The 3-3-1 model with $S_4$ flavor symmetry

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We construct a 3-3-1 model based on family symmetry $S_4$ responsible for the neutrino and quark masses. The tribimaximal neutrino mixing and the diagonal quark mixing have been obtained. The new lepton charge $L$ related to the ordinary lepton charge $L$ and a SU(3) charge by $L = \frac{2}{\sqrt{3}} T_8 + L$ and the lepton parity $P_l = (-)^L$ known as a residual symmetry of $L$ have been introduced which provide insights in this kind of model. The expected vacuum alignments resulting in potential minimization can origin from appropriate violation terms of $S_4$ and $L$. The smallness of seesaw contributions can be explained from the existence of such terms too. If $P_l$ is not broken by the vacuum values of the scalar fields, there is no mixing between the exotic and the ordinary quarks at the tree level.

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I. INTRODUCTION

In the standard model the fundamental fermions come in families. In writing down the theory one may start by first introducing just one family, then one may repeat the same procedure by introducing copies of the first family. Why do quarks and leptons come in repetitive structures—families? How many families are there? How to understand the interrelation and mass-hierarchy between the families? In addition, the standard model cannot explain the tiny masses and mixing profile of neutrinos, and the close-to-unity of quark mixing matrix as well [1]. These have been the central puzzles known as the flavor question in particle physics beyond the standard model.

The current neutrino experimental data are consistent with the tribimaximal form proposed by

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Harrison-Perkins-Scott (HPS), which apart from the phase redefinitions, is given by

\[ U_{\text{HPS}} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \]

where the large mixing angles are completely different from the quark mixing ones defined by the Cabibbo-Kobayashi-Maskawa (CKM) matrix. It is an interesting challenge to formulate dynamical principles that can lead to the flavor mixing patterns for quarks and leptons given in a completely natural way as first approximations. A fascinating way seems to be the use of some discrete non-Abelian groups \[ 3 \] as family symmetries added to the standard model gauge group. There is a series of models based on the group \[ A_4 \] \[ 4, 5 \], \[ T' \] \[ 6 \], and more recently \[ S_4 \] \[ 7, 8 \]—the group of permutations of four objects, which is also the symmetry group of the cube.

We would like to extend the above application to the SU(3)_C \( \otimes \) SU(3)_L \( \otimes \) U(1)_X (3-3-1) gauge model \[ 9–11 \] because of the following. The \([\text{SU}(3)_L]^3\) anomaly cancelation in the model requires the number of SU(3)_L fermion triplets to equal that of antitriplets. Taking into account an unrestricted number of standard model families with corresponding extensions of lepton and quark representations, the number of families results in a multiple of 3. Furthermore the QCD asymptotic condition constrains the number of quark families to be lesser than or equal to 5. The family number is exact 3. The model thus provides a partial explanation of the family number, as also required by flavor symmetries such as \( S_4 \) for 3-dimensional representations. In addition, due to the anomaly cancelation one family of quarks has to transform under SU(3)_L differently from the two others. We should look for a family symmetry group with 2- and 3-dimensional irreducible representations respectively acting on the 2- and 3-family indices, the simplest of which is just \( S_4 \). Note that \( S_4 \) has not been considered before in the kind of the 3-3-1 model. For the similar works on \( A_4 \), let us call the reader’s attention to Refs. \[ 5 \].

There are two typical variants of the 3-3-1 model as far as lepton sectors are concerned. In the minimal version, three SU(3)_L lepton triplets are of the form \((\nu_L, l_L, l_R^c)\), where \( l_R \) are ordinary right-handed charged-leptons \[ 9 \]. In the second version, the third components of lepton triplets include right-handed neutrinos, respectively, \((\nu_L, l_L, \nu_R^c)\) \[ 10 \]. In trying to recover the tribimaximal form in present work, by analysis a possibility close to the typical versions is when we replace the right-handed neutrinos by new standard model fermion singlets \((N_R)\) with vanishing lepton-number \[ 12 \]. The resulting model is near that of our previous work in \[ 5 \]. The neutrinos thus gain masses from only contributions of SU(3)_L scalar antisextets. The antisextets contain tiny vacuum expectation values (VEVs) in the first components, similar to the cases of the standard model with
scalar triplets. To avoid the decay of $Z$ into the Majorons associated with these components, the lepton-number violating potential should be turned on. The lepton charge is therefore no longer of an exact symmetry; thereby the Majorons can get large enough masses to escape from the $Z$ decay \[12\]. Assuming the antisextets very heavy, the potential minimization can provide a natural explanation of the expected vacuum alignments as well as the smallness of seesaw contributions responsible for neutrino mass.

The rest of this article is organized as follows. In Sec. II we propose the model with $S_4$. The masses and mixing matrices of leptons and quarks are obtained then. In Sec. IV we consider the Higgs potential and minimization conditions. We summarize our results and make conclusions in Sec. V. Appendix A is devoted to $S_4$ group with its Clebsch-Gordan coefficients. Appendix B presents the lepton numbers and lepton parities of model particles.

II. THE MODEL

The fermions in this model under $[SU(3)_L, U(1)_X, U(1)_L, S_4]$ symmetries, respectively, transform as

$$\psi_L \equiv \psi_{1,2,3L} = \left( \begin{array}{c} \nu_{1,2,3L} \\ l_{1,2,3L} \\ N_{1,2,3R} \end{array} \right) \sim [3, -1/3, 2/3, 3],$$

$$l_{1R} \sim [1, -1, 1, \underline{1}], \quad l_R \equiv l_{2,3R} \sim [1, -1, 1, 2],$$

$$Q_{3L} = \left( \begin{array}{c} u_{3L} \\ d_{3L} \\ U_L \end{array} \right) \sim [3, 1/3, -1/3, 1], \quad Q_L \equiv Q_{1,2L} = \left( \begin{array}{c} d_{1,2L} \\ -u_{1,2L} \\ D_{1,2L} \end{array} \right) \sim [3^*, 0, 1/3, 2],$$

$$u_R \equiv u_{1,2,3R} \sim [1, 2/3, 0, 2], \quad d_R \equiv d_{1,2,3R} \sim [1, -1/3, 0, 2],$$

$$U_R \sim [1, 2/3, -1, \underline{1}], \quad D_R \equiv D_{1,2R} \sim [1, -1/3, 1, 2],$$

where the numbered subscripts on field indicate to respective families which also in order define components of their $S_4$ multiplet representation. The reader can see in Appendix A for more details of the $S_4$ group representations. As usual, the $X$ charge is related to the electric charge operator as $Q = T_3 - \frac{1}{\sqrt{3}} T_8 + X$ where $T_a$ ($a = 1, 2, ..., 8$) are $SU(3)_L$ charges, satisfying $\text{Tr}[T_a T_b] = \frac{1}{2} \delta_{ab}$.

The $N_R$ as above mentioned are exotic neutral fermions having the lepton number $L(N_R) = 0$ \[8, 12\]. Hence the lepton number $L$ in this model does not commute with the gauge symmetry. We can therefore search for a new conserved charge $\mathcal{L}$ as given in the square brackets above, which is defined in terms of the ordinary lepton number by $L = \frac{2}{\sqrt{3}} T_8 + \mathcal{L}$ \[5, 13\]. This definition is
only convenient one for accounting the global lepton numbers of the model particles, because the $T_8$ is a gauged charge, and thus $L$ consequently gauged. The gauging of the $L$ charge deserves further studies, where in the present work we will take it globally. This is possible since the $T_8$ can be considered as the charge of a group replication of SU$(3)_L$ but taken globally, thus $L$ is not gauged. Finally, the lepton charge arranged in the way is to suppress unwanted interactions (due to U$(1)_{L}$ symmetry) to yield the tribimaximal form as shown below. $U$ and $D_{1,2}$ as supplied are exotic quarks carrying lepton numbers $L(U) = -1$ and $L(D_{1,2}) = 1$, known as leptoquarks.

The lepton parity is introduced as follows $P_l = (-)^L$, which is a residual symmetry of $L$. The particles possess $L = 0, \pm 2$ such as $N_R$, ordinary quarks and bileptons having $P_l = 1$; the particles with $L = \pm 1$ such as ordinary leptons and exotic quarks have $P_l = -1$. Any non-zero VEV with odd parity, $P_l = -1$, will break this symmetry spontaneously. For convenience in reading, the numbers $L$ and $P_l$ of the component particles are given in Appendix B.

In the following, we consider possibilities of generating the masses for the fermions. The scalar multiplets needed for the purpose are introduced accordingly.

### III. FERMION MASS

#### A. Lepton mass

To generate masses for the charged leptons, we need two scalar multiplets:

$$\phi = \begin{pmatrix} \phi^+_1 \\ \phi_2^0 \\ \phi^+_3 \end{pmatrix} \sim [3, 2/3, -1/3, 3], \quad \phi' = \begin{pmatrix} \phi'^+_1 \\ \phi'^+_2 \\ \phi'^+_3 \end{pmatrix} \sim [3, 2/3, -1/3, 3']$$

with the VEVs $\langle \phi \rangle = (v, v, v)$ and $\langle \phi' \rangle = (v', v', v')$ written as those of $S_4$ components respectively (these will be derived from the potential minimization conditions). Here and after, the number subscripts on the component scalar fields are indices of SU$(3)_L$. The $S_4$ indices are discarded and should be understood. The Yukawa interactions are

$$-\mathcal{L}_l = h_1 (\bar{\psi}_L \phi) l_{1R} + h_2 (\bar{\psi}_L \phi) l_{2R} + h_3 (\bar{\psi}_L \phi') l_{3R} + h.c. \quad (8)$$

The mass Lagrangian of the charged leptons reads $-\mathcal{L}_l^{\text{mass}} = (\bar{l}_{1L}, \bar{l}_{2L}, \bar{l}_{3L}) M_l (l_{1R}, l_{2R}, l_{3R})^T + h.c.$,

$$M_l = \begin{pmatrix} h_1 v & h_2 v - h_3 v' & h_2 v + h_3 v' \\ h_1 v & (h_2 v - h_3 v') \omega & (h_2 v + h_3 v') \omega^2 \\ h_1 v & (h_2 v - h_3 v') \omega^2 & (h_2 v + h_3 v') \omega \end{pmatrix} \quad (9)$$
The mass matrix is then diagonalized,

\[
U_L^\dagger M_l U_R = \begin{pmatrix}
\sqrt{3}h_1v & 0 & 0 \\
0 & \sqrt{3}(h_2v - h_3v') & 0 \\
0 & 0 & \sqrt{3}(h_2v + h_3v')
\end{pmatrix} = \begin{pmatrix} m_\mu & 0 & 0 \\
0 & m_\tau & 0 \\
n & 0 & m_\tau
\end{pmatrix},
\]

where

\[
U_L = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\
1 & \omega & \omega^2 \\
1 & \omega^2 & \omega
\end{pmatrix}, \quad U_R = 1.
\]

We see that the masses of muon and tauon are separated by the \(\phi'\) triplet. This is the reason why we introduce \(\phi'\) in addition to \(\phi\).

Notice that the couplings \(\bar{\psi}^c_L \psi_L \phi\) and \(\bar{\psi}^c_L \psi_L \phi'\) are suppressed because of the \(L\)-symmetry violation. Therefore \(\bar{\psi}^c_L \psi_L\) can couple to SU(3)_L antisextets instead to generate masses for the neutrinos.

The antisextets in this model transform as

\[
\sigma = \begin{pmatrix} \sigma^0_{11} & \sigma^+_{12} & \sigma^0_{13} \\
\sigma^+_{12} & \sigma^+_{22} & \sigma^+_{23} \\
\sigma^0_{13} & \sigma^+_{23} & \sigma^0_{33}
\end{pmatrix} \sim [6^*, 2/3, -4/3, 1],
\]

\[s = \begin{pmatrix} s^0_{11} & s^+_{12} & s^0_{13} \\
s^+_{12} & s^+_{22} & s^+_{23} \\
s^0_{13} & s^+_{23} & s^0_{33}
\end{pmatrix} \sim [6^*, 2/3, -4/3, 3].
\]

The Yukawa interactions are

\[
- L_\nu = \frac{1}{2} x (\bar{\psi}^c_L \psi_L)_{1\sigma} + \frac{1}{2} y (\bar{\psi}^c_L \psi_L)_{3s} + h.c.
\]

\[= \frac{1}{2} x (\bar{\psi}^c_L \psi_{1L} + \bar{\psi}^c_L \psi_{2L} + \bar{\psi}^c_L \psi_{3L}) \sigma
\]

\[+ y (\bar{\psi}^c_{2L} \psi_{3L} s_1 + \bar{\psi}^c_{3L} \psi_{1L} s_2 + \bar{\psi}^c_{1L} \psi_{2L} s_3)
\]

\[+ h.c.
\]

The VEV of \(s\) is set as \((\langle s_1 \rangle, 0, 0)\) under \(S_4\) (which is also a natural minimization condition for the scalar potential), where

\[
\langle s_1 \rangle = \begin{pmatrix} \lambda_s & 0 & v_s \\
0 & 0 & 0 \\
v_s & 0 & \Lambda_s
\end{pmatrix}.
\]

(14)
The VEV of $\sigma$ is

$$
\langle \sigma \rangle = \begin{pmatrix} 
\lambda_{\sigma} & 0 & v_{\sigma} \\
0 & 0 & 0 \\
v_{\sigma} & 0 & \Lambda_{\sigma}
\end{pmatrix}.
$$

(16)

The mass Lagrangian for the neutrinos is defined by

$$
- \mathcal{L}_{\mu}^{\text{mass}} = \frac{1}{2} \bar{\nu} \chi L M_{\nu} \chi L + h.c., \quad \chi L \equiv \begin{pmatrix} \nu_L \\
N^c_R
\end{pmatrix}, \quad M_{\nu} \equiv \begin{pmatrix} M_L & M_D^T \\
M_D & M_R
\end{pmatrix},
$$

(17)

where $\nu = (\nu_1, \nu_2, \nu_3)^T$ and $N = (N_1, N_2, N_3)^T$. The mass matrices are then obtained by

$$
M_{L,R,D} = \begin{pmatrix} 
a_{L,R,D} & 0 & 0 \\
0 & a_{L,R,D} & b_{L,R,D} \\
0 & b_{L,R,D} & a_{L,R,D}
\end{pmatrix},
$$

(18)

with

$$
a_L = x\lambda_{\sigma}, \quad a_D = xv_{\sigma}, \quad a_R = x\Lambda_{\sigma}, \quad b_L = y\lambda_{s}, \quad b_D = yv_{s}, \quad b_R = y\Lambda_{s}.
$$

(19)

The VEVs $\Lambda_{\sigma,s}$ break the 3-3-1 gauge symmetry down to that of the standard model, and provide the masses for the neutral fermions $N_R$ and the new gauge bosons: the neutral $Z'$ and the charged $Y^\pm$ and $X^{0,0\ast}$. The $\lambda_{\sigma,s}$ and $v_{\sigma,s}$ belong to the second stage of the symmetry breaking from the standard model down to the $SU(3)_C \otimes U(1)_Q$ symmetry, and contribute the masses to the neutrinos. Hence, to keep a consistency we assume that $\Lambda_{\sigma,s} \gg v_{\sigma,s}, \lambda_{\sigma,s}$. The natural smallness of the lepton number violating VEVs $\lambda_{\sigma,s}$ and $v_{\sigma,s}$ will be explained in Section IV. Three active-neutrinos ($\sim \nu_L$) therefore gain masses via a combination of type I and type II seesaw mechanisms derived from (17) as

$$
M_{\text{eff}} = M_L - M_D^T M_R^{-1} M_D = \begin{pmatrix} 
a' & 0 & 0 \\
0 & a & b \\
0 & b & a
\end{pmatrix},
$$

(20)

where

$$
a' = a_L - \frac{a_D^2}{a_R},
$$

$$
a = a_L + 2a_Db_D \frac{b_R}{a_R^2 - b_R^2} - (a_D^2 + b_D^2) \frac{a_R}{a_R^2 - b_R^2},
$$

$$
b = b_L - 2a_Db_D \frac{a_R}{a_R^2 - b_R^2} + (a_D^2 + b_D^2) \frac{b_R}{a_R^2 - b_R^2}.
$$

(21)
We can diagonalize the mass matrix (20) as follows:
\[
U_\nu^T M_{\text{eff}} U_\nu = \begin{pmatrix} a + b & 0 & 0 \\ 0 & a' & 0 \\ 0 & 0 & b - a \end{pmatrix} \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix},
\]
(22)
where
\[
U_\nu = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -i \end{pmatrix}. 
\]
(23)
Combined with (11), the lepton mixing matrix yields the tribimaximal form:
\[
U_L^\dagger U_\nu = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \end{pmatrix} = U_{\text{HPS}},
\]
(24)
which is a main result of the paper.

If the lepton parity is an exact and spontaneously unbroken symmetry, the \(a_D\) and \(b_D\) vanish. The neutrinos then gain masses only from the type II seesaw due to the VEVs of first components of \(\sigma\) and \(s\), as we can see from (21) with \(a_D = b_D = 0\). If this parity is broken, there is no reason to prevent the 13 and 31 components of \(\sigma\) and \(s\) from getting nonzero VEVs as given in \(a_D, b_D\). The neutrino masses therefore gain additional contributions from the type I seesaw as well. Deviations from the tribimaximal form if required can be further explained by \(S_4\) breaking soft-terms, or if \(L\) was slightly violated, the terms breaking this charge as mentioned would also give contributions.

**B. Quark mass**

To generate masses for quarks, we additionally acquire the following scalar multiplets:
\[
\chi = \begin{pmatrix} \chi_1^0 \\ \chi_2^0 \\ \chi_3^- \\ \chi_3^0 \end{pmatrix} \sim [3, -1/3, 2/3, 1],
\]
(25)
\[
\eta = \begin{pmatrix} \eta_1^0 \\ \eta_2^- \\ \eta_3^0 \end{pmatrix} \sim [3, -1/3, -1/3, 3], \quad \eta' = \begin{pmatrix} \eta_1^0 \\ \eta_2^- \\ \eta_3^0 \end{pmatrix} \sim [3, -1/3, -1/3, 3'].
\]
(26)
The Yukawa interactions are

\[- \mathcal{L}_q = f_3 \bar{Q}_L \chi U_R + f \bar{Q}_L \chi^* D_R \]

\[+ h^d_3 \overline{Q}_3 (\phi d_R)_1 + h^u_3 \overline{Q}_3 (\eta u_R)_1 \]

\[+ h^u_3 \overline{Q}_3 (\phi^* u_R)_2 + h^d_3 \overline{Q}_3 (\eta^* d_R)_2 \]

\[+ h^n_3 \overline{Q}_3 (\phi^n u_R)_2 + h^n_3 \overline{Q}_3 (\eta^n d_R)_2 \]

\[+ h.c. \quad \text{(27)} \]

Suppose that the VEVs of \( \eta, \eta' \) and \( \chi \) are \((u, u, u)\), \((u', u', u')\) and \(w\), where \( u = \langle \eta_1^0 \rangle, u' = \langle \eta'_1^0 \rangle, \)

\( w = \langle \chi_3^0 \rangle \). The other VEVs \( \langle \eta_3^0 \rangle, \langle \eta'_3^0 \rangle, \langle \chi_1^0 \rangle \) vanish if the lepton parity is conserved. Otherwise they can develop VEVs. In addition, the VEV \( w \) also breaks the 3-3-1 gauge symmetry down to that of the standard model, and provides the masses for the exotic quarks \( U \) and \( D \) as well as the new gauge bosons. The \( u, u' \) as well as \( v, v' \) break the standard model symmetry, and give the masses for the ordinary quarks, charged leptons and gauge bosons. To keep a consistency with the effective theory, we assume that \( w \) is much larger than those of \( \phi \) and \( \eta \). In the following we consider the first case of the unbroken parity.

The exotic quarks get masses \( m_U = f_3 w \) and \( m_{D_{1,2}} = fw \), where the \( U \) and \( D_{1,2} \) by themselves are the mass eigenstates. The mass matrices for ordinary up-quarks and down-quarks are, respectively, obtained as follows:

\[
M_u = \begin{pmatrix}
-(h^u v + h^n v') & -(h^u v + h^n v') \omega & -(h^u v + h^n v') \omega^2 \\
(h^u v + h^n v') & -(h^u v + h^n v') \omega & -(h^u v + h^n v') \omega^2 \\
h^u_2 u & h^u_3 u & h^u_3 u
\end{pmatrix}, \quad \text{(28)}
\]

\[
M_d = \begin{pmatrix}
h^d u + h^n u' & (h^d u + h^n u') \omega & (h^d u + h^n u') \omega^2 \\
h^d u - h^n u' & (h^d u - h^n u') \omega & (h^d u - h^n u') \omega^2 \\
h^d_2 v & h^d_3 v & h^d_3 v
\end{pmatrix}. \quad \text{(29)}
\]

Let us define

\[
A = \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & 1 & 1 \\
\omega & \omega^2 & 1 \\
\omega^2 & \omega & 1
\end{pmatrix}. \quad \text{(30)}
\]

We have then

\[
M_u A = \begin{pmatrix}
-\sqrt{3}(h^u v + h^n v') & 0 & 0 \\
0 & -\sqrt{3}(h^u v - h^n v') & 0 \\
0 & 0 & \sqrt{3} h^u_3 u
\end{pmatrix} = \begin{pmatrix}
m_u & 0 & 0 \\
0 & m_c & 0 \\
0 & 0 & m_t
\end{pmatrix},
\]
\[
M_dA = \begin{pmatrix}
\sqrt{3}(h^d u + h'^d u') & 0 & 0 \\
0 & \sqrt{3}(h^d u - h'^d u') & 0 \\
0 & 0 & \sqrt{3}h'^d v
\end{pmatrix} = \begin{pmatrix}
m_d & 0 & 0 \\
0 & m_s & 0 \\
0 & 0 & m_b
\end{pmatrix}.
\]

In similarity to the charged leptons, the \( u \) and \( c \) quarks are also separated by the \( \phi' \) scalar. We see also that the introduction of \( \eta' \) is necessary to provide the different masses for \( d \) and \( s \) quarks. The unitary matrices, which couple the left-handed up- and down-quarks to those in the mass bases, are \( U^u_L = 1 \) and \( U^d_L = 1 \), respectively. Therefore we get the quark mixing matrix

\[
U_{\text{CKM}} = U^d_L U^u_L = 1.
\]

This is also an important result of our paper since the experimental quark mixing matrix is close to the unit matrix. In this case, the flavor changing neutral current (FCNC) can arise from one-loop processes with the exchange of heavy exotic quarks: see, for example, a contribution to the \( K^0 - \bar{K}^0 \) mixing due to the box diagram in Fig. 1. The amplitude after integrating out the heavy particles

\[
\text{is proportional to } \left[ \frac{(h^d)^4}{(16\pi^2 m_R^2)} \right] (d_R \gamma^\mu s_R)(d_R \gamma^\mu s_R), \text{ which is strongly suppressed by the loop factor and the exotic quark mass. The deviation of the CKM matrix from the identity can be given by the FCNC effects with the left-handed quarks, but such deviations are highly suppressed by the mass of the extra quarks also.}
\]

If the lepton parity is spontaneously broken, i.e. \( \langle \eta^0_R \rangle, \langle \eta'^0_R \rangle, \langle \chi^0_1 \rangle \neq 0 \), then there exist the following effects: (i) the mixings between ordinary quarks and exotic quarks (namely, \( u_{1,2,3} \) mix with \( U \) and \( d_{1,2,3} \) with \( D_{1,2} \)) which can lead to FCNC processes at the tree level; (ii) the result (32) is no longer correct, and the CKM is not unitary. A small mixing among the ordinary quarks may exist due to this violation. Let us recall that in the ordinary 3-3-1 model without \( S_4 \), the Yukawa interactions like (27) might additionally contain \( \mathcal{L} \) explicitly-violating terms [13], which can be also
the residual symmetry $P_1$ as in the first case should be more natural.

### IV. VACUUM ALIGNMENT

We can separate the general scalar potential into

$$V_{\text{total}} = V_{\text{tri}} + V_{\text{sext}} + V_{\text{tri-sext}} + \mathbf{V},$$  \hspace{1cm} (33)

where $V_{\text{tri}}$ and $V_{\text{sext}}$ respectively consist of the SU(3)$_L$ scalar triplets and sextets, whereas $V_{\text{tri-sext}}$ contains the terms connecting the two sectors. Moreover $V_{\text{tri-sext}}$ conserves $S_4$ symmetry, while $\mathbf{V}$ includes possible soft-terms explicitly violating these charges. Here the soft-terms as we meant include the trilinear and quartic ones as well. The reason for imposing $\mathbf{V}$ will be shown below.

The details on the potentials are given as follows. We first denote $V(X \to X_1, Y \to Y_1, \cdots) \equiv V(X, Y, \cdots | x=x_1, y=y_1, \cdots$ Notice also that $(\text{Tr} A)(\text{Tr} B) = \text{Tr}(A \text{Tr} B)$. $V_{\text{tri}}$ is a sum of

$$V(\chi) = \mu_X^2 \phi_1^+ \phi + \lambda_X^1 (\phi_1^+ \phi)^2,$$

$$V(\phi) = \mu_\phi^2 (\phi^+ \phi)^1 + \lambda_\phi^1 (\phi^+ \phi)^1 \phi + \lambda_\phi^2 (\phi^+ \phi)^2 \phi \phi,$$

$$V(\phi^+ \phi) = \lambda_\phi^3 (\phi^+ \phi)^3 + \lambda_\phi^4 (\phi^+ \phi)^4 \phi \phi \phi,$$

$$V(\phi, \chi) = \lambda_\phi \phi (\phi^+ \phi)^2 \phi \phi,$$

$$V(\phi, \chi) = \lambda_\phi \phi_1^+ \phi (\phi_1^+ \phi)^2 \phi \phi,$$

$$V(\phi, \chi) = \lambda_\phi \phi (\phi_1^+ \phi)^2 \phi \phi,$$

$$V(\phi, \chi) = \lambda_\phi \phi \phi (\phi_1^+ \phi)^2 \phi \phi,$$

$$V(\phi, \chi) = \lambda_\phi \phi_1^+ \phi (\phi_1^+ \phi)^2 \phi \phi,$$

$$V(\phi, \chi) = \lambda_\phi \phi_1^+ \phi (\phi_1^+ \phi)^2 \phi \phi,$$

$$V(\phi, \chi) = \lambda_\phi \phi_1^+ \phi (\phi_1^+ \phi)^2 \phi \phi,$$

$$V(\phi, \chi) = \lambda_\phi \phi_1^+ \phi (\phi_1^+ \phi)^2 \phi \phi,$$

$$V(\phi, \chi) = \lambda_\phi \phi_1^+ \phi (\phi_1^+ \phi)^2 \phi \phi,$$

$$V(\phi, \chi) = \lambda_\phi \phi_1^+ \phi (\phi_1^+ \phi)^2 \phi \phi,$$

$$V(\phi, \chi) = \lambda_\phi \phi_1^+ \phi (\phi_1^+ \phi)^2 \phi \phi,$$

$$V(\phi, \chi) = \lambda_\phi \phi_1^+ \phi (\phi_1^+ \phi)^2 \phi \phi,$$

$$V(\phi, \chi) = \lambda_\phi \phi_1^+ \phi (\phi_1^+ \phi)^2 \phi \phi,$$

$$V(\phi, \chi) = \lambda_\phi \phi_1^+ \phi (\phi_1^+ \phi)^2 \phi \phi,$$

$$V(\phi, \chi) = \lambda_\phi \phi_1^+ \phi (\phi_1^+ \phi)^2 \phi \phi,$$

$$V(\phi, \chi) = \lambda_\phi \phi_1^+ \phi (\phi_1^+ \phi)^2 \phi \phi,$$
\[ V(\phi,\eta) = V(\phi \to \phi' \eta' \to \eta), \]
\[ V(\phi,\phi') = V(\phi \to \phi, \eta' \to \phi') + \left[ \lambda_{10}^{\phi}\phi\phi\phi(\phi')_2 + \lambda_{10}^{\phi}\phi(\phi')_\eta + \lambda_{12}^{\phi}\phi\phi\phi(\phi')_2 + \lambda_{14}^{\phi}\phi(\phi')_\eta + h.c. \right], \]
\[ V(\eta,\eta') = V(\phi \to \eta, \phi' \to \eta'), \]
\[ V_{\chi\phi\phi',\eta'} = \mu_1 \chi(\phi' \eta') \]
\[ + \lambda_{1}^{\phi\phi'}(\eta^+ \eta')_1 + \lambda_{2}^{\phi\phi'}(\eta^+ \eta')_\eta + \lambda_{3}^{\phi\phi'}(\eta^+ \eta')_\phi + \lambda_{4}^{\phi\phi'}(\eta^+ \eta')_\lambda + h.c. \]
\[ + \lambda_{15} \phi^\dagger \phi' + \lambda_{16} \phi'^\dagger \phi + \lambda_{17} \eta^\dagger \eta + \lambda_{18} \eta'^\dagger \eta' + \lambda_{19} \eta^\dagger \eta' + \lambda_{20} \eta'^\dagger \eta \]
\[ + \phi^\dagger \sigma^s (\lambda_{21} \phi + \lambda_{22} \phi') + \phi'^\dagger \sigma^s (\lambda_{23} \phi + \lambda_{24} \phi') + \eta^\dagger \sigma^s (\lambda_{25} \eta + \lambda_{26} \eta') \]
\[ + \eta'^\dagger \sigma^s (\lambda_{27} \eta + \lambda_{28} \eta') + \lambda_{29} (\phi^\dagger s^\dagger_2 (s \phi')_2 + \lambda_{30} (\phi'^\dagger s^\dagger_2 (s \phi')_2 \]
\[ + \lambda_{31} (s^\dagger_3 (s \phi')_2 + \lambda_{32} (\eta^\dagger s^\dagger_3 (s \eta')_2 + \lambda_{33} (\eta'^\dagger s^\dagger_3 (s \eta')_2 \]
\[ + \lambda_{34} (\eta^\dagger s^\dagger_3 (s \eta')_2 + h.c. \]  

(54)

And, the \( \nabla \) up to quartic interactions is given by

\[ \bar{V} = (\mu_1 \eta \phi + \mu'_1 \eta' \phi')_2\sigma + (\mu_2 \eta \phi + \mu'_2 \eta' \phi')_2 \sigma + \mu_3 \chi \eta \]  

where all the terms in this potential violate the \( L \)-charge, but conserving \( S_4 \). Yet we have not pointed out, but there must additionally exist the terms in \( V \) explicitly violating the only \( S_4 \) symmetry or both the \( S_4 \) and \( L \)-charge too. In the following, most of them will be omitted, only the terms of the kind of interest are provided.

There are the several scalar sectors corresponding to the expected VEV directions: \((1, 0, 0)\) for \( s \) and \((1, 1, 1)\) for \( \phi, \phi', \eta, \eta' \), as written out before. However if these sectors are strongly coupled through the potential \( V_{\text{tri-sext}} \neq 0 \), such vacuum misalignment cannot be given from the potential.
minimization. To overcome the difficulty, as in the literature we might include the extradimensions or supersymmetry, or using additional discrete symmetries. However, in this paper we will provide an alternative explanation, following the works in Refs. [4] of Ma and/or collaborations in 2001, 2004, and 2010. We thus suppose that $\sigma$ and $s$ are all very heavy (see also [12]) with masses $\mu_\sigma$ and $\mu_s$ respectively, so that all of them (as given in $V_{\text{tri-sext}}$) are integrated away. They therefore have the only interactions among themselves as given in $V_{\text{sext}}$. They do not appear as physical particles at or below the TeV scale. Only their imprint at the low energy is a resulting effective potential, which consists of only the fields $\phi$, $\phi'$, $\eta$, $\eta'$ and $\chi$, up to the fourth orders having the same form as $V_{\text{tri}}$.

Consider the potential $V_{\text{tri}}$. The flavons $\phi$, $\phi'$, $\eta$, $\eta'$ with their VEVs aligned in the same direction $(1,1,1)$ are a automatical solution from the minimization conditions of $V_{\text{tri}}$. To see this obviously, in the system of minimization equations let us put $v_1 = v_2 = v_3 = v$, $v'_1 = v'_2 = v'_3 = v'$, $u_1 = u_2 = u_3 = u$, and $u'_1 = u'_2 = u'_3 = u'$, which reduces to

$$
\begin{align*}
(\mu_\phi^2 + \lambda_1^\phi v_\chi^2)u + (3\lambda_1^\phi \phi + 4\lambda_3^\phi \phi)u^2v + (3\lambda_1^\phi \phi' + 4\lambda_3^\phi \phi')u^2v + (6\lambda_1^\phi + 8\lambda_3^\phi)u^3v^3 \\
+(3\lambda_1^\phi \phi' + 4\lambda_3^\phi \phi' + 3\lambda_5^\phi \phi' + 4\lambda_8^\phi \phi')uv^2 + (3\lambda_1^\phi + 4\lambda_3^\phi + 3\lambda_5^\phi + 4\lambda_8^\phi)uu^2v' + (6\lambda_1^\phi + 8\lambda_3^\phi)uv^3 \\
+(3\lambda_1^\phi + 4\lambda_3^\phi + 3\lambda_5^\phi + 4\lambda_8^\phi)uv^2v' + (3\lambda_1^\phi + 4\lambda_3^\phi + 3\lambda_5^\phi + 4\lambda_8^\phi)uu^2v' = 0,
\end{align*}
$$

(56)

$$
\begin{align*}
(\mu_{\phi'}^2 + \lambda_1^\phi v_\chi^2)u + (3\lambda_1^\phi \phi + 4\lambda_3^\phi \phi)u^2v + (3\lambda_1^\phi \phi' + 4\lambda_3^\phi \phi')u^2v + (6\lambda_1^\phi + 8\lambda_3^\phi)u^3v^3 \\
+(3\lambda_1^\phi \phi' + 4\lambda_3^\phi \phi' + 3\lambda_5^\phi \phi' + 4\lambda_8^\phi \phi')uv^2 + (3\lambda_1^\phi + 4\lambda_3^\phi + 3\lambda_5^\phi + 4\lambda_8^\phi)uu^2v' + (6\lambda_1^\phi + 8\lambda_3^\phi)uv^3 \\
+(3\lambda_1^\phi + 4\lambda_3^\phi + 3\lambda_5^\phi + 4\lambda_8^\phi)uv^2v' + (3\lambda_1^\phi + 4\lambda_3^\phi + 3\lambda_5^\phi + 4\lambda_8^\phi)uu^2v' = 0,
\end{align*}
$$

(57)

This system always give the solution $(u, v, u', v')$ as expected, even though it is complicated. It is also noted that the aligned $(1,1,1)$ as given is only one solution. The other directions such as $(1,0,0)$ are also the solution of the potential minimization. We have thus imposed the first case to have the desirable results.

Now we consider the potential $V^{s\sigma}$ concerning the antisextets. To obtain the desirable solution $\langle \sigma \rangle \neq 0$, $\langle s_1 \rangle \neq 0$, and $\langle s_2 \rangle = \langle s_3 \rangle = 0$, the $L$-charge as well as the $S_4$ symmetry must be broken as spoken of around (55). Assume the following choice of soft scalar trilinear and quartic terms as given in the general potential expression $\tilde{V}$ works in $V^{s\sigma}$:

$$
V^{s\sigma} = V_{\text{sext}} + [\bar{\mu}_1(\eta\eta)\sigma + \bar{\mu}_2(\eta\eta)s_1 + \bar{\lambda}_1\eta^\dagger\sigma^\dagger(\phi\eta) + \bar{\lambda}_2s_1^\dagger(\phi\eta) + h.c.] 
$$

(60)
To understand this, note first that in order for \( \sigma \) or \( s_{1,2,3} \) to have a VEV, \( \mathcal{L} \) must be broken and that can only be achieved through the terms of \( \tilde{V} \). However, as in one of the works of Ma cited above, we can introduce a protect symmetry \( Z_2 \) so that the \( s_2 \) and \( s_3 \) are only connected to the terms in the potentials or the Yukawa couplings, which always preserve the symmetry \( \psi_{2,3} \rightarrow -\psi_{2,3} \), where \( \psi \) is any \( S_4 \) triplet appearing in the text such as \( s, \phi, \psi_L \) and so on. Hence they always appear together and protect each other from getting a VEV if neither has one to begin with.

From \( V^{s\sigma} \), the unique solution to the minimization conditions is \( \langle s_2 \rangle = \langle s_3 \rangle = 0 \) and nonzero but very small values of \( \lambda_{\sigma, s} \) and \( v_{\sigma, s} \) as induced in \( \langle s_1 \rangle \) and \( \langle \sigma \rangle \) of Eqs. (15,16) being the root of the \( \frac{\partial V^{s\sigma}_{\text{min}}}{\partial \langle s_1 \rangle} = 0 \) and \( \frac{\partial V^{s\sigma}_{\text{min}}}{\partial \langle \sigma \rangle} = 0 \) (with \( V^{s\sigma}_{\text{min}} \) the minimum of \( V^{s\sigma} \)). First, the equations \( \frac{\partial V^{s\sigma}_{\text{min}}}{\partial \lambda_{\sigma, s}^*} = 0 \) and \( \frac{\partial V^{s\sigma}_{\text{min}}}{\partial v_{\sigma, s}^*} = 0 \) imply that \( \lambda_{\sigma} \) and \( \lambda_{s} \) are in the scale of the antisextets’ masses \( \mu_{\sigma} \) and \( \mu_{s} \). Let us denote a characteristic scale \( M \) so that \( \Lambda_{\sigma}, \Lambda_{s}, \mu_{\sigma}, \mu_{s} \sim M \). The remaining equations \( \frac{\partial V^{s\sigma}_{\text{min}}}{\partial \lambda_{\sigma, s}^*} = 0 \) and \( \frac{\partial V^{s\sigma}_{\text{min}}}{\partial v_{\sigma, s}^*} = 0 \) provide the small VEVs induced by the standard model electroweak scale \( u \sim v \):

\[
\lambda_{\sigma} \sim \bar{\lambda}_1 v^2 M^2, \quad \lambda_{s} \sim \bar{\lambda}_2 v^2 M^2, \quad v_{\sigma} \sim \bar{\mu}_1 v^2 \frac{M^2}{2}, \quad v_{s} \sim \bar{\mu}_2 v^2 \frac{M^2}{2}.
\]

The parameters \( \bar{\mu}_{1,2} \) and \( \bar{\lambda}_{1,2} v \) (which have the dimension of mass) may be naturally small in comparison with \( v \), because its absence enhances the symmetry of \( V^{s\sigma} \). We remark that the VEVs of the type II seesaw mechanism \( \lambda_{\sigma}, \lambda_{s} \) work because from (61) the spontaneous breaking of electroweak symmetry is already accomplished by \( v \), the \( \lambda_{\sigma}, \lambda_{s} \) may be small, as long as \( M \) is large. On the other hand, \( v_{\sigma} \) and \( v_{s} \) are the VEVs of the type I seesaw mechanism which are also small for the same reason; therefore, in this model the seesaw scale \( M \) may be much lower than that of the unusual type I seesaw. These are also the important results of our paper.

Along the model, as mentioned the new particles are: \( N_R \) getting masses in \( \Lambda_{\sigma, s} \) scale, \( U \) and \( D \) with masses proportional to \( w \), and \( Z', X, Y \) having masses as combinations of \( w \) and \( \Lambda_{\sigma, s} \), where \( w \) and \( \Lambda_{\sigma, s} \) are the scales of 3-3-1 gauge symmetry breaking down to the standard model [9,10]. If the antisextets \( \sigma, s \) are heaviest, i.e. \( \Lambda_{\sigma, s}^2 \gg w^2 \), the new gauge bosons and \( N_R \) will have large masses ranging in this scale accordingly, however \( U \) and \( D \) can gain masses much smaller than (for example, in some hundreds of GeV). In the case of \( w \sim \Lambda_{\sigma, s} \), the masses of \( U \) and \( D \) will be picked up in the same order with those of the new gauge bosons and \( N_R \). By the way, the \( \chi \) scalar may be also integrated out like the antisextets. This will explain why the parity breaking parameters \( \langle \eta_{3}^{0} \rangle, \langle \eta_{5}^{0} \rangle, \langle \chi_{1}^{0} \rangle \) are small, in similarity to \( v_{\sigma, s} \). The mixings among the ordinary quarks and exotic quarks and the tree-level FCNC as mentioned can be suppressed by this mechanism.
There are a lot of SU(2)$_L$ scalar doublets and triplets in the model, under which they can lead to modifications for the precision electroweak data (see [15] for a detailed analysis on this problem). The most serious one comes from tree-level corrections for the $\rho$ parameter. In the effective theory limit, the mass of $W$ boson and $\rho$ are evaluated by

$$m_W^2 = \frac{g^2}{2} v_w^2, \quad \rho = \frac{m_W^2}{c_W^2 m_Z^2} = 1 - \frac{2(\lambda_\sigma^2 + \lambda_s^2)}{v_w^2},$$

where $v_w^2 \simeq 3(v^2 + v'^2 + u^2 + u'^2) = (174 \text{ GeV})^2$ is a natural approximation due to $v_\sigma^2, v_s^2, \langle \chi_1^0 \rangle^2 \ll v^2, v'^2, u^2, u'^2$, as given above. Because $\lambda_{\sigma,s}$ are in eV order responsible for the neutrino masses, the $\rho$ parameter is absolutely close to 1, which is in good agreement with the data [1].

V. CONCLUSIONS

As a result of anomaly cancelation, the 3-3-1 model accepts discrepancy of one family of quarks from other two. We have therefore searched for a symmetry group acting on both 2-family and 3-family indices, the simplest of which is $S_4$—the symmetry group of a cube as a flavor symmetry. Corresponding to the lepton number, the new lepton charge $L$ and its residual symmetry—the lepton parity $P_l$ have been introduced into the model.

If $P_l$ is conserved, the neutrino masses come from small VEVs of first components of scalar antisextets, known as type II seesaw contributions. If $P_l$ is broken there are additional contributions from type I seesaw due to suppression of 3-3-1 symmetry breaking VEVs of just the antisextets. The tribimaximal mixing arises as a result under $S_4$ and $L$ symmetries. A deviation from this mixing can result from $L$ small violating terms or $S_4$ breaking soft-terms. By imposing appropriate $L$ and $S_4$ violating potential, the VEV alignments have been obtained. Also, the smallness of the seesaw contributions have been explained.

Quark mixing matrix is unity at the tree-level only if $P_l$ is exact, not spontaneously broken. A breaking of the charge will lead to mixings between exotic quarks and ordinary quarks. It can also provide mixings among the ordinary quarks. In this case the CKM is not unitary. There are contributions to flavor changing neutral currents at the tree-level.

The model can provide interesting candidates for dark matter without supersymmetry as stored in the antisextet flavons as well as in the $\chi$ triplet if the lepton parity is conserved (see also the notes as sketched in [16]), and the model’s phenomenology is very rich. They are worthy to be devoted to further studies.
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Appendix A: $S_4$ group and Clebsch-Gordan coefficients

$S_4$ is the permutation group of four objects, which is also the symmetry group of a cube. It has 24 elements divided into 5 conjugacy classes, with $1, 1', 2, 3,$ and $3'$ as its 5 irreducible representations. Any element of $S_4$ can be formed by multiplication of the generators $S$ and $T$ obeying the relations $S^4 = T^3 = 1, ST^2S = T$. Without loss of generality, we could choose $S = (1234), T = (123)$ where the cycle $(1234)$ denotes the permutation $(1, 2, 3, 4) \rightarrow (2, 3, 4, 1)$, and $(123)$ means $(1, 2, 3, 4) \rightarrow (2, 3, 1, 4)$. The conjugacy classes generated from $S$ and $T$ are

- $C_1 : 1$
- $C_2 : (12)(34) = TS^2T^2, (13)(24) = S^2, (14)(23) = T^2S^2T$
- $C_3 : (123) = T, (132) = T^2, (124) = T^2S^2, (142) = S^2T,$
  $(134) = S^2TS^2, (143) = STS, (234) = S^2T^2, (243) = TS^2$
- $C_4 : (1234) = S, (1243) = T^2ST, (1324) = ST,$
  $(1342) = TS, (1423) = TST^2, (1432) = S^3$
- $C_5 : (12) = STS^2, (13) = TSTS^2, (14) = ST^2,$
  $(23) = S^2TS, (24) = TST, (34) = T^2S$

The character table of $S_4$ is given as follows

| Class | $n$ | $h$ | $\chi_1$ | $\chi_1'$ | $\chi_2$ | $\chi_3$ | $\chi_3'$ |
|-------|-----|-----|----------|----------|----------|----------|----------|
| $C_1$ | 1   | 1   | 1        | 1        | 2        | 3        | 3        |
| $C_2$ | 3   | 2   | 1        | 1        | 2        | -1       | -1       |
| $C_3$ | 8   | 3   | 1        | 1        | -1       | 0        | 0        |
| $C_4$ | 6   | 4   | 1        | -1       | 0        | -1       | 1        |
| $C_5$ | 6   | 2   | 1        | -1       | 0        | 1        | -1       |
where \( n \) is the order of class and \( h \) the order of elements within each class. Let us note that 
\( C_{1,2,3} \) are even permutations, while \( C_{4,5} \) are odd permutations. The two three-dimensional representations differ only in the signs of their \( C_4 \) and \( C_5 \) matrices. Similarly, the two one-dimensional representations behave the same.

We will work in basis where \( 3, 3' \) are real representations whereas \( 2 \) is complex. One possible choice of generators is given as follows

\[
\begin{align*}
\mathbb{1} : & \quad S = 1, \quad T = 1 \\
\mathbb{1}' : & \quad S = -1, \quad T = 1 \\
\mathbb{2} : & \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} \\
\mathbb{3} : & \quad S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\
\mathbb{2}' : & \quad S = - \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
\end{align*}
\]

(A1)

where \( \omega = e^{2\pi i / 3} = -1/2 + i\sqrt{3}/2 \) is the cube root of unity. Using them we calculate the Clebsch-Gordan coefficients for all the tensor products as given below.

First, let us put \( \mathbb{3}(1,2,3) \) which means some \( \mathbb{3} \) multiplet such as \( x = (x_1, x_2, x_3) \sim \mathbb{3} \) or \( y = (y_1, y_2, y_3) \sim \mathbb{2} \) or so on, and similarly for the other representations. Moreover, the numbered multiplets such as \( (... , ij , ...) \) mean \( (... , x_i y_j , ...) \) where \( x_i \) and \( y_j \) are the multiplet components of different representations \( x \) and \( y \), respectively. In the following the components of representations in l.h.s will be omitted and should be understood, but they always exist in order in the components of decompositions in r.h.s:

\[
\begin{align*}
\mathbb{1} \otimes \mathbb{1} & = \mathbb{1}(11), \quad \mathbb{1}' \otimes \mathbb{1}' = \mathbb{1}(11), \quad \mathbb{1} \otimes \mathbb{1}' = \mathbb{1}'(11), \\
\mathbb{1} \otimes \mathbb{2} & = \mathbb{2}(11,12), \quad \mathbb{1}' \otimes \mathbb{2} = \mathbb{2}(11,-12), \\
\mathbb{1} \otimes \mathbb{3} & = \mathbb{3}(11,12,13), \quad \mathbb{1}' \otimes \mathbb{3} = \mathbb{3}'(11,12,13), \\
\mathbb{1} \otimes \mathbb{3}' & = \mathbb{3}'(11,12,13), \quad \mathbb{1}' \otimes \mathbb{3}' = \mathbb{3}(11,12,13), \\
\mathbb{2} \otimes \mathbb{2} & = \mathbb{1}(12+21) \oplus \mathbb{1}'(12-21) \oplus \mathbb{2}(22,11), \\
\mathbb{2} \otimes \mathbb{3} & = \mathbb{3} \left( (1+2)1, \omega(1+\omega2)2, \omega^2(1+\omega^22)3 \right)
\end{align*}
\]

(A2)
\[ \oplus 3' \left( (1-2)1, \omega(1-\omega)2, \omega^2(1-\omega^2)3 \right) \]  
\[ 2 \otimes 3' = 3' \left( (1+2)1, \omega(1+\omega)2, \omega^2(1+\omega^2)3 \right) \]
\[ \oplus 2 \left( (1-2)1, \omega(1-\omega)2, \omega^2(1-\omega^2)3 \right), \]  
\[ 3 \otimes 3 = 1(11+22+33) \oplus 2(11+\omega^222+\omega33, 11+\omega22+\omega^233) \]
\[ \oplus 3_s(23+32, 31+13, 12+21) \oplus 3'_s(23-32, 31-13, 12-21), \]  
\[ 3' \otimes 3' = 1(11+22+33) \oplus 2(11+\omega^222+\omega33, 11+\omega22+\omega^233) \]
\[ \oplus 3_s(23+32, 31+13, 12+21) \oplus 3'_s(23-32, 31-13, 12-21), \]  
\[ 3 \otimes 3' = 1'(11+22+33) \oplus 2(11+\omega^222+\omega33, -11-\omega22-\omega^233) \]
\[ \oplus 3'_s(23+32, 31+13, 12+21) \oplus 3_s(23-32, 31-13, 12-21), \]
\[ \text{where the subscripts } s \text{ and } a \text{ respectively refer to their symmetric and antisymmetric product combinations as explicitly pointed out. We also notice that many group multiplication rules above have similar forms as those of } S_3 \text{ and } A_4 \text{ groups.} \]

In the text we usually use the following notations, for example, \( (xy)_3 = [xy]_3 \equiv (xy_3 - x3y_2, x2y_3, x1y_2 - x2y_1) \) which is the Clebsch-Gordan coefficients of \( 3 \) in the decomposition of \( 3 \otimes 3' \), where as mentioned \( x = (x_1, x_2, x_3) \sim 3 \) and \( y' = (y_1, y_2, y_3) \sim 3' \).

The rules to conjugate the representations \( \Lambda, \Lambda', 2, 3, \) and \( 3' \) are given by

\[ 2^*(1^*, 2^*) = 2(2^*, 1^*), \quad 1^*(1^*) = 1(1^*), \quad 1'^*(1^*) = 1'(1^*), \]  
\[ 2^*(1^*, 2^*, 3^*) = 3(1^*, 2^*, 3^*), \quad 2'^*(1^*, 2^*, 3^*) = 3'(1^*, 2^*, 3^*), \]

where, for example, \( 2^*(1^*, 2^*) \) denotes some \( 2^* \) multiplet of the form \( (x_1^*, x_2^*) \sim 2^* \).

**Appendix B: The numbers**

In the following we will explicitly point out the lepton number \( L \) and lepton parity \( P_L \) of the model particles (notice that the family indices are suppressed):

| Particles | \( L \) | \( P_L \) |
|-----------|--------|---------|
| \( N_R, u, d, \phi_1^+, \phi_1^+, \phi_1^+, \phi_2^0, \phi_2^0, \eta_1^+, \eta_1^+, \eta_2^+, \eta_2^+, \chi_3^0, \sigma_3^0, s_3^0 \) | 0 | 1 |
| \( \nu_L, l, U, D^*, \phi_3^+, \phi_3^+, \phi_3^+, \eta_3^+, \eta_3^+, \lambda_0^+, \lambda_0^+, \lambda_1^+, \lambda_2^+, \sigma_1^0, \sigma_1^0, \sigma_2^0, \sigma_2^0, s_2^0, s_2^0 \) | -1 | -1 |
| \( \sigma_0^0, \sigma_1^0, \sigma_2^+, s_1^+, s_2^+, s_2^+ \) | -2 | 1 |

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