Quantum Stationary Hamilton Jacobi Equation in 3-D for Symmetrical Potentials.
Introduction of the Spin.

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Abstract
We establish the quantum stationary Hamilton-Jacobi equation in 3-D and its solutions for three symmetrical potentials: Cartesian symmetry potential, spherical symmetry potential and cylindrical symmetry potential. For the two last potentials, a new interpretation of the spin is proposed within the framework of trajectory representation.

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1- Introduction

Since twenty years, the trajectory representation has been proposed by Floyd as a new approach of quantum mechanics. In fact, Floyd took up the QSHJE in one dimension, solved it \[1, 2, 3\] and constructed quantum trajectories from the Jacobi’s theorem \[4, 5\]
\[
\frac{\partial S_0}{\partial E} = t - t_0.
\] (1)

After fifteen years, Faraggi and Matone derived one dimensional QSHJE from the equivalence postulate \[6, 7, 8, 9\]. They took up the Jacobi’s theorem and established an equation of quantum motion \[8\]. Then with Bertoldi \[8, 9\], they derived the N-dimensional QSHJE. Later, Bouda \[10\] re-investigated the QSHJE and proposed to write the reduced action as
\[
S_0 = \bar{h} \arctan \left( \frac{\mu \theta + \phi}{\theta + \nu \phi} \right) + e \bar{h},
\] (2)
where \(\mu, \nu\) and \(e\) are real constants identified to the integration constants of the QSHJE.

Recently, Bouda and Djama have criticized the use of Jacobi’s theorem as written in Eq. (1), to derive trajectories equation since this theorem is applied in classical mechanics for a first order differential equation while the QSHJE is a third order one \[11\]. They have constructed, in one dimension, a quantum Lagrangian and formulated a quantum version of Jacobi’s theorem from which they obtained the relation connecting velocity \(\dot{x}\) and conjugate momentum \(\partial S_0/\partial x\).

Then, in a recent paper \[12\], Bouda and Djama have plotted quantum trajectories for several potentials, and have given a new physical meaning of the de Broglie wavelength.

The construction of the Hamilton-Jacobi equation in one dimension within the framework of the quantum mechanics \[1, 3\], is an important step for the elaboration of the deterministic quantum theory introduced by Floyd \[1, 2, 3, 4\], and reformulated by Bouda and Djama \[11, 12\]. Nevertheless, such equation is not sufficient for an adequate description of reality. Indeed, physical phenomena are often realized in three dimensional space. For this reason, a generalization of the quantum stationary Hamilton-Jacobi equation (QSHJE) into 3-D prove to be indispensable. For this purpose, we introduce an attempt of such generalization. Thus, we construct in this letter the 3-D QSHJE for three symmetrical potentials. In Sec. 2, the Cartesian symmetry potential is studied. Then, in Sec. 3 we investigate the case of the spherical symmetry potential. After, in Sec. 4, we approach the cylindrical symmetry potential. Finally, in Sec. 5, we comment on our results and propose a new interpretation of the spin and its physical meaning in trajectory interpretation of quantum mechanics.
2- Cartesian Symmetry Potential

Let us begin by a potential with Cartesian symmetry whose expression is given by

\[ V(\vec{r}) = V_x(x) + V_y(y) + V_z(z) = \sum_q V(q) ; q = x, y, z . \]  \hspace{1cm} (3)

An example of such a potential is the spatial harmonic oscillator’s potential. The stationary Schrödinger equation is

\[ -\frac{\hbar^2}{2m} \Delta \psi(x) + V(\vec{r}) \psi(x) = E \psi(x) . \]  \hspace{1cm} (4)

For a potential given by Eq. (3), the wave function is written as the product of three functions of one variable

\[ \psi(x) = \prod_q \phi_q(q) ; q = x, y, z , \]  \hspace{1cm} (5)

and the corresponding energy is

\[ E = \sum_q E_q ; q = x, y, z . \]  \hspace{1cm} (6)

By substituting Eqs. (3), (5) and (6) in Eq. (4), one get to the one dimensional Schrödinger equations

\[ -\frac{\hbar^2}{2m} \frac{d^2 \phi_q}{dq^2} + V_q(q) \phi_q(q) = E_q \phi_q(q) , \quad q = x, y, z . \]  \hspace{1cm} (7)

As in the one dimensional approach [8, 9, 10] where the same form as (8) is used for the one dimensional wave function \( \psi(x) \). Then, it is obvious to take up this form for each one dimensional function \( \phi_q \). Taking Eq. (8) into Eq. (7), we obtain

\[ \frac{1}{2m} \left( \frac{\partial S_{0q}(q)}{\partial q} \right)^2 - \frac{\hbar^2}{2m} \frac{\partial^2 A_q(q)/\partial q^2}{A_q(q)} + V_q(q) - E_q = 0 . \]  \hspace{1cm} (9)

which is satisfied when the two following equations are so

\[ \frac{1}{2m} \left( \frac{\partial S_{0q}(q)}{\partial q} \right)^2 - \frac{\hbar^2}{2m} \frac{\partial^2 A_q(q)/\partial q^2}{A_q(q)} + V_q(q) - E_q = 0 . \]  \hspace{1cm} (10)
\[ \frac{\partial}{\partial q} \left( A_q \frac{\partial S_{0q}}{\partial q} \right) = 0 . \]  

(11)

Now, as it is the case in one dimension, Eqs (10) and (11) lead to

\[ \frac{1}{2m} \left( \frac{\partial S_{0q}(q)}{\partial q} \right)^2 - \frac{\hbar^2}{2m} \{ S_{0q}, q \} + V_q = E_q , \]  

(12)

where \( \{ S_{0q}, q \} \) is the Schwarzian derivative of \( S_{0q} \) with respect to \( q \) and given by

\[ \{ S_{0q}, q \} = \left[ \frac{3}{2} \left( \frac{\partial^2 S_{0q}}{\partial q^2} / \partial q \right) - \frac{\partial^3 S_{0q}}{\partial q^3} \right] , \]  

(13)

and \( \partial S_{0}/\partial q \) representing the conjugate momentum along the direction \( q \).

Eqs. (12) represents the components of the QSHJE along each of the three directions \( x, y \) and \( z \). If we set

\[ S_0 = \sum_q S_{0q} , \quad q = x, y, z , \]  

(14)

where \( S_0 \) define the reduced action in 3-D, then summing the three Eqs. (12) and taking in account Eqs. (3), (6) and (14), we find

\[ \frac{1}{2m} (\nabla S_0)^2 - \frac{\hbar^2}{4m} \sum_q \{ S_{0q}, q \} + V(\vec{r}) = E . \]  

(15)

Eq. (15) represents the QSHJE in 3-D for a potential with Cartesian symmetry. It is a third order differential equation since the Schwarzian contains the third derivative of \( S_0 \) with respect to \( q \). The terms of more than one order of the derivative with respect to \( q \) are proportional to \( \hbar^2 \) meaning that at the classical limit (\( \hbar \rightarrow 0 \)) these terms vanish and Eq. (15) goes to the classical stationary Hamilton-Jacobi equation (CSHJE)

\[ \frac{1}{2m} (\nabla S_0)^2 + V(\vec{r}) = E . \]  

(16)

These results are already presented by Faraggi and Matone \[6, 8\] within the framework of the equivalence postulate and using differential geometry.

It is useful to remark that through our equations for Cartesian symmetry potential, a dynamical approach of the motion, as it is done in one dimension \[11\], seems to be possible by treating each one of the variables \( q \) separately from the two others

3. The Spherical Symmetry Potential

Let us now examine the case of the spherical symmetry potential. For this purpose, we take up the stationary Schrödinger equation written with spherical coordinates

\[ \cot \vartheta \frac{\partial}{\partial \vartheta} \psi + \frac{1}{r^2 \sin^2 \vartheta} \frac{\partial^2 \psi}{\partial \varphi^2} + \frac{2m}{\hbar^2} (E - V(r))\psi = 0 , \]  

(17)
for which the potential depends only on the radius $r$, $V(\vec{r}) = V(r)$. The Schrödinger wave function can be written in the following form

$$\psi(\vec{r}) = R(r) T(\theta) F(\varphi),$$

where $R(r)$ is the radial wave function, $T(\theta)$ and $F(\varphi)$ are the angular wave functions. An example of such a problem is the dynamics of an electron in hydrogenoid atoms. If one substitute Eq. (18) in Eq. (17), one gets to the following relations

$$\frac{r^2}{R} \frac{d^2 R}{dr^2} + \frac{2r}{R} \frac{dR}{dr} + \frac{2m\ell^2}{\hbar^2} (E - V(r)) = \lambda,$$

(19)

$$\frac{d^2 T}{d\theta^2} + \cot \theta \frac{dT}{d\theta} + \left( \lambda - \frac{m^2\ell^2}{\sin^2 \theta} \right) T = 0,$$

(20)

$$\frac{d^2 F}{d\varphi^2} + m^2\ell F = 0,$$

(21)

where $\lambda$ is a constant such as

$$\lambda = \ell(\ell + 1),$$

$\ell$ being positive integer or vanish, and $m\ell$ is an integer satisfying to

$$-\ell \leq m\ell \leq \ell.$$

$\ell(\ell+1)$ is the eigen value of $L^2$, $L$ representing the angular momentum operator. $m\ell$ represents the eigen value of $L_z$, the projection operator of $L$ along the $z$ axis.

Let us write Eqs. (19), (20) and (21) in a form analogous to the one dimensional Schrödinger equation (Eq. (7)). First, take up Eq. (19) and write that

$$R(r) = \frac{\mathcal{X}(r)}{r},$$

(22)

where $\mathcal{X}(r)$ is a function of the radius. Then by taking twice the derivative of Eq. (22) with respect to $r$, we obtain

$$\frac{dR}{dr} = \frac{1}{r} \frac{d\mathcal{X}}{dr} - \frac{\mathcal{X}}{r^2},$$

(23)

$$\frac{d^2 R}{dr^2} = \frac{1}{r} \frac{d^2 \mathcal{X}}{dr^2} - \frac{2}{r^2} \frac{d\mathcal{X}}{dr} + \frac{2}{r^3} \mathcal{X}.$$

(24)

After substituting Eqs. (22) and (23) in Eq. (19), one gets

$$\frac{-\hbar^2}{2m} \frac{d^2 \mathcal{X}}{dr^2} + \left[ V(r) + \frac{\lambda\hbar^2}{2m\ell^2} \right] \mathcal{X} = E\mathcal{X}.$$

(25)

This last equation has the same form as the one dimensional Schrödinger equation for $\mathcal{X}(r)$ with an energy $E$ and a fictive potential

$$V'(r) = V(r) + \frac{\lambda\hbar^2}{2m\ell^2}.$$

(26)
Likewise, the form of Eq. (20) can be reduced to the form of Eq. (7). This can be done by introducing the function $T(\vartheta)$ defined as

$$T(\vartheta) = \sin \frac{1}{2} \vartheta T(\vartheta).$$

Taking twice the derivative of this last equation with respect to $\vartheta$, we deduce

$$\frac{d^2T}{d\vartheta^2} = \frac{1}{\sin^2 \vartheta} \frac{d^2T}{d\vartheta^2} - \cos \vartheta \frac{dT}{d\vartheta} + \frac{3}{4} \cos^2 \vartheta + \frac{1}{2} \frac{T}{\sin \vartheta}. \quad (28)$$

If we replace Eqs. (28) and (29) in Eq. (20), we obtain

$$\frac{d}{d\vartheta} \left( \frac{d^2T}{d\vartheta^2} \right) + \left( \lambda + \frac{1}{4} \right) T + \frac{(1/4 - m^2)}{\sin^2 \vartheta} T = 0. \quad (30)$$

Remark that Eq. (30) has the form of the one dimensional Schrödinger equation with a potential

$$V(\vartheta) = \frac{\hbar^2}{2m} \frac{(m^2 - 1/4)}{\sin^2 \vartheta},$$

and an energy

$$E_\vartheta = \left( \lambda + \frac{1}{4} \right) \frac{\hbar^2}{2m}. \quad (32)$$

Remark also that Eq. (21) has the same form as Eq. (7) with a vanishing potential and an energy equal to $(m^2 \hbar^2/2m)$.

Because Eqs. (19), (20) and (21) come to the form of the one dimensional Schrödinger equation, it is legitimate to take up the form (8) for the function $X(r)$, $T(\vartheta)$ and $F(\varphi)$ so to write them as

$$X(r) = A(r) \left[ \alpha e^{\pm iZ(r)} + \beta e^{-\pm iZ(r)} \right], \quad (33)$$

$$T(\vartheta) = \xi(\vartheta) \left[ \gamma e^{\pm iL(\vartheta)} + \epsilon e^{-\pm iL(\vartheta)} \right], \quad (34)$$

$$F(\varphi) = \eta(\varphi) \left[ \sigma e^{\pm iM(\varphi)} + \omega e^{-\pm iM(\varphi)} \right]. \quad (35)$$

By replacing Eqs. (33), (34) and (35) respectively in Eqs. (25), (30) and (21), we get

$$\frac{1}{2m} \left( \frac{dZ}{dr} \right)^2 - \frac{\hbar^2}{2m} \frac{1}{A \frac{dA}{dr^2}} + V(r) + \frac{\lambda \hbar^2}{2m} r^2 = E, \quad (36)$$

$$\frac{d}{dr} \left( A^2 \frac{dZ}{dr} \right) = 0, \quad (37)$$

$$\left( \frac{dL}{d\vartheta} \right)^2 - \frac{\hbar^2}{\xi} \frac{d^2 \xi}{d\vartheta^2} - \left( \lambda + \frac{1}{4} \right) \frac{\hbar^2}{\sin^2 \vartheta} = 0, \quad (38)$$

$$\frac{d}{d\vartheta} \left( \xi^2 \frac{dL}{d\vartheta} \right) = 0, \quad (39)$$

$$\left( \frac{dM}{d\varphi} \right)^2 - \frac{\hbar^2}{\eta(\varphi)} \frac{d^2 \eta}{d\varphi^2} - m^2 \hbar^2 = 0, \quad (40)$$
\[
\frac{d}{d\varphi} \left( \eta^2 \frac{dM}{d\varphi} \right) = 0. \tag{41}
\]

As in one dimensional case, these equations lead to

\[
\frac{1}{2m} \left( \frac{dZ}{dr} \right)^2 - \frac{\hbar^2}{4m} \{Z, r\} + V(r) + \frac{\lambda \hbar^2}{2mr^2} = E, \tag{42}
\]

\[
\left( \frac{dL}{d\vartheta} \right)^2 - \frac{\hbar^2}{2} \{L, \vartheta\} + \frac{1}{r^2 \sin^2 \vartheta} \hbar^2 = (\lambda + \frac{1}{4}) \hbar^2, \tag{43}
\]

\[
\left( \frac{dM}{d\varphi} \right)^2 - \frac{\hbar^2}{2} \{M, \varphi\} = m^2 \ell \hbar^2. \tag{44}
\]

Eqs. (42), (43) and (44) represent the components of the QSHJE in 3-D for a spherical symmetry potential. Note that these equations contain the Schwarzian derivatives of the functions \(X, T\) and \(F\) respectively, and the conjugate momentums \(\frac{dZ}{dr}, \frac{dL}{d\vartheta}\) and \(\frac{dM}{d\varphi}\). Now, by deducing the value of \(\lambda\) from (42) and the value of \(m \ell\) from (44), then substituting in Eq. (43), one obtain

\[
\frac{1}{2m} \left( \frac{dZ}{dr} \right)^2 + \frac{1}{r^2} \left( \frac{dL}{d\vartheta} \right)^2 + \frac{1}{r^2 \sin^2 \vartheta} \left( \frac{dM}{d\varphi} \right)^2 -
\frac{\hbar^2}{4m} \{Z, r\} + \frac{1}{r^2 \sin^2 \vartheta} \{L, \vartheta\} + \frac{1}{r^2 \sin^2 \vartheta} \{M, \varphi\} +
V(r) - \frac{\hbar^2}{8m r^2} - \frac{\hbar^2}{8m r^2 \sin^2 \vartheta} = E. \tag{45}
\]

Let us define the reduced action in this case as

\[
S_0(r, \vartheta, \varphi) = Z(r) + L(\vartheta) + M(\varphi). \tag{46}
\]

Taking up this equation in Eq. (45), we find

\[
\frac{1}{2m} \left( \nabla_{r, \vartheta, \varphi} S_0 \right)^2 - \frac{\hbar^2}{4m} \{S_0, r\} + \frac{1}{r^2} \{S_0, \vartheta\} +
\frac{1}{r^2 \sin^2 \vartheta} \{S_0, \varphi\} + V(r) - \frac{\hbar^2}{8m r^2} - \frac{\hbar^2}{8m r^2 \sin^2 \vartheta} = E. \tag{46}
\]

Eq. (46) represents the QSHJE in 3-D for a spherical symmetry potential. At the classical limit \(\hbar \to 0\), Eq. (46) goes to

\[
\frac{1}{2m} \left( \nabla_{r, \vartheta, \varphi} S_0 \right)^2 + V(r) = E. \tag{47}
\]

which is the CSHJE. Note that taking the classical limit in Eqs. (42), (43) and (44) making \(dL/d\vartheta\) and \(dM/d\varphi\) vanish, then, one cannot obtain the CSHJE (Eq.(47)). The classical limit must be taken in Eq. (46). Note also that considering the QSHJE, and separate variables we get to Eqs. (42), (43) and (44) which lead to Eqs. (19), (20) and (21). After separating variables in Eq. (46), three constants of motion appear. They are the energy \(E\), \(\lambda\) and \(m_\ell\). These two last constants will be linked in Sec. 5 with the quantum angular momentum.
Before introducing the solutions of Eqs. (42), (43) and (44), remark that these equations are identical to the one dimensional QSHJE with particular potentials and energies. These three equations are solvable when Eqs. (19), (20) and (21) are so. Then, as the solution of the one dimensional QSHJE is written in terms of two independent solutions of the stationary Schrödinger equation, the solutions of Eqs (42), (43) and (44) are written in terms of two independent solutions of Eqs. (19), (20) and (21) respectively. These solutions are

\[ Z(r) = \hbar \arctan \left\{ \frac{\mu_r R_1(r) + R_2(r)}{R_1(r) + \nu_r R_2(r)} \right\}, \]

\[ L(\vartheta) = \hbar \arctan \left\{ \frac{\mu_\vartheta T_1(\vartheta) + T_2(\vartheta)}{T_1(\vartheta) + \nu_\vartheta T_2(\vartheta)} \right\}, \]

\[ M(\phi) = \hbar \arctan \left\{ \frac{\mu_\phi \sin(m_\ell \phi) + \cos(m_\ell \phi)}{\sin(m_\ell \phi) + \nu_\phi \cos(m_\ell \phi)} \right\}, \]

in which we have used Bouda’s notation \(^{(10)}\), and where \( \mu_r, \nu_r, \mu_\vartheta, \nu_\vartheta, \mu_\phi \) and \( \nu_\phi \) are real constants. \( R_1 \) and \( R_2 \) are two real independent solutions of Eq. (19). \( T_1 \) and \( T_2 \) are two real independent solutions of Eq. (20). \( \sin(m_\ell \phi) \) and \( \cos(m_\ell \phi) \) are two independent solutions of Eq. (21).

4 The Cylindrical Symmetry Potential

Now, let us consider a potential with cylindrical symmetry whose expression is

\[ V(\vec{r}) = V(\rho) . \]

The stationary Schrödinger equation written with the cylindrical coordinates \((\rho, \phi, z)\) \(^{(13)}\) is

\[ -\frac{\hbar^2}{2m} \left( \frac{d^2 \psi}{d\rho^2} + \frac{1}{\rho} \frac{d \psi}{d\rho} + \frac{1}{\rho^2} \frac{d^2 \psi}{d\phi^2} + \frac{d^2 \psi}{dz^2} \right) + V(\rho) = E \psi . \]

In this case, we can write the wave function \( \psi(\rho, \phi, z) \) as

\[ \psi(\rho, \phi, z) = G(\rho) \ N(\phi) \ U(z) . \]

\( G(\rho) \) is the radial wave function. \( N(\phi) \) is the angular wave function while \( U(z) \) is the axial wave function. By substituting relation (53) in Eq. (52), one obtain

\[ -\frac{\hbar^2}{2m} \left( \frac{d^2 G}{d\rho^2} + \frac{1}{\rho} \frac{d G}{d\rho} \right) + \left[ \frac{m_\phi^2 \hbar^2}{2m\rho^2} - E + V(\rho) - \frac{\beta}{2m\rho^2} \right] G(\rho) = 0 . \]

\[ \frac{d^2 N}{d\phi^2} + m_\phi^2 N(\phi) = 0 \]

\[ \frac{d^2 U}{dz^2} - \beta U(z) = 0 \]

\( \beta \) is a real constant and \( m_\phi \) is an integer one. Eqs. (55) and (56) have the same form as the one of Schrödinger equation with vanishing potential, while Eq. (54) can be written in the Schrödinger equation form by setting

\[ G(\rho) = \frac{H(\rho)}{\sqrt{\rho}} . \]
Taking twice the derivative of Eq. (57) with respect to \( \rho \), we get to
\[
\frac{dG}{d\rho} = \frac{dH/d\rho}{\sqrt{\rho}} - \frac{H(\rho)}{2\rho^2}.
\] (58)
\[
\frac{d^2G}{d\rho^2} = \frac{d^2H/d\rho^2}{\sqrt{\rho}} - \frac{dH/d\rho}{\rho\sqrt{\rho}} + \frac{H(\rho)}{4\rho^2} + \frac{3H(\rho)}{4\rho^2}.
\] (59)

If we inject these two last relations in Eq. (54), we deduce
\[
- \bar{h}^2 \frac{d^2H}{d\rho^2} + \left[ \frac{(m^2_{\phi} - 1/4)h^2}{2m\rho^2} - \beta \frac{h^2}{2m} + V(\rho) \right] H(\rho) = E \frac{H(\rho)}{H(\rho)}.
\] (60)

Now, just as spherical symmetry and Cartesian symmetry cases, the functions
\[ H(\rho) = h(\rho) \left[ \alpha_{\rho} e^{iZ_{\rho}(\rho)} + \beta_{\rho} e^{-iZ_{\rho}(\rho)} \right] , \] (61)
\[ N(\phi) = n(\phi) \left[ \alpha_{\phi} e^{iM_{\phi}(\phi)} + \beta_{\phi} e^{-iM_{\phi}(\phi)} \right] , \] (62)
\[ U(z) = u(z) \left[ \alpha_z e^{iK(z)} + \beta_z e^{-iK(z)} \right] , \] (63)
where \( \alpha_{\rho}, \beta_{\rho}, \alpha_{\phi}, \beta_{\phi}, \alpha_z \) and \( \beta_z \) are complex constants. \( h(\rho), n(\phi) \) and \( u(z) \) are real functions of one variable. Injecting Eqs. (61), (62) and (63) in Eqs. (60), (55) and (56), we get, with same procedure as in one dimension, to the following equations
\[
\frac{1}{2m} \left( \frac{dZ_{\rho}}{d\rho} \right)^2 - \frac{h^2}{4m} \{ Z_{\rho}, \rho \} + V(\rho) + \frac{(m^2_{\phi} - 1/4)h^2}{2m\rho^2} = E + \frac{\beta h^2}{2m} ,
\] (64)
\[
\left( \frac{dM_{\phi}}{d\phi} \right)^2 - \frac{h^2}{4m} \{ M_{\phi}, \phi \} = m^2_{\phi} h^2 ,
\] (65)
\[
\left( \frac{dK}{dz} \right)^2 - \frac{h^2}{4m} \{ K, z \} + \beta h^2 = 0 ,
\] (66)
Eqs. (64), (65) and (66) are the components of QSHJE in 3-D for a potential with cylindrical symmetry. They contain the Schwarzian derivatives of respectively \( Z_{\rho}, M_{\phi} \) and \( K(z) \). By deducing the value of \( m_{\phi} \) from Eq. (65) and \( \beta \) from (66) and replacing in Eq. (64), we obtain
\[
\frac{1}{2m} \left[ \left( \frac{dZ_{\rho}}{d\rho} \right)^2 + \frac{1}{\rho^2} \left( \frac{dM_{\phi}}{d\phi} \right)^2 + \left( \frac{dK}{dz} \right)^2 \right] - \frac{h^2}{4m} \{ Z_{\rho}, \rho \} + \frac{1}{\rho^2} \{ M_{\phi}, \phi \} + \{ K, z \} + V(\rho) - \frac{h^2}{8m\rho^2} = E .
\] (67)

Let us define the reduced action in 3-D for a cylindrical symmetry potential as
\[ S_0(\rho, \phi, z) = Z_{\rho}(\rho) + M_{\phi}(\phi) + K(z) .
\]
Injecting this last relation in Eq. (67), we get
\[
\frac{1}{2m} \left( \vec{\nabla}_{\rho,\phi,z} S_0 \right)^2 - \frac{\hbar^2}{4m} \left[ \{S_0, \rho\} + \frac{1}{\rho^2} \{S_0, \phi\} + \{S_0, z\} \right] + V(\rho) - \frac{\hbar^2}{8m \rho^2} = E.
\]  
(68)

Eq. (68) represents the QSHJE in 3-D for a cylindrical symmetry potential. At the classical limit \((\hbar \to 0)\), Eq. (68) goes to
\[
\frac{1}{2m} \left( \vec{\nabla}_{\rho,\phi,z} S_0 \right)^2 + V(\rho) = E,
\]  
(69)

which is the CSHJE. As in the spherical symmetry case, taking the classical limit in the three components of the QSHJE does not lead to the CSHJE because the conjugate momenta \(dM_\phi/d\phi\) and \(dK/dz\) vanish. The classical limit must be taken in Eq. (68).

Note that for this case, after having separate variables in Eq. (68), three constants of motion appear, \(m_\phi\), \(\beta\) and energy \(E\). The constant \(m_\phi\) will be linked, in the following section, with the angular momentum of the particle.

5- Trajectory Representation and the Spin

Now, let us make some comments on the results obtained above. First, remark that, for each of the three cases studied in this paper, quantum terms appear in the 3-D QSHJE. The Schwarzian derivatives appear for the three cases (Eqs. (15), (46) and (68)), which means that they take the fundamental role in the motion of the particle. These derivatives can have, for the dynamical motion, the same role as the Schwarzian derivative in one dimension (see Ref. [11]). For the spherical and cylindrical symmetry cases, we observe in addition of Schwarzian derivatives, more quantum terms in the 3-D QSHJE.

For the spherical symmetry potential, the QSHJE (Eq. (46)) contains two quantum terms
\[
ter_1 = -\frac{\hbar^2}{8mr^2} = -\frac{(\pm 1/2)^2 \hbar^2}{2mr^2}
\]
and
\[
ter_2 = -\frac{\hbar^2}{8mr^2 \sin^2 \vartheta} = -\frac{(\pm 1/2)^2 \hbar^2}{8mr^2}.
\]

These two terms are purely quantum’s, since at the classical limit they vanish. \(ter_1\) has, as we can see in Eq. (42), the form of \(-\lambda \hbar^2 / 2mr^2\) in which \(\lambda = \ell(\ell+1)\) represents the eigen value of \(L^2\) (\(L\) being angular momentum operator). For these reasons, it is obvious to link the quantity \(1/4 = (\pm 1/2)^2\) appearing in \(ter_1\) with a residual angular momentum which may be connected, as a first approach, with the spin momentum of the particle. This can be done since the semi-integer value \(1/2\) appears only for spin momentum. The term \(ter_2\) has the form of \(m_\ell^2 / 2mr^2 \sin^2 \vartheta\) in which \(m_\ell\) is the eigen value of \(L_z\). This suggest that the quantities \((\pm 1/2)^2\) are related to the two projections of the residual angular momentum or spin. Otherwise, the quantum terms \((-\hbar^2 / 8mr^2)\) and \((-\hbar^2 / 8mr^2 \sin^2 \vartheta)\) in Eq. (46) can be seen, in a first approach, as the manifestation of the spin momentum. And, the vanishing of these terms at the classical limit \((\hbar \to 0)\) is in agreement with the fact that the Spin is a quantum characteristic of particles.
The cylindrical symmetry case shows the same property of residual angular momentum (Spin) since a quantum term $-\hbar^2/8\hbar r^2$ appears in the 3-D QSHJE (Eq. (68)).

**Conclusions** In the present article, we have introduced a generalization of the one dimensional QSHJE -introduced and investigated by Floyd [1, 2, 3, 5]- into three dimensions for three cases, Cartesian, spherical and cylindrical symmetry potentials. The aim of such as generalization is to make possible the study of three dimensional motions. Hence, Eqs. (15), (46) and (68) can be considered as the first step of a dynamical approach in 3-D. For the Cartesian symmetry case, such approach is obvious, since we can separate the three directions of motion, and define for each direction the corresponding Lagrangian in the same manner as it is done in Ref. [11]. For the other cases, it is not obvious to make a dynamical approach of the particle’s motion. But, the presence of the Schwarzian derivatives in the 3-D QSHJE and the analogy between the form of the 3-D reduced actions (Eqs. (46), (47), (48)) and the one dimensional reduced action [2, 10] indicates that such an approach is possible.

Another point which is important is the fact that to reduce Eqs. (46) and (68) to the 3-D CSHJE, one must take the limit in these equations, not in the separated equations (Eqs. (42), (43), (44), (62), (63) and (64) respectively).

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