ALGEBRAIC AND GEOMETRIC
METHODS IN REPRESENTATION THEORY

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1. Being awarded the Shaw Prize was a (pleasant) surprise for me; I feel very honored by it.

This paper is an expanded version of the Shaw Prize Lecture given on September 25, 2014 at the Chinese University of Hong Kong; it is concerned with some of my work in the theory of group representations, with emphasis on representations of finite groups of Lie type. Throughout this paper I have tried to explain not only what the results are but also the origin of those results.

I wish to thank David Vogan for his comments on an earlier version of this paper.

2. In mathematics, groups are everywhere. They also appear in the natural sciences: in physics, the motions of space-time form a (continuous) group; in chemistry, the symmetries of a crystal form a (finite) group. We shall be mainly interested in finite groups and more specifically, in finite simple groups, which form the building blocks for all finite groups. Rather surprisingly, up to finitely many known exceptions, and up to extensions by finite commutative groups, the finite noncommutative simple groups are of Lie type that is, they can be described in terms of continuous (or Lie) groups as follows.

Start with a compact connected Lie group; by complexification (Chevalley, 1946) this gives rise to a complex (reductive connected) Lie group and by a careful choice of integral structure (Chevalley, 1955 and 1960), to a reductive connected algebraic group $G$ over any given field, for example over $K$, an algebraic closure of the finite field with $p$ elements (with $p$ a prime number). For any field $k$ we shall denote by $GL_n(k)$ the group of automorphisms of $k^n$ (or the group of invertible $n \times n$ matrices with entries in $k$); let $F_1 : GL_n(K) \to GL_n(K)$ be the homomorphism given by $(a_{ij}) \mapsto (a_{ij}^p)$. We now choose an endomorphism $F : G \to G$ such that for some $n, s, s'$ in $\mathbb{Z}_{>0}$ and some imbedding $i : G \to GL_n(K)$ as a closed subgroup, we have $i(F^{s'}(g)) = F_1^s(i(g))$ for any $g \in G$. Then the fixed point set $G^F$ of $F$ is a finite group, said to be of Lie type. When $s' = 1$, $G^F$ is the group
representations of various subgroups. \(G(\mathbb{F}_q)\) for an \(\mathbb{F}_q\)-rational structure on \(G\) with Frobenius map \(F\) (here \(\mathbb{F}_q\) is the subfield of \(K\) with \(q = p^s\) elements). The Weyl group \(W\) of \(G\) should be also regarded as a finite group of Lie type.

For example, when \(G = GL_n(K)\), the finite group \(GL_n(\mathbb{F}_q)\) is of Lie type; \(W\) can be identified with the group of all permutations of \(\{1, 2, \ldots, n\}\). As another example, let \(G = Sp_{2n}(K)\) be the group of all \(g \in GL_{2n}(K)\) which preserve the symplectic form

\[
x_1y_{2n} - x_{2n}y_1 + x_{2y_{2n-1}} - x_{2n-1}y_2 + \cdots + x_ny_{n+1} - x_{n+1}y_n
\]

(a symplectic group) and let \(F : G \rightarrow G\) be the restriction of \(F^* : GL_{2n}(K) \rightarrow F_1 : GL_{2n}(K)\). Then \(G^F = Sp_{2n}(\mathbb{F}_q)\) is a finite group of Lie type (here \(q = p^s\)). In this case \(W\) can be identified with the group of all permutations of \(\{1, 2, \ldots, 2n\}\) which commute with the involution \(1 \mapsto 2n, 2 \mapsto 2n - 1, 3 \mapsto 3n - 2, \ldots, 2n \mapsto 1\).

**3.** A representation of a finite group \(\Gamma\) is a homomorphism \(\rho : \Gamma \rightarrow GL(V)\) of \(\Gamma\) into the group of automorphisms of a \(\mathbb{C}\)-vector space \(V\) (all vector spaces in this paper are assumed to be of finite dimension). We say that \((\rho, V)\) is irreducible if \(V \neq 0\) and there is no vector subspace \(V'\) of \(V\) (with \(0 \neq V' \neq V\)) such that \(\rho(g)V' = V'\) for any \(g \in \Gamma\). (We will occasionally replace \(\mathbb{C}\) by other fields.) These definitions appeared in a groundbreaking paper of F. G. Frobenius in 1896 which marks the birth of representation theory. The study of irreducible representations of a finite group is a key to understanding the finite group itself in the same way as understanding an object can be achieved by analyzing pictures of that object from many different angles.

For any representation \((\rho, V)\) of \(\Gamma\), the character \(\chi_\rho\) of \(\rho\) is the function \(\Gamma \rightarrow \mathbb{C}\) given by \(g \mapsto \text{tr}(\rho(g), V)\). We have \(\chi_\rho(g) = \chi_{\rho'}(gg'g'^{-1})\) for any \(g, g' \in \Gamma\); thus \(\chi_\rho\) is constant on each conjugacy class of \(\Gamma\).

Let \(\text{Irr}\Gamma\) be the set of isomorphism classes of irreducible representations of \(\Gamma\). In his 1896 paper, Frobenius showed that, if \(\text{Irr}\Gamma = \{\rho_1, \rho_2, \ldots, \rho_e\}\), then \(e\) is also the number of conjugacy classes of \(\Gamma\) and the functions \(\chi_{\rho_1}, \chi_{\rho_2}, \ldots, \chi_{\rho_e}\) (the “irreducible characters”) form a basis for the \(\mathbb{C}\)-vector space of functions \(\Gamma \rightarrow \mathbb{C}\) which are constant on the conjugacy classes \(C_1, C_2, \ldots, C_e\) of \(\Gamma\). (If \(g_i \in C_i\), then the invertible \(e \times e\)-matrix \((\chi_{\rho_j}(g_i))\) is called the character table of \(\Gamma\).) In the same paper Frobenius defined the notion of representation of \(\Gamma\) induced by a representation of a subgroup and several years later, in 1900, he showed that the irreducible characters of the symmetric group in \(n\) letters can be expressed as explicit \(\mathbb{Z}\)-linear combinations of characters of representations induced by the unit representations of various subgroups.

In this paper we are interested in understanding as much as possible about the representations of \(G^F\) where \(G\), a connected reductive group over \(K\), and \(F : G \rightarrow G\) (as in no.2) are fixed; we assume that \(F\) is the Frobenius map for an \(\mathbb{F}_q\)-rational structure on \(G\) where \(q = p^s\) for some \(s \in \mathbb{Z}_{>0}\) so that \(G^F = G(\mathbb{F}_q)\). As in no.2, \(W\) will denote the Weyl group of \(G\).
4. In 1955, J. A. Green published a remarkable paper in which he described completely the character table of the general linear groups $GL_n(F_q)$. (For $n = 2$ this was essentially done by Frobenius in 1896.) Green used Frobenius’s method of taking induced representations from proper subgroups (in this case the subgroups are stabilizers of flags of subspaces in $F^n_q$) and some rather nontrivial combinatorics coming from Hall algebras. But then he was faced with the problem of constructing the “discrete series”, a family of irreducible representations which are not contained in any of the induced representations above and for which a new method was needed. His approach was based on the following result that he proved using R. Brauer’s characterization of characters.

(a) Let $u$ be an isomorphism of the group $K^*$ with the group of roots of 1 of order prime to $q$ in $C$. Let $\Gamma$ be a finite group and let $\phi : \Gamma \to GL_N(K)$ be a group homomorphism. Define a function $\chi : \Gamma \to C$ by $\chi(g) = u(\lambda_1) + \cdots + u(\lambda_N)$ where $\lambda_1, \ldots, \lambda_N$ are the eigenvalues (in $K^*$) of $\phi(g) : K^N \to K^N$; here $g \in \Gamma$. Then $\chi$ is a $Z$-linear combination of irreducible characters of $\Gamma$.

(Green called $\chi$ the “Brauer lifting” of $\phi$.) (a) allowed Green to find the characters of the discrete series representations but not the representations themselves. It turns out that they are all of dimension $(q - 1)(q^2 - 1)\ldots(q^{n-1} - 1)$.

It is interesting that Green’s work on the representations of $GL_n(F_q)$ was almost at the same time as the work of Harish-Chandra on representations of real semisimple Lie groups. In both cases, most irreducible representations were associated with characters of maximal tori defined over the ground field and in both cases there was at most one “elliptic” maximal torus up to conjugacy.

The next progress in this field was achieved in 1968, when B. Srinivasan computed the character table of the symplectic group $Sp_4(F_q)$ for $q$ odd. In this case there are two conjugacy classes of elliptic maximal tori and an unexpected irreducible discrete series representation appears.

5. In 1970, at the Institute for Advanced Study, I attended a talk by D. Quillen, in which he explained his solution to a problem in homotopy theory, the Adams conjecture. That solution used essentially the Brauer lifting 4(a) of the natural $n$-dimensional representation of $GL_n(F_q)$. After the talk, I have asked M. Atiyah whether the irreducible representations which enter in that Brauer lifting were explicitly known, and he told me that only their characters were known. I became very interested to understand the representations which enter, not only their characters. In 1971 I accepted a lectureship at the University of Warwick where J. A. Green was a professor. At Warwick, I learned much about algebraic groups from discussions with R. W. Carter with whom I also wrote two papers. In one of those papers I became familiar with Hecke algebras (with parameter 0) and how to use them to construct irreducible modular representations. The experience with Hecke algebras was to become very useful in my later research.

In 1972 and early 1973, I found a way to describe explicitly the Brauer lifting of the natural $n$-dimensional representation of $GL_n(F_q)$. I will try to explain some of the ideas involved, assuming for simplicity that $q = p$ is a prime number and
that $\mathcal{C}$ is replaced by a maximal unramified extension of $\mathbb{Q}_p$, the $p$-adic numbers 
(in this case there is a natural choice for $u$ in 4(a) whose image is contained in the 
units of $\mathbb{Z}_p$, the $p$-adic integers).

Let $V$ be an $n$-dimensional $\mathbb{F}_p$-vector space, $n \geq 1$. Let $E$ be the set of complete 
flags $V_0 \subset V_1 \subset \ldots \subset V_n$ in $V$ ($\dim V_i = i$ for all $i$). Let $X_V$ be the $\mathbb{F}_p$-vector 
space consisting all functions which to each $(V_0 \subset V_1 \subset \ldots \subset V_n) \in E$ associate a 
vector $f(V_0 \subset V_1 \subset \ldots \subset V_n) \in V_1$ in such a way that for any $i \in [1, n]$ and 
any almost complete flag of the form $(V_0 \subset V_1 \subset \ldots \subset V_{i-1} \subset V_{i+1} \subset \ldots \subset V_n)$, 
the sum
\[
\sum_{V_i} f(V_0 \subset V_1 \subset \ldots \subset V_{i-1} \subset V_i \subset V_{i+1} \subset \ldots \subset V_n)
\]
is 0 (here $V_i$ runs over all $i$-dimensional subspaces which fit between $V_{i-1}$ and 
$V_{i+1}$); the sum is taken in $V_1$ if $i \geq 2$ and in $V_2$ if $i = 1$. Note that $X_V$ has a 
natural action of $GL(V)$. (If in the definition of $X_V$ one takes functions $f$ with 
values in $\mathcal{C}$ instead of $V_1$ and satisfying the similar equations, one obtains the 
Steinberg representation of $GL(V)$ of dimension $p^{1+2+\cdots+(n-1)}$.) One shows that 
dim $X_V = (p-1)(p^2-1)\ldots(p^{n-1}-1)$ and that $X_V$ is naturally the reduction 
modulo $p$ of a free $\mathbb{Z}_p$-module $\tilde{X}_V$ of rank $(p-1)(p^2-1)\ldots(p^{n-1}-1)$ with natural 
action of $GL(V)$ (this is the most nontrivial part of the story). Moreover, if for 
any $j \in [1, n]$ we set $\tilde{X}_V^j = \oplus_{V_j} \tilde{X}_V$ (sum over the $j$-dimensional subspaces of 
$V$) then $\tilde{X}_V^j \otimes \mathbb{Q}_p$ has a natural action of $GL(V)$ and these provide the required 
components of the Brauer lifting, which is equal to $\sum_{j=1}^{n} (-1)^{j-1} \chi_{\tilde{X}_V^j}$.

In late 1973 I found the explicit structure of the Brauer lifting of the standard 
representation of the various classical groups over $\mathbb{F}_q$, and I thus obtained in each 
case some discrete series representations which were new (at that time). But it 
seemed to be difficult to obtain other discrete series by this method.

6. In the early 1960's A. Grothendieck defined $l$-adic cohomology spaces for algebraic 
varieties in characteristic $p$ (here $l$ is a prime number other than $p$) which 
shared many properties with the usual rational cohomology of algebraic varieties 
over $\mathcal{C}$. For example, the smooth curve $C_0$ with equation $x^{q+1} + y^{q+1} + z^{q+1} = 0$ in 
the projective plane over $K$ has $l$-adic cohomology group in degree 1 of dimension 
$q^2 - q$, the same as it would have (in ordinary cohomology) if it was viewed as a 
curve over $\mathcal{C}$. But the curve over $\mathcal{C}$ has no interesting automorphism, while $C_0$ 
has a natural action of a finite unitary group (a subgroup of $GL_3(\mathbb{F}_q^2)$) which, 
as Tate and Thompson have shown in a 1965 paper (by J. Tate), induces an irreducible 
representation of dimension $q^2 - q$ of this unitary group on the first $l$-adic 
cohomology space. (This representation is difficult to obtain by other means.)

Towards the end of 1973, T. A. Springer explained to me some work of V. 
Drinfeld about which he learned during a recent visit to Moscow. Namely, Drinfeld 
(who was 19 years old at the time) considered the plane curve $C = \{(x, y) \in K^2; x^qy - xy^q = 1\}$ on which $SL_2(\mathbb{F}_q)$ acts naturally (an old observation of L. E. 
Dickson) and $\mathcal{T} := \{\lambda \in K; \lambda^{q+1} = 1\}$ acts (freely) by homothety, commuting with
the $SL_2(F_2)$-action. (Note that $SL_2(F_2) = GF$ in the case where $G = SL_2(K)$ and $F : G \to G$ is given by $F(a_{ij}) = (a_{ij}^q)$. ) Then $SL_2(F_2) \times T$ acts on $H^1_c(C, \overline{Q}_l)$, the $l$-adic cohomology with compact support in degree 1 of $C$. (Here $\overline{Q}_l$ is an algebraic closure of the field of $l$-adic numbers.) Note that $SL_2(F_2) \times T$ can be viewed as a subgroup of the unitary group above and $C$ can be viewed as the complement of a finite set in $C_0$ above so that the action of $SL_2(F_2) \times T$ on $C$ becomes the restriction of the action of the unitary group on $C$ of $B$ a subvariety of $O$ the element of order 2, is the complement of the diagonal). See also no.8. We write $W$ naturally indexed by the elements of $T$ (commutative) group $F$.

Each $X$ is stable under the conjugation action of $w$ and $w \cap F$ for the orbit indexed by $w \in B$. For any $w \in W$ we set $X_w = \{ B \in B; (B, F(B)) \in O_w \}$, a subvariety of $B$. The subvarieties $X_w (w \in W)$ form a partition of $B$ which in the case where $G = SL_2(K)$ coincides with the partition of $B$ described earlier. Each $X_w$ is stable under the conjugation action of $G(F_2)$. Now for each $w \in W$ we can find an $F$-stable maximal torus $T_w$ of $G$ which is contained in some $B \in X_w$. Moreover, $X_w$ has a natural finite principal covering $\tilde{X}_w \to X_w$ with group $T_w^F = T_w \cap GF$, to which the $GF$-action can be lifted. (In the case where $G = SL_2(K)$ and $w \neq 1$, $\tilde{X}_w$ can be taken to be the curve $C$ considered by Drinfeld.) It follows that $GF$ acts naturally on the $l$-adic cohomology spaces $H^i_c(\tilde{X}_w, \overline{Q}_l)$. The (commutative) group $T^*_w$ also acts naturally on $H^i_c(\tilde{X}_w, \overline{Q}_l^*)$, commuting with the $GF$-action; hence for any homomorphism $\theta : T^*_w \to \overline{Q}_l^*$, $GF$ acts naturally on the

7. During the spring of 1974 I was at the Institut des Hautes Études Scientifiques in Bures-sur-Yvette and in the later part of my stay I started a joint work with P. Deligne, trying to generalize Drinfeld’s construction in no.6 to a general $GF$; this was written up as a joint paper in the first half of 1975 (it appeared in 1976).

In the setup of no.6, the map $(x, y) \mapsto [x, y]$ identifies the quotient $T \backslash C$ with $\{ L \in B; F(L) \neq L \}$ where $B$ denotes the set of lines in $K^2$ and $F : B \to B$ is induced by $(x, y) \mapsto (x^q, y^q)$. Thus $B$ has a natural partition $\{ L \in B; F(L) = L \} \sqcup \{ L \in B; F(L) \neq L \}$ in which the second piece is responsible for the irreducible representations of dimension $q - 1$ and similarly the first piece is responsible for the irreducible representations of dimension $q + 1$. (Note also that the first piece has Euler characteristic $q + 1$ and the second piece has Euler characteristic $1 - q$.)

In the general case, $B$ can be interpreted as the variety of all Borel subgroups (maximal closed connected solvable subgroups) of $G$ (when $G = SL_2(K)$, a line $L$ in $K^2$ can be identified with its stabilizer of $L$ in $SL_2(K)$). Now $G$ acts on $B \times B$ by simultaneous conjugation; it is known that this action has finitely many orbits naturally indexed by the elements of $W$ (when $G = SL_2(K)$ there are two such orbits: one, indexed by the unit element, is the diagonal and the other, indexed by the element of order 2, is the complement of the diagonal). See also no.8. We write $O_w$ for the orbit indexed by $w \in W$. Now $F : G \to G$ induces an endomorphism of $B$ denoted again by $F$. For any $w \in W$ we set $X_w = \{ B \in B; (B, F(B)) \in O_w \}$, a subvariety of $B$. The subvarieties $X_w (w \in W)$ form a partition of $B$ which in the case where $G = SL_2(K)$ coincides with the partition of $B$ described earlier. Each $X_w$ is stable under the conjugation action of $G(F_2)$. Now for each $w \in W$ we can find an $F$-stable maximal torus $T_w$ of $G$ which is contained in some $B \in X_w$. Moreover, $X_w$ has a natural finite principal covering $\tilde{X}_w \to X_w$ with group $T^*_w = T_w \cap GF$, to which the $GF$-action can be lifted. (In the case where $G = SL_2(K)$ and $w \neq 1$, $\tilde{X}_w$ can be taken to be the curve $C$ considered by Drinfeld.) It follows that $GF$ acts naturally on the $l$-adic cohomology spaces $H^i_c(\tilde{X}_w, \overline{Q}_l)$. The (commutative) group $T^*_w$ also acts naturally on $H^i_c(\tilde{X}_w, \overline{Q}_l^*)$, commuting with the $GF$-action; hence for any homomorphism $\theta : T^*_w \to \overline{Q}_l^*$, $GF$ acts naturally on the
\(\theta\)-eigenspace \(H^i_w(\tilde{X}_w, \mathbb{Q}_l)_{\bar{w}}\) of the \(T^F_w\)-action. In the paper with Deligne we proved that any irreducible representation \(\rho\) of \(G^F\) appears in the virtual representation \(\sum_i (-1)^i H^i_w(\tilde{X}_w, \mathbb{Q}_l)_{\bar{w}}\) for some \(w, \theta\). Now from each pair \((w, \theta)\) as above one can produce in a natural way a conjugacy class \(\gamma_{w,\theta}\) defined over \(\textbf{F}_q\) of elements of order prime to \(q\) in a connected reductive group \(G^*\) over \(K\) (of Langlands dual type to that of \(G\)) with a natural \(\textbf{F}_q\)-structure and in the paper we prove that if \(w, \theta\) corresponds to \(\rho\) as above, then \(\gamma_{w,\theta}\) depends only on \(\rho\), not on \(w, \theta\) hence it can be denoted by \(\gamma_{\rho}\). Thus we obtain a natural (surjective) map \(\Phi: \rho \mapsto \gamma_{\rho}\) from the set of irreducible representations of \(G^F\) (up to isomorphism) to the set of \(G^*\)-conjugacy classes (defined over \(\textbf{F}_q\)) of elements of order prime to \(q\) in \(G^*\). The irreducible representations of \(G^F\) in \(\Phi^{-1}(1)\) are called the unipotent representations. They are precisely the irreducible representations which appear in \(H^i_w(X_w, \mathbb{Q}_l)\) for some \(i, w\) or, equivalently, which appear in the virtual representation \(\sum_i (-1)^i H^i_w(X_w, \mathbb{Q}_l)\) for some \(w\). At the other extreme, for almost any \(G^*\)-conjugacy class \(\gamma\) defined over \(\textbf{F}_q\) of elements of order prime to \(q\) in \(G^*\), \(\Phi^{-1}(\gamma)\) is a single irreducible representation of \(G^F\); it is equal to \(\pm \sum_i (-1)^i H^i_w(\tilde{X}_w, \mathbb{Q}_l)_{\bar{w}}\) for some \(w\) and some \(\theta\). Almost all irreducible representations of \(G^F\) are obtained in this way. (In particular this proved a conjecture formulated by I. G. Macdonald in 1968.)

8. I will now describe in more detail how the \(G\)-orbits on \(\mathcal{B} \times \mathcal{B}\) are parametrized by \(W\) when \(G\) is \(GL_n(K)\) or \(Sp_{2n}(K)\).

Assume first that \(G = GL_n(K)\); then \(\mathcal{B}\) can be identified with the set of complete flags of subspaces \(V_0 \subset V_1 \subset \ldots \subset V_n\) in \(K^n\) (\(\dim V_i = i\) for all \(i\)). Given two complete flags \(V_0 \subset V_1 \subset \ldots \subset V_n\) and \(V'_0 \subset V'_1 \subset \ldots \subset V'_n\), we can define uniquely a permutation \(w\) of \(\{1, 2, \ldots, n\}\) by the following requirement: there exists a basis \(\{v_1, v_2, \ldots, v_n\}\) of \(K^n\) such that for \(i = 1, 2, \ldots, n, \{v_1, v_2, \ldots, v_i\}\) is a basis of \(V_i\) and \(\{v_{w(1)}, v_{w(2)}, \ldots, v_{w(i)}\}\) is a basis of \(V'_i\). The set of pairs of complete flags whose associated permutation is \(w\) form the orbit \(O_w\).

Assume next that \(G = Sp_{2n}(K)\); then \(\mathcal{B}\) can be identified with the set of complete flags of subspaces \(V_0 \subset V_1 \subset \ldots \subset V_{2n}\) in \(K^{2n}\) (\(\dim V_i = i\) for all \(i\)) such that for any \(i = 0, 1, \ldots, 2n\), \(V_i\) is the perpendicular to \(V_{2n-i}\) with respect to the symplectic form. To two such complete flags \(V_0 \subset V_1 \subset \ldots \subset V_{2n}\) and \(V'_0 \subset V'_1 \subset \ldots \subset V'_{2n}\), we associate a permutation \(w\) of \(\{1, 2, \ldots, 2n\}\) as for \(GL_{2n}(K)\); this permutation commutes with the involution \(1 \mapsto 2n, 2 \mapsto 2n-1, 3 \mapsto 3n-2, \ldots, 2n \mapsto 1\) hence is an element of \(W\). The set of pairs of complete flags as above whose associated permutation is \(w\) form the orbit \(O_w\).

9. In late 1975, I wrote a paper (which appeared in 1976) in which I analyzed the variety \(X_w\) in no.7 in the case where \(G\) modulo its centre is simple and \(X_w\) is irreducible, of minimum possible dimension. For simplicity we shall assume here that \(G\) is split over \(\textbf{F}_q\) that is, there exists a maximal torus \(T\) of \(G\) such that \(F(t) = t^q\) for all \(t \in T\). In this case, \(w\) is a “Coxeter element of minimal length” of \(W\). For example when \(G = GL_n(K)\), one can take \(X_w\) to be the set of complete flags \(V_0 \subset V_1 \subset \ldots \subset V_n\) in \(K^n\) (see no.8) such that \(V_i \neq F(V_i) \subset V_{i+1}\) for
In the case of classical groups, the corresponding series of representations is the discrete series which can be approached using Brauer lifting (see the last paragraph in no.5). Note that in our case, \( X_y \) is stable under \( F : B \to B \) for any \( y \in W \). The main result of the paper is that the Frobenius map acts on \( \bigoplus_i H_i^c(X_w, \mathbb{Q}_l) \) semisimply and that its eigenspaces are distinct irreducible representations of \( G^F \). When \( G \) is of exceptional type one finds several (unipotent) discrete series representations among these eigenspaces. Moreover, the eigenvalues of the Frobenius map are explicitly computed. They are most of the time roots of \( 1 \) times integer powers of \( q \), but in some cases (type \( E_7, E_8 \)) they can be odd powers of \( \sqrt{-q} \). This shows that the topology of \( X_w \) can be quite complicated and that I was quite lucky that these eigenvalues could be computed at all. The results in this paper (in the case of exceptional groups) played an essential role in my later classification of unipotent representations of \( G^F \).

10. In the summer and fall of 1976 I wrote a paper (published in 1977) in which I found a complete classification (including dimensions) of the irreducible representations of \( G^F \) in the case where \( G \) is a classical group with connected centre, other than \( GL_n(K) \). (Thus, for example, instead of \( Sp_{2n}(K) \), I considered the group of symplectic similitudes.) The method I used was based on an extension of the method in no.7, which allowed cohomological induction from subgroups more general than a maximal torus, and on the use of the dimension formulas for the irreducible representations of Hecke algebras of type B with two parameters due to Hoefsmit (1974). This paper establishes what in no.17 is called “quasi-induction” for the representations of classical groups (with connected centre). It also establishes the parametrization of unipotent representations for these groups in terms of some new combinatorial objects, the “symbols”, and the classification of unipotent discrete series representations of classical groups. Moreover, it is shown that the endomorphism algebra of the representation induced from an isolated discrete series representation to a larger classical group is an Iwahori-Hecke algebra (anticipating a later result of Howlett-Lehrer, 1980) and giving also precise information on the values of the parameters of that Iwahori-Hecke algebra. For this we need to count in terms of generating functions the number of conjugacy classes in a classical group with connected centre. This together with an inductive hypothesis and the methods outlined above give a way to predict the number of isolated discrete series representations. The degrees of these representations can be guessed using the technique of symbols by “interpolation” from the degrees of already known representations. To prove that these guesses are correct we need to calculate the sum of squares of the (guessed) degrees of unipotent representations which is perhaps the most interesting part of this paper. To do this I find explicit formulas (for each irreducible representation \( E \) of \( W \)) of the fake degrees \( d_E(q) \) (see no.11). Then I show that the (guessed) degree polynomials can be expressed as linear combinations of the \( d_E(q) \) with constant coefficients of the form plus or minus \( 1/2^s \). This anticipates
the notion of family of representations of the Weyl group and the role of the non-abelian Fourier transform, see no.11, (which in this case happens to be abelian.) Here the use of the technique of symbols is crucial.

11. I obtained the classification of unipotent representations of $G^F$ (assuming that $G$ is simple of exceptional type and $q$ is large enough) in late 1977 (for types $\neq E_8$) and in the spring of 1978 (for type $E_8$) when I was already at MIT; this has appeared in 1978 (resp. 1979). In both these papers, as well as in that in no.10, the notion of fake degree of an irreducible representation $E$ of a Weyl group $W$ is considered; it is a certain polynomial $d_E(X)$ with coefficients in $\mathbb{N}$ whose value at $X = 1$ is the dimension of $E$. In the second of these papers (on $E_8$) it is observed that, for any simple $G$, both the set of unipotent representations of $G^F$ and the set $\text{Irr}W$ can be naturally partitioned into families (the families being indexed by the same set in both cases). Moreover, to each family $F$ one can attach a finite group $\Gamma_F$ such that the unipotent representations of $G^F$ in $F$ are in bijection with the set $\mathcal{M}(\Gamma_F)$ defined below, the irreducible representations of $W$ in the family corresponding to $F$ are indexed by a subset of $\mathcal{M}(\Gamma_F)$ and such that the dimension of fake degree of an irreducible representation $E$ of a Weyl group $W$ are linear combinations of the fake degrees $d_E(q)$ with $E$ running through the family of irreducible representations of $W$ corresponding to $F$. The coefficients in this linear combination are given by the entries $\{(x, \sigma), (y, \tau)\}$ of a certain matrix indexed by $\mathcal{M}(\Gamma_F) \times \mathcal{M}(\Gamma_F)$ (see below).

We now define the set $\mathcal{M}(\Gamma)$ for any finite group $\Gamma$. It consists of pairs $(x, \sigma)$ where $x$ is an element of $\Gamma$ defined up to conjugacy and $\sigma \in \text{Irr}Z(x)$ ($Z(x)$ is the centralizer of $x$ in $\Gamma$). Consider the matrix with entries $\{(x, \sigma), (y, \tau)\}$ indexed by $\mathcal{M}(\Gamma) \times \mathcal{M}(\Gamma)$ given by

$$\{(x, \sigma), (y, \tau)\} = \sum_{g \in \Gamma; xgyg^{-1} = gyyg^{-1}g} \frac{\text{tr}(g^{-1}x^{-1}g, \tau)\text{tr}(gyg^{-1}, \sigma)}{|Z(x)||Z(y)|}.$$  

This matrix has properties very similar to that of a Fourier transform matrix (it is unitary and involutive). In fact, when $\Gamma = \mathbb{F}_2^n$ so that $\mathcal{M}(\Gamma) = \mathbb{F}_2^{2n}$, this is actually a Fourier transform matrix.

The fact that this nonabelian generalization of Fourier transform appears in representation theory ($\Gamma$ can be the symmetric group $S_5$ for $G$ of type $E_8$) was one of the most unexpected things in my research.

12. Let $\tilde{W}$ be a Coxeter group with canonical set of generators $S$ and length function $\ell: \tilde{W} \to \mathbb{N}$. (We could take for example $\tilde{W} = W$ with $\ell(w) = \dim \mathcal{O}_w - \dim \mathcal{O}_1$, notation of no.7.) Following Iwahori (1964) and Bourbaki (1968) we recall the definition of the Hecke algebra $H$ associated to $\tilde{W}$. It is an algebra over $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ where $v$ is an indeterminate. As an $\mathcal{A}$-module, it is free with basis $\{T_w; w \in \tilde{W}\}$. The (associative) multiplication is defined by the rules

$$T_w T_w' = T_{ww'} \text{ if } \ell(ww') = \ell(w) + \ell(w'),$$  

and
Note that $T_1$ is the unit element of $H$. Moreover, $\tilde{W}$ has a natural partial order. (In the case where $\tilde{W} = W$, we choose a point $B_0 \in B$ and for $w \in W$ we set, with notation of no.7, $o_w = \{B \in B; (B_0, B) \in O_w\}$. We have $y \leq w$ if and only if $o_y$ is contained in the closure $\bar{o}_w$ of $o_w$.)

13. We keep the setup of no.12. Motivated by a result of R. Kilmoyer (1969) in which he described a $v$-analogue of the reflection representation of $W$ which was a representation of $H$ in which the generators $T_s$ act by particularly simple formulas, I tried (in 1977) to find similar $v$-analogues for other irreducible representations of $W$. (I succeeded to do that in only a small number of examples.) This has led me to find (in the fall of 1978) a new basis of $H$ itself (with $\tilde{W}$ as in no.12). First I observed that the map $v^nT_w \mapsto v^nT_w = v^{-n}T_{w^{-1}}$ (with $n \in \mathbb{Z}, w \in \tilde{W}$) defines a ring involution $h \mapsto \bar{h}$ of $H$. Next I showed that for any $w \in \tilde{W}$ there is a unique element $C_w \in H$ such that $C_w = C_w$, and $C_w = v^{-l(w)} \sum_{y \leq w} P_{y,w}T_y$ where $P_{y,w} \in A$ is a polynomial in $v^2$ of degree $\leq (l(w) - l(y) - 1)/2$ if $y < w$ and $P_{w,w} = 1$. The proof of existence showed also that the polynomials $P_{y,w}$ can be computed inductively by a (rather complicated) algorithm.

Clearly, $\{C_w; w \in \tilde{W}\}$ is an $A$-basis of $H$. It has the property that there are many two-sided ideals of $H$ which are spanned as $A$-modules by subsets of $\tilde{W}$. The subsets of $\tilde{W}$ which appear in this way are unions of subsets (called two-sided cells) in a certain partition of $\tilde{W}$.

Consider for example the case where $\tilde{W}$ is the Weyl group $W$ of $GL_4(K)$ so that $W$ is the group of permutations of $\{1, 2, 3, 4\}$ and let $s_1 = (12), s_2 = (23), s_3 = (34)$. In this case, $P_{y,w} = 1$ for any $y \leq w$ except when $w = s_2 s_1 s_3 s_2$ and $y \in \{1, s_2\}$, or when $w = s_1 s_3 s_2 s_3 s_1$ and $y \in \{1, s_1, s_3, s_1 s_3\}$, in which case $P_{y,w} = 1 + v^2$. On the other hand, $o_w$ (see no.12) is nonsingular except when $w = s_2 s_1 s_3 s_2$ (and its singular locus is $o_1 \cup o_{s_2}$) or when $w = s_1 s_3 s_2 s_3 s_1$ (and its singular locus is $o_1 \cup o_{s_1} \cup o_{s_3} \cup o_{s_1 s_3}$).

After I told D. Kazhdan about the polynomials $P_{y,w}$ and their apparent connection with the singularities of $o_w$, he suggested a cohomological formula for $P_{y,w}$, assuming that $\tilde{W} = W$ and that $P_{y',w} = 1$ for all $y'$ such that $y < y' \leq w$. The construction of the polynomials $P_{y,w}$ and the proof of the cohomological formula became part of my joint paper with Kazhdan (published in 1979). That paper also contains the conjectural equality

\[(a) \quad L_w = \sum_{y \leq w} (-1)^{l(w)+l(y)} P_{y,w}(1) M_y\]

in the theory of highest weight representations of a simple Lie algebra with Weyl group $W$. ($M_w$ are certain Verma modules and $L_w$ is the simple quotient of $M_w$.)
14. In 1977, M. Goresky and R. MacPherson introduced intersection cohomology for singular complex algebraic varieties which (in the projective case) satisfied Poincaré duality. In early 1979, R. Bott suggested to Kazhdan that in the case where $\tilde{W} = W$, the $P_{y,w}$ in no.13 might have something to do with intersection cohomology. I have separately arrived at the same conclusion, having attended a talk by MacPherson on intersection cohomology at Warwick in 1977; moreover, I knew that (as a consequence of the definition) for any $w \in W$, the polynomial $\Pi(X) = \sum_{y \leq w} X^{l(y)} P_{y,w}(X)$ satisfies $X^{l(w)} \Pi(X^{-1}) = \Pi(X)$, which looks like a manifestation of Poincaré duality. The cohomological formula for $P_{y,w}$ mentioned in no.13 seemed to be related to a formula in the paper of Goresky and MacPherson but there was a discrepancy. By talking to MacPherson (in early 1979), Kazhdan and I found out that the discrepancy was due to a misprint and also that Deligne has defined intersection cohomology in the $l$-adic setting in arbitrary characteristic. After writing to Deligne, we received his letter (in the spring of 1979) in which he explained his approach to intersection cohomology using derived categories and $l$-adic sheaves. Using Deligne’s results, Kazhdan and I were able (in the summer of 1979) to establish the interpretation of the coefficients of $P_{y,w}$ in no.13 as dimensions of stalks of cohomology sheaves of the intersection cohomology complex of $\tilde{o}_w$ (in the $l$-adic setting). This appeared in our joint paper published in 1980.

The results of this paper were a first step in the proof of the conjecture in no.13(a). The remaining steps in that proof were achieved in 1981 by Beilinson-Bernstein and Brylinski-Kashiwara. A direct proof of 13(a), which avoids intersection cohomology, has been found in 2013 by Elias and Williamson, building on work of Soergel.

15. In July 1979 I gave a talk at AMS conference in Santa Cruz where I stated several conjectures. I will list here two of them.

(a) I stated a multiplicity formula for the unipotent representations of $G^F$ in the virtual representations $\sum_i (-1)^i H^c_i(X_w, \mathbb{Q}_l)$ (notation of no.7) in terms of the nonabelian Fourier transform in no.11. (See no.16.)

(b) I stated a conjecture relating the decomposition of $\text{Irr} W$ into families (see no.11) with the two-sided cells (see no.13) of $W$. (This was proved by D. Barbasch and D. Vogan in 1982 and 1983, based on the solution of the conjecture 13(a).)

16. In early 1980 I found a proof of the multiplicity formula in 15(a) in the case where $G$ is of exceptional type and $q$ is sufficiently large. (This has appeared in a paper in 1980.) In early 1981, while visiting the Australian National University, Canberra, I found a proof of the multiplicity formula in 15(a) in the case where $G$ is a symplectic or odd orthogonal group and $q$ is sufficiently large; later that year I proved the analogous result for even orthogonal groups. (These appeared in papers in 1981 and 1982.) The methods I used in the 1980 paper were not strong enough to deal with the classical groups treated in the 1981 and 1982 papers. In the last two papers I used the following new approach: instead of studying the decomposition into irreducible $G^F$-modules of the $l$-adic cohomology with compact
support of $X_w$ in no.7 (a very difficult question since $X_w$ is in general not proper) we can try to study the decomposition into irreducible $G^F$-modules of the $l$-adic intersection cohomology of the closure $\bar{X}_w$ of $X_w$ in $\mathcal{B}$ (which is better behaved due to the fact that Deligne’s theory of weights is applicable so that we can obtain information about individual cohomology spaces rather than alternating sums of them, which makes a crucial difference). The passage from the first problem to the second problem could be controlled at the level of Euler characteristic, using the knowledge of the local intersection cohomology of $\bar{X}_w$ (since $\bar{X}_w$ looks locally like $\tilde{o}_w$, see no.12, it has the same local intersection cohomology as $\tilde{o}_w$ which, by no.14, is expressible in terms of the polynomials $P_{y,w}$ in no.13). These two papers also rely on a method which was new at the time, namely the determination in many cases of the leading coefficients of the character values of the Hecke algebra.

17. From late 1981 to the middle of 1982 I worked on what was to become my 1984 book on the representation theory of $G^F$ where $G$ is assumed to have connected centre. In this book I obtained the classification of not necessarily unipotent representations and the computation of their multiplicities in the various $\sum_i (-1)^i H^i_c(\tilde{X}_w, \mathbb{Q}_l)\theta$ (see no.7). For classical groups the classification was already essentially known, see no.10, but for exceptional groups it was new for the non-unipotent case and also for the unipotent case with $q$ small, see no.11. The multiplicity formulas in the non-unipotent cases and in the unipotent case with $q$ small were also new. To study not necessarily unipotent representations, I had to study the local and global intersection cohomology of $\tilde{X}_w$ with coefficients in certain local systems and for this I first had to generalize the results in no.14 to include “monodromic systems” on varieties like $\tilde{o}_w$, see no.12. But the use of intersection cohomology of $\tilde{X}_w$ was along the same lines as in no.16. Another new ingredient which was not used in the papers in no.16 was the use of the Barbasch-Vogan results 15(b); moreover, a result of A. Joseph on Goldie rank representations of Weyl groups was used in the proof. (Note that the use of these two ingredients requires the validity of 13(a).)

One of the main results of the book was that any fibre of the map $\Phi$ in no.7 is in a bijection (which could be called “quasi-induction”) with the set of unipotent representations of a smaller $G$ and that the multiplicity formulas for the representations in the fibre look similar to those for the corresponding unipotent representations. (At the present time there is no a priori proof of this fact.)

The book also contains results for $G$ with not necessarily connected centre. But the case where $G = \text{Spin}_{4k}(\mathbb{F}_q)$ with $q$ odd could not be treated in the book. It required extensive additional computations which I carried out after the book was written, in the summer of 1983, so that in my ICM talk (1983) I did not have to make any assumption on $G$. (My paper with the details of these computations appeared only in 2008.)

18. A unipotent representation $\rho$ of $G^F$ is cuspidal (or discrete series) if the following condition is satisfied: whenever $w \in W$ is such that $\rho$ appears in
\[ \sum_i (-1)^i H_c^j(X_w, \mathbb{Q}_l) \] where \( w \) must be elliptic (that is, its conjugacy class does not meet any proper subgroup of \( W \) generated by elements of length 1). In some sense, the classification of arbitrary irreducible representations of \( G^F \) can be reduced to the classification of unipotent cuspidal representations of \( G^F \) and of smaller groups. (All other irreducible representations are obtained either by quasi-induction, see no.17, at least when \( G \) has connected centre, or by decomposing some induced representations governed by explicitly known Hecke algebras.) In this sense the unipotent cuspidal representation are the most basic of all irreducible representations.

Let \( \U_q^0 \) be the set of unipotent cuspidal representations of \( G^F \) (up to isomorphism). I will describe a parametrization of the set \( \U_q^0 \) which follows from results in my 1984 book and results in one of my papers which appeared in 2002; I shall assume that \( G \) is simple and split over \( \mathbb{F}_q \) (see no.9).

Let \( \rho \in \U_q^0 \). There is a unique a pair \((C, \mu)\) where \( C \) is a conjugacy class in \( W \) and \( \mu \in \mathbb{Q}_l^* \) are such that the three conditions below are satisfied (we denote by \( w \) any element of minimal length of \( C \)):

- the multiplicity of \( \rho \) in \( H_c^i(X_w, \mathbb{Q}_l) \) is 1 if \( i = |w| \) and is 0 if \( i \neq |w| \);
- if \( z \in W - C \) satisfies \( |z| \leq |w| \) then the multiplicity of \( \rho \) in \( H_c^i(X_z, \mathbb{Q}_l) \) is 0 for any \( i \in \mathbb{Z} \);
- the Frobenius map acts on the \( \rho \)-isotypic component of \( H_c^{l(w)}(X_w, \mathbb{Q}_l) \) as multiplication by \( \mu \).

Note that \( \mu \) is necessarily a root of 1 times \( q^{|w|/2} \) with \( w \) as above. Let \( \mathfrak{S}_q \) be the set of all pairs \((C, \mu)\) that are attached to some \( \rho \in \U_q^0 \) as above. The map \( \U_q^0 \to \mathfrak{S}_q, \rho \mapsto (C, \mu), \) is bijective. (Moreover, two elements of \( \U_q^0 \) which have the same associated \( \mu \) coincide.)

We now describe the set \( \mathfrak{S}_q \) in each case; we will specify \( G \) by its type \( A_n, B_n, \ldots, E_8 \). (For exceptional types we specify an elliptic conjugacy class \( C \) in \( W \) by the characteristic polynomial \(|C|\) of one of its elements in the reflection representation of \( W \), written as a product of cyclotomic polynomials \( \Phi_d \); we denote by \( \theta, \sqrt{-1}, \zeta \) a primitive root of 1 of order 3, 4, 5 in \( \mathbb{Q}_l \).)

- \( G \) of type \( A_n \) \((n \geq 1)\): we have \( \mathfrak{S}_q = \emptyset \).
- \( G \) of type \( B_n \) or \( C_n \) \((n \geq 2)\): if \( n = k^2 + k \) for some integer \( k \geq 1 \), then
  \[ \mathfrak{S}_q = \{ (C, (-1)^{n/2} q^{k(k+1)(2k+1)/3}) \} \]
  where \( C \) consists of the elements of \( W \) which, as permutations of \( \{1, 2, \ldots, 2n\} \) are a product of cycles of length 4, 8, 12, \ldots, 4k; if \( n \) is not of this form, then \( \mathfrak{S}_q = \emptyset \).
- \( G \) of type \( D_n \) \((n \geq 4)\): if \( n = k^2 \) for some even integer \( k \geq 2 \), then
  \[ \mathfrak{S}_q = \{ (C, (-1)^{n/4} q^{2k(k^2-1)/3}) \} \]
  where \( C \) consists of the elements of \( W \) which, when viewed as permutations of \( \{1, 2, \ldots, 2n\} \), are a product of cycles of length 2, 6, 10, \ldots, 4k - 2; if \( n \) is not of this form, then \( \mathfrak{S}_q = \emptyset \).
G of type $E_6$: $\mathfrak{S}_q$ consists of $(C, \theta q^3), (C, \theta^2 q^3)$ where $|C| = \Phi_{12} \Phi_3$.
G of type $E_7$: $\mathfrak{S}_q$ consists of $(C, \sqrt{-1}q^{7/2}), (C, -\sqrt{-1}q^{7/2})$ where $|C| = \Phi_{18} \Phi_2$.
G of type $E_8$: $\mathfrak{S}_q$ consists of

$(C_{30}, -\theta q^4), (C_{30}, -\theta^2 q^4), (C_{30}, \zeta q^4), (C_{30}, \zeta^2 q^4), (C_{30}, \zeta^3 q^4), (C_{30}, \zeta^4 q^4),$

$(C_{24}, \sqrt{-1}q^5), (C_{24}, -\sqrt{-1}q^5), (C_{18}, \theta q^7), (C_{18}, \theta^2 q^7),$

$(C_{12}, q^{10}), (C_{12}', -q^{11}), (C_{6}, q^{20}),$

where

$|C_{30}| = \Phi_{30}, |C_{24}| = \Phi_{24}, |C_{18}| = \Phi_{18} \Phi_6, |C_{12}| = \Phi_{12}^2, |C_{12}'| = \Phi_{12} \Phi_6^2, |C_{6}| = \Phi_6^4.$

G of type $F_4$: $\mathfrak{S}_q$ consists of

$(C_{12}, \sqrt{-1}q^2), (C_{12}, -\sqrt{-1}q^2), (C_{12}, \theta q^2), (C_{12}, \theta^2 q^2),$

$(C_{8}, -q^3), (C_{6}, q^4), (C_{4}, q^6),$

where $|C_{12}| = \Phi_{12}, |C_{8}| = \Phi_8, |C_{6}| = \Phi_6^2, |C_{4}| = \Phi_4^2.$

G of type $G_2$: $\mathfrak{S}_q$ consists of $(C_6, \theta q), (C_6, \theta^2 q), (C_6, -q), (C_3, q^2)$ where $|C_6| = \Phi_6, |C_3| = \Phi_3.$

19. Since the group $G^F$ has a counterpart over $\mathbb{C}$ one can ask whether the set of unipotent (or unipotent cuspidal) representations of $G^F$ has a counterpart over $\mathbb{C}$. The answer is yes, as we will explain below.

In a paper written in 1977 (which appeared in 1979) I tried to see what happens if, in the definition of the variety $X_w$ in no.7, one replaces the Frobenius map by conjugation by $t$, a regular semisimple element of $G$. (Thus, we can define $^tY_w = \{ B \in B; (B, tBt^{-1}) \in \mathcal{O}_w \}.$) These varieties again form a partition of $B$ into pieces which can be shown to be nonempty and smooth. Now there is no longer an action of an interesting group on $^tY_w$ but, if $t$ is allowed to vary, the cohomologies of $^tY_w$ form interesting local systems on the variety of regular semisimple elements in $G$. In this paper I showed that from these local systems one can recover information about the unipotent representations $\rho$ of $G^F$ (with $G$ split over $\mathbb{F}_q$) such that $\rho$ appears in the space of functions on the finite set $X_1$. In particular, I could show that for such $\rho$ the multiplicity of $\rho$ in $\sum_i (-1)^i H^i_\mathbb{C}(X_w, \mathbb{Q}_l)$ is independent of $q$ and in fact is expressible in terms of geometry over $\mathbb{C}$ (since $^tY_w$, unlike $X_w$, made sense over $\mathbb{C}$). This paper was for me the beginning of a geometric theory of characters of $G$.

20. In 1980, I tried to compute in the case where $G = SL_2(K)$, the intersection cohomology complex $\mathcal{K}$ of $G$ with coefficients in the nontrivial local system of rank 1 on the set of regular semisimple elements given by the natural double covering of this set. To do this, I noticed that this double covering, when extended to the
whole of \( G \) as the Grothendieck-Springer resolution \( \{(g, B) \in G \times B; g \in B\} \to G \), is a small map (in the sense of Goresky-MacPherson) which allowed me to perform calculations. These calculations showed that the alternating sum of traces of the Frobenius map on the stalks of the cohomology sheaves of \( K \) (the “characteristic function” of \( K \)) was giving exactly the values of the character of the \( q \)-dimensional (or Steinberg) irreducible representation of \( SL_2(F_q) \). This was for me a strong indication that for a general \( G \), the characters of irreducible representations of \( GF \) are closely related to the characteristic functions of certain simple perverse sheaves on \( G \) and I thus started to look for those perverse sheaves. As a first step, I observed that the Grothendieck-Springer resolution was small for general \( G \) and, as an application, I found a new construction of the Springer representations of the Weyl group \( W \) in terms of intersection cohomology which, unlike Springer’s original definition, made sense in arbitrary characteristic. This appeared in my paper in 1981. At the time (1982) when my 1984 book was written, I had the definition of the required collection of simple perverse sheaves (or character sheaves) in the case where \( G \) was \( GL_n \), but I knew that for other groups the analogous collection was not complete and I conjectured that it can be completed.

Later, in 1983, I found two definitions of the collection of character sheaves on a general \( G \). One, which appeared in a paper in 1984, was describing the character sheaves as explicit intersection cohomology complexes on certain subvarieties of \( G \); this involved a generalization of the Springer representations of \( W \) and a generalization of the Grothendieck-Springer resolution which satisfied the definition of a small map except that it was not proper in general (but it could still be used). A second definition was given in a series of papers on character sheaves which appeared in 1985-1987. It used the idea in no.19 but applied to not necessarily regular semisimple elements (that series of papers also contains a proof of the equivalence of the two definitions, except for exceptional groups in small characteristic where the proof was completed only in a 2012 paper). The second definition has the virtue that it is in many ways similar to the constructions in no.7; in particular the notion of unipotent character sheaf (analogous to that of unipotent representation of \( GF \)) is defined.

21. We now give the definition of unipotent character sheaves on \( G \). For each \( w \in W \) we define

\[
\pi_w : \tilde{Y}_w = \{(g, B) \in G \times B; (B, gBg^{-1}) \in \tilde{O}_w\} \to G
\]

by \( (g, B) \mapsto g \); here \( \tilde{O}_w \) is the closure of \( O_w \) in \( B \times B \). Let \( K_w \) be the direct image under \( \pi_w \) of the intersection cohomology complex of \( \tilde{Y}_w \) with coefficients in \( Q_l\). By a theorem of Beilinson, Bernstein, Deligne and Gabber (1982), \( K_w \) is a direct sum of simple perverse sheaves (with shifts) on \( G \). The simple perverse sheaves which appear in this way (for some \( w \)) are called the unipotent character sheaves of \( w \). Assume that \( G \) is simple. A unipotent character sheaf \( A \) on \( G \) is said to be cuspidal if the following condition is satisfied: the set of \( g \in G \) such that \( A|\{g\} \neq 0 \) is a union of finitely many conjugacy classes (it is then a single conjugacy class).
We now describe the classification of unipotent cuspidal character sheaves of $G$. Let $\mathcal{G}_1$ be the set of pairs $(C, \mu_1)$ obtained by replacing each pair $(C, \mu) \in \mathcal{G}_q$ (see no.18) by $(C, \mu|_{q=1})$. For example, if $G$ is of type $A_n$ ($n \geq 1$) we have $\mathcal{G}_1 = \emptyset$. For $G$ of type $B_n$ or $C_n$ ($n \geq 2$), if $n = k^2 + k$ for some integer $k \geq 1$, then $\mathcal{G}_1 = \{(C, (-1)^{n/2})\}$ where $C$ is as in no.18 and $\mathcal{G}_1 = \emptyset$ if $n$ is not of this form.

One can show that the set of unipotent cuspidal character sheaves of $G$ is in natural bijection with $\mathcal{G}_1$. Since $\mathcal{G}_1$ is in natural bijection with $\mathcal{G}_q$ in no.18 (in type $E_7$ this bijection depends on a fixed choice of $\sqrt{q}$) we see that the set of unipotent cuspidal character sheaves of $G$ is in natural bijection with the set of unipotent cuspidal representations of $G^F$. (A more satisfactory explanation of this fact is given in a preprint I posted in 2014.) In this way, the unipotent cuspidal character sheaves of $G$ (which can be defined also over $\mathbb{C}$) appear as limits as $q$ tends to 1 of unipotent cuspidal representations of $G(\mathbb{F}_q)$.

22. The simple Lie group of type $E_8$ (over $\mathbb{C}$) has an almost mythical status in mathematics. It is the simple Lie group for which the dimension divided by the square of its rank is maximum possible (namely $\frac{248}{8} = 4 - \frac{1}{8}$) which makes it the most noncommutative simple Lie group. It is the simple Lie group for which the dimension of the smallest nontrivial irreducible representation divided by the dimension of the group is maximum possible (namely $\frac{248}{248} = 1$). It is the only simple Lie group such that the centralizer of one of its elements has a nonsolvable group of connected components. It is the only simple Lie group $G$ of maximal dimension with the following property: there exist conjugacy classes $C_2, C_3, C_5$ of elements of order 2, 3, 5 respectively such that $\dim C_2 + \dim C_3 + \dim C_5 = 2 \dim G$. (In fact, $C_2, C_3, C_5$ are uniquely determined; they have dimension 248 - 240/2, 248 - 240/3, 248 - 240/5 respectively.) It is the only simple Lie group $G$ of maximal dimension such that the following set $G_*$ is nonempty: $G_*$ is the set of triples $(x_2, x_3, x_5) \in G^3$ such that $x_2^2 = x_3^3 = x_5^5 = x_2 x_3 x_5 = 1$ and such that $\{g \in G; gx_2 = x_2 g, gx_3 = x_3 g, gx_5 = x_5 g\}$ is finite. (The fact that $G_* \neq \emptyset$ in type $E_8$ was first shown by A. V. Borovik in 1989.) Note that $G_*$ can be viewed as the set of homomorphisms $A_5 \to G$ (where $A_5$ is the alternating group in 5 letters) whose image has a finite centralizer. Let $G$ be a simple Lie group of type $E_8$ over $\mathbb{C}$. The following question was raised by D. D. Frey and J.-P. Serre in 1998 (I have learned about it from R. Griess in 2001.) What is the number of orbits of the simultaneous conjugation action of $G$ on $G_*$? Eventually I found that this number is 1 (this appeared in my paper in 2003.) It is interesting that the solution of this problem relies on the results on the representation theory of the group of type $E_8$ over a finite field $\mathbb{F}_q$, explained earlier in this paper. Frey and Serre have shown (in 1998) that $G_* = \{(x_2, x_3, x_5) \in C_2 \times C_3 \times C_5; x_2 x_3 x_5 = 1\}$ where $C_n$ are as above. Moreover, from the definitions, the $G$-action on $G_*$ has finite stabilizers. It is then enough to show that if $G$ is replaced by the corresponding group $E_8(\mathbb{F}_q)$ over $\mathbb{F}_q$ (with $q$ large and not divisible by 2, 3, 5) and $C_2, C_3, C_5$ by the corresponding conjugacy classes in $E_8(\mathbb{F}_q)$ (denoted again by $C_2, C_3, C_5$) then (a) $\left|\{(x_2, x_3, x_5) \in C_2 \times C_3 \times C_5; x_2 x_3 x_5 = 1\}\right|$
is approximately equal to $|E_8(F_q)|$. By a general result of W. Burnside, the number (a) is equal to

$$ (b) \frac{|C_2||C_3||C_5|}{|E_8(F_q)|} \sum_{\rho} \frac{\rho(g_2)\rho(g_3)\rho(g_5)}{\rho(1)} $$

where $\rho$ runs over the irreducible characters of $E_8(F_q)$ and $g_2, g_3, g_5$ are elements of $C_2, C_3, C_5$. It is then enough to show that the sum (b) is approximately equal to $q^{248}$. This can be done using the classification of irreducible characters of $E_8(F_q)$ and the knowledge of their values at $g_2, g_3, g_5, 1$. While it is too difficult to compute the sum (b) exactly, it is possible to compute it approximately and this is enough for the desired result. This gives an example of the use of representation theory of reductive groups over $F_q$ to obtain information on the structure of the corresponding groups over $C$.

23. Another direction of my research is the representation theory of semisimple groups over $p$-adic fields. In a 1983 paper I found new examples of square integrable representations of such groups associated with subregular unipotent elements. This relied on some complicated calculations which had the side benefit that they suggested a way to refine the Deligne-Langlands conjecture (for representations of affine Hecke algebras) from being a rough parametrization to being an exact parametrization. They also suggested that the representations of the affine Hecke algebras were intimately related to the geometry of “Springer fibres” in the dual group, in particular that the weight structure of the representations could be predicted from the geometry of Springer fibres. In my paper written in the summer of 1984, which appeared in 1985, I formulated the idea that equivariant $K$-theory of Springer fibres should be used to construct representations of affine Hecke algebras and also that the parameter $q$ of the affine Hecke algebra should be interpreted as the generator of the equivariant $K$-theory of a point with respect to the circle group action. This idea was further developed in my two papers with Kazhdan (one, written in the fall of 1984 and one written in the summer of 1985) where the irreducible representations of affine Hecke algebras with equal parameters were classified, thus providing a proof of the Deligne-Langlands conjecture in the refined form stated in my 1983 paper.

In my 1983 paper I also gave a definition of unipotent representations of a simple split adjoint group over a $p$-adic field and stated a conjectural parametrization for them extending the Deligne-Langlands conjecture. This parametrization was established in my paper in 1995 where the unipotent representations of finite reductive groups and the theory of character sheaves come together in a rather unexpected way; I did this by using character sheaves mixed with equivariant homology to realize representations of various graded affine Hecke algebras from which the representations of affine Hecke algebras with unequal parameters could be reconstructed.
24. There are several instances where a bar operator similar to the one mentioned in no.13 (in the construction of the new basis of a Hecke algebra) has been used to construct new bases after its first use in 1978; I will mention some of them.

(a) In a 1981 paper I constructed periodic analogues of the polynomials $P_{y,w}$ for affine Weyl groups. In my 1999 paper I extended this to a conjecture (proved by R. Bezrukavnikov and I. Mirkovic in 2012) defining canonical bases in certain equivariant K-theory groups, related to the representation theory of simple Lie algebras in characteristic $> 0$.

(b) In a 1983 paper I extended the method in no.13 to include Hecke algebras with unequal parameters.

(c) In another 1983 paper (with D. Vogan) we used a bar operator to generalize the polynomials $P_{y,w}$ in no.13 to the setting of symmetric spaces.

(d) In my 1990 paper I defined a canonical basis in the plus part of the quantized enveloping algebra of type $A, D$ or $E$ using a bar operator method similar to that in no.13. (Later, M. Kashiwara gave another construction of the canonical basis, using the bar operator from my 1990 paper.)

(e) In my 2012 paper with D. Vogan we defined a canonical splitting of $P_{y,w}$ as a sum of two polynomials (when $y, w$ are involutions in $W$) using a bar operator.