Asymptotically Anti-de Sitter Space-times:
Conserved Quantities

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Abstract

Asymptotically anti-de Sitter space-times are considered in a general dimension $d \geq 4$. As one might expect, the boundary conditions at infinity ensure that the asymptotic symmetry group is the anti-de Sitter group (although there is an interesting subtlety if $d = 4$). Asymptotic field equations imply that, associated with each generator $\xi$ of this group, there is a quantity $Q_\xi$ which satisfies the expected ‘balance equation’ if there is flux of physical matter fields across the boundary $\mathcal{I}$ at infinity and is absolutely conserved in absence of this flux. Irrespective of the dimension $d$, all these quantities vanish if the space-time under considerations is (globally) anti-de Sitter. Furthermore, this result is required by a general covariance argument. However, it contradicts some of the recent findings based on the conjectured ADS/CFT duality. This and other features of our analysis suggest that, if a consistent dictionary between gravity and conformal field theories does exist in fully non-perturbative regimes, it would have to be more subtle than the one used currently.

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I. INTRODUCTION

In the early to mid eighties, interest in asymptotically anti-de Sitter space-times was sparked by the discovery that anti-de Sitter space-times generically arise as ground states in certain supergravity theories which at the time were considered to be among the most promising candidates for quantum gravity. In particular, the relevant boundary conditions at infinity, asymptotic symmetries and associated conserved quantities were discussed by a number of authors. (See, e.g., [1–7]). Some of the early investigations used intuitively natural but somewhat imprecise boundary conditions based on the behavior of metric coefficients in certain coordinate systems. These coordinates referred (sometimes implicitly) to an underlying anti-de Sitter metric to which the given physical metric was to approach. Unfortunately, given a candidate asymptotically anti-de Sitter space-time, there are ambiguities in picking out ‘the’ anti-de Sitter metric to which the given metric approaches. As a result, the definitions of conserved quantities had certain ambiguities.\footnote{There is an analogous problem at null infinity in the asymptotically Minkowskian context. In presence of generic gravitational radiation, it is not possible to single out ‘the’ Minkowski metric to which a physical metric approaches and this leads to the well-known super-translation ambiguities in the definition of angular-momentum at null infinity \[8\]. However, that problem is ‘intrinsic’ in the sense that it is tied to the enlargement of the asymptotic symmetry group from the Poincaré group to the Bondi-Metzner-Sachs group \[8\]. In the asymptotically anti-de Sitter context, all treatments (correctly) assumed that the asymptotic symmetry group was the anti-de Sitter group and the ambiguities arose simply because of imprecision in techniques; limits to infinity are now delicate because some of the metric coefficients diverge there.}

In [6], it was pointed out that these problems can be avoided by using Penrose’s conformal techniques [10]. If, in addition to the restrictions required in Penrose’s conformal completion, one imposes ‘reflective boundary conditions’, the boundary \(\mathcal{I}\) becomes conformally flat and the asymptotic symmetry group reduces to the group of its conformal isometries, i.e., the anti-de Sitter group \(O(d−1,2)\) in four dimensions [3,6]. A detailed analysis of the asymptotic field equations showed that it is possible to define conserved quantities as 2-sphere integrals involving the electric part of the Weyl tensor and conformal Killing fields on \(\mathcal{I}\) representing infinitesimal asymptotic symmetries. In particular, since the Weyl tensor vanishes identically in the anti-de Sitter space-time, all conserved quantities vanish as well. Finally, there exists a rigorous analysis of the mixed initial-value boundary-value problem within Einstein’s theory which shows that there exists a large class of asymptotically anti-de Sitter solutions satisfying these boundary conditions [11].

The investigations mentioned above were carried out in four space-time dimensions. During the last three years, higher dimensional asymptotically anti-de Sitter space-times have attracted a great deal of attention because of a bold conjecture put forward by Maldecena [12]. In particular, inspired by the anti-de Sitter/conformal field theory (ADS/CFT) correspondence proposed in this conjecture, the issue of conserved quantities in such space-times was investigated in higher dimensions [13–15]. The starting point of these analyses is the introduction of an appropriate action. As in asymptotically Minkowski space-times, the
Einstein-Hilbert action of asymptotically anti-de Sitter space-times diverges \cite{[16]} (even when one restricts oneself to a space-time region bounded by two space-like surfaces with a finite time-like separation). The strategy \cite{[14],[15]} is to use a ‘counter-term subtraction method’, where the counter-terms depend on the geometry of the space-time boundary, which in turn encodes the property that the space-time under considerations is asymptotically anti-de Sitter. By varying the action with respect to the boundary metric, one obtains an ‘effective stress-energy tensor’ on the boundary. Note that, contrary to the normal usage of the term ‘stress-energy’ in the general relativity literature, this tensor field does not refer to any physical matter fields near the boundary, but is meant to encode the purely gravitational contribution to the conserved quantities. More precisely, the conserved quantities are obtained by first computing \((d - 2)\)-sphere integrals of this effective stress energy tensor, contracted with vector fields representing asymptotic symmetries, and then taking the limit, as the boundary goes to infinity.

These methods have led to some intriguing results on the relation between conserved quantities calculated in general relativity and analogous quantities calculated using an appropriate conformal field theory on the boundary. In particular, while the energy of the pure anti-de Sitter space-time does vanish when \(d = 4\) as in earlier analyses \cite{[6]}, it fails to vanish for \(d = 5\). Thus, using these methods, from purely gravitational considerations one finds that there is a ‘vacuum energy’ in 5-dimensional asymptotically anti-de Sitter space-times. This seems puzzling from the conventional general relativity standpoint. However, it has been argued that the ADS/CFT correspondence illuminates the origin of this vacuum energy: its value agrees precisely with the Casimir energy in the dual \((\mathcal{N} = 4, \text{super-symmetric})\) Yang-Mills theory on the boundary used in the Maldecena conjecture. This agreement is striking because the two calculations are so different. The calculation on the gravity side is non-perturbative; it refers to the full non-linear theory rather than the linearized theory. The Casimir energy on the Yang-Mills side, on the other hand, uses only linear quantum fields. Even more surprising is the feature that the first calculation is purely classical while the second refers to a quintessential quantum effect. In particular, the value of the Casimir energy is proportional to \(\bar{\hbar}\) and has no analog in the classical Yang-Mills theory. And yet, the available dictionary in the conjectured ADS/CFT correspondence converts the relevant parameters in such a way that the quantum Casimir energy of gauge fields is translated precisely to the classical vacuum energy in general relativity. Therefore, the agreement has been used as a clear example of the power and subtlety of the ADS/CFT duality which interpolates between a gravity theory in the bulk and a Yang-Mills theory on the boundary.

It is therefore desirable to understand various aspects of this calculation as thoroughly and as deeply as possible. The purpose of this note is to re-examine the gravity side of the calculation using Penrose’s conformal methods which, as discussed above, avoid a number of pitfalls associated with boundary conditions and limiting procedures.

In Sections II and III, we extend the analysis of \cite{[6]} to higher dimensions. We find that all results of \cite{[6]} directly generalize to \(d > 4\) (although there is a small but interesting difference in the treatment of asymptotic symmetries). In particular, in any dimension \(d \geq 4\), we find that all conserved quantities at \(\mathcal{I}\) vanish identically in pure anti-de Sitter space-times. At first sight, whether these quantities vanish or not may seem to depend on how one chooses their ‘ground state values’ which may in turn depend on one’s approach to the problem. However, one can give a simple argument to show that, if the definition of
these quantities is covariant, they must vanish in the anti-de Sitter space-time, irrespective of the method used to define them. At its crux, it is the same reasoning that one invokes to conclude that the angular momentum of a spherically symmetric configuration must vanish.

There is thus a tension between our results and those inspired by the ADS/CFT correspondence. To clarify the issue, in Section [V] we explicitly calculate the difference between our conserved quantities and those constructed using the ‘counter-term subtraction procedure’. Although the starting points are quite different, we show that, if \( d = 4 \), the two definitions agree for general asymptotically anti-de Sitter space-times (i.e., not just for Kerr-anti-de Sitter space-times for which explicit calculations are available). If \( d = 5 \), as one would expect, the two definitions disagree but the difference is finite in all asymptotically anti-de Sitter space-times. We provide an explicit expression for the difference. In the Schwarzschild-anti-de Sitter family, the difference is a constant shift by the vacuum energy. However, as far as we can tell, for more general asymptotically anti-de Sitter space-times, there are some additional terms. These differences suggest that the ADS/CFT dictionaries currently used to go between gravity theories in the bulk and conformal field theories on the boundary have certain unsatisfactory features in the non-perturbative regimes. This issue is discussed in Section [V].

This note is addressed to both general relativity and string theory communities. In the hope of bridging some apparent gaps, we have included introductory remarks on the ADS/CFT conjecture and some details on higher dimensional gravity and conformal techniques.

II. ASYMPTOTICALLY ANTI-DE SITTER SPACE-TIMES

This section is divided into two parts. In the first, we present the basic definition of asymptotically anti-de Sitter space-times and illustrate the Penrose completion with a simple example. In the second, we work out the basic consequences of the conditions introduced in the definition. Throughout this section, we work in \( d \)-dimensional space-times with \( d \geq 4 \). (The \( d = 3 \) case needs a special treatment already in the asymptotically Minkowskian context \([\mathcal{I}]\) and will be discussed elsewhere.) For simplicity, all fields will be assumed to be smooth (i.e., \( C^\infty \)) on the domain of their definitions.

A. Definition

A \( d \)-dimensional space-time \((\hat{M}, \hat{g}_{ab})\) will be said to be asymptotically anti-de Sitter if there exists a manifold \( M \) with boundary \( \mathcal{I} \), equipped with a metric \( g_{ab} \) and a diffeomorphism from \( \hat{M} \) onto \( M - \mathcal{I} \) of \( M \) (with which we identify \( \hat{M} \) and \( M - \mathcal{I} \)) and the interior of \( M \) such that:

1. there exists a function \( \Omega \) on \( M \) for which \( g_{ab} = \Omega^2 \hat{g}_{ab} \) on \( \hat{M} \);

2. \( \mathcal{I} \) is topologically \( S^{d-2} \times R \), \( \Omega \) vanishes on \( \mathcal{I} \) but its gradient \( \nabla_a \Omega \) is nowhere vanishing on \( \mathcal{I} \).
3. On \( \hat{M} \), \( \hat{g}_{ab} \) satisfies \( \hat{R}_{ab} - \frac{1}{2} \hat{R} \hat{g}_{ab} + \Lambda \hat{g}_{ab} = 8\pi G_{(d)} \hat{T}_{ab} \), where \( \Lambda \) is a negative constant, \( G_{(d)} \) is Newton’s constant in \( d \)-dimensions, and the matter stress-energy \( \hat{T}_{ab} \) is such that \( \Omega^{2-d} \hat{T}_{ab} \) admits a smooth limit to \( \mathcal{I} \).

4. The Weyl tensor of \( g_{ab} \) is such that \( \Omega^{4-d} C_{abcd} \) is smooth on \( M \) and vanishes at \( \mathcal{I} \).

The meaning of the conditions in the definition is as follows. The first condition ensures that the physical metric \( \hat{g}_{ab} \) is conformally related to the new metric \( g_{ab} \) which is mathematically more convenient. The second condition ensures that the topology of the boundary is the one suggested by the geometry of anti-de Sitter space-times. Since the conformal factor \( \Omega \) vanishes on \( \mathcal{I} \), the boundary attached in the conformal completion is at infinity with respect to the physical metric \( \hat{g}_{ab} \). The condition that \( \nabla_a \Omega \) is nowhere vanishing on \( \mathcal{I} \) ensures that \( \Omega \) can be used as a ‘good (radial) coordinate’ in a neighborhood of \( \mathcal{I} \) in the completed space-time \( (M, g_{ab}) \). (In terms of the physical metric, heuristically, the condition tells us that \( \Omega \) ‘falls off as \( 1/r^2 \).’) The conformal rescaling brings the boundary (at infinity of \( (M, g_{ab}) \)) to a finite ‘distance’ (with respect to \( g_{ab} \)). Therefore, the delicate operations involving limits of various fields in \( \hat{M} \) can now be reduced to simple local differential geometry of fields at \( \mathcal{I} \) in the completed space-time. This simple fact removes ambiguities in the limiting procedures and streamlines manipulations at infinity.

The third condition restricts the asymptotic behavior of matter fields. (In the analyses based on the ‘counter-term subtraction’ it is implicitly assumed that there is no physical matter in a neighborhood of infinity, whence this condition is trivially met.) The fall-off condition on \( \hat{T}_{ab} \) is motivated by the analysis of test fields in anti-de Sitter space-time; it ensures that the fluxes of conserved quantities associated with test fields across \( \mathcal{I} \) are well-defined. We will see that for the coupled Einstein-matter field system, it will also ensure that we obtain the physically expected ‘balance equations’ for conserved quantities when there is flux of matter fields across \( \mathcal{I} \). Finally, since the Weyl tensor vanishes identically in the anti de-Sitter space-time, it is natural to expect it to fall off at an appropriate rate in asymptotically anti-de Sitter space-times. The precise rate is given in the fourth condition. It is suggested by the following dimensional considerations. One expects the leading non-zero contribution to the components of the Weyl tensor in an orthonormal tetrad to behave asymptotically as \( G_{(d)} M/r^n \) for some \( n \), where \( M \) is a measure of the total mass. Physical dimensions of the Weyl tensor and \( G_{(d)} \) dictate \( n = d - 1 \). Using the behavior \( \Omega \sim 1/r \) and the conformal rescalings of orthonormal tetrads in the passage from \( \hat{g}_{ab} \) to \( g_{ab} \) we arrive at the fall off given in the fourth condition. However, this condition can be weakened because the first three conditions already constrain the asymptotic Weyl curvature. These and other details will be discussed elsewhere.

Next, let us consider a simple example to illustrate how this definition works. Let \( (\hat{M}, \hat{g}_{ab}) \) be Schwarzschild-anti-de Sitter space-time. Then, in an asymptotic region (away from the horizon) the metric \( \hat{g}_{ab} \) can be taken to be

\[
\hat{g}_{ab}dx^a dx^b = -\hat{F}^2(r) dt^2 + \hat{F}^{-2}(r) dr^2 + r^2 d\sigma^2,
\]

with

\[
\hat{F}^2(r) = 1 + \frac{r^2}{\ell^2} - \left( \frac{r_o}{r} \right)^{d-3}
\]

where \( d\sigma^2 \) is the line-element of the round, unit \( d-2 \)-sphere metric. Following the usual conventions we have set...
\[ \Lambda = -\frac{(d-1)(d-2)}{2\ell^2} \quad \text{and} \quad M_{(d)} = \frac{(d-2)(d-3)A_{(d-2)}r_0^{d-3}}{16\pi G_d} \]

(2.2)

where \( M_{(d)} \) is the Schwarzschild mass and \( A_{(d-2)} \) the area of the unit \( (d-2) \)-sphere. To show that this space-time satisfies our boundary conditions, we need to first pick a conformal factor \( \Omega \) and then carry out the completion by attaching to \( \hat{M} \) the \( \Omega = 0 \) surface. Set \( \Omega = 1/r \).

By replacing \( r \) by \( 1/\Omega \) in (2.1) we obtain

\[ g_{ab}dx^a dx^b := \Omega^2 \hat{g}_{ab} dx^a dx^b = -F^2(\Omega) dt^2 + F^{-2}(\Omega) d\Omega^2 + d\sigma^2, \]

with \( F^2(\Omega) = \Omega^2 \hat{F}^2(r) = \left( \frac{1}{\ell^2} + \Omega^2 - r_0^{d-3} \Omega^{d-1} \right) \)

(2.3)

The surface \( \Omega = 0 \) did not belong to \( \hat{M} \) since it corresponds to \( r = \infty \); indeed, if we simply attached it to \( \hat{M} \), the metric \( \hat{g}_{ab} \) would be singular there. However, in the \( (t, \Omega, \text{angles}) \) chart, it is trivial to extend \( \hat{M} \) by attaching to it a boundary \( \mathcal{I} \) defined by \( \Omega = 0 \). The rescaled metric \( g_{ab} \) is well-defined at this boundary.

Let us now set \( M = \hat{M} \cup \mathcal{I} \) and verify that the conditions of the definition are satisfied. Clearly, \( g_{ab} \) is smooth on \( M \) (i.e., its components in the chart \( (t, \Omega, \text{angles}) \) are all smooth). \( \mathcal{I} \) is topologically \( S^{d-2} \times \mathbb{R} \); the conformal factor \( \Omega \) vanishes at \( \mathcal{I} \) but its gradient \( \nabla_{\Omega} \) does not (indeed, \( g^{ab} \nabla_a \Omega \nabla_b \Omega = 1/\ell^2 \) on \( \mathcal{I} \)). The third condition is trivially satisfied since the stress energy tensor \( \hat{T}_{ab} \) of physical matter fields vanishes identically on \( \hat{M} \). Finally, a direct calculation or the dimensional argument given above shows that the condition on Weyl curvature is also satisfied.

We will conclude with some remarks on the conformal freedom. First, note that we could choose the conformal factor \( \Omega \) to be \( 1/r \) in all dimensions; there is no \( d \)-dependence in the power of \( r \). However, there is considerable freedom in the choice of \( \Omega \). In particular, \( \hat{\Omega} = \omega \Omega \) is an equally good conformal factor provided \( \omega \) is a smooth, nowhere vanishing function on \( M \). In Schwarzschild-anti-de Sitter space-times, the choice \( \Omega = 1/r \) is adapted to the isometries and, in particular, to the rest frame selected by the time translation Killing field \( \partial/\partial t \). In the anti-de Sitter space-time, on the other hand, there is no natural rest frame and choices of conformal factors associated with distinct frames are related by a \( \omega \) which is non-vanishing on \( \mathcal{I} \). Because of this conformal freedom, for physical quantities defined at \( \mathcal{I} \), covariance with respect to the physical space-time is tied to conformally invariance at \( \mathcal{I} \). This point will play an important role in Section III.

**B. Asymptotic fields and their equations**

We now wish to analyze physical fields near \( \mathcal{I} \) and explore equations they satisfy. These equations will lead us to the asymptotic symmetry group and conserved quantities in Section III.

Let us begin by introducing some notation. The Riemann tensor of the metric \( g_{ab} \) can be decomposed in to the Weyl and Ricci tensors as follows:

\[ R_{abmn} = C_{abmn} + \frac{2}{d-2} \left( g_{a[m} S_{n]b} - g_{b[m} S_{n]a} \right), \]

(2.4)
where \( S_{ab} \) is given by
\[
S_{ab} = R_{ab} - \frac{R}{2(d-1)} g_{ab}. \tag{2.5}
\]

Next, under the conformal rescaling \( g_{ab} = \Omega^2 \hat{g}_{ab} \) on \( \hat{M} \), the Weyl tensor transforms trivially, \( \hat{C}_{abc}{}^d = C_{abc}{}^d \), while the Ricci pieces transform via
\[
\hat{S}_{ab} = S_{ab} + (d-2)\Omega^{-1} \nabla_a n_b - \left( \frac{d-2}{2} \right) \Omega^{-2} g_{ab} n^c n_c, \quad \text{where} \quad n_a := \nabla_a \Omega \tag{2.6}
\]

Field equations enable us to express \( \hat{S}_{ab} \) in terms of the cosmological constant and the matter stress-energy as
\[
\hat{S}_{ab} - \frac{\Lambda}{d-1} \hat{g}_{ab} = 8\pi G_{(d)} \left( \hat{T}_{ab} - \frac{\hat{T}}{d-1} \right) \hat{g}_{ab} \tag{2.7}
\]

Multiplying (2.6) by \( \Omega^2 \), taking the limit as \( \Omega \) tends to 0 and using the fact that \( \Omega^2 - d \hat{T}_{ab} \) admits a smooth limit to \( I \), we obtain:
\[
n \cdot n \equiv \frac{-2\Lambda}{(d-2)(d-1)} \equiv \frac{1}{\ell^2}, \tag{2.8}
\]

where \( \equiv \) denotes equality restricted to \( I \). (We will use this notation extensively in the rest of the paper.) Thus, \( n^a \) is space-like at \( I \), whence \( I \) is a time-like surface. Now, if a smooth field \( f \) vanishes on \( I \), then \( f = \Omega^{-1} f \) admits a smooth limit to \( I \). Hence it follows that \( \Omega^{-1} (n \cdot n - 1/\ell^2) \) admits a smooth limit. Now, multiplying (2.7) by \( \Omega \) and taking limit to \( I \) we obtain
\[
2\nabla_a n_b \equiv \left[ \lim_{\rightarrow I} \Omega^{-1} \left( n^c n_c - \frac{1}{\ell^2} \right) \right] g_{ab} \equiv \frac{2}{d} (\nabla c n^c) g_{ab} \tag{2.9}
\]

Next, it is easy to verify that under a conformal rescaling \( \Omega = \omega \Omega \), we have
\[
\nabla_a \bar{n}_b \equiv \frac{1}{d} \omega (\nabla c n^c) g_{ab} + (n^c \nabla c \omega) g_{ab} \tag{2.10}
\]

where \( \nabla \) is the derivative operator compatible with \( \bar{g}_{ab} = \Omega^2 \hat{g}_{ab} \) and \( \pi_a = \nabla_a \Omega \). Hence, by suitable rescaling, we can always obtain a conformal completion in which
\[
\nabla_a n_b \equiv 0 \tag{2.11}
\]

From now on, we will only use such conformal completions. (The explicit completions of the Schwarzschild-anti-de Sitter space-times given in Section [IIA] meet this condition.) There is still a remaining conformal freedom: \( \Omega \mapsto \omega \Omega \) where \( n^a \nabla_a \omega \equiv 0 \). Note that the value of \( \omega \) at \( I \) is unrestricted.

The first key equation for obtaining conserved quantities results from taking the derivative of (2.6) and using Einstein’s equation (2.7).
\[ \Omega \nabla_{[\alpha} S_{\beta]c} + \left( \frac{d-2}{2} \right) C_{\alpha bc}^d n_d = 8\pi G_{(d)} \tilde{T}_{d[a} g_{b]c} n^d + 8\pi G_{(d)} \nabla_{[\alpha} \tilde{T}_{b]c} \]  

(2.12)

where \( \tilde{T}_{\alpha\beta} = \tilde{T}_{\alpha\beta} - (\tilde{T}/d - 1)g_{\alpha\beta} \). The second key equation is the (contracted) Bianchi identity

\[ \nabla^d C_{\alphabcd} + \frac{2(d-3)}{(d-2)} \nabla_{[\alpha} S_{\beta]c} = 0 \]  

(2.13)

We can now eliminate the \( S_{\alpha\beta} \) term in (2.12) using (2.13). Since the fourth condition in the main definition ensures that

\[ K_{\alphabcd} := \lim_{\Omega \to I} \Omega^{3-d} C_{\alphabcd} \]  

(2.14)

admits a smooth limit to \( I \), using the fall-off condition on \( \hat{T}_{\alpha\beta} \) given in the main definition, we obtain

\[ \nabla^d K_{\alphabcd} \equiv \lim_{\Omega \to I} \frac{2(d-3)}{(d-2)} \Omega^{2-d} 8\pi G_{(d)} \left[ \tilde{T}_{d[a} g_{b]c} n^d + \nabla_{[\alpha} (\Omega \tilde{T}_{b]c}) \right] \]  

(2.15)

Finally, by projecting this equation via \( n^a n^c h^b_m \), where \( h_{\alpha\beta} = g_{\alpha\beta} - \ell^2 n_a n_b \) is the intrinsic metric on \( I \), we obtain,

\[ D^d \mathcal{E}_{md} = -8\pi G_{(d)} (d-3) T_{ab} n^a h^b_m . \]  

(2.16)

Here, \( D \) is the intrinsic derivative operator on \( I \) compatible with \( h_{\alpha\beta} \), \( \mathcal{E}_{ab} \equiv \ell^2 K_{amnb} n^m n^n \) is the electric part of the (leading order) Weyl tensor \( K_{amnb} \) at \( I \) and \( T_{ab} = \lim \Omega^{2-d} \tilde{T}_{ab} \). (Note that \( \mathcal{E}_{ab} \) and \( T_{ab} \) are smooth fields at \( I \).) This is the key identity that will enable us to introduce conserved quantities at infinity.

### III. SYMMETRIES AND ASSOCIATED CHARGES

This section is divided into two parts. In the first, we discuss asymptotic symmetries and in the second we introduce the conserved quantities associated with these symmetries.

#### A. Asymptotic symmetries

In the physical space-time picture, the asymptotic symmetry group \( \mathcal{G} \) is the quotient \( \text{Diff}(\hat{M})/\text{Diff}_{0}(\hat{M}) \) of the group of diffeomorphism which preserve the boundary conditions by its sub-group consisting of asymptotically identity diffeomorphisms. In the conformally completed space-time, \( \mathcal{G} \) is the subgroup of the diffeomorphism group on \( I \) which preserves the ‘universal structure’ at \( I \), i.e., the structure that is common to boundaries of all asymptotically anti-de Sitter space-times. Let us begin by exploring this structure.

Since \( \nabla_a n_b \equiv 0 \), it follows that the extrinsic curvature \( K_{\alpha\beta} \) of \( I \) vanishes. Hence, the intrinsic curvature tensor \( \hat{R}_{abcd} \) of \( (I, h_{\alpha\beta}) \) is given by: \( \hat{R}_{abcd} = h^m_h^a n^c h^s_d R_{mnsht} \). Now,
our boundary conditions imply that the Weyl tensor $C_{abcd}$ of $g_{ab}$ vanishes on $I$ for all $d \geq 4$. Hence, using (2.4) it now follows that the intrinsic Weyl tensor $C_{abcd}$ of $I$ vanishes. Thus, if $d > 4$, $(I, h_{ab})$ is conformally flat in all asymptotically anti-de Sitter space-times. Hence $\mathcal{G}$ is just the $d(d+1)/2$ dimensional conformal group, i.e., the anti-de Sitter group in $d$ dimensions.

If $d = 4$, however, $I$ is a 3-manifold and vanishing of $C_{abcd}$ imposes no restriction at all. Let us examine this case in some detail. Note first that, in any dimension $d > 4$, we used only the first three conditions in the main definition to derive all equations up to (2.13). Therefore, using (2.12) and the fall-off conditions on the stress energy $\hat{T}_{ab}$, we can conclude $C_{abcd}n^d \equiv 0$ without any reference to the fourth condition in the definition on the fall-off of Weyl tensor. This in particular implies that the electric and magnetic parts of the Weyl tensor $C_{abcd}$ must vanish at $I$. Now, in four dimensions, the Weyl tensor has no other independent components and hence we can conclude $C_{abcd} \equiv 0$. Therefore, if $d = 4$, the fourth condition in the definition is redundant. However, as noted above, in this case, the definition does not imply that the intrinsic geometry of $I$ is conformally flat. In the anti-de Sitter space-time, on the other hand, this geometry is indeed conformally flat.

Therefore, to fully capture the idea that $(\hat{M}, \hat{g}_{ab})$ is asymptotically anti-de Sitter in four dimensions, we need to impose a new condition which may be regarded as a replacement of the fourth condition in the definition. Let us suppose that the magnetic part

$$B_{ab} := *K_{ambn}n^m n^n \equiv \lim_{\Omega \to I} \Omega^{-1} *C_{ambn}n^m n^n$$

(3.1)

of the asymptotic Weyl curvature $K_{ambn}$ vanishes on $I$. Then, one can show that $(I, h_{ab})$ is conformally flat and $G$ is the anti-de Sitter group in four dimensions. (A completely analogous condition is necessary to obtain Poincaré group as the asymptotic symmetry group at spatial infinity of asymptotically Minkowskian space-times.) The condition $B_{ab} \equiv 0$ is sometimes referred to as the ‘reflective boundary condition’.

Thus, as one might intuitively expect, the asymptotic symmetry group is the anti-de Sitter group in all cases (i.e., for all $d \geq 4$). Note that, since $I$ is not endowed with additional universal structure, $\mathcal{G}$ can not be further reduced. In any one completion, of course, one can single out the isometry group of $h_{ab}$ as a sub-group of $\mathcal{G}$. However, this group, and indeed even its dimension, will vary with the choice of the conformal factor (which determines $h_{ab}$). It is only the equivalence class $[h_{ab}]$ of conformally flat metrics on $I$ and the full conformal isometry group $\mathcal{G}$ that have an invariant meaning. On the other hand, if one restricts oneself to a specific space-time $(\hat{M}, \hat{g}_{ab})$ admitting isometries, in any conformal completion of that space-time, one can select a preferred subgroup of $\mathcal{G}$ (since every Killing vector of $\hat{g}_{ab}$ admits an extension to $I$ and is a conformal Killing field of $h_{ab}$ irrespective of the choice of $\Omega$.)

**B. Conserved quantities**

Eq (2.16) is a differential conservation law which immediately leads us to the required ‘conserved’ charges. Given any infinitesimal asymptotic symmetry (i.e., a conformal Killing field $\xi^a$ on $I$) and a $d-2$ sphere cross section $C$ on $I$ we can now define a ‘conserved’ quantity

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\[ Q_\xi[C] := -\frac{1}{8\pi G_d} \frac{\ell}{n-3} \oint_C \mathcal{E}_{ab}\xi^a dS^b \]  

It now follows immediately from (2.20) that \( Q_\xi \) satisfies a balance equation: Given two cross-sections \( C_1 \) and \( C_2 \) which form boundaries of a \( d-1 \) dimensional region \( \Delta I \) of \( I \), we have

\[ Q_\xi[C_2] - Q_\xi[C_1] = \int_{\Delta I} T_{ab}\xi^a dS^b \]  

where we have used the fact that \( \mathcal{E}_{ab} \) is trace-free. Thus the difference between values of \( Q_\xi \) evaluated at the two cross-sections is precisely the flux of that physical quantity, associated with matter, across the portion \( \Delta I \) of \( I \). It is only when the matter flux vanishes that the quantity is absolutely conserved (whence the use of inverted commas in the label ‘conserved’).

As in Section II A, let us consider the example of a Schwarzschild-anti-de Sitter space-time. Since there is no physical matter field anywhere, all the \( Q_\xi \) are absolutely conserved. It is straightforward to evaluate them explicitly. If the conformal Killing field \( \xi^a \) corresponds to the time-translation Killing field on \((\hat{M}, \hat{g}_{ab})\), then \( Q_\xi = M \), the mass parameter in the metric. All other conserved quantities vanish. In particular, if \((\hat{M}, \hat{g}_{ab})\) is the anti-de Sitter space-time, all \( Q_\xi \) vanish identically, irrespective of the dimension \( d \).

We will now present a general argument based on covariance to show that this last conclusion is robust and not tied to the specific procedure we used to define conserved quantities. Let us suppose that one has a covariant procedure that leads to a conserved quantity \( \tilde{Q}_\xi \) associated with each conformal Killing field \( \xi^a \) on \( I \) (i.e., a procedure which uses only the boundary conditions and does not refer to additional structures such as preferred coordinates or background fields.) Let us first fix a general asymptotically anti-de Sitter space-time \((\hat{M}, \hat{g}_{ab})\) in which there are no matter fields near infinity and compute its \( \frac{d(d+1)}{2} \) absolutely conserved charges \( \tilde{Q}_\xi \). Apply a diffeomorphism \( \varphi \) which is an asymptotic symmetry, i.e., defines an element of \( \mathcal{G} \). The image \( \hat{g}'_{ab} = \varphi^*(\hat{g}_{ab}) \) of \( \hat{g}_{ab} \) is again asymptotically anti-de Sitter and, by covariance, values \( \tilde{Q}' \) of its conserved charges are \( \tilde{Q}'_{\xi} = \tilde{Q}'_{\varphi, \xi} \) where \( \varphi, \xi \) is given by the action of \( \mathcal{G} \) on \( I \). (More precisely, the \( \frac{d(d+1)}{2} \) charges \( \tilde{Q}' \xi \) define an ‘anti-de Sitter momentum’ which lies in the dual of the Lie algebra of \( \mathcal{G} \) and the anti-de Sitter momenta of \( \hat{g}_{ab} \) and \( \hat{g}'_{ab} \) are related by the natural action of \( \mathcal{G} \) on the dual of its Lie algebra.) Now, let \((\hat{M}, \hat{g}_{ab})\) be the anti-de Sitter space-time itself and choose for \( \varphi \) an isometry of \( \hat{g}_{ab} \). Then \( \hat{g}'_{ab} = \hat{g}_{ab} \). Hence we must have \( Q'_{\varphi, \xi} = Q_{\xi} \) for all \( \varphi \) in \( \mathcal{G} \). This is possible only if \( Q_{\xi} = 0 \).

The crux of the argument is quite simple. In the asymptotically Minkowskian context, the same argument tells us that if the total 4-momentum at spatial infinity is defined by a

\[ \text{Note that (like the rest of our analysis) this argument does not go through if } d = 3. \text{ In this case, there are two possible sets of boundary conditions at infinity} \]  

10
covariant procedure, then it must vanish in Minkowski space since the Minkowski metric is invariant under Lorentz transformations and there is no non-zero Lorentz invariant 4-vector.

We will conclude this section with three remarks.

1. The anti-de Sitter space-time is conformally flat. Therefore, it was natural to require the Weyl curvature of asymptotically anti-de Sitter space-times to vanish at infinity. The physical observables $Q_{\xi}$ are defined using its ‘residue’ near infinity. By contrast, the extraction of this physical information from metric components is much more difficult and delicate: since the components of the anti-de Sitter metric diverge at infinity in standard charts, the extraction involves a comparison between two infinite quantities. Note also that even in the asymptotically Minkowskian context, one can define energy-momentum and angular-momentum at spatial infinity in terms of the asymptotic behavior of the Weyl tensor \[18,19\]. Furthermore, this approach offers perhaps the clearest way to weed out the unwanted super-translations.

2. If there are no physical matter fields near $\mathcal{I}$, all $d(d+1)/2$ quantities $Q_{\xi}$ are absolutely conserved. This is in striking contrast with the situation at null infinity of asymptotically Minkowskian space-times, where, even in the absence of matter, gravitational radiation can and does carry away energy-momentum and angular momentum (see, e.g., \[3,10,21\]). On the other hand, matter fields can carry away these quantities in both cases. Thus, in certain respects the boundary $\mathcal{I}$ in the asymptotically anti-de Sitter context is analogous to spatial infinity \[18,19\] in the asymptotically Minkowskian space-times and in other respects it is analogous to null infinity \[3,10,21\].

3. Finally, note that since $\xi^a$ are conformal Killing fields on $(\mathcal{I}, h_{ab})$, $Q_{\xi}$ would not be absolutely conserved even in absence of matter if $E_{ab}$ had not been trace-free. Thus, in that case, conserved charges would not exist. This is quite distinct from the issue of whether or not the Poisson algebra between conserved charges yields a ‘normal’ or an ‘anomalous’ representation of the Lie algebra of the asymptotic symmetry group.

IV. COMPARISON

In this section we will present an explicit comparison between the conserved charges $Q_{\xi}$ of Section \[1\] and the quantities $Q_{\xi}$ obtained by the ‘counter-term subtraction method’. As noted in the Introduction, the quantities $Q_{\xi}$ are constructed from an ‘effective stress-energy tensor’ which we will denote by $\hat{T}_{ab}$ (to distinguish it from the stress energy-tensor $\hat{T}_{ab}$ of physical matter fields). $Q_{\xi}$ are defined by

$$Q_{\xi}[\mathcal{C}] = \lim_{\mathcal{I}\rightarrow\mathcal{C}} \int_\mathcal{C} \hat{T}_{ab} \hat{\xi}^a d\hat{S}^b \quad (4.1)$$

where $\hat{C}$ are $(d-2)$-spheres in $\hat{M}$ converging to the cross-section $\mathcal{C}$ of $\mathcal{I}$ and $\hat{\xi}^a$ is a vector field in $\hat{M}$ which tends to an infinitesimal asymptotic symmetry $\xi^a$ on $\mathcal{I}$. In this work it is (implicitly) assumed that there is no physical matter field in a neighborhood of infinity. Therefore, in this section, we set $\hat{T}_{ab} = 0$. Also, since the expressions for $\hat{T}_{ab}$ become rapidly complicated in higher dimensions, we restrict ourselves to $d = 4$ and $d = 5$, although our qualitative conclusions hold also in higher dimensions.
Let \((\hat{M}, \hat{g}_{ab})\) be asymptotically anti-de Sitter in the sense of our definition and let \((M, g_{ab}, \Omega)\) be a conformal completion satisfying our conditions. Denote by \(\mathcal{I}_\Omega\) the \((d - 1)\)-dimensional sub-manifolds of \(M\) on which \(\Omega\) is constant. In a sufficiently small but finite neighborhood of \(\mathcal{I}\), each \(\mathcal{I}_\Omega\) is a time-like surface. Denote by \(h_{ab}\) the intrinsic metric induced on \(\mathcal{I}_\Omega\) by \(\hat{g}_{ab}\), by \(\hat{G}_{ab}\) its Einstein tensor and by \(\hat{K}_{ab}\) the extrinsic curvature. (Following the standard sign conventions used in general relativity \([21]\), we will set \(\hat{K}_{ab} = \hat{h}^m_a \hat{h}^n_b \nabla_m \hat{\eta}_n\), where \(\hat{h}_{ab}\) is the intrinsic metric and \(\hat{\eta}^a\) is the unit outward normal on \(\mathcal{I}_\Omega\). This sign convention is opposite to the one used in \([14]\).) Then, motivated by considerations of \([22]\), an explicit expression of \(\hat{\tau}_{ab}\) was obtained in \([14]\):

\[
\hat{\tau}_{ab} = \frac{1}{8\pi G(d)} \left[ \frac{\ell}{n-3} \hat{g}_{ab} - \frac{n-2}{\ell} \hat{h}_{ab} - \hat{K}_{ab} + \hat{\kappa} \hat{h}_{ab} \right] \tag{4.2}
\]

We will now relate \(\hat{\tau}_{ab}\) to the field \(\mathcal{E}_{ab}\) we used to define our conserved charges. Note first that, using the relation between the Riemann tensors of \(\hat{g}_{ab}\) and \(\hat{h}_{ab}\), we can express the electric part \(\hat{E}_{ab}\) of the Weyl tensor \(\hat{C}_{amnb}\) of \(\hat{g}_{ab}\) as:

\[
\hat{E}_{ab} := \hat{C}_{amnb} \hat{\eta}^m \hat{\eta}^n = -\hat{\nabla}_a \hat{K} \hat{K}_{ac} - \hat{K}_{ab} \hat{K}^b_c + \frac{2}{d-1} \Lambda \hat{h}_{ab} \tag{4.3}
\]

Hence, it follows that

\[
8\pi G(d) \hat{\tau}_{ab} + \frac{\ell}{d-3} \hat{E}_{ab} = \frac{\ell}{d-3} \left[ \hat{K} \hat{K}_{ac} - \hat{K}^c_a \hat{K}_{bc} + \frac{1}{2} (\hat{K}^{mn} \hat{K}_{mn} - \hat{K}^2) \hat{h}_{ab} \right] - \frac{d-2}{2\ell} \hat{h}_{ab} - \hat{K}_{ab} + \hat{\kappa} \hat{h}_{ab} \tag{4.4}
\]

To take limit to \(\mathcal{I}\), let us express the relevant ‘hatted’ fields (constructed from \(\hat{g}_{ab}\)) in terms of the corresponding ‘unhatted’ fields (constructed from \(g_{ab}\)). First, it is straightforward to show that

\[
\hat{K}_{ab} = \Omega^{-1} K_{ab} - \Omega^{-2} (\eta \cdot n) h_{ab} \tag{4.5}
\]

where, as before, \(n_a = \nabla_a \Omega\) and \(\eta^a\) is the unit outward normal to \(\mathcal{I}_\Omega\) with respect to \(g_{ab}\). Next, using \((2.11)\) and \((2.8)\), it is easy to show that the bold-faced fields in

\[
K_{ab} = \Omega^{-1} K_{ab}, \quad \text{and} \quad f = \Omega^{-2} (l(\eta \cdot n) + 1) \tag{4.6}
\]

admit smooth limits to \(\mathcal{I}\). Using these facts, one can rewrite \((1.4)\) as:

\[
8\pi G(d) \hat{\tau}_{ab} + \frac{\ell}{d-3} \hat{E}_{ab} = \Omega^2 \left[ \frac{\ell}{d-3} \left( KK_{ab} - K_c^a K_{bc} + \frac{1}{2} (K^{mn} K_{mn} - K^2) h_{ab} \right) \right. \\
+ \frac{f(K_{ab} - K h_{ab}) - d-2}{2\ell} f^2 h_{ac} \biggr] = \Omega^2 \Delta_{ab}, \quad \text{say}. \tag{4.7}
\]

By inspection, \(\Delta_{ab}\) admits a smooth limit to \(\mathcal{I}\). Finally, since \(\hat{\xi}^a\) admits a smooth limit \(\xi^a\) to \(\mathcal{I}\) and \(dS^b = \Omega^{3-d} dS^b\), we can express the difference between the two sets of charges as:
\[ Q_\xi[C] - Q_\xi[C] = \frac{1}{8\pi G_{(d)}} \oint_c \Omega^{d-5} \Delta_{ab}\xi^a ds^b \] \tag{4.8}

It follows immediately that the two charges agree if \( d = 4 \) but they do not if \( d = 5 \). To our knowledge, in four dimensions the equality had been established only in the case of Kerr-anti-de Sitter space-times and the focus was on energy, i.e., on the case when the infinitesimal symmetry \( \xi^a \) corresponds to the time translation isometry of this space-time. Eq (4.8) establishes the result in all asymptotically anti-de Sitter space-times and for all asymptotic symmetries.

Let us investigate the situation in \( d = 5 \) further. Let \( (\hat{M}, \hat{g}_{ab}) \) be the anti-de Sitter space-time itself. Choose a rest-frame, express \( \hat{g}_{ab} \) in the adapted chart, set \( \Omega = 1/r \) and consider the resulting conformal completion as in Section 1A. Then, it is easy to verify that

\[ K_{ab} = \frac{1}{\ell} (1 + \ell^2 \Omega^2)^{1/2} \nabla_a t \nabla_b t \quad \text{and} \quad f = \Omega^{-2} \left[ 1 - (1 + \ell^2 \Omega^2)^{1/2} \right]. \tag{4.9} \]

It therefore follows that \( \Delta_{ab} = -(3/8)\ell^3 h_{ab} \). Since \( \hat{E}_{ab} = 0 \) on \( \hat{M} \), as noted before, all \( Q_\xi[C] \) vanish identically. Hence, if we choose \( \hat{\xi} = \partial/\partial t \) the time translation adapted to our initial choice of rest frame, we obtain

\[ Q_t[C] = \frac{\ell}{8\pi G} \frac{3}{8} A_{(3)} \tag{4.10} \]

where, as before, \( A_{(3)} \) is the area of the unit 3-sphere. This is the ‘ground state energy’ of the ‘counter-term subtraction method’ [14].

In the string theory literature [14,15], quantities \( Q_\xi \) (and their higher dimensional versions) are generally interpreted as the gravitational analogs of the symmetry generators in the conform field theory. It is important to examine if they are viable as candidate Hamiltonians generating asymptotic symmetries on the gravitational side. Now, as noted in Section 112 the fact that \( Q_t \) fails to vanish in pure anti-de Sitter space-time implies that \( Q_\xi \) are not defined in a covariant fashion. Indeed, if there is a well-defined, non-zero ground state energy, the obvious question is: what rest frame does it refer to? There is no preferred rest-frame in pure anti-de Sitter space-time and covariance prevents us from saying that the right side of (4.10) is the ground state energy in every rest frame. Finally, from the perspective of classical general relativity, it would have been rather strange if there were a ground state energy in five dimensions but not in four. If this energy is to have physical significance, one should be able to measure it in terms of ‘planetary’ (i.e. test particle) motions and it is difficult to imagine such a qualitative difference in four and five dimensions.

In general asymptotically anti-de Sitter space-times, the remainder term \( \Delta_{ab} \) has a more complicated form. Although we do not have a definitive result, it seems unlikely that the extrinsic curvature terms will always conspire to cancel each other out. If they do not, the difference between \( Q_\xi \) and \( Q_\xi \) would not be just a ‘constant shift’. Then, there would be two independent sets of conserved quantities in asymptotically anti-de Sitter space-times. This seems implausible. And indeed, there is another problem with the definition of \( Q_\xi \). As noted in Section 3.3, the trace-free property of \( E_{ab} \) is essential for \( Q_\xi \) to be conserved. Unfortunately, \( \Delta_{ab} \) is not trace-free. Hence, even though there is no physical matter fields...
near infinity, the charges $Q_\xi [C]$ fail to be conserved; their values depend on the choice of the cross-section $[C]$. This fact can go unnoticed if one focuses on a specific conformal factor and on the asymptotic symmetries $\xi^a$ which are Killing fields of the resulting $h_{ab}$. However, this procedure introduces an additional structure and thus breaks covariance. As emphasized in Sections II A and III A, in general there is no preferred metric on $\cal{I}$, and we must deal squarely with the full equivalence class $[h_{ab}]$ of conformally flat metrics.

V. DISCUSSION

In its original version [12], the ADS/CFT conjecture suggests that two quite distinct theories are equivalent: i) $\cal{N} = 4$ supersymmetric $SU(N)$ Yang-Mills theory in four dimensional Minkowski space-time; and, ii) type-IIB string theory in ten space-time dimensions subject to the boundary conditions that the space-time is asymptotically a product of a (metric) 5-sphere and 5-dimensional anti-de Sitter space-time, where the radius of the 5-spheres equals the anti-de Sitter radius $\ell$. (The parameters of the two theories are related by $\ell = (4\pi g_s N)^{1/4} \sqrt{\alpha'/\bar{\alpha}}$ where $g_s$ is the string coupling constant and $T = 1/2\pi\alpha'$ is the string tension.) Although the boundary conditions used on the gravitational side are not of direct physical interest, the conjecture is fascinating from a mathematical physics perspective. In this respect it has had remarkable successes, especially in the perturbative regimes [12]. In particular, all low energy states of supergravity (corresponding to the type IIB string theory in the conjecture) can be mapped to states in the Yang-Mills Hilbert space. Furthermore, in the large $N$ limit with $g_s N$ constant, several interactions also agree: in this limit, 3-point functions can be calculated either in the supersymmetric Yang-Mills theory or in supergravity. Finally, it was possible to show that the entropy of Schwarzschild-anti-de Sitter black holes is proportional to their horizon area in the limit $r_s \gg \ell$ where $r_s$ is the Schwarzschild radius.

Because of such surprising agreements, hopes have been expressed that this duality may enable one in particular to describe classical gravity (with asymptotically anti-de Sitter boundary conditions) entirely in terms of the flat space, supersymmetric Yang-Mills theory. To test these ideas, one needs an appropriate dictionary between the two theories. A natural strategy is to begin with simple, basic observables which have ‘obvious’ correspondence in the two theories. Since the two theories share symmetry groups, observables generating these symmetries are natural candidates. One needs to first calculate these observables separately within each theory and then analyze if it is consistent to map one set to the other.

In this note, we obtained conserved charges $Q_\xi$ corresponding to asymptotic symmetries $\xi$ purely from the gravitational perspective. The field equations and the boundary conditions defining asymptotically anti-de Sitter space-times naturally led us to differential identities which, upon integration, yielded the expression of $Q_\xi$. In four space-time dimensions, our charges $Q_\xi$ agree with the quantities $Q_\xi$ obtained by the ‘counter-term subtraction method’ [14]. However, in five (or higher) dimensions they do not. Furthermore, we were able to give a general argument to show that the ‘ground state energies’ obtained in that method can not result from any covariant procedure. Therefore, from the perspective of classical general relativity, $Q_\xi$ appear to be unacceptable as Hamiltonians generating asymptotic symmetries. On the other hand, it has been argued that the results of the ‘counter term subtraction
method’ on the gravitational side agree with those obtained from the supersymmetric Yang-Mills theory and are in fact related to such fundamental features of that theory as the existence of a Casimir energy. Let us suppose this is the case. Then, one would have to conclude that there is a tension between gravity and gauge theories: the ‘natural’ dictionary would be incorrect. This is somewhat unsettling because the problem occurs for the most basic of observables on both sides. Furthermore, these are essentially the only ‘explicitly known’ diffeomorphism invariant observables in non-perturbative general relativity.

An outstanding issue then is whether a consistent dictionary does exist. Since any two separable Hilbert spaces are isomorphic, a useful correspondence between two theories has to achieve much more than an isomorphism between their Hilbert spaces: the isomorphism should map a complete set of observables in one theory to a complete set of corresponding observables in the other. Therefore, it would not suffice to just compare the ground states and their energies in the two theories. One must also account for other differences between $Q_{\xi}$ and $Q_{\zeta}$. Indeed, the difference $\Delta_{ab}$ in the integrands of the two expressions does not appear to be just a constant shift but may vary from one space-time to another, especially in the non-stationary context. More importantly, while all ten $Q_{\xi}$ are conserved in the absence of fluxes of physical matter fields across $\mathcal{I}$, the ten $Q_{\zeta}$ are not. Thus, our results seem to provide guidelines as well as constraints on possible dictionaries.

We will conclude by listing two open problems on the gravitational side. First, it would be desirable to probe the structure of $\Delta_{ab}$ in detail. Are there asymptotic equations which we have overlooked that force a cancellation between some of the terms? A simplification in the form of $\Delta_{ab}$ may be necessary in the construction of a consistent dictionary. Second, our analysis is based directly on asymptotic field equations. It should not be difficult to examine the situation using a covariant phase space formulation [23, 24]. It should also be possible to construct these quantities starting with a manifestly finite action and performing the Legendre transform; indeed it would be surprising if infinite subtractions are essential in purely classical treatments of Hamiltonians. In the asymptotically Minkowskian context, this seems to be possible. The asymptotically anti-de Sitter case is under investigation.

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\footnote{However, it is possible that these arguments overlooked some subtlety in the gauge theory. In the calculation of the Casimir energy of a field confined inside a physical box, the full anti-de Sitter invariance is broken because the box defines a rest frame. However, it is not obvious that there is such a preferred rest frame in the supersymmetric Yang-Mills theory used in the ADS/CFT conjecture.}
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