On boundedness and growth of unsteady solutions under the double porosity/permeability model

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Figure 1. This figure shows that the evolution of an unsteady solution under the DPP model satisfies the theoretical bound derived in this paper. $\|Y\|_V$ denotes a norm defined in terms of the velocities in the two pore-networks.
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double porosity/permeability model

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Abstract. There is a recent surge in research activities on modeling the flow of fluids in porous media with complex pore-networks. A prominent mathematical model, which describes the flow of incompressible fluids in porous media with two dominant pore-networks allowing mass transfer across them, is the double porosity/permeability (DPP) model. However, we currently do not have a complete understanding of unsteady solutions under the DPP model. Also, because of the complex nature of the mathematical model, it is not possible to find analytical solutions, and one has to resort to numerical solutions. It is therefore desirable to have a procedure that can serve as a measure to assess the veracity of numerical solutions. In this paper, we establish that unsteady solutions under the transient DPP model are stable in the sense of Lyapunov. We also show that the unsteady solutions grow at most linear with time. These results not only have a theoretical value but also serve as valuable a posteriori measures to verify numerical solutions in the transient setting and under anisotropic medium properties, as analytical solutions are scarce for these scenarios under the DPP model.

1. OPENING STATEMENT

The study of the flow of fluids through porous media is central to various technological applications such as geological carbon sequestration, water purification, bioremediation, and enhanced oil recovery. Mathematical modeling often plays a crucial role in understanding the underlying dynamics in these applications, as the interior of the medium is inaccessible, notably, in subsurface applications. Also, the practical problems in these application areas are so complicated that they are not tractable via an analytical approach, and numerical solutions serve as the only viable tool.

Traditionally, Darcy equations have been used to model the flow of fluids through porous media. Because of the inherent simplicity of Darcy equations, and driven by their popularity, one can find in the literature analytical solutions for many problems [Strack, 2017] and several robust numerical formulations [Chen et al., 2006; Masud and Hughes, 2002]. Researchers have also established a myriad of mathematical properties that the solutions to Darcy equations satisfy, and these properties can serve as a posteriori measures to assess the accuracy of numerical solutions; for example, see [Shabouei and Nakshatrala, 2016]. However, it is vital to realize that the Darcy model is valid under a plethora of assumptions [Nakshatrala and Rajagopal, 2011; Rajagopal,

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An assumption relevant to this paper is that the Darcy model assumes that the porous medium comprises a single dominant pore-network. Because of the recent interest in the exploration of unconventional hydrocarbons (e.g., oil and gas extraction from tight shale), the attention has shifted towards porous media with complex pore-networks. The latest revolution in manufacturing, such as additive manufacturing, has enabled us to build porous media with complicated pore structures and networks tailored to specific needs. It should be clear that Darcy equations are not adequate to model the flow of fluids in porous media with such complex pore-networks. The flow dynamics in a porous medium with two pore-networks can be complex and differ from that of a single pore-network. This inadequacy has resulted in the development of mathematical models, more complicated than Darcy equations, to address porous media with complex pore-networks. In particular, there has been a tremendous focus on flows in porous media with two or more dominant pore-networks with mass transfer across them; for example, see [Barenblatt et al., 1960; Borja and Koliji, 2009; Choo et al., 2015; Nakshatrala et al., 2018]. We will consider one such model, which has been proposed in [Nakshatrala et al., 2018], and refer to it as the double porosity/permeability (DPP) model from here onwards.

Recently, researchers have developed analytical solutions for steady-state responses under the DPP model [Nakshatrala et al., 2018]. However, analytical solutions for the DPP model under transient conditions and anisotropic medium properties are scarce. The primary reasons for the scarcity are (i) the mathematical model is complex, comprising four coupled partial differential equations expressed in terms of four field variables, and (ii) anisotropy gives rise to tensorial (rather than scalar) permeabilities, and tensorial quantities are more challenging to deal with than scalars. Other studies have developed numerical formulations to solve the governing equations under the DPP model [Joodat et al., 2018; Joshaghani et al., 2019]. Although some of these studies have addressed the transient DPP model, their focus has been narrow, primarily aimed at obtaining numerical solutions for specific initial-boundary value problems. To the best of the author’s knowledge, there is no study on the general nature of unsteady solutions. It is fitting to recall the words of Truesdell in his book on Six Lectures on Modern Natural Philosophy [Truesdell, 1966], “A mathematical theory is empty if it does not go beyond a few postulates, definitions, and routine calculations. Theorems must be proved, theorems, good theorems.” Motivated by these words, this paper takes a modest step towards filling the lacuna in the theory of DPP.

Returning to the other focus of this paper—regarding numerical solutions—it is imperative that a numerical simulator has to be well-tested by performing a series of checks before using it to carry out predictive numerical simulations. To put it another way, one needs to perform verification of solutions on the numerical simulator. Two popular strategies for verification are a comparison of the numerical solution with the analytical solution and the method of manufactured solutions [Oberkampf and Blottner, 1998; Roache, 2002]. However, as mentioned earlier, analytical solutions are scarce for the DPP model, and the method of manufactured solutions uses unrealistic boundary conditions and forcing functions. It is therefore desirable to have an alternate technique to check a numerical implementation so that one can use the formulation to solve other problems with confidence. It is also useful if we know the nature of the unsteady solutions and bounds on the growth or decay of solution fields with the time.

In the rest of this paper, we shall show that the solutions under the transient DPP model are stable in the sense of Lyapunov. We also show that the growth of the unsteady solutions can be at most linear in time under homogeneous boundary conditions. We will illustrate how one can utilize
Let us consider a porous medium comprising two dominant pore-networks, referred to as the macro- and micro-pore networks. Each of these pore-networks has its hydromechanical properties; however, there could be a transfer of mass across the pore-networks. We denote the spatial domain by \( \Omega \subset \mathbb{R}^{nd} \), where “\( nd \)” denotes the number of spatial dimensions. A spatial point is denoted by \( x \). The divergence and gradient operators with respect to \( x \) are, respectively, denoted by \( \text{div}[\cdot] \) and \( \text{grad}[\cdot] \). We denote the time by \( t \in [0,T] \), where \( T \) denotes the length of the time interval of interest.

For convenience, the quantities associated with the macro- and micro-pore networks will be indicated with subscripts 1 and 2, respectively. We denote the volume fractions by \( \phi_1 \) and \( \phi_2 \), the true (seepage) velocities by \( v_1(x,t) \) and \( v_2(x,t) \), the pressures by \( p_1(x,t) \) and \( p_2(x,t) \), and the bulk densities by \( p_1 \) and \( p_2 \). We denote the coefficient of viscosity and true density of the fluid by \( \mu \) and \( \gamma \), respectively. The bulk densities are related to the true density of the fluid as follows:

\[
\rho_1 = \gamma \phi_1 \quad \text{and} \quad \rho_2 = \gamma \phi_2 \quad (2.1)
\]

It is also common to work in terms of the Darcy (discharge) velocities, which are defined as follows:

\[
u_1(x) = \phi_1(x) v_1(x) \quad \text{and} \quad u_2(x) = \phi_2(x) v_2(x) \quad (2.2)\]

However, herein, we will work with the true velocities, and extending the framework based on the Darcy velocities is straightforward.

The transient governing equations of the DPP model take the following form:

\[
\rho_1 \frac{\partial v_1}{\partial t} + \mu \phi_1^2 K_1^{-1} v_1 + \phi_1 \text{grad}[p_1] = \rho_1 b_1 \quad (2.3)
\]

\[
\rho_2 \frac{\partial v_2}{\partial t} + \mu \phi_2^2 K_2^{-1} v_2 + \phi_2 \text{grad}[p_2] = \rho_2 b_2 \quad (2.4)
\]

\[
\text{div}[\phi_1 v_1] = -\frac{\beta}{\mu} (p_1 - p_2) \quad (2.5)
\]

\[
\text{div}[\phi_2 v_2] = +\frac{\beta}{\mu} (p_1 - p_2) \quad (2.6)
\]

where \( b_1 \) and \( b_2 \) denote the specific body force in the pore-networks, and \( \beta \) is a characteristic parameter of the porous medium. We often have \( b_1 = b_2 \) in practical situations; for example, the specific body force in each of the pore-networks is the acceleration due to gravity. It is important to note that \( \phi_1, \phi_2, p_1, p_2 \) and \( \mu \) are all positive, and \( \beta \) is non-negative. The permeabilities \( K_1 \) and \( K_2 \) are symmetric and positive definite tensors. The quantity \( -\frac{\beta}{\mu} (p_1 - p_2) \) is the rate of volumetric transfer from the micro-pore network to the macro-pore network. The corresponding rate of mass transfer will then be \( -\frac{\beta \gamma}{\mu} (p_1 - p_2) \).

The boundary conditions take the following form:

\[
v_1(x,t) \cdot \hat{n}(x) = v_{n1}(x,t) \quad \text{on } \Gamma_1^n \quad (2.7a)
\]

\[
v_2(x,t) \cdot \hat{n}(x) = v_{n2}(x,t) \quad \text{on } \Gamma_2^n \quad (2.7b)
\]

\[
p_1(x,t) = p_{1b}(x,t) \quad \text{on } \Gamma_1^b \quad (2.7c)
\]

\[
p_2(x,t) = p_{2b}(x,t) \quad \text{on } \Gamma_2^b \quad (2.7d)
\]
where $\Gamma_v^1$ and $\Gamma_p^1$ are the *complementary* partitions of the boundary $\partial \Omega$, and likewise with $\Gamma_v^2$ and $\Gamma_p^2$ partitions. The initial conditions are prescribed as follows:

$$v_1(x,0) = v_1^0(x) \quad \text{and} \quad v_2(x,0) = v_2^0(x) \quad \forall x \in \Omega$$ (2.8)

We now mention two main assumptions behind the DPP model, and these assumptions are crucial in establishing the mathematical results presented in the subsequent sections. First, the volume fraction of each pore-network is independent of the time. That is,

$$\frac{\partial \phi_1}{\partial t} = 0 \quad \text{and} \quad \frac{\partial \phi_2}{\partial t} = 0$$ (2.9)

Second, the true density of the fluid, $\gamma$, is independent of the time. Note that this assumption is a stronger condition than the fluid is incompressible. In lieu of equations (2.1) and (2.9), the bulk density in each pore-network is independent of time. That is,

$$\frac{\partial \rho_1}{\partial t} = 0 \quad \text{and} \quad \frac{\partial \rho_2}{\partial t} = 0$$ (2.10)

A derivation along with a complete list of assumptions behind the DPP model are presented in [Nakshatrala et al., 2018].

### 3. Stability

We now show that the unsteady solutions under the transient DPP model are stable in the sense of a dynamical system. In particular, the solutions are Lyapunov stable [Dym, 2002; Hale and Koçak, 2012]. We assume the velocity boundary conditions to be homogeneous (i.e., $v_{n1} = 0$ on $\Gamma_v^1$ and $v_{n2} = 0$ on $\Gamma_p^2$). However, we allow the pressure boundary conditions to be non-homogeneous.

For convenience, let

$$\Upsilon = \{ v_1(x,t) \} = \{ v_2(x,t) \}$$ (3.1)

We denote the equilibrium solution as follows:

$$\Upsilon_{eq} = \{ 0 \}$$ (3.2)

We consider the following functional, as potential candidate for Lyapunov functional:

$$V(\Upsilon) := \int_\Omega \left( \frac{1}{2} \rho_1 v_1 \cdot v_1 + \frac{1}{2} \rho_2 v_2 \cdot v_2 \right) d\Omega + \Pi_{ext} - \Pi_{ext}^{eq}$$ (3.3)

where $\Pi_{ext}$ denotes the potential energy due to external loadings and $\Pi_{ext}^{eq}$ is the potential energy due to external loadings under an equilibrium state at a given instance of time. We assume the external loadings are conservative, which allows us to write the following:

$$\frac{d\Pi_{ext}}{dt} = -\int_\Omega (\rho_1 b_1 \cdot v_1 + \rho_2 b_2 \cdot v_2) d\Omega + \int_{\Gamma_v^1} \phi_1 p_1^0 v_1 \cdot \hat{n} \ d\Gamma + \int_{\Gamma_p^2} \phi_2 p_2^0 v_2 \cdot \hat{n} \ d\Gamma$$ (3.4)

It is important to note that at a given instance of time, say $t = t_0$, we have

$$\Pi_{ext|t=t_0} = \Pi_{ext|t=t_0}^{eq}$$ (3.5)

but

$$\frac{d\Pi_{ext}}{dt} \bigg|_{t=t_0} \neq 0 \quad \text{and} \quad \frac{d\Pi_{ext}^{eq}}{dt} \bigg|_{t=t_0} = 0$$ (3.6)
Figure 2. This figure illustrates properties of $\Pi_{\text{ext}}$ and $\Pi_{\text{ext}}^{\text{eq}}$, given by equations (3.5) and (3.6). We have denoted the one-parameter family of equilibrium states by $\Pi_{\text{ext}}^{\text{eq}}$.

Figure 2 gives a pictorial description of the above equations.

We now show that $V$ is a Lyapunov functional for the DPP model. The task at hand is to show that the functional $V$ satisfies the following three properties:

(i) $V(\Upsilon = \Upsilon_{\text{eq}}) = 0$,
(ii) $V(\Upsilon \neq \Upsilon_{\text{eq}}) > 0$, and
(iii) $dV/dt < 0$ for all $\Upsilon \neq \Upsilon_{\text{eq}}$.

The first two conditions are direct consequences of equation (3.5) and the definition of $\Upsilon_{\text{eq}}$. To establish the third condition, we proceed as follows:

$$
\frac{dV}{dt} = \int_{\Omega} \left( \rho_1 v_1 \cdot \frac{\partial v_1}{\partial t} + \rho_2 v_2 \cdot \frac{\partial v_2}{\partial t} \right) \, d\Omega + \frac{d\Pi_{\text{ext}}}{dt}.
$$

(3.7)

Using the balance of linear momentum in each pore-network, given by equations (2.3) and (2.4), we obtain the following:

$$
\frac{dV}{dt} = -\int_{\Omega} v_1 \cdot (\mu \phi_1^2 K_1^{-1} v_1 + \phi_1 \text{grad}[p_1] - \rho_1 b_1) \, d\Omega
$$

$$
- \int_{\Omega} v_2 \cdot (\mu \phi_2^2 K_2^{-1} v_2 + \phi_2 \text{grad}[p_2] - \rho_2 b_2) \, d\Omega + \frac{d\Pi_{\text{ext}}}{dt}.
$$

(3.8)

Using the Green’s identity and equation (3.4), we get the following:

$$
\frac{dV}{dt} = -\int_{\Omega} (v_1 \cdot \mu \phi_1^2 K_1^{-1} v_1 + v_2 \cdot \mu \phi_2^2 K_2^{-1} v_2) \, d\Omega + \int_{\Omega} \text{div}[\phi_1 v_1] p_1 \, d\Omega + \int_{\Omega} \text{div}[\phi_2 v_2] p_2 \, d\Omega
$$

$$
- \int_{\partial \Omega} \phi_1 p_1 v_1 \cdot \hat{n} \, d\Gamma - \int_{\partial \Omega} \phi_2 p_2 v_2 \cdot \hat{n} \, d\Gamma + \int_{\Gamma_1} \phi_1 p_1^0 v_1 \cdot \hat{n} \, d\Gamma + \int_{\Gamma_2} \phi_2 p_2^0 v_2 \cdot \hat{n} \, d\Gamma
$$

(3.9)
Using the boundary conditions for the pressures, and invoking the assumption that the velocity boundary conditions are homogeneous, we obtain the following:

\[
\frac{dV}{dt} = -\int_\Omega (v_1 \cdot \mu \phi_1^2 \mathbf{K}^{-1} v_1 + v_2 \cdot \mu \phi_2^2 \mathbf{K}^{-1} v_2) \, d\Omega + \int_\Omega \text{div}[\phi_1 v_1] p_1 \, d\Omega + \int_\Omega \text{div}[\phi_2 v_2] p_2 \, d\Omega
\]  

(3.10)

Using the balance of mass in each pore-network, given by equations (2.5) and (2.6), we obtain the following:

\[
\frac{dV}{dt} = -\int_\Omega (v_1 \cdot \mu \phi_1^2 \mathbf{K}^{-1} v_1 + v_2 \cdot \mu \phi_2^2 \mathbf{K}^{-1} v_2) \, d\Omega - \int_\Omega \beta \mu (p_1 - p_2)^2 \, d\Omega
\]  

(3.11)

Noting that \( \beta \geq 0, \mu > 0, \mathbf{K}_1 \) and \( \mathbf{K}_2 \) are positive definite, we conclude that

\[
\frac{dV}{dt} < 0 \quad \forall \mathbf{\Upsilon} \neq \mathbf{\Upsilon}_{eq}
\]  

(3.12)

This implies that \( V \) is a non-increasing functional along the flow field, and this establishes that it is a Lyapunov functional for the DPP model. From the theory of dynamical systems [Dym, 2002], we conclude that the solutions under the DPP model are Lyapunov stable.

### 4. GROWTH OF UNSTEADY SOLUTIONS

The governing equations, presented in Section 2, can be compactly written as the following constrained evolution problem:

\[
\frac{d\mathbf{\Upsilon}}{dt} = \mathcal{L}[\mathbf{\Upsilon}, p_1, p_2] + \mathbf{f} \\
0 = \mathcal{G}[\mathbf{\Upsilon}, p_1, p_2]
\]

(4.1) (4.2)

where the linear operators \( \mathcal{L}[] \) and \( \mathcal{G}[] \), and \( \mathbf{f} \) are, respectively, defined as follows:

\[
\mathcal{L}[\mathbf{\Upsilon}, p_1, p_2] = \left\{ \frac{-\mu \phi_1^2 \mathbf{K}^{-1}_1 v_1 - \phi_1 \text{grad}[p_1]}{p_1} \right\} \\
\mathcal{G}[\mathbf{\Upsilon}, p_1, p_2] = \left\{ \text{div}[\phi_1 v_1] + \frac{\beta}{\mu} (p_1 - p_2) \right\}
\]

\[
\mathbf{f} = \left\{ \begin{array}{l} b_1 \\ b_2 \end{array} \right\}
\]

(4.3) (4.4) (4.5)

We assume homogeneous boundary conditions are enforced on the entire boundary.

We denote the standard \( L_2 \) inner-product for scalar and vector fields defined on \( \Omega \) by \( \langle ; ; \rangle \). That is, for given scalar fields \( a \) and \( b \) and vector fields \( \mathbf{a} \) and \( \mathbf{b} \) we have

\[
\langle a; b \rangle = \int_\Omega ab \, d\Omega \quad \text{and} \quad \langle \mathbf{a}; \mathbf{b} \rangle = \int_\Omega \mathbf{a} \cdot \mathbf{b} \, d\Omega
\]

(4.6)

We consider the following product function space:

\[
\mathcal{V} = (L_2(\Omega))^{nd} \times (L_2(\Omega))^{nd}
\]

(4.7)

A natural inner-product on \( \mathcal{V} \) will be

\[
\langle \mathbf{\Upsilon}; \mathbf{\bar{\Upsilon}} \rangle = \int_\Omega (v_1 \cdot \mathbf{\bar{v}}_1 + v_2 \cdot \mathbf{\bar{v}}_2) \, d\Omega \quad \forall \mathbf{\Upsilon}, \mathbf{\bar{\Upsilon}} \in \mathcal{V}
\]

(4.8)
where
\[ \tilde{\Upsilon} = \left\{ \begin{array}{l} \tilde{v}_1(x, t) \\ \tilde{v}_2(x, t) \end{array} \right\} \] (4.9)

The norm corresponding to the inner-product \( \langle \cdot ; \cdot \rangle \) is defined as follows:
\[ \| \Upsilon \| := \sqrt{\langle \Upsilon; \Upsilon \rangle} \] (4.10)

However, noting that \( \rho_1 > 0 \) and \( \rho_2 > 0 \), we choose the following convenient inner-product on the function space \( V \):
\[ \langle \Upsilon; \tilde{\Upsilon} \rangle_V = \int_{\Omega} \left( \rho_1 v_1 \cdot \tilde{v}_1 + \rho_2 v_2 \cdot \tilde{v}_2 \right) d\Omega \quad \forall \Upsilon, \tilde{\Upsilon} \in V \] (4.11)

The associated norm on \( V \) is defined as follows:
\[ \| \Upsilon \|_V := \sqrt{\langle \Upsilon; \Upsilon \rangle_V} \] (4.12)

Noting that the bulk densities are bounded below and bounded above by finite positive constants, the norms \( \| \cdot \| \) and \( \| \cdot \|_V \) are equivalent. To wit, if
\[ 0 < \rho_{\text{min}} \leq \rho_1, \rho_2 \leq \rho_{\text{max}} < \infty \] (4.13)

then we have
\[ \rho_{\text{min}} \| \Upsilon \| \leq \| \Upsilon \|_V \leq \rho_{\text{max}} \| \Upsilon \| \] (4.14)

We first show that operator \( L \) is dissipative on \( V \) which is used then to establish that the growth of the unsteady solutions is at most linear with time.

To establish that the operator \( L \) is dissipative on \( V \), we need to show the following:
\[ \langle \Upsilon; L \rangle_V \leq 0 \quad \forall \Upsilon \in V \] (4.15)

We proceed by substituting the definition of \( L \), equation (4.3), into the left hand side of (4.15):
\[ \langle \Upsilon; L \rangle_V = -\langle v_1; \mu \phi_1 \Delta v_1 \rangle - \langle v_2; \mu \phi_2 \Delta v_2 \rangle \] (4.16)

Noting that \( K_1 \) and \( K_2 \) are positive definite tensors, \( \mu > 0, \phi_1 > 0 \) and \( \phi_2 > 0 \), we conclude the following:
\[ \langle \Upsilon; L \rangle_V \leq -\langle v_1; \phi_1 \Delta [p_1] \rangle - \langle v_2; \phi_2 \Delta [p_2] \rangle \] (4.17)

Invoking Green’s identity and noting that the boundary conditions are homogeneous, we obtain the following:
\[ \langle \Upsilon; L \rangle_V \leq \langle \text{div}[\phi_1 v_1]; p_1 \rangle + \langle \text{div}[\phi_2 v_2]; p_2 \rangle \] (4.18)

Using the incompressibility constraints, given by equations (4.2) and (4.4), we obtain the following:
\[ \langle \Upsilon; L \rangle_V \leq -\left\langle \frac{\beta}{\mu} (p_1 - p_2); (p_1 - p_2) \right\rangle \] (4.19)

Noting that \( \beta \geq 0 \) and \( \mu > 0 \), we obtain the desired result: \( \langle \Upsilon; L \rangle_V \leq 0 \).

We now address the growth of the unsteady solutions. We proceed as follows:
\[ \| \Upsilon \|_V \frac{d}{dt} \left( \| \Upsilon \|_V \right) = \frac{d}{dt} \left( \frac{1}{2} \| \Upsilon \|_V^2 \right) = \frac{d}{dt} \left( \frac{1}{2} \langle \Upsilon; \Upsilon \rangle_V \right) \]
\[ = \langle \Upsilon; \partial \Upsilon/\partial t \rangle_V \]
\[ = \langle \Upsilon; L \Upsilon, p_1, p_2 \rangle_V + \langle \Upsilon; f \rangle_V \] (4.20)
Noting that \( L \) is dissipative, inequality (4.15), we obtain the following:
\[
\|\mathbf{\Upsilon}\|_{V} \frac{d}{dt} (\|\mathbf{\Upsilon}\|_{V}) \leq \langle \mathbf{\Upsilon}; f \rangle_{V}
\] (4.21)

By invoking the Cauchy-Schwarz inequality, we obtain the following:
\[
\|\mathbf{\Upsilon}\|_{V} \frac{d}{dt} (\|\mathbf{\Upsilon}\|_{V}) \leq \|\mathbf{\Upsilon}\|_{V} \|f\|_{V}
\] (4.22)

For \( \|\mathbf{\Upsilon}\|_{V} \neq 0 \), we conclude:
\[
\frac{d}{dt} (\|\mathbf{\Upsilon}\|_{V}) \leq \|f\|_{V}
\] (4.23)

By integrating both sides with time, we establish:
\[
\|\mathbf{\Upsilon}\|_{V} \leq tf_{\text{max}} + c
\] (4.24)

where
\[
f_{\text{max}} = \max_{t \in [0,T]} \|f(x,t)\|_{V}
\] (4.25)
\[
c = \|\mathbf{\Upsilon}(x, t = 0)\|_{V}
\] (4.26)

Since \( f_{\text{max}} \) and \( c \) are constants and finite, we conclude that the unsteady solutions under the DPP model grow at most linear with time if the driving forcing functions are bounded.

5. A REPRESENTATIVE NUMERICAL RESULT

We now show how one can use the bound on the growth of unsteady solutions to check the veracity of numerical solutions. We proceed by outlining an initial-boundary value problem (IBVP) under the DPP model and get numerical solutions of the IBVP using stable numerical formulation and discretization techniques. The norm \( \|\mathbf{\Upsilon}\|_{V} \) will be calculated using the resulting numerical solutions and compared with the derived theoretical bound on the norm.

The computational domain is a unit square: \( \Omega = (0,1) \times (0,1) \). The boundary conditions are no flow on the entire boundary for both the pore-networks (i.e., \( v_{n1}(x) = 0 \) and \( v_{n2}(x) = 0 \)). Since the boundary conditions are no flow on the entire boundary for both the pore-networks (i.e., homogeneous velocity boundary conditions), we prescribe pressure at a point in the domain for one of the pore-networks to ensure uniqueness of solutions. For further details on the uniqueness of solutions, see [Joodat et al., 2018; Nakshatrala et al., 2018].

The time interval of interest for the numerical study is \([0,2]\). The backward Euler, which is an unconditionally stable time-stepping scheme, is used with a time-step of 0.001. Table 1 provides the parameters used in the simulation. We considered two cases, each of which has a different specific body force. We chose different anisotropic permeabilities for the macro- and micro-pore networks, as we want the flow dynamics to be a characteristic of the DPP model and different from that of the Darcy model. See [Nakshatrala et al., 2018] for a discussion on the scenarios under which Darcy equations can capture the solutions of the DPP model.

Figure 3 shows the three-node triangular mesh used in the numerical simulation. We used the continuous Galerkin formulation with cubic interpolation for the velocity fields and linear interpolation for the pressure fields. This combination of interpolation functions—the so-called \( P_3 P_1 \) interpolation on triangular elements—satisfies the Ladyzhenskaya-Babuška-Brezzi (LBB) condition.
Figure 3. This figure shows the mesh used in the numerical simulation.

[Brezzi and Fortin, 2012]; see Figure 4. For the initial conditions, the intercept for the bound will be \( c = 1.5811 \). The slope of the bound for the two cases will be:

\[
\text{case 1: } f_{\text{max}} = 5.5902 \\
\text{case 2: } f_{\text{max}} = 5
\]

Figure 5 shows that the numerical results satisfy the theoretical bound, given by equation (4.24).

| Table 1. Parameters used in the numerical simulations. |
|----------------------------------|---|
| **Quantity**                     | **Value** |
| True density, \( \gamma \)       | 1        |
| Coefficient of viscosity, \( \mu \) | 1        |
| Mass transfer parameter, \( \beta \) | 0.5      |
| Macro volume fraction, \( \phi_1 \) | 0.2      |
| Micro volume fraction, \( \phi_2 \) | 0.05     |
| Macro drag coefficient, \( \mu K_1^{-1} \) | \[
\begin{bmatrix}
1 & 0.1 \\
0.1 & 0.9
\end{bmatrix}
\] |
| Micro drag coefficient, \( \mu K_2^{-1} \) | \[
\begin{bmatrix}
100 & 5 \\
5 & 100
\end{bmatrix}
\] |
| Macro-velocity initial condition  | \( \mathbf{u}_1^0 = \phi_1 \mathbf{v}_1^0 = (\sin(\pi x) \cos(\pi y), -\cos(\pi x) \sin(\pi y)) \) |
| Micro-velocity initial condition  | \( \mathbf{u}_2^0 = \phi_2 \mathbf{v}_2^0 = (0, 0) \) |
| Case 1: specific body force, \( \mathbf{b}_1 = \mathbf{b}_2 = \mathbf{b} \) | \( (10 \sin(\pi x t), 5 \sin(2 \pi x y t)) \) |
| Case 2: specific body force, \( \mathbf{b}_1 = \mathbf{b}_2 = \mathbf{b} \) | \( (0, -10) \) |

6. CONCLUDING REMARKS

We have presented two mathematical properties that the solutions under the transient DPP model satisfy: the unsteady solutions are Lyapunov stable, and they grow at most linear with time under homogeneous boundary conditions. Using a representative numerical example, we have
Figure 4. This figure shows the profiles of rates of volumetric transfer from the micro-pore network to the macro-pore network at time $t = 1$. The numerical results are stable and do not display any spurious oscillations which are typical of LBB violations.

Figure 5. This figure shows the evolution of the norm $\|\mathbf{Y}\|_V$ with time for the two cases considered in this paper. The corresponding bounds are plotted. The growth of the norm under the numerical simulations respect the theoretical bound derived in the paper.

shown that the second property—the nature of the growth—can serve as a verification procedure to check computer implementation of a numerical formulation. The attractive features are that the verification technique is easy to implement, in the form of a posteriori measure, non-intrusive (i.e., one need not rewrite the computer code), and valid even under anisotropic medium properties. The results presented in this paper enlarge the repository of verification techniques to assess the accuracy of numerical simulations for the DPP model.

One potential future work could be towards using the tools from functional analysis to get other mathematical properties and devise an array of verification techniques to assess the accuracy of numerical solutions under the transient DPP model.
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