Turbulence origination initial stage analysis for plane-parallel straight fluid flow, the rheological model of which takes into account the threshold “addition” of the transverse viscosity factor

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Abstract. A plane-parallel straight fluid flow is considered, the rheological model of which takes into account the threshold “addition” of the transverse viscosity factor. For the case of the velocity profile power dependence on the transverse coordinate in a small neighborhood of the flow region point in question, the conditions for “generating” and, possibly, the subsequent unlimited increase in the transverse velocity component are obtained. The simultaneous fulfillment of these two conditions is proposed to be interpreted as the laminar-turbulent transition initial stage. The analysis of the fulfillment of these conditions for the simplest special cases with linear, parabolic and cubic velocity profiles has been carried out. It is shown that the beginning of the transition in the inflection point vicinity on the cubic velocity profile is determined by the strain rate tensor second invariant modulus exceedance in this point of the threshold level corresponding to the transverse viscosity factor addition.

1. Introduction
At present, the main approaches to predicting the laminar-turbulent transition onset are based on the hydrodynamic stability theory [1-3]. These approaches are about the imposition of the velocity and pressure perturbations pulsation background with subsequent identification of conditions under which such perturbations amplitudes or the perturbations kinetic energy will subsequently increase without limit on the initially laminar flow satisfying the Navier-Stokes equations. Such conditions are imposed on the Reynolds number for the flow in question and the background wave number. In this case, the perturbations background is considered to exist immediately in the entire flow region. This does not allow, directly, to indicate locally that flow region zone in which the turbulence actually originates.

Along with this approach, there is a number of papers, where attempts were made [4–10] to construct conditions for the beginning of the transition based on several other hypotheses. As a rule, these papers postulate that there are some dimensionless complexes that are distributed continuously over the flow region and can register subtle differential properties of the velocity and pressure fields. Despite the variety of the dimensionless complexes that characterize the beginning of the transition, these papers are based on the following assumption. Let a dimensionless complex maximum value, which is reached at a certain spatial point of the flow region, exceed a certain threshold value. Then
such a spatial point and its neighborhood begin to act as the transition “initiators” of the laminar flow to the turbulent one.

In [11], a rationale was proposed for the origination of the laminar-turbulent transition initial stage based on the use of the following rheological model with a threshold “addition” of the transverse viscosity factor.

\[ \tau_{ij} = -P \cdot \delta_{ij} + 2 \cdot \mu \cdot e_{ij} + 4 \cdot \eta_c(I_2) \cdot \xi_{ij}; \quad i, j = 1, 2, 3; \]

\[ e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right); \quad \xi_{ij} = \sum_{k=1}^{3} e_{ik} \cdot e_{kj}; \quad i, j = 1, 2, 3; \]

\[ I_2 = e_{11} \cdot e_{22} + e_{22} \cdot e_{33} + e_{33} \cdot e_{11} - e_{12}^2 - e_{23}^2 - e_{31}^2; \]

\[ \eta_c(I_2) = \begin{cases} 0; & |I_2| < I_{2\eta}; \\ \eta_q \cdot |I_2| - I_{2\eta}; & |I_2| \geq I_{2\eta}; \end{cases} \]

where \( \tau_{ij}, e_{ij} \) are the stress and strain rate tensors components respectively; \( P \) is hydrostatic pressure; \( u_i \) are velocity vector components; \( \delta_{ij} \) is the Kronecker symbol; \( x_i \) are coordinates in the Cartesian frame of reference, \( \mu \) is taken as a constant the dynamic fluid viscosity; \( \eta_c(I_2) \) is a function expressing the transverse viscosity dependence on the strain rate tensor second invariant \( I_2 \); \( \eta_q, q \) are the rheological model empirical parameters; \( I_{2\eta} \) is the threshold value of the strain rate second invariant tensor modulus.

A similar model, but for the case of a constant transverse viscosity, was considered in [12].

It is well-known that the hydrodynamics problems for liquids with the consideration of the transverse viscosity lead to the solutions that demonstrate secondary flows [13]. These flows are characterized, first of all, by the presence of transverse velocity components with respect to the streamlines, which could be obtained by solving the same problems, but without taking into account the transverse viscosity.

We note the following analogy. The laminar-turbulent transition initial stage is also characterized by the transverse velocity components formation with respect to the previously existing laminar flow streamlines. The postulation of the transverse viscosity threshold addition corresponds quite well to the threshold nature of the transition to turbulence. However, it is necessary to require that the condition of an unlimited increase in the “generated” transverse velocity components should be satisfied, at least at the model level.

The rheological model proposed in [11] at a certain range of variation of the strain rate tensor second invariant coincides with the classical Newtonian fluid model, and when it exceeds a certain critical level, the transverse viscosity factor begins to be taken into account. With the use of this rheological model, the equations of the liquid dynamics of this kind were represented in a dimensionless form. In this connection, all the scale values of these equations were determined with the involvement of the velocity and pressure fields local characteristics. As a result, it become possible to introduce into consideration a dimensionless complexes set that turned out to be composed solely of the flow invariant values in question.

The obtained equations analysis showed that in the neighborhood of that spatial point where the condition of the transverse viscosity factor “addition” is satisfied, the process of “generating” the transverse velocity components begins. If there is an unlimited increase in these “generated” transverse velocity components, then such a situation is proposed to be interpreted as the initial stage of the laminar-turbulent transition. With the involvement of a sufficiently large number of known experimental data, related, inter alia, to the formation of the secondary flows, which are caused by the transverse velocity components onset, certain empirical conditions were proposed in [11]. These
conditions are imposed on the invariant dimensionless complexes and determine the beginning of laminar turbulent transition.

In this paper, based on the results of [11], we analyze the conditions for the beginning of the transition using the example of a plane-parallel straight flow and some of its most well-known special cases.

2. A general case of the plane-parallel straight flow

Let us consider a plane-parallel straight laminar flow of an incompressible fluid with a Newtonian model applicability limit in the vicinity of a spatial point, by the volume forces field among them. We will assume that the rheological model of this fluid implies a threshold "addition" of the transverse viscosity factor [11].

We introduce a local Cartesian coordinate system with the origin at the point in question. In this case, one of the axes, for example, the axis $Ox_1$, is directed tangentially to the streamline at this point of the flow region.

Let us assume that at the initial moment of time a flow is formed in a small neighborhood of some point. The velocity vector components of this flow and its module in the dimensional form can be represented as follows

$$U^{(0,1)} = V(x_2); \quad U^{(0,2)} = 0; \quad U^{(0,3)} = 0;$$

$$U = \sqrt{U^{(0,1)^2} + U^{(0,2)^2} + U^{(0,3)^2}} = V(x_2).$$

As a scale for measuring velocity, we take its value at the point in question

$$U_s = V(0) = V_0.$$

Hereinafter, the subscript "s" indicates that the corresponding characteristics are determined at the initial moment of time at the origin.

Vector modules

$$\bar{E} = \text{grad}\left\{ P + \frac{\rho \cdot U^2}{2} \right\}; \quad \bar{D} = \text{grad}\left\{ 2 \cdot \mu \cdot \sqrt{V_2} \right\},$$

that were entered into consideration in [11] are defined for such a flow as follows

$$E = \left[ F_1 + \mu \cdot \frac{d^2V}{d(x_2)^2} \right]^2 + \left[ F_2 + \rho \cdot V \cdot \frac{dV}{dx_2} \right]^2; \quad D = \mu \cdot \frac{d^2V}{d(x_2)^2},$$

where $\rho$ is the fluid density; $F_1$, $F_2$ are the volume force projections on the corresponding coordinate axes. Hereinafter, it is naturally assumed that $F_3 = 0$.

Note that the vector $\bar{E}$ characterizes the direction and "rapidity" (in the spatial coordinates terms) of the maximum increase in the density of the fluid flow total mechanical energy at the flow region point in question. Similarly, the vector $\bar{D}$ characterizes the direction and "rapidity" of the maximum increase of the viscous dissipation factor at the flow region point in question.

Then, the main dimensionless complexes [11], which determine the conditions for "generating" the transverse velocity components and the possible subsequent transition of the flow initial laminar state to the turbulence, will be determined as follows

$$K_1 = \sqrt{f(x_2)} \bigg|_{x_2} = 0; \quad K_2 = \left\{ \frac{K_6^2}{4 \cdot f(x_2)} \right\} \bigg|_{x_2} = 0; \quad K_3 = \frac{K_7 \cdot K_6}{\sqrt{f(x_2)}} \bigg|_{x_2} = 0. \quad (1)$$
For brevity sake, the following notation for the parameters is introduced in the last relationships.

\[
 f(x_2) = \left[ F^{(0.1)} + \text{sign}\left( \frac{d^2V}{d(x_2)^2} \right) \right]^2 + \left[ F^{(0.2)} + K_6 \right] ;
\]

\[
 K_6 = \left( \frac{\rho \cdot V}{\mu} \cdot \frac{dV}{dx_2} \right) \left( \frac{d^2V}{d(x_2)^2} \right)^{-1} \bigg|_{x_2 = 0} ; \quad K_7 = \frac{\rho \cdot V^2}{\mu} \left( \frac{dV}{dx_2} \right)^{-1} \bigg|_{x_2 = 0} ;
\]

\[
 I_{2s} = \left( \frac{E_s}{\rho \cdot U_s} \right)^2 ; \quad F^{(0.1)} = \frac{F_1}{D_s} ; \quad F^{(0.2)} = \frac{F_2}{D_s} .
\]

Note that the parameter \( K_6 \) exactly repeats in its form the dimensionless complex adopted in [7] to characterize the laminar – turbulent transition. In addition, \( K_6 \) in its structure turns out to be similar to the corresponding dimensionless complex from [6], as if it were presented relative to a flow in a flat channel.

The dimensionless complexes (1) can be taken as the basis for modeling of the laminar-turbulent transition initial stage within an approach based on the use of the rheological model with the transverse viscosity factor threshold “addition”. In particular, it was shown in [11] that when the empirical conditions are fulfilled at the point (origin) in question

\[
 K_2 > K_{2G} = k_0 + \frac{k_1}{K_3 - k_2} ; \quad K_3 > k_2 ; \quad k_0 = 0.221 ; \quad k_1 = 8.572 ; \quad k_2 = 84.015 ,
\]

in its small neighborhood, the transverse viscosity factor is added, and, accordingly, the transverse velocity components generation begins.

Since the “generated” transverse velocity components do not always lead to the beginning of the transition of a laminar flow to a turbulent one, conditions (3) should be considered only as necessary, although insufficient for a transition to occur. Therefore, in addition to (3), another empirical condition was proposed in [11] in the form of the following

\[
 K_{1\text{max}} > q_0 \cdot (K_3)^q_1 ; \quad q_0 = 9.857 \cdot 10^{-4} ; \quad q_1 = 2 .
\]

In this case, \( K_{1\text{max}} \) represents the dimensionless complex maximum value, which is achieved at the corresponding spatial point of that flow region part where the following condition (3) is satisfied.

The last condition fulfillment (4) assumes that the “generated” transverse velocity components will increase indefinitely. Such a situation, due to the fulfillment of the combination of the conditions (3), (4), is proposed to be interpreted as the beginning of a laminar-turbulent transition.

3. A special case of the cubic velocity profile. The inflection point role

Let us consider a hypothetical version of a plane-parallel straight flow with a cubic velocity profile under the action of both the pressure drop and the presence of a corresponding volume force.

Suppose that by the initial instant in the neighborhood of the spatial point in question (the origin), a current has been formed, whose velocity components in the dimensional form are defined as follows

\[
 U^{(0,1)} = V(x_2) = V_0 + b_1 \cdot x_2 + b_2 \cdot x_2^2 + b_3 \cdot x_2^3 ; \quad U^{(0,2)} \equiv 0 ; \quad U^{(0,3)} \equiv 0 ,
\]

where \( b_1, b_2, b_3 \) are the parameters defining the velocity profile.
The velocity distribution in the form of (5), by analogy with the Kolmogorov flow, can occur if we assume that the volume forces density components are determined from the expressions

\[ F_1(x_2) = -\delta \cdot \mu \cdot b_3 \cdot x_2; \quad F_2 = 0; \quad F_3 = 0. \]  
(6)

Taking into account (5), (6) from (1), (2) we get

\[ K_1 = \sqrt{1 + K_0^2}; \quad K_2 = \frac{K_0^2}{4 \cdot (1 + K_0^2)}; \quad K_3 = \frac{K_0 \cdot K_0}{\sqrt{1 + K_0^2}}; \]  
(7)

Here

\[ K_0 = \frac{\rho \cdot V_0}{2 \cdot \mu \cdot |b_2|}; \quad K_7 = \frac{\rho \cdot V_0^2}{\mu \cdot b_1}. \]  
(8)

Taking into account (7), the condition (3) of "generating" the transverse velocity component at the point in question (the origin) can be represented as follows

\[ |K_7| > \frac{\sqrt{1 + K_0^2}}{|K_6|} \cdot \left( k_2 + \frac{4 \cdot k_1 \cdot (1 - K_0^2)}{K_0^2 - 4 \cdot k_0 \cdot (1 + K_0^2)} \right). \]  
(9)

The condition of the type (4), written not for the entire flow area maximum value \( K_{1,\text{max}} \), but for the current complex value \( K_1 \) at the point in question (the origin) with regard to (7) takes the following form

\[ |K_7| < \frac{1}{\sqrt{q_0} \cdot |K_6|} \cdot (1 + K_0^2)^{3/2}. \]  
(10)

Note that the velocity profiles described by cubic polynomials should demonstrate the inflection point. In this regard, let us consider the question of the inflection point role on the velocity profile.

In the aspect of the hydrodynamic theory of the ideal fluid flows stability with such a point on the velocity profile, this question was posed in the Rayleigh theorem [14–16]. According to this theorem, flows with similar profiles are unstable.

The analysis (5) shows that at \( b_2 = 0 \) the point in question (the origin) on the velocity profile corresponds to the inflection point.

Let's pass to the limit in the relationships presented above at \( b_2 \to 0 \).

In this case, based on (8), we get

\[ |K_6| \to \infty; \]  
(11)

and, therefore, in view of (7) we have

\[ K_1 \to \infty; \quad K_2 \to 0.25; \quad K_3 \to |K_7|. \]  
(12)

The last result (12) means that the condition (4) of an unlimited increase in the “generated” transverse velocity components will certainly be fulfilled for any finite value \( K_7 \). Therefore, in such a situation, the very question of the laminar-turbulent transition start, in fact, comes down to fulfilling the condition of “generating” the transverse velocity components in the form (3). Taking into account (11), the inequality (9), which is a condition (3) written for a cubic velocity profile, can be represented as
Thus, to initiate a laminar – turbulent transition in a small neighborhood of the velocity profile inflection point, it is sufficient to require only the fulfillment of the condition (13) to “generate” the transverse velocity components. This is due to the fact that the condition (10) of their unlimited increase in the neighborhood of such a point, taking into account (11), is always fulfilled here. The latter means that the fact that there is an inflection point on the velocity profile is not yet the final condition for the transition start. In particular, if the condition (13) is not satisfied, then the “generation”, as the main “preliminary” factor of the laminar-turbulent transition start, simply will not take place. And, therefore, despite the unconditional fulfillment of the inequality (4), the laminar flow pattern will remain in the channel.

4. A parabolic velocity profile special case. Poiseuille flow

In the case when the condition \( b_3 = 0 \) from (5) is satisfied, we get a parabolic velocity profile in the vicinity of the flow region spatial point in question. The velocity profile of this type is realized, in particular, during Poiseuille flow in a flat channel and in the traditional recording form is described by the following relationship

\[
u(y) = \frac{3}{2} \cdot V_{\text{aver}} \left( 1 - \left( \frac{y}{h} \right)^2 \right),
\]

where \( V_{\text{aver}} \) is the average fluid velocity in the channel; \( y \) is the transverse coordinate measured from the longitudinal symmetry axis of the channel; \( h \) is half the width of the channel.

Let us consider a point with the coordinate \( y = y_0 \) in the flow region. We introduce here a local coordinate system with the origin at this point. In this case, the axis \( Ox_2 \) is oriented in the same direction as the main coordinate system axis \( Y \) associated with the channel. Then, in the local coordinate system, the velocity distribution (5) taking into account (14) takes the form

\[
U^{(0,1)} = V(x_2) = V_0 + b_1 \cdot x_2 + b_2 \cdot x_2^2; \quad U^{(0,2)} \equiv 0; \quad U^{(0,3)} \equiv 0;
\]

\[
V_0 = \frac{3}{2} \cdot V_{\text{aver}} \left[ 1 - \left( \frac{y_0}{h} \right)^2 \right]; \quad b_1 = -\frac{3 \cdot V_{\text{aver}} \cdot y_0}{h^2}; \quad b_2 = -\frac{3 \cdot V_{\text{aver}}}{2 \cdot h^2}.
\]

Taking into account (15), repeating the previous section rationales, we get the following result. In order for the considered flow region point in the channel with the coordinate \( y = y_0 \) to act as the laminar-turbulent transition initiator, it is necessary to require simultaneous fulfillment of the conditions (9), (10). At the same time, the dimensionless complexes entering into these inequalities should be defined with the account for (15) as follows

\[
K_6 = -\frac{3}{4} \cdot \text{Re} \left( 1 - y_0^2 \right) y_0'; \quad K_7 = -\frac{3}{8} \cdot \text{Re} \left( 1 - y_0^2 \right)^2; \quad \text{Re} = \frac{2 \cdot \rho \cdot V_{\text{aver}} \cdot h}{\mu} \quad y_0' = \frac{y_0}{h}.
\]

Here \( \text{Re} \) is the Reynolds number recorded in the traditional form for Poiseuille flow in a flat channel.

Note that with the fixed \( \text{Re} \), the maximum parameter value \( K_6 \), and, consequently, the dimensionless complex \( K_7 \) is achieved at the points with the coordinates \( y_0' = \pm 1 / \sqrt{3} \).
Calculating the main parameters (16) at these points, it is possible with the help of (9), (10) to show that the critical Reynolds number in this case takes on the value $Re_{\text{crit}} \approx 3510$. At the same time, the additional requirement fulfillment was taken into account, to ensure the maximum value of the ratio $K_2 / K_2G$, which corresponds to the greatest manifestation of the velocity transverse components “generating” effect. Estimating the result obtained, we note that there are laminar – turbulent transition experimental studies for Poiseuille flow in a flat channel. These studies show the Reynolds number critical values (depending on the intensity of the pulsating background) from the range $Re_{\text{crit}} = 2100 \div 6500$ [17].

5. A linear velocity profile special case. Couette flow

In the case when the conditions $b_2 \equiv 0$ and $b_3 \equiv 0$ are fulfilled and, from (5) we get a linear velocity profile in the vicinity of the flow region spatial point in question. Such a velocity profile occurs, in particular, when Couette flow in a flat channel and in the traditional notation is determined by the following ratio

$$u(y) = V_w \cdot \frac{y}{h},$$

where $V_w$ is the speed of the channel moving wall; $y$ is the transverse coordinate measured from the fixed channel wall; $h$ is the channel width.

The occurrence of the laminar-turbulent transition for Couette flow in a flat channel has been studied in detail on the basis of the hydrodynamic stability theory. In spite of the extremely simple velocity distribution described by the linear function of the transverse coordinate, the analysis of the transition condition from the standpoint of the hydrodynamic stability theory has turned out to be not so simple. The complexity and special situation when considering laminar-turbulent transition from the standpoint of stability or instability for Couette flow in a flat channel were pointed out by Rayleigh and Lamb. A detailed bibliography on this subject is given in [2]. Note that well-known theoretical results demonstrate the stability of Couette flows with respect to small perturbations [18–20]. At the same time, the well-known experimental data indicate the possibility of the turbulence development for flows of this kind [21–24].

Let us introduce at a flow region point with the coordinate $y = y_0$ a local coordinate system with the origin at this point. In this case, as in the previous section, the axis $Ox_2$ will be oriented in the same direction as the axis $Y$ of the main coordinate system. Then, in the local coordinate system, the velocity distribution (5) taking into account (17) takes the form of

$$U^{(0,1)} = V(x_2) = V_0 + b_1 \cdot x_2; \quad U^{(0,2)} \equiv 0; \quad U^{(0,3)} \equiv 0; \quad V_0 = V_w \cdot \frac{y_0}{h}; \quad b_1 = \frac{V_w}{h}.$$

Taking into account the last relationships we get

$$K_6 \to \infty; \quad K_7 = Re \cdot \frac{y_0^2}{\mu}; \quad Re = \frac{\rho \cdot V_w \cdot h}{\mu}; \quad y'_0 = \frac{y_0}{h},$$

which ultimately leads to the result that coincides with (12) up to the definition of the parameter $K_7$.

In other words, in the sense of the laminar-turbulent transition onset, the considered flow variant is identical to the case with the inflection point on the velocity profile. In this case, the condition (13) remains valid, which, taking into account (18), for the point $y'_0 = 1$ on the channel wall (where the maximum parameter value $K_7$ is reached) leads to the result $Re_{\text{crit}} \approx 379.6$. The calculated value for $Re_{\text{crit}}$ is in satisfactory agreement with the results of the experimental work [22–24], according to which the critical Reynolds number is within the range $Re_{\text{crit}} = 325 \div 370$.  

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6. Conclusion
Using the previously obtained results for the plane-parallel straight fluid flow, whose rheological model takes into account the transverse viscosity factor threshold “addition”, the conditions for “generating” the transverse velocity components as well as their possible subsequent unlimited increase are considered for any spatial point of the flow area. The fulfillment of the combination of these two conditions is proposed to be interpreted as the initial stage of the laminar-turbulent transition origination.

For the particular case of the cubic velocity profile, it is shown that at the inflection point the condition of an unlimited increase in the “generated” transverse velocity components always holds. Therefore, to initiate in the small neighborhood of this point the transition of a laminar flow to a turbulent one, it is only necessary to require that the strain rate tensor second invariant modulus exceeds the threshold value at the inflection point.

A similar analysis was also carried out for particular cases of the parabolic and linear profiles in the vicinity of the considered point of the flow region.

A comparison of the predicted results with the known experimental data of other authors demonstrates satisfactory agreement.

The examples discussed above for the particular, well-known cases of plane-parallel straight fluid flow demonstrate the potential of the proposed approach to analyzing the conditions for the laminar-turbulent transition onset. Such an approach can be proposed for further use in the study of more complex flow patterns.

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