Non-Reidemeister Knot Theory and Its Applications in Dynamical Systems, Geometry, and Topology

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Abstract

Classical knot theory deals with diagrams and invariants. By means of horizontal trisecants, we construct a new theory of classical braids with invariants valued in pictures. These pictures are closely related to diagrams of the initial object.

The main tool is the notion of free k-braid group. In the simplest case, for free 2-braids, the word problem and the conjugacy problem can be solved by finding the minimal representative, which can be thought of as a graph, and is unique, as such.

We prove a general theorem about invariants of dynamical systems which are valued in such groups and hence, in pictures.

We describe various applications of the above theory: invariants of weavings (collections of skew lines in $\mathbb{R}^3$), and many other objects in geometry and topology.

Keywords: Dynamical System, Braid, Knot, Graph, Trisecant, Group, Picture, Invariant, Reidemeister Move, Weave.

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1 Introduction

Usually, invariants of mathematical objects are valued in numerical or polynomial rings, rings of homology groups, etc. In the present paper, we prove a general theorem about invariants of dynamical systems which are valued in pictures, the so-called free k-braids.

Formally speaking, free k-braids form a group presented by generators and relations; it has lots of picture-valued invariants; these invariants (in our new terminology, 2-braids). For free 2-braids, the following principle can be realized

if a braid diagram $D$ is complicated enough, then it realizes itself as a subdiagram of any diagram $D'$ contained in $D$.

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Our invariant is constructed by using horizontal trisecant lines. Herewith, the set of critical values (corresponding to these trisecants) leads to a certain picture which appears in all diagrams equivalent to the initial picture.

The main theorem of the present paper has various applications in knot theory, geometry, and topology.

It is based on the following important principle:

\textit{if in some science an object has intersections of multiplicity }k\textit{, then such an object can be studied by }k\textit{-braids and their generalizations.}

In topology, this principle was first demonstrated in terms of parity for the case of virtual knots, see [Ma1].

Given a topological space \(\Sigma\), called the \textit{configuration space}; the elements of \(\Sigma\) will be referred to as \textit{particles}. The topology on \(\Sigma^N\) defines a natural topology on the space of all continuous mappings \([0, 1] \to \Sigma^N\); we shall also study mappings \(S^1 \to \Sigma^N\), where \(S^1 = [0, 1]/\{0 = 1\}\) is the circle; these mappings will naturally lead to \textit{closures} of \(k\)-braids.

Let us fix positive integers \(n\) and \(k\).

The \textit{space of admissible dynamical systems} \(\mathcal{D}\) is a closed subset in the space of all maps \([0, 1] \to \Sigma^n\).

By a \textit{dynamical system} we mean an element of \(\mathcal{D}\). By a \textit{state} of \(\mathcal{D}\) we mean the ordered set of particles \(D(t) \in \Sigma^n\). Herewith, \(D(0)\) and \(D(1)\) are called \textit{the initial state} and \textit{the terminal state}.

We shall also deal with \textit{cyclic dynamical system}, where \(D(0) = D(1)\).

As usual, we shall fix the initial state and the terminal state and consider the set of admissible dynamics with such initial and terminal states.

We say that a property \(P\) of defined for subsets of the set of \(n\) particles from \(\Sigma\) is \textit{k-good}, if the following conditions hold:

1. if this property holds for some set of particles then it holds for every subset of this set;
2. this property holds for every set consisting of \(k - 1\) particles among \(n\) ones (hence, for every smaller set);
3. fix \(k\) pairwise distinct numbers \(i_1, \ldots, i_k, i_j = 1, \ldots, n\); if the property \(P\) holds for particles with numbers \(i_1, \ldots, i_{k-1}\) and for the set of particles with numbers \(i_2, \ldots, i_k\), then it holds for the set of all \(k\) particles \(i_1, \ldots, i_k\).

\textbf{Remark 1.} Besides \textit{statically good properties} \(P\), which are defined for subsets of the set of particles regardless the state of the dynamics, one can talk about \textit{dynamically good} properties, which can be defined for states considered in time; however, we shall not need it in the present paper.

The simplest example is the motion of points in the Euclidian space \(\mathbb{R}^N\). Here for \(k\)-good property we take the property of points to lie on a fixed \((k - 2)\)-plane \((k \leq N)\).

\textbf{Remark 2.} Our main example deals with the case \(k = 3\), where for particles we take different points on the plane, and \(P\) is the property of points to belong to the same line. In general, particles may be more complicated objects than just points.
Let $\mathcal{P}$ be a $k$-good property defined on a set of $n$ particles $n > k$. For each $t \in [0, 1]$ we shall fix the corresponding state of particles, and pay attention to those $t$ for which there is a set $k$ particles possessing $\mathcal{P}$; we shall refer to these moments as $\mathcal{P}$-critical (or just critical).

**Definition 1.** We say that a dynamical system $D$ is pleasant, if the set of its critical moments is finite whereas each for critical moment there exists exactly one $k$-index set for which the set $\mathcal{P}$ holds (thus, for larger sets the property $\mathcal{P}$ does not hold). Such an unordered $k$-tuple of indices will be called a multiindex of critical moments.

With a pleasant dynamical system we associate its type $\tau(D)$, which will of the set of multiindices $m^1, \ldots, m^N$, written as $t$ increases from $t = 0$ to $t = 1$.

For each dynamics $D$ and each multiindex $m = (m_1, \ldots, m_k)$, let us define the $m$-type of $D$ as the ordered set $t_1 < t_2 < \ldots$ of values of $t$ for which the set of particles $m_1, \ldots, m_k$ possesses the property $\mathcal{P}$.

Notation: $\tau_m(D)$. If the number $l$ of values $t_1 < \cdots < t_l$ is fixed, then the type can be thought of as a point in $\mathbb{R}^l$ with coordinates $t_1, \ldots, t_l$.

**Definition 2.** By the type of a dynamical system, we mean the set of all its types $\tau_m(D)$. Notation: $\tau(D)$.

If $D$ is pleasant, then these set are pairwise disjoint.

**Definition 3.** Fix a number $k$ and a $k$-good property $\mathcal{P}$. We say that $D$ is $\mathcal{P}$-stable, if there is a neighbourhood $U(D)$, where each dynamical system $D' \in U$ is pleasant, whereas for each multiindex $m$ (consisting of $k$ indices), the number $l$ of critical values corresponding to this multiindex $U(D)$, the type $\tau_m$ is a continuous mapping $U(D) \to \mathbb{R}^l$.

We shall often say pleasant dynamical systems or stable dynamical systems without referring to $\mathcal{P}$ if it is clear from the context which $\mathcal{P}$ we mean.

**Definition 4.** By a deformation we mean a continuous path $s : [0, 1] \to \mathcal{D}$ in the space of admissible dynamical systems, from a stable pleasant dynamics $s(0)$ to another pleasant stable dynamical system $s(1)$.

**Definition 5.** We say that a deformation $s$ is admissible if:

1. the set of values $u$, where $s(u)$ is not pleasant or is not stable, is finite, and for those $u$ where $s(u)$ is not pleasant, $s(u)$ is stable.
2. inside the stability intervals, the $m$-types are continuous for each multiindex $m$.
3. for each value $u = u_0$, where $s(u_0)$ is not pleasant, exactly one of the two following cases occurs:
   
   (a) There exists exactly one $t = t_0$ and exactly one $(k+1)$-tuple $m = (m_1, \ldots, m_{k+1})$ satisfying $\mathcal{P}$ for this $m$ (hence, $\mathcal{P}$ does not hold for larger sets).
   
   Let $\tilde{m}_j = m \setminus \{m_j\}, j = 1, \ldots, k + 1$ For types $\tau_{\tilde{m}_j}$, choose those coordinates $\zeta_j$, which correspond to the value $t = t_0$. It is required that for all these values $u_0$ all functions $\zeta_j(u)$ are smooth, and all derivatives $\frac{\partial \zeta_j}{\partial u}$ are pairwise distinct;

   (b) there exists exactly one value $t = t_0$ and exactly two multiindices $m = \{m_1, \ldots, m_k\}$ and $m' = \{m'_1, \ldots, m'_k\}$ for which $\mathcal{P}$ holds; we require that $\text{Card}(m \cap m') < k - 1$. 

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4. For each value \( u \), where the dynamical system \( s(u) \) is not stable, there exists a value \( t = t_0 \), which is not critical for \( D_u \), and a multiindex \( \mu = (\mu_1, \ldots, \mu_k) \), for which the following holds.

For some small \( \varepsilon \) all dynamical systems \( D_{u_0} \) for \( u_0 \in (u - \varepsilon, u) \) and \( u_0 \in (u, u + \varepsilon) \) are stable (for \( \delta < \varepsilon \)), and the type \( \tau_\mu(D_{u+\delta}) \) differs from the type \( \tau_\mu(D_{u-\delta}) \) by an addition/removal of two identical multiindices \( \mu \) in the position corresponding to \( t_0 \).

For the space of deformation, one defines an induced topology.

**Definition 6.** We say that a k-good property \( P \) is k-correct for the space of admissible dynamical systems, if the following conditions hold:

1. In each neighbourhood of any dynamical system \( D \) there exists a pleasant dynamical system \( D' \).

2. For each deformation \( s \) there exists an admissible deformation with the same ends \( s'(0) = s(0), s'(1) = s(1) \).

**Definition 7.** We say that two dynamical systems \( D_0, D_1 \) are equivalent, if there exists a deformation \( s, s(0) = D_0, s(1) = D_1 \).

Thus, if we talk about a correct \( P \)-property, we can talk about an admissible deformation when defining the equivalence.

## 2 Free \( k \)-braids

Let us now pass to the definition of the \( n \)-strand free \( k \)-braid group \( G_{n,k} \).

Consider the following \( \binom{n}{k} \) generators \( a_m \), where \( m \) runs the set of all unordered \( k \)-tuples \( m_1, \ldots, m_k \), whereas each \( m_i \) are pairwise distinct numbers from \( \{1, \ldots, n\} \).

For each \( (k + 1) \)-tuple \( U \) of indices \( u_1, \ldots, u_{k+1} \in \{1, \ldots, n\} \), consider the \( k + 1 \) sets \( m^j = U\setminus\{u_j\}, j = 1, \ldots, k + 1 \). With \( U \), we associate the relation

\[
a_{m^1} \cdot a_{m^2} \cdots a_{m^{k+1}} = a_{m^{k+1}} \cdots a_{m^2} \cdot a_{m^1};
\]  

for two tuples \( U \) and \( \bar{U} \), which differ by order reversal, we get the same relation.

Thus, we totally have \( \frac{(k+1)!}{2} \binom{n}{k+1} \) relations.

We shall call them the tetrahedron relations.

For \( k \)-tuples \( m, m' \) with \( \text{Card}(m \cap m') < k - 1 \), consider the far commutativity relation:

\[
am_m a_{m'} = a_{m'} a_m
\]  

Note that the far commutativity relation can occur only if \( n > k + 1 \).

Besides that, for all multiindices \( m \), we write down the following relation:

\[
a_m^2 = 1
\]  

Define \( G_{n,k} \) as the quotient group of the free group generated by all \( a_m \) for all multiindices \( m \) by relations (1), (2) and (3).
Example. The group $G_2^3$ is $\langle a, b, c | a^2 = b^2 = (abc)^2 = 1 \rangle$, where $a = a_{12}, b = a_{13}, c = a_{23}$.

Indeed, the relation $(abc)^2 = 1$ is equivalent to the relation $abc = cab = 1$ because of $a^2 = b^2 = c^2 = 1$. This obviously yields all the other tetrahedron relations.

Example. The group $G_3^4$ is isomorphic to $\langle a, b, c, d | a^2 = b^2 = c^2 = 1, (abcd)^2 = 1, (adbc)^2 = 1, (acdb)^2 = 1 \rangle$.

Here $a = a_{123}, b = a_{124}, c = a_{134}, d = a_{234}$.

It is easy to check that instead of $4!^2 = 12$ relations, it suffices to take only $3!^2 = 9$ relations.

By the complexity of a word we mean the number of letters in this word, by the complexity of a free $k$-braid we mean the minimal complexity of all words representing it. Such words will be called minimal representatives. The tetrahedron relations (in the case of free 2-braids we call them the triangle relations) and the far commutativity relations do not change the complexity, and the relation $a_m^2 = 1$ increases or decreases the complexity by 2.

As usual in the group theory, it is natural to look for minimal complexity words representing the given free $k$-braid.

If we deal with conjugacy classes of free $k$-braids, one deals with the complexity of cyclic words.

The number of words of fixed complexity in a finite alphabet is finite; $k$-braids and their conjugacy classes are the main invariant of the present work.

Let us define the following two types of homomorphisms for free braids. For each $l = 1, \ldots, n$, there is an index forgetting homomorphism $f_l : G_n^k \to G_{n-1}^{k-1}$; this homomorphism takes all generators $a_m$ with multiindex $m$ not containing $l$ to the unit element of the group, and takes the other generators $a_m$ to $a_{m'}$, where $m' = m \setminus \{l\}$; this operation is followed by the index renumbering.

The strand-deletion homomorphism $d_j$ is defined as a homomorphism $G_n^k \to G_{n-1}^{k-1}$; it takes all generators $a_m$ having multiindex containing $j$ to the unit element; after that we renumber indices.

The free 2-braids (called also pure free braids) were studied in [Ma1, MW, Ma2].

For free 2-braids, the following theorem holds.

**Theorem 1.** Let $b'$ be a word representing a free 2-braid $\beta$. Then every word $b$ which is a minimal representative of $\beta$, is equivalent by the triangle relations and the far commutativity relation to some subword of the word $b'$.

Every two representatives $b_1$ and $b_2$ of the same free 2-braid $\beta$ are equivalent by the triangle relations and the far commutativity relations.

Thus, for free 2-braids, the recognition problem can be solved by means of considering its minimal representative.

The main idea of the proof of this theorem is similar to the classification of homotopy classes of curves in 2-surfaces due to Hass and Scott [HS]: in order to find a minimal representative, one looks for “bigon reductions” until possible, and the final result is unique up to third Reidemeister moves for the exception of certain special cases (multiple curves etc). For free 2-braids “bigon reductions” refer to some cancellations of generators similar generators $a_m$ and $a_m$ in good position with respect to each other, see Fig. 2; third Reidemeister moves correspond to the triangle relations, and the far commutativity does not change the picture at all [Ma2].

For us, it is crucial to know that when looking at a free 2-braid, one can see which pairs of crossings can be cancelled. Once we cancel all possible crossings, we get an invariant picture.

Thus, we get a complete picture-valued invariant of a free 2-braid.
Theorem 1 means that this picture (complete invariant) occurs as a sub-picture in every picture representing the same free 2-braid.

A complete proof of Theorem 1 will be given in a separate publication.

However, various homomorphisms \( G^k_n \to G^{k-1}_{n-1} \), whose combination lead to homomorphisms of type \( G^k_n \to G^2_{n-k+2} \), allow one to construct lots of invariants of groups \( G^k_n \) valued in picture.

In particular, these pictures allow one to get easy estimate for the complexity of braids and corresponding dynamics.

Free \( k \)-braids will be considered in a separate publication.

3 The Main Theorem

Let \( \mathcal{P} \) be a \( k \)-correct property on the space of admissible dynamical systems with fixed initial and final states.

Let \( D \) be a pleasant stable dynamical system describing the motion of \( n \) particles with respect to \( \mathcal{P} \). Let us enumerate all critical values \( t \) corresponding to all multiindices for \( D \), as \( t \) increases from 0 to 1. With \( D \) we associate an element \( c(D) \) of \( G^k_n \), which is equal to the product of \( a_{m} \), where \( m \) are multiindices corresponding to critical values of \( D \) as \( t \) increases from 0 to 1.

**Theorem 2.** Let \( D_0 \) and \( D_1 \) be two equivalent stable pleasant dynamics with respect to \( \mathcal{P} \). Then \( c(D_0) = c(D_1) \) are equal as elements of \( G^k_n \).

**Proof.** Let us consider an admissible deformation \( D_s \) between \( D_0 \) and \( D_1 \). For those intervals of values \( s \), where \( D_s \) is pleasant and stable, the word representing \( c(D_s) \), does not change by construction. When passing through those values of \( s \), where \( D_s \) is not pleasant or is not stable, \( c(D(s)) \) changes as follows:

1. Let \( s_0 \) be the value of the parameter deformation, for which the property \( \mathcal{P} \) holds for some \( (k+1) \)-tuple of indices at some time \( t = t_0 \). Note that \( D_{s_0} \) is stable.

   Consider the multiindex \( m = (m_1, \ldots, m_{k+1}) \) for which \( \mathcal{P} \) holds at \( t = t_0 \) for \( s = s_0 \). Let \( \tilde{m}_{ij} = m \setminus m_j, j = 1, \ldots, k+1 \) For types \( \tau_{\tilde{m}_{ij}} \), let us choose those coordinates \( \zeta_j \), which correspond to the intersection \( t \) at \( s = s_0 \). As \( s \) changes, these type are continuous functions with respect to \( s \).

   Then for small \( \varepsilon \), for \( u = u_0 + \varepsilon \), the word \( c(D_s) \) will contain a sequence of letters \( a_{\tilde{m}_{ij}} \) in a certain order. For values \( s = s_0 - \varepsilon \), the word \( c(D_s) \) will contain the same set of letters in the reverse order. Here we have used the fact that \( D_s \) is stable.

2. If for some \( s = s_0 \) we have a critical value with two different \( k \)-tuples \( m, m' \) possessing \( \mathcal{P} \) and \( \text{Card}(m \cap m') < k - 1 \), then the word \( c(D_s) \) undergoes the relation (2) as \( s \) passes through \( s_0 \); here we also require the stability of \( D_{s_0} \).

3. If at some \( s = s_0 \), the deformation \( D_s \) is unstable, then \( c(D_s) \) changes according to (3) as \( s \) passes through \( s_0 \). Here we use the fact that the deformation is admissible.

\( \square \)
Let us now pass to our main example, the classical braid group. Here distinct points on the plane are particles. We can require that their initial and final positions they are uniformly distributed along the unit circle centered at 0.

For \( P \), we take the property to belong to the same line. This property is, certainly, 3-good. Every motion of points where the initial state and the final state are fixed, can be approximated by a motion where no more than 3 points belong to the same straight line at once, and the set of moments where three points belong to the same line, is finite, moreover, no more than one set of 3 points belong to the same line simultaneously. This means that this dynamical system is pleasant.

Finally, the correctedness of \( P \) means that if we take two isotopic braids in general position (in our terminology: two pleasant dynamical systems connected by a deformation), then by a small perturbation we can get an admissible deformation for which the following holds. There are only with finitely many values of the parameter \( s \) with four points on the same line or two triples of points on the same line at the same moment; moreover, for each such \( s \) only one such case occurs exactly for one value of \( t \).

In this example, as well as in the sequel, the properties of being pleasant and correct are based on the fact that every two general position state can be connected by a curve passing through states of codimension 1 (simplest generation) finitely many times, and every two paths with fixed endpoints, which can be connected by a deformation, can be connected by a general position deformation where points of codimensions 1 and 2 occur, the latter happen only finitely many times.

In particular, the most complicated condition saying that the set of some \((k + 1)\) particles satisfies the property \( P \), the corresponding derivatives are all distinct, is also a general position argument. For example, assuming that some 4 points belong to the same horizontal line (event of codimension 2), we may require there is no coincidence of any further parameters (we avoid events of codimension 3).

From the definition of the invariant \( c \), one easily gets the following

**Theorem 3.** Let \( D \) be a dynamical system corresponding to a classical braid. Then the number of horizontal trisecants of the braid \( D \) is not smaller than the complexity of the free 3-braid \( \beta = c(D) \).

Analogously, various geometrical properties of dynamical systems can be analyzed by looking at complexities of corresponding groups of free \( k \)-braids, if one can define a \( k \)-correct property for these dynamics, which lead to invariants valued in free \( k \)-braids.

Let us now collect some situations where the above methods can be applied.

1. An evident invariant of closed pure classical braids is the conjugacy class of the group \( G_n^3 \). To pass from arbitrary braids to pure braids, one can take some power of the braid in question.

2. Note that the most important partial case for \( k = 2 \) is the classical Reidemeister braid theory. Indeed, for a set of points on the plane \( Oxy \), we can take for \( P \) the property that the \( y \)-coordinates of points coincide. Then, considering a braid as motion of distinct points in the plane \( z = 1 - t \) as \( t \) changes from 0 to 1, we get a set of curves in space whose projection to \( Oxz \) will have intersections exactly in the case when the property \( P \) holds. The additional information coming from the \( x \) coordinate, leads one to the classical braid theory.
3. For classical braids, one can construct invariants for $k = 4$ in a way similar to $k = 3$. In this case, we again take ordered sets of $n$ points on the plane, and the property $\mathcal{P}$ means that the set of points belongs to the same circle or straight line; for three distinct points this property always holds, and the circle/straight line is unique.

4. With practically no changes this theory can be used for the study of weavings [Viro], collections of projective lines in $\mathbb{R}P^3$ considered up to isotopy. Here $\mathcal{P}$ is the property of a set of points to belong to the same projective line. The main difficulty here is that the general position deformation may contain three lines having infinitely many common horizontal trisecants. Another difficulty occurs when one of our lines becomes horizontal; this leads to some additional relations to our groups $G_n^3$, which are easy to handle.

5. In the case of points on a 2-sphere we can define $\mathcal{P}$ to be the property of points to belong to the same geodesic. This theory works with an additional restriction which forbids antipodal points. Some constraints should be imposed in the case of 2-dimensional Riemannian manifolds: for the space of all dynamical systems, we should impose the restrictions which allow one to detect the geodesic passing through two points in a way such that if two geodesics chosen for $a, b$ and for $b, c$ coincide, then the same geodesic should be chosen for $a, c$.

6. In the case of $n$ non-intersecting projective $m$-dimensional planes in $\mathbb{R}P^{m+2}$ considered up to isotopy, the theory works as well. Here, in order to define the dynamical system, we take a one-parameter family of projective hyperplanes in general position, for particles we take $(m - 1)$-dimensional planes which appear as intersections of the initial planes with the hyperplane.

   The properties of being good, correct etc. follow from the fact that in general position “particles” have a unique same secant line (in a way similar to projective lines, one should allow the projective planes not to be straight).

   For $m = 1$ one should take $k = 3$, in the general case one takes $k = 2m + 1$.

   This theory will be developed in a separate publication.

7. This theory can be applied to the study of fundamental groups of various discriminant spaces, if such spaces can be defined by several equalities and subsets of these equalities can be thought of as property $\mathcal{P}$.

8. The case of classical knots, unlike classical links, is a bit more complicated: it can be considered as a dynamical system, where the number of particles is not constant, but it is rather allowed for two particles to be born from one point or to be mutually contracted.

   The difficulty here is that knots do not possess a group structure, thus, we don’t have a natural order on the set of particles. Nevertheless, it is possible to construct a map from classical knots to free 3-knots (or free 4-knots) and study them in a way similar to free 3-braids (free 4-braids).

   This theory will be developed in detail in a separate publication.

**Remark 3.** In the case of sets of points in space of dimension 3, the property of some points to belong to a 2-plane (or higher-dimensional plane) is not correct. Indeed, if three points belong to the
same line, then whatever fourth point we add to them, the four points will belong to the same plane, thus we can get various multiindices of 4 points corresponding to the same moment.

The “triviality” of such theory taken without any additional constraints has the simple description that the configuration space of sets of points in $\mathbb{R}^3$ has trivial fundamental group.

4 Pictures

The free $k$-braids can be depicted by strands connecting points $(1, 0), \ldots, (n, 0)$ to points $(1, 1), \ldots, (n, 1)$; every strand connects $(i, 0)$ to $(i, 1)$; its projection to the second coordinate is a homeomorphism. We mark crossings corresponding $a_{i,j,k}$ by a solid dot where strands #i, #j, #k intersect transversally.

All other crossings on the plane are artifacts of planar drawing; they do not correspond to any generator of the group, and they are encircled. View Fig. 1

The clue for the recognition of free 2-braids is the bigon reduction shown in Fig. 2

Here we reduce the bigon whose vertices vertices are $X, Y$.

The graph which appears after all possible bigon cancellations (with opposite edge structure at vertices) is the picture which is the complete invariant of the free 2-braid (see also [Ma2]).

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Figure 2: The bigon reduction

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