Massless Thirring fermion fields in the boson field representation

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Abstract

We show that the boson field representation of the massless fermion fields, suggested by Morchio, Pierotti and Strocchi in J. Math. Phys. 33, 777 (1992) for the operator solution of the massless Thirring model, agrees completely with the existence of the chirally broken phase in the massless Thirring model revealed in EPJC 20, 723 (2001) and hep–th/0112183, when the free massless boson fields are described by the quantum field theory, free of infrared divergences in 1+1–dimensional space–time, formulated in hep–th/0112184 and hep–th/0204237.

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1 Introduction

In our recent publications [1–4] we have revealed a non–perturbative phase of spontaneously broken continuous symmetry in the massless Thirring model [1,4] and the free massless (pseudo)scalar field theory defined in 1+1–dimensional space–time. We have shown that the massless Thirring model [5], invariant under chiral $U(1) \times U(1)$ symmetry, is unstable under chiral symmetry breaking [1,4]. We have proved that the spontaneously broken phase of the massless Thirring model is fully characterized by a non–vanishing fermion condensate and the wave function of the ground state of the chirally broken phase coincides with the wave function of the ground state of the superconducting phase of the BCS–theory of superconductivity [1]. In the quantum field theory of a free massless (pseudo)scalar field the symmetry broken phase is characterized by a non–vanishing spontaneous magnetization and the wave function of the ground state is infinitely degenerate [2,3]. Goldstone bosons are the quanta of the free massless (pseudo)scalar field [1–3,6].

We would like to emphasize that these results cannot be considered as counterexamples to the well–known Mermin–Wagner–Hohenberg theorem [7] asserting the vanishing of long range order in quantum field theories in two dimensions. Indeed, as has been pointed out by the authors [7], the absence of the long–range order can be inferred only for non–zero temperature $T \neq 0$ [7] and no conclusion about its value can be derived for $T = 0$ [7]. Unlike the quantum field theories treated by Mermin, Wagner and Hohenberg [7], the massless Thirring model [1,4] and the quantum field theory of a free massless (pseudo)scalar field [1–3] are formulated at zero temperature $T = 0$ and fermion condensation and spontaneous magnetization have been found at $T = 0$. Hence, these results go beyond the scope of the applicability of the Mermin–Wagner–Hohenberg theorem [7].

Coleman’s theorem, asserting the absence of Goldstone bosons and a spontaneously broken continuous symmetry by example of the quantum field theory of a free massless (pseudo)scalar field [8], has been recently critically discussed in our paper [3]. We have shown that the fulfillment of this theorem would lead to the vanishing not only vacuum expectation value of the variation of the free massless (pseudo)scalar field, required by the Goldstone theorem [9] in the case of the absence of a spontaneously broken continuous symmetry, but of all Wightman functions [3].

The problem of infrared divergences is one of the main problems in the quantum field theory of a free massless (pseudo)scalar field [10–12]. The infrared divergences appear in the two–point Wightman functions

\[
D^{(+)}(x) = \langle 0 | \vartheta(x) \vartheta(0) | 0 \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk^1}{2k^0} e^{-i k \cdot x} = -\frac{1}{4\pi} \ln[-\mu^2 x^2 + i 0 \cdot \varepsilon(x^0)],
\]

\[
D^{(-)}(x) = \langle 0 | \vartheta(0) \vartheta(x) | 0 \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk^1}{2k^0} e^{i k \cdot x} = -\frac{1}{4\pi} \ln[-\mu^2 x^2 - i 0 \cdot \varepsilon(x^0)],
\]

(1.1)

where $\varepsilon(x^0)$ is the sign function, $x^2 = (x^0)^2 - (x^1)^2$, $k \cdot x = k^0 x^0 - k^1 x^1$, $k^0 = |k^1|$ is the energy of a free massless (pseudo)scalar quantum with momentum $k^1$ and $\mu$ is the infrared cut–off reflecting the infrared divergence of the Wightman function (1.1). For the calculation of the two–point Wightman functions (1.1) one should use the expansion

\[
D^{(\pm)}(x) \rightarrow D^{(\pm)}(x; \mu),
\]

Further in order to underscore the dependence of the two–point Wightman functions on the scale $\mu$ we will denote $D^{(\pm)}(x) \rightarrow D^{(\pm)}(x; \mu)$. 

2
of the massless (pseudo)scalar field $\vartheta(x)$ into plane waves [1–3]

$$\vartheta(x) = \int_{-\infty}^{\infty} \frac{dk^1}{2\pi} \frac{1}{2k^0} \left( a(k^1) e^{-ik \cdot x} + a^\dagger(k^1) e^{ik \cdot x} \right),$$  \hspace{1cm} (1.2)

where $a(k^1)$ and $a^\dagger(k^1)$ are annihilation and creation operators obeying the standard commutation relation

$$[a(k^1), a^\dagger(q^1)] = (2\pi) 2k^0 \delta(k^1 - q^1).$$  \hspace{1cm} (1.3)

Klaiber was the first [10] who asserted the importance of the problem of infrared divergences of the two–point Wightman functions (1.1) for the solution of the massless Thirring model: *If one wants to solve the Thirring model, one has to overcome this problem.*

In Ref.[2] we have suggested a recipe for the struggle against the infrared divergences of the two–point Wightman functions (1.1). This recipe concerns the constraint on the external source $J(x)$ of a free massless (pseudo)scalar field $\vartheta(x)$ in the path integral formulation of the quantum field theory of a free massless (pseudo)scalar field. Let $Z[J]$, the generating functional of Green functions of a free massless (pseudo)scalar field $\vartheta(x)$, be defined by

$$Z[J] = \langle 0 | T \left( e^{i \int d^2x \vartheta(x)J(x)} \right) | 0 \rangle = \int D\vartheta e^{i \int d^2x [\mathcal{L}(x) + \vartheta(x)J(x)]},$$  \hspace{1cm} (1.4)

where $T$ is a time–ordering operator. The Lagrangian of the $\vartheta$–field $\mathcal{L}(x)$ reads

$$\mathcal{L}(x) = \frac{1}{2} \partial_\mu \vartheta(x) \partial^\mu \vartheta(x).$$  \hspace{1cm} (1.5)

It is invariant under field translations [1–3,6]

$$\vartheta(x) \rightarrow \vartheta'(x) = \vartheta(x) + \alpha,$$  \hspace{1cm} (1.6)

where $\alpha$ is an arbitrary parameter $\alpha \in \mathbb{R}^1$ [1–3].\footnote{As has been shown in [1] the parameter $\alpha = -2\alpha_A$ is related to the chiral phase $\alpha_A$ of global chiral rotations of Thirring fermion fields.} Any correlation function of the free massless $\vartheta$–field can be calculated in terms of functional derivatives of $Z[J]$ with respect to the external source $J(x)$ [2]

$$G(x_1, \ldots, x_n; y_1, \ldots, y_p) = \langle 0 | F(\vartheta(x_1), \ldots, \vartheta(x_n); \vartheta(y_1), \ldots, \vartheta(y_p)) | 0 \rangle =$$

$$= F \left( -i \delta \frac{\delta}{\delta J(x_1)}, \ldots, -i \delta \frac{\delta}{\delta J(x_n)}; -i \delta \frac{\delta}{\delta J(y_1)}, \ldots, -i \delta \frac{\delta}{\delta J(y_p)} \right) Z[J] \bigg| J = 0. \hspace{1cm} (1.7)$$

One encounters the problem of the calculation of these correlation functions in connection with the calculation of correlations functions in the massless Thirring model [1,2], since the bosonized version of the massless Thirring model reduces to the quantum field theory of the free massless pseudoscalar field $\vartheta(x)$ described by the Lagrangian (1.2) [1].

The functional $Z[J]$ is invariant under the field translations (1.6) if the external source $J(x)$ obeys the constraint [2]

$$\int d^2x J(x) = 0.$$  \hspace{1cm} (1.8)
In momentum representation

\[ J(x) = \int \frac{d^2k}{(2\pi)^2} \tilde{J}(k) e^{-ik \cdot x} \]  

(1.9)

the constraint (1.8) corresponds to the vanishing of the Fourier transform of the external source at \( k = 0 \), i.e., \( \tilde{J}(0) = 0 \). This means that (i) the external source belongs to the class functions obeying the constraint \( \tilde{J}(0) = 0 \)[11,12] and (ii) an external perturbation does not excite a zero–mode response of the quanta of the \( \vartheta \)–field [2].

In Ref.[2] we have shown that the zero–mode of the free massless (pseudo)scalar field \( \vartheta(x) \) describes the motion of the “center of mass” of the \( \vartheta \)–field. This can be easily comprehended by using a mechanical analog of the free massless (pseudo)scalar field \( \vartheta(x) \) in terms of a chain of \( N \) linear self–coupled oscillators [2,13]. In Ref.[3] we have quantized these oscillators in terms of the operators of creation and annihilation of the quanta of the \( \vartheta \)–field.

Integrating over the \( \vartheta \)–field in (1.4) we get [2]

\[ Z[J] = \lim_{\mu \to 0} \exp \left\{ \frac{i}{2} \int d^2x d^2y J(x) \Delta(x - y; \mu) J(y) \right\}, \]

(1.10)

where \( \Delta(x - y; \mu) \), the causal two–point Green function, is related to the two–point Wightman functions [2]

\[ \Delta(x - y; \mu) = i \theta(x^0 - y^0) D^{(+)}(x - y; \mu) + i \theta(y^0 - x^0) D^{(-)}(x - y; \mu) = \]

\[ = -\frac{i}{4\pi} \ell n[-\mu^2(x - y)^2 + i 0]. \]  

(1.11)

Due to the constraint (1.8) the infrared scale \( \mu \) can be replaced by the finite scale \( M \) [2]. This yields

\[ Z[J] = \exp \left\{ \frac{i}{2} \int d^2x d^2y J(x) \Delta(x - y; M) J(y) \right\}, \]

(1.12)

where \( \Delta(x - y; M) \) is equal to

\[ \Delta(x - y; M) = i \theta(x^0 - y^0) D^{(+)}(x - y; M) + i \theta(y^0 - x^0) D^{(-)}(x - y; M) = \]

\[ = -\frac{i}{4\pi} \ell n[-M^2(x - y)^2 + i 0]. \]  

(1.13)

This gives the Wightman functions \( D^{(\pm)}(x; M) \), defined by the momentum integrals [2]

\[ D^{(\pm)}(x; M) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk^1}{2k^0} \left( e^{\mp i k \cdot x - \cos(k^1 \lambda_M)} \right), \]

(1.14)

where \( \lambda_M = 1/M \), which are obviously convergent in the infrared region \( k^1 \to 0 \) and Lorentz invariant. This solves the infrared problem of the free massless (pseudo)scalar field theory pointed out by Klaiber [10].

\[ \text{It does not mean that this is Schwartz’s class of test functions } S_0(\mathbb{R}^2) = \{ J(x) \in S(\mathbb{R}^2); \tilde{J}(0) = 0 \}. \]
Since the free massless (pseudo)scalar field theory free of infrared divergences describes the bosonized version of the massless Thirring model in the chirally broken phase, when spontaneous magnetization of the free massless (pseudo)scalar field defines fully fermion condensate in the massless Thirring model \([1,2]\), one can expect that the fermion fields constructed directly from this (pseudo)scalar field should be equivalent to the Thirring fermion fields quantized in the chirally broken phase.

The idea to construct fermion fields in terms of boson fields has been suggested by Skyrme \([14]\). The first attempts in this direction have been undertaken by Lieb and Mattis \([15]\) and Streater and Wilde \([16]\) within axiomatic quantum field theory. At the standard quantum field theoretic level the realization of the fermion field operators in terms of the boson field has been developed by Mandelstam in connection with the proof of the equivalence between the massive Thirring model and the sine–Gordon model \([17]\). Then, it has been reconsidered and applied to the solution of the massless Thirring model by different authors \([18–20]\).

The main problem of the construction of fermion fields from the free massless boson ones \([19,20]\) is in the infrared divergences. The canonical dimension of a free fermion field is \(\sqrt{\mu}\), where \(\mu\) is a typical energy scale. In the free massless (pseudo)scalar field theory, suffering with infrared divergences, a typical energy scale is the infrared cut–off \(\mu\), which should be taken in the limit \(\mu \to 0\) in the final expressions. In the boson field approach to the solution of the massless Thirring model \([19,20]\) the fermion fields, either implicitly or explicitly, are proportional to the infrared cut–off \(\sqrt{\mu}\), and, therefore, should vanish in the limit \(\mu \to 0\). At an intermediate step of calculations, when the infrared scale \(\mu\) is kept finite but infinitesimally small, the proportionality of the fermion fields to the infrared cut–off \(\mu\) leads to the violation of the constant of motion \([1]\). It is obvious, since the constant does not depend on the infrared cut–off.

The main aim of this paper is to show that the removal of the infrared divergences from the quantum field theory of a free massless (pseudo)scalar field \(\vartheta(x)\) described by the Lagrangian \([1,3]\) \([2,3]\) allows to construct fermion field operators in terms of the boson fields possessing the properties of the Thirring fermion fields quantized in the chirally broken phase \([1]\).

The paper is organized as follows. In Section 2 we give a cursory outline of the massless Thirring model. We write down the Lagrangian of the self–coupled fermion fields, the equations of motion, the constant of motion and canonical anti–commutation relations. In Section 3 we discuss the properties of the free massless (pseudo)scalar fields defined in the quantum field theory free of infrared divergences formulated in \([2,3]\). In Section 4, following mainly the results obtained by Morchio, Pierotti and Strocchi \([20]\), we discuss the properties of the free massless fermion fields in the boson field representation. We show that the fermion field operators obey all standard canonical anti–commutation relations required for the free fermion quantum field theory. The self–consistency of all results is controlled by the normalization scale of the two–point Wightman functions \([2,3]\). In Section 5 we discuss the properties of the massless Thirring fermion fields in the boson field representation constructed form the boson fields according to the recipe suggested by Morchio, Pierotti and Strocchi \([20]\). We show that the fermion field operators invented by Morchio, Pierotti and Strocchi \([20]\) and regularized by the finite normalization scale \(M\) of the two–point Wightman functions \([2,3]\) obey all anti–commutation relations and non–perturbative properties of the Thirring fermion fields \([1–4]\). In Section 6 we construct
the operator of the scalar fermion density \( \bar{\psi}(x)\psi(x) \) in the boson field representation of the massless Thirring fermion fields discussed in Section 5. We show that the vacuum expectation value of this operator, i.e. the fermion condensate, does not vanish. This means that the fermion fields, constructed from the boson fields, possess the properties of the massless Thirring fermion fields quantized in the chirally broken phase. This confirms the results obtained in [1,4] pointing out the existence of the non–perturbative chirally broken phase in the massless Thirring model. In Section 7 we show that the Thirring fermion field operators in the boson field representation obey the constant of motion revealed in [1]. In the Conclusion we discuss the obtained results. In the Appendix we formulate the procedure for the normal ordering of the exponential operators depending on the free massless boson fields defined by the quantum field theory without infrared divergences.

2 Massless Thirring model. Cursory outline

The massless Thirring model [5] is a theory of a self–coupled Dirac field \( \psi(x) \)

\[
L_{\text{Th}}(x) = \bar{\psi}(x)i\gamma^\mu \partial_\mu \psi(x) - \frac{1}{2} g \bar{\psi}(x)\gamma^\mu \psi(x)\bar{\psi}(x)\gamma_\mu \psi(x),
\]

(2.1)

where \( g \) is a dimensionless coupling constant that can be either positive or negative. Since for \( g > 0 \) the fermion system described by the Lagrangian (2.1) is unstable under chiral symmetry breaking [1–4], we will consider only \( g > 0 \).

The field \( \psi(x) \) is a spinor field with two components \( \psi_1(x) \) and \( \psi_2(x) \). The \( \gamma \)–matrices are defined in terms of the well–known 2 \( \times \) 2 Pauli matrices \( \sigma_1, \sigma_2 \) and \( \sigma_3 \)

\[
\gamma^0 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0\gamma^1 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

(2.2)

These \( \gamma \)–matrices obey the standard relations

\[
\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu},
\gamma^\mu \gamma^5 + \gamma^5 \gamma^\mu = 0.
\]

(2.3)

We use the metric tensor \( g^{\mu\nu} \) defined by \( g^{00} = -g^{11} = 1 \) and \( g^{01} = g^{10} = 0 \). The axial–vector product \( \gamma^\mu \gamma^5 \) can be expressed in terms of \( \gamma^\nu \)

\[
\gamma^\mu \gamma^5 = -\epsilon^{\mu\nu} \gamma_\nu,
\]

(2.4)

where \( \epsilon^{\mu\nu} \) is the anti–symmetric tensor defined by \( \epsilon^{01} = -\epsilon^{10} = 1 \). Further, we also use the relation \( \gamma^\mu \gamma^\nu = g^{\mu\nu} + \epsilon^{\mu\nu} \gamma^5 \).

The Lagrangian (2.1) is obviously invariant under the chiral group \( U_\chi(1) \times U_\Lambda(1) \)

\[
\psi(x) \overset{V}{\rightarrow} \psi'(x) = e^{i\alpha_V} \psi(x),
\]

\[
\psi(x) \overset{\Lambda}{\rightarrow} \psi'(x) = e^{i\alpha_\Lambda \gamma^5} \psi(x),
\]

(2.5)

where \( \alpha_V \) and \( \alpha_\Lambda \) are real parameters defining global rotations.
Due to invariance under chiral group $U_V(1) \times U_A(1)$ the vector and axial–vector current $j^\mu(x)$ and $j_5^\mu(x)$, induced by vector (V) and axial–vector (A) rotations and defined by

$$j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x),$$
$$j_5^\mu(x) = \bar{\psi}(x)\gamma^\mu\gamma^5\psi(x),$$

are conserved

$$\partial_\mu j^\mu(x) = \partial_\mu j_5^\mu(x) = 0.$$  

(2.7)

Recall, that in 1+1–dimensional field theories the vector and axial–vector currents are related by $j_5^\mu(x) = -\varepsilon^{\mu\nu}j_\nu(x)$ due to the properties of Dirac matrices (2.4).

Using the Lagrangian (2.1) we derive the equations of motion

$$i\gamma^\mu\partial_\mu\psi(x) = g j^\mu(x)\gamma_\mu\psi(x),$$
$$-i\partial_\mu\bar{\psi}(x)\gamma^\mu = g \bar{\psi}(x,t)\gamma_\mu j^\mu(x).$$

(2.8)

Due to the peculiarity of 1+1–dimensional quantum field theory of the Thirring fermion fields [1] these equations are equivalent to

$$i\partial_\mu\psi(x) = a j_\mu(x)\psi(x) + b \varepsilon_{\mu\nu} j^\nu(x)\gamma^5\psi(x),$$
$$-i\partial_\mu\bar{\psi}(x) = a \bar{\psi}(x)j_\mu(x) + b \bar{\psi}(x)\gamma^5j^\nu(x)\varepsilon_{\nu\mu}.$$ 

(2.9)

The parameters $a$ and $b$ are constrained by the condition $a + b = g$ and $a - b = 1/c$ where $c$ is the Schwinger term [1]. Multiplying equations (2.9) by $\gamma^\mu$ and summing over $\mu = 0, 1$ we end up with the equations of motion (2.8). The Thirring fermion fields evolving according to the equations of motion (2.8) and (2.9) obey the constant of motion [1]

$$[\bar{\psi}(x)\psi(x)]^2 + [\bar{\psi}(x)i\gamma^5\psi(x)]^2 = C,$$ 

(2.10)

where $C$ is a non–vanishing constant. As has been shown in Ref.[1] $C = M_f^2/g^2$, where $M_f$ is a dynamical mass of the Thirring fermion fields quantized in the chirally broken phase and defining the fermion condensate $(0|\bar{\psi}(x)\psi(x)|0) = -M_f/g$ [1,4]. In terms of the components of the fermion field the constant of motion (2.10) reads

$$\psi_1^\dagger(x)\psi_2(x)\psi_2^\dagger(x)\psi_1(x) + \psi_2^\dagger(x)\psi_1(x)\psi_1^\dagger(x)\psi_2(x) = \frac{1}{2}C.$$ 

(2.11)

The conjugate momenta of the fields $\psi_1(x)$ and $\psi_2(x)$ are $i\psi_1^\dagger(x)$ and $i\psi_2^\dagger(x)$, respectively. Therefore, they obey the canonical anti–commutation relations

$$\psi_1(x^0, x^1)\psi_1^\dagger(x^0, y^1) + \psi_1^\dagger(x^0, y^1)\psi_1(x^0, x^1) = \delta(x^1 - y^1),$$
$$\psi_2(x^0, x^1)\psi_2^\dagger(x^0, y^1) + \psi_2^\dagger(x^0, y^1)\psi_2(x^0, x^1) = \delta(x^1 - y^1).$$ 

(2.12)

We would like to emphasize that such canonical anti–commutation relations are valid for fermion fields with a canonical dimension $D_\psi = 1/2$\footnote{The total dimension of the fermion field $D_\psi$ is equal to $D_\psi = d_\psi + 1/2$, where $d_\psi$ is a dynamical dimension [4].}. In 1+1–dimensional space time these are only free fermion fields. In the case of the Thirring model with self–coupled
fermion fields the total dimension of the fermion fields differs from the canonical one $D_\psi \neq 1/2$ [4]. Therefore, for the Thirring fermion fields the canonical anti-commutation relation (2.12) is valid only for bare fermion fields having the dimension $D_\psi = 1/2$. For the derivation of the canonical anti-commutation relations for the Thirring fermion fields in the boson field representation one has to construct the products of the fermion field operators $\psi_i(x^0, x^1)\psi_i^\dagger(x^0, y^1)$ for $(i = 1, 2)$ and to subtract the dynamical dimensions by multiplying the products by the factors $(x^1 - y^1 - i0)^2D_\psi - 1$. This yields
\[
\psi_i(x^0, x^1)\psi_i^\dagger(x^0, y^1) \rightarrow \Psi_i(x^0, x^1)\Psi_i^\dagger(x^0, y^1) \propto (x^1 - y^1 - i0)^2D_\psi - 1 \psi_i(x^0, x^1)\psi_i^\dagger(x^0, y^1).
\]
(2.13)

For the $\Psi$–fields one would get the canonical equal–time anti-commutation relations
\[
\Psi_i(x^0, x^1)\Psi_i^\dagger(x^0, y^1) + \Psi_i^\dagger(x^0, y^1)\Psi_i(x^0, x^1) = \delta(x^1 - y^1), \quad (i = 1, 2).
\]
(2.14)

However, considering the spatial coordinates $x^1$ and $y^1$ separated by an infinitesimal distance, $x^1 \sim y^1$, one can show that the canonical anti-commutation relations for the massless Thirring fermions fields $\psi_i(x^0, x^1)$ and $\psi_i^\dagger(x^0, y^1)$ can be written in the form
\[
\psi_i(x^0, x^1)\psi_i^\dagger(x^0, y^1) + \psi_i^\dagger(x^0, y^1)\psi_i(x^0, x^1) = Z_2 \delta(x^1 - y^1), \quad (i = 1, 2).
\]
(2.15)

where $Z_2$ is the renormalization constant of the wave function of the massless Thirring fermion fields [4]. The expression (2.13) agrees with Mandelstam’s relation (see Eq.(3.1) of Ref.[17]).

3 Free massless boson fields as a basis for the representation of massless fermion fields

For the construction of fermion fields from the boson fields we use two free massless local fields $\varphi(x)$ and $\tilde{\varphi}(x)$ related by
\[
\frac{\partial \varphi(x)}{\partial x^0} = -\frac{\partial \tilde{\varphi}(x)}{\partial x^1}, \quad \frac{\partial \varphi(x)}{\partial x^1} = -\frac{\partial \tilde{\varphi}(x)}{\partial x^0}.
\]
(3.1)

Since the fields $\varphi(x)$ and $\tilde{\varphi}(x)$ describe free massless (pseudo)scalar fields and their conjugate momenta $\Pi(x) = \tilde{\varphi}(x)$ and $\tilde{\Pi}(x) = \dot{\varphi}(x)$, they can be represented in the form of expansions into plane waves [1–3]
\[
\varphi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk^1}{2k^0} \left[ c(k^1) e^{-ik^0x^0 + ik^1x^1} + c^\dagger(k^1) e^{+ik^0x^0 - ik^1x^1} \right],
\]
\[
\Pi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk^1}{2i} \left[ c(k^1) e^{-ik^0x^0 + ik^1x^1} - c^\dagger(k^1) e^{+ik^0x^0 - ik^1x^1} \right],
\]

\footnote{In this paper we will use as close as it is possible the notations accepted in the paper by Morchio, Pierotti, and Strocchi [20]. Therefore, the $\varphi$–field should be an analogy of the $\tilde{\varphi}$–field.}
\[ \tilde{\varphi}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk^1}{2k^0} \left[ \tilde{c}(k^1) e^{-ik^0x^0 + ik^1x^1} + \tilde{c}^\dagger(k^1) e^{+ik^0x^0 - ik^1x^1} \right], \]

\[ \tilde{\Pi}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk^1}{2i} \left[ \tilde{c}(k^1) e^{-ik^0x^0 + ik^1x^1} - \tilde{c}^\dagger(k^1) e^{+ik^0x^0 - ik^1x^1} \right]. \quad (3.2) \]

The operators of annihilation and creation of quanta of the fields \( \varphi(x) \) and \( \tilde{\varphi}(x) \) obey standard commutation relations

\[ [c(k^1), c^\dagger(q^1)] = (2\pi) 2k^0 \delta(k^1 - q^1), \]

\[ [\tilde{c}(k^1), \tilde{c}^\dagger(q^1)] = (2\pi) 2k^0 \delta(k^1 - q^1), \quad (3.3) \]

which give the equal–time commutation relations

\[ [\varphi(x^0, x^1), \Pi(x^0, y^1)] = i \delta(x^1 - y^1), \]

\[ [\tilde{\varphi}(x^0, x^1), \tilde{\Pi}(x^0, y^1)] = i \delta(x^1 - y^1). \quad (3.4) \]

According to the definition (3.1) the operators \( c(k^1) \) and \( \tilde{c}(k^1) \) are related by

\[ k^1 c(k^1) = k^0 \tilde{c}(k^1), \quad k^1 c^\dagger(k^1) = k^0 \tilde{c}^\dagger(k^1). \quad (3.5) \]

These relations can be rewritten in terms of the sign function \( \varepsilon(k^1) \)

\[ \tilde{c}(k^1) = \varepsilon(k^1) c(k^1), \quad \tilde{c}^\dagger(k^1) = \varepsilon(k^1) c^\dagger(k^1). \quad (3.6) \]

The fields \( \varphi(x) \) and \( \tilde{\varphi}(x) \) are supplemented by the charges \( Q(x^0) \) and \( \tilde{Q}(x^0) \)

\[ Q(x^0) = \int_{-\infty}^{\infty} dx^1 \Pi(x^0, x^1), \]

\[ \tilde{Q}(x^0) = \int_{-\infty}^{\infty} dx^1 \tilde{\Pi}(x^0, x^1). \quad (3.7) \]

We suggest to regularize these charges according to the procedure developed in Refs.[1–3] and get

\[ Q(x^0) = \lim_{V \to \infty} Q(x^0; V) = \]

\[ = \lim_{V \to \infty} \frac{-iV}{4\sqrt{\pi}} \int_{-\infty}^{\infty} dk^1 \left[ c(k^1) e^{-i|k^1|x^0} - c^\dagger(k^1) e^{+i|k^1|x^0} \right] e^{-V^2(k^1)^2/4}, \]

\[ \tilde{Q}(x^0) = \lim_{V \to \infty} \tilde{Q}(x^0; V) = \]

\[ = \lim_{V \to \infty} \frac{-iV}{4\sqrt{\pi}} \int_{-\infty}^{\infty} dk^1 \left[ \tilde{c}(k^1) e^{-i|k^1|x^0} - \tilde{c}^\dagger(k^1) e^{+i|k^1|x^0} \right] e^{-V^2(k^1)^2/4}, \quad (3.8) \]

where \( V \) is the spatial extent of the system. The charge operators \( Q(x^0) \) and \( \tilde{Q}(x^0) \) are the generators of a \( U(1) \times U(1) \) symmetry related to the shifts of the fields \( \varphi(x) \) and \( \tilde{\varphi}(x) \)

\[ \varphi(x) \to \varphi'(x) \overset{Q}{=} \varphi(x) + \alpha, \]

\[ \tilde{\varphi}(x) \to \tilde{\varphi}'(x) \overset{\tilde{Q}}{=} \tilde{\varphi}(x), \]

\[ \varphi(x) \to \varphi'(x) \overset{\tilde{Q}}{=} \varphi(x), \]

\[ \tilde{\varphi}(x) \to \tilde{\varphi}'(x) \overset{Q}{=} \tilde{\varphi}(x) + \beta, \quad (3.9) \]
where $\alpha$ and $\beta$ are the group parameters $\alpha, \beta \in \mathbb{R}$. As we have shown in Refs.[1–3] these symmetries are spontaneously broken and the Goldstone bosons are the quanta of the fields $\varphi(x)$ and $\tilde{\varphi}(x)$.

For the construction of fermion field operators from the boson operators it is convenient to introduce left and right fields $\varphi_L(x)$ and $\varphi_R(x)$ defined by [20]

$$\varphi_L(x) = \frac{1}{2} [\varphi(x) - \tilde{\varphi}(x)] , \quad \varphi_R(x) = \frac{1}{2} [\varphi(x) + \tilde{\varphi}(x)],$$

which commute $[\varphi_L(x), \varphi_R(y)] = 0$ for arbitrary space–time points $x$ and $y$.

The expansions of these fields and their conjugate momenta $\Pi_L(x)$ and $\Pi_R(x)$ into plane waves read

$$\varphi_L(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \theta(-k) \left[ c^\dagger(k) e^{-ik^0 x^0 + ik^1 x^1} + c(k) e^{ik^0 x^0 - ik^1 x^1} \right],$$

$$\Pi_L(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \theta(-k) \left[ c^\dagger(k) e^{-ik^0 x^0 + ik^1 x^1} - c(k) e^{ik^0 x^0 - ik^1 x^1} \right],$$

$$\varphi_R(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \theta(+k) \left[ c^\dagger(k) e^{-ik^0 x^0 + ik^1 x^1} + c(k) e^{ik^0 x^0 - ik^1 x^1} \right],$$

$$\Pi_R(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \theta(+k) \left[ c^\dagger(k) e^{-ik^0 x^0 + ik^1 x^1} - c(k) e^{ik^0 x^0 - ik^1 x^1} \right],$$

where $\theta(\pm k)$ are the Heaviside functions. The charge operators $Q_L(x^0)$ and $Q_R(x^0)$ are defined by

$$Q_L(x^0) = \lim_{V \to \infty} Q_L(x^0; V) =$$

$$= \lim_{V \to \infty} \frac{-iV}{\sqrt{\pi}} \int_{-\infty}^{\infty} dk \theta(-k) \left[ c^\dagger(k) e^{-ik^1|x^0|} - c(k) e^{ik^1|x^0|} \right] e^{-V^2(k^1)^2/4},$$

$$Q_R(x^0) = \lim_{V \to \infty} Q_R(x^0; V) =$$

$$= \lim_{V \to \infty} \frac{-iV}{2\sqrt{\pi}} \int_{-\infty}^{\infty} dk \theta(+k) \left[ c^\dagger(k) e^{-ik^1|x^0|} - c(k) e^{ik^1|x^0|} \right] e^{-V^2(k^1)^2/4}.$$  

The fields $\varphi_L(x)$ and $\varphi_R(x)$ and their charge operators have the following commutation relations [20]

$$[\varphi_L(x), Q_L(x^0)] = [\varphi_R(x), Q_R(x^0)] = [\varphi_L(x), Q_L(0)] = [\varphi_R(x), Q_R(0)] = -i.$$  

The two–point Wightman functions of the fields $\varphi_L(x)$ and $\varphi_R(x)$ are equal to

$$D_L^{(+)}(x - y; M) = \langle 0 | \varphi_L(x) \varphi_L(y) | 0 \rangle = -\frac{1}{4\pi} \ell n [i M (x_+ - y_+ - i 0)],$$

$$D_R^{(+)}(x - y; M) = \langle 0 | \varphi_R(x) \varphi_R(y) | 0 \rangle = -\frac{1}{4\pi} \ell n [i M (x_- - y_- - i 0)].$$
where we have denoted $x_{\pm} = x^0 \pm x^1$ and $y_{\pm} = y^0 \pm y^1$, and $M$ is a finite normalization scale $[2,3]$. The sum of the Wightman functions (3.14) is equal to $D^{(+)}(x - y; M)$

$$D^{(+)}(x - y; M) = D^{(+)}_L(x - y; M) + D^{(+)}_R(x - y; M) = -\frac{1}{4\pi} \ln[iM(x_+ - y_+ - i0)] - \frac{1}{4\pi} \ln[iM(x_- - y_- - i0)] = -\frac{1}{4\pi} \ln[-M^2(x - y)^2 + i0 \cdot \varepsilon(x^0 - y^0)].$$ (3.15)

In turn, the difference of the Wightman functions (3.14) does not depend on $M$ and amounts to

$$D^{(+)}_L(x - y; M) - D^{(+)}_R(x - y; M) = -\frac{1}{4\pi} \ln\left(\frac{x_+ - y_+ - i0}{x_- - y_- - i0}\right).$$ (3.16)

The r.h.s. of (3.16) is the well–known Wightman function $D^{(+)}_5(x - y; M)$ defined by [4]

$$D^{(+)}_5(x - y; M) = \langle 0|\varphi(x)\bar{\varphi}(y)|0\rangle = \langle 0|\bar{\varphi}(x)\varphi(y)|0\rangle = -\frac{1}{4\pi} \ln\left(\frac{x_+ - y_+ - i0}{x_- - y_- - i0}\right).$$ (3.17)

Therefore, for further analysis we will denote

$$D^{(+)}_L(x - y; M) - D^{(+)}_R(x - y; M) = D^{(+)}_5(x - y; M).$$ (3.18)

Now we are able to construct the fermion fields from the free massless boson fields $\varphi_L(x)$, $\varphi_R(x)$ and their charges $Q_L(0)$ and $Q_R(0)$.

## 4 Free fermion fields in the boson field representation

Following Morchio, Pierotti and Strocchi [20] we represent the free massless fermion fields $\Psi_1(x)$ and $\Psi_2(x)$ in terms of the fields $\varphi_L(x)$ and $\varphi_R(x)$ and their charge operators $Q_L(0)$ and $Q_R(0)$

$$\Psi_1(x^0, x^1) = +i \sqrt{\frac{M}{2\pi}} e^{-i(\sqrt{\pi}/4)Q_R(0)} :e^{2i\sqrt{\pi} \varphi_L(x^0, x^1)}:,$$

$$\Psi_2(x^0, x^1) = -i \sqrt{\frac{M}{2\pi}} e^{+i(\sqrt{\pi}/4)Q_L(0)} :e^{2i\sqrt{\pi} \varphi_R(x^0, x^1)}:,$$ (4.1)

where $M$ is the normalization scale in the Wightman functions (3.14).

The anti–commutativity of the fermion field operators can be verified by analysing the products $\Psi_i(x)\Psi_j(y)$ and $\Psi_i(x)\Psi_j^\dagger(y)$. For the product $\Psi_1(x^0, x^1)\Psi_1(y^0, y^1)$ we get

$$\begin{align*}
\Psi_1(x^0, x^1)\Psi_1(y^0, y^1) &= -\frac{M}{2\pi} e^{-i(\sqrt{\pi}/4)Q_R(0)} :e^{2i\sqrt{\pi} \varphi_L(x^0, x^1)} : e^{-i(\sqrt{\pi}/4)Q_R(0)} \\
\times :e^{2i\sqrt{\pi} \varphi_L(y^0, y^1)}: &=-\frac{M}{2\pi} e^{-i(\sqrt{\pi}/2)Q_R(0)} :e^{2i\sqrt{\pi} \varphi_L(x^0, x^1)} : e^{2i\sqrt{\pi} \varphi_L(y^0, y^1)}: \\
&= e^{-4\pi [\varphi_L^{(+)}(x^0, x^1), \varphi_L^{(-)}(y^0, y^1)]} \\
\times (-1)^{\frac{M}{2\pi}} e^{-i(\sqrt{\pi}/2)Q_R(0)} :e^{2i\sqrt{\pi} (\varphi_L(x^0, x^1) + \varphi_L(y^0, y^1))}:
\end{align*}$$

(4.2)

In the Appendix we formulate the procedure for the normal ordering of the exponential operators depending on the free massless boson fields defined in the quantum field theory without infrared divergences.
where we have used the relation [17]

\[ :e^{2i\sqrt{\pi}\varphi_L(x^0, x^1)}:e^{2i\sqrt{\pi}\varphi_L(y^0, y^1)} := e^{-4\pi\left[\varphi_L^{(+)}(x^0, x^1), \varphi_L^{(-)}(y^0, y^1)\right]} \]
\[ \times :e^{2i\sqrt{\pi}(\varphi_L(x^0, x^1) + \varphi_L(y^0, y^1))} :. \]  

(4.3)

The commutator \([\varphi_L^{(+)}(x^0, x^1), \varphi_L^{(-)}(y^0, y^1)]\) is the Wightman function \(D_L^{(+)}(x - y; M)\)

\[ D_L^{(+)}(x - y; M) = [\varphi_L^{(+)}(x^0, x^1), \varphi_L^{(-)}(y^0, y^1)] = \langle 0|\varphi_L(x^0, x^1)\varphi_L(y^0, y^1)|0\rangle. \]  

(4.4)

Thus, we get (see the Appendix)

\[ -4\pi [\varphi_L^{(+)}(x^0, x^1), \varphi_L^{(-)}(y^0, y^1)] = \ell n[iM (x_+ - y_+ - i0)]. \]  

(4.5)

Substituting (1.3) in (1.2) we obtain

\[ \Psi_1(x^0, x^1)\Psi_1(y^0, y^1) = -\Psi_1(y^0, y^1)\Psi_1(x^0, x^1) = \]

\[ = \frac{M^2}{2\pi i} (x_+ - y_+ - i0) e^{-i(\sqrt{\pi}/2)Q_R(0)} :e^{2i\sqrt{\pi}(\varphi_L(x^0, x^1) + \varphi_L(y^0, y^1))} :. \]  

(4.6)

Thus, the operators \(\Psi_1(x^0, x^1)\) and \(\Psi_1(y^0, y^1)\) obey the anti–commutation relation. For \(x = y\) the product of the fermion fields \(\Psi_1^\dagger(x^0, x^1) = 0\) vanishes as required for fermion fields [5].

Now let us show that the fields \(\Psi_1(x^0, x^1)\) and \(\Psi_1^\dagger(x^0, y^1)\) satisfy the canonical anti–commutation relations (2.12). We get

\[ \Psi_1(x^0, x^1)\Psi_1^\dagger(x^0, y^1) = \frac{M}{2\pi} e^{-i(\sqrt{\pi}/4)Q_R(0)} :e^{2i\sqrt{\pi}\varphi_L(x^0, x^1)} :e^{+i(\sqrt{\pi}/4)Q_R(0)} \]
\[ \times :e^{-2i\sqrt{\pi}\varphi_L(x^0, y^1)} :. \]
\[ = \frac{M}{2\pi} e^{+4\pi [\varphi_L^{(+)}(x^0, x^1), \varphi_L^{(-)}(x^0, y^1)]} :e^{+2i\sqrt{\pi}[\varphi_L(x^0, x^1) - \varphi_L(x^0, y^1)]} :. \]
\[ = \frac{M}{2\pi} e^{-\ell n[-iM (x^1 - y^1 + i0)]} :e^{+2i\sqrt{\pi}[\varphi_L(x^0, x^1) - \varphi_L(x^0, y^1)]} :. \]
\[ = \frac{i}{2\pi} \frac{1}{x^1 - y^1 + i0} :e^{+2i\sqrt{\pi}[\varphi_L(x^0, x^1) - \varphi_L(x^0, y^1)]} :. \]  

(4.7)

For the product \(\Psi_1^\dagger(x^0, y^1)\Psi_1(x^0, x^1)\) we get

\[ \Psi_1^\dagger(x^0, y^1)\Psi_1(x^0, x^1) = \frac{i}{2\pi} \frac{1}{y^1 - x^1 + i0} :e^{+2i\sqrt{\pi}[\varphi_L(x^0, x^1) - \varphi_L(x^0, y^1)]} :. \]  

(4.8)

Using relations (4.7) and (4.8) we derive the canonical anti–commutation relation

\[ \{\Psi_1(x^0, x^1), \Psi_1^\dagger(x^0, y^1)\} = \frac{i}{2\pi} \left[\frac{1}{x^1 - y^1 + i0} - \frac{1}{x^1 - y^1 - i0}\right] \]
\[ \times :e^{+2i\sqrt{\pi}[\varphi_L(x^0, x^1) - \varphi_L(x^0, y^1)]} :\delta(x^1 - y^1). \]  

(4.9)

The other standard anti–commutation relations

\[ \{\Psi_1(x^0, x^1), \Psi_2(y^0, y^1)\} = 0, \]
\[ \{\Psi_2(x^0, x^1), \Psi_2(y^0, y^1)\} = 0, \]
\[ \{\Psi_2(x^0, x^1), \Psi_2^\dagger(x^0, y^1)\} = \delta(x^1 - y^1) \]  

(4.10)
can be derived in a similar way.

We would like to emphasize that the canonical anti–commutation relations (4.9) and (4.10), proportional to the $\delta$–function $\delta(x^1 - y^1)$, can be derived within the boson field representation only for a free fermion field with a canonical dimension $D_\psi = 1/2$. In the case of the massless Thirring model, where the fermion fields have a total dimension $D_\psi \neq 1/2$, the derivation of the canonical anti–commutation relations like (4.9) can be obtained in the boson representation of the massless Thirring fermion fields only after the subtracting the dynamical dimension [4].

5 Massless Thirring fermion fields in the boson field representation

The free massless Thirring fermion fields $\psi_1(x)$ and $\psi_2(x)$ in the boson field representation [20] are defined by

$$\psi_1(x) = +i\sqrt{\frac{M}{2\pi}} e^{-i(\pi/4)(b^{-1}Q_L(0) + a^{-1}Q_R(0))} :e^{i(a \varphi_L(x) + b \varphi_R(x))} :,$$

$$\psi_2(x) = -i\sqrt{\frac{M}{2\pi}} e^{i(\pi/4)(a^{-1}Q_L(0) + b^{-1}Q_R(0))} :e^{i(b \varphi_L(x) + a \varphi_R(x))} :. \quad (5.1)$$

The parameters $a$ and $b$ will be adjusted to fulfill the constraints imposed by the anti–commutativity of the Thirring fermion field operators $\psi_1(x)$ and $\psi_2(x)$ and their conjugate momenta.

For the verification of anti–commutativity of the fermion operators and the canonical anti–commutation relations (2.12) we have to consider the products of the fermion field operators $\psi_i(x)\psi_j(y)$ and $\psi_j^\dagger(x)\psi_i(y)$ for ($i, j = 1, 2$).

In terms of the fermion field operators (5.1) the product $\psi_1(x)\psi_1(y)$ is defined by

$$\psi_1(x)\psi_1(y) = -\frac{M}{2\pi} e^{-i(\pi/4)(b^{-1}Q_L(0) + a^{-1}Q_R(0))} :e^{i(a \varphi_L(x) + b \varphi_R(x))} : \times e^{-i(\pi/4)(b^{-1}Q_L(0) + a^{-1}Q_R(0))} :e^{i(a \varphi_L(y) + b \varphi_R(y))} : =$$

$$= -\frac{M}{2\pi} e^{i(\pi/2)(b^{-1}Q_L(0) + a^{-1}Q_R(0))}$$

$$\times e^{i(\pi/4)[a \varphi_L(x) + b \varphi_R(x), b^{-1}Q_L(0) + a^{-1}Q_R(0)]}$$

$$\times e^{-[a \varphi_L^{(+)}(x) + b \varphi_R^{(+)}(x), a \varphi_L^{(-)}(y) + b \varphi_R^{(-)}(y)]}$$

$$\times :e^{ia(\varphi_L(x) + \varphi_L(y))} :e^{ib(\varphi_R(x) + \varphi_R(y))} :,$$

where we have used the relations [17]

$$e^A e^B = e^{[A, B]} e^A e^B, \quad :e^A : :e^B := e^{[A^{(+)}], B^{(-)}} :e^{A + B} :. \quad (5.3)$$

The commutators $[\varphi_L^{(+)}(x), \varphi_L^{(-)}(y)]$ and $[\varphi_R^{(+)}(x), \varphi_R^{(-)}(y)]$ are defined by the Wightman functions $D_L^{(+)}(x - y; M)$ and $D_R^{(+)}(x - y; M)$ (see the Appendix)

$$D_L^{(+)}(x - y; M) = [\varphi_L^{(+)}(x), \varphi_L^{(-)}(y)] = \langle 0 | \varphi_L(x) \varphi_L(y) | 0 \rangle = -\frac{1}{4\pi} \ell n[iM(x_+ - y_+ - i0)],$$
\[ D_{R}^{(+)}(x - y; M) = [\varphi_{R}^{(+)}(x), \varphi_{R}^{(-)}(y)] = \langle 0|\varphi_{R}(x)\varphi_{R}(y)|0\rangle = -\frac{1}{4\pi}\ell n[iM(x_+ - y_+ - i0)] \] (5.4)

and

\[ [a\varphi_{L}(x) + b\varphi_{R}(x), b^{-1}Q_{L}(0) + a^{-1}Q_{R}(0)] = -i\left(\frac{a}{b} + \frac{b}{a}\right). \] (5.5)

Substituting (5.4) and (5.5) in (5.2) we get

\[ \psi_{1}(x)\psi_{1}(y) = -\frac{M}{2\pi} e^{-i\pi (a^2 + b^2)/4ab} e^{-i\pi/2(b^{-1}Q_{L}(0) + a^{-1}Q_{R}(0))} \times [iM(x_+ - y_+ - i0)] [a^2/4\pi [iM(x_+ - y_+ - i0)] b^2/4\pi \times :e^{ia(\varphi_{L}(x) + \varphi_{R}(y)) + ib(\varphi_{R}(x) + \varphi_{R}(y))}\] (5.6)

The important property of this expression is its vanishing for \( x = y, \psi_{1}^{2}(x) = 0 \), as it is required for fermion fields.

Now let us verify the anti–commutativity of the operators \( \psi_{1}(x) \) and \( \psi_{1}(y) \). For this aim we represent the r.h.s. of (5.2) as follows

\[ \psi_{1}(x)\psi_{1}(y) = e^{(\pi/4)[a\varphi_{L}(x) + b\varphi_{R}(x), b^{-1}Q_{L}(0) + a^{-1}Q_{R}(0)]} \times -[a\varphi_{L}^{(+)}(x) + b\varphi_{R}^{(+)}(x), a\varphi_{L}^{(-)}(y) + b\varphi_{R}^{(-)}(y)] \times e^{(\pi/4)[b^{-1}Q_{L}(0) + a^{-1}Q_{R}(0), a\varphi_{L}(y) + b\varphi_{R}(y)]} \times e^{-[a\varphi_{L}(y) + b\varphi_{R}(y), a\varphi_{L}(x) + b\varphi_{R}(x)]}\psi_{1}(y)\psi_{1}(x) = \]

\[ = e^{-a^2(D_{L}^{(+)}(x - y; M) - D_{L}^{(+)}(y - x; M)) - b^2(D_{R}^{(+)}(x - y; M) - D_{R}^{(+)}(y - x; M))} \times \psi_{1}(y)\psi_{1}(x). \] (5.7)

Since the differences of the Wightman functions are equal to

\[ D_{L}^{(+)}(x - y; M) - D_{L}^{(+)}(y - x; M) = \frac{i}{4}\varepsilon(x_+ - y_+), \]

\[ D_{R}^{(+)}(x - y; M) - D_{R}^{(+)}(y - x; M) = \frac{i}{4}\varepsilon(x_- - y_-), \] (5.8)

substituting (5.8) in (5.7) we obtain

\[ \psi_{1}(x)\psi_{1}(y) = e + i(a^2/4)\varepsilon(x_+ - y_+) + i(b^2/4)\varepsilon(x_- - y_-)\psi_{1}(y)\psi_{1}(x). \] (5.9)

This expression agrees fully with the result obtained by Morchio, Pierotti and Strocchi [20]. According to Morchio, Pierotti and Strocchi [20] the exponential in (5.9) can get \((-1)\) for a certain choice of the parameters \( a \) and \( b \).

Now let us show that the fields \( \psi_{1}(x^0, x^1) \) and \( \psi_{1}^{2}(x^0, y^1) \) satisfy the canonical anti–commutation relations (2.12). The product of the operators \( \psi_{1}(x^0, x^1)\psi_{1}^{2}(x^0, y^1) \) is equal to

\[ \psi_{1}(x^0, x^1)\psi_{1}^{2}(x^0, y^1) = \]
Using relations (5.13) we derive the anti–commutation relation

\[ e^{+i(a \varphi_L(x^0, x^1) + b \varphi_R(x^0, x^1))} : e^{-i(a \varphi_L(x^0, y^1) + b \varphi_R(x^0, y^1))} : = \]

\[ M \frac{e^{-i(\pi/4) (b^{-1} Q_L(0) + a^{-1} Q_R(0))}}{2\pi} e^{-i(\pi/4) (b^{-1} Q_L(0) + a^{-1} Q_R(0))} = \]

\[ = M \frac{e^{+a^2 D_L^{(+)}(x^1 - y^1; M) + b^2 D_R^{(+)}(x^1 - y^1; M)} e^{-i(\pi/4) (b^{-1} Q_L(0) + a^{-1} Q_R(0))}}{2\pi} e^{+i(a \varphi_L(x^0, x^1) + b \varphi_R(x^0, x^1))} : e^{-i(a \varphi_L(x^0, y^1) + b \varphi_R(x^0, y^1))} : . \] (5.10)

For the product \( \psi_1^\dagger(x^0, y^1) \psi_1(x^0, x^1) \) we get

\[ \psi_1^\dagger(x^0, y^1) \psi_1(x^0, x^1) = M \frac{e^{+a^2 D_L^{(+)}(y^1 - x^1; M) + b^2 D_R^{(+)}(y^1 - x^1; M)}}{2\pi} e^{+i(a \varphi_L(x^0, x^1) + b \varphi_R(x^0, x^1))} : e^{-i(a \varphi_L(x^0, y^1) + b \varphi_R(x^0, y^1))} : . \] (5.11)

Hence, the equal–time anti–commutation relation reads

\[ \{ \psi_1(x^0, x^1), \psi_1^\dagger(x^0, y^1) \} = \psi_1(x^0, x^1) \psi_1^\dagger(x^0, y^1) + \psi_1^\dagger(x^0, y^1) \psi_1(x^0, x^1) = \]

\[ = M \frac{e^{+a^2 D_L^{(+)}(x^1 - y^1; M) + b^2 D_R^{(+)}(x^1 - y^1; M)} + e^{+a^2 D_L^{(+)}(y^1 - x^1; M) + b^2 D_R^{(+)}(y^1 - x^1; M)}}{2\pi} e^{+i(a \varphi_L(x^0, x^1) + b \varphi_R(x^0, x^1))} : e^{-i(a \varphi_L(x^0, y^1) + b \varphi_R(x^0, y^1))} : . \] (5.12)

The Wightman functions give the following contribution

\[ a^2 D_L^{(+)}(x^1 - y^1; M) + b^2 D_R^{(+)}(x^1 - y^1; M) = \]

\[ = -\frac{a^2}{4\pi} \ln[iM(x^1 - y^1 - i0)] - \frac{b^2}{4\pi} \ln[-iM(x^1 - y^1 - i0)] = \]

\[ = -i \frac{a^2 - b^2}{8} \frac{a^2 + b^2}{4\pi} \ln[M(x^1 - y^1 - i0)], \]

\[ a^2 D_L^{(+)}(y^1 - x^1; M) + b^2 D_R^{(+)}(y^1 - x^1; M) = \]

\[ = -\frac{a^2}{4\pi} \ln[iM(y^1 - x^1 - i0)] - \frac{b^2}{4\pi} \ln[-iM(y^1 - x^1 - i0)] = \]

\[ = -i \frac{a^2 - b^2}{8} \frac{a^2 + b^2}{4\pi} \ln[M(y^1 - x^1 - i0)]. \] (5.13)

Using relations (5.13) we derive the anti–commutation relation

\[ \{ \psi_1(x^0, x^1), \psi_1^\dagger(x^0, y^1) \} = \psi_1(x^0, x^1) \psi_1^\dagger(x^0, y^1) + \psi_1^\dagger(x^0, y^1) \psi_1(x^0, x^1) = M \frac{e^{-i(a^2 - b^2)/8}}{2\pi} e^{-i(\pi/4) (b^{-1} Q_L(0) + a^{-1} Q_R(0))} \]

\[ \times \left[ (x^1 - y^1 - i0) - (a^2 + b^2)/4\pi + (y^1 - x^1 - i0) - (a^2 + b^2)/4\pi \right] M^{-i(a^2 + b^2)/4\pi} \]

\[ \times : e^{-i(a \varphi_L(x^0, x^1) + b \varphi_R(x^0, x^1))} : e^{-i(a \varphi_L(x^0, y^1) + b \varphi_R(x^0, y^1))} : . \] (5.14)
The r.h.s. of (5.14) is not equal to $\delta(x^1 - y^1)$ due to the distinction of the dynamical dimension of the massless Thirring fermion fields from the canonical dimension $D_{\psi} = 1/2$. In order to get the canonical anti-commutation relations we have to subtract the contribution of the dynamical dimension. For this aim we have to introduce the products
\[
\Psi_1(x^0, x^1) \Psi_1^\dagger(x^0, y^1) = (-1)^n [M(x^1 - y^1 - i0)](a^2 + b^2 - 4\pi)/4\pi \psi_1(x^0, x^1)\psi_1^\dagger(x^0, y^1),
\]
\[
\Psi_1^\dagger(x^0, y^1)\Psi_1(x^0, x^1) = (-1)^n [M(y^1 - x^1 - i0)](a^2 + b^2 - 4\pi)/4\pi \psi_1^\dagger(x^0, y^1)\psi_1(x^0, x^1),
\]
(5.15)
where $n \in \mathbb{Z}$. Following [20] and setting $a^2 - b^2 = 4\pi(2n + 1)$ we get for the fermion fields $\Psi_1(x^0, x^1)$ and $\Psi_1^\dagger(x^0, y^1)$ the canonical equal-time anti-commutation relation
\[
\{\Psi_1(x^0, x^1), \Psi_1^\dagger(x^0, y^1)\} = \delta(x^1 - y^1).
\]
(5.16)
The other standard anti-commutation relations are
\[
\{\psi_1(x^0, x^1), \psi_2(y^0, y^1)\} = 0,
\]
\[
\{\psi_2(x^0, x^1), \psi_2(y^0, y^1)\} = 0,
\]
\[
\{\Psi_2(x^0, x^1), \Psi_2^\dagger(x^0, y^1)\} = \delta(x^1 - y^1)
\]
(5.17)
and can be derived in a similar way.

For $x^1 \sim y^1$ the Wightman functions (5.13) can be rewritten as follows
\[
a^2 D_L^{(+)}(x^1 - y^1; M) + b^2 D_R^{(+)}(x^1 - y^1; M) =
\]
\[
= -\frac{a^2}{4\pi} \ell n[iM(x^1 - y^1 - i0)] - \frac{b^2}{4\pi} \ell n[-iM(x^1 - y^1 - i0)]
\]
\[
= -i\frac{a^2 - b^2}{8} - \frac{a^2 + b^2 - 4\pi}{4\pi} \ell n\left(\frac{M}{\Lambda}\right) - \ell n[M(x^1 - y^1 - i0)],
\]
\[
a^2 D_L^{(+)}(y^1 - x^1; M) + b^2 D_R^{(+)}(y^1 - x^1; M) =
\]
\[
= -\frac{a^2}{4\pi} \ell n[iM(y^1 - x^1 - i0)] - \frac{b^2}{4\pi} \ell n[-iM(y^1 - x^1 - i0)]
\]
\[
= -i\frac{a^2 - b^2}{8} - \frac{a^2 + b^2 - 4\pi}{4\pi} \ell n\left(\frac{M}{\Lambda}\right) - \ell n[M(y^1 - x^1 - i0)],
\]
(5.18)
where $\Lambda$ is the ultra-violet cut-off [4]. Using (5.18) the anti-commutator (5.12) is represented by
\[
\{\psi_1(x^0, x^1), \psi_1^\dagger(x^0, y^1)\} = \frac{1}{2\pi} \left(\frac{\Lambda}{M}\right) (a^2 + b^2 - 4\pi)/4\pi e^{-i(a^2 - b^2)/8}
\]
\[
\times \left[\frac{1}{x^1 - y^1 - i0} - \frac{1}{y^1 - x^1 - i0}\right]
\]
\[
\times e^{-i(a\varphi_L(x^0, x^1) + b\varphi_R(x^0, x^1))} - i(a\varphi_L(x^0, y^1) + b\varphi_R(x^0, y^1)), =
\]
\[
= \left(\frac{\Lambda}{M}\right) (a^2 + b^2 - 4\pi)/4\pi \delta(x^1 - y^1).
\]
(5.19)
where we have set $a^2 - b^2 = 4\pi$ [20]. Introducing the renormalization constant $Z_2$ of the wave functions of the massless Thirring fermion fields [4]
\[
Z_2 = \left(\frac{\Lambda}{M}\right) (a^2 + b^2 - 4\pi)/4\pi,
\]
(5.20)
The sum of the Wightman functions
\[ \{ \psi_1(x^0, x^1), \psi_1^+(x^0, y^1) \} = Z_2 \delta(x^1 - y^1). \]

This completes the analysis of the Thirring fermion fields in the boson field representation.

\[ d_\psi = \frac{a^2 + b^2}{8\pi} - \frac{1}{2}. \]

This agrees fully with Mandelstam’s relation (see Eq.(3.1) of Ref.[17]). Recall, that according to [4] the dynamical dimension of the massless Thirring fermion fields in the boson representation defined by (5.1) is equal to

\section{Fermion condensation in the boson field representation of the massless Thirring fermion fields}

In order to show that the Thirring fermion fields in the boson field representation (5.1) are quantized in the chirally broken phase, it is sufficient to calculate the vacuum expectation value of the fermion operator \( \bar{\psi}(x)\psi(x) \), expressed in terms of the boson fields.

In the boson field representation (5.1), the operator \( \bar{\psi}(x)\psi(x) \) reads (see the Appendix)

\begin{align*}
\bar{\psi}(x)\psi(x) &= \psi_2^+(x)\psi_1(x) + \psi_1^+(x)\psi_2(x) = -\frac{M}{2\pi}e^{i(b\varphi_L(x) + a\varphi_R(x))} \times e^{-i\pi((a + b)/4ab)[Q_L(0) + Q_R(0)] + i(a\varphi_L(x) + b\varphi_R(x))} \\
&= -\frac{M}{2\pi}e^{i\pi((a + b)/4ab)} e^{[(D^{(+) L}/2\pi) + D^{(+) R}/2\pi] + e^{-i\pi((a + b)/4ab)} e^{[(D^{(+) L}/2\pi) + D^{(+) R}/2\pi]}}
\end{align*}

For the calculation of the vacuum expectation value of the operator (6.1) we have to use the fermion vacuum \( |\Omega_T\rangle \) introduced by Morchio, Pierotti and Strocchi [20]. Using the rules for the calculation of vacuum expectation values with respect to the fermion vacuum \( |\Omega_T\rangle \), formulated by Morchio, Pierotti and Strocchi [20], we get

\begin{equation}
\langle \Omega_T | \bar{\psi}(x)\psi(x) | \Omega_T \rangle = -\frac{M}{\pi} \cos \left( \frac{(a + b)^2}{4ab} \right) e^{ab[D^{(+) L}/2\pi] + D^{(+) R}/2\pi].
\end{equation}

The sum of the Wightman functions \( D^{(+) L}/2\pi) + D^{(+) R}/2\pi]) \) is equal to [4]

\begin{equation}
D^{(+) L}/2\pi) + D^{(+) R}/2\pi] = \frac{1}{2\pi} \log \left( \frac{\Lambda}{M} \right),
\end{equation}

17
where \( \Lambda \) is the ultra–violet cut–off [4].

Substituting (6.3) in (6.2) we arrive at the fermion condensate

\[
\langle \Omega_T | \bar{\psi}(x) \psi(x) | \Omega_T \rangle = -\frac{M}{\pi} \cos \left( \frac{\pi}{4} \frac{(a + b)^2}{4ab} \right) \left( \frac{\Lambda}{M} \right)^{ab/2\pi}.
\] (6.4)

For the calculation of the vacuum expectation value of the operator \( \bar{\psi}(x) \psi(x) \) with respect to the vacuum constructed in [2,3] it is convenient to rewrite the expression (6.1) as follows

\[
\bar{\psi}(x) \psi(x) = -\frac{M}{2\pi} e^{i\pi (a + b)^2/4ab} e^{i\pi ((a + b)/2ab) Q(0)} \phi(x) : e^{i\pi ((a + b)/2ab) Q(0)}.
\] (6.5)

The vacuum expectation value of the operator \( \bar{\psi}(x) \psi(x) \) is defined by

\[
\langle 0 | \bar{\psi}(x) \psi(x) | 0 \rangle = -\frac{M}{2\pi} e^{i\pi (a + b)^2/4ab} e^{i\pi ((a + b)/2ab) Q(0)} \phi(x) : 0 \rangle e^{i\pi (a + b)/2ab} Q(0) | 0 \rangle.
\] (6.6)

Using the properties of the vacuum wave function |0\rangle investigated in [2,3] we get

\[
\langle 0 | \bar{\psi}(x) \psi(x) | 0 \rangle = -\frac{M}{\pi} \cos \left( \frac{\pi}{4} \frac{(a + b)^2}{4ab} \right) \left( \frac{\Lambda}{M} \right)^{ab/2\pi} e^{-\pi^2 (a + b)^2/16a^2b^2},
\] (6.7)

where we have applied (6.3).

The obtained results (6.4) and (6.7) testify the equivalence of the solution of the massless Thirring model with fermion fields constructed from the free massless boson fields to the massless Thirring model with fermion fields quantized in the chirally broken phase.

### 7 Constant of motion of the massless Thirring model in the boson field representation

In the boson field representation the fermion operator products defining the constant of motion of the massless Thirring model read (see the Appendix)

\[
\psi_1^\dagger(x) \psi_2(x) \bar{\psi}_1(x) \psi_1(x) + \psi_2^\dagger(x) \bar{\psi}_1(x) \psi_1^\dagger(x) \psi_2(x) =
\]

\[= M^2 \lambda - i (a \varphi_L(x) + b \varphi_R(x)) : e^{-i (\pi/4) (b^{-1}Q_L(0) + a^{-1}Q_R(0)) + i (\pi/4) (a^{-1}Q_L(0) + b^{-1}Q_R(0))}.
\]

\times \lambda + i (b \varphi_L(x) + a \varphi_R(x)).
\]
\[ \times e^{-i (b \varphi_L(x) + a \varphi_R(x))} ; \quad e^{-i (\pi/4) (a^{-1} Q_L(0) + b^{-1} Q_R(0))} \]
\[ \times e^{-i (\pi/4) (b^{-1} Q_L(0) + a^{-1} Q_R(0))} ; \quad e + i (a \varphi_L(x) + b \varphi_R(x)) ; \]
\[ + \frac{M^2}{4\pi^2} e^{-i (b \varphi_L(x) + a \varphi_R(x))} ; \quad e^{-i (\pi/4) (a^{-1} Q_L(0) + b^{-1} Q_R(0))} \]
\[ \times e^{-i (\pi/4) (b^{-1} Q_L(0) + a^{-1} Q_R(0))} ; \quad e + i (a \varphi_L(x) + b \varphi_R(x)) ; \]
\[ \times e + i (a \varphi_L(x) + b \varphi_R(x)) ; \quad e^{-i (\pi/4) (a^{-1} Q_L(0) + b^{-1} Q_R(0))} \]
\[ \times e + i (b \varphi_L(x) + a \varphi_R(x)) ; . \quad (7.1) \]

One can easily show that the r.h.s. of (7.1) is equal to
\[ \psi_1^\dagger(x) \psi_2(x) \psi_3^\dagger(x) \psi_1(x) + \psi_2^\dagger(x) \psi_1(x) \psi_1^\dagger(x) \psi_2(x) = \frac{M^2}{2\pi^2} e^{(a^2 + b^2)} D^{(+)}(0; M). \quad (7.2) \]

Using the definition of the Wightman function \( D^{(+)}(0; M) \) given by (6.3) we get
\[ \psi_1^\dagger(x) \psi_2(x) \psi_3^\dagger(x) \psi_1(x) + \psi_2^\dagger(x) \psi_1(x) \psi_1^\dagger(x) \psi_2(x) = \frac{M^2}{2\pi^2} \left( \frac{\Lambda}{M} \right)^{(a^2 + b^2)/2\pi}. \quad (7.3) \]

This defines the constant \( C \) in terms of the normalization scale \( M \) and the ultra–violet cut–off \( \Lambda \)
\[ C = \frac{M^2}{\pi^2} \left( \frac{\Lambda}{M} \right)^{(a^2 + b^2)/2\pi}. \quad (7.4) \]

Thus, we have shown that the boson field representation, developed by Morchio, Pierotti and Strocchi [20] and supplemented by our recipe [2,3] for the removal of the infrared divergences in the quantum field theory of the free massless (pseudo)scalar fields defined in 1+1-dimensional space–time, confirms completely our assertion concerning the existence of the constant of motion for the evolution of the fermion fields described by the massless Thirring model [1].

**8 Conclusion**

We have analysed the boson field representation of the massless fermion fields defined in 1+1-dimensional space–time. We have shown that the formulation of the quantum field theory of the free massless (pseudo)scalar field, free of infrared divergences in 1+1-dimensional space–time, has given the basis for the self–consistent description of the fermion field operators in terms of the free massless (pseudo)scalar fields. In our version of the approach developed by Morchio, Pierotti and Strocchi [20] the fermion field operators in the boson field representation do not depend on the infrared cut–off \( \mu \), which should be taken finally in the limit \( \mu \to 0 \). We would like to emphasize that such a limit would make all the results ill–defined. The most important disagreement, which could be produced by the limit \( \mu \to 0 \), would be the violation of the constant of motion revealed in [1].

As has been shown in [1] the product of the massless Thirring fermion field operators \( \tilde{\psi}(x) \psi(x) \) and \( \tilde{\psi}(x) i \gamma^5 \psi(x) \) is conserved in the evolution of the fermion fields obeying the equations of motion for the massless Thirring model. The constant \( C \) should not vanish in the limit \( \mu \to 0 \).
We have shown that the Thirring fermion field operators defined in the boson field representation by the boson fields, described by a quantum field theory free of infrared divergences [2,3], satisfy the constant of motion with the constant $C$ independent of the infrared cut–off.

The non–vanishing value of the fermion condensate obtained within the boson field representation agrees fully with our assertions concerning (i) the existence of the chirally broken phase in the massless Thirring model [1,4] and (ii) the equivalence of the fermion fields, represented by the massless boson fields, to the Thirring fermion fields quantized in the chirally broken phase.

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**Appendix. Normal ordering of exponential operators depending on free massless boson fields**

The normal ordering of the exponential operators depending on the free massless boson fields defined in the quantum field theory without infrared divergences, where the zero–mode configurations are removed by the constraint on the external sources

$$\int d^2 x \, J(x) = \tilde{J}(0) = 0,$$

(A.1)

can not be understood and performed naively. In the case of the naive performance of the normal ordering of the exponential operators in the massless boson field representation one should encounter the problem of the appearance of the infrared cut–off $\mu$ in the final expressions. This should contradict the results of the calculation of the vacuum expectation values of these operators. Therefore, the required normal ordering of the exponential operators in the massless boson field representation should be carried out in the way agreeing with the calculation of vacuum expectation values, which do not depend on the infrared cut–off $\mu$ [2].

Let us describe the required procedure of the normal ordering of the exponential operators. We suggest to start with the operator $O^{(1)}[\varphi]$ defined by

$$O^{(1)}[\varphi] = e^{i\beta \varphi(x)}.$$  

(A.2)

The direct application of Wick’s theorem [21] allows to represent the operator (A.2) in the following normal–ordered form

$$e^{i\beta \varphi(x)} = _{\text{\normal order}} e^{i\beta \varphi(x)} : e^{i\frac{1}{2} \beta^2 i\Delta(0)},$$  

(A.3)

where the symbol $:\ldots:\$ stands for the normal ordering and the Green function $i\Delta(0)$ will be defined later.

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7see also Glimm and Jaffe, p.107 of Ref.[11] and Eq.(2.6) of Ref.[22].
The vacuum expectation value of the both sides of the relation (A.3) is equal to
\[ \langle 0 | e^{i\beta \varphi(x)} | 0 \rangle = \langle 0 | : e^{i\beta \varphi(x)} : | 0 \rangle e^{\frac{1}{2} \beta^2 i\Delta(0)}, \] (A.4)
Since the vacuum expectation value of the normal–ordered exponential is a unity, the relation (A.4) reads
\[ \langle 0 | e^{i\beta \varphi(x)} | 0 \rangle = e^{\frac{1}{2} \beta^2 i\Delta(0)}, \] (A.5)
The l.h.s. of this relation can be calculated in terms of the generating functional of Green functions \( Z[J] \) (1.12) which does not depend on the infrared cut–off \( \mu \). We get \[ \langle 0 | e^{i\beta \varphi(x)} | 0 \rangle = \langle 0 | e^{i\beta \varphi(0)} | 0 \rangle = \exp \left\{ \beta \frac{\delta}{\delta J(0)} \right\} Z[J] \bigg|_{J=0} = e^{\frac{1}{2} \beta^2 i\Delta(0; M)}. \] (A.6)
The Green function \( i\Delta(0; M) \) is defined by \[ i\Delta(0; M) = -\frac{1}{4\pi} \log \left( \frac{\Lambda^2}{M^2} \right). \] (A.7)
Substituting (A.6) in (A.5) we obtain that
\[ i\Delta(0) = i\Delta(0; M) = -\frac{1}{4\pi} \log \left( \frac{\Lambda^2}{M^2} \right). \] (A.8)
Hence, the normal–ordered form of the operator \( e^{i\beta \varphi(x)} \) is defined by
\[ e^{i\beta \varphi(x)} := : e^{i\beta \varphi(x)} : e^{\frac{1}{2} \beta^2 i\Delta(0; M)}. \] (A.9)
The inverse expression reads
\[ : e^{i\beta \varphi(x)} := e^{-\frac{1}{2} \beta^2 i\Delta(0; M)} e^{i\beta \varphi(x)}. \] (A.10)
Now let us consider the operator \( O^{(2)}[\varphi] \) given by
\[ O^{(2)}[\varphi] = e^{i\alpha \varphi(x)} e^{i\beta \varphi(y)}. \] (A.11)
Using (A.9) we get
\[ e^{i\alpha \varphi(x)} e^{i\beta \varphi(y)} = : e^{i\alpha \varphi(x)} : e^{i\beta \varphi(y)} : e^{\frac{1}{2} (\alpha^2 + \beta^2) i\Delta(0; M)}. \] (A.12)
The product of the normal–ordered exponentials is equal to
\[ : e^{i\alpha \varphi(x)} : e^{i\beta \varphi(y)} := e^{-\frac{1}{2} (\alpha^2 + \beta^2) i\Delta(0; M)} e^{i\alpha \varphi(x)} e^{i\beta \varphi(y)}. \] (A.13)
Taking time–ordered expressions and calculating the vacuum expectation values of the both sides we get
\[ \langle 0 | T \left( : e^{i\alpha \varphi(x)} : e^{i\beta \varphi(y)} : \right) | 0 \rangle = e^{-\frac{1}{2} (\alpha^2 + \beta^2) i\Delta(0; M)} \langle 0 | T \left( e^{i\alpha \varphi(x)} e^{i\beta \varphi(y)} \right) | 0 \rangle. \] (A.14)
The vacuum expectation value of the time–ordered operator in the r.h.s. of (A.14) can be calculated in terms of $Z[J]$. The result reads [2]

$$
\langle 0 | \mathcal{T} \left( e^{i\alpha \varphi(x)} e^{i\beta \varphi(y)} \right) | 0 \rangle = \exp \left\{ -i\alpha \frac{\delta}{\delta J(x)} - i\beta \frac{\delta}{\delta J(y)} \right\} Z[J] \big|_{J=0} =
$$

$$
e^{\frac{1}{2} \left( \alpha^2 + \beta^2 \right) i\Delta(0; M)} e^{+\alpha \beta i\Delta(x-y; M)}. \quad (A.15)
$$

Substituting (A.15) in (A.14) we get

$$
\langle 0 | \mathcal{T} \left( : e^{i\alpha \varphi(x)} :: e^{i\beta \varphi(y)} : \right) | 0 \rangle = e^{+\alpha \beta i\Delta(x-y; M)}. \quad (A.16)
$$

This yields

$$
\langle 0 | : e^{i\alpha \varphi(x)} :: e^{i\beta \varphi(y)} : | 0 \rangle = e^{+\alpha \beta D(+) (x-y; M)} \cdot e^{i\alpha \varphi(x) + i\beta \varphi(y)} . \quad (A.17)
$$

Hence, taking into account that the vacuum expectation value of the normal–ordered exponential is equal to unity, the operator form of the relation (A.17) reads

$$
: e^{i\alpha \varphi(x)} :: e^{i\beta \varphi(y)} : e^{+\alpha \beta D(+) (x-y; M)} : e^{i\alpha \varphi(x) + i\beta \varphi(y)} : . \quad (A.18)
$$

This relation can be rewritten in the form

$$
: e^{i\alpha \varphi(x)} :: e^{i\beta \varphi(y)} := e^{+\alpha \beta [\varphi^{(+)}(x), \varphi^{(-)}(y)]} : e^{i\alpha \varphi(x) + i\beta \varphi(y)} : . \quad (A.19)
$$

where the commutator $[\varphi^{(+)}(x), \varphi^{(-)}(y)]$ is defined by

$$
[\varphi^{(+)}(x), \varphi^{(-)}(y)] = D(+) (x-y; M). \quad (A.19)
$$

Thus, we have formulated the procedure for the normal ordering exponential operators in the boson field representation of fermion fields with the boson fields described by the quantum field theory without infrared divergences [2]. Following this procedure one can calculate any product of the normal–ordered exponential operators depending on the free massless boson fields described by the quantum field theory without infrared divergences.

We would like to emphasize that according to the normal ordering procedure, formulated above, no infrared cut–off $\mu$ can appear in the products of fermion field operators. All expressions should depend only on the finite arbitrary scale $M$. 

22
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