Powers of Brownian Green Potentials

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Abstract
In this article we study stability properties of \( \mathcal{G}_\Omega \), the standard Green kernel for \( \Omega \) an open regular set in \( \mathbb{R}^d \). In \( d \geq 3 \) we show that \( \mathcal{G}_\Omega^\beta \) is again a Green kernel of a Markov Feller process, for any power \( \beta \in (1, d/(d - 2)) \). In dimension \( d = 2 \), we show the same result for \( \mathcal{G}_\Omega^\beta \), for any \( \beta \geq 1 \) and for kernels \( \exp(\alpha \mathcal{G}_\Omega) \), \( \exp(\alpha \mathcal{G}_\Omega) - 1 \), for \( \alpha \in (0, 2\pi) \), when \( \Omega \) is an open Greenian regular set whose complement contains a ball.

Keywords Green potentials · Markov processes · Brownian motion

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1 Introduction and Main Results
In this paper we study powers, in the sense of Hadamard, of \( \mathcal{G}_\Omega \), the standard Green potential associated to Brownian Motion (BM) on a regular open set \( \Omega \subset \mathbb{R}^d \), killed when exiting \( \Omega \). These operators have a kernel which are powers of the standard Green kernel in \( \Omega \).
Most of the time, we will assume that \( d \geq 3 \), and some extensions will be given for \( d = 2 \). So, unless we say the contrary, \( d \) will be greater than or equal to 3. In what follows we denote by \( G = G_d, \) \( g = g_d \) the Green potential and kernel for standard \( \text{BM} \) in \( \mathbb{R}^d \), that is, for \( x \neq y \in \mathbb{R}^d \)

\[
g(x, y) = g(0, x - y) = C(d) \|x - y\|^{2-d},
\]

where \( C(d) = \frac{\Gamma(d/2-1)}{2\pi^{d/2}} \), and for any \( F \in C_{\mathcal{K}} \) we have

\[
GF(x) = \int F(y)g(x, y)\,dy.
\]

We recall that \( GF(x) = \mathbb{E}_x \left( \int_0^\infty F(B_t)\,dt \right) \), where \( (B_t) \) is a \( d \)-dimensional \( \text{BM} \).

If \( \mathcal{O} \) is a regular (for \( \text{BM} \)) open set and \( T_{\mathcal{O}} = \inf\{t > 0 : B_t \in \mathcal{O}^c\} \) is the exiting time for \( \mathcal{O} \), then for \( F \in C_{\mathcal{K}}(\mathcal{O}) \)

\[
G_{\mathcal{O}} F(x) = \mathbb{E}_x \left( \int_0^{T_{\mathcal{O}}} F(B_t)\,dt \right) = \mathbb{E}_x \left( \int_0^\infty F(B_{t \wedge T_{\mathcal{O}}})\,dt \right).
\]

We denote by \( g_{\mathcal{O}}(x, y) \) the density of \( G_{\mathcal{O}} \) with respect to Lebesgue measure, which for \( x \neq y \in \mathcal{O} \) is given by

\[
g_{\mathcal{O}}(x, y) = g(x, y) - \mathbb{E}_x(g(B_{T_{\mathcal{O}}}, y)).
\]

For a parameter \( \beta \in \mathbb{R}_+ \) we denote by \( g_{\mathcal{O}}^\beta \) the \( \beta \)-power of \( g_{\mathcal{O}} \), that is, \( g_{\mathcal{O}}^\beta(x, y) = (g_{\mathcal{O}}(x, y))^\beta \) and the corresponding operator \( G_{\mathcal{O}}^{(\beta)} \) defined as

\[
G_{\mathcal{O}}^{(\beta)} F(x) = \int F(y)g_{\mathcal{O}}^\beta(x, y)\,dy.
\]

In what follows, we denote by \( \hat{\mathcal{O}} = \mathcal{O} \cup \{\partial\} \) the one point compactification of \( \mathcal{O} \), and given a function \( f \) defined on \( \mathcal{O} \), we extend it to \( \hat{\mathcal{O}} \) by putting \( f(\partial) = 0 \), unless we say the contrary. Now, we can state our main results.

**Theorem 1.1** Let \( \mathcal{O} \) be a regular open set in \( \mathbb{R}^d \), with \( d \geq 3 \), and \( \beta \in [1, \frac{d}{d - 2}] \). Then, the operator \( G_{\mathcal{O}}^{(\beta)} \) is the Green potential of a unique Feller semigroup in \( \mathbb{B}_b(\hat{\mathcal{O}}) \). That is, there exists a unique (in law) Feller process \( (X_t)_t \), with càdlàg paths on \( \hat{\mathcal{O}} = \mathcal{O} \cup \{\partial\} \), such that for any \( F \in C_b(\mathcal{O}) \cap L^1(\mathcal{O}, dx) \) and all \( x \in \mathcal{O} \)

\[
G_{\mathcal{O}}^{(\beta)}(F)(x) = \mathbb{E}_x \left( \int_0^\infty F(X_t)\,dt \right).
\]

**Remark** In case \( \mathcal{O} \) is bounded, of course the condition \( F \in C_b(\mathcal{O}) \cap L^1(\mathcal{O}, dx) \) is just \( F \in C_b(\mathcal{O}) \). We also note that there is a big difference between \( G_{\mathcal{O}}^{(\beta)} \) and the operator \( H \) whose kernel is given by

\[
h(x, y) = g^{\beta}(x, y) - \mathbb{E}_x(g^{\beta}(X_{T_{\mathcal{O}}}, y)),
\]

where \( X \) is the Feller process with Green kernel \( g^{\beta} \). The fact that \( H \) is a Green potential is part of a general result, and it is the Green potential for the killed process: \( Y_t = X_t, t < T_{\mathcal{O}} \).
In dimension $d = 2$, we have a similar result, but given the logarithmic singularity of Green kernels in this dimension, we can consider more general transformations. We define, for $x, y \in \mathcal{O}$

$$g^{\exp, \alpha}_{\mathcal{O}}(x, y) = \exp(\alpha g_{\mathcal{O}}(x, y)),$$

$$g^{\exp^{-1}, \alpha}_{\mathcal{O}}(x, y) = \exp(\alpha g_{\mathcal{O}}(x, y)) - 1.$$ 

and the associated operators $G^{\exp, \alpha}_{\mathcal{O}}, G^{\exp^{-1}, \alpha}_{\mathcal{O}}$.

When $\mathcal{O}$ is unbounded, we need some extra hypothesis to have a control near $\infty$. We denote by $\partial \mathcal{O}$ the boundary of $\mathcal{O}$ as a subset of $\mathbb{C}$, and we will make explicit conditions about $\infty$ as a boundary point in $\mathbb{C}_\infty$, the Riemann sphere. This time, there are two natural compactifications of $\mathcal{O}$. As before we consider $\hat{\mathcal{O}} = \mathcal{O} \cup \{\infty\}$ and also the compactification of $\mathcal{O}$ in $\mathbb{C}_\infty$. The main hypothesis we need is the following.

**Hypothesis H2d**

(i) The complement of $\mathcal{O}$ contains an open ball.

(ii) Every $z \in \partial \mathcal{O}$ is a regular point and the following limit exists for all $y \in \mathcal{O} \cup \partial \mathcal{O}$:

$$\lim_{y \to \infty, z \in \mathcal{O} \cup \partial \mathcal{O}} g_{\mathcal{O}}(z, y) = 0.$$

**Remark** We notice that in hypothesis (H2d.ii), the behaviour at $\infty$ is exactly that $\infty$ is a regular boundary point (see for example [14], Theorem 4.4.9). In particular $\infty$ is not an isolated boundary point.

**Theorem 1.2** Let $\mathcal{O}$ be a regular open set in $\mathbb{C}$, which satisfies (H2d) if it is unbounded. Then, each one of the operators $G^{(\beta)}_{\mathcal{O}}$, for $\beta \in [1, \infty)$, $G^{(\exp, \alpha)}_{\mathcal{O}}$ and $G^{(\exp^{-1}, \alpha)}_{\mathcal{O}}$, for $\alpha \in (0, 2\pi)$, are the zero potential of a unique (in law) Feller process, with càdlàg paths on $\hat{\mathcal{O}}$. We denote by $X^{(\beta)}, X^{(\exp, \alpha)}$ and $X^{(\exp^{-1}, \alpha)}$ the respective Feller processes.

(i) For all $F \in C_b(\mathcal{O}) \cap L^1(\mathcal{O}, (\log(|y|) + 1)^\beta \, dy)$ it holds

$$G^{(\beta)}_{\mathcal{O}} F(x) = \mathbb{E}_x \left( \int_0^\infty F \left( X^{(\beta)}_t \right) \, dt \right).$$

(ii) Let $1 < p < 2\pi/\alpha$ and $q$ the conjugated index. Then, for all $F \in C_b(\mathcal{O}) \cap L^1(\mathcal{O}, |y|^{2\alpha/p} \, dy) \cap L^q(\mathcal{O}, |y|^{3\alpha/q} \, dy)$ it holds

$$G^{(\exp, \alpha)}_{\mathcal{O}} F(x) = \mathbb{E}_x \left( \int_0^\infty F \left( X^{(\exp, \alpha)}_t \right) \, dt \right),$$

$$G^{(\exp^{-1}, \alpha)}_{\mathcal{O}} F(x) = \mathbb{E}_x \left( \int_0^\infty F \left( X^{(\exp^{-1}, \alpha)}_t \right) \, dt \right).$$

**Remark** In the previous theorem, when $\mathcal{O}$ is bounded, we can drop the extra integrability conditions in representations (1.2), (1.3) or (1.4)

**Remark** Notice that $G^{(\exp, \alpha)}_{\mathcal{O}}$ and $G^{(\exp^{-1}, \alpha)}_{\mathcal{O}}$ are related by

$$G^{(\exp, \alpha)}_{\mathcal{O}} F(x) = G^{(\exp^{-1}, \alpha)}_{\mathcal{O}} F(x) + \int_{\mathcal{O}} F(y) \, dy.$$ 

Using Theorem 1.2 and the inversion $G(z) = z^{-1}$, we can extend this result to contain some unbounded domains for which $\infty$ is an isolated boundary point.
Proposition 1.3 Assume \( K \) is a compact set, with non empty interior and consider the open set \( \mathcal{O} = \mathbb{C} \setminus K \), for which we assume that \( \partial \mathcal{O} \) is regular. Then, \( G_{\mathcal{O}}^{(\beta)} \), for \( \beta \in [1, \infty) \), \( G_{\mathcal{O}}^{(\exp, \alpha)} \) and \( G_{\mathcal{O}}^{(\exp, -\alpha)} \), for \( \alpha \in (0, 2\pi) \), are the zero potential of a unique (in law) Feller process, with càdlàg paths on \( \mathcal{O} \). Representations (1.2), (1.3) and (1.4) hold under the same integrability assumptions.

Remark Notice that under the assumptions of this Proposition, \( \infty \) is not a regular boundary point (it is an isolated boundary point). Nevertheless, the limit

\[
h(y) = \lim_{\varepsilon \to \infty, z \in \mathcal{O} \cup \partial \mathcal{O}} g_{\mathcal{O}}(z, y),
\]

exists for all \( y \in \mathcal{O} \cup \partial \mathcal{O} \), and it is a nonnegative, non trivial harmonic function, that vanishes at \( \partial \mathcal{O} \). Assuming for example that \( B(0, 1) \subset K \), this function is given by the relation

\[
h(y) = gu(0, 1/y),
\]

where \( 0 \in U \subset B(0, 1) \), is the inversion of \( \mathcal{O} \cup \{\infty\} \).

The general question we are addressing, is when a function of \( g_{\mathcal{O}} \) is again the Green potential of a Markov process. The above theorems show this happens for powers. In dimension \( d \geq 3 \), there is a restriction on these powers, due to integrability conditions. In dimension \( d = 2 \), we even have that some exponentials of \( g_{\mathcal{O}} \) are Green potentials. As we will see, these results are consequence of similar results shown in [3] (see also [4] and [5]) for potential matrices (potentials of transient finite continuous time Markov chains). In addition to powers and exponentials, we know that some few polynomial with positive coefficients, like \( F(a) = a + a^2 \) and by iterating \( G(a) = a^\beta + a^{2\beta} \), preserve potential matrices and so this can be transfered to the setting of this article. The general problem is open even in the case of matrices. The authors of [5] conjecture that this is true for any absolutely monotone function, in particular, for any function which is the Laplace transform of a positive measure, that is, \( F(a) = \int_0^\infty e^{at} d\mu(\xi) \). We mention here that using the results about potential matrices, this stability under powers was extended in [7] in the context of continuous and bounded Green kernels.

We mention here that the results of this article can be extended to potentials of other transient diffusions \( X \), or even more general transient Markov processes, taking values on a bounded open set of \( \mathbb{R}^d \). One way to achieve such an extension, is to use an approximation of \( X \) by finite Markov chains with enough control on the respective approximated potentials.

The paper is organized as follows. In Section 2 we consider \( \mathcal{O} = \mathbb{R}^d \), for which the result is well known. In particular, Theorem 1.1 was proved essentially by Frostman in [8] and Riesz in [15, 16]. The operator \( G_{\mathcal{O}}^{(\alpha)} = G_{\mathcal{R}^d}^{(\alpha)} \) is proportional to what is called a Riesz potential, which corresponds to the Green potential of a standard Brownian Motion subordinated to a \( \alpha \)-stable process where \( \alpha = d - \beta(d - 2) \in (0, 2) \) (see for example [10]), that is, for all \( F \in C_\mathcal{K} \), \( x \in \mathbb{R}^d \)

\[
G^{(\beta)}(F)(x) = D \mathbb{E}_x \left( \int_0^\infty F(B_t) dt \right) = D \int_0^\infty \int_{\mathbb{R}^d} F(y) |x - y|^{-\alpha} dy dt,
\]

\[ \int_{\mathbb{R}^d} f(y) dy = \frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)2^{\frac{d-\alpha}{2}}\pi^{\frac{d-\alpha}{2}}}. \]
where $(B_t)_t$ is a standard $d$-dimensional BM, $(\eta_t)_t$ is a $\xi$-stable subordinator, normalized to $\mathbb{E}(e^{-\lambda \eta_t}) = e^{-\lambda t^{\alpha/2}}$, $(B_t)_t$, and $(\eta_t)_t$, are independent, and

$$D = \left( \frac{\Gamma(\frac{d-2}{2})}{2\pi^{d/2}} \right)^{\frac{d-2}{2}} \frac{\Gamma(\frac{d-\alpha}{2})}{\Gamma(\alpha/2)2^{\alpha/2}\pi^{d/2}}.$$  

The constant $D$ can be removed by using a linear time change $t' = Dt$.

**Example**  $d = 3$, $\alpha = 1$, $\beta = 2$. The Green kernel is in this case $g^2_{R^3}(x, y) = C^2(3)||x - y||^{-2}$, for $x, y \in \mathbb{R}^3$. The process whose potential is $g^3_{R^3}$ can be constructed as follows. Take $W$ a standard one dimensional BM and consider the passage times, for $t \geq 0$

$$\tau_t = \inf\{s \geq 0 : W_s > t\}.$$  

The Laplace transform of $\tau_t$ is $\mathbb{E}(e^{-\lambda \tau_t}) = e^{-\sqrt{2}\lambda t^{1/2}}$ and then $\eta_t = \tau_{t/\sqrt{2}}$ is a normalized $\frac{1}{2}$-stable subordinator. So, if we take an independent three dimensional BM $B$ and subordinate it

$$X_t = B_{\tau_{t/\sqrt{2}}}$$

we get a Feller Process, whose Green potential is $G_{R^3}^{(2)}$. Its Green kernel is proportional to $||x||^{-2}$, which formally is the standard Green kernel in $\mathbb{R}$ at the point $z = (x, 0)$, and so it is proportional to the density of the amount of time the 4 dimensional BM $(B, W)$ spends around $z$. We are not aware if there is a pathwise explanation of this interpretation.

The case of $\mathcal{O} = \mathbb{R}^d$ is simpler because $G^{(d)}$ is a convolution operator and Fourier analysis can be used to show the result. In Section 2, we provide the basic estimations we need and we shall prove directly that $G^{(d)}$ satisfies a suitable version of the Complete Maximum Principle (CMP, see Definition 2.1) on $C_{\mathcal{K}}$, which is one of the main ingredients to prove that a positive operator is the potential of a Feller semigroup. The proof of this CMP is based on a new inequality characterizing potential matrices (see Proposition A.1 in the Appendix or in the article [6]).

In Section 3, we extend the results, of the previous section, to a general regular bounded domain $\mathcal{O}$. Following the tools developed in Section 2, we shall prove that $G^{(d)}_{\mathcal{O}}$ is the potential of a Ray process. Then, an extra argument is necessary to show that the set of branching points is empty, to conclude that actually $G^{(d)}_{\mathcal{O}}$ is the potential of a Feller process.

In Section 4, we treat the unbounded case, proving the general result in $d = 3$. In Section 5, we indicate how to prove the case $d = 2$. In Section 6, we prove that these semigroups have a density with respect to Lebesgue measure. In the Appendix we summarize the tools we need from the theory of $M$-matrices and their inverses.

The main questions in this article, have some relevance in applications. When using Markov chains (or more general Markov processes) to model some phenomena, we usually fit the transition probability $P$ (or the infinitesimal generator). That is, we put a model on $P$, which in general should be a nonnegative matrix, whose row sums are bounded by one. Then, we impose other restrictions given by the particular problem. What if we cannot measure $P$ directly, but we can only measure $U = (I - P)^{-1}$, the potential of the associated Markov chain? This happens, for example in electrical networks. Then, one should give a model for $U$. This is complicated, because it is not simple to describe which nonnegative matrices $U$ are potentials. This is part of what is known in Linear Algebra as the inverse $M$-matrix problem. So, if we have a large class of functions that leave invariant the set of
potential matrices, we can model the problem by putting a parametric family in such class of functions.

We denote by \( C(\Omega) \) the set of continuous functions defined on \( \Omega \), \( C_0(\Omega) \) the subset of continuous functions vanishing at \( \infty \), \( C_K(\Omega) \) the subset of continuous functions with compact support contained in \( \Omega \). Notice that \( C(\tilde{\Omega}) \) is naturally identified to \( C_0(\Omega) \oplus 1 \), where the decomposition is \( F = F - F(\partial) + F(\partial) \) and \( F - F(\partial) \in C_0(\Omega) \). Given a function \( f \) defined on \( \Omega \), we extend it to \( \tilde{\Omega} \) by putting \( f(\partial) = 0 \). We remark that \( F \in C_0(\Omega) \) iff \( \overline{F} \in C(\tilde{\Omega}) \) if \( \Omega \) is bounded, then \( F \in C_0(\Omega) \) iff \( F \in C(\Omega) \) and for all \( (x_n)_n \subset \Omega \) such that \( x_n \to x \in \partial \Omega \), then \( F(x_n) \to 0 \). If \( \Omega \) is unbounded, we need to prove also that for all \( (x_n)_n \subset \Omega \), such that \( ||x_n|| \to \infty \) then \( F(x_n) \to 0 \). We denote by \( \mathbb{B}_b(\Omega) \) the set of bounded measurable functions and \( C_b(\Omega) \) the set of continuous bounded functions. Finally, in sums and integrals a restriction of the form \( A \cap p(z) \), where \( z \) is the variable of integration and \( p \) is a functional proposition, it is understood, as usual, as \( A \cap \{ z : p(z) \text{ is true} \} \).

## 2 Powers of the Green Potential in \( \mathbb{R}^d \), \( d \geq 3 \)

In this section we consider \( \Omega = \mathbb{R}^d \). In what follows we denote by \( g = g_d \) the Green kernel for the simple random walk in \( \mathbb{Z}^d \). We recall that \( G \) is the Green potential associated to a BM in \( \mathbb{R}^d \), its density with respect to Lebesgue measure is

\[
g(x, y) = g(0, x - y) = C(d) ||x - y||^{-d} \quad \text{for} \quad x \neq y \in \mathbb{R}^d,
\]

where \( C(d) = \frac{\Gamma(d/2 - 1)}{2\pi^{d/2}} \). The following proposition summarizes some well known relations between \( g \) and \( G \) (see [11] Theorem 4.3.1).

**Lemma 2.1**

(i) The Green function \( g \) is bounded and moreover \( g(0, x) \leq g(0, 0) < \infty \) for all \( x \in \mathbb{Z}^d \).

(ii) \( g \) has the following decay, for \( x \in \mathbb{Z}^d \), \( x \neq 0 \)

\[
g(0, x) = dC(d)||x||^{-d} + O(||x||^{-d}) = d g(0, x) + O(||x||^{-d}).
\]

In particular, there exists a constant \( c_0 = c_0(d) \), such that, for all \( x \in \mathbb{Z}^d \)

\[
g(0, x) \leq c_0 ||x||^{-d}.
\]

Consider now \( x \in \mathbb{R}^d \) and the normalized simple random walk starting from \( x \)

\[
S_{i,n}^x = x + \frac{d}{n} \sum_{k=1}^{[nt]} \xi_k,
\]

where the random variables \( (\xi_k : k \geq 1) \) are i.i.d. with common distribution

\[
P(\xi = e) = \frac{1}{2d},
\]

for \( e \in \{ \pm e_1, \ldots, \pm e_d \} \), with \( (e_1, \ldots, e_d) \) the canonical basis for \( \mathbb{R}^d \). We also denote by \( \mathcal{S}_t = \sum_{k=1}^{t} \xi_k \), with \( \mathcal{S}_0 \in \mathbb{Z}^d \).

In what follows we will repeatedly use the following notation \( \mathbb{Z}^d,n = \sqrt{d/n} \mathbb{Z}^d \). We also denote by \( B_{\infty}(x, r) = \{ y \in \mathbb{R}^d : ||x - y||_{\infty} < r \} \), the open ball in the \( \infty \)-norm of \( \mathbb{R}^d \), centered at \( x \) with radius \( r \). The corresponding ball for the Euclidean distance is denoted by \( B(x, r) \). Similarly \( d_{\infty}(y, A) \) is the distance between \( y \) and a set \( A \), with respect to the infinite norm.

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For a set $A \subset \mathbb{R}^d$ we denote by
\[ A^\square = \{ y \in \mathbb{R}^d : d_{\infty}(y, A) \leq \sqrt{d} \}. \]

Assume that $F \in C_\mathcal{K}$ is a nonnegative function with support $\mathcal{K} = \text{supp}(F)$. Then, for $x = 0$, we have
\[
\mathbb{E}\left( \int_0^\infty F(\mathcal{S}_{t,n})dt \right) = \sum_{\ell \in \mathbb{N}} \mathbb{E}\left( \int_{\ell/n}^{(\ell+1)/n} F(\sqrt{d/n} \mathcal{S}_t) dt \right) = \frac{1}{n} \mathbb{E}\left( F(\sqrt{d/n} \mathcal{S}_t) \right)
\]
\[
= \sum_{z \in \mathbb{Z}^d} F(z/\sqrt{d/n}) \frac{1}{n} \sum_{\ell \in \mathbb{N}} \mathbb{P}(\mathcal{S}_\ell = z) = \sum_{z \in \mathbb{Z}^d} F(z/\sqrt{d/n}) \frac{1}{n} g(0, z)
\]
\[
= \sum_{w \in \mathbb{Z}^d, n} F(w) \frac{1}{n} g(0, \sqrt{n/d}) = \sum_{w \in \mathbb{Z}^d, n} F(w) \left( \frac{d}{n} \right)^{d/2} g(0, \sqrt{n/d}) \frac{n^{d/2-1}}{d^{d/2}}
\]
\[
= \frac{F(0)g(0,0)}{n} + \int_{\mathbb{R}^d} H_n(y)dy,
\]
where $H_n$ is the simple function given by
\[
H_n(y) = \sum_{w \in \mathbb{Z}^d, w \neq 0} F(w) g(0, \sqrt{n/d}) \frac{n^{d/2-1}}{d^{d/2}} \mathbb{1}_{B_\infty \left( w, \frac{1}{\sqrt{n}} \sqrt{d/n} \right)}(y).
\]

Lemma 2.1 gives the following bound, with $c_1 = c_1(d) = c_0(d)/d$
\[
H_n(y) \leq c_1 \| F \|_\infty \frac{1}{\mathcal{K}^{\square}}(y) \sum_{w \in \mathbb{Z}^d, w \neq 0} \| w \|^{2-d} \mathbb{1}_{B_\infty \left( w, \frac{1}{\sqrt{n}} \sqrt{d/n} \right)}(y),
\]
Now, if $y \in B_\infty \left( w, \frac{1}{\sqrt{n}} \sqrt{d/n} \right)$ we have $\| y \| = \| y - w + w \| \leq \| w \| + \frac{\sqrt{d}}{2} \sqrt{d/n}$. The fact that $w \in \mathbb{Z}^d, w \neq 0$ implies that $\| w \| \geq \sqrt{d/n}$ and so $\| y \| \leq \left( 1 + \frac{\sqrt{d}}{2} \right) \| w \|$, which gives for $c_2 = c_2(d) = c_1(d) \left( 1 + \frac{\sqrt{d}}{2} \right)^{d/2}$
\[
H_n(y) \leq c_2 \| F \|_\infty \mathbb{1}_{\mathcal{K}^{\square}}(y) \| y \|^{2-d} \in L^1(dy).
\]

The asymptotic for $g$ gives also the pointwise convergence, for all $y \neq 0$
\[
H_n(y) \to F(y)g(0, y).
\]

We conclude, that
\[
\mathbb{E}\left( \int_0^\infty F(\mathcal{S}_{t,n})dt \right) = \sum_{w \in \mathbb{Z}^d, n} F(w) \left( \frac{d}{n} \right)^{d/2} g(0, \sqrt{n/d}) \frac{n^{d/2-1}}{d^{d/2}}
\]
\[
\overset{n \to \infty}{\longrightarrow} \int F(y)g(0, y)dy = G(F)(0) = \mathbb{E}_0 \left( \int_0^\infty F(\mathcal{B}_t)dt \right),
\]
a well known fact. These arguments have been included because they serve as a model for the general case in Theorem 1.1.

We denote by $\mathcal{G}^{n, \beta}$ the operator
\[
\mathcal{G}^{n, \beta}(F)(x) = \sum_{w \in \mathbb{Z}^d, n} F(w + x) \left( \frac{d}{n} \right)^{d/2} \left[ g(0, \sqrt{n/d}) \frac{n^{d/2-1}}{d^{d/2}} \right]^\beta,
\]
\[\text{Powers of Brownian Green Potentials}\]
which is well defined for all $F \in C_{\mathcal{K}}$ and all $x \in \mathbb{R}^d$ because $\mathcal{G}_{n,\beta}(F)(x)$ contains, for every $x$, a finite number of terms. Notice that if $x \in \mathbb{Z}^{d,n}$ then

$$
\mathcal{G}_{n,\beta}(F)(x) = \sum_{w \in \mathbb{Z}^{d,n}} F(w) \left( \frac{d}{n} \right)^{d/2} \left[ g \left( x \sqrt{n/d}, w \sqrt{n/d} \right) \frac{n^{d/2-1}}{d^{d/2}} \right]^\beta,
$$

Recall that, we have defined $\mathcal{G}_{\beta}$ as the operator acting in $C_{\mathcal{K}}$ given by

$$
\mathcal{G}_{\beta}(F)(x) = \int_{\mathbb{R}^d} F(y + x) [g(0, y)]^\beta \, dy = \int_{\mathbb{R}^d} F(y) [g(x, y)]^\beta \, dy.
$$

In what follows, we denote by $\text{osc} = \text{osc}_F$ the oscillation of $F$, which is given by

$$
\text{osc}(\delta) = \sup_{x,y: \|x-y\|_\infty \leq \delta} \{|F(x) - F(y)|\} \leq 2\|F\|_\infty
$$

for any $\delta > 0$. The fact that $F$ is uniformly continuous on $\mathbb{R}^d$ implies that $\text{osc}(\delta) \to 0$ as $\delta \downarrow 0$. Using the ideas developed before, we prove the following important result.

**Proposition 2.2** Assume that $1 \leq \beta < \frac{d}{d-2}$ and $F \in C_{\mathcal{K}}$ with support $\mathcal{K} = \text{supp}(F)$. Given $x, y \in \mathbb{R}^d$ such that $\|x - y\|_\infty < \frac{1}{\sqrt{d}}$. Then,

$$
|\mathcal{G}_{n,\beta}(F)(x) - \mathcal{G}_{n,\beta}(F)(y)| \leq \text{osc}(\|x - y\|_\infty) \Gamma(F),
$$

where $\Gamma(F) = \left( g^{\beta}(0, 0) d^{\frac{d}{2}} \right)^{1-\beta} + c_3 \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^{d/2}} \|x-y\|^{\beta(2-d)} \, dy < \infty$, $c_3 = c_3(d, \beta) = \left( \frac{c_3(d)}{d} \right)^{2} (2 + \sqrt{d})^{d-2}$ and $\mathcal{K}^{\text{support}} = (\mathcal{K}^{\text{support}})^{\beta}$.

Similarly, it holds for all $y$

$$
|\mathcal{G}_{n,\beta}(F)(y)| \leq \Gamma(F) \|F\|_\infty.
$$

Finally, we have the convergence: If $(z_n)_{n}$ is any sequence converging to $y$, then

$$
limit_{n \to \infty} \mathcal{G}_{n,\beta}(F)(z_n) = \mathcal{G}_{\beta}(F)(y) = \int F(w) (g(y, w))^\beta \, dw.
$$

**Proof** Let us prove (2.3). Assume $w \in \mathbb{Z}^{d,n}$. If $w + x \notin \mathcal{K}$ and $w + y \notin \mathcal{K}$ then $F(w + x) - F(w + y) = 0$. So, if this difference is not zero then $w \in \mathbb{Z}^{d,n} \cap ((\mathcal{K} - x) \cup (\mathcal{K} - y))$. Consider now, $y(n) \in \mathbb{Z}^{d,n}$ one of the closets elements of $\mathbb{Z}^{d,n}$ to $y$, in the infinity norm. In particular, $\|y - y(n)\|_\infty \leq \frac{1}{\sqrt{d}} \leq \frac{1}{\sqrt{d}}$. It is straightforward to show that

$$(\mathcal{K} - x) \cup (\mathcal{K} - y) \subset (\mathcal{K}^{\text{support}} - y(n)).$$

Denote by $A_n = \mathbb{Z}^{d,n} \cap \mathcal{K}^{\text{support}}$. Since $|F(w + x) - F(w + y)| \leq \text{osc}(\|x - y\|_\infty)$, we get

$$
|\mathcal{G}_{n,\beta}(F)(x) - \mathcal{G}_{n,\beta}(F)(y)| \leq \text{osc}(\|x - y\|_\infty) \sum_{w \in A_n} \left( \frac{d}{n} \right)^{d/2} \left[ g \left( 0, w \sqrt{n/d} \right) \frac{n^{d/2-1}}{d^{d/2}} \right]^\beta
$$

$$
\leq \text{osc}(\|x - y\|_\infty) \left( \frac{\|F(0, 0)\|}{n^\beta} d^{\frac{d}{2}} \right)^{1-\beta} + \sum_{w \in A_n, \|w\|_\infty \neq y(n)} \left( \frac{d}{n} \right)^{d/2} \left[ g \left( y(n) \sqrt{n/d}, w \sqrt{n/d} \right) \frac{n^{d/2-1}}{d^{d/2}} \right]^\beta
$$

$$
\leq \text{osc}(\|x - y\|_\infty) \left( \frac{\|F(0, 0)\|}{n^\beta} d^{\frac{d}{2}} \right)^{1-\beta} + \sum_{w \in A_n, \|w\|_\infty \neq y(n)} \left( \frac{d}{n} \right)^{d/2} \|y(n) - w\|^{\beta(2-d)}.
$$
where \( \gamma = \frac{d-\beta(d-2)}{2} > 0 \). For \( z \in B_\infty(w, \frac{1}{2}\sqrt{d/n}) \), we have
\[
\|y(n) - z\| \leq \|y(n) - w\| + \|w - z\| \leq \frac{\sqrt{d}}{2} \sqrt{d/n} + \|y(n) - w\|.
\]
Since \( y(n) \neq w \) and both belong to \( \mathbb{Z}^{d,n} \), we conclude that \( \|y(n) - w\| \geq \sqrt{d/n} \) and
\[
\|y(n) - z\| \leq \left(1 + \frac{\sqrt{d}}{2}\right)\|y(n) - w\|.
\]
Thus, we obtain (2.3) from
\[
|G^{n,(\beta)}(F)(x) - G^{n,(\beta)}(F)(y)| \leq \osc(\|x - y\|) \left(\frac{G^{\beta}(0,0,d/2)^{1-\beta}}{n} + c_3 \sum_{w \in A_n} \int_{B_\infty(w, \frac{1}{2}\sqrt{d/n})} \|y(n) - z\|^{\beta(2-d)} dz\right)
\]
\[
\leq \osc(\|x - y\|) \left(\frac{G^{\beta}(0,0,d/2)^{1-\beta}}{n} + c_3 \int_{X \mathbb{Z}^d} \|y(n) - z\|^{\beta(2-d)} dz\right)
\]
\[
\leq \osc(\|x - y\|) \left(\frac{G^{\beta}(0,0,d/2)^{1-\beta}}{n} + c_3 \sup_{u \in \mathbb{R}^d} \int_{X \mathbb{Z}^d} \|u - z\|^{\beta(2-d)} dz\right)
\]
\[
= \osc(\|x - y\|) \Gamma(F),
\]
where \( \mathcal{X} \mathbb{Z}^d = (\mathcal{X} \mathbb{Z}^d)^\oplus \), and \( c_3 = c_3(d, \beta) = c_1(\beta) \left(1 + \frac{\sqrt{d}}{2}\right)^{\beta(d-2)} \). In particular, we have
\[
|G^{n,(\beta)}(F)(x) - G^{n,(\beta)}(F)(y)| \leq 2\|F\|_\infty \Gamma(F).
\]
In a similar way we obtain
\[
|G^{n,(\beta)}(F)(y)| \leq \|F\|_\infty \left(\frac{G^{\beta}(0,0,d/2)^{1-\beta}}{n} + c_3 \sup_{u \in \mathbb{R}^d} \int_{X \mathbb{Z}^d} \|u - z\|^{\beta(2-d)} dz\right)
\]
\[
\leq \Gamma(F) \|F\|_\infty,
\]
and inequality (2.4) is shown.

Following the same ideas as in the proof of Eq. 2.2, we get for any \( y \)
\[
\lim_{n \to \infty} G^{n,(\beta)}(F)(y(n)) = G^{(\beta)}(F)(y).
\]
The result is shown by using Eq. 2.3. \(\square\)

**Corollary 2.3** Under the hypothesis of Proposition 2.2 we have
\[
\lim_{n \to \infty} \sum_{x \in \mathbb{Z}^{d,n}} \left(\frac{G^{(n,\beta)}(F)(x) - 1}{n}\right)^{d/2} F(x) \left(\frac{d}{n}\right)^{d/2} = \int_{\mathbb{R}^d} \left(\frac{G^{(\beta)}(F)(x) - 1}{n}\right)^{d/2} F(x) \, dx.
\]

**Proof** Proposition 2.2 shows that the sequence of functions
\[
\left(\frac{G^{(n,\beta)}(F)(\bullet) - 1}{n}\right)^{d/2} F(\bullet)
\]
converge pointwise to \( (G^{(\beta)}(F)(\bullet) - 1)^{d/2} F(\bullet) \), and Eq. 2.4 provides the domination we need to show that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} \left(\frac{G^{(n,\beta)}(F)(x) - 1}{n}\right)^{d/2} F(x) \, dx = \int_{\mathbb{R}^d} \left(\frac{G^{(\beta)}(F)(x) - 1}{n}\right)^{d/2} F(x) \, dx.
\]
On the other hand, if \( x \in \mathbb{Z}^d \) and \( y \in B_\infty(x, \frac{1}{\sqrt{d/n}}) \) we have from Eq. 2.3

\[
|\mathcal{G}^n_{\beta(F)}(x) - \mathcal{G}^n_{\beta(F)}(y)| \leq \text{osc}(\|x - y\|_\infty) \Gamma(F),
\]

which together with Eq. 2.4 gives, for \( A_n = \mathbb{Z}^d \cap \mathcal{K} \)

\[
\left| \int (\mathcal{G}^{(n, \beta)}(F)(y) - 1)^+ F(y) \, dy - \sum_{x \in \mathbb{Z}^d} (\mathcal{G}^{(n, \beta)}(F)(x) - 1)^+ F(x) \left( \frac{d}{n} \right)^{d/2} \right|
\]

\[
\leq \sum_{x \in A_n} \left| \int (\mathcal{G}^{(n, \beta)}(F)(y) - 1)^+ F(y) - (\mathcal{G}^{(n, \beta)}(F)(x) - 1)^+ F(x) \right| \mathbb{1}_{B_\infty(x, \frac{1}{\sqrt{d/n}})}(y) \, dy
\]

\[
\leq 4 \Gamma(F) \| F \| \text{osc}(\frac{1}{\sqrt{d/n}}) \mu(\mathcal{K}),
\]

where \( \mu \) is the Lebesgue measure and the result follows.

Our next step it to show \( \mathcal{G}^n(\mathcal{K}) \) for all \( x, \beta \geq 1 \) and all \( \mathcal{F} \in C_\mathcal{K} \). This will be a consequence of results on potential matrices (see [3] or [5]) and Proposition A.1 in Appendix A. To use them, we first need to prove the following Lemma.

**Lemma 2.4** Assume that \( A \subset \mathbb{Z}^d \) is a finite nonempty set. Then, the matrix \( U \) defined as, for \( x, w \in A \)

\[
U_{x,w} = g(x \sqrt{n/d}, w \sqrt{n/d}),
\]

is a nonsingular symmetric potential.

As usual we have taken a particular order on \( A \) to define \( U \), for example the lexicographical order.

**Proof** Consider a large integer \( N \) such that \( E = \sqrt{n/d} A \subset [-\,(N - 1), (N - 1)]^d \cap \mathbb{Z}^d \).

We define the following symmetric transition matrix \( P \) indexed by \( J = [-N, N]^d \cap \mathbb{Z}^d \)

\[
P_{jk} = \mathbb{P}_j(R_J < \infty, \mathcal{J}_{R_J} = k),
\]

where \( R_J = \inf\{p \geq 1 : \mathcal{J}_p \in J\} \) is the first strict hitting time to \( J \), for the process \( \mathcal{J}_p \), and \( \mathbb{P}_j \) indicates that \( \mathcal{J}_0 = j \). If \( \text{int}(J) \) denotes the points in \( J \) for which all their neighbours in \( \mathbb{Z}^d \) belong to \( J \) (the interior of \( J \),) then \( P_{jk} = \mathbb{1}_E \) for \( j \in \text{int}(J), k \in J, \| j - k \|_\infty = 1 \), while \( P_{jk} > 0 \) for all \( j, k \in \partial J = J \setminus \text{int}(J) \). We also notice that

\[
\sum_{k \in J} P_{jk} = \begin{cases} 1 & \text{if } j \in \text{int}(J) \\ \mathbb{P}_j(R_J < \infty) < 1 & \text{if } j \in \partial J \end{cases}
\]

Notice that \( P \) is irreducible and strictly substochastic at least at one vertex. Thus, the matrix \( M = \mathbb{I} - P \) is nonsingular (is an M-matrix) and its inverse \( V = (\mathbb{I} - P)^{-1} \) is just the Green potential, restricted to \( J \), for the standard random walk

\[
V_{jk} = \mathbb{E}_j \left( \sum_{\ell} \mathbb{1}_k(\mathcal{J}_{\ell}) \right) = g(j, k).
\]

Our matrix \( U \) is a principal submatrix of \( V \), that is \( U = V|_{E \times E} \). Thus \( U \) is a nonsingular potential, which corresponds to the potential of the standard random walk on \( \mathbb{Z}^d \), restricted to \( E \).
Proposition 2.5 For all $n \geq 1$, $\beta \geq 1$ and all $F \in \mathcal{C}_K$ it holds
\[
\sum_{x \in \mathbb{Z}^{d,n}} \left( G^{(n,\beta)}(F)(x) - 1 \right)^+ F(x) \left( \frac{d}{n} \right)^{d/2} \geq 0.
\]

Proof Consider $F \in \mathcal{C}_K$ and denote by $\mathcal{K}$ its support. We consider a large $a > 0$ such that $\mathcal{K} \subset [-a,a]^d$. Consider $U$ the symmetric matrix indexed by $A = \mathbb{Z}^{d,n} \cap [-a,a]^d$, given by
\[
U_{x,w} = c(n, d, \beta) \mathcal{g} \left( x \sqrt{n/d}, w \sqrt{n/d} \right),
\]
with
\[
c(n, d, \beta) = \left( \frac{d}{n} \right)^{\frac{d}{2}} \frac{n^{d/2-1}}{d^{d/2}}.
\]
This is a nonsingular symmetric potential matrix according to the previous Lemma. Then, Proposition A.2 shows that its Hadamard power $U^{(\beta)}$ is again a nonsingular symmetric potential matrix. Finally, Proposition A.1 allow us to conclude that for all $v \in \mathbb{R}^A$
\[
0 \leq \left( \frac{d}{n} \right)^{\frac{d}{2}} \left( (U^{(\beta)}v - 1)^+ , v \right) = \left( \frac{d}{n} \right)^{\frac{d}{2}} \sum_{x \in A} \left( \sum_{w \in A} (U_{x,w})^\beta v_w - 1 \right)^+ v_x.
\]
The result follows by taking $v \in \mathbb{R}^A$ with $v_x = F(x)$.

Now, we introduce the version of the maximum principle suitable for our purposes.

Definition 2.1 A positive linear bounded operator $V$ defined on $\mathcal{C}_K$ is said to satisfy the Complete Maximum principle on $\mathcal{C}_K$ if for any $F \in \mathcal{C}_K$ it holds: if $V(F)(x) \leq 1$ whenever $F(x) \geq 0$, then $V(F)(x) \leq 1$ for all $x \in \emptyset$.

We are in a position to prove the main result of this section.

Theorem 2.6 Assume that $1 \leq \beta < \frac{d}{d-2}$, then the operator $G^{(\beta)}$ satisfies the CMP on $\mathcal{C}_K$.

Proof Consider $F \in \mathcal{C}_K$. It is clear that $G^{(\beta)}(F)$ is a continuous function vanishing at $\infty$. Assume that $G^{(\beta)}(F)(x) \leq 1$ for those $x \in \mathbb{R}^d$ such that $F(x) \geq 0$. From Corollary 2.3 and Proposition 2.5, we have
\[
0 \leq \int_{\mathbb{R}^d} (G^{(\beta)}(F)(x) - 1)^+ F(x) \, dx = \int_{x : F(x) < 0} (G^{(\beta)}(F)(x) - 1)^+ F(x) \, dx.
\]
We conclude that $G^{(\beta)}(F)(x) \leq 1$ hold a.s. The continuity of $G^{(\beta)}(F)$ shows that $G^{(\beta)}(F) \leq 1$ and the result follows.

3 Powers of the Green Potential in Bounded Regular Open Sets

In this section we consider $\emptyset \subset \mathbb{R}^d$ a bounded regular open set. Let us introduce some of the basic notation we need. The Green kernel for Brownian Motion in $\emptyset$ is
\[
g_\emptyset(x, y) = g(x, y) - \mathbb{E}_x(g(B_{T_\emptyset}, y)),
\]
for \( x, y \in \emptyset \), where \( T_{\emptyset} = T_{\emptyset}(B) \) is the exiting time of \( \emptyset \) for the Brownian Motion \( B \). The associated Green operator is denoted by \( G_{\emptyset} \).

In the same spirit, if \( E \subset \mathbb{Z}^d \) we denote by \( g_E \) the Green kernel associated to the random walk, killed upon leaving \( E \), which is given by

\[
g_E(x, y) = \mathbb{E}_x \left( \sum_{k=0}^{R_{E}-1} \mathbb{I}_y(S_k) \right).
\]

defined for \( x, y \in E \), where \( R_E = \inf\{k \geq 0 : S_k \in E^c\} \). We extend this function by 0, that is \( g_E(x, y) = 0 \) if \( x \) or \( y \) belong to \( E^c \). We have a similar formula

\[
g_E(x, y) = g(x, y) - \mathbb{E}_x (g(S_{R_E}, y)),
\]

which is valid for all \( x, y \in \mathbb{Z}^d \).

We consider the \( \beta \)-powers of these functions: \( g^\beta\_\emptyset \) and \( g^\beta \) and we shall prove similar results as in Section 2.

### 3.1 Cubic Open Sets

In this section we consider the following family of simple open sets: cubic open sets, which are constructed as follows. Fix a positive integer \( m \) and a finite set \( E \subset \mathbb{Z}^d \) and consider the following set

\[
Q = \bigcup_{k \in E} B_{\infty}(k\sqrt{d/m}, 1/2\sqrt{d/m}).
\]

We take \( \emptyset = \text{int}(Q) \) and call the cubic open set (CO) with height \( m \) and basis \( E \). We point out that a CO set \( \emptyset \) can be described using different couples \( (m, E) \), as we will see. These open sets are bounded and regular for the Brownian Motion (they satisfy for example the exterior cone condition). In general \( \emptyset \) is not connected, but a finite union of connected components, which are also CO of the same height.

Now, fix \( \emptyset \) a CO with height \( m \geq 1 \) and basis \( E \). For every \( \ell \in \mathbb{N} \), we denote by \( n = n_\ell = m^{2\ell} \) and consider the set \( A_n = \sqrt{d/n} \mathbb{Z}^d \cap \emptyset \subset \mathbb{Z}^{d,n} \). We denote by \( E_n = \sqrt{n/d} A_n \subset \mathbb{Z}^d \). We point out that \( \emptyset \) is also a CO with height \( n \) and basis \( E_n \). Each point \( z \in A_n \) generates \( 3^\ell \) points in \( A_{n+1} : z + \frac{1}{3}\sqrt{d/n} e \), with \( e \in \mathbb{E} = \{-1, 0, 1\}^d \). Also, \( E_{n+1} = 3E_n + \mathbb{E} \). We point out that \( A_n \) is the set of centers of cubes forming \( \emptyset \) at height \( n \).

For \( x \in \mathbb{R}^d \), we choose \( x(n) \in \mathbb{Z}^{d,n} = \sqrt{d/n} \mathbb{Z}^d \) any of the closests elements in \( \mathbb{Z}^{d,n} \) to \( x \), in \( \infty \)-norm. To avoid any ambiguity, if there are more than one, we take \( x(n) \) the smallest such elements in the lexicographical order. We claim that if \( x \in \emptyset \) then \( x(n) \in A_n \).

Indeed, if \( x \in B_\infty(k\sqrt{d/n}, 1/2\sqrt{d/n}) \), for some \( k \in E_n \), then \( x(n) = k\sqrt{d/n} \in A_n \) and the claim holds in this case. So, assume that \( x \) does not belong to any of the open cubes \( B_\infty(k\sqrt{d/n}, 1/2\sqrt{d/n}) \) with \( k \in E_n \). Then, necessarily \( x \) belongs to a \( p \)-dimensional face of some closed ball \( K_1 = \overline{B_\infty(k_1\sqrt{d/n}, 1/2\sqrt{d/n})} \), with \( k_1 \in E_n \), and \( p \in \{0, \ldots, d-1\} \). We choose \( p \) the smallest such dimensions. Consider the contiguous closed balls that share the same face with \( K_1 : K_i = \overline{B_\infty(k_i\sqrt{d/n}, 1/2\sqrt{d/n})} \), \( k_i \in \mathbb{Z}^d \) for \( i = 2, \ldots, 2^{d-p} \). Since \( x \in \emptyset \), and \( \emptyset \) is open, we have that \( \emptyset \) intersects the interior of all these balls and therefore \( B_\infty(k_i\sqrt{d/n}, 1/2\sqrt{d/n}) \subset \emptyset \), for \( i = 1, \ldots, 2^{d-p} \). This implies that \( N = \{k_i\sqrt{d/n} : i = 1, \ldots, 2^{d-p}\} \subset \emptyset \), and therefore \( N \subset A_n \). Finally, we notice that the set of closests points in \( \mathbb{Z}^{d,n} \) to \( x \) is exactly \( N \), showing the claim.
Similar to Section 2 we define for all $F \in \mathcal{C}_K(\emptyset)$ and all $x \in \mathbb{R}^d$

$$\mathcal{G}_{\emptyset}^{n,(\beta)}(F)(x) = \sum_{w \in A_n} F(w + x - x(n)) \left( \frac{d}{n} \right)^{d/2} \left[ \mathcal{g}_{E_n} \left( x(n) \sqrt{n/d}, w \sqrt{n/d} \right) \frac{n^{d/2-1}}{d^{d/2}} \right]^\beta.$$ 

When $\beta = 1$, we denote $\mathcal{G}_{\emptyset}^{n,(1)} = \mathcal{G}_{\emptyset}^{n}$. Notice that $\mathcal{G}_{\emptyset}^{n} F(x)$, for $x \in A_n$, is just the Green potential of $F$, for the normalized random walk $(\delta_{t,n})_t$ killed when exiting $\emptyset$ (which is exactly exiting $A_n$) starting from $x$.

We observe that for $x, w \in A_n$

$$\mathcal{g}_{E_n} \left( x \sqrt{n/d}, w \sqrt{n/d} \right) \frac{n^{d/2-1}}{d^{d/2}} =$$

$$= \mathcal{g} \left( x \sqrt{n/d}, w \sqrt{n/d} \right) \frac{n^{d/2-1}}{d^{d/2}} - \mathbb{E}_x \mathcal{g} \left( \mathcal{R}_{E_n}, w \sqrt{n/d} \right) \frac{n^{d/2-1}}{d^{d/2}}$$

$$= \mathcal{g} \left( x \sqrt{n/d}, w \sqrt{n/d} \right) \frac{n^{d/2-1}}{d^{d/2}} - \mathbb{E}_x \mathcal{g} \left( \mathcal{S}_{T_{\emptyset}, n \sqrt{n/d}}, w \sqrt{n/d} \right) \frac{n^{d/2-1}}{d^{d/2}},$$

where the normalized random walk starts from $x$, and $T_{\emptyset} = T_{\emptyset}(\mathcal{S}_{*,n})$ is the exiting time from $\emptyset$.

For any $x, w$, we have $\mathcal{g}(x \sqrt{n/d}, w \sqrt{n/d}) \frac{n^{d/2-1}}{d^{d/2}} \rightarrow \mathcal{g}(x, w)$, as $n = m_{3\ell}^3$ converges to infinity. So, in order to prove that $\mathcal{G}_{\emptyset}^{n,(\beta)}(F)(x) \rightarrow \mathcal{G}_{\emptyset}^{(\beta)}(F)(x)$, we need the following Lemma, which is a consequence of the weak convergence of $(\delta_{t,n})_t$ to $(B_t)_t$ (see for example Theorem 3.5.1 in [11]).

**Lemma 3.1** For any $x, y \in \emptyset$ we have

$$\lim_{n \to \infty} \mathbb{E}_{x(n)} \left( \mathcal{g} \left( \mathcal{S}_{T_{\emptyset}, n \sqrt{n/d}, y(n) \sqrt{n/d}} \right) \frac{n^{d/2-1}}{d^{d/2}} \right) = \mathbb{E}_x(\mathcal{g}(B_{T_{\emptyset}}, y)).$$

**Proof** From Lemma 2.1, it is enough to show that

$$\lim_{n \to \infty} \mathbb{E}_{x(n)} \left( \mathcal{g}(\mathcal{S}_{T_{\emptyset}, n, y}) \right) = \mathbb{E}_x(\mathcal{g}(B_{T_{\emptyset}}, y)).$$

The function $h : \partial \emptyset \to \mathbb{R}_+$ defined by $h(z) = g(z, y)$ is continuous and bounded, since $d(y, \partial \emptyset) > 0$. In order to apply the weak convergence of $(\delta_{t,n})_t$ to $(B_t)_t$, we fix $t_0 > 0$ and consider, for every $\varepsilon > 0$, a continuous and bounded function $\psi : \mathbb{R}_+ \to [0, 1]$ such that $\psi(t) = 1$ for all $t \leq t_0$ and $\psi(t) = 0$ for all $t > t_0 + \varepsilon$. Then,

$$\lim_{n \to \infty} \mathbb{E}_{x(n)} \left( \mathcal{g} \left( \mathcal{S}_{T_{\emptyset}, n, y} \right) \psi(T_{\emptyset}) \right) = \mathbb{E}_x(\mathcal{g}(B_{T_{\emptyset}}, y) \psi(T_{\emptyset})).$$

Here, we have used that $\mathcal{g}(B_{T_{\emptyset}}, y)\psi(T_{\emptyset})$ is $\mathcal{F}_{t_0 + \varepsilon}$-measurable, bounded and $B$-continuous ($\emptyset$ is regular so $T_{\emptyset}(B) = T_{\emptyset}(B)$ for $\mathbb{P}_x$-a.s.). Recall that a measurable function $h$ is said to be $B$-continuous if $\mathbb{P}(B \in D_h) = 0$, where $D_h$ is the set of discontinuities of $h$. This implies that

$$\limsup_{n \to \infty} \mathbb{E}_{x(n)} \left( \mathcal{g}(\mathcal{S}_{T_{\emptyset}, n, y}) \mathbb{1}_{T_{\emptyset} \leq t_0} \right) \leq \mathbb{E}_x \left( \mathcal{g}(B_{T_{\emptyset}}, y) \mathbb{1}_{T_{\emptyset} \leq t_0} \right).$$

Similarly, we show that

$$\mathbb{E}_x(\mathcal{g}(B_{T_{\emptyset}}, y) \mathbb{1}_{T_{\emptyset} > t_0}) \leq \liminf_{n \to \infty} \mathbb{E}_{x(n)} \left( \mathcal{g}(\mathcal{S}_{T_{\emptyset}, n, y}) \mathbb{1}_{T_{\emptyset} \leq t_0} \right).$$

If we take $t_0$ a continuity point for the distribution of $T_{\emptyset}(B)$ we conclude that

$$\lim_{n \to \infty} \mathbb{E}_{x(n)} \left( \mathcal{g}(\mathcal{S}_{T_{\emptyset}, n, y}) \mathbb{1}_{T_{\emptyset} \leq t_0} \right) = \mathbb{E}_x(\mathcal{g}(B_{T_{\emptyset}}, y) \mathbb{1}_{T_{\emptyset} \leq t_0}).$$

(3.1)
To finish the proof, let $\varepsilon > 0$ be arbitrary, and choose $t_0$, a continuity point for $T_0(B)$, such that $P_x(T_0(B) > t_0) < \varepsilon$. This is possible since, as $\emptyset$ is bounded, we have $\lim_{t \to \infty} P_x(T_0(B) > t) = 0$. By repeating the main argument of this proof for $h(z) = 1$, we obtain that

$$\lim_{n \to \infty} P_x(T_0(S_{\ast,n}) \leq t_0) = P_x(T_0(B) \leq t_0) > 1 - \varepsilon,$$

from where it follows that $P_x(n(T_0(S_{\ast,n}) > t_0) < 2\varepsilon$ for large enough $n$. Since the Green kernel $g(\cdot, y)$ is bounded away from $y$, we can set $C_y = \sup_{x \in \partial \emptyset} g(x, y) < \infty$. The set $\emptyset$ is cubic open, thus regular, which implies that both $S_{T_0,n}$ and $B_{T_0}$ belong to $\partial \emptyset$. This in turn implies that both $g(S_{T_0,n}, y)$ and $g(B_{T_0}, y)$ are bounded above by $C_y$. Using all of this together with Eq. 3.1, it is straightforward to obtain that

$$\limsup_{n \to \infty} \left| E_x(n) \left( g(S_{T_0,n}, y) \right) - E_x(g(B_{T_0}, y)) \right| \leq 3C_y \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this finishes the proof.

It is straightforward to generalize Lemma 2.4 and Proposition 2.5 to the present setting. Also notice that $\partial E \subseteq g$ for any set $E$, then we can use the bounds developed in Section 2 (in particular (2.3), (2.4)) to prove the following result.

**Proposition 3.2** Let $\emptyset$ be a CO, then for all $\beta \in [1, \frac{d}{d-2})$, $x \in \emptyset$ and all $F \in C_K(\emptyset)$ the following limits exist

(i) $\lim_{n \to \infty} G^n_\emptyset(\beta)(F)(x) = G^{\emptyset}(F)(x)$;

(ii) $0 \leq \lim_{n \to \infty} \int \left( G^n_\emptyset(\beta)(F)(x) - 1 \right)^+ F(x) \, dx = \int \left( G^{\emptyset}(F)(x) - 1 \right)^+ F(x) \, dx$.

### 3.2 General Bounded Regular Open Sets

The purpose of this section is to generalize Proposition 3.2 to the case of a general bounded regular open set $\emptyset \subset \mathbb{R}^d$. To this end, we take the following approximations of $\emptyset$. For every positive $m$ consider

$$A^i_m = \{ x \in \sqrt{d/m} \mathbb{Z}^d : d_{\infty}(x, \emptyset) > \sqrt{d/m} \};$$

$$A^e_m = \{ x \in \sqrt{d/m} \mathbb{Z}^d : d_{\infty}(x, \emptyset) < \sqrt{d/m} \}.$$

We assume that $m_0$ is large enough so $A^i_{m_0}$ is not empty, and consider only integers of the form $m = m \varepsilon = m_0 2^\ell$. Notice that $A^i_m \subset A^e_m$ and $A^i_m \subset A^i_n$ if $m \leq n$. We define the CO sets

$$\emptyset^i_m = \text{int} \left( \bigcup_{x \in A^i_m} B_{\infty}(x, 1/2\sqrt{d/m}) \right) \subset \emptyset;$$

$$\emptyset^e_m = \text{int} \left( \bigcup_{x \in A^e_m} B_{\infty}(x, 1/2\sqrt{d/m}) \right) \subset \overline{\emptyset},$$

where here $i, e$ means interior and exterior respectively (similarly for $A^i_m, A^e_m$). We also denote by $I_m = \sqrt{m/d} A^i_m, E_m = \sqrt{m/d} A^e_m$. We point out that $\emptyset^i_m \subset \emptyset^i_{m+1} \subset \emptyset^e_{m+1} \subset \emptyset^e_n$, for all $m, n$.

For any fixed $m$, Lemma 3.1 can be applied to $\emptyset^i_m, \emptyset^e_m$. For $x, y \in \emptyset^i_m$ and $m \leq n$, we obviously have

$$\emptyset^n_{\emptyset^i_m}(\mathbbm{1}_{\{y(n)\}})(x(n)) \leq \emptyset^n_{\emptyset^e_m}(\mathbbm{1}_{\{y(n)\}})(x(n)) \leq \emptyset^n_{\emptyset^i_m}(\mathbbm{1}_{\{y(n)\}})(x(n)).$$
which gives
\[ \mathbb{E}_x(n) \left( g \left( S_{T_{O_m}^i, n\sqrt{n/d}}, y(n)\sqrt{n/d} \right) \right) \geq \mathbb{E}_x(n) \left( g \left( S_{T_{O_m}^\epsilon, n\sqrt{n/d}}, y(n)\sqrt{n/d} \right) \right). \]

Thus,
\[
\mathbb{E}_x (g(B_{T_{O_m}^i}, y)) = \lim_{n \to \infty} \frac{n^{d/2-1}}{d^{d/2}} \mathbb{E}_x(n) \left( g \left( S_{T_{O_m}^i, n\sqrt{n/d}}, y(n)\sqrt{n/d} \right) \right) \\
\geq \limsup_{n \to \infty} \frac{n^{d/2-1}}{d^{d/2}} \mathbb{E}_x(n) \left( g \left( S_{T_{O_m}^\epsilon, n\sqrt{n/d}}, y(n)\sqrt{n/d} \right) \right) \\
\geq \liminf_{n \to \infty} \frac{n^{d/2-1}}{d^{d/2}} \mathbb{E}_x(n) \left( g \left( S_{T_{O_m}^\epsilon, n\sqrt{n/d}}, y(n)\sqrt{n/d} \right) \right) \\
= \mathbb{E}_x (g(B_{T_{O_m}^\epsilon}, y)).
\]

Since \( O_m^i \uparrow \varnothing \) and \( O_m^\epsilon \downarrow \overline{\varnothing} \), as \( m \to \infty \), then the regularity of \( \varnothing \) is used to show the analogous of Lemma 3.1: For all \( x, y \in \varnothing \), with \( a = e, i \)

\[
\lim_{n \to \infty} \frac{n^{d/2-1}}{d^{d/2}} \mathbb{E}_x(n) \left( g \left( S_{T_{O_m}^\epsilon, n\sqrt{n/d}}, y(n)\sqrt{n/d} \right) \right) = \mathbb{E}_x (g(B_{T_{O}^\epsilon}, y)). \quad (3.2)
\]

We have essentially shown the following result.

**Proposition 3.3** Let \( \varnothing \) be a bounded regular open set. Then, for all \( \beta \in [1, \frac{d}{d-2}) \), \( x \in \varnothing \) and all \( F \in C_{\mathcal{K}}(\varnothing) \) the following limits hold

(i) \( \lim_{n \to \infty} \mathcal{O}_{n(\beta)}^x(F)(x) = G_{\varnothing}^{(\beta)}(F)(x); \)

(ii) \( 0 \leq \lim_{n \to \infty} \int_{\varnothing} \left( G_{\varnothing}^{(\beta)}(F)(z) - 1 \right)^+ F(z) \, dz = \int_{\varnothing} \left( G_{\varnothing}^{(\beta)}(F)(x) - 1 \right)^+ F(z) \, dz. \)

The positive operator \( G_{\varnothing}^{(\beta)} \) can be extended to the set \( \mathbb{B}_b = \mathbb{B}_b(\varnothing) \) of bounded measurable functions, because if \( f \in \mathbb{B}_b \), we have

\[
|G_{\varnothing}^{(\beta)}(f)(x)| \leq \int_{\varnothing} |f(y)| \, g^{\beta}_\varnothing(x, y) \, dy \leq \|f\|_{\infty} \int_{\varnothing} g^{\beta}_\varnothing(x, y) \, dy \\
\leq \|f\|_{\infty} \int_{B(0,R)} g^{\beta}(0, y) \, dy = D(d, \beta, \varnothing) \|f\|_{\infty},
\]

where \( D(d, \beta, \varnothing) = \left( \frac{\Gamma(d/2-1)}{2\pi^{d/2}} \right)^\beta S(d) \frac{\theta^{\beta(d-2)}}{d^{\beta(d-2)}}, \) with \( S(d) \) the surface of the \( (\text{Euclidean}) \) unit ball in \( \mathbb{R}^d \) and \( R = \text{diameter}(\varnothing). \)

It is also straightforward to show that \( G_{\varnothing}^{(\beta)} \) maps \( \mathbb{B}_b \) into \( C_0(\varnothing) \). Indeed, take any \( x \in \varnothing, \varepsilon > 0 \) small enough and \( (x_n)_n \subset \varnothing \) such that \( x_n \to x \). Let us start with

\[
G_{\varnothing}^{(\beta)}(f)(x_n) = \int_{B(x, \varepsilon)} f(y) \, g^{\beta}_\varnothing(x_n, y) \, dy + \int_{y \cdot d(y, x) > \varepsilon} f(y) \, g^{\beta}_\varnothing(x_n, y) \, dy.
\]

For every fixed \( \varepsilon \) the second integral converges to \( \int_{y \cdot d(y, x) > \varepsilon} f(y) \, g^{\beta}_\varnothing(x, y) \, dy. \)
On the other hand \(|\int_{B(x, \varepsilon)} f(y) g^\beta(x_n, y) \, dy| \leq \|f\|_\infty \int_{B(x, \varepsilon)} g^\beta(x, y) \, dy|\) which is uniformly bounded by \(\|f\|_\infty \int_{B(x, 2\varepsilon)} g^\beta(x, y) \, dy|\) if \(|x - x_n| \leq \varepsilon\). This last integral converges to 0 when \(\varepsilon \downarrow 0\), proving that \(G^{(\beta)}_\emptyset(f)\) is a continuous function on \(\emptyset\). We already know it is also a bounded function. In particular \(G^{(\beta)}_\emptyset: C_b(\emptyset) \to C_b(\emptyset)\) is a bounded positive linear operator. Now, let us show that \(G^{(\beta)}_\emptyset(f) \in C_0(\emptyset)\), for any \(f \in B_\emptyset\). For that purpose, we notice that

\[
\limsup_{z \to \partial\emptyset} G^{(\beta)}_\emptyset(f)(z) \leq \|f\|_\infty \int_{\emptyset} g^\beta_\emptyset(z, y) \, dy,
\]

which is finite (it suffices to replace \(g^\beta_\emptyset\) by \(g^\beta\)). So, it is enough to show that \(a = \limsup_{z \to \partial\emptyset} \int_{\emptyset} g^\beta_\emptyset(z, y) \, dy\) is zero. We take any sequence \((z_k)_k \subset \emptyset\) such that \(z_k \to \partial\emptyset\) and

\[
a = \lim_{k \to \infty} \int_{\emptyset} g^\beta_\emptyset(z_k, y) \, dy.
\]

Since \(\emptyset\) is bounded we can assume further that \(z_k \to z \in \partial\emptyset\). For \(\varepsilon > 0\), we have

\[
\int_{\emptyset} g^\beta_\emptyset(z_k, y) \, dy \leq \int_{\emptyset \cap \|z - y\| \leq 2\varepsilon} g^\beta_\emptyset(z_k, y) \, dy + \int_{\emptyset \cap \|z - y\| > 2\varepsilon} g^\beta_\emptyset(z_k, y) \, dy.
\]

The second term converges to 0 from the Dominated Convergence Theorem. Indeed, the regularity of \(\emptyset\) shows that for all \(y \in \emptyset\) we have \(g^\beta_\emptyset(z_k, y) \to 0\), when \(k \to \infty\). On the other hand, for large \(k\) we can assume that \(\|z_k - y\| > \varepsilon\) and then \(g^\beta_\emptyset(z_k, y) \leq C(d)\beta(2-d)\), providing the desired domination.

For the first term, if \(k\) is large enough such that \(\|z - z_k\| \leq \varepsilon\), we get

\[
\int_{\emptyset \cap \|z - y\| \leq 2\varepsilon} g^\beta_\emptyset(z_k, y) \, dy \leq \int_{\emptyset \cap \|z - y\| \leq 2\varepsilon} g^\beta(z_k, y) \, dy \leq \int_{\|y\| \leq 3\varepsilon} g^\beta(0, y) \, dy.
\]

We conclude that \(a \leq \lim_{\varepsilon \downarrow 0} \int_{\|y\| \leq 3\varepsilon} g^\beta(0, y) \, dy = 0\), and the claim is shown.

So far we have proved that \(G^{(\beta)}_\emptyset: B_\emptyset(\emptyset) \to B_\emptyset(\emptyset)\) is a bounded positive linear operator, which also satisfies \(G^{(\beta)}_\emptyset: B_\emptyset(\emptyset) \subset C_0(\emptyset)\). We now extend \((ii)\) in Proposition 3.3 for \(f \in B_\emptyset(\emptyset)\), that is \(G^{(\beta)}_\emptyset\) satisfies the CMP in \(B_\emptyset(\emptyset)\).

**Lemma 3.4** Assume \(f \in B_\emptyset(\emptyset)\) then

\[
0 \leq \int_{\emptyset} \left( G^{(\beta)}_\emptyset(f)(x) - 1 \right)^+ f(x) \, dx.
\]

**Proof** Similar to Eq. 3.3, for any \(f \in B_\emptyset(\emptyset)\) and every \(\varepsilon > 0\), we have

\[
|G^{(\beta)}_\emptyset(f)(x)| \leq \|f\|_\infty \int_{B(0, \varepsilon)} g^\beta(0, y) + \left( \frac{C(d)}{\varepsilon^{d-2}} \right)^\beta \|f\|_1.
\]
So, if \((F_k)_k \subset C_0(\Omega)\) is a sequence of functions such that \(\|F_k - f\|_1 \to 0\) and \(\|F_k\|_\infty \leq \|f\|_\infty\), then
\[
\|G_\Omega^{(\beta)}(f) - G_\Omega^{(\beta)}(F_k)\|_\infty \leq 2\|f\|_\infty \int_{B(0,\varepsilon)} g^\beta(0, y) + \left(\frac{C(d)}{\varepsilon^{d-2}}\right)^\beta \|F_k - f\|_1.
\]
Therefore, taking limits in \(k\) and then in \(\varepsilon\), we conclude that \((G_\Omega^{(\beta)}(F_k))_k\) converges uniformly to \(G_\Omega^{(\beta)}(f)\). The rest of the proof is an application of Dominated Convergence Theorem and Proposition 3.3 (ii).

In summary, \(G_\Omega^{(\beta)} : \mathbb{B}_b(\Omega) \to \mathbb{B}_b(\Omega)\) is a positive, bounded linear operator and from the previous Lemma it satisfies the CMP in \(\mathbb{B}_b(\Omega)\). Hence, \(G_\Omega^{(\beta)}\) is the potential of a unique contraction resolvent \(\mathcal{U} = (U^\lambda)_{\lambda \geq 0}\), defined on \(\mathbb{B}_b(\Omega)\). In particular, for each \(\lambda\) we have \(U^\lambda : \mathbb{B}_b(\Omega) \to \mathbb{B}_b(\Omega)\) is a bounded positive linear operator such that \(\|\lambda U^\lambda\| \leq 1\) and
\[
U^0 = G_\Omega^{(\beta)}.
\]
See for example Lemma 4.1.9 in [12] and Remark 4.1.10 applied to the Banach space \((\mathbb{B}_b(\Omega), \|\cdot\|_\infty)\). We recall that \(G_\Omega^{(\beta)}(\mathbb{B}_b(\Omega)) \subset C_0(\Omega)\), and the way \(\mathcal{U}\) is constructed implies that for all \(\lambda\) one has \(U^\lambda(\mathbb{B}_b(\Omega)) \subset C_0(\Omega)\) (recall that \(U^\lambda = G_\Omega^{(\beta)}\left(1 + \lambda G_\Omega^{(\beta)}\right)^{-1}\)). We prove now an extra property of this resolvent.

**Proposition 3.5** The resolvent \(\mathcal{U}\) is continuous on \(\lambda\), that is, for all \(\lambda \geq 0\)
\[
\lim_{\lambda' \to \lambda} \|U^\lambda - U^{\lambda'}\| = 0.
\]
In case \(\lambda = 0\), the limit is taken as \(\lambda' > 0\). In particular, for all \(f \in \mathbb{B}_b(\Omega)\) we have
\[
\|U^\lambda (f) - G_\Omega^{(\beta)} (f)\|_\infty \to 0,
\]
as \(\lambda \downarrow 0\).

**Proof** We start with the case \(\lambda = 0\). The resolvent equation shows that \(U^{\lambda'} \leq U^0\) on \(\mathbb{B}_b(\Omega)\). Using inequality (3.3) and the resolvent equation, we get for all \(x \in \Omega\)
\[
|U^0(f)(x) - U^{\lambda'}(f)(x)| \leq \lambda' U^{\lambda'}(U^0(|f|))(x) \leq \lambda' U^0(U^0(|f|))(x) \leq \lambda' D^2 \|f\|_\infty,
\]
and the result is shown in this case.

For \(\lambda > 0\), the result is a consequence of the resolvent equation and the fact that the resolvent is a contraction
\[
\|U^\lambda - U^{\lambda'}\| \leq \frac{\|\lambda - \lambda'\|}{\lambda \lambda'} \|U^{\lambda'}\| \|\lambda' U^{\lambda'}\| \leq \frac{|\lambda - \lambda'|}{\lambda \lambda'}.
\]

Our next step is to show that this resolvent comes from a Ray process (see Chapter 4 in [12]). To this end, we recall that a function \(f \in \mathbb{B}_b^+\) is supermedian if
\[
a U^\alpha(f)(x) \leq f(x),
\]
for all \(x \in \Omega\) and some (all) \(\alpha > 0\). We denote by \(\mathcal{M}^+\) the set of supermedian functions and \(\mathcal{M} = \mathcal{M}^+ - \mathcal{M}^+\) the linear space generated. It is well known that \(\mathcal{M}^+\) is closed under monotone pointwise convergence, \(\mathcal{M}\) is a lattice and contains the constants. Using the resolvent
equation it is straightforward to show that $G^{(q)}_{\mathbb{B}_{b}^{+}} \subset \mathcal{M}^{+}$. The following technical result is needed.

**Lemma 3.6** The set $\mathcal{H} = \mathcal{M} \cap C_{0}(\mathcal{O})$ is dense in $C_{0}(\mathcal{O})$.

**Proof** Recall that $\hat{\mathcal{O}} = \mathcal{O} \cup \{\vartheta\}$ is the one point compactification of $\mathcal{O}$, and that $C(\hat{\mathcal{O}})$ is identified with $C_{0}(\mathcal{O}) \oplus 1$. The set $\hat{\mathcal{H}} = \mathcal{M} \cap C(\hat{\mathcal{O}})$ is a lattice that contains the constants and according to the Stone-Weierstrass Theorem is dense in $C(\hat{\mathcal{O}})$, as soon as it separates points.

Take first, $x, y \in \mathcal{O}$ and consider $f_{\varepsilon} = f = \mathbb{1}_{B(x, \varepsilon)}$ for small $0 < \varepsilon < 1$ to be determined later. At least we assume for the moment that $B(x, 2\varepsilon) \subset \mathcal{O}$ and $\varepsilon < \|x - y\|$. The function $F = G^{(q)}_{\mathcal{O}}(f)$ belongs to $\mathcal{M}^{+} \cap C_{0}(\mathcal{O})$ and we have for $z \in B(x, \varepsilon), \xi \in \partial \mathcal{O}$

$$g(\xi, z) = C(d)\|\xi - z\|^{2-d} \leq d(z, \partial \mathcal{O})^{2-d}\|x - z\|^{2-d}g(x, z)$$

$$\leq d(z, \partial \mathcal{O})^{2-d}d^{2-d}g(x, z) \leq (d(x, \partial \mathcal{O}) - \varepsilon)\|x - z\|^{2-d}g(x, z) \quad (3.4)$$

and therefore $g^{\beta}_{\mathcal{O}}(x, z) \geq g^{\beta}(x, z)\left(1 - d(x, \partial \mathcal{O})^{2-d}(2\varepsilon)^{d-2}\right)^{\beta}$. On the other hand, $g^{\beta}_{\mathcal{O}}(y, z) \leq g^{\beta}(y, z) = C^{\beta}(d)\|y - z\|^{\beta(2-d)}$, moreover

$$\|y - z\| \geq \|x - y\| - \varepsilon \geq \frac{\|y - x\| - \varepsilon}{\varepsilon}\|x - z\|,$$

and we conclude

$$g^{\beta}_{\mathcal{O}}(y, z) \leq (\|y - x\| - \varepsilon)\|x - z\|\varepsilon^{\beta(2-d)}g^{\beta}(x, z).$$

Thus,

$$F(y) = G^{(q)}_{\mathcal{O}}(f)(y) \leq (\|y - x\| - \varepsilon)\varepsilon^{\beta(2-d)}\int_{B(x, \varepsilon)} g^{\beta}(x, z)dz \leq \left(\frac{\|y - x\| - \varepsilon}{1 - d(x, \partial \mathcal{O})^{2-d}(2\varepsilon)^{d-2}}\right)^{\beta}\varepsilon^{\beta(d-2)}G^{(q)}_{\mathcal{O}}(f)(x).$$

By taking $\varepsilon$ small enough, we prove that $G^{(q)}_{\mathcal{O}}(f)(y) < G^{(q)}_{\mathcal{O}}(f)(x)$. The function $F \in \mathcal{M}^{+} \cap C_{0}(\mathcal{O})$ separates the points $x, y$.

Now, we separate $x \in \mathcal{O}$ and $\partial$. For this we need the constant functions. If we take now $F = G^{(q)}_{\mathcal{O}}(1) + 1$. Clearly we have $F(\mathcal{O}) > 1 = F(\partial)$.

So far we have shown that $\mathcal{M} \cap C(\hat{\mathcal{O}})$ is dense in $C(\hat{\mathcal{O}})$. In particular is dense in $C_{0}(\mathcal{O})$. So, for any $F \in C_{0}(\mathcal{O})$ there exists a sequence $(F_{n})_{n} \subset \mathcal{M} \cap C(\hat{\mathcal{O}})$, which converges uniformly on $\hat{\mathcal{O}}$ to $F$ (extended by 0 on $\partial$). The sequence defined by $H_{n}(x) = F_{n}(x) - F_{n}(\partial) \in \mathcal{M} \cap C_{0}(\mathcal{O})$ converges uniformly to $F$ and the result is shown.$\square$

We can apply now Theorem 4.7.1 in [12] to show the following result, which is closer to Theorem 1.1.

**Theorem 3.7** Assume that $\mathcal{O} \subset \mathbb{B}^{d}$ is a bounded regular open set. Then, for all $\beta \in [1, \frac{d}{d-2})$ there exists a sub-Markov Ray-semigroup $\mathcal{P} = (P_{t})_{t}$, defined on $\mathbb{B}_{b}(\mathcal{O})$, such that
(i) for all $F \in \mathcal{C}_b(\emptyset)$ the function $(t, x) \mapsto P_t(F)(x)$ is, for every $x$, right continuous in $t \in [0, \infty)$ and for every $t$, a Borel measurable function on $x$. Therefore, $P_t(F)(x)$ it is a Borel measurable function of $(t, x)$.

(ii) for all $F \in \mathcal{C}_b(\emptyset)$, $x \in \emptyset$, $\lambda \geq 0$

$$U^\lambda(F)(x) = \int_0^\infty e^{-\lambda t} P_t(F)(x) \, dt.$$ 

Proof Theorem 4.7.1 in [12] shows (i) and (ii) except for the case $\lambda = 0$, which we now prove. Proposition 3.5 and the Monotone Convergence Theorem shows that for all $F \in \mathcal{C}_b(\emptyset)^+$ it holds

$$G^P_{\emptyset}(F)(x) = U^0(F)(x) = \lim_{\lambda \downarrow 0} U^\lambda(F)(x) = \lim_{\lambda \downarrow 0} \int_0^\infty e^{-\lambda t} P_t(F)(x) \, dt$$

$$= \int_0^\infty P_t(F)(x) \, dt. \quad \Box$$

Recall that a sub-Markov Ray-semigroup $\mathcal{P} = (P_t)_t$ is like a sub-Markov semigroup, except that $P_0$ may not be the identity. So,

1. for all $t, s \in \mathbb{R}^+$, $f \in \mathbb{B}_b$, $x \in \emptyset$ we have $P_{t+s}(f)(x) = P_t(P_s(f))(x) = P_s(P_t(f))(x)$;
2. each $P_t$ is a positive operator and $P_t(1) \leq 1$, so it is a contraction. Thus, for all $t \geq 0$, we can decompose $P_t(f)(x) = \int f(y) P_t(x, dy)$, where $P_t(x, dy)$ is a sub-probability measure, for $(t, x)$ fixed, which is measurable in $(t, x)$;
3. for all $F \in \mathcal{C}_b(\emptyset)$, $x \in \emptyset$ the function $t \mapsto P_t(F)(x)$ is right continuous in $[0, \infty)$, and if $f \in \mathcal{M}^+$ then $P_t(f)(x)$ is decreasing in $t$.

The fact that $P_0$ is not the identity has important consequences. For example, $P_t(F)(x)$ is, in general, a discontinuous function of $x$, even if $F \in \mathcal{C}_b(\emptyset)$.

In [12], a Ray process $X$ is constructed taking values in $\hat{\emptyset}$ with c\'adl\'ag paths and associated semigroup an extension of $(P_t)_t$. The semigroup is extended to $(\hat{\emptyset}, \mathcal{B}(\hat{\emptyset}))$ by simply putting $\overline{P}_t(x, \partial) = 1 - P_t(x, \emptyset)$ and $\overline{P}_t(\partial, \partial) = 1$. Similarly, we extend the resolvent $(\overline{U}^\lambda)_{\lambda \geq 0}$ and for any function defined on $\emptyset$ we set $\overline{f}$ the extension to $\hat{\emptyset}$ given by $\overline{f}(\partial) = 0$. Notice that $\overline{f} \in \mathcal{C}_b(\hat{\emptyset})$ iff $f \in \mathcal{C}_0(\emptyset)$. We also have

$$\overline{P}_t(\overline{f})(x) = \overline{P}_t(\overline{f})(x), \quad \overline{U}^\lambda(\overline{f})(x) = \overline{U}^\lambda(\overline{f})(x).$$

We remark that $\overline{U}^\lambda(\overline{f})$ is in general not continuous at $\partial$. On the other hand, for all $f \in \mathbb{B}_b(\hat{\emptyset})$ and $t, s \geq 0$ it holds

$$\mathbb{E}(f(X_{t+s})|\mathcal{F}_t) = \overline{P}_s(f)(X_t).$$

For what it follows, a distinguished set is the set of branching points denoted by

$$N = \{ x \in \emptyset : P_0(x, dy) \neq \delta_x(dy) \},$$

and we put $D = \emptyset \setminus N$. We denote by $\widehat{D} = D \cup \partial$. It is known that

1. $D$ is a Borel set;
2. for all $x \in \widehat{D}$ we have $P_0(x, dy) = \delta_x(dy)$;
3. for all $t \geq 0$ and all $x \in \emptyset$, we have $\overline{P}_t(x, N) = 0$. 

\[ \text{Springer} \]
An interesting result (see Lemma 4.7.9 in [12]) is that \( X_t(\omega) \in \hat{D} \) for all \( t \geq 0 \), \( \mathbb{P} \)-a.s. Nevertheless, at some times \( t \) the left limit \( X_{t-}(\omega) \) may belong to \( N \), where the process branches again.

In what follows, we prove that \( X \) is a Feller process. This is equivalent to show that \( N \) is empty, and this will finish the proof of Theorem 1.1, when \( \mathcal{O} \) is a bounded regular open set.

**Lemma 3.8** \( N = \emptyset \).

**Proof** Consider \( F \in C_b(\mathcal{O})^+ \). Since \( G_{\mathcal{O}}^{(\beta)}(F)(x) = \int P_t(F)(x) \, dt \), Fubini’s theorem and the semigroup property shows for all \( x \)

\[
P_0(G_{\mathcal{O}}^{(\beta)}(F))(x) = \int G_{\mathcal{O}}^{(\beta)}(F)(y) P_0(x, dy) = \int_0^\infty P_t(F)(y) dt \, P_0(x, dy) = \int_0^\infty P_t(F)(x) dt = G_{\mathcal{O}}^{(\beta)}(F)(x).
\]

Hence, \( P_0(G_{\mathcal{O}}^{(\beta)}(F)) = G_{\mathcal{O}}^{(\beta)}(F) \), for all \( F \in C_b(\mathcal{O}) \). Another way to say this is, if we take \( H \in G_{\mathcal{O}}^{(\beta)}(C_b(\mathcal{O})) \), then \( P_0(H)(x) = H(x) \) holds for all \( x \in \mathcal{O} \). From here we cannot conclude that \( P_0(x, dz) = \delta_x(dz) \), unless we can prove that \( G_{\mathcal{O}}^{(\beta)}(C_b(\mathcal{O})) \) is dense in \( C_0(\mathcal{O}) \), which turns out to be equivalent to the result we are trying to prove.

Let us continue with the proof. For \( x \in \mathcal{O} \), we consider, \( f = 1_{\mathcal{B}} \), where \( B = B(x, \varepsilon) \). We assume that \( \varepsilon > 0 \) is small enough such that \( \overline{B}(x, 2\varepsilon) \subset \mathcal{O} \). We define \( F = G_{\mathcal{O}}^{(\beta)}(f) \). Take a decreasing sequence \( (H_k)_k \) of continuous functions with compact support in \( B(x, 2\varepsilon) \), taking values on \([0, 1]\), such that \( H_k \downarrow f \). Since \( G_{\mathcal{O}}^{(\beta)}(H_k) \downarrow G_{\mathcal{O}}^{(\beta)}(f) \) and \( P_0(x, \bullet) \) is a finite measure, we conclude from the Monotone Convergence Theorem \( P_0(F) = P_0(G_{\mathcal{O}}^{(\beta)}(f)) = G_{\mathcal{O}}^{(\beta)}(f) = F \), which is one of the main ingredients we need.

We obtain the following lower estimate (see Eq. 3.4 in the proof of Lemma 3.6)

\[
P_0(F)(x) = F(x) \geq (1 - d(x, \partial \mathcal{O})^{2-d}(2\varepsilon)^{d-2})^\beta \int_B g^{(\beta)}(x, z) \, dz
\]

\[
= (1 - d(x, \partial \mathcal{O})^{2-d}(2\varepsilon)^{d-2})^\beta G^{(\beta)}(f)(x).
\]

On the other hand, \( F(z) = G_{\mathcal{O}}^{(\beta)}(f)(z) \leq \int_B g^{(\beta)}(z, w) \, dw = G^{(\beta)}(f)(z) \). It is straightforward to show that \( G^{(\beta)}(f)(z) \leq G^{(\beta)}(f)(x) \). When \( 0 < \|z - x\| \leq 2\varepsilon \) this property can be shown using a reflection with respect to the hyperplane with normal \((x - z)/\|x - z\|\) passing through \( z = 1/2(z + x) \) (this is just the reflection principle for BM). Indeed, take the regions

\( R_1 = \{w \in B : \|w - z\| \leq \|w - x\|\}, \)

\( R_2 \) the reflection of \( R_1 \) with respect to the hyperplane and \( R_3 = B \setminus (R_1 \cup R_2) \). Then

\[
\int_{R_1} g^{(\beta)}(z, w) \, dw = \int_{R_2} g^{(\beta)}(x, w) \, dw
\]

\[
\int_{R_2} g^{(\beta)}(z, w) \, dw = \int_{R_1} g^{(\beta)}(x, w) \, dw
\]

\[
\int_{R_3} g^{(\beta)}(z, w) \, dw < \int_{R_3} g^{(\beta)}(x, w) \, dw,
\]

and the claim follows.
Now, if \( k \geq 2 \) and \( \|z - x\| > k\varepsilon \), we obtain \( \|z - w\| \geq (k - 1)\|x - w\| \), for any \( w \in B \), showing that 

\[
F(z) \leq G^{(\beta)}(f)(z) \leq (k - 1)^{\beta(2-d)} G^{(\beta)}(f)(x) \leq G^{(\beta)}(f)(x).
\]

So, for fixed \( k \geq 3 \), we get 

\[
P_0(F)(x) = \int_{B(x,k\varepsilon)} F(z) P_0(x, dz) + \int_{\Omega \cap (B(x,k\varepsilon))^c} F(z) P_0(x, dz) \\
\leq G^{(\beta)}(f)(x) \left( \int_{B(x,k\varepsilon)} P_0(x, dz) + (k - 1)^{\beta(2-d)} \int_{\Omega \cap (B(x,k\varepsilon))^c} P_0(x, dz) \right) \\
\leq G^{(\beta)}(f)(x) \left( P_0(x, B(x,k\varepsilon)) + (k - 1)^{\beta(2-d)} (1 - P_0(x, B(x,k\varepsilon))) \right).
\]

In the last inequality we have used that \( P_0(x, \bullet) \) is a measure whose total mass is at most 1. Using the lower bound obtained in Eq. 3.5 we conclude that 

\[
\left( 1 - d(x, \partial\Omega)^{2-d}(2\varepsilon)^{d-2} \right)^{\beta} \leq P_0(x, B(x,k\varepsilon)) + (k - 1)^{\beta(2-d)} (1 - P_0(x, B(x,k\varepsilon)))
\]

and taking \( \varepsilon \downarrow 0 \) yields 

\[
1 \leq P_0(x, \{x\})(1 - (k - 1)^{\beta(2-d)}) + (k - 1)^{\beta(2-d)},
\]

which is possible only if \( P_0(x, \{x\}) = 1 \) and therefore \( x \notin N \). The result is shown. \( \square \)

This finishes the proof of Theorem 1.1, when \( \Omega \) is a bounded regular open set.

### 4 Powers of the Green Potential in Unbounded Regular Open Sets

In this section we shall prove Theorem 1.1 for \( \Omega \) an unbounded regular open set. We shall use the same notation of previous sections. The first thing we shall prove is that \( G^{(\beta)}_\Omega \) satisfies the CMP on \( C_K \). For that purpose, we approximate \( \Omega \) by an increasing sequence of bounded regular open sets. For every \( n \geq 1 \) we define \( \Omega_n = \Omega \cap B(0, n) \) and we assume \( n \) is large enough, so \( \Omega_n \) is not empty. It is straightforward to show that \( \Omega_n \) is also regular.

For each \( n \) consider \( G^{(\beta)}_{\Omega_n} \), which is a positive bounded linear operator defined on \( C_b(\Omega_n) \) that satisfies there the CMP. Moreover, for every \( F \in C_b(\Omega_n) \) we have 

\[
\int_{\Omega_n} \left( G^{(\beta)}_{\Omega_n}(F)(x) - 1 \right)^+ F(x) dx \geq 0.
\]

The Green kernel of \( G^{(\beta)}_{\Omega_n} \) is \( g^{\beta}_{\Omega_n}(x, y) = \left( g(x, y) - \mathbb{E}_x (g(B_{\Omega_n}, y)) \right)^\beta \), which converges pointwise, for all \( x \neq y \in \Omega \), to \( g^{\beta}_\Omega(x, y) = \left( g(x, y) - \mathbb{E}_x (g(B_T\Omega, y)) \right)^\beta \), as \( n \to \infty \).

Consider, now \( F \in C_K(\Omega) \) and \( n \) large enough such that \( \text{supp}(F) \subset \Omega_n \). Then, for \( \varepsilon > 0 \) we have 

\[
G^{(\beta)}_{\Omega_n}(F)(x) = \int_{B(x,\varepsilon) \cap \Omega_n} F(y) g^{\beta}_{\Omega_n}(x, y) dy + \int_{\Omega_n \setminus B(x,\varepsilon)} F(y) g^{\beta}_{\Omega_n}(x, y) dy.
\]

The first integral is bounded by \( \|F\|_\infty \int_{B(x,\varepsilon)} g^{\beta}(x, y) dy = \|F\|_\infty \int_{B(0,\varepsilon)} g^{\beta}(0, y) dy \). The second integral converges, for every fixed \( \varepsilon \), as \( n \to \infty \) to 

\[
\int_{\Omega \setminus B(x,\varepsilon)} F(y) g^{\beta}_\Omega(x, y) dy,
\]
because the Dominated Convergence Theorem. Here the domination is given by the function \((C(d)\varepsilon^{2-d})^{\beta}|F|\). Thus,

\[
\limsup_{n \to \infty} |G_{\Omega_n}^{(\varepsilon)}(F)(x) - G_{\Omega}^{(\varepsilon)}(F)(x)| \leq 2 \|F\|_\infty \int_{B(0, \varepsilon)} g^\beta(0, y) \, dy,
\]
which converges to 0 as \(\varepsilon \downarrow 0\). Notice that we also have the uniform domination for \(G_{\Omega_n}^{(\varepsilon)}(F)(x)\) given by

\[
|G_{\Omega_n}^{(\varepsilon)}(F)(x)| \leq \|F\|_\infty \int_{B(0, \varepsilon)} g^\beta(0, y) \, dy + \left(\frac{C(d)}{a^{d-2}}\right)^\beta \int_{\Omega} |F(y)| \, dy,
\]
which is also inherited by \(G_{\Omega}^{(\varepsilon)}\)

\[
|G_{\Omega}^{(\varepsilon)}(F)(x)| \leq \|F\|_\infty \int_{B(0, \varepsilon)} g^\beta(0, y) \, dy + \left(\frac{C(d)}{a^{d-2}}\right)^\beta \|F\|_1, \tag{4.1}
\]
valid for every \(a > 0\).

Again an application of the Dominated Convergence Theorem, we conclude that

\[
0 \leq \lim_{n \to \infty} \int_{\Omega_n} \left( G_{\Omega_n}^{(\varepsilon)}(F)(x) - 1 \right)^+ F(x) \, dx = \int_{\Omega} \left( G_{\Omega}^{(\varepsilon)}(F)(x) - 1 \right)^+ F(x) \, dx.
\]

We have proved the following result.

**Proposition 4.1** \(G_{\Omega}^{(\varepsilon)}\) satisfies the CMP in \(C_{\Omega}(\Omega)\). Moreover, for every \(F \in C_{\Omega}(\Omega)\) we have

\[
0 \leq \int_{\Omega} \left( G_{\Omega}^{(\varepsilon)}(F)(x) - 1 \right)^+ F(x) \, dx. \tag{4.2}
\]

For reasons that will be clear later on, we need to extend \(G_{\Omega}^{(\varepsilon)}\) to \(B_b(\Omega) \cap L^1\), and prove it satisfies the CMP there (here \(L^1 = L^1(\Omega, dx)\)).

Inequality (4.1) allow us to extend \(G_{\Omega}^{(\varepsilon)}\) to \(B_b(\Omega) \cap L^1\). Indeed, for a fixed \(f \in B_b(\Omega) \cap L^1\) consider a sequence of functions \((F_k)_{k \in \mathbb{C}_{\Omega}(\Omega)}\), such that \(\|F_k\|_\infty \leq \|f\|_\infty\) and \(\|f - F_k\|_1 \to 0\).

Then, for every \(\varepsilon > 0\), choose \(a = a(\varepsilon) > 0\) such that

\[
2 \|f\|_\infty \int_{B(0, a)} g^\beta(0, y) \, dy \leq \varepsilon/2
\]
and then choose \(n_0 = n_0(\varepsilon)\) such that \(\left(\frac{C(d)}{a^{d-2}}\right)^\beta \|F_k - F_m\|_1 \leq \varepsilon/2\), for all \(k, m \geq n_0\).

Then \(\|G_{\Omega}^{(\varepsilon)}(F_k) - G_{\Omega}^{(\varepsilon)}(F_m)\|_\infty \leq \varepsilon\), which means that \(G_{\Omega}^{(\varepsilon)}\) is well defined on \(B_b(\Omega) \cap L^1\) and it satisfies inequality (4.1) there. Using the Dominated Convergence Theorem, we prove this extension also satisfies the CMP on \(B_b(\Omega) \cap L^1\).

Now, we prove that for \(f \in B_b(\Omega) \cap L^1\), the function \(G_{\Omega}^{(\varepsilon)}(f) \in C_{\Omega}(\Omega)\). Take \((x_n)_{n \in \mathbb{C}} \subset \Omega\). We consider first the case where \(x_n \to x \in \Omega\). Fix a positive \(\varepsilon\) and assume that \(\|x - x_n\| \leq \varepsilon\), then

\[
|G_{\Omega}^{(\varepsilon)}(f)(x) - G_{\Omega}^{(\varepsilon)}(f)(x_n)| \leq 2 \|f\|_\infty \int_{B(0, 3\varepsilon)} g^\beta(0, y) \, dy + \int_{\Omega \setminus B(x, 2\varepsilon)} |f(y)||g_{\Omega}^{(\varepsilon)}(x_n, y) - g_{\Omega}^{(\varepsilon)}(x, y)| \, dy.
\]
The second integral converges to 0 as $n \to \infty$ by the Dominated Convergence Theorem. Then, by taking $\varepsilon \downarrow 0$ the first term converges to 0, and the continuity of $G^{(\rho)}_{\partial} (F)$ is shown in $\partial$.

Assume now $x \in \partial \Omega$. We need to show that $G^{(\rho)}_{\partial} (f)(x_n) \to 0$. This is done in a similar way, by decomposing the integral as above (here we use that $\partial$ is regular). Finally, we assume that $\|x_n\| \to \infty$. Consider $M > 0$, $\varepsilon > 0$ and decompose

$$|G^{(\rho)}_{\partial} (f)(x_n)| \leq \|f\| \int_{\partial \Omega \cap B(0,M)} g^{\rho}_{\partial} (x_n, y) \, dy + \|f\| \int_{B(x_n, \varepsilon) \cap \partial} g^{\rho}_{\partial} (x_n, y) \, dy \leq \|f\| \int_{B(0,M)} g^{\rho} (x_n, y) \, dy + \|f\| \int_{B(0,\varepsilon)} g^{\rho} (0, y) \, dy + (C(d)\varepsilon^{2-d})^\rho \int_{\partial \Omega \setminus B(0,M)} |f(y)| \, dy.$$  

For $M$ fixed, the first term converges to 0 as $n \to \infty$, by the Dominated Convergence Theorem. Hence,

$$\limsup_{n \to \infty} |G^{(\rho)}_{\partial} (f)(x_n)| \leq \|f\| \int_{B(0,\varepsilon)} g^{\rho} (0, y) \, dy + (C(d)\varepsilon^{2-d})^\rho \int_{\partial \Omega \setminus B(0,M)} |f(y)| \, dy.$$  

Now, we let $M \to \infty$ to get $\limsup_{n \to \infty} |G^{(\rho)}_{\partial} (f)(x_n)| \leq \|f\| \int_{B(0,\varepsilon)} g^{\rho} (0, y) \, dy$. Finally, the claim follows by making $\varepsilon \downarrow 0$.

Now, we prove existence and uniqueness of a Feller semigroup associated to $G^{(\rho)}_{\partial}$. For that, consider $\Psi \in C_0(\partial)^{+} \cap L^1$, such that $0 < \Psi(x)$ for all $x \in \partial$. Without loss of generality we can assume that $\|\Psi\|_{\infty} \leq 1$. With the aid of this function, we construct a new potential operator defined on $C_b(\partial)$ as

$$V_{\Psi}(F)(x) = G^{(\rho)}_{\partial} (\Psi F)(x).$$

We point out that $\Psi F \in C_0(\partial) \cap L^1$. Then, $V_{\Psi}$ is a well defined, positive bounded linear operator $V_{\Psi} : C_b(\partial) \to C_0(\partial) \subset C_b(\partial)$, with norm

$$\|V_{\Psi}\|_{C_b(\partial)} \leq \inf_{a > 0} \left\{ \int_{B(0,a)} g^{\rho} (0, y) \, dy + \left( \frac{C(d)}{a^{d-2}} \right)^\rho \|\Psi\|_{1} \right\}.$$  

We also have that $V_{\Psi}$ satisfies the CMP in $C_b(\partial)$, because

$$\int (V_{\Psi}(F)(x) - 1)^+ \Psi(x) F(x) \, dx = \int \left( G^{(\rho)}_{\partial} (\Psi F)(x) - 1 \right)^+ \Psi(x) F(x) \, dx \geq 0.$$  

Thus, if $V_{\Psi}(F)(x) \leq 1$ on the set $\{ z \in \partial : F(z) \leq 0 \}$, we conclude that $V_{\Psi}(F)(x) \leq 1$ on the set $\{ z \in \partial : \Psi(z) F(z) < 0 \}$, which is exactly the set where $F$ is negative. Thus, $V_{\Psi}$ satisfies the CMP.

Then, there exists a unique contraction resolvent $V_{\Psi} = (V_{\Psi}^0)^\lambda$, of positive continuous linear operators on $C_b(\partial)$, such that $V_{\Psi} = V_{\Psi}^0$. Again, we can prove that this resolvent is continuous on $\lambda$ (see Proposition 3.5). In particular

$$V_{\Psi} = \lim_{\lambda \downarrow 0} V_{\Psi}^\lambda.$$
On the other hand, following the proof of Lemma 3.6, we have the density of $\mathcal{M} \cap \mathcal{C}_0(\mathcal{O})$ in $\mathcal{C}_0(\mathcal{O})$. Then, there exists a Ray semigroup $\mathcal{M}^\Psi = (P_t^\Psi)_{t \geq 0}$ and the associated Ray process $X^\Psi$, such that for all $F \in \mathcal{C}_b(\mathcal{O})$ and all $\lambda \geq 0$

$$V_\lambda^\Psi(F)(x) = \int e^{-\lambda t} F_t^\Psi(F)(x) \, dt = \mathbb{E}_x \left( \int e^{-\lambda t} F(X_t^\Psi) \, dt \right).$$

The important case is $\lambda = 0$, which gives for all $F \in \mathcal{C}_b(\mathcal{O})$, $x \in \mathcal{O}$

$$G_0^{(\Psi)}(F)(x) = \int P_t^\Psi(F)(x) \, dt = \mathbb{E}_x \left( \int F(X_t^\Psi) \, dt \right).$$

Recall that $X^\Psi$ has càdlàg paths on $\hat{\mathcal{O}}$, the one point compactification of $\mathcal{O}$, and the semigroup can be assumed to be extended to $\mathbb{B}_b(\hat{\mathcal{O}})$.

Finally, $X^\Psi$ is a Feller process as soon as we prove that the set of branching points is empty. This is done exactly in the same way as we did it in Lemma 3.8. So we summarize this in the next proposition.

**Proposition 4.2** For every $\Psi \in \mathcal{C}_0^+(\mathcal{O}) \cap L^1$, which we assume is strictly positive, there exists a unique Feller process $X^\Psi$, with càdlàg paths, taking values in $\hat{\mathcal{O}}$ such that its 0-potential is $V_\Psi$, that is, for all $F \in \mathcal{C}_b(\mathcal{O})$ we have

$$G_0^{(\Psi)}(F)(x) = V_\Psi(F)(x) = \mathbb{E}_x \left( \int F(X_t^\Psi) \, dt \right).$$

Next, we study the dependence on $\Psi$ for the resolvent, semigroup and process. This is done throughout a time change, which we explain it now. Assume that $\Psi_1 \leq \Psi_2$, both functions satisfying the above requirements. We denote by $X^\Psi_i$ for $i = 1, 2$, the associated processes.

Consider the increasing process $A_t = \int_0^t \frac{\Psi_1}{\Psi_2} (X_s^\Psi_2) \, ds$, where we assume that $0/0 = 0$. Since $\Psi_1 \leq \Psi_2$, then $A$ is an increasing continuous process. If $\xi$ is the hitting time of $\sigma$ for $X^\Psi_2$, then $A$ is strictly increasing on $[0, \xi]$ and it is constant on $[\xi, \infty]$. We define $(\tau_t)_t$, the right continuous inverse of $A$, which is continuous and strictly increasing on $[0, A_t)$ and $\tau_t = \infty$ for $t \geq A_t$. Given that $A_t \leq t$, we have $\tau_t \geq t$.

Let $\tilde{X}$ be the process obtained by time change from $X^\Psi_2$, that is $\tilde{X}_t = X_{\tau_t}^\Psi$. $\tilde{X}$ is a Feller process, with càdlàg paths on $\hat{\mathcal{O}}$. The potential associated to $\tilde{X}$ is for $F \in \mathcal{C}_b(\mathcal{O})^+$ (we extend $F$ to $\hat{\mathcal{O}}$ by $F(\emptyset) = 0$

$$\tilde{V}(F)(x) = \mathbb{E}_x \left( \int F(\tilde{X}_t) \, dt \right) = \mathbb{E}_x \left( \int F(X_{\tau_t}^\Psi) \, dt \right) = \mathbb{E}_x \left( \int F(X_{\tau_t}^\Psi) \, dA_t \right) = \mathbb{E}_x \left( \int F(X_{\tau_t}^\Psi) \psi_2(\psi_2(X_{\tau_t}^\Psi)dt \right) = V_{\Psi_2}(F_{\Psi_1/\Psi_2}) = G_0^{(\Psi)}(\Psi_1 F) = V_{\Psi_1}(F).$$

Therefore, $\tilde{V} = V_{\Psi_1}$ and from the uniqueness of the resolvent associated to this potential, we obtain that $\tilde{V}_\lambda = V_{\Psi_1}^\lambda$, for all $\lambda$. From here, we get that the Laplace transform of the semigroups associated to $X^\Psi_1$ and $\tilde{X}$ coincide, which implies that both processes have the same distribution: $X^\Psi_1 \overset{\mathcal{L}}{=} \tilde{X}$.

We proceed now to construct the Feller process associated to $G_0^{(\Psi)}$. For that, take any function $\Psi$ as above. The time change we propose is $A_t = \int_0^t (\Psi(X^\Psi_s))^{-1} \, ds$. We point out that $A$ is an increasing process, continuous in the interval $[0, \xi)$, where $\xi$ is the hitting time of $\sigma$ for the process $X^\Psi$. We put $A_t = A_{t-}$ for $t \geq \xi$, whenever $\xi < \infty$. Again, we consider
(\gamma_t), the right continuous inverse of \(A\). Then, \((\gamma_t)\) is strictly increasing and continuous on \([0, A_{t-})\).

As before, we consider the Feller process \(X^* = X^*_{\Psi} = (X_t^\Psi)_{t},\) taking values on \(\hat{\Omega}\) and lifetime \(\xi^* = A_{t-}\). In principle this process \(X^*\) depends on \(\Psi\). As in the previous computation, we get for every \(F \in C_{\mathcal{K}}(\Omega)\)

\[
V^*(F)(x) = \mathbb{E}_x \left( \int F(X_t^*) \, dt \right) = \mathbb{E}_x \left( \int F(X_t^\Psi) \, dt \right) = \mathbb{E}_x \left( \int F(X_t^\Psi) \frac{1}{\Psi}(X_t^\Psi) \, dt \right) \tag{4.3}
\]

Existence in Theorem 1.1 is shown, with the Feller process \(X^*\).

We remark that the law of \(X^*\), constructed above does not depend on the choice of \(\Psi\). Indeed, if we have two functions \(\Psi_1, \Psi_2\), we consider \(\Psi_3 = \Psi_1 \vee \Psi_2\) and we proceed as before. Let us call \(Z = X^{\Psi_3}, W = X^{\Psi_1}, Y = X^{\Psi_2}\). As above, \(W\) is a time change of \(Z\). More precisely, \(\tilde{W} = (Z_{\eta})_{t}\) has the same law as \(W\), where \(\eta\) is the inverse of the increasing process \(dA_t = \frac{1}{\Psi_1}(\tilde{W}_t)dt\).

On the other hand, we denote by \(\tilde{W}^*\) obtained from \(\tilde{W}\) with a time change \((\tau_t)\), which is the inverse of \(dA_t = \frac{1}{\Psi_1}(\tilde{W}_t)dt\). Notice that \(\tilde{W}^* = F(\tilde{W})\) for some fixed measurable transformation \(F\), and \(X^* = F(X_t^\Psi)\). Therefore \(\tilde{W}^*\) and \(X^*\) have the same law. We can see \(\tilde{W}^*\) as a time change of \(Z\). This time change is just the composition of the two time changes, which is the inverse of \(C = (A_{B_t})_{t}\), and

\[
dC_t = (dA_t)_{B_t}, \quad dC_t = \frac{1}{\Psi_1}(\tilde{W}^*_{B_t})dt = \frac{1}{\Psi_1}(Z_{\eta_{B_t}})dt = \frac{1}{\Psi_3}(Z_t)dt,
\]

which is the time change from \(Z\) to \(Z^*\). This shows that the law of \(X^*\) and \(Z^*\) are the same. Analogously, the law of \(Y^*\) and \(Z^*\) coincide, proving the claim.

With the same ideas we can prove uniqueness in Theorem 1.1. Assume that \(Y\) is a Feller process, with càdlàg paths on \(\hat{\Omega}\), such that, for all \(F \in C_{\mathcal{K}}(\Omega)\) it holds

\[
G^{(\Psi)}(F)(x) = \mathbb{E}_x \left( \int F(Y_t) \, dt \right) .
\]

Then, using a time change with \((\tau_t)_t\), the inverse of \(dA_t = \Psi(Y_t) \, dt\), we see that \(W = (Y_{\tau_t})_{t}\) is a Feller process whose 0-potential is, for \(F \in C_{\mathcal{K}}(\Omega)\),

\[
U(F)(x) = G^{(\Psi)}(F)(x) = V_{\Psi}(F)(x),
\]

and therefore, \(V_{\Psi}\) is an extension of \(U\) to \(C_b(\Omega)\). This shows that \(W\) has the same law as \(X^{\Psi}\). Finally, \(Y\) has the same law as \(X^*\) and uniqueness is shown.

### 5 Dimension \(d = 2\), Proof of Theorem 1.2

In this section we shall prove Theorem 1.2. We first consider \(\Omega\) a bounded regular open set in \(\mathbb{R}^2\). The main ideas are already exposed in previous sections. We need some bounds for
the Green potential on bounded domains of $\mathbb{R}^2$. For that purpose we take first the case of a ball. Consider $\Omega = B(0, 1)$, then it is well known (see for example [9]) that
\[ g_\Omega(x, y) = -\frac{1}{\pi} \left[ \log(\|x - y\|) - \log(\|x\| \|y - x^*_1\|) \right], \]
where $x^*_1 = \frac{x}{\|x\|^2}$ is the point dual of $x$ with respecto to $\partial B(0, 1)$. By a scaling argument we obtain the Green kernel for $B(0, R)$ is
\[ g_R(x, y) = g_\Omega \left( \frac{x}{R}, \frac{y}{R} \right) = -\frac{1}{\pi} \left[ \log(\|x - y\|) - \log(\|x\| \|y - x^*_R\|) + \log(R) \right], \]
where now $x^*_R = R^2 \frac{x}{\|x\|^2}$.

Now, for the simple random walk in $\mathbb{Z}^2$, we consider
\[ a(x) = \sum_{n=0}^{\infty} p_n(0, 0) - p_n(0, x), \]
which gives, roughly, the difference between the expected number of visits to 0 minus the expected number of visits to $x$. Notice that $a(0) = 0$. The estimate we need on $a(x)$ is the following (see [11], Theorem 4.4.4), for all $x \in \mathbb{Z}^2$
\[ \left| a(x) - \left( c_x \log(\|x\| + 1) + \frac{2y + \log(8)}{\pi} \right) \right| \leq \Psi(\|x\|), \]
where $c_x = \frac{2}{\pi}$, $\gamma$ is the Euler constant and $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded decreasing function, such that $\Psi(r)r^2$ is also bounded.

Now, the Green kernel for the simple random walk of a finite set $E$ is
\[ g_E(x, y) = \mathbb{E}_x \left( \sum_{k=0}^{R^E - 1} 1_{\gamma(\mathcal{R}^k)} \right) = \mathbb{E}_x (a(\mathcal{R}^E - y)) - a(x - y). \]

Consider as before, $\mathbb{Z}^{2,n} = \sqrt{\frac{n}{2}} \mathbb{Z}^2$. Take $A_n = \mathbb{Z}^{2,n} \cap B(0, 1)$ and $E_n = \mathbb{Z}^{2,n} \cap B(0, \sqrt{\frac{n}{2}})$, so for $u, v \in A_n$ we have
\[ g_{E_n}(u, v) = \mathbb{E}_{u, v}(a(\mathcal{R}_{E_n} - u, v)) - a(\sqrt{\frac{n}{2}}(u - v)) \]
\[ = \sum \left( a(\sqrt{\frac{n}{2}}(u - v)) \right) \mathbb{P}_{u, v}(\mathcal{R}_{E_n} = \sqrt{\frac{n}{2}}(u - v)) \]
\[ = \sum c_2(\log(\|u - v\|) - \log(\|u - v\|)) \mathbb{P}_{u, v}(\mathcal{R}_{E_n} = \sqrt{\frac{n}{2}}(u - v)) + \mathcal{R}(u, v, n), \]
where $\sup_{n, u, v, z} \|\mathcal{R}(w, z, n)\| < 2\Psi(0)$ and for every $\varepsilon > 0$ we have
\[ D(\varepsilon) = \sup_{n \geq 1} \sup_{u, v, z, \in A_n} \|\mathcal{R}(w, z, n)\| : w, z \in \mathbb{Z}^2, \|w - z\| > \varepsilon, \|w\| \leq 1 - \varepsilon, \|z\| \leq 1 - \varepsilon < \infty \]

Now, for any $x, y \in B(0, 1)$, we take $u_n, v_n \in A_n$ any pair of sequences such that $u_n \to x, v_n \to y$. Then, if $x \neq y$, $\|x\| < 1$, $\|y\| < 1$, we have as $n \to \infty$.
\[ \frac{1}{\pi} g_{E_n}(u_n, v_n) - \frac{1}{\pi} f(\log(\frac{\|x\|}{\|x - y\|})) \mathbb{P}_{x}(W_{\mathcal{R}^E} = \mathcal{R}(u, v, n)) = g_\Omega(x, y), \]
where $W$ is a BM. For the last equality see for instance [9], Example 1.5.1.

In what follows we need some extra domination, to use the Dominated Convergence Theorem. For that purpose we consider $C_n = \mathbb{Z}^{2,n} \cap B(0, 2)$ and $F_n = \sqrt{\frac{n}{2}} C_n$. Take $n_0$
large enough, such that \( d(A_n, \partial C_n) \geq 1/2 \) for all \( n \geq n_0 \) \( (n_0 = 1 \text{ actually works}) \). If we take \( n \geq n_0 \) and \( u, v \in A_n \), using the previous computations we have

\[
\frac{1}{2} g_{E_n}(u \sqrt{n/2}, v \sqrt{n/2}) \leq \frac{1}{2} g_{F_n}(u \sqrt{n/2}, v \sqrt{n/2}) \leq \frac{1}{\pi} \log(\|u - v\|) + C, \tag{5.2}
\]

with \( C = \frac{1}{\pi} \log(3) + \Psi(0) \).

Now, we define the Green potential associated to scaled random walk. We assume that \( \beta \geq 1 \) and \( \alpha \in (0, 2\pi) \)

\[
\begin{align*}
\mathcal{G}_{A_n}^{n, (\beta)}(F)(x) &= \sum_{w \in A_n} \frac{2}{n} F(w + x - x(n)) \left[ \frac{1}{2} g_{E_n} \left( x(n) \sqrt{n/2}, w \sqrt{n/2} \right) \right]^\beta \tag{5.3} \\
\mathcal{G}_{A_n}^{n, (\exp, \alpha)}(F)(x) &= \sum_{w \in A_n} \frac{2}{n} F(w + x - x(n)) e^{\frac{\alpha}{n} \log g_{E_n}(x(n) \sqrt{n/2}, w \sqrt{n/2})} \tag{5.4} \\
\mathcal{G}_{A_n}^{n, (\exp-1, \alpha)}(F)(x) &= \sum_{w \in A_n} \frac{2}{n} F(w + x - x(n)) \left( e^{\frac{\alpha}{n} \log g_{E_n}(x(n) \sqrt{n/2}, w \sqrt{n/2})} - 1 \right) \tag{5.5}
\end{align*}
\]

Using the convergence (5.1) and domination (5.2) we prove the following lemma.

**Lemma 5.1** Consider \( \mathcal{O} = B(0, 1) \). For \( \beta \geq 1, \alpha \in (0, 2\pi), F \in C_b(\mathcal{O}) \) and \( x \in \mathcal{O} \). For \( (\mathcal{G}_n) \) any sequence of the tree discrete potentials \( (\mathcal{G}_{A_n}^{n, (\beta)})_n, (\mathcal{G}_{A_n}^{n, (\exp, \alpha)})_n \) and \( (\mathcal{G}_{A_n}^{n, (\exp-1, \alpha)})_n \), and \( \mathcal{G}_\mathcal{O} \) the corresponding continuous analogues \( \mathcal{G}_{\mathcal{O}}^{(\beta)}, \mathcal{G}_{\mathcal{O}}^{(\exp, \alpha)} \) and \( \mathcal{G}_{\mathcal{O}}^{(\exp-1, \alpha)} \), we have

1. \( \lim_{n \to \infty} \mathcal{G}_n(F)(x) = \mathcal{G}_\mathcal{O}(F)(x) \).
2. \( 0 \leq \lim_{n \to \infty} \int (\mathcal{G}_n(F)(x) - 1)^+ F(x) \, dx = \int_\mathcal{O} \left( \mathcal{G}_\mathcal{O}(F)(x) - 1 \right)^+ F(x) \, dx \).

**Remark** We can extend operators \( \mathcal{G}_{\mathcal{O}}^{(\beta)}, \mathcal{G}_{\mathcal{O}}^{(\exp, \alpha)} \) and \( \mathcal{G}_{\mathcal{O}}^{(\exp-1, \alpha)} \) to \( \mathbb{B}_b(\mathcal{O}) \), by using the density of \( C_b(\mathcal{O}) \) in \( L^1(\mathcal{O}, dx) \). These extensions are positive, bounded operators that map \( \mathbb{B}_b(\mathcal{O}) \) into \( C(\mathcal{O}) \) and they also satisfy the CMP in \( \mathbb{B}_b(\mathcal{O}) \).

The immediate consequence of this lemma is that

**Proposition 5.2** For \( \mathcal{O} = B(0, 1), \beta > 1, \alpha \in (0, 2\pi) \) the bounded positive linear operators \( \mathcal{G}_{\mathcal{O}}^{(\beta)}, \mathcal{G}_{\mathcal{O}}^{(\exp, \alpha)} \) and \( \mathcal{G}_{\mathcal{O}}^{(\exp-1, \alpha)} \) satisfy the CMP in \( C_b(\mathcal{O}) \).

The rest of the proof of Theorem 1.2, for \( \mathcal{O} = B(0, 1) \), is obtained by following the lines in Section 3, showing the analogues of Proposition 3.5, Lemma 3.6, Theorem 3.7 and finally Lemma 3.8. The key to prove Lemma 3.6 and Lemma 3.8 is the logarithmic singularity that \( g_{\mathcal{O}} \) has: for all \( x \in \mathcal{O} \) and all \( \varepsilon > 0 \), there is a \( 0 < \rho < 1/2 \wedge d(x, \partial \mathcal{O}) \) such that for all \( y \in B(x, \rho) \)

\[
(1 - \varepsilon) \log(\|x - y\|^{-1}) \leq g_{\mathcal{O}}(x, y) \leq (1 + \varepsilon) \log(\|x - y\|^{-1}). \tag{5.6}
\]

See for example [14] Theorem 4.4.2.

Finally, we sketch the proof of Theorem 1.2 for a general bounded regular open set \( \mathcal{O} \subset \mathbb{R}^2 \). For that purpose, we consider a large ball \( \tilde{\mathcal{O}} = B(0, R) \), such that \( \mathcal{O} \subset B(0, R/2) \). We take \( A_n = \mathbb{Z}^{2,n} \cap \tilde{\mathcal{O}}, E_n = \sqrt{n/2} A_n \) and \( C_n = \mathbb{Z}^{2,n} \cap \tilde{\mathcal{O}}, F_n = \sqrt{n/2} C_n \). The idea is to use \( g = g_{\tilde{\mathcal{O}}}, \mathcal{G} = \mathcal{G}_{F_n} \), as we did with \( g_{\mathbb{R}^2}, \mathcal{G}_{\mathbb{R}^2} \) in Section 3.

The main relations we need are: for \( u, v \in A_n, x, y \in \mathcal{O} \)

\[
\begin{align*}
\mathcal{G}_{E_n}(u, v) &= \mathcal{G}_{F_n}(u, v) - \mathcal{E}_x(g_{F_n}(\mathcal{S}_{E_n}, v))) \leq \mathcal{G}_{F_n}(u, v) \\
g_{\mathcal{O}}(x, y) = g_{\tilde{\mathcal{O}}}(x, y) - \mathcal{E}_x(g_{\tilde{\mathcal{O}}}(W_{T_{\mathcal{O}}}, y))) \leq g_{\mathcal{O}}(x, y).
\end{align*}
\]
These representations provide the convergence and domination we need to finish the proof when \( \mathcal{O} \) is a bounded regular open set.

### 5.1 Dimension \( d = 2 \), Unbounded Regular Domains

In this subsection we will provide the necessary details to prove Theorem 1.2 for the case in which \( \mathcal{O} \) is an unbounded open set in \( \mathbb{C} \), that satisfies (H2d). We start by developing some consequences of this hypothesis.

The main idea is to use the inversion \( z \to f(z) = z^{-1} \). Without loss of generality, we assume that \( \mathcal{B} = \overline{B}(0, 1) \) is compactly contained in \( \overline{\mathcal{O}} \) and the distance \( d(\mathcal{B}, \mathcal{O}) \geq 2 \). This can be achieved with a translation and an homothecy of \( \mathcal{O} \). We denote by \( U = f(\mathcal{O}) \subseteq \mathcal{B} \), which is a regular open set. We denote by \( g_U \) its Green kernel. Using a change of variables, we get for any nonnegative bounded and measurable function \( h \) (we assume \( h(y) = 0 \) for \( y \in \partial \mathcal{O} \)) that

\[
\int_{\mathcal{O}} h(y) g^\beta_{\mathcal{O}}(x, y) \, dy = \int_{U} h(u^{-1}) g^\beta_{\mathcal{O}}(x, u^{-1}) \|u\|^{-d} \, du.
\]

By the conformal invariance, we have \( g_{\mathcal{O}}(x, u^{-1}) = g_U(x^{-1}, u) \), which gives

\[
\int_{\mathcal{O}} h(y) g^\beta_{\mathcal{O}}(x, y) \, dy = \int_{U} h(u^{-1}) g^\beta_U(x^{-1}, u) \|u\|^{-d} \, du = \mathbb{E}_{x^{-1}} \left( \int_0^\infty h(X_t^{-1}) \|X_t\|^{-d} \, dt \right),
\]

where \( (X_t) \) is the unique (in law) Feller process associated to \( g^\beta_{U^{-1}} \).

Now, we consider the increasing and continuous process \( A_t = \int_0^t \|X_s\|^{-d} \, ds \), for \( t \leq \xi \), where \( \xi \) is the lifetime of \( X \). We also define \( A_t = A_\xi \), whenever \( t \geq \xi \). We take, \( (\tau_t) \), the right continuous inverse of \( (A_t) \), which is continuous and strictly increasing on the interval \( [0, A_\xi) \) and jumps to \( \infty \) afterwards.

Then, performing a time change we get

\[
\int_{\mathcal{O}} h(y) g^\beta_{\mathcal{O}}(x, y) \, dy = \mathbb{E}_{x^{-1}} \left( \int_0^\infty h(X_t^{-1}) \, dt \right) = \mathbb{E}_x \left( \int_0^\infty h(Z_t) \, dt \right),
\]

where \( Z = (Z_t) \) is the Feller process given by \( Z_t = X_t^{-1} \) for \( t < A_t \) and \( Z_t = \partial \), for \( t \geq A_t \) (\( A_t \) is the lifetime for \( Z \)). This gives \( g^\beta_{\mathcal{O}} \) as the 0-potential for \( Z \). In the same way it can be proved the uniqueness (in law) of \( Z \).

Similar arguments can be used for operators \( G^\text{(exp,a)}_{\mathcal{O}} \) and \( G^\text{(exp-1,a)}_{\mathcal{O}} \). We note that the time change has the same form for all these processes.

The rest of the section is devoted to control these integrals and to ensure they are finite, as stated in Theorem 1.2. To do that we can rely on the inversion as above. The main problem with that is to have an explicit control on \( g_U \). We prefer to do so, by a direct control using the Green kernel for the complement of a ball, which will provide the sufficient conditions on representations (1.2), (1.3) and (1.4).

The Green kernel of \( \mathcal{B}^c \) will be denoted by \( g \) and will serve as a uniform bound for \( g_{\mathcal{O}} \). The main properties that we will use of \( g \) are summarized in the next lemma. Notice that we are assuming implicitly that for all \( x \in \mathcal{O} \), we have \( \|x\| \geq 2 \).

**Lemma 5.3**

1. For all \( x, y \in \mathcal{B}^c \), we have

\[
g(x, y) = \frac{1}{\pi} \left( -\log(\|x - y\|) + \log \left( \left\| x \right\| \left\| y \right\| - \frac{y}{\|y\|} \right) \right).
\]

2. For all \( x, y \in \partial \mathcal{O} \), it holds \( g_{\mathcal{O}}(x, y) \leq g(x, y) \).
3. For all $0 < a < 1$, all $x, y \in \overline{U}$ and all $\beta \geq 1$, $\alpha \geq 0$:

If $\|x - y\| \leq a$

\[ g_{\beta}(x, y) \leq \frac{1}{\pi} \left( -\log(\|x - y\|) + 3 \log(\|y\|) \right). \]

Hence,

\[ g_{\beta}^{\exp}(x, y) \leq \frac{2^{\beta-1}}{\pi \beta} \left( |\log(\|x - y\|)\|^\beta + 3^\beta |\log(\|y\|)\|^\beta \right). \quad (5.7) \]

\[ g_{\exp}(x, y) = \exp(\alpha g_{\beta}(x, y)) - 1 \leq g_{\exp}(x, y) \leq \|y\|^{3\alpha/\pi} \frac{1}{\|x - y\|^\alpha/\pi}. \quad (5.8) \]

If $\|x - y\| > a$

\[ g_{\beta}(x, y) \leq \frac{1}{\pi} \left( 2 \log(\|y\|) + \log(2/a) \right), \]

and

\[ g_{\exp}(x, y) \leq \frac{1}{\pi \beta} \left( 2^\beta |\log(\|y\|)|^{\beta} + (\log(2/a))^{\beta} \right). \quad (5.9) \]

\[ g_{\exp}(x, y) \leq (\frac{2}{\alpha})^{\alpha/\pi} \|y\|^{2\alpha/\pi}. \quad (5.10) \]

**Proof** (1) This is a standard formula and it is obtained for example using the inversion. (2) is a consequence of the domination of domains property of Green kernels.

(3) This follows from (2). First assume that $\|x - y\| \leq a$, then

\[ \|x\| \|y\| - \frac{y}{\|y\|} = \|(x - y)\| |y| + (\|y\| - \|y\|^{-1})| \leq \|(x - y)\| \|y\| + \|y\|(|y| - \|y\|^{-1}) \]

\[ \leq \|y\|^2 + \|y\|a \leq 2\|y\|^2 \leq \|y\|^3, \]

where we have used that $\|y\| \geq 2$. This gives the first inequality. For the second inequality, assume that $\|x - y\| > a$. Following the same ideas, this time we need to dominate

\[ \log(\|(x - y)\| \|y\| + \|y\|(|y| - \|y\|^{-1})) \leq \log(\|(x - y)\| \|y\| + \|y\|^2). \]

To do this, we consider two cases. First assume that $\|x - y\| \leq \|y\|$, then

\[ \|x - y\| \|y\| + \|y\|^2 \leq 2\|y\|^2 \leq \frac{a}{\alpha} \|x - y\| \|y\|^2, \]

which gives the bound

\[ \log(\|(x - y)\| \|y\| + \|y\|(|y| - \|y\|^{-1})) \leq \log(\|x - y\|) + 2\log(\|y\|) + \log(2/a). \]

On the other hand, if $\|x - y\| > \|y\|$, we get

\[ \|x - y\| \|y\| + \|y\|^2 \leq 2\|x - y\| \|y\|. \]

Therefore, we obtain

\[ \log(\|(x - y)\| \|y\| + \|y\|(|y| - \|y\|^{-1})) \leq \log(\|x - y\|) + \log(\|y\|) + \log(2) \]

\[ \leq \log(\|x - y\|) + 2\log(\|y\|) + \log(2/a). \]

This finishes the proof. \( \square \)

Now, we can prove the following properties of $G^{(6)}_{\beta}$. 
Proposition 5.4 Assume that \( F \in \mathbb{B}_b \cap L^1(\mathcal{O}, |\log(\|y\|)|^\beta d y) \). Then, for all \( 0 < a \leq 1 \) and all \( x \in \mathcal{O} \), we have
\[
|G^{(\beta)}_\mathcal{O} F(x)| \leq C_1(\beta, a) \|F\|_\infty + C_2(\beta, a) \|F\|_1
\] (5.11)
where \( \|F\|_1 = \int_\mathcal{O} |F(y)| |\log(\|y\|)|^\beta d y \) and
\[
C_1(\beta, a) = 2 \left( \frac{2}{\pi} \right)^{\beta-1} a \int_0^a |\log(r)|^\beta r dr,
\]
\[
C_2(\beta, a) = \frac{2^{\beta-1}}{\pi^\beta} \left( 3^\beta + \frac{1}{\log(2)a) \log(2/a) \|y\| \right).
\]
In particular, we have \( G^{(\beta)}_\mathcal{O} F \in \mathbb{B}_b(\mathcal{O}) \). Moreover, it holds \( G^{(\beta)}_\mathcal{O} F \in C_0(\mathcal{O}) \).

Remark Notice that the assumption on \( \mathcal{O} \), that is, for all \( x \in \mathcal{O} \) we have \( \|x\| \geq 2 \) implies that \( L^1(\mathcal{O}, |\log(\|y\|)|^\beta d y) \subset L^1(\mathcal{O}, d y) \).

Proof Inequality (5.11) is a consequence of Eqs. 5.7 and 5.9, which gives us the boundedness of \( G^{(\beta)}_\mathcal{O} F \). To prove it is a continuous function, we use the Domination Convergence Theorem. For every fixed \( 0 < a \leq 1/2 \), as done previously, we consider \( x_n \to x \in \mathcal{O} \cup \partial \mathcal{O} \) and decompose (assuming that \( \|x_n - x\| \leq a/2 \), for example)
\[
G^{(\beta)}_\mathcal{O} F(x_n) = \int_{\|x-y\| \leq a} F(y) g^{\beta}_\mathcal{O}(x_n, y) d y + \int_{\|x-y\| > a} F(y) g^{\beta}_\mathcal{O}(x_n, y) d y.
\]
In the second term we can use the Dominated Convergence Theorem to pass to the limit and we obtain that
\[
\limsup_{n \to \infty} \left| G^{(\beta)}_\mathcal{O} F(x_n) - G^{(\beta)}_\mathcal{O} F(x) \right| \leq 2^{\beta-1} \frac{a^\beta}{\pi^\beta} \|F\|_\infty \left[ 2\pi \int_0^a |\log(r)|^\beta r dr + \pi a^2 3^\beta |\log(\|x\| + 1)|^\beta \right],
\]
which converges to zero as \( a \downarrow 0 \). The same procedure, can be used to show: if \( x_n \to x \in \partial \mathcal{O} \), then \( G^{(\beta)}_\mathcal{O} F(x_n) \to 0 \).

Finally, we consider the case \( x = \infty \). This time we decompose
\[
G^{(\beta)}_\mathcal{O} F(x_n) = \int_{\|x_n - y\| \leq a} F(y) g^{\beta}_\mathcal{O}(x_n, y) d y + \int_{\|x_n - y\| > a} F(y) g^{\beta}_\mathcal{O}(x_n, y) d y.
\] (5.12)
We propose the following upper bound, for the first term on the RHS
\[
\left| \int_{\|x_n - y\| \leq a} F(y) g^{\beta}_\mathcal{O}(x_n, y) d y \right| \leq 2^{\beta-1} \frac{a^\beta}{\pi^\beta} \left[ 2\pi \|F\|_\infty \int_0^a |\log(r)|^\beta r dr + 3^\beta \int_{\|x_n - y\| \leq a} |\log(\|y\|)|^\beta |F(y)| d y \right]
\leq \|F\|_\infty \frac{2^{\beta-1}}{\pi^\beta} \int_0^a |\log(r)|^\beta r dr + \frac{2^{\beta-1}3^\beta}{\pi^\beta} \int_{\|y\| \geq \|x_n\|} |\log(\|y\|)|^\beta |F(y)| d y.
\]
This upper bound converges to 0 when \( n \to \infty, a \to 0 \).

On the other hand, for the second term in Eq. 5.12, we use the Dominated Convergence Theorem. By hypothesis (H2d.ii), the integrand of this term converges pointwise to 0 and it is dominated by \( |F(y)| 2^{\beta-1} \left( 2^{\beta} |\log(\|y\|)|^\beta + (\log(2/a))\beta \right) \) (see Eq. 5.9). Hence, the result is shown.

Now, we prove that \( G^{(\beta)}_\mathcal{O} \) satisfies the CMP.
Proposition 5.5  Assume \( F \in \mathbb{B}_b(\Omega) \cap L^1(\Omega, |\log(\|y\|)|^\beta \, dy) \), then
\[
\int_\Omega \left( G_{\Omega}^{(\beta)} F(x) - 1 \right)^+ F(x) \, dx \geq 0,
\]
and \( G_{\Omega}^{(\beta)} \) satisfies the CMP in \( \mathbb{B}_b(\Omega) \cap L^1(\Omega, |\log(\|y\|)|^\beta \, dy) \).

Proof Consider the sequence of bounded regular open sets \( \Omega_n = \emptyset \cap B(0, n) \). Using the Dominated Convergence Theorem, we get the pointwise convergence for all \( x \in \emptyset \)
\[
\lim_{n \to \infty} G_{\Omega_n}^{(\beta)} F(x) = G_{\emptyset}^{(\beta)} F(x).
\]
To use, again, the Dominated Convergence Theorem, we use Proposition 5.4 (with \( a = 1 \))
\[
| \left( G_{\Omega_n}^{(\beta)} F(x) - 1 \right)^+ F(x) | \leq (G_{\Omega_n}^{(\beta)} |F|(x) + 1) |F(x)| \leq (G_{\emptyset}^{(\beta)} |F|(x) + 1) |F(x)| \leq C(\|F\|_\infty + \|F\|_1 + 1) |F(x)| \in L^1(\emptyset, dx),
\]
for some finite constant \( C = C(\beta) \) and \( \|F\|_1 = \int_\emptyset |F(y)| |\log(\|y\|)|^\beta \, dy \). The result follows from Lemma 5.1 (see the Remark after that Lemma) applied to the bounded regular open sets \( (\Omega_n)_n \).

In the case of operators \( G_{\emptyset}^{(\exp, \alpha)} \), \( G_{\emptyset}^{(\exp^{-1}, \alpha)} \), for \( \alpha \in (0, 2\pi) \), we have a similar result.

Proposition 5.6  Consider \( 1 \leq p < 2\pi/\alpha \) and \( q < \infty \) the conjugated index. Given \( F \in \mathbb{B}_b(\emptyset) \cap L^1(\emptyset, \|y\|^{2\alpha/p} \, dy) \cap L^q(\emptyset, \|y\|^{3\alpha/q} \, dy) \), we have for \( \mathcal{H} = G_{\emptyset}^{(\exp, \alpha)} \) and \( \mathcal{H} = G_{\emptyset}^{(\exp^{-1}, \alpha)} \), the following properties hold:

1. For all \( x \in \emptyset \)
\[
|\mathcal{H} F(x)| \leq C_3(\beta, \alpha) \|F\|_q + C_4(\beta, \alpha) \|F\|_1,
\]
where \( \|F\|_1 = \int_\emptyset |F(y)| \|y\|^{2\alpha/p} \, dy \), \( \|F\|_q = \int_\emptyset |F(y)|^q \|y\|^{3\alpha/q} \, dy \), and
\[
C_3(\beta, \alpha) = (2\pi \int_0^\alpha r^{-\alpha/p} \, r \, dr)^{1/p},
\]
\[
C_4(\beta, \alpha) = \left( \frac{3\alpha}{p} \right)^{\alpha/p}.
\]

2. \( \mathcal{H} F \in C(\emptyset) \). More precisely, if \( \mathcal{H} = G_{\emptyset}^{(\exp^{-1}, \alpha)} \), and we take any sequence \( (x_n)_n \subset \emptyset \cup \emptyset \) such that \( x_n \to x \in \emptyset \cup \emptyset \cup \{\infty\} \), then \( G_{\emptyset}^{(\exp^{-1}, \alpha)} F(x_n) \to 0 \). Thus \( G_{\emptyset}^{(\exp^{-1}, \alpha)} F \in C_0(\emptyset) \).

For the operator \( G_{\emptyset}^{(\exp, \alpha)} \) we have \( G_{\emptyset}^{(\exp, \alpha)} F(x_n) \to \int_\emptyset F(y) \, dy \).

3. Finally, we obtain
\[
\int_\emptyset (\mathcal{H} F(x) - 1)^+ F(x) \, dx \geq 0,
\]
and \( \mathcal{H} \) satisfies the CMP on \( \mathbb{B}_b(\emptyset) \cap L^1(\emptyset, \|y\|^{2\alpha/p} \, dy) \cap L^q(\emptyset, \|y\|^{3\alpha/q} \, dy) \).

This proves Theorem 1.2 using the inversion. Nevertheless, we also provide here the details of a direct proof in the spirit of Section 4, as we did in higher dimensions. Take for example \( G_{\emptyset}^{(\beta)} \), and consider a function \( \Psi \) such that \( 0 < \Psi \leq 1 \) in \( \emptyset \) and \( \Psi \in C_0(\emptyset) \cap L^1(\emptyset, |\log(\|y\|)|^\beta \, dy) \). The positive linear operator \( V_\Psi : \mathbb{B}_b(\emptyset) \to C_0(\emptyset) \subset \mathbb{B}_b(\emptyset) \) defined as
\[
V_\Psi(F)(x) = G_{\emptyset}^{(\beta)} (\Psi F)(x),
\]
is continuous with a norm

\[ \| V_\Psi \|_{B_b(\Omega)} \leq \inf_{0 < a \leq 1} \left\{ C_1(\beta, a) + C_2(\beta, a) \int_\Omega \Psi(y)\|y\|^{2a/\pi} \, dy \right\}, \]

where \( C_1, C_2 \) are given in Proposition 5.4.

Moreover, we claim that the operator \( V_\Psi \) satisfies the CMP in \( B_b(\Omega) \). Indeed, if we consider \( F \in B_b(\Omega) \) then

\[ 0 \leq \int_\Omega \left( G_\Omega^{(\beta)}(\Psi F)(x) - 1 \right)^+ \Psi(x) F(x) \, dx, \]

and therefore, if \( V_\Psi(F)(x) \leq 1 \) on the set \( \{ \xi : F(\xi) > 0 \} = \{ \xi : \Psi(\xi) F(\xi) > 0 \} \), then \( V_\Psi(F)(x) \leq 1 \) for a.a. \( x \). The continuity of \( V_\Psi(F) \) shows the desired claim.

Hence, \( V_\Psi \) is the 0-potential of a unique contraction resolvent \( V_\Psi = (V_\Psi^0) \), in \( B_b(\Omega) \). As before, we denote by \( \mathcal{M}^+ \) the set of supermedian functions for this resolvent, and \( \mathcal{M} \) the linear space generated by it. The proof of the analogue to Lemma 3.6 for \( V_\Psi \), that is, \( \mathcal{M} = \mathcal{M} \cap C_0(\Omega) \) is dense in \( C_0(\Omega) \), is done in a similar fashion by using that \( g_\theta^\beta(x, \bullet) \) has a logarithmic singularity (see Eq. 5.6), which implies that: For every \( \varepsilon > 0 \) there exists \( 0 < \rho \leq 1 \) and \( \rho < \frac{1}{2} d(x, \partial \Omega) \) such that for all \( y \in \Omega, 0 < \|x - y\| \leq \rho \)

\[ |\log(\|x - y\|)|^\beta (1 - \varepsilon) \leq g_\theta^\beta(x, y) \leq |\log(\|x - y\|)|^\beta (1 + \varepsilon), \]

Again, this logarithmic singularity of \( g_\theta^\beta \), allow us also to show the analogue of Lemma 3.8, that is, the set of branching points is empty.

In conclusion, for every function \( \Psi \) as above, there exists a unique Feller semigroup \( X_\Psi \), with càdlàg paths, taking values in \( \partial \Omega \cup \{\partial\} \), such that its 0-potential is \( V_\Psi \), that is, for all \( F \in C_b(\Omega) \) we have

\[ C_\Omega^{(\beta)}(\Psi F)(x) = V_\Psi(F)(x) = \mathbb{E}_x \left( \int F(X_t^\Psi) \, dt \right). \]

The rest of the proof is identical to the case \( d \geq 3 \) using time change.

### 6 Existence of a Density for the Semigroup

In this section we show that the semigroup \( \mathcal{P} = (P_t)_t \), whose 0-potential is \( G_\Omega^{(\beta)} \), has a density \( p(t, x, y) \) with respect to Lebesgue measure, that is, for all \( f \in C_b(\Omega) \) it holds

\[ P_t(f)(x) = \int_\Omega f(y) p(t, x, y) \, dy. \]

We restrict ourselves to the case \( \Omega \) is a bounded regular open set of \( \mathbb{R}^d \) and \( d \geq 2 \). The arguments given below, with some minor modifications, will work as well for unbounded regular open set \( \Omega \subset \mathbb{R}^d \). In dimension \( d = 2 \), under the hypothesis of Theorem 1.2 (or Proposition 1.3), we also have the existence of densities for the semigroups associated to \( G_\Omega^{(\exp, \alpha)} \) and \( G_\Omega^{(\exp^{-1}, \alpha)} \).

In what follows, we shall use some results for symmetric Markov process given in [9]. We denote by \( \mathcal{P} = (U^\lambda)_\lambda \) the associated resolvent. We know that \( \mathcal{P} \) is a Feller semigroup and \( \mathcal{P} \) is a continuous resolution in \( \lambda \). In particular, for every \( f \in C_b(\Omega) \) the function \( (t, x) \rightarrow P_t(f)(x) \) is continuous. Recall that the infinitesimal generator of \( \mathcal{P} \) is the (unbounded) operator \( \mathcal{L} \), whose domain \( \mathcal{D} \subset C_b(\Omega) \) is defined as the collection of functions \( f \in C_b(\Omega) \) for which the following limit exists uniformly

\[ \lim_{t \downarrow 0} \frac{P_t(f) - f}{t} = \mathcal{L}(f). \]

**Proposition 6.1** There exists a function \( u : \mathbb{R}^+ \times \Omega \times \Omega \rightarrow (0, \infty] \), such that

1. \( u(\lambda, \bullet, \bullet) \) is a symmetric function, bounded above by \( g_\Omega^\beta \).
2. $u$ is continuous on $[0, \infty) \times (0 \times 0 \setminus \{(x, x) : x \in 0\})$, and $u(0, x, y) = g_0^\beta(x, y)$ for all $x, y$.

3. $u(\lambda, x, \bullet)$ is a density for $U^\lambda$, that is, for all $f \in C_b(0)$ it holds

$$U^\lambda (f)(x) = \int_0^\infty f(y) u(\lambda, x, y) \, dy.$$ 

4. The semigroup $\mathcal{P}$ has an extension to $L^2(0, dx)$, which is symmetric and continuous.

5. For every $t > 0, x \in 0$ the measure $P_t(x, dx)$ has a density $p(t, x, y)$ with respect to Lebesgue measure. The semigroup $\mathcal{P}$ has a continuous extension to $L^p(0, dx)$ for all $1 \leq p \leq \infty$. Every $P_t$ is a contraction in $L^p(0, dx)$.

6. For all $t, x \neq y$, the function $P_t(g_0^\beta(\bullet, x))(y)$ is well defined and finite. This function is decreasing and right continuous in $t$ and it is measurable in the three variables. The function $v(t, x, y) = g_0^\beta(y, x) - P_t(g_0^\beta(\bullet, x))(y)$ is increasing and right continuous on $t$. For all $\lambda \geq 0$ and $x, y$, we have

$$u(\lambda, x, y) = \int_0^\infty e^{-\lambda t} v(dt, x, y).$$

(6.1)

In particular,

$$g_0^\beta(x, y) = \int_0^\infty v(dt, x, y).$$

Also, for all $f \in C_b(0)$ it holds

$$U^\lambda (f)(x) = \int_0^\infty \int_0^\infty e^{-\lambda t} f(y) v(dt, x, y) \, dy.$$ 

The measure $v(dt, x, y)$ is absolutely continuous in $t$, for $x$ and $dy$-a.e. Its density (with respect to $t$) is

$$\frac{\partial}{\partial t} v(t, x, y) = p(t, x, y).$$

In particular, for all $x$ and $dy$-a.e. it holds

$$g_0^\beta(x, y) = \int_0^\infty p(t, x, y) dt.$$ 

7. For all $f \in C_b(0), x \in 0$ the function $t \to \int_0^\infty P_t(g_0^\beta(\bullet, x))(y)f(y) \, dy$ is $C^1([0, \infty))$ and

$$-\frac{\partial}{\partial t} P_t(G_0^\beta(f))(x) = -\frac{\partial}{\partial t} \int_0^\infty P_t(g_0^\beta(\bullet, x))(y)f(y) \, dy = P_t(f)(x).$$

As a special case we obtain

$$\lim_{h \downarrow 0} \frac{P_h(G_0^\beta(f)) - G_0^\beta(f)}{h} = -f,$$

holds uniformly. This means that $G_0^\beta(f)$ is in the domain of $\mathcal{L}$, the infinitesimal generator of $(P_t)_t$, and

$$\mathcal{L}(G_0^\beta(f)) = -f.$$ 

Proof (1)–(3). From the resolvent equation, we have for all $f \in C_b^+(0)$ and all $x$

$$0 \leq U^\lambda (f)(x) \leq G_0^\beta(f)(x).$$

By the Riesz representation theorem, there exists a measure $\rho(dy)$, which depends on $\lambda, x$ such that $U^\lambda (f)(x) = \int_0 f(y) \rho(dy) \leq \int_0 f(y) g_0^\beta(x, y) \, dy$. The conclusion is that $\rho$ is absolutely
continuous with respect to the Lebesgue measure, with a density \( u(y) = u(\lambda, x, y) \), which is bounded by \( g_\lambda^\rho (x, y) \). Again the resolvent equation shows that, for \( x \neq y \)

\[
g_\lambda^\rho (x, y) - u(\lambda, x, y) = \lambda \int_0^\rho u(\lambda, x, z) g_\lambda^\rho (z, y) \, dz = \lambda U^\lambda (g_\lambda^\rho (y, \bullet))(x),
\]

which also proves that \( u(\lambda, x, \bullet) \) has a continuous version on \( \{0\} \setminus \{x\} \). Here we notice that

\[
U^\lambda (g_\lambda^\rho (y, \bullet))(x) \leq G_\rho (g_\lambda^\rho (y, \bullet))(x) = \int_0^\rho g_\lambda^\rho (y, z) g_\lambda^\rho (z, x) \, dz \leq \int_0^\rho g^\rho (y, z) g^\rho (z, x) \, dz,
\]

which is finite for \( y \neq x \). This representation also proves the continuity of \( u \) on

\[
[0, \infty) \times (0 \times 0 \setminus \{(x, x) : x \in 0\}).
\]

The symmetry of \( u(\lambda, \bullet, \bullet) \), follows from the symmetry of the operator \( U^\lambda \) with respect to Lebesgue measure: for all \( f, g \in C_b(0) \) it holds

\[
\int_0^\rho U^\lambda (f)(x)g(x) \, dx = \int_0^\rho U^\lambda (g)(x)f(x) \, dx.
\]

This can be shown by using that

\[
U^\lambda = (I + \lambda G_\rho )^{-1} G_\rho .
\]

Indeed, for small \( \lambda < \|G_\rho \|C_b(0) \), we can use the expansion

\[
U^\lambda = \sum_{n=0}^\infty (-1)^n \lambda^n (G_\rho )^{n+1},
\]

which proves that \( U^\lambda \) satisfies the desired symmetry for small \( \lambda \), given that \( G_\rho \) has a symmetric kernel \( g_\rho \). The resolvent equation allows us to extend this property to all \( \lambda \).

(4) We first show that for all \( f, g \in C_b(0) \) and all \( t \geq 0 \), we have the desired symmetry

\[
\int_0^\rho U^\lambda (f)(x)g(x) \, dx = \int_0^\rho e^{-\lambda t} \int_0^\rho P_t (f)(x)g(x) \, dx = \int_0^\rho U^\lambda (g)(x)f(x) \, dx.
\]

The uniqueness of Laplace transform implies that

\[
\int_0^\rho P_t (f)(x)g(x) \, dx = \int_0^\rho P_t (g)(x)f(x) \, dx
\]

holds for almost all \( t \). Since the functions \( t \rightarrow \int_0^\rho P_t (f)(x)g(x) \, dx \), \( t \rightarrow \int_0^\rho P_t (g)(x)f(x) \, dx \) are continuous, we conclude they are equal.

Now, we show that \( P \) has an extension to \( L^2 \). Using the Cauchy-Schwarz inequality, we show that for all \( f \in C_b(0) \)

\[
\int_0^\rho (P_t (f)(x))^2 \, dx \leq \int_0^\rho (P_t (1)(x))^2 \, dx \leq \int_0^\rho P_t (f^2)(x) \, dx
\]

This shows that \( P \) has a continuous extension to \( L^2 (0, dx) \).

(5) The existence of densities is a straightforward consequence of the symmetry of \( P \) in \( L^2 (0, dx) \), and Theorem 4.1.2 and Theorem 4.2.4 in [9].

Now, consider \( p = 1 \) and \( f \in C_b(0) \), we have again

\[
\int_0^\rho |P_t (f)(x)| \, dx \leq \int_0^\rho P_t (|f|)(x) \, dx = \int_0^\rho P_t (1)(x)|f(x)| \, dx \leq \|f\|_1.
\]

This shows that \( P_t (f) \in L^1 (0, dx) \) and \( \|P_t \|_1 \leq 1 \). The case \( p = \infty \) is obvious since for all \( f \in L^\infty (0, dx) \) we have

\[
|P_t (f)(x)| \leq \int_0^\rho p(t, x, y)|f(y)| \, dy \leq \|f\|_L^\infty P_t (1)(x) \leq \|f\|_L^\infty.
\]
Moreover, in this case we have
\[ \|P_t(f)\|_\infty \leq \|f\|_{L^\infty}.\]
For a general \(1 < p < \infty\) and \(f \in C_b(\mathbb{C})\), we have from H"older's inequality with \(q\) the conjugated of \(p\)
\[
\int_O |P_t(f)(x)|^p \, dx \leq \int_O (P_t(|f|^p)(x))(P_t(\mathbb{I})(x))^{p/q} \, dx \\
\leq \int_O (P_t(|f|^p))(x) \, dx = \int_O (P_t(\mathbb{I})(x))|f|^p(x) \, dx \leq \int_O |f|^p(x) \, dx.
\]
This proves the claim.

\(6\) Consider \(f_k \in C_0^+(\mathbb{C})\) whose support is contained in \(B(y, 1/k)\), and \(\int f_k(x)dx = 1\), that is, \(f_k\) is an approximation of \(\delta_y\). The function \(G_\rho(f_k)(x)\) is a nonnegative supermedian continuous function and also it is \(G_\rho(f_k)(x) \wedge N\). Then, it is known that \((t, z) \to P_t(G_\rho(f_k) \wedge N)(z)\) is a jointly continuous function of \((t, z)\) and it is decreasing in \(t\). We notice that \(G_\rho(f_k) \wedge N\) converges in \(k\) to the continuous and bounded function \(g_\rho(y, \bullet) \wedge N\) and therefore \((t, z) \to P_t(g_\rho(y, \bullet) \wedge N)(z)\) is also jointly continuous and decreasing in \(t\) (continuity follows from the Feller property).

The monotone convergence Theorem allows us to conclude that
\[(t, z) \to P_t(g_\rho(y, \bullet))(z)\]
is jointly measurable, decreasing in \(t\) and bounded by \(g_\rho(y, z)\), that is, for all \(t, x, z\)
\[P_t(g_\rho(y, \bullet))(z) \leq g_\rho(y, z).
\]
The function \(t \to P_t(g_\rho(y, \bullet))(z)\) is decreasing and lower semicontinuous, which implies that it is right continuous.

The same reasoning shows that
\[
U^\lambda(g_\rho(y, \bullet))(x) = \int_0^\infty e^{-\lambda t} P_t(g_\rho(y, \bullet))(x) dt.
\]
Thus, from Eq. \(6.2\)
\[g_\rho(x, y) - u(\lambda, x, y) &= \lambda \int e^{-\lambda t} P_t(g_\rho(y, \bullet))(x) dt,
\]
or equivalently
\[u(\lambda, x, y) = \lambda \int e^{-\lambda t} (g_\rho(x, y) - P_t(g_\rho(y, \bullet))(x)) dt.
\]
The symmetry between \(x\) and \(y\) also shows that
\[u(\lambda, x, y) = \lambda \int e^{-\lambda t} (g_\rho(x, y) - P_t(g_\rho(x, \bullet))(y)) dt. \quad (6.3)
\]
Integrating by parts the RHS we obtain
\[u(\lambda, x, y) = \int e^{-\lambda t} v(dt, x, y),\]
which proves \((6.1)\). Notice that \(v(dt, x, y)\) is also symmetric in \(x, y\).

We can take the limit as \(\lambda \downarrow 0\) in Eq. \(6.2\), to show that for \(x \neq y\)
\[g_\rho(x, y) = \lim_{\lambda \downarrow 0} u(\lambda, x, y) = \int v(dt, x, y) = v(\mathbb{R}^+, x, y)\]
which gives the desired result. That is, for \(x \neq y\) the measure \(v(dt, x, y)\) is a finite measure. Also we conclude that for \(x \neq y\) the limit
\[\lim_{t \to \infty} P_t(g_\rho(y, \bullet))(x) = 0.
\]
On the other hand, for any \( f \in C_b(\Omega)^+ \) we have

\[
U^\lambda(f)(x) = \int_0^\infty e^{-\lambda t} P_t(f)(x) \, dt = \int_0^\infty \int_\Omega e^{-\lambda t} p(t, x, y) \, dt \, f(y) \, dy.
\]

This implies that for all \( x, \lambda \geq 0 \) and for \( dy \)-a.e. it holds

\[
u(\lambda, x, y) = \int_0^\infty e^{-\lambda t} p(t, x, y) \, dt.
\]

Integrating by parts the RHS we get

\[
u(\lambda, x, y) = \lambda \int_0^\infty e^{-\lambda t} \int_0^t p(s, x, y) \, ds \, dt.
\]

Comparing with Eq. 6.3, we get that \( dt \)-a.s. it holds

\[
g^\phi_\lambda(x, y) - P_t(g(x, \bullet))(y) = \int_0^t p(s, x, y) \, ds.
\]

Since the left side is monotone in \( t \) and the right side is continuous in \( t \), we conclude they are equal for all \( t \). So, in this situation the measure \( \nu(dt, x, y) \) is absolutely continuous and its derivative is \( p(t, x, y) \) (of course \( dt \)-a.s.). In particular, for all \( x, \lambda \) and \( dy \)-a.e., we have

\[
g^\phi_\lambda(x, y) = \int_0^\infty p(t, x, y) \, dt,
\]

\[
u(\lambda, x, y) = \int_0^\infty e^{-\lambda t} p(t, x, y) \, dt.
\]

(7) Notice that

\[
\int P_t(g^\phi_\lambda(\bullet, x))(y) f(y) \, dy = \int P_t(g^\phi_\lambda(\bullet, y))(x) f(y) \, dy = P_t(G^\phi_\lambda(f))(x)
\]

\[
= \int_0^\infty P_s(f)(x) \, ds.
\]

The function \( t \to P_t(f)(x) \) is continuous, so \( P_t(G^\phi_\lambda(f))(x) \) is \( C^1([0, \infty)) \).

To get the uniform convergence we have

\[
P_h(G(f))(y) - P_0(G(f))(y) = -\int_0^h P_s(f)(y) \, dy.
\]

Since \( P_t(f) \) converges uniformly to \( f \), as \( s \downarrow 0 \), we get the result.

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Appendix A: Some Matrix Results

In this appendix, we discuss the results for potential matrices associated to finite state Markov chains (or continuous time Markov chains) used in this article.

Definition A.1  A nonnegative matrix \( U \), indexed by finite set \( E \), is called a potential if it satisfies the Complete Maximum Principle (CMP): for all \( v \in \mathbb{R}^E \) if it is verified that \((Uv)_j \leq 1\) on the coordinates \( j \) where \( v_j \geq 0 \), then it holds that \((Uv)_j \leq 1\) at all coordinates \( j \).
A nonnegative nonsingular matrix is a potential if and only if its inverse is a row diagonally dominant \( M \)-matrix, that is,

1. for all \( i \neq j \in E \), we have \((U^{-1})_{ij} \leq 0\) and
2. for all \( i \in E \) the row sum \( \sum_j (U^{-1})_{ij} \geq 0 \).

This is part of the fundamental Choquet-Deny paper see [1] (see also Theorem 2.9 in [5]).

If \( U \) is a nonsingular potential then \( Q = -U^{-1} \) is the generator of a continuous time transient Markov chain \( X = (X_t : t \geq 0) \) with state space \( E \), such that

\[
U_{ij} = \mathbb{E}_i \left( \int_0^\infty 1_j(X_t) \, dt \right),
\]

that is, \( U \) is the Green potential of \( X \). This property characterizes nonsingular potential matrices (see for example Theorem 2.27 in [5]).

On the other hand, it is straightforward to show that \( U^{-1} = k(\mathbb{I} - P) \) for some constant \( k \) and a sub-stochastic kernel \( P \), and the spectral radius of \( P \) is strictly bounded by one. This last property is equivalent to the convergence of the series \( \sum_{n \in \mathbb{N}} P^n \) and moreover \( U = \frac{1}{k} \sum_{n \in \mathbb{N}} P^n \). Notice that since \( U = \frac{1}{k} \sum_{n \in \mathbb{N}} P^n \), we get that \( P \) is irreducible if and only if \( U > 0 \). On the other hand, if we can choose \( k = 1 \), then \( U \) is the potential of a Markov chain, whose transition kernel is exactly \( P \).

The following result is a characterization of potential matrices, which is crucial for our work (see [6]). Here we denote by \( \langle \cdot, \cdot \rangle \) the Euclidean inner product in \( \mathbb{R}^E \).

**Proposition A.1** Assume that \( U \) is an entrywise nonnegative matrix indexed by a finite set \( E \). If

\[
\forall v \in \mathbb{R}^E : \langle (Uv - 1)^+, v \rangle = \sum_{j \in E} \left( \sum_{k \in E} U_{jk} v_k - 1 \right)^+ v_j \geq 0, \tag{A.1}
\]

then \( U \) is a potential.

Conversely, if \( U, U^t \) are potential matrices (for example if \( U \) is a symmetric potential) then (A.1) holds.

**Proof** We prove that (A.1) is sufficient for \( U \) to be a potential. Take a vector \( v \in \mathbb{R}^E \) that satisfies \( \forall j, v_j \geq 0 \Rightarrow (Uv)_j \leq 1 \). Then, we have

\[
0 \leq \langle (Uv - 1)^+, v \rangle = \langle (Uv - 1)^+, -v^- \rangle \leq 0.
\]

This implies that if \( v_j < 0 \) then \( ((Uv)_j - 1)^+ = 0 \) and therefore \( (Uv)_j \leq 1 \) proving that \( U \) satisfies the CMP.

Conversely, assume that \( U, U^t \) are potential matrices. Consider \( a > 0 \), then the matrix \( U(a) = aI + U \) is nonsingular and satisfies the CMP. So, \( M(a) = (U(a))^{-1} = k(a)(\mathbb{I} - P(a)) \), for some constant \( k(a) \) and a double substochastic matrix \( P(a) \) (here we have used that \( U, U^t \) are potentials). We define \( \mu(a) = M(a) \mathbb{I} \geq 0 \) and \( \xi(a) = U(a)(v - \mu(a)) = U(a)v - 1 \) to get

\[
\langle (U(a)v - 1)^+, v \rangle = \langle (U(a)v - U(a)\mu)^+, v \rangle = \langle \xi^+(a), M(a)\xi(a) + \mu(a) \rangle \\
= \langle \xi^+(a), k(a)\xi(a) + \mu(a) \rangle - k(a)\langle \xi^+(a), P(a)\xi(a) \rangle \\
= k(a)\left( \langle \xi^+(a), \xi^+(a) - \langle \xi^+(a), P(a)\xi(a) \rangle \rangle + \langle \xi^+(a), \mu(a) \rangle \right).
\]
Since $P(a) \geq 0$, we get
\[
\langle \xi^+(a), P(a)\xi(a) \rangle \leq \langle \xi^+(a), P(a)\xi^+(a) \rangle = \langle \xi^+(a), \frac{1}{2}(P(a) + P^t(a))(\xi)^+(a) \rangle \\
\leq \langle \xi^+(a), \xi^+(a) \rangle.
\]
The last inequality holds because the nonnegative symmetric matrix $\frac{1}{2}(P(a) + P^t(a))$ is sub-stochastic and therefore its spectral radius is smaller than 1, which implies that for all $z \in \mathbb{R}^d$ it holds $\langle z, \frac{1}{2}(P(a) + P^t(a))z \rangle \leq \langle z, z \rangle$. We get
\[
\langle (U(a)v - 1)^+, v \rangle \geq \langle (U(a)v - 1)^+, \mu(a) \rangle \geq 0.
\]
The result follows by taking $a \downarrow 0$.

The next result is the analogous of Theorems 1.1 and 1.2 for matrices and it can be found in Theorem 6.5 in [5] (see also [3]), Theorem 2.6 and Corollary 3.4 in [4]. Recall that given a matrix $A$ and a real function $F$, the $F$-Hadamard function of $A$ is defined entrywise as $(F(A))_{ij} = F(A_{ij})$. When $F(x) = x^\beta$ is a power function, we denote $A^{(\beta)} = F(A)$.

**Proposition A.2** Assume that $U$ is a (nonsingular) potential matrix. Then,
1. if $\beta \geq 1$, the Hadamard power $U^{(\beta)}$ is also a (nonsingular) potential matrix.
2. For all $\alpha > 0$ the Hadamard exponential $\exp(\alpha U)$ and $\exp(\alpha U) - 1$ are also (nonsingular) potential matrices.

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