CONVERGENCE OF THE SPECTRAL RADIUS OF A RANDOM MATRIX
THROUGH ITS CHARACTERISTIC POLYNOMIAL

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ABSTRACT. Consider a square random matrix with independent and identically distributed entries of mean
zero and unit variance. We show that as the dimension tends to infinity, the spectral radius is equivalent
to the square root of the dimension in probability. This result can also be seen as the convergence of the
support in the circular law theorem under optimal moment conditions. In the proof we establish the con-
vergence in law of the reciprocal characteristic polynomial to a random analytic function outside the unit
disc, related to a hyperbolic Gaussian analytic function. The proof is short and differs from the usual ap-
proaches for the spectral radius. It relies on a tightness argument and a joint central limit phenomenon for
traces of fixed powers.

1. INTRODUCTION AND MAIN RESULTS

Let \( \{a_{jk}\}_{j,k \geq 1} \) be independent and identically distributed complex random variables with mean
zero and unit variance, namely \( E[a_{11}] = 0 \) and \( E[|a_{11}|^2] = 1 \). For all \( n \geq 1 \), let
\[
A_n = (a_{jk})_{1 \leq j,k \leq n}.
\] (1.1)

We call it a Girko matrix \([13]\). When \( a_{11} \) is Gaussian with independent and identically distributed real
and imaginary parts then \( A_n \) has density proportional to \( e^{-\text{Tr}(A_n^2)} \) and belongs to the complex Ginibre
ensemble \([11]\). We are interested in the matrix \( \frac{1}{\sqrt{n}} A_n \) for which each row and each column has a unit
mean squared Euclidean norm. Its characteristic polynomial at point \( z \in \mathbb{C} \) is
\[
p_n(z) = \det(z - \frac{A_n}{\sqrt{n}})
\] (1.2)

where \( z \) stands for \( z \) times the identity matrix. The \( n \) roots of \( p_n \) in \( \mathbb{C} \) are the eigenvalues of \( \frac{1}{\sqrt{n}} A_n \). They
form a multiset \( \Lambda_n \) which is the spectrum of \( \frac{1}{\sqrt{n}} A_n \). The spectral radius of \( \frac{1}{\sqrt{n}} A_n \) is defined by
\[
\rho_n = \max_{\lambda \in \Lambda_n} |\lambda|.
\] (1.3)

The circular law theorem states that the empirical measure of the elements of \( \Lambda_n \) tends weakly as \( n \to \infty \)
to the uniform distribution on the closed unit disc: almost surely, for every nice Borel set \( B \subset \mathbb{C} \),
\[
\lim_{n \to \infty} \frac{\text{card}(B \cap \Lambda_n)}{n} = \frac{\text{area}(B \cap \overline{D})}{\pi},
\] (1.4)

where “area” stands for the Lebesgue measure on \( \mathbb{C} \), and where \( \overline{D} = \{z \in \mathbb{C} : |z| \leq 1\} \) is the closed unit
disc, see \([12, 13, 21, 6]\). The circular law \([14]\), which involves weak convergence, does not provide the
convergence of the spectral radius, it gives only that almost surely
\[
\lim_{n \to \infty} \rho_n \geq 1.
\] (1.5)

Theorem 1.1 provides the convergence of the spectral radius, without extra assumptions on the entries.
This result was conjectured in \([5]\), and improves over \([10, 16, 1, 5, 3]\). The moments assumptions are
optimal, and the \( \frac{1}{\sqrt{n}} \) scaling is no longer adequate for entries of infinite variance, see for instance \([4]\).

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\text{Date: Autumn 2020, revised Summer 2021. Preprint, compiled July 8, 2021.}
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2010 Mathematics Subject Classification. Primary: 30C15, 60B20; Secondary: 60F05.

Key words and phrases. Random Matrix; Spectral Radius; Gaussian Analytic Function; Central Limit Theorem; Combinatorics;
Digraph; Circular Law.

CB and DGZ are supported by the grant ANR-16-CE40-0024.
We have $\rho_n \leq \sigma_n$ where $\sigma_n$ is the operator norm of $\frac{1}{\sqrt{n}} A_n$, its largest singular value. It is known that the condition $E[|a_{11}|^4] < \infty$ is necessary and sufficient for the convergence of $\sigma_n$ as $n \to \infty$, see \cite{2}. A striking aspect of the spectral radius is that it converges without any extra moment condition.

**Theorem 1.1 (Spectral radius).** We have $\lim_{n \to \infty} \rho_n = 1$ in probability, in the sense that for all $\varepsilon > 0$,

$$
\lim_{n \to \infty} P(|\rho_n - 1| \geq \varepsilon) = 0.
$$

The proof of Theorem 1.1 is given in Section 2. It relies on Theorem 1.2 below, which is of independent interest. It does not involve any Hermitization or norms of powers in the spirit of Gelfand’s spectral radius formula. The idea is to show that on $\mathbb{C} \cup \{\infty\} \setminus \overline{\mathbb{D}}$, the polynomial $z^{-n} p_n(z)$ tends as $n \to \infty$ to a random analytic function which does not vanish. The first step for mathematical convenience is to convert $\mathbb{C} \cup \{\infty\} \setminus \overline{\mathbb{D}}$ into $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ by noting that $p_n(z) = z^n q_n(1/z)$, $z \not\in \overline{\mathbb{D}}$, where for all $z \in \mathbb{D}$,

$$
q_n(z) = \det \left(1 - z A_n \sqrt{n}\right)
$$

is the reciprocal polynomial of the characteristic polynomial $p_n$. Let $H(\mathbb{D})$ be the set of holomorphic or complex analytic functions on $\mathbb{D}$, equipped with the topology of uniform convergence on compact subsets, the compact-open topology, see for instance \cite{20}. This allows to see $q_n$ as a random variable on $H(\mathbb{D})$ and gives a meaning to convergence in law of $q_n$ as $n \to \infty$, namely, $q_n$ converges in law to some random element $q$ of $H(\mathbb{D})$ if for every bounded real continuous function $f$ on $H(\mathbb{D})$, $E[f(q_n)] \to E[f(q)]$.

**Theorem 1.2 (Convergence of reciprocal characteristic polynomial).** We have

$$
q_n \xrightarrow{\text{law}} \kappa e^{-F},
$$

where $F$ is the random holomorphic function on $\mathbb{D}$ defined by

$$
F(z) = \sum_{k=1}^{\infty} X_k z^k,\sqrt{k
$$

where $(X_k)_{k \geq 1}$ is a sequence of independent complex Gaussian random variables such that

$$
E[X_k] = 0, \quad E[|X_k|^2] = 1 \quad \text{and} \quad E[X_k^2] = E[|a_{11}|^2],
$$

and where $\kappa : \mathbb{D} \to \mathbb{C}$ is the holomorphic function defined for all $z \in \mathbb{D}$ by

$$
\kappa(z) = \sqrt{1 - z^2 E[|a_{11}|^2]}.
$$

The square root defining $\kappa$ is the one such that $\kappa(0) = 1$. Notice that it is a well-defined holomorphic function on the simply connected domain $\mathbb{D}$ since the function $z \mapsto 1 - z^2 E[|a_{11}|^2]$ does not vanish on $\mathbb{D}$ which is true due to the fact that $E[|a_{11}|^2] \leq E[|a_{11}|^2] = 1$.

The proof of Theorem 1.2 is given in Section 3. It is partially inspired by \cite{3} and relies crucially on a joint combinatorial central limit theorem for traces of fixed powers (Lemma 3.4) inspired from \cite{17}. Unlike previous arguments used in the literature for the analysis of Girko matrices, the approach does not rely on Girko Hermitization, Gelfand spectral radius formula, high order traces, resolvent method or Cauchy–Stieltjes transform. The first step consists in showing the tightness of $(q_n)_{n \geq 1}$, by using a decomposition of the determinant into orthogonal elements related to determinants of submatrices, as in \cite{3}. Knowing this tightness, the problem is reduced to show the convergence in law of these elements. A reduction step, inspired by \cite{17}, consists in truncating the entries, reducing the analysis to the case of bounded entries. The next step consists in a central limit theorem for product of traces of powers of fixed order. It is important to note that we truncate with a fixed threshold with respect to $n$, and the order of the powers in the traces are fixed with respect to $n$. This is in sharp contrast with the usual Füredi–Komlós truncation-trace approach related to the Gelfand spectral radius formula used in \cite{10,16,15}.

1.1. Comments and open problems.
1.1.1. Moment assumptions. The universality for the first order global asymptotics (1.4) depends only on the trace $E[|a_{11}|^2]$ of the covariance matrix of $R a_{11}$ and $\Im a_{11}$. The universality stated by Theorem 1.2 just like for the central limit theorem, depends on the whole covariance matrix. Since

$$E[|a_{11}|^2] = E[(R a_{11})^2] - E[(\Im a_{11})^2] + 2iE[R a_{11} \Im a_{11}],$$

we can see that $E[|a_{11}|^2] = 0$ if and only if $E[(R a_{11})^2] = E[(\Im a_{11})^2]$ and $E[R a_{11} \Im a_{11}] = 0$. Moreover, we cannot in general get rid of $E[|a_{11}|^2]$ by simply multiplying the matrix $A_n$ by a phase.

1.1.2. Hyperbolic Gaussian analytic function. When $E[|a_{11}|^2] = 0$ then $\kappa = 1$ while the random analytic function $F$ which appears in the limit in Theorem 3 is a degenerate case of the well-known hyperbolic Gaussian Analytic Functions (GAFs) [14, Equation (2.3.5)]. It can also be obtained as the antiderivative of the $L = 2$ hyperbolic GAF which is 0 at $z = 0$. This $L = 2$ hyperbolic GAF is related to the Bergman kernel and could be called the Bergman GAF. These GAFs appear also at various places in mathematics and physics and, in particular, in the asymptotic analysis of Haar unitary matrices, see [15] [18].

1.1.3. Cauchy–Stieltjes transform. If $E[|a_{11}|^2] = 0$ then by returning to $p_n$, taking the logarithm and the derivative with respect to $z$ in Theorem 1.2 we obtain the convergence in law of the Cauchy–Stieltjes transform (complex conjugate of the electric field) minus $n/z$ towards $z \mapsto F'(1/z)/z^2$ which is a Gaussian analytic function on $\mathbb{C} \setminus \mathbb{D}$ with covariance given by a Bergman kernel.

1.1.4. Central Limit Theorem. We should see Theorem 1.2 as a global second order analysis, just like the central limit theorem (CLT) for linear spectral statistics [19] [8]. Namely for all $z \in \mathbb{D}$, we have

$$|d_n(1/z)| = \exp[-n (U_n(z) - U(z))]|,$$

where $U_n(z) = -\frac{1}{2} \log|p_n(z)|$ is the logarithmic potential at the point $z$ of the empirical spectral distribution of $\frac{1}{\sqrt{n}} A_n$ and $U(z) = -\log|z|$ is the logarithmic potential at the point $z$ of the uniform distribution on the unit disc $\mathbb{D}$.

Moreover, it is possible to extract from Theorem 1.2 a CLT for linear spectral statistics with respect to analytic functions in a neighborhood of $\mathbb{D}$. This can be done by using the Cauchy formula for an analytic function $f$,

$$\int f(\lambda) \mu(d\lambda) = \frac{1}{2\pi i} \int \left( \frac{f(z)}{z - \lambda} dz \right) \mu(d\lambda) = \frac{1}{2\pi i} \int f(z) \left( \frac{\mu(d\lambda)}{z - \lambda} \right) dz = \frac{1}{2\pi i} \int f(z) (\log \det (z - A))^t dz,$$

where $\mu$ is the counting measure of the eigenvalues of $A$, where the contour integral is around a centered circle of radius strictly larger than 1, and where we have taken any branch of the logarithm. The approach is purely complex analytic. In particular, it is different from the usual approach with the logarithmic potential of $\mu$ based on the real function given by $z \mapsto \int \log|z - \lambda| \mu(d\lambda) = \log |\det (z - A)|$.

1.1.5. Wigner case and elliptic interpolation. The finite second moment assumption of Theorem 1.1 is optimal. We could explore its relation with the finite fourth moment assumption for the convergence of the spectral edge of Wigner random matrices, which is also optimal. Heuristic arguments tell us that the interpolating condition on the matrix entries should be $E[|a_{jk}|^2|a_{kj}|^2] < \infty$ for $j \neq k$, which is a finite second moment condition for Girko matrices and a finite fourth moment condition for Wigner matrices. This is work in progress.

1.1.6. Coupling and almost sure convergence. For simplicity, we define in (1.1) our random matrix $A_n$ for all $n \geq 1$ by truncating from the upper left corner the infinite random matrix $\{a_{jk}\}_{j,k=1}$. This imposes a coupling for the matrices $\{A_n\}_{n \geq 1}$. However, since Theorem 1.1 involves a convergence in probability, it remains valid for an arbitrary coupling, in the spirit of the triangular arrays assumptions used for classical central limit theorems. In another direction, one could ask about the upgrade of the convergence in probability into almost sure convergence in Theorem 1.1. This is an open problem.

1.1.7. Heavy tails. An analogue of (1.4) in the heavy-tailed case $E[|a_{11}|^2] = \infty$ is considered in [4] but requires another scaling than $\frac{1}{\sqrt{n}}$. The spectral radius of this model tends to infinity as $n \to \infty$ but it could be possible to analyze the limiting point process at the edge as $n \to \infty$ and its universality. This is an open problem.
2. Proof of Theorem 1.1

Let $f = ke^{-F}$ be as in Theorem 1.2. We observe that the equation $f(z) = 0$, $z \in \mathbb{D}$ is equivalent to $k(z) = 0$, $z \in \mathbb{D}$, which has no solution, because $|E|a_1^2| \leq E|a_1|^2 = 1$. In particular, for every $r \in (0, 1)$,

$$\inf_{z \in \bar{D}_r} |f(z)| = \inf_{z \in \bar{D}_r} \left\{|k(z)|e^{-R(F(z))}\right\} > 0,$$

where $\bar{D}_r = \{z \in \mathbb{C} : |z| \leq r\}$ is the closed disc of radius $r$. On the other hand, the convergence in law provided by Theorem 1.2, together with the continuous mapping theorem used for the continuous function $f \in H(\mathbb{D}) \mapsto \inf_{z \in \bar{D}_r} |f(z)|$, gives, for every $r \in (0, 1)$,

$$\inf_{z \in \bar{D}_r} |q_n(z)| \xrightarrow{\text{law}} \inf_{z \in \bar{D}_r} \left\{|k(z)|e^{-R(F(z))}\right\}.$$

Now, since $q_n(z) = z^n p_n(1/z)$ for every $z \in \mathbb{D}$, we obtain, by combining these two facts, for every $r \in (0, 1)$,

$$P \left( \rho_n < \frac{1}{r} \right) = P \left( \inf_{|z| \leq \frac{1}{2}} |p_n(z)| > 0 \right) = P \left( \inf_{z \in \bar{D}_r} |q_n(z)| > 0 \right) \xrightarrow{n \to \infty} P \left( \inf_{z \in \bar{D}_r} \left\{|k(z)|e^{-R(F(z))}\right\} > 0 \right) = 1.$$

In other words, for all $\varepsilon > 0$,

$$\lim_{n \to \infty} P(\rho_n \geq 1 + \varepsilon) = 0.$$

Combined with (1.5), this leads to the desired result

$$\lim_{n \to \infty} P(|\rho_n - 1| > \varepsilon) = 0.$$

Note that it could be possible to obtain the result by using the Radon measures of the zeros and the Hurwitz phenomenon, see [20, Lemma 2.2] and [7, Lemma 5.2], but this would be more complicated!

3. Proof of Theorem 1.2

By developing the determinant we can see that

$$q_n(z) = \det \left( 1 - \frac{A_n}{\sqrt{n}}\right) = 1 + \sum_{k=1}^n (-z)^k p_k^n,$$

where

$$p_k^n = \sum_{I=\{1, \ldots, n\}} \sum_{|I|=k} n^{-k/2} \det(A_n(I))$$

and $A_n(I) = \{a_{ij}\}_{j,k \in I}$.

The following lemma is essentially contained in [3, Appendix A]. It is proved in Section 4.1.

**Lemma 3.1** (Tightness). The sequence $\{q_n\}_{n \geq 1}$ is tight.

For completeness, let us recall that the sequence $\{q_n\}_{n \geq 1}$ is tight if for every $\varepsilon > 0$, there exists a compact subset of $H(\mathbb{D})$ such that $P(q_n \not\in K) > 1 - \varepsilon$ for every $n$.

Now that we know that $\{q_n\}_{n \geq 1}$ is tight, it is enough to understand, for each $k \geq 1$, the limit of $(p_1^n, \ldots, p_k^n)$ as $n \to \infty$. Indeed, we have the following Lemma 3.2 close to [20, Second part of Proposition 2.5]. For the reader's convenience and for completeness, we give a proof in Section 4.2.

**Lemma 3.2** (Reduction to convergence of coefficients). Let $\{f_n\}_{n \geq 1}$ be a tight sequence of random elements of $H(\mathbb{D})$, and let us write, for every $n \geq 1$, $f_n(z) = \sum_{k=0}^\infty z^k p_k^n$. If for every $m \geq 0$,

$$(p_0^n, \ldots, p_m^n) \xrightarrow{\text{law}} (p_0, \ldots, p_m),$$

for a common sequence of random variables $(p_m)_{m \geq 0}$ then $f = \sum_{k=0}^\infty z^k p_k$ is well-defined in $H(\mathbb{D})$ and

$$f_n \xrightarrow{\text{law}} f.$$

The first simplification we shall make is to assume that $a_{11}$ is bounded. This is motivated by [17, Proof of Lemma 7]. We write this in the following lemma, proved in Section 4.3.
Lemma 3.3 (Reduction to bounded entries by truncation). For $M > 0$ let us define

$$A_n^{(M)} = \left\{ a_{ij}^{(M)} \right\}_{1 \leq i,j \leq n} \quad \text{where} \quad a_{ij}^{(M)} = a_{ij}1_{|a_{ij}| < M} - E \left[ a_{ij}1_{|a_{ij}| < M} \right]$$

and

$$p_{1,k}^{(n,M)} = \sum_{l \in \{1, \ldots, n\}} n^{-k/2} \det(A_n^{(M)}(I)) \quad \text{where} \quad A_n^{(M)}(I) = \left\{ a_{ij}^{(M)} \right\}_{i,j \in I}.$$

Let $k \geq 1$. If there exists \( \left( Y_1^{(M)}, \ldots, Y_k^{(M)} \right) \) and a random vector \( (Y_1, \ldots, Y_k) \) such that for all \( M \geq 1 \),

$$\left( p_{1,1}^{(n,M)}, \ldots, p_{1,k}^{(n,M)} \right) \xrightarrow{n \to \infty} (Y_1^{(M)}, \ldots, Y_k^{(M)}), \quad \text{and} \quad \left( Y_1^{(M)}, \ldots, Y_k^{(M)} \right) \xrightarrow{M \to \infty} (Y_1, \ldots, Y_k),$$

then

$$\left( p_{1,1}^{(n)}, \ldots, p_{1,k}^{(n)} \right) \xrightarrow{n \to \infty} (Y_1, \ldots, Y_k).$$

To simplify the study of \( p_{1,k}^{(n)} \) we notice the following. For each integer \( n \geq 1 \), the series

$$B_n = - \sum_{k=1}^{\infty} \left( \frac{A_n}{\sqrt{n}} \right)^k \frac{z^k}{k}$$

converges for \( z \) small enough and its exponential is \( \left( 1 - \frac{z^2 A_n^2}{2} \right)^{1/2} \). This can be shown in the standard way if \( A_n \) is diagonalizable and can be extended to non-diagonalizable matrices by continuity. Then, since \( \det(e^{B_n}) = e^{n B_n} \), we obtain

$$q_n(z) = \exp \left( - \sum_{k=1}^{\infty} \frac{\text{Tr}(A_n^k) z^k}{n^{k/2}} \right)$$

for \( z \) small enough. In particular, \( (p_{1,1}^{(n)}, \ldots, p_{1,k}^{(n)}) \) is a polynomial function of \( \left( \frac{\text{Tr}(A_n)}{n^{1/2}}, \ldots, \frac{\text{Tr}(A_n^k)}{n^{k/2}} \right) \) that does not depend on \( n \) and vice versa. The idea is to study, by the method of moments, the quantity

$$\text{Tr}(A_n^k) = \sum_{1 \leq i_1, \ldots, i_k \leq n} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k} a_{i_k i_1}.$$ 

That is why we preferred to have \( a_{11} \) bounded (or at least having all its moments finite). Note that we have used the determinantal terms \( p_{1,k}^{(n)} \) to perform this truncation step, it would have been much more challenging to justify this truncation directly for the traces \( \text{Tr}(A_n^k) \). On the other hand, it would have been much more difficult to prove directly the convergence of the determinantal terms \( p_{1,k}^{(n)} \) thanks to the method of moments since these terms are asymptotically neither independent nor Gaussian.

We decompose the above sum in two sums,

$$\text{Tr}(A_n^k) = \sum_{1 \leq i_1, \ldots, i_k \leq n} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k} a_{i_k i_1} + \sum_{1 \leq i_1, \ldots, i_k \leq n} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k} a_{i_k i_1}. \quad (3.1)$$

The first term in the right-hand side of (3.1) has zero expected value and gives rise to the random part of the limit. The second term in the right-hand side of (3.1) gives the deterministic part.

We begin by looking at the term

$$\sum_{1 \leq i_1, \ldots, i_k \leq n} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k} a_{i_k i_1}. \quad (3.2)$$

Notice that the sum in (3.2) is indexed by sequences \( i = (i_1, \ldots, i_k) \) of pairwise distinct elements of \( \{1, \ldots, n\} \). However, if two sequences are cyclic permutations of each other we obtain the same term. To deal with this fact, we should consider sequences up to cyclic permutations or, what is the same, directed cycles in \( \{1, \ldots, n\} \). More precisely, let us consider \( \{1, \ldots, n\} \) as the complete directed graph with no loops and let us consider the graph \( G = (V, E) \) with vertex and edge sets

$$V = \{1, \ldots, k\} \quad \text{and} \quad E = \{(1,2), (2,3), \ldots, (k-1,k), (k,1)\}.$$
A $k$-directed cycle in $\{1, \ldots, n\}$ is a subgraph $g$ of $\{1, \ldots, n\}$ that is isomorphic to $G$. The sum in (3.2) is better indexed by the set of $k$-directed cycles in $\{1, \ldots, n\}$ that we shall call $\mathcal{C}_k^{(n)}$. For $g \in \mathcal{C}_k^{(n)}$, we define

$$a_g = \prod_{e \text{ edge of } g} a_e,$$

where $a_e = a_{ij}$ if $e = (i, j)$. Now, we can write

$$\sum_{\substack{1 \leq i_1, \ldots, i_k \leq n \\ \text{card}\{i_1, \ldots, i_k\} = k}} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k} a_{i_k i_1} = k \sum_{g \in \mathcal{C}_k^{(n)}} a_g$$

so that the term we have to study is

$$r_k^{(n)} = \sum_{g \in \mathcal{C}_k^{(n)}} a_g.$$

The following lemma is a sort of combinatorial joint central limit theorem. It provides the $e^{-F}$ part of the limiting random analytic function in Theorem 1.2. It is proved in Section 4.4.

**Lemma 3.4** (Convergence to a Gaussian object). For any $k_1, \ldots, k_m \geq 1$ and any sequence $s_1, \ldots, s_m \in \{*, \#\}$,

$$E \left[ \sum_{k=1}^{k_m} \frac{r_k^{(n)}}{n^{k/2}} \right]^{s_m} \xrightarrow{n \to \infty} E \left[ \left( \frac{X_{k_1}}{\sqrt{k_1}} \right)^{s_1} \cdots \left( \frac{X_{k_m}}{\sqrt{k_m}} \right)^{s_m} \right],$$

where we have used the notation $x^* = x$ and $x^\# = \bar{x}$, and where $\{X_k\}_{k \geq 1}$ are independent complex Gaussian random variables such that $E[X_k] = 0$, $E[|X_k|^2] = 1$, and $E[X_k^2] = E[a_{11}^2]$ for all $k \geq 1$.

The term that is left to understand is

$$r_k^{(n)} = \sum_{\substack{1 \leq i_1, \ldots, i_k \leq n \\ \text{card}\{i_1, \ldots, i_k\} < k}} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k} a_{i_k i_1}.$$

The following lemma, proved in Section 4.5, provides the $\kappa$ part of the limit in Theorem 1.2.

**Lemma 3.5** (Deterministic limit part).

$$r_k^{(n)} = \frac{r_k^{(n)}}{n^{k/2}} \xrightarrow{n \to \infty} \begin{cases} E[a_{11}^2]^{k/2} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}$$

To sum up, if $(X_k)_{k \geq 1}$ is a sequence of independent complex Gaussian random variables such that

$$E[|X_k|^2] = 1 \quad \text{and} \quad E[X_k^2] = E[a_{11}^2],$$

and if

$$\text{mean}_k = \begin{cases} E[a_{11}^2]^{k/2} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}$$

then

$$\left( \frac{\text{Tr}(A_n)}{\sqrt{n}}, \frac{\text{Tr}(A_n^2)}{n}, \ldots, \frac{\text{Tr}(A_n^k)}{n^{k/2}} \right) \xrightarrow{n \to \infty} \left( X_1, \sqrt{2}X_2, \ldots, \sqrt{k}X_k \right) + (\text{mean}_1, \text{mean}_2, \ldots, \text{mean}_k).$$

This implies the convergence of $(p_k^{(n)}, \ldots, p_k^{(n)})$ to the corresponding polynomials of $X_i$ and mean$_i$. Moreover, by Lemma 3.2 and since the limit depends continuously on the second moment of the variable, the assertion also holds for non-bounded $a_{11}$. We have found that $(q_{k})_{k \geq 1}$ has a limit that can be written as a Maclaurin series whose coefficients are polynomials of $X_i$ and mean$_i$. By construction, the joint law of these coefficients is the same as the joint law of the coefficients of the random holomorphic function $\kappa \exp(-F)$ so that the proof of the theorem is complete.
4. Proofs of the lemmas used in the proof of Theorem 1.2

4.1. Proof of Lemma 3.1  Recall that if \( \{f_n\}_{n \geq 1} \) is a sequence of random variables on \( \mathbb{H(D)} \) such that for every compact set \( K \subset \mathbb{D} \) the sequence of random variables \( \|f_n\|_K \) is tight, \( \|f_n\|_K = \max_K |f_n| \), then \( \{f_n\}_{n \geq 1} \) is tight, see for instance [20 Proposition 2.5].

By [20 Lemma 2.6], it is enough to bound \( \mathbb{E}[|q_n(z)|^2] \) by a deterministic function of \( z \) that does not depend on \( n \). Recall the notation \( A_n(I) \) from the beginning of Section 3. Notice that each \( \det(A_n(I)) \) has mean zero and that

\[
\mathbb{E}\left[ \left| \det(A_n(I)) \right| \right] = 0 \quad \text{if } I \neq J \quad \text{and} \quad \mathbb{E}[|\det(A_n(I))|^2] = \text{card}(I)!. 
\]

In particular,

\[
\mathbb{E}\left[ |P_{k}^{(n)}|^2 \right] = n^{-k} \mathbb{E}\left[ |\det A_k|^2 \right] = n^{-k} \frac{n!}{(n-k)!} \leq 1 \quad \text{and} \quad \mathbb{E}\left[ \frac{P_{k}^{(n)}\overline{P_{k}^{(m)}}}{\sqrt{M}} \right] = 0 \quad \text{if } k \neq l.
\]

So, we have

\[
\mathbb{E}[|q_n(z)|^2] \leq 1 + \sum_{k=1}^{n} |z|^{2k} \mathbb{E}[|P_{k}^{(n)}|^2] \leq \sum_{k=0}^{n} |z|^{2k} \leq \frac{1}{1-|z|^2}.
\]

4.2. Proof of Lemma 3.2  The statement is close to [20 Proposition 2.5].

Take two subsequences \( \{f_{n_{1}}\}_{n_{1} \geq 1} \) and \( \{f_{n_{2}}\}_{n_{2} \geq 1} \) of random functions that converge, in law, to some random functions \( g \) and \( \tilde{g} \) in \( \mathbb{H(D)} \). We want to show that the distributions of \( g \) and \( \tilde{g} \) coincide. By Remark 4.1 below, we can write \( g(z) = \sum_{k=0}^{\infty} Q_k z^k \) for \( z \in \mathbb{H(D)} \) and \( \tilde{g}(z) = \sum_{k=0}^{\infty} \tilde{Q}_k z^k \) for \( z \in \mathbb{H(D)} \), where \( \{Q_k\}_{k \geq 0} \) and \( \{\tilde{Q}_k\}_{k \geq 0} \) are two sequences of complex random variables. By the same remark, we have that for any \( m \geq 0 \), the limit in law as \( \ell \to \infty \) of \( \{P_{0}^{(n)}\}, \ldots, P_{m}^{(n)} \) is \( \{Q_0, \ldots, Q_m\} \) while the limit in law as \( \ell \to \infty \) of \( \{P_{0}^{(n)}\}, \ldots, P_{m}^{(n)} \) is \( \{\tilde{Q}_0, \ldots, \tilde{Q}_m\} \). In particular, \( \{Q_0, \ldots, Q_m\} \) and \( \{\tilde{Q}_0, \ldots, \tilde{Q}_m\} \) have the same distribution as \( \{P_0, \ldots, P_m\} \) for every \( m \geq 0 \) so that \( \{Q_k\}_{k \geq 0} \), \( \{\tilde{Q}_k\}_{k \geq 0} \) and \( \{P_k\}_{k \geq 0} \) have the same distribution as random elements of \( \mathbb{C}^{\mathbb{Z}_{\geq 0}} \). By Remark 4.1 again, \( g \) and \( \tilde{g} \) have also the same distribution. Moreover, the random function \( z \in \mathbb{D} \to f(z) = \sum_{k=0}^{\infty} P_k z^k \) is well-defined as a random variable on \( \mathbb{H(D)} \) and its distribution is the unique limit point of the sequence of distributions of \( \{f_{n} \}_{n \geq 1} \). Finally, since \( \{f_{n} \}_{n \geq 1} \) is tight and since, by Prokhorov’s theorem, tightness means that its sequence of distributions is sequentially relatively compact in the space of probability measures on \( \mathbb{H(D)} \), we conclude that \( \{f_n\}_{n \geq 1} \) converges in law to \( f \) as \( n \to \infty \).

Remark 4.1.  The \( P_k \)’s are related to the successive derivatives of \( f \) at point 0. Due to the properties of analytic functions, the map \( T : \mathbb{H(D)} \to \mathbb{C}^{\mathbb{Z}_{\geq 0}} \) defined for all \( h \in \mathbb{H(D)} \) and all \( k \in \mathbb{Z}_{\geq 0} \) by

\[
T(h)_k = \frac{1}{k!} \frac{d^k h}{dz^k}(0)
\]

is continuous and injective. The inverse map \( T^{-1} : \{a_k \}_{k \geq 0} \in \mathbb{C}^{\mathbb{Z}_{\geq 0}} : \lim_{k \to -\infty} |a_k|^{1/k} \leq 1 \} \to \mathbb{H(D)} \) given by

\[
(T^{-1}(a))(z) = \sum_{k=0}^{\infty} a_k z^k
\]

is measurable. Denoting \( \mathcal{P}(E) \) the set of probability measures on \( E \), it follows that the pushforward map

\[
T_\ast : \mathcal{P}(\mathbb{H(D)}) \to \mathcal{P}(\mathbb{C}^{\mathbb{Z}_{\geq 0}})
\]

is injective in the sense that for all \( \mu \) and \( \nu \) in \( \mathcal{P}(\mathbb{H(D)}) \), if \( T_\ast \mu = T_\ast \nu \) then \( \mu = \nu \).

4.3. Proof of Lemma 3.3  It is enough to notice that, for each \( k \geq 1 \), there exists a sequence \( \{C_M\}_{M \geq 1} \) that goes to zero such that

\[
\mathbb{E}\left[ \left| P_k^{(n,M)} - P_k^{(n)} \right|^2 \right] \leq C_M
\]
for every $n, M \geq 1$. But

$$\mathbb{E} \left[ |p_{k}^{(n,M)} - p_{k}^{(n)}|^2 \right] = n^{-k} \sum_{l \in \{1, \ldots, n\} \atop |l| = k} \mathbb{E} \left[ \left| \det(A_{n}(I)^{(M)}) - \det(A_{n}(I)) \right|^2 \right]$$

$$= n^{-k} \binom{n}{k} \mathbb{E} \left[ \left| a_{11}^{(M)} \cdots a_{1k}^{(M)} - a_{11} \cdots a_{1k} \right|^2 \right] k!$$

$$\leq \mathbb{E} \left[ \left| a_{11}^{(M)} \cdots a_{1k}^{(M)} - a_{11} \cdots a_{1k} \right|^2 \right]$$

so that $C_{M} = \mathbb{E} \left[ \left| a_{11}^{(M)} \cdots a_{1k}^{(M)} - a_{11} \cdots a_{1k} \right|^2 \right]$ works.

4.4. **Proof of Lemma 3.4** As it is usual, the idea is to understand which terms are dominant. We have

$$\mathbb{E} \left[ \frac{\binom{n}{k_{1}}^{s_{1}} \cdots \binom{n}{k_{m}}^{s_{m}}}{n^{(k_{1}+\cdots+k_{m})/2}} \right] \leq \frac{1}{n^{(k_{1}+\cdots+k_{m})/2}} \mathbb{E} \left[ \left( \binom{n}{k_{1}} \right)^{s_{1}} \cdots \left( \binom{n}{k_{m}} \right)^{s_{m}} \right]$$

$$= \frac{1}{n^{(k_{1}+\cdots+k_{m})/2}} \sum_{G_{1} \in \mathcal{E}^{(n)}_{k_{1}}} \cdots \sum_{G_{m} \in \mathcal{E}^{(n)}_{k_{m}}} \mathbb{E} \left[ \prod_{i=1}^{m} \left( a_{G_{i}} \right)^{s_{i}} \right]$$

We say that $(g_{1}, \ldots, g_{m})$ is equivalent to $(\bar{g}_{1}, \ldots, \bar{g}_{m})$ if there is a bijection $\theta : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ such that

$$\bar{g}_{i} = \theta_{*}(g_{i}) \text{ for } i \in \{1, \ldots, m\},$$

where $\theta_{*}$ denotes the map induced by $\theta$ on the subgraphs of $\{1, \ldots, n\}$. So,

$$\mathbb{E} \left[ \prod_{i=1}^{m} \left( a_{G_{i}} \right)^{s_{i}} \right] = \mathbb{E} \left[ \prod_{i=1}^{m} \left( a_{\bar{G}_{i}} \right)^{s_{i}} \right]$$

if $\Gamma = (g_{1}, \ldots, g_{m})$ is equivalent to $\bar{\Gamma} = (\bar{g}_{1}, \ldots, \bar{g}_{m})$. Hence, if we denote by $\mathcal{T}^{(n)}_{(k_{1}, \ldots, k_{m})}$ the set of equivalence classes, we can define

$$W_{\Gamma} = \mathbb{E} \left[ \prod_{i=1}^{m} \left( a_{G_{i}} \right)^{s_{i}} \right],$$

where $\left\{ \Gamma \right\}$ is the class of $\Gamma$. We can then write

$$\frac{1}{n^{(k_{1}+\cdots+k_{m})/2}} \sum_{G_{1} \in \mathcal{E}^{(n)}_{k_{1}}} \cdots \sum_{G_{m} \in \mathcal{E}^{(n)}_{k_{m}}} \mathbb{E} \left[ \prod_{i=1}^{m} \left( a_{G_{i}} \right)^{s_{i}} \right] \leq \frac{1}{n^{(k_{1}+\cdots+k_{m})/2}} \sum_{v \in \mathcal{T}^{(n)}_{(k_{1}, \ldots, k_{m})}} \text{card}(v) W_{v},$$

where $\text{card}(v)$ is the cardinality of $v$ seen as a subset of $\mathcal{E}_{k_{1}}^{(n)} \times \cdots \times \mathcal{E}_{k_{m}}^{(n)}$. There is a natural inclusion map from $\mathcal{T}^{(n)}_{(k_{1}, \ldots, k_{m})}$ into $\mathcal{T}^{(n+1)}_{(k_{1}, \ldots, k_{m})}$ induced by the inclusion $\{1, \ldots, n\} \subset \{1, \ldots, n +1\}$ and these inclusions are surjective if $n \geq k_{1} + \cdots + k_{m}$. With the help of these inclusions we can write, for $n \geq k_{1} + \cdots + k_{m},$

$$\frac{1}{n^{(k_{1}+\cdots+k_{m})/2}} \sum_{v \in \mathcal{T}^{(n)}_{(k_{1}, \ldots, k_{m})}} \text{card}(v) W_{v} = \frac{1}{n^{(k_{1}+\cdots+k_{m})/2}} \sum_{\mu \in \mathcal{T}^{(n+1)}_{(k_{1}, \ldots, k_{m})}} \text{card}_{n}(\mu) W_{\mu},$$

**Figure 1.** An example of a multigraph $E^{P}$ constructed from one 3-directed cycle, one 4-directed cycle and one 5-directed cycle.
where \( \text{card}_n (\mu) \) denotes the cardinality of \( \mu \) when seen as a subset of \( \mathcal{C}^{(n)}_{k_1} \times \cdots \times \mathcal{C}^{(n)}_{k_m} \). So, it is enough to find the limit, as \( n \to \infty \), of
\[
\text{card}_n (\mu) \frac{1}{n^{(k_1 + \cdots + k_m)/2}}
\]
for any \( \mu \in \mathcal{T}^{(k_1 + \cdots + k_m)} \). To understand better this cardinality, to each \( \mu = [(g_1, \ldots, g_m)] \in \mathcal{T}^{(k_1 + \cdots + k_m)} \) we associate the oriented multigraph \( G^\mu \) consisting of the union of the \( g_i \)'s with edges counted multiple times (see Figure 1). More precisely, if \( V^g_i \) and \( E^g_i \) are the vertex set and the edge set of \( g_i \), then the vertex set \( V^\mu \) and the edge set \( E^\mu \) of \( G^\mu \) are
\[
V^\mu = \bigcup_{i=1}^m V^{g_i} \quad \text{and} \quad E^\mu = \bigcup_{i=1}^m (I \times E^{g_i})
\]
with the source and target maps, \( s : E^\mu \to V^\mu \) and \( t : E^\mu \to V^\mu \), defined by
\[
s(l, (i, j)) = i \quad \text{and} \quad t(l, (i, j)) = j.
\]
If there is an edge \( e \in E^\mu \) that is not multiple, in other words such that \( (s, t)(e) \neq (s, t)(e') \) for every other edge \( e' \neq e \), then \( \mathcal{W}_\mu = 0 \). So we consider only graphs where all edges are multiple. If for each \( v \in V^\mu \) the outer degree \( \text{deg}(v) \) is defined by
\[
\text{deg}(v) = \text{card} \{ e \in E^\mu : s(e) = v \}
\]
we have that \( \text{deg}(v) \geq 2 \) for every \( v \in V^\mu \). By using the handshaking lemma, we have
\[
\sum_{v \in V^\mu} \text{deg}(v) = \text{card}(E^\mu) = k_1 + \cdots + k_m.
\]
We notice that if, moreover, \( \text{deg}(v_*) \geq 3 \) for some \( v_* \in V^\mu \) then
\[
k_1 + \cdots + k_m = \sum_{v \in V^\mu \setminus \{v_*\}} \text{deg}(v) + \text{deg}(v_*) \geq 2(\text{card}(V^\mu) - 1) + 3 = 2\text{card}(V^\mu) + 1
\]
so that
\[
\text{card}(V^\mu) < \frac{k_1 + \cdots + k_m}{2}.
\]
But \( \text{card}_n (\mu) \leq n^{\text{card}(V^\mu)} \), which implies that
\[
\frac{\text{card}_n (\mu)}{n^{(k_1 + \cdots + k_m)/2}} \to 0 \quad n \to \infty.
\]

**Figure 2.** A graph formed by two 4-directed cycles and two 3-directed cycles satisfying the condition in (4.1).

Then, we suppose that \( \text{deg}(v) = 2 \) for every \( v \in V^\mu \). Choose \( (g_1, \ldots, g_m) \) such that \( [(g_1, \ldots, g_m)] = \mu \). By using that all edges of \( G^\mu \) are multiple and that every vertex has degree exactly 2 we can see that there must be a partition into pairs of \( \{1, \ldots, m\} \) such that (see Figure 2)
\[
g_i = g_j \quad \text{if} \ i \sim j \quad \text{and} \quad V^{g_i} \cap V^{g_j} = \emptyset \quad \text{if} \ i \neq j
\] (4.1)
where the relation ∼ denotes if the elements belong to the same set of the partition and \( V_{g_l} \) denotes the vertex set of \( g_l \) as before. Necessarily \( m \) is even and for each such \( \mu \) we have

\[
\frac{\text{card}_n(\mu)}{n^{(k_1+\cdots+k_m)/2}} \xrightarrow{n \to \infty} \frac{1}{\sqrt{k_1 \cdots \sqrt{k_m}}}
\]

where the term \( \sqrt{k_1} \cdots \sqrt{k_m} \) appears because we are counting cycles with no distinguished vertex. There is precisely one \( \mu \) associated to any partition \( P \) into pairs of \( \{1, \ldots, m\} \) such that \( k_i = k_j \) if \( i \sim j \).

Then, we shall use the notation \( W_\rho = W_\mu \) to notice that

\[
\frac{1}{n^{(k_1+\cdots+k_m)/2}} \sum_{\mu \in \mathcal{T}(k_1, \ldots, k_m)} \text{card}_n(\mu) W_\mu \xrightarrow{n \to \infty} \frac{1}{\sqrt{k_1} \cdots \sqrt{k_m}} \sum_{\rho} W_\rho
\]

where the sum is over all partition into pairs of \( \{1, \ldots, m\} \) such that (4.2) happens. Since

\[
W_\rho = \prod \mathbb{E} \left[ a_{i_j}^{x_j} a_{j_l}^{x_j} \right],
\]

where the product runs over all the pairs \( (i, j) \) with \( i \sim j \) and \( i \neq j \), and where \( \alpha_k = a_{12} a_{23} \ldots a_{(k-1)k} a_{k1} \).

We may use Isserlis/Wick theorem to conclude.

4.5. **Proof of Lemma 3.5** We start by checking that

\[
\mathbb{E} \left[ \frac{t_k^{(n)}}{n^{k/2}} \right] \xrightarrow{n \to \infty} \begin{cases} \mathbb{E}[a_{11}^2]^{k/2} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd}. \end{cases}
\]

We may use the same kind of counting argument as in Lemma 3.4. Given a sequence \( (i_1, \ldots, i_k) \), we construct a multigraph from it and notice that every edge must be multiple for the graph to contribute. Next, if some vertex has outer degree greater or equal than three then that graph does not contribute neither. Finally, if the graph constructed from \( (i_1, \ldots, i_k) \) has every edge multiple and every vertex has outer degree two we can show that it is a double cycle (see Figure 3), in other words \( k \) is even, and for \( l < l' \) we have that

\[
i_l = i_{l'} \quad \text{if and only if} \quad l' = l + \frac{k}{2}.
\]

The expectation \( \mathbb{E}[a_k] \) for such double cycle \( g \) is equal to \( \mathbb{E}[a_{11}^2]^{k/2} \). Since there are

\[
n(n-1) \ldots \left(n-\frac{k}{2}+1\right)
\]

of those \( (i_1, \ldots, i_k) \), we have checked that (4.3) holds.

![Figure 3. A single graph formed by a double cycle that counts for obtaining the expected value.](image)

Now, for any pair of square-integrable complex random variables \( X \) and \( Y \) let us use the notation

\[
\text{cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(\overline{Y} - \mathbb{E}[\overline{Y}])] \quad \text{while} \quad \text{var}(X) = \text{cov}(X, X) = \mathbb{E}[(X - \mathbb{E}[X])^2].
\]
To complete the proof of Lemma 3.5 it is sufficient to prove that
\[
\var\left(\frac{r_k(n)}{n^{k/2}}\right) \xrightarrow{n \to \infty} 0. \tag{4.4}
\]
To this end, if \( i = (i_1, \ldots, i_k) \), we set
\[
a_i = a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_{k-1} i_k} a_{i_k i_1}.
\]
By construction, we have
\[
\var\left(\frac{r_k(n)}{n^{k/2}}\right) = \sum_{i,j} \cov(a_i, a_j), \tag{4.5}
\]
where the sum is over all pairs \((i, j)\) of \(k\)-tuples such that both \(i\) and \(j\) have less than \(k\) distinct elements.

From Cauchy–Schwarz inequality, the following crude bound holds:
\[
|\cov(a_i, a_j)| \leq 4M^{2k},
\]
where \(M\) is such that the support of \(a_{11}\) is contained in the ball of radius \(M\). Also, as above, we may identify each \(k\)-tuple \(i\) with a path of length \(k\). Setting \(l_{k+1} = i_1\), let us introduce the set of visited vertices and the set of directed edges by
\[
V_i = \{i_1, \ldots, i_k\} \quad \text{and} \quad E_i = \{(i_l, i_{l+1}) : l = 1, \ldots, k\}.
\]
Then \(G_i = (V_i, E_i)\) is the directed graph associated to \(i\) (self-loop edges allowed). We define its excess as
\[
\chi_i = \card(E_i) - \card(V_i) + 1 \geq 0.
\]
It is the minimal number of edges to be removed such that the remaining subgraph has no undirected cycle (with the convention that \((u, u)\) is a cycle of length 1 and for \(u \neq v\), \(((u, v), (v, u))\) forms a cycle of length 2). Since \(G_i\) is the graph associated to a path of length \(k\), the assumption \(\card(V_i) < k\) implies that
\[
\chi_i \geq 2.
\]
Similarly, if \(i, j\) are two \(k\)-tuples, we consider their associated graph with vertex and directed edge sets
\[
V_{ij} = V_i \cup V_j \quad \text{and} \quad E_{ij} = E_i \cup E_j.
\]
The excess of the corresponding graph \(G_{ij} = (V_{ij}, E_{ij})\) is
\[
\chi_{ij} = \card(E_{ij}) - \card(V_{ij}) + c
\]
where \(c \in \{1, 2\}\) is the number of weak connected components of \(G_{ij}\): \(c = 1\) if \(V_i \cap V_j \neq \emptyset\) and \(c = 2\) otherwise. Since \(G_{ij}\) is the union of \(G_i\) and \(G_j\), we have
\[
\chi_{ij} \geq \max(\chi_i, \chi_j) \geq 2.
\]
Now, from the independence of the entries of the matrix \(A_n\), we have \(\cov(a_i, a_j) = 0\) unless \(E_i \cap E_j\) is not empty. Thus \(G_{ij}\) is connected for such \(i, j\). Moreover, \(\cov(a_i, a_j) = 0\) unless all edges of \(E_{ij}\) are visited at least twice by the union of paths \(i\) and \(j\). Hence, for such \(i, j\), \(\card(E_{ij}) \leq k\) and thus
\[
\card(V_{ij}) = 1 - \chi_{ij} + \card(E_{ij}) \leq k - 1.
\]
We thus have checked that
\[
\var\left(\frac{r_k(n)}{n^{k/2}}\right) \leq 4C_k M^{2k} n^{k-1},
\]
where \(C_k\) bounds the number of possibilities for the pair of \(k\)-tuples \((i, j)\) once the set \(V_{ij}\) is chosen. This gives (4.4), which concludes the proof of the lemma.
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