Corrigendum: On the spatial asymptotic decay of a suitable weak solution to the Navier–Stokes Cauchy problem (2016 Nonlinearity 29 1355–83)

F Crispo and P Maremonti

Dipartimento di Matematica e Fisica, Seconda Università degli Studi di Napoli, via Vivaldi 43, 81100 Caserta, Italy

E-mail: francesca.crispo@unina2.it and paolo.maremonti@unina2.it

Received 6 September 2016
Accepted for publication 12 October 2016
Published 21 October 2016

The corrections listed below only concern misprints that might make the reading of some proofs in the paper difficult.

Errata
Page 1364, line 16 : ...it is bounded
Page 1365, line 19 : $-(\nabla v, \nabla (\varphi h_j) \otimes w)$
Page 1365, line 22 : $v \in L^t(0,T; L^q(\mathbb{R}^3 \setminus B_{M_j R_0}))$
Page 1371, line 3 : $\int_\varepsilon^{s-\varepsilon} \xi^{-\frac{4}{p}} + \xi^{-1-\frac{2}{p}} \, d\tau$

Corrige
Page 1364, line 16 : ...it is finite
Page 1365, line 19 : $-(\nabla v, w \otimes \nabla (\varphi h_j))$
$\quad \in L^t(\varepsilon,T; L^q(\mathbb{R}^3 \setminus B_{M_j R_0}))$
$\quad \int_\varepsilon^{s-\varepsilon} (s-\tau)^{-\frac{4}{p}} + (s-\tau)^{-1-\frac{2}{p}} \, d\tau$

Page 1371, line 8 : $\|v\|_{L^2(\Omega)}^2$
Page 1373, line 2 : $(H_j(t,x), \varphi)$
Page 1373, line 4 : $(v \cdot \nabla T(t,x), v)$
Page 1373, line 7 : can be written
Page 1376, line 1 : $\|v(t)\|_{L^{\frac{2(\varepsilon+1)}{\varepsilon}}(\{|y|>2M_j R_0\})}$
Page 1376, line 3 : $\|v\|_{L^{\frac{2(\varepsilon+1)}{\varepsilon}}(\{|y|>2M_j R_0\})}^2$

Page 1376, line 1 : $\|v(t)\|_{L^{\frac{2(\varepsilon+1)}{\varepsilon}}(\{|y|>2M_j R_0\})}$
Page 1376, line 3 : $\|v\|_{L^{\frac{2(\varepsilon+1)}{\varepsilon}}(\{|y|>2M_j R_0\})}^2$
On the spatial asymptotic decay of a suitable weak solution to the Navier–Stokes Cauchy problem

F Crispo and P Maremonti

Dipartimento di Matematica e Fisica, Seconda Università degli Studi di Napoli, via Vivaldi 43, 81100 Caserta, Italy

E-mail: francesca.crispo@unina2.it and paolo.maremonti@unina2.it

Received 21 January 2015, revised 17 November 2015
Accepted for publication 14 January 2016
Published 9 March 2016

Recommended by Professor Edriss S Titi

Abstract

We prove space–time decay estimates of suitable weak solutions to the Navier–Stokes Cauchy problem, corresponding to a given asymptotic behavior of the initial data of the same order of decay. We use two main tools. The first is a result obtained in [7] for the behavior of the solution in a neighborhood of \( t = 0 \) in the \( L^\infty_{\text{loc}} \)-norm, which enables us to furnish a representation formula for a suitable weak solution. The second is the asymptotic behavior of \( \|u(t)\|_{L^2(R^n \setminus B)} \) for \( R \to \infty \). Following Leray’s point of view, roughly speaking our result proves that a possible space–time turbulence does not perturb the asymptotic spatial behavior of the initial data of a suitable weak solution.

Keywords: Navier–Stokes equations, suitable weak solutions, space–time asymptotic behavior
Mathematics Subject Classification numbers: 35B35, 35B65, 76D03

1. Introduction

Because of its physical interest, in mathematical fluid dynamics much effort has been devoted to studying the stability and attractivity of the unperturbed motion of a fluid already in the easier case of perturbations to the rest state. This problem can be approached from a global point of view, in the sense of the \( L^2 \)-norm, and from a pointwise point of view. Concerning the time asymptotic behavior of the kinetic energy of the perturbation (the \( L^2 \)-norm of the solutions), a problem left open by Leray [16], pioneering contributions were made in [3, 12, 17, 18, 26, 27, 29]. The questions of the sharpness of the asymptotic behaviors and the characterization of the corresponding set of solutions are strictly connected to this study, and for them we refer to...
As far as the pointwise asymptotic behavior is concerned, one should distinguish between the space–time asymptotic behavior and the behavior in the time of the $L^\infty$-norm of the solution and its derivatives. For this type there is, for instance, [24], where the decay of the $L^\infty$-norm of the solutions can be obtained via Sobolev embedding. Independently, and earlier, Knightly began the study of space–time pointwise estimates, basically for the Navier–Stokes Cauchy problem corresponding to small data, and continued, in the half-space, in [6]. These last papers essentially resort to a representation formula of the solutions through the Oseen tensor or the Green function. An alternative study of pointwise estimates, whose setting can be ascribed to Nash [23], starts from the equations of moments for either the asymptotic behavior in time or the asymptotic behavior in space [1, 2, 14, 15]. However, this approach does not furnish sharp spatial behaviors. Finally, we recall the special case of the spatial behavior related to a nonzero forcing term, as in [10, 22, 28].

The quoted results on pointwise estimates are related to the decay of solutions and their derivatives, hence one is forced to deal with smooth solutions corresponding to small data or weak solutions satisfying some additional assumptions that ensure regularity. Therefore, the issue of the sharp pointwise stability of weak solutions in space and time—without burdening the initial data or the solution itself with extra assumption—remains an open problem.

Addressing this is the chief task of this paper.

In this paper, we study the initial value problem:

\[
\begin{align*}
\psi + v \cdot \nabla v + \nabla \pi &= \Delta v, \\
\nabla \cdot v &= 0, \quad \text{in } (0, T) \times \mathbb{R}^3, \\
v(0, x) &= \psi_0(x) \text{ on } [0] \times \mathbb{R}^3.
\end{align*}
\]  

(1.1)

In system (1.1) $v$ is the kinetic field, $\pi$ is the pressure field, $\psi := \frac{\partial}{\partial t} v$ and $v \cdot \nabla v := \psi \frac{\partial}{\partial t} v$. For brevity, we assume zero body force. We set $J^q(\mathbb{R}^3) := \text{completion of } \mathcal{C}_0(\mathbb{R}^3)$ with respect to the $L^q$-norm, $q \in (1, \infty)$. The symbol $\mathcal{C}_0(\mathbb{R}^3)$ denotes the subset of $\mathcal{C}_0^\infty(\mathbb{R}^3)$ whose elements are divergence free. By $P_q$ (the index $q$ is omitted when there is no danger of confusion) we mean the projector from $L^q$ into $J^q$. For the properties and details of these spaces see, for instance, [9]. Moreover, we set $J^{q, \alpha}(\mathbb{R}^3) := \text{completion of } \mathcal{C}_0(\mathbb{R}^3)$ with respect to the $W^{q, \alpha}$-norm. For a nonnegative integer $m$, if $X$ is a Banach space, the symbols $C^m(a, b; X)$ and $L^q(a, b; X)$ mean the spaces of functions defined in $(a, b) \subseteq \mathbb{R}$ with values in the Banach space $X$, that are $m$-times continuous differentiable in $[a, b]$ and $L^q$-integrable on $(a, b)$, respectively.

We use the same symbol to denote vector or scalar functions and function spaces. We set $(u, g) := \int_\Omega u \cdot g \, dx$.

**Definition 1.1.** A pair $(v, \pi)$, such that $v : (0, \infty) \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and $\pi : (0, T) \times \mathbb{R}^3 \rightarrow \mathbb{R}$, is said to be a weak solution to problem (1.1) if

(i) for all $T > 0$, $v \in L^2(0, T; J^{1,2}(\mathbb{R}^3))$ and $\pi \in L^2(0, T) \times \mathbb{R}^3$,

\[ \|v(t)\|_2^2 + 2 \int_0^t \|\nabla v\|^2_2 \, d\tau \leq \|v(s)\|_2^2, \quad \forall t \geq s, \quad \text{for } s = 0 \text{ and a.e. in } s \geq 0; \]

(ii) $\lim_{t \to 0} \|v(t) - v_0\|_2 = 0$;

(iii) for all $t, s \in (0, T)$, the pair $(v, \pi)$ satisfies the equation:

\[
\int_s^t \left[ (v, \varphi) - (\nabla v, \nabla \varphi) + (v \cdot \nabla v, \varphi) + (\pi, \nabla \cdot \varphi) \right] \, d\tau + (v(s), \varphi(s)) = (v(t), \varphi(t)),
\]

for all $\varphi \in C_0^1([0, T) \times \mathbb{R}^3)$. 

1356
As is known, the regularity and uniqueness of weak solutions are still open problems. However, in order to improve the results of regularity, in the fundamental paper \cite{4}, Caffarelli, Kohn and Nirenberg introduce the notion of the suitable weak solution:

**Definition 1.2.** A pair \((\pi, v]\) is said to be a suitable weak solution if it is a weak solution in the sense of definition 1.1 and, moreover,

\[
\int_{\mathbb{R}^3} |v(t)|^2 \phi(t) \, dx + 2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\nabla v|^2 \phi \, dx \, d\tau \\
+ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v|^2 (\phi_x + \Delta \phi) \, dx \, d\tau \\
+ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (|v|^2 + 2\pi_x) v \cdot \nabla \phi \, dx \, d\tau,
\]

for all \(t \geq s\), for \(s = 0\) and a.e. in \(s \geq 0\), and for all nonnegative \(\phi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^3)\).

In \cite{4} and \cite{25} the following existence result is proved:

**Theorem 1.1.** For all \(v_0 \in L^2(\Omega)\) there exists a suitable weak solution.

Further, in the same paper \cite{4} it is proved that in the set of suitable weak solutions a partial regularity result holds. Among others things, in \cite{4} the authors prove that under the further assumption \(\nabla v_0 \in L^2(\mathbb{R}^3 \setminus B_{R_0})\), a suitable weak solution is regular in a neighborhood of infinity, that is for \(|x| > M_0 R_0\), \(M_0 > 1\). Roughly speaking, this result is the counterpart of the one related to Leray’s structure theorem, where the regularity of a weak solution is in a neighborhood of \(t = +\infty\). Still, in light of the analogy, roughly speaking and following Leray, we say that the possible turbulence of a weak solution not only appears in a finite time, but also in a bounded region of space, whose parabolic one-dimensional Hausdorff measure is null (this last statement is true as soon as a suitable weak solution exists). In fact, the smallness of the data for large \(|x|\)—although given by means of integrability conditions outside a ball—preserves the regularity of the weak solution (as for ‘small data’).

In the wake of the previous results, in \cite{7} we proved a result concerning the behavior in time of the \(L^\infty_{loc}\) norm of the solution in a neighborhood of \(t = 0\) for suitable weak solutions, corresponding to suitably small data (see theorem 2.1 below).

This paper follows the direction of the last claims. We prove that not only does the possible turbulence not perturb the regularity of a weak solution in a neighborhood of \(t = +\infty\), but, if an asymptotic spatial behavior of the initial data \(v_0\) is given, then the same behavior holds for a suitable weak solution for all \(t > 0\).

The theorem we are going to state is the main result of the paper.

**Theorem 1.2.** Let \((v, \pi)\) be a suitable weak solution to the Navier–Stokes Cauchy problem. If for some \(\alpha \in [1, 3)\) and \(R_0 > 0\), \(|v_0(x)| \leq V_0 |x|^{-\alpha}\), for \(|x| > R_0\), then there exists a constant \(M > 1\) such that

\[
|v(t, x)| \leq c(v_0) |x|^{-\alpha}, \text{ for all } (t, x) \in (0, \infty) \times \mathbb{R}^3 \setminus B_{MR_0},
\]

where \(M\) is independent of \(v_0\) and \(c(v_0)\) depends on \(V_0\) and \(\|v_0\|\).

**Corollary 1.1.** For a solution of theorem 1.2, for all \(\beta \in [0, \alpha]\), we obtain:

\[
|v(t, x)| \leq c(v_0) |x|^{-\alpha + \beta} t^{-\frac{\beta}{3}}, \text{ for all } (t, x) \in (0, \infty) \times \mathbb{R}^3 \setminus B_{MR_0},
\]

and
\[ |v(t, x)| \leq c(t_0) |x|^{-\alpha + \beta} t^{-\beta/\alpha}, \text{ for all } (t, x) \in (T_0, \infty) \times \mathbb{R}^3, \quad (1.5) \]

where \( T_0 \leq c \|v_0\|_{L^2}^2 \).

Here we state theorem 1.2 for solutions to the Navier–Stokes Cauchy problem just for the sake of brevity. The result of theorem 1.2 can be seen as a continuous dependence of the null solution. In this sense the theorem is a continuation of the one proved in [7], which we employ here for regularity, see theorem 2.1 below.

In a forthcoming paper, the same result will be proved for a three-dimensional exterior domain \( \Omega \) and for weak solutions with initial data in \( J^2(\Omega) \), hence not necessarily with finite energy. This assumption seems to be more coherent with the assumption \( |v_0(x)| \leq c_0 |x|^{-\alpha}, \alpha \in [1, 3) \).

To the best of our knowledge, the technique employed to prove theorem 1.2 is original in the framework of those employed to prove a spatial asymptotic behavior of a solution \( v(t, x) \) and, more generally, those employed for a parabolic equation. Indeed, it essentially follows from two properties. The former, which is well known, concerns the spatial behavior of the solution \( w(t, x) \) to the heat equation. The latter is connected with the asymptotic behavior of the functional:

\[ \int_{|y| > |x|} |u(t,y)|^2 \, dy \leq c(t) |x|^{-1}, \quad (1.6) \]

where \( u = v - w \), which we think is an original tool for spatial behavior. Estimate (1.6) comes from estimate (1.2) for a suitable \( \phi(x) \), written for the difference \( u \). For the special \( \phi \), we like to call the above functional Leray’s generalized energy inequality. It was used by Leray in [16] for a compactness property.

We conclude the introduction by quoting [8], where a similar result is given. More precisely, for the Navier–Stokes initial boundary value problem in an exterior domain \( \Omega \subseteq \mathbb{R}^3 \) it is proved:

Assume that the initial data \( v_0 \in J^2(\Omega) \cap J^\infty(\Omega) \) with \( |v_0(x)| \leq c_0 |x|^{-\alpha}, \) for some \( \alpha \in [\frac{3}{2}, 3) \) and \( |x| > R_0 > 0 \). Assume that, \( \tau \in (0, \frac{1}{4}], \quad v_0 \in D(A_2^{1+\tau}) \cap J^\infty(\Omega). \) Then, there are suitable constants \( c_0 \) and \( R_0 \) such that

\[ |v(t, x)| \leq c_0 |x|^{-\min\{\alpha, \frac{1}{2}\}}, \quad \text{for all } (t, x) \in (0, \infty) \times \mathbb{R}^3 \setminus B_{R_0}. \]

Here, \( A_2 := \mathcal{P}_2 \Delta \) is the Stokes operator, and \( D(A_2^{1+\tau}) \) is the domain of the definition of the fractional power \( s \) of \( A_2 \).

We point out that the key ideas, the technique and the proofs in this paper are completely different from those in [8].

The plan of the paper is as follows. In order to perform pointwise estimates, in section 2 we establish results for partial regularity based on the results of [7]. In section 3 we recall classical results concerning solutions to the Stokes Cauchy problem. In section 4 we prove estimate (1.6), which is strategic for our aims. In section 5 we give the representation formula of the solutions to the Navier–Stokes Cauchy problem that we employ in sections 6 and 7 to prove our results.

2. Preliminary results on the partial regularity of a suitable weak solution

Throughout this paper, where appropriate, we give an explicit dependence for the constants from the \( L^2 \)-norm of the data. In the other cases, the dependence will be referred to simply by \( c(v_0) \).
Lemma 2.1. In the hypotheses of theorem 1.2, for all \( \varepsilon > 0 \),
\[
\int_{\mathbb{R}^3} \frac{|v_0(y)|^2}{|x-y|} \, dy \leq \frac{c}{\varepsilon} \|v_0\|_2^2 + \frac{c}{|x|^{2}} \|V_0\|_2 \|v_0\|_2, |x| > \frac{R_0}{1 - \varepsilon},
\] (2.1)
with \( c \) independent of \( x \) and \( v_0 \).

**Proof.** We start by proving (2.1) for \( \alpha = 1 \). Then, *a fortiori*, it holds for \( \alpha \in (1, 3) \). Given \( \varepsilon > 0 \) and \( x \in \mathbb{R}^3 \), by virtue of our assumption, we easily deduce that
\[
\int_{\mathbb{R}^3} \frac{|v_0|^2}{|x-y|} \, dy \leq \int_{|y| < (1-\varepsilon)|x|} \frac{|v_0|^2}{|x-y|} \, dy + c \int_{|y| > (1-\varepsilon)|x|} \frac{|v_0|^2}{|x-y|(1+|y|)} \, dy
\]
\[
\leq \frac{\|v_0\|_2^2}{\varepsilon|x|} + c \left( \int_{|y| > (1-\varepsilon)|x|} \frac{(1+|y|)^{-2}}{|x-y|^2} \, dy \right)^{\frac{1}{2}} \|V_0\|_2 \|v_0\|_2,
\] (2.2)
which implies the thesis. \( \square \)

Let \( x_0 \in \mathbb{R}^3 \) and \( R_0 > 0 \). Let \( v_0 \in \mathcal{F}(\mathbb{R}^3) \). We set
\[
\mathcal{E}_0(x_0, R_0) := \text{ess sup}_{B(x_0, R_0)} \|v_0\|_{w(x_0)} := \left\| \int_{\mathbb{R}^3} \frac{|v_0(y)|^2}{|x-y|} \, dy \right\|_{L^\infty(B(x_0, R_0))}.
\]

**Theorem 2.1.** Let \((v, \pi_v)\) be a suitable weak solution corresponding to \( v_0 \in \mathcal{F}(\mathbb{R}^3) \). There exist absolute constants \( \varepsilon_1, C_1 \) and \( C_2 \) such that, if
\[
C_1 \mathcal{E}_0(x_0, R_0) < 1 \quad \text{and} \quad C_2 (\varepsilon_0^{\frac{3}{4}} + \varepsilon_0^{\frac{5}{4}}) \leq \varepsilon_1,
\] (2.3)
then
\[
|v(t, x)| \leq c (\varepsilon_0^{\frac{3}{4}} + \varepsilon_0^{\frac{5}{4}})^{\frac{1}{2}},
\] (2.4)
provided that \((t, x)\) is a Lebesgue point with \( \|v_0\|_{w(x_0)} < \infty \) and \( x \in B(x_0, R_0) \).

**Proof.** See [7], theorem 1.2. \( \square \)

We complete the results concerning a suitable weak solution with the following lemma on the regularity and asymptotic behavior of the solutions:

**Lemma 2.2.** If \( v_0 \in \mathcal{F}(\mathbb{R}^3) \cap \mathcal{F}(\mathbb{R}^3), p \in (1, 2], \) then there exists a \( T_0 \leq \varepsilon \|v_0\|_2^4 \) such that
\[
|v(t)|_2 = \begin{cases} \alpha(1) & \text{if } p = 2, \\ c(v_0)t^{-\frac{3}{2}\left(\frac{1}{p} - \frac{1}{2}\right)}, t > 0 & \text{if } p \in (1, 2).
\end{cases}
\] (2.5)

**Proof.** Estimates (2.5)\(_{1,2}\) can be found in [18], while (2.5)\(_3\) is well known (see Leray [16]). \( \square \)

**Lemma 2.3.** In the hypotheses of theorem 1.2 there exists a constant \( M_0 > 1 \) such that
\[
|v(t, x)| \leq c (\varepsilon_0^{\frac{3}{4}} + \varepsilon_0^{\frac{5}{4}})^{\frac{1}{2}}, \text{ a.e. in } t > 0 \text{ and } |x| > M_0 R_0,
\] (2.6)
provided that \((t, x)\) is a Lebesgue point.
Proof. It is enough to verify that the hypotheses of theorem 2.1 are satisfied. To this end, we employ lemma 2.1, which ensures the existence of \( R \in \mathbb{C}^2 \), such that, for \( |x| > R \),
\[
\left[ \int_{\mathbb{R}^3} \frac{|y|}{|x-y|} \frac{1}{2} dy \right]^2 \]

satisfies (2.3). Setting \( R := M_0 R_0 \), we have proved the lemma. \( \blacksquare \)

As a consequence of the above lemmas on the \( L^\infty \)-norm of a suitable weak solution we can claim:

**Corollary 2.1.** In the hypotheses of theorem 1.2, we obtain
\[
\|v(t)\|_{L^\infty(|x| > M_0 R_0)} \leq c(v_0, T_0) t^{-\frac{3}{2}}, \quad t > 0.
\] (2.7)

**Lemma 2.4.** If \((v, \pi)\) is a suitable weak solution, then the pressure field admits the representation formula:
\[
\pi(t, x) = -D_x D_y \int_{\mathbb{R}^3} \mathcal{E}(x-y)v^i(y)v^j(y)dy =: \mathbb{H}[v, v](t, x),
\] a.e. in \((t, x) \in (0, \infty) \times \mathbb{R}^3\).

**Proof.** See [7], lemma 2.3. \( \blacksquare \)

### 3. Some lemmas on the Stokes Cauchy problem

We consider the Stokes Cauchy problem
\[
w_t - \Delta w = -\nabla \pi, \quad \nabla \cdot w = 0 \text{ in } (0, T) \times \mathbb{R}^3, \quad w(0, x) = v_0(x) \text{ on } \{0\} \times \mathbb{R}^3.
\] (3.1)

We denote by
\[
H_b(s, z) := \delta_b(4\pi s) \frac{3}{\pi e} \frac{s^2}{4}
\]
the general component of the heat kernel tensor, and set
\[
\mathbb{H}[w(t-s)](t, x) := \int_{\mathbb{R}^3} H(t-s, x-y)w(s, y)dy.
\]

Then, for the solution of (3.1) we have \( w(t, x) := \mathbb{H}[v_0](t, x) \). Moreover, we recall the estimate \((k \text{ nonnegative integer and } \beta \text{ multi-index})\):
\[
|D^\beta_t D^\beta_x H(s, z)| \leq c(|z| + s^2)^{-3-2k-|\beta|}.
\] (3.3)

We are interested in the following result\(^1\):

**Lemma 3.1.** In the hypotheses of theorem 1.2,

for all \( T > 0 \), \( w \in C(0, T; L^2(\mathbb{R}^3)) \),

for all \( \eta > 0, k \geq 0, |\beta| \geq 0, D^\beta_t D^\beta_x w \in C(\eta, T; C_b(\mathbb{R}^3)) \cap C(\eta, T; L^2(\mathbb{R}^3)) \),
\[
\|w(t)\| + 2 \int_{\eta}^t \|\nabla w\|^2 \, dt = \|w(s)\|^2, \text{ for all } t \geq s > 0.
\] (3.4)

Moreover, there holds

\(^1\)The symbol \( C_b(\mathbb{R}^3) \) means the set of bounded and \( m \)-times continuous differentiable functions.
\[ |w(t, x)| \leq c \min \{ \|v_0\|_2, L_r \} \min \left\{ \frac{1}{(1 + t)^{\frac{1}{2} + \alpha}}, \frac{1}{(1 + |x|)^{\alpha}} \right\}, t > 0, |x| > \max \{ 2R_0, 1 \}. \] (3.5)

**Proof.** Properties (3.4) are well known. To prove (3.5), we employ the representation formula and (3.3), so that

\[ |w(t, x)| \leq \int_{\mathbb{R}^3} H(t, x - y)\|v_0(y)\|dy = \int_{B(0, R_0)} H(t, x - y)\|v_0(y)\|dy + \int_{\mathbb{R}^3 \setminus B(0, R_0)} H(t, x - y)(1 + |y|)^{-\alpha}dy \]

\[ \leq c(R_0)\|v_0\|\{ |x| + t^{\frac{1}{2} + \alpha} + \epsilon V_0 \min \{ (1 + t)^{-\frac{1}{2}}, (1 + |x|)^{-\alpha} \} \}, \]

provided that \(|x| > \max \{ 2R_0, 1 \} \), which proves the lemma. \( \square \)

We also approach problem (3.1) in a form weaker than the usual one for the initial value problem for the Stokes equations. This weak formulation, introduced in [19], allows us to consider the initial data in the Lebesgue spaces \( L^p, p \in [1, \infty] \), and not in the space of the hydrodynamics \( J^p, p \in (1, \infty) \). Its interest is connected with the possibility of deducing estimates in \( L^r \)-spaces with \( r \in (1, \infty) \) by means of duality arguments. Of course, for an initial datum in \( J^p \) we come back to the classical Stokes solutions.

We have the following special result, for a general formulation see [19] (such a solution is denoted by \( (\theta, \eta) \)):

**Lemma 3.2.** Let \( \theta_0 \in C_0(\mathbb{R}^3) \). Then, to the data \( \theta_0 \) there corresponds a unique smooth solution \((\theta, \eta)\) to the Cauchy problem (3.1) such that \( \theta \in \cap_{q \geq 1} C(0, T; J^q(\mathbb{R}^3)), \theta \in \cap_{q \geq 1} L^q(\eta, T; W^{1, q}(\mathbb{R}^3)) \) and \( \eta \in \cap_{q \geq 1} L^q(\eta, T; L^q(\mathbb{R}^3)), \eta > 0 \). Moreover, for \( q \in (1, \infty) \),

\[ \|\theta(t)\|_q \leq c\|\theta_0\|_r^{-\mu}, \quad \mu = \frac{3}{2} \left( 1 - \frac{1}{q} \right), \quad t > 0; \]

\[ \|\nabla \theta(t)\|_q \leq c\|\theta_0\|_r^{-\mu_1}, \quad \mu_1 = \frac{1}{2} + \mu \]

\[ \|\theta(t)\|_r \leq c\|\theta_0\|_r^{-\mu_2}, \quad \mu_2 = 1 + \mu, \quad t > 0; \]

(3.6)

with \( c \) independent of \( \theta_0 \). Finally, \( \lim_{t \to 0} (\theta(t) - \theta_0, \varphi) = 0 \) holds for any \( \varphi \in C^1(\mathbb{R}^3) \cap J^1(\mathbb{R}^3) \).

**Proof.** See [19], lemma 3.2. \( \square \)

**Corollary 3.1.** In the hypotheses of lemma 3.2, for all \( \lambda \in (0, 1) \), the following estimates hold:

\[ \|\theta(t) - \theta(s)\|_q \leq c\xi^{-\lambda - \frac{3}{2}}(\frac{1}{2})(t - s)^{\lambda}\|\theta_0\|_r, \]

\[ \|\nabla \theta(t) - \nabla \theta(s)\|_q \leq c\xi^{-\lambda - \frac{3}{2}}(\frac{1}{2})(t - s)^{\lambda}\|\theta_0\|_r, \]

(3.7)

\( t, s > 0 \), where \( \xi := \min \{ s, t \} \).

**Proof.** Let \( \xi := \min \{ s, t \} \). From (3.6)_1, \( \|\theta(t)\|_q \leq c\|\theta_0\|_r^{-\mu} \). On the other hand, from the representation formula, one has \( \theta(t, x) = H(\theta(t))(t - \frac{x}{2}, x) \). Hence, using the \( L^q \)-Hölder properties of \( \theta \), for all \( \lambda \in (0, 1) \), we obtain
\[ \| \theta(t) - \theta(s) \|_q \leq c \xi^{-\lambda}|t-s|^\lambda \| \theta(s) \|_q \leq c \xi^{-\lambda-\frac{3}{2}(1-\frac{1}{p})}|t-s|^\lambda \| \theta_0 \|. \]

Similar arguments lead to estimate (3.7)\textsuperscript{2}. \[ \square \]

4. A space–time behavior of Leray’s generalized energy inequality

We start by proving the following interpolation inequality, which is of the same kind as the one by Gagliardo and Nirenberg. It is a particular case of the more general result for exterior domains obtained in [5]. The difference with respect to the usual result is that the function \( u \) does not belong to a completion space of \( C_0^\infty(\mathbb{R}^3) \).

\textbf{Lemma 4.1.} Let \( u \in W^{1,2}(\mathbb{R}^3 \setminus B_R) \). Then there exists a constant \( c \) independent of \( u \) and \( R \) such that for any \( p \in [2, 6] \),

\[ \| u \|_{L^p(\Omega, |x| \geq R)} \leq c \| \nabla u \|_{L^2(|x| \geq R)}^{\alpha} \| u \|_{L^2(|x| \geq R)}^{1-\alpha}, \quad \alpha = \frac{3(p-2)}{2p}. \] (4.1)

\textbf{Proof.} Let \( x \in \mathbb{R}^3 \setminus B_R \) be the vertex of an infinite cone \( C_x \subset \mathbb{R}^3 \setminus B_R \) of fixed aperture independent of \( x \). Let \( (r, \theta) \) be spherical polar coordinates with origin at \( x \), assume that the cone \( C_x \) is given by \( r \in (0, \infty) \) and \( \theta \in \Theta \), and let \( r^2 \omega(\theta) \, dr \, d\theta \) be the volume element. Let \( \{ h_\rho(r) \} \) be a sequence of smooth cut-off functions such that \( h_\rho(r) \in [0, 1] \), \( h_\rho(r) = 1 \) for \( r \leq \rho \), \( h_\rho(r) = 0 \) for \( r \geq 2\rho \), and \( |h_\rho'(r)| \leq c \rho^{-1} \). Then

\[ |u(x)| = |u(0, \theta)| = - \int_0^\infty \frac{\partial}{\partial r} (u(r, \theta) h(r)) \, dr \leq \int_0^\infty |\nabla u(r, \theta) h(r)| \, dr + \frac{c}{\rho} \int_0^\infty |u(r, \theta)| \, dr. \]

Multiplying by \( \omega(\theta) \) and integrating over \( \Theta \) we obtain

\[ 4\pi |u(x)| \leq \int_{C_x} \frac{|\nabla u(y)|}{|x-y|^2} \, dy + \frac{c}{\rho} \int_{C_x} \frac{|u(y)|}{|x-y|^2} \, dy \leq \int_{|y| \geq R} \frac{|\nabla u(y)|}{|x-y|^2} \, dy + \frac{c}{\rho} \int_{|y| \geq R} \frac{|u(y)|}{|x-y|^2} \, dy. \]

We let \( \rho \) tend to infinity and then apply the Hardy–Littlewood–Sobolev theorem, and we find

\[ \| u \|_{L^p(|x| \geq R)} \leq c \| \nabla u \|_{L^2(|x| \geq R)}, \] (4.2)

with \( c \) independent of \( R \). Using the interpolation between Lebesgue spaces

\[ \| u \|_{L^p(|x| \geq R)} \leq c \| u \|_{L^q(|x| \geq R)}^{\alpha} \| u \|_{L^2(|x| \geq R)}^{1-\alpha}, \quad \alpha = \frac{3(p-2)}{2p}, \]

and then estimate (4.2), we arrive at (4.1). \[ \square \]

We set

\[ u := v - w \quad \text{and} \quad \pi_u := \pi_v, \] (4.3)

where \( (v, \pi_v) \) is a suitable weak solution to the Navier–Stokes Cauchy problem and \( w \) is the solution to the Stokes Cauchy problem. Both the solutions assume the initial data \( v_0 \). From now on, without loss of generality, we assume that \( 2R_0 > 1 \).
We define
\[ \| \cdot \|_{L^k(R)} := \| \cdot \|_{L^k(|y| > \frac{k}{R + k})}, \quad \text{for all } k \geq 0. \] (4.4)

**Lemma 4.2.** In the hypotheses of theorem 1.2, for all \( k \geq 2 \) and for \( R > 4R_0 \), there exists a \( c(k) \) such that
\[ \| \pi_i \|_{L^k(R)} \leq c \| u \|_{L^k(|y| > \frac{k}{R + k})} \| \nabla u \|_{L^k(|y| > \frac{k}{R + k})} + \frac{V_0}{R^2} \| u \|_{L^k(|y| > \frac{k}{R + k})} + c \frac{V_0^2}{R^{2\alpha + 3}} + \frac{1}{R} \| v_2 \|_{L^2}^2. \] (4.5)
almost everywhere in \( t > 0 \).

**Proof.** From (4.3) and the representation formula (2.8) of the pressure field we obtain
\[ \pi_i = E(v \otimes v) = E(u \otimes u) + E(u \otimes w) + E(w \otimes u) + E(w \otimes w) = \sum_{j=1}^4 \pi_i^j. \] (4.6)

For \( i = 1, \ldots, 4 \) and \( |x| > \frac{k}{R + k} \), we obtain, with obvious meaning of the symbols, the estimate
\[ |\pi_i^j(x)| \leq c \int_{|y| < \frac{k}{R + k}} \frac{|a| |b|}{|x - y|} \, dy + \left| D_{v_i} \int_{|y| > \frac{k}{R + k}} \phi^j(x - y) a_i b_i \, dy \right| =: \pi_i^1 + \pi_i^2. \]

By the assumption on \( x \) there holds
\[ |x - y| \geq |x| - |y| \geq \frac{|x|}{k^2}, \quad \text{for all } |y| < \frac{k - 1}{k} R, \]
and we obtain
\[ |\pi_i^j(x)| \leq c(k) \frac{1}{|x|^2} \int_{|y| < \frac{k}{R + k}} |u| |w| \, dy \leq \frac{c(k)}{|x|^3} \| u \|_2 \| w \|_2, \quad i = 2, 3, \]
which implies
\[ \| \pi_i^j \|_{L^k(R)} \leq \frac{c(k)}{R^2} \| u \|_2 \| v_2 \|_2, \quad i = 2, 3. \]

For the term \( \pi_i^3 \), by applying the Calderón–Zigmund theorem and then lemma 3.1, we obtain
\[ \| \pi_i^3 \|_{L^k(R)} \leq c \| |u| |w| \|_{L^k(|y| > \frac{k}{R + k})} \leq c \frac{V_0}{R^2} \| u \|_{L^k(|y| > \frac{k}{R + k})}. \]

Repeating the above arguments for the term \( \pi_i^4 \), we obtain
\[ |\pi_i^4(x)| \leq \frac{c(k)}{|x|^2} \| v_2 \|_2^2, \]
hence
\[ \| \pi_i^4 \|_{L^k(R)} \leq \frac{c(k)}{R^2} \| v_2 \|_2^2. \]
Since $\alpha \geq 1$ and $R > 4R_0 > 1$, from lemma 3.1 we easily deduce
\[
\|\pi_2\|_{L^2(\mathbb{R}, \mathbb{R})} \leq c\|w^2\|_{L^2(\mathbb{R}, \mathbb{R})} \leq cV_2R^{2\alpha - \frac{1}{2}}.
\]
Finally, we estimate $\pi_1$. For $\pi_1$ we obtain the same estimate, that is
\[
|\pi_1(x)| \leq \frac{c(k)}{|x|^3} \|\nu\|_2^2 \leq \frac{c(k)}{|x|^3} \|\nu_0\|_2^2.
\]
Hence, we obtain
\[
\|\pi_1\|_{L^2(\mathbb{R}, \mathbb{R})} \leq \frac{c(k)}{R^2} \|\nu_0\|_2^2.
\]
Applying the Calderón–Zigmund theorem for singular integrals and estimate (4.1), we obtain
\[
\|\pi_2\|_{L^2(\mathbb{R}, \mathbb{R})} \leq c\|u\|_{L^2(\mathbb{R}, \mathbb{R})}^2 \leq c\|\nu_0\|_{L^2(\mathbb{R}, \mathbb{R})}^2.
\]

**Lemma 4.3.** In the hypotheses of theorem 1.2, for all $k \geq 2$ and for $R > 2M_0R_0$, there exists a constant $c$ such that
\[
\|\pi_k\|_{L^2(\mathbb{R}, \mathbb{R})} \leq c(\|\nu\|_2^2 + \|\nu_0\|_{L^2(\mathbb{R}, \mathbb{R})}^2),
\]
almost everywhere in $t > 0$.

**Proof.** Estimate (4.7) is obtained via the same arguments used in the proof of lemma 4.2, provided that we consider $E[v, v]$ and not its decomposition by means of $u, w$. The constant $c$ depends on $k$ and $R$, however, it is bounded with respect to either $k$ and $R$.

**Lemma 4.4.** Assume that $(v, \pi_v)$ is a suitable weak solution. Then it satisfies the following inequality
\[
\int_{\mathbb{R}^d}|v|^2 \psi(t)dx + 2\int_s^t \int_{\mathbb{R}^d} |\nabla v|^2 \psi \, dx \, d\tau \leq \int_{\mathbb{R}^d}|v(s)|^2 \psi(s)dx + \int_s^t \int_{\mathbb{R}^d} |v|^2 (\psi_t + \Delta \psi) \, dx \, d\tau + \int_s^t \int_{\mathbb{R}^d}(|v|^2 + 2\pi_v) \cdot \nabla \psi \, dx \, d\tau,
\]
for all $t \geq s$, for $s = 0$ and a.e. in $s \geq 0$, and for all nonnegative $\psi \in C_0^\infty(\mathbb{R}; C_0^\infty(\mathbb{R}^3))$.

**Proof.** Taking into account the integrability properties of a suitable weak solution, all the terms in (4.8) make sense. Let us define a sequence of smooth cut-off functions $h_\eta(x)$ with $h_\eta(x) \in [0, 1]$, $h_\eta(x) = 1$ for $|x| \leq \rho$, $h_\eta(x) = 0$ for $|x| \geq 2\rho$. Then for any $\eta > 0$ and $\rho > 0$ the function $\phi^{\rho, \eta}(t, x) := J_\eta(h_\rho(x)\psi(t, x))$, where $J_\eta$ is a Friedrichs mollifier, belongs to $C_0^\infty(\mathbb{R} \times \mathbb{R}^3)$. Hence, from (1.2) written with $\phi$ replaced by $\phi^{\rho, \eta}$, passing to the limit as $\eta \to 0$ and, subsequently, as $\rho \to \infty$, by the integrability properties of $(v, \pi_v)$ and the Lebesgue dominated convergence theorem, we obtain the result.

**Lemma 4.5.** In the hypotheses of theorem 1.2, for all $t > 0$ and for all nonnegative function $\varphi \in C_0^\infty(\mathbb{R}^3)$, such that $\varphi = 0$ for $|x| \leq \max\{2R_0, 1\}$, the following inequality holds
\[
\frac{1}{2} \int_{\mathbb{R}^3} |u(t)|^2 \varphi \, dx + \int_0^t \int_{\mathbb{R}^3} \left| \nabla u \right|^2 \varphi \, dx \, dt + \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} |u|^2 \Delta \varphi \, dx \, dt + \int_0^t \int_{\mathbb{R}^3} u \cdot \nabla \varphi \pi_\rho \, dx \, dt
\]
\[
+ \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} [u^2 \cdot \nabla \varphi + |u|^2 w \cdot \nabla \varphi] \, dx \, dt
\]
\[
+ \int_0^t \int_{\mathbb{R}^3} (u \cdot \nabla u \cdot w \varphi + u \cdot \nabla \varphi u \cdot w + w \cdot \nabla u \cdot w \varphi + w \cdot \nabla \varphi u \cdot w) \, dx \, dt.
\]

**Proof.** The proof is an easy consequence of the Leray–Serrin technique. Indeed, from lemma 4.4 we can consider the generalized energy inequality (4.8) for a weak solution \(v\) with \(\psi(\tau, x) := h(\tau)\varphi(x)\), with \(\varphi \in C_0([0, 1])\) nonnegative, such that \(\varphi = 0\) for \(|x| \leq \max\{2R_0, 1\}\), and \(h \in [0, 1]\) smooth cut-off function such that \(h(\tau) = 1\) for \(\tau \in [s, t]\), \(t > s > 2\varepsilon\), \(h(\tau) = 0\) for \(\tau < \varepsilon\) and in a neighborhood of \(T\). Then the following inequality holds:

\[
\int_{\mathbb{R}^3} |v(t)|^2 \varphi \, dx + 2 \int_0^t \int_{\mathbb{R}^3} \left| \nabla v \right|^2 \varphi \, dx \, dt \leq \int_{\mathbb{R}^3} |v(s)|^2 \varphi \, dx
\]
\[
+ \int_s^t \int_{\mathbb{R}^3} |v|^2 \Delta \varphi \, dx \, dt + \int_s^t \int_{\mathbb{R}^3} (|v|^2 + 2\pi_\rho) v \cdot \nabla \varphi \, dx \, dt.
\]

(4.10)

Reasoning in an analogous way for \(w\), but recalling the regularity of \(w\) and the linear character of the equations, we deduce

\[
\int_{\mathbb{R}^3} |w(t)|^2 \varphi \, dx + 2 \int_0^t \int_{\mathbb{R}^3} \left| \nabla w \right|^2 \varphi \, dx \, dt = \int_{\mathbb{R}^3} |w(s)|^2 \varphi \, dx + \int_s^t \int_{\mathbb{R}^3} |w|^2 \Delta \varphi \, dx \, dt.
\]

(4.11)

In the weak formulation of \((v, \pi_\rho)\) we can replace the test function \(\varphi(\tau, x)\) by the function \(\psi(\tau, x)h_\rho(x)w(\tau, x)\) with \(\{h_\rho(x)\} \subset [0, 1]\) sequence of smooth cut-off functions, \(h_\rho(x) = 1\) for \(|x| \leq \rho\), \(h_\rho(x) = 0\) for \(|x| \geq 2\rho\). Then for any \(\rho > 0\) the function \(\psi(\tau, x)h_\rho(x)w(\tau, x)\) belongs to \(C_0([0, T) \times \mathbb{R}^3)\). Hence, we obtain

\[
(v(t), w(t))h_\rho = \int_{\mathbb{R}^3} \nabla v \cdot \nabla w \varphi h_\rho \, dx \, dt = (v(s), w(s))h_\rho
\]
\[
+ \int_s^t \left[ (v, w \varphi h_\rho) - (\nabla v, \nabla (\varphi h_\rho) \otimes w) + (v \cdot \nabla (\varphi h_\rho), v \cdot w) + (v \cdot \nabla w, \varphi h_\rho) + (\pi_\rho, w \cdot \nabla (\varphi h_\rho)) \right] \, dx \, dt.
\]

Recalling that \(w = \Delta w\), and observing that, by interpolation, estimate (2.6) and the energy inequality imply \(v \in L^4(0, T; L^4(\mathbb{R}^3 \setminus B_{2R_0}))\), we have

\[
(v(t), w(t))h_\rho = \int_{\mathbb{R}^3} \nabla v \cdot \nabla w \varphi h_\rho \, dx \, dt = (v(s), w(s))h_\rho
\]
\[
+ \int_s^t \left[ (v, w \Delta (\varphi h_\rho)) + (v \cdot \nabla (\varphi h_\rho), v \cdot w) + (v \cdot \nabla w, \varphi h_\rho) + (\pi_\rho, w \cdot \nabla (\varphi h_\rho)) \right] \, dx \, dt.
\]

By using the integrability properties of \(v\) and \(w\), passing to the limit as \(\rho\) tends to infinity, we find
(v(t), w(t)ϕ) + 2 \int_s^t \int_{\mathbb{R}^3} \nabla v \cdot \nabla w \varphi \, dy \, d\tau = (v(s), w(s)\varphi) \\
+ \int_s^t \left[ (v, w \Delta \varphi) + (v \cdot \nabla \varphi, v \cdot w) + (v \cdot \nabla w, \varphi v) + (\pi_v, w \cdot \nabla \varphi) \right] \, d\tau.

Multiplying this last relation by \(-2\) and summing to the inequalities (4.10)–(4.11), and also recalling that \(\varphi\) is null for \(|x| \leq \max\{2R_0, 1\}\) to ensure that \(w(t, x)\) satisfies estimate (3.5), we deduce
\[
\frac{1}{2} \int_{\mathbb{R}^3} |u(t)|^2 \varphi \, dy + \frac{1}{2} \int_{\mathbb{R}^3} |u| s \varphi \, dy + \frac{1}{2} \int_{\mathbb{R}^3} |u|^2 \Delta \varphi \, dy \\
+ \int_s^t \int_{\mathbb{R}^3} u \cdot \nabla \varphi \, dy + \int_s^t \int_{\mathbb{R}^3} [u^2 w \cdot \nabla \varphi] \, dy \\
+ \int_s^t \int_{\mathbb{R}^3} u \cdot \nabla u \cdot w + u \cdot \nabla \varphi \cdot w + w \cdot \nabla u \cdot w + w \cdot \nabla \varphi \cdot w \, dy \, d\tau. \quad (4.12)
\]

Thanks to the integrability properties of \(u\) and the regularity of \(w\) for \(|x| > 2R_0\), given in lemma 3.1, we can pass to the limit as \(\varepsilon \to 0\) and we obtain (4.9). □

**Remark 4.1.** This lemma is a relevant tool for proving lemma 4.6 on the asymptotic behavior in \(\mathbb{R}\) of the \(L^2\)-norm of \(u = v - w\) outside the ball \(B_{2R}\), uniformly in \(\alpha\). In fact, if we start from the energy inequality, we can only obtain an asymptotic behavior in \(\mathbb{R}\) for \(\alpha > \frac{3}{2}\).

Assume that \(\varphi \in C^2(\mathbb{R}^3)\) is defined as follows:
\[
m \geq 2, \quad \varphi := \varphi_R(m) := \begin{cases} \\
1 & \text{if } |x| \geq \frac{m}{m + 1} R, \\
\in [0, 1] & \text{if } |x| \in \left[ \frac{m - 1}{m} R, \frac{m}{m + 1} R \right], \\
0 & \text{if } |x| \leq \frac{m - 1}{m} R, \\
\end{cases}
\]

(4.13)

We define \(\phi_R(m)(\tau, x) := k(\tau) \varphi_R(m)(x)\) with \(k(\tau) \in [0, 1]\) smooth cut-off function such that \(k(\tau) = 1\) for \(|\tau| \leq t\) and \(k(\tau) = 0\) for \(|\tau| \geq 2t\). Since \(\phi_R(m)\) belongs to \(C^2(\mathbb{R}; C^2(\mathbb{R}^3))\), by lemma 4.4 we can use \(\phi_R(m)\) as the test function in (1.2), and we obtain the following generalized energy inequality (4.11), which we will call the generalized Leray energy inequality:
\[
\int_{\mathbb{R}^3} |v(t)|^2 \phi_R(m) \, dx + 2 \int_s^t \int_{\mathbb{R}^3} |\nabla v|^2 \phi_R(m) \, dx \, d\tau \leq \int_{\mathbb{R}^3} |v(s)|^2 \phi_R(m) \, dx \\
+ \int_s^t \int_{\mathbb{R}^3} |\nabla \phi_R(m)| \, dx \, d\tau + \int_s^t \int_{\mathbb{R}^3} (|\nabla^2 + 2\pi_v| v \cdot \nabla \phi_R(m)) \, dx \, d\tau, \quad (4.14)
\]

for all \(t > s\), for \(s = 0\) and a.e. in \(s \geq 0\).

**Lemma 4.6.** The following estimate holds:
\[
\|u(t)\|_{L^{16/5}(\mathbb{R}^3)} \leq c R^{1/2} t C(v_0), \quad R > 4R_0, t > 0,
\]

(15.14)

with \(C(v_0) := V_0^4 + V_0^2 \|v_0\|_2^2 + V_0^2 \|v_0\|_2 + V_0 \|v_0\|_2 + \|v_0\|_2^2 + \|v_0\|_2^3\).
Proof. We consider the sequence of functions (4.13). In (4.9) we replace \( \varphi \) by \( \varphi^2(k) \). With obvious meanings for the symbols on the right-hand side, we obtain

\[
\frac{1}{2} \| u(t, \varphi^2(k)) \|_{L^2}^2 + \int_0^T \int_{\mathbb{R}^N} |\nabla u|^2 \varphi^2(k) \, dy \, dt \leq \sum_{i=1}^8 |I_i(t, k)|.
\]

We estimate each \( I_i(t, k), i = 1, \ldots, 8 \). Recalling the definition of \( \varphi^2(k) \), we obtain

- \( |I_1| \leq c(k)R^{-2} \int_0^t \| u \|_{L^2(k-1,R)}^2 \, dt \);  
- by virtue of estimate (4.5), applying Hölder’s inequality, we obtain
  \[
  |I_2| \leq c(k)R^{-1} \int_0^t \| \tau_0 \|_{L^2(k-1,R)} \| u \|_{L^2(k-1,R)} \, dt \leq c(k)R^{-1} \int_0^t \| \tau_0 \|_{L^2(k-1,R)} \| u \|_{L^2(k-1,R)} \, dt \]
  \[
  \leq c(k)R^{-1} \int_0^t \left[ \| u \|_{L^2(k-2,R)}^2 \| \nabla u \|_{L^2(k-2,R)}^2 + \frac{V_0}{R^2} \| u \|_{L^2(k-2,R)}^2 \right] \| u \|_{L^2(k-1,R)} \, dt ;
  \]

- by virtue of estimate (4.1), we obtain
  \[
  |I_3| \leq c(k)R^{-1} \int_0^t \| u \|_{L^2(k-1,R)}^2 \, dt \leq c(k)R^{-1} \int_0^t \| u \|_{L^2(k-1,R)}^2 \, dt ;
  \]

- by virtue of (3.5), we have
  \[
  |I_4 + I_5| \leq c(k)R^{-1-\alpha}V_0 \int_0^t \| u \|_{L^2(k-1,R)}^2 \, dt ;
  \]

- by virtue of (3.5), applying the Hölder inequality and then the Cauchy inequality, we obtain
  \[
  |I_6| \leq cR^{-2\alpha}V_0^2 \int_0^t \| u \|_{L^2(k-1,R)}^2 \, dt \leq cR^{-2\alpha}V_0^2 \int_0^t \| u \|_{L^2(k-1,R)}^2 \, dt ;
  \]

- by virtue of (3.5), applying the Hölder inequality and then the Cauchy inequality, we obtain
  \[
  |I_7| \leq cV^jR^{-2\alpha+3\gamma} \int_0^t \| u \|_{L^2(k-1,R)}^2 \, dt ;
  \]

The above estimates allow one to deduce the following one:
\[
\frac{1}{2} \|u(t)\varphi_R(k)\|_2^2 + \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 \varphi_R^2(k) \, dy \, dt \\
\leq c(k)R^{-1} \left[ C_0(v_0) t + \int_0^t \|u\|_{L^2(k^{-2} - 2, R)}^3 \|\nabla u\|_{L^2(k^{-2} - 2, R)}^3 \, dt \right],
\]

(4.16)

with \(C_0(v_0) := V_0^d + V_0^2\|v_0\|_2^2 + V_0^3\|v_0\|_2^2 + \|v_0\|_2^2\). Writing estimate (4.16) with \(k = 2\) gives
\[
\frac{1}{2} \|u(t)\varphi_R(2)\|_2^2 + \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 \varphi_R^2(2) \, dy \, dt \\
\leq c(2)R^{-1}(C_0(v_0) + \|v_0\|_2^2) t, \quad \text{for } t \in (0, 1),
\]

(4.17)

and
\[
\frac{1}{2} \|u(t)\varphi_R(2)\|_2^2 + \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 \varphi_R^2(2) \, dy \, dt \\
\leq c(2)R^{-1}(C_0(v_0) + \|v_0\|_2^2) t, \quad \text{for } t > 1.
\]

(4.18)

which proves (4.15) for \(t \geq 1\). Taking into account estimate (4.17), we evaluate (4.16) for \(k = 4\) and \(t \in (0, 1)\), and we obtain
\[
\frac{1}{2} \|u(t)\varphi_R(4)\|_2^2 + \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 \varphi_R^2(4) \, dy \, dt \\
\leq c(4)R^{-1} R^{\frac{8}{3}}(C_0(v_0) + \|v_0\|_2^2), \quad \text{for } t \in (0, 1).
\]

(4.19)

Taking into account (4.19) and evaluating (4.16) for \(k = 6\) and \(t \in (0, 1)\), we obtain
\[
\frac{1}{2} \|u(t)\varphi_R(6)\|_2^2 + \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 \varphi_R^2(6) \, dy \, dt \\
\leq c(6)R^{-1} R^{\frac{19}{4}}(C_0(v_0) t + \|v_0\|_2^2), \quad \text{for } t \in (0, 1),
\]

which gives (4.15) for \(t \in (0, 1)\). This last estimate and (4.18) complete the proof. \(\square\)

5. Pointwise representation of the weak solution for \((t, x) \in (0, T) \times (\mathbb{R}^3 \setminus B_R)\)

The following result is similar to lemma 3.5 in [20], but we replace a \(J^2\)-continuity assumption with a \(J^2\)-weak continuity one.

**Lemma 5.1.** Let \(v(t)\) be a \(J^2(\mathbb{R}^3)\)-weakly continuous function on \(0, T\), \(\psi \in C(0, T; J^2(\mathbb{R}^3))\), and \(\psi_0 \in L^2(\mathbb{R}^3)\) such that \(\lim_{\delta \to 0} \langle v(t), \varphi \rangle = \langle \psi_0, \varphi \rangle\) for all \(\varphi \in J^2(\mathbb{R}^3)\). Then, for all \(t \in (0, T)\), the following limit property holds:
\[
\lim_{\delta \to 0} \langle v(t - \delta), \psi(\delta) \rangle = \langle v(t), \psi_0 \rangle.
\]

**Proof.** From the limit assumption on \(\psi(t)\) as \(t \to 0\) and from the \(J^2(\mathbb{R}^3)\)-strong continuity of \(\psi(t)\), we can infer that \(\psi(0) = \psi_0\). Since for any \(t \in [0, T]\), \(v(t) \in J^2(\mathbb{R}^3)\), for any \(\delta > 0\) we have
\[
|\langle v(t - \delta), \psi(\delta) \rangle - \langle v(t), \psi(0) \rangle| = |\langle v(t - \delta) - v(t), \psi_0 \rangle + \langle v(t - \delta), \psi(\delta) - \psi_0 \rangle|
\leq |\langle v(t - \delta) - v(t), \psi(0) \rangle| + \|v(t - \delta)\| \|\psi(\delta) - \psi(0)\|.
\]

Using the \(J^2\)-weak continuity for the first term on the right-hand side, and the \(J^2\)-strong continuity of \(\psi(\delta)\) for the second one, we obtain the result. \(\square\)
We premise the following regularity result of the weak solution \( v(t,x) \):

**Lemma 5.2.** In the hypotheses of theorem 1.2 we obtain \( v(t,x) \in C(0,T;L^\infty(\mathbb{R}^3 \setminus B_0)) \), with \( R := 2M_0R_0, M_0 \) given in lemma 2.3.

**Proof.** Let \( \{ h_\eta \} \subset [0,1] \) be a sequence of smooth cut-off functions with \( h_\eta(t-\tau) = 1 \) for \( t-\tau > 2\eta \), \( h_\eta(t-\tau) = 0 \) for \( t-\tau < \eta \). Let us consider the Navier–Stokes weak formulation corresponding to a solution \((v,\pi)\) written on the interval \((0,t-2\eta)\), with \( h_\eta(t-\tau)\varphi_\eta(4)\theta(\tau,x) \) as the test function. The function \( \varphi_\eta(4) \) is defined in (4.13), with \( R = 2M_0R_0 \), while \( \theta := \theta(t-\tau,x) \) for \( \tau \in (0,t) \), with the \( \theta(\sigma,x) \) solution to the Stokes Cauchy problem (3.1) given in lemma 3.2. It is known that \( \theta(\sigma) \) is a backward in time solution to the Stokes Cauchy problem on \((0,t) \times \mathbb{R}^3 \). In the following, since there is no danger of confusion, we denote \( \varphi_\eta(4) \) simply by \( \varphi \). Hence, after substituting, we obtain

\[
(v(t-2\eta),\varphi\theta(2\eta)) = (v_\omega,\varphi\theta(t)) + 2\int_0^{t-2\eta} (v, \nabla \varphi \cdot \nabla \theta) d\tau + \int_0^{t-2\eta} (v, \theta \Delta \varphi) d\tau \]

\[
- \int_0^{t-2\eta} (v \cdot \nabla \varphi, \varphi \theta) d\tau + \int_0^{t-2\eta} (\pi, \nabla \varphi \cdot \theta) d\tau. \tag{5.1}
\]

The same relation written on the interval \((0,s-2\eta)\) and with the test function \( h_\eta(s-\tau)\varphi_\eta(4)\theta(\tau,x) \) furnishes

\[
(v(s-2\eta),\varphi\theta(2\eta)) = (v_\omega,\varphi\theta(s)) + 2\int_0^{s-2\eta} (v, \nabla \varphi \cdot \nabla \theta) d\tau + \int_0^{s-2\eta} (v, \theta \Delta \varphi) d\tau \]

\[
- \int_0^{s-2\eta} (v \cdot \nabla \varphi, \varphi \theta) d\tau + \int_0^{s-2\eta} (\pi, \nabla \varphi \cdot \theta) d\tau. \tag{5.2}
\]

From lemma 5.1 one has

\[
\lim_{\eta \to 0}(v(t-2\eta),\theta(2\eta)) = (v(t),\theta_0), \text{ and } \lim_{\eta \to 0}(v(s-2\eta),\theta(2\eta)) = (v(s),\theta_0).
\]

Passing to the limit as \( \eta \) tends to 0 on the right-hand side of (5.1) and (5.2), we can use the Lebesgue dominated convergence theorem, observing that the integrals on \((0,t)\), such as the integrals on \((0,s)\), are finite thanks to estimate (3.6) with \( q \leq \frac{3}{2} \). Hence, we obtain

\[
(v(t),\varphi\theta_0) = (v_\omega,\varphi\theta(t)) + 2\int_0^t (v, \nabla \varphi \cdot \nabla \theta) d\tau + \int_0^t (v, \theta \Delta \varphi) d\tau \]

\[
- \int_0^t (v \cdot \nabla \varphi, \varphi \theta) d\tau + \int_0^t (\pi, \nabla \varphi \cdot \theta) d\tau,
\]

and

\[
(v(s),\varphi\theta_0) = (v_\omega,\varphi\theta(s)) + 2\int_0^s (v, \nabla \varphi \cdot \nabla \theta) d\tau + \int_0^s (v, \theta \Delta \varphi) d\tau \]

\[
- \int_0^s (v \cdot \nabla \varphi, \varphi \theta) d\tau + \int_0^s (\pi, \nabla \varphi \cdot \theta) d\tau.
\]

Then, we deduce
\[(v(t) - v(s), \varphi_0) = (v_0, \varphi(\theta(t) - \theta(s))) + 2 \int_s^t (v, \nabla \varphi \cdot \nabla \theta') d\tau + \int_s^t (v, \theta' \Delta \varphi) d\tau \]

\[- \int_s^t (v \cdot \nabla v, \varphi\theta') d\tau + \int_s^t (\pi_s, \nabla \varphi \cdot \theta') d\tau \]

\[+ \sum_{i=1}^3 2 \int_{\chi_{i+1}}^\phi (v, \nabla \varphi \cdot (\nabla \theta' - \nabla \theta')) d\tau + \int_{\chi_{i+1}}^\phi (v, (\theta' - \theta') \Delta \varphi) d\tau \]

\[- \int_{\chi_{i+1}}^\phi (v \cdot \nabla v, \varphi(\theta' - \theta^\prime)) d\tau + \int_{\chi_{i+1}}^\phi (\pi_s, \nabla \varphi \cdot (\theta' - \theta')) d\tau \], \quad (5.3)

with \(s_1 = 0, s_2 = \varepsilon, s_3 = s - \varepsilon\) and \(s_4 = s\). We limit ourselves to estimating the first term, all the integrals involving the nonlinear term \(v \cdot \nabla v\) and the pressure field, as the others are simpler to discuss.

(i) Applying Hölder’s inequality and the semigroup properties (3.7) of the solution \(\theta\), we obtain

\[|v_0, \varphi(\theta(t) - \theta(s))| \leq \|v_0\|_2 \|\theta(t) - \theta(s)\|_2 \leq c \|v_0\|_2 \xi^{-1} (t - s) \|\theta_0\|;\]

(ii) applying Hölder’s inequality and the semigroup properties (3.6), and recalling estimate (2.6) for the weak solution \(v\), for \(p = 6\) we obtain

\[\int_s^t (v \cdot \nabla v, \varphi\theta') d\tau \leq \int_s^t \left[\||v|^2 \nabla \varphi||\theta'||_H + \||v|^2 \varphi|\nabla \theta'||_H\right] d\tau \]

\[\leq \int_s^t \left[\|v\|_{L^2(\Omega)} \|\theta'||_H + \|v\|_{L^2(\Omega)} \|\nabla \theta'||_H\right] d\tau \]

\[\leq \int_s^t \left[\frac{2}{5} \|v\|_{L^2(\Omega)} \left(\|\theta'||_H + \|\nabla \theta'||_H\right)\right] d\tau \]

\[\leq \left\|v_0\right\|_2 \left(c \left\|v_0\right\|_2 \frac{s}{5} \left|t - s\right| + \left|t - s\right|^\frac{1}{2}\right) \|\theta_0\|;\]

(iii) applying Hölder’s inequality and the semigroup properties (3.6) of \(\theta\), and taking into account that \(t - \tau > s - \tau\), we obtain

\[\int_{\tau_{s_1}}^{\tau_{s_2}} (v \cdot \nabla v, \varphi(\theta' - \theta^\prime)) d\tau \leq \int_{\tau_{s_1}}^{\tau_{s_2}} \|v \cdot \nabla \theta'||_H + \|v \cdot \nabla \theta^\prime||_H d\tau \]

\[\leq \int_{\tau_{s_1}}^{\tau_{s_2}} \|v\|_2 \|\nabla \theta'||_H + \|v\|_2 \|\nabla \theta^\prime||_H d\tau \]

\[\leq c \left\|v_0\right\|_2 \left[\int_0^{\tau_{s_1}} (s - \tau)^{-3} d\tau\right]^\frac{1}{2} \left[\int_0^{\tau_{s_1}} \|\nabla v\|_2^2 d\tau\right]^\frac{1}{2} \left\|\theta_0\right\|_2 \leq c \left\|v_0\right\|_2 \left(\frac{2(s - \varepsilon)^{-\frac{1}{2}}}{s(s - \varepsilon)}\right) \|\theta_0\|;\]

(iv) applying Hölder’s inequality and the semigroup properties (3.7) of \(\theta\), and recalling estimate (2.6) for \(v\) and that \(s - \tau < t - \tau\), for \(p = 6\) we obtain

1370
\[
\left| \int_{t_2}^{t_1} (v \cdot \nabla \varphi, v \cdot (\theta' - \theta')) \, dt \right| + \left| \int_{t_2}^{t_1} (v \cdot \nabla(\theta' - \theta'), v \varphi) \, dt \right|
\leq \int_{t_2}^{t_1} \left[ \|v\|_{L^p}^2 \|\nabla(\theta' - \theta')\|_{L^q} + \|v\|_{L^p}^2 \|\theta' - \theta\|_{L^r} \right] \, dt
\leq c \|v\|_{L^2}^2 c(v_0)^2 \|\theta_0\| \int_0^{t_1} \left[ \frac{\xi^{1-p} + \xi^{1-r}}{\tau^2} \right] \, d\tau
\leq c \|v\|_{L^2}^2 c(v_0)^2 \int_0^{t_1} (t - s) \|\theta_0\| \, ds.
\]

(v) applying Hölder’s inequality and the semigroup properties (3.6), and recalling estimate (2.6) for the weak solution \(v\) and that \(s - \tau < t - \tau\), for \(p = 6\) we obtain

\[
\left| \int_{t_2}^{t_1} (v \cdot \nabla \varphi, v \cdot (\theta' - \theta')) \, dt \right| + \left| \int_{t_2}^{t_1} (v \cdot \nabla(\theta' - \theta'), v \varphi) \, dt \right|
\leq c \|v\|_{L^2}^2 c(v_0)^2 \left[ \int_0^{t_1} \frac{\|\nabla \theta'\|_{L^q} + \|\nabla \theta'\|_{L^q} + \|\theta'\|_{L^q} + \|\theta'\|_{L^q}}{\tau^2} \, d\tau \right]
\leq c \|v\|_{L^2}^2 c(v_0)^2 \left[ \frac{1}{\tau^2} + \frac{3}{\tau^2} \right] \|\theta_0\|.
\]

Finally, we estimate the terms with the pressure field \(\pi\). To this end, taking into account that, by definition, \(\nabla \varphi\) is nonnull on \(B_{2g} \setminus B_{2g}\), we can use estimate (4.7) for \(k = 3\):

\[\|\pi\|_{L^2(\Omega)} \leq c \|v\|_{L^2}^2 + c(v_0)^2 c(v_0)^2 \|\theta_0\|_{\Omega}, \quad \text{a.e. in } t > 0.\]

By using (2.6) and the energy relation, we obtain

\[\|\pi\|_{L^2(\Omega)} \leq c \left( \|v\|_{L^2}^2 + c(v_0) c(v_0) \right) \|\theta_0\| \tag{5.4}\]

We start with the estimate on \((s, t)\):

(j) applying Hölder’s inequality, the semigroup properties (3.6) for \(\theta\) and estimate (5.4), we deduce

\[\int_s^t \|\pi\|_{L^2(\Omega)} \, ds \leq c \left( \|v\|_{L^2}^2 + c(v_0)^2 c(v_0) \right) (t - s) \|\theta_0\| \tag{5.4} \]

(jj) using the same arguments as before, and taking into account that \(s - \tau < t - \tau\), we obtain

\[\int_{s_1}^{s_2} \|\pi\|_{L^2(\Omega)} \, ds \leq c \int_0^\tau \|\theta_0\| \left[ \int_0^\tau \left( \|v\|_{L^2}^2 + c(v_0) \right) \left( t - s - \frac{1}{2} \right) \, ds \right] \, d\tau
\leq c \left( \frac{s - \tau}{2} \right) \|\theta_0\| \int_0^\tau \left( \|v\|_{L^2}^2 + c(v_0) \right) \left( t - s - \frac{1}{2} \right) \, d\tau
\leq c \left( \frac{s - \tau}{2} \right) \left( \|v\|_{L^2}^2 + c(v_0) \right) \left( t - s - \frac{1}{2} \right) \|\theta_0\| \tag{5.4}.\]
(jjj) using the same arguments and employing estimate (3.7), we obtain
\[
\left| \int_{s_2}^s (\pi_\tau, \nabla \varphi \cdot (\theta^\tau - \theta^0)) d\tau \right| \leq c \int_{s_2}^s \| \pi_\tau \|_{L^2(\mathbb{R}^d)} \| \theta^\tau - \theta^0 \|_2 d\tau \\
\leq c \left( \| v_\tau \|_2^2 + \| v_\tau \|_2 c(v_\tau) (s - s)^{-\frac{1}{2}} \right) \int_{s_2}^s \| \theta^\tau - \theta^0 \|_2 d\tau \\
\leq c \left( \| v_\tau \|_2^2 + \| v_\tau \|_2 c(v_\tau) (s - s)^{-\frac{1}{2}} \right) \varepsilon^2 x (t - s) \| \theta_0 \|_1 ;
\]
(jv) finally,
\[
\left| \int_{s_2}^s (\pi_\tau, \nabla \varphi \cdot (\theta^\tau - \theta^0)) d\tau \right| \leq c \int_{s_2}^s \| \pi_\tau \|_{L^2(\mathbb{R}^d)} \| \theta^\tau - \theta^0 \|_2 d\tau \\
\leq c \left( \| v_\tau \|_2^2 + \| v_\tau \|_2 c(v_\tau) (s - s)^{-\frac{1}{2}} \right) \int_{s_2}^s \| \theta^\tau - \theta^0 \|_2 d\tau \\
\leq c \left( \| v_\tau \|_2^2 + \| v_\tau \|_2 c(v_\tau) (s - s)^{-\frac{1}{2}} \right) \varepsilon^2 x (t - s) \| \theta_0 \|_1 .
\]

Considering estimates (i)–(v) and (j)–(jv), and the corresponding estimates for the linear terms of relation (5.3), we deduce
\[
\| (v(t) - v(s), \varphi(\theta_0)) \| \leq F(\varepsilon, s, t, v_\tau) \| \theta_0 \|_1 , \text{ for arbitrary } \theta_0 \in C_0(\mathbb{R}^d),
\]
with a clear meaning for function $F$, hence
\[
\| (v(t) - v(s)) \varphi \|_\infty \leq F(\varepsilon, s, t, v_\tau).
\]

Since we can assume $t - s < \varepsilon^4$, for fixed $s > 0$ and $v_\tau$ the above estimates ensure $\lim_{\varepsilon \to 0} F(\varepsilon, s, t, v_\tau) = 0$, and the lemma is proved. \qed

In the following lemma we give the representation formula for the weak solution $(v, \pi_\tau)$ provided that $|x| \geq 2M_d R_0$. To this end, we recall that the representation formula is given by means of the fundamental heat kernel $H$ (for its definition and properties see (3.2)–(3.3)) and the Oseen tensor $T$ (see [13]):
\[
T_j(t - \tau, x - y) := -\Delta \phi(t - \tau, |x - y|) \delta_{ij} + D^2 \phi(t - \tau, |x - y|)
= H_j(t - \tau, x - y) + D^2 \phi(t - \tau, |x - y|),
\]
\[
\phi(t, r) = \frac{1}{2} \frac{1}{\sqrt{\pi}} \frac{1}{r} \int_0^{rt} e^{-r^2 d\rho}.
\]
We denote by $T_j(t - \tau, x - y)$ the $j$-th column of the matrix $T_j$. The pair $(T_j(t - \tau, x - y), \rho)$, with $p = 0$, for $t - \tau > 0$ is a solution in the $(t, x)$ variables of the Stokes system, and in the $(\tau, y)$ variables of its adjoint system:
\[
\hat{\omega} + \Delta \hat{\omega} + \nabla p = 0, \quad \nabla \cdot \hat{\omega} = 0.
\]

The following estimate holds:
\[
|D^j\pi_\tau^0 T(s, z)| \leq c(|z| + s^2)^{-3 - 2k - |\beta|}.
\]
Finally, we recall that from the definition of $T$ we obtain

$$(T_j(t, x), \varphi) = (H_j(t, x), \varphi), \text{ for all } \varphi \in L^p(\mathbb{R}^3), \ p > 1.$$ 

We set

$$T[v \otimes v](s, t, x) := \int_s^t (v \cdot \nabla T_j(t, x), v) d\tau.$$ 

If $s = 0$, we simply write $T[v \otimes v](t, x)$.

**Lemma 5.3.** Let $(v, \pi)$ be a suitable weak solution to the Navier–Stokes equations. Then, for all $t > 0$ and $s \geq 0$, and almost a.e. in $x \in \mathbb{R}^3 \setminus B_{2M}^c$, 

$$v(t, x) = \mathbb{H}[v(s)](t - s, x) + T[v \otimes v](s, t, x). \quad (5.6)$$ 

Moreover, for $t > T_0$ and for $x \in \mathbb{R}^3$, 

$$v(t, x) = \mathbb{H}[v(T_0)](t - T_0, x) + T[v \otimes v](T_0, t, x). \quad (5.7)$$

**Proof.** Let us consider the weak formulation with a divergence free test function:

$$(v(t), \varphi(t)) = (v(s), \varphi(s)) - \int_s^t (v, \varphi_t + \Delta \varphi) d\tau + \int_s^t (v \cdot \nabla \varphi, v) d\tau. \quad (5.8)$$

We set, for all $i = 1, 2, 3$, and $t - \varepsilon > s$,

$$\varphi(\tau, y) := h(\tau)J_0[T(t - \tau, x)](y), \text{ with } h(\tau) := \begin{cases} 
1 & \text{if } \tau \leq t - \varepsilon, \\
\in [0, 1] & \text{if } \tau \in [t - \varepsilon, t - \frac{\varepsilon}{4}], \\
0 & \text{if } \tau > t - \frac{\varepsilon}{4},
\end{cases}$$

where $J_0$ is a Friedrichs mollifier. Hence, inserting such a $\varphi$ in (5.8) with $t$ replaced by $t - \varepsilon$, and recalling that $T_j$ is a solution backward in time with respect to $(\tau, y) \in (0, t) \times \mathbb{R}^3$, we obtain

$$(v_j(t - \varepsilon), J_0[H(\varepsilon, .)]) = (v_j(s), J_0[H(t - s, .)]) + \int_s^{t-\varepsilon} (v \cdot \nabla J_0[T(t - \tau, x)], v) d\tau. \quad (5.9)$$

We perform the limit as $\varepsilon \to 0$. To this end, we deal separately with the terms of the last integral equation.

(i) The first term can be written as:

$$
(v_j(t - \varepsilon), J_0[H(\varepsilon, .)]) - J_0[v_j(t)](x) = 
\int_{\mathbb{R}^3} v_j(t - \varepsilon, y) \int_{\mathbb{R}^3} J_0(y - z) H(\varepsilon, x - z) dz dy 
- \int_{\mathbb{R}^3} J_0(x - y) v_j(t, y) dy
$$

$$
= \int_{\mathbb{R}^3} H(\varepsilon, x - z) \left[ \int_{\mathbb{R}^3} J_0(y - z) v_j(t - \varepsilon, y) dy 
- \int_{\mathbb{R}^3} J_0(x - y) v_j(t, y) dy \right] dz
$$

$$
= \int_{|x - z| > \eta} H(\varepsilon, x - z) \left[ \int_{\mathbb{R}^3} J_0(y - z) v_j(t - \varepsilon, y) dy 
- \int_{\mathbb{R}^3} J_0(x - y) v_j(t, y) dy \right] dz
$$

$$
+ \int_{|x - z| < \eta} H(\varepsilon, x - z) \int_{\mathbb{R}^3} J_0(y - z) [v_j(t - \varepsilon, y) - v_j(t, y)] dy dz
$$

$$
+ \int_{|x - z| < \eta} H(\varepsilon, x - z) \int_{\mathbb{R}^3} [J_0(y - z) - J_0(x - y)] v_j(t, y) dy dz
= \sum_{i=1}^3 I(\varepsilon, \eta).$$
Thanks to the energy inequality, the $L^2$-norm of $v$ is finite for all $t > 0$. Hence, for all $n \in \mathbb{N}$, 
\[
\left[ \int_{\mathbb{R}^3} J_n(y - z) v(t - \varepsilon, y) \mathrm{d}y - \int_{\mathbb{R}^3} J_n(x - y) v(t, y) \mathrm{d}y \right] \text{ belongs to } L^\infty(\mathbb{R}^3), \text{ uniformly with respect to } \varepsilon.
\]
Therefore it is easy to deduce that for all $x \in \mathbb{R}^3$ and $\eta > 0$, 
\[
\lim_{\varepsilon \to 0} I^1(\varepsilon, \eta) = 0.
\]
For the term $I_2$, for $\eta > 0$ sufficiently small and $n$ sufficiently large, we obtain 
\[
|I_2(\varepsilon, \eta)| \leq \|H(\varepsilon, x)\|_{L^3(B(0, \eta))} \|J_n[v(t - \varepsilon) - v(t)]\|_{L^\infty(\mathbb{R}^3)} \\
\leq \|H(\varepsilon, x)\|_{L^3(B(0, \eta))} \|v(t - \varepsilon) - v(t)\|_{L^\infty(\mathbb{R}^3)},
\]
and, by the continuity proved in lemma 5.2, we deduce, for all $|x| > 2M_0 R_0$ and $\eta > 0$, 
\[
\lim_{\varepsilon \to 0} I_2(\varepsilon, \eta) = 0.
\]
Finally, there holds 
\[
|I(\varepsilon, \eta)| \leq \|H(\varepsilon, x)\|_{L^3(B(0, \eta))} \|J_n[v(t)](z) - J_n[v(t)](x)\|_{C(B(0, \eta))} \\
\leq \|J_n[v(t)](z) - J_n[v(t)](x)\|_{C(B(0, \eta))}, \text{ for all } \varepsilon > 0.
\]
Since, for all $n \in \mathbb{N}$, $J_n[v(t)](z)$ is a continuous function of $z$, we deduce that 
\[
\lim_{\eta \to 0, \varepsilon \to 0} |I(\varepsilon, \eta)| \leq \lim_{\eta \to 0} \|J_n[v(t)](z) - J_n[v(t)](x)\|_{C(B(0, \eta))} = 0.
\]
Hence, for all $n \in \mathbb{N}$, we have proved that 
\[
\lim_{\varepsilon \to 0} (v(t - \varepsilon), J_n[H(\varepsilon, x)]) = J_n[v(t)](x). \tag{5.10}
\]
(ii) Trivially, the second term admits a limit for $\varepsilon \to 0$.
(iii) For the last term it is enough to note that for all $n \in \mathbb{N}$, 
\[
\nabla J_n[T(t - \tau, x)] \in L^1(0, t; L^\infty(\mathbb{R}^3))
\]
and $v \in L^\infty(0, T; L^2(\mathbb{R}^3))$; then, applying the Lebesgue dominated convergence theorem, 
we deduce the limit property 
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} (v \cdot \nabla J_n[T(t - \tau, x)]), v) \mathrm{d}\tau = \int_{\mathbb{R}^3} (v \cdot \nabla J_n[T(t - \tau, x)]), v) \mathrm{d}\tau. \tag{5.11}
\]
So that from (5.9), via (5.10)-(5.11), we have proved 
\[
J_n[v(t)](x) = \mathbb{H}[J_n[v(s)]](t - s, x) + \int_s^t (v \cdot \nabla J_n[T(t - \tau, x)]), v) \mathrm{d}\tau,
\]
for $(t, x) \in (0, T) \times (\mathbb{R}^3 \setminus B_{2M_0 R_0})$. \tag{5.12}
Now, we perform the limit as $n \to \infty$. We begin by remarking that for all $t$ and $s$, \{\{J_n[v(t)]\}\} and \{J_n[v(s)]\} converge in $L^2(\mathbb{R}^3)$. So there exists a subsequence, labeled again by $n$, converging almost everywhere in $(t, x)$ to $v(t, x)$, and to $v(s, x)$ in $L^2(\mathbb{R}^3)$. Therefore, recalling that for all $t - s > 0$, 
$H(t - s, x - y) \in L^2(\mathbb{R}^3)$, almost everywhere in $x \in \mathbb{R}^3 \setminus B_{2M_0 R_0}$ the following limit properties hold:
\[
\lim_n J_n[v(t)](x) = v(t, x) \text{ and } \lim_n \mathbb{H}[J_n[v(s)]](t - s, x) = \mathbb{H}[v(s)](t - s, x). \tag{5.13}
\]
Since for $|x| > 2M_0R_0$

$$
\lim_{n} J_n[\nabla T(t - \tau, x)](y) = \nabla T(t - \tau, x - y) \quad \text{and} \quad |\nabla T(t - \tau, x - y)| \leq C(x),
$$
for all $(\tau, y) \in (0, t) \times B_{2M_0R_0}$,

and $v \in L^\infty(0, T; L^2(\mathbb{R}^3))$, the following limit holds for all $|x| > 2M_0R_0$:

$$
\lim_{n} \int_{s}^{t} \int_{|y| < 2M_0R_0} v(\tau, y) \cdot J_n[\nabla T(t - \tau, x)](y) \cdot v(\tau, y) dy d\tau
= \int_{s}^{t} \int_{|y| < 2M_0R_0} v(\tau, y) \cdot \nabla T(t - \tau, x - y) \cdot v(\tau, y) dy d\tau.
$$

(5.14)

Since $\nabla T(s, z) \in L^1(0, T; L^1(\mathbb{R}^3))$, then $J_n[\nabla T(t - \tau, x)](y)$ converges to $\nabla T(t - \tau, x - y)$ in $L^1(0, T; L^1(\mathbb{R}^3))$. Moreover, since by (2.6)

$$
\|v(t)\|_{L^\infty(|y| > 2M_0R_0)} \leq c(v_0)^{\frac{1}{\tau}},
$$

the following limit holds for all $|x| > 2M_0R_0$:

$$
\lim_{n} \int_{s}^{t} \int_{|y| > 2M_0R_0} v(\tau, y) \cdot J_n[\nabla T(t - \tau, x)](y) \cdot v(\tau, y) dy d\tau
= \int_{s}^{t} \int_{|y| > 2M_0R_0} v(\tau, y) \cdot \nabla T(t - \tau, x - y) \cdot v(\tau, y) dy d\tau.
$$

(5.15)

So that passing to the limit in (5.12), thanks to (5.13)–(5.15), we obtain

$$
v_{\bar{v}}(t, x) = \mathbb{H}[v_{\bar{v}}(x)](t - s, x) + \int_{s}^{t} (v \cdot \nabla T(t, x), v) d\tau,
\quad \text{for } (t, x) \in (0, T) \times (\mathbb{R}^3 \setminus B_{2M_0R_0}).
$$

(5.16)

This proves (5.6) for $s > 0$. Let us show its validity for $s = 0$. Since for $t > 0$ a solution to the heat equation is a continuous function, and since the weak solution $v$ is a continuous function in $s = 0$ in the $L^2$-norm, we easily obtain

$$
\lim_{s \to 0} \mathbb{H}[v_{\bar{v}}(s)](t - s, x) = \lim_{s \to 0} \mathbb{H}[v_{\bar{v}}(s)](t - s, x) + \lim_{s \to 0} \mathbb{H}[v_{\bar{v}}(s) - v_{\bar{v}}](t - s, x)
= \mathbb{H}[v_{\bar{v}}(t, x)], \quad \text{a.e. in } x \in \mathbb{R}^3.
$$

(5.17)

For the integral term we have to verify that the integral is well posed on $(0, t) \times \mathbb{R}^3$. Noting that

$$
|x| > 2M_0R_0 \Rightarrow |\nabla T(t - \tau, x - y)| \leq c(x), \quad \text{for all } (\tau, y) \in (0, t) \times B_{2M_0R_0},
$$

we deduce

$$
\int_{|y| < 2M_0R_0} |v \cdot \nabla T(t, x) \cdot v| dy \leq c(x) \|v\|_{L^2}^2 \leq c \|v_0\|_{L^2}^2, \quad \text{for all } \tau \in (0, t).
$$

Moreover, by virtue of (2.6) we have
\[\|v(t)\|_{L^{\frac{3}{2}}(\mathbb{R}^d)} \leq \|v(t)\|_{L^{\frac{3}{4}}(\mathbb{R}^d)}^{\frac{1}{2}} \leq c\|v(0)\|_{L^{\frac{3}{4}}(\mathbb{R}^d)}^{\frac{1}{2}}\|v(t)\|_{L^{\frac{3}{2}}(\mathbb{R}^d)}^{\frac{1}{2}} \leq c\|v_0\|_{L^{\frac{3}{4}}(\mathbb{R}^d)}^{\frac{1}{2}} c(\|v_0\|_{L^{\frac{3}{2}}(\mathbb{R}^d)}^{\frac{1}{2}} t^{-\frac{1}{2}} )^{-\frac{1}{2}}.\]

Therefore, applying Hölder’s inequality, we deduce
\[
\int_{|y| > 2M_0 R_0} |v \cdot \nabla T_t(t, x) \cdot v| \, dy \leq \|\nabla T_t(t, x)\|_{L^1} \|v\|_{L^{\frac{3}{2}}(\mathbb{R}^d)}^{\frac{1}{2}} \leq c(t - \tau)^{-\frac{3}{2} - \frac{2d}{3} - \frac{2}{3} - \frac{2}{3} - 2}.\]

The above estimates and the limit property (5.17) allow us to make the limit in (5.16) \(s \to 0\), and complete the proof of estimate (5.6).

Finally, since the solution is regular for \(t > T_0\) (see [16]), we can repeat the above argument lines with obvious simplifications, starting from (5.9) and obtaining (5.7).

6. Spatial behavior of the weak solution: proof of theorem 1.2

For all \(a, b < 1\), we set
\[A := \int_0^1 \tau^{-\alpha}(1 - \tau)^{-b} \, d\tau.\]

Hence, we obtain
\[
\int_0^t \tau^{-\alpha}(t - \tau)^{-b} \, d\tau = At^{-1-a-b}, \text{ for all } t > 0.\tag{6.1}
\]

To prove theorem 1.2, as a first step we prove estimate (1.3) only on the interval \((0, 1)\) and \(|x| > R_1 := \frac{14}{3} M_0 R_0\).

Our starting point is the representation formula (5.6), which we write as follows:
\[v(t, x) = w(t, x) + \mathbb{T}[w \otimes u](t, x) + \mathbb{T}[u \otimes w](t, x) + \mathbb{T}[w \otimes w](t, x),\tag{6.2}
\]

where \(w\) is the solution of the Stokes problem whose properties were established in lemma 3.1, and \(u = v - w\). Recall that thanks to lemma 4.6, \(u\) satisfies estimate (4.15). We introduce the following decomposition:
\[\mathbb{T}[a \otimes b](t, x) := \mathbb{T}^{(1)}[a \otimes b](t, x) + \mathbb{T}^{(2)}[a \otimes b](t, x),\tag{6.3}\]

where
\[
\mathbb{T}^{(1)}[a \otimes b](t, x) := \int_0^t \int_{|\tau - y| < \frac{1}{2}|x|} a(\tau, y) \otimes b(\tau, y) \cdot \nabla T(t - \tau, x - y) \, dy \, d\tau,
\]
\[
\mathbb{T}^{(2)}[a \otimes b](t, x) := \int_0^t \int_{|\tau - y| < \frac{1}{2}|x|} a(\tau, y) \otimes b(\tau, y) \cdot \nabla T(t - \tau, x - y) \, dy \, d\tau.
\]

We estimate the terms on the right-hand side of (6.2). Our task is to prove that all the terms satisfy the bound given in (1.3).

(i) For the solution \(w\) estimate (1.3) follows from lemma 3.1.

(ii) For \(|y| < \frac{6}{7}|x|\), if \(a\) and \(b \in L^\infty(0, T; L^2(\mathbb{R}^3))\), employing estimate (5.5), we easily deduce...
\[ |T^{(1)}[a \otimes b](t,x)| \lesssim c \int_0^t \int_{|y| \leq \frac{2}{\tau} |x|} (|x-y| + (t-\tau)^\frac{1}{2})^{-4} |a| |b| dyd\tau \lesssim c |x|^{-1.5} \sup_{(0,t)} ||a|| \cdot ||b||. \]

Since \( u \) and \( w \) belong to \( L^\infty(0,T;L^2(\mathbb{R}^3)) \) and \( \alpha \in [1,3) \), the above estimate ensures that
\[
|T^{(1)}[u \otimes u](t,x)| + |T^{(1)}[u \otimes w](t,x)| + |T^{(1)}[w \otimes u](t,x)| + |T^{(1)}[w \otimes w](t,x)| \lesssim c(v_0)(1 + |x|)^{-\alpha}, (t,x) \in (0,1) \times \mathbb{R}^3 \backslash B_R.
\]

(iii) First of all, we note that
\[
|T^{(2)}[u \otimes w](t,x) + T^{(2)}[w \otimes u](t,x)| \leq 2 \int_0^t \int_{|y| > \frac{2}{\tau} |x|} (|x-y| + (t-\tau)^\frac{1}{2})^{-4} |u| |w| dyd\tau.
\]

Also, since \(|y| > \frac{6}{7} |x| > 4M_0 R_0\), then (2.6) and (3.5) imply in particular
\[
|u(t,y)| \lesssim c(v_0) r^{-\frac{1}{2}}.
\]

Hence, from the above integral inequality and again using (3.5) for \( w \), and recalling (6.1), it follows that
\[
|T^{(2)}[u \otimes w](t,x) + T^{(2)}[w \otimes u](t,x)| \leq 2c(v_0)(1 + |x|)^{-\alpha} \int_0^t \int_{|y| > \frac{2}{\tau} |x|} (|x-y| + (t-\tau)^\frac{1}{2})^{-4} \tau^{-\frac{1}{4}} dyd\tau = c(v_0)(1 + |x|)^{-\alpha},
\]

\((t,x) \in (0,T) \times \mathbb{R}^3 \backslash B_R\).

(iv) In this step, we estimate \( T^{(2)}[u \otimes u](t,x) \) by applying a bootstrap argument, starting from estimate (6.4).

(iv) Recalling definition (4.4) and using the interpolation between \( L^q \)-spaces, estimates (4.15) and (6.4) give
\[
\|u\|_{L^\infty(0,1;L^6(|x|))} \lesssim \|u\|_{L^6(0,1;L^6(|x|))} \lesssim c(v_0) t^{-\frac{11+2\epsilon}{3-2\epsilon}} |x|^{-\frac{1}{2} - \frac{2\epsilon}{2-3\epsilon}}, t > 0,
\]

where, here and in the following, \( \epsilon \in (0,\frac{1}{2}) \) and \(|x| > R\). Applying H"older’s inequality, and then estimates (6.5) and (6.1), we obtain
\[
|T^{(2)}[u \otimes u](t,x)| \leq c \int_0^t \|\nabla T(t,x)\|_L^2 \frac{1}{\tau} \|u\|_{L^\infty(0,1;L^6(|x|))}^2 d\tau \lesssim c(v_0) |x|^{-\frac{1}{2} - \frac{2\epsilon}{2-3\epsilon}} \int_0^t (t-\tau)^{-\frac{3}{3-2\epsilon}} \tau^{-\frac{3}{3-2\epsilon}} d\tau \lesssim c(v_0) r^{-\frac{1}{3-2\epsilon}} |x|^{-\frac{1}{2} - \frac{2\epsilon}{2-3\epsilon}}.
\]

(iv) Now, from formula (6.2) and estimates (i)-(iv), we obtain
\[
|v(t,x)| \lesssim c(v_0) t^{-\frac{1}{3-2\epsilon}} |x|^{-\frac{1}{2} - \frac{2\epsilon}{2-3\epsilon}}, (t,x) \in (0,1) \times \mathbb{R}^3 \backslash B_R.
\]

On the other hand, taking into account (3.5) and that \( u = v - w \),
\[
|u(t,x)| \lesssim c(v_0) t^{-\frac{1}{3-2\epsilon}} |x|^{-\frac{1}{2} - \frac{2\epsilon}{2-3\epsilon}}, (t,x) \in (0,1) \times \mathbb{R}^3 \backslash B_R.
\]
Employing estimates (4.15) and (6.6), we modify (6.5) as follows:

$$
\|u\|_{L^\infty((6,|x|))} \leq \|u\|_{L^2((6,|x|))}^{1-2\varepsilon} \frac{1}{\varepsilon} \|\nabla T(t,x)\|_{L^\infty((6,|x|))}^{1-2\varepsilon} \frac{2}{\varepsilon^2} \\
\leq c(v_0) \frac{1}{\varepsilon} \|u\|_{L^\infty((6,|x|))}^{1-2\varepsilon} \frac{1}{\varepsilon} \frac{4-8\varepsilon}{(3-2\varepsilon)^2} \frac{8-16\varepsilon}{(3-2\varepsilon)^3} \frac{1}{\varepsilon^2}, \ t \in (0,1).
$$

(6.7)

We evaluate $T^2[u \otimes u](t,x)$ via (6.7):

$$
|T^2[u \otimes u](t,x)| \leq c \int_0^t \|\nabla T(t,x)\|_{L^\infty((6,|x|))}^{\frac{1}{2}} \|u\|_{L^\infty((6,|x|))}^{\frac{1}{2}} \frac{1}{\varepsilon} \frac{4-8\varepsilon}{(3-2\varepsilon)^2} \frac{8-16\varepsilon}{(3-2\varepsilon)^3} d\tau \\
\leq c(v_0) \frac{1}{\varepsilon} \frac{1-2\varepsilon}{3-2\varepsilon} \frac{2}{\varepsilon^2} \frac{4-8\varepsilon}{(3-2\varepsilon)^2} \frac{8-16\varepsilon}{(3-2\varepsilon)^3} \frac{1}{\varepsilon^2}, \ (t,x) \in (0,1) \times \mathbb{R}^3 \setminus B_R.
$$

(iv)\, Now, from formula (6.2) and estimates (i)–(iii) and (iv)\,2, we obtain

$$
|v(t,x)| \leq c(v_0) \frac{1}{\varepsilon} \frac{1-2\varepsilon}{3-2\varepsilon} \frac{1}{\varepsilon^2} \frac{4-8\varepsilon}{(3-2\varepsilon)^2} \frac{8-16\varepsilon}{(3-2\varepsilon)^3}, \ (t,x) \in (0,1) \times \mathbb{R}^3 \setminus B_R,
$$

and, taking into account (3.5) and $u = v - w$, then

$$
|u(t,x)| \leq c(v_0) \frac{1}{\varepsilon} \frac{1-2\varepsilon}{3-2\varepsilon} \frac{1}{\varepsilon^2} \frac{4-8\varepsilon}{(3-2\varepsilon)^2} \frac{8-16\varepsilon}{(3-2\varepsilon)^3}, \ (t,x) \in (0,1) \times \mathbb{R}^3 \setminus B_R.
$$

(6.8)

Employing estimates (4.15) and (6.8), we modify (6.7) as follows:

$$
\|u\|_{L^\infty((6,|x|))} \leq \|u\|_{L^2((6,|x|))}^{\frac{1}{2}} \|u\|_{L^\infty((6,|x|))}^{\frac{1}{2}} \frac{1}{\varepsilon} \frac{1-2\varepsilon}{3-2\varepsilon} \frac{1}{\varepsilon^2} \frac{4-8\varepsilon}{(3-2\varepsilon)^2} \frac{8-16\varepsilon}{(3-2\varepsilon)^3} \frac{1}{\varepsilon^2}, \ t \in (0,1).
$$

(6.9)

By the same arguments we evaluate $T^2[u \otimes u](t,x)$ as

$$
|T^2[u \otimes u](t,x)| \leq c \int_0^t \|\nabla T(t,x)\|_{L^\infty((6,|x|))}^{\frac{1}{2}} \|u\|_{L^\infty((6,|x|))}^{\frac{1}{2}} \frac{1}{\varepsilon} \frac{4-8\varepsilon}{(3-2\varepsilon)^2} \frac{8-16\varepsilon}{(3-2\varepsilon)^3} d\tau \\
\leq c(v_0) \frac{1}{\varepsilon} \frac{1-2\varepsilon}{3-2\varepsilon} \frac{1}{\varepsilon^2} \frac{4-8\varepsilon}{(3-2\varepsilon)^2} \frac{8-16\varepsilon}{(3-2\varepsilon)^3} \frac{1}{\varepsilon^2}, \ (t,x) \in (0,1) \times \mathbb{R}^3 \setminus B_R,
$$

with $\sigma := \frac{5+4\varepsilon^2-20\varepsilon}{(3-2\varepsilon)^2}$ and $\gamma := \frac{1-2\varepsilon}{3-2\varepsilon} + \frac{1}{\varepsilon} \frac{4-8\varepsilon}{(3-2\varepsilon)^2} \frac{8-16\varepsilon}{(3-2\varepsilon)^3}$. Exponent $\sigma$ is non-negative for all $\varepsilon \in (0,\frac{5}{2} - \sqrt{5}]$. Note that any further iteration will give a larger exponent for time $t$, which can be increased by $t = 1$.

(iv)\, For $\varepsilon \in (0,\frac{5}{2} - \sqrt{5})$ we make a final iteration. Now, from formula (6.2) and estimates (i)–(iii) and (iv)\,3, we obtain

$$
|v(t,x)| \leq c(v_0) \frac{1}{\varepsilon} \frac{1-2\varepsilon}{3-2\varepsilon} \frac{1}{\varepsilon^2} \frac{4-8\varepsilon}{(3-2\varepsilon)^2} \frac{8-16\varepsilon}{(3-2\varepsilon)^3}, \ (t,x) \in (0,1) \times \mathbb{R}^3 \setminus B_R,
$$

and, taking into account (3.5) and $u = v - w$, then

$$
|u(t,x)| \leq c(v_0) \frac{1}{\varepsilon} \frac{1-2\varepsilon}{3-2\varepsilon} \frac{1}{\varepsilon^2} \frac{4-8\varepsilon}{(3-2\varepsilon)^2} \frac{8-16\varepsilon}{(3-2\varepsilon)^3}, \ (t,x) \in (0,1) \times \mathbb{R}^3 \setminus B_R.
$$

(6.10)
Employing estimates (4.15) and (6.10), we obtain

\[ \|u\|_{L^6(\mathbb{R}^3)} \leq \|u\|_{L^6(\mathbb{R}^3)} \|u\|_{L^6(\mathbb{R}^3)} \leq c(v_0) \left( \frac{1 - 2\varepsilon}{6 - 4\varepsilon} + \frac{2\varepsilon}{3 - 2\varepsilon} \right) \left[ \frac{1 - 2\varepsilon}{6 - 4\varepsilon} + \frac{2\varepsilon}{3 - 2\varepsilon} \right], \; t \in (0, 1). \]  

We obtain

\[ |T^{(2)}[u \otimes u](t, x)| \leq c \int_0^t \|\nabla T(t, x)\| \frac{1}{t} \|u\|_{L^6(\mathbb{R}^3)}^2 \mathrm{d}t \leq c(v_0) \left( |x|^{-\alpha} + |(t, x) / \mathbb{R}^3 \setminus B_{R_0} \right). \]

This concludes the bootstrap argument if \( \alpha \leq \frac{3}{2} < -\gamma \): recalling (6.2) and using estimate (3.5) for \( w \) and estimates (i)–(iii) and (iv), we obtain

\[ |v(t, x)| \leq c(v_0)|x|^{-\alpha}, \; \text{for all } (t, x) \in (0, 1) \times \mathbb{R}^3 \setminus B_{R_0}. \]  

By virtue of (5.6) we have

\[ |v(t, x)| \leq |w(t, x)| + |T[v \otimes v](t, x)| \leq |w(t, x)| + |T^{(1)}[v \otimes v](t, x)| + |T^{(2)}[v \otimes v](t, x)|. \]

If \( \alpha > \frac{3}{2} \), thanks to (ii) the term \( T^{(1)}[v \otimes v] \) admits the estimate:

\[ |T^{(1)}[v \otimes v](t, x)| \leq c(v_0)|x|^{-3}, \; \text{for all } t \in (0, 1) \text{ and } |x| > R_1. \]  

We consider \( T^{(2)}[v \otimes v] \). Thanks to (6.12) with \( \alpha = \frac{3}{2} \), the term \( T^{(2)}[v \otimes v](t, x) \) admits the estimate:

\[ |T^{(2)}[v \otimes v](t, x)| \leq \int_0^t \|\nabla T(t - \tau, x)\| \|v(\tau)\|_{L^6(\mathbb{R}^3)}^2 \mathrm{d}\tau \leq c(v_0)|x|^{-3}, \; \text{for all } t \in (0, 1). \]

Since \( \alpha < 3 \), the above estimates for \( T^{(1)} \) and (3.5) for \( w \) prove, for \( \alpha \in [1, 3) \),

\[ |v(t, x)| \leq c(v_0)|x|^{-\alpha}, \; \text{for all } (t, x) \in (0, 1) \times \mathbb{R}^3 \setminus B_{R_0}. \]  

We complete the proof of theorem 1.2 for \( \alpha > 1 \).

We consider the representation formula (5.6) for \( s = 1 \). Since the previous arguments ensure that \( |v(1, x)| \leq c(v_0)|x|^{-\alpha} \), for all \( x \in \mathbb{R}^3 \setminus B_{R_0} \), then, applying lemma 3.1, we easily deduce that

\[ |H[|v(1)(t - 1, x)|] \leq c(v_0)|x|^{-\alpha}, \; \text{for all } t > 1 \text{ and } x \in \mathbb{R}^3 \setminus B_{R_0}. \]  

Let us evaluate \( T[v \otimes v](t, x) \), whose decomposition is now

\[ T[v \otimes v](t, x) := T^{(1)}[v \otimes v](t, x) + T^{(2)}[v \otimes v](t, x) \]

with

\[ T^{(1)}[v \otimes v](t, x) := \int_0^t \int_{|y| < \frac{1}{2} |x|} |\nabla T(t - \tau, x - y)| |v(\tau, y)|^2 \mathrm{d}y \mathrm{d}\tau \]

\[ T^{(2)}[v \otimes v](t, x) := \int_0^t \int_{|y| \geq \frac{1}{2} |x|} |\nabla T(t - \tau, x - y)| |v(\tau, y)|^2 \mathrm{d}y \mathrm{d}\tau, \; \text{for all } (t, x) \in (1, T) \times \mathbb{R}^3 \setminus B_{R_0}. \]
We initially consider $\alpha \in [1, 2]$. Taking into account (5.5), the term $\mathcal{T}^{(1)}[v \otimes v]$ admits the estimate:

$$|\mathcal{T}^{(1)}[v \otimes v](t, x)| \leq c \int_1^t (|x|^2 + t - \tau)^{-2} \|v\|_L^2 \, d\tau \leq c(v_0)|x|^{-2}, \quad t > 1.$$ 

If $\alpha \in (2, 3)$, then, in particular, we obtain $v_0 \in J^H(\mathbb{R})$. Hence, by lemma 2.2,

$$\|v(t)\|_2 \leq c(v_0) t^{-\frac{1}{4}}, \quad t > 0. \quad (6.16)$$

So that, employing (5.5) and (6.16), we are able to estimate $\mathcal{T}^{(1)}[v \otimes v](t, x)$ in the following way:

$$|\mathcal{T}^{(1)}[v \otimes v](t, x)| \leq c \int_1^t \|\nabla T(t - \tau, x)\|_{L^6[0,1]} \|v(\tau)\|_L^2 \, d\tau$$

$$\leq c|x|^{-3} \int_1^t (t - \tau)^{-\frac{1}{2}} \tau^{-\frac{1}{2}} \|v(\tau)\|_L^2 \, d\tau = c(v_0)|x|^{-3}, \quad t > 1.$$ 

Therefore, we have proved that

$$\text{for all } \alpha \in [1, 3), \quad |\mathcal{T}^{(1)}[v, v](t, x)| \leq c(v_0)|x|^{-\alpha}, \quad t > 0. \quad (6.17)$$

For the estimate of $\mathcal{T}^{(2)}$, we consider the decomposition again:

$$\mathcal{T}^{(2)}[v, v](t, x) = \mathcal{T}^{(2)}[u, u](t, x) + \mathcal{T}^{(2)}[u \otimes w](t, x) + \mathcal{T}^{(2)}[w \otimes u](t, x) + \mathcal{T}^{(2)}[w \otimes w](t, x).$$

Moreover, we recall that the estimate of item (iii) holds uniformly with respect to $t$. Hence

$$\mathcal{T}^{(2)}[u \otimes w](t, x) + |\mathcal{T}^{(2)}[w \otimes u](t, x)| + |\mathcal{T}^{(2)}[w \otimes w](t, x)| \leq c(v_0)|x|^{-\alpha}, \quad \text{for } \alpha \in [1, 3), \quad t > 1 \text{ and } x \in \mathbb{R}^3 \setminus B_R.$$ 

(6.18)

We have to consider $\mathcal{T}^{(2)}[u \otimes u](t, x)$, for which we argue as in item (iv). Employing the interpolation between the $L^q$-spaces, estimates (4.15) and (6.4), uniformly in $|x| > R$, give

$$\|u\|_{L^\frac{1+2\alpha}{3\alpha}(6, |x|)} \leq \|u\|_{L^\frac{1+2\alpha}{2\alpha}(6, |x|)} \|u\|_{L^\frac{1+2\alpha}{2\alpha}(6, |x|)} \leq c(v_0) t^{-\frac{1+2\alpha}{3\alpha}}|x|^{-\frac{1+2\alpha}{3\alpha}}, \quad t > 0. \quad (6.19)$$

By using estimate (5.5) for $T(s, z)$ and estimate (6.19) for $u$, and applying Hölder’s inequality, we obtain

$$|\mathcal{T}^{(2)}[u \otimes u](t - \tau, x)| \leq c \int_1^t \|\nabla T(t - \tau, x)\|_{L^\frac{3}{2\alpha}(6, |x|)} \|u(\tau)\|_{L^\frac{1+2\alpha}{2\alpha}(6, |x|)}^2 \, d\tau$$

$$\leq c(v_0)|x|^{-\frac{1+2\alpha}{3\alpha}} \int_1^t (t - \tau)^{-\frac{3+4\alpha}{3\alpha}} \tau^{-\frac{1+2\alpha}{3\alpha}} \, d\tau \leq c(v_0) t^{-\frac{1+2\alpha}{3\alpha}}|x|^{-\frac{1+2\alpha}{3\alpha}}, \quad \text{for } \alpha \in [1, 3), \quad t > 1 \text{ and } x \in \mathbb{R}^3 \setminus B_R.$$ 

The last estimate, together with estimates (6.17) and (6.18), furnishes

$$|v(t, x)| \leq c(v_0)|x|^{-\frac{1+2\alpha}{3\alpha}}, \quad t > 1, \quad x \in \mathbb{R}^3 \setminus B_R. \quad (6.19)$$

Now, for the term $\mathcal{T}^{(2)}[v \otimes v]$, we employ a bootstrap argument to realize the exponent $\alpha$ of spatial decay. The last estimate and estimate (2.7) give
\[
|v(t, x)|^2 \leq |v(t, x)|^\frac{2}{3} |v(t, x)|^\frac{2}{3} \leq c(v_0)|x|^{\frac{2}{3}} \|v(t)|^2_{\infty} \\
\leq c(v_0)|x|^{\frac{4}{3} - \frac{2\alpha}{3}} t^{-\frac{1}{2}}, t > 0, x \in \mathbb{R}^3 \setminus B_R. \tag{6.20}
\]

Hence, recalling that \(\|\nabla T(t - \tau, x)\|_1 \leq c(t - \tau)^{-\frac{2\alpha}{3}}\) and (6.1), we obtain
\[
|\nabla^2 [v \otimes v]| \leq c(v_0)|x|^{\frac{4}{3} - \frac{2\alpha}{3}} \int_0^t \|\nabla T(t - \tau, x)\|_1 \tau^{-\frac{1}{2}} d\tau \leq c(v_0)|x|^{\frac{4}{3} - \frac{2\alpha}{3}}, t > 1, x \in \mathbb{R}^3 \setminus B_R.
\]

The last estimate, together with estimates (6.17) and (6.18), furnishes us with
\[
|v(t, x)| \leq c(v_0)|x|^{\frac{4}{3} - \frac{2\alpha}{3}}, t > 1, x \in \mathbb{R}^3 \setminus B_R.
\]

Thanks to the last estimate, we modify (6.20) as
\[
|v(t, x)|^2 \leq c(v_0)|x|^{\frac{16}{3} - \frac{2\alpha}{3}} \|v(t)|^2_{\infty} \leq c(v_0)|x|^{\frac{16}{3} - \frac{2\alpha}{3}} t^{-\frac{1}{2}}, t > 0, x \in \mathbb{R}^3 \setminus B_R. \tag{6.21}
\]

If we compare (6.20) and (6.21), then we find that the exponent of spatial decay is increased by a factor \(\frac{4}{3}\). Since this can be made in sequence, after a finite number of steps (actually, at most eight) we arrive at an exponent that is greater than or equal to \(\alpha\), proving the final estimate (1.3). The theorem is completely proved.

\[\square\]

7. Asymptotic time behavior: proof of corollary 1.1

Thanks to estimate (1.3), to achieve the proof of corollary 1.1 we can limit ourselves to proving estimate (1.5). Further, it is enough to prove (1.5) for \(\beta = \alpha\). The instant \(T_0 \leq c\|v_0\|_2^4\), given in corollary 1.1, is the same as in lemma 2.2 and is due to Leray in [16]. Moreover, thanks to estimate (2.5), we can prove (1.5) for \(\alpha \in \left(\frac{3}{2}, 3\right)\). To this end, we start from formula (5.7) written for \(t > 2T_0\):
\[
v(t, x) = \mathbb{F}[v(T_0)](t - T_0, x) + \int_0^t (v \cdot \nabla T(t, x), v) d\tau, \quad \text{for} \ (t, x) \in (0, T) \times \mathbb{R}^3. \tag{7.1}
\]

From theorem 1.2 and lemma 2.2 we obtain
\[
|v(t, x)| \leq c(v_0)(1 + |x|)^{-\alpha}, \quad \text{for all} \ t \geq T_0, \quad \text{for all} \ x \in \mathbb{R}^3. \tag{7.2}
\]

Then, thanks to (7.2) and the proprieties of the solutions to the Stokes Cauchy problem (see e.g. [13]), we have \(\|\mathbb{F}[v(T_0)](t - T_0, x)\| \leq c(v_0)\tau^{-\frac{\alpha}{2}}\). Thus, to achieve the result we only need to estimate the nonlinear term in (7.1). We set
\[
\int_0^t (v \cdot \nabla T(t, x), v) d\tau = \int_0^{\tau_1} (v \cdot \nabla T(t, x), v) d\tau + \int_{\tau_1}^t (v \cdot \nabla T(t, x), v) d\tau =: I_1 + I_2.
\]

By virtue of (5.5) and (7.2), for all \(\alpha \in \left(\frac{3}{2}, 3\right)\) we easily deduce the estimate:
\[
|I_1| \leq c(v_0) t \int_0^{\tau_1} |x - y| \left(t - \tau \right)^{-\frac{1}{2}} (1 + |y|)^{-2\alpha} dy \leq c(v_0) t^{-\frac{\alpha}{2}}.
\]

F Crispo and P Maremonti
Nonlinearity 29 (2016) 1355
Instead, for the term $I_2$ we achieve the result in two steps. For $\alpha \in \{\frac{3}{2}, 2\}$, recalling (2.5) and (7.2) and observing that $|v(t,x)|^2 = |v(t,x)|^{\frac{2}{2}}|v(t,x)|^{\frac{2}{2}}$, we obtain

$$|I_2| \leq c(v_0) \int_0^T \int_{\mathbb{R}^3} \left(|x-y| + (t-t')^{\frac{3}{2}}(1 + |y|)^{-2 + \frac{3}{2} \alpha} \tau^{\frac{3}{2} \alpha} \right) dy \, dt'$$

$$\leq c(v_0) \int_0^T \int_{\mathbb{R}^3} \left(|x-y|^{2} (1 + |y|)^{-\alpha} \tau^{\frac{3}{2} \alpha} \right) dy \leq c(v_0) \tau^{-\alpha}.$$

Hence, by the first step for $I_2$ and the result for $I_1$ and for the linear part, we conclude

$$|v(t,x)| \leq c(v_0) \tau^{-\frac{\alpha}{2}}, \text{ for all } \alpha \in [1, 2].$$

For $\alpha \in (2, 3)$, we invoke this last estimate. Hence, the estimate for $I_2$ becomes:

$$|I_2| \leq c(v_0) \int_0^T \int_{\mathbb{R}^3} \left(|x-y| + (t-t')^{\frac{3}{2}}(1 + |y|)^{-2 + \frac{3}{2} \alpha} \tau^{\frac{3}{2} \alpha} \right) dy \, dt'$$

$$\leq c(v_0) \tau^{-\frac{\alpha}{2}} \int_{\mathbb{R}^3} |x-y|^{2} (1 + |y|)^{-\alpha} \tau^{\frac{3}{2} \alpha} dy \leq c(v_0) \tau^{-\frac{\alpha}{2}}.$$

Therefore, the proof is completed.

Acknowledgment

The authors are grateful to the referees and an associate editor, whose suggestions made the paper more readable.

This research was partly supported by GNFM-INdAM, and by MIUR via the PRIN 2012 ‘Nonlinear Hyperbolic Partial Differential Equations, Dispersive and Transport Equations: Theoretical and Applicative Aspects’.

References

[1] Amrouche C, Girault V, Schonbek M E and Schonbek T 2000 Pointwise decay of solutions and of higher derivatives to Navier–Stokes equations SIAM J. Math. Anal. 31 740–53
[2] Bae H-O and Jin B J 2005 Upper and lower bounds of temporal and spatial decays for the Navier–Stokes equations J. Differ. Equ. 209 365–91
[3] Borchers W and Miyakawa T 1988 $L^2$ decay for the Navier–Stokes flow in halfspaces Math. Ann. 282 139–55
[4] Caffarelli L, Kohn R and Nirenberg L 1982 Partial regularity of suitable weak solutions of the Navier–Stokes equations Commun. Pure Appl. Math. 35 771–831
[5] Crispo F and Maremonti P 2004 An interpolation inequality in exterior domains Rend. Sem. Mat. Univ. Padova 112 11–39
[6] Crispo F and Maremonti P 2004 On the $(x, t)$ asymptotic properties of solutions of the Navier–Stokes equations in the half-space Zap. Nauchn. Sem. POMI 318 147–202
[7] Crispo F and Maremonti P 2006 J. Math. Sci. 136 3735–67 (Engl. translation)
[8] Deuring P 2015 Pointwise spatial decay of weak solutions to the Navier–Stokes system in 3D exterior domains J. Math Fluid Mech. 17 199–232
[9] Galdi G P 2011 An Introduction to the Mathematical Theory of the Navier–Stokes Equations, Steady-State Problems (Springer Monographs in Mathematics) 2nd edn (New York: Springer)
[10] Galdi G P and Sohr H 2004 Existence and uniqueness of time-periodic physically reasonable Navier–Stokes flow past a body Arch. Ration. Mech. Anal. 172 363–406

[11] Gallay T and Wayne C E 2002 Long-time asymptotics of the Navier–Stokes and vorticity equations on $\mathbb{R}^3$. Recent developments in the mathematical theory of water waves (Oberwolfach, 2001) R. Soc. Lond. Phil. Trans. A 360 2155–88

[12] Kajikiya R and Miyakawa T 1986 On $L^2$ decay of weak solutions of the Navier–Stokes equations in $\mathbb{R}^n$ Math. Z. 192 135–48

[13] Knightly G H 1966 On a class of global solutions of the Navier–Stokes equations Arch. Ration. Mech. Anal. 21 211–45

[14] Kukavica I 2001 Space–time decay for solutions of the Navier–Stokes and vorticity equations on $\mathbb{R}^3$, Recent developments in the mathematical theory of water waves (Oberwolfach, 2001) R. Soc. Lond. Phil. Trans. A 360 2155–88

[15] Kukavica I and Torres J J 2007 Weighted $L^p$ decay for solutions of the Navier–Stokes equations Commun. PDE 32 819–31

[16] Leray J 1934 Sur le mouvement d’un liquide visqueux emplissant l’espace Acta Math. 63 193–248

[17] Maremonti P 1985 Asymptotic stability in the mean for viscous fluid motion in exterior domains Ann. Mat. Pura Appl. 142 57–75

[18] Maremonti P 1988 On the asymptotic behavior of the $L^2$-norm of suitable weak solutions to the Navier–Stokes equations in three-dimensional exterior domains Commun. Math. Phys. 198 385–400

[19] Maremonti P 2011 A remark on the Stokes problem with initial data in $L^1$ J. Math. Fluid Mech. 13 469–80

[20] Maremonti P 2014 On the Stokes problem in exterior domains: the maximum modulus theorem Discrete Contin. Dyn. Syst. A 34 2135–71

[21] Miyakawa T and Schonbek M E 2001 On optimal decay rates for weak solutions to the Navier–Stokes equations in $\mathbb{R}^n$ Math. Bohem. 126 443–55

[22] Mizumachi R 1984 On the asymptotic behavior of incompressible viscous fluid motions past bodies J. Math. Soc. Japan 36 497–522

[23] Nash J 1958 Continuity of solutions of parabolic and elliptic equations Am. J. Math. 80 931–54

[24] Oliver M and Titi E S 2000 Remark on the rate of decay of higher order derivatives for solutions to the Navier–Stokes equations in $\mathbb{R}^n$ J. Funct. Anal. 172 1–18

[25] Scheffer V 1977 Hausdorff measure and the Navier–Stokes equations Commun. Math. Phys. 55 97–112

[26] Schonbek M E 1985 $L^2$ decay for weak solutions of the Navier–Stokes equations Arch. Ration. Mech. Anal. 88 209–22

[27] Schonbek M E 1986 Large time behaviour of solutions to the Navier–Stokes equations Commun. PDE 11 733–63

[28] Takahashi S 1999 A weighted equation approach to decay rate estimates for the Navier–Stokes equations Nonlinear Anal. 37 751–89

[29] Wiegner M 1987 Decay results for weak solutions of the Navier–Stokes equations on $\mathbb{R}^n$ J. Lond. Math. Soc. 35 303–13