Higher order corrections to the large scale matter power spectrum in the presence of massive neutrinos

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Abstract. We present the first systematic derivation of the one-loop correction to the large scale matter power spectrum in a mixed cold+hot dark matter cosmology with subdominant massive neutrino hot dark matter. Starting with the equations of motion for the density and velocity fields, we derive perturbative solutions to these quantities and construct recursion relations for the interaction kernels, noting and justifying all approximations along the way. We find interaction kernels similar to those for a cold dark matter-only universe, but with additional dependences on the neutrino energy density fraction \( f_\nu \) and the linear growth functions of the incoming wavevectors. Compared with the \( f_\nu = 0 \) case, the one-loop corrected matter power spectrum for a mixed dark matter cosmology exhibits a decrease in small scale power exceeding the canonical \( \sim 8 f_\nu \) suppression predicted by linear theory, a feature also seen in multi-component \( N \)-body simulations.

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1. Introduction

Standard big bang theory predicts a background of relic neutrinos, permeating the universe at an average of 112 neutrinos per cubic cm per neutrino flavour. This enormous abundance means that even a sub-eV to eV neutrino mass \( m_\nu \) will render these otherwise elusive particles a significant dark matter component, \( \Omega_\nu h^2 = \sum m_\nu / (93 \text{ eV}) \). Experimentally, a lower limit of \( \sum m_\nu \gtrsim 0.05 \text{ eV} \) on the sum of the neutrino masses has been established by neutrino oscillation experiments (e.g., [1]). On the other hand, tritium \( \beta^- \)-decay end-point spectrum measurements point to an upper limit of \( \sum m_\nu \lesssim 6 \text{ eV} \) (e.g., [1]). The corresponding energy densities, \( 0.001 \lesssim \Omega_\nu \lesssim 0.12 \), make for an interesting prediction to be tested against cosmological observations.

Importantly, neutrino dark matter is of the hot variety because of their inherent thermal velocity, which prevents them from clustering gravitationally on scales smaller than their free-streaming length. This free-streaming effect then feedbacks into the evolution of the dominant cold dark matter (CDM) component via the gravitational source term, leading to a suppressed rate in the formation of structures on small length scales that is in principle manifest in the power spectrum of the large scale structure distribution [2–6]. At present, observations of the cosmic microwave background anisotropies, galaxy clustering, and type Ia supernovae together engender a conservative upper limit on the contribution of neutrino dark matter, or equivalently, on the neutrino masses, of \( \sum m_\nu \lesssim 1 \text{ eV} \) within the \( \Lambda \text{CDM} \) framework. See [7, 8] for recent reviews.

Many future cosmological probes will continue to improve on this limit, or perhaps even detect neutrino dark matter (e.g., [7,9–17]). To this end, it is important to note that many of these probes, particularly high redshift galaxy surveys and weak gravitational lensing, will derive most of their constraining power at wavenumbers \( k \sim 0.1 \rightarrow 1 \text{ h Mpc}^{-1} \), where the evolution of density perturbations has become weakly nonlinear. Coincidentally, these are also the scales at which neutrino free-streaming is expected to produce the largest effect on the large scale matter power spectrum. Thus, it is crucial that we understand the nonlinear evolution of density perturbations at these scales, in order to maximise our gain in detecting/constraining neutrino dark matter.

A number of recent works have attempted to model the effects of massive neutrinos on the matter power spectrum at nonlinear scales using various different techniques. These include multi-component \( N \)-body simulations [18] and semi-analytic halo models [11,19,20]. However, with the exception of \( N \)-body simulations, these methods all contain elements that require prior calibration against simulation results, and not surprisingly, all calibrations to date have been performed in a \( \Lambda \text{CDM} \) setting. This renders these nonlinear models at present not completely satisfactory for use with cosmologies containing massive neutrinos. Recalibration against simulations in the appropriate cosmology will, however, enhance/restore our confidence in these methods.

Recently, Saito et al. [21] proposed an alternative method based on higher order cosmological perturbation theory. Since its inception in the 1980s [22–25], higher order cosmological perturbation theory has found numerous applications ranging from
computation of the weakly nonlinear power spectrum and gravity-induced bispectrum, to the exploration of nonlinear galaxy bias. See reference [26] for a review. In the perturbative approach, one envisages nonlinear evolution as the outcome of interactions of a collection of linear waves (perturbations). The equations of motion of the system define the interaction kernels.

Saito et al.’s recipe is simple: assume the neutrino density perturbations remain linear at all times, and apply nonlinear modelling only to the CDM+baryon component. This basic scheme has also been adopted in some earlier nonlinear models including massive neutrinos [11]. For $\sum m_\nu \lesssim 0.6$ eV, multi-component N-body simulations have confirmed that a linear approximation for the neutrino density contrast is sound [18].

For the CDM+baryon component, Saito et al. calculated the one-loop correction to the CDM+baryon auto-correlation power spectrum, using the correctly computed linear waves (perturbations), but interaction kernels that have been developed for a CDM-only universe. This amounts to ignoring additional mode-coupling effects between the linear growth functions at different wavenumbers. Although, as we shall show, this approximation does lead to a considerable simplification in the final form of the nonlinear power spectrum, it is nonetheless reminiscent of the mismatch between the cosmology to which we apply and that on which we calibrate some of the semi-analytic methods discussed above, and therefore calls for closer scrutiny.

In this paper we present a systematic derivation of the one-loop correction to the matter power spectrum in the presence of massive neutrinos from first principles. As in [11, 21], we assume linearity for the neutrino component. For CDM+baryons, however, we begin with the relevant equations of motion, and solve them (approximately) perturbatively in an Einstein–de Sitter $\Omega_m = 1$ background at high redshifts ($z \gtrsim 0.5$). From these solutions we derive interaction kernels that capture also the physics of mode-coupling between linear growth functions at different wavenumbers. With these kernels we compute the correct nonlinear power spectrum.

In section 2 we give the relevant equations of motion. We discuss briefly the linear order solutions in section 3, before formulating our higher order perturbative approach in section 4. In section 5 we outline our scheme to obtain approximate solutions to the equations of motion for higher order perturbations, while in section 6 we derive recursion relations for the interaction kernels and thus generalise our approximate solutions to arbitrary orders in perturbative expansion. Section 7 deals with the calculation of the one-loop correction to the matter power spectrum, which we evaluate numerically for realistic cosmological models and discuss in detail in section 8. We conclude in section 9.

2. Equations of motion

We begin with the standard set of equations of motion for the density perturbations $\delta(x, \tau)$ and the peculiar velocity $u(x, \tau)$ of the CDM+baryon component (e.g., [26]),

$$\frac{\partial \delta(x, \tau)}{\partial \tau} + \nabla \cdot \{ [1 + \delta(x, \tau)] u(x, \tau) \} = 0,$$
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\[ \frac{\partial \mathbf{u}(\mathbf{x}, \tau)}{\partial \tau} + \mathcal{H}(\tau) \mathbf{u}(\mathbf{x}, \tau) + [\mathbf{u}(\mathbf{x}, \tau) \cdot \nabla] \mathbf{u}(\mathbf{x}, \tau) + \nabla \Phi(\mathbf{x}, \tau) = 0, \]  

where \( \tau \) is the conformal time, \( \mathbf{x} \) the comoving coordinates, \( \mathcal{H} \equiv d \ln a/d\tau = Ha \) the conformal expansion rate, and the stress tensor has been set explicitly to zero. The Poisson equation

\[ \nabla^2 \Phi(\mathbf{x}, \tau) = \frac{3}{2} \mathcal{H}^2(\tau) \Omega_m(\tau)[(1 - f_\nu)\delta(\mathbf{x}, \tau) + f_\nu \delta^\nu(\mathbf{x}, \tau)] \]  

relates the Newtonian gravitational potential \( \Phi(\mathbf{x}, \tau) \) to the density perturbations. Perturbations from both CDM+baryons \( \delta(\mathbf{x}, \tau) \) and neutrinos \( \delta^\nu(\mathbf{x}, \tau) \) contribute to this gravitational source term, although in the latter case the contribution is suppressed by the neutrino density fraction \( f_\nu \equiv \Omega_\nu(\tau)/\Omega_m(\tau) = \Omega_{\nu,0}/\Omega_{m,0} \). This fraction is constant in time once the neutrinos have become nonrelativistic.

It is common to rewrite the equations of motion in Fourier space. Defining the Fourier transform

\[ \tilde{\phi}(\mathbf{k}, \tau) = \int \frac{d^3x}{(2\pi)^3} \exp(-i\mathbf{k} \cdot \mathbf{x}) \phi(\mathbf{x}, \tau) \]  

for some field \( \phi(\mathbf{x}, \tau) \), and the divergence of the velocity field

\[ \theta(\mathbf{x}, \tau) \equiv \nabla \cdot \mathbf{u}(\mathbf{x}, \tau), \]  

equations (2.1) and (2.2) can be equivalently expressed as

\[ \frac{\partial \tilde{\delta}(\mathbf{k}, \tau)}{\partial \tau} + \tilde{\theta}(\mathbf{k}, \tau) = -\int d^3q_1d^3q_2 \delta_D(\mathbf{k} - \mathbf{q}_12)\alpha(q_1, q_2)\tilde{\theta}(q_1, \tau)\tilde{\delta}(q_2, \tau), \]

\[ \frac{\partial \tilde{\theta}(\mathbf{k}, \tau)}{\partial \tau} = \mathcal{H}(\tau)\tilde{\theta}(\mathbf{k}, \tau) + \frac{3}{2} \mathcal{H}^2(\tau) \Omega_m(\tau)[(1 - f_\nu)\tilde{\delta}(\mathbf{k}, \tau) + f_\nu \tilde{\delta}^\nu(\mathbf{k}, \tau)] = -\int d^3q_1d^3q_2 \delta_D(\mathbf{k} - \mathbf{q}_12)\beta(q_1, q_2)\tilde{\theta}(q_1, \tau)\tilde{\theta}(q_2, \tau), \]

where

\[ \alpha(q_1, q_2) \equiv \frac{q_12 \cdot q_1}{q_1^2}, \quad \beta(q_1, q_2) \equiv \frac{q_12 \cdot q_2}{2q_1^2q_2^2}. \]  

Here, we have defined \( q \equiv |\mathbf{q}| \), and \( q_{i-j} \equiv q_i + \cdots + q_j \). The function \( \delta_D \) denotes the Dirac delta function. As usual, we assume the vorticity of the velocity field \( \mathbf{w}(\mathbf{x}, \tau) \equiv \nabla \times \mathbf{u}(\mathbf{x}, \tau) \) to be negligible at all times.

For the neutrino component, a full treatment requires that we track the evolution of the neutrino phase space density \( f^\nu(\mathbf{x}, \mathbf{p}, \tau) \) by solving the Vlasov equation (e.g., [26]),

\[ \frac{\partial f^\nu}{\partial \tau} + \frac{\mathbf{p}}{m_\nu c} \cdot \nabla f^\nu - am_\nu \nabla \Phi \cdot \frac{\partial f^\nu}{\partial \mathbf{p}} = 0, \]  

where \( m_\nu \) is the neutrino mass, and we have assumed the neutrinos to be nonrelativistic. Integrating over the neutrino momentum \( \mathbf{p} \),

\[ \int d^3\mathbf{p} f^\nu(\mathbf{x}, \mathbf{p}, \tau) \equiv \bar{\rho}^\nu(\tau)[1 + \delta^\nu(\mathbf{x}, \tau)], \]

yields the neutrino density contrast \( \delta^\nu(\mathbf{x}, \tau) \).
3. Linear theory

Linearising the equations of motion (2.5), i.e., dropping all terms on the r.h.s. of the equal sign, the growing mode solutions for the CDM+baryon density contrast and peculiar velocity at some wavevector $k$ can be written as

$$\delta_1(k, \tau) = D_1(k, \tau) \delta(k, \tau_0),$$
$$\theta_1(k, \tau) = -\mathcal{H}(\tau) f(k, \tau) D_1(k, \tau) \delta(k, \tau_0),$$

where $\tau_0$ denotes some initial time well in the matter-domination epoch, $D_1(k, \tau)$ is the linear growth function, its logarithmic derivative

$$f(k, \tau) \equiv \frac{\partial \ln D_1(k, \tau)}{\partial \ln a} = \frac{1}{H} \frac{\partial \ln D_1(k, \tau)}{\partial \tau},$$

and we have dropped the tildes on $\tilde{\delta}_1(k, \tau)$ and $\tilde{\theta}_1(k, \tau)$ for convenience. A similar expression,

$$\delta_\nu_1(k, \tau) = D_\nu_1(k, \tau) \delta_\nu(k, \tau_0),$$

describes the growth of the neutrino density perturbations in the linear regime.

The inclusion of massive neutrinos in the matter content of the universe introduces a new length scale to the problem, the neutrino free-streaming scale $k_{\text{FS}}$ (e.g., [27]),

$$k_{\text{FS}}(\tau) \equiv \sqrt{\frac{3\Omega_m(\tau)\mathcal{H}^2(\tau)}{2c_\nu^2}} \simeq 1.5 \sqrt{a(\tau)\Omega_m,0} \left(\frac{m_\nu}{\text{eV}}\right) h \text{ Mpc}^{-1},$$

where

$$c_\nu \equiv \frac{T_\nu(\tau)}{m_\nu} \sqrt{\frac{3\zeta(3)}{2\ln(2)}} \simeq \frac{81}{a(\tau)} \left(\frac{\text{eV}}{m_\nu}\right) \text{ km s}^{-1}.$$

At wavenumbers $k \ll k_{\text{FS}}$, neutrinos cluster gravitationally and behave essentially like CDM. In the other limit $k \gg k_{\text{FS}}$, the neutrinos’ inherent thermal velocity $c_\nu$ prevents efficient infall into gravitational potential wells, thereby suppressing the neutrino density perturbations relatively to their CDM+baryon counterparts.

This qualitative picture is generally true for any cosmology. However, if we restrict our considerations to an Einstein–de Sitter universe, then using the relations $a \propto \tau^2$ and $\mathcal{H}(\tau) = 2/\tau$ it is easy to show that

$$D_1(k, \tau) \propto a, \quad f(k, \tau) = 1, \quad k \ll k_{\text{FS}},$$
$$D_1(k, \tau) \propto a^{1-\mu}, \quad f(k, \tau) = 1 - \mu, \quad k \gg k_{\text{FS}},$$

with

$$\mu = \frac{5}{4} - \sqrt{\frac{1 + 24(1 - f_\nu)}{4}} \simeq \frac{3}{5} f_\nu.$$

The $k \gg k_{\text{FS}}$ solution is obtained by setting $\delta_\nu(k, \tau)$ explicitly to zero in the equations of motion. For the neutrino component, the expressions

$$D_1'(k, \tau) \simeq D_1(k, \tau) \frac{\delta(k, \tau_0)}{\delta_\nu(k, \tau_0)}, \quad k \ll k_{\text{FS}},$$
$$D_1'(k, \tau) \simeq D_1(k, \tau) \frac{k_{\text{FS}}^2(\tau)(1 - f_\nu)}{k^2 - k_{\text{FS}}^2(\tau)f_\nu}, \quad k \gg k_{\text{FS}}$$

(3.8)
have been shown to be asymptotic solutions to equation (2.7) [27]. For intermediate
\( k \) values, the linear growth functions for both the CDM+baryon and the neutrino
components must be calculated numerically with a Boltzmann code such as CAMB [28],
which also gives the neutrino sector a full general relativistic treatment [29].

3.1. Fitting formulae

We shall need estimates of the linear growth function and particularly its logarithmic
derivative later in the analysis. For this we resort to fitting formulae.

For the CDM+baryon component, we have [30]

\[
D_1(k, \tau) \simeq [1 + r^c(k, \tau)]^{\mu/c} D_1^{1-\mu}(\tau),
\]

and hence

\[
f(k, \tau) \simeq 1 - \frac{\mu}{1 + r^c(k, \tau)},
\]

where \( c \simeq 0.7 \), and

\[
r(k, \tau) \equiv \frac{D_1(\tau)}{1 + y_{FS}(k)},
\]

\[
y_{FS}(k) = 17.2 f_\nu (1 + 0.488 f_\nu^{-7/6}) (p N_\nu / f_\nu)^2,
\]

\[
p = \left( \frac{k}{\text{Mpc}^{-1}} \right) \Theta_{2.7}^2 (\Omega_{m,0} h^2)^{-1}.
\]

Here, \( D_1(\tau) \) is the growth function in the absence of massive neutrinos, normalised such
that \( D_1(\tau) = a/a_{eq} \) in an Einstein–de Sitter universe. The quantity \( N_\nu \) is the number
of massive neutrinos (assuming equal masses) which we generally take to be three, and
\( \Theta_{2.7} \) is defined through \( T_{\text{CMB}} = 2.7 \Theta_{2.7} \) K. The claimed accuracy of the formula (3.9)
is 1 to 2 % [30].

For the neutrino component, we find

\[
D_1'(k, \tau) \simeq D_1(k, \tau) \delta(k, \tau_0) \left[ \frac{k_{FS}^2(\tau)(1 - f_\nu)}{[k + k_{FS}(\tau)]^2 - k_{FS}^2(\tau)f_\nu} \right],
\]

to be a good interpolation between the \( k \ll k_{FS} \) and \( k \gg k_{FS} \) limits, accurate to better
than 5 %.

3.2. Linear power spectrum

The linear matter power spectrum is defined as

\[
P^L(k, \tau) \equiv \langle \delta^T(k, \tau) \delta^T(k', \tau) \rangle,
\]

where \( \delta^T(k, \tau) \equiv f_{cb} \delta_1(k, \tau) + f_\nu \delta_2(k, \tau) \), with \( f_{cb} = (\Omega_c + \Omega_b)/\Omega_m \), counts both the
CDM+baryon and the neutrino density contrast. In terms of the linear growth functions,

\[
P^L(k, \tau) = f_{cb}^2 P^L_{cb}(k, \tau) + 2 f_{cb} f_\nu P^L_{c\nu}(k, \tau) + f_\nu^2 P^L_\nu(k, \tau),
\]
with

$$P_{cb}^L(k, \tau) = [D_1(k, \tau)T_{cb}(k, \tau_0)]^2 P_I(k),$$
$$P_{\nu}^L(k, \tau) = [D_{\nu 1}(k, \tau)T_{\nu}(k, \tau_0)]^2 P_I(k),$$
$$P_{cb\nu}^L(k, \tau) = D_1(k, \tau)D_{\nu 1}(k, \tau)T_{cb}(k, \tau_0)T_{\nu}(k, \tau_0)P_I(k),$$

(3.15)

where $T_i(k, \tau_0)$ are the linear transfer functions mapping the initial fluctuations (from, e.g., inflation) through the epochs of horizon crossing and matter–radiation equality to time $\tau_0$, i.e., $\delta(k, \tau_0) = T(k, \tau_0)\delta_i(k)$. We have also assumed in equation (3.15) adiabatic initial conditions so that $P_{cb}^I(k) = P_{\nu}^I(k) = P_{cb\nu}^I(k) \equiv P_I(k)$. Note that the power spectrum defined in this manner contributes $4\pi k^3 P(k)$ per logarithmic wavenumber to the variance, in contrast to, e.g., the default output of CAMB, which contributes $k^3 P(k)/(2\pi^2)$ between $\ln k$ and $\ln (k + d\ln k)$.

4. Beyond linear theory

We wish to find a higher order perturbative description for the CDM+baryon density contrast and peculiar velocity. To do so it is convenient to define a new time variable

$$s \equiv \ln a(\tau) = \mathcal{H}(\tau)d\tau,$$

(4.1)

and the vectors

$$\Psi(k, s) \equiv \begin{bmatrix} \delta(k, s) \\ -\frac{1}{\mathcal{H}(s)}\theta(k, s) \end{bmatrix}, \quad \Psi^\nu(k, s) \equiv \begin{bmatrix} \delta^\nu(k, s) \\ 0 \end{bmatrix}. $$

(4.2)

The equations of motion (2.5) can then be written in a more compact form,

$$\partial_s \Psi_a(k, s) + K_{ab} \Psi_b(k, s) + N_{ab} \Psi^\nu_b(k, s) =$$
$$\int d^3q_1d^3q_2 \delta D(k - q_{12})\gamma_{abc}(q_1, q_2)\Psi_b(q_1, s)\Psi_c(q_2, s),$$

(4.3)

where $a, b, c = 1, 2$, and repeated indices imply summation.

In the homogeneous part of equation (4.3), the matrix $K$ is given by

$$K = \begin{bmatrix} 0 & -1 \\ -\frac{3}{2}(1 - f_\nu) & \frac{3}{2} \end{bmatrix},$$

(4.4)

where we have assumed explicitly an Einstein–de Sitter universe. The inhomogeneous part is specified by the matrix,

$$N = \begin{bmatrix} 0 & 0 \\ -\frac{3}{2}f_\nu & 0 \end{bmatrix},$$

(4.5)

and the tensor $\gamma_{abc}(q_1, q_2)$ is zero except for $\gamma_{121} = \alpha(q_1, q_2)$ and $\gamma_{222} = \beta(q_1, q_2)$.

We seek a perturbative solution of the form

$$\Psi(k, s) = \sum_{n=1}^{\infty} \psi^{(n)}(k, s).$$

(4.6)

For the neutrinos, we assume only the linear solution is nonzero, i.e.,

$$\Psi^\nu(k, s) = \begin{bmatrix} \delta^\nu(k, s) \\ 0 \end{bmatrix}. $$

(4.7)
Then the equation of motion for the $n$th order solution, where $n \geq 2$, is
\[
\partial_a \psi_a^{(n)}(k, s) + K_{ab} \psi_b^{(n)}(k, s) = B_a^{(n)}(k, s),
\]
with
\[
B_a^{(n)}(k, s) = \int d^3q_1 d^3q_2 \delta_D(k - q_{12}) \gamma_{abc}(q_1, q_2) \\
\times \sum_{m=1}^{n-1} \psi_{b}^{(n-m)}(q_1, s) \psi_{c}^{(m)}(q_2, s).
\]
Observe how the term proportional to $\Psi''(k, s)$ has disappeared for $n > 1$ because of the assumption that neutrino density perturbations remain linear at all times.

5. Approximate solutions

Equation (4.8) can be solved by first defining a transformation matrix $U$ via
\[
\tilde{K} \equiv U^{-1} K U = \text{Diag}(\kappa_1, \kappa_2),
\]
so that in the diagonal basis, the equation of motion becomes
\[
\partial_a \tilde{\psi}_a^{(n)}(k, s) = -\tilde{K}_{ab} \tilde{\psi}_b^{(n)}(k, s) + \tilde{B}_a^{(n)}(k, s),
\]
where $\tilde{\psi}^{(n)} = U^{-1} \psi^{(n)}$, and $\tilde{B}^{(n)} = U^{-1} B^{(n)}$. The formal solution to equation (5.2) is simple,
\[
\tilde{\psi}_a^{(n)}(k, s) = e^{-\int_{s_0}^s \kappa_a ds'} \tilde{\psi}_a^{(n)}(k, s_0) + \int_{s_0}^s e^{-\int_{s_0}^{s'} \kappa_a ds''} \tilde{B}_a^{(n)}(k, s') ds',
\]
where we have used $\tilde{\psi}_a^{(n)}(k, s_0) = 0$ in the second line, i.e., the perturbations at the initial time $s_0$ are completely described by linear theory. Within our formulation of the problem the solution (5.3) is exact. To gain further ground, however, we must make some approximations, as we show order by order below.

(i) The $n = 1$ solution is by definition
\[
\psi_1^{(1)}(k, s) = \int d^3q \delta_D(k - q) Q_1^{(1)}(q, s) \delta_1(q, s),
\]
where $Q_1^{(1)}(q, s) = 1$, and $Q_2^{(1)}(q, s) = f(q, s)$.

(ii) For $n = 2$, we have
\[
\tilde{B}_a^{(2)}(k, s) = \int d^3q_1 d^3q_2 \delta_D(k - q_{12}) D_1(q_1, s) D_1(q_2, s) \\
\times U_{ab}^{-1} \gamma_{bcd}(q_1, q_2) Q_c^{(1)}(q_1, s) Q_d^{(1)}(q_2, s) \delta(q_1, s_0) \delta(q_2, s_0).
\]

The solution in the diagonal basis is then
\[
\tilde{\psi}_a^{(2)}(k, s) = \int_{s_0}^s ds' \int d^3q_1 d^3q_2 e^{-\int_{s_0}^{s'} \kappa_a ds''} e^{[h(q_1, s') + h(q_2, s')]} ds' \\
\times \delta_D(k - q_{12}) U_{ab}^{-1} \gamma_{bcd}(q_1, q_2) \\
\times Q_c^{(1)}(q_1, s') Q_d^{(1)}(q_2, s') \delta(q_1, s_0) \delta(q_2, s_0),
\]
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where we have defined \( h(q, s) \equiv \ln D_1(q, s)/\ln a(s) \).

The integration over \( s' \) can be performed using integration by parts, i.e.,

\[
\int_{s_0}^{s} ds' Q^{(1)}_c(q_1, s') Q^{(1)}_d(q_2, s') e^{-\int_{s_0}^{s} \kappa_a ds'' e^{[h(q_1, s') + h(q_2, s')] s'}} =
\]

\[
\frac{Q^{(1)}_c(q_1, s') Q^{(1)}_d(q_2, s')}{\kappa_a + f(q_1, s') + f(q_2, s')} e^{-\int_{s_0}^{s} \kappa_a ds'' e^{[h(q_1, s') + h(q_2, s')] s'}}
- \int_{s_0}^{s} ds' Q^{(1)}_c(q_1, s') Q^{(1)}_d(q_2, s') e^{-\int_{s_0}^{s} \kappa_a ds'' e^{[h(q_1, s') + h(q_2, s')] s'}}
\times \frac{1}{Q^{(1)}_c(q_1, s') Q^{(1)}_d(q_2, s')} \frac{\partial}{\partial s'} \left[ Q^{(1)}_c(q_1, s') Q^{(1)}_d(q_2, s') \right],
\tag{5.7}
\]

and we have used the relation \( h(q, s) + (\partial h/\partial s)s = f(q, s) \). Compared with the original integrand on the l.h.s., the integrand on the r.h.s. of equation (5.7) is suppressed by the time derivative of \( f(q, s) \), whose size we estimate using the fitting formulae (3.9) and (3.10) to be

\[
\frac{\partial f(q, s)}{\partial s} \simeq \frac{c \mu r c}{(1 + r^2)^2},
\tag{5.8}
\]

and \( \partial f/\partial s \geq 0 \) for all \( s \), meaning that the linear growth rate cannot decrease in an Einstein–de Sitter cosmology. For a given \( q \) value, \( \partial f/\partial s \) peaks at \( r = 1 \), or \( a = [1 + y_{FS}(q)] a_{eq} \), when the neutrinos transit from a non-clustering to a clustering dark matter, thereby enhancing the (linear) gravitational source term. Here, \( \partial f/\partial s \rvert_{r=1} \simeq c \mu/4 \simeq 0.1 f_c \). Since the integrand is positive definite, we can conclude based on this estimate that the absolute fractional error incurred by dropping the second term of the integral (5.7) is bounded from above by \( \sim 0.1 f_c \) (worst case: \( \kappa_a \simeq -1, c = 1, d = 2, q_1 = q_2 \)); the actual error is likely to be much less. Hence we shall proceed by keeping only the first term of equation (5.7).

Of what remains of the time integral (5.7), we retain as usual only the non-decaying part, leading to

\[
\dot{\psi}^{(2)}_a(k, s) = \int d^3 q_1 d^3 q_2 \frac{D_1(q_1, s) D_1(q_2, s) f(q_1, s)}{\kappa_a + f(q_1, s) + f(q_2, s)}
\times \delta_D(k - q_{12}) U_{ab}^{-1} \gamma_{cde}(q_1, q_2)
Q^{(1)}_c(q_1, s) Q^{(1)}_d(q_2, s) \delta(q_1, s_0) \delta(q_2, s_0).
\tag{5.9}
\]

Rotating back to the original basis gives us

\[
\psi^{(2)}_a(k, s) = \int d^3 q_1 d^3 q_2 \delta_D(k - q_{12}) Q^{(2)}_a(q_1, q_2; s) \delta_1(q_1, s) \delta_1(q_2, s),
\tag{5.10}
\]

where

\[
Q^{(2)}_a(q_1, q_2; s) \equiv \frac{U_{ab} U_{bc}^{-1} \gamma_{cde}(q_1, q_2) Q^{(1)}_d(q_1, s) Q^{(1)}_c(q_2, s)}{\kappa_b + f(q_1, s) + f(q_2, s)}
\tag{5.11}
\]

is the interaction kernel.
(iii) For $n = 3$, the same procedure and approximation scheme yield

\[
\psi_{a}^{(3)}(k, s) = \int d^3q_1 d^3q_2 d^3q_3 \delta_D(k - q_{123}) \\
\times Q_{a}^{(3)}(q_1, q_2, q_3; s) \delta_1(q_1, s) \delta_1(q_2, s) \delta_1(q_3, s),
\]

where the interaction kernel $Q_{a}^{(3)}(q_1, q_2, q_3; s)$ is again estimated at a negligible $q, s$

\[
\begin{align*}
Q_{a}^{(3)}(q_1, q_2, q_3; s) &\equiv \frac{U_{ab} U_{bc}^{-1}}{\kappa_b + f(q_1, s) + f(q_2, s) + f(q_3, s)} \\
&\times \left[ \gamma_{cde}(q_1, q_2) Q_{d}^{(2)}(q_1, q_2; s) Q_{e}^{(1)}(q_3, s) \\
&\quad + \gamma_{cde}(q_1, q_2, q_3; s) Q_{d}^{(1)}(q_1, s) Q_{e}^{(2)}(q_2, q_3; s) \right].
\end{align*}
\]

The maximum absolute fractional error incurred by neglecting the time derivatives of $f(q, s)$ is again estimated at a negligible $\sim 0.1 f_\nu$ for the $n = 3$ solution.

6. General $n$th order solution and recursion relations

Generalising to the $n$th order, we obtain the approximate solution

\[
\psi_{a}^{(n)}(k, \tau) = \int d^3q_1 \cdots d^3q_n \delta_D(k - q_{1\cdots n}) \\
\times Q_{a}^{(n)}(q_1, \cdots, q_n; \tau) \delta_1(q_1, \tau) \cdots \delta_1(q_n, \tau),
\]

where the interaction kernel $Q_{a}^{(n)}(q_1, \cdots, q_n; \tau)$ can be constructed from the recursion relation

\[
Q_{a}^{(n)}(q_1, \cdots, q_n; \tau) = \sigma_{ab}^{(n)}(q_1, \cdots, q_n; \tau) \\
\times \sum_{m=1}^{n-1} \gamma_{cde}(q_{1\cdots m}, q_{m+1\cdots n}) Q_{c}^{(m)}(q_1, \cdots, q_m; \tau) \\
\times Q_{d}^{(n-m)}(q_{m+1}, \cdots, q_n; \tau),
\]

with

\[
\sigma_{ab}^{(n)}(q_1, \cdots, q_n) \equiv \frac{U_{ab} U_{cb}^{-1}}{\kappa_c + \omega^{(n)}(q_1, \cdots, q_n; \tau)}.
\]

\[
\delta \equiv \frac{1}{N^{(n)}} \left[ \frac{2\omega^{(n)} + 1}{3(1 - f_\nu)} \right].
\]

and

\[
N^{(n)}(q_1, \cdots, q_n; \tau) \equiv (2\omega^{(n)} + 3)(\omega^{(n)} - 1) + 3 f_\nu,
\]

\[
\omega^{(n)}(q_1, \cdots, q_n; \tau) \equiv \sum_{m=1}^{n} f(q_m, \tau) = \sum_{m=1}^{n} \frac{\partial \ln D_1(q_m; \tau)}{\partial \ln a(\tau)}.
\]

Equations (6.1) to (6.4) should be compared with, e.g., equations (41) to (44), (83) and (84) of reference [26] for a CDM-only universe.

In the limit $f_\nu \to 0$, we have $\omega^{(n)}(q_1, \cdots, q_n; \tau) \to n$, so that the $\sigma^{(n)}$ matrix (6.3) depends only on $n$ such as in equation (84) of [26]. This renders the interaction
kernel (6.2) into the standard expressions $F_n(q_1, \ldots, q_n)$ and $G_n(q_1, \ldots, q_n)$ for a CDM-only universe given in equations (43) and (44) of [26], i.e.,

$$Q_1^{(n)}(q_1, \ldots, q_n; \tau) \to F_n(q_1, \ldots, q_n),$$
$$Q_2^{(n)}(q_1, \ldots, q_n; \tau) \to G_n(q_1, \ldots, q_n).$$

(6.5)

We caution at this point that the terms proportional to $f_\nu$ in equations (6.3) and (6.4) count only the explicit dependences on $f_\nu$; the function $\omega^{(n)}$ also depends implicitly on $f_\nu$ through the linear growth function.

6.1. Symmetrised kernels

In practice we use the symmetrised kernels $\bar{Q}^{(n)}(q_1, \ldots, q_n; \tau)$, constructed by summing $Q^{(n)}(q_1, \ldots, q_n; \tau)$ over all permutations of the momenta $q_1, \ldots, q_n$ and then dividing by $n!$. For future reference, we give here explicit expressions for $\bar{Q}^{(2)}(q_1, q_2; \tau)$ and $\bar{Q}^{(3)}(q_1, q_2, q_3; \tau)$, cast in a form as close to the standard CDM-only ones as possible:

$$\bar{Q}_1^{(2)}(q_1, q_2; \tau) = \frac{5}{7} A_1 + \frac{1}{2} \frac{q_1 \cdot q_2}{q_1 q_2} \left[ A_2 \frac{q_1}{q_2} + A_3 \frac{q_2}{q_1} \right] + \frac{2}{7} A_4 \frac{(q_1 \cdot q_2)^2}{q_1^2 q_2^2},$$
$$\bar{Q}_2^{(2)}(q_1, q_2; \tau) = \frac{3}{7} C_1 + \frac{1}{2} \frac{q_1 \cdot q_2}{q_1 q_2} \left[ C_2 \frac{q_1}{q_2} + C_3 \frac{q_2}{q_1} \right] + \frac{4}{7} C_4 \frac{(q_1 \cdot q_2)^2}{q_1^2 q_2^2},$$

(6.6)

with

$$A_1 = \frac{7}{10} \sigma^{(2)}_{11}(q_1, q_2) \left[ f(q_1) + f(q_2) \right],$$
$$A_2 = f(q_2) \left[ \sigma^{(2)}_{11}(q_1, q_2) + \sigma^{(2)}_{12}(q_1, q_2) f(q_1) \right],$$
$$A_3 = f(q_1) \left[ \sigma^{(2)}_{11}(q_1, q_2) + \sigma^{(2)}_{12}(q_1, q_2) f(q_2) \right],$$
$$A_4 = \frac{7}{2} \sigma^{(2)}_{12}(q_1, q_2) f(q_1) f(q_2),$$
$$C_1 = \frac{7}{6} \sigma^{(2)}_{21}(q_1, q_2) \left[ f(q_1) + f(q_2) \right],$$
$$C_2 = f(q_2) \left[ \sigma^{(2)}_{21}(q_1, q_2) + \sigma^{(2)}_{22}(q_1, q_2) f(q_1) \right],$$
$$C_3 = f(q_1) \left[ \sigma^{(2)}_{21}(q_1, q_2) + \sigma^{(2)}_{22}(q_1, q_2) f(q_2) \right],$$
$$C_4 = \frac{7}{4} \sigma^{(2)}_{22}(q_1, q_2) f(q_1) f(q_2),$$

(6.7)

where we have dropped the $\tau$ label in $\sigma^{(2)}(q_1, q_2; \tau)$ and $f(q, \tau)$ for convenience. The factors $A_{1,2,3,4}$ and $C_{1,2,3,4}$ have been defined so that they tend to unity as $f_\nu \to 0$.

The symmetrised form of $Q^{(3)}$ can be written as

$$\bar{Q}_1^{(3)}(q_1, q_2, q_3; \tau) = \frac{1}{3} \left\{ \frac{7}{18} X_{123} \left( \frac{q_{123} \cdot q_3}{q_3^2} \right) \bar{Q}_1^{(2)}(q_1, q_2; \tau) + \left[ \frac{7}{18} Y_{123} \left( \frac{q_{123} \cdot q_{12}}{q_{12}^2} \right) + \frac{1}{9} Z_{123} \left( \frac{q_{123}^2 (q_{12} \cdot q_3)}{q_{12}^2 q_3^2} \right) \right] \bar{Q}_2^{(2)}(q_1, q_2; \tau) + \text{cyclic permutations} \right\},$$

(6.8)
7. Power spectra

We are interested in the total matter power spectrum $P(k, \tau)$. As in the case for the linear power spectrum (3.13), it can be expressed as a weighted sum of the CDM+baryon and the neutrino density contrast auto- and cross-correlation power spectra,

$$P(k, \tau) = f_c^2 P_{cb}(k, \tau) + 2 f_c f_{\nu} P_{cb\nu}(k, \tau) + f_{\nu}^2 P_{\nu}(k, \tau),$$

where

$$P_{cb}(k, \tau) = P_{cb}^L(k, \tau) + [P_{cb}^{(22)}(k, \tau) + 2 P_{cb}^{(13)}(k, \tau)],$$

$$P_{\nu}(k, \tau) = P_{\nu}^L(k, \tau),$$

$$P_{cb\nu}(k, \tau) = P_{cb\nu}^L(k, \tau) + P_{cb\nu}^{(13)}(k, \tau),$$

(7.3)

taking the neutrino density perturbations to be linear at all times.

The one-loop terms for the CDM+baryon auto-correlation are

$$P_{cb}^{(22)}(k, \tau) = 2 \int d^3 q \ 2 \left[ \bar{Q}_1^{(2)}(k - q, q; \tau) \right]^2 P_{cb}^L(k - q, \tau) P_{cb}^L(q, \tau),$$

$$P_{cb}^{(13)}(k, \tau) = 3 \ P_{cb}^L(k, \tau) \int d^3 q \ \bar{Q}_1^{(3)}(k, q, -q; \tau) P_{cb}^L(q, \tau),$$

(7.4a)

and

$$P_{cb\nu}^{(13)}(k, \tau) = 3 \ P_{cb\nu}^L(k, \tau) \int d^3 q \ \bar{Q}_1^{(3)}(k, q, -q; \tau) P_{cb\nu}^L(q, \tau),$$

(7.4b)

where the prefactors “2” and “3” arise from summing all possible groupings of $\langle \delta^I(k) \delta^I(k') \rangle$ pairs to give $\langle \delta^I(k) \delta^I(k') \delta^I(k'') \rangle$ (e.g., [31]). The cross-correlation between CDM+baryons and neutrinos also contains a one-loop correction,

$$P_{cb\nu}^{(13)}(k, \tau) = 3 \ P_{cb\nu}^L(k, \tau) \int d^3 q \ \bar{Q}_1^{(3)}(k, q, -q; \tau) P_{cb\nu}^L(q, \tau),$$

(7.5)

which is identical to the $P_{cb\nu}^{(13)}(k, \tau)$ correction to the CDM+baryon auto-correlation except for the prefactor $P_{cb\nu}^L(k, \tau)$ instead of $P_{cb}^L(k, \tau)$. This term is missing from Saito et al.’s formulation, which assumed explicitly $P_{cb\nu}(k, \tau) = P_{cb\nu}^L(k, \tau)$ [21]. This assumption is not self-consistent, since a one-loop correction must be present in the CDM+baryon-neutrino cross-correlation, if the CDM+baryon density contrast has indeed been calculated to third order in perturbative expansion.
7.1. Explicit forms

Using equations (6.6) and (6.7) to evaluate \( \hat{Q}_1^{(2)}(\mathbf{k} - \mathbf{q}, \mathbf{q}; \tau) \), and defining \( x \equiv k \cdot \mathbf{q} / (kq) \), \( r \equiv q/k \), and \( \eta \equiv \sqrt{1 + r^2 - 2rx} \), we find for the “22” correction term (7.4a)

\[
P_{cb}^{(22)}(k, \tau) = \frac{2\pi k^3}{98} \int_0^\infty dr P_{cb}^L(kr, \tau) \int_{-1}^1 dx P_{cb}^L(k \sqrt{1 + r^2 - 2rx}, \tau) \times \frac{[3S_1 r + 7(S_2 + S_3r^2)x - 10S_4r^2x^2]^2}{(1 + r^2 - 2rx)^2},
\]

(7.6)

with

\[
S_1 = \frac{7}{3} f(k\eta) [\sigma_{11}^{(2)}(k\eta, kr) - \sigma_{12}^{(2)}(k\eta, kr)f(kr)],
\]

\[
S_2 = f(kr)[\sigma_{11}^{(2)}(k\eta, kr) + \sigma_{12}^{(2)}(k\eta, kr)f(kr)],
\]

\[
S_3 = \sigma_{11}^{(2)}(k\eta, kr)[f(kr) - f(k\eta)],
\]

\[
S_4 = \frac{7}{5}\sigma_{11}^{(2)}(k\eta, kr)f(kr),
\]

(7.7)

and the functions \( f(q) \) and \( \sigma_{ab}^{(n)}(q_1, \ldots, q_n) \) are defined in equations (3.2) and (6.3) respectively. In the \( f_v \to 0 \) limit, we have \( S_{1,2,4} \to 1 \) and \( S_3 \to 0 \), which is the standard CDM-only result first given in references [32, 33] and adopted in the analysis of [21].

Similarly, evaluating \( \hat{Q}_1^{(3)}(k, \mathbf{q}, -\mathbf{q}; \tau) \) with the aid of equations (6.8) and (6.9), we obtain for the “13” terms (7.4b) and (7.5)

\[
2P_{cb,cb}^{(13)}(k, \tau) = \frac{2\pi k^3}{63} P_{cb,cb}^{L}(k, \tau) \int_0^\infty dr P_{cb}^L(kr, \tau) \int_{-1}^1 dx \left[ \frac{V}{\eta^2} + W \right],
\]

(7.8)

where

\[
V = \left[ 6I_1 r - 7(I_2 + I_3r^2)x + 8I_4r^2x^2 \right]\left[ (7M - 2R)r - (7Mr^2 - 2R)x \right],
\]

\[
W = 7L \left[ 10H_1rx - 7(H_2 + H_3r^2)x^2 + 4H_4rx^3 \right],
\]

(7.9)

and

\[
H_1 = \frac{7}{10}\sigma_{11}^{(2)}(k, kr)[f(k) + f(kr)],
\]

\[
H_2 = f(kr)[\sigma_{11}^{(2)}(k, kr) + \sigma_{12}^{(2)}(k, kr)f(k)],
\]

\[
H_3 = f(k)[\sigma_{11}^{(2)}(k, kr) + \sigma_{12}^{(2)}(k, kr)f(kr)],
\]

\[
H_4 = \frac{7}{2}\sigma_{12}^{(2)}(k, kr)f(k)f(kr),
\]

\[
I_1 = \frac{7}{6}\sigma_{11}^{(2)}(k, kr)[f(k) + f(kr)],
\]

\[
I_2 = f(kr)[\sigma_{11}^{(2)}(k, kr) + \sigma_{12}^{(2)}(k, kr)f(k)],
\]

\[
I_3 = f(k)[\sigma_{11}^{(2)}(k, kr) + \sigma_{22}^{(2)}(k, kr)f(kr)],
\]

\[
I_4 = \frac{7}{4}\sigma_{22}^{(2)}(k, kr)f(k)f(kr),
\]

\[
L = \frac{18}{7}\sigma_{11}^{(3)}(k, kr, kr)f(kr),
\]

\[
M = \frac{18}{7}\sigma_{11}^{(3)}(k, kr, kr),
\]
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\[ R = 9\sigma_{12}^{(3)}(k, kr, kr)f(kr). \]  

(7.10)

Again, these factors have been defined so that \( H_{1,2,3,4}, I_{1,2,3,4}, \) and \( L, M, R \) tend to unity in the \( f_\nu = 0 \) limit. Since these factors have no \( x \)-dependence, we can further simplify equation (7.8) by performing the integration over \( x \) to obtain

\[
\int_{-1}^{1} dx \left[ \frac{V}{\eta^2} + W \right] = \frac{1}{4} \left[ \frac{12T_1}{r^2} - 158T_2 + 100T_3 r^2 - 42T_4 r^4 + \frac{3}{r^3}(r^2 - 1)^2(T_5 r^2 - T_6)(7T_7 r^2 + 2T_8) \ln \left| \frac{1 + r}{1 - r} \right| \right],
\]

(7.11)

where

\[
T_1 = 21\sigma_{12}^{(3)}(k, kr, kr)\sigma_{21}^{(2)}(k, kr)f^2(kr),
\]

\[
T_2 = \frac{42}{79}f(kr) \left\{ \sigma_{11}^{(3)}(k, kr, kr) \left[ 4\sigma_{11}^{(2)}(k, kr)f(kr) + 4\sigma_{12}^{(2)}(k, kr)f(kr) - 3\sigma_{21}^{(2)}(k, kr) \right] + \sigma_{12}^{(3)}(k, kr, kr) \left[ \sigma_{21}^{(2)}(k, kr)[3f(k) + 9f(kr)] + 4\sigma_{22}^{(2)}(k, kr)f(k) + 9f(kr) \right] \right\},
\]

\[
T_3 = \frac{21}{25} \left\{ -3\sigma_{12}^{(3)}(k, kr, kr)\sigma_{21}^{(2)}(k, kr)f(k)f(kr) + \sigma_{11}^{(3)}(k, kr, kr) \left[ \sigma_{21}^{(2)}(k, kr)[3f(kr) + 9f(k)] + 4[\sigma_{22}^{(2)}(k, kr) - \sigma_{11}^{(2)}(k, kr)]f(k) + 9f(kr) \right] - 4\sigma_{12}^{(2)}(k, kr)f(k) f^2(kr) \right\},
\]

\[
T_4 = 6\sigma_{11}^{(3)}(k, kr, kr)\sigma_{21}^{(2)}(k, kr)f(k),
\]

\[
T_5 = f(k),
\]

\[
T_6 = f(kr),
\]

\[
T_7 = 6\sigma_{11}^{(3)}(k, kr, kr)\sigma_{21}^{(2)}(k, kr),
\]

\[
T_8 = 21\sigma_{12}^{(3)}(k, kr, kr)\sigma_{21}^{(2)}(k, kr)f(kr),
\]

(7.12)

and \( T_{1,\ldots,8} \to 1 \) as \( f_\nu \to 0 \). The \( f_\nu = 0 \) limit of equations (7.8) and (7.11) was first derived in [32,33] and then used in [21].

8. Results and discussions

8.1. One-loop corrected power spectra

Figure 1 shows various contributions to the total matter power spectrum at \( z = 1 \) and \( z = 3 \) in a \( \Lambda \) mixed cold+hot dark matter (\( \Lambda \)CDM) cosmology, assuming a total matter density \( \Omega_m h^2 = 0.14 \), neutrino fraction \( f_\nu = 0.1 \), Hubble rate \( h = 0.72 \), and scalar spectral index \( n_s = 0.963 \). The total matter power spectrum has been normalised at the pivot scale \( k_0 = 0.002 \) Mpc\(^{-1}\) to the vanilla best-fit from the Wilkinson Microwave Anisotropy Probe five-year data, i.e., the best-fit amplitude of the curvature perturbations \( \Delta_R^2 = 2.4 \times 10^{-9} \) [34]. Note that at the chosen redshifts the universe is very
Figure 1. Contributions to the total matter power spectrum (units: \(k^3\) Mpc\(^{-3}\)) at \(z = 1\) (left) and \(z = 3\) (right) for a ΛCHDM cosmology with \(f_\nu = 0.1\). Top: CDM+baryon auto-correlation \(P_{cb}(k, \tau)\). Middle: CDM+baryon–neutrino cross-correlation \(P_{cb\nu}(k, \tau)\). Bottom: Total matter power spectrum \(P(k, \tau)\). In all cases the linear contribution is shown in red/thin solid, the one-loop correction in green/thick solid (green/short dash when negative), and their sum in blue/long dash (blue/dotted when negative). All spectra have been divided by \(a^2\) to facilitate comparison. The three vertical lines demarcate, from left to right, the upper limits in \(k\) for which the linear matter power spectrum is accurate to 1\% or less, the one-loop corrected spectrum to 1\% or less, and the one-loop corrected spectrum to 5\% or less.

nearly described by an Einstein–de Sitter model. However, the vacuum energy Λ term still has some residual effects, particularly on the logarithmic derivative of the linear growth function, \(f(k, \tau)\). It is therefore necessary to rescale \(f(k, \tau)\) so that \(f(k, \tau) \to 1\) as \(k \to 0\), in order for the expressions (7.6) to (7.8), (7.11) and (7.12) to be applicable.
Observe that the $P^{(22)}_{cb}(k,\tau)$ term derives from a positive definite integrand and is therefore always positive. The $P^{(13)}_{cb}(k,\tau)$ term, on the other hand, is negative at most $k$ values. The net effect of their sum on the CDM+baryon auto-correlation is that the power spectrum is first suppressed at $k \sim 0.01 \rightarrow 0.1 \ h \ Mpc^{-1}$, and then enhanced as we move beyond $k \sim 0.1 \ h \ Mpc^{-1}$.

Contrastingly, the cross-correlation spectrum between the CDM+baryon and neutrino components receives a one-loop correction only from $P^{(13)}_{cb\nu}(k,\tau)$. Like the $P^{(13)}_{cb}(k,\tau)$ term, $P^{(13)}_{cb\nu}(k,\tau)$ is mostly negative. This causes the already free-streaming-suppressed power spectrum—recall the neutrino linear growth function falls off like $k^{-2}$ at $k \gg k_{FS}$, see equation (3.12)—to be further suppressed, before turning negative at $k \sim 0.2 \ h \ Mpc^{-1}$. Beyond $k \sim 0.2 \ h \ Mpc^{-1}$, however, the one-loop correction dominates over the linear term, resulting in an enhancement in the magnitude of $P_{cb\nu}(k,\tau)$, but in the negative direction.

The net correction to the total matter power spectrum follows essentially the same trend as the correction to $P_{cb}(k,\tau)$, beginning with a small suppression at $k \lesssim 0.1 \ h \ Mpc^{-1}$, and culminating in an enhancement at $k \gtrsim 0.1 \ h \ Mpc^{-1}$.

8.2. Regions of validity

Perturbation theory is not expected to describe reality at very large $k$ values, since any perturbative expansion must break down when the evolution of structures becomes fully nonlinear. A good rule of thumb is to take as the valid regime the range of $k$ values at which the one-loop correction is smaller than the linear contribution [36]. More recent analyses show that for a ΛCDM cosmology, the one-loop corrected power spectrum deviates by less than 1% from N-body simulation results provided that the dimensionless power spectrum, defined in this work as $\Delta^2(k,\tau) \equiv 4\pi k^3 P(k,\tau)$, does not exceed $\sim 0.4$ [37]. Applying this criterion to the total matter power spectrum in our ΛCCHDM scenario, we expect our one-loop corrected power spectrum to be accurate to better than 1% for $k \lesssim 0.2 \ h \ Mpc^{-1}$ at $z = 1$ and $k \lesssim 0.4 \ h \ Mpc^{-1}$ at $z = 3$. For completeness we also estimate a 5% accurate region following figure 2 of [37]: $k \lesssim 0.4 \ h \ Mpc^{-1}$ at $z = 1$, and $k \lesssim 1 \ h \ Mpc^{-1}$ at $z = 3$.

It is also interesting to ask up to what values of $k$ is linear theory expected to provide an accurate description of the matter power spectrum. Comparing the total matter power spectrum computed from linear theory to that including the one-loop correction, we find 1% agreement between the two only at $k \lesssim 0.09 \ h \ Mpc^{-1}$ for $z = 1$ and $k \lesssim 0.1 \ h \ Mpc^{-1}$ for $z = 3$. From this we conclude that the one-loop correction improves on linear theory to better than 1% accuracy in the $k$ ranges

$$0.09 \lesssim k/(h \ Mpc^{-1}) \lesssim 0.2, \quad z = 1,$$

$$0.1 \lesssim k/(h \ Mpc^{-1}) \lesssim 0.4, \quad z = 3.$$  \hspace{1cm} (8.1)

Figure 1 shows the various regions of validity discussed in this section.
Higher order corrections to the large scale matter power spectrum with neutrinos

Figure 2. Relative differences between the total matter power spectra for a pure ΛCDM cosmology and three ΛCHDM models with $f_\nu = 0.1$ (red/solid), 0.05 (blue/dotted), and 0.01 (green/dash) at $z = 1$ (left) and $z = 3$ (right). Thick lines indicate results including the one-loop correction, while the linear results are represented by the thin lines. The three vertical lines indicate the maximum $k$ values at which the linear and the one-loop corrected matter power spectra are accurate to better than 1% and 5%.

8.3. Suppression due to neutrino free-streaming

Figure 2 shows the suppression in the total matter power spectrum due to neutrino hot dark matter for three ΛCHDM models with $f_\nu = 0.1, 0.05, 0.01$, relative to the case with $f_\nu = 0$, i.e.,

$$\frac{\Delta P(k,a)}{P(k,a)} \equiv \frac{P_{f_\nu \neq 0}(k,a) - P_{f_\nu = 0}(k,a)}{P_{f_\nu = 0}(k,a)}.$$ (8.2)

Observe that the relative decrease of small scale power due to neutrino free-streaming in general exceeds the amount of suppression predicted by linear theory, once nonlinear corrections have been included. Within the 5% accurate region, it is clear that the suppression easily exceeds the canonical linear suppression factor of $\sim 8f_\nu$. This enhanced suppression has been observed in multi-component $N$-body simulations [18], which support an asymptotic suppression factor of $\sim 9.8f_\nu$. Furthermore, compared to the linear results, the one-loop correction leads to a small increase in relative power at just below $k \sim 0.1$ h Mpc$^{-1}$, before the enhanced suppression sets in, a feature that seems to be present also in figure 4 of [18].

A more rigorous comparison between our perturbation theory results and the $N$-body results of [18] is not possible at present, since our results are valid only for $z \gtrsim 1$, while reference [18] gives their results at $z = 0$.

§ The linear suppression factor reaches a maximum of $\sim 8f_\nu$ only for small values of $f_\nu$. For, e.g., $f_\nu = 0.2$, the linear suppression factor is $\sim 4.5f_\nu$ [35].
Higher order corrections to the large scale matter power spectrum with neutrinos

Figure 3. One-loop corrections to the CDM+baryon auto-correlation (top) and the CDM+baryon–neutrino cross-correlation (middle) at \( z = 1 \) (left) and \( z = 3 \) (right) for a ΛCHDM cosmology with \( f_\nu = 0 \). In the top and middle rows, the red/thin solid lines indicate the linear contribution, while the green/thick solid lines denote the one-loop correction (green/short dash when negative). The blue/long dash lines represent an approximation to the one-loop correction (blue/dotted when negative), computed by setting \( S_{1,2,4} = T_{1,\ldots,8} = 1 \) and \( S_3 = 0 \). The fractional error incurred in the total matter power spectrum by this approximation is shown in the bottom row in red/solid. Also plotted in the bottom row in blue/long dash is the fractional error incurred by neglecting the one-loop correction to the CDM+baryon–neutrino cross-correlation.

8.4. Further approximations?

There are two sources of deviation from the standard CDM-only case in the one-loop correction terms (7.6), (7.8) and (7.11). The first is encapsulated in the linear power spectra \( P_{\text{cb}}(k,\tau) \) and \( P_{\text{cbo}}^L(k,\tau) \); the second in the factors \( S_{1,\ldots,4} \) and \( T_{1,\ldots,8} \) defined in
Higher order corrections to the large scale matter power spectrum with neutrinos

Figure 4. Same as figure 3, but for $f_\nu = 0.05$.

equations (7.7) and (7.12), which depend on the linear growth functions. The linear power spectra are readily calculable with a sophisticated Boltzmann code such as CAMB. With some extra but minor effort the same is true also for the factors $S_{1,2,4}$ and $T_{1,\ldots,8}$. Nonetheless, it is tempting to simply assume $S_{1,2,4} = T_{1,\ldots,8} = 1$ and $S_3 = 0$, as was done in [21]. In this section we examine the validity of this approximation.

Figures 3, 4 and 5 show the correction terms $P^{(22)}_{cb} + 2P^{(13)}_{cb}$ and $P^{(13)}_{cb\nu}$ computed under the assumption of $S_{1,2,4} = T_{1,\ldots,8} = 1$ and $S_3 = 0$ for three ΛCHDM models with $f_\nu = 0.1, 0.05, 0.01$ respectively. Compared with their correct forms, we find deviations as large as a factor of twenty, sometimes even accompanied by a sign flip. However, such large deviations are confined to a region at $k \ll 0.1 \ h \ Mpc^{-1}$, where the one-loop correction terms are in any case subdominant to the linear contribution; the net fractional error incurred in the total matter power spectrum turns out to never exceed
the 1% level for our choices of $f_\nu$ and redshifts. Future cosmological probes will generally require an accuracy of $\sim 1\%$ in the matter power spectrum in order not to bias parameter estimation. On this basis, we conclude that setting $S_{1,2,4} = T_1, \ldots, 8 = 1$ and $S_3 = 0$ is a tolerable simplification.

Also shown in figures 3, 4 and 5 are the consequences of dropping the $P_{\text{chb}}^{(13)}(k, \tau)$ correction to the CDM+baryon–neutrino cross-correlation. As expected, the importance of this term increases as we increase $f_\nu$. For $f_\nu = 0.1$, the maximum contribution of $P_{\text{chb}}^{(13)}(k, \tau)$ to the $z = 1$ total matter power spectrum is 2.5% at $k \sim 0.4 \, h \, \text{Mpc}^{-1}$. For $f_\nu = 0.05$, the contribution is essentially halved. Thus for $f_\nu < 0.05$, dropping the $P_{\text{chb}}^{(13)}(k, \tau)$ term will unlikely bias our results. Nonetheless, $P_{\text{chb}}^{(13)}(k, \tau)$ can be computed at essentially no extra expense to the user because of its similarity to the (mandatory) $P_{\text{cb}}^{(13)}(k, \tau)$ contribution, cf. equation (7.8). Hence there is no real need to resort to
approximations in this instance.

9. Conclusions

In this paper we have presented the first rigorous and systematic derivation of the one-loop correction to the large scale matter power spectrum in a mixed dark matter cosmology with subdominant massive neutrino hot dark matter.

Beginning with the relevant equations of motion, we find that by invoking an “adiabatic” approximation, accurate to better than $\sim 0.1 f_\nu$, higher order corrections to the CDM+baryon density contrast and velocity field can be rendered into a form nearly identical to that for a pure-CDM cosmology. The interaction kernels and their recursion relations also exhibit striking similarities to their standard CDM-only counterparts, but contain additional dependences on the neutrino energy density fraction $f_\nu$ and the linear growth functions of the incoming wavevectors. These results, generalised to $n$th order in perturbative expansion, are summarised in equations (6.1) to (6.4).

Using these approximate solutions we compute the usual “22” and “13” one-loop correction terms to the matter power spectrum. As in the standard CDM-only case, these correction terms take the form of integrals over the wavevector $q$ of the linear power spectrum $P_L(q, \tau)$ multiplied by the interaction kernels. In addition to the corrections to the CDM+baryon auto-correlation, we also find a one-loop correction term for the cross-correlation between the CDM+baryon and the neutrino components which was previously neglected. These correction terms appear in their evaluated and most simplified form in equations (7.6) to (7.8), (7.11) and (7.12).

Evaluating these expressions numerically, we find that nonlinear corrections to the large scale matter power spectrum can enhance the suppression of small scale power due to neutrino free-streaming relative to the $f_\nu = 0$ case to beyond the canonical linear suppression factor of $\sim 8 f_\nu$. This enhanced suppression has been observed in multi-component $N$-body simulations [18].

As said, the interaction kernels contain hitherto unaccounted dependences on $f_\nu$ and the linear growth functions. Neglecting these dependences in principle generates large deviations in the one-loop corrections. However, since these deviations occur at wavenumbers at which the linear contribution dominates over the correction terms, their net effect on the total matter power spectrum never exceeds 1%. Future cosmological probes will require an accuracy of $\sim 1\%$ in the matter power spectrum in order not to bias parameter estimation. We have thus verified the validity of the approach of [21].

An important assumption in our present treatment is that the neutrino density perturbations have been taken to remain linear at all times. Although for realistic values of $f_\nu$ this can be justified by $N$-body simulation results [18], a truly complete analysis of higher order corrections to the clustering statistics of the large scale structure distribution in the presence of massive neutrinos should include also a proper account of nonlinear neutrino evolution. We defer this investigation to a future publication.

Finally, as noted in reference [21], although higher order perturbation theory
appears at first glance to have a limited range of validity—we expect our one-loop corrections to improve on linear theory to better than 1% accuracy in the region $0.1 \lesssim k/(h \text{Mpc}^{-1}) \lesssim 0.4$ at $z = 3$, it does enable an approximate factor of four increase in the maximum usable wavenumber in a data set. This is equivalent to a factor of 64 gain in the number of independent Fourier modes, or an eight-fold gain in statistical power for a fixed survey volume. Such an improvement is no small feat, and may very well be just what we need to detect neutrino dark matter.

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