A self-similar inhomogeneous dust cosmology

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Abstract

A detailed study of an inhomogeneous dust cosmology contained in a $\gamma$-law family of perfect-fluid metrics recently presented by Mars and Senovilla is performed. The metric is shown to be the most general orthogonal transitive, Abelian, $G_2$ on $S_2$ solution admitting an additional homothety such that the self-similar group $H_3$ is of Bianchi type VI and the fluid flow is tangent to its orbits. The analogous cases with Bianchi types I, II, III, V, VIII and IX are shown to be impossible thus making this metric privileged from a mathematical viewpoint. The differential equations determining the metric are partially integrated and the line-element is given up to a first order differential equation of Abel type of first kind and two quadratures. The solutions are qualitatively analyzed by investigating the corresponding autonomous dynamical system. The space-time is regular everywhere except for the big bang and the metric is complete both into the future and in all spatial directions. The energy-density is positive, bounded from above at any instant of time and with an spatial profile (in the direction of inhomogeneity) which is oscillating with a rapidly decreasing amplitude. The generic asymptotic behaviour at spatial infinity is a homogeneous plane wave. Well-known dynamical system results indicate that this metric is very likely to describe the asymptotic behaviour in time of a much more general class of inhomogeneous $G_2$ dust cosmologies.

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1 Introduction

Solutions of Einstein’s field equations for dust with the energy-momentum tensor

\[ T_{\alpha\beta} = \rho u_\alpha u_\beta, \]

where \( \rho \) and \( \vec{u} \) are the energy-density and the velocity vector of the fluid, respectively, are adequate for describing cosmological models at late times, i.e. when the electromagnetic radiation becomes dynamically negligible and the evolution of the universe is dominated by matter. The standard model describing the geometry of the universe at late times is the spatially homogeneous and isotropic Friedman-Lemaitre-Robertson-Walker metric with vanishing pressure. However, since the universe is not exactly isotropic and homogeneous, it becomes necessary to consider also less symmetric dust cosmologies in order to describe those irregularities. The first natural generalization are the spatially homogeneous Bianchi (including locally rotationally symmetric - LRS-) cosmologies \([1]\) (see also \([2]\) for more recent developments).

The next step forward is generalizing the models so as to describe inhomogeneous cosmologies. The simplest case here is the class of inhomogeneous models admitting a three-dimensional group of isometries acting on spacelike two-surfaces \((G_3 \text{ on } S_2)\) models). That class of dust solutions includes the spherically symmetric case (Tolman–Bondi metric) first found by Lemaitre \([3]\) and Tolman \([4]\) (see also Krasiński’s review book \([\text{?}]\) for a comprehensive discussion on this topic) and the plane and hyperbolic symmetric ones first discovered by Ellis \([5]\). The study of more general inhomogeneous models has been much less systematic and only very partial and incomplete results are known. Among them, we find the remarkable Szekeres \([7]\) class of inhomogeneous irrotational dust solutions with Petrov type D and vanishing magnetic part of the Weyl tensor (along the fluid congruence). A detailed analysis of the physical and mathematical aspects of this class of solutions can be found in \([8]\) and references therein.

In a systematic approach for investigating inhomogeneous dust cosmologies, the next step is the study of spacetimes admitting two linearly independent spacelike Killing vector fields, the so-called \(G_2\) cosmologies. Very few solutions are known within this class. The first one is contained in a \(\gamma\)-law family of solutions obtained recently in \([8]\). The solution was given up to a coupled system of non-linear, ordinary differential equations. As we shall see below, this metric has interesting mathematical and physical properties and will be carefully studied in this paper. Very recently, the first completely \(G_2\) dust solutions have been obtained \([9]\). It is worth mentioning here that the situation is completely different when the spacetime is stationary and axially symmetric (so that the isometry group is still two-dimensional and Abelian, but acting on timelike surfaces). See \([10]\) for a review of dust solutions for this case.

In this paper we study in detail the inhomogeneous dust solution of Einstein’s field equations contained in the \(\gamma\)-law family of perfect fluids given in \([8]\). Its maximal isometry group is two-dimensional, Abelian and acting orthogonally transitively on spacelike orbits. The metric is non-diagonal and turns out to be self-similar, i.e. the spacetime possesses also a homothetic Killing vector (see \([11]\) for a definition). The
self-similar group $H_3$ (spanned by the Killing vector fields and the homothetic Killing vector) acts on timelike hypersurfaces and the fluid velocity is tangent to its orbits (we will use the term **tangent self-similar** solution to denote those solutions for which the fluid velocity is tangent to the self-similar orbits). As we shall see below, this metric is the most general orthogonally transitive $G_2$ tangent self-similar solution such that the $H_3$ is of Bianchi type VI. Furthermore, no analogous solutions for Bianchi types I, II, III or V exist. Since Bianchi types VIII and IX are impossible (they do not contain an Abelian two-dimensional subgroup), the only remaining cases which could in principle admit self-similar tangent solutions are Bianchi types IV or VII (these cases are not considered in this paper). As explained below, the tangent self-similar inhomogeneous models are known to be suitable candidates to describe the behaviour of more general cosmologies at late times or near the big bang singularity. Therefore, the solution studied here is very likely to describe the asymptotic behaviour in time of a large class of general inhomogeneous $G_2$ dust cosmologies. The metric is algebraically general, sharing this property with the explicit solutions recently found in [9] (both Szekeres and $G_3$ on $S_2$ are of Petrov type D). The system of coupled, ordinary differential equations is partially solved so that the metric can be given up to one first order ordinary differential equation of Abel type and two quadratures. The most relevant properties of the solution are obtained by performing a detailed qualitative study of the dynamical system associated with the Abel equation. In particular, we show that the metric is free of singularities except for the big bang, the spacetime is complete both into the future and all spatial directions and the energy-density is positive and bounded from above at each instant of time. The number of essential parameters is two.

The plan of the paper is as follows. In section 2, the line-element given in [8] and the differential equations to be solved are written down. The results stating the uniqueness of the metric are presented and a summary of known results about the relevance of self-similar cosmologies as asymptotic states for more general models is included. We then rewrite the metric so that only a first order differential equation and two quadratures remain to be solved and we give some properties of the spacetime, like the kinematical quantities of the fluid, the energy-density, the Petrov type and the Killing generators of the self-similar group, as well as the Bianchi type of the Lie algebra they generate. In section 3, a detailed qualitative analysis of the two-dimensional dynamical system, including a description of the topology of the phase space, the set of equilibrium points and its character and a proof of the non-existence of periodic orbits is performed. We then consider the asymptotic behaviour of the solution at spatial infinity which turns out to be a homogeneous vacuum plane wave of Petrov type N. The coordinate transformation which brings this plane wave metric into the standard form is then given, thus allowing us to partially interpret the original coordinates in which the dust metric is written. Finally, the limiting cases for the family are analyzed. In section 4 we describe how this dust metric fits into the general framework developed by Hewitt and Wainwright to describe Abelian $G_2$ perfect-fluids with a $\gamma$-law equation of state. This is particularly interesting due to the special character of this solution.
as an equilibrium point of the dynamical system in terms of expansion normalized, dimensionless variables.

## 2 General dust solution

Finding perfect-fluid solutions of Einstein’s field equations for $G_2$ Abelian cosmologies is a difficult task and several simplifications have been used in order to obtain particular solutions. One of the most fruitful ansätze has been assuming that the metric coefficients in comoving coordinates are products of functions of only one variable (the so-called separable cosmologies). Within the class $A(ii)$ in Wainwright’s classification [12] for $G_2$ Abelian cosmologies, the separable case with a $\gamma$-law equation of state (i.e. $p = \gamma \rho$ where $p$ is the pressure of the fluid) has been studied in [13]. From results in that paper it is easy to infer that the particular case $\gamma = 0$ implies vacuum and so no dust solutions is possible. The separable diagonal case was studied in [14] (see also [15] for the stiff fluid solutions and the pioneering work in [16] where the solutions with the three-spaces orthogonal to the fluid velocity being conformally flat were found). Again, no dust solutions are contained in this family. The next case in complexity (class B(i) in Wainwright’s classification) has been recently analyzed in [8] where the complete list of solutions has been given (for some families a set of ordinary differential equations remains to be solved). One of these families describes a perfect fluid with $\gamma$-law equation of state. The particular case $\gamma = 0$ (dust) is now possible. From the results in [8] it is not difficult to see that the most general dust solution for space times admitting an Abelian maximal two-dimensional isometry group acting orthogonally transitively on spacelike surfaces with separable metric coefficients in comoving coordinates can be written in the form

$$ds^2 = e^{2b\hat{t}} \left( -d\hat{t}^2 + d\hat{x}^2 \right) + G Pe^{(b+1)i} dy^2 + \frac{G}{P} e^{(b-1)i} \left( dz + We^i dy \right)^2,$$

(2)

where $b$ is an arbitrary non-vanishing constant and $G$, $P$ and $W$ are functions of $\hat{x}$ which must satisfy the ordinary system of differential equations (prime denotes derivative with respect to $\hat{x}$)

$$b\frac{G'}{G} - \frac{P'}{P} - \frac{WW'}{P^2} = 0,$$

(3)

$$\frac{G'}{G} \frac{P'}{P} + \frac{P''}{P} - \frac{P'^2}{P^2} + \frac{W^2}{P^2} - \frac{W'}{P} - b = 0,$$

$$\frac{G''}{G^2} - \frac{P'^2}{P^2} - \frac{W'^2}{P^2} - \frac{W^2}{P^2} + b^2 - 1 = 0.$$

The symmetry assumptions in deriving this metric involved only a maximal isometry group of dimension two. However, it turns out that the metric (2) possesses also a homothetic Killing vector (in fact, this also holds for the full class of solutions with $\gamma$-law equation of state in [8]). The three-dimensional homothety group $H_3$ belongs
to the Bianchi type VI and its orbits are timelike everywhere with the the fluid flow tangent to them.

Spacetimes admitting a three-dimensional homothety group have been investigated by several authors in the last few years (see e.g. \cite{17} and references therein). In particular, a recent paper by Carot et al \cite{18} gives the explicit time dependence for this kind of metrics. Furthermore, in \cite{17} it is stated that tangent self-similar perfect fluids of Bianchi types I and II have to satisfy $\rho + 3p = 0$ thus excluding dust solutions. Combining their results with the uniqueness of the metric (3) as the only dust solution with separable coefficients in comoving coordinates, the following proposition holds

**The line-element (3) satisfying the equations (3) is the most general dust solution of Einstein’s field equation for spacetimes of class B in Wainwright’s classification of inhomogeneous cosmologies, admitting one homothety and such that the fluid flow is tangent to the self-similar group orbits and the Bianchi type of the three-dimensional homothetic group is VI. Furthermore, the analogous cases in which the self-similar group is of Bianchi types I, II, III and V are impossible.**

Additionally, $G_2$ Abelian cosmologies with an additional homothetic vector field such that the three-dimensional self-similar group $H_3$ is of Bianchi types VIII and IX are clearly impossible because these Bianchi types do not contain Abelian two-dimensional subalgebras. In principle, $G_2$ dust solutions with an $H_3$ of Bianchi types IV and VII could admit tangent solutions but none of these algebras admit diagonal metrics (unless the spacetimes possess more than two isometries). Furthermore, the diagonal limit of the metric (2) (obtained for $W \equiv 0$) can be trivially seen to be either Minkowski or a Bianchi cosmology. Thus, the following result also holds

**No dust solutions of Einstein’s field equations exist for diagonal inhomogeneous cosmologies (B(ii) in Wainwright’s classification) admitting a homothetic symmetry such that the fluid flow is tangent to the self-similar orbits.**

Tangent self-similar cosmologies play a prominent role in describing the asymptotic behaviour (either near the big bang singularity or at $t \to \infty$) for more general cosmological models. This result is one of the main consequences of the dynamical system analysis of cosmological models which has been undertaken in recent years (see \cite{2} and references therein). The dynamical system approach consists of writing Einstein’s field equations as first order partial differential equations for some kinematical quantities associated with an orthonormal tetrad which must be chosen appropriately \cite{19}. Then, an expansion normalized set of variables is introduced and a qualitative analysis of the resulting dynamical system is performed. This approach for analyzing cosmological models started by considering the simplest cases, namely homogeneous cosmologies in which the Einstein field equations become ordinary differential equations and therefore the associated phase space is finite dimensional. The mathematical theory for describing finite dimensional dynamical systems is well-developed and a good number
of properties of the solutions can be derived without the need of solving the equations explicitly. One of the most relevant results which were proven in that context \[20\], \[21\] is that all equilibrium points of the dynamical system associated with non-tilted Bianchi cosmologies (i.e. such that the fluid velocity is orthogonal to the group orbits) satisfying a $\gamma$-law equation of state correspond to self-similar cosmologies, i.e. spatially homogeneous spacetimes admitting a homothetic Killing vector. Thus, this class of self-similar Bianchi models plays a relevant role as asymptotic states for more general cosmologies.

For the case of $G_2$ cosmologies, the phase space becomes, in general, infinite dimensional since the tetrad form of Einstein’s equations are first order partial differential equations. The mathematical theory in this case is less developed and the results which can be obtained are more restricted. However, Hewitt and Wainwright \[22\] reformulated the field equations as a dynamical system for orthogonally transitive $G_2$ perfect fluid cosmologies with a $\gamma$-law equation of state. In \[22\] it has been proven that the equilibrium points of that dynamical system also correspond to self-similar models. They admit a three-dimensional self-similar group $H_3$ (spanned by the two Killing vectors and one homothetic Killing vector) acting on timelike hypersurfaces. The fluid velocity vector is tangent to the three-dimensional similarity orbits. These two results suggest that tangent self-similar models are important in describing the asymptotic behaviour of a large class of inhomogeneous cosmologies. Thus, the unique (for the Bianchi types I, II, III, V and VI of the $H_3$ group) tangent self-similar solution \[1\] is very likely to be relevant for describing the asymptotic state of general dust cosmologies.

All these results above show that the metric \[1\] is privileged from a mathematical point of view, and, thus, it deserves a more detailed analysis. Furthermore, as we shall see below, it turns out that the metric is also well-behaved from a physical viewpoint. Let us then proceed with the analysis of the metric \[1\] and field equations \[3\]. First of all, it is immediate to see that the transformation

$$
b \longleftrightarrow -b, \quad \hat{t} \longleftrightarrow -\hat{t}, \quad z \longleftrightarrow y, \quad P \longleftrightarrow \frac{P}{P^2 + W^2}, \quad W \longleftrightarrow \frac{W}{P^2 + W^2}
$$

leaves the metric \[1\] invariant. Thus, the constant $b$ can be restricted to be positive without loss of generality. Furthermore, the energy-density of the fluid is well-behaved only for $b \geq 1$ (otherwise it is negative everywhere) so that we can restrict the analysis to $b \geq 1$ and introduce another constant $\alpha$ defined as

$$
\sin \alpha = \frac{1}{b}, \quad \cos \alpha \geq 0.
$$

The system of differential equations we have to deal with constitutes (after substituting $\frac{\mathcal{G}}{\mathcal{T}}$ obtained from the first equation in \[3\] into the other two equations) a non-linear coupled system of two ordinary differential equations for two unknowns. One of the equations is of second order and linear in the highest derivative and the other is of the
first order but quadratic in the highest derivatives. Thus, the system is very com-

plex and it must be simplified. To that end we rewrite the metric, after redefining
the coordinates \( \hat{t} \) and \( \hat{x} \) and rescaling \( y \) and \( z \), as

\[
\begin{align*}
ds^2 &= -dt^2 + t^2 \frac{dx^2}{L(x)} + G \frac{t^{1+\sin \alpha}}{P} dy^2 + \frac{G}{P} t^{1-\sin \alpha} \left( dz + W t^{\sin \alpha} dy \right)^2
\end{align*}
\]

where \( L(x) \) is an arbitrary non-vanishing function at our disposal. This function has

to be chosen appropriately in order to simplify the field equations without introducing
unnecessary coordinate singularities in the metric. This is not a trivial task and a

careful investigation of the equations (3) is involved. It turn out that the choice

\[
L(x) = \sin^2 \alpha + \cos^2 \alpha \cos^2 \left( \frac{H(x)}{2} \right),
\]

(4)

allows for an explicit integration of the function \( W \) and decomposes the remaining
field equations into a first order ordinary differential equation for the function \( H \)
and two quadratures for \( G \) and \( P \) respectively, which is a substantial simplification of the
problem. The metric can be written as

\[
\begin{align*}
ds^2 &= -dt^2 + \frac{t^2 dx^2}{\sin^2 \alpha + \cos^2 \alpha \cos^2 \left( \frac{H}{2} \right)} + t^{1+\sin \alpha} G \frac{dy^2}{P} \\
&\quad + t^{1-\sin \alpha} \frac{G}{P} \left( dz + \cos \alpha \sin \alpha t^{\sin \alpha} P \cos \left( \frac{H}{2} \right) dy \right)^2.
\end{align*}
\]

(5)

The first order differential equation for the function \( H \) is best analyzed when written
as an autonomous dynamical system in two dimensions. It reads, explicitly,

\[
\begin{align*}
\frac{dH}{dx} &= 2 \cos Q, \\
\frac{dQ}{dx} &= \cos Q + \frac{\sin Q \sin H}{2 \tan^2 \alpha + 1 + \cos H}.
\end{align*}
\]

(6)

where we have introduced a new function \( Q(x) \). The change \( Q \equiv \arctan |M(H)| \)
transforms this dynamical system back into the first order ordinary differential equation

\[
\frac{dM}{dH} = \frac{1}{2} \left( 1 + M^2 \right) \left[ 1 + \frac{M \sin H}{2 \tan^2 \alpha + 1 + \cos H} \right],
\]

which is an Abel equation of the first kind \[23\]. It does not belong to the known inte-
grable cases \[24\]. Although apparently simple, solving this Abel equation is still very
difficult and no particular solutions have been found. However, a qualitative analysis
of the dynamical system (3) is enough for a good understanding of the solutions. This
will be performed in the next section.
The other two metric coefficients can then be calculated by the quadratures

\[
\frac{G_x}{G} = \sin Q, \quad (7)
\]

\[
\frac{P_x}{P} = \frac{2 \sin \alpha \sin Q + \cos^2 \alpha \cos Q \sin H}{2 \sin^2 \alpha + \cos^2 \alpha (1 + \cos H)}. \quad (8)
\]

The number of essential parameters in the family (5) is two; the explicit constant \(\alpha\) in the line-element and the initial condition in the dynamical system (6). The integration constants in \(G\) and \(P\) are superfluous and can be set equal to one by trivial redefinitions of \(y\) and \(z\).

Let us now describe some of the properties of the dust solution we are considering. The fluid velocity vector is \(\vec{u} = \frac{\partial}{\partial t}\) and the energy-density reads

\[
\rho = \frac{\cos^2 \alpha \sin^2 \left(\frac{H}{2}\right)}{t^2}, \quad (9)
\]

which is positive everywhere and bounded from above at any instant of time \(t\). The metric (5) has a big bang singularity at \(t = 0\), where the energy-density blows up. Regarding the kinematical quantities, the fluid velocity is obviously geodesic and irrotational and the expansion is spatially homogeneous and reads

\[
\theta = \frac{2}{t},
\]

showing that the fluid never recollapses. The shear tensor is highly anisotropic and the shear scalar is

\[
\sigma_{\alpha \beta} \sigma^{\alpha \beta} = \frac{4 - 3 \cos^2 \alpha \sin^2 \left(\frac{H}{2}\right)}{6t^2}.
\]

All fluid elements are expanding and a comoving observer would measure red shift in every direction. Regarding the Petrov type, the metric is algebraically general (Petrov type I) except for the particular case \(\alpha = \frac{\pi}{2}\) when it degenerates to conformally flat. In this case the energy-density (9) also vanishes and the metric becomes flat. The magnetic part of the Weyl tensor along the fluid velocity vector in the spacetime (5) is non-vanishing. Except for some particular subcases described below, the metric (5) has the only two Killing vectors \(\xi_2 = \partial_y\) and \(\xi_3 = \partial_z\). The homothetic Killing vector is

\[
\xi_1 = t \frac{\partial}{\partial t} + \frac{(1 - \sin \alpha)}{2} y \frac{\partial}{\partial y} + \frac{(1 + \sin \alpha)}{2} z \frac{\partial}{\partial z},
\]

and the three-dimensional homothetic Lie algebra spanned by these three vectors is of Bianchi type \(VI_h\), where \(h\) is given by

\[
h = \frac{-1}{\sin^2 \alpha} = -b^2.
\]
This dust solution can be matched to a vacuum solution with two commuting spacelike Killing vectors (which can therefore be interpreted as a gravitational wave), but the vacuum metric cannot possess an additional homothetic Killing vector any longer. Indeed, it is easy to see that the matching between the dust solution and any vacuum metric must be performed across one of the homothetic group orbits in the dust solution. If we assume that the homothetic Killing vector extends to the vacuum region, the time dependence of the vacuum metric becomes determined everywhere and must coincide (due to the continuity of the first fundamental forms on the matching hypersurface) with the time dependence in the dust region. Thus, the vacuum metric must be a homogeneous plane wave. However, the remaining set of matching conditions forbids the matching of a homogeneous plane wave and the self-similar dust cosmology at finite distances. The field equations for the vacuum solution matching the metric (5) are still quite complicated, and no explicit solution has been found.

3 Phase space analysis and limiting solutions

In this section we will analyze the spatial behaviour of the dust metrics (5). To this aim, we have to study the dynamical system (6). This is a two-dimensional dynamical system, depending on an arbitrary parametric constant \( \alpha \), which possesses the two discrete symmetries \( Q \to Q + 2\pi \) and \( H \to H + 2\pi \). There is another discrete symmetry \( \{ Q \to -Q, H \to -H \} \). Thus, it is sufficient to examine the domain \( \mathcal{U} = (-\pi, \pi] \times (-\pi, \pi] \) in the phase space diagram for \( H \) and \( Q \) and use the above periodicity. More strictly, the phase space of the dynamical system (6) is topologically a two-dimensional torus \( S^1 \times S^1 \) which can be described by the coordinates \( (H, Q) \in \mathcal{U} \) by identifying the two vertical and the two horizontal boundaries respectively. The qualitative behaviour of the dynamical system is independent of the value of the parameter \( \alpha \) and no bifurcations occur. A plot for the phase space portrait with a typical value \( \alpha = \pi/4 \) (which will be assumed for all Figures) is given in Fig.1. In order to understand the behaviour of the solutions we must consider the set of fixed points of the dynamical system. There are four fixed points; two of them are saddle points, one is an attractor and the remaining one is a repellor. They are given by

\[
\text{Saddle points: } \left( Q = \frac{\pi}{2}, H = \pi \right), \quad \left( Q = -\frac{\pi}{2}, H = \pi \right) \tag{11}
\]

\[
\text{Attractor: } \left( Q = \frac{\pi}{2}, H = 0 \right), \quad \text{Repellor: } \left( Q = -\frac{\pi}{2}, H = 0 \right). \tag{12}
\]

The two solutions corresponding to the two-saddle points are equivalent and represent a spatially homogeneous self-similar cosmology (with the three-dimensional isometry group of Bianchi type VI). The two solutions corresponding to the attractor and repellor are also equivalent and represent vacuum spacetimes. Since the homotopy group of the torus is non-trivial (it is isomorphic to the additive group \( \mathbb{Z} \)), any solution starting at the repellor and finishing at the attractor can be classified by the number of times the trajectory goes around the torus before reaching its endpoint (i.e. they can be
classified by the homotopy class they belong to). In order to understand the possible behaviour of the solutions, it is convenient to draw the set of trajectories finishing or starting at the saddle points. The plot is given in Fig. 2. These curves divide the phase space diagram into basins of attraction which give a clear description of the behaviour of the solutions. In particular, we can conclude from this diagram that no periodic solutions of (6) exist. This follows from the fundamental fact that two different orbits in any dynamical system cannot intersect anywhere and that any closed orbit which is contractible to a point must contain an equilibrium point in its interior.

Thus, all the solutions of (6), except those starting and finishing on the saddle points, approach asymptotically the repellor (for \( x \to -\infty \)) and the attractor (for \( x \to +\infty \)). Regarding the different homotopy classes the solutions can belong to, it follows from Fig 2. that two different cases are possible. Either the solutions go from the repellor to the attractor without going around the torus, or they go around it at most once. In the second case, the solutions can belong to two different basins of attraction, which can be distinguished by the number of times the boundary of \( \mathcal{U} \) is crossed. This boundary can be crossed either once (in the \( H \) direction) or twice (one in the \( H \) and one in the \( Q \) direction). A typical solution for each one of the three possible cases is shown in Fig. 3, together with the energy-density \( \rho \), which oscillates with rapidly decreasing amplitude.

A numerical analysis of the solutions of (6) shows that the asymptotic approach to the stable fixed points (12) is quite fast and that the solutions oscillate around those points in their asymptotic approach. Since every crossing of the value \( H = 0 \) corresponds to a vanishing value for the energy-density (see (9)), it turns out that the energy-density profile of the solutions is oscillating in the positive range, reaching the zero value at a discrete number of points and with a rapidly decreasing amplitude (see the energy-density plots in Fig.3 for examples of this behaviour).

Let us now study in detail the line-elements corresponding to the fixed points, since this will allow us for a partial interpretation of the coordinate system \( \{t, x, y, z\} \) in (6).

The asymptotic vacuum solution is represented in the coordinate system (6) by the line-element

\[
ds^2 = -dt^2 + t^2 dx^2 + (te^{\epsilon x})^{1+\sin \alpha} dy^2 + (te^{\epsilon x})^{1-\sin \alpha} \left( dz + \frac{\cos \alpha}{\sin \alpha} (te^{\epsilon x})^{\sin \alpha} dy \right)^2
\]

where \( \epsilon = +1 \) (\( \epsilon = -1 \)) for \( x \to +\infty \) (\( x \to -\infty \)). This vacuum spacetime is a homogeneous plane wave with a six-dimensional isometry group acting transitively on \( V_4 \). The Petrov type is N and the only repeated principal null direction is \( \vec{l} = t\partial_t - \epsilon \partial_x \). Thus, the inhomogeneous dust metric contains a matter and a radiative part. The radiative part dominates at large distances in the spatial direction perpendicular to the two-planes spanned by the isometries. In a cosmological context it could be interpreted as a gravitational wave background superposed to the expanding dust. This asymptotic behaviour at spacelike infinity seems to be quite common in this kind of situations for perfect fluids. Hewitt et al [25] performed a dynamical system analysis of tangent self-similar diagonal \( G_2 \) perfect fluids with a \( \gamma \)-law equation of state. They concluded
that the spacelike asymptotic behaviour can be either matter dominated (i.e. with a
non-zero value for the energy-density at spacelike infinity) or vacuum dominated (in
which the energy-density tends to zero at infinity). In this second case, the vacuum
asymptotic behaviour is also a homogeneous plane wave.

The very neat behaviour of the solution at spatial infinity can provide us with
a partial interpretation for the coordinate system \( \{ t, x, y, z \} \). In order to do so, we
consider the coordinate change which brings (13) into the standard form for plane
waves [1]

\[
ds^2 = -2du dv + 2\left[ A(u) \left( Z^2 - Y^2 \right) + 2B(u)YZ \right] du^2 + dY^2 + dZ^2. \tag{14}
\]

It can be easily seen that this coordinate transformation is given by

\[
y = -u^{\frac{1+\sin \alpha}{2}} \left[ Y \sin(\beta - \alpha) + Z \cos(\beta - \alpha) \right], \quad z = \frac{u^{-\frac{1+\sin \alpha}{2}}}{\sin \alpha} \left( Y \sin \beta + Z \cos \beta \right), \tag{15}
\]

\[
te^{cx} = u, \quad te^{-cx} = 2v - \frac{1}{2u} \left[ (Z^2 - Y^2) \sin(2\beta - \alpha) - 2YZ \cos(2\beta - \alpha) + Y^2 + Z^2 \right],
\]

where

\[
\beta \equiv \frac{\cos \alpha}{2} \log u. \tag{16}
\]

The two function \( A(u) \) and \( B(u) \) in (14) read

\[
A(u) = \frac{\cos \alpha}{4u^2} \cos(2\beta - \alpha), \quad B(u) = \frac{\cos \alpha}{4u^2} \sin(2\beta - \alpha)
\]

So, near spatial infinity a partial interpretation of the coordinate system (5) is possible.
The coordinate \( x \) measures a spacelike distance along the direction of propagation of
the plane wave. The coordinates \( y \) and \( z \) are straight lines lying in a plane tangent to
the null hypersurface corresponding to the plane wave front. They are not orthogonal
lines, the angle between them being directly related to the parameter \( \alpha \). The lines
\( y = \text{const.} \) and \( z = \text{const.} \) are scaled by different units, so there is no rotational
symmetry in the corresponding plane. Moreover, the straight lines associated with \( y \)
and \( z \) are rotated by an angle \( \beta \) (which depends on \( u \) and therefore is different for
different wave fronts) with respect to the orthogonal straight lines \( Y, Z \) associated
with the plane of symmetry of the wave. It follows then that the coordinates \( y, z \) are
rotating with respect to the homogeneous plane wave background.

The solution corresponding to the saddle points is a diagonal spatially homogeneous
spacetime of Bianchi type VI \((-\sin^2 \alpha)\). The line-element (5) takes the form

\[
ds^2 = -dt^2 + t^2 dx^2 + (te^{cx})^{1+\sin \alpha} dy^2 + (te^{-cx})^{-\sin \alpha} dz^2 \tag{17}
\]

which possesses the additional killing vector field

\[
\xi_1 = \frac{\partial}{\partial x} - \frac{1 + \sin \alpha}{2} y \frac{\partial}{\partial y} + \frac{1 - \sin \alpha}{2} z \frac{\partial}{\partial z}
\]
and the energy density is \( \rho = \cos^2 \alpha \). This metric was first obtained by Collins [26] and appears listed in the catalogue of self-similar solutions by Hsu and Wainwright [27]. This solution also appears as a saddle point in a dynamical system investigation of tangent self-similar, \( G_2 \) diagonal perfect fluids with a \( \gamma \)-law equation of state [25].

For \( \alpha = \frac{\pi}{2} \) the line-elements (14) and (17) are exactly the same. Moreover, the expressions for \( A(u) \) and \( B(u) \) vanish when \( \alpha = \frac{\pi}{2} \) and therefore the metric becomes flat. The transformation (15) particularized to \( \alpha = \frac{\pi}{2} \) simply reads

\[
y = \frac{Y}{u}, \quad z = Z, \quad te^x = u, \quad te^{-x} = 2v - \frac{Y^2}{u},
\]

which can be rewritten using the standard orthogonal coordinates \( T, X, Y, Z \) for Minkowski spacetime as

\[
t = \sqrt{T^2 - X^2 - Y^2}, \quad e^x = \frac{(T + X)}{\sqrt{2\sqrt{T^2 - X^2 - Y^2}}}, \quad y = \frac{\sqrt{2}Y}{T + X}, \quad z = Z.
\]

The four velocity \( u = -dt \) is transformed to

\[
u = -\frac{1}{\sqrt{T^2 - X^2 - Y^2}}(TdT + YdY + XdX).
\]

Let us finish the discussion on the solution (3) by considering another limiting case. The family (3) is defined for every value of \( \alpha \) except when \( \sin \alpha = 0 \) (this corresponds to \( b = \infty \) in the metric (2)). It is therefore convenient to consider the limiting case \( \sin \alpha \to 0 \). It can be easily seen that rescaling the coordinates in order to reabsorb the diverging behaviour in the coefficient \( dydz \) in (3) leads necessarily to a degenerate metric. Therefore, the only possibility is that \( \cos \left( \frac{H}{2} \right) \) tends to zero when \( \sin \alpha \to 0 \).

Since the metric coefficient in \( dx^2 \) then diverges, it is necessary to perform a coordinate transformation to obtain a regular metric. Thus, we define

\[
\cos \left( \frac{H}{2} \right) = \delta(r) \sin \alpha, \quad \frac{dx}{\sin \alpha \sqrt{1 + \delta^2}} = dr
\]

which brings both the metric and the field equations into a form allowing the limit \( \sin \alpha \to 0 \). The field equations turn out to be completely integrable and the resulting line-element can be written as

\[
ds^2 = -dt^2 + t^2dr^2 + \frac{t}{\cosh r}dy^2 + t \cosh r \left( dz + \tanh r \, dy \right)^2,
\]

which can be transformed into the particular case \( \sin \alpha = 0 \) in (17) by the coordinate transformation

\[
r \to x, \quad \frac{y + z}{\sqrt{2}} \to y, \quad \frac{y - z}{\sqrt{2}} \to z.
\]
Thus the solution (17) is the correct spatially homogeneous limiting case for all values of $\alpha$.

As we have emphasized above, the dust model we study in this paper corresponds to an equilibrium point in the dynamical system approach for $G_2$ Abelian perfect-fluids developed by Hewitt and Wainwright [22]. It is, therefore, interesting to analyze how this solution fits into that framework. This will be the task of the next section.

4 Description of the solution using the tetrad approach

Throughout this section, we shall adopt the same notation and definitions as in [22]. One of the main results in that paper is that every equilibrium point of the dynamical system they construct (in terms of expansion normalized, dimensionless connection coefficients) corresponds to an orthogonally transitive $G_2$ perfect-fluid (assuming a linear equation of state) which admits a homothetic vector field lying in the three-plane spanned by $\vec{u}$ and the two Killing vectors (i.e. a tangent self-similar solution). Since these conditions are met by the metric we analyze, we can assume $\partial_0 X = 0$ where $X$ represents any dimensionless variable. Furthermore, the fluid is dust and hence the four velocity is geodesic. Imposing these conditions on the dynamical system in [22], we readily obtain

$$q = \frac{1}{2}, \quad \Sigma_+ = -\frac{1}{4}, \quad r = 0,$$

which, in particular, imply that $\partial_0$ and $\partial_1$ commute. Thus, every tangent self-similar dust solution has constant and positive deceleration parameter $q$ and the metric is never inflating. The defining equations for $q$ and $r$ provide now

$$\Omega = \frac{3}{4} - 12\left(\tilde{\Sigma}_{22}^2 + \tilde{\Sigma}_{23}^2\right), \quad A = 12\left(\tilde{\Sigma}_{22}\tilde{N}_{23} - \tilde{\Sigma}_{23}\tilde{N}_{22}\right),$$

indicating a very simple relationship between the density parameter $\Omega$ and the dimensionless shear scalar. Using all this information, the dynamical system $\Omega$ splits into a set of ordinary differential equations

$$\partial_1 R = -18(\tilde{\Sigma}_{22}\tilde{N}_{22} + \tilde{\Sigma}_{23}\tilde{N}_{23}),$$

$$\partial_1 \tilde{\Sigma}_{23} = -2\left(N_+\tilde{\Sigma}_{22} + R\tilde{N}_{23}\right), \quad \partial_1 \tilde{\Sigma}_{22} = 2\left(N_+\tilde{\Sigma}_{23} - R\tilde{N}_{22}\right),$$

$$\partial_1 \tilde{N}_{23} = 24\tilde{N}_{23}(\tilde{\Sigma}_{22}\tilde{N}_{23} - \tilde{\Sigma}_{23}\tilde{N}_{22}) - 2N_+\tilde{N}_{22} + 2R\tilde{\Sigma}_{23} - \frac{3}{2}\tilde{\Sigma}_{22},$$

$$\partial_1 \tilde{N}_{22} = 24\tilde{N}_{22}(\tilde{\Sigma}_{22}\tilde{N}_{23} - \tilde{\Sigma}_{23}\tilde{N}_{22}) + 2N_+\tilde{N}_{23} + 2R\tilde{\Sigma}_{22} + \frac{3}{2}\tilde{\Sigma}_{23},$$

and the algebraic constraint

$$1 - 16(\tilde{N}_{22}^2 + \tilde{N}_{23}^2 + \tilde{\Sigma}_{22}^2 + \tilde{\Sigma}_{23}^2) + 16^2(\tilde{\Sigma}_{23}\tilde{N}_{22} - \tilde{\Sigma}_{22}\tilde{N}_{23})^2 = 0,$$
which is a first integral of (19). It is a matter of simple calculation to check that
\[ R^2 = 9(\tilde{\Sigma}_{22}^2 + \tilde{\Sigma}_{23}^2) + \kappa \]  
(20)
(where \( \kappa \) is an arbitrary constant) is also a first integral of (19). The sign of this constant can be related to the Bianchi type of the homothetic algebra. It turns out that \( \kappa < 0 \) corresponds to Bianchi VI, \( \kappa = 0 \) to Bianchi IV and \( \kappa > 0 \) to Bianchi VII. Thus, the dynamical system is six-dimensional (in the variables \( R, \tilde{\Sigma}_{22}, \tilde{\Sigma}_{23}, \tilde{N}_{22}, \tilde{N}_{23} \) and \( N_+ \)) with two polynomic constraints. In order to simplify it further we take advantage of the rotational freedom in the \( e_2, e_3 \) plane (the rotation angle \( \phi \) must be constant both along the group orbits and along the integral lines of the fluid velocity in order to comply with all the conditions we already imposed). Under such a transformation, \( R \) behaves as an scalar, \( N_+ \) transforms as \( N_+ \to N_+ + \partial_1 \phi \), \( \tilde{\Sigma}_{AB} \) and \( \tilde{N}_{AB} \) as symmetric rank two tensors. The Lie algebra we study in this paper is of Bianchi type VI and therefore \( \kappa \) is strictly negative, which will be assumed from now on. Hence, the first integral (20) implies \( R^2 < 9(\tilde{\Sigma}_{22}^2 + \tilde{\Sigma}_{23}^2) \), which allows us to fix \( \phi \) so that the condition
\[ R = -3\tilde{\Sigma}_{23} \]  
(21)
is fulfilled. Notice that this choice cannot be imposed in the case of Bianchi type VII (for which \( \kappa > 0 \)). The choice (21) immediately implies \( \tilde{\Sigma}_{22} = \sqrt{-\kappa/3} \). The non-negativity of the energy-density demands \( 4\sqrt{\kappa} \leq 1 \) and we can set \( 4\sqrt{\kappa} \equiv \sin \alpha \). Furthermore, the differential equation for \( \tilde{\Sigma}_{22} \) above fixes \( N_+ = -3\tilde{N}_{22} \). Inserting all this information into the dynamical system, we obtain the following set of equations
\[ \partial_1 \tilde{\Sigma}_{23} = \frac{3}{2} \left( 4\tilde{\Sigma}_{23} \tilde{N}_{23} + \sin \alpha \tilde{N}_{22} \right), \]
\[ \partial_1 \tilde{N}_{23} = 6 \left( \tilde{N}_{23}^2 \sin \alpha - 4\tilde{N}_{22} \tilde{N}_{23} \tilde{\Sigma}_{23} + \tilde{N}_{22}^2 - \tilde{\Sigma}_{23}^2 \right) - \frac{3 \sin \alpha}{8}, \]  
(22)
and the constraint
\[ \cos^2 \alpha \left( 1 - 16\tilde{N}_{23}^2 \right) - 16\tilde{\Sigma}_{23}^2 \left( 1 - 16\tilde{N}_{22}^2 \right) - 16\tilde{N}_{22} \left( \tilde{N}_{22} + 8\tilde{N}_{23} \tilde{\Sigma}_{23} \sin \alpha \right) = 0. \]  
(23)
Hence, the dynamical system is two-dimensional as the motion happens on the surface \( S \) defined by (23). In order to obtain an adequate description of the problem we should find appropriate global coordinates on the surface \( S \) such that the dynamical system (22) takes a simple form, which is not a trivial problem. Since we have already analyzed the solution from the coordinate perspective, it is a matter of simple calculation to obtain which coordinates in \( S \) provide the dynamical system studied in section 2. Since \( \partial_0 \) and \( \partial_1 \) commute we can introduce coordinates \( t \) and \( x \) such that \( \partial_0 = 3t/2 \partial_t \) and \( \partial_1 = 3/2 \sqrt{L(x)} \partial_x \), where \( L(x) \) is an arbitrary non-vanishing function. Parametrizing
\[ \tilde{\Sigma}_{23}, \tilde{N}_{23} \text{ and } \tilde{N}_{22} \text{ as} \]

\[
\tilde{\Sigma}_{23} = \frac{\cos \alpha \cos \left(\frac{H}{2}\right)}{4}, \quad \tilde{N}_{23} = -\frac{2 \sin \alpha \sin Q - \cos^2 \alpha \cos Q \sin H}{8 \sqrt{\sin^2 \alpha + \cos^2 \alpha \cos^2 \left(\frac{H}{2}\right)}},
\]

\[
\tilde{N}_{22} = -\frac{\cos \alpha}{4 \sin \alpha} \left[ 4 \cos \left(\frac{H}{2}\right) \tilde{N}_{23} + \cos Q \sin \left(\frac{H}{2}\right) \sqrt{\sin^2 \alpha + \cos^2 \alpha \cos^2 \left(\frac{H}{2}\right)} \right],
\]

in terms of \( H \) and \( Q \) (which are global coordinates on the surface \( S \)) the dynamical system (22) takes exactly the form (11) whenever we also choose the free function \( L(x) \) as in (11). Clearly, finding this parametrization of \( S \) by simply inspecting the dynamical system (22) is very difficult. If we had started from the tetrad approach we would probably have found a different parametrization of \( S \) but the description of the problem would have been essentially equivalent. Actually, it is remarkable that both methods lead quite naturally to a two-dimensional dynamical system. This implies that both the level of final understanding and the amount of the required work are similar for the two approaches. Performing the comparison of them is, however, very interesting because the dimensionless variables of the tetrad formulation have a very clear physical interpretation. Furthermore, the tetrad approach is coordinate independent and hence it characterizes the solutions intrinsically. Finally, a combined use of both methods can provide us with a powerful tool for solving more difficult problems. In particular, the analysis of Bianchi type VII tangent self-similar dust cosmologies has proven to be very difficult following the coordinate approach. It is possible that using and combining the information from the coordinate and the tetrad methods can help solving this case. This matter is now under current investigation and the results will be reported elsewhere.

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Figure 1: Phase space portrait of the dynamical system \( \mathbb{R} \) for \( \alpha = \frac{1}{4}\pi \). The fixed points are marked by circles.
Figure 2: Phase space diagram for the dynamical system (3) showing the set of trajectories starting or finishing on the saddle points $\left(\pi, \frac{\pi}{2}\right)$ and $\left(\pi, -\frac{\pi}{2}\right)$ (which correspond to four points in the diagram due to the identification of the two vertical lines). The direction of increasing $x$ for each solution is shown.
Figure 3: Phase space portraits and energy-densities for three different solutions of the dynamical system with $\alpha = \pi/4$. The thin-line solution corresponds to the initial values $Q(0) = \frac{1}{3}$ and $H(0) = 0$. It never crosses the boundary of $\mathcal{U}$ and it does not wrap the torus. The medium-line solution corresponds to the initial values $Q(0) = 0$ and $H(0) = \frac{4}{5}\pi$. This solution wraps the torus once and the boundary of $\mathcal{U}$ is crossed once in the $H$ direction. The thick-line solution corresponds to $Q(0) = \frac{2}{3}\pi$ and $H(0) = \frac{3}{4}\pi$. It wraps the torus once and the boundary of $\mathcal{U}$ is crossed twice, once in the $H$ and once in the $Q$ direction.