The conformal Penrose limit and the resolution of the pp-curvature singularities

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Abstract
We consider the exact solutions of the supergravity theories in various dimensions in which the spacetime has the form \( M_d \times S^{D-d} \) where \( M_d \) is an Einstein space admitting a conformal Killing vector and \( S^{D-d} \) is a sphere of an appropriate dimension. We show that if the cosmological constant of \( M_d \) is negative and the conformal Killing vector is spacelike, then such solutions will have a conformal Penrose limit: \( M_d^{(0)} \times S^{D-d} \) where \( M_d^{(0)} \) is a generalized \( d \)-dimensional AdS plane wave. We study the properties of the limiting solutions and find that \( M_d^{(0)} \) has 1/4 supersymmetry as well as a Virasoro symmetry. We also describe how the pp-curvature singularity of \( M_d^{(0)} \) is resolved in the particular case of the \( D6 \)-branes of \( D=10 \) type IIA supergravity theory. This distinguished case provides an interesting generalization of the plane waves in \( D=11 \) supergravity theory and suggests a duality between the \( SU(2) \) gauged \( d=8 \) supergravity theory of Salam and Sezgin on \( M_8^{(0)} \) and the \( d=7 \) ungauged supergravity theory on its pp-wave boundary.

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1. Introduction

A considerable portion of our present understanding of the properties of the string theories and of M-theory is based on two different notions of limits. The first of these is the notion of the near-horizon limit of a BPS \( p \)-brane [1]. In this limit, one usually ends up with a \( \text{AdS}_{p+2} \times S^{D-p-2} \) type of geometry which is a product of an anti de Sitter (AdS) space and a sphere \( S \) of appropriate dimensions that add up to the total dimension \( D \). The second notion is that of the Penrose limit [2] which implies, when suitably generalized to supergravity theories [3], that any supergravity solution has a plane wave solution as a limit. Recently, it was found that the Penrose limits of \( \text{AdS}_{p+2} \times S^{D-p-2} \) types of geometries are the maximally supersymmetric plane waves [4] and these provide in \( D=10 \) a convenient setting for the

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quantization of superstrings with non-trivial Ramond–Ramond (RR) fields [5]. Remarkably, a string theory counterpart of the Penrose limit also exists [6] and these developments led to new insights about the AdS/CFT correspondence in a regime which is beyond the supergravity approximation.

In the light of these developments, it is of interest to explore the possible generalizations of the Penrose limit together with the neighbouring geometries of the AdS\(_p+2 \times S^{D-p-2}\) spacetimes. The purpose of the present paper is to carry out such a programme which is based on the existence of conformal Killing vectors. We shall be interested primarily in the spacelike conformal Killing vectors and although most of our considerations will be valid for all \(D > 3\), we shall be mainly concerned with the \(D = 10\) and \(D = 11\) solutions.

In four dimensions AdS\(_4\) is known to be the unique spacetime which admits a spacelike conformal Killing vector (CKV) and a negative cosmological constant [7]. It is, however, possible to construct a two-parameter family of Einstein spaces which admits a spacelike CKV as well as a negative cosmological constant when one moves to higher dimensions. These geometries have a warped product structure and give rise to a class of exact solutions of the supergravity theories which are of the Freund–Rubin type. We shall be concerned with such classes of solutions that include the AdS\(_p+2 \times S^{D-p-2}\) spacetimes as special cases.

We shall start by considering supergravity solutions in the dual frame assuming that the spacetime is of the form \(M_{p+2} \times S^{D-p-2}\) where \(M_{p+2}\) is an Einstein space admitting a CKV. These solutions will have in general non-zero \((D-p-3)\)-form fluxes and possess dilaton fields that may play the role of conformal Killing potentials on \(M_{p+2}\). We shall show, by studying the null geodesic congruences of such spacetimes and employing the limiting procedure of [8], that these solutions have limits that are of the form \(M(0)_{p+2} \times S^{D-p-2}\) where \(M(0)_{p+2}\) is a generalized \((p+2)\)-dimensional AdS plane wave [8] which reduces to the nonlinear version of the Randall–Sundrum zero mode [9] when the CKV is hypersurface orthogonal. We shall also see that these limits exist only when the CKV of \(M_{p+2}\) is spacelike and the cosmological constant of \(M_{p+2}\) is negative.

As in the case of the Kaigorodov spacetime [10], AdS plane waves are known to preserve 1/4 supersymmetries and possess a Virasoro symmetry [11]. These spacetimes have been extensively studied [12, 13] and were found to suffer from pp-curvature singularities. In this paper, we shall show that all of these properties are shared by the general limiting solution \(M(0)_{p+2}\). We shall also describe how the pp-curvature singularity is resolved in the case of \(p = 6\), \(D = 10\) type IIA supergravity by lifting up the solution to the \(D = 11\) supergravity theory. This particular case will be seen to play a distinguished role because the CKV will be forced to be spacelike by the field equations themselves and will lead us to an interesting generalization of plane waves in \(D = 11\) as well as to a duality conjecture between the \(SU(2)\)-gauged \(d = 8\) supergravity theory of Salam and Sezgin [14] on the limiting background \(M(0)_{8}\) and the ungauged supergravity theory on its \(d = 7\) pp-wave boundary.

2. Dual frame products and Penrose limits

The part of the supergravity Lagrangians that is relevant to the study of AdS/CFT and DW/QFT dualities in various dimensions [15, 16] can be written as

\[
\mathcal{L}_D = \frac{1}{2\kappa_D} e^{\delta\phi} \left[ -R \ast 1 + \tilde{\gamma} \ast d\phi \wedge \ast d\phi + \frac{1}{2} \ast F_{D-p-2} \wedge F_{D-p-2} \right],
\]

where \(\kappa_D\) is the gravitational coupling constant in \(D\) dimensions and \(\delta, \tilde{\gamma}\) are two parameters that will be specified below. The independent fields are the metric \(g_{MN}\), the dilaton \(\phi\) and a \((D-p-3)\)-form potential whose field strength is \(F_{D-p-2}\). The integer \(p\) corresponds to the
spatial dimension of the brane in the case of the $p$-brane solutions, $R$ is the scalar curvature of $g_{MN}$ and $*I$ with the Hodge dual $*$ denotes the volume $D$-form. Our spacetime conventions and the field equations that follow from (2.1) are given in the appendix.

The parameters $\delta$ and $\gamma$ that appear in (2.1) are not independent:

$$\delta = \frac{-(D-2)a}{2(D-p-3)}, \quad \gamma = \frac{D-1}{D-2} \delta^2 - \frac{4}{D-2},$$

but are determined in terms of $D$ together with a choice of a constant $a$ that specifies the relevant theory. For $D = 11$ supergravity $a = 0$ and the dilaton also vanishes. In $D = 10$, one finds that $a = (3-p)/2$ when $F_{D-p-2}$ is chosen to belong to the RR sector of type IIA or IIB supergravity theory and in these cases (2.1) gives rise to the $Dp$-brane solutions. On the other hand, when $F_{D-p-2}$ is in the $D = 10$ Neveu–Schwarz (NS) sector one has $a = -1$ for $p = 1$ and $a = 1$ for $p = 5$. Other values of $a$ that are relevant to the lower dimensional supergravity theories can be found in [16].

An interesting feature of (2.1) is that $g_{MN}$ is the dual frame metric. The Einstein frame and the string frame Lagrangians can be obtained from (2.1) by the conformal mappings:

$$g_{MN}^{\text{Einstein}} = \exp(-[a/(D-p-3)]\phi)g_{MN}, \quad g_{MN}^{\text{String}} = \exp([1/(D-2)]\phi)g_{MN}^{\text{Einstein}}$$

and these three frames, of course, coalesce in $D = 11$. (Note that in (2.1) we are working with the magnetic potentials. In the particular case $D = 11$, $p = 2$, it is $F_7 = *F_4$ that appears as the field strength.) One should also keep in mind that, when $D = 10$, $p = 3$ and $F_5$ is taken to be in the RR sector, the field equations of (2.1) must be supplemented with the self-duality condition: $*F_5 = F_5$.

Consider now the solutions of (2.1) under the assumption that the spacetime $M_D$ is topologically and metrically a product, $M_D = M_{p+2} \times K$, of a $(p+2)$-dimensional spacetime $M_{p+2}$ and a $(D-p-2)$-dimensional compact manifold $K$. Note that metrically this is a frame-dependent condition within the conformal class of $g_{MN}$, and depending on the nature of $\phi$, the metric may lose its product form in other frames. For example, if one takes as $\phi$ the lift up of a function which is defined either on $M_{p+2}$ or on $K$, then one ends up with warped product metrics in other frames. If one assumes that the dilaton is a constant, the product structure of $g_{MN}$ is, of course, preserved in all the frames. In such cases one can readily construct exact solutions by taking $F_{D-p-2}$ to be proportional to the volume-form $\text{Vol}(K)$ of $K$. This is, of course, the well-known Freund–Rubin mechanism which reduces the field equations to the requirement that $M_{p+2}$ and $K$ are both Einstein spaces. With this reduction the cosmological constant of $M_{p+2}$ turns out to be non-positive.

In $D = 11$ supergravity this procedure is known to give rise to two distinct families which can be conveniently displayed in terms of $F_3$. The first family is obtained by taking $F_3 = \frac{1}{7} \text{Vol}(M_4)$ in which case the field equations reduce to

$$^4 R_{\mu \nu} = -\frac{3}{l^2} g_{\mu \nu}, \quad ^7 R_{mn} = \frac{3}{2l^2} g_{mn},$$

and admit $\text{AdS}_4 \times S^7$ as a distinguished solution. The dual geometry $\text{AdS}_7 \times S^4$ belongs to the second family for which $F_3 = \frac{6}{7} \text{Vol}(K)$ and this implies

$$^7 R_{\mu \nu} = -\frac{6}{l^2} g_{\mu \nu}, \quad ^4 R_{mn} = \frac{12}{l^2} g_{mn}.$$  

Here and in the following, Greek $(\mu, \nu, \ldots)$ and Latin $(m, n, \ldots)$ indices refer to the bases of $M_{p+2}$ and $K$, respectively, and $l$ is a real constant. The left superscripts denote the dimensions of the spaces. In any dimension $D$, the cosmological constant $\Lambda$ is defined to be $\Lambda = [\epsilon(D-1) (D-2)]/2 l_D^2$, where $l_D$ is the corresponding real constant and $\epsilon = \pm 1$ is introduced to allow both signs for $\Lambda$. 

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A similar family is encountered in the specialization of (2.1) to the $D = 10$, $p = 3$ Lagrangian where $F_5$ is the self-dual RR field. After setting $\phi$ to a constant and taking $F_5 = \frac{3}{2} [\operatorname{Vol}(M_3) - \operatorname{Vol}(K)]$, all the $D = 10$ field equations are satisfied if
\[
5 R_{\mu\nu} = - \frac{4}{\ell^2} g_{\mu\nu}, \quad 5 R_{\mu\nu} = \frac{4}{\ell^2} g_{\mu\nu}.
\] (2.6)

The distinguished member of this family is $\text{AdS}_5 \times S^5$.

Although all of these examples involve a constant dilaton and are familiar from the near-horizon limits of non-dilatonic branes, similar families exist even when $M_D$ on the coordinates of $M_D$ is taken to be a RR field strength. When $p = 4$, letting $F_3 = \frac{1}{2} \operatorname{Vol}(K)$ and demanding at the same time that $\psi = e^{2\phi/3}$ is a conformal Killing potential on $M_6$:
\[
\nabla_\mu \nabla_\nu \psi = - \frac{1}{\ell^2} \psi g_{\mu\nu},
\] (2.7)
gives
\[
6 R_{\mu\nu} = - \frac{5}{\ell^2} g_{\mu\nu}, \quad 4 R_{\mu\nu} = \frac{12}{\ell^2} g_{\mu\nu}.
\] (2.8)

The near-horizon $\text{AdS}_6 \times S^4$ geometry of the $D4$-brane is in this family.

Some interesting changes occur when one moves to higher values of $p$ in the $Dp$-brane-type Lagrangian. By letting $\psi = e^{3\phi/4}$ for $p = 5$, it can be inferred that, when $F_5$ is chosen on $K$ in the above manner, $\nabla_\mu \psi$ must be a Killing vector on $M_7$: $\nabla_\mu \nabla_\nu \psi = 0$. This gives a Ricci-flat $M_7$ while keeping $K$ as an Einstein space. On the other hand, in the $p = 6$ case one finds that if $F_3 = \frac{1}{2} \operatorname{Vol}(K)$ and
\[
8 R_{\mu\nu} = - \frac{7}{\ell^2} g_{\mu\nu}, \quad 2 R_{\mu\nu} = \frac{4}{\ell^2} g_{\mu\nu},
\] (2.9)
then all field equations will be satisfied provided $\psi = e^{-2\phi/3}$ is a solution of (2.7) on $M_8$ that also obeys the condition
\[
\nabla_\mu \psi \nabla^\mu \psi = - \frac{1}{\ell^2} \psi^2.
\] (2.10)

Hence, in this case one is forced on a particular CKV which must be spacelike. The $\text{AdS}_8 \times S^2$ near-horizon limit of the $D6$-branes is in this category\(^2\).

Suppose now we wish to take the Penrose limit of such an $M_D$ using the $D$-dimensional scaling rules [3]. Recall that the Penrose limit is a local procedure which requires first the introduction of the appropriate coordinates in a conjugate point-free portion of a null geodesic congruence of $M_D$. Since $M_D$ has the form $M_D = M_d \times K$ and the dual frame line element $ds_D^2 = ds_d^2 + ds_K^2$ where $d = p + 2$ and $ds_K^2$ is the line element of $K$, the set of all null geodesics of $M_D$ splits into a union of two disjoint subsets. Because $K$ has a Riemannian metric, the first subset consists of the timelike geodesics of $M_d$ together with the geodesics

\(^2\) It may be useful to note that the variable $\Phi = e^{\phi}$, which is used in the field equations in appendix A, is related to $\psi$ by $\psi = \Phi, \frac{\Phi^{1/8}}{\Phi^{-1/9}}$ for $p = 4, 5, 6$, respectively.
of $K$ and both of these are parametrized by the same $D$-dimensional affine parameter. (On the underlying spaces these are, of course, not unit-speed curves.) In the second subset, one has the null geodesics of $M_d$ that are passing from fixed points of $K$. In this case, the affine parameter of $M_d$ coincides with that of $M_D$. It turns out that the outcome of the Penrose limit depends crucially on the subset that contains the chosen null congruence.

For a congruence that belongs to the first subset, the Penrose coordinates of $M_D$ can be constructed by choosing a synchronous coordinates system [19] on $M_d$ together with a set of geodesic coordinates on $K$. The two null coordinates of the Penrose patch are then defined by the sum and the difference of two coordinates which measure the proper time on the timelike geodesic and the length of the Riemannian geodesic. Using the scaling rules for the supergravity fields it is then easy to see that the limit of $M_D$ is a plane wave spacetime with a non-zero flux $F_{D-d}$ and $\phi$. The well-known plane wave limits of the near-horizon geometries of the non-dilatonic branes [4] are in this category.

A completely different situation arises when the congruence is chosen from the second subset. In this case, the usual Penrose limit cannot result in a solution that has a non-zero flux. This is because the Penrose coordinates of $M_D$ consist of the Penrose coordinates of $M_d$ together with a set of suitable coordinates on $K$ and the limit of the product $M_d \times K$ can be seen to be the product of the separate limits of $M_d$ and $K$. The two null coordinates of the Penrose patch are now on $M_d$ and since $F_{D-d}$ is a $(D-d)$-form on $K$, it picks up the $(D-d)$ powers of the scaling parameter from the coordinates of $K$. However, according to the supergravity scaling rules, it should have scaled with a power which is one less than its degree in order to survive in the limit. Therefore, $F_{D-d}$ goes to zero in the standard limit. This was observed for $\text{AdS}_{p+2} \times S^{D-p-2}$ in [20] and its limit was found to be the $D$-dimensional Minkowski space.

The vanishing of the flux within the second subset is related to the fact that the Penrose procedure cannot give rise to a non-zero cosmological constant on the limit of $M_d$. Moreover, the standard Penrose limit does not take into account the possibility of having a metric that contains functions homogeneous of degree zero in the coordinates. Let us assume that $K$ admits a coordinate system in which its metric is a homogeneous function of degree zero in $l$ and of these coordinates. For example, let us take $K = S^{D-d}$ with the standard metric and adopt the stereographic coordinates of $S^{D-d}$ as part of the Penrose patch of $M_D$. On the other hand, let us assume that $M_d$ admits a spacelike CKV. Under these assumptions, a generalization of the Penrose limit, which allows a non-zero cosmological constant on $M_d$ as well as a non-zero flux and also involves a scaling of $l$, is available [8]. Let $\Omega$ be the real scaling parameter. Since one of the scaling rules of [8] is $l \to \Omega l$, it can be checked that $S^{D-d}$ and $F_{D-d}$ are not affected by this generalized limit. The corresponding limit $M_d^{(0)}$ of $M_d$ was found to be a generalized AdS plane wave spacetime [8]. Therefore, when one chooses a null congruence that belongs to the second subset and takes $K = S^{D-d}$, the $D$-dimensional solution $M_D$ will have a limit which is the product of a $d$-dimensional AdS plane wave and $S^{D-d}$ in the dual frame. This limit is also an exact solution of the field equations with a non-zero flux and will be called the conformal Penrose limit of $M_D$. Note that in the particular case $M_D = \text{AdS}_d \times S^{D-d}$ the solution is mapped into itself under the conformal Penrose limit.

3. The conformal Penrose limit

As in the above $K = S^{D-d}$ example, we shall assume in general that $K$ and $F_{D-d}$ are not affected by the conformal Penrose limit and examine in detail its action on $M_d$. Noting also that the only spacetime which admits a spacelike CKV and has $\Lambda < 0$ in $d = 4$ is the AdS$_4$ itself [7], we shall concentrate on the cases $d > 4$. 
The implications of the existence of a CKV field on \( M_d \) is a well-studied subject, dating back to the classical work of Brinkmann [21], although the global results are quite recent for the Lorentzian metrics, see e.g. [22–25]. When \( M_d \) is an Einstein space:

\[
R_{\mu \nu} = [\epsilon (d - 1)/l^2] g_{\mu \nu},
\]

and a smooth vector field \( V^\mu \) satisfying

\[
\mathcal{L}_V g_{\mu \nu} = 2\psi g_{\mu \nu},
\]

exists, where \( \mathcal{L} \) is the Lie derivative and \( \psi \) is a smooth function on \( M_d \), it can be deduced that \( \nabla_\mu \psi \) must itself be a CKV

\[
\nabla_\mu \nabla_\nu \psi = \frac{\epsilon}{l^2} \psi g_{\mu \nu},
\]

(we shall assume \( \psi \neq \text{const} \) to exclude the homotheties) and that

\[
V_\mu = \epsilon l^2 (\xi_\mu + \nabla_\mu \psi),
\]

(3.4)

where \( \xi_\mu \) is an ordinary Killing vector. The algebra of conformal vector fields on \( M_d \) therefore decomposes into the direct sum of the algebra of the Killing vectors and the algebra of closed CKVs that are locally gradients.

An interesting distinction between the Riemannian and Lorentzian metrics is the number of fixed points of \( V^\mu \) and these points play an important role globally [24, 25]. Since each \( V^\mu \) gives rise to a closed CKV, it is convenient to concentrate on the fixed points of \( \nabla_\mu \psi \). These are the critical points of \( \psi \) which can be shown to be isolated points [24]. Around any point with \( \nabla_\mu \psi \nabla_\mu \psi \neq 0 \), one can find a neighbourhood where \( g_{\mu \nu} \) is a warped product metric and in this neighbourhood a coordinate system \( \{y, x^a\}, a = 1, \ldots, (d - 1) \), exists where

\[
\nabla_\mu \psi = U(y) \delta_\mu^a, \quad U = d\psi/dy
\]

and consequently,

\[
ds^2 = \eta dy^2 + U^2(y)g_{ab}(x) \, dx^a \, dx^b.
\]

(3.5)

Here, \( g_{ab}(x) \) is a metric on the \( (d - 1) \)-dimensional fibres and \( \eta = \pm 1 \) is chosen according to the type of the CKV: \( \nabla_\mu \psi \nabla_\mu \psi = \eta U^2 \). Note that whereas \( \epsilon \) is the sign of the cosmological constant, \( \eta \) is the sign of the pseudo-norm of \( \nabla_\mu \psi \) and these quantities are, of course, independent. For a spacelike CKV, one has \( \eta = -1 \) for any sign of \( \Lambda \). It is also useful to note that the critical points of \( \psi \) are now located at the zeros of \( U \) where the coordinate system breaks down.

For (3.5), the field equations (3.1) require

\[
U'' = \frac{\epsilon \eta}{l^2} U,
\]

(3.6)

and consequently,

\[
U = A \cosh(y/l) + B \sinh(y/l) \quad (\epsilon = \eta),
\]

\[
U = A \cos(y/l) + B \sin(y/l) \quad (\epsilon = -\eta),
\]

(3.7)

where \( A \) and \( B \) are real constants and a prime denotes differentiation with respect to \( y \). Moreover, according to (3.1) the fibre metric \( g_{ab}(x) \) must also be Einstein:

\[
(d-1)R_{ab} = [\epsilon (d - 2)(A^2 - B^2)/l^2]g_{ab} \quad (\epsilon = \eta),
\]

\[
(d-1)R_{ab} = [\epsilon (d - 2)(A^2 + B^2)/l^2]g_{ab} \quad (\epsilon = -\eta),
\]

(3.8)

which now guarantees that (3.1) is fully satisfied. Choosing \( \epsilon = \eta = -1, A = -B = 1 \) and taking \( g_{ab}(x) \) as the \( (d - 1) \)-dimensional flat Minkowski metric gives AdS\(_d\) in Poincaré coordinates. In this case, the zero of \( U \) is located at the AdS horizon.
According to (3.5), $M_d$ is a warped product: $M_d = \bar{I} \times_U N$, where $I$ is a real interval and the fibre $N$ is a $(d-1)$-dimensional manifold, and the structure of the set of all null geodesics of $M_d$ as well as the nature of $N$ depend on $\eta$ in an obvious manner. If $\eta = -\bar{I}$, $N$ is a Lorentzian manifold and the set of all null geodesics of $M_d$ is again a union of two disjoint subsets. The first subset is composed of the null geodesics along which $y$ is not constant and for these $\psi(y)$ is an affine parameter. The second subset consists of the null geodesics of $N$ that are passing from fixed points of $I$ so that $y = \text{const}$ for these geodesics. In both subsets, the points $U = 0$ are conjugate points for the null geodesics. Note that when $\eta = 1$ so that the CKV is timelike, $N$ must be a Riemannian manifold and then the second subset is obviously not available.

Before considering the conformal Penrose limits relative to these subsets let us momentarily specialize to the $\epsilon = \eta = -\bar{I}$ solutions which are our main concern and note that such $M_d$ are conformally compactifiable provided $U > 0$. The conformal boundary corresponds to $U \to \infty$ and the inverse of $U$ is a defining function. Letting $r = U^{-1}$, one can find a manifold $\bar{M}$ with boundary $\partial \bar{M} = N$ such that $M_d$ is diffeomorphic to $\bar{M} - \partial \bar{M}$. The Einstein metric $\bar{g}_{\mu\nu}$ of $\bar{M}$ is $\bar{g}_{\mu\nu} = r^2 g_{\mu\nu}$ and on $\partial \bar{M}$, $r = 0$ but $\bar{g}^{\mu\nu} \partial_r \bar{g}_{\nu r} = -r^{-2}$.

In general, $\partial \bar{M}$ is a timelike boundary and may not be connected. Since $\partial \bar{M}$ need not be topologically $\mathbb{R} \times S^{(d-2)}$ and $r^{(d-4)}\bar{C}_{\mu\nu\kappa\sigma}$ need not vanish at the boundary, in general $M_d$ will not be asymptotically AdS$_d$. Here, $\bar{C}_{\mu\nu\kappa\sigma}$ is the Weyl tensor of $\bar{M}_d$. In fact assuming $\Lambda < 0$, it is easy to see that the only $M_d$ which admits a spacelike closed CKV and is asymptotically AdS$_d$ is the AdS$_d$ space itself. Hence, AdS$_d$ can be uniquely singled out by imposing the appropriate boundary conditions together with the presence of a CKV when $d > 4$.

Returning to (3.4), let us write the Killing vector as $\xi^\mu = (\beta, \xi^a)$. The Killing equation then implies that $\beta$ must be a scalar field on $N$ and if we define $\xi_a = \bar{g}_{ab}\bar{\xi}^b$ with the decomposition $\xi_a = f(y)\nabla_a\beta(x) + \xi_a(x)$ it follows that either $\nabla_a\beta = 0$ or $U^2 f'(y) = -\eta$. In the latter case, $f(y)$ can be easily determined and $\nabla_a\beta$ can be seen to be a non-homothetic CKV of $N$. Then, $\xi_a$ is an ordinary Killing vector on $N$. On the other hand, if $\nabla_a\beta = 0$, one can show that $\beta = 0$ as long as $U'/U \neq \text{const}$ and $\xi_a$ is again a Killing vector on $N$. In the particular case $U'/U = \text{const}$, a non-zero constant $\beta$ can exist and $\xi_a$ becomes a homothetic Killing vector on $N$. In this manner, the isometries of $M_d$ give rise to the conformal motions on the boundary $N$.

Consider now the conformal Penrose limit of $M_d$ for all values of $\epsilon$ and $\eta$. First, let us suppose that $\epsilon = \eta = -\bar{I}$ and $M_d$ admits a Killing vector $\xi^\mu$ associated with $V^\mu$. If one is then interested in setting up the Penrose coordinates around a congruence that belongs to the first subset, the coordinate system of (3.5) is not a convenient starting point. In this case, the procedure described in [8] can be taken over and a neighbourhood in which $V^\mu$ has no fixed points can be blown up to get

$$ds_d^2 = \frac{\lambda^2}{\epsilon z/\lambda + b_k(u)x^k + c_i(u)}[2 du dv - h_{ij}(u)x^i x^j du^2 - \delta_{ij} dx^i dx^j - (dz - \gamma du)^2],$$

(3.9)

as the conformal Penrose limit of $M_d$. Here, $x^i, i = 1, \ldots, d-3$, is a set of $(d-3)$-transverse coordinates, $\gamma = -2\lambda^2 \dot{\gamma}^b + \dot{\lambda} z/\lambda$ and due to (3.1), the metric functions satisfy

$$\lambda^2 + b_j b_j = 1,$$

$$\dot{\gamma} = 0,$$

(3.10)

$$\dot{b}_j = h_{jk} b_k,$$

(3.11)

$$h_{jj} = -\dot{\lambda}/\lambda - 2\lambda^2 \dot{b}_j \dot{b}_j,$$

(3.12)

and

$$h_{jj} = -\dot{\lambda}/\lambda - 2\lambda^2 \dot{b}_j \dot{b}_j,$$
where a dot denotes differentiation with respect to the null coordinate $u$ and repeated indices are summed with $\delta_{ij}$. This is just the metric of a generalized AdS plane wave for which $V^\mu = \lambda(u) \delta^\mu_z$ is the CKV inherited from $M_d$. The vector $b_j(u)$ characterizes the twist of $V^\mu$; if $\dot{b}_j(u) = 0$, $V^\mu$ is hypersurface orthogonal in which case it is reducible to a closed vector field. Another useful representation of (3.9), in terms of a new coordinate $\tilde{z}$, is

$$\mathrm{d}s^2 = \frac{l^2}{\tilde{z}^2} \left[ 2 \, \mathrm{d}u \, \mathrm{d}v - h_{ij}(u)x^i x^j \, \mathrm{d}u^2 - \delta_{ij} \, \mathrm{d}x^i \, \mathrm{d}x^j - \lambda^2 (\mathrm{d}\tilde{z} + A)^2 \right], \quad (3.14)$$

where the Kaluza–Klein (KK) gauge field is given by $A = (b_j x^j - \dot{c}) \, \mathrm{d}u - b_j \, \mathrm{d}x^j$.

It should be noted that the conformal Penrose limit of $M_d$ always has the metric (3.9) whenever $\eta = -1$ for the above choice of the null congruence and the CKV. However, when the cosmological constant of $M_d$ is taken to be positive, (3.1) requires in the limit in place of (3.10) that

$$\lambda^{-2} + b_j b_j = -1, \quad (3.15)$$

and it follows that $M_d$ does not have a conformal Penrose limit if $\eta = -1$, $\epsilon = 1$ and the congruence is chosen from the first subset.

On the other hand, if one would start from the premise that $\eta = 1$, but $\epsilon = \pm 1$, then the gauge choice on $M_d$ which leads to (3.9) would not be available [8]. Hence, such an $M_d$ would not have any conformal Penrose limit because the second subset of the null geodesics would also be absent. We may therefore conclude that $M_d$ must admit a spacelike CKV in order to have a well-defined conformal Penrose limit.

For a null geodesic congruence that belongs to the second subset when $\eta = -1$, the Penrose coordinates of $M_d$ can be taken to be the Penrose coordinates of $N$ together with the coordinate $y$. Consequently, the limit involves the standard scalings of the Penrose coordinates of $N$ together with the scalings: $y \to \Omega y$ and $l \to \Omega l$ which leave $U$ invariant. This means that on $N$ the procedure will amount to the standard Penrose limit which will force the limit of $N$ to be a Ricci flat, ordinary plane wave spacetime. Since one would like to keep the $d$-dimensional $\Lambda \neq 0$ in the limit and (3.8) must hold, this will only be possible if $\epsilon = \eta$ and $A^2 = B^2$. When these hold, one gets as the limit

$$\mathrm{d}s^2 = \frac{l^2}{z^2} \left[ 2 \, \mathrm{d}u \, \mathrm{d}v - h_{ij}(u)x^i x^j \, \mathrm{d}u^2 - \delta_{ij} \, \mathrm{d}x^i \, \mathrm{d}x^j - \mathrm{d}\tilde{z}^2 \right], \quad (3.16)$$

which can be interpreted as the nonlinear version of the Randall–Sundrum zero mode [9] and corresponds to the $\dot{b}_j = 0$ specialization of (3.9).

The above discussion which led us to the limit (3.16), of course, disregards once again the possible existence of a spacelike CKV, this time on $N$. If $N$ admits such a CKV, then the above arguments can be applied recursively to $N$ in which case $N$ will again have a warped product structure.

Also note that so far we have considered only the neighbourhoods of $M_d$ without fixed points. It is known [24] that in a neighbourhood which contains a critical point of $\psi$, $M_d$ is isometric to the Poincaré patch of AdS$_d$ and, consequently, this neighbourhood of $M_d$ will be invariant under the conformal Penrose limit. The remaining possibility is the case of a $\nabla_\mu \psi$ that is null on a neighbourhood of $M_d$. In this case, $M_d$ is already a Ricci-flat pp-wave spacetime [21].

4. Properties of the limiting solution

We have thus seen that $M_d$ has a conformal Penrose limit only when its cosmological constant is negative and the CKV is spacelike. Since it is precisely these types of $M_d$ that solve
the supergravity equations in the dual frame, we now examine the properties of the limiting solution \( M_d^{(0)} \). In the particular case \( b_j(u) = 0 \), it is already known that \( M_d^{(0)} \) preserves 1/4 of the maximal supersymmetry [13] and has a Virasoro symmetry [11]. Our first goal is to see whether these properties can be extended to the general solution (3.9) which allows a non-zero twist.

We start by considering the existence of the Killing spinors that satisfy

\[
D_{\mu}\bar{\varepsilon} = \frac{i}{2} \Gamma_{\mu}\bar{\varepsilon} = 0,
\]

(4.1)

where \( D_{\mu} \) is the spinor covariant derivative and \( \Gamma_{\mu} \) are the Dirac matrices on \( M_d^{(0)} \). In the coordinate system of (3.9), it will be convenient to let

\[
W = \left[ z/\lambda + b_k(u)x^k + l_c(u) \right]
\]

and choose the orthonormal basis 1-forms as

\[
e_0 = \frac{1}{\sqrt{2W}} \left[ du + \gamma dz \right] + \frac{1}{2} \left[ h_{ij}x^ix^j + \gamma^2 \right] du,
\]

\[
e_1 = \frac{1}{\sqrt{2W}} \left[ du + \gamma dz \right] - \frac{1}{2} \left[ h_{ij}x^ix^j + \gamma^2 \right] du,
\]

(4.2)

\[
e_j = \frac{1}{W} dx^j, \quad (j = 1, \ldots, d - 3),
\]

\[
e^d = \frac{1}{W} dz.
\]

If one then defines

\[
\Gamma_{\pm} = \frac{1}{\sqrt{2}} (\Gamma_0 \pm \Gamma_1),
\]

(4.3)

and

\[
\bar{\varepsilon} = \sqrt{1/W} \bar{\varepsilon},
\]

(4.4)

the integrability condition for (4.1) can be seen to be

\[
[(h_{jk} + 3\lambda^2 b_j b_k)\Gamma^k - (\lambda b_j + 3\lambda b_i)\Gamma^d]\Gamma_{\pm}\bar{\varepsilon} = 0.
\]

(4.5)

Therefore, whenever \( h_{jk} \neq 0 \) and the matrix within the square brackets is invertible, the necessary condition for the existence of a Killing spinor is

\[
\Gamma_{\pm}\bar{\varepsilon} = 0.
\]

(4.6)

The vanishing of \( h_{jk} \) implies that \( b_j = 0 \) in which case \( M_d^{(0)} = \text{AdS}_d \) and (4.5) is identically satisfied. On the other hand, if one imposes rather than (4.6) the condition

\[
[(h_{jk} + 3\lambda^2 b_j b_k)\Gamma^k - (\lambda b_j + 3\lambda b_i)\Gamma^d]\bar{\varepsilon} = 0,
\]

(4.7)

then it is easy to see that such a spinor does not allow a \( b_j = 0 \) specialization as long as \( h_{jk} \) is an invertible \((d - 3) \times (d - 3)\) matrix. For this reason, we impose (4.6) as a necessary condition and its effect is to eliminate 1/2 of the AdS\(_d\) supersymmetry.

When (4.6) holds, the components of (4.1) can be cast into the form

\[
\partial_u\bar{\varepsilon} = 0, \quad \partial_j\bar{\varepsilon} - \frac{1}{2W} \Gamma_j Q_+\bar{\varepsilon} = 0, \quad \partial_d\bar{\varepsilon} - \frac{1}{2W} \Gamma_d Q_+\bar{\varepsilon} = 0,
\]

\[
\partial_z\bar{\varepsilon} - \frac{1}{2W} \Gamma_z Q_+\bar{\varepsilon} = 0, \quad \partial_d\bar{\varepsilon} + \frac{1}{2} \lambda b_j \Gamma_j Q_+\bar{\varepsilon} - \frac{1}{2W} \Gamma_d Q_+\bar{\varepsilon} = 0,
\]

(4.8)

where \( \partial_{\mu} \) are the partial derivatives with respect to the coordinates and \( Q_\pm \) are the operators

\[
Q_\pm = \Gamma^\mu \partial_{\mu} W \pm iI,
\]

(4.9)

with \( I \) denoting now the identity operator. These operators can depend only on the coordinate \( u \) and satisfy

\[
Q_\pm^2 = \pm 2i Q_\pm, \quad Q_+ Q_- = Q_- Q_+ = 0,
\]

(4.10)
as a consequence of (3.10). One can impose
\[ Q\bar{\epsilon} = 0, \]  
(4.11)
as an additional algebraic condition on the Killing spinor and this can be ensured by setting
\[ \bar{\epsilon} = Q^\lambda \alpha(u). \]  
(4.12)
With these conditions Killing spinor equations now reduce to an equation that just fixes the \( u \)-dependence of the spinor \( \alpha \):
\[
\frac{d}{du}[Q^\lambda \alpha(u)] + \frac{1}{2} \bar{\epsilon} b_I \nabla_I Q^\lambda \alpha(u) = 0.
\]  
(4.13)
Since (4.13) always has a solution and (4.11) eliminates one-half of the remaining supersymmetry, we conclude that \( M_0 \) possesses 1/4 of the maximal supersymmetry. Note
\[ Q^\lambda + \bar{\epsilon} = i I - (\lambda^{-1} \Gamma_d + b_k \Gamma_k), \]  
(4.14)
and when \( \dot{b}_I = 0 \), (4.11) becomes \( \bar{\epsilon} = i \bar{\epsilon} \) and \( \alpha \) reduces to a constant spinor. Hence, in this particular case Killing spinors are given simply by (4.4) where \( \bar{\epsilon} \) is a constant spinor that
\[ \{ f, U \} \]  
(4.15)
denotes the Schwarzian derivative:
\[ \{ f, U \} = \frac{f'''f'}{(f')^2} - \frac{3}{2}(f')^2, \]  
(4.16)
and \( e(U) = c(u)/\sqrt{f} \) satisfies
\[ e'' + \frac{1}{2} \{ f, U \} e = 0. \]  
(4.17)
The diffeomorphism (4.15) is precisely the one that was utilized in [11] to derive the Virasoro symmetry and (4.17) reduces to the corresponding transformation of the metric function when \( \dot{b}_J = 0 \). In our context, (4.15) is a mapping within the family of generalized AdS plane waves.

Finally, let us consider the pp-curvature singularity of \( M_0 \). Most of the \( \epsilon = \eta = -1 \) spaces \( M_d \) that we had studied in the previous section locally are not geodesically complete. In fact, one can show that if \( M_d \) is a geodesically complete Einstein space, then it is either AdS \( U \) or must have the form \( M_d = \mathbb{R} \times_U N \) where \( U = A \cosh(y/l) \) and \( N \) is a complete Einstein space [24]. It is easy to see that, unless \( N \) is allowed to possess a spacelike CKV, geodesically complete \( \mathbb{R} \times_U N \) spacetimes will not have a conformal Penrose limit. When \( N \) is assumed to have the desired CKV, taking the limit recursively shows that the conformal
Penrose limit of this particular $\mathbb{R} \times U$ $N$ spacetime is again itself. Hence, the AdS plane waves are obtained as the conformal Penrose limits of $M_d$ that are not geodesically complete.

In particular, the $A^2 = B^2$ subset of solutions which have $U = A e^{(a+b)/3}$ and tend to (3.16) are not null complete even when $N$ is complete [26]. The limits (3.16) of these solutions are known to have pp-curvature singularities [12, 13] at $z = \infty$ whenever $h_{i\bar{j}}(u) \neq 0$ and $d > 3$. It can be checked, by examining the components of the Riemann tensor in a frame that is parallelly transported along a timelike geodesic, that the same behaviour persists even when the CKV vector is not hypersurface orthogonal. In other words, the metric (3.9) also has a pp-curvature singularity, this time when $z/\lambda + b_k(u)x^k + lc(u)$ tends to infinity. In the coordinates of (3.14), the pp-curvature singularity is located at $\tilde{z} = \infty$. We therefore see that $M_d$ has two boundaries; there is a conformal boundary $N$ which is positioned at $\tilde{z} = 0$ and which is a perfectly well-behaved spacetime with the metric
\[
\mathrm{ds}_N^2 = 2 \mathrm{d}u \mathrm{d}v - [h_{i\bar{j}}(u)x^i x^j + \lambda^2 (b_j x^j - lc)(b_k x^k - lc)] \, \mathrm{d}u^2 + 2\lambda^2 (b_j x^j - lc) b_k \, \mathrm{d}x^k \, \mathrm{d}u - (\delta_{jk} + \lambda^2 b_j b_k) \, \mathrm{d}x^j \, \mathrm{d}x^k. \tag{4.20}
\]
This shows that in general $N$ is a $(d - 1)$-dimensional, Ricci-flat pp-wave spacetime with a special dependence on the transverse coordinates. When $b_j = 0$, $N$ possesses the standard plane wave metric in the harmonic coordinates. On the other hand, $M_d$ also has a pp-curvature singularity at $\tilde{z} = \infty$ which should be regarded as a physical boundary when viewed from the $d$-dimensional perspective. Since $M_d$ has the limit $M_d(0) \times S^{D-d}$, it is clear that $\tilde{z} = \infty$ is also a genuine singularity from the $D$-dimensional viewpoint. This raises the question whether these pp-curvature singularities can be resolved by some means.

It is known that certain scalar polynomial singularities of the Riemann tensor can be resolved when $p = d - 2$ is odd by viewing the same solution in a higher dimension [27]. Remarkably, it turns out that the above pp-curvature singularities of the $D = 10$ solutions can be resolved only for a particular even value of $p$ when the solutions are lifted up to $D = 11$. This occurs when $p = 6$ and the field equations of type IIA supergravity theory reduce to (2.9). Recall that this is the only case in which the CKV is forced to be spacelike by the field equations.

In the following, let us distinguish the $D = 11$ supergravity fields with hats and note that the field content of (2.1) in the case of $D = 10$ type IIA supergravity implies for $p = 6$
\[
\mathrm{d}\tilde{s}^2 = e^{4\psi/3} \left[ \mathrm{d}s_{10}^2 - (\mathrm{d}Y + \hat{A})^2 \right] \tag{4.21}
\]
and $\tilde{F}_4 = 0$. Here, $Y$ is the $D = 11$ Killing coordinate used in the reduction, $\hat{A}$ is the KK 1-form with $F_2 = \mathrm{d} \hat{A}$ and $\mathrm{d}s_{10}^2$ denotes the $D = 10$ line element in the dual frame. Since $\tilde{F}_4 = 0$, the oxidation of the family (2.9), which includes its conformal Penrose limit, gives rise to Ricci-flat solutions of $D = 11$ supergravity.

Let us also recall that for the present $p = 6$ case, $\psi = e^{-2\psi/3}$ and $e^{4\psi/3} = \tilde{z}^2$ in the coordinate system of (3.14). According to (4.21) and (3.14), the $D = 11$ vacuum solution which is obtained by oxidizing the metric of $M_d(0) \times S^2$ is therefore
\[
\mathrm{d}s^2 = \frac{1}{2} \left[ 2 \mathrm{d}u \mathrm{d}v - h_{i\bar{j}}(u)x^i x^j \, \mathrm{d}u^2 - \delta_{ij} \, \mathrm{d}x^i \, \mathrm{d}x^j - \lambda^2 (\tilde{d}^2 + \hat{A})^2 \right] - \tilde{z}^2 \left[ (\mathrm{d}Y + \hat{A})^2 + \mathrm{d}\Omega_2^2 \right], \tag{4.22}
\]
where $\mathrm{d}\Omega_2^2$ is the metric of a sphere of radius 1/2 and for the transverse coordinates $x^i$ one now has $i = 1, \ldots, 5$. Since $F_2 = \frac{2}{3} \mathrm{Vol}(S^2)$, it follows that the $D = 11$ manifold is the warped product: $M = M_8 \times \hat{B}_3$ where $B_3$ is a $U(1)$ bundle over $S^2$ and $M_8$ is a plane wave spacetime. Using the spherical coordinates $(\theta, \varphi)$ of $S^2$ and defining $\chi = 2Y/l$, one can locally write
\[
(\mathrm{d}Y + \hat{A})^2 + \mathrm{d}\Omega_2^2 = \frac{l^2}{4} \left[ (\mathrm{d}\chi + (1 - \cos \theta) \, \mathrm{d}\varphi)^2 + (\mathrm{d}\theta^2 + \sin^2 \theta \, \mathrm{d}\varphi^2) \right]. \tag{4.23}
\]
If the coordinate \( \chi \) on the \( U(1) \) fibres is identified with a period \( 4\pi \), \( B_3 = S^3 \) and one has a Hopf fibration \( S^3 \to S^2 \). More generally, \( B_3 \) is a cyclic lens space. When the gravitational degrees of freedom are switched off by setting \( h_{ij}(u) = 0 \), the solution (4.22) reduces to the flat \( D = 11 \) metric which was studied in [28] as the lift up of the near-horizon limit of the \( D6 \)-branes.

We have checked that all the components \( \hat{R}_{ABCD} \) of the Riemann tensor, in a frame which is parallelly transported along a timelike geodesic of (4.22), are regular everywhere provided \( \tilde{z} > 0 \). In particular, the geodesics are perfectly well behaved as \( \tilde{z} \to \infty \). We therefore conclude that the pp-curvature singularity of \( M^{(0)}_8 \times S^2 \) is resolved in \( D = 11 \) supergravity theory.

5. Discussion

In this paper, we have seen how the limiting procedure of [8] finds a natural application in the supergravity theories as conformal Penrose limits of solutions in the dual frame. For this purpose, we have considered the solutions in which the spacetime is of the form \( M_D = M_J \times K \) where \( M_J \) is an Einstein space admitting a CKV and \( K \) is a compact Riemannian manifold. By studying the null geodesics in all possible neighbourhoods, we have seen that \( M_D \) has a conformal Penrose limit \( M_d^{(0)} \times K \) if the cosmological constant of \( M_d \) is negative and the CKV is spacelike. Under these conditions, \( M_d^{(0)} \) was found to be a generalized AdS plane wave that possesses 1/4 of the maximal supersymmetry as well as a Virasoro symmetry.

The basic requirement on \( K \) was that it admits a coordinate system in which its metric is a homogeneous function of degree zero in \( l \) and of these coordinates. Although the limit was seen to apply to any such \( K \), we have concentrated on the case \( K = S^{D-d} \) because this choice had the virtue that AdS_\( _d \times S^{D-d} \) is a member of the family and, moreover, such a coordinate system was explicitly known on \( S^{D-d} \). Since it is also known that \( S^{D-d} \) admits a CKV but this property was not referred to in our discussion, it will be interesting to determine the set of all \( K \) that is left invariant by the conformal Penrose limit.

We have also studied the global structure of \( M_d^{(0)} \) and noted that it has a conformal boundary \( N \) which is located at \( \tilde{z} = 0 \) and a pp-curvature singularity at \( \tilde{z} = \infty \). We have found that in the \( D = 10, p = 6 \) case, the pp-curvature singularity of \( M_8^{(0)} \) can be resolved by lifting up the solution to the \( D = 11 \) supergravity theory. In this case, one can verify that (4.22) implies \( \hat{R}_{ABCD} \hat{R}^{ABCD} = 0 \) and \( \hat{M} \) can be viewed as an interesting generalization of the ordinary plane wave spacetimes. What one now has in \( D = 11 \) is an asymptotically locally Euclidean (ALE) plane wave with an \( A_{N-1} \) singularity [29] at \( \tilde{z} = 0 \) which depends on the nature of the identifications on \( S^3 \). Since we already know that in the \( D = 10 \) picture \( \tilde{z} = 0 \) defines the conformal boundary of \( M^{(0)}_8 \) and the boundary is a regular, \( d = 7 \) pp-wave spacetime, the \( A_{N-1} \) singularity is resolved in turn in the corresponding eight-dimensional theory.

When the type IIA supergravity is compactified on \( S^2 \) or the \( D = 11 \) supergravity is compactified on a \( U(1) \) bundle over \( S^2 \), one obtains the \( SU(2) \) gauged \( d = 8 \) supergravity theory of Salam and Sezgin [14]. The domain wall solution of this theory was constructed in [30] and is known [15] to correspond to the \( h_{ij}(u) = 0 \) specialization of \( M_8^{(0)} \). Even when \( h_{ij}(u) \neq 0 \), the metric (3.9) and the dilaton of \( M_8^{(0)} \) constitute a solution of the gravity/dilaton sector of the \( SU(2) \) gauged theory. Hence, the \( A_{N-1} \) singularity is resolved in the \( SU(2) \) gauged \( d = 8 \) supergravity.

Note that the \( SU(2) \) gauged \( d = 8 \) supergravity theory has a consistent braneworld KK reduction to \( d = 7 \) ungauged supergravity [31] and the pp-wave metric of the boundary \( N \).
is a Ricci-flat solution of this $d = 7$ supergravity theory. This raises the question whether
the ungauged $d = 7$ supergravity on the pp-wave background $N$ can be in some sense dual
to the $SU(2)$ gauged supergravity theory on $M_8^{(0)}$. Since the $D6$-brane worldvolume theory
does not decouple from the bulk [28], it is not unreasonable to expect a generalized sort of a
duality between two supergravities in the present context and an attractive feature of the plane
wave spacetimes is that they have a very restricted set of quantum corrections [32, 33]. It will
therefore be an interesting problem to see whether such a duality can be established between
these two theories with the aid of the ALE plane waves of the $D = 11$ supergravity theory.

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Appendix

Our conventions are as follows:

In all $D \geq 2$, we use the ‘mostly minus’ signature $(+, -, \ldots, -)$ and the orientation
$\epsilon_{012 \ldots D-1} = 1$. The Ricci tensor is defined as $R_{MN} = R_{MNL}^K$ and the Riemann curvature
obeys $(\nabla_N \nabla_M - \nabla_M \nabla_N)T_K = R^L_{KMN}T_L$ for an arbitrary $T_M$. The Hodge dual of a $p$-form
$(p \leq D)$ is defined by

$$ (* (W_{A_1} \wedge \ldots \wedge W_{A_p}) = (-1)^{(D-1)} (D-p)! \epsilon^{A_1 \ldots A_p A_{p+1} \ldots A_D} W_{A_{p+1}} \wedge \ldots \wedge W_{A_D} ,$$

in terms of an orthonormal basis $\{W^A\}$.

The field equation that follow from (2.1) are

$$ d(* \Phi F_{D-p-2}) = 0,$$

$$ \Delta \Phi + \frac{8^2 (D-p-3)(-1)^{p(D-p)}}{8(D-p-2)!} F^2 \Phi = 0,$$

$$ R_{MN} = -\Phi^{-1} \nabla_M \nabla_N \Phi + \hat{\gamma}^2 \Phi^{-2} \nabla_M \Phi \nabla_N \Phi + \frac{(-1)^{p(D-p)}}{2(D-p-3)!} F_{A_1 \ldots A_{D-p-3} M} F^{A_1 \ldots A_{D-p-3} N} \frac{8}{8(D-2)(D-p-2)!} F^2 g_{MN},$$

where $\Phi = e^{\delta \phi}$ and $F^2 = F_{A_1 \ldots A_{D-p-3}} F^{A_1 \ldots A_{D-p-3}}$.

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