A modal logic amalgam of classical and intuitionistic propositional logic

Steffen Lewitzka*

May 7, 2014

Abstract

We present a modal extension of classical propositional logic in which intuitionistic propositional logic is mirrored by means of the modal operator. In this sense, the modal extension combines classical and intuitionistic propositional logic avoiding the collapsing problem. Adopting ideas from a recent paper (S. Lewitzka, A denotational semantics for a Lewis-style modal system close to S1, 2013), we define a non-Fregean-style semantics such that propositional identity $\varphi \equiv \psi$ is strict equivalence $\square(\varphi \rightarrow \psi) \land \square(\psi \rightarrow \varphi)$. It turns out that the modal operator $\square$, when restricted to propositional formulas, can be regarded as an intuitionistic truth predicate. As a main result, we prove that a propositional formula $\varphi$ is an intuitionistic theorem iff $\square\varphi$ is a theorem of our modal logic. Moreover, we show the existence of models which, for any propositional formula $\varphi$, satisfy $\square\varphi$ iff $\varphi$ is an intuitionistic theorem. In the context of such models, the modal operator can be seen as a predicate for intuitionistic validity. Finally, we characterize the class of models as the class of certain Heyting algebras with an operator.

Keywords: combining classical and intuitionistic logic, collapsing problem, modal logic, non-Fregean logic, Heyting algebra

---

*Universidade Federal da Bahia – UFBA, Instituto de Matemática, Departamento de Ciência da Computação, Campus de Ondina, 40170-110 Salvador – BA, Brazil, e-mail: steffen@dcc.ufba.br
1 Introduction

Given two logics $\mathcal{L}_1$ and $\mathcal{L}_2$, one may ask for the existence and the properties of a logic $\mathcal{L}$ which, in some reasonable and precise sense, can be regarded as a combination of $\mathcal{L}_1$ and $\mathcal{L}_2$. Several techniques of combining and decomposing logics, such as fusion, products, fibering, algebraic fibering, cryptofibering, possible translation semantics, and others, have been studied in recent years. For an excellent survey of the area we refer the reader to [3]. Most combining techniques, such as fibering, assume two different object languages (possibly sharing some elements) that have to be composed. In the present paper, we deviate from that and some other assumptions underlying the traditional approach. Our purpose here is to propose a solution to the particular problem of combining classical and intuitionistic propositional logic applying ideas from two recent papers [12, 13] where modal logics with non-Fregean-style semantics are investigated. The study of combinations of classical and intuitionistic propositional logic is of interest because it involves a phenomenon which is known as the collapsing problem: combining both logics, the intuitionistic logic collapses into the classical one such that the resulting classical logic contains two equivalent copies of each connective. Let us explain the phenomenon from a semantic point of view. By a theory we mean a consistent and deductively closed set of formulas. Models in intuitionistic and classical logic correspond to prime theories, and vice-versa (recall that in classical logic the prime theories are exactly the maximal theories), and properties of connectives can be described by means of truth conditions over prime theories. For instance, intuitionistic negation $\neg$ can be characterized as follows: for all formulas $\varphi$ and all prime theories $T$, $\sim \varphi \in T \iff T \cup \{$$\varphi$$\}$ is inconsistent ($\iff \varphi \notin T'$ for every prime theory $T' \supseteq T$). Classical negation $\bar{\neg}$ is characterized by the following truth condition: for all formulas $\varphi$ and all prime theories $T$, $\sim \varphi \in T \iff \varphi \notin T$. If the logic contains both intuitionistic and classical negation, then both truth conditions must hold. Suppose $T$ is a prime theory which is not maximal, then $T$ extends to a (maximal) theory $T' \supseteq T$, and there is a $\varphi \in T' \sim T$. By the classical truth condition, $\bar{\neg} \varphi \in T$. Thus, $\varphi, \bar{\neg} \varphi \in T'$ — a contradiction. It follows that every prime theory is maximal. That is, the logic is classical and the truth conditions for both negations $\sim$ and $\bar{\neg}$ are equivalent (of course, also the truth conditions of intuitionistic and classical implication etc. are now equivalent).

We may overcome the problem if, instead of combining two different propositional languages, we work with only one propositional language. This approach relies on the view that intuitionistic and classical propositional logic can be defined over exactly the same object language — the logics then differ only in their
sets of (prime) theories. Now we augment this propositional language with the modal operator □ which, roughly speaking, separates intuitionistic from classical truth. For any propositional formula ϕ we expect that the formula □ϕ is classically true (i.e., is contained in a maximal theory T of our classical modal logic) iff ϕ is intuitionistically true (i.e., is contained in an associated prime theory P ⊆ T of intuitionistic propositional logic). That is, □ϕ can be read as “ϕ is intuitionistically true”, for any propositional formula ϕ. Consequently, a propositional formula ϕ is a theorem of intuitionistic propositional logic iff □ϕ is a theorem of our modal logic. That is, our modal extension contains a copy of intuitionistic propositional logic: the modal operator □ marks a propositional formula as an intuitionistic theorem. Of course, we also expect that our modal logic is a conservative extension of classical propositional logic. In order to develop an adequate semantics for a logic with these properties we adopt ideas from [13] and define non-Fregean-style models where propositional identity is given as strict equivalence. We are able to characterize this semantics algebraically by means of certain Heyting algebras: the algebraic reducts of our models are Heyting algebras with the property that the smallest filter is a prime filter (because of a similar situation in intuitionistic propositional logic, we call this property the disjunction property); on the other hand, for every Heyting algebra H with the disjunction property there is a model of our logic whose reduct is H.

The collapsing problem in the literature: In the context of fibring techniques, the collapsing problem can be studied as a general problem which is not restricted to the combination of intuitionistic and classical propositional logic. In its special form, the problem was first identified in [6, 7]. A logical system that combines classical and intuitionistic logic is found in [4]. A first general solution to the collapsing problem, based on modulated fibring, is presented in [14]. In [2] the authors propose cryptofibring semantics as a generalization of fibring semantics and show that this provides a solution to the collapsing problem. Other solutions to the problem are given by graph-theoretic fibring [15, 16] and, more recently, by a new method called meet combination of logics [17, 18].

\[\text{\textsuperscript{1}}\text{There might be contributions we are not aware of. We apologize any omission.}\]
2 The deductive system

The set $Fm$ of formulas is inductively defined in the usual way over an infinite set $V = \{x_0, x_1, x_2, \ldots\}$ of propositional variables, logical connectives $\bot, \rightarrow, \lor, \land$, and the modal operator $\Box$. If $x$ is a variable and $\varphi, \psi$ are formulas, then we write $\varphi[x := \psi]$ for the formula that we obtain by substituting $\psi$ for $x$ in $\varphi$. By $Fm_0$ we denote the set of those formulas which do not contain the modal operator $\Box$. We call the elements of $Fm_0$ propositional formulas, and we call the elements of $Fm \setminus Fm_0$ modal formulas. We use the following abbreviations:

- $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi)$
- $\neg \varphi := \varphi \rightarrow \bot$
- $\varphi \equiv \psi := \Box(\varphi \rightarrow \psi) \land \Box(\psi \rightarrow \varphi)$ (strict equivalence)
- $\Box \Phi := \{\Box \varphi \mid \varphi \in \Phi\}$, for any set $\Phi \subseteq Fm$.

The set of axioms is given by the following schemes (i)--(xvi):

(i) $\Box(\varphi \rightarrow (\psi \rightarrow \varphi))$
(ii) $\Box((\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)))$
(iii) $\Box((\varphi \land \psi) \rightarrow \varphi)$
(iv) $\Box((\varphi \land \psi) \rightarrow \psi)$
(v) $\Box(\varphi \rightarrow (\psi \rightarrow (\varphi \land \psi)))$
(vi) $\Box(\varphi \rightarrow (\varphi \lor \psi))$
(vii) $\Box(\psi \rightarrow (\varphi \lor \psi))$
(viii) $\Box((\varphi \rightarrow \psi) \rightarrow ((\chi \rightarrow \psi) \rightarrow ((\varphi \lor \chi) \rightarrow \psi)))$
(ix) $\Box(\bot \rightarrow \psi)$
(x) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box(\psi \rightarrow \chi) \rightarrow \Box(\varphi \rightarrow \chi))$
(xi) $\Box \varphi \rightarrow \varphi$
(xii) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$

4
(xiii) \((\Box \varphi \land \Box \psi) \rightarrow \Box (\varphi \land \psi)\)

(xiv) \(\Box (\varphi \lor \psi) \rightarrow (\Box \varphi \lor \Box \psi)\)

(xv) \((\psi \equiv \psi') \rightarrow (\varphi[x := \psi] \equiv \varphi[x := \psi'])\)

(xvi) \(\varphi \lor \neg \varphi\)

Note that (i)–(ix) constitute a complete axiomatization of intuitionistic propositional logic if in each axiom one drops the operator \(\Box\) and only considers propositional formulas \(\varphi, \psi, \chi \in Fm_0\). Also notice that the axioms of the form (xiii) are derivable from (v) and (xii). The notion of derivation is given in the usual way:

**Definition 2.1** For \(\Phi \subseteq Fm\) we define \(\Phi^+\) as the smallest set that contains \(\Phi\) together with all axioms and is closed under the rule of Modus Ponens. If \(\varphi \in \Phi^+\), then we write \(\Phi \vdash \varphi\) and say that \(\varphi\) is derivable from \(\Phi\). For a set \(\Phi \cup \{\varphi\} \subseteq Fm_0\) we write \(\Phi \vdash_{\text{int}} \varphi\) if \(\varphi\) is derivable from \(\Phi\) in intuitionistic logic by means of the complete calculus which is given by the axioms (i)–(ix) above (of course, without the modal operator \(\Box\)) together with Modus Ponens.

In view of the axioms (xi) and (xvi) one recognizes that our system contains all theorems of classical propositional logic. In Corollary 3.7 below we will show that our system is in fact a conservative extension of classical propositional logic. Moreover, our system can be seen as a particular modal interpretation of basic non-Fregean logic SCI [1]. In fact, the axioms (a), (b), (c1) and (c2) of Definition 1.1 in [1] represent an axiomatization of classical propositional logic and are therefore derivable in our system. Axiom (d) follows from Lemma 2.4 below. The axioms (e), (f) and (g) are special cases of our axiom (xv). Finally, (h) derives from axiom (xi). Instead of taking \(\varphi \equiv \psi\) as an abbreviation for the formula \(\Box (\varphi \rightarrow \psi) \land \Box (\psi \rightarrow \varphi)\) we could add the identity connective \(\equiv\) to our object language and augment our set of axioms with the scheme \((\varphi \equiv \psi) \equiv (\Box (\varphi \rightarrow \psi) \land \Box (\psi \rightarrow \varphi))\). The resulting system would be a specific modal theory of SCI, and it would differ from our system only in the aspect that the identity connective belongs to the language while in our system the identity connective is defined in terms of the modal operator.

The Deduction Theorem can be shown by induction on a derivation.

**Lemma 2.2** (Deduction Theorem) If \(\Phi \cup \{\varphi\} \vdash \psi\), then \(\Phi \vdash \varphi \rightarrow \psi\).

**Lemma 2.3** Let \(\Phi \cup \{\varphi\} \subseteq Fm_0\). Then \(\Phi \vdash_{\text{int}} \varphi\) implies \(\Box \Phi \vdash \Box \varphi\).
Proof. Suppose $\Phi \vdash_{int} \varphi$. We show the assertion by induction on a derivation. If $\varphi$ is an intuitionistic axiom, then we may assume that $\Box \varphi$ is one of the axioms (i)–(ix) of our calculus. If $\varphi \in \Phi$, then obviously $\Box \varphi \in \Box \Phi$. Finally, suppose $\varphi$ is obtained by Modus Ponens from $\psi$ and $\psi \rightarrow \varphi$. By induction hypothesis, $\Box \Phi \vdash \Box \psi$ and $\Box \Phi \vdash \Box (\psi \rightarrow \varphi)$. Axiom scheme (xii) and Modus Ponens yield $\Box \Phi \vdash \Box \varphi$. q.e.d.

We shall prove the converse of Lemma 2.3 in our Main Theorem 5.1 below. Let $\chi \in Fm_0$ be a theorem of intuitionistic propositional logic, and let $\chi' \in Fm$ be the result of uniformly replacing some variables in $\chi$ with formulas from $Fm$. Then $\chi'$ has still the logical form of an intuitionistic theorem even if it contains $\Box$ at some places. We believe that $\vdash \Box \chi'$ although we do not aim for a proof here. Instead, we present the proof of the following particular example which will be useful later.

Lemma 2.4 $\vdash \Box (\varphi \rightarrow \varphi)$, for any $\varphi \in Fm$.

Proof. By axiom (ii):
$\vdash \Box ((\varphi \rightarrow ((\varphi \rightarrow \varphi) \rightarrow \varphi)) \rightarrow (((\varphi \rightarrow (\varphi \rightarrow \varphi)) \rightarrow (\varphi \rightarrow \varphi)))$, for any formula $\varphi \in Fm$. By axiom (xii) and Modus Ponens:
$\vdash \Box (\varphi \rightarrow ((\varphi \rightarrow \varphi) \rightarrow \varphi)) \rightarrow \Box (((\varphi \rightarrow (\varphi \rightarrow \varphi)) \rightarrow (\varphi \rightarrow \varphi)))$. Axiom (i) and Modus Ponens yield:
$\vdash \Box (\varphi \rightarrow (\varphi \rightarrow \varphi)) \rightarrow (\varphi \rightarrow \varphi)$. By axiom (xii) and Modus Ponens:
$\vdash \Box (\varphi \rightarrow (\varphi \rightarrow \varphi)) \rightarrow \Box (\varphi \rightarrow \varphi)$. Finally, by axiom (i) and Modus Ponens:
$\vdash \Box (\varphi \rightarrow \varphi)$. q.e.d.

3 Semantics

The definition of semantics is inspired by modal non-Fregean semantics [12] and, in particular, by the approach presented in [13] where a denotational semantics for a Lewis-style modal logic is developed by interpreting strict equivalence as propositional identity.

Definition 3.1 A model $\mathcal{M} = (M, TRUE, P, f_\bot, f_\Box, f_\rightarrow, f_\vee, f_\wedge, \leq)$ is determined by the following ingredients and conditions. The universe $M$ is a set whose elements are called propositions, $P \subseteq TRUE \subseteq M$, $\leq$ is a partial ordering on $M$, $f_\bot \in M \setminus TRUE$, and $f_\Box : M \rightarrow M$, $f_c : M \times M \rightarrow M$, where $c \in \{\vee, \wedge, \rightarrow\}$, are functions such that for all $m, m', m'' \in M$ the following hold:
(i) \( m \leq f \rightarrow (m', m) \).

(ii) \( f \rightarrow (m, f \rightarrow (m', m'')) \leq f \rightarrow (f \rightarrow (m, m'), f \rightarrow (m, m'')) \).

(iii) \( f \land (m, m') \leq m \).

(iv) \( f \land (m, m') \leq m' \).

(v) \( m \leq f \rightarrow (m', f \rightarrow (m, m')) \).

(vi) \( m \leq f \lor (m, m') \).

(vii) \( m' \leq f \lor (m, m') \).

(viii) \( f \rightarrow (m, m') \leq f \rightarrow (f \rightarrow (m'', m'), f \rightarrow (f \rightarrow (m, m''), m')) \).

(ix) \( f \bot \leq m \).

(x) \( m \in P \text{ and } m \leq m' \Rightarrow m' \in P \).

(xi) \( m, m' \in P \Rightarrow f \land (m, m') \in P \).

(xii) \( f \lor (m, m') \in P \Rightarrow m \in P \text{ or } m' \in P \).

(xiii) \( f \rightarrow (m, m') \in P \Leftrightarrow m \leq m' \).

(xiv) \( f \land (m) \in \text{TRUE} \Leftrightarrow m \in P \).

(xv) \( f \rightarrow (m, m') \in \text{TRUE} \Leftrightarrow m \notin \text{TRUE} \text{ or } m' \in \text{TRUE} \).

(xvi) \( f \lor (m, m') \in \text{TRUE} \Leftrightarrow m \in \text{TRUE} \text{ or } m' \in \text{TRUE} \).

(xvii) \( f \land (m, m') \in \text{TRUE} \Leftrightarrow m \in \text{TRUE} \text{ and } m' \in \text{TRUE} \).

An assignment of a model \( \mathcal{M} \) is a function \( \gamma : V \cup \{ \bot \} \rightarrow M \) which maps \( \bot \) to \( f \bot \) and extends in the canonical way to a function \( \gamma : Fm \rightarrow M \). That is, \( \gamma(\neg \varphi) = f_{\neg}(\gamma(\varphi)), \gamma(\square \varphi) = f_{\square}(\gamma(\varphi)) \) and \( \gamma(\varphi \ast \psi) = f_{\ast}(\gamma(\varphi), \gamma(\psi)) \), where \( \ast \in \{ \rightarrow, \vee, \land \} \).

Note that the conditions (i)–(ix) of a model correspond exactly to the “intuitionistic” axiom schemes (i)–(ix) of our calculus. In the last chapter we will show that \( P \) contains only one element, namely the greatest one w.r.t. the ordering \( \leq \). This will simplify the definition in the sense that conditions (x) and (xi) can be omitted.
It would be more accurate to use for each model $\mathcal{M}$ a specific notation for its ingredients, for example: $TRUE^{\mathcal{M}}$, $f^M_\perp$, $f^M_\rightarrow$, etc. However, for the sake of simplicity we refer to these ingredients always as $TRUE$, $f_\perp$, $f_\rightarrow$, etc., independently of the given model. Sometimes it is useful to have an extra operation on the propositional universe $M$ which corresponds to the connective of negation. We define this operation $f_\neg$ by $m \mapsto f_\rightarrow(f_\rightarrow(m, f_\perp))$.

**Definition 3.2** An interpretation is a tuple $(\mathcal{M}, \gamma)$ consisting of a model $\mathcal{M}$ and an assignment $\gamma : V \cup \{\perp\} \to M$. An interpretation $(\mathcal{M}, \gamma)$ is a model of a formula $\varphi$, notation: $(\mathcal{M}, \gamma) \vDash \varphi$, if $\gamma(\varphi) \in TRUE$. If $(\mathcal{M}, \gamma)$ is a model of $\varphi$, then we say that $(\mathcal{M}, \gamma)$ satisfies $\varphi$ or $\varphi$ is true in $(\mathcal{M}, \gamma)$. These notions extend in the usual way to sets of formulas. For a set of formulas $\Phi$ we define $\text{Mod}(\Phi) := \{(\mathcal{M}, \gamma) \mid (\mathcal{M}, \gamma) \vDash \Phi\}$. The relation of logical consequence is defined in the following way: $\Phi \vDash \varphi :\iff \text{Mod}(\Phi) \subseteq \text{Mod}(\{\varphi\})$.

Independently of the given logic, by a theory we mean a consistent and deductively closed set of formulas. A prime theory is a theory $T$ such that $\varphi \lor \psi \in T$ implies $\varphi \in T$ or $\psi \in T$. In intuitionistic and classical propositional logic, to each model corresponds a prime theory and vice-versa. In classical logic, the prime theories are precisely the maximal theories. Also recall that truth and falsity are concepts which in intuitionistic and classical logic can be characterized relative to a given prime theory $T$ in the following way: a formula $\varphi$ is true iff $\varphi \in T$; and a formula $\varphi$ is false iff $\neg \varphi \in T$.

**Proposition 3.3** For every interpretation $(\mathcal{M}, \gamma)$ there is a unique prime theory $\Phi_p \subseteq Fm_0$ of intuitionistic propositional logic such that for all $\varphi \in Fm_0$,

$$(\mathcal{M}, \gamma) \vDash \Box \varphi \iff \varphi \in \Phi_p.$$  

That is, the modal operator $\Box$, restricted to propositional formulas, can be seen as a predicate for intuitionistic truth.

**Proof.** Consider $\Phi_p := \{\varphi \in Fm_0 \mid \gamma(\varphi) \in P\}$. Obviously, $(\mathcal{M}, \gamma) \vDash \Box \varphi \iff \varphi \in \Phi_p$, for any $\varphi \in Fm_0$. We show by induction on a derivation that $\Phi_p \vdash_{int} \psi$ implies $\psi \in \Phi_p$, for any $\psi \in Fm_0$. If $\psi$ is an intuitionistic axiom, then we may assume that $\Box \psi$ has the form of one of the axioms (i)–(ix) of our calculus. Then $\psi$ corresponds to one of the conditions (i)–(ix) of Definition 3.1 of a model. By condition (xiii) of Definition 3.1, $\gamma(\psi) \in P$, i.e., $\psi \in \Phi_p$. Finally, suppose $\psi$ is
obtained by Modus Ponens from formulas \( \varphi \) and \( \varphi \rightarrow \psi \). By induction hypothesis, \( \gamma(\varphi) \in P \) and \( \gamma(\varphi \rightarrow \psi) \in P \). By condition (xiii) and (x) of Definition 3.1 \( \gamma(\psi) \in P \), i.e., \( \psi \in \Phi_p \). That is, \( \Phi_p \) is deductively closed in intuitionistic logic. Now it also follows that \( \Phi_p \) is consistent in intuitionistic logic. Thus, \( \Phi_p \) is an intuitionistic theory. Suppose \( \varphi \lor \psi \in \Phi_p \), for \( \varphi, \psi \in \text{Fm}_0 \). Then \( \gamma(\varphi \lor \psi) = f_\lor(\gamma(\varphi), \gamma(\psi)) \in P \). Thus, \( \gamma(\varphi) \in P \) or \( \gamma(\psi) \in P \). That is, \( \varphi \in \Phi_p \) or \( \psi \in \Phi_p \), and \( \Phi_p \) is a prime theory. It is clear that \( \Phi_p \) is unique. q.e.d.

Later we will see that also the converse of Proposition 3.3 holds true. Moreover, we shall prove the existence of models where the modal operator \( \Box \), restricted to propositional formulas, is not only a predicate for intuitionistic truth but also for intuitionistic validity. Of course, this is the case when the underlying prime theory is the smallest theory, i.e., the set of all intuitionistic theorems.

The next result follows readily from the definitions. It says that two formulas denote the same proposition iff they are strictly equivalent in the ambient model.

**Lemma 3.4** \( (M, \gamma) \models \varphi \equiv \psi \iff \gamma(\varphi) = \gamma(\psi) \).

By means of the defined identity connective \( \equiv \) we are able to express propositional (self-) references (see, e.g., [8, 9, 10, 11] for a discussion). If we regard \( \Box \) as an intuitionistic truth predicate for propositional formulas in the sense of Proposition 3.3 then such self-referential statements may also involve truth and negation. An (intuitionistic) liar proposition, for example, can be asserted by the equation \( x \equiv \neg \Box x \). A proposition \( m \in M \) that satisfies that equation says “This proposition is not intuitionistically true.” We would like to point out that the existence of such an “intuitionistic” liar \( m \) is by no means paradoxical: \( m = \gamma(x) = \gamma(\neg \Box x) = f_\neg(f_\Box(m)) \in TRUE \) does not imply any contradiction. It only implies that \( m \in TRUE \) (i.e., \( m \) is classically true) and \( m \notin P \) (i.e., \( m \) is not intuitionistically true) which is quite possible. The simple reason for the absence of the paradox, as it arises in the classical case, is the fact that \( \varphi \notin \Phi_p \), for a prime theory \( \Phi_p \), in intuitionistic logic does not imply \( \neg \varphi \in \Phi_p \). That is, a formula may be neither true nor false. In other words: \( \varphi \lor \neg \varphi \) is not valid in intuitionistic logic. The liar proposition in the form “This proposition is intuitionistically false” can be asserted by the equation \( x \equiv \Box \neg x \). If a model contains such a liar proposition \( m \in M \), then \( m \notin TRUE \). An equation asserting an intuitionistic truth-teller is the following: \( x \equiv \Box x \).

\(^2\)A model of non-Fregean logic that contains exactly two self-referential propositions, a true
Lemma 3.5 (Substitution Principle) For all $\varphi, \psi, \psi' \in Fm$:
$$\vDash (\psi \equiv \psi') \rightarrow (\varphi[x := \psi] \equiv \varphi[x := \psi']).$$

Proof. Let $(M, \gamma) \vDash \psi \equiv \psi'$. That is, $\gamma(\psi) = \gamma(\psi')$. By induction on $\varphi$ one shows that $\gamma(\varphi[x := \psi]) = \gamma(\varphi[x := \psi'])$. That is, $(M, \gamma) \vDash \varphi[x := \psi] \equiv \varphi[x := \psi']$. q.e.d.

Now one easily checks that every interpretation satisfies the axioms. It is also clear that the rule of Modus Ponens is sound. Soundness of the calculus now follows by induction on a derivation.

Theorem 3.6 (Soundness) For any set $\Phi \cup \{\varphi\} \subseteq Fm$, if $\Phi \vdash \varphi$, then $\Phi \vDash \varphi$.

It is not hard to construct a trivial, two-element model $M_B$ which is uniquely determined by its universe of two propositions $M = \{\top, \bot\}$. In fact, it follows that $TRUE = P = \{\top\}$, $\leq = \{(\top, \top), (\bot, \top), (\bot, \bot)\}$ and the function $f_\Box$ is the identity on $M$. Furthermore, the functions $f_\lor, f_\land$ and $f_\rightarrow$ reduce here to the usual boolean functions. That is, the two propositions can be identified with the classical truth-values, and an assignment of $M_B$ can be seen as a truth-value assignment. Notice that for any truth-value assignment $\beta : V \cup \{\bot\} \rightarrow \{\top, \bot\}$ and any formula $\varphi \in Fm$, $(M_B, \beta) \vDash \varphi$ iff $(M_B, \beta) \vDash \Box \varphi$. That is, the model does not distinguish between classical and intuitionistic truth.

Corollary 3.7 Our modal logic is a conservative extension of classical propositional logic. That is, for any $\varphi \in Fm_0$: $\varphi$ is a theorem of classical propositional logic iff $\vdash \varphi$.

Proof. In view of the axioms one recognizes immediately that our system contains all tautologies of classical propositional logic. Now let $\varphi$ be any propositional formula such that $\vdash \varphi$. Suppose $\varphi$ is not a theorem of classical logic. Then for some truth-value assignment $v$, $v(\varphi) = 0$. Consider the two-element model $M_B$ and an assignment $\gamma$ such that $\gamma(x) = \top$ if $v(x) = 1$, for all variables $x$ that occur in $\varphi$. One shows inductively that $\gamma(\varphi) \notin TRUE$. That is, $(M_B, \gamma) \not\vDash \varphi$ and thus $\not\vDash \varphi$. By soundness, $\not\vDash \varphi$. This is a contradiction to our hypothesis. Thus, $\varphi$ is a theorem of classical logic. q.e.d.

truth-teller and a false truth-teller, is constructed in [8].
4 The Completeness Theorem

Our proof of the Completeness Theorem is very similar to that of [13]. For the convenience of the reader we repeat some details. The notions of consistent set, inconsistent set and maximal consistent set are defined as usual. For a maximal consistent set \( \Phi \) we define a relation \( \approx_\Phi \) on \( Fm \) by \( \phi \approx_\Phi \psi \) if \( \Phi \vdash \phi \equiv \psi \).

Lemma 4.1 Let \( \Phi \) be a maximal consistent set. The relation \( \approx_\Phi \) is an equivalence relation on \( Fm \) with the following properties:

- If \( \phi_1 \approx_\Phi \psi_1 \) and \( \phi_2 \approx_\Phi \psi_2 \), then \( \neg \phi_1 \approx_\Phi \neg \psi_1 \), \( \Box \phi_1 \approx_\Phi \Box \psi_1 \) and \( \phi_1 \ast \psi_2 \approx_\Phi \psi_1 \ast \psi_2 \), where \( \ast \in \{\lor, \land, \to\} \).
- If \( \phi \approx_\Phi \psi \), then \( \phi \in \Phi \iff \psi \in \Phi \).
- If \( \phi \approx_\Phi \psi \), then \( \Box \phi \in \Phi \iff \Box \psi \in \Phi \).

Proof. Symmetry of \( \approx_\Phi \) follows from axioms of propositional logic. By Lemma 2.4 \( \Box (\phi \to \psi) \in \Phi \). It follows that \( \approx_\Phi \) is reflexive. Transitivity follows from axiom (x). In order to show the first item of the Lemma suppose \( \phi_1 \approx_\Phi \psi_1 \) and \( \phi_2 \approx_\Phi \psi_2 \). Let \( x \neq y \) be variables such that \( x \) does not occur in \( \psi_2 \) and \( y \) does not occur in \( \phi_1 \). Then by axiom (xv) and Modus Ponens: \( (\phi_1 \ast \phi_2) = (\phi_1 \ast y)[y := \phi_2] \approx_\Phi (\phi_1 \ast y)[y := \psi_2] = (\phi_1 \ast \psi_2) = (x \ast \psi_2)[x := \phi_1] \approx_\Phi (x \ast \psi_2)[x := \psi_1] = (\psi_1 \ast \psi_2) \). The assertion now follows from the transitivity of \( \approx_\Phi \). The remaining cases follow similarly. The second item of the Lemma follows from axiom (xi) and Modus Ponens. The third item follows from the previous items of the Lemma or, alternatively, from axiom (xii). q.e.d.

Lemma 4.2 A maximal consistent has a model.

Proof. Let \( \Phi \subseteq Fm \) be maximal consistent. For \( \phi \in Fm \) let \( \phi \) be the equivalence class of \( \phi \) modulo \( \approx_\Phi \). The ingredients of our model are given by:

- \( M := \{\phi \mid \phi \in Fm\} \)
- \( TRUE := \{\phi \mid \phi \in \Phi\} \)
- \( P := \{\phi \mid \Box \phi \in \Phi\} \)
- functions \( f_\bot, f_\Box, f_\ast \), where \( \ast \in \{\lor, \land, \to\} \), defined by \( f_\bot := \top, f_\Box(\phi) := \Box \phi, f_\ast(\phi, \psi) := \phi \ast \psi \), respectively
a binary relation $\leq$ on $M$ defined by $\varphi \leq \psi :\Leftrightarrow \Box(\varphi \rightarrow \psi) \in \Phi$.

By Lemma 4.1, these ingredients are well-defined. By Lemma 2.4, $\Box(\varphi \rightarrow \varphi) \in \Phi$. Now it follows from the axioms that $\leq$ is a partial ordering on $M$, $P \subseteq TRUE \subseteq M$ and the conditions of Definition 3.1 are satisfied. That is, $\mathcal{M} = (M, TRUE, P, f_\bot, f_\Box, f_\rightarrow, f_\lor, f_\land, \leq)$ is a model. Let us consider the assignment $\gamma : V \cup \{\bot\} \rightarrow M$ defined by $x \mapsto \overline{x}$. One recognizes that $\gamma(\varphi) = \overline{\varphi}$, for any formula $\varphi$. We get the following for the interpretation $(\mathcal{M}, \gamma)$:

$$\varphi \in \Phi \iff \overline{\varphi} \in TRUE \iff \gamma(\varphi) \in TRUE \iff (\mathcal{M}, \gamma) \models \varphi.$$  

Thus, $(\mathcal{M}, \gamma) \models \Phi$. q.e.d.

**Corollary 4.3 (Completeness Theorem)** $\Phi \models \varphi \Rightarrow \Phi \vdash \varphi$.

**Proof.** $\Phi \not\models \varphi$ implies the consistency of $\Phi \cup \{\neg \varphi\}$. Using standard arguments, one shows that this set extends to a maximal consistent set which, by Lemma 4.2, has a model. It follows that $\Phi \not\models \varphi$. q.e.d.

**Lemma 4.4** Let $\mathcal{M}_B$ be the two-element model. Then for any model $\mathcal{M} = (M, TRUE, ...)$, any assignment $\gamma$ of $\mathcal{M}$, and any formula $\varphi \in Fm_0$ it holds the following:

$$(\mathcal{M}, \gamma) \models \varphi \iff (\mathcal{M}_B, \gamma_B) \models \varphi \iff (\mathcal{M}_B, \gamma_B) \models \Box \varphi,$$

where $\gamma_B : V \cup \{\bot\} \rightarrow \{f_\bot, f_\top\}$ is the truth-value assignment defined by $\gamma_B(x) = f_\top$ iff $\gamma(x) \in TRUE$.

**Proof.** The assertion follows by induction on $\varphi \in Fm_0$ and from the fact that truth is the same as necessity in the two-element model. q.e.d.

**Corollary 4.5** Let $\Phi \cup \{\varphi\} \subseteq Fm_0$. If $\Box \Phi \vdash \Box \varphi$, then $\Phi \vdash \varphi$.

**Proof.** Suppose $\Phi \not\models \varphi$. By the Completeness Theorem, there is a model $(\mathcal{M}, \gamma)$ such that $(\mathcal{M}, \gamma) \models \Phi$ and $(\mathcal{M}, \gamma) \not\models \varphi$. By Lemma 4.4, $(\mathcal{M}_B, \gamma_B) \models \Box \Phi$ and $(\mathcal{M}_B, \gamma_B) \not\models \Box \varphi$, where $(\mathcal{M}_B, \gamma_B)$ is the two-element model together with the assignment defined in the Lemma. By soundness of the calculus, $\Box \Phi \not\models \Box \varphi$. q.e.d.

**Corollary 4.6** Let $\Phi \subseteq Fm_0$. $\Phi$ is consistent iff $\Box \Phi$ is consistent.
Proof. By soundness and completeness, satisfiability and consistency are equivalent concepts. Of course, every model of $\Box \Phi$ also satisfies $\Phi$. On the other hand, if $\Box \Phi$ is inconsistent, then $\Box \Phi \vdash \Box \bot$. By Corollary 4.5, $\Phi \vdash \bot$ and $\Phi$ is inconsistent. q.e.d.

We were not able to show the assertion of Corollary 4.6 for the general case $\Phi \subseteq Fm$.

5 The Main Theorem

The expression $\Box \Phi \vdash \Box \varphi$ is an abbreviation of the assertion that there is a derivation of $\Box \varphi$ from $\Box \Phi$ in our logic. However, in the special case $\Phi \cup \{ \varphi \} \subseteq Fm_0$ there is a further intuition behind that expression, namely that $\varphi$ is derivable from $\Phi$ in intuitionistic propositional logic. This intuition is confirmed by the following result.

Theorem 5.1 Let $\Phi \cup \{ \chi \} \subseteq Fm_0$. Then $\Box \Phi \vdash \Box \chi \iff \Phi \vdash \text{int} \chi$.

Proof. The direction from right to left is Lemma 2.3. Suppose $\Phi \not\vdash \text{int} \chi$. Using a standard construction (see, e.g., [5]), we may extend $\Phi$ to a prime theory $\Phi_p \subseteq Fm_0$ such that $\Phi_p \not\vdash \text{int} \chi$. From an application of Zorn’s Lemma it follows that $\Phi_p$ is contained in a maximal theory $\Phi_{\text{max}} \subseteq Fm_0$, i.e., a maximal consistent set of intuitionistic propositional logic. Such a set is also maximal consistent in classical propositional logic (a Kripke model of $\Phi_{\text{max}}$ is a singleton). Note that possibly $\chi \in \Phi_{\text{max}}$. We now construct a model of $\Phi_{\text{max}}$ that identifies precisely those propositional formulas $\varphi, \psi \in Fm_0$ which are intuitionistically equivalent modulo the prime theory $\Phi_p \subseteq \Phi_{\text{max}}$. For this we define a relation $\approx$ on $Fm_0$ by

$\varphi \approx \psi \iff \Phi_p \vdash \text{int} \varphi \leftrightarrow \psi$.

Note that $\approx$ is defined on the set $Fm_0 \subseteq Fm$ and not on the whole set $Fm$. From intuitionistic propositional logic it follows that $\approx$ is a congruence relation on $Fm_0$, that is, it is an equivalence relation, and $\varphi_1 \approx \psi_1, \varphi_2 \approx \psi_2$ imply $(\varphi_1 \ast \varphi_2) \approx (\psi_1 \ast \psi_2)$ for $\ast \in \{ \lor, \land, \rightarrow \}$. Let $\overline{\varphi}$ denote the congruence class of
formula \( \varphi \in F_{m0} \) modulo \( \approx \). We define

\[
M := \{ \varphi \mid \varphi \in F_{m0} \}
\]

\[
TRUE := \{ \varphi \mid \varphi \in \Phi_{\text{max}} \}
\]

\[
P := \{ \varphi \mid \varphi \in \Phi_p \}
\]

\[
f_\perp := \bot
\]

\[
f_\top := \bot \rightarrow \bot
\]

\[
f_\ast(\varphi, \psi) := \varphi \ast \psi \text{ for } * \in \{ \lor, \land, \rightarrow \}
\]

\[
f_{\Box}(\varphi) := f_\top \text{ if } \varphi = f_\top, \text{ and } f_{\Diamond}(\varphi) := f_\perp \text{ if } \varphi \neq f_\top
\]

\[
\varphi \leq \psi \iff \varphi \rightarrow \psi \in P
\]

It is clear that \( \varphi, \psi \in \Phi_p \) implies \( \Phi_p \models \varphi \iff \psi \). Thus, \( P \) consists of exactly one element, namely \( f_\top \). This enables us to define the function \( f_{\Box} \) in an appropriate way: \( f_{\Box}(\varphi) \in \{ f_\top, f_\perp \} \) instead of \( f_{\Box}(\varphi) := \Box \varphi \) — note that there are no elements of the form \( \Box \varphi \) in the universe \( M \). Now one easily checks that \( \mathcal{M} := (M, TRUE, P, f_\perp, f_\top, f_\lor, f_\land, \leq) \) satisfies the conditions of a model given in Definition \( 3.1 \). We consider the assignment \( \varepsilon : V \cup \{ \bot \} \rightarrow M \) defined by \( x \mapsto \varpi \). By induction one shows that \( \varepsilon(\varphi) = \varphi \), for any formula \( \varphi \in F_{m0} \). By construction, \( \Phi \subseteq \Phi_p \) and \( \chi \notin \Phi_p \). Furthermore, for every \( \varphi \in F_{m0} \):

\[
(M, \varepsilon) \models \Box \varphi \iff \varepsilon(\Box \varphi) = f_{\Box}(\varphi) \in TRUE \iff \varphi = f_\top \iff \varphi \in P \iff \varphi \in \Phi_p.
\]

Hence, \( (M, \varepsilon) \models \Box \Phi \) and \( (M, \varepsilon) \not\models \Box \chi \). Thus, \( \Box \Phi \not\models \Box \chi \). Soundness of the calculus yields \( \Box \Phi \not\models \Box \chi \). q.e.d.

**Corollary 5.2** For any \( \varphi \in F_{m0} \),

\[
\vdash \Box \varphi \iff \vdash_{\text{int}} \varphi.
\]

The model construction in the proof of Theorem \( 5.1 \) yields the converse of Proposition \( 3.3 \).

**Corollary 5.3** For every prime theory \( \Phi_p \subseteq F_{m0} \) of intuitionistic propositional logic there is an interpretation \( (M, \varepsilon) \) such that for all \( \varphi \in F_{m0} \),

\[
(M, \varepsilon) \models \Box \varphi \iff \varphi \in \Phi_p.
\]

This ensures the existence of a specific model with the following interesting property.

**Corollary 5.4** There is an interpretation \( (M_{\text{int}}, \varepsilon) \) such that for all \( \varphi \in F_{m0} \),

\[
(M_{\text{int}}, \varepsilon) \models \Box \varphi \iff \vdash_{\text{int}} \varphi.
\]
Proof. By the disjunction property of intuitionistic propositional logic, the smallest theory of intuitionistic logic, i.e., the set of all intuitionistic theorems, is a prime theory $\Phi_p$. Now apply Corollary 5.3. q.e.d.

In the context of model $(\mathcal{M}_{int}, \varepsilon)$ of Corollary 5.4, the modal operator $\Box$, restricted to propositional formulas, can be seen as a predicate for intuitionistic validity. This predicate, however, has only local character. That is, $\Box$ is a validity predicate only in the context of a specific model such as the model $\mathcal{M}_{int}$ above.

**Corollary 5.5** There is an interpretation $(\mathcal{M}_{int}, \varepsilon)$ such that for all $\varphi, \psi \in Fm_0$,

$$(\mathcal{M}_{int}, \varepsilon) \models \varphi \equiv \psi \iff \vdash_{int} \varphi \leftrightarrow \psi.$$  

**Proof.** One recognizes that the interpretation whose existence is guaranteed by Corollary 5.4 has the desired property. q.e.d.

Recall that for any interpretation $(\mathcal{M}, \gamma)$ and formulas $\varphi, \psi \in Fm$ we have $(\mathcal{M}, \gamma) \models \varphi \equiv \psi$ iff $\varphi$ and $\psi$ denote the same proposition in $\mathcal{M}$ (see Lemma 3.4). By Corollary 5.2 $(\mathcal{M}, \gamma) \models \varphi \equiv \psi$ whenever $\vdash_{int} \varphi \leftrightarrow \psi$, for any $\varphi, \psi \in Fm_0$. In general, however, an interpretation satisfies more equations. The two-element model, for example, identifies all formulas with the same truth-value, in particular all classically equivalent formulas such as $\varphi$ and $\neg\neg\varphi$. Corollary 5.5 shows the existence of a model $(\mathcal{M}_{int}, \varepsilon)$ which identifies precisely those formulas $\varphi, \psi \in Fm_0$ which are logically equivalent in intuitionistic logic.

A much stronger result than Corollary 5.5 would be the existence of a model $(\mathcal{M}, \gamma)$ which for all $\varphi, \psi \in Fm$ satisfies the following:

$$(\mathcal{M}, \gamma) \models \varphi \equiv \psi \iff \vdash \varphi \equiv \psi.$$  

We call such an interpretation a canonical model. A canonical model satisfies only those equations which are theorems of the underlying theory, i.e. those equations which are true in every model. We were unable to prove the existence of such a model and leave this as an open problem. It is also an open problem in the context

---

**3** An intuitionistic non-Fregean logic with a global predicate for validity on the object level is presented in [10]: a formula $\varphi$ is valid iff the formula $(\varphi : valid)$ is true in every model. As a consequence of such a global predicate, all models satisfy the same equations. The set of these equations must be determined by axioms of the theory. One may choose the set of trivial equations $\varphi \equiv \varphi$. 

---

15
of the modal logic studied in [13]. In general, one may ask for a canonical model whenever one deals with a logic that contains a connective for propositional identity (the connective may be definable, as in our case, or it may be an element of the language as it is the case in non-Fregean logics). It is not hard to find a canonical model of the basic non-Fregean logic SCI studied in [1]. Its construction (see, e.g., [8]) is a model-theoretic proof of the fact that in SCI we have: $\vdash_{SCI} \varphi \equiv \psi$ iff $\varphi = \psi$. If one adds propositional quantifiers to the language, then one expects that a canonical model identifies two sentences iff they are alpha-congruent, i.e. they differ at most on their bound variables. We require additionally that a canonical model of a language with propositional quantifiers does not contain non-standard elements.\footnote{A non-standard element is a proposition $m$ such that there is no sentence denoting $m$. A sentence is a formula that contains no free variables.} The construction of such a model turns out to be a very complex task (see, e.g., [11]).

6 Models and Heyting algebras

In this last chapter we consider the algebraic structure of our models and establish a representation result with respect to a certain class of Heyting algebras. Recall that a Heyting algebra is a bounded lattice $\mathcal{H} = (H, f_{\top}, f_{\bot}, f_{\lor}, f_{\land})$ with an additional binary operation $f_{\rightarrow}$ which maps any two elements $m, m' \in H$ to the greatest element $x \in H$ with the property $f_{\land}(m, x) \preceq m'$, where $\preceq$ is the ordering induced by the lattice. A (proper) filter of an Heyting algebra $\mathcal{H}$ is a non-empty subset $F \subseteq H$ such that for all $m, m' \in H$:

(a) $m \in F$ and $m \preceq m'$ implies $m' \in F$,
(b) $m \in F$ and $m' \in F$ implies $f_{\land}(m, m') \in F$,
(c) $f_{\bot} \notin F$.

Notice that by a filter we always mean a proper filter, i.e. a proper subset $F \subset H$. A filter $F$ is a prime filter, if it satisfies for all $m, m' \in H$:

(d) $f_{\lor}(m, m') \in F$ implies $m \in F$ or $m' \in F$.

Finally, an ultra-filter is a filter which is maximal with respect to inclusion. A filter $F$ is an ultra-filter if it has the following property: $m \notin F$ iff $f_{\rightarrow}(m) \in F$, for all $m \in H$. The intersection of any non-empty set of filters is a filter. The smallest filter is the set $\{f_{\top}\}$. It follows from Zorn’s Lemma that every filter extends to an ultra-filter.
Lemma 6.1 Let $\mathcal{M} = (M, \text{TRUE}, P, f_\bot, f\Box, f\rightarrow, f\vee, f\land, \leq)$ be a model. There is a greatest element with respect to $\leq$, i.e. there is a $f_\top \in M$ such that $m \leq f_\top$, for all $m \in M$. It follows that $f_\top \in P$.

Proof. Define $\top := \bot \rightarrow \bot$. Then for every assignment $\gamma$ we have $\gamma(\top) = f_\rightarrow(\gamma(\bot), \gamma(\bot)) = f_\rightarrow(f_\bot, f_\bot) =: f_\top$. Since $\leq$ is a partial order, $f_\bot \leq f_\bot$. Thus, $f_\top \in P$. Let $m \in M$, $x \in V$ and let $\beta$ be an assignment such that $\beta(x) = m$. By soundness of axiom scheme (i), $(\mathcal{M}, \beta) \models \Box(\top \rightarrow (x \rightarrow \top))$. That is, $f_\top \leq f_\rightarrow(m, f_\top)$. From $f_\top \in P$ it follows that $f_\rightarrow(m, f_\top) \in P$. That is, $m \leq f_\top$. q.e.d.

Proposition 6.2 Let $\mathcal{M} = (M, \text{TRUE}, P, f_\bot, f\Box, f\rightarrow, f\vee, f\land, \leq)$ be a model. Then $P = \{f_\top\}$, where $f_\top$ is the greatest element w.r.t. $\leq$.

Proof. We already know that $f_\top \in P$. Suppose $m, m' \in P$. We show $m = m'$.

Let $\gamma$ be an assignment such that $\gamma(x) = m$ and $\gamma(y) = m'$, for variables $x, y \in V$. Then $(\mathcal{M}, \gamma) \models \Box x \land \Box y$. Thus, $(\mathcal{M}, \gamma) \models \Box(x \land y)$. Since $x \land y \vdash_{\text{int}} x \rightarrow y$ and $x \land y \vdash_{\text{int}} y \rightarrow x$, Lemma 2.3 yields $\Box(x \land y) \vdash \Box(x \rightarrow y)$ and $\Box(x \land y) \vdash \Box(y \rightarrow x)$. By soundness, $(\mathcal{M}, \gamma) \models \Box(x \rightarrow y)$ and $(\mathcal{M}, \gamma) \models \Box(y \rightarrow x)$. That is, $(\mathcal{M}, \gamma) \models x \equiv y$. Hence, $m = \gamma(x) = \gamma(y) = m'$. q.e.d.

With these results we may simplify the notion of a model. For a model $\mathcal{M}$ we write in the following $\mathcal{M} = (M, \text{TRUE}, f_\top, f_\bot, f\Box, f\rightarrow, f\vee, f\land, \leq)$ replacing the subset $P$ by its unique element $f_\top$. Conditions (x) and (xi) of Definition 3.1 are now trivial.

Notice that the smallest filter $\{f_\top\}$ of a Heyting algebra is not necessarily a prime filter. For instance, in a Boolean algebra we have $f\vee(m, f_\neg(m)) = f_\top$, for any element $m$. If the Boolean algebra is not the trivial two-element algebra, then not necessarily $m = f_\top$ or $f_\neg(m) = f_\top$.

Definition 6.3 Let $\mathcal{H}$ be an Heyting algebra and let $f_\top$ be its greatest element. We say that $\mathcal{H}$ has the disjunction property if the smallest filter $P = \{f_\top\}$ is a prime filter.

In an Heyting algebra with the disjunction property it holds the following property for all elements $m, m'$: $f\vee(m, m') = f_\top \iff m = f_\top \text{ or } m' = f_\top$. Among all Heyting algebras there is, up to isomorphism, only one Boolean algebra with the disjunction property:
Lemma 6.4 If a Boolean algebra \( B \) has the disjunction property, then \( B \) is the two-element Boolean algebra.

Proof. Suppose \( B \) has the disjunction property. Since \( B \) is a Boolean algebra, 
\[ f_\lor(a, f_\lnot(a)) = f_\top, \text{ for all elements } a. \]
Thus, \( a = f_\top \) or \( f_\lnot(a) = f_\bot \), for all elements \( a \). In a Boolean algebra, the latter equation is equivalent with \( a = f_\bot \). That is, an element of \( B \) is either the greatest element or it is the smallest element.
q.e.d.

Theorem 6.5 Let \( M = (M, \text{TRUE}, f_\top, f_\bot, f_\Box, f_\to, f_\lor, f_\land, \leq) \) be a model. Then the reduct given by

\[ H(M) := (M, f_\top, f_\bot, f_\to, f_\lor, f_\land) \]

is an Heyting algebra with the disjunction property, and \( \text{TRUE} \) is an ultra-filter of \( H(M) \).

Proof. It is well-known that a partial ordering \((M, \leq)\) with a smallest element \( f_\bot \), a greatest element \( f_\top \), and binary operations \( f_\to \), \( f_\lor \), \( f_\land \) is a Heyting algebra iff the following hold for all \( m, m', m'' \in M \):
(a) \( m \leq f_\top \)
(b) \( f_\land(m, m') \leq m \)
(c) \( f_\land(m, m') \leq m' \)
(d) \( m \leq m' \) and \( m \leq m'' \) implies \( m \leq f_\land(m', m'') \)
(e) \( f_\bot \leq m \)
(f) \( m \leq f_\lor(m, m') \)
(g) \( m' \leq f_\lor(m, m') \)
(h) \( m' \leq m \) and \( m'' \leq m \) implies \( f_\lor(m', m'') \leq m \)
(i) \( m \leq f_\lor(m', m'') \) iff \( m \land m' \leq m'' \).

(a), (e), (b), (c), (f), (g) are true by definition of a model. We consider (d). Suppose \( m \leq m' \) and \( m \leq m'' \). There are variables \( x, y, z \) and an assignment \( \gamma \) such that \( \gamma(x) = m, \gamma(y) = m' \) and \( \gamma(z) = m'' \). Thus, \( (M, \gamma) \models \Box\{x \rightarrow y, x \rightarrow z\} \vdash_{\text{int}} x \rightarrow (y \land z) \). By Lemma 2.3 \( \Box\{x \rightarrow y, x \rightarrow z\} \vdash \Box(x \rightarrow (y \land z)) \). By soundness of our calculus, \( \Box\{x \rightarrow y, x \rightarrow z\} \models \Box(x \rightarrow (y \land z)) \). Thus, \( (M, \gamma) \models \Box(x \rightarrow (y \land z)) \). That is, \( m \leq f_\land(m', m'') \). The conditions (h) and (i) follow similarly. Let us show that the partial ordering \( \leq \) coincides with the lattice-theoretic ordering \( \leq \) given by \( m \leq m' \Leftrightarrow f_\land(m, m') = m \). Suppose \( m \leq m' \), i.e. \( f_\land(m, m') = f_\top \).
(recall that \( P = \{ f_\perp \} \)). Let \( x, y \) be variables and \( \gamma \) be an assignment such that \( \gamma(x) = m \) and \( \gamma(y) = m' \). Then \( (M, \gamma) \models \Box \{ x \rightarrow x, x \rightarrow y \} \). Since \( \{ x \rightarrow x, x \rightarrow y \} \vdash_{\text{int}} x \rightarrow (x \land y) \), we get \( \Box \{ x \rightarrow x, x \rightarrow y \} \vdash \Box (x \rightarrow (x \land y)) \).

By soundness, \( (M, \gamma) \models \Box (x \rightarrow (x \land y)) \). That is, \( f_\rightarrow (m, f_\land (m, m')) \in P \) and \( m \leq f_\land (m, m') \). Thus, \( m \preceq m' \). Similarly, one shows the implication \( m \preceq m' \Rightarrow m \leq m' \). Now it follows easily from the definition of a model that \( P = \{ f_\top \} \) is a prime filter and \( TRUE \) is an ultra-filter of the Heyting algebra \( H(M) \). q.e.d.

Theorem 6.5 defines an operation \( H \) that associates to each model \( M \) an Heyting algebra \( H(M) \) which has the disjunction property. \( H(M) \) is, in general, not a Boolean algebra. In order to see this, consider, e.g., the formula \( \neg \neg x \rightarrow x \), which is not a theorem of intuitionistic logic. By Corollary 5.2 and the Completeness Theorem, there is an interpretation \( (M, \gamma) \) such that \( (M, \gamma) \not\models \Box (\neg \neg x \rightarrow x) \). Thus, \( \gamma(\neg \neg x \rightarrow x) \notin P = \{ f_\top \} \) and therefore \( f_\rightarrow (f_\neg (\gamma(x))) \notin \gamma(x) \). If \( H(M) \) were a Boolean algebra, then we would have \( f_\rightarrow (f_\neg (m)) = m \), for every \( m \in M \); in particular \( f_\rightarrow (f_\neg (\gamma(x))) \leq \gamma(x) \).

Observe that our modal logic is classical in the sense that the logical connectives have a classical behavior. In this regard it might be surprising that the algebraic reducts of our models are, in general, not Boolean algebras but only Heyting algebras. In fact, instead of \( f_\rightarrow (f_\neg (m)) = m \), for all elements \( m \), it holds, in general, only the weaker condition \( f_\rightarrow (f_\neg (m)) \in TRUE \Leftrightarrow m \in TRUE \). That is, the propositions \( f_\rightarrow (f_\neg (m)) \) and \( m \) have the same truth value but are not necessarily equal. The same is true, for example, for the propositions \( f_\rightarrow (m, m') \) and \( f_\lor (f_\neg (m), m') \). This, however, is in perfect accordance with the principles of non-Fregean logic.

Corollary 6.6 Let \( M = (M, TRUE, f_\top, f_\perp, f_\Box, f_\rightarrow, f_\lor, f_\land, \leq) \) be a model and let \( H(M) \) be the associated Heyting algebra. Then \( H(M) \) is a Boolean algebra iff \( H(M) \) is the two-element Boolean algebra with universe M = \{ f_\perp, f_\top \}.

Proof. This follows immediately from Lemma 6.4 and Theorem 6.5. It also follows from Theorem 6.5 alone, if we consider the fact that in a Boolean algebra every prime filter is an ultra-filter (in particular: \( P = \{ f_\top \} \subseteq TRUE \) collapses with \( TRUE \), hence \( M = \{ f_\top, f_\perp \} \)). q.e.d.
Definition 6.7 Let $\mathcal{H}$ be an Heyting algebra and let $\text{TRUE}$ be any ultra-filter of $\mathcal{H}$. An operator of $\mathcal{H}$ w.r.t. the ultra-filter $\text{TRUE}$ is a function $f_\Box: H \to H$ satisfying $f_\Box(m) \in \text{TRUE} \iff m = f_\top$, for all $m \in H$. If $\mathcal{H} = (H, f_\top, f_\bot, f_\to, f_\vee, f_\wedge)$ is an Heyting algebra, $\text{TRUE} \subseteq H$ is an ultra-filter, and $f_\Box$ is an operator w.r.t. $\text{TRUE}$, then we call the structure $\mathcal{H} = (H, \text{TRUE}, f_\top, f_\bot, f_\Box, f_\to, f_\vee, f_\wedge)$ an Heyting algebra with an operator.

Theorem 6.8 Suppose $\mathcal{H} = (H, f_\top, f_\bot, f_\to, f_\vee, f_\wedge)$ is an Heyting algebra with the disjunction property. Let $\text{TRUE}$ be any ultra-filter of $\mathcal{H}$, $f_\Box$ an operator w.r.t. $\text{TRUE}$, and let $\leq$ be the ordering induced by the lattice. Then the expansion of $\mathcal{H}$ given by

$$\mathcal{M} = (H, \text{TRUE}, f_\top, f_\bot, f_\Box, f_\to, f_\vee, f_\wedge, \leq)$$

is a model.

Proof. Put $P := \{f_\top\}$. We must show that $\mathcal{M}$ satisfies the conditions of Definition 3.1. For this we apply a series of well-known facts. The first one is that the lattice ordering $\leq$ in a Heyting algebra is given by $m \leq m' \iff f_\to(m, m') = f_\top$ (this follows from the fact that in a Heyting algebra the element $f_\to(m, m')$ is the greatest element $x$ such that $f_\wedge(m, x) \leq m'$. Hence, condition (xiii) of a model is satisfied. A further known result is that the class of Heyting algebras constitutes a sound and complete algebraic semantics for intuitionistic propositional logic in the sense that a formula $\varphi$ is an intuitionistic theorem iff in every Heyting algebra $\mathcal{A}$, $\varphi$ is mapped by any assignment to the greatest element of $\mathcal{A}$. The conditions (i)--(ix) of Definition 3.1 now follow from condition (xiii) together with the fact that (i)--(ix) represent intuitionistic theorems. Conditions (x) and (xi) are trivially satisfied, condition (xii) holds true because $\mathcal{H}$ has the disjunction property, (xiv) is true by definition. Also it is known that in a distributive lattice with a smallest element every ultra-filter is a prime filter. Conditions (xvi) and (xvii) then follow from the hypothesis that $\text{TRUE}$ is an ultra-filter. Let us look at condition (xv). Suppose $f_\wedge(m, m') \in \text{TRUE}$ and $m \in \text{TRUE}$. Then $f_\wedge(m, f_\to(m, m')) \in \text{TRUE}$, since $\text{TRUE}$ is a filter. But $f_\wedge(m, f_\to(m, m')) \leq m'$ in every Heyting algebra. Since $\text{TRUE}$ is a filter, it follows that $m' \in \text{TRUE}$. Thus, the implication from left to right of (xv) holds true. Now suppose $m \notin \text{TRUE}$. Since $\text{TRUE}$ is a maximal filter, $f_\wedge(m) \in \text{TRUE}$. $\neg x \to (x \to y)$ is a theorem of intuitionistic logic and is mapped to $f_\top$ by any assignment. If we consider an assignment $\gamma$ such that $\gamma(x) = m$ and $\gamma(y) = m'$, then $f_\to(f_\wedge(m), f_\to(m, m')) = f_\top$. Thus,
f_\neg (m) = f_\land (f_\neg (m), f_\top) = f_\land (f_\neg (m), f_\neg (f_\neg (m), f_\rightarrow (m, m'))) \leq f_\rightarrow (m, m').

Since \textit{TRUE} is a filter, \( f_\rightarrow (m, m') \in \text{TRUE}. \) Similarly, if \( m' \in \text{TRUE}, \) then one uses the intuitionistic theorem \( y \rightarrow (x \rightarrow y) \) in order to show that \( f_\rightarrow (m, m') \in \text{TRUE}. \) q.e.d.

Of course, the Heyting algebra \( H(\mathcal{M}) \), as the reduct of a given model \( \mathcal{M} \), is uniquely determined by \( \mathcal{M} \). On the other hand, for a given Heyting algebra \( \mathcal{H} \) with disjunction property there is, in general, more than one expansion \( \mathcal{M} \) such that \( H(\mathcal{M}) = \mathcal{H} \). Such an expansion \( \mathcal{M} \) depends on the particular choice of the ultra-filter \( \text{TRUE} \) of \( \mathcal{H} \) and the concrete definition of the associated operator \( f_\Box \).

Theorem \[6.5\] along with Theorem \[6.8\] (and Definition \[6.7\]) constitute a representation result which can be expressed in the following way.

\textbf{Corollary 6.9} The class of models is given by the class of all Heyting algebras with an operator and with the disjunction property.

By Lemma \[6.4\] among the Heyting algebras which represent models is, up to isomorphism, exactly one Boolean algebra, namely the two-element one.

Recall that the class of all Heyting algebras constitutes a sound and complete algebraic semantics of intuitionistic propositional logic. By Corollary \[5.2\] soundness and completeness of our calculus, and Corollary \[6.9\] it is enough to consider all Heyting algebras with an operator and with the disjunction property:

\textbf{Corollary 6.10} Let \( \varphi \in Fm_0 \). Then \( \varphi \) is a theorem of intuitionistic propositional logic iff in any Heyting algebra with an operator and with the disjunction property, every assignment maps \( \varphi \) to the greatest element \( f_\top \).

It is known that the set \( H \) of equivalence classes \( \varphi \) of propositional formulas \( \varphi \) modulo intuitionistic equivalence form an Heyting algebra \( H_{\text{int}} \). Since intuitionistic propositional logic satisfies the disjunction property, the greatest element \( f_\top \) of \( H_{\text{int}} \), i.e. the set of all intuitionistic theorems, is a prime filter of \( H_{\text{int}} \). Thus, \( H_{\text{int}} \) is a Heyting algebra with the disjunction property in the sense of Definition \[6.3\]. Of course, every ultra-filter of \( H_{\text{int}} \) represents a maximal consistent set in intuitionistic logic. Such a set is also maximal consistent in classical logic. On the other hand, every maximal consistent set is represented by an ultra-filter of \( H_{\text{int}} \). Define an operator \( f_\Box \) on \( H \) by \( f_\Box (m) = f_\top \) if \( m = f_\top \), and \( f_\Box (m) = f_\bot \) otherwise. Note that this is an operator w.r.t. any ultra-filter. Choose an ultra-filter \( \text{TRUE} \) of \( H_{\text{int}} \) (\( \text{TRUE} \) corresponds to a maximal consistent set of intuitionistic
or classical propositional logic). Let $M_{int}$ be the model whose existence is guaranteed by Theorem 6.8. Let $\varepsilon : V \cup \{ \bot \} \to H$ be the assignment $x \mapsto \overline{x}$. Then $\varepsilon(\varphi) = \overline{\varphi}$, for each $\varphi \in Fm_0$. It follows that $(M_{int}, \varepsilon) \models \varphi \iff \overline{\varphi} \in TRUE$, for every $\varphi \in Fm_0$. Moreover, from the definitions it follows immediately that for all $\varphi \in Fm_0$:

\[
(M_{int}, \varepsilon) \models \Box \varphi \iff \varepsilon(\Box \varphi) = f_\Box(\varepsilon(\varphi)) \in TRUE
\]

\[
\iff \varepsilon(\varphi) = \overline{\varphi} = f_\top
\]

\[
\iff \vdash_{int} \varphi, \text{i.e., } \varphi \text{ is an intuitionistic theorem}
\]

That is, we get an alternative proof of Corollary 5.4 and Corollary 5.5.

References

[1] S. L. Bloom and R. Suszko, *Investigation into the sentential calculus with identity*, Notre Dame Journal of Formal Logic 13(3), 289 – 308, 1972.

[2] C. Caleiro and J. Ramos, *From fibring to cryptofibring: A solution to the collapsing problem*, Logica Universalis, 1(1): 71 – 92, 2007.

[3] W. Carnielli and M. E. Coniglio, *Combining Logics*, The Stanford Encyclopedia of Philosophy (Winter 2011 Edition), Edward N. Zalta (ed.), [http://plato.stanford.edu/archives/win2011/entries/logic-combining/](http://plato.stanford.edu/archives/win2011/entries/logic-combining/)

[4] L. F. del Cerro and A. Herzig, *Combining classical and intuitionistic logic, or: Intuitionistic implication as a conditional*, in F. Baader and K. Schulz, eds., *Frontiers of Combining Systems: Proceedings of the 1st International Workshop*, Munich (Germany), volume 3 of Applied Logic Series, pp. 93 – 102, Kluwer Academic Publishers, 1996.

[5] D. van Dalen, *Logic and Structure*, 4th ed., Springer, 2004.

[6] D. Gabbay, *Fibred semantics and the weaving of logics: Part 1*, Journal of Symbolic Logic, 61(4): 1057 – 1120, 1996.

[7] D. Gabbay, *An overview of fibred semantics and the combination of logics*, in F. Baader and K. Schulz, eds., *Frontiers of Combining Systems: Proceedings of the 1st International Workshop*, Munich (Germany), volume 3 of Applied Logic Series, pp. 1 – 55, Kluwer Academic Publishers, 1996.
[8] S. Lewitzka, $\in_I$: An intuitionistic logic without Fregean Axiom and with predicates for truth and falsity, Notre Dame Journal of Formal Logic 50(3): 275 – 301, 2009.

[9] S. Lewitzka, $\in_K$: a non-Fregean logic of explicit knowledge, Studia Logica 97(2), 233 – 264, 2011.

[10] S. Lewitzka, Semantically closed intuitionistic abstract logics, Journal of Logic and Computation, 22 (3): 351 – 374, 2012.

[11] S. Lewitzka, Construction of a canonical model for a first-order non-Fregean logic with a connective for reference and a total truth predicate, Logic Journal of the IGPL 20(6), 1083 – 1109, 2012.

[12] S. Lewitzka, Denotational semantics for normal modal logics with propositional quantifiers and identity, 2013, [http://arxiv.org/abs/1212.6576](http://arxiv.org/abs/1212.6576)

[13] S. Lewitzka, A denotational semantics for a Lewis-style modal system close to $S1$, 2013, [http://arxiv.org/abs/1304.6983](http://arxiv.org/abs/1304.6983)

[14] C. Sernadas, J. Rasga and W. Carnielli, Modulated fibring and the collapsing problem, Journal of Symbolic Logic, 67(4): 1541 – 1569, 2002.

[15] A. Sernadas, C. Sernadas, J. Rasga and M. Coniglio, A graph-theoretic account of logics, Journal of Logic and Computation, 19(6): 1281 – 1320, 2009.

[16] A. Sernadas, C. Sernadas, J. Rasga and M. Coniglio, On graph-theoretic Fibring of logics, Journal of Logic and Computation, 19(6): 1321 – 1357, 2009.

[17] A. Sernadas, C. Sernadas, and J. Rasga, On combined connectives, Logica Universalis, 5(2): 205 – 224, 2011.

[18] A. Sernadas, C. Sernadas, and J. Rasga, On meet-combination of logics, Journal of Logic and Computation 22(6): 1453 – 1470, 2012.