UNDERLYING FLAG POLYMATROIDS

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Abstract. We describe a natural geometric relationship between matroids and underlying flag matroids by relating the geometry of the greedy algorithm to monotone path polytopes. This perspective allows us to generalize the construction of underlying flag matroids to polymatroids. We show that the polytopes associated to underlying flag polymatroids are simple by proving that they are normally equivalent to certain nestohedra. We use this to show that polymatroids realized by subspace arrangements give rise to smooth toric varieties in flag varieties and we interpret our construction in terms of toric quotients. We give various examples that illustrate the rich combinatorial structure of flag polymatroids. Finally, we study general monotone paths on polymatroid polytopes, that relate to the enumeration of certain Young tableaux.

1. Introduction

Many exciting recent developments have benefited from the discrete geometric perspective on matroids: For a matroid \( M \) on ground set \( E \) and independent sets \( I \), its matroid base polytope is

\[
B_M = \text{conv}\{ e_B : B \in \mathcal{I} \text{ basis} \} \subset \mathbb{R}^E.
\]

This is a 0/1-polytope with edge directions in the type-A roots \( \{ e_i - e_j : i \neq j \} \). The geometric perspective was pioneered by Gelfand, Goresky, MacPherson, and Serganova [27], who showed that these geometric properties characterize matroids. Matroid base polytopes play an important role in the interplay of combinatorics and algebraic geometry [1] as well as in tropical algebraic geometry [35]; see also Section 6.

Another important polytope associated to \( M \) comes from flag matroids. Borovik, Gelfand, Vince, and White [12] introduced the underlying flag matroid \( \mathcal{F}_M \) of \( M \) as the collection of maximal chains of independent sets and studied them via their flag matroid polytopes

\[
\Delta(\mathcal{F}_M) = \text{conv}\{ e_{I_0} + e_{I_1} + \cdots + e_{I_r} : I_0 \subset I_1 \subset \cdots \subset I_r \in \mathcal{I} \text{ maximal chain} \}.
\]

Underlying flag matroids were also called truncation flag matroids in [4]. Like matroid base polytopes, flag matroid polytopes are also generalized permutahedra [41] that occur in connection with torus-orbit closures in flag varieties [15] and they are key in understanding tropical flag varieties [14, 31]. Underlying flag matroids are special cases of general flag matroids and Coxeter matroids [13, Section 1.7].

A first goal of our paper is to describe a natural geometric relationship between these two classes of polytopes that allows us to generalize the notion of underlying flag matroids to polymatroids. As our notation emphasizes the reference to the underlying matroid \( M \), we will simply speak of flag matroids henceforth.
Geometry of the greedy algorithm. The well-known greedy algorithm solves linear programs over $B_M$. Edmonds [22] interpreted the greedy algorithm geometrically and extended it to polymatroids. Polymatroids are certain submodular functions $f : 2^E \to \mathbb{R}_{\geq 0}$ that generalize rank functions of matroids and that naturally emerge in combinatorial optimization [25] as well as in the study of subspace arrangements [8]. To a polymatroid $(E, f)$, Edmonds associated the polymatroid polytope

$$P_f = \left\{ x \in \mathbb{R}^E : x \geq 0, \sum_{e \in A} x_e \leq f(A) \text{ for all } A \subseteq E \right\},$$

If $f$ is the rank function of a matroid $M$, then $P_f = \text{conv}\{e_I : I \in \mathcal{I}\}$ is the independence polytope of $M$, which we denote by $P_M$.

The base polytope $B_f$ is the face of $P_f$ that maximizes the linear function $1(x) = \sum_{e \in E} x_e$ and Edmonds showed that $B_f$ completely determines $f$. Up to translation, base polytopes $B_f$ are precisely Postnikov’s generalized permutahedra [41]. Edmonds’ greedy algorithm combinatorially solves the problem of maximizing $w \in \mathbb{R}^E$ over $B_f$ by tracing a 1-monotone path from 0 to a $w$-optimal vertex of $B_f$; see Figure 1 for an example.

So-called Baues posets capture the combinatorics of monotone paths on polytopes and considerable attention was devoted to the topology of Baues posets; see Section 2.2 and [6]. The subposet of coherent 1-cellular strings on $P_f$ is isomorphic to the face lattice of the monotone path polytope $\Sigma_1(P_f)$ of Billera–Sturmfels [7]. We show in Theorem 3.1 that all 1-cellular strings on $P_f$ are coherent and arise from the greedy algorithm. Hence the geometry of the greedy algorithm is completely captured by $\Sigma_1(P_f)$. Applied to matroids, this yields the relationship between matroid base polytopes and flag matroid polytopes.

**Theorem 1.** Let $M$ be a matroid. The flag matroid polytope $\Delta(\mathcal{F}_M)$ is normally equivalent to the monotone path polytope $\Sigma_1(P_M)$ of 1-cellular strings on $P_M$.

Normal equivalence, reviewed in Section 2, means that $\Delta(\mathcal{F}_M)$ and $\Sigma_1(P_M)$ have the same underlying normal fan and, in particular, is a strong form of combinatorial equivalence. The case of partial flag matroids associated to a matroid $M$ is settled by rank-selected independence polytopes that we define in Section 5. Theorem 1 is then a special case of Theorem 5.3.

**Flag polymatroids.** Flag matroid polytopes are polymatroid base polytopes. Hence the behavior of the greedy algorithm on a matroid $M$ is governed by an associated polymatroid.

**Theorem 2 (Theorem 3.8).** Let $(E, f)$ be a polymatroid. Then $\Sigma_1(P_f)$ is a polymatroid base polytope for the polymatroid

$$\widehat{f} := 2f(E) \cdot f - f^2.$$
We call \((E, \tilde{f})\) the **underlying flag polymatroid** of \((E, f)\).

A flat of \((E, f)\) is a subset \(F \subseteq E\) such that \(f(F \cup e) > f(F)\) for all \(e \in E \setminus F\). The lattice of flats \(\mathcal{L}(f)\) is the collection of flats partially ordered by inclusion. For matroids, lattices of flats are geometric lattices that completely determine the combinatorial structure of \(B_M\). For general polymatroids, this is not true. However, for flag polymatroids it turns out that \(\Sigma_1(P_f)\) is completely determined by \(\mathcal{L}(f)\). To make this more transparent, we relate flag polymatroids to yet another class of generalized permutahedra. Postnikov \([41]\) and Feichtner–Sturmfels \([23]\) introduced nestohedra, a rich class of simple generalized permutahedra associated to building sets \(B \subseteq 2^E\); see Section 4.

**Theorem 3.** Let \((E, f)\) be a polymatroid. The base polytope \(B_f\) of the flag polymatroid \((E, \tilde{f})\) is normally equivalent to the nestohedron for the building set

\[
\mathcal{U}(f) := \{E \setminus F : F \text{ flat of } f\}.
\]

In particular, flag (poly)matroid polytopes are simple polytopes.

In order to prove Theorem 3, we make a detour via max-slope pivot rule polytopes \([11]\). We show that the greedy algorithm on \(P_f\) coincides with the simplex algorithm on \(P_f\) with respect to the max-slope pivot rule. The behavior of the max-slope pivot rule on a fixed linear program such as \((P_f, 1)\) is encoded by an arborescence. The arborescence represents the choices made by the pivot rule along the simplex path started at a vertex \(v\) of \(P_f\) to an optimal vertex. Pivot rule polytopes geometrically encode these arborescences. We show that \(\Sigma_1(P_f)\) is normally equivalent to the max-slope pivot rule polytope \(\Pi_{P_f, 1}\). From the optimization perspective, this says the greedy path completely determines the behavior of the max-slope pivot rule on \(P_f\). Lemma 4.5 makes that precise and might be of independent interest.

**1.1. Realizable polymatroids and toric quotients.** A polymatroid \((E, f)\) is realizable over \(\mathbb{C}\) if there are linear subspaces \((U_e)_{e \in E}\) of some common vector space such that \(f(A) = \dim_{\mathbb{C}} \sum_{e \in A} U_e\) for all \(A \subseteq E\). If all subspaces are 1-dimensional, then \(f\) is the rank function of a (realizable) matroid \(M\). Choosing an ordered bases for each \(U_e\) determines a point \(L_f\) in the Grassmannian \(\text{Gr}(N, r)\) for \(N = \sum_{e} \dim U_e\) and \(r = f(E)\). We describe the action of the algebraic torus \(T^n = (\mathbb{C}^*)^n\) for \(n = |E|\) on \(\text{Gr}(N, r)\) for which the closure of the torus orbit \(T^n \cdot L_f\) is a projective toric variety \(X_f\) whose moment polytope is \(B_f\) (Theorem 6.7). If \((E, f)\) is a matroid, then this goes back to \([27]\).

We show how a realization determines a point in the flag variety \(\text{Fl}(N, r)\). With respect to a suitable action of \(T^n\), the torus-orbit closure \(\Sigma_1(f)\) yields a projective toric variety \(Y_f \subseteq \text{Fl}(N, r)\) with moment polytope normally equivalent to \(\Sigma_1(f)\). If \((E, f)\) is a matroid, then the moment polytope is precisely \(\Delta(\mathcal{F}_M)\). There is a 1-dimensional subtorus \(H \subseteq T^n\) for which \(X_f / H\) is isomorphic to \(Y_f\) as topological spaces. Kapranov, Sturmfels, and Zelevinsky \([32]\) showed that quotients of toric varieties by subtori are again toric varieties. The associated fan is called the **quotient fan** and toric varieties with this fan are called **combinatorial quotients**.

**Theorem 4** (Theorem 6.8). The toric variety \(Y_f\) is a combinatorial quotient for the action of \(H\) on \(X_f\) and the moment polytope of \(Y_f\) is normally equivalent to \(\Sigma_1(f)\). In particular, \(Y_f\) is a smooth toric variety for every realizable polymatroid.

This gives an algebro-geometric explanation for the relationship between matroid base polytopes and flag matroid polytopes.

**Algebraic combinatorics of monotone paths on polymatroids.** A simple variant of the greedy algorithm allows for optimization over \(P_f\) and gives rise to partial greedy paths. We show that the corresponding monotone path polytopes are again polymatroid base polytopes. As an application,
we completely resolve a conjecture of Heuer–Striker [30] on the face structure of partial permutation polytopes (Theorem 7.5).

Partial greedy paths on $P_f$ can be seen as paths on polymatroid base polytopes. In Section 8, we investigate the combinatorics of monotone paths on base polytopes $B_f$ with respect to the special linear functions $1_S(x) = \sum_{i \in S} x_i$ for $S \subseteq E$. A case that we study in some detail are $1_S$-monotone paths on the permutahedron $\Pi_{n-1}$. Let $\text{SYT}(m, n)$ be the set of standard Young tableaux of rectangular shape $m \times n$. Following Mallows and Vanderbei [36], we call a rectangular standard Young tableau realizable if it can be obtained from a tropical rank-1 matrix; see Section 8 for details. Let us denote by $\mathcal{S}_m$ the symmetric group on $m$ letters.

**Theorem 5.** Let $S \subseteq [n]$ with $k = |S|$. The $1_S$-monotone paths on the permutahedron $\Pi_{n-1}$ are in bijection with $\mathcal{S}_k \times \mathcal{S}_{n-k} \times \text{SYT}(k, n-k)$. A path is coherent if and only if it corresponds to a realizable standard Young tableau.

For $k = 2$, Mallows and Vanderbei showed that all $2 \times n$ rectangular standard Young tableaux are realizable. We give a short proof of this fact by relating realizable standard Young tableaux to regions of the Shi arrangement contained in the fundamental region.

**Organization of the paper.** In Section 2, we recall notation and results on polytopes, (poly)matroids, and monotone path polytopes. In Section 3, we show that all cellular strings of $P_f$ are coherent and that the monotone path polytope $\Sigma_1(P_f)$ is a polymatroid base polytope. We also determine the vertices and facets. To that end, we show that $f$ and $\widehat{f}$ have the same lattice of flats but all flats of $\widehat{f}$ are facet-defining. We illustrate the construction on Loday’s associahedron (Example 3.15).

In Section 4 we study max-slope pivot rule polytopes of $(P_f, 1)$ and show that they are normally equivalent to $\Sigma_1(P_f)$. We also show that they are nestohedra for certain union-closed building sets. In Section 5, we show that partial flag matroids arise as monotone path polytopes of rank-selected independence polytopes. In Section 6 we treat realizable polymatroids from the viewpoint of toric varieties in Grassmannians and flag-varieties. Section 7 extends our results to partial greedy paths and treats a conjecture of Heuer–Striker [30] on the face structure of partial permutation polytopes (Theorem 7.5).

A case that we study in some detail are $1_S$-monotone paths on the permutahedron $\Pi_{n-1}$. Let $\text{SYT}(m, n)$ be the set of standard Young tableaux of rectangular shape $m \times n$. Following Mallows and Vanderbei [36], we call a rectangular standard Young tableau realizable if it can be obtained from a tropical rank-1 matrix; see Section 8 for details. Let us denote by $\mathcal{S}_m$ the symmetric group on $m$ letters.

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## 2. Background

In this section, we briefly recall the necessary background on polytopes, (poly)matroids, and monotone path polytopes. For more background on polytopes, we refer to [29] and [48]. For a finite set $E$, the elements of $\mathbb{R}^E$ are vectors $(x_a)_{a \in E}$. We will sometimes abuse notation and identify $\mathbb{R}^E \cong \mathbb{R}^{|E|}$ with standard basis $(e_a)_{a \in E}$ and standard inner product $\langle x, y \rangle = \sum_{a \in E} x_a y_a$. For a $A \subseteq E$, we denote by $e_A = \sum_{a \in A} e_a$ the characteristic vector of $A$ and for any $x \in \mathbb{R}^E$, we write $x(A) = 1_A(x) = \sum_{a \in A} x_a$. We also abbreviate $1 = 1_E$.

A polytope $P \subset \mathbb{R}^d$ is the convex hull of finitely many points $P = \text{conv}\{v_1, \ldots, v_n\}$. For $w \in \mathbb{R}^d$, we write

$$P^w := \{x \in P : \langle w, x \rangle \geq \langle w, y \rangle \text{ for all } y \in P\}$$
for the face in direction \( w \), that is, the set of maximizers of the linear function \( x \mapsto \langle w, x \rangle \) over \( P \).

If \( P^w = \{v\} \), then \( v \) is a vertex of \( P \) and we write \( V(P) \) for the set of vertices. The face lattice \( \mathcal{L}(P) \) of \( P \) is the collection of faces of \( P \) partially ordered by inclusion. Two polytopes \( P, Q \) are combinatorially isomorphic if they have isomorphic face lattices.

The Minkowski sum of two polytopes \( P, Q \subset \mathbb{R}^d \) is the polytope
\[
P + Q = \{p + q : p \in P, q \in Q\} = \text{conv}\{u + v : u \in V(P), v \in V(Q)\}.
\]

A polytope \( Q \) is a **Minkowski summand** of \( P \) if there is a polytope \( R \) such that \( Q + R = P \). More generally, \( Q \) is a **weak Minkowski summand** of \( P \) if \( P \) is a Minkowski summand of \( \mu P \) for some \( \mu > 0 \).

We will use the following characterization of weak Minkowski summands.

**Proposition 2.1** ([29, Thm. 15.1.2]). Let \( P, Q \subset \mathbb{R}^d \) be polytopes. Then \( Q \) is a weak Minkowski summand of \( P \) if and only if for all \( w \in \mathbb{R}^d \) it holds that \( Q^w \) is a vertex whenever \( P^w \) is.

If \( P \) is also a weak Minkowski summand of \( Q \), then \( P \) and \( Q \) are **normally equivalent**. If \( P \) and \( Q \) are normally equivalent, then \( P \) and \( Q \) are combinatorially isomorphic and the isomorphism between face lattices is given by \( P^w \mapsto Q^w \). Note that any two full-dimensional axis-parallel boxes in \( \mathbb{R}^d \) are normally equivalent but in general not affinely isomorphic.

### 2.1. Matroids and Polymatroids

There is a vast literature on matroids and polymatroids and we refer the reader to [39] and [25] for more.

Let \( E \) be a finite set. A **polymatroid** [22] is a monotone and submodular function \( f : 2^E \to \mathbb{R}_{\geq 0} \). That is, \( f(\emptyset) = 0 \) and for all \( A, B \subseteq E \)
\[
f(A) \leq f(A \cup B) \leq f(A) + f(B) - f(A \cap B).
\]
The **polymatroid (independence) polytope** of \( f \) is
\[
P_f := \{x \in \mathbb{R}^E : x \geq 0, 1_A(x) \leq f(A) \text{ for all } A \subseteq E\}.
\]
The polytope \( P_f \) is of full dimension \( |E| \) if and only if \( f(\{e\}) > 0 \) for all \( e \in E \). Note that if \( y \in P_f \) and \( x \in \mathbb{R}^E \) satisfies \( 0 \leq x_e \leq y_e \) for all \( e \in E \), then \( x \in P_f \). Polytopes satisfying this condition are called **anti-blocking** polytopes [26].

Edmonds [22] originally defined polymatroids as those anti-blocking polytopes for which all points \( y \in P_f \) maximal with respect to the componentwise order have the same coordinate sum \( 1(y) \).

Theorem 14 in [22] shows the equivalence to our definition above. The **base polytope** \( B_f \) of \( f \) is the face \( P_f^1 \). Edmonds’ definition implies that \( P_f = \mathbb{R}_{\geq 0}^E \cap (-\mathbb{R}_{\geq 0}^E + B_f) \) and thus \( B_f \) completely determines the polymatroid. It follows from submodularity that
\[
B_f = \{x \in P_f : 1(x) = f(E)\}.
\]

Up to translation, base polytopes are characterized as precisely those polytopes \( B \subset \{x : 1(x) = c\} \) for some \( c \) and such that if \([u, v]\) is an edge of \( B \), then \( u - v = \mu(e_i - e_j) \) for some \( \mu \in \mathbb{R} \) and \( i, j \in E \); see [22]. In the context of geometric combinatorics, such polytopes were studied by Postnikov [41] under the name **generalized permutahedra**. The prototypical examples are permutahedra: A **permutahedron** is a polytope of the form
\[
\Pi(a_1, \ldots, a_d) = \text{conv}\{(a_\sigma(1), \ldots, a_\sigma(d)) : \sigma \text{ permutation of } [d]\}
\]for \( a_1, \ldots, a_d \in \mathbb{R}^d \); see also Example 3.13. The **standard** permutahedron is \( \Pi_{n-1} := \Pi(1, 2, \ldots, n) \).

The most well-known polymatroids are matroids. A **matroid** is a pair \( M = (E, \mathcal{I}) \), where \( E \) is a finite set and \( \mathcal{I} \subseteq 2^E \). The collection \( \mathcal{I} \) is a nonempty hereditary set system (or simplicial complex) that satisfies the augmentation property: if \( I, J \in \mathcal{I} \) such that \( |I| < |J| \), then there is \( e \in J \setminus I \) such that \( I \cup e \in \mathcal{I} \). The sets in \( \mathcal{I} \) are called **independent** and the inclusion-maximal sets are called
bases. The rank function of $M$ is $r_M : 2^E \to \mathbb{Z}_{\geq 0}$ given by $r_M(X) := \max\{|I| : I \in \mathcal{I}, I \subseteq X\}$. The rank function is a polymatroid with the additional property that $r_M(X) \leq |X|$ and this characterizes matroid rank functions among polymatroids.

For $A \subseteq E$, let $e_A \in \{0, 1\}^E$ be its characteristic vector. The independence polytope of a matroid $M$ is

$$P_M := P_{r_M} = \text{conv}\{e_I : I \in \mathcal{I}\}.$$ 

The base polytope of $M$ is then

$$B_M := P_M^{\mathcal{I}} = \text{conv}\{e_B : B \text{ basis of } M\}.$$ 

The uniform matroid on $n$ elements of rank $k$ is the matroid $U_{n,k} = ([n], \mathcal{I})$ for which a set $A \subseteq [n]$ is independent if and only if $|A| \leq k$. The corresponding base polytope is the $(n,k)$-hypersimplex

$$\Delta(n,k) := B_{U_{n,k}} = \text{conv}\{e_A : A \subseteq [n], |A| = k\}.$$ 

A set $F \subseteq E$ is closed or a flat with respect to $f$ if $f(F \cup e) > f(F)$ for all $e \in E \setminus F$. For $A \subseteq E$, the closure of $A$ is the flat $\overline{A} := \{e \in E : f(A \cup e) = f(A)\}$. Note that $\dim P_f = |E|$ if and only if $\varnothing$ is a flat. We call a flat proper if $F \neq \mathbb{Z}$ and $F \neq E$. The lattice of flats $\mathcal{L}(f)$ is the collection of flats of $f$, partially ordered by inclusion. A flat $F$ is separable if $F = F_1 \cup F_2$ for two disjoint, nonempty flats $F_1, F_2$ with $f(F) = f(F_1) + f(F_2)$.

**Theorem 2.2** ([22, Thm. 28]). Let $(E, f)$ be a polymatroid such that $\varnothing$ is closed. An irredundant inequality description of $P_f$ is given by

$$P_f = \{x \in \mathbb{R}^E : x \geq 0, 1_F(x) \leq f(F) \text{ for all proper and inseparable } F \in \mathcal{L}(f)\}.$$ 

An operation that will be used later is the truncation of a polymatroid: For $0 \leq \alpha \leq f(E)$, the truncation [25, Sect. 3.1(d)] of $f$ by $\alpha$ is the polymatroid $f_\alpha$ with

$$f_\alpha(A) = \min(\alpha, f(A)).$$ 

The base polytope of $f_\alpha$ is $B_{f_\alpha} = P_f \cap \{x : 1_F(x) = \alpha\}$.

A matroid $M$ is realizable over $\mathbb{C}$ if there are 1-dimensional linear subspaces $U_e \subset \mathbb{C}^n$ for $e \in E$ such that $r_M(X) = \dim \sum_{e \in X} U_e$. If $U_1, \ldots, U_n$ is any collection of linear subspaces, then $f(X) = \dim \sum_{e \in X} U_e$ defines an integral polymatroid, that we call a realizable polymatroid. In this case $P_f$ and hence $B_f$ is a lattice polytope.

### 2.2. Monotone path polytopes

Let $P \subset \mathbb{R}^d$ be a polytope and $c \in \mathbb{R}^d$ a linear function that is not constant on $P$. Let $P_{\min} = P^{\leq c}$ and $P_{\max} = P^{\geq c}$ be the faces on which $c$ is minimized and maximized, respectively. A cellular string of $(P, c)$ is a sequence of faces $F_s = (F_0, F_1, F_2, \ldots, F_r)$ of $P$ such that $c$ is not constant on $F_i$, $F_0^{\leq c} \subseteq P_{\min}$, $F_r^{\geq c} \subseteq P_{\max}$, and

$$F_i^{\leq c} = F_i \cap F_{i+1} = F_{i+1}^{\geq c}$$

for all $0 \leq i < r$. If $c$ is edge generic, that is, $(c, u) \neq (c, v)$ whenever $[u, v]$ is an edge of $P$, then the condition simplifies to $F_0^{\leq c} = P_{\min}$, $F_r^{\geq c} = P_{\max}$ and $F_i^{\leq c} = F_{i+1}^{\geq c}$. Cellular strings for generic $c$ were introduced and studied in [6]. A partial order on cellular strings is given by refinement, for which some $F_i$ are replaced by a cellular string of $F_i$. For general $c$, the collection of cellular strings is still partially ordered by refinement and we continue to call the partially ordered set the Baues poset $\text{Baues}(P, c)$. The minimal elements are the c-monotone paths. They correspond to sequences of vertices $v_s = (v_0, v_1, \ldots, v_k)$ such that $v_0 \in V(P_{\min})$, $v_k \in V(P_{\max})$, and $[v_i, v_{i+1}] \subset P$ is an edge with $(c, v_i) < (c, v_{i+1})$ for all $0 \leq i < k$. Figure 2 gives an illustration.

Let $w \in \mathbb{R}^d$. The projection $\pi : \mathbb{R}^d \to \mathbb{R}^2$ given by $x \mapsto ((c, x), (w, x))$ maps $P$ to a (degenerate) polygon $\pi(P)$. The projections $\pi(P_{\min}), \pi(P_{\max})$ are faces of $\pi(P)$. The set of points $(s, t) \in \pi(P)$ with $(s, t + \varepsilon) \notin \pi(P)$ for all $\varepsilon > 0$ is a vertex-edge path from the vertex $\pi(P_{\min}^w)$ to the vertex $\pi(P_{\max}^w)$.
The preimage of every edge of this path is a cellular string, called a coherent cellular string. If \( w \) is generic, then this is a \( c \)-monotone path \( v_s = (v_0, v_1, \ldots, v_k) \), called the shadow vertex path of \((P, c)\) with respect to \( w \). A \( c \)-monotone path \( v_s \) of \( P \) is called coherent if \( v_s \) is a shadow vertex path with respect to some \( w \). We refer to [37, Section 4] for an illustration of non-coherent monotone paths.

Let \( I := \{ \langle c, x \rangle : x \in P \} \subset \mathbb{R} \). A section of \((P, c)\) is a continuous map \( \gamma : I \to P \) such that \( \langle c, \gamma(s) \rangle = s \) for all \( s \in I \). The collection of sections is a convex body and Billera–Sturmfels [7] showed that the projection

\[
\Sigma_c(P) = \left\{ 2 \int_I \gamma ds : \gamma \text{ section} \right\} \subset \mathbb{R}^d
\]

is a convex polytope, called the monotone path polytope of \((P, c)\). Every \( c \)-monotone path \( v_s \) gives rise to a piecewise-linear section \( \gamma_{v_s} \) of \((P, c)\) and

\[
\Psi_{P,c}(v_s) := 2 \int_I \gamma_{v_s} dt = \sum_{j=1}^k \langle c, v_j - v_{j-1} \rangle (v_{j-1} + v_j). \tag{2}
\]

Billera–Sturmfels [7] showed that a \( c \)-monotone path \( v_s \) is coherent with respect to \( w \) if and only if \( \Sigma_c(P)^w = \Psi_{P,c}(v_s) \).

**Theorem 2.3** ([7]). The poset of coherent cellular strings is isomorphic to the face lattice of \( \Sigma_c(P) \).

We remark that the definition given in [7] is actually \( \frac{1}{\text{vol}_d(I)} \Sigma_c(P) \). This does not change the combinatorics and has the benefit that if \( c \in \mathbb{Z}^d \) and \( P \) is a lattice polytope, then \( \Sigma_c(P) \) as well.

For \( s \in I \) define \( P_s := \{ x \in P : \langle c, x \rangle = s \} \). The monotone path polytope \( \Sigma_c(P) \) is equivalently given by the Minkowski integral

\[
\Sigma_c(P) = 2 \int_I P_s ds. \tag{3}
\]

Let \( I' = \{ \langle c, v \rangle : v \in V(P) \} = \{ t_0 < t_1 < \cdots < t_m \} \). For \( 0 \leq i < m \) and \( t_i < s < t_{i+1} \), the polytope \( P_s \) is normally equivalent to \( P_{t_i} + P_{t_{i+1}} \). The additivity of the integral gives a simple construction for a polytope normally equivalent to the monotone path polytope.

**Proposition 2.4.** The monotone path polytope \( \Sigma_c(P) \) is normally equivalent to \( \sum_{s \in I'} P_s \).

We give a useful local criterion of when a monotone path is coherent. Let \( P \) be a polytope and \( c \in \mathbb{R}^d \). For a vertex \( v \in V(P) \), we write

\[
\text{Nb}_{P,c}(v) := \{ u \in V(P) : [u, v] \text{ edge of } P, \langle c, u \rangle > \langle c, v \rangle \}
\]

for the \( c \)-improving neighbors of \( v \).

**Lemma 2.5.** Let \( v_s = (v_0, v_1, \ldots, v_k) \) be a \( c \)-monotone path on \((P, c)\). Then \( v_s \) is coherent if and only if there is a weight \( w \in \mathbb{R}^d \) such that \( v_0 = (P_{\text{min}})^w \) and for every \( i = 1, \ldots, k \):

\[
\frac{\langle w, v_i - v_{i-1} \rangle}{\langle c, v_i - v_{i-1} \rangle} > \frac{\langle w, u - v_{i-1} \rangle}{\langle c, u - v_{i-1} \rangle} \quad \text{for all } u \in \text{Nb}_{P,c}(v_{i-1}) \setminus \{ v_i \}. \tag{4}
\]

**Proof.** Let \( w \in \mathbb{R}^d \) such that \( w \) is not constant on \( P \). The projection \( P' = \pi(P) = \{ (\langle c, x \rangle, \langle w, x \rangle) : x \in P \} \) is a convex polygon and the upper hull \( U \) of \( P' \) is the set of points \( p \in P' \) such that \( p + (0, \epsilon) \notin P' \) for all \( \epsilon > 0 \). The upper hull is a union of edges and the corresponding coherent cellular string consists of the preimages of the edges of \( U \) under \( \pi \). If the cellular string is a monotone path \( v_s = (v_0, v_1, \ldots, v_k) \) in \( P \), then \( \pi([v_{i-1}, v_i]) \subseteq U \) implies that the stated conditions are necessary.

Conversely, if \( u \in P \) is a vertex such that \( \pi(u) \) is a vertex in the upper hull, then its neighbor to the right, provided it exists, is given by \( \pi(v) \) with \( \langle c, v \rangle > \langle c, u \rangle \) and such that \( e' := [\pi(u), \pi(v)] \) has
maximal slope. Now, in order for $\pi^{-1}(e')$ to be an edge, $u$ and $v$ have to be unique. This means that $v_0$ is the unique maximizer of $w$ over $P_{\text{min}}$ and Equation (4) has to be satisfied for all $i = 1, \ldots, k$.

### 3. Monotone Paths on Polymatroid Polytopes

Let $(E, f)$ be a fixed polymatroid and let $1(x) := \sum_{i \in E} x_i$. The first goal of this section is to show that all 1-cellular strings on $P_f$ are coherent. We write $\Sigma_1(f) := \Sigma_1(P_f)$ for the monotone path polytope of $P_f$ with respect to 1.

**Theorem 3.1.** Let $(E, f)$ be a polymatroid. Every 1-cellular string on $P_f$ is coherent. In particular, the Baues poset $\text{Baues}(P_f, \mathbf{1})$ is isomorphic to the face lattice of $\Sigma_1(P_f)$.

We will identify $E = \{1, \ldots, n\}$. Edmonds [21] showed that the following geometric version of the greedy algorithm can be used on polymatroids.

**Theorem 3.2** (Greedy Algorithm). Let $(E, f)$ be a polymatroid and $w \in \mathbb{R}^E$. Let $\sigma$ be a permutation such that $w_{\sigma(1)} \leq w_{\sigma(2)} \leq \cdots \leq w_{\sigma(n)}$. For $i = 0, \ldots, n$ define $A_i := \{\sigma(1), \ldots, \sigma(i)\}$ and $x \in \mathbb{R}^E$ by

$$x_{\sigma(i)} := f(A_i) - f(A_{i-1})$$

for $i = 1, \ldots, n$. Then $x$ maximizes $w$ over the base polytope $B_f$. If the greedy algorithm is stopped at $w_{\sigma(i)} < 0$ and $x_{\sigma(j)} := 0$ for $j \geq i$, then $x$ maximizes $w$ over $P_f$.

In particular every vertex of $P_f$ and $B_f$ can be found using the greedy algorithm. For a vertex $v \in V(P_f)$, the support $I(v) := \{i \in E : v_i > 0\}$ is called the basis of $v$. This is rarely a closed set. For example, if $v \in B_f$, then $\overline{I(v)} = E \setminus \mathcal{B}$.

Let $v$ be the vertex of $B_f$ obtained from the greedy algorithm with respect to $w$. We can assume that $w$ is generic. Let $I(v) = \{j_1, j_2, \ldots, j_k\}$ so that $w_{j_1} > w_{j_2} > \cdots > w_{j_k}$. Define $F_0 \subset F_1 \subset \cdots \subset F_k$ by $F_i := \{j_1, \ldots, j_i\}$ for $i = 0, \ldots, k$ and define $0 = v_0, v_1, \ldots, v_k = v$ by setting

$$(v_i)_j = \begin{cases} f(F_{jh}) - f(F_{jh-1}) & \text{if } j = j_h, h \leq i \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.2 implies that $v_0, \ldots, v_k$ are distinct vertices of $P_f$ such that $\overline{I(v_i)} = F_i, \sum_j (v_i)_j = f(F_i)$, and $[v_i, v_{i+1}]$ is a 1-increasing edge of $P_f$. Note that the 1-monotone path is completely determined by the ordered sequence $j_* = (j_1, j_2, \ldots, j_k)$. We call $v_0, \ldots, v_k$ or, equivalently, $j_*$ a greedy path of $P_f$.

**Proposition 3.3.** Let $(E, f)$ be a polymatroid and $w \in \mathbb{R}^E$ generic. The greedy path associated to $w$ is a coherent 1-monotone path.

**Proof.** Let $v' \in V(P_f)$ be a vertex. If $u$ is a neighbor of $v$ with $1(u) > 1(v)$, then $u - v' = \delta e_i$ for some $i = 1, \ldots, n$ and $\delta > 0$. Hence,

$$\frac{\langle w, u - v' \rangle}{1(u - v')} = \frac{\delta w_i}{\delta} = w_i$$

and (4) implies that the coherent monotone path $0 = v_0, v_1, \ldots, v_k = v$ of $P_f$ is precisely the path obtained from the greedy algorithm.

**Proposition 3.4.** Every 1-monotone path on $P_f$ is a greedy path.

The paper was again published in the Edmonds Festschrift [22] and throughout we will reference the results there.
Proof. Let \( 0 = v_0, v_1, \ldots, v_s \) be a 1-monotone path on \( P_f \). Then \( v_i - v_{i-1} = \delta_i e_j \) for \( i = 1, \ldots, s \). Choose a weight \( w \) with \( w_{j_1} > w_{j_2} > \cdots > w_{j_s} > w_h \) for \( h \notin \{j_1, \ldots, j_s\} \). Since \( v_i \) is a vertex of the truncation \( P_{f, \alpha} \) for \( \alpha = 1(v_i) \), the greedy path with respect to \( w \) will be precisely the given monotone path. \( \square 

Proof of Theorem 3.1. \) Let \( 0 = F_0, F_1, F_2, \ldots, F_k \) be a 1-cellular string on \( P_f \). For \( h = 1, \ldots, k \) define
\[
I_h := \{ i \in E : p + \delta e_i \in F_i \text{ for some } p \in F_{h-1} \cap F_h \text{ and } \delta > 0 \}.
\]

We claim that the cellular string is completely determined by \( I_1, \ldots, I_k \). Indeed, let \( L_h = \text{span}\{e_i : i \in I_h\} \). Then \( F_1 = P_f \cap L_1 \). If, by induction, \( F_h \) is determined, then we can employ the greedy algorithm to find a point \( p \) in \( F_h^1 = F_h \cap F_{h+1} \) and \( F_{h+1} = P_f \cap (p + L_{h+1}) \). Again the greedy algorithm shows that \( F_0, F_1, \ldots, F_k \) is precisely the coherent cellular string for \( w = ke + (k-1)e_2 + \cdots + e_k - e_{E \setminus k} \). \( \square 

Remark 3.5. \) Note that the linear function \( c = 1 \) is essential for the validity of Theorem 3.1. Consider, for example the uniform matroid \( U_{4,2} \) with rank function \( f(A) = \min(|A|, 2) \) for \( A \subseteq [4] \). The polymatroid polytope \( P_f \) is the convex hull of all \( v \in \{0, 1\}^4 \) with at most two entries equal to 1. The linear function \( c = (-10, -5, 7, 8) \) is generic on the polymatroid (independence) polytope \( P_f \) with minimum \( v_{\min} = (1, 1, 0, 0) \) and maximum \( v_{\max} = (0, 0, 1, 1) \). It can be checked that, for example using Lemma 2.5, that the \( c \)-monotone path \((1, 1, 0, 0), (1, 0, 1, 0), (0, 1, 0, 1), (0, 0, 1, 1)\) is not coherent. Note that the monotone path is contained in the base polytope \( B_f \). In Section 8 we focus on monotone paths in base polytopes.

We give a complete combinatorial description of \( \Sigma_1(f) \) in Section 4. Here, we only describe the vertices and facet-defining inequalities. The greedy algorithm readily gives a purely combinatorial description of the vertices of \( \Sigma_1(f) \).

**Corollary 3.6.** Let \((E, f)\) be a polymatroid. The vertices of \( \Sigma_1(f) \) are in correspondence with sequences \( j_* = (j_1, j_2, \ldots, j_k) \) of distinct elements of \( E \) such that
\[
\emptyset \subset \{j_1\} \subset \{j_1, j_2\} \subset \cdots \subset \{j_1, j_2, \ldots, j_k\} = E
\]
is a maximal chain of flats in \( \mathcal{L}(f) \).

Corollary 3.6 also prompts an organizing principle to group vertices which produce the same maximal chain of flats. For a sequence \( j_* = (j_1, j_2, \ldots, j_k) \) define \( F_i(j_*) : = \{j_1, j_2, \ldots, j_i\} \) for \( i = 0, 1, \ldots, k \). Using (2), a direct computation yields the vertices of \( \Sigma_1(f) \).

**Corollary 3.7.** Let \( j_* = (j_1, \ldots, j_k) \) be a 1-monotone path of \( P_f \) and let \( F_i = F_i(j_*) \) for \( i = 0, \ldots, k \). The vertex \( \Psi(j_*) \) of \( \Sigma_1(f) \) corresponding to the greedy path \( j_* \) satisfies \( \Psi(j_*) = 0 \) if \( r \notin \{j_1, \ldots, j_k\} \) and
\[
\Psi(j_* j_i) = (f(F_i) - f(F_{i-1}))(f(E) - f(F_i) + f(E) - f(F_{i-1})).
\]

Our next goal is to show that monotone path polytopes of polymatroid polytopes are polymatroid base polytopes. Figure 2 gives a first illustration.

**Theorem 3.8.** Let \((E, f)\) be a polymatroid. Then \( \Sigma_1(f) \) is a polymatroid base polytope for the polymatroid
\[
\hat{f} := 2f(E) \cdot f - f^2.
\]

Theorem 3.8 is a first justification of calling \( \hat{f} \) a flag polymatroid associated to \( f \). Note that the transformation \( f \mapsto \hat{f} \) is homogeneous of degree 1, that is, \( \hat{\alpha f} = \alpha \hat{f} \) for all \( \alpha > 0 \).
Proof. By the characterization (3)

\[ \Sigma_1(f) = 2 \int_0^{f(E)} P_f \cap \{ x : 1(x) = t \} \, dt = \int_0^{f(E)} 2B_{f_t} \, dt, \]

where \( B_{f_t} \) is the base polytope of the truncation \( f_t(A) = \min(f(A), t) \); see Section 2.1. Since polymatroid base polytopes are closed under Minkowski sums, it follows that \( \Sigma_1(P_f) = B_g \) for some submodular function \( g \). In order to determine \( g \), we compute for \( S \subseteq E \)

\[ g(S) = \int_0^{f(S)} 2f_t(S) \, dt = \int_0^{f(S)} 2t \, dt + \int_0^{f(E)} 2f(S) \, dt = f(S)^2 + 2f(S)f(E) - 2f(S)^2, \]

which finishes the proof. \( \square \)

Via (1), Theorem 3.8 gives an inequality description. We next determine the inseparable flats.

Proposition 3.9. Let \((E, f)\) be a polymatroid. For \( A, B \subseteq E \) we have

\( \hat{f}(A) = \hat{f}(B) \) if and only if \( f(A) = f(B) \).

In particular, \( f \) and \( \hat{f} \) have the same lattices of flats.

Proof. Consider the function \( g(t) := 2t - t^2 \), which is an injective function on \([0, 1]\). We may assume that \( f(E) = 1 \) so that \( \hat{f} = g(f) \) and the result follows. \( \square \)

Proposition 3.10. Let \((E, f)\) be a polymatroid and \( \hat{f} \) its flag polymatroid. Every flat \( A \) of \( \hat{f} \) is inseparable.

Proof. We may assume that \( f(E) = 1 \). Let \( A \) be a fixed flat with \( a = f(A) \leq 1 \) and assume that \( A \) is separable with respect to \( \hat{f} \). That is, there are disjoint flats \( A_1, A_2 \subseteq A \) such that \( \hat{f}(A) = \hat{f}(A_1) + \hat{f}(A_2) \). Then \( (a_1, a_2) = (f(A_1), f(A_2)) \) satisfies

\[ 2a_1 - a_1^2 + 2a_2 - a_2^2 = 2a - a^2 \iff (1-a_1)^2 + (1-a_2)^2 = (1-a)^2 + 1. \]

Monotonicity and submodularity yield \( 0 \leq a_1, a_2 \leq a \) and \( a \leq a_1 + a_2 \). Reparametrizing \((a_1, a_2, a) = (1 - b_1, 1 - b_2, 1 - b)\), we are thus looking at pairs \((b_1, b_2)\) such that

\[ b \leq b_1, b_2 \leq 1 \quad \text{and} \quad b_1 + b_2 \leq 1 + b \quad \text{and} \quad b_1^2 + b_2^2 = 1 + b^2. \]
The linear inequalities describe a triangle in the plane contained in the disc with radius $\sqrt{1+b^2}$ and meeting the bounding circle in the points $(1, b)$ and $(b, 1)$. This, however, implies that $\tilde{f}(A_1) = \tilde{f}(A)$ or $\tilde{f}(A_2) = \tilde{f}(A)$ and hence $A = A_1$ or $A = A_2$. This shows that $A$ is inseparable.

Theorem 3.8 together with the last two propositions give an irredundant inequality description:

$$\Sigma_1(f) = \left\{ x \in \mathbb{R}^E : x \geq 0, 1(x) = f(E)^2, 1_F(x) \leq 2f(E)f(F) - f(F)^2 \text{ for all proper } F \in \mathcal{L}(f) \right\}.$$ 

Corollary 3.11. Let $F \in \mathcal{L}(f)$ be a flat. The vertices of the facet $\Sigma_1(f)^{e_F}$ are precisely the greedy paths that pass through the flat $F$.

Moreover, let $\varnothing = F_0 \subset \cdots \subset F_k = E$ a maximal chain of flats in $\mathcal{L}(f)$. The collection of vertices $\varnothing$, with $F_i(j_x) = F_i$ for $i = 0, \ldots, k$ form a face of $\Sigma_1(f)$ combinatorially isomorphic to a product of simplices of dimensions $|F_i \setminus F_{i-1}| - 1$ for $i = 1, \ldots, k$.

Example 3.12 (Matroids). Let $M$ be a rank-$r$ matroid on ground set $E$. It follows from Corollary 3.6 that a sequence $j_x = (j_1, \ldots, j_k)$ is a greedy path if and only if $k = r$ and $\{j_1, \ldots, j_k\}$ is independent in $M$ for $i = 0, \ldots, r$. In particular, $(j_1, \ldots, j_r)$ is an ordered basis of $M$. Using Corollary 3.7 together with the fact that $r_M((j_1, \ldots, j_l)) = i$, we find that the vertex of $\Sigma_1(P_M)$ corresponding to the greedy path is

$$(2r - 1)e_{j_1} + (2r - 3)e_{j_2} + \cdots + 3e_{j_{r-1}} + e_{j_r}$$

If $B \subseteq E$ is a basis of $M$, then the face $\Sigma_1(M)^{e_B}$ is linearly isomorphic to $-1 + 2\Pi_{r-1}$. We will come back to this example in the next sections.

Example 3.13 (Cubes and permutahedra). For $E = [n]$, let $f : 2^E \to \mathbb{Z}_{\geq 0}$ be the polymatroid given by $f(A) = |A|$. This is the rank function of the uniform matroid $\mathcal{U}_{n,n}$ and $P_f = \{0,1\}^n$ with $B_f = \{(1, \ldots, 1)\}$. The greedy paths are given by all permutations $(\sigma(1), \ldots, \sigma(n))$. The flag polymatroid is $\tilde{f}(A) = n^2 - (n - |A|)^2$ and using Corollary 3.7, we see that the vertices of $\Sigma_1(f)$ are the permutations of $(1, 3, \ldots, 2n - 1)$. Hence $\Sigma_1(f) = -1 + 2\Pi_{n-1}$.

Notice that if $(E, f)$ is a polymatroid with $\mathcal{L}(f) = 2^E$, then $\Sigma_1(f)$ has $2^{|E|} - 2$ facet-defining inequalities and hence is normally equivalent to the permutahedron.

Proposition 3.14. If $f(E) - f(E \setminus i) > 0$ for all $i \in E$, then $\Sigma_1(f)$ is normally equivalent to the permutahedron.

Proof. Assume that $B_f \subset \mathbb{R}^E_{\geq 0}$. Any two subsets $A \subset A'$ with $|A'| = |A| + 1$ occur in some execution of the greedy algorithm (Theorem 3.2) and lead to a vertex $v \in B_f$. It thus follows that $f(A) < f(A')$ and $\mathcal{L}(f) = 2^E$. Now $B_f \subset \mathbb{R}^E_0$ if and only if the maximum of the linear function $x \mapsto -x_i$ is positive over $B_f$ for all $i$. The greedy algorithm shows that this is the case if and only if $f(E) - f(E \setminus i) > 0$ for all $i \in E$.

We call a polymatroid $f$ 	extit{tight} if $f(E \setminus i) = f(E)$ for all $i \in E$.

Example 3.15 (Associahedra). Let $n \geq 1$. For $1 \leq i \leq j \leq n$, we write $\Delta_{[i,j]} = \text{conv}\{e_i, e_{i+1}, \ldots, e_j\}$. The 	extbf{Loday Associahedron} [34] is the polymatroid base polytope

$$\text{Ass}_{n-1} := \sum_{1 \leq i \leq j \leq n} \Delta_{[i,j]}.$$ 

More precisely, $\text{Ass}_{n-1}$ is a nestohedron; see next section and [41, Sect. 8.2]. The underlying polymatroid $(\{n\}, f_{\text{Ass}})$ is given by

$$f_{\text{Ass}}(A) := |\{1 \leq i \leq j \leq n : \{i, \ldots, j\} \cap A \neq \varnothing\}|$$
for $A \subseteq [n]$. The vertices of $\text{Ass}_{n-1}$ are in bijection with plane binary trees. For a generic weight $w \in \mathbb{R}^n$, let $i \in [n]$ with $w_i$ maximal. The vertex $v$ of $\text{Ass}_{n-1}$ maximizing $w$ corresponds to the plane binary tree $T$ with root $i$ and left and right subtree recursively determined by $(w_1, \ldots, w_{i-1})$ and $(w_{i+1}, \ldots, w_n)$, respectively. Let $\sigma$ be the unique permutation such that $w_{\sigma^{-1}(n)} > w_{\sigma^{-1}(n-1)} > \cdots > w_{\sigma^{-1}(1)}$. Then, viewed as a linear function $\sigma \in \mathbb{R}^n$, $\sigma$ determines the same binary tree. The permutation determines how $T$ is built up. Hence every permutation represents a different greedy path and hence $\Sigma_1(f_{\text{Ass}})$ is normally equivalent to a permutahedron.

To see this differently, let $T$ be a plane binary tree and let $L_j$ and $R_j$ be the number of nodes in the left, respectively, right subtree of $T$ rooted at $j$. The vertex $v$ of $\text{Ass}_{n-1}$ corresponding to $T$ has coordinates $v_j = (L_j + 1)(R_j + 1)$ [41, Cor. 8.2]. In particular, $f_{\text{Ass}}$ is not tight.

The number of greedy paths that lead to a fixed tree $T$ can be computed as follows. View $T$ as a poset where the minimal elements are precisely the leaves of $T$. A linear extension is a permutation $\sigma$ with $\sigma(i) < \sigma(j)$ whenever $j$ is on the path from $i$ to the root. The greedy paths leading to $T$ are precisely the linear extensions of $T$. The number of linear extensions can be computed by the tree hook-length formula [33, Exercise 5.1.4.(20)] [44, Prop. 22.1]

$$e(T) = n! \prod_{i=1}^{n} \frac{1}{(L_i + R_i + 1)}.$$ 

Consider the polytope

$$\overline{\text{Ass}}_{n-1} := \sum_{1 \leq i < j \leq n} \Delta_{i,j} = -1 + \text{Ass}_{n-1}.$$ 

This is a tight version of the associahedron with $f_{\text{Ass}}(A) = f_{\text{Ass}}(A) - |A|$. The polytopes $\text{Ass}_{n-1}$ and $\overline{\text{Ass}}_{n-1}$ differ only by a translation but their polymatroid polytopes and their flag polymatroids are different. For a binary tree $T$ let $T'$ be the tree obtained from $T$ by removing all leaves. Two permutations $\sigma^1$ and $\sigma^2$ yield the same greedy path on $P_{f_{\text{Ass}}}$ if and only if both are linear extensions of $T$ and they yield the same linear extension of $T'$ after relabelling. The number of vertices of $\Sigma_1(f_{\text{Ass}})$ is then

$$\sum_T e(T'),$$

where the sum is over all plane binary trees on $n$ nodes. The first few numbers starting with $n = 2$ are 2, 5, 14, 46, 176, 766, 3704, 19600, 112496.

Let us close this section with the observation that the flag polymatroid defines a nonlinear transformation on the space of polymatroids. For instance, let $\mathcal{P}^1_n$ be the compact convex set of polymatroids $f : 2^{[n]} \to \mathbb{R}_{\geq 0}$ with $f([n]) = 1$. Then $f \mapsto \hat{f} = 2f - f^2$ defines a discrete dynamical system on $\mathcal{P}^1_n$.

**Proposition 3.16.** Let $f \in \mathcal{P}^1_n$ be a polymatroid for which $\emptyset$ is closed. Define $f^0 := f$ and $f^{i+1} := 2\hat{f}^i$. The sequence $(f^i)_{i \geq 0}$ converges to the function $f^\infty \in \mathcal{P}^1_n$ with $f^\infty(A) = 1$ for all $A$.

**Proof.** It follows from Theorem 3.8 that $f^{i+1} \in \mathcal{P}^1_n$. Since $g(t) = 2t - t^2$ is strictly increasing on the interval $(0,1)$, we have $f^n(A) \leq f^{n+1}(A) = g(f^n(A))$ for all $A \subseteq [n]$ and with strict inequality unless $f(A) = 1$. Now if $\emptyset$ is closed, this implies $f(A) > 0$ for all $A \neq \emptyset$.

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4. Pivot Polytopes and Nestohedra

In the context of linear optimization, the authors, De Loera, and Lütjeharms introduced **pivot polytopes** in [11]. Let $P \subseteq \mathbb{R}^n$ be a fixed polytope with vertex set $V(P)$. Recall that $\text{Nb}_{P,c}(v)$ is the collection of $c$-improving neighbors of $v \in V(P)$. For a fixed linear function $c$, a memory-less pivot rule for the pair $(P,c)$ is a map $\mathcal{A} : V(P) \to V(P)$ such that $\mathcal{A}(v) = v$ for all vertices $v$ maximizing
Corollary 4.4. From the perspective of optimization, Theorem 4.3 implies the following.

Theorem 4.3. We now show that the converse relation also holds for the max-slope pivot rule polytope determined by

\[ \mathcal{A}_P(c)(v) = \arg\max \left\{ \langle w, u - v \rangle : u \in \text{Nb}_P(c)(v) \right\}. \]

(5)

For an arborescence \( \mathcal{A} \), define

\[ \psi(\mathcal{A}) := \sum_v \mathcal{A}(v) - v \langle c, \mathcal{A}(v) - v \rangle, \]

(6)

where we tacitly assume that \( \frac{0}{\langle c, 0 \rangle} = 0 \). The max-slope pivot rule polytope is the polytope

\[ \Pi_P(c) := \text{conv}\{\psi(\mathcal{A}) : \mathcal{A} \text{ arborescence of } (P, c)\}. \]

(7)

**Theorem 4.1** ([11, Theorem 1.4]). The vertices of \( \Pi_P(c) \) are in one-to-one correspondence to the max-slope arborescences of \((P, c)\).

We can canonically decompose \( \Pi_P(c) \) into a Minkowski sum

\[ \Pi_P(c) = \sum_{v \in V(P)} \Pi_{P,c}(v), \]

(8)

where

\[ \Pi_{P,c}(v) := \text{conv}\left\{ \frac{u - v}{\langle c, u - v \rangle} : u \in \text{Nb}_P(c)(v) \right\}. \]

(9)

The max-slope pivot rule polytope is intimately related to the monotone path polytope \( \Sigma_c(P) \). For a generic \( w \), let \( v_0 = (P-c)^w \) and define \( v_i := \mathcal{A}_P(c)(v_{i-1}) \) for \( i \geq 1 \). If \( k \) is minimal with \( v_k = v_{k+1} \), then \( v_0, v_1 \ldots, v_k \) is the coherent monotone path of \((P, c)\) with respect to \( w \). From this, we deduced the following geometric implication.

**Proposition 4.2** ([11, Theorem 1.6]). The monotone path polytope \( \Sigma_c(P) \) is a weak Minkowski summand of the max-slope pivot rule polytope \( \Pi_P(c) \).

We now show that the converse relation also holds for \((P_f, 1)\).

**Theorem 4.3.** Let \((E, f)\) be a polymatroid. Then \( \Sigma_1(P_f) \) is normally equivalent to \( \Pi_{P_f, 1} \).

From the perspective of optimization, Theorem 4.3 implies the following.

**Corollary 4.4.** The greedy algorithm on \( P_f \) corresponds to linear optimization on \((P_f, 1)\) with respect to the max-slope pivot rule.

We start by making an observation about the behavior of the greedy algorithm. Any generic \( w \in \mathbb{R}^E \) induces a total order \( \preceq \) on \( E \) by setting \( i < j \) if \( w_i > w_j \). The greedy algorithm on \( P_f \) with respect to \( w \) produces a vertex \( u \in V(B_f) \). We call \( I(u) = \{b_1 < b_2 < \ldots < b_k\} \) the optimal basis of \( f \) with respect to \( w \).

**Lemma 4.5.** Let \((E, f)\) be a polymatroid with total order \( \preceq \) and optimal basis \( B \). Let \( v \) be a vertex of \( P_f \) and \( j \in E \) \( \prec \)-minimal with the property that \( v + \lambda e_j \in P_f \) for some \( \lambda > 0 \). Then \( j \in B \).
where the sum is over all saturated chains in simplex $\Delta$. For any $j$, Theorem 4.3 also implies that $\rho_j$ is minimal with $j \not\in I$ for $\rho_j$ not contained in $P_f$.

**Proof.** Assume that $j \not\in B$. Let $B^+ := \{b \in B : b < j\}$. Since $j$ is not added to $B$ by the greedy algorithm, we have $B^+ \not= \emptyset$ and $f(B^+ \cup j) = f(B^+)$. Let $I = I(v) = \{i \in E : v_i > 0\}$ be the basis of $v$. Since $j$ is the next direction chosen at $v$, $f(I \cup b) = f(I)$ for each $b \in B^+$. Monotonicity and submodularity implies $f(I \cup B^+) = f(I)$. Again by monotonicity and submodularity,

$$f(I) \leq f(I \cup j) \leq f(I \cup B^+ \cup j) \leq f(I \cup B^+) + f(B^+ \cup j) - f(B^+) = f(I \cup B^+) = f(I)$$

which contradicts the fact that $v + \lambda e_j \in P_f$ for $\lambda > 0$. \qed

**Proof of Theorem 4.3.** We may assume that $P_f$ is full-dimensional. We need to show for every weight $w$ that $\Pi_{f,1}^w$ is a vertex whenever $\Sigma_1(f)^w$ is a vertex. To that end, let $\Sigma_1(f)^w$ be a vertex corresponding to a coherent monotone path of $P_f$ with respect to $w$. The path is encoded by the optimal basis $B = (j_1 < j_2 < \cdots < j_k)$ of $P_f$ with respect to $w$. We need to show that $B$ completely determines the max-slope arborescence $A_{P_f,1}^w$. Let $v$ be a vertex of $P_f$ not contained in $P_f^1 = B_f$ and let $I = I(v)$. It follows from the structure of polymatroid polytopes and (5) that

$$A_{P_f,1}^w(v) = \arg\max\left\{\frac{\langle w, u - v \rangle}{1(u - v)} : u \in \operatorname{Nb}_{P_f,1}(v)\right\} = v + \langle f(I) - f(I)\rangle e_j,$$

where $j$ is minimal with $j \not\in I$. Now Lemma 4.5 implies that $j = j_i$, where $i$ is minimal with $j_i \not\in I$. This shows the claim. \qed

Geometrically, the lemma states that if we start the geometric greedy algorithm at a vertex $v$, then the set of directions taken is a subset of the directions taken from the vertex 0 along the greedy path. Figure 3 shows this for the polymatroid polytope of the permutahedron $\Pi_2$.

**Figure 3.** The figures left and right show two max-slope arborescences of the polymatroid polytope of the 2-dimensional permutahedron. The red paths are the greedy paths. The arborescences are adjacent on the pivot rule polytope. The middle figure shows the multi-arborescence corresponding to the edge. The cellular string is shown in red.

Thus, describing the monotone path polytope is equivalent to describing the max-slope pivot polytope. Theorem 4.3 also implies that $\Pi_{P_f,1}$ is a generalized permutahedron. In fact, we can give a nice presentation as a Minkowski sum of standard simplices. For any $S \subseteq E$, we define the standard simplex $\Delta_S = \operatorname{conv}(e_s : s \in S)$. For a flat $F \in \mathcal{L}(f)$, let us define

$$\rho(F) := \sum_{\emptyset = F_0 \subset \cdots \subset F_i \subset F} \prod_{i=1}^{l-1} |F_i \setminus F_{i-1}|,$$

where the sum is over all saturated chains in $\mathcal{L}(f)$ ending in $F$. 

Proposition 4.6. Let \((E, f)\) be a polymatroid with lattice of flats \(\mathcal{L}(f)\). Then
\[
\Pi_{P_j, 1} = \sum_{F \in \mathcal{L}(f) \setminus \{E\}} \rho(F) \Delta_{E \setminus F}.
\]

Proof. For a vertex \(v \in P_j\) not contained in \(B_f\), we infer from (9) that
\[
\Pi_{P_j, 1}(v) := \text{conv} \left\{ (f(I(v) \cup j) - f(I(v))) e_j : j \notin I(v) \right\}.
\]
Now \(f(I(v) \cup j) - f(I(v)) > 0\) if and only if \(j \notin \overline{I(v)}\). In particular \(\Pi_{P_j, 1}(v)\) only depends on the flat \(F = \overline{I(v)}\). For every vertex \(v\) with \(F = \overline{I(v)}\) there is a unique chain of flats \(\emptyset = F_0 \subset \cdots \subset F_l \subset F\) and \(i_s \in F_s \setminus F_{s-1}\). Thus, the number of vertices with \(\overline{I(v)} = F\) is precisely \(\rho(F)\). The representation then follows from (8).

A nonempty collection \(B \subseteq 2^E\) is a building set \([41]\) if for all \(S, T \in B\)
\[
S \cap T \neq \emptyset \implies S \cup T \in B.
\]
Let \(y_S \in \mathbb{R}_{>0}\) for all \(S \in B\). The generalized permutahedron
\[
\Delta(B) := \sum_{S \in B} y_S \Delta_S
\]
is called a nestohedron. Building sets and nestohedra were introduced by Postnikov \([41]\) and independently by Feichtner–Sturmfels \([23]\). In \([41]\), the definition of building sets requires \(\{i\} \in B\) for every \(i \in E\). This only adds a translation by \(e_E = (1, \ldots, 1)\) but is quite handy for the combinatorial description of \(\Delta(B)\). We leave it out for the following reason.

Proposition 4.7. Let \(\emptyset \neq U \subseteq 2^E\) be a union-closed family of sets, that is, \(S \cup T \in U\) for all \(S, T \in U\). Then \(U\) is a building set.

Edmonds \([22, \text{Thm. 27}]\) showed that \(\mathcal{L}(f) \subseteq 2^E\) is closed under intersections. We define for a polymatroid \((E, f)\)
\[
U(f) := \{ E \setminus F : F \in \mathcal{L}(f) \}.
\]
Let \(B \subseteq 2^E\) be a building set. A nested set is a subset \(N \subseteq \overline{B} := B \cup \binom{E}{1}\) such that
\begin{enumerate}[(N1)]
\item For any \(S, T \in N\), we have \(S \subseteq T\), \(T \subseteq S\), or \(S \cap T = \emptyset\);
\item For any \(S_1, \ldots, S_k \in N\) with \(k \geq 2\) if \(S_1 \cup \cdots \cup S_k \in \overline{B}\), then \(S_i \cap S_j \neq \emptyset\) for some \(i < j\);
\item If \(S \in \overline{B}\) is inclusion-maximal, then \(S \in N\).
\end{enumerate}
The collection \(\mathcal{N}(B)\) of nested sets of \(B\) is called the nested set complex.

Proposition 4.8 \(([41, \text{Thm. 7.4}])\). Let \(B \subseteq 2^E\) be a building set. Then the face lattice of \(\Delta(B)\) is anti-isomorphic to the nested set complex \(\mathcal{N}(B)\). In particular \(\Delta(B)\) is a simple polytope.

Proof of Theorem 3. Theorem 4.3 shows that \(\Sigma_1(P_f)\) is normally equivalent to \(\Pi_{P_f, 1}\). It now follows from Proposition 4.6 that \(\Pi_{P_f, 1}\) equals \(\Delta(U(f))\) for \(y_{E \setminus F} = \rho(F)\) for all \(F \in \mathcal{L}(f)\). Since nestohedra are simple, this holds true for \(\Pi_{P_f, 1}\) as well as for \(\Sigma_1(P_f)\).

Corollary 4.9. For every polymatroid \((E, f)\), the monotone path polytope \(\Sigma_1(f)\) as well as the max-slope pivot polytope \(\Pi_{P_j, 1}\) are simple polytopes.

The facial structure of a nestohedron is determined by the maximal nested sets of \(B\). Postnikov \([41]\) gave a nice description in terms of certain rooted forests. We encode a rooted forest \(\mathcal{F}\) on \(E\) by the map \(\text{desc}_{\mathcal{F}} : E \to 2^E\) such that \(\text{desc}_{\mathcal{F}}(i)\) is the collection of nodes (including \(i\)) in the subtree rooted at
i. That is, \( \text{desc}_F(i) \) are the descendants of \( i \). Two nodes \( i, j \) are comparable if \( \text{desc}_F(i) \subseteq \text{desc}_F(j) \) or \( \text{desc}_F(j) \subseteq \text{desc}_F(i) \).

For a building set \( \mathcal{B} \), a B-forest is a rooted forest \( F \) on \( E \) such that

(F1) \( \text{desc}_F(i) \in \mathcal{B} \) for all \( i \in E \);

(F2) If \( s_1, \ldots, s_k \in E \) for \( k \geq 2 \) satisfy \( \bigcup_j \text{desc}_F(s_j) \in \mathcal{B} \), then \( s_i, s_j \) are comparable for some \( i < j \);

(F3) For every inclusion-maximal \( S \in \mathcal{B} \) there is \( i \in E \) with \( \text{desc}_F(i) = S \).

The maximal nested set corresponding to a B-tree is \( \{ \text{desc}_F(i) : i \in E \} \).

**Proposition 4.10.** Let \( \mathcal{U} \subseteq 2^E \) be a union-closed family such that \( \bigcup \mathcal{U} = E \). The B-forests are in bijection to a collection \( (t_i, S_i) \) for \( i = 1, \ldots, k \) such that \( S_1 \subset S_2 \subset \cdots \subset S_k = E \) is a chain in \( \mathcal{U} \), \( t_i \in S_i \setminus S_{i-1} \) with \( S_0 := \emptyset \), and for any \( i \geq 0 \) and nonempty \( R \in E \setminus (S_i \cup \{t_{i+1}, \ldots, t_k\}) \) it holds that \( S_i \cup R \notin \mathcal{U} \).

**Proof.** Let \( F \) be a B-forest for \( \mathcal{U} \). Since \( E \in \mathcal{U} \), it follows that \( F \) is a tree. If \( s_1, s_2 \in E \) are not leaves, then \( \text{desc}_F(s_j) \in \mathcal{U} \) for \( j = 1, 2 \) and \( \text{desc}_F(s_1) \cup \text{desc}_F(s_2) \in \mathcal{U} \) implies that \( s_1 \) and \( s_2 \) are comparable. It follows that every node has at most one non-leaf child. Let \( t_1, \ldots, t_k \) be the non-leaves and set \( S_i := \text{desc}_F(t_i) \). Then \( S_1 \subset \cdots \subset S_k \) is a chain in \( \mathcal{U} \). The leaves are \( L = \bigcup_{i \geq 1} S_i \setminus (S_{i-1} \cup t_i) \) and \( T \) of nodes is incomparable iff \( T \subseteq L \cup \{t_i\} \) for some \( i \) and \( T \setminus t_i \subseteq L \setminus S_i \); see also Figure 4. This shows that \( t_i \in S_i \) for \( i = 1, \ldots, k \) satisfies the condition and it is straightforward to check that every such collection yields a B-tree. \( \square \)

For the union closed family \( \mathcal{U}(f) = \{ \emptyset \} \cup \mathcal{L}(f) \) associated to a polymatroid \((E, f)\) Proposition 4.10 recovers the greedy paths. Every chain \( \emptyset = S_0 \subset S_1 \subset \cdots \subset S_k = E \) corresponds to a maximal chain of flats \( \emptyset = F_0 \subset \cdots \subset F_k = E \) with \( F_i = E \setminus S_{k-i} \) and \( t_i \in F_i \setminus F_{i-1} \). From this description, we can also deduce adjacency.

**Proposition 4.11.** Let \( (j_1, \ldots, j_k) \) be a greedy path for \((E, f)\) and \( v \in \Sigma_1(f) \) the corresponding vertex. The neighbors of \( v \) correspond to the greedy paths \((j_1, \ldots, j'_t, \ldots, j_k)\) for some \( 1 \leq t \leq k \) and \( j'_t \in \{j_1, \ldots, j_t\} \setminus (\{j_1, \ldots, j_{t-1}\} \cup \{j_t\}) \) or to greedy paths derived from the sequences \((j_1, \ldots, j_{s+1}, j_s, \ldots, j_k)\) for some \( 1 \leq s < k \).

**Proof.** Let \( v \) be the vertex of \( \Sigma_1(f) \) corresponding to the greedy path \((j_1, \ldots, j_k)\) and let \( F \) be the associated B-tree. For a weight \( w \in \mathbb{R}^E \) it follows from Proposition 7.10 of [41] that \( v \in \Sigma_1(f)^w \) if and only if \( w_i \geq w_j \) for all \( i, j \in E \) with \( j \in \text{desc}_F(i) \). That is, \( w \) is an order preserving map from the poset \( F \) into the real numbers. The cone of such \( w \) is simplicial. The facets of the cone are given by the edges of the B-tree and correspond to neighbors of \( v \) in \( \Sigma_1(f) \). The description of B-trees above now yields the claim. \( \square \)
Let us finally note that although Theorem 4.3 states that $\Pi_{P_f,1}$ and $\Sigma_1(f)$ are normally equivalent, they are not homothetic in general.

**Proposition 4.12.** $\Sigma_1(f)$ is in general not a sum of simplices. In particular, $\Sigma_1(f)$ is not necessarily a nestohedron.

**Proof.** Let $f$ be the rank function of the uniform matroid $U_{n,k}$, that is, $f(I) = \min(|I|, k)$. The polymatroid polytope $P_f$ is the convex hull of all $u \in \{0,1\}^n$ with $\sum_i u_i \leq k$. The monotone path polytope satisfies

$$\Sigma_1(f) = 2\Delta_{n,1} + 2\Delta_{n,2} + \cdots + 2\Delta_{n,k-1} + \Delta_{n,k}.$$ 

This is the permutahedron for the point $v = (0, \ldots, 0, 1, 3, \ldots, 2k - 1)$; see also [37]. Assume that there are $y_I \geq 0$ for all $I \subseteq [n]$ such that

$$\Sigma_1(f) = \sum_I y_I \Delta_I.$$ 

Note that the left-hand side is invariant under the symmetric group. Hence, we can symmetrize to get

$$\Sigma_1(f) = \sum_{j=1}^n c_j S_j \quad \text{where} \quad S_j := \sum_{I : |I| = j} \Delta_I$$

and $c = (c_1, \ldots, c_n) \geq 0$. The vertex maximizing the right-hand side for the linear function $w = (1, 2, \ldots, n)$ is given by $Mc$, where $M_{ij} = \binom{i-1}{j-1}$. In particular $c = M^{-1}v$. For $n = 4$ and $k = 3$, we get $v = (0, 1, 3, 5)$ and $c = (0, 1, 1, -1).$ \hfill $\Box$

## 5. Flag Matroids

Let $M = (E, \mathcal{I})$ be a matroid of rank $r$ and let $k = (k_1, \ldots, k_s)$ be a vector of integers satisfying $0 \leq k_1 < k_2 < \cdots < k_s \leq r$. The **flag matroid** $\mathcal{F}_M^k$ of $M$ of rank $k$ is the collection of chains

$$I_s : I_1 \subset I_2 \subset \cdots \subset I_s$$

of independent sets of $M$ with $|I_j| = k_j$ for $j = 1, \ldots, s$. Borovik, Gelfand, Vince, and White [12] introduced flag matroids more generally in terms of strong maps. In this paper, we only treat the special case of flag matroids of a matroid $M$. We refer to [13] for relations to Coxeter matroids and to Section 6 for the algebro-geometric point of view. We call $\mathcal{F}_M := \mathcal{F}_M^{(0,1,\ldots,r)}$ the **underlying flag matroid** of $M$.

For a flag $I_s$, define $\delta(I_s) := e_{I_1} + e_{I_2} + \cdots + e_{I_s} \in \mathbb{Z}^E$ and with it the **flag matroid polytope** [12]

$$\Delta(\mathcal{F}_M^k) := \text{conv}\{\delta(I_s) : I_s \in \mathcal{F}_M^k\}.$$ 

In this section, we relate flag matroid polytopes and monotone path polytopes of matroids via a generalization of the independence polytope.

For $k = (k_1, \ldots, k_s)$ define the **rank-selected independent sets**

$$\mathcal{I}^k := \{I \in \mathcal{I} : |I| = k_i \text{ for some } i = 1, \ldots, s\}$$

and the **rank-selected independence polytope** $\mathcal{P}_M^k := \text{conv}\{e_I : I \in \mathcal{I}^k\}$.

**Lemma 5.1.** Let $I, J \in \mathcal{I}^k$ with $|I| < |J|$. Then $[e_I, e_J]$ is an edge of $\mathcal{P}_M^k$ if and only if $I \subset J$ and $|I| = k_i$, $|J| = k_{i+1}$ for some $1 \leq i < s$.

**Proof.** Assume that $[e_I, e_J] = (\mathcal{P}_M^k)^w$ for some $w \in \mathbb{R}^E$. Let $\emptyset = I_0 \subset I_1 \subset \cdots \subset I_r$ be the sequence of independent sets obtained from the greedy algorithm on $\mathcal{P}_M$ with respect to $w$. Let $|I| = k_i$ and $|J| = e_j$ with $i < j$. Since $e_I$ is the unique maximizer over the base polytope of the restriction $M_{k_i}$, we have $I_{k_i} = I$ and likewise $I_{k_j} = J$. Now, since $\langle w, e_I \rangle = \langle w, e_J \rangle = w(I) + w(J \setminus I)$, it follows
that \( w(K) = w(I) \) for all \( I \subseteq K \subseteq J \). Hence, \( j = i + 1 \). For the converse, take the linear function \( w = e_I - e_{E \setminus J} \).

In the same way as in Section 3, one shows that every cellular string of \( P^k_M \) is coherent.

**Theorem 5.2.** Let \( M = (E, \mathcal{I}) \) be a matroid and \( k \) a rank vector. Then every \( 1 \)-cellular string of \( P^k_M \) is coherent.

Lemma 5.1 in particular implies that \( 1 \)-monotone paths on \( P^k_M \) are precisely the elements of the flag matroid \( \mathcal{F}^k_M \).

**Theorem 5.3.** Let \( M = (E, \mathcal{I}) \) be a matroid of rank \( r \) and \( k \) a rank vector. The monotone path polytope \( \Sigma_1(P^k_M) \) is normally equivalent to the flag matroid polytope \( \Delta(\mathcal{F}^k_M) \).

**Proof.** Note that the distinct values of the linear function \( 1 \) on the vertices of \( P^k_M \) are precisely \( k_1 < k_2 < \cdots < k_s \). For \( i = 1, \ldots, s \), the fiber \( \{ x : 1(x) = k_i \} \cap P^k_M \) is the base polytope of the truncation \( M_{k_i} \) that we denote by \( B_{k_i} \). It follows from Proposition 2.4 that \( \Sigma_1(P^k_M) \) is normally equivalent to

\[
B_{k_1} + B_{k_2} + \cdots + B_{k_s}.
\]

This is precisely the decomposition of \( \Delta(\mathcal{F}^k_M) \) given in Corollary 1.13.5 of [13].

It seems likely that the obvious generalization of rank-selected independence polytopes to the setting of general flag matroids [12] will generalize Theorem 5.3.

**Remark 5.4.** For a rank-\( r \) matroid \( M \) with rank function \( r_M \), Theorem 5.3 states that the base polytope of the flag polymatroid \( \mathcal{F}_M \) is normally equivalent to the base polytope of the underlying flag matroid \( \mathcal{F}_M \). This gives another justification for calling \( \mathcal{F}_M \) (underlying) flag polymatroid: Theorem 4.3 and Theorem 3 imply that the facial structure of \( P_f \) only depends on the flags of \( \mathcal{L}(f) \).

This prompts the question as to a notion of partial flag polymatroid. The rank vectors of flag matroids are subsets of the values \( \{1(v) : v \in V(P_M)\} = \{r_M(A) : A \in \mathcal{L}(M)\} \). The important property for the description of flag matroid polytopes is that for every flat \( A \in \mathcal{L}(M) \) the vertices of \( P_f \cap \{ x : 1(x) = f(A) \} \) are vertices of \( P_f \). This happens if and only if there are no long edges: If \([u, v] \subset P_f \) is an edge with \( 1(u) < 1(v) \) then \( f(A) \leq 1(v) = 1(v) \leq f(A) \) for all flats \( A \in \mathcal{L}(f) \). Note that the greedy algorithm implies \( 1(u) = f(I(v)) = f(T(v)) \). The next result implies that this is characteristic for matroids.

For flats \( A, B \in \mathcal{L}(f) \), we write \( A \rightarrow B \) if \( A \) is covered by \( B \), that is, if \( A \subset B \) and there is no flat \( C \) with \( A \subset C \subset B \).

**Proposition 5.5.** Let \( (E, f) \) be a polymatroid such that for all closed sets \( A, B, C \in \mathcal{L}(f) \) with \( A \rightarrow B \) we have \( f(B) \leq f(C) \) or \( f(C) \leq f(A) \), then \( f \) is a multiple of a matroid rank function.

**Proof.** For any \( A \in \mathcal{L}(f) \) choose \( B \in \mathcal{L}(f) \) with \( A \rightarrow B \) and \( f(B) \) minimal. If \( A \rightarrow B' \), then \( f(A) < f(B) \leq f(B') \) and the condition implies \( f(B) = f(B') \). Now, for \( A \rightarrow B \rightarrow C \) and \( A \rightarrow B' \rightarrow C' \) with \( f(C) < f(C') \), our condition implies \( f(C) \leq f(B') = f(B) < f(C) \). Thus \( f(C) = f(C') \). Iterating the argument then shows that given two maximal chains if \( A_0 \rightarrow A_1 \rightarrow \cdots \rightarrow A_k \) and \( A'_0 \rightarrow A'_1 \rightarrow \cdots \rightarrow A'_l \), we have \( f(A_i) = f(A'_i) \) for all \( i \) and, in particular, \( k = l \). This implies that \( \mathcal{L}(f) \) is a graded poset. Assuming that \( \{i\} \) is closed for every \( i \in E \), we can scale \( f \) so that \( f(\{i\}) = 1 \) for all \( i \in E \). This implies that for \( A \in \mathcal{L}(f) \), \( f(A) \in \mathbb{Z}_{\geq 0} \) and submodularity shows

\[
f(A) \leq \sum_{i \in A} f(\{i\}) = |A|.
\]
Example 5.6 (S-hypersimplices). Let $M$ be the uniform matroid $U_{n,n}$ on $n$ elements for which every subset $I \subseteq E$ is independent. The independence polytope $P_M$ is the unit cube and $\mathcal{L}(M) = 2^E$ is the Boolean lattice. The base polytope of a truncation of $M$ to $k$ is the $(n,k)$-hypersimplex, that is, the convex hull of all $v \in \{0,1\}^E$ with $\sum_i v_i = k$.

For $0 \leq k_1 < \cdots < k_s \leq n$, the rank-selected independence polytope $P^k_M$ is an S-hypersimplex [37] with $S = \{k_1, \ldots, k_s\}$. For $k_i = i$, these are also the line-up polytopes for the cube introduced in [16, Sect. 6.2.2]. The corresponding monotone path polytope $\Sigma(I(P^k_M))$ is homothetic to the permutahedron $\mathcal{P}(s, \ldots, s, s-1, \ldots, s-1, \ldots, 1)$ with multiplicities given by $k_1, \ldots, k_s - k_{s-1}$. The proof of Proposition 4.12 shows that these need not be nestohedra. \hfill \Box

The facial structure of $\Delta(\mathcal{F}_M)$ and hence of $\Sigma(I(P^k_M))$ is given in Exercise 1.14.26 of [13]. For the underlying flag matroid we can give an alternative description. Recall that a set $K \subseteq E$ is a \textbf{cocircuit} of $M$ if it is inclusion-minimal with the property that it meets every basis of $M$.

**Corollary 5.7.** For any matroid $M$, the flag matroid polytope $\Delta(\mathcal{F}_M)$ is a simple polytope normally equivalent to a nestohedron for the building set

$$U(M) = \{K_1 \cup \cdots \cup K_m : m \geq 0, K_1, \ldots, K_m \text{ cocircuits}\}.$$ 

**Proof.** It follows from Theorem 4.3, Theorem 3, and Theorem 5.3 that $\Delta(\mathcal{F}_M)$ is normally equivalent to the nestohedron for the union-closed family of sets $E \setminus F$ where $F$ ranges over all flats of $M$. Now $F$ is a flat if and only if $E \setminus F$ is a union of cocircuits [39, Ex. 2.1.13(a)]. \hfill \Box

We close this section with a few thoughts on the max-slope pivot polytopes of rank-selected independence polytopes. If $I$ is an independent set of rank $\|I\| = k_i$ for $i < s$, then Lemma 5.1 yields that the 1-improving neighbors correspond to independent sets $J$ with $I \subset J$ and $\|J\| = k_{i+1}$. From (8), we infer that

$$\Pi_{\mathcal{F}^k_M}(e_I) = \frac{1}{k_{i+1} - k_i} \text{conv}\{e_J - e_I : I \subset J \in \mathcal{I}, \|J\| = k_{i+1}\}.$$ 

The independent sets of the contraction $M/I$ are precisely those $K \subseteq E \setminus I$ with $I \cup K$ independent in $M$. Hence

$$(k_{i+1} - k_i) \cdot \Pi_{\mathcal{F}^k_M}(e_I) = B_{(M/I)_{k_{i+1} - k_i}},$$

where $(M/I)_{k_{i+1} - k_i}$ is the contraction of $M/I$ to rank $k_{i+1} - k_i$. Consequently, the max-slope pivot polytope $\Pi_{\mathcal{F}^k_M}$ is normally equivalent to

$$\sum_{i=1}^{s-1} \sum_{F \in \mathcal{L}(f) \atop \text{rk}(F) = k_i} B_{(M/F)_{k_{i+1} - k_i}}.$$ 

If $k_{i+1} = k_i + 1$, then $B_{(M/F)_{k_{i+1} - k_i}}$ is the convex hull of all $e_j$ such that $I \cup j \in \mathcal{I}$ and hence a standard simplex. This prompts a generalization of nestohedra where standard simplices are replaced by matroid base polytopes.

It is still true that $\Delta(\mathcal{F}_M)$ is a weak Minkowski summand of $\Pi_{\mathcal{F}^k_M}$ but normal equivalence does not hold in general. We suspect that the refinement of the normal cone of $\Delta(\mathcal{F}_M)$ corresponding to a flag $I_s$ reflects the freedom of the greedy algorithm to order the elements in $I_{j+1} \setminus I_j$.

6. Toric varieties in Grassmannians and flag varieties

In this section, we give a toric perspective on the monotone path polytopes of realizable polymatroids and the relation between Grassmannians and flag varieties.
For $1 \leq r \leq n$, let $\text{Gr}(n, r)$ be the Grassmannian of $r$-dimensional linear subspaces in $\mathbb{C}^n$. We can view a point $L \in \text{Gr}(n, r)$ as the row span of a full rank matrix $A \in \mathbb{C}^{r \times n}$. The algebraic torus $T^n = (\mathbb{C}^*)^n$ acts on $\text{Gr}(n, r)$ as follows. If $L$ is represented by $A = (a_1, \ldots, a_n)$ and $t = (t_1, \ldots, t_n) \in T^n$, then $t$ sends $L$ to $t \cdot L = \text{row span}(t \cdot A)$, where $t \cdot A = (t_1 a_1, t_2 a_2, \ldots, t_n a_n)$. The fixed points of this action are precisely the $r$-dimensional coordinate subspaces of $\mathbb{C}^n$. In its Plücker embedding, a subspace $L$ is identified with its Plücker vector $p(L) \in \mathbb{P}(\bigwedge^k \mathbb{C}^n) \cong \mathbb{P}((n)_k)$ with $p(L)_j = \det(A_j) = \det(a_{j1}, \ldots, a_{jk})$, where $J = \{j_1 < \cdots < j_k\}$ is an ordered $r$-subset of $[n]$. The fixed points then correspond to Plücker vectors $p$ of the form $p_J \neq 0$ for a fixed $r$-subset $J \neq 0$.

The moment map $\mu : \text{Gr}(n, r) \to \mathbb{R}^n$ of the action of $T^n$ on $\text{Gr}(n, r)$ is given by

$$\mu(L)_j = \frac{\sum_{J \subseteq [n]} |p(L)_J|^2}{\sum_{J} |p(L)_J|^2},$$

where $J$ ranges over all $r$-subsets of $[n]$; see [27, Sect. 2.1]. The image of $\text{Gr}(n, r)$ under $\mu$ is precisely the $(n, r)$-hypersimplex $\Delta(n, r)$, whose vertices correspond to the fixed points.

Let $M = M(L) = ([n], I)$ be the rank-$r$ matroid with $I \in I$ if and only if $(a_i)_{i \in I}$ is linearly independent. Note that this only depends on $L$ and not on $A$.

**Theorem 6.1 ([27, Sect. 2.4]).** Let $L \in \text{Gr}(n, r)$ be a subspace with matroid $M$. The Zariski closure of $T^n \cdot L$ is a projective toric variety in $\text{Gr}(n, r)$ with moment polytope $B_M$.

The independence polytope can also be obtained as a moment polytope. Choose a representation $A$ of $L$ such that $e_1, \ldots, e_r$ is in general position with respect to $a_1, \ldots, a_n$. That is, every linearly independent collection $(a_i : i \in I)$ can be completed to a basis of $\mathbb{C}^r$ by any choice of $r - |I|$ vectors from $e_1, \ldots, e_r$. Define $\widehat{A} := (A, E) = (a_1, \ldots, a_n, e_1, \ldots, e_r) \in \mathbb{C}^{r \times (n+r)}$ and $\widehat{L} := \text{row span}(\widehat{A})$. We can view $T^n$ as a subtorus of $T^{n+r}$ acting on $\widehat{L}$ by

$$t \cdot \widehat{A} = (t_1 a_1, \ldots, t_n a_n, e_1, \ldots, e_r).$$

**Corollary 6.2.** The Zariski closure of the orbit of $\widehat{L} = \text{row span}(\widehat{A})$ under $T^n$ is a projective toric variety $X_L \subseteq \text{Gr}(n + r, r)$ with moment polytope $P_M$.

**Proof.** Let $\lambda^w(t) = (t^{w_1}, \ldots, t^{w_n})$ be a one-parameter subgroup. On the level of Plücker vectors, it can be seen that $\lim_{t \to \infty} \lambda^w(t) \cdot \widehat{L}$ is fixed by $T^n$ if and only if there is a unique $I \in I$ such that $w(I) = \sum_{i \in I} w_i$ is maximal. Let $p$ be the Plücker vector of the limit point for some $I \in I$. Then $p_J \neq 0$ if and only if $I \subseteq J$ and $J \setminus I \subseteq \{n + 1, \ldots, n + r\}$. The representation of the moment map above yields $\mu(p) = e_I$ and shows $\mu(X_L) = P_M$. \hfill $\square$

For $1 \leq r \leq n$, let $\text{Fl}(n, r)$ be the flag variety of complete flags $0 = F_0 \subset F_1 \subset \cdots \subset F_r \subseteq \mathbb{C}^n$ with $\dim F_i = i$ for $i = 1, \ldots, r$. Any such flag can be represented by a full-rank matrix $A \in \mathbb{C}^{r \times n}$. If $A_i \in \mathbb{C}^{i \times n}$ is the submatrix obtained from $A$ by taking the first $i$ rows, then $F_i = \text{row span}(A_i)$ for $i = 0, \ldots, r$ defines a complete flag $F_i = (F_i)_{i=0,\ldots,r}$. If $A$ and $A'$ define the same flag, then $A' = gA$, where $g \in B \subseteq \text{GL}(\mathbb{C}^r)$, the (standard) Borel subgroup of invertible lower-triangular matrices.

Notice that $M(F_{i-1})$ is a quotient of $M(F_i)$ and Theorem 1.7.3 of [13] asserts that $(M(F_1), M(F_2), \ldots, M(F_r))$ is a general flag matroid. We call the flag $F_i$ very general if $M(F_i)$ is the $i$-th truncation of $M(F_r)$ for each $i = 1, \ldots, r - 1$.

$\text{Fl}(n, r)$ is naturally a subvariety of $\prod_{i=1}^{r} \text{Gr}(n, i)$ and the diagonal action of $T^n$ extends to $\text{Fl}(n, r)$. Thus, any flag $F_i$ yields a toric subvariety

$$Y_{F_i} \subseteq X_{F_1} \times X_{F_2} \times \cdots \times X_{F_r}.$$

Theorem 6.19 in [15] asserts that the moment polytope of $Y_{F_i}$ is $B_{M(F_1)} + B_{M(F_2)} + \cdots + B_{M(F_r)}$, the polytope of the flag matroid $(M(F_1), M(F_2), \ldots, M(F_r))$. 


Theorem 6.3. If $F_\bullet$ is very general, then the moment polytope of $Y_{F_\bullet}$ is $\Delta(\mathcal{F}_M)$, where $M = M(F_r)$. Conversely, if $M$ is a rank-$r$ matroid realizable over $\mathbb{C}$, then there is very general flag $F_\bullet$ with $M(F_r) = M$.

Proof. The first statement follows directly from the preceding discussion and the definition of very general flag. For the second statement, let $L = \text{rowspan}(A)$ be a realization of $M$ with $A \in \mathbb{C}^{r \times n}$. Projecting $L$ onto a general linear subspace $L' \subset L$ of dimension $r - 1$ yields a realization of the first truncation of $M$. Iterating this yields a very general flag. Up to a change of coordinates this is means that the flag $F_\bullet$ associated to $gA$ for any general $g \in \text{Gl}(\mathbb{C}^r)$ is very general and since $L = \text{rowspan}(gA)$, this proves the claim.

There is a rational map $\phi : \text{Gr}(n + r, r) \rightarrow \text{Fl}(n, r)$. Let $\tilde{L} \in \text{Gr}(n + r, r)$ such that $\tilde{L}$ is represented by a matrix of the form $\tilde{A} = (A, E)$, where $A \in \mathbb{C}^{r \times n}$ is of full rank and $E = (e_1, \ldots, e_r)$. Then $\phi$ takes $\tilde{L}$ to the flag $F_\bullet$ with $F_i = \text{rowspan}(A_i)$ as above. The set of such $\tilde{L}$ is Zariski open and $\phi$ is a rational surjective map. The fibers of $F_\bullet$ are represented by $(gA, E)$ with $g \in B$.

Note that $\phi$ is equivariant with respect to the action of $T^n$.

Proposition 6.4. Let $\tilde{L} \in \text{Gr}(n + r, r)$ such that $\phi(\tilde{L}) = F_\bullet$ is defined. Then $\phi$ is a regular map on $X_{\tilde{L}}$ with image $Y_{F_\bullet}$. The preimage of $F_\circ$ in $X_{\tilde{L}}$ are the linear subspaces $t\tilde{L}$, where $t = (t, t, \ldots, t) \in T^n$.

Proof. The flag variety $\text{Fl}(n, r)$ is embedded in the projective space over $\bigoplus_{k=1}^{r} \mathbb{C}^n$ with coordinates $(p_K)_{K}$, where $K$ ranges over all non-empty subsets of $[n]$ of size $|K| \leq r$. If $F_\bullet$ is represented by $A$, then it is represented by the flag minors $(p(F_\bullet)_{K})_{K}$ with $p(F_\bullet)_K = \det((A_k)_K)$, where $k = |K|$; see [38, Ch. 14.1]. Let $\tilde{L} = \text{rowspan}(\tilde{A})$, where $\tilde{A} = (A, E)$. On the level of Plücker vectors, the map $\phi$ is given by a coordinate projection: For $K \subseteq [n]$ and $|K| = k$, $p(\phi(\tilde{L}))_K = p(L)_{K \cup \{n+1, \ldots, n+k\}}$. Let $(gA, E)$ represent a preimage of $F_\circ$. Then $(gA, E) = t \cdot (A, E)$ if and only if $g$ is a multiple of the identity matrix.

Kapranov, Sturmfels, and Zelevinsky [32] studied quotients of toric varieties by subtori. Let $X \subset \mathbb{P}^{n-1}$ be a projective toric variety with $n$-dimensional torus $T$ and fan $\mathcal{N}$ in $\mathbb{R}^n$. A subtorus $H \subset T$ is represented by a rational subspace $U \subset \mathbb{R}^n$. Define an equivalence relation on $\mathbb{R}^n/U$ by setting $q + U \sim q' + U$ if $q + U$ meets the same cones of $\mathcal{N}$ as $q' + U$. The equivalence classes form a fan $\mathcal{N}/U$ in $\mathbb{R}^n/U$ called the quotient fan. A toric variety $Y$ with fan $\mathcal{N}/U$ is called a combinatorial quotient.

We can now state the relationship between $X_{\tilde{L}}$ and $Y_{F_\bullet}$.

Theorem 6.5. Let $F_\bullet$ be a very general flag. Then the toric variety $Y_{F_\bullet}$ is a combinatorial quotient for the action of $H$ on $X_{\tilde{L}}$. Moreover, $Y_{F_\bullet}$ is a smooth toric variety.

In [32], the authors construct a canonical combinatorial quotient associated to $X$ and $H$, called the Chow quotient $X//H$. This is a toric variety associated to the Chow form of the closure of $H \cdot E_0$, where $E_0$ is the distinguished point of $X$. The embedding $H \subset T$ yields a linear projection $\pi : \mathbb{R}^n \rightarrow U$. Let $\Sigma_{x}(P)$ be the fiber polytope [7] of the pair $(P, \pi)$; see also [32, Section 2]. The following is a consequence of Theorem 2.1, Proposition 2.3, and Lemma 2.6 of [32].

Theorem 6.6. Let $X$ be the toric variety associated to the lattice polytope $P$. Then the Chow quotient $X//H$ is the toric variety associated to the fiber polytope $\Sigma(P, \pi)$.

Proof of Theorem 6.5. By Corollary 6.2, the polytope associated to $X_{\tilde{L}}$ is the independence polytope $P_M$ of the matroid $M = M(L)$ for $L = \text{rowspan}(A)$. The linear subspace associated to the subtorus $H$ is $U = \{(u, u, \ldots, u) : u \in \mathbb{R}\}$. The linear projection $\pi$ is $\pi(x) = x_1 + \cdots + x_n$. Hence the fiber polytope $\Sigma(P, \pi)$ is the monotone path polytope $\Sigma_1(P_M)$. If $F_\bullet$ is very general, then the moment
polytope of \( Y_{F} \) is \( \Delta(F_M) \) by Theorem 6.3. The first claim now follows from Theorem 5.3 and the fact that normally equivalent polytopes have the same underlying fan.

As for the second claim, we note from Theorem 3 (see also Corollary 4.9) that \( \Sigma_1(P_M) \) is a simple generalized permutahedron. This implies that at every vertex, there are precisely \( \dim \Sigma_1(P_M) \) many incident edges and primitive vectors along the edge directions are of the form \( e_i - e_j \) and hence provide a lattice basis. This is equivalent to \( Y_{F} \) being smooth. \( \square \)

We can extend this relation to realizable polymatroids; see end of Section 2.1. Let \( f : 2^{[n]} \to \mathbb{Z}_{\geq 0} \) be an integral polymatroid realized by linear subspaces \( V_1, \ldots, V_n \subset \mathbb{C}^r \) so that \( f(I) = \dim \mathbb{C} \sum_{i \in I} V_i \). For \( i = 0, \ldots, n \) define \( s_i = \sum_{j=1}^i \dim V_j \). We can represent \( f \) by a full-rank matrix \( A = (a_1, \ldots, a_{sn}) \in \mathbb{C}^{r \times sn} \) by letting \( a_{si-1+1}, \ldots, a_{si} \) be a basis of \( V_i \). Let \( L_f = \text{rowspan}(A, E) \subset \text{Gr}(sn + r, r) \). We view \( T^n \) as a subtorus of \( T^{sn+r} \)

\[
T^n = \{(t_1, \ldots, t_1, t_2, \ldots, t_2, \ldots, t_n, \ldots, t_n, 1, \ldots, 1) : t_1, \ldots, t_n \in \mathbb{C}^r \}.
\]

The same argument as before then shows

**Theorem 6.7.** Let \( L_f \subset \text{Gr}(sn + r, r) \) as above. The Zariski closure of the orbit \( T^n \cdot L_f \) is a projective toric variety \( X_f \) with moment polytope \( P_f \).

The matrix \( A \) also defines a flag \( F \in \text{Fl}(sn, r) \) and a toric variety \( Y_f \) with respect to the action of \( T^n \). The constituents in every \( \text{Gr}(n, i) \) are not so easy to describe as they depend on the choice of a basis for each \( V_i \). However, the relation between the toric varieties stays intact and the same proof as for Theorem 6.5 yields the following.

**Theorem 6.8.** Let \( F \) be a very general flag. Then the toric variety \( Y_f \) is a combinatorial quotient for the action of \( H \) on \( X_f \) and the moment polytope of \( Y_f \) is normally equivalent to \( \Sigma_1(f) \). In particular, \( Y_f \) is a smooth toric variety for every realizable polymatroid.

7. **Independent Set Greedy Paths and Partial Permutahedra**

The base polytope of the flag polymatroid \( \Sigma_1(f) \) of Section 3 is a polytope whose vertices encode the different greedy paths for optimizing on the base polytope \( B_f \). The greedy algorithm (Theorem 3.2) can also be used to optimize linear functions over \( P_f \) by simply stopping when \( w_{\sigma(i)} < 0 \). It turns out that up to a simple modification of the polymatroid, the space of partial greedy paths may also be represented by a flag polymatroid. We apply this extension to resolve a conjecture on partial permutation polytopes of Heuer–Striker [30].

For a polymatroid \( f : 2^E \to \mathbb{R} \) with \( E = [n] \), define \( f' : 2^{E'} \to \mathbb{R} \) with \( E' := [n+1] \) by

\[
f'(A) := \begin{cases} f(A) & \text{if } n+1 \notin A \\ f(E) & \text{otherwise.} \end{cases}
\]

**Proposition 7.1.** Let \( f \) be a polymatroid and \( f' \) as defined above. Then \((E', f')\) is a polymatroid with base polytope

\[
B_{f'} = \{(x, f(E) - 1(x)) : x \in P_f \} \cong P_f.
\]

**Proof.** For \( x \in \mathbb{R}^E \), let \( x' := (x, f(E) - 1(x)) \). Let \( B'_f := \{x' : x \in P_f\} \), which is linearly isomorphic to \( P_f \). Every edge of \( B'_f \) is of the form \([u', v']\) where \([u, v] \subseteq P_f\) is an edge. If \( u - v = \mu(e_i - e_j) \) for some \( \mu \neq 0 \), then \( 1(u) = 1(v) \) and hence \( u' - v' = \mu(e_i - e_j) \). If \( u - v = \mu e_i \), then \( u' - v' = \mu(e_i - e_{n+1}) \). Hence \( B'_f = B_g \) is a generalized permutahedron or polymatroid base polytope for some polymatroid \( g : 2^E \to \mathbb{R} \). For \( A \subseteq E' \) we have \( g(A) = \max \{1_A(x) : x \in B_g\} \). If \( n+1 \notin A \), then \( g(A) = \max \{1_A(x) : x \in P_f\} = f(A) = f'(A) \). If \( A = S \cup \{n+1\} \), then we maximize
For a weight $w$ between largest and smallest nonempty subsets is at most the convex hull of all points partial permutahedra. For $m, n$ maximal value is

Note that from the convex hull description, it is apparent that the polytope using our new tools to endow the partial permutahedron with a nested set structure.

**Conjecture 5.24**. The vertices of $P(m, n)$ for the general case with nestohedra, and they use the nested set structure to verify the desired bijection. Their missing link as optimizing $\tilde{w} : = (w, 0) \in \mathbb{R}^{E'}$. Then optimizing $w$ over $P_f$ is precisely the same as optimizing $\tilde{w}$ over $P_f$, which can be done with the usual greedy algorithm. If $(j_1, j_2, \ldots, j_k)$ represents a partial greedy path on $P_f$, then $(j_1, j_2, \ldots, j_k, n + 1)$ is the corresponding greedy path on $P_f$.

From the definition we get that

$$\mathcal{L}(f') = (\mathcal{L}(f) \setminus \{E\}) \cup \{E'\}.$$  

**Corollary 7.2.** Let $(E, f)$ be a polymatroid. The 1-monotone paths on $P_f$ from 0 to some vertex are in bijection to 1-monotone paths on $P_{f'}$ from 0 to a vertex of $B_{f'}$. All these greedy paths are coherent and the monotone path polytope $\Sigma_1(f')$ is normally equivalent to the nestohedron with building set

$$\mathcal{U}(f') = \{(E \setminus F) \cup \{n + 1\} : F \in \mathcal{L}(f)\}.$$  

As an application of these tools, we completely resolve a conjecture of Heuer and Striker [30] about partial permutahedra. For $m, n \geq 1$ the $(m, n)$-partial permutahedron $\mathcal{P}(m, n) \subset \mathbb{R}^m$ is the convex hull of all points $x \in \{0, 1, \ldots, n\}^m$ such that the non-zero entries are all distinct.

**Conjecture 5.24 ([30]).** Faces of $\mathcal{P}(m, n)$ are in bijection with flags of subsets of $[m]$ whose difference between largest and smallest nonempty subsets is at most $n - 1$. A face of $\mathcal{P}(m, n)$ is of dimension $k$ if and only if the corresponding flag has $k$ missing ranks.

In their paper, they prove the case when $m = n$ via the observation that $\mathcal{P}(m, n)$ is the graph associahedron for the star graph, the so-called stellohedron. Graph associahedra are in particular nestohedra, and they use the nested set structure to verify the desired bijection. Their missing link for the general case with $m \neq n$ was a lack of a nested set structure. We resolve their conjecture by using our new tools to endow the partial permutahedron with a nested set structure.

Note that from the convex hull description, it is apparent that the polytope $\mathcal{P}(m, n)$ is anti-blocking. The vertices of $\mathcal{P}(m, n)$ are the points $v \in \mathbb{R}^m$ with $0 \leq k \leq \max(m - n, 0)$ zero entries and the remaining entries a permutation of $\{n, n - 1, \ldots, n - (m - k + 1)\}$. The face $F$ of $\mathcal{P}(m, n)$ that maximizes $1$ is the convex hull of permutations of $(0, 1, \ldots, n)$ if $n \leq m$ and $(n - m + 1, \ldots, n)$ if $n > m$. Since $F$ is a permutahedron and $\mathcal{P}(m, n) = \mathbb{R}_{\geq 0}^m \cap (F - \mathbb{R}_{\leq 0}^m)$, we conclude that the $(m, n)$-partial permutahedron $\mathcal{P}(m, n)$ is a polymatroid polytope.

For $n > m$, $\mathcal{P}(m, n)$ is normally equivalent to the polymatroid of the permutahedron and hence combinatorially (even normally) equivalent to $\mathcal{P}(m, m)$. Thus the only relevant case is $m > n$.

For $1 \leq n \leq m$, let $U_{m,n}$ be uniform matroid on $[m]$ of rank $n$. The partial greedy paths for $U_{m,n}$ are precisely sequences $(j_1, j_2, \ldots, j_k)$ with $j_1, \ldots, j_k \in [m]$ distinct and $k \leq n$. The corresponding chain of flats is $\emptyset = A_0 \subset A_1 \subset \cdots \subset A_k$, where $A_i = \{j_1, \ldots, j_i\}$ for $i < k$. If $k < n$, then $A_k = \{j_1, \ldots, j_k\}$ and $A_k = [m]$ otherwise.

The rank function of $U_{m,n}$ is given by $r_{m,n}(A) = \min(|A|, n)$ and we let $f_{m,n} : = r_{m,n}'$ as defined above. That is, $f_{m,n} : 2^{[m+1]} \to \mathbb{Z}_{\geq 0}$ with $f_{m,n}(A) = \min(|A|, n)$ if $m + 1 \not\in A$ and $n$ otherwise.

Let

$$\mathcal{P}'(m, n) = \{(x, {\binom{n+1}{2}} - 1(x)) : x \in \mathcal{P}(m, n)\} \subset \mathbb{R}^{m+1}.$$  

be the embedding of $\mathcal{P}(m, n)$ into the hyperplane $\{y \in \mathbb{R}^{m+1} : y_1 + \cdots + y_{m+1} = 1 + \cdots + n\}$.

**Theorem 7.3.** The partial permutahedron $\mathcal{P}'(m, n)$ is normally equivalent to the polymatroid polytope of the flag polymatroid $\Sigma_1(f_{m,n})$.  

Proof. Let $c \in \mathbb{R}^{m+1}$ be a general linear function. We show that $\Sigma_1(f_{m,n})^c$ is a vertex if and only if $\mathcal{P}'(m,n)^c$ is a vertex. Equivalently, we show that $c$ determines a 1-monotone path on $P_{f_{m,n}}$ from 0 to some vertex $u \in P_{f_{m,n}}$ if and only if $\mathcal{P}'(m,n)^c$ is a vertex. Let $\sigma$ be a permutation of $[m]$ such that $c_{\sigma(1)} \geq c_{\sigma(2)} \geq \cdots \geq c_{\sigma(k)} > c_{m+1} \geq c_{\sigma(k+1)} \geq \cdots \geq c_{\sigma(m)}$. Now, $c$ determines a greedy path on $P_{f_{m,n}}$ if and only if $c_{\sigma(i)} \neq c_{\sigma(j)}$ for $1 \leq i < j \leq \min(k,n)$. The face $\mathcal{P}'(m,n)^c$ is linearly isomorphic to the face $\mathcal{P}(m,n)^{\tilde{c}}$ for the function $\tilde{c} = (c_1 - c_{m+1}, c_2 - c_{m+1}, \ldots, c_m - c_{m+1})$. From the definition of vertices of $\mathcal{P}(m,n)$, we see that $\mathcal{P}(m,n)^{\tilde{c}}$ is a vertex if and only if the same condition is satisfied. □

Corollary 7.4. For $m \geq n \geq 1$, the $(m,n)$-partial permutahedron is combinatorially isomorphic to the nestohedron $\Delta(U(m,n))$ for the union closed set $U(m,n) = \{S \cup \{m+1\} : S \subseteq [m], |S| > m - n \text{ or } S = \emptyset\}$.

With this description, we are able to prove Conjecture 5.24. Our proof is analogous to their proof of the case in which $m = n$.

Theorem 7.5. Faces of $\mathcal{P}(m,n)$ are in bijection with flags of subsets of $[m]$ whose difference between largest and smallest nonempty subsets is at most $n - 1$. A face of $\mathcal{P}(m,n)$ is of dimension $k$ if and only if the corresponding flag has $k$ missing ranks.

Proof. It suffices to describe the nested set complex. We first define a bijection between nested sets and flags. Let $N \subseteq \mathcal{B}$ be a nested set. Consider the set $S = \{X \in N : m + 1 \in X\}$. Since a nested set contains the maximal element of the building set, $[m+1] \in N$ meaning that $S$ is nonempty. Furthermore, $N \setminus S$ must consist entirely of singletons, since every set in $\mathcal{B}$ of size at least 2 contains $m+1$. Let $S_0 = \{y : \{y\} \in N \setminus S\}$. Let $x \in S_0$, and let $T \subseteq S$. Then, by the nested set axioms, either $x \in T$ or $T \cup x \notin \mathcal{B}$. However, $|T \cup x| \geq |T| \geq m + n - 1$ and $m + 1 \in T \cup x$, so $T \cup x \in \mathcal{B}$. Hence, for all $x \in S_0$, we must have that $x \in T$ for all $T \subseteq S$.

With these observations about the nested set structure in mind, we are prepared to define our bijection. Note that any two sets in $S$ must intersect, since they all contain $m+1$, so by the nested set axioms, $S$ must be a flag of subsets $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k = [m + 1]$ in $[m + 1]$. Then the flag we associate to $N$ is exactly $S_1 \setminus (S_0 \cup m + 1) \subseteq S_2 \setminus (S_0 \cup m + 1) \subseteq \cdots \subseteq S_k \setminus (S_0 \cup m + 1) = [m + 1] \setminus (S_0 \cup m + 1)$.

Note that $|S_1| \geq m - n + 1$, so $|[m + 1] \setminus S_1| \leq n - 1$. Hence, this map is well-defined. To see it is a bijection, start with a chain $T_1 \subseteq T_2 \subseteq \cdots \subseteq T_k$. Then the corresponding nested set is the following: $N = \{T_i \cup [m + 1] \setminus T_k : i \in [k]\} \cup \{\{x\} : x \in [m] \setminus T_k\}$.

Since $|T_k \setminus T_1| \leq n - 1$, $|T_1 \cup ([m] \setminus T_k)| \geq m - (n - 1) = m - n + 1$, which is precisely the condition necessary to ensure that each $T_i \cup ([m + 1] \setminus T_k)$ is in the building set. It remains to show that $N$ satisfies the nested set axioms, but this is immediate since a flag of building sets and collection of singletons contained in every set will always satisfy the nested set axioms so long as the union of the singletons is not the minimal element of the flag. The union of the singletons cannot be a minimal element of the flag, since $m + 1$ is not contained in the singletons. Hence, this map is well-defined and is clearly of the inverse of the previous map. Thus, we have a bijection between the face lattices.

It remains to understand how the grading is mapped via this bijection using the notion of missing rank. Note that each face in the nested set complex is given by adding sets to $[m+1]$. In that sense, the nested set $\{m+1\}$ corresponds to the trivial $m$ dimensional face. Each face is attained by adding compatible sets to $[m+1]$ one at a time. A compatible set to a given system is either a singleton that appears in all sets containing $n+1$ or a set in the building set containing $m+1$ that is contained in or contains all the sets containing $m+1$. Under the bijection, adding a singleton corresponds to removing a single element from every set in the current chain, and adding a new set containing $m+1$
corresponds to adding a set in the chain. Removing a single element from every chain reduces the number of missing ranks by 1 by decreasing the size of the maximal rank set without affecting any of the subsets considered for missing ranks. Similarly, adding a new set to the chain decreases the number of missing ranks by 1 by filling up a rank. Therefore, both of these operations reduce the numbers of missing ranks by precisely 1 and equivalently reduce the dimension by 1. Hence, this bijection takes the dimension statistic to the missing ranks statistic, which finishes the proof. □

8. Paths on Base Polytopes

In the previous section, we showed that for a polymatroid \([n], f\), the linear embedding

\[\{(x, f(E) - 1(x)) : x \in P_f\}\]

is the base polytope of an associated polymatroid \(f'\). Thus, \(1\)-monotone paths on \(P_f\) correspond to \(1_{[n]}\)-monotone paths on \(B_{f'} \subset \mathbb{R}^{n+1}\). This suggests the investigation of (coherent) monotone paths on polymatroid base polytopes or, equivalently, generalized permutahedra. In this section, we make some first observations on this very interesting but widely unexplored subject.

We begin with the permutahedron \(\Pi_{n-1}\). For a generic linear function \(c\), the oriented graph of the permutahedron is the Hasse diagram of the weak Bruhat order of the symmetric group \(S_n\); cf. [9]. The monotone paths are precisely maximal chains in the weak Bruhat order, which correspond to reduced words for the longest element \((n, n-1, \ldots, 2, 1)\). In this language, the results of Edelmann–Greene [19] can be interpreted as a bijection between the set of monotone paths on the permutahedron for a generic orientation and standard Young tableaux of staircase shape. These monotone paths have also appeared in the literature under the name of allowable sequences or stretchable/geometrically realizable sorting networks [2, 28, 40]. Furthermore, there is a canonical method of drawing a sorting network as a wiring diagram, which has appeared in the study of cluster algebras for describing the cluster remarkable description [3, 18]. Furthermore, there is a canonical method of drawing a sorting network as a wiring diagram, which has appeared in the study of cluster algebras for describing the cluster monotone paths are precisely maximal chains in the weak Bruhat order, which correspond to reduced expressions. Describing the coherent monotone paths on the permutahedron was left open in Billera–Sturmfels [7]. Coherent monotone paths for generic orientations on the permutahedron have also appeared in the literature previously under the name of sorting networks, for which their random behavior has a remarkable description [3, 18].

Even for matroids, not all \(c\)-monotone paths on base polytopes will be coherent. For example, the base polytope for the uniform matroid \(U_{4,2}\) is the hypersimplex \(\Delta(4, 2)\), which is linearly isomorphic to the octahedron \(C_3^\triangle = \text{conv}(\pm e_1, \pm e_2, \pm e_3)\). Monotone paths on cross-polytopes are studied in detail in [10] and Theorem 1.1 in this paper yields that there is always at least one incoherent monotone path for every generic linear function.

In this section, we will focus on (coherent) monotone paths on \(B_f\) with respect to the special linear functions \(1_S(x) = \sum_{i \in S} x_i\) for \(S \subseteq [n]\). We start with the case of matroids.

**Theorem 8.1.** Let \(M\) be a matroid with ground set \(E\) and \(S \subseteq E\). Every \(1_S\)-monotone path on the base polytope \(B_M\) is coherent.

**Proof.** Let \(M\) be a matroid rank \(r\) matroid on \(E\) and \(S \subseteq E\) a set of size \(s\). Two bases \(B, B'\) are adjacent on \(B_M\) if and only if \(B' = (B \setminus a) \cup b\) for \(a \in B\setminus B'\) and \(b \in B'\setminus B\). Moreover, \(B'\) is a \(1_S\)-improving neighbor of \(B\) if and only if \(a \not\in S\) and \(b \in S\). In particular \(1_S(1_B - 1_{B'}) = 1\).

It follows that an \(1_S\)-monotone path on \(B_M\) is a sequence of bases \(B_0, B_1, \ldots, B_m\) such that \(|B_0 \cap S|\) is minimal, \(|B_m \cap S|\) is maximal and \(B_i = (B_{i-1} \setminus a_i) \cup b_i\) for \(a_i \in B_{i-1} \setminus S\) and \(b_i \in S\) for \(i = 1, \ldots, m\).
The elements $a_1, \ldots, a_m, b_1, \ldots, b_m$ are all distinct. For $0 < \varepsilon \ll 1$ and we can define $w \in \mathbb{R}^E$ by

$$w_e := \begin{cases} 1 & \text{if } e \in B_0 \\ \varepsilon^i & \text{if } e = a_i \\ -\varepsilon^i & \text{if } e = b_i \\ 0 & \text{otherwise.} \end{cases}$$

Then $1_{B_0}$ is the unique maximizer of $w$ over $B_M \cap \{1_S(x) = |S \cap B_0|\}$ and Lemma 2.5 asserts that $B_0, \ldots, B_m$ is the coherent path with respect to $w$. \hfill \Box

Example 8.2 (Hypersimplices). Let $M = U_{n,k}$ be the uniform matroid of rank $k$ on $n$ elements and $S \subseteq [n]$ of cardinality $s$. The base polytope is the $(n, k)$-hypersimplex $\Delta(n, k)$ and the monotone path polytope $\Sigma_{1_S}(B_M) = \Sigma_{1_S}(\Delta(n, k))$ is normally equivalent to

$$\sum_{j=\max(0,k+s-n)}^{\min(k,s)} \Delta(s, j) \times \Delta(n-s, k-j).$$

Indeed, for every $k$-element subset $B \subseteq [n]$, we have $\max(0, k+s-n) \leq |B \cap S| \leq \min(k,s)$ and every value can be attained. For any $j$ in that range, $\Delta(n, k) \cap \{1_S(x) = j\}$ is the convex hull of $1_G + 1_H$, where $G \in \binom{\mathbb{N}}{k}^S$ and $H \in \binom{\mathbb{N} \setminus S}{k-j}$.

If $s = k$ and $n = 2k$, then $\Sigma_{1_S}(\Delta(n, k))$ is normally equivalent to $\Pi_k \times \Pi_k$.

It would be very interesting to further understand the combinatorics of $\Sigma_{1_S}(B_M)$. Observe that for $|S| \geq 2$, the polytopes $B_M \cap \{1_S(x) = j\}$ for $j \in \mathbb{Z}$ are 0/1-polytopes with edge directions $e_i - e_j + e_k - e_l$. Such polytopes were studied by Castillo and Liu [17] in the context of nested braid fans.

For $S = \{e\}$, where $e$ is not a loop or coloop, the monotone path polytope $\Sigma_{1_S}(B_M)$ is normally equivalent to $B_{M \setminus e} + B_{M/e}$, which is again a polymatroid base polytope. This holds in general.

Proposition 8.3. Let $(E, f)$ be a polymatroid and $S \subseteq E$ such that $|S| = 1$ or $|S| = |E| - 1$. Then $\Sigma_{1_S}(B_f)$ is a polymatroid base polytope.

Proof. Since $B_f \subset \{1(x) = f(E)\}$, the linear functions $1_{\{e\}}$ and $1_{E \setminus \{e\}}$ induce the same monotone paths. We may thus assume that $S = \{e\}$.

Let $\alpha \in \mathbb{R}$ such that $H_\alpha = \{x \in \mathbb{R}^E : x_e = \alpha\}$ meets $B_f$ in the relative interior. An edge of $B_f \cap H_\alpha$ is of the form $F \cap H_\alpha$, where $F \subset B_f$ is a face of dimension 2 that meets $H_\alpha$ in its relative interior. Now, $F$ is itself the base polytope of a polymatroid, which is either the Cartesian product of two 1-dimensional base polytopes or the base polytope of a polymatroid on three elements. In both cases, it follows that $F \cap H_\alpha = [u, v]$ and $u - v = \lambda(e_i - e_j)$ for some $i, j \in E \setminus \{e\}$. Since base polytopes are polytopes all whose edge directions are of the form $e_i - e_j$ for some $i, j \in E$, this proves the claim. \hfill \Box

We do not know if for a general polymatroid $(E, f)$ and $|S| = 1$, all $1_S$-monotone paths of $B_f$ are coherent nor what the corresponding polymatroid is.

Example 8.4 (Monotone paths on the associahedron). Let $\text{Ass}_{n-1} \subset \mathbb{R}^n$ be the Loday associahedron; cf. Example 3.15. Let $i \in [n]$. A binary tree $T$ corresponding to a vertex $v$ of $\text{Ass}_{n-1}$ maximizes $v_i$ (with value $i(n-i+1)$) if and only if $i$ is the root of $T$. It minimizes $v_i$ (with value 1) if and only if $i$ is a leaf. It is easy to see that for $S = \{i\}$, $\text{Ass}_{n-1}^{1_S}$ is linearly isomorphic to $\text{Ass}_{n-2}$. By removing the leaf $i$ from $T$ and relabelling every node $j > i$ to $j - 1$, this yields a plane binary tree on $n-1$ nodes and every such tree arises uniquely this way. Two trees $T$ and $T'$ correspond to adjacent vertices of $\text{Ass}_{n-1}$ if they differ by a rotation of two adjacent nodes $x$ and $y$. The tree $T'$ corresponds to a $1_S$-improving neighbor of $T$ iff the rotation decreased the distance of $i$ to the root. It follows
that for every tree $T$ with $i$ a leaf, $T$ is the starting point of a unique monotone path. The nodes that are rotated along the path, readily yield a weight $w$ which certifies that the path is coherent. In particular, since the starting point of the path determines the whole path, this shows that the polytopes $\text{Ass}_{n-1} \cap \{x_i = \alpha\}$ are weak Minkowski summands of $\text{Ass}_{n-1} \cap \{x_i = 1\} = \text{Ass}_{n-1}^{-1}$. Thus $\Sigma_1(\text{Ass}_{n-1})$ is normally equivalent to the associahedron $\text{Ass}_{n-2}$.

We conclude the section with a discussion of $1_S$-monotone paths on the permutahedron. Recall that a standard Young tableau (SYT) of shape $m \times n$ is a rectangular array filled with numbers from $1, \ldots, mn$ without repetitions and such that rows and columns are increasing top-to-bottom and left-to-right, respectively. Let $\text{SYT}(m,n)$ denote the collection of all such standard Young tableaux.

**Proposition 8.5.** For $1 \leq k < n$ let $S = \{1, \ldots, k\}$. The $1_S$-monotone paths on the permutahedron $\Pi_{n-1}$ starting from $(1, 2, \ldots, n)$ are in bijection with standard Young tableaux of shape $k \times (n-k)$.

**Proof.** A **rectangular lattice permutation** of size $k \times (n-k)$ is a sequence $a_1, a_2, \ldots, a_{k(n-k)} \in [k]$ such that

(i) The number of occurrences of $j \in [k]$ is $n - k$,

(ii) For any $1 \leq m \leq k(n-k)$, the number of occurrences of $i$ in $a_1, \ldots, a_m$ is at least as large as the number of occurrences of $i + 1$.

For a rectangular lattice permutation, one associates a rectangular SYT by starting with an empty rectangular array and appending the number $k$ in row $a_k$. Proposition 7.10.3 in [45] yields that this is a bijection from rectangular lattice permutations of size $k \times (n-k)$ to rectangular SYT of shape $k \times (n-k)$. We give an explicit bijection between monotone paths and rectangular lattice permutations.

Let $(1, 2, \ldots, n) = \sigma_0, \sigma_1, \ldots, \sigma_M$ be a $1_S$-monotone path on $\Pi_{n-1}$. For every $1 \leq h \leq M$, we have $\sigma_h - \sigma_{h-1} = e_i - e_j$ with $1 \leq i < j \leq n$ and we define $a_1, a_2, \ldots, a_M$ by $a_h := k + 1 - i$.

Each step along the path will always increase the value of an element of the first $k$ coordinates by precisely $1$. Since the first $k$ coordinates move from $(1, 2, \ldots, k)$ to $(n-k+1, \ldots, n)$, the total length of the path is $M = k(n-k)$. It also shows that every $i = 1, \ldots, k$ occurs exactly $n-k$ times in $a_1, \ldots, a_M$, which verifies (i). Moreover, $\sigma_h$ is a permutation for every $h$ and it can be seen that the first $k$ and the last $n-k$ entries of $\sigma_h$ are always increasing. This shows that (ii) is satisfied and hence $a_1, \ldots, a_M$ is a lattice permutation. This also shows that $\sigma_M$ is $(n-k+1, \ldots, n, 1, \ldots, n-k)$.

For a given rectangular lattice permutation $a_1, \ldots, a_{k(n-k)}$, we define a sequence of permutations $\sigma_0, \ldots, \sigma_{k(n-k)}$ as follows. We set $\sigma_0 := (1, \ldots, n)$ and for $h \geq 1$, we define $\sigma_h$ by swapping the values $\sigma_{h-1}(k+1-a_h)$ and $\sigma_{h-1}(k+1-a_h)+1$. Since $1 \leq k+1-a_h \leq k$, this increases the values on the first $k$ coordinates. The sequence is well-defined since by (i), every coordinate is swapped $n-k$ times for a larger one. Moreover, condition (ii) ensures that the first $k$ coordinates are increasing, which implies that the sequence is a $1_S$-monotone path.

For $n = 5$ and $k = 3$, a monotone path is given by

$$12345 \xrightarrow{e_3 - e_4} 12435 \xrightarrow{e_3 - e_5} 12534 \xrightarrow{e_2 - e_4} 13524 \xrightarrow{e_1 - e_4} 23514 \xrightarrow{e_2 - e_5} 24513 \xrightarrow{e_1 - e_5} 34512.$$ 

The corresponding rectangular lattice permutation is $1, 1, 2, 3, 2, 3$ and the rectangular SYT is

$$\begin{bmatrix}
1 & 2 \\
3 & 5 \\
4 & 6
\end{bmatrix}$$

As observed by Postnikov in Example 10.4 of [42], $\text{SYT}(k, n-k)$ is also in bijection with the *longest* monotone paths on the hypersimplex $\Delta(n, k)$ for a generic orientation. Thus, the monotone paths on
the permutohedron for these special orientations correspond exactly to the longest monotone paths on hypersimplices for generic orientations.

Not all rectangular SYT correspond to coherent monotone paths. Coherent $1_S$-monotone paths of $\Pi_{n-1}$ are related to realizable SYT by work of Mallows and Vanderbei [36]. For vectors $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$, the outer sum or tropical rank-1 matrix is the matrix $u \oplus v \in \mathbb{R}^{m \times n}$ with $(u \oplus v)_{ij} := u_i + v_j$. If $u_1 < u_2 < \cdots < u_m$ and $v_1 < v_2 < \cdots < v_n$ are sufficiently generic, then all entries of $u \oplus v$ are distinct and strictly increasing along rows and columns. Replacing every entry in $u \oplus v$ with its rank yields a SYT of shape $m \times n$, that Mallows and Vanderbei call realizable. For example, for $u = (0, 10, 11)$ and $v = (1, 3)$, this yields the SYT above. In [36], they ask which SYT are realizable. We write $\text{rSYT}(m, n) \subseteq \text{SYT}(m, n)$ for the collection of realizable SYT. Realizable SYT are closely related to coherent monotone paths.

To prove Theorem 5, we relate the monotone path polytopes to another class of polytopes. For a finite set $T \subset \mathbb{R}^d$ the sweep polytope [40] is defined as
\[
\text{SP}(T) := \frac{1}{2} \sum_{a,b \in T} [a - b, b - a].
\]
This is a zonotope whose vertices record possible orderings of $T$ induced by generic linear functions. Let us also write $Z(T) = \sum_{a \in T} [0, a]$ for the zonotope associated to $T$.

**Proposition 8.6** ([40, Prop. 2.10]). Let $T \subset \mathbb{R}^d$. For $c \in \mathbb{R}^d$ define $T' := \{ \frac{a}{\langle c, a \rangle} : a \in T, \langle c, a \rangle \neq 0 \}$ and $T'' := T \setminus T'$. Then the monotone path polytope $\Sigma_c(Z(T))$ is normally equivalent to $\text{SP}(T') + Z(T'')$.

In the case of the permutohedron and special orientations, we can be more explicit.

**Proposition 8.7.** Let $n \geq 1$ and $\emptyset \neq S \subseteq [n]$. The monotone path polytope $\Sigma_{1_S}(\Pi_{n-1})$ is normally equivalent to the sweep polytope $\text{SP}(T')$ for $T' = \{ e_i - e_j : i \in S, j \notin S \}$.

**Proof.** The permutohedron is normally equivalent to the zonotope $Z = Z(T)$ for $T = \{ e_i - e_j : i, j \in [n], i \neq j \}$. By Proposition 8.6, $\Sigma_{1_S}(Z(T))$ is normally equivalent to $\text{SP}(T') + Z(T'')$. Note that $T'$ consists of all vectors $e_i - e_j$ for $i \in S$ and $j \in S^c := [n] \setminus S$. For $i, k \in S$ and $j \in S^c$ arbitrary, $e_i - e_k = (e_i - e_j) - (e_k - e_j)$ is a generator for $\text{SP}(T')$ and hence $Z(T'')$ is a weak Minkowski summand of $\text{SP}(T')$. This means that $\Sigma_{1_S}(Z(T))$ is normally equivalent to $\text{SP}(T')$. 

Note that for $S \subseteq [n]$ of size $k \geq 1$, the set $T' = \{ e_i - e_j : i \in S, j \notin S \} \subset \mathbb{R}^n$ is linearly isomorphic to \{$(e_i, e_j) : i \in [k], j \in [n - k]$\}, that is, the vertices of the polytope $\Delta_{k-1} \times \Delta_{n-k-1}$.

**Corollary 8.8.** Let $n \geq 1$ and $S \subseteq [n]$ of size $k = |S| \geq 1$. Then $\Sigma_{1_S}(\Pi_{n-1})$ is combinatorially equivalent to the sweep polytope of the product of simplices $\Delta_{k-1} \times \Delta_{n-k-1}$.

Let us note Proposition 8.7 also implies that $\Sigma_{1_S}(\Pi_{n-1})$ is normally equivalent to $\Sigma_{1_S}(Z(T'))$ and, by Theorem 1.7 in [11], to the max-slope pivot polytope of $\Pi_{n-1}$ with respect to $1_S$.

**Proof of Theorem 5.** Up to symmetry, we may assume that $S = \{ 1, \ldots, k \}$. If $\sigma_0, \ldots, \sigma_M$ is a $1_S$-monotone path of $\Pi_{n-1}$, then the first $k$ coordinates of $\sigma_0$ are a permutation of $1, \ldots, k$. Likewise, the last $n - k$ coordinates are a permutation of $k + 1, \ldots, n$. Up to symmetry, we can assume that $\sigma_0 = (1, \ldots, n)$. Proposition 8.5 now proves the first claim.

As for the second claim, note that by Proposition 8.7, it suffices show that the vertices of $\text{SP}(T')$ with $T' = \{ e_i - e_j : 1 \leq i \leq k < j \leq n \}$ are in bijection to $\mathcal{S}_k \times \mathcal{S}_{n-k} \times \text{rSYT}(k, n - k)$.

Let $w \in \mathbb{R}^n$ be a generic linear function. The segments $[e_i - e_j, e_j - e_i]$ for $i, j \in S$ or $i, j \in S^c$ are subsumed by $\text{SP}(T')$. Hence, the sweep of $T'$ induced by $w$ totally orders $S$ and $S^c$. Without
loss of generality, we may assume that the ordering is the natural ordering on \( S \) and \( S^c \). Let us write \( w = (u, -v) \) with \( u \in \mathbb{R}^k \) and \( v \in \mathbb{R}^{n-k} \). Then \( \langle w, e_i - e_j \rangle = u_i + v_j \). The sweep of \( T' \) is thus determined by the ranks of \( u \oplus v \) and hence determines a unique element in \( \text{rSYT}(k, n-k) \). Conversely, every element in \( \text{rSYT}(k, n-k) \) is determined by some \((u, -v)\) up to a total order on \( S \) and \( S^c \). This proves the claim. 

This perspective in terms of coherent \( 1_S \)-monotone paths provides an alternative geometric perspective on realizable SYT.

**Theorem 8.9** ([36]). All rectangular standard Young Tableaux of shape \( 2 \times (n-2) \) are realizable.

**Corollary 8.10.** For \( S \subseteq [n] \) and \( |S| = 2 \), all \( 1_S \)-monotone paths on \( \Pi_{n-1} \) are coherent. The number of such paths is \( 2(n-2)! C_{n-2} \), where \( C_k \) denotes the \( k \)th Catalan number.

We give a short geometric proof of Theorem 8.9. To that end, we make the observation that, since the sweep polytopes are zonotopes, the normal fan of \( \Sigma_{1_S}(\Pi_{n-1}) \) for \( S = \{1, \ldots, k\} \) is given by the arrangement of hyperplanes 

\[
\left\{ (x, y) \in \mathbb{R}^k \times \mathbb{R}^{n-k} : x_i + y_k = x_j + y_l \right\}
\]

for \( i, j \in [m], k, l \in [n] \) with \( i \neq j \) and \( k \neq l \). Inspecting the proof of Theorem 5, we arrive at the following conclusion.

**Corollary 8.11.** For \( m, n \geq 1 \), realizable SYT of shape \( m \times n \) are in bijection to the regions of the arrangement of hyperplanes \( \left\{ (x, y) \in \mathbb{R}^m \times \mathbb{R}^n : x_i + y_k = x_j + y_l \right\} \) for \( i, j \in [m], k, l \in [n] \) with \( i \neq j \) and \( k \neq l \) restricted to the cone \( \left\{ (x, y) : x_1 \leq \cdots \leq x_m, y_1 \leq \cdots \leq y_n \right\} \).

**Proof of Theorem 8.9.** Let \( m = 2 \). the cone \( \left\{ (x, y) : x_1 \leq x_2, y_1 \leq \cdots \leq y_n \right\} \) has a 2-dimensional lineality space given by adding a constant to all coordinates of \( x \) and, independently, to \( y \). Since all the hyperplanes are linear, we may thus assume that \( x_1 = 0 \). We may also restrict to \( x_2 = 1 \) and count the number of regions in the cone \( C = \left\{ y \in \mathbb{R}^n : y_1 \leq \cdots \leq y_n \right\} \) induced by the hyperplanes 

\[
y_k - y_l = \pm 1
\]

for \( k \neq l \). The cone \( C \) is the fundamental cone for the braid arrangement and hyperplanes constitute the so called **Catalan arrangement**. The number of regions in \( C \) is well-known to be \( C_n \); see, for example, [46, Prop. 5.14] or [5] for the connection to Shi arrangements. 

Mallows and Vanderbei also discuss realizability of general rectangular SYT and show that the tableau 

\[
\begin{bmatrix}
1 & 2 & 6 \\
3 & 5 & 7 \\
4 & 8 & 9
\end{bmatrix}
\]

is not realizable.

**Corollary 8.12.** For \( |S| \geq 3 \) not all \( 1_S \)-monotone paths on \( \Pi_{n-1} \) are coherent.

All monotone paths being coherent on zonotopes in general is a strong restriction. In [20] it is shown that for a generic objective function all monotone paths being coherent implies that all cellular strings are coherent and they provide a complete characterization of the cases in which this arises. However, in this special case, the objective function \( 1_{\{1,2\}} \) is not generic, so their tools do not apply.

The combinatorics of the monotone path polytope for the permutahedron in other cases remains complicated. However, it is surprisingly natural and connected to applications through the motivation of Mallows and Vanderbei in [36]. We end on the open question of whether we can obtain a more robust description of the \( 1_S \)-monotone path polytopes of large classes of generalized permutahedra or the permutahedron itself.
Open Problem 1. For fixed $m, n \geq 3$, determine the (number of) realizable SYT of shape $m \times n$. Equivalently, determine the (number of) coherent monotone paths of $\Pi_{n-1}$ for special directions $1_S$ with $|S| \geq 3$.

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