Aggregation of network traffic and anisotropic scaling of random fields

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Abstract

We discuss joint spatial-temporal scaling limits of sums $A_{\lambda, \gamma}$ (indexed by $(x, y) \in \mathbb{R}_+^2$) of large number $O(\lambda^\gamma)$ of independent copies of integrated input process $X = \{X(t), t \in \mathbb{R}\}$ at time scale $\lambda$, for any given $\gamma > 0$. We consider two classes of inputs $X$: (I) Poisson shot-noise with (random) pulse process, and (II) regenerative process with random pulse process and regeneration times following a heavy-tailed stationary renewal process. The above classes include several queueing and network traffic models for which joint spatial-temporal limits were previously discussed in the literature. In both cases (I) and (II) we find simple conditions on the input process in order that normalized random fields $A_{\lambda, \gamma}$ tend to an $\alpha$-stable Lévy sheet $(1 < \alpha < 2)$ if $\gamma < \gamma_0$, and to a fractional Brownian sheet if $\gamma > \gamma_0$, for some $\gamma_0 > 0$. We also prove an ‘intermediate’ limit for $\gamma = \gamma_0$. Our results extend previous work \cite{20, 9} and other papers to more general and new input processes.

Keywords: heavy tails, long-range dependence, self-similarity, shot-noise process, regenerative process, superimposed network traffic, joint spatial-temporal limits, anisotropic scaling of random fields, scaling transition, intermediate limit, Telecom process, stable Lévy sheet, fractional Brownian sheet, renewal process, large deviations, ON/OFF process, M/G/$\infty$ queue, M/G/1/0 queue, M/G/1/$\infty$ queue

1 Introduction

Broadband network traffic exhibits three distinctive properties - heavy tails, self-similarity and long-range dependence \cite{23, 21, 40}. To explain these empirical facts, several mathematical models of network traffic were proposed and studied \cite{23, 38, 20, 14, 15, 21}. The usual framework can be described as joint temporal-spatial scaling (aggregation) of independent ‘sources’ or ‘inputs’ each described by a stationary process $X = \{X(t), t \in \mathbb{R}\}$ with long-range dependence (LRD). The aggregation procedure involves summation of $M$ independent integrated copies $X_i, i = 1, \ldots, M$ of $X$ at time scale $T$ when $T, M$ increase jointly to infinity, possibly at a different rate. Accordingly, we may consider the limit distribution of

$$A(T, M) := \sum_{i=1}^{M} \int_{0}^{T} X_i(s)ds, \quad \text{where} \quad T = x\lambda, \quad M = \lfloor y\lambda^\gamma \rfloor, \quad (x, y) \in \mathbb{R}_+^2 := (0, \infty)^2 \quad (1.1)$$
increase jointly with \( \lambda \to \infty \) and \( \gamma > 0 \) is an arbitrary fixed number. Then \( A_{\lambda, \gamma}(x, y) := A(x\lambda, [y\lambda^\gamma]) \) is a random field (RF) indexed by \( (x, y) \in \mathbb{R}_+^2 \) and the limit problem becomes anisotropic scaling of RFs:

\[
d_{\lambda, \gamma}^{-1}\{A_{\lambda, \gamma}(x, y) - E A_{\lambda, \gamma}(x, y)\} \xrightarrow{\text{fdd}} V_\gamma(x, y),
\]

where \( d_{\lambda, \gamma} \to \infty \) is a normalization. In most of the cited works, the limit in (1.2) is restricted to the case \( y = 1 \) and a limit random process with one-dimensional time \( x \in \mathbb{R}_+ \); however the results can be easily extended to the RF set-up in (1.2) using the fact that the summands in (1.1) are independent, see [28].

A ‘typical’ result in the heavy-tailed aggregated traffic research (see, e.g. [20, 9, 15]) says that there exists a critical ‘connection rate’ \( M_0(T) \to \infty (T \to \infty) \) such that the (normalized) ‘aggregated input’ \( A(xT, M(T)) \) tends to an \( \alpha \)-stable Lévy process or a Fractional Brownian Motion depending on whether \( M(T)/M_0(T) \) tends to 0 or \( \infty \); the critical growth \( M(T)/M_0(T) \to c \in (0, \infty) \) results in a different ‘intermediate’ limit which is neither Gaussian nor stable. In the afore-mentioned works, \( M_0(T) = T^{\gamma_0} \) is a power function of \( T \) implying that for any \( y > 0, yT^{\gamma}/T^{\gamma_0} \to 0 \) or \( \infty \) for \( \gamma < \gamma_0 \) and \( \gamma > \gamma_0 \), respectively. Therefore, by restricting to power connection rates, we may expect that the above-mentioned research leads to the scaling limits in (1.2)

\[
V_\gamma(x, y) = \begin{cases} 
B_{H,1/2}(x, y), & \gamma > \gamma_0, \\
L_\alpha(x, y), & \gamma < \gamma_0, \\
J(x, y), & \gamma = \gamma_0,
\end{cases}
\]

where \( B_{H,1/2} \) is a Fractional Brownian Sheet (FBS), \( L_\alpha \) is an \( \alpha \)-stable Lévy sheet, and \( J \) is an ‘intermediate’ RF (all three defined rigorously below in Section 3). For \( \gamma = \gamma_0 \) the limit \( J \) in (1.3) was proved in [3, 8] to be an infinitely divisible ‘intermediate’ Telecom RF, (defined rigorously in Section 3). Following [20], we refer to the cases \( \gamma > \gamma_0 \) and \( \gamma < \gamma_0 \) as fast and slow growth for the connection rate, respectively.

Related scaling trichotomy (termed scaling transition) was observed for a large class of planar RFs with long-range dependence (LRD), see [34, 33, 27, 28, 37, 29]. In these works, \( A_{\lambda, \gamma}(x, y) \) correspond to a sum or integral of values of a stationary RF (indexed by \( \mathbb{Z}^2 \) or \( \mathbb{R}^2 \)) over large rectangle \( (0, \lambda x] \times (0, \lambda^\gamma y] \) whose sides increase as \( O(\lambda) \) and \( O(\lambda^\gamma) \), for a given \( \gamma > 0 \). Therefore, limit theorems as in (1.2) relate teletraffic research to limit theorems for RFs and vice versa. Related scaling results were also obtained for aggregation of random-coefficient AR(1) process [25, 26, 18, 24].

The present paper extends (1.2)–(1.3) to two classes of ‘input’ processes. Class (I) (shot-noise inputs) has a form

\[
X(t) = \sum_j W_j(t - T_j) \mathbf{1}(t > T_j), \quad t \in \mathbb{R},
\]

where \( \{T_j\} \) is a standard Poisson point process and \( \{W_j\} \) are random ‘pulses’ (i.i.d. copies of generic ‘pulse’ \( W = \{W(t), t \in \mathbb{R}\} \)) satisfying certain conditions. Class (II) (regenerative inputs) have a somewhat similar form

\[
X(t) = \sum_j W_j(t - T_j-1) \mathbf{1}(T_{j-1} < t < T_j), \quad t \in \mathbb{R},
\]
except that \{T_j\} in (1.5) is a stationary renewal process with heavy-tailed inter-renewal intervals \(Z_j = T_j - T_{j-1}\) and ‘pulses’ \(W_j\) in (1.5) are generally not independent of \(Z_j\); the regeneration being a consequence of independence of \((Z_j, \{W_j(t)\}_{t \in [0,Z_j)}, j \in \mathbb{Z}.

The aim of this research is to obtain simple general conditions on \{T_j\} and generic pulse \(W\) guaranteeing the existence of Gaussian/stable fast/slow growth limits as in (1.3), and to verify these conditions for classical network traffic models, including some new ones for which the convergences in (1.2) were not established previously.

Let us describe the content of our work in more detail. Sec. 2 introduces the concept of \((\gamma, H)\)-scaling measure \(\nu\) on \(L^1(\mathbb{R}_+)\) and relates to it a general class of ‘intermediate’ RFs \(J_\nu = \{J_\nu(x,y), (x,y) \in \mathbb{R}_+^2\}\) satisfying an (anisotropic) \((\gamma, H(\gamma))\)-self-similarity property in (2.5). We also establish asymptotic self-similarity properties of the random process \(\{J_\nu(x,1), x > 0\}\) (Theorem 1). Theorem 2 (Section 3) obtains sufficient conditions for the existence of the scaling limits in (1.2) and shot-noise inputs as in (I). We show that the LRD property of the covariance function \(\text{Cov}(X(0), X(t)) \sim c_X t^{-2(1-H)}, t \to \infty (H \in (\frac{1}{2},1))\) together with a Lyapunov condition guarantee the Gaussian convergence in (1.2) towards a FBS \(B_{H,1/2}\) in (1.3). For \(\alpha\)-stable limit in (1.3) the crucial condition in the above model seems to be the requirement that the distribution of the integral \(W = \int_0^\infty W(t)dt\) belongs to the domain of attraction of an \(\alpha\)-stable law, complemented by a decay rate of \(E|W(t)|^{\alpha'}, t \to \infty\) for suitable \(\alpha' < \alpha\) (see eqs. (3.10), (3.11) of Theorem 2). In contrast to the rather general slow or fast connection rate assumptions (cases (i) and (ii) of Theorem 2), the existence of ‘intermediate’ limit at \(\gamma = \gamma_0\) in (1.3) (case (iii) of Theorem 2) requires an asymptotic scaling form of the pulse process \(W\) which enters the Poisson stochastic integral representation in (2.6).

Section 4 provides several examples of pulse process \(W\) verifying all three cases (i)–(iii) of the above theorem. Example 2 includes the most simple (probably, the most important) shot-noise model – the infinite source Poisson process or M/G/\(\infty\) queue studied in [20], as well as its extension – the finite variance continuous flow reward model discussed in [15]. Example 3 (deterministically related transmission rate and duration model) is part of the network traffic models whose scaling behavior was studied in [27]. Examples 4 and 5 refer to shot-noises whose scaling behavior in (1.2) was not established previously, namely the exponentially damped transmission rate model with \(W(t) = e^{-At}1(t < R)\) with random \(A\) and \(R\), and the Brownian pulse model with \(W(t) = B(t)1(t < R)\) and Brownian motion \(B\).

Let us turn to our limit results for regenerative processes (class (II)). This class is more difficult to study than (I) but also more interesting for applications since it contains many queueing models with limited service capacity. In contrast to class (I) which are completely specified by the distribution of \(W\), scaling properties of regenerative processes depend both on pulse \(W\) and the length \(Z\) of the generic inter-renewal interval \(Z_j = T_j - T_{j-1}\). It is usually assumed that the latter length has a heavy-tailed regularly varying distribution, viz.,

\[
P(Z > x) \sim c_Z x^{-\alpha}, \quad x \to \infty
\]

with some \(\alpha \in (1,2), c_Z > 0\). Condition \(\alpha > 1\) guarantees \(E Z < \infty\) which is necessary for stationarity of
\{T_\gamma\} while \(\alpha < 2\) implies that \(\{T_\gamma\}\) is LRD [20, 9]. The most studied case of such processes \(X\), from the point of view of the limits in (1.2), is the ON/OFF model with \(W(t) = 1(t < Z_{\text{on}})\) and \(Z = Z_{\text{on}} + Z_{\text{off}}\), where \(Z_{\text{on}}\) and \(Z_{\text{off}}\) are (random) durations of ON and OFF intervals. As noted in the seminal work [20], the limit results for heavy-tailed ON/OFF and infinite source Poisson inputs have striking similarities, the limits in (1.2) for \(\gamma \neq \gamma_0\) being virtually the same for both models. The coincidence of the ‘intermediate’ limits (the Telecom RF) at \(\gamma = \gamma_0\) for the above classes of inputs was proved in [9, 7, 15]. See also Sections 2 and 5 below. For more general pulse processes as in the present paper, differences between limit results for classes (I) and (II) appear which are especially notable in Examples 4 and 12. The main results of Section 5 are Theorems 4–6 providing conditions on the regenerative inputs for fast/slow/intermediate limits in (1.3), together with examples of regenerative processes satisfying all these conditions. Finally, Section 6 provides a new proof of one-dimensional convergence in the intermediate Telecom limit for heavy-tailed renewal process based on large deviations complementing the earlier result [9].

The present work can be improved and extended in several directions. All our results assume finite variance inputs \(\mathbb{E}X^2(t) < \infty\) which is further strengthened to \(|X(t)| < C\) in the regenerative case. As shown in [30, 15] inputs with infinite variance may lead to non-Gaussian limit RF under fast growth assumptions \((\gamma > \gamma_0)\). Another restrictive condition is (5.50) (Theorem 6), restricting the intermediate limit for class (II) to the Telecom RF, and leaving open this limit when (5.50) is not satisfied. We also expect that our results remain valid if the pure power law asymptotics in our theorems (1.6), (3.8), (3.10), etc.) are generalized to include slowly varying factors. Finally, strengthening (1.2) to a functional convergence on the plane is a natural open problem.

**Notation.** In this paper, \(\rightarrow, \text{ fdd, d, fdd} \) denote the weak convergence and equality of (finite dimensional) distributions. \(C\) stands for a generic positive constant which may assume different values at various locations and whose precise value has no importance. \(\mathbb{R}_+ := (0, \infty), \mathbb{R}_+^2 := (0, \infty)^2, \mathbb{Z}_+ := \{0, 1, \ldots\}, \mathbb{N} := \{1, 2, \ldots\}, (x)_+ := x \vee 0, (x)_- := (-x) \vee 0 (x \in \mathbb{R})\). \(1(A)\) denotes the indicator function of a set \(A\).

## 2 Limit random fields

Let \(\nu\) be a \(\sigma\)-finite measure on \(\mathbb{W} := \{w \in L^1(\mathbb{R}) : w(t) = 0 \text{ for } t < 0\}\) equipped with the \(\sigma\)-algebra \(\mathcal{B}(\mathbb{W})\) of Borel subsets of \(\mathbb{W}\) and such that for any \(x > 0\)

\[
\int_{\mathbb{R} \times \mathbb{W}} \left( \left| \int_{0}^{x} w(t-u)dt \right| \wedge \left| \int_{0}^{x} w(t-u)dt \right| \right) d\nu(dw) < \infty. \tag{2.1}
\]

With the above \(\nu\) we can associate a centered Poisson random measure \(Z_{\nu}(dw, dv, dw)\) on \(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{W}\) with control measure \(dudv\nu(dw)\). The stochastic integral \(\int_{\mathbb{R} \times \mathbb{R}_+ \times \mathbb{W}} f(u,v,w)Z_{\nu}(dw, dv, dw) \equiv \int f dZ_{\nu}\) is well-defined for any \(f\) with \(\int_{\mathbb{R} \times \mathbb{R}_+ \times \mathbb{W}} (|f(u,v,w)| \wedge |f(u,v,w)|^2) dudv\nu(dw) < \infty\) and its characteristic function is given by

\[
\mathbb{E}e^{i\theta \int fdZ_{\nu}} = \exp \left\{ \int_{\mathbb{R} \times \mathbb{R}_+ \times \mathbb{W}} (e^{i\theta f(u,v,w)} - 1 - i\theta f(u,v,w)) dudv\nu(dw) \right\}, \quad \theta \in \mathbb{R}. \tag{2.2}
\]
Moreover, \( E|\int f dZ_\nu| < \infty \) and \( E\int f dZ_\nu = 0 \).

With given \( \lambda > 0 \) and \( H \in \mathbb{R} \) we associate a one-to-one mapping \( \phi_{\lambda,H} \) on \( \mathbb{W} \) with inverse \( \phi_{\lambda,H}^{-1} = \phi_{\lambda^{-1},H} \) defined as

\[
\phi_{\lambda,H} w(t) := \lambda^{H^{-1}}w(\lambda^{-1}t), \quad t \in \mathbb{R}, \ \lambda > 0, \ w \in \mathbb{W}.
\] (2.3)

Note \( \{\phi_{\lambda,H}, \lambda > 0\} \) form a group of scaling transformations on \( \mathbb{W} \).

**Definition 1.** Let \( (\gamma, H) \in \mathbb{R}_+ \times \mathbb{R} \).

(i) A \( \sigma \)-finite measure \( \nu \) on \( \mathbb{W} \) is said \( (\gamma, H)\)-scaling if

\[
\lambda^{1+\gamma}\nu \circ \phi_{\lambda,H} = \nu, \quad \forall \lambda > 0.
\] (2.4)

(ii) A RF \( J = \{J(x,y),(x,y) \in \mathbb{R}^2_+\} \) is said \( (\gamma, H)\)-self-similar if

\[
\{J(\lambda x, \lambda^\gamma y), (x,y) \in \mathbb{R}^2_+\} \overset{fdd}{=} \{\lambda^HJ(x,y), (x,y) \in \mathbb{R}^2_+\}, \quad \forall \lambda > 0.
\] (2.5)

Some comments regarding Definition 1 are in order. \( (\gamma, H)\)-self-similarity property is a particular case of the operator self-similarity property introduced in [5], corresponding to scaling \( (x,y) \to \lambda^E(x,y) \) with diagonal matrix \( E = \text{diag}(1,\gamma) \). \( (1,H)\)-self-similarity coincides with the usual \( H\)-self-similarity property for RFs on \( \mathbb{R}^2_+ \) [35]. The notion of \( (\gamma, H)\)-scaling measure seems to be new and applies to infinite measures; indeed, (2.4) implies \( \lambda^{1+\gamma}\nu(\mathbb{W}) = \nu(\mathbb{W}) (\forall \lambda > 0) \), meaning that \( \nu \) cannot be a finite measure. Such measures are used to construct ‘intermediate’ \((\gamma, H)\)-self-similar RFs \( J_\nu \) through Poisson stochastic integrals in (2.2), see (2.6) below. To avoid any confusion with the scaling set-up in (1.2), we note that in Definition 1 (i) \( \gamma > 0 \) is fixed: the above-mentioned RF \( J_\nu \) appear as ‘intermediate’ limits in (1.2) for a particular \( \gamma = \gamma_0 \) only. On the other hand, \( (\gamma, H)\)-self-similarity is a general property shared by limit RFs in (1.2) for any \( \gamma > 0 \) [33]. A RF \( J = \{J(x,y),(x,y) \in \mathbb{R}^2_+\} \) is said \((H_1, H_2)\)-multi-self-similar with indices \( H_1, H_2 \in \mathbb{R} \) if \( \{J(\lambda_1 x, \lambda_2 y), (x,y) \in \mathbb{R}^2_+\} \overset{fdd}{=} \{\lambda_1^{H_1}\lambda_2^{H_2}J(x,y), (x,y) \in \mathbb{R}^2_+\} \) for any \( \lambda_i > 0, i = 1, 2 \) [10]. Clearly, a \((H_1, H_2)\)-multi-self-similar RF satisfies (2.5) for any \( \gamma > 0 \), with \( H = H_1 + \gamma H_2 \) a linear function of \( \gamma \). The class of \((H_1, H_2)\)-multi-self-similar RF contains FBS but else is quite restrictive and does not include \((\gamma, H)\)-self-similar RFs. Somewhat surprisingly, a given measure \( \nu \) can be \((\gamma, H)\)-scaling with different \((\gamma, H)\), see Example 2, but then the corresponding RF \( J_\nu \) lacks the important ‘intermediate’ property of Theorem 1 and apparently cannot appear as ‘intermediate’ scaling limit in (1.2).

Given a \((\gamma, H)\)-scaling measure \( \nu \) satisfying (2.1) we define a RF \( J_\nu = \{J_\nu(x,y),(x,y) \in \mathbb{R}^2_+\} \) as stochastic integral

\[
J_\nu(x,y) := \int_{\mathbb{R} \times (0,y) \times \mathbb{W}} \left\{ \int_0^x w(t-u)du \right\} Z_\nu(du,dv,dw)
\] (2.6)

w.r.t. to (centered) Poisson random measure in (2.2). We extend the definition in (2.6) to \( \mathbb{R}^2_+ := [0,\infty)^2 \) by setting \( J_\nu(x,y) := 0 \) for \( x \land y = 0 \). Probably, the most simple and important case of RFs in (2.6) is the Telecom RF defined as

\[
J_R(x,y) := \int_{\mathbb{R} \times (0,y) \times \mathbb{R}_+} \left\{ \int_0^x 1(u < t \leq u + r)dt \right\} Z_R(du,dv,dr),
\] (2.7)
which corresponds to $\nu = \nu_R$ concentrated on indicator functions $1(0 < t \leq r)$, $r \in \mathbb{R}_+$, and given by

$$\nu_R(dr) := \rho c_{\rho} r^{-\rho - 1} dr, \quad r \in \mathbb{R}_+, \quad (2.8)$$

where $\rho \in (1,2)$, $c_{\rho} > 0$ are parameters. Note $\nu_R$ in (2.8) is a $\sigma$-finite measure which is $(\gamma, H)$-scaling for $\gamma = \rho - 1$, $H = 1$, and satisfies (2.1). The Telecom RF in (2.7) satisfies $(\gamma, H)$-self-similarity property, see below, but it is not $(H_1, H_2)$-multi-self-similar for any $H_i, i = 1, 2$. Indeed, the latter property would imply that the restriction $\{J_{R}(1, \lambda^{H_2} y), y \in \mathbb{R}_+\} \stackrel{fdd}{=} \{\lambda^{H_2} J_{R}(1, y), y \in \mathbb{R}_+\}$ to the vertical line $x = 1$ is a $H_2$-self-similar Lévy process with independent increments, in other words, an $\alpha$-stable process, but this is not true, see e.g. [8].

The reader might ask what is the advantage of discussing RF in (2.6) indexed by $(x, y)$ instead of random process of argument $x$ alone. Particularly, the Telecom process discussed in [9, 8, 15] is defined as the restriction of (2.7) to $y = 1$. The behavior of (2.6) in $y$ is rather simple, the primary interest being that in the horizontal direction $x$. On the other hand, the RF set-up used in this paper seems more natural from scaling point of view: the Telecom process is not self-similar (only asymptotically self-similar) while the corresponding Telecom RF in (2.7) satisfies the self-similarity property in Definition 1 (ii). The latter property seems to be related to the notion of aggregate self-similarity with rigidity index introduced in [11]. See also [25, p.1022-1023].

The following proposition gives some properties of RF $J_{\nu}$ in (2.6). Recall that a RF $V = \{V(x, y), (x, y) \in \mathbb{R}_+^2\}$ has stationary rectangular increments if for any fixed $(x_0, y_0) \in \mathbb{R}_+^2$ and any rectangle $K := (x_0, x] \times (y_0, y] \subset \mathbb{R}_+^2$ its rectangular increment $V(K) := V(x, y) - V(x_0, y) - V(x, y_0) + V(x_0, y_0)$ satisfies

$$\{V(x, y) - V(x_0, y) - V(x, y_0) + V(x_0, y_0), x \geq x_0, y \geq y_0\} \stackrel{fdd}{=} \{V(x - x_0, y - y_0) - V(0, y - y_0) - V(x - x_0, 0) + V(0, 0), x \geq x_0, y \geq y_0\}.$$

We say that $V$ has independent rectangular increments in the vertical direction if $V(K)$ and $V(K')$ are independent for any rectangles $K$ and $K'$ which are separated by a horizontal line.

**Proposition 1.** Let $\nu$ be a $(\gamma, H)$-scaling measure on $\mathbb{W}$ satisfying (2.1) an let $J_{\nu}$ be as in (2.6). Then:

(i) $J_{\nu}(x, y)$ is well-defined for any $(x, y) \in \mathbb{R}_+^2$ as stochastic integral w.r.t. Poisson random measure $Z_{\nu}$ and has infinitely divisible finite-dimensional distributions and zero mean $E J_{\nu}(x, y) = 0$. Moreover, RF $J_{\nu}$ has stationary rectangular increments and independent rectangular increments in the vertical direction;

(ii) RF $J_{\nu}$ is $(\gamma, H)$-self-similar.

(iii) If, in addition,

$$C_{\nu}^2 := \int_{\mathbb{R} \times \mathbb{W}} \left( \int_0^1 w(t - u) dt \right)^2 du \nu(dw) < \infty, \quad (2.9)$$

then $E |J_{\nu}(x, y)|^2 < \infty$, $(x, y) \in \mathbb{R}_+^2$, and

$$E J_{\nu}(x_1, y_1) J_{\nu}(x_2, y_2) = \frac{C_{\nu}^2}{2} \left[ x_1^{2H_1} + x_2^{2H_1} - |x_1 - x_2|^{2H_1} (y_1 \land y_2), \quad (x_i, y_i) \in \mathbb{R}_+^2, \ i = 1, 2, \quad (2.10)$$

where $H_1 := H - \frac{\gamma}{2}$. 
Hence, (2.10) follows from (2.11) and properties of rectangular increments in (i).

(ii) For any \( \theta \in \mathbb{R} \), \((x_j, y_j) \in \mathbb{R}^2_+\), \( j = 1, \ldots, m \), \( m, \in \mathbb{N} \) we have by (2.2) that \( \mathbb{E} \exp \{ i \sum_{j=1}^{m} \theta_j J_{\nu}(\lambda x_j, \lambda^\gamma y_j) \} = e^{I_{\lambda}} \), where

\[
I_{\lambda} = \int_{\mathbb{R} \times \mathbb{R} \times \mathcal{W}} \Psi \left( \sum_{j=1}^{m} \theta_j 1(\nu \leq \lambda^\gamma y_j) \int_{0}^{\lambda x_j} w(t-u)dt \right) du dv \nu(dw)
\]

and \( \Psi(z) = e^{iz} - 1 - iz, \ z \in \mathbb{R} \). In \( I_{\lambda} \) change the variables \( w \to \phi_{\lambda,H} u', u \to \lambda u', v \to \lambda^\gamma v' \) and observe that \( \int_{0}^{\lambda x_j} (\phi_{\lambda,H} w')(t-\lambda u')dt = \lambda H \int_{0}^{x_j} w'(t-u')dt \) and \( \lambda^{\gamma+1}(\nu \circ \phi_{\lambda,H})(dw') = \nu(dw') \) according to (2.4). This yields the equality of characteristic functions: \( \mathbb{E} \exp \{ i \sum_{j=1}^{m} \theta_j J_{\nu}(\lambda x_j, \lambda^\gamma y_j) \} = \mathbb{E} \exp \{ i \sum_{j=1}^{m} \theta_j \lambda^H J_{\nu}(x_j, y_j) \} \).

(iii) Using (2.4) with \( \lambda = x \) similarly as in part (ii) we get that

\[
\mathbb{E}|J_{\nu}(x,y)|^2 = y \int_{\mathbb{R} \times \mathbb{W}} \left( \int_{0}^{x} w(t-u)dt \right)^2 du \nu(dw)
\]

\[
= y \int_{\mathbb{R} \times \mathbb{W}} \left( \int_{0}^{x} (\phi_{x,H} w)(t-u)dt \right)^2 du \nu(\phi_{x,H})(dw)
\]

\[
= yx^{2H-\gamma} \int_{\mathbb{R} \times \mathcal{W}} \left( \int_{0}^{1} w(t-u)dt \right)^2 du \nu(dw) = C_{\nu} x^{2H} y.
\]

Hence, (2.10) follows from (2.11) and properties of rectangular increments in (i).

The natural question is what additional conditions the exponent \( H \) should satisfy in order that \( \nu \) and \( J_{\nu} \) in Proposition 1 exist. From (2.10) it is clear that \( H \) cannot be arbitrary in general.

Proposition 2. Let \( \nu \) and \( J_{\nu} \) be as in Proposition 1. Then \( 0 \leq H \leq 1 + \gamma \). Moreover, if \( \nu \) satisfies (2.9) then \( 0 \leq H_1 = H - \frac{\gamma}{2} \leq 1 \).

Proof. By stationarity of rectangular increments, for any \( n \geq 1 \) we have \( \mathbb{E}|J_{\nu}(n, n^\gamma)| \leq n[n^\gamma] \mathbb{E}|J_{\nu}(1,1)| \leq n^{1+\gamma} \mathbb{E}|J_{\nu}(1,1)| \). On the other hand, \( \mathbb{E}|J_{\nu}(n, n^\gamma)| = n^H \mathbb{E}|J_{\nu}(1,1)| \) by (2.5). Hence, \( H \leq 1 + \gamma \). To show \( H \geq 0 \) we follow [35, Lemma 8.2.2]. Assume \textit{ad absurdum} that \( H < 0 \), we will show that \( J_{\nu}(x,y) = 0 \) for any \((x,y) \in \mathbb{R}^2_+\). Let us first prove that for any \( K > 0 \)

\[
\lim_{x \to \infty} \mathbb{E} \sup_{0 < y < K x^\gamma} |J_{\nu}(x,y)| = 0.
\]

Indeed, by (2.5), for any fixed \( x > 0 \) we have that \( \{ J_{\nu}(x,y), 0 < y < K x^\gamma \} \overset{\text{fdd}}{=} \{ x^H J_{\nu}(1, \frac{y}{x}), 0 < y < K x^\gamma \} \). Therefore,

\[
\mathbb{E} \sup_{0 < y < K x^\gamma} |J_{\nu}(x,y)| \leq x^H \mathbb{E} \sup_{0 < y < K} |J_{\nu}(1,y)| \to 0 \quad (x \to \infty)
\]

provided \( \mathbb{E} \sup_{0 < y < K} |J_{\nu}(1,y)| < C \). The last fact follows from properties of Lévy processes, see e.g. [36, Remark 25.19], since \( \{ J_{\nu}(1,y), y > 0 \} \) is a homogeneous Lévy process with \( \mathbb{E}|J_{\nu}(1,y)| < \infty \). This proves (2.12).

Next, by stationarity of rectangular increments,

\[
J_{\nu}(x,y) \overset{d}{=} J_{\nu}(x + \lambda, y + \lambda^\gamma) - J_{\nu}(\lambda, y + \lambda^\gamma) - J_{\nu}(x + \lambda, \lambda^\gamma) + J_{\nu}(\lambda, \lambda^\gamma),
\]
where each term on the r.h.s. tends to 0 (in the sense of $L^1$-convergence) as $\lambda \to \infty$, according to (2.12). This proves $E|J_\nu(x,y)| = 0$, completing the proof of nonnegativity $H \geq 0$. The second statement of the proposition is a consequence of (2.10) and the classical Schoenberg’s theorem concerning covariance functions, see e.g. [6, Cor.2.1.1]. □

**Remark 1.** Assumption (2.1) guaranteeing $E|J_\nu(x,y)| < \infty$ is satisfied in Theorem 2 which uses an even stronger assumption (3.2) on the pulse process. We expect that (2.1) can be relaxed and Propositions 1-2 extended to include the case when $E|J_\nu(x,y)| = \infty$.

Theorem 1 establishes asymptotic local and global self-similarity, in spirit of [3, 4, 8, 15, 25, 24], of the random process $J_\nu(x) := \{J_\nu(x,1), x > 0\}$ under some additional conditions on $(\gamma, H)$-scaling measure $\nu$. (The terminology is reminiscent of the fact that the limit processes in (2.13) and (2.16) are self-similar with respective parameters $H_1$ and $1/\alpha$.) Denote $W(w) := \int_0^\infty w(t)dt, w \in \mathbb{W}$.

**Theorem 1.** Let $\nu$ be a $(\gamma, H)$-scaling measure on $\mathbb{W}$ satisfying (2.1).

(i) Let $\nu$ satisfy (2.9) and $\frac{\gamma}{2} < H \leq 1 + \frac{\gamma}{2}$. Then

$$
\lambda^{-H_1} J_\nu(\lambda x) \xrightarrow{\text{fdd}} C_\nu B_{H_1}(x), \quad \lambda \to 0,
$$

where $B_{H_1}$ is FBM with $H_1 = H - \frac{\gamma}{2}$.

(ii) Let $\frac{1+\gamma}{2} < H < 1 + \gamma$, $\alpha := \frac{1+\gamma}{H}$ and

$$
\int_{\mathbb{W}} (|W(w)| \wedge |W(w)|^2) \nu(dw) < \infty.
$$

In addition, assume that

$$
\int_{\mathbb{W}} |w(t)|^\alpha' \nu(dw) \leq Ct^{-(\alpha'+\delta)}, \quad t > 0, \quad (\exists \alpha' \in [1, \alpha), \ 1 > \delta > 1 - \alpha'/\alpha).
$$

Then

$$
\lambda^{-1/\alpha} J_\nu(\lambda x) \xrightarrow{\text{fdd}} L_\alpha(x), \quad \lambda \to \infty,
$$

where $L_\alpha = \{L_\alpha(x), x > 0\}$ is an $\alpha$-stable Lévy process with ch.f.

$$
Ee^{i\theta L_\alpha(x)} = e^{x(\theta)\alpha D_\alpha^c + (\theta)^\alpha D_\alpha}, \quad \theta \in \mathbb{R}, \quad D_\alpha^\pm := \int_{\mathbb{W}} \{e^{\pm iW(w)} - 1 \mp iW(w)\} \nu(dw).
$$

**Proof.** We use characteristic function in (2.2) and notation $\Psi(z) = e^{iz} - 1 - iz$ as above. Fix $(x, \theta) = ((x_j, \theta_j), j = 1, \ldots, m), (x_j, \theta_j) \in \mathbb{R}^+ \times \mathbb{R}, 0 =: x_0 < x_1 < \cdots < x_m$.

(i) Recall (2.11). It suffices to prove that

$$
I_\lambda(x, \theta) := \int_{\mathbb{R} \times \mathbb{W}} \Psi\left(\lambda^{-H_1} \sum_{j=1}^m \theta_j \int_0^{\lambda x_j} w(t)dt\right)du \nu(dw)
$$

$$
\to -\frac{1}{2} \int_{\mathbb{R} \times \mathbb{W}} \left\{ \sum_{j=1}^m \theta_j \int_0^{x_j} w(t)dt \right\}^2 du \nu(dw),
$$

8
as $\lambda \to 0$. In view of (2.23), $I_\lambda(x, \alpha, \theta) = \lambda^{-\gamma} \int_{\mathbb{R} \times \mathbb{W}} \Psi(\lambda^{\gamma/2} \sum_{j=1}^{m} \theta_j \int_0^{x_j} w(t-u)dt) d\nu(dw)$, where $\lambda^{-\gamma} \Psi(\lambda^{\gamma/2} z) \to -z^2/2$ implies (2.18) by $|\Psi(z)| \leq |z|^2/2$, $z \in \mathbb{R}$, and (Pratt’s) Lemma 3.

(ii) Note first that

\[
\int \Psi(\theta W(w)) \nu(dw) = (\theta)^\alpha D_\nu^+ + (\theta)^\alpha D_\nu^- = \log \mathbb{E} e^{\theta L_\alpha(1)}, \quad \theta \in \mathbb{R}
\]

is the log ch.f. of $\alpha$-stable r.v. $L_\alpha(1)$ in (2.17). Indeed, let $\theta > 0$ then $\theta W(w) = W(\phi_{\theta \lambda, H, H} w)$ and $\int \Psi(\theta W(w)) \nu(dw) = \int \Psi(W(\phi_{\theta \lambda, H, H} w)) \nu(dw) = \theta^\alpha \int \Psi(W(w)) \nu(dw) = \theta^\alpha D_\nu^+$ follows by the $(\gamma, H)$-scaling property in (2.4); for $\theta < 0$ relation (2.19) follows analogously.

Let us prove (2.16). Rewrite $I_\lambda(x, \alpha, \theta) := \int_{\mathbb{R} \times \mathbb{W}} \Psi(\lambda^{-1/\gamma} \sum_{j=1}^{m} \theta_j \int_0^{x_j} w(t-u)dt) d\nu(dw) = I_{\lambda}(x, \tilde{\theta})$, where $\tilde{\theta} = (\tilde{\theta}_1, \ldots, \tilde{\theta}_m)_\alpha$, $\tilde{\theta}_j := \sum_{i=j}^{m} \theta_i$, and

\[
I_{\lambda}(x, \tilde{\theta}) := \int_{\mathbb{R} \times \mathbb{W}} \Psi(\lambda^{-1/\gamma} \sum_{i=j}^{m} \tilde{\theta}_j \int_0^{x_j-x_{j-1}} w(t-u)dt) d\nu(dw).
\]

Further on, rewrite $I_j(\lambda)$ as $I_j(\lambda) = I_j(\lambda) - R_j(\lambda)$, where

\[
I_j(\lambda) := \int_{(0, \lambda(x_j-x_{j-1})]} \Psi\left(\lambda^{-1/\gamma} \tilde{\theta}_j \int_0^{\infty} w(t-u)dt\right) d\nu(dw),
\]

\[
R_j(\lambda) := \int_{(0, \lambda(x_j-x_{j-1})]} \left\{ \Psi\left(\lambda^{-1/\gamma} \sum_{i=j}^{m} \tilde{\theta}_i \int_{\lambda(x_i-x_{i-1})-u}^\infty w(t)dt\right) - \Psi\left(\lambda^{-1/\gamma} \tilde{\theta}_j \int_0^{\infty} w(t)dt\right) \right\} d\nu(dw).
\]

According to (2.19), $I_j(\lambda) = (x_j - x_{j-1})(\tilde{\theta}_j)_{\alpha} D_\nu^+ + (\tilde{\theta}_j)_{\alpha} D_\nu^-$ does not depend on $\lambda > 0$ and

\[
\sum_{j=1}^{m} I_j(\lambda) = \sum_{j=1}^{m} ((\tilde{\theta}_j)_{\alpha} D_\nu^+ + (\tilde{\theta}_j)_{\alpha} D_\nu^-) = \log \mathbb{E} \exp \left\{ \sum_{j=1}^{m} \tilde{\theta}_j (L_\alpha(x_j) - L_\alpha(x_{j-1})) \right\}
\]

is the log ch.f. of the limit $\alpha$-stable process in (2.16). Hence, (ii) follows from

\[
I_-(\lambda) \to 0, \quad R_j(\lambda) \to 0, \quad j = 1, \ldots, m.
\]

Consider the first relation in (2.20). Using (2.15), $|\Psi(z)| \leq (2|z|) \wedge (1/2 |z|^2) \leq C|z|^\alpha$, $z \in \mathbb{R}$ and Minkowski’s inequality we obtain

\[
|I_-(\lambda)| \leq C \lambda^{-\alpha/\alpha} \sum_{j=1}^{m} \int_{\mathbb{R} \times \mathbb{W}} \left| \int_0^{\lambda(x_j-x_{j-1})} w(t+u)dt \right|^{\alpha'} d\nu(dw)
\]

\[
\leq C \lambda^{-\alpha/\alpha} \sum_{j=1}^{m} \int_0^{\lambda(x_j-x_{j-1})} \left( \int_{\mathbb{R} \times \mathbb{W}} |w(t+u)|^{\alpha'} d\nu(dw) \right)^{1/\alpha'} dt
\]

\[
\leq C \lambda^{-(\alpha'/\alpha) - \delta + 1} = o(1).
\]
To evaluate $R_j(\lambda)$, we need the inequality: for any $1 \leq \alpha' \leq 2$,

$$|\Psi(z) - \Psi(z')| \leq C(|z' - z|^{\alpha'} + (|z| \wedge 1)|z - z'|), \quad z, z' \in \mathbb{R},$$

(2.21)

which follows from $|\Psi(z) - \Psi(z')| = |e^{iz}\Psi(z'-z) + i(e^{iz} - 1)(z'-z)|$ and $|\Psi(z) - z| \leq C|z' - z|^{\alpha'}$. Use (2.21) with $z = \lambda^{-1/\alpha} \lambda J_{(x_j-x_j)} w(t) dt = \lambda^{-1/\alpha} \lambda J_{(x_j-x_j)} w(t) dt$. Note $|z' - z| \leq C \lambda^{-1/\alpha} \lambda J_{(x_j-x_j)} w(t) dt$. Then $|R_j(\lambda)| \leq C(R'_j(\lambda) + R''_j(\lambda))$, where by assumption (2.15) and Minkowski's inequality,

$$R'_j(\lambda) := \lambda^{-\alpha'/\alpha} \left( \int_0^\lambda \int_{(x_j-x_j)}^\lambda |w(t)| dt \right) \lambda^{1/\alpha} \nu(dw)
$$

$$ \leq C \lambda^{-\alpha'/\alpha} \left( \int_0^\lambda \int_{(x_j-x_j)}^\lambda |w(t)| dt \right) \lambda^{1/\alpha} \nu(dw) = C \lambda^{-\alpha'/\alpha - \delta + 1} = o(1).$$

(2.22)

Finally, by Hölder’s inequality with $\frac{1}{\alpha'} + \frac{1}{\alpha''} = 1$, we get

$$R''_j(\lambda) \leq \lambda^{-2/\alpha} \int_0^\lambda \left( \lambda^{1/\alpha} \nu(dw) \right)^{1/\alpha'} \lambda^{1/\alpha} \nu(dw) \leq \lambda^{-2/\alpha} J_1(\lambda) J_2(\lambda),$$

where

$$J_1(\lambda) := \int_0^\lambda \left( \int_0^\lambda \int_0^\lambda |w(t)| dt \right) \lambda^{1/\alpha} \nu(dw) \leq C \lambda^{-\delta/\alpha'}$$

similarly to (2.22) above, and

$$J_2(\lambda) := \int_0^\lambda \left( |w(t)| \wedge \lambda^{1/\alpha} \nu(dw) \right)^{1/\alpha''} \lambda^{1/\alpha} \nu(dw) = \lambda^{1/\alpha} \int_0^\lambda \left( |w(t)| \wedge \lambda^{1/\alpha} \nu(dw) \right)^{1/\alpha''} \lambda^{1/\alpha} \nu(dw) \leq C \lambda^{(1/\alpha) + (1/\alpha'') - 1}$$

since $\alpha'' > 2$ and the last integral is finite in view of (2.13). Whence, $R''_j(\lambda) \leq C \lambda^{-(\alpha'/\alpha) + \delta - 1/\alpha'} = o(1)$ since $(\alpha'/\alpha) + \delta - 1 > 0$ according to the condition on $\delta$ in (2.15). This proves (2.20), thereby completing the proof of Theorem 1. □

**Lemma 3.** [32] Let $(\mathcal{X}, \mu)$ be a measure space and $L^p(\mathcal{X})$ be the Banach space of all measurable functions $f : \mathcal{X} \to \mathbb{R}$ with $\int |f|^p \equiv \int_\mathcal{X} |f(x)|^p \mu(dx) < \infty, \ p \geq 1$.

Let $f_\lambda, g_\lambda \in L^p(\mathcal{X}), \lambda > 0$ satisfying the following conditions (j)-(jjj): (j) $\lim_{\lambda \to \infty} f_\lambda(x) = f(x), \lim_{\lambda \to \infty} g_\lambda(x) = g(x)$ exists a.e. in $\mathcal{X}$, (jj) $|f_\lambda(x)| \leq |g_\lambda(x)|$ a.e. in $\mathcal{X}, \lambda > 0$, and (jjj) $\lim_{\lambda \to \infty} \int |g_\lambda|^p = \int |g|^p < \infty$. Then $f \in L^p(\mathcal{X})$ and $\lim_{\lambda \to \infty} \int |f_\lambda - f|^p = 0$.

**Example 1.** Let us verify the conditions of Theorem 1 for the Telecom process $J_R(x) = \{J_R(x, 1), x > 0\}$ of (2.7) corresponding to the measure $\nu_R$ in (2.8), for $\gamma = \sigma - 1, H = 1$ and $\varrho \in (1, 2)$. By elementary integration, $C_{\nu_R}^2 = C \int_{[0, 1]^2} t_1 - t_2 |^{1-\sigma} dt_1 dt_2 < \infty$. Hence (2.4) and the assumptions of part (i) are satisfied. Since $W(t) = \int_0^\infty w(t) dt = r$ for $w(t) = 1(t < r)$, (2.14) is satisfied as well, while (2.15) holds with
\[ \alpha = \varrho, \alpha' = 1, \delta = \varrho - 1 \text{ since } \varrho - 1 > 1 - \frac{1}{\varrho}. \] We also have that \( D_{\nu}^+ = c'e^{\frac{\pi \varrho}{2} t} \), hence the characteristic function in (2.17) in this example writes as
\[
\text{E}e^{i\theta L_\alpha(x)} = \exp\left\{c'x|\theta|^\alpha \left( \cos\left(\frac{\pi \theta}{2}\right) - i \operatorname{sgn}(\theta) \sin\left(\frac{\pi \theta}{2}\right) \right) \right\}, \quad \theta \in \mathbb{R},
\]
where \( c' := \frac{c_{\varrho} \Gamma(2-\varrho)}{\varrho(\varrho-1)} \). We note that the convergences (2.13) and (2.16) for the Telecom process were earlier established in [3].

**Example 2.** ([5] (40), (41)] Let \( \nu(dw) \) be concentrated on ‘rectangular’ functions \( w(t) \equiv \tilde{w}_{a,r}(t) := a1(0 < t < r), (a, r) \in \mathbb{R}^2_+ \) and given by
\[
\nu(da, dr) := c_\nu a^{-\kappa-1} r^{-\varrho-1} da \, dr, \quad (a, r) \in \mathbb{R}^2_+, \tag{2.23}
\]
where \( c_\nu > 0, \kappa > 0, \varrho > 0 \) are parameters satisfying
\[ 1 < \varrho < \kappa < 2. \tag{2.24} \]
Then \( \nu \) satisfies condition (2.1) and the corresponding RF \( J_\nu = \{ J_\nu(x, y), (x, y) \in \mathbb{R}^2_+ \} \) is well-defined. Moreover, \( J_\nu \) has \( \kappa \)-stable finite-dimensional distributions and a stochastic integral representation
\[
J_\nu(x, y) \overset{\text{fdd}}{=} \int_{\mathbb{R} \times (0, y] \times \mathbb{R}^+} \left\{ \int_0^x 1(u < t < r + u) \, dt \right\} Z_\kappa(du, dv, dr), \quad (x, y) \in \mathbb{R}^2_+, \tag{2.25}
\]
where \( Z_\kappa(du, dv, dr) \) is a \( \kappa \)-stable random measure on \( \mathbb{R} \times \mathbb{R}^2_+ \) with ch.f.
\[
\text{E}e^{i\theta Z_\kappa(B)} = \exp \left\{ \left( \theta \right)_+^\kappa D_\kappa^+ + \left( \theta \right)_-^\kappa D_\kappa^- \right\} \int_B \frac{du \, dv \, dr}{r^{1+\varrho}} \tag{2.26}
\]
where \( D_\kappa^+ := c_\nu \int_{\mathbb{R}_+^2} (e^{ia} - 1 + ia)a^{-1-\kappa}da, \quad \tilde{c}_\nu := \frac{c_{\nu} \Gamma(2-\kappa)}{(\kappa-1)\kappa}, \) and \( B \subset \mathbb{R} \times \mathbb{R}^2_+ \) is any Borel set with \( \int_B du \, dv \, dr / r^{1+\varrho} < \infty \). Condition (2.1) follows by elementary integration. We have
\[
\int_0^1 \tilde{w}_{a,r}(t-u) \, dt = \begin{cases} 
(r+u) \wedge 1, & \text{if } r < u \leq 0, \\
r, & \text{if } 0 < u \leq 1, 0 < r < 1 - u, \\
1 - u, & \text{if } 0 < u \leq 1, r \geq 1 - u, \\
0, & \text{elsewhere.}
\end{cases}
\]
Hence,
\[
\int_{-\infty}^0 du \int_{\mathbb{R}^+} \left( | \int_0^1 w(t-u) \, dt | \wedge | \int_0^1 w(t-u) \, dt |^2 \right) \nu(dw) = \frac{c_\nu}{\varrho} \int_{\mathbb{R}^2_+} \left( (a(u \wedge 1)) \wedge (a(u \wedge 1))^2 \right) a^{-\kappa-1} u^{-\varrho} \, da \, du = \frac{c_\nu}{\varrho} \left( \frac{1}{2 - \kappa} + \frac{1}{\kappa - 1} \right) \left( \frac{1}{1 + \kappa - \varrho} + \frac{1}{\varrho - 1} \right) < \infty
\]
and
\[
\int_0^1 du \int_{\mathcal{W}} \left( | \int_0^1 w(t-u) dt | \wedge | \int_0^1 w(t-u) dt |^2 \right) \nu(dw) \\
= c_\nu \int_0^1 du \int_{\mathbb{R}_+^2} \left( (a(u \wedge r)) \wedge (a(u \wedge r))^2 \right) a^{-\kappa-1} r^{-\varrho-1} da \, dr \\
= c_\nu \left( \frac{1}{2 - \kappa} + \frac{1}{\kappa - 1} \right) \left( \frac{1}{\kappa - \varrho} + \frac{1}{\varrho} \right) \frac{1}{\kappa - \varrho + 1} < \infty
\]
proving (2.1) when (2.24) holds. To show (2.25), consider the log-ch.f.
\[
\Phi(\theta) := \log \mathbb{E} \exp \left\{ \sum_{j=1}^m \theta_j J_\nu(x_j, y_j) \right\} = \int_{\mathbb{R}^2 \times \mathbb{R}_+^2} \Psi(a \sum_{j=1}^m \theta_j 1(s \leq y_j) \int_0^x 1(u < t < r + u) dt) d\nu(da, dr), \quad (x_j, y_j) \in \mathbb{R}_+^2, \quad \theta_j \in \mathbb{R}, \quad j = 1, \ldots, m, \ m \geq 1, \text{ with } \nu(da, dr) \text{ in (2.23)}.
\]
By integrating this expression over \( a \in (0, \infty) \) and using (2.26) we obtain
\[
\Phi(\theta) = \int_{\mathbb{R}^2 \times \mathbb{R}_+^2} \left\{ \left( \sum_{j=1}^m \theta_j 1(v \leq y_j) \int_0^{x_j} 1(u < t < r + u) dt \right)^\kappa D_\kappa^+ \\
+ \left( \sum_{j=1}^m \theta_j 1(v \leq y_j) \int_0^{x_j} 1(u < t < r + u) dt \right)^\kappa D_\kappa^- \right\} \frac{du \, dv \, dr}{r^{1+\varrho}}
\]
where \( J_\nu(x, y) \) is the \( \kappa \)-stable RF on the r.h.s. of (2.25). By changing the variables in the last integral:
\[
t \to \lambda_1 H_1 t, \ u \to \lambda_1 H_1 u, \ r \to \lambda_1 H_1 r, \ v \to \lambda_2 H_2 v, \ \text{with arbitrary } \lambda_i > 0, \ i = 1, 2 \text{ and}
\]
\[
H_1 := \frac{1 + \kappa - \varrho}{\kappa}, \quad H_2 := \frac{1}{\kappa},
\]
we see that RF \( J_\nu \overset{\text{fdd}}{=} \tilde{J}_\nu \) is \((H_1, H_2)\)-multi-self-similar hence also \((\gamma, H)\)-self-similar for any \( \gamma > 0 \) with \( H = H_1 + \gamma H_2 \). Particularly, the random process \( \{J_\nu(x, 1), x > 0\} \) is \( H_1 \)-self-similar and \( \kappa \)-stable. Obviously, this \( J_\nu \) does not satisfy Theorem 11 (neither (i), nor (ii)) and may serve as a justification for the necessity of additional conditions such as (2.9) and (2.14) in this theorem. Actually, the RF in (2.25) does not arise in [15] under ‘intermediate’ scaling but appears in the fast connection rate limit for ‘continuous flow reward’ model (Example 3) with reward distribution \( A > 0 \) varying regularly at infinity with exponent \( \kappa \). Example 2 is interesting since it provides a negative answer to the question about the uniqueness of the pair \((\gamma, H)\) in Definition 14(i).

### 3 Shot-noise inputs

Let
\[
X(t) := \sum_{j \in \mathbb{Z}} W_j(t - T_j), \quad t \in \mathbb{R},
\]
be a shot-noise process, where \( \{T_j, j \in \mathbb{Z}\} \) is a homogeneous Poisson point process with unit rate and \( \{W_j, j \in \mathbb{Z}\} \) is a sequence of independent copies of a ‘pulse process’ \( W = \{W(t), t \in \mathbb{R}\} \) with trajectories in
\[\mathbb{W} = \{w \in L^1(\mathbb{R}) : w(t) = 0 \text{ for } t < 0\}\] and distribution \(P_W = P \circ W^{-1}\), i.e. \(P_W(A) = P(W \in A)\) for any Borel subset \(A \subset \mathbb{W}\). Moreover, we assume that \(\{T_j\}\) and \(\{W_j\}\) are independent, and

\[
\int_0^\infty (E|W(t)| + E|W(t)|^2)dt < \infty. \tag{3.2}
\]

Then \(X(t)\) in (3.1) can be written as

\[
X(t) = \int_{\mathbb{R} \times \mathbb{W}} w(t - u)p(du, dw), \quad t \in \mathbb{R}, \tag{3.3}
\]

where \(p(du, dw)\) is Poisson random measure on \(\mathbb{R} \times \mathbb{W}\) with mean \(duP_W(dw)\). It follows that \(\{X(t), t \in \mathbb{R}\}\) is a stationary process with

\[
EX(t) = \int_0^\infty EW(u)du, \quad \text{Cov}(X(0), X(t)) = \int_0^\infty E[W(u)W(t + u)]du, \quad t \geq 0. \tag{3.4}
\]

Then

\[
A_{\lambda, \gamma}(x, y) = \int_{\mathbb{R} \times [0, |\lambda \gamma|] \times \mathbb{W}} \left\{ \int_0^{\lambda \gamma} w(t - u)dt \right\} q(du, dv, dw), \quad (x, y) \in \mathbb{R}_+^2, \tag{3.5}
\]

where \(q(du, dv, dw)\) is the centered Poisson random measure on \(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{W}\) with control measure \(du \, dv \, P_W(dw)\). Note \(\text{Var}(A_{\lambda, \gamma}(x, y)) = 2[\lambda^2 \gamma] \int_0^{\lambda \gamma} \lambda x - t \text{ Cov}(X(0), X(t))dt\). In as follows, the normalization in (1.2) takes the form \(d_{\lambda, \gamma} = \lambda H(\gamma)\) with some \(H(\gamma) > 0\).

We recall the definitions of FBS \(B_{H_1, H_2}\) and \(\alpha\)-stable Lévy sheet \(L_\alpha\) used below. Gaussian process \(B_{H_1, H_2} = \{B_{H_1, H_2}(x, y), (x, y) \in \mathbb{R}_+^2\}\) with parameters \(H_1, H_2 \in (0, 1]\) is called FBS if it has zero mean and covariance

\[
EB_{H_1, H_2}(x, y)B_{H_1, H_2}(x', y') = \frac{1}{4}(x^2H_1 + x'^2H_1 - |x - x'|^2H_1)(y^2H_2 + y'^2H_2 - |y - y'|^2H_2),
\]

where \((x, y), (x', y') \in \mathbb{R}_+^2\). Clearly, \(B_{H_1, H_2}\) has stationary rectangular increments while \(B_{H_1, \frac{1}{2}}\) has independent rectangular increments in the vertical direction. Also, for \(\alpha \in (1, 2)\) we introduce \(\alpha\)-stable Lévy sheet \(\{L_\alpha(x, y) := W_\alpha((0, x] \times (0, y]), (x, y) \in \mathbb{R}_+^2\}\) as a stochastic integral w.r.t. an \(\alpha\)-stable random measure \(\sigma \, du \, dv\) (\(\sigma > 0\) ), skewness parameter \(\beta \in [-1, 1]\) such that for every bounded Borel subset \(A \subset \mathbb{R}_+^2\),

\[
E \exp\{i\theta W_\alpha(A)\} = \exp\left\{ -\text{Leb}(A)\sigma^\alpha|\theta|^\alpha(1 - i\beta \text{ sgn}(\theta) \tan(\frac{\pi\alpha}{2})) \right\}, \quad \theta \in \mathbb{R}. \tag{3.6}
\]

Alternatively, parameters \(\sigma > 0, \beta \in [-1, 1]\) in (3.6) are uniquely determined by parameters \(c_\pm \geq 0, c_+ + c_- > 0\) as

\[
\sigma^\alpha = \frac{\Gamma(2 - \alpha)}{1 - \alpha} \cos\left(\frac{\pi\alpha}{2}\right)c_+ + c_-, \quad \beta = \frac{c_+ - c_-}{c_+ + c_-}. \tag{3.7}
\]

**Theorem 2.** Let \(\gamma > 0\) and let \(X\) be a shot-noise process in (3.3) satisfying (3.2),

(i) Let

\[
\text{Cov}(X(0), X(t)) \sim c_X t^{-2(1-H_1)}, \quad t \to \infty \quad (\exists c_X > 0, \ H_1 \in (\frac{1}{2}, 1]), \tag{3.8}
\]

and

\[
\lambda^{1+\gamma} \int_{\mathbb{R}} \left| \lambda^{1-H} \int_0^1 W(\lambda(t - u))du \right|^{2+\delta} du \to 0, \quad \lambda \to \infty \quad (\exists \delta > 0), \tag{3.9}
\]
where \( H := H_1 + \frac{\gamma}{2} \). Then the convergence in (1.2) holds with \( d_{\alpha, \gamma} = \lambda^H \) and \( V_{\gamma} \overset{\text{fdd}}{=} \{ C_W B_{H,1} (x,y), (x,y) \in \mathbb{R}_+^2 \} \), where \( B_{H,1} \) is a FBS, \( C_W^2 := \frac{C_X}{(2H-1)(H+1)} \).

(ii) Let \( \alpha \in (1,2) \) and let the distribution of \( W := \int_0^\infty W(t)\,dt \) satisfy

\[
P(W > x) = (c_+ + o(1))x^{-\alpha}, \quad P(W \leq -x) = (c_- + o(1))x^{-\alpha}, \quad x \to \infty, \tag{3.10}
\]

for some \( c_\pm \geq 0, c_+ + c_- > 0 \). Moreover, let

\[
E|W(t)|^{\alpha'} \leq C(1 \vee t)^{-(\alpha'+\delta)}, \quad t > 0 \quad (\exists \alpha' \in [1, \alpha), \ 1 \wedge \delta > (1 - \frac{\alpha'}{\alpha})(1 + \gamma)). \tag{3.11}
\]

Then the convergence in (1.2) holds with \( d_{\alpha, \gamma} = \lambda^H \), \( H := \frac{1+\gamma}{\alpha} \) and \( V_{\gamma} \overset{\text{fdd}}{=} \{ L_\alpha(x,y), (x,y) \in \mathbb{R}_+^2 \} \), where \( L_\alpha \) is \( \alpha \)-stable Lévy sheet corresponding to \( \alpha \)-stable random measure \( \mathcal{W}_\alpha \) defined by (3.6)-(3.7).

(iii) Assume there exists a \((\gamma, H)\)-scaling measure \( \nu_W \) on \( \mathbb{W} \) satisfying (2.1) and such that for any \( \lambda > 1 \) the measure \( \lambda^{1+\gamma} P \circ \phi_{\lambda, H} \) is absolutely continuous w.r.t. \( \nu_W \) with bounded Radon-Nikodym derivative \( g_{\lambda, \gamma, H}(w) := \frac{d(\lambda^{1+\gamma} P \circ \phi_{\lambda, H})}{d\nu_W}(w) \) tending to 1 as \( \lambda \to \infty \), viz.,

\[
g_{\lambda, \gamma, H}(w) \to 1, \quad |g_{\lambda, \gamma, H}(w)| \leq C \quad \text{for } \nu_W \text{-a.e. } w \in \mathbb{W}. \tag{3.12}
\]

Then the convergence in (1.2) holds with \( d_{\alpha, \gamma} = \lambda^H \) and \( V_{\gamma} \overset{\text{fdd}}{=} J_{\nu_W} \), where RF \( J_{\nu_W} \) is defined in (2.6).

Let us comment on conditions of Theorem 2. (3.8) is typical for LRD of \( X \). It can be replaced by a weaker variance condition, viz.,

\[
\text{Var}(\lambda^{-H} A_{\lambda, \gamma}(1,1)) = \lambda |\gamma| \int_{\mathbb{R}} E \left( \lambda^{1-H(\gamma)} \int_{0}^{1} W(\lambda(t-u))\,dt \right)^2 \, du \to C_W^2. \tag{3.13}
\]

(3.9) is a Lyapunov type condition implying asymptotic normality of the sum \( A_{\lambda, \gamma}(x,y) \) of \( |y\lambda| \) i.i.d. inputs. If \( H > 1 \) and the pulse process is bounded: \( \sup_{t \geq 0}|W(t)| \leq C \), where \( C < \infty \) is non-random, then (3.9) is automatically satisfied in view of (3.13).

Conditions (3.10), (3.11) are rather simple and easily verifiable in concrete situations, see Section 4. When \( \gamma < H \), condition (3.11) with \( \alpha' = 1 \) becomes

\[
E|W(t)| \leq C(1 \vee t)^{-1-\delta}, \quad t > 0 \quad (\exists \delta > (1 - \frac{1}{\alpha})(1 + \gamma)), \tag{3.14}
\]

whereas in the case \( \gamma \geq H \) it requires \( \alpha' > 1 \). The examples in Section 4 show that the bounds on \( \alpha', \delta \) in (3.11) are sharp and cannot be replaced by lower quantities in general.

On the other hand, the conditions in (iii) (particularly, (3.12)) are much stronger and require certain scaling property of \( W \).

**Remark 2.** [12, 13] studied scaling limits for a large class of random processes with immigration which include (integrated) Poisson shot-noises and regenerative processes discussed in our paper as special cases. The above mentioned works refer to a single input process and do not apply to aggregated sums as in (1.1). On the other hand, these results and the approaches developed therein could be useful for possible extension of our work to more general inputs and/or a richer class of limit RFs than [13].
Proof of Theorem 2. Let $\tilde{A}_{\lambda,\gamma}(x, y) := \lambda^{-H}(A_{\lambda,\gamma}(x, y) - EA_{\lambda,\gamma}(x, y))$, $(x, y) \in \mathbb{R}^d_+$. Since it has stationary rectangular increments, independent in the vertical direction, moreover, $\tilde{A}_{\lambda,\gamma}(x, y) = 0$, $x \land y = 0$, we only need to prove the convergence of ch.f.s of its f.d.d.s for $y = 1$ and any $x \in \mathbb{R}^d_+$, $\theta \in \mathbb{R}^d$, $d \in \mathbb{N}$:

$$
E \exp \left\{ i \sum_{j=1}^d \theta_j \hat{A}_{\lambda,\gamma}(x_j, 1) \right\} \to E \exp \left\{ i \sum_{j=1}^d \theta_j V_{\gamma}(x_j, 1) \right\}, \quad \lambda \to \infty.
$$

(3.15)

In (3.15) the l.h.s. equals $e^{i\lambda \gamma}$ with

$$
I_{\lambda,\gamma} := [\lambda \gamma] \int_\mathbb{R} \mathbb{E} \left( \lambda^{-H} \sum_{j=1}^d \theta_j \int_0^{\lambda x_j} W(t-u)dt \right) du,
$$

where $\Psi(z) := e^{iz} - 1 - iz$, $z \in \mathbb{R}$.

(i) In (3.15) the limit equals $E \exp\{i \sum_{j=1}^d \theta_j C_W B_{\lambda,\gamma}(x_j, 1)\} = e^{-\frac{1}{2} J}$, where $J := E[ \sum_{j=1}^d \theta_j C_W B_{\lambda,\gamma}(x_j, 1)^2 ]$.

Firstly, let us prove $E[ \sum_{j=1}^d \theta_j \hat{A}_{\lambda,\gamma}(x_j, 1)^2 ] =: J_{\lambda,\gamma} \to J$ which is equivalent to $E\hat{A}_{\lambda,\gamma}(x, 1)\hat{A}_{\lambda,\gamma}(x, 1) \to C_W^2 EB_{\lambda,\gamma}(x, 1)^2B_{\lambda,\gamma}(x, 1)$ for every $i, j$. Since $\{\hat{A}_{\lambda,\gamma}(x, 1), x \in \mathbb{R}^d_+\}$ has stationary increments, the last-mentioned convergence follows from $E[\hat{A}_{\lambda,\gamma}(x, 1)^2] = [\lambda \gamma] \int_\mathbb{R} \mathbb{E}[\lambda^{-H} \int_0^{\lambda x} W(t-u)dt]^2 du \to C_W^2 x^{2H}$ for every $j$, which in turn follows from (3.14), (3.8). Secondly, for $\delta \in (0, 1)$, use of $|\Psi(z) + \frac{1}{2} z^2| \leq z^2 \wedge (\frac{1}{2} |z|^3) \leq C |z|^{2+\delta}$, $z \in \mathbb{R}$ gives $|I_{\lambda,\gamma} + \frac{1}{2} J_{\lambda,\gamma}| \leq C \gamma \int_\mathbb{R} \mathbb{E}[\lambda^{-H} \sum_{j=1}^d \theta_j \int_0^{\lambda x_j} W(t-u)dt]^2 du = o(1)$ by (3.9), which completes the proof of (3.15) in part (i).

(ii) Let us prove (3.15), where $0 := x_0 < x_1 < \cdots < x_d$ and $V_{\gamma} = L_\alpha$. In (3.15) we rewrite the limit as $E \exp\{i \sum_{j=1}^d \hat{\theta}_j (\hat{L}_\alpha(x_j, 1) - L_\alpha(x_j-1, 1))\} = \exp\{\sum_{j=1}^d (x_j - x_{j-1})J_j\}$, where $J_j := -\sigma^\alpha |\hat{\theta}_j|^\alpha (1 - i\beta \text{sgn}(|\hat{\theta}_j|) \tan(\frac{\pi \alpha}{2}))$ and $\hat{\theta}_j := \sum_{i=j}^d \theta_i$ for every $j$. To prove (3.15), we rewrite it to the convergence of log ch.f.s. We define $\tilde{I}_{\lambda,\gamma} := \sum_{j=1}^d (x_j - x_{j-1})J_{\lambda,\gamma,j}$, where

$$
J_{\lambda,\gamma,j} := [\lambda \gamma] \lambda \mathbb{E} \Psi(\hat{\theta}_j \lambda^{-H} W)
$$

for every $j$ with $H = \frac{1+\gamma}{\alpha}$. In view of (3.10), the distribution of $W$ belongs to the normal domain of attraction of $\alpha$-stable law, which in turn yields

$$
J_{\lambda,\gamma,j} \to \int_\mathbb{R} i \hat{\theta}_j (e^{i \hat{\theta}_j u} - 1) (-c_- 1(u < 0) + c_+ 1(u > 0))|u|^{-\alpha} du = J_j
$$

for every $j$, whence $\tilde{I}_{\lambda,\gamma} \to \sum_{j=1}^d (x_j - x_{j-1})J_j$. It remains to prove

$$
I_{\lambda,\gamma} - \tilde{I}_{\lambda,\gamma} \to 0, \quad \lambda \to \infty.
$$

(3.16)

We decompose the log ch.f. $I_{\lambda,\gamma}$ given above into a sum of $d+1$ integrals:

$$
I_{\lambda,\gamma} = [\lambda \gamma] \left( \sum_{j=1}^d \int_{\lambda x_{j-1}}^{\lambda x_j} + \int_{-\infty}^0 \right) \mathbb{E} \Psi \left( \sum_{i=1}^d \theta_i \lambda^{-H} \int_0^{\lambda x_i} W(t-u)dt \right) du = \sum_{j=1}^d I_{\lambda,\gamma,j} + I_{\lambda,\gamma,0}.
$$

Then (3.16) follows from

$$
I_{\lambda,\gamma,j} - (x_j - x_{j-1})J_{\lambda,\gamma,j} \to 0, \quad j = 1, \ldots, d, \quad \text{and} \quad I_{\lambda,\gamma,0} \to 0.
$$

(3.17)
Using \(|\Psi(z)| \leq (2|z|) \wedge (\frac{1}{2}|z|^2) \leq C|z|^\alpha, z \in \mathbb{R}\) and Minkowski’s inequality we obtain

\[
|I_{\lambda, \gamma, 0}| \leq C\lambda^{\gamma - \alpha H} \int_0^\infty E\left|\int_0^{\lambda x_d} |W(t-u)|dt\right|^{\alpha'} du \leq C\lambda^{\gamma - \alpha H} \left(\int_0^\infty E|W(v)|^{\alpha'} dv\right)^{\frac{1}{\alpha'}} dt^{\alpha'}
\]

\[
\leq C\lambda^{\gamma - \alpha H} \left(\int_0^1 dt + \int_1^\infty \left(\int_t^\infty v^{-(\alpha' + \delta)} dv\right)^{\frac{1}{\alpha'}} dt\right) = o(1)
\]

(3.18)

since \((1 + \gamma)(1 - \frac{\alpha'}{\alpha}) < \delta \wedge 1\). Next, let us prove the first relation in (3.17) for any \(j\). For this purpose, rewrite

\[
I_{\lambda, \gamma, j} = [\lambda^\gamma] \int_0^\infty E\Psi\left(\sum_{i=j}^d \theta_i 1_{\lambda^-} \int_0^{\lambda (x_i - x_j - 1)} W(s) ds\right) dv,
\]

where for any \(v, j \leq i\) with \(W = \int_0^\infty W(s) ds\) note \(|W - \int_0^{\lambda (x_i - x_j) + v} W(s) ds| \leq \int_0^\infty |W(s)| ds = \mathcal{W}(v)\). Use inequality (2.21). Hence, \(|I_{\lambda, \gamma, j} - (x_j - x_j - 1)J_{\lambda, \gamma, j}| \leq C(K_{\lambda, \gamma} + K''_{\lambda, \gamma})\), where \(K_{\lambda, \gamma}' := \lambda^{\gamma - \alpha H} \int_0^{\lambda (x_j - x_j - 1)} E|\mathcal{W}(v)|^{\alpha'} dv \ll 1\) since \(E|\mathcal{W}(v)|^{\alpha'} \leq C(\int_v^\infty (E|\mathcal{W}(s)|^{\alpha'})^{\frac{1}{\alpha'}} ds)^{\alpha'} \leq C(1 \vee v)^{-\delta/2}\), whereas \(\alpha''\) such that \(\frac{1}{\alpha'} + \frac{1}{\alpha''} = 1\),

\[
K''_{\lambda, \gamma} := \lambda^{\gamma - 2H} \int_0^{\lambda (x_j - x_j - 1)} E\left[(\mathcal{W}(v) \wedge \lambda^H)^{\alpha''}\right] \frac{1}{\alpha''} dv\leq \lambda^{\gamma - 2H} \left(\mathcal{E}(\mathcal{W} \wedge \lambda^H)^{\alpha''}\right) \int_0^{\lambda (x_j - x_j - 1)} (E|\mathcal{W}(v)|^{\alpha'})^{\frac{1}{\alpha'}} dv.
\]

(3.19)

Relation (3.10) together with integration by parts implies \(E(|\mathcal{W} \wedge \lambda^H|^{\alpha''}) \leq C \lambda^{H(\alpha'' - \alpha)}\). Then \(\gamma - 2H + H(1 - \frac{\alpha'}{\alpha''}) < \frac{\lambda \delta}{\alpha''} - 1\) according to the bound on \(\alpha', \delta\) in (3.11) and \(\int_0^{\lambda (x_j - x_j - 1)} (E|\mathcal{W}(v)|^{\alpha'})^{\frac{1}{\alpha'}} dv \leq C \int_0^{\lambda (x_j - x_j - 1)} (1 \vee v)^{-\frac{\delta}{2}} dv\) lead to \(K''_{\lambda, \gamma} = o(1)\) for every \(i\). This proves (3.17) and part (ii), too.

(iii) We use the criterion in [31] Thm. 1. Using (3.5) we can write \(\sum_{i=1}^d \theta_i \tilde{A}_{\lambda, \gamma}(x_i, 1) = \int_S f_\lambda(s; \theta, x) q(ds)\), where \(S := \{s = (u, v, w) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{W}\}\),

\[
f_\lambda(s; \theta, x) := 1(v \in (0, [\lambda^\gamma])) \lambda^{-H} \sum_{i=1}^d \theta_i \int_0^{\lambda x_i} w(t-u) dt
\]

and \(q(ds) = q(du, dv, dw)\) is the same centered Poisson random measure as in (3.5) with control measure \(\mu(ds) = dudvP_W(dw)\). We extend the mapping \(\phi_{\lambda, H}\) (2.3) to \(\mathbb{W}\) to \(S\) by setting \(\phi_{\lambda, \gamma, H} := (\lambda u, \lambda^\gamma v, \phi_{\lambda, H} w)\), and let

\[
h(s; \theta, x) := 1(v \in (0, 1)) \sum_{i=1}^d \theta_i \int_0^{x_i} w(t-u) dt
\]

be the integrand of the stochastic integral \(\sum_{i=1}^d \theta_i J_r(x_i, 1)\) in (2.6). Then

\[
f_\lambda(\tilde{\phi}_{\lambda, \gamma, H} s; \theta, x) = 1(\lambda^\gamma v \in (0, [\lambda^\gamma])) \lambda^{-H} \sum_{i=1}^d \theta_i \int_0^{\lambda x_i} \lambda^{H-1} w(t-\frac{\lambda t}{\lambda}) dt \to h(s; \theta, x)
\]

point-wise on \(S\). Note \(\tilde{\phi}_{\lambda, \gamma, H}\) is a one-to-one mapping on \(S\) with inverse \(\tilde{\phi}^{-1}_{\lambda, \gamma, H} = \tilde{\phi}_{\lambda, \gamma, H}^{-1}\). Due to assumptions in (3.12) the conditions of [31] Thm. 1 (ii) are satisfied, yielding the convergence in (3.15) in part (iii) of the theorem.

A natural question is can we recognize the trichotomy in (1.3) from Theorem 2? The limits in (i) and (ii) clearly agree with the first two limits in (1.3) while the limit in (iii) is apparently related to the third

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limit in (1.3). Moreover, under the second moment assumption (2.9), all three limit RFs in (i)–(iii) are different. (Indeed, the limit in (ii) is different from those in (i) and (iii) having finite variance, while (i) and (iii) are different since the distribution in (2.2) is non-Gaussian by the uniqueness of the Lévy-Khinchine representation.) Obviously, for a given \( \gamma \), only one of the three possibilities (i)–(iii) may occur. This leads to the two following questions: a) can all three limits (i)- (iii) occur for the same shot-noise process \( X \), in different regions of \( \gamma \)? and, if the answer to a) is affirmative, b) does there exist a \( \gamma_0 > 0 \) separating these regions as in (1.3)? For examples treated in the following section, the answer to a) and b) positive. The following theorem shows that a positive answer to b) holds in the general case of Theorem 2 under some additional conditions. Note that conditions (3.9) and (3.11) are ‘monotone’ in \( \gamma \): indeed, if (3.9) holds for some \( \gamma > 0 \) then it is also satisfied for any \( \gamma' > \gamma \) as \( 1 + \gamma + (1 - H_1 - \frac{\gamma}{2}) (2 + \delta) \) decreases with \( \gamma \), whereas (3.11) extends to any \( \gamma' < \gamma \) as \( (1 - \alpha') (1 + \gamma) \) decreases with \( \gamma \).

\[
\gamma_0^+ := \inf \{ \gamma > 0 : \text{ (3.9) holds} \}, \quad \gamma_0^- := \sup \{ \gamma > 0 : \text{ (3.11) holds} \}.
\]

the infimum taken for a fixed \( H_1 \in (\frac{2}{2}, 1) \) and some \( \delta > 0 \), and the supremum for a given \( \alpha \in (1, 2) \) and some \( \alpha', \delta \) satisfying (3.11). Then \( \gamma_0^- \leq \gamma_0^+ \) and a positive answer to b) follows if the interval \([\gamma_0^-, \gamma_0^+]\) containing \( \gamma_0 \) consists of a single point.

**Theorem 3.** Assume that \( X \) and \( (\gamma, H, \nu) : \nu \sim \nu_W \) satisfy Theorem 2 (iii). Moreover, let \( \gamma, H, \nu \) satisfy (2.9), (2.14) and

\[
\frac{1 + \gamma}{2} < H < 1 + \frac{\gamma}{2}, \tag{3.20}
\]

Then such \( (\gamma, H) \) is unique; in other words, the intermediate limit \( V_\gamma = J_\nu \) in in Theorem 2 (iii) occurs at a single point \( \gamma_0 = \gamma \).

**Proof.** Let \( (\gamma', H', \nu') : \nu' \sim \nu_W \) be another triplet satisfying the conditions of this theorem. It suffices to prove that

\[
\frac{1 + \gamma'}{H'} = \frac{1 + \gamma}{H} \quad \text{and} \quad \frac{\gamma' - \gamma}{2} = H' - H. \tag{3.21}
\]

Indeed, (3.21) imply either \( (\gamma', H') = (\gamma, H) \), or \( 1 + \gamma' = H \), the second possibility excluded by (3.20).

It remains to prove (3.21). By (3.12), (2.9),

\[
I_\lambda := \lambda^{\gamma'} \int_\mathbb{R} \mathbb{E} \left( \theta \lambda^{H'} \int_0^\lambda W(t - u) dt \right) du \rightarrow \int_{\mathbb{R} \times W} \left( \theta \int_0^1 w(t - u) dt \right) du \nu'(dw), \tag{3.22}
\]

analogous relation holds with \( (\gamma', H', \nu') \) replaced by \( (\gamma, H, \nu) \). Let \( H' > H \) w.l.g. Rewrite the l.h.s. of (3.22) as

\[
I_\lambda = \lambda^{\gamma'} \int_{\mathbb{R} \times W} \left( \theta \lambda^{H'} \int_0^1 w(t - u) dt \right) du \left( \lambda^{1 + \gamma} P_W \circ \phi_{\lambda, H} \right)(dw)
\]

\[
= \lambda^{\gamma'} \int_{\mathbb{R} \times W} \left( \theta \lambda^{H'} \int_0^1 w(t - u) dt \right) g_{\lambda, \gamma, H}(w) du \nu'(dw)
\]

\[
\sim \lambda^{\gamma'} \int_{\mathbb{R} \times W} \left( \theta \lambda^{H'} \int_0^1 w(t - u) dt \right) du \nu'(dw)
\]

\[
\sim (\theta^2/2) C_\gamma^2 \lambda^{\gamma'} \lambda^{2(H - H')},
\]

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which tends to a finite limit if and only if the second equality in (3.21) is true.

The proof of the first equality in (3.21) is similar. Let $\tilde{I}_\lambda := \lambda^{1+\gamma} \int E\Psi(\theta \lambda^{-H} \int_0^\infty W(t) dt)$. Using (3.12), (2.14),

$$\tilde{I}_\lambda = \int W \Psi(\theta W(w)) \left(\lambda^{1+\gamma} P_W \circ \phi_{\lambda,H'}(dw) \right)$$

analogous relation holds with $(\gamma', H', \nu')$ replaced by $(\gamma, H, \nu)$. On the other hand, $\tilde{I}_\lambda$ can be written as

$$\tilde{I}_\lambda = \int W \Psi(\theta W(w)) \left(\lambda^{1+\gamma} P_W \circ \phi_{\lambda,H'}(dw) \right)$$

$$\sim \lambda^{\gamma'-\gamma} \int W \Psi(\theta W(w)) (1+\gamma P_W \circ \phi_{\lambda,H'}(dw)$$

$$= \lambda^{\gamma'-\gamma} \int W \Psi(\theta W(w)) g_{\lambda,\gamma,H}(w) \nu(dw)$$

$$\sim \lambda^{\gamma'-\gamma} \int W \Psi(\theta W(w)) \nu(dw) = \lambda^{\gamma'-\gamma+\alpha(H-H')} \int W \Psi(\theta W(w)) \nu(dw),$$

$\lambda \to \infty$, where $\alpha = \frac{1+\gamma}{\gamma'}$, see (2.19). The two expressions for $\tilde{I}_\lambda$ agree as $\lambda \to \infty$ if and only if $\gamma'-\gamma+\alpha(H-H') = 0$, or the first relation in (3.21) holds. This proves (3.21) and the theorem, too. □

4 Examples

We present four examples of pulse process $W$ satisfying the conditions of Theorem 2.

Example 3. (‘Independent transmission rate and duration’, or ‘continuous flow rewards’ model, see [15].) Let

$$W(t) = A1(0 < t \leq R), \quad (4.1)$$

where r.v. $R > 0$ satisfies

$$P(R > x) \sim c_\varphi x^{-\varphi}, \quad x \to \infty, \quad \text{for some } \varphi > 0, \ c_\varphi > 0,$$

$A > 0$ is a r.v. with $EA^2 < \infty$ independent of $R$. The particular case of (4.1) corresponding to $A = 1$ is known as M/G/$\infty$ queue or the infinite source Poisson model [20]. Let

$$1 < \varphi < 2, \quad \gamma_0 := \varphi - 1, \quad \alpha := \varphi, \quad H := \begin{cases} \frac{3+\gamma-\varphi}{2}, & \gamma > \gamma_0, \\ 1, & \gamma = \gamma_0, \\ \frac{1+\gamma}{\varphi}, & 0 < \gamma < \gamma_0. \end{cases} \quad (4.3)$$

Let us check that, under some additional conditions on the distribution of r.v.s $A$, $R$ the pulse process in (4.1) satisfies Theorem 2 with $H$ in (4.3) and the measure $\nu_W$ defined a line below (4.4).

(i) $(\gamma > \gamma_0)$ Relation (3.5) with $2(1-H_1) = \varphi - 1$ follows by (3.14) and (4.1): $\text{Cov}(X(0), X(t)) = E[A^2] \int_0^\infty P(R > t + u) du \sim (c_\varphi/(\varphi - 1)) E[A^2] t^{-(\varphi-1)} (t \to \infty)$. To show (3.9), assume $EA^{2+\delta} < \infty$. Then the l.h.s. of (3.9)
does not exceed
\[
\lambda e^{-(1+\gamma-\delta)(\delta/2)} \mathbb{E}A^2 \int_{-\infty}^{1} E \left| \int_{0}^{1} \mathbbm{1}(0 < \lambda(t-u) < R)dt \right|^{2+\delta} du
\]
\[
\leq C\lambda e^{-(1+\gamma-\delta)(\delta/2)} \int_{-\infty}^{1} E \left| \int_{0}^{1} \mathbbm{1}(0 < \lambda(t-u) < R)dt \right|^{2} du \leq C\lambda^{-1+\gamma-\delta}(\delta/2) \to 0
\]
since \((1 + \gamma - \delta)(\delta/2) > 0\) for \(\gamma > \gamma_0 = \varrho - 1\).

(ii) \((\gamma < \gamma_0)\) Condition (3.10) for \(W = AR\) with \(c_+ = c_0 \mathbb{E}A^\alpha, c_- = 0\) is immediate by (4.2) and Breiman’s lemma, while (3.11) follows for any \(0 < \gamma < \gamma_0\) since \(\mathrm{E}[W(t)] = \mathbb{E}[A] P(R > t) \leq C(1 + t)^{-\varrho}\) and \(\delta = \varrho - 1 > \frac{\varrho - 1}{\varrho}(1 + \gamma)\).

(iii) \((\gamma = \gamma_0)\) Assume additionally that that d.f. \(P_R\) of \(R\) has a density \(f_R(r), r > 0\), satisfying
\[
f_R(r) \leq Cr^{\varrho - 1} \quad (\forall r > 0) \quad \text{and} \quad \lim_{r \to \infty} r^{1+\varrho} f_R(r) = \varrho c_0.
\]

Let \(\nu_W := P_A \times \nu_R\), where \(\nu_R\) defined in (2.8). The measure \(P_W\) can be identified with the distribution \(P_A \times P_R\) of \((A, R)\) on \(\mathbb{R}^2_+\) and \(\lambda^{1+\gamma_0} P_W \circ \phi_{\lambda,H(\gamma_0)}\) with the measure \(P_A \times \lambda^{\varrho} P_{R/\lambda}\) where \(\lambda^{\varrho} P_{R/\lambda}(dr) = \lambda^{\varrho + 1} f_R(\lambda r)dr\).

Conditions (4.4) guarantee the fulfillment of (3.12), or part (iii) of Theorem 2. The intermediate RF \(J_{\nu_W}\) in this example has a similar form to the Telecom RF in (2.7) and satisfies Theorem 1 following Example 1.

Essentially, all facts in this example are part of the results in [15]. Finally, we note that the condition \(\mathbb{E}A^2 < \infty\) is nearly crucial as its violation may lead to drastically different limits in [12] (a different trichotomy from [13]) [15].

**Example 4.** (‘Deterministically related transmission rate and duration’ model, see [27].) Let
\[
W(t) = R^{1-p} \mathbbm{1}(0 < t \leq R^p),
\]
where \(R > 0\) is a r.v. satisfying (4.2) and \(p \in (0, 1]\) is a (shape) parameter. (For \(p = 1\) (4.3) coincides with (4.1), \(A = 1\).) Let
\[
1 < \varrho < 2, \quad \gamma_0 := \frac{\varrho - 1}{p}, \quad \alpha := \varrho, \quad \lambda := \begin{cases} \frac{\gamma}{2} + \frac{2+\varrho-\varrho}{2p}, & \gamma > \gamma_0, \\ \frac{1}{p}, & \gamma = \gamma_0, \\ \frac{1+\gamma}{\varrho}, & 0 < \gamma < \gamma_0. \end{cases}
\]

Condition (3.2) holds if \(2 - p < \varrho < 2\). Let us check that this example satisfies further conditions of Theorem 2 in respective parameter ranges.

(i) \((\gamma > \gamma_0)\) For \(2 - p < \varrho < 2\) using
\[
\mathbb{E}R^{2-2p} \mathbbm{1}(R^p > t) = t^{2-2p} P(R > t^p) + (2 - 2p) \int_{t^p}^{\infty} u^{1-2p} P(R > u)du \sim \frac{c_0 \varrho}{2p - 2 + \varrho} t^{2-2p-2} \quad (t \to \infty),
\]
we see that condition (3.8) is satisfied with \(H_1 = \frac{2 + \varrho - \varrho}{2p} \in (\frac{1}{2}, 1]\) as
\[
\text{Cov}(X(0), X(t)) = t \int_{0}^{\infty} \mathbb{E}R^{2(1-p)} \mathbbm{1}(R^p > t(1 + u))du \sim c_X t^{\frac{2-2p-\varrho}{p}},
\]
where \( c_X = \frac{c_{\rho \theta}}{e - 2(1-p)} \int_0^\infty (1 + u)^{2(1-p) - \theta} \, du < \infty \).

To prove (3.9) write

\[
E \left| \int_0^1 R^{1-p} \mathbf{1}(0 < \lambda(t-u) < R^p) \, dt \right|^{2+\delta} \leq E \left[ R^{(1-p)(2+\delta)} \left( \int_0^1 \mathbf{1}(0 < \lambda(t-u) < R^p) \, dt \right)^2 \right] \\
\leq C \left( E \left[ R^{(1-p)(2+\delta)} \left( - \frac{R^p}{\lambda} < u \leq 1 \right) \right] \right. \\
+ \left. E \left[ R^{(1-p)(2+\delta)} \left( \frac{R^p}{\lambda} \land 1 \right)^2 \right] \right) 1(-1 < u \leq 1).
\]

Then similarly to (4.7) we find that the l.h.s. of (3.9) does not exceed \( C \lambda^{-\delta'} \), where \( \delta' := \delta \left( - \frac{2+p-\theta}{p} + \frac{2+2p-\theta}{2p} - 1 \right) > 0 \) for \( \gamma > \gamma_0 = \frac{\rho}{p} - 1 \), see (4.6), and \( \delta > 0 \) such that \( (1-p)(2+\delta) < \theta \).

(ii) \( (\gamma < \gamma_0) \) Since \( W = R \) for \( W(t) \) in (4.5), condition (3.10) holds by (4.2). Condition (3.11) for any \( \gamma < \gamma_0 \) and \( 1 < \gamma' < \gamma \) sufficiently close to \( \rho \) holds in view of \( E|W(t)|^{\gamma'} = ER^{\gamma'(1-p)} \mathbf{1}(R^p > t) \leq C(1 \vee t)^{\gamma'(1-p) - \rho} \)

since \( \delta = \frac{\theta - \gamma'}{\rho} > (1 + \gamma)(1 - \frac{\gamma'}{\rho}) \) is equivalent to \( \gamma < \gamma_0 \).

(iii) \( (\gamma = \gamma_0) \) The distribution \( P_W \) is induced by the mapping \( r \mapsto r^{1-p} \mathbf{1}(0 < t < R^p) \) from \( \mathbb{R}_+ \) equipped with \( P_R \) to \( L^1(\mathbb{R}_+) \). Assuming the existence of density \( f_R(r) \), \( r > 0 \), as in (4.4), relation (3.12) follows as in the previous example, with \( \nu_W = \nu_R \) given by (2.8). For \( 2-p < \theta < 2 \) condition (2.1) holds in view of (4.7). One can also verify (2.1) for any \( p \in (0,1] \), \( \rho \in (1,2) \), see [27].

(iv) Let us check that the above measure \( \nu_W \) \( (\gamma = \gamma_0) \) satisfies the conditions of Theorem 11. Indeed, those of part (i) are satisfied for \( 2-p < \rho < 2 \) since \( C_{\nu_W}^2 = C \int_{[0,1]^2} |t_1 - t_2|^{2(p-\theta)\frac{3}{2}} \, dt_1 dt_2 < \infty \) as in Example 1 and \( H(\gamma_0) = \frac{1}{p} \). As for part (ii), we note that \( W(w) = r \) and so (2.14) holds, moreover, (2.15) holds since \( \int_{\mathbb{R}_+} |w(t)|^{\gamma'} \nu_W(dw) = C t^{\gamma'(1-p) - \rho} \) with \( 1 < \gamma' < \rho = \alpha < 2 \), \( 1 - \frac{\gamma'}{\rho} < \frac{\gamma - \gamma'}{\rho} = \delta < 1 \).

Example 5. (‘Exponentially damped transmission rate’ model.) Let

\[
W(t) = e^{-At} \mathbf{1}(0 < t \leq R), \quad (4.8)
\]

where \( R > 0 \), \( A > 0 \) are independent r.v.s with

\[
P(R > r) \sim c_\rho r^{-\rho} \quad (r \to \infty), \quad P(A \leq a) \sim c_\kappa a^{-\kappa} \quad (a \to 0) \quad (4.9)
\]

for some positive exponents \( \rho > 0 \), \( \kappa > 0 \) and asymptotic constants \( c_\rho > 0 \), \( c_\kappa > 0 \). Let

\[
1 < \rho + \kappa < 2, \quad \gamma_0 := \rho + \kappa - 1, \quad H := \begin{cases} \frac{\gamma + 3 - \rho - \kappa}{2}, & \gamma > \gamma_0, \\ 1, & \gamma = \gamma_0, \\ \frac{1 + \gamma}{\rho + \kappa}, & 0 < \gamma < \gamma_0. \end{cases} \quad (4.10)
\]

The corresponding distribution \( P_W \) is identified with the distribution induced by the mapping \( (a,r) \mapsto J(t,a,r) := e^{-at} \mathbf{1}(0 < t \leq r), \) \( t > 0 \), from \( (\mathbb{R}_+^2, P_A \times P_R) \) to \( L^1(\mathbb{R}_+) \).
Proposition 4. Let $W$ be as in (4.9), (1.8), where $\gamma + \kappa \in (1,2)$. Moreover, for $\gamma = \gamma_0$ assume that $P_R, P_A$ have densities $f_R(r), r > 0, f_A(a), a > 0$ satisfying (4.4) and $f_A(a) \leq C a^{\kappa - 1}, a > 0, \lim_{a \to 0} a^{1-\kappa} f_A(a) = \kappa c_\kappa$. Then:

(i) conditions (3.8) and (3.9) hold for any $\gamma > \gamma_0$ with $2(1 - H_1) = \varrho + \kappa - 1$ and

$$c_X = \Gamma(\kappa + 1)c_\kappa c_\varrho c_{\kappa, \varrho}, \quad c_{\kappa, \varrho} := \int_0^\infty (1 + z)^{-\varrho}(1 + 2z)^{-\kappa}dz = \frac{2F_1(\kappa, 1; \kappa + \varrho; -1)}{\kappa + \varrho - 1}, \quad (4.11)$$

where $2F_1(a, b; c; z)$ denotes the hypergeometric function;

(ii) conditions (3.10) and (3.11) hold for any $0 < \gamma < \gamma_0$ with $\alpha = \varrho + \kappa$ and

$$c_+ := \kappa c_\kappa \int_0^1 (1 - u)^{\kappa+\varrho-1}\left(\log \frac{1}{u}\right)^{-\varrho}du < \infty, \quad c_- = 0; \quad (4.12)$$

(iii) conditions (2.1) and (3.12) hold for $\gamma = \gamma_0$ with the measure $\nu_W$ induced by the mapping $J(\cdot ; a, r)$ from $(\mathbb{R}_+, \nu_A \times \nu_R)$ to $L^1(\mathbb{R}_+)$ with $\nu_R(dr), r > 0$ as in (2.8) and $\nu_A(da) := \kappa c_\kappa a^{\kappa-1}da, a > 0$;

(iv) the above measure $\nu_W (\gamma = \gamma_0)$ satisfies the conditions in (i)-(ii) of Theorem 7.

Proof. (i) Let $\alpha = \varrho + \kappa$. Using (4.9) and (4.10) we get

$$\text{Cov}(X(0), X(t)) t^{\alpha-1} = t^{\alpha-1} \int_0^\infty \left[P(R > u + t) E e^{-A(2u+t)} - P(R > u + t)\right]du$$

$$= t^{\alpha-1} \int_0^\infty (2u + t)P(R > u + t)du \int_0^\infty P(A \leq z)e^{-z(2u+t)}dz$$

$$= t^{\alpha} \int_0^\infty (2u + 1)P(R > t(u + 1))du \int_0^\infty P(A \leq t^{-1}z)e^{-z(2u+1)}dz$$

$$\sim c_\kappa c_\varrho \int_0^\infty (2u + 1)(u + 1)^{-\varrho}du \int_0^\infty z^\kappa e^{-z(2u+1)}dz = c_X$$

from the dominated convergence theorem. Since $|W(t)| \leq 1$ in (4.5), condition (3.9) is satisfied when $H(\gamma) > 1$ or $\gamma > \gamma_0$, see (4.10).

(ii) Note the integral in (4.12) converges since $(1 - u)^{\kappa+\varrho-1}(\log(1/u))^{-\varrho} \sim (1 - u)^{\kappa-1} (u \nearrow 1)$ where $\kappa > 0$. Consider the tail of the d.f. of $W = \int_0^R e^{-Ax}dt = (1 - e^{-AR})/A$. Then assuming that $1/x > 0$ is a continuity point of the d.f. of $A$ we can write

$$J(x) := x^{\varrho+\kappa}P(W > x) = x^{\varrho+\kappa} \int_0^{1/x} P\left(R > \frac{1}{a} \log \frac{1}{1-ax}\right)dP(A \leq a)$$

as a sum of

$$J_1(x, \epsilon) := x^{\varrho+\kappa}c_\kappa \int_{\epsilon/x}^{(1-\epsilon)/x} a^{\varrho} \left(\log \frac{1}{1-ax}\right)^{-\varrho}dP(A \leq a),$$

$$J_2(x, \epsilon) := x^{\varrho+\kappa} \int_0^{(1-\epsilon)/x} P\left(R > \frac{1}{a} \log \frac{1}{1-ax}\right)dP(A \leq a),$$

$$J_3(x, \epsilon) := x^{\varrho+\kappa} \int_{(1-\epsilon)/x}^{1/x} P\left(R > \frac{1}{a} \log \frac{1}{1-ax}\right)dP(A \leq a)$$
and $J_i(x, \epsilon) := J(x) - \sum_{i=1}^3 J_i(x, \epsilon)$ for a small $\epsilon > 0$. Let us prove that
\[
\lim_{\epsilon \to 0} \limsup_{x \to \infty} J_i(x, \epsilon) = 0, \quad i = 2, 3, \quad \lim_{x \to \infty} J_4(x, \epsilon) = 0 \quad (\forall \epsilon > 0) \tag{4.13}
\]
and
\[
\lim_{x \to \infty} J_1(x, \epsilon) := J(\epsilon) = \kappa c_0 c_\kappa \int_0^1 a^{\kappa + \gamma - 1} \left( \log \frac{1}{1 - a} \right)^{-\theta} da \quad as \ \epsilon \to 0, \tag{4.14}
\]
where $J(\epsilon)$ is given in (4.15) and $J = c_+ < \infty$, see (4.12). Relations (4.13) and (4.14) imply (3.10).

Consider $J_2(x, \epsilon)$. Using log(1/(1 - ax)) ≥ Cax (0 < a < \epsilon/x, C > 0) and P(R > x) ≤ Cx^{-\theta} (\forall x > 0) we obtain
\[
J_2(x, \epsilon) \leq Cx^\epsilon P\left( A \leq \frac{\epsilon}{x} \right) \leq Cx^\epsilon \to 0 \quad (\epsilon \to 0).
\]
Similarly, using log(1/(1 - ax)) ≥ log(1/ε) ((1 - \epsilon)/x < a < 1/x) we obtain
\[
J_2(x, \epsilon) \leq \left( \log \frac{1}{\epsilon} \right)^{-\theta} x^{\kappa + \gamma} \int_0^{1/x} \epsilon^\theta \alpha^\theta dP(A \leq a) \leq C \left( \log \frac{1}{\epsilon} \right)^{-\theta} \to 0 \quad (\epsilon \to 0),
\]
thus proving the first relation in (4.13). The second relation in (4.13) follows similarly using the uniform convergence lim_{x \to \infty} \sup_{a \in [c, d]} \epsilon^\theta |P(R > ax) - c_\theta(ax)^{-\theta}| = 0 on each compact interval [c, d] ∈ (0, \infty).

Let us prove (4.14). Using condition (4.9) and integrating by parts we infer the existence of the following limit as $x \to \infty$:
\[
J_1(x, \epsilon) = x^\kappa c_\theta \left[ P\left( A \leq \frac{1 - \epsilon}{x} \right) \right] \frac{(1 - \epsilon)^\theta}{\log^\theta(1/(1 - \epsilon))} - P\left( A \leq \frac{\epsilon}{x} \right) \frac{\epsilon^\theta}{\log^\theta(1/(1 - \epsilon))}
\]
\[
- \int_{\epsilon}^{1-\epsilon} P\left( A \leq \frac{a}{x} \right) \left( a^\epsilon \left( \log \frac{1}{1 - a} \right)^{-\theta} \right) da'a^\theta \left( \log \frac{1}{1 - a} \right)^{-\theta} \to \int_{\epsilon}^{1-\epsilon} a^\kappa \left( a^\epsilon \left( \log \frac{1}{1 - a} \right)^{-\theta} \right) da = J(\epsilon).
\]
Then (4.14) follows by the convergence of the integral $J$. This proves (3.10). Relation (3.11) follows from E[W(t)] = E[e^{-At}1(R > t)] ≤ C(1 + t)^{-\theta - \kappa}$ since since $\delta = \alpha - 1 + 1 + \gamma - H(\gamma)$ is equivalent to $\gamma < \gamma_0$.

(iii) Follows similarly as in Examples 1,3,4 above.

(iv) As in Example 1, we find that $C_\nu^2 = C \int_{[0,1]} |t_1 - t_2|^{1-\kappa-\theta} dt_1 dt_2 < \infty$, hence (2.28) holds. Since $H(\gamma_0) = 1$, we see that all conditions of part (i) are satisfied. Next, since $W(w) = (1 - e^{-ar}) / a \leq (1 + ar) / a$ and $\int_w |w(t)| \nu(w) (dw) = C \int_{\mathbb{R}^+} e^{-at} 1(t < r) a^{\kappa-1} r^{\kappa-1} dadr = C t^{\kappa-\theta}$ for $w(t) = e^{-at} 1(0 < t \leq r)$, conditions (2.14) and (2.15) hold with $\alpha = \theta + \kappa \in (1,2), \alpha' = 1$ and $\delta = \alpha - 1 > 1 - 1/\alpha$.

Example 6. (‘Brownian pulse’ model.) Let
\[
W(t) = B(t)1(0 < t \leq R),
\]
where $B = \{B(t), t \geq 0\}$ is a standard Brownian motion and $R > 0$ is a r.v. independent of $B$ and satisfying (4.12). Consider the measure space $C(\mathbb{R}^+) \times \mathbb{R}^+$ equipped with $P_B \times P_R$, where $P_B$ is the Wiener measure and $P_R$ the distribution of r.v. $R$. Then, the distribution $P_W$ on $L^1(\mathbb{R}^+)$ can be identified with the distribution induced by the mapping $J : C(\mathbb{R}^+) \times \mathbb{R}^+ \to L^1(\mathbb{R}^+), J(g,r)(t) := g(t)1(0 < t < r), t > 0, (g,r) \in C(\mathbb{R}^+) \times \mathbb{R}^+$.
Proposition 5. Let $W$ be as in (4.16), (4.2), where $2 < \varrho < 3$. Let $\alpha := \frac{2\varrho}{3}$, $H_1 := 2 - \frac{\varrho}{2}$, 

$$
\gamma_0 := \varrho - 1, \quad H := \begin{cases} 
\frac{2}{3} + H, & \gamma < \gamma_0, \\
\frac{3}{2}, & \gamma = \gamma_0, \\
\frac{14+\delta}{\alpha}, & \gamma < \gamma_0.
\end{cases}
$$

(4.17)

Moreover, for $\gamma = \gamma_0$ assume that $P_R$ has a density $f_R(r)$, $r > 0$, satisfying (4.4). Then:

(i) conditions (3.8) and (3.9) hold for any $\gamma > \gamma_0$ with $c_X = \frac{C_\varrho}{(q-1)(q-2)}$;

(ii) conditions (3.10) and (3.11) hold for any $0 < \gamma < \gamma_0$;

(iii) conditions (2.1) and (3.12) hold for $\gamma = \gamma_0$ with the measure $\nu_W$ induced by the mapping $J$ on $C(\mathbb{R}_+) \times \mathbb{R}_+$ equipped with the measure $P_B \times \nu_R$, with $\nu_R(dr)$, $r > 0$, in (2.8);

(iv) the above measure $\nu_W$ ($\gamma = \gamma_0$) satisfies the conditions in (i)-(ii) of Theorem 4.

Proof. (i) Note for $H$ in (4.17) condition $H \in \left(\frac{1+1}{2}, \frac{1+2}{2}\right)$ translates to $\varrho \in (2, 3)$. Assumption (4.2) implies $\sup_{x>0} x^\varrho P(R > x) \leq C$. Then by (4.2) and dominated convergence theorem we get that as $t \to \infty$,

$$
\text{Cov}(X(0), X(t)) = \int_0^\infty u P(R > t + u)du = t^2 \int_0^\infty u P(R > t(1 + u))du \sim c_X t^{2 - \varrho},
$$

where expression of $c_X = c_\varrho \int_0^\infty u(1 + u)^{-\varrho}du$ follows by elementary integration. This proves (3.13).

To show (3.9) note

$$
\sigma_X^2(u, r) := E\left[\left(\int_0^1 B(\lambda(t-u))1(0 < \lambda(t-u) < R)dt\right)^2 \mid R = r\right] 
= 2\lambda \int_{-u}^t \int_{t_2}^{t_1} t_1(0 < t_1 < t_2 < \frac{r}{X})dt_1dt_2 
\leq C\lambda \left(\frac{\varrho}{X} < u < -1\right) + \left(1 - \frac{r}{X}\right)^3 \left(1(1 \leq u < 1)\right).
$$

(4.18)

Therefore the l.h.s. of (3.9) does not exceed $C\lambda^{1+\varrho+(2+\delta)(1-H(\varrho))}J_\lambda$, where

$$
J_\lambda := \int_{-\infty}^1 E\left[\sigma_X^2(u, R)^{1+\frac{\varrho}{2}}\right]du \leq C\lambda^{1+\frac{\varrho}{2}} \int_1^\infty u^{1+\frac{\varrho}{2}} P(R > \lambda u)du + E\left[\left(\frac{R}{X} \lambda \right)^{3(1+\frac{\varrho}{2})}\right] \leq C\lambda^{1+\frac{\varrho}{2} - \varrho}
$$

follows from (4.2) and (4.18) using integration by parts for $0 < \delta < 2(\varrho - 2)$. Since the resulting exponent of $\varrho$, viz., $1 + \varrho + (2 + \delta)(1 - H(\varrho)) + 1 + \varrho - \varrho = \varrho - \varrho + 1 < 0$ for $\gamma > \gamma_0 = \varrho - 1$ this proves (3.9) and part (i), too.

(ii) The conditional distribution of $W := \int_0^R B(t)dt$ given $R$ is Gaussian with variance $\sigma^2(R) = R^3/3$. Since $W \overset{d}{=} \sqrt{Z^3/3}$, where $Z \sim N(0, 1)$ is independent of $R$, (3.10) holds by (4.2) and Breiman’s lemma with $\alpha = 2\varrho/3$ and $c_\alpha = (1/2)c_\varrho E|Z|^{2\varrho/3-\varrho/3}$. Condition (3.11) is satisfied by $E|W(t)|^{2\varrho/3} = E|B(t)|^{2\varrho/3}P(R > t) \leq C(1 + \varrho)(1 + \varrho)^{\varrho/3} - \varrho$ for $1 \leq \varrho' < \varrho$ sufficiently close to $\varrho$ and $\delta = \varrho - \varrho' > (1 + \varrho)(1 - \varrho)\varrho$ equivalent to $\gamma < \varrho - 1 = \gamma_0$, proving part (ii).

(iii) Note $H(\gamma_0) - 1 = 1/2$, $1 + \gamma_0 = \varrho$. From the scaling invariance $P_B \circ \phi_{\lambda, \gamma_0} = P_B$ of the Wiener measure we see that $\lambda^{1+\gamma_0}P_W \circ \phi_{\lambda, \gamma_0}$ can be identified with the measure on $C(\mathbb{R}_+) \times \mathbb{R}_+$ equipped with
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conditions in (3.12) are satisfied. Condition (2.1) holds since for \( x = 1 \) the l.h.s. of (2.1) does not exceed \( \int_{(-\infty,1)\times\mathbb{R}^+} \sigma_I^T(u,r)\,d\nu_R(dr) < \infty. \)

(iv) Similarly as the proofs of Propositions 3 and 4, part (iv), \( C_{\ast \ast}^2 = C \int_{[0,1]}^2 |t_1-t_2|^2d\tau_1d\tau_2 < \infty \) and (2.9) holds. Since \( H(\gamma_0) = \frac{2}{3} \), the conditions in part (i) are satisfied. As for part (ii), condition (2.14) translates to \( \int_0^\infty \mathbb{E}[(|Z|^{\frac{3}{2}}) \wedge (Z^2)^{-\gamma}r^{-1}]\,dr = \mathbb{C}\mathbb{E}|Z|^{2\gamma/3} < \infty \) (\( Z \sim N(0,1) \)) while (2.15) holds for any \( \gamma' \geq 3/2 \) and \( 0 < \gamma' < \gamma \) since \( \int_{\mathbb{R}} |w(t)|^{2\gamma'/3}\nu_W(dw) = \mathbb{E}|B(t)|^{2\gamma'/3}\nu_R(t,\infty) = \mathbb{C}\mathbb{E}|\gamma'/3|^{-\gamma}, t > 0. \)

\( 5 \) Regenerative inputs

We follow [17] with some notational changes. Let \( Z > 0 \) be a r.v. with \( \mu := EZ < \infty \) and \( \{W(t)\}_{t \in [0,Z]} \) be a real-valued process with \( \int_0^Z |W(t)|\,dt < \infty \) a.s., defined on the same probability space. Let

\[
(Z_j, \{W_j(t)\}_{t \in [0,Z_j]}), \quad j = 1, 2, \ldots
\]

be i.i.d. copies of \((Z, \{W(t)\}_{t \in [0,Z]}))\). Let \((Z_0, \{W_0(t)\}_{t \in [0,Z_0]})\) be independent of \((5.1)\), whose distribution will be specified below. Define \( T_0 := Z_0, T_n := \sum_{i=0}^n Z_i, n \geq 1 \) and consider a (stationary) regenerative process with regeneration points \( T_0, T_1, \ldots \) defined as

\[
X(t) := W_0(t)1(t < T_0) + \sum_{j=1}^\infty W_j(t - T_{j-1})1(T_{j-1} \leq t < T_j), \quad t \geq 0.
\]

The ‘initial’ distribution \((Z_0, \{W_0(t)\}_{t \in [0,Z_0]})\) guaranteeing stationarity of (5.2) is given by

\[
P(W_0(\cdot) \in A, Z_0 \in ds_0) := \mu^{-1} \int_{-\infty}^0 ds_{-1}P(W(\cdot - s_{-1}) \in A, Z + s_{-1} \in ds_0)
\]

for any \( s_0 > 0 \) and any (Borel measurable) set \( A \subset L^1([0,s_0]). \) See the monograph [39] ch. 10.2.1. Particularly, (5.3) yields the well-known distribution of \( T_0 = Z_0 \) in a stationary renewal process:

\[
P(T_0 > t) = \mu^{-1} \int_0^\infty P(Z > s)\,ds, \quad t \geq 0.
\]

The class of regenerative processes is very large and includes many queueing models. We present three examples of such processes discussed in the literature.

\begin{example} \text{(ON/OFF process.)} Let \( Z = Z_{\text{on}} + Z_{\text{off}}, W(t) = 1(Z_{\text{on}} > t) \) where \( Z_{\text{on}} > 0 \) and \( Z_{\text{off}} > 0 \) are respective durations of ON and OFF intervals which can be mutually dependent. See [11] [20]. This process arises in some queueing models with single server, with \( X(t) \) representing the number of customers served at time \( t. \)

\end{example}

\begin{example} \text{(Workload process.)} Let \( Z = Z_{\text{on}} + Z_{\text{off}} \) as in ON/OFF process and \( W(t) := (Z_{\text{on}} - t)_+ \).

Then \( X(t) \) in (5.2) is the forward recurrence time of the busy period. In the G/G/1/0 queue, the process \( X(t) \) represents the current workload in the system.

\end{example}
Example 8. (Renewal-reward process.) Let $W(t) = W1(t \in [0, Z])$ where $W \in \mathbb{R}$ is a (random) constant which can be dependent or independent of duration $Z > 0$. See [30] and the references therein.

5.1 Gaussian limit

The question about FBS limit in [12] relies on the convergence of the variances, or the regular decay of the covariance function as in [5, 8]. LRD property of ON/OFF process with independent ON and OFF durations was analyzed in [11] using renewal methods. The last study was extended in [17] to some other regenerative processes (Examples 6 and 7). The last paper derived the representation of the covariance function of stationary regenerative process in [5.2] assuming the existence of $EX^2(0) < \infty$. Let

$$U(t) := 1 + E[\sum_{j=1}^{\infty} 1(T_j \leq t) | T_0 = 0] = \sum_{j=0}^{\infty} F^j*(t), \quad t \geq 0$$

be the renewal function, where $F(t) := P(Z \leq t)$ is the p.d.f. of $Z$ and $F^j*(t)$ its $j$th fold convolution, see [11, 1]. According to [17, Lemma 2.1],

$$Cov(X(0), X(t)) = R(t) + h(t), \quad t \geq 0,$$

where

$$R(t) := E[X(0)X(t)1(t < T_0)] = E[W_0(0)W_0(t)1(t < T_0)], \quad (5.7)$$

$$h(t) := \mu^{-1}\int_{0}^{t} z(t-s)U(ds) - (EX(0))^2$$

and

$$z(t) := \int_{0}^{t}G^0(s)G^1(t-s)ds, \quad G^0(t) := E[W(Z-t)1(t < Z)], \quad G^1(t) := E[W(t)1(t < Z)]$$

In [17], functions $G^0(t)$ and $G^1(t)$ are respectively called the backward and forward tour mean. From [5.3] we have the following relations:

$$EX(t) = \mu^{-1}\int_{0}^{\infty} G^0(s)ds = \mu^{-1}\int_{0}^{\infty} G^1(s)ds,$$

$$EX^2(0) = \mu^{-1}\int_{0}^{\infty} E[W^2(s)1(s < Z)]ds = R(0), \quad R(t) = \mu^{-1}\int_{0}^{\infty} E[W(s)W(s+t)1(Z > t+s)]ds.$$

As in most related work, we shall assume regular variation of the duration interval $Z$, viz.,

$$P(Z > t) \sim c_Zt^{-\alpha}, \quad t \to \infty \quad (\exists \alpha \in (1, 2), \ c_Z > 0).$$

Under assumption (5.10), we can expect that the covariance function in (5.6) decays as $t^{-1-\alpha}$ with $t \to \infty$; moreover, such decay can be due to one of the terms $R(t)$ and $h(t)$ in (5.7), or to both of them. Intuitively, the LRD decay of $R(t)$ is due to long ‘initial’ renewal interval $T_0 > t$, while $h(t)$ reflects the behavior after $T_0$. The latter behavior concerns the rate of convergence in the key renewal theorem in the heavy tailed case.
Then seems to be well-known but we include a short proof for the sake of completeness.

**Proof.** According to [11, Thm. 3.1 (iii)], relations (5.10), (5.11), (5.12) and (5.13) imply the asymptotics

\[ z(t) \sim c^*_z t^{-\alpha}, \quad t \to \infty \quad (\exists c^*_z \geq 0). \]

and

\[ F^{j*}(t) \text{ is nonsingular for some } j \geq 1. \] (5.12)

The above mentioned study [11] and the discussion in [17, p. 385] leads to the following result.

**Proposition 6.** Let \( \{X(t), t \geq 0\} \) be a stationary regenerative process in (5.2) with finite variance and covariance as in (5.6) satisfying conditions (5.10), (5.11), (5.12) and

\[ z(t) \sim c^*_z t^{-\alpha}, \quad t \to \infty \quad (\exists c^*_z \geq 0). \]

Then

\[ h(t) \sim \frac{1}{(\alpha - 1)\mu^2} \left( \frac{czm}{\mu} - c^*_z \right) t^{-(\alpha-1)}, \quad t \to \infty, \text{ where } m := \int_0^\infty z(s)ds. \] (5.14)

(i) (‘R(t) dominates’). Assume additionally that

\[ R(t) \sim c_R t^{-\beta}, \quad t \to \infty \quad (\exists 0 < \beta < \alpha - 1, c_R > 0) \] (5.15)

Then

\[ \text{Cov}(X(0), X(t)) \sim R(t) \sim c_R t^{-\beta}, \quad t \to \infty. \] (5.16)

(ii) (‘h(t) dominates’) Assume additionally \( R(t) = o(t^{-(\alpha-1)}), t \to \infty \). Then

\[ \text{Cov}(X(0), X(t)) \sim h(t) \sim \frac{1}{(\alpha - 1)\mu^2} \left( \frac{czm}{\mu} - c^*_z \right) t^{-(\alpha-1)}, \quad t \to \infty. \] (5.17)

**Proof.** According to [11] Thm. 3.1 (iii)], relations (5.10), (5.11), (5.12) and (5.13) imply the asymptotics of \( h(t) \) as in (5.14). Then, part (i) follows by (5.7) and (5.6). (ii) Follows directly from (5.14) and \( R(t) = o(t^{-(\alpha-1)}) \). \( \Box \)

**Remark 3.** While conditions of right continuity of \( z(t) \) and \( \lim_{t \to \infty} z(t) = 0 \) in (5.11) are mild, the two other conditions in (5.11) are more restrictive. Clearly, \( z(t) \geq 0 \) holds if either \( G^i(t) \geq 0, i = 1, 2 \) for all \( t \geq 0 \), or \( G^1(t) = E[W(t)1(t < Z)] = 0 (\forall t > 0) \) hold. The bounded variation condition in (5.11) holds if \( G^0(t) \) is integrable and \( G^1(t) \) has bounded variation on \( [0, \infty) \), \( \lim_{t \to \infty} G^1(t) = 0 \) (or vice versa), which follows from Proposition 7 below.

Let \( (g_1 * g_2)(t) = \int_0^t g_1(s)g_2(t-s)ds, t \geq 0 \) denote the convolution of \( g_i = g_i(t), t \geq 0, i = 1, 2 \) and \( V(f) \) denote the variation of a function \( f \) on \( [0, \infty) \) defined as \( V(f) := \sup \{ \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)| \} \), where the supremum is taken over all partitions \( 0 = t_0 < t_1 < \cdots < t_n < \infty, n = 1, 2, \ldots \) of \( [0, \infty) \). The following fact seems to be well-known but we include a short proof for the sake of completeness.
Proposition 7. Let \( \|g\|_1 \defeq \int \|g_1(t)\|dt \leq \infty \) and \( V(g_2) < \infty \), \( \lim_{t \to \infty} g_2(t) = 0 \). Then \( V(g_1 * g_2) \leq 2V(g_2)\|g_1\|_1 < \infty \).

Proof. For any \( 0 = t_0 < t_1 < \cdots < t_n < \infty \), we have that \( \|(g_1 * g_2)(t_{i+1}) - (g_1 * g_2)(t_i)\| \leq \int_{t_i}^{t_{i+1}} \|g_1(s)\|g_2(t_{i+1} - s) - g_2(t_i - s)\|ds + \int_{t_i}^{t_{i+1}} |g_1(s)|g_2(t_{i+1} - s) - g_2(t_i - s)|ds, \) \( 0 \leq i < n \) implying

\[
\sum_{i=0}^{n-1} \|(g_1 * g_2)(t_{i+1}) - (g_1 * g_2)(t_i)\| \leq \int_0^\infty |g_1(s)|\|g_2(t_{i+1} - s) - g_2(t_i - s)\|ds + \|g_2\| \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |g_1(s)|ds \leq V(g_2)\|g_1\|_1 + \|g_2\|_\infty \|g_1\|_1 \leq 2V(g_2)\|g_1\|_1 \text{ since } \|g_2\|_\infty \defeq \sup_{t \geq 0} \|g_2(t)\| \leq V(g_2) \text{ due to } \lim_{t \to \infty} g_2(t) = 0. \quad \square
\]

Theorem 4. Let \( X_i = \{X_i(t), t \geq 0\} \) be i.i.d. copies of stationary regenerative process in \( [5,2] \) with finite variance and a covariance function

\[
\text{Cov}(X(0), X(t)) \sim c_X t^{-(\alpha - 1)}, \quad t \to \infty \quad (\exists \ c_X > 0, \ 1 < \alpha < 2).
\]

Assume in addition that

\[
\sup_{t \in [0,2]} |W(t)| \leq C
\]

with a nonrandom constant \( C < \infty \). Then for any \( \gamma > \alpha - 1 \)

\[
\lambda^{-H(\gamma)}(A_{\lambda,\gamma}(x,y) - \mathbb{E}A_{\lambda,\gamma}(x,y)) \xrightarrow{f.d.l.} C_W B_{H,1/2}(x,y)
\]

where \( H(\gamma), C_W, H \) and the limit in \( (5.20) \) are the same as in Theorem 2 (i).

Proof. The convergence of the corresponding variances:

\[
\lambda^{-2H(\gamma)}\text{Var}(A_{\lambda,\gamma}(x,y)) \sim y\lambda^{-(3-\alpha)}\text{Var}\left(\int_0^{\lambda x} X(t)dt\right) \to C_W^2 y x^{3-\alpha} = C_W^2 \mathbb{E}B_{H,1/2}^2(x,y)
\]

follows as in Theorem 2 (i). In a similar way,

\[
\lambda^{-2H(\gamma)}\text{Cov}(A_{\lambda,\gamma}(x,y), A_{\lambda,\gamma}(x',y')) \to \frac{C_W^2}{2}(x^{3-\alpha} + (x')^{3-\alpha} - |x - x'|^{3-\alpha})(y \wedge y') = C_W^2 \mathbb{E}B_{H,1/2}(x,y)B_{H,1/2}(x',y')
\]

for any \((x,y),(x',y')) \in \mathbb{R}_+^2 \).

Let us prove \( (5.20) \) at a given point \((x,y) \in \mathbb{R}_+^2 \). Note \( (5.19) \) implies \( \sup_{t \geq 0} |X(t)| < C \). As noted in \( [20, \text{p.} 62] \), the last fact together with independence of \( \{X_i(t)\} \) and \( H(\gamma) > 1 \) guarantees asymptotic normality in \( (5.20) \). E.g., the Lyapunov condition \( \lambda^\gamma \mathbb{E}|\lambda^{-H(\gamma)}\int_0^\lambda X(t) - \mathbb{E}X(t)dt|^{2+\delta} \to 0 \) \( (\exists \delta > 0) \) for \( H(\gamma) = (3 - \alpha + \gamma)/2 > 1 \) \( (\gamma > \alpha - 1) \) translates to \( \mathbb{E}|\int_0^\lambda X(t) - \mathbb{E}X(t)dt|^{2+\delta} = o(\lambda^\delta) \) for some \( \delta' > 0 \) which follows from the asymptotics of the variance since \( |\int_0^\lambda X(t) - \mathbb{E}X(t)dt|^{\delta} \leq C\lambda^\delta \) in view of boundedness of \( \{X(t)\} \). Theorem 4 is proved. \( \square \)

Remark 4. The boundedness condition \( (5.19) \) on pulse process is quite restrictive and is not satisfied in Example 7. (The same condition is also imposed in Theorems 5 and 6.) In the case of renewal reward process (Example 8) it requires boundedness of reward variable \( W \). The proof in \( [30] \) of the Gaussian limit for renewal-reward process when \( W \) is unbounded but has finite variance uses approximation by bounded \( W \), suggesting that a similar approximation might help to weaken \( (5.19) \) in Theorem 2.
5.2 Stable limit

Given a stationary regenerative process in (5.2) with generic pair $(Z, \{W(t)\}_{t \in [0, Z])}$ denote

$$W_Z := \int_0^Z W(s)ds, \quad \mu_W := EW_Z, \quad \widetilde{W}_Z := W_Z - (\mu_W/\mu)Z. \quad (5.21)$$

Note $E\widetilde{W}_Z = 0$.

**Theorem 5.** Let $X_i = \{X_i(t), t \geq 0\}$ be i.i.d. copies of stationary regenerative process in (5.2). Assume that $(Z, \{W(t)\}_{t \in [0, Z])}$ satisfy (5.10), (5.19) and

$$P(\widetilde{W}_Z > x) = (c_+ + o(1)) x^{-\alpha}, \quad P(\widetilde{W}_Z < -x) = (c_- + o(1)) x^{-\alpha}, \quad x \to \infty, \quad (5.22)$$

for some $c_+ \geq 0, c_+ + c_- > 0$. Then

$$\lambda^{-(1+\gamma)/\alpha}(A_{\lambda, \gamma}(x, y) - E A_{\lambda, \gamma}(x, y)) \xrightarrow{\text{r.d.}} L_\alpha(x, y), \quad \forall 0 < \gamma < \alpha - 1, \quad (5.23)$$

where $L_\alpha$ is $\alpha$-stable Lévy sheet.

**Proof.** We prove one-dimensional convergence in (5.28) at $x = y = 1$ only. Let $N(t) := \sum_{i=0}^\infty 1(T_i \in (0, t])$ be the number of regeneration points in $(0, t]$. From (5.2) following [20, p. 41] we have the basic decomposition

$$\int_0^\Lambda X(t)dt = \int_0^{\Lambda \wedge T_0} W_0(t)dt + \left( \int_0^{T_N(\lambda)} - \int_\lambda^{T_N(\lambda)} \right) \sum_{i=1}^\infty W_i(t - T_{i-1})1(T_i - 1 \leq t < T_i)dt$$

$$= \int_0^{\Lambda \wedge T_0} W_0(t)dt + \sum_{i=1}^{N(\lambda)} \int_0^{Z_i} W_i(t)dt - \int_\lambda^{T_N(\lambda)} W_N(\lambda)(t - T_{N(\lambda)-1})dt. \quad (5.24)$$

Then, since $EX := EX(t) = \mu_W/\mu$ (see (5.9)), so $\int_0^\Lambda (X(t) - EX)dt = I_1(\lambda) + I_2(\lambda) - I_3(\lambda)$, where

$$I_1(\lambda) := \int_0^{\Lambda \wedge T_0} W_0(t)dt - (EX)(\lambda \wedge T_0), \quad (5.25)$$

$$I_2(\lambda) := \sum_{i=1}^{N(\lambda)} \widetilde{W}_{Z,i}, \quad \text{with} \quad \widetilde{W}_{Z,i} := \int_0^{Z_i} W_i(t)dt - (EX)Z_i,$$

$$I_3(\lambda) := \int_\lambda^{T_N(\lambda)} W_N(\lambda)(t - T_{N(\lambda)-1})dt - (EX)(T_{N(\lambda)} - \lambda),$$

with the convention that $I_2(\lambda) = I_3(\lambda) := 0$ if $N(\lambda) = 0$. Thus,

$$A_{\lambda, \gamma}(1, 1) - E A_{\lambda, \gamma}(1, 1) = A_1(\lambda) + A_2(\lambda) - A_3(\lambda), \quad A_k(\lambda) = \sum_{j=1}^{[\lambda^\gamma]} I_k^{(j)}(\lambda), \quad k = 1, 2, 3, \quad (5.26)$$

where $(I_1^{(j)}(\lambda), I_2^{(j)}(\lambda), I_3^{(j)}(\lambda)), j = 1, 2, \ldots$ are independent copies of $(I_1(\lambda), I_2(\lambda), I_3(\lambda))$ in (5.24). If $N(\lambda)$ in $I_2(\lambda)$ can be replaced by its mean $\lambda \mu / \mu$, the behavior of $I_2(\lambda)$ and the middle term $A_2(\lambda)$ in the r.h.s. of (5.25) is similar to the sum of $[\lambda^\gamma] \times [\lambda / \mu]$ i.i.d. copies of $\widetilde{W}_{Z,i}^{(j)}$, $1 \leq i \leq [\lambda / \mu], 1 \leq j \leq [\lambda^\gamma]$ of $\widetilde{W}_Z$ in the domain of attraction of $\alpha$-stable law, see (5.22). Accordingly, let

$$A_k'(\lambda) := \sum_{j=1}^{[\lambda^\gamma]} I_k^{(j)}(\lambda), \quad I_k^{(j)}(\lambda) := \sum_{i=1}^{[\lambda / \mu]} \widetilde{W}_{Z,i}^{(j)}, \quad j = 1, 2, \ldots, [\lambda^\gamma]. \quad (5.27)$$
To justify (5.28) and proceeding as in [20, proof of Thm. 2] we need to verify Steps 1 and 2 below.

Step 1. \( A_1(\lambda) \) and \( A_3(\lambda) \) are asymptotically negligible:

\[
A_k(\lambda) = o_p(\lambda^{(1+\gamma)/\alpha}), \quad k = 1, 3, \quad 0 < \gamma < \alpha - 1.
\]

(5.27)

It suffices to prove

\[
E|A_k(\lambda)|^p = o(\lambda^{(1+\gamma)p/\alpha}), \quad k = 1, 3
\]

(5.28)

for some \( p > 0 \). Let \( 0 < p \leq 1 \) then \( E|A_k(\lambda)|^p \leq \lambda^\gamma E|I_k(\lambda)|^p \) and (5.28) follows for \( \gamma < \alpha - 1 \) and \( 0 < p < \alpha - 1 < 1 \) sufficiently close to \( \alpha - 1 \) provided

\[
E|I_k(\lambda)|^p < C < \infty, \quad k = 1, 3.
\]

(5.29)

Integrating by parts: \( E|I_k(\lambda)|^p = p \int_0^\infty x^{p-1}P(|I_k(\lambda)| > x)dx \) we see that (5.28) holds if

\[
P(|I_k(\lambda)| > x) \leq Cx^{-(\alpha-1)}, \quad x > 0, \quad k = 1, 3.
\]

(5.30)

Note (5.19) implies \( \sup_{t \geq 0} |W_0(t)| \leq C \) with the same \( C > 0 \) (the random shift in (5.3) does not change the supremum). Hence for \( k = 1 \), (5.30) is immediate by \(|I_1(\lambda)| \leq CT_0 \sup_{t \geq 0} |W_0(t)| \leq CT_0 \) and \( P(T_0 > x) \leq Cx^{-(\alpha-1)}, \quad x > 0 \), see (5.4), (5.10). For \( k = 3 \), note by (5.2) and stationarity of \( X \)

\[
I_3(\lambda) \overset{d}{=} \left( \int_0^T X(t)dt - (EX)T_0 \right)1(T_1 > -\lambda),
\]

(5.31)

hence \( P(|I_3(\lambda)| > x) \leq P(CT_0 > x) \) and (5.30) follows exactly as in the case \( k = 1 \) above. This proves (5.27), or Step 1.

Step 2. Approximation of \( A_2(\lambda) \) by the sum \( A'_2(\lambda) \) in (5.26) of a fixed number of terms:

\[
A_2(\lambda) - A'_2(\lambda) = \sum_{j=1}^{\lfloor \lambda \gamma \rfloor} (I_2^{(j)}(\lambda) - I_2^{(j)}(\lambda)) = o_P(\lambda^{(1+\gamma)/\alpha}), \quad 0 < \gamma < \alpha - 1.
\]

(5.32)

Let \( J(\lambda) := \lambda^{-(1+\gamma)/\alpha}(I_2(\lambda) - I'_2(\lambda)) \). Since \( E\hat{W}Z = 0 \) so \( EJ'_2(\lambda) = 0 \) while \( EJ_2(\lambda) = 0 \) follows by Wald’s identity, c.f. [20, p.?]. Therefore, \( EJ(\lambda) = 0 \). By i.i.d. property of the summands on the l.h.s. in (5.32) it is equivalent to

\[
\left( 1 + \frac{\lfloor \lambda \gamma \rfloor E[e^{i\theta J(\lambda)} - 1]}{\lfloor \lambda \gamma \rfloor} \right)^{\lfloor \lambda \gamma \rfloor} \to 0
\]

or

\[
\lfloor \lambda \gamma \rfloor E[e^{i\theta J(\lambda)} - 1] = \lfloor \lambda \gamma \rfloor E[e^{i\theta J(\lambda)} - 1 - i\theta J(\lambda)] \to 0.
\]

(5.33)

In turn, (5.33) follows from

\[
\lambda^\gamma E[|J(\lambda)|^21(|J(\lambda)| \leq 1)] = o(1)
\]

(5.34)

and

\[
\lambda^\gamma E[|J(\lambda)|1(|J(\lambda)| > 1)] = o(1).
\]

(5.35)
Proof of (5.34)-(5.35) uses the following Lemma 1 whose proof is postponed to the end of this subsection. We introduce auxiliary quantities $\kappa$ and $p > 1$ satisfying the following three (sets of) inequalities:

$$\frac{1 + \gamma}{\alpha} < \kappa < 1, \quad \frac{\alpha(\gamma + \kappa)}{\gamma + 1} \sqrt{\frac{1 + \gamma}{\kappa}} \sqrt{\frac{\alpha(2 + \gamma - \kappa)}{1 + \gamma + \alpha(1 - \kappa)}} < p < \alpha, \quad \text{and} \quad \kappa > \frac{1}{\alpha} - \frac{(\alpha - 1)(\alpha - 1 - \gamma)}{\alpha}.$$  

(5.36)

The existence of such $\kappa, p$ follows from $0 < \gamma < \alpha - 1$ and $\frac{2 + \gamma}{\gamma + 1} \sqrt{\frac{1 + \gamma}{\alpha}} \sqrt{\frac{(1 + \gamma)(1 - \kappa)}{\gamma + \alpha(1 - \kappa)}} < 1$ superseding the choice of $\kappa$. In Lemma 1 and its proof we set $\mu = 1$ for brevity of notation.

**Lemma 1.** Under the assumptions of Theorem 3 for any $\kappa, p$ satisfying (5.36)

$$P(|J(\lambda)| > x) \leq C \left\{ \frac{\lambda^{\kappa - \frac{(1 + \gamma)p}{\alpha}}}{x^p - 1} + \frac{\lambda^{1 - \kappa p}}{x^p} \right\}, \quad 0 < x \leq 1 \quad (5.37)$$

and

$$P(|J(\lambda)| > x, N(\lambda) \leq 2\lambda) \leq C \left\{ \left( \frac{x^\kappa}{x^p - 1} \right) + \frac{\lambda^{1 - \kappa p}}{x^p} + \frac{\lambda^{\kappa - \kappa p} \mathbb{1} \{x \leq \lambda^{1 - \kappa}\}}{x^{\alpha - 1}} \right\}, \quad x \geq 1 \quad (5.38)$$

Moreover,

$$E[J(\lambda) \mathbb{1} \{J(\lambda) > 1, N(\lambda) > 2\lambda\}] \leq C \lambda^{2 - \alpha - (1 + \gamma)/\alpha}. \quad (5.39)$$

**Proof of (5.34).** Using $E[J(\lambda)|^2 \mathbb{1} \{J(\lambda) \leq 1\}] \leq 2 \int_0^1 x P(|J(\lambda)| > x) dx$ and (5.37) we get that

$$\lambda^\gamma E[J(\lambda)|^2 \mathbb{1} \{J(\lambda) \leq 1\}] \leq C \left( \int_0^1 (x^{2 - p} + x^{1 - p}) dx \right) \left( \lambda^{\gamma + \kappa - \frac{(1 + \gamma)p}{\alpha}} + \lambda^{1 + \gamma - \kappa p} \right) = o(1),$$

since the exponents $\gamma + \kappa - \frac{(1 + \gamma)p}{\alpha} < 0$ and $1 + \gamma - \kappa p < 0$ in view of (5.36).

**Proof of (5.35).** We have $E[J(\lambda)|^1 \{J(\lambda) > 1\}] = E[J(\lambda) \mathbb{1} \{J(\lambda) > 1, N(\lambda) > 2\lambda\}] + E[J(\lambda)|^1 \{J(\lambda) > 1, N(\lambda) \leq 2\lambda\}]$. With (5.39) in mind, it suffices to show $\lambda^\gamma E[J(\lambda)|^1 \{J(\lambda) > 1, N(\lambda) > 2\lambda\}] = o(1)$, or

$$\lambda^\gamma \int_1^\infty P(|J(\lambda)| > x, N(\lambda) \leq 2\lambda) dx = o(1). \quad (5.40)$$

Clearly, by (5.38), it suffices to prove the above relation for each term on the r.h.s. of (5.38). For the first term, we have that the corresponding quantity does not exceed

$$C \lambda^{\gamma + \frac{(1 + \gamma)p}{\alpha}} \int_1^{\lambda^{1 - \kappa}} x^{1 - p} dx + \lambda \int_{\lambda^{1 - \kappa}}^\infty x^{-p} dx \leq C \lambda^{\gamma + \frac{(1 + \gamma)p}{\alpha} + (1 + \gamma) + (2 - p)(1 - \kappa)}$$

due to (5.36).

Using the second term on the r.h.s. of (5.38) we see that the corresponding quantity does not exceed $C \lambda^{1 + \gamma - p\kappa} = o(1)$ in view of (5.36).

Finally, using the last term on the r.h.s. of (5.38) we see that the corresponding quantity does not exceed

$$C \lambda^{\gamma - \kappa - \kappa p} \int_1^{\lambda^{1 - \kappa}} x^{1 - \alpha} dx = C \lambda^{\gamma + 2 - \kappa - \alpha} = o(1).$$
since $\gamma + 2 - \kappa - \alpha < 0$ due to (5.36). This proves (5.35) and completes the proof of (5.33) or Step 2. This also ends the proof of Theorem 5 since the convergence in (5.23) with $A_{\lambda,\gamma}(1,1)$ replaced by $A'_{2}(\lambda)$ of (5.26) is classical by assumption (5.22). □

**Proof of Lemma 1.** Let us prove (5.38), with $x$ on the l.h.s. replaced by $4x$. Split $J(\lambda) 1(N(\lambda) \leq 2\lambda) = \sum_{k=1}^{4} J_{k}(\lambda, x)$, where

\[
J_{1}(\lambda, x) := J(\lambda) 1(\lambda < N(\lambda) \leq \lambda + (x\lambda^{\kappa}) \wedge \lambda), \quad (5.41)
\]
\[
J_{2}(\lambda, x) := J(\lambda) 1(\lambda - (x\lambda^{\kappa}) \wedge \lambda \leq N(\lambda) < \lambda),
\]
\[
J_{3}(\lambda, x) := J(\lambda) 1(\lambda + x\lambda^{\kappa} < N(\lambda) < 2\lambda, x\lambda^{\kappa} < \lambda),
\]
\[
J_{4}(\lambda, x) := J(\lambda) 1(0 \leq N(\lambda) < \lambda - x\lambda^{\kappa}, x\lambda^{\kappa} < \lambda).
\]

Note $J_{k}(\lambda, x) = 0, x > \lambda^{1-\kappa}, k = 3, 4$. We have $P(|J(\lambda)| > 4x, N(\lambda) \leq 2\lambda) \leq \sum_{k=1}^{4} P(J_{k}(\lambda, x) > x)$.

Consider the last probability for $k = 1$. Let $\lambda_{+}(x) := \lambda + (x\lambda^{\kappa}) \wedge \lambda$. Note $\{S_{k} := \sum_{i=1}^{k} \tilde{W}_{Z,i}, k = 1, 2, \ldots \}$ is a $p$-integrable martingale, since $\{\tilde{W}_{Z,i}, i = 1, 2, \ldots \}$ is an i.i.d. sequence with $E\tilde{W}_{Z,1} = 0$ and $E|\tilde{W}_{Z,1}|^{p} < \infty$ by assumption (5.22). Then, using Kolmogorov’s inequality for martingales for any $x > 0$ we obtain

\[
x^{p}\lambda^{(1+\gamma)p/\alpha}P(|J_{1}(\lambda, x)| > x) \leq E[|S_{N(\lambda)} - S_{\lfloor \lambda \rfloor}|^{p} 1(\lambda < N(\lambda) \leq \lambda_{+}(x))] \leq E \max_{1 \leq k \leq [\lambda_{+}(x)] - [\lambda]} |S_{k}|^{p} \leq C \max_{1 \leq k \leq [\lambda_{+}(x)] - [\lambda]} E|S_{k}|^{p} \leq C([\lambda_{+}(x)] - [\lambda])E|\tilde{W}_{Z,1}|^{p} \leq C(x\lambda^{\kappa}) \wedge \lambda,
\]

which agrees with the first term on the r.h.s. in (5.38). The proof of the same bound for $P(|J_{2}(\lambda, x)| > x)$ is completely analogous.

Next, consider the above bound for the term $J_{3}(\lambda, x)$ in (5.41). Denote $S_{j} := Z_{1} + \cdots + Z_{j}, j = 1, 2, \ldots$. Since $\lambda_{+}(x) \leq 2\lambda, x \leq \lambda^{1-\kappa}$ we have by Chebyshev and von Bahr–Esseen inequalities that We have

\[
P(|J_{3}(\lambda, x)| > x) \leq P(N(\lambda) > \lambda_{+}(x), x \leq \lambda^{1-\kappa}) \leq P(S_{[\lambda_{+}(x)]} - ES_{[\lambda_{+}(x)]} \leq -x\lambda^{\kappa}/2) 1(x \leq \lambda^{1-\kappa}) \leq C E|S_{[\lambda_{+}(x)]} - ES_{[\lambda_{+}(x)]}|^{p} x^{p}\lambda^{p\kappa} 1(x \leq \lambda^{1-\kappa}) \leq C x^{p}\lambda^{1-p\kappa} 1(x \leq \lambda^{1-\kappa})
\]

which agrees with the second term on the r.h.s. in (5.38).

Next, consider the last term $J_{4}(\lambda, x)$ in (5.41). We have with $\lambda_{-}(x) := \lambda - x\lambda^{\kappa}$ and $p \in (1, \alpha)$

\[
P(|J_{4}(\lambda, x)| > x) \leq P(N(\lambda) < \lambda_{-}(x)) 1(x \leq \lambda^{1-\kappa}) \leq P(T_{0} + Z_{1} + \cdots + Z_{[\lambda_{-}(x)]-1} > \lambda) 1(x \leq \lambda^{1-\kappa}) \leq (P(T_{0} > x\lambda^{\kappa}/2) + P(S_{[\lambda_{-}(x)]-1} > \lambda - x\lambda^{\kappa}/2)) 1(x \leq \lambda^{1-\kappa}) \leq C \left(\lambda^{-\kappa(\alpha-1)}x^{-(\alpha-1)} + x^{-p}\lambda^{1-p\kappa}\right) 1(x \leq \lambda^{1-\kappa})
\]

(5.44)
which agrees with (5.38) and completes the proof of (5.38).

The proof of (5.37) resembles that of (5.38) but is simpler. Let \( \kappa, p \) satisfy (5.36) and \( x \in (0, 1] \). \( \lambda \pm (x) := \lambda \pm x \lambda^\kappa \). We use the decomposition \( J(\lambda) = \sum_{k=1}^3 J'_k(\lambda, x) \) where

\[
J'_1(\lambda, x) := J(\lambda)1(\{N(\lambda) - \lambda | x \lambda^\kappa\}),
J'_2(\lambda, x) := J(\lambda)1(N(\lambda) > \lambda_+(x)),
J'_3(\lambda, x) := J(\lambda)1(0 \leq N(\lambda) < \lambda_-(x)).
\]

Then similarly to (5.42), \( x^p \lambda^{(1+\gamma)p/\alpha} P(J'_1(\lambda, x) > x) \leq C x \lambda^\kappa \) which agrees with the first term on the r.h.s. of (5.37). The proofs of \( P(J'_i(\lambda, x) > x) \leq C x^{-p} \lambda^{1-\kappa p} \), \( i = 2, 3 \) follow (5.43), (5.44) and use the fact that \( x^{-(\alpha-1)} \lambda^{-\kappa(\alpha-1)} \leq x^{-p} \lambda^{1-\kappa p} \) for \( \kappa \leq 1, p \leq \alpha, 0 < x \leq 1 \). This proves (5.37).

Let us prove (5.39). By the boundedness condition (5.19) of the pulse process, \( |J(\lambda)|1(N(\lambda) > 2\lambda) \leq C \lambda^{-(1+\gamma)/\alpha} \sum_{i=1}^{N(\lambda)} [|Z_{i,1}] \leq C \lambda^{-(1+\gamma)/\alpha} \sum_{i=1}^{N(\lambda)} Z_i1(N(\lambda) > 2\lambda) \), where

\[
\sum_{i=1}^{N(\lambda)} Z_i1(N(\lambda) > 2\lambda) = (T_{N(\lambda)} - T_0)1(N(\lambda) > 2\lambda) \leq (\lambda + (T_{N(\lambda)} - T_{N(\lambda)-1}))1(N(\lambda) > 2\lambda)
\]

Therefore

\[
\lambda^\gamma E[|J(\lambda)|1(N(\lambda) > 2\lambda)] \leq \lambda^{\gamma - \frac{1+\gamma}{\alpha}} \{ \lambda P(N(\lambda) > 2\lambda) + E[(T_{N(\lambda)} - T_{N(\lambda)-1})1(N(\lambda) > 0)] \}
\] (5.45)

To evaluate the first term on the r.h.s. of (5.45) we use the argument as in (5.43) yielding

\[
P(N(\lambda) > 2\lambda) \leq P(S_{2\lambda} - ES_{2\lambda} < -\lambda) \leq \frac{E[S_{2\lambda} - ES_{2\lambda}]^p}{\lambda^p} \leq C \lambda^{1-p}
\] (5.46)

for \( p \in (1, \alpha) \). Therefore, the first term on the r.h.s. of (5.45) does not exceed \( C \lambda^{2+\gamma - \frac{1+\gamma}{\alpha} - p} = o(1) \) provided the last exponent is negative, which follows from the last inequality in (5.36).

Next, consider the second term on the r.h.s. of (5.45). By stationarity of the renewal process \( E[(T_{N(\lambda)} - T_{N(\lambda)-1})1(N(\lambda) > 0)] = E[(T_{N(\lambda)} - T_{N(\lambda)-1})1(0 \leq T_{N(\lambda)-1} < \lambda)] = E[(T_0 - T_{-1})1(-\lambda \leq T_{-1} < 0)] \), where \( \ldots < T_{-1} < 0 < T_0 < T_1 < \ldots \) is the stationary renewal process on \( \mathbb{R} \). We use the fact that the joint distribution of \( (T_{-1}, T_0) \) in such process is given by

\[
P(T_{-1} \in ds_{-1}, T_0 \in ds_0) = \mu^{-1} ds_{-1} P(Z \in ds_0), \quad s_{-1} < 0 \leq s_0,
\] (5.47)

see e.g. [39] (recall that \( \mu = EZ = 1 \) in Lemma 1). Whence,

\[
E[(T_{N(\lambda)} - T_{N(\lambda)-1})1(N(\lambda) > 0)] = \int_0^\lambda E[Z1(Z > s)]ds = \int_0^\lambda ds s P(Z > s) + \int_s^\infty P(Z > x)dx \leq C \int_0^\lambda s^{1-\alpha} ds \leq C \lambda^{2-\alpha}
\] (5.48)

and hence the second term on the r.h.s. of (5.45) does not exceed \( C \lambda^{2+\gamma - \frac{1+\gamma}{\alpha} - \alpha} = o(1) \) for \( \gamma < \alpha - 1 \). This proves (5.39), thereby completing the proof of Lemma 1. \( \square \)
5.3 Intermediate limit

The intermediate limit or the convergence to the Telecom process was proved in [9] for heavy-tailed renewal process and extended to ON/OFF process (with independent ON and OFF intervals) in [7]. In this section we extend this result to more general regenerative processes. Observe from (5.22) and the decomposition in (5.24) that the stable limit in Theorem 5 is due to \( \alpha \)-tail behavior of the difference \( \tilde{W}_Z = W_Z - Z(EW_Z/\mu) \).

Since \( Z \) is assumed to have the same \( \alpha \)-tail, it follows that \( P(|W_Z| > x) = O(|x|^{-\alpha}) \) but of course this does not exclude \( P(|W_Z| > x) = o(|x|^{-\alpha}) \). Particularly, for ON/OFF process with tail parameters \( \alpha_{on}, \alpha_{off} \) we have both possibilities depending on whether \( \alpha_{on} < \alpha_{off} \), or \( \alpha_{on} > \alpha_{off} \). The case of ON/OFF inputs is rather special since ON and OFF intervals can be exchanged, see [20].

In this paper the discussion of the intermediate limit is restricted to the situation when \( W_Z \) has a lighter tail than \( Z \) and the tail behavior of \( \tilde{W}_Z \) is determined by the term \(-Z(\mu_W/\mu)\). In this case, the problem can be reduced to the intermediate limit for the renewal process \( N(t) \) similarly as for ON/OFF process [7], leaving open the case when \( W_Z \) and \( Z \) have the same \( \alpha \)-tail.

We start with the following decomposition for \( \int_0^\lambda X(t)dt - \lambda(\mu_W/\mu) \):

\[
\int_0^\lambda X(t)dt - \lambda(\mu_W/\mu) = \sum_{k=1}^4 \tilde{I}_k(\lambda), \quad \text{where}
\]

\[
\tilde{I}_1(\lambda) := \mu_W(N(\lambda) - EN(\lambda)), \quad \tilde{I}_2(\lambda) := \sum_{i=1}^{N(\lambda)}(W_{Z,i} - \mu_W),
\]

\[
\tilde{I}_3(\lambda) := \int_0^{\lambda\wedge T_0} W_0(t)dt, \quad \tilde{I}_4(\lambda) := -\int_\lambda^{T_{N(\lambda)}} W_{N(\lambda)}(t - T_{N(\lambda)-1})dt,
\]

where \( T_{N(\lambda)-1} < \lambda \leq T_{N(\lambda)} \) are the renewal points before and after time \( \lambda \). (5.49) is similar but different from (5.24). (5.49) also agrees with the decomposition in the ON/OFF case in [7, p.38] (with ON and OFF intervals exchanged). We will prove that \( \tilde{I}_1(\lambda) \) in (5.49) is the main term while the remaining terms \( \tilde{I}_k(\lambda) \), \( k = 2,3,4 \) are negligible under the intermediate scaling at \( \gamma = \gamma_0 = \alpha - 1 \).

**Theorem 6.** Let \( X_i = \{X_i(t), t \geq 0\} \) in (1.1) be i.i.d. copies of stationary regenerative process in (5.2). Assume that \((Z,\{W(t)\}_{t \in [0,Z]})\) satisfy (5.10), (5.19) and

\[
P(|W_Z| > x) \leq Cx^{-\alpha\gamma - \delta} \quad \text{and} \quad E|W(t)|1(t < Z) \leq Ct^{-\alpha\gamma - \delta} \quad \forall \ x,t \geq 1 \quad (\exists \ \delta, C > 0).
\]

Then with \( \gamma_0 = \alpha - 1 \)

\[
\lambda^{-1}(A_{\lambda,\gamma_0}(x,y) - E A_{\lambda,\gamma_0}(x,y)) \overset{\text{fdd}}{=} -(\mu_W/\mu)J(x,y), \quad (5.51)
\]

where \( J = \{ J(x,y), (x,y) \in \mathbb{R}^2_+ \} \) is the Telecom RF in (2.7)-(2.8) with \( \varrho \) replaced by \( \alpha \).

**Proof.** Using (5.49) similarly to (5.25) we can write

\[
A_{\lambda,\gamma_0}(1,1) - EA_{\lambda,\gamma_0}(1,1) = \sum_{k=1}^4 \sum_{j=1}^{\lfloor \lambda/\gamma_0 \rfloor} \tilde{I}_k^{(j)}(\lambda) =: \sum_{k=1}^4 \tilde{A}_k(\lambda), \quad (5.52)
\]
where \((\tilde{I}_1^{(j)}(\lambda), \tilde{I}_2^{(j)}(\lambda), \tilde{I}_3^{(j)}(\lambda), \tilde{I}_4^{(j)}(\lambda))\), \(j = 1, 2, \ldots\) are independent copies of \((\tilde{I}_1(\lambda), \tilde{I}_2(\lambda), \tilde{I}_3(\lambda), \tilde{I}_4(\lambda))\) in (5.49). Then, one-dimensional convergence in (5.51) at \((x, y) = (1, 1)\) follows from Steps 1–3.

Step 1. \(\tilde{A}_k(\lambda) = o_P(\lambda), k = 3, 4.\) From (5.3) and the second condition in (5.50) we get

\[
E|\tilde{I}_3(\lambda)| = \mu^{-1} \int_0^\infty ds E \left| \int_0^\lambda W(t+s)1(t+s < Z)dt \right| \leq C \int_0^\infty ds \int_0^\lambda (1 \vee (t+s)^{-\alpha-\delta})dt \leq C\lambda^{2-\alpha-\delta},
\]

implying \(E|\tilde{A}_3(\lambda)| \leq |\lambda^{\alpha-1} E|\tilde{I}_3(\lambda)| \leq C\lambda^{1-\delta} = o(\lambda).\) Next, consider \(k = 4.\) By stationarity of \(\{X(t)\}\) and (5.3) we see that \(\tilde{I}_4(\lambda) \overset{d}{=} - \int_0^{T_0} W_0(t)dt =: \tilde{I}_4\) does not depend on \(\lambda\) in distribution and

\[
P(|\tilde{I}_4| > x) = \mu^{-1} \int_0^\infty P \left( \left| \int_0^\lambda W(t+s)1(Z > t+s)dt \right| > x \right)ds.
\]

Write \(\tilde{I}_4 = \tilde{I}_4'(\lambda) + \tilde{I}_4''(\lambda)\) and \(\tilde{A}_4(\lambda) = \tilde{A}_4'(\lambda) + \tilde{A}_4''(\lambda)\) accordingly, where

\[
\tilde{I}_4'(\lambda) := \tilde{I}_4(\tilde{I}_4 \leq \hat{\lambda}), \quad \tilde{I}_4''(\lambda) := \tilde{I}_4(\tilde{I}_4 > \hat{\lambda}), \quad \hat{\lambda} := \lambda^{\alpha-1}. \lambda
\]

Let us check that

\[
P(|\tilde{I}_4| > x) \leq Cx^{-\frac{\alpha-1}{1-\delta}}, \quad x > 0.
\]

Indeed, split the r.h.s. of (5.53) as \(C(\int_{x'}^\infty ds + \int_{x'}^\infty ds) =: C(g_1(x) + g_2(x)),\) with \(x' := x^{1-\delta}.\) Then, using (5.39) and (5.40) we obtain

\[
g_2(x) \leq C \int_{x'}^\infty P(C(Z-s)_+ > x) ds = C \int_{x'}^\infty P \left( Z > s + \frac{x}{C} \right) ds \leq C \int_{x'}^\infty s^{-\alpha} ds \leq C(x')^{-(\alpha-1)},
\]

and, by the second assumption in (5.51),

\[
g_1(x) \leq Cx^{-1} \int_{x'}^\infty ds \int_s^\infty E|W(t)|1(t < Z) dt \leq Cx^{-1} \int_{x'}^\infty ds \int_s^\infty t^{-\alpha-\delta} dt \leq Cx^{-1} \int_0^{x'} s^{-\alpha-\delta} ds = C(x')^{2-\alpha-\delta}x^{-1},
\]

proving (5.54). Using it, \(E[\tilde{A}_4'(\lambda)] \leq C\lambda^{\alpha-1} E[\tilde{I}_4'(\lambda)] \leq C\lambda^{\alpha-1}\hat{\lambda}^{1-\frac{\alpha-1}{1-\delta}} = o(\lambda)\) follows from the definition of \(\hat{\lambda}\) since \(\alpha - 1 + \frac{\alpha-1}{\alpha-1-\delta}(1 - \frac{\alpha-1}{1-\delta}) < 1\) reduces to \(\alpha > 1.\) Finally, \(P(\tilde{A}_4''(\lambda) \neq 0) \leq C\lambda^{\alpha-1} P(|\tilde{I}_4| > \hat{\lambda}) = o(1)\) according to (5.51). This proves \(\tilde{A}_4(\lambda) = o_P(\lambda)\) and completes the proof of Step 1.

Step 2. \(\tilde{A}_2(\lambda) = o_P(\lambda).\) Write \(\tilde{A}_2(\lambda) = \tilde{A}_2'(\lambda) + \tilde{A}_2''(\lambda),\) where

\[
\tilde{A}_2'(\lambda) = \sum_{j=1}^{[\lambda^{\alpha-1}]} \sum_{i=1}^{[\lambda/\mu]} (W^{(j)}_{Z,i} - EW^{(j)}_{Z,i})
\]

(5.55)

is a sum of \([\lambda^{\alpha-1}] [\lambda/\mu] = O(\lambda^\alpha)\) i.i.d. r.v.s with tail behavior as in (5.50). These facts for \(1 < \alpha + \delta < 2\) entail \(\tilde{A}_2'(\lambda) = O_P(\lambda^{\alpha+\delta}) = o_P(\lambda)\). Indeed, split

\[
W_Z - EW_Z = (W_Z 1(|W_Z| \leq \lambda^{\alpha+\delta}) - EW_Z 1(|W_Z| \leq \lambda^{\alpha+\delta})) + (W_Z 1(|W_Z| > \lambda^{\alpha+\delta}) - EW_Z 1(|W_Z| > \lambda^{\alpha+\delta})) =: \eta^- + \eta^+,
\]

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and let \( \{ \eta^+_i, i \in \mathbb{N}_+ \} \) be i.i.d. copies of \( \eta^+ \). Then using (5.50) we obtain
\[
E|\sum_{i=1}^{N(\lambda)} \eta_i^+|^2 \leq \lambda^\alpha E|\eta^+|^2 \leq C\lambda^\alpha \int_0^{\lambda^{\alpha/\alpha}} \lambda P(|W_Z| > x)dx \leq C\lambda^{\alpha/\alpha+\delta}
\]
and similarly, \( E|\sum_{i=1}^{N(\lambda)} \eta_i^-|^2 \leq 2\lambda^\alpha E|W_Z|1(|W_Z| > \lambda^{\alpha/\alpha+\delta}) \leq C\lambda^{\alpha/\alpha+\delta} \), proving that (5.55) is \( O_P(\lambda^{\alpha/\alpha+\delta}) \) hence negligible. Next, let us prove
\[
\tilde{A}_2^\alpha(\lambda) = \tilde{A}_2(\lambda) - \tilde{A}_2^\alpha(\lambda) = o_P(\lambda). 
\]
(5.56)
We have \( \lambda^{-1} \tilde{A}_2^\alpha(\lambda) \overset{d}{=} \sum_{j=1}^{\lfloor \lambda^{-1} \rfloor} (\tilde{J}^{(j)}(\lambda), \tilde{J}^{(j)}(\lambda), j \geq 1 \) are i.i.d. copies of \( \tilde{J}(\lambda) := \lambda^{-1} (\sum_{i=1}^{N(\lambda)} - \sum_{i=1}^{\lfloor \lambda/\beta \rfloor} (W_{Z,i} - EW_{Z,i}) \). The proof of (5.50) resembles that of Step 2 of Theorem 6 but is simpler due to tail assumption in (5.50). Since \( E\tilde{J}(\lambda) = 0 \) as in Theorem 6, (5.56) follows from \( B_i(\lambda) = o_P(1), i = 1, 2, \) where
\[
B_1(\lambda) := \sum_{k=1}^{\lfloor \lambda^{-1} \rfloor} (\tilde{J}^{(k)}(\lambda)1(|N(\lambda) - \lambda/\mu| > \lambda^\kappa) - E\tilde{J}^{(k)}(\lambda)1(|N(\lambda) - \lambda/\mu| > \lambda^\kappa),
\]
\[
B_2(\lambda) := \sum_{k=1}^{\lfloor \lambda^{-1} \rfloor} (\tilde{J}_{k}^{(k)}(\lambda)1(|N(\lambda) - \lambda/\mu| \leq \lambda^\kappa) - E\tilde{J}^{(k)}(\lambda)1(|N(\lambda) - \lambda/\mu| \leq \lambda^\kappa)
\]
and where \( \kappa > 1 \) is specified below. Since \( B_2(\lambda) \) is a sum of i.i.d. r.v.s with zero mean, so
\[
E|B_2(\lambda)|^{p'} \leq 2\lambda^{\alpha-1}E|\tilde{J}(\lambda)1(|N(\lambda) - \lambda/\mu| \leq \lambda^\kappa) - E\tilde{J}(\lambda)1(|N(\lambda) - \lambda/\mu| \leq \lambda^\kappa)|^{p'} \leq C\lambda^{\alpha-1}E|\tilde{J}(\lambda)|^{p'} 1(|N(\lambda) - \lambda/\mu| \leq \lambda^\kappa)
\]
for any \( 1 \leq p' \leq 2 \). By taking \( \alpha < p' < \alpha + \delta \) and using condition (5.50) and the same martingale argument as in (5.42) we obtain
\[
E|\tilde{J}(\lambda)|^{p'} 1(|N(\lambda) - \lambda/\mu| \leq \lambda^\kappa) \leq C\lambda^{\alpha-p'} E|W_Z|^{p'} \leq C\lambda^{\alpha-p'},
\]
so that \( E|B_2(\lambda)|^{p'} \leq C\lambda^{\alpha+\kappa-1-p'} = o(1) \) provided \( \kappa < 1 + (p' - \alpha) > 1 \). Next, \( P(B_1(\lambda) \neq 0) \leq \lambda^{-1}P(|N(\lambda) - \lambda/\mu| > \lambda^\kappa) \), where \( P(|N(\lambda) - \lambda/\mu| > \lambda^\kappa) = P(N(\lambda) > \lambda/\mu + \lambda^\kappa) \leq C\lambda^{-\kappa(p-1)} \) for any \( p \in (1, \alpha) \) as in (5.46). Whence,
\[
P(B_1(\lambda) \neq 0) \leq C\lambda^{-(\alpha-1)/(\alpha-1)+\kappa(\alpha-p)} = o(1)
\]
follows for \( \kappa > 1 \) by choosing \( p \) sufficiently close to \( \alpha \). This proves (5.56) and completes the proof of Step 2.

**Step 3.** \( \lambda^{-1} \tilde{A}_1(\lambda) \overset{d}{\to} - (\mu \nu\mu/\mu)J(1,1) \): follows from (9) (see (6.1) below).

### 5.4 Examples

Below, we present some examples of regenerative processes satisfying the conditions of Theorems 4 and 6. Examples 9 and 10 are particular cases of general ON/OFF process (Example 6).

**Example 9.** M/G/1/0 queue. The queueing system with standard Poisson arrivals and general service times with single server without waiting room (customers arriving when the system is busy are rejected). Let \( 0 \leq T_0 < T_1 < \ldots \) be consecutive times when the arriving customer finds the system empty, \( Z_i := T_i - T_{i-1} = \ldots \)
\(Z_i^{on} + Z_i^{off}, i = 1, 2, \ldots\), where \(Z_i^{on} > 0\) are service times and \(Z_i^{off} > 0\) idle periods. Assume that the generic service distribution is regularly varying with tail parameter \(\alpha \in (1, 2)\), viz.,

\[
P(Z^{on} > x) \sim c_{on} x^{-\alpha}, \quad x \to \infty, \quad (\exists c_{on} > 0, \ \alpha \in (1, 2)).
\]  

(5.57)

The idle periods \(Z_i^{off}\) in the above M/G/1/0 queue have a standard exponential distribution with mean \(\mu_{off} = 1\). By the memoryless property of exponential distribution, \(Z^{on}\) and \(Z^{off}\) are independent and the p.d.f. of \(Z\) is nonsingular, so that condition (5.12) is satisfied. Let \(X(t) \in \{0, 1\}\) be the number of customers in service at time \(t\). We assume that the queueing system is stationary, then \(\{X(t), t \geq 0\}\) is a stationary regenerative process with generic pair \((Z, \{W(t)\}_{t \in [0, Z]}), W(t) = 1 (t \in [0, Z^{on}])\). Note \(EX(t) = \mu_{on}/\mu, \mu_{on} := EZ^{on}, \mu = EZ = \mu_{on} + 1\) and \(\tilde{W}_Z = \mu^{-1}(Z^{on} - \mu_{on} Z^{off})\) satisfies (5.22) with \(c_+ = \mu^{-1}c_{on}, c_- = 0\). Moreover, \(Cov(X(0), X(t))\) satisfies (5.18) with \(c_X = \frac{c_{on}}{(\alpha - 1)\mu^2}\), see [11, 17]. We see that the above ON/OFF process satisfies the conditions of Theorems 4 and 5 and the corresponding aggregated input RF \(A_{\lambda, \gamma}(x, y)\) tends to the Gaussian and stable limits in these theorems in respective regions \(\gamma > \alpha - 1\) and \(\gamma < \alpha - 1\). As noted above, these facts essentially follow from [20, Theorems 3, 2] dealing with general ON/OFF process which include the above M/G/1/0 queue as a particular case. The intermediate convergence at \(\gamma = \alpha - 1\) in Theorem 5 is also valid in this example as \(X(t) - EX(t) = -(X'(t) - EX'(t))\) where \(X'(t) := 1 - X(t)\) is a regenerative ON/OFF process with ON and OFF distributions exchanged and hence satisfying (5.59), since the corresponding \(W'_Z \overset{d}{=} Z^{off}\) has exponential distribution. Again, this result is part of [2] dealing with the general ON/OFF processes under the intermediate scaling.

**Example 10.** M/G/1/\(\infty\) queue. The queueing system with Poisson arrivals and general service times with single server and infinite waiting room. Similarly as in the previous example, the number \(X(t) \in \{0, 1\}\) of customers at time \(t\) in this system can be regarded as a regenerative ON/OFF process which starts anew at the every moment \(T_j\) when the new arrival finds the system empty. The first question is whether in this model the regeneration intervals \(Z_j = T_{j+1} - T_j\) can have a heavy-tailed distribution with parameter \(\alpha \in (1, 2)\). This problem is classical in the queueing theory, see [1, 2, 11, 14] and the references therein.

To be more explicit, let \(\tau_i > 0, i = 1, 2, \ldots\) (inter-arrival times) be standard exponential i.i.d. r.v.s with \(E\tau_i = 1\), and \(\sigma_i > 0, i = 1, 2, \ldots\) (service times) be i.i.d. r.v.s independent of \(\{\tau_i\}\) with mean \(E\sigma := E\sigma_i < 1\) and tail d.f.

\[
P(\sigma_i > x) \sim c_{\sigma} x^{-\alpha}, \quad x \to \infty, \quad (\exists c_{\sigma} > 0, \ \alpha < 2).
\]  

(5.58)

The generic distribution distribution of the busy period \(Z^{on}\) in M/G/1/\(\infty\) queue satisfies

\[
Z^{on} = \sigma_1 + \sigma_2 (\tau_1 < \sigma_1) + \sigma_3 (\tau_1 < \sigma_1, \tau_1 + \tau_2 < \sigma_1 + \sigma_2) + \cdots.
\]

The last series converges a.s. and \(\mu_{on} := EZ^{on} = E\sigma E\tau_i^S < \infty\) where \(S_n := \sum_{i=1}^n (\sigma_i - \tau_i), n \geq 1\) is the associated random walk with negative drift and \(\tau_0^S \geq 1\) is the first time it hits \((-\infty, 0]\), see [14]. It was proved in [14] that under the above assumptions

\[
P(Z^{on} > x) \sim c_{on} x^{-\alpha}, \quad x \to \infty, \quad \text{where} \quad c_{on} := \frac{c_{\sigma}}{(1 - E\sigma)^{1+\alpha}}.
\]  

(5.59)
From (5.59) we obtain exactly the same conclusions for the aggregated RF \( A_{\lambda, \gamma}(x, y), (x, y) \in \mathbb{R}^2 \) from M/G/1/\( \infty \) inputs as in Example 9 above.

**Example 11.** Renewal-reward process (Example 8). Gaussian and stable convergences of the aggregate \( A(T, M) \) in (1.1) for discrete-time renewal-reward inputs were investigated in [30], with particular emphasis on infinite variance reward case. The above-mentioned work additionally assumes that \( W \) and \( Z \) are independent and \( EW = 0 \) when \( E|W| < \infty \). In contrast, the results of the present paper are applicable to renewal-reward model with bounded \( |W| < C \) but the independence of \( W \) and \( Z \) is not required (more precisely, it is replaced by some sort of asymptotic independence as \( Z \to \infty \); see below).

To be more specific, let \( \{X(t), t \in \mathbb{R}\} \) be a stationary regenerative process in (5.2) with pulse process \( W(t) = W_1(t < Z) \), where \( Z > 0 \) is regularly varying as in (5.10) and satisfying (5.12), and \( |W| \leq C \) is a bounded r.v. and \( (Z, W) \) possibly dependent. In order to apply Proposition 6, we assume that

\[
G(t) := E[W_1(t < Z)] \geq 0 \quad (\forall t \geq 0) \quad \text{and} \quad G(t) \text{ is monotone non-increasing on } [0, \infty).
\]

(5.60)

Then \( G^0(t) = G^1(t) = G(t) \leq CP(Z > t) = O(t^{-\alpha}) \). Also note \( \int_0^\infty G(t)dt = E[WZ], \ R(t) = \mu^{-1}E[W^2(Z - t)_+] = O(t^{-(\alpha - 1)}) \). We also assume the existence of limits of conditional moments:

\[
\lim_{x \to \infty} E[W^i|Z = x] = \omega_i \geq 0, \quad i = 1, 2.
\]

(5.61)

Note (5.61) and (5.10) imply

\[
G(t) \sim \omega_1 P(Z > t) \sim \omega_1 c_Z t^{-\alpha},
\]

(5.62)

\[
R(t) \sim \omega_2 \mu^{-1}E[(Z - t)_+] \sim \left(\frac{c_Z \omega_2}{\alpha \mu}\right) t^{-(\alpha - 1)}, \quad t \to \infty
\]

In turn, (5.62) and the DCT imply

\[
z(t) = \int_0^t G(s)G(t - s)ds \sim c^*_z t^{-\alpha}, \quad t \to \infty, \quad c^*_z := 2c_Z\omega_1 E[WZ].
\]

Moreover, \( z(t) \) is nonnegative, continuous and has bounded variation on \([0, \infty)\) according to Remark 4. Hence, \( z(t) \) satisfies the conditions of Proposition 6 according to which

\[
h(t) \sim \frac{1}{(\alpha - 1)\mu^2} \left(\frac{c_Z \mu}{\mu} - c^*_z\right) t^{-(\alpha - 1)}, \quad t \to \infty, \quad m = (E[WZ])^2.
\]

implying

\[
\text{Cov}(X(0), X(t)) \sim c_X t^{-(\alpha - 1)}, \quad t \to \infty, \quad c_X := \frac{c_Z \omega_2}{\alpha \mu} + \frac{1}{(\alpha - 1)\mu^2} \left(\frac{c_Z \mu}{\mu} - c^*_z\right).
\]

Next, consider the tail behavior of \( W_Z = WZ \). We use the following generalization of Breiman’s lemma for products of dependent r.v.s. from [16] Lemma 4.1].

**Proposition 8.** Let \( Z > 0 \) satisfy (5.10) and \( |W| \leq C \) be bounded. Moreover, assume that

\[
\sup_{\substack{y \in \mathbb{R}}} |P\{W \leq y | Z = x\} - P\{W^\infty \leq y\}| \to 0, \quad x \to \infty.
\]

(5.63)
Then, as $x \to \infty$,

$$
P(WZ > x) = (c_Z E(W^\infty_+ + o(1)) x^{-\alpha}, \quad P(WZ \leq -x) = (c_Z E(W^\infty_- + o(1)) x^{-\alpha}. \quad (5.64)
$$

Particularly, the distribution of $\tilde{W}Z = (W - (\mu_W/\mu))Z$, where $\mu_W = EWZ, \mu = EZ$, satisfies (5.22) with $c_+ = c_Z E(W^\infty_+ - (\mu_W/\mu))_+$, $c_- = c_Z E(W^\infty_- - (\mu_W/\mu))_-.$

**Corollary 1.** Let $X = \{X(t), t \in \mathbb{R}\}$ be a stationary renewal-reward process with inter-renewal distribution $Z > 0$ regularly varying with exponent $\alpha \in (1, 2)$ as in (5.10) and bounded reward variable $|W| \leq K$.

(i) Assume (5.12), (5.60) and (5.61). Then $X$ satisfies the conditions of Theorem 4 and the convergence in (5.20) towards a (multiple) of FBS $B_{H,1/2}$, $H = (3 - \alpha)/2$ holds for any $\gamma > \alpha - 1$;

(ii) Assume (5.63). Then $X$ satisfies the conditions of Theorem 5 and the convergence in (5.23) towards $\alpha$-stable Lévy sheet $L_\alpha$, defined using (3.6), (3.7) with $c_+, c_-$ in Proposition 8 holds for any $0 < \gamma < \alpha - 1$;

(iii) Assume

$$
E[|W||Z = y] \leq Cy^{\delta}, \quad y > 0 \quad (\exists \delta > 0).
$$

Then $X$ satisfies the conditions of Theorem 6 and the convergence in (5.51) towards a (multiple) of the Telecom RF $J$ holds for $\gamma_0 = \alpha - 1$.

**Proof.** Verification of conditions of Theorems 4 and 5 (parts (i) and (ii) of this corollary) was discussed before its formulation. It remains to check conditions (5.50) of Theorem 6. Using (5.10), (5.65) and $|W| \leq K$ we obtain

$$
P(|W|Z > x) \leq \int_{x/K}^{\infty} P(|W| > x/y|Z = y)dP(Z \leq y) \leq x^{-1} \int_{x/K}^{\infty} yE[|W||Z = y]dP(Z \leq y)
$$

$$
\leq Cx^{-1} \int_{x/K}^{\infty} y^{1-\delta}dP(Z \leq y) = Cx^{-1}E[Z^{1-\delta}1(Z > x/K)] \leq Cx^{-\alpha-\delta},
$$

proving the first relation in (5.50) and the second one follows similarly. \[ \square \]

Let us note that for independent $W$ and $Z$, conditions (5.61) and (5.63) of Corollary 1 (i)-(ii) are satisfied with $\omega_i = EW^i, i = 1, 2, W^\infty \overset{d}{=} W$. These conditions are also satisfied for many dependent pairs $(W, Z)$, e.g. $W = g(Z, U)$, where $U$ is a r.v. independent of $Z$ and $g$ is a bounded function having $\lim_{z \to \infty} g(z, u) =: g_\infty(u)$, in which case $\omega_i = \mathbb{E}g_\infty(U), i = 1, 2, W^\infty \overset{d}{=} g_\infty(U)$. Particularly, if $g_\infty \equiv 0$ then $W^\infty = 0$ and $P(|W|Z > x) = P(|W|Z > x) = o(x^{-\alpha}) \quad (x \to \infty); \text{ however the tail d.f. of } \tilde{W}Z \text{ satisfies (5.22) with } c_+ = c_Z E(WZ/\mu)_+, \quad c_- = 0 \text{ if } EWZ < 0, \text{ and } c_+ = 0, \quad c_- = c_Z EWZ/\mu \text{ otherwise. On the other hand, conditions (5.50) and/or (5.65) are not satisfied for independent } W \text{ and } Z, \text{ suggesting that in this case the intermediate limit should be different from the Telecom RF.}

**Example 12.** (A ‘regenerative version’ of shot-noise in Example 4.) Let $\{X(t)\}$ be a stationary regenerative process corresponding to $Z = R, W(t)1(t < Z) = e^{-At}1(t < R)$, where r.v.s $A > 0$ and $R > 0$ are independent and distributed as in (4.9). The following proposition shows that the above ‘regenerative version’ behaves
differently from the ‘shot-noise version’ of Example 4, in the sense of scaling limits of the aggregated inputs.

Particularly, the critical point $\gamma_0$ separating fast and slow growth regimes, as well as the parameters $H, \alpha$ of the limit distributions in (1.3) are different in Examples 4 and 12 under the same distributional assumptions on $(A,R)$.

**Proposition 9.** Let $\{X(t)\}$ be a stationary regenerative process with $Z = R, W(t) = e^{-A(t)} 1(t < R)$, where $A > 0$ and $R > 0$ are independent r.v.s satisfying (4.10). Moreover, in case (i) we assume (5.12). Let

\[ 1 < \rho < \rho + \kappa < 2. \]  

(5.66)

Then:

(i) conditions (5.18) and (5.19) of Theorem 4 hold for any $\gamma > \rho - 1$ and hence

\[ \lambda^{-(3-\rho+\gamma)/2} (A_{\lambda,\gamma}(x,y) - \mathbb{E}A_{\lambda,\gamma}(x,y)) \stackrel{\text{fdd}}{\rightarrow} C_W B_{H,1/2}(x,y), \quad \forall \gamma > \rho - 1, \]  

(5.67)

where $H = (3 - \rho)/2$ and $C_W^2 = \frac{\ell X}{(2\pi)^{1/2} \Gamma(H)}$, $c_X$ given in (5.71) below.

(ii) conditions (5.10), (5.19) and (5.22) of Theorem 5 with $\alpha = \rho$ hold for any $0 < \gamma < \rho - 1$ and hence

\[ \lambda^{-(1+\gamma)/\rho} (A_{\lambda,\gamma}(x,y) - \mathbb{E}A_{\lambda,\gamma}(x,y)) \stackrel{\text{fdd}}{\rightarrow} L_\rho(x,y) \quad \forall \gamma < \rho - 1, \]  

(5.68)

where $L_\rho$ is a completely asymmetric $\rho$-stable Lévy sheet, defined using (3.6), (3.7) with parameters $c_+ = 0, c_- > 0$ in (5.72) below.

(iii) conditions (5.10), (5.19) and (5.50) of Theorem 6 hold with $\alpha = \rho$ for $\gamma_0 = \rho - 1$ and hence

\[ \lambda^{-1} (A_{\lambda,\gamma_0}(x,y) - \mathbb{E}A_{\lambda,\gamma_0}(x,y)) \stackrel{\text{fdd}}{\rightarrow} - (\mu Y/\mu) J(x,y), \]  

(5.69)

where $J = \{(J(x,y), (x,y)) \in \mathbb{R}^2_+ \}$ is the Telecom RF in (2.4) - (2.8).

**Proof.** (i) Consider (5.18) ((5.19) is obvious). Recall (5.6) and (5.9). Here, $R(t) = \mu^{-1} \int_0^\infty \mathbb{E}[W(s)W(s+t)1(Z > s + t)] ds = \mu^{-1} \int_0^\infty P(R > s + t) e^{-A(2s+t)} ds = O(t^{-((\rho+\kappa)-1)}) = o(t^{-((\rho-1)-1)}) (t \to \infty)$ follows from Proposition 4 (i). We also see that $G_1(t) = \mathbb{E}[e^{-A(t)} P(R > t)]$ is monotone and has bounded variation on $[0,\infty)$. Let us check that

\[ G_1(t) \sim c_\rho c_\alpha \Gamma(\kappa + 1)t^{-\rho-\kappa} \quad \text{and} \quad G^0(t) = o(t^{-\rho}) \quad (t \to \infty). \]  

(5.70)

The first relation in (5.70) is easy from (4.9). To show the second relation in (5.70), split $G^0(t) = \mathbb{E}[e^{-A(t)} 1(t < R)] = \mathbb{E}[e^{-A(R-t)} 1(t < R, A > \epsilon)] + \mathbb{E}[e^{-A(R-t)} 1(t < R, A \leq \epsilon)] =: G'_\epsilon(t) + G''_\epsilon(t)$, where $t^\rho G''_\epsilon(t) \leq P(A \leq \epsilon) t^\rho P(R > t)$ can be made arbitrary small uniformly in $t \geq 1$ by an appropriate choice of $\epsilon > 0$. On the other hand, integrating by parts we see that for any $\epsilon > 0$

\[ t^\rho G'_\epsilon(t) \leq t^\rho \mathbb{E}[e^{-\epsilon(R-t)} 1(t < R)] \]

\[ = t^\rho P(R > t) - \epsilon \int_0^\infty \frac{t^\rho}{(t+z)^\rho} t^\rho P(R > t + z) e^{-\epsilon z} dz \]

\[ \to c_\rho - c_\rho \epsilon \int_0^\infty e^{-\epsilon z} dz = 0 \]
as \( t \to \infty \) according to (4.9) and the DCT, proving (5.70). Therefore, \( 0 \leq z(t) \leq G^1(t/2) \int_0^{t/2} G^0(s)ds + C t^{1/2} G^0(s)(1 \vee (t-s))^{-\rho-\nu}ds = o(t^{-\rho}). \) Whence and by Remark 4, we see that \( z(t) \) in this example satisfies all conditions of Proposition 5 with \( \alpha = \rho \) and part (ii) of the latter proposition applies, yielding

\[
\text{Cov}(X(0), X(t)) \sim c_X t^{-(\rho-1)} \text{, } t \to \infty \text{ as in (5.17) or (5.18), with}
\]

\[
c_X = \frac{czm}{(\rho-1)\mu^2}, \quad m := \int_0^{\infty} G^0(s)ds \int_0^{\infty} G^1(t)dt = (E[(1 - e^{-AR})/A])^2.
\] (5.71)

This proves part (i).

(ii) From Proposition 4(ii) we have that \( \mathcal{W}_Z = \int_0^{R} e^{-A t}dt \) satisfies \( P(\mathcal{W}_Z > x) \sim c_W x^{-\rho-\kappa} = o(x^{-\rho}), \text{ } x \to \infty. \)

Then, the tail behavior of \( \tilde{\mathcal{W}}_Z \) in (5.22) is determined by \( -(\mu_W/\mu)R, \text{ viz.,} \)

\[
P(\tilde{\mathcal{W}}_Z \leq -x) \sim P(R > (\mu/\mu_W)x) \sim c_\rho(\mu_W/\mu)^\rho x^{-\rho}, \quad P(\tilde{\mathcal{W}}_Z > x) = o(x^{-\rho}), \quad x \to \infty.
\]

Hence, \( \tilde{\mathcal{W}}_Z \) satisfies (5.22) with

\[
c_\rho = 0, \quad c_- = c_\rho(\mu_W/\mu)^\rho = c_\rho\frac{E[(1 - e^{-AR})/A]}{ER} \rho.
\] (5.72)

(iii) Relations (5.10), (5.19) obviously hold. The first relation in (5.70) follows from the proof of part (ii) since \( \kappa > 0. \) The second one follows from (5.70) since \( E[|W(t)| 1(t < Z)] = G^1(t). \)

\[\Box\]

6 The Telecom process and large deviations

As noted above, the proof of the intermediate limit in (7) and in Theorem 6 relies on (9) where this limit is established for stationary renewal process \( N(x) \) with regularly varying inter-renewal distribution. More precisely, (9) proved

\[
\lambda^{-1} \sum_{j=1}^{\lfloor \lambda^{-1} \rfloor} (N_j(\lambda x) - E N_j(\lambda x)) \xrightarrow{fd} -\mu^{-1} J(x),
\] (6.1)

where \( \{N_j(x), x > 0\}, j = 1, 2, \ldots \) are independent copies of \( N = \{N(x), x > 0\}, \) where \( N(x) := \sum_{n=0}^{\infty} 1(T_n \in (0, x]) \) and \( 0 < T_0 < T_1 < \ldots \) is a stationary renewal process as in (5.2) and elsewhere in this paper, with generic duration distribution \( Z \) satisfying (5.10). The limit (Telecom) process in (6.1) is defined as

\[
J(x) = \int_{\mathbb{R} \times \mathbb{R}_+} \{ \int_0^{x} 1(u < s < u + r)ds \} q(du, dr), \quad x > 0
\] (6.2)

where \( q(du, dr) \) is centered Poisson random measure on \( \mathbb{R} \times \mathbb{R}_+ \) with control measure \( \alpha c_Z \mu^{-1} x^{-\alpha} \text{d}udr, \alpha \in (1, 2), c_Z, \mu > 0 \) so that \( \{J(x), x > 0\} \xrightarrow{fd} \{J(x, 1), x > 0\} \) agrees with the Telecom RF in (2.7) with \( \rho, c_\rho \) replaced by \( \alpha, c_Z \mu^{-1}. \)

The proof of (6.1) in (9) using moment argument is quite involved. The aim of this section is to give a different and possibly simpler proof of one-dimensional convergence in (6.1), viz.,

\[
\lambda^{-1} \sum_{j=1}^{\lfloor \lambda^{-1} \rfloor} (N_j(\lambda x) - E N_j(\lambda x)) \xrightarrow{d} -\mu^{-1} J(x) \quad (\forall x > 0)
\] (6.3)
using large deviations for sums of heavy-tailed i.i.d. r.v.s. (however, extending it to finite-dimensional convergence in (6.1) does not seem easy). The basic large deviation result - Theorem 7 - has long history and has been attributed to several authors, see [22]. For our purposes, the formulation of Theorem 7 following [22, Thm. 1.1] is most convenient.

**Theorem 7.** Let \( S_n := \sum_{i=1}^n Z_i, n \geq 1, S_0 := 0, \) where \( Z, Z_i > 0, i = 1, 2, \ldots \) are i.i.d. r.v.s with d.f. as in (5.10), \( 1 < \alpha < 2. \) Let \( b_n := n^{\frac{1}{\alpha} + \delta} (\delta > 0). \) Then

\[
\lim_{n \to \infty} \sup_{x > b_n} \left| \frac{P(S_n - E S_n > x)}{nP(Z > x)} - 1 \right| = 0, \quad \lim_{n \to \infty} \sup_{x > b_n} \frac{P(S_n - E S_n < -x)}{nP(Z > x)} = 0. \tag{6.4}
\]

Let \( P_0 \) denote the distribution of the pure renewal process \( N(t) = \sum_{n=0}^\infty 1(\sum_{k=1}^n Z_k \leq t), t \geq 0 \) starting at \( T_0 = 0, \) with inter-renewal time \( Z \) as in (5.10).

**Corollary 2.** Let \( Z \) be as in (5.10). Then for any \( \kappa \in (\frac{1}{\alpha}, 1) \) with \( \bar{b}_\lambda := \lambda^\kappa, \, \bar{b}_\lambda := \lambda - \lambda^{2 - \alpha} \) as \( \lambda \to \infty \)

\[
\sup_{u \geq \bar{b}_\lambda} \left( \frac{u^\alpha}{\lambda} \right) P_0 (N(\lambda) - \lambda/\mu > u/\mu) \to 0, \tag{6.5}
\]

\[
\sup_{u \in (\bar{b}_\lambda, \bar{b}_\lambda)} \left( \frac{u^\alpha}{\lambda} \right) P_0 \left( N(\lambda) - \lambda/\mu \leq -u/\mu \right) - ((\lambda - u)/\lambda)(cz/\mu) \to 0. \tag{6.6}
\]

**Proof.** Under \( P_0, \) \( \{N(\lambda) > j\} = \{S_j \leq \lambda, j \geq 0 \} \) and \( \{N(\lambda) \leq x\} = \{S_x > \lambda\}, x > 0. \) Thus, \( P_0(N(\lambda) \leq (\lambda - u)/\mu) = P(S_{[(\lambda - u)/\mu]} - ES_{[(\lambda - u)/\mu]} > u^*_\lambda), \) where

\[
u^*_\lambda := \lambda - ((\lambda - u)/\mu), \quad 0 < u < \lambda.
\]

Then, \( u \leq u^*_\lambda \leq u + \mu \) and \( u^*_\lambda \geq [(\lambda - u)/\mu]^{\frac{1}{\alpha} + \delta} \) hold for any \( u \in (\bar{b}_\lambda, \bar{b}_\lambda), \) large enough \( \lambda \geq 1, \, \kappa \in (\frac{1}{\alpha}, 1) \) and some small enough \( \delta > 0. \) Indeed, there exists such a \( \delta > 0 \) that \( \lambda - (\lambda - u) \geq ((\lambda - u)/\mu)^{\frac{1}{\alpha} + \delta} \) holds uniformly on \( (\bar{b}_\lambda, \lambda) \) for all large enough \( \lambda \geq 1, \) moreover, \( \lambda - u \) uniformly on \( (0, \bar{b}_\lambda) \) tends to infinity as \( \lambda \to \infty. \) Whence, using (5.10) and the first relation in (6.4), we get

\[
P_0(N(\lambda) \leq (\lambda - u)/\mu) = [(\lambda - u)/\mu]P(Z > u^*_\lambda)(1 + o(1)) = ((\lambda - u)/\mu)(cz/\mu)(1 + o(1))
\]

uniformly on \( (\bar{b}_\lambda, \bar{b}_\lambda). \) This proves (6.6).

The proof of (6.5) follows similarly. We have

\[
P_0(N(\lambda) > (\lambda + u)/\mu) = P_0(N(\lambda) > (\lambda + u)/\mu)) = P(S_{[(\lambda + u)/\mu]} - ES_{[(\lambda + u)/\mu]} \leq -u^{**}_\lambda)
\]

where \( u^{**}_\lambda := -\lambda + [(\lambda + u)/\mu]. \) Note \( u - \mu \leq u^{**}_\lambda \leq u \) and \( u^{**}_\lambda \geq [(\lambda + u)/\mu]^{\frac{1}{\alpha} + \delta} \) hold for any \( u > \lambda^\kappa, \lambda \geq 1, \, \frac{1}{\alpha} < \kappa < 1 \) and some \( \delta > 0 \) small enough. Hence,

\[
(u^\alpha/(\lambda + u))P_0(N(\lambda) > (\lambda + u)/\mu) = (u^\alpha/(\lambda + u))((\lambda + u)/\mu)P(Z > u^{**}_\lambda) o(1) = o(1). \tag{6.7}
\]

**Proof of (6.5) using Corollary 2.** We use characteristic functions. By independence of the summands in (6.1), it suffices to prove

\[
\lambda^{\alpha - 1} \log P e^{i 0 \lambda^{\alpha - 1}(N(\lambda x) - EN(\lambda x))} \to \log P e^{-i 0 \mu^{-1} J(x)}, \quad \lambda \to \infty, \quad \forall \theta \in \mathbb{R}. \tag{6.7}
\]
Using \( \log(1 + x) \sim x, x \to 0 \) and notation \( \Psi(z) := e^{iz} - 1 - iz, \ z \in \mathbb{R} \), relation (6.7) follows from
\[
\lambda^\alpha - 1 E\Psi(\theta \lambda^{-1}(N(\lambda x) - EN(\lambda x))) \rightarrow \log Ee^{-i\theta \mu^{-1}J_-(x)} + \log Ee^{-i\theta \mu^{-1}J_+(x)}, \tag{6.8}
\]
where \( J_-(x) := \int_{(-\infty, 0) \times \mathbb{R}^+} \ldots J_+(x) := \int_{(0, x) \times \mathbb{R}^+} \ldots \) are integrals taken over ‘past’ and ‘present’ subregions in \( J_-(x) \), \( J_+(x) \) independent of \( J_+(x) \). Then
\[
\log Ee^{-i\theta \mu^{-1}J_-(x)} = c_{Z\mu^{-1}} \int_{-\infty}^{0} du \int_{0}^{\infty} \frac{adr}{r^{1+\alpha}} \Psi(-\theta \mu^{-1} \int_{0}^{\infty} t < u + r) ds
\]
\[
= c_{Z\mu^{-1}} \int_{0}^{\infty} du \int_{0}^{\infty} \frac{adr}{r^{1+\alpha}} \Psi(-\theta \mu^{-1}(x \wedge u)) du = c_{Z\mu^{-1}} \int_{0}^{\infty} \Psi(-\theta \mu^{-1}(x \wedge u)) r^{-\alpha} dr,
\]
\[
\log Ee^{-i\theta \mu^{-1}}J_+(x) = c_{Z\mu^{-1}} \int_{0}^{\infty} du \int_{0}^{\infty} \frac{adr}{r^{1+\alpha}} \Psi(-\theta \mu^{-1}(x \wedge u)) du = c_{Z\mu^{-1}} \int_{0}^{\infty} \Psi(-\theta \mu^{-1}(r \wedge u)) du = -i\theta c_{Z\mu^{-1}} \int_{0}^{\infty} (e^{-i\theta \mu^{-1}r} - 1)(x - r) r^{-\alpha} dr.
\]
For \( \epsilon_\lambda := \lambda^{1-\alpha} \), split \( N(\lambda x) = N(\lambda x)1(T_0 \leq \lambda \epsilon_\lambda) + N(\lambda x)1(T_0 > \lambda \epsilon_\lambda) \). Let us prove that for any \( \theta \in \mathbb{R} \), as \( \lambda \to \infty \),
\[
\lambda^{\alpha - 1} E\Psi(\theta \lambda^{-1}(N(\lambda x) - EN(\lambda x)))1(T_0 > \lambda \epsilon_\lambda) \to \log Ee^{-i\theta \mu^{-1}J_-(x)}.
\tag{6.10}
\]
Recall \( E\Psi(\theta \lambda^{-1}(N(\lambda x) - EN(\lambda x)))1(T_0 > \lambda \epsilon_\lambda) \to \log Ee^{-i\theta \mu^{-1}J_-(x)} \).
\[
\lambda^{\alpha - 1} E\Psi(\theta \lambda^{-1}(N(\lambda x) - EN(\lambda x)))1(T_0 > \lambda \epsilon_\lambda) \to \log Ee^{-i\theta \mu^{-1}J_-(x)}.
\tag{6.10}
\]
Next, consider the case \( T_0 \leq \lambda \epsilon_\lambda \). We shall prove a similar fact to (6.10), viz.,
\[
I_+(\lambda) := \lambda^{\alpha - 1} E\Psi(\theta \lambda^{-1}(N(\lambda x) - EN(\lambda x)))1(T_0 \leq \lambda \epsilon_\lambda) \to \log Ee^{-i\theta \mu^{-1}J_+(x)} \tag{6.13}
\]
for any \( \theta \in \mathbb{R} \). The proof of (6.13) is more involved than (6.10) and uses Corollary 2 (which was not used in (6.10)). Similarly to (6.11) write the l.h.s. of (6.13) as
\[
I_+(\lambda) = \lambda^{\alpha - 1} \mu^{-1} \int_{0}^{\lambda \epsilon_\lambda} P(Z > t) E_0[\Psi(\theta \lambda^{-1}(\lambda x - t) - \theta \mu^{-1} x)] dt. \tag{6.14}
\]
Let \( \psi_\lambda(\theta; x, t) := \lambda^{\alpha-1} E_0 \Psi(\theta \lambda^{-1} N(\lambda x - t) - \theta \mu^{-1} x) \). Below we prove that
\[
\lim_{\lambda \to \infty} \psi_\lambda(\theta; x, t) = \log E e^{-i\theta \mu^{-1} J_+(x)}, \quad \forall \theta \in \mathbb{R}, \ t \geq 0, \ x > 0.
\] (6.15)

We restrict the proof of (6.15) to that of \( \psi_\lambda(\theta) := \psi_\lambda(\theta; 1, 0) \) and assume \( \mu = c_Z = 1 \) for brevity of notation.

For \( 1 > \kappa > 1/\alpha \) split \( \psi_\lambda(\theta) = \sum_{i=0}^3 \psi_\lambda^{(i)}(\theta), \) where
\[
\psi_\lambda^{(0)}(\theta) := \lambda^{\alpha-1} E_0 \Psi(\theta \lambda^{-1}(N(\lambda) - \lambda))1(\lambda \epsilon_\lambda < N(\lambda) \leq \lambda - \lambda^\kappa),
\psi_\lambda^{(1)}(\theta) := \lambda^{\alpha-1} E_0 \Psi(\theta \lambda^{-1}(N(\lambda) - \lambda))1(N(\lambda) \leq \lambda \epsilon_\lambda),
\psi_\lambda^{(2)}(\theta) := \lambda^{\alpha-1} E_0 \Psi(\theta \lambda^{-1}(N(\lambda) - \lambda))1(N(\lambda) \geq \lambda + \lambda^\kappa),
\psi_\lambda^{(3)}(\theta) := \lambda^{\alpha-1} E_0 \Psi(\theta \lambda^{-1}(N(\lambda) - \lambda))1(|N(\lambda) - \lambda| < \lambda^\kappa).
\]

(6.15) follows from
\[
\psi_\lambda^{(0)}(\theta) \to \log E e^{-i\theta J_+(x)}, \quad |\psi_\lambda^{(i)}(\theta)| \to 0, \ i = 1, 2, 3, \ (\lambda \to \infty) \quad (6.16)
\]

Relation (6.16) for \( i = 3 \) is immediate by \( |\Psi(z)| \leq |z|^2/2, \) \( z \in \mathbb{R}. \) Indeed, since \( 3 - (2/\alpha) - \alpha = (\alpha - 1)(2 - \alpha)/\alpha > 0 \) so \( |\psi_\lambda^{(3)}(\theta)| \leq C_\lambda^{\alpha-1}\lambda^{2\kappa-1} = o(1) \) provided \( \kappa - (1/\alpha) > 0 \) is small enough.

Next, consider (6.16) for \( i = 2. \) Write \( \tilde{F}_\lambda(u) := P_0(N(\lambda) - \lambda > u), \) \( u \in \mathbb{R}. \) Fix a large \( K > 1. \) Then integrating by parts and using \( |\Psi(z)| \leq (2|z|) \wedge (|z|^2/2), \) \( z \in \mathbb{R}. \) we get
\[
|\psi_\lambda^{(2)}(\theta)| \leq -C\lambda^{\alpha-1} \int_{\lambda^\kappa}^{\infty} \left( \frac{u}{\lambda} \wedge \left( \frac{u}{\lambda} \right)^2 \right) d\tilde{F}_\lambda(u)
\leq C\lambda^{\alpha-1} \left\{ \frac{\lambda^{2\kappa}}{\lambda^2} \tilde{F}_\lambda(\lambda^\kappa) + \lambda^{-2} \int_{\lambda^\kappa}^{K\lambda} u \tilde{F}_\lambda(u) du + \lambda^{-1} \int_{K\lambda}^{\infty} \tilde{F}_\lambda(u) du \right\}. \quad (6.17)
\]

According to (6.5), \( \tilde{F}_\lambda(\lambda^\kappa) = o(\lambda^{-1-\kappa}) \) implying \( \lambda^{\alpha+2\kappa-3} \tilde{F}_\lambda(\lambda^\kappa) \leq C\lambda^\alpha \lambda^{2\kappa-1} = o(1) \) for \( \kappa < 1. \) The same relation, viz., \( \sup_{\lambda^\kappa < u < K\lambda} (u^\alpha/(\lambda + u)) \tilde{F}_\lambda(u) = o(1), \) also implies that \( \lambda^{-2} \int_{K\lambda}^{\infty} u \tilde{F}_\lambda(u) du \to o(1) \) for any \( K > 1 \) fixed. To evaluate the third term on the r.h.s. of (6.17), use the Chebyshev inequality and the fact that \( E_0(N(\lambda) - \lambda)^2 \leq C\lambda^{2-\alpha}, \lambda \geq 1, \) see e.g. [9] (28), (26)]. Then \( \lambda^{-1} \int_{K\lambda}^{\infty} \tilde{F}_\lambda(u) du \leq C\lambda^{2-\alpha} \int_{K\lambda}^{\infty} u^{-2} du \leq CK^{-1}\lambda^{1-\alpha} \) and hence the corresponding term in (6.17) can be made arbitrarily small uniformly in \( \lambda \geq 1 \) by choosing \( K > 1 \) sufficiently large, proving (6.16) for \( i = 2. \)

Consider (6.16) for \( i = 1. \) We have \( |\Psi(\theta \lambda^{-1}(N(\lambda) - \lambda))1(N(\lambda) \leq \lambda \epsilon_\lambda)| \leq C \mathbf{1}(N(\lambda) \leq \lambda \epsilon_\lambda). \) Hence using (6.6) with \( u = \tilde{\theta}_\lambda = \lambda(1 - \epsilon_\lambda) \) we get \( |\psi_\lambda^{(1)}(\theta)| \leq C\lambda^{\alpha-1} P_0(N(\lambda) \leq \lambda \epsilon_\lambda) \leq C\lambda\alpha-1(\lambda \epsilon_\lambda)(\lambda(1 - \epsilon_\lambda))^{-\alpha} \leq C\epsilon_\lambda = o(1), \) proving (6.16) for \( \psi_\lambda^{(1)}(\theta). \)

Finally, consider \( \psi_\lambda^{(0)}(\theta) \) (the main term). We have
\[
\psi_\lambda^{(0)}(\theta) = \lambda^{\alpha-1} \sum_{j=[\lambda \epsilon_\lambda] + 1}^{[\lambda - \lambda^\kappa]} \Psi(\theta \left( \frac{j}{\lambda} - 1 \right)) P_0(N(\lambda) = j)
\leq \Psi(\theta \left( \frac{\lambda - \lambda^\kappa}{\lambda} - 1 \right)) \lambda^{\alpha-1} P_0(N(\lambda) \leq \lambda - \lambda^\kappa) - \Psi(\theta \left( \frac{\lambda \epsilon_\lambda + 1}{\lambda} - 1 \right)) \lambda^{\alpha-1} P_0(N(\lambda) \leq \lambda \epsilon_\lambda)
+ \sum_{j=[\lambda \epsilon_\lambda] + 1}^{[\lambda - \lambda^\kappa] - 1} \left\{ \Psi(\theta \left( \frac{j}{\lambda} - 1 \right)) - \Psi(\theta \left( \frac{j+1}{\lambda} - 1 \right)) \right\} \lambda^{\alpha-1} P_0(N(\lambda) \leq j). \quad (6.18)
\]
The first term on the r.h.s. of (6.18) vanishes with \( \lambda \to \infty \) in the same way as \( \psi_\lambda^{(3)}(\theta) \) does above. The second term on the r.h.s. of (6.18) is bounded by \( C\epsilon \lambda^\alpha \) as in the case of \( \psi_\lambda^{(1)}(\theta) \). Consider the last term – the sum over \( [\lambda \epsilon \lambda] < j < \lfloor \lambda - \lambda \epsilon \rfloor \) – on the r.h.s. of (6.18), denoted by \( \tilde{\psi}_\lambda^{(0)}(\theta) \). Using

\[
\Psi(\theta(\frac{1}{\lambda} - 1)) - \Psi(\theta(\frac{1}{\lambda} + 1)) = e^{i\theta(\frac{1}{\lambda} - 1)}(1 - e^{i\theta}) + i\theta \quad \text{and (6.6) (with } c_Z = \mu = 1 \text{)} \text{ we get that}
\]

\[
\tilde{\psi}_\lambda^{(0)}(\theta) = \sum_{j = [\lambda \epsilon \lambda] + 1}^{[\lambda - \lambda \epsilon \lambda] - 1} \left( e^{i\theta(\frac{1}{\lambda} - 1)} \lambda(1 - e^{i\theta}) + i\theta \right) \lambda^{\alpha - 2} \mathbb{P}(N(\lambda) \leq j)
\]

\[
= -i\theta \sum_{j = [\lambda \epsilon \lambda] + 1}^{[\lambda - \lambda \epsilon \lambda] - 1} \left( e^{i\theta(\frac{1}{\lambda} - 1)} - 1 \right) \lambda^{\alpha - 2} j \left( \lambda - j \right)^{\alpha - 1} (1 + o(1))
\]

\[
= -i\theta \sum_{j = [\lambda \epsilon \lambda] + 1}^{\lfloor \lambda - \lambda \epsilon \lambda \rfloor - 1} \left( e^{i\theta(\frac{1}{\lambda} - 1)} - 1 \right) \left( \frac{j}{\lambda} \right) \left( 1 - \frac{j}{\lambda} \right)^{\alpha - 2} \left( \frac{1}{\lambda} \right) (1 + o(1))
\]

\[
\rightarrow -i\theta \int_0^1 \left( e^{i\theta(z - 1)} - 1 \right) z(1 - z)^{-\alpha} \, dz = -i\theta \int_0^1 (e^{-\theta r} - 1)(1 - r)r^{-\alpha} \, dr
\]

as \( \lambda \to \infty \), where the last integral agrees with \( \log \mathbb{E} e^{-\theta \mu^{-1} J_1(x)} \) in (6.9) for \( x = \mu = c_Z = 1 \). This proves the first relation in (6.16), completing the proof of (6.15) for \( t = 0 \). For \( t > 0 \), \( |\psi_\lambda(\theta; x, 0) - \psi_\lambda(\theta; x, t)| = O(\lambda^{\alpha - 2}) = o(1) \) follows using \( |\Psi(z) - \Psi(z')| \leq 2|z - z'|, \ z, z' \in \mathbb{R} \), and \( \mathbb{E}_0[N(\lambda x) - N(\lambda x - t)] \to \mu^{-1}t \) by the renewal theorem. Using \( |\Psi(z)| \leq |z|^2/2, \ z \in \mathbb{R} \), and \( \mathbb{E}_0(N(\lambda x) - (\lambda x - t))^2 \leq C\lambda^{3-\alpha} \) implies \( |\psi_\lambda(\theta; x, t)| \leq C \) for all \( 0 < t < \lambda \epsilon \lambda \), hence (6.13) follows by the DCT in view of \( \int_0^\infty \mathbb{P}(Z > t) \, dt = \mu \). This proves (6.8) and completes the proof of (6.3).

\[\square\]

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