Self-stabilizing Localization of the Middle Point of a Line Segment by an Oblivious Robot with Limited Visibility

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Abstract. This paper poses a question about a simple localization problem, which is arisen from self-stabilizing location problems by oblivious mobile autonomous robots with limited visibility. The question is if an oblivious mobile robot on a line-segment can localize the middle point of the line-segment in finite steps observing the direction (i.e., Left or Right) and distance to the nearest end point. This problem is also akin to (a continuous version of) binary search, and could be closely related to computable real functions. Contrary to appearances, it is far from trivial if this simple problem is solvable or not, and unsettled yet. This paper is concerned with three variants of the original problem, minimally relaxing, and presents self-stabilizing algorithms for them. We also show an easy impossibility theorem for bilaterally symmetric algorithms.

Key words: self-stabilization, oblivious mobile autonomous robot with limited visibility, computable real functions, continuous binary search

1 Introduction

1.1 Background

Motivated by real applications such as wireless sensor networks with mobile nodes, or by computability of a distributed system from the theoretical point of view, designing self-stabilizing distributed algorithms for mobile autonomous robots has been intensively investigated in distributed computing on various problems such as pattern formation [7,19,20,21,22], gathering [1,12,13,14], self-deployment [3,5,10,13,18] including scattering and coverage. Mainly from the view point of self-stabilization, robots in those research have few memory, or often no memory (oblivious), and hence they are required to solve problems from “geometric” information observing. As an extreme case for theoretical tractability, robots are often assumed to have (infinitely) large visibility such that they respectively observe the whole robots, which corresponds to a situation that sensor nodes are congested with in there sensing area in the practical sense. However, real sensor nodes often do not have enough power of sensing, and limited visibility could be a more practical model and a challenging target. The
limitation of the visibility causes many intractability not only in practice but also in theory, then the theory of distributed algorithms for autonomous mobile robots with limited visibility is recently developing [1,2,7,9,10,11,12,22].

1.2 Problem: localization of the midpoint

A localization, inferring a place in a known environment, is clearly a fundamental and significant task [10], especially for an autonomous mobile robot under limited visibility. This paper is concerned with a very simple localization problem by an oblivious mobile autonomous robot with limited visibility: Suppose that a robot is located on a line-segment, then the goal is to localize eventually the middle point of the line-segment (see Fig. 1 and also Section 2 for detail). The robot has minimally sufficient visibility, precisely the visibility range is exactly the half length of the line-segment, meaning that the robot can observe the both ends only when it is located exactly at the midpoint, and observes only the nearest end-point in the other location. The robot distinguishes left and right. Then, the robot observes the direction to the nearest end-point, Left (L) or Right (R), and the distance to there. The robot is oblivious, meaning that it does not have a memory of the previous steps. However, it has very strong computability, beyond Turing computable, in each step to deal with reals. Then, the question of the paper is if there is a self-stabilizing algorithm to solve the localization problem for any length of the line-segment and for any initial position of the robot.

If a robot has a memory of the previous position and the motion to the current position, then the problem is trivially solved: Suppose that the robot observed the left-end in distance \( d \) at the previous position, moved to right by distance 1, and observes the right-end in distance \( d' \). Then, it is easy to see that the midpoint is left to the current position by distance \( \frac{d+1+d'}{2} - d' \). Thus, the oblivious is clearly a difficulty of the problem.
1.3 Our results

Contrary to its simple appearance of the problem, especially there is a single robot in a 1-D space, it is far from trivial if the problem is solvable or not, and the question is unsettled yet. This paper is concerned with three relaxed versions of the basic problem, and shows the solvability of them by giving self-stabilization algorithms, respectively. The first version is a convergence problem, which relaxes the visibility condition such that the robot around the midpoint observes the both ends, instead of exactly at the midpoint. Thus, the goal of the problem is to localize a point in the line-segment near the midpoint, instead of exactly localizing the midpoint (Section 3). The second version assumes a condition that the length of the line-segment is rational, instead of an arbitrary finite real. The algorithm is the most technical in the three versions in this paper (see Section 4). The third version allows the robot a small memory. As we stated, if a robot has a memory of the previous position, then the problem is easily solved. We show that only a single-bit of memory is sufficient to solve the problem, meaning that the robot localizes the midpoint in finite steps from arbitrary initial position without any initialization of memory. The algorithm is simpler than the other two cases, using some parity tricks, and could be practical (Section 5).

Above reasonably minimal relaxations of the problem are solvable, nevertheless we conjecture that the original problem is unsolvable. Concerning the impossibility, we also give an easy impossibility theorem, where we assume that an algorithm is restricted to be mirror symmetric at the midpoint (Section 6). In Concluding Remark (Section 7) we also refer to some interesting versions unsettled.

1.4 Related works

Closely related problems, or a direct motivation of the paper, are scattering or coverage over a line or a ring by autonomous mobile robots with limited visibility [6,7,10,11]. Cohen and Peleg [6] were concerned with spreading of autonomous mobile robots over a line (1-D space) where a robot observes the nearest neighboring robot in each of left and right side. They presented a local algorithm leading to equidistant spreading on a line, and showed convergence and convergence rate for fully synchronous (FSYNC) and semi-synchronous (SSYNC) models. They also gave an algorithm to solve exactly when each robot has enough size of memory, that is linear to the number of robots. Eftekhari et al. [10] studied the coverage of a line segment by autonomous mobile robots placing grid points with minimum visibility to solve the problem. They gave two local distributed algorithms, one is for oblivious robots and it terminates in time quadratic to the number of robots, while the other is for robots with a constant memory and it terminates in linear time. Eftekhari et al. [9] showed the impossibility of the coverage of a line-segment by robots with limited visibility in SSYNC model when robots do not share left-right direction. Whereas, they showed that it is solvable even in ASYNC model if robots shares left-right direction, have a
visibility range strictly greater than mobility range, and know the size of visibility range.

Flocchini et al. [11] were concerned with equidistant covering of a circle by oblivious robots with limited visibility. They showed the impossibility of exact solution if they do not share a common orientation of the ring. They also showed the possibility by oblivious asynchronous robots with almost minimum visibility when robots share a common orientation. Defago and Konagaya [7] were concerned with circle formation in 2-D space by oblivious robots with limited visibility, where robots do not know the size of their visibility range. In the paper, they also dealt with equidistant covering of a circle, and gave an algorithm for convergence.

2 Problem Description

The goal of this section is to describe our problem, Problem 1 in Section 2.2, in a form of an existence of a function. Before explaining the formal description, Section 2.1 explains the problem as an algorithm for an autonomous mobile robot. Let \( \mathbb{Z}, \mathbb{Z}_{\geq 0}, \mathbb{Z}_{> 0}, \mathbb{Q} \) and \( \mathbb{R} \) respectively denote the whole set of integers, nonnegative integers, positive integers, rationals and reals, in the paper.

2.1 Self-stabilizing localization of the midpoint by a robot

Suppose that a robot is in a closed real interval \( [a, b] \subseteq \mathbb{R} \) \((a < b)\), where the robot is a point and \( x \in [a, b] \) denotes a position of the robot. The robot does not know neither \( a, b, b - a \), nor \( x \). Our goal is to design an algorithm according to which a robot eventually localize the point \( (a + b)/2 \in [a, b] \) in finite steps. The robot repeats executing a “Look-Compute-Move” cycle. In a Look phase, it observes the nearest end-point SIDE \( \in \{L, R\} \), and the distance \( d \in \mathbb{R} \) to the end-point. The robot also observes \( q_{\text{mid}} \) if it places exactly at the midpoint \( (a + b)/2 \). In a Compute phase, the robot deterministically decides the next point only using SIDE and \( d \). The robot also does not have any memory of the previous “Look-Compute-Move” cycles, while we assume that the robot is able to deal with reals, meaning that the computability in a Compute phase is much stronger than Turing machine. In a Move phase, the robot moves to the point which is computed in the Compute phase. Any move is rigid, meaning that the robot arrives at the point without any failure.

Then, the question is if there is a universal algorithm by which the robot localizes the midpoint \( (a + b)/2 \in [a, b] \) in finite steps for any \( a, b \in \mathbb{R} \) \((a < b)\) and \( x \in [a, b] \), where the algorithm is universal means that it is described homogeneous to any \( a, b \) and \( x \).

\footnote{We conjecture that Problem 1 is unsolvable. To avoid ambiguity, especially for an impossibility proof (in the future), we give there a formal description of the problem.}
2.2 Formal description of the problem

In order to avoid a confusion on the (computational) ability of the robot, we give a simple and formal description of the problem, which is formulated as an existence of a function describing the motion of the robot.

Problem 1 (Basic problem). As given a real $D \ (1 < D < \infty)$, and a real $x$ in the closed real interval $[-D, D]$, an observation function $\phi : \mathbb{R} \times [-D, D] \to \mathcal{O}$, where $\mathcal{O} := \{(R, L) \times [0, D]) \cup \{q_{\text{Mid}}\}$, is defined by

$$\phi(D, x) = \begin{cases} (R, D - x) & \text{if } x > 0 \\ (L, D + x) & \text{if } x < 0 \\ q_{\text{Mid}} & \text{otherwise, i.e., } x = 0. \end{cases}$$

(1)

For convenience, we denote $\phi(D, x) = (\text{SIDE}_D(x), d_D(x))$ when $x \neq 0$. A map $f : \mathcal{O} \to [-D, D]$ is a transition map if $f(\phi(D, x)) - x$ only depends on $d_D(x)$ and $\text{SIDE}_D(x)$ (but independent of $D$ or $x$), and $f(\phi(D, x)) \in [-D, D]$ for any $x \in [-D, D]$. The goal of the problem is to design a transition map $f : \mathcal{O} \to [-D, D]$ for which an integer $n \ (0 \leq n < \infty)$ exists for any real $D \ (1 < D < \infty)$ and $x_0 \in [-D, D]$ such that $x_n = 0$ where $x_{i+1} = f(\phi(D, x_i))$ for $i = 0, 1, 2, \ldots$. More precisely, let $\Psi : \mathbb{R} \times [-D, D] \to \mathbb{Z}_{\geq 0}$ be a potential function defined by

$$\Psi(D, x) = \min\{n \in \mathbb{Z}_{\geq 0} \mid x_0 = x, \ x_n = 0, \ x_{i+1} = f(\phi(D, x_i))\}$$

(2)

for any $D \ (1 < D < \infty)$ and $x \in [-D, D]$, then $\Psi(D, x)$ needs to be bounded (may depend on $D$).

In terms of the localization by an autonomous mobile robot, the robot at $x \in [-D, D]$ observes the nearest end-point $\text{SIDE}(x)$, and the distance $d(x)$ to the end, where $\text{SIDE}(x)$ and $d(x)$ respectively abbreviate $\text{SIDE}_D(x)$ and $d_D(x)$ without a confusion. Notice that $x = D - d(x)$ if $\text{SIDE}(x) = R$, while $x = -D + d(x)$ if $\text{SIDE}(x) = L$. Then, the robot moves to the right by distance $f(\text{SIDE}(x), d(x)) - x$, where $f(\text{SIDE}(x), d(x)) - x < 0$ implies that the robot actually moves to the left by distance $|f(\text{SIDE}(x), d(x)) - x|$. Since the robot observes only $\text{SIDE}(x)$ and $d(x)$, meaning that it does not know neither $D$ nor $x$, $f(\text{SIDE}(x), d(x)) - x = f(\text{SIDE}(x), d(x)) - D + d(x)$ when $\text{SIDE} = R$ (resp. $f(\text{SIDE}(x), d(x)) - x = f(\text{SIDE}(x), d(x)) + D - d(x)$ when $\text{SIDE} = L$) depends only on $\text{SIDE}(x)$ and $d(x)$, but must be independent of $D$ or $x$. The potential function $\Psi(D, x)$ represents the number of steps to localize the midpoint by the algorithm given by the transition function $f$.

3 Relaxation 1: Convergence

To begin with, this section shows that a convergence version of Problem 1 is solvable. To be precisely, we are concerned with the following problem

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Problem 2 (Convergence). As given a real $D \ (1 < D < \infty)$, a real $\epsilon \ (0 < \epsilon \leq D)$ and a real $x \in [-D, D]$, the observation function $\phi: \mathbb{R} \times \mathbb{R} \times [-D, D] \rightarrow \mathcal{O}$ is given by

$$\phi(D, \epsilon, x) = \begin{cases} (R, D-x) & \text{(if } x > \epsilon) \\ (L, D+x) & \text{(if } x < -\epsilon) \\ \text{qMid} & \text{(otherwise, i.e., } -\epsilon \leq x \leq \epsilon) \end{cases} \quad (3)$$

A map $f$ is a transition map if $f(\phi(D, x)) - x$ only depends on $d_D(x)$ and $\text{SIDE}_D(x)$ (but independent of $D, x, \epsilon$), and $f(\phi(D, x)) \in [-D, D]$ for any $x \in [-D, D]$. The goal of the problem is to design a transition map $f: \mathcal{O} \rightarrow [-D, D]$ for which an integer $n \ (0 \leq n < \infty)$ exists for any reals $D \ (1 < D < \infty)$, $\epsilon \ (0 < \epsilon \leq D)$ and $x_0 \in [-D, D]$ such that $-\epsilon \leq x_n \leq \epsilon$ where $x_{i+1} = f(\phi(D, \epsilon, x_i))$ for $i = 0, 1, 2, \ldots$.

The condition that $f(\phi(D, x)) - x$ is independent of $\epsilon$ means the robot does not know $\epsilon$. Thus, an algorithm is required two conflicting functions: The step-lengths are (preferably) decreasing, otherwise the robot misses the small interval $[-\epsilon, \epsilon]$. On the other hand, the total length of the moves should diverge as increasing the number of steps, otherwise the robot stops before reaching at the midpoint when $D$ is larger than the upper bound of the total length of the moves, like “Achilles and the Tortoise.”

The rest of this section shows the following theorem.

Theorem 1. Problem 2 is solvable.

3.1 Preliminary

As a preliminary step of the proof of Theorem 1 as well as for Theorem 2 in Section 4 here we briefly remark three properties on the reciprocals of the square roots of primes, which are versions of well-known fundamental facts. Let $\mathbb{P}$ denote the whole set of prime numbers, and let $\pi_i \in \mathbb{P} \ (i = 1, 2, 3, \ldots)$ denote the $i$-th smallest prime number, i.e., $\pi_1 = 2, \pi_2 = 3, \pi_3 = 5, \pi_4 = 7, \ldots$.

First, we remark the following (almost) trivial fact.

Proposition 1. $1/\sqrt{\pi_k}$ is monotone decreasing and asymptotic to 0 with respect to $k$. \hfill $\square$

Second, we remark that the sum of $1/\sqrt{\pi_i}$ diverges, using the well-known fact that the sum of the reciprocals of all prime numbers diverges. For convenience of the later argument, let

$$\sigma_k = \sum_{i=1}^{k} \frac{1}{\sqrt{\pi_i}} \quad (4)$$

for each $k \in \mathbb{Z}$. We also define $\sigma_0 = 0$ for convenience.

Proposition 2. $\sum_{i=j}^{\infty} \frac{1}{\sqrt{\pi_i}} = \infty$ for any finite $j \in \mathbb{Z}_{>0}$.
Proof. It is known, due to Euler \cite{Euler}, that $\sum_{i=1}^{\infty} \frac{1}{\pi_i} = \infty$ (cf. \cite{17}). Clearly, $\frac{1}{\sqrt{\pi_i}} > \frac{1}{\pi_i}$ holds for each $i = 1, 2, \ldots$, and we obtain that $\sum_{i=1}^{\infty} \frac{1}{\sqrt{\pi_i}} = \lim_{k \to \infty} \sigma_k = \infty$. Since the finite sum $\sum_{i=1}^{j-1} \frac{1}{\sqrt{\pi_i}}$ is upper bounded for any finite $j$, we obtain the claim. \qed

Third, we remark the fact that $\frac{1}{\sqrt{\pi_i}}$ are bases of $\mathbb{R}$ with rational coefficients (see e.g., \cite{15}).

**Proposition 3.** Let

$$\Sigma_i = \{\alpha + \beta \sigma_i \mid \alpha, \beta \in \mathbb{Q}, \beta \neq 0\}$$

for $i \in \mathbb{Z}_{\geq 0}$. Then, $\Sigma_i \cap \Sigma_j = \emptyset$ when $i \neq j$.

**Proof (Sketch).** Notice that $1/\sqrt{\pi_k} = (1/\pi_k)\sqrt{\pi_k}$. Let $\mathbb{F}_0 = \mathbb{Q}$, and let $\mathbb{F}_{k+1} = \mathbb{F}_k(\pi_{k+1})$ for $k = 0, 1, 2, \ldots$ be the extension field adjoining $\{\sqrt{\pi_{k+1}}\}$ to $\mathbb{F}_k$. We claim that $\left\{\sqrt{\prod_{i \in I} \pi_i} \mid I \subseteq \{1, \ldots, k\}\right\}$, where $\prod_{i \in \emptyset} \pi_i = 1$, is a basis of $\mathbb{F}_k$ over $\mathbb{Q}$. The proof is an induction with respect to $k$. In case that $k = 1$, it is easy to see that $\{1, \sqrt{2}\}$ is a basis of $\mathbb{F}_1$ over $\mathbb{Q}$. Suppose that claim holds for $k$, then we claim that $\{1, \sqrt{\pi_{k+1}}\}$ is a basis of $\mathbb{F}_{k+1}$ over $\mathbb{F}_k$. Assume for a contradiction that

$$\sqrt{\pi_{k+1}} = \alpha_0 + \alpha_1 \sqrt{w_1} + \cdots + \alpha_{2^k-1} \sqrt{w_{2^k-1}},$$

holds where $\alpha_0, \ldots, \alpha_{2^k-1} \in \mathbb{Q}$. Suppose $\alpha_i \neq 0$ and $\alpha_j \neq 0$ holds for a distinct $i, j$. Then,

$$\pi_{k+1} = (\alpha_0 + \alpha_1 \sqrt{w_1} + \cdots + \alpha_{2^k-1} \sqrt{w_{2^k-1}})^2$$

implies a contradiction since the left-hand-side is rational but the right-hand-side is irrational since $\alpha_i\alpha_j \sqrt{w_iw_j} \neq 0$ remains there, where we use the inductive hypothesis that $\sqrt{w_iw_j}$ is a basis of $\mathbb{F}_k$ over $\mathbb{Q}$. Suppose there uniquely exists $i$ satisfying $\alpha_i \neq 0$. Then, $\pi_{k+1} = \alpha_i^2 w_i$, which implies $\alpha_i = \pm \sqrt{\pi_{k+1}/w_i}$. It contradicts to $\alpha_i \in \mathbb{Q}$ since $\pi_{k+1}$ and $w_i$ are coprime. Thus, $\mathbb{F}_{k+1} = \{a_1 + a_2 \pi_{k+1} \mid a_1, a_2 \in \mathbb{F}_k\}$ holds, and we obtain the claim. \qed

### 3.2 Proof of Theorem \cite{11}

Now, we prove Theorem \cite{11}.

**Proof (Proof of Theorem \cite{11}).** The proof is constructive. For convenience, let $\Delta_k = \{\sigma_k - z \mid z \in \mathbb{Z}_{\geq 0}\}$ for $k = 1, 2, \ldots$ (recall the definition \cite{11} of $\sigma_k$). We define a transition function $f: \mathcal{O} \to [-D, D]$ to solve Problem \cite{2} by

$$f((L, d)) = \begin{cases} x + \frac{1}{\sqrt{\pi_{k+1}}} & \text{(if } d \in \Delta_k \text{ for some } k \in \mathbb{Z}_{\geq 0}), \\ x - d + \frac{1}{\sqrt{2}} & \text{(otherwise, i.e., } d \notin \Delta_k \text{ for any } k \in \mathbb{Z}_{\geq 0}), \end{cases}$$

$$f((R, d)) = x - 1,$$

$$f(q_{\text{Mid}}) = x$$

where $\mathcal{O} = \{L, R\}$ is an ordered set of $1$ and $-1$, and $d$ is the transition direction.
This implies that $k$ step size $x - D$. If Problem 3 is solvable.

We remark that an arbitrary $x$, $D$. Precisely, we are concerned with the following problem. $D$.

Secondly, we observe that if SIDE$(t) = R$, then there exists $t' (t' > t)$ such that SIDE$(t') = L$, or $x = f(D, x_r)$ holds for a finite $n \in \mathbb{Z}_{\geq 0}$ exists such that $-\varepsilon < x_n < \varepsilon$ where $x_t = f(D, x_{t-1})$ for $t = 1, 2, \ldots$. For convenience, let $(\text{SIDE}(t), d(t)) = \phi(D, x_t)$.

Firstly, we observe that if SIDE$(t) = R$, then there exists $t' (t' > t)$ such that SIDE$(t') = L$, or $x = f(D, x_r)$, since the sum of $-1$’s diverges (to $-\infty$). Secondly, we observe that if SIDE$(t) = L$ and $d(t) \notin \Delta_k$ for any $k = 1, 2, \ldots$, then SIDE$(t+1) = L$ and $d(t+1) = 1/\sqrt{2} \in \Delta_1$. Thus, without loss of generality, we may assume that SIDE$(0) = L$ and $d(0) \in \Delta_k$ for some $k = 1, 2, \ldots$, where notice that $k$ is uniquely determined by Proposition 3.

Suppose SIDE$(t) = L$ and $d(t) \notin \Delta_k$. We remark that $-D + x_t \in \Delta_k$ since $d(t) = -D + x_t$ when SIDE$(t) = L$. Then, $-D + x_{t+1} = -D + x_t + 1/\sqrt{\pi_{k+1}} \in \Delta_{k+1}$ by the definition of $f$. Since $\sum_{j=1}^{\infty} 1/\sqrt{\pi_j}$ diverges by Proposition 2, there exists $t' (t' > t)$ such that SIDE$(t') = R$, or $x = f(D, x_r) = q_{\text{Mid}}$. Here, we specially remark that $-D + x_{t'} \in \Delta_{k'}$ holds for some $k'$ even in the case that SIDE$(t') = R$.

If $-D + x_{t'} \notin \Delta_{k'}$ and SIDE$(t') = R$, then $-D + x_{t+1} = -D + (x_{t'} - 1) \in \Delta_{k'}$. This implies that $k$ is monotone nondecreasing with respect to $t$, and hence the step size $x_t - x_t \in 1/\sqrt{\pi_k}$ when SIDE$(t) = L$ is monotone decreasing with respect to $t$ by Proposition 1. Particularly, we note that the step size $x_t - x_t$ when SIDE$(t) = L$ is smaller than $\varepsilon$ if $1/\sqrt{\pi_k} < \varepsilon$ holds for $k$. Thus, eventually we obtain the situation $-\varepsilon \leq x_t \leq \varepsilon$ for a finite $t^* \in \mathbb{Z}_{\geq 0}$.

\section{Relaxation 2: $D$ Is Rational}

Problem 1 is solved under some assumptions or conditions. As a nontrivial and interesting example, this section presents an algorithm for any rational $D$, where we remark that an arbitrary real point of the interval is given as an initial position. Precisely, we are concerned with the following problem.

\textbf{Problem 3 (Rational $D$).} As given an observation function $\phi: \mathbb{R} \times [-D, D] \to \mathbb{O}$ defined by 1, the goal of the problem is to design a transition map $f: \mathbb{O} \to [-D, D]$, for which the potential function $\Psi(D, x)$, defined by 2, is bounded for any rational $D$ ($1 < D < \infty$) and any real $x \in [-D, D]$.

\textbf{Theorem 2.} Problem 3 is solvable.
Algorithm 1 (for convergence)

1: loop
2: observe (SIDE, d) or qMid
3: if SIDE = L then
4: if d ∈ ∆k then
5: move to the right by distance \( \frac{1}{\sqrt{k+1}} \)
6: else
7: move to the point distance \( \frac{1}{\sqrt{2}} \) right from the left-end
8: end if
9: else if SIDE = R then
10: move to the left by distance 1
11: else
12: (i.e., qMid is observed) stay there
13: end if
14: end loop

Proof. The proof is constructive. We define a transition function \( f : \mathcal{O} \to [-D, D] \) to solve Problem 3 by

\[
\begin{align*}
    f((L, d)) &= \begin{cases} 
        x + \frac{1}{\sqrt{k+1}} & \text{if } d = \sigma_k \text{ for some } k \in \mathbb{Z}_{\geq 0} \\
        x - d & \text{if } d \neq \sigma_k \text{ for any } k \in \mathbb{Z}_{\geq 0}
    \end{cases} \\
    f((R, d)) &= \begin{cases} 
        x - \min \left\{ \frac{\sigma_k - d}{2}, d \right\} & \text{if } d + \sigma_k \in \mathbb{Q} \text{ for some } k \in \mathbb{Z}_{> 0} \\
        x - d & \text{if } d + \sigma_k \notin \mathbb{Q} \text{ for any } k \in \mathbb{Z}_{> 0}
    \end{cases} \\
    f(q_{\text{Mid}}) &= x
\end{align*}
\]

in each case of \( \phi(D, x) = (L, d), (R, d) \) or \( q_{\text{Mid}} \) for any \( x \in [-D, D] \) (see also Algorithm 2). It is not difficult to observe that \( f \) is a transition function (recall Problem 1). For convenience, let \( (\text{SIDE}(t), d(t)) = \phi(D, x_t) \).

First, we show for any \( x_0 \in [-D, 0) \) that a finite \( n \in \mathbb{Z}_{> 0} \) exists such that \( x_n = 0 \) where \( x_t = f(\phi(D, x_{t-1})) \) for \( t = 1, 2, \ldots \). If \( d(0) \neq \sigma_k \) for any \( k \in \mathbb{Z}_{> 0} \), then \( x_1 = -D \), meaning that \( d(1) = 0 = \sigma_0 \), thus it is reduced to the case \( d(0) = \sigma_k \) for some \( k \). We also remark that \( \text{SIDE}(t) = L \) and \( d(t) = \sigma_k \) imply that \( x_t = -D + \sigma_k \). Suppose that \( \text{SIDE}(t) = L \) and \( x_t = -D + \sigma_k \) then \( x_{t+1} = -D + \sigma_k + 1/\sqrt{k+1} = -D + \sigma_{k+1} \). This implies that we have a finite \( \tau = \min \{ t' \in \mathbb{Z}_{> 0} \mid \text{SIDE}(t') = R \} \) since \( \lim_{k \to \infty} \sigma_k = \infty \) by Proposition 2. Notice that \( x_{\tau} = -D + \sigma_{k'} \), for some \( k' \in \mathbb{Z}_{> 0} \) where \( k' \) is uniquely determined by Proposition 3. Furthermore, \( d(\tau) + \sigma_{k'} = (D - x_{\tau}) + \sigma_k = 2D \in \mathbb{Q} \) by the hypothesis \( D \in \mathbb{Q} \). Therefore, \( x_{\tau} - \frac{\sigma_{k'} - d(\tau)}{2} = 0 \) holds, and we obtain the claim in this case.

Next, we are concerned with the case that \( x_0 \in (0, D) \), and show that there is \( t \in \mathbb{Z}_{> 0} \) such that \( x_t \leq 0 \), then it is reduced to the case that \( x_0 \in [-D, 0) \), or the trivial case \( x_0 = 0 \). Notice that if \( d(s) + \sigma_k \notin \mathbb{Q} \) then \( d(s+1) = 2d(s) \), which implies that if the case occurs at most finite times, we eventually obtain
the desired case that $x_t < 0$. In fact, we claim that the case occurs at most once before $x_t < 0$. Without loss of generality, we may assume that $d(0) + \sigma_k \in \mathbb{Q}$, then we claim that $d(s) + \sigma_i \notin \mathbb{Q}$ for any $s \in \{t \in \mathbb{Z}_{>0} | \forall t' \leq t, x_{t'} > 0\}$ and for any $i \in \mathbb{Z}_{\geq 0}$. By the definition of $f$, if $\frac{\sigma_k - d(0)}{2} \leq d(0)$ then

$$
d(1) = D - x_1
= D - \left( x_0 - \frac{\sigma_k - d(0)}{2} \right)
= d(0) + \frac{\sigma_k - d(0)}{2}
= \frac{d(0) + \sigma_k}{2}
$$

and hence $d(1) = d(0) + \sigma_k \in \mathbb{Q}$ by the hypothesis of the case. This implies that $d(1) + \sigma_i \notin \mathbb{Q}$ for any $i = 1, 2, 3, \ldots$. Clearly, $d(2) = 2d(1) \in \mathbb{Q}$, and recursively we obtain the claim. \hfill \Box

5 Relaxation 3: With a Single-bit Memory

Memoryless is definitely a property which makes the problem difficult because Problem 1 is easily solved if the robot has enough memory (recall Section 1.2). Interestingly, this section shows that only a single-bit memory is sufficient for a self-stabilizing localization of the midpoint. The problem, with which this section is concerned, is formally described as follows.

**Problem 4 (With a single-bit memory).** As given an observation function $\phi : \mathbb{R} \times [-D, D] \rightarrow \mathcal{O}$ defined by (1), the goal of the problem is to design a transition map with memory $f : \mathcal{O} \times \{0, 1\} \rightarrow [-D, D] \times \{0, 1\}$ for which an integer $n \ (0 \leq n < \infty)$ exists for any real $D \ (1 < D < \infty)$, real $x_0 \in [-D, D]$ and $b_0 \in \{0, 1\}$ such that $x_n = 0$ where $(x_{i+1}, b_{i+1}) = f(\phi(D, x_i), b_i)$ for $i = 0, 1, 2, \ldots$.

**Theorem 3.** Problem 4 is solvable.
Algorithm 2 (rational $D$)

1: loop
2: observe (SIDE, $d$) or $q_{\text{mid}}$
3: if SIDE = $L$ then
4: if $d = \sigma_k$ for some $k = 0, 1, 2, \ldots$ then
5: move to the right by distance $1/\sqrt{\sigma_{k+1}}$
6: else
7: move to the left-end
8: end if
9: else if SIDE = $R$ then
10: if $d + \sigma_k \in \mathbb{Q}$ for some $k = 1, 2, \ldots$, and $\frac{\sigma_k - d}{2} \leq d$ then
11: move to the left by distance $\frac{\sigma_k - d}{2}$
12: else
13: move to the left by distance $d$
14: end if
15: else
16: (i.e., $q_{\text{mid}}$ is observed) stay there
17: end if
18: end loop

Proof. The proof is constructive. We define a transition function $f: \mathcal{O} \times \{0, 1\} \rightarrow [-D, D] \times \{0, 1\}$ to solve Problem 4 by

$$f((L, d), b) = \begin{cases} (x - d, 0) & \text{if } d \notin \mathbb{Z}_{\geq 0} \\ (x + 1, (d + 1) \mod 2) & \text{if } d \in \mathbb{Z}_{\geq 0} \end{cases}$$

$$f((R, d), b) = \begin{cases} \left( \frac{D - d + [d]}{2}, (b + 1) \mod 2 \right) & \text{if } b \equiv [d] \pmod{2} \\ \left( \frac{D - d + [d] + 1}{2}, (b + 1) \mod 2 \right) & \text{if } b \not\equiv [d] \pmod{2} \end{cases}$$

$$f(q_{\text{mid}}, b) = (x, b)$$

in each case of $\phi(D, x) = (L, d)$, $(R, d)$ or $q_{\text{mid}}$ for any $x \in [-D, D]$ (see also Algorithm 3). It is not difficult to observe that $f$ is a transition function (recall Problem 1), especially considering that $D = x + d$ when $\phi(D, x) = (R, d)$.

First, we show for any $x_0 \in [-D, 0)$ that a finite $n \in \mathbb{Z}_{>0}$ exists such that $x_n = 0$ where $(x_t, b_t) = f(\phi(D, x_{t-1}), b_{t-1})$ for $t = 1, 2, \ldots$. For convenience, let $(\text{SIDE}(t), d(t)) = \phi(D, x_t)$. Let $\tau = \min\{t \in \mathbb{Z}_{>0} \mid \text{SIDE}(t) = R\}$. Then, we can observe that $x_\tau = -D + \lfloor D \rfloor$, and hence $d(\tau) = D - x_\tau = D - (-D + \lfloor D \rfloor) = 2D - \lfloor D \rfloor$. Thus, $[d(\tau)] = [2D - \lfloor D \rfloor] = [2D] - [D]$, meaning that $[d(\tau)] + [D] = [2D]$. It is not difficult from the property of the ceiling function to see that $2[D] - 1 \leq [2D] \leq 2[D]$ holds. Note that $b \equiv [D] \pmod{2}$, then

$$[d(\tau)] = \begin{cases} \lfloor D \rfloor & \text{if } b \equiv [d(\tau)] \pmod{2} \\ \lfloor D \rfloor - 1 & \text{if } b \not\equiv [d(\tau)] \pmod{2} \end{cases}$$
holds. Since \( d(\tau) = 2D - \lfloor D \rfloor \),
\[
D = \begin{cases} 
\frac{d(\tau) + \lfloor d(\tau) \rfloor}{2} & \text{if } b \equiv \lfloor d(\tau) \rfloor \pmod{2} \\
\frac{d(\tau) + \lfloor d(\tau) \rfloor + 1}{2} & \text{if } b \not\equiv \lfloor d(\tau) \rfloor \pmod{2}
\end{cases}
\]
holds. Now it is not difficult to observe that we obtain \( x_{\tau+1} = 0 \) by the definition of \( f \).

Next, we claim that if \( \text{SIDE}(t) = R \) then there is \( t' \) \((t' > t)\) such that \( \text{SIDE}(t') = L \) or \( x_{t'} = 0 \), meaning that it is reduced to the case \( x_0 \leq 0 \). In fact, we show that \( x_{t+3} \leq x_t - \frac{1}{2} \) holds for any \( t \) as long as \( \text{SIDE}(t) = \text{SIDE}(t+1) = \text{SIDE}(t+2) = R \), and hence it implies the claim. We remark that \( x_{t+1} \leq x_t \) holds when \( \text{SIDE}(t) = R \) by the definition of the transition function \( f \). Suppose \( \text{SIDE}(t) = \text{SIDE}(t+1) = \text{SIDE}(t+2) = R \). In case that \( b(s) \not\equiv \lfloor d(s) \rfloor \pmod{2} \) holds for some \( s \in \{ t, t+1, t+2 \} \), then \( x_{s+1} = D - \frac{d(s) + \lfloor d(s) \rfloor + 1}{2} \leq D - d(s) - \frac{1}{2} = x(s) - \frac{1}{2} \), and we obtain the claim in the case. In the other case, i.e., \( b(s) \equiv \lfloor d(s) \rfloor \pmod{2} \) hold for each \( s \in \{ t, t+1, t+2 \} \). Since the parities of \( b(t) \), \( b(t+1) \) and \( b(t+2) \) alternately changes, the parities of \( \lfloor d(t) \rfloor \), \( \lfloor d(t+1) \rfloor \) and \( \lfloor d(t+2) \rfloor \) alternately changes, too. This implies \( \lfloor d(t) \rfloor \equiv \lfloor d(t+2) \rfloor \pmod{2} \) but \( \lfloor d(t) \rfloor \not\equiv \lfloor d(t+2) \rfloor \). Accordingly, \( d(t+2) - d(t) > 1 \) holds in the case. We obtain the claim.

\[\square\]

6 Impossibility of a Symmetric Algorithm

We conjecture Problem [¶] is unsolvable under some appropriate axiomatic system. This section gives an easy impossibility theorem for Problem [¶] assuming a (very strong) condition. We say a transition function is symmetric if \( f(\phi(D, -x)) = -f(\phi(D, x)) \) holds for any \( x \in [-D, D] \) for any \( D \in \mathbb{R} \).

**Theorem 4.** No symmetric algorithm solves Problem [¶].

*Proof.* Assume for a contradiction that \( f \) is a symmetric transition function which solves Problem [¶]. Then, there is \( x^* \in [-D, D] \setminus \{ 0 \} \) such that \( f(\phi(D, x^*)) = 0 \), meaning that \( \Psi(D, x^*) = 1 \). Since \( f \) is symmetric, \( f(\phi(D, -x^*)) = -f(\phi(D, x^*)) = 0 \) holds, too. Thus, we may assume \( x^* > 0 \) without loss of generality.

Here, we remark on an observation function that \( \phi(D - u, x - u) = \phi(D, x) \) holds for any \( D, x \) and \( u \) \((u < x)\) when \( x > 0 \), as well as that \( \phi(D - u, x + u) = \phi(D, x) \) when \( x < 0 \). Since \( f \) is a transition function, meaning that \( f(\phi(D, x)) - x \) is independent of \( x \),
\[
f\left( \phi\left( D - \frac{x^*}{2}, x^* - \frac{x^*}{2} \right) \right) - \left( x^* - \frac{x^*}{2} \right) = f\left( \phi(D, x^*) \right) - \frac{x^*}{2} = -\frac{x^*}{2} \tag{8}
\]
Algorithm 3 (with a single-bit memory)

1: given initial memory bit \( b \in \{0, 1\} \) (adversarially) arbitrarily
2: loop
3: observe (SIDE, \( d \)) or \( q_{\text{Mid}} \)
4: if SIDE = L then
5: \hspace{1em} if \( d \in \mathbb{Z} \) then
6: \hspace{2em} move to the right by distance 1
7: \hspace{2em} set \( b := d + 1 \pmod{2} \)
8: \hspace{1em} else
9: \hspace{2em} move to the left-end
10: \hspace{2em} set \( b := 0 \)
11: \hspace{1em} end if
12: else if SIDE = R then
13: \hspace{1em} if \( b \equiv \lceil d \rceil \pmod{2} \) then
14: \hspace{2em} move to the left by distance \( \frac{d + \lceil d \rceil}{2} \)
15: \hspace{2em} set \( b := b + 1 \pmod{2} \)
16: \hspace{1em} else
17: \hspace{2em} move to the left by distance \( \frac{d + \lceil d \rceil + 1}{2} \)
18: \hspace{2em} set \( b := b + 1 \pmod{2} \)
19: \hspace{1em} end if
20: \hspace{1em} else
21: \hspace{2em} (i.e., \( q_{\text{Mid}} \) is observed) stay there
22: \hspace{1em} end if
23: end loop

Fig. 4. Impossibility by a symmetric algorithm
holds by the assumption \( f(\phi(D, x^*)) = 0 \). On the other hand,

\[
f \left( \phi \left( D - \frac{x^*}{2}, \frac{x^*}{2} \right) \right) = -f \left( \phi \left( D - \frac{x^*}{2}, \frac{x^*}{2} \right) \right) = \frac{x^*}{2}
\]

holds since the assumption that \( f \) is symmetric. It is not difficult to see that (8) and (9) imply \( \Psi(D - \frac{x^*}{2}, \frac{x^*}{2}) = \Psi(D - \frac{x^*}{2}, -\frac{x^*}{2}) = \infty \). Contradiction. \( \Box \)

7 Concluding Remark

Motivated by the theoretical difficulty of self-stabilization of autonomous mobile robots with limited visibility, this paper is concerned with a very simple localization problem. The techniques used in Sections 3 and 4 are theoretically interesting, and may indicate why the impossibility proofs of this topic are often difficult. On the other hand, the parity tricks used in Section 5 for a robot with a single-bit memory could be reasonably simple and practically useful.

Problem 1 remains as unsettled, and we conjecture that it is unsolvable under an appropriate axiom system. There are many possible variants of Problem 1. A mathematically interesting version is a restriction to the rational interval, formally described as follows.

Problem 5 (Rational interval). As given an observation function \( \phi: \mathbb{Q} \times [-D, D]_Q \rightarrow O \), the goal is to design a rational transition function \( f: O \rightarrow [-D, D]_Q \) such that the potential function \( \Psi(D, x) \) is bounded for any rational \( D \) (1 < \( D < \infty \)), and rational \( x \in [-D, D]_Q \), where \([-D, D]_Q \) denotes \([-D, D] \cap \mathbb{Q}\).

For the version, a diagonal argument may work.

Clearly, self-stabilizing coverage, spreading, pattern formation etc. by many robots with limited visibility are important future works.

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