Hammocks to Visualize
the Support of Finitely Presented Functors

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Abstract: Many properties of a module can be expressed in terms of the dimension of the vector space obtained by applying a finitely presented functor to that module. For example, the dimension of the kernel, image or cokernel of the multiplication map given by an algebra element; or the number of summands of a certain type when the module is considered a module over a subalgebra. When the indecomposable modules over the algebra are arranged in the Auslander-Reiten quiver, the support of the finitely presented functor typically has the shape of a hammock, spanned between sources and sinks. There may also be tangents which are meshes where the hammock function at the middle term exceeds the sum of the values at the start and end terms. We describe how sources, sinks and tangents of the hammock relate to the modules which define the projective resolution of the finitely presented functor. The key tool is the Cokernel Complex Lemma which links the values of the hammock function to the Auslander-Reiten structure of the category. We are also interested in exact subcategories of module categories which have Auslander-Reiten sequences. Our examples include quiver representations and invariant subspaces of nilpotent linear operators.

Keywords: Hammock, Auslander-Reiten quiver, finitely presented functor, invariant subspace.

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1. Introduction

Let \( \Lambda \) be a finite dimensional algebra over a field \( k \). Our interest is in the category \( \text{mod}\Lambda \) of finite dimensional (right) \( \Lambda \)-modules, or more general, in a full exact extension-closed subcategory \( \mathcal{C} \) of \( \text{mod}\Lambda \) which, in particular, has Auslander-Reiten sequences. Thus, we can picture the category \( \mathcal{C} \) as a directed graph, the Auslander-Reiten quiver \( \Gamma = (\Gamma_0, \Gamma_1, \tau) \), which comprises the information how homomorphisms in \( \mathcal{C} \) factor locally. The vertex set \( \Gamma_0 = \text{ind}\mathcal{C} \) consists of the isomorphism classes of indecomposable objects, the arrows in \( \Gamma_1 \) represent
irreducible morphisms, and the translation $\tau$ is a partial map on $\Gamma_0$. Each Auslander-Reiten sequence $A : 0 \to A \to \bigoplus B_i \to C \to 0$ contributes a mesh in $\Gamma$ in the sense that $A = \tau C$ and the irreducible morphisms starting at $A$ correspond to irreducible morphisms ending in $C$. For more details and examples see [2, Chapter VII].

Our aim is to visualize the support of a finitely presented functor $E : \text{mod}\Lambda \to \text{mod}\ k$ as a subset of $\Gamma_0$. Such a functor is given by an exact sequence of Hom-functors,

$$0 \to (Z, -) \to (Y, -) \to (X, -) \to E \to 0,$$

where $X, Y, Z \in \text{mod}\Lambda$ are finitely generated modules and where we write $(X, -)$ as abbreviation for $\text{Hom}_\Lambda(X, -)$. The sequence is induced by a short right exact sequence

$$\mathcal{E} : X \xrightarrow{u} Y \to Z \to 0$$

of $\Lambda$-modules.

1.1. Example: Before we state our main result, we illustrate this aim in an example. Let $\Lambda$ be the path algebra of the quiver $E_6$ in bipartite orientation as in Section 4.2. Then the category of $\Lambda$-modules is equivalent to the category of systems $V = (V_i, V_\alpha)$ consisting of six vector spaces and five linear maps, arranged as follows.

It turns out that there are 36 indecomposable $\Lambda$-modules, each is determined uniquely by its dimension type $(\dim V_i)_i$; for example, the indecomposable module $X$ which is zero in each component except the second where $X_2 = k$ has dimension type $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. In the Auslander-Reiten quiver in Figure 1, the indecomposable objects are represented by their dimension types, the irreducible morphisms by arrows, and the translation $\tau$ maps each non-projective object $C$ to the object $A$ on the left so that the Auslander-Reiten sequence $0 \to A \to \bigoplus B_i \to C \to 0$ gives rise to a mesh in the graph.

In this example we are interested in the (finitely presented) functor $E : \text{mod}\Lambda \to \text{mod}\ k$ which assigns to each object $V$ the cokernel $\text{Cok} V_\alpha$ of the leftmost linear map. The collection of modules $V$ which satisfy $E(V) \neq 0$ is encircled. The encircled region is a hammock: It has a source at $X$, similarly, there is a sink at the module $Z'$ of dimension type $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, and we will see that the functor $E$ is additive on each mesh except the ones involving
call the map $\tau_C$.

Definition: the module $\tau_C$ has projective presentation $0 \to E \to Z$, respectively. A hammock is the support of a hammock function.

For $A, C \in \Gamma_0$, we denote by $A^+$ and $C^-$ the multisets of successors and predecessors of $A$ and $C$ with respect to irreducible morphisms in $\Gamma_1$, respectively. A mesh is a sequence in $\Gamma$ that has the form $0 \to \bigoplus_{B \in C^-} B \to C$ if $C$ is an indecomposable projective object in $\mathcal{C}$, or $A \to \bigoplus_{B \in A^+} B \to 0$ if $A$ is indecomposable injective. Otherwise, the mesh $\tau C \to \bigoplus_{B \in C^-} B \to C$ is given by an Auslander-Reiten sequence.

$X, Y$ and $Z'$. We call the mesh ending at the module marked $Y$ a tangent for $E$. Denote by $Z = \tau^{-1}Z'$ the module on the right hand side of $Z'$, of dimension type $000_01$.

Our main result relates sinks, sources and tangents of the hammock with the modules defining the presentation of the functor. In fact, in this case the functor $E$ has projective presentation

$$0 \to (Z, -) \to (Y, -) \to (X, -) \to E \to 0.$$

Definition: Given a finitely presented functor $E : \text{mod} \Lambda \to \text{mod} k$, we call the map

$$e_* : \Gamma_0 \to \mathbb{N}_0, \ M \mapsto \dim EM,$$

the hammock function on $\Gamma$ (or on $\mathcal{C}$) for the functor $E$ (or for the sequence $\mathcal{E}$). A hammock is the support of a hammock function.

1. A vertex $C \in \Gamma_0$ is an (isolated) source for $e_*$ if $e_*(C) > \sum_{B \in C^-} e_*(B)$.
2. A vertex $A \in \Gamma_0$ is an (isolated) sink for $e_*$ if $e_*(A) > \sum_{B \in A^+} e_*(B)$.
3. A mesh $M : A \to \bigoplus B_i \to C$ in $\Gamma$ is an (isolated) tangent for $e_*$ if $\sum e_*(B_i) > e_*(A) + e_*(C)$.

Figure 1. The hammock for the functor $E : V \mapsto \text{Cok} V\alpha$
Here is our main result about hammock functions.

1.2. **Theorem.** Let $E : X \to Y \to Z \to 0$ be a short right exact sequence and $e_*$ the hammock function for $E$ on $\Gamma$.

1. (1) If $C$ is an isolated source for $e_*$ then $C$ is a direct summand of $X$.
2. (2) If $A$ is an isolated sink for $e_*$ then $A$ is an injective object in $C$ or $C = \tau^{-1}A$ is a direct summand of $Z$.
3. (3) If $M : A \to \bigoplus B_i \to C$ is an isolated tangent for $e_*$ then either $A$ is indecomposable injective or $C$ is an indecomposable direct summand of $Z$.
4. (4) If $M : A \to \bigoplus B_i \to C$ is a mesh in $\Gamma$ with $C \neq 0$ such that $C$ does not occur as a direct summand of $X \oplus Y \oplus Z$ then the hammock function $e_*$ is additive on $M$ in the sense that $e_*(A) + e_*(C) = \sum e_*(B_i)$ holds.

The theorem describes how the terms of $E : X \to Y \to Z \to 0$ control the shape of the hammock given by the support of the functor $E$ on $\Gamma$. The hammock is clearly visible if $\mathcal{C}$ is representation-directed (so there are no oriented cycles in $\Gamma$):

Tracing the hammock function $e_*$ on $\Gamma$ from left to right, the source or sources of the hammock occur at the indecomposable summands of $X$ (1). The values of $e_*$ can be computed successively by using the mesh relation (4), except when summands of $X \oplus Y \oplus Z$ are encountered. Summands of $Y$ are endpoints of meshes where a middle term is touched by the hammock (3). Finally, the hammock terminates at one or several vertices which are injective or of the form $\tau Z'$ where $Z'$ is a summand of $Z$ (2).

We use the prefix *isolated* since indecomposable objects may occur as direct summands in several of the terms $X$, $Y$, $Z$ in the resolution for $E$. Such sources, sinks and tangents may not be identifiable by the numerical condition in the definition alone. See Section 5.6 for an example.

Theorem 1.2 deals with hammocks which visualize the support of a finitely presented covariant functor. The corresponding dual result for finitely presented contravariant functors is stated as Theorem 3.1.

There is a different way to describe the modules in the hammock. Suppose $E : 0 \to X \to Y \to Z \to 0$ is given by a non-split short exact sequence with $Z$ indecomposable non-projective. Then the modules $M \in \mathcal{C}$ which occur in the hammock given by $E$ (which hence satisfy
by definition that $0 \neq \text{Cok} \text{Hom}(u, M) : \text{Hom}(Y, M) \to \text{Hom}(X, M))$
are characterized by lying between one of the summands of $X$ and of
$	au Z$ in the sense that there is a map $f : X \to M$ which does not factor
over $u$ and a map $g : M \to \tau Z$ such that the product $gf : X \to \tau Z$
does not factor over $u$. Our result generalizes, but is weaker than, the
corresponding statement [12, Corollary 5].

1.3. Proposition. Let $E : 0 \to X \xrightarrow{u} Y \to Z \to 0$ be a non-split short
exact sequence in $\mathcal{C}$ with $Z$ indecomposable. Then for any $M \in \mathcal{C}$, the
bilinear form given by composition

$$\text{Hom}(M, \tau Z) \times \text{Cok}(u, M) \longrightarrow \text{Cok}(u, \tau Z)$$

is right non-degenerate.

The dual result for finitely presented contravariant functors is stated
as Proposition 3.2.

In the proof of Theorem 1.2, we relate the Auslander-Reiten structure
of the category $\mathcal{C}$ with the modules in the projective resolution of the
functor $E$. The following result is our key tool. It will be shown in
Section 2.

1.4. Proposition (Cokernel Complex Lemma). Consider the commu-
tative diagram with exact rows and columns.

\[
\begin{array}{ccc}
0 & \rightarrow & C' \rightarrow C \rightarrow C'' \\
& \gamma & \downarrow \gamma'' \\
& \beta & \downarrow \beta'' \\
& \alpha & \downarrow \alpha'' \\
A & \rightarrow & A'' \\
& \downarrow g \downarrow g' \downarrow g'' \\
B' & \rightarrow & B \rightarrow B'' \\
& \downarrow g' \downarrow g'' \\
& \downarrow \beta' \\
& \downarrow \beta'' \\
& \downarrow 0 \\
& A' & \rightarrow & A'' \\
& \downarrow g' \downarrow g'' \\
& \downarrow \beta' \\
& \downarrow \beta'' \\
& \downarrow 0 \\
& 0 & \rightarrow & C' \rightarrow C \rightarrow C''
\end{array}
\]

The complexes $0 \to \text{Cok} \alpha \to \text{Cok} \beta \to \text{Cok} \gamma \to 0$ and $0 \to \text{Cok} g' \to \text{Cok} g \to \text{Cok} g'' \to 0$ given by taking the cokernels of the horizontal
maps at the right and the vertical maps at the bottom, respectively,
have in each position isomorphic homology.

The second part of this paper consists of examples. We consider a
variety of quantities related to modules which can be measured by
hammock functions.
In Section 4 we consider multiplication maps. For an element \( a \) in an algebra \( \Lambda \) and \( M \) a finite dimensional \( \Lambda \)-module, the multiplication by \( a \) gives rise to a linear map \( \mu_a : M \rightarrow M \). We describe the hammock functions given by the kernel, \( \dim \ker \mu_a \), the image, \( \dim \im \mu_a \), and the cokernel, \( \dim \text{Cok} \mu_a \). If \( \Lambda = kQ \) is the path algebra of a quiver, \( i \in Q_0 \) a vertex, \( \alpha \in Q_1 \) an arrow and \( M = ((M_i)_{i \in Q_0}, (M_\alpha)_{\alpha \in Q_1}) \) a representation, then quantities like \( \dim M_i \), \( \dim \ker M_\alpha \), \( \dim \im M_\alpha \), \( \dim \text{Cok} M_\alpha \) give rise to hammock functions; some of them are pictured. Suppose \( p,q \) are composable paths in the quiver, say meeting at the vertex \( i \), such that \( M_p \circ M_q = 0 \). Then also the homology \( \dim \ker M_p / \im M_q \) (which is a subspace of \( M_i \)) defines a hammock function.

In Section 5 we deal with invariant subspaces of nilpotent linear operators; they are pairs \((V,U)\) where the linear operator \( V \) is a module over some bounded polynomial ring \( k[T]/(T^n) \) and the invariant subspace \( U \) is a \( k[T] \)-submodule of \( V \). Such systems form a full exact subcategory of a module category which has Auslander-Reiten sequences. Note that if \( n \) is large enough, then the \( k[T] \)-modules \( U, V, V/U \) may have many indecomposable direct summands, even if the pair \((V,U)\) itself is indecomposable. For each \( 1 \leq m < n \), we exhibit a hammock function which counts the number of summands in \( V \) (and similarly in \( U, V/U \)) which are isomorphic to \( k[T]/(T^m) \). For \( n = 4 \) we picture some such functions as hammocks in the Auslander-Reiten quiver. Our last example is a hammock for which the source is not isolated.

Remark: Hammocks have been introduced by Sheila Brenner \[3\] to study, for a representation directed algebra, the collection of indecomposable modules which contain a given simple module as a composition factor. Regarding finitely presented functors, there has been interest in their support, see for example the paper \[1\] on uniserial functors. The link to poset representations is established in \[12\]; a decomposition of the poset given by the hammock is studied in \[13\]. The minimal elements in hammocks are described in \[14\]. Y. Lin \[6, 7\] uses hammocks to trace algorithms introduced by Nazarova and Roiter, and by Zavadskii. In \[4\], kit algebras are introduced as representation-directed algebras for which each hammock given by the modules which contain a certain simple composition factor is a garland.

We briefly describe the contents of this paper.

The Cokernel Complex Lemma is shown in Section 2. For the proof we use relations as they may be occur in a proof of the Snake Lemma.
In Section 3 we show the above results about the modules in the hammock and state in Theorem 3.1 and Proposition 3.2 the dual versions which deal with finitely presented contravariant functors. Our first example are hammocks given by Auslander-Reiten sequences, they are characterized by having only one vertex.

In the last two sections we present applications which show that hammock functions exhibit a variety of meaningful properties of modules.

2. THE COKERNEL COMPLEX LEMMA

Our main tool to study hammocks is the Cokernel Complex Lemma, stated in the introduction as Proposition 1.4. For its proof we introduce relations following [11, Section 26].

Recall that for modules $X, Y$, a relation on $Y \times X$ is a submodule of $Y \oplus X$. In particular, a map $f : X \to Y$ gives rise to relations $f = \{(f(x), x) | x \in X\} \subset Y \oplus X$ and $f^{-1} = \{(x, f(x)) | x \in X\} \subset X \oplus Y$. Given two relations $u \subset Y \oplus X$ and $v \subset Z \oplus Y$, the composition $v \circ u$ is the relation given by the submodule

$$v \circ u = \{(z, x) | (y, x) \in u, (z, y) \in v \text{ for some } y \in Y\} \subset Z \oplus X.$$ 

Clearly, a relation $u$ on $Y \times X$ is given by a map if for every $x \in X$ there is a unique $y \in Y$ with $(y, x) \in u$; this map is an isomorphism if for every $y \in Y$ there is a unique $x \in X$ with $(y, x) \in u$.

The proof of the following lemma is straightforward and easy.

2.1. LEMMA. Consider the diagram with one exact row and one exact column.

$$\begin{array}{ccc}
A & \xrightarrow{f} & B' \\
\downarrow & & \downarrow \beta' \\
B & \xrightarrow{\beta} & B'' \\
\downarrow g & & \downarrow \\
C & & \\
\end{array}$$

(1) Let $u \subset C \oplus B''$ be the relation given by $g \circ \beta^{-1}$. For $(c, b'') \in u$ we have $b'' \in \text{Im } \beta f$ if and only if $c \in \text{Im } g \beta'$.

(2) Let $v \subset B' \oplus A$ be the relation given by $\beta'^{-1} \circ f$.

(a) Let $a \in A$. Then $a \in \text{Ker } \beta f$ if and only if there is $b' \in B'$ with $(b', a) \in v$.

(b) Given $b' \in B'$, we have $b' \in \text{Ker } g \beta'$ if and only if there is $a \in A$ with $(b', a) \in v$. \qed
The Cokernel Complex Lemma is a consequence of the following three claims. In each, we consider parts of the following commutative diagram in which all rows and columns are exact except the rightmost column and the row at the bottom (which may be proper complexes).

\[
\begin{array}{c}
\[
\begin{array}{c}
0 \\
A \xrightarrow{\alpha} A'' \xrightarrow{\alpha''} \tilde{A} \xrightarrow{} 0 \\
B' \xrightarrow{\beta'} B \xrightarrow{\beta} B'' \xrightarrow{\beta''} \tilde{B} \xrightarrow{} 0 \\
C' \xrightarrow{\gamma'} C \xrightarrow{\gamma} C'' \xrightarrow{\gamma''} \tilde{C} \xrightarrow{} 0 \\
D' \xrightarrow{\delta'} D \xrightarrow{\delta} D'' \xrightarrow{} 0 \\
\end{array}
\end{array}
\]

2.2. Claim. The relation \(w = h'' \circ \gamma'^{-1} \subset D'' \times \tilde{C}\) as in the diagram yields an isomorphism \(\tilde{C}/\text{Im } \tilde{g} \cong D''/\text{Im } \delta\).

\[
\begin{array}{c}
\[
\begin{array}{c}
0 \\
B'' \xrightarrow{\beta''} \tilde{B} \xrightarrow{} 0 \\
C \xrightarrow{\gamma} C'' \xrightarrow{\gamma''} \tilde{C} \xrightarrow{} 0 \\
D \xrightarrow{\delta} D'' \xrightarrow{} 0 \\
\end{array}
\end{array}
\]

Proof. The relation \(w\) has the following properties:

(1) For every \(d'' \in D''\) there is \(\tilde{c} \in \tilde{C}\) such that \((d'', \tilde{c}) \in w\).
(2) For every \(\tilde{c} \in \tilde{C}\) there is \(d'' \in D''\) such that \((d'', \tilde{c}) \in w\).
(3) For \((d'', \tilde{c}) \in w\), \(d'' \in \text{Im } \delta\) if and only if \(\tilde{c} \in \text{Im } \tilde{g}\). Namely, \(\tilde{c} \in \text{Im } \tilde{g}\) is equivalent to \(\tilde{c} \in \text{Im } \tilde{g} \beta'' = \text{Im } \gamma'' g''\) since \(\beta''\) is onto. By the Lemma, Part (1), this is equivalent to \(d'' \in \text{Im } h'' \gamma = \text{Im } \delta h\) and hence to \(d'' \in \text{Im } \delta\).
We obtain from (2) and (3) that there is a map $\tilde{C} \to D''/\text{Im}\ \delta$ which clearly is a homomorphism. By (1), this map is onto and by (3), the map has kernel $\text{Im}\ \tilde{g}$, and hence induces an isomorphism $\tilde{C}/\text{Im}\ \tilde{g} \to D''/\text{Im}\ \delta$.

2.3. CLAIM. The relation $v = h \circ \gamma^{-1} \circ g'' \circ \beta''^{-1} \subset D \times \tilde{B}$ as in the diagram gives rise to an isomorphism $\text{Ker}\ \tilde{g}/\text{Im}\ \tilde{f} \cong \text{Ker}\ \delta/\text{Im}\ \delta'$.

\[
\begin{array}{cccccc}
A'' & \xrightarrow{\alpha''} & \tilde{A} & \rightarrow & 0 \\
B & \xrightarrow{\beta} & B'' & \xrightarrow{\beta''} & \tilde{B} & \rightarrow & 0 \\
C' & \xrightarrow{\gamma'} & C & \xrightarrow{\gamma} & C'' & \xrightarrow{\gamma''} & \tilde{C} \\
D' & \xrightarrow{\delta'} & D & \xrightarrow{\delta} & D'' & & \\
0 & & 0 & & & & \\
\end{array}
\]

Proof. The relation $v$ has the following properties.

(1) Suppose $\tilde{b} \in \text{Ker}\ \tilde{g}$. There is $d \in \text{Ker}\ \delta$ with $(d, \tilde{b}) \in v$. Namely, since $\beta''$ is onto, there is $b''$ with $\beta''(b'') = \tilde{b}$. We apply the Lemma, Part (2a), center at $C''$, to the relation $\gamma^{-1} \circ g''$: Since $0 = \tilde{g} \beta''(b'') = \gamma'' g''(b'')$, there is $c \in C$ with $(c, b'')$ in the relation given by $\gamma^{-1} \circ g''$ and $h'' \gamma(c) = 0$. Then $(h(c), \tilde{b}) \in v$ and $h(c) \in \text{Ker}\ \delta$.

(2) Similarly, for $d \in \text{Ker}\ \delta$ there is $\tilde{b} \in \text{Ker}\ \tilde{g}$ with $(d, \tilde{b}) \in v$.

(3) For $(d, \tilde{b}) \in v$, $d \in \text{Im}\ \delta'$ if and only if $\tilde{b} \in \text{Im}\ \tilde{f}$: Let $c'' \in C''$ such that $(d, c'') \in h \circ \gamma^{-1}$ and $(c'', \tilde{b}) \in g'' \circ (\beta'')^{-1}$. Then $\tilde{b} \in \text{Im}\ \tilde{f}$ is equivalent to $\tilde{b} \in \text{Im}\ \tilde{f} \alpha'' = \text{Im}\ \beta'' f''$, since $\alpha''$ is onto. By (1) in the Lemma, center at $B''$, this is equivalent to $c'' \in \text{Im}\ g'' \beta = \text{Im}\ \gamma g$, which by (1) in the Lemma, center at $C$, is equivalent to $d \in \text{Im}\ h \gamma' = \text{Im}\ \delta' h'$ and hence to $d \in \text{Im}\ \delta'$.

From (1) and (3) we obtain that there is a map $\text{Ker}\ \tilde{g} \to \text{Ker}\ \delta/\text{Im}\ \delta'$; this map is onto by (2) and has kernel $\text{Im}\ \tilde{f}$ by (3). Hence it induces the desired isomorphism $\text{Ker}\ \tilde{g}/\text{Im}\ \tilde{f} \cong \text{Ker}\ \delta/\text{Im}\ \delta'$. \hfill $\square$
2.4. **Claim.** The relation \( u = h' \circ \gamma'^{-1} \circ g \circ \beta^{-1} \circ f'' \circ \alpha''^{-1} \) as in the diagram induces an isomorphism \( \text{Ker } \tilde{f} \to \text{Ker } \delta' \).

\[
\begin{array}{c}
0 \\
\downarrow \downarrow \downarrow \\
A \xrightarrow{\alpha} A'' \xrightarrow{\alpha''} \tilde{A} \xrightarrow{\tilde{f}} 0 \\
\downarrow f \downarrow \downarrow f'' \\
B' \xrightarrow{\beta'} B \xrightarrow{\beta} B'' \xrightarrow{\beta''} \tilde{B} \\
\downarrow g' \downarrow \downarrow g'' \\
0 \xrightarrow{\gamma'} C' \xrightarrow{\gamma} C \xrightarrow{\gamma} C'' \\
\downarrow h' \downarrow \downarrow h \\
D' \xrightarrow{\delta'} D \\
\end{array}
\]

**Proof.** Using the Lemma we obtain the following properties:

1. For \( \tilde{a} \in \text{Ker } \tilde{f} \) there is \( d' \in \text{Ker } \delta' \) with \( (d', \tilde{a}) \in u \): Let \( \tilde{a} \in \text{Ker } \tilde{f} \). Since \( \alpha'' \) is onto, there is \( a'' \in A'' \) with \( \alpha''(a'') = \tilde{a} \). By Part (2a), since \( \beta''f''(a'') = 0 \), there is \( b \in B \) with \( b \in \text{Ker } g''\beta \) and \( \beta(b) = f''(a'') \). Again by Part (2a), since \( b \in \text{Ker } \gamma g = \text{Ker } g''\beta \), there is \( c' \in C' \) with \( c' \in \text{Ker } h'\gamma' \) and \( \gamma'(c') = g(b) \). Then \( h'(c) \in \text{Ker } \delta' \) and \( (h'(c), \tilde{a}) \in u \).

2. For \( d' \in \text{Ker } \delta' \) there is \( \tilde{a} \in \text{Ker } \tilde{f} \) with \( (d', \tilde{a}) \in u \). The proof is similar, using Part (2b) in the Lemma.

3. For \( (d', \tilde{a}) \in u, d' = 0 \) if and only if \( \tilde{a} = 0 \). Namely, let \( c \in C \) and \( b'' \in B'' \) such that \( (d', c) \in h' \circ \gamma'^{-1}, (c, b'') \in g \circ \beta^{-1} \) and \( (b'', \tilde{a}) \in f'' \circ \alpha''^{-1} \). The following statements are equivalent: \( d' = 0; d' \in \text{Im } h' \circ 0; c \in \text{Im } \gamma g' = \text{Im } g\beta' \) (use (1) in the Lemma, center at \( C'' \)); \( b'' \in \text{Im } \beta f = \text{Im } f''\alpha \) (again by (1), center at \( B' \)); \( \tilde{a} \in \text{Im } \alpha'' \circ 0; \tilde{a} = 0 \).

Again, (1) and (3) define a map \( \text{Ker } \tilde{f} \to \text{Ker } \delta' \); this map is onto by (2) and one-to-one by (3).

The Proposition applies in particular to commutative diagrams given by “multiplying” a short right exact sequence and a short left exact sequence using the Hom-functor.
2.5. **Theorem.** Suppose $\mathcal{E} : X \to Y \to Z \to 0$ is a short right exact sequence and $\mathcal{A} : 0 \to A \to B \to C$ a short left exact sequence. The complexes
\[
0 \to \text{Cok}(u, A) \to \text{Cok}(u, B) \to \text{Cok}(u, C) \to 0
\]
and
\[
0 \to \text{Cok}(Z, g) \to \text{Cok}(Y, g) \to \text{Cok}(X, g) \to 0
\]
have isomorphic homology in each position.

**Proof.** The following diagram is commutative with exact rows and columns.

\[
\begin{array}{ccc}
0 & \to & (Z, A) \\
\downarrow & & \downarrow \\
0 & \to & (Y, A) & \to & (X, A) & \to & \text{Cok}(u, A) & \to & 0 \\
\downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow & & \downarrow \\
0 & \to & (Z, B) & \to & (Y, B) & \to & (X, B) & \to & \text{Cok}(u, B) & \to & 0 \\
\downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow & & \downarrow \\
0 & \to & (Z, C) & \to & (Y, C) & \to & (X, C) & \to & \text{Cok}(u, C) & \to & 0 \\
\downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow & & \downarrow \\
\text{Cok}(Z, g) & \to & \text{Cok}(Y, g) & \to & \text{Cok}(X, g) & \to & 0 & \to & 0 & \to & 0
\end{array}
\]

The result follows from Proposition 1.4. $\square$

3. **Hammock Functions**

In this section we give the proofs for Theorem 1.2 and Proposition 1.3 from the Introduction. Then we state the dual results.

**Proof of Theorem 1.2.** Let $B \to C$ be a sink map for $C$ in $\mathcal{C}$, so the expression $\dim \text{Cok}(M, g)$ determines the multiplicity of $C$ as a summand of $M \in \mathcal{C}$. (Namely, if $d = \dim k \text{End}(C)/\text{rad End}(C)$ then $\dim k \text{Cok}(M, g) = d \cdot \mu_C(M)$.) If $C$ is an indecomposable non-projective object in $\mathcal{C}$, then it occurs as the end term of an Auslander-Reiten sequence $\mathcal{A} : 0 \to A \to B \to C \to 0$ in $\mathcal{C}$. In case $C$ is indecomposable projective in $\mathcal{C}$, the sink map $g$ is still part of a short left exact sequence $\mathcal{A} : 0 \to A \to B \to C$. 
In the statement of the theorem, the short right exact sequence is given by $\mathcal{E} : X \xrightarrow{u} Y \rightarrow Z \rightarrow 0$. We apply Theorem 2.5 to $\mathcal{A}$ and $\mathcal{E}$, then the following two complexes have isomorphic homology.

$$
0 \longrightarrow \text{Cok}(u, A) \longrightarrow \text{Cok}(u, B) \longrightarrow \text{Cok}(u, C) \longrightarrow 0
$$

$$
0 \longrightarrow \text{Cok}(Z, g) \longrightarrow \text{Cok}(Y, g) \longrightarrow \text{Cok}(X, g) \longrightarrow 0
$$

Note that the dimensions of the vector spaces in the first sequence are the values of the hammock function. Up to a factor of $d$, the dimensions of the modules in the second sequence are the multiplicities of $C$ as a direct summand of $Z$, $Y$, or $X$, respectively.

1. The hammock function $e_*$ has an isolated source $C$ by definition if $\dim \text{Cok}(u, C) > \dim \text{Cok}(u, B)$. In this case, $\text{Cok}(X, g) \neq 0$, so $X$ has a summand isomorphic to $C$.

2. Assume that $A$ is not injective. Then $e_*$ has an isolated sink $A$ if $\dim \text{Cok}(u, A) > \dim \text{Cok}(u, B)$. In this situation, $Z$ has a summand isomorphic to $C = \tau^{-1}A$.

3. Moreover, $e_*$ has a tangent point at a mesh $\mathcal{M} : A \rightarrow \bigoplus B_i \rightarrow C$ where $C$ is indecomposable if $\sum \dim \text{Cok}(u, B_i) > \dim \text{Cok}(u, A) + \dim \text{Cok}(u, C)$. Hence $Y$ has a summand isomorphic to $C$, the end point of the mesh.

4. The condition that $C$ does not occur as a direct summand of $X \oplus Y \oplus Z$, is equivalent to the second sequence being constant zero, which means that the first sequence is exact, so $e_*$ is additive on $\mathcal{A}$. □

Proof of Proposition 1.3. Let $\mathcal{E} : 0 \rightarrow X \xrightarrow{u} Y \rightarrow Z \rightarrow 0$ be the short non-split exact sequence in the Proposition, and let $f : X \rightarrow M$ be a morphism which does not factor over $u$. Since $\mathcal{E}$ is non-split, the indecomposable module $Z$ is not a projective object in $\mathcal{C}$ and the Auslander-Reiten sequence $\mathcal{A} : 0 \rightarrow \tau Z \rightarrow W \rightarrow Z \rightarrow 0$ in $\mathcal{C}$ is defined. We show that there exists a morphism $g : M \rightarrow \tau Z$ such that $gf : X \rightarrow \tau Z$ does not factor over $u$.

\[
\begin{align*}
\mathcal{A} : & \quad 0 \longrightarrow \tau Z \xrightarrow{w'} W \xrightarrow{w} Z \longrightarrow 0 \\
\mathcal{E} : & \quad 0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{v} Z \longrightarrow 0 \\
\mathcal{E}f : & \quad 0 \longrightarrow M \xrightarrow{p'} P \xrightarrow{p} Z \longrightarrow 0
\end{align*}
\]
Since $v : Y \to Z$ is not a split epimorphism, there exists $h' : Y \to W$ such that $v = wh'$. Then the kernel map $h : X \to \tau Z$ does not factor over $u$ since otherwise $1_Z$ would factor over $w$ which is not possible since $A$ is non-split. In particular, the target $\text{Cok}(u, \tau Z)$ of the map in the Proposition is non-zero.

Since $f$ does not factor over $u$, the map $1_Z$ does not factor over $p$ so the induced sequence $\mathcal{E}f$ is non-split. Since $C$ is extension-closed, the middle term in $\mathcal{E}f$ is an object in $C$. Hence there is a map $g' : P \to W$ with $p = wg'$. Let $g : M \to \tau Z$ be the kernel map. We show $gf - h$ factors over $u$, so $gf$ does not factor over $u$.

Since $w(g'f' - h') = pf' - v = v - v = 0$, there is a map $d : Y \to \tau Z$ with $w'd = g'f' - h'$. Then $w'(gf - h) = g'p'f - h'u = (g'f' - h')u = w'du$ implies that $gf - h = du$ since $w'$ is a monomorphism. In particular, $gf - h$ factors over $u$. We have seen that $h$ does not factor over $u$, so $gf$ does not factor over $u$. □

We state the dual results for Theorem 1.2 and Proposition 1.3.

Suppose $E : \text{mod}\Lambda \to \text{mod}k$ is a finitely presented contravariant functor, say with projective resolution

$$0 \to (-, X) \to (-, Y) \to (-, Z) \to E \to 0,$$

given by a short left exact sequence $\mathcal{E} : 0 \to X \to Y \to Z$. We call the map

$$e^* : \Gamma_0 \to \mathbb{N}_0, \ A \mapsto \dim \text{Hom Cok}(A, v)$$

the hammock function for the contravariant functor $E$ (or for the sequence $\mathcal{E}$).

Here is the dual version of Theorem 1.2.

3.1. **Theorem.** Let $\mathcal{E} : 0 \to X \to Y \to Z$ be a short left exact sequence and $e^* : \Gamma_0 \to \mathbb{N}_0$ the corresponding hammock function.

(1) If $A$ is an isolated sink for $e^*$ then $A$ is a direct summand of $Z$.

(2) If $C$ is an isolated source for $e^*$ then $C$ is projective in $C$ or $A = \tau C$ is a direct summand of $X$.

(3) If the mesh $\mathcal{M} : A \to \bigoplus B_i \to C$ is an isolated tangent for $e^*$ then $C$ is indecomposable projective or $A$ is an indecomposable direct summand of $Y$.

(4) If $\mathcal{M} : A \to \bigoplus B_i \to C$ is a mesh such that $A$ is indecomposable and does not occur as a direct summand of $X \oplus Y \oplus Z$ then the hammock function $e^*$ is additive on $\mathcal{M}$. □
For the dual version of Proposition 1.3, suppose $E : 0 \to X \to Y \xrightarrow{v} Z \to 0$ is a non-split short exact sequence with $X$ indecomposable non-injective. Then the modules $M$ in the hammock are characterized by lying between $\tau^{-1}X$ and one of the summands of $Z$ in the sense that there is a map $f : M \to Z$ which does not factor over $v$ and a map $g : \tau^{-1}X \to M$ such that the product $fg : \tau^{-1}X \to Z$ does not factor over $v$:

3.2. Proposition. Let $E : 0 \to X \to Y \xrightarrow{v} Z \to 0$ be a non-split short exact sequence in $C$ with $X$ indecomposable. Then for any $M \in C$, the bilinear form given by composition,

$$Cok(M, v) \times \text{Hom}(\tau^{-1}X, M) \longrightarrow \text{Cok}(\tau^{-1}X, v),$$

is left non-degenerate. □

3.3. Example: We illustrate the two results in the introduction and their dual versions in the example of a hammock that contain exactly one vertex. Let $M$ be an indecomposable module, neither projective nor injective in $C$.

For Theorem 1.2, consider the Auslander-Reiten sequence $E : 0 \to X \xrightarrow{u} Y \to Z \to 0$ starting at $X = M$. The hammock function $e_*$ counts the multiplicity of $M$ as a direct summand of its object, hence $e_*$ vanishes on $\Gamma$, except at the vertex $M$. In fact, $M = X$ is an isolated source; for each indecomposable summand $D$ of $Y$, the mesh ending at $D$ (which contains $M$ as a summand of the middle term) is an isolated tangent; and the sink of the hammock is also $M$ since $M = \tau Z$.

Regarding Proposition 1.3, note that $M$ is the only object in $\Gamma$ between $X$ and $\tau Z$.

For Theorem 3.1, take the Auslander-Reiten sequence $\tilde{E} : 0 \to \tilde{X} \xrightarrow{\tilde{u}} \tilde{Y} \xrightarrow{\tilde{v}} \tilde{Z} \to 0$ ending at $\tilde{Z} = M$. Then $e^*$ vanishes on $\Gamma$ except at $M$. We have $M$ as an isolated sink since $M = \tilde{Z}$; for each indecomposable summand $D$ of $\tilde{Y}$, the mesh starting at $D$ (which contains $M$ as a summand of the middle term) is an isolated tangent; and the source of the hammock is $M$ since $M = \tau^{-1}\tilde{X}$.

This also illustrates Proposition 3.2 since $M$ is the only object in $\Gamma$ between $\tau^{-1}\tilde{X}$ and $\tilde{Z}$.

4. Applications I: Modules

In this section we use hammocks to visualize some properties of modules, in particular properties related to multiplication functions.
4.1. The multiplication map for modules. Let $R$ be an algebra, $M$ a finite dimensional (right) $R$-module and $a \in R$. The multiplication map

$$\mu_{M,a} : M \to M, \ m \mapsto ma,$$

gives rise to subspaces of $M$, in particular, $\ker \mu_{M,a}$, $\mathrm{im} \mu_{M,a}$ and $\mathrm{cok} \mu_{M,a}$, the dimension of which is a hammock function.

As a left $\text{End}_M$-module, the kernel of the multiplication map is given as a $\text{Hom}$ space by the isomorphism

$$\text{Hom}_R(R/aR, M) \cong \ker \mu_{M,a}, \ f \mapsto f(1).$$

Hence the dimension of the kernel is the covariant defect $\dim \ker \mu_{M,a} = e_\ast(M)$ where $\mathcal{E}$ is the short right exact sequence

$$\mathcal{E} : \ R/aR \longrightarrow 0 \longrightarrow 0 \longrightarrow 0.$$

Considering Theorem 1.2, the hammock function has sources at the indecomposable direct summands of $R/aR$, no non-injective isolated sinks, and no tangents with indecomposable end point.

As a special case, consider $a = 1 - e$ for an idempotent $e \in R$. Then the above hammock function yields $\dim Me = \dim \ker \mu_{M,a}$. The arising hammocks have already been studied in [3].

The image, being isomorphic to the factor of $M$ modulo the kernel of the multiplication map, is obtained as the cokernel of the sequence

$$0 \longrightarrow (R/aR, M) \longrightarrow (R, M) \longrightarrow E(M) \longrightarrow 0 \quad \cong \quad 0 \longrightarrow \ker \mu_a \longrightarrow M \longrightarrow M/\ker \mu_a \cong \mathrm{im} \mu_a \longrightarrow 0$$

(where we abbreviate $\text{Hom}(A, B)$ by $(A, B)$), thus $\dim \mathrm{im} \mu_{M,a} = e_\ast(M)$ is the covariant defect of the short right exact sequence

$$\mathcal{E} : \ R \longrightarrow R/aR \longrightarrow 0 \longrightarrow 0.$$

For the cokernel of the multiplication map, consider the short right exact sequence

$$\mathcal{E} : \ R \longrightarrow R \longrightarrow R/aR \longrightarrow 0$$

where $a$ is the left multiplication by $a$, which gives rise to the commutative diagram

$$0 \longrightarrow (R/aR, M) \longrightarrow (R, M) \longrightarrow (R, M) \longrightarrow E(M) \longrightarrow 0 \quad \cong \quad 0 \longrightarrow \ker \mu_a \longrightarrow M \overset{\mu_a}{\longrightarrow} M \longrightarrow \mathrm{cok} \mu_a \longrightarrow 0.$$
hence $\dim \text{Cok} \mu_{M,a} = e_*(M)$, as desired.

4.2. Quiver representations. We use hammocks to visualize some data derived from quiver representations.

Let $\Lambda = kQ$ be the path algebra of a quiver $Q = (Q_0, Q_1, s, t)$. For a $k$-linear representation $M$ of $Q$, a vertex $v \in Q_0$, an arrow $\alpha \in Q_1$, and a path $\pi = \alpha_1 \cdots \alpha_s$ in $Q$, we denote by $M_v$ the vector space in position $v$, by $M_{\alpha}$ the linear map $M_s(\alpha) \to M_t(\alpha)$, and by $M_\pi$ the linear map $M_{\alpha_s} \cdots M_{\alpha_1} : M_{s(\pi)} \to M_{t(\pi)}$.

In particular, we consider the path algebra of the quiver $E_6$ in bipartite orientation:

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\alpha & \beta & & & & &
\end{array}
\]

In this section, we picture the hammocks given by the kernel of the linear map $M_{\beta} : M_3 \to M_2$, by the image of $M_{\beta}$, and by the cokernel of $M_{\alpha}$.

First, we consider the kernel at an arrow or a path. For vertices $i, j \in Q_0$ and an arrow $\alpha : i \to j$ in $Q_1$, denote by $P(i)$ the indecomposable projective module corresponding to vertex $i$ and by $(\alpha)$ the submodule of $P(i)$ given by the image of the map $P(\alpha) : P(j) \to P(i)$; similarly, for a path $\pi = \alpha_1 \cdots \alpha_s : i \to j$, we write $(\pi)$ for the submodule of $P(i)$ given by the image of the composition $P(\pi) = P(\alpha_1) \cdots P(\alpha_s) : P(j) \to P(i)$. We claim that the short right exact sequence

\[ E : \ P(i)/(\pi) \to 0 \to 0 \to 0 \]

defines the hammock function $e_*(M) = \dim \text{Ker} \ M_\pi$.

Namely, applying the functor $\text{Hom}(\_, M)$ to the short exact sequence $0 \to P(j) \to P(i) \to P(i)/(\pi) \to 0$ gives the long exact sequence starting with

\[ 0 \to (P(i)/(\pi), M) \to (P(i), M) \xrightarrow{g} (P(j), M) \to \cdots. \]

Identifying $(P(i), M)$ with $M_i$, $(P(j), M)$ with $M_j$ and $g$ with $M_\pi$, we obtain $\text{Ker} M_\pi \cong \text{Hom}(P(i)/(\pi), M)$ and the claim follows.

We obtain from Theorem 1.2: In the Auslander-Reiten quiver, $e_*$ is the hammock function with source $P(i)/(\pi)$, no tangents with indecomposable end point, and no non-injective sinks.

4.1. Example: Figure 2 shows the Auslander-Reiten quiver for the path algebra $kQ$ of type $E_6$ given above. Each object is represented by its dimension vector. The region encircled by the dotted curve is the...
hammock for the function \( e_\ast(M) = \dim \ker M_\beta \). The hammock has source \( X = P(3)/(\beta) \), no tangents containing the end point, but there is a tangent at the mesh starting at \( \tilde{Y} \), and no non-injective sinks, but there is an injective sink at \( \tilde{Z} = I(3) \).

**Figure 2.** The hammock for the functor \( E : M \mapsto \ker M_\beta \)

The path \( \pi \) also defines a canonical map between indecomposable injective modules, \( \nu : I(j) \to I(i) \). Using the contravariant defect and Theorem 3.1, one can obtain \( \ker M_\pi \cong \cok(M_\nu) \cong \cok \hom(M, \nu) \).

In the above example, take \( \pi = \beta \) and consider the short left exact sequence

\[
\tilde{E} : 0 \to P(2) \to I(2) \xrightarrow{\nu} I(3).
\]

Then \( \tilde{Z} = I(3) \) is the sink for the hammock function \( e_\ast \); there is a tangent at the mesh starting at \( \tilde{Y} = I(2) \); and the source occurs at \( \tau^{-1}\tilde{X} \) where \( \tilde{X} = \ker \nu \cong P(2) \). The modules \( \tilde{X}, \tilde{Y}, \tilde{Z} \) are indicated in the above Auslander-Reiten quiver.

We consider the **image at an arrow or at a path.** For \( \alpha : i \to j \) an arrow (or \( \pi = \alpha_1 \cdots \alpha_s : i \to j \) a path), consider the short right exact sequence

\[
\mathcal{E} : P(i) \xrightarrow{u} P(i)/(\alpha) \xrightarrow{} 0 \to 0,
\]

and the exact sequence obtained by applying the functor \( \hom(-, M) \).

\[
\begin{array}{ccccccccc}
0 & \to & (P(i)/(\alpha), M) & \xrightarrow{(u,M)} & (P(i), M) & \to & \cok(P(\alpha), M) & \to & 0 \\
\downarrow & \cong & \downarrow & \cong & \downarrow & \cong & \downarrow & \cong & \downarrow \\
0 & \to & \ker M_\alpha & \xrightarrow{\text{incl}} & M_i & \xrightarrow{\text{can}} & \im M_\alpha & \to & 0
\end{array}
\]
Thus, the dimension of the image of the map associated with an arrow $\alpha$ is given by a hammock function with source $P(i)$, tangent at $P(i)/(\alpha)$ (or, in case of a path $\pi$, at $P(i)/(\pi)$), and no non-injective sink.

4.2. Example: The indecomposable modules $M$ in the Auslander-Reiten quiver for $kQ$ for which $\text{Im } M_{\beta} \neq 0$ are pictured in Figure 3. The hammock is given by the sequence $\mathcal{E} : P(3) \xrightarrow{u} P(3)/(\beta) \rightarrow 0 \rightarrow 0$, it has source $X = P(3)$, one tangent before $Y = P(3)/(\beta)$ and injective sink $I(2)$. Note that there are five indecomposables $M$ in the hammock with $e_*(M) = \text{dim } \text{Im } M_\beta = 2$; they are characterized here by dim $M_2 = 2$.

![Figure 3. The hammock for the functor $E : M \mapsto \text{Im } M_{\beta}$](image)

The same hammock can be obtained dually using Theorem 3.1 and the short left exact sequence $\tilde{\mathcal{E}} : 0 \rightarrow 0 \rightarrow \text{Ker}_{I(2)}(\beta) \xrightarrow{\text{incl}} I(2)$. Thus, the hammock has a sink at $\tilde{Z} = I(2)$, a tangent at the mesh starting at $\tilde{Y} = \text{Ker}_{I(2)}(\beta)$, and a source at some projective module or modules.

Next, the cokernel at an arrow or at a path. For $\alpha : i \rightarrow j$ an arrow, the right exact sequence

$$\mathcal{E} : \quad P(j) \xrightarrow{v} P(i) \rightarrow P(i)/(\alpha) \rightarrow 0$$

where $v = P(\alpha)$ gives rise to the top sequence in the diagram

$$\begin{array}{cccccc}
\text{Hom}(P(i), M) & \xrightarrow{(v, M)} & \text{Hom}(P(j), M) & \rightarrow & \text{Cok}(v, M) & \rightarrow 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & \\
M_i & \xrightarrow{M_\alpha} & M_j & \rightarrow & \text{Cok } M_\alpha & \rightarrow 0
\end{array}$$
Thus, \( \dim \text{Coker } M_\alpha = e_*(M) \) is given by the hammock function for \( \mathcal{E} \). The hammock has a source at \( P(j) \), a tangent at the mesh ending at \( P(i) \) and a sink at \( \tau P(i)/(\alpha) \).

4.3. Example: In the Auslander-Reiten quiver in Example 1.1, the hammock consists of those indecomposable \( kQ \)-modules \( M \) for which \( \text{Coker } M_\alpha \neq 0 \). It is a hammock given by the sequence \( \mathcal{E} : P(2) \xrightarrow{\pi} P(1)/(\alpha) \rightarrow 0 \). The hammock has source at \( X = P(2) \), one tangent before \( Y = P(1) \) and a non-injective sink at \( Z = \tau P(1)/(\alpha) = \tau I(1) \), as indicated.

4.3. The homology at two composable arrows or paths. Let \( \pi : i \rightarrow j \) and \( \rho : j \rightarrow k \) be paths in \( Q \), and \( M \) a representation of \( Q \) such that the composition \( M_\pi \rho \) of \( M_\pi : M_i \rightarrow M_j \) and \( M_\rho : M_j \rightarrow M_k \) is the zero map.

We are interested in the subquotient \( \text{Ker } M_\rho/\text{Im } M_\pi \) of \( M_j \).

The following sequence is short right exact.

\[
\mathcal{E} : \quad P(j)/(\rho) \xrightarrow{P(\pi)} P(i)/(\pi \rho) \rightarrow P(i)/(\pi) \rightarrow 0
\]

Applying the functor \( \text{Hom}(\cdot, M) \) we obtain

\[
0 \rightarrow (P(i)/(\pi), M) \rightarrow (P(i)/(\pi \rho), M) \xrightarrow{(P(\pi), M)} (P(j)/(\rho), M) \rightarrow \text{Coker}(P(\pi), M) \rightarrow 0
\]

where the first and second modules are isomorphic to \( \text{Ker } M_\pi \) and \( M_i = \text{Ker } M_\pi \); and the third to \( \text{Ker } M_\rho \). Thus,

\[
\dim \text{Coker } (P(\pi), M) = \dim \text{Ker } M_\rho - (\dim M_i - \dim \text{Ker } M_\pi) = \dim \text{Ker } M_\rho/\text{Im } M_\pi.
\]

Thus, the hammock given by the homology of two composable paths \( \pi \) and \( \rho \) has source at \( P(j)/(\rho) \), a unique non-injective sink at \( \tau P(i)/(\pi) \) and a unique tangent with indecomposable end term at \( P(i)/(\pi \rho) \).

5. Applications II: Invariant subspaces

In this section, let \( R = k[T]/(T^n) \) be the bounded polynomial ring, then the finite dimensional \( R \)-modules form the category \( \mathcal{N}(n) = \text{mod}R \) of nilpotent linear operators with nilpotency index at most \( n \). Each indecomposable \( R \)-module has the form \( P^\ell = k[T]/(T^\ell) \) for some \( 1 \leq \ell \leq n \).

The category \( \mathcal{S}(n) \) or \( \mathcal{S}(R) \) of invariant subspaces of nilpotent linear operators has as objects the pairs \( (V, U) \) where \( V \in \mathcal{N}(n) \) and \( U \) is a submodule of \( V \). For example, if \( \ell \leq m \), then the pair \( P^m_\ell = (P^m, \text{soc}^\ell P^m) \)
is an object in $\mathcal{S}(n)$. Such objects, called pickets, give rise to many interesting hammock functions. They also turn out to be decisive in the study of the geometry of the representation space, see [5].

We picture an object $V = e_1 P^{\ell_1} \oplus \cdots e_s P^{\ell_s}$ in $\mathcal{N}(n)$ as $s$ columns of boxes, where the $i$-th column has length $\ell_i$. Thus, the $j$-th box in the $i$-th column represents the basic element $e_i T^{j-1}$ in $V$. Each submodule $U$ of $V$ considered in this paper has the property that $U$ has a set of generators which are sums (not arbitrary linear combinations) of basic elements of $V$. Hence we can picture each generator as a connected row of dots. Here are some examples.

$$P_4^4 = \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\end{array}, \quad P_3^4 = \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}, \quad P_{31}^{42} = (e_1 P^4 \oplus e_2 P^2, \langle e_1 T + e_2, e_2 T \rangle) : 
\begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}.$$

The triangular matrix ring $\Lambda = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$ is a Gorenstein algebra, so the Gorenstein-projective objects form a full exact extension-closed subcategory which has Auslander-Reiten sequences. This category is equivalent to $\mathcal{S}(n)$, see [8]. Thus we are dealing with a Frobenius category with two indecomposable projective-injective objects, $P_0^n$ and $P^n_n$.

In Figure 4, we picture the Auslander-Reiten quiver $\Gamma$ for $\mathcal{S}(4)$, taken from [11, (6,4)]. There are 20 indecomposable objects in $\Gamma$; note that the quiver is periodic. We picture here one fundamental domain, so the objects on the left and on the right are to be identified, up to a reflection.

In this section, we consider hammock functions which, for an object $(V, U) \in \mathcal{S}(n)$, count the multiplicity of a module $P^\ell \in \mathcal{N}(n)$ as a direct summand of $V$ (or of $U$, or of $V/U$). In our last example, the hammock function counts the number of indecomposable direct summands of $U$.

5.1. **Our tool: Adjoint functors.** In this section we assume that $\mathcal{C}$ and $\mathcal{D}$ are exact $k$-categories with Auslander-Reiten sequences.

5.1. **Proposition.** Suppose $R$ and $L$ form a pair of adjoint functors

\begin{equation}
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{R} & \mathcal{D} \\
\xleftarrow{L} \\
\end{array}
\end{equation}

and $M$ is an indecomposable object in $\mathcal{D}$ with source map $s : M \rightarrow N$. Put $d = \dim \text{End } M/\text{rad } \text{End } M$. The short right exact sequence

\begin{equation}
\mathcal{E} : \quad LM \xrightarrow{Ls} LN \rightarrow L \text{Cok}(s) \rightarrow 0
\end{equation}
We can now compute using the adjoint isomorphism \( D \) sequence in 5.2.

right exact sequence in and hence, since a left adjoint functor is always right exact, to the short

\[
\mathcal{E} : \quad LM \xrightarrow{Ls} LN \rightarrow L\text{Cok}(s) \rightarrow 0.
\]

We can now compute using the adjoint isomorphism

\[
d \cdot \mu_M(RA) = \dim \text{Cok} \text{Hom}_\mathcal{D}(s, RA)
\]

\[
= \text{hom}_\mathcal{D}(\text{Cok}(s), RA) - \text{hom}_\mathcal{D}(N, RA) + \text{hom}_\mathcal{D}(M, RA)
\]

\[
= \text{hom}_\mathcal{C}(L\text{Cok}(s), A) - \text{hom}_\mathcal{C}(LN, A) + \text{hom}_\mathcal{C}(LM, A)
\]

\[
= \dim \text{Cok} \text{Hom}_\mathcal{C}(Ls, X).
\]

5.2. The multiplicity of \( P^m \) as a summand of \( V \). Consider the categories \( \mathcal{C} = \mathcal{S}(n) \) and \( \mathcal{D} = \mathcal{N}(n) \), and the functors \( R : \mathcal{S}(n) \rightarrow \mathcal{N}(n), (V, U) \mapsto V \) and \( L : \mathcal{N}(n) \rightarrow \mathcal{S}(n), V \mapsto (V, 0) \) which form an adjoint pair:

\[
\mathcal{S}(4) \xrightarrow{(U \subset V) \mapsto V} \mathcal{N}(4) \xleftarrow{(0 \subset V) \mapsto V}
\]
Starting with an Auslander-Reiten sequence for $P^m$ in $\mathcal{N}(n)$,

\[(*) \quad 0 \to P^m \to P^{m+1} \oplus P^{m-1} \to P^m \to 0\]

if $m < n$ (where $P^0 = 0$ in case $m = 1$) or with the source map $P^n \to P^{n-1}$, we obtain by applying $L$ the corresponding sequence in $\mathcal{S}(n)$:

$\mathcal{E} : P^m_0 \xrightarrow{u} P^{m+1}_{0} \oplus P^{m-1}_{0} \to P^m_0 \to 0$

for $m < n$ or $\mathcal{E} : P^n_0 \xrightarrow{u} P^n_1$ if $m = n$.

It follows from the Proposition that for an object $A = (V, U)$ in $\mathcal{S}(n)$, the multiplicity of $P^m$ as a direct summand of $V$ is

$\mu_{P^m}(V) = \dim \text{Cok} \text{Hom}(u, A) = e_*(A)$.

**5.2. Example:** We picture in Figure 5 the hammock in the category $\mathcal{S}(4)$ given by those objects $(V, U)$ where $V$ contains a direct summand isomorphic to $P^3$. For clarity, we present two copies of the fundamental domain of the above Auslander-Reiten quiver.

The objects in the hammocks which we consider in sections 5.2, 5.3 and 5.5 are marked by a large semicircle on the left, it is solid, dotted or dashed, depending on the section: The ambient space is marked by a solid, the subspace by a dotted, and the factor by a dashed semicircle.

**Figure 5.** Hammocks for summands of $U$, $V$ and $V/U$

The large semicircle indicates a summand of type $P^3$; some other objects are marked with a small semicircle, which indicates a summand of
type $P^1$. If the semicircle is solid, the summand occurs in the ambient space $V$; if it is dotted, it occurs in the subspace $U$; and the dashed semicircle represents a summand of the factor $V/U$.

The five objects marked with a large solid semicircle on the left are characterized by having a summand of type $P^3$ in the ambient space $V$. The short right exact sequence $E: P^3_0 \to P^2_0 \oplus P^4_0 \to P^3_0 \to 0$ in this section defines a hammock with source $X^a = P^3_0$, two tangents at the meshes ending at $Y_1^a = P^2_0$ and $Y_2^a = P^4_0$ and a sink at $\tau Z^a$ where $Z^a = P^3_0$. There is another tangent at the mesh starting at the injective $P^4_4$ labeled $Y_2^a$ which has no indecomposable end term.

Note that in the picture, the semicircle indicates the position of $P^3$ as a direct summand of the ambient space.

5.3. The multiplicity of $P^m$ as a summand of $U$. Here the functors $R: S(n) \to \mathcal{N}(n), (V,U) \mapsto U$, and $L: \mathcal{N}(n) \to S(n), U \mapsto (U,U)$ form an adjoint pair as follows:

$$
\begin{array}{ccc}
S(4) & \xrightarrow{(U \subset V) \mapsto U} & \mathcal{N}(4) \\
(U = U) \mapsto U
\end{array}
$$

The right exact sequence obtained from the above source map (*) in $\mathcal{N}(n)$ is

$$E: \begin{array}{c}
P^m_m \xrightarrow{u} P^{m+1}_{m+1} \oplus P^{m-1}_{m-1} \longrightarrow P^m_m \longrightarrow 0
\end{array}$$

if $m < n$ and $E: P^m_n \xrightarrow{u} P^{n-1}_{n-1}$ otherwise.

We obtain from the Proposition that for an object $A = (V,U)$, the multiplicity of $P^m$ as a direct summand of $U$ is

$$\mu_{P^m}(U) = \dim \text{Cok} \text{Hom}(u, A) = e_*(A).$$

5.3. Example: Returning to the example where $n = 4$ and $m = 3$, the five objects in the Auslander-Reiten quiver marked with a large dotted semicircle have $P^3$ as a direct summand of the subspace. In fact, the dotted semicircle indicates the position of the subspace. The sequence in this section, $E: P^3_3 \to P^2_2 \oplus P^4_4 \to P^3_3 \to 0$ defines a hammock with source $X^s = P^3_3$, tangents at the meshes ending in $Y_1^s = P^2_2$ and $Y_2^s = P^4_4$; and a sink at $\tau Z^s$ where $Z^s = P^3_3$. There is one other mesh, starting at the injective $P^4_4$ on the left which has no indecomposable end term.
5.4. **The adjoint isomorphism between covers and cokernels.**

This adjoint isomorphism will be used to identify objects \((V, U)\) where the factor \(V/U\) contains a summand of type \(P^\ell\).

In the submodule category \(\mathcal{S}(R)\), let \(\mathcal{I}\) be the categorical ideal of all maps which factor through an object of the form \((A = A)\) where \(A \in \text{mod} R\). It is easy to see that

\[
\mathcal{I}((U \subset V), (X \subset Y)) = \{ f \in \text{Hom}_R(V, Y) : f(V) \subset X \}.
\]

5.4. **Lemma.** A collection of assignments \(M \mapsto (K \subset P)\) where \(0 \to K \to P \to M \to 0\) is a projective resolution for \(M \in \text{mod} R\) defines a functor

\[
\text{Cov} : \text{mod} R \to \mathcal{S}(R)/\mathcal{I}.
\]

**Proof.** Under \(\text{Cov}\), a map \(h : M \to M'\) is sent to the class of the commutative square given by a lifting. This assignment is well defined. Namely, given a projective resolution \(0 \to K \to P \to M \to 0\), a short exact sequence \(0 \to U \to V \to W \to 0\), a map \(h : M \to W\) and two liftings \(g_1, g_2 : P \to V, f_1, f_2 : K \to U\), then their difference \(g = g_2 - g_1, f = f_2 - f_1\) gives rise to a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & K & \to & P & \to & M & \to & 0 \\
\downarrow f & & \downarrow g & & \downarrow 0 & \\
0 & \to & U & \overset{u}{\to} & V & \to & W & \to & 0
\end{array}
\]

where the map \(g\) factors over \(u\). Hence \((f, g)\), as a map in \(\mathcal{S}(R)\), factors through the object \((U = U)\). \(\square\)

5.5. **Lemma.** Let \(0 \to A \to B \to C \to 0\) be a short exact sequence in \(\text{mod} R\) and

\[
\begin{array}{ccccccccc}
0 & \to & K_A & \to & K_B & \to & K_C & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & P_A & \to & P_A \oplus P_C & \to & P_C & \to & 0
\end{array}
\]

be the corresponding sequence in \(\mathcal{S}(R)\), say obtained by using the Horsehoe Lemma. Then for any embedding \(X : (U \subset V)\), the following sequence is split exact

\[0 \to \mathcal{I}((K_A \subset P_A), X) \to \mathcal{I}((K_B \subset P_A \oplus P_C), X) \to \mathcal{I}((K_C \subset P_C), X) \to 0.\]

**Proof.** The last sequence simplifies to

\[0 \to \text{Hom}_R(P_A, U) \to \text{Hom}_R(P_A \oplus P_C, U) \to \text{Hom}_R(P_C, U) \to 0.\]

\(\square\)
5.6. **Proposition.** The functors $R = \text{Cok, L = Cov}$ form an adjoint pair

$$\begin{array}{c}
\mathcal{S}(R)/\mathcal{I} \\
\xleftarrow{\text{Cok}} \\
\xrightarrow{\text{Cov}} \\
\text{mod} R
\end{array}$$

**Proof.** Let $0 \to K \to P \to M \to 0$ be a projective cover and $(U \subset V)$ an embedding with cokernel $W$. We show that the map

$$\alpha : \text{Hom}_{\mathcal{S}/\mathcal{I}}(\text{Cov}(M), (U \subset V)) \to \text{Hom}_R(M, \text{Cok}(U \subset V))$$

is an isomorphism. The map $\alpha$ is defined since if a map $(f, g)$ in $\mathcal{S}(R)$ factors through an object of the form $(U, U)$, then the cokernel map is zero (since $\text{Cok} : \mathcal{S}(R) \to \text{mod} R$ is a functor).

To see that $\alpha$ is invertible, consider $h : M \to W$. We have seen in the proof of the first lemma above that the difference of any two liftings factors through the embedding $(U, U)$. Hence if $(f, g)$ is a lifting for $h$ in the sense that the diagram is commutative

$$\begin{array}{ccc}
0 & \longrightarrow & K \\
\downarrow f & & \downarrow g \\
0 & \longrightarrow & U
\end{array} \quad \begin{array}{ccc}
& \longrightarrow & P \\
& \downarrow h & \\
& \longrightarrow & V
\end{array} \quad \begin{array}{ccc}
M & \longrightarrow & M \\
\downarrow & & \downarrow \\
0 & \longrightarrow & W
\end{array} \quad 0$$

then the morphism $(f, g)$ in $\mathcal{S}(R)/\mathcal{I}$ is determined uniquely. \hfill \square

5.5. **The multiplicity of $P^n$ as a summand of $V/U$.** We return to the case where $R = k[T]/(T^n)$ and consider the Auslander-Reiten sequence (*) in $\text{mod} R$ from Section 5.2,

$$0 \longrightarrow A \overset{f}{\longrightarrow} B \longrightarrow C \longrightarrow 0,$$

and an embedding $(U \subset V)$ with cokernel $W$. We have seen that the multiplicity of $A$ as a direct summand of the $R$-module $W$ is

$$\mu_A W = \dim \text{Cok} \text{Hom}_R(f, W).$$

Using the Horseshoe Lemma as in the previous subsection, we obtain the sequence

$$\mathcal{E} : \quad (K_A \subset P_A) \overset{i}{\longrightarrow} (K_B \subset P_A \oplus P_C) \longrightarrow (K_C \subset P_C) \longrightarrow 0;$$

we denote the objects in this sequence $\bar{A}$, $\bar{B}$, and $\bar{C}$ and write $X = (U \subset V)$. 
From the adjoint isomorphisms in Proposition 5.1 and Proposition 5.6 and from Lemma 5.5 we obtain

\[ \mu_A(W) = \dim \text{Cok} \text{Hom}_R(f, W) \]

\[ = \text{hom}_R(C, W) - \text{hom}_R(B, W) + \text{hom}_R(A, W) \]

\[ = \text{hom}_{S/I}(\tilde{C}, X) - \text{hom}_{S/I}(\tilde{B}, X) + \text{hom}_{S/I}(\tilde{A}, X) \]

\[ = \text{hom}_S(\tilde{C}, X) - \text{hom}_S(\tilde{B}, X) + \text{hom}_S(\tilde{A}, X) \]

\[ = \dim \text{Cok} \text{Hom}_S(\tilde{f}, X). \]

where we write hom for dim Hom. This shows that the multiplicity of \( A \) as a summand of the factor is the covariant defect of the sequence \( E \) at the embedding \((U \subset V)\):

\[ \mu_A(V/U) = e_*(U \subset V). \]

5.7. Example: We return to the example in Section 5.2. In the case where \( n = 4 \) and \( m = 3 \), The Auslander-Reiten sequence in \( \mathcal{N}(4) \) starting at \( P_3 \),

\[ 0 \to P^3 \to P^2 \oplus P^4 \to P^3 \to 0, \]

gives rise to the short right exact sequence in \( \mathcal{S}(4) \),

\[ \mathcal{E} : P^4_1 \xrightarrow{\tilde{f}} P^4_2 \oplus P^4_0 \to P^4_1 \to 0, \]

hence for an object \( X = (U \subset V) \), the multiplicity of \( P^3 \) as a summand of \( V/U \) is \( e_*(X) \).

The five objects \((V, U)\) marked with a large dashed semicircle have \( P^3 \) as a direct summand in the factor space \( V/U \). The above sequence \( \mathcal{E} : P^4_1 \to P^4_2 \oplus P^4_0 \to P^4_1 \to 0 \) gives rise to the hammock with source \( X^f = P^4_1 \), tangents at the meshes ending in \( Y^f_1 = P^4_1 \) and \( Y^f_2 = P^4_0 \) and sink at \( \tau Z^f \) where \( Z^f = P^4_1 \). There is an additional tangent with no indecomposable end term at the injective object \( P^4_0 \) labeled \( Y^a_2 \).

We would like to point out that the marked objects visualize a periodicity in the stable part of the Auslander-Reiten quiver: If the object \( X \) is marked with a large solid semicircle, then \( \tau X \) has a large dotted semicircle, \( \tau^2 X \) a small dashed semicircle, \( \tau^3 X \) a small solid semicircle, \( \tau^4 X \) a small dotted semicircle, and \( \tau^5 X \) a large dashed semicircle. Finally, \( \tau^6 X \cong X \) is again marked with a large solid semicircle. This illustrates [10, Theorem 5.1]:

Consider the indecomposable non-projective object \( X = (U \subset V) \) as a triangle in the stable category \( \overline{\mathcal{V}(n)} \) of the Frobenius category \( \mathcal{V}(n) \),

\[ X : U \to V \to V/U \to \Omega U \to \]
then the Auslander-Reiten translate for $X$, considered as a triangle in
the stable category, is given by rotating the above triangle:

$$\tau_S X : \quad V \to V/U \to \Omega U \to \Omega V \to$$

Thus, a non-projective summand $A$ of the ambient space in $X$ becomes
a summand of the subspace of $\tau_S X$, then $\Omega \mathcal{N} A$ becomes a summand of
the factor space of $\tau_S^2 X$ etc.

5.6. Example: A hammock with no isolated source or sink.
Consider the hammock function indicated by small numbers in the
Auslander-Reiten quiver studied before, it is pictured in Figure 6.

![Hammock function with no isolated source or sink](image)

The function is given by counting for an object $(V, U)$ the number
of indecomposable direct summands of the subspace $U$. This is the
number of generators of $U$ as a $k[T]$-module which are pictured as dots
or connected sequences of dots, see the introduction in Section 5.

Note that this hammock function is additive on each complete mesh
except the one marked (*). However, the hammock does not have an
isolated source or an isolated sink.

As the hammock function counts the number of indecomposable sum-
mands of $U$, it is given by the short right exact sequence $\mathcal{E} : P^1 \to
0 \to 0 \to 0$. Hence the hammock has a non-isolated source at $P^1$, no
tangent with an indecomposable end term, and no non-injective sink.
The picture in Figure 7 shows this hammock as a dotted region, the source is labeled by $Z$. Since it stretches over several copies of the fundamental domain, there is an overlap. We picture a shifted copy of the hammock, encircled by a dotted line. The union of the two regions yields the hammock function in the above picture, note that the value is 2 where the two regions overlap. The one object which occurs in both regions has been pictured as $P_{31}^2$ in the introduction to this section. It is the unique indecomposable object in $S(4)$ for which the subspace as a $k[T]$-module has two indecomposable summands.

Figure 7. Hammock counting the indecomposable summands of the subspace $U$

Note that the hammock is symmetric with respect to the vertical line through (*). The same hammock function is obtained from the contravariant functor given by the short left exact sequence $\mathcal{E}: 0 \to 0 \to P_0^4 \to P_1^4$. It has a sink at $\tilde{Z} = P_1^4$, a tangent starting at the injective module $\tilde{Y} = P_0^4$, and no non-projective source.

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