On the Galerkin approximation and strong norm bounds for the stochastic Navier-Stokes equations with multiplicative noise

Igor Kukavica, Kerem Uğurlu, and Mohammed Ziane

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Department of Mathematics, University of Southern California, Los Angeles, CA 90089
e-mails: kukavica@usc.edu, kugurlu@usc.edu, ziane@usc.edu

Abstract

We investigate the convergence of the Galerkin approximation for the stochastic Navier-Stokes equations in an open bounded domain $O$ with the non-slip boundary condition. We prove that

$$E \left[ \sup_{t \in [0,T]} \phi_1(\|u(t) - u^n(t)\|_V^2) \right] \to 0$$

as $n \to \infty$ for any deterministic time $T > 0$ and for a specified moment function $\phi_1(x)$ where $u^n(t, x)$ denotes the Galerkin approximation of the solution $u(t, x)$. Also, we provide a result on uniform boundedness of the moment $E[\sup_{t \in [0,T]} \phi(\|u(t)\|_V^2)]$ where $\phi$ grows as a single logarithm at infinity. Finally, we summarize results on convergence of the Galerkin approximation up to a deterministic time $T$ when the $V$-norm is replaced by the $H$-norm.

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1 Introduction

In this paper, we address the convergence properties of the Galerkin approximation to the stochastic Navier-Stokes equations and obtain new estimates on the convergence in the strong norm.

The stochastic Navier-Stokes equations (SNSE) in a smooth bounded domain $O \subseteq \mathbb{R}^2$ with a multiplicative white noise read

$$du + ((u \cdot \nabla)u - \nu \Delta u + \nabla p)dt = f dt + g(u)dW$$
$$\nabla \cdot u = 0$$
$$u(0) = u_0$$

(1.1)
viscosity, whereas $f$ stands for the deterministic force. Also, $g(u)W = \sum_k g_k(u)e_k W_k$ stands for an infinite dimensional Brownian motion, where each $W_k$ is the standard one dimensional Brownian motion and $g_k(u)$ are the corresponding Lipschitz coefficients.

The study of the SNSSE was initiated by Bensoussan and Temam in 1973 [BT], and the equations have been extensively studied since then ([BF, F, GTW, Ku, KS, PR]). The well-posedness in $L^2$ was considered by Breckner [B], while the existence in Sobolev spaces $W^{1,p}$, where $p > 2$, was obtained by Brzezniak and Peszat [BP] as well as by Mikulevicius and Rozovsky [MR1]. Finally, the local existence in $H^1$ was proven in [GZ], where a method was introduced which extends also to less regular Sobolev spaces. For a comprehensive treatment of the subject of SNSSE, we refer the reader to the books by Vishik and Fursikov [FV], Capinski and Cutland [CC], and Flandoli [F].

As in the case of the deterministic NSE, the solutions are commonly constructed as a limit of solutions of the Galerkin system [BS, CF1, T]. In [B], Breckner proved that the solutions $u$ of the SNSSE can be approximated by solutions $u^n$ of the corresponding Galerkin systems. Namely, she proved that for all $t > 0$, we have
\[
E \left[ \|u(t) - u^n(t)\|^2_H + \int_0^t \|u(s) - u^n(s)\|^2_V \, ds \right] \to 0
\] as $n \to \infty$ (cf. (2.1) and (2.2) for the definitions of the spaces $H$ and $V$). In the absence of boundaries, her results extend easily to the case of stronger norms. More specifically, using the cancellation property
\[(B(u, u), Au) = 0\]
where $B$ is the bilinear form and $A$ the Stokes operator, which is valid in the case of periodic boundary conditions, one can easily obtain a stronger convergence result
\[
E \left[ \|u(t) - u^n(t)\|^2_V + \int_0^t \|u(s) - u^n(s)\|^2_{H^2} \, ds \right] \to 0
\] as $n \to \infty$, under suitable assumptions on the noise.

The goal of this paper is to address the convergence of the Galerkin approximation pointwise in time for the $V$ norm in the case of the Dirichlet boundary conditions when the cancellation property does not hold. In this case, it is easy to obtain results in this direction up to a suitable stopping time. However, the finiteness of the expected value of the second moment of the norm $\|u(t)\|^2_V$ for any fixed non-random time $t$ is an open problem. By the same token, it is not known whether the expected value of the supremum of $\|u(t) - u^n(t)\|^2_V$ up to a deterministic time converges to 0 as $n \to \infty$. A positive result in this direction was obtained in [KV], where it was proven that
\[
E \left[ \sup_{0 \leq t \leq T} \hat{\phi}(\|u\|^2_V) \right] < \infty
\] where
\[
\hat{\phi}(\tau) = \log(1 + \log(1 + \tau)), \quad \tau \in (0, \infty).
\]

The aim of this paper is twofold. First, we strengthen the main result in [KV] by showing that holds with
\[
\phi(\tau) = \log(1 + \tau)
\]
instead of $\tilde{\phi}$ (cf. Theorem 3.2 below). The second goal is to obtain the convergence of the Galerkin approximation in the $V$ norm. Namely, we prove that

$$E \left[ \sup_{[0,T]} \phi(\|u - u^n\|_V^2)^{1-\epsilon} \right] \to 0 \quad (1.8)$$

as $n \to \infty$ for all $\epsilon > 0$.

The paper is organized as follows. In Section 2 we give the theoretical background along with the deterministic and the stochastic settings. In Section 3 we state the main results on the convergence of the Galerkin approximations in the $V$-norm and on the finiteness of the logarithmic moment functions. In Remark 3.3 we summarize results on convergence when the $V$-norm is replaced by the $H$-norm. Finally, Section 4 contains the proof of the convergence of the Galerkin approximation to the original solutions. The proof uses the new moment estimate provided in Theorem 3.2.

2 Functional Setting

First, we recall the deterministic and the probabilistic frameworks used throughout the paper.

2.1 Deterministic Framework

Let $O$ be a smooth bounded open connected subset of $\mathbb{R}^2$, and let $V = \{ u \in C_0^\infty(O) : \nabla \cdot u = 0 \}$. Denote by $H$ and $V$ the closures of $V$ in $L^2(O)$ and $H^1(O)$ respectively. The spaces $H$ and $V$ are identified by

$$H = \{ u \in L^2(O) : \nabla \cdot u = 0, u \cdot N|_{\partial O} = 0 \},$$

$$V = \{ u \in H_0^1(O) : \nabla \cdot u = 0 \}$$

(cf. [CF2, T]). Here $N$ is the outer pointing normal to $\partial O$. On $H$ we denote the $L^2(O)$ inner product and the norm as

$$\langle u, v \rangle = \int_O u \cdot v \, dx$$

$$\|u\|_H = \sqrt{\langle u, u \rangle}. \quad (2.3)$$

Let $P_H$ be the Leray-Hopf projector of $L^2(O)$ onto $H$. Recall that for $u \in L^2(O)$ we have $P_H u = (1 - Q_H) u$ where $Q_H u = \nabla \pi_1 + \nabla \pi_2$ and $\pi_1, \pi_2 \in H^1(O)$ are solutions of the problems

$$\Delta \pi_1 = \nabla \cdot u \text{ in } O$$

$$\pi_1 = 0 \text{ on } \partial O \quad (2.4)$$

and

$$\Delta \pi_2 = 0 \text{ in } O$$

$$\nabla \pi_2 \cdot N = u - \nabla \pi_1 \text{ on } \partial O \quad (2.5)$$

(cf. [CF2, T]). Let

$$A = -P_H \Delta \quad (2.6)$$
be the Stokes operator with the domain $D(A) = V \cap H^2(\mathcal{O})$. The dual of $V = D(A^{1/2})$ with respect to $H$ is denoted by $V' = D(A^{-1/2})$. Here $A$ is defined as a bounded, linear map from $V$ to $V'$ via

$$\langle Au, v \rangle = \int_\mathcal{O} \nabla u \cdot \nabla v \, dx, \quad u, v \in V,$$

with the corresponding norm defined as

$$\|u\|^2_V = \langle Au, u \rangle = \langle A^{1/2}u, A^{1/2}u \rangle, \quad u \in V.$$

By the theory of symmetric, compact operators applied to $A^{-1}$, there exists an orthonormal basis $\{e_k\}$ for $H$ consisting of eigenfunctions of $A$. The corresponding eigenvalues $\{\lambda_k\}$ form an increasing, unbounded sequence

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots.$$

We also define the nonlinear term as a bilinear mapping $V \times V$ to $V'$ via

$$B(u, v) = \mathcal{P}_H(u \cdot \nabla v).$$

The deterministic force $f$ is assumed to be bounded with values in $H$. Note that the cancellation property $\langle B(u, v), v \rangle = 0$ holds for $u, v \in V$.

### 2.2 Stochastic Framework

In this section, we recall the necessary background material for stochastic analysis in infinite dimensions needed in this paper (cf. [DZ, DGT, F, PR]). Fix a stochastic basis $S = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}, \mathcal{W})$, which consists of a complete probability space $(\Omega, \mathbb{P})$, equipped with a complete right-continuous filtration $\mathcal{F}_t$, and a cylindrical Brownian motion $\mathcal{W}$, defined on a separable Hilbert space $U$ adapted to this filtration.

Given a separable Hilbert space $X$, we denote by $L_2(U, X)$ the space of Hilbert-Schmidt operators from $U$ to $X$, equipped with the norm $\|G\|_{L_2(U, X)} = (\sum_k \|G_k\|^2_X)^{1/2}$ (cf. [DZ]). For an $X$-valued predictable process $G \in L^2(\Omega; L^2_{\text{loc}}([0, \infty]; L_2(U, X)))$, we define the Itô stochastic integral

$$\int_0^t G \, d\mathcal{W} = \sum_k \int_0^t G_k \, d\mathcal{W}_k \quad (2.7)$$

which lies in the space $\mathcal{O}_X$ of $X$-valued square integrable martingales. We also recall the Burkholder-Davis-Gundy inequality: For all $p \geq 1$ we have

$$\mathbb{E} \left[ \sup_{t \in [0, T]} \left\| \int_0^t G \, d\mathcal{W} \right\|^p_X \right] \leq C \mathbb{E} \left[ \int_0^T \|G\|^2_{L_2(U, X)} \right]^{p/2} \quad (2.8)$$

for some $C = C(p) > 0$.

### 2.3 Conditions on the noise

Given a pair of Banach spaces $X$ and $Y$, we denote by $\text{Lip}_u(X, Y)$ the collection of continuous functions $h: [0, \infty) \times X \to Y$ which are sublinear

$$\|h(t, x)\|_Y \leq K_Y (1 + \|x\|_X), \quad t \geq 0, x \in X \quad (2.9)$$
Assume that the initial data  \( u \) with  \( V \), square integrable in time with values in  \( X \), and if  \( v \) holds for every pathwise strong solution of the system if \( \tau \) is a strictly positive stopping time and there exists a non-decreasing sequence of stopping times  \( \{ \tau_n \} \) on the set  \( \{ \xi < \infty \} \). Such a solution is called global if  \( P(\xi < \infty) = 0 \).
We proceed with the definition of the Galerkin system.

Definition 2.2. An adapted process $u^n$ in $C([0, T]; H_n)$, where $H_n = L\{e_1, \ldots, e_n\}$, is a solution to the Galerkin system of order $n$ if for any $v$ in $H_n$

\[
\begin{align*}
d(u^n, v) + \langle \nu Au^n + B(u^n), v \rangle dt &= \langle f, v \rangle dt + \sum_{k=1}^{\infty} \langle g_k(u^n), v \rangle dW_k \\
\langle u^n(0), v \rangle &= \langle u_0, v \rangle.
\end{align*}
\] (2.19)

We may also rewrite (2.19) as equations in $H_n$, i.e.,

\[
\begin{align*}
du^n + (\nu Au^n + P_nB(u^n))dt &= P_n f dt + \sum_{k=1}^{\infty} P_n g_k(u^n)dW_k \\
u^n(0) &= P_n u_0 = u^n_0.
\end{align*}
\] (2.20)

3 The Main Results

Our main result establishes the convergence of Galerkin approximations in the $V$ norm up to any deterministic time $T$.

Theorem 3.1. Let $\epsilon \in (0, 1)$ and let $T > 0$ be arbitrary. Suppose that $u$ is a solution to the equation (1.1), and let $u^n$ be the corresponding Galerkin approximation. Then we have

\[
E \left[ \sup_{[0,T]} \phi_1(\|u - u^n\|^2_V) \right] \to 0
\] (3.1)

as $n \to \infty$, where $\phi_1(x) = (\log(1 + x))^{1-\epsilon}$.

The main tool used in the proof is the following improvement of the main result in [KV] of independent interest.

Theorem 3.2. Let $u_0$, $f$, and $g$ be as in Definition 2.1 and suppose that $u$ is the solution to the equation (1.1). Then we have

\[
E \left[ \sup_{[0,T]} \phi(\|u\|^2_V) \right] \leq C(f, g, u_0, T),
\] (3.2)

where $\phi(x) = \log(1 + x)$.

Remark 3.3. When considering the convergence of the Galerkin approximations $u^n$ in $H$, a stronger results may be obtained. Namely, let $u$ be the solution to the equation (1.1), and let $u^n$ be the corresponding Galerkin approximation. Assume that $f \in L^{2k}(\Omega; L^{2k}([0, \infty); V'))$ and $u_0 \in L^{2k+2}(\Omega; H) \cap L^2(\Omega; V)$ for all $k \in \mathbb{N}$. Then we have

\[
E \left[ \sup_{[0,T]} \|u - u^n\|^m_{H} \right] \to 0 \quad \text{as} \quad n \to \infty, \quad m \in \mathbb{N}
\] (3.3)
for any deterministic time $T > 0$. Indeed, let $k \in \mathbb{N}$. By [FG], we have

$$E \left[ \sup_{[0,T]} \| u \|_{H}^{2k} \right] + E \left[ \int_{0}^{T} \| u \|_{V}^{2k} \| u \|_{H}^{2k-2} ds \right] \leq C(k, \| u_0 \|_{H}^{2k}, \| f \|_{V'}^{2k}, T).$$

(3.4)

Also, by the same argument applied to the Galerkin system, we get

$$E \left[ \sup_{[0,T]} \| u^n \|_{H}^{2k} \right] + E \left[ \int_{0}^{T} \| u^n \|_{V} \| u^n \|_{H}^{2k-2} ds \right] \leq C(k, \| u_0 \|_{H}^{2k}, \| f \|_{V'}^{2k}, T).$$

(3.5)

Then, we have using

$$\log(1 + x) \leq x, \quad x \geq 0 \quad (3.6)$$

Recall that, by [B], we have

$$P \left( \sup_{t \in [0,T]} \| u - u^n \|_{H} \geq \delta \right) \to 0 \quad (3.7)$$

while, by (3.4) and (3.5),

$$E \left[ \sup_{[0,T]} \| u - u^n \|_{H}^{2k} \right] \leq 2^{2k} \left( E \left[ \sup_{[0,T]} \| u \|_{H}^{2k} \right] + E \left[ \sup_{[0,T]} \| u^n \|_{H}^{2k} \right] \right).$$

(3.8)

Using the uniform integrability principle with (3.7) and (3.8), we get

$$E \left[ \sup_{[0,T]} \| u - u^n \|_{H}^{2k(1-\alpha)} \right] \to 0 \quad (3.9)$$

as $n \to \infty$, for every $k \in \mathbb{N}$ and $\alpha \in (0, 1)$, and (3.3) is proven.

It is possible to obtain more precise information regarding the convergence in $H$. Assume first that

$$\| g(t,x) \|_{H} \leq C. \quad (3.10)$$

Then estimating $E[\sup_{[0,T]} \| u \|_{H}^{2k}]$ for $k = 1, 2, 3, \ldots$ and keeping the dependence on $k$, we get

$$E \left[ \sup_{[0,T]} \exp(\| u \|_{H}/K) \right] \leq C \quad (3.11)$$

for a sufficiently large constant $K$ (cf. also [G] Lemma 3.1 and [KS] for a different approach). As in [B], we get

$$E \left[ \sup_{[0,T]} \exp(\| u - u_n \|_{H}/K') \right] \to 0 \quad (3.12)$$

as $n \to \infty$, where $K'$ is any constant larger than $K$. More generally, if

$$\| g(t,x) \|_{H} \leq C(1 + \| x \|_{H}^{\alpha}) \quad (3.13)$$

where $\alpha \in [0, 1)$, then instead

$$E \left[ \sup_{[0,T]} \exp(\| u - u_n \|_{H}^{2(1-\alpha)}/K') \right] \to 0 \quad (3.14)$$

as $n \to \infty$. 7
4 Galerkin Convergence in V

In this section, we prove the first main result, Theorem 3.1. We first recall the existence result from [GZ].

Theorem 4.1. [GZ] Let \( \{u^n\} \) be the sequence of solutions of (2.19), and let \( u \) be the solution to the equation (1.1) with \( g, f, \) and \( u_0 \) as in Definition 2.1. Then there exists a global, maximal pathwise strong solution \((u, \xi)\). Namely, there exists an increasing sequence of strictly positive stopping times \( \{\tau_m\}_{m \geq 0} \) converging to \( \xi \), for which \( P(\xi < \infty) = 0 \). Moreover, there exists an increasing sequence of measurable subsets \( \{\Omega_s\}_{s \geq 1} \) with \( \Omega_s \uparrow \Omega \) as \( s \to \infty \) so that on any \( \Omega_s \) we have

\[
\lim_{n \to \infty} \mathbb{E} \left[ 1_{\Omega_s} \left( \sup_{t \in [0, \tau_m]} \|u - u^n\|_V^2 + \nu \int_0^{\tau_m} \|A(u - u^n)\|_H^2 dt \right) \right] = 0 \quad (4.1)
\]
as \( n \to \infty \) for any \( \tau_m \).

First, we establish the convergence of the Galerkin approximations in probability.

Lemma 4.2. Let \( u \) and \( u^n \) be defined as in Definitions 2.1 and 2.2. Then for any deterministic time \( T > 0 \), the Galerkin approximations \( u^n \) converge in probability with respect to the \( V \) norm to the solution of the equation (1.1), i.e., for any \( \delta > 0 \) we have

\[
P \left( \sup_{t \in [0, T]} \|u - u^n\|_V^2 \geq \delta \right) \to 0 \quad (4.2)
\]
as \( n \to \infty \).

Proof of Lemma 4.2. Let \( \epsilon > 0 \). With \( \{\tau_n\}_{n \geq 1} \) the stopping times as in Theorem 4.1 denote \( \tau_n = \tilde{\tau}_n \wedge T \). Then there exists \( N_0 \) such that \( P(\tau_{N_0} < T) \leq \epsilon/4 \). Now, choose an \( s \) such that \( P(\Omega_s) > 1 - \epsilon/2 \), where \( \Omega_s \) is as in Theorem 4.1. By Theorem 4.1 we have

\[
\lim_{n \to \infty} \mathbb{E} \left[ 1_{\Omega_s} \sup_{t \in [0, \tau_{N_0}]} \|u - u^n\|_V^2 \right] = 0 \quad (4.3)
\]
which implies the convergence in probability, i.e.,

\[
\lim_{n \to \infty} P \left( 1_{\Omega_s} \sup_{t \in [0, \tau_{N_0}]} \|u - u^n\|_V^2 \geq \delta \right) = 0, \quad (4.4)
\]
for any \( \delta > 0 \). Hence, we have

\[
P \left( 1_{\Omega_s} \sup_{t \in [0, T]} \|u - u^n\|_V^2 \geq \delta \right)
= P \left( \left\{ \sup_{t \in [0, T]} \|u - u^n\|_V^2 \geq \delta \right\} \cap \{\tau_{N_0} < T\} \cap \{\omega \in \Omega_s\} \right)
+ P \left( \left\{ \sup_{t \in [0, T]} \|u - u^n\|_V^2 \geq \delta \right\} \cap \{\tau_{N_0} = T\} \cap \{\omega \in \Omega_s\} \right)
\leq P(\tau_{N_0} < T) + P \left( 1_{\Omega_s} \sup_{t \in [0, \tau_{N_0}]} \|u - u^n\|_V^2 \geq \delta \right) \quad (4.5)
\]
and thus
\[ \mathbb{P} \left( \sup_{t \in [0,T]} \| u - u^n \|^2_V^T \geq \delta \right) \leq \mathbb{P} (\tau_{N_0} < T) + \mathbb{P} \left( \mathbb{1}_{\Omega_m} \sup_{t \in [0,\tau_{N_0}]} \| u - u^n \|^2_V \geq \delta \right) + \mathbb{P} (\Omega_s^c) \]
\[ \leq \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon \]
(4.6)
for \( n \) sufficiently large, and the proof is concluded.

**Proof of Theorem 3.2.** From the infinite dimensional version of Itô’s lemma we get
\[ d(\phi(\|u\|^2_V)) + 2\nu \phi'(\|u\|^2_V) \|Au\|^2_H dt \]
\[ = \phi'(\|u\|^2_V) \left( 2\langle f, Au \rangle - 2\langle B(u, u), Au \rangle + \phi'(|u|^2_V) \| g(u) \|^2_V \right) dt \]
\[ + 2\phi''(|u|^2_V) \sum_k (g_k(u), Au)^2 dt + 2\phi''(|u|^2_V) \langle g(u), Au \rangle dW. \]
(4.7)
We take the supremum up to the stopping time \( \tilde{\tau}_m = \tau_m \wedge T \), where \( \tau_m \) is introduced in Theorem 4.1. Denoting \( \Omega_m = \{ \omega \in \Omega : \tilde{\tau}_m = T \} \), we see that \( \Omega_m \uparrow \Omega \) as \( m \to \infty \) by Theorem 4.1. By taking the expectation on \( \Omega_m \) and, suppressing \( \mathbb{1}_{\Omega_m} \) for simplicity of notation, we get
\[ \mathbb{E} \left[ \sup_{[0,\tilde{\tau}_m]} \phi(|u|^2_V) \right] + 2\nu \mathbb{E} \left[ \int_0^{\tilde{\tau}_m} \phi'(\|u\|^2_V) \|Au\|^2_H ds \right] \]
\[ \leq \phi'(\|u_0\|^2_V) + \mathbb{E} \left[ \int_0^{\tilde{\tau}_m} (T_1 + T_2 + T_3 + T_4) ds \right] + 2\nu \mathbb{E} \left[ \sup_{s \in [0,\tilde{\tau}_m]} \left| \int_0^s T_0 dW \right| \right] \]
(4.8)
where we denoted
\[ T_0 = 2\phi'(|u|^2_V) \langle g(u), Au \rangle \]
(4.9)
\[ T_1 = 2\phi'(|u|^2_V) \langle B(u, u), Au \rangle \]
(4.10)
\[ T_2 = 2\phi'(|u|^2_V) \langle f, Au \rangle \leq 2\phi'(|u|^2_V) \| f \|_H \| Au \|_H \leq C \phi'(|u|^2_V) \| f \|^2_H + \frac{\nu}{8} \phi'(|u|^2_V) \| Au \|^2_H \]
(4.11)
\[ T_3 = \phi'(|u|^2_V) \langle g(u) \rangle \leq \phi'(|u|^2_V) (1 + \| u \|^2_V) \]
(4.12)
\[ T_4 = 2\phi''(|u|^2_V) \langle g(u), Au \rangle \leq C \phi''(|u|^2_V) |u|^2_V (1 + \| u \|^2_V) \]
(4.13)
where \( C \) is allowed to depend on \( K_j, \) for \( j = 0, 1, 2, \) and \( K_Y. \) Appealing to the BDG inequality, we have
\[ \mathbb{E} \left[ \sup_{s \in [0,\tilde{\tau}_m]} \left| \int_0^s T_0 dW \right| \right] \leq C \mathbb{E} \left[ \left( \int_0^{\tilde{\tau}_m} \phi'(|u|^2_V) |g(u)|^2 \|u\|^2_V ds \right)^{1/2} \right] \]
(4.14)
and thus, using the Lipschitz condition on \( g(u), \)
\[ \mathbb{E} \left[ \sup_{s \in [0,\tilde{\tau}_m]} \left| \int_0^s T_0 dW \right| \right] \leq C \mathbb{E} \left[ \left( \int_0^{\tilde{\tau}_m} \frac{1}{(1 + \| u \|^2_V)^2} (1 + \| u \|^2_V) \|u\|^2_V ds \right)^{1/2} \right] \leq C(T). \]
(4.15)
Next, we estimate the term $T_1$ as
\[ T_1 = 2\phi'(\|u\|^2_V) |(B(u, u), Au)| \]
\[ \leq 2\phi'(\|u\|^2_V)\|u\|^1/2\|u\|^1/2\|u\|^1/2\|Au\|^3/2 \]
\[ \leq C\phi'(\|u\|^2_V)\|u\|^1/2 + \frac{1}{4}\phi'(\|u\|^2_V)\|Au\|^3/2 \]
\[ \leq C\|u\|^2_H\|u\|^1/2 + \frac{1}{4}\phi'(\|u\|^2_V)\|Au\|^3/2, \]
where we note that by (3.5)
\[ E\left[ \int_0^T \|u\|^2_V\|u\|^2_H dt \right] \leq M(\|u_0\|^4_H, \|f\|^4_V, T). \]  
(4.17)

By combining all the estimates and writing out $\mathbb{I}_{\Omega_m}$ explicitly, we obtain
\[ E\left[ \mathbb{I}_{\Omega_m} \sup_{[0, \tau_m]} \phi(\|u\|^2_V) \right] \leq C(f, g, u_0, T). \]  
(4.18)

By letting $m \to \infty$ and appealing to the monotone convergence theorem, we get
\[ E\left[ \sup_{[0, T]} \phi(\|u\|^2_V) \right] \leq C(f, g, u_0, T) \]  
(4.19)
and the proof is concluded. \hfill \square

**Lemma 4.3.** Let $u^n$ be as in Definition 2.2. Then we have
\[ E\left[ \sup_{[0, T]} \log(1 + \|u^n\|^2_V) \right] \leq C(f, g, u_0, T) \]  
(4.20)
and
\[ E\left[ \sup_{[0, T]} \log(1 + \|u - u^n\|^2_V) \right] \leq C(f, g, u_0, T), \]  
(4.21)
for all $n \in \mathbb{N}$.

*Proof of Lemma 4.3.* The proof of (4.20) follows the same steps as the proof of Theorem 3.1 and it is thus omitted. The inequality (4.21) is a consequence of (3.2) and (4.20). \hfill \square

Now, we are ready to prove the first stated main result, Theorem 3.1.

*Proof of Theorem 3.1.* Let $\epsilon \in (0, 1)$. By (4.21), we have
\[ \sup_{[0, T]} \log(1 + \|u - u^n\|^2_V)^{1-\epsilon} \to 0 \]  
(4.22)
in probability as $n \to \infty$. Moreover, using Lemma 4.3
\[ E\left[ \sup_{[0, T]} \log(1 + \|u - u^n\|^2_V) \right] \leq M(u_0, f, g, T). \]  
(4.23)
Denoting
\[ U_n = \sup_{[0,T]} \log(1 + \|u - u^n\|_V^2)^{1-\epsilon} \]
we have by (4.23)
\[ \mathbb{E} \left[ U_n^{1/(1-\epsilon)} \right] \leq M(u_0, f, g, T) \]
while (4.21) gives
\[ U_n^{1/(1-\epsilon)} \to 0 \]
in probability. Using the de la Vallée-Poussin criterion for uniform integrability (see e.g. [D]), we get that
\[ U_n \to 0 \text{ in } L^1 \text{ as } n \to \infty \] and Theorem 3.1 is proven. \(\square\)

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