Second Type Almost Geodesic Mappings of Special Class and Their Invariants

Nenad O. Vesić*, Mića S. Stanković*

*Department of Mathematics, Faculty of Sciences and Mathematics, Niš

Abstract. Invariants of almost geodesic mappings of a generalized Riemannian space are discussed in this paper. As a special case, invariants of equitorsion almost geodesic mappings of this type are discussed here.

1. Introduction and preliminaries

Geodesic lines and their generalizations are important for applications of differential geometry in physics. A diffeomorphism \( f : \mathbb{R}^N \to \mathbb{R}^N \) of Riemannian spaces \( \mathbb{R}^N \) and \( \mathbb{R}^N \) endowed with symmetric metric tensor \( g_{ij} \) is called the almost geodesic mapping if it maps any geodesic line of the space \( \mathbb{R}^N \) into an almost geodesic line of the space \( \mathbb{R}^N \). Sinyukov involved this concept of research for the mappings between affine connected spaces without torsion (see [16]). J. Mikeš [1, 7–9] significantly contributed to the study of geodesic and almost geodesic mappings of affine connected, Riemannian and Einstein spaces. Invariants of almost geodesic mappings of a generalized Riemannian space will be searched in this paper. The almost geodesic mappings of generalized Riemannian spaces and of spaces with non-symmetric affine connection as well are discussed in [17–20, 25, 26].

An \( N \)-dimensional manifold endowed with metric tensor \( g_{ij} \) non-symmetric in indices \( i \) and \( j \) is the generalized Riemannian space \( \text{GR}_N \) in the sense of Eisenhart definition [3–5].

Because of the non-symmetry \( g_{ij} \neq g_{ji} \), the symmetric and anti-symmetric part of metric tensor \( g \) are defined as

\[
g_{ij} = \frac{1}{2}(g_{ij} + g_{ji}) \quad \text{and} \quad g_{ij} = \frac{1}{2}(g_{ij} - g_{ji}).
\]

We assume that is \( \det[g_{ij}] \neq 0 \). Tensor \( g^{ij}_{\alpha} \) is determined by the condition \( g^{ij}_{\alpha} g_{\alpha \beta} = \delta^i_j \) where \( \delta^i_j \) is a Cronecker’s symbol. Affine connection coefficients of the space \( \text{GR}_N \) are generalized Christoffel symbols \( \Gamma^i_{jk} \) of this space defined as

\[
\Gamma^i_{jk} = \frac{1}{2} g^{\alpha i}_{\beta j} (g_{\alpha k, \beta} - g_{\alpha k, \beta} + g_{\alpha k, j}).
\]
for partial derivation $\partial f/\partial x^k$ denoted by comma. These coefficients are non-symmetric by indices $j$ and $k$. For this reason, their symmetric and anti-symmetric parts are:

$$\Gamma^i_{jk} = \frac{1}{2}(\Gamma^i_{jk} + \Gamma^i_{kj}) \quad \text{and} \quad \Gamma^i_{jk} = \frac{1}{2}(\Gamma^i_{jk} - \Gamma^i_{kj}).$$

(3)

The symmetric parts $\Gamma^i_{jk}$ are the affine connection coefficients of the associated Riemannian space $\mathbb{R}^n$ [10, 11]. The anti-symmetric part $\Gamma^i_{jk}$ of the Christoffel symbol $\Gamma^i_{jk}$ is the torsion tensor of the space $\mathbb{R}^n$. It also holds

$$\Gamma^i_{jk} = \frac{1}{2}(\ln |g|)_j \quad \text{and} \quad \Gamma^i_{aj} = 0.$$  (4)

One kind of covariant derivation with regard to the affine connection $\mathbb{R}^n$ is defined as (see [7–9, 16]). For example, for the tensor $a'_{i}$, we have

$$a'_{i} = a'_{i} + \Gamma^i_{ak}a'_{k} = \Gamma^i_{ak}a'_{k}.$$  (5)

Unlike the affine connection of a non-symmetric affine connection spaces, one may discover four kinds of affine connection of a generalized Riemannian space $\mathbb{R}^n$. With regard to these kinds of affine connection, S. M. Minčić obtained twelve curvature tensors [10, 11]

$$K^i_{jmn} = R^i_{jmn} + u\Gamma^i_{jmn} + u'\Gamma^i_{jmn} + v\Gamma^i_{jmn} + v'\Gamma^i_{jmn} + w\Gamma^i_{jmn} + w'\Gamma^i_{jmn}.$$

(6)

for real constants $u, u', v, v', w$ and

$$R^i_{jmn} = \Gamma^i_{jmn} - \Gamma^i_{jm,n} + \Gamma^i_{jn,m} - \Gamma^i_{jm,n}.$$  (7)

Many books and research papers are dedicated to the study of spaces with torsion, generalized Riemannian spaces and mappings between them [2–6, 10–15, 17–25, 27–30]. The aim of this paper is to obtain invariants of special almost geodesic mappings of the second type.

2. Invariants of second type almost geodesic mappings

A mapping $f : \mathbb{G}R_N \rightarrow \mathbb{G}R_N$ determined with the equations

$$\Gamma^i_{jk} = \Gamma^i_{jk} + \psi_j \delta^i_k + \psi_k \delta^i_j + 2F^i_j \sigma_k + 2F^i_k \sigma_j + \xi^i_{jk}$$

(8)

$$F^i_{jk} = F^i_{kj} + 2F^i \sigma_k + 2F^i_k \sigma_j + 2\xi^i_{jk} \sigma_k + 2\xi^i_{kj} \sigma_j = \mu^i \sigma_k + \mu_k ^i \sigma_j + v_j \delta^i_k + v_k \delta^i_j.$$  (9)

$p = 1, 2$, for 1-forms $\psi_j, \sigma_j, \mu_j, v_j$, affinor structure $F^i_j$, and the tensor $\xi^i_{jk}$ anti-symmetric in indices $j$ and $k$ is called the almost geodesic mapping of the second type and the $p$-th kind.

Second type almost geodesic mapping satisfies the property of reciprocity if it preserves the affinor structure $F^i_j$ and the corresponding inverse mapping $f^{-1}$ is the second type almost geodesic mapping of the $p$-th kind. This mapping satisfies the property of reciprocity if and only if $F^i_a F^a_j = \delta^i_j, e = \pm 1, 0$. These mappings are elements of the class $\pi^*_p(e)$. 
2.1. Generalized Thomas projective parameter

Let us consider an almost geodesic mapping $f : \mathcal{GR}_N \to \mathcal{GR}_N$ of a type $\pi_2(e)$ determined by the affinor $F^i_j = \frac{1}{2}g^{\alpha\beta}g_{\alpha j}^\beta$.

We have that is

$$\Gamma^i_j = F^i_j - F^i_k - \frac{1}{2}g^{\alpha\beta}g_{\beta k}^\gamma.$$ (10)

From this equation, one obtains that is

$$\tilde{\Gamma}^i_j - \Gamma^i_j = \tau^i_j - \tau^i_{jk}.$$ (11)

$p = 1, \ldots, 4$, for

$$\tau^i_j = \Gamma^i_j - \Gamma^i_k - \frac{1}{2}g^{\alpha\beta}g_{\beta k}^\gamma,$$ (12)

$$\tau^i_{jk} = \frac{1}{N+1}(\delta^i_{\lambda}(\Gamma^\alpha_{\alpha j} + (N+1)\alpha\sigma_j) - \delta^i_{\lambda}(\Gamma^\alpha_{\alpha k} + (N+1)\alpha\sigma_k))$$

$$+ (F^i_j - F^i_k)\sigma_j - \frac{1}{2}g^{\alpha\beta}g_{\beta j}^\gamma - \frac{1}{N+1}(F^i_j(\Gamma^\alpha_k + \alpha\sigma_j) - F^i_k(\Gamma^\gamma_{\gamma j} + \sigma_j)).$$ (13)

$$\Gamma^i_{jk} = \epsilon\delta^i_{\lambda}\sigma_j + \frac{1}{N+1}\delta^i_{\lambda}(\Gamma^\alpha_{\alpha j} + \alpha\sigma_j)$$

$$+ (F^i_j + F^i_k)\sigma_j - \frac{1}{2}g^{\alpha\beta}g_{\beta j}^\gamma - \frac{1}{N+1}(F^i_j(\Gamma^\gamma_{\gamma j} + \sigma_j),$$ (14)

$$\Gamma^i_{jk} = -\epsilon\delta^i_{\lambda}\sigma_j + \frac{1}{N+1}\delta^i_{\lambda}(\Gamma^\alpha_{\alpha j} + \alpha\sigma_j)$$

$$- (F^i_j - F^i_k)\sigma_j - \frac{1}{2}g^{\alpha\beta}g_{\beta k}^\gamma + \frac{1}{N+1}(F^i_j(\Gamma^\gamma_{\gamma j} + \sigma_j).$$ (15)

Moreover, it holds

$$\tilde{\Gamma}^i_j - \Gamma^i_j = \tau^i_j - \tau^i_{jk}.$$ (16)

$q = 1, 2$, for

$$\tau^i_j = \Gamma^i_j - \alpha^i_j,$$ (17)

$$\tau^i_{jk} = \frac{1}{N+1}\delta^i_{\lambda}(\Gamma^\alpha_j + \alpha\sigma_j) + \frac{1}{N+1}\delta^i_{\lambda}(\Gamma^\gamma_{\gamma j} + \sigma_j).$$ (18)

Lemma 2.1. [27] Let $f : \mathcal{GR}_N \to \mathcal{GR}_N$ be an almost geodesic mapping of a type $\pi(e), t = 1, 2$ determined with $F^i_j = \frac{1}{2}g^{\alpha\beta}g_{\alpha j}^\beta$. The geometrical objects

$$\tau^i_{jk} = \Gamma^i_j - \alpha^i_{jk}.$$ (19)

are invariants of the mapping $f$. □
2.2. Generalized Weyl projective tensor

We have that is

\[
\mathring{\Gamma}^{i}_{p} = \nabla^{i}_{p} = \mathring{\Gamma}^{i}_{p} = \mathring{\Gamma}^{i}_{p} - \Gamma^{i}_{p} + \frac{1}{2} \left( \nabla^{i}_{p} \right)^{r} \nabla_{r} = \frac{1}{2} \left( \nabla^{i}_{p} \right)^{r} \nabla_{r}
\]

for \( q = \{ q_{1}, q_{2}, q_{3}, q_{4}, q_{5}, q_{6}, q_{7}, q_{8} \} \in \{ 1, 2 \} \) and

\[
\mathring{\zeta}^{i}_{p} = \frac{1}{2} \left( \nabla^{i}_{p} \right)^{r} \nabla_{r} = \frac{1}{2} \left( \nabla^{i}_{p} \right)^{r} \nabla_{r}
\]

From this equation, we get it holds the following equation

\[
\mathring{\Gamma}^{i}_{p} = \Gamma^{i}_{p} + \frac{1}{2} \left( \nabla^{i}_{p} \right)^{r} \nabla_{r} = \frac{1}{2} \left( \nabla^{i}_{p} \right)^{r} \nabla_{r}
\]

From the equality \( \mathring{\Gamma}^{i}_{p} = \mathring{\Gamma}^{i}_{p} \mathring{\Pi}^{i}_{p} \mathring{\Pi}^{i}_{p} \), we obtain that is

\[
\mathring{\Pi}^{i}_{p} = \Gamma^{i}_{p} + \frac{1}{2} \left( \nabla^{i}_{p} \right)^{r} \nabla_{r} = \frac{1}{2} \left( \nabla^{i}_{p} \right)^{r} \nabla_{r}
\]

It is obtained [27] that the geometrical objects

\[
\mathring{W}^{i}_{p} = R^{i}_{p} + \mathring{W}^{i}_{p} \quad \text{and} \quad W^{i}_{p} = R^{i}_{p} + \mathring{W}^{i}_{p}
\]

for

\[
\mathring{W}^{i}_{p} = \sigma^{i}_{F} + \sigma^{i}_{F} + \sigma^{i}_{F} + \sigma^{i}_{F}
\]

and

\[
W^{i}_{p} = \sigma^{i}_{F} + \sigma^{i}_{F} + \sigma^{i}_{F} + \sigma^{i}_{F}
\]
are invariants of an almost geodesic mapping \( f : \mathcal{GR}_N \to \mathcal{GR}_N \) of a type \( \tau_2(e), t = 1, 2 \) determined by affinor
\[
2F^i_j = g^{2i}g_{ja}.
\]

**Theorem 2.2.** Let \( f : \mathcal{GR}_N \to \mathcal{GR}_N \) be an almost geodesic mapping of a type \( \tau_2(e), t = 1, 2 \) determined by affinor
\[
2F^i_j = g^{2i}g_{ja}.
\] The families of geometrical objects

\[
\begin{align*}
\mathcal{W}^{ij}_{(2)(p,q),jmn} = K^{ij}_{jmn} + \overline{W}^{ij}_{(2)(p,q),jmn} - \sigma\left(\tau^{ij}_{(p'j)jmn} + \zeta^{ij}_{(p^n)jmn}\right) - \sigma'\left(\tau^{ij}_{(p'n)jmn} + \zeta^{ij}_{(p^n)jmn}\right) \\
&- \nu\Theta^{ij}_{(p'j)jmn} - \nu'\Theta^{ij}_{(p'j)jmn} - \sigma\Theta^{ij}_{(p'j)jmn} \quad (q^n)
\end{align*}
\]

are families of invariants of the mapping \( f \).

**Proof.** We have that is

\[
\begin{align*}
\overline{K}^{ij}_{jmn} & = K^{ij}_{jmn} + (\overline{R} - R)^{ij}_{jmn} + \sigma\left(\overline{\Gamma}^{ij}_{jmn} - \Gamma^{ij}_{jmn}\right) + \sigma'\left(\overline{\Gamma}^{ij}_{jmn} - \Gamma^{ij}_{jmn}\right) \\
&+ \nu\left(\overline{\Gamma}^{ij}_{jmn} - \Gamma^{ij}_{jmn}\right) + \nu'\left(\overline{\Gamma}^{ij}_{jmn} - \Gamma^{ij}_{jmn}\right) + \sigma\left(\overline{\Gamma}^{ij}_{jmn} - \Gamma^{ij}_{jmn}\right) \quad (q^n)
\end{align*}
\]

From the equalities \( \overline{W}^{ij}_{(2)(p,q),jmn} = \mathcal{W}^{ij}_{(2)(p,q),jmn} \) and \( \overline{W}^{ij}_{(2)(p,q),jmn} = \mathcal{W}^{ij}_{(2)(p,q),jmn} \) such as the equations (22, 23) as well, we obtain that is

\[
\begin{align*}
\overline{W}^{ij}_{(2)(p,q),jmn} = \mathcal{W}^{ij}_{(2)(p,q),jmn} \quad \text{and} \quad \overline{W}^{ij}_{(2)(p,q),jmn} = \mathcal{W}^{ij}_{(2)(p,q),jmn}
\end{align*}
\]

which proves this theorem.  \( \Box \)
Corollary 2.3. The invariants (26, 27, 28, 29) satisfy the following equations

\[
\mathcal{W}_i^{(2)}(p,q) = \mathcal{W}_i^{(2)} + u\left(\mathcal{T}_i^{(q)}(p) - \Theta_i^{(q)}\right) - u'\left(\mathcal{T}_i^{(q)}(p) - \Theta_i^{(q)}\right) + v\left(\Gamma_i^{(p)}\Theta_i^{(q)} - \Theta_i^{(q)}\right) + \mathcal{W}_i^{(2)}(p,q)
\]

for the above defined \(\mathcal{W}_i^{(2)}\). □

Corollary 2.4. Let \(f : \mathcal{GR}_N \rightarrow \mathcal{GR}_N\) be an equitorsion almost geodesic mapping of a type \(\pi_3(e), t = 1, 2\), determined with the affinor \(2F_i^j = \gamma^a_{\alpha j}g_{\alpha}^i\). The families of geometrical objects

\[
\mathcal{E}_i^{(q)}(1), jmn = K_i^{j} - \mathcal{W}_i^{(2)} - u\left(\mathcal{T}_i^{(q)}\right) - u'\left(\mathcal{T}_i^{(q)}\right) + \mathcal{W}_i^{(2)}(q)
\]

for \(q = (q_1, q_2) = \left((q_1^1, q_2^1, q_3^1), (q_1^2, q_2^2, q_3^2)\right)\) and

\[
\mathcal{b}^{i}_{jmn} = \omega_i^{(q)}\Gamma_j^{(q)} - \omega_j^{(q)}\Gamma_i^{(q)} - \omega_{mn}^{(q)}\Gamma_i^{(q)}
\]
Corollary 2.5. The invariants (26, 27, 34, 35) satisfy the following equations

\[
\begin{align*}
\mathcal{E}^{(2),(1),jmn}_{2} &= W^{(2)}_{jmn} + u\left(\Gamma^{j}_{\nu} - \zeta^{j}_{\nu} \right) + u'\left(\Gamma^{j}_{\nu} - \zeta^{j}_{\nu} \right) \\
&\quad + v\Gamma^{\nu}_{jmn} + v'\Gamma^{\nu}_{jmn} + w\Gamma^{\nu}_{jmn}
\end{align*}
\]

\[
\begin{align*}
\mathcal{E}^{(2),(1),jmn}_{2} &= W^{(2)}_{jmn} - \Gamma^{(2)}_{jmn} + u\left(\Gamma^{j}_{\nu} - \zeta^{j}_{\nu} \right) + u'\left(\Gamma^{j}_{\nu} - \zeta^{j}_{\nu} \right) \\
&\quad + v\Gamma^{\nu}_{jmn} + v'\Gamma^{\nu}_{jmn} + w\Gamma^{\nu}_{jmn}
\end{align*}
\]

\[
\begin{align*}
\mathcal{E}^{(2),(2),jmn}_{2} &= W^{(2)}_{jmn} + \Gamma^{(2)}_{jmn} + u\left(\Gamma^{j}_{\nu} - \zeta^{j}_{\nu} \right) + u'\left(\Gamma^{j}_{\nu} - \zeta^{j}_{\nu} \right) \\
&\quad + v\Gamma^{\nu}_{jmn} + v'\Gamma^{\nu}_{jmn} + w\Gamma^{\nu}_{jmn}
\end{align*}
\]

for the above defined \( \tilde{W}^{(2)}_{jmn} \). \( \square \)

References

[1] V. Berezovski, J. Mikeš, On a classification of almost geodesic mappings of affine connection spaces, Acta Univ. Palacki. Olomuc Fac. rer. nat. Mathematica 35 (1996) 21–24.

[2] M. S. Čirić, M. Lj. Zlatanović, M. S. Stanković, Lj. S. Velimirović, On geodesic mappings of equidistant generalized Riemannian spaces, Applied Mathematics and Computation 218 (12) (2012) 6648–6655.

[3] L. P. Eisenhart, Non-Riemannian geometry, vol 8, Amer. Math. Soc. Colloq. Publ., New York, 1927.

[4] L. P. Eisenhart, Generalized Riemannian spaces, Proc. Natl. Acad. Sci. USA 37 (1951) 311–315.

[5] S. M. Ivanov, M. Lj. Zlatanović, Connections on a non-symmetric (generalized) Riemannian manifold and gravity, Class. Quantum Grav. 33 (2016), No. 7.

[6] J. Mikeš, V. Kiosak, A. Vanžurová, Geodesic mappings of manifolds with affine connection, Palacký University, Olomouc, 2008.

[7] J. Mikeš, E. Stepanova, A. Vanžurová, et all, Differential geometry of special mappings, Palacký University, Olomouc, 2015.

[8] J. Mikeš, A. Vanžurova, I. Hinterleitner, Geodesic mappings and some generalizations, Palacký University, Olomouc, 2009.

[9] S. M. Minčić, On curvature tensors and pseudotensors of the spaces with non-symmetric affine connection, Math. Balkanica (N.S.) 76 (1974) No. 4 427–430.

[10] S. M. Minčić, Independent curvature tensors and pseudotensors of spaces with non-symmetric affine connexion, Colloquia Mathematica Societatis János Bolyai 30 (1979) 445–460.

[11] S. M. Minčić, M. S. Stanković, On geodesic mappings of general affine connexion spaces and of generalized Riemannian spaces, Mat. Vesn. 49 (1997) No. 2 27–33.

[12] S. M. Minčić, M. S. Stanković, Equitorsion geodesic mappings of generalized Riemannian spaces, Publ. Inst. Math. (Beograd) (N. S.) 61 (75) (1997) 97–104.

[13] M. Prvanović, On two tensors in a locally decomposable Riemannian space, Review of Research of Science University of Novi Sad, Volume 6 (1976).

[14] M. Prvanović, Product Semi-Symmetric Connections of the Locally Decomposable Riemannian Spaces, Bulletin (Académie serbe des sciences et des arts. Classe des sciences mathématiques et naturelles. Sciences mathématiques) No. 10 (1979) pp. 17–27.

[15] S. M. Stanković, First almost geodesic mappings of general affine connection spaces, Novi Sad J. Math. 29 No. 3 (1999) 313–323.

[16] M. S. Stanković, On a canonic almost geodesic mappings of the second type of affine spaces FİLOMAT 13 (1999) 105–114.

[17] M. S. Stanković, On a special almost geodesic mappings of third type of affine spaces, Novi Sad J. Math. Vol. 31 No. 2 2001 125–135.

[18] M. S. Stanković, Special equitorsion almost geodesic mappings of the third type of non-symmetric affine connection spaces, Applied Mathematics and Computation 244 (2014) 695–701.
[21] M. S. Stanković, S. M. Minčić, New special geodesic mappings of generalized Riemannian space, Publ. Inst. Math. (Beograd) (N. S) 67(81) (2000) 92–102.
[22] M. S. Stanković, S. M. Minčić, Lj. S. Velimirović, On holomorphically projective mappings of generalized Kahlerian spaces, Matematicki vesnik 54 (2002) 195–202.
[23] M. S. Stanković, S. M. Minčić, Lj. S. Velimirović, On equitorsion holomorphically projective mappings of generalised Kahlerian spaces, Czechoslovak Mathematical Journal 54 (129) (2004) No. 3 701-715.
[24] M. S. Stanković, M. Lj. Zlatanović, Lj. S. Velimirović, Equitorsion holomorphically projective mappings of generalized Kahlerian space of the second kind, International Electronic Journal of Geometry Vol. 3 No. 2 (2010) 26–39.
[25] M. S. Stanković, M. Lj. Zlatanović, N. O. Vesić, Basic equations of G-almost geodesic mappings of the second type, which have the property of reciprocity, Czechoslovak Mathematical Journal (2015) Vol. 65 No. 3 pp. 787–799.
[26] V.M. Stanković, Certain properties of generalized Einstein spaces, Filomat 32:13 (2018), 4603–4810.
[27] N. O. Vesić, M. S. Stanković, Invariants of Special Second-Type Almost Geodesic Mappings of Generalized Riemannian Space, Mediterr. J. Math. (2018) 15: 60. https://doi.org/10.1007/s00009-018-1110-3.
[28] M. Lj. Zlatanović, New projective tensors for equitorsion geodesic mappings, Appl. Math. Lett. 25 (2012) 890–897.
[29] M. Lj. Zlatanović, V.M. Stanković, Some invariants of holomorphically projective mappings of generalized Kahlerian spaces, J. Math. Anal. Appl. Volume 450 (2017) 601–610.
[30] M. Lj. Zlatanović, V.M. Stanković, Geodesic mapping onto Kahlerian space of the third kind, J. Math. Anal. Appl. Vol. 458 (2018) 601–610.