On the FitzHugh - Nagumo Model

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Abstract

The initial value problem $\mathcal{P}_0$, in all of the space, for the spatio - temporal FitzHugh - Nagumo equations is analyzed. When the reaction kinetics of the model can be outlined by means of piecewise linear approximations, then the solution of $\mathcal{P}_0$ is explicitly obtained. For periodic initial data are possible damped travelling waves and their speed of propagation is evaluated. The results imply applications also to the non linear case.

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1 Introduction

One of the reaction diffusion systems which models various important biological phenomena is given by

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \varepsilon \frac{\partial^2 u}{\partial x^2} - v + f(u) \\
\frac{\partial v}{\partial t} &= bu - \beta v
\end{align*}
\]

(1.1)

where the appropriate class of functions $f(u)$ depends on the reaction kinetics of the model [1] - [4]. In the theory of nerve membranes, for example, the

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system (1.1) is related to the spatio-temporal FitzHugh - Nagumo equations (FHN) with

\[(1.2) \quad f(u) = u(a - u)(u - 1) \quad (0 < a < 1)\]

and where \(u(x,t)\) models the transmembrane voltage of a nerve axon at distance \(x\) and time \(t\), \(v(x,t)\) is an auxiliary variable that acts as recovery variable. Further, the diffusion coefficient \(\varepsilon\) and the parameters \(b, \beta\) are all non negative [5],[9].

In addition to the propagation of nerve action potentials, (1.1) - (1.2) govern other several biological and biochemical phenomena; the list of references is very long and the variety of analytical aspects examined is wide. [10] - [15]. Further, we must remark travelling pulses and periodic wave trains obtained by means of piecewise linear approximations of \(f(u)\) as

\[(1.3) \quad f(u) = \eta(u - a) - u \quad (0 < a < 1)\]

where \(\eta\) denotes the unit- step function.[16]- [21].

Typical boundary value problems related to the linear case (1.3) can be explicitly solved. Aim of this paper is the analysis of the initial value problem \(P_0\) in all of the space; the fundamental solution and the explicit solution of \(P_0\) are determined when (1.3) holds. More, in the non linear case of the FHN model given by (1.1)-(1.2), the problem \(P_0\) is reduced to appropriate integral equations whose kernels are functions characterized by basic properties. All this implies existence and uniqueness properties, together with a priori estimates.

2 Statement of the problem and results

Both the non linear source (1.2) and the linear approximation (1.3) involve a linear term \(-ku\) with \(k = a\) for (1.2), and \(k = 1\) for (1.3). As consequence, the system (1.1) becomes
\begin{align}
\left\{ \begin{array}{ll}
\psi_t - \varepsilon \psi_{xx} + k \psi + \psi = \varphi(\psi) \\
v_t + \beta \psi - b \psi = 0,
\end{array} \right. (x, t) \in \mathbb{R}
\end{align}

where \( \varphi(u) = u^2(a + 1 - u) \) when \( f \) is given by (1.2), while, in the linear case, \( \varphi(u) \) is equal to the constant \( \bar{\eta} \) that holds zero or one.

The initial-value problem \( P_0 \) related to (2.1) with \( \varphi(u) = \bar{\eta} \) is analyzed in the set

\begin{align}
\Omega_T &= \{(x, t) : x \in \mathbb{R}, \ 0 < t \leq T\},
\end{align}

with the conditions

\begin{align}
u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \mathbb{R}.
\end{align}

When the linear approximation of \( f \) holds, then the explicit solution of the problem \( P_0 \) can be obtained by means of functional transforms (Fourier with respect to \( x \) and Laplace as for \( t \)). If one puts formally

\begin{align}
\hat{u}(x, s) &= \int_{\mathbb{R}} u(x, t) e^{-st} dt, \quad \hat{v}(x, s) = \int_{\mathbb{R}} v(x, t) e^{-st} dt,
\end{align}

from (2.1) (2.3) one deduces:

\begin{align}
\hat{u}(x, s) &= \int_{\mathbb{R}} [u_0(\xi) + \bar{\eta} s] \hat{K}_0(\xi, s) d\xi - \int_{\mathbb{R}} v_0(\xi) \hat{K}_1(x - \xi, s) d\xi,
\end{align}

\begin{align}
\hat{v}(x, s) &= b \int_{\mathbb{R}} [u_0(\xi) + \bar{\eta} s] \hat{K}_1(x - \xi, s) d\xi - b \int_{\mathbb{R}} v_0(\xi) \hat{K}_2(x - \xi, s) d\xi + \frac{v_0(x)}{s + \beta},
\end{align}

where
\[ K_n(x, s) = \frac{e^{-\frac{|x|}{\sigma}}}{2\sqrt{\varepsilon} \sigma (s + \beta)^n} \quad (n = 0, 1, 2) \]

with \( \sigma^2 = s + k + \frac{b}{s + \beta} \).

Now, let us consider the fundamental solution

\[ g(x, y) = \frac{1}{2\sqrt{\pi \varepsilon y}} e^{-\frac{k y - \frac{y^2}{4\varepsilon y}}{}} \]

of the heat equation \( \varepsilon g_{xx} - g_y - kg = 0 \) and, for \( n = 0, 1, 2 \), let

\[ G_n(x, t) = \int_0^t e^{-\beta(t-\tau)} g(x, \tau) \left( \frac{t - \tau}{by} \right)^{n-1} J_{n-1} \left( 2\sqrt{b(t-\tau)y} \right) dy, \]

where \( J_n(z) \) denotes the Bessel function of first kind. Then, if one puts

\[ (2.10) \quad K_0 = g(x, t) + G_0(x, t), \quad K_i = G_i(x, t) \quad (i = 1, 2), \]

the following theorems hold.

**Theorem 2.1** - In the half-plane \( \Re s > \max(-k, -\beta) \), the Laplace integrals of \( K_n(x, t) \) \( (n = 0, 1, 2) \) converge absolutely for all \( |x| > 0 \), and one has \( \mathcal{L}_t K_n(x, t) = \hat{K}_n(x, s) \).

**Theorem 2.2** - The functions \( K_0, K_1, K_2 \) are \( C^\infty(\Omega_T) \) solutions of the integro differential equation:

\[ z_t - \varepsilon z_{xx} + kz + b \int_0^t e^{-\beta(t-\tau)} z(x, \tau) d\tau = 0 \]

and have the same basic properties of the fundamental solution (2.8) of the heat operator. Further, for \( i = 0 \) and \( i = 1 \), it results:
\[ K_i(x, t) = (\partial_t + \beta) K_{i+1}, \quad \lim_{t \to 0} K_{i+1} = 0 \quad (i = 0, 1) \]

while \( \lim_{t \to 0} K_0(x, t) = 0 \) only for \(|x| > 0\).

These properties assure the convergence of the convolutions

\[ K_n \ast \psi = \int_{\mathbb{R}} K_n(x - \xi, t) \psi(\xi) \, d\xi \quad (n = 0, 1, 2) \]

for all the functions that satisfy a growth condition of the form

\[ |\psi(x)| < c_1 \exp\left[c_2 |x|^\alpha + 1\right], \quad 0 < \alpha < 1 \]

with \( c_1 \) and \( c_2 \) positive constants.

Further, let consider the following functions

\[ N_1(t) = \frac{\bar{\eta} \beta}{b + \beta} \left[ 1 - e^{-1+\beta t} \cos(\gamma t) \right] + \frac{\bar{\eta}(2b + \beta - \beta^2)}{2\gamma(b + \beta)} e^{-1+\beta t} \sin(\gamma t) \]

\[ N_2(t) = \frac{\bar{\eta}}{b + \beta} \left[ 1 - e^{-1+\beta t} \cos(\gamma t) \right] - \frac{\bar{\eta}(1 + \beta)}{2\gamma(b + \beta)} e^{-1+\beta t} \sin(\gamma t), \]

where \( \gamma = \left[b - (\beta - 1)^2/4\right]^{1/2} \). Then, by (2.5)-(2.6) and the foregoing statements, the explicit solution of the linear problem \( P_0 \) is given by

\[
\begin{aligned}
  u &= N_1(t) + u_0 \ast K_0 - v_0 \ast K_1 \\
v &= N_2(t) + u_0 \ast K_1 - v_0 \ast K_2 + v_0(x) e^{-\beta t},
\end{aligned}
\]

and the following conclusion is deduced.
Theorem 2.3 - When the data \((u_0, v_0)\) are continuous functions that satisfy the growth condition (2.14), then the formulae (2.17) represent the unique solution of the problem \(\mathcal{P}_0\) in the class of solutions compatible with (2.14).

3 Travelling waves and a priori estimates

A first example of applications is related to the linear case and concerns the analysis of travelling waves. By the explicit solution (2.17), for instance, when \(\bar{\eta} = 0, v_0 = 0, u_0 = A \cos(wx)\), then one obtains:

\[
(3.1) \quad u = (\partial_t + \beta)w, \quad v = bw
\]

where

\[
(3.2) \quad w = \frac{A}{2\alpha} e^{-\frac{1+\epsilon w^2-\beta}{2} t} \left[ \sin(wx + \alpha t) - \sin(wx - \alpha t) \right]
\]

with \(\alpha = \left[ b - \left( \frac{1+\epsilon w^2-\beta}{2} \right)^2 \right]^{1/2} \). So, when \(b > \left( \frac{1+\epsilon w^2-\beta}{2} \right)^2 \), there exist damped travelling waves with speed equal to \(\alpha/w\).

Moreover, in the non linear case, by (2.5),(2.6) one deduces the following integral equations

\[
(3.3) \quad \begin{cases} 
    u = u_0 \ast K_0 - v_0 \ast K_1 + \int_0^t \varphi(u) \ast K_0 \, dt \\
    v = b u_0 \ast K_1 - b v_0 \ast K_2 + v_0(x) e^{-\beta t} + b \int_0^t \varphi(u) \ast K_1 \, dt
\end{cases}
\]

that imply a priori estimates. In fact, in the class of bounded solutions, if one puts
by means of the basic properties of the kernels $K_0, K_1 K_2$, one obtains:

$$
\begin{align*}
|u| < c_1 ||\varphi|| + c_2 ( ||u_0|| t + ||v_0|| ) E(t) \\
|v| < c_1 ||\varphi|| + c_2 ( ||u_0|| + ||v_0|| t ) E(t)
\end{align*}
$$

(3.5)

where

$$
E(t) = \frac{e^{-kt} - e^{-\beta t}}{\beta - k}
$$

(3.6)

and where the constants $c_1, c_2$ depend on the parameters $b, k, \beta$. Worthy of remark is the fact that the estimates (3.5) hold for all $t$, also when $T \to \infty$.

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