Isoperimetric Problems on Time Scales with Nabla Derivatives

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Abstract: We prove a necessary optimality condition for isoperimetric problems under nabla-differentiable curves. As a consequence, the recent results of Caputo (2008), that put together seemingly dissimilar optimal control problems in economics and physics, are extended to a generic time scale. We end with an illustrative example of the application of our main result to a dynamic optimization problem from economics.

Keywords: Control, isoperimetric problems, nabla derivatives, time scales.

1. INTRODUCTION

The calculus on time scales is a recent field of mathematics, introduced by Aulbach and Hilger (1988), which unifies the theory of difference equations with the theory of differential equations. It has found applications in several fields that require simultaneous modeling of discrete and continuous data, in particular in control theory (Bartosiewicz et al., 2007; Mozyrska and Bartosiewicz, 2007; Bartosiewicz and Pawluszewicz, 2008) and the calculus of variations (Bohner, 2004; Atici et al., 2006; Ferreira and Torres, 2007, 2008a; Malinowska and Torres, 2009).

In Section 2 we present a short introduction to time scales and nabla derivatives. Section 3 is the main core of the paper: we prove a necessary optimality condition for the isoperimetric problem on time scales with nabla derivatives. Differently from Ferreira and Torres (2008b), where the minimizing curve is assumed not to be an extremal of the delta-integral constraint, here both normal (Theorem 1) and abnormal extremals (Theorem 2) are considered. As a result of Theorem 2, we extend the recent result of Caputo (2008) to a generic time scale (Proposition 1). Finally, in Section 4 we illustrate the application of Theorem 1 to an isoperimetric problem on time scales motivated by Atici et al. (2006).
2. PRELIMINARIES

For an introduction to time scales we refer the reader to the comprehensive books by Bohner and Peterson (2001) and Bohner and Peterson (2003). Here we just recall the results and notation needed in what follows.

By $\mathbb{T}$ we denote a time scale, i.e., a nonempty closed subset of $\mathbb{R}$. The backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ if $t \neq \inf \mathbb{T}$, and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$. Analogously, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ if $t \neq \sup \mathbb{T}$, and $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$.

A point $t \in \mathbb{T}$ is said to be right-dense, right-scattered, left-dense, and left-scattered if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, and $\rho(t) < t$, respectively. Let $\mathbb{T}_k$ be the set defined in the following way: if $m$ is a right-scattered minimum of $\mathbb{T}$, then $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$; if not, $\mathbb{T}_k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be nabla differentiable at $t \in \mathbb{T}_k$ if there exists a number $f^\nabla(t)$ (called the nabla derivative of $f$ at $t$) such that for every $\epsilon > 0$ there exists some neighborhood $U$ of $t$ at $\mathbb{T}$ with

$$|f(\rho(t)) - f(s) - f^\nabla(t)(\rho(t) - s)| \leq \epsilon|\rho(t) - s|$$

for all $s \in U$. If $f$ is nabla differentiable at all $t \in \mathbb{T}_k$, then we say that $f$ is nabla differentiable.

When $\mathbb{T} = \mathbb{R}$, $f$ is nabla differentiable at $t$ if and only if is differentiable at $t$. If $\mathbb{T} = \mathbb{Z}$, then $f$ is always nabla differentiable and

$$f^\nabla(t) = \frac{f(t) - f(\rho(t))}{t - \rho(t)} = f(t) - f(t - 1).$$

Throughout the text we abbreviate $f \circ \rho$ by $f^\rho$ and $[a, b] \cap \mathbb{T}$ by $[a, b]$. We use $C^1([a, b], \mathbb{R})$ to denote the set

$$C^1([a, b], \mathbb{R}) := \{y : [a, b] \to \mathbb{R} \mid y^\nabla \text{ exists and is continuous on } [a, b]\}.$$

We say that $F : \mathbb{T} \to \mathbb{R}$ is a nabla antiderivative of $f$ if

$$F^\nabla(t) = f(t), \quad \forall t \in \mathbb{T}_k.$$

We then define the nabla integral of $f$ by

$$\int_a^b f(t) \nabla t := F(b) - F(a).$$

The following two formulas of nabla integration by parts hold:

$$\int_a^b f^\rho(t) g^\nabla(t) \nabla t = \left[(fg)(t)\right]_a^b - \int_a^b f^\nabla(t) g(t) \nabla t.$$
and
\[
\int_a^b f(t)g^\nabla(t) \nabla t = \left[ (fg)(t) \right]_{t=a}^{t=b} - \int_a^b f^\nabla(t)g^\rho(t) \nabla t.
\]

We end our brief review of the calculus on time scales via nabla derivatives by recalling a fundamental lemma of the calculus of variations recently proved in Martins and Torres (2008).

**Lemma 1.** Let \( f(\cdot) \in C([a, b], \mathbb{R}) \). If \( \int_a^b f(t)\eta^\nabla(t) \nabla t = 0 \) for every curve \( \eta(\cdot) \in C^1([a, b], \mathbb{R}) \) satisfying \( \eta(a) = \eta(b) = 0 \), then \( f(t) = c, \ c \in \mathbb{R}, \) for all \( t \in [a, b] \).

### 3. MAIN RESULTS

We study the isoperimetric problem on time scales with a nabla-integral constraint both for normal and abnormal extremizers. The problem consists of minimizing or maximizing

\[
I[y(\cdot)] = \int_a^b f(t, y^\rho(t), y^\nabla(t)) \nabla t
\]

in the class of functions \( y(\cdot) \in C^1([a, b], \mathbb{R}) \) satisfying the boundary conditions

\[
y(a) = \alpha \quad \text{and} \quad y(b) = \beta
\]

and the nabla-integral constraint

\[
J[y(\cdot)] = \int_a^b g(t, y^\rho(t), y^\nabla(t)) \nabla t = \Lambda,
\]

where \( \alpha, \beta, \) and \( \Lambda \) are given real numbers. We assume that functions \( (t, x, v) \rightarrow f(t, x, v) \) and \( (t, x, v) \rightarrow g(t, x, v) \) possess continuous partial derivatives with respect to the second and third variables, and we denote them by \( f_\alpha, f_\nu, g_\alpha, \) and \( g_\nu \).

**Definition 1.** We say that \( y(\cdot) \in C^1([a, b], \mathbb{R}) \) is a (weak) local minimizer (respectively local maximizer) for the isoperimetric problem in equations 1–3 if there exists \( \delta > 0 \) such that \( I[y(\cdot)] \leq I[\hat{y}(\cdot)] \) (respectively \( I[y(\cdot)] \geq I[\hat{y}(\cdot)] \)) for all \( \hat{y}(\cdot) \in C^1([a, b], \mathbb{R}) \) satisfying the boundary conditions in equation 2, the isoperimetric constraint equation 3, and \( \| \hat{y}^\rho(\cdot) - y^\rho(\cdot) \| + \| \hat{y}^\nabla(\cdot) - y^\nabla(\cdot) \| < \delta, \) where \( \| \hat{y}(\cdot) \| := \sup_{t \in [a, b]} |\hat{y}(t)|. \)

**Definition 2.** We say that \( y(\cdot) \in C^1([a, b], \mathbb{R}) \) is an extremal for \( J[\cdot] \) if

\[
g_\nu (t, y^\rho(t), y^\nabla(t)) = \int_a^t g_\alpha (\tau, y^\rho(\tau), y^\nabla(\tau)) \nabla \tau = c
\]
for some constant $c$ and for all $t \in [a, b]_\kappa$. An extremizer (i.e., a local minimizer or a local maximizer) to the problem in equations 1–3 that is not an extremal for $J[\cdot]$ is said to be a normal extremizer; otherwise (i.e., if it is an extremal for $J[\cdot]$), the extremizer is said to be abnormal.

**Theorem 1. (Necessary optimality condition for normal extremizers of the problem in equations 1–3).** Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$ with $a < b$, and $y(\cdot) \in C^1([a, b], \mathbb{R})$. Suppose that $y(\cdot)$ gives a local minimum or a local maximum to the functional $I[\cdot]$ subject to the boundary conditions $y(a) = \alpha$ and $y(b) = \beta$ and the integral constraint $J[y(\cdot)] = \Lambda$, where $\alpha, \beta,$ and $\Lambda$ are prescribed real values. If $y(\cdot)$ is not an extremal for $J[\cdot]$, then there exists a real $\lambda$ such that

$$F^\nabla_v(t, y^\rho(t), y^\nabla(t)) - F^\nabla_x(t, y^\rho(t), y^\nabla(t)) = 0$$

for all $t \in [a, b]_\kappa$, where $F = f - \lambda g$.

**Proof.** Consider a variation of $y(\cdot)$, say $\hat{y}(\cdot) = y(\cdot) + \epsilon_1 \eta_1(\cdot) + \epsilon_2 \eta_2(\cdot)$, where $\eta_i(\cdot)$ is a curve in $C^1([a, b], \mathbb{R})$ satisfying $\eta_i(a) = \eta_i(b) = 0$, $i = 1, 2$. Define the real functions

$$\hat{I}(\epsilon_1, \epsilon_2) := I[\hat{y}(\cdot)] \quad \text{and} \quad \hat{J}(\epsilon_1, \epsilon_2) := J[\hat{y}(\cdot)] - \Lambda.$$

Integration by parts gives

$$\left. \frac{\partial \hat{J}}{\partial \epsilon_2} \right|_{(0, 0)} = \int_a^b \left( \eta_2^\rho g_x(t, y^\rho, y^\nabla) + \eta_2^\nabla g_v(t, y^\rho, y^\nabla) \right) \nabla t$$

$$= \int_a^b \left[ \eta_2^\rho \left( \int_a^t g_x(\tau, y^\rho, y^\nabla) \nabla \tau \right)^\nabla + \eta_2^\nabla g_v(t, y^\rho, y^\nabla) \right] \nabla t$$

$$= \int_a^b \left( - \int_a^t g_x(\tau, y^\rho, y^\nabla) \nabla \tau + g_v(t, y^\rho, y^\nabla) \right) \eta_2^\nabla \nabla t$$

since $\eta_2(a) = \eta_2(b) = 0$. By Lemma 1, there exists a curve $\eta_2(\cdot)$ such that $\left. \frac{\partial \hat{J}}{\partial \epsilon_2} \right|_{(0, 0)} \neq 0$.

Since $\hat{J}(0, 0) = 0$, by the implicit function theorem there exists a function $\epsilon_2(\cdot)$ defined in a neighborhood of zero, such that $\hat{J}(\epsilon_1, \epsilon_2(\epsilon_1)) = 0$, i.e., we may choose a subset of the variation curves $\hat{y}(\cdot)$ satisfying the isoperimetric constraint.

In summary, $(0, 0)$ is an extremal of $\hat{I}$ subject to the constraint $\hat{J} = 0$ and $\text{grad}(\hat{J}) = 0$, i.e., by the Lagrange multiplier rule, there exists some $\lambda$ such that $\text{grad}(\hat{I} - \lambda \hat{J}) = 0$. Analogously,

$$\left. \frac{\partial \hat{I}}{\partial \epsilon_1} \right|_{(0, 0)} = \int_a^b \left( - \int_a^t f_x(\tau, y^\rho, y^\nabla) \nabla \tau + f_v(t, y^\rho, y^\nabla) \right) \eta_1^\nabla \nabla t$$
and

\[
\frac{\partial \hat{J}}{\partial \epsilon_1} \bigg|_{(0,0)} = \int_a^b \left( - \int_a^t g_x(\tau, y^\rho, y^\nabla) \nabla \tau + g_v(t, y^\rho, y^\nabla) \right) \eta_1^\nabla \nabla t.
\]

Therefore,

\[
\int_a^b \left[ - \int_a^t f_x(\tau, y^\rho, y^\nabla) \nabla \tau + f_v(t, y^\rho, y^\nabla) \right.
\]

\[
- \lambda \left(- \int_a^t g_x(\tau, y^\rho, y^\nabla) \nabla \tau + g_v(t, y^\rho, y^\nabla) \right) \eta_1^\nabla \nabla t = 0. \tag{4}
\]

Since equation 4 holds for any curve \( \eta_1(\cdot) \), again by Lemma 1,

\[
- \int_a^t f_x(\tau, y^\rho, y^\nabla) \nabla \tau + f_v(t, y^\rho, y^\nabla)
\]

\[
- \lambda \left(- \int_a^t g_x(\tau, y^\rho, y^\nabla) \nabla \tau + g_v(t, y^\rho, y^\nabla) \right) = \text{const}. \tag{5}
\]

Applying the nabla derivative to both sides of equation 5, we get

\[- f_x(t, y^\rho, y^\nabla) + f_v(t, y^\rho, y^\nabla) - \lambda \left(- g_x(t, y^\rho, y^\nabla) + g_v(t, y^\rho, y^\nabla) \right) = 0,
\]

i.e.,

\[F_v(t, y^\rho, y^\nabla) - F_x(t, y^\rho, y^\nabla) = 0,
\]

where \( F = f - \lambda g \).

Theorem 2. (Necessary optimality condition for normal and abnormal extremizers of the problem in equations 1–3). Let \( \mathbb{T} \) be a time scale, \( a, b \in \mathbb{T} \) with \( a < b \). If \( y(\cdot) \in \mathcal{C}^1([a, b], \mathbb{R}) \) is a local minimizer or maximizer for \( I[\cdot] \) subject to the boundary conditions \( y(a) = \alpha \) and \( y(b) = \beta \) and to the integral constraint \( J[y(\cdot)] = \Lambda \), then there exist two constants \( \lambda_0 \) and \( \lambda \), not both zero, such that

\[K_v^\nabla(t, y^\rho(t), y^\nabla(t)) - K_x(t, y^\rho(t), y^\nabla(t)) = 0
\]

for all \( t \in [a, b]_\kappa \), where \( K = \lambda_0 f - \lambda g \).

Remark 1. If \( y(\cdot) \) is a normal extremizer, then one can choose \( \lambda_0 = 1 \) in Theorem 2, thus obtaining Theorem 1. For abnormal extremizers, Theorem 2 holds with \( \lambda_0 = 0 \). The condition \( (\lambda_0, \lambda) \neq 0 \) guarantees that Theorem 2 is a useful necessary condition.
Proof. Following the proof of Theorem 1, since \( (0,0) \) is an extremal of \( \hat{I} \) subject to the constraint \( \hat{J} = 0 \), the abnormal Lagrange multiplier rule (cf., e.g., van Brunt (2004)) asserts the existence of reals \( \lambda_0 \) and \( \lambda \), not both zero, such that \( \text{grad}(\lambda_0 I - \lambda \hat{J}) = 0 \). Therefore,

\[
\lambda_0 \frac{\partial \hat{I}}{\partial \epsilon_1} \bigg|_{(0,0)} - \lambda \frac{\partial \hat{J}}{\partial \epsilon_1} \bigg|_{(0,0)} = 0
\]

\[
\Leftrightarrow \int_a^b \left[ \lambda_0 \left( -\int_a^t f_\xi(\tau, y^\rho, y^\gamma) \nabla \tau + f_\upsilon(t, y^\rho, y^\gamma) \right) \\
- \lambda \left( -\int_a^t g_\xi(\tau, y^\rho, y^\gamma) \nabla \tau + g_\upsilon(t, y^\rho, y^\gamma) \right) \right] \eta_1^\gamma \nabla t = 0.
\]

From the arbitrariness of \( \eta_1(\cdot) \), we conclude by Lemma 1 that

\[
\lambda_0 \left( -\int_a^t f_\xi(\tau, y^\rho, y^\gamma) \nabla \tau + f_\upsilon(t, y^\rho, y^\gamma) \right) \\
- \lambda \left( -\int_a^t g_\xi(\tau, y^\rho, y^\gamma) \nabla \tau + g_\upsilon(t, y^\rho, y^\gamma) \right) = \text{const}.
\]

The desired condition follows by nabla differentiation.

In the recent paper (Caputo, 2008) the classical isoperimetric problem (i.e., the problem in equations 1–3 with \( \mathbb{T} = \mathbb{R} \)) was studied for a very particular functional in equation 1 and a very particular constraint equation 3 that often occurs in economics and physics. For that particular class of problems, the extremal curve is shown to be a constant. Here we remark that the main result of Caputo (2008) is still valid on a generic time scale \( \mathbb{T} \).

**Proposition 1.** Let \( \mathbb{T} \) be a time scale, \( a, b \in \mathbb{T} \) with \( a < b \), \( f : \mathbb{R}^2 \to \mathbb{R} \) be a \( C^2 \) function and \( \xi \) a real parameter. Assume that \( (\xi, \zeta) \to f(\xi, \zeta) \) satisfies the conditions \( f_\xi(y^\rho(t), \zeta) \neq 0 \), in some interval \([c, d]\) \( \subseteq [a, b] \), for all \( \zeta \) and for all admissible function \( y(\cdot) \), and \( f_{\xi\xi}(y^\rho(t), \zeta) \leq 0 \), for all \( t \in [a, b] \) and for all \( \zeta \) over an open interval containing all the admissible values of \( y^\rho(\cdot) \). Then, there exists a unique solution \( y(\cdot) \) for the isoperimetric problem

\[
I[y(\cdot)] = \int_a^b f(y^\rho(t), \zeta) \nabla t \to \text{extr}
\]

\[
J[y(\cdot)] = \int_a^b y^\rho(t) \nabla t = \Lambda
\]

\[
y(a) = \alpha, \quad y(b) = \beta.
\]

Moreover, \( y^\rho(t) = \text{constant}, \ t \in [a, b]_\mathbb{T} \).
Proof. The proof follows from Theorem 2 and is the same, *mutatis mutandis*, as that of Proposition 1 of Caputo (2008).

Remark 2. Proposition 1 asserts that if \( y(\cdot) \) is a solution of the problem in equations 6–8, then \( y(t) \) is constant in \([a, \rho(b)] \cap \mathbb{T}\), say \( y(t) = c \). If \( \mathbb{T} = \mathbb{R} \), then by the isoperimetric constraint, one has \( \gamma^\rho(t) = y(t) = \Lambda / (b - a) \). The constant \( c = \Lambda / (b - a) \) is precisely that obtained in Caputo (2008).

Remark 3. Let \( \mathbb{T} = \mathbb{Z} \). Then, \( y(t) \) is constant in \([a, \ldots, b - 1]\). By the isoperimetric constraint,

\[
\int_a^b y^\rho(t) \nabla t = \sum_{t=a+1}^b y^\rho(t) = \sum_{t=a}^{b-1} y(t) = (b - a)c = \Lambda.
\]

Similar to \( \mathbb{T} = \mathbb{R} \), one has \( c = \Lambda / (b - a) \).

4. AN EXAMPLE

We exemplify the use of Theorem 1 with an economic model. Assume we wish to maximize the functional

\[
I[y(\cdot)] = \int_a^b u(y^\rho(t)) l^t \nabla t,
\]

where \( y^\rho(t) \) is the consumption during period \( t \in [a, b] \), \( u(\cdot) \) is the one-period utility, \( l \in (1/2, 1) \) is the discount factor, and \( I[\cdot] \) is the lifetime utility. For a detailed explanation of the economic model we refer the reader to Atici et al. (2006). We consider the maximization problem subject to the constraint

\[
\int_a^b t y^t(t) \nabla t = \text{const}.
\]

Let \( g(t, x, v) = tv \). Since

\[
g_v(t, y^\rho, y^\psi) - \int_a^t g_\tau(\tau, y^\rho, y^\psi) \nabla \tau = t,
\]

there are no abnormal extremals for the problem. The augmented Lagrange function is

\[
F(t, x, v, \lambda) = u(x) l^t - \lambda tv.
\]

By Theorem 1, \( F_x = F_v^\psi \), and so

\[
u'(x) l^t = - (\lambda t)^\psi = -\lambda.
\]
It follows that \( u(x) = \lambda l^{-x} / \ln l + C \). If \( u(\cdot) \) is invertible, the extremal is given by

\[
y^\varphi(t) = u^{-1} \left( \lambda l^{-t} / \ln l + C \right).
\]

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