Classes of free group extensions.

Noam M.D. Kolodner

Abstract

In this paper we identify different classes of free group extension using core graphs, by further developing machinery from \[3\]. We show that every free group extension \(H \leq K \leq F\) has a base \(B\) such that the associated pointed graph morphism \(\Gamma_B(H) \to \Gamma_B(K)\) is onto. But if we examine graphs without base points, there is an extension \(\langle b \rangle \leq \langle b, aba^{-1} \rangle < F_{\langle a, b \rangle}\) such that for every base of \(F_{\langle a, b \rangle}\) the associated graph morphisms are injective.

1 Introduction

In this paper we identify different classes of free group extension using core graphs, by further developing machinery from \[3\]. Leveraging the theory of topological cover spaces, Stallings \[9\] established a correspondence between subgroups of a free group with labeled graphs called core graphs. Let \(F\) be a free group and \(B\) a base. For every subgroup \(H \leq F\) we associate a pointed labeled graph \(\Gamma_B(H)\) and for subgroups \(H \leq K \leq F\) a graph morphism \(\Gamma_B(H) \to \Gamma_B(K)\). Thus we realize the category of subgroups of free groups ordered by inclusion as a subcategory of the category of labeled graphs. When we order subgroups by inclusion, these inclusion have no “flavor”. But as morphisms in the category of labeled graphs they have easily detectable properties. Labeled graph morphisms can be injective or surjective for instance. The problem is that these properties are incidental and are dependent on the arbitrary base chosen for constructing the correspondence, so if one wants to leverage these properties to study algebraic properties of free group extensions one must look at invariant properties.

The first attempt to do this was made by Miasnikov Ventura and Weil \[5\], who conjectured that extensions \(H \leq K \leq F\), such that the morphism of the corresponding core graphs is surjective for every base, are algebraic extensions (i.e. such that \(H\) is not included in any proper free factor of \(K\)). Puder and Parzanchevski showed this to be false for subgroups of a free group with two generators but conjectured that it is still true for free groups with more generators, or that it is true if one allows automorphisms of free extension of the ambient free group \(F\).

The author of this paper found a counter example for the revised conjectures \[3\] and thus showed that algebraic extensions are strictly included in extensions that are onto on all bases, which form a separate extension class. Parzanchevski and Puder \[7\] suggested another class of extensions: one where there exists a base for which the graph morphism is onto. They asked if this is true for all extensions. Another suggestion proposed by Verdú in his masters thesis \[1\] is an extension class where graph morphisms are injective for every base.

In Theorem \[21\] we show that for every extension \(H \leq K \leq F\) there exists a base such that the morphism of the corresponding labeled graphs is onto; moreover one obtains this base by conjugation. Thus the extension class where there exists a base that is onto includes all extensions and the extension class where every base is injective is empty. If instead of the category of pointed labeled graphs we look at graphs without base points, and instead of automorphisms we look at outer automorphisms we show that in this setting these classes of extensions are non-trivial. Using the methods developed in \[3\] we show that the extension \(\langle b \rangle \leq \langle b, aba^{-1} \rangle\) is injective for every outer automorphism. Thus in the new setting there is a class of extension whose graph morphism is injective on every base and there is also a class of extensions s.t. there exists a base where the graph morphism is surjective.
In order to prove that extension \( \langle b \rangle \leq \langle b, aba^{-1} \rangle \) has the desired property we further develop in this paper the 
machinery from \cite{3}, to deal with cases where a graph does not have stencil finiteness.

2 Preliminaries

In this paper we use the machinery developed in \cite{3}. We present the definitions we use in the paper, merging the 
machinery from \cite{3}, to deal with cases where a graph does not have stencil finiteness.

**Definition 1 (Graphs).** A graph \( \Gamma \) is a set \( V(\Gamma) \) of vertices and a set \( E(\Gamma) \) of edges with a function \( \iota: E(\Gamma) \to V(\Gamma) \) called the initial vertex map and an involution \( \overline{\iota}: E(\Gamma) \to E(\Gamma) \) with \( \overline{\iota}(e) \neq e \) and \( \overline{\overline{\iota}(e)} = e \). A graph morphism \( f: \Gamma \to \Delta \) is a pair of set functions \( f^E: E(\Gamma) \to E(\Delta) \) and \( f^V: V(\Gamma) \to V(\Delta) \) that commute with the structure functions. A path \( \pi \) in \( \Gamma \) is a finite sequence \( e_1, \ldots, e_n \in E(\Gamma) \) with \( \iota(\overline{e_k}) = \iota(e_{k+1}) \) for every \( 1 \leq k < n \). The path is closed or circuit if \( \iota(\overline{e_n}) = \iota(e_1) \), and reduced if \( e_{k+1} \neq \overline{e_k} \) for all \( k \). All the graphs in the paper are assumed to be connected unless specified otherwise, namely, for every \( e, f \in E(\Gamma) \) there is a path \( e, e_1, \ldots, e_n, f \).

**Definition 2 (Labeled graphs).** Let \( X \) be a set and let \( X^{-1} \) be the set of its formal inverses. We define \( R_X \) to be the graph with \( E(R_X) = X \cup X^{-1}, V(R_X) = \{ \ast \} \), \( \overline{\ast} = \ast^{-1} \) and \( \iota(\ast) = \ast \). An \( X \)-labeled graph is a graph \( \Gamma \) together with a graph morphism \( l: \Gamma \to R_X \). A morphism of \( X \)-labeled graphs \( \Gamma \) and \( \Delta \) is a graph morphism \( f: \Gamma \to \Delta \) that commutes with the label functions. Let \( \mathcal{P}(\Gamma) \) be the set of all the paths in \( \Gamma \), let \( F_X \) be the free group on \( X \) and let \( P = e_1, \ldots, e_n \) be a path. The edge part of the label function \( l^E: E(\Gamma) \to E(R_X) \) can be extended to a function \( l: \mathcal{P}(\Gamma) \to F_X \) by the rule \( l(P) = l(e_1) \ldots (e_n) \).

A pointed \( X \)-labeled graph is an \( X \)-labeled graph that has a distinguished vertex called the base point. A morphism of a pointed labeled graph \( \Gamma \) to \( \Gamma' \) sends the base point to the base point. This constitutes a category called \( X \)-Grph. For a pointed \( X \)-labeled graph \( \Gamma \) we define \( \pi_1(\Gamma) \) to be

\[
\pi_1(\Gamma) = \{ l(P) \in F_X \mid P \text{ is a closed path beginning at the base point} \}.
\]

**Definition 3.** Let \( F_X \) be the free group on the set \( X \). We define a category \( \text{Sub}(F_X) \), whose objects are subgroups \( H \leq F_X \) and there is a unique morphism in \( \text{Hom}(H, K) \) iff \( H \leq K \). It is easy to verify that \( \pi_1 \) is a functor from \( X \)-Grph to \( \text{Sub}(F_X) \).

Note: The functor \( \pi_1 \) defined above is not the fundamental group of \( \Gamma \) as a topological space. Rather, if one views \( \Gamma \) and \( R_X \) as topological spaces and \( l \) as a continuous function, then \( \pi_1 \) is the image of the fundamental group of \( \Gamma \) in that of \( R_X \) under the group homomorphism induced by \( l \).

**Definition 4 (Folding).** A labeled graph \( \Gamma \) is folded if \( l(e) \neq l(f) \) holds for every two edges \( e, f \in E(\Gamma) \) with \( \overline{l(e)} = \overline{l(f)} \). We notice that there is at most one morphism between two pointed folded labeled graphs. If \( \Gamma \) is not folded, there exist \( e, f \in E(\Gamma) \) s.t. \( \overline{l(e)} = \overline{l(f)} \) and \( l(e) = l(f) \); Let \( \Gamma' \) be the graph obtained by identifying the vertex \( \overline{l(\overline{e})} \) with \( \overline{l(\overline{f})} \), the edges \( e \) with \( f \) and \( \overline{e} \) with \( \overline{f} \). We say \( \Gamma' \) is the result of folding \( e \) and \( f \). The label function \( l \) factors through \( \Gamma' \), yielding a label function \( l' \) on \( \Gamma' \), and we notice that \( \pi_1(\Gamma') = \pi_1(\Gamma) \).

**Definition 5 (Core graph).** A core graph \( \Gamma \) is a labeled, folded, pointed graph s.t. every edge in \( \Gamma \) belongs in a closed reduced path around the base point. In case of finite graphs this is equivalent to every \( v \in V(\Gamma) \) having \( \deg(v) := |\iota^{-1}(v)| > 1 \) except the base point which can have any degree.

**Definition 6.** Let \( X \)-CGrph be the category of connected, pointed, folded, \( X \)-labeled core graphs. Define a functor \( \Gamma_X: \text{Sub}(F_X) \to X \)-CGrph that associates to the subgroup \( H \leq F_X \) a graph \( \Gamma_X(H) \) (which is unique up to a unique isomorphism) s.t. \( \pi_1(\Gamma_X(H)) = H \).
Fact 7 ([9] [2]). The functors $\pi_1$ and $\Gamma$ define an equivalence between the categories $X$-$\text{Grph}$ and $\text{Sub}(F_X)$.

The correspondence between the categories of $X$-$\text{Grph}$ and $\text{Sub}(F_X)$ follows from the theory of cover spaces. Let us sketch a proof. We regard $R_X$ as a topological space and look at the category of connected pointed cover spaces of $R_X$. This category is equivalent to $\text{Sub}(F_X)$, following from the fact that $R_X$ has a universal cover. Let $\Gamma$ be a connected folded $X$-labeled core graph, viewed as a topological space and $l$ as a continuous function. There is a unique way up to cover isomorphism to extend $\Gamma$ to a cover of $R_X$. There is also a unique way to associate a core graph to a cover space of $R_X$. This gives us an equivalence between the category of connected pointed cover spaces of $R_X$ and pointed connected folded $X$-labeled core graphs.

Definition 8. By uniqueness of the core graph of a subgroup we can define a functor Core: $X$-$\text{Grph}$ $\rightarrow$ $X$-$\text{Grph}$ that associates to a graph $\Gamma$ a core graph s.t. $\pi_1(\text{Core}(\Gamma)) = \pi_1(\Gamma)$.

Definition 9. Let $\Gamma$ be a graph with a vertex $v$ of degree one which is not the base-point. For $e = \iota^{-1}(v)$, let $\Gamma'$ be the graph with $V(\Gamma') = V(\Gamma) \setminus \{v\}$ and $E(\Gamma') = E(\Gamma) \setminus \{e, \pi\}$. We say that $\Gamma'$ is the result of trimming $e$ from $\Gamma$, and we notice that $\pi_1(\Gamma) = \pi_1(\Gamma')$.

Remark 10. For a finite graph $\Gamma$, after both trimming and folding $|E(\Gamma')| < |E(\Gamma)|$. If no foldings or trimmings are possible then $\Gamma$ is a core graph. This means that after preforming a finite amount of trimmings and foldings we arrive at Core$(\Gamma)$. It follows from the uniqueness of Core$(\Gamma)$ that the order in which one performs the trimmings and foldings does not matter.

Definition 11 (Whitehead graph). A 2-path in a graph $\Gamma$ is a pair $(e, f) \in E(\Gamma) \times E(\Gamma)$ with $\iota(f) = \iota(\pi)$ and $f \neq \pi$. If $\Gamma$ is $X$-labeled, the set

$$W(\Gamma) = \{ \{l(e), l(\pi)\} \mid (e, f) \text{ is a 2-path in } \Gamma \}$$

forms the set of edges of a combinatorial (undirected) graph whose vertices are $X \cup X^{-1}$, called the Whitehead graph of $\Gamma$. If $w \in F_X$ is a cyclically reduced word, the Whitehead graph of $w$ as defined in [11, 10] and the Whitehead graph of $\Gamma((w))$ defined here coincide. Let $W_X = W(R_X)$ be the set of edges of the Whitehead graph of $R_X$, which we call the full Whitehead graph. Let $x, y \in X \cup X^{-1}$ and let $\{x, y\} \in W_X$ be an edge. We denote $x.y = \{x, y\}$ (this is similar to the notation in [9]).

Definition 12. A homomorphism $\varphi: F_Y \rightarrow F_X$ is non-degenerate if $\varphi(y) \neq 1$ for every $y \in Y$.

Definition 13. Let $w \in F_X$ be a reduced word of length $n$. Define $\Gamma_w$ to be the $X$-labeled graph with $V(\Gamma_w) = \{1, \ldots, n+1\}$ forming a path $P$ labeled by $l(P) = w$. Notice that $\Gamma_w \cong \Gamma_{w^{-1}}$.

Definition 14. Let $\varphi: F_Y \rightarrow F_X$ be a non-degenerate homomorphism. We define a functor $F_{\varphi}$ from $Y$-labeled graphs to $X$-labeled graphs by sending $y$-labeled edges to $\varphi(y)$-labeled paths. Formally, let $\Delta$ be a $Y$-labeled graph and let $E_0 = \{e \in E(\Delta) \mid l(e) \in Y\}$ be an orientation of $\Delta$, namely, $E(\Delta) = E_0 \cup \{\pi \mid e \in E_0\}$. For every $e \in E(\Delta)$ let $n_e \in \mathbb{N}$ be the length of the word $\varphi(l(e)) \in F_X$ plus one. We consider $V(\Delta)$ as a graph without edges, take the disjoint union of graphs $\bigsqcup_{e \in E_0} \Gamma_{\varphi(l(e))} \sqcup V(\Delta)$ and for every $e \in E_0$ glue $1 \in V(\Gamma_{\varphi(l(e))})$ to $\iota(e) \in V(\Delta)$, and $n_e \in V(\Gamma_{\varphi(l(e))})$ to $\iota(\pi) \in V(\Delta)$. Let $\Delta$ and $\Xi$ be $Y$-labeled graphs, and $f: \Delta \rightarrow \Xi$ a graph morphism. As for functionality, if $f: \Delta \rightarrow \Xi$ is a morphism of $Y$-labeled graphs, $F_{\varphi}.f$ is defined as follows: every edge in $F_{\varphi}(\Delta)$ belongs to a path $F_{\varphi}(e)$ for some $e \in E(\Delta)$, and we define $(F_{\varphi}.f)(F_{\varphi}(e)) = F_{\varphi}(f(e))$.

Remark 15. For $H \leq F_Y$ we notice that Core$((F_{\varphi}.\Gamma_Y(H))) = \Gamma_X(\varphi(H))$.

Definition 16 (Stencil). Let $\Gamma$ be an $Y$-labeled graph, and $\varphi: F_Y \rightarrow F_X$ a non-degenerate homomorphism. We say that the pair $(\varphi, \Gamma)$ is a stencil iff $F_{\varphi}(\Gamma)$ is a folded graph. Notice that if $\Gamma$ is not folded, then $F_{\varphi}(\Gamma)$ is not folded for any $\varphi$. 3
Definition 17. Let $\Gamma$ be a $Y$-labeled folded graph. An FGR object $(Y, N_Y)$ is said to be a stencil space of $\Gamma$ if $W(\Gamma) \subseteq N_Y$. The reason for the name is that for any object $(X, N_X)$ and morphism $\varphi \in \text{Hom}((Y, N_Y), (X, N_X))$, the pair $(\varphi, \Gamma)$ is a stencil.

Definition 18. Let $\tau : F_X \setminus \{1\} \to X \cup X^{-1}$ be the function returning the last letter of a reduced word. For reduced words $u, v$ in a free group, we write $u \cdot v$ to indicate that there is no cancellation in their concatenation, namely $\tau(u) \neq \tau(v^{-1})$.

Definition 19 (FGR). The objects of the category Free Groups with Restrictions (FGR) are pairs $(X, N)$ where $X$ is a set of “generators” and $N \subseteq W_X$ a set of “restrictions”. A morphism $\varphi \in \text{Hom}_{\text{FGR}}((X, N), (Y, M))$ is a group homomorphism $\varphi : F_Y \to F_X$ with the following properties:

(i) For every $x \in Y$, $\varphi(x) \neq 1$ ($\varphi$ is non-degenerate).

(ii) For every $x \in Y$, $W(\varphi(x)) \subseteq M$.

(iii) For every $x, y \in N$, $\varphi(x) \cdot \varphi(y)^{-1}$ (i.e. $\tau(\varphi(x)) \neq \tau(\varphi(y))$).

(iv) For every $x, y \in N$, $\tau(\varphi(x)) \cdot \tau(\varphi(y)) \in M$.\footnote{Technically, (iv) implies (iii), as $M \subseteq W_Y$ and $x \cdot y \notin W_Y$.}

3 There is always a base where the graph morphism is onto

All lemmas and propositions used here can be found in the section titled The core functor in [3].

Proposition 20. Let $H \leq F_Y$ a subgroup let $w \in F_Y$ be a word. We can obtain the core graph $\Gamma_Y(wHw^{-1})$ from the graph $\Gamma_Y(H)$ by the following process:

1. Attach a reduced path labeled $w$ to the base point in $\Gamma_Y(H)$

2. Set the new base point to be at the beginning of the path labeled by $w$.

3. Fold and trim if necessary

Theorem 21. Let $F_Y$ be a free group with a finite set of generators $Y$ and let $H \leq K \leq F_Y$ be finitely generated subgroups. Then there is a basis $B$ of $F_Y$ s.t. the morphism of Stallings graphs $\Gamma_B(H) \to \Gamma_B(K)$ is onto. Moreover this basis can be obtained by conjugation.

Proof. We will prove the equivalent statement. There is an automorphism $\varphi \in \text{Inn}(F_Y)$ s.t. $\Gamma_Y(\varphi(H)) \to \Gamma_Y(\varphi(K))$ is onto. Without loss of generality we can assume that the base point of $\Gamma_Y(H)$ has degree greater or equal to two (if not this can be corrected by conjugation). Let $u \in F_Y$ be the label of a reduced circuit in $\Gamma_Y(K)$ at the base point that traverses each edge of $\Gamma_Y(H)$ at least once (we are not bothered if it traverses some edges multiple times). Let $\varphi \in \text{Inn}(F_Y)$ be conjugation by $u$. By construction $u \in K$ therefore $\text{Core}_{\varphi}(\Gamma_Y(K)) = \Gamma_Y(\varphi(K)) = \Gamma_Y(uKu^{-1}) = \Gamma_Y(K)$. We construct $\Gamma_Y(uHu^{-1})$ by the process described in proposition 20 but stop at stage 2 (before preforming folding and trimming), we denote this graph by $\Gamma'$. By construction the graph morphism $\Gamma' \to \Gamma_Y(K)$ is onto. Because $\Gamma_Y(H)$ is folded and its base point has degree at least 2 the graph $\Gamma'$ satisfies the conditions of lemma 2.1 from [3] this means that Core$(\Gamma')$ is obtained without trimming. Without trimming the morphism remains onto when one takes its core (remark 3.16 from [3]). Thus we get that Core$(\Gamma' \to \Gamma_Y(K))$ is onto but Core$(\Gamma' \to \Gamma_Y(K)) = \Gamma_Y(\varphi(H)) \to \Gamma_Y(\varphi(K))$.\footnote{Technically, (iv) implies (iii), as $M \subseteq W_Y$ and $x \cdot y \notin W_Y$.}
Corollary 22. Let $H < K \leq F_Y$ (H strictly contained in K). There is a basis $B$ of $F_Y$ s.t. the graph morphism $\Gamma_B(H) \rightarrow \Gamma_B(K)$ is not injective.

Proof. There exists a basis $B$ s.t. $\Gamma_B(H) \rightarrow \Gamma_B(K)$ is onto. Because $H$ is strictly contained in $K$ the graph morphism $\Gamma_B(H) \rightarrow \Gamma_B(K)$ cannot be an isomorphism therefore it is not injective.

4 Example

In the category of pointed labeled core graphs we can’t have a free group extension s.t. for every automorphism the graph morphism is injective. We saw that the obstruction was conjugation we will show that this is the only obstruction. For this we focus our attention on labeled graphs without a base point. Miasnikov and Kapovich [2] called this “the type of a graph”. To get the type of a graph we forget the base point and trim again. We can look at the category of core X-labeled graphs without base points. Let $\Gamma, \Delta$ be X-labeled core graphs. There is a graph morphism $\Gamma \rightarrow \Delta$ iff there exists a $u \in F_X$ s.t. $u\pi_1 (\Gamma) u^{-1} < \pi_1 (\Delta)$ (technically one has to choose a base point for $\pi_1$ to be well defined. But it is well defined up to conjugation which is what we are using here). We notice that without a base point there is no longer a unique graph morphism between two graphs. Because we have an action of Out$(F_X)$ on the set of subgroups of $F_X$ up to conjugation this gives us an action of Out$(F_X)$ on morphisms $\Gamma \rightarrow \Delta$ in the category of labeled graphs. In this setting there is a graph morphism $\Gamma \rightarrow \Delta$ that is injective in its whole orbit under outer automorphism. We give the example of $\langle b \rangle < \langle b,aba^{-1} \rangle$. We will use the tools developed in [3].

Definition 23. We define a functor

$\text{Trimf} : \text{PLCGraphs} \rightarrow \text{LCGraphs}$

from pointed labeled core graphs to labeled core graphs. The functor forgets the base point and trims the “tail” (trimf stands for forget then trim). It takes graph morphisms to their restrictions. This definition is indeed a legal functor: Let $\Gamma \rightarrow \Delta$ be a morphism of pointed labeled core graphs. Let $v \in V(\Delta)$ be the base point of $\Delta$ and suppose it is of valency one (otherwise no trimming occurs and the definition is clearly legal). Let $w \in V(\Gamma)$ be the inverse image of $v$, it is the base point of $\Gamma$. Since $\Gamma$ is folded the morphism $\Gamma \rightarrow \Delta$ is locally injective therefore $w$ must also be of valency one. Let $\Gamma', \Delta'$ be the graphs obtained by trimming the edges incident to $v$ and $w$ respectively. We see that the morphism $\Gamma' \rightarrow \Delta'$ obtained by restricting $\Gamma \rightarrow \Delta$ to $\Gamma'$ is well defined. By induction we can trim the whole “tails” of $\Gamma$ and $\Delta$. ($\text{Trimf}$ would not be defined if we include graphs that aren’t folded).

We denote $\Gamma = \Gamma_{\langle a,b \rangle} (\langle b \rangle)$ and $\Delta = \Gamma_{\langle a,b \rangle} (\langle b,aba^{-1} \rangle)$ and let $X$ be a countably infinite set.

Theorem 24. All the morphism in set $\{ \text{Trimf} \circ \text{Core} \circ \mathcal{F}_\varphi (\Gamma \rightarrow \Delta) | \varphi \in \text{Hom}(\langle \{a,b\}, \emptyset \rangle, (X,W_X)) \}$ are injective.

Theorem shows that for every morphism $\Gamma \rightarrow \Delta$ in the orbit under Out$(F_{\langle a,b \rangle})$ is injective: Let $u \in F_X$ and let $\varphi_1, \varphi_2 : F_{\langle a,b \rangle} \rightarrow F_X$ non-degenerate homomorphisms s.t. $u\varphi_1 u^{-1} = \varphi_2$ then clearly $\text{Trimf} \circ \text{Core} \circ \mathcal{F}_{\varphi_1} (\Gamma \rightarrow \Delta) = \text{Trimf} \circ \text{Core} \circ \mathcal{F}_{\varphi_2} (\Gamma \rightarrow \Delta)$. Without loss of generality we can assume that $\{a,b\} \subseteq X$ so $\text{Hom}(\langle \{a,b\}, \emptyset \rangle, (X,W_X))$ includes Aut$(F_{\langle a,b \rangle})$ it includes also all automorphisms of free extensions of $F_{\langle a,b \rangle}$ and non free extensions as well. We will use the method presented in [3] with modifications to account to the fact that we are now interested in injective not surjective morphisms.

Remark 25. Let $\Gamma \rightarrow \Delta$ be a morphism of $U$-labeled graphs and let $N_U$ be a set of restrictions. Suppose $(U,N_U)$ is a stencil space of $\Delta$.

1. $(U,N_U)$ is also a stencil space of $\Gamma$
2. If \( \Gamma \rightarrow \Delta \) is injective then \( F_{\varphi}(\Gamma \rightarrow \Delta) \) is injective for every \( \varphi \in \text{Hom}((U, N_U), (X, W_X)) \). (This is true generally the assumption that \( (U, N_U) \) is a stencil space is unnecessary)

3. \( \text{Core} \circ F_{\varphi}(\Gamma \rightarrow \Delta) = F_{\varphi}(\Gamma \rightarrow \Delta) \)

We can use a surjectivity problem \( (\Gamma \rightarrow \Delta, (U, N_U)) \) from \([3]\) as an injectivity problem. We say that an injectivity problem resolves positively if all morphisms in \( \mathcal{P} = \{ \text{Trimf} \circ \text{Core} \circ F_{\varphi}(\Gamma \rightarrow \Delta) | \varphi \in \text{Hom}((U, N_U), (X, W_X)) \} \) are injective. we distinguish three cases

1. \( \Gamma \rightarrow \Delta \) is not injective: clearly \( \mathcal{P} \) resolves negatively.

2. \( \Gamma \rightarrow \Delta \) is injective and \( (U, N_U) \) is a stencil space of \( \Delta \): following Remark \( [25] \) \( \mathcal{P} \) resolves positively.

3. \( \Gamma \rightarrow \Delta \) is injective and \( W(\Delta) \setminus N_U \neq \emptyset \): in this case we cannot resolve \( \mathcal{P} \) immediately. We call this the ambiguous case.

If \( \mathcal{P} \) is of the ambiguous case we can split to five cases using FGR. We examine the five new cases and then split again if necessary. Because of Theorem 3.14 in \([3]\) every morphisms \( \varphi \in \text{Hom}((U, N_U), (X, W_X)) \) either \( F_{\varphi}(\Gamma \rightarrow \Delta) \) isn’t injective or it ends up in a stencil case. Therefore we try to classify all possible stencil cases that my arise in this process and determine that they are all positive. In contrasted to the example in \([3]\) the graph \( \Delta \) does not have stencil finitness therefore we end this process differently. We notice the by conjugation we can assume that \( b \) is cyclically reduced so instead of \( (\Gamma \rightarrow \Delta, (\{a,b\}, \emptyset)) \) we consider the problem \( (\Gamma \rightarrow \Delta, (\{a,b\}, \{b,b^{-1}\})) \). We preform a change of coordinates (see \([3]\)). Let \( V = \{a,b\}, N_V = \{b,b^{-1}\} \), and

\[
\sigma : F_{\{a,b\}} \rightarrow F_{\{a,b\}}, \quad \sigma(\alpha) = b, \quad \sigma(\beta) = aba^{-1}.
\]

We notice that \( \langle b, aba^{-1} \rangle \leq \text{Im} \sigma \). For any non-degenerate \( \varphi : F_{\{a,b\}} \rightarrow F_X \), the words \( \varphi(b) = \varphi \circ \sigma(\alpha) \) and \( \varphi(aba^{-1}) = \varphi \circ \sigma(\beta) \) are conjugate and \( b \) is cyclically reduced, hence there exist reduced words \( \overline{y}, \overline{\pi}, \overline{\tau} \in F_X \) such that \( \varphi(b) = \overline{\pi} \cdot \overline{\tau}, \varphi(aba^{-1}) = \overline{y} \cdot \overline{\pi} \cdot \overline{y}^{-1} \) (in particular, \( \overline{\pi} \) and \( \overline{\tau} \) are cyclically reduced). By non-degeneracy we can also assume \( \overline{\tau} \neq 1 \), and if \( \overline{\tau} = 1 \) then \( \overline{\tau} \) is cyclically reduced. We perform a change of coordinates according to four possible cases, with \( (U_i, N_i) \), \( \psi_i \) and \( \sigma_i \) being:

| \# | \( \overline{\tau} \) | \( \overline{\pi} \) | \( U_i \) | \( N_i \) | \( \psi_i(\alpha), \psi_i(\beta) \) | \( \sigma_i(\alpha), \sigma_i(\beta) \) | \( \Gamma_i \) | \( \Delta_i \) |
|---|---|---|---|---|---|---|---|---|
| 1 | \( = 1 \) | \( = 1 \) | \( u \) | \( u, u^{-1} \) | \( u, u \) | \( u \) | \( u \) |
| 2 | \( \neq 1 \) | \( = 1 \) | \( y, u \) | \( y, u, u^{-1} \) | \( u, yuy^{-1} \) | \( y, u \) | \( u \) |
| 3 | \( = 1 \) | \( \neq 1 \) | \( v, u \) | \( v, v^{-1}, u, v^{-1} \) | \( uv, vu \) | \( u^{-1}, uv \) | \( u \) | \( v \) |
| 4 | \( \neq 1 \) | \( \neq 1 \) | \( v, u, y \) | \( v, u^{-1}, y, v^{-1}, u, y \) | \( uv, yuvy^{-1} \) | \( yv, uv \) | \( u \) | \( v \) |
Case 3 is the problematic case it splits in to new cases indefinitely this shows that $\langle b, aba^{-1} \rangle$ does not have stencil finitess. In order to treat it we define two auxiliary cases $x$ and $x'$. Cases $x$ and $x'$ include all the subcase of case 3 but they are more general and includes many more cases that are irrelevant to the original question. The advantage of using cases $x$ and $x'$ is that they split into a finite set of ambiguous cases in contrast to case 3 that splits into new ambiguous cases indefinitely. In the table is an analysis of all the different cases.
| #  | FGR split | Homo. | $N_i$ | $\Gamma_i$ | $\Delta_i$ | $W(\Delta_i) \setminus N_i$ | Comment |
|----|-----------|-------|-------|----------|-----------|--------------------------|---------|
| 2' |           |       | $u.y, y.u^{-1}$ | $u^{-1}.y^{-1}, u.u^{-1}$ | $u$ | $u.y^{-1}$ | triangle rule + symmetry |
| 2.1 | $u.y^{-1.1}$ | Id | $u.y, y.u^{-1}$ | $u.y^{-1}, u.u^{-1}$ | $u$ | $\emptyset$ | ✓ |
| 2.2 | $u.y^{-1.2}$ | $u \mapsto ut$ | $t.y, y.u^{-1}$ | $t.u^{-1}, u.t^{-1}$ | $u \mapsto t$ | $\emptyset$ | ✓ |
| 2.3 | $u.y^{-1.3}$ | $u \mapsto uy^{-1}$ | $u.y, y.u^{-1}$ | $u^{-1}.y^{-1}, y.y^{-1}$ | $u \mapsto y$ | $u.u^{-1}, u.y^{-1}$ | contained in case 3 via $u \mapsto u$ $v \mapsto y^{-1}$ |
| 2.4 | $u.y^{-1.4}$ | $u \mapsto u$ | $u.u^{-1}, y.u^{-1}$ | $u^{-1}.y^{-1}, u.y$ | $u$ | $u.y^{-1}$ | equivalent to case 2' |
| 4' |           |       | $v.u^{-1}, u.v^{-1}$ | $y.v^{-1}, u.y$ | $u \mapsto v$ | $v.y^{-1}$ | triangle rule + symmetry |
| 4.1 | $v.y^{-1.1}$ | Id | $v.u^{-1}, u.v^{-1}$ | $y.v^{-1}, u.y$ | $u \mapsto v$ | $\emptyset$ | ✓ |
|   |   |
|---|---|
| 4.2 | $v.y^{-1}$, $u \mapsto vt$, $y \mapsto t^{-1}y$ |
| 4.3 | $v.y^{-1}$, $u \mapsto vy$, $y \mapsto y$ |
| 4.4 | $v.y^{-1}$, $u \mapsto vu$, $y \mapsto y$ |
| 3.1 | $u.v$, $id$, $v.u^{-1}$, $u.v^{-1}$ |
| 3.2 | $v \mapsto vt$, $u \mapsto ut$ |
| 3.3 | $v \mapsto v$, $u \mapsto uv$ |
| 3.4 | $v \mapsto vu$, $u \mapsto u$ |
| $x$ | $v.u^{-1}$, $x.v^{-1}$, $x.u^{-1}$, $x.v^{-1}$ |
| $x'$ | $t.u^{-1}$, $t.v^{-1}$, $u.t^{-1}$, $x.u^{-1}$, $x.v^{-1}$ |
|   | $u.v.1$ | Id | $v.u^{-1}, u.v^{-1}$ | $x.u^{-1}, x.v^{-1}$ | $u.v^{-1}$ | $u^{-1}.v^{-1}$ | ambiguous |
|---|---|---|---|---|---|---|---|
| $x.1$ |   |   |   |   | $x$ | $v \xrightarrow{x} u$ | $u^{-1}.v^{-1}$ |
| $x.2$ | $u.v.2$ | $v \mapsto vt$ | $u \mapsto ut$ | $t.u^{-1}, t.v^{-1}$ | $x.u^{-1}, x.v^{-1}$ | $u.v^{-1}$ | $u^{-1}.v^{-1}$ | contained in $x'$ via $x \mapsto tx$
| $x.3$ | $u.v.3$ | $v \mapsto vu$ | $u \mapsto u$ | $u.u^{-1}, u.v^{-1}$ | $x.u^{-1}, x.v^{-1}$ | $u.v^{-1}$ | $u^{-1}.v^{-1}$ | contained in $x$ via $x \mapsto ux$
| $x.4$ | $u.v.4$ | $v \mapsto v$ | $u \mapsto uv$ | $v.u^{-1}, v.v^{-1}$ | $x.u^{-1}, x.v^{-1}$ | $u.v^{-1}$ | $u^{-1}.v^{-1}$ | contained in $x$ via $x \mapsto vx$
| $x.5$ | $u.v.5$ | $v \mapsto u$ | $u \mapsto u$ | $u.u^{-1}, u.x^{-1}$ | $x.u^{-1}, x.v^{-1}$ | $u.v^{-1}$ | $u^{-1}.v^{-1}$ | $\emptyset$ | $\checkmark$
| $x.1.1$ | $u^{-1}.v^{-1}.1$ | Id | $v.u^{-1}, u.v^{-1}$ | $x.u^{-1}, x.v^{-1}$ | $u.v^{-1}$ | $u^{-1}.v^{-1}$ | $\emptyset$ | $\checkmark$
| $x.1.2$ | $u^{-1}.v^{-1}.2$ | $v \mapsto tv$ | $u \mapsto tu$ | $v.t^{-1}, u.t^{-1}$ | $x.t^{-1}, x.t^{-1}$ | $u.v^{-1}$ | $u^{-1}.v^{-1}$ | contained in $x'.1$ via $x \mapsto xt$
| $x.1.3$ | $u^{-1}.v^{-1}.3$ | $v \mapsto uv$ | $u \mapsto u$ | $v.u^{-1}, v.v^{-1}$ | $x.u^{-1}, x.v^{-1}$ | $u.v^{-1}$ | $u^{-1}.v^{-1}$ | contained in $x.1$ via $x \mapsto xu$
| $x.1.4$ | $u^{-1}.v^{-1}.4$ | $v \mapsto v$ | $u \mapsto vu$ | $v.v^{-1}, u.v^{-1}$ | $x.v^{-1}, x.v^{-1}$ | $u.v^{-1}$ | $u^{-1}.v^{-1}$ | contained in $x.1$ via $x \mapsto xv$
| $x'.1$ | $v^{-1}.u^{-1}.1$ | Id | $t.u^{-1},\ t.v^{-1},\ v.t^{-1},\ u.t^{-1},\ x.t^{-1},\ u.v^{-1},\ v.x^{-1},\ x.v^{-1},\ u.x^{-1},\ v^{-1}.u^{-1}$ | $u \xrightarrow{t} v$ | $v \xrightarrow{u} t$ | $u.v^{-1},\ u.v,\ u.t^{-1},\ u.v^{-1},\ v.t^{-1},\ x.t^{-1},\ u.v^{-1},\ u.x^{-1},\ v^{-1}.u^{-1}$ | $v^{-1}.u^{-1}$ | ✓ |
|---|---|---|---|---|---|---|---|
| $x'.2$ | $v^{-1}.u^{-1}.2$ | $v \mapsto sv\ \ u \mapsto su$ | $t.s^{-1},\ t.s,\ t.s^{-1},\ x.s^{-1},\ v.x^{-1},\ s.v^{-1},\ u.s^{-1}$ | $u \xrightarrow{s} v$ | $v \xrightarrow{u} s$ | $t.s^{-1},\ t.s,\ x.s^{-1},\ s.v^{-1},\ s.u^{-1}$ | $v^{-1}.u^{-1}$ | ✓ contained in $x'.1$ via $x \mapsto xs\ \ u \mapsto ts$ |
| $x'.3$ | $v^{-1}.u^{-1}.3$ | $v \mapsto uw\ \ u \mapsto u$ | $t.u^{-1},\ u.v^{-1},\ u.t^{-1},\ u.v^{-1},\ x.u^{-1},\ v.x^{-1}$ | $u \xrightarrow{v} t$ | $v \xrightarrow{u} t$ | $t.u^{-1},\ u.v^{-1},\ u.t^{-1},\ x.u^{-1},\ v.x^{-1}$ | $v^{-1}.u^{-1}$ | ✓ contained in $x.1$ via $x \mapsto xu\ \ u \mapsto tu$ |
| $x'.4$ | $v^{-1}.u^{-1}.4$ | $v \mapsto v\ \ u \mapsto vu$ | $t.v^{-1},\ v.t^{-1},\ v.u^{-1},\ v.t^{-1},\ x.v^{-1},\ v.x^{-1},\ u.v^{-1},\ u.x^{-1}$ | $u \xrightarrow{v} t$ | $v \xrightarrow{u} t$ | $t.v^{-1},\ v.t^{-1},\ v.u^{-1},\ x.v^{-1},\ v.x^{-1},\ u.v^{-1},\ u.x^{-1}$ | $v^{-1}.u^{-1}$ | ✓ contained in $x.1$ via $x \mapsto xv\ \ v \mapsto tv$ |

| 3.1.1 | $v^{-1}.u^{-1}.1$ | Id | $v.u^{-1},\ u.v^{-1},\ v.v^{-1},\ u.v$ | $u \xrightarrow{v} v$ | $u \xrightarrow{u} v$ | $u.u^{-1},\ v.u^{-1},\ v.v^{-1},\ v.v^{-1}$ | ambiguous |
|---|---|---|---|---|---|---|---|
| 3.1.1.2 | $v.v^{-1}.2$ | $v \mapsto t^{-1}.vt\ \ u \mapsto u$ | $t.u^{-1},\ u.t^{-1},\ v.t^{-1},\ v^{-1},\ t^{-1}.v$ | $u \xrightarrow{v} t$ | $v \xrightarrow{t} t$ | $t.u^{-1},\ u.t^{-1},\ v.t^{-1},\ t^{-1}.v,\ u.u^{-1}$ | $u.u^{-1}$ | ambiguous |
| 3.1.1.2.1 | $u.u^{-1}.1$ | Id | $t.u^{-1},\ u.t^{-1},\ v.t^{-1},\ t^{-1}.v,\ u.u^{-1}$ | $u \xrightarrow{v} t$ | $v \xrightarrow{u} t$ | $t.u^{-1},\ u.t^{-1},\ v.t^{-1},\ t^{-1}.v,\ u.u^{-1}$ | $u.u^{-1}$ | ✓ |
| 3.1.1.2.2 | $u.u^{-1}.2$ | $v \mapsto v\ \ u \mapsto s^{-1}.us$ | $v.s,\ v.s^{-1},\ u.s^{-1},\ u.s^{-1},\ u.u^{-1}$ | $u \xrightarrow{s} v$ | $v \xrightarrow{u} s$ | $v.s,\ v.s^{-1},\ u.s^{-1},\ u.s^{-1},\ u.u^{-1}$ | $u.u^{-1}$ | ✓ |

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