..., jn are pairwise different.

To this set of axiom schemes we add the following rules of inference:

From A, A → B infer B; (R1)

From A infer KA; (R2)

From KA → (∼Proven(t1, B1) ∨ ... ∨ ¬Proven(tn, Bn))

infer KA → (∩ j∈Ag ¬Prove(j, t1, B1) ∨ ... ∨ ∩ j∈Ag ¬Prove(j, tn, Bn)). (S4)

The different notation styles present in the above sets of axioms and inference rules are meant to underscore that the axioms (A0)–(A8) and rules (R1), (R2) are shared by Π with other axiomatizations for logics combining justification and stit modalities, including the axiomatization of the implicit jstit logic given in [7] and the axiomatization of JA-STIT given in [8].

A standard way to obtain extensions of Π is by adding to it constant specifications, which basically ensure that one has enough pre-assigned proofs for the axioms of this system. More precisely, a constant specification is a set CS such that:

• CS ⊆ {cn: . . . c1:A | c1, ..., cn ∈ PConst, A an instance of (A0)–(A8), (B9)–(B13);

• Whenever cn+1:cn: . . . c1:A ∈ CS, then also cn: . . . c1:A ∈ CS.

A given constant specification can be added to Π by appending the following inference rule (RCS) to its set of rules:

If cn: . . . c1:A ∈ CS, infer cn: . . . c1:A. (RCS)
The resulting axiomatic system is then called \( \Pi(\mathcal{C}S) \). Note that \( \emptyset \) is clearly one example of constant specification and that we have \( \Pi(\emptyset) = \Pi \). Whenever \( \mathcal{C}S \neq \emptyset \), the system \( \Pi(\mathcal{C}S) \) ends up proving some formulas which are not valid over the class of jstit models. Nevertheless, restriction on jstit models which comes with a commitment to a given \( \mathcal{C}S \) is relatively straightforward to describe. We say that a jstit model \( \mathcal{M} \) is \( \mathcal{C}S \)-normal iff it is true that:

\[
(\forall c \in P\text{Const})(\forall m \in \text{Tree})(\{A \mid c \in \mathcal{C}S\} \subseteq \mathcal{E}(m, c)),
\]

where \( \mathcal{E} \) is the \( \mathcal{M} \)'s admissible evidence function. Again, it is easy to see that the class of \( \emptyset \)-normal jstit models is just the whole class of jstit models so that the representation \( \Pi(\emptyset) = \Pi \) does not place any additional restrictions on the class of intended models of \( \Pi \).

For a given constant specification \( \mathcal{C}S \), we define that a proof in \( \Pi(\mathcal{C}S) \) as a finite sequence of formulas such that every formula in it is either an instance of one of the schemes (A0)–(A8), (B9)–(B12) or is obtained from earlier elements of the sequence by one of the inference rules (R1), (R2), (\( R_{\mathcal{C}S} \)), (S4). A proof is a proof of its last formula. If an \( A \in \text{Form}^{Ag} \) is provable in \( \Pi(\mathcal{C}S) \), we will write \( \vdash_{\mathcal{C}S} A \). We say that \( \Gamma \subseteq \text{Form}^{Ag} \) is \( \mathcal{C}S \)-inconsistent iff for some \( A_1, \ldots, A_n \in \Gamma \) we have \( \vdash_{\mathcal{C}S} (A_1 \land \ldots \land A_n) \rightarrow \bot \), and we say that \( \Gamma \) is \( \mathcal{C}S \)-consistent iff it is not \( \mathcal{C}S \)-inconsistent. \( \Gamma \) is \( \mathcal{C}S \)-maxiconsistent iff it is \( \mathcal{C}S \)-consistent and no \( \mathcal{C}S \)-consistent subset of \( \text{Form}^{Ag} \) properly extends \( \Gamma \).

We observe that this definition allows for the standard operations with consistent and maxiconsistent sets. Namely, every \( \mathcal{C}S \)-consistent set \( \Gamma \) can be extended to a \( \mathcal{C}S \)-maxiconsistent set \( \Delta \supseteq \Gamma \), and \( \mathcal{C}S \)-maxiconsistent sets are regular relative to the propositional connectives in that for every \( \mathcal{C}S \)-maxiconsistent set \( \Gamma \) and all \( A, B \in \text{Form}^{Ag} \) all of the following holds:

- Exactly one element of \( \{A, \neg A\} \) is in \( \Gamma \).
- \( A \lor B \in \Gamma \) iff \( (A \in \Gamma \lor B \in \Gamma) \).
- If \( A, (A \rightarrow B) \in \Gamma \), then \( B \in \Gamma \).
- \( A \land B \in \Gamma \) iff \( (A \in \Gamma \land B \in \Gamma) \).

Our goal is now to obtain, for any given constant specification \( \mathcal{C}S \), a strong completeness theorem for \( \Pi(\mathcal{C}S) \), and we start by establishing some soundness claims:

**Theorem 1.** Let \( \mathcal{C}S \) be an arbitrary constant specification and let \( A \in \text{Form}^{Ag} \) be such that \( \vdash_{\mathcal{C}S} A \). Then \( A \) is valid over the class of \( \mathcal{C}S \)-normal jstit models.

**Proof.** Given the above notion of proof, it is sufficient to show that every instance of (A0)–(A3), (B9)–(B12) is valid over the class of \( \mathcal{C}S \)-normal jstit models and that every application of rules (R1), (R2), (\( R_{\mathcal{C}S} \)), and (S4) to formulas which are valid over the class of \( \mathcal{C}S \)-normal jstit models yields a formula which is valid over the class of \( \mathcal{C}S \)-normal jstit models.

First, note that if \( \mathcal{M} = (\text{Tree}, \preceq, \text{Choice}, \text{Act}, R, R_c, \mathcal{E}, V) \) is a normal jstit model, then \( (\text{Tree}, \preceq, \text{Choice}, V) \) is a model of stit logic. Therefore, axioms (A0)–(A3), which were copy-pasted from the standard axiomatization of dstit logic\(^3\), must be

\(^3\)See, e.g. [4, Ch. 17], although \( \Pi \) uses a simpler format closer to that given in [3, Section 2.3].
valid. Second, note that if $\mathcal{M} = \langle \text{Tree}, \preceq, \text{Choice}, \text{Act}, R, R_e, \mathcal{E}, V \rangle$ is a normal jstit model, then $\mathcal{M} = \langle \text{Tree}, R, R_e, \mathcal{E}, V \rangle$ is what is called in [2] Section 6] a justification model. This means that also all of the $\{A4\} - \{A7\}$ must be valid, given that all of them were borrowed from the standard axiomatization of justification logic. The validity of other axioms will be motivated below in some detail. In what follows, $\mathcal{M} = \langle \text{Tree}, \preceq, \text{Choice}, \text{Act}, R, R_e, \mathcal{E}, V \rangle$ will always stand for an arbitrary CS-normal jstit model, and $(m, h)$ for an arbitrary element of $MH(\mathcal{M})$.

As for $\{A8\}$, assume for reductio that $\mathcal{M}, m, h \models KA \land K\lnot A$. Then (using Lemma 2 to omit the histories) $\mathcal{M}, m \models KA$ and also $\mathcal{M}, m \models K\lnot A$. By reflexivity of $R$, it follows that $\lnot A$ will be satisfied at $m$ in $\mathcal{M}$. The latter means that, for some $h' \in H_m$, $A$ must fail at $(m, h')$ and therefore, again by reflexivity of $R$, $KA$ must fail at $m$ in $\mathcal{M}$, a contradiction.

Consider $\{B9\}$ and assume that $\mathcal{M}, m, h \models \text{Prove}(j, t, A)$. Then $t \notin \text{Act}_m$, which immediately implies that:

$$\mathcal{M}, m, h \not\models \text{Proven}(t, A).$$

Next, we must have, just by the satisfaction of $\text{Prove}(j, t, A)$, that:

$$\mathcal{M}, m \models t : A. \quad (14)$$

Further, note that for every $g \in \text{Choice}_m^j(h)$ we will have $\text{Choice}_j^m(g) = \text{Choice}_j^m(h)$, so that for every such $g$ we will have $t \in \text{Act}_{(m,g,j)}$. Adding this up with (14) and the fact that $t \notin \text{Act}_m$, we get that $\text{Prove}(j, t, A)$ is satisfied at $(m, g)$ for every $g \in \text{Choice}_j^m(h)$, or, in other words, that we have:

$$\mathcal{M}, m, h \models [j] \text{Prove}(j, t, A).$$

Finally, given $t \notin \text{Act}_m$, consider $h' \in H_m$ such that $t \notin \text{Act}(m, h')$. Given that $h' \in \text{Choice}_j^m(h')$, we know that $t \notin \text{Act}_{(m,h',j)}$, which means that $\text{Prove}(j, t, A)$ fails at $(m, h')$ so that, in view of $h' \in H_m$, we get:

$$\mathcal{M}, m, h \models \lnot \Box \text{Prove}(j, t, A).$$

Summing up (13)–(16), we see that $\{B9\}$ is satisfied at $(m, h)$.

As for $\{B10\}$, assume that $\text{Prove}(j, t, A) \land t : B$ is satisfied at $(m, h)$. By the satisfaction of the first conjunct we get that $t \in \text{Act}_{(m,h,j)}k t \notin \text{Act}_m$, which, together with the satisfaction of $t : B$, yields that $\mathcal{M}, m, h \models \text{Prove}(j, t, B)$.

Next we consider $\{B11\}$. Assuming that $\mathcal{M}, m, h \models \text{Proven}(t, A)$, we immediately get that:

$$t \in \text{Act}_m \quad (17)$$

and that:

$$\mathcal{M}, m \models t : A. \quad (18)$$

Assume that $m' \in \text{Tree}$ is such that $R(m, m')$. Then we must have:

$$\mathcal{M}, m' \models t : A \quad (19)$$

\[\text{The format for the variable assignment } V \text{ is slightly different, but this is of no consequence for the present setting.}\]
by (17) and \( R \subseteq R_e \), and we must also have:

\[ t \in \text{Act}_{m'} \]  

(20)

by the epistemic transparency of presented proofs. Thus we will have \( M, m' \models \text{Proven}(t, A) \) for an arbitrary \( R \)-successor \( m' \) of \( m \), which means, by definition, that we will have:

\[ M, m \models K \text{Proven}(t, A). \]  

(21)

Taken together, (18) and (21) show that (B11) is satisfied at \((m, h)\).

As for (B12), assume that \( \text{Proven}(t, A) \land t : B \) is satisfied at \( m \). By the satisfaction of the first conjunct we get that \( t \in \text{Act}_m \), which together with the satisfaction of \( t : B \) yields that \( M, m \models \text{Proven}(t, B) \).

The last axiomatic scheme is (B13). Assume that \( \text{Prove}(j, t, A) \) fails at \((m, h)\). This can happen for different reasons, therefore, we have to distinguish between three cases:

Case 1. \( M, m, h \not\models t : A \). Then, by the validity of (B9) we must have \( M, m, h \models \bigwedge_{i \in A} \neg \text{Prove}(i, t, A) \). Given that \( h \in \text{Choice}^m_j(h) \), we further obtain that \( M, m, h \models (j) \bigwedge_{i \in A} \neg \text{Prove}(i, t, A) \), and the axiom is satisfied.

Case 2. \( M, m, h \models t : A \) and \( t \notin \text{Act}_m \). Then we must have \( M, m, h \models \text{Proven}(t, A) \) and, again by (B9), we must have \( M, m, h \models \bigwedge_{i \in A} \neg \text{Prove}(i, t, A) \). Reasoning as in Case 1, we again see that the axiom is satisfied.

Case 3. \( M, m, h \models t : A \) and \( t \notin \text{Act}_m \). But then, given the failure of \( \text{Prove}(j, t, A) \) at \((m, h)\), there must be some \( h' \in \text{Choice}^m_i(h) \) such that \( t \notin \text{Act}(m, h') \). Notice that we will have then \( h' \in \text{Choice}^m_i(h) \) for every \( i \in A_{-} \). Therefore, for every \( i \in A_{-} \) we will have \( t \notin \text{Act}(m, h', i) \), whence it follows that \( M, m, h' \models (j) \bigwedge_{i \in A_{-}} \neg \text{Prove}(i, t, A) \). Since \( h' \in \text{Choice}^m_i(h) \), this further means that \( M, m, h \models (j) \bigwedge_{i \in A} \neg \text{Prove}(i, t, A) \), and the axiom is satisfied.

Taking up the rules of inference, we immediately see that (11) and (12) can only return \( C_\$ \)-validities when given another \( C_\$ \)-validities as premises. As for (\( R_{<CS} \)), assume that \( B = c_{n+1} : c_n : \ldots : c_1 : A \in C_\$ \). We argue by induction on \( n \geq 0 \).

Basis. If \( n = 0 \) then \( B = c_1 : A \in C_\$ \). Since \( M \) is \( C_\$ \)-normal, this means that \( A \in \mathcal{E}(m, c_1) \). Also \( A \) must be an instance of one of the above axiomatic schemes which were all shown to be \( C_\$ \)-validities above, which means that \( A \) must hold at every moment-history pair in \( M \), including the pairs where the moment is some \( R_e \)-successor of \( m \). Therefore, we must have \( M, m, h \models c_1 : A = B \).

Induction step. \( n = k + 1 \). Then \( B = c_{k+2} : c_{k+1} : \ldots : c_1 : A \in C_\$ \). By definition of constant specifications, we will also have then \( c_{k+1} : \ldots : c_1 : A \in C_\$ \). By induction hypothesis, we know that \( c_{k+1} : \ldots : c_1 : A \) is a \( C_\$ \)-validity, hence must hold in every moment-history pair of \( M \), including those pairs where the moment is some \( R_e \)-successor of \( m \). By \( C_\$ \)-normality of \( M \) we also know that \( c_{k+1} : \ldots : c_1 : A \in \mathcal{E}(m, c_{k+2}) \), which shows that \( M, m, h \models B \).

The hardest part is to show the soundness of the rule (S4). Assume that \( KA \rightarrow \neg \text{Proven}(t_1, B_1) \lor \ldots \lor \neg \text{Proven}(t_n, B_n) \) is valid over \( C_\$ \)-normal jstit models, and assume also that for some \( C_\$ \)-normal jstit model \( M = (Tree, \leq, \text{Choice}, Act, R, R_e, \mathcal{E}, V) \) and for some \((m, h) \in MH(M)\) we have:

\[ M, m, h \models KA \land \left( \bigvee_{j \in A_{-}} \text{Prove}(j, t_1, B_1) \land \ldots \land \bigvee_{j \in A_{-}} \text{Prove}(j, t_n, B_n) \right). \]  

(22)
Then we can choose $j_1, \ldots, j_n$ in such a way that we have:

$$
\mathcal{M}, m, h \models KA \land (Prove(j_1, t_1, B_1) \land \ldots \land Prove(j_n, t_n, B_n)).
$$

(23)

By the definition of satisfaction relation, we obtain that:

$$
\mathcal{M}, m, h \models t_1:B_1 \land \ldots \land t_n:B_n.
$$

(24)

The latter basically means two things:

$$
B_1 \in \mathcal{E}(m, t_1), \ldots, B_n \in \mathcal{E}(m, t_n).
$$

(25)

and

$$(\forall m_0 \in \text{Tree})(\forall h_0 \in H_{m_0})(R_e(m, m_0) \Rightarrow \mathcal{M}, m_0, h_0 \models B_1 \land \ldots \land B_n).$$

(26)

On the other hand, we obtain from (23), also by the definition of satisfaction relation and $h \in \text{Choice}_j^m(h)$, that:

$$
t_1, \ldots, t_n \in \text{Act}(m, h).
$$

(27)

We also know that we can choose a $g \in H_m$ such that $t_1 \notin \text{Act}(m, g)$. This means that $h \neq g$. By Lemma 1.2, it follows that we can choose an $m' \in h$ such that $m' \succ m$. So we choose such an $m'$. By Lemma 1.3 $H_{m'} \subseteq H_m$, and, moreover, every history in $H_{m'}$ is undivided from $h$ at $m$. By the presenting a new proof makes histories divide constraint, this means that:

$$(\forall g \in H_{m'})(\text{Act}(m, g) = \text{Act}(m, h)).$$

(28)

By (27) and (28), this means that:

$$
t_1, \ldots, t_n \in \bigcap_{g \in H_{m'}} \text{Act}(m, g).
$$

(29)

Note that it follows from $m' \succ m$ and the expansion of presented proofs constraint that $\bigcap_{g \in H_{m'}} \text{Act}(m, g) \subseteq \text{Act}_{m'}$, so that we must have, by (29), that:

$$
t_1, \ldots, t_n \in \text{Act}_{m'}.
$$

(30)

Next, it follows from (25) by the monotonicity of evidence that:

$$
B_1 \in \mathcal{E}(m', t_1), \ldots, B_n \in \mathcal{E}(m', t_n).
$$

(31)

and it follows from $m' \succ m$ by the future always matters constraint and the inclusion $R \subseteq R_e$ that $R_e(m, m')$. From the latter fact we get, by (26) and transitivity of $R_e$ that:

$$(\forall m_0 \in \text{Tree})(\forall h_0 \in H_{m_0})(R_e(m', m_0) \Rightarrow \mathcal{M}, m_0, h_0 \models B_1 \land \ldots \land B_n).$$

(32)

In their turn, (31) and (32) yield that:

$$
\mathcal{M}, m' \models t_1:B_1 \land \ldots \land t_n:B_n.
$$

(33)
by the definition of satisfaction relation. Adding this up with (30) we get that:

\[ M, m', h \models \text{Proven}(t_1, B_1) \land \ldots \land \text{Prove}(t_n, B_n). \]  

(34)

Finally, by \( m' \not\preceq m \) and the future always matters constraint, we get that \( R(m, m') \), whence, by transitivity of \( R \) and (23), we obtain that:

\[ M, m', h \models KA. \]  

(35)

Taken together, (34) and (35) contradict the assumed validity of \( KA \rightarrow (\neg \text{Proven}(t_1, B_1) \lor \ldots \lor \neg \text{Proven}(t_n, B_n)) \), which shows that (22) cannot be true for any moment-history pair in any CS-normal jstit model.

Before treating completeness, we make some elementary observations about provability in the systems of the form \( \Pi(\mathcal{CS}) \). We first state some theorems and derivable rules of \( \Pi \).

**Lemma 3.** Let \( A \in \text{Form}^{Ag}, j, i_1, \ldots, i_n \in Ag \) and \( t \in \text{Pol} \). Then all of the following theorems and derived rules are provable in \( \Pi \):

\[ KA \rightarrow \Box A \quad \text{(T0)} \]

From \( A \) infer \( \Box A \) \quad \text{(R'1)}

From \( A \) infer \( [j]A \) \quad \text{(R'2)}

\( t: A \rightarrow Kt:A \quad \text{(T1)} \)

\( t: A \rightarrow \Box t:A \quad \text{(T2)} \)

\( KA \rightarrow \Box KA \quad \text{(T3)} \)

\( \text{Proven}(t, A) \rightarrow \Box \text{Proven}(t, A) \quad \text{(T4)} \)

\( \neg \Box (\text{Prove}(i_1, t, A) \lor \ldots \lor \text{Prove}(i_n, t, A)) \quad \text{(T5)} \)

**Proof.** (T0). We use the transitivity of implication w.r.t. the following set of formulas:

\[ KA \rightarrow \Box KA \quad \text{(by (A8))} \]  

(36)

\[ \Box K \Box A \rightarrow K \Box A \quad \text{(by (A1))} \]  

(37)

\[ K \Box A \rightarrow \Box A \quad \text{(by (A7))} \]  

(38)

(R'1). From \( A \) we infer \( KA \) by (R2) and then use (T0) and modus ponens to get \( \Box A \).

(R'2). From \( A \) we infer \( \Box A \) by (R1) and then apply (A2) and modus ponens.

We pause to note that by (R1) and (R2) we know that every modality in the set \( \{ \Box \} \cup \{ [j] \mid j \in Ag \} \) is an S5-modality.

(T1). We have both \( t: A \rightarrow !t:(t:A) \) and \( !t:(t:A) \rightarrow Kt:A \) by (A9) so that we get (T1) by transitivity of implication.

(T2). By (T1) and (T0).

(T3). By \( KA \rightarrow KKA \) (a part of (A7)) and (T0).

(T4). By (T1) and (T0).

(T5). We first prove the theorem for \( n = 1 \). In this case, note that we have \( \Box \text{Prove}(i_1, t, A) \rightarrow \text{Prove}(i_1, t, A) \) by (A1) whence by contraposition we get
\[ \neg \text{Prove}(i_1, t, A) \rightarrow \Box \neg \text{Prove}(i_1, t, A). \] We also have \[ \text{Prove}(i_1, t, A) \rightarrow \Box \neg \text{Prove}(i_1, t, A) \] by (B9). By classical propositional logic we get then:

\[
(\text{Prove}(i_1, t, A) \lor \neg \text{Prove}(i_1, t, A)) \rightarrow \Box \neg \text{Prove}(i_1, t, A),
\]

and, further \( \neg \Box \text{Prove}(i_1, t, A) \), as desired. We now turn to the general case and sketch the derivation as follows:

\[
\neg \text{Prove}(i_1, t, A) \rightarrow \langle i_1 \rangle (\bigwedge_{j \in Ag} \neg \text{Prove}(j, t, A)) \quad \text{(by (B13)) (39)}
\]

\[
\langle i_1 \rangle (\bigwedge_{j \in Ag} \neg \text{Prove}(j, t, A)) \rightarrow \langle i_1 \rangle (\bigwedge_{k=1}^{n} \neg \text{Prove}(i_k, t, A)) \quad \text{([i_1] is S5)} (40)
\]

\[
\neg \text{Prove}(i_1, t, A) \rightarrow \langle i_1 \rangle (\bigwedge_{k=1}^{n} \neg \text{Prove}(i_k, t, A)) \quad \text{(by (39) and (40)) (41)}
\]

\[
[i_1](\bigvee_{k=1}^{n} \text{Prove}(i_k, t, A)) \rightarrow \text{Prove}(i_1, t, A) \quad \text{(by (41), contrap.) (42)}
\]

\[
\Box (\bigvee_{k=1}^{n} \text{Prove}(i_k, t, A)) \rightarrow \text{Prove}(i_1, t, A) \quad \text{(by (42), (A2)) (43)}
\]

\[
\Box (\bigvee_{k=1}^{n} \text{Prove}(i_k, t, A)) \rightarrow \Box \text{Prove}(i_1, t, A) \quad \Box \text{is S5} (44)
\]

From (44), (T5) follows by the case for \( n = 1 \) and classical propositional logic. \( \square \)

Our second point is that the rule [S4] can be substituted by an infinite array of axiomatic schemes without affecting the set of provable formulas, which gives us, in effect, an alternative axiomatization for the systems of the form \( \Pi(\mathcal{CS}) \). More precisely, the following lemma holds:

**Lemma 4.** Let \( \mathcal{CS} \) be a constant specification. Consider the following axiomatic scheme:

\[
K (\neg \text{Proven}(t_1, B_1) \lor \ldots \lor \neg \text{Proven}(t_n, B_n)) \rightarrow \neg \text{Proven}(t_1, B_1) \lor \ldots \lor \neg \text{Proven}(t_n, B_n)) \quad (A_{S4})
\]

for arbitrary \( t_1, \ldots, t_n \in \text{Pol} \) and \( B_1, \ldots, B_n \in \text{Form}^{Ag} \). Let \( \Pi'(\mathcal{CS}) \) be the axiomatic system obtained from \( \Pi(\mathcal{CS}) \) by replacing [S4] with \( A_{S4} \). Then, for every \( A \in \text{Form}^{Ag} \), it is true that \( A \) is provable in \( \Pi(\mathcal{CS}) \) iff \( A \) is provable in \( \Pi'(\mathcal{CS}) \)

**Proof.** (\( \Leftarrow \)) Assume that \( t_1, \ldots, t_n \in \text{Pol} \) and \( B_1, \ldots, B_n \in \text{Form}^{Ag} \). Then, setting:

\[
B := \neg \text{Proven}(t_1, B_1) \lor \ldots \lor \neg \text{Proven}(t_n, B_n),
\]

it suffices to note that the respective instance of \( A_{S4} \) is provable in \( \Pi(\mathcal{CS}) \) by one application of [S4] to \( KB \rightarrow B \), which itself is an axiom by [A7].
(⇒). Assume that \( t_1, \ldots, t_n \in Pol \) and \( B_1, \ldots, B_n \in Form^Ag \). Then, let \( B \) be as above and set:

\[
C := \bigwedge_{j \in Ag} \neg \text{Prove}(j, t_1, B_1) \lor \ldots \lor \bigwedge_{j \in Ag} \neg \text{Prove}(j, t_n, B_n).
\]

Then note that every transition from \( KD \to B \) to \( KD \to C \) according to (S4) can be replaced with a proof in \( \Pi'CS \) sketched below:

1. \( KD \to B \) (premise) \( (45) \)
2. \( K(KD \to B) \) (by 45 and (R2)) \( (46) \)
3. \( KD \to KB \) (by (46), (A7), and (R1)) \( (47) \)
4. \( KB \to C \) (by (A5)) \( (48) \)
5. \( KD \to C \) (by (47), (48), and (A0)) \( (49) \)

We are now prepared to formulate our main result:

**Theorem 2.** Let \( \Gamma \subseteq Form^Ag \) and let \( CS \) be a constant specification. Then \( \Gamma \) is \( CS \)-consistent iff it is satisfiable in a \( CS \)-normal jstit model iff it is satisfiable in a unirelational \( CS \)-normal jstit model.

One part of the completeness results we have, of course, right away, as a consequence of Theorem 1:

**Corollary 1.** Let \( \Gamma \subseteq Form^Ag \) and let \( CS \) be a constant specification. If \( \Gamma \) is satisfiable in a \( CS \)-normal (unirelational) jstit model, then \( \Gamma \) is \( CS \)-consistent.

**Proof.** Let \( \Gamma \subseteq Form^Ag \) be satisfiable in a \( CS \)-normal jstit model \( \mathcal{M} \), either unirelational or not. Then for some \((m, h) \in MH(\mathcal{M})\) we have \( \mathcal{M}, m, h \models \Gamma \). If \( \Gamma \) were \( CS \)-inconsistent this would mean that for some \( A_1, \ldots, A_n \in \Gamma \) we would have \( \vdash_{CS} (A_1 \land \ldots \land A_n) \to \bot \). By Theorem 1 this would mean that:

\[
\mathcal{M}, m, h \models (A_1 \land \ldots \land A_n) \to \bot,
\]

whence clearly \( \mathcal{M}, m, h \models \bot \), which is impossible. Therefore, \( \Gamma \) must be \( CS \)-consistent.

\[\square\]

### 4 The canonical model

We begin by fixing an arbitrary constant specification \( CS \) throughout the present section. The main aim of the section is to prepare the proof of the inverse of Corollary 1. The method used is a variant of the canonical model technique, but, due to the complexity of the case, we do not define our model in one sweeping definition. Rather, we proceed piecewise, defining elements of the model one by one, and checking the relevant constraints as soon, as we have got enough parts of the model in place. The last subsection proves the truth lemma for the defined model. As we have already indicated, the model to be built will be a normal unirelational jstit model, so that \( R_e \).
will be omitted, or, equivalently, assumed to coincide with \( R \). It should also be noted that our definitions of stit- and justifications-related components of the canonical model are borrowed to the last letter from the construction of the canonical model for JA-STIT given in [8]. Even though the basic building blocks for our current case are somewhat different from those used for JA-STIT case, this does not affect the proofs of the respective lemmas in the least. Therefore, we omit the proofs of almost every lemma claimed in subsection 4.1 and replace them with the following table bringing the lemmas in question into correspondence with the respective lemmas of [8] so that the reader may look up the proofs whenever this is called for.

| Numbering given in subsection 4.1 | Reference to [8] |
|-----------------------------------|------------------|
| Lemma 8                           | Lemma 4          |
| Lemma 9                           | Lemma 5          |
| Lemma 10                          | Lemma 9          |
| Lemma 11                          | Lemma 10         |
| Lemma 12                          | Lemma 12         |
| Lemma 16                          | Lemma 14         |
| Lemma 17                          | Lemma 15         |

The canonical model to be constructed below will be a \( \mathcal{CS} \)-normal jstit model named \( M_{Ag}^{CS} \). The ultimate building blocks of \( M_{Ag}^{CS} \) we will call elements. Before going on with the definition of \( M_{Ag}^{CS} \), we define what these elements are and explore some of their properties.

**Definition 1.** An element is a sequence of the form \((\Gamma_1, \ldots, \Gamma_n)\) for some \( n \in \omega \) with \( n \geq 1 \) such that:

- For every \( k \leq n \), \( \Gamma_k \) is a \( \mathcal{CS} \)-maxiconsistent subset of \( Form^{Ag} \);
- For every \( k < n \), for all \( A \in Form^{Ag} \), if \( KA \in \Gamma_k \), then \( KA \in \Gamma_{k+1} \);
- For every \( k < n \), for all \( t \in Pol, A \in Form^{Ag} \), and \( j \in Ag \), if \( Prove(j, t, A) \in \Gamma_k \), then \( Proven(t, A) \in \Gamma_{k+1} \).

We prove the following lemma:

**Lemma 5.** Whenever \((\Gamma_1, \ldots, \Gamma_n)\) is an element, there exists a \( \Gamma_{n+1} \subseteq Form^{Ag} \) such that the sequence \((\Gamma_1, \ldots, \Gamma_{n+1})\) is also an element.

**Proof.** Assume \((\Gamma_1, \ldots, \Gamma_n)\) is an element and consider the following set:

\[ \Delta := \{ KA \mid KA \in \Gamma_n \} \cup \{ Proven(t, A) \mid (\exists j \in Ag)(Prove(j, t, A) \in \Gamma_n) \} \]

We show that \( \Delta \) is \( \mathcal{CS} \)-consistent. Of course, the set \( \{ KA \mid KA \in \Gamma_n \} \) is \( \mathcal{CS} \)-consistent since it is a subset of \( \Gamma_n \), and the latter is assumed to be \( \mathcal{CS} \)-consistent. Further, if \( \Delta \) is \( \mathcal{CS} \)-inconsistent, then, wlog, for some \( B_1, \ldots, B_k, C_1, \ldots, C_l \in Form^{Ag} \), \( t_1, \ldots, t_l \in Ag \), and \( j_1, \ldots, j_l \in Ag \) such that \( KB_1, \ldots, KB_k \) and \( Prove(j_1, t_1, C_1), \ldots, Prove(j_l, t_l, C_l) \) are in \( \Gamma_n \), we will have:

\[ \vdash_{\mathcal{CS}} (KB_1 \land \ldots \land KB_r) \rightarrow (\neg Proven(t_1, C_1) \lor \ldots \lor \neg Proven(t_l, C_l)) \]
Proof. We first show that (\(\Gamma\) \& \(\Delta\)) has \(\text{\(\neg\)maxiconsistency}\) of \(\Gamma\), whence \(\text{\(\neg\)proven}\) of \(\Gamma\). But then \(\text{\(\neg\)proven}\) of \(\Delta\). Therefore, \(\Delta\) must be \(\text{\(\neg\)CS}\)-inconsistent which contradicts the assumption that \((\Gamma_1, \ldots, \Gamma_n)\) is an element.

Therefore, \(\Delta\) must be \(\text{\(\neg\)CS}\)-consistent, and is also extendable to a \(\text{\(\neg\)CS}\)-maxiconsistent \(\Gamma_{n+1}\). By the choice of \(\Delta\), this means that \((\Gamma_1, \ldots, \Gamma_n, \Gamma_{n+1})\) must be an element. □

The structure of elements will be important in what follows. If \(\xi = (\Gamma_1, \ldots, \Gamma_n)\) is an element and an element \(\tau\) is of the form \((\Gamma_1, \ldots, \Gamma_k)\) with \(k < n\), we say that \(\tau\) is a \textit{proper} initial segment of \(\xi\). Moreover, if \(k = n - 1\), then \(\tau\) is the \textit{greatest} proper initial segment of \(\xi\). We define \(n\) to be the \textit{length} of \(\xi\). Furthermore, we define that \(\Gamma_n\) is the end element of \(\xi\) and let \(\Gamma_n = \text{end}(\xi)\).

We now define the canonical model using elements as our building blocks. We start by defining the following relation \(\equiv\):

\[
(\Gamma_1, \ldots, \Gamma_n, \Gamma_{n+1}) \equiv (\Delta_1, \ldots, \Delta_n, \Delta_{n+1}) \iff (\Gamma_1 = \Delta_1 \& \ldots \& \Gamma_n = \Delta_n \& (\forall A \in \text{Form}^A) (\Box A \in \Gamma_{n+1} \Rightarrow A \in \Delta_{n+1})).
\]

It is routine to check that \(\equiv\) is an equivalence relation given that \(\Box\) is an S5 modality. The notation \([\Gamma_1, \ldots, \Gamma_n]\) denotes the \(\equiv\)-equivalence class generated by \((\Gamma_1, \ldots, \Gamma_n)\). Since all the elements inside a given \(\equiv\)-equivalence class are of the same length, we may extend the notion of length to these classes setting that the length of \([\Gamma_1, \ldots, \Gamma_n]\) also equals \(n\).

The next two lemmas will be repeatedly used in what follows:

**Lemma 6.** Let \((\Gamma_1, \ldots, \Gamma_n, \Gamma)\) be an element, let \(\Delta \subseteq \text{Form}^A\) be \(\text{\(\neg\)CS}\)-maxiconsistent and let:

\[
\{\Box A \mid \Box A \in \Gamma\} \subseteq \Delta.
\]

Then \((\Gamma_1, \ldots, \Gamma_n, \Delta)\) is also an element, and, moreover:

\[
(\Gamma_1, \ldots, \Gamma_n, \Gamma) \equiv (\Gamma_1, \ldots, \Gamma_n, \Delta).
\]

**Proof.** We first show that \((\Gamma_1, \ldots, \Gamma_n, \Delta)\) is an element. Indeed, if \(\Box A \in \Gamma_n\), then \(\Box A \in \Gamma\) by definition of an element. But then \(\Box K A \in \Gamma\) by \(\Box A \in \Gamma\) and \(\text{\(\neg\)CS}\)-maxiconsistency of \(\Gamma\), whence \(\Box K A \in \Delta\). By \(\Box A \equiv \Box K A \in \Delta\) and \(\text{\(\neg\)CS}\)-maxiconsistency of \(\Delta\) we get then \(\Box K A \in \Delta\).

Similarly, if \(\text{\(\neg\)Prove}(j, t, A) \in \Gamma_n\), then \(\text{\(\neg\)Proven}(t, A) \in \Gamma\) by definition of an element. But then \(\Box \text{\(\neg\)Proven}(t, A) \in \Gamma\) by \(\Box A \equiv \Box K A \in \Delta\) and \(\text{\(\neg\)CS}\)-maxiconsistency of \(\Gamma\), whence \(\Box \text{\(\neg\)Proven}(t, A) \in \Delta\).

Given the inclusion \(\{\Box A \mid \Box A \in \Gamma\} \subseteq \Delta\), the other part of the lemma is straightforward. □
Lemma 7. Let $\xi$ be an element and let $A \in \text{Form}^Ag$. Then the following statements hold:

1. $A \in \bigcap_{\tau \equiv \xi} \text{end}(\tau) \Leftrightarrow \Box A \in \text{end}(\xi) \Leftrightarrow A \in \bigcap_{\tau \equiv \xi} \text{end}(\tau)$.

2. If $\vdash_{\text{CS}} A \rightarrow \Box A$, then $A \in \text{end}(\xi) \Leftrightarrow A \in \bigcap_{\tau \equiv \xi} \text{end}(\tau) \Leftrightarrow \Box A \in \bigcap_{\tau \equiv \xi} \text{end}(\tau)$.

Proof. (Part 1) If $\Box A \notin \bigcap_{\tau \equiv \xi} \text{end}(\tau)$, then (of course) $\Box A \notin \text{end}(\xi)$, whence, by definition of $\equiv$ we get that $A \in \bigcap_{\tau \equiv \xi} \text{end}(\tau)$. On the other hand, if $A \in \bigcap_{\tau \equiv \xi} \text{end}(\tau)$, then choose an arbitrary $\tau$ such that $\tau \equiv \xi$. If $\Box A \notin \text{end}(\tau)$, then we are done. Otherwise, we can obtain a contradiction as follows. Consider the set:

$$\Gamma = \{ \Box B \mid \Box B \in \text{end}(\tau) \} \cup \{ \neg A \}.$$ 

If $\Gamma$ were $\text{CS}$-inconsistent, then we would have:

$$\vdash_{\text{CS}} (\Box B_1 \land \ldots \land \Box B_n) \rightarrow A$$

for some $\Box B_1, \ldots, \Box B_n \in \text{end}(\tau)$, whence by S5 reasoning for $\Box$ we would also have that:

$$\vdash_{\text{CS}} (\Box B_1 \land \ldots \land \Box B_n) \rightarrow \Box A.$$

By $\text{CS}$-maxiconsistency of $\text{end}(\tau)$ it would follow then that $\Box A \in \text{end}(\tau)$, contrary to our assumption. Therefore, $\Gamma$ is $\text{CS}$-consistent and we can extend it to a $\text{CS}$-maxiconsistent $\Delta$. Consider the inner structure of $\tau$. We must have $\tau = (\Gamma_1, \ldots, \Gamma_n, \text{end}(\tau))$ for appropriate $n \geq 0$ and $\Gamma_1, \ldots, \Gamma_n \subseteq \text{Form}^Ag$. But then, by Lemma 6 we must also have that $(\Gamma_1, \ldots, \Gamma_n, \Delta)$ is an element, and, moreover, that $(\Gamma_1, \ldots, \Gamma_n, \Delta) \equiv \tau \equiv \xi$.

But then, note that by $\Gamma \subseteq \Delta$ we have $\neg A \in \Delta$, whence by $\text{CS}$-consistency $A \notin \Delta$ in contradiction with our assumption that $A \in \bigcap_{\tau \equiv \xi} \text{end}(\tau)$. This contradiction shows that for no $\tau \equiv \xi$ can we have $\Box A \notin \text{end}(\tau)$ so that we must end up having $\Box A \in \bigcap_{\tau \equiv \xi} \text{end}(\tau)$.

(Part 2). If $\vdash_{\text{CS}} A \rightarrow \Box A$ then, by $[A1]$ and $\text{CS}$-maxiconsistency of $\text{end}(\xi)$, we will have $\Box A \in \text{end}(\xi) \Leftrightarrow A \in \text{end}(\xi)$, and the rest follows by Part 1. \qed

We now proceed to definitions of components for the canonical model.

4.1 Stit and justification components

The first two components of the canonical model $M^Ag_{\text{CS}}$ are as follows:

- $\text{Tree} = \{ \dagger \} \cup \{ ([\xi]_|= n) \mid n \in \omega, \xi \text{ is an element} \}$. Thus the elements of $\text{Tree}$, with the exception of the special moment $\dagger$, are $\equiv$-equivalence classes of elements coupled with natural numbers. Such moments we will call standard moments, and the left projection of a standard moment $m$ we will call its core (and write $\overline{m}$), while the right projection of such moment we will call its height (and write $|m|$). In this way, we get the equality $m = (\overline{m}, |m|)$ for every standard $m \in \text{Tree}$.

We further define that the length of a standard moment $m$ is the length of its core. For the sake of completeness, we extend the above notions to $\dagger$ setting both length and height of this moment to 0 and defining that $\overline{\dagger} = \overline{\dagger} = 0$. 

We set that $(\forall m \in \text{Tree} \setminus \{ \top \})(\top \not< m \land m \neq \top)$. We further set that for any two standard moments $m$ and $m'$, we have that $m < m'$ if and only if (1) there exists a $\xi \in \overline{m}$ such that for every $\tau \in \overline{m'}$, $\xi$ is a proper initial segment of $\tau$, or (2) $\overline{m} = \overline{m'}$ and $|m'| < |m|$. The relation $\leq$ is then defined as the reflexive companion to $\not<$.

With this settings, we claim that the restraints imposed by our semantics on $\text{Tree}$ and $\leq$ are satisfied:

**Lemma 8.** The relation $\leq$, as defined above, is a partial order on $\text{Tree}$, which satisfies both historical connection and no backward branching constraints.

Before we move on to the other components of the canonical model $\mathcal{M}$ to be defined in this section, we formulate some important facts about the structure of $\text{Hist}(\mathcal{M}_{C/S}^{Ag})$ as induced by the above-defined $\text{Tree}$ and $\leq$. We start by defining basic sequences of elements. A basic sequence of elements is a set of elements of the form ${\xi_1, \ldots, \xi_n, \ldots}$ such that for every $n \geq 1$:

- $\xi_n$ is of length $n$;
- $\xi_n$ is the greatest proper initial segment of $\xi_{n+1}$.

Basic sequences will be denoted by capital Latin letters $S$, $T$, and $U$ with subscripts when needed. Every given basic sequence $S$ induces the following $[S] \subseteq \text{Tree}$:

$$[S] = \{ \top \} \cup \bigcup_{n \in \omega} \{ ([\xi_n]\equiv, k) \mid k \in \omega \}.$$  

It is immediate that every basic sequence $S$ induces a unique $[S] \subseteq \text{Tree}$ in this way. It is, perhaps, less immediate that the mapping $S \mapsto [S]$ is injective:

**Lemma 9.** Let $S, T$ be basic sequences of elements. Then:

$$[S] = [T] \iff S = T.$$  

Another striking fact is that basic sequences can be used to characterize $\text{Hist}(\mathcal{M}_{C/S}^{Ag})$ through this injection:

**Lemma 10.** The following statements hold:

1. If $S = \{\xi_1, \ldots, \xi_n, \ldots\}$ is a basic sequence, then $[S] \in \text{Hist}(\mathcal{M}_{C/S}^{Ag})$, and the following presentation gives $[S]$ in the $\leq$-ascending order:

$$\top, \ldots, ([\xi_1]\equiv, k), \ldots, ([\xi_1]\equiv, 0), \ldots, ([\xi_n]\equiv, k), \ldots, ([\xi_n]\equiv, 0), \ldots,$$

2. $\text{Hist}(\mathcal{M}_{C/S}^{Ag}) = \{ [S] \mid S$ is a basic sequence $\}$.  

It follows from Lemmas 9 and 10 that not only every basic sequence generates a unique $h \in \text{Hist}(\mathcal{M}_{C/S}^{Ag})$, but also for every $h \in \text{Hist}(\mathcal{M}_{C/S}^{Ag})$ there exists a unique basic sequence $S$ such that $h = [S]$. We will denote this unique $S$ for a given $h$ by $|h|$. It is immediate from Lemmas 9 and 10 that for every $h \in \text{Hist}(\mathcal{M}_{C/S}^{Ag})$, $h = [|h|]$. Likewise, for every basic sequence $S$, we have $S = [|S|]$. As a further useful piece of notation, we
introduce the notion of *intersection* of a standard moment \( m \) with a history \( h \in H_m \).

Assume that \( m \) is of the length \( n \) and that |\( h|\{\xi_1, \ldots, \xi_n, \ldots, \} \). Then \( m \) must be of the form \( (\xi_n)_{\equiv k} \) for some \( k \in \omega \), and we will also have \( m \cap h = \{\xi_n\} \). We now define the only member of the latter singleton as the result \( m \cap h \) of the intersection of \( m \) and \( h \), setting \( m \cap h = \xi_n \). It can be shown that for any element \( \xi \) in the core of a given standard moment \( m \) there exists an \( h \in H_m \) such that \( \xi = m \cap h \):

**Lemma 11.** Let \((\Gamma_1, \ldots, \Gamma_k)\) be an element. Then, for every \( n \in \omega \) there is at least one history \( h \in H_{([\Gamma_1, \ldots, \Gamma_k])_{\equiv n}} \) such that \( ([\Gamma_1, \ldots, \Gamma_k])_{\equiv n} \cap h = (\Gamma_1, \ldots, \Gamma_k) \).

An immediate but important corollary of Lemma 11 is that the core of a given moment \( m \) is exactly the set of \( m \)'s intersections with the histories passing through \( m \):

**Corollary 2.** Let \( m \in \text{Tree} \). Then \( \{\xi \mid \xi \in \overline{m}\} = \{m \cap h \mid h \in H_m\} \).

We offer some general remarks on what we have shown thus far. Lemma 11 shows that every history in the canonical model has a uniform order structure, namely, it consists of \( \uparrow \) followed by \( \omega \) copies of the set of negative integers. Another general observation is that histories in \( M^Ag_\omega \) can only branch off at moments of height 0, so that at moments of other heights all the histories remain undivided. This last fact does not follow from the lemmas proved thus far but it can be proved in the same way as we have proved the similar fact in [8, Corollary 3].

We now define the choice function for our canonical model:

- **Choice**\(^m\)\(_j\)(\( h \)) = \{\( h' \in H_m \mid (\forall A \in \text{Form})([j]A \in \text{end}(h \cap m) \Rightarrow A \in \text{end}(h' \cap m))\},

  if \( m \neq \uparrow \) and |\( m| = 0;

- **Choice**\(^m\)\(_j\) = \( H_m \), otherwise.

Since for every \( j \in Ag \), \([j]\) is an S5-modality, **Choice** induces a partition on \( H_m \) for every given \( m \in \text{Tree} \). We claim that the choice function verifies the relevant semantic constraints:

**Lemma 12.** The tuple \((\text{Tree}, \leq, \text{Choice})\), as defined above, verifies both independence of agents and no choice between undivided histories constraints.

The next two lemmas can be viewed as ‘stit versions’ of Lemmas \( \square \) and \( \square \)

**Lemma 13.** Let \( j \in Ag \), let \((\Gamma_1, \ldots, \Gamma_n, \Gamma)\) be an element, let \( \Delta \subseteq \text{Form}^Ag \) be \( \text{CS-maxiconsistent} \) and let:

\[ \{[j]A \mid [j]A \in \Gamma\} \subseteq \Delta. \]

Then \((\Gamma_1, \ldots, \Gamma_n, \Delta)\) is also an element, and for \( m = (\langle \Gamma_1, \ldots, \Gamma_n, \Gamma \rangle_{\equiv 0}) \), whenever \( m \cap h = (\Gamma_1, \ldots, \Gamma_n, \Gamma) \), there exists a \( g \in \text{Choice}^m_j(h) \) such that:

\[(\Gamma_1, \ldots, \Gamma_n, \Delta) = m \cap g.\]

**Proof.** First of all, note that whenever \( \square A \in \Gamma \) we have, in virtue of \( \text{CS-maxiconsistency} \) of \( \Gamma \), that \( \square A \in \Gamma \) (by \( A1 \)) and, further, that \([j] \square A \in \Gamma \) (by \( A2 \)). Therefore, we must have \([j] \square A \in \Delta \) and in view of \( \text{CS-maxiconsistency} \) of \( \Delta \) and \( A1 \) we will also have \( \square A \in \Delta \). Thus we have established that \( \{\square A \mid A \in \Gamma\} \subseteq \Delta \) and it follows, by Lemma \( \square \) that \((\Gamma_1, \ldots, \Gamma_n, \Delta)\) is an element and that \((\Gamma_1, \ldots, \Gamma_n, \Delta) \equiv \overline{m} \). Now, if \( h \) is chosen as in the lemma, use Lemma 11 to pick a \( g \in H_m \) such that \((\Gamma_1, \ldots, \Gamma_n, \Delta) = m \cap g \) holds. Recall that we have |\( m\| = 0. By the construction of \( \Delta \) and \( A1 \) we must then have \( g \in \text{Choice}^m_j(h) \).

\( \Box \)
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Before moving on with the lemmas, we introduce two further notations, which are similar to the notations we used for Act, but refer to the inner structures of $\mathcal{M}^{Ag}$:

$$
\text{end}_m = \bigcap_{h \in H_m} \text{end}(m \cap h);
\text{end}_{(m,h,j)} = \bigcap_{g \in \text{Choice}_j^m(h)} \text{end}(m \cap g).
$$

**Lemma 14.** Let $m \in \text{Tree}$ be such that $|m| = 0$, and let $A \in \text{Form}^{Ag}$. Then:

$$
A \in \text{end}_{(m,h,j)} \iff [j]A \in \text{end}(m \cap h) \iff [j]A \in \text{end}_{(m,h,j)}.
$$

*Proof.* If $[j]A \in \text{end}_{(m,h,j)}$, then, by $h \in \text{Choice}_j^m(h)$, $[j]A \in \text{end}(m \cap h)$, whence, by $|m| = 0$ we get that $A \in \text{end}_{(m,h,j)}$. On the other hand, if $A \in \text{end}_{(m,h,j)}$, then choose an arbitrary $g \in \text{Choice}_j^m(h)$. If $[j]A \in m \cap g$, then we are done. Otherwise, we can obtain a contradiction as follows. Consider the set

$$
\Gamma = \{[j]B \mid [j]B \in \text{end}(m \cap g)\} \cup \{\neg A\}.
$$

If $\Gamma$ were CS-inconsistent, then we would have:

$$
\vdash_{\text{CS}} ([j]B_1 \land \ldots \land [j]B_n) \rightarrow A,
$$

for some $[j]B_1, \ldots, [j]B_n \in \text{end}(m \cap g)$, whence by S5 reasoning for $[j]$ we would also have that:

$$
\vdash_{\text{CS}} ([j]B_1 \land \ldots \land [j]B_n) \rightarrow [j]A.
$$

By CS-maxiconsistency of $\text{end}(m \cap g)$ it would follow then that $[j]A \in \text{end}(m \cap g)$, contrary to our assumption. Therefore, $\Gamma$ is CS-consistent and we can extend it to a CS-maxiconsistent $\Delta$. Consider the inner structure of $m \cap g$. We must have $m \cap g = (\Gamma_1, \ldots, \Gamma_n, \text{end}(m \cap g))$ for appropriate $n \geq 0$ and $\Gamma_1, \ldots, \Gamma_n \subseteq \text{Form}^{Ag}$. But then, by Lemma 13 we must also have that $(\Gamma_1, \ldots, \Gamma_n, \Delta)$ is an element, and, moreover, that $(\Gamma_1, \ldots, \Gamma_n, \Delta) = m \cap h'$ for some $h' \in \text{Choice}_j^m(g) = \text{Choice}_j^m(h)$ (the latter equality obtains by $g \in \text{Choice}_j^m(h)$). But then, note that by $\Gamma \subseteq \Delta$ we have $\neg A \in \Delta$, whence by CS-consistency $A \not\in \Delta = \text{end}(m \cap h')$ in contradiction with our assumption that $A \in \text{end}_{(m,h,j)}$. This contradiction shows that for no $g \in \text{Choice}_j^m(h)$ can we have $[j]A \not\in \text{end}(\tau)$ so that we must end up having $[j]A \in \text{end}_{(m,h,j)}$.

We sum up the implications of the above lemmas for our modalities as follows:

**Corollary 3.** Let $m \in \text{Tree}$, $h \in H_m$, $A \in \text{Form}^{Ag}$, $t \in P o l$, and $j \in A g$. Then:

1. $\alpha \in \text{end}(m \cap h) \iff \alpha \in \text{end}_m$, for all $\alpha \in \{\Box A, t \vdash A, KA, \text{Proven}(t, A)\}$;

2. $\alpha \in \text{end}(m \cap h) \iff \alpha \in \text{end}_{(m,h,j)}$, for all $\alpha \in \{[j]A, \text{Prove}(j, t, A)\}$.

*Proof.* Part 1 we get by $(A1)$, $(T2)$–$(T4)$, Lemma 7, and Corollary 2. Part 2 we get by $(A1)$, $(B9)$, CS-maxiconsistency of end sets of elements, and Lemma 14.

Turning to the justifications-related components, we first define $R$ as follows:

- $R(\langle[\Gamma_1, \ldots, \Gamma_n, \Gamma]\rangle_k, m') \iff$
  - $(m' \neq \bot) \land (\forall A \in \text{Form}^{Ag})(KA \in \Gamma \Rightarrow KA \in \text{end}_{m'})$;
• \(R(\dagger, m)\), for all \(m \in \text{Tree}\).

Now, for the definition of \(\mathcal{E}\):

• For all \(t \in \text{Pol}\): \(\mathcal{E}(\dagger, t) = \{A \in \text{Form}^\mathcal{Ag} \mid \vdash t: A\};\)

• For all \(t \in \text{Pol}\) and \(m \neq \dagger\): \((\forall A \in \text{Form}^\mathcal{Ag})(A \in \mathcal{E}(m, t) \Leftrightarrow t: A \in \text{end}_m)\).

We start by mentioning a straightforward corollary to the above definition:

**Lemma 15.** For all \(m \in \text{Tree}\) and \(t \in \text{Pol}\) it is true that \(\{A \in \text{Form}^\mathcal{Ag} \mid \vdash t: A\} \subseteq \mathcal{E}(m, t)\).

**Proof.** This holds simply by the definition of \(\mathcal{E}\) when \(m = \dagger\). If \(m \neq \dagger\), then, for every \(\xi \in \vec{m}\), \(\text{end}(\xi)\) is a maxiconsistent subset of \(\text{Form}^\mathcal{Ag}\) and must contain every provable formula. \(\square\)

Note that it follows from Lemma 15 that the above-defined function \(\mathcal{E}\) satisfies the \(\mathcal{CS}\)-normality condition on \jstit\ models. We now mention the respective adequacy claims:

**Lemma 16.** The relation \(R\), as defined above, is a preorder on \(\text{Tree}\), and, together with \(\preceq\), verifies the future always matters constraint.

**Lemma 17.** The function \(\mathcal{E}\), as defined above, satisfies both monotonicity of evidence and evidence closure properties.

### 4.2 Act and \(V\)

It remains to define \(\text{Act}\) and \(V\) for our canonical model, and we define them as follows:

• \((m, h) \in V(p) \iff p \in \text{end}(m \cap h)\), for all \(p \in \text{Var}\);

• \(\text{Act}(\dagger, h) = \emptyset\) for all \(h \in \text{Hist}(\mathcal{M})\);

• \(\text{Act}(m, h) = \{t \in \text{Pol} \mid (\exists A \in \text{Form}^\mathcal{Ag}, j \in \mathcal{Ag})(\text{Proven}(t, A) \in \text{end}(m \cap h) \vee \text{Prove}(j, t, A) \in \text{end}(m \cap h))\}\), if \(m \neq \dagger\), \(|m| = 0\) and \(h \in \mathcal{H}_m\);

• \(\text{Act}(m, h) = \{t \in \text{Pol} \mid (\exists A \in \text{Form}^\mathcal{Ag})(\text{Proven}(t, A) \in \text{end}(m \cap g))\}\), if \(m \neq \dagger\), \(|m| > 0\) and \(h \in \mathcal{H}_m\).

Since in the definition of \(\text{Act}\) we have used the proving modalities not available in \jajstit\, we can no longer rely on constructions carried over for the canonical model of \[8\].

We first draw some of the immediate consequences of the above definitions:

**Lemma 18.** Assume that \(m \in \text{Tree} \setminus \{\dagger\}\), \(h \in \mathcal{H}_m\), and \(t \in \text{Pol}\). Then the following statements are true:

1. \((\exists A \in \text{Form}^\mathcal{Ag})(\text{Proven}(t, A) \in \text{end}(m \cap h)) \Leftrightarrow t \in \text{Act}_m\);

2. If \(|m| > 0\) and \(h, h' \in \mathcal{H}_m\), then \(\text{Act}(m, h) = \text{Act}(m, h')\);

3. If \(h, h' \in \mathcal{H}_m\) and \(m \cap h = m \cap h'\), then \(\text{Act}(m, h) = \text{Act}(m, h')\).
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Proof. (Part 1). Assume that for some \( A \in \text{Form}^Ag \) we have \( \text{Proven}(t, A) \in \text{end}(m \cap h) \). Then, by Corollary \([31]\), we must also have \( \text{Proven}(t, A) \in \text{end}_m \), whence, by definition of \( \text{Act} \), it follows that \( t \in \text{Act}_m \). In the other direction, let \( t \in \text{Act}_m \). If for some \( g \in H_m \) and some \( A \in \text{Form}^Ag \) we have \( \text{Proven}(t, A) \in \text{end}(m \cap g) \), then, by Corollary \([31]\), we must also have \( \text{Proven}(t, A) \in \text{end}_m \), whence also \( \text{Proven}(t, A) \in \text{end}(m \cap h) \). Otherwise, we obtain a contradiction.

Indeed, if for no \( g \in H_m \) and \( A \in \text{Form}^Ag \) we have \( \text{Proven}(t, A) \in \text{end}(m \cap g) \), then, by definition of \( \text{Act} \), for every \( g \in H_m \) there must be an \( A_g \in \text{Form}^Ag \) and a \( j_g \in Ag \) such that \( \text{Prove}(j_g, t, A_g) \in \text{end}(m \cap g) \). Now, consider \( A_h \). We have \( \text{Prove}(j_h, t, A_h) \in \text{end}(m \cap h) \), therefore, by \([139]\) and \( CS\)-maxiconsistency of \( \text{end}(m \cap h) \), we must also have \( t: A_h \in \text{end}(m \cap h) \). It follows by Lemma \([31]\), that \( t: A_h \in \text{end}_m \). Therefore, given the \( CS\)-maxiconsistency of end sets in elements, we must have \( \text{Prove}(j_g, t, A_g) \land t: A_h \in \text{end}_m \).

By \([150]\), we further get that \( \text{Prove}(j_g, t, A_h) \in \text{end}_m \). Note that \( \{j_g \mid g \in H_m\} \subseteq Ag \) and hence must be finite. Therefore \( \bigvee_{g \in H_m} \text{Prove}(j_g, t, A_g) \) is in fact a finite disjunction, and, again using \( CS\)-maxiconsistency of of end sets in elements, we obtain that \( \bigvee_{g \in H_m} \text{Prove}(j_g, t, A_h) \in \text{end}_m \), whence, by Lemma \([71]\) and Corollary \([2]\) it follows that \( \bigvee_{g \in H_m} \text{Prove}(j_g, t, A_h) \subseteq \text{end}(m \cap h) \). By \([155]\), the latter is in contradiction with \( CS\)-maxiconsistency of \( \text{end}(m \cap h) \), which finishes the proof of Part 1.

(Part 2). In the assumptions of this part, we get, by Corollary \([31]\), that:

\[
\begin{align*}
t \in \text{Act}(m, h) & \iff (\exists A \in \text{Form}^Ag)(\text{Proven}(t, A) \in \text{end}(m \cap h)) \\
& \iff (\exists A \in \text{Form}^Ag)(\text{Proven}(t, A) \in \text{end}_m) \\
& \iff (\exists A \in \text{Form}^Ag)(\text{Proven}(t, A) \in \text{end}(m \cap h')) \\
& \iff t \in \text{Act}(m, h'),
\end{align*}
\]

for an arbitrary \( t \in \text{Pol} \).

(Part 3). Note that \( \text{Act}(m, h) \) and \( \text{Act}(m, h') \) are fully determined by \( \text{end}(m \cap h) \) and \( \text{end}(m \cap h') \), respectively, and that, by our assumptions, we must have \( \text{end}(m \cap h) = \text{end}(m \cap h') \).

\[\square\]

We now check that the remaining semantic constraints on jstit models are satisfied:

Lemma 19. \( M^Ag_{CS} \) satisfies the constraints as to the expression of presented proofs, no new proofs guaranteed, presenting a new proof makes histories divide, and epistemic transparency of presented proofs.

Proof. We consider the expansion of presented proofs first. Let \( m' \prec m \) and let \( h \in H_m \). If \( m' = \dagger \), then we have \( \text{Act}(\dagger, h) = \emptyset \), so that the expansion of presented proofs holds. If \( m' \neq \dagger \), then \( m \) is also standard. Consider then \( m' \cap h \) and \( m \cap h \). Both these elements must be in the basic sequence \( \|h\| \), therefore, one of them must be an initial segment of another. By \( m' \prec m \) we know that \( m' \cap h \) must be a proper initial segment of \( m \cap h \). So we may assume that \( m' \cap h = (\Gamma_1, \ldots, \Gamma_k) \) and \( m \cap h = (\Gamma_1, \ldots, \Gamma_n) \) for some appropriate \( \Gamma_1, \ldots, \Gamma_n \subseteq \text{Form}^Ag \) and \( n > k \). Now, if \( t \in \text{Act}(m', h) \), then for some \( A \in \text{Form}^Ag \) we must have either that \( \text{Prove}(j, t, A) \in \text{end}(m' \cap h) = \Gamma_k \) for some \( j \in Ag \), or that \( \text{Proven}(t, A) \in \text{end}(m' \cap h) \). In the latter case we will also have \( K\text{Proven}(t, A) \in \text{end}(m' \cap h) \) by \([1311]\) and \( CS\)-maxiconsistency of \( \Gamma_k \). Then, since \( (\Gamma_1, \ldots, \Gamma_n) \) is an element, we must have \( K\text{Proven}(t, A) \in \Gamma_n \), whence, by \([A, 7]\) and
\(\mathcal{CS}\)-maxiconsistency of \(\Gamma_n\), we further obtain that \(\text{Proven}(t, A) \in \Gamma_n = \text{end}(m \cap h)\). Hence we must have \(t \in \text{Act}(m, h)\).

In the former case we also invoke the fact that \((\Gamma_1, \ldots, \Gamma_n)\) is an element, which in this case directly entails that \(\text{Proven}(t, A) \in \Gamma_{k+1}\) and, given that \(k+1 \leq n\), the rest is the same as in the previous case.

We consider next the no new proofs guaranteed constraint. Let \(m \in \text{Tree}\).

If \(m = \top\), then \(\text{Act}_m = \bigcup_{n < m, h \in H_m} (\text{Act}(m', h)) = \emptyset\) and the constraint is trivially satisfied. Assume that \(m \neq \top\) and that \(t \in \text{Act}_m\) and choose an arbitrary \(h \in H_m\). Consider \(m \cap h\), say \(m \cap h = (\Gamma_1, \ldots, \Gamma_n)\). We get then that \(m = ([\Gamma_1, \ldots, \Gamma_n])_\equiv, k\) for some \(k \in \omega\). By Lemma 18.1, we further obtain that for some \(A \in \text{Form}^{Ag}\) we will have \(\text{Proven}(t, A) \in \Gamma_n\). Now, consider \(m' = ([\Gamma_1, \ldots, \Gamma_n])_\equiv, k+1\). We clearly have \(m' \not< m\), therefore, by Lemma 18.3, we immediately get \(m' \cap h = (\Gamma_1, \ldots, \Gamma_n)_\equiv, \Gamma_n\). Moreover, it is clear that \(m' \cap h\) also equals \((\Gamma_1, \ldots, \Gamma_n)\), so that we get \(\text{Proven}(t, A) \in \Gamma_n = \text{end}(m' \cap h)\), whence \(t \in \text{Act}(m', h)\) as desired.

We turn next to the presenting a new proof makes histories divide constraint. Consider \(m, m' \in \text{Tree}\) such that \(m < m'\) and arbitrary \(h, h' \in H_{m'}\). We immediately get then that \(h, h' \in H_m\). If \(m = \top\), then the constraint is verified trivially. If \(m \neq \top\), then we have two cases to consider:

Case 1. \(\overrightarrow{m} = m'\) and \(|m| > |m'|\). Then we must have \(|m| > 0\), and by Lemma 18.2 it follows that in this case for all \(h, h' \in H_m\) we will have \(\text{Act}(m, h) = \text{Act}(m, h')\) so that the constraint is verified.

Case 2. There is a \(\xi \in \overrightarrow{m}\) such that \(\xi\) is a proper initial segment of every \(\tau \in \overrightarrow{m'}\). Consider then \(m' \cap h\) and \(m' \cap h'\). These are elements in \(m'\), and hence \(\xi\) is a proper initial segment of both \(m' \cap h\) and \(m' \cap h'\). Moreover, we know that \(m' \cap h = \emptyset\) and hence must be an initial segment of \(m' \cap h\) of the length equal to the length of \(\xi\). The same holds for \(m' \cap h\), respectively. It follows that \(m \cap h = m' \cap h' = \xi\)

whence, by Lemma 18.3, we immediately get \(\text{Act}(m, h) = \text{Act}(m, h')\).

It remains to check the epistemic transparency of presented proofs constraint. Assume that \(m, m' \in \text{Tree}\) are such that \(R(m, m')\). If we have \(m = \top\), then, by definition, we must have \(\text{Act}_m = \emptyset\), and the constraint is verified in a trivial way. If, on the other hand, \(m \neq \top\), then, by \(R(m, m')\), we must also have \(m' \neq \top\). Assume that \(t \in \text{Act}_m\). Then, by Lemma 18.1, we also have \(\text{Proven}(t, A) \in \text{end}(m \cap h)\) for some \(A \in \text{Form}^{Ag}\). By \(R(m, m')\) and Corollary 2 it follows that \(\text{Proven}(t, A) \in \bigcap_{g \in H_{m'}} \text{end}(m' \cap g)\), whence, again by Lemma 18.1, we know that \(t \in \text{Act}_{m'}\).

\[\square\]

4.3 The truth lemma

It follows from Lemmas 5, 12, 17 and 19 that our above-defined canonical model \(\mathcal{M}_{\mathcal{CS}}^{Ag}\) is a unirelational jstit model for \(Ag\). By Lemma 15 we know that \(\mathcal{M}_{\mathcal{CS}}^{Ag}\) is \(\mathcal{CS}\)-normal. Now we need to supply a truth lemma:

**Lemma 20.** Let \(A \in \text{Form}\), let \(m \in \text{Tree} \setminus \{\top\}\) be such that \(|m| = 0\), and let \(h \in H_m\). Then:

\[\mathcal{M}_{\mathcal{CS}}^{Ag}, m, h \models A \iff A \in \text{end}(m \cap h)\]

**Proof.** As is usual, we prove the lemma by induction on the construction of \(A\). The basis of induction with \(A = p \in \text{Var}\) we have by definition of \(V\), whereas Boolean cases for the induction step are trivial. The cases for \(A\) being of the form \(\Box B, [j]B, KB\) or
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It follows by Lemma 14, that we can choose a

This immediately gives us, by the induction case for

Next, we observe that

then, by Lemma 18.1, there

must be some

such that

By (B12) and

CS-maxiconsistency of

, this means that

contrary to our assumption. The obtained contradiction shows that

, whence by our assumption. The obtained contradiction shows that

In the other direction, assume that

We have then two

subcases to consider:

Case 1.1. \( t:B \not\in end(m \cap h) \). Then, by the induction case for \( t:B \) it follows that

whence, by definition of satisfaction relation, we get that

\( \mathcal{M}_{CS}^{Ag}, m, h \not\models t:B \).

Case 1.2. \( t:B \in end(m \cap h) \). Now, if \( t \in Act_m \), then, by Lemma 15, there

must be some \( C \in \text{Form}^{Ag} \) such that \( \text{Proven}(t, C) \in end(m \cap h) \). By (B12) and

CS-maxiconsistency of \( end(m \cap h) \), this means that \( \text{Proven}(t, B) \in end(m \cap h) \) contrary to our assumption. The obtained contradiction shows that \( t \not\in Act_m \), whence

, as desired.

Case 2. \( A = \text{Prove}(j, t, B) \) for some \( j \in Ag \) and \( t \in Pol \). If \( \text{Prove}(j, t, B) \in end(m \cap h) \), then, by CS-maxiconsistency of \( end(m \cap h) \) and (B12), we must have

This immediately gives us, by the induction case for \( t:B \), that:

\[ \mathcal{M}_{CS}^{Ag}, m, h \models t:B. \]  

Moreover, we can infer by Corollary 3.2 that \( \text{Prove}(j, t, B) \in end(m, h, j) \), whence it follows, by \( |m| = 0 \) and the definition of \( Act \), that:

\[ t \in Act_{m, h, j}. \]

Next, we observe that \( \neg \Box \text{Prove}(j, t, B) \in end(m \cap h) \) yields, by CS-maxiconsistency of \( end(m \cap h) \), that \( \Box \text{Prove}(j, t, B) \not\in end(m \cap h) \), whence, by Lemma 14 and Corollary 3, we get that \( \text{Prove}(j, t, B) \not\in end_m. \) Therefore, we choose a \( g \in H_m \) such that

. By CS-maxiconsistency of \( end(m \cap g) \) and (B13) we get that:

\[ \langle j \rangle \bigwedge_{i \in Ag} \neg \text{Prove}(i, t, B) \in end(m \cap g). \]  

It follows by Lemma 14 that we can choose a \( g' \in \text{Choice}_{j}^{m}(g) \subseteq H_m \) such that:

\[ \bigwedge_{i \in Ag} \neg \text{Prove}(i, t, B) \in end(m \cap g'). \]

We now show that assuming \( t \in Act(m, g') \) leads to a contradiction. Indeed, if \( t \in Act(m, g') \), then we would have either \( \text{Prove}(i, t, C) \) or \( \text{Proven}(t, C) \) in \( end(m \cap g') \) for some \( i \in Ag \) and \( C \in \text{Form}^{Ag} \). In the former case note that, by Corollary 3, it would follow from (50) that \( t:B \in end(m \cap g') \), whence by CS-maxiconsistency of
end(m \cap g') and (B10) we would further obtain that \( \text{Prove}(i, t, B) \in \text{end}(m \cap g') \), contrary to (54). In the latter case we would have \( \text{Proven}(t, C) \in \text{end}(m \cap h) \), by \( g', h \in H_m \), and Corollary 3.1. In virtue of CS-maxiconsistency of \( \text{end}(m \cap h) \), (51), and (B12) this would further imply that \( \text{Proven}(t, B) \in \text{end}(m \cap h) \), in contradiction with (50). The obtained contradiction shows that we must have:

\[
t \notin \text{Act}(m, g').
\]  

(55)

Adding up (51), (52), and (55), yields that \( \mathcal{M}^g_{\text{CS}}, m, h \models \text{Prove}(j, t, B) \).

In the other direction, assume that \( \text{Prove}(j, t, B) \notin \text{end}(m \cap h) \). We have to consider three further subcases.

Case 2.1. \( t:B \notin \text{end}(m \cap h) \). Then, by induction case for \( t:B \), it follows that \( \mathcal{M}^g_{\text{CS}}, m, h \not\models t:B \), whence by (B9) we get that \( \mathcal{M}^g_{\text{CS}}, m, h \not\models \text{Prove}(j, t, B) \).

Case 2.2. \( t:B, \text{Proven}(t, B) \in \text{end}(m \cap h) \). Then it follows, by Case 1, that \( \mathcal{M}^g_{\text{CS}}, m, h \models \text{Proven}(t, B) \), whence, again by (B9), we get that \( \mathcal{M}^g_{\text{CS}}, m, h \not\models \text{Prove}(j, t, B) \).

Case 2.3. \( t:B, \neg \text{Proven}(t, B) \in \text{end}(m \cap h) \). Then, by Corollary 3.1 and CS-maxiconsistency of \( \text{end}(m \cap h) \), it follows that \( t:B \in \text{end}_m \). Now, observe that \( \text{Prove}(j, t, B) \notin \text{end}(m \cap h) \) implies, by CS-maxiconsistency of \( \text{end}(m \cap h) \) and (B13), that:

\[
\langle j \rangle \bigwedge_{i \in Ag} \neg \text{Prove}(i, t, B) \in \text{end}(m \cap h).
\]  

(56)

It follows by Lemma 14 that we can choose a \( g \in \text{Choice}^m_j(h) \) such that:

\[
\bigwedge_{i \in Ag} \neg \text{Prove}(i, t, B) \in \text{end}(m \cap g).
\]  

(57)

We now show that assuming \( t \in \text{Act}(m, g) \) leads to a contradiction. Indeed, if \( t \in \text{Act}(m, g) \), then we would have either \( \text{Prove}(i, t, C) \) or \( \text{Proven}(t, C) \) in \( \text{end}(m \cap g) \) for some \( i \in Ag \) and \( C \in \text{Form}^Ag \). In the former case note that we also have \( t:B \in \text{end}(m \cap g) \), whence by CS-maxiconsistency of \( \text{end}(m \cap g) \) and (B10) we would further obtain that \( \text{Prove}(i, t, B) \in \text{end}(m \cap g') \), contrary to (57). In the latter case, again using \( t:B \in \text{end}(m \cap g) \), we would have \( \text{Proven}(t, B) \in \text{end}(m \cap g) \), by CS-maxiconsistency of \( \text{end}(m \cap g) \) and (B12), which, by Corollary 3.1 and the fact that \( g \in H_m \) would mean that \( \text{Proven}(t, B) \in \text{end}(m \cap h) \), making \( \text{end}(m \cap h) \) CS-inconsistent. The obtained contradiction shows that we must have:

\[
t \notin \text{Act}(m, g),
\]  

(58)

and since \( g \in \text{Choice}^m_j(h) \), this immediately implies that \( \mathcal{M}^g_{\text{CS}}, m, h \not\models \text{Prove}(j, t, B) \), as desired.

This finishes the list of the modal induction cases at hand, and thus the proof of our truth lemma is complete.

\[\square\]

5 The main result and conclusions

We are now in a position to prove Theorem 2. The proof proceeds as follows. One direction of the theorem was proved as Corollary 1. In the other direction, assume that
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\( \Gamma \subseteq Form^{Ag} \) is \( \mathcal{CS} \)-consistent. Then \( \Gamma \) can be extended to a \( \mathcal{CS} \)-maxiconsistent \( \Delta \). But then consider \( M^{Ag}_{\mathcal{CS}} = (Tree, \preceq, Choice, Act, R, \mathcal{E}, V) \), the canonical model defined in Section 4. It is clear that \( (\Delta) \) is an element, therefore \( m = \left( ([\Delta])_{\mathcal{E}}, 0 \right) \in Tree \). By Lemma 11 there is a history \( h \in H_m \) such that \( (\Delta) = \left( ([\Delta])_{\mathcal{E}}, 0 \right) \cap h \). For this \( h \), we will also have \( \Delta = \text{end} \left( \left( ([\Delta])_{\mathcal{E}}, 0 \right) \cap h \right) \). By Lemma 20 we therefore get that:

\[ M, \left( ([\Delta])_{\mathcal{E}}, 0 \right), h \models \Delta \supseteq \Gamma, \]

and thus \( \Gamma \) is shown to be satisfiable in a \( \mathcal{CS} \)-normal jstit unirelational model, hence in a normal jstit model.

**Remark.** Note that the canonical model used in this proof is universal in the sense that it satisfies every subset of \( Form^{Ag} \) which is \( \mathcal{CS} \)-consistent.

As an obvious corollary of Theorem 2 we get the compactness property. Thus, building up on an earlier work on justification stit formalisms, we have defined the explicit fragment of basic jstit logic introduced in [9]. For this logic we have presented a strongly complete axiomatization which is stable relative to extensions with constant specifications. This result is similar to the completeness theorem obtained earlier for JA-STIT in [8] and also borrows from this paper some techniques and results related to the construction of canonical model. We also note that Proposition 11 proven above in Section 2 shows that explicit jstit logic, just like JA-STIT, can distinguish between the class of models with a discrete temporal substructure and the general class of models, even though it apparently has less expressive powers. We observe that the formula \( A \) used in the proof of Proposition 11 is clearly related to the formula

\[ K(\neg \Box Ex \lor \Box Ey) \rightarrow (\neg Ex \lor Ey) \]

used to prove a similar proposition for JA-STIT in [8]. The latter formula was shown to admit of an easy generalization which led to an axiomatization of JA-STIT over the class of jstit models based on discrete time. It would be natural to look for an axiomatization of explicit jstit logic over the same class of models. However, this time the reduced expressive power of explicit jstit logic may actually prove to be an obstacle, since the generalization pattern which we applied in the case of JA-STIT does not seem to work in the case of explicit jstit logic.

Another natural, but even more uphill, task for the future research would be to try and attempt an axiomatization of full basic jstit logic now that we have positive experience with axiomatizations of both its implicit and its explicit fragment.

### 6 Acknowledgements

To be inserted.

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