A model of dynamic reflection of a longitudinal shear crack from a stressed boundary

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Abstract. The modeling of the phenomenon of reflection of the front edge of a longitudinal shear crack from a stressed boundary is considered. The front edge of crack is modeled by an extreme along the edge rupture of rates and stresses exceeding the plastic limit propagating as an orthogonal trajectory of a longitudinal wave of plastic loading. The dependence of the intensity of the reflected cracks of a simple shear and detachment (as a function of the angle of incidence of the initiating crack) on the stress at the boundary at the moment of reflection and on the physical constants of the material is investigated. The conditions for the absence of reflected cracks of shear and detachment and criteria for boundary stresses initiating the cracks of various types are obtained.

1. Introduction
The surface $S$ bordered by the spatial line $L$ named the edge of the crack is defined as physical crack in solid materials. The discontinuities of displacements and stresses are possible on the surface $S$. So, the surfaces $\wedge S$ and $\vee S$ belonging to two parts of the body separated by the surface $S$ are distinguished.

The crack propagates by growing of its surface $S$ along the crack edge $L$. The behavior of a solid body far from the edge $L$ as a geometric peculiarity of surfaces $\wedge S$ and $\vee S$ conjugated along the line $L$ is assumed to be elastic according to the mechanics of a deformable solid. In the neighborhood of the crack edge $L$, i.e., near the crack tip, the material rupture goes by shift or detachment, and the model of the elastic material does not correspond to the phenomenon of real deforming at the crack tip [1, 2].

In the work of V. M. Ievlev [3] it is shown that at the nanoscale at the crack tip there are interactions of a film nature with the physical and chemical interactions of material particles, leading to irreversible deformations during the material shape changing.

According to solid mechanics, the static deformation of a material flow near tip of the front edge $L$ of a spatial crack $S$ can be described by a model of material plastic flow [1, 2]. In conditions of dynamic deformation, the plastic flow of the material goes depending on the strain rate, which in the case of a linear approximation leads to a viscoplastic flow of the material [4, 5]. Experiments to determine the crack propagation rate $c$, i.e., the propagation in the space of its edge $L$, have shown that the rate is difficult to accurately determine because of its big value and...
rather small size of the experimental samples, and that the value of rate is close to the rates $c_1$ or $c_2$ propagation of longitudinal $\Sigma_1$ or shear $\Sigma_2$ elastic waves [2].

One of the permissible models of the dynamic deformation of the material in the neighborhood of the crack tip, taking into account the properties of elasticity, plasticity and viscosity, is the Bingham model.

2. Materials and methods

2.1. Mathematical model of spatial deformation of elasto-viscoplastic material in the neighborhood of the crack tip

Let’s consider the stress-strain state (SSS) of a solid deformable material in the neighborhood of the tip of a detachment crack. The model (figure 1) of an elasto-viscoplastic (EVP) material [5–8] admits elastic deformations only until reaching the von Mises plasticity condition with the plastic limit $K$, and then plastic deformations must be taken into account in total deformations with viscous resistance on the rates of plastic deformation

$$\sigma_{ij} = \lambda e_{kk} \delta_{ij} + 2\mu e_{ij} - 2\mu e_{ij}^p; \quad e_{ij} = e_{ij}^e + e_{ij}^p; \quad e_{ij}^p = e_{ij}^v; \quad \varepsilon_{ij}^p = \frac{\partial e_{ij}^p}{\partial t} = \frac{(I_2 - K\sqrt{2})\sigma_{ij}^p}{I_2\eta}. \quad (1)$$

Here $e_{ij} = \frac{(u_{ij} + u_{ji})}{2}$ are total Cauchy deformations; $u_i$ are displacements of material particles; $u_{i,j} = \frac{\partial u_i}{\partial x_j}$; $I_2 = \left(\sigma_{ij}^p \sigma_{ij}^p\right)^{\frac{3}{2}}$, where $\sigma_{ij}^p = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij}$ are the deviator components of the stress tensor $\sigma_{ij}$; $\lambda$, $\mu$ are Lame elastic parameters; $\eta$ is the viscosity coefficient; $t$ is the time; the top indices $e$, $p$ and $v$ correspond to elastic, plastic and viscous components, respectively.

![Figure 1](image.png)

**Figure 1.** The scheme of an element of Bingham material with elastic, plastic and viscous properties

A single “stress-strain” curve does not exist for the model of this material. The strain rate acts as a parameter of dynamic strengthening. That can be observed in dynamic experiments, when the dynamic plastic limit increases with the strain rate increasing [4].

The three-dimensional graph of possible curves $\sigma$, $e$ in the plane $\sigma$, $e$ (stress, strain) is presented in figure 2 in the space $\sigma$, $e$, $\varepsilon = \dot{\varepsilon}$ (stress, strain, strain rate).

Rheological equations (1) are a system of equations that includes linear partial differential equations (PDE) and non-linear equations for plastic strain rates $\varepsilon_{ij}^p = \frac{\partial e_{ij}^p}{\partial t}$. The von Mises plasticity condition acts as a criterion of the plastic strain rate arising. The condition can be formulated in the form

$$\varepsilon_{ij}^p = 0 \quad \text{when} \quad \sigma_{ij}^p \sigma_{ij}^p - 2K^2 < 0; \quad \varepsilon_{ij}^p \neq 0 \quad \text{when} \quad \sigma_{ij}^p \sigma_{ij}^p - 2K^2 \geq 0. \quad (2)$$
Figure 2. The scheme of dynamic loading experiments, where in the plane $\sigma, e$ the observed dynamic loading diagram is a projection of a specific spatial curve $f(\sigma, e, \dot{e})$

The mathematical model of the dynamic deformation of an EVP material in the neighborhood of the front edge of the crack of detachment presented as a complete system of equations in which the number of equations equals to the number of functions determining the SSS is given by rheological equations and by equations of motion in stresses

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j} + b_i$$  \hspace{1cm} (3)

and by continuity equation

$$\rho = \rho_0 = \text{const.}$$

Let’s consider the system of equations (1)–(3) at the frontal front of plastic precursor of the crack.

Let’s write the system of equations (1)–(3) containing the partial derivatives with respect to coordinate $x$ and time $t$ in a movable coordinate system [1,9]

$$\frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial n_i} + g_{\alpha \beta} \frac{\partial x_i}{\partial y_\alpha} \frac{\partial f}{\partial y_\beta}, \quad \frac{\partial f}{\partial t} = \frac{\delta t}{\delta t} - c \frac{\partial f}{\partial n}, \quad (i = 1, 2, 3; \ \alpha, \beta = 1, 2).$$  \hspace{1cm} (4)

Here $\frac{\delta f}{\delta t}$ is the local derivative with respect to time $t$ of the function defined on the moving surface $\Sigma$; $g^{\alpha \beta} = \frac{\partial x_i}{\partial y_\alpha} \cdot \frac{\partial x_i}{\partial y_\beta}$ is a metric tensor.

Rheological equations (1) and equations of motion in stresses (3) take the following form

$$\sigma_{ij} = \lambda e_{kk} \delta_{ij} + 2\mu \left( e_{ij} - e_{ij}^p \right),$$

$$e_{ij}^p = \frac{\partial e_{ij}^p}{\partial t} = \frac{\delta e_{ij}^p}{\delta t} - c \frac{\partial e_{ij}^p}{\partial n},$$

$$e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial n} n_j + \frac{\partial u_j}{\partial n} n_i + g^{\alpha \beta} \frac{\partial x_j}{\partial y_\beta} \frac{\partial u_i}{\partial y_\alpha} + g^{\alpha \beta} \frac{\partial x_i}{\partial y_\alpha} \frac{\partial u_j}{\partial y_\beta} \right),$$

$$v_i = \frac{\partial u_i}{\partial t} = \frac{\delta u_i}{\delta t} - c \frac{\partial u_i}{\partial n},$$

$$\rho \frac{\partial v_i}{\partial t} = \rho \frac{\delta v_i}{\delta t} - c \frac{\partial v_i}{\partial n} = \frac{\partial \sigma_{ij}}{\partial n} n_j + g^{\alpha \beta} \frac{\partial x_i}{\partial y_\alpha} \frac{\partial \sigma_{ij}}{\partial y_\beta} + b_i.$$  \hspace{1cm} (5)
Representation of solutions for rates and stresses in the form of power series

\[ f(n, y_1, y_2, t) = \sum_{l=1}^{\infty} \frac{1}{l!} \frac{\partial^l f(0, y_1, y_2, t)}{\partial n^l} n^l \]  

with a discontinuity of the first kind of function \( f \) at \( n = 0 \) belongs to the class of generalized functions. These functions have no normal derivative at the point \( n = 0 \) (figure 3) and therefore must satisfy differential laws (5) in integral form.

**Figure 3.** The scheme of the generalized function \( f(n) \), having a discontinuity \( (f(0^+) \neq f(0^-)) \) at \( n = 0 \), by the continuous function \( \tilde{f}(n) \) on the interval \( n \in [-\varepsilon, +\varepsilon] \)

Integration of PDE (5) with respect to \( n \) from \( -\varepsilon \) to \( +\varepsilon \) and passing to the limit for \( \varepsilon \to 0 \) give the following equations

\[ -\rho c [v_i] = [\sigma_{ij}] n_j, \quad [\epsilon_{ij}^p] = 0, \quad \left[ \frac{\partial \epsilon_{ij}^p}{\partial n} \right] \neq 0. \]  

The second equation demonstrates the physical fact that on the surface \( \Sigma_\delta \) of the crack precursor the plastic strains \( \epsilon_{ij}^p \) are continuous and only the gradient of plastic strains arises

\[ [\epsilon_{ij}^p] = 0; \quad \left[ \frac{\partial \epsilon_{ij}^p}{\partial n} \right] \neq 0. \]  

The rheological equations in (5) and the expressions for strains \( \epsilon_{ij} \) and rates \( v_i \), taken as a difference between the values to the right and left of the surface of the precursor \( \Sigma_\delta \), lead to the following expressions

\[ [\sigma_{ij}] = \lambda [\epsilon_{kk}] \delta_{ij} + 2\mu ([\epsilon_{ij}] - [\epsilon_{ij}^p]); \quad [v_i] = \frac{\delta [u_i]}{\delta t} - c \left[ \frac{\partial u_i}{\partial n} \right]; \]  

\[ [\epsilon_{ij}] = \frac{1}{2} \left( [u_{i,j}] + [u_{j,i}] + g^{\alpha\beta} x_{j,\alpha} \left[ \frac{\partial u_i}{\partial y_\alpha} \right] + g^{\alpha\beta} x_{i,\beta} \left[ \frac{\partial u_j}{\partial y_\alpha} \right] \right). \]  

The system of equations (7)–(9) is a linear homogeneous system of algebraic equations for jumps of the rates \([v_i]\), the displacement gradients \(\left[ \frac{\partial u_i}{\partial n} \right]\), the stresses \([\sigma_{ij}]\), the displacements \([u_i]\), and the displacement gradients \(\left[ \frac{\partial u_i}{\partial y_\alpha} \right]\) tangent along \(y_\alpha\).
The condition of continuity of displacements along $\Sigma_\delta$ along $y_\alpha$ and the condition of material continuity, the absence of discontinuity of displacements in the precursor lead to the equalities

$$\frac{\delta [u_i]}{\delta t} = 0; \quad \left[ \frac{\partial u_i}{\partial y_\alpha} \right] = \left[ \frac{\partial [u_i]}{\partial y_\alpha} \right] = 0. \quad (10)$$

Excluding $[\sigma_{ij}]$, $[e_{ij}]$, $[u_{i,j}]$ from the system of equations (7)–(9), we obtain

$$\rho c^2 [v_i] = (\lambda + \mu) [v_j] n_i n_j + \mu [v_i]. \quad (11)$$

There are only two nonzero solutions for $[v_i]$

$$[v_i] n_i = \omega_n \quad \text{when} \quad \rho c_1^2 = \lambda + 2\mu; \quad (12)$$

$$[v_i] \tau_i = \omega_{\tau} \quad \text{when} \quad \rho c_2^2 = \mu. \quad (13)$$

**Figure 4.** The image of the surface of the wave $\Sigma_1$ of longitudinal to $\Sigma_1$ rupture of rate and stress, the surface $S$ of a longitudinal shear crack with a front edge $L$ lying on the front $\Sigma_1$, and the behavior of the material near the tip of the longitudinal shear crack

Permissible discontinuous solutions with jumps in the normal rate component $[v_n] = \omega_n$ at the front $\Sigma_1$ moving with rate $c_1$ (figure 4) and the tangent rate component $[v_{\tau}] = \omega_{\tau}$ at the front $\Sigma_2$ moving with rate $c_2$ (figures 5 and 6) take place in a neighborhood of front edge of the crack.

**Figure 5.** The image of the surface $\Sigma_2$ of the wave of the rapture of rate and stresses tangent to $\Sigma_2$ and $L$, the surface $S$ with the front edge $L$ lying on the front $\Sigma_2$ and the behavior of the material near the tip of the antiplane shear crack

The correspondence of the kinematics of material displacements in the neighborhood of longitudinal shear crack, detachment crack and antiplane deformation crack allows constructing a mathematical model of material motion near the edge of a spatial crack in terms of the medium material points displacements rates.
2.2. Mathematical model of the front edge of the crack

Let’s select a $\delta$-neighborhood of the crack front edge $L$ in the form of a cylinder $S_\delta$ with a curvilinear axis $L$. The surface $\Sigma_\delta$ is the surface of a weak discontinuity generated by the initial disturbance at crack edge $L_0$ initiation. Let’s regard the curve $L_\delta$ on $\Sigma_\delta$ as the precursor of crack edge $L$.

Let’s consider the trace of crack front edge at the front $\Sigma_\delta$ of the wave precursor, which is represented by the spatial curve $L_\delta$ belonging to the surface $\Sigma_\delta$ (figure 7).

The correspondence of the material behavior kinematics in displacements near the crack tip and the material behavior kinematics in the displacement rates in the neighborhood of the fronts of longitudinal and shear wave allows to conclude:

1. The front edge of the precursor of a spatial crack $L$ propagates as an orthogonal trajectory of the wave front $\Sigma_\delta$ with its line $L_\delta$ with discontinuities in rates and stresses when crossing across this line.

1.1. The front edge of a longitudinal shear crack propagates at a rate $c_1$ of longitudinal strain waves in an elastic body (figure 4).

1.2. The front edges of the antiplane shear crack and the detachment crack propagate with shear wave rate $c_2$ (figures 5 and 6).
2. It is convenient to choose a jump in rate when crossing through the front edge \( L_\delta \) of the crack \( S \) as the intensity of the cracks front edges.

2.1. For a longitudinal shear crack at \( \Sigma_1 \) (figure 4)

\[
\omega_{nL} = \left( \hat{v}_n - \overline{v}_n \right) |_{L} = [v_n]_L. \tag{14}
\]

2.2. For an antiplane shear crack at \( \Sigma_2 \) (figure 5)

\[
\omega_{2L} = \left( \hat{v}_{\tau_2} - \overline{v}_{\tau_2} \right) |_{L} = [v_{\tau_2}]_L. \tag{15}
\]

2.3. For a detachment crack at \( \Sigma_2 \) (figure 6)

\[
\omega_{1L} = \left( \hat{v}_{\tau_1} - \overline{v}_{\tau_1} \right) |_{L} = [v_{\tau_1}]_L. \tag{16}
\]

Thus, the type of crack is determined by the type of the precursor \( \Sigma_1 \) or \( \Sigma_2 \), and the existence of the front edge is determined by the frontal front intensity, i.e., when the dynamic second invariant of the stress tensor deviator exceeds the static plasticity limit \( K: \sigma_i^j \sigma_i^j \geq K^2 \).

Let's show that the crack intensity \( \omega_L \) (14)–(16) is determined by the intensity of the corresponding waves of strong discontinuity \( \Sigma_1 \) and \( \Sigma_2 \).

In the neighborhood of the cracks precursor \( L \) on \( \Sigma \) the following expressions take place

\[
\hat{v} = v^+ - [\hat{v}]; \quad \overline{v} = v^+ - [\overline{v}];
\]

\[
\omega_L = \hat{v} - \overline{v} = - [\hat{v}] + [\overline{v}].
\]

Let's assume that the existence of a crack is determined by the presence of plastic deformation on at least one of the sides of front edge \( L \). Let's consider the problem of analyzing the intensity of the edges of cracks as the problem of analyzing the plastic stress state behind the fronts of waves of strong discontinuity.

2.3. Kinematics of reflection of a limiting plastic wave as an orthogonal precursor of a crack tip of a longitudinal shear

The front edges of the cracks lying on the wave fronts are orthogonal trajectories to these fronts. It is theoretically and experimentally proved [2] that the front edges of cracks propagate with rates of longitudinal or shear waves

\[
c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_2 = \sqrt{\frac{\mu}{\rho}}. \tag{17}
\]

The front edge of the crack is modeled by a small \( \delta \)-neighborhood of the plastic behavior of the material, so that the direction of motion of the crack is orthogonal to the front of the plastic loading wave. For the case of the front edge of a longitudinal shear crack (figure 4), we assume the rates distribution behind the front of the longitudinal wave, having a rate discontinuity across the front edge.

As the intensity of the front edge of the crack, we mean the difference between the longitudinal rates (14) taken above and below the plane of the crack. The values of the rates behind the wave fronts themselves satisfy the basic laws of the model of EVP deformation of the material (11) and the laws of reflection of wave fronts from the boundary.
From the condition of the simultaneous existence at the point of three waves $\Sigma_1$, $\Sigma_1'$ and $\Sigma_2'$ (the prime indicates the front of the reflected wave), the reflection angles $\varphi'$ and $\psi'$ are related by Snell’s law (figure 8)

$$\frac{c_1}{\sin \varphi} = \frac{c_1}{\sin \varphi'} = \frac{c_2}{\sin \psi'}.$$  \hfill (18)

From here we derive

$$\sin \psi' = \frac{c_2}{c_1} \sin \varphi, \quad \varphi' = \varphi.$$  \hfill (19)

The compatibility conditions for stresses and rates at the front of a strong wave ahead $\Sigma_1$ take the form

$$-c_1 [\sigma_{ij}] = (\lambda \delta_{ij} + 2\mu n_i n_j) \omega_n,$$  \hfill (20)

where $\omega_n$ is the jump of the rate normal component at the wave front and $\omega_n = [v_i] n_i$.

**Figure 8.** The image of the reflection of a plane longitudinal wave $\Sigma$ from the boundary of half-space in a coordinate system moving along the boundary

Conditions (12), (13) demonstrate the fact that the front $\Sigma_\delta$ of the precursor of a spatial crack can be a longitudinal deformation wave for a longitudinal shear crack (figure 4) propagating with rate $c_1$ ($\rho c_1^2 = \lambda + 2\mu$) and a shear deformation wave for detachment or transverse shear cracks (figure 6) propagating with rate $c_2$ ($\rho c_2^2 = \mu$).

The stress jumps on $\Sigma_\delta$ are expressed with the use of rate jumps as follows

$$[\sigma_{ij}] = -\frac{1}{c} \lambda [v_n] \delta_{ij} - \frac{1}{c} \mu ([v_i] n_j + [v_j] n_i).$$

Consider the trace of the front edge of the crack on the front $\Sigma_\delta$ of the wave precursor, which is represented by the spatial curve $L_\delta$ belonging to the surface $\Sigma_\delta$ (figure 7).

The jumps of the rates and stresses on the reflected waves $\Sigma_1'$ and $\Sigma_2'$ are related by the relations

$$-c_1 [\sigma_{ij}]_1 = (\lambda \delta_{ij} + 2\mu n_i^{(1)} n_j^{(1)}) \omega_1 n_i, \quad [v_i]_1 = \omega_1 n_i^{(1)};$$  \hfill (21)

$$-c_2 [\sigma_{ij}]_2 = \mu \left( \tau_i^{(2)} n_j^{(2)} + \tau_j^{(2)} n_i^{(2)} \right) \omega_2, \quad [v_i]_2 = \omega_2 \tau_i^{(2)}.$$  \hfill (22)
Let’s add to the system of equations (20), (21), (22) the boundary conditions

\[ \sigma_{ij}^{(3)} \cdot N_j = P_i, \]  

(23)

where \( P_i \) is the stress vector at the crack reflection boundary.

Here are the expressions for \( n, n^{(1)}, n^{(2)}, \tau^{(2)}, N, T \)

\[
\begin{align*}
    n &= (\sin \phi, \cos \phi, 0); & n^{(1)} &= (\sin \phi', - \cos \phi', 0); \\
    n^{(2)} &= (\sin \psi', - \cos \psi', 0); & \tau^{(2)} &= (\cos \psi', \sin \psi', 0); \\
    N &= (0, 1, 0); & T &= (1, 0, 0).
\end{align*}
\]

(24)

Let’s assume that

\[ [\sigma_{ij}] = - \sigma_{ij}^{(1)}; \quad [\sigma_{ij}]_1 = \sigma_{ij}^{(1)} - \sigma_{ij}^{(2)}; \quad [\sigma_{ij}]_2 = \sigma_{ij}^{(2)} - \sigma_{ij}^{(3)}. \]  

(25)

From (20), (21), (22) we obtain

\[ -[\sigma_{ij}]_1 = \frac{1}{c_1} (\lambda \delta_{ij} + 2 \mu n_i n_j) \omega_n; \]

\[ -[\sigma_{ij}]_2 = \frac{1}{c_2} \mu \left( \tau_{ij}^{(2)} n_j^{(2)} + \tau_{ij}^{(2)} n_i^{(2)} \right) \omega_2. \]  

(26)

Excluding stress jumps from (25), (26), we obtain

\[ \sigma_{ij}^{(1)} - \sigma_{ij}^{(1)} + \sigma_{ij}^{(2)} - \sigma_{ij}^{(2)} + \sigma_{ij}^{(3)} = \frac{1}{c_1} (\lambda \delta_{ij} + 2 \mu n_i n_j) \omega_n + \]

\[ + \frac{1}{c_1} \left( \lambda \delta_{ij} + 2 \mu n_i^{(1)} n_j^{(1)} \right) \omega_{1n} + \frac{\mu}{c_2} \left( \tau_{ij}^{(2)} n_j^{(2)} + \tau_{ij}^{(2)} n_i^{(2)} \right) \omega_2. \]  

(27)

We satisfy the boundary conditions (23) in zone \( \{3\} \), multiplying (27) by \( N_j \)

\[ (\lambda \delta_{ij} + 2 \mu n_i n_j) \omega_n N_j + \left( \lambda \delta_{ij} + 2 \mu n_i^{(1)} n_j^{(1)} \right) \omega_{1n} N_j + \]

\[ + \frac{\mu}{c_2} \left( \tau_{ij}^{(2)} n_j^{(2)} + \tau_{ij}^{(2)} n_i^{(2)} \right) \omega_2 N_j = c_1 (F_n N_i + F_\tau T_i) N_j, \]  

(28)

where \( F_n \) and \( F_\tau \) are the normal and tangential stresses at the reflection boundary.

Projecting equality (28) on the \( x_1 \) and \( x_2 \) axes, taking into account (24), we obtain for the \( x_2 \) axis:

\[ (\lambda + \mu \sin 2 \phi) \omega_n + (\lambda - \mu \sin 2 \phi') \omega_{1n} + \mu \frac{c_1}{c_2} \cos 2 \psi' \cdot \omega_2 = c_1 F_n, \]  

(29)

for the \( x_1 \) axis:

\[ \mu \sin 2 \phi \cdot \omega_n - \mu \sin 2 \phi' \cdot \omega_{1n} + \mu \frac{c_1}{c_2} \cos 2 \psi' \cdot \omega_2 = c_1 F_\tau. \]  

(30)

Using equalities (29), (30), we compose a system of equations for determining the intensities of the reflected waves \( \omega_1 \) and \( \omega_2 \)

\[
\begin{cases}
    a_1 \omega_{1n} + b_1 \omega_2 = d_1 \omega_n + F_n; \\
    a_2 \omega_{1n} + b_2 \omega_2 = d_2 \omega_n + F_\tau.
\end{cases}
\]  

(31)
Here
\[ a_1 = \lambda - \mu \cdot \sin 2\varphi; \quad a_2 = -\mu \cdot \sin 2\varphi; \quad b_1 = b_2 = \frac{c_1}{c_2} \mu \cdot \cos 2\psi; \]
\[ d_1 = -(\lambda + \mu \cdot \sin 2\varphi); \quad d_2 = -\mu \cdot \sin 2\varphi.\]

Using system (31), we express the intensities \( \omega_{1n} \) and \( \omega_2 \) of the reflected waves with the intensity of the incident wave \( \omega_n \).

Denoting \( k_1 = \frac{d_1b_2 - d_2b_1}{a_1b_2 - a_2b_1}; \quad D_1 = \frac{b_2F_n - b_1F_r}{a_1b_2 - a_2b_1} \), we obtain
\[ \omega_{1n} = k_1\omega_n + D_1. \tag{32} \]

Denoting \( k_2 = \frac{d_1a_2 - d_2a_1}{a_2b_1 - a_1b_2}; \quad D_2 = \frac{a_2F_n - a_1F_r}{a_2b_1 - a_1b_2} \), we obtain
\[ \omega_2 = k_2\omega_n + D_2. \tag{33} \]

Let’s express \( k_1, k_2, D_1, D_2 \) with the use of \( c_1, c_2 \) and angle \( \varphi \)
\[ k_1 = \frac{c_1^2 + 2c_2^2}{c_1^2 - c_2^2} = \frac{1 + 2\frac{c_2^2}{c_1^2}}{1 - \frac{c_2^2}{c_1^2}}; \tag{34} \]
\[ k_2 = \frac{-3c_2^3 \sin 2\varphi}{c_1(c_1^2 - c_2^2)}; \tag{35} \]
\[ D_1 = \frac{F_n - F_r}{\rho(c_1^2 - c_2^2)}; \tag{36} \]
\[ D_2 = \frac{\frac{c_2^2}{c_1^2}(\sin 2\varphi F_n + (1 + \sin 2\varphi)F_r) - F_r}{\rho c_1 c_2 \left( \frac{1}{2} - \frac{c_2^2}{c_1^2} \sin^2 \varphi \right) \left( \frac{c_2^2}{c_1^2} - 1 \right)}. \tag{37} \]

Let’s substitute the expressions (34)–(37) for \( k_1, k_2, D_1, D_2 \) in (32), (33) and obtain
\[ \omega_{1n} = \frac{1 + 2\frac{c_2^2}{c_1^2}}{1 - \frac{c_2^2}{c_1^2}} \cdot \omega_n + \frac{F_n - F_r}{\rho(c_1^2 - c_2^2)} = \frac{1 + 2\tilde{c}}{1 - \tilde{c}} \cdot \omega_n + \frac{F_n - F_r}{\rho c_1^2 (1 - \tilde{c})}, \tag{38} \]
where \( \tilde{c} = \frac{c_2^2}{c_1^2} \).

The graphs of the following dependences are presented below: shear wave reflection coefficient on boundary conditions (figure 9); shear wave reflection coefficient on angle of incidence and on material properties (figure 11); the relative rate of the longitudinal wave on the properties of the material (figure 10).

Taking into account that on the limiting longitudinal plastic wave \( \omega_n = \sqrt{3}K/2\mu \), we obtain the expression for \( \omega_n \) with the use of \( \tilde{c} = \frac{c_2^2}{c_1^2} \)
\[ \omega_n^2 = \frac{K^2}{\rho^2 \tilde{c} c_1^2 \left( \frac{10}{\tilde{c}} - 1 \right)}. \tag{39} \]
Figure 9. The graph of the reflection coefficient \( \omega_{1n}/\omega_n \) of a longitudinal shear wave from a boundary with predetermined stresses \((F_n - F_\tau)\)

Figure 10. The graph of the dependence of the relative rate \( \omega_n \) at the front \( \Sigma_1 \) of the longitudinal plastic incident wave on the physical properties of the material \( \tilde{c} = c_2^2/c_1^2 \)

The intensity of the reflected shear wave \( \omega_2 \) is conveniently represented in the following two forms

\[
\begin{align*}
\omega_2 &= -\frac{3\tilde{c}^2 \sin 2\varphi}{c_1^3 \left( \frac{c_2}{c_1} - 1 \right) \left( \frac{1}{2} - \frac{c_2^2}{c_1^2} \sin^2 \varphi \right)} \cdot \omega_n - \frac{c_2^2}{c_1^2} \left( \sin 2\varphi F_n + (1 + \sin 2\varphi) F_\tau \right) - F_\tau \quad ; \\
\omega_2 &= -\frac{3\tilde{c} \sin 2\varphi}{\tilde{c} - 1} \cdot \frac{K}{\rho c_1 \sqrt{\frac{10}{3} \tilde{c} - 1}} - \frac{\tilde{c} \left( \sin 2\varphi F_n + (1 + \sin 2\varphi) F_\tau \right) - F_\tau}{\rho c_1 c_2 \left( \frac{1}{2} - \tilde{c} \sin^2 \varphi \right) \left( \tilde{c} - 1 \right)} .
\end{align*}
\]
Figure 11. The graph of the dependence of the shear wave reflection coefficient $\omega_2/\omega_n$ on the angle of incidence $\varphi$ and material properties $\bar{c} = c_2^2/c_1^2$ without taking into account normal and tangential stresses applied at the boundary.

2.4. The stress state behind the fronts of the precursors of reflected longitudinal shear crack and detachment crack

The fronts of the reflected longitudinal $\Sigma_1$ and shear $\Sigma_2$ waves are the precursors of longitudinal shear and detachment cracks that exist as long as the stress state remains in the plastic zone.

Let’s study the deformation of the material in zones $\{2\}$ and $\{3\}$ shown in figure 8.

Let’s compute there the intensity of the maximum tangential stresses $I^2$ and evaluate the possibility of plastic deformation depending on the angles of incidence $\varphi$ of the longitudinal wave $\Sigma_1$ and wave propagation rate $\bar{c}$.

Knowing the reflection coefficients $k_1$, $k_2$ and $D_1$, $D_2$ for longitudinal and shear waves, we can construct expressions for stresses in zones $\{2\}$ and $\{3\}$ with the use of the intensity of the incident longitudinal wave. This intensity we can determine from the plasticity condition

$$I^2 = \sigma'_{ij}\sigma'_{ij} = \sigma_{ij}\sigma_{ij} - \frac{1}{3}\sigma_{kk}^2 = 2K^2.$$ (42)

The second invariant of the tensor $\sigma'_{ij}$ is defined behind the wave $\Sigma_1$ as follows

$$\sigma^1_{ij}\sigma^1_{ij} = [\sigma_{ij}][\sigma_{ij}] = \frac{1}{c_1^2}\omega^2_n[3\lambda^2 + 2\lambda\mu + 4\mu^2]; \quad \sigma^2_{kk} = \frac{1}{c_1^2}\omega^2_n(3\lambda + 2\mu)^2;$$

$$I^2_{\{1\}} = \sigma'_{ij}\sigma'_{ij} - \frac{1}{3}\sigma_{kk}^2 = \left(\frac{\omega_n}{c_1}\right)^2 (-2\lambda\mu + \frac{8}{3}\mu^2) = 2K^2.$$ (43)

The second invariant $I^2_{\{2\}}$ of the tensor $\sigma'^2_{ij}$ are calculated behind the wave $\Sigma_2$ as

$$I^2_{\{2\}} = \sigma'^2_{ij}\sigma'^2_{ij} - \frac{1}{3}\sigma'^2_{kk};$$
\[ \sigma_{ij}^2 = \sigma_{ij}^1 - [\sigma_{ij}]_1 = [\sigma_{ij}] - [\sigma_{ij}]_1 = \frac{1}{c_1} \left[ \left( \lambda \delta_{ij} + 2 \mu n_i^{(1)} n_j^{(1)} \right) \omega_{1n} - \left( \lambda \delta_{ij} + 2 \mu n_i n_j \right) \omega_n \right], \]

here \( \omega_{1n} \) is determined by formula (38) and \( \omega_n \) is determined by formula (40).

\[ I_{(2)}^2 = \frac{2 \rho^2 c_1^2 2K^2 c_2^2}{2 \mu \left( \frac{4}{3} \mu - \lambda \right)} \cdot \left( 2(1 - 2\tilde{c})(3 - 2\tilde{c}) + \frac{1}{3} (3 - 4\tilde{c})^2 - 4\tilde{c}^2 \cdot \cos 2\varphi \right) \]

or

\[ I_{(2)}^2 = \frac{6K^2}{\tilde{c}(10\tilde{c} - 3)} \cdot \left( 2(1 - 2\tilde{c})(3 - 2\tilde{c}) + \frac{1}{3} (3 - 4\tilde{c})^2 - 4\tilde{c}^2 \cdot \cos 2\varphi \right). \]  

(44)

Figures 12 and 13 show the values of the second invariant of the deviator of the stress tensor.

\[ I_{(2)}^2 = \frac{2 \rho^2 c_1^2 c_2^2 \omega_n^2}{2 \mu \left( \frac{4}{3} \mu - \lambda \right)} \cdot \left( 2(1 - 2\tilde{c})(3 - 2\tilde{c}) + \frac{1}{3} (3 - 4\tilde{c})^2 - 4\tilde{c}^2 \cdot \cos 2\varphi \right) \]

(45)

Thus, in zone \( \{1\} \)

\[ I_{(1)}^2 = 2 \rho^2 c_1^2 c_2^2 \omega_n^2 \left( \frac{10\tilde{c}^2}{3} - 1 \right). \]

In zone \( \{2\} \) behind the reflected longitudinal wave we have

\[ I_{(2)}^2 = 2 \omega_n^2 \rho^2 c_1^2 c_2^2 \left( \frac{10\tilde{c}^2}{3} - 1 \right). \]  

(46)

Taking into account external normal \( F_n \) and tangent \( F_t \) forces, the expression for \( I_{(2)}^2 \) takes the form

\[ I_{(2)}^2 = \left( \frac{1 + 2\tilde{c}}{1 - \tilde{c}} \right)^2 \cdot \omega_n^2 + 2 \omega_n \cdot \left( \frac{1 + 2\tilde{c}}{1 - \tilde{c}} \cdot \frac{F_n - F_t}{\rho c_1^2 \sqrt{1 - \tilde{c}}} + \frac{\left( F_n - F_t \right)^2}{\rho^2 c_1^2 (1 - \tilde{c})^2} \right) \cdot \tilde{c} \rho^2 \left( \frac{10\tilde{c}^2}{3} - 1 \right). \]  

(46)

In zone \( \{3\} \)

\[ I_{(3)}^2 = \frac{6 \mu^2}{c_2^2} \omega_2^2 = 6 \rho c_2^2 \left( \frac{3 \sqrt{\tilde{c}} \sin 2\varphi}{(\tilde{c} - 1) \left( \frac{1}{2} - \tilde{c} \sin^2 \varphi \right)} \cdot \frac{K}{\rho c_1 \sqrt{\tilde{c}}} \cdot \frac{10\tilde{c}^2}{3} - 1 \right. \]
\[ + \bar{c} \cdot F_n \sin 2\varphi + F_\tau (1 + \sin 2\varphi) + F_\tau \left( \frac{1}{2} - \bar{c} \cdot \sin \varphi \right) \left( \bar{c} - 1 \right) \right) ^2. \]

(47)

Figure 13. The graph of the value of the second invariant in zone \{3\} behind the reflected shear wave without taking into account normal and tangential stresses

The graph of the intensity of tangential stresses in zone \{3\} demonstrates the fact of increasing their intensity with increasing angle \( \varphi \) and value \( \bar{c} = \frac{c_2}{c_1} \) without taking into account boundary stresses (figure 14).

The applying of boundary stresses to the reflection surface leads to a significant increase in the intensity of shear stresses.

Figure 14. The graph of the value of the second invariant in zone \{3\} behind the reflected shear wave taking into account the boundary stresses \( F_n \) and \( F_\tau \)
3. Results and analysis

The kinematic laws of reflection of the limiting plastic wave of longitudinal deformation $\Sigma_1$ demonstrate:

1. The rate $\omega_n$ required to achieve the limiting plastic state of the material in zone $\{1\}$ increases with increasing plasticity limit $K$ and decreases with increasing shear wave rate $c_2$ (figure 12).

2. The intensity $\omega_{1n}/\omega_n$ of the longitudinal reflected wave $\Sigma'_1$ does not depend on the angle of incidence $\varphi$, and with the applying of normal $F_n$ and tangential $F_\tau$ stresses to the boundary, it can increase (when $F_n - F_\tau > 0$) or decrease (when $F_n - F_\tau < 0$), depending on the sign of the difference between the normal and tangential stresses (figures 9 and 10).

3. The intensity $\omega_{2r}/\omega_n$ of the shear reflected wave $\Sigma'_1$ increases with increasing angle of incidence $\varphi$ and increasing rate $c_2$ of shear waves (figure 11).

4. The value of the second invariant $I^2_{[2]}$ of the deviator of the stress tensor in zone $\{2\}$ increases with increasing angle of incidence of the front $\varphi$ and decreases with increasing $\tilde{c} = \frac{c_2^2}{c_1^2}$ (figure 13).

5. The incident front edge of the longitudinal shear crack following the front of the incident wave $\Sigma_1$ can generate the front edges of the longitudinal shear crack following the reflected longitudinal wave and the front edge of the detachment crack behind the front of the reflected shear wave. This is possible only in cases of plastic deformation behind the fronts of reflected waves in zones $\{2\}$ and $\{3\}$.

6. The intensity $\omega'_{nL}$ of the reflected longitudinal shear crack does not depend on the angle of incidence $\varphi$ of the initiating crack, and its value is determined by the difference between the normal and tangential stresses $(F_\tau - F_n)$ at the reflection boundary.

The intensity $\omega_{1L}$ of the reflected detachment crack increases with increasing angle $\varphi$ of incidence of the initiating crack and depends on the value and direction of the boundary stresses $F_\tau$ and $F_n$.

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