Dynamical phase transition of two-component Bose–Einstein condensate with nonlinear tunneling in an optomechanical cavity-mediated double-well system

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Abstract
We investigate the dynamical phase transition of two-component Bose–Einstein condensate with nonlinear tunneling, which is trapped inside a double-well and dispersively coupled to a single mode of a high-finesse optical cavity with one moving end mirror driven by a single mode standing field. In the mean-field approximation, the Hamiltonian of the system with nonlinear tunneling is characterized by the population difference of the particles and relative phase in double wells. The characteristics of the system’s stability points are determined by the corresponding Hessian matrices. However, the number of the system’s stability points depends on the roots of the first derivative of the energy. In the absence of the nonlinearity, the Hamiltonian of the system is a binary quadratic equation. When there is nonlinear tunneling, the Hamiltonian of the system is a binary quartic equation. Therefore, from a mathematical point of view, it can be found that the roots of the equation will increase, which means that the number of stability points in the system will increase. We also build the phase portrait from classical analysis. Considering the strong pair tunneling case, we confirm that nonlinear tunneling interaction leads to an increment of stability points and enriches the phase diagram of the system. We further analyze the phase portrait with the moving end mirror. It is shown that the semiclassical dynamics of the system undergoes a bifurcation of stability points with an increase in the coupling strength or detuning. In other words, the system experiences a phase transition.

Keywords: dynamical phase transition, nonlinear tunneling, optomechanical cavity-mediated double-well

(Some figures may appear in colour only in the online journal)

1. Introduction
The system of Bose–Einstein condensates (BECs) in double-well (DW) potentials is an important platform for quantum manipulation due to its highly controllable experimental parameters [1, 2], which has great potential to demonstrate a wide range of fundamental quantum phenomena with regard to manipulating the tunneling dynamics governed by the local two-body interactions and single particle tunneling strength between wells. Some exciting rich phase-space dynamics in theoretical and experimental studies have been revealed. These include the dynamics of spin–orbit-coupled condensates [3–5],...
the existence of a nonlinear steady state [6], the cross structure of the level [7, 8], nonlinear Landau–Zener tunneling [9], and the nonlinear Josephson oscillation [1, 10]. Most recently, the tunneling probabilities of few bosons [11], nonequilibrium dynamical ion transfer [12], asymmetric many-body loss [13], dynamical phase transition of binary species BECs [14], interaction blockade [15] and interaction-modulated tunneling dynamics [16] have also been explored, respectively.

To obtain strong atom–field nonlinearity and tailor atom–field coupling effectively, two groups succeeded independently in coupling BECs to a single-cavity mode in experiment [17, 18]. In this sense, BEC systems trapped in double well coupled to the cavity fields have been discussed previously. Homodyne measurements [19], the interplay dynamics [20] and the mean-field dynamics [21, 22] of a Bose–Josephson junction, the outcomes of the atom–field nonlinearity [23] are investigated. Nondemolition measurement based on this system [24] have also been proposed. In recent years, optomechanical cavities have emerged and become another ideal and irreplaceable system to study strong matter-field interaction [25]. Such a system demonstrates the interaction between the movable oscillator and the cavity field via the radiation pressure and becomes a new platform for the study of the ground-state cooling of the vibrational modes of a mechanical oscillator [26], coherent quantum noise cancellation [27], steady-state bipartite entanglement and quadrature squeezing [28], bistability [29], electromagnetically induced transparency (EIT) and Fano resonances [30], laser phase noise [31], and the emergences of the entanglement [32, 33]. Given the wealth of effects resulting from the hybrid system of BECs in an optomechanical cavity, it is natural to ask for the new phenomena stemming from an optomechanical cavity-mediated BECs DW system.

Note that nonlinear tunneling can be omitted as it is several orders of magnitude smaller than the linear tunneling strength in the weak interaction range. Actually, new phenomena will occur when one varies the interaction strength from the weak to the strong limit. Correlated pair tunneling was firstly observed in a sample of rubidium atoms in the few-atom and strong coupling limit [34]. It is shown that the two atoms evolve from Rabi oscillations to correlated pair tunneling with the increase in interaction strength [35]. Following from this finding, there have been a great deal of efforts devoted to the Bose–Hubbard model with nonlinear tunneling, which are trapped in a high-finesse optical cavity with a moving end mirror due to the reasons that much more complicated and achievable states can be obtained because of the interplay of intra-species and interspecies interaction of two-component BECs. We find that nonlinear tunneling increases the number of the stability points and enriches the phase diagram of the system. Furthermore, the coupling strength between the cavity and the moving end mirror and the detuning between the pumping field and the moving end mirror can regulate the number of the stability points of the system, and then control the dynamics of the system.

The paper is organized as follows: the Hamiltonian of the system and the dynamical equation are presented in section 2. In section 3, we derive the classical model of the system using mean-field theory. Section 4 is devoted to discussing the stability points and energy contours of the BECs DW. Finally, the conclusion is summarized in section 5.

2. System Hamiltonian and the dynamical equation

We consider an optomechanical cavity consisting of a fixed mirror and a moving end mirror which is driven by a pumping field with frequency $\omega_p$. The frequency of the moving end mirror is $\omega_m$. The optomechanical cavity is coupled to the system which is constructed by two weakly linked condensates trapped in a double well, as is schematically shown.
in figure 1. Each condensate has two-component atoms being two isotopes of an alkali metal, such as $^{85}$Rb and $^{87}$Rb. The numbers of each component are $N$. In the large-detuning and low-excitation limit, atomic spontaneous emission can be neglected. The Hamiltonian of the system can be written as

$$H = H_a + H_F + H_M + H_{couple},$$

where $H_a$ describes the behavior of the atomic modes (BECs) and their interactions with each other, $H_F$ gives the energy of the single mode cavity and its association with the pumping field, $H_M$ is related to the moving end mirror, and $H_{couple}$ accounts for the interaction of the single mode cavity with the mechanical resonator and the atoms. The atom-pair tunneling term (nonlinear tunneling) is also included in this Hamiltonian [38]. In the two-mode approximation [1], the canonical Hamiltonian of BECs DW in the optomechanical mediated cavity reads (assuming $\hbar = 1$)

$$H_a = -\Omega_1 (b_1^\dagger b_1 + b_2^\dagger b_2) - \Omega_2 (c_1^\dagger c_1 + c_2^\dagger c_2) + \frac{V_1}{2} (b_1^\dagger b_1 b_1 b_1 + b_1^\dagger b_1 b_2 b_2) + \frac{V_2}{2} (c_1^\dagger c_1 c_1 c_1 + c_2^\dagger c_2 c_2 c_2) + \frac{V_1'}{2} b_1^\dagger c_1 c_1 c_2 + \frac{V_2'}{2} b_2^\dagger c_2 c_1 c_2 - \frac{S_1}{2} (b_1^\dagger b_1 b_2 b_2 + b_1^\dagger b_1 b_1 b_1) - \frac{S_2}{2} (c_1^\dagger c_2 c_2 c_2 + c_2^\dagger c_1 c_1 c_1).$$

For simplicity, we assume that $\Omega_1 = \Omega_2 = \Omega$, $V_1 = V_2 = V$, $V_1' = V_2' = V'$ [10], $S_1 = S_2 = S$, the subscripts 1 and 2 represent the localized modes in the left and right potential wells, respectively. $b_i^\dagger (b_i)$ and $c_i^\dagger (c_i)$ are the creation (annihilation) operators of the two atom modes. $\Omega$ means the parameter of tunneling between two modes. The parameter $V$ ($V'$) denotes the interaction between atoms of the homogeneous (heterogeneous) species. $S$ is the coupling strength for the atom-pair tunneling. The two-mode model is assumed two stability wave functions are such that the two lowest states are closely spaced and well separated from higher levels of the potential, and that many-particle interactions do not significantly a change in DW [1]. An atom-pair tunneling term does not significantly influence the wave function of the atoms in the DW system, two-mode approximation is still valid and a large number of studies are conducted based on this assumption [36, 38]. As for the optomechanical cavity-mediated system, the intracavity light intensity and frequency depend on the length of the cavity, which will be changed due to the vibration of the moving end mirror at the equilibrium position. However, when the Bose–Josephson junction is coupled to an optomechanical cavity, the frequency of the cavity only can slightly tilt the DW under certain parameter conditions and does not change the system structure, so the two-mode approximation is still fulfilled. Correspondingly, the Hamiltonian of the single-mode optical field is

$$H_F = \omega_c a^\dagger a + \eta(t) e^{-i\omega_p t} d^\dagger + \eta(t)^* e^{i\omega_p t} a,$$

where $\omega_c$ and $\omega_p$ are the cavity and pump frequencies, respectively. $\eta(t)$ represents the optical amplitude of the pumping field. Here, we assume that the amplitude of the pumping field varies slowly, that is $|\eta|/\eta \approx \omega_p$ [39]. The Hamiltonian of the moving end mirror $H_M$ can be read as [40]

$$H_M = \omega_m d^\dagger d,$$

where $\omega_m$ is the frequency of the moving end mirror. In the two-mode approximation, due to the coupling strength between the cavity mode and the atomic tunneling being much smaller than the overlaps between the atomic modes and the cavity mode, that is $J_2(J_2') \ll J_1(J_1')$. Therefore, we have dropped some terms $u_0 a^\dagger a J_2(b_1^\dagger b_2^\dagger + b_1^\dagger b_2 b_1)$ and $u_0 a^\dagger a J_2'(c_1^\dagger c_2 + c_2^\dagger c_1)$ [21]. Now, we rewrite the Hamiltonian $H_{couple}$ as [40, 41]

$$H_{couple} = U_0 a^\dagger a (J_1 m_1 + J_2 n_2 + J_1 m'_1 + J_2 n'_2) - G_0 (a^\dagger + a) (d^\dagger + d),$$

where $U_0 = \omega_0^2 / \omega_a$ is the light shift per photon, $\omega_0$ is the atom–field coupling strength at an antinode. $\delta_0 = \omega_c - \omega_p$ is the far-off detuning between the atoms and field frequency [40, 42], $J_1(J'_1)$ and $J_2(J'_2)$ account for the overlap between the atomic modes and the cavity mode [39], $G_0 = \omega_0 / \sqrt{\hbar M_\omega} = \omega_0 / M_\omega^{1/2}$ is the coupling strength between the cavity and the moving end mirror. In this section, we do not discuss the case that $J_1(J'_1) = J_2(J'_2)$ because it illustrates that the atoms do not interact with the single-mode field. In other words, the cavity cannot influence the distribution of the atoms in DW. Therefore, we focus on another case, $J_1(J'_1) \neq J_2(J'_2)$. In the rotating-wave approximation, the total Hamiltonian leads to coupled quantum Langevin equations for the annihilation operators of BEC, the cavity and the moving end mirror, viz.,

$$i \dot{b}_1 = -\Omega b_2 + V b_1 b_1 b_1 + \frac{V'}{2} c_1^\dagger c_1 b_1 + J_1 U_0 a^\dagger a b_1 - S b_2^\dagger b_2,$$

$$i \dot{b}_2 = -\Omega b_1 + V b_1 b_2 b_2 + \frac{V'}{2} c_2^\dagger c_2 b_2 + J_2 U_0 a^\dagger a b_2 - S b_2^\dagger b_2,$$

$$i \dot{c}_1 = -\Omega c_2 + V c_1^\dagger c_1 c_1 + \frac{V'}{2} c_1^\dagger c_1 b_1 + J'_1 U_0 a^\dagger a c_1 - S c_2^\dagger c_2,$$

$$i \dot{c}_2 = -\Omega c_1 + V c_2^\dagger c_2 c_2 + \frac{V'}{2} b_2^\dagger b_2 c_2 + J'_2 U_0 a^\dagger a c_2 - S c_2^\dagger c_2,$$

$$i \dot{a} = [\omega_c + U_0 (J_1 m_1 + J_2 n_2 + J'_1 m'_1 + J'_2 n'_2)] a - G_0 d - i s a + \eta(t) e^{-i\omega_p t},$$

$$i \dot{d} = \omega_m d - G_0 a.$$
the varying pump frequency to regulate the property of the BJJ.

3. The model of the system

Under the mean-field approximation, we consider atomic and photonic operators to be classical quantities, namely $b_1 = \sqrt{N_1} e^{it_1}$, $b_2 = \sqrt{N_2} e^{it_2}$, $c_1 = \sqrt{N'_1} e^{i\theta_1}$, $c_2 = \sqrt{N'_2} e^{i\theta_2}$, $d = \beta$, $\theta_1$, $\theta_2$ and $\theta_1'$, $\theta_2'$ describe the corresponding phase of the atom. Moreover, $N_{1,2}$ and $N'_{1,2}$ are the total atomic numbers of two species in two wells, respectively. In the model, we assume that the total atomic numbers of two species are equal, i.e., $N_b = N_c = N$ [43].

In the system, it is clear from equation (6) that the relaxation time scale of the cavity mode is of the order of $1/\kappa$, $\kappa \sim 2\pi \times 10^7$ Hz, which is much shorter than the oscillation period of a bare BECs DW [44]. The relaxation time scale of the BECs DW is of the order of $1/\Delta$, usually, $\Delta \sim 2\pi \times 10^{1-2}$ Hz in the real experiment [2]. The moving end mirror with the frequency $\omega_m$ is of the order of $2\pi \times 10^7$ Hz [45]. This implies that the cavity field follows the motion of the condensates adiabatically [46]. Thus, from equation (6), one reads

$$\langle a \rangle = \alpha(t) = \frac{\eta(t)e^{-i\omega_m t}}{G_0^2 - [\omega_e + U_0(N_1 + N_2) + J_2(N_1' + N_2')] - i \kappa - \omega_m]}(\omega_m - \omega_p),$$

and the mean photon number is

$$\langle a^\dagger a \rangle = |\alpha(t)|^2 = \frac{\eta(t)^2 \Delta^2}{G_0^2 - 2G_0^2 \Delta(\Delta - T + T') + \Delta^2 (\Delta - T - T')^2 + \kappa^2},$$

where $T = \delta U_0(N_1 - N_2)/2$, $T' = \delta U_0(N'_1 - N'_2)/2$, $\Delta = \omega_p - \omega_e - (J_1 + J_2)N_0/2 + (J'_1 + J'_2)N'_0/2$, $\Delta' = \omega_p - \omega_m$, $\delta = J_1 - J_2$, $\delta' = J'_1 - J'_2$, and $\delta = \delta'$. Therefore, we rewrite equation (8) as

$$\alpha(z_b, z_c, t) = \frac{A(t)^2 E^2}{D^3 - 2D^2E(Z_b + Z_c - B) + E^2 [(Z_b + Z_c - B)^2 + C^2]},$$

where $A(t)$, $B$, $C$, $D$, $E$ are all scaled by $\delta NU_0/2$. We regard $A(t)$, $B$, $C$ as the reduced pump strength, reduced detuning and reduced loss rate, respectively. Additionally, we may understand $D$ as the reduced coupling strength between the cavity and the moving end mirror. It can be found from equation (9) that the mean photon number is a Lorentzian at $z_b + z_c = B + D^2/E$ with a width $2C$, which is a function of $z_b$ and $z_c$. This is because atoms and mirror are forced to vibrate due to the cavity mode. The addition of the mirror makes the peak position of the photon number distribution move to the left by $D^2/E$.

Substituting equation (9) into equation (6) and defining the relative phases of the atoms as $\phi_b = \theta_1 - \theta_2$ and $\phi_b' = \theta_1' - \theta_2'$, equation (6) can be rewritten in terms of $z_b(z_c)$ and $\phi_b(\phi_b')$ as

$$\phi_b = \frac{2z_b}{\sqrt{1 - z_b^2}} \cos \phi_b + \frac{r_{bc}z_c}{2} + \Lambda z_b \cos(2\phi_b)$$

$$+ \frac{\delta U_0}{2\Omega} |\alpha|^2,$$

$$z_b = -\frac{1 - z_b^2}{2} \sin \phi_b - \Lambda(1 - z_b^2) \sin(2\phi_b),$$

$$\phi_c = \frac{2z_c}{\sqrt{1 - z_c^2}} \cos \phi_c + \frac{r_{bc}z_c}{2} + \Lambda z_c \cos(2\phi_c)$$

$$+ \frac{\delta U_0}{2\Omega} |\alpha|^2,$$

$$z_c = -\frac{1 - z_c^2}{2} \sin \phi_c - \Lambda(1 - z_c^2) \sin(2\phi_c),$$

where the time has been rescaled in units of the Rabi oscillation time $1/(2\Omega)$, $2\Omega t \rightarrow t$. $r_b = r_c = r = NV/2\Omega$ denotes the ratio of the homologous atoms, $r_{bc} = NV/2\Omega$ denotes the ratio of the interspecies atoms, and $r \geq r_{bc}$. $\Lambda = S/2\Omega$ expresses the ratio of nonlinear tunnelling. We further define a Hamiltonian as a function of two conjugate variables $z_b(n = b, c)$ and $\phi_b(n = b, c)$, i.e., $z_n = \frac{\partial H}{\partial \phi_b}$, $\phi_n = \frac{\partial H}{\partial \phi_b}$, therefore the Hamiltonian is

$$H_b(z_b, \phi_b, t) = -\frac{1 - z_b^2}{2} \cos \phi_b + \frac{r_{bc}z_c}{2} + \frac{r_{bc}z_c}{2}$$

$$- \Lambda z_b^2 - \Lambda(1 - z_b^2) \cos^2 \phi_b$$

$$\frac{\delta U_0}{2\Omega} F(z_b, z_c,t),$$

$$H_c(z_c, \phi_c, t) = -\frac{1 - z_c^2}{2} \cos \phi_c + \frac{r_{bc}z_c}{2} + \frac{r_{bc}z_c}{2}$$

$$- \Lambda z_c^2 - \Lambda(1 - z_c^2) \cos^2 \phi_c$$

$$\frac{\delta U_0}{2\Omega} F(z_b, z_c,t),$$

with

$$F(z_b, z_c,t) = \frac{A(t)^2}{C} \arctan \frac{-D^2 + E(z_b + z_c - B)}{CE}. $$

The first five terms of the Hamiltonian in equation (11) are the Hamiltonian of a bare BECs DW. Among them the first three terms describe the energy cost due to the phase twisting between the two condensates, the interaction between atoms of
the homogeneous species and the interaction between atoms of the heterogeneous species, respectively. Compared with [22], the added fourth and fifth terms of Hamiltonian indicate atom-pair tunneling in a DW potential [38], which is caused by the nonlinear tunneling coupling and the phase twisting between the two condensates. In the standard Bose–Hubbard model, the nonlinear tunneling terms are neglected as they are small compared with the hopping energy and the on-site interaction. However, equation (11) reveals that the nonlinear tunneling changes the distribution of the energy contour of the system and influences the dynamical of the atoms. In addition, the last term of the two Hamiltonian is regarded as the cavity-field-induced tilt [21]. If the pump strength changes with time, the Hamiltonian can be made explicitly time dependent. However, in this work, we will focus on the case that the pump strength is a constant, viz., \( \eta(t) \equiv \eta \). So, the Hamiltonian is preserved in time. In a simple mechanical analogy, \( H_b(H_c) \) describes a nonrigid pendulum with a tilted angle \( \phi_b(\phi_c) \) and length proportional to \( \sqrt{1-z_b^2} \) \((\sqrt{1-z_c^2})\) which decreases with the angular momentum \( z_b(z_c) \) increasing. But in this paper, \( H_b(H_c) \) describes the stack of nonrigid pendulum, which is tilted angle \( 2\phi_b(2\phi_c) \) and length proportional to \( \frac{A}{2}(\sqrt{1-z_b^2})^2 \).

As is well known, the energy of the system can be obtained for the conservative system. It has been demonstrated that the eigenstates of the system are related to the stability points of phase-space level curves. Next, we will explore the dynamics of a BECs DW in the perspective of the phase-space level curves. First of all, we need to figure out the stability points of the system by the equations \( \frac{\partial H}{\partial z_b} = 0, \frac{\partial H}{\partial z_c} = 0 \) \((n = b, c)\). The second equation suggests that \( \phi = 0 \) or \( \phi = \pi \). Then, we can get the following expression from the first equation:

\[
\begin{align*}
 f_1(z_b) &= \frac{z_b}{\sqrt{1-z_b^2}} + r_b z_b + \frac{r_b c z_c}{2} + \Lambda z_b \\
 &+ \frac{D^4 - 2D^2E(z_b + z_c - B) + E^2[(z_b + z_c - B)^2 + C^2]}{\bar{A} E^2} = 0, \\
 f_2(z_c) &= \frac{z_c}{\sqrt{1-z_c^2}} + r_c z_c + \frac{r_b c z_c}{2} + \Lambda z_c \\
 &+ \frac{D^4 - 2D^2E(z_b + z_c - B) + E^2[(z_b + z_c - B)^2 + C^2]}{\bar{A} E^2} = 0, \\
 f_1(z_b) &= \frac{z_b}{\sqrt{1-z_b^2}} + r_b z_b + \frac{r_b c z_c}{2} + \Lambda z_b \\
 &+ \frac{D^4 - 2D^2E(z_b + z_c - B) + E^2[(z_b + z_c - B)^2 + C^2]}{\bar{A} E^2} = 0, \\
 f_2(z_c) &= \frac{z_c}{\sqrt{1-z_c^2}} + r_c z_c + \frac{r_b c z_c}{2} + \Lambda z_c \\
 &+ \frac{D^4 - 2D^2E(z_b + z_c - B) + E^2[(z_b + z_c - B)^2 + C^2]}{\bar{A} E^2} = 0, \\
 f_3(z_b) &= \frac{z_b}{\sqrt{1-z_b^2}} + r_b z_b + \frac{r_b c z_c}{2} + \Lambda z_b \\
 &+ \frac{D^4 - 2D^2E(z_b + z_c - B) + E^2[(z_b + z_c - B)^2 + C^2]}{\bar{A} E^2} = 0, \\
 f_4(z_c) &= \frac{z_c}{\sqrt{1-z_c^2}} + r_c z_c + \frac{r_b c z_c}{2} + \Lambda z_c \\
 &+ \frac{D^4 - 2D^2E(z_b + z_c - B) + E^2[(z_b + z_c - B)^2 + C^2]}{\bar{A} E^2} = 0,
\end{align*}
\]

(13)

here \( \bar{A} = \delta U_0 A(t)^2/2\Omega \); other stability points of the system are discussed using the numerical method, which can be worked out by setting \( \frac{\partial H}{\partial z_b} = 0, \frac{\partial H}{\partial z_c} = 0 \) \((n = b, c)\) when \( \phi \neq 0 \) and \( \phi \neq \pi \). For simplicity, we concentrate on the case that \( z_b = z_c = z \) and \( \phi_b = \phi_c = \phi \). The characteristics (minimum, saddle, or maximum) of the possible stability points determined by the corresponding Hessian matrices are analyzed, i.e., the square matrix of second-order partial derivatives of the Hamiltonian on \( z_n \) and \( \phi_n \).

4. Stability points and energy contours of the Bose–Josephson junction

The Bose–Josephson junction is developed from the superconducting Josephson junction. A so-called superconducting Josephson junction is a tunnel junction formed by adding a layer of insulator between two superconductors. Josephson found that when the insulating layer is very thin, the electron pairs can form an electric current across the insulating layer, and there is no voltage at both ends of the tunneling junction. The Josephson effects can be explained successfully by the tunneling principle of quantum mechanics. The two superconductors are regarded as two potential wells, and the insulating layer is equivalent to constructing a barrier between the two wells. When the insulating layer is thick, the pairs of electrons in the two conductors are independent. As the insulating layer becomes thin, the wave functions of the two conductors begin to overlap. The overlap provides a quantum correlation for the electron pairs in different conductors, so that the electron pairs can successfully pass through the insulating layer to reach another conductor by the tunneling effect, which forms a superconducting current. This model is called a superconducting Josephson junction. Instead of conductors, BECs are used to study the tunneling of the atoms in DW. Thereby, such a model is also called a Bose–Josephson junction.

Without the moving end mirror. In order to focus on the impact of the nonlinear tunneling on the system dynamics, we study the energy contours and the stability points of the system with \( D = 0, E = \omega \mu \). As can be seen from equations (11) and (13), nonlinear tunneling contributes a term linear in \( z^2 \) to the Hamiltonian \( H_b(H_c) \), in turn, a term linear in \( z \) to the functions \( f_{1,2} \). It is natural to expect that this term can produce new roots and potentially make the phase portrait of the system quantitatively or even qualitatively different from that without nonlinear tunneling. For the case of interplay between the cavity and the BECs, the roots of equation (13) have to be solved numerically. When the nonlinear tunneling strength is small, the corresponding phase diagram is similar to the one without a nonlinear tunneling term [22], as shown in figure 2(a). There are three stability points (two minima and one saddle point) along the line \( \phi = 0 \), and five stability points (three maxima and two saddle points) along the line \( \phi = \pi \). In the strong pair tunneling case, the phase diagram of the system undergoes tremendous change, as illustrated in figure 2(b), and the map of
the corresponding stability points are given in figures 2(c)–(e). At first, a stability point (a maximum) appears along near the line $\phi = \pi$; its specific location can be seen in figure 2(d) (the stability point appears along near $\phi = \pi$ from figure 2(b)). Secondly, the system experiences a transition from oscillating-phase-type self-trapping to running-phase-type self-trapping [47] near $z = -1$ and $z = 1$ along the line $\phi = \pi$, and the stability points have a transition from two saddle points to two minima and from one maximum to a saddle point.

In figure 3, we plot the time evolution of the population imbalance with different nonlinear tunneling parameters. Initially, assuming that $(\phi(0), z(0)) = (0, -0.6)$, it is obvious that the system presents a Josephson oscillation evolution, which has a stable amplitude. However, as the nonlinear tunneling effect increases, the period of the oscillation becomes smaller. Therefore, one can come to a conclusion that the nonlinear tunneling cannot influence the population of the atoms in DW, but speed up the tunneling of atoms. In other words, we can experimentally control the tunneling rate of atoms in a DW system by changing the nonlinear tunneling coupling strength.

With the moving end mirror. In order to compare the difference of the dynamical of BECs DW between the moving end mirror and without the moving end mirror, we assume $D = 0$, $E \approx \omega_p$. First of all, we numerically solve the roots of equation (13) with different parameters, as shown in figures 4(a)–(d). Figure 4(a) indicates that the stability points of system can move upwards with the increasing coupling strength between the cavity and the mirror along the line $\phi = 0$, when the detuning between the pumping field and the mirror has a fixed value. However, while the coupling strength between the cavity and the mirror has a fixed value, an opposite result appears with an increase in detuning, as shown in figure 4(b). So, in the specific sets of parameters, we can flexibly control the number of stability points. From figures 4(c) and (d), we can obtain the same results along the line $\phi = \pi$. As is well known, the loss of stability of semi-classical stability points is associated with an entanglement in the steady state of the full quantum system, the semiclassical dynamics of the system undergoes a bifurcation of the stability point corresponding to the quantum steady state, and the maximum entanglement occurs at the parameter values about

![Energy contours of a Bose-Josephson junction](image_url)

**Figure 2.** Energy contours of a Bose-Josephson junction, with the nonlinear tunneling term (a) $S = 0.1$, (b) $S = 3.87$. (c), (d) and (e) Gradient of the energy along the lines $\phi = 0, \phi = \pi$ and $\phi = \pi$, respectively. The other parameters are $NV/(2\Omega) = r = 3, NV^\prime/(2\Omega) = r_{bc} = 0.1, \bar{A} = 0.02, B = -0.65$, and $C = 0.07$. 

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bifurcation of the stability point in a dissipative many-body system [48].

It can be found that the number of stability points can be changed by increasing the coupling strength and detuning of the moving end mirror in figure 4. In other words, the stability points diverge under certain parameters. In fact, we discuss the bifurcation of the stability points in detail in the phase diagram. The relationship between the bifurcation of the stability points and the quantum entanglement of the system has been discussed in [48]. Therefore, we conjecture that adjusting the coupling strength and detuning between the moving end mirror and the pumping field can control entanglement in the steady state of the full quantum system well.

As the stability point of the system has a great relationship with the phase diagram, we further plot the phase diagram with different coupling strengths and detunings between the cavity and the mirror to discuss the occurrence of the bifurcation of the stability point. The left-hand column shows the changes of the phase diagram with the increasing coupling strength between the cavity and the mirror for a fixed detuning in figure 5. There are three stability points for \( \phi = 0 \), one stability point near by \( \phi = \pi/2 \), and three stability points for \( \phi = \pi \), when the small coupling strength between the cavity and the moving end mirror, as shown in figure 5(a). The typical characteristic of the level curves undergoes tremendous changes with the coupling strength increasing. For example, in figure 5(c), there are one stability point for \( \phi = 0 \), one stability point near by \( \phi = \pi/2 \), and one stability point for \( \phi = \pi \) while the coupling strength between the cavity and the moving end mirror is strong. This means that the system experiences a stability point bifurcation for certain choices of the coupling parameters. From equations (11), (13) and figure 4, one can see that the distribution of the phase diagram is symmetric about \( z = 0 \) without the cavity-field-induced tilt. However, this symmetry is broken when the coupling strength between the atoms and the cavity with the moving end mirror exists. In this case, there are three stability points along the line \( \phi = 0 \) or \( \phi = \pi \) in this system, which satisfies \(-B + D^2/E < 0\). Additionally, there is one stability point when \(-B + D^2/E > 0\). In addition, we can obtain that the absolute value of \(-B + D^2/E\) represents the distance of two maxima (two minima) in phase diagram from figure 4 (figure 5). One of the maxima (minima) is localized in \( z = 0 \) and another location of the maximum (minimum) depends on the absolute value of \(-B + D^2/E\). Therefore, when adding the moving end mirror, we can adjust the values of \( D \) and \( E \) to control the distance between two maxima (minima) in the phase diagram and the number of stability points of the system. The right-hand column shows that the phase diagram is changed with the increased detuning between the pumping field and the mirror, when the coupling strength has a fixed value in figure 5. Obviously, one gets an

![Figure 3](image-url)
Figure 4. The gradient of the energy along the line $\phi = 0$ and $\phi = \pi$, respectively. The left-hand column shows changes in the location and number of fixed points with parameters $E = 0.1$ and $D = 0.1$ (black dotted lines), $D = 0.2$ (red dashed lines), $D = 0.4$ (blue solid lines), along the line $\phi = 0$ in figure 4(a) and $\phi = \pi$ in figure 4(c). The right-hand column shows changes in the location and number of fixed points with parameter $D = 0.3$ and $E = 0.1$ (black dotted lines), $E = 0.3$ (red dashed lines), $E = 1.5$ (blue solid lines), along the line $\phi = 0$ of figure 4(b) and $\phi = \pi$ of figure 4(d). $S = 3.87$. The same parameters as in figure 2.

Figure 5. Energy contours of a Bose–Josephson junction. The left-hand column shows changes of the phase diagram with $E = 0.1$ and different parameter $D$ in (a) $D = 0.1$, (b) $D = 0.2$, and (c) $D = 0.4$. The right-hand column shows changes of the phase diagram with $D = 0.3$ and different $E$ in (d) $E = 0.1$, (e) $E = 0.3$, and (f) $E = 1.5$. The same parameters as in figure 4.
opposite variation tendency compared with the result of the left-hand column.

The output of the cavity mode carries a lot of information about the population of the atoms between the two wells as it leaks out of the cavity. From equations (9)–(11), we note that the distribution of photon numbers is influenced by two factors. One is the initial conditions of system evolution. Another is the evolution of the conjugate variables $z$ and $\phi$ of the energy, which is determined by the initial conditions. The different energy curves correspond to different distributions of photon numbers. For example, when the system is in a stability state, the distribution of photon numbers is constant. When the state of the system evolves along the energy curves around the stability point, the distribution of photon numbers has a small change during the period. When the evolved state of the system is a Josephson oscillation over the range of the population imbalance, the distribution of photon numbers has an enormous change. Compared with condition of no moving end mirror, the coupling strength between the cavity and the moving end mirror and the detuning between the pumping field and the moving end mirror make the energy curve change, the distribution of photon number is also changed. As shown in figures 6 and 7, we plot the number of intracavity photons with different coupling strengths between the pumping field and the moving end mirror, and different detuning between the cavity and the moving end mirror respectively. It can be found that although the variety of detuning and coupling strength are small, the change of the difference in the output of the cavity has an enormous difference. Compared with [21], Lorenzian appears six peaks in figures 6 and 7, as can be seen from equations (10) and (11) due to the nonlinear tunneling term. In addition, the coupling strength and the detuning between the cavity and mirror only change the distribution of photon numbers, but they do not influence the maximum of the one.

Figure 6. Intracavity photon number $|\alpha|^2$ (in units of $2\Omega/(\delta U_0)$) versus the reduced time $2\Omega t$ with different parameters (a) $D = 0.1$, (b) $D = 0.2$ and (c) $D = 0.4$. The initial conditions are $(\phi(0), z(0)) = (0, -0.6), E = 0.1, S = 1.37$. The same parameters as in figure 2.
5. Conclusion

In this paper, we investigate two-component BECs DW in the optomechanical cavity with nonlinear tunneling interaction and moving end mirror. We use the mean-field method to obtain the dynamical equation of BECs DW based on the two-mode approximation and find that the model exhibits abundant dynamical information on the BECs DW. The introduction of nonlinear tunneling leads to an increase in the number of the system’s stability points along near $\phi = \pi/2$, and the distribution of photon numbers is very different from ones without the nonlinear tunneling term, which makes the phase diagram of the system more rich. In addition, when the nonlinear tunneling interaction strength increases, the distribution period of the number of particles becomes smaller and smaller. It is clear that the moving end mirror has little effect on the phase diagram of the system and the population of the atoms between the two traps, but the coupling strength of the moving end mirror can regulate the number of the stability points as the degree of freedom of the system, so can the detuning. We control a bifurcation of the stability points of the system by changing the parameters of the mirror-cavity interaction. This is also very helpful for the study of the entanglement of the system.

Experimentally, it is a mature technology that BECs are trapped into an intracavity optical lattice [17, 18]. Meanwhile, the coupling of optical and movable mirror degrees of freedom can be performed in the experiments [49, 50]. Therefore, the model we propose may be experimentally realized if one integrates the process of the two points mentioned above. This work may provide a possibility for corresponding research and the application of BECs in an optomechanical cavity-mediated DW system beyond the scope of the current experiment.

Figure 7. Intracavity photon number $|\alpha|^2$ (in units of $2\Omega/(\delta U_0)$) versus the reduced time $2\Omega t$ with different parameters (a) $E = 0.1$, (b) $E = 0.4$ and (c) $E = 0.9$. The initial conditions are $(\phi(0), z(0)) = (0, -0.6)$, $D = 0.3$, $S = 1.37$. The same parameters as in figure 2.
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