A generic new platform for topological quantum computation using semiconductor heterostructures

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We show that a film of a semiconductor in which \( s \)-wave superconductivity and a Zeeman splitting are induced by proximity effect, supports zero-energy Majorana fermion modes in the ordinary vortex excitations. Since time reversal symmetry is explicitly broken, the edge of the film constitutes a chiral Majorana wire. The heterostructure we propose -- a semiconducting thin film sandwiched between an \( s \)-wave superconductor and a magnetic insulator -- is a generic system which can be used as the platform for topological quantum computation by virtue of the existence of non-Abelian Majorana fermions.

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Introduction. In two spatial dimensions, where permutation and exchange are not necessarily equivalent, particles can have quantum statistics which are strikingly different from the familiar statistics of bosons and fermions. In situations where the many body ground state wave-function is a linear combination of states from a degenerate subspace, a pairwise exchange of the particle coordinates can unitarily rotate the ground state wave-function in the degenerate subspace. In this case, the exchange statistics is given by a multi-dimensional unitary matrix representation (as opposed to just space. In this case, the exchange statistics is given by a multi-dimensional unitary matrix representation (as opposed to just

\[
\bar{H} = \frac{\hbar^2}{2m^*} + \mu \gamma_1 + \frac{V_z}{2} \gamma_2 + \sigma_x \frac{e}{c} \hat{\sigma} \cdot \hat{p} - \frac{e}{c} \hat{k} \cdot \hat{B}
\]

Here, \( m^* \), \( V_z \) and \( \mu \) are the conduction-band effective mass of an electron, effective Zeeman coupling induced by proximity to a magnetic insulator (we neglect the direct coupling of the electrons with the magnetic field from the magnetic insulator), and chemical potential, respectively. The coefficient \( \alpha \) describes the strength of the Rashba spin-orbit coupling and \( \sigma_x \) are the Pauli matrices. Despite the similarity in the spin-orbit-coupling terms, \( \bar{H}_0 \) and the Hamiltonian for the TI surface in Ref. \( [8] \) differ by the existence of a spin-diagonal kinetic energy term in Eq. \( \bar{H}_0 \). Because of the spin-diagonal kinetic energy, there are in general two spin-orbit-split Fermi surfaces in the present system, in contrast to the surface of a TI in which an odd number of bands cross the Fermi level \( [9] \). In Eq. \( \bar{H}_0 \), for out-of-plane Zeeman coupling such that \( |V_z| > |\mu| \), a single band crosses the Fermi level. Thus, analogous to a strong TI surface (but arising from qualitatively different physics), the system has a single Fermi surface, which is suggestive of
non-Abelian topological order if s-wave superconductivity is induced in the film. We show below that this is indeed the case by analyzing the Bogoliubov de Gennes (BdG) equations for a vortex in the superconductor in the heterostructure shown in Fig. 1.

The proximity-induced superconductivity in the semiconductor can be described by the Hamiltonian,

$$\hat{H}_p = \int dr \left\{ \Delta_0(r) \hat{c}_\uparrow(r) \hat{c}_\downarrow(r) + H.c. \right\}, \quad (2)$$

where $\hat{c}_\sigma(r)$ are the creation operators for electrons with spin $\sigma$ and $\Delta_0(r)$ is the proximity-induced gap. The corresponding BdG equations written in Nambu space become,

$$\begin{pmatrix} \hat{H}_0 & \Delta_0(r) \\ \Delta_0(r) & -\sigma_y \hat{H}_0 \sigma_y \end{pmatrix} \Psi(r) = E \Psi(r), \quad (3)$$

where $\Psi(r)$ is the wave function in the Nambu spinor basis, $\Psi(r) = (u_\uparrow(r), u_\downarrow(r), v_\uparrow(r), -v_\downarrow(r))^T$. Using the solutions of the BdG equations, one can define Bogoliubov quasiparticle operators as

$$\hat{c}_\sigma(r) = \int dr \sum_\sigma u_\sigma(r) \hat{c}_\sigma(r) + v_\sigma(r) \hat{c}_\sigma(r).$$

The bulk excitation spectrum of the BdG equations with $\Delta(r) = \Delta_0$ has a gap for non-vanishing spin-orbit coupling.

**BdG equations for a vortex.** We now consider the vortex in the heterostructure shown in Fig 1 and take the vortex-like configuration of the order parameter: $\Delta_0(r, \theta) = \Delta_0(r) e^{i\theta}$. Because of the rotational symmetry, the BdG equations can be decoupled into angular momentum channels indexed by $m$ with the corresponding spinor wave-function,

$$\Psi_m(r, \theta) = e^{im\theta} \left( u_\uparrow(r), u_\downarrow(r) e^{i\theta}, v_\uparrow(r) e^{-i\theta}, -v_\downarrow(r) \right)^T. \quad (4)$$

Because of the particle-hole symmetry of the BdG equations, if $\Psi(r)$ is a solution with energy $E$ then $i\sigma_y \tau_y \Psi^*(r)$ is also a solution at energy $-E$. Here $\tau_y$ is defined to be the Pauli matrix in Nambu spinor space. In particular, zero-energy solutions of the BdG equations must come in pairs, $\Psi(r)$ and $i\sigma_y \tau_y \Psi^*(r)$, unless these two wave-functions refer to the same state. Thus, a zero-energy solution in an angular momentum channel $m$ is always paired with another zero-energy solution in the channel $-m$ and, therefore, can be non-degenerate only if it corresponds to the $m = 0$ angular momentum channel.

The radial BdG equations describing the zero-energy state in the $m = 0$ channel can be written as

$$\begin{pmatrix} H_0 & \Delta_0(r) \\ \Delta_0(r) & -\sigma_y H_0 \sigma_y \end{pmatrix} \Psi(r) = 0, \quad (5)$$

with $\eta = \frac{1}{2m^2}$. Since the BdG matrix is real, there are two solutions $\Psi(r)$ and $\Psi^*(r)$ with $E = 0$. Furthermore, it follows from the particle-hole symmetry of the BdG equations that $i\sigma_y \tau_y \Psi(r)$ is also a solution. Thus, any non-degenerate $E = 0$ solution must satisfy the property $\sigma_y \tau_y \Psi(r) = i\lambda \Psi(r)$. Moreover, because $(i\sigma_y \tau_y)^2 = -1$, the possible values of $\lambda$ are $\lambda = \pm 1$. The value of $\lambda$ sets a constraint on the spin degree of freedom of $\Psi(r)$, such that $v_\uparrow(r) = \lambda u_\uparrow(r)$ and $u_\downarrow(r) = \lambda v_\downarrow(r)$. This allows one to eliminate the spin degree of freedom in $\Psi(r)$ and define a reduced spinor $\Psi_0(r) = (u_\uparrow(r), u_\downarrow(r))^T$. The corresponding reduced BdG equations take the form of a $2 \times 2$ matrix differential equation:

$$\begin{pmatrix} -\eta(\partial_r^2 + \frac{1}{r} \partial_r) + V_z - \mu & \lambda \Delta(r) + \alpha(\partial_r + \frac{1}{r}) \\ -\lambda \Delta(r) - \alpha \partial_r & -\eta(\partial_r^2 + \frac{1}{r} \partial_r - \frac{1}{r^2}) - V_z - \mu \end{pmatrix} \Psi_0(r) = 0. \quad (6)$$

We now approximate the radial dependence of $\Delta_0(r)$ by $\Delta_0(r) = 0$ for $r < R$ and $\Delta_0(r) = \Delta_0$ for $r \geq R$. In view of the stability of the putative Majorana zero-energy solution to local changes in the Hamiltonian, such an approximation can be made without loss of generality. For $r < R$ the analytical solution to Eq. (6) is given by $\Psi_0(r) = (u_\uparrow J_0(\alpha z), u_\downarrow J_1(\alpha z))^T$ with the constraint

$$\begin{pmatrix} \eta \alpha z^2 + V_z - \mu & \alpha z \\ \alpha z & \eta \alpha z^2 - V_z - \mu \end{pmatrix} \begin{pmatrix} u_\uparrow \\ u_\downarrow \end{pmatrix} = 0. \quad (7)$$

Here $J_n(r)$ are the Bessel functions of the first kind. The characteristic equation for $z$ is

$$(\eta \alpha z^2 - \mu)^2 - V_z^2 - \alpha^2 z^2 = 0. \quad (8)$$

In the case $V_z > V_x$ the roots of Eq. (8) are real: $z_{1,2} = \pm \sqrt{w_2}$ and $z_{3,4} = \pm i \sqrt{|w_2|}$ with $w_{1,2} > 0$ being the solutions for $z^2$. In the opposite limit, $0 < \mu < V_z$, there are two real solutions $z_{1,2} = \pm \sqrt{|w_1|}$ and two imaginary solutions $z_{3,4} = \pm i \sqrt{|w_2|}$. Since the Bessel functions are symmetric, we find two solutions which are well-behaved at the origin: $\Psi_1(r) = (u_\uparrow J_0(z r), u_\downarrow J_1(z r))^T$ and $\Psi_2(r) = (u_\uparrow J_0(z r), u_\downarrow J_1(z r))^T$. Therefore, the full solution at $r < R$ is $\Psi_0(r) = c_1 \Psi_1(r) + c_2 \Psi_2(r)$.

At large distances $r > R$, where $\Delta_0(r) = \Delta_0$, the solution to Eq. (6) is complicated. Nevertheless, one can write the solution as a series expansion in $1/r$:

$$\Psi_0(r) = \frac{e^{i z r}}{\sqrt{r}} \sum_{n=0,1,2...} \frac{a_n}{r^n}, \quad (9)$$

Here $c_1$ is the coefficient of $\Psi_1(r)$ and $c_2$ is the coefficient of $\Psi_2(r)$.
different values of $\mu$ and $\lambda$ with $\Delta_0 \neq 0$. Lower panel: Numerical solution for the Majorana zero-energy state $\Psi_0(r) = (u(r), v(r))^T$ for $\lambda = -1$ and $\mu < V_z$. The dashed (red) and solid (blue) lines correspond to $u(r)$ and $v(r)$, respectively. Here we used the following parameters: $\eta = \alpha = V_z = 1$, $\mu = 0$, $\Delta_0 = 0.1$ and $R = 1$. The boundary conditions used are $\Psi_0(0) = (0, 0)^T$ and $\Psi_0(r = 40) = (0, 0)^T$.

where $a_n$ are the corresponding spinors. The zeroth order coefficient $a_0$ satisfies the following equation:

$$
\begin{pmatrix}
\eta z^2 + V_z - \mu & \lambda \Delta_0 + iz\alpha \\
-\lambda \Delta_0 - iz\alpha & \eta z^2 - V_z - \mu
\end{pmatrix} a_0 = 0.
$$

(10)

The higher order coefficients $a_n$ can be calculated from $a_0$ using a set of recursion relations [11]. The characteristic equation for Eq. (10) has 4 complex roots for $z$, which are shown in Fig. 2. Physical solutions of Eq. (6) at $r > R$, $\Psi_0^n(r) = \sum_{n \geq 0} c_n \Psi_n(r)$, require that $\text{Im}(z_n) > 0$. (Here $\Psi_n(r)$ is the solution corresponding to the eigenvalue $z_n$.) Thus, for $(\mu^2 + \Delta_0^2) > V_z^2$, there are two solutions for $\lambda = \pm 1$. On the other hand, for $(\mu^2 + \Delta_0^2) < V_z^2$ there are three solutions for $\lambda = -1$ and only one for $\lambda = 1$.

In order to obtain a unique solution for the zero-energy state, the wavefunctions $\Psi_0^n(r)$ and $\Psi_0^n(r)$ should satisfy boundary conditions at $r = R$. Since we are matching 2-component wave-functions and their derivatives at $r = R$, the continuity of $\Psi_0(R)$ and $\Psi_0'(R)$ leads to 4 independent equations. One additional constraint comes from the normalization of the wavefunction in all space. Thus, there are five independent constraints for the coefficients $c_n$. A unique zero-energy solution exists if the number of unknown coefficients $c_n$ is five, which is the case for $(\mu^2 + \Delta_0^2) < V_z^2$ and $\lambda = -1$. In this case, the wavefunctions $\Psi_0^n(r) = \sum_{n = 1, 2} c_n \Psi_n(r)$ and $\Psi_0^n(r) = \sum_{n = 3, 4, 5} c_n \Psi_n(r)$ have 2 and 3 unknown coefficients, respectively. In all other cases the number of unknowns $c_n$ is smaller than five, and thus, as we have checked explicitly, solutions for the zero-energy eigenfunction do not exist. From these arguments one can conclude that, for $(\mu^2 + \Delta_0^2) < V_z^2$, an ordinary vortex in the superconducting condensate contains a unique non-degenerate $E = 0$ solution in the $m = 0$ angular momentum channel. The numerical solution for the zero-energy state is shown in Fig. 2. It is straightforward to check that the zero-energy solution corresponds to a self-hermitian second-quantized operator $\gamma = \gamma^\dagger$: it is a Majorana fermion excitation.

We can also consider the special case with $\eta = 0$, $V_z = 0$ in the above equations, which describes the recent proposal for TQC [9] using zero-energy Majorana bound states at vortices on the interface of a TI and an s-wave superconductor. In this case, we find a single solution for $r < R$ and a pair of independent solutions for $r > R$. Since the BdG differential equation is now only first order, we need only match the 2-component spinors themselves (derivatives need not match) which yields 3 equations for the 3 coefficients. This leads to a unique Majorana fermion solution at the vortex, which is consistent with Ref. [9]. Interestingly, in contrast to our Hamiltonian for $\eta > 0$, the condition for the existence of a Majorana fermion for $\eta = 0$ is given by $V_z^2 < (\Delta_0^2 + \mu^2)$. The model considered in Ref. [9] and our present system have a similar order parameter structure. In both cases the order parameter component $(c_\uparrow(r)c_\downarrow(r'))$ has an s-wave orbital symmetry while the order parameter components $(c_\uparrow(r)c_\uparrow(r'))$ and $(c_\downarrow(r)c_\downarrow(r'))$ have $p_x + ip_y$ and $p_x - ip_y$ orbital symmetries, respectively. On the surface of a TI, because of time-reversal invariance, $\langle c_\uparrow(r)c_\downarrow(r')\rangle = \langle c_\downarrow(r)c_\uparrow(r')\rangle$. In our system, the ratio of the order parameter components in the two spin sectors is different from 1, and approaches 1 in the limit $\alpha^2/\eta \gg V_z$. In both cases, however, the superconducting pairing potential is s-wave, and is induced by proximity effect. Therefore, the superconducting state and the associated non-Abelian topological character are expected to be robust in the presence of finite disorder.

**Topological phase transition.** We have shown above that a non-degenerate Majorana state exists in a vortex in the superconductor only in the parameter regime $(\mu^2 + \Delta_0^2) < V_z^2$. This suggests that there must be a quantum phase transition (QPT) separating the parameter regimes $(\mu^2 + \Delta_0^2) < V_z^2$ and $(\mu^2 + \Delta_0^2) > V_z^2$, even though the system in both regimes is a $s$-wave superconductor. A non-degenerate zero-energy solution cannot disappear unless a continuum of energy levels appears around $E = 0$. Such a continuum of states at $E = 0$ can only appear if the bulk gap closes, which can be used to define a topological quantum phase transition. In the present system, such a phase transition can be accessed by varying either the Zeeman splitting or the chemical potential. A similar topological quantum phase transition has already been predicted for ultra-cold atoms with vortices in the spin-orbit coupling [12].

The bulk gap of the present system can be calculated from the bulk excitation spectrum:

$$
E^2 = V_z^2 + \Delta_0^2 + \tilde{\epsilon}^2 + \alpha^2 k^2 \pm 2\sqrt{V_z^2\Delta_0^2 + \tilde{\epsilon}^2(V_z^2 + \alpha^2 k^2)}
$$

(11)

where $\tilde{\epsilon} = \eta k^2 - \mu$. As seen in Fig. 3 the excitation gap first increases as a function of $\Delta_0$ (proximity-induced pairing) and then decreases and vanishes at a critical point, $\Delta_{0c} = \sqrt{V_z^2 - \mu^2}$, before re-opening and increasing with
any point. This is the phase without Majorana Fermion excitations. In fact, this phase can be reached from the conventional local state in a vortex in the analyzed in a way that closely follows our derivation of the low energy between the superconductors is ports a pair of zero-energy excitations when the phase differ-

which can be deposited on the semiconductor thin film, sup-

we find that an interface between two superconductor layers, in our system can be studied experimentally using non-local braiding in a way completely analogous to Ref. [9] to perform TQC. Majorana bound states as well as Majorana edge modes in our system can be studied experimentally using non-local Andreev reflection [13] and electrically detected interferome-

The experimental implementation of this proposal involves a heterostructure of a magnetic insulator (e.g. EuO), a strong spin-orbit coupled semiconductor (e.g. InAs) and an s-wave superconductor with a large $T_c$ (e.g. Nb). Using these materials, it is possible [11] to induce an effective superconducting pairing potential $\Delta_0 \sim 0.5$ meV and a tunneling-induced effective Zeeman splitting $V_z \sim 1$ meV. Additionally, the strength of spin-orbit interaction $\alpha$ in InAs heterostructures is electric-field tunable and can be made as large as $\alpha \approx 50$ meV-Å [16]. With these estimates, the quasiparticle gap $E_g$ is of the order of 1 K. Given that the chemical potential is gate-tunable and can be of the of the order of $\Delta_0$, we numerically estimate the magnitude of the excitation energy for the bound states in a vortex core of size $\sim 20$ nm to be of the order of 0.1 K [11], which sets the temperature scale for TQC in this system.

**Conclusion.** Our proposed TQC platform should be simpler to implement experimentally than any of the TQC candidates proposed in the literature so far, since it involves a standard heterostructure with a magnetic insulator, a semiconductor film, and an ordinary s-wave superconductor. We believe that the proposed scheme provides the most straightforward method for the solid-state realization of non-Abelian Majorana fermions. A significant practical advantage of the proposed TQC scheme is its generic simplicity: it requires neither special samples or materials nor ultra-low temperatures or high magnetic fields.

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