Moishezon Manifolds

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Abstract. Let $X$ be a compact Moishezon manifold which becomes projective after blowing up a smooth subvariety $Y \subset X$. We assume also that there exists a proper map $\rho : X \to X'$ onto a projective variety $X'$ with $\rho(Y)$ a point, such that $\text{Pic}(X/X') = \mathbb{Z}$ and $K_X$ is $\rho$-big. We prove some inequalities between the dimensions of $Y$ and $X$ and we construct examples which shows the optimality of the inequalities. Then we discuss some differential geometry properties of these examples which lead to a conjecture.

Résumé Soit $X$ une variété de Moishezon compacte que est rendue projective après éclatement le long d’une sous-variété $Y \subset X$ lisse. Supposons de plus que existe une application propre $\rho : X \to X'$ dans une variété projective $X'$ avec $\rho(Y)$ un point, tel que $\text{Pic}(X/X') = \mathbb{Z}$ et $K_X$ est $\rho$-gros. Nous montrons quelques inégalités entre les dimensions de $Y$ et $X$ et nous construisons des exemples qui montrent que les inégalités sont les meilleures possibles. Après nous décrivons quelques propriétés de géométrie différentielle de ces exemples qui donnent origine à une conjecture.

Let $X$ be a smooth compact complex manifold of dimension $n$; $X$ is said to be a Moishezon manifold if the transcendent degree of the field of meromorphic functions over $\mathbb{C}$ is equal to $n$. It was B.Moishezon who proved that every such manifold becomes projective after a finite number of monoidal transformations (i.e. blow-up) with non singular center (see [Moi]).

In this paper we consider some particular Moishezon manifold $X$, namely we will make two assumptions. First we assume that $X$ can be made projective with only a simple blow-up; that is there exists a smooth submanifold $Y \subset X$ such that if we blow-up $X$ along $Y$, $\pi : \tilde{X} := \text{Bl}_Y(X) \to X$, we obtain a projective manifold $\tilde{X}$. Let $m = \text{dim} Y$.

Secondly we assume that there is a proper map $\rho : X \to X'$ onto a projective variety $X'$ with $\rho(Y)$ a point and such that $\text{Pic}(X/X') = \mathbb{Z}$ (for instance if $\text{Pic}(X) = \mathbb{Z}$, taking $\rho$ to be a constant). Moreover if $K_X := \Lambda^n T^* X$ denotes the canonical bundle of $X$, we also assume that $K_X$ is $\rho$-big.

**Theorem 1.** Under the above hypothesis $2\text{dim} Y \geq n - 1$; moreover the equality holds if and only if $Y \simeq \mathbb{P}^m$, $N_{Y/X} = \mathcal{O}(-1)^{\oplus (m+1)}$ and $K_X$ is $\rho$-nef.

If $K_X$ is not $\rho$-nef then $2\text{dim} Y \geq n$; the equality holds if and only if $Y \simeq \mathbb{P}^m$ and $N_{Y/X} = \mathcal{O}(-1)^{\oplus m}$.

The lower bounds of the theorem are actually attained as the two following examples show (the notation we use is standard in the field of Minimal Model Theory; however we provide some basic definitions in the next section).
Example 1. Assume that \( \varphi : \tilde{X} \to Z \) is an elementary extremal contraction of a smooth projective variety of dimension \( 2k + 1 \) which contracts a divisor \( E \simeq \mathbb{P}^k \times \mathbb{P}^k \) to a point; assume also that the normal bundle of \( E \) into \( X \) is the line bundle \( L \) given by \( \pi_i^*(\mathcal{O}(-1)) \otimes \pi_{\tilde{X}\setminus E}^*(\mathcal{O}(-1)) \), where \( \pi_i \) are the projections on the two factors.

By the Nakano criterium (see [Na]), we can contract \( \tilde{X} \) along \( \pi_1 \) (or symmetrically along \( \pi_2 \)); that is there exists a bimeromorphic map \( \pi : \tilde{X} \to X \) such that \( \pi_1 \circ \pi = \pi_1 \) and \( \pi_{\tilde{X}\setminus E} \) is an isomorphism. \( X \) is a Moishezon non projective manifold since every curve in \( \pi(\pi_2^{-1}(p)) \simeq \mathbb{P}^k := Y \), for any \( p \), is homologous to zero. Note that \( N_{Y/X} = \mathcal{O}(-1)^{\oplus(k+1)} \) and thus \( K_{X|Y} \) is nef (it is actually trivial). Moreover there exist a map \( \rho : X \to Z \) such that \( \varphi = \rho \circ \pi \) with \( Y \) as only non trivial fiber, thus \( K_X \) is \( \rho \)-big.

In order to construct an elementary, extremal, divisorial contraction as at the beginning of the example we will generalize a construction of S. Mori (which gives the case \( k = 1 \); see [Mor], example 3.44.2); I like to thank J. Wiśniewski for suggesting it to me.

Let \( V \) be a smooth \( 2k + 1 \)-fold and \( Z \subset V \) a subvariety of dimension \( k \) with an isolated singular point \( p \) in which two branches of \( Z \) intersect transversally. Let \( \pi : W \to Z \) be the blow-up of \( Z \). \( \rho(W/Z) := \text{dimPic}(W/Z) = 1 \), since the exceptional divisor is irreducible. One now check with local calculation that \( W \) has only an isolated singularity which is the vertex of the cone over \( \mathbb{P}^k \times \mathbb{P}^k \) (embedded via the Segre embedding).

Finally let \( g : U \to W \) be the blow-up of \( W \) at the singular point; \( U \) is smooth and \( g \) is a desingularization of \( W \) with exceptional locus \( \mathbb{P}^k \times \mathbb{P}^k \). The composition \( g \circ \pi := f \) is a birational map between two smooth varieties and it is easy to check that \( \rho(U/Z) = 2 \). Thus \( \rho(U/W) = 1 \) and \( g \) is the elementary contraction we wanted.

Example 2. Let \( \phi : X' \to Z' \) be an elementary extremal small contraction of a smooth projective manifold \( X' \) of dimension \( 2k \) whose exceptional locus is a disjoint union of \( E_i \simeq \mathbb{P}^k \) with \( N_{E_i/X'} = \mathcal{O}(-1)^{\oplus(k)} \) and \( i > 2 \).

Now we "flip" all the \( \mathbb{P}^k \) in \( X' \) except one, say \( E_1 := Y \); that is we blow-up all the \( E_i \) with \( i \neq 1 \) and then we blow-down the exceptional divisor into a disjoint union of \( D_1 \simeq \mathbb{P}^{k-1} \) contained in a smooth compact manifold \( X \). We obtain a Moishezon non projective manifold: in fact on one hand \( \text{Pic}(X/Z') = \mathbb{Z} \) and on the other \( K_{X|Y} = \mathcal{O}(-1) \) and \( K_{X|D_1} = \mathcal{O}(1) \). Note that we can make \( X \) projective just blowing-up \( Y \). Finally one can see that there is a natural map \( \rho : X \to Z' \) whose fibers are the \( D_i \) and \( Y \) and thus \( K_X \) is \( \rho \)-big.

A contraction as at the beginning can be constructed in the following way (this is due to Kawamata, [Ka], and slightly generalized in [A-B-W1] in order to obtain examples like this one): take a smooth projective variety \( V \) of dimension \( 2k \) and with \( K_V \) ample. Let \( U \), resp. \( W \), a \( k - 1 \), resp. a \( k \), dimensional smooth subvariety of \( V \). Assume that \( U \) and \( W \) intersect transversally at points \( p_i, i > 1 \). Let \( \alpha : V' \to V \) be the blow-up with center \( U \) and \( \beta : X' \to V' \) the blow-up with center \( W' \), the strict transform of \( W \) by \( \alpha \). The strict transform by \( \beta \) of \( \alpha^{-1}(p_i) \) are isomorphic to \( \mathbb{P}^k \) with normal bundle \( \mathcal{O}(-1)^{\oplus(k)} \); the contraction of these \( \mathbb{P}^k \) is the small contraction we wanted.

We will actually prove a more general, but perhaps less immediately clear, version of the theorem 1, namely:
**Theorem 2.** In the above hypothesis we have the following inequalities according to the positivity of $K_X$.

(a) If $K_X$ is $\rho$-nef then there exist a morphism $\psi : X \to Z$ (over $X'$) into a projective variety of the same dimension which is an isomorphism outside $Y$. Let $C := \psi(Y)$, $c = \dim C$ and $F$ be a general fiber of $\psi$. Then $C$ is a normal irreducible projective variety and $\dim Y > \dim C$, moreover

i) $2\dim Y \geq n + c - 1$,

ii) If the equality holds then $F \simeq \mathbb{P}^{(m-c)}$ and $N_{Y/X|F} = \mathcal{O}(-1) \oplus (n-m)$

iii) If $2\dim Y = n + c$ then $F \simeq \mathbb{P}^{(m-c)}$ or $\mathbb{Q}^{(m-c)}$ and $N_{Y/X} = \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus (m-c-1)$ or $\mathbb{TP}^{(m-c)}$, respectively $\mathcal{O}(1) \oplus (m-c)$.

(b) If $K_X$ is not $\rho$-nef then there exists an extremal birational contraction $\varphi : \tilde{X} \to Z$ (over $X'$) such that any fiber of $\varphi$ meets any fiber of $\pi$ at most in a zero dimensional set. Moreover, let $\tilde{E}$ be the exceptional locus and $F$ a fiber of $\varphi$, then $\dim Y \geq \dim F \geq n - \dim Y + \text{codim} \tilde{E} - 1$ and the equality holds in the last inequality if and only if $F' = \mathbb{P}^k$ where $F'$ is the normalization of the fiber $F$.

**Theorem 3.** In the theorem (2.a.ii) if for all fibers $F$ we have $\dim F = m - c$ and if also $\dim Y - 2\dim C + 1 \geq 0$, then $C$ is smooth and for any fiber $F \simeq \mathbb{P}^{(m-c)}$ and $N_{Y/X|F} = \mathcal{O}(-1) \oplus (n-m)$. In particular if $\dim C \leq 1$ then $Y$ is a $\mathbb{P}^{(m-c)}$.-bundle over $C$ and thus $N_{Y/X|F} = \mathcal{O}(-1) \oplus (n-m) \oplus \mathcal{O}^{\oplus c}$. In the theorem (2.b) if moreover $\dim F \leq n - \dim Y$ for every fiber then $Z$ is smooth and $\varphi$ is the blow-up of a smooth subvariety of $Z$. In particular if $2\dim Y = n$ then $Y$ is $\mathbb{P}^{n/2}$ and $N^*_Y/X = \mathcal{O}(1)^{\oplus n/2}$

**Remark 1.** The above theorems hold also if we allow mild singularities on $Y$; for instance if $Y$ has log terminal singularities.

**Remark 2.** The part a) of the theorems 2 and 3 holds actually in slight more general assumption: namely one can drop the assumptions that $\text{Pic}(X/X') = \mathbb{Z}$ and that $K_X$ is $\rho$-big and suppose instead that $K_X \cdot C > 0$ for every curve $C$ not contained in $Y$ and in a fiber of $\rho$ (see (1.2) and (1.2.1) for the inclusion between the two assumptions). The stronger assumption is used in the prove of part b) only for proving the Claim (b). A very nice example by J. Kollár (which disprove a conjecture of Moishezon) shows that we cannot drop all these assumptions; in this example he construct non-projective, compact Moishezon manifold of dimension 3 which can be made projective by blowing-up a smooth curve of genus $g$, with $g$ arbitrary (see (4.3.1) in [Ko]).

**Remark 3.** The idea to prove the above theorems comes from reading the Thèse de doctorat of Laurent Bonavero, see [Bo]. Assuming that $\text{Pic}(X) = \mathbb{Z}$ and that $K_X$ is big he proves the second part of theorem 1 (namely that if $K_X$ is not nef then $2\dim Y \geq n$).
1. Some definitions.

Definition (1.0). A line bundle \( L \) on a compact complex manifold \( X \) is said to be

1. Nef (in a metric sense) if for every \( \epsilon > 0 \) there exists a metric \( h_\epsilon \) on \( L \) such that
\[
\Theta_{h_\epsilon} \geq -\epsilon \omega.
\]
2. Nef (in the curve sense) if for every \( C \subset X \) complex compact curve in \( X \) it is \( L \cdot C \geq 0 \).

(More generally if \( \rho : X \to X' \) is a proper map \( L \) is said to be \( \rho \)-nef if \( L \cdot C \geq 0 \), for every \( C \subset X \) complex compact curve which is mapped to a point by \( \rho \).)

M. Paun recently proved (see [Pa]):

Proposition (1.1). On a Moishezon manifold the two definitions are equivalent.

Definition (1.2). A line bundle \( L \) on a compact complex manifold \( X \) is said to be big if
its Kodaira-Iitaka dimension, denoted by \( \kappa(L) \), is equal to the dimension of \( X \) (see for instance [K-M-M] for the definition of \( \kappa(L) \)). More generally if \( \rho : X \to X' \) is a proper map \( L \) is said to be \( \rho \)-big if \( L \cdot C \geq 0 \) for the generic \( x' \in X' \).

Remark (1.2.1). Let \( \rho : X \to X' \) be a proper map between complex compact manifolds with \( \text{Pic}(X/X') = \mathbb{Z} \); suppose that there exists a smooth subvariety \( Y \subset X \) such that if \( \pi : \tilde{X} \to X \) is the blow-up of \( X \) along \( Y \), then \( \tilde{X} \) is projective. Then we can choose a generator \( O(1) \) of \( \text{Pic}(X/X') \) such that \( O(k) \) is effective for some positive integer \( k \); moreover, for every curve \( C \) in \( X \) which is mapped into a point via \( \rho \) and which is not contained in \( Y \), we have \( O(1) \cdot C > 0 \). If \( L \) is a \( \rho \)-big line bundle then there exists a positive integer \( m \) such that \( L \equiv O(m) \); thus for every curve \( C \) as above \( L \cdot C > 0 \). The proof of this remark is straightforward, see for instance the section 5 of [Ko].

The following definitions assume implicitly many deep facts of the Minimal Model Theory like the Cone Theorem and the Base Point Free Theorem; our notation agree with that of [K-M-M] to which we refer the reader for any further details.

Definition (1.3). A proper surjective map \( \varphi : \tilde{X} \to Z \) from a smooth complex manifold \( \tilde{X} \) onto a normal variety \( Z \) with connected fiber and such that \( -K_{\tilde{X}} \) is \( \varphi \)-ample is called an extremal contraction. If \( \text{Pic}(\tilde{X}/Z) = \mathbb{Z} \) then \( \varphi \) is said to be elementary.

The contraction \( \varphi \) is said to be birational if \( \dim \tilde{X} = \dim Z \); it is said to be small if it is birational and it is an isomorphism in codimension 2.

If \( \varphi \) is an extremal contraction the set of curves \( C \subset \tilde{X} \) such that \( \varphi(C) \) is a point is an extremal face of the cone \( \overline{\text{NE}}(\tilde{X}) \), i.e. the closure of the cone of curves in \( \tilde{X} \). If \( \varphi \) is elementary this face is one dimensional, i.e. a ray \( R \); it is called an extremal ray. We say that \( \varphi \) is the extremal contraction of the extremal ray \( R \).

The inequality contained in the next proposition is a fundamental tool in the theory of Minimal Model and a first basic version of it was introduced by S. Mori in [Mor]; the proofs of our theorems consists basically in reducing to it. The version we give was proved in [Wi] (see also [A-W2] for a survey of recent results on extremal contractions of smooth projective varieties).
Proposition (1.4). If \( \varphi \) is the contraction of the extremal ray \( R \) one has the following inequality:
\[
dim(\text{Exc}(\varphi)) + \dim F \geq n + l(R) - 1
\]
where \( l(R) := \min\{-K_X \cdot C : C \text{ is a rational curve in } R\} \), \( F \) is a fiber of \( \varphi \) and \( \text{Exc}(\varphi) \) is the exceptional locus of \( \varphi \).

Let \( \pi : \tilde{X} \to X \) be the blow-up of \( X \) along \( Y \). We will denote with \( E \) the exceptional divisor of \( \pi \), that is \( E \simeq \mathbb{P}(N_{Y/X}^*) \). We have that \( K_{\tilde{X}} = \pi^*(K_X) + (r - 1)E \) where \( r = n - m = \dim X - \dim Y \) and \( K_E = \pi^*(K_Y + \text{det}N_{Y/X}^*) - r\mathcal{O}(1) \), where \( N_{Y/X}^* \) is the conormal bundle of \( Y \) in \( X \) and \( \mathcal{O}(1) \) is the tautological bundle of the projectivization.

2. Proof of the theorems 2 and 3.

We will prove the theorems under the hypothesis that \( Pic(X) = \mathbb{Z} \), i.e. \( \rho \) is constant; the proof for the relative case is exactly the same but using the relative Minimal Model Theory (i.e. the relative Cone Theorem, see [K-M-M]).

By construction the projective variety \( \tilde{X} \) is not minimal, that is \( K_{\tilde{X}} \) is not nef; in fact \( K_{\tilde{X}} \) is not positive on curves contained in the fibers of \( \pi \). Therefore we have some extremal rays on \( X \).

As implicit in the theorems there are two different behaviors according to the positivity of the canonical bundle.

First we assume that \( K_X \) is nef; therefore \( \pi^*(K_X) \) is nef (actually it is equivalent, see also [Pa]). By the cone theorem and the Kleiman criterium of ampleness this implies that \( \mathbb{R}^+[C] \), where \( C \) is a rational curve in a fiber of \( \pi \), is contained in an extremal face on which \( \pi^*(K_X) \) is trivial. Let \( \varphi : \tilde{X} \to Z \) be the extremal contraction of this face; note that this contraction is not necessarily given by a multiple of \( \pi^*(K_X) \) which can be not semiample. However \( \pi^*(K_X) \) is trivial on the fiber of \( \varphi \). Thus, since \( K_{\tilde{X}} = \pi^*K_X + (r - 1)E \), the divisor \(-E\) is \( \varphi \)-ample.

By our assumption \( K_X \cdot R > 0 \), for every curve \( R \) not contained in \( Y \); (see (1.2.1)); this implies that the exceptional locus of \( \varphi \) is contained in \( E \), and therefore it is \( E \); that is \( \varphi \) is a birational divisorial contraction. Note also that \( \varphi \) factors through \( \pi \), i.e. there exists a morphism \( \psi : X \to Z \) such that \( f = \psi \circ \pi \). Let \( C := \psi(Y) = \varphi(E) \); \( C \) is a normal irreducible projective variety. If, by contradiction, \( \dim Y = \dim C \) then \( \psi \) is a finite map and therefore \( \tilde{X} \) would be projective.

Claim (a). The conormal bundle of \( Y \) in \( X \), \( N_{Y/X}^* \) is \( \psi \)-ample (by abuse we call again \( \psi \) the restriction of \( \psi \) to \( Y \), that is \( \psi : Y \to C \)). Moreover \( K_Y + \text{det}N_{Y/X}^* \) is \( \psi \)-trivial.

Proof. Since \( Z \) is \( \mathbb{Q} \)-Gorenstein, there exists an integer \( m \) such that \( mK_X = \psi^*(mK_Z) \) as Cartier divisors. In particular \( K_X \) is trivial on the fibers of \( \psi \), that is \( K_Y + \text{det}N_{Y/X}^* \) is \( \psi \)-trivial.

The formula \( K_E = \pi^*(K_Y + \text{det}N_{Y/X}^*) - r\mathcal{O}(1) = \pi^*(K_X|_{Y}) - r\mathcal{O}(1) \) implies that \( -K_E = r\mathcal{O}(1) \) relatively on \( \varphi : E \to C \). Thus, since \( -K_E = -K_{\tilde{X}|E} - E_E \) is \( \varphi \)-ample, also \( \mathcal{O}(1) \) is \( \varphi \)-ample. By definition therefore \( N_{Y/X}^* \) is \( \psi \)-ample.

The theorem (2.(a)) follows now directly from the claim and the next theorem.
Theorem. Let \( \psi : Y \to C \) be a proper map between complex variety and assume that \( Y \) is smooth and \( C \) is normal; assume also that \( \dim Y > \dim C \). Let \( \mathcal{E} \) be a \( \psi \)-ample vector bundle of rank \( r \) such that \( K_Y + \det \mathcal{E} \) is \( \psi \)-trivial. Let \( F \) be the general fiber of \( \psi \); then

\[
\dim Y - \dim C = \dim F \geq r - 1.
\]

If the equality holds then \( F \simeq \mathbb{P}^{r-1} \) and \( \mathcal{E}|_F = \mathcal{O}(\mathbb{P}^{r-1}) \). If \( \dim Y - \dim C = \dim F = r \), then \( F \simeq \mathbb{P}^r \), respectively \( \mathbb{Q}^r \), and \( \mathcal{E}|_F = \mathcal{O}(2) \oplus \mathcal{O}(1)^{(r-1)} \) or \( TP^r \), respectively \( \mathcal{O}(1)^{(r)} \).

Proof. The general fiber \( F \) is a smooth projective variety with an ample vector bundle \( \mathcal{E}|_F \) such that \( K_F = c_1(\mathcal{E}|_F) \). Now apply the results in [Pe]. (which are generalizations of the Hartshorne-Frenkel conjecture proved by S. Mori.)

Assume now that \( K_X \) is not nef, that is there exist a curve \( R \) such that \( -K_X \cdot R > 0 \); note that this curve, by our assumption, is contained in \( Y \) (see (1.2.1)).

Claim (b). There exists \( C \subset \tilde{X} \) an extremal rational curve which is in the convex space \(-\pi^*K_X > 0 \) and \(-E > 0 \), i.e. such that \(-\pi^*K_X \cdot C > 0 \) and \(-E \cdot C > 0 \).

Proof. The existence of \( C \) follows from the cone theorem and the next three facts, strongly using that \( \rho(X/X') = 1 \), thus that \( \rho(\tilde{X}/X') = 2 \) (this is the only place in the paper where we use this assumpton):

i) there exists curves on which \(-K_{\tilde{X}} \) is positive (that is there are extremal rays) and curves on which \(-\pi^*K_X \) is positive (this is the non nefness of \( K_X \)) and other in which it is negative (i.e. \(-\pi^*K_X \) "crosses" the cone of effective curves)

ii) there exist curves on which \(-\pi^*K_X \), \(-E \) and \(-K_{\tilde{X}} \) are all together negative (any curve not contained in \( E \)),

iii) there exists a curve on which \(-K_{\tilde{X}} \), \(-E \) are positive and \(-\pi^*K_X \) is zero (the curve in the fiber of \( \pi \)).

Let \( \varphi : \tilde{X} \to Z \) be the extremal contraction associated to \( \mathbb{R}^+[C] \); let \( Exc(\varphi) \) be the exceptional locus of \( \varphi \) and \( F \) be a general fiber.

Such a curve is not numerically equivalent to any curve out of \( E \) by our assumptition and it is not numerically equivalent to a curve in the fiber of \( \pi \). This implies that \( Exc(\varphi) := \tilde{E} \subset E \) and that any non trivial fiber of \( \varphi \) has intersection of finite dimension with the fiber of \( \pi \). In particular \( \dim Y \geq \dim F \).

Since \(-\pi^*K_X \cdot C > 0 \) and \(-E \cdot C > 0 \) we have immediately, from the formula \( K_{\tilde{X}} = (\pi^*K_X + (r-1)E) \), that \( l(R) \geq r \).

Thus, applying to \( \varphi \) the inequality (1.4), we obtain the inequality of the theorem (2.b):

\[
\dim F \geq \text{codim} \tilde{E} + r - 1 = n - \dim Y + \text{codim} \tilde{E} - 1.
\]

(In the relative case, i.e. with \( \rho \) non constant, perhaps it is worthy to observe that by assumption \( Y \) is contained in a fiber of \( \rho \) and thus that \( E \) and \( \tilde{E} \) are contained in a fiber of \( \rho \circ \pi \); thus we can apply the inequality (1.4) which is local).
If the equality holds then \(-E \cdot C = 1\) and \(-K_X \cdot C = r\); thus \(K_X + r(-E)\) is a good supporting divisor for \(\varphi\) (possibly adding the pull back of some very ample line bundle on \(Z\)). Then we apply the Lemma (1.1) in [A-B-W2] to conclude that \(F' \simeq \mathbb{P}^k\).

This conclude the proof of theorem 2.

As for the proof of the theorem 3 assume that in (2.b) the dimension of \(F\) is \(\leq r = n - \text{dim} Y\) for all fibers \(F\). We apply the theorems (4.1) and (5.1) in [A-W1] (see also (2.1) in [An2]); they imply that \(-E\) is \(\varphi\)-spanned (actually \(\varphi\)-very ample), that \(Z\) is smooth and that \(\varphi\) is the blow-up of \(Z\) along a smooth subvariety. In particular if \(2\text{dim} Y = n\) we restrict \(\pi\) to any fiber of \(\varphi\) and we have a finite morphims from \(\mathbb{P}^r\) into \(Y\); this implies that \(Y\) is \(\mathbb{P}^r\). It is straightforward to see now that \(N^*_{Y/X} = \mathcal{O}(1)^{\otimes n/2}\).

Finally assume that in (2.a.ii) \(\text{dim} F = m - c\) for all fibers and that \(\text{dim} Y - 2\text{dim} C + 1 \geq 0\). The theorem 3 follows then from the theorem (3.2) in [A-M] applied at the map \(\psi: Y \to C\) and with \(E = N^*_{Y/X}\).

3. Some remarks and a conjecture.

To study the geometry of complex manifolds by means of submanifolds, or more generally by means of analytic currents, and in the spirit of Calibrated Geometry introduced in [H-L], in the paper [A-A] the following definitions were given.

**Definition (3.1).** Let \(\omega\) be a complex valued differentiable form on a complex manifold \(X\) of bidegree \((p, p)\). \(\omega\) is said to be *real* if \(\omega = \overline{\omega}\). A real \((p, p)\)-form is said to be *transverse* if \(\omega_x \left( \frac{2^p}{\sqrt{p!^2}} V \wedge \overline{V} \right) > 0\) for every \(x \in X\) and every decomposable \(V \in \bigwedge^p (T'_x X)\).

**Remark (3.1.1).** The word "transverse" is often used in differential geometry; we took it from the paper [Su] where the complex case was discussed. The forms in our definition are transverse to the cone structure given in every (real) tangent spaces by the complex subspaces.

It is easy to prove that the set of transverse \((p, p)\)-forms is the interior of the cone \(WP^{(p, p)}\) of weakly positive \((p, p)\)-forms; for this last definition as well as for the more general definition of weakly positive \((p, p)\)-currents we refer the reader to [De] or [Ha].

**Definition (3.2).** A complex manifold \(X\) is *\(p\)-kähler* if it admits a closed complex transverse \((p, p)\)-form (called the \(p\)-kähler form).

**Remark (3.2.1).** Note that a kähler manifold is exactly a 1-kähler manifold and that every complex manifold is \(n\)-kähler, where \(n = \text{dim} X\). A \((n - 1)\)-kähler manifold is a balanced manifold, i.e. there exists an hermitian metric with kähler form \(\omega\) such that \(d\omega^{n-1} = 0\). Note also that the \(p\)-kähler form restricted to any \(p\)-dimensional submanifold become a volume form on it.

In [A-A] we tested the definition on the compact holomorphically parallelizable manifolds, i.e. on homogeneous manifolds \(G/\Gamma\) where \(G\) is a complex Lie group and \(\Gamma\) a discrete uniform subgroup. One can prove that those manifolds are kähler if and only if they are complex tori and that they are always \((n - 1)\)-kähler (see section 3 in [A-A]). Fixed a positive integer \(p\) with some simple multilinear algebra one can then construct holomorphically parallelizable manifold which are \(p\)-kähler but not kähler.
This definition is an interesting tool in non-kähler geometry and our actual purpose is to show that (most of, conjecturally all) Moishezon manifolds are $p$-kähler for some integer $p$ smaller then the dimension. A well known theorem of Kodaira states that a Moishezon manifold is kähler if and only if it is projective.

The following is the main structure theorem proved in [A-A] (see (1.17)).

**Theorem (3.3).** A complex manifold $X$ is $p$-kähler if and only if there are no non trivial (weakly) positive currents of bidimension $(p, p)$ which are $(p, p)$-components of boundaries.

As an application of the structure theorem we now prove that a very simple condition, satisfied by many Moishezon manifolds, implies the existence of a $p$-kähler form (this was first done in the unpublished manuscript [An1]).

**Proposition (3.4).** Let $X$ be a complex compact manifold with a semipositive closed real $(1,1)$ form $\omega$ which is semipositive and actually positive outside an analytic subset of $2p$-Hausdorff measure zero. (That is $\omega_x(v, \overline{v}) \geq 0$, for every $x \in X$ and $v \in T'_x X$, and $\mathcal{H}^{2p}(A) = 0$, where $A = \{ a \in X : \omega_a(v, \overline{v}) = 0$ for some $v \in T'_a X \}$, and $\mathcal{H}^{2p}$ denotes the $2p$-hausdorff measure.) Then $X$ is $p$-kähler.

**Proof.** We must show that on $X$ there are no non trivial (weakly) positive currents of bidimension $(p, p)$ which are $(p, p)$-components of boundaries. Assume by contradiction that $T$ is such a current and that $T = d_{p,p}S$ (we use the notation introduced in [A-A]: in particular we denote by $d_{p,p}$ the differential operator dual to $d : [\mathcal{E}^{p,p}(X)]_\mathbb{R} \to [\mathcal{E}^{p+1,p+1}(X) \oplus \mathcal{E}^{p,p+1}(X)]_\mathbb{R}$, where $[\mathcal{E}^{p,p}(X)]_\mathbb{R}$ is the space of real $(p,p)$-forms on $X$).

Let $\omega^p$ be the $p$ exterior power of $\omega$: we have the following equalities

$$0 = <d\omega^p, S> = <\omega^p, d_{p,p}S> = <\omega^p, T>,$$

where $<,>$ denotes the pairing between currents and forms and the second equality follows from Stokes theorem. Since $T$ is strictly positive on transverse forms we have that the support of $T$ has to be contained in $A$; thus $\mathcal{H}^{2p}(\text{supp}T) = 0$.

The proposition follows now immediately by the next lemma.

**Lemma (3.4.1).** Let $T$ be a weakly positive current of bidimension $(p, p)$ on a complex manifold $X$ which is $\partial \bar{\partial}$ closed (i.e. $\partial \bar{\partial}(T) = 0$). If $\mathcal{H}^{2p}(\text{supp}T) = 0$ then $T \equiv 0$.

**Proof.** If $T$ is a locally flat current this is the support theorem in [Fe], (4.1.20), p.378. The current in our hypothesis is not necessarily locally flat however the proof in [ibidem] applies in the same way after we notice the following fact (which will substitute (4.1.18) in the proof of ibidem). A positive current of bidegree $(m, m)$ on $\mathbb{C}^m$ can be identified with a plurisubharmonic distribution on $\mathbb{C}^n$ (i.e. $T = f \Omega$ where $\Omega$ is a volume form and $f$ is the distribution). Since $\partial \bar{\partial}(T) = 0$ also $\partial \bar{\partial}f = 0$; thus $f$, and $T$ as well, is smooth. Smooth positive currents are flat and for them the support theorem holds by [ibidem].

The Moishezon manifolds in the examples 1) and 2) have the nice property of proposition (3.3).
In fact the subvariety $Y \subset X$ along which one must blow-up $X$ in order to get a projective manifold can be contracted to a point in a projective manifold $Z$; i.e. there is an holomorphic map $\rho : X \to Z$ which is an isomorphism on $X \setminus Y$ and such that $\rho(Y)$ is a point.

But this implies that on $X$ there exists a closed $(1,1)$-form $\omega$ which is positive defined on every point of $X \setminus Y$ and semipositive on $Y$; in fact one just take the curvature of a line bundle $L$ which is the pull back of a very ample line bundle on $Z$. Note also that $\mathcal{H}^{2p}(Y) = 0$ for every $p > \dim Y$.

**Corollary (3.5).** The Moishezon manifolds of the examples 1) and 2) are $p$-kähler for $p > \dim Y$. In particular we have, for every integer $p$, a non kähler Moishezon manifold which is $p$-kähler.

We like finally to ask the following:

**Conjecture** Let $X$ be a Moishezon manifold which can be made projective after blowing up a finite number of smooth subvarieties of dimension $< p$. Then $X$ is $p$-kähler.

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