An $L_\infty$ algebra structure on polyvector fields

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Abstract

In this paper we construct an $L_\infty$ structure on polyvector fields on a vector space $V$ over $\mathbb{C}$ where $V$ may be infinite-dimensional. We prove that the constructed $L_\infty$ algebra of polyvector fields is $L_\infty$ equivalent to the Hochschild complex of polynomial functions on $V$, even in the infinite-dimensional case. For a finite-dimensional space $V$, our $L_\infty$ algebra is equivalent to the classical Schouten-Nijenhuis Lie algebra of polyvector fields. For an infinite-dimensional $V$, it is essentially different. In particular, we get the higher obstructions for deformation quantization in infinite-dimensional case.

1 The set-up

Let $V = \bigoplus_{i \geq 0} V_i$ be an infinite-dimensional non-negatively graded vector space over $\mathbb{C}$ with finite-dimensional components $V_i$. Our goal in this paper is to establish the Kontsevich formality theorem for the Hochschild complex of the algebra $S(V^*) = S(\bigoplus_{i \geq 0} V_i^*)$.

We start the paper with defining what are the right versions of polyvector fields on $V$ and of the cohomological Hochschild complex of $S(V^*)$ for an infinite-dimensional vector space $V$ with finite-dimensional graded components. Then we prove an analog of the Hochschild-Kostant-Rosenberg theorem in our setting.

1.1 The polyvector fields $T_{\text{fin}}(V)$

First of all, let us define an appropriate Lie algebra of polyvector fields $T_{\text{poly}}(V)$. We want to allow some infinite sums. Here is the precise definition. An $i$-linear polyvector field $\gamma$ of inner degree $k$ is a (possibly infinite) sum in $S(V^*) \hat{\otimes} \Lambda^i(V)$ such that all summands have degree $k$ (here we set $\deg V_a^* = -a$). With a fixed element in $\Lambda^i(V)$, its “coefficient” maybe only a finite sum because the vector space $V$ is non-negatively graded. We denote the space of $i$-linear polyvector fields of inner degree $k$ on the space $V$ by $T_{\text{fin}}^{i,k}(V)$. Now the space of $i$-linear polyvector fields $T_{\text{fin}}^i(V)$ is by definition a direct sum $T_{\text{fin}}^i(V) = \bigoplus_k T_{\text{fin}}^{i,k}(V)$. That is, any polyvector field has a finite number of non-zero inner degrees. The Schouten-Nijenhuis bracket of two such polyvector fields is well-defined. We denote this graded Lie algebra by $T_{\text{fin}}(V)$. 

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1.2 The Hochschild complex \( \text{Hoch}^\ast_{\text{fin}}(S(V^*)) \)

Now we define some Hochschild cohomological complex of the algebra \( S(V^*) = S(\oplus_i V_i^*) \) which has as its cohomology the graded Lie algebra \( T_{\text{fin}}(V) \).

We define an \( \ell \)-cochain as an element in \( \text{Hom}(S(V^*)^{\otimes \ell}, S(V^*)) \) having only a finite number of different inner gradings. Here we set, as before, \( S(V^*) = S(\oplus_i V_i^*) \). We denote this Hochschild complex by \( \text{Hoch}^\ast_{\text{fin}}(S(V^*)) \). It is a dg Lie algebra with the Gerstenhaber bracket.

The cohomology of this Hochschild complex can be interpreted as a derived functor, as follows.

Denote by \( \mathcal{A} \) the category whose objects are graded (with respect to the inner grading) \( S(V^*) \)-bimodules, and whose morphisms are graded maps between them. Then \( \mathcal{A} \) is an Abelian category. For an object \( X \in \text{Ob}(\mathcal{A}) \), denote by \( X(\langle k \rangle) \) the object of \( \mathcal{A} \) whose inner grading defined as usual: \( (X(\langle k \rangle))^i = X^{k+i} \).

We have the following lemma:

**Lemma.** The cohomology \( H^\ell(\text{Hoch}^\ast_{\text{fin}}(S(V^*))) \) is equal to the \( \text{Ext}^\ell_{\mathcal{A}}(S(V^*), \oplus_{k \in \mathbb{Z}} S(V^*)(\langle k \rangle)) \). Here in the r.h.s. we have the direct sum, an element of it is a finite linear combination of elements of the summands.

**Proof.** The bar-resolution of \( S(V^*) \) is clearly a projective resolution of the tautological bimodule \( S(V^*) \) in \( \mathcal{A} \). We compute the Exts functors using this resolution. The complex \( \text{Hom}_{\mathcal{A}}(\text{Bar}^\ast(S(V^*)), \oplus_{k \in \mathbb{Z}} S(V^*)(\langle k \rangle)) \) is exactly the complex \( \text{Hoch}^\ast_{\text{fin}}(S(V^*)) \). \( \square \)

1.3 The Hochschild-Kostant-Rosenberg theorem

Define the Hochschild-Kostant-Rosenberg map \( \varphi_{\text{HKR}}: T_{\text{fin}}(V) \to \text{Hoch}^\ast_{\text{fin}}(V) \) as

\[
\varphi_{\text{HKR}}(\gamma) = \frac{1}{k!} \left\{ f_1 \otimes \cdots \otimes f_k \mapsto \gamma(df_1 \wedge \cdots \wedge df_k) \right\}
\]

for \( \gamma \in T_{\text{fin}}^k(V) \).

We have the following result:

**Theorem.** The map \( \varphi_{\text{HKR}}: T_{\text{fin}}(V) \to \text{Hoch}^\ast_{\text{fin}}(S(V^*)) \) is a quasi-isomorphism of complexes.

**Proof.** Consider the following Koszul complex \( K^\ast \):

\[
\ldots \xrightarrow{d} K_3 \xrightarrow{d} K_2 \xrightarrow{d} K_1 \xrightarrow{d} K_0 \to 0
\]

where

\[
K_i = \Lambda^i(V) \hat{\otimes} S(V^* \oplus V^*)
\]

\[
1.2 \, \text{The Hochschild complex} \quad \text{Hoch}^\ast_{\text{fin}}(S(V^*))
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where

\[
K_i = \Lambda^i(V) \hat{\otimes} S(V^* \oplus V^*)
\]
where all infinite tensor products are allowed, and the differential is

\[ d((\xi_{i_1} \wedge \cdots \wedge \xi_{i_k}) \otimes f) = \sum_{j=1}^{k} (-1)^{j-1}(\xi_{i_1} \wedge \cdots \wedge \hat{\xi}_{i_j} \wedge \cdots \wedge \xi_{i_k}) \otimes ((x_j - y_j)f) \]  

where \( \{x_i\} \) is a basis in \( V^* \) compatible with the grading \( V = \oplus_i V_i \), \( \{y_i\} \) is the same basis in the second copy of \( V \), and \( \{\xi_i\} \) is the corresponded basis in \( V[1] \).

This Koszul complex is clearly a resolution of the tautological \( S(V^*) \)-bimodule \( S(V^*) \) by free bimodules, it is a resolution in the category \( \mathcal{A} \) because the differential \( d \) preserves the inner grading. When we compute \( \text{Hom}_\mathcal{A}(K^*, \oplus_{k \in \mathbb{Z}} S(V^* \langle k \rangle)) \), we get exactly \( T_{\text{fin}}(V) \) with zero differential.

It remains to note that the image of the Hochschild-Kostant-Rosenberg map coincides with the cohomology classes in \( \text{Hoch}_{\text{fin}}^*(V) \) given by the Koszul resolution.

1.4 Towards the formality for an infinite-dimensional space

Let us comment here why the Kontsevich’s formality theorem is not clear in the infinite-dimensional setting. Let us try to prove it using the Kontsevich’s graphical technique from [K97]. Then, if a graph \( \Gamma \) contains an oriented cycle between some its vertices of the first type, the corresponding polydifferential operator \( \mathcal{U}_\Gamma \) may be ill-defined for an infinite-dimensional vector space \( V \).

The reason for this is that an oriented cycle looks like trace which may be ill-defined for an operator on an infinite-dimensional space.

We have, however, the following lemma:

**Lemma.** Let \( \Gamma \) be a Kontsevich admissible graph in the sense of [K97] with \( n \) vertices of the first type and \( m \) vertices of the second type. Suppose that the graph \( \Gamma \) has no any oriented cycles between vertices of the first type. Let \( V \) be as above, and let \( \gamma_1, \ldots, \gamma_n \in T_{\text{fin}}(V) \). Then the Kontsevich’s polydifferential operator \( \mathcal{U}_\Gamma(\gamma_1 \wedge \cdots \wedge \gamma_n) \) is a well-defined element of \( \text{Hoch}_{\text{fin}}^*(S(V^*)) \).

The proof is clear.

Thus, only the graphs with oriented cycles cause a problem for us, and we should exclude them from the graphical computations. We really succeed to do that by constructing of a new propagator 1-form. However, we should pay for it: the Kontsevich’s lemma [K97], Section 6.6 is not anymore true. It means that we get a new \( L_\infty \) structure on \( T_{\text{fin}}(V) \), such that the second Taylor component is the Schouten-Nijenhuis bracket, and there are higher non-vanishing Taylor components.

We describe this \( L_\infty \) structure first.
2 An $L_\infty$ structure on $T_{\text{fin}}(V)$

2.1 Configuration spaces $C_{n,\Gamma}$

For an oriented graph $\Gamma$ we denote by $V(\Gamma)$ the set of its vertices, by $E(\Gamma)$ the set of its edges, and for a vertex $v \in V(\Gamma)$ we denote by $\text{Star}(v)$ the set of outgoing edges from the vertex $v$, and by $\text{In}(v)$ the set of outgoing edges from the vertex $v$.

Throughout this Section, by an admissible graph $\Gamma$ we mean an oriented graph without oriented loops. Its vertices are labeled by $\{1, \ldots, n\}$, and it may have any number of edges, in particular, it is not necessarily connected. (The ordering of the vertices is a part of the data of an admissible graph). The ordering should obey the following relation. Denote by $n(v)$ the label of the vertex $v$, $n(v) \in [1, \ldots, \#V(\Gamma)]$. Then one should have:

$$n(v_1) < n(v_2) \text{ if there is an edge } \overrightarrow{v_1v_2} \quad (5)$$

We denote the set of admissible graphs $\Gamma$ with $n$ vertices and $E$ edges by $G(n, E)$.

Any graph without loops has at least one labeling obeying (5), but some of them clearly may have several admissible labeling, see Figure 1.

![Figure 1: The two possible labelings of a graph with 4 vertices](image)

Let $\Gamma \in G(n, E)$. Define the configuration space $C_{n,\Gamma}$, as follows:

$$C_{n,\Gamma} = \{z_1, \ldots, z_n, z_i \neq z_j \text{ for } i \neq j, \text{ and } \text{Im}(z_j - z_i) < 0 \text{ when } \overrightarrow{ij} \text{ is an edge of } \Gamma\} / G^3 \quad (6)$$

where $G^3 = \{z \mapsto az + c, a \in \mathbb{R}_{>0}, c \in \mathbb{C}\}$ is a 3-dimensional group of transformations.

In particular, if $E = 0$ we recognize the Kontsevich’s configuration space $C_n$ from [K97]. Dimension $\dim C_{n,\Gamma}$ is equal to $2n - 3$ and does not depend on $\Gamma$.

Now suppose that $\sharp E(\Gamma) = 2n - 3$. If $e = \overrightarrow{ij}$ is an edge of $\Gamma$, we associate with it the differential 1-form

$$\phi_e = d\text{Arg}(z_j - z_i) = \frac{1}{2i}d\text{Log} \frac{z_j - z_i}{\overline{z}_j - \overline{z}_i} \quad (7)$$
The numeration of the vertices of $\Gamma$ gives a corresponding numeration of the edges: firstly we count the edges outgoing from the vertex 1, in the order of the end-vertex, then we count the edges outgoing from the vertex 2, and so on.

Define

$$W_\Gamma = \frac{1}{\pi^{2n-3}} \int_{C_{n,\Gamma}} \bigwedge_{e \in E(\Gamma)} \phi_e$$

(8)

Here in the wedge product we use the order of the 1-forms $\phi_e$ corresponding to the ordering of the edges of $\Gamma$ described above.

**Remark.** In [K97], Section 6.2, M.Kontsevich has the combinatorial factor $\prod_{v \in V(\Gamma)} \frac{1}{2^{\text{Star}(v)}}$ before an analogous formula. The reason why we omit this factor here is the following. In [K97] in the definition of an admissible graph not only the vertices are labeled, but also the sets $\text{Star}(v)$ for all vertices $v$ are ordered. It is not necessary because as soon as the vertices are enumerated, there appears a canonical ordering, by the number of the end-vertex. Therefore, we omit here and thereafter this ingredient from the data of an admissible graph. As a consequence, we have not the mentioned above combinatorial factor.

We prove firstly that $W_\Gamma$ may be nonzero only for an even $n$.

**Lemma.** For an odd $n$, the integral $W_\Gamma = 0$ for any number of edges $\sharp E(\Gamma)$.

**Proof.** Fix one vertex $p$ using the group $G^3$ to the point $0 + 0 \cdot i$, and let another point $q$ move along the unit circle around $p$. Draw the vertical line $\ell$ through $p$ and consider the symmetry $\sigma$ with respect to $\ell$. We have the following general formula:

$$\int_c \sigma^* \omega = \int_{\sigma \cdot c} \omega$$

(9)

where $c$ is an oriented chain.

In our case $c = C_{n,\Gamma}$ and $\omega = \bigwedge_{e \in E(\Gamma)} \phi_e$. We have: $\sigma \cdot \phi_e = -\phi_e$ (there are $2n-3$ edges), and at each movable point (there is $n-1$ of such points) the orientation changes to $-1$. Finally, if $I$ is the integral, we have from (9): $(-1)^{2n-3} I = (-1)^{n-1} I$ which implies that $I = 0$ for an odd $n$. \qed

It is clear that when $\Gamma$ is the graph with two vertices and one edge, $W_\Gamma = 1$. Maxim Kontsevich found an example of a graph with 4 vertices which has a nonzero weight $W_\Gamma$. Consider firstly a slightly different example, which we will use in the sequel.

**Example 1:** Consider the graph $\Gamma$ shown in Figure 2. Let us compute $W_\Gamma$ for this graph. Fix the vertex 2 to the point $0 + 0 \cdot i$ by the action of group $G^3$, and let the vertex 3 move along the unit lower half-circle around 2. Let $x$ be the angle of the arrow $(2,3)$, $-\pi \leq x \leq 0$. Then we can integrate over positions of the vertices 1 and 4 separately. Let us firstly integrate (for a fixed $x$) over 1, denote it $z$. 

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Figure 2: An admissible graph $\Gamma$ with $n = 4$ and nonzero $W_\Gamma$.

We need to compute the integral

$$I_1 = \int_{\text{Im} z \geq 0} d\text{Arg}(-z) \wedge d\text{Arg}(-z + \exp(ix))$$

(10)

We use the Stokes formula:

$$I_1 = \int_{\partial \text{Im} z \geq 0} \text{Arg}(-z) d\text{Arg}(-z + \exp(ix))$$

(11)

The Stokes formula is applied to the domain $D = \{\text{Im} z \geq 0, \varepsilon \leq |z| \leq R\}$ where $\varepsilon \to 0$ and $R \to \infty$. There are two boundary strata which contribute to the integral:

$$I_1 = -\lim_{\varepsilon \to 0} \int_{|z| = \varepsilon, \text{Im} z \geq 0} \text{Arg}(-z) d\text{Arg}(-z + \exp(ix)) + \lim_{R \to \infty} \int_{|z| = R, \text{Im} z \geq 0} \text{Arg}(-z) d\text{Arg}(-z + \exp(ix))$$

(12)

The first integral in the limit $\varepsilon \to 0$ is equal to $\pi x$, and the second one in the limit $R \to \infty$ is equal to $\frac{1}{2}\pi^2$. The total answer is

$$I_1 = -\pi x + \frac{1}{2}\pi^2$$

(13)

Analogously we compute the integral $I_2$ over position of the vertex 4. We get the same answer:

$$I_2 = -\pi x + \frac{1}{2}\pi^2$$

(14)

Finally,

$$W_\Gamma = \frac{1}{\pi^5} \int_{-\pi}^{0} I_1(x) I_2(x) dx = \frac{13}{12}$$

(15)
Example 2: More generally, if we have the graph $\Gamma$ shown in Figure 3, the same computation shows that

$$W_\Gamma = \frac{1}{\pi^{2m+2n+1}} \int_{-\pi}^{0} (\pi x - \frac{1}{2} \pi^2)^{m+n} dx = \frac{1}{\pi^{2m+2n+1}} \cdot \frac{1}{\pi m + n + 1} \cdot \left( \frac{1}{\pi m + n + 1} (\pi x - \frac{1}{2} \pi^2)^{m+n+1} \right) |_{0}^{\pi} = (-1)^{m+n} \frac{3^{m+n+1} - 1}{(m + n + 1)2^{m+n+1}}$$

(16)

It is clear that $W_\Gamma \neq 0$ for any $m, n$.

2.2 The $L_\infty$ structure

Let $\gamma_1, \ldots, \gamma_n$ be homogeneous polyvector fields. We are going to define the $n$-th Taylor component $L_n(\gamma_1 \wedge \cdots \wedge \gamma_n)$.

First of all, we define with each admissible graph $\Gamma$ with $n$ vertices and the ordered set $\gamma_1, \ldots, \gamma_n$ of polyvector fields a new polyvector field $L_\Gamma(\gamma_1 \otimes \cdots \otimes \gamma_n)$ which is $(\sum_{i=1}^{n} \deg \gamma_i + n - \sharp E(\Gamma))$-vector field. (Here we denote by $\deg \gamma$ the Lie algebra degree, that is, if $V$ is concentrated in the cohomological degree zero, $\deg \gamma = k - 1$ for a $k$-vector field $\gamma$).

When the target vector space $V$ is finite-dimensional, the polyvector field $L_\Gamma(\gamma_1 \otimes \cdots \otimes \gamma_n)$ is the sum

$$L_\Gamma(\gamma_1 \otimes \cdots \otimes \gamma_n) = \sum_{I: E(\Gamma) \rightarrow \{1, 2, \ldots, \dim V\}} L_\Gamma^I(\gamma_1 \otimes \cdots \otimes \gamma_n)$$

(17)
The polyvector field $L^I_I(\gamma_1 \otimes \cdots \otimes \gamma_n)$ is the product over the vertices of $\Gamma$:

$$L^I_I(\gamma_1 \otimes \cdots \otimes \gamma_n) = \bigwedge_{v \in V(\Gamma)} \Psi^I_v$$

where the vertices come in the order fixed by the labeling of the vertices. Each $\Psi^I_v$ is defined as

$$\Psi^I_v = \left( \prod_{e \in I(n(v))} \frac{\partial}{\partial x_{I(e)}} (\gamma_{n(v)}(\wedge_{e \in \text{Star}(v)} dx_{I(e)})) \right)$$

where $n(v)$ is the label of the vertex $v$, and the order in the wedge-product is fixed by the ordering of the set $\text{Star}(v)$, as above (see Remark in Section 2.1). This completes the definition of $L_I(\gamma_1 \otimes \cdots \otimes \gamma_n)$ for a finite-dimensional vector space $V$.

Now suppose that $V = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} V_i$ where all graded components $V_i$ are finite-dimensional, as in Section 1. We leave to the reader the proof of the following lemma:

**Lemma.** The polyvector field $L_I(\gamma_1 \otimes \cdots \otimes \gamma_n)$ is well-defined in the infinite-dimensional case of Section 1, when all $\gamma_i \in T_{\text{fin}}(V)$, and the graph $\Gamma$ does not contain oriented cycles (which holds for any admissible graph).

Now we give our main definition. Define for $\gamma_1, \ldots, \gamma_n \in T_{\text{fin}}(V)$ the polyvector field

$$L_n(\gamma_1 \wedge \cdots \wedge \gamma_n) = \text{Alt}_{\gamma_1, \ldots, \gamma_n} \left( \sum_{\Gamma \in G_{n,2n-3}} W_{\Gamma} \cdot L_I(\gamma_1 \otimes \cdots \otimes \gamma_n) \right)$$

Here $\text{Alt}$ is the sum over all permutations of $\gamma_1, \ldots, \gamma_n$ with signs, such that when we permute $\gamma_i$ and $\gamma_j$ the sign $(-1)^{(\deg \gamma_i + 1)(\deg \gamma_j + 1)}$ appears, and $G_{n,2n-3}$ is the set of admissible connected graphs with $n$ vertices and $2n-3$ edges.

**Example.** $L_2(\gamma_1 \wedge \gamma_2) = \{\gamma_1, \gamma_2\}$ is the Schouten-Nijenhuis bracket.

**Remark.** In what follows, we deal with dg Lie algebras, and, more generally, $L_\infty$ algebras. Here we meet some sign problem. Indeed, for a dg Lie algebra we have two possible notions of skew-commutativity of the bracket: the first is $[a, b]_1 = (-1)^{\deg a \deg b + 1} [b, a]_1$, and the second is $[a, b]_2 = (-1)^{\deg a + 1}(\deg b + 1) [b, a]_2$. In what follows we will use the second concept of skew-commutativity.

Recall that an $L_\infty$ algebra structure on a $\mathbb{Z}$-graded vector space $g$ is a coderivation $Q$ of degree +1 of the free coalgebra $S(g[1])$ such that $Q^2 = 0$. If $g$ is a Lie algebra, such a $Q$ is the differential in the chain complex of the Lie algebra $g$.

In coordinates, an $L_\infty$ structure on $g$ is the collection of maps (”Taylor components” of the $L_\infty$ morphism) $L_n : \Lambda^n g \to g[2-n], n \geq 1$
which satisfy for each \( N \geq 1 \) the following quadratic equation:

\[
\operatorname{Alt}_{g_1, \ldots, g_N} \sum_{a+b=N+1, a, b \geq 1} \pm \frac{1}{a!b!} \mathcal{L}_b \left( (\mathcal{L}_a (g_1 \wedge \cdots \wedge g_a) \wedge g_{a+1} \wedge \cdots \wedge g_N) \right) = 0 \quad (22)
\]

Turn back now to our polyvector fields \( \mathcal{L}_\Gamma (\gamma_1 \otimes \cdots \otimes \gamma_n) \). This polyvector field has degree \( \sum_{i=1}^n \deg \gamma_i + n - \sharp E(\Gamma) - 1 \). When \( \sharp E(\Gamma) = 2n - 3 \), this degree is \( \sum \deg \gamma_i - n + 2 \), exactly as in (21).

**Theorem.** The maps \( \mathcal{L}_n : \Lambda^n T_{\text{fin}}(V) \to T_{\text{fin}}(V)[2-n] \) are the Taylor components of an \( L_\infty \) algebra structure on \( T_{\text{fin}}(V) \).

**Proof.** Consider the relation (22) for some fixed \( N \). It can be written as

\[
\text{the l.h.s. of (22) } = \sum_{\Gamma \in G_{N,2N-4}} c_\Gamma \cdot \operatorname{Alt}_{\gamma_1,\ldots,\gamma_N} \mathcal{L}_\Gamma (\gamma_1 \otimes \cdots \otimes \gamma_N) \quad (23)
\]

where the summation is taken over the admissible connected graphs with \( N \) vertices and \( 2N - 4 \) edges (for 1 edge less than in (22)), and \( c_\Gamma \) are some (real) numbers. We need to prove that \( c_\Gamma = 0 \) for any \( \Gamma \in G_{N,2N-4} \).

For, consider the integral

\[
\int_{C_{N,\Gamma}} d( \bigwedge_{e \in E(\Gamma)} \phi_e ) \quad (24)
\]

This integral is clearly equal to 0, because all 1-forms \( \phi_e \) are closed (moreover, they are exact). Now we want to apply the Stokes’ formula. For this we need to construct the compactifications \( \overline{C}_{n,\Gamma} \) of the spaces \( C_{n,\Gamma} \) which is a smooth manifold with corners, and such that the forms \( \phi_e \) can be extended to a smooth forms on \( \overline{C}_{n,\Gamma} \). It can be done in the standard way, see [K97], Section 5. Here we describe the boundary strata of codimension 1 which are the only strata which contribute to the integrals we consider. Here is the list of the boundary strata of codimension 1:

1) some \( S \) points \( p_{i_1}, \ldots, p_{i_S} \) among the \( n \) points approach each other, such that \( 2 \leq \sharp S \leq n - 1 \); in this case let \( \Gamma_1 \) be the restriction of the graph \( \Gamma \) into these \( S \) points, and let \( \Gamma_2 \) be the graph obtained from contracting of the \( S \) vertices into a single new vertex. Thus, \( \Gamma_1 \) has \( S \) vertices, and \( \Gamma_2 \) has \( n - S + 1 \) vertices. In this case the boundary stratum is isomorphic to \( C_{S,\Gamma_1} \times C_{n-S+1,\Gamma_2} \);

2) a point \( q \) connected by an edge \( \overrightarrow{pq} \) is placed on the horizontal line passing through the point \( p \).

We continue:

\[
0 = \int_{C_{N,\Gamma}} d( \bigwedge_{e \in E(\Gamma)} \phi_e ) = \int_{\overline{C}_{N,\Gamma}} d( \bigwedge_{e \in E(\Gamma)} \phi_e ) = \int_{\partial \overline{C}_{N,\Gamma}} \bigwedge_{e \in E(\Gamma)} \phi_e \quad (25)
\]
Only the boundary strata of codimension 1 do contribute to the r.h.s. integral. The strata of type T2) do not contribute because the form \( \wedge e \phi_e \) vanishes there. We can therefore consider only the strata of type T1).

For these strata we have the following factorization principle: it says that the integral over a stratum \( T \) of type T1) is the product:

\[
\int_T \wedge e \phi_e = \left( \int_{C_{S,T_1}} \wedge e \phi_e \right) \times \left( \int_{C_{n-S+1,T_2}} \wedge e \phi_e \right)
\]  

(26)

The same factorization holds therefore for the weights \( W_\Gamma \).

We get the following identity:

\[
0 = \int_{\partial T \cap \nabla_\Gamma} \wedge \phi_e = \sum_{T \in \partial T_1} \left( \int_{C_{S,T_1}} \wedge \phi_e \right) \times \left( \int_{C_{n-S+1,T_2}} \wedge \phi_e \right)
\]  

(27)

where the strata \( T \) come with its orientation.

The summands in the r.h.s. are in 1-1 correspondence with the summands in (22) which contribute to \( c_\Gamma \). Therefore, all \( c_\Gamma = 0 \).

It follows from Lemma 2.1 that this \( L_\infty \) structure has only even non-zero components \( \mathcal{L}_2, \mathcal{L}_4, \mathcal{L}_6, \ldots \). On the other hand, Example 2 in Section 2.1 shows that the higher components \( \mathcal{L}_{2n}, n \geq 2 \), are nonzero.

We denote the \( L_\infty \) algebra \( T_{\text{fin}}(V) \) with the constructed \( L_\infty \) structure by \( T_{\text{fin}}^\mathcal{L}(V) \).

### 2.3 Infinite-dimensional formality

We are now ready to state the main result of this paper.

**Main Theorem.** Let \( V = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} V_i \) be a non-negatively graded vector space with \( \dim V_i < \infty \). Then there is an \( L_\infty \) morphism from the \( L_\infty \) algebra \( T_{\text{fin}}^\mathcal{L}(V) \) constructed in Section 2.2 to the Hochschild complex \( \text{Hoch}_{\text{fin}}^\bullet(S(V^*)) \) with the Gerstenhaber bracket. Its first Taylor component is the Hochschild-Kostant-Rosenberg map.

We have an immediate corollary:

**Corollary.** When \( V \) is finite-dimensional, the \( L_\infty \) algebra \( T_{\text{fin}}^\mathcal{L}(V) \) on polyvector fields is \( L_\infty \) quasi-isomorphic to the classical one, which is the Lie algebra with the Schouten-Nijenhuis bracket.

**Proof.** By the Kontsevich’s formality [K97], there is an \( L_\infty \) quasi-isomorphism \( U: T_{\text{poly}}(V) \to \text{Hoch}^\bullet(S(V^*)) \). By our Main Theorem, there is an \( L_\infty \) quasi-isomorphism \( F: T_{\text{fin}}^\mathcal{L}(V) \to \text{Hoch}^\bullet(S(V^*)) \). It implies that the two \( L_\infty \) structures on polyvector fields are \( L_\infty \) quasi-isomorphic. \( \square \)
However, for an infinite-dimensional space $V$, the two structures may be completely different. For example, suppose we have a bivector $\alpha \in T_{\text{fin}}(V)$ for an infinite-dimensional $V$. The right concept what is that $\alpha$ is Poisson is given by our Main Theorem as follows:

$$\frac{1}{2} \mathcal{L}_2(\alpha \wedge \alpha) + \frac{1}{24} \mathcal{L}_4(\alpha \wedge \alpha \wedge \alpha \wedge \alpha) + \cdots = 0 \quad (28)$$

For a fixed $\alpha$ which is polynomial in coordinates the sum is actually finite. We see, in particular, that if $\alpha$ satisfies (28), the bivector field $\lambda \cdot \alpha$, $\lambda \in \mathbb{C}$, may not satisfy. That is, the equation (28) is not homogeneous. We call (28) the quasi-Poisson equation.

In the case when $\alpha$ is a linear bivector field the quasi-Poisson equation (28) coincides with the classical one:

**Lemma.** Let $\alpha$ be a linear bivector on $V$. Then the higher components $\mathcal{L}_{2n}(\alpha^{\wedge 2n}) = 0$, $n \geq 2$. That is, (28) is equivalent to the Poisson equation $\{\alpha, \alpha\} = 0$.

**Proof.** Any admissible graph with $k$ vertices which contributes to the $L_\infty$ algebra $\mathcal{L}$ has $2k - 3$ edges. If $k > 2$ it implies that there is at least one vertex with at least two incoming edges. Then the corresponding operator $\mathcal{L}_\Gamma(\alpha^{\wedge k})$ is zero because $\alpha$ has linear coefficients.

We expect that there are Poisson bivectors which are of degree $\geq 2$ on an infinite-dimensional vector space which are impossible to quantize in the sense of deformation quantization. As the condition (28) is non-homogeneous, our formality theorem gives a deformation quantization only if all $\mathcal{L}_{2n}(\alpha^{\wedge 2n}) = 0$ separately, so we have a sequence of homogeneous equations. We expect that this series of equations gives the higher obstructions for deformation quantization problem in the infinite-dimensional case.

Compute now the first obstruction $\mathcal{L}_4(\alpha^{\wedge 4})$. The list of all 7 admissible graphs (up to the labeling) with 4 vertices and $2 \cdot 4 - 3 = 5$ edges is shown in Figure 4.

Their weights are $2 \times \frac{13}{12}$, $2 \times \frac{13}{12}$, $2 \times \frac{1}{12}$, $\frac{1}{12}$, $\frac{7}{12}$, $\frac{7}{12}$, respectively. (Here $2 \times \ldots$ counts the two possible labelings). Clearly only $\Gamma_1$, $\Gamma_4$, $\Gamma_5$ contribute to $\mathcal{L}_4(\alpha^{\wedge 4})$, because other graphs has a vertex with 3 outgoing edges. We get the following equation for the vanishing of the first obstruction in the infinite-dimensional case:

$$\left(\frac{13}{6} \mathcal{L}_{\Gamma_1} + \frac{13}{12} \mathcal{L}_{\Gamma_4} + \frac{7}{12} \mathcal{L}_{\Gamma_5}\right) (\alpha \wedge \alpha \wedge \alpha \wedge \alpha) = 0$$

When $\alpha$ is a quadratic Poisson bivector, only $\mathcal{L}_{\Gamma_4}$ survives because other graphs have a vertex with 3 incoming edges. Then in the quadratic case the first obstruction reads

$$\mathcal{L}_{\Gamma_4}(\alpha \wedge \alpha \wedge \alpha \wedge \alpha) = 0 \quad (29)$$
3 A proof of the Main Theorem 2.3

3.1 The new propagator

Recall that the main configuration space in [K97] is the following space $C_{n,m}$. First of all, $\text{Conf}_{n,m}$ is the configuration space of pairwise distinct points $n$ of which belong to the upper half-plane, and the remaining $m$ are placed to the real line, which is thought as the boundary of the upper-half plane. Then

$$C_{n,m} = \text{Conf}_{n,m}/G$$

where $G$ is the two-dimensional group of symmetries of the upper half-plane of the form \{ $z \mapsto az + b$, $a \in \mathbb{R}_+, b \in \mathbb{R}$ \}. Here a point $z$ of the upper half-plane is considered as a complex number with positive imaginary part.

Kontsevich introduces in [K97] some compactification $\overline{C}_{n,m}$ of these spaces and constructs a top degree differential forms on them, depending on a graph $\Gamma$. We refer the reader to [K97], Section 5, for a detailed description of the compactification.

The top degree differential form on $\overline{C}_{n,m}$ is constructed as follows. Firstly one constructs some closed 1-form $\phi$ on the space $\overline{C}_{2,0}$ ("the propagator"). Now suppose that exactly $2n + m - 2$ oriented edges of $\Gamma$ connect our $n + m$ points. For each edge $e$ of $\Gamma$ we have the forgetful map $t_e: \overline{C}_{n,m} \to \overline{C}_{2,0}$. Now if the 1-form $\phi$ is fixed, we define the
top degree form on $C_{n,m}$ associated with an admissible graph $\Gamma$ as

$$\phi_\Gamma = \bigwedge_{e \in E(\Gamma)} t^*_e(\phi)$$

(we should have some ordering of the edges to define this wedge-product of 1-forms; this ordering is a part of the data of an admissible graph).

We see that all the game depends only on the propagator 1-form $\phi$. Our construction differs from the one from [K97] only by this 1-form $\phi$. By an admissible graph in this Section we mean almost the Kontsevich’s admissible graph from [K97], Section 6.1, without oriented cycles. Let us give a precise definition.

An admissible graph in this Section is an oriented graph $\Gamma$ with two types of vertices, called the vertices of the first and of the second type, such that:

1) there are no oriented cycles in $\Gamma$;

2) the vertices of the first type are labeled by $\{1, 2, \ldots, n\}$, and the vertices of the second type are labeled by $\{1, 2, \ldots, m\}$, $2n + m \geq 2$, we denote for a vertex $v$ by $n(v)$ its labeling; this labeling should obey the following property:

For two vertices of the first type $n(v_1) < n(v_2)$ if there is an edge $\overrightarrow{v_1v_2}$ (32)

3) each edge starts at a vertex of the first type.

As in Section 2, we do not include an ordering of the sets Star($v$) of outgoing edges from the vertices $v$ of $\Gamma$ in the data of an admissible graph. The reason is that as soon as the vertices are labeled, there are the canonical orderings by the label of the end-point; here we count first the end-points which are of the first type, then the end-points of the second type.

Recall firstly what the space $C_{2,0}$ is. It looks like an eye (see the left picture in Figure 5). The two boundary lines comes when one of the two points $z_1$ or $z_2$ approaches the real line, which is the boundary of the upper half-plane. The circle comes when the two points $z_1$ and $z_2$ approach each other and are far from the real line. The role of the two points $z_1$ and $z_2$ here is completely symmetric. Now we are going to break this symmetry.

Divide the space $\overline{C_{2,0}}$ into two parts, as follows. Let the point $z_1$ be fixed. Draw the half-circle orthogonal to the real line (a geodesics in the Poincaré model of hyperbolic geometry) with $z_1$ as the top point, see Figure 6. This half-circle is a geodesic, and the group $G$ in (30) is the group of symmetries in hyperbolic geometry. This proves that this half-circle transforms to the analogous half-circle under any $g \in G$. Therefore, the image of this half-circle is well-defined on the eye $\overline{C_{2,0}}$.

We show this image schematically in Figure 5 as the border between the light and the dark parts. Now we want to vanish the 1-form $\phi$ when the oriented pair $(z_1, z_2)$ is in the light area in Figure 6 ($z_2$ is outside the half-circle). We contract all light area in
the left picture in Figure 5 to a point, and we get the right picture in Figure 5. Here the both vertices from the left picture, and the upper boundary are contracted to a one point, as well as the upper half of the circle. The entire external boundary in the right picture is formed from the low boundary component on the left picture.

Here we give the following definition:

**Definition.** A modified angle function is any map from the modified (contracted) $C_{2,0}$ to the circle unit $S^1$ such that the internal circle (in the right picture in Figure 5) maps isomorphically to $S^1$ as the Euclidean angle, and the external boundary component maps also to $S^1$ (in the homotopically unique way, with the period $\pi$).

Here the angle in the internal circle is the Euclidian angle when $z_2$ approaches $z_1$. The period of this angle inside the half-circle is $\pi$, not $2\pi$.

The Kontsevich’s harmonic angle is an example of such a map, see (36) below.

Let us recall that the angle function in [K97], Section 6.2, is a map of the eye from the left picture in Figure 5 such that the inner circle maps isomorphically as angle, the upper boundary is contracted to a point, and the lower boundary is mapped with period $2\pi$.

Let us emphasize a difference between the two definitions: in our case, any angle function $\theta$ is a function in a proper sense, while in the Kontsevich’s definition it is a multi-valued function defined only up to $2\pi$. Therefore, in our case the de Rham derivative $\phi = d\theta$ is an exact 1-form, while in the Kontsevich’s case it is only closed. But a more serious breakdown is that in our case the propagator $\phi$ is not a smooth function on the manifold with corners $C_{2,0}$. This will be the main source of problems in the next Subsection, where we prove the $L_{\infty}$ property using the Stokes’ theorem.

We define the weight of an admissible graph $\Gamma$ as

$$W_{\Gamma} = \frac{1}{\pi^{2n+m-2}} \int_{C_{n,m}} \bigwedge_{e \in E(\Gamma)} f_e^* \phi$$

(33)

where the forgetful map $f_e$ associated with an edge $e \in E(\Gamma)$ is defined just above (31).

This definition has a small defect caused by non-smoothness of the 1-form $\phi$ on $C_{2,0}$. We advice to the reader skip it for the moment till Section 2.2.2, where we give a rigorous definition of the weights in (37).

This definition completely coincides with [K97], Section 6.2, but here we use the different definition of the angle function.

Let us note that, although our 1-form $\phi$ is exact, the integrals $W_{\Gamma}$ are in general nonzero, because the spaces $C_{n,m}$ are manifolds (with corners) with boundary.

### 3.2 The proof

**3.2.1 Formulation of the result**

Let $V$ be a $\mathbb{Z}_{\geq 0}$-graded vector space over $\mathbb{C}$ with finite-dimensional components $V_i$. 


Figure 5: The Kontsevich’s configuration space $\mathcal{C}_{2,0}$ (left) and our modified angle function (right)

Figure 6: The height ordering of two points, $z_2 \leq z_1$

See Section 3.1 for our definition of an admissible graph in this Section. Let $G_{n,m,2n+m-2}$ denotes the set of all connected admissible graphs with $n$ vertices of the first type, $m$ vertices of the second type, and $2n + m - 2$ edges. Let $U_\Gamma(\gamma_1 \wedge \cdots \wedge \gamma_n)$, $\Gamma \in G_{n,m}$, be the polydifferential operator associated with the graph $\Gamma$ and polyvector fields $\gamma_1, \ldots, \gamma_n$, see [K97], Section 6.3. Define

$$F_n(\gamma_1 \wedge \cdots \wedge \gamma_n) = \sum_{\Gamma \in G_{n,m}} W_\Gamma \times U_\Gamma(\gamma_1 \wedge \cdots \wedge \gamma_n)$$

(34)

where the weight $W_\Gamma$ is defined in (33) via the modified angle function. We have the following two cases: either $\Gamma$ contains an oriented cycle (and in this case clearly our $W_\Gamma = 0$), or it does not contain (and in this case $U_\Gamma$ is well-defined by Lemma 1.4). Therefore, the cochain $F_n(\gamma_1 \wedge \cdots \wedge \gamma_n)$ is well-defined.

Our main result is the following

**Theorem.** Let $V$ be a $\mathbb{Z}_{\geq 0}$-graded vector space over $\mathbb{C}$ with finite-dimensional graded
components $V_i$. Then the maps $F_n$ are well-defined and are the Taylor components of some $L_\infty$ quasi-isomorphism $F: T^*_\text{fin}(V) \to \text{Hoch}_\text{fin}(S(V^*))$ where the l.h.s. is the $L_\infty$ algebra introduced in Section 2. Its first Taylor component $F_1$ is the Hochschild-Kostant-Rosenberg map \((I)\).

This Theorem is clearly a specification of the Main Theorem 2.3, we prove here this Theorem.

### 3.2.2 Configuration spaces

The main problem in proving Theorem 3.2.1 is that the de Rham derivative of the modified angle function is \textit{not} a smooth differential form on the compactified space $C_{2,0}$. As a corollary, we cannot apply the Stokes’ formula in the same way as in [K97]. There are two ways to overcome this problem. The first one is to subdivide the configuration spaces $C_{n,m}$ such that the wedge-product of the angle 1-forms would be a smooth differential form on the subdivision. In particular, it is clear how to subdivide $C_{2,0}$: the subdivision is shown in Figure 7. This way is, however, not acceptable, because the subdivided spaces are not manifold with corners anymore, and we can not apply the Stokes’ formula as well.

![Figure 7: A possible subdivision $C_{2,0}$ (which is not a manifold with corners)](image)

The second way is to define different configuration spaces, cutting off the area when $z_1 \leq z_2$ (see Figure 6) if one have an edge from $z_1$ to $z_2$. The only problem with this solution is that the configuration space will depend on the graph $\Gamma$, for each $\Gamma$ we have its own configuration space. This solution works, as we will see in the rest of this Section.

Let $\Gamma$ be an admissible graph with $n$ vertices of the first type and $m$ vertices of the second type. Recall that this means, in particular, that all vertices of the first type are ordered and labeled as $\{1,2,\ldots,n\}$, and all vertices of the second type are ordered and
labeled as \( \{ \overline{1}, \overline{2}, \ldots, \overline{m} \} \). Define the configuration space \( C_{n,m,\Gamma} \) as follows:

\[
C_{n,m,\Gamma} = \{ z_1, \ldots, z_n \in \mathcal{H}; t_1, \ldots, t_n \in \mathbb{R} \}
\]

\[
z_i \neq z_j \text{ for } i \neq j, \quad t_1 < \cdots < t_m, \quad z_j \leq z_i \text{ if } (i, j) \text{ is an edge in } \Gamma
\]

and \( t_j \leq z_i \) if \( (i, j) \) is an edge in \( \Gamma \)

Here \( \mathcal{H} = \{ z \in \mathbb{C}, \text{Im} z > 0 \} \) is the upper half-plane, and the relations \( z_j \leq z_i \) and \( t_j \leq z_i \) in the r.h.s. are understood in the sense of the ordering with half-circle, shown in Figure 6. The group \( G \) in the r.h.s. consists from the transformations \( G = \{ z \mapsto az + b, a \in \mathbb{R}_{>0}, b \in \mathbb{R} \} \).

Let us note that the graph \( \Gamma \) in the definition of \( C_{n,m,\Gamma} \) may have an arbitrary (not necessarily \( 2n + m - 2 \)) number of edges. If \( \Gamma \) has no edges at all, we recognize the Kontsevich’s original space \( C_{n,m} \).

It is easy to construct a Kontsevich-type compactification \( \overline{C}_{n,m,\Gamma} \) which is a manifold with corners, with projections \( p_{\Gamma,\Gamma'}: \overline{C}_{n,m,\Gamma} \to \overline{C}_{n',m',\Gamma'} \) defined when \( \Gamma' \) is a subgraph of \( \Gamma \). The space \( \overline{C}_{2,0,\Gamma_0} \), where \( \Gamma_0 \) is just one oriented edge connecting point 1 with point 2, is shown in Figure 8.

![Figure 8: The "Eye" \( \overline{C}_{2,0,\Gamma_0} \)](image)

Here the lower component of the boundary comes when the point \( z_2 \) (see Figure 6) approaches the real line, the left and the right upper boundary component comes when the point \( z_2 \) approaches the left and the right parts of the geodesic half-circle in Figure 6, and the third upper boundary component (the half-circle) comes when \( z_2 \) approaches \( z_1 \) inside the geodesic half-circle.

We repeat the definition of the modified angle function from this new point of view:

**Definition.** A modified angle function is a map \( \theta \) of \( \overline{C}_{2,0,\Gamma_0} \) to a unit circle \( S^1 \) such that \( \theta \) is the Euclidean angle varying from \( -\pi \) to 0 on the upper half-circle, and \( \theta \) contracts the two other upper boundary components to a point \( 0 \in S^1 \).

**An example** of the modified angle function is the doubled Kontsevich’s harmonic angle:
\[ f(z_1, z_2) = \frac{1}{i} \log \frac{(z_1 - z_2)(z_1 - \overline{z}_2)}{(z_1 - \overline{z}_2)(\overline{z}_1 - \overline{z}_2)} \] where \( z_1 \geq z_2 \) in the sense of Figure 6 \hspace{1cm} (36)

It is clear that the function \( f(z_1, z_2) \) defined in this way is well-defined when \( z_1 \geq z_2 \) and is equal to 0 on the "border" circle, see Figure 6.

Now we define for an edge \( e \) of an admissible graph \( \Gamma \) the 1-form \( \phi_e \) on \( \overline{C}_{n,m,\Gamma} \) as \( p^*_{\Gamma,\Gamma_0}(d\theta) \), where \( \Gamma_0 \) is the graph with two vertices and one edge \( e \). Finally, we give a rigorous definition of the weight:

\[ W_{\Gamma} = \frac{1}{\pi^{2n+m-2}} \int_{\overline{C}_{n,m,\Gamma}} \bigwedge_{e \in E(\Gamma)} \phi_e \] \hspace{1cm} (37)

Let us describe the boundary strata of codimension 1 of \( \overline{C}_{n,m,\Gamma} \). We will need also the space \( C_{n,\Gamma} \) defined in Section 2.

Now is the list of the boundary strata of codimension 1 in \( \overline{C}_{n,m,\Gamma} \):

S1) some points \( p_1, \ldots, p_S \in \mathcal{H}, S \geq 2, \) approach each other and remain far from the geodesic half-circles of all points \( q \neq p_1, \ldots, p_S \) connected by an edge with any of the approaching \( S \) points. In this case we get the boundary stratum of codimension 1 isomorphic to \( C_{n-S+1,m,\Gamma_1} \times C_{S,\Gamma_2} \) where \( \Gamma_2 \) is the subgraph of \( \Gamma \) of the edges connecting the point \( p-1, \ldots, p_S \) with each other, and \( \Gamma_1 \) is obtained from \( \Gamma \) by collapsing the graph \( \Gamma_2 \) into a new vertex;

S2) some points \( p_1, \ldots, p_S \in \mathcal{H} \) and some points \( q_1, \ldots, q_R \in \mathbb{R}, 2S + R \geq 2, 2S + R \leq 2m + n - 1, \) approach each other and a point of the real line, but are far from the geodesic half-circle of any other point connected with these points by an edge. In this case we get a boundary stratum of codimension 1 isomorphic to \( C_{S,R,\Gamma_1} \times C_{n-S,m-R+1,\Gamma_2} \) where \( \Gamma_1 \) is the graph formed from the edges between the approaching points, and \( \Gamma_2 \) is obtained from \( \Gamma \) by collapsing the subgraph \( \Gamma_1 \) into a new vertex of the second type;

S3) some point \( p \) is placed on the geodesic half-circle of exactly one point \( q \neq p \) which is far from \( p \), such there is an edge \( (q, p) \). These boundary strata will be irrelevant for us because they do not contribute to the integrals we consider.

Now we pass to a proof of Theorem 3.2.1.
3.2.3 Application of the Stokes’ formula and the boundary strata

We need to prove the \( L_\infty \) morphism quadratic relations on the maps \( F_n \). Recall, that for each \( k \geq 1 \) and for any polyvector fields \( \gamma_1, \ldots, \gamma_k \in T_{fin}(V) \) it is the relation:

\[
d_{Hoch}(F_k(\gamma_1 \wedge \cdots \wedge \gamma_k)) + \sum_{2 \leq N \leq k} \frac{1}{k!(N-k)!} \sum_{\sigma \in S_k} \pm F_{k-N}(L_N(\gamma_{\sigma(1)} \wedge \cdots \wedge \gamma_{\sigma(N)})) \wedge \gamma_{\sigma(k-N+1)} \wedge \cdots \wedge \gamma_{\sigma(k)}) + \frac{1}{2} \sum_{a,b \geq 1, a+b=k} \frac{1}{a!b!} \sum_{\sigma \in S_k} \pm [F_a(\gamma_{\sigma(1)} \wedge \cdots \wedge \gamma_{\sigma(a)}), F_b(\gamma_{\sigma(a+1)} \wedge \cdots \wedge \gamma_{\sigma(k)})] = 0
\]

The l.h.s. of (38) is a sum over the admissible graphs \( \Gamma' \in G_{n,m,2n+m-3} \) with \( n \) vertices of the first type, \( m \) vertices of the second type, and \( 2n + m - 3 \) edges (that is, for 1 edge less than in the graphs contributing to \( F_n \)). This sum is of the form:

\[
\text{l.h.s} = \sum_{\Gamma' \in G_{n,m,2n+m-3}} \alpha_{\Gamma'} \mathcal{U}_{\Gamma'}
\]

where \( \alpha_{\Gamma'} \) are some complex numbers, and \( \mathcal{U}_{\Gamma'} \) are the Kontsevich’s polydifferential operators from [K97]. It is clear that all \( \mathcal{U}_{\Gamma'} \) do not contain any oriented cycle.

Our goal is to prove that all numbers \( \alpha_{\Gamma'} = 0 \).

Each \( \alpha_{\Gamma'} \) is a quadratic-linear combination of our weights \( W_{\Gamma} \) for \( \Gamma \in G_{n,m,2n+m-2} \). There is a way how one can get some quadratic-linear combinations of \( W_{\Gamma} \) which are equal to 0. This way is the following:

Consider some \( \Gamma' \in G_{n,m,2n+m-3} \). We associated with it a differential form \( \bigwedge_{e \in E(\Gamma')} \phi_e \) on \( C_{n,m,\Gamma'} \), as before. Now consider

\[
\int_{C_{n,m,\Gamma'}} d\left( \bigwedge_{e \in E(\Gamma')} \phi_e \right)
\]

This expression is 0 because the form \( \bigwedge_{e \in E(\Gamma')} \phi_e \) is closed (moreover, it is exact). Next, by the Stokes’ theorem, we have

\[
0 = \int_{C_{n,m,\Gamma'}} d\left( \bigwedge_{e \in E(\Gamma')} \phi_e \right) = \int_{\partial C_{n,m,\Gamma'}} \left( \bigwedge_{e \in E(\Gamma')} \phi_e \right)
\]

Now only strata of codimension 1 in \( \partial C_{n,m,\Gamma'} \) do contribute to the r.h.s. They are given by the list (S1)-S3) in Section 3.2.2.

The strata S3) clearly do not contribute because of our boundary conditions on the modified angle function.

For strata S1) and S2) we have the following factorization lemma: the integral over the product is equal to the product of integrals.

This allows us to prove the following key-lemma:
**Key-Lemma.** For any graph $\Gamma \in G_{n,m,2n+m-3}$, the coefficient $\alpha_{\Gamma}$ in (39) is equal to $\int_{\partial C_{n,m,\Gamma}} (\wedge_{e \in E(\Gamma')} \phi_e)$ (which is zero by the Stokes’ formula).

Theorem 3.2.1 clearly follows from this Lemma.

To prove the Key-Lemma, we express

$$\int_{\partial C_{n,m,\Gamma}} (\wedge_{e \in E(\Gamma')} \phi_e) = \int_{\partial S_1 C_{n,m,\Gamma'}} (\wedge_{e \in E(\Gamma')} \phi_e) + \int_{\partial S_2 C_{n,m,\Gamma'}} (\wedge_{e \in E(\Gamma')} \phi_e) + \int_{\partial S_3 C_{n,m,\Gamma'}} (\wedge_{e \in E(\Gamma')} \phi_e)$$

the integral over the boundary $\partial C_{n,m,\Gamma'}$ as the sum of the integrals over the three types of boundary strata $S_1$-$S_3$ of codimension 1.

As we already mentioned, the integral $\int_{\partial S_3 C_{n,m,\Gamma'}} (\wedge_{e \in E(\Gamma')} \phi_e)$ is 0 by the boundary conditions for the propagator.

The summand $\int_{\partial S_2 C_{n,m,\Gamma'}} (\wedge_{e \in E(\Gamma')} \phi_e)$ corresponds exactly to the first and to the third summands of the l.h.s. of (38) containing the Hochschild differential and the Gerstenhaber bracket, by the factorization lemma.

It remains to associate the boundary strata $S_1$ with the second summand of (38), containing operations $L_N$. It is clear that the part of the second summand of (38) for a fixed $N$ is in 1-1 correspondence with that summands in $\int_{\partial S_1 C_{n,m,\Gamma'}} (\wedge_{e \in E(\Gamma')} \phi_e)$ where $S = N$ points in the upper half-plane approach each other.

**Remark.** In [K97] where this kind of computation is originated analogous strata of type $S_1$ give vanishing integrals for $N > 2$. This is why Kontsevich gets the formality theorem there. This vanishing is proven by a rather non-trivial computation. As we have seen in Section 2.1, Examples 1 and 2, the analogous vanishing fails for our construction.

Theorem 3.2.1, and the Main Theorem 2.3, are proven \( \blacksquare \)

**Acknowledgements**

I am very grateful to Borya Feigin for sharing with me around ’98-’99 some his conjecture which stimulated my work on the infinite-dimensional formality in general. Maxim Kontsevich found a mistake in Lemma 2.2.4 of the previous version which failed Theorem 2.2.4 of that version. This problem is fixed here by introduction of a new $L_\infty$ structure on polyvector fields in Section 2. I am very grateful to Maxim for his interest in my work and for fruitful correspondence. I am thankful to Pasha Etingof for his interest and suggestions.

The author is grateful to the research grant R1F105L15 of the University of Luxembourg for partial financial support.
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