IDEALS AND STRONG AXIOMS OF DETERMINACY

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Abstract

Θ is the least ordinal α with the property that there is no surjection \( f : \mathbb{R} \to \alpha \). \( \text{AD}_{\mathbb{R}} \) is the Axiom of Determinacy for games played on the reals. It asserts that every game of length \( \omega \) of perfect information in which players take turns to play reals is determined. For a sentence \( \phi \), we say that \( M \) is the minimal model of \( \text{ZF} + \text{AD}_{\mathbb{R}} + \phi \) if \( M \) is a transitive model of \( \text{ZF} + \text{AD}_{\mathbb{R}} + \phi \) containing all reals and ordinals, and whenever \( N \) is a transitive model of \( \text{ZF} + \text{AD}_{\mathbb{R}} + \phi \) containing all reals and ordinals then \( M \subseteq N \). We consider the theories

\((T_1)\) \( \text{ZFC} + \text{CH} + \text{"There is an } \omega_1 \text{-dense ideal on } \omega_1 \text{ such that if } j \text{ is the generic embedding associated with it then } j \upharpoonright \text{Ord is independent of the generic object".}\)

\((T_2)\) \( \text{ZF} + \text{AD}_{\mathbb{R}} + \text{"} \Theta \text{ is a regular cardinal."} \)

The main result of this paper is that \( T_1 \) implies that the minimal model of \( T_2 \) exists. Woodin, in unpublished work, showed that the consistency of \( T_2 \) implies the consistency of \( T_1 \). We will also give a proof of this result, which, together with our main theorem, establishes the equiconsistency of \( T_1 \) and \( T_2 \).

Woodin conjectured that a variant of \( T_1 \) above, without the additional assumption on \( j \upharpoonright \text{Ord} \), is equiconsistent with \( T_2 \) (see [34, Conjecture 12]). Currently, the best method for attacking such conjectures is the Core Model Induction (CMI). Ketchersid [6] introduced CMI (based on ideas due mainly to Woodin) and initiated the project of using it to attack Woodin’s conjecture. The extra hypothesis that \( j \upharpoonright \text{Ord} \) is a class of \( V \) featured prominently in [6], and it continues to do so in our current work. Thus, our work completes the work that started in [6] some 20 years ago. The aforementioned theorem is the only known equiconsistency at the level of \( \text{AD}_{\mathbb{R}} + \text{"} \Theta \text{ is a regular cardinal."} \) We also establish other theorems of a similar nature.

1. INTRODUCTION

Famously, Ulam’s investigations of the Measure Problem, which asks whether there is a measure on \([0,1]\), led him to prove that there is no countably complete 0-1 measure, that is an ultrafilter, on \( \omega_1 \)

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Ulam’s theorem is often presented as showing that \( \omega_1 \) is not a measurable cardinal, where we say that \( \kappa \) is a measurable cardinal if there is a \( \kappa \)-complete ultrafilter \( U \) on \( \kappa \).

Ulam’s theorem and the Measure Problem in general has been a source of great ideas in set theory, and one of these ideas has been the study of ideals that could induce nice ultrafilters on uncountable cardinals. Suppose, for example, that \( \mathcal{I} \subseteq \wp(\kappa) \) is an ideal on \( \kappa \). Let \( \mathbb{P}_\mathcal{I} = \wp(\omega_1)/\mathcal{I} \) be the corresponding boolean algebra induced by \( \mathcal{I} \). One can also think of \( \mathbb{P}_\mathcal{I} \) as a poset ordered by inclusion. It is not hard to see that if \( U \) is a generic object for \( \mathbb{P}_\mathcal{I} \) then the function \( U^* : (\wp(\kappa))^V \to \{0,1\} \) given by \( U^*(A) = 0 \iff A \notin U \) satisfies many of the properties of being a 0-1 measure with two major deficiencies. First \( U^* \) may not measure all subsets of \( \kappa \) that exist in \( V[U] \), and second, \( U^* \) may not be countably complete. It is then unclear exactly in what way this approach could lead to a reasonable study of the Measure Problem.

The concept of ultrapower introduced the necessary formalism to eliminate the aforementioned issues. It is a well-known fact that a cardinal \( \kappa \) is a measurable cardinal if and only if there is an elementary embedding \( j : V \to M \) such that \( M \) is a transitive class of \( V \), \( j \neq id \), \( j \upharpoonright \kappa = id \) and \( j(\kappa) > \kappa \). If \( \kappa \) is a measurable cardinal then one obtains the \( M \) above as an ultrapower of \( V \) by a \( \kappa \)-complete ultrafilter on \( \kappa \). The same can be done with our generic \( U \) above, and for the start one can only demand the well-foundedness of \( \text{Ult}(V,U) \).

An ideal \( \mathcal{I} \) is called precipitous if whenever \( U \subseteq \mathbb{P}_\mathcal{I} \) is a generic ultrafilter, the generic ultrapower of \( V \) by \( U \), \( \text{Ult}(V,U) \), is well-founded. This approach to the Measure Problem has been incredibly fruitful and has lead to many great discoveries. The story has been partially told in Foreman’s long manuscript [2]. The study of precipitous ideals has led to solutions of problems considered not just by set theorists but by wider mathematical community. For example, Theorem 5.42 of [2] states that the existence of a certain nice ideal implies among other things that every projective set of reals is Lebesgue measurable.

Shelah, Jensen, and Steel have established the following famous theorem, which is one of the first equiconsistency results that connects ideals and large cardinals.

**Theorem 1.1.** The following theories are equiconsistent.

1. \( \text{ZFC} + \text{There is a pre-saturated ideal on } \omega_1 \).

2. \( \text{ZFC} + \text{There is a Woodin cardinal}. \)

Claverie and Schindler [1] have improved the above result and shown that in fact theory 2 above is equiconsistent with the theory “there is a strong ideal on \( \omega_1 \)”.

Using a stronger property of ideals, Woodin established the following remarkable theorem. Below we say that the ideal \( \mathcal{I} \subseteq \wp(\omega_1) \) is \( \omega_1 \)-dense if \( \mathbb{P}_\mathcal{I} \) has a dense set of size \( \omega_1 \). \( L(\mathbb{R}) \) is the minimal model of \( \text{ZF} \) that contains all the ordinals and the real numbers (see [16, Theorem 2.11.1]).

\[ \text{1.e. intersects all dense open subsets of } \mathbb{P}_\mathcal{I} . \]

\[ \text{2} \text{. Shelah proves the consistency of 2 from the consistency of 1 by forcing techniques. Jensen and Steel proves the converse using inner model theoretic techniques, in particular core model theory. See, for example, [21, 4]. Pre-saturated ideals are precipitous, see [3].} \]

\[ \text{3} \text{. See Section 2 for the relevant definitions. The property of being strong is weaker than being pre-saturated. Strong ideals are precipitous.} \]
Theorem 1.2 (Woodin). The following theories are equiconsistent.

1. $\text{ZFC} + \text{"There is an } \omega_1\text{-dense ideal on } \omega_1\text{"}.$

2. The Axiom of Determinacy ($\text{AD}$) holds in $L(\mathbb{R})$.\footnote{We note that it had been established by Woodin that the theory $\text{AD}^{L(\mathbb{R})}$ is equiconsistent with the theory $\text{ZFC} + \text{"there are } \omega\text{ many Woodin cardinals"}.}$

In fact, the theorem gives more. The forward direction of the theorem uses a technique known as the Core Model Induction (CMI), which is an inductive method for proving determinacy in canonical models such as $L(\mathbb{R})$. The reverse direction is achieved in a $\mathbb{P}_{\text{max}}$ extension of $L(\mathbb{R})$. The proof of the forward direction can be found in \cite[Theorem 2.11.1]{16} and the reverse direction is presented in \cite[Chapter 6.2]{34}. The aforementioned theorems of Shelah, Jensen, Steel, Woodin, and others demonstrate intimate connections between different branches of set theory, namely the study of precipitous ideals and the study of models of $\text{AD}$. It seems that the connections that theorems like above establish are rooted in the naturalness of the constructions that produce the models of one theory given a model of another, and this naturalness — the feeling of having no barriers to naturally drift from one theory to another as if they were one and the same theory — is not expressed in the formal statement of the theorem, namely that the two theories are equiconsistent. The main theorem of this paper, Theorem 1.4, has the same spirit as Woodin’s theorem above.

Definition 1.3 (DI). Let $\text{DI}$ be the conjunction of CH with the statement that there is an $\omega_1$-dense ideal $I$ on $\omega_1$ such that letting $j_G : V \to \text{Ult}(V,G)$ denote the ultrapower embedding associated to any generic filter $G \subset \mathbb{P}_I$,

$$(\dagger) \quad \text{for all ordinals } \alpha, j_G \restriction \alpha \in V \text{ and does not depend on the choice of } G.$$\footnote{We thank John Krueger for pointing out the following argument. Assume for all $\alpha, j \restriction \alpha \in V$. By the pigeonhole principle, there exists some nonzero $b$ in $\mathbb{P}_I$ such that the class $X = \{\alpha : b \text{ decides the value of } j \restriction \alpha\}$ is not bounded in the ordinals. But $X$ is closed downwards, so $b$ decides $j \restriction \alpha$ for all $\alpha$. Now replace $I$ with the ideal $J = I \restriction b$. Then $J$ is $\omega_1$-dense and the value of $j \restriction \alpha$ does not depend on $G$ for all $\alpha$.}

The following theorem is the main result of this paper.

Theorem 1.4. $\text{ZFC} + \text{DI}$ implies that the minimal model of $\text{ZF} + \text{AD}_R + \text{"}\Theta\text{ is a regular cardinal"}$ exists.

Remark 1.5. \begin{itemize}
\item We note that if $I$ is $\omega_1$-dense, then $\mathbb{P}_I$ is homogeneous and $I$ is presaturated. In practice, e.g. in core model induction applications that make use of $\omega_1$-dense ideals, homogeneity and presaturation are the main consequences used in arguments. However, we seem to need a bit more of the density assumption and $(\dagger)$ for our proof.
\item The second clause of $(\dagger)$ is redundant and follows from the first clause.\footnote{In fact, we just need $j \restriction \omega_3^V \in V$ for the proof of Theorem 1.4.} \end{itemize}
As was mentioned in the abstract, a theory $T$ extending $ZF + AD_R$ has a minimal model if it has a transitive model $M$ containing the reals and ordinals such that it is contained in any other transitive model of $T$ containing the ordinals and the reals. The proof of [9, Theorem 6.26] explicitly establishes that the existence of divergent models of $AD^+$ implies their common part is beyond a model of $ZF + AD_R + \text{\"{O} is a regular cardinal\"}$. Thus, if there is a model of $ZF + AD_R + \text{\"{O} is a regular cardinal\"}$ then there is a minimal one.

As a result of this theorem and Woodin’s unpublished work, which we will present in Section 2, we obtain the following equiconsistency result.

**Theorem 1.6.** The following theories are equiconsistent.

1. $ZFC + DI$
2. $ZF + AD_R + \text{\"{O} is a regular cardinal\"}$

Below we give some more motivations for proving such theorems.

**Motivations**

Motivated by the success of the generic elementary embeddings induced by ideals or other similar structures, Foreman has suggested them as a possible foundational framework, and explicated his ideas in [2, Chapter 11]. As is well known, the basic foundational issue that set theory is facing is its inability to produce a single foundational framework that is accepted by all and at the same time solves all fundamental problems including the Continuum Hypothesis. Several successful foundational frameworks, such as Forcing Axioms, Canonical Inner Models and Generic Embeddings, have been proposed and developed, but they all seem to disagree on basic questions such as whether the Continuum Hypothesis is true or whether the universe is a ground (i.e., cannot be obtained as a non-trivial forcing extension of an inner model) and on many other such fundamental questions.

One of the main goals of CMI is to unify all of these frameworks by showing that each can be naturally interpreted in another. Given such bi-interpretations, disagreements on fundamental questions can be traced to subjective preferences in one framework over another, or preferences in one type of formalism over another.

For example, Woodin’s theorem (Theorem 1.2) and Theorem 1.6 show how to interpret natural ideas occurring in the study of generic embeddings in models of determinacy and vice versa. The reason is that, in both cases, the forcing notion used to obtain the models carrying such ideals are natural forcing notions, and in the other direction, the models of determinacy built in both cases are natural canonical models of AD. This sort of bi-interpretability demonstrates that one cannot have scientifically objective reasons for preferring generic embeddings over, say, determinacy axioms, as they are deeply interconnected: commitment to one entails commitment to the other. A bias towards a particular formalism can be justified by other more pragmatic ways, for example by insisting on the shortest or clearest or most natural possible proofs of certain desired theorems. The ideas exposted above are the motivational ideas behind proving theorems like the main theorem of
The history behind the paper.

The first written presentation of CMI is Ketchersid’s PhD thesis [6], which motivated Ralf Schindler and John Steel to work on a book presenting the Core Model Induction (see [16]). In 2006 they organized a seminar in Berlin covering the basics of CMI. As one can see by flipping through [16], one of the main directions pursued by the community at this time was to complete Ketchersid’s project. See John Steel’s [23] for a conjecture along the same vein.

One of the main reasons this was believed to be important was that it was not known and still is not known how to force DI, clause 1 of Theorem 1.6, from conventional large cardinals that are weaker than supercompact cardinals. Woodin forced DI both over the models of $\text{AD}_R + \langle \Theta \rangle$ and from an almost huge cardinal (see [2, Chapter 7.14]). In [34], Woodin also forced $\text{MM}^{++}(c)$, Martin’s Maximum for forcing posets of size at most the continuum, over a model of $\text{AD}_R + \langle \Theta \rangle$ (see [34, Theorem 9.40]), and just like with DI, it is not known how to force $\text{MM}^{++}(c)$ from conventional large cardinals weaker than a supercompact cardinal. These and other results of Woodin from [34] seem to suggest that the theory $\text{AD}_R + \langle \Theta \rangle$ is in the region of supercompact cardinals, and the project of getting a model of it via CMI seemed to be equivalent to getting canonical inner models that could have supercompact cardinals in it, which has been one of the Holy Grails of set theory.

However, [9] showed that in fact the theory $\text{AD}_R + \langle \Theta \rangle$ is much weaker than a supercompact cardinal: it is weaker than a Woodin cardinal that is a limit of Woodin cardinals (see [9, Theorem 6.26]). This theorem seems to suggest the existence of a gap in our understanding of models of set theory. On the one hand, the conventional forcing and large cardinal technology that is needed to force statements such as DI or $\text{MM}^{++}(c)$ requires the complexity of a supercompact cardinal or beyond, and on the other hand, equally natural but different technologies based on [34] place the complexity far below a supercompact cardinal. This phenomenon has not yet found a proper explanation.

While [9] did show that finishing Ketchersid’s project will not lead to one of the Holy Grails of set theory, the importance of the project didn’t diminish, as it was perceived to be one of the main guiding problems for developing the CMI to a technique for producing models of $\text{AD}_R + \langle \Theta \rangle$ and beyond\footnote{See for example [12] for an analysis of determinacy models stronger than those of $\text{AD}_R + \langle \Theta \rangle$ and core model induction techniques for constructing such models from strong theories like PFA.}. In this direction, the last chapter of the first author’s thesis [14] gave a rough outline of producing models of $\text{AD}_R + \langle \Theta \rangle$ from DI, but later on a substantial error was discovered in the proof by Steel and the second author. The concept of *embeddings with condensation* introduced in [10] (see [10, Definition 11.14, Lemma 11.15]) and further developed in [29] (see [29, Definition 3.81, Lemma 3.82]) and [12] seemed good enough for correcting the aforementioned error, which is what we will do in this paper (see Theorem 4.34).
Furthermore, the third author, in his thesis \cite{32}, developed techniques for handling the successor stages of CMI that avoid the famous “A-iterability” proofs (see \cite[Theorem 5.4.8]{16} or \cite[Theorem 1.46]{22}) and various other complicated arguments originally due to Woodin. We adapt the third author’s arguments to our current context (see Section 4.3.1).

As mentioned above, it is a well-known unpublished theorem of Woodin that one can force DI over models of AD\_R + \(\Theta\) is a regular cardinal. The third author forced some more general statements about ideals in his thesis, and we will use his argument to give a proof of this theorem of Woodin in Subsection 2.3 below. Thus, this paper presents a self-contained proof of Theorem 1.6, giving the proof of both directions in as much detail as it is possible to do in a research article. Theorem 1.6 and Theorem 1.10 are currently the only known equiconsistency results at the level of AD\_R + “\(\Theta\) is a regular cardinal”, and it completes Ketchersid’s project.

We do not know the consistency strength of the theory ZFC + CH + \(\text{there is an } \omega_1\text{-dense ideal on } \omega_1\),” so \cite[Conjecture 12]{34} still stands, but we expect that refinements of our techniques here would settle the problem.

**Some definitions and more results.**

Let \(\mathcal{I}\) be an ideal on \(\omega_1\) and as above, let \(\mathbb{P}_{\mathcal{I}} = \mathcal{P}(\omega_1)/\mathcal{I}\) be the corresponding boolean algebra induced by \(\mathcal{I}\). We write \(\mathcal{I}^+\) for the collection of \(\mathcal{I}\)-positive sets and \(\mathcal{F}_{\mathcal{I}}\) for the dual filter of \(\mathcal{I}\). All ideals considered in this paper are proper, countably complete, and normal (see Section 2 for more discussions on basic properties of ideals).

For any set \(X\), let \(\mathcal{P}(X)\) be the set of countable subsets of \(X\). Let \(\mathcal{I}\) be an ideal on \(\mathcal{P}(\omega_1(\mathbb{R}))\). We let \(\mathcal{I}^+\) and \(\mathcal{F}_{\mathcal{I}}\) be as before and let \(\mathbb{P}_{\mathcal{I}}\) be the boolean algebra \(\mathcal{P}(\mathcal{P}(\omega_1(\mathbb{R}))/\mathcal{I}\). Let \(c\) denote the size of the continuum.

**Definition 1.7.** An ideal \(\mathcal{I}\) on \(\omega_1\) or on \(\mathcal{P}(\omega_1(\mathbb{R}))\) is **precipitous** if whenever \(G \subseteq \mathbb{P}_{\mathcal{I}}\) is a \(V\)-generic ultrafilter, the generic ultrapower \(\text{Ult}(V,G)\) induced by \(G\) is well-founded. \(\dashv\)

**Definition 1.8.** An ideal \(\mathcal{I}\) on \(\mathcal{P}(\omega_1(\mathbb{R}))\) is **strong** if

(a) \(\mathcal{I}\) is precipitous, and

(b) whenever \(G \subseteq \mathbb{P}_{\mathcal{I}}\) is \(V\)-generic, letting \(j_G : V \to \text{Ult}(V,G)\) be the ultrapower map, then \(j_G(\omega_1) = c^+\). \(\dashv\)

**Definition 1.9.** An ideal \(\mathcal{I}\) on \(\mathcal{P}(\omega_1(\mathbb{R}))\) is **pseudo-homogeneous** if for every \(\alpha \in \text{ON}, s \in \text{ON}^\omega, \lambda < c^+, A \subseteq \lambda^\omega\), and formula \(\theta\) in the language of set theory, letting \(G \subseteq \mathbb{P}_{\mathcal{I}}\) be a \(V\)-generic filter and \(j_G : V \to \text{Ult}(V,G)\) the corresponding ultrapower map, the truth of the statement

\[
\text{Ult}(V,G) \models \theta[\alpha, j_G(s), j_G[A]]
\]

is independent of the choice of \(G\). \(\dashv\)
We obtain an equiconsistency regarding strong, pseudo-homogeneous ideals on $\mathcal{P}_{\omega_1}(\mathbb{R})$.

**Theorem 1.10.** The following are equiconsistent.

1. ZFC + “The nonstationary ideal on $\mathcal{P}_{\omega_1}(\mathbb{R})$ is strong and pseudo-homogeneous.”

2. ZF + AD$_{\mathbb{R}}$ + “$\Theta$ is a regular cardinal.”

In Section 2, we summarize basic facts about ideals and AD$^+$ we need in this paper and show that $\text{DI}$ and the existence of a strong, pseudo-homogeneous ideal on $\mathcal{P}_{\omega_1}(\mathbb{R})$ are consistent relative to AD$_{\mathbb{R}}$ + “$\Theta$ is a regular cardinal.”

In Section 3, we summarize preliminaries and basic notions we need for CMI. Section 4 obtains models of “AD$_{\mathbb{R}}$ + $\Theta$ is a regular cardinal” from DI. In Section 5, we outline the argument obtaining models of “AD$_{\mathbb{R}}$ + $\Theta$ is a regular cardinal” from the assumption that the nonstationary ideal on $\mathcal{P}_{\omega_1}(\mathbb{R})$ is strong and pseudo-homogeneous. Since the argument is very similar to the argument from DI, we simply focus on the main changes, leaving the details to the reader. In the following, we will often write “$\Theta$ is regular” for “$\Theta$ is a regular cardinal.”

**Acknowledgments.** The work here is greatly influenced by Ketchersid’s work in his thesis [6], which in turn is greatly influenced by Woodin’s early work in the CMI. We are grateful to them for their inspiring work in this direction. We are also grateful to Woodin for his permission to include the proof of his unpublished work which shows that Con(ZF + AD$_{\mathbb{R}}$ + “$\Theta$ is a regular cardinal”) implies Con(ZFC + DI). The second author is grateful to the NSF for its generous support via Career Award DMS-1945592.

2. DENSE IDEALS AND STRONG PSEUDO-HOMOGENEOUS IDEALS FROM MODELS OF AD$_{\mathbb{R}}$ + $\Theta$ IS REGULAR

In this section, we show the consistency of DI and of the existence of a strong, pseudo-homogeneous ideal on $\mathcal{P}_{\omega_1}(\mathbb{R})$ from AD$_{\mathbb{R}}$ + “$\Theta$ is regular.” We first review basic facts about AD$^+$ and ideals. In Subsection 2.3, we will give the consistency proof.

2.1. Basic facts about AD$^+$

We start with the definition of Woodin’s theory of AD$^+$. In this paper, we identify $\mathbb{R}$ with $\omega^\omega$. We use $\Theta$ to denote the sup of ordinals $\alpha$ such that there is a surjection $\pi : \mathbb{R} \to \alpha$. Under AC, $\Theta$ is just the successor cardinal of the continuum. In the context of AD, the cardinal $\Theta$ is shown to be the supremum of $w(A)^9$ for $A \subseteq \mathbb{R}$ (cf. [20]). The definition of $\Theta$ relativizes to any determined pointclass $\Gamma$ with sufficient closure properties, and we may write $\Theta^\Gamma$ for the supremum of ordinals $\alpha$ such that there is a surjection from $\mathbb{R}$ onto $\alpha$ coded by a set of reals in $\Gamma$.

**Definition 2.1.** AD$^+$ is the theory ZF + AD + DC$_{\mathbb{R}}$ plus the following two statements:

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8 We adapt the proof given in the third author’s thesis here. We note the result that Con(DI) follows from Con(AD$_{\mathbb{R}}$ + “$\Theta$ is regular”) is due to Woodin.

9 $w(A)$ is the Wadge rank of $A$. 

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1. For every set of reals $A$, there are a set of ordinals $S$ and a formula $\varphi$ such that $x \in A \iff L[S, x] \models \varphi[S, x]$. The pair $(S, \varphi)$ is called an $\infty$-Borel code for $A$.

2. For every $\lambda < \Theta$, every continuous $\pi : \lambda^\omega \to \omega^\omega$, and every set of reals $A$, the set $\pi^{-1}[A]$ is determined.

$\text{AD}^+$ is equivalent to $\text{AD} + \text{“the set of Suslin cardinals is closed below } \Theta.\text{”}$ Another, perhaps more useful, characterization of $\text{AD}^+$ is $\text{AD} + \text{“} \Sigma_1\text{” statements reflect into the Suslin co-Suslin sets} \text{”} (\text{see } [26]) \text{ for the precise statement).}$

For $A \subseteq \mathbb{R}$, we let $\theta_A$ be the supremum of all $\alpha$ such that there is an $\text{OD}(A)$ surjection from $\mathbb{R}$ onto $\alpha$. If $\Gamma$ is a determined pointclass and $A \in \Gamma$, we write $\Gamma \upharpoonright A$ for the set of all $B \in \Gamma$ that are Wadge reducible to $A$. If $\alpha < \Theta^{\Gamma}$, we write $\Gamma \upharpoonright \alpha$ for the set of all $A \in \Gamma$ with Wadge rank strictly less than $\alpha$.

**Definition 2.2 ($\text{AD}^+$).** The **Solovay sequence** is the sequence $\langle \theta_\alpha \mid \alpha \leq \lambda \rangle$ where

1. $\theta_0$ is the supremum of ordinals $\beta$ such that there is an $\text{OD}$ surjection from $\mathbb{R}$ onto $\beta$;

2. if $\alpha > 0$ is limit, then $\theta_\alpha = \sup \{ \theta_\beta \mid \beta < \alpha \}$;

3. if $\alpha = \beta + 1$ and $\theta_\beta < \Theta$ (i.e. $\beta < \lambda$), fixing a set $A \subseteq \mathbb{R}$ of Wadge rank $\theta_\beta$, $\theta_\alpha$ is the sup of ordinals $\gamma$ such that there is an $\text{OD}(A)$ surjection from $\mathbb{R}$ onto $\gamma$, i.e. $\theta_\alpha = \theta_A$.

Note that the definition of $\theta_\alpha$ for $\alpha = \beta + 1$ in Definition 2.2 does not depend on the choice of $A$. One can also make sense of the Solovay sequence of pointclasses that may not be constructibly closed. Such pointclasses show up in core model induction applications. The Solovay sequence $\langle \theta_\alpha : \alpha < \gamma \rangle$ of a pointclass $\Omega$ with the property that if $A \in \Omega$, then $L(A, \mathbb{R}) \models \text{AD}^+$ and $\varphi(\mathbb{R}) \cap L(A, \mathbb{R}) \subseteq \Omega$ is defined as follows. First, $\theta_0$ is the supremum of all $\alpha$ such that there is some $A \in \Omega$ and some $\text{OD}^{L(A, \mathbb{R})}$ surjection $\pi : \mathbb{R} \to \alpha$. If $\lambda < \gamma$ is limit, then $\theta_\gamma = \sup_{\alpha < \lambda} \theta_\alpha$. If $\theta_\alpha$ has been defined and $\alpha + 1 < \gamma$, then letting $A \in \mathbb{R}$ be of Wadge rank $\theta_\alpha$, $\theta_{\alpha + 1}$ is the supremum of $\beta$ such that there is some $B \in \Omega$ and some $\text{OD}(A)^{L(B, \mathbb{R})}$ surjection $\pi : \mathbb{R} \to \beta$.

Roughly speaking, the longer the Solovay sequence is, the stronger the associated $\text{AD}^+$-theory is. The minimal model of $\text{AD}^+$ is $L(\mathbb{R})$, which satisfies $\Theta = \theta_0$. The theory $\text{AD}^+ + \text{AD}_\mathbb{R}$ implies that the Solovay sequence has limit length. The theory $\text{AD}_\mathbb{R} + \text{DC}$ is strictly stronger than $\text{AD}_\mathbb{R}$ since by [20], $\text{DC}$ implies $\text{cof}(\Theta) > \omega$ whereas the minimal model\footnote{From here on, whenever we talk about “models of $\text{AD}^+$”, we always mean transitive models of $\text{AD}^+$ that contain all reals and ordinals.} of $\text{AD}_\mathbb{R}$ satisfies $\Theta = \theta_\omega$. The theory “$\text{AD}_\mathbb{R} + \Theta$ is regular” is much stronger still, as it implies the existence of many models of $\text{AD}_\mathbb{R} + \text{DC}$. We end this section with a theorem of Woodin, which produces models with Woodin cardinals from $\text{AD}^+$. The theorem is important in the HOD analysis of such models.
Theorem 2.3 (Woodin, see [7]). Assume AD⁺. Let \( \langle \theta_\alpha \mid \alpha \leq \Omega \rangle \) be the Solovay sequence. Suppose \( \alpha = 0 \) or \( \alpha = \beta + 1 \) for some \( \beta < \Omega \). Then HOD \( \models \theta_\alpha \) is Woodin.

2.2. Basic properties of ideals

We summarize standard facts about ideals that we will need in this paper. See for example [34] and [3] for a more detailed discussion.

Suppose \( I \) is an ideal on a set \( X \). We say that \( I \) is countably complete if whenever \( \{ A_n : n < \omega \} \) are sets in \( I \) then \( \bigcup_{n<\omega} A_n \in I \). Supposing \( X \) is a cardinal (e.g. \( X = \omega_1 \)), we say \( I \) is normal if whenever \( \{ A_x : x \in X \} \subset I \) then the diagonal union \( \{ x \in X : \exists y \in x (x \in A_y) \} \in I \). All ideals \( I \) on a cardinal considered in this paper will be assumed countably complete and normal.

Suppose \( I \) is an \( \omega_1 \)-dense ideal on \( \omega_1 \). The following are standard facts; see [34, Definition 6.19] and the discussion after it.

Fact 2.4. (i) \( \mathbb{P}_I \) is a homogeneous forcing.\(^\dagger\)

(ii) There is a boolean isomorphism \( \pi : \mathbb{P}_I \to RO(\text{Coll}(\omega, \omega_1)) \). In particular, \( \mathbb{P}_I \) is forcing equivalent to \( \text{Coll}(\omega, \omega_1) \).

(iii) For any \( V \)-generic filter \( G \subset \text{Coll}(\omega, \omega_1) \), \( \pi \) induces a \( V \)-generic filter \( H \subset \mathbb{P}_I \), and letting \( j : V \to M =_{\text{def}} \text{Ult}(V, H) \subset V[H] \) be the associated generic ultrapower map, we have:

(a) \( j(f)(\omega_1^V) = G \) for some \( f : \omega_1 \to H_{\omega_1} \); in particular, \( V[H] = V[G] \).

(b) \( j(\omega_1^V) = \omega_2^V \).

(c) \( M \) is well-founded and \( M^\omega \subset M \) in \( V[H] \).

Suppose \( X = \wp_{\omega_1}(Y) \), where \( \wp_{\omega_1}(Y) \) is the collection of all countable subsets of \( Y \), for some set \( Y \) (e.g. \( Y = \mathbb{R} \)). We say \( I \) is fine if for any \( y \in Y \), the set \( \{ \sigma \in \wp_{\omega_1}(Y) : y \notin \sigma \} \in I \). We say \( I \) is normal if whenever \( \{ A_y : y \in Y \} \subset I \), the diagonal union \( \{ \sigma \in \wp_{\omega_1}(Y) : \exists y \in \sigma (\sigma \in A_y) \} \in I \). \( I \) is \(|Y|\)-dense if there is a dense subset of \( \mathbb{P}_I \) of size \( |Y| \). All ideals on sets of the form \( \wp_{\omega_1}(Y) \) considered in this paper will be assumed countably complete, normal, and fine.

Lemma 2.5. Suppose \( I \) is a pseudo-homogeneous ideal on \( \wp_{\omega_1}(\mathbb{R}) \). Let \( G \subset \mathbb{P}_I \) be \( V \)-generic and let \( j_G : V \to \text{Ult}(V, G) \) be the associated generic embedding. Then:

(a) For any ordinal \( \alpha \), \( j_G \upharpoonright \alpha \) does not depend on \( G \); in particular, \( j_G \upharpoonright \alpha \in V \).

(b) If \( \lambda < c^+ \) and \( A \subseteq \lambda^\omega \), then \( j_G[A] \) does not depend on \( G \) and \( j_G[A] \in V \).

(c) If \( A \) is a set of ordinals that is definable in \( V \) from a countable sequence of ordinals, then \( j_G(A) \) does not depend on \( G \) and \( j_G(A) \in V \).

\(^\dagger\)A forcing \( \mathbb{P} \) is homogeneous if whenever \( p, q \in \mathbb{P} \), there is an automorphism \( \sigma : \mathbb{P} \to \mathbb{P} \) such that \( \sigma(p) \) is compatible with \( q \).

\(^\dagger\dagger\)\( RO(\text{Coll}(\omega, \omega_1)) \) is the regular open algebra of \( \text{Coll}(\omega, \omega_1) \).
Proof. We give the proof for (a). The other items are similar. Let \( \theta(u,v,w) \) be the formula “\( u = v(0) \)”. Let \( \alpha \) be an ordinal. Let \( s : \omega \rightarrow \text{Ord} \) be the constant function \( s(n) = \alpha \) for all \( n \in \omega \). For each ordinal \( \beta \) the truth of the statement \( \text{Ult}(V,G) \models \theta[\beta,j_G(s),\emptyset] \) is independent of \( G \) by pseudo-homogeneity, so the value of \( j_G(\alpha) \) is independent of \( G \).

2.3. Ideals from determinacy

We assume \( \text{AD}_R + \text{“\( \Theta \) is regular”} \) and \( V = L(\wp(R)) \). Let \( P \) be a poset with the following properties:

• \( P \) is coded by a set of reals.
• \( P \) is \( \sigma \)-closed.
• \( P \) is homogeneous.
• \( 1 \Vdash_P \mathbb{R} \) is wellorderable.
• \( 1 \Vdash_P \text{c-DC}, \) dependent choices for \( \epsilon \)-sequences.

Examples of such \( P \) are \( \text{Coll}(\omega_1,\mathbb{R}) \) and \( P_{\text{max}} \).

Let \( G \subseteq P \) be \( V \)-generic and let \( H \subset \text{Coll}(\Theta,\wp(\mathbb{R})) \). Note that by the properties of \( P \) and the assumption \( V = L(\wp(\mathbb{R})) \), in \( V[G][H] \), ZFC holds and \( \Theta = \mathfrak{c}^+ \).

Definition 2.6. In \( V[G][H] \) an ideal \( I \) on \( \wp(\omega_1,\mathbb{R}) \) is said to have the ordinal covering property with respect to \( V \) if for every function \( F : \wp(\omega_1,\mathbb{R}) \rightarrow \text{Ord} \) and every \( I \)-positive set \( S \), there is some \( I \)-positive set \( S_0 \subseteq S \) and some \( F_0 : \wp(\omega_1,\mathbb{R}) \rightarrow \text{Ord} \) in \( V \) such that \( F \upharpoonright S_0 = F_0 \upharpoonright S_0 \).

We will show that in \( V[G][H] \), there is an ideal \( I \) with the ordinal covering property with respect to \( V \). Let \( \mu \) be the Solovay measure on \( \wp(\omega_1,\mathbb{R}) \), so \( A \in \mu \) if and only if \( A \) contains a club set in \( \wp(\omega_1,\mathbb{R}) \). A set \( A \) is club if and only if there is a function \( F : \mathbb{R}^{<\omega} \rightarrow \mathbb{R} \) such that

\[
\sigma \in A \iff F[\sigma] \subseteq \sigma.
\]

We say that \( A \) is the club set generated by \( F \).

The measure \( \mu \) induces an ultrapower map on the ordinals, \( j_\mu : \text{Ord} \rightarrow \text{Ord} \). By the basic theory of \( \text{AD}^+ \),

\[
j_\mu(\omega_1) = \Theta.
\] \hfill (2.1)

See, for example, [32, Section 1.2] for a proof of this fact.

Lemma 2.7. Suppose \( V,G,H \) are as above. Suppose \( I \) is an ideal on \( \wp(\omega_1,\mathbb{R}) \) with the ordinal covering property with respect to \( V \). Let \( K \subset P_I \) be a \( V[G][H] \)-generic filter. Then:

(a) The generic embedding \( j_K \upharpoonright \text{Ord} = j_\mu \upharpoonright \text{Ord} \). In particular, \( j_K \upharpoonright \alpha \in V[G][H] \) for every ordinal \( \alpha \) and doesn’t depend on the choice of \( K \).

(b) \( I \) is strong.
Proof. For (a), for any \( F : \wp_{\omega_1}(\mathbb{R}) \to \text{Ord} \) in \( V[G][H] \), the covering property gives some \( S \in K \) and \( F_0 \in V \) such that \( F \upharpoonright S = F_0 \upharpoonright S \). Also, \( K \cap V = \mu \) since \( K \) is normal; this gives
\[
\{ F : \wp_{\omega_1}(\mathbb{R}) \to \text{Ord} \}^{V[G][H]/K} = \{ F : \wp_{\omega_1}(\mathbb{R}) \to \text{Ord} \}^{V/\mu}
\]
and \( j_K \upharpoonright \text{Ord} = j_\mu \upharpoonright \text{Ord} \). Part (b) follows from (a) and (2.1).

Lemma 2.8. In \( V[G][H] \), if \( I \) has the ordinal covering property relative to \( V \), then \( I \) is pseudo-homogeneous.

Proof. Let \( K \subset \mathbb{P}_I \) be a \( V[G][H] \)-generic filter. Let \( \alpha \in \text{Ord}, s \in \text{Ord}^\omega, \lambda < c^+ \), and let \( \theta \) be a formula in the language of set theory. It suffices to show that the statement \( \text{Ult}(V[G][H], K) \vDash \theta[\alpha, j_K(s), j_K[\lambda^\omega]] \) is independent of \( K \). By the ordinal covering property, we can find \( F_0 \in V \) that represents \( \alpha \) in both \( \text{Ult}(V, \mu) \) and \( \text{Ult}(V[G][H], K) \). In both ultrapowers, \( j(s) \) is represented by the constant function \( F_1(\sigma) = s \) for all \( \sigma \in \wp_{\omega_1}(\mathbb{R}) \). Fix a surjection \( \pi : \mathbb{R} \to \lambda^\omega \) in \( V \). Then \( j_K[\lambda^\omega] \) is represented by the function \( F_2 \in V \) given by \( F_2(\sigma) = \pi[\sigma] \). So we have \( \text{Ult}(V[G][H], K) \vDash \theta[\alpha, j_K(s), j_K[\lambda^\omega]] \) if and only if the set
\[
S = \{ \sigma : V[G][H] \vDash \theta[F_0(\sigma), F_1(\sigma), F_2(\sigma)] \}
\]
is in \( K \). By homogeneity of \( \mathbb{P} \), \( S \in V \). But then we have \( S \in K \) if and only if \( S \in \mu \), as desired.

Theorem 2.9. In \( V[G][H] \), the nonstationary ideal \( I = NS_{\omega_1,\mathbb{R}} \) on \( \wp_{\omega_1}(\mathbb{R}) \) has the ordinal covering property with respect to \( V \).

To establish the covering property of \( I \) in \( V[G][H] \), or equivalently in \( V[G] \), we will need the following lemma.

Lemma 2.10. Let \( \dot{S} \) be a \( \mathbb{P} \)-name for a subset of \( \wp_{\omega_1}(\mathbb{R}) \). The following statements are equivalent for any given \( p \in \mathbb{P} \):

(a) \( p \Vdash \text{“}\dot{S} \text{ contains a club.”} \)

(b) For a club of \( \sigma \in \wp_{\omega_1}(\mathbb{R}) \),
\[
(\dagger) \quad \forall^* g \in \mathbb{P} \upharpoonright \sigma \text{ containing } p \forall q \leq g \ q \Vdash \sigma \in \dot{S}.
\]

Here \( \forall^* g \) stands for “for a comeager set of filters \( g \)”\(^{13}\) and \( q \leq g \) means \( \forall r \in g \ q \vDash r \).

Proof. Fix \( p \in \mathbb{P} \). Assume (a) holds for \( p \). Let \( \dot{f} \) be a \( \mathbb{P} \)-name for a function from \( \mathbb{R}^{<\omega} \) into \( \mathbb{R} \) such that \( p \) forces \( \dot{S} \) to contain the club set generated by \( \dot{f} \). We may assume \( \mathbb{P} \subseteq \mathbb{R} \). To see (b), note that there is a club set of \( \sigma \) such that for all \( t \in \sigma^{<\omega} \), the set
\[
D_t = \{ q \in \mathbb{P} \cap \sigma : (\exists x \in \sigma) \ (q \vDash \dot{f}(t) = x) \}.
\]

\(^{13}\)By \( \mathbb{P} \upharpoonright \sigma \), we mean the set of conditions in \( \mathbb{P} \) coded by a real in \( \sigma \). Note that \( \mathbb{P} \upharpoonright \sigma \) is countable, so the category quantifier over the set of all filters on it makes sense.
is dense below \( p \) in \( \mathbb{P} \cap \sigma \). This easily gives (\( \dagger \)) for \( \sigma \) as there are countably many dense sets \( D_t \) and hence there is a comeager set of filters \( g \subset \mathbb{P} \cap \sigma \) meeting all the \( D_t \)'s.

Assume (b) holds for \( p \). Let

\[
A = \{(q, x) : x \text{ codes } \sigma \in \varphi_{\omega_1}(R) \text{ and } q \vDash \sigma \in \hat{S}\}.
\]

Take \( N = L_\alpha(P_\beta(R)) \) satisfying \( \text{ZF}^- + \text{AD}_R + \"\Theta \text{ is regular}" \), containing \( A \), and admitting a surjection \( F : R \to N \).\(^{14}\) Let \( B \subset R \) code the first order theory of the structure \((V_{\omega+1}, \in, A)\). Because \( \text{AD}_R \) implies that every set of reals is \( R \)-universally Baire (see e.g. [32, Section 1.2]), in particular \( A \) and \( B \) are \( R \)-universally Baire. There is then a club \( C \) of \( \sigma \in \varphi_{\omega_1}(N) \) having the following properties:

- (\( \dagger \)) holds for \( \sigma \cap R \).
- \( \sigma < N \).
- Defining \( \pi_\sigma : \sigma \to N_\sigma \) as the transitive collapse of \( \sigma \), we have

\[
(V_{\omega+1} \cap N_\sigma[h], \in, A \cap N_\sigma[h]) < (V_{\omega+1}, \in, A)
\]

for any \( N_\sigma \)-generic filter \( h \subset \text{Coll}(\omega, \sigma \cap R) \).

The last item follows from the \( R \)-universal Baireness of \( B \).

All \( \sigma \in C \) have the following property:

\[
N_\sigma \vDash p \vDash^g_{\mathbb{P}|(R \cap \sigma)} (1 \vDash^h_{\text{Coll}(\omega, R \cap \sigma)} (\forall q \leq g)((q, \sigma_h) \in \pi_\sigma(A)_{g \times h})). \tag{2.2}
\]

In (2.2), \( \sigma_h \) denotes the real generally coding \( \sigma \cap R \) relative to \( h \) and \( \pi_\sigma(A)_{g \times h} \) denotes the unique extension of \( \pi_\sigma(A) \) to a set of reals in \( N_\sigma[g][h] \), which can be construed as a generic extension of \( N_\sigma \) by \( \text{Coll}(\omega, \sigma \cap R) \); the extension is given by the universal Baireness of \( A \).

Now suppose \( G \subset \mathbb{P} \) is \( V \)-generic and \( p \in G \). There is a club set \( D \) of \( \sigma \in C \) such that \( \sigma[G] < N[G] \) and \( \sigma[G] \cap V = \sigma \). Take a \( \sigma \) in this club and \( g = G \cap \sigma \). Note that any lower bound \( q \leq g \) forces \( \sigma \in \hat{S} \) by (2.2) and there is \( q \leq g \) in \( G \); so \( \sigma \cap R \in \hat{S}_G \). Therefore, the club set \( \{\sigma \cap R : \sigma \in D\} \) witnesses (a).

**Proof of Theorem 2.9.** Suppose \( p_0 \) forces “\( \hat{F} : \hat{S} \to \text{Ord} \) and \( \hat{S} \subset \varphi_{\omega_1}(R) \) is stationary.” Using (\( \dagger \)), the latter part of this statement is equivalent to the following statement. For stationary many (equivalently by \( \text{AD}_R \), for club many) countable \( \sigma \subset R \),

\[
\exists g \subset \mathbb{P} \mid \sigma \text{ containing } p_0 \exists q \leq g \ q \vDash \sigma \in \hat{S}.
\]

Under \( \text{AD} \), a well-ordered union of meager sets is meager, so let \( F_0(\sigma) \) be the least \( \alpha \) such that

\[
\exists g \subset \mathbb{P} \mid \sigma \text{ containing } p_0 \exists q \leq g \ q \vDash \hat{F}(\sigma) = \alpha.
\]

\(^{14}\)Here \( P_\beta(R) \) is the set \( \{B \subset R : B \text{ has Wadge rank less than } \beta\} \).
By the above, \( p_0 \) forces that the set of \( \sigma \in \dot{S} \) such that \( F(\sigma) = F_0(\sigma) \) is stationary. \( \square \)

Theorem 2.9 and Lemmas 2.7 and 2.8 immediately give one direction of Theorem 1.10.

**Corollary 2.11.** Con\((\text{ZF } + \text{AD}_\mathbb{R} \ + \ \text{“\( \Theta \) is regular”}) \) implies Con\((\text{ZFC } + \ \text{“the nonstationary ideal on \( \wp_{\omega_1}(\mathbb{R}) \) is strong and pseudo-homogeneous”}) \).

Now we proceed to prove one direction of Theorem 1.6. We show Con\((\text{AD}_\mathbb{R} \ + \ \text{“\( \Theta \) is regular”}) \) implies Con\((\text{DI}) \). We fix objects \( V, \mathbb{P}, G, H \) as before. The following is the main theorem.

**Theorem 2.12.** In \( V[G][H] \), there is a \( \mathfrak{c} \)-dense ideal on \( \wp_{\omega_1}(\mathbb{R}) \) with the ordinal covering property relative to \( V \).

We review some facts regarding generic ultrapowers by \( \text{Coll}(\omega, \mathbb{R}) \)-generics. See [32] for a more detailed discussion. Let \( h \subset \text{Coll}(\omega, \mathbb{R}) \) be \( V \)-generic and

\[
U_h = \{ A \subseteq \mathbb{R}^\omega : A \text{ is weakly comeager below some } p \in h \}.
\]

Here \( A \subseteq \mathbb{R}^\omega \) is weakly comeager below a condition \( p \in \text{Coll}(\omega, \mathbb{R}) \) if for a club set of \( \sigma \in \wp_{\omega_1}(\mathbb{R}) \), \( A \cap \sigma^\omega \) is comeager below \( p \) in \( \sigma^\omega \).

15We equip \( \sigma^\omega \) with the product of the discrete topologies on \( \sigma \), so it is homeomorphic to the Baire space.

16We need this property for the following argument because this is the model in which \( j_h(\mathbb{P}) \) is countably closed.
By the assumptions on \( \mathbb{P}, \mathbb{P} \times \text{Coll}(\omega, \mathbb{R}) \) is forcing equivalent to \( \text{Coll}(\omega, \mathbb{R}) \); therefore, we can find an \( h \) satisfying the hypothesis of Claim 2.13. By Claim 2.13, forcing with \( \text{Coll}(\omega, \mathbb{R}) \) adds an \( \text{Ult}(V, U_h) \)-generic filter \( G' \subset j_h(\mathbb{P}) \) extending \( j^*G \). We can then extend \( j_h \) to an elementary embedding

\[
j_h^*: V[G] \to \text{Ult}(V, U_h)[G']
\]

by defining \( j_h^*(\tau_G) = j_h(\tau_{G'}) \).

Now in \( V[G][H] \), define an ideal \( \mathcal{I} \) on \( \wp_\omega(\mathbb{R}) \) by

\[
S \in \mathcal{I} \iff \emptyset \models_{\text{Coll}(\omega, \mathbb{R})} \vec{R} \notin j_h^*(\vec{S}).
\]

So \( \mathbb{P}_\mathcal{I} \) is isomorphic to the subalgebra \( \mathcal{B} = \{||\vec{R} \in j_h^*(\vec{S})|| : S \subseteq \wp_\omega(\mathbb{R})\} \) of the regular-open algebra \( \text{RO}(\text{Coll}(\omega, \mathbb{R})) \).

\( \mathcal{I} \) is fine: for any \( x \in \mathbb{R} \), the set \( T_x = \{\sigma : x \notin \sigma\} \in \mathcal{I} \) because clearly \( \emptyset \models_{\text{Coll}(\omega, \mathbb{R})} \vec{R} \notin j_h^*(\vec{T}_x) \).

\( \mathcal{I} \) is normal: suppose \( (S_x : x \in \mathbb{R}) \) is a family of subsets of \( \wp_\omega(\mathbb{R}) \) and \( S \) is the diagonal union, i.e. \( \sigma \in S \) if and only if there is some \( x \in \sigma \) such that \( x \in S_x \). Then

\[
||\vec{R} \in j_h^*(S)|| = ||\exists x \in \mathbb{R} (\vec{R} \in j_h^*(S_x))|| = \sup_x ||\vec{R} \in j_h^*(S_x)||.
\]

This verifies normality of \( \mathcal{I} \) and also verifies \( \mathcal{B} \) is a \( c \)-complete subalgebra of \( \text{RO}(\text{Coll}(\omega, \mathbb{R})) \). Since in \( V[G][H] \), \( \text{RO}(\text{Coll}(\omega, \mathbb{R})) \) has size \( c^+ \), has the \( c^+ \)-chain condition, and is \( c \)-dense, \( \mathcal{B} \) is \( c \)-dense and is a complete subalgebra of \( \text{RO}(\text{Coll}(\omega, \mathbb{R})) \).

We now show \( \mathcal{I} \) has the covering property relative to \( V \). In \( V[G][H] \), suppose \( F : S \to \text{Ord} \) where \( S \in \mathcal{I}^+ \). Note that \( F \in V[G] \). Let \( p \in \text{Coll}(\omega, \mathbb{R}) \) force “\( \vec{R} \in j_h^*(S) \)” and \( q \leq p \) force “\( j^*_h(F)(\vec{R}) = \alpha \)” for some ordinal \( \alpha \). In \( V \), let \( F_0 : \wp_\omega(\mathbb{R}) \to \text{Ord} \) such that \( [F_0]_\mu = \alpha \). By the discussion above, before the proof of the theorem,

\[
\emptyset \models_{\text{Coll}(\omega, \mathbb{R})} [F_0]_\mu = j_h(F_0)(\vec{R}) = j_h^*(F_0)(\vec{R}).
\]

Therefore,

\[
q \models_{\text{Coll}(\omega, \mathbb{R})} j_h^*(F_0)(\vec{R}) = j_h^*(F)(\vec{R}).
\]

This means the set \( \{\sigma \in S : F(\sigma) = F_0(\sigma)\} \) is \( \mathcal{I} \)-positive. \( \square \)

Now, let \( \mathbb{P} \) be such that \( \mathcal{CH} \) holds in \( V[G][H] \). For example, we can take \( \mathbb{P} = \text{Coll}(\omega_1, \mathbb{R}) \). So in \( V[G][H] \), \( c = \omega_1 \) and \( \Theta^V = \omega_2 \). By Theorem 2.12, in \( V[G][H] \), there is an \( \omega_1 \)-dense ideal \( \mathcal{I} \) on \( \wp_\omega(\mathbb{R}) \) that has the covering property with respect to \( V \). Since \( |\wp_\omega(\mathbb{R})| = \omega_1 \) in \( V[G][H] \), we easily obtain an \( \omega_1 \)-dense ideal on \( \omega_1 \) with the ordinal covering property. This and Lemma 2.7 give us one direction of Theorem 1.6.

**Corollary 2.14.** \( \text{Con}(\mathbf{ZF} + \mathbf{AD}_\mathbb{R} + \text{"\( \Theta \) is regular") implies } \text{Con}(\mathbf{ZFC} + \mathbf{DI}). \)
3. PRELIMINARIES ON THE CORE MODEL INDUCTION

This section, consisting of several subsections, develops some terminology and framework for the core model induction. The first subsection gives a brief summary of the theory of $\mathcal{F}$-premice and strategy premice developed in [18]. For a full development of these concepts, the reader should consult [18]. These concepts and notations will be used in the next subsection, which defines core model induction operators, which are the operators that we construct during the course of the core model induction in this paper. The last two sections briefly summarize the theory of hod mice and the HOD analysis in $\text{AD}^+$ models (see [9] for a more detailed discussions of these topics). The reader who wishes to see the main argument can skip them on the first read, and go back when needed.

3.1. $\mathcal{F}$-premice and strategy premice

**Definition 3.1.** Let $\mathcal{L}_0$ be the language of set theory expanded by unary predicate symbols $\dot{E}, \dot{B}, \dot{S}$, and constant symbols $\dot{a}, \dot{P}$. Let $\mathcal{L}_0^- = \mathcal{L}_0 \setminus \{\dot{E}, \dot{B}\}$.

Let $a$ be transitive. Let $\varrho : a \to \text{rank}(a)$ be the rank function. We write $\hat{a} = \text{trancl}((\{a, \varrho\})$. Let $\mathfrak{P} \in J_1(\hat{a})$.

A $\mathcal{J}$-structure over $a$ (with parameter $\mathfrak{P}$) (for $\mathcal{L}_0$) is a structure $\mathcal{M}$ for $\mathcal{L}_0$ such that $a^\mathcal{M} = a$, $(\mathfrak{P}^\mathcal{M} = \mathfrak{P})$, and there is $\lambda \in [1, \text{Ord})$ such that $|\mathcal{M}| = J_\lambda^{S^\mathcal{M}}(\hat{a})$.

Here we also let $l(\mathcal{M})$ denote $\lambda$, the length of $\mathcal{M}$, and let $\hat{a}^\mathcal{M}$ denote $\hat{a}$.

For $\alpha \in [1, \lambda]$ let $\mathcal{M}_\alpha = J_{\alpha}^{S^\mathcal{M}}(\hat{a})$. We say that $\mathcal{M}$ is acceptable iff for each $\alpha < \lambda$ and $\tau < o(\mathcal{M}_\alpha)$, if

$$\mathcal{P}(\tau^{<\omega} \times \hat{a}^{<\omega}) \cap \mathcal{M}_\alpha \neq \mathcal{P}(\tau^{<\omega} \times \hat{a}^{<\omega}) \cap \mathcal{M}_{\alpha + 1},$$

then there is a surjection $\tau^{<\omega} \times \hat{a}^{<\omega} \to \mathcal{M}_\alpha$ in $\mathcal{M}_{\alpha + 1}$.

A $\mathcal{J}$-structure (for $\mathcal{L}_0$) is a $\mathcal{J}$-structure over $a$, for some $a$.

As all $\mathcal{J}$-structures we consider will be for $\mathcal{L}_0$, we will omit the phrase “for $\mathcal{L}_0$”. We also often omit the phrase “with parameter $\mathfrak{P}$”. Note that if $\mathcal{M}$ is a $\mathcal{J}$-structure over $a$ then $|\mathcal{M}|$ is transitive and rud-closed, $\hat{a} \in M$, and $o \cap M = \text{rank}(M)$. This last point is because we construct from $\hat{a}$ instead of $a$.

$\mathcal{F}$-premice will be $\mathcal{J}$-structures of the following form.

**Definition 3.2.** A $\mathcal{J}$-model over $a$ (with parameter $\mathfrak{P}$) is an acceptable $\mathcal{J}$-structure over $a$ (with parameter $\mathfrak{P}$), of the form

$$\mathcal{M} = (M; E, B, S, a, \mathfrak{P})$$

where $\dot{E}^\mathcal{M} = E$, etc., and letting $\lambda = l(\mathcal{M})$, the following hold.

1. $\mathcal{M}$ is amenable.

2. $S = \langle S_\xi \mid \xi \in [1, \lambda) \rangle$ is a sequence of $\mathcal{J}$-models over $a$ (with parameter $\mathfrak{P}$).
3. For each $\xi \in [1, \lambda)$, $\hat{S}^S_\xi = S \upharpoonright \xi$ and $M_\xi = |S_\xi|$.

4. Suppose $E \neq \emptyset$. Then $B = \emptyset$ and there is an extender $F$ over $M$ which is $(\hat{a} \times \gamma)$-complete for all $\gamma < \text{crt}(F)$ and such that the premouse axioms [32, Definition 2.2.1] hold for $(M, F)$, and $E$ codes $\hat{F} \cup \{G\}$ where: (i) $\hat{F} \subseteq M$ is the amenable code for $F$ (as in [27]); and (ii) if $F$ is not type 2 then $G = \emptyset$, and otherwise $G$ is the “longest” non-type $Z$ proper segment of $F$ in $M$.\footnote{We use $G$ explicitly, instead of the code $\gamma^M$ used for $G$ in [8, Section 2], because $G$ does not depend on which (if there is any) wellorder of $M$ we use. This ensures that certain pure mouse operators are forgetful.}

Our notion of a “$J$-model over $a$” is a bit different from the notion of “model with parameter $a$” in [16] or [32, Definition 2.1.1] in that we build into our notion some fine structure and we do not have the predicate $l$ used in [32, Definition 2.1.1]. Note that with notation as above, if $\lambda$ is a successor ordinal then $M = J(S^{M}_{\lambda-1})$, and otherwise, $M = \bigcup_{\alpha < \lambda} |S_\alpha|$. The predicate $B$ will be used to code extra information such as a (partial) branch of a tree in $M$.

**Definition 3.3.** Let $\mathcal{M}$ be a $J$-model over $a$ (with parameter $\mathcal{P}$). Let $E^M$ denote $E^M$, etc. Let $\lambda = l(\mathcal{M})$, $S_0^M = a$, $S^M_\lambda = M$, and $\mathcal{M}|\xi = S^M_\xi$ for all $\xi \leq \lambda$. An (initial) segment of $\mathcal{M}$ is just a structure of the form $\mathcal{M}|\xi$ for some $\xi \in [1, \lambda]$. We write $\mathcal{P} \preceq \mathcal{M}$ iff $\mathcal{P}$ is a segment of $\mathcal{M}$, and $\mathcal{P} \prec \mathcal{M}$ iff $\mathcal{P} \preceq \mathcal{M}$ and $\mathcal{P} \neq \mathcal{M}$. Let $\mathcal{M}|\xi$ be the structure having the same universe and predicates as $\mathcal{M}|\xi$, except that $E^M|^\xi = \emptyset$. We say that $\mathcal{M}$ is $E$-active iff $E^M \neq \emptyset$, and $B$-active iff $B^M \neq \emptyset$. Active means either $E$-active or $B$-active; $E$-passive means not $E$-active; $B$-passive means not $B$-active; and passive means not active.

Given a $J$-model $\mathcal{M}_1$ over $b$ and a $J$-model $\mathcal{M}_2$ over $\mathcal{M}_1$, we write $\mathcal{M}_2 \downarrow b$ for the $J$-model $\mathcal{M}$ over $b$, such that $\mathcal{M}$ is “$\mathcal{M}_1 \sim \mathcal{M}_2$”. That is, $|\mathcal{M}| = |\mathcal{M}_2|$, $a^\mathcal{M} = b$, $E^\mathcal{M} = E^{M_2}$, $B^\mathcal{M} = B^{M_2}$, and $\mathcal{P} \prec \mathcal{M}$ iff $\mathcal{P} \preceq \mathcal{M}_1$ or there is $Q \prec \mathcal{M}_2$ such that $\mathcal{P} = Q \downarrow b$, when such an $M$ exists. Existence depends on whether the $J$-structure $\mathcal{M}$ is acceptable.

In the following, the variable $i$ should be interpreted as follows. When $i = 0$, we ignore history, and so $\mathcal{P}$ is treated as a coarse object when determining $\mathcal{F}(0, \mathcal{P})$. When $i = 1$ we respect the history (given it exists).

**Definition 3.4.** An operator $\mathcal{F}$ with domain $D$ is a function with domain $D$, such that for some cone $C = C_{\mathcal{F}}$, possibly self-wellordered (sword),\footnote{$C$ is a cone if there are a cardinal $\kappa$ and a transitive set $a \in H_{\kappa}$ such that $C$ is the set of $b \in H_{\kappa}$ such that $a \in L_1(b)$; $a$ is called the base of the cone. A set $a$ is self-wellordered if there is a well-ordering of $a$ in $L_1(a)$. A set $C$ is a self-wellordered cone if $C$ is the restriction of a cone $C'$ to its own self-wellordered elements.} $D$ is the set of pairs $(i, X)$ such that either:

- $i = 0$ and $X \subseteq C$, or

- $i = 1$ and $X$ is a $J$-model over $X_1 \subseteq C$,

and for each $(i, X) \in D$, $\mathcal{F}(i, X)$ is a $J$-model over $X$ such that for each $\mathcal{P} \preceq \mathcal{F}(i, X)$, $\mathcal{P}$ is fully sound. (Note that $\mathcal{P}$ is a $J$-model over $X$, so soundness is in this sense.)

\[\]
Let $F, D$ be as above. We say $F$ is **forgetful** iff $F(0, X) = F(1, X)$ whenever $(0, X), (1, X) \in D$, and whenever $X$ is a $J$-model over $X_1$, and $X_1$ is a $J$-model over $X_2 \in C$, we have $F(1, X) = F(1, X \downarrow X_2)$. Otherwise we say $F$ is **historical**. Even when $F$ is historical, we often just write $F(X)$ instead of $F(i, X)$ when the nature of $F$ is clear from the context. We say $F$ is **basic** iff for all $(i, X) \in D$ and $\mathcal{P} \supseteq F(i, X)$, we have $E_\mathcal{P} = \emptyset$. We say $F$ is **projecting** iff for all $(i, X) \in D$, we have $\rho_{\mathcal{F}(i, X)} = X$.

Here are some illustrations. Strategy operators (to be explained in more detail later) are basic, and as usually defined, projecting and historical. Suppose we have an iteration strategy $\Sigma$ and we want to build a $J$-model $\mathcal{N}$ (over some $\alpha$) that codes a fragment of $\Sigma$ via its predicate $\dot{B}$. We feed $\Sigma$ into $\mathcal{N}$ by always providing $b = \Sigma(T)$, for the $\prec\mathcal{N}$-least tree $T$ for which this information is required. So given a reasonably closed level $\mathcal{P} \ll \mathcal{N}$, the choice of which tree $T$ should be processed next will usually depend on the information regarding $\Sigma$ already encoded in $\mathcal{P}$ (its history). Using an operator $F$ to build $\mathcal{N}$, then $F(i, \mathcal{P})$ will be a structure extending $\mathcal{P}$ and over which $b = \Sigma(T)$ is encoded. The variable $i$ should be interpreted as follows. When $i = 1$, we respect the history of $\mathcal{P}$ when selecting $T$. When $i = 0$, we ignore history when selecting $T$. The operator $F(X) = X^\#$ is forgetful and projecting, and not basic; here $F(X) = F(0, X)$.

**Definition 3.5.** For any $P$ and any ordinal $\alpha \geq 1$, the operator $\mathcal{J}_\alpha^m(\cdot; P)$ is defined as follows.\(^{19}\)

For $X$ such that $P \in \mathcal{J}_1(\dot{X})$, let $\mathcal{J}_\alpha^m(X; P)$ be the $J$-model $\mathcal{M}$ over $X$, with parameter $P$, such that $|\mathcal{M}| = \mathcal{J}_\alpha(\dot{X})$ and for each $\beta \in [1, \alpha]$, $\mathcal{M}|_\beta$ is passive. Clearly $\mathcal{J}_\alpha^m(\cdot; P)$ is basic and forgetful. If $P = \emptyset$ or we wish to suppress $P$, we just write $\mathcal{J}_\alpha^m(\cdot)$.

**Definition 3.6** (Potential $F$-premouse, $\mathcal{C}_F$). Let $F$ be an operator with domain $D$ of self-wellordered sets. Let $b \in \mathcal{C}_F$, so there is a well-ordering of $b$ in $L_{\alpha}[b]$. A **potential $F$-premouse over $b$** is an acceptable $J$-model $\mathcal{M}$ over $b$ such that there is an ordinal $\iota > 0$ and an increasing, closed sequence $\langle \zeta_\alpha \rangle_{\alpha \leq \iota}$ of ordinals such that for each $\alpha \leq \iota$, we have:

1. $0 = \zeta_0 \leq \zeta_\iota \leq \zeta_\iota = l(\mathcal{M})$ (so $\mathcal{M}|_{\zeta_0} = b$ and $\mathcal{M}|_{\zeta_\iota} = \mathcal{M}$).
2. If $1 < \iota$ then $\mathcal{M}|_{\zeta_1} = F(0, b)$.
3. If $1 = \iota$ then $\mathcal{M} \subseteq F(0, b)$.
4. If $1 < \alpha + 1 < \iota$ then $\mathcal{M}|_{\zeta_{\alpha + 1}} = F(1, \mathcal{M}|_{\zeta_\alpha}) \downarrow b$.
5. If $1 < \alpha + 1 = \iota$, then $\mathcal{M} \subseteq F(1, \mathcal{M}|_{\zeta_\iota}) \downarrow b$.
6. Suppose $\alpha$ is a limit. Then $\mathcal{M}|_{\zeta_\alpha}$ is $B$-passive, and if $E$-active, then $\text{crt}(E(\mathcal{M}|_{\zeta_\iota}) > \text{rank}(b)$.

We say that $\mathcal{M}$ is a **(F-)whole** if $\iota$ is a limit or else, $\iota = \alpha + 1$ and $\mathcal{M} = F(\mathcal{M}|_{\zeta_\alpha}) \downarrow b$.

A (potential) $F$-premouse is a (potential) $F$-premouse over $b$, for some $b$.

\(^{19}\)The “m” is for “model”.

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Definition 3.7. Let \( \mathcal{F} \) be an operator and \( b \in C_{\mathcal{F}} \). Let \( \mathcal{N} \) be a whole \( \mathcal{F} \)-premouse over \( b \). A potential continuing \( \mathcal{F} \)-premouse over \( \mathcal{N} \) is a \( \mathcal{J} \)-model \( \mathcal{M} \) over \( \mathcal{N} \) such that \( \mathcal{M} \downarrow b \) is a potential \( \mathcal{F} \)-premouse over \( b \). (Therefore \( \mathcal{N} \) is a whole strong cutpoint of \( \mathcal{M} \).)

We say that \( \mathcal{M} \) (as above) is whole iff \( \mathcal{M} \downarrow b \) is whole.

A (potential) continuing \( \mathcal{F} \)-premouse is a (potential) continuing \( \mathcal{F} \)-premouse over \( b \), for some \( b \).

Definition 3.8. \( Lp^\mathcal{F}(a) \) denotes the stack of all countably \( \mathcal{F} \)-iterable \( \mathcal{F} \)-premice \( \mathcal{M} \) over \( a \) such that \( \mathcal{M} \) is fully sound and projects to \( a \).\(^{20}\)

Let \( \mathcal{N} \) be a whole \( \mathcal{F} \)-premouse over \( b \), for \( b \in C_{\mathcal{F}} \). Then \( Lp^\mathcal{F}(\mathcal{N}) \) denotes the stack of all countably \( \mathcal{F} \)-iterable (above \( o(\mathcal{N}) \)) continuing \( \mathcal{F} \)-premice \( \mathcal{M} \) over \( \mathcal{N} \) such that \( \mathcal{M} \downarrow b \) is fully sound and projects to \( \mathcal{N} \).\(^{21}\)

We say that \( \mathcal{F} \) is uniformly \( \Sigma_1 \) iff there are \( \Sigma_1 \) formulas \( \varphi_1 \) and \( \varphi_2 \) in \( \mathcal{L}_0^- \) such that whenever \( \mathcal{M} \) is a (continuing) \( \mathcal{F} \)-premouse, then the set of whole proper segments of \( \mathcal{M} \) is defined over \( \mathcal{M} \) by \( \varphi_1 \) (\( \varphi_2 \)). For such an operator \( \mathcal{F} \), let \( \varphi_{wh}^\mathcal{F} \) denote the least such \( \varphi_1 \).

Definition 3.9 (Mouse operator). Let \( Y \) be a projecting, uniformly \( \Sigma_1 \) operator. A \( Y \)-mouse operator \( \mathcal{F} \) with domain \( D \) is an operator with domain \( D \) such for each \( (0, X) \in D \), \( \mathcal{F}(0, X) \triangleleft Lp^Y(X) \), and for each \( (1, X) \in D \), \( \mathcal{F}(1, X) \triangleleft Lp^Y_+(X) \).\(^{22}\) (So any \( Y \)-mouse operator is an operator.) A \( Y \)-mouse operator \( \mathcal{F} \) is called first-order if there are formulas \( \varphi_1 \) and \( \varphi_2 \) in the language of \( Y \)-premice such that \( \mathcal{F}(0, X) (\mathcal{F}(1, X)) \) is the first \( \mathcal{M} \triangleleft Lp^Y(X) (Lp^Y_+(X)) \) satisfying \( \varphi_1 \) (\( \varphi_2 \)).

A mouse operator is a \( \mathcal{J}^m \)-mouse operator.

We can then define \( \mathcal{F} \)-solidity, the \( L^\mathcal{F}[E] \)-construction etc. as usual (see \cite{18} for more details).

We now define the kind of condensation that mouse operators need to satisfy to ensure for example that the \( L^\mathcal{F}[E] \)-construction converges. We define the coarse version of condensation (condense coarsely) here for illustrative purposes. The finer version (condense finely), which is more technical, is discussed in detail in \cite{18}. The core model induction operators, which form a subclass of the \( Y \)-mouse operators, will have these condensation properties.

Definition 3.10. Let \( Y \) be an operator. We say that \( Y \) condenses coarsely iff for all \( i \in \{0, 1\} \) and \( (i, \bar{X}), (i, X) \in \text{dom}(Y) \), and all \( \mathcal{J} \)-models \( \mathcal{M}^+ \) over \( \bar{X} \), if \( \pi : \mathcal{M}^+ \to Y_i(X) \) is fully elementary and fixes the parameters in the definition of \( Y \), then

1. if \( i = 0 \) then \( \mathcal{M}^+ \trianglelefteq Y_0(\bar{X}) \); and

2. if \( i = 1 \) and \( X \) is a sound whole \( Y \)-premouse, then \( \mathcal{M}^+ \trianglelefteq Y_1(\bar{X}) \).

We now proceed to defining \( \Sigma \)-premice, for an iteration strategy \( \Sigma \). We first define the operator to be used to feed in \( \Sigma \).

\(^{20}\)Countable substructures of \( \mathcal{M} \) are \( (\omega, \omega_1 + 1) \)-\( \mathcal{F} \)-iterable, i.e. all iterates are \( \mathcal{F} \)-premice. See \cite[Section 2]{18} for more details on \( \mathcal{F} \)-iterability.

\(^{21}\)Often times in this paper, when the context is clear, we will use the notation \( Lp \) for \( Lp^+ \).

\(^{22}\)This restricts the usual notion defined in \cite{16}.
Definition 3.11 \((\mathcal{B}(a, T, b), b^N)\). Let \(a, \mathcal{P}\) be transitive, with \(\mathcal{P} \in J_1(\bar{a})\). Let \(\lambda > 0\) and let \(T\) be an iteration tree\(^{23}\) on \(\mathcal{P}\), of length \(\omega \lambda\), with \(T \restriction \beta \in a \) for all \(\beta \leq \omega \lambda\). Let \(b \subseteq \omega \lambda\). We define \(N = \mathcal{B}(a, T, b)\) recursively on \(lh(T)\), as the \(J\)-model \(N\) over \(a\) with parameter \(\mathcal{P}\)\(^{24}\) such that:

1. \(l(N) = \lambda\),
2. for each \(\gamma \in (0, \lambda)\), \(N \restriction \gamma = \mathcal{B}(a, T \restriction \omega \gamma, [0, \omega \gamma]_T)\),
3. \(b^N\) is the set of ordinals \(o(a) + \gamma\) such that \(\gamma \in b\),
4. \(E^N = \emptyset\).

We also write \(b^N = b\).

It is easy to see that every initial segment of \(N\) is sound, so \(N\) is acceptable and is indeed a \(J\)-model (not just a \(J\)-structure).

In the context of a \(\Sigma\)-premouse \(M\) for an iteration strategy \(\Sigma\), if \(T\) is the \(<_{\mathcal{M}}\)-least tree for which \(M\) lacks instruction regarding \(\Sigma(T)\), then \(M\) will already have been instructed regarding \(\Sigma(T \restriction \alpha)\) for all \(\alpha < lh(T)\). Therefore if \(lh(T) > \omega\) then \(\mathcal{B}(M, T, \Sigma(T))\) codes redundant information (the branches already in \(T\)) before coding \(\Sigma(T)\). This redundancy seems to allow one to prove slightly stronger condensation properties, given that \(\Sigma\) has nice condensation properties (see [18]). It also simplifies the definition.

Definition 3.12. Let \(\Sigma\) be a partial iteration strategy. Let \(C\) be a class of iteration trees, closed under initial segment. We say that \((\Sigma, C)\) is suitably condensing iff for every \(T \in C\) such that \(T\) is via \(\Sigma\) and \(lh(T) = \lambda + 1\) for some limit \(\lambda\), either (i) \(\Sigma\) has hull condensation with respect to \(T\), or (ii) \(b^T\) does not drop and \(\Sigma\) has branch condensation with respect to \(T\), that is, any hull \(U \dashv c\) of \(T \dashv b\) is according to \(\Sigma\).

When \(C\) is the class of all iteration trees according to \(\Sigma\), we simply omit it from our notation.

Definition 3.13. Let \(\varphi\) be an \(L_0\)-formula. Let \(\mathcal{P}\) be transitive. Let \(\mathcal{M}\) be a \(J\)-model (over some \(a\)), with parameter \(\mathcal{P}\). Let \(T \in \mathcal{M}\). We say that \(\varphi\) selects \(T\) for \(\mathcal{M}\), and write \(T = T^\varphi_{\mathcal{M}}\), iff

(a) \(T\) is the unique \(x \in \mathcal{M}\) such that \(\mathcal{M} \models \varphi(x)\),
(b) \(T\) is an iteration tree on \(\mathcal{P}\) of limit length,
(c) for every \(N \triangleleft \mathcal{M}\), we have \(N \not\models \varphi(T)\), and
(d) for every limit \(\lambda < lh(T)\), there is \(N \triangleleft \mathcal{M}\) such that \(N \models \varphi(T \restriction \lambda)\).

One instance of \(\phi(\mathcal{P}, T)\) is, in the case \(a\) is self-wellordered, the formula “\(T\) is the least tree on \(\mathcal{P}\) that doesn’t have a cofinal branch”, where least is computed with respect to the canonical well-order of the model.

\(^{23}\)We formally take an iteration tree to include the entire sequence \(\langle M^T_\alpha \rangle_{\alpha < lh(T)}\) of models. So it is \(\Sigma_0(T, \mathcal{P})\) to assert that “\(T\) is an iteration tree on \(\mathcal{P}\)”.

\(^{24}\)\(\mathcal{P} = M^T_{\delta}\) is determined by \(T\).
Definition 3.14 (Potential \(\mathcal{P}\)-strategy-premouse, \(\Sigma^M\)). Let \(\varphi \in \mathcal{L}_0\). Let \(\mathcal{P}, a\) be transitive with \(\mathcal{P} \in \mathcal{J}_1(\hat{a})\). A potential \(\mathcal{P}\)-strategy-premouse \(\text{(over } a\text{, of type } \varphi)\) is a \(\mathcal{J}\)-model \(M\) over \(a\), with parameter \(\mathcal{P}\), such that the \(\mathfrak{B}\) operator is used to feed in an iteration strategy for trees on \(\mathcal{P}\), using the sequence of trees naturally determined by \(S^M\) and selection by \(\varphi\). We let \(\Sigma^M\) denote the partial strategy coded by the predicates \(B^M|\eta\), for \(\eta \leq l(M)\).

In more detail, there is an increasing, closed sequence of ordinals \((\eta_\alpha)_{\alpha \leq \iota}\) with the following properties. We will also define \(\Sigma^M|\eta\) for all \(\eta \in [1, l(M)]\) and \(T_\eta = T^M_\eta\) for all \(\eta \in [1, l(M)]\).

1. \(1 = \eta_0\) and \(M|1 = \mathcal{J}_1^m(a; \mathcal{P})\) and \(\Sigma^M|1 = \emptyset\).

2. \(l(M) = \eta_\iota\), so \(M|\eta_\iota = M\).

3. Given \(\eta \leq l(M)\) such that \(B^M|\eta = \emptyset\), we set \(\Sigma^M|\eta = \bigcup_{\eta' < \eta} \Sigma^M|\eta'\).

Let \(\eta \in [1, l(M)]\). Suppose there is \(\gamma \in [1, \eta]\) and \(T \in M|\gamma\) such that \(T = T^M_\eta\) and \(T\) is via \(\Sigma^M|\eta\), but no proper extension of \(T\) is via \(\Sigma^M|\eta\). Taking \(\gamma\) minimal such, let \(T_\eta = T^M_\eta\). Otherwise let \(T_\eta = \emptyset\).

4. Let \(\alpha + 1 \leq \iota\). Suppose \(T_{\eta_\alpha} = \emptyset\). Then \(\eta_{\alpha + 1} = \eta_\alpha + 1\) and \(M|\eta_{\alpha + 1} = \mathcal{J}_1^m(M|\eta_\alpha; \mathcal{P}) \downarrow a\).

5. Let \(\alpha + 1 \leq \iota\). Suppose \(T = T_{\eta_\alpha} \neq \emptyset\). Let \(\omega \lambda = \text{lh}(T)\). Then for some \(b \subseteq \omega \lambda\), and \(S = \mathfrak{B}(M|\eta_\alpha, T, b)\), we have:
   (a) \(M|\eta_{\alpha + 1} \subseteq S\).
   (b) If \(\alpha + 1 < \iota\) then \(M|\eta_{\alpha + 1} = S\).
   (c) If \(S \subseteq M\) then \(b\) is a \(T\)-cofinal branch.\(^{25}\)
   (d) If \(\eta \in [\eta_\alpha, l(M)]\) such that \(\eta < l(S)\), \(\Sigma^M|\eta = \Sigma^M|\eta_\alpha\).
   (e) If \(S \subseteq M\) then \(\Sigma^S = \Sigma^M|\eta_\alpha \cup \{(T, b^S)\}\).

6. For each limit \(\alpha \leq \iota\), \(B^M|\eta_\alpha = \emptyset\). \(\dagger\)

Definition 3.15 (Whole). Let \(M\) be a potential \(\mathcal{P}\)-strategy-premouse of type \(\varphi\). We say \(\mathcal{P}\) is \(\varphi\)-whole \(\text{(or just whole if } \varphi\text{ is fixed)}\) iff for every \(\eta < l(M)\), if \(T_\eta \neq \emptyset\) and \(T_\eta \neq T_{\eta'}\) for all \(\eta' < \eta\), then for some \(b\), \(\mathfrak{B}(M|\eta, T_\eta, b) \subseteq M\).\(^{26}\) \(\dagger\)

Definition 3.16 (Potential \(\Sigma\)-premouse). Let \(\Sigma\) be a (partial) iteration strategy for a transitive structure \(\mathcal{P}\). A potential \(\Sigma\)-premouse \(\text{(over } a\text{, of type } \varphi)\) is a potential \(\mathcal{P}\)-strategy premouse \(M\) \(\text{(over } a\text{, of type } \varphi)\) such that \(\Sigma^M \subseteq \Sigma\).\(^{27}\) \(\dagger\)

Definition 3.17. Let \(\mathcal{P}\) be transitive and \(\Sigma\) a partial iteration strategy for \(\mathcal{P}\). Let \(\varphi \in \mathcal{L}_0\). Let \(\mathcal{F} = \mathcal{F}_{\Sigma, \varphi}\) be the operator such that:

\(^{25}\)We allow \(M^*_\eta\) to be illfounded, but then \(T \upharpoonright b\) is not an iteration tree, so is not continued by \(\Sigma^M\).

\(^{26}\)\(\varphi\)-whole depends on \(\varphi\) as the definition of \(T_\eta\) does.

\(^{27}\)If \(M\) is a model all of whose proper segments are potential \(\Sigma\)-premice, and the rules for potential \(\mathcal{P}\)-strategy premice require that \(B^M\) code a \(T\)-cofinal branch, but \(\Sigma(T)\) is not defined, then \(M\) is not a potential \(\Sigma\)-premouse, whatever its predicates are.
1. $\mathcal{F}_0(a) = J_1^\mu(a; \mathcal{P})$, for all transitive $a$ such that $\mathcal{P} \in J_1(\check{a})$;

2. Let $\mathcal{M}$ be a sound branch-whole $\Sigma$-premouse of type $\varphi$. Let $\lambda = l(\mathcal{M})$ and with notation as in 3.14, let $\mathcal{T} = \mathcal{T}_b$. If $\mathcal{T} = \emptyset$ then $\mathcal{F}_1(\mathcal{M}) = J_1^\mu(\mathcal{M}; \mathcal{P})$. If $\mathcal{T} \neq \emptyset$ then $\mathcal{F}_1(\mathcal{M}) = \mathcal{B}(\mathcal{M}, \mathcal{T}, b)$ where $b = \Sigma(\mathcal{T})$.

We say that $\mathcal{F}$ is a strategy operator.

**Lemma 3.18.** Let $\mathcal{P}$ be countable and transitive. Let $\varphi$ be a formula of $\mathcal{L}_0$. Let $\Sigma$ be a partial strategy for $\mathcal{P}$. Let $D_\varphi$ be the class of iteration trees $\mathcal{T}$ on $\mathcal{P}$ such that for some $\mathcal{J}$-model $\mathcal{M}$, with parameter $\mathcal{P}$, we have $\mathcal{T} = \mathcal{T}_\varphi^\mathcal{M}$. Suppose that $(\Sigma, D_\varphi)$ is suitably condensing. Then $\mathcal{F}_{\Sigma, \varphi}$ is uniformly $\Sigma_1$, projecting, and condenses finely.

**Definition 3.19.** Let $a$ be transitive and let $\mathcal{F}$ be an operator. We say that $\mathcal{M}_{1}^{\mathcal{F}, \#}(a)$ exists if there is a $(0, |a|, |a| + 1)$-$\mathcal{F}$-iterable, non-$1$-small $\mathcal{F}$-premouse over $a$. We write $\mathcal{M}_{1}^{\mathcal{F}, \#}(a)$ for the least such sound structure. For $\Sigma, \mathcal{P}, a, \varphi$ as in Definition 3.17, we write $\mathcal{M}_{1}^{\Sigma, \varphi, \#}(a)$ for $\mathcal{M}_{1}^{\mathcal{F}_{\Sigma, \varphi}, \#}(a)$.

Let $\mathcal{L}_0^+$ be the language $\mathcal{L}_0 \cup \{\check{\Sigma}, \check{\Sigma}\}$, where $\check{\Sigma}$ is the binary relation defined by “$\check{\Sigma}$ is self-wellordered, with ordering $\prec_{\check{\Sigma}}$, and $\check{\Sigma}$ is the canonical wellorder of the universe extending $\check{\Sigma}$”, and $\check{\Sigma}$ is the partial function defined by “$\check{\Sigma}$ is a transitive structure and the universe is a potential $\check{\Sigma}$-strategy premouse over $\check{\Sigma}$ and $\check{\Sigma}$ is the associated partial putative iteration strategy for $\check{\Sigma}$”. Let $\varphi_{\text{all}}(\mathcal{T})$ be the $\mathcal{L}_0$-formula “$\mathcal{T}$ is the $\check{\Sigma}$-least limit length iteration tree $\mathcal{U}$ on $\check{\Sigma}$ such that $\mathcal{U}$ is via $\check{\Sigma}$, but no proper extension of $\mathcal{U}$ is via $\check{\Sigma}$”. Then for $\Sigma, \mathcal{P}, a$ as in Definition 3.17, we sometimes write $\mathcal{M}_{1}^{\Sigma, \#}(a)$ for $\mathcal{M}_{1}^{\mathcal{F}_{\Sigma, \varphi_{\text{all}}}, \#}(a)$.

Let $\kappa$ be a cardinal and suppose that $\mathcal{M} = \mathcal{M}_{1}^{\mathcal{F}, \#}(a)$ exists and is $(0, \kappa^+ + 1)$-iterable. We write $\Lambda_\mathcal{M}$ for the unique $(0, \kappa^+ + 1)$-iteration strategy for $\mathcal{M}$ (given that $\kappa$ is fixed).

### 3.2. Core model induction operators

In core model induction applications, we often have a pair $(\mathcal{P}, \Sigma)$ where $\mathcal{P}$ is a hod premouse and $\Sigma$ is $\mathcal{P}$’s strategy with branch condensation and is fullness preserving (relative to mice with strategies in some pointclass) or $\mathcal{P}$ is a sound (hybrid) premouse projecting to some countable set $a$ and $\Sigma$ is the unique (normal) $(\omega_1 + 1)$-strategy for $\mathcal{P}$. Let $\mathcal{F}$ be the operator corresponding to $\Sigma$ (using the formula $\varphi_{\text{all}}$) and suppose $\mathcal{M}_{1}^{\mathcal{F}, \#} \exists$. Then [18, Lemma 4.8] shows that $\mathcal{F}$ condenses finely and $\mathcal{M}_{1}^{\mathcal{F}, \#}$ generically interprets $\mathcal{F}$. Also, the core model induction will give us that $\mathcal{F} \upharpoonright \mathbb{R}$ is self-scaled (defined below). In the following, we will write $\mathcal{M}_{1}^{\Sigma, \#}$ for $\mathcal{M}_{1}^{\mathcal{F}, \#}$.

In this section, our main goal is to introduce the main concepts that one uses in the core model induction through the hierarchy $L^\mathcal{P}_{1}(\mathbb{R}, \Sigma \upharpoonright HC)$.

28 An equivalent way to define this is to first fix a canonical coding function $\text{Code}: HC \rightarrow \mathbb{R}$ and consider $L^\mathcal{P}_{1}(\mathbb{R}, \text{Code}(\Sigma \upharpoonright HC))$.

29 Instead of feeding $\Sigma$ into the hierarchy, we feed in $\Lambda$, the canonical strategy of $\mathcal{M}_{1}^{\Sigma, \#}$, into the hierarchy. Roughly speaking, the trees according to $\Lambda$ that we feed into $L^\mathcal{P}_{1}(\mathbb{R}, \text{Code}(\Sigma \upharpoonright HC))$ are those making the local HOD of $L^\mathcal{P}_{1}(\mathbb{R}, \text{Code}(\Sigma \upharpoonright HC))$ generically generic, for appropriately chosen ordinals $\alpha$. See [18].
The Θ-

For the purpose of this paper, it will not be important to see 

hierarchy and for the Θ-

g manner as the scales analysis in \( L^p(\mathbb{R}) \) (instead of the traditional “least branch” hierarchy of \( \Sigma \)-ice) because the scales analysis under optimal hypotheses can be carried out in \( L^p(\mathbb{R}) \). It is not clear how one can perform the full scales analysis for the hierarchy \( L^p \). We note that the full constructions from \( [17] \) work out nicely for this hierarchy.

The \( \Theta \)-g-organized hierarchy, which is a slight modification of the \(\delta \text{-}\)organized \(\Sigma\)-mice, is considered because the scales analysis under optimal hypotheses can be carried out in \( L^p(\mathbb{R}, \Sigma \upharpoonright HC) \) in much the same manner as the scales analysis in \( L^p(\mathbb{R}) \). For the purpose of this paper, it will not be important to go into the detailed definitions of these hierarchies. Whenever it makes sense to define \( L^p(\mathbb{R}) \) and \( L^p(\mathbb{R}) \), \([18]\) shows that \( \varphi(x) \cap L^p(\mathbb{R}) = \varphi(x) \cap L^p(\mathbb{R}) \) (and similarly for \( L^p(\mathbb{R}) \)); also in the case it is not clear how to make sense of \( L^p(\mathbb{R}) \) (say for instance when \( x = \mathbb{R} \)), it still makes sense to define \( L^p(\mathbb{R}) \) and \( L^p(\mathbb{R}) \) and in that case, \([18]\) shows that \( \varphi(x) \cap L^p(\mathbb{R}) = \varphi(x) \cap L^p(\mathbb{R}) \).

In the paragraph below, we briefly remark on how the S-constructions work for the \( g \)-organized hierarchy and for the \( \Theta \)-g-hierarchy.

Suppose \( \mathcal{F} \) is a nice operator (with parameter \( \Psi \)) and suppose \( \mathcal{M} \) is a \( \mathcal{G} \)-mouse (over some transitive \( a \)), where \( \mathcal{G} \) is either \( \mathcal{G}^* \) or \( \mathcal{G}^* \). Suppose \( \delta \) is a cutpoint of \( \mathcal{M} \) and suppose \( \mathcal{N} \) is a transitive structure such that \( \delta \subseteq \mathcal{N} \subseteq \mathcal{M} \delta \) and \( \Psi \in \mathcal{N} \). Suppose \( \mathcal{P} \in \mathcal{J}_\omega[\mathcal{N}] \) is such that \( \mathcal{M} \delta \) is \( \mathcal{P} \)-generic over \( \mathcal{J}_\omega[\mathcal{N}] \) and suppose whenever \( \mathcal{Q} \in \mathcal{J}_\omega[\mathcal{N}] \) is a \( \mathcal{G} \)-mouse over \( \mathcal{N} \) such that \( H^\mathcal{P}_\mathcal{Q} = \mathcal{N} \) then \( \mathcal{M} \delta \) is \( \mathcal{P} \)-generic over \( \mathcal{Q} \). Then the S-constructions (or P-constructions) from \([17]\) give a \( \mathcal{G} \)-mouse \( \mathcal{R} \) over \( \mathcal{N} \) such that \( \mathcal{R}[\mathcal{M} \delta] = \mathcal{M} \). The S-constructions give the sequence \( (\mathcal{R}_\alpha : \delta < \alpha \leq \lambda) \) of \( \mathcal{G} \)-premouse over \( \mathcal{N} \), where

(i) \( \mathcal{R}_{\delta+1} = \mathcal{J}_\omega^m(\mathcal{N}) \);

(ii) if \( \alpha \) is limit then let \( \mathcal{R}_\alpha = \bigcup_{\beta < \alpha} \mathcal{R}_\beta \). If \( \mathcal{M} \alpha \) is passive, then let \( \mathcal{R}_\alpha = \mathcal{R}_\alpha^n \). So \( \mathcal{R}_\alpha \) is passive. If \( B^{\mathcal{M} \alpha} \neq \emptyset \), then let \( \mathcal{R}_\alpha = (\{\mathcal{R}_\alpha^n, B^{\mathcal{M} \alpha}, \bigcup_{\beta < \alpha} S^R \beta, \mathcal{N}, \Psi\}) \). Suppose \( E^{\mathcal{M} \alpha} \neq \emptyset \); let \( E^* = E^{\mathcal{M} \alpha} \cap (\bigcup \mathcal{R}_\alpha^n) \), then let \( \mathcal{R}_\alpha = (\{\mathcal{R}_\alpha^n, E^*, \bigcup_{\beta < \alpha} S^R \beta, \mathcal{N}, \Psi\}) \). By the hypothesis, we have \( \mathcal{R}_\alpha \delta = \mathcal{M} \delta \).

(iii) Suppose we have already constructed \( \mathcal{R}_\alpha \) and (by the hypothesis) maintain that \( \mathcal{R}_\alpha \delta = \mathcal{M} \delta \). Then \( \mathcal{R}_{\alpha+1} = \mathcal{J}_\omega^m(\mathcal{R}_\alpha) \).

(iv) \( \lambda \) is such that \( \mathcal{R}_\lambda \delta = \mathcal{M} \). We set \( \mathcal{R}_\lambda = \mathcal{R} \).

We note that the full constructions from \([17]\) do not require that \( \delta \) is a cutpoint of \( \mathcal{M} \) but we don’t need the full power of the S-constructions in our paper. Also, the fact that \( \mathcal{M} \) is \( g \)-organized

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\(^{30}\)This means whenever \( \mathcal{T} \) is an iteration tree according to \( \Lambda \) with last model \( \mathcal{N} \), then \( \mathcal{N} \) is a \( \Sigma \)-premouse.

\(^{31}\)It is not clear how one can perform S-constructions over the least branch hierarchy.

\(^{32}\)\([18]\) generalizes Steel’s scales analysis in \([25, 24]\) to \( L^p(\mathbb{R}, \Sigma \upharpoonright HC) \) for various classes of nice strategies \( \Sigma \). It is not clear that one can carry out the full scales analysis for the hierarchy \( L^p(\mathbb{R}, \Sigma \upharpoonright HC) \).

\(^{33}\)Nice is defined in \([18], \text{Definition 3.8}\). Roughly speaking, these are operators that condense well and determine themselves on generic extensions. CMI operators defined in this section are nice.
(or $\Theta$-g-organized) is important for our constructions above because it allows us to get past levels $\mathcal{M}|\alpha$ for which $B^{\mathcal{M}|\alpha} \neq \emptyset$. Because of this fact, in this paper, hod mice are reorganized into the g-organized hierarchy, that is if $\mathcal{P}$ is a hod mouse then $\mathcal{P}(\alpha + 1)$ is a g-organized $\Sigma_{\mathcal{P}(\alpha)}$-premouse for all $\alpha < \lambda^\mathcal{P}$. The S-constructions are also important in many other contexts. One such context is the local HOD analysis of levels of $Lp^{G,F}(\mathbb{R}, F \upharpoonright \mathbb{R})$, which features in the scales analysis of $Lp^{G,F}(\mathbb{R}, F \upharpoonright \mathbb{R})$ (cf. [18]).

In the following, a transitive structure $N$ is closed under an operator $\Omega$ if whenever $x \in \text{dom}(\Omega) \cap N$, then $\Omega(x) \in N$. We are now in a position to introduce the core model induction operators that we will need in this paper. These are particular kinds of mouse operators (in the sense of [19, Example 3.41]) that are constructed during the course of the core model induction. These operators can be shown to satisfy the sort of condensation described in [19, Section 3] (e.g. condense coarsely and condense finely), relativize well, and determine themselves on generic extensions.

**Definition 3.20** (relativizes well). Let $\Omega$ be an a $Y$-mouse operator for some operator $Y$.\footnote{Y may be the rud operator, in which case $\Omega$ is just a mouse operator in the usual sense.} We say that $\Omega$ relativizes well if there is a formula $\phi(x, y, z)$ such that for any $a, b \in \text{dom}(\Omega)$ such that $a \in L_1(b)$, whenever $N$ is a transitive model of $ZFC^-$ such that $N$ is closed under $Y$ and $a, b, \Omega(b) \in N$, then $\Omega(a) \in N$ and is the unique $x \in N$ such that $N \models \phi[x, a, \Omega(b)]$.

**Definition 3.21** (determines itself on generic extensions). Suppose $\Omega$ is an operator. We say that $\Omega$ determines itself on generic extensions if there is a formula $\phi(x, y, z)$ and a parameter $c \in HC$ such that for any countable transitive structure $N$ of $ZFC^-$ such that $N$ contains $c$ and is closed under $\Omega$, for any generic extension $N[g]$ of $N$ in $V$, $\Omega \cap N[g] \in N[g]$ and is definable over $N[g]$ via $(\phi, c)$, i.e. for any $e \in N[g] \cap \text{dom}(\Omega)$, $\Omega(e) = d$ if and only if $d$ is the unique $d' \in N[g]$ such that $N[g] \models \phi[c, d', e]$.

**Definition 3.22.** Let $\Gamma$ be an inductive-like pointclass. For $x \in \mathbb{R}$, $C_{\Gamma}(x)$ denotes the set of all $y \in \mathbb{R}$ such that for some ordinal $\gamma < \omega_1$, $y$ (as a subset of $\omega$) is $\Delta_\gamma(\{\gamma, x\})$.

Let $x \in HC$ be transitive and let $f: \omega \to x$ be a surjection. Then $c_f \in \mathbb{R}$ denotes the code for $(x, e)$ determined by $f$. And $C_{\Gamma}(x)$ denotes the set of all $y \in HC \cap \varphi(x)$ such that for all surjections $f: \omega \to x$ we have $f^{-1}(y) \in C_{\Gamma}(c_f)$.

We say that $\vec{A}$ is a self-justifying-system (sjs) if for any $A \in \text{rng}(\vec{A})$, $\neg A \in \text{rng}(\vec{A})$ and there is a scale $\varphi$ on $A$ such that the set of premwellorders associated with $\varphi$ is a subset of $\text{rng}(\vec{A})$. A set $Y \subseteq \mathbb{R}$ is self-scaled if there are scales on $Y$ and $\mathbb{R}\setminus Y$ which are projective in $Y$.

In the following, $\eta$ is a strong cutpoint of $\mathcal{N}$ if there is no extender $E$ on the sequence of $\mathcal{N}$ such that $\text{crt}(E) \leq \eta \leq \text{lh}(E)$.

**Definition 3.23.** Let $(\Omega, A)$ be as above and let $t \in HC$ with $\mathfrak{M} \in J_1(t)$. Let $1 \leq k < \omega$. A premouse $\mathcal{N}$ over $t$ is $\Omega$-$\Gamma$-$k$-suitable (or just $k$-suitable if $\Gamma$ and $\Omega$ are clear from the context) iff there is a strictly increasing sequence $\langle \delta_i \rangle_{i<k}$ such that

1. $\forall \delta \in \mathcal{N}, \mathcal{N} \models \"\delta \text{ is Woodin}\"$ if and only if $\exists i < k (\delta = \delta_i)$. 

\footnote{Y may be the rud operator, in which case $\Omega$ is just a mouse operator in the usual sense.}

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2. \( o(\mathcal{N}) = \sup_{i<\omega}(\delta_{k-1}^{\mathcal{N}}) \).

3. If \( \mathcal{N}|\eta \) is a strong cutpoint of \( \mathcal{N} \) then \( \mathcal{N}|(\eta^+)\mathcal{N} = L_{p}^{\mathcal{N}}(\mathcal{N}|\eta) \).

4. Let \( \xi < o(\mathcal{N}) \), where \( \mathcal{N} \models \text{“} \xi \text{ is not Woodin} \text{“} \). Then \( C_{\Gamma}(\mathcal{N}|\xi) \models \text{“} \xi \text{ is not Woodin} \text{“} \).

We write \( \delta^{\mathcal{N}} = \delta \); also let \( \delta_{-1}^{\mathcal{N}} = 0 \) and \( \delta_{0}^{\mathcal{N}} = o(\mathcal{N}) \).\(^{35}\)

If \( \mathcal{N} \) is 1-suitable, we simply say \( \mathcal{N} \) is suitable, and we write \( \delta^{\mathcal{N}} \) for \( \delta_{0}^{\mathcal{N}} \).

Let \( \mathcal{N} \) be 1-suitable and let \( \xi \in o(\mathcal{N}) \) be a limit ordinal such that \( \mathcal{N} \models \text{“} \xi \text{ isn’t Woodin} \text{“} \). Let \( Q \triangleleft \mathcal{N} \) be the \( Q \)-structure for \( \xi \). Let \( \alpha \) be such that \( \xi = o(\mathcal{N}|\alpha) \). If \( \xi \) is a strong cutpoint of \( \mathcal{N} \) then \( Q \triangleleft L_{p}^{\mathcal{N}}(\mathcal{N}|\xi) \) by clause 3 of the definition. Assume now that \( \mathcal{N} \) is reasonably iterable. If \( \xi \) is a strong cutpoint of \( Q \), our mouse capturing hypothesis combined with clause 4 gives that \( Q \triangleleft L_{p}^{\mathcal{N}}(\mathcal{N}|\xi) \). If \( \xi \) is an \( \mathcal{N} \)-cardinal then indeed \( \xi \) is a strong cutpoint of \( Q \), since \( \mathcal{N} \) has only finitely many Woodins. If \( \xi \) is not a strong cutpoint of \( Q \), then by definition, we do not have \( Q \triangleleft L_{p}^{\mathcal{N}}(\mathcal{N}|\xi) \). However, using \( * \)-translation (see [23]), one can find a level of \( L_{p}^{\mathcal{N}}(\mathcal{N}|\xi) \) which corresponds to \( Q \) (and this level is in \( C_{\Gamma}(\mathcal{N}|\xi) \)).

If \( \Omega \) is a nice operator (in the sense of [18]) and \( \Sigma \) is an iteration strategy for a \( \Omega \)-\( \Gamma \)-1-suitable premouse \( \mathcal{P} \) such that \( \Sigma \) has branch condensation and is \( \Gamma \)-fullness preserving (for some pointclass \( \Gamma \)), then we say that (\( \mathcal{P}, \Sigma \)) is a \( \Omega \)-\( \Gamma \)-suitable pair or just \( \Gamma \)-suitable pair or just suitable pair if the pointclass and/or the operator \( \Omega \) is clear from the context.

**Definition 3.24** (Core model induction operators). Suppose (\( \mathcal{P}, \Sigma \)) is a \( G \)-\( \Omega \)-suitable pair for some nice operator \( G \) or a hod pair such that \( \Sigma \) has branch condensation and is \( \Omega \)-fullness preserving. Let \( \Omega = \Sigma \). Assume \( \text{Code}(\Omega) \) is self-scaled. We say \( J \) is a \( \Sigma \)-core model induction operator or just a \( \Sigma \)-cmi operator if one of the following holds:

1. \( J \) is a nice \( \Omega \)-mouse operator (or \( g \)-organized \( \Omega \)-mouse operator) defined on a cone of \( HC \) above some \( a \in HC \). Furthermore, \( J \) condenses finely, relativizes well and determines itself on generic extensions.

2. For some \( \alpha \in OR \) such that \( \alpha \) ends either a weak or a strong gap in the sense of [24] and [18], letting \( M = L_{p}^{\mathcal{N}}(\mathcal{R}, \Omega \upharpoonright HC)|\alpha \) and \( \Gamma = (\Sigma_{1})^{M} \), \( M \models \text{AD}^{+} + \text{MC}(\Sigma) \).\(^{36}\) For some transitive \( b \in HC \) and some 1-suitable (or more fully \( \Omega \)-\( \Gamma \)-1-suitable) \( \Omega \)-premouse \( Q \) over \( b \), \( J = \Lambda \), where \( \Lambda \) is an \( (\omega_{1}, \omega_{1}) \)-iteration strategy for \( Q \) which is \( \Gamma \)-fullness preserving, has branch condensation and is guided by some self-justifying-system (sjs) \( \bar{A} = (A_{i} : i < \omega) \) such that for some real \( x \), for each \( i, A_{i} \in \text{OD}_{b, \Sigma, x}^{M} \) and \( \bar{A} \) seals the gap that ends at \( \alpha \).

\(^{35}\)We could also define a suitable premouse \( \mathcal{N} \) as a \( \Theta \)-\( g \)-organized \( F \)-premouse and all the results that follow in this paper will be unaffected.

\(^{36}\)\( \text{MC}(\Sigma) \) stands for Mouse Capturing relative to \( \Sigma \) which says that for \( x, y \in R \), \( x \in \text{OD}(\Sigma, y) \) (or equivalently \( x \) is \( \text{OD}(\Omega, y) \)) if \( x \) is in some \( g \)-organized \( \Omega \)-mouse over \( y \). \( \text{SMC} \) is the statement that for every hod pair (\( \mathcal{P}, \Sigma \)) such that \( \Sigma \) is fullness preserving and has branch condensation, \( \text{MC}(\Sigma) \) holds.
Remark 3.25. Let \( \Gamma, M \) be as in clause 2 above. The (lightface) envelope of \( \Gamma \) is defined as: \( A \in \text{Env}(\Gamma) \) iff for every countable \( \sigma \subset \mathbb{R} \) there is some \( A' \) such that \( A' \) is \( \Delta_1 \)-definable over \( M \) from ordinal parameters and \( A \cap \sigma = A' \cap \sigma \). For a real \( x \), we define \( \text{Env}(\Gamma(x)) \) similarly: here \( \Gamma(x) = \Sigma_1(x)^M \) and \( A \in \text{Env}(\Gamma(x)) \) iff for every countable \( \sigma \subset \mathbb{R} \) there is some \( A' \) that is \( \Delta_1(x) \)-definable over \( M \) from ordinal parameters such that \( A \cap \sigma = A' \cap \sigma \). We now let \( \text{Env}(\Gamma) = \bigcup_{x \in \mathbb{R}} \text{Env}(\Gamma(x)) \). Note that \( \text{Env}(\Gamma) = \wp(\mathbb{R})^M \) if \( \alpha \) ends a weak gap and \( \text{Env}(\Gamma) = \wp(\mathbb{R})^{Lp(\mathbb{R})}(\alpha+1) \) if \( \alpha \) ends a strong gap.

In clause 2 above, \( \tilde{A} \) is Wadge cofinal in \( \text{Env}(\Gamma) \) where \( \Gamma = \Sigma_1^M \).

The following definitions are obvious generalizations of those defined in [16].

Definition 3.26. We say that the coarse mouse witness condition \( W_{\gamma,\delta}^{\omega,\delta} \) holds if, whenever \( U \subseteq \mathbb{R} \) and both \( U \) and its complement have scales in \( L_p^{\omega,\delta}(\mathbb{R}, \Omega \upharpoonright \text{HC})|\gamma \), then for all \( k < \omega \) and \( x \in \mathbb{R} \) there is a coarse \( (k, U) \)-Woodin mouse \( M \) containing \( x \) and closed under the strategy \( \Lambda \) of \( M_{\gamma,\delta}^{\omega,\delta} \) with an \((\omega_1+1)\)-iteration strategy whose restriction to \( \text{HC} \) is in \( L_p^{\omega,\delta}(\mathbb{R}, \Omega \upharpoonright \text{HC})|\gamma \).\(^{37}\)

Remark 3.27. By the proof of [16, Lemma 3.3.5], \( W_{\gamma,\delta}^{\omega,\delta} \) implies \( L_p^{\omega,\delta}(\mathbb{R}, \Omega \upharpoonright \text{HC})|\gamma \vdash \text{AD}^+ \).

Definition 3.28. An ordinal \( \gamma \) is a critical ordinal in \( L_p^{\omega,\delta}(\mathbb{R}, \Omega \upharpoonright \text{HC}) \) if there is some \( U \subseteq \mathbb{R} \) such that \( U \) and \( \mathbb{R} \setminus U \) have scales in \( L_p^{\omega,\delta}(\mathbb{R}, \Omega \upharpoonright \text{HC})|\gamma \) but not in \( L_p^{\omega,\delta}(\mathbb{R}, \Omega \upharpoonright \text{HC})|\gamma \). In other words, \( \gamma \) is critical in \( L_p^{\omega,\delta}(\mathbb{R}, \Omega \upharpoonright \text{HC}) \) just in case \( W_{\gamma+1}^{\omega,\delta} \) does not follow trivially from \( W_{\gamma}^{\omega,\delta} \).

To any \( \Sigma_1 \) formula \( \theta(v) \) in the language of \( L_p^{\omega,\delta}(\mathbb{R}, \Omega \upharpoonright \text{HC}) \) we associate formulae \( \theta_k(v) \) for \( k \in \omega \), such that \( \theta_k \) is \( \Sigma_k \), and for any \( \gamma \) and any real \( x \),

\[
L_p^{\omega,\delta}(\mathbb{R}, \Omega \upharpoonright \text{HC})|\gamma+1 \vdash \theta[x] \iff \exists k < \omega \ L_p^{\omega,\delta}(\mathbb{R}, \Omega \upharpoonright \text{HC})|\gamma \vdash \theta_k[x].
\]

Definition 3.29. Suppose \( \theta(v) \) is a \( \Sigma_1 \) formula (in the language of set theory expanded by a name for \( \mathbb{R} \) and a predicate for \( G(\Omega) \), and \( z \) is a real; then a \( \langle \theta, z \rangle \)-prewitness is an \( \omega \)-sound \( g \)-organized \( \Omega \)-premouse \( N \) over \( z \) in which there are \( \delta_0 < \cdots < \delta_9 \), \( S \), and \( T \) such that \( N \) satisfies the formulae expressing

(a) ZFC,

(b) \( \delta_0, \ldots, \delta_9 \) are Woodin,

(c) \( S \) and \( T \) are trees on some \( \omega \times \eta \) which are absolutely complementing in \( V^{\text{Col}(\omega,\delta_9)} \), and

(d) For some \( k < \omega \), \( p[T] \) is the \( \Sigma_{k+3} \)-theory (in the language with names for each real and predicate for \( G(\Omega) \)) of \( L_p^{\omega,\delta}(\mathbb{R}, \Omega \upharpoonright \text{HC})|\gamma \), where \( \gamma \) is least such that \( L_p^{\omega,\delta}(\mathbb{R}, \Omega \upharpoonright \text{HC})|\gamma \vdash \theta_k[z] \).

If \( N \) is also \( (\omega, \omega_1, \omega_1+1) \)-iterable (as a \( g \)-organized \( \Omega \)-mouse), then we call it a \( \langle \theta, z \rangle \)-witness.\(^{25}\)
**Definition 3.30.** We say that the fine mouse witness condition $W^*_\gamma^{\langle \Omega \rangle}$ holds if whenever $\theta(v)$ is a $\Sigma_1$ formula (in the language $\mathcal{L}^+$ of g-organized $\Omega$-premice (cf. [18])), $z$ is a real, and $L^\emptyset(g \upharpoonright \Omega \upharpoonright HC) | \gamma \vDash \theta(z)$, then there is a $\langle \theta, z \rangle$-witness $\mathcal{N}$ whose $\emptyset\Omega$-iteration strategy, when restricted to countable trees on $\mathcal{N}$, is in $L^\emptyset(g \upharpoonright \Omega \upharpoonright HC) | \gamma$. $\hdashline$

**Lemma 3.31.** $W^*_\gamma^{\langle g \emptyset \Omega \rangle}$ implies $W^*_\gamma^{\emptyset \Omega}$ for limit $\gamma$.

The proof of the above lemma is a straightforward adaptation of that of [16, Lemma 3.5.4]. One main point is the use of the $g$-organization: $g$-organized $\Omega$-mice behave well with respect to generic extensions in the sense that if $P$ is a $g$-organized $\Omega$-mouse and $h$ is set generic over $P$ then $P[h]$ can be rearranged to a $g$-organized $\Omega$-mouse over $h$.

**Remark 3.32.** In light of the discussion above, the core model induction (through $L^\emptyset(g \upharpoonright \Omega \upharpoonright HC)$) inductively shows $L^\emptyset(g \upharpoonright \Omega \upharpoonright HC) | \gamma \vDash \text{AD}^+$ by showing that $W^*_\gamma^{\langle g \emptyset \Omega \rangle}$ holds for critical ordinals $\gamma$. This, in turn, is done by constructing appropriate $\Omega$-cmi operators “capturing” the theory of those levels (as specified in Definitions 3.26 and 3.30).

Finally, as in [31], the maximal model of $\Theta = \theta_\Omega$ is $sL^\emptyset(g \upharpoonright \Omega, \text{Code}(\Omega))$, an initial segment (possibly strict) of $L^\emptyset(g \upharpoonright \Omega \upharpoonright HC)$. $\hdashline$

**Definition 3.33.** We define $sL^\emptyset(g \upharpoonright \Omega, \text{Code}(\Omega))$ to be the union of those $M \subset L^\emptyset(g \upharpoonright \Omega \upharpoonright HC)$ such that whenever $\pi : M^* \to M$ is elementary, $P \in \pi^{-1}(HC)$, and $M^*$ is countable and transitive, then $M^*$ is $X-(\omega_1 + 1)$-iterable with unique strategy $\Lambda$ such that $\Lambda \upharpoonright HC \in M$.

We will outline the core model induction in the next section, showing that $L^\emptyset(g \upharpoonright \Omega, \text{Code}(\Omega)) \models \text{AD}^+ + \text{MC}(\Omega)$ in Theorem 4.8 for sufficiently nice $\Omega$. We note that by [11], if $M$ is a model of $\text{AD}^+ + \text{MC}(\Omega)$ satisfying $\Theta = \Theta_\Omega$ and $V = L(\wp(R))$, then $M$ satisfies that every set of reals $A$ belongs to $sL^\emptyset(g \upharpoonright \Omega, \text{Code}(\Omega))$. So in fact, in the situation of this paper, $sL^\emptyset(g \upharpoonright \Omega, \text{Code}(\Omega)) = L^\emptyset(g \upharpoonright \Omega, \text{Code}(\Omega))$.

For notational simplicity, from now on, we denote $L^\emptyset(g \upharpoonright \Omega, \Sigma \upharpoonright HC)$ by $L^\emptyset(g \upharpoonright \Omega)$. $\hdashline$

### 3.3. A Brief Introduction To Hod Mice

In this paper, a hod premouse $\mathcal{P}$ is one defined as in [9]. The reader is advised to consult [9] for basic results and notations concerning hod premice and mice. Let us mention some basic first-order properties of a hod premouse $\mathcal{P}$. There are an ordinal $\lambda^\mathcal{P}$ and sequences $\langle (\mathcal{P}(\alpha), \Sigma^\mathcal{P}_\alpha) \mid \alpha < \lambda^\mathcal{P} \rangle$ and $\langle \delta^\mathcal{P}_\alpha \mid \alpha \leq \lambda^\mathcal{P} \rangle$ such that

1. $\langle \delta^\mathcal{P}_\alpha \mid \alpha \leq \lambda^\mathcal{P} \rangle$ is increasing and continuous and if $\alpha$ is a successor ordinal then $\mathcal{P} \models \delta^\mathcal{P}_\alpha$ is Woodin.

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38 $\text{MC}(\Omega)$ states that if $x, y \in R$ and $x \in OD(y, \Omega)$, then there is a $\Omega$-mouse $\mathcal{M}$ over $y$ such that $\mathcal{M}$ is sound, $\rho_\omega(\mathcal{M}) = \omega$, and $x \in \mathcal{M}$. 

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2. every Woodin cardinal or limit of Woodin cardinals of $\mathcal{P}$ is of the form $\delta^P_\alpha$ for some $\alpha$;

3. $\mathcal{P}(0) = L_{\omega}(\mathcal{P}|\delta_0)^P$; for $\alpha < \lambda^P$, $\mathcal{P}(\alpha + 1) = (L_{\omega}\mathcal{P}(\delta_{\alpha + 1})^P);^{39}$ for limit $\alpha \leq \lambda^P$, $\mathcal{P}(\alpha) = (L_{\omega}^{\oplus_{\beta<\alpha}\Sigma_{\beta}}(\mathcal{P}|\delta_\alpha))^P$;

4. $\mathcal{P} \models \Sigma^P_\alpha$ is a $(\omega, o(\mathcal{P}), o(\mathcal{P}))^{40}$-strategy for $\mathcal{P}(\alpha)$ with hull condensation;

5. if $\alpha < \beta < \lambda^P$ then $\Sigma^P_\beta$ extends $\Sigma^P_\alpha$.

We will write $\delta^P$ for $\delta^P_{\lambda^P}$ and $\Sigma^P = \oplus_{\beta<\lambda^P}\Sigma^P_\beta$. Note that $\mathcal{P}(0)$ is a pure extender model. Suppose $\mathcal{P}$ and $\mathcal{Q}$ are two hod premice. Then $\mathcal{P} \subseteq_{\text{hod}} \mathcal{Q}$ if there is $\alpha \leq \lambda^Q$ such that $\mathcal{P} = \mathcal{Q}(\alpha)$. We say then that $\mathcal{P}$ is a hod initial segment of $\mathcal{Q}$. We say $(\mathcal{P}, \Sigma)$ is a hod pair if $\mathcal{P}$ is a hod premouse and $\Sigma$ is a strategy for $\mathcal{P}$ (acting on countable stacks of countable normal trees) such that $\Sigma^P \subseteq \Sigma$ and this fact is preserved under $\Sigma$-iterations. Typically, we will construct hod pairs $(\mathcal{P}, \Sigma)$ such that $\Sigma$ has hull condensation, branch condensation, and is $\Gamma$-fullness preserving for some pointclass $\Gamma$.

We say that $\gamma$ is a cutpoint of $\mathcal{P}$ if there is no extender $E$ on the $\mathcal{P}$-sequence such that $\text{crt}(\kappa) < \gamma < \text{lh}(E)$, and that $\gamma$ is a strong cutpoint of $\mathcal{P}$ if there is no extender $E$ on the $\mathcal{P}$-sequence such that $\text{crt}(\kappa) \leq \gamma < \text{lh}(E)$.

Suppose $(\mathcal{Q}, \Sigma)$ is a hod pair such that $\Sigma$ has hull condensation. We say $\mathcal{P}$ is a $(\mathcal{Q}, \Sigma)$-hod premouse if there are an ordinal $\lambda^P$ and sequences $((\mathcal{P}(\alpha), \Sigma^P_\alpha) | \alpha < \lambda^P)$ and $(\delta^P_\alpha | \alpha \leq \lambda^P)$ such that

1. $(\delta^P_\alpha | \alpha \leq \lambda^P)$ is increasing and continuous and if $\alpha$ is a successor ordinal then $\mathcal{P} \models \delta^P_\alpha$ is Woodin;

2. every Woodin cardinal or limit of Woodin cardinals of $\mathcal{P}$ is of the form $\delta^P_\alpha$ for some $\alpha$;

3. $\mathcal{P}(0) = L_{\omega}(\mathcal{P}|\delta_0)^P$ (so $\mathcal{P}(0)$ is a $\Sigma$-premouse built over $\mathcal{Q}$); for $\alpha < \lambda^P$, $\mathcal{P}(\alpha + 1) = (L_{\omega}\mathcal{P}(\delta_{\alpha + 1})^P);^{39}$ for limit $\alpha \leq \lambda^P$, $\mathcal{P}(\alpha) = (L_{\omega}^{\oplus_{\beta<\alpha}\Sigma_{\beta}}(\mathcal{P}|\delta_\alpha))^P$;

4. $\mathcal{P} \models \Sigma \cap \mathcal{P}$ is a $(\omega, o(\mathcal{P}), o(\mathcal{P}))$-strategy for $\mathcal{Q}$ with hull condensation;

5. $\mathcal{P} \models \Sigma^P_\alpha$ is a $(\omega, o(\mathcal{P}), o(\mathcal{P}))$-strategy for $\mathcal{P}(\alpha)$ with hull condensation;

6. if $\alpha < \beta < \lambda^P$ then $\Sigma^P_\beta$ extends $\Sigma^P_\alpha$.

Inside $\mathcal{P}$, the strategies $\Sigma^P_\alpha$ act on stacks above $\mathcal{Q}$ and every $\Sigma^P_\alpha$ iterate is a $\Sigma$-premouse. Again, we write $\delta^P$ for $\delta^P_{\lambda^P}$ and $\Sigma^P = \oplus_{\beta<\lambda^P}\Sigma^P_\beta$. We say $(\mathcal{P}, \Lambda)$ is a $(\mathcal{Q}, \Sigma)$-hod pair if $\mathcal{P}$ is a $(\mathcal{Q}, \Sigma)$-hod premouse and $\Lambda$ is a strategy for $\mathcal{P}$ such that $\Sigma^P \subseteq \Lambda$ and this fact is preserved under $\Lambda$-iterations. The reader should consult [9] for the definition of $B(\mathcal{Q}, \Sigma)$ and $I(\mathcal{Q}, \Sigma)$. Roughly speaking, $B(\mathcal{Q}, \Sigma)$ is the collection of all hod pairs which are strict hod initial segments of a $\Sigma$-iterate of $\mathcal{Q}$ and $I(\mathcal{Q}, \Sigma)$ is the collection of all $\Sigma$-iterates of $\mathcal{Q}$. In the case $\lambda^P$ is limit, the pointclass $\Gamma(\mathcal{Q}, \Sigma)$ is the collection of $A \subseteq \mathbb{R}$ such that $A$ is Wadge reducible to some $\Psi$ for which there is some $\mathcal{R}$ such

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39 $\mathcal{P}(\alpha + 1)$ is a $(g$-organized) $\Sigma_\alpha$-premouse in the sense defined above.

40 This just means $\Sigma^P_\alpha$ acts on all stacks of $\omega$-maximal, normal trees in $\mathcal{P}$. 

27
that \((\mathcal{R}, \Psi) \in B(Q, \Sigma)\). See [9] for the definition of \(\Gamma(Q, \Sigma)\) in the case \(\lambda^Q\) is a successor ordinal. If \((\mathcal{P}, \Sigma)\) is a hod pair, and \(\vec{T}\) is according to \(\Sigma\) with last model \(Q\), then we write \(\Sigma_{Q, \vec{T}}\) for the \(\vec{T}\)-tail strategy of \(Q\) induced by \(\Sigma\), i.e. \(\Sigma_{Q, \vec{T}}(\vec{U}) = \Sigma(\vec{T}^{-\vec{U}})\).

**Definition 3.34** (Branch condensation). Let \((\mathcal{P}, \Sigma)\) be a hod pair. We say that \(\Sigma\) has branch condensation if for any \(\Sigma\) iterate \(Q\) of \(P\), letting \(k : P \to Q\) be the iteration map, for any stack \(\vec{T}\) according to \(\Sigma\) and any cofinal non-dropping branch \(b\) of \(\vec{T}\), letting \(i = i_b\), if there is an embedding \(j : M^\vec{T}_b \to Q\) such that \(k = j \circ i\), then \(b = \Sigma(\vec{T})\).

**Definition 3.35** (\(\Gamma\)-Fullness preservation). Suppose \((\mathcal{P}, \Sigma)\) is a hod pair such that \(\mathcal{P} \in HC\) and \(\Gamma\) is a pointclass. We say \(\Sigma\) is \(\Gamma\)-fullness preserving if the following holds for all \((Q, \vec{T}) \in I(\mathcal{P}, \Sigma)\).

1. For all limit \(\alpha < \lambda^Q\), letting \(R = Q(\alpha)\), then
   \[
   \mathcal{R} = L_{p^\omega}^{\Gamma, \beta < \alpha \Sigma_{R(\beta)}, \vec{T}}(\mathcal{R}|\delta \mathcal{R}).
   \]
2. For all successor \(\alpha < \lambda^Q\), letting \(R = Q(\alpha)\) and \(\beta = \alpha - 1\),
   \[
   \mathcal{R} = L_{p^\omega}^{\Gamma, \Sigma_{R(\beta)}, \vec{T}}(\mathcal{R}|\delta \mathcal{R}).
   \]
3. If \(\eta\) is a cardinal strong cutpoint of \(Q\), letting \(\alpha\) be the largest such that \(Q(\alpha) \smallfrown Q|\eta\) and \(\mathcal{R} = Q(\alpha)\), then
   \[
   Q|(\eta^+)^Q = L_{p^\omega}^{\Gamma, \Sigma_{R(\beta)}, \vec{T}}(Q|\eta).
   \]
4. Furthermore, letting for \(\alpha + 1 \leq \lambda^Q\),
   \[
   U_{Q(\alpha), \Sigma} = \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{R} \text{ codes a countable set } a \text{ and } y \text{ codes a sound } \Sigma_{Q(\alpha)}\text{-mouse } M \text{ over } a \text{ whose unique strategy is in } \Gamma \text{ such that } \rho(M) = a\},
   \]
   and
   \[
   W_{Q(\alpha), \Sigma} = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in U_{Q(\alpha), \Sigma} \text{ and } z \text{ codes an iteration tree on the mouse } M \text{ coded by } y\},
   \]
   then whenever \((\vec{U}, \mathcal{R}) \in I(Q(\alpha + 1), \Sigma_{Q(\alpha + 1)}, \vec{T})\) such that \(\vec{U}\) only uses extenders with critical points above \(\delta^Q_\alpha\) and its images along branch embeddings of \(\vec{U}\), we have
   \[
   \pi(\vec{U})(f_A(Q)) = f_A(\mathcal{R}),
   \]
   where \(A = U_{Q(\alpha), \Sigma} \oplus W_{Q(\alpha), \Sigma}\) and \(f_A\) is defined in (3.1) below.
Remark 3.36. In [9], clauses (1)–(3) comprise the definition of fullness preservation of Σ; if in addition, clause (4) holds for Σ, then Σ is said to be super fullness preserving (with respect to Γ). We simplify the terminology by combining these two notions into one definition.

Under AD and the hypothesis that there are no models of AD+$\Theta$ is regular,” [9] constructs hod pairs that are fullness preserving and have branch condensation (see [9] for a full discussion of these notions). Such hod pairs are particularly important for our computation as they are points in the direct limit system giving rise to HOD of AD+$\Theta$ models. Under AD+$\Theta$, for hod pairs $(M_\Sigma, \Sigma)$, if Σ is a strategy with branch condensation and $\vec{T}$ is a stack on $M_\Sigma$ with last model $N$, then $\Sigma_{N,\vec{T}}$ is independent of $\vec{T}$. Therefore, later on we will omit the subscript $\vec{T}$ from $\Sigma_{N,\vec{T}}$ whenever Σ is a strategy with branch condensation and $M_\Sigma$ is a hod mouse. In a core model induction, at the moment $(M_\Sigma, \Sigma)$ is constructed we don’t quite have an AD+$\Theta$-model $M$ such that $(M_\Sigma, \Sigma) \in M$, but we do know that every $(R, \Lambda) \in B(M_\Sigma, \Sigma)$ belongs to such a model. We then can show (using our hypothesis) that $(M_\Sigma, \Sigma)$ belongs to an AD+$\Theta$-model.

We briefly review definitions and notations related to the analysis of stacks in [9, Section 6.2]; see [9, Section 6.2] for a more detailed discussion. These notions will be useful in Section 4.4. Suppose $\mathcal{P}$ is a hod premouse and $\vec{T}$ is a stack on $\mathcal{P}$. Let $\mathcal{S}$ be a model that appears in $\vec{T}$. By $\vec{T}_{\leq S}$ we mean the part of $\vec{T}$ up to and including $S$ (according to the tree order of $\vec{T}$), we define $\vec{T}_{\geq S}, \vec{T}_{< S}, \vec{T}_{> S}$ similarly. We let $(M_\alpha, T_\alpha : \alpha < \eta)$ be the normal components of $\vec{T}$, i.e. $M_0 = \mathcal{P}$, $T_\alpha$ is a normal tree on $M_\alpha$, and $M_\alpha+1 = M_{T_\alpha}$. We say $R$ is a terminal node of $\vec{T}$ if for some $\alpha, \beta$, $R = M_{T_\beta}^{T_\alpha}$ and $\pi_{T_\alpha}^{T_\beta}$ is defined. We say $R$ is a non-trivial terminal node of $\vec{T}$ if letting $(\alpha, \beta)$ witness that $R$ is a terminal node of $\vec{T}$, the extender $E_{T_\beta}^{T_\alpha}$ is applied to $R$ in the tree $T_\alpha$ to obtain the model $M_{T_\beta+1}^{T_\alpha}$. We write $tn(\vec{T})$ for the set of terminal nodes of $\vec{T}$ and $ntn(\vec{T})$ for the set of non-trivial terminal nodes of $\vec{T}$.

For $Q, R \in tn(\vec{T})$, we write $Q \prec \vec{T} R$ if the Q-to-R iteration embedding in $\vec{T}$ exists, and we write $\pi_{Q, R}$ for this embedding. We write $Q \prec^{T, s} R$ if letting $\vec{U}$ be the part of $\vec{T}$ between $Q$ and $R$, then $\vec{U}$ is an iteration on $Q$. We write $\pi_{Q, R}$ for $\vec{U}$.

Let $C \subseteq tn(\vec{T})$. We say C is linear (strongly linear respectively) if C is linearly ordered by $\prec^{\vec{T}}$ ($\prec^{T, s}$ respectively). We say C is closed if C is strongly linear and whenever $\alpha$ is a limit point of C, then letting $R$ be the direct limit of $C \upharpoonright \alpha$ (under the iteration embeddings), we have $R \in C$. We say C is cofinal if for every $S \in \vec{T}$, there are $Q, R \in C$ such that $Q \prec^{T, s} R$ and S is in $\vec{T}_{Q, R}$. Note that if $\vec{T}$ doesn’t have a last model, but there is a strongly closed and cofinal $C \subseteq tn(\vec{T})$, then C uniquely determines a cofinal branch of $\vec{T}$. If such a C doesn’t exist, then $\eta$ is a successor ordinal, say $\eta = \alpha + 1$. Let $U = \vec{T}_\alpha$ and $D = \{S \in tn(U) : U_{\geq S}$ is a tree on $S\}$. In this case $U$ has a $\prec^{T, s}$-largest element and we write $S_{T_\alpha}$ for this element. Then $\vec{T}_{S_{T_\alpha}}$ is a normal tree based on $S_{T_\alpha}(\beta + 1)$ and above $\delta_{\beta}^{S_{T_\alpha}}$ for some $\beta < \lambda_{S_{T_\alpha}}$.

3.4. HOD and HOD$_\Sigma$ under AD$^+$

Suppose $\Sigma$ is an iteration strategy of some hod mouse $Q$ and suppose $\Sigma$ is fullness preserving (see [9]) and has branch condensation. Assume further that $V = L(\emptyset(\mathbb{R}))$ and $MC(\Sigma)$ holds and $\Theta = \theta_{\Sigma}$. 29
Definition 3.37 \((S(\Gamma, \Sigma) \text{ and } F(\Gamma, \Sigma))\). Suppose \(\Gamma\) is a pointclass. Let \(S(\Gamma, \Sigma) = \{Q : Q \text{ is } \Sigma\text{-suitable}\}\). Also, we let \(F(\Gamma, \Sigma)\) be the set of functions \(f\) such that \(\text{dom}(f) = S(\Gamma, \Sigma)\) and for each \(P \in S(\Gamma, \Sigma)\), \(f(P) \subseteq P\) and \(f(P)\) is amenable to \(P\), i.e., for every \(X \in P\), \(X \cap f(P) \in P\).

We let \(\Gamma = \varphi(\mathbb{R})\) and for the duration of this subsection, we drop \(\Gamma\) from our notation whenever it is unambiguous to do so. Thus, a \(\Sigma\)-suitable premouse is a \(\Sigma\text{-}\Gamma\)-suitable premouse etc. We remark that by \([11]\),

\[
V = L(Lp^\Sigma(\mathbb{R})).
\]

Also, we allow for the case \((P, \Sigma) = (\emptyset, \emptyset)\), in which case \(V = L(Lp(\mathbb{R}))\) and \(\text{HOD}_\Sigma = \text{HOD}\).

Suppose \(P\) is \(\Sigma\)-suitable and \(A \subseteq \mathbb{R}\) is \(\text{OD}_\Sigma\). We say \(P\) weakly term captures \(A\) if letting \(\delta = \delta^P\), for each \(n < \omega\) there is a term relation \(\tau \in P^{\text{Coll}(\omega, (\delta+n)^P)}\) such that for comeager many \(P\)-generics \(g \subseteq \text{Coll}(\omega, (\delta+n)^P)\), we have \(\tau_g = P[g] \cap A\). We say \(P\) term captures \(A\) if the equality holds for all generics. The following lemma is essentially due to Woodin and the proof for mice can be found in \([16]\).

Lemma 3.38. Suppose \(P\) is \(\Sigma\)-suitable and \(A \subseteq \mathbb{R}\) is \(\text{OD}_\Sigma\). Then \(P\) weakly term captures \(A\). Moreover, there is a \(\Sigma\)-suitable \(Q\) which term captures \(A\).

Given a \(\Sigma\)-suitable \(P\) and an \(\text{OD}_\Sigma\) set of reals \(A\), we let \(\tau_{A,n}^P\) be the standard name for a set of reals in \(P^{\text{Coll}(\omega, (\delta+n)^P)}\) witnessing the fact that \(P\) weakly captures \(A\). We let

\[
f_A(P) = \{\tau_{A,n}^P : n < \omega\}. \tag{3.1}
\]

Let \(F_{\Sigma, \text{od}} = \{f_A : A \subseteq \mathbb{R} \land A \in \text{OD}_\Sigma\}\). Clearly, \(F_{\Sigma, \text{od}} \subseteq F(\Gamma, \Sigma)\).

The following lemma is one of the most fundamental lemmas used to compute \(\text{HOD}\) and it is originally due to Woodin. Again, the proof can be found in \([16]\). See also \([16]\) for detailed discussions of related standard notions like \(f\)-iterability and \(f\)-quasi-iterability.

Theorem 3.39. For each \(f \in F_{\Sigma, \text{od}}\), there is a \(\Sigma\)-suitable premouse \(P\) which is strongly \(f\)-iterable.

To save some ink, in what follows, we will sometimes say \(A\)-iterable instead of \(f_A\)-iterable and similarly for other notions. Also, we will use \(A\) in our subscripts instead of \(f_A\).

Given \(P \in S(\Gamma, \Sigma)\) and \(f \in F_{\Sigma, \text{od}}\) we let \(f_n(P) = f(P) \cap P|((\delta^P+n)^P}\). Then \(f(P) = \bigcup_{n < \omega} f_n(P)\). We also let

\[
\gamma_f^P = \sup(\delta^P \cap \text{Hull}_f^P(\{f_n(P) : n < \omega\})).
\]

Notice that

\[
\gamma_f^P = \delta^P \cap \text{Hull}_f^P(\gamma_f^P \cup \{f_n(P) : n < \omega\}).
\]

We then let

\[
H_f^P = \text{Hull}_f^P(\gamma_f^P \cup \{f_n(P) : n < \omega\}).
\]
If $\mathcal{P} \in S(\Gamma, \Sigma)$, $f \in F_{\Sigma, \text{od}}$, and $i : \mathcal{P} \to \mathcal{Q}$ is an embedding, then we let $i(f(\mathcal{P})) = \bigcup_{n < \omega} i(f_n(\mathcal{P}))$.

The following are the next block of definitions that routinely generalize into our context: (1) $(f, \Sigma)$-iterability, (2) $\vec{b} = (b_k : k < m)$ witnesses $(f, \Sigma)$-iterability for $\vec{T} = (T_k, P_k : k < m)$, and (3) strong $(f, \Sigma)$-iterability.

If $\mathcal{P}$ is strongly $(f, \Sigma)$-iterable and $\vec{T}$ is a $(\Gamma, \Sigma)$-correctly guided finite stack on $\mathcal{P}$ with last model $\mathcal{R}$ then we let

$$
\pi^\Sigma_{P, \mathcal{R}, f} : H^P_f \to H^\mathcal{R}
$$

be the embedding given by any $\vec{b}$ which witnesses the $(f, \Sigma)$-iterability of $\vec{T}$, i.e., fixing $\vec{b}$ which witnesses $f$-iterability for $\vec{T}$,

$$
\pi^\Sigma_{P, \mathcal{R}, f} = \pi_{\vec{T}, \vec{b}}|_{H^P_f}.
$$

Clearly, $\pi^\Sigma_{P, \mathcal{R}, f}$ is independent of $\vec{T}$ and $\vec{b}$. Here we keep $\Sigma$ in our notation for $\pi^\Sigma_{P, \mathcal{R}, f}$ because it depends on a $(\Gamma, \Sigma)$-correct iteration. It is conceivable that $\mathcal{R}$ might also be a $(\Gamma, \Lambda)$-correct iterate of $\mathcal{P}$ for another $\Lambda$, in which case $\pi^\Sigma_{P, \mathcal{R}, f}$ might be different from $\pi^\Lambda_{P, \mathcal{R}, f}$. However, the point is that these embeddings agree on $H^P_f$.

Given a finite sequence of functions $\vec{f} = (f_i : i < n)$ in $F_{\Sigma, \text{od}}$, we let $\oplus_{i<n} f_i \in F_{\Sigma, \text{od}}$ be the function given by $(\oplus_{i<n} f_i)(\mathcal{P}) = (f_i(\mathcal{P}) : i < n)$. We set $\oplus \vec{f} = \oplus_{i<n} f_i$.

We let $F = F_{\Sigma, \text{od}}$ and

$$
\mathcal{I}_{F, \Sigma} = \{ (\mathcal{P}, \vec{f}) : \mathcal{P} \in S(\Gamma, \Sigma), \vec{f} \in F^{<\omega} \text{ and } \mathcal{P} \text{ is strongly } \oplus \vec{f}-\text{iterable} \}
$$

and

$$
\mathcal{F}_{F, \Sigma} = \{ H^P_f : (\mathcal{P}, f) \in \mathcal{I}_{F, \Sigma} \}.
$$

We then define $\preceq_{F, \Sigma}$ on $\mathcal{I}_{F, \Sigma}$ by letting $(\mathcal{P}, \vec{f}) \preceq_{F, \Sigma} (\mathcal{Q}, \vec{g})$ iff $\mathcal{Q}$ is a $\Sigma$-correct iterate of $\mathcal{P}$ and $\vec{f} \subseteq \vec{g}$. Given $(\mathcal{P}, \vec{f}) \preceq_{F, \Sigma} (\mathcal{Q}, \vec{g})$, we have

$$
\pi^\Sigma_{P, \mathcal{Q}, \vec{f}} : H^P_{\oplus \vec{f}} \to H^\mathcal{Q}_{\oplus \vec{f}}
$$

Notice that $\preceq_{F, \Sigma}$ is directed. Let then

$$
\mathcal{M}_{\infty, F, \Sigma}
$$

be the direct limit of $(\mathcal{F}_{F, \Sigma}, \preceq_{F, \Sigma})$ under the maps $\pi^\Sigma_{P, \mathcal{Q}, \vec{f}}$. Given $(\mathcal{P}, \vec{f}) \in \mathcal{I}_{F, \Sigma}$, we let $\pi^\Sigma_{P, \mathcal{Q}, f_\infty} : H^P_{\oplus \vec{f}} \to \mathcal{M}_{\infty, F, \Sigma}$ be the direct limit embedding.

Let $\mathcal{M}_{\infty} = \mathcal{M}_{\infty, F, \Sigma}$.

**Theorem 3.40** (Woodin, [16]). $\delta^{\mathcal{M}_{\infty}} = \Theta$, $\mathcal{M}_{\infty} \in \text{HOD}_\Sigma$, and

$$
\mathcal{M}_{\infty}|\Theta = (V^{\text{HOD}_\Sigma}_{\Theta}, \bar{E}^{\mathcal{M}_{\infty}}|_{\Theta}, S^{\mathcal{M}_{\infty}}, \in),
$$

where $S^{\mathcal{M}_{\infty}}$ is the predicate of $\mathcal{M}_{\infty}$ describing $\Sigma$. 31
Remark 3.41. In some of the arguments below, for convenience, we actually use the “one cardinal” version of suitability. More precisely, for \((P, f) \in I_F\), we consider direct limits of \((\hat{P}, \hat{f})\) where \(\delta = \delta^P, \hat{P} = P|_{(\delta^+)^P}, \) and \(\hat{f} = \tilde{f}(P) \cap P|_{(\delta^+)^P}.\) We define \(\gamma_{\hat{P}, \hat{f}} = \sup(\delta^P \cap Hull_1^P(\{f_0(P)\}))\) etc.

We let \(\hat{M}_\infty\) be the direct limit of such pairs \((\hat{P}, \hat{f})\). Then it is easy to see also that \(\hat{M}_\infty|_\Theta = (V_{\Theta}^{HOD}, \hat{E}_{\infty}|_{\Theta}, S_{\infty}|_{\Theta})\) etc.

Finally, if \(a \in H_{\omega_1}\) is self-wellordered then we could define \(M_\infty(a)\) by working with \(\Sigma\)-suitable premice over \(a\). Everything we have said about \(\Sigma\)-suitable premice can also be said about \(\Sigma\)-suitable premice over \(a\), and in particular the equivalent of Theorem 3.40 can be proven using \(\text{HOD}(\Sigma, a) \cup \{a\}\) instead of \(\text{HOD}\) and \(M_\infty(a)\) instead of \(M_\infty\).

[9] computes \(\text{HOD}\) (up to \(\Theta\)) in models of \((V = L(\wp(\mathbb{R}))) + \text{SMC} + \text{AD}_\mathbb{R}\) below \(\text{AD}_\mathbb{R} + \text{“}\Theta \text{ is regular”}\) by exhibiting a hod premouse \(M_\infty\) satisfying

1. \(M_\infty \in \text{HOD}\).
2. \(M_\infty\) is a hod premouse.
3. \(M_\infty|_\Theta = (V_\Theta^{HOD}, \hat{E}_{\infty}|_{\Theta}, S_{\infty}|_{\Theta})\), where \(S_{\infty}|_{\Theta}\) is the predicate for strategies of hod initial segments of \(M_\infty|_\Theta\).

Here \(\text{SMC}\) is Strong Mouse Capturing, which is the statement that for any \(x, y \in \mathbb{R}, \) if \(x \in OD_{y, \Sigma}\) where \((P, \Sigma)\) is a hod pair such that \(\Sigma\) has branch condensation and is fullness preserving, then \(x\) is in a \(\Sigma\)-mouse \(M\) over \(y\). We call \(M_\infty\) the hod limit. Here \(M_\infty = \bigcup_{(Q, \Lambda)} M_\infty(Q, \Lambda)\), where \((Q, \Lambda)\) is a hod pair with branch condensation and is fullness preserving and \(M_\infty(Q, \Lambda)\) is the direct limit of all (non-dropping) \(\Lambda\)-iterates of \(Q\).

4. THE PROOF OF THEOREM 1.6

We proceed with the proof of Theorem 1.6. (In the next section we will outline the proof of Theorem 1.10, focusing only on the changes needed to make this proof work with the hypothesis of that theorem.) Throughout this section, we assume DI. We assume for contradiction:

\((\dagger):\) There is no inner model of \(\text{AD}_\mathbb{R} + \text{“}\Theta \text{ is regular”}\) in \(V\).

4.1. The maximal pointclass of \(\text{AD}_\mathbb{R}^+\)

Definition 4.1 (Maximal Pointclass of \(\text{AD}_\mathbb{R}^+\)). \(\Gamma = \{A \subseteq \mathbb{R} : L(A, \mathbb{R}) \models \text{AD}_\mathbb{R}^+\}\). \(\dagger\)

\((\dagger)\) implies that for any \(A, B \in \Gamma,\) either \(A \in L(B, \mathbb{R})\) or \(B \in L(A, \mathbb{R});\) in particular, \(A\) and \(B\) are Wadge comparable. Since we assume \((\dagger)\), the theory of hod mice in [9] applies throughout our constructions. All notions related to hod mice used in this paper come from [9]; some of those notions have been summarized in the previous section. [9] shows that for any \(A \in \Gamma,\)

\[L(A, \mathbb{R}) \models \text{SMC}.\]
Remark 4.2. From [9] and our assumption (†), there are no sets $A, B \subset \mathbb{R}$ such that $L(A, \mathbb{R}) \models \text{AD}^+$, $L(B, \mathbb{R}) \models \text{AD}^+$, $A \notin L(B, \mathbb{R})$ and $B \notin L(A, \mathbb{R})$. We call such models divergent models of $\text{AD}^+$. In fact, [9] shows that if there are such divergent models of $\text{AD}^+$, then letting $\Lambda = L(A, \mathbb{R}) \cap L(B, \mathbb{R})$, then there is a Wadge initial segment $\Omega$ of $\Lambda$ such that $L(\Omega, \mathbb{R}) \models \text{AD}^R + \text{“}\Theta \text{ is regular.”}$

Suppose $(\mathcal{P}, \Sigma)$ is a hod pair or a $\Lambda$-hod pair for some hod mouse strategy $\Lambda$, and $\mathcal{M}$ is a $\Sigma$-mouse, for example $\mathcal{M} = M_{\Sigma}^{\Sigma_1 \sharp}$. We will tacitly assume that $\mathcal{P}$ and $\mathcal{M}$ are in the $g$-organized hierarchy; the main reason for this is so that generic extensions of these objects can be organized into $g$-organized premice. For instance, suppose $\mathcal{M}[g]$ can be regarded as a $g$-organized $\Sigma$-mouse (a $g^{\Sigma}$-mouse) over $g$ (see [9] or [18]). The only exception is when $\mathcal{M} = M_{\Sigma}^{\Sigma_1 \sharp}$; then $\mathcal{M}$ is a (standard) $\Sigma$-premouse.$^{41}$

Again, let us mention that we will write $L^p_{\Sigma}(\mathbb{R})$ for $L^p_{\Sigma}(\mathbb{R}, \Sigma \upharpoonright \text{HC})$. $^{42}$

4.2. Outline

In the next two sections we will analyze the complexity of $\Gamma$, ultimately showing that there is some Wadge initial segment $\Omega$ of $\Gamma$ (possibly $\Omega = \Gamma$) such that $L(\Omega, \mathbb{R}) \models \text{AD}^R + \text{“}\Theta \text{ is regular.”}$ There are two major cases. We summarize the key points of each case below before jumping into the details.

(i) The successor case (Section 4.3): we first show that if $(\mathcal{P}, \Sigma) \in \Gamma$ (possibly $\Sigma$ may be $\emptyset$) is a hod pair such that $\Sigma$ is $\Gamma$-fullness preserving and has branch condensation, then $L^p_{\Sigma}(\mathbb{R}) \models \text{AD}^+$, and therefore $\varphi(\mathbb{R}) \cap L^p_{\Sigma}(\mathbb{R}) \subset \Gamma$. This is via a standard core model induction argument similar to that showing $\text{AD}$ holds in $L(\mathbb{R})$ under either hypothesis ([16, 32]). One wrinkle that appears in the case that $\Sigma \neq \emptyset$ is that one needs to show $\mathcal{M}^{\Sigma_1 \sharp}_1$ exists before being able to define $L^p_{\Sigma}(\mathbb{R})$ as done in [18]. The argument showing that $\mathcal{M}^{\Sigma_1 \sharp}_1$ exists is given in Theorem 4.5. The model $W$, a canonical model over $\mathbb{R}^{\Sigma}$ (a non-self-wellordered set) closed under $\Sigma$ when we construct the core model $K^{\Sigma}$ defined in the proof of 4.5, is crucial. In the case where $\lambda^{\mathcal{P}}$ is a limit ordinal of measurable cofinality in $\mathcal{P}$, there does not seem to be an analog of $W$ in [16, 32].$^{43}$

As part of the induction, we maintain the hypothesis (**) defined in Section 4.3 for $\Sigma$-cmi operators $J$. This allows us to adapt the standard arguments in [16, 32] to show $L^p_{\Sigma}(\mathbb{R}) \models \text{AD}^+$.

In Section 4.3.1, we adapt the argument in [32] to show that there is a self-justifying system $\mathcal{A}$ consisting of sets Wadge cofinal in $L^p_{\Sigma}(\mathbb{R})$, and a $\Sigma$-suitable pair $(Q, \Lambda)$ where $\Lambda$ is the

$^{41}$We need this standard $\Sigma$-premouse to even define the $g$-organized $\Sigma$-premice hierarchy.

$^{42}$We need to know also that $\Sigma$ is self-scaled so that the scales analysis works, but see the proof of Theorem 4.8.

$^{43}$In the case that $\lambda^{\mathcal{P}} = 0$ or is a successor ordinal, one can define the model $W$ by closing under the term relations for a self-justifying system guiding $\Sigma$. 
strategy guided by $\mathcal{A}$. Therefore, $\Lambda$ is $\Gamma$-fullness preserving and has branch condensation and $\Lambda \not\in \text{Lp}^\Sigma(\mathbb{R})$. We can then show $\text{Lp}^\Lambda(\mathbb{R}) \models \text{AD}^+$ and therefore $\Lambda \in \Gamma$.

(ii) The limit case (Section 4.4): assuming $(\dagger)$ and letting $\mathcal{H}$ and $\Sigma$ be defined as in Section 4.4, we use the generic embedding $j : V \rightarrow M$ induced by a $V$-generic $G \subseteq \text{Coll}(\omega, \omega_1)$ to derive a nice strategy $\Lambda$ for $\mathcal{H}$ in $M$. The strategy $\Lambda$ is $j(\Gamma)$-fullness preserving, has branch condensation, and most importantly, if $\Gamma(\mathcal{H}, \Lambda) \subseteq j(\Gamma)$, then letting $\mathcal{M}_\infty(\mathcal{H}, \Lambda)$ be the direct limit of non-dropping iterates of $(\mathcal{H}, \Lambda)$ in $j(\Gamma)$, we have $\mathcal{M}_\infty(\mathcal{H}, \Lambda) = \mathcal{H}(\delta)$ where $\delta = \delta^{\mathcal{M}_\infty(\mathcal{H}, \Lambda)}$, and there is a factor map $\sigma : \mathcal{M}_\infty(\mathcal{H}, \Lambda) \rightarrow j(\mathcal{H})$ such that $\text{crt}(\sigma) = \delta$. This property is a consequence of the $j$-condensation lemma, Theorem 4.34. This result is crucial here and its variations are important in many other arguments (cf. [10, 12, 29]).

Now there are two cases. Suppose first that $\Gamma(\mathcal{H}, \Lambda) = j(\Gamma)$. Then by elementarity, in $V$ there is a hod pair $(\mathcal{P}, \Sigma)$ such that $\Gamma(\mathcal{P}, \Sigma) = \Gamma$; in particular, $\Sigma \notin \Gamma$. By a core model induction as in the successor case, $\text{Lp}^\Sigma(\mathbb{R}) \models \text{AD}^+$. This implies $\Sigma \in \Gamma$. Contradiction. Otherwise, $\Gamma(\mathcal{H}, \Lambda) \subseteq j(\Gamma)$. Therefore $\sigma$ exists and $\delta$ is a regular cardinal which is a limit of Woodin cardinals in $\mathcal{M}_\infty(\mathcal{H}, \Lambda)$. By standard arguments, $L(j(\Gamma) \upharpoonright \delta, \mathbb{R}^M) \models \text{AD}_\mathbb{R} + \text{“$\Theta$ is regular.”}$ This is again a contradiction, so $(\dagger)$ fails.

4.3. The Successor Case

Suppose $(\mathcal{P}, \Sigma) \in \Gamma$ is a hod pair such that $\Sigma$ has branch condensation and is fullness preserving in $\Gamma$. The main goal of this section is to outline the proof that

$$\varphi(\mathbb{R}) \cap \text{Lp}^\Sigma(\mathbb{R}) \subseteq \Gamma.$$  \hspace{1cm} (4.1)

See [31], [32], or [29] for a detailed proof of the core model induction through $\text{Lp}^\Sigma(\mathbb{R})$, showing that $\text{Lp}^\Sigma(\mathbb{R}) \models \text{AD}^+$. Here we give an outline, focusing on the main points of the argument that are particular to our hypothesis.

First, we need to extend $\Sigma$ to a strategy with domain $H^V_{c^+}$, to make sense of $\text{Lp}^\Sigma(\mathbb{R})$; at this point, we just know that $\Sigma$ acts on countable stacks on $\mathcal{P}$ (and $\mathcal{P}$ is countable). Let $\mathbb{P} = \mathcal{P}_\mathcal{I}$ for the ideal $\mathcal{I}$ as in $\text{DI}$. Again, recall that under $\text{DI}$, $c^+ = \omega_2$. Let $G \subseteq \mathbb{P}$ be $V$-generic and $j = j_G : V \rightarrow \text{Ult}(V, G) = M$

be the corresponding ultrapower map; also let $g \subseteq \text{Coll}(\omega, \omega_1)$ be the $V$-generic that corresponds to $G$ (see Section 2.2). We note that $V[G] = V[g]$, $j(\omega_1) = c^+ = \omega_2$, and $(M^{\omega_2})^{V[G]} \subseteq M$. Let $\mathcal{M}(\mathcal{P}, \Sigma)$ be the direct limit of $\Sigma$-iterates of $\mathcal{P}$ and $i^{V}_{\mathcal{P}, \Sigma} : \mathcal{P} \rightarrow \mathcal{M}(\mathcal{P}, \Sigma)$ be the direct limit map.

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\textsuperscript{44} This argument allows us to construct $(\mathcal{Q}, \Lambda)$ without the technical hypothesis $\text{HI}(c)$ in Ketchersid’s thesis. See [16, 6] for an alternative argument constructing $(\mathcal{Q}, \Lambda)$ that uses a seemingly stronger hypothesis.

\textsuperscript{45} In [31], we use the notation $\text{Lp}^\Sigma(\mathbb{R}, \Sigma \upharpoonright \text{HC})$ for $\text{Lp}^\Sigma(\mathbb{R})$. We simplify the notations here for brevity. In [32], the third author deals with the case $(\mathcal{P}, \Sigma) = \emptyset$, showing that $\text{Lp}(\mathbb{R}) \models \text{AD}^+$ and therefore $\Gamma \neq \emptyset$. 

34
Lemma 4.3. \( j(\Sigma) \cap V^{46} \) is in \( V \) and does not depend on the choice of generic \( G \).

Proof. First note that by DI, letting \( \mathcal{H} = \mathcal{M}(\mathcal{P}, \Sigma) \), the restriction \( j \upharpoonright \mathcal{H} \) is in \( V \) and does not depend on \( G \). Furthermore,

\[ j(\mathcal{H}) \in V. \]

This is because \( \mathbb{R}^M = \mathbb{R}^{V[G]} \), and \( j(\Gamma) \) and hence \( j(\mathcal{H}) \) is ordinal definable in \( V[G] \), so by homogeneity we have \( j(\mathcal{H}) \in V \). We note that \( j(\Gamma) \) is definable in \( V[G] \) as the unique \( \text{AD}^+ \) model whose Wadge rank is \( j(\Theta^\Gamma) \); this follows from our smallness assumption (†).47

This implies \( j(\Sigma) \cap V \in V \). This is because for any \( \bar{T} \in M \) according to \( j(\Sigma) \),

\[ j(\Sigma)(\bar{T}) = b \quad \text{and} \quad i_b^\mathcal{T} \text{ exists} \iff \exists \sigma : \mathcal{M}_b^\mathcal{T} \to j(\mathcal{H}) \sigma \circ i_b^\mathcal{T} = j \upharpoonright \mathcal{H} \circ i_\Sigma^\mathcal{T}. \quad (4.2) \]

To see the equivalence, first note that \( j \upharpoonright \mathcal{H} \circ i_\Sigma^\mathcal{T} \) is the direct limit map of \( \mathcal{P} \) into \( j(\mathcal{H}) \) according to \( j(\Sigma) \), and by DI, \( j \upharpoonright \mathcal{H} \circ i_\Sigma^\mathcal{T} \) does not depend on \( G \). Now note that \( \Sigma \subseteq j(\Sigma) \) and \( j(\Sigma) \) has branch condensation in \( M = \text{Ult}(V, G) \).

Let \( H \subseteq j(\mathcal{P}) \) be \( M \)-generic and let \( k : M \to \text{Ult}(M, H) = N \) be the associated ultrapower embedding. Then \( k(j(\Sigma)) \) extends \( j(\Sigma) \) and hence acts on trees (stacks) of length \( < \Theta \) in \( j(L(\Gamma, \mathbb{R})) \); this is because these trees (stacks) are countable in \( N \).

If \( b = j(\Sigma)(\bar{T}) \) then \( \mathcal{H} \) is simply the direct limit map from \( \mathcal{M}_b^\mathcal{T} \) into \( j(\mathcal{H}) \). Conversely, suppose \( b \) and \( \sigma \) are such that \( \sigma \circ i_b^\mathcal{T} = j \upharpoonright \mathcal{H} \circ i_\Sigma^\mathcal{T} \). Choose a stack \( \bar{U} : \mathcal{P} \to j(\mathcal{H}) \) according to \( k(j(\Sigma)) \) with iteration embedding \( j \upharpoonright \mathcal{H} \circ i_\Sigma^\mathcal{T} \). Then by branch condensation of \( j(\Sigma) \), we have \( b = k(j(\Sigma))(\bar{T}) \) and hence \( b = j(\Sigma)(\bar{T}) \).48 The right hand side of the equivalence gives a definition of \( j(\Sigma) \upharpoonright V \) in \( j(L(\Gamma, \mathbb{R})) \) from \( j \upharpoonright \mathcal{H} \circ i_\Sigma^\mathcal{T} \), which can be construed as \( j(s) \) for some countable sequence of ordinals \( s \). By DI, it follows that the restriction \( j(\Sigma) \upharpoonright V \) is in \( V \) and does not depend on \( G \).

In the case \( i_b^\mathcal{T} \) doesn’t exist, one can compute \( b \) from the relevant \( \mathcal{Q} \)-structures in \( j(\Gamma) \). For instance, suppose \( \mathcal{T} = \mathcal{T}_0 \upharpoonright \mathcal{T}_1 \) where \( \mathcal{T}_0 \) is nondropping with last model \( \mathcal{P} \) and \( \mathcal{T}_1 \) is a normal tree based on \( \mathcal{P}(\alpha) \) for some successor \( \alpha \). Let \( \gamma \) be the Wadge rank of \( j(\Sigma)_{Q(\alpha)} \) in \( j(\Gamma) \) and let \( \Omega = \Sigma_1^2(j(\Sigma)_{Q(\alpha - 1)}) \). By our assumption, \( \mathcal{T}_1 \) is short and has \( \mathcal{Q} \)-structure \( \mathcal{Q}(\mathcal{T}_1) \subseteq C_{\Omega}(\mathcal{M}(\mathcal{T}_1)) \). Let \( c \) be the unique cofinal branch of \( \mathcal{T}_1 \) whose \( \mathcal{Q} \)-structure \( \mathcal{Q}(c, \mathcal{T}_1) = \mathcal{Q}(\mathcal{T}_1) \) and determines \( b \). The general case can be handled similarly.49

We have shown that \( j(\Sigma) \) is definable in \( M \) from \( j(\Gamma) \) (which is itself definable in \( M \)) and a countable sequence of ordinals, namely \( j \upharpoonright \mathcal{H} \circ i_\Sigma^\mathcal{T} \). \( \square \)

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46 This is equal to \( j(\Sigma) \cap H^V \), because the ideal is strong.

47 From (†) and elementarity, we know that there are no divergent models of \( \text{AD}^+ \) in \( M \). Therefore, if there are divergent models of \( \text{AD}^+ \) in \( V[G] \), they have to “diverge above \( j(\Gamma) \).” More precisely, suppose \( L(A, \mathbb{R}), L(B, \mathbb{R}) \) are divergent models of \( \text{AD}^+ \) in \( V[G] \), then \( j(\Gamma) \subseteq L(A, \mathbb{R}) \cap L(B, \mathbb{R}) \); this follows from [9] as in Remark 4.2. So regardless of whether there are divergent models of \( \text{AD}^+ \) in \( V[G] \), we can still ordinal define in \( V[G] \) \( j(\Gamma) \) as above.

48 Note that \( \bar{U} \) need not be in \( M \); generally, \( \bar{U} \in N \) and is countable there. Also, \( j(\Sigma) \) agrees with \( k(j(\Sigma)) \) on all stacks in their common domains, so since \( \mathcal{T} \) is countable in \( M \), it follows that \( j(\Sigma) \) and \( k(j(\Sigma)) \) agree on \( \mathcal{T} \).

49 See [9] for more details on essential components and cut-points of a stack. These concepts are useful for the proof of the general case.
From the proof of the above lemma, we also get:

\((\ast):\) \quad \Sigma \text{ is definable in } V \text{ from } \mathcal{P}^\infty_{\mathcal{P},\infty}.

It implies that \(\Sigma\) is definable in \(V\) from a countable sequence of ordinals. As a consequence,

\((\ast\ast):\) \quad \text{if } J \text{ is a } \Sigma\text{-cmi operator, then } J \text{ is definable in } V \text{ from a countable sequence of ordinals.}

\((\ast\ast)\) will be part of our inductive hypothesis. The following theorem is proved in [32, Theorem 2.4.4] under the existence of a strong, pseudo-homogeneous ideal on \(\mathcal{P}_{\omega_1}(\mathbb{R})\) and in [16] under DI for operators that are ordinal definable in \(V\). Therefore, we will sketch the proof of Theorem 4.5 under the hypothesis DI for \(\Sigma\)-cmi operators. These operators in general are (by the induction hypothesis) definable in \(V\) from a countable sequence of ordinals.

First, by standard arguments, we have the following lemma, which will be used in the proof of Theorem 4.5.

**Lemma 4.4.** Suppose \(J\) is a \(\Sigma\)-cmi operator as in \((\ast\ast)\). Suppose \(J\) is defined on a cone over \(HC\) above some \(a \in HC\), i.e. \(\text{dom}(J) = \{b \in HC : a \in L_1[b]\}\). Then \(J\) can be uniquely extended to an operator \(J^+\) on a cone over \(H_{c^+}\) above \(a\). Furthermore, the operator \(J^\sharp\) exists and \(\text{dom}(J^\sharp) = \text{dom}(J)\).

**Proof sketch.** The proof is standard. Details have been given in [32, 16]. We only mention some key points here. The operator \(J^+\), the unique extension of \(J\) in \(V\), is simply \(j(J) \upharpoonright H^V_{c^+}\). Since \(J\) satisfies \((\ast\ast)\), the hypothesis DI will imply that \(j(J) \upharpoonright V\) is in \(V\) and doesn’t depend on \(G\).

To see \(J^\sharp(x)\) is defined for each \(x \in \text{dom}(J)\), note that from \(j\), one can define an ultrafilter \(\mu\) over \(L^J[x]\)\footnote{This is the model \(L^J_{c^+}[x]\).} as follows: for each \(A \in \varphi(\omega^V_1) \cap L^J[x]\),

\[ A \in \mu \iff \omega_1 \in j(A). \]

By a standard argument, \(\mu\) is a countably complete, normal measure over \(L^J[x]\) that is amenable to \(L^J[x]\) in the sense that for any \(Y\) of size \(\omega^V_1\) in \(L^J[x]\), we have \(\mu \cap Y \in L^J[x]\). Furthermore, by condensation properties of \(J\) we have \(\text{Ult}(L^J[x], \mu) = L^J[x]\) as it embeds into \(j(L^J[x])\). By standard arguments due to Kunen, the amenable structure \((L^J[x], \mu)\) is iterable. This implies \(J^\sharp(x)\) exists. \(\Box\)

**Theorem 4.5.** Suppose \(\mathcal{F}\) is a \(\Sigma\)-cmi operator as in \((\ast\ast)\) defined on a cone above \(a \in HC\). Then the \(\Sigma\)-cmi operator \(H = \mathcal{M}_{1}^{F,\sharp}\) is also defined on the same cone. Furthermore, the operator \(j(H) \cap V\) is in \(V\) and is the unique extension of \(H\) to \(H^V_{c^+}\).

**Proof.** Let \((\mathcal{P}, \Sigma)\) be as above. Let \(\Sigma^+ = j(\Sigma) \upharpoonright V\). By Lemma 4.3, \(\Sigma^+ \in V\). We let \(\mathcal{F} = \mathcal{F}_{\Sigma, \varphi}\) be the operator induced by \(\Sigma\) and with \(\varphi = \varphi_{\text{all}}\). See Definition 3.17.

We will prove the theorem for this particular \(\mathcal{F}\). The proof for other \(\Sigma\)-cmi operators is similar, albeit somewhat simpler. We note that \(\mathcal{F}\) is indeed a \(\Sigma\)-cmi operator.\footnote{Since \(\Sigma\) has hull and branch condensation, \(\mathcal{F}\) condenses finely. [9, Lemma 3.35] implies \(\mathcal{F}\) determines itself on generic extensions. That \(\mathcal{F}\) relativizes well is clear from the definition.}
We define the following model \( W \) by induction on \( \alpha < \omega_2^V \): \( W_0 = (HC^V, \in) \),
\[
W_{\alpha+1} = J_\omega(W_\alpha \cup \{(T, b) : b = \Sigma^+(T) \wedge T \in W_\alpha \wedge T \text{ is according to } \Sigma^+\}),
\]
and for \( \alpha \) limit, \( W_\alpha = \bigcup_{\beta < \alpha} W_\beta \). Finally, \( W = \bigcup_{\alpha < \omega_2^V} W_\alpha \). Note that \( W \in V \) and \( \Sigma^+ \models W_\alpha \in W \) for all \( \alpha < \omega_2^V \).

By the proof of [9, Lemma 3.35], we have the following.

**Fact 4.6.** For any poset \( P \in W \) and any \( W \)-generic \( g \in P \) such that \( g \in V \) (or \( g \in M \)), \( W[g] \) is closed under \( \Sigma^+ \) (respectively \( j(\Sigma) \)).\(^{52}\)

Let \( F^+ \) be the canonical extension of \( F \) to \( H_{c^+} \) (\( c^+ = \omega_2 \) by Di). Suppose \( g \in V \) (or in \( M \)) is a generic enumeration of \( \mathbb{R}^V \) in order type \( \omega_1 \), let \( X_g = \bigcup_{\alpha < \omega_2^V} X_\alpha \), where \( X_0 = \text{tr.cl.}(g \cup \{g\}) \), \( X_1 = F_0^+(g) \), and for \( \alpha \geq 1 \), \( X_{\alpha+1} = F_1^+(X_\alpha) \), and \( X_\lambda = \bigcup_{\alpha < \lambda} X_\alpha \) for \( \lambda \) a limit ordinal (see Definition 3.17 for the notations we use here). We note that \( X_g \) contains \( \mathbb{R}^V \) and is closed under \( \Sigma^+ \). Now, if we let \( W^X_g \) be the structure \( W \) defined as above, but the definition is carried out inside \( X_g \), then
\[
W^X_g = W.
\]

This means that the model \( W \) is independent of \( g \).

In a similar manner, letting \( G = (F^+)^2 \),\(^{53}\) we define \( X_g = \bigcup_{\alpha < \omega_2^V} X_\alpha \), where \( X_0 = \text{tr.cl.}(g \cup \{g\}) \), \( X_1 = G_0(X_0) \), and for \( \alpha \geq 1 \), \( X_{\alpha+1} = G_1(X_\alpha) \), and \( X_\lambda = \bigcup_{\alpha < \lambda} X_\alpha \) for \( \lambda \) a limit ordinal\(^{54}\) and let \( W \) be the model defined in \( X_g \) as above, but using \( G \) instead of \( F \). In particular, it is easy to verify that \( W \) now has the following properties:

- \( o(W) = \omega_2 \) and \( W \) is a model over \( \mathbb{R}^V \);
- for any \( a \in H_{\omega_2} \cap W \), we have \( G(a) \in W \); in particular, \( W \) is closed under \( \Sigma^+ \) and if \( h \in V \) (or in \( M \)) is \( W \)-generic, then \( W[h] \) is closed under \( \Sigma^+ \);
- \( W \) is independent of \( g \); in other words, suppose \( g_1, g_2 \in V \) (or in \( M \)) are two enumerations of \( \mathbb{R} \) in order type \( \omega_1 \), then \( W^{X_{g_1}} = W^{X_{g_2}} \);
- If \( g \in V \) (or in \( M \)) is \( \text{Coll}(\omega_1, \mathbb{R}^V) \)-generic over \( W \), then the universe of \( X_g \) is just the universe of \( W[g] \).

Suppose that on a cone of \( x \in HC \), \( M_1^{\Sigma^+}(x) \)\(^{55}\) does not exist. Then in \( W[g] \) where \( g \in V \) is \( \text{Coll}(\omega_1, \mathbb{R}^V) \)-generic over \( W \), the core model \( K = \text{def} K^{\Sigma^+}(x) \) exists\(^{56}\). Here \( K \) is a \( F \)-mouse and is in \( W \).

\(^{52}\) [9, Lemma 3.35] indeed implies that \( J \) determines itself on generic extensions. It is also easy to see that \( J \) relativizes well.

\(^{53}\) \((F^+)^2 \) exists by Lemma 4.4.

\(^{54}\) \( X_g \) is a potential \( G \)-premouse over \( g \) and it is closed under \( G \) because \( G \) relativizes well.

\(^{55}\) Recall that this is just \( M_1^{\Sigma^+}(x) \).

\(^{56}\) Here the core model relative to \( \Sigma^+ \) is defined in the sense of [4] and \( o(K) = \xi < o(W) \) and \( \omega_1^V < \xi \) is a sufficiently large indiscernible relative to \(( J^+ )^2 \).
Claim 4.7. \( j(K) \in V \).

Proof of Claim 4.7. To see that \( j(K) \in V \), it suffices to show that \( j(W) \) is definable in \( V[G] \) from parameters in \( V \). To see this, first note that \( j(j(\Sigma) \upharpoonright V) \) is definable in \( \text{Ult}(V,G) \) as the unique extension of \( j(\Sigma) \) to \( H_{\omega_2} \) that condenses well. Let \( \Lambda = j(j(\Sigma) \upharpoonright V) \). Note that \( \Lambda \) may not be definable in \( V[G] \); the main wrinkle is that \( H_{\omega_2}^{V[G]} \) may differ from \( H_{\omega_2}^{\text{Ult}(V,G)} \). But in \( V[G] \) we can define \( \Psi \), the maximal (partial) strategy on \( H_{\omega_2} \) that extends \( j(\Sigma) \) with the property that whenever \( T \) is according to \( \Psi \), the branch \( \Psi(\xi) \) (if defined) is the unique well-founded branch \( b \) such that whenever \((U,c)\) is a countable hull of \((T,b)\), then \( U \) is according to \( j(\Sigma) \) and \( c = j(\Sigma)(U) \). Note that if \( \Lambda_1 \) and \( \Lambda_2 \) are partial strategies extending \( j(\Sigma) \) satisfying the above properties, then whenever \( T \in \text{dom}(\Lambda_1) \cap \text{dom}(\Lambda_2) \), we have \( \Lambda_1(T) = \Lambda_2(T) \). As a result, \( \Psi \) is simply the union of all such partial strategies, and since \( \Lambda \) is one such partial strategy,

\[ \Lambda \subseteq \Psi. \]

This easily implies that \( j(W) \) is definable in \( V[G] \) from \( \Psi \) as \( \Psi \upharpoonright W = \Lambda \upharpoonright W \). Hence \( j(W) \) is definable in \( V[G] \) from \( j(\Sigma) \), but \( j(\Sigma) \) is definable in \( V[G] \) from \( j(s) \in V \), for some countable sequence of ordinals \( s \in V \). By homogeneity, \( j(W) \in V \). \( \square \)

Given this claim, the rest of the proof proceeds as in [16, Theorem 2.10.2] by showing that for the \((\omega_1^V,\omega_2^V)\)-extender \( E \) derived from \( j \), we have \( E \upharpoonright \alpha \in j(K) \) for all \( \alpha < \omega_2^V \). This implies that \( \omega_2^V \) is Shelah in \( j(K) \), contradiction. Fixing \( \alpha < \omega_2^V \), we give a sketch of \( E \upharpoonright \alpha \in j(K) \). We note again that \( W \) is closed under \( G \). We need to see that the phalanx \((j(K), \text{Ult}(j(K), \text{Ult}(j(K), E \upharpoonright \alpha), \alpha)\) is iterable in \( j(W) \).\(^{57}\) Otherwise in \( j(W) \) there is a countable \( F \)-premouse \( \bar{K} \) and a map \( \sigma : \bar{K} \to \text{Ult}(K, E \upharpoonright \alpha) \) with \( \text{crt}(\sigma) = \alpha \) and

\[ j(W) \models (j(K), \bar{K}, \alpha) \text{ is not } \omega_1 \text{-iterable.} \]

We have a factor map \( k : \text{Ult}(K, E \upharpoonright \alpha) \to j(j(K)) \) with \( k \upharpoonright \alpha = \text{id} \) and

\[ k \circ \sigma : \bar{K} \to j(j(K)) \]

such that \( k \circ \sigma \upharpoonright \alpha = \text{id} \).

Let \( \psi = k \circ \sigma \) and \( \psi = [\beta \mapsto \psi_\beta]_G \). Let \( \bar{K} = [\beta \mapsto \bar{K}_\beta]_G \) and \( \alpha = [\beta \mapsto \alpha_\beta]_G \). We need to see that for \( G \)-almost all \( \beta \),

\[ W \models (K, K_\beta, \alpha_\beta) \text{ is } \omega_1 \text{-iterable.} \]

By absoluteness, in \( j(W) \) there is some \( \psi_\beta : K_\beta \to j(K) \) such that \( \psi_\beta \upharpoonright \alpha = \text{id} \). Then in \( W \) there is some \( \tilde{\psi} : K_\beta \to \bar{K} \) such that \( \tilde{\psi} \upharpoonright \alpha_\beta = \text{id} \). But this means \( (K, K_\beta, \alpha_\beta) \) is iterable in \( W \).

Finally, the operator \( H : \alpha \mapsto \mathcal{M}_1^{\Sigma}[x] \) is definable from \( \Sigma \). Since \( j(\Sigma) \upharpoonright V \in V \), we have \( j(H) \upharpoonright V \in V \) also. \( \square \)

\(^{57}\)Iterability here is with respect to trees of length \( < j(\xi) \) in \( j(W) \).
From the above lemma, we easily get that the canonical strategy $\Lambda$ of $\mathcal{M}^{\Sigma, \delta}_1$ can be extended uniquely to a strategy $\Lambda^+$ acting on trees in $H_{c^+}$. This then allows us to make sense of $\text{Lp}^\Sigma(\mathbb{R})$ as defined in [18] because $\Lambda^+$ is what we use to define the $\Theta$-g-organized $\Sigma$-mice hierarchy $\text{Lp}^\Sigma(\mathbb{R})$. The reader should consult [31, Section 4] for the terminology of the core model induction through $\text{Lp}^\Sigma(\mathbb{R})$ (many of the major terms and notations have been summarized in the previous section); our argument follows the one given in [32], which shows from the same hypothesis that $\text{Lp}(\mathbb{R}) \cap \mathfrak{U}(\mathbb{R}) \subseteq \Gamma$.

**Theorem 4.8.** $\text{Lp}^\Sigma(\mathbb{R}) \models \text{AD}^+ + \text{MC}(\Sigma)$.

**Proof sketch.** Our hypothesis implies that for every $\Sigma$-cmi operator $H$ defined on a cone above some $a \in HC$ such that $j(H) \upharpoonright V \in V$, the operator $F : x \mapsto \mathcal{M}_{1}^{H, \sharp}(x)$ is defined for all $x$ in the same cone, by Theorem 4.5. We can then extend $F$ to an operator $F^+$ on the cone above $a$ in $H_c^V$; $F^+$ is just $j(F) \upharpoonright V$. Furthermore, $F^+$ relativizes well, condenses finely, and determines itself on generic extensions since $H$ is assumed to have those properties.58 We sometimes write $\mathcal{M}_{1}^{H, \sharp}$ for the operator $F^+$. This outlines the general “successor” case of the core model induction. In particular, this gives us that $\Sigma$ is self-scaled and the determinacy of projective-in-$\Sigma$ sets, i.e. $\text{Lp}^\Sigma(\mathbb{R})|_1 \models \text{AD}^+$. Next, we address the “limit case”.

Let $\alpha$ be the strict supremum of the ordinals $\gamma$ such that

1. the coarse mouse witness condition $W_{\gamma + 1}^{x, \Sigma}$ holds,59 and

2. $\gamma$ is a critical ordinal in $\text{Lp}^\Sigma(\mathbb{R})$ (i.e. $\gamma + 1$ begins a gap in $\text{Lp}^\Sigma(\mathbb{R})$).

Using the fact that $\mathcal{M}_{1}^{H, \sharp}$ exists for every $\Sigma$-cmi operator $H$, it’s easily seen that $\alpha$ is a limit ordinal. By essentially the same proof as that in [31], we can advance past inadmissible gaps and admissible gaps in $\text{Lp}^\Sigma(\mathbb{R})|_{\alpha}$.

We deal with the inadmissible case first. Suppose $\alpha$ is such that $\alpha$ is inadmissible and $\text{cf}(\alpha) = \omega$. In this case, let $\Omega = \Sigma^\text{Lp}^\Sigma(\mathbb{R})|_{\alpha}$. Then $C_\Omega = \bigcup_{n < \omega} C_{\Omega_n}$ for some increasing sequence of scaled pointclasses $(\Omega_n | n < \omega)$. By $W_{\alpha}^{x, \Sigma}$, for each $n$, we have $\Sigma$-cmi operators $(K_m | m < \omega)$ that collectively witness Det$(\bigcup_n \Omega_n)$. Say each $K_m$ is defined on a cone above some fixed $a \in HC$. The desired mouse operator $J_0$ is defined as follows: For each transitive and self-wellordered $A \in HC$ coding $a$, $J_0(A)$ is the shortest initial segment $\mathcal{M} \triangleleft \text{Lp}^{\ell_1, \Sigma}(A)$ such that $\mathcal{M} \models \text{ZFC}^-$ and $\mathcal{M}$ is closed under $K_m$ for all $m$. Then $J_0$ is total and trivially relativizes well and determines itself on generic extensions because the $K_m$’s have these properties. We then use Theorem 4.5 to get that $J_1 = \mathcal{M}^{\Sigma, J_0}_1$ is defined on the cone above $a$ by arguments in the previous section. Inductively, we get that $J_{m+1} = \mathcal{M}_{1}^{\Sigma, J_m}$ is defined on the cone above $a$ for all $n$ and one easily gets that these operators are $\Sigma$-cmi operators. By Lemma 4.1.3 of [16], this implies $W_{\alpha+1}^{x, \Sigma}$.

Now we’re on to the case where $\alpha$ is inadmissible and $\text{cf}(\alpha) > \omega$. Let $\phi(v_0, v_1)$ be a $\Sigma_1$ formula and $x \in \mathbb{R}$ be such that

58 The base case is when $H = F_{v_{11}, \Sigma}$. Then $H$ has these properties by the basic theory of hod mice in [9] and the work in [18].

59 See Definition 3.26 and the remark after.
\[ \forall y \in \mathbb{R} \exists \beta < \alpha \text{ Lp}^\Sigma(\mathbb{R}) | \beta \models \phi[x, y], \]

and letting \( \beta(x, y) \) be the least such \( \beta \),

\[ \alpha = \sup \{ \beta(x, y) \mid y \in \mathbb{R} \}. \]

So \( \phi \) witness the inadmissibility of \( \alpha \). We first define \( J_0 \) on transitive and self-wellordered \( A \in \text{HC} \) coding \( x \). For \( n < \omega \), let

\[ \phi^*_n \equiv \exists \gamma \text{ (Lp}^\Sigma(\mathbb{R}) | \gamma \models \forall i \in \omega (i > 0 \Rightarrow \phi((v)_0, (v)_i) \land (\gamma + \omega n) \text{ exists})). \]

For such an \( A \) as above, let \( \mathcal{M} \) be an \( A \)-premouse and let \( G \) be a Col(\( \omega \), \( A \))-generic filter over \( \mathcal{M} \). Then \( \mathcal{M}[G] \) can be regarded as a g-organized \( \Sigma \)-mouse over \( z(G, A) \) where \( z(G, A) \) is a real coding \( G, A \) and is obtained from \( G, A \) in some simple fashion.\(^{60}\) Also, let \( \sigma_A \) be a term defined uniformly (in \( \mathcal{M} \)) from \( A, x \) such that

\[ (\sigma^G_A)_0 = x \]

and

\[ \{(\sigma^G_A)_i \mid i > 0\} = \{\rho^G \mid \rho \in L_1(A) \land \rho^G \in \mathbb{R}\}. \]

Let \( \varphi \) be a sentence in the language of \( A \)-premice such that for any \( A \)-premouse \( \mathcal{M} \), we have \( \mathcal{M} \models \varphi \) iff for every \( \mathcal{M} \)-generic filter \( G \subseteq \text{Col}(\omega, A) \) and every \( n < \omega \) there is a \( \gamma < \sigma(\mathcal{M}) \) such that

\[ \mathcal{M}[z(G, A)] | \gamma \text{ is a } \langle \phi^*_n, \sigma^G_A \rangle \text{-prewitness}. \]

Then \( J_0(A) \) is the shortest initial segment of \( \text{Lp}^\Sigma(\mathbb{R}) \) which satisfies \( \varphi \), if it exists, and is undefined otherwise. Using the fact that \( W^\varphi_\alpha \) holds, we get that \( J_0(A) \) exists for all \( A \in \text{HC} \) coding \( x \) because \( \alpha \) has uncountable cofinality and there are only countably many \( \langle \phi^*_n, \rho^G_A \rangle \). Also we can then define \( J_n \) as before. It’s easy to show again that the \( J_n \)'s relativize well and determine themselves on generic extensions, so they are \( \Sigma \)-cmi operators. This implies \( W^{\varphi}_\alpha \).

Now suppose that \( \alpha \) is admissible. So \( \alpha \) begins a gap \( [\alpha, \alpha^*] \) that is either a weak gap or a strong gap. We let \( \beta = \alpha^* \) if the gap is weak and \( \beta = \alpha^* + 1 \) if the gap is strong. Let \( \Omega = \Sigma^1_T \text{Lp}^\Sigma(\mathbb{R}) | \alpha \).

Then \( \Omega \) is inductive-like. If \( [\alpha, \alpha^*] \) is a weak gap, then by the scales analysis at the end of a weak gap from \( [25] \) and \( [18] \), we can construct a self-justifying system (sjs) \( \mathcal{A} \) that is Wadge-cofinal in \( \varphi(\mathbb{R}) \cap \text{Lp}^\Sigma(\mathbb{R}) | \alpha^* \).\(^{61}\) If \( [\gamma, \alpha^*] \) is a strong gap, then \( \text{AD}^+ \) holds in \( \text{Lp}^\Sigma(\mathbb{R}) | \beta \) by the Kechris–Woodin theorem, and again by results of \( [25], [18] \), and \( [33] \), we also get a self-justifying system \( \mathcal{A} \) Wadge-cofinal in \( \text{Lp}^\Sigma(\mathbb{R}) | \alpha \cap \varphi(\mathbb{R}) \). See the next section for an alternative proof that constructs \( \mathcal{A} \).\(^{62}\)

\(^{60}\)This is one of the main reasons that we consider \( \# \Sigma \)-mice; this is so that generic extensions of \( \# \Sigma \)-mice can be rearranged to \( \# \Sigma \)-mice.

\(^{61}\)This means \( \mathcal{A} \) is a countable collection containing a universal \( \Omega \) set and closed under complements, and every set \( A \in \mathcal{A} \) has a scale whose individual norms are coded by sets in \( \mathcal{A} \).

\(^{62}\)The scales analysis in \( [25], [18] \) would give us the desired sjs \( \mathcal{A} \) if \( [\alpha, \alpha^*] \) is not the “last gap” in \( \text{Lp}^\Sigma(\mathbb{R}) \). The argument we present in the next section will construct such an \( \mathcal{A} \) regardless of whether \( [\alpha, \alpha^*] \) is the last gap, however, unlike \( [25], [18] \) which doesn’t use any particular hypothesis, we will need to use our hypothesis (either
From \(\mathcal{A}\) and arguments in [16, Section 5], there is a pair \((Q, \Lambda)\) such that \(Q\) is \(\Gamma\)-suitable and \(\Lambda\) is the \((\omega_1, \omega_1)\)-strategy for \(Q\) guided by \(\mathcal{A}\) (see the next section for more details on self-justifying systems). Let \(J_0\) be the operator defined from \(\Lambda\) as in the proof of Theorem 4.5 (i.e., \(J_0 = F_{\varphi_{\text{all}}(\Lambda)}\)). As remarked in Theorem 4.5, \(J_0\) relativizes well and determines itself on generic extensions, and \(J_0\) can be uniquely extended to an operator \(J_0^+\) on \(H_{\epsilon^+}\). If \(\mathcal{A}\) (or equivalently \(J_0\)) is in \(Lp^{\Sigma}(\mathbb{R})\) (i.e., \([\alpha, \beta]\) is not the last gap of \(Lp^{\Sigma}(\mathbb{R})\)), then we can show \(W_{\beta+1}^{\ast, \Sigma}\) by constructing a sequence of operators \((J_n : n < \omega)\) where \(J_{n+1} = \mathcal{M}_1^{J_n \ast}\) for all \(n\) like before.

Now suppose \([\alpha, \alpha^*]\) is the last gap of \(Lp^{\Sigma}(\mathbb{R})\). It is easy to see that the gap is weak. Recall that the (lightface) pointclass \(\Omega = \sum_1 \mathcal{L}p^{\Sigma}(\mathbb{R})\) is inductive-like and we let \(\Delta_\Omega = \varphi(\mathbb{R}) \cap Lp^{\Sigma}(\mathbb{R})\). Since \(\Delta_\Omega\) is inductive-like and \(\Delta_\Omega\) is determined, \(\mathcal{E}nv(\Omega)\) is determined by Theorem 3.2.4 of [32]. Since whenever \(\gamma\) is a critical ordinal in \(Lp^{\Sigma}(\mathbb{R})\) and \(W_{\gamma+1}^{\ast, \Sigma}\) holds then \(AD^+\) holds in \(Lp^{\Sigma}(\mathbb{R})\), we have that \(AD^+\) holds in \(Lp^{\Sigma}(\mathbb{R})\). We also have \(MC(\Sigma)\) in \(Lp^{\Sigma}(\mathbb{R})\), which is a consequence of \(W_{\gamma+1}^{\ast, \Sigma}\) for critical ordinals \(\gamma < \alpha\). Now we claim that

**Claim 4.9.** \(\mathcal{E}nv(\Omega) \subseteq \varphi(\mathbb{R}) \cap Lp^{\Sigma}(\mathbb{R})\).

**Proof.** We first show \(\mathcal{E}nv(\Omega) \subseteq \varphi(\mathbb{R}) \cap Lp^{\Sigma}(\mathbb{R})\). Let \(A \in \mathcal{E}nv(\Omega)\), say \(A \in \mathcal{E}nv(\Omega(x))\) for some \(x \in \mathbb{R}\). By definition of \(\mathcal{E}nv(\Omega(x))\), for each countable \(\sigma \subseteq \mathbb{R}\) we have \(A \cap \sigma = A \cap \sigma\) for some \(A'\) that is \(\Delta_1\)-definable over \(Lp^{\Sigma}(\mathbb{R})\) from \(x\) and some ordinal parameter. By the fact that \(\mathbb{R}^V\) is countable in \(M\) and \(j(A) \cap \mathbb{R}^V = A\), we have that \(A\) is \(\Delta_1\)-definable from \(\{x, \mathbb{R}^V\}\) in \(j(Lp^{\Sigma}(\mathbb{R}))(\alpha)\). Therefore, by \(MC(j(\Sigma))\) in \(j(Lp^{\Sigma}(\mathbb{R}))(\alpha)\), \(A \in Lp^{\Sigma}(\mathbb{R}^V)\). So \(A \in Lp^{\Sigma}(\mathbb{R})\) in \(V\) by the following argument.\(^63\)

Let \(\mathcal{M}\) be the least level of \(Lp(j(\Sigma)(\mathbb{R}^V))\) that defines \(A\). It suffices to show \(\mathcal{M} \cap Lp^{\Sigma}(\mathbb{R}^V)\). First, note that since \(\mathcal{M}\) (equivalently \(A\)) is definable in \(Ult(V, G)\) from \(\{\mathbb{R}^V\}\) and a countable-in-\(V\) sequence of ordinals \(s\) \((s = j(\mathcal{H} \cap \Sigma_x \cap \mathcal{M}))\), we have \(\mathcal{M} \in V\) by homogeneity. Let \(\mathcal{M}\) be countable in \(V\) such that there is an elementary embedding \(\pi : \mathcal{M} \rightarrow \mathcal{M}\). Therefore \(\mathcal{M}\) is \(j(\omega_1)\)-iterable as a \(j(\Sigma)\)-mouse in \(\mathcal{M}\), or equivalently in \(V[G]\) by elementarity of \(j\), \(\mathcal{M}\) is \(\omega_1\)-iterable as a \(\Sigma\)-mouse in \(V\); using \(j\) and absoluteness, one can uniquely extend this strategy to an \((\omega_1 + 1)\)-strategy in \(V\). This shows \(\mathcal{M} \cap Lp^{\Sigma}(\mathbb{R})\).\(^64\)

Now assume toward a contradiction that \(\mathcal{E}nv(\Omega) \not\subseteq \varphi(\mathbb{R}) \cap Lp^{\Sigma}(\mathbb{R})\). Hence \(\alpha < \Theta^{Lp^{\Sigma}(\mathbb{R})}\). Let \(\beta^*\) be the end of the gap starting at \(\alpha\) in \(Lp^{\Sigma}(\mathbb{R})\). Let \(\beta = \beta^*\) if the gap is weak and \(\beta = \beta^* + 1\) if the gap is strong. Note that \(\alpha \leq \beta\), \(\varphi(\mathbb{R}) \cap Lp^{\Sigma}(\mathbb{R})(\beta) = \mathcal{E}nv(\Omega) \cap Lp^{\Sigma}(\mathbb{R})\). Hence \(\beta < \Theta^{Lp^{\Sigma}(\mathbb{R})}\) and \(Lp^{\Sigma}(\mathbb{R})(\beta)\) projects to \(\mathbb{R}\). Furthermore, \(Lp^{\Sigma}(\mathbb{R})(\beta) \vdash AD + \Omega - MC(\Sigma)\), where \(\Omega - MC(\Sigma)\) is the statement: for any countable transitive \(a\), \((Lp^{\Sigma}(\mathbb{R})(\beta) \cap \varphi(a) = C_{\Omega}(a)\). The fact that \(Lp^{\Sigma}(\mathbb{R})(\beta) \vdash \Omega - MC(\Sigma)\) is clear; if \(\beta = \beta^*\), \(Lp^{\Sigma}(\mathbb{R})(\beta) \vdash AD\) by the fact that \([\alpha, \beta^*]\) is a \(\Sigma_1\)-gap; otherwise, \(Lp^{\Sigma}(\mathbb{R})(\beta) \vdash AD\) by the Kechris–Woodin transfer theorem (see [5]). Since \(Lp^{\Sigma}(\mathbb{R})(\beta)\) projects to \(\mathbb{R}\), every countable sequence from \(\mathcal{E}nv(\Omega) \cap Lp^{\Sigma}(\mathbb{R})\) is in \(Lp^{\Sigma}(\mathbb{R})((\beta + 1)\). As mentioned

\(^{63}\)In applying \(MC(\Sigma)\), we need that definability is done without referencing the extender sequence and we can do this since we are inside \(j(Lp^{\Sigma}(\mathbb{R}^V))\), where the self-iterability condition helps us define the extender sequence.

\(^{64}\)One can argue, using the fact that \(\mathcal{M}\) is iterable in \(j(Lp^{\Sigma}(\mathbb{R}))\), that the unique \(\Sigma\)-strategy of \(\mathcal{M}\) is in \(Lp^{\Sigma}(\mathbb{R})\). Furthermore, as in [32, Lemma 4.5.1], \(\mathcal{M}\)'s strategy can be extended to a \(\epsilon^+\)-strategy that is \(\epsilon^+\)-universally Baire.

\(^{65}\)We get equality in this case but we don’t need this fact.
above, the scales analysis of \cite{18} and Theorem 4.3.2 and Corollary 4.3.4 of \cite{32} together imply that there is a self-justifying-system $\mathcal{A} = \{A_i \mid i < \omega\} \subseteq \text{Env}(\Omega)^{Lp^\Sigma(\mathbb{R})}$ containing a universal $\Omega$ set (see the next section for details on this argument). By a theorem of Woodin (cf. \cite[Theorem 5.4.8]{16}), we can get a pair $(\mathcal{N}, \Lambda)$ such that $\mathcal{N} \in V$, $\mathcal{N}$ is a $\Gamma$-suitable $^*\Sigma$-premouse, and $\Lambda$ is the strategy for $\mathcal{N}$ guided by $\mathcal{A}$. Again, if $\text{DI}$ holds, we can extend $\Lambda$ to a unique $(c^+, c^+)$-iteration strategy $\Lambda^+$ (just let $\Lambda^+ = j(\Lambda) \upharpoonright V$).

As above, we can get a sequence of nice operators $(F_n : n < \omega)$ where each $F_n$ is in $Lp^\Sigma(\mathbb{R})|_{\beta+1}$. Each $F_n$ is first defined on a cone in $HC^V$; using $\text{DI}$, we can extend $F_n$ to $H_{\zeta^+}$ and furthermore, each $F_n$ is nice (i.e. condenses and relativizes well and determines itself on generic extensions). These operators are all projective in $\mathcal{A}$ and are cofinal in the projective-like hierarchy containing $\mathcal{A}$, or equivalently in the Levy hierarchy of sets of reals definable from parameters over $Lp^\Sigma(\mathbb{R})|\beta$. Together these model operators can be used to establish the coarse mouse witness condition $W_{\beta+1}^{*\Sigma}$. Therefore $\beta < \alpha$ by the definition of $\alpha$, which is a contradiction. 

By basic facts about envelopes, cf. \cite{32}, we get that $\text{AD}^+$ holds in $\text{Env}(\Omega)$, hence in $Lp^\Sigma(\mathbb{R})$. Finally, since $\text{MC}(\Sigma)$ holds in $Lp^\Sigma(\mathbb{R})|\alpha$ and $Lp^\Sigma(\mathbb{R})|\alpha \prec R^\Sigma 1 Lp^\Sigma(\mathbb{R})$\footnote{See \cite[Definition 5.12]{18}.}, $\text{MC}(\Sigma)$ holds in $Lp^\Sigma(\mathbb{R})$.

\textbf{Lemma 4.10.} $o(Lp^\Sigma(\mathbb{R})) < j(\omega_1^Y) = \omega_2^Y$.

\textit{Proof.} This follows from the proof of Claim 4.9. The point is that in $\text{Ult}(V,G)$, the set $\mathbb{R}^V$ is countable and each $\mathcal{M} \triangleleft Lp^\Sigma(\mathbb{R})$ is $OD(j(\Sigma), \mathbb{R}^V)$ in $j(Lp^\Sigma(\mathbb{R}))$. Since $j(Lp^\Sigma(\mathbb{R})) \models \text{AD}^+$, it follows that $Lp^\Sigma(\mathbb{R})$ is countable there. \hfill $\square$

We have shown that

$$\wp(\mathbb{R}) \cap Lp^\Sigma(\mathbb{R}) \subseteq \Gamma.$$ 

To complete the proof of (4.1), we need to construct a set $B \in \Gamma \setminus Lp^\Sigma(\mathbb{R})$. In the next section, we give an argument that is based on a game as done in \cite{32}. In this argument, we construct a self-justifying system consisting of sets Wadge cofinal in $\text{Env}(\Omega)$. This self-justifying system gives rise to such a $B$.

This argument, as done originally in \cite{32} uses the existence of a strong, pseudo-homogenous ideal on $\wp_{\omega_1}(\mathbb{R})$. Below, we give the argument assuming $\text{DI}$.

\subsection{4.3.1. Sealing the envelope of an inductive-like pointclass}

We assume $\text{DI}$ throughout this section. Again, we let $P = Lp^\Sigma(\mathbb{R})$ and $\Omega = \Sigma^P_1(\Sigma)$. We assume (**\footnote{See \cite[Definition 5.12]{18}.}) for $\Sigma$ and $\Sigma$-cmi operators throughout this section. We outline the argument before giving details. We first want to construct a scale on a universal $\Omega$-set whose norm relations are in the boldface envelope $\text{Env}(\Omega)$. Suppose we were able to show this claim. Then by standard results (cf. \cite[Section 4.3]{32}), there is a self-justifying system (sjs) $\vec{A} = \{A_i : i < \omega\} \in \text{Env}(\Omega)^\omega$ such
From the results of the previous section, there is a $\Omega$-suitable \( k\Sigma \)-premouse \( P \) with an \( \bar{A} \)-guided iteration strategy \( \Lambda \). By sjs term-condensation, we conclude that \( \Lambda \) has branch condensation (and condenses well in the sense of \cite{16}), cf. \cite[Section 5.4]{16}. Thus, we can make sense of \( L^\Lambda(\mathbb{R}) \) and by a core model induction argument as above, we can show that \( L^\Lambda(\mathbb{R}) \models AD^+ \). This gives us that \( \Lambda \in \Gamma \setminus L^\Sigma(\mathbb{R}) \) as desired.

The following lemma will be useful for us (see \cite[Proposition 3.2.5]{32} for a proof).

**Lemma 4.11.** The lightface envelope, \( Env(\Omega) \), has a well-ordering that is definable from \( \Sigma \).

**Definition 4.12.** Suppose \( \Omega \) is an inductive-like pointclass and \( \Delta_\Omega \) is determined. Let \( \kappa \) be the supremum of the lengths of wellorderings in \( \Delta_\Omega \). Let \( T \) be the tree of a \( \Omega \)-scale on a universal \( \Omega \) set.

1. \( \wp^\Omega(\kappa) \) is the \( \sigma \)-algebra consisting of subsets \( Y \subseteq \kappa \) such that \( Y \in L[T, z] \) for some real \( z \).
2. \( \text{meas}^\Omega(\kappa) \) is the set of countably complete measures on \( \wp^\Omega(\kappa) \).
3. Using the canonical bijection \( \kappa \rightarrow \kappa^{<\omega} \), we can define \( \wp^\Omega(\kappa^n) \), \( \text{meas}^\Omega(\kappa^n) \), \( \wp^\Omega(\kappa^{<\omega}) \), and \( \text{meas}^\Omega(\kappa^{<\omega}) \) in a similar fashion.

Let \( T \) be the tree of a \( \Omega \)-scale on a universal \( \Omega \) set; \( T \) is a tree on \( \omega \times \kappa \), where \( \kappa \) is the largest Suslin cardinal of \( P \). Let

\[ \sigma = j'' \text{meas}^\Omega(\kappa^{<\omega}) \cdot \]

By Lemma 4.10, \( \sigma \in M \).

Let \( \mu \in \sigma \). Suppose \( \mu \) concentrates on \( j(\kappa)^n \) and let \( \langle \mu_i \mid i \leq n \rangle \) be the projections of \( \mu \), meaning \( A \in \mu_i \iff \{ s \in j(\kappa)^n \mid s \upharpoonright i \in A \} \in \mu \). Note that \( \mu_0 \) is the trivial measure.

In \( \text{Ult}(V, G) \), we define the following putative scale \( \{ \varphi_\mu : \mu \in \sigma \} \) on \( \mathbb{R} \setminus p[j(T)] \) as follows. For each \( \mu \in \sigma \), and for each \( x \in \mathbb{R} \setminus p[j(T)] \) (so \( j(T)_x \) is well-founded),

\[ \varphi_\mu(x) = [\text{rank}_{j(T)_x}]_\mu. \]

We now define the following closed game \( G^{j(\mu)}_{\text{ult}(V)} \) in \( \text{Ult}(V, G) \) (equivalently in \( V[G] \), recalling that \( \mathbb{R}^{V[G]} = \mathbb{R}^{\text{Ult}(V, G)} \) and by (\( \dagger \)), the pointclass \( j(T) \) is ordinal definable in \( V[G] \): player I starts by playing \( m_0, \ldots, m_n \) and \( s_n, h_n \), and player II responds by playing a measure \( \mu_{n+1} \). In each subsequent move (numbered \( i > n \)) player I plays \( m_i, s_i, h_i \), and player II plays a measure \( \mu_{i+1} \).

**Rules for player I:**

- \( m_k < \omega \) for all \( k < \omega \)

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\(^{67}\) \( \bar{A} \) is a sjs if \( \bar{A} \) is closed under complements and scales, i.e. if \( A \in \text{rng}(\bar{A}) \) then \( \mathbb{R} \setminus A \in \text{rng}(\bar{A}) \) and there is a scale whose individual prewellorderings associated with that scale are in \( \text{rng}(\bar{A}) \).

\(^{68}\) \( \text{rank}_{j(T)_x}(t) \) denotes the rank of the node \( t \) in the tree \( j(T)_x \), and is considered to be zero if \( t \notin j(T)_x \) and undefined if \( j(T)_x \) is illfounded below \( t \).
• \( j(T)_{(m_0, \ldots, m_{n-1})} \in \mu = \mu_n \)
• \( s_i \in j_{\mu_i}(j(T)_{(m_0, \ldots, m_i)}) \), and in particular \( s_i \in j_{\mu_i}(j(\kappa))^{i+1} \) for all \( i \geq n \)
• \( s_n \supseteq [\text{id}]_{\mu_n} \)
• \( j_{\mu_i, \mu_{i+1}}(s_i) \subseteq s_{i+1} \) for all \( i \geq n \)
• \( h_i \in \text{OR} \) for all \( i \geq n \)
• \( j_{\mu_i, \mu_{i+1}}(h_i) > h_{i+1} \) for all \( i \geq n \)

Rules for player II:

• \( \mu_{i+1} \in \sigma \) is a measure on \( j(\kappa)^{i+1} \) projecting to \( \mu_i \)
• \( \mu_{i+1} \) concentrates on the set \( j(T)_{(m_0, \ldots, m_i)} \subset j(\kappa)^{i+1} \).

The first player that violates one of these rules loses, and if both players follow the rules for all \( \omega \) moves, then player I wins.

The game is closed, hence determined by the Gale–Stewart theorem. Intuitively, player I is building a real \( x = (m_0, m_1, \ldots) \), player II is trying to build a tower \( \bar{\mu} \) of measures in \( \sigma \) concentrating on \( j(T)_x \), and player I is trying to build a continuous witness \( \bar{h} \) to the ill-foundedness of \( \bar{\mu} \) as well as a special kind of branch \( (j_{i, \infty}(s_i) : i \geq n) \) through the direct limit \( j_{0, \infty}(j(T)_x) \) of \( j(T)_x \) along \( \bar{\mu} \). The following is the main lemma.

**Lemma 4.13** (Di). **Player II has a winning strategy in the game** \( G_{j(T)}^{\sigma, \mu} \) **for each** \( \mu \in \sigma \).

**Proof.** First note that by Di, \( j(T) \in V \) and is independent of \( G \); this is because \( T \) is definable in \( V \) from \( \Sigma \) and a countable sequence of ordinals and \( \Sigma \) is determined in \( V \) from a countable sequence of ordinals, namely the iteration map \( j_{\mathcal{P}, \infty}^\Sigma \) (see \((*)\), \((***)\) and the proof of the lemma before it). The parameter defining \( j(T) \) in \( V[G] \) has the form \( j(s) \) for some countable sequence of ordinals \( s \in V \). Therefore, \( j(s) \in V \) and is independent of \( G \) (by \((\xi)\)) and by homogeneity of \( \mathcal{P} \) and the fact that \( j(\Gamma) \) is ordinal definable in \( V[G] \), \( j(T) \in V \) and is independent of \( G \).

Let \( \lambda \) be the length of the well-ordering of \( \text{Env}(\Omega) \) in Lemma 4.11. By Lemma 4.10 we have \( \lambda < j(\omega_1^\nu) = \omega_1^\nu \). It follows that \( j''\lambda \) (and hence also \( \sigma \)) is in \( \text{Ult}(V, G) \) and is countable there.

Fix \( \mu \in \sigma \). We define a winning strategy for player II in \( G_{j(T)}^{\sigma, \mu} \) in \( \text{Ult}(V, G) \). Let \( \mu_0, \ldots, \mu_n \) be the projections of \( \mu \) in order (here \( \mu_n = \mu \)). Let \( j(\bar{\mu}_i) = \mu_i \) for \( i = 0, \ldots, n \). Note that for all \( i \),

\[ j_{\mu_i} \circ j = j \circ j_{\bar{\mu}_i} \]

Suppose player I starts the game by playing integers \( m_0, \ldots, m_n \), a finite sequence of ordinals \( s_n \in j_{\mu_n}(j(T_{m_0, \ldots, m_n})) \cap j_{\mu_n}(j(\kappa)^{n+1}) \), and an ordinal \( h_n \). Define the measure \( \bar{\mu}_{n+1} \in \text{meas}^\Omega(\kappa^{<\omega}) \) as follows.

\[ X \in \bar{\mu}_{n+1} \iff s_n \in j_{\mu_n}(j(X)) \].
Notice that \( j_{\mu_n}(j(X)) = j(j_{\mu_n}(X)) \) does not depend on the generic \( G \): \( X \) is definable in \( P \) from \( T \) and a real, \( \mu_n \) is definable in \( P \) from \( T \) and a finite set of ordinals (cf. [32, Lemma 3.5.4]), and \( T \) itself is definable in \( P \) from a countable sequence of ordinals, so \( j_{\mu_n}(X) \) is definable in \( j(P) \), hence in \( V[G] \), from a countable sequence of ordinals. The argument above shows \( j_{\mu_n} \circ j[\Omega(\kappa^{n+1})] \) does not depend on \( G \). Therefore we have \( \bar{\mu}_{n+1} \in V \). Also note that \( \bar{\mu}_{n+1} \) concentrates on \( T_{m_0,...,m_n} \) and projects to \( \bar{\mu}_n \). Let player II play the measure \( \mu_{n+1} = j(\bar{\mu}_{n+1}) \).

For \( i > n \), suppose player I has played an integer \( m_i \), a finite sequence of ordinals \( s_i \in j_{\mu_i}(j(T_{m_0,...,m_i})) \cap j_{\mu_i}(j(\kappa^{i+1})) \), and an ordinal \( h_i \). Define the measure \( \bar{\mu}_{i+1} \in \text{meas}^{\Omega(\kappa^{<\omega})} \) as follows.

\[
X \in \bar{\mu}_{i+1} \iff s_i \in j_{\mu_i}(j(X)).
\]

As before, the measure \( \bar{\mu}_{i+1} \) is in \( V \), concentrates on \( T_{m_0,...,m_i} \), and projects to \( \bar{\mu}_i \). Let player II play the measure \( \mu_{i+1} = j(\bar{\mu}_{i+1}) \).

Assume for contradiction that player I is able to play \( \omega \) many moves, following all the rules of the game. We get a real \( x = (m_0,m_1,...) \), a tower of measures \( (\mu_i : i < \omega) \) in \( \sigma \), and a countable sequence of ordinals \( (h_i : i < \omega) \) witnessing the illfoundedness of this tower. By elementarity, the tower \( (\bar{\mu}_i : i < \omega) \) is also illfounded.

Take a wellfounded tree \( W \in \bigcup_{x \in R} L[T,x] \) on \( \kappa \) on which each measure \( \bar{\mu}_i \) in this tower concentrates, and such that the function \( \bar{h} : \omega \to \text{Ord} \) defined by \( \bar{h}(i) = [\text{rank}_W,\bar{\mu}_i] \) is a pointwise minimal witness to the illfoundedness of the tower \( (\bar{\mu}_i : i < \omega) \) (see [32, Lemma 3.5.9]). Then by the elementarity of \( j \), the function \( h = j(\bar{h}) \) is a pointwise minimal witness to the illfoundedness of the tower \( (\mu_i : i < \omega) \).\(^{69}\) Because \( \bar{\mu}_i \) concentrates on \( W \) we have \( s_i \in j_{\mu_i}(j(\bar{W})) \) for all \( i < \omega \). Define a function \( h' : \omega \to \text{Ord} \) by \( h'(i) = \text{rank}_{j_{\mu_i}(j(W))}(s_i) \). Then from the rules for player I concerning the finite sequences \( s_i \) we have \( j_{\mu_i,\mu_{i+1}}(h'(i)) > h'(i+1) \) and also \( h'(n) < \text{rank}_{j_{\mu_n}(j(W))}(\text{id} \mu_n) = h(n) \), contradicting the minimality of \( h(n) \).

\[\square\]

**Lemma 4.14 (Di).** In Ult\((V,G)\), the set of norms \( \{ \varphi_\mu : \mu \in \sigma \} \) defined by \( \varphi_\mu(x) = [\text{rank}_{j(T),\mu}] \) (or more precisely, any enumeration of this countable set of norms in order type \( \omega \)) is a scale on the complement of \( p[j(T)] \).

**Proof.** Work in Ult\((V,G)\). Let \( \mu \in \sigma \). We say that \( \sigma \) stabilizes\(^{70}\) \( \mu \) if, whenever \( (x_k : k < \omega) \) is a sequence of reals in \( R \setminus p[j(T)] \) converging to a limit \( x \) and such that for each \( \mu' \in \sigma \), the ordinals \( \varphi_{\mu'}(x_k) \) are eventually constant, we have \( \varphi_\mu(x) \leq \lim_{k \to \omega} \varphi_\mu(x_k) \). (In particular, \( \varphi_\mu(x) < \infty \).

It is clear from the definition that if \( \sigma \) stabilizes every \( \mu \in \sigma \), then \( \{ \varphi_\mu : \mu \in \sigma \} \) is a scale. So fix a measure \( \mu \in \sigma \). We want to show \( \sigma \) stabilizes \( \mu \). Suppose not. We describe a winning strategy for player I in \( G^\mu,\sigma_{j(T)} \). Let \( (x_k : k < \omega) \) witness that \( \sigma \) does not stabilize \( \mu \). That is, \( x_k \in R \setminus p[j(T)] \) for each \( k < \omega \), and the sequence of ordinals \( \varphi_\mu(x_k) : k < \omega \) has an eventually constant value \( h(\nu) \) for each measure \( \nu \in \sigma \) but the limit \( x \) of the sequence \( (x_k : k < \omega) \) satisfies \( \varphi_\mu(x) > \lim_{k \to \omega} \varphi_\mu(x_k) \).

(This includes the possibility that \( \varphi_\mu(x) = \infty \).

---

\(^{69}\)Actually we only need the minimality of \( h(n) \).

\(^{70}\)The idea of this definition comes from a similar notion of stability used in unpublished work of S. Jackson.
Define \( m_i = x(i) \) and \( h(\nu) = \lim_{k \to \omega} \varphi_{\nu}(x_k) \). Let \( n \) be the unique integer such that \( \mu \) concentrates on \( j(\kappa)^n \) and let \( \mu_i \) be the projection of \( \mu \) onto \( j(\kappa)^i \) for all \( i \leq n \). In particular, \( \mu_n = \mu \). By definition,
\[
\varphi_{\mu_n}(x) = [s \mapsto \text{rank}_{j(T)x}(s)]_{\mu_n} = \text{rank}_{j_{\mu_n}(j(T)x)}([id]_{\mu_n}) > h(\mu_n).
\]
So there is a finite sequence \( s_n \supseteq [id]_{\mu_n} \) with rank \( \geq h(\mu_n) \) in the tree \( j_{\mu_n}(j(T)x) \). Let player I play as his first move the integers \( m_0, \ldots, m_n \), the ordinal \( h_n = h(\mu_n) \), and \( s_n \), where \( s_n \) is the least such sequence. For \( i \geq n \), we will show inductively that player I can maintain the inequality
\[
\text{rank}_{j_{\mu_i}(j(T)x)}(s_i) \geq h(\mu_i).
\]
Whenever player II plays a measure \( \mu_{i+1} \) according to the rules of the game, we have
\[
\text{rank}_{j_{\mu_{i+1}}(j(T)x)}(j_{\mu_i,\mu_{i+1}}(s_i)) = j_{\mu_i,\mu_{i+1}}(\text{rank}_{j_{\mu_i}(j(T)x)}(s_i)) \geq j_{\mu_i,\mu_{i+1}}(h_i) > h_{i+1}.
\]
To show the last step \( j_{\mu_i,\mu_{i+1}}(h_i) > h_{i+1} \), we argue as follows. Recall that for each \( l \) we have \( h_l = h(\mu_l) = \lim_{k \to \omega} \varphi_{\mu_l}(x_k) \). Since the measure \( \mu_{i+1} \) concentrates on \( j(T)x_{(i+1)} \) and projects to \( \mu_i \), for each \( k \) we have
\[
j_{\mu_i,\mu_{i+1}}(\varphi_{\mu_l}(x_k)) = j_{\mu_{i+1}}(\text{rank}_{j_{\mu_l}(j(T)x_k)}(x_k)) = [\text{ext}_{i+1} \text{rank}_{j_{\mu_l}(j(T)x_k)}]_{\mu_{i+1}}(x_k)
\]
where the “extension” of a function \( F : j(\kappa)^i \to \text{Ord} \) to \( j(\kappa)^{i+1} \) is defined by \( \text{ext}_{i+1}F(s) = F(s \upharpoonright i) \) for all \( s \in j(\kappa)^{i+1} \). Note that
\[
[\text{ext}_{i+1} \text{rank}_{j_{\mu_l}(j(T)x_k)}]_{\mu_{i+1}} > [\text{rank}_{j_{\mu_l}(j(T)x_k)}]_{\mu_{i+1}} = \varphi_{\mu_{i+1}}(x_k).
\]
Finally, since for each \( l \) the ordinal \( h_l \) is the eventual value of \( \varphi_{\mu_l}(x_k) \) as \( k \to \omega \), consideration of sufficiently large \( k \) gives \( j_{\mu_i,\mu_{i+1}}(h_i) > h_{i+1} \).

This shows that player I can choose a successor \( s_{i+1} \supseteq j_{\mu_i,\mu_{i+1}}(s_i) \) of rank at least \( h(\mu_{i+1}) \) in the tree \( j_{\mu_{i+1}}(j(T)x) \), thereby maintaining the desired inequality (4.3) for one more step. Then player I can play the integer \( m_{i+1} = x(i+1) \), the least such finite sequence \( s_{i+1} \), and the ordinal \( h_{i+1} = h(\mu_{i+1}) \). By playing in this way, player I can follow the rules forever. This contradicts the previous lemma, which showed that player II has a winning strategy.

By elementarity, there is a scale on \( \mathbb{R} \setminus p[T] \) whose norms are in \( \text{Env}(\Omega) \) as desired. By [32, Section 4.3], there is a sjs \( \vec{A} = \langle A_i : i < \omega \rangle \) such that \( A_0 \) is the universal \( \bar{\Omega} \)-set and the set \( \{ A_i : i < \omega \} \) is Wadge cofinal in \( \text{Env}(\Omega) \).

4.3.2. Beyond \( L^{\Sigma}(\mathbb{R}) \)

The following theorem, due to Woodin [16], gives the existence of strong \( A \)-quasi-iterable \( \Sigma \)-premouse in \( P \).

**Theorem 4.15.** Assume \( \text{ZF + AD + MC}(\Sigma) \). Then for every \( OD(\Sigma) \) set of reals \( A \), there is a strongly \( A \)-quasi-iterable suitable \( \Sigma \)-premouse.
Theorem 4.15, gives the existence of strong $A$-quasi-iterable $\Sigma$-premouse in $P$ for each $A \in \text{rng}(\tilde{A})$. Using the theorem and the fact that $\text{rng}(\tilde{A})$ is countable and Wadge cofinal in $\text{Env}(\Omega) = \wp(\mathbb{R})^P$ (cf. Theorem 4.8 and the subsequent remark), we obtain a countable, suitable $\Sigma$-premouse $Q$ and a $\tilde{A}$-guided iteration strategy $\Lambda$ for $Q$. $\Lambda$ is defined as: for any maximal tree $T$, letting $R$ be the last model of $T$, $\Lambda(T)^71$ is the unique branch $b$ that moves the term relation $\tau^Q_A$ correctly for each $A \in \text{rng}(\tilde{A})$, i.e. $i^b_\delta(\tau^Q_A) = \tau^R_A$; for short trees $T$, $\Lambda(T)$ is the unique branch $b$ such that $Q(b,T) = Q(T)$ is the $Q$-structure for $T$ in $C_\Omega(M(T))$. We define $\Lambda$ on stacks similarly.

Now we use sjs term-condensation (cf. [16, Theorem 5.4.3]), we get that $\Lambda$ has branch condensation as follows: suppose $\tilde{T}$ is according to $\Lambda$ with last model $S$ such that the iteration embedding $i : Q \to S$ exists, and $U$ is according to $\Lambda$ and $c$ is a cofinal branch of $U$ such that there is some embedding $\sigma : M^{\tilde{U}}_c \to S$ such that $i = \sigma \circ i^c_\delta$, then $c = \Lambda(U)$. This is because for each $A \in \text{rng}(\tilde{A})$,

$$\sigma^{-1}(\tau^S_A) = i^c_\delta(\tau^P_A) = \tau^{M^{\tilde{U}}_c}_A$$

by sjs term condensation and the commutativity of the relevant maps. Since the hull $Hull^P_Q(\tau^P_A : A \in \text{rng}(\tilde{A}))$ is cofinal in $\delta^P$, the above equality uniquely determines $c$ (as $\Lambda(U)$).

Finally, we can show that $Lp^\Sigma(\mathbb{R}) \equiv AD^+$ by a core model induction as above. This gives us that $\Lambda \in \Gamma \setminus Lp^\Sigma(\mathbb{R})$ as desired.

Remark 4.16. The proof of Theorem 4.15 is really a simple reflection argument (cf. see [9, 30] for detailed arguments), because we are at the end of a weak gap in this situation. In the situation where we are at a strong gap (e.g. the very first strong gap in $Lp^\Sigma(\mathbb{R})$), where the aforementioned reflection arguments do not apply, we will need Woodin’s full argument in [16] in general. Fortunately, from our particular hypotheses, we can avoid using this theorem. We have replaced that by the argument in Section 4.3.1. This was remarked in the introduction.

4.4. The Limit Case

Recall we let $G \subseteq \mathbb{P}_T$ be $V$-generic and $j = j_G : V \to M = \text{Ult}(V,G)$ be the corresponding ultrapower map; by $\text{Di}$, $G$ corresponds to a $V$-generic $g \subset \text{Coll}(\omega,\omega_1)$. We also let $k : M \to N = \text{Ult}(M,K)$ for some $M$-generic $K \subseteq j(\mathbb{P}_T)$. We will fix these objects from now on.

Let $\langle \theta_\alpha : \gamma < \gamma \rangle$ be the Solovay sequence computed in $\Gamma$ and $\Theta = \sup_\alpha \theta_\alpha$. By the previous section, $\gamma$ is a limit ordinal and $\Theta$ is the Wadge ordinal of $\Gamma$. For $\alpha \leq \Theta$, by $\Gamma \upharpoonright \alpha$, we mean the set of $B \in \Gamma$ such that the Wadge rank of $B$ is less than $\alpha$. First we claim

$$|\Gamma| \leq \mathfrak{c}.$$  

Lemma 4.17. Suppose $|\Gamma| = \mathfrak{c}^+$. Then $\Gamma = \wp(\mathbb{R}) \cap L(\Gamma, \mathbb{R})$.

Proof. Suppose not. Let $\alpha$ be the least such that $\rho_\alpha(J_\alpha(\Gamma, \mathbb{R})) = \mathbb{R}$, i.e. $J_\alpha(\Gamma, \mathbb{R})$ defines a set of reals $A$ such that $A \notin \Gamma$. Hence $\alpha \geq \mathfrak{c}^+$ by our assumption. Let $f : \alpha \times \Gamma \to J_\alpha(\Gamma, \mathbb{R})$ be a surjection that is definable over $J_\alpha(\Gamma, \mathbb{R})$ (from parameters).

71Any short tree $T$ is guided by the unique $Q$-structures in $Lp^\Sigma(\mathcal{M}(T))$ and hence is according to $\Lambda$. A maximal tree has a last model, independent of the cofinal branches of $T$. 

47
We first define a sequence $\langle H_i \mid i < \omega \rangle$ as follows. Let $H_0 = \mathbb{R}$. By induction, suppose $H_n$ is defined and there is a surjection from $\mathbb{R} \to H_n$. Suppose $(\psi, a)$ is such that $a \in H_n$ and $J_\alpha(\Gamma, \mathbb{R}) \models \exists x \psi[x, a]$. Let $(\gamma_{a, \psi}, \beta_{a, \psi})$ be the $\leq_{\text{lex}}$-least pair such that there is a $B \in \Gamma$ with Wadge rank $\beta_{a, \psi}$ such that

$$J_\alpha(\Gamma, \mathbb{R}) \models \psi[f(\gamma_{a, \psi}, B), a].$$

Let then $H_{n+1} = H_n \cup \{f(\gamma_{a, \psi}, B) \mid J_\alpha(\Gamma, \mathbb{R}) \models \exists x \psi[x, a] \wedge w(B) = \beta_{a, \psi} \wedge a \in H_n\}$. It’s easy to see that there is a surjection from $\mathbb{R} \to H_{n+1}$. This uses the fact that $\Theta^\Gamma = \varepsilon^+$ is regular, which implies $\sup \{\beta_{a, \psi} \mid a \in H_n \wedge L_\alpha(\Gamma, \mathbb{R}) \models \exists x \psi[x, a]\} < \Theta = \varepsilon^+$. Let $H = \bigcup_n H_n$. By construction, $H \preceq J_\alpha(\Gamma, \mathbb{R})$. Finally, let $M$ be the transitive collapse of $H$.

Say $M = J_\beta(\Gamma^*, \mathbb{R})$. By construction, it is easy to see that $\Gamma^* = \Gamma \upharpoonright \theta_\gamma$ for some $\gamma$ such that $\theta_\gamma < \Theta$. But then $\rho_\omega(J_\beta(\Gamma^*, \mathbb{R})) = \mathbb{R}$. This contradicts that $\Gamma^*$ is constructibly closed. 

The lemma gives $\Gamma = \wp(\mathbb{R}) \cap L(\Gamma, \mathbb{R})$ and in fact, $L(\Gamma, \mathbb{R}) \models \text{AD} + \Theta$ is regular”. This is because $\Theta = \varepsilon^+$ in this case. This contradicts (t). Therefore, $|\Gamma| \leq \varepsilon$ as desired.

Let $\mathcal{H}$ be the direct limit of hod pairs $(\mathcal{P}, \Sigma) \in \Gamma$ such that $\Sigma$ has branch condensation and is fullness preserving under iteration embeddings by $\Sigma$. So $\lambda^\mathcal{H}$ is a limit ordinal. For each $\alpha < \lambda^\mathcal{H}$, let $\Sigma_\alpha$ be the strategy of $\mathcal{H}(\alpha)$ in $j(\Gamma)$ obtained as a tail of some (any) $j(\Sigma)$, where $(\mathcal{P}, \Sigma)$ is a hod pair in $\Gamma$ with branch condensation and is fullness preserving such that $M(\mathcal{P}, \Sigma) = \mathcal{H}(\alpha)$. Let

$$\Sigma = \oplus_{\alpha < \lambda^\mathcal{H}} \Sigma_\alpha.$$

Now suppose $j$ is continuous at $\lambda^\mathcal{H}$. This implies

$$\text{cof}^V(\lambda^\mathcal{H}) = \omega.$$

This is because $j \upharpoonright \omega^V_1 \in M$ and if $j$ is continuous at $\omega^V_1$, then $j(\omega^V_1)$ is singular in $M$. This contradicts the fact that $j(\omega^V_1)$ is a successor cardinal, hence regular, in $M$. This implies $\text{cof}^V(\lambda^\mathcal{H}) \neq \omega^V$ and hence $\text{cof}^V(\lambda^\mathcal{H}) = \omega$.

Then by fixing a sequence $(\theta_{\alpha_n} : n < \omega \wedge \exists \xi \alpha_n = \xi + 1)$ cofinal in $\Theta$, and by choosing a self-justifying system $\mathcal{A}_n$ of sets cofinal in $\theta_{\alpha_n}$ for each $n$, we let $\mathcal{A} = \oplus_n \mathcal{A}_n$. So $\mathcal{A}$ is a self-justifying system and $\mathcal{A} \notin \Gamma$. Using the core model induction as in [32], we can show $L_p^\mathcal{A}(\mathbb{R}) \models \text{AD}^+$. Contradiction.\footnote{For instance, to see that $\Gamma \upharpoonright \theta_0 \subseteq \Gamma^*$, let $\alpha \in \Gamma$ be $OD$ in $J_\alpha(\Gamma, \mathbb{R})$ from a real $x$. Suppose $A \notin M$. By minimizing the Wadge rank of $A$ and minimizing the ordinal parameters defining $A$, we may assume $A$ is definable in $J_\alpha(\Gamma, \mathbb{R})$ from $x$. By elementarity, $A$ is definable in $M$ from $x$, so $\alpha \in \Gamma^*$. Contradiction.}

From now on, we assume $j$ is discontinuous at $\lambda^\mathcal{H}$. This implies $\Sigma \in j(\Gamma)$. Furthermore,

\begin{lemma}
\[\Sigma \upharpoonright V \in V \text{ and } \Sigma \text{ does not depend on } G.\]
\end{lemma}

\footnote{See [32] for the definition of $L_p^\mathcal{A}(\mathbb{R})$. Alternatively, we can define a $\Theta$-organized hierarchy $L_p^{\oplus_n \Sigma_n}(\mathbb{R})$, where for each $n < \omega$, $(\mathcal{P}_n, \Sigma_n)$ is a hod pair such that $\Sigma_n$ has branch condensation, is fullness preserving and $M_\infty(\mathcal{P}_n, \Sigma_n) = \mathcal{H}(\alpha_n)$ and show $L_p^{\oplus_n \Sigma_n}(\mathbb{R}) \models \text{AD}^+$. The definition of $L_p^{\oplus_n \Sigma_n}(\mathbb{R})$ is a routine adaptation of $L_p^\mathcal{A}(\mathbb{R})$ for a single strategy $\Sigma$ and the scales analysis goes through as well.}
Proof. This follows from Lemma 4.3, which shows that for each $\alpha$, $\Sigma_\alpha \Vdash V \in V$ and is not dependent on the choice of $G$. \hfill \square

Let

\[
H^+ = \begin{cases} 
   L_{P_{\Sigma_\alpha}}(H) & \text{if } \forall M < L_{P_{\Sigma_\alpha}}(H) \rho_\omega(M) \geq \Theta \\
   \mathcal{P} & \text{where } \mathcal{P} < L_{P_{\Sigma_\alpha}}(H) \text{ is least } N \text{ such that } \rho_\omega(N) < \Theta.
\end{cases}
\] (4.4)

To be technically correct, by $L_{P_{\Sigma_\alpha}}(H)$ we mean $L_{P_\Sigma}(H)$ defined inside $L(j(\mathbb{R}), C)$ for some $C \in j(\Gamma)$. This makes sense as $\Sigma \in j(\Gamma)$ and the Solovay sequence of $j(\Gamma)$ has limit length. By Lemma 4.18, we get that $H^+ \in V$.

**Proposition 4.19.** $|H^+| \leq \mathfrak{c}$. Therefore, $j \upharpoonright H^+ \in M$

Proof. It is clear that the statements of the proposition hold if we are in the second case of 4.4. So assume we have $H^+ = L_{P_{\Sigma_\alpha}}(H)$. If $|H^+| = \mathfrak{c}^+$, we would get an $\omega_1$-sequence of distinct reals in $j(\Gamma)$, noting that $(\mathfrak{c}^+)^V = \omega_2^V$ is $\omega_1$ in $M$ by the strongness of $\mathcal{I}$. Contradiction. \hfill \square

Using the embedding $j$, the fact that $j \upharpoonright H^+ \in M$, and the construction in [10, Section 11], we obtain a strategy $\Lambda$ for $H^+$ such that

1. $\Lambda$ extends $\Sigma$;
2. for any $\Lambda$-iterate $\mathcal{P}$ of $H^+$ via a stack $\bar{T}$ such that $i^{\bar{T}}$ exists, there is an embedding $\sigma : \mathcal{P} \to j(H^+)$ such that $j \upharpoonright H^+ = \sigma \circ i^{\bar{T}}$. Furthermore, letting $\Lambda_\mathcal{P}$ be the $\bar{T}$-tail of $\Lambda$, for all $\alpha < \lambda^\mathcal{P}$, $\Lambda_\mathcal{P}(\alpha) \in j(\Gamma)$ has branch condensation.
3. $\Lambda$ is $\Gamma(H^+, \Lambda)$-fullness preserving.

We outline the construction here. The reader can see [10] for more details. See Section 3.3 for a brief discussion on definitions and notations related to stacks of normal trees on hod mice; these are discussed in details in [9, Section 6.2]. We first describe the $j$-realizable iterations.

**Definition 4.20** ($j$-realizable iterations). Let $\bar{T} \in HC^M$ be a stack on $H^+$. We say $\bar{T}$ is $j$-realizable if there is a sequence $\langle \sigma_\mathcal{R} : \mathcal{R} \in tn(\bar{T}) \rangle$ such that

1. $\sigma_{\mathcal{H}^+} = j \upharpoonright \mathcal{H}^+$; for all $\mathcal{R} \in tn(\bar{T})$, $\sigma_\mathcal{R} : \mathcal{R} \to j(\mathcal{H}^+)$.
2. For $\mathcal{R}, \mathcal{Q} \in tn(\bar{T})$ such that $\mathcal{R} \sim^{\bar{T}, s}_{\mathcal{Q}} \mathcal{Q}$, $\sigma_\mathcal{R} = \sigma_\mathcal{Q} \circ \pi^{\bar{T}, \mathcal{Q}}_{\mathcal{R}, \mathcal{Q}}$.
3. For every $\mathcal{R} \in ntn(\bar{T})$, there is a hod pair $(S_\mathcal{R}, \Lambda_\mathcal{R}) \in j(\Gamma)$ that is $j(\Gamma)$-fullness preserving and has branch condensation such that $\sigma_\mathcal{R}[\mathcal{R}(\xi^{\bar{T}, \mathcal{R}} + 1)] \subset \text{rng}(\pi^{\Lambda_\mathcal{R}}_{S_\mathcal{R}, \infty})$.
4. For every $\mathcal{R} \in ntn(\bar{T})$, letting $(S_\mathcal{R}, \Lambda_\mathcal{R})$ be as above, and letting $k_\mathcal{R} : \mathcal{R}(\xi^{\bar{T}, \mathcal{R}} + 1) \to S_\mathcal{R}$ be given by: $k_\mathcal{R}(x) = y$ if and only if $\sigma_\mathcal{R}(x) = \pi^{\Lambda_\mathcal{R}}_{S_\mathcal{R}, \infty}(y)$ and $k_\mathcal{R}^{\bar{T}, \mathcal{R}}$ is according to $\Lambda_\mathcal{R}$.

\[74\text{From DI, we use homogeneity of } \mathbb{P}_T \text{ and } \mathbb{R}^M = \mathbb{R}^{V[G]} \text{ to define } H^+ \text{ from } j(\Gamma), \Sigma \mid V \text{ in } V[G]; \text{ this is possible since by our smallness assumption, no divergent models of } \mathbf{AD}^+ \text{ exist. See the argument in Footnote 47.}\]
5. For every $\mathcal{R} \in ntn(\vec{T})$, let $S^*_\mathcal{R}$ be the last model of $k_\mathcal{R}\vec{T}_\mathcal{R}$ and let $Q_\mathcal{R}$ be the last model $\vec{T}_\mathcal{R}$. Suppose $\pi^{\vec{T}_\mathcal{R}}$ is defined (hence, $Q_\mathcal{R} \in tn(\vec{T})$) and $\mathcal{R} \not\prec^{\vec{T}_s} Q_\mathcal{R}$). Let $k^*_\mathcal{R} : Q_\mathcal{R} \to S^*_\mathcal{R}$ be the natural map that comes from the copying construction. Then for all $x \in Q_\mathcal{R}$,

$$\sigma_{Q_\mathcal{R}}(x) = \sigma_{\mathcal{R}}(f)(\pi^{\vec{T}_\mathcal{R}}(k^*_\mathcal{R}(a))),$$

where $f \in \mathcal{R}$, and $a \in [Q(\pi^{\vec{T}_\mathcal{R}}_{Q_\mathcal{R}}(\xi^{\vec{T}_\mathcal{R}} + 1) < \omega]$ are such that $x = \pi^{\vec{T}_\mathcal{R}}_{Q_\mathcal{R}}(f)(a)$; here $\Lambda = (\Lambda_\mathcal{R})_{k_\mathcal{R}\vec{T}_\mathcal{R}, S^*_\mathcal{R}}$.

6. For every trivial terminal node $\mathcal{R}$, for every $\xi < \lambda^\mathcal{R}$, there is a hod pair $(S_\mathcal{R}, \Lambda_\mathcal{R}) \in j(\Gamma)$ where $\Lambda$ is $j(\Gamma)$-fullness preserving, and has branch condensation and $\sigma_{\mathcal{R}}(\xi + 1) \subset \text{rng}(\pi^{S^*_\mathcal{R}}_{S^*_\mathcal{R}, \infty})$.

The maps $(\sigma_{\mathcal{R}} : \mathcal{R} \in tn(\vec{T}))$ are the $j$-realizable embeddings of $\vec{T}$. In the above, we may also choose $(S_\mathcal{R}, \Lambda_\mathcal{R})$ such that letting $j(\mathcal{H})(\alpha) = M_{\infty}(S_\mathcal{R}, \Lambda_\mathcal{R})$, then $\alpha$ is minimal. \[\square\]

Now we define the domain of the strategy $\Lambda$. Basically, it consists of $j$-realizable stacks. See [10, Definition 11.5].

**Definition 4.21.** Let $\vec{T} \in HC^M$ be a stack of on $\mathcal{H}^+$. We let $\vec{T} \in \text{dom}(\Lambda)$ iff $\vec{T}$ is $j$-realizable. Define $\Lambda(\vec{T}) = b$ iff $\vec{T} \prec b$ is $j$-realizable. \[\square\]

**Lemma 4.22.** Whenever $\vec{T} \in \text{dom}(\vec{T})$, then $\Lambda(\vec{T})$ is defined.

See [10, Lemma 11.6] for a similar argument. In other words, the lemma states that if $\vec{T}$ is $j$-realizable and has no last model, then we can find a cofinal branch $b$ of $\vec{T}$ so that $\vec{T} \prec b$ is $j$-realizable. We sketch the argument here.

**Proof.** Suppose there is a strongly closed, cofinal $C \subset tn(\vec{T})$. In this case $\vec{T}$ has a unique, cofinal, non-dropping branch $b$ determined by $C$. Let $Q = M^\vec{T}_b$ and $\sigma_Q : Q \to j(\mathcal{H}^+)$ be the direct limit of the maps $\{\sigma_{\mathcal{R}} : \mathcal{R} \in C\}$; more precisely, let $\sigma_Q(x) = y$ if and only if there is some $x^* \in R$ for some $R \in C$ such that $\pi^\vec{T}_{\mathcal{R}, Q}(x^*) = x$ and $\sigma_{\mathcal{R}}(x^*) = y$. It is easy to see that $\sigma_Q$ is well-defined and satisfies the clauses of Definition 4.20 (note that in this case, $Q$ is a trivial terminal node).

Otherwise, we are looking for a branch of $\vec{T}_{S^*_p}$. Let $\mathcal{R} = S^*_\vec{T}$ and $\mathcal{U} = \vec{T}_{S^*_p}$. By our hypothesis, objects like $\sigma_{\mathcal{R}}, k_{\mathcal{R}}, (S_\mathcal{R}, \Lambda_\mathcal{R})$ have been defined. Let then $b = \Lambda_\mathcal{R}(k_{\mathcal{R}} \mathcal{U})$, $Q = M^\mathcal{U}_b$, $S^* = M^k_{\mathcal{R} \mathcal{U}}$, and $k : Q \to S^*$ be the natural map induced by the copying process. In the following, we assume $Q$ is a terminal node; otherwise, we’re done.

We let $\Lambda = (\Lambda_\mathcal{R})_{k_\mathcal{R} \mathcal{U}, S^*}$ and $\sigma_Q = \pi^{S^*}_{S^*_\mathcal{R}, \infty} \circ k$. We need to verify clause (6) in the case $Q$ is a trivial terminal node. Fix $\xi < \lambda^Q$. Let $(\mathcal{W}, \Psi) \in j(\Gamma)$ be a hod pair such that $\Psi$ is fullness preserving and has branch condensation and such that $M_{\infty}(\mathcal{W}, \Psi) = j(\mathcal{H}^+)(\sigma_{\mathcal{R}}(\xi + 1))$. We can then find $(S, \Psi_S) \in I(\mathcal{W}, \Psi)$ such that $\sigma_Q[Q(\xi + 1)] \subset \text{rng}(\pi^{S^*_S}_{S^*_S, \infty})$. We are done. \[\square\]

\[75\] $\vec{T}$ either has a strongly linear, closed and cofinal set $C \subseteq tn(\vec{T})$ or $\vec{T}_{S^*_p}$ is of limit length.
**Remark 4.23.** Suppose $\tilde{T} \in \text{dom}(\Lambda)$, then there is at most one $b$ such that $\tilde{T}^b$ is $j$-realizable. In the proof of Lemma 4.22, the only case to verify is when $S_\mathcal{T}$ exists. Let $\mathcal{R}, \mathcal{U}, \sigma_\mathcal{R}, k_\mathcal{R}, (S_\mathcal{R}, \lambda_\mathcal{R})$ be as there. Suppose $(S_\mathcal{R}, \lambda_\mathcal{R})$ and $l_\mathcal{R} : \mathcal{R}(\tilde{e}_\mathcal{R} + 1) \to S_\mathcal{R}$ are such that $l_\mathcal{R}(x) = y$ if and only if $\sigma_\mathcal{R}(x) = \pi_{S_\mathcal{R},\infty}(y)$, $l_\mathcal{R}T_\mathcal{R}$ is according to $\lambda_\mathcal{R}$ and $c = \lambda_\mathcal{R}(l_\mathcal{R}U)$. To see $b = c$, we let $(S, \Psi)$ be the common iterate of $(S_\mathcal{R}, \lambda_\mathcal{R})$ and $(S_\mathcal{R}, \lambda_\mathcal{R})$. Let $\sigma_0 : S_\mathcal{R} \to S$ and $\sigma_1 : S_\mathcal{R} \to S$ be the iteration maps. So $\Lambda_\mathcal{R} = (\Psi)^{\sigma_0} \circ \sigma_1$ because these strategies are pullback consistent. It is also easy to verify that $\sigma_0 \circ k_\mathcal{R} = \sigma_1 \circ l_\mathcal{R}$. Therefore, $b = \Psi^{\sigma_0 \circ k_\mathcal{R}}(U) = \Psi^{\sigma_1 \circ l_\mathcal{R}}(U) = c$.

Clearly, if $\Lambda$ is a $j$-realizable strategy, then $\Lambda$ satisfies (1) and the first clause of (2); by basic hod mice theory (cf. [9]), $\Lambda$ also satisfies the “Furthermore” clause. By the proof of [10, Lemma 11.8], we can choose $\Lambda$ so that $\Gamma(\mathcal{H}^+, \Lambda)$ is Wadge minimal (amongst all strategies $\Lambda$ constructed this way) and this particular choice of $\Lambda$ satisfies (3) as well.

**Remark 4.24.** The construction in [10] is nontrivial in the case that $\mathcal{H}^+ \models \text{cof}(\Theta)$ is measurable; otherwise, $\Lambda$ is simply $\Sigma$ but because of $j$, it acts on all of $\mathcal{H}^+$ by an argument as in the proof of Lemma 4.25.

**Lemma 4.25.** $\mathcal{H}^+ = L_P^{\Sigma^j(\Gamma)}(\mathcal{H})$ and $\mathcal{H}^+ \models \text{cof}(\lambda^\mathcal{H})$ is measurable.

**Proof.** The second clause follows from the first clause and the case assumption that $j$ is discontinuous at $\lambda^\mathcal{H}$. If $\mathcal{H}^+ \models \text{“}\lambda^\mathcal{H} \text{ is regular”}$, then by standard results on Vopenka forcing (cf. [28]), $L[\mathcal{H}^+](\Gamma) \cap \mathcal{P}(\mathbb{R}) = \Gamma$ and therefore, $L(\Gamma, \mathcal{R}) \models \text{“AD}_\mathcal{R} + \Theta \text{ is regular”}$, contradicting our smallness assumption (1). If $\mathcal{H}^+ \models \text{“}\lambda^\mathcal{H} \text{ is singular”}$, then letting $\kappa = \text{cof}\mathcal{H}(\lambda^\mathcal{H})$, then $\kappa$ must be measurable in $\mathcal{H}^+$. This is because $j \restriction (\kappa + 1)$ is the iteration embedding of $\mathcal{H}(\alpha)$ according to $\Psi = \text{def} \Sigma_{\mathcal{H}(\alpha)}$ in $M$ for some (equivalently any) $\alpha$ such that $\kappa \in \mathcal{H}(\alpha)$; therefore, $i^\mathcal{H}_{\mathcal{H}(\alpha), \infty}(\psi) \in \lambda^\mathcal{H}$, implying $\kappa$ is measurable in $\mathcal{H}(\alpha)$, hence in $\mathcal{H}^+$.

Now, suppose for contradiction that there is a $\mathcal{P} \in \mathcal{H}^+ \text{ such that } \rho_{\omega}(\mathcal{P}) < \Theta$. Let $\mathcal{P}$ be the least such. Let $\beta < \lambda^\mathcal{H}$ be least such that $\rho_\omega(\mathcal{P}) \leq \delta^\mathcal{P}_\beta$ and $\delta^\mathcal{P}_\beta > \text{cof}^\mathcal{P}(\lambda^\mathcal{P})$, here $\lambda^\mathcal{P} = \lambda^\mathcal{H}$ and $\delta^\mathcal{P}_\alpha = \delta^\mathcal{H}_\alpha$ for all $\alpha < \lambda^\mathcal{P}$. $\mathcal{P}$ can be considered a hod premouse over $(\mathcal{H}(\beta), \Sigma_\beta)$. Using $j$ and the construction in [10, Section 11] discussed above, we can define a strategy $\Lambda$ for $\mathcal{P}$ such that $\Lambda$ acts on stacks above $\delta^\mathcal{P}_\beta$ and extends $\oplus_{\alpha < \lambda^\mathcal{P}} \Sigma_\alpha$ (the strategy is simply $\oplus_{\alpha < \lambda^\mathcal{P}} \Sigma_\alpha$ for stacks based on $\mathcal{H}$ (above $\delta^\mathcal{P}_\beta$)), but the point is that it also acts on all of $\mathcal{P}$ because of $j$. This is because given a stack $\tilde{T}$ according to $\Lambda$, by a similar calculation as done in 4.2, there is a map $\sigma : \mathcal{M}_{\tilde{T}} \to j(\mathcal{P})$ such that $\sigma \circ i_{\tilde{T}} = j \restriction \mathcal{P}$, where for any $f \in \mathcal{P}$, any generator $a$ used along the main branch of $\tilde{T}$, say $a \in \mathcal{M}_{\tilde{T}}(\gamma)$ and $\mathcal{M}_{\tilde{T}}(\gamma)$ is the image of $\mathcal{P}(\gamma^*)$, then letting $\Psi = \Sigma_\gamma^*$, $\quad \sigma(i_{\tilde{T}}(f)(a)) = j(f)(i_{\tilde{T},\mathcal{M}_{\tilde{T}}(\gamma)}(a))$.

In the above, we note that $i_{\tilde{T}}$ is continuous at $\lambda^\mathcal{P}$, so we can find $\gamma, \gamma^*$. Note that $\Lambda$ has branch condensation. By a core model induction as in the successor case, we get that $\Lambda \in j(\Gamma)$. In $j(\Gamma)$, let $\mathcal{F}$ be the direct limit system of $\Sigma_{\beta'}$-hod pairs $(\mathcal{Q}, \Psi)$ Dodd-Jensen...
equivalent to \((\mathcal{P}, \Lambda)^{76}\). \(\mathcal{F}\) can be characterized as the direct limit system of \(\Sigma_{\beta}\)-hod pairs \((\mathcal{Q}, \Psi)\) in \(j(\Gamma)\) such that \(\Psi\) is \(\Gamma(\mathcal{P}, \Lambda)\)-fullness preserving and has branch condensation and \(\Gamma(\mathcal{Q}, \Psi) = \Gamma(\mathcal{P}, \Lambda)\). \(\mathcal{F}\) only depends on \(\Sigma_{\beta}\) and the Wadge rank of \(\Gamma(\mathcal{P}, \Lambda)\) and hence is \(OD_{\Sigma_{\beta}}^{\mathcal{F}(j(\mathcal{R}), C)}\) for some \(C_j \in j(\Gamma)\).

Fix such a \(C\) and note that \(L(j(\mathbb{R}), C) \models AD^+ + SMC\). Let \(A \subseteq \delta_{\beta}^\mathcal{P}\) witness \(\rho_\omega(\mathcal{P}) \leq \delta_{\beta}^\mathcal{P}\), that is, \(A \notin \mathcal{P}\) and there is a formula \(\phi\) such that for all \(\alpha \in \delta_{\beta}^\mathcal{P}\),

\[
\alpha \in A \iff \mathcal{P} \models \phi[\alpha, p],
\]

where \(p\) is the standard parameter of \(\mathcal{P}\). Now \(A\) is \(OD_{\Sigma_{\beta}}\) in \(L(j(\mathbb{R}), C)\); this is because letting \(\mathcal{M}_{\infty}\) be the direct limit of \(\mathcal{F}\) under iteration maps, then in \(L(j(\mathbb{R}), C), \mathcal{M}_{\infty} \in \text{HOD}_{\Sigma_{\beta}}\) and \(A\) witnesses that \(\rho_\omega(\mathcal{M}_{\infty}) \leq \delta_{\beta}^\mathcal{P}\). By SMC in \(L(j(\mathbb{R}), C)\) and the fact that \(\mathcal{H}(\beta + 1)\) is \(j(\Gamma)\)-full, we get that \(A \notin \mathcal{P}\). This is a contradiction.

\[\Box\]

**Lemma 4.26.** \(j\) is continuous at \(o(\mathcal{H}^+)\), i.e. letting \(\gamma = o(\mathcal{H}^+)\),

\[
j(\gamma) = \sup j[\gamma].
\]

**Proof.** We first claim \(j \upharpoonright \mathcal{H}^+ \in V\). Let \(<\) be the canonical well-order of \(\mathcal{H}^+\); \(<\) is definable over \(\mathcal{H}^+\). We think of \(<\) as a bijection from \(o(\mathcal{H}^+)\) onto \(\mathcal{H}^+\). Note that \(j(\mathcal{H}^+) \in V\) (equivalently \(j(<) \in V\)) and \(j \upharpoonright o(\mathcal{H}^+) \in V\) (this follows DI). \(j \upharpoonright \mathcal{H}^+\) can be easily computed from \(j \upharpoonright o(\mathcal{H}^+), j(\mathcal{H}^+), j(<)\).

Therefore, \(j \upharpoonright \mathcal{H}^+ \in V\).

Suppose for contradiction that \(j(\gamma) > \sup j[\gamma]\). Let \(\nu = \sup j[\gamma]\). Let \(\tilde{C} = (C_\alpha : \alpha < \gamma)\) be the canonical \(\Box_\phi\)-sequence defined over \(\mathcal{H}^+\) (see [15] for a construction of such a sequence). Let \(D = j(\tilde{C})\). Since \(\nu < j(\gamma)\), \(D\) is defined and is club in \(\nu\). Furthermore, since \(j \upharpoonright \mathcal{H}^+ \in V\),

\[
\text{cof}^V(\nu) = \text{cof}^V(\gamma) > \omega.
\]

Since \(j(\mathcal{H}^+) \in V\), \(\text{cof}^V(\mathcal{H}^+)(\nu) > \omega\). This, in particular, implies that the set of limit points of \(D\) is non-empty and in fact a club in \(\nu\). By the property of \(\Box\)-sequences, for each limit point \(\alpha \in D\),

\[
D \cap \alpha = j(\tilde{C})_\alpha.
\]

Since \(j \upharpoonright \mathcal{H}^+ \in V\), \(E =_{\text{def}} j^{-1}[D] \in V\) is an \(\omega\)-club in \(\nu\) with the property: for all limit point \(\alpha\) of \(E\) with \(\text{cof}^V(\alpha) = \omega\),

\[
E \cap \alpha = C_\alpha.
\]

By the construction of \(\tilde{C}\), \(E\) induces a \(\mathcal{P} \triangleleft Lp^{\Sigma_{j(\Gamma)}^{\mathcal{H}}}(\mathcal{H})\), but also that every \(\mathcal{M} \triangleleft \mathcal{H}^+ = Lp^{\Sigma_{j(\Gamma)}^{\mathcal{H}}}(\mathcal{H})\) is an initial segment of \(\mathcal{P}\). So \(\mathcal{P} \notin Lp^{\Sigma_{j(\Gamma)}^{\mathcal{H}}}(\mathcal{H})\). Contradiction.

\[\Box\]

\(^{76}(\mathcal{P}, \Lambda)\) is an anomalous hod pair in the terminology of [9]. \((\mathcal{Q}, \Psi)\) is Dodd-Jensen equivalent to \((\mathcal{P}, \Lambda)\) means that there are non-dropping iterates \((\mathcal{Q}', \Psi')\) of \((\mathcal{Q}, \Psi)\) and \((\mathcal{P}', \Lambda')\) of \((\mathcal{P}, \Lambda)\) such that \((\mathcal{Q}', \Psi') = (\mathcal{P}', \Lambda')\).
4.4.1. Fullness preservation

Let $\Lambda$ be a strategy for $\mathcal{H}^+$ satisfying (1)-(3) above. Now we show $\Lambda$ is fullness preserving in $j(\Gamma)$. Let $E$ be the (long) extender of length $j(\Theta)$ derived from $j$ and $\pi_E : V \to \Ult(V, E)$ be the canonical extender. We note that by Lemma 4.26

$$\pi \upharpoonright \mathcal{H}^+ \cup \{\mathcal{H}^+\} = j \upharpoonright \mathcal{H}^+ \cup \{\mathcal{H}^+\}$$

and

$$\pi_E(\Gamma) = j(\Gamma)$$

**Lemma 4.27.** $\Lambda$ is $j(\Gamma)$-fullness preserving.

**Proof.** Suppose not. Let $\bar{T}$ be according to $\Lambda$ with end model $\mathcal{Q}$ such that $\mathcal{Q}$ is not $j(\Gamma)$-full. This means there is a strong cut point $\gamma$ such that letting $\alpha \leq \lambda^{\mathcal{Q}}$ be the largest such that $\delta^{\mathcal{Q}}_\alpha \leq \gamma$, then without loss of generality, in $j(\Gamma)$, there is a mouse $\mathcal{M} \subset \mathcal{L}p_{\Sigma^{(\alpha)}}(\mathcal{Q}|\gamma)$ such that $\mathcal{M} \not\in \mathcal{Q}$. Let $l : \mathcal{Q} \to j(\mathcal{H}^+)$ be such that $j \upharpoonright \mathcal{H}^+ = l \circ \bar{T}$; here by Lemma 4.26, $j(\mathcal{H}^+) = \pi_E(\mathcal{H}^+) = \{j(f)(a) : a \in [j(\Theta)]^{<\omega} \land f \in \mathcal{H}^+\}$ and $l$ is defined as:

$$l(i \bar{T}(f)(a)) = j(f)(l_{\delta^{\mathcal{Q}}}(\alpha)(a)),$$

where $f \in \mathcal{H}^+$, $a \in [\delta^{\mathcal{Q}}]^{<\omega}$. We use $i$ to denote $i \bar{T}$ from now on.

**Claim 4.28.** There is a $\Sigma$-hod pair $(\mathcal{P}, \Phi)$ such that

(a) $\mathcal{P} \in V$, $\Phi \upharpoonright V \in V$, and $\Phi \in j(\Gamma)$ is fullness preserving and has branch condensation.

(b) $\mathcal{P}$ is countable in $M$, $\lambda^{\mathcal{P}}$ is limit and $\text{cof}(\lambda^{\mathcal{P}})$ is not measurable in $\mathcal{P}$.

(c) in $j(\Gamma)$, $(\mathcal{P}, \Phi)$ witnesses $\Lambda$ is not fullness preserving.

**Proof.** First note that in $M$, there is some $\alpha$ such that the canonical strategy of $\mathcal{M}$ is in $j(\Gamma)|\delta^{\mathcal{P}}_\alpha$, where $\mathcal{P}^* = \text{HOD}^{j(\Gamma)}(\alpha)$ and $\mathcal{P}^* \models \exists \beta \alpha = \beta + \omega$. Such $\mathcal{P}^*$ and $\alpha$ exists by our assumptions on $\Gamma$. $\mathcal{P}^* \subset V$ follows from $\text{DI}$. Let $\Psi$ be the strategy of $\mathcal{P}^*$ which is the tail of some (equivalently, all) $\Sigma$-hod pair $(\mathcal{R}, \Psi) \in j(\Gamma)$ $\Psi$ is fullness preserving and has branch condensation in $j(\Gamma)$ and $\mathcal{M}_\infty(\mathcal{R}, \Psi) = \mathcal{P}^*$. Note that $\Psi$ is fullness preserving and has branch condensation in $k(j(\Gamma))$. It follows that $\Psi \upharpoonright V \in V$. From $\text{DI}$, we can ordinal define $\Psi \upharpoonright V$ in $V[G]$ from $\Sigma$ and $\mathcal{P}$ with the prescription above, using the fact that $j(\Gamma)$ is $OD$ in $V[G]$ and $j(\Theta)$, the Wadge rank of $j(\Gamma)$, doesn’t depend on the choice of $G$; so by homogeneity, $\Psi \upharpoonright V \in V$. By an argument similar to that of Lemma 4.26, $j \upharpoonright \mathcal{P}^* \in V$. We want to find a countable-in-$M$ version of $\mathcal{P}^*$ in $V$.

Let $(\bar{T}, \bar{Q}, \bar{M}, \bar{\Lambda})$ be $\mathbb{P}_\mathcal{Z}$-names for $(\bar{T}, \bar{Q}, \bar{M}, \bar{\Lambda})$ and let $p \in \mathbb{P}_\mathcal{Z}$ force all relevant facts about these objects. Let $X \prec (H_\lambda, \in)$ where

---

77. The case where $\gamma = \delta_\alpha$ and $\mathcal{M} \subset \mathcal{L}p_{\beta^{\mathcal{Q}}\subseteq \delta^{\mathcal{Q}}}(\mathcal{Q}|\gamma)$ is similar.

78. By $\Phi \upharpoonright V$, we mean $\Phi \upharpoonright H^{\lambda_\infty}_{\epsilon_0}$.

79. We identify $\text{HOD}^{j(\Gamma)}$ with the direct limit of $\Sigma$-hod pairs $(\mathcal{R}, \Psi)$ and $\Psi$ is fullness preserving and has branch condensation in $j(\Gamma)$.
• \( \lambda > \epsilon^+ \) is regular,
• \( X^\omega \subseteq X \),
• \( \epsilon \cup \Gamma \cup \mathcal{H}^+ \cup \{ \tilde{T}, \tilde{Q}, \tilde{M}, \Gamma, (\mathcal{P}^*, \Psi \upharpoonright V), (j(\mathcal{P}^*), j \upharpoonright \mathcal{P}^*) \} \subseteq X \), and
• \( |X| = \omega_1 \).

Let \( \pi : M_X \rightarrow X \) be the transitive uncollapse map and for any \( x \in X \), let \( \bar{x} = \pi^{-1}(x) \). Let \( \mathbb{P} = \mathbb{P}_I \).

Note that \( (\mathcal{P}, \mathcal{H}^+) = (\mathcal{P}, \mathcal{H}^+) \), and \( \pi \) extends to an elementary map, which we also call \( \pi \), from \( M_X[g] \) to \( H_X[g] \).

Work in \( M_X[g] \), let \( (\tilde{T}, \tilde{Q}, \tilde{M}, \bar{\Lambda}) \) be the interpretation of \( (\mathcal{T}, \mathcal{Q}, \mathcal{M}, \Lambda) \). Note that we can and do choose \( X \) so that \( (\mathcal{Q}, \mathcal{T}, \mathcal{M}) = (\mathcal{Q}, \mathcal{T}, \mathcal{M}) \). Let \( \sigma = j \upharpoonright \mathcal{P}^* \); so \( \bar{\sigma} : \mathcal{P}^* \rightarrow j(\mathcal{P}^*) \).

Let \( \mathcal{R} \) be the image of \( \mathcal{P}^* \) under the extender \( F \) of length \( \delta_{\mathcal{Q}} \) derived from \( \bar{i} \), i.e.
\[
\mathcal{R} = \{ \bar{i}^T(f)(a) : f \in \mathcal{P}^* \wedge a \in [\delta_{\mathcal{Q}}]^{<\omega} \}.
\]
Let \( i_{F} : \mathcal{P}^* \rightarrow \mathcal{R} \) be the associated ultrapower map, and let \( \bar{l} : \mathcal{R} \rightarrow j(\mathcal{P}^*) \).

Let \( \tau : \mathcal{R} \rightarrow j(\mathcal{P}^*) \) be \( \tau = \pi \circ \bar{l} \). Note that

- \( i_{F} \upharpoonright \mathcal{H}^+ = i_{\mathcal{T}} \).
- \( Q \triangleleft \mathcal{R} \) and \( \mathcal{R} \models \text{"} o(Q) \text{ is a cardinal."} \)
- \( \sigma \circ \pi = \tau \circ i_{F} = \pi \circ \bar{\sigma} \).

The first two items follow from the continuity of \( i_{\mathcal{T}} \) at \( o(\mathcal{H}^+) \); the third item follows from elementarity of \( \pi \). Let \( \Upsilon = j(\Psi \upharpoonright V) \) and \( \Psi^* = \pi^{-1}(\Psi \upharpoonright V) \). In \( M_X[g] \), \( \bar{\Lambda} \) is not full as witnessed by \( \overline{\mathcal{T}, \mathcal{Q}, \mathcal{M}} \) inside \( j(\Gamma)|\bar{\alpha} \), where \( j \) is the generic ultrapower induced by \( g \). Therefore, letting \( \Upsilon^{roi_{F}} = \Sigma_1 \) and \( \Upsilon^\tau = \Sigma_2 \), note that
\[
\Sigma_1 \leq_w \Sigma_2.
\]

Therefore, in \( M \),
\[
\Gamma(\mathcal{P}^*, \Sigma_1) \subseteq \Gamma(\mathcal{R}, \Sigma_2),
\]
and letting \( \Sigma_3 = j(\Sigma)^\tau \),
\[
L(\Gamma(\mathcal{P}^*, \Sigma_1)) \models \text{"} \mathcal{M} \text{ is a } \Sigma_3\text{-mouse and } \neg(\mathcal{M} \triangleleft \mathcal{Q}).\text{"}
\]
Finally, note that \( \mathcal{T} \) is according to \( \Lambda \) as \( \mathcal{T} \) is \( j \)-realizable. It is easy then to see that (a),(b), (c) hold for \( (\mathcal{P}^*, \Sigma_1) \). Therefore, the pair \( (\mathcal{P}^*, \Sigma_1) \) is the desired \( (\mathcal{P}, \Phi) \). See Figure 1.
Let $(P, \Phi)$ be as in the claim. We assume that $L(\Gamma(P, \Phi))$ satisfies the statement: “$Q$ is not full as witnessed by $M$”, i.e. we reuse the notation for $\bar{T}, Q, M, l$. By arguments similar to that used in Lemma 4.25, no levels of $P$ projects across $\Theta$ and in fact, $o(\mathcal{H}^+)$ is a cardinal of $P$. The second clause follows from the following argument. Suppose not and for simplicity, let $H^+ \bigcap \mathcal{N} \bigcap P$ be least such that $\rho_1(\mathcal{N}) = \Theta$. Let $f: \kappa^* \rightarrow \Theta$ be an increasing and cofinal map in $H^+$, where $\kappa^* = \text{cof}(\mathcal{H}^+(\Theta))$. $\mathcal{N}$ is intercomputable with the sequence $g = \langle N_\alpha | \alpha < \kappa^* \rangle$, where $N_\alpha = \text{Th}_i^\mathcal{N}(\delta^\mathcal{H}_f(\alpha) \cup \{p_N\})$. Note that $N_\alpha \in H^+$ for each $\alpha < \kappa^*$.

Now let $R_0 = \text{Ult}_0(H^+, \mu)$, $R_1 = \text{Ult}_1(N, \mu)$, where $\mu \in H^+$ is the (extender on the sequence of $H^+$ coding a) measure on $\kappa^*$ with Mitchell order 0. Let $i_0: H^+ \rightarrow R_0$, $i_1: N \rightarrow R_1$ be the ultrapower maps. Letting $\delta = \delta_\lambda^{H^+} = \Theta$, it’s easy to see that $i_0 \upharpoonright (\kappa^* + 1) = i_1 \upharpoonright (\kappa^* + 1)$ and $\varphi(\delta)^{R_0} = \varphi(\delta)^{R_1}$. The second equality follows from the fact that $R_0$ is full in $j(\Gamma)$ (and hence in $k(j(\Gamma))$).

This means $\langle i_1(N_\alpha) | \alpha < \kappa^* \rangle \in \varphi(\delta)^{R_0}$. By fullness of $H^+$ in $j(\Gamma)$, $\langle i_1(N_\alpha) | \alpha < \kappa^* \rangle \in H^+$. Similarly, $\langle i_0(N_\alpha) | \alpha < \kappa^* \rangle \in H^+$. Using these and the fact that $i_0 \upharpoonright H^+|\Theta = i_1 \upharpoonright N|\Theta \in H^+$, we can get $\mathcal{N} \in H^+$ as follows. For any $\alpha < \Theta, \beta < \kappa^*$, $\alpha \in N_\beta$ if and only if $i_0(\alpha) \in i_1(N_\beta) = i_0(N_\beta)$. Since $H^+$ can compute the right hand side of the equivalence, it can compute the sequence $\langle N_\alpha | \alpha < \kappa^* \rangle$. Contradiction.

In other words, $P$ thinks $H^+$ is full. Let $\Psi = \Phi \upharpoonright V$ and let

$$i^*: P \rightarrow R$$

be the ultrapower map by the extender induced by $i$ of length $\delta^Q$. Note that $Q \bigcap R$ and $R$ is wellfounded since there is a natural map

$$l^*: R \rightarrow \mathcal{P}_E$$

80Any $A \subset \delta$ in $R_0$ is $OD^L_{\Sigma}$, this means $OD^L_{\Sigma}(\delta^R, C)$ for some $C \in j(\Gamma))$ and so by Strong Mouse Capturing (SMC, see [9]), $A \in H^+$.
extending \( l \) and \( \pi_E | P = l^* \circ i^* \); here \( l^*(i^*(f)(a)) = \pi_E(f)(i_{Q,\alpha}(a)) \) for \( f \in P \) and \( a \in [Q]^{\omega} \) and \( P = \{ \pi_E(f)(a) : f \in P \land a \in [j(\Theta)]^{<\omega} \} \). We note here that since \( \pi_E \) is continuous at \( o(H^+) \), \( j(H^+) \) is a cardinal initial segment of \( \mathcal{P}_E \). Furthermore, there is a natural embedding \( \sigma^- : \mathcal{P}_E \to j(\mathcal{P}) \) such that

\[
j \upharpoonright \mathcal{P} = \sigma^- \circ l^* \circ i^*.
\]

Here \( \sigma^- (\pi_E(f)(a)) = j(f)(a) \) for all \( f \in P \) and \( a \in [j(\Theta)]^{<\omega} \). The equality above just comes from the fact that \( E \) is an extender derived from \( j \).

By the choice of \((\mathcal{P}, \Phi)\), \( M \)'s unique strategy \( \Sigma_M \leq M \Phi \) and \( \Sigma_M \in L(\Gamma(\mathcal{P}, \Phi)) \); so in particular, \( L(\Gamma(\mathcal{P}, \Phi)) \) knows \( Q \) is not full as witnessed by \((M, \Sigma_M)\).

Let \( W = M_{\omega}^{\Phi^\omega} \) and \( \Lambda^* \) be the unique strategy of \( W \); again \( W \in V \), \( W \) is countable in \( M \), and \( \Lambda^* \upharpoonright V \in V \). Furthermore, by fullness of \( \mathcal{P} \), \( o(\mathcal{P}) \) is a cardinal of \( W \). Let \( W^* \) be a \( \Lambda^* \)-iterate of \( \mathcal{W} \) below its first Woodin cardinal that makes \((W, \mathcal{T})\) generic via the \((W, \mathcal{T})\)-genericity iteration. Letting \( K \) be the generic for the extender algebra of \( W^* \) at its first Woodin cardinal such that \((W, \mathcal{T}) \in W^*[K] \), then the derived model \( D(W^*[K]) \) (at the supremum of the Woodin cardinals of \( W^*[K] \)) satisfies

\[
L(\Gamma(\mathcal{P}, \Phi), \mathbb{R}) \models Q \text{ is not full}^{81,82}
\]

So the above fact is forced over \( W^*[K] \).

Now further extend \( i^* \) to \( i^+ : W \to V \) and extend \( l^* \) to \( l^+ : Y \to W_E \) so that \( \pi_E | W = l^+ \circ i^+ ; i^+, l^+, W_E \) are defined in a similar manner as above. Again, there is a natural map \( \sigma : W_E \to j(W) \) such that \( \sigma \circ l^+ \circ i^+ = j \upharpoonright W \). Note that \((Y, \sigma \circ l^+)\) are countable in \( M \); this is the key reason we need \( \mathcal{P} \) is countable in \( M \). Therefore, it makes sense to pullback in \( M \) via \( \sigma \circ l^+ \). Let

\[
\Psi^* = j(\Lambda^*)^{\sigma \circ l^+}.
\]

Now note that \( \Phi = (\pi_E(\Psi)^i)^* \) and \( \Lambda^* = (\Psi^*)^{i^+} \), so

\[
\Gamma(\mathcal{P}, \Phi) \subseteq \Gamma(\mathcal{R}, \pi_E(\Psi)^i) \tag{4.5}
\]

and

\[
\Lambda \leq \Psi^* \tag{4.6}
\]

Now iterate \( \mathcal{Y} \) using \( \Psi^* \) to \( \mathcal{Y}^* \) above \( Q \) to make \( \mathbb{R}^M \) generic\(^83\). From 4.5 and 4.6, we get that in

\[81\] Here we abuse notations a bit, by using the same notation for \( \Phi \) and its various restrictions.

\[82\] This is because we can continue iterating \( \mathcal{Y}^* \) above the first Woodin cardinal to \( \mathcal{Y}^{**} \) such that letting \( \lambda \) be the sup of the Woodin cardinals of \( \mathcal{Y}^{**} \), then there is a \text{Col}(\omega, \mathbb{R}^M) \) such that \( \mathbb{R}^{[\mathcal{G}]} \) is the symmetric reals for \( \mathcal{Y}^{**}[\lambda] \). And in \( \mathcal{Y}^{**}[\mathbb{R}^{[\mathcal{G}]}] \), the derived model satisfies \( L(\Gamma(\mathcal{P}, \Phi)) \models Q \) is not full. In the above, we have used the fact that the interpretation of the UB-code of the strategy for \( \mathcal{P} \) in \( \mathcal{Y}^{**} \) to its derived model is \( \Phi \upharpoonright \mathbb{R}^{[\mathcal{G}]} \); this key fact is proved in [9, Theorem 3.26].

\[83\] We write \((\delta^\mathcal{Y}_i : i < \omega) \) for the Woodin cardinals of \( \mathcal{Y} \) and a similar notation applies to iterates of \( \mathcal{Y} \). We work in \( M[E] \) where \( L \subseteq \text{Coll}(\omega, \mathbb{R}^M) \). We have a generic enumeration \((x_n : n < \omega) \) of \( \mathbb{R}^M \) and we have a sequence of normal trees and models \((T_n, \mathcal{Y}_n : n < \omega) \) according to \( \mathcal{Y}^* \), where \( T_0 \) is on \( \mathcal{Y} = \mathcal{Y}_0 \), \( T_n \) is a \( x_n \)-genericity iteration tree on \( \mathcal{Y}_n \) on the window \((\delta^\mathcal{Y}_{n-1}, \delta^\mathcal{Y}_n) \) according to the \( T_{n-1} \)-tail of \( \mathcal{Y}^* \), here \( \delta^\mathcal{Y}_0 = 0 \). Letting \( \mathcal{Y}_\infty \) be the direct limit, then \( \mathbb{R}^M \) is the symmetric reals of \( \mathcal{Y}_\infty \) for some \( \mathcal{Y} \subseteq \text{Coll}(\omega, < \lambda) \), where \( \lambda \) is the supremum of the Woodin cardinals of \( \mathcal{Y}_\infty \).
Figure 2: Diagram for the proof of Lemma 4.27.

\[
\begin{array}{c}
W & \xrightarrow{\pi_E} & W_E \\
\downarrow{j} & \quad & \downarrow{\sigma} \\
\downarrow{i^+} & \quad & \downarrow{l^+} \\
\mathcal{R}
\end{array}
\]

\[
D(\mathcal{Y}^\star),
\]

\[
L(\Gamma(\mathcal{R}, \pi_E(\Psi)^\mathcal{R})) \models Q
\]
is not full as witnessed by \(\mathcal{M}\).

This gives \(\mathcal{M}\) is OD\(\mathcal{Y}^\star\), so \(\mathcal{M} \in \mathcal{Y}^\star\) and so \(\mathcal{M} \in \mathcal{R}\) since \(\mathcal{R}\) is a cardinal initial segment of \(\mathcal{Y}^\star\). This contradicts the internal fullness of \(Q\) inside \(\mathcal{R}\) (\(P\) thinks \(H^+\) is full, so by elementarity, \(\mathcal{R}\) thinks \(Q\) is full).

\[\square\]

**Remark 4.29.** In fact, the proof above shows more (cf. [10, Theorem 10.3]). We can show that \(j\) has the weak condensation property in the sense that, whenever \(\mathcal{R} \in HC^M\) is a hod premouse, \(\tau : \mathcal{R} \to j(\mathcal{H}^+), \pi : \mathcal{H}^+ \to \mathcal{R}\) are such that \(\tau, \pi \in M\) and \(\pi_E | \mathcal{H}^+ = \tau \circ \pi\), then letting \(\Psi\) be the \(\tau\)-pullback of \(j(\Sigma)\) in \(M\), then \(\mathcal{R} = L^\Psi(\mathcal{R}|_{\delta\mathcal{R}})\) in \(j(\Gamma)\).

By a ZFC-diamond comparison argument as in [9, Theorem 2.47] using the fact that \(\Lambda\) satisfies (3), there is a (non-dropping) \(\Lambda\)-iterate \((\mathcal{Q}, \Psi)\) of \((\mathcal{H}^+, \Lambda)\) such that

- \(\Psi\) has branch condensation,
- \(\Gamma(\mathcal{Q}, \Psi) = \Gamma(\mathcal{H}^+, \Lambda)\) and therefore, \(\Psi\) is \(j(\Gamma)\)-fullness preserving.

### 4.4.2. Condensation

Recall we let \(G \subseteq \mathbb{P}_\mathcal{I}\) be \(V\)-generic, \(j : V \to \text{Ult}(V, G) = M\) be the associated ultrapower, \(H \subseteq j(\mathbb{P}_\mathcal{I})\) be \(M\)-generic and \(k : M \to \text{Ult}(M, H) = N\) be the associated ultrapower embedding. Let \(\mathcal{F}\) be the directed system of hod pairs \((\mathcal{P}, \Sigma) \in \Gamma\) such that \(\Sigma\) is fullness preserving and has branch condensation. So \(\mathcal{H}\) is the direct limit of \(\mathcal{F}\). From our hypothesis, \(j \upharpoonright \mathcal{H}^+ \in V \cap M\) and is independent of \(G\). These facts were shown in 4.19 and 4.26.

\[\text{Note that } \Sigma \upharpoonright V \in V, \text{ so } j(\Sigma) \text{ acts on stacks in } \mathcal{H}^+ \text{ in } M.\]
Definition 4.30. In $M$, suppose $X \prec (H_{i+}, \in)$ is countable.\(^{85}\) $X$ is good if letting $\pi_X : M_X \to X$ be the uncollapse map,

(a) $j[\mathcal{H}^+] \cup \{j(\mathcal{H}^+)\} \subset \text{rng}(\pi_X)$;

(b) $\mathcal{H}^+ \cup \{\mathcal{H}^+\} \subset M_X$;

(c) letting $\mathcal{P}_X = \pi_X^{-1}(j(\mathcal{H}^+))$, then $\mathcal{P}_X$ is $j(\Gamma)$-full and for any $\alpha < \lambda^{\mathcal{P}_X}$, $\pi_X \upharpoonright \mathcal{P}_X(\alpha) = i^{\Lambda_X^\mathcal{P}_X(\alpha),\infty}$, where $\Lambda_X$ is a tail of $\Lambda$ for some (equivalently any) hod pair $(Q, \Lambda) \in j(\mathcal{F}) \cap X$ such that $\Lambda$ is $j(\Gamma)$-fullness preserving and has branch condensation and $(\mathcal{M}_\infty(Q, \Lambda))^{M_X} = \mathcal{P}_X(\alpha)$.

\(\square\)

Remark 4.31. (a) Note that if $X$ is good, then $\mathcal{P}_X$ is the transitive collapse of $\text{Hull}^{j(\mathcal{H}^+)}(j[\mathcal{H}^+]) \cup \oplus_{\alpha < \lambda^{\mathcal{P}_X}} i^{\Lambda_X^\mathcal{P}_X(\alpha),\infty}$.

(b) Letting $X^* = \text{Hull}^{H^+}(\mathcal{H}^+)$ and $X = j[X^*]$, then $X$ is good.

(c) Any good $X$ is cofinal in $o(j(\mathcal{H}^+))$ by Lemma 4.26.

Lemma 4.32. In $M$, the set $\{X \cap \mathbb{R} : X \text{ is good}\}$ is in $j(\mathcal{F}_\text{L})$ and the set of good $X$ is closed and unbounded.

Proof. Let $X$ be as in Remark 4.31(b) and let $Y \prec (H_{i+}, \in)$ be countable in $M$, $X \prec Y$, and $\mathcal{H}^+ \cup \{\mathcal{H}^+\} \subset Y$. Since $\mathcal{H}^+$ is countable in $M$, there is a club of such $Y$. Clearly, (a) and (b) in Definition 4.30 hold for $Y$. For (c), using the notation above and Remark 4.29, we have that $\mathcal{P}_Y$ is $j(\Gamma)$-full. Furthermore, for all $\alpha < \lambda^{\mathcal{P}_Y}$, $\pi_Y \upharpoonright \mathcal{P}_Y(\alpha) = i^{\Lambda_X^\mathcal{P}_Y(\alpha),\infty}$ by elementarity of $\pi_Y$. \(\square\)

Suppose $X$ is a good hull, we let $j_X : \mathcal{H}^+ \to \mathcal{P}_X$ be $j_X = \pi_X^{-1} \circ j$. We let $\Lambda_X$ be the strategy for $\mathcal{P}_X$ defined from $\pi_X$ the same way $\Lambda$ is defined from $j$ for $\mathcal{H}^+$. By Lemma 4.27 and the fact that $X$ is good, $\Lambda_X$ is $j(\Gamma)$-fullness preserving. By [9], there is an iterate $(T_X, Q_X)$ of $(\mathcal{P}_X, \Lambda_X)$ such that letting $\Psi_X = (\Lambda_X)_{T_X, Q_X}$, $\Psi_X$ has branch condensation. Let now $\mathcal{M}_\infty^X = \mathcal{M}_\infty(Q_X, \Psi_X)$. Note that $\mathcal{M}_\infty^X = j(\mathcal{H}^+)$$^\infty$ for some $\gamma < j(\lambda^H)$ and $\mathcal{M}_\infty^X$ does not depend on the choice of $(Q_X, \Psi_X)$.

By construction of $\Lambda_X$, there is a map $m_X : \mathcal{M}_\infty^X \to j(\mathcal{H}^+)$ such that

$\pi_X \upharpoonright \mathcal{P}_X = m_X \circ i_{Q_X, \infty} \circ i_{T_X}$.

We need a strong form of condensation to show $\mathcal{H}^+ \models \text{"\Theta is regular";}$ basically, this form of condensation will imply that if $m_X$ is nontrivial, then

$\text{crt}(m_X) = \delta^{\mathcal{M}_\infty^X}$.\(^{86}\)

\(^{85}\)Sometimes, we just write $H^+$ for $(H_{i+}, \in)$ for brevity. Also, note that $c^+ = \omega_2$ in $M$ by elementarity.

\(^{86}\)It could be that $\mathcal{M}_\infty^X = j(\mathcal{H}^+)$ and $m_X$ is the identity map. In which case, we cannot conclude $\Theta$ is regular in $\mathcal{H}^+$. In this case, $\Gamma(\mathcal{H}^+, \Lambda) = j(\Gamma)$. We then simply continue the core model induction. See Section 4.2.
Therefore, $\mathcal{M}_\infty^X \models \"\delta^{\mathcal{M}_\infty^X}\"$ is a regular cardinal which is a limit of Woodin cardinal.” This easily implies $\Theta$ is regular in $\mathcal{H}^+$.

The following definition originates from [10, Definition 11.14]. Let $\mathcal{G}$ be the set of good hulls. For each $X \in \mathcal{G}$, let $\Theta_X = j_X(\Theta)$.

**Definition 4.33.** Suppose $X \in \mathcal{G}$ and $A \in \mathcal{P}_X \cap \mathcal{V}(\Theta_X)$. We say that $\pi_X$ has $A$-condensation if whenever there are elementary embeddings $\nu : \mathcal{P}_X \to \mathcal{Q}$, $\tau : \mathcal{Q} \to j(\mathcal{H}^+)$ such that $\mathcal{Q}$ is countable in $M$ and $\pi_X = \tau \circ \nu$, then

$$v(T_{\mathcal{P}_X,A}) = T_{\mathcal{Q},\tau,A},$$

where

$$T_{\mathcal{P}_X,A} = \{(\phi, s) \mid s \in [\Theta_X]^{<\omega} \land \mathcal{P}_X \models \phi[s, A]\},$$

and

$$T_{\mathcal{Q},\tau,A} = \{(\phi, s) \mid s \in [\delta^\mathcal{Q}]^{<\omega} \text{ for some } \alpha < \lambda_\mathcal{Q} \land j(\mathcal{H}^+) \models \phi[i_{\mathcal{Q}(s),\xi}(s), \pi_X(A)]\},$$

where $\Sigma^{\tau^-}_\mathcal{Q}$ is the $\tau$-pullback strategy of $j(\Sigma)$.  

We say $\pi_X$ has condensation if it has $A$-condensation for every $A \in \mathcal{P}_X \cap \mathcal{V}(\Theta_X)$. \hfill \text{-} 1

In the following, if $X \prec Y$ are good, then let $\pi_{X,Y} = \pi_Y^{-1} \circ \pi_X$. Recall the extender $E$ defined in the previous section.

**Theorem 4.34 (j-condensation lemma).** Let $X^* = \text{Hull}^{\mathcal{H}_Y^+(\mathcal{H}^+)}$ and $X = j[X^*]$; so $\mathcal{P}_X = \mathcal{H}^+$, $\Theta_X = \Theta$, and $\pi_X \upharpoonright \mathcal{P}_X = j \upharpoonright \mathcal{P}_X$. Then $\pi_X$ has condensation.

**Proof.** Fix $A \in \mathcal{P}_X \cap \mathcal{V}(\Theta_X)$. We show that $\pi_X$ has $A$-condensation. Suppose not.

We first claim that if $Y \in \mathcal{G}$ is such that $X \prec Y$ and $\pi_Y$ has $\pi_{X,Y}(A)$-condensation, then $\pi_X$ has $A$-condensation. Fix such a $Y$. Note that $k(\pi_X) = k(\pi_Y) \circ \pi_{X,Y}$ and $k(\pi_Y) = k \upharpoonright j(\mathcal{P}_X) \circ \pi_Y$. By elementarity, $k(\pi_Y)$ has $\pi_{X,Y}(A)$-condensation in $N$ and hence $k \upharpoonright j(\mathcal{P}_X)$ has $j(A)$-condensation in $N$, by the following calculations: for any countable $\mathcal{R}$ in $N$, suppose there are embeddings $i : j(\mathcal{P}_X) \to \mathcal{R}$ and $\tau : \mathcal{R} \to k(j(\mathcal{P}_X))$ such that $k \upharpoonright j(\mathcal{P}_X) = \tau \circ i$, then

$$i(T_{j(\mathcal{P}_X), j(A)}) = i((\pi_Y(T_{\mathcal{P}_X, \pi_{X,Y}(A)})))$$

$$= T_{\mathcal{R}, \tau, \pi_{X,Y}(A)}$$

$$= T_{\mathcal{R}, \tau, j(A)};$$

the second equality uses the fact that $k(\pi_Y)$ has $\pi_{X,Y}(A)$-condensation in $N$ and $k(\pi_Y) = \tau \circ i \circ \pi_Y$. Therefore, $\pi_X$ has $A$-condensation (in $M$) by the elementarity of $j$.

Suppose now for every $Y \in \mathcal{G}$ such that $X \prec Y$, $\pi_Y$ does not have $\pi_{X,Y}(A)$-condensation. Recall that if $(\mathcal{P}, \Sigma)$ is a hod pair such that $\delta^\mathcal{P}$ has measurable cofinality then we let $\Sigma^\mathcal{P} = \oplus_{\alpha < \lambda^\mathcal{P}} \Sigma_{\mathcal{P}(\alpha)}$. We say that a tuple $\{(\mathcal{P}_i, \mathcal{Q}_i, \tau_i, \xi_i, \pi_i, \sigma_i \mid i < \omega), \mathcal{M}_\infty^Y\}$ is a bad tuple (see Figure 3) if

\footnote{\(\Sigma^\mathcal{Q}_{\mathcal{Q}} = \oplus_{\alpha < \lambda^\mathcal{Q}} j(\Sigma)_{\mathcal{Q}(\alpha)}\).}
Figure 3: A bad tuple

1. \( Y \in \mathcal{G} \);
2. \( P_i = P_{X_i} \) for all \( i \), where \( X_i \in \mathcal{G} \);
3. \( X_0 = X \) and for all \( i < j \), \( X_i < X_j < Y \);
4. for all \( i \), \( \xi_i : P_i \rightarrow Q_i \), \( \sigma_i : Q_i \rightarrow \mathcal{M}_\infty^Y \), \( \tau_i : P_{i+1} \rightarrow \mathcal{M}_\infty^Y \), and \( \pi_i : Q_i \rightarrow P_{i+1} \);
5. for all \( i \), \( \tau_i = \sigma_i \circ \xi_i \), \( \sigma_i = \tau_{i+1} \circ \pi_i \), and \( \pi_{X_i} X_{i+1} \mid P_i = \text{def} \phi_{i,i+1} = \pi_i \circ \xi_i \);
6. \( \phi_{i,i+1}(A_i) = A_{i+1} \), where \( A_i = \pi_{X_i} X_i(A) \);
7. for all \( i \), \( \xi_i(T_{P_i,A_i}) \neq T_{Q_i,\sigma_i,A_i} \).

In (7), \( T_{Q_i,\sigma_i,A_i} \) is computed relative to \( \mathcal{M}_\infty^Y \), that is

\[
T_{Q_i,\sigma_i,A_i} = \{ (\phi, s) \mid s \in [\delta^{\mathcal{Q}_i}_\alpha]^{<\omega} \text{ for some } \alpha < \lambda^{\mathcal{Q}_i} \land \mathcal{M}_\infty^Y \models \phi[i^{\mathcal{Q}_i(\alpha)}_{(\alpha)}(s), \tau_i(A_i)] \}
\]

**Claim:** There is a bad tuple.

**Proof.** For brevity, we first construct a bad tuple \( \{ (P_i, Q_i, \tau_i, \xi_i, \pi_i, \sigma_i \mid i < \omega), j(\mathcal{H}^+) \} \) with \( j(\mathcal{H}^+) \) playing the role of \( \mathcal{M}_\infty^Y \). We then simply choose a sufficiently large \( Y \in \mathcal{G} \) and let \( i_Y : \mathcal{P}_Y \rightarrow \mathcal{M}_\infty^Y \) be the direct limit map, \( m_Y : \mathcal{M}_\infty^Y \rightarrow \mathcal{H}^+ \) be the natural factor map, i.e. \( m_Y \circ i_Y = \pi_Y \). It’s easy to see that for all sufficiently large \( Y \), the tuple \( \{ (P_i, Q_i, m_Y^{-1} \circ \tau_i, m_Y^{-1} \circ \xi_i, m_Y^{-1} \circ \pi_i, m_Y^{-1} \circ \sigma_i \mid i < \omega), \mathcal{M}_\infty^Y \} \) is a bad tuple. But the existence of such a tuple \( \{ (P_i, Q_i, \tau_i, \xi_i, \pi_i, \sigma_i \mid i < \omega), j(\mathcal{H}^+) \} \) follows from our assumption.

By essentially the same proof as in Claim 4.28, we have a \( \Sigma^{-}_{\mathcal{P}_0} \)-hod pair \( (\mathcal{P}_0^{+}, \Pi)^{88} \) such that

(a) \( \lambda_{\mathcal{P}_0^{+}} \) is limit ordinal of the form \( \alpha' + \omega \), and such that \( \Lambda_Y \leq \omega \Pi_{\mathcal{P}_0^{+}}(\alpha') \) (so \( \Lambda_{X_i} \leq \omega \Pi_{\mathcal{P}_0^{+}}(\alpha') \) for all \( i \)).

(b) \( (\mathcal{P}_0^{+}, \Pi \mid V) \in V \).

---

\(^{88}\Sigma^{-}_{\mathcal{P}_0} \) is just \( \Sigma \) since \( \mathcal{P}_0 = \mathcal{H}^+ \).
(c) In $M$, $\mathcal{P}_0^+$ is countable and $\Gamma(\mathcal{P}_0^+(\alpha'), \Pi_{\mathcal{P}_0^+(\alpha')}) \models \mathcal{A}$ is a bad tuple.

This type of reflection is possible because we replace $j(\mathcal{H}^+)$ by $\mathcal{M}_{\infty}^Y$. If $Z$ is the result of iterating $\mathcal{P}_0^+$ via $\Pi$ above $\delta_{0,0}^+ \mathcal{P}_0$ to make $R^M$ generic (see Footnote 83), then letting $h$ be $Z$-generic for the Levy collapse of the sup of $Z$’s Woodin cardinals such that $R^M$ is the symmetric reals of $Z[h]$, then in $Z(R^M)$,

$$\Gamma(\mathcal{P}_0^+(\alpha'), \Pi_{\mathcal{P}_0^+(\alpha')}) \models \mathcal{A}$$

is a bad tuple.

Now we define by induction $\xi_i^+: \mathcal{P}_i^+ \to \mathcal{Q}_i^+$, $\pi_i^+: \mathcal{Q}_i^+ \to \mathcal{P}_{i+1}^+$, $\phi_{i,i+1}^+: \mathcal{P}_i^+ \to \mathcal{P}_{i+1}^+$ as follows. $\phi_{0,1}^+: \mathcal{P}_0^+ \to \mathcal{P}_1^+$ is the ultrapower map by the extender of length $\Theta_X$, derived from $\pi_{X_0,X_1}$. Note that $\phi_{0,1}^+$ extends $\phi_{0,1}$. Let $\xi_0^+: \mathcal{P}_0^+ \to \mathcal{Q}_0^+$ extend $\xi_0$ be the ultrapower map by the $(\text{crt}(\xi_0), \delta_{00})$-extender derived from $\xi_0$. Finally let $\pi_0^+ = (\phi_{0,1}^+)^{-1} \circ \xi_0^+$. The maps $\xi_i^+, \pi_i^+, \phi_{i,i+1}^+$ are defined similarly. Let also $\mathcal{M}_Y = \text{Ult}(\mathcal{P}_0^+, F)$, where $F$ is the extender of length $\Theta_Y$ derived from $\pi_{X,Y}$. There are maps $\epsilon_n: \mathcal{P}_i^+ \to \mathcal{M}_Y$, $\epsilon_0 +1: \mathcal{Q}_i^+ \to \mathcal{M}_Y$ for all $i$ such that $\epsilon_i = \epsilon_{i+1} \circ \xi_i^+$, $\epsilon_i = \epsilon_{i+1} \circ \phi_{i,i+1}^+$, and $\epsilon_0 = \epsilon_{i+1} \circ \pi_i^+$.

Let $\pi: \mathcal{M}_Y \to j(\mathcal{P}_0^+)^{89}$ be the factor map. When $i = 0$, $\epsilon_0$ is simply $\pi_F$, the ultrapower map by $F$. Letting $\Sigma_i = \Sigma_{\mathcal{P}_i}$ and $\Psi_i = \Sigma_{\mathcal{Q}_i}$, there is a finite sequence of ordinals $t$ and a formula $\theta(u, v)$ such that in $\Gamma(\mathcal{P}_0^+, \Pi)$

8. for every $i < \omega$, $(\phi, s) \in T_{\mathcal{P}_i, \mathcal{A}_i} \iff \theta[i, \Sigma_i] \in (s_i, t)$, where $\alpha$ is least such that $s \in \delta_{0,1}^\omega$;

9. for every $i$, there is $(\phi_i, s_i) \in T_{\mathcal{Q}_i, \xi_i(\mathcal{A}_i)}$ such that $-\theta[i, \Psi_i] \in (s_i, t)$, where $\alpha$ is least such that $s_i \in \delta_{0,1}^\omega$.

The pair $(\theta, t)$ essentially defines a Wadge-initial segment of $\Gamma(\mathcal{P}_0^+, \Pi)$ that can define the pair $(\mathcal{M}_Y^{\omega}, A^*)$, where $\tau_i(\mathcal{A}_i) = A^*$ for some (any) $i$. In fact, these parameters are inside $\Gamma(\mathcal{P}_0^+(\alpha'), \Pi)$.

Let $\Pi_i$ be the $\pi \circ \epsilon_i$-pullback of $j(\Pi)$. Hence,

$$\Sigma_\infty \leq_w \Pi_0 \leq_w \Pi_1 \cdots \leq_w j(\Pi \cap V)^\pi.$$

By a similar argument as in [30, Lemma 4.3]^{90}, we can use the strategies $\Pi_i$’s to simultaneously execute a $R^M$-genericity iterations. We outline the process here. First we rename $(\mathcal{P}_i^+, \xi_i^+, \phi_{i,i+1}^+, \pi_i^+, \epsilon_i^+ \mid i < \omega)$ to $(\mathcal{P}_i^0, \xi_i^0, \phi_i^0, \pi_i^0, \epsilon_i^0 \mid i < \omega)$. We fix in $M^\text{Col}(\omega, R)$, $\langle x_i \mid i < \omega \rangle$, a generic enumeration of $R^M$. We get $\langle \mathcal{P}_i^0, \xi_i^0, \phi_i^0, \pi_i^0, \tau_i^0, k_i^0 \mid n \leq \omega \land i < \omega \rangle$ such that

(i) $\mathcal{P}_i^0$ is the direct limit of the $\mathcal{P}_i^0$’s under maps $\tau_i^0$’s for all $i < \omega$.

(ii) $\mathcal{Q}_i^0$ is the direct limit of the $\mathcal{Q}_i^0$’s under maps $k_i^0$’s for all $i < \omega$.

(iii) $\mathcal{P}_i^\omega$ is the direct limit of the $\mathcal{P}_i^\omega$’s under maps $\tau_i^\omega$’s.

(iv) for all $n \leq \omega$, $i < \omega$, $\phi_i^n: \mathcal{P}_i^\omega \to \mathcal{P}_{i+1}^\omega$, $\xi_i^n: \mathcal{P}_i^\omega \to \mathcal{Q}_{i+1}^\omega$, $\pi_i^n: \mathcal{Q}_i^\omega \to \mathcal{P}_{i+1}^\omega$ and $\phi_i^n = \pi_i^n \circ \xi_i^n$.

89 As in the proof of 4.27, $\pi = \sigma_1 \circ \sigma_0$, where $\sigma_0: \mathcal{M}_Y \to \pi_E(\mathcal{P}_0^+)$ is given by $\sigma_0(\pi_X(\mathcal{H}^+)) = \pi_E(\mathcal{H}^+) \mathcal{H}^+$ for $f \in \mathcal{P}_0^+$ and $a \in \mathcal{P}_0^\omega$ and $\sigma_1: \pi_E(\mathcal{P}_0^+) \to j(\mathcal{P}_0^+)$ is defined as: $\sigma_1(\pi_E(\mathcal{H}^+) = j(\mathcal{H}^+) \mathcal{H}^+$ for $f \in \mathcal{P}_0^+$ and $a \in \mathcal{P}_0^\omega$.

90 This is sometimes called the “three dimensional argument”.

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Then we start by iterating $\mathcal{P}_0^\omega$ above $\delta^\mathcal{P}_0^\omega$ to $\mathcal{P}_1^\omega$ to make $x_0$-generic at $\delta_{\alpha+1}^\mathcal{P}_0^\omega$; say the tree is $T_0$. We let $\tau_0^0 : \mathcal{P}_0^0 \to \mathcal{P}_1^0$ be the iteration map. During this process, we lift $T_0$ to all $\mathcal{P}_n^0, Q_n^0$ for $n < \omega$ using the maps $\xi_0^0, \phi_0^0$. We pick branches for the trees on $\mathcal{P}_1^0, Q_1^0$ according to the strategies $\Pi_i$. We describe this process for the models $Q_n^0, \mathcal{P}_i^0$.

Note that the tree $\xi_0^0T_0$ is according to $\Pi_1$. We then iterate $\mathcal{W}$ to $Q_1^0$ (using $(\Pi_1)_\mathcal{W}$) to make $x_0$ generic at $\delta^\mathcal{Q}_1^0$. Let $\xi_0^1 : \mathcal{P}_1^0 \to Q_1^0$ be the iteration map. During this process, we lift $T_1$ be the $x_0$-genericity iteration tree on $\mathcal{W}$ just described and $\mathcal{W}^\omega$ be the last model of $\phi_0^0T_0, \xi T_1$, where $\xi$ is the natural map from $\mathcal{W}$ to the last model of $\phi_0^0T_0$. We then iterate the end model of the lifted stack $\phi_0^0T_0, \xi T_1$ on $Q_1^0$, noting that this stack is according to $\Pi_2$, to $Q_1^1$ to make $x_0$ generic at $\delta^\mathcal{Q}_1^1$. Let $\eta_0^1 \mathcal{Q}_1^0 \to Q_1^1$, $\tau_1^0 : \mathcal{P}_1^0 \to \mathcal{P}_1^1$ be the iteration embeddings, $\pi_0^1 : Q_1^1 \to \mathcal{P}_1^1$ be the natural map, and $\phi_1^0 = \pi_0^1 \circ \xi_0^1$.

Continue this process of making $x_0$ generic for the later models $Q_n^0$'s and $\mathcal{P}_n^0$'s for $n < \omega$. We then start at $\mathcal{P}_1^0$ and repeat the above process, iterating above $\delta^\mathcal{P}_1^0$ to make $x_1$ generic at images of $\delta_{\alpha+2}^\mathcal{P}_1^0$ etc. This whole process defines models and maps $\langle \mathcal{P}_i^n, Q_i^n, \xi_i^n, \phi_i^n, \pi_i^n, \tau_i^n, k_i^n | n \leq \omega \land i < \omega \rangle$ as described above. See Figure 4.

The process yields a sequence of models $\langle \mathcal{P}_{i+1,\omega}^+, Q_{i+1,\omega}^+, \xi_{i+1,\omega}^+ | i < \omega \rangle$ and maps $\eta_{i+1,\omega}^+ : \mathcal{P}_{i+1,\omega}^+ \to Q_{i+1,\omega}^+$, $\pi_{i+1,\omega}^+ = \pi_{i,\omega}^+ : Q_{i,\omega}^+ \to \mathcal{P}_{i+1,\omega}^+$, $\phi_{i+1,\omega}^+ = \phi_{i,\omega}^+ \circ \pi_{i,\omega}^+$. Furthermore, each $\mathcal{P}_{i,\omega}^+, Q_{i,\omega}^+$

Figure 4: The $(x_n : n < \omega)$ genericity iteration process

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embeds into a \(j(\Pi \upharpoonright V)^{\pi}\)-iterate of \(M_{\Pi}\) and hence the direct limit \(P_{\infty}\) of \(\langle P_{i,\omega}^+, Q_{j,\omega}^+ \mid i, j < \omega \rangle\) under maps \(\pi_{i,\omega}^+\)'s and \(\xi_{i,\omega}^+\)'s is wellfounded.

Let \(C_i\) be the derived model of \(P_{i,\omega}^+\), \(D_i\) be the derived model of \(Q_{i,\omega}^+\) (at the sup of the Woodin cardinals of each model), then \(R^N = R^{C_i} = R^{D_i}\). Furthermore, \(C_i \cap \varphi(\mathbb{R}) \subseteq D_i \cap \varphi(\mathbb{R}) \subseteq C_{i+1} \cap \varphi(\mathbb{R})\) for all \(i\).

(8), (9) and the construction above give us that there is a \(t \in [\text{OR}]^{<\omega}\), a formula \(\theta(u, v)\) such that

10. for each \(i\), in \(C_i\), for every \((\phi, s)\) such that \(s \in \delta P_i\), \((\phi, s) \in T_{P_i,A_i} \Leftrightarrow \theta[i^{Q_i}_{\varphi(\mathbb{R}),\infty}(s), t]\) where \(\alpha\) is least such that \(s \in [\delta^{P_i}_\alpha]^{<\omega}\).

Let \(n\) be such that for all \(i \geq n\), \(\xi_{i,\omega}^+(t) = t\). Such an \(n\) exists because the direct limit \(P_{\infty}\) is wellfounded.\(^{91}\) By elementarity of \(\xi_{i,\omega}^+\) and the fact that \(\xi_{i,\omega}^+ \upharpoonright P_i = \xi_i\).

11. for all \(i \geq n\), in \(D_i\), for every \((\phi, s)\) such that \(s \in \delta Q_i\), \((\phi, s) \in T_{Q_i,\xi_i(A_i)} \Leftrightarrow \theta[i^{Q_i}_{\varphi(\mathbb{R}),\infty}(s), t]\) where \(\alpha\) is least such that \(s \in [\delta^{Q_i}_\alpha]^{<\omega}\).

However, using (9), we get

12. for every \(i\), in \(D_i\), there is a formula \(\phi_i\) and some \(s_i \in [\delta^{Q_i}]^{<\omega}\) such that \((\phi_i, s_i) \in T_{Q_i,\xi_i(A_i)}\) but \(\neg \phi_i[i^{Q_i}_{\varphi(\mathbb{R}),\infty}(s_i), t]\) where \(\alpha\) is least such that \(s \in [\delta^{Q_i}_\alpha]^{<\omega}\).

Clearly (11) and (12) give us a contradiction. This shows that \(\pi_X\) has \(A\)-condensation. Since \(A\) is arbitrary, \(\pi_X\) has condensation. This completes the proof of the theorem.

\section{4.5. The equiconsistency}

Let \(X\) be as in the statement of Theorem 4.34. If \(M_{\infty}^X = j(\mathcal{H})\), then \(j(\Gamma) = I(P_X, \Lambda_X)\). In this case, we see that \(\Lambda_X\) is definable in \(M\) from \(j(\mathcal{H})\) and \(j(\Gamma)\) (see the proof of Lemma 4.3). Therefore, by elementarity, there is a hod pair \((P, \Lambda) \in V\) such that \(\Lambda\) is definable in \(V\) from a countable sequence of ordinals and \(I(P, \Lambda) = \Gamma\). In particular, \(\Lambda \notin \Gamma\). A core model induction as above shows that \(L^{P_A}(\mathbb{R}) \models \text{AD}^+\). This contradicts the maximality of \(\Gamma\).

Suppose \(M_{\infty}^X = j(\mathcal{H})(\alpha)\) for some \(\alpha < j(\lambda^H)\). In this case, \(m_X\) is not the identity map and we want to show \(P_X \models \Theta_X\) is regular. Suppose \((Q, \tilde{T}) \in I(P_X, \Lambda_X)\) is such that \(i^{\tilde{T}} : P_X \rightarrow Q\) exists. Let \(\gamma^{\tilde{T}}\) be the sup of the generators of \(\tilde{T}\). For each \(x \in Q\), say \(x = i^{\tilde{T}}(f)(s)\) for \(f \in P_X\) and \(s \in [\delta^{Q}_\alpha]^{<\omega}\), where \(\delta^{Q}_\alpha \leq \gamma^{\tilde{T}}\) is least such, then let

\[\tau_Q(x) = \pi_X(f)(i^{Q, \varphi, \tau}_{\varphi(\mathbb{R}),\infty}(s)).\]

\textbf{Remark 4.35.} By Theorem 4.34 and [9, Theorem 3.26], \(\tau_Q\) is elementary and \(\tau_Q \upharpoonright \delta^Q = i^{Q, \varphi, \tau}_{\varphi(\mathbb{R}),\infty} \upharpoonright \delta^Q = i^{\tilde{T}}_{\varphi(\mathbb{R}),\infty} \upharpoonright \delta^Q\), where \(\gamma\) is the \(\tau_Q\)-pullback strategy of \(Q\upharpoonright \delta^Q\). We note that the function \(f\) in the definition above can be coded by a set \(A \in P_X \cap \varphi(\Theta_X)\).

\(^{91}\)We can arrange that \(P_{\infty}\) embeds into a \((j(\Pi))^+\)-iterate of \(j(P_0^+)\), where \(j(\Pi)^+\) is the canonical extension of \(j(\Pi)\) in \(N\).
Lemma 4.36. Suppose \((Q, \tilde{T}) \in I(\mathcal{P}_X, \Lambda_X)\) and \((R, \tilde{U}) \in I(Q, \Sigma_Q, \tilde{T})\) are such that \(i_{\tilde{T}}, i_{\tilde{U}}\) exist and \(\Sigma_Q, \tilde{T}\) and \(\Sigma_R, \tilde{U}\) have branch condensation. Then \(\tau_Q = \tau_R \circ i_{\tilde{U}}\).

Proof. Let \(x \in Q\). There are some \(f \in \mathcal{P}_X\) and \(s \in [\gamma_{\tilde{T}}]^{<\omega}\) such that \(x = i_{\tilde{T}}(f)(s)\). So \(\tau_Q(x) = \pi_X(f)(i_{\Sigma_Q, \tilde{T}}(s))\). On the other hand, \(\tau_R \circ i_{\tilde{U}}(x) = \tau_R \circ i_{\tilde{U}}(i_{\tilde{T}}(f)(s)) = \pi_X(f)(i_{\Sigma_R, \tilde{U}} \circ i_{\tilde{U}}(s)) = \pi_X(f)(i_{\Sigma_Q, \tilde{T}}(s)) = \tau_Q(x)\).

The remark and lemma imply that the map \(\sigma : \mathcal{M}_\infty^X \to j(\mathcal{H}^+)\) defined as:

\[
\sigma(x) = y \text{ iff whenever } (R, \Psi) \in I(\mathcal{H}^+, \Lambda) \text{ is such that } \Psi \text{ has branch condensation, and } i_{\Psi}^{\mathcal{M}_\infty^X}(x^*) = x \text{ for some } x^*, \text{ then } y = \tau_R(x^*)
\]

is elementary and

\[
\text{crt}(\sigma) = \delta =_{\text{def}} \delta^{\mathcal{M}_\infty^X}.
\]

This implies that \(\mathcal{M}_\infty^X \models \text{"\(\delta\) is regular"}\). Let \((Q, \Psi) \in I(\mathcal{P}_X, \Lambda_X)\) be such that \(\Psi\) has branch condensation. By a similar argument as those used before, we get \(\Psi \in \Omega\) and in fact since \(Q \models \text{"\(\delta^Q\) is regular"}\), we easily get that \(N = \mathcal{L}(\Gamma(Q, \Psi), \Lambda) \models \text{"\(\Theta\) is regular"}\) (note that \(\Theta^N\) is the image of \(\delta^Q\) under the direct limit map into the direct limit of all \(\Psi\)-iterates). This contradicts the assumption that there is no model \(M\) satisfying \(\text{"AD}_\mathbb{R} + \Theta\) is regular"\). Such an \(M\) has to exist after all. This finishes this subsection and the proof of Theorem 1.6.

5. OUTLINE OF THE PROOF OF THEOREM 1.10

We outline the argument constructing models of \(\text{"AD}_\mathbb{R} + \Theta\) is regular"\) from the assumption that the non-stationary ideal on \(\varphi_{\omega_1}(\mathbb{R})\) is strong and pseudo-homogeneous. We let \(\mathcal{I}\) be the non-stationary ideal on \(\varphi_{\omega_1}(\mathbb{R})\). Let \(G \subseteq \mathcal{P}_\mathcal{I}\) be \(V\)-generic and \(j = j_G : V \to M = \text{Ult}(V, G) \subseteq V[G]\) be the generic embedding. Let \(k : M \to N\) be the generic embedding given by an \(M\)-generic \(H \subset j(\mathcal{P}_\mathcal{I})\). We note that

- \(j(\omega_1) = \mathfrak{c}^+\) (by the strength of the ideal).
- The properties in Lemma 2.5 hold for \(j\).
- Letting \(M = \text{Ult}(V, G)\). \(M\) need not be closed under \(\omega\)-sequences in \(V[G]\). In particular, \(\mathbb{R}^M\) may differ from \(\mathbb{R}^{V[G]}\). Also, \(\mathfrak{c}^+\) may be \(\omega_2^V\).

We let \(\Gamma\) be defined as in Definition 4.1 and operate under the smallness assumption \(\dagger\) as before. \(\dagger\) is our inductive hypothesis. The core model induction is almost verbatim to the one given in the previous section. We mention some key points below. The details are left to the reader.

- If \(J\) is a \(\Sigma\)-cmi operator on \((a \text{ cone above some } a \text{ in } H_{\omega_1}^V\) that satisfies \(\dagger\), then by pseudo-homogeneity, we can show \(j(J) \restriction V \subseteq V\) and by strongness, \(j(J) \restriction V \subseteq V\) has domain the cone above \(a\) in \(H_{\omega_1}^V\). The definability calculations are done in \(M\) and \(V[G]\) plays no role in the argument.
From pseudo-homogeneity, we can define in $M \langle \mathcal{H}^+ \rangle$ (see 4.4) from $j \upharpoonright \mathcal{H}$ and $j[\bar{\lambda}_\infty] = \lambda^M$, where for each such $\alpha$, $M_\infty(\mathcal{Q}_\infty, \Lambda_\infty) = \mathcal{H}(\alpha)$ and these parameters are of the form $j[A]$ for some $A \subseteq \lambda^\infty$ and some $\lambda < c^+$. Therefore, $\mathcal{H}^+ \in V$ by pseudo-homogeneity.

- $j(\mathcal{H}^+) \in V$ by pseudo-homogeneity as $j(\mathcal{H}^+)$ is definable in $M$ by a countable sequence of ordinals.

- By Lemma 2.5 and arguments in Proposition 4.19 and Lemma 4.26, we get that $j \upharpoonright \mathcal{H}^+ \in V \cap M$.

- Lemma 4.3 then follows from our hypothesis.

- Lemma 4.4 and Theorem 4.5 are proved by an argument as in [32, Theorem 2.4.4]. Theorem 4.8 then follows with the same proof.

- The proof of Lemma 4.10 still goes through, which gives us $o(Lp^1_\omega(\mathbb{R})) < j(\omega^+)$. Though in this case, $j(\omega^+) = c^+ = \omega^+$. The proof of Lemmata 4.13 and 4.14 is given in [32].

- We can show the corresponding claim in Section 4.4 that continuity of $j$ at $\lambda^\mathcal{H}$ implies $\text{cof}^V(\lambda^\mathcal{H}) = \omega$ as follows. If $\kappa \in [\omega_1, c]$ is a successor cardinal or a weakly inaccessible cardinal, then $j$ is discontinuous at $\kappa$. This is because $j \upharpoonright \kappa \in M$ and if $j$ is continuous at $\kappa$, then $j(\kappa)$ is singular in $M$. This contradicts the fact that $j(\kappa)$ is successor or weakly inaccessible, hence regular, in $M$. This implies $\text{cof}^V(\lambda^\mathcal{H}) = \omega$. The proof that $|\mathcal{H}^+| \leq c$, $\Sigma \in V \in V$ and does not depend on $G$, $\mathcal{H}^+ = Lp^{\Sigma,j(\Gamma)}(\mathcal{H}) \models \text{"cof}^V(\lambda^\mathcal{H})$ is measurable" (if $j$ is discontinuous at $\lambda^\mathcal{H}$), and $j$ is continuous at $o(\mathcal{H}^+)$ is the same.

- The proof of Lemma 4.27 is the same modulo Claim 4.28, so we outline the proof of the claim here (with notations as in the proof of the lemma). Let $(\mathcal{P}^*, \Phi)$ be as in the claim. By pseudo-homogeneity, we easily see that $\mathcal{P}^* \subseteq V$, $\Phi \upharpoonright V$ and independent of $G$. We also have that $j(\mathcal{P}^*) \subseteq V$. This is because $j(\mathcal{P}^*)$ is definable in $M$ from $\{j(\alpha), j(\mathcal{I}), j(\mathcal{H}^+)\}$, but $j(\mathcal{I})$ and $j(\mathcal{H}^+)$ are both definable in $M$.

Let $(\bar{T}, \bar{Q}, \bar{M}, \Lambda)$ be $\mathbb{P}_T$-names for $(T, Q, M, \Lambda)$ and let $p \in \mathbb{P}_T$ force all relevant facts about these objects. Let $X < (H_\infty, \infty)$ where

- $\lambda = c^+$ is regular,
- $X^\omega \subseteq X$,
- $c \cup \Gamma \cup \mathcal{H}^+ \cup \{\bar{T}, \bar{Q}, \bar{M}, \Gamma, (\mathcal{P}^*, \Phi \upharpoonright V), (j(\mathcal{P}^*), j \upharpoonright \mathcal{P}^*)\} \subseteq X$, and
- $|X| \leq c$.

---

This is one place where we use the ideal $\mathcal{I}$ is the non-stationary ideal, or just that it is definable in $V$. 65
Let $\pi : M_X \to X$ be the transitive uncollapse map and for any $x \in X$, let $\bar{x} = \pi^{-1}(x)$. Note that

$$\overline{\mathcal{H}^+} = \mathcal{H}^+.$$ 

Let $\mathbb{P} = \mathbb{P}_{\mathcal{I}}$ and $h \subset \mathbb{P}$ be $M_X$-generic such that $h \in M$. Such an $h$ exists by the properties of $X$.\footnote{We do not have a way of lifting $\pi$ to all of $M_X[h]$ like in the proof of Lemma 4.27.}

Work in $M_X[h]$, let $(\mathcal{T}, \mathcal{Q}, \mathcal{M}, \Lambda)$ be the interpretation of $(\check{T}, \check{Q}, \check{M}, \check{\Lambda})$. Let $\sigma = j \upharpoonright \mathcal{P}^*$; so $\sigma : \mathcal{P}^* \to \mathcal{J}(\mathcal{P}^*)$. Let $R$ be the image of $\mathcal{P}^*$ under the extender $F$ derived from $\check{i} \check{\bar{T}}$, i.e.

$$R = \{ i \check{T}(f)(a) : f \in \mathcal{P}^* \land a \in [\check{\mathcal{Q}}]^\omega \}.$$ 

Let $i_F : \mathcal{P}^* \to \mathcal{R}$ be the associated ultrapower map, and let $\bar{\check{\mathcal{T}}} : \mathcal{R} \to \mathcal{J}(\mathcal{P}^*)$. Let $\tau : \mathcal{R} \to \mathcal{J}(\mathcal{P}^*)$ be $\tau = \pi \circ \bar{\check{\mathcal{T}}}$. Note that $\sigma \circ \pi = \tau \circ i_F$.

Let $\Upsilon = j(\Psi \upharpoonright V)$ and $\Upsilon^* = \pi^{-1}(\Psi \upharpoonright V)$. In $M_X[h]$, $\Lambda$ is not full as witnessed by $\mathcal{T}, \mathcal{Q}, \mathcal{M}$ inside $\check{j}(\mathcal{T})|\check{\alpha}$, where $\check{j}$ is the generic ultrapower induced by $h$. Therefore, letting $j(\Psi \upharpoonright V)^{\tau \circ i_F} = \Sigma_1$ and $j(\Psi \upharpoonright V)^{\tau} = \Sigma_2$, note that

$$\Sigma_1 \leq_w \Sigma_2.$$ 

In $M$,

$$\Gamma(\mathcal{P}^*, \Sigma_1) \subset \Gamma(\mathcal{R}, \Sigma_2),$$

and letting $\Sigma_3 = j(\Sigma)^\tau$,

$$L(\Gamma(\mathcal{P}^*, \Sigma_1)) \models \text{"\check{\mathcal{M}} is a } \Sigma_3\text{-mouse and } \neg(\mathcal{M} \prec \check{\mathcal{Q}})."$$

Finally, note that $\mathcal{T}$ is according to $\Lambda$ as $\mathcal{T}$ is $j$-realizable. It is easy then to see that (a),(b), (c) hold for $(\mathcal{P}^*, \Sigma_1)$. Therefore, the pair $(\mathcal{P}^*, \Sigma_1)$ is the desired $(\mathcal{P}, \Phi)$.

• Regarding the proof of the $j$-condensation lemma (Theorem 4.34), the following are the main changes we need. Fix a bad tuple $\mathcal{A} = \{ (\mathcal{P}_i, \mathcal{Q}_i, \tau_i, \xi_i, \pi_i, \sigma_i \mid i < \omega), \mathcal{M}^\mathcal{Y}_\infty \}$ as in the proof of Theorem 4.34; note that $k(\mathcal{A}) = \{ (\mathcal{P}_i, \mathcal{Q}_i, \tau_i, \xi_i, \pi_i, \sigma_i \mid i < \omega), k(\mathcal{M}^\mathcal{Y}_\infty) \}$ is also a bad tuple in $N$ because $k$ fixes all these objects.

We let $(\mathcal{P}^+_0, \Pi)$ be such that

(a) $\mathcal{P}^+_0 = \text{HOD}_\Sigma^{j(\Gamma)}(\alpha' + \omega)$ for some limit ordinal $\alpha'$ such that $\mathcal{A} \in j(\Gamma)|\theta_{\alpha'}$. Note that $\mathcal{P}^+_0$ is countable in $N$ and $\{ k(\mathcal{P}^+_0), k \upharpoonright \mathcal{P}^+_0 \} \in M$.

(b) $\Pi$ is the natural strategy of $\mathcal{P}^+_0$ and is the tail of any $\Sigma$-hod pair $(\mathcal{R}, \Psi)$ such that $\mathcal{M}_\infty(\mathcal{R}, \Psi) = \mathcal{P}^+_0$. 

93 We do not have a way of lifting $\pi$ to all of $M_X[h]$ like in the proof of Lemma 4.27.
(c) \( \Pi \upharpoonright M \in M \) and \( \Pi \upharpoonright M \subseteq k(\Pi \upharpoonright M)^k \). We will also denote \( \Pi \) for \( k(\Pi \upharpoonright M)^k \) when interpreted in \( N \).

(d) \( \Lambda_Y \leq_w \Pi_{P_0^+(\alpha')} \) (so \( \Lambda_{X_i} \leq_w \Pi_{P_0^+(\alpha')} \) for all \( i \)) in \( N \). Note that we can extend \( \Lambda_Y \) (similarly \( \Lambda_{X_i} \) for all \( i \)) in \( N \) as the realizable strategy (which we also call \( \Lambda_Y \) ) of \( P_Y \) into \( k(j(H^+)) \) using the map \( k \circ \pi_Y \).

(e) In \( M \), \( P_0^+ \) is countable and \( \Gamma(P_0^+(\alpha'), \Pi_{P_0^+(\alpha')}) \models A \) is a bad tuple.

The rest of the proof is essentially the same as before, but now we run the “three dimensional argument” using \( k \) (instead of \( j \)) and the argument takes place in \( N \) (instead of in \( M \)). We leave the details to the reader.

• Finally, Section 4.5 goes through verbatim. This finishes our outline.

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