Global asymptotics of particle transport in porous medium

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Abstract. Particle transport in a porous medium occurs in environmental, chemical and industrial technologies. The transport of suspended concrete grains in a liquid grout through porous soil is used in construction industry to strengthen foundations. When particles are transported by a fluid flow in a porous medium, some particles are retained in the pores and form a deposit. The aim of the work is the construction and study of a one-dimensional mathematical model of particle transport and retention in the porous medium, taking into account the simultaneous action of several particle capture mechanisms. The model consists of mass balance equation and the kinetic equation of deposit growth. The deposit growth rate is proportional to the filtration function of the retained particles concentration, and the nonlinear concentration function, which depends on the concentration of suspended particles. The use of a new parameter, depending on the distance to the porous medium inlet allows to construct a global asymptotic solution in the entire area of the mathematical model. Asymptotics is obtained as a series in two small parameters. The global asymptotics is close to the numerical solution at all points of the porous medium at any time.

1. Introduction

Particle transport in a porous medium occurs during the purification of drinking water, the filtration of industrial and municipal waste water, the spread of bacteria and viruses in groundwater [1]. In the construction industry, to strengthen loose soil, a liquid grout is pumped into it under pressure. Grains of the grout are transported in the pores of the soil and, after solidification, create a solid foundation [2].

Particle transport in a porous medium is accompanied by their retention and transformation into a stationary deposit. Various physical particle capture mechanisms are known. If the pore and particle size distributions overlap, a geometric size-exclusion mechanism prevails. Particles pass freely through large pores and get stuck in the inlet of small pores, whose sizes are smaller than the diameter of the particles. At high speeds, hydrodynamic forces bring together particles moving in the fluid flow, and they can form arch bridges that block the pore inlets, which are larger than the particle sizes. Particles moving near the walls of the pores can get into the recesses of the uneven surface of the pores and linger in them. Electrostatic forces can attract particles to the walls of large pores. Some particles can diffuse into dead-end pores. Depending on the structure and composition of the porous medium, suspended particles, and the carrier fluid, the main role is played by one or more particle capture mechanisms [3].

The fluid flow with suspended particles moves in a porous medium due to the pressure difference at the inlet and outlet (Darcy's law). It is assumed that at the porous medium inlet the concentration of
particles is constant; at the initial time, the porous medium does not contain suspended and retained particles. The flow of the suspension moves at a constant speed from inlet to outlet. The boundary line between the empty porous medium and the suspension is called the concentrations front of suspended and retained particles.

The one-dimensional mathematical model of particle transport in a porous medium includes the equation of mass balance of suspended and retained particles and the kinetic equation of deposit growth. Deposit growth is proportional to the filtration function $\Lambda(S)$, which depends on the concentration of retained particles $S$, and the function of the suspended particles concentration $C$ [4]. The mass balance equation and the kinetic equation of deposit growth form a closed hyperbolic system of first-order equations. Unknown functions are the suspended and retained particles concentrations, depending on the time and distance from the porous medium inlet.

In most particle transport models, the linear decreasing filtration function (Langmuir coefficient) is most often used. At a low concentration of suspended particles, the growth of the deposit is proportional to the first degree of $C$. This case was studied in detail in [5]. Solutions to the problem with a nonlinear filtration function for monodisperse and polydisperse suspensions were obtained in [6, 7]. With the simultaneous action of several mechanisms of particle capture and a high concentration of suspended particles, the function $F(C)$ is nonlinear [8]. In [9], an asymptotic solution was constructed for a problem with a nonlinear concentration function of suspended particles for almost constant filtration functions and for small Langmuir coefficients. Exact solutions for the constant filtration function and local solutions on the characteristics of a hyperbolic system were obtained in [10]. However, global analytical solutions with nonlinear filtration and concentration functions were not previously known.

The global asymptotic solution of the particle transport problem in a homogeneous porous medium with nonlinear filtration and concentration functions is constructed. The transition to a new spatial variable, small throughout the porous medium, allows to construct an asymptotic solution in explicit form. The results of analytical calculations coincide with a numerical solution of the problem.

2. Mathematical model

In the domain

$$\Omega = \{(x,t): 0 < x < 1, t > 0\}$$

dimensionless concentrations of suspended $C(x,t)$ and retained $S(x,t)$ particles satisfy the system of equations

$$\frac{\partial C}{\partial t} + \frac{\partial C}{\partial x} + \frac{\partial S}{\partial t} = 0;$$

$$\frac{\partial S}{\partial t} = \Lambda(S)F(C). \quad (2)$$

Here $\Lambda(S), F(C)$ are non-negative smooth functions, $F(0) = 0$.

Boundary and initial conditions

$$C|_{x=0} = 1; \quad (3)$$

$$C|_{t=0} = 0; \quad S|_{t=0} = 0; \quad (4)$$

determine the unique solution of system (1), (2). Condition (3) corresponds to the injection of a suspension with constant suspended particles concentration at the porous medium inlet; according to condition (4), at the initial moment, the porous medium does not contain any suspended and retained particles.

The concentrations front $t = x$ moves in a porous medium from inlet $x = 0$ to outlet $x = 1$ at a speed $v = 1$. Before the front in the domain

$$\Omega_0 = \{(x,t): 0 < x < 1, 0 < t < x\}$$
the solution is zero; after the front in the domain
\[ \Omega_S = \{(x,t): 0 < x < 1, t > x\} \]
the solution is positive. The solution \( C(x,t) \) has a gap on the concentrations front \( t = x \); the solution \( S(x,t) \) is continuous in the domain \( \Omega \) and
\[ S|_{t=x} = 0. \tag{5} \]

Transition to characteristic variables
\[ \tau = t - x, \ y = x \]
converts a wedge-shaped domain \( \Omega_S \) into a half-strip
\[ \Omega^*_S = \{(x,t): 0 < x < 1, \tau > 0\} \]
and simplifies the equations:
\[ \frac{\partial C}{\partial y} + \frac{\partial S}{\partial \tau} = 0; \tag{6} \]
\[ \frac{\partial S}{\partial \tau} = \Lambda(S)F(C). \tag{7} \]
The solution of the Goursat problem \((6), (7)\) with conditions \((3), (5)\) coincides with the solution of the problem \((1) - (4)\) in the domain \( \Omega_S \).

For a blocking linear filtration coefficient (Langmuir coefficient)
\[ \Lambda(S) = a - bS \tag{8} \]
and linear concentration function
\[ F(C) = C \]
this problem in the domain \( \Omega^*_S \) has an exact solution
\[ C(y, \tau) = \frac{e^{\beta \tau}}{e^{\beta \tau} + e^{\alpha y} - 1}; \quad S(y, \tau) = \frac{a}{b} \cdot \frac{e^{\beta \tau} - 1}{e^{\beta \tau} + e^{\alpha y} - 1}. \tag{9} \]
In the case of nonlinear functions \( \Lambda(S), F(C) \), the exact analytical solution is not available.

3. Global asymptotic solution
Consider equations \((6), (7)\) with conditions \((3), (5)\) and nonlinear coefficients
\[ \Lambda(S) = a - bS + \varepsilon \Delta(S), \quad a > 0, \ b > 0; \quad F(C) = C + \varepsilon f(C), \tag{10} \]
where \( \varepsilon > 0 \) is a small positive parameter.

To construct the global asymptotics by methods of \([11, 12]\), we make the change
\[ z = 1 - e^{-\alpha y}, \quad T = 1 - e^{-\beta \tau}. \tag{11} \]
In the new variables in the domain
\[ \Omega_{z,T} = \{0 \leq z \leq e^{-\alpha y}, \ 0 \leq T < 1\} \]
problem \((6), (7)\) with conditions \((3), (5)\) takes the form
\[ a(1-z)\frac{\partial C}{\partial z} + (a-bS+\varepsilon \Delta(S))(C+\varepsilon f(C)) = 0; \tag{12} \]
\[ b(1-T)\frac{\partial S}{\partial T} = (a-bS+\varepsilon \Delta(S))(C+\varepsilon f(C)); \tag{13} \]
\[ C|_{z=0} = 1; \tag{14} \]
\[ S|_{T=0} = 0. \tag{15} \]
Since in the domain \( \Omega_{z,T} \) the variable \( z \) satisfies the inequality
\[ 0 \leq z \leq 1 - e^{-\alpha y} < 1, \tag{16} \]
it can be used as a small parameter for construction of global asymptotics в \( \Omega_{z,T} \).
The asymptotics of problem (12) - (15) is constructed in the form of a series in two small parameters $\varepsilon$ and $z$:

$$
C(\varepsilon, z, T) = 1 + \sum_{i=0}^{\infty} \sum_{j=0}^{n} c_{i,j}(T) \varepsilon^i z^j / j! = 1 + z(c_{0,1} + \varepsilon c_{1,1} + \varepsilon^2 c_{2,1} + \ldots) + \\
+ \frac{\varepsilon^2}{2}(c_{0,2} + \varepsilon c_{1,2} + \varepsilon^2 c_{2,2} + \ldots) + \ldots;
$$

$$
S(\varepsilon, z, T) = \sum_{i=0}^{\infty} \sum_{j=0}^{m} s_{i,j}(T) \varepsilon^i z^j / j! = (s_{0,0} + \varepsilon s_{1,0} + \varepsilon^2 s_{2,0} + \ldots) + \\
+ z(s_{0,1} + \varepsilon s_{1,1} + \varepsilon^2 s_{2,1} + \ldots) + \frac{\varepsilon^2}{2}(s_{0,2} + \varepsilon s_{1,2} + \varepsilon^2 s_{2,2} + \ldots) + \ldots.
$$

(17)  
(18)

Here, the coefficients $c_{i,j}$, $s_{i,j}$ depend only on $T$.

Substitution of expansions (17), (18) into equations (12), (13), and equating the terms for the same powers of $\varepsilon$, $z$, yields a recurrent system of ordinary differential equations for the coefficients $s_{i,j}$ and algebraic equations for finding $c_{i,j}$. The first equations are given below:

$$
e^0 z^0 : \quad b(1-T)s'_{0,0} = (a-bs_{0,0}) \cdot 1; \quad ac_{0,1} + (a-bs_{0,0}) \cdot 1 = 0;
$$

$$
e^0 z^1 : \quad b(1-T)s'_{0,1} = -bs_{0,1} + (a-bs_{0,0})c_{0,1}; \quad ac_{0,2} - ac_{0,1} - bs_{0,1} + (a-bs_{0,0})c_{0,1} = 0; \quad (19)
$$

$$
e^1 z^0 : \quad b(1-T)s'_{0,0} = (bs_{0,0} + \Delta(s_{0,0})) + af(1); \quad ac_{1,1} + (bs_{0,0} + \Delta(s_{0,0})) + af(1) = 0.
$$

The initial conditions for differential equations follow from condition (15):

$$
s_{0,0} = 0. \quad (20)
$$

Solution of the equations (19) with conditions (20)

$$
s_{0,0} = \frac{a}{b}T; \quad c_{0,1} = T - 1; \quad s_{0,1} = \frac{a}{b}T(T - 1); \quad c_{0,2} = 2T(T - 1);
$$

$$
s_{0,0} = \frac{1-T}{b} \int_{0}^{(t-1)} \Delta(at/b) \, dt + \frac{T}{b} \int_{0}^{(t-1)} af(1); \quad c_{1,1} = \frac{1-T}{a} \int_{0}^{(t-1)} \Delta(at/b) \, dt + \frac{T}{a} \Delta(at/b) + (T - 1) f(1). \quad (21)
$$

The following asymptotic coefficients are obtained similarly. Substitution of solutions (21) into expansions (17), (18) gives global asymptotic solution to the problem (12) - (15).

4. Numerical calculation

Experiments with a monodisperse suspension in a homogeneous porous medium made it possible to determine the filtration coefficient for suspended particles of radius $r = 3.168 \mu m$ [15]

$$
\Lambda(S) = 1.551 - 3.467 \cdot 10^{-3} S - 1.16 \cdot 10^{-4} S^2 - 1.16 \cdot 10^{-7} S^3. \quad (22)
$$

Representation of function (22) in the form (10) gives the following parameter values

$$
a = 0.51; \quad b = 5.956 \cdot 10^{-3}; \quad \varepsilon = 0.1; \quad \Delta(S) = mS^2 + nS^3; \quad m = 2.29 \cdot 10^{-5}, \quad n = 1.35 \cdot 10^{-6}. \quad (23)
$$

Denote

$$
F(C) = C + 0.001C^3,
$$

then

$$
\varepsilon = 0.1; \quad f(C) = 0.01C^3
$$

The asymptotic solution (17), (18) takes the form

$$
C = 1 + z(-1 + t + \varepsilon c_{1,1}) + \frac{z^2}{2} (2t(-1 + t) + \varepsilon c_{1,2}) + \ldots,
$$

$$
S = \frac{at}{b} + \varepsilon s_{1,0} + z \left( \frac{at(-1 + t)}{b} + \varepsilon s_{1,1} \right) + \ldots,
$$

where
The numerical calculation of problem (12) - (15) with coefficients (23) was performed by the finite difference method in an explicit scheme with steps $h_i = h = 0.01$. Using the trapezoidal method in constructing difference equations allowed us to obtain a scheme with approximation of the second order of smallness [13, 14].

The figures depict asymptotic graphs of first-order in $\varepsilon$ (blue line), exact solution (9) for the simplified problem with linear filtration coefficient (8) and linear function $F(C) = C$ (green line) and the numerical solution (red line). The graphs show break-through concentrations at the porous medium outlet ($x=1$) and concentration profiles at large time.

![Graph](image_url)

**Figure 1.** Suspended concentration $C(1,t)$ at the outlet $x=1$ a) general form; b) enlarged form.
Figure 2. Retained concentration $S(l,t)$ at outlet $x=1$ a) general form; b) enlarged form.

When $t \leq 100$ the graphs of the two asymptotics and the numerical solution are practically the same. Figures 3 and 4 show graphs of the suspended and retained particles concentrations at $t = 500$ and $t = 1000$.

Figure 3. Particles concentrations at $t = 500$ a) suspended $C(x,500)$; b) retained $S(x,500)$.

Figure 4. Particles concentrations at $t = 1000$ a) suspended $C(x,1000)$; b) retained $S(x,1000)$. 
5. Discussion
In the mathematical model of particle transport in a porous medium with several capture mechanisms, a nonlinear filtration function is used [15]. Different types of filtration functions correspond to different distributions of pore sizes in a porous rock [16]. The nonlinear function of the suspended particles concentration models not only large concentrations of particles, but also particle transport during chemical reactions in a porous medium [17].
Analytical methods for solving the particle transport problem give way to regularization of the inverse problem: finding the filtration function by the known suspended particles concentration at the porous medium outlet [18]. With the known solution of the direct problem, the coefficients of the equations are determined from the experimental data [19]. The solutions of the particle transport problem make it possible to select the optimal sizes and density of reinforcing grains for soil strengthening [20].
The system of equations (1), (2) with zero boundary conditions at the inlet of the porous medium simulates the problem of separation of the deposit from the framework of the porous medium with subsequent transport and retention of particles [21]. This problem will be studied separately.

6. Conclusions
A model of particle transport with nonlinear filtration and concentration functions is studied. The model describes the simultaneous action of several particle capture mechanisms at a high concentration of suspended particles.
A new method is proposed for constructing a global asymptotic solution in the entire domain of nonzero suspended and retained particles concentrations. A new spatial variable is used as an asymptotic parameter, which is small throughout the porous medium.
An asymptotic solution to the particle transport problem is constructed in the form of a series in powers of two small parameters. Numerical calculation of the problem shows that the asymptotic solution at all points of the porous medium at all time is closer to the solution than the exact solution of the linearized problem.
The maximum errors of the asymptotics and the exact solution are achieved at the porous medium outlet.

References
[1] Elimelech M, Gregory J, Jia X and Williams R A F 2013 Particle Deposition and Aggregation: Measurement, Modelling and Simulation (revised ed. Butterworth-Heinemann, NY)
[2] Yoon J and Mohtar C S El 2014 Groutability of Granular Soils Using Bentonite Grout Based on Filtration Model Transport in Porous Media 102(3) pp 365–385
[3] Bradford S A, Torkzaban S, Leij F and Simunek J 2015 Equilibrium and kinetic models for colloid release under transient solution chemistry conditions J. of Contaminant Hydrology 181 pp. 141–152
[4] Xu J 2016 Propagation behavior of permeability reduction in heterogeneous porous media due to particulate transport Europhysics Letters 114(1) 14001
[5] Herzig J P, Leclerc D M and Legoff P 1970 Flow of suspensions through porous media - application to deep filtration Industrial and Engineering Chemistry 62 pp 8-35
[6] Vyazmina E A, Bedrikovetskii P G and Polyamin A D 2007 New classes of exact solutions to nonlinear sets of equations in the theory of filtration and convective mass transfer Theoretical Foundations of Chemical Engineering 41(5) pp. 556-564
[7] Goldberg E, Scheringer M, Bucheli T D and Hungerbühler K 2014 Critical assessment of models for transport of engineered nanoparticles in saturated porous media Environmental Science & Technology 48(21) pp. 12 732-12 741
[8] Civan F 2015 Reservoir formation damage (Norman, OK: Gulf Professional Publishing)
[9] Kuzmina L I, Osipov Yu V and Zheglova Yu G 2018 Analytical model for deep bed filtration with multiple mechanisms of particle capture *International J. of Non-Linear Mechanics* 105 pp. 242–248
[10] Kuzmina L I and Osipov Yu V 2018 Deep bed filtration with multiple pore-blocking mechanisms *MATEC Web of Conferences* 196 04003
[11] Kuzmina L I, Osipov Yu V and Zheglova Yu G 2019 Global asymptotics of filtration in porous media *E3S Web of Conferences* 97 05002
[12] Kuzmina L I, Osipov Yu V and Zheglova Yu G 2019 Global asymptotics of the filtration problem in a porous medium *International J. for Computational Civil and Structural Engineering* 15(2) pp. 77-85
[13] Galaguz Yu P and Safina G L 2016 Modeling of Particle Filtration in a Porous Medium with Changing Flow Direction *Procedia Engineering* 153 pp. 157-161
[14] Safina G L 2019 Numerical solution of filtration in porous rock *E3S Web of Conferences* 97 05016
[15] Guedes R G, Al-Abduwani F, Bedrikovetsky P and Currie P K 2009 Deep-Bed Filtration Under Multiple Particle-Capture Mechanisms *SPE J.* 14(3) pp. 477-487
[16] Bedrikovetsky P 2008 Upscaling of stochastic micro model for suspension transport in porous media *Transport in Porous Media* 75 pp. 335-369
[17] Fogler H S 2006 *Elements of chemical reaction engineering*. 4th ed. (Prentice Hall: Upper Saddle River, NJ)
[18] Alvarez A C, Hime G, Marchesin D and Bedrikovetsky P G 2007 The inverse problem of determining the filtration function and permeability reduction in flow of water with particles in porous media *Transport in Porous Media* 70(1) pp. 43-62
[19] Vaz A, Bedrikovetsky P, Fernandes P D, Badalyan A and Carageorgos T 2017 Determining model parameters for non-linear deep-bed filtration using laboratory pressure measurements *J. of Petroleum Science and Engineering* 151 pp 421–433
[20] Faramarzi L, Rasti A and Abtahi S M 2016 An experimental study of the effect of cement and chemical grouting on the improvement of the mechanical and hydraulic properties of alluvial formations *J. of Construction & Building Materials* 126 pp. 32-43
[21] Yang Y, Siqueira F D, Vaz A, You Z and Bedrikovetsky P 2016 Slow migration of detached fine particles over rock surface in porous media *J. of Natural Gas Science and Engineering* 34 pp. 1159-1173