Fuzzy Logic, Informativeness and Bayesian Decision-Making Problems

Peter V. Golubtsov

Department of Physics, Moscow State Lomonosov University
119899, Moscow, Russia

E-mail: P.V.G@mail.ru

Stepan S. Moskaliuk

Bogolyubov Institute for Theoretical Physics
Metrolohichna Str., 14-b, Kyiv-143, Ukraine, UA-03143
e-mail: mss@bitp.kiev.ua

Abstract

This paper develops a category-theoretic approach to uncertainty, informativeness and decision-making problems. It is based on appropriate first order fuzzy logic in which not only logical connectives but also quantifiers have fuzzy interpretation. It is shown that all fundamental concepts of probability and statistics such as joint distribution, conditional distribution, etc., have meaningful analogs in new context. This approach makes it possible to utilize rich conceptual experience of statistics. Connection with underlying fuzzy logic reveals the logical semantics for fuzzy decision making. Decision-making problems within the framework of IT-categories and generalizes Bayesian approach to decision-making with a prior information are considered. It leads to fuzzy Bayesian approach in decision making and provides methods for construction of optimal strategies.
1 Introduction

The theory of fuzzy sets is used more and more widely in the description of uncertainty. Indeed, very often some poorly formalizable notions or expert knowledge are readily expressed in terms of fuzzy sets. In particular, fuzzy sets are extremely convenient in descriptions of linguistic uncertainties [1]. On the other hand, fuzzy notions themselves often admit flexible linguistic interpretations. This makes the exploitation of fuzzy sets especially natural and illustrative.

It is well known that fuzzy set theory suggests a wide range of specific approaches to decision problems in a fuzzy environment [2,3]. However, it still cannot easily compete with probability theory in dealing with uncertainty. Certainly, mathematical statistics (more precisely, statistical decision theory) has accumulated a rich conceptual experience.

It was shown in [4, 5] that the basic constructions and propositions of probability theory and statistics playing the fundamental role in decision-making problems have meaningful counterparts in fuzzy theories. It makes it possible to use the methodology of statistical decision-making in the fuzzy context. In particular, the fuzzy variant of the Bayes principle derived in this paper plays the same role in fuzzy decision-making problems as its probabilistic prototype in the theory of statistical games [7].

In contrast to the approach of [4] where all the notions were introduced “operationally” in order to be more close to the similar notions in statistics, in this paper all the notions are introduced “logically,” i.e., by the corresponding formulas in an appropriate fuzzy logic of first order. In this logic not only logical connectives but also quantifiers have fuzzy interpretation.

Connection with underlying fuzzy logic provides an interesting logical semantics of fuzzy decision making. Indeed in this approach a priori information may be represented by a fuzzy predicate and an experiment – by certain fuzzy relation. The loss function is replaced by fuzzy relation “good decision for,” mathematical expectation operator – by fuzzy universal quantifier, etc. As a result the notion “good decision strategy” is expressed by a first order formula in this logic.

It is convenient to consider different systems that take place in information acquiring and processing as particular cases of so-called information
transformers (ITs). Besides, it is useful to work with families of ITs in which certain operations, e.g., sequential and parallel compositions are defined.

It was noticed fairly long ago [9–13], that the adequate algebraic structure for describing information transformers (initially for the study of statistical experiments) is the structure of category [14–17].

Analysis of general properties for the classes of linear, multivalued, and fuzzy information transformers, studied in [4, 13, 18–23], allowed to extract general features shared by all these classes. Namely, each of these classes can be considered as a family of morphisms in an appropriate category, where the composition of information transformers corresponds to their “consecutive application.” Each category of ITs (or IT-category) contains a subcategory (of so called, deterministic ITs) that has products. Moreover, the operation of morphism product is extended in a “coherent way” to the whole category of ITs.

The work [24, 25] undertook an attempt to formulate a set of “elementary” axioms for a category of ITs, which would be sufficient for an abstract expression of the basic concepts of the theory of information transformers and for study of informativeness, decision problems, etc. This paper proposes another, significantly more compact axiomatic for a category of ITs. According to this axiomatic a category of ITs is defined in effect as a monoidal category [14, 16], containing a subcategory (of deterministic ITs) with finite products.

Among the basic concepts connected to information transformers there is one that plays an important role in the uniform construction of a wide spectrum of IT-categories — the concept of distribution. Indeed, fairly often an IT \( a: \mathcal{A} \to \mathcal{B} \) can be represented by a mapping from \( \mathcal{A} \) to the “space of distributions” on \( \mathcal{B} \) (see, e.g., [4, 5, 21]). For example, a probabilistic transition distribution (an IT in the category of stochastic ITs) can be represented by a certain measurable mapping from \( \mathcal{A} \) to the space of distributions on \( \mathcal{B} \). This observation suggests to construct a category of ITs as a Kleisli category [14, 26, 27], arising from the following components: an obvious category of deterministic ITs; a functor that takes an object \( \mathcal{A} \) to the object of “distributions” on \( \mathcal{A} \); and a natural transformation of functors, describing an “independent product of distributions”.

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The approach developed in this paper allows to express easily in terms of IT-categories such concepts as distribution, joint and conditional distributions, independence, and others. It is shown that on the basis of these concepts it is possible to formulate fairly general statement of decision-making problem with a prior information, which generalizes the Bayesian approach in the theory of statistical decisions. Moreover, the Bayesian principle, derived below, like its statistical prototype [28], reduces the problem of optimal decision strategy construction to a significantly simpler problem of finding optimal decision for a posterior distribution.

Among the most important concepts in categories of ITs is the concept of (relative) informativeness of information transformers. There are two different approaches to the concept of informativeness.

One of these approaches is based on analyzing the “relative positions” of information transformers in the corresponding mathematical structure. Roughly speaking, one information transformer is regarded as more informative than another one if with the aid of an additional information transformer the former one can be “transformed” to an IT, which is similar to (or more “accurate” than) the letter one. In fact, this means that all the information that can be obtained from the latter information transformer can be extracted from the former one as well.

The other approach to informativeness is based on treating information transformers as data sources for decision-making problems. Here, one information transformer is said to be semantically more informative than another if it provides better quality of decision making. Obviously, the notion of semantical informativeness depends on the class of decision-making problems under consideration.

In the classical researches of Blackwell [29, 30], the correspondence between informativeness (Blackwell sufficiency) and semantical informativeness (Blackwell informativeness) were investigated in a statistical context. These studies were extended by Morse, Sacksteder, and Chentsov [9–12], who applied the category theory techniques to their studies of statistical systems.

It is interesting, that under very general conditions the relations of informativeness and semantical informativeness (with respect to a certain class of decision-making problems) coincide. Moreover, in some categories of ITs it
is possible to point out one special decision problem, such that the resulting semantical informativeness coincides with informativeness.

Analysis of classes of equivalent (with respect to informativeness) information transformers shows that they form a partially ordered Abelian monoid with the smallest (also neutral) and the largest elements.

One of the objectives of this paper is to show that the basic constructions and propositions of probability theory and statistics playing the fundamental role in decision-making problems have meaningful counterparts in terms of IT-categories. Furthermore, some definitions and propositions (for example, the notion of conditional distribution and the Bayesian principle) in terms of IT-categories often have more transparent meanings. This provides an opportunity to look at the well known results from a different angle. What is even more significant, it makes it possible to apply the methodology of statistical decision-making in an alternative (not probabilistic) context.

Approaches, proposed in this work may provide a background for construction and study of new classes of ITs, in particular, dynamical nondeterministic ITs, which may provide an adequate description for information flows and information interactions evolving in time. Besides, a uniform approach to problems of information transformations may be useful for better understanding of information processes that take place in complex artificial and natural systems.

2 Categories of information transformers

2.1 Common structure of classes of information transformers

It is natural to assume that for any information transformer \( a \) there are defined a couple of spaces: \( \mathcal{A} \) and \( \mathcal{B} \), the space of “inputs” (or input signals) and the space of “outputs” (results of measurement, transformation, processing, etc.). We will say that \( a \) “acts” from \( \mathcal{A} \) to \( \mathcal{B} \) and denote this as \( a: \mathcal{A} \rightarrow \mathcal{B} \). It is important to note that typically an information transformer not only transforms signals, but also introduces some “noise”. In this case it is nondeterministic and cannot be represented just by a mapping from \( \mathcal{A} \).
to $B$.

It is natural to study information transformers of similar type by aggregating them into families endowed by a fairly rich algebraic structure [13,18]. Specifically, it is natural to assume that families of ITs poses the following properties:

(a) If $a: \mathcal{A} \to B$ and $b: B \to \mathcal{C}$ are two ITs, then their composition $b \circ a: \mathcal{A} \to \mathcal{C}$ is defined.

(b) This operation of composition is associative.

(c) There are certain neutral elements in these families, i.e., ITs that do not introduce any alterations. Namely, for any space $\mathcal{B}$ there exist a corresponding IT $i_{\mathcal{B}}: \mathcal{B} \to \mathcal{B}$ such that $i_{\mathcal{B}} \circ a = a$ and $b \circ i_{\mathcal{B}} = b$.

Algebraic structures of this type are called categories [14,16].

Furthermore, we will assume, that to every pair of information transformers, acting from the same space $\mathcal{D}$ to spaces $\mathcal{A}$ and $\mathcal{B}$ respectively, there corresponds a certain IT $a \ast b$ (called product of $a$ and $b$) from $\mathcal{D}$ to $\mathcal{A} \times \mathcal{B}$. This IT in a certain sense “represents” both ITs $a$ and $b$ simultaneously. Specifically, ITs $a$ and $b$ can be “extracted” from $a \ast b$ by means of projections $\pi_{\mathcal{A},\mathcal{B}}$ and $\nu_{\mathcal{A},\mathcal{B}}$ from $\mathcal{A} \times \mathcal{B}$ to $\mathcal{A}$ and $\mathcal{B}$, respectively, i.e., $\pi_{\mathcal{A},\mathcal{B}} \circ (a \ast b) = a$, $\nu_{\mathcal{A},\mathcal{B}} \circ (a \ast b) = b$. Note, that typically, an IT $c$ such that $\pi_{\mathcal{A},\mathcal{B}} \circ c = a$, $\nu_{\mathcal{A},\mathcal{B}} \circ c = b$ is not unique, i.e., a category of ITs does not have products (in category-theoretic sense [14–17]). Thus, the notion of a category of ITs demands for an accurate formalization.

Analysis of classes of information transformers studied in [4,5,13,18–23,25], gives grounds to consider these classes as categories that satisfy certain fairly general conditions.

### 2.2 Categories: basic concepts

Recall that a category (see, for example, [14–17]) $\mathbf{C}$ consists of a class of objects $\text{Ob}(\mathbf{C})$, a class of morphisms (or arrows) $\text{Ar}(\mathbf{C})$, and a composition operation $\circ$ for morphisms, such that:

(a) To any morphism $a$ there corresponds a certain pair of objects $\mathcal{A}$ and $\mathcal{B}$ (the source and the target of $a$), which is denoted $a: \mathcal{A} \to \mathcal{B}$. 


(b) To every pair of morphisms \( a: \mathcal{A} \to \mathcal{B} \) and \( b: \mathcal{B} \to \mathcal{C} \) their composition \( b \circ a: \mathcal{A} \to \mathcal{C} \) is defined.

Moreover, the following axioms hold:

(c) The composition is associative:

\[
    c \circ (b \circ a) = (c \circ b) \circ a.
\]

(d) To every object \( \mathcal{R} \) there corresponds an (identity) morphism \( i_\mathcal{R}: \mathcal{R} \to \mathcal{R} \), so that

\[
    \forall a: \mathcal{A} \to \mathcal{B}, \quad a \circ i_\mathcal{A} = a = i_\mathcal{B} \circ a.
\]

A morphism \( a: \mathcal{A} \to \mathcal{B} \) is called isomorphism if there exists a morphism \( b: \mathcal{B} \to \mathcal{A} \) such that \( a \circ b = i_\mathcal{B} \) and \( b \circ a = i_\mathcal{A} \). In this case objects \( \mathcal{A} \) and \( \mathcal{B} \) are called isomorphic.

Morphisms \( a: \mathcal{D} \to \mathcal{A} \) and \( b: \mathcal{D} \to \mathcal{B} \) are called isomorphic if there exists an isomorphism \( c: \mathcal{A} \to \mathcal{B} \) such that \( c \circ a = b \).

An object \( \mathcal{Z} \) is called terminal object if for any object \( \mathcal{A} \) there exists a unique morphism from \( \mathcal{A} \) to \( \mathcal{Z} \), which is denoted \( z_\mathcal{A}: \mathcal{A} \to \mathcal{Z} \) in what follows.

A category \( \mathcal{D} \) is called a subcategory of a category \( \mathcal{C} \) if \( \text{Ob}(\mathcal{D}) \subseteq \text{Ob}(\mathcal{C}) \), \( \text{Ar}(\mathcal{D}) \subseteq \text{Ar}(\mathcal{C}) \), and morphism composition in \( \mathcal{D} \) coincide with their composition in \( \mathcal{C} \).

It is said that a category has (pairwise) products if for every pair of objects \( \mathcal{A} \) and \( \mathcal{B} \) there exists their product, that is, an object \( \mathcal{A} \times \mathcal{B} \) and a pair of morphisms \( \pi_{AB}: \mathcal{A} \times \mathcal{B} \to \mathcal{A} \) and \( \nu_{AB}: \mathcal{A} \times \mathcal{B} \to \mathcal{B} \), called projections, such that for any object \( \mathcal{D} \) and for any pair of morphisms \( a: \mathcal{D} \to \mathcal{A} \) and \( b: \mathcal{D} \to \mathcal{B} \) there exists a unique morphism \( c: \mathcal{D} \to \mathcal{A} \times \mathcal{B} \), that yields a commutative diagram:
i.e., satisfies the following conditions:

\[
\pi_{A,B} \circ c = a, \quad \nu_{A,B} \circ c = b. \tag{1}
\]

We call such morphism \(c\) the product of morphisms \(a\) and \(b\) and denote it \(a \ast b\).

It is easily seen that existence of products in a category implies the following equality:

\[
(a \ast b) \circ d = (a \circ d) \ast (b \circ d). \tag{2}
\]

In a category with products, for two arbitrary morphisms \(a: A \to C\) and \(b: B \to D\) one can define the morphism \(a \times b:\)

\[
a \times b: A \times B \to C \times D, \quad a \times b \overset{\text{def}}{=} (a \circ \pi_{A,B}) \ast (b \circ \nu_{A,B}). \tag{3}
\]

This definition and (1) obviously imply that the morphism \(c = a \times b\) satisfy the following conditions:

\[
\pi_{C,D} \circ c = a \circ \pi_{A,B}, \quad \nu_{C,D} \circ c = b \circ \nu_{A,B} \tag{4}
\]

Moreover, \(c = a \times b\) is the only morphism satisfying conditions (1).

It is also easily seen that (2) and (3) imply the following equality:

\[
(a \times b) \circ (c \ast d) = (a \circ c) \ast (b \circ d). \tag{5}
\]

Suppose \(A \times B\) and \(B \times A\) are two products of objects \(A\) and \(B\) taken in different order. By the properties of products, the objects \(A \times B\) and \(B \times A\) are isomorphic and the natural isomorphism is

\[
\sigma_{A,B}: A \times B \to B \times A, \quad \sigma_{A,B} \overset{\text{def}}{=} \nu_{A,B} \ast \pi_{A,B}. \tag{6}
\]

i.e., a unique morphism that makes the following diagram commutative:
Moreover, for any object \( D \) and for any morphisms \( a: D \to A \) and \( b: D \to B \), the morphisms \( a \ast b \) and \( b \ast a \) are isomorphic, that is,

\[
\sigma_{A,B} \circ (a \ast b) = b \ast a. \tag{7}
\]

Similarly, by the properties of products, the objects \((A \times B) \times C\) and \(A \times (B \times C)\) are isomorphic. Let

\[
\alpha_{A,B,C}: (A \times B) \times C \to A \times (B \times C)
\]

be the corresponding natural isomorphism. Its “explicit” form is:

\[
\alpha_{A,B,C} \circ ((a \ast b) \ast c) = a \ast (b \ast c). \tag{9}
\]

### 2.3 Elementary axioms for categories of information transformers

In this subsection we set forward the main properties of categories of ITs. All the following study will rely exactly on these properties.
In [13, 18, 19, 21, 23, 25] it is shown that classes of information transformers can be considered as morphisms in certain categories. As a rule, such categories do not have products, which is a peculiar expression of nondeterministic nature of ITs in these categories. However, it turns out that deterministic information transformers, which are usually determined in a natural way in any category of ITs, form a subcategory with products. This point makes it possible to define a “product” of objects in a category of ITs. Moreover, it provides an axiomatic way to describe an extension of the product operation from the subcategory of deterministic ITs to the whole category of ITs.

**Definition 2.1** We shall say that a category $C$ is a category of information transformers if the following axioms hold:

1. There is a fixed subcategory of deterministic ITs $D$ that contains all the objects of the category $C$ ($\text{Ob}(D) = \text{Ob}(C)$).

2. The classes of isomorphisms in $D$ and in $C$ coincide, that is, all the isomorphisms in $C$ are deterministic.

3. The categories $D$ and $C$ have a common terminal object $Z$.

4. The category $D$ has pairwise products.

5. There is a specified extension of morphism product from the subcategory $D$ to the whole category $C$, that is, for any object $D$ and for any pair of morphisms $a: D \to A$ and $b: D \to B$ in $C$ there is certain information transformer $a \ast b: D \to A \times B$ (which is also called a product of ITs $a$ and $b$) such that

$$\pi_{A,B} \circ (a \ast b) = a, \quad \nu_{A,B} \circ (a \ast b) = b.$$  

6. Let $a: A \to C$ and $b: B \to D$ are arbitrary ITs in $C$, then the IT $a \times b$ defined by Eq. (3) satisfies Eq. (5):

$$(a \times b) \circ (c \ast d) = (a \circ c) \ast (b \circ d).$$
7. Equality (7) holds not only in \( D \) but in \( C \) as well, that is, \( \text{product of information transformers is “commutative up to isomorphism”} \).

8. Equality (9) also holds in \( C \). In other words, \( \text{product of information transformers is “associative up to isomorphism”} \) too.

Now let us make several comments concerning the above definition.

We stress that in the description of the extension of morphism product from the category \( D \) to \( C \) (Axiom 5) we \textit{do not require the uniqueness} of an IT \( c: D \rightarrow A \times B \) that satisfies conditions (1).

Nevertheless, it is easily verified, that the equations (4) are valid for \( c = a \times b \) not only in the category \( D \), but in \( C \) as well, that is,

\[
\pi_{C,D} \circ (a \times b) = a \circ \pi_{A,B} \quad \nu_{C,D} \circ (a \times b) = b \circ \nu_{A,B}.
\]

However, the IT \( c \) that satisfy the equations (4) may be not unique. Note also that in the category \( C \) Eq. (2) in general does not hold.

Further, note that Axiom (4) immediately implies

\[
(a \times b) \circ (c \times d) = (a \circ c) \times (b \circ d).
\]

Finally, note that any category that has a terminal object and pairwise products can be considered as a category of ITs in which all information transformers are deterministic.

### 3 Fuzzy logic and fuzzy quantifiers

In this section we shall introduce the fuzzy interpretation of formulas of first order logic which is an extension of the classical interpretation.

#### 3.1 Truth values and operations of fuzzy logic

Let \( \mathcal{I} \) be the closed interval \( \mathcal{I} = [0, 1] \) – the set of \textit{fuzzy truth values}. For the sake of simplicity of notations we shall denote any fuzzy proposition \( a \) and its truth value (evaluation of \( a \)) by the same symbol, i.e., \( a \in \mathcal{I} \), but we shall mark fuzzy logical operations by dot in order to avoid ambiguity.
Let $a$ and $b$ be any fuzzy propositions, i.e., $a, b \in \mathcal{I}$. We put by definition for fuzzy disjunction, conjunction, negation, and implication:

\[ a \lor b \overset{\text{def}}{=} \max(a, b), \quad \neg a \overset{\text{def}}{=} 1 - a, \]
\[ a \land b \overset{\text{def}}{=} \min(a, b), \quad a \Rightarrow b \overset{\text{def}}{=} \neg \neg a \lor b. \]

Let $A(x)$ be any $x$-indexed family of propositions $x \in \mathcal{X}$, then define multivalent disjunction and conjunction:

\[ \bigvee_x A(x) \overset{\text{def}}{=} \sup_x A(x), \quad \bigwedge_x A(x) \overset{\text{def}}{=} \inf_x A(x). \]

### 3.2 Fuzzy sets as fuzzy predicates

When we consider some family of fuzzy propositions $A(x), x \in \mathcal{X}$ we may say that $A$ is a fuzzy property (predicate, characteristics) for elements of the set $\mathcal{X}$. That is an element $x$ of $\mathcal{X}$ satisfies $A$ with the degree $A(x) \in \mathcal{I}$. On the other hand any fuzzy property of elements of $\mathcal{X}$ may be considered as a fuzzy subset of $\mathcal{X}$.

In what follows we shall identify a fuzzy property, the corresponding fuzzy set and its characteristic function and hence use the same notation for all three (conceptually different!) notions. We shall also find it very convenient to denote the grade of membership of $x$ to a fuzzy set $A$ by $\in A$, that is $x \in A \overset{\text{def}}{=} A(x)$.

### 3.3 Fuzzy quantifiers

Quantifier in classical logic may be treated as conjunction or disjunction of a family of propositions. Specifically, let $A(x)$ be any (classical) predicate, then the formula $\forall x A(x)$ takes the truth value $1$ (true) iff $A(x) = 1$ for all $x$.

Now for any fuzzy predicate $A(x), x \in \mathcal{X}$ define fuzzy universal and existential quantifiers:

\[ \forall x A(x) \overset{\text{def}}{=} \bigwedge_x A(x), \quad \exists x A(x) \overset{\text{def}}{=} \bigvee_x A(x). \]
3.4 Bounded quantifiers

Bounded quantifiers are convenient for manipulations with the statements of the kind: “all \( x \) such that... satisfy...” and “there exists \( x \) such that... satisfying...”. We shall need fuzzy quantifiers with fuzzy bounds.

By analogy with the classical logic let us put by definition:

\[
\forall x \in A \, B(x) \overset{\text{def}}{=} \forall x \, (x \in A \Rightarrow B(x)),
\]

\[
\exists x \in A \, B(x) \overset{\text{def}}{=} \exists x \, (x \in A \land B(x)).
\]

Note that the bounded quantifiers are related in the same way as the unbounded ones:

\[
\forall (\forall x \in A \, B(x)) = \exists x \in A \, \forall B(x).
\]

4 Algebra of fuzzy sets

The algebra of sets in the classical set theory is determined by the underlying classical logic. The systematic extension of set theoretic notations allows to introduce the algebra of fuzzy sets in a very natural way.

4.1 Definitions of fuzzy sets by comprehension

Let \( \varphi \) be any fuzzy property for elements \( x \) of \( X \). This property may be any formula in the fuzzy logic discussed above. By \( A = \{ x \mid \varphi(x) \} \) we shall denote the fuzzy set such that the level of membership of \( x \) to \( A \) is equal to the truth value of \( \varphi(x) \), i.e.:

\[
A = \{ x \mid \varphi(x) \} \quad \text{iff} \quad x \in A = \varphi(x). \quad (10)
\]

4.2 Fuzzy sets operations

Suppose that \( A \) and \( B \) are any fuzzy sets over the space \( X \). The principle (10) provides definitions of union, intersection, and complement for fuzzy
sets:

\[ A \cup B \overset{\text{def}}{=} \{ x | x \in A \lor x \in B \}, \]
\[ A \cap B \overset{\text{def}}{=} \{ x | x \in A \land x \in B \}, \]
\[ \overline{A} \overset{\text{def}}{=} \{ x | \neg (x \in A) \}. \]

Now let \( \alpha(y) \) be any family of fuzzy sets on \( \mathcal{X} \) indexed by \( y \in \mathcal{Y} \). By definition, put

\[ \bigcup_y \alpha(y) \overset{\text{def}}{=} \{ x | \exists y x \in \alpha(y) \}, \]
\[ \bigcap_y \alpha(y) \overset{\text{def}}{=} \{ x | \forall y x \in \alpha(y) \}. \]

Finally, if the index \( y \) itself varies in a fuzzy set \( Y \), then operations for fuzzy families of fuzzy sets are easily described by bounded fuzzy quantifiers:

\[ \bigcup_{y \in Y} \alpha(y) \overset{\text{def}}{=} \{ x | \exists y \in Y x \in \alpha(y) \}, \]
\[ \bigcap_{y \in Y} \alpha(y) \overset{\text{def}}{=} \{ x | \forall y \in Y x \in \alpha(y) \}. \]

### 4.3 Containment of fuzzy sets

In contrast to the traditional definition of containment of fuzzy sets: “\( A \subset B \) iff \( A(x) \subseteq B(x) \) for all \( x \)” we shall adopt another definition of \textit{containment}, which is the natural extension of containment of crisp sets. In our approach “\( A \) is contained in \( B \)” is a \textit{fuzzy property}. Hence it may take fuzzy truth values. By definition, put

\[ A \subset B \overset{\text{def}}{=} \forall x \in A \ x \in B, \]

that is “\( A \) is \textit{contained} in \( B \)” with the degree in which “\textit{all} the elements \( \textit{contained} \) in \( A \) \textit{belong} to \( B \)” (in these phrases we emphasized fuzzy notions).

### 5 Fuzzy sets and distributions

We shall say that a \textit{fuzzy distribution} (or \textit{possibility distribution} [3]) \( X \) on the space \( \mathcal{X} \) is any fuzzy set in \( \mathcal{X} \). Note that we consider a fuzzy distribution as an analog of a probabilistic distribution.
5.1 Joint distributions

We shall say that any fuzzy distribution $C$ on $\mathcal{X} \times \mathcal{Y}$ determines a fuzzy joint distribution \cite{6} of the pair $(x, y)$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$.

Sometimes it is convenient to interpret a joint distribution as a fuzzy relation, i.e., to consider $C(x, y)$ as the “degree in which $x$ and $y$ satisfy the relation $C$”.

The distribution $C^\bullet$ on $\mathcal{Y} \times \mathcal{X}$ is called the converse of $C$ if

$$C^\bullet(y, x) = C(x, y).$$

The joint distribution $C$ induces the marginal distributions \cite{4, 5} $X$ and $Y$ on the spaces $\mathcal{X}$ and $\mathcal{Y}$ respectively:

$$X = \{x | \exists y C(x, y)\}, \quad Y = \{y | \exists x C(x, y)\}.$$  

5.2 Transition distributions

The notion of a transition distribution \cite{8} is of great importance in mathematical statistics.

We shall say that a fuzzy transition distribution from $\mathcal{X}$ to $\mathcal{Y}$ is any map $\alpha$ from $\mathcal{X}$ to the family of all fuzzy distributions on $\mathcal{Y}$, that is, $\alpha$ takes each element $x \in \mathcal{X}$ to some fuzzy distribution $\alpha(x)$ on the space $\mathcal{Y}$.

Note that the condition $A(x, y) = \alpha(x)(y)$ for all $x$ in $\mathcal{X}$ for all $y$ in $\mathcal{Y}$ determines the one-to-one correspondence between $A$ and $\alpha$. Using set theoretic notations we may write:

$$A = \{(x, y) | \alpha(x)(y)\}, \quad \alpha(x) = \{y | A(x, y)\}.$$  

By $\overline{\alpha}$ we shall denote the transition distribution, such that $\overline{\alpha}(x) \overset{\text{def}}{=} \overline{\alpha(x)}$. Obviously, $\overline{\alpha}$ corresponds to $\overline{A}$.

The correspondence between joint distributions and transition distributions reveals the algebraic semantics of the marginal distributions. Let $\alpha$ and $\beta$ be transition distributions corresponding to $A$ and $A^\bullet$ respectively. Then

$$X = \{x | \exists y \beta(y)(x)\} = \bigcup_y \beta(y),$$
$$Y = \{y | \exists x \alpha(x)(y)\} = \bigcup_x \alpha(x).$$
5.3 Images of distributions

As a probabilistic transition distribution transforms one probability distribution to another [8], in the same way a fuzzy transition distribution takes some fuzzy distribution to another one.

Let $X$ be any distribution on $\mathcal{X}$ and $\alpha$ is some transition distribution from $\mathcal{X}$ to $\mathcal{Y}$. We say that a distribution $\alpha X$ on $\mathcal{Y}$ is the image of the distribution $X$ induced by the transition distribution $\alpha$ if

$$\alpha X \overset{\text{def}}{=} \{ y \mid \exists x \in X \ y \in \alpha(x) \} = \bigcup_{x \in \mathcal{X}} \alpha(x).$$

There is an interesting notion dual to the notion of the image. We say that a distribution $\alpha \circ X$ on $\mathcal{Y}$ is the lower image of the distribution $X$ induced by the transition distribution $\alpha$ if

$$\alpha \circ X \overset{\text{def}}{=} \{ y \mid \forall x \in X \ y \in \alpha(x) \} = \bigcap_{x \in \mathcal{X}} \alpha(x).$$

It may be proved that $\alpha \circ X = \alpha X$.

5.4 Generated joint distributions

Suppose that $X$ is an arbitrary distribution in $\mathcal{X}$ and $\alpha$ is some transition distribution from $\mathcal{X}$ to $\mathcal{Y}$. We say that the distribution $\alpha \ast X$ in $\mathcal{X} \times \mathcal{Y}$ is the joint distribution generated by $\alpha$ and $X$ if

$$\alpha \ast X \overset{\text{def}}{=} \{ (x, y) \mid x \in X \land y \in \alpha(x) \}.$$

It can be easily verified that the image $\alpha X$ coincides with the $\mathcal{Y}$-marginal distribution of the generated joint distribution $\alpha \ast X$.

5.5 Fuzzy conditional distributions

We shall define this notion in the same way as its probabilistic prototype is defined in statistics [4, 5, 8].

Let $A$ be any joint distribution on $\mathcal{X} \times \mathcal{Y}$. We say that some transition distribution $\alpha$ from $\mathcal{X}$ to $\mathcal{Y}$ is (a variant of) the conditional distribution for...
A with respect to $\mathcal{X}$ if $A$ is generated by its $\mathcal{X}$-marginal distribution $X$ together with $\alpha$:

$$A = \alpha \ast X.$$  

Similarly we say that $\beta$ from $\mathcal{X}$ to $\mathcal{Y}$ is a conditional distribution for $A$ with respect to $\mathcal{Y}$ if

$$A^\bullet = \beta \ast Y.$$  

**Theorem 5.1** For any joint distribution $A$ in $\mathcal{X} \times \mathcal{Y}$, conditional distributions always exist. Some variants of conditional distributions $\alpha$ and $\beta$ are determined by the following expressions:

$$\alpha(x) = \{x| A(x, y)\}, \quad \beta(y) = \{y| A(x, y)\}.$$  

### 5.6 Iterated quantifiers

The possibility of representation of any joint distribution by conditional distributions has interesting consequences that will be used in the Bayesian decision analysis.

**Theorem 5.2** Let $A$ be any joint distribution on $\mathcal{X} \times \mathcal{Y}$, $X$ and $Y$ — its marginal distributions, $\alpha$ and $\beta$ — conditional distributions, that is $A = \alpha \ast X = (\beta \ast Y)^\bullet$. Then for any formula $\varphi$

$$\forall (x, y) \in A \varphi = \forall x \in X \forall y \in \alpha(x) \varphi = \forall y \in Y \forall x \in \beta(y) \varphi,$$

$$\exists (x, y) \in A \varphi = \exists x \in X \exists y \in \alpha(x) \varphi = \exists y \in Y \exists x \in \beta(y) \varphi.$$  

### 6 Semantics of decision making problems

#### 6.1 Fuzzy games with a priory information

Let $\mathcal{X}$ be some space of objects of interest; let $\mathcal{D}$ be some space of decisions; and let $G$ be a fuzzy relation between $\mathcal{X}$ and $\mathcal{D}$, i.e., a fuzzy set in $\mathcal{X} \times \mathcal{D}$ that determines the notion "good decision," i.e., $G = \{(x, d)|d$ is a good decision for $x\}$. 

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We are dealing in fact with the two-person game \( \langle X, D, G \rangle \), where \( X \) is the decision space of the first player, \( D \) is the decision space of the second player, and the fuzzy notion of the quality of decisions for the second player \( G \) replaces the classical loss function. Hence we shall call \( \langle X, D, G \rangle \) the fuzzy game \([4, 5]\). If it is known a priori that all choices of the first player are restricted by a fuzzy set \( X \), then we are dealing with the fuzzy game with the a priori information \([4, 5]\).

### 6.2 Good decisions

Let us now define the notion \( G_X = \text{“for all } x \text{ in } X \text{ decision } d \text{ is good for } x \” \)
i.e., formally:

\[
G_X \overset{\text{def}}{=} \forall x \in X G(x, d).
\]

Truth value of this formula determines the quality of the decision \( d \) for \( X \). Note that the distribution of good decisions permits the following interpretation. Let the transition distribution \( \gamma \) from \( X \) to \( D \) is determined by \( \gamma(x) = \{ d | G(x, d) \} \), i.e., the distribution of “good decisions for \( x \” \). Let \( \gamma^* \) be the opposite to \( \gamma \). Then

\[
G_X = \{ d | \forall x \in X x \in \gamma^*(d) \} = \{ x | X \subseteq \gamma^*(d) \}.
\]

Another interpretation for \( G_X \) — its algebraic semantics — comes from the representation:

\[
G_X = \bigcap_{x \in X} \gamma(x) = \gamma \circ X.
\]

Note that in statistics the loss function is often used. Suppose that some joint fuzzy distribution \( B \) on \( X \times D \) is fixed and let \( \beta \) be the associated transition distribution \( \beta(x) = \text{“bad decisions for } x \” = \{ d | (x, d) \in B \} \). Then the fuzzy set of bad decisions with respect to the a priori distribution \( X \) may be defined by:

\[
B_X = \{ d | \exists x \in X B(x, d) \} = \bigcup_{x \in X} \beta(x) = \beta X.
\]

However if \( B = \overline{G} \) (it is quite natural) then we have \( \beta = \overline{\gamma} \) and

\[
B_X = \beta X = \overline{\gamma} X = \overline{\gamma \circ X} = \overline{G_X},
\]

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so the two approaches are dual to each other.

Let us consider an optimal decision problem for the second player with respect to the a priori information $X$. The statement “there exists an optimal decision for the second player” may be expressed by

$$\exists d \forall x \in X G(x, d) = \exists d \in G_X.$$  \hfill (12)

We shall say that a decision $d_X$, is optimal (or a Bayes decision) with respect to the a priori distribution $X$ (or simply $X$-optimal) if $G_X(d)$ takes on its maximum at $d_X$.

### 6.3 Optimal decision strategies

Now consider problems of constructing optimal decision strategies in fuzzy experiments. Suppose that a fuzzy experiment from $X$ to $Y$ is determined by a fuzzy transition distribution $\alpha$. A decision strategy (or simply strategy) $r$ is any mapping from the observation space $Y$ to the decision space $D$.

Let us define the relation $H(x, r) = “\text{decision strategy } r \text{ is good for } x”$ by

$$H(x, r) \overset{\text{def}}{=} \forall y \in \alpha(x) \ r(y) \in \gamma(x).$$

It may be shown that $H(x, r) = r\alpha(x) \subseteq \gamma(x)$.

In fact, we have come to a new two-person game $\langle X, D^Y, H \rangle$, where the decision space of the second player is replaced by the set of all strategies, and the goodness relation $G$ for decisions is replaced by the goodness for strategies. Such games are analogous to classical statistical games [7].

Finally for a given a priori distribution $X$ we define the Bayes goodness for a strategy with respect to $X$, i.e., the fuzzy set on $D^Y$, $H_X(r) = “r$ is good for all $x$ in $X”$

$$H_X(r) \overset{\text{def}}{=} \forall x \in X H(x, r).$$  \hfill (13)

The optimal decision strategy problem is one of obtaining a mapping $r_X$, called a Bayes strategy, such that the level of goodness for $r_X$ in (13) is the highest.
6.4 Bayes principle

A well-known result of the theory of statistical games states that an optimal decision strategy problem for a statistical game may be reduced to a similar problem for an original game [7]. This reduction involves conditional distributions.

**Theorem 6.1** Let $X$ be any a priori distribution in $\mathcal{X}$, $\alpha$ any transition distribution from $\mathcal{X}$ to $\mathcal{Y}$, and $\beta$ a variant of a conditional distribution for $\alpha \ast X$ with respect to $\mathcal{Y}$. Assume that for every $y$ in $\mathcal{Y}$ there exists a Bayes decision $d_{\beta(y)}$, which is optimal for the original decision problem with respect to the distribution $\beta(y)$. Then the optimal decision strategy $r_X$ and the corresponding goodness $H_X(r_X)$ are determined by

$$r_X(y) = d_{\beta(y)}, \quad H_X(r_X) = \forall y \in \mathcal{Y} \ G_{\beta(y)}(d_{\beta(y)}).$$

For an arbitrary decision strategy $r$

$$H_X(r) = \forall x \in \mathcal{X} \ \forall y \in \alpha X \ r(y) \in \gamma(x) = \forall y \in \alpha X \ Y \ r(y) \in \gamma(x) = \forall y \in \alpha X \ r(y) \in G_{\beta(y)}(y).$$

In the calculations above we utilize the definition of the conditional distribution and in the last step we use the definition of $G_{\beta(y)}(y)$ for $X = \beta(y)$.

Hence, by virtue of the optimality of the Bayes decision $d_{\beta(y)}$, we have for any $y$

$$G_{\beta(y)}(r(y)) G_{\beta(y)}(d_{\beta(y)}) = G_{\beta(y)}(r_X(y)).$$

Therefore we obtain

$$H_X(r) = \bigvee_{y \in \alpha X} G_{\beta(y)}(r(y)) = H_X(r_X).$$

Note that the notion of conditional distribution and tightly related with it “interchange” of quantifiers play the principal role in the proof.

The Bayes principle is very intuitive. It asserts that to construct (or calculate) an optimal strategy $r_X$ for a given observation $y$ one has to find the conditional distribution of $x$ for a given $y$, i.e., $\beta y$ and then take a decision $d_{\beta(y)}$, which is optimal with respect to this distribution. In other words, the observation of $y$ results in the passage from the a priori information $X$ to the a posteriory information $\beta(y)$. 

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7 Informativeness of information transformers

7.1 Accuracy relation

In order to define informativeness relation we will need to introduce first the following auxiliary notion.

Definition 7.1 We will say that $\triangleright$ is an accuracy relation on an IT-category $C$ if for any pair of objects $A$ and $B$ in $C$ the set $C(A, B)$ of all ITs from $A$ to $B$ is equipped with a partial order $\triangleright$ that satisfies the following monotonicity conditions:

$$a \triangleright a', b \triangleright b' \implies a \circ b \triangleright a' \circ b',$$
$$a \triangleright a', b \triangleright b' \implies a * b \triangleright a' * b'.$$

Thus, the composition and the product are monotone with respect to the partial order $\triangleright$. For a pair of ITs $a, b \in C(A, B)$ we shall say that $a$ is more accurate than $b$ whenever $a \triangleright b$.

It obviously follows from the very definition of the operation $\times$ and from the monotonicity conditions that the operation $\times$ is monotone as well:

$$a \triangleright a', b \triangleright b' \implies a \times b \triangleright a' \times b'.$$

It is clear that for any IT-category there exists at least a “trivial variant” of the partial order $\triangleright$, namely, one can choose an equality relation for $\triangleright$, that is, one can put $a \triangleright b \overset{\text{def}}{\iff} a = b$. However, many categories of ITs (for example, multivalued and fuzzy ITs) provide a “natural” choice of the accuracy relation, which is different from the equality relation.

7.2 Definition of informativeness relation

Suppose $a: D \to A$ and $b: D \to B$ are two information transformers with a common source $D$. Assume that there exists an IT $c: A \to B$ such that $c \circ a = b$. Then any information that can be obtained from $b$ can be obtained from $a$ as well (by attaching the IT $c$ next to $a$). Thus, it is natural to
consider the information transformer $a$ as being more informative than the IT $b$ and also more informative than any IT less accurate than $b$.

Now we give the formal definition of the informativeness relation in the category of information transformers.

**Definition 7.2** We shall say that an information transformer $a$ is more *informative* (better) than $b$ if there exists an information transformer $c$ such that $c \circ a \triangleright b$, that is,

$$a \succ b \overset{\text{def}}{\iff} \exists c \ c \circ a \triangleright b.$$ 

It is easily verified that the informativeness relation $\succ$ is a preorder on the class of information transformers in $\mathbf{C}$. This preorder $\succ$ induces an equivalence relation $\sim$ in the following way:

$$a \sim b \overset{\text{def}}{\iff} a \succ b \land b \succ a.$$ 

Obviously, the relation “more informative” extends the relation “more accurate,” that is,

$$a \triangleright b \implies a \succ b.$$ 

### 7.3 Main properties of informativeness

It can be easily verified that the informativeness relation $\succ$ satisfies the following natural properties.

**Lemma 7.3** Consider all information transformers with a fixed source $\mathcal{D}$.

(a) The identity information transformer $i_\mathcal{D}$ is the most informative and the terminal information transformer $z_\mathcal{D}$ is the least informative:

$$\forall a \ i_\mathcal{D} \succ a \succ z_\mathcal{D}.$$ 

(b) Any information transformer $a: \mathcal{D} \to \mathcal{B} \times \mathcal{C}$ is more informative than its parts $\pi_{\mathcal{B},\mathcal{C}} \circ a$ and $\nu_{\mathcal{B},\mathcal{C}} \circ a$. 

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(c) The product \( a * b \) is more informative than its components

\[ a * b \succ a, b. \]

Furthermore, the informativeness relation is compatible with the composition and the product operations.

**Lemma 7.4** (a) If \( a \succeq b \), then \( a \circ c \succeq b \circ c \).
(b) If \( a \succeq b \) and \( c \succeq e \), then \( a * c \succeq b * e \).

### 7.4 Structure of the family of informativeness equivalence classes

Let \( a \) be some information transformer. We shall denote by \([a]\) the equivalence (with respect to informativeness) class of \( a \). We shall also use boldface for equivalence classes, that is, \( a \in \mathbf{a} \) is equivalent to \( a = [a] \).

**Theorem 7.5** Let \( \mathfrak{J}(D) \) be the family of informativeness equivalence classes for the class of all information transformers with a fixed domain \( D \). The family \( \mathfrak{J}(D) \) forms a partial ordered Abelian monoid \( \langle \mathfrak{J}(D), \succeq, *, 0 \rangle \) with the smallest element \( 0 \) and the largest element \( 1 \), where

\[
[a] \succeq [b] \iff a \succeq b, \quad [a] * [b] \defeq [a * b], \quad 0 \defeq [\varepsilon_D], \quad 1 \defeq [i_D].
\]

Moreover, the following properties hold:

(a) \( 0 * a = a \),
(b) \( 1 * a = 1 \),
(c) \( 0 \preceq a \preceq 1 \),
(d) \( a * b \succ a, b \),
(e) \((a \succeq b) \& (c \succeq e) \implies a * c \succeq b * e \).
7.5 Informativeness structure of concrete IT-categories

7.5.1 Linear stochastic ITs

In addition to the trivial relation of accuracy (which coincides with the equality relation) one can define the accuracy relation in the following way:

\[ (A_a, \Sigma_a) \triangleright (A_b, \Sigma_b) \iff A_a = A_b, \Sigma_a \leq \Sigma_b. \]

However, it can be proved that the informativeness relations corresponding these different accuracy preorders, actually coincide.

**Theorem 7.6** In the category of linear information transformers every equivalence class \([a]\) corresponds to a pair \((\mathcal{Q}, S)\), where \(\mathcal{Q} \subseteq \mathcal{D}\) is an Euclidean subspace and \(S: \mathcal{Q} \rightarrow \mathcal{Q}\) is nonnegative definite operator, that is, \(S \succeq 0\). In these terms

\[ (\mathcal{Q}_1, S_1) \triangleright (\mathcal{Q}_2, S_2) \iff \mathcal{Q}_1 \supseteq \mathcal{Q}_2, S_1 \upharpoonright \mathcal{Q}_2 \leq S_2. \]

Here \(S_1 \upharpoonright \mathcal{Q}_2\) (the restriction of \(S_1\) on \(\mathcal{Q}_2\)) is defined by the expression

\[ S_1 \upharpoonright \mathcal{Q}_2 \stackrel{\text{def}}{=} P_2 I_1 S_1 P_1 I_2, \]

where \(I_j: \mathcal{Q}_j \rightarrow \mathcal{D}\) is the subspace inclusion, and \(P_j: \mathcal{D} \rightarrow \mathcal{Q}_j\) is the orthogonal projection (cf. [18, 33]).

7.5.2 The category of sets as a category of ITs

It is not hard to prove that for a given set \(\mathcal{D}\), the class of equivalent informativeness for an IT \(a\) with the set \(\mathcal{D}\) being its domain, is completely determined by the following equivalence relation \(\approx_a\) on \(\mathcal{D}\):

\[ x \approx_a y \iff a x = a y \quad \forall x, y \in \mathcal{D}. \]

Furthermore, \(a \succeq b\) if and only if the equivalence relation \(\approx_a\) is finer than \(\approx_b\), that is,

\[ a \succeq b \iff \forall x, y \in \mathcal{D} \ (x \approx_a y \implies x \approx_b y). \]

It is clear that we have the following
Theorem 7.7 The partially ordered monoid of equivalence classes for ITs with the source $\mathcal{D}$, is isomorphic to the monoid of all equivalence relations on $\mathcal{D}$ equipped with the order “finer” and with the product:

$$x (\approx_a \ast \approx_b) y \iff (x \approx_a y, x \approx_b y) \quad \forall x, y \in \mathcal{D}.$$ 

7.5.3 Multivalued ITs

In addition to the trivial accuracy relation in the category of multivalued ITs one can put

$$a \triangleright b \iff \forall x \in \mathcal{D} ax \subseteq bx.$$ 

These two accuracy relations lead to different informativeness relations [21], called (strong) informativeness $\gtrsim$ and weak informativeness $\gtrless$.

For the both informativeness relations the classes of equivalent ITs with a fixed source $\mathcal{D}$ can be described explicitly.

Theorem 7.8 In the category of multivalued ITs with weak informativeness every class of equivalent ITs corresponds to a certain covering $\mathcal{P}$ of the set $\mathcal{D}$, such that if $\mathcal{P}$ contains some set $B$ then it contains all its subsets:

$$(\exists B \in \mathcal{P} \ (A \subseteq B)) \implies A \in \mathcal{P}.$$ 

Moreover, a covering $\mathcal{P}_1$ is more (weakly) informative than $\mathcal{P}_2$ (namely, $\mathcal{P}_1$ corresponds to a class of more (weakly) informative ITs than $\mathcal{P}_2$) if $\mathcal{P}_1$ is contained in $\mathcal{P}_2$, that is,

$$\mathcal{P}_1 \gtrsim \mathcal{P}_2 \iff \mathcal{P}_1 \subseteq \mathcal{P}_2.$$ 

Theorem 7.9 In the category of multivalued ITs with strong informativeness every class of equivalent ITs corresponds to a covering $\mathcal{P}$ of the set $\mathcal{D}$, that satisfy the following condition:

$$\left( (\exists B \in \mathcal{P} \ A \subseteq B) \ & (\exists B \subseteq \mathcal{P} \ A = \bigcup B) \right) \implies A \in \mathcal{P}.$$ 

In this case

$$\mathcal{P}_1 \gtrsim \mathcal{P}_2 \iff \left( (\forall A \in \mathcal{P}_1 \ \exists B \in \mathcal{P}_2 \ A \subseteq B) \right. \ & (\forall B \in \mathcal{P}_2 \ \exists A \subseteq \mathcal{P}_1 \ B = \bigcup A) \bigg).$$
8 Informativeness and decision-making problems

In this section, we consider an alternative (with respect to the above) approach to informativeness comparison. This approach is based on treating information transformers as data sources for decision-making problems.

8.1 Decision-making problems in categories of ITs

Results of observations, obtained on real sources of information (e.g. indirect measurements) are as a rule unsuitable for straightforward interpretation. Typically it is assumed that observations suitable for interpretation are those into a certain object \( \mathcal{U} \) which in what follows will be called object of interpretations or object of decisions.

By an interpretable information transformer for signals from an object \( \mathcal{D} \) we mean any information transformer \( a: \mathcal{D} \rightarrow \mathcal{U} \).

It is usually thought that some interpretable information transformers are more suitable for interpretation (of obtained results) than others. Namely, on a set \( C(\mathcal{D}, \mathcal{U}) \) of information transformers from \( \mathcal{D} \) to \( \mathcal{U} \), one defines some preorder relation \( \succ \), which specifies the relative quality of various interpretable information transformers. Typically the relation \( \succ \) is predetermined by the specific formulation of a problem of optimal information transformer synthesis (that is, decision-making problem).

We shall say that an abstract decision-making problem is determined by a triple \( \langle \mathcal{D}, \mathcal{U}, \succ \rangle \), where \( \mathcal{D} \) is an object of studied (input) signals, \( \mathcal{U} \) is an object of decisions (or interpretations), and \( \succ \) is a preorder on the set \( C(\mathcal{D}, \mathcal{U}) \).

We shall call a preorder \( \succ \) monotone if for any \( a, b \in C(\mathcal{D}, \mathcal{U}) \)

\[
 a \succ b \implies a \succ b,
\]

that is, more accurate IT provides better quality of interpretation.

For a given information transformer \( a: \mathcal{D} \rightarrow \mathcal{A} \) we shall also say that an IT \( b \) reduces \( a \) to an interpretable information transformer if \( b \circ a: \mathcal{D} \rightarrow \mathcal{U} \), that is, if \( b: \mathcal{A} \rightarrow \mathcal{U} \). Such an information transformer \( b \) will be called a decision strategy.
The set of all interpretable information transformers obtainable on the
basis of \( a: \mathcal{D} \to \mathcal{A} \) will be denoted \( \mathcal{U}_a \subseteq \mathcal{C}(\mathcal{D}, \mathcal{U}) \):

\[
\mathcal{U}_a \overset{\text{def}}{=} \{ b \circ a \mid b: \mathcal{A} \to \mathcal{U} \}.
\]

We shall call a decision strategy \( r: \mathcal{A} \to \mathcal{U} \) optimal (for the IT \( a \) with
respect to the problem \( \langle \mathcal{D}, \mathcal{U}, \gg \rangle \)) if the IT \( r \circ a \) is a maximal element
in \( \mathcal{U}_a \) with respect to \( \gg \). Thus, a decision-making problem for a given
information transformer \( a \) is stated as the problem of constructing optimal
decision strategies.

### 8.2 Semantical informativeness

The relation \( \gg \) induces a preorder relation \( \sqsupseteq \) on a class of information
transformers operating from \( \mathcal{D} \) in the following way.

Assume that \( a \) and \( b \) are information transformers with the source \( \mathcal{D} \),
that is, \( a: \mathcal{D} \to \mathcal{A} \), \( b: \mathcal{D} \to \mathcal{B} \). By definition, put

\[
a \sqsupseteq b \overset{\text{def}}{\iff} \forall b': \mathcal{B} \to \mathcal{U} \exists a': \mathcal{A} \to \mathcal{U} \quad a' \circ a \gg b' \circ b.
\]

In other words, \( a \sqsupseteq b \) if for every interpretable information transformer
\( d \) derived from \( b \) there exists an interpretable information transformer \( c \)
derived from \( a \) such that \( c \gg d \), that is,

\[
a \sqsupseteq b \iff \forall d \in \mathcal{U}_b \exists c \in \mathcal{U}_a \quad c \gg d.
\]

It can easily be checked that the relation \( \sqsupseteq \) is a preorder relation.

It is natural to expect that if one information transformer is more in-
formative than the other, then the former will be better than the latter in
any context. In other words, for any preorder \( \gg \) on the set of interpretable
information transformers the induced preorder \( \sqsupseteq \) is dominated by the infor-
mativeness relation \( \succeq \) (that is, \( \sqsupseteq \) is weaker than \( \succeq \)). The converse is also
true.

**Definition 8.1** We shall say that an information transformer \( a \) is *semanti-
cally more informative* than \( b \) if for any interpretation object \( \mathcal{U} \) and for any
preorder \( \gg \) (on the set of interpretable information transformers) \( a \sqsupseteq b \) for
the induced preorder \( \sqsupseteq \).
The following theorem is in some sense a “completeness” theorem, which establishes a relation between “structure” \((b\) can be “derived” from \(a\)) and “semantics” \((a\) is uniformly better then \(b\) in decision-making problems).

**Theorem 8.2** For any information transformers \(a\) and \(b\) with a common source \(D\), information transformer \(a\) is more informative than \(b\) if and only if \(a\) is semantically more informative than \(b\).

Let us remark that the above proof relies heavily on the extreme extent of the class of decision problems involved. This makes it possible to select for any given pair of ITs \(a, b\) an appropriate decision-making problem \(⟨D, U_b, ≿_b⟩\) in which the interpretation object \(U_b\) and the preorder \(≿_b\) depend on the IT \(b\). However, in some cases it is possible to point out a concrete (universal) decision-making problem such that

\[ a ≿ b \iff a ≱ b. \]

**Theorem 8.3** Assume that for a given object \(D\) there exists an object \(\tilde{D}\) such that for every information transformer acting from \(D\) there exists an equivalent (with respect to informativeness) IT acting from \(D\) to \(\tilde{D}\), that is,

\[ \forall B \forall b: D \to B \exists b': D \to \tilde{D} \ b ∼ b'. \]

Let us choose the decision object \(U \overset{\text{def}}{=} \tilde{D}\) and the preorder \(≿\), defined by

\[ c ≿ d \overset{\text{def}}{=} c > d. \]

Then \(a ≿ b\) if and only if \(a ≱ b\).

Note that in general case an optimal decision strategy (if exists) can be nondeterministic. However, in many cases it is sufficient to search optimal strategies among deterministic ITs. Indeed, in some categories of information transformers the relation of “accuracy” satisfies the following condition: every IT is dominated by some deterministic IT, that is, for every IT there exists a more accurate deterministic IT.
Proposition 8.4 Assume that \( \langle D, U, \gg \rangle \) is a monotone decision-making problem in a category of ITs \( C \). Assume also that the following condition holds:

\[
\forall c \in \text{Ar}(C) \quad \exists d \in \text{Ar}(D) \quad d \gg c.
\]

Then for any IT \( a: D \to R \) and for any decision strategy \( r: R \to U \) there exists a deterministic strategy \( r_0: R \to U \) such that \( r_0 \circ a \gg r \circ a \).

8.3 Decision-making problems with a prior information

In this section we formulate in terms of categories of information transformers an analogy for the classical problem of optimal decision strategy construction for decision problems with a prior information (or information \textit{a priori}). We also prove a counterpart of the Bayesian principle from the theory of statistical games [7, 28]. Like its statistical prototype it reduces the problem of constructing an optimal decision strategy to a much simpler problem of finding an optimal decision for a posterior information (or information \textit{a posteriori}).

First we define in terms of categories of information transformers some necessary concepts, namely, concepts of distribution, conditional information transformer, decision problem with a prior information, and others.

8.4 Distributions in categories if ITs

We shall say that a \textit{distribution} on an object \( A \) (in some fixed category of ITs \( C \)) is any IT \( f: Z \to A \), where \( Z \) is the terminal object in \( C \).

The concept of distribution corresponds to the general concept of an element of some object in a category, namely, a morphism from the terminal object (see, e.g., [17]).

Any distribution of the form \( h: Z \to A \times B \) will be called a \textit{joint distribution} on \( A \) and \( B \). The projections \( \pi_{A,B} \) and \( \nu_{A,B} \) on the components \( A \) and \( B \) respectively, “extract” marginal distributions \( f \) and \( g \) of the joint distribution \( h \), that is,

\[
f = \pi_{A,B} \circ h: Z \to A,
\]

\[
g = \nu_{A,B} \circ h: Z \to B.
\]
We say that the components of a joint distribution \( h: \mathcal{Z} \to \mathcal{A} \times \mathcal{B} \) are independent whenever this joint distribution is completely determined by its marginal distributions, that is,

\[
g = \nu_{A,B} \circ h: \mathcal{Z} \to \mathcal{B}.
\]

Let \( f \) be an arbitrary distribution on \( \mathcal{A} \) and let \( a: \mathcal{A} \to \mathcal{B} \) be some information transformer. Then the distribution \( g = a \circ f \) in some sense “contains an information about \( f \).” This concept can be expressed precisely of one consider the joint distribution generated by the distribution \( f \) and the IT \( a \):

\[
h: \mathcal{Z} \to \mathcal{A} \times \mathcal{B}, \quad h = (i_A \ast a) \circ f.
\]

Let \( h \) be a joint distribution on \( \mathcal{A} \times \mathcal{B} \). We shall say that \( a: \mathcal{A} \to \mathcal{B} \) is a conditional IT for \( h \) with respect to \( \mathcal{A} \) whenever \( h \) is generated by the marginal distribution \( \pi_{A,B} \circ h \) and the IT \( a \), that is,

\[
h = (i_A \ast a) \circ \pi_{A,B} \circ h.
\]

Similarly, an IT \( b: \mathcal{B} \to \mathcal{A} \) such that

\[
h = (b \ast i_B) \circ \nu_{A,B} \circ h
\]

will be called a conditional IT for \( h \) with respect to \( \mathcal{B} \).

### 8.5 Bayesian decision-making problems

Suppose that there are fixed two objects \( \mathcal{D} \) and \( \mathcal{U} \) in some category of ITs, namely, the object of signals and the object of decisions, respectively. In a decision-making problem with a prior distribution \( f \) on \( \mathcal{D} \) one fixes some
preorder $\gg_f$ on the set of joint distributions on $\mathcal{D} \times \mathcal{U}$ for which $\mathcal{D}$-marginal distribution coincides with $f$.

Informally, any joint distribution $h$ on $\mathcal{D} \times \mathcal{U}$ of this kind can be considered as a joint distribution of a studied signal (with the distribution $f = \pi_{\mathcal{D},\mathcal{U}} \circ h$ on $\mathcal{D}$) and a decision (with the distribution $g = \nu_{\mathcal{D},\mathcal{U}} \circ h$ on $\mathcal{U}$). The preorder $\gg_f$ determines how good is the “correlation” between studied signals and decisions.

Formally, an abstract decision problem with a prior information is determined by a quadruple $\langle \mathcal{D}, \mathcal{U}, f, \gg_f \rangle$, where $\mathcal{D}$ is an object of studied signals, $\mathcal{U}$ is an object of decisions (or interpretations), $f: \mathcal{Z} \rightarrow \mathcal{D}$ is a prior distribution (or distribution a priori), and $\gg_f$ is a preorder on the set of ITs $h: \mathcal{Z} \rightarrow \mathcal{D} \times \mathcal{U}$ that satisfy the condition $\pi_{\mathcal{D},\mathcal{U}} \circ h = f$.

Furthermore, suppose that there is a fixed IT $a: \mathcal{D} \rightarrow \mathcal{R}$ (which determines a measurement; $\mathcal{R}$ can be called an object of observations). An IT $r: \mathcal{R} \rightarrow \mathcal{U}$ is called optimal (for the IT $a$ with respect to $\gg_f$) if the distribution $(i * r \circ a) \circ f$ is a maximal element with respect to $\gg_f$. The set of all optimal information transformers is denoted $\text{Opt}_f(a \circ f)$.

**Theorem 8.5 (Bayesian principle).** Let $f$ be a given prior distribution on $\mathcal{D}$, let $a: \mathcal{D} \rightarrow \mathcal{R}$ be a fixed IT, and let $b: \mathcal{R} \rightarrow \mathcal{D}$ be a conditional information transformer for $(i * a) \circ f$ with respect to $\mathcal{R}$. Then the set of optimal ITs $r: \mathcal{R} \rightarrow \mathcal{U}$, namely, the set of optimal decision strategies for $f$ over $a \circ f$ coincides with the set of optimal decision strategies for $b \circ g$ over $g$, where $g = a \circ f$:

$$\text{Opt}_f(a \circ f) = \text{Opt}_{b \circ g}(g).$$

In a wide class of decision problems (e.g., in linear estimation problems) an optimal IT $r$ happens to be deterministic and is specified by the “deterministic part” of the IT $b$.

For many categories of information transformers (for example, stochastic, multivalued, and fuzzy ITs [4,5,20,28]) an optimal decision strategy $r$ can be constructed “pointwise” according to the following scheme. For the given “result of observation” $y \in \mathcal{R}$ consider the conditional (posterior) distribution $b(y)$ for $f$ under a fixed $g = y$, and put

$$r(y) \overset{\text{def}}{=} d_{b(y)},$$

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where \( d_{b(y)} \) is an optimal decision with respect to the posterior distribution \( b(y) \).

### 8.6 Decision making problems in concrete categories of ITs

#### 8.6.1 Stochastic ITs

Let us demonstrate here that the basic concepts of mathematical statistics are adequately described in terms of this IT-category. Namely, we shall verify that the concepts of distribution, conditional distribution, etc. (introduced above in terms of IT-categories), in the category of stochastic ITs lead to the corresponding classical concepts.

Indeed, any probability distribution \( Q \) on a given measurable space \( \mathcal{A} = (\Omega_\mathcal{A}, \mathcal{S}_\mathcal{A}) \) is uniquely determined by the morphism \( f: \mathcal{Z} \to \mathcal{A} \) from the terminal object \( \mathcal{Z} = (\{0\}, \{\emptyset, \{0\}\}) \) (a one-point measurable space) such that

\[
P_f(0, A) = Q(A) \quad \forall A \in \mathcal{S}_\mathcal{A}.
\]

In what follows we shall omit the first argument in \( P_f(0, A) \) and write just \( P_f(A) \) instead.

A statistical experiment is described by a family of probability measures \( Q_\theta \) on some measurable space \( \mathcal{B} \). This family is usually parametrized by elements of a certain set \( \Omega_\mathcal{A} \). Sometimes (especially when statistical problems with a prior information are studied) it is additionally assumed that the set \( \Omega_\mathcal{A} \) is equipped by some \( \sigma \)-algebra \( \mathcal{S}_\mathcal{A} \) and that \( Q_\theta(B) \) is a measurable function of \( \theta \in \Omega_\mathcal{A} \) for all \( B \in \mathcal{S}_\mathcal{B} \) (and thus, \( Q_\theta(B) \) is a transition probability function [8]). Therefore, such statistical experiment is determined by the stochastic information transformer

\[
a: \mathcal{A} \to \mathcal{B},
\]

where

\[
P_a(\theta, B) = Q_\theta(B) \quad \forall \theta \in \Omega_\mathcal{A}, \quad \forall B \in \mathcal{S}_\mathcal{B}.
\]

In the case when no \( \sigma \)-algebra on the set \( \Omega_\mathcal{A} \) is specified, one can put \( \mathcal{S}_\mathcal{A} = \mathcal{P}(\Omega_\mathcal{A}) \), that is, the \( \sigma \)-algebra of all the subsets of the set \( \Omega_\mathcal{A} \). It is clear that in this case the function \( P_a(\theta, B) = Q_\theta(B) \) is a measurable
function of $\theta \in \Omega_A$ for every fixed $B \in \mathcal{S}_B$ and thus (being a transition probability function), is described by a stochastic IT $a: A \rightarrow B$.

Note also, that any statistic, being a measurable function, is represented by a certain deterministic IT. Decision strategies also correspond to deterministic ITs. At the same time, nondeterministic (mixed) decision strategies are adequately represented by stochastic information transformers of general kind.

Now, let $f$ be some fixed distribution on $A$ and let $a: A \rightarrow B$ be some IT. The joint distribution $h$ on $A \times B$, generated by $f$ and $a$ (from the IT-categorical point of view, see Section 8.4) is

$$h = (i \ast a) \circ f.$$  

It means that for every set $A \times B$, where $A \in \Omega_A$ and $B \in \Omega_B$,

$$P_h(A \times B) = \int_{\Omega_A} P_{i \ast a}(\omega, A \times B) P_f(d\omega) =$$

$$= \int_{\Omega_A} P_i(\omega, A) P_a(\omega, B) P_f(d\omega) =$$

$$= \int_A P_a(\omega, B) P_f(d\omega).$$

Thus we come to the well known classical expression for the generated joint distribution (see, for example, [8]).

Now assume that $P_f$ is considered as some probability prior distribution (or distribution a priori) on $A$. Then for a given transition probability function $P_a$, a posterior (or conditional) distribution $P_b(\omega', \cdot)$ on $A$ for a fixed $\omega' \in \Omega_B$ is determined, accordingly to [8] by a transition probability function $P_b(\omega', A)$, $\omega' \in \Omega_B$, $A \in \mathcal{S}_A$ such that

$$P_h(A \times B) = \int_B P_b(\omega', A) P_g(d\omega') \quad \forall A \in \mathcal{S}_A, \quad \forall B \in \mathcal{S}_B,$$

where

$$P_g(B) = \int_{\Omega_A} P_a(\omega, B) P_f(d\omega) \quad \forall B \in \mathcal{S}_B.$$
It is easily verified that in terms of ITs the above expressions have the following forms:

\[ h = (b \ast i) \circ g, \]

where

\[ g = a \circ f. \]

This shows, that the classical concept of conditional distribution is adequately described by the concept of conditional IT in terms of categories of information transformers.

### 8.6.2 Linear stochastic ITs

Note that in the category of linear information transformers every IT is dominated (in the sense of the preorder relation \( \triangleright \)) by a deterministic IT. Hence, according to Proposition 8.4, in any monotone decision-making problem without loss of quality one can search optimal decision strategies in the class of deterministic ITs.

According to section 8.4 any joint distribution in \( A \times B \) is an IT \( h: Z \to A \times B \), where \( Z \) is a terminal object in the category of linear ITs, i.e., \( Z = \{0\} \) is a 0-dimensional linear space. Thus, \( h = \langle 0, \Sigma_h \rangle \), where \( \Sigma_h \) is a self-adjoint nonnegative operator in \( h: Z \to A \times B \). Operator \( \Sigma_h \) can be represented in the following “matrix” form:

\[
\Sigma_h = \left( \begin{array}{cc} \Sigma_f & \Sigma_{f,g} \\ \Sigma_{g,f} & \Sigma_g \end{array} \right)
\]  

(14)

where \( \Sigma_{f,g} = -\Sigma_{g,f} \).

It is shown in [19], that in the category of linear ITs for any joint distribution there always exist conditional distributions.

**Theorem 8.6** For any joint distribution \( h: Z \to A \times B \) there exists conditional information transformers \( a: A \to B \) and \( b: B \to A \).

Variants of conditional information transformers are given by the formulas

\[
a = \langle \Sigma_{g,f} \Sigma_f, \Sigma_g - \Sigma_{g,f} \Sigma_f \Sigma_{f,g} \rangle, \\
b = \langle \Sigma_{f,g} \Sigma_g, \Sigma_f - \Sigma_{f,g} \Sigma_g \Sigma_{g,f} \rangle.
\]
Here $A^-$ we denotes the pseudoinverse operator for $A$ [34].

If $\Sigma_f > 0$ or $\Sigma_g > 0$ (in this case this operator is nonsingular and its pseudoinverse coincides with its inverse $\Sigma_f^{-1} = \Sigma_f^{-1}$), then the corresponding conditional information transformer $a$ or $b$ is unique.

Thus in problems with a prior information one can apply Bayesian principle. Its direct proof in the category of linear ITs as well as the explicit expression for conditional information transformers can be found in [19].

### 8.6.3 Multivalued ITs

In the category of multivalued information transformers every IT is dominated (in the sense of the partial order $\triangleright$) by a deterministic IT. Thus, in the monotone decision-making problem one can search optimal decision strategies in the class of deterministic ones.

For every joint distribution in the category of multivalued ITs there exist conditional distributions [20].

It is clear, that any joint distribution in $\mathcal{A} \times \mathcal{B}$, (i.e., an IT $h: \mathcal{Z} \rightarrow \mathcal{A} \times \mathcal{B}$) is specified by a subset $H$ of $\mathcal{A} \times \mathcal{B}$, since a terminal object $\mathcal{Z}$ in the categories of multivalued ITs is a 1-element set. It is easy to see that for every joint distribution in the category of multivalued ITs there exist conditional distributions [20].

**Theorem 8.7** For any joint distribution $H$ in $\mathcal{A} \times \mathcal{B}$, conditional information transformers $a: \mathcal{A} \rightarrow \mathcal{B}$ and $b: \mathcal{B} \rightarrow \mathcal{A}$ always exist. Some variants of conditional ITs are determined by the following expressions:

$$
ax = \begin{cases} 
\{ y \in \mathcal{B} \mid \langle x, y \rangle \in H \} , & \text{if } x \in p_A H , \\
\mathcal{B} , & \text{if } x \notin p_A H ,
\end{cases}$$

$$
b_y = \begin{cases} 
\{ x \in \mathcal{A} \mid \langle x, y \rangle \in H \} , & \text{if } y \in p_B H , \\
\mathcal{A} , & \text{if } y \notin p_B H .
\end{cases}
$$

Here $p_A H$ and $p_B H$ denote projections of $H$ on $\mathcal{A}$ and $\mathcal{B}$ respectively, e.g., $p_A H \overset{\text{def}}{=} \{ x \in \mathcal{A} \mid \exists y \in \mathcal{B} \langle x, y \rangle \in H \}$. 

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Therefore, in decision problems with a prior information, the Bayesian approach can be effectively applied.

### 8.6.4 Categories of fuzzy information transformers

Here we define two categories of fuzzy information transformers $\text{FMT}$ and $\text{FPT}$ that correspond to different fuzzy theories [4].

Objects of these categories are arbitrary sets and morphisms are everywhere defined fuzzy maps, namely, maps that take an element to a normed fuzzy set (a fuzzy set $A$ is normed if supremum of its membership function $\mu_A$ is 1). Thus, an information transformer $a: \mathcal{A} \to \mathcal{B}$ is defined by a membership function $\mu_{ax}(y)$ which is interpreted as the grade of membership of an element $y \in \mathcal{B}$ to a fuzzy set $ax$ for every element $x \in \mathcal{A}$.

The category $\text{FMT}$. Suppose $a: \mathcal{A} \to \mathcal{B}$ and $b: \mathcal{B} \to \mathcal{C}$ are some fuzzy maps. We define their composition $b \circ a$ as follows: for every element $x \in \mathcal{A}$ put

$$
\mu_{(b \circ a)x}(z) \overset{\text{def}}{=} \sup_{y \in \mathcal{B}} \min \left( \mu_{ax}(y), \mu_{by}(z) \right).
$$

For a pair of fuzzy information transformers $a: \mathcal{D} \to \mathcal{A}$ and $b: \mathcal{D} \to \mathcal{B}$ with the common source $\mathcal{D}$, we define their product as the IT that acts from $\mathcal{D}$ to the Cartesian product $\mathcal{A} \times \mathcal{B}$, such that

$$
\mu_{(a \times b)x}(y, z) \overset{\text{def}}{=} \min \left( \mu_{ax}(y), \mu_{by}(z) \right).
$$

The category $\text{FPT}$. Define the composition and the product by the following expressions:

$$
\mu_{(b \circ a)x}(z) \overset{\text{def}}{=} \sup_{y \in \mathcal{B}} \left( \mu_{ax}(y) \mu_{by}(z) \right),
$$

$$
\mu_{(a \times b)x}(y, z) \overset{\text{def}}{=} \mu_{ax}(y) \mu_{by}(z).
$$

In the both defined above categories of fuzzy information transformers the subcategory of deterministic ITs is (isomorphic to) the category of sets $\text{Set}$. Let $g: \mathcal{A} \to \mathcal{B}$ be some map (morphism in $\text{Set}$). Define the corresponding fuzzy IT (namely, a fuzzy map, which is obviously, everywhere
defined) \( \tilde{g} : \mathcal{A} \to \mathcal{B} \) in the following way:

\[
\mu_{\tilde{g}(x)}(y) \overset{\text{def}}{=} \delta_{g(x), y} = \begin{cases} 
1, & \text{if } g(x) = y, \\
0, & \text{if } g(x) \neq y.
\end{cases}
\]

Concerning the choice of accuracy relation, note, that in these IT-categories, like in the category of multivalued ITs, apart from the trivial accuracy relation one can put for \( a, b : \mathcal{A} \to \mathcal{B} \)

\[
a \triangleright b \overset{\text{def}}{=} \forall x \in \mathcal{A} \; \forall y \in \mathcal{B} \; \mu_{ax}(y) \leq \mu_{bx}(y).
\]

In each fuzzy IT-category these two choices lead to two different informativeness relations, namely the strong and the weak ones.

Like in the categories of linear and multivalued ITs discussed above, monotone decision-making problems admit restriction of the class of optimal decision strategies to deterministic ITs without loss of quality.

It is clear, that any joint distribution in \( \mathcal{A} \times \mathcal{B} \), (i.e., an IT \( h : \mathcal{Z} \to \mathcal{A} \times \mathcal{B} \)) is, in fact a normed fuzzy subset of \( \mathcal{A} \times \mathcal{B} \), since a terminal object \( \mathcal{Z} \) in the categories of fuzzy ITs is a 1-element set. Denote this fuzzy set by \( H \). It is shown in [4, 5] that for every joint distribution in the categories of fuzzy ITs there exist conditional distributions.

**Theorem 8.8** For any joint distribution \( H \) in \( \mathcal{A} \times \mathcal{B} \), conditional information transformers \( a : \mathcal{A} \to \mathcal{B} \) and \( b : \mathcal{B} \to \mathcal{A} \) always exist. Some variants of conditional ITs are determined by the following expressions:

(a) in the category FMT

\[
\begin{align*}
\mu_{ax}(y) &= \mu_H(x, y), \\
\mu_{by}(x) &= \mu_H(x, y); \\
\mu_{ax}(y) &= \begin{cases} 
\mu_H(x, y), & \text{if } \mu_H(x, y) \neq \sup_z \mu_H(x, z), \\
1, & \text{if } \mu_H(x, y) = \sup_z \mu_H(x, z),
\end{cases} \\
\mu_{by}(x) &= \begin{cases} 
\mu_H(x, y), & \text{if } \mu_H(x, y) \neq \sup_z \mu_H(z, y), \\
1, & \text{if } \mu_H(x, y) \neq \sup_z \mu_H(z, y).
\end{cases}
\end{align*}
\]
(b) in the category FPT

\[ \mu_{ax}(y) = \begin{cases} \frac{\mu_H(x, y)}{\sup_z \mu_H(x, z)}, & \text{if } \sup_z \mu_H(x, z) \neq 0, \\ 1, & \text{if } \sup_z \mu_H(x, z) = 0, \end{cases} \]

\[ \mu_{bg}(x) = \begin{cases} \frac{\mu_H(x, y)}{\sup_z \mu_H(z, y)}, & \text{if } \sup_z \mu_H(z, y) \neq 0, \\ 1, & \text{if } \sup_z \mu_H(z, y) = 0. \end{cases} \]

This allows Bayesian approach and makes use of Bayesian principle in decision problems with a prior information for fuzzy ITs [4,5,22,41], where connections between fuzzy decision problems and the underlying fuzzy logic are studied.

The present paper, while very much a first step, lays the basis for number of further applications. In paper [41] we propose some realizations for above categories, which we believe can be the basis for some interesting new directions in quantum computation and bioinformatics.

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