Optimal management of website under adverse impacts conditions

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Abstract. The paper considers a mathematical model for managing a website of a company operating in a competitive environment. Management is carried out in order to maximize profits. Optimal control actions is determined using Pontryagin’s maximum principle. Numerical solution is obtained using iteration algorithm based on gradient descent method. The obtained results form the basis of a software application that can be used as a management decision support system.

1. Introduction

The modern realities of market relations, high competition and a decrease in the effectiveness of traditional methods of advertising and promotion, force many companies to use new information technologies in their activities that are directly connected using the web. Currently, this environment is constantly evolving due to new components and services. Data from international analytical agencies [1] show that more than 4 billion people currently use the Internet around the world. This is a huge audience and the vast majority of the companies, regardless of the scope of the provided services, are fighting for its attention, in one way or another.

In domestic and foreign literature, it will be seen different opinions about the role played by the company own site in its communication and marketing processes. According to V.I. Kholmogorov, “every modern commercial company striving to achieve maximum efficiency in generating financial profit must have its own website”, which is an effective tool for solving the problem of increasing profit volumes, conducting research on consumers and organizing customer feedback. [2]

A. Wheeler in his works concludes that sites, more than other attributes and methods of advertising, tell about the individual characteristics of companies, and also create the effect of direct interaction with it. “A visit to the company’s website gives the same effect as a real visit to it, and sometimes it’s more effective, easier and faster, because it allows the client to make a decision in a relaxed atmosphere, without pressure and haste.” [2]

The official website of the company may function as: [3]
1. advertising platform, its own media channel and the information media, controlled by the company;
2. a tool for increasing recognition, shaping the image of the company as a market leader;
3. factor in the development of foreign economic relations of the company [4];
4. trading platform, sales promotion channel [5];
5. a tool for promoting products and services of a company [6];

However, having own site, in addition to the described advantages, becomes the cause of a number of problems associated with its maintenance and proper exploitation. The implementation of concepts such as e-commerce, digital libraries, video on demand and distance learning lead to an increase in web traffic. Some popular websites receive millions of requests per day, often without providing a sufficiently short response time. The long response time is becoming a source of frustration for many users and a problem for managers of many web sites. One of the problems for site administrators is associated with the adequate determination of the size of the infrastructure in terms of information technology for a site that can provide the proper quality of service needed by users. Administrators must monitor the performance of their websites and the services they offer, i.e. evaluate load intensity, detect network bottlenecks, and predict future performance reductions. In addition, the actions of competitors can also be a cause of additional problems and become an obstacle to the normal functioning of the resource. In [7], the main methods of the negative impact that third parties can have on the site are considered.

The speed and performance of sites is very important to users. If the web-resource does not process requests quickly enough, then the company will not only lose virtual visitors, but also potential customers, and therefore profit. Search engines such as Google, Yandex take into account the response time of a website in the search ranking, therefore, when optimizing speed, you need to take everything into account. The following factors affect the page loading speed of a website: the page size with all the items to be loaded (images, banners, counters, etc.); congestion of the web server on which the site is located; user internet connection speed. The usual conversation about increasing the speed of web-applications comes down to discussing delays in transferring information between the server and the client, optimizing the database and the time spent by the server for processing and sending HTML to the client. These processes, however, only partially address the time that the client spends waiting for the page to load in his browser window. Most of the time is spent on loading, caching, and displaying JavaScript, CSS, and images. In work [7] the basic techniques for speeding up the Internet resource are listed. Among other methods, this includes updating and upgrading of server hardware, which serves as a hardware platform for the site functioning. Each improvement entails an increased expense of funds, which ultimately affects the profit. Often the question arises of the appropriateness of a particular modernization action in terms of the ratio of planned profits to implementation costs. In [8], several approaches were considered, which are guided in practice for solving the problem of company’s website modernization to attract new customers. For the most part, these methods are based on expert assessment and are subjective in nature and may not bring the desired result.

For this reason, the question of the applicability of the mathematical methods to solve the problem of optimizing the management of the website of a company operating in a competitive environment seems to be relevant.

2. Materials and methods
In this part, the mathematical model and methods of problem solving will be considered.

2.1 Problem statement
Consider $m$ web sites that provide similar services and compete for the same group of users, which by definition is much larger than the number of sites. Each site $i$ uses different strategies to increase its customer base $x_i$ ($x_i \geq 0$): from advertising to lower prices. Note that if $x_i$ is a part of the population that is a client of website $i$, it can, in general, be considered as a part of the population that knows about the existence of the site. The customer base volume can be determined by the number of users who saved the site in the bookmarks of the Internet browser.

The evolution over the time of the $x_i$ customer base corresponding to site $i$ is determined by two main factors. In the case when there is no competition with other sites, the client base grows at a rate
of $\alpha_i$ ($\alpha_i \geq 0$), reaching saturation at the level of $N$ ($N \geq 0$). These values are determined by the throughput of the site, i.e. the ability to serve a given number of visitors per time unit.

On the other hand, if other sites provide similar services, the level of competition determines whether the user visits several competing sites (low level of competition) or, having visited one, does not want to go to competing sites (high level of competition). In particular, the conditions of competition can be represented as follows: if both sites provide the same services, then some users will stop using one of them.

Let construct a mathematical model that describes the change of the volume of the client base of each firm $x_i(t), i = (1, m)$ on a given time interval. Given the introduced notation, $x_i(t)$ is the volume of the client base of the $i$-th site at time $t$, $i = (1, m)$; $\alpha_i(t) \geq 0$ – is a coefficient that characterizes the rate of change of the client base of the $i$-th site, $N$ is the market saturation level of sites, which is determined based on the economic situation in the region where the companies are located. We introduce the control action - the indicator $u_i(t) \geq 0$, which characterizes the consumption of resources aimed at the upgrade of the $i$-th site, for example, to increase the speed of its work per time unit. Thus the dynamic of the potential customer base of the $i$-th site $x_i(t), i = (1, m)$ is described by a system of nonlinear differential equations on a given time interval:

$$\dot{x}_i(t) = \alpha_i x_i(t) \left( N - \sum_{i=1}^{m} x_i(t) \right) + u_i(t)x_i(t), i = 1, m \tag{1}$$

It is needed to solve the problem of optimal management of the activity of the first site in order to optimize the profit of its owner. Let $\rho(t)$ be the profit that the client brings to the site at time $t$. Then the value of the profit of the first site obtained during time $T$ is determined by the ratio:

$$I(u) = \int_{0}^{T} (\rho(t) - u_1(t))x_1(t)dt \tag{2}$$

The solution of the problem is the set of the optimal controls $\bar{u}(t) = (\bar{u}_1(t), \bar{u}_2(t), \ldots, \bar{u}_m(t))$, which maximizes the profit functional:

$$I(u) = \int_{0}^{T} (\rho(t) - u_1(t))x_1(t)dt \rightarrow \text{max} \tag{3}$$

with dynamic constraints that describes system state change:

$$\dot{x}_i(t) = \alpha_i x_i(t) \left( N - \sum_{i=1}^{m} x_i(t) \right) + u_i(t)x_i(t), i = 1, m, t \in [0, T]. \tag{4}$$

close values constraints:

$$0 \leq u_i(t) \leq u_{\text{max}}, i = 1, m, t \in [0, T] \tag{5}$$

Initial conditions:

$$x_i(0) = x_i^0, i = 1, m. \tag{6}$$

2.2 Solution method
To solve the problem of finding optimal control, the Pontryagin’s maximum principle was applied. The scalar Pontryagin’s function [9] has the form:

$$H(t, x, u, p, \psi, \lambda_0) = \lambda_0(\rho(t) - u_1(t))x_1(t) +$$
According to the maximum principle, the Pontryagin’s function on optimal control reaches a maximum:

\[
H(t, \bar{x}, \bar{u}, \rho, \psi, \lambda_0) = \lambda_0 (\rho(t) - \bar{u}_i(t)) \bar{x}_i(t) + \sum_{i=1}^{m} \left( \psi_i(t) \left( \alpha_i \bar{x}_i(t) \left( N - \sum_{i=1}^{m} x_i(t) \right) + \bar{u}_i(t) \bar{x}_i(t) \right) \right) = \\
= \max_{0 \leq u_i \leq u_{\text{max}}, i} \left( \lambda_0 (\rho(t) - u_i(t)) x_i(t) + \sum_{i=1}^{m} \left( \psi_i(t) \left( \alpha_i x_i(t) \left( N - \sum_{i=1}^{m} x_i(t) \right) + u_i(t) x_i(t) \right) \right) \right) = \tag{7}
\]

Adjoint functions satisfy the conditions:

\[
\dot{\psi}_i(t) = -\frac{\partial H}{\partial x_i} = \begin{cases} 
-\lambda_0 (\rho(t) - u_1(t)) - \psi_1(t) \alpha_1 \left( N - 2 x_1(t) - \sum_{i=2}^{m} x_i(t) \right) - \\
-\psi_1(t) u_1(t), i = 1, \\
-\psi_i(t) \alpha_i \left( N - 2 x_i(t) - \sum_{i \neq i}^{m} x_i(t) \right) - \psi_i(t) u_i(t), i = 2, m 
\end{cases} \tag{8}
\]

\[
\psi_i(T) = 0, i = 1, m.
\]

Then introduce the switching functions:

\[
\phi_1(t) = -\lambda_0 x_1(t) + \psi_1(t) x_1(t), \\
\phi_i(t) = -\psi_i(t) x_i(t), i = 2, m. \tag{9}
\]

From the maximum principle equation, we obtain expressions for the components of optimal control:

\[
\bar{u}_i(t) = \begin{cases} 
u_{\text{max}}, & \text{if } \phi_i(t) < 0, \\
0, & \text{if } \phi_i(t) > 0, \\
\gamma_1 \in [0, u_{\text{max}}], & \text{if } \phi_i(t) = 0, \\
i = 1, m.
\end{cases} \tag{10}
\]

Thus, the boundary value problem of the maximum principle takes the form:
The system is bounded by the control defined by relations (10). To solve the boundary value problem, various technics can be used: gradient descent method, sweep (double-sweep) method, multiple shunting, genetic algorithm, etc.

To further solve the problem, it is advisable to use numerical methods [10] to find an approximate numerical solution. In this paper, we will use the gradient descent method, according to which an iterative process of constructing an approximate numerical solution of the problem is organized. For this, it will be necessary to proceed to a discrete approximation of the original problem. Using the relations obtained when considering discrete approximations, an iterative process is organized that gives an approximate numerical solution. The gradient descent method allows you to get a minimum of functions by performing optimization in the direction of the anti-gradient.

Since the constraints on the control of the parallelepiped type (5) are chosen in the problem, when improving the control actions, the gradient projection method will be used. When using other restrictions, they should be included in the functional using the penalty function method.

In the future, the cost of sales of goods $\rho$ can be considered a constant for fairly small periods of time during which a computational experiment is conducted. To take into account phase constraints, we use the method of external penalty functions and introduce additional terms into the functional. The method of external penalty functions was chosen because of the existence of restrictions of the type of equalities in the problem under consideration. As will be seen from the parameters of the computational experiments, the correction introduced by the value $x_i(t)$ is very significant relative to the entire value of the functional and, therefore, the exponent of the term of the penalty function is enough to choose equal to 2. The objective function will take the form:

$$I(u) = -\int_0^T (\rho(t) - u_1(t))x_1(t)dt + \int_0^T \left(\sum_{i=1}^{m} \max\{-x_i(t),0\}\right)^2 dt \to \min.$$  

(12)

2.3 Discrete approximation

The approximation of the system of differential equations is carried out according to the Euler scheme, the integrals are replaced by the expression of the sums according to the rule of left rectangles. We divide the interval $[0,T]$ uniformly with $q$ points $t_k, k = (0, q-1)$, denoting $t_k = k\Delta t$, $\Delta t = \frac{T}{q}$.

$$x_i^k = x_i(t_k), u_i^k = u_i(t_k).$$

The objective functional will take the form:

$$I = -\sum_{k=0}^{q-1} (\rho^k - u_1^k)x_1^k \Delta t + \left(\sum_{k=0}^{q-1} \sum_{i=1}^{m} \max\{-x_i^k,0\}\right)^2 \cdot \Delta t \to \min.$$  

(13)
We approximate the dynamic constraints by the Euler formula:

\[
\dot{x}_i^{k+1} = \frac{x_i^{k+1} - x_i^k}{\Delta t}, k = 0, q - 1,
\]

\[
x_i^{k+1} = x_i^k + \Delta t (a_i x_i^k \left( N - \sum_{i=1}^m x_i^k \right) + u_i^k x_i^k), k = 0, q - 1, i = 1, m.
\]  

To solve the discrete optimal control problem, we use the method of Lagrange [10] method of multipliers. We compose the Lagrange function:

\[
L(u) = -\lambda_0 \sum_{k=0}^{q-1} \left( \rho^k - u_i^k \right) x_i^k - \left( \sum_{i=1}^m \max\{-x_i^k, 0\} \right)^2 \Delta t + \\
+ \sum_{k=0}^{q-1} \sum_{i=1}^m \psi_i^{k+1} \left( x_i^{k+1} - x_i^k - \Delta t \left( a_i x_i^k \left( N - \sum_{i=1}^m x_i^k \right) + u_i^k x_i^k \right) \right)
\]

Stationary conditions:

\[
\frac{\partial L}{\partial u_i^j} = \begin{cases} 
\lambda_0 \Delta t x_i^s - \psi_i^{s+1} x_i^s \Delta t, l = 1, \\
-\psi_i^{s+1} x_i^s \Delta t, l = 2, m, 
\end{cases}
\]

\[
-\lambda_0 \Delta t (\rho^s - u_i^s) + \lambda_0 \Delta t \frac{\partial (\max\{-x_i^s, 0\})^2}{\partial x_i^s} + \psi_i^s - \psi_i^{s+1} - \\
- \psi_i^{s+1} \Delta t \left( a_i N - 2 a_i x_i^s - a_i \sum_{r \neq i} x_r^s + u_i^s \right) = 0, l = 1, s = 0, q - 1,
\]

\[
\frac{\partial L}{\partial x_i^j} = \begin{cases} 
\lambda_0 \Delta t \frac{\partial (\max\{-x_i^s, 0\})^2}{\partial x_i^s} + \psi_i^s - \psi_i^{s+1} - \\
- \psi_i^{s+1} \Delta t \left( a_i N - 2 a_i x_i^s - a_i \sum_{r \neq i} x_r^s + u_i^s \right) = 0, l = 2, m, s = 0, q - 1,
\end{cases}
\]

\[
\frac{\partial L}{\partial x_i^l} = 0, l = 1, m.
\]

From the expressions of stationary conditions, one can obtain recurrence relations for recalculating the adjoint vectors \([\psi]\) and control actions \([u]\).

Using these relations, we compose an algorithm for constructing an approximate numerical solution of the optimal control problem:

1. Set the initial values for external control actions \([u]^{(0)}\) for zero iteration. Establish system initial state components \([x]\). Further \(r = 0, 1, 2, 3, \ldots\) describes iteration number.
2. Calculate client base state on \(r\)-th iteration \([x]^{(r)}\) according to (14).
3. Find the value of objective function on \(r\)-th iteration \(l^{(r)}\) according to (13).
4. Calculate values of the adjoint vectors on \(r\)-th iteration \([\psi]^{(r)}\) according to (17), starting with the component on the last discretization layer \(q\).
5. Using equation 16 define the external control actions on \((r + 1)\)-th iteration:

\[
[u]^{(r+1)} = [u]^{(r)} - \alpha \left( \frac{\partial L}{\partial u} \right)^{\left( r \right)},
\]

where \(\alpha\) – gradient descent step.
6. Further, with improved set of controls repeat steps 2-4 to find \([x]^{(r+1)}, l^{(r+1)}\).
7. Compare objective function values \( I^{(r)}, I^{(r+1)} \). If \( I^{(r+1)} < I^{(r)} \), then assume that iteration was successful and it’s necessary to check termination condition. In this case we assume termination condition to be:

\[
|I^{(r)} - I^{(r+1)}| < \varepsilon,
\]

where \( \varepsilon \) is responsible for calculation accuracy. This value should be chosen as the smallest possible and it is limited only by the computing power of the used hardware platform.

If during comparison of objective function values on step 7 the condition \( I^{(r+1)} > I^{(r)} \) is fulfilled, then the descent step is adjusted and steps 5-7 are repeated. These actions are repeated until the condition in step 7 is met.

Value of step \( \alpha \) is chosen so that the correction is proportional to the adjustable values. There are several basic ways to select the gradient descent step:

1. \( \alpha \) can be chosen by a constant number, in this case, the convergence of the method is not guaranteed,
2. \( \alpha \) can be fractional, for this, when switching to the next iteration, it is divided by a certain number,
3. \( \alpha \) can be found using the steepest descent method, in this case:

\[
\alpha = \arg\min_{\alpha} I(u[k] - \alpha f[k]),
\]

where \( k \) – iteration number.

The block diagram of the algorithm for constructing an approximate optimal solution to the problem is presented in figure 1.

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**Figure 1.** Block diagram of approximate numerical solution constructing algorithm.

In process of writing a program that implements the algorithm for constructing an approximate optimal solution to the optimal control problem described above, the C++ programming language was used. As a development environment, the Qt 5.9 platform was chosen, which allows you to create applications with a graphical interface.

An additional qcustomplot library was used to plot the functions, connecting to the application as a separate module.

Graphical user interface of the developed application is provided in figure 2:
Figure 2. Graphical user interface of developed software application. Parameters input window.

The “Parameters” tab provides the user an opportunity to enter parameters of the optimization method, such as accuracy of calculations and step of gradient descent, as well as parameters of the mathematical model.

3. Results and discussion

We now proceed to the description of the performed computational experiments. In the examples under consideration, we set the parameter $m$ (number of sites) equal to 2. This value was chosen due to the available statistical data at the time of writing this article. Next, a description will be given of some possible situations of interaction of two companies that provide similar services (equipment sales). We will consider cases when companies have similar or different growth rates of the client base $a_1$ and $a_2$, initial volumes of the client base $x_1^0$ and $x_2^0$, restrictions on the maximum level of cash injections for self-development $u_1^{max}$ and $u_2^{max}$, respectively.

The first example demonstrates the situation when there is already an well-established company on the market (under number 2) with an established clientele and a new company enters the market (under number 1). Accordingly, companies have different volumes of client bases and the rate of these change. Suppose that company 1 decides to invest more in its development in order to “poach” part of customers from the leading company. After performing the calculations, the graphs were obtained that reflect the dynamic of changes in the volume of customer bases. These results show how the new company (at number 2) was able to pick up and retain part of the clientele from a company that was already on the market and did not begin to invest additional funds in development cause management team have underrated their opponents.

Initial parameters: $a_1 = 0.05$, $a_2 = 0.3$, $x_1^0 = 0.3$, $x_2^0 = 15.2$, $u_1^{max} = 2$, $u_2^{max} = 1$, $N=10$, $\rho = 0.1$, $T=12$, $q=1000$. The calculated trajectories of the state of client bases and external control actions have the form shown in figures 3, 4.

Figure 3. External control actions $u_1(t)$, $u_2(t)$.

Figure 4. Trajectories of the state of client bases $x_1(t)$, $x_2(t)$.
In the next experiment, the situation when company 2 has large resources to attract customers \((u_2 = 2)\) is shown. The initial parameters of the client base (growth coefficient \(\alpha_1, \alpha_2\) and quantity \(x_1^0, x_2^0\) respectively) are equal. The trajectory of client base state dynamic shows that spending money on attracting users was not in vain. The amplitude of the graph of the second site is greater than the first. However, given that the growth rate initially coincided, financial injections gave only a short-term advantage.

Initial parameters: \(\alpha_1 = 0.05, \alpha_2 = 0.05, x_1^0 = 0.1, x_2^0 = 0.1, u_1^{\text{max}} = 1, u_2^{\text{max}} = 2, N=10, \rho = 0.3, T = 12, q=1000\). The calculated trajectories of the state of client bases and external control actions have the form shown in figures 5, 6.

In this experiment, we assume that company 2 also has great resources for attracting customers \((u_2 = 2)\), and in addition, initially it showed higher growth rates of the client base. It can be seen that the infusion of money allowed not only to poach some customers, but also to keep them. This suggests that in a situation where the company already has an advantage, additional expenses for expanding the client base will give more tangible results. This can clearly be seen in the example of young and mature companies.

Initial parameters: \(\alpha_1 = 0.05, \alpha_2 = 0.3, x_1^0 = 0.2, x_2^0 = 0.3, u_1^{\text{max}} = 1, u_2^{\text{max}} = 2, N=10, \rho = 0.2, T = 12, q=1000\). The calculated trajectories of the state of client bases and external control actions have the form shown in figures 7, 8.

4. Conclusion
In this work the problem of optimal management of the website of a company in a competitive environment to achieve the highest possible level of profit was considered. To solve the problem, the Pontryagin maximum principle was applied, a boundary-value problem was compiled, and a general view of the optimal control was obtained. To obtain an approximate numerical solution of the problem, the gradient descent method was applied, and a discrete approximation of the original problem was considered. An algorithm for constructing a numerical solution is presented.
An application is obtained that can be used in modeling the processes of website interaction in a competitive environment. The results of computational experiments showing the performance of the application and correctness of the operation of the algorithm for finding a numerical solution are presented. The considered model of the interaction of two sites can be refined by entering and varying additional parameters, which will lead to an increase in the level of estimation accuracy.

A further improvement of the algorithm will be the transition to more accurate methods for choosing the step of gradient descent, such as the steepest descent method. As a refinement of a software tool, it may be its development into a management decision support system for maintaining a web-site.

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