Two problems in graph Ramsey theory

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Abstract

We study two problems in graph Ramsey theory. In the early 1970’s, Erdős and O’Neil considered a generalization of Ramsey numbers. Given integers \(n, k, s\) and \(t\) with \(n \geq k \geq s, t \geq 2\), they asked for the least integer \(N = f_k(n, s, t)\) such that in any red-blue coloring of the \(k\)-subsets of \(\{1, 2, \ldots, N\}\), there is a set of size \(n\) such that either each of its \(s\)-subsets is contained in some red \(k\)-subset, or each of its \(t\)-subsets is contained in some blue \(k\)-subset. Erdős and O’Neil found an exact formula for \(f_k(n, s, t)\) when \(k \geq s + t - 1\). In the arguably more interesting case where \(k = s + t - 2\), they showed \(2^{-\binom{k}{2}}n < \log f_k(n, s, t) < 2n\) for sufficiently large \(n\). Our main result closes the gap between these lower and upper bounds, determining the logarithm of \(f_{s+1-t}(n, s, t)\) up to a multiplicative factor.

Recently, Damásdi, Keszegh, Malec, Tompkins, Wang and Zamora initiated the investigation of saturation problems in Ramsey theory, wherein one seeks to minimize \(n\) such that there exists an \(r\)-edge-coloring of \(K_n\) for which any extension of this to an \(r\)-edge-coloring of \(K_{n+1}\) would create a new monochromatic copy of \(K_k\). We obtain essentially sharp bounds for this problem.

1 Introduction

The Ramsey number \(R(n)\) is the smallest natural number \(N\) such that every two-coloring of the edges of \(K_N\) contains a monochromatic clique of size \(n\). The existence of these numbers is guaranteed by Ramsey’s theorem [13]. Classic results of Erdős and Szekeres [8] and Erdős [6] imply \(2^{n/2} < R(n) < 4^n\) for every \(n \geq 3\). While there have been several improvements on these bounds (see [3, 14, 16]), the constant factors in the exponents have remained unchanged for over seventy years. Given these difficulties, it is natural that the field has stretched in different directions. One such direction is to try to generalize Ramsey’s theorem.

1.1 A generalization of Ramsey numbers

In the early 1970’s, Erdős and O’Neil [7] considered the following generalization of Ramsey numbers. Given integers \(n, k, s\) and \(t\) with \(n \geq k \geq s, t \geq 2\), define \(N = f_k(n, s, t)\) to be the minimum integer with the property that, in any red-blue coloring of the \(k\)-subsets of an \(N\)-element set, there exists a subset of size \(n\) for which either each of its \(s\)-subsets is contained in some red \(k\)-subset, or each of its \(t\)-subsets is contained in some blue \(k\)-subset. For example, \(f_2(n, 2, 2)\) is the Ramsey number \(R(n)\). It is worth noting that a variant of \(f_k(n, 2, 2)\) has been studied recently in [10, 15]. Erdős and O’Neil [7] showed \(f_k(n, s, t) = 2n - s - t + 1\) when \(k = s + t - 1\). The next interesting case occurs for \(k = s + t - 2\), where they proved \(2^{\binom{k}{2}}n < f_k(n, s, t) < 4^n\) assuming \(n\) is sufficiently large. They claimed that the upper bound can be improved further to \(f_k(n, s, t) < 2^{\binom{k}{2}}n\). This result is interesting as it indicates the separation between \(f_{s+t-2}(n, s, t)\) and \(R(n)\).

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The aforementioned lower and upper bounds for \( f_{s+t-2}(n, s, t) \) are far apart if we allow \( s \) and \( t \) to grow with \( n \). In this paper, we close this gap, determining the logarithm of \( f_{s+t-2}(n, s, t) \) asymptotically.

**Theorem 1.1.** For all sufficiently large \( n \) and all \( s, t \) with \( 2 \leq s \leq t \leq \log n/(120 \log \log n) \), we have

\[
2^{\frac{n}{\log n}} \log \left( \frac{2^{n^2}}{n^2} \right) \leq f_{s+t-2}(n, s, t) \leq 2^{\frac{n}{\log n}} \log \left( \frac{2^{n^2}}{n^2} \right).
\]

One can reduce finding \( f_{s+t-2}(n, s, t) \) to a graph Ramsey problem. Let \( g(n, s, t) \) be the smallest integer \( N \) such that in any \( N \)-vertex graph \( G \) one can find an \( n \)-element vertex set which does not contain both a size-\( s \) clique and a size-\( t \) independent set. The key point (see Lemma 2.1) is

\[
f_{s+t-2}(n, s, t) = g(n, s, t) \text{ for } n \geq s + t - 2.
\]

Theorem 1.1 then follows from the following bounds on \( g(n, s, t) \), which may be of independent interest.

**Proposition 1.2.** Suppose that \( n \) is sufficiently large and \( 2 \leq s \leq t \leq \log n/(120 \log \log n) \). Then

\[
2^{\frac{n}{\log n}} \log \left( \frac{2^{n^2}}{n^2} \right) \leq g(n, s, t) \leq 2^{\frac{n}{\log n}} \log \left( \frac{2^{n^2}}{n^2} \right).
\]

### 1.2 Semisaturated Ramsey numbers

Given an integer \( r \geq 2 \), let \( G_r \) denote the family of complete graphs whose edges are colored with \( r \) colors \( 1, 2, \ldots, r \). For \( G, G' \in G_r \), we say \( G' \) extends \( G \) if \( G' \) can be obtained from \( G \) by iteratively adding a new vertex and colored edges connecting the new vertex with the existing ones. A member \( G \) of \( G_r \) is called \((r, K_k)\)-semisaturated if every \( G' \in G_r \) that extends \( G \) must contain a new monochromatic \( K_k \).

We are interested in the smallest size an \((r, K_k)\)-semisaturated graph can have, that is,

\[
\text{ssat}_r(K_k) := \min \{|V(G)| : G \in G_r \text{ is } (r, K_k)\text{-semisaturated}\}.
\]

From the definition, it is clear that \( \text{ssat}_2(K_k) < R(k) \), and more generally, \( \text{ssat}_r(K_k) \) is less than the \( r \)-color Ramsey number of \( K_k \). Damásdi et al. [4] initiated the study of \( \text{ssat}_r(K_k) \), establishing the following result.

**Theorem 1.3** (Damásdi, Keszegh, Malec, Tompkins, Wang, Zamora).

(i) \( (r - 1)k^2 - (3r - 4)k + (2r - 3) \leq \text{ssat}_r(K_k) \), and the equality holds when \( r = 2 \).

(ii) \( \text{ssat}_r(K_k) \leq (k - 1)^r \).

(iii) \( \text{ssat}_r(K_k) \leq 48k^2r^{-k^2} \).

The upper bound from (ii) is polynomial in \( k \) for fixed \( r \), while the upper bound from (iii) is polynomial in \( r \) for fixed \( k \). The second main result of this paper gives rather sharp estimates for \( \text{ssat}_r(K_k) \) in these regimes.

**Theorem 1.4.**

(i) For fixed \( r \geq 2 \), one has

\[
\text{ssat}_r(K_k) = \Theta(k^2).
\]
(ii) For all \( k \geq 3 \), there exists a constant \( C = C(k) > 0 \) such that for all \( r \geq 3 \),
\[
\frac{1}{2}r^2 \leq \ssat_r(K_k) \leq C(\log r)^8(k-1)^2r^2.
\]

The proof of the upper bound in Theorem 1.4 (i) requires \( k \) to be large with respect to \( r \). Moreover, the exponent of the \( \log r \) factor in the second upper bound depends on the size of the clique. Therefore, we also prove an upper bound on \( \ssat_r(K_k) \) which is polynomial in both parameters.

**Theorem 1.5.** For every \( k \geq 2 \) and every \( r \geq 3 \),
\[
\ssat_r(K_k) \leq 8k^3r^3.
\]

Our proofs of Theorems 1.4 and 1.5 exploit connections between \( \ssat_r(K_k) \) and the vertex Folkman numbers. To learn more about the vertex Folkman numbers, we refer the interested readers to [12] and the references therein.

### 1.3 Organization and notation

The paper is organized as follows. In Section 2 we prove Theorem 1.1. In Section 3 we justify Theorems 1.4 and 1.5. We close this paper with some concluding remarks.

We write \([N]\) for the set \( \{1, 2, \ldots, N\} \), while \( \binom{X}{k} \) is the family of all \( k \)-element subsets of a set \( X \). Given two functions \( f \) and \( g \) of some underlying parameter \( n \), we employ the following asymptotic notation: \( f = o(g) \) or \( g = \omega(f) \) means that \( \lim_{n \to \infty} f(n)/g(n) = 0 \), while \( f = O(g) \) means that \( |f| \leq Cg \) for some absolute constant \( C > 0 \). We omit floor and ceiling signs where the argument is unaffected. Finally, all logarithms are to the base 2.

### 2 Proof of Theorem 1.1

Recall that \( g(n, s, t) \) is the smallest integer \( N \) such that in any \( N \)-vertex graph \( G \) one can find an \( n \)-element vertex set which does not contain both a size-\( s \) clique and a size-\( t \) independent set. On the other hand, \( f_k(n, s, t) \) is the minimum integer \( N \) with the property that, in any red-blue coloring of the \( k \)-subsets of an \( N \)-element set, there exists a subset of size \( n \) for which either each of its \( s \)-subsets is contained in some red \( k \)-subset, or each of its \( t \)-subsets is contained in some blue \( k \)-subset. Our starting point is the following connection between \( f_k(n, s, t) \) and \( g(n, s, t) \).

**Lemma 2.1.** Provided \( s, t \geq 2 \) and \( n \geq s + t - 2 \) we have
\[
f_{s+t-2}(n, s, t) = g(n, s, t).
\]

**Proof.** Throughout the proof, set \( k = s + t - 2 \). In order to show \( f_k(n, s, t) \leq g(n, s, t) \), one needs to justify that in any red-blue coloring \( \chi \) of the \( k \)-subsets of \( [g(n, s, t)] \) there exists a set of size \( n \) such that either each of its \( s \)-subsets is contained in some red \( k \)-subset, or each of its \( t \)-subsets is contained in some blue \( k \)-subset. For this purpose, we define a graph \( G_\chi \) on the vertex set \( [g(n, s, t)] \) as follows. Consider a pair \( \{x, y\} \) of vertices. If \( \{x, y\} \) is contained in some \( s \)-subset \( S \) for which all \( k \)-subsets containing \( S \) are blue, then we assign \( \{x, y\} \) to \( E(G_\chi) \). If \( \{x, y\} \) is contained in some \( t \)-subset \( T \) such that all \( k \)-subsets containing \( T \) are colored red, then we declare \( \{x, y\} \notin E(G_\chi) \). Finally, if \( \{x, y\} \) satisfies neither of the previous two conditions we arbitrarily decide whether \( \{x, y\} \) belongs to \( E(G_\chi) \) or not. The graph \( G_\chi \) is well-defined: if \( \{x, y\} \) satisfies both conditions, then \( |S \cup T| \leq s + t - 2 = k \) and every \( k \)-subset containing \( S \cup T \) receives two colors, which is impossible.
By the definition of $g(n, s, t)$, one can find an $n$-element vertex set $V$ which does not contain either a size-$s$ clique, or a size-$t$ independent set. Without loss of generality we can assume that the former case occurs. We claim that every $s$-subset $S$ of $V$ is contained in some red $k$-subset of $\lfloor g(n, s, t) \rfloor$, implying $f_k(n, s, t) \leq g(n, s, t)$. Indeed, if $S$ is contained in blue $k$-subsets only, then $S$ induces a clique of size $s$ in $G_X$, a contradiction.

What’s left is to show that $f_k(n, s, t) \geq g(n, s, t)$. To this end, let $N = g(n, s, t) - 1$. According to the definition of $g(n, s, t)$, there exists a graph $G$ on $[N]$ so that every set of $n$ vertices contains both a clique of size $s$ and an independent set of size $t$. Consider a red-blue coloring of the $k$-subsets of $[N]$ defined as follows. Given a $k$-subset $K$, we color $K$ blue if $G[K]$ contains a clique of size $s$, color $K$ red if $G[K]$ contains an independent set of size $t$, and color $K$ arbitrarily otherwise. The coloring is well-defined: a $k$-subset cannot host both a clique of size $s$ and an independent set of size $t$ as they would intersect at two (or more) vertices. Obviously, a size-$s$ clique is contained in blue $k$-subsets only, and a size-$t$ independent set is contained in red $k$-subsets only. It thus follows from the assumption on $G$ that every set of $n$ vertices contains an $s$-subset all of whose containing $k$-subsets are blue and a $t$-subset all of whose containing $k$-subsets are red. Hence $f_k(n, s, t) \geq N + 1 = g(n, s, t)$, completing our proof. \hfill $\square$

Theorem 1.1 clearly follows Lemma 2.1 and Proposition 1.2. For reader’s convenience we restate Proposition 1.2 here.

**Proposition 1.2.** Suppose that $n$ is sufficiently large and $2 \leq s \leq \log n / (120 \log \log n)$. Then

$$2^{\frac{n}{\log(\frac{2}{s})}} \leq g(n, s, t) \leq 2^{\frac{n}{\log(\frac{2}{t})}}.$$ 

The rest of this section is devoted to the proof of Proposition 1.2. Naturally, the proof is broken into two lemmas corresponding to the upper and lower bounds of $g(n, s, t)$.

**Lemma 2.2.** Provided $n$ is sufficiently large and $2 \leq s \leq t \leq n / \log n$, we have

$$g(n, s, t) \leq 2^{\frac{n}{\log(\frac{2}{s})}}.$$ 

**Lemma 2.3.** Suppose $n$ is sufficiently large and $2 \leq s \leq t \leq \log n / (120 \log \log n)$. Then

$$g(n, s, t) \geq 2^{\frac{n}{\log(\frac{2}{t})}}.$$ 

We will prove Lemma 2.2 using the Erdős-Szekeres bound for the off-diagonal Ramsey numbers.

**Proof of Lemma 2.2.** The off-diagonal Ramsey number $R(a, b)$ is the smallest natural number $N$ such that any $N$-vertex graph $G$ contains either a clique of size $a$ or an independent set of size $b$. The Erdős-Szekeres bound [8] says $R(a, b) \leq \left(\frac{a+b}{a}\right)$ for any $a, b \geq 1$. In the case $1 \leq a \leq b$, this implies $R(a, b) \leq \left(\frac{2b}{a}\right) < 2^{a \log(\frac{2b}{a})}$. Thus, for any integer $m \geq 1$ we have

$$R\left(\frac{\log m}{\log(\frac{2}{s})}, \frac{t \log m}{s \log(\frac{2}{t})}\right) < 2^{\frac{\log m}{\log(\frac{2}{s})} \log(\frac{2}{t})} = 2^{\frac{m}{\log(\frac{2}{s})}}$$



We conclude that in any $m$-vertex graph one can find either a clique of size $\frac{\log m}{\log(\frac{2}{s})}$ or an independent set of size $\frac{t \log m}{s \log(\frac{2}{t})}$.
Let \( N := 2 \cdot 2^{3n \log(2t)} \), \( a := \frac{\log(N/2)}{\log(2t)} = \frac{3n}{t} \), and \( b := \frac{t \log(N/2)}{s \log(2t)} = \frac{3n}{s} \). Since \( 2 \leq s \leq t \leq n/\log n \), we have \( N \geq 2^{\frac{3n}{t}} \geq 2^{3\log n} = n^3 \). Moreover, as \( 2 \leq s \leq t \), we see that \( a \leq b \leq \frac{t \log N}{4} \). Consider any graph \( G \) on \( N \) vertices. From the above discussion, we know that \( G \) must contain either a clique of size \( a \) or an independent set of size \( b \). By removing this set and repeating \( s + t - 3 \) times, we obtain either \( t - 1 \) vertex disjoint cliques of the same size \( a \) or \( s - 1 \) vertex disjoint independent sets of the same size \( b \), because the number of removed vertices is always at most

\[
(s + t - 3) \cdot \max\{a, b\} \leq 2t \cdot \frac{t \log N}{4} \leq N/2
\]

for \( t \leq n/\log n \) and \( N \geq n^3 \). In the latter case, the union of these sets has at least \( (s - 1)b = 3(s - 1)n/s \geq 1.5n \) vertices and does not contain a clique of size \( s \). In the former, the union of those sets has cardinality at least \( (t - 1)a = 3(t - 1)n/t \geq 1.5n \) and does not contain an independent set of size \( t \). Therefore, \( g(n, s, t) \leq N = 2 \cdot 2^{3n \log(2t)} \leq 2^{4n \log(2t)/s(1+o(1))} \).

The proof of Lemma 2.3 uses a container theorem of Balogh and Samotij [2, Proposition 6.1].

**Lemma 2.4** (Balogh and Samotij). For all sufficiently large \( n \) and all \( r \) with \( 3 \leq r \leq \log n/(120 \log \log n) \), there exists a collection \( C \) of at most \( 2^{2r-1/(8r)} \) subgraphs of \( K_n \) such that:

(a) Each \( K_r \)-free subgraph of \( K_n \) is contained in some member of \( C \),

(b) Each \( G \in C \) has fewer than \( (1 - \frac{1}{2r}) \binom{n}{2} \) edges.

**Remark.** In [2], the above statement is proved with property (b) replaced by

(b') Each \( G \in C \) either has less than \( n^2/8 \) edges or it contains a subgraph \( G' \) with \( e(G') > e(G) - o(n^2) \) that has fewer than \( n^{r-1/2} \) copies of \( K_r \).

A supersaturation result of Balogh, Bushaw, Collares, Liu, Morris and Sharifzadeh [1, Theorem 1.2] shows property (b) follows from property (b').

We are now in a position to justify Lemma 2.3.

**Proof of Lemma 2.3.** Let \( N = 2^{3n \log(2t)} \), and \( p = \frac{s}{2rt} \log(2t) \). According to Lemma 2.4, there exists a collection \( C \) of at most \( 2^{2r-1/(8r)} \) subgraphs of \( K_n \) satisfying:

(a) Each \( K_s \)-free subgraph of \( K_n \) is contained in some member of \( C \),

(b) Each \( G \in C \) has fewer than \( (1 - \frac{1}{2s}) \binom{n}{2} \) edges.

Let \( G(n, p) \) denote the Erdős-Rényi random graph with \( n \) vertices and edge density \( p \). It follows from properties (a) and (b) that

\[
\mathbb{P}(G(n, p) \text{ is } K_s\text{-free}) \leq \sum_{G \in C} \binom{n}{2}^{-e(G)} \leq 2^{2r-1/(8s)} (1 - p)^{(1+o(1))n^2/(4s)}.
\]

By the same argument, we get \( \mathbb{P}(G(n, p) \text{ has no independent sets of size } t) \leq 2^{2r-1/(8t)} (1 - p + o(1))n^2/(4t) \).

Thus in \( G(N, p) \) the expected number of vertex sets of size \( n \) which does not contain both a clique
of size $s$ and an independent set of size $t$ is at most
\[
\binom{N}{n} \cdot 2^{n^{2-1/(8s)}} (1-p)^{(1+o(1))n^2/(4s)} + \binom{N}{n} \cdot 2^{n^{2-1/(8t)}} p^{(1+o(1))n^2/(4t)}
\leq \exp_2 \left( n \log N + n^2 \frac{1}{8s} - (1 + o(1)) \frac{p n^2}{4s} \right) + \exp_2 \left( n \log N + n^2 \frac{1}{8t} + (1 + o(1)) \log(p) n^2 \frac{n^2}{4t} \right)
= \exp_2 \left( \frac{epn}{16s} + n^2 \frac{1}{8s} - (1 + o(1)) \frac{p n^2}{4s} \right) + \exp_2 \left( \log \left( \frac{2et}{s} \right) + n^2 \frac{1}{8t} + (1 + o(1)) \log(p) n^2 \frac{n^2}{4t} \right)
= o(1),
\]
where in the second line we used the inequalities $\binom{N}{n} \leq N^n$ and $1 - p \leq 2^{-p}$, in the third line we substituted $\log N = \frac{n}{32t} \log(\frac{2et}{s})$ and $p = \frac{s}{2et} \log(\frac{2et}{s})$, and in the last passage we used the estimate $\log p \leq -\frac{1}{4} \log(\frac{2et}{s})$. This completes our proof.

\section{Proofs of Theorems 1.4 and 1.5}

We will use the following observation several times, often without referring to it explicitly.

\begin{observation}
Let $k \geq 3$ and $s \geq r \geq 2$. Suppose that $G_1, \ldots, G_s$ are edge-disjoint subgraphs of the complete graph on $V$ such that for every $i \in [s]$ and for every $U \in \binom{V}{\lfloor |V|/r \rfloor}$ the graph $G_i[U]$ contains a copy of $K_{k-1}$. Then
\[
\ssat_r(K_k) \leq |V|.
\]
We prove Part (i) of Theorem 1.4 via an explicit construction.

\begin{theorem}
\begin{enumerate}[label=(oman*)]
\item For fixed $r \geq 2$, one has \[\ssat_r(K_k) = \Theta(k^2).\]
\item For all $k \geq 3$, there exists a constant $C = C(k) > 0$ such that for all $r \geq 3$, \[\frac{1}{4} r^2 \leq \ssat_r(K_k) \leq C(\log r)^{8(k-1)^2} r^2.\]
\end{enumerate}
\end{theorem}

\begin{proof}[Proof of Theorem 1.4 (i)] From Theorem 1.3 (i), we learn that $\ssat_r(K_k) = \Omega_r(k^2)$. To complete the proof, it remains to show that $\ssat_r(K_k) = O_r(k^2)$, a task we now begin.
Without loss of generality we may assume that $k \geq 6r$. By Chebyshev’s Theorem there exists a prime $q$ with $3rk \leq q \leq 6rk$. Let $\mathbb{F}_q^2$ be the affine plane over the $q$-element field $\mathbb{F}_q$, with point set $\mathcal{P}$ and line set $\mathcal{L}$. The common vertex set of our graphs $G_1, \ldots, G_r$ is $V := \mathcal{P}$. Note that $n := |V| = q^2 \leq 36r^2 k^2$. We partition $\mathcal{L}$ arbitrarily into $r$ families $\mathcal{L}_1, \ldots, \mathcal{L}_r$, each consists of $\frac{q^2 + q}{r}$ lines. As the edges of $G_i$, we take exactly those pairs $u, v \in \mathcal{P}$ which lie in a line from $\mathcal{L}_i$.
Since there is exactly one line passing through two given points, the graphs $G_i$ form an $r$-edge-coloring of the complete graph on $V$. To finish the proof, it suffices to show that for every $i \in [r]$ and every subset $U \in \binom{V}{\lfloor |V|/r \rfloor}$ the graph $G_i[U]$ contains a copy of $K_k$.
For a line $\ell \in \mathcal{L}_i$, let $p_{\ell}$ be the number of points $p \in U$ such that $p \in \ell$. It follows from a point-line
following \[9\], we call a sequence of pairwise edge-disjoint graphs $G_1, \ldots, G_r$ on the same vertex set $V$, a color pattern on $V$; we assign color $i$ to all the edges of $G_i$. A color pattern $G_1, \ldots, G_r$ is called $K_k$-free if none of the $G_i$ contains $K_k$ as a subgraph. A graph with colored vertices and edges is called strongly monochromatic if all its vertices and edges have the same color. An $r$-coloring is a function $\chi: V \to [r]$.

Define $P_r(k-1)$ to be the least integer $n$ such that there exists a $K_k$-free color pattern $G_1, \ldots, G_r$ on an $n$-element vertex set $V$ with the property that any $r$-coloring of $V$ contains a strongly monochromatic $K_{k-1}$.

Inspecting the definitions of $\text{ssat}_r(K_k)$ and $P_r(k-1)$, we see that $\text{ssat}_r(K_k) \leq P_r(k-1)$. Moreover, it follows from [9, Lemma 4.3] that $P_r(k-1) \leq C(\log r)^{8(k-1)^2} r^2$, where $C = C(k) > 0$ is a constant.

It is worth pointing out that [11, Theorem 1] implies $P_r(2) = \Theta(r^2 \log r)$. Therefore, we get the desired bound

$$\text{ssat}_r(K_k) \leq C(\log r)^{8(k-1)^2} r^2.$$ 

For the lower bound, we first establish the following recursion

$$\text{ssat}_r(K_k) \geq \text{sat}_{r-1}(K_k) + r/2 \quad \forall r \geq 2, k \geq 3. \quad (1)$$

Clearly, (1) forces $\text{ssat}_r(K_k) \geq \frac{1}{2} \sum_{i=2}^r i \geq \frac{1}{4} r^2$.

Our remaining task is to justify (1). Take a smallest $(r, K_k)$-semisaturated graph $G \in \mathcal{G}_r$. Then $G$ has $n = \text{sat}_r(K_k)$ vertices. Theorem 1.3 (i) implies $n \geq (r-1)k^2 - (3r-4)k + (2r-3) \geq r$.

For $i \in [r]$, denote by $G_i$ the $i$-th color class of $G$. Some color class, say $G_1$, has at most $\frac{1}{r} \binom{n}{2}$ edges. By Turán’s theorem applied to the complement of $G_1$, $G_1$ contains an independent set $I$ of cardinality $|I| \geq \frac{n^2}{n + 2e(G_1)} \geq \frac{n^2}{n + n(n-1)/r} \geq r/2$, as $n \geq r$. Since $I$ does not contain $K_{k-1}$ at all, the restriction of $G$ to the vertex set $V(G) \setminus I$ is $(r-1, K_k)$-semisaturated. It follows that

$$\text{sat}_r(K_k) = |V(G) \setminus I| + |I| \geq \text{sat}_{r-1}(K_k) + r/2,$$

completing the proof.

We close this section with a proof of Theorem 1.5. Once again it is done via an explicit construction.
Theorem 1.5. For every $k \geq 2$ and every $r \geq 3$, 

$$\text{ssat}_r(K_k) \leq 8k^3r^3.$$ 

Proof. We learn from Chebyshev’s Theorem that the interval $[kr, 2kr]$ contains a prime number $q$. Let $\mathbb{F}_q$ be the finite field of order $q$. The vertex set of our graph will be $V := \mathbb{F}_q^3$. By the choice of $q$, $n := |V| = q^3 \leq 8k^3r^3$. For each $\lambda \in \mathbb{F}_q$, we will define an incidence structure $\mathcal{I}_\lambda = (V, \mathcal{L}_\lambda)$ where $\mathcal{L}_\lambda$ is a family of lines in $\mathbb{F}_q^3$. For $\lambda \in \mathbb{F}_q$, let 

$$C_\lambda := \{(1, \lambda, \mu) : \mu \in \mathbb{F}_q\}.$$ 

Clearly, $C_{\lambda_1} \cap C_{\lambda_2} = \emptyset$ if $\lambda_1 \neq \lambda_2$.

A line in $\mathbb{F}_q^3$ is a set of the form $\ell_{s,v} = \{\beta s + v : \beta \in \mathbb{F}_q\}$ for some $s \in \mathbb{F}_q^3 \setminus \{(0,0,0)\}$ and $v \in \mathbb{F}_q^3$, where $s$ is called the slope. We define 

$$\mathcal{L}_\lambda := \{\ell_{s,v} : s \in C_\lambda, v \in \mathbb{F}_q^3\}.$$ 

Since each line contains exactly $q$ points, $|\mathcal{L}_\lambda| = |C_\lambda|q^3 = q^3$. Each structure $\mathcal{I}_\lambda$ enjoys the following properties:

(P1) Every point $v \in V$ is contained in $q$ lines from $\mathcal{L}_\lambda$;

(P2) Any two points lie in at most one line;

Furthermore, for $\lambda_1 \neq \lambda_2$, we have

(P3) $\mathcal{L}_{\lambda_1} \cap \mathcal{L}_{\lambda_2} = \emptyset$.

Each incidence structure $\mathcal{I}_\lambda = (V, \mathcal{L}_\lambda)$ gives rise to a graph $G_\lambda$ with vertex set $V$, and edge set 

$$E(G_\lambda) = \{xy : x, y \in \ell \text{ for some } \ell \in \mathcal{L}_\lambda\}.$$ 

By (P3), the graphs $G_\lambda$ form a partial $q$-edge-coloring of the complete graph on $V$. To establish the theorem, it suffices to prove that for every $\lambda \in \mathbb{F}_q$ and every subset $U \subseteq \binom{V}{\lfloor |V|/r \rfloor}$ the graph $G_\lambda[U]$ contains a copy of $K_k$. Indeed, according to (P1), each point $v \in U$ is contained in $q$ lines from $\mathcal{L}_\lambda$. By the pigeonhole principle, we thus can find a line $\ell \in \mathcal{L}_\lambda$ such that 

$$|\ell \cap U| \geq \frac{q|U|}{|\mathcal{L}_\lambda|} \geq \frac{q \cdot (q^3/r)}{q^3} = \frac{q}{r} \geq k.$$ 

It follows that $G_\lambda[\ell \cap U]$ contains a copy of $K_k$. This completes the proof. \qed

4 Concluding remarks

We have obtained good bounds for two Ramsey-type problems by exploring their connections to other well-studied problems. In particular, we have shown that

$$\frac{1}{4}r^2 \leq \text{ssat}_r(K_k) \leq C_k(\log r)^{8(k-1)^2}r^2,$$

which is tight up to the logarithmic factor. However, it is not clear whether the lower bound or the upper bound is closer to the truth. This naturally leads us to the following modest questions.
Question 4.1. Is it true that $\text{ssat}_r(K_k) = \omega(r^2)$ as $r \to \infty$?

Question 4.2. Does there exist a constant $C$ (independent of $k$) such that $\text{ssat}_r(K_k) = O_k((\log r)^C r^2)$?

In [4], Damásdi et al. also studied the behavior of a sibling of the function $\text{ssat}_r(K_k)$. Let $\mathcal{G}_r$ denote the family of complete graphs whose edges are colored with $r$ colors (numbered by $1, 2, \ldots, r$). A member $G$ of $\mathcal{G}_r$ is called $(r, K_k)$-saturated if for every $i \in \{r\}$, the graph $G$ does not contain a monochromatic $K_k$ of color $i$, but every $G' \in \mathcal{G}_r$ that extends $G$ must contain a monochromatic $K_k$.

Let

$$\text{sat}_r(K_k) := \min\{|V(G)| : G \in \mathcal{G}_r \text{ is } (r, K_k)\text{-saturated}\}.$$ 

From the definitions of $\text{sat}_r(K_k)$ and $\text{ssat}_r(K_k)$, we see that $\text{ssat}_r(K_k) \leq \text{sat}_r(K_k)$. Unlike $\text{ssat}_r(K_k)$, our understanding of $\text{sat}_r(K_k)$ is still in its infancy state: the only known upper bound, obtained by Damásdi et al. [4], says that $\text{sat}_r(K_k) \leq (k-1)^r$. The obvious open questions here are the following.

Question 4.3. Does there exist an absolute constant $C > 0$ such that $\text{sat}_r(K_k) = O_r(k^C)$?

Question 4.4. Is there an absolute constant $C > 0$ such that $\text{sat}_r(K_k) = O_k(r^C)$?

The triangle case of Question 4.4 is of particular interest to us.

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