GENERICITY FOR NON-WANDERING SURFACE FLOWS

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Abstract. Consider the set $\chi_{nw}^r$ of non-wandering $C^r$ flows on a closed surface ($r = 0$ or $1$). Then such a flow can be approximated by regular non-wandering flows without heteroclinic connections nor locally dense orbits in $\chi_{nw}^r$. Moreover we state the characterization of topologically stable non-wandering flow on $S^2$ and the non-existence of structurally stable non-wandering flows on closed surfaces.

1. Introduction and preliminaries

In [NZ], they have shown the characterization of the non-wandering flows on compact surfaces with finitely many singularities. In [Mar], one has given a description near orbits of the non-wandering flow with the set of singularities which is totally disconnected. On the other hand, in [MW], they have characterized structurally stable divergence free vector fields on connected compact orientable surfaces. In this paper, we study genericity of non-wandering flows with arbitrary singularities on closed surfaces.

By flows, we mean continuous $\mathbb{R}$-actions on compact surfaces. Let $v : \mathbb{R} \times M \to M$ be a flow. Put $v_t(\cdot) := v(t, \cdot)$ and $O_v(\cdot) := v(\mathbb{R}, \cdot)$. Recall that a point $x$ of $M$ is non-wandering if for each neighborhood $U$ of $x$ and each positive number $N$, there is $t \in \mathbb{R}$ with $|t| > N$ such that $v_t(U) \cap U$ (resp. $v_t(x) \cap U$) is not empty, and that $v$ is non-wandering if every point is non-wandering. An orbit is proper if it is embedded, locally dense if the closure of it has nonempty interior, and exceptional if it is neither proper nor locally dense. A point is proper (resp. locally dense, exceptional) if so is its orbit. Denote by $\Omega(v)$ the set of non-wandering points of $v$ and by LD (resp. E, P) the union of locally dense orbits (resp. exceptional orbits, non-singular non-periodic proper orbits). Notice that LD need not open (see Example 1). Denote by $\chi^r_{nw} = \chi^r_{nw}(M)$ the set of non-wandering $C^r$ flows on a surface $M$ with respect to the $C^r$-topology for $r \geq 0$. Recall that a (degenerate) saddle is a fixed point with $N \neq 2$ saddle-like sectors bounded by prongs as separatrices. For $N = 2k$ it is topologically an ordinary saddle of multiplicity $k - 1$, for odd $N$ it is locally non-orientable. A saddle connection is an orbit of a regular orbit whose $\omega$-limit (resp. $\alpha$-limit) set is a singleton of a saddle point.

2. Structural stability

Lemma 2.1. Each flow $v$ can be approximated by a flow without locally dense orbits in $\chi^r_{nw}$ such that it preserves $\text{Sing}(v) \cup \text{Per}(v)$. Moreover if $\text{Sing}(v)$ is finite, then this perturbation can be chosen such that it preserves $\text{Sing}(v) \cup \text{Per}(v) \cup P$.
Proof. Fix any $v \in \chi_{\text{nw}}^r$. By Theorem 2.3 [Y], there are no exceptional orbits and $\text{Per}(v)$ is open. Then $\text{Per}(v) \subseteq \text{Per}(v) \cup \text{Sing}(v) \cup P$ and so $\text{Per}(v) \cup \text{Sing}(v) \cap \text{LD} = \emptyset$. By taking the doubling of $M$, we may assume that $M$ is closed. By induction on the genus $g = g(M)$ of $M$, we show the assertion. Suppose that $g = 0$. Trivially there are no locally dense orbits. Suppose that $g > 0$. For any locally dense orbit $O$, we can take a small transversal arc $\gamma$ to a point of $O$ such that $\gamma \cap (\text{Sing}(v) \cup \text{Per}(v)) = \emptyset$. If $\text{Sing}(v)$ is finite, then Corollary 2.4 [Y] implies that we can choose $\gamma$ which is disjoint from $\text{Sing}(v) \cup \text{Per}(v) \cup P$, since $\text{Sing}(v) \cup \text{Per}(v) \cup P = M - \text{LD}$ is closed. Considering the orientable double covering of $M$ if necessary, we can construct a closed transversal $T \subseteq \text{int}O$. Moreover we can choose that $T$ is disjoint from $\text{Sing}(v) \cup \text{Per}(v) \cup P$ if $\text{Sing}(v)$ is finite. Then $T \cap O$ is dense in $T$. Recall the following fact [BS] that for any countable dense subsets $A$ and $B$ of $\mathbb{R}$, there is an entire function $f : \mathbb{C} \to \mathbb{C}$ such that $f(A) = B$. Taking an arbitrary small perturbation by a rotation near $v$, we can make the $O \cap T \subseteq \text{Per}(v')$ for the resulting flow $v'$. Now we show that $v'$ is non-wandering. Indeed, for any $x \in M$, if $x \in \mathbb{O}_v(T)$, then $x \in \mathbb{Per}(v')$. If $x \notin \mathbb{O}_v(T)$, then the fact $v|_{M - \mathbb{O}_v(T)} = v'|_{M - \mathbb{O}_v(T)}$ implies that $x$ is non-wandering for $v'$. By construction, we have $v = v'$ on $\mathbb{Sing}(v) \cup \text{Per}(v) \cup P$ if $\text{Sing}(v)$ is finite. Finally we break locally dense orbits. Fix $y \in O \cap T \subseteq \text{Per}(v')$. Consider $M - \mathbb{O}_{v'}(y)$ and adding two center disks to the boundaries of $M - \mathbb{O}_{v'}(y)$. Let $M'$ be the resulting closed surface and $w'$ the resulting non-wandering flow on $M'$. By inductive hypothesis, $w'$ can be approximated by a non-wandering flow $w''$ without locally dense orbits which preserves $\text{Sing}(w') \cup \text{Per}(w')$. Moreover $P$ is also preserved if $\text{Sing}(v)$ is finite. Therefore $w''$ can be lifted to such a flow $w'''$ on $M$. This $w'''$ is desired.

Recall that a $C^r$ flow $(r > 0)$ is regular if it has no degenerate singularities and that a continuous flow is regular if each singularity has a neighborhood which is topologically equivalent to a neighborhood of a non-degenerate singularity.

**Lemma 2.2.** $C^r$ regular non-wandering flows form a dense subset of $\chi_{\text{nw}}^r$ for each $r = 0$ or 1.

**Proof.** Fix any non-identical $v \in \chi_{\text{nw}}^r$ and any distance $d$ induced by a Riemannian metric. If there is an essential periodic orbit, then we can obtain a new non-wandering $C^r$ flow by removing it and pasting two center disks. Thus we may assume that there are no essential periodic orbits. By Lemma 2.1 we may assume that $v$ has no locally dense orbits. Since $v$ is non-wandering, we have that $\text{Per}(v) \supseteq \text{Sing}(v)$ and so that $M - (\text{Sing}(v) \cup \text{Per}(v)) = P$ is nowhere dense. Fix a small number $\varepsilon > 0$ and a large number $T > 0$. We show that there are a number $\delta' > 0$ and a compact neighborhood $U$ of $\text{Sing}(v)$ whose boundary has finitely many tangencies for $v$ such that $\max_{t \in [-T,T]} d^\prime_t(v_t|_{U}, \text{id}) < \varepsilon/2$ and $\max_{t \in [-T,T]} d(v_t(x), y) < \varepsilon/2$ for any points $x, y \in U'$ with $d(x, y) < \delta'$, where $d^\prime_t(v_t|_{U}, \text{id}) := \max_{x \in U} d(v_t(x), x)$ and $d^\prime_t(v_t|_{U}, \text{id}) := \max_{x \in U} d(v_t(x), x), \| D(v_t(x)) \| - 1$. Here $\| . \|$ is the operation norm and a tangency to a curve $C$ means a point $p \in C$ which has a small neighborhood $N$ of it and a small number $s$ such that $C \cap \cup_{t \in (-s,s)} v_t(p) = \{ p \}$ and a connected component of $N - C$ contains $\cup_{t \in (-s,s)} v_t(p) \setminus C$. Indeed, for $x \in \text{Sing}(v)$, let $B_x(x) := \{ z \in M | d(x, z) < \varepsilon \}$. Since $v : [-T,T] \times M \to M$ is uniformly continuous, there is a large number $n > 0$ such that $\max_{t \in [-T,T]} d^\prime_t(v_t|_{B_x(x)}, \text{id}) < \varepsilon/2$. Let $U_x := \{ y \in B_x/\varepsilon(x) | \max_{t \in [-T,T]} d(v_t(y), y) < \varepsilon/2 \}$. By construction of $n$, we have $U_x \neq \emptyset$. Let $U_x' \subset U_x$ be an open neighborhood of $x$ such that $\partial U_x'$ is
homeomorphic to a circle and that $\overline{U}_x \subset U_x$. Since $\text{Sing}(v)$ is compact, there are finitely many points $x_1, \ldots, x_k \in \text{Sing}(v)$ such that $U := U_{x_1} \cup \cdots \cup U_{x_k} \supset \text{Sing}(v)$. Then $U$ is an open submanifold of $M$ and $\partial U$ consists of finitely many disjoint circles. By the flow box theorem, we may assume that $U$ has no periodic orbit connecting different boundaries of $U$, by removing the small neighborhood of the periodic orbit if there is a periodic orbit connecting different boundaries of $U$. We may assume that there is no arc $\gamma$ in $U$ which is a subset of an periodic orbit and connects $\partial U$ such that $U - \gamma$ is connected (i.e. $[\gamma] \neq 0 \in \pi_1(U, \partial U)$), by removing the small open neighborhood of $\gamma$. Moreover there is an open neighborhood $V$ of $\partial U$ which is homeomorphic to a finite disjoint union of open annuli such that $V \cap \text{Sing}(v) = \emptyset$. Since $\text{Per}(v)$ is open, we obtain $\partial U \setminus \text{Per}(v)$ is compact. Then there are connected open transverses $\gamma_1, \ldots, \gamma_l$ in $V$ such that the image $W$ of $\gamma_1 \cup \cdots \cup \gamma_l$ by $v|_V$ has at most finitely many connected component and contains $\partial U \setminus \text{Per}(v)$. Then $\partial U \setminus W \subset \text{Per}(v)$. Hence there is a simple closed curve $C_i$ for each connected component of $\partial U$ such that $C_i \subseteq (W \cup \text{Per}(v)) \cap (V \cup U)$ has at most finitely many tangencies such that $\partial_i C_i$ is homotopic to $\partial U$ in $V \cup U$. Since $V$ is small, replacing $U$ with $U \cup V$, we may assume that $U$ has at most finitely many tangencies.

Since $v|_U$ is small, if we replace $v|_U$ with a $C^r$ flow on $U$ which is near to the identity and corresponds to $v$ near $\partial U$, then the resulting flow is near $v$. Therefore it suffices to show that there is a non-wandering regular $C^r$ flow on a small neighborhood of $U$ whose orbit near $\partial U$ corresponds to orbits of $v$ such that $\partial U \cap \text{Per}(v)$ consists periodic orbits with at most finite exceptions for the resulting flow and their saturation in $U$ is dense in $U$. Fix a connected component $U'$ of $U$ and a connected component $C$ of $\partial U'$. Then $C$ has at most finitely many tangencies and $C \cap \text{Per}(v)$ is dense in $C$. We show that there is a sufficiently slow $C^r$ flow $v'$ which is corresponded to $v$ near $\partial U'$ such that the resulting $C^r$ flow is regular non-wandering by replacing $v|_{U'}$ with $v'$. Indeed, by induction on the number $c_0$ of tangencies, suppose that $c_0 = 0$. The non-wandering property implies the existence of essential periodic orbits, which contradicts the hypothesis. Suppose $c_0 > 0$. Then the numbers of inward and outward transverse intervals are equal and so $c_0$ is even. Suppose that there is an outer tangency $p$. For each point $q \in \text{Per}(v)$ outside of $U'$ and near $p$, the complement of a small neighborhood of the orbit of $q$ in $U'$ has $p$ as an inner tangency and consists of at least three connected components each of which contains at least one tangency. Removing a small neighborhood of the orbit of $q$ in $U'$, we may assume that there are no outer tangencies. Suppose that there is a periodic orbit intersecting $U'$ such that each connected component of the complement in $U'$ has at least two tangencies. By inductive hypothesis, the proof also can be done. Thus we may assume that each periodic point in $U'$ has a connected component of the complement of the orbit in $U'$ which has exactly one tangency. First, consider the case $U'$ has genus 0. Suppose that $U'$ has just one boundary. Then we can fill up $U'$ by the flow which has just one singularity, which is a $c_0$-saddle (see Figure 2). By a small perturbation, we obtain a flow $v'$ with $(c_0 - 2)/2$ (usual) saddles, which is desired. Suppose that $U'$ has two boundaries with $c, c'$ tangencies. Consider two flows $v_c$, $v_{c'}$ on disks with tangencies $c, c'$. Fix curves $\gamma_c, \gamma_{c'}$ contained in regular orbits of $v_c, v_{c'}$. Replace $\gamma_c$ (resp. $\gamma_{c'}$) with a saddle with two separatrices and one homoclinic saddle connection which bounds a center disk. Removing small center disks and pasting the boundaries of center disks, the resulting flow is desired. When $U'$ has at least three boundaries, iterating
this process at $\#\partial U' - 1$, we can obtain a desired flow. Finally, consider the case with genus $g > 0$. Recall that $[\gamma] = 0 \in \pi_1(U, \partial U')$ and one connected component of $\partial U' - \partial \gamma$ contains exactly one tangency for any connected component $\gamma$ of the intersection of $U'$ and an periodic orbit. Consider a flow $v'$ on a disk. Fix $2g$ curves $\gamma_i$, $\gamma'_i$ contained in $2g$ regular orbits. Replace $\gamma_i$ (resp. $\gamma'_i$) with a saddle with two separatrices and one homoclinic saddle connection which bounds a center disk. Removing small center disks and pasting the boundaries of center disks for each $\gamma_i$ and $\gamma'_i$, the resulting flow is desired.

Applying the above construction finitely many times, we can obtain a regular non-wandering $C^r$ flow near $v$. □

Lemma 2.3. Each regular non-wandering flow can be approximated by a sequence of regular flows without heteroclinic connections.

Proof. Fix any regular $v \in \chi_{nw}$. Then each singularity is either a center or a saddle. By Lemma 2.1, an arbitrary small perturbation makes $v$ into a flow without locally dense orbits. Thus we may assume that $v$ has no locally dense orbits. Then $\overline{\text{Per}(v)} = M$ and $\partial \overline{\text{Per}(v)} = \text{Sing}(v) \cup P$. Now we break heteroclinic connections by an arbitrary small perturbation. Indeed, fix a saddle point $p$ and an arbitrary small neighborhood of $p$. Then we can identify $U$ with an open disk $\{(x, y) \mid x^2 + y^2 < 4\}$. Also we may assume that $p$ is the origin $(0, 0)$ on $U$, the $x$-axis is the local stable manifold, and the $y$-axis is the local unstable manifold, and that the set of orbits of $v$ is origin symmetric and axial symmetric with respect to $I_{\pm}$ on $U$, where $I_{\pm} := \{(x, \pm x) \mid x \in \mathbb{R}\}$. We also may assume that there is an arbitrary small $a_0 > 0$ such that $(b, 1) \in O_v(1, b)$ for $b \neq 0 \in (-a_0, a_0)$ and $U \cap (W_u^o(p) \cup W^o_v(p)) = \{(0, y) \mid y \in (-2, 2)\} \cup \{(x, 0) \mid x \in (-2, 2)\}$, and $U \cap \bigcup_{r, \sigma, \epsilon \in \{+, -\} O_v(\sigma 1, \sigma' a_0)}$ consists of just four connected components, by taking $U$ small. Here $W^o_u(p)$ (resp. $W^o_v(p)$) is the stable (resp. unstable) manifold of $p$ for $v$. Take a small open annulus $A$ centered at the origin. In fact, $A := \{(x, y) \mid x^2 + y^2 \in (\epsilon^2/2, \epsilon)\}$ for some small $\epsilon > 0$. Let $v_{\text{rot}}$ be a flow whose support is $A$ and each of whose regular orbits $O_v$ moves at a constant
velocity and is a circle centered at the origin (i.e. \( \{(x, y) \mid x^2 + y^2 = s\} \) for some \( s \in (\varepsilon/2, \varepsilon) \)) such that the regular orbits of \( v_{\text{rot}} \) move clockwise and \( v' := v + v_{\text{rot}} \) is origin symmetric (see Figure 2). Applying the above fact [BS] to a loop around \( p \), we may assume \( v_{\text{rot}} \) is \( C^r \). Put \( v'_U := v' \big| U \). Then there is a small number \( a \in (0, a_0) \) such that \( \pm (1, b) \in W^{s}_{v'_U}(p) \), \( \pm (a, 1) \in W^{u}_{v'_U}(p) \), \( (a, -1) \in O_{v'_U}(1, -a) \), and \( (-b, -1) \in O_{v'_U}(1, b) \) for any \( b \in (-a, a) \). Since \( O_v(\pm (1, \pm a)) \) are periodic, we have that \( W^{s}_{v'_U}(p) \) contains either \( \pm (1, \pm a) \). Since \( O_{v'}(\pm (1, \pm a)) \) goes to either \( p \) or \( \pm (1, \pm a) \), we have that \( W^{u}_{v'}(p) \) must go back to \( p \). This means that \( p \) is a homoclinic saddle connection. We show that \( v' \) is non-wandering. By symmetry, it suffices to show that each point contains in \( \{(1, b) \mid b \in (0, a)\} \) is also periodic. By construction, we have that \( (-b, -1) \in O_{v'}(1, b) \) and that \( (-b, -1) \) returns to either \( \pm (1, \pm b) \). The periodicity of \( \pm (1, \pm b) \) by \( v' \) implies that \( (1, b) \) is periodic for \( v' \). This shows that \( v' \) is also non-wandering. Applying the above perturbation for all saddles, the resulting flow has no heteroclinic connections and is non-wandering. \( \square \)

Now we summarize the previous lemmas.

**Theorem 2.4.** Let \( M \) be a connected closed surface and \( r = 0 \) or \( 1 \). The set of regular non-wandering flows on \( M \) without heteroclinic connections nor locally dense orbits is dense in \( \chi_{nw}^0 \).

The \( C^r \) (\( r = 0, 1 \)) structurally stability does not hold. Indeed, the regular singularity can be replaced by a closed ball with nonempty interior. However there are a few topologically stable flows on \( S^2 \). Recall that \( v \) is topologically stable in \( \chi_{nw}^0 \) if for any \( w \in \chi_{nw}^0 \) \( C^0 \)-near \( v \), there is a surjective continuous map which takes the orbits of \( w \) onto the orbits of \( v \). The surjective continuous map is called the semi-conjugacy.

**Corollary 2.5.** Let \( v \) be a non-wandering continuous flow on the sphere \( S^2 \). Then \( v \) is topologically stable in \( \chi_{nw}^0 \) if and only if \( v \) consists of two centers and other periodic points.
Proof. Suppose that \( v \) consists of two centers and other periodic points. Taking any small perturbation, let \( v' \) be the resulting non-wandering flow. Then \( \text{Per}(v') \) is not empty and so we can easily construct a semi-conjugacy, by collapsing non-periodic orbits into centers. Conversely, suppose that \( v \) is topologically stable in \( \chi_{\text{nw}}^0 \). Then Theorem 2.3 implies \( v \) is regular and has no heteroclinic connections. By Theorem 2.3 [Y], we have \( \text{Per}(v) = M \). Suppose there are saddles. Then there is a connected component \( A \) of \( \text{Per}(v) \) one of the whose boundaries is a figure eight saddle connection \( C \) with a saddle \( x \). By any small perturbation, replace countably many periodic orbits \( O_n' \) in \( A \) converging \( x \) with homoclinic 2-saddle connections \( \{x_n\} \cup O_n \) (where \( O_n := O_n' - \{x_n\} \)) such that the \( \omega \)-limit set (resp. \( \alpha \)-limit set) of \( O_n \) is \( x_n \). Then \( (x_n) \) converges to \( x \). Note that there is no orbit-preserving continuous surjection from a periodic orbit to a homoclinic saddle connection. Since \( \text{Sing}(v) \) is finite, for any semi-conjugacy \( h \), there is \( k \in \mathbb{Z}_{>0} \) such that \( h(x_k) = x \). However, this is impossible because \( \partial A \) containing \( x \) is a figure eight saddle connection \( C \) and there are no orbit-preserving continuous surjections \( \{x_k\} \cup O_k \to C \). Thus there are no saddle points and so each singularity is a center. Poincaré-Hopf theorem implies that the singularities are just two centers. This implies that \( v \) consists of two centers and other periodic points. \( \Box \)

Now we consider the non-spherical case.

Corollary 2.6. Let \( M \) be a connected closed orientable surface with positive genus. For any \( r = 0 \) or \( 1 \), non-wandering flows on \( M \) can be approximated by regular non-wandering flows with locally dense orbits and without heteroclinic connections.

Proof. Fix a non-wandering flow \( v \) on \( M \). Since non-wandering flows without singularities are (ir)rational rotations, we may assume that \( v \) has singularities. By Theorem 2.3 we may assume that \( v \) is regular, and has neither locally dense orbits nor heteroclinic orbits. First we show that if there are singularities, then there are essential saddle connections. Otherwise we may assume that there are only centers and contractible saddle connections. By the finiteness of singularities, Poincaré-Hopf theorem implies that there are saddle points. Since each saddle connection is a figure eight, the sum of indices is positive. This contradicts that the Euler number is non-positive.

Second, by induction on the genus \( g \) of \( M \), we show the assertion. Suppose that \( M \) is a torus. Since there are singularities, there is an essential saddle connection \( \gamma \). Since each connected component of the complement of \( \gamma \) is an annulus or a disk, we have that all the other saddle connections are homologically linearly dependent to \( \gamma \) and that there is an essential periodic orbit \( \gamma' \) of \( v \) which is homologically linearly dependent to \( \gamma \) (see Figure 3). Let \( A \) be the union of the saddle connections which bound the center disks each of whose closure is essential, and let \( B \) be the union of open center disks bounded by \( A \). Take a simple closed curve \( C \) which intersects \( \gamma \) and is homologically linearly independent to \( \gamma \) such that \( C \) meets neither centers nor center disks whose closures are contractible and that \( C \cap M' \) is connected for any connected component \( M' \) of \( M - A \). Then we can construct a flow \( v' \) whose support is a small annular neighborhood of \( C \) and which consists of periodic orbits such that each connected component of \( D - \text{Supp}(v') \) containing a center is a subset of a center disk of \( v + v' \). We may assume that \( v + v' \) has no locally dense orbits. Let \( D' \) be the union of center disks for \( v + v' \). By construction, there is a closed transversal \( \gamma'' \) for \( v + v' \) which is near and homotopic to \( \gamma \) such that \( \gamma'' \cap D' = \emptyset \) and
each essential periodic orbit of $v + v'$ in $\text{Per}(v + v')$ intersects $\gamma''$. By an arbitrary small perturbation along $\gamma''$, we may assume that $v + v'$ is non-wandering and contains locally dense orbits. This implies that $v$ can be approximated by regular non-wandering flows with locally dense orbits and without heteroclinic connections. Suppose that $M$ has genus $g(M) > 1$. Since $v$ is regular and has a singularity, there is an essential saddle connection. Since $\partial \text{Per}(v) \supset P$ and $\text{Per}(v)$ is the union of disjoint open annuli, there is an essential periodic orbit $\gamma$. Adding two center disks to the new two boundaries of $M - \gamma$, we obtain the new surface $M'$ and the new non-wandering flow $w'$ on $M'$. Since the genus $g(M')$ is less than $g(M)$, by inductive hypothesis, $w'$ can be approximated by a regular non-wandering flow $w''$ with locally dense orbits and without heteroclinic connections. By above construction, $w'$ and $w''$ coincide on a small neighborhood of the union of centers. Then we may assume that the small neighborhood contains the new two center disks. Therefore $w''$ can be lifted to the non-wandering flow on $M$ with locally dense orbits which approximates $v$. This completes the proof.

3. An example

We construct a non-wandering flow on $T^2$ such that $P$ and $\text{LD}$ are dense. In particular, $\text{LD}$ is not open. This example also shows that the finiteness condition in Lemma 2.1 is necessary.

**Example 1.** Consider an irrational rotation $v$ on $T^2$. Fix any points $p, q \in T^2$ with $O_v(p) \neq O_v(q)$. Using bump functions, replace $O_v(q)$ with one singularity and two locally dense orbits and $O_v(p)$ with a union of countably many singularities $p_i$ ($i \in \mathbb{Z}$) and countably many proper orbits such that $(p_i)$ converges to $q$ as $i \to \infty$ (resp. $i \to -\infty$). Moreover we choose that there is a sequence $(t_i)$ such that $p_i = O_v(t_i, p_0)$ and that $t_i \to \infty$ as $i \to \infty$ (resp. $t_i \to -\infty$ as $i \to -\infty$). Let $v'$ be the resulting vector field. For any point $x \in T^2 - (O(p) \cup O(q))$, we have $O_v(x) = O_{v'}(x)$. Moreover $O_v(p) \setminus \text{Sing}(v')$ is the union of proper orbits of $v'$.
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