Communication-Efficient Distributed Estimator for Generalized Linear Models with a Diverging Number of Covariates

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Abstract
Distributed statistical inference has recently attracted immense attention. Herein, we study the asymptotic efficiency of the maximum likelihood estimator (MLE), the one-step MLE, and the aggregated estimating equation estimator for generalized linear models with a diverging number of covariates. Then a novel method is proposed to obtain an asymptotically efficient estimator for large-scale distributed data by two rounds of communication between local machines and the central server. The assumption on the number of machines in this paper is more relaxed and thus practical for real-world applications. Simulations and a case study demonstrate the satisfactory finite-sample performance of the proposed estimators.

Keywords: Generalized linear models, Large-scale distributed data, Asymptotic efficiency, One-step MLE, Diverging p

MSC: 62J12

1. Introduction
In modern times, large-scale data sets have become increasingly common, and they are often stored across multiple machines. This necessitates a reconsideration of statistical inference (Jordan et al., 2019). Since communication cost between machines is considerably higher than the cost of conducting statistical analysis on a single machine (Jaggi et al., 2014; Smith et al., 2018), it is inefficient to calculate a global estimator by the transmission of the local data to a central machine. Further, the application of the traditional iterative algorithms in a distributed system, such as the Fisher-scoring algorithm for maximum likelihood estimator (MLE) in generalized linear models (GLMs), cannot avoid multiple rounds of communication that incurs exorbitant costs. Therefore, communication-efficient distributed algorithms must be developed to accommodate the new features of modern data sets.

In recent years, various parallel and distributed procedures have been proposed, such as divide and conquer algorithms and “one-shot” distributed algorithms or “embarrassingly parallel” approaches which only require one round of communication (see Zhang et al., 2013 and references therein). For these methods, the estimations are made in parallel on local machines, then these local results are transmitted to the central node to get an aggregated estimator (Fan et al., 2007; Lin & Xie, 2011; Chen & Xie, 2014). To guarantee the statistical properties of the aggregated estimators, it requires constraints on the divergence speed of the number of machines, such as \( K = o(\sqrt{n}) \) in Lin & Xie (2011), where \( K \) is the number of machines and \( n \) is the sample size. However, Smith et al. (2018) pointed out that the assumptions of the diverging speed of \( K \) in the existing distributed estimators by one round of communication are too restrictive to be in accordance

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with the common practice whereby a huge number of machines are in use relative to the sample size. Jordan et al. (2019) showed that average-based one-shot estimators do not perform well for nonlinear cases and proposed an iterative estimation algorithm to reduce the approximation error. However, the algorithm involves multiple rounds of communication. Our work is inspired by the one-step distributed estimator used to surrogate M-estimators in Huang & Huo (2015). Based on the averaging estimator, it is developed by adding a single Newton-Raphson update with one additional round of communication. Provided that the averaging estimators are \(\sqrt{n}\)-consistent, asymptotic properties of the one-step estimator can be assured under a weak assumption on the diverging speed of \(K\). In this paper, we will focus on GLMs, the MLE of which is a special case of M-estimators. It is worth pointing out that nearly all existing distributed estimators used to surrogate the MLE (not penalized MLE), such as Lin & Xi (2011), Huang & Huo (2015), and Jordan et al. (2019), are studied under a fixed number of covariates. However, the “big data” in the modern era are characterized not only by huge sample sizes but also by high dimensions. Hence, we will propose a communication-efficient distributed estimator with two rounds of communication for GLMs with a diverging number of covariates.

As far as we know, limited research has been focused on the MLE in GLMs under a diverging dimension, although related work does exist. For example, Liang & Du (2012) attempted to use the MLE for logistic regression with a diverging dimension, which was proved wrong in Zhang (2018). He & Shao (2000) built the asymptotic normality of M-estimators for general parametric models when dimension \(p\) increases with the sample size \(n\) but under a relatively strong assumption, i.e., \(p^2\log p = o(n)\). Wang (2011) gave consistency of the GEE estimator when \(p^2 n = o(n)\) and its asymptotic efficiency when \(p^2 = o(n)\). Thus, in this paper, we first show the asymptotic efficiency of the MLE in GLMs under the assumption of \(n = o(\sqrt{n})\). Based on the same assumption on the diverging speed of \(p_n\), the proposed distributed estimator is then shown to be asymptotically efficient. We further demonstrate through simulations and a case study that the proposed method outperforms existing distributed estimators with one round of communication, including the simple average method and the aggregated estimating equation (AEE) method, when the number of machines is relatively large.

The main contributions of this paper are as follows. First, we propose a distributed estimator with two rounds of communication for GLMs under a diverging dimension. Unlike the one-step estimator proposed in Huang & Huo (2015), the dimension of our estimator diverges with the sample size. To show that the proposed estimator has the same asymptotic efficiency as the MLE based on the full data set, we examine the consistency of the averaging estimator for GLMs with a diverging dimension and extend the one-step method to GLMs under a diverging dimension. Besides, our method updates the average estimator by a single round of Fisher-scoring iteration instead of Newton-Raphson iteration in Huang & Huo (2015), and thus requires less computation.

Second, compared with the restrictive assumptions on the number of machines for the distributed estimators with one round of communication, the assumption of the proposed distributed estimator is relaxed at the expense of an additional round of communication. The proposed method is shown to have greater advantages when data are distributed across relatively more machines.

The rest of this article is organized as follows. Section 2 presents the basic notations used in this paper. In Section 3, we introduce the distributed estimators in GLMs as well as their asymptotic efficiency. Simulation studies are given in Section 4 to show the finite sample performance of the proposed method. Conclusions are presented in Section 5. Some technical details are relegated to the Appendix.

2. Notations

Let \(Y = (y_1, \ldots, y_n)^T\) be the response vector. In GLMs, the density function of \(y_i\) is
\[
f(y_i|\theta_i, \phi) = c(y_i, \phi) \exp \{ (y_i - b(\theta_i))/\phi \}, \quad i = 1, \ldots, n,
\]
where \(\theta_i\) is the canonical parameter, \(\phi\) is the dispersion parameter, and \(b(\cdot)\) and \(c(\cdot)\) are known functions. It can be easily shown that
\[
E_{\theta_i}(y_i) = \mu(\theta_i) = b'(\theta_i) \quad \text{and} \quad \text{Var}_{\theta_i}(y_i) = \sigma^2(\theta_i) = \phi b''(\theta_i).
\]
Let \( Z = (z_1, \ldots, z_n)^T \) be a \( n \times p_n \) design matrix and \( \beta_n \) be an unknown \( p_n \times 1 \) vector of regression coefficients. The true value of \( \beta_n \) is \( \beta_{0n} \). Assume that \( \mu \) is related to \( z_i \) through
\[
\mu(\theta_i) = h(z_i^T \beta_n),
\]
where \( h(\cdot) \) is the inverse of the strictly monotone link function. Define \( \theta_i = u(z_i^T \beta_n) = \mu^{-1}(h(z_i^T \beta_n)) \) and \( v(z_i^T \beta_n) = \sigma^2(u(z_i^T \beta_n)) \), where \( v(\cdot) \) is a positive function. Then, the log-likelihood function of \( Y \) is
\[
L_n(\beta_n) = \sum_{i=1}^{n} \left\{ y_i u(z_i^T \beta_n) - h(z_i^T \beta_n) \right\}/\phi + \log(c(y_i, \phi)) \right\}
\]
Denote the parts related with \( \beta_n \) in the first and second derivatives of \( L_n(\beta_n) \) as
\[
S_n(\beta_n) = \sum_{i=1}^{n} z_i u'(z_i^T \beta_n)[y_i - h(z_i^T \beta_n)],
\]
and
\[
H_n(\beta_n) = \sum_{i=1}^{n} z_i u''(z_i^T \beta_n)[y_i - h(z_i^T \beta_n)]z_i - \sum_{i=1}^{n} z_i w(z_i^T \beta_n)z_i^T,
\]
respectively, where
\[
 w(z_i^T \beta_n) = [h'(z_i^T \beta_n)]^2 v^{-1}(z_i^T \beta_n).
\]
The \( p_n \times 1 \) vector \( S_n(\beta_n) \) is the score function, while its covariance, the \( p_n \times p_n \) matrix
\[
F_n(\beta_n) = \sum_{i=1}^{n} z_i w(z_i^T \beta_n)z_i^T,
\]
is the Fisher information matrix.

In a distributed system, suppose that the full data set is distributed across \( K \) machines and the \( k \)th (\( k = 1, \ldots, K \)) machine contains \( n_k \) observations denoted by \( (Y_k, Z_k) \). For the \( k \)th machine, the corresponding log-likelihood function is denoted by \( L_{nk}(\beta_{nk}) \), and its first and second derivatives are \( S_{nk}(\beta_{nk}) \) and \( H_{nk}(\beta_{nk}) \), respectively.

### 3. Asymptotically efficient distributed estimation

#### 3.1. MLEs when \( p_n \to \infty \)

In this subsection, we first show the asymptotic existence of the MLE in GLMs and its asymptotic efficiency when \( p_n \) diverges with \( n \). Then we motivate the construction of the proposed one-step estimator.

The MLE of \( \beta_n \) is defined as
\[
\hat{\beta}_n = \arg \max_{\beta_n \in \Theta_n} L_n(\beta_n),
\]
where \( \Theta_n \) is the parameter space.

In the following statement, the maximum and minimum eigenvalue of matrix \( A \) are denoted by \( \lambda_{\max}(A) \) and \( \lambda_{\min}(A) \), respectively. Let \( \|\alpha\| \) denote the Euclidean norm of a vector \( \alpha \in \mathbb{R}^{p_n} \), i.e., \( \|\alpha\| = \sqrt{\alpha^T \alpha} \), and \( S_{p_n} = \{ \alpha \in \mathbb{R}^{p_n} : \|\alpha\| = 1 \} \).

We establish the asymptotic results with the following assumptions.

**Assumption 1.** The \( n \times p_n \) fixed design matrix \( Z \) satisfies
\[
0 < C_{\min} \leq \lambda_{\min}(\frac{1}{n} \sum_{i=1}^{n} z_i z_i^T) \leq \lambda_{\max}(\frac{1}{n} \sum_{i=1}^{n} z_i z_i^T) \leq C_{\max} < \infty,
\]
and

\[
\sup_{\alpha_1, \alpha_2 \in S_{p_n}} \sum_{i=1}^{n} |\alpha_1^T z_i|^2 |\alpha_2^T z_i|^2 = O(n),
\]

where we use \( C_{\min} \) and \( C_{\max} \) to denote constants which may vary from case to case.

**Assumption 2.** \( \sup_{i \geq 1} \mathbb{E}(e_i^r) < \infty \) for some \( r > 2 \), where \( e_i = y_i - h(z_i^T \beta_{0n}) \).

**Assumption 3.** \( \max_{1 \leq i \leq n} |z_i^T \beta_{0n}| \leq M_0 \).

**Assumption 4.** The function \( w(\cdot) \) has continuous third-order derivative and \( w(\cdot) \) has continuous first-order derivative.

**Remark 1.** In Assumption 4, two conditions are imposed on the design matrix so that it has a reasonably good behavior. Similar assumptions can also be found in [He & Shao (2000)] and [Lin & Xi (2011)]. Assumptions 3 and 4 imply that \( w(z_i^T \beta_{0n})(i = 1, \ldots, n) \) are bounded. Let

\[
W_{\min} = \min_{1 \leq i \leq n} w(z_i^T \beta_{0n}),
\]

and

\[
W_{\max} = \max_{1 \leq i \leq n} w(z_i^T \beta_{0n}).
\]

Since \( h(\cdot) \) is strictly monotone and \( w(\cdot) \) is positive, we have \( W_{\min} > 0 \). Therefore,

\[
0 < C_{\min} W_{\min} \leq \lambda_{\min}(F_n(\beta_{0n})/n) \leq \lambda_{\max}(F_n(\beta_{0n})/n) \leq C_{\max} W_{\max} < \infty.
\]

**Theorem 1.** Suppose Assumptions 2-4 hold.

(i) If \( p_n = O(\sqrt{n}) \), then there exists a sequence of estimators \( \{\hat{\beta}_n\} \) such that

\[
P(S_n(\hat{\beta}_n) = 0) \to 1 \quad \text{as} \quad n \to \infty,
\]

where \( S_n(\hat{\beta}_n) \) is the score function evaluated at \( \hat{\beta}_n \).

(ii) If \( p_n = o(\sqrt{n}) \), then

\[
\alpha^T F_n^{1/2}(\beta_{0n})(\hat{\beta}_n - \beta_{0n}) \to_d N(0, 1) \quad \text{as} \quad n \to \infty,
\]

where \( \alpha \in S_{p_n} \) and \( F_n(\beta_{0n}) \) is the Fisher information matrix evaluated at the true parameter value \( \beta_{0n} \).

Theorem 1 assures the asymptotic existence and the asymptotic efficiency of an MLE for diverging dimensions. Thus, we can construct one in the distributed framework. [Huang & Huo (2015)] developed a one-step MLE based on the Newton-Raphson method for a fixed \( p \), and it inspires our work. Likewise, the efficient one-step MLE for a fixed \( p \) can also be obtained by the Fisher-scoring method with an initial value \( \hat{\beta}_{0n}^{(0)} \) which is consistent but not asymptotically efficient (Shao 2003). Such a one-step estimator requires less computation and it is defined as

\[
\hat{\beta}_n^{(1)} = \hat{\beta}_{0n}^{(0)} + F_n^{-1}(\hat{\beta}_{0n}^{(0)}) S_n(\hat{\beta}_{0n}^{(0)}).
\]

In the following theorem, we study the asymptotic efficiency of the one-step estimator when \( p_n \) diverges.

**Theorem 2.** Suppose that Assumptions 2-4 hold and \( \hat{\beta}_{0n}^{(0)} \) is a \( \sqrt{n/p_n} \)-consistent estimator of \( \beta_{0n} \). If \( p_n = o(\sqrt{n}) \), (i) there exists \( \hat{\beta}_n^{(1)} \) such that

\[
P(F_n(\hat{\beta}_n^{(1)})(\hat{\beta}_n^{(1)} - \hat{\beta}_{0n}^{(0)})) \to 1 \quad \text{as} \quad n \to \infty,
\]

and (ii) \( \hat{\beta}_n^{(1)} \) satisfies

\[
\alpha^T F_n^{1/2}(\beta_{0n})(\hat{\beta}_n^{(1)} - \beta_{0n}) \to_d N(0, 1) \quad \text{as} \quad n \to \infty,
\]

where \( \alpha \in S_{p_n}, \hat{\beta}_n^{(1)} \) is a one-step estimator based on the Fisher-scoring iteration with \( \hat{\beta}_n^{(0)} \) as the initial value, and \( F_n(\hat{\beta}_{0n}) \) is the Fisher information matrix evaluated at \( \beta_{0n} \).
Theorem 2 shows that the one-step MLE is well defined in probability, and it is constructed based on a $\sqrt{n/p_n}$-consistent initial estimator $\hat{\beta}_n^{(0)}$. The following subsections make use of the one-step method to construct estimators in the distributed system, and comparisons are made with the existing method in the literature.

### 3.2. Aggregated distributed estimator

This subsection reviews the AEE estimator in a fixed dimension given in [Lin & Xi (2011)] and further examines its asymptotic behaviors for GLMs with a diverging number of covariates.

The AEE estimator is defined as

$$\hat{\beta}_n^F = \sum_{k=1}^{K} \sum_{k=1}^{K} F_{nk}(\hat{\beta}_{nk})^{-1} F_{nk}(\hat{\beta}_{nk}) \hat{\beta}_{nk},$$

where $\hat{\beta}_{nk}$ is the local MLE on the $k$th machine,

$$\beta_{nk} = \arg \max_{\beta_{nk} \in \Theta_n} L_{nk}(\beta_{nk}), \quad k = 1, \ldots, K.$$

More assumptions are required to analyze the asymptotic behavior of the AEE estimator under the diverging dimension:

**Assumption 5.** The design matrix for the data in the $k$th subset $Z_k$ satisfies conditions

$$0 < C_{\min} \leq \lambda_{\min}(\frac{1}{n_k} \sum_{i=1}^{n_k} z_i z_i^T) \leq \lambda_{\max}(\frac{1}{n_k} \sum_{i=1}^{n_k} z_i z_i^T) \leq C_{\max} < \infty$$

and

$$\sup_{\alpha_1, \alpha_2 \in S_{p_n}} \sum_{i=1}^{n_k} \|\alpha_1^T z_i\|^2 \|\alpha_2^T z_i\|^2 = O(n_k).$$

**Assumption 6.** The number of observations stored on the $k$th local machine satisfies $n_k = O(n/K)(k = 1, \ldots, K)$; in other words,

$$\max_{1 \leq k \leq K} n_k / \min_{1 \leq k \leq K} n_k = O(1).$$

**Remark 2.** Similar to Assumption 7, Assumption 5 imposes some mild constraints on the design matrix for each node in the distributed system. Assumption 6 is similar to that for the AEE estimator in the literature. Like (6), the following can be derived:

$$0 < C_{\min} W_{\min} \leq \lambda_{\min}(F_{nk}(\beta_{0n})/n_k) \leq \lambda_{\max}(F_{nk}(\beta_{0n})/n_k) \leq C_{\max} W_{\max} < \infty.$$

Then, we have the following theorem.

**Theorem 3.** Suppose Assumptions 2, 5 hold. If $p_n = o(\sqrt{n})$ and the number of machines $K = o(\sqrt{n}/p_n)$, then (i) there exists an AEE estimator $\hat{\beta}_n^F$ such that

$$P\left( \left[ \sum_{k=1}^{K} F_{nk}(\hat{\beta}_{nk}) \right] \beta_n^F = \sum_{k=1}^{K} F_{nk}(\hat{\beta}_{nk}) \hat{\beta}_{nk} \right) \rightarrow 1 \quad \text{as} \quad n \rightarrow \infty,$$

and (ii) $\hat{\beta}_n^F$ satisfies

$$\alpha^T F_n^{1/2}(\beta_{0n})(\hat{\beta}_n^F - \beta_{0n}) \rightarrow_d N(0, 1) \quad \text{as} \quad n \rightarrow \infty,$$

where $\alpha \in S_{p_n}$, $\hat{\beta}_{nk}(k = 1, \ldots, K)$ are local estimators, and $F_{nk}(\hat{\beta}_{nk})$ is the Fisher information matrix evaluated at $\hat{\beta}_{nk}$.

The AEE method offers an asymptotically efficient distributed estimator $\hat{\beta}_n^F$ when $p_n = o(\sqrt{n})$ as shown in Theorem 3. However, its restrictive assumption on the number of machines, $K = o(\sqrt{n}/p_n)$, limits its widespread application.
### 3.3. One-step distributed estimator

Inspired by [Huang & Huo (2015)](citation), this section proposes a communication-efficient one-step estimator for diverging-dimensional GLMs with a more relaxed assumption for \( K \).

Our one-step estimator first takes the weighted average of the local estimates \( \hat{\beta}_{nk} \) as in (9), which is denoted by

\[
\bar{\beta}_n = \sum_{k=1}^{K} \frac{n_k}{n} \hat{\beta}_{nk}. \tag{11}
\]

Theorem 4 shows that it is \( \sqrt{n/p_n} \)-consistent under mild conditions.

**Theorem 4.** Suppose \( \lim_{n \to \infty} \log(p_n)/\log(n) = \gamma \). Under Assumptions [3][5], if \( p_n = O(\sqrt{n}) \) and the number of machines satisfies

\[
K = \begin{cases} 
O(\sqrt{n/p_n}) & \text{if } 1/3 \leq \gamma < 1/2, \\
O(n/p_n^2) & \text{if } 0 \leq \gamma < 1/3,
\end{cases}
\]

then the one-step distributed estimator \( \hat{\beta}_n = O(\sqrt{n/p_n}) \).

The proof of Theorem 4 requires the consistency of \( \hat{\beta}_{nk} \), which needs to assume \( p = O(\sqrt{n}) \). Then, given Assumption 6, we have the second part of the condition on \( K \), i.e., \( K = O(n/p_n^2) \). By contrast, the consistency of \( \hat{\beta}_{nk} \) will not hold if \( K \) is too large. It requires \( K = O(\sqrt{n/p_n}) \) to make the consistency valid.

Further, the weighted average is transmitted to local machines to get \( S_n(\bar{\beta}_n) \) and \( F_n(\bar{\beta}_n) \), which are then sent back to the central node. Finally, a Fisher-scoring iteration is performed to compute the one-step estimator \( \bar{\beta}_n^{(1)} \), which is asymptotically efficient by Theorem 2.

**Algorithm: Asymptotically Efficient One-step Distributed Estimation**

1. Compute local GLM estimators \( \hat{\beta}_{nk} (k = 1, \ldots, K) \) on each local machine, and transmit them to the central machine.
2. Take the weighted average of all local estimators at the central machine,

\[
\bar{\beta}_n = \sum_{k=1}^{K} \frac{n_k}{n} \hat{\beta}_{nk},
\]

and transmit it to the local machines.
3. Calculate the local score functions \( S_n(\bar{\beta}_n) \) and the observed Fisher information \( F_n(\bar{\beta}_n) \) on \( \bar{\beta}_n \) at each local machine, and transmit the results to the central node.
4. Calculate the global score function and the observed Fisher information at the central machine

\[
S_n(\bar{\beta}_n) = \sum_{k=1}^{K} S_n(\hat{\beta}_{nk}), \quad F_n(\bar{\beta}_n) = \sum_{k=1}^{K} F_n(\hat{\beta}_{nk}).
\]
5. Perform a single Fisher-scoring iteration at the central machine to get the one-step distributed estimator

\[
\bar{\beta}_n^{(1)} = \bar{\beta}_n + F_n^{-1}(\bar{\beta}_n) S_n(\bar{\beta}_n).
\]

**Remark 3.** The weighted-average \( \bar{\beta}_n \) can be calculated with less time cost than \( \bar{\beta}_n^F \), for it reduces the computation and communication of the \( p_n \times p_n \) matrix \( F_n(\hat{\beta}_{nk}) \).

### 4. Simulation

In this section, simulations are demonstrated to study the performance of the proposed one-step distributed estimator \( \bar{\beta}_n^{(1)} \) compared with that of other distributed estimators, including the simple average distributed estimator \( \bar{\beta}_n \) and the AEE estimator \( \beta_n^F \), and that of the global estimator \( \beta_n^{global} \), which is computed directly using the full data set. These estimators are implemented for three classical generalized linear models presented as follows.
Example 1 (Probit regression). Probit regression can be used to model binary response data. The response $Y_i$ ($i = 1, \ldots, n$) is generated independently from the Bernoulli distribution with the probability of success being

$$P(Y_i = 1|X_i) = \Phi(X_i\beta_0),$$

(12)

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. The true value of $\beta$, denoted by $\beta_0$, is assigned to be a $p \times 1$ vector of $(-0.25, 0.25, \ldots, -0.25, 0.25)^T$. We set the true parameter vector $\beta_0$ as $(-0.25, 0.25, \ldots, -0.25, 0.25)^T$.

Example 2 (Logistic regression). Logistic regression is also a commonly used model to deal with binary response data sets. Given the design matrix and the true value of $\beta$, the binary response $Y_i$ ($i = 1, \ldots, n$) is generated independently from the Bernoulli distribution as

$$P(Y_i = 1|X_i) = \frac{\exp\{X_i\beta_0\}}{1 + \exp\{X_i\beta_0\}}.$$  

(13)

We set the true parameter vector $\beta_0$ as $(-0.25, 0.25, \ldots, -0.25, 0.25)^T$.

Example 3 (Poisson regression). To model counts as a response, we consider the Poisson regression model whose response $Y_i$ ($i = 1, \ldots, n$) is generated from Poisson distribution as

$$P(Y_i|X_i) = \frac{\lambda^Y_i e^{-\lambda_i}}{Y_i!},$$

(14)

where $\lambda = \exp\{X_i\beta_0\}$ and the true parameter is set as $\beta_0 = (0.5, -0.5, \ldots, 0.5, -0.5)^T$.

For each model, we generate the i.i.d. observations $(x_i, y_i)_{i=1}^n$ for a fixed sample size $n = 2^{17}$. Each $x_i \in \mathbb{R}^p$ with $p \in \{16, 32, 64\}$ is sampled from $N(0, \Sigma)$, where $\Sigma_{ij} = 0.75^{|i-j|}$. For each $p$, to evaluate the behavior of the proposed estimator when $p$ gets higher relative to the subset sample size $n_k = \frac{n}{2^k}$ in each machine, we vary the number of machines $K$ from $2^2$ to $2^8$. It may provide some insight into the condition of $K$ required to guarantee the $\sqrt{n}/p_n$-consistency of the weighted average estimator in Theorem 4. Based on $T = 1000$ trials under each setting, we compare the performance of the three distributed estimators by computing the root-mean-square error (RMSE) for every parameter. As for the proposed one-step estimator, the RMSE for the $j$th parameter is given by

$$\text{RMSE}_j = \sqrt{\frac{1}{T} \sum_{t=1}^{T} (\hat{\beta}_{njt} - \beta_{0j})^2}, \quad j = 1, \ldots, p,$$

(15)

where $\hat{\beta}_{njt}^{(1)}$ is the $j$th element of the one-step estimator $\hat{\beta}_{nj}^{(1)}$ in the $t$th trial and $\beta_{0j}$ is the $j$th element of true parameter $\beta_0$. Given that the global estimator remains the same for a fixed $n$, let the relative efficiency (RE) of the distributed estimator concerning the global estimator be $\text{RE}_j = \text{RMSE}_j / \text{RMSE}_j^{\text{global}}$ to measure the estimation error of the current distributed method compared with that of the MLE using the full data set. We further investigate the performance of the proposed method by the coverage probability of the 95% confidence interval (CPCI). It is estimated by the proportion that the confidence interval

$$[\hat{\beta}_{njt}^{(1)} - 1.96F_n^{-1/2}(\hat{\beta}_{njt}^{(1)}), \hat{\beta}_{njt}^{(1)} + 1.96F_n^{-1/2}(\hat{\beta}_{njt}^{(1)})], \quad t = 1, \ldots, T,$$

(16)

covers the true value of $\beta_j$ in the 1000 repeated trials. Let relative coverage probability (RC) be $\text{RC}_j = \text{CPCI}_j / \text{CPCI}_j^{\text{global}}$ where the denominator $\text{CPCI}_j^{\text{global}}$ is the CPCI of the $j$th parameter for the global estimator. RMSEs and PCIs for the aforementioned three distributed estimators are calculated in the same way.

Because the results of the three examples are similar, we only report figures for the probit regression in this section, while results for other models are given in the appendix. It can be easily seen in Figure 7 that the proposed one-step distributed estimator outperforms the AEE estimator and the simple average estimator concerning RE, especially when $p$ is large. Unlike other methods, the RE of the one-step estimator remains
Figure 1: RE of the three distributed methods for the probit model as $K$ varies. The three rows of subplots illustrate how RE varies as $p$ increases from 16 to 32 and to 64. For each row, the minima, median, and maxima (from left to right) of the $R_{E,j}$ ($j = 1, \ldots, p$) are plotted against the number of machines $K$.

Figure 2: RC of the three distributed methods for the probit model as $K$ varies. The three rows of subplots illustrate how the relative coverage of the CI varies as $p$ increases from 16 to 32 and to 64. For each row, the minima, median, and maxima (from left to right) of the $R_{C,j}$ ($j = 1, \ldots, p$) are plotted against the number of machines $K$. 
to be approximately 1 as $K$ varies, suggesting that its RMSE resembles that of the global estimator even when $K$ is relatively large. As shown in Figure 2, the one-step estimator is highly competitive with regard to the CPCI. In comparison, RC for the simple average method being the lowest under each setting decays rapidly as $K$ increases. The CPCI of the AEE estimator do not perform well as $K$ increases, especially when $p$ is large. It is worth noting out that the performance of the one-step estimator becomes worse when $p$ increases, as shown in Figure 3 and it is sensitive to the choice of $K$.

![Figure 3](image)

Figure 3: Comparison of the RE and RC of the one-step estimator for different $p$ as $K$ varies in the probit model. The first row gives the minima, median, and maxima (from left to right) of the RE$_j$ as the number of machines $K$ increases; the second row provides the relative coverage of CI$_j$ ($j = 1, \ldots, p$).

5. Case Study

This section examines the performance of the proposed distributed method on a public supersymmetric (SUSY) benchmark data set (available at https://archive.ics.uci.edu/ml/datasets/SUSY). We compare the proposed distributed method with other distributed methods and the existing methods dealing with this data set in the literature, i.e., the deep learning techniques in Baldi et al. (2014) and the subsampling algorithm in Wang et al. (2018). Although the data set is not stored distributively, the dimension after the preprocess is relatively large, which could demonstrate the advantage of our method. The data set involves a two-class classification problem that aims at distinguishing a signal process from a background process based on 18 numeric covariates. The sample size is five million. We preprocess the data on the basis of the relationship between the response and each covariate. The range of covariate values is first equally divided into 1000 intervals on which the proportions of the signal process were calculated, as shown in Figure 4. All subplots show nonlinear relationships except those for $X_3, X_6,$ and $X_8$ (lepton 1 phi, lepton 2 phi, and missing energy phi), which exhibit no clear trend.

We propose an additive model for this data set. Given the non-linearity shown in Figure 4, the b-spline method is more suitable than using the linear combination of the covariates directly for this data set. Therefore, the linear forms of $X_3, X_6,$ and $X_8$ are directly used in the model, while fourth-order b-spline basis expansions are employed for other covariates. Interior knots for each covariate are set as the first, second, and third quartiles from the empirical distribution of the covariate. Then, a linear combination of
90 b-spline basis, 3 covariates, and an intercept makes the dimension in this case 94. We randomly distribute
data across $K$ local nodes, and sample sizes for each local node are the same.

To illustrate the performance of the proposed method, we calculate the average of the AUCs of the
proposed method for 100 trials, which is 87.4% ($sd < 0.001$). The average AUCs remain the same when
$K$ increases from 10 to 200. The AUC of the full data MLE is also 87.4%. They are extremely close
to the AUC of 87.6% ($sd < 0.001$) using the deep learning (DL) method given in [Baldi et al. (2014)],
suggesting that the basis expansion captures almost as many nonlinear relations among the covariates as
the “black box” deep learning. [Wang et al. (2018)] pointed out that this DL method is so complex that it
“requires special computing resources and coding skills” to build. They instead applied a logistic regression
using a linear combination of the 18 covariates and showed that AUCs of their subsampling methods are
around 85.0% ($sd \approx 0.3$), which are slightly lower than the full data AUC (85.8%). Although our method
may require more computation, it outperforms these subsampling methods in terms of AUC and takes
the distributed system into consideration, which is more practical for large-scale data sets in real-world
applications. Moreover, the proposed one-step method can be used in hypothesis tests, whereas DL and
subsampling methods cannot. The performance of the proposed method can be further improved by adding
interactions between covariates to the model and choosing the number and the locations of the knots.

RMSEs and SEs are also calculated to compare the distributed methods, including the simple average
method, the AEE estimator, and the proposed method. We repeat the trials 100 times and calculate the
SE of each $\hat{\beta}_j$ for each method as follows:

$$SE_j = \sqrt{\frac{1}{99} \sum_{i=1}^{100} (\hat{\beta}_j^{(i)} - \bar{\hat{\beta}}_j)^2}, \quad j = 1, \ldots, p,$$

(17)

where $\hat{\beta}_j^{(i)} (i = 1, \ldots, 100)$ is the estimate of the $j$th coefficient $\beta_j$ for the $i$th trial and $\bar{\hat{\beta}}_j = \sum_{i=1}^{100} \hat{\beta}_j^{(i)}/100$. As shown in Figure 5, when the number of the local nodes $K$ is as small as 10, the three distributed
estimators are found to have similar good performance. Our one-step method outperforms other methods
when $K$ increases.
6. Conclusion

Herein, we studied the asymptotic efficiency of the MLE, the one-step MLE, and the AEE estimator for GLMs with a diverging number of covariates, and proposed a novel method to build an asymptotically efficient estimator with two rounds of communication for large-scale distributed data sets. The assumption on $K$ is more relaxed in our method and is more practical than that imposed on the AEE estimator for asymptotic efficiency. The properties of the proposed method are further illustrated by our numerical experiments and the case study.

The theorems and results of this paper are all based on the assumption that $p_n$ diverges with $n$. In this case, we may wonder how to develop a distributed estimator when only a few covariates have real influence on the response. Future studies should involve consideration of variable selection into the framework.

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Appendix

A Proof of Theorem

Lemma 1. (Chen et al. 1999) Let $G$ be a smooth map from $\mathbb{R}^{p_n}$ to $\mathbb{R}^{p_n}$ such that $G(x_0) = y_0$ and $\inf_{\|x-x_0\|=\delta} \|G(x) - y_0\| \geq r$. Then, for any $y \in \{y: \|y - y_0\| \leq r\}$, there exists an $x_1 \in \{x: \|x - x_0\| \leq \delta\}$ such that $G(x_1) = y$.

Lemma 2. Suppose that Assumptions hold. If $p_n = O(\sqrt{n})$ and $K = O(n/p^2_n)$, there exists a finite $M > 0$ such that

$$\max_{1 \leq i \leq n_k} \|z_i^T \beta_{n_k}\| < M$$

for all $\beta_{n_k} \in B_{n_k}(\delta_k) = \left\{ \beta_{n_k} : p_n^{-1/2} \left\| F_n^{-1/2}(\beta_{0n})(\beta_{n_k} - \beta_{0n}) \right\| \leq \delta_k \right\}$, ($k = 1, \ldots, K$).

Proof. Note that

$$\max_{1 \leq i \leq n_k} \|z_i^T \beta_{n_k}\| \leq \max_{1 \leq i \leq n_k} \|z_i^T \beta_{0n}\| + \max_{1 \leq i \leq n_k} \|z_i^T (\beta_{n_k} - \beta_{0n})\|.$$

Since $\max_{1 \leq i \leq n_k} \|z_i^T \beta_{0n}\| \leq M_0$ according to Assumption, the lemma follows if $\max_{1 \leq i \leq n_k} \|z_i^T (\beta_{n_k} - \beta_{0n})\|$ is bounded.

From Assumption 6 and Assumption 1,

$$\max_{1 \leq i \leq n_k} \|z_i^T (\beta_{n_k} - \beta_{0n})\|^2 \leq \max_{1 \leq i \leq n_k} \left\| z_i^T F_n^{-1/2}(\beta_{0n}) \right\|^2 \left\| F_n^{-1/2}(\beta_{0n})(\beta_{n_k} - \beta_{0n}) \right\|^2 \leq C^{-1} \min_k n_k^{-1} p_n \delta_k^2 \max_{1 \leq i \leq n_k} \|z_i\|^2 = O(K p_n^2 / n) = O(1).$$

This completes the proof.

Remark 4. Lemma 2 shows that when $p_n = O(\sqrt{n})$ and $K = O(n/p^2_n)$, $\|z_i^T \beta_{n_k}\|$ is always bounded for all $\beta_{n_k} \in B_{n_k}(\delta_k)$. Then, from Assumption 4, $u''(z_i^T \beta_{n_k})$, $u'''(z_i^T \beta_{n_k})$, and $w'(z_i^T \beta_{n_k})$ ($i = 1, \ldots, n_k$, $k = 1, \ldots, K$) are also bounded.

Proof of Theorem

(i) Let $\alpha \in S_{p_n}$. Denote $\hat{\beta}_n \in B_\delta(\delta) = \left\{ \beta_{n_k} : p_n^{-1/2} \left\| F_n^{-1/2}(\beta_{0n})(\beta_{n_k} - \beta_{0n}) \right\| \leq \delta \right\}$, $\delta > 0$. At first, we prove

$$\alpha^T F_n^{-1/2}(\beta_{0n}) S_n(\beta_{0n}) \rightarrow_d N(0, 1).$$

(A.1)
Let $\xi_i = \mathbf{a}^T F_n^{-1/2}(\mathbf{\beta}_0 n) z_i u'(z_i^T \mathbf{\beta}_0) e_i$. It is easy to check that $E(\xi_i) = 0$ and $\text{Var}(\sum_{i=1}^n \xi_i) = 1$. Then from the Lindeberg central limit theorem, (A.1) follows if

$$
g_n(\varepsilon) = \sum_{i=1}^n E\{ |\xi_i|^2 I(|\xi_i| > \varepsilon) \} \to 0$$

for any $\varepsilon > 0$ as $n \to \infty$. By Assumptions 1 and (6),

$$
\sum_{i=1}^n \left| \mathbf{a}^T F_n^{-1/2}(\mathbf{\beta}_0) z_i \right|^2 = \sum_{i=1}^n \mathbf{a}^T F_n^{-1/2}(\mathbf{\beta}_0) z_i z_i^T \mathbf{a}^{-1/2}(\mathbf{\beta}_0) \mathbf{a}^T \leq \alpha \left( \sum_{i=1}^n z_i z_i^T \right)^{-1/2} \sum_{i=1}^n z_i z_i^T \left[ \sum_{i=1}^n z_i^T z_i \right]^{-1/2} \mathbf{a}^{-1/2} (A.2)
$$

Let $a_i = \mathbf{a}^T F_n^{-1/2}(\mathbf{\beta}_0) z_i u'(z_i^T \mathbf{\beta}_0)$. By Assumptions 3 and 4, we have $\max_{1 \leq i \leq n} u'(z_i^T \mathbf{\beta}_0) = O(1)$, then similar to (A.2), we can show that $\max_{1 \leq i \leq n} |a_i| = O(1)$. Also (A.2) implies that $\sum_{i=1}^n |a_i|^2$ is bounded. Thus, by Assumption 2, we get for $\alpha = r - 2 > 0$

$$
g_n(\varepsilon) = \sum_{i=1}^n |a_i|^2 E\{ |e_i|^2 I(|e_i| > \varepsilon/|a_i|) \}
\leq \sum_{i=1}^n |a_i|^2 |a_i|^\alpha \varepsilon^{-\alpha} E|e_i|^\alpha
\leq \varepsilon^{-\alpha} \max_{1 \leq i \leq n} |a_i|^\alpha \sup_{1 \leq i \leq n} E|e_i|^\alpha \sum_{i=1}^n |a_i|^2 \to 0,
$$

which implies (A.1). According to the assumption, $S_n(\mathbf{\beta}_n)$ is continuously differentiable with respect to $\mathbf{\beta}_n$, then we have

$$
S_n(\mathbf{\beta}_n) - S_n(\mathbf{\beta}_0 n) = H_n^*(\mathbf{\beta}_n) (\mathbf{\beta}_n - \mathbf{\beta}_0 n), (A.3)
$$

where $H_n^*(\mathbf{\beta}_n) = \int_0^1 H_n(\mathbf{\beta}_n + t(\mathbf{\beta}_n - \mathbf{\beta}_0 n)) dt$. Now, we proceed to prove

$$
\sup_{\mathbf{\beta}_n, \mathbf{\beta}_0 n \in B_n(\delta)} \left| \mathbf{a}^T F_n^{-1/2}(\mathbf{\beta}_0 n) H_n^*(\mathbf{\beta}_n) F_n^{-1/2}(\mathbf{\beta}_0 n) \mathbf{a} + 1 \right| \to 0 (A.4)
$$

and

$$
\sup_{\mathbf{\beta}_1 n, \mathbf{\beta}_2 n \in B_n(\delta)} \left| \mathbf{a}^T F_n^{-1/2}(\mathbf{\beta}_0 n) H_n^*(\mathbf{\beta}_1 n, \mathbf{\beta}_2 n) F_n^{-1/2}(\mathbf{\beta}_0 n) \mathbf{a} + 1 \right| \to 0, (A.5)
$$

where $H_n^*(\mathbf{\beta}_1 n, \mathbf{\beta}_2 n) = \int_0^1 H_n(\mathbf{\beta}_1 n + t(\mathbf{\beta}_2 n - \mathbf{\beta}_1 n)) dt$. It is enough to prove, as $n \to \infty$,

$$
\sup_{\mathbf{\beta}_n \in B_n(\delta)} \left| \mathbf{a}^T F_n^{-1/2}(\mathbf{\beta}_0 n) H_n(\mathbf{\beta}_n) F_n^{-1/2}(\mathbf{\beta}_0 n) \mathbf{a} + 1 \right| \to 0. (A.6)
$$

By a direct computation and decomposition, we get

$$
\mathbf{a}^T F_n^{-1/2}(\mathbf{\beta}_0 n) H_n(\mathbf{\beta}_n) F_n^{-1/2}(\mathbf{\beta}_0 n) \mathbf{a} + 1 = K_{11} + K_{12} + K_{13} + K_{14}, (A.7)
$$

where

$$
K_{11} = \mathbf{a}^T F_n^{-1/2}(\mathbf{\beta}_0 n) \left[ F_n(\mathbf{\beta}_n) - F_n(\mathbf{\beta}_0 n) \right] F_n^{-1/2}(\mathbf{\beta}_0 n) \mathbf{a},
$$

and
\[ K_{12} = \alpha^T F_n^{-1/2}(\beta_{0n}) \sum_{i=1}^{n} z_i \nu''(z_i^T \beta_n) c_i z_i^T F_n^{-1/2}(\beta_{0n}) \alpha, \]
\[ K_{13} = \alpha^T F_n^{-1/2}(\beta_{0n}) \sum_{i=1}^{n} z_i \left[ u''(z_i^T \beta_n) - u''(z_i^T \beta_{0n}) \right] c_i z_i^T F_n^{-1/2}(\beta_{0n}) \alpha, \]
\[ K_{14} = \alpha^T F_n^{-1/2}(\beta_{0n}) \sum_{i=1}^{n} z_i u''(z_i^T \beta_n) \left[ h(z_i^T \beta_{0n}) - h(z_i^T \beta_n) \right] z_i^T F_n^{-1/2}(\beta_{0n}) \alpha. \]

By (6), we have
\[ \| \alpha^T F_n^{-1/2}(\beta_{0n}) \|^2 \alpha = C_{\min}^{-1} W_{\min}^{-1} n^{-1} \| \alpha \|^2 \]

and
\[ \| \beta_n - \beta_{0n} \|^2 = \left\| F_n^{-1/2}(\beta_{0n}) F_n^{-1/2}(\beta_{0n})(\beta_n - \beta_{0n}) \right\|^2 \]
\[ \leq C_{\min}^{-1} W_{\min}^{-1} n^{-1} \left\| F_n^{1/2}(\beta_{0n})(\beta_n - \beta_{0n}) \right\|^2 \]

then by the Cauchy-Schwarz inequality, and Assumptions 1 and (A.2), we obtain that
\[ \sum_{i=1}^{n} \left| \alpha^T F_n^{-1/2}(\beta_{0n}) z_i \right|^2 \left| z_i^T (\beta_n - \beta_{0n}) \right| \]
\[ \leq \left\{ \sum_{i=1}^{n} \left| \alpha^T F_n^{-1/2}(\beta_{0n}) z_i \right|^2 \left| z_i^T (\beta_n - \beta_{0n}) \right|^2 \right\}^{1/2} \left\{ \sum_{i=1}^{n} \left| \alpha^T F_n^{-1/2}(\beta_{0n}) z_i \right|^2 \right\}^{1/2} \]

\[ = O(\sqrt{p_n/n}). \]

Note that,
\[ |K_{11}| = \left| \alpha^T F_n^{-1/2}(\beta_{0n}) \sum_{i=1}^{n} z_i w'(z_i^T \beta_n^*) z_i^T (\beta_n - \beta_{0n}) \right| \]
\[ \leq \max_{1 \leq i \leq n} \left| w'(z_i^T \beta_n^*) \sum_{i=1}^{n} \alpha^T F_n^{-1/2}(\beta_{0n}) z_i \right|^2 \left| z_i^T (\beta_n - \beta_{0n}) \right|, \]

where \( \beta_n^* \) is between \( \beta_{0n} \) and \( \beta_n \), and according to Remark 4, \( \max_{1 \leq i \leq n} \left| w'(z_i^T \beta_n^*) \right| = O(1) \), then by (A.10) we have
\[ \sup_{\beta_n \in B_n(\delta)} |K_{11}| = O(\sqrt{p_n/n}) \to 0. \]

To prove \( |K_{12}| \to 0 \), we only need to prove that \( \text{Var}(K_{12}) \to 0 \) as \( n \to \infty \) because \( \text{E}(K_{12}) = 0 \). By Assumptions 3 and 4 we get \( \max_{1 \leq i \leq n} \left| u''(z_i^T \beta_{0n}) \right|^2 v(z_i^T \beta_{0n}) = O(1) \). Then by (A.8) and Assumption 1 we have
\[ \text{Var}(K_{12}) = \sum_{i=1}^{n} \left| \alpha^T F_n^{-1/2}(\beta_{0n}) z_i \right|^4 \left| u''(z_i^T \beta_{0n}) \right|^2 v(z_i^T \beta_{0n}) \]
\[ \leq \max_{1 \leq i \leq n} \left| u''(z_i^T \beta_{0n}) \right|^2 v(z_i^T \beta_{0n}) \sum_{i=1}^{n} \left| \alpha^T F_n^{-1/2}(\beta_{0n}) z_i \right|^4 \]
\[ = O(n^{-1}), \]
which implies

\[ |K_{12}| = O_p(n^{-1/2}) \to 0. \]  

(A.12)

As for the term \( K_{13} \), we have

\[
E \sup_{\beta_n \in B_n(\delta)} |K_{13}| \leq \sup_{1 \leq i \leq n} E|e_i| \sup_{\beta_n \in B_n(\delta)} \sum_{i=1}^{n} |\alpha^T F_n^{-1/2}(\beta_{0n})z_i| |u''(\beta_n^*)z_i^T(\beta_n - \beta_{0n})| .
\]

where \( \beta_n^* \) is between \( \beta_{0n} \) and \( \beta_n \), \(|e_i|\) is bounded with probability tending to 1 and according to Remark 4, \( \max_{1 \leq i \leq n} |u''(\beta_n^*)| = O(1) \). Thus, by (A.16), we get

\[
\sup_{\beta_n \in B_n(\delta)} |K_{13}| = O_p(\sqrt{p_n/n}) \to 0. \]  

(A.13)

Similar to the arguments for \( K_{11} \), we may prove

\[
\sup_{\beta_n \in B_n(\delta)} |K_{14}| = O(\sqrt{p_n/n}) \to 0. \]  

(A.14)

By (A.11), (A.12), (A.13), and (A.14), we get (A.4), (A.5), and (A.6).

Now, we prove that, for any \( \varepsilon > 0 \), there exist a \( \delta > 0 \) such that when \( n \) is large enough,

\[
P \left( \text{there is } \hat{\beta}_n \in B_n(\delta) \text{ s.t. } S_n(\hat{\beta}_n) = 0 \right) > 1 - \varepsilon. \]  

(A.15)

Let \( \partial B_n(\delta) = \left\{ \beta_n : p_n^{-1/2} \| F_n^{-1/2}(\beta_{0n})(\beta_n - \beta_{0n}) \| = \delta \right\} \). Note that \( p_n^{-1/2} \| F_n^{-1/2}(\beta_{0n})(\beta_n - \beta_{0n}) \| / \delta = 1 \) for \( \beta_n \in \partial B_n(\delta) \). By the Cauchy-Schwartz inequality, we have that for any \( \delta > 0 \),

\[
\inf_{\beta_n \in \partial B_n(\delta)} (\beta_n - \beta_{0n})^T H_n^*(\beta_n) p_n^{-1/2}(\beta_{0n}) H_n^*(\beta_n)(\beta_n - \beta_{0n}) \\
\geq \inf_{\beta_n \in \partial B_n(\delta)} \left[ p_n^{-1/2} (\beta_n - \beta_{0n})^T H_n^*(\beta_n)(\beta_n - \beta_{0n}) \right]^2. \]  

(A.16)

By (A.4), for any \( \varepsilon > 0 \) and \( \delta > 0 \), there is a \( c_0 \in (0,1) \) independent of \( \delta \), such that

\[
P \left( \inf_{\beta_n \in \partial B_n(\delta)} \left| \alpha^T F_n^{-1/2}(\beta_{0n}) H_n^*(\beta_n) F_n^{-1/2}(\beta_{0n}) \alpha \right| > c_0 \right) > 1 - \varepsilon/4. \]  

(A.17)

By (A.3), (A.16), and (A.17), it follows that, for any \( \delta > 0 \), when \( n \) is large enough,

\[
P \left( \inf_{\beta_n \in \partial B_n(\delta)} p_n^{-1/2} \| F_n^{-1/2}(\beta_{0n}) \left( S_n(\beta_n) - S_n(\beta_{0n}) \right) \| \geq c_0 \delta \right) > 1 - \varepsilon/4. \]  

(A.18)

Taking \( \delta = \sqrt{(4/\varepsilon)/c_0} \), by the Markov inequality, we have

\[
P \left( p_n^{-1/2} \| F_n^{-1/2}(\beta_{0n}) S_n(\beta_{0n}) \| \leq c_0 \delta \right) \\
= 1 - P \left( p_n^{-1/2} \| F_n^{-1/2}(\beta_{0n}) S_n(\beta_{0n}) \|^2 / (c_0 \delta)^2 \right) \\
= 1 - \varepsilon/4. \]  

(A.19)

Let \( E_n = \left\{ \| F_n^{-1/2}(\beta_{0n}) S_n(\beta_{0n}) \| \leq \inf_{\beta_n \in \partial B_n(\delta)} \| F_n^{-1/2}(\beta_{0n}) \left( S_n(\beta_n) - S_n(\beta_{0n}) \right) \| \right\} \). By (A.18) and (A.19), when \( n \) is large enough,

\[
P(E_n) > 1 - \varepsilon/2. \]

Let \( E_n^* = \left\{ \det \{ H_n^*(\beta_{0n}) \} \neq 0 \text{ for all } \beta_{0n}, \beta_n \in B_n(\delta) \right\} \). By (A.5), when \( n \) is large enough,

\[
P(E_n^*) > 1 - \varepsilon/2. \]
Using Lemma [1] we know that, on $E_n \cap E'_n$, there is $\hat{\beta}_n \in B_n(\delta)$ such that $S_n(\hat{\beta}_n) = 0$. Then (A.15) holds. (ii) Finally, we prove
\begin{equation}
\alpha^T F_n^{-1/2}(\beta_{0n})(\hat{\beta}_n - \beta_{0n}) \to_d N(0, 1). \tag{A.20}
\end{equation}
From (A.3) and (A.15),
\begin{equation}
P(\text{there exists a } \hat{\beta}_n \in B_n(\delta) \text{ s.t. } S_n(\beta_{0n}) + H_n^*(\hat{\beta}_n)(\hat{\beta}_n - \beta_{0n}) = 0) \to 1.
\end{equation}
Then, we can focus on the set of $\hat{\beta}_n \in B_n(\delta)$ on which $S_n(\beta_{0n}) + H_n^*(\hat{\beta}_n)(\hat{\beta}_n - \beta_{0n}) = 0$. Note that
\begin{equation}
\alpha^T F_n^{-1/2}(\beta_{0n})(\hat{\beta}_n - \beta_{0n}) = \alpha^T F_n^{-1/2}(\beta_{0n})S_n(\beta_{0n}) + \alpha^T F_n^{-1/2}(\beta_{0n})[H_n^*(\hat{\beta}_n) + F_n(\beta_{0n})](\hat{\beta}_n - \beta_{0n}).
\end{equation}
Similar to the arguments for $\sup_{\beta_n \in B_n(\delta)} |\alpha^T F_n^{-1/2}(\beta_{0n})[H_n(\beta_n) + F_n(\beta_{0n})](\beta_n - \beta_{0n})|$, we can get
\begin{equation}
\sup_{\beta_n \in B_n(\delta)} |\alpha^T F_n^{-1/2}(\beta_{0n})[H_n(\beta_n) + F_n(\beta_{0n})](\beta_n - \beta_{0n})| = O_p(p_n/\sqrt{n}). \tag{A.21}
\end{equation}
Denote $K_2(\beta_n) = |\alpha^T F_n^{-1/2}(\beta_{0n})[H_n(\beta_n) + F_n(\beta_{0n})](\beta_n - \beta_{0n})|$, and (A.21) implies for all $\varepsilon' > 0$, there exist a $\delta' > 0$ such that when $n$ is large enough
\begin{equation}
P \left( \left| p_n^{-1/2} K_2(\hat{\beta}_n) \right| < \delta' \left| \hat{\beta}_n \in B_n(\delta) \right| \right) > 1 - \varepsilon',
\end{equation}
from which we can obtain
\begin{equation}
P \left( \left| p_n^{-1/2} K_2(\hat{\beta}_n) \right| \geq \delta' \left| \hat{\beta}_n \in B_n(\delta) \right| \right)
\leq P \left( \left| p_n^{-1/2} K_2(\hat{\beta}_n) \right| \geq \delta' \left| \hat{\beta}_n \in B_n(\delta) \right| \right) + P \left( \hat{\beta}_n \notin B_n(\delta) \right)
\leq \varepsilon' + \varepsilon.
\end{equation}
Thus,
\begin{equation}
\alpha^T F_n^{-1/2}(\beta_{0n})[H_n^*(\hat{\beta}_n) + F_n(\beta_{0n})](\hat{\beta}_n - \beta_{0n}) = O_p(p_n/\sqrt{n}).
\end{equation}
Since $p_n^2 = o(n)$, we have
\begin{equation}
\alpha^T F_n^{-1/2}(\beta_{0n})(\hat{\beta}_n - \beta_{0n}) = \alpha^T F_n^{-1/2}(\beta_{0n})S_n(\beta_{0n}) + o_p(1); \tag{A.22}
\end{equation}
therefore, (A.20) follows from (A.1) and (A.22).

\section*{B Proof of Theorem \[\text{2}\]}

\textbf{Lemma 3.} Suppose that Assumptions [3][4] hold. If $p_n = O(\sqrt{n})$ and $K = O(n/p_n^2)$, then
\begin{equation}
P \left( \frac{1}{K} \sum_{k=1}^{K} F_n(\beta_{nk}) \text{ is positive definite} \right) \to 1.
\end{equation}

\textbf{Proof.} According to Assumption [4]
\begin{equation}
p_n^2/n_k = O(Kp_n^2/n) = O(1).
\end{equation}
Theorem [1] shows that the local MLE $\hat{\beta}_{nk}(k = 1, \ldots, K)$ satisfies
\begin{equation}
P \left( \hat{\beta}_{nk} \in B_{nk}(\delta_k) \right) \to 1. \tag{A.23}
\end{equation}
Let $\lambda_j(A)$ be the $j$th eigenvalue of matrix $A$. For the symmetric matrix $[F_{n_k}(\beta_{n_k}) - F_{n_k}(\beta_{0n})]$, there exists an $\alpha_j \in S_{p_n}$ such that

$$\lambda_j \left( F_{n_k}(\beta_{n_k}) - F_{n_k}(\beta_{0n}) \right) = \left| \sum_{i=1}^{n_k} \alpha_j^T z_{i+} (z_{i+}^T \beta_{n_k}^*) z_{i+}^T (\beta_{n_k} - \beta_{0n}) z_{i+} \alpha_j \right| \leq \max_{1 \leq i \leq n_k} \left| w^T (z_{i+}^T \beta_{n_k}^*) \right| \left| \sum_{i=1}^{n_k} \alpha_j^T z_{i+}^T (\beta_{n_k} - \beta_{0n}) z_{i+} \alpha_j \right|, \quad j = 1, \ldots, p_n,$$

where $\beta_{n_k} \in B_{n_k}(\delta_k)$ and $\beta_{n_k}^*$ is between $\beta_{0n}$ and $\beta_{n_k}$. Using the argument similar to that in (A.9), we can show that

$$||\beta_{n_k} - \beta_{0n}||^2 = O(p_n/n_k), \quad k = 1, \ldots, K. \quad (A.24)$$

Then the Cauchy-Schwarz inequality and Assumption 5 imply that

$$\left| \sum_{i=1}^{n_k} \alpha_j^T z_{i+} (\beta_{n_k} - \beta_{0n}) z_{i+} \alpha_j \right| \leq \left\{ \sum_{i=1}^{n_k} |\alpha_j^T z_{i+}|^2 \right\}^{1/2} \left\{ \sum_{i=1}^{n_k} |z_{i+} \alpha_j|^2 \right\}^{1/2} = O(\sqrt{p_n n_k}). \quad (A.25)$$

This result and Remark 4, max$_{1 \leq i \leq n_k} |w^T (z_{i+}^T \beta_{n_k}^*)| = O(1)$, show that

$$\lambda_j \left( F_{n_k}(\beta_{n_k}) - F_{n_k}(\beta_{0n}) \right) = O(\sqrt{p_n n_k}), \quad j = 1, \ldots, p_n. \quad (A.26)$$

Furthermore, by (10), we can obtain that

$$\lambda_{\min} \left( n^{-1} \sum_{k=1}^{K} F_{n_k}(\beta_{n_k}) \right) \geq n^{-1} \sum_{k=1}^{K} \lambda_{\min} \left( F_{n_k}(\beta_{0n}) \right) + n^{-1} \sum_{k=1}^{K} \lambda_{\min} \left( F_{n_k}(\beta_{n_k}) - F_{n_k}(\beta_{0n}) \right) \geq W_{\min} C_{\min} + O(\sqrt{p_n K / n_k}),$$

and

$$\lambda_{\max} \left( n^{-1} \sum_{k=1}^{K} F_{n_k}(\beta_{n_k}) \right) \leq n^{-1} \sum_{k=1}^{K} \lambda_{\max} \left( F_{n_k}(\beta_{0n}) \right) + n^{-1} \sum_{k=1}^{K} \lambda_{\max} \left( F_{n_k}(\beta_{n_k}) - F_{n_k}(\beta_{0n}) \right) \leq W_{\max} C_{\max} + O(\sqrt{p_n K / n_k}),$$

which implies

$$W_{\min} C_{\min} + o(1) \leq \lambda_{\min} \left( n^{-1} \sum_{k=1}^{K} F_{n_k}(\beta_{n_k}) \right) \leq \lambda_{\max} \left( n^{-1} \sum_{k=1}^{K} F_{n_k}(\beta_{n_k}) \right) \leq W_{\max} C_{\max} + o(1). \quad (A.27)$$

Thus, combining (A.23) and (A.27), we ascertain that the matrix $\sum_{k=1}^{K} F_{n_k}(\beta_{n_k})$ is positive definite with a probability approaching one.

**Proof of Theorem 2**

(i) Since $\beta_{n}^{(0)}$ is $\sqrt{n}/p_n$-consistent,

$$P \left( \beta_{n}^{(0)} \in B_n(\delta) \right) \to 1, \quad (A.28)$$

17
and similar to Lemma 3, we can easily deduce that $F_n(\hat{\beta}_{n0}^{(0)})$ is also positive definite in probability. Therefore, there exists a $\beta_n^{(1)}$ such that

$$P \left( F_n(\hat{\beta}_{n}^{(0)})(\beta_n^{(1)} - \hat{\beta}_{n0}^{(0)}) = S_n(\beta_n^{(0)}) \right) \rightarrow 1.$$ 

(ii) Now, we start to prove

$$\alpha^T F_n^{1/2}(\beta_{n0})(\beta_n^{(1)} - \beta_{n0}) \rightarrow_d N(0,1).$$  \hspace{1cm} (A.29)

Since

$$S_n(\hat{\beta}_{n}^{(0)}) = S_n(\beta_{n0}) + H_n(\beta_{n0}^{(0)})(\hat{\beta}_{n}^{(0)} - \beta_{n0}),$$

where $H_n(\beta_{n0}^{(0)}) = \int_0^1 H_n(\beta_{n0} + t(\hat{\beta}_{n}^{(0)} - \beta_{n0}))dt$, by a direct computation, we have

$$\alpha^T F_n^{1/2}(\beta_{n0})(\beta_n^{(1)} - \beta_{n0}) = \alpha^T F_n^{1/2}(\beta_{n0})S_n(\beta_{n0})$$

$$+ \alpha^T F_n^{1/2}(\beta_{n0})F_n^{-1}(\beta_{n0}) \left[ F_n^{-1}(\beta_{n0}) - F_n^{-1}(\beta_{n0}) \right] S_n(\beta_{n0})$$

$$+ \alpha^T F_n^{1/2}(\beta_{n0})F_n^{-1}(\beta_{n0}) \left[ H_n^{*}(\beta_{n0}^{(0)}) + F_n(\beta_{n0}^{(0)}) \right] (\beta_n^{(0)} - \beta_{n0}).$$  \hspace{1cm} (A.30)

We first show that the norm of the second part of the right-hand side of (A.30) is $O_p(p_n/\sqrt{n})$. For any $\beta_n \in B_n(\delta)$, the Cauchy-Schwarz inequality implies that

$$\left\| \alpha^T F_n^{1/2}(\beta_{n0}) \left[ F_n^{-1}(\beta_{n0}) - F_n^{-1}(\beta_{n0}) \right] S_n(\beta_{n0}) \right\|$$

$$\leq \left\| \alpha^T F_n^{1/2}(\beta_{n0})F_n^{-1}(\beta_{n0}) \left[ F_n(\beta_{n0}) - F_n(\beta_{n0}) \right] \right\| \left\| F_n^{-1}(\beta_{n0}) S_n(\beta_{n0}) \right\|.$$ 

Since $E[S_n(\beta_{n0})] = 0$, we obtain that

$$E\|S_n(\beta_{n0})\|^2 = \text{Tr} \{ F_n(\beta_{n0}) \} = O(p_n),$$

which implies that

$$\| S_n(\beta_{n0}) \| = O_p(\sqrt{n}).$$  \hspace{1cm} (A.31)

and furthermore,

$$\| F_n^{-1}(\beta_{n0}) S_n(\beta_{n0}) \| \leq (C_{\text{min}} W_{\text{min}} n)^{-1} \| S_n(\beta_{n0}) \| = O_p(\sqrt{n}).$$  \hspace{1cm} (A.32)

Similar to the arguments for (A.26), we may prove

$$\lambda_j \left\{ F_n(\beta_{n0}) - F_n(\beta_{n0}) \right\} = O(\sqrt{n}), \quad j = 1, \ldots, p_n,$$  \hspace{1cm} (A.33)

and then

$$C_{\text{min}} W_{\text{min}} + o(1) \leq \lambda_{\text{min}} \left( n^{-1} F_n(\beta_{n0}) \right) \leq \lambda_{\text{max}} \left( n^{-1} F_n(\beta_{n0}) \right) \leq C_{\text{max}} W_{\text{max}} + o(1).$$  \hspace{1cm} (A.34)

By (A.31), (A.33), and (A.34),

$$\left\| \alpha^T F_n^{1/2}(\beta_{n0}) F_n^{-1}(\beta_{n0}) \left[ F_n(\beta_{n0}) - F_n(\beta_{n0}) \right] \right\|^2$$

$$\leq \lambda_{\text{max}}^{-2} \left( F_n(\beta_{n0}) - F_n(\beta_{n0}) \right) \left( C_{\text{min}} W_{\text{min}} + o(1) \right)^{-2} C_{\text{max}} W_{\text{max}} n^{-1} \| \alpha \|^2$$  \hspace{1cm} (A.35)

$$= O(p_n).$$

Hence, (A.32) and (A.35) imply that

$$\left\| \alpha^T F_n^{1/2}(\beta_{n0}) \left[ F_n^{-1}(\beta_{n0}) - F_n^{-1}(\beta_{n0}) \right] S_n(\beta_{n0}) \right\| = O(p_n/\sqrt{n}).$$  \hspace{1cm} (A.36)
In addition, similar as \((A.21)\), we obtain that
\[
\left| \alpha^T F_n^{1/2}(\beta_{bn}) F_n^{-1}(\beta_n) [H_n(\beta_n) + F_n(\beta_n)](\beta_n - \beta_{bn}) \right| = O_p\left(\frac{p_n}{\sqrt{n}}\right). \tag{A.37}
\]
Thus, by \((A.28), (A.36), (A.37)\), and the assumption \(p_n = o(\sqrt{n})\), \((A.29)\) follows from
\[
\alpha^T F_n^{1/2}(\beta_{bn}) (\hat{\beta}_n(1) - \beta_{bn}) = \alpha^T F_n^{-1/2}(\beta_{bn}) S_n(\beta_{bn}) + o_p(1).
\]
This proves Theorem 2.

### C Proof of Theorem 3

(i) According to Lemma 3, the matrix \(\sum_{k=1}^{K} F_{nk}(\hat{\beta}_{nk})\) is positive definite in probability, so there exists a \(\beta_n^F\) such that
\[
\mathbb{P}\left(\sum_{k=1}^{K} F_{nk}(\hat{\beta}_{nk}) \beta_n^F = \sum_{k=1}^{K} F_{nk}(\hat{\beta}_{nk}) \beta_n^F - \beta_{bn}\right) \to 1. \tag{A.38}
\]

(ii) Note that
\[
S_n(\beta_{bn}) = S_n(\beta_{bn}) + H_{nk}^* (\hat{\beta}_{nk})(\hat{\beta}_{nk} - \beta_{bn}), \tag{A.39}
\]
where \(H_{nk}^* (\hat{\beta}_{nk}) = \int_0^1 H_{nk}(\beta_{bn} + t(\beta_{nk} - \beta_{bn})) dt\). Summing over \(k\) on both sides of \((A.38)\), we obtain
\[
S_n(\beta_{bn}) + \sum_{k=1}^{K} H_{nk}^* (\hat{\beta}_{nk})(\hat{\beta}_{nk} - \beta_{bn}) = 0.
\]
Then it follows that
\[
\alpha^T F_n^{1/2}(\beta_{bn})(\hat{\beta}_n^F - \beta_{bn}) = \alpha^T F_n^{-1/2}(\beta_{bn}) S_n(\beta_{bn})
\]
\[
+ \sum_{k=1}^{K} \alpha^T F_n^{1/2}(\beta_{bn}) \left\{ \left[ \sum_{k=1}^{K} F_{nk}(\hat{\beta}_{nk}) \right]^{-1} - F_n^{-1}(\beta_{bn}) \right\} F_{nk}(\hat{\beta}_{nk})(\hat{\beta}_{nk} - \beta_{bn})
\]
\[
+ \sum_{k=1}^{K} \alpha^T F_n^{-1/2}(\beta_{bn}) \left[ H_{nk}^* (\hat{\beta}_{nk}) + F_{nk}(\hat{\beta}_{nk}) \right](\hat{\beta}_{nk} - \beta_{bn}).
\]

For any \(\beta_{nk} \in B_{nk}(\delta_k)(k = 1, \ldots, K)\), by the Cauchy-Schwarz inequality, we obtain that
\[
\left\| \alpha^T F_n^{1/2}(\beta_{bn}) \left[ \sum_{k=1}^{K} F_{nk}(\beta_{nk}) \right]^{-1} - F_n^{-1}(\beta_{bn}) \right\| F_{nk}(\beta_{nk})(\beta_{nk} - \beta_{bn})
\]
\[
= \left\| \alpha^T F_n^{1/2}(\beta_{bn}) \left[ \sum_{k=1}^{K} F_{nk}(\beta_{nk}) \right]^{-1} \right\| \left\| \sum_{k=1}^{K} F_{nk}(\beta_{nk}) - F_{nk}(\beta_{nk}) \right\| F_n^{-1}(\beta_{bn}) F_{nk}(\beta_{nk})(\beta_{nk} - \beta_{bn}).
\]
By \((3), (A.24), (A.26), (A.27)\), we have
\[
\left\| \alpha^T F_n^{1/2}(\beta_{bn}) \left[ \sum_{k=1}^{K} F_{nk}(\beta_{nk}) \right]^{-1} \right\|^2 \leq \left( C_{\min} W_{\min} + o(1) \right)^{-2} C_{\max} W_{\max} n^{-1} \| \alpha \|^2
\]
\[
= O(n^{-1})
\]
19
and
\[
\left\| \sum_{k=1}^{K} \left[ F_{n_k}(\beta_{0n}) - F_{n_k}(\beta_{nk}) \right] F_n^{-1}(\beta_{0n}) F_{n_k}(\beta_{nk}) (\beta_{nk} - \beta_{0n}) \right\|^2
\]
\leq \sum_{k=1}^{K} \left\| \left[ F_{n_k}(\beta_{0n}) - F_{n_k}(\beta_{nk}) \right] F_n^{-1}(\beta_{0n}) F_{n_k}(\beta_{nk}) (\beta_{nk} - \beta_{0n}) \right\|^2
\leq \sum_{k=1}^{K} \lambda_{\text{max}}^2 \left( F_{n_k}(\beta_{0n}) - F_{n_k}(\beta_{nk}) \right) C_{\text{min}}^{-2} W_{\text{min}}^{-2} \left( C_{\text{max}} W_{\text{max}} + o(1) \right) n_k^2 \| \beta_{nk} - \beta_{0n} \|^2
= O(p_n^2).

Then,
\[
\left\| \alpha^T F_n^{1/2}(\beta_{0n}) \left\{ \left( \sum_{k=1}^{K} F_{n_k}(\beta_{nk}) \right)^{-1} - F_n^{-1}(\beta_{0n}) \right\} F_{n_k}(\beta_{nk}) (\beta_{nk} - \beta_{0n}) \right\| = O(p_n/\sqrt{n}). \tag{A.39}
\]

In addition, using a similar argument to the one used for \( \text{(A.21)} \), we may prove
\[
\left\| \alpha^T F_n^{-1/2}(\beta_{0n}) \left[ H_{n_k}(\beta_{nk}) + F_{n_k}(\beta_{nk}) \right] (\beta_{nk} - \beta_{0n}) \right\| = O(p_n/\sqrt{n}). \tag{A.40}
\]

Since \( p_n = o(\sqrt{n}) \) and \( K = o(\sqrt{n}/p_n) \), combining \( \text{(A.39)} \), \( \text{(A.40)} \), and the fact that \( P \left( \hat{\beta}_{nk} \in B_{nk}(\delta_k) \right) \rightarrow 1, k = 1, \ldots, K \), we find that
\[
\alpha^T F_n^{1/2}(\beta_{0n})(\hat{\beta}_n - \beta_{0n}) = \alpha^T F_n^{-1/2}(\beta_{0n}) S_n(\beta_{0n}) + o_p(1).
\]

This proves Theorem 3.

**D Proof of Theorem 4**

By the definition of \( \hat{\beta}_n \),
\[
\hat{\beta}_n - \beta_{0n} = \sum_{k=1}^{K} \frac{n_k}{n} (\hat{\beta}_{nk} - \beta_{0n}).
\]

Note that
\[
\hat{\beta}_{nk} - \beta_{0n} = F_n^{-1}(\beta_{0n}) S_{nk}(\beta_{0n}) + F_n^{-1}(\beta_{0n}) \left[ H_{nk}^*(\hat{\beta}_{nk}) + F_{nk}(\beta_{0n}) \right] (\hat{\beta}_{nk} - \beta_{0n})
\]
according to \( \text{(A.38)} \); then it follows that
\[
\hat{\beta}_n - \beta_{0n} = \sum_{k=1}^{K} \frac{n_k}{n} F_n^{-1}(\beta_{0n}) S_{nk}(\beta_{0n}) + \sum_{k=1}^{K} \frac{n_k}{n} F_n^{-1}(\beta_{0n}) \left[ H_{nk}^*(\hat{\beta}_{nk}) + F_{nk}(\beta_{0n}) \right] (\hat{\beta}_{nk} - \beta_{0n}). \tag{A.41}
\]

We first take a look at the first part of the right-hand side of \( \text{(A.41)} \). Note that \( \text{Tr} [n_k F_n^{-1}(\beta_{0n})] = O(p_n) \) by \( \text{(10)} \). Since \( E \left[ \sum_{k=1}^{K} \frac{n_k}{n} F_n^{-1}(\beta_{0n}) S_{nk}(\beta_{0n}) \right] = 0 \),
\[
E \left\| \sum_{k=1}^{K} \frac{n_k}{n} F_n^{-1}(\beta_{0n}) S_{nk}(\beta_{0n}) \right\|^2 = \text{Tr} \left[ \sum_{k=1}^{K} \frac{n_k^2}{n^2} F_n^{-1}(\beta_{0n}) \right].
\]
\[
= O(p_n/n),
\]

20
which implies that
\[
\left\| \sum_{k=1}^{K} \frac{n_k}{p} F_{n_k}^{-1}(\beta_{on}) S_{n_k}(\beta_{on}) \right\| = O_p(\sqrt{p/n}).
\] (A.42)

Then, we examine the second part of the right-hand side of (A.41). For any \( \beta_{nk} \in B_{nk}(\delta_k) \), denote
\[K_3(\beta_{nk}) = \left[ H_{nk}^*(\beta_{nk}) + F_{nk}(\beta_{on}) \right](\beta_{nk} - \beta_{on}).\]
To obtain \( \|K_3(\beta_{nk})\| \), we first analyze the term
\[\sup_{\beta_{nk}^* \in B_{nk}(\delta_k)} \left[ H_{nk}(\beta_{nk}^*) + F_{nk}(\beta_{on}) \right](\beta_{nk} - \beta_{on}).\]

Note that
\[\left[ H_{nk}(\beta_{nk}^*) + F_{nk}(\beta_{on}) \right](\beta_{nk} - \beta_{on}) = K_{31} + K_{32} + K_{33} + K_{34},\]
where
\[K_{31} = \left[ F_{nk}(\beta_{on}) - F_{nk}(\beta_{nk}^*) \right](\beta_{nk} - \beta_{on}),\]
\[K_{32} = \sum_{i=1}^{n_k} z_i u''(z_i^T \beta_{nk}^*) \epsilon_i z_i^T (\beta_{nk} - \beta_{on}),\]
\[K_{33} = \sum_{i=1}^{n_k} z_i u''(z_i^T \beta_{nk}^*) \left[ h(z_i^T \beta_{on}) - h(z_i^T \beta_{nk}^*) \right] z_i u^-(z_i^T \beta_{on})\]
\[K_{34} = \sum_{i=1}^{n_k} z_i u''(z_i^T \beta_{nk}^*) \left[ h(z_i^T \beta_{on}) - h(z_i^T \beta_{nk}^*) \right] z_i u^-(z_i^T \beta_{on}).\]

Similar to the proof for (A.26), one can show that
\[\lambda_j \left( F_{nk}(\beta_{nk}^*) - F_{nk}(\beta_{nk}) \right) = O(\sqrt{p/n_k}),\] (A.43)
\[\lambda_j \left( \sum_{i=1}^{n_k} z_i \left[ u''(z_i^T \beta_{nk}^*) - u''(z_i^T \beta_{on}) \right] z_i \right) = O(\sqrt{p/n_k}),\] (A.44)
\[\lambda_j \left( \sum_{i=1}^{n_k} z_i \left[ h(z_i^T \beta_{on}) - h(z_i^T \beta_{nk}^*) \right] z_i \right) = O(\sqrt{p/n_k}), \quad j = 1, \ldots, p_n.\]

Recall that \( |\epsilon_j| \) is bounded with probability tending to 1, and \( \max_{1 \leq i \leq n_k} u''(z_i^T \beta_{nk}^*) = O(1) \) according to Remark 4. Then, by (A.24),
\[\sup_{\beta_{nk}^* \in B_{nk}(\delta_k)} \|K_{31}\| = O(p_n), \quad \sup_{\beta_{nk} \in B_{nk}(\delta_k)} \|K_{32}\| = O(p_n), \quad \sup_{\beta_{nk}^* \in B_{nk}(\delta_k)} \|K_{33}\| = O(p_n), \quad \sup_{\beta_{nk}^* \in B_{nk}(\delta_k)} \|K_{34}\| = O(p_n).\] (A.45)

Since \( \text{E}(K_{32}) = 0, \) \( K_{32} \) satisfies
\[\text{E}\|K_{32}\|^2 = \text{Tr} \left[ \sum_{i=1}^{n_k} z_i u''(z_i^T \beta_{nk}^*)^2 \epsilon_i z_i^T (\beta_{nk} - \beta_{on})^2 z_i \right] \leq \left( \max_{1 \leq i \leq n_k} u''(z_i^T \beta_{nk})^2 \epsilon_i z_i^T \beta_{nk} \right) \text{Tr} \left[ \sum_{i=1}^{n_k} z_i^T (\beta_{nk} - \beta_{on})^2 z_i \right],\]
where \( \max_{1 \leq i \leq n_k} u''(z_i^T \beta_{nk})^2 \epsilon_i z_i^T \beta_{nk} = O(1) \) according to Assumptions 3 and 4. Note that \( \sum_{i=1}^{n_k} z_i z_i^T (\beta_{nk} - \beta_{nk})^2 z_i \) is a symmetric matrix. Then, there exists a sequence \( \{\alpha_j \in S_{p_n} : j = 1, \ldots, p_n\} \) such that
\[\lambda_j \left( \sum_{i=1}^{n_k} z_i \left| z_i^T (\beta_{nk} - \beta_{on}) \right|^2 z_i \right) = \sum_{i=1}^{n_k} \alpha_j^2 z_i \left| z_i^T (\beta_{nk} - \beta_{on}) \right|^2 z_i \alpha_j.\]
By Assumptions 5 and (A.24),
\[ \sum_{i=1}^{n_k} |\alpha_i^T z_i|^2 |z_i^T (\beta_{n_k} - \beta_{0n})|^2 = O(p_n). \]

Then,
\[ \lambda_j \left( \sum_{i=1}^{n_k} |z_i|^2 |z_i^T (\beta_{n_k} - \beta_{0n})|^2 z_i^T \right) = O(p_n), \quad j = 1, \ldots, p_n. \]

and \( \operatorname{E}\|K_{32}\|^2 = O(p_n^2) \). This implies \( \|K_{32}\| = O(p_n) \). Combining this result with (A.45), we obtain that \( \|K_3(\beta_{n_k})\| = O(p_n) \). Thus,
\[ \left\| \sum_{k=1}^{K} \frac{n_k}{n} F^{-1}_{n_k} (\beta_{0n}) \left[ H_{n_k} (\beta_{n_k}) + F_{n_k} (\beta_{n_k}) \right] (\beta_{n_k} - \beta_{0n}) \right\| = O_p(K p_n/n). \tag{A.46} \]

Since \( K \leq O(\sqrt{n/p_n}) \), it follows from (A.42), (A.46), and \( P(\beta_{n_k} \in B_{n_k}(\delta_k)) \to 1 \) \((k = 1, \ldots, K)\) that \( \|\beta_n - \beta_{0n}\| = O_p(\sqrt{p_n/n}) \). That is, \( \beta_n \) is \( \sqrt{n/p_n} \)-consistent. This completes the proof.

Figure 6: RE of the three distributed methods for the logit model as \( K \) varies. The three rows of subplots illustrate how RE varies as \( p \) increases from 16 to 32 and to 64. For each row, the minima, median, and maxima (from left to right) of the \( \text{RE}_j \) \((j = 1, \ldots, p)\) are plotted against the number of machines \( K \).

22
Figure 7: RC of the three distributed methods for the logit model as $K$ varies. The three rows of subplots illustrate how the relative coverage of the CI varies as $p$ increases from 16 to 32 and to 64. For each row, the minima, median, and maxima (from left to right) of the RC$_j$ ($j = 1, \ldots, p$) are plotted against the number of machines $K$.

Figure 8: Comparison of the RE and RC of the one-step estimator for different $p$ in the logit model. The first row gives the minima, median, and maxima (from left to right) of the RE$_j$ as the number of machines $K$ increases and the second row gives the relative coverage of CI$_j$ ($j = 1, \ldots, p$).
Figure 9: RE of the three distributed methods for the Poisson model as $K$ varies. The three rows of subplots illustrate how RE varies as $p$ increases from 16 to 32 and to 64. For each row, the minima, median, and maxima (from left to right) of the $RE_j$ ($j = 1, \ldots, p$) are plotted against the number of machines $K$.

Figure 10: RC of the three distributed methods for the Poisson model as $K$ varies. The three rows of subplots illustrate how the relative coverage of the CI varies as $p$ increases from 16 to 32 and to 64. For each row, the minima, median, and maxima (from left to right) of the $RC_j$ ($j = 1, \ldots, p$) are plotted against the number of machines $K$. 

24
Figure 11: Comparison of the RE and RC of the one-step estimator for different $p$ in the Poisson model. The first row gives the minima, median, and maxima (from left to right) of the $RE_j$ as the number of machines $K$ increases and the second row gives the relative coverage of $CL_j$ ($j = 1, \ldots, p$).