FACTORS OF HOMFLY POLYNOMIALS

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Abstract. We study factorizations of HOMFLY polynomials of certain knots and oriented links. We begin with a computer analysis of knots with at most 12 crossings, finding 17 non-trivial factorizations. Next, we give an irreducibility criterion for HOMFLY polynomials of oriented links associated to 2-connected plane graphs.

Introduction

Several properties of knots and links are encoded using polynomial invariants. Many of the properties of these polynomials are of a combinatorial nature, such as the degree, or coordinate dependent, such as special evaluations. For a few examples, see the Morton-Franks-Williams inequality [FW87, Mor86, Mor88], the slope conjecture [Gar11], some evaluations of link polynomials [LM86], degree computations [vdV].

In this paper, we propose to study a geometric property: irreducibility of the HOMFLY polynomial. Thus, we view the HOMFLY polynomial of an oriented link as a plane algebraic curve and we ask if the curve is irreducible. Since the HOMFLY polynomial is really a Laurent polynomial, we disregard the coordinate axes in our analysis.

First, we perform a computer analysis of HOMFLY polynomials of the 2977 knots with at most 12 crossings: we find 17 non-trivial factorizations (Table 1). To obtain the polynomials, we consulted the databases KnotInfo [LM20] and KnotAtlas [BNM]. To factor them, we used the computer algebra program MAGMA [BCP97].

Second, we give a sufficient criterion for irreducibility of the HOMFLY polynomials of oriented links associated to plane graphs by Jaeger in [Jae88]. A standard construction of Jaeger ([Jae88, page 649]) associates to each connected plane graph $G$ an oriented link diagram $D(G)$. Jaeger shows that the HOMFLY polynomial $P(D(G), x, y, z)$ can be computed from the Tutte polynomial $T_G(x, y)$ of $G$ using the formula

$$P(D(G), x, y, z) = \left(\frac{y}{z}\right)^{|V(G)|-1} \left(\frac{-z}{x}\right)^{|E(G)|} T_G\left(-\frac{x}{y}, 1 - \frac{(x + y)y}{z^2}\right).$$

Thus, ignoring powers of $x, y, z$, the irreducibility of $T_G(x, y)$ is a necessary condition for the irreducibility of $P(D(G), x, y, z)$. The Tutte polynomial of a 2-connected graph is irreducible by a result of Merino, de Mier and Noy ([MdMN01, Theorem 1]). Hence, we reduce the study of the irreducibility of the HOMFLY polynomial $P(D(G), x, y, z)$ to understanding how the substitution in (1) interacts with the Tutte polynomial.

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To achieve our goal, Proposition 1 simplifies formula (1). The verification of the identity is entirely mechanical, but our arguments hinge on the existence of such a simple final result. Next, Lemma 2 gives a sufficient criterion for irreducibility of polynomials, adapted to our needs. We combine these statements in Theorem 5, the main criterion for irreducibility of HOMFLY polynomials of this paper.

**Notation.** Let \( K \) be a knot. To simplify our formulas, we denote by \((K)\) the HOMFLY polynomial of the knot \( K \), defined using the convention of \([FYH+85, \text{MAIN THEOREM}]\). Thus, \((K)\) is a homogeneous Laurent polynomial of degree 0 in \( Z[x^\pm 1, y^\pm 1, z^\pm 1] \): the numerator of \((K)\) is a homogeneous polynomial in \( x, y, z \) and the denominator of \((K)\) is a monomial of the same degree as the numerator. We denote by \( \overline{K} \) the mirror image of the knot. Recall that the HOMFLY polynomial of the mirror image of a knot \( K \) satisfies the identity
\[
(\overline{K})(x, y, z) = (K)(y, x, z).
\]

To identify knots, we follow the notation of KnotInfo \([LM20]\). For convenience, we reproduce the part of the convention that is relevant for us:

"For knots with 10 or fewer crossings, we use the classical names, as tabulated for instance by Rolfsen, eliminating the duplicate 10_{162} from the count. For 11 crossing knots, we use the Dowker-Thistlethwaite name convention, based on the lexicographical ordering of the minimal Dowker notation for each knot."

For instance, \( 4_1 \) is the Figure-eight knot, while
\[
(3_1) = \frac{z^2}{y^2} - 2\frac{x}{y} - \frac{x^2}{y^2}
\]
is the HOMFLY polynomial of the left-handed Trefoil knot.

**Caution.** The convention for the HOMFLY polynomial used in \([LM20]\] differs from the one that we use. We obtain the HOMFLY polynomial \((K)_{KI}\) of the knot \( K \) tabulated in \([LM20]\) by the substitution
\[
(K)_{KI} = (K)(v^{-1}, v, -z).
\]

This happens in the background and plays almost no role in the arguments.

1. **Knots with up to 12 crossings**

We started this project wondering about irreducibility of HOMFLY polynomials. A quick calculation with a computer, shows that the HOMFLY polynomial of the knot \( 9_{12} \) is the product of the HOMFLY polynomials of the knots \( 4_1 \) (Figure-eight knot) and \( 5_2 \) (3-twist knot):
\[
(9_{12}) = (4_1)(5_2).
\]

Similarly, also the identity
\[
(11a_{175}) = (3_1)(8_{16})
\]
holds. Systematizing these results, we analyzed the knots with up to 12 crossings, using the database \([LM20]\). Out of these 2977 HOMFLY polynomials, 17 are reducible. Each one of these 17 reducible polynomials is a product of previous members of the database. When checking for divisibility, we work in the Laurent polynomial ring \( Z[x^\pm 1, y^\pm 1, z^\pm 1] \), that is, we disregard powers, positive or negative, of \( x, y, z \). Still, the factorizations that we find are correct as stated: there is no need to adjust by multiplying by a unit. We collect this data in Table 1.
In particular, the HOMFLY polynomial $(12n_{462})$ is the only one having a repeated irreducible factor. We observe also that the Kauffman polynomials of the 2977 knots with at most 12 crossings are all irreducible.

2. Graphs and oriented link diagrams

In this section, we prove a criterion for the irreducibility of the HOMFLY polynomials of certain oriented links associated to plane graphs.

To argue irreducibility, we exploit the morphism $(\mathbb{C}^*)^2 \to (\mathbb{C}^*)^2$ appearing in [Jae88, Proposition 1]:

$$J_0: \quad (\mathbb{C}^*)^2 \to (\mathbb{C}^*)^2$$

$$(x, y) \mapsto \left(-\frac{x}{y}, 1 - (x + y)y\right).$$

We simplify the expression of $J_0$ by changing coordinates on the domain and codomain of $J_0$. Denote by $\Xi: \mathbb{P}_\mathbb{C}^1 \times \mathbb{P}_\mathbb{C}^1 \to (\mathbb{C}^*)^2$ the birational map

$$\Xi: \quad \mathbb{P}_\mathbb{C}^1 \times \mathbb{P}_\mathbb{C}^1 \to (\mathbb{C}^*)^2$$

$$([x_0, x_1], [y_0, y_1]) \mapsto \left(\frac{y_0}{y_1} - \frac{x_1 y_0}{x_0 y_1}, \frac{y_0}{x_0 y_1}\right)$$

with birational inverse

$$\Xi^{-1}: \quad (\mathbb{C}^*)^2 \to \mathbb{P}_\mathbb{C}^1 \times \mathbb{P}_\mathbb{C}^1$$

$$(x_l, y_l) \mapsto \left([x_l + y_l, y_1], [x_l + y_l, 1]\right).$$

Denote by $\Sigma: (\mathbb{C}^*)^2 \to \mathbb{P}_\mathbb{C}^1 \times \mathbb{P}_\mathbb{C}^1$ the birational morphism

$$\Sigma: \quad (\mathbb{C}^*)^2 \to \mathbb{P}_\mathbb{C}^1 \times \mathbb{P}_\mathbb{C}^1$$

$$(x_g, y_g) \mapsto \left([1 - x_g, 1], [(1 - x_g)(1 - y_g), 1]\right)$$

with birational inverse

$$\Sigma^{-1}: \quad \mathbb{P}_\mathbb{C}^1 \times \mathbb{P}_\mathbb{C}^1 \to (\mathbb{C}^*)^2$$

$$([x_0, x_1], [y_0, y_1]) \mapsto \left(1 - \frac{x_0}{x_1}, 1 - \frac{x_1 y_0}{x_0 y_1}\right).$$

Define the morphism $J: \mathbb{P}_\mathbb{C}^1 \times \mathbb{P}_\mathbb{C}^1 \to \mathbb{P}_\mathbb{C}^1 \times \mathbb{P}_\mathbb{C}^1$ by setting

$$J: \quad \mathbb{P}_\mathbb{C}^1 \times \mathbb{P}_\mathbb{C}^1 \to \mathbb{P}_\mathbb{C}^1 \times \mathbb{P}_\mathbb{C}^1$$

$$([x_0, x_1], [y_0, y_1]) \mapsto \left([x_0, x_1], [y_0^2, y_1^2]\right).$$

Table 1. Factorizations of HOMFLY polynomials

| $9a_{12}$ | $(41) (52)$ |
|-----------|-------------|
| $11a_{175}$ | $(31) (8_{16})$ |
| $11a_{176}$ | $(31) (8_{17})$ |
| $11a_{220}$ | $(41) (7_{5})$ |
| $11a_{306}$ | $(31) (8_{16})$ |
| $12a_{151}$ | $(52) (7_{7})$ |
| $12a_{165}$ | $(52) (10_{136})$ |
| $12a_{259}$ | $(41) (11_{20})$ |
| $12a_{300}$ | $(41) (8_{14})$ |
| $12a_{471}$ | $(41) (8_{3})$ |
| $12a_{505}$ | $(31) (9_{33})$ |
| $12a_{506}$ | $(41) (8_{17})$ |
| $12a_{515}$ | $(31) (11_{124})$ |
| $12a_{517}$ | $(41) (10_{150})$ |
| $12a_{535}$ | $(41) (8_{16})$ |
| $12n_{362}$ | $(41)^2$ |
| $12n_{500}$ | $(31) (7_{5})$ |
Proposition 1. The rational maps $J$ and $\Sigma \circ J_0 \circ \Xi$ coincide.

Proof. This is a matter of a simple substitution, using the definition of the involved maps. $\square$

The morphism $J$ is finite of degree 2 and it is branched over the divisor $R \subset \mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}$, with equation $y_0y_1 = 0$.

In our argument for irreducibility, we exploit the following easy algebraic lemma.

Lemma 2. Let $C \subset \mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}$ be an irreducible curve, defined by the equation $F(x_0, x_1, y_0, y_1) = 0$. Assume that the polynomial $F$ is bihomogeneous of degree $a$ in $x_0, x_1$ and of degree $b$ in $y_0, y_1$. If the curve with equation $F(x_0, x_1, y_0^2, y_1^2) = 0$ is reducible, then the two polynomials $F(x_0, x_1, 1, 0)$ and $F(x_0, x_1, 0, 1)$ are squares. In particular, the degree $a$ is even.

Proof. We cover $\mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}$ by 4 standard affine charts isomorphic to $\mathbb{A}^2$, by setting one among $x_0$ or $x_1$ to 1 and also one among $y_0$ or $y_1$ to 1. Fix one of these charts. The bihomogeneous polynomial $F$ becomes an irreducible polynomial $f(x, y) \in \mathbb{C}[x, y]$. To prove the result, we show that if the polynomial $f(x, y^2)$ is reducible, then $f(x, 0) \in \mathbb{C}[x]$ is a square.

Let $g(x, y) \in \mathbb{C}[x, y]$ be an irreducible factor of $f(x, y^2)$. Since $f(x, y^2)$ is not irreducible, we deduce the inequality $g(x, y) \neq f(x, y^2)$. Separating the terms of $g(x, y)$ with an even and an odd exponent of $y$, we find polynomials $g_0(x, y^2)$ and $g_1(x, y^2)$ such that the identity

$$g(x, y) = g_0(x, y^2) + yg_1(x, y^2)$$

holds. If $g_0(x, y^2)$ vanishes, then we are done. Suppose therefore that $g_0(x, y^2)$ is not the zero polynomial. If $g_1(x, y^2)$ vanishes, then $g_0(x, y^2)$ is a proper factor of $f(x, y^2)$; as a consequence, $g_0(x, y)$ is a proper factor of $f(x, y)$, contradicting the irreducibility of $f(x, y)$. It follows that $g(x, -y)$ is a polynomial that is not proportional to $g(x, y)$ and that also divides $f(x, y^2)$. By irreducibility of $g(x, y)$, we deduce that the product $g(x, y)g(x, -y)$ divides $f(x, y^2)$. By irreducibility of $f(x, y)$, we deduce that the product $g(x, y)g(x, -y)$ actually equals $f(x, y^2)$. We therefore find

$$f(x, y^2) = g_0(x, y^2)^2 - y^2g_1(x, y^2).$$

Setting $y$ to 0, we conclude that the identity $f(x, 0) = g_0(x, 0)^2$ holds, as needed. $\square$

Remark 3. The statement above still holds replacing the complex numbers by any field $k$. If the characteristic of $k$ is different from 2, then the given proof goes through essentially unchanged. If the characteristic of $k$ is 2, then the statement follows from [Sta20, Tag 0BRA]: in this case, the morphism $J$ is purely inseparable and hence a homeomorphism.

Suppose that $G$ is a connected plane graph and denote by $T_G(x, y)$ the Tutte polynomial of $G$. Denote by $D(G)$ the associated link diagram constructed by Jaeger [Jae88]. We do not reproduce here the construction of $D(G)$: we refer the interested reader to [Jae88, Section 2]. All that we need is that the identity

$$(D(G)) = \left(\frac{y}{z}\right)^{|V(G)|-1} \left(-\frac{z}{x}\right)^{|E(G)|} T_G \left(-\frac{x}{y}, 1 - \frac{(x+y)y}{z^2}\right)$$

holds (see [Jae88, Proposition 1]).
We are interested in the irreducibility of the HOMFLY polynomial of the link diagram $D(G)$.

We view HOMFLY polynomials as elements of the Laurent polynomial ring $L = \mathbb{C}[x^{±1}, y^{±1}, z^{±1}]$. Thus, irreducibility of a non-zero element $f \in L$ means that any factorisation $f = gh$, with $g, h \in L$, implies that either $g$ or $h$ has the form $\alpha x^a y^b z^c$, with $\alpha \in \mathbb{C}$ and $a, b, c \in \mathbb{Z}$.

For a polynomial $t(x_g, y_g)$ in the coordinates $x_g, y_g$ of $(\mathbb{C}^*)^2$, we want to read the information about the ramification of the morphism $J$. Thus, we take the strict transform under $\Sigma^{-1}$ of the vanishing set of $t(x_g, y_g)$ and intersect the closure of this locus with $y_0 = 0$ and $y_1 = 0$. We summarize the outcome of this easy computation in the following lemma for future reference.

**Lemma 4.** Let $t(x_g, y_g) = \sum_{i,j} t_{ij} x_g^i y_g^j$ be a polynomial in $\mathbb{C}[x_g, y_g]$ and let $T \subset (\mathbb{C}^*)^2$ be the curve defined by the equation $t(x_g, y_g) = 0$. Let $d \in \mathbb{N}$ be the largest exponent of $y_g$ among the monomials appearing in $t(x_g, y_g)$ with non-zero coefficient.

- An equation for the intersection $\Sigma(T) \cap \{y_0 = 0, \; x_0 x_1 \neq 0\}$ is
  
  $$t \left(1 - \frac{x_0}{x_1}, 1\right) = 0.$$

- An equation for the intersection $\Sigma(T) \cap \{y_1 = 0, \; x_0 x_1 \neq 0\}$ is
  
  $$\sum_i t_{id} \left(1 - \frac{x_0}{x_1}\right)^i = 0.$$

**Proof.** We obtain an equation vanishing of the curve $\Sigma(T)$ by setting to 0 the numerator of the evaluation $t \left(1 - \frac{x_0}{x_1}, 1 - \frac{x_1 y_0}{x_0 y_1}\right)$. It is now a matter of a straightforward computation to check that the stated identities hold. \(\square\)

Let $G$ be a connected plane graph. We define two curves $P_G \subset (\mathbb{C}^*)^2$ and $\mathcal{P}_G \subset \mathbb{P}_C^1 \times \mathbb{P}_C^1$. We set

$$P_G : \quad (D(G)) (x_l, y_l, 1) = 0 \subset (\mathbb{C}^*)^2,$$

and

$$\mathcal{P}_G = \Sigma^{-1}(P_G) \subset \mathbb{P}_C^1 \times \mathbb{P}_C^1.$$

Similarly, we define two curves $T_G \subset (\mathbb{C}^*)^2$ and $\mathcal{T}_G \subset \mathbb{P}_C^1 \times \mathbb{P}_C^1$. We set

$$T_G : \quad T_G (x_g, y_g) = 0 \subset (\mathbb{C}^*)^2,$$

and

$$\mathcal{T}_G = \Sigma(T_G) \subset \mathbb{P}_C^1 \times \mathbb{P}_C^1.$$

Thus, $\mathcal{P}_G$ and $\mathcal{T}_G$ are, essentially, the vanishing of the HOMFLY polynomial of the oriented link $D(G)$ and of the Tutte polynomial of $G$, respectively.

As a consequence of the definitions and of [Jae88, Proposition 1], we deduce that there is a diagram

$$\begin{array}{ccc}
\mathbb{P}_C^1 \times \mathbb{P}_C^1 & \xrightarrow{J} & \mathbb{P}_C^1 \times \mathbb{P}_C^1 \\
\cup \ & \ & \cup \\
\mathcal{P}_G & \rightarrow & \mathcal{T}_G
\end{array}$$

and $J|_{\mathcal{P}_G} : \mathcal{P}_G \rightarrow \mathcal{T}_G$ is therefore a branched double cover.
Theorem 5. Let $G$ be a 2-connected plane graph. If the HOMFLY polynomial $(D(G))$ is reducible then

- the number of edges of $G$ is even;
- the number of vertices of $G$ is even;
- the polynomial $T_G(x, 1)$ is a square.

Proof. Let $\tilde{T}_G(x_0, x_1, y_0, y_1) \in \mathbb{Z}[x_0, x_1, y_0, y_1]$ be the numerator of the Laurent polynomial $T_G \circ \Sigma^{-1}$. By construction, the polynomial $\tilde{T}_G$ is bihomogeneous of degree $h_1(G)$ in $y_0, y_1$ and of degree

$$\deg_x T_G + \deg_y T_G = \#V(G) - 1 + h_1(G) = \#E(G)$$

in $x_0, x_1$. The vanishing set of $\tilde{T}_G$ in $\mathbb{P}^1 \times \mathbb{P}^1$ is the curve $\mathcal{P}_G$.

Because the graph $G$ is 2-connected, the Tutte polynomial $T_G(x, y)$ is irreducible by [MdMN01, Theorem 1]: even though the cited paper states irreducibility over $\mathbb{Z}$, the authors mention, and their argument shows, that $T_G(x, y)$ is also irreducible in $\mathbb{C}[x, y]$. Since $\Sigma$ is a birational map, the polynomial $\tilde{T}_G$ is also irreducible, and hence so is the curve $\mathcal{P}_G$. An equation of curve $\mathcal{P}_G$ is $\tilde{T}_G(x_0, x_1, y_0^2, y_1^2) = 0$. If $\mathcal{P}_G$ is reducible, then we are in a position to apply Lemma 2. We deduce that the number of edges of $G$, the degree of $\tilde{T}_G$ with respect to $x_0, x_1$, is even. Using Lemma 4, we evaluate $\tilde{T}_G(x, 1, 0, 1)$ and we find

$$\tilde{T}_G(x, 1, 0, 1) = T_G(1 - x, 1).$$

We obtain that the evaluation $T_G(1 - x, 1)$, or, equivalently, $T_G(x, 1)$, is a square. Finally, since the degree of $T_G(x, 1)$ is $\#E(G) - h_1(G) - 1$, and we already argued that $\#E(G)$ is even, we deduce that $h_1(G) - 1$ is even. Since $G$ is connected, the identity $h_1(G) - 1 = \#E(G) - \#V(G)$ holds. As we already showed that $\#E(G)$ is even, we conclude that $\#V(G)$ is even and the proof is complete. \hfill \Box

Remark 6. Let $G$ be a finite graph. Define a simplicial complex $F(G)$ on the edges of $G$ by letting $\sigma \subset E(G)$ be a face of $F(G)$ if and only if $\sigma$ contains no cycle. The evaluation $T_G(1 - x, 1)$ is the face polynomial of the simplicial complex $F(G)$.

3. Further directions

We found that every reducible HOMFLY polynomial of a knot with at most 12 crossings is itself the product of irreducible HOMFLY polynomials of knots. We would find it surprising if this was always the case. Nevertheless, it would be interesting to study further the divisibility properties of HOMFLY polynomials of knots (or even of links). At an experimental level, extensive tables of HOMFLY polynomials of knots and links are available, so gathering further evidence is easily within reach. At a conceptual level, we would find it very interesting to predict factorizations of HOMFLY polynomials, without having to look them up in tables.

We could not find a 2-connected plane graph $G$ with an even number of vertices and of edges and such that the evaluation $T_G(x, 1)$ is a square, nor we could prove that they do not exist. Our expectation is that such graphs do not exist. If this were the case, then it would follow from Theorem 5 that the HOMFLY polynomials of the oriented links associated to 2-connected plane graphs are all irreducible. Using Remark 6 we can reformulate one of the conditions on the graph saying that the
face polynomial of a simplicial complex is a square. We have never come across a similar condition.

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