COARSE OBSTRUCTIONS TO POSITIVE SCALAR CURVATURE METRICS IN
NONCOMPACT ARITHMETIC MANIFOLDS

STANLEY S. CHANG

Abstract. Block and Weinberger show that an arithmetic manifold can be endowed with a positive scalar curvature metric if and only if its \( \mathbb{Q} \)-rank exceeds 2. We show in this article that these metrics are never in the same coarse class as the natural metric inherited from the base Lie group. Furthering the coarse \( C^* \)-algebraic methods of Roe, we find a nonzero Dirac obstruction in the \( K \)-theory of a particular operator algebra which encodes information about the quasi-isometry type of the manifold as well as its local geometry.

I. Introduction

In the course of showing that no manifold of non-positive sectional curvature can be endowed with a metric of positive scalar curvature, Gromov and Lawson \[12\] were led to consider what we would now call restrictions on the coarse equivalence type of complete noncompact manifolds of such positively curved metrics. In particular, they showed that such metrics cannot exist in manifolds for which there exists a degree one proper Lipschitz map from the universal cover to \( \mathbb{R}^n \), now understood to be essentially a coarse condition. Block and Weinberger \[4\] investigate the situation in which no coarse conditions are imposed upon the complete metric, focusing on quotients \( \Gamma \backslash G/K \) of symmetric spaces associated to a lattice \( \Gamma \) in an irreducible semisimple Lie group \( G \). They show that the space \( M = \Gamma \backslash G/K \) can be given a complete metric of uniformly positive scalar curvature \( \kappa \geq \varepsilon > 0 \) if and only if \( \Gamma \) is an arithmetic group of \( \mathbb{Q} \)-rank exceeding 2.

Note that the theorem of Gromov and Lawson \[12\] mentioned above establishes this theorem in the case of \( \text{rank}_G \Gamma = 0 \). In the higher rank cases, for which the resulting quotient space is noncompact, the metrics constructed by Block and Weinberger are however wildly different in the large when compared to the natural one on \( M \) inherited from the base Lie group \( G \). In fact, their examples are all coarse quasi-isometric to rays. Their theory evokes a natural question: Can the metric be chosen so that it is simultaneously uniformly positively curved and coarsely equivalent to the natural metric induced by \( G \)?

One of the important developments in analyzing positive scalar curvature in the context of non-compact manifolds, especially when restricted to the coarse quasi-isometry type, is introduced by Roe \[21\], \[20\], who considers a higher index, analogous to the Novikov higher signature, that lives naturally in the \( K \)-theory of the \( C^* \)-algebra \( C^*(M) \) of operators on \( M \) with finite propagation speed. He describes a map from the \( K \)-theory group \( K_*(C^*(M)) \) to the \( K \)-homology \( K_*(\nu M) \) of the Higson corona space which admits a dual transgression map \( H^*(\nu M) \rightarrow HX^*(M) \). If the Dirac operator on \( M \) is invertible, then the image of its index in \( K_*(\nu M) \) vanishes, leading to vanishing theorems for the index paired with coarse classes from the transgression of \( \nu M \). Roe’s
II. The Generalized Roe Algebra

The coarse category is defined to contain metric spaces as its objects and maps \( f : (X,d_X) \rightarrow (Y,d_Y) \) between metric spaces as its morphisms satisfying the following expansion and properness conditions:

- Expansion: For every point \( x \in X \), there exists a neighborhood \( U_x \) such that for every point \( y \in U_x \), there exists a neighborhood \( V_y \) such that \( d_X(x,y) \geq d_Y(f(x),f(y)) \).
- Properness: For every compact subset \( K \) of \( Y \), the preimage \( f^{-1}(K) \) is bounded in \( X \).

The usual Roe algebra, however, is unsuited to provide information about the existence of positive scalar curvature metrics that exist on arithmetic manifolds, because their coronae are too anemic. For example, the space at infinity of a product of punctured two-dimensional tori is a simplicial complex and therefore contractible. As a coarse object, the \( K \)-theory of the Roe algebra associated to this multi-product space can be identified with \( K_\ast(C^\ast(\mathbb{R}_0^\geq n)) \). Yet Higson, Roe and Yu [15] have shown that the Euclidean cone \( cP \) on a single simplex \( P \) must satisfy \( K_\ast(C^\ast(cP)) = 0 \). Since the Euclidean hyperoctant \( \mathbb{R}_0^\geq n \) is simply the cone on an \((n-1)\)-simplex, we find that \( K_\ast(C^\ast(\mathbb{R}_0^\geq n)) \) is the trivial group and hence no obstructions are detectable. Even by considering the fundamental group of the manifold by tensoring the Roe algebra with \( C^\ast\pi_1(M) \) is this detection process unfruitful, since the \( K \)-theory group \( K_\ast(C^\ast(M) \otimes C^\ast\pi_1(M)) \) vanishes as well. What seems to be critical is how different elements of the fundamental group at infinity can be localized to different parts of the space at infinity.

In this article, we shall provide coarse indicial obstructions in the following noncompact manifolds: a finite product of punctured two-dimensional tori, a finite product of hyperbolic manifolds, and more generally the double quotient space \( \Gamma \backslash G/K \), where \( G \) is an irreducible semisimple Lie group, \( K \) its maximal compact subgroup and \( \Gamma \) an arithmetic subgroup of \( G \). Note that the first two do not correspond to irreducible quotients, but an analysis of these spaces gives us the proper insight to attack the more general cases. A further research project will analyze this problem without the irreducibility assumption. The key feature in these particular manifolds \( M \) is that they contain hypersurfaces \( V \) that are coarsely equivalent to a product \( E \times U \) of Euclidean space \( E \) with some iterated circle bundle \( U \) (i.e. a torus, Heisenberg group, or more generally a group of unipotent matrices). Moreover such a hypersurface decomposes the manifold \( M \) into a coarsely excise pair \((A,B)\) for which \( A \cup B = M \) and \( A \cap B = V \). A generalized form of the Mayer-Vietoris sequence constructed by Higson, Roe and Yu [15] provides the following:

\[ \cdots \rightarrow K_\ast(C_G^\ast(A)) \oplus K_\ast(C_G^\ast(B)) \rightarrow K_\ast(C_G^\ast(M)) \rightarrow K_{\ast-1}(C_G^\ast(V)) \rightarrow \cdots \]

The boundary map \( \partial : K_\ast(C_G^\ast(M)) \rightarrow K_{\ast-1}(C_G^\ast(E \times U)) \) sends \( \text{Ind}_{M}(D) \), the index of the spinor Dirac bundle on the universal cover lifted from that on \( M \), to \( \text{Ind}_{E \times U}(D) \). To see that these indices are indeed nonzero, we note that there is a boundary map \( K_{\ast-1}(C_G^\ast(E \times U)) \rightarrow K_{\ast-\dim E}(C_G^\ast(\mathbb{R} \times U)) \), which sends index to index. We show that the index of the Dirac operator in the latter group, however, is nonzero by noting that the Gromov-Lawson-Rosenberg conjecture is true for nilpotent groups and hence provides an appropriate nonzero obstruction.

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II. The Generalized Roe Algebra

The coarse category is defined to contain metric spaces as its objects and maps \( f : (X,d_X) \rightarrow (Y,d_Y) \) between metric spaces as its morphisms satisfying the following expansion and properness conditions:
conditions: (a) for each \( R > 0 \) there is a corresponding \( S > 0 \) such that, if \( d_X(x_1, x_2) \leq R \) in \( X \), then \( d_Y(f(x_1), f(x_2)) \leq S \), (b) the inverse image \( f^{-1}(B) \) under \( f \) of each bounded set \( B \subseteq Y \) is also bounded in \( X \). Such a function will be designated a coarse map, and two coarse maps \( f, g : X \to Y \) are said to be coarsely equivalent if their mutual distance of separation \( d_Y(f(x), g(x)) \) is uniformly bounded in \( x \). Naturally two metric spaces are coarsely equivalent if there exist maps from one to the other whose compositions are coarsely equivalent to the appropriate identity maps. Two metrics \( g_1 \) and \( g_2 \) on the same space \( M \) are said to be coarsely equivalent if \((M,g_1)\) and \((M,g_2)\) are coarsely equivalent metric spaces.

Following Roe [20], we recall a Hilbert space \( H \) is an \( M \)-module for a manifold \( M \) if there is a representation of \( C_0(M) \) on \( H \), that is, a \( C^* \)-homomorphism \( C_0(M) \to B(H) \). We will say that an operator \( T : H \to H \) is locally compact if, for all \( \varphi \in C_0(M) \), the operators \( T\varphi \) and \( \varphi T \) are compact on \( H \). We define the support of \( \varphi \) in an \( M \)-module \( H \) to be the smallest closed set \( K \subseteq M \) such that, if \( f \in C_0(M) \) and \( f\varphi \neq 0 \), then \( f|_K \) is not identically zero. Consider the \( M \)-module \( H = L^2(\tilde{M}) \), where \( \tilde{M} \) is the universal cover of \( M \) endowed with the appropriate metric lifted from the base space. Let \( \pi : \tilde{M} \to M \) be the usual projection map and for any \( \varphi, \psi \in C_0(\tilde{M}) \) consider the collection \( \Gamma(\varphi, \psi) \) of paths \( \gamma : [0, 1] \to \tilde{M} \) originating in \( \operatorname{Supp}(\varphi) \) and ending in \( \operatorname{Supp}(\psi) \). Denote by \( L[\gamma] \), for \( \gamma \in \Gamma(\varphi, \psi) \), the maximum distance of any two points on the projection of the curve \( \gamma \) in \( M \) by \( \pi \), i.e. \( L[\gamma] = \sup_{x,y \in [0,1]} d(\pi \circ \gamma(x), \pi \circ \gamma(y)) \).

**Definition:** Let \( M \) be a manifold with universal cover \( \tilde{M} \). We say that an operator \( T \) on \( L^2(\tilde{M}) \) has generalized finite propagation if there is a constant \( R > 0 \) such that \( \varphi T\psi \) is identically zero in \( B(H) \) whenever \( \varphi, \psi \in C_0(\tilde{M}) \) satisfies

\[
\inf_{\gamma \in \Gamma(\varphi, \psi)} L[\gamma] > R.
\]

The infimum of all such \( R \) will be the generalized propagation speed of the operator \( T \). If \( G = \pi_1(M) \) is the fundamental group of \( M \), we denote by \( D^*_G(M) \) to be the norm closure of the \( C^* \)-algebra of all locally compact, \( G \)-equivariant, generalized finite propagation operators on \( H \).

Let \( M \) be a manifold and \( \tilde{M} \) its universal cover. Let \( T : H \to H \) be an operator on \( H = L^2(\tilde{M}) \). Consider the subset \( Q \subseteq \tilde{M} \times \tilde{M} \) of pairs \((m, m')\) for which there exist functions \( \varphi, \psi \in C_0(\tilde{M}) \) such that \( \varphi(m) \neq 0, \psi(m') \neq 0 \) and \( \varphi T\psi \) does not identically vanish. We will say that the support of \( T \) is the complement in \( \tilde{M} \times \tilde{M} \) of \( Q \). For such two points \( m, m' \in \tilde{M} \), let \( \gamma_{mm'} : [0, 1] \to \tilde{M} \) be the path of least length joining \( m \) and \( m' \) in \( \tilde{M} \). We consider the projection of this path into \( M \) by \( \pi \) and take the greatest distance between two points on this projected path. Then it is easy to see that an operator \( T \) has generalized finite propagation, as previously defined, if

\[
\sup_{m,m'} \sup_{x,y \in [0,1]} d(\pi \circ \gamma_{mm'}(x), \pi \circ \gamma_{mm'}(y)) < \infty.
\]

**Definition:** Consider the norm closure \( I \) of the ideal in \( D^*_G(M) \) generated by operators \( T \) whose matrix representation, parametrized by \( M \times \tilde{M} \), satisfies the condition that \( (\pi \times \pi)(\operatorname{Supp} T) \) is bounded in \( M \times M \). Then the generalized Roe algebra, denoted by \( C^*_G(M) \), is obtained as the quotient \( D^*_G(M)/I \). Two operators in \( D^*_G(M) \) belong to the same class in \( C^*_G(M) \) if their nonzero entries differ on at most a bounded set when viewed from the perspective of the base space.

**Examples:**
(1) Let $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be operator on $L^2$-functions on the real line given by $(Tg)(x) = g(x + 1)$ for all $g \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$. Then for any $\varphi, \psi \in C_0(\mathbb{R})$, $(\varphi T \psi)g(x) = \varphi(x + 1)g(x + 1)\psi(x)$. If $\varphi$ is supported at $m = 1$ and $\psi$ is supported at $m' = 0$, then $(\varphi T \psi)g$ is nonzero for any $g$ supported at $x = 1$. Hence $(0, 1) \in \text{Supp} T$. It is easy to see that $(m, m') \in \text{Supp} T$ if and only if $m' - m = 1$. The propagation speed of $T$ is $1$. If we write $T$ as a matrix parametrized by $\mathbb{R} \times \mathbb{R}$, all the nonzero entries will lie at distance one from the diagonal.

(2) Let $M$ be the cylinder $S^1 \times \mathbb{R}$ with its universal cover $\tilde{M} = \mathbb{R}^2$. An operator in the algebra $D_G^*(M)$ will be some $T : H \to H$ on $L^2(\mathbb{R}^2)$, which is of finite propagation speed (in the usual sense) in the direction projecting down to the noncompact direction in $\tilde{M}$, but has no such condition in the orthogonal direction corresponding to the compact direction of $M$. In this direction, however, the operator is controlled by the condition that it be $\mathbb{Z}$-equivariant. It is apparent that the operator, when restricted to individual fibers, has finite propagation speed, there is no requirement that the speed to be uniformly bounded across all fibers.

(3) Let $M = [\mathbb{R}^n, n \geq 3$, the once-punctured real projective space, expressible as the quotient $(S^{n-1} \times \mathbb{R})/\mathbb{Z}_2$. Certainly $M$ is coarsely equivalent to the ray $[0, \infty)$ and is covered by the space $\tilde{M} = S^{n-1} \times \mathbb{R}$, where the points $(s, r)$ and $(-s, -r)$ are identified by the projection map to $M$. Let $T : L^2(\tilde{M}) \to L^2(\tilde{M})$ be given by the reflection $(Tf)(s, r) = f(s, -r)$. Consider $\varphi_i, \psi_i \in C_0(\tilde{M})$ compactly supported on $S^{n-1} \times [-i - 1, -i]$ and $S^{n-1} \times [i, i + 1]$, respectively. Notice that $\varphi T \psi$ will never be identically zero, and yet the length $L_i[\gamma]$ associated to $\varphi_i$ and $\psi_i$ will always be at least $i$. Hence the operator $T$ is not of generalized finite propagation speed and therefore not an element of the generalized Roe algebra $C_G^*(M)$.

**Lemma 1:** Let $D$ a generalized elliptic operator in $L^2(M, S)$. Suppose that $\tilde{D}$ is the lifted operator on $\tilde{M}$. If $\Phi : \mathbb{R} \to \mathbb{R}$ is compactly supported, then $\Phi(\tilde{D})$ lies in the generalized Roe algebra $C_G^*(M)$.

**Proof:** (cf. [20], [11]) Suppose that $\Phi$ has compactly supported Fourier transform and denote by $\hat{\Phi}$ the Fourier transform of $\Phi$. We may write

$$
\Phi(\tilde{D}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\Phi}(t) e^{it\tilde{D}} dt.
$$

It is known that $e^{it\tilde{D}}$ has finite propagation speed, and since $\hat{\Phi}$ is compactly supported, the integral is defined and has a generalized propagation bound. Moreover, by construction $\tilde{D}$ is $\pi_1(M)$-equivariant. So $\Phi(\tilde{D})$ is $\pi_1(M)$-equivariant as well. Therefore if $\Phi$ is compactly supported, then $\Phi(\tilde{D})$ lies in $D_G^*(M)$ and passes to an element of the quotient $C_G^*(M)$. However, functions with compactly supported Fourier transform form a dense set in $C_0(\mathbb{R})$ and the functional calculus map $f \mapsto f(\tilde{D})$ is continuous, so the result holds for all $\Phi \in C_0(\mathbb{R})$.

Let $\chi : \mathbb{R} \to \mathbb{R}$ be a chopping function on $\mathbb{R}$, i.e. an odd continuous function with the property that $\chi(x) \to \pm 1$ as $x \to \pm \infty$. In addition, denote by $B_G^*(M)$ the multiplier algebra of $C_G^*(M)$, that is, the collection of all operators $S$ such that $ST$ and $TS$ belong to $C_G^*(M)$ for all $T \in C_G^*(M)$. Then $B_G^*(M)$ contains $C_G^*(M)$ as an ideal. If $D$ is a generalized elliptic operator on $\tilde{M}$ and $\tilde{D}$ its lift to $\tilde{M}$, then $\chi(\tilde{D})$ belongs to $B_G^*(M)$. In addition, since $\chi^2 - 1 \in C_0(\mathbb{R})$, we have
\( \chi(\tilde{D})^2 - 1 \in C^*_G(M) \). Moreover, since the \( \mathbb{Z}_2 \)-grading renders the decompositions

\[
\chi(\tilde{D}) = \begin{pmatrix}
0 & \chi(\tilde{D})_-
\chi(\tilde{D})_+ & 0
\end{pmatrix},
\varepsilon = \begin{pmatrix}
1 & 0
0 & -1
\end{pmatrix},
\]

it follows that \( \varepsilon \chi(\tilde{D}) + \chi(\tilde{D}) \varepsilon = 0 \). By the discussion in [20], it follows that \( F = \chi(\tilde{D}) \) is a Fredholm operator and admits an index \( \text{Ind} F \in K_0(C_G^*(M)) \). In addition, any two chopping functions \( \chi_1 \) and \( \chi_2 \) differ by an element of \( C_0(\mathbb{R}) \). By the lemma above, we have \( \chi_1(\tilde{D}) - \chi_2(\tilde{D}) \in C_G^*(M) \), so they define the same elements of \( K \)-theory. The common value for \( \text{Ind} F \) is denoted \( \text{Ind}(\tilde{D}) \) and called the \textit{generalized coarse index} of \( \tilde{D} \). We write \( C^*_G(M) \) and \( \text{Ind}(\tilde{D}) \) instead of \( C^*_G(M) \) and \( \text{Ind}(\tilde{D}) \) to indicate that the construction is initiated by a generalized Dirac operator on the base space. The following statements are standard results of index theory; one may consult [20] and [21] for the essentially identical proof in the nonequivariant case.

**Proposition 1**: Let \( D \) be a generalized elliptic operator in \( L^2(M, S) \). If \( 0 \) does not belong to the spectrum of \( \tilde{D} \), then the generalized coarse index \( \text{Ind} D \) vanishes in \( K_0(C^*_G(M)) \).

**Proposition 2**: Let \( \tilde{D} \) the lift of a generalized elliptic operator in \( L^2(\tilde{M}, S) \). In the ungraded case, if there is a gap in the spectrum of \( \tilde{D} \), then the index \( \text{Ind} D \) vanishes in \( K_1(C^*_G(M)) \).

**Corollary**: Let \( M \) be a complete spin manifold. If \( M \) has a metric of uniformly positive scalar curvature in some coarse class, then the generalized coarse index of the spinor Dirac operator vanishes.

We now embark on the task of computing the \( K \)-theory of this algebra and of coarse indices.

Let \( (M, d) \) be a proper metric space. For any subset \( U \subset M \) and \( R > 0 \), we denote by \( \text{Pen}(U, R) \) the open neighborhood of \( U \) consisting of points \( x \in M \) for which \( d(x, U) < R \). Let \( A \) and \( B \) be closed subspaces of \( M \) with \( M = A \cup B \). We then say that the decomposition \( (A, B) \) is a \textit{coarsely excisive pair} if for each \( R > 0 \) there is an \( S > 0 \) such that

\[
\text{Pen}(A, R) \cap \text{Pen}(B, R) \subseteq \text{Pen}(A \cap B, S).
\]

We wish to analyze this decomposition in the following context.

Given \( C^* \)-algebras \( A, B \) and \( \mathcal{M} \) for which \( \mathcal{M} = A + B \), we have the Mayer-Vietoris sequence

\[
\cdots \to K_{j+1}(\mathcal{M}) \to K_j(A \cap B) \to K_j(A) \oplus K_j(B) \to K_j(\mathcal{M}) \to \cdots
\]

The standard proof for the existence of such a sequence is developed from the isomorphism \( K_* (\mathcal{T}) \cong K_* -1(\mathcal{M}) \), where \( \mathcal{T} \) is the suspension of \( \mathcal{M} \). A short discussion of this construction is given in [15]. We are in particular interested in exploiting the boundary map \( \partial : K_j(\mathcal{M}) \to K_{j-1}(A \cap B) \) to transfer information about the index of the Dirac operator on a complete noncompact manifold \( M \) to information about that on some hypersurface \( V \). For our purposes, we wish to set \( \mathcal{M} \) to be the generalized Roe algebra \( C^*_G(M) \) on \( M \), while \( A \) and \( B \) represent analogous operator algebras on closed subsets \( A \) and \( B \), where \( (A, B) \) form a coarsely excisive decomposition of \( M \). To construct the boundary map in question, we require a few technical lemmas and notion of equivariant operators with generalized finite propagation on a subset of \( M \). The proof of the first lemma follows the same argument as that in [15] and is stated without proof.
**Definition:** Let \( A \) be a closed subspace of a proper metric space \( M \). Denote by \( D^*_G(A, M) \) the \( C^* \)-algebra of all operators \( T \) in \( D^*_G(M) \) such that \( \text{Supp} T \subseteq \text{Pen}(\pi^{-1}(A), R) \times \text{Pen}(\pi^{-1}(A), R) \), for some \( R > 0 \). Let \( C^*_G(A, M) \) be the quotient \( D^*_G(A, M)/I \).

**Lemma 2:** Let \((A, B)\) be a decomposition of \( M \). Then
1. \( C^*_G(A, M) + C^*_G(B, M) = C^*_G(M) \).
2. \( C^*_G(A, M) \cap C^*_G(B, M) = C^*_G(A \cap B, M) \) if in addition we assume that \((A, B)\) is coarsely excisive.

**Lemma 3:** Suppose \( V \subset M \) and \( \pi_1(V) \) injects into \( \pi_1(M) \). There is an isomorphism \( K_*(C^*_G(V)) \cong K_*(C^*_G(V, M)) \).

**Proof:** Let \( \pi: \tilde{M} \to M \) be the projection map. Consider the \( C^* \)-algebra \( C^*(\text{Pen}(\pi^{-1}(V), n), \pi_1(M)) \) given by the quotient by \( I \) of the \( C^* \)-algebra of locally compact, \( \pi_1(M) \)-equivariant operators on the \( n \)-neighborhood penumbra \( \text{Pen}(\pi^{-1}(V), n) \). Then
\[
C^*_G(V, M) = \lim_{\to} C^*(\text{Pen}(\pi^{-1}(V), n), \pi_1(M)).
\]

The inclusion map \( i: \pi^{-1}(V) \to \text{Pen}(\pi^{-1}(V), n) \) is a coarse equivalence. Since by the construction the generalized Roe algebra its operators are defined up to their bounded parts, the map \( i \) induces a series of isomorphisms
\[
K_*(C^*(\pi^{-1}(V), \pi_1(M))) \cong K_*(C^*(\text{Pen}(\pi^{-1}(V), n), \pi_1(M)) \cong K_*(\lim_{\to} C^*(\text{Pen}(\pi^{-1}(V), n), \pi_1(M)) \cong K_*(C^*_G(V, M)).
\]

Since \( \pi_1(V) \hookrightarrow \pi_1(M) \) is an injection, the inverse image \( \pi^{-1}(V) \subseteq \tilde{M} \) is a disjoint union of isomorphic copies of \( \tilde{V} \), parametrized by the coset space \( \pi_1(M)/\pi_1(V) \). Therefore, there is a one-to-one correspondence between \( \pi_1(M) \)-equivariant operators on \( \pi^{-1}(V) \) and \( \pi_1(V) \)-equivariant operators on \( \tilde{V} \). Hence \( C^*(\pi^{-1}(V), \pi_1(M)) \cong C^*_G(V) \). We then have \( K_*(C^*_G(V)) \cong K_*(C^*_G(V, M)) \), as desired.

Let \((A, B)\) be a coarsely excisive decomposition of \( M \) such that \( V = A \cap B \) satisfies \( \pi_1(V) \hookrightarrow \pi_1(M) \). The boundary operator \( \partial: K_1(C^*_G(A, M) + C^*_G(B, M)) \to K_{1-1}(C^*_G(A, M) \cap C^*_G(B, M)) \) arising from the coarse Mayer-Vietoris sequence is by the previous lemmas truly a map
\[
\partial: K_*(C^*_G(M)) \to K_{*-1}(C^*_G(V)).
\]

**Theorem: (Boundary of Dirac is Dirac)** Consider a coarsely excisive decomposition \((A, B)\) of \( M \) and let \( V = A \cap B \). If \( \partial: K_*(C^*_G(M)) \to K_{*-1}(C^*_G(V)) \) is the boundary map from the Mayer-Vietoris sequence derived above, then we have \( \partial (\text{Ind}_M(D)) = \text{Ind}_V(D) \).

**Remark:** Here \( \text{Ind}_M(D) \) and \( \text{Ind}_V(D) \) represent the generalized coarse indices of the spinor Dirac operators on \( M \) and \( V \), respectively. We will continue to use a subscript if the space to which the index is related is ambiguous. The “boundary of Dirac is Dirac” principle is essentially equivalent to Bott periodicity in topological \( K \)-theory. In all cases considered here, there are commutative diagrams relating topological boundary to the boundary operator arising in the \( K \)-theory of \( C^* \)-algebras, and on the topological side, a consideration of symbols suffices. See [22], [13], [21] and [52].
Theorem 1: The $n$-fold product $M$ of punctured two-dimensional tori does not have a metric of uniform positive scalar curvature in the same coarse equivalence class as the positive hyperoctant with its standard Euclidean metric.

Proof: Consider the projection map $p: M \to \mathbb{R}_{\geq 0}^n$ from the multifold product $M = \tilde{T} \times \cdots \times \tilde{T}$ to the positive hyperoctant, where each component $p_i$ is the quasi-isometric projection of the punctured torus onto the positive reals numbers. Take a hypersurface $S \subset \mathbb{R}_{\geq 0}^n$ sufficiently far from the origin so that the inverse image of every point on $S$ is an $n$-torus, and so the space $V$ is coarsely equivalent to the $(2n-1)$-dimensional noncompact manifold $\mathbb{R}^{n-1} \times T^n$. The complement of the hypersurface $V$ consists of two noncompact components. Define $A$ to be the closure of the component containing the inverse image of $p^{-1}(0)$ of the origin in $\mathbb{R}_{\geq 0}^n$. Take $B$ the closure of $M \setminus A^o$. Then the pair $(A, B)$ forms a coarsely excisive decomposition of the space $M$ whose intersection is $A \cap B = V$.

Consider the generalized coarse index $\text{Ind}_M(D) \in K_n(C^*_G(M))$ of the lifted classical Dirac operator on the pullback spinor bundle of the universal cover $M$. Note that $\pi_1(M)$ is the $n$-fold product $F_2 \times \cdots \times F_2$ of free groups, and that $\pi_1(V) \cong \pi_1(\mathbb{R}^{n-1} \times T^n) \cong \mathbb{Z}^n$. Hence there is an injection $\pi_1(V) \to \pi_1(M)$ and the $K$-theoretic Mayer-Vietoris sequence applies. The boundary map $\partial$ of this sequence satisfies $\partial(\text{Ind}_M(D)) = \text{Ind}_V(D) \in K_n(C^*_G(V))$. However, $V$ is coarsely equivalent to the hypersurface $\mathbb{R}^{n-1} \times T^n$, so the index $\text{Ind}_V(D)$ can be taken to live in $K_n(C^*_G(\mathbb{R}^{n-1} \times T^n))$. Note that $n$ will be taken to be at least 2. There is yet another boundary map $K_{n-1}((C^*_G(\mathbb{R}^{n-1} \times T^n)) \to K_{n-2}((C^*_G(\mathbb{R} \times T^n))$ by peeling off $n - 2$ copies of the real line. This boundary map (or composition of $n - 2$ boundary maps) preserves index.

Recall that $D^*_G(M)$ is the norm closure of the $C^*$-algebra of all locally compact, $\pi_1(M)$-equivariant, generalized finite propagation operators on $L^2(M)$, and $I \subseteq D^*_G(M)$ is the closure of the ideal of such operators $T$ that satisfies the condition that $(\pi \times \pi)(\text{Supp}T)$ is bounded in $M \times M$. The short exact sequence $0 \to I \to D^*_G(\mathbb{R} \times T^n) \to C^*_G(\mathbb{R} \times T^n) \to 0$ gives rise to the six-term exact sequence in $K$-theory:

\[
\begin{array}{ccccccc}
K_0(I) & \longrightarrow & K_0(D^*_G(\mathbb{R} \times T^n)) & \longrightarrow & K_0(C^*_G(\mathbb{R} \times T^n)) & \\
\downarrow & & \downarrow & & \downarrow & \\
K_1(C^*_G(\mathbb{R} \times T^n)) & \longleftarrow & K_1(D^*_G(\mathbb{R} \times T^n)) & \longleftarrow & K_1(I)
\end{array}
\]
Theorem 2: An $n$-fold product of hyperbolic manifolds has no uniform positive scalar curvature metric coarsely equivalent to the usual Euclidean metric on the positive Euclidean hyperoctant.

Proof: Without loss of generality, it suffices to consider the case when the noncompact hyperbolic spaces have only one cusp. Let $m$ be the dimension of this product manifold. As in the multifold product of tori, there is a positive $b \in \mathbb{R}$ such that on each hyperbolic space $\mathcal{H}_i$ the inverse image of each point $x \geq b$ under the projection $\mathcal{H}_i \to \mathbb{R}_{\geq 0}$ is by Margulis’ lemma a flat compact connected Riemannian manifold of finite dimension. Consider the inverse image $\mathcal{V}$ under the induced product map $p : \mathcal{H}_1 \times \cdots \times \mathcal{H}_m \to \mathbb{R}_{\geq 0}^m$ of the same hypersurface as described in the previous theorem. By Bieberbach’s theorem, every flat compact connected Riemannian manifold admits a normal Riemannian covering by a flat torus of the same dimension. Hence $\mathcal{V}$ is covered by some product of Euclidean space and a higher-dimensional torus. Any metric of positive scalar curvature on $\mathcal{V}$ would certainly lift to such a metric in this covering space. Using the same induction argument as before, we show that such a metric is obstructed by the presence of a nonzero Dirac class. \[\Box\]

III. Noncompact Quotients of Symmetric Spaces: A Special Case

The Iwasawa decomposition gives a unique way of expressing the group $\text{SL}_n(\mathbb{R})$ as a product $\text{SL}_n(\mathbb{R}) = NAK$, where $N$ is the subgroup of standard unipotent matrices (upper triangular matrices with all diagonal entries equal to 1), $A$ the subgroup of $\text{SL}_n(\mathbb{R})$ consisting of diagonal matrices with positive entries, and $K$ the orthogonal subgroup $\text{SO}_n(\mathbb{R})$. The effect of taking the double quotient of $\text{SL}_n(\mathbb{R})$ by both $\text{SO}_n(\mathbb{R})$ and $\text{SL}_n(\mathbb{Z})$ on the Iwasawa decomposition is that we
are left with classes of matrices represented by those of the form \( n^* a^* \), where \( n^* \) is represented by a unipotent matrix and \( a^* = \text{diag}(a_1, \ldots, a_n) \) is diagonal with weakly increasing entries \( a_1 \leq a_2 \leq \cdots \leq a_n \). Let \( N^* \) be the iterated circle bundle that arises upon taking a quotient of \( N \) by \( \text{SL}_n(\mathbb{Z}) \) (for example, when \( n = 3 \), the group \( N \) is a Heisenberg group). Let \( A^* \) be the semigroup of matrices (note that it is not closed under inversion) with such an increasing condition on the entries. Consider the map \( \rho : \text{SL}_n(\mathbb{Z})/\text{SL}_n(\mathbb{R})/\text{SO}_n(\mathbb{R}) \to A^* \) given by \( n^* a^* \mapsto a^* \). This map denotes a fiber bundle with fiber \( N^* \) over every point \( a^* \in A^* \). Notice that the Iwasawa decomposition gives another way of observing the dimension of \( \text{SL}_n(\mathbb{R})/\text{SO}_n(\mathbb{R}) \), the sum of \( n-1 \) dimensions from \( A \) and \( \frac{n(n-1)}{2} \) compact directions from \( N^* \).

The space \( A^* \) is identifiable with the subset of \((n-1)\)-dimensional Euclidean space given by \( \{(a_1, \ldots, a_n) : 0 < a_1 \leq \cdots \leq a_n, \ a_1 \cdots a_n = 1 \} \). We wish to construct a hypersurface in \( A^* \) whose inverse image under \( \rho \) is an iterated circle bundle over Euclidean space. We recall that a geodesic in the quotient space \( \text{SL}_n(\mathbb{R})/\text{SO}_n(\mathbb{R}) \) is of the form \( t \mapsto e^{tA} \cdot \text{SO}_n(\mathbb{R}) \), where \( \Lambda \) is an \( n \times n \) symmetric matrix with zero trace. We shall construct an appropriate hypersurface in \( A^* \) by taking the union of sufficiently many geodesics.

Consider a geodesic \( t \mapsto e^{tA} \cdot \text{SO}_n(\mathbb{R}) \) in the quotient \( \text{SL}_n(\mathbb{R})/\text{SO}_n(\mathbb{R}) \) where \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \) with \( \lambda_1 \leq \cdots \leq \lambda_n \) and \( \lambda_1 + \cdots + \lambda_n = 0 \). Then the geodesic is a map \( t \mapsto \text{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t}) \cdot \text{SO}_n(\mathbb{R}) \). Because two symmetric traceless matrices of the form \( \Lambda_1 = \text{diag}(\lambda_1, \ldots, \lambda_n) \) and \( \Lambda_2 = \text{diag}(\alpha \lambda_1, \ldots, \alpha \lambda_n) \) give the same geodesic image for \( \alpha > 0 \), we can normalize the \( \lambda \)-vector so that \( \lambda_1 = -1 \). In addition, let \( \mu_2 = 1 \) and \( \mu_i \in [i - 1, i] \) for \( i = 3, \ldots, n \) and let \( m = \sum_{i=2}^n \mu_i \). Set \( \lambda_i = \mu_i/m \). Then \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \) is symmetric and traceless with weakly increasing entries.

**Lemma 4:** Each ordered \((n-1)\)-tuple \((\mu_2, \ldots, \mu_n) \in [2, 3] \times \cdots \times [n, n+1] \) gives rise to a unique geodesic \( t \mapsto e^{tA} \cdot \text{SO}_n(\mathbb{R}) \), up to reparametrization.

**Proof:** Suppose \( (\mu_2^{(1)}, \ldots, \mu_n^{(1)}), (\mu_2^{(2)}, \ldots, \mu_n^{(2)}) \in [2, 3] \times \cdots \times [n, n+1] \) give rise to the same geodesic. The two vectors correspond to the traceless matrices \( \Lambda_1 = \text{diag}(-1, \lambda_2^{(1)}, \ldots, \lambda_n^{(1)}) \) and \( \Lambda_2 = \text{diag}(-1, \lambda_2^{(2)}, \ldots, \lambda_n^{(2)}) \), respectively. Let \( \nu_1 = (-1, \lambda_2^{(1)}, \ldots, \lambda_n^{(1)}) \) and \( \nu_2 = (-1, \lambda_2^{(2)}, \ldots, \lambda_n^{(2)}) \). Obtain the normalized matrices \( \Lambda_1^\ast \) and \( \Lambda_2^\ast \) by dividing the entries by the Euclidean norms \( ||\nu_1|| \) and \( ||\nu_2|| \). By assumption, \( \Lambda_1^\ast = \Lambda_2^\ast \); in other words, \( \frac{\nu_1}{||\nu_1||} = \frac{\nu_2}{||\nu_2||} \). But then \( \theta = \cos^{-1}\frac{\nu_1 \cdot \nu_2}{||\nu_1|| \cdot ||\nu_2||} = 0 \), so \( \nu_1 \) and \( \nu_2 \) are parallel. Since their first coordinates coincide, they are identical. Hence \( \frac{\mu_2^{(1)} \cdots \mu_n^{(1)}}{\mu_2^{(2)} + \cdots + \mu_n^{(2)}} = \frac{\mu_2^{(2)} \cdots \mu_n^{(2)}}{\mu_2^{(2)} + \cdots + \mu_n^{(2)}} \). Since each \( \mu_i^{(j)} \) is positive, it can be written as the square of some other positive number. Arguing as before, we see that \( (\mu_2^{(1)}, \ldots, \mu_n^{(1)}) = \beta (\mu_2^{(2)}, \ldots, \mu_n^{(2)}) \) for some \( \beta > 0 \). Since \( \mu_2^{(1)} \) and \( \mu_2^{(2)} \) both equal one, the vectors are coincident.

The space \( A^* \) of diagonal matrices in \( \text{SL}_n(\mathbb{R}) \) with weakly increasing entries is itself simply-connected of dimension \( n-1 \), and its space at infinity is an \((n-2)\)-dimensional simplex \( P \). Let \( W \) be the union of geodesics constructed above, with \( (\mu_2, \ldots, \mu_n) \) ranging in the product \([2, 3] \times \cdots \times [n, n+1] \) of intervals. The subset \( W' \subset A^* \) is an \((n-1)\)-dimensional space with boundary \( V' = \partial W' \). The \((n-2)\)-dimensional hypersurface \( V' \) is also simply-connected whose
space at infinity is homeomorphic to an \((n-3)\)-sphere that is disjoint with \(\partial P\). The space \(V'\) itself is coarsely equivalent to \(\mathbb{R}^{n-2}\).

**Theorem 3:** The double quotient group \(\text{SL}_n(\mathbb{Z})\setminus\text{SL}_n(\mathbb{R})/\text{SO}_n(\mathbb{R})\) does not have a uniform positive scalar curvature metric that is coarsely equivalent to the natural one inherited from \(\text{SL}_n(\mathbb{R})\).

**Proof:** Let \(M = \text{SL}_n(\mathbb{Z})\setminus\text{SL}_n(\mathbb{R})/\text{SO}_n(\mathbb{R})\) and recall the projection map \(\rho : M \to A^*\) given by \(n^*a^* \mapsto a^*\). Since the fiber over each point is an arithmetic quotient of the group of unipotent matrices, the inverse image \(V \equiv \rho^{-1}(V')\) is coarsely equivalent to an iterated circle bundle over Euclidean space. Moreover, \(V\) partitions the space into a coarsely excisive pair whose closures \((A, B)\) satisfy the equalities \(A\cup B = M\) and \(A\cap B = V'\) (note that \(B\) can be taken as \(\rho^{-1}(W')\) and \(A\) the closure of its complement). If \(\text{Ind}_M(D)\) denotes the generalized coarse index of the classical spinor Dirac operator on \(\tilde{M}\), then the Mayer-Vietoris map \(\partial : K_*(C^*_G(M)) \to K_{*-1}(C^*_G(V))\) defined in the previous chapter satisfies \(\partial(\text{Ind}_M(D)) = \text{Ind}_{V'}(D) = \text{Ind}_{\mathbb{R}^{n-2}\times U^m}(D)\), where \(U^m\) is the compact fiber of the iterated circle bundle of dimension \(m = \frac{n(n-1)}{2}\). Applying the same argument as before, it suffices to show that the index of the Dirac operator in \(K_*(D^*_G(\tilde{U}^m))\) is nonzero. However, \(\tilde{U}^m\) is a quotient of a nilpotent group with a cocompact lattice, and hence by Gromov and Lawson [12] has no metric of positive scalar curvature at all. As with the theorem for punctured tori, there is a nonvanishing Rosenberg index \(\alpha(\tilde{U}^m) \in K_*(C^*(\pi))\), where \(\pi = \pi_1(\tilde{U}^m)\), which maps to the generalized coarse index in \(K_*(D^*_G(\tilde{U}^m))\), as desired. Here the Gromov-Lawson-Rosenberg conjecture is true since \(\tilde{U}^m\) is a nilmanifold. \(\square\)

IV. The General Noncompact Arithmetic Case

To understand how we might achieve a similar result for general double quotient spaces \(\Gamma\setminus G/K\), we appeal to the following.

**Picture from Reduction Theory:** There is a compact polyhedron \(P\) and a Lipschitz map \(\pi : M \to cP\), where \(cP\) is the open cone on \(P\) so that (1) every point inverse deform retracts to an arithmetic manifold, (2) \(\pi\) respects the radial direction, and (3) all point inverses have uniformly bounded size.

Indeed, the polyhedron \(P\) is the geometric realization of the category of proper \(\mathbb{Q}\)-parabolic subgroups of \(G\), modulo the action of \(\Gamma\), and \(\pi^{-1}\) of the barycenter of a simplex is the arithmetic symmetric space associated to that parabolic. Concretely, for \(\text{SL}_n(\mathbb{Z}) \subset \text{SL}_n(\mathbb{R})\), the space \(P\) is an \(n-2\) simplex, the parabolics correspond to flags, and the associated arithmetic groups have a unipotent normal subgroup with quotient equal to a product of \(\text{SL}_{m_i}(\mathbb{Z})\), where the \(m_i\) are sizes of the blocks occurring in the flag. As one goes to infinity, the unipotent directions shrink in diameter and are responsible for the finite volume property of the lattice quotient, while the other parabolic directions remain of bounded size.

Alternatively, for any choice of basepoint in the homogeneous space, there are constants \(C\) and \(D\) that satisfy the following condition: if \(x\) is a given point and \(Q_x\) is the largest parabolic subgroup associated with a simplex whose cone contains \(x\) within its \(C\)-neighborhood, then the orbit of \(x\) under \(Q_x\) has diameter less than \(D\). Note the empty simplex means that there is a compact core which is stabilized by the whole group.
This picture can essentially be ascertained from [6], [29]; the fact that $K \backslash G / \Gamma$ has finite Gromov-Hausdorff distance from $cP$ is asserted in [10]. However, one needs a key estimate about the “coarse isotropy.” Details are given in unpublished work of Eskin [9]; some are given below.

As a guide the reader should think through the picture suggested by a product of hyperbolic manifolds. Each hyperbolic manifold contributes to $cP$ either a point, in the compact case, or the open cone on a finite set of points, in the case of cusps. Thus $P$ is a join of some number of finite sets. Using this model, we find that the inverse image of any point in the interior of any simplex is exactly a product of closed hyperbolic manifolds, cores of hyperbolic manifolds, and flat manifolds.

Let us now consider the unique decomposition [3] of a semisimple Lie group $G = N_a \cdot A_a \cdot K$, where $x$ is a point on the space of infinity of $G / K$. If $\Gamma$ is an arithmetic lattice in $G$, we are interested in knowing how $\Gamma$ acts on $G / K$. In other words, we ask how $\Gamma$ acts on this particular coordinate system. Let $g = nak \in G$. If $\gamma \in \Gamma$, let $\gamma g = n'a'k'$. Notice that $N_a \cap \Gamma$ is a lattice in $N_a$, and acts cocompactly on $N_a$. Consider the projection $\rho : \Gamma \backslash G / K \rightarrow A^*_a$ given by $n'a* \rightarrow a^*$, where $A^*_a$ is a fundamental domain of $A_a$ under the action of $\Gamma$. The fiber above each point is a compact manifold arising from the action of $N_a \cap \Gamma$ on $N_a$. Let $f + p$ be the usual Cartan decomposition and let $X \in p$ be the element such that $dp(X) = \gamma'_p x(0)$. If $Z(X)$ is defined by $Z(X) = \{Y \in g : [X, Y] = 0\}$, then $a = Z(X) \cap p$ is the unique maximal abelian subspace of $p$ that contains $X$. By definition $A_a = \exp(a)$.

We are interested in constructing a hypersurface $V'$ in the space $A^*_a$ whose inverse image $V \equiv \rho^{-1}(V')$ under the projection map $\rho$ provides us with an appropriate excisive decomposition of $\Gamma \backslash G / K$ for which the $K$-theoretic Mayer-Vietoris sequence is applicable. Consider the chamber decomposition of $a$. The Weyl group $W$ acts on these chambers via the hyperplanes. Consider the Bruhat decomposition $G = \coprod_{w \in W} BwB$ of $G$, and let $\gamma \in BwB$ for some $w \in W$. Recall that, if $g = nak$, we write $\gamma g = n'a'k'$. Denote by $\sum^+$ the set of positive roots of $a^*$ and $\sum^-$ the set of negative roots. Let $R = \sum^- \cap w \sum^+$ be the set of roots that start out positive but are made negative under the action of $w$. For some positive reals constants $c_\alpha$, the following equation holds [11],[3]:

$$a' = wa - \sum_{\alpha \in R} c_\alpha \alpha(a) + O(1),$$

where $a$ and $a'$ are viewed as elements of the Lie algebra $a$. The implications of this equation are as follows. Consider an element $a$ in the positive Weyl chamber $C(a)$. Then the intersection $\Gamma(a) \cap C(a)$ of the orbit $\Gamma(a)$ of $a$ under $\Gamma$ and the Weyl chamber $C(a)$ containing $a$ has a bounded diameter, uniformly in $a$. In other words, if $\gamma(a)$ stays in the same positive Weyl chamber, then $w$ is the identity and $R$ is empty. Hence $a' = a + a(1)$, implying that $a'$ can be found at a uniformly bounded distance from $a$ itself. In this event, the action of $\gamma$ corresponds to a translation of $a$ to (possibly) the compact fiber direction of $\Gamma \backslash G / K$. It is also a general fact that the action of any $g \in \Gamma$ will take the vertex of any subsector (as drawn in the previous figure) to the vertex of an analogous subsector. With this machinery, we are able to prove the following.

**Theorem 4:** The double quotient space $M = \Gamma \backslash G / K$ has no metric of uniform positive scalar curvature in the same coarse class as the natural metric inherited from $G$.

**Proof:** The picture from reduction theory provides a polyhedron $P$ in the space of infinity of the positive Weyl chamber and an open cone $W = cP$ on $P$ in $C^+$ oriented so that the distance from $W$ to any hyperplane $\alpha = 0$ will exceed the quantity $\sup_{a \in C^+} \text{diam}(\Gamma(a) \cap C^+)$. This
Let $V' = \partial W$ and $V = \rho^{-1}(V')$ in $M = \Gamma\backslash G/K$. Then $V$ is a hypersurface that induces a decomposition $(A, B)$ of $M$. The fundamental group $\pi_1(V) = N \cap \Gamma$ injects into $\pi_1(\Gamma\backslash G/K) = \Gamma$, satisfying the requirement needed in the construction of the Mayer-Vietoris sequence. In the most general case, the space $V$ is coarsely equivalent to a bundle over Euclidean space whose fiber consists of two components: a nilmanifold $N$ and (possibly) a compact homogeneous manifold $H$. In the absence of such an $H$, the argument follows exactly as it does for $\text{SL}_n(\mathbb{Z})/\text{SL}_n(\mathbb{R})/\text{SO}_n(\mathbb{R})$. In the presence of a compact homogeneous manifold, we may pass the coarse index of the Dirac operator to $\mathbb{R} \times H$ and use the usual Rosenberg obstruction on $H$ as in Theorem 1 to obtain our desired result.

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Rice University, Houston, TX 77005

E-mail address: sschang@math.rice.edu