Hydroelastic response of three-layered beam resting on winkler foundation

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Abstract. The hydroelastic oscillations problem of the sandwich beam resting on elastic foundation and interacting with the pulsating fluid layer was investigated. The three-layered beam with an incompressible lightweight filler was considered. The broken normal hypothesis for the three-layered beam and the model of viscous incompressible fluid, as well as Winkler model for elastic foundation, was chosen to study hydroelastic problem. A mathematical model of the considered mechanical system consists of dynamic equations of the three-layered beam with incompressible lightweight filler and the Navier-Stokes equations with continuity equation for the pulsating viscous fluid layer. The no-slip conditions and pressure coincidence at the edges with the given pressure into edges cavities were selected as boundary ones. The plane problem of hydroelastic beam bending oscillations for the regime of steady harmonic ones were considered. The solution of the hydroelastic problem was carried out by the perturbations method using proposed small parameters of the hydroelastic problem. As a result, the laws of three-layered beam elastic deflections and the hydrodynamic parameters of pulsating viscous fluid layer were obtained.

Keywords: Hydroelastic oscillations, Sandwich beam, Viscous fluid, Pulsating pressure.

1. Introduction
Nowadays, the multi-layered elastic elements of structures are widely used in the civil construction, aerospace, mechanical and energy engineering and other industries. Therefore, the static and dynamic study of elastic multilayer beams and plates are of theoretical and practical interest. For example, a historical review of different approaches to analysing the behaviour of multilayer structural elements based on zig-zag theories is carried out in [1]. Reference [2] is devoted to the approach to studying layered beams and plates with incompressible filler based on the broken normal hypothesis. The static and dynamic problems of layered elements of structures with compressible filler under various loadings are considered in this monograph too. References [3,4] consider the deformation of sandwich and three-layered elastoplastic beam with a compressible filler in a temperature field under the action of uniformly distributed [3] and sinusoidal local loads [4].

The static and dynamic problems of elastic elements of structures resting on an elastic foundation are importance for theory and practical applications. Reference [5] is one of the first fundamental studies devoted to this problem. Reference [6] considers the behaviour of beams, plates and shells mounted on an elastic base. In this study, an elastic foundation is considered to be as a single- or double-layer model whose properties are described by two or more elastic characteristics. Contemporary researches on development of elastic foundation models, as well as various approaches to analytical and numerical research of beams and plates interaction with elastic foundation, are considered in review [7]. References [8-10] are devoted to studying the statics [10] and dynamics...
problems of three-layered beams [8] and plates [9] resting on elastic foundation under the local and
distributed loads of various natures. However, the hydroelasticity problems of beams and plates are of
great importance, too. Reference [11] deals with the axisymmetric free oscillations of the circular plate
interacting with water. It is one of the first investigations on hydroelasticity. The circular plate
vibration on a free surface of an ideal incompressible liquid contained in a rigid cylinder is made in
[12]. Reference [13] considers the dynamic behaviour of annular channel elastic walls interacting with
layer of an ideal fluid. Reference [14] is devoted to the vibration response study of infinite length plate
interacting with viscous liquid layer. The hydroelastic response of the elastic-fixed channel wall
interacting with viscous liquid layer was investigated in [15]. The hydroelastic problem of vibrating
cantilever beam, plunged into Newtonian fluid, is solved in [16]. The hydroelastic oscillations of discs
interacting with viscous incompressible liquid layer between them are studied in [17]. The analogous
study for the two vibrating channel walls of finite sizes is made in [18]. Reference [19] is devoted to
the investigation of hydroelastic oscillations of the piezo-electric beam in a viscous liquid flow. The
problem of the bending hydroelastic oscillations of the plate interacting with a viscous liquid pulsating
layer was considered in [20]. The hydroelastic oscillations of the plate, resting on Winkler foundation
was studied in [21] and the forced oscillations of the elastic fixed stamp and the plate, resting on
Pasternak foundation was investigated in [22].

The investigations of elastic three-layered plate dynamic interaction with a liquid are of theoretical
and practical interest for applied mechanics problems and studying dynamic mechanical systems. But
there is a shortage of studies in that field. For example, the free hydroelastic oscillations of composite
plates interacting with ideal liquid are considered in [23]. The forced hydroelastic oscillations of the
three-layered circular plate, interacting with viscous liquid are studied in [24]. However, the
hydroelastic response of a three-layered beam resting on elastic foundation is of theoretical and
practical interest, too.

2. Statement of the Problem
Let us consider the narrow channel formed by two parallel walls and a pulsating liquid between them
(Fig.1). The channel’s length is $2\ell$ and we will study the plane hydroelastic problem. The upper
channel wall is an absolutely solid body. The bottom channel wall is an elastic three-layered beam
resting on elastic foundation. We propose the model of viscous incompressible fluid for the liquid
layer and Winkler model for an elastic foundation. The three-layered beam is the package consisting
of outer layers 1, 2, their thickness being $h_1$ and $h_2$, and incompressible lightweight filler 3. The filler
thickness is $2c$. The broken normal hypothesis for the three-layered beam is accepted, i.e. Kirchhoff
hypothesis is valid for outer layers, as well as the normal in the beam filler remains straight and
turning by the angle $\varphi$ [2]. Let us introduce Cartesian coordinate $Oxz$, which center connecting with
the beam filler center in an unperturbed state. The rigid diaphragms, hindering the relative layers shift,
but not impeding the deformation from its plane, are supposed to be situated at the beam edges [2].
The three-layered beam oscillations are caused by pulsating liquid pressure, while deformations of the
plate are considered to be small. The three-layered beam is simply supported at the edges. The liquid
layer thickness in the channel is $h_0 \ll \ell$. The liquid pressure at the right and left edges consists of the
constant component $p_0$ and the pulsating component $p^* (\omega t)$. We can consider only the steady-state
harmonic oscillations, because the liquid layer damping due to its viscosity leads to the transient
processes quick going down [25]. According to [26] the inertial forces in a longitudinal direction are
not considered and we study the three-layer beam bending oscillations only.
Figure 1. The narrow channel with a three-layered wall resting on Winkler foundation.

3. The Theory and Solution

The pulsating component pressure law at the channel edges is presented as:

$$ p^* (\omega t) = p_n f_p (\omega t), \quad f_p (\omega t) = \sin \omega t \quad (1) $$

where $p_n$ is the amplitude of pressure pulsating; $\omega$ is the pulsating frequency.

The dynamic equations of three-layered beam resting on Winkler foundation are presented in the form of

$$ a_1 \frac{\partial^2 u}{\partial x^2} + a_6 \frac{\partial^2 \phi}{\partial x^2} - a_7 \frac{\partial^3 w}{\partial x^3} = q_x, \quad a_6 \frac{\partial^2 u}{\partial x^2} + a_7 \frac{\partial^3 \phi}{\partial x^3} - a_1 \frac{\partial^3 w}{\partial x^3} = 0, \quad (2) $$

$$ a_7 \frac{\partial^3 u}{\partial x^3} + a_3 \frac{\partial^3 \phi}{\partial x^3} - a_4 \frac{\partial^4 w}{\partial x^4} - \kappa w - m_0 \frac{\partial^2 w}{\partial t^2} = -q_{zz}, $$

$$ a_1 = K_1^+ h_1 + K_2^+ h_2 + 2 K_3^+ c, \quad a_2 = c^2 \left[ K_1^+ h_1 + K_2^+ h_2 + \frac{2}{3} K_3^+ c \right], $$

$$ a_3 = c \left[ K_1^+ h_1 \left( c + \frac{1}{2} h_1 \right) + K_2^+ h_2 \left( c + \frac{1}{2} h_2 \right) + \frac{2}{3} K_3^+ c^2 \right], $$

$$ a_4 = K_1^+ h_1 \left( c^2 + c h_1 + \frac{1}{3} h_1^2 \right) + K_2^+ h_2 \left( c^2 + c h_2 + \frac{1}{3} h_2^2 \right) + \frac{2}{3} K_3^+ c^3, \quad a_6 = c (K_1^+ h_1 - K_2^+ h_2), $$

$$ a_7 = K_1^+ h_1 \left( c + \frac{1}{2} h_1 \right) - K_2^+ h_2 \left( c + \frac{1}{2} h_2 \right), \quad K_j^+ = K_j + \frac{4}{3} G_j, \quad m_0 = \rho_1 h_1 + \rho_2 h_2 + 2 \rho_3 c, \quad K_j^+ = K_j + \frac{4}{3} G_j, $$

here $j = 1, 2, 3$ are the layer numbers, $G_j$ is the shear modulus of $j$-th layer, $K_j$ is the bulk modulus of $j$-th layer, $\rho_j$ is the density of $j$-th layer material, $u$ is the longitudinal beam displacement, $w$ is the beam deflection, $\phi$ is the angle of rotation of the deformed normal in the beam filler, $\kappa$ is the foundation modulus, $q_{zz}$ is the normal stress in the viscous liquid layer, $q_{xx}$ is the shear stress in the viscous liquid layer.

The boundary conditions of Eq. (2) are

$$ u = \varphi = w = \frac{\partial^2 w}{\partial x^2} = 0 \text{ at } x = \pm \ell. \quad (3) $$

The movement of viscous liquid layer in a narrow channel can be considered as a creeping one [27] and the liquid layer dynamic equations are:

$$ \frac{1}{\rho} \frac{\partial \rho}{\partial x} = \rho \left( \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial z^2} \right), \quad \frac{1}{\rho} \frac{\partial \rho}{\partial z} = \frac{\partial^2 u_x}{\partial x^2} \frac{\partial u_x}{\partial x} + \frac{\partial^2 u_x}{\partial z^2} \frac{\partial u_x}{\partial z} = 0, \quad (4) $$

where $u_x, u_z$ are liquid velocity projections on the coordinate axis, $\rho$ is the liquid density, $\nu$ is the kinematic coefficient of the liquid viscosity, $p$ is the pressure.

The boundary conditions of Eq. (4) are the no-slip conditions and the ones for the pressure at the edges:

$$ u_x = 0, \quad u_z = 0 \text{ at } z = h_0 + c + h_1, \quad u_x = \frac{\partial u}{\partial t}, \quad u_z = \frac{\partial w}{\partial t} \text{ at } z = w + c + h_1, \quad (5) $$

$$ p = p_0 + p^* (\omega t) \text{ at } x = \pm \ell. \quad (6) $$


Let us introduce small parameters and dimensionless variables:
\[ \lambda = w_m/h_0 \ll 1, \quad \psi = h_0/\ell \ll 1, \quad \zeta = (z - c - h_1)/h_0, \quad \xi = x/\ell, \quad \tau = \omega t, \quad w = w_m W, \]
\[ u = u_m U, \quad \varphi = \varphi_m \Phi, \quad p = p_0 + p^* (\tau) + \rho \nu w_m \omega (h_0\nu^2)^{-1} P, \quad u_z = w_m \omega U_\zeta, \quad u_z = (w_m \omega/\psi)U_\xi. \]

Bearing in mind (7), the Eq. (4)-(6) in the dimensionless variables in zero approximation on \( \psi \) and \( \lambda \) will be presented in the form of:
\[ \frac{\partial \psi}{\partial \xi} \approx 0, \quad \frac{\partial \psi}{\partial \zeta} = 0, \quad \frac{\partial \psi}{\partial \tau} = 0, \quad \frac{\partial \psi}{\partial \zeta} = 0, \quad \frac{\partial \psi}{\partial \tau} = 0, \]
\[ \frac{\partial \psi}{\partial \xi} \approx 0, \quad \frac{\partial \psi}{\partial \zeta} = 0, \quad \frac{\partial \psi}{\partial \tau} = 0, \quad \frac{\partial \psi}{\partial \zeta} = 0, \quad P = 0 \text{ at } \xi = \pm 1. \]

By solving the problems of Eq. (8)-(9) we obtained:
\[ \psi_\xi = 0, \quad \psi_\zeta = 0 \quad \text{at } \zeta = 1, \quad \psi_\psi = 0, \quad \psi_\psi = \frac{\partial \psi}{\partial \tau} \quad \text{at } \zeta = 0, \quad P = 0 \text{ at } \xi = \pm 1. \]

The normal and shear stresses of the viscous liquid layer in variables (7) are:
\[ q_{zz} = -\left(p_0 + p^* (\tau)\right) + \frac{\rho \nu w_m \omega}{h_0 \psi^2} P \quad \text{at } \zeta = 0, \]
\[ q_{zx} = \frac{\rho \nu w_m \omega}{h_0 \psi} \frac{\partial \psi_\psi}{\partial \zeta} \quad \text{at } \zeta = 0. \]

According to (12), (13) we get \( q_{zz} \gg q_{zx} \), i.e. in comparison with normal stress, the shear one can be neglected, and by substituting (12) and neglecting (13) in Eq. (2) we obtain:
\[ a_1 \frac{\partial^2 u}{\partial x^2} + a_2 \frac{\partial^2 \varphi}{\partial x^2} - a_3 \frac{\partial^2 \psi}{\partial x^2} = 0, \]
\[ a_4 \frac{\partial^2 \psi}{\partial x^2} + a_5 \frac{\partial^2 \varphi}{\partial x^2} = 0, \]
\[ a_6 \frac{\partial^2 u}{\partial x^2} + a_7 \frac{\partial^2 \varphi}{\partial x^2} = 0, \]
\[ a_8 \frac{\partial^2 \psi}{\partial x^2} - a_9 \frac{\partial^2 \psi}{\partial x^2} - \kappa W - m_0 \frac{\partial^2 \psi}{\partial \tau^2} = p_0 + p^* (\tau) + \frac{\rho \nu w_m \omega}{h_0 \psi^2} P. \]

By using Eq. (14), we find that:
\[ \frac{\partial^2 \psi}{\partial x^2} = b_1 \frac{\partial^2 \varphi}{\partial x^2}, \quad \frac{\partial^2 \psi}{\partial x^2} = b_2 \frac{\partial^2 \varphi}{\partial x^2}, \quad b_1 = (a_2 a_7 - a_3 a_8)/(a_1 a_2 - a_3 a_5), \quad b_2 = (a_1 a_4 - a_2 a_5)/(a_1 a_2 - a_3 a_5), \]
and obtain hydroelastic bending oscillations equation of three-layered beam resting on Winkler foundation:
\[ D^* \frac{\partial^4 \psi}{\partial \tau^4} + \kappa W = -\left(p_0 + p^* (\tau)\right) - \frac{\rho \nu w_m \omega}{h_0 \psi^2} P, \quad D^* = a_4 - a_7 b_1 - a_3 b_2. \]

The solution of the Eq. (16) is presented the form of:
\[ w = w_m \sum_{k=1}^{\infty} \left( R_k^0 + R_k(\tau) \right) \cos \left( \frac{2k-1}{2} \pi \xi \right). \] \hspace{1cm} (17) 

Here the upper index 0 means the solution, corresponding to the static pressure \( p_0 \).

By substituting Eq. (7), (11), (17) into Eq. (16) and expanding the pressure \( p_0 + p^*(\omega t) \) in the series of trigonometric functions of the longitudinal coordinate \( x \), we obtained the system of linear algebraic equations for the definition \( R_k^0 \) and the system of the linear differential equations for the definition \( R_k(\tau) \). As a result of solving these equations systems, we obtain the following expression for three-layered beam elastic deflection:

\begin{equation}
\begin{aligned}
& w = -p_0 \frac{2\ell^4}{D^*} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)} \cos \left( \frac{2k-1}{2} \pi \xi \right) \left( \frac{2}{(2k-1)\pi} \right)^5 + p_m A(\xi, \omega) \sin(\tau + \varphi(\xi, \omega)), \\
& A(\xi, \omega) = \sqrt{C(\xi, \omega)^2 + G(\xi, \omega)^2}, \quad \varphi(\xi, \omega) = \arctg \left( \frac{C(\xi, \omega)}{G(\xi, \omega)} \right), \\
& C(\xi, \omega) = \sum_{k=1}^{\infty} \frac{4(-1)^{k+1}}{(2k-1)\pi} A_k \cos \left( \frac{2k-1}{2} \pi \xi \right), \\
& G(\xi, \omega) = \sum_{k=1}^{\infty} \frac{4(-1)^{k+1}}{(2k-1)\pi} B_k \cos \left( \frac{2k-1}{2} \pi \xi \right), \\
& A_k = -a_{2k}(a_{1k}^2 + a_{2k}^2)^{-1}, \quad B_k = a_{1k}(a_{1k}^2 + a_{2k}^2)^{-1}, \quad a_{2k} = K_k \omega / D^*, \\
& K_k = \frac{12\rho \nu}{h_{0y}/2} \left( \frac{2}{(2k-1)\pi} \right)^2, \quad a_{1k} = \left( D^* \right)^{-1} \left[ \left( \frac{(2k-1)\pi}{2\ell} \right)^4 D^* + \kappa - m_0 \omega^2 \right].
\end{aligned}
\end{equation} \hspace{1cm} (18)

4. Summary and Conclusion
The function \( A(\xi, \omega) \) in Eq. (18) is the frequency dependent function of three-layered beam deflection distribution along the channel. Therefore, investigating this function gives the opportunity to study the hydroelastic response of the channel wall resting on Winkler foundation. Thus, the presented mathematical model can be used for investigating the three-layered beam resonance oscillations, in the case of the three-layered beam being a wall of the channel with pulsating viscous incompressible liquid inside it. Furthermore, the model can be used to investigate the tense-deformed state of three-layered beam resting on elastic foundation, as well as to define its resonance bending oscillations. The obtained results can be used to analyse the causes of hydroelastic vibrations in the three-layered elements of heat exchange systems, hydraulic drives, fuel supply and lubrication systems and so on.

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