Theoretical and computational aspects of entanglement

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Abstract

We show that the two notions of entanglement: the maximum of the geometric measure entanglement and the maximum of the nuclear norm is attained for the same states. We affirm the conjecture of Higuchi-Sudberry on the maximum entangled state of four qubits. We introduce the notion of $d$-density tensor for mixed $d$-partite states. We show that $d$-density tensor is separable if and only if its nuclear norm is 1. We suggest an alternating method for computing the nuclear norm of tensors. We apply the above results to symmetric tensors.

Keywords: Entanglement, geometric measure of entanglement, spectral and nuclear norms of tensors, symmetric tensors, $d$-qubits, symmetric $d$-qubits, density tensors, computation of spectral norm.

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1 Introduction

The most important notion in quantum mechanics is the notion of (quantum) entanglement of $d$-partite systems [13, 43, 44]. (Recall that a $d$-partite state is represented by a $d$-mode tensor $T$ of Hilbert-Schmidt norm one: $\|T\| = 1.$) A state $T$ is called entangled if it is not a product state, (rank one tensor). One of the quantitative ways to measure the entanglement of a $d$-partite state $T$ is the geometric measure of entanglement of $T$ [50]. It is given by the distance of $T$ to the variety of product states. In mathematical terms the geometric measure of entanglement of $T$ is equal to $\sqrt{2(1 - \|T\|_\infty)}$, where $\|T\|_\infty$ is the spectral norm of the state $T$ [24]. Thus, $T$ is entangled if and only if $\|T\|_\infty < 1$.

Another important notion in quantum mechanics is a mixed state [15], which is represented by a hermitian nonnegative definite matrix of trace one. A pure state is represented by a rank one density matrix. Mathematically, a $d$-partite quantum state is described by a $2d$-mode tensor $T$, which was called density tensor in [18, 20]. A mixed density tensor $T$ corresponding to the mixture of product states is called separable, or separable state [40]. Thus separable density tensors are generalizations of product states, and inseparable density tensors, i.e., density tensors which are not separable, are analogous to the entangled states. The following result for the mixed density tensor was discovered in [18]: Let $\|T\|_1$ be the nuclear norm of a $d$-partite
tensor, which is the dual norm of the spectral norm \[20\]. Then the nuclear norm of a density tensor is at least one, and equality holds if and only if the density is separable. Hence \(\|H\|_1\) measures the inseparability of the corresponding mixed states for \(d \geq 2\). (For separable bipartite states, i.e., \(d = 2\), this result was discovered in [41].)

The aim of this paper is to discuss further theoretical and numerical aspects of the nuclear norm of tensors initiated in [18, 20] and their relationship to entanglement. We first describe our main theoretical results which are related to entanglement. With respect to the geometric measure of entanglement, the most entangled state is a \(d\)-partite state with the minimal spectral norm [46, 24]. We propose here another measure of entanglement a \(d\)-partite state \(T\): the value of the nuclear norm \(\|T\|_1\). Clearly, \(\|T\|_1 \geq 1\) and equality holds if and only if \(T\) is a product state. Hence a maximum entangled state with respect to this measure is a state with maximum \(\|T\|_1\). We show that a state has maximum geometric measure of entanglement if and only if it has maximum nuclear norm. Similarly, the most inseparable density tensor is the one with the maximum nuclear norm. We show that the nuclear norm of the most inseparable density tensor is achieved for all pure states which are maximally entangled.

As pointed out in [26] most qubit states are too entangled to use for quantum computations. On the other hand, the symmetric \(d\)-qubits, are much less entangled for large values of \(d\) [17], and their geometric measure of entanglement is polynomially computable [24]. Furthermore, the symmetric qubits are actually available in current designs for quantum computers [1]. Therefore we also discuss in this paper the maximum entangled and maximum inseparable states corresponding to symmetric tensors, also known as Bosons in physics.

The second part of this paper is devoted to the computational aspects of the nuclear norm. We first propose a simple numerical algorithm to compute the nuclear norm of a tensor, which is an analog of the alternating method for computing the spectral norm of a tensor [11, 21, 23, 33]. Note that this algorithm gives an upper bound on the nuclear norm. It will usually converge to a local minimum or at least to a critical point. We remark that the computation of the spectral and nuclear norm of tensors is NP-hard (for \(d \geq 3\)) [30, 20]. In general, one would not expect to have a polynomial-time algorithm to compute the spectral norm, unless \(P=NP\).

Next we consider the case of symmetric tensors. We propose a variation of our alternating algorithm to symmetric tensors. We compare our algorithm to a different approach suggested by J. Nie in [36], where the Lasserre hierarchy of semi-definite relaxations based on moments is applied to the non-convex polynomial optimization problem. The iterations of the Nie’s algorithm yield a lower bound for the nuclear norm.

We also try to find most entangled states and most entangled symmetric states using software. For 3-qubits our software confirms that the \(W\)-state is the most entangled state. For 4-qubits we prove that the conjectured 4-qubit given in [29] is the most entangled one. For symmetric \(d\)-qubits we also confirm numerically that the \(d\)-symmetric qubits suggested in [2] are the most entangled ones.

We now survey briefly the contents of this paper. In §2 we recall the definitions and properties of the spectral and nuclear norms. We discuss the distortion constants of these two norms with respect to the Hilbert-Schmidt norm. In §3 we discuss similar notions and results for symmetric tensors. In §4 we discuss the no-
tion of density tensors related to the mixed $d$-partite state. We show that a density tensor is separable if and only if its nuclear norm is 1. In §5 we discuss the density tensors corresponding to the mixed symmetric states, which are called bisymmetric density tensors. In §6 we discuss the notion of the most entangled states and mixed states with respect to spectral and nuclear norms of tensors. In §7 we discuss the most entangled 3 and 4 qubits. We show that the 4-qubit state given by Higuchi-Sudbery [29] is the most entangled with respect to the spectral and nuclear norm. In §8 we recall two well-known minimization problems: the minimum sum of Euclidean norms [3] and a Second Order Cone Programming (SOCP) [6], which are the foundations for proposing the alternating method for nuclear norm calculation. In §9 and §10 we give alternating methods for computing the nuclear norm of nonsymmetric and symmetric tensors respectively. In §11 we give some numerical examples to demonstrate the performance of Algorithms 9.1 and 10.1.

2 The spectral and the nuclear norms of tensors

Assume that $d$ is a positive integer and let $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$. In this paper we assume that we are dealing with a field $F$ which is either the field of complex numbers $\mathbb{C}$, which is fundamental in quantum mechanics, or the field of real numbers $\mathbb{R}$, which appears frequently in engineering applications. Denote by $\mathbb{F}^n$ the $d$-dimensional tensor product $\otimes_{i=1}^d \mathbb{F}^{n_i}$ and by $[d]$ the set of positive integers $\{1, \ldots, d\}$. Note that the dimension of the vector space $\mathbb{F}^n$ is $N(n) = \prod_{i=1}^d n_i$. Recall that $x = (x_1, \ldots, x_n)^T \in \mathbb{F}^n$, $A = [a_{i,j}] \in \mathbb{F}^{m \times n}$, $T = [t_{i_1,\ldots, i_d}] \in \mathbb{F}^n$ are called vector, matrix and $d$-mode tensor (for $d \geq 3$), with the entries $x_i, a_{i,j}, t_{i_1,\ldots, i_d}$ respectively.

Assume that $J = \{j_1, \ldots, j_k\} \subseteq [d]$, where $1 \leq j_1 < \cdots < j_k \leq d$. Let $n' = (n_{j_1}, \ldots, n_{j_k})$ and $\mathcal{Y} = \{y_{i_1,\ldots, i_k}\} \subseteq \mathbb{C}^{n'}$. Then

$$\mathcal{T} \times \mathcal{Y} = \sum_{i_{j_1},\ldots, i_{j_k}} t_{i_{j_1,\ldots, i_{j_k}}} y_{i_{j_1,\ldots, i_{j_k}}}$$

is $d-k$ mode tensor. In particular, for for $k = d$ one has that the standard inner product on $\mathbb{F}^n$ is given by $\langle \mathcal{T}, \mathcal{Y} \rangle = \mathcal{T} \times \mathcal{Y}$. Then $\|\mathcal{T}\| = \sqrt{\langle \mathcal{T}, \mathcal{T} \rangle}$ is the Hilbert-Schmidt norm of $\mathcal{T}$.

We now recall the two important norms on $\mathbb{F}^n$, which are of major importance in quantum mechanics for $F = \mathbb{C}$. Let

$$\Pi^a(\mathbb{F}) = \{\otimes_{i=1}^d x_i, \ x_i \in \mathbb{F}^{n_i}, \|x_i\| = 1, \ i \in [d]\}. \tag{2.1}$$

$\Pi^a(\mathbb{F})$, or its projectivization $\mathbb{P}\Pi^a(\mathbb{F})$, is called the Segre variety. The spectral norm of a tensor is given by

$$\|\mathcal{T}\|_{\infty, \mathbb{F}} = \max\{\|\langle \mathcal{T}, \mathcal{X} \rangle\|, \ \mathcal{X} \in \Pi^a(\mathbb{F})\}, \text{ for } \mathcal{T} \in \mathbb{F}^n. \tag{2.2}$$

Clearly, $\|\mathcal{T}\|_{\infty, \mathbb{F}} \leq \|\mathcal{T}\|$, and for a nonzero tensor $\mathcal{T}$ the equality $\|\mathcal{T}\|_{\infty, \mathbb{F}} = \|\mathcal{T}\|$ if and only if $\mathcal{T}$ is a rank one tensor. That is $\mathcal{T} = \otimes_{j=1}^d x_j$, where $x_j \in \mathbb{F}^{n_j} \setminus \{0\}$ for $j \in [d]$. We let $\|\mathcal{T}\|_{\infty} = \|\mathcal{T}\|_{\infty, \mathbb{C}}$. It is shown in [24] that

$$\|\mathcal{T}\|_{\infty} = \max\{\Re(\langle \mathcal{T}, \mathcal{X} \rangle), \ \mathcal{X} \in \Pi^a(\mathbb{C})\}.$$
The nuclear norm of $\mathcal{T} \in \mathbb{F}^n$ is defined as follows:

$$\|\mathcal{T}\|_{1,\mathbb{F}} = \min \left\{ \sum_{i=1}^{\rho} \prod_{j=1}^{d} \|x_{i,j}\|, \mathcal{T} = \sum_{i=1}^{\rho} \otimes_{j=1}^{d} x_{i,j}, x_{i,j} \in \mathbb{F}^{m_{ij}}, i \in [\rho], j \in [d] \right\}.$$ (2.3)

Again we let $\|\mathcal{T}\|_1 = \|\mathcal{T}\|_{1,\mathbb{C}}$. It is shown in [20] that we can choose in the characterization (2.3) $\rho = N(n)$ for $\mathbb{F} = \mathbb{R}$. As $\mathbb{C}^n$ can be viewed as $\mathbb{R}^n \oplus \mathbb{R}^n$ it follows that we can choose in the characterization (2.3) $\rho = 2N(n)$ for $\mathbb{F} = \mathbb{C}$.

Furthermore, it is known that the nuclear norm is the dual norm to the spectral norm over $\mathbb{F}$ [20]. That is,

$$\|\mathcal{T}\|_{q,\mathbb{F}} = \max \{ |\langle \mathcal{T}, Y \rangle|, \|Y\|_{p,\mathbb{F}} = 1 \}, \quad \frac{1}{p} + \frac{1}{q} = 1, \ p \in \{1, \infty\}. \quad (2.4)$$

Hence the following well known inequality holds

$$\|\mathcal{T}\|_2 \leq \|\mathcal{T}\|_{1,\mathbb{F}} \|\mathcal{T}\|_{\infty,\mathbb{F}} \quad \text{for all } \mathcal{T} \in \mathbb{F}^n. \quad (2.5)$$

(Assume for example that $\|\mathcal{T}\|_{1,\mathbb{F}} = 1$ and take in the maximal characterization of $\|\mathcal{T}\|_{\infty,\mathbb{F}}$, $Y = \mathcal{T}$.)

Note that the characterization of spectral norm and the characterization (2.4) yield that the extreme points of the unit ball of the nuclear norm is the set $\Pi^n(\mathbb{F})$. Hence $\|\mathcal{T}\|_{1,\mathbb{F}} \geq \|\mathcal{T}\|$. Equality for a nonzero $\mathcal{T}$ holds if and only if $\mathcal{T}$ is a rank one tensor.

Observe that by the definition

$$\|\mathcal{T}\|_{\infty,\mathbb{R}} \leq \|\mathcal{T}\|_{\infty,\mathbb{F}}, \quad \|\mathcal{T}\|_{1,\mathbb{R}} \geq \|\mathcal{T}\|_1, \quad \text{for } \mathcal{T} \in \mathbb{R}^n.$$ For $d \geq 3$ one may have strict inequalities [20]. However for matrices, $d = 2$, we have always equalities in the above inequalities, since the spectral norm and the nuclear norm of a matrix $\mathcal{T}$ is the maximal singular value and the sum of singular values respectively.

Let $\alpha(n, \mathbb{F})$ and $\beta(n, \mathbb{F})$ be the best constants for comparison of the norms $\|\mathcal{T}\|_{1,\mathbb{F}}$, $\|\mathcal{T}\|$ and $\|\mathcal{T}\|_{\infty,\mathbb{F}}$:

$$\frac{1}{\alpha(n, \mathbb{F})} \|\mathcal{T}\|_{1,\mathbb{F}} \leq \|\mathcal{T}\| \leq \frac{1}{\beta(n, \mathbb{F})} \|\mathcal{T}\|_{\infty,\mathbb{F}}, \quad \text{for all } \mathcal{T} \in \mathbb{F}^n. \quad (2.6)$$

Thus

$$\alpha(n, \mathbb{F}) = \max \{ \|\mathcal{T}\|_{1,\mathbb{F}}, \mathcal{T} \in \mathbb{F}^n, \|\mathcal{T}\| = 1 \}, \quad (2.7)$$

$$\beta(n, \mathbb{F}) = \min \{ \|\mathcal{T}\|_{\infty,\mathbb{F}}, \mathcal{T} \in \mathbb{F}^n, \|\mathcal{T}\| = 1 \}. \quad (2.8)$$

The following result is well known for matrices.

**Lemma 2.1** Let $1 < m \leq n$ be integers. Let $\mathcal{T} \in \mathbb{F}^{m \times n}$ and assume $\|\mathcal{T}\| = 1$. Then

1. The equality $\|T\|_1 = \alpha(m, n)$ holds if and only if $m$ singular values of $T$ are $\frac{1}{\sqrt{m}}$.  

4
2. The equality \( \|T\|_\infty = \beta(m, n) \) holds if and only if \( m \) singular values of \( T \) are \( \frac{1}{\sqrt{m}} \).

In particular
\[
\alpha(m, n) = \sqrt{m}, \quad \beta(m, n) = \frac{1}{\sqrt{m}}.
\]

Proof. Let \( \sigma_1 \geq \cdots \geq \sigma_m \geq 0 \). Then \( 1 = \|T\|^2 = \sum_{i=1}^{m} \sigma_i^2 \). Use Cauchy-Schwarz inequality to deduce that \( \|T\|^2 = (\sum_{i=1}^{m} \sigma_i^2)^2 \leq m \left( \sum_{i=1}^{m} \sigma_i^2 \right) = m \). Equality holds if and only if all singular values of \( T \) are \( \frac{1}{\sqrt{m}} \).

Observe next that from the equality \( 1 = \sum_{i=1}^{m} \sigma_i^2 \) we deduce that \( 1 \leq m \sigma_1^2 \). Hence \( \|T\|_\infty = \sigma_1 \geq \frac{1}{\sqrt{m}} \). Equality holds if and only if all singular values of \( T \) are \( \frac{1}{\sqrt{m}} \). \( \square \)

The following theorem generalizes the above lemma to tensors, \( d \geq 3 \). Its first part is a simple consequence of the fact that the spectral and the nuclear norms are dual.

Theorem 2.2 Let \( d \geq 3 \). Then
\[
\alpha(n, F) \beta(n, F) = 1.
\] (2.10)

Assume furthermore that \( T \in F^n \) and \( \|T\| = 1 \). If either \( \|T\|_{1, F} = \alpha(n, F) \) or \( \|T\|_{\infty, F} = \beta(n, F) \) then
\[
\|T\|_{1, F} \|T\|_{\infty, F} = \|T\|^2 = 1.
\] (2.11)

That is, \( \|T\|_{1, F} = \alpha(n, F) \) if and only if \( \|T\|_{\infty, F} = \beta(n, F) \).

Proof. The dual characterization of \( \|T\|_{\infty, F} \) (2.4) yields
\[
\|T\|_{\infty, F} = \max_{\|Y\|_{1, F} \leq 1} \frac{\langle T, Y \rangle}{\|Y\|_{1, F}} \geq \max_{\|Y\|_{1, F} \neq 0} \frac{\langle T, Y \rangle}{\alpha(n, F)\|Y\|} = \frac{1}{\alpha(n, F)} \|T\|.
\]

Hence \( \beta(n, F) \geq \frac{1}{\alpha(n, F)} \). The maximal characterization of \( \|T\|_{1, F} \) yields
\[
\|T\|_{1, F} = \max_{\|Y\|_{\infty, F} \leq 1} \frac{\langle T, Y \rangle}{\|Y\|_{\infty, F}} \leq \max_{\|Y\|_{\infty, F} \neq 0} \frac{\langle T, Y \rangle}{\beta(n, F)\|Y\|} = \frac{1}{\beta(n, F)} \|T\|.
\]

Hence \( \alpha(n, F) \leq \frac{1}{\beta(n, F)} \). This proves (2.10).

We now prove the second part of the theorem. Assume first the case \( F = C \). Let \( T^* \in C^n \) satisfy \( \|T^*\| = 1 \) and \( \|T^*\|_1 = \alpha(n, C) \). Assume that \( B \in C^n \) and \( \Re(\langle B, T^* \rangle) = 0 \). Set \( T(\varepsilon) = T^* + \varepsilon B \). Here \( \varepsilon \) is a small real number. So \( \|T(\varepsilon)\| = 1 + O(\varepsilon^2) \). Assume that \( \|S\|_{\infty} = 1 \) and \( \langle T^*, S \rangle = \|T^*\|_1 \). Hence
\[
\Re\left(\frac{1}{\|T(\varepsilon)\|} \langle T(\varepsilon), S \rangle \right) \leq \|\frac{1}{\|T(\varepsilon)\|} \langle T(\varepsilon), S \rangle\|_{1, F} \leq \alpha(n, C).
\]

From the maximality of \( \|T^*\|_{1, F} \) it follows that \( \Re(\langle S, B \rangle) \leq 0 \). By replacing \( B \) by \( -B \) we deduce that \( \Re(\langle S, B \rangle) = 0 \).

Consider the hyperplane \( \Re(\langle X, \|T^*\|_1 T^* \rangle) = \|T^*\|_1 \). This hyperplane passes through \( T^* \). Consider the balanced convex set \( C := \{X \in C^n, \|X\|_1 \leq \|T^*\|_1 \} \). We
claim that the hyperplane \( \mathbb{R} (\langle \mathcal{X}, \| T^* \|_1 T^* \rangle) = \| T^* \|_1 \) supports this convex set at \( T^* \).

If not, there exists \( B, \mathbb{R} (\langle B, T^* \rangle) = 0 \) and \( T^*(\varepsilon) \) is in the interior of \( C \) for each small positive \( \varepsilon \). Recall that a supporting hyperplane of \( C \) at \( T^* \) is \( \mathbb{R} (\langle \mathcal{X}, \mathcal{S} \rangle) \leq \| T^* \|_1 \), for some \( \mathcal{S} \in \mathbb{C}^d \), where \( \| \mathcal{S} \|_\infty = 1 \) and \( \langle T^*, \mathcal{S} \rangle = \| T^* \|_1 \). As \( T(\varepsilon) \) is in the interior of \( C \) for small enough \( \varepsilon \) it follows that \( \mathbb{R} (\langle B, \mathcal{S} \rangle) < 0 \). This will contradict the previous observation. Hence \( \mathbb{R} (\langle \mathcal{X}, \| T^* \|_1 T^* \rangle) = \| T^* \|_1 \) is a supporting hyperplane to \( C \) at \( T^* \). Therefore \( 1 = \| \| T^* \|_1 T^* \|_\infty = \| T^* \|_1 \| T^* \|_\infty \).

Other cases of the second part of the theorem are proved similarly. \( \square \)

Let \( \mathbf{n} \in \mathbb{N}^d, \mathbf{n}' \in \mathbb{N}^d \). Then \( \mathbb{F}^n \otimes \mathbb{F}^{n'} = \mathbb{F}^m \), where \( \mathbf{m} = (\mathbf{n}, \mathbf{n}') \in \mathbb{N}^{d+d'} \). In what follows we will need the following lemma.

**Lemma 2.3** Let \( d, d' \in \mathbb{N} \) and assume that \( \mathbf{n} \in \mathbb{N}^d, \mathbf{n}' \in \mathbb{N}^{d'} \). Suppose that \( \mathcal{T} \in \mathbb{F}^n, \mathcal{T}' \in \mathbb{F}^{n'} \). Then

\[
\| \mathcal{T} \otimes \mathcal{T}' \|_{1,F} = \max \{ |\langle \mathcal{T} \otimes \mathcal{T}', \mathcal{Z} \rangle|, \mathcal{Z} \in \mathbb{F}^m, \| \mathcal{Z} \|_{\infty,F} = 1 \}.
\]

Consider the subset of all \( \mathcal{Z} \in \mathbb{F}^m \) of spectral norm one, of the form \( \mathcal{X} \otimes \mathcal{X}' \), where \( \mathcal{X} \in \mathbb{F}^n, \mathcal{X}' \in \mathbb{F}^{n'} \) and \( \| \mathcal{X} \|_{\infty,F} = \| \mathcal{X}' \|_{\infty,F} = 1 \). Hence

\[
\| \mathcal{T} \otimes \mathcal{T}' \|_{1,F} \geq (\max \{ |\langle \mathcal{T}, \mathcal{X} \rangle|, \| \mathcal{X} \|_{\infty,F} = 1 \} \) (\max \{ |\langle \mathcal{T}', \mathcal{X}' \rangle|, \| \mathcal{X}' \|_{\infty,F} = 1 \}) = \| \mathcal{T} \|_{1,F} \| \mathcal{T}' \|_{1,F}.
\]

The value of \( \beta(\mathbf{n}, \mathbb{C}) \), and hence of \( \alpha(\mathbf{n}, \mathbb{C}) \) is known for \( \mathbf{n} = (2, 2, 2) \), see §7.

## 3 Symmetric tensors

A tensor \( \mathcal{S} = [s_{i_1, \ldots, i_d}] \in \otimes^d \mathbb{F}^n \) is called symmetric if \( s_{i_1, \ldots, i_d} = s_{\omega(i_1), \ldots, \omega(i_d)} \) for every permutation \( \omega : [d] \rightarrow [d] \). Denote by \( \mathcal{S}^d \mathbb{F}^n \subset \otimes^d \mathbb{F}^n \) the vector space of \( d \)-mode symmetric tensors on \( \mathbb{F}^n \). It is well known that \( \dim \mathcal{S}^d \mathbb{F}^n = (n+d-1) \) [17]. In what
follows we assume that $\mathcal{S}$ is a symmetric tensor and $d \geq 2$, unless stated otherwise. A tensor $\mathcal{S} \in \mathbb{S}^{d \times \mathbb{F}^n}$ defines a unique homogeneous polynomial of degree $d$ in $n$ variables

$$f(\mathbf{x}) = \mathcal{S} \times \otimes^d \mathbf{x} = \sum_{0 \leq j_k \leq d, k \in [n], j_1 + \cdots + j_n = d} \frac{d!}{j_1! \cdots j_n!} f_{j_1, \ldots, j_n} x_1^{j_1} \cdots x_n^{j_n}. \quad (3.1)$$

Conversely, a homogeneous polynomial $f(\mathbf{x})$ of degree $d$ in $n$ variables defines a unique symmetric $\mathcal{S} \in \mathbb{S}^{d \times \mathbb{F}^n}$ by the following relation. Consider the multiset \{$i_1, \ldots, i_d\}, \text{ where each } i_i \in [n]. \text{ Let } j_k \text{ be the number of times the integer } k \in [n] \text{ appears in the multiset } \{i_1, \ldots, i_d\}. \text{ Then } \mathcal{S}_{i_1, \ldots, i_d} = f_{j_1, \ldots, j_n}. \text{ Furthermore}

$$\|\mathcal{S}\|^2 = \sum_{0 \leq j_k \leq d, k \in [n], j_1 + \cdots + j_n = d} \frac{d!}{j_1! \cdots j_n!} |f_{j_1, \ldots, j_n}|^2, \quad (3.2)$$

where $\mathcal{S}_{i_1, \ldots, i_d} = f_{j_1, \ldots, j_n}$.

The remarkable result of Banach [4] claims that the spectral norm of a symmetric tensor can be computed as a maximum on the set of rank one symmetric tensors:

$$\|\mathcal{S}\|_{\sigma, \mathbb{F}} = \max\{\|\mathcal{S} \times \otimes^d \mathbf{x}\|, \mathbf{x} \in \Pi^n(\mathbb{F}), \mathcal{S} \in \mathbb{S}^{d \times \mathbb{F}^n}\}. \quad (3.3)$$

This result was rediscovered several times since 1938. In quantum information theory (QIT), for the case $\mathbb{F} = \mathbb{C}$, it appeared in [32]. In mathematical literature, for the case $\mathbb{F} = \mathbb{R}$, it appeared in [9, 16]. (Observe that a natural generalization of Banach’s theorem to partially symmetric tensors is given in [16].)

The analog of Banach’s theorem for the nuclear norm of symmetric tensors was stated in [20]. Namely, for $\mathcal{S} \in \mathbb{S}^{d \times \mathbb{F}^n}$ we have the following minimal characterization

$$\|\mathcal{S}\|_{1, \mathbb{F}} = \min\{\sum_{i=1}^M \|\mathbf{x}_i\|^d, \mathcal{S} = \sum_{i=1}^M \varepsilon_i \otimes^d \mathbf{x}_i, \mathbf{x}_i \in \mathbb{F}^n, \varepsilon_i \in \{1, -1\} \text{ for } i \in [r]\}. \quad (3.4)$$

We can assume that $\varepsilon_i = 1$ unless $\mathbb{F} = \mathbb{R}$ and $d$ is even. Furthermore, we can assume that $r = \binom{n+d-1}{d}$ for $\mathbb{F} = \mathbb{R}$ and $r = 2 \binom{n+d-1}{d}$ for $\mathbb{F} = \mathbb{C}$.

For symmetric tensors we can improve the inequalities (2.6) to

$$\frac{1}{\alpha'(n, d, \mathbb{F})} \|\mathcal{S}\|_{1, \mathbb{F}} \leq \|\mathcal{S}\| \leq \frac{1}{\beta'(n, d, \mathbb{F})} \|\mathcal{S}\|_{\infty, \mathbb{F}}, \text{ for all } \mathcal{S} \in \mathbb{S}^{d \times \mathbb{F}^n}. \quad (3.5)$$

Here

$$\alpha'(n, d, \mathbb{F}) = \max\{\|\mathcal{S}\|_{1, \mathbb{F}}, \mathcal{S} \in \mathbb{S}^{d \times \mathbb{F}^n}, \|\mathcal{S}\| = 1\}, \quad (3.5)$$

$$\beta'(n, d, \mathbb{F}) = \min\{\|\mathcal{S}\|_{\infty, \mathbb{F}}, \mathcal{S} \in \mathbb{S}^{d \times \mathbb{F}^n}, \|\mathcal{S}\| = 1\}. \quad (3.6)$$

We state an analog of Theorem 2.2 and Lemma 2.1. The proof of this theorem is similar to Theorem 2.2 and Lemma 2.1 and we leave it to the reader.

**Theorem 3.1** Let $n, d \geq 2$ be integers. Then

1. $\alpha'(n, d, \mathbb{F}) \beta'(n, d, \mathbb{F}) = 1$.

2. Assume that $\mathcal{S} \in \mathbb{S}^{d \times \mathbb{F}^n}$ and $\|\mathcal{S}\| = 1$. Then $\|\mathcal{S}\|_{1, \mathbb{F}} = \alpha'(n, d, \mathbb{F})$ if and only if $\|\mathcal{S}\|_{\infty, \mathbb{F}} = \beta'(n, d, \mathbb{F})$.

3. $\alpha'(n, 2, \mathbb{F}) = \sqrt{n}$, $\beta(n, 2, \mathbb{F}) = \frac{1}{\sqrt{n}}$.

4. Assume that $S$ is an $n \times n$ complex valued symmetric matrix having Frobenius norm one, $\|S\| = 1$. Then $\|\mathcal{S}\|_{1} = \alpha'(n, 2, \mathbb{F})$ if and only if $\sqrt{n} S$ is a unitary matrix.
4 Separability and nuclear norm

In quantum physics, a state is a normalized vector $\mathbf{u} \in \mathbb{C}^n$ of length one: $\|\mathbf{u}\| = 1$. Furthermore, the state $\zeta \mathbf{u}$ is identified with $\mathbf{u}$ for each $\zeta \in \mathbb{C}, |\zeta| = 1$. Suppose that we have a number of sources that emit states $\mathbf{u}_i \in \mathbb{C}^n$, independently, each with probability $p_i > 0$ for $i \in [n]$. The resulting physical system is called a mixed state.

A standard model of von J. Neumann and L. Landau associates the above mixed state, or unentangled, if we have a number of sources that emit states $\mathbf{u}_i \in \mathbb{C}^n$, independently, each with probability $p_i > 0$ for $i \in [n]$. The resulting physical system is called a mixed state.

Furthermore, the state $\zeta \mathbf{u}$ is identified with $\mathbf{u}$ for each $\zeta \in \mathbb{C}, |\zeta| = 1$. Suppose that we have a number of sources that emit states $\mathbf{u}_i \in \mathbb{C}^n$, independently, each with probability $p_i > 0$ for $i \in [n]$. The resulting physical system is called a mixed state.

Let $H$ be the real vector subspace of $2^n$-dimensional tensor products, each with entries $a_{i_1,\ldots,i_d}$ where $i_k \notin [n]$ for $k \in [d]$. The hermitian condition is $a_{j_1,\ldots,j_d}(i_1,\ldots,i_d) = a_{i_1,\ldots,i_d}(j_1,\ldots,j_d)$. Furthermore $A$ is a nonnegative definite matrix with trace 1:

$$\sum_{i_1=\ldots=i_d=1}^{n_1,\ldots,n_d} a_{i_1,\ldots,i_d}(i_1,\ldots,i_d) = 1.$$ 

Consider 2d-mode tensors $B = [b_{i_1,\ldots,i_{2d}}] \in \mathbb{C}^m$. Viewing $B$ as a matrix over $\mathbb{C}^n$ we define the trace as

$$\text{tr}(B) := \sum_{i_1=\ldots=i_{2d}=1}^{n_1,\ldots,n_d} b_{i_1,\ldots,i_{2d}}.$$  \hfill (4.1)

It is straightforward to see that

$$\text{tr}(\mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_{2d}) = \prod_{j=1}^{d} \mathbf{x}_{j+d}^\top \mathbf{x}_j$$  \hfill (4.2)

for every $\mathbf{x}_j, \mathbf{x}_{j+d} \in \mathbb{C}^{n_j}, j = 1,\ldots,d$.

We call $B$ a hermitian tensor if $b_{j_1,\ldots,j_d}(i_1,\ldots,i_d) = \overline{b_{i_1,\ldots,i_d}(j_1,\ldots,j_d)}$ for all indices.

Let $\mathbb{H}^{n \times n} \subset \mathbb{C}^{n \times n}$ be the real vector subspace of 2d-hermitian tensors. A hermitian tensor $B \in \mathbb{H}^{n \times n}$ is nonnegative definite if the corresponding $N(n) \times N(n)$ hermitian matrix $B$ with entries $b_{i_1,\ldots,i_d}(j_1,\ldots,j_d)$ is nonnegative definite. The convex set of nonnegative definite hermitian tensors with trace 1 are identified with density tensors on $\mathbb{C}^n$, and denoted by $\mathbb{H}^{n \times n}_{+1}$. With the density matrix $A$ as above we associate the density tensor $A = [a_{i_1,\ldots,i_{2d}}] \in \mathbb{C}^m$.

A state $\mathcal{T} \in \mathbb{C}^n$ induces the density tensor $\mathcal{A} = \mathcal{T} \otimes \mathcal{T}$, which we also call pure state. A density tensor $\mathcal{A}$ is a convex combination of pure states. A product state $\mathcal{T} = \otimes_{i=1}^d x_i$ induces the pure product state $\otimes_{i=1}^d (x_i x_i^*)$, which we also identify with $\otimes_{i=1}^d (x_i x_i^*)$, i.e., the tensor product of pure states. A density
tensor corresponding to a mixed state of product states is called separable. That is,
\( A \in \mathbb{H}^{n \times n}_{sep} \) is separable if it is of the form
\[
A = \sum_{i=1}^{r} p_i (\otimes_{j=1}^{d} x_{j,i}) \otimes (\otimes_{j=1}^{d} x_{j,i}^*),
\]
where \( x_{j,i} \in \mathbb{C}^{n_j}, x_{j,i}^* x_{j,i} = 1, j \in [d], p_i \geq 0, i \in [r], \sum_{i=1}^{r} p_i = 0. \)

We denote by \( \mathbb{H}^{n \times n}_{sep} \subset \mathbb{H}^{n \times n} \) the convex set of separable density tensors. The following separability criterion was stated and proved in [18] (unpublished):

**Lemma 4.1** Let \( A \in \mathbb{C}^{n \times n}. \) Then
\[
|\text{tr}(A)| \leq \|A\|_1, \quad (4.4)
\]
and equality holds if and only if \( A = zB \) for some \( z \in \mathbb{C} \) and \( B \in \mathbb{H}^{n \times n}_{sep}. \) Assume furthermore that \( A \) is a density tensor. Then \( \|A\|_1 \geq 1 \) and equality holds if and only if \( A \) is separable.

**Proof.** Let \( A = \sum_{i=1}^{r} x_{1,i} \otimes \cdots \otimes x_{2d,i}, \) where \( \|A\|_1 = \sum_{i=1}^{r} \prod_{j=1}^{d} \|x_{j,i}\| > 0. \) In view of (4.2), \( \text{tr}(A) = \sum_{i=1}^{r} \prod_{j=1}^{d} (x_{j+d,i}^T x_{j,i}). \) The Cauchy–Schwarz inequality yields that \( |x_{j+d,i}^T x_{j,i}| \leq \|x_{j+d,i}\| \|x_{j,i}\|. \) Equality holds if and only if \( x_{j+d,i} = z_{j,i} x_{j,i} \) for some \( z_{j,i} \in \mathbb{C}. \) Thus
\[
|\text{tr}(A)| \leq \sum_{i=1}^{r} \prod_{j=1}^{d} \|x_{j+d,i}\| \leq \sum_{i=1}^{r} \prod_{j=1}^{d} \|x_{j,i}\| = \|A\|_1.
\]
This establishes (4.4). Suppose that equality holds in (4.4). Then \( x_{1,i} \otimes \cdots \otimes x_{2d,i} \) is of the form \( z_{j,i}(x_{1,i}x_{1,i}^*) \otimes \cdots \otimes (x_{d,i}x_{d,i}^*). \) Observe that
\[
\text{tr}(z_{j,i}(x_{1,i}x_{1,i}^*) \otimes \cdots \otimes (x_{d,i}x_{d,i}^*)) = z_{j,i} \prod_{j=1}^{d} \|x_{j,i}\|^2.
\]
Without loss of generality we may assume that \( \|x_{j,i}\| = 1 \) for \( j = 1, \ldots, d. \) Since equality holds in the triangle inequality it follows that all \( z_{j,i} \) must have the same arguments. Hence \( A = zB \) where
\[
B = \sum_{i=1}^{r} t_i (x_{1,i}x_{1,i}^*) \otimes \cdots \otimes (x_{d,i}x_{d,i}^*), \quad (4.5)
\]
where \( x_{j,i}^* x_{j,i} = 1 \) for \( j = 1, \ldots, d, \) and \( \sum_{i=1}^{r} t_i = 1, t_i > 0, \) for \( i = 1, \ldots, r. \)

Conversely, suppose \( B \) is separable. Hence \( B \) is of the above form. Therefore
\[
\|B\|_1 \leq \sum_{i=1}^{r} t_i \prod_{j=1}^{d} \|x_{j,i}\|^2 = 1.
\]
Clearly, \( \text{tr}(B) = 1. \) In view of (4.4), it follows that \( \|B\|_1 = 1. \) Hence a decomposition (4.5) of \( B \) is minimal with respect to the nuclear norm.

Assume now that \( A \) is a density tensor. Then \( \text{tr}(A) = 1 \) and (4.4) yields that \( \|A\|_1 \geq 1. \) The above arguments show that \( \|A\|_1 = 1 \) if and only if \( A \) is separable. \( \square \)

For \( d = 2 \) this result is due to [41]. We will use the following hardness result from [27] (see also [25]).
Theorem 4.2 (Gurvits) Deciding whether a given density tensor is bipartite separable \((d = 2)\), is an NP-hard problem.

From this and Lemma 4.1, we immediately deduce the hardness result for tensor nuclear norm [20]:

Corollary 4.3 Deciding whether a given 4-tensor is in the nuclear norm unit ball is an NP-hard problem.

A simple, (polynomially computable), necessary condition for separability of density tensors, is the positivity of the partial transpose [40]. For bipartite density tensors it is also sufficient if and only if \(n = (2, 2), n = (2, 3), n = (3, 2)\) [31].

5 Bisymmetric density tensors

Assume that \(n_1 = \cdots = n_d = n\) and denote \(n^{\times d} = (n, \ldots, n) \in \mathbb{N}^d\). A hermitian tensor \(A = [a_{i_1 \ldots i_d, i_{d+1} \ldots i_{2d}}] \in \mathbb{H}^{n^{\times d} \times n^{\times d}}\) is called bisymmetric if it is symmetric with respect to the \(d\) indices \((i_1, \ldots, i_d)\) and \((i_{d+1}, \ldots, i_{2d})\) (separately). Denote by \(\mathbb{H}_{bsym}^{n^{\times d} \times n^{\times d}}\) the space of hermitian symmetric tensors. We first observe the following result.

Lemma 5.1 Assume that \(n, d \geq 2\) are integers. Then

\[
\dim \mathbb{H}_{bsym}^{n^{\times d} \times n^{\times d}} = \binom{n + d - 1}{d}^2.
\] (5.1)

Furthermore, a hermitian tensor \(A \in \mathbb{H}^{n^{\times d} \times n^{\times d}}\) is bisymmetric if and only it has the spectral decomposition.

\[
A = \sum_{i=1}^{(n+d-1\choose d}} \lambda_i T_i \otimes T_i, \quad T_i \in \mathcal{S}^{d \times n}, \langle T_i, T_j \rangle = \delta_{ij}, \lambda_i \in \mathbb{R}, i, j \in \left[\binom{n + d - 1}{d}\right].
\] (5.2)

Proof. Equality (5.1) follows from counting the set of indices \((i_1, \ldots, i_d)\), invariant under the action the symmetric group on \([d]\). (Just consider the indices \((i_1, \ldots, i_d)\) where \(1 \leq i_1 \leq \cdots \leq i_d \leq n\).

Clearly, a tensor \(A\) of the form (5.2) is a hermitian bisymmetric tensor. Consider a spectral decomposition of a hermitian bisymmetric tensor \(B\), as a Hermitian matrix.

\[
B = \sum_{i=1}^{n^d} \mu_i X_i \otimes X_i, \quad X_i \in \mathbb{C}^{n \times n}, \langle X_i, X_j \rangle = \delta_{ij}, \mu_i \in \mathbb{R}, i, j \in [n^d].
\]

Assume that \(\mu_i \neq 0\). Then \(X_i = \mu_i^{-1} B \times X_i\). Hence \(X_i \in \mathcal{S}^{d \times n}\). In view of (5.1) we can have at most \((n + d - 1\choose d)\) orthonormal vectors in \(\mathcal{S}^{d \times n}\). Therefore \(B\) has representation (5.2).

Clearly, the real space of hermitian bisymmetric states is a real subspace of \(\mathcal{S}^{d \times n} \otimes \mathcal{S}^{d \times n}\), the subspace of \(\otimes^{2d} \mathcal{C}^n\) tensors which are bisymmetric, i.e. symmetric in the first \(d\) and the last \(d\) indices. We first consider the restriction of the spectral and the nuclear norms on \(\mathcal{S}^{d \times n} \otimes \mathcal{S}^{d \times n}\).
Theorem 5.2 Let $C \in \mathbb{S}^{d_C n} \otimes \mathbb{S}^{d_C n}$. Then

$$\|C\|_{\infty} = \max \{ \Re \left( \langle C, (\otimes^d x) \otimes (\otimes^d y) \rangle \right) : x, y \in \mathbb{C}^n, \|x\| = \|y\| = 1 \} \quad (5.3)$$

$$\|C\|_1 = \min \{ \sum_{i=1}^M \|x_i\| \|y_i\| : C = \sum_{i=1}^M (\otimes^d x_i) \otimes (\otimes^d y_i) \}. \quad (5.4)$$

Suppose that $A \in \mathbb{H}^{n \times d \times n \times d}_{bsym}$. Then

$$\|A\|_{\infty} = \max \{ \Re \left( \langle A, \frac{1}{2}((\otimes^d x) \otimes (\otimes^d y) + (\otimes^d y) \otimes (\otimes^d x)) \rangle \right) : x, y \in \mathbb{C}^n, \|x\| = \|y\| = 1 \} \quad (5.5)$$

$$\|A\|_1 = \min \{ \sum_{i=1}^M \|x_i\| \|y_i\| : A = \sum_{i=1}^M \frac{1}{2}((\otimes^d x_i) \otimes (\otimes^d y_i) + (\otimes^d y_i) \otimes (\otimes^d x)) \}. \quad (5.6)$$

In particular, the set of the extreme points of the restriction of the nuclear norm to $\mathbb{H}^{n \times d \times n \times d}_{bsym}$ is of the form

$$\frac{1}{2}((\otimes^d x) \otimes (\otimes^d y) + (\otimes^d y) \otimes (\otimes^d x)), \quad x, y \in \mathbb{C}^n, \|x\| = \|y\| = 1. \quad (5.7)$$

Proof. The characterization (5.3) follows from Banach’s theorem. The characterization (5.4) follows from the arguments of the proof of the generalization of the Banach theorem to nuclear norm [20]. The characterization (5.5) follows from (5.3) and the equality

$$\Re \left( \langle A, (\otimes^d x) \otimes (\otimes^d y) \rangle \right) = \Re \left( \langle A, (\otimes^d y) \otimes (\otimes^d x) \rangle \right).$$

The characterization (5.5) yields that the set of the extreme points of the restriction of the nuclear norm to $\mathbb{H}^{n \times d \times n \times d}_{bsym}$ is given by (5.6). The arguments in [20] yield the characterization (5.6). $\Box$

Combine the proof of Lemma 4.1 with (5.6) to deduce

Corollary 5.3 Let $A \in \mathbb{H}^{n \times d \times n \times d}_{bsym}$. Then the following statements are equivalent:

1. $A$ is separable.
2. $\|A\|_1 = 1$.
3. $A = \sum_{i=1}^r p_i (\otimes^d x_i) \otimes (\otimes^d y_i), \quad x_i \in \mathbb{C}^n, \|x_i\| = 1, p_i > 0, \sum_{i=1}^r p_i = 1. \quad (5.8)$

Another criterion to check if a bipartite ($d = 2$) density tensor is separable is given in [39].
6 Maximally entangled states

One of the main notions in quantum physics is the notion of entanglement. The entanglement of \( \mathcal{T} \) can be measured in many different ways. One of them that we discuss here is the geometric measure of entanglement. Let \( \Pi^n := \Pi^n(\mathbb{C}) \) be the space of the product states (2.1). Then the geometric measure of entanglement is the distance of a state \( \mathcal{T} \) to \( \Pi^n \):

\[
\text{dist}(\mathcal{T}, \Pi^n) = \min\{\|\mathcal{T} - \mathcal{Y}\|, \mathcal{Y} \in \Pi^n\}.
\]

As \( \|\mathcal{T}\| = \|\mathcal{Y}\| = 1 \) it follows that \( \text{dist}(\mathcal{T}, \Pi^n(F))^2 = 2(1 - \|\mathcal{T}\|_{\infty,F}) \). Hence an equivalent notion of the geometric measure of entanglement is

\[
\eta(\mathcal{T}) := -\log_2 \|\mathcal{T}\|_\infty^2. \tag{6.1}
\]

Thus \( \mathcal{T} \) is entangled if and only if \( \eta(\mathcal{T}) > 0 \).

Recall that for a \( d \)-partite state \( \|\mathcal{T}\|_1 \geq \|\mathcal{T}\|_\infty = 1 \). Furthermore, \( \mathcal{T} \) is a product state if and only if \( \|\mathcal{T}\|_1 = 1 \). Hence another way to measure the entanglement of \( \mathcal{T} \) is:

\[
\omega(\mathcal{T}) := \log_2 \|\mathcal{T}\|_1^2. \tag{6.2}
\]

Theorem 2.2 yields:

\[
\omega(n) := \max\{\omega(\mathcal{T}), \mathcal{T} \in \mathbb{C}^n, \|\mathcal{T}\|_1 = 1\} = \max\{\eta(\mathcal{T}), \mathcal{T} \in \mathbb{C}^n, \|\mathcal{T}\|_1 = 1\} = 2\log_2 \alpha(n, \mathbb{C}) = -2\log_2 \beta(n, \mathbb{C}). \tag{6.3}
\]

Furthermore, the most entangled states have the minimal spectral norm and maximal nuclear norm.

Similarly, for density tensor \( \mathcal{A} \in \mathbb{H}_{+1}^{n \times n} \) we can define the inseparability measure as \( \log_2 \|\mathcal{A}\|_1 \). Lemma 4.1 yields that \( \log_2 \|\mathcal{A}\|_1 \geq 0 \), and \( \mathcal{A} \) is separable if and only if \( \log_2 \|\mathcal{A}\|_1 = 0 \). Hence maximum inseparable density has the maximum value of \( \log_2 \|\mathcal{A}\|_1 \). We now show that any maximum entangled state in \( \mathbb{C}^n \) induces a maximum inseparable density tensor:

**Lemma 6.2** A density tensor \( \mathcal{A} \in \mathbb{H}_{+1}^{n \times n} \) satisfies inequality

\[
\log_2 \|\mathcal{A}\|_1 \leq 2\log_2 \alpha(n, \mathbb{C}). \tag{6.4}
\]

Equality holds if \( \mathcal{A} = \mathcal{T} \otimes \overline{\mathcal{T}} \) and \( \mathcal{T} \) is maximum entangled.

**Proof.** Suppose that \( \mathcal{T} \in \mathbb{C}^n \) is a state. Then \( \|\mathcal{T}\|_1 = \|\overline{\mathcal{T}}\|_1 \leq \alpha(n, \mathbb{C}) \). Hence \( \mathcal{B} = \mathcal{T} \otimes \overline{\mathcal{T}} \) is a pure density tensor, and \( \|\mathcal{B}\|_1 = \|\mathcal{T}\|_1^2 \leq \alpha(n, \mathbb{C})^2 \). Therefore \( \log_2 \|\mathcal{B}\| \leq 2\log_2 \alpha(n, \mathbb{C}) \). Clearly, \( \|\mathcal{B}\|_1 = \alpha(n, \mathbb{C})^2 \) if \( \mathcal{T} \) is maximum entangled. Assume that \( \mathcal{A} \in \mathbb{H}_{+1}^{n \times n} \). The spectral decomposition of \( \mathcal{A} \) gives a decomposition of \( \mathcal{A} \) as a mixed tensor:

\[
\mathcal{A} = \sum_{i=1}^{r} \lambda_i \mathcal{T}_i \otimes \overline{\mathcal{T}}_i, \quad \lambda_i > 0, i \in [r], \sum_{i=1}^{r} = 1.
\]
As the nuclear norm is a convex function it follows that
\[ \|A\|_1 \leq \sum_{i=1}^{r} \lambda_i \|T_i \otimes \overline{T}_i\|_1 = \sum_{i=1}^{r} \lambda_i \|T_i\|^2 \leq \alpha(n, \mathbb{C})^2 \sum_{i=1}^{r} \lambda_i = \alpha(n, \mathbb{C})^2. \]

We conjecture that equality in (6.4) implies that $A$ is a pure density state corresponding to maximum entangled state.

7 Maximum entangled 3 and 4 qubits

We start with the following simple lemma.

Lemma 7.1 We have
\[ \beta((n_1, n_2, \ldots, n_{d+1}), \mathbb{F}) \geq \frac{\beta((n_1, n_2, \ldots, n_d), \mathbb{F})}{\sqrt{n_{d+1}}} \]
and
\[ \alpha((n_1, n_2, \ldots, n_{d+1}), \mathbb{F}) \leq \sqrt{n_{d+1}} \alpha((n_1, n_2, \ldots, n_d), \mathbb{F}). \]

Proof. If $T$ is a unit tensor of type $(n_1, n_2, \ldots, n_{d+1})$ then we can write
\[ T = \lambda_1 T_1 \otimes e_1 + \lambda_2 T_2 \otimes e_2 + \cdots + \lambda_{n_{d+1}} T_{n_{d+1}} \otimes e_{n_{d+1}}, \]
where $T_1, \ldots, T_{n_{d+1}}$ are unit tensors of type $(n_1, \ldots, n_d)$ and $\lambda_1, \ldots, \lambda_{n_{d+1}} \in \mathbb{F}$ such that $|\lambda_1|^2 + \cdots + |\lambda_{n_{d+1}}|^2 = 1$. For some $i$ we have $|\lambda_i| \geq \frac{1}{\sqrt{n_{d+1}}}$. There exist a simple tensor $\mathcal{X}$ with $|\langle T_i, \mathcal{X} \rangle| \geq \beta((n_1, \ldots, n_d), \mathbb{F})$. Now we get
\[ \|T\|_{\infty, \mathbb{F}} \geq |\langle T_i, \mathcal{X} \otimes e_i \rangle| = |\lambda_i| \cdot |\langle T_i, \mathcal{X} \rangle| \geq \frac{1}{\sqrt{n_{d+1}}} \beta((n_1, \ldots, n_d), \mathbb{F}). \]

In what follows we use Dirac’s notation in this section. Namely, let $e_1 = (1, 0)^T, e_2 = (0, 1)^T$ be the standard orthonormal basis in $\mathbb{C}^2$. Recall that $\otimes^d \mathbb{C}^2$ has the standard basis $\otimes_{j=1}^{d} e_{i_j}$, where $i_1, \ldots, i_d \in [2]$. Then Dirac’s notation is
\[ |(i_1 - 1) \cdots (i_d - 1)\rangle = \otimes_{j=1}^{d} e_{i_j}, \quad i_1, \ldots, i_d \in [2]. \]

Consider the tensor the $W$ 3-tensor [12]
\[ W = |W\rangle = \frac{|100\rangle + |010\rangle + |001\rangle}{\sqrt{3}}. \]

By Banach’s theorem, the spectral norm is achieved on a symmetric state
\[ \langle S\rangle = (x(0) + y(1))^\otimes^3 = (x(0) + y(1)) \otimes (x(0) + y(1)) \otimes (x(0) + y(1)) \]
with value
\[ |\langle S|W\rangle| = |\sqrt{3}yx^2| = \sqrt{3}|y||x|^2. \]
where \( x, y \in \mathbb{C} \) with \( |x|^2 + |y|^2 = 1 \). An easy calculus exercise shows that the maximum is achieved when \( |x| = \sqrt{2}/\sqrt{3} \) and \( |y| = 1/\sqrt{3} \). We obtain

\[
\|W\|_{\infty, \mathbb{C}} = \sqrt{3} \left( \frac{\sqrt{2}}{\sqrt{3}} \right)^2 \frac{1}{\sqrt{3}} = \frac{2}{3} \text{ and } \|W\|_{1, \mathbb{C}} = \frac{3}{2}.
\]

(See also [20, \$6].) It is shown in [10] that 3-qubit state \( T \in \otimes^3 \mathbb{C}^2 \) is the most entangled if and only if it is locally unitary equivalent to \( W \). That is, let \( U(2) \subset \mathbb{C}^{2 \times 2} \) be the unitary group of \( 2 \times 2 \) complex valued matrices. Then orbit of \( W \) is defined as

\[
\text{orb}(W) = \{ T, T = (A_1 \otimes A_2 \otimes A_3)W, A_1, A_2, A_3 \in U(2) \}.
\]

Thus, \( T \) is maximally entangled if and only if \( T \in \text{orb}(W) \). In particular,

\[
\beta(2, 2, 2, \mathbb{C}) = \frac{2}{3}, \quad \alpha(2, 2, 2, \mathbb{C}) = \frac{3}{2}.
\]

(7.1)

A rank one decomposition that achieves the nuclear norm of \( W \) is:

\[
W = \frac{1}{6\sqrt{3}} \left[ \left( \sqrt{2} \langle 0 | + | 1 \rangle \right)^{\otimes 3} + \zeta^2 \left( \sqrt{2} \langle 0 | + \zeta | 1 \rangle \right)^{\otimes 3} + \zeta \left( \sqrt{2} \langle 0 | + \zeta^2 | 1 \rangle \right)^{\otimes 3} \right].
\]

Combine Lemma 7.1 with Lemma 2.1 and (7.1) to deduce

**Corollary 7.2**

\[
\beta(n^{x(d+1)}, \mathbb{F}) \geq \frac{1}{\sqrt{n}} \beta(n^{x(d)}, \mathbb{F}) \quad \text{for } d \geq 2.
\]

(7.2)

In particular

\[
\beta(n^{x(d)}, \mathbb{R}) \geq n^{\frac{1-d}{2}} \quad d \geq 2,
\]

(7.3)

\[
\beta(2^{x(d)}, \mathbb{C}) \geq \left( \frac{2}{3} \right)^{2^{\frac{(d-3)}{2}}} d \geq 3.
\]

(7.4)

We remark that Theorem 7.4 shows that the inequality (7.3) is sharp for \( n = 2 \) and each \( d \geq 2 \).

For \( n \geq 1 \) and \( \lambda \in \mathbb{C} \) with \( |\lambda| = 1 \), we define a tensor

\[
T_{n, \lambda} = \frac{1}{\sqrt{2}} \left( \lambda \left( \frac{|0 \rangle + i|1 \rangle}{\sqrt{2}} \right)^{\otimes n} + \overline{\lambda \left( \frac{|0 \rangle - i|1 \rangle}{\sqrt{2}} \right)^{\otimes n}} \right)
\]

and we will use the convention \( T_n = T_{n, 1} \). For example, we have

\[
T_3 = \frac{|000 \rangle - |110 \rangle - |101 \rangle - |011 \rangle}{\sqrt{2}},
\]

\[
T_4 = \frac{|0000 \rangle - |1100 \rangle - |1010 \rangle - |1001 \rangle - |0110 \rangle - |0101 \rangle - |0011 \rangle + |1111 \rangle}{2\sqrt{2}}
\]

and

\[
T_{4, -i} = \frac{|1000 \rangle + |0100 \rangle + |0010 \rangle + |0001 \rangle - |1110 \rangle - |1101 \rangle - |1011 \rangle - |0111 \rangle}{2\sqrt{2}}.
\]

As a complex tensor, the state \( T_{n, \lambda} \) is not much entangled in the sense of the nuclear or spectral norm.
Lemma 7.3 Let $\mathcal{T}_{n,\lambda}$ be defined as above. Then
\[
\|\mathcal{T}_{n,\lambda}\|_{\infty,C} = 1/\sqrt{2}, \quad \|\mathcal{T}_{n,\lambda}\|_{1,C} = \sqrt{2}. \tag{7.6}
\]

Proof. Clearly, $\mathcal{T}_{n,\lambda}$ is symmetric:
\[
\mathcal{T}_{n,\lambda} = \frac{1}{\sqrt{2}}(\otimes^n u + \otimes^n \bar{u}), \quad u = \frac{1}{\sqrt{2}}(1,i)^{\top}.
\]
Furthermore, $u$ and $\bar{u}$ is an orthonormal basis in $\mathbb{C}^2$. For $d \geq 2$ view the tensor $\mathcal{T}_{n,\lambda}$ as a matrix $T$ of dimension $2^{d_1} \times 2^{d_2}$, where $d_1 = \left\lceil \frac{n}{d} \right\rceil$, $d_2 = \left\lceil \frac{n}{d} \right\rceil$. Hence (7.5) is a singular value decomposition of $T$. Thus $\sigma_1(T) = \frac{1}{\sqrt{2}}$ and $\|T\|_1 = \sqrt{2}$. Therefore, $\|\mathcal{T}_{n,\lambda}\|_{\infty} \leq \sigma_1(T)$ and $\|\mathcal{T}_{n,\lambda}\|_1 \geq \|T\|_1$. However, this singular value decomposition are realized by left and right singular vectors which are rank one tensors. Hence (7.6) holds. \(\square\)

Theorem 7.4 For every mixed real $n$-qubit state $\mathcal{T}$ we have $\|\mathcal{T}\|_{\infty,R} \geq 2^{(1-n)/2}$ and $\|\mathcal{T}\|_{1,R} \leq 2^{(n-1)/2}$ and these inequalities are tight when $\mathcal{T} = \mathcal{T}_{n,\lambda}$. In particular, we have
\[
\alpha\left(\left(\frac{2,2,\ldots,2}{n}\right), \mathbb{R}\right) = 2^{(n-1)/2} \quad \text{and} \quad \beta\left(\left(\frac{2,2,\ldots,2}{n}\right), \mathbb{R}\right) = 2^{(1-n)/2}.
\]

Proof. We prove the theorem by induction on $n$. The case $n = 1$ is clear. If $\mathcal{T}$ is a $n$-qubit tensor of unit length, then we can write
\[
\mathcal{T} = |0\rangle \otimes S_0 + |0\rangle \otimes S_1
\]
with $\|S_0\|^2 + \|S_1\|^2 = 1$. For some $i \in \{0,1\}$ we have $\|S_i\|_{\infty,R}^2 \geq \frac{1}{2}$, so we get
\[
\|\mathcal{T}\|_{\infty,R} \geq \|S_i\|_{\infty,R} \geq 2^{(2-n)/2} \|S_i\| \geq 2^{(1-n)/2}.
\]
To calculate $\|\mathcal{T}_{n,\lambda}\|_{\infty,R}$, we use Banach’s theorem. We have to optimize Banach’s theorem. We have to optimize
\[
\left\langle (x|0\rangle + y|0\rangle)^{\otimes n}, \mathcal{T}_{n,\lambda} \right\rangle = \sqrt{2} \left| \Re \left( \lambda^{\frac{x+iy}{\sqrt{2}}} \right)^n \right|
\]
under the constraint $x^2 + y^2 = 1$. The optimal value clearly is equal to $2^{(1-n)/2}$ which shows that $\|\mathcal{T}_{n,\lambda}\|_{\infty,R} = 2^{(1-n)/2}$. It follows now that $\mathcal{T}_{n,\lambda}$ also has an optimal nuclear norm, which must be equal to $2^{(n-1)/2}$. \(\square\)

Theorem 7.5 Let $\mathcal{M}_4 = |M_4\rangle$ be the state given in is [29]:
\[
\frac{1}{\sqrt{6}}\left| (e_1 \otimes e_1 \otimes e_2 \otimes e_2 + e_2 \otimes e_2 \otimes e_1 \otimes e_1 + \omega(e_2 \otimes e_1 \otimes e_2 \otimes e_1 + e_1 \otimes e_2 \otimes e_1 \otimes e_2) + \omega^2(e_2 \otimes e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_2 \otimes e_1) ) \right\rangle = \mathcal{M}_4, \quad \omega = e^{2\pi i/3}. \tag{7.7}
\]
Then $\|\mathcal{M}_4\|_{\infty} = \frac{\sqrt{7}}{3}$. Hence
\[
\beta(2^{\times 4}, \mathbb{C}) = \frac{\sqrt{2}}{3}, \quad \alpha(2^{\times 4}, \mathbb{C}) = \frac{3}{\sqrt{2}}. \tag{7.8}
\]
and $\mathcal{M}_4$ is the most entangled state.

**Proof.** Let $\phi_3 : U(2) \to \otimes^3 U(2)$ be the diagonal map $A \mapsto \otimes^3 A$. So $\phi_3(U(2))$ acts on 3-qubits. We claim that the two dimensional space $\mathcal{W} = \text{span}(\mathcal{W}_0, \mathcal{W}_1)$ is invariant under the action $\phi_3(U(2))$. Consider first a diagonal unitary matrix: $A = \text{diag}(a, b)$, where $|a| = |b| = 1$. Then

$$(\otimes^3 A)\mathcal{W}_0 = ab^2 \mathcal{W}_0, \quad (\otimes^3 A)\mathcal{W}_1 = a^2 b \mathcal{W}_1.$$ 

Hence $\phi_3(A)\mathcal{W} = \mathcal{W}$. It is left to show that $\mathcal{W}$ is invariant under the action of $\phi_3(SU(2))$, where $SU(2)$ is the special unitary group. Hence, it is enough to show $\mathcal{W}$ is invariant under the action of Lie group of $\phi_3(SU(2))$. The generators of the $SU(2)$ are $i$ times the Pauli matrices:

$$B_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, B_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$ 

The generators of $\phi_3(SU(2))$ are

$$C_j = B_j \otimes I_2 \otimes I_2 + I_2 \otimes B_j \otimes I_2 + I_2 \otimes I_2 \otimes B_j, \quad quadj \in [3].$$

As $\mathcal{W}$ invariant under the action $\phi_3(A)$, where $A$ is a diagonal unitary matrix, we deduce that $\mathcal{W}$ is invariant under the action of $C_1$. A straightforward calculation shows that $\mathcal{W}$ is invariant under the action of $C_2$ and $C_3$. Furthermore, the action of $\phi_3(SU_2)$ on $\mathcal{W}$ is identical to the action of $SU(2)$ on the two dimensional subspace $\mathcal{W}$, with respect to the orthogonal basis $\mathcal{W}_0, \mathcal{W}_1$. That is, given a state $(a, b)^\top \in \mathbb{C}^2$, there exists $A$,

$$A = \begin{bmatrix} a & -b \\ b & \bar{a} \end{bmatrix}$$

such that $\phi_2(A)\mathcal{W}_0 = a\mathcal{W}_0 + b\mathcal{W}_1$. Hence

$$\|a\mathcal{W}_0 + b\mathcal{W}_1\|_\infty = \|\phi_2(A)\mathcal{W}_0\|_\infty = \frac{2}{3} \quad \text{for } |a|^2 + |b|^2 = 1.$$ 

Observe that

$$\mathcal{M}_4 = \frac{1}{\sqrt{2}}(\mathcal{W}_0 \otimes |0\rangle + \mathcal{W}_1 \otimes |1\rangle).$$

Let $\mathcal{X} = x \otimes y \otimes u \otimes v = \mathcal{Y} \otimes v$, where $\|x\| = \|y\| = \|u\| = \|v\| = 1$, be a product state. Then

$$\sqrt{2}|(\mathcal{M}_4, \mathcal{X})| = |\langle a\mathcal{W}_0 + b\mathcal{W}_1, \mathcal{Y} \rangle|, \quad a = \langle |0\rangle, v \rangle, b = \langle |1\rangle, v \rangle.$$ 

Clearly $|a|^2 + |b|^2 = \|v\|^2 = 1$. Thus if we maximize on all product states $\mathcal{Y}$ and keep $v$ fixed we get this this maximum is $\|a\mathcal{W}_0 + b\mathcal{W}_1\|_\infty = \frac{2}{3}$. This shows that $\|\mathcal{M}_4\|_\infty = \frac{\sqrt{2}}{3}$. The inequality (7.4) for $d = 4$ yields that $\beta(2 \times 4, \mathbb{C}) = \frac{\sqrt{2}}{3}$. This equality yields the second equality in (7.8). Furthermore, $\mathcal{M}_4$ is the most entangled 4-qubit. 

Numerical simulations point out that that (7.4) is not sharp for $d = 5$.

Inequality (7.4) yields that

$$\omega(2 \times d) \leq d - 5 + 2 \log_2 3 \text{ for } d \geq 3.$$
The concentration result of [26] claims that most of $d$ quibits, with respect to the corresponding Haar measure, satisfy the inequality

$$\eta(T) \geq d - 2\log_2 d - 3 \text{ for most of } T \otimes d^2 \in \mathbb{C}^2 \text{ for } d \gg 1.$$ 

A Boson is a symmetric state in $S \in S^d \mathbb{C}^n$, where $||S|| = 1$. The maximum entanglement of Bosons in $S^d \mathbb{C}^n$ is $-2\log_2 \alpha'(n, d, \mathbb{C})$. For $d = 3$ the most entangled state $W$ is symmetric. The most entangled symmetric 4-qubit is conjectured to be [2, Example 6.1]

$$\frac{1}{\sqrt{3}}(\otimes^4 e_1 + \frac{1}{\sqrt{2}}(e_1 \otimes \otimes^3 e_2 + e_2 \otimes e_1 \otimes e_2 + \otimes^2 e_2 \otimes e_1))$$

Its spectral norm is $\frac{1}{\sqrt{3}} \approx 0.5774$ [2]. It is shown in [17] that

$$-2\log_2 \alpha'(2^x d, \mathbb{C}) \leq \log_2 (d + 1).$$

(Note that this is totally different from the results of Theorem 7.4.) Furthermore

$$\eta(S) \geq \log_2 d - \log_2 \log_2 d - 3, \text{ for most } S \in S^d \mathbb{C}^2 \text{ for } d \gg 1.$$ 

8 Preliminaries to computational part of the paper

In this section, we recall two well-known minimization problems, which are the foundations for proposing the alternating method for nuclear norm calculation.

8.1 Matrix Decomposition

Let us consider the following matrix decomposition problem:

Problem 8.1 Let $m, n \geq 2$ and $Q \geq \min(m, n)$ be given integers. Assume that $A \in \mathbb{F}^{n \times m} \setminus \{0\}$ is given. Suppose furthermore that there exists a following decomposition of matrix $A$:

$$A = \sum_{i=1}^{Q} u_i v_i^\top, \quad v_i \neq 0 \text{ for } i \in [Q]. \quad (8.1)$$

For fixed vectors $v_i, i \in [Q]$, how to find a minimal decomposition with respect to the absolute norm sums of the components?

A minimal decomposition (8.1) is a solution of the following minimization problem:

$$\min_{y_i \in \mathbb{F}^n, i \in [Q]} \sum_{i=1}^{Q} ||v_i|| \sum_{i=1}^{Q} ||y_i|| \quad \text{s.t.} \quad \sum_{i=1}^{Q} y_i v_i^\top = A. \quad (8.2)$$

Especially, if the given $v_i$ are unit vectors, we need to solve the following problem:

$$\min_{y_i \in \mathbb{F}^n, i \in [Q]} \sum_{i=1}^{Q} ||y_i|| \quad \text{s.t.} \quad \sum_{i=1}^{Q} y_i v_i^\top = A. \quad (8.3)$$
This is a well known problem called the minimum sum of Euclidean norms [3], which can be solved efficiently by reformulating it as a Second Order Cone Programming (SOCP) [6]. By bringing extra $Q$ variables $t_i$, we reformulate (8.3) as:

$$\min_{y_i \in \mathbb{F}^n, i \in [Q]} \sum_{i=1}^{Q} t_i \quad \text{s.t.} \quad \|y_i\| \leq t_i, \quad i \in [Q], \quad \sum_{i=1}^{Q} y_i v_i^T = A. \quad (8.4)$$

Instead of solving problem (8.2) with $Q$ vector variables $y_i, i \in [Q]$, we can further reformulate it as an unconstrained minimization problem. Consider the linear constraint:

$$\sum_{i=1}^{Q} y_i v_i^T = A, \quad (8.5)$$

which is a non-homogeneous linear system with $nQ$ variables $y_i$ and $mn$ equations. Let $w = (w_1, \ldots, w_s)^T$ be the vector of the free variables of this system. Hence the general solution of (8.5) is

$$y_i = u_i + B_i w, \quad i \in [Q], \quad (8.6)$$

where $u_i$ is a special solution of the linear system (8.5), and $B_i$ is the basis of the null space. Then the minimum objective function $\sum_{i=1}^{N} \|v_i\|\|y_i\|$ of problem (8.2) boils down to

$$\min_{w \in \mathbb{R}^s} \sum_{i=1}^{Q} \|b_i + A_i w\|, \quad i \in [N].$$

(8.7)

where $b_i = \|v_i\|u_i$, $A_i = \|v_i\|B_i$. By bringing in extra variable $w_{s+i} \in \mathbb{R}$ for each $i \in [Q]$, we reformulate problem (8.7) as the following SOCP:

$$\min_{w \in \mathbb{F}^s, w_{s+i}, i \in [Q]} \sum_{i=1}^{N} w_{s+i}, \text{ subject to } \|b_i + A_i w\| \leq w_{s+i}, \quad i \in [Q]. \quad (8.8)$$

### 8.2 Second Order Cone Programming

We claim that each step minimization step in an alternating method for computing the nuclear norm given in §9 is equivalent to the solution of the following minimum problem in matrix decomposition:

Indeed, consider the minimization problem in §9 for $k = 1$. Then

$$n = n_1, m = \prod_{j=2}^{d} n_j, Q = N(\mathbb{F}), y_i = y_{1,i}, v_i = \otimes_{j=2}^{d} y_{j,i}, i \in Q,$$

$$\phi(y_{1,1}, \ldots, y_{d,Q}) = \sum_{i=1}^{Q} \|v_i\|\|y_i\| = \sum_{i=1}^{Q} \|y_i\|.$$

Let $N = N(\mathbb{F})$ and $y := (y_1^T, \ldots, y_N^T)^T$. Then (8.5) is a solvable system of linear non-homogeneous system of linear equations.

We now restate the minimization problem (8.8) as SDPT3, MATLAB software for semidefinite-quadratic-linear programming. With each $y_i \in \mathbb{F}^n$ we associate a vector $z_i = (w_i, y_i^T)^T \in K_n$, where $K_n \subset \mathbb{R}^{n+1}$ is the Lorentzian cone

$$K_n := \{ z = (w, y^T)^T \in \mathbb{R}^{n+1}, \ w \geq \|y\| \}.\quad 18$$
In the notation of [49] in our problem we have only the variables \( z_i \in K_n \) for \( i \in [N] \). The condition (8.5), can be restated as the following condition second condition (P) in [49]:

\[
\sum_{i=1}^{N} A_i z_i = \hat{A}. \tag{8.9}
\]

Here \( \hat{A} \in \mathbb{R}^{mn} \) is a vector formed from the matrix \( A = [a_1, \ldots, a_m] \in \mathbb{R}^{n \times m} \) as follows: \( \hat{A} = (a_1^T \cdots a_m^T)^T \). Then

\[
A_i = [v_{1,i} B^T \cdots v_{m,i} B^T]^T \in \mathbb{R}^{mn \times (n+1)}, \quad B = [0 \ I_0] \in \mathbb{R}^{n \times (n+1)}, \tag{8.10}
\]

\[
v_i = (v_{1,i}, \ldots, v_{m,i})^T, \quad i \in [N].
\]

The minimizing function is

\[
\sum_{i=1}^{N} \|v_i\| w_i = \sum_{i=1}^{N} c_i^T z_i, \quad c_i = (\|v_i\|, 0, \ldots, 0)^T \in \mathbb{R}^{n+1}, i \in [N]. \tag{8.11}
\]

### 9 An alternating method for computing the nuclear norm of nonsymmetric tensors

Assume that \( d \geq 3 \) and \( n_1, \ldots, n_d \geq 2 \) are positive integers. Let \( T \in \mathbb{F}^{n_1 \times \cdots \times n_d} \) be a given tensor. Recall that the unfolding of a tensor \( T \) in the \( j \)-th mode is a matrix \( T_j \in \mathbb{F}^{n_j \times n_1 \cdots \hat{n}_j \cdots n_d} \). The entries of \( T_j \) are indexed by the rows \( i_j \in [n_j] \) and the columns by a \( d - 1 \) tuple \( k = (i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_d) \), where \( i_p \in [n_p] \) for \( p \in [d] \setminus \{j\} \). Furthermore the entry \((i_j, k)\) of \( T_j \) is \( t_{i_1, \ldots, i_d} \). Let \( r_j = r_j(T) = \text{rank} \ T_j \). If rank \( T_j \) \( < n_j \) one can use Gram-Schmidt process to find an orthonormal basis \( b_{1,j}, \ldots, b_{r_j,j} \) of the columns space of \( T_j \), denoted as \( \mathbf{V}_j \subseteq \mathbb{F}^{n_j} \). We assume that if \( r_j = n_j \) then \( b_{1,j}, \ldots, b_{n_j,j} \) is the standard orthonormal basis \( e_{1,j}, \ldots, e_{n_j,j} \) in \( \mathbb{F}^{n_j} \).

(It is well known that there exists \( T \in \mathbb{F}^{n_1 \times \cdots \times n_d} \) such that \( r_j = n_j \), which are most the tensors, if and only if \( n_j \leq \frac{n_1 \cdots n_d}{n_j} \) [22].) Thus, \( T \in \otimes_{j=1}^{d} \mathbf{V}_j \). Equivalently, \( T \) has the Tucker representation

\[
T = \sum_{l_j \in [r_j], j \in [d]} t'_{l_1, \ldots, l_d} \otimes_{j=1}^{d} b_{l_j,j}, \quad \text{where} \ t'_{l_1, \ldots, l_d} \in \mathbb{F}. \tag{9.1}
\]

For simplicity of the exposition we assume that rank \( T_j = n_j \) for \( j \in [d] \).

We now describe the alternating method for computing the nuclear norm of tensor \( T \). The iteration process is designed over vector variables \( y_{k,1}, \ldots, y_{k,N} \in \mathbb{F}^{n_k} \) for each \( k \in [d] \) in an alternating scheme. Let

\[
N(n_1, \ldots, n_d, \mathbb{R}) = \prod_{i=1}^{d} n_i, \quad N(n_1, \ldots, n_d, \mathbb{C}) = 2 \prod_{i=1}^{d} n_i. \tag{9.1}
\]

**Extension Step:** Assume first that we have a decomposition of the tensor \( T \) as a sum of rank one (nonzero) tensors

\[
T = \sum_{i=1}^{N'} \otimes_{j=1}^{d} x_{j,i}, \quad N' \leq N(n_1, \ldots, n_d, \mathbb{F}).
\]
If $N' < N(n_1, \ldots, n_d, \mathbb{F})$ we first extend the above decomposition to
\begin{equation}
T = \sum_{i=1}^{N} \otimes_{j=1}^{d} y_{j,i}, \quad N = N(n_1, \ldots, n_d, \mathbb{F}).
\end{equation}
as follows:

1. $y_{j,i} = \frac{1}{\|x_{j,i}\|} x_{j,i}$ for $j \in [d] \setminus \{k\}, i \in [N']$.
2. $y_{k,i} = (\prod_{j\in[d]\setminus\{k\}} \|x_{j,i}\|) x_{k,i}$ for $i \in [N']$.
3. The vectors $y_{j,i} \in \mathbb{F}^{n_j}$ for $j \in [d] \setminus \{k\}$ and $i = N' + 1, \ldots, N$ are random norm one vectors.
4. $y_{k,i} = 0$ for $i = N' + 1, \ldots, N$.

**Minimization Step:** We fix the vectors $y_{j,i}$ for $j \in [d] \setminus \{k\}, i \in [N]$, and view the equality (9.2) as a system of $N(n_1, \ldots, n_d, \mathbb{F})$ scalar equations in $N$ vector variables $y_{k,1}, \ldots, y_{k,N} \in \mathbb{F}^{n_k}$. Define the objective function
\begin{equation}
\phi_k(y_{k,1}, \ldots, y_{k,N}) = \sum_{i=1}^{N} \left( \prod_{j\in[d]\setminus\{k\}} \|y_{j,i}\| \right) \|y_{k,i}\|.
\end{equation}

Observe that $\phi_k(y_{k,1}, \ldots, y_{k,N})$ is an upper bound on $\|T\|_1,\mathbb{F}$ induced by the decomposition (9.2). Then our minimization problem is
\begin{equation}
\min \{ \phi_k(y_{k,1}, \ldots, y_{k,N}), y_{k,1}, \ldots, y_{k,N} \in \mathbb{F}^{n_k} \text{ subject to conditions (9.2)} \}.
\end{equation}

As $\prod_{j\in[d]\setminus\{k\}} \|y_{j,i}\| > 0$ for each $i \in [N]$ the function $\phi_k$ is a strict convex function in variables $y_{k,i}, i \in [N]$. Hence the above minimum is achieved at the unique $y_{k,1}^*, \ldots, y_{k,N}^*$. This gives rise to another decomposition of $T$
\begin{equation}
T = \sum_{i=1}^{N} \left( \otimes_{j=1}^{k-1} y_{j,i} \right) \otimes y_{k,i}^* \otimes \left( \otimes_{j=k+1}^{d} y_{j,i} \right).
\end{equation}

We now repeat the above **Extension Step** and **Minimization Step** for $k' \in [d]$ until the relative decrease of the objective function $\phi_k$ is smaller than the determined threshold $\epsilon$ (a tiny positive number). We stop the algorithm and output the last value of the target function $\phi_k$ as the nuclear norm of $T$ and the corresponding (9.5) is a decomposition of tensor $T$.

**Algorithm 9.1** (Nonsymmetric Tensor Nuclear Norm Computation)

**Input:** Nonsymmetric tensor $T \in \mathbb{F}^{n_1 \times \cdots \times n_d}$, tolerance $\epsilon > 0$, iteration $I = 1$, maximum iteration $I_{\text{max}}$, $N$, and initial point $x_{j,i} \in \mathbb{F}^{n_j}, i \in [N]$, let $\phi_k^0 = +\infty$ for all $k \in [d]$.

**Step 1:** For $k = 1 : d$,
(a) do **Extension Step** 1-4 and **Minimization Step** by solving (9.4);
(b) if $|\phi_k^I - \phi_k^{I-1}| < \epsilon$, then break and go to **Step 2**, otherwise go to **Step 3**.

**Step 2:** Output $\|T\|_1,\mathbb{F} = \phi_k^I$, and decomposition (9.5).

**Step 3:** Set $I = I + 1$, if $I \leq I_{\text{max}}$, go to **Step 1**; otherwise, go to **Step 2**.
Note that the value of $\phi$ is a better lower bound for $\|T\|_{1,F}$ then the lower bound $\phi(y_{1,1}, \ldots, x_{d,N})$ induced by the decomposition (9.2). Note that it is possible that exactly $N - \hat{N}$ vectors $y^{*}_{k,i} = 0$. Hence the decomposition (9.5) gives rise to decomposition of $T$ to $\hat{N}$ rank one tensors.

10 An alternating method for computing the nuclear norm of symmetric tensors

Denote by $\Sigma_d$ the group of all permutations of $[d]$. Recall that the cardinality of $\Sigma_d$ is $d!$. For each $x_1, \ldots, x_d \in \mathbb{F}$ let us denote

$$\text{sym}_d(x_1, \ldots, x_d) = \frac{1}{d!} \sum_{\sigma \in \Sigma_d} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(d)} \in S^d \mathbb{F}^n.$$ 

Observe the basic equality

$$\|x_1\| \cdots \|x_d\| = \frac{1}{d!} \sum_{\sigma \in \Sigma_d} \|x_{\sigma(1)}\| \cdots \|x_{\sigma(d)}\|.$$ 

Fix $x_1, \ldots, x_{d-1}$. Then $L(x_1, \ldots, x_{d-1})$ is a linear operator from $\mathbb{F}$ to $S^d \mathbb{F}^n$ given by the equality

$$L(x_1, \ldots, x_{d-1})(x_d) = \text{sym}_d(x_1, \ldots, x_d).$$ 

Observe the equality

$$L(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_d)(x_k) = \text{sym}_d(x_1, \ldots, x_d)$$ 

for each $k \in [d]$.

Suppose we have a decomposition of a symmetric tensor $S \in S^d \mathbb{F}^n$ to a sum of the rank one tensor

$$S = \sum_{i=1}^{K} \otimes_{j=1}^{d} x_{j,i} \in S^d \mathbb{F}^n. \quad (10.1)$$ 

The decomposition (10.1) induces a symmetric decomposition

$$S = \sum_{i=1}^{K} L(x_{1,i}, \ldots, x_{k-1,i}, x_{k+1,i} \ldots x_{d,i})(x_{k,i}) \text{ for each } k \in [d].$$ 

(10.2)

Note that if span($x_{1,i}$) = $\cdots$ = span($x_{d,i}$) = span($x_i$) $\subset \mathbb{F}^n$ for $i = 1, \ldots, K$. Then it follows that $S = \sum_{i=1}^{K} \varepsilon_i \otimes^d z_i$, where $z_i \in \text{span}(x_i)$ and $\varepsilon_i = \pm 1$ for $i \in [K]$. (We can always assume that $\varepsilon_i = 1$ if $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$ and $d$ is odd.) The extension of Banach’s theorem for the nuclear norm of symmetric tensors [20] implies the following decomposition to a sum of rank one symmetric tensors

$$S = \sum_{i=1}^{K} \varepsilon_i \otimes^d x_i, x_i \in \mathbb{F}^n, \varepsilon_i = \pm 1, i \in [K], \|S\|_{1,F} = \sum_{i=1}^{K} \|x_i\|^d, K \leq M(\mathbb{F}).$$ 

(10.3)

Here

$$M(\mathbb{R}) = \binom{n + d - 1}{d - 1}, \quad M(\mathbb{C}) = 2 \binom{n + d - 1}{d - 1}.$$ 

(10.4)
Hence for finding the nuclear norm of \( S \in S^d \mathbb{F}^n \) using an alternating method we need consider only the decomposition (10.2) of \( S \) of where each \( x_{j,i} \neq 0 \) and \( K \leq M(\mathbb{F}) \).

Let \( r_j(S) \) be the rank of the unfolded matrix \( T_j(S) \) as in §9. Clearly, \( r_1(S) = \cdots = r_d(S) = r \). If \( r < n \) then \( V_1 = \cdots = V_r = V \) is the columns space of each \( T_j(S) \). As in §9 it follows that \( S \in S^d \mathbb{V} \). In what follows we assume that \( r_j(S) = n \).

Recall that \( S^d \mathbb{F}^n \) has a standard orthogonal basis consisting of \( \text{sym}(e_{i_1}, \ldots, e_{i_d}) \), where \( 1 \leq i_1 \leq i_2 \leq \cdots \leq i_d \leq n \) [17]. This representation gives rise to a representation of the form (10.2). Here \( K = M(\mathbb{R}) - K \), where \( K \) is the number of zero coordinates of \( S = [s_{i_1, \ldots, i_d}] \) satisfying \( 1 \leq i_1 \leq i_2 \leq \cdots \leq i_d \leq n \). As in §9 each representation (10.2) with \( K \leq M(\mathbb{F}) \), where all \( x_{i,j} \neq 0 \) induces the following representation for a given \( k \in [d] \)

\[
S = \sum_{i=1}^{M} L(y_{1,i}, \ldots, y_{k-1,i}, y_{k+1,i} \cdots y_{d,i})(y_{k,i}), \quad M = M(\mathbb{F}),
\]

(10.5)

which satisfies the following conditions.

1. \( y_{j,i} = \frac{1}{|x_{j,i}|} x_{j,i} \) for \( j \in [d] \setminus \{k\} \) and \( i \in [K] \).
2. \( y_{k,i} = (\prod_{j \in [d] \setminus \{k\}} |x_{j,i}|) x_{k,i} \) for \( i \in [K] \).
3. \( y_{j,i} \) is a random vector in \( \mathbb{F}^n \) of norm one for \( j \in [k] \setminus \{i\} \) and \( i = K+1, \ldots, M \).
4. \( y_{k,i} = 0 \) for \( i = K+1, \ldots, M \).

Then the upper bound for the nuclear norm induced by (10.5) is given by

\[
\psi(y_{1,1}, \ldots, y_{d,M}) = \sum_{i=1}^{M} \prod_{j=1}^{d} \|y_{j,i}\|.
\]

(10.6)

Not that the above decomposition gives rise to a symmetric decomposition of \( S \) to a sum of rank one matrix. The upper bound of the nuclear norm for this decomposition is also \( \phi(x_{1,1}, \ldots, x_{d,M}) \).

Hence the alternating method for computing the nuclear norm of a given symmetric tensor is given by the basic minimum step

\[
\min\{\psi(y_{1,1}, \ldots, y_{d,M}), \text{ on } y_{k,1}, \ldots, y_{k,M} \in \mathbb{F}^n, \text{ subject to (10.5)}\}.
\]

(10.7)

The advantage of this method versus the method in §9 is that we replace \( N(\mathbb{F}) \) by much smaller number \( M(\mathbb{F}) \). Furthermore the number of linear conditions is \( \binom{n+d-1}{d} \) versus \( n^d \). As in the nonsymmetric case, the minimum problem (10.7) can be solved by SDPT3 software. In the setting (8.9), the vector \( \hat{\phi} \) has \( \binom{n+d-1}{d} \) coordinates.

**Algorithm 10.1 Symmetric Tensor Nuclear Norm Computation**

**Input:** Symmetric tensor \( S \in \mathbb{F}^{n \times \cdots \times n} \), tolerance \( \epsilon > 0 \), iteration \( I = 1 \), maximum iteration \( I_{\text{max}} \), \( N \), and initial point \( x_{i,j} \in \mathbb{F}^{n_1}, i \in [N], \) let \( \phi_0^I = +\infty \) for all \( k \in [d] \).

**Step 1:** For \( k = 1 : d \),
(a) do **Extension Step** 1-4 and **Minimization Step** by solving (10.7);
(b) if \( |\phi_0^I - \phi_0^{I-1}| < \epsilon \), then break and go to Step 2, otherwise go to Step 3.

**Step 2:** Output \( \|T\|_1, \phi_0^I = \phi_0^0 \), and decomposition (9.5).

**Step 3:** Set \( I = I + 1 \), if \( I \leq I_{\text{max}} \), go to Step 1; otherwise, go to Step 2.
11 Numerical Experiments

In this section, we give some numerical examples to demonstrate the performance of Algorithms 9.1 and 10.1. All the computation are implemented with Matlab R2012a on a MacBook Pro 64-bit OSX (10.9.5) system with 16GB memory and 2.3 GHz Intel Core i7 CPU. The SOCP subproblem is formulated with software Yalmip [34] and solved with SDPT3 [48]. We use the default values of the parameters in SDPT3.

In the tables, $K_{\text{rand}}$ stands for the number of random examples; $\text{Min}_{F,\text{Iter}}$, $\text{Avg}_{F,\text{Iter}}$, $\text{Max}_{F,\text{Iter}}$ stand for the minimum, average, maximum iterations for the $K_{\text{rand}}$ random examples over field $F = \mathbb{R}$ or $\mathbb{C}$. Similarly, $\text{Min}_{F,\text{Time}}$, $\text{Avg}_{F,\text{Time}}$, $\text{Max}_{F,\text{Time}}$ stand for the minimum, average, maximum computational CPU time for the $K_{\text{rand}}$ random examples over field $F = \mathbb{R}$ or $\mathbb{C}$. For cleanness of the paper, we keep four digits for all numerical results.

For different starting point $x_0$, Algorithms 9.1 and 10.1 might converge to different local minimizer, so we consider to implement Algorithms 9.1 and 10.1 with random starting points $x_0$ for 30 times, and choose the one with smallest objective function value as the nuclear norm.

Equation (2.11) is a necessary condition that holds for maximum entangled states. We are interested to find these maximum entangled states over field $F$, so in the tables, we report both spectral norm and complex norms for each tensor, and we also report the product of these two norms over field $F$ as $P_F$, i.e., $\|T\|_F \|T\|_{\infty,F}$.

For each real state, we report its spectral norm and nuclear norm over field $F = \mathbb{R}$ and $\mathbb{C}$, so there are four norms. For each complex state, we only report its spectral norm and nuclear norm over $F = \mathbb{C}$. In the following, we list all the numerical methods used to calculate these norms:

1. For nonsymmetric tensor, we calculate its nuclear norm by Algorithm 9.1.

2. For symmetric tensor, we use two numerical methods to calculate its nuclear norm. The first one is Algorithm 10.1, another one is semidefinite relaxation method which is proposed by Nie [36]. We implement semidefinite relaxation method with software Gloptipoly [28] and the formulated SDP problem is solved by Sedumi [45].

3. For real nonsymmetric state, we calculate its real spectral norm by using semidefinite relaxation method [37].

4. For symmetric state (either real or complex), we calculate its real and complex spectral norms by using the method proposed in [24], which is numerically implemented with Software Bertini [5] (version 1.5, released in 2015).

5. For complex nonsymmetric state, we calculate its complex spectral norm by using the semidefinite relaxation method [37], which is originally designed for real spectral norm computation. For complex spectral norm calculation, we can replace each complex variable with two real variables, then (3.3) can be easily reformulated as a homogeneous polynomial optimization problem [38]. The method discussed in [37] can also be applied to find the complex spectral norm, however, the global optimality certification condition may not always hold. In this case, the complex spectral norm provided by the semidefinite
relaxation method might only be an upper bound. Please refer to [37] for details.

### 11.1 Density Tensor Separability Checking

In the following, we first test the alternating algorithm 9.1 on some density tensors whose separability is known in advance, i.e., its nuclear norm is equal to 1 or not is known. Also, we randomly generate some separable density tensors, and calculate their nuclear norm by Algorithm 9.1 to see if their nuclear norm is equal to 1 or not.

**Example 11.1** [35, Example 2.5] Let us consider the following density tensor $\mathcal{T} \in \mathbb{C}^{2 \times 2 \times 2 \times 2}$ with $b \in [0, 1]$, which is known to be inseparable for any $\frac{1}{3} < b \leq 1$. For $b \in \{1, 3/4, 2/3, 1/2, 1/3, 1/4, 1/5, 0\}$, we calculate the nuclear norm by Algorithm 9.1, and the results are shown in Table 1. For $b \in [0, 1/3]$, we get the numerical nuclear norms of these density tensor are 1, which match Lemma 4.1. For most $b$, we do get the nuclear norm is bigger than 1. However, there is one special case, for $b = 1/2$, we numerically find the nuclear norm of tensor $\mathcal{T}$ is equal to 1, which contradict to Lemma 4.1. We conjecture here, the nuclear norm for $b = 1/2$ should be a number that is very close to 1, but numerically, we might not able to detect this fact in our implementation.

$$
\mathcal{T}_{1,1,1,1} = \mathcal{T}_{2,2,2,2} = \frac{1-b}{4}, \quad \mathcal{T}_{1,2,1,2} = \mathcal{T}_{2,1,2,1} = \frac{1+b}{4}, \quad \mathcal{T}_{1,2,2,1} = \mathcal{T}_{2,1,1,2} = -\frac{b}{2}.
$$

| $b$   | 1    | 3/4  | 2/3  | 0.60 | 0.55 | 0.52 | 1/2  | 1/3  | 1/4  | 1/5  | 0    |
|-------|------|------|------|------|------|------|------|------|------|------|------|
| $\|\mathcal{T}\|_1$ | 2.0000 | 1.5000 | 1.3333 | 1.2000 | 1.0000 | 1.04000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

Table 1: Nuclear norm of example 11.1.

**Example 11.2** [35, Example 2.6] Let us consider the following density tensor $\mathcal{T} \in \mathbb{C}^{2 \times 4 \times 2 \times 4}$ with parameter $b \in [0, 1]$, which is known to be inseparable for $b \in (0, 1]$, and separable for $b = 0$. In Table 2, we list the nuclear norms for $b = \{1, 3/4, 2/3, 1/2, 1/3, 1/4, 1/5, 0\}$, which are calculated by Algorithm 9.1. For $b = 0$, tensor $\mathcal{T}$ is separable, by Lemma 4.1, we know its nuclear norm is equal to 1. From Table 2, we can see our numerical result also certifies this fact. For each $b > 0$, we get $\|\mathcal{T}\|_{1,\mathcal{C}} > 1$. Since for any $b$, $\text{tr}(\mathcal{T}) = 1$, by Lemma 4.1, we have $\|\mathcal{T}\|_{1,\mathcal{C}} > 1$ since $\mathcal{T}$ is inseparable for any $b > 0$. Numerical results in Table 2 also certify this fact.

$$
\mathcal{T}_{1,1,1,1} = \mathcal{T}_{1,2,1,2} = \mathcal{T}_{1,3,1,3} = \mathcal{T}_{1,4,1,4} = \mathcal{T}_{1,1,2,2} = \mathcal{T}_{1,2,2,3} = \frac{b}{7b+1},
$$

$$
\mathcal{T}_{2,2,2,2} = \mathcal{T}_{2,3,2,3} = \mathcal{T}_{1,3,2,4} = \mathcal{T}_{2,2,1,1} = \mathcal{T}_{2,3,1,2} = \mathcal{T}_{2,4,1,3} = \frac{b}{7b+1},
$$

$$
\mathcal{T}_{2,1,2,1} = \mathcal{T}_{2,4,2,4} = \frac{1+b}{2(7b+1)}, \quad \mathcal{T}_{2,1,2,4} = \mathcal{T}_{2,4,2,1} = \frac{\sqrt{1-b^2}}{2(7b+1)}.
$$
Table 2: Nuclear norm of example 11.2

Example 11.3 (Random Separable Density Tensor Examples) We test Algorithm 9.1 on random density tensors $T \in \mathbb{C}^{n_1 \times \cdots \times n_d \times n_{d+1} \times \cdots \times n_{2d}}$, which are generated as follows: (i) let $r \in [2 \prod_{j=1}^{d} n_j]$ be a random integer number; (ii) randomly generate $r$ positive numbers $p_i$ satisfy $\sum_{i=1}^{r} p_i = 1$; (iii) randomly generate nonzero vectors $x_{j,i} \in \mathbb{C}^{n_j}$, $j \in [d], i \in [r]$, and normalize them as length 1 vectors; (iv) calculate tensor $T$ by formula (4.3). Computationally, for each density tensor, we will get its nuclear norm close to 1, with tiny numerical error. In Table 3, we list the average iteration and average computational time.

Table 3: Computational results for random density tensors

| $d$ | $(n_1, \ldots, n_d)$ | $K_{\text{rand}}$ | Type   | AvgIter | AvgTime   |
|-----|---------------------|-------------------|--------|---------|-----------|
| 2   | (2,4)               | 20                | nonsym | 6.10    | 0:00:28   |
| 2   | (2,5)               | 20                | nonsym | 4.45    | 0:00:54   |
| 2   | (2,6)               | 20                | nonsym | 5.55    | 0:03:30   |
| 2   | (3,4)               | 20                | nonsym | 5.50    | 0:02:42   |
| 2   | (4,4)               | 10                | nonsym | 7.25    | 0:19:22   |
| 3   | (2,2,2)             | 20                | nonsym | 11.00   | 0:01:38   |
| 3   | (2,2,3)             | 20                | nonsym | 10.25   | 0:20:01   |
| 3   | (2,3,3)             | 5                 | nonsym | 7.60    | 3:19:09   |
| 2   | (2,2)               | 20                | sym    | 2       | 0:00:06   |
| 2   | (3,3)               | 20                | sym    | 2       | 0:06:11   |
| 3   | (2,2,2)             | 10                | sym    | 3.5     | 0:37:25   |

Table 11.2 Nonsymmetric Tensors

In this subsection, we report the performance of Algorithm 9.1 on nonsymmetric tensors. We will test Algorithm 9.1 on the following tensors: (1) explicit nonsymmetric $d$-qubits found from references [7, 29]; (2) random nonsymmetric $d$-qubits; (3) random nonsymmetric tensors.

11.2.1 Nonsymmetric $d$-qubits

We test Algorithm 9.1 on some nonsymmetric $d$-qubits that we can find from references. The tensors and their four norms are reported in Table 4. Examples No.1-3 are found from [7]. Example No.4 is conjectured in [29] as the maximum entangled state for $d = 4$ and $\mathbb{F} = \mathbb{C}$. Example No.4 is a complex state, so its real nuclear norm and spectral norm do not exist, we use “–” in the table. We also report the product of the nuclear norm and spectral norm over field $\mathbb{F}$ for each tensor. In Table 4, we certify that the necessary condition (2.11) holds for Example No.4 over field $\mathbb{F} = \mathbb{C}$. The equation (2.11) also holds for Examples No. 1-3 over field $\mathbb{F} = \mathbb{C}$, however, their complex nuclear norm is smaller than Example No. 4, which shows that condition (2.11) is only necessary but not sufficient for maximum entangled state.
Table 4: Computational results for nonsymmetric $d$-qubits

| No. | $d$ | Tensor | $\|T\|_1$ | $\|T\|_1$ | $\|T\|_{\infty}$ | $\|T\|_{\infty}$ | $P_R$ | $P_C$ |
|-----|-----|---------|-----------|-----------|-------------|-------------|-------|-------|
| 1   | 4   | $T_{1,1,1,1} = T_{1,2,2,2} = \frac{1}{\sqrt{2}}$ $T_{2,1,1,1} = T_{2,2,2,1} = \frac{1}{\sqrt{2}}$ | 2.0005 | 2.0002 | 0.5000 | 0.5000 | 1.0003 | 1.0001 |
| 2   | 4   | $T_{1,1,1,1} = T_{2,1,2,1} = \frac{1}{\sqrt{2}}$ $T_{2,1,2,2} = T_{2,1,2,1} = \frac{1}{\sqrt{2}}$ $T_{1,1,2,2} = \frac{1}{\sqrt{2}}$ $T_{2,2,2,1} = -\frac{1}{\sqrt{2}}$ | 2.0002 | 2.0001 | 0.5000 | 0.5000 | 1.0001 | 1.0001 |
| 3   | 4   | $T_{1,1,1,1} = T_{1,2,1,2} = \frac{1}{\sqrt{2}}$ $T_{2,1,2,1} = T_{2,2,2,2} = \frac{1}{\sqrt{2}}$ | 2.0000 | 2.0000 | 0.5000 | 0.5000 | 1.0000 | 1.0000 |
| 4   | 4   | $T_{1,1,2,2} = T_{2,2,1,1} = \frac{1}{\sqrt{2}}$ $T_{2,1,2,1} = T_{1,2,1,2} = \frac{1}{\sqrt{2}}$ $\zeta = 1+i \sqrt{2}$ | – | 2.1216 | – | 0.4714 | – | 1.0001 |
| 5   | 5   | $T_{2,2,1,1,1} = \frac{1}{\sqrt{2}}$ $T_{1,1,1,1} = \frac{2}{\sqrt{2}}$ $T_{1,1,2,2} = \frac{2}{\sqrt{2}}$ $T_{1,2,1,2} = \frac{2}{\sqrt{2}}$ $T_{2,2,2,2} = \frac{2}{\sqrt{2}}$ | 2.8284 | 2.8281 | 0.3536 | 0.3536 | 1.0001 | 1.0000 |
| 6   | 6   | $T_{i,j,k,i,j,k} = \frac{1}{\sqrt{2}}$ for $i,j,k \in \{1,2\}$ | 2.8283 | 2.8283 | 0.3536 | 0.3536 | 1.0001 | 1.0001 |

Table 5: The maximal nuclear norm for randomly 500 nonsymmetric examples

| $d$ | $\|T\|_1$ | $\|T\|_{\infty}$ | $P_R$ | $\|T\|_1$ | $\|T\|_{\infty}$ | $P_R$ |
|-----|-----------|-------------|-------|-----------|-------------|-------|
| 3   | $0.9473$  | $0.5901$    | $1.4797$ | $0.8982$  | $1.1700$    | $1.0987$ |
| 4   | $2.2665$  | $0.6882$    | $1.5771$ | $0.6187$  | $1.1928$    | $1.1576$ |
| 5   | $2.8233$  | $0.5533$    | $1.4011$ | $0.5213$  | $1.1978$    | $1.2088$ |
| 6   | $3.2290$  | $0.4383$    | $1.4757$ | $0.4502$  | $1.3088$    | $1.3888$ |

11.2.2 Random nonsymmetric $d$-qubits

It is interesting to find the maximum entangled states. In this example, we consider to randomly generate nonsymmetric states, and calculate their nuclear norm by implementing Algorithm 9.1 over field $\mathbb{F}$. We generate a nonsymmetric tensor with each entry being a random variable obeying Gaussian distribution (by `randn` in Matlab), then we normalize the generated random tensor and get a random $d$-qubit $T$ with $\|T\| = 1$. For $d = 3, 4, 5, 6$, we randomly generate 500 states over field $\mathbb{F} = \mathbb{R}$ and $\mathbb{C}$, and report the maximal nuclear norm we find over these 500 randomly generated states. The computational results are shown in Table 5. The corresponding real states that get the maximal real nuclear norm are shown in Table 6, and the corresponding complex states that get the maximal complex nuclear norm are shown in Table 7.

11.2.3 Random nonsymmetric tensors

We explore the performance of Algorithms 9.1 on calculating the nuclear norm for randomly generated nonsymmetric tensors $T \in \mathbb{F}^{n \times \cdots \times n}$. The computational results are shown in Table 8. For each $(n, d)$ pair, we randomly generate $K_{\text{rand}}$ tensors. For each tensor, we run 30 times Algorithm 9.1 with random initial points, and we choose the smallest objective value as the nuclear norm. The maximal (resp. minimal, average) iteration and time are calculated over these $30K_{\text{rand}}$ random calculation of
We test Algorithm 10.1 on several symmetric tensors. Table 9 shows the computational results of Algorithm 10.1 with the semidefinite relaxation method [36]. The computational results are shown in Table 9. In the table, \( ||T||_{1,F}(S) \) is the nuclear norm calculated by implementing Algorithm 10.1, as shown in Table 9. For large examples, Algorithm 10.1 needs longer time to solve.

**11.3 Symmetric Tensors**

In this subsection, we test Algorithm 10.1 on symmetric tensors. Semidefinite relaxation method [36] generally can find the nuclear norm for symmetric tensors, while Algorithm 10.1 works well in finding the nuclear norm. However, it might not be as fast as semidefinite relaxation method, since we need to solve several SOCP problems before it converges. We will test Algorithm 10.1 on the following tensors: explicit symmetric \( d \)-qubits found from references [2, 20] and random symmetric \( d \)-qubits.

### 11.3.1 Symmetric \( d \)-qubits

We test Algorithm 10.1 on several symmetric \( d \)-qubits found from references [2, 20]. The symmetric \( d \)-qubits are shown in Table 9.

#### Table 6: The most entangled real states for 500 nonsymmetric examples

| \( d = 3 \) | \( d = 4 \) | \( d = 5 \) | \( d = 6 \) |
|-------------|-------------|-------------|-------------|
| \( F = \mathbb{R} \) | \( F = \mathbb{R} \) | \( F = \mathbb{R} \) | \( F = \mathbb{R} \) |
| \( \mathcal{T}(\cdot, 1) = \begin{bmatrix} -0.3947 & -0.3663 \\ -0.3316 & 0.3077 \end{bmatrix} \) | \( \mathcal{T}(\cdot, 1) = \begin{bmatrix} -0.3863 & 0.0504 \\ -0.3620 & 0.2986 \end{bmatrix} \) | \( \mathcal{T}(\cdot, 1) = \begin{bmatrix} 0.1924 & 0.2460 \\ 0.1419 & 0.0317 \end{bmatrix} \) | \( \mathcal{T}(\cdot, 1, 1, 1) = \begin{bmatrix} -0.1354 & -0.0111 \\ 0.1972 & -0.1357 \end{bmatrix} \) |
| \( \mathcal{T}(\cdot, 1, 2) = \begin{bmatrix} -0.3065 & 0.2301 \\ 0.2369 & 0.0084 \end{bmatrix} \) | \( \mathcal{T}(\cdot, 1, 2) = \begin{bmatrix} -0.3002 & 0.2366 \\ 0.2369 & 0.0084 \end{bmatrix} \) | \( \mathcal{T}(\cdot, 1, 2, 1) = \begin{bmatrix} -0.0895 & -0.0111 \\ 0.1972 & -0.1357 \end{bmatrix} \) | \( \mathcal{T}(\cdot, 1, 2, 1, 1) = \begin{bmatrix} -0.1354 & -0.0111 \\ 0.1972 & -0.1357 \end{bmatrix} \) |
| \( \mathcal{T}(\cdot, 2) = \begin{bmatrix} -0.2170 & 0.2405 \\ 0.1368 & 0.4426 \end{bmatrix} \) | \( \mathcal{T}(\cdot, 2, 1) = \begin{bmatrix} -0.2657 & 0.1729 \\ 0.0526 & 0.2609 \end{bmatrix} \) | \( \mathcal{T}(\cdot, 2, 1, 1) = \begin{bmatrix} 0.1198 & 0.1609 \\ 0.1198 & 0.0605 \end{bmatrix} \) | \( \mathcal{T}(\cdot, 2, 1, 2, 1) = \begin{bmatrix} -0.0895 & -0.0111 \\ 0.1972 & -0.1357 \end{bmatrix} \) |

| \( \mathcal{T}(\cdot, 2, 2) = \begin{bmatrix} -0.2170 & 0.2405 \\ 0.1368 & 0.4426 \end{bmatrix} \) | \( \mathcal{T}(\cdot, 2, 2) = \begin{bmatrix} -0.2657 & 0.1729 \\ 0.0526 & 0.2609 \end{bmatrix} \) | \( \mathcal{T}(\cdot, 2, 2, 1) = \begin{bmatrix} 0.1198 & 0.1609 \\ 0.1198 & 0.0605 \end{bmatrix} \) | \( \mathcal{T}(\cdot, 2, 2, 1, 1) = \begin{bmatrix} -0.0895 & -0.0111 \\ 0.1972 & -0.1357 \end{bmatrix} \) | \( \mathcal{T}(\cdot, 2, 2, 1, 2, 1) = \begin{bmatrix} -0.0895 & -0.0111 \\ 0.1972 & -0.1357 \end{bmatrix} \) | \( \mathcal{T}(\cdot, 2, 2, 2, 1, 2) = \begin{bmatrix} -0.0895 & -0.0111 \\ 0.1972 & -0.1357 \end{bmatrix} \) | \( \mathcal{T}(\cdot, 2, 2, 2, 2, 1) = \begin{bmatrix} -0.0895 & -0.0111 \\ 0.1972 & -0.1357 \end{bmatrix} \) | \( \mathcal{T}(\cdot, 2, 2, 2, 2, 2) = \begin{bmatrix} -0.0895 & -0.0111 \\ 0.1972 & -0.1357 \end{bmatrix} \) |}
and \(\|T\|_{1,F}(N)\) is the nuclear norm calculated by semidefinite relaxation method [36]. For all the symmetric \(d\)-qubits list in the Table 9, the semidefinite relaxation method found exact nuclear norms with certification. For the certification conditions of semidefinite relaxation method, please refer to [36] for details. From the Table 9, we can see that Algorithm 10.1 also finds all nuclear norms with tiny numerical errors. The spectral norms of these symmetric \(d\)-qubits list in Table 9 are shown in Table 10. The computational time comparison results are shown in Table 11 in seconds. \(\text{Avg}_{\text{Fast,Time}}(S)\) is the average computational time of Algorithm 10.1 over field \(F\), and \(\text{Avg}_{\text{Iter,Time}}(S)\) is the average iteration of Algorithm 10.1 over field \(F\). Time\(N,F)\) is the computational time of semidefinite relaxation method [36] over field \(F\).

| \(d\) | \(F = C\) | \(T(:,1,1)\) | \(T(:,2,2)\) | \(d\) | \(F = C\) | \(T(:,1,1)\) | \(T(:,2,2)\) | \(d\) | \(F = C\) | \(T(:,1,1)\) | \(T(:,2,2)\) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 0.3643 ± 0.0337i, 0.2643 ± 0.3760i | 0.1555 ± 0.0832i, 0.2076 ± 0.3741i | 0.1481 ± 0.0609i, 0.2237 ± 0.0730i | 0.2045 ± 0.1013i, 0.2143 ± 0.2667i | 0.1583 ± 0.0379i, 0.0913 ± 0.2778i |
| 4 | 0.9552 ± 0.2873i, 0.1329 ± 0.0183i | 0.1937 ± 0.1979i, 0.2413 ± 0.2667i | -0.0470 ± 0.1859i, 0.2708 ± 0.0454i | -0.1674 ± 0.0091i, 0.2443 ± 0.0936i |
| 5 | -0.1292 ± 0.0647i, 0.1299 ± 0.1403i | -0.0781 ± 0.0165i, 0.0567 ± 0.0179i | -0.1138 ± 0.0326i, 0.2077 ± 0.1496i |
| 6 | 0.0162 ± 0.0526i, 0.1556 ± 0.0521i | 0.0969 ± 0.0981i, 0.2059 ± 0.2059i | -0.0762 ± 0.1214i, 0.0390 ± 0.0843i |

Table 7: The most entangled complex states for 500 nonsymmetric examples

| \(d\) | \(F = C\) | \(T(:,1,1,1)\) | \(T(:,2,1,1)\) | \(d\) | \(F = C\) | \(T(:,1,1,1)\) | \(T(:,2,1,1)\) | \(d\) | \(F = C\) | \(T(:,1,1,1)\) | \(T(:,2,1,1)\) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 0.0106 ± 0.1409i, 0.0443 ± 0.0007i | -0.0529 ± 0.1296i, 0.1138 ± 0.0376i | -0.0286 ± 0.1434i, 0.2077 ± 0.1496i |
| 4 | 0.1019 ± 0.0332i, 0.0174 ± 0.0142i | 0.0913 ± 0.0154i, 0.0321 ± 0.0973i | 0.1153 ± 0.0326i, 0.0276 ± 0.0703i |
| 5 | -0.1374 ± 0.0612i, 0.0321 ± 0.0973i | -0.0427 ± 0.1427i, 0.0279 ± 0.0873i | -0.0762 ± 0.1214i, 0.0390 ± 0.0843i |
| 6 | 0.1131 ± 0.0779i, 0.1079 ± 0.1387i | 0.0919 ± 0.0154i, 0.0567 ± 0.0179i | 0.1153 ± 0.0326i, 0.0276 ± 0.0703i |

11.3.2 Random symmetric \(d\)-qubits

Similar like nonsymmetric case, we are also interested in finding the maximum entangled states. Here we randomly generate 5000 symmetric states over field \(F\) for \(d = 3, 4, 5, 6\), and calculate their nuclear norms over field \(F\) by using alternating Algorithm 10.1 and semidefinite relaxation method [36]. Algorithm 10.1 is implemented with random starting point for 30 times, and choose the one with smallest objective value as the nuclear norm. For all these 5000 randomly examples, two algorithms find the same nuclear norm with tiny numerical errors for Algorithm 10.1. So Algorithm 10.1 also works well for calculating nuclear norms for symmetric tensors. We show the most entangled symmetric states among the 5000 random examples in Table 12 and Table 13 for \(F = \mathbb{R}\) and \(F = C\) respectively.

28
Table 8: Computational results for random nonsymmetric tensors

| No. | d | Tensor | $\|T\|_{1,R}(S)$ | $\|T\|_{1,C}(S)$ | $\|T\|_{1,R}(N)$ | $\|T\|_{1,C}(N)$ |
|-----|---|--------|-----------------|-----------------|-----------------|-----------------|
| 1   | 3 | $T_{1,1,1} = T_{2,2,2} = \sqrt{2}$ | 1.4149 | 1.4141 | 1.4142 | 1.4142 |
| 2   | 3 | $T_{1,1,2} = T_{1,2,2} = \sqrt{2}$ | 1.7321 | 1.5000 | 1.7321 | 1.5000 |
| 3   | 3 | $T_{1,1,1} = T_{2,2,2} = \sqrt{3}$ | 1.9235 | 1.9235 | 1.9235 | 1.9235 |
| 4   | 3 | $T_{1,1,1} = T_{2,2,2} = \sqrt{3}$ | 1.9903 | 1.9903 | 1.9903 | 1.9903 |
| 5   | 3 | $T_{1,1,2} = T_{1,2,2} = \sqrt{3}$ | 2.0000 | 2.0000 | 2.0000 | 2.0000 |
| 6   | 3 | $T_{1,1,1} = T_{2,2,2} = \sqrt{3}$ | 2.0000 | 2.0000 | 2.0000 | 2.0000 |
| 7   | 3 | $T_{1,1,1} = T_{2,2,2} = \sqrt{3}$ | 2.0000 | 2.0000 | 2.0000 | 2.0000 |
| 8   | 3 | $T_{1,1,1} = T_{2,2,2} = \sqrt{3}$ | 2.0000 | 2.0000 | 2.0000 | 2.0000 |
| 9   | 3 | $T_{1,1,1} = T_{2,2,2} = \sqrt{3}$ | 2.0000 | 2.0000 | 2.0000 | 2.0000 |
| 10  | 3 | $T_{1,1,1} = T_{2,2,2} = \sqrt{3}$ | 2.0000 | 2.0000 | 2.0000 | 2.0000 |

Table 9: Computational results for symmetric $d$-qubits

| No. | $\|T\|_{\infty,R}$ | $\|T\|_{\infty,C}$ | $P_R$ | $P_C$ |
|-----|-----------------|-----------------|--------|--------|
| 1   | 0.7071          | 0.6667          | 1.0005 | 1.0000 |
| 2   | 0.6938          | 0.6938          | 1.3345 | 1.0000 |
| 3   | 0.5785          | 0.5785          | 1.3345 | 1.0000 |
| 4   | 0.5000          | 0.5000          | 1.3345 | 1.0000 |
| 5   | 0.6495          | 0.6495          | 1.3345 | 1.0000 |
| 6   | 0.5774          | 0.5774          | 1.3345 | 1.0000 |
| 7   | 0.4472          | 0.4472          | 1.3345 | 1.0000 |
| 8   | 0.5492          | 0.5492          | 1.3345 | 1.0000 |
| 9   | 0.4714          | 0.4714          | 1.3345 | 1.0000 |
| 10  | 0.4714          | 0.4714          | 1.3345 | 1.0000 |

Table 10: Spectral norms of symmetric $d$-qubits in Table 9

| No. | Avg$_R$,Time(S) | Avg$_C$,Time(S) | Avg$_R$,Iter(S) | Avg$_C$,Iter(S) | Time(N,R) | Time(N,C) |
|-----|-----------------|-----------------|-----------------|-----------------|-----------|-----------|
| 1   | 2.05            | 4.69            | 3.1            | 13.1           | 0.07      | 0.15      |
| 2   | 1.76            | 2.22            | 1.9            | 5.9            | 0.10      | 0.13      |
| 3   | 1.51            | 3.59            | 1.5            | 10.2           | 0.11      | 0.12      |
| 4   | 1.50            | 3.95            | 1.5            | 10.9           | 0.10      | 0.11      |
| 5   | 1.58            | 2.73            | 1.5            | 6.8            | 0.07      | 0.10      |
| 6   | 1.89            | 8.74            | 2.6            | 6.5            | 0.11      | 1.11      |
| 7   | 4.32            | 6.51            | 5.8            | 6.7            | 0.13      | 1.31      |
| 8   | 1.70            | 3.98            | 3.6            | 8.6            | 0.11      | 0.34      |
| 9   | 1.88            | 4.35            | 3.6            | 8.8            | 0.11      | 0.36      |
| 10  | 4.79            | 4.82            | 7.3            | 8.2            | 0.19      | 0.37      |

Table 11: Computational time comparison for symmetric $d$-qubits in Table 9
| $d$ | Tensor | $\|T\|_{1,R}$ | $\|T\|_{\infty,R}$ | $P_R$ |
|-----|--------|----------------|-----------------|-----|
| 3   | $T_{1,1,1} = -0.4950, T_{1,2,1} = -0.1078$; $T_{2,2,2} = 0.4864, T_{2,2,2} = 0.1018$ | 1.9999 | 0.5001 | 1.0000 |
| 4   | $T_{1,1,1,1} = -0.3132; T_{1,1,1,2} = 0.1703; T_{1,1,2,2} = 0.3089; T_{1,2,2,2} = -0.1737; T_{2,2,2,2} = -0.3042$ | 2.8283 | 0.3581 | 1.0128 |
| 5   | $T_{1,1,1,1,1} = 0.2388; T_{1,1,1,1,2} = 0.0952; T_{1,1,2,2,2} = -0.2304; T_{1,2,2,2,2} = -0.0947; T_{2,2,2,2,2} = 0.2289; T_{2,2,2,2,2} = 0.1223$ | 3.9984 | 0.2716 | 1.0860 |
| 6   | $T_{1,1,1,1,1,1} = 0.0541; T_{1,1,1,1,1,2} = 0.1972; T_{1,1,1,1,2,2} = -0.0598; T_{1,1,1,2,2,2} = -0.170; T_{1,1,2,2,2,2} = 0.2073; T_{1,2,2,2,2,2} = 0.1411; T_{2,2,2,2,2,2} = 0.0274$ | 5.5931 | 0.2049 | 1.1464 |

Table 12: The most entangled real symmetric states for 5000 random examples.

| $d$ | Tensor | $\|T\|_{1,C}$ | $\|T\|_{\infty,C}$ | $P_C$ |
|-----|--------|----------------|-----------------|-----|
| 3   | $T_{1,1,1} = 0.3029 - 0.3436i, T_{1,2,1} = -0.3423 - 0.2033i$; $T_{2,2,2} = 0.0727 - 0.3054i, T_{2,2,2} = 0.1365 + 0.0183i$ | 1.4985 | 0.6869 | 1.0293 |
| 4   | $T_{1,1,1,1} = 0.2679 - 0.2422i; T_{1,1,1,2} = 0.3086 + 0.0143i; T_{1,1,2,2} = -0.1385 - 0.1266i; T_{1,2,2,2} = -0.0986 - 0.0715i; T_{2,2,2,2} = 0.1396 - 0.4449i$ | 1.7247 | 0.6136 | 1.0583 |
| 5   | $T_{1,1,1,1,1} = 0.2025 - 0.1845i; T_{1,1,1,1,2} = 0.1868 - 0.2069i; T_{1,1,2,2,2} = 0.0060 + 0.1202i; T_{1,2,2,2,2} = 0.0177 - 0.0913i; T_{2,2,2,2,2} = -0.0953 - 0.1989i; T_{2,2,2,2,2} = 0.1964 - 0.1606i$ | 1.8112 | 0.6236 | 1.1295 |
| 6   | $T_{1,1,1,1,1,1} = 0.0967 - 0.04477i; T_{1,1,1,1,1,2} = -0.1934 - 0.2027i; T_{1,1,1,1,2,2} = -0.0655 + 0.0558i; T_{1,1,1,2,2,2} = 0.0196 + 0.0043i; T_{1,1,2,2,2,2} = 0.0371 - 0.0115i; T_{1,2,2,2,2,2} = 0.1735 - 0.1746i; T_{2,2,2,2,2,2} = -0.0595 + 0.0973i$ | 2.0312 | 0.6271 | 1.2738 |

Table 13: The most entangled complex symmetric states for 5000 random examples.
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