The Solutions of the NLS Equations with Self-Consistent Sources

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Abstract

We construct the generalized Darboux transformation with arbitrary functions in time \( t \) for the AKNS equation with self-consistent sources (AKNSESCS) which, in contrast with the Darboux transformation for the AKNS equation, provides a non-auto-Bäcklund transformation between two AKNSESCSs with different degrees of sources. The formula for N-times repeated generalized Darboux transformation is proposed. By reduction the generalized Darboux transformation with arbitrary functions in time \( t \) for the Nonlinear Schrödinger equation with self-consistent sources (NLSESCS) is obtained and enables us to find the dark soliton, bright soliton and positon solutions for NLS+ ESCS and NLS− ESCS. The properties of these solution are analyzed.

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1 Introduction

The nonlinear Schrödinger equation with self-consistent sources (NLSESCS) describes the Soliton propagation in a medium with both resonant and nonresonant nonlinearities [1-4], and it also describes the nonlinear interaction of high-frequency electrostatic wave with ion acoustic waves in plasma [5]. Some soliton solution for the NLSESCS was obtained by inverse scattering transformation in [1]. Since the explicit time part of the Lax representation of the NLSESCS was not found, the method to solve the NLSESCS by inverse scattering transformation in [1] was quite complicated.

Due to the important role played by the soliton equations with self-consistent sources (SESCSs) in many fields of physics, such as hydrodynamics, solid state physics, plasma physics, SESCSs have attracted some attention [6-16]. In recent years we have presented method to find the explicit time part of the Lax representation for SESCSs and to construct generalized binary Darboux transformations with arbitrary functions in time \( t \) for SESCSs which, in contrast with the Darboux transformation for soliton equations [17, 18], offer a non-auto-Bäcklund transformation between two SESCSs with different degrees of sources and can be used to obtain N-soliton, positon and negaton solutions [19-21].

The positon solution for many soliton equations and their physical application have been widely studied, for example, the positon solutions for KdV and mKdV equations were investigated in [23, 24], for the nonlinear Schrödinger equation in [25], for the sine-Gordon equation in [26]. However positon solutions for SESCSs except for KdV equation with self-consistent sources in [19,20] have not been studied.

In this paper, we develop the method presented in [19,20] to study the NLSESCS. First we construct the generalized Darboux transformation with arbitrary functions in time \( t \) for the AKNS equation with self-consistent sources (AKNSESCS) which offers a non-auto-Bäcklund transformation between two AKNSESCSs with different degrees of sources. Then by reduction we obtained the generalized Darboux transformation with arbitrary functions in time \( t \) for the NLSESCS which also provides a non-auto-Bäcklund transformation between two NLSECSs with
different degrees of sources. Some interesting solutions of NLSESCS such as dark soliton, bright soliton and positon solutions for NLS$^+$ESCS and NLS$^-$ESCS are found. The properties of these solutions are analyzed.

2 Binary Darboux transformations for the AKNS equation with self-consistent sources

The AKNSESCS is defined as [15,16]

$$q_t = -i(q_{xx} - 2q^2r) + \sum_{j=1}^{n}(\varphi_j^{(1)})^2, \quad r_t = i(r_{xx} - 2r^2) + \sum_{j=1}^{n}(\varphi_j^{(2)})^2, \quad (2.1a)$$

$$\varphi_{j,x} = \begin{pmatrix} -\lambda_j & q \\ r & \lambda_j \end{pmatrix} \varphi_j, \quad j = 1, \cdots, n, \quad (2.1b)$$

where $\lambda_j$’s are $n$ distinct complex constants, $\varphi_j = (\varphi_j^{(1)}, \varphi_j^{(2)})^T$ (hereafter, we use superscripts (1) and (2) to denote the first and second element of a two dimensional vector respectively).

The Lax pair for Eqs. (2.1) is given by [15,16]

$$\psi_x = U \psi, \quad U := U(\lambda, q, r) = \begin{pmatrix} -\lambda & q \\ r & \lambda \end{pmatrix}, \quad (2.2a)$$

$$\psi_{t} = R(n) \psi, \quad R(n) := V + \sum_{j=1}^{n} \frac{H(\varphi_j)}{\lambda - \lambda_j}, \quad (2.2b)$$

where

$$V := V(\lambda, q, r) = i \begin{pmatrix} -2\lambda^2 + qr & 2\lambda q - qr \\ 2qr + r_x & 2\lambda^2 - qr \end{pmatrix}, \quad H(\varphi_j) = \frac{1}{2} \begin{pmatrix} -\varphi_j^{(1)} \varphi_j^{(2)}(\varphi_j^{(1)})^2 \\ -(\varphi_j^{(2)})^2 \varphi_j^{(1)} \varphi_j^{(2)} \end{pmatrix}$$

2.1 Binary Darboux transformation with an arbitrary constant

It is known [16] that based on the Darboux transformation for the AKNS equation [22], the AKNSESCS admits two elementary Darboux transformations $T_{1,2} : (q, r, \varphi_1, \cdots, \varphi_n) \mapsto (\tilde{q}, \tilde{r}, \tilde{\varphi}_1, \cdots, \tilde{\varphi}_n)$. Given two arbitrary complex numbers $\mu$ and $\nu$, $\mu \neq \nu$, let $f = f(\mu)$ and $g = g(\nu)$ be two solutions of (2.2) with $\lambda = \mu$ and $\lambda = \nu$ respectively, and define $T_1[f]$:

$$\tilde{\psi} = T_1 \psi, \quad T_1 = T_1(\lambda, f) = \begin{pmatrix} \lambda - \mu + qf^{(2)}/(2f^{(1)}) & -q/2 \\ -f^{(2)}/f^{(1)} & 1 \end{pmatrix},$$

$$\tilde{q} = -q_x/2 - \mu q + q^2 f^{(2)}/(2f^{(1)}), \quad \tilde{r} = 2f^{(2)}/f^{(1)}, \quad \tilde{\varphi}_j = \frac{T_1(\lambda_j, f)\varphi_j}{\sqrt{\lambda_j - \mu}}, \quad j = 1, \cdots, n;$$

$T_2[g]$:

$$\tilde{\psi} = T_2 \psi, \quad T_2 = T_2(\lambda, g) = \begin{pmatrix} 1 & -g^{(1)}/g^{(2)} \\ r/2 & \lambda - \nu - rg^{(1)}/(2g^{(2)}) \end{pmatrix},$$

$$\tilde{q} = -2g^{(1)}/g^{(2)}, \quad \tilde{r} = r_x/2 - \nu r - r^2 g^{(1)}/(2g^{(2)}), \quad \tilde{\varphi}_j = \frac{T_2(\lambda_j, g)\varphi_j}{\sqrt{\lambda_j - \nu}}, \quad j = 1, \cdots, n.$$
Theorem 2.1 The linear system (2.2) is covariant with respect to (w.r.t) the two Darboux transformations $T_1$, $T_2$, i.e., the new variables $\tilde{\psi}$, $\tilde{\varphi}, \tilde{\varphi}_j$ and $\tilde{\varphi}_j$ satisfy

\begin{equation}
\tilde{\psi}_x = \tilde{U}\tilde{\psi}, \quad \tilde{U} = U(\lambda, \tilde{q}, \tilde{r}),
\end{equation}

(2.3a)

\begin{equation}
\tilde{\psi}_t = \tilde{R}^{(n)}\tilde{\psi} := \left[V^{(n)}(\lambda, \tilde{q}, \tilde{r}) + \sum_{j=1}^{n} \frac{H(\tilde{\varphi}_j)}{\lambda - \lambda_j}\right] \tilde{\psi}.
\end{equation}

(2.3b)

We now construct a new Darboux transformation based on $T_1$ and $T_2$. Our method is similar to that for the KdV equation with self-consistent sources [20]. Define

\[ \sigma(f, g) := -\frac{W(f, g)}{2(\mu - \nu)}, \quad \sigma(f, f) := \lim_{\lambda \to \mu} -\frac{W(f(\lambda), f(\mu))}{2(\lambda - \mu)} = \frac{1}{2} W(f, \partial_\mu f). \]

where $W(f, g)$ is the Wronskian $W(f, g) := f^{(1)}(g) - f^{(2)}(g)$. We assume that we have obtained $(\tilde{\psi}, \tilde{q}, \tilde{r}, \tilde{\varphi}_1, \cdots, \tilde{\varphi}_n)$ satisfying (2.3) by applying $T_1[f]$ to $(\psi, q, r, \varphi_1, \cdots, \varphi_n)$. Then we derive two linearly independent solutions of (2.3) with $\lambda = \mu$ and in terms of $f$ only.

**First solution.** Let $f_1 = f_1(\mu)$ be a solution of (2.2) with $\lambda = \mu$, and $W(f, f_1) \neq 0$ (i.e., $f$ and $f_1$ are linearly independent). Then applying $T_1[f]$ to $f_1$ gives a solution of (2.2) with $\lambda = \mu$:

\[ \tilde{f}_1 := T_1(\mu, f) f_1 = \frac{W(f, f_1)}{2f^{(1)}} \begin{pmatrix} -q \\ 2 \end{pmatrix}. \]

Since $W(f, f_1)$ is independent of both $x$ and $t$, we assume $W(f, f_1) \equiv 1$. Thus, we obtain the first solution of (2.2):

\[ \tilde{f}_1 = \frac{1}{2f^{(1)}} \begin{pmatrix} -q \\ 2 \end{pmatrix}. \]

**Second solution.** Note that $\psi_1(\lambda) := f(\lambda)/(\lambda - \mu)$ is a solution of (2.2). Applying $T_1[f]$ to $\psi_1$ gives a solution of (2.2):

\[ \tilde{\psi}_1(\lambda) = T_1(\lambda, f) \psi_1 = \begin{pmatrix} f^{(1)}(\lambda) \\ 0 \end{pmatrix} + \frac{W(f(\mu), f(\lambda))}{2f^{(1)}(\mu)(\lambda - \mu)} \begin{pmatrix} -q \\ 2 \end{pmatrix}. \]

Taking the limit, we find a second solution of (2.2) with $\lambda = \mu$:

\[ \tilde{f} := \lim_{\lambda \to \mu} \tilde{\psi}_1(\lambda) = \begin{pmatrix} f^{(1)}(\mu) \\ 0 \end{pmatrix} + \frac{\sigma(f, f)}{f^{(1)}} \begin{pmatrix} -q \\ 2 \end{pmatrix}. \]

Let $C$ be an arbitrary constant, then the linear combination of the above solutions

\[ \tilde{h} := \tilde{f} + 2C\tilde{f}_1 = \begin{pmatrix} f^{(1)}(\mu) \\ 0 \end{pmatrix} + \frac{C \sigma(f, f)}{f^{(1)}} \begin{pmatrix} -q \\ 2 \end{pmatrix}, \]

is also a solution of (2.2) with $\lambda = \mu$. Apply $T_2[\tilde{h}]$ to $(\tilde{\psi}/(\lambda - \mu), \tilde{q}, \tilde{r}, \tilde{\varphi}_1, \cdots, \tilde{\varphi}_n)$, i.e., define

\[ \hat{\psi} = T_2(\lambda, \tilde{h}) \frac{\tilde{\psi}}{\lambda - \mu} = \psi - \frac{f}{C + \sigma(f, f)} \sigma(f, \psi), \]

(2.4a)

\[ \hat{q} = -\frac{\tilde{h}_1}{\tilde{h}_2} = q - \frac{2(f^{(1)})^2}{C + \sigma(f, f)}, \quad \hat{r} = \tilde{r} - \mu \tilde{r} - r^2 \tilde{h}_1 2\tilde{h}_2 = r - \frac{(f^{(2)})^2}{C + \sigma(f, f)}. \]

(2.4b)

\[ \hat{\varphi}_j = \frac{T_2(\lambda, \tilde{h}) \tilde{\varphi}_j}{\sqrt{\lambda - \mu}} = \varphi_j - \frac{f}{C + \sigma(f, f)} \sigma(f, \varphi_j), \]

(2.4c)
then the new variables \( \hat{\psi}, \hat{\bar{q}}, \hat{\bar{r}}, \hat{\phi}_j \) satisfy

\[
\hat{\psi}_t = \hat{U}\hat{\psi},
\]

(2.5a)

\[
\hat{\psi}_t = \hat{R}^{(n)}\hat{\psi},
\]

(2.5b)

where \( \hat{U} = U(\lambda, \hat{\bar{q}}, \hat{\bar{r}}) \) and \( \hat{R}^{(n)} = V(\lambda, \hat{\bar{q}}, \hat{\bar{r}}) + \sum_{j=1}^{n} H(\hat{\phi}_j)/(\lambda - \lambda_j) \).

**Proposition 2.1** Let \( f \) be a solution of (2.2) with \( \lambda = \mu \), and \( C \) be an arbitrary constant, then \( \hat{\psi}, \hat{\bar{q}}, \hat{\bar{r}} \) and \( \hat{\phi}_j \) given by (2.4) present a binary Darboux transformation with an arbitrary constant for (2.2), and \( \hat{\phi}_1, \hat{\phi}_2, \ldots, \hat{\phi}_n \) is a new solution of (2.2). Moreover, we have

\[
\hat{\bar{r}} = q\hat{r} - \partial_x^2 \log[C + \sigma(f, f)].
\]

2.2 Binary Darboux transformation with an arbitrary function of \( t \)

Substituting (2.6b) into the left side of Eq. (2.6b), we have a polynomial of \( [C + \sigma(f, f)]^{-1} \):

\[
\hat{\psi}_t = \frac{\partial}{\partial t} \left[ \psi - \frac{f}{C + \sigma(f, f)} \sigma(f, \psi) \right] = \psi_t - \frac{f_t}{C + \sigma(f, f)} \sigma(f, \psi) - \frac{f_t W(f_t, \psi) + W(f, \psi)}{2(\mu - \lambda)(C + \sigma(f, f))}
\]

\[
+ \frac{f \sigma(f, \psi)[W(f_t, f_t) + W(f, f_t, \mu)]}{2[C + \sigma(f, f)]} = \sum_{j=0}^{3} L_j [C + \sigma(f, f)]^{-j},
\]

where \( L_j \)'s are two-dimensional vector functions defined by the last equality. We can expect that substituting (2.4) into the right side of (2.5b) will also give a polynomial of \( [C + \sigma(f, f)]^{-1} \), but it will be more complicated. So we just write it as

\[
\hat{R}^{(n)}\hat{\psi} = \sum_{j=0}^{3} R_j [C + \sigma(f, f)]^{-j},
\]

where \( R_j \)'s are also two-dimensional vector functions dependent on \( \psi, q, r, \phi_j \) and \( f \) and their derivatives w.r.t. \( x \). Since (2.5b) holds for any constant \( C \), we have the following lemma.

**Lemma 2.1** Assume that \( \psi, q, r \) and \( \phi_j \) satisfy (2.4), and let \( f \) be a solution of (2.2) with \( \lambda = \mu \), then we have

\[
L_j = R_j, \quad j = 0, 1, 2, \quad R_3 = 0,
\]

for all \( x \) and \( t \).

We now replace the constant \( C \) with an arbitrary function of \( t \), say \( c(t) \). Since there is no derivatives w.r.t. \( t \) in the expression of \( \hat{R}^{(n)} \), if we replace \( C \) with \( c(t) \) in the definition of (2.4), we will have

\[
\hat{R}^{(n)}\hat{\psi} = \sum_{j=0}^{3} R_j [c(t) + \sigma(f, f)]^{-j}.
\]

But we will not have \( \hat{\psi}_t = \sum_{j=0}^{3} L_j [c(t) + \sigma(f, f)]^{-j} \) under this replacement. However, this replacement will lead to a non-auto-Bäcklund transformation.

**Proposition 2.2** Let \( f \) be a solution of (2.2) with \( \lambda = \lambda_{n+1} \), and \( c(t) \) be an arbitrary function of \( t \). If we define

\[
\tilde{\psi} = \psi - \frac{f}{c(t) + \sigma(f, f)} \sigma(f, \psi),
\]

(2.6a)
\[ \bar{q} = q - \frac{(f^{(1)})^2}{c(t) + \sigma(f, f)}, \quad \bar{r} = r - \frac{(f^{(2)})^2}{c(t) + \sigma(f, f)}, \] (2.6b)

\[ \bar{\varphi}_j = \varphi_j - \frac{f}{c(t) + \sigma(f, f)} \sigma(f, \varphi_j), \quad j = 1, \ldots, n, \] (2.6c)

and

\[ \bar{\varphi}_{n+1} = \frac{\sqrt{\sigma(t)} f}{c(t) + \sigma(f, f)} \sigma(f, \varphi_j), \] (2.6d)

then the new variables \( \bar{\psi}, \bar{q}, \bar{r}, \bar{\varphi}_1, \ldots, \bar{\varphi}_{n+1} \) satisfy a new system

\[ \bar{\psi}_t = \bar{U} \bar{\psi}, \quad \bar{U} = U(\lambda, \bar{q}, \bar{r}), \] (2.7a)

\[ \bar{\psi}_k = \bar{R}^{(n+1)} \bar{\psi}, \quad \bar{R}^{(n+1)} = V(\lambda, \bar{q}, \bar{r}) + \sum_{j=1}^{n+1} \frac{H(\bar{\varphi}_j)}{\lambda - \lambda_j}, \] (2.7b)

and \((\bar{q}, \bar{r}, \bar{\varphi}_1, \ldots, \bar{\varphi}_{n+1})\) is a solution of (2.1) with \( n \) replaced by \( n + 1 \). Moreover, we have

\[ \bar{q} \bar{r} = qr - \partial_t^2 \log[c(t) + \sigma(f, f)]. \]

**Proof.** Since no derivatives w.r.t. \( t \) appear in Eq. (2.7a), it is covariant w.r.t. the transformation defined by (2.6). Substitution of (2.6a) into the left side of (2.7b) gives

\[ \tilde{\psi}_t = \frac{\partial}{\partial t} \left[ \psi - \frac{f}{c(t) + \sigma(f, f)} \sigma(f, \psi) \right] = \psi_t - \frac{f_t}{c(t) + \sigma(f, f)} \sigma(f, \psi) \]

\[ - f [W(f_t, \psi) + W(f, \psi_t)] + \frac{f \sigma(f, \psi) [2 \dot{c}(t) + W(f_t, f_\mu) + W(f, f_{\mu, \nu})]}{2(c(t) + \sigma(f, f))^2} \]

\[ = \sum_{j=0}^{2} L_j [c(t) + \sigma(f, f)]^{-j} + \frac{\dot{c}(t) f \sigma(f, \psi)}{c(t) + \sigma(f, f))^2} \]

\[ = \sum_{j=0}^{2} R_j [c(t) + \sigma(f, f)]^{-j} + \frac{\dot{c}(t) \sigma(f, \psi)}{c(t) + \sigma(f, f))} \bar{\varphi}_j \]

\[ = \bar{R}^{(n)} \bar{\psi} + \frac{H(\bar{\varphi}_{n+1})}{2(\lambda - \lambda_{n+1})} \bar{\psi} = \bar{R}^{(n+1)} \bar{\psi} \]

This completes the proof.

**Example of solution.** We start from the Eqs. (2.2) with \( n = 0 \), and the initial solution \( q = r = 0 \). Choose a solution of (2.2) with \( n = 0 \), \( q = r = 0 \) as \( f = (e^{-\lambda_1 x - 2i\lambda_1^2 t}, e^{\lambda_1 x + 2i\lambda_1^2 t})^T \), then by Proposition 2.2, we obtain a solution of (2.1) with \( n = 1 \):

\[ q = -e^{-2\lambda_1 x - 4i\lambda_1^2 t} \left( \frac{c(t)}{x + 4i\lambda_1 t + c(t)} \right), \quad r = -e^{2\lambda_1 x + 4i\lambda_1^2 t} \left( \frac{c(t)}{x + 4i\lambda_1 t + c(t)} \right), \quad \varphi_1 = \frac{\sqrt{c(t)}}{x + 4i\lambda_1 t + c(t)} \left( \frac{e^{-\lambda_1 x - 2i\lambda_1^2 t}}{e^{\lambda_1 x + 2i\lambda_1^2 t}} \right), \]

where \( c(t) \) is an arbitrary function.

**Remark.** The binary Darboux transformation (2.6), in fact, provides a non-auto-Bäcklund transformation between the AKNS equation with sources of different degrees of freedom. Since a function \( c(t) \) is involved, we call it a binary Darboux transformation with an arbitrary function of \( t \). This transformation is dependent on two elements, \( c(t) \) and \( f \), so we just write them together as a pair \( \{c, f\} \).
2.3 Multi-times repeated binary Darboux transformation with arbitrary functions

It is evident that the binary Darboux transformation with an arbitrary function can be applied \(N\) times, and we will obtain the \(N\)-times repeated binary Darboux transformation with \(N\) arbitrary functions. Let \(f_1, f_2, \ldots,\) be a series of solutions of (2.8) with \(\lambda = \lambda_1, \lambda_2, \ldots,\) and let \(c_1, c_2, \ldots,\) be a series of arbitrary functions of \(t\). Let \(\psi[N], \ g[N], r[N], \ \varphi_j[N]\) and \(f_j[N]\) denote the \(N\)-times Darboux transformed variables.

We define some symmetric forms. Let \(c_j\) and \(g_j, \ j = 1, 2, \cdots\) be a series of scaler and two-dimensional vectors, \(u\) be a scaler, \(h\) be a two-dimensional vector, and \(\sigma(g_j, g_j)\) and \(\sigma(g_j, h)\) are defined. For \(N = 1, 2, \cdots,\) we define five forms \(W_0, W_1^{(i)}\) and \(W_2^{(i)}\), \(i = 1, 2,\) which are symmetric for the \(N\) pairs \(\{c_j, g_j\}\), as follows:

\[
W_0(c_1, g_1), \cdots, (c_N, g_N) = \det A,
\]

\[
W_1^{(i)}(c_1, g_1), \cdots, (c_N, g_N); h) = \det \begin{pmatrix} A & b \\ \alpha^{(i)} & h^{(i)} \end{pmatrix}, \quad \text{quadi = 1, 2,}
\]

\[
W_2^{(i)}(c_1, g_1), \cdots, (c_N, g_N); u) = \det \begin{pmatrix} A & (\alpha^{(i)})^T \\ \alpha^{(i)} & u \end{pmatrix}, \quad i = 1, 2,
\]

where

\[
A = (\delta_{ij} c_i + \sigma(g_i, g_j))_{N \times N}, \quad b = (\sigma(g_1, h) + \sigma(g_N, h))^T, \quad \alpha^{(i)} = (g_i^{(i)}, \ldots, g_N^{(i)}).
\]

For convenience, we define

\[
W_1(c_1, g_1), \cdots, (c_N, g_N); h) = \begin{pmatrix} W_1^{(1)}(c_1, g_1), \cdots, (c_N, g_N); h \\ W_2^{(2)}(c_1, g_1), \cdots, (c_N, g_N); h \end{pmatrix}
\]

Lemma 2.2 Let \(F_i[j] = \{c_i, f_i[j]\}, i, j = 1, 2, \ldots,\) then for \(l, k = 1, 2, \ldots,\) we have

\[
W_0(F_l[l-1], \ldots, F_{l+k}[l-1]) = \frac{W_0(F_l[l-1], \ldots, F_{l+k}[l-1])}{W_0(F_l[l-1])} \tag{2.8a}
\]

\[
W_1(F_l[l-1], \ldots, F_{l+k}[l]; \psi[l-1]) = \frac{W_1(F_l[l-1], \ldots, F_{l+k}[l-1]; \psi[l-1])}{W_0(F_l[l-1])} \tag{2.8b}
\]

\[
W_2^{(1)}(F_l[l-1], \ldots, F_{l+k}[l]; q[l]) = \frac{W_2^{(1)}(F_l[l-1], \ldots, F_{l+k}[l-1]; q[l-1])}{W_0(F_l[l-1])} \tag{2.8c}
\]

\[
W_2^{(2)}(F_l[l-1], \ldots, F_{l+k}[l]; r[l]) = \frac{W_2^{(2)}(F_l[l-1], \ldots, F_{l+k}[l-1]; r[l-1])}{W_0(F_l[l-1])} \tag{2.8d}
\]

Proof. Let \(a_{ij} = \delta_{ij} c_{i+j} + \sigma(f_{i+j}[l-1], f_{i+j}[l-1]), i, j = 1, 2, \ldots,\) Direct calculation yields

\[
\delta_{ij} c_{i+j} + \sigma(f_{i+j}[l], f_{i+j}[l]) = a_{ij} = a_{00} a_{i0}^{-1} a_{0j} = \bar{a}_{ij}, \quad i, j = 1, 2, \ldots
\]

Note that

\[
\begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0k} \\ a_{10} & a_{11} & \cdots & a_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k0} & a_{k1} & \cdots & a_{kk} \end{pmatrix} \begin{pmatrix} 1 & -a_{01} & \cdots & -a_{0k} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} a_{00} & 0 & \cdots & 0 \\ a_{10} & a_{11} & \cdots & a_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k0} & a_{k1} & \cdots & a_{kk} \end{pmatrix}.
\]

Taking determinant for both sides, we have

\[
W_0(F_l[l-1], \ldots, F_{l+k}[l-1]) = W_0(F_l[l-1]) W_0(F_{l+1}[l], \ldots, F_{l+k}[l]),
\]

which is just the eq. (2.8a). Similarly, we can prove (2.8b), (2.8c) and (2.8d).
Proposition 2.3 For $N = 1, 2, 3, \ldots$, we have

$$
\psi[N] = \frac{1}{\Delta} W_1(\{c_1, f_1\}, \ldots, \{c_N, f_N\}; \psi),
$$
\tag{2.9a}

$$
q[N] = \frac{1}{\Delta} W_2(\{c_1, f_1\}, \ldots, \{c_N, f_N\}; q),
$$
\tag{2.9b}

$$
r[N] = \frac{1}{\Delta} W_2(\{c_1, f_1\}, \ldots, \{c_N, f_N\}; r),
$$
\tag{2.9c}

$$
\varphi_j[N] = \frac{1}{\Delta} W_1(\{c_1, f_1\}, \ldots, \{c_N, f_N\}; \varphi_j), \quad j = 1, \ldots, n,
$$
\tag{2.9d}

$$
\varphi_{n+j}[N] = \frac{\sqrt{c_j}}{c_j \Delta} W_1(\{c_1, f_1\}, \ldots, \{c_N, f_N\}; f_j), \quad j = 1, \ldots, N,
$$
\tag{2.9e}

and

$$
q[N]r[N] = qr - \partial^2 \log \Delta
$$
\tag{2.9f}

where $\Delta = W_0(\{c_1, f_1\}, \ldots, \{c_N, f_N\})$.

Proof. By the definition of $\psi[N]$ and Lemma 2.2, we have

$$
\psi[N] = \frac{W_1(\{c_N, f_N[N-1]\}; \psi[1-2])}{W_0(\{c_N, f_N[N-1]\})} = \frac{W_1(\{c_{N-1}, f_N[N-2]\}; \psi[1-2])}{W_0(\{c_{N-1}, f_N[N-2]\})} = \cdots
$$

which gives rise to the eq. (2.9a). Similarly, we can prove (2.9b), (2.9c), (2.9d) and (2.9e).

3 Binary Darboux transformations for the NLS equations with self-consistent sources

It is well known that from the ordinary AKNS equation

$$
q_t = -i(q_{xx} - 2q^2r), \quad r_t = i(r_{xx} - 2qr^2).
$$
\tag{3.1}

if we set $r = \varepsilon q^*, \varepsilon = \pm 1$, then Eqs. (3.1) are reduced to the ordinary NLS equation

$$
q_t = i(2\varepsilon|q|^2q - q_{xx}).
$$
\tag{3.2}

We call the equation with $\varepsilon = +1$ the NLS$^+$ equation and the equation with $\varepsilon = -1$ the NLS$^-$ equation.

Similarly, we can reduce the AKNS/ESCS into the NLS$^\pm$ equations with self-consistent sources (NLS$^\pm$/ESCS), but the reductions are more complicated since the sources need to be reduced as well. First, we define two linear maps $S_+$ and $S_-$ by

$$
S_{\pm} : \left(\begin{array}{c}
z^{(1)}_1 \\
z^{(2)}_1 \\
\end{array}\right) \mapsto \left(\begin{array}{c}
\pm z^{(2)*}_1 \\
z^{(1)*}_1 \\
\end{array}\right).
$$
\tag{3.3}

For the reduced AKNS spectral problem, i.e., the NLS$^+$ spectral problem:

$$
\psi_x = U(\lambda, q, q^*)\psi
$$
\tag{3.4}
and the NLS⁻ spectral problem:

\[ \psi_x = U(\lambda, q, -q^*) \psi, \quad (3.5) \]

we have the following lemma.

**Lemma 3.1** (1) If \( f \) is a solution of (3.4) with \( \lambda = \lambda_1 \), then \( S_+ f \) is a solution of (3.4) with \( \lambda = -\lambda_1^* \); there exists a solution \( f \) of (3.4) with \( \lambda = \lambda_1 \) satisfying \( f^{(2)} = f^{(1)*} \) if and only if \( \text{Re} \lambda_1 = 0 \). (2) If \( f \) is a solution of (3.4) with \( \lambda = \lambda_1 \), then \( S_- f \) is a solution of (3.4) with \( \lambda = -\lambda_1^* \); there exists no solution \( f \) of (3.4) satisfying \( f^{(2)} = f^{(1)*} \) if \( q \neq 0 \).

The NLSESCS are reduced from the AKNSESCS defined by

\[ \varphi_{j,x} = U(\lambda_j, q, r) \varphi_j, \quad \varphi'_{j,x} = U(\lambda'_j, q, r) \varphi'_j, \quad j = 1, \ldots, m, \quad (3.6a) \]

\[ \phi_{j,x} = U(\zeta_j, q, r) \phi_j, \quad j = 1, \ldots, n, \quad (3.6b) \]

\[ q_t = -i(q_{xx} - 2q^2 r) + \sum_{j=1}^{m} \left( (\varphi_j^{(1)})^2 + (\varphi'_j^{(1)})^2 \right) + \sum_{j=1}^{n} (\phi_j^{(1)})^2, \quad (3.6c) \]

\[ r_t = i(r_{xx} - 2qr^2) + \sum_{j=1}^{m} \left( (\varphi_j^{(2)})^2 + (\varphi'_j^{(2)})^2 \right) + \sum_{j=1}^{n} (\phi_j^{(2)})^2. \quad (3.6d) \]

where \( \lambda_1, \ldots, \lambda_n, \lambda'_1, \ldots, \lambda'_n, \zeta_1, \ldots, \zeta_m \) are \( 2n + m \) distinct constants. The corresponding Lax pair is

\[ \psi_x = U(\lambda, q, r) \psi, \quad \psi_t = V(\lambda, q, r) \psi + \sum_{j=1}^{m} \left[ \frac{H(\varphi_j)}{\lambda - \lambda_j} + \frac{H(\varphi'_j)}{\lambda - \lambda'_j} \right] \psi + \sum_{j=1}^{n} \frac{H(\phi_j)}{\lambda - \zeta_j} \psi. \quad (3.7) \]

(1) **Reductions to the NLS⁺ ESCS**

Let

\[ r = q^*, \quad \lambda_j = -\lambda_j^* \quad \varphi_j = \pm S_+ \varphi_j, \quad j = 1, \ldots, m, \quad (3.8a) \]

\[ \text{Re} \zeta_j = 0, \quad \phi_j^{(2)*} = \phi_j^{(1)} \equiv w_j, \quad j = 1, \ldots, n, \quad (3.8b) \]

then Eqs. (3.6) are reduced to the NLS⁺ ESCS

\[ \varphi_{j,x} = U(\lambda_j, q, q^*) \varphi_j, \quad j = 1, \ldots, m, \quad (3.9a) \]

\[ w_{j,x} = \zeta_j w_j + qw'_j, \quad \text{Re} \zeta_j = 0, \quad j = 1, \ldots, n, \quad (3.9b) \]

\[ q_t = i(2|q|^2 q - q_{xx}) + \sum_{j=1}^{m} \left( (\varphi_j^{(1)})^2 + (\varphi'_j^{(2)})^2 \right) + \sum_{j=1}^{n} w_j. \quad (3.9c) \]

And the system (3.7) is reduced to the Lax pair for the NLS⁺ ESCS

\[ \psi_x = U(\lambda, q, q^*) \psi, \quad \psi_t = V(\lambda, q, q^*) \psi + \sum_{j=1}^{m} \left[ \frac{H(\varphi_j)}{\lambda - \lambda_j} + \frac{H(S_+ \varphi_j)}{\lambda + \lambda_j^*} \right] \psi + \sum_{j=1}^{n} H((w_j, w'_j)^T) \frac{1}{\lambda - \zeta_j} \psi. \quad (3.10) \]
(2) Reductions to the NLS−ESCS

Take \( n = 0 \) in \((3.10)\) and let

\[
    r = -q^*, \quad \lambda'_j = -\lambda'_j \quad \varphi'_j = \pm i S_- \varphi_j, \quad j = 1, \ldots, m,
\]

then Eqs. \((3.10)\) with \( n = 0 \) are reduced to the NLS−ESCS

\[
    \varphi_{j,x} = U(\lambda_j, q, -q^*) \varphi_j, \quad j = 1, \ldots, m, \tag{3.12a}
\]

\[
    q_t = i (-2 |q|^2 q - q_{xx}) + \sum_{j=1}^{m} \left[ (\varphi_j^{(1)})^2 - (\varphi_j^{(2)*})^2 \right]. \tag{3.12b}
\]

Correspondingly, the system \((3.7)\) with \( n = 0 \) is reduced to the Lax pair for the NLS−SCS

\[
    \psi_x = U(\lambda, q, -q^*) \psi, \quad \psi_t = V(\lambda, q, -q^*) \psi + \sum_{j=1}^{m} \left[ \frac{H(\varphi_j)}{\lambda - \lambda_j} - \frac{H(S_- \varphi_j)}{\lambda + \lambda_j^*} \right] \psi. \tag{3.13}
\]

We now reduce the Darboux transformations for the AKNSESCS to the NLSESCS. It is easy to verify the following statements.

**Lemma 3.2**

(1) Let \( f \) and \( g \) be two solutions of the NLS\(^+\) spectral problem \( \psi_x = U(\lambda, q, q^*) \psi \) with \( \lambda = \mu, \nu \) respectively, and let \( C \) be a complex constant w.r.t. \( x \), then we have

\[
    \sigma(f, S_+ g)^* = \sigma(S_+ f, g), \quad \sigma(S_+ f, S_+ g)^* = \sigma(f, g),
\]

\[
    \sigma(f, S_+ f)^* = \sigma(S_+ f, f), \quad \sigma(S_+ f, S_+ f)^* = \sigma(f, f);
\]

\[
    W_0([{C}, {C^*}, {S_+ f}]); 0)^* = W_0([{C}, {C^*}, {S_+ f}]); 0),
\]

\[
    W_1([{C}, {C^*}, {S_+ f}]); 0)^* = W_1([{C}, {C^*}, {S_+ f}]); 0),
\]

Moreover, if \( g \) satisfies \( g^{(2)} = g^{(1)*} \) \( (\Rightarrow \text{Re} \nu = 0) \), then

\[
    W_1^{(2)}([{C}, {C^*}, {S_+ f}]); g)^* = W_1^{(1)}([{C}, {C^*}, {S_+ f}]); g).
\]

(2) Let \( f \) and \( g \) be two solutions of the NLS\(^−\) spectral problem \( \psi_x = U(\lambda, q, -q^*) \psi \) with \( \lambda = \mu, \nu \) respectively, and let \( C \) is a complex constant w.r.t. \( x \), then we have

\[
    \sigma(f, S_- g)^* = \sigma(S_- f, g), \quad \sigma(S_- f, S_- g)^* = -\sigma(f, g),
\]

\[
    \sigma(f, S_- f)^* = \sigma(S_- f, f), \quad \sigma(S_- f, S_- f)^* = -\sigma(f, f);
\]

\[
    W_0([{C}, {C^*}, {S_- f}]); 0)^* = W_0([{C}, {C^*}, {S_- f}]); 0),
\]

\[
    W_1([{C}, {C^*}, {S_- f}]); g)^* = S_- W_1([{C}, {C^*}, {S_- f}]); g),
\]

\[
    W_2^{(2)}([{C}, {C^*}, {S_- f}]); 0)^* = -W_2^{(1)}([{C}, {C^*}, {S_- f}]); 0).
\]

Using this lemma, we can reduce binary Darboux transformations for the AKNSESCS to binary Darboux transformations for the NLSESCS.

(1) **Darboux transformations for the NLS\(^+\)ESCS**

The binary Darboux transformation \((2.6)\) for the AKNSSCS is reduced to a binary Darboux transformation with an arbitrary function for the NLS\(^+\)ESCS as follows:
Proposition 3.1 Given a solution \((g, \varphi_1, \ldots, \varphi_m, w_1, \ldots, w_n)\) of the NLS+ ESCS \((3.9)\), let \(c(t)\) be a real function satisfying \(\dot{c}(t) \geq 0\), and let \(f\) be a solution of the linear system \((3.10)\) with \(\lambda = \zeta_{n+1}\), \(\Re \zeta_{n+1} = 0\) and satisfy \(f^{(1)} = f^{(2)}*\). Define

\[
\psi = \psi - \frac{f}{c(t) + \sigma(f, f)} \sigma(f, \psi), \quad \bar{q} = q - \frac{(f^{(1)})^2}{c(t) + \sigma(f, f)},
\]

(3.14a)

\[
\varphi_j = \varphi_j - \frac{f}{c(t) + \sigma(f, f)} \sigma(f, \varphi_j), \quad j = 1, \ldots, m,
\]

(3.14b)

\[
\bar{w}_j = w_j - \frac{f^{(1)}}{c(t) + \sigma(f, f)} \sigma(f, (w_j, w_j^*)^T), \quad j = 1, \ldots, n,
\]

(3.14c)

\[
\bar{w}_{n+1} = \sqrt{\frac{c(t) f^{(1)}}{c(t) + \sigma(f, f)}},
\]

(3.14d)

then the new variables \(\bar{\psi}, \bar{q}, \bar{\varphi}_1, \ldots, \bar{\varphi}_m\) and \(\bar{w}_1, \ldots, \bar{w}_{n+1}\) satisfy the system \((3.10)\) with \(n\) replaced by \(n + 1\). Hence \((\bar{q}, \bar{\varphi}_1, \ldots, \bar{\varphi}_m, \bar{w}_1, \ldots, \bar{w}_{n+1})\) is a solution of the NLS+ ESCS \((3.9)\) with \(n\) replaced by \(n + 1\). Moreover, we have

\[
|\bar{q}|^2 = |q|^2 - \partial_{\gamma^2} \log [c(t) + \sigma(f, f)].
\]

(3.15)

The two times repeated binary Darboux transformation for the AKNSESCS can be reduced to a second binary Darboux transformation with an arbitrary function for the NLS+ ESCS as follows:

Proposition 3.2 Given a solution \((g, \varphi_1, \ldots, \varphi_m, w_1, \ldots, w_n)\) of the NLS+ ESCS \((3.9)\), let \(c(t)\) be an arbitrary complex function, and \(f\) be a solution of the linear system \((3.10)\) with \(\lambda = \lambda_{m+1}\), \(\Re \lambda_{m+1} \neq 0\). Let \(\Delta = W_0\{c, f\}, \{c^*, S_+ f\}\), and define

\[
\bar{\psi} = \Delta^{-1} W_1\{c, f\}, \{c^*, S_+ f\}; \psi),
\]

(3.16a)

\[
\bar{q} = q + \Delta^{-1} W_2^{(1)}\{c, f\}, \{c^*, S_+ f\}; 0),
\]

(3.16b)

\[
\bar{\varphi}_j = \Delta^{-1} W_1\{c, f\}, \{c^*, S_+ f\}; \varphi_j), \quad j = 1, \ldots, m
\]

(3.16c)

\[
\bar{w}_j = \Delta^{-1} W_1^{-1}\{c, f\}, \{c^*, S_+ f\}; (w_j, w_j^*)^T), \quad j = 1, \ldots, n,
\]

(3.16d)

\[
\bar{\varphi}_{m+1} = \sqrt{c} (c \Delta)^{-1} W_1\{c, f\}, \{c^*, S_+ f\}; f)
\]

(3.16e)

then the new variables \(\bar{\psi}, \bar{q}, \bar{\varphi}_1, \ldots, \bar{\varphi}_{m+1}\) and \(\bar{w}_1, \ldots, \bar{w}_n\) satisfy the system \((3.10)\) with \(m\) replaced by \(m + 1\). Hence \((\bar{q}, \bar{\varphi}_1, \ldots, \bar{\varphi}_{m+1}, \bar{w}_1, \ldots, \bar{w}_m)\) is a solution of the NLS+ ESCS \((3.9)\) with \(m\) replaced by \(m + 1\). Moreover, we have

\[
|\bar{q}|^2 = |q|^2 - \partial_{\gamma^2} \log \Delta.
\]

(3.17)

If we repeat Darboux transformation \((3.14)\) for \(N\) times and Darboux transformation \((3.16)\) for \(M\) times, then we have a general multi-times repeated Darboux transformation with \(N + M\) arbitrary functions as follows:
**Proposition 3.3** Given a solution \((q, \varphi_1, \ldots, \varphi_m, w_1, \ldots, w_n)\) of the NLS+ ESCS \(3.32\), let \(f_j\) be a solution of the linear system \(3.10\) with \(\lambda = \zeta_{n+j}\), \(\Re \zeta_{n+j} = 0\), and satisfy \(f_j^{(1)} = f_j^{(2)*}\), \(j = 1, \ldots, N\), and let \(g_j\) be a solution of the linear system \(3.11\) with \(\lambda = \lambda_{m+j}\), \(\Re \lambda_{m+j} \neq 0\), \(j = 1, \ldots, M\). Let \(c_j(t)\) be an arbitrary real function satisfying \(c_j(t) > 0\), \(j = 1, \ldots, M\), and let \(d_j(t)\) be an arbitrary complex function, \(j = 1, \ldots, M\). Let \(F_j = \{c_j, f_j\}\), \(G_j = \{d_j, g_j\}\), \(G_k = \{d_k^*, S_+ g_j\}\), and \(\Delta = W_0(F_1, \ldots, F_N, G_1, G'_1, \ldots, G_M, G'_M)\), and define

\[
\tilde{\psi} = \Delta^{-1}W_1(F_1, \ldots, F_N, G_1, G'_1, \ldots, G_M, G'_M; \psi),
\]

(3.18a)

\[
\tilde{q} = q + \Delta^{-1}W_2^{(1)}(F_1, \ldots, F_N, G_1, G'_1, \ldots, G_M, G'_M; 0),
\]

(3.18b)

\[
\tilde{\varphi}_j = \Delta^{-1}W_1(F_1, \ldots, F_N, G_1, G'_1, \ldots, G_M, G'_M; \varphi_j), \quad j = 1, \ldots, m,
\]

(3.18c)

\[
\tilde{\varphi}_{m+j} = \sqrt{c_j}(c_j \Delta)^{-1}W_1(F_1, \ldots, F_N, G_1, G'_1, \ldots, G_M, G'_M; g_j), \quad j = 1, \ldots, M,
\]

(3.18d)

\[
\tilde{w}_j = \Delta^{-1}W_1^{(1)}(F_1, \ldots, F_N, G_1, G'_1, \ldots, G_M, G'_M; (w_j, w_j^*)^T), \quad j = 1, \ldots, n,
\]

(3.18e)

\[
\tilde{w}_{n+j} = \sqrt{d_j}(d_j \Delta)^{-1}W_1(F_1, \ldots, F_N, G_1, G'_1, \ldots, G_M, G'_M; f_j), \quad j = 1, \ldots, N,
\]

(3.18f)

then the new variables \(\tilde{\psi}, \tilde{q}, \tilde{\varphi}_1, \ldots, \tilde{\varphi}_{m+M}\) and \(\tilde{w}_1, \ldots, \tilde{w}_{n+N}\) satisfy the system \(3.14\) with \(m, n\) replaced by \(m + M, n + N\), respectively. Hence \((\tilde{q}, \tilde{\varphi}_1, \ldots, \tilde{\varphi}_{m+M}, \tilde{w}_1, \ldots, \tilde{w}_{n+N})\) is a solution of the NLS+ ESCS \(3.32\) with \(m, n\) replaced by \(m + M, n + N\). Moreover, we have

\[
|\tilde{q}|^2 = |q|^2 - \partial_x^2 \log \Delta.
\]

(3.19)

**Proposition 3.4** Given a solution \((q, \varphi_1, \ldots, \varphi_m)\) of the NLS+ ESCS \(3.12\), let \(f\) be a solution of the linear system \(3.11\) with \(\lambda = \lambda_{m+1}\), \(\Re \lambda_{m+1} \neq 0\). Let \(c(t)\) be an arbitrary complex function, \(\Delta = W_0([c, f], [-c^*, S_- f])\), and define

\[
\tilde{\psi} = \Delta^{-1}W_1([c, f], [-c^*, S_- f]; \psi),
\]

(3.20a)

\[
\tilde{q} = q + \Delta^{-1}W_2^{(1)}([c, f], [-c^*, S_- f]; 0),
\]

(3.20b)

\[
\tilde{\varphi}_j = \Delta^{-1}W_1([c, f], [-c^*, S_- f]; \varphi_j), \quad j = 1, \ldots, m
\]

(3.20c)

\[
\tilde{\varphi}_{n+1} = \sqrt{c}(c \Delta)^{-1}W_1([c, f], [-c^*, S_- f]; f),
\]

(3.20d)

then the new variables \(\tilde{\psi}, \tilde{q}, \tilde{\varphi}_1, \ldots, \tilde{\varphi}_{m+1}\) satisfy the system \(3.13\) with \(m\) replaced by \(m + 1\). Moreover, we have

\[
|\tilde{q}|^2 = |q|^2 + \partial_x^2 \log \Delta.
\]

(3.21)
Repeating the above Darboux transformation for \( N \) times gives rise to a general \( N \)-times repeated binary Darboux transformation with \( N \) arbitrary functions for the NLS^−ESCS.

**Proposition 3.5** Given a solution \((q, \varphi_1, \ldots, \varphi_m)\) of the NLS^− equations with sources \(0, M\), let \( f_j \) be a solution of the linear system \( 0, M \) with \( \lambda = \lambda_{m+j} \), \( \text{Re} \lambda_{m+j} \neq 0 \), \( j = 1, \ldots, N \). Let \( c_j(t) \) be an arbitrary complex function, \( F_j = \{c_j, f_j\} \), \( F'_j = \{-c_j', S-f_j\} \), \( j = 1, \ldots, N \), \( \Delta = W_0(F_1, F'_1, \ldots, F_N, F'_N) \), and define

\[
\tilde{\psi} = \Delta^{-1}W_1(F_1, F'_1, \ldots, F_N, F'_N; \psi),
\]

\(\tilde{q} = q + \Delta^{-1}W_2^{(1)}(F_1, F'_1, \ldots, F_N, F'_N; 0),\)

\[
\varphi_j = \Delta^{-1}W_1(F_1, F'_1, \ldots, F_N, F'_N; \varphi_j), \quad j = 1, \ldots, m
\]

\[
\varphi_{m+j} = \sqrt{e_j} (e_j \Delta)^{-1}W_1(F_1, F'_1, \ldots, F_N, F'_N; f_j), \quad j = 1, \ldots, N
\]

then the new variables \( \tilde{\psi}, \tilde{q}, \varphi_1, \ldots, \varphi_{m+1} \) satisfy the system \( 0, M \) with \( m \) replaced by \( m + N \), and hence \((\tilde{q}, \varphi_1, \ldots, \varphi_{m+N})\) is a solution of the NLS^+ESCS \( 0, M \) with \( m \) replaced by \( m + N \). Moreover, we have

\[
|\tilde{q}|^2 = |q|^2 + \partial^2_t \log \Delta.
\]

### 4 Solutions of the NLS equations with sources

This section is devoted to the obtaining some examples of the solutions of the NLSESCS by Darboux transformations and the analysis for these solutions. We use subscripts \( z_R \) and \( z_I \) to indicate the real part and the imaginary part of a complex number \( z \). For \( \forall z = |z|e^{i\theta} \in \mathbb{C} \) with \( \theta \in (-\pi, \pi] \), we define \( \sqrt{z} = \sqrt{|z|} e^{i\theta/2} \).

#### 4.1 Solutions of the NLS^+ESCS

We only consider the NLS^+ESCS \( 0, M \) with \( m = 0 \). We start from the NLS^+ESCS (i.e., \( m = n = 0 \))

\[
q_t = i(2|q|^2 q - q_{xx})
\]

and its solution

\[
q = \rho e^{2i\rho^2 t},
\]

where \( \rho \in \mathbb{R}_+ \) is a constant. We need to solve the linear system

\[
\psi_x = U(\lambda, \rho e^{2i\rho^2 t}, \rho e^{-2i\rho^2 t})\psi, \quad \psi_t = V(\lambda, \rho e^{2i\rho^2 t}, \rho e^{-2i\rho^2 t})\psi.
\]

The fundamental solution matrix for the linear system \( 0, M \) is

\[
\Psi = \begin{pmatrix}
\rho e^{\kappa(x+2i\lambda t)+i\rho^2 t} & (\kappa + \rho) e^{-\kappa(x+2i\lambda t)+i\rho^2 t} \\
(\kappa + \lambda) e^{\kappa(x+2i\lambda t)-i\rho^2 t} & -\rho e^{-\kappa(x+2i\lambda t)-i\rho^2 t}
\end{pmatrix},
\]

where \( \kappa = \kappa(\lambda) \) satisfies \( \kappa^2 = \lambda^2 + \rho^2 \).
4.1.1 Solutions of the NLS$^+$ ESCS with $m = 0$ and $n = 1$.

The NLS$^+$ ESCS with $m = 0$ and $n = 1$ reads

$$w_{1,x} = i\ell w_1 + qu_1^*, \quad (4.5a)$$

where $\ell \neq 0$ is a real constant. Let $f$ be a solution of the system \ref{eq:4.3} with $i = \ell t$, and let $c(t)$ be an arbitrary real function with $c(t) \geq 0$, then by Proposition \ref{prop:3.3}, a solution to the equation is given by

$$q = \rho e^{2i\rho^2 t} - \frac{(f^{(1)})^2}{c(t) + \sigma(f, f)}, \quad w_1 = \frac{\sqrt{c(t)} f^{(1)}}{c(t) + \sigma(f, f)}.$$

Moreover, we have

$$|q|^2 = \rho^2 - \partial_x^2 \log[c(t) + \sigma(f, f)]. \quad (4.7)$$

For the two cases: $\rho > |\ell|$ and $\rho < |\ell|$, formulas \ref{eq:4.6} will give two different classes of solutions respectively: dark one-soliton solution and one-positon solution.

(1) Dark one-soliton solution and scattering property.

We take $\rho > |\ell|$ and let $\kappa_1 = \kappa(i\ell)$. We choose $\kappa = \sqrt{\lambda^2 + \rho^2}$, then $\kappa$ and $\sqrt{\lambda^2 + \rho^2}$ are analytic at $\lambda = i\ell$, and $\kappa_1 = \sqrt{\rho^2 - \ell^2} > 0$. Taking into account that the equality $\rho = \sqrt{\kappa - \lambda / \sqrt{\kappa + \lambda}}$ holds near $\lambda = i\ell$, we choose $f$ as

$$f = \left[\Psi \left(\begin{array}{c} \sqrt{\kappa - \lambda} / \rho \\ 0 \end{array}\right)\right]_{\lambda = i\ell} = \left[\begin{array}{c} \sqrt{\kappa - \lambda} e^{(x+2i\lambda t)+i\rho^2 t} \\ \sqrt{\kappa + \lambda} e^{(x+2i\lambda t)-i\rho^2 t} \end{array}\right]_{\lambda = i\ell}.$$

Then one finds that $f^{(2)} = f^{(1)*}$. Calculation yields

$$\sigma(f, f) = \frac{1}{2} \left| f^{(1)} \partial_{(i)\ell} f^{(1)} - f^{(2)} \partial_{(i)\ell} f^{(2)} \right| = \frac{\rho}{2\kappa_1} e^{2\kappa_1(x-2\ell t)}.$$

Let $c(t) = (2\kappa_1)^{-1} \rho e^{2\kappa_1(at+b)}$ with $a \in \mathbb{R}_+$, $b \in \mathbb{R}$ being constants, then formulas \ref{eq:4.6} give a dark one-soliton solution

$$q = \rho e^{2i\rho^2 t} - \frac{2\kappa_1(\kappa_1 - i\ell)e^{2\kappa_1(x-2\ell t)+2i\rho^2 t}}{\rho(e^{2\kappa_1(at+b)} + e^{2\kappa_1(x-2\ell t)})} = \frac{1 - e^{-4\theta} e^{2\xi}}{1 + e^{2\xi}} \rho e^{2i\rho^2 t}, \quad (4.8a)$$

$$w_1 = \sqrt{\rho e^{2\kappa_1(x-2\ell t)+2i\rho^2 t}} = \frac{2\sqrt{\rho} \kappa_1 e^{\xi - i\theta}}{1 + e^{2\xi}} e^{i\rho^2 t},$$

where

$$\xi = \kappa_1[x - (2\ell + a)t - b], \quad \theta = \frac{1}{2} \arcsin \frac{\ell}{\rho}.$$ 

By formula \ref{eq:4.7}, one obtains

$$|q|^2 = \rho^2 - \partial_x^2 \log(1 + e^{2\xi}) = \rho^2 - \frac{\kappa_1^2}{\cosh^2 \xi}, \quad (4.9)$$

which shows that $|q|^2$ describes the propagation of a dark soliton on the constant background $\rho$. The soliton is localized around $\xi = 0$, so the location of the soliton is $x(t) = (2\ell + a)t + b$, and the the soliton velocity is $2\ell + a$. If $a = 0$, then $w_1 \equiv 0$, and $q$ defined by \ref{eq:4.8a} becomes a dark one-soliton solution \ref{eq:27} of the NLS$^+$ equation \ref{eq:11}. 

\[\]
We fix a solution of the system \( q \) as
\[
\psi_0(x, t; \lambda) = \left( \frac{\rho e^{i\alpha t}}{(\kappa + \lambda) e^{-i\beta t}} \right) e^{\kappa(x+2i\lambda t)},
\]
(4.10)
Then a solution of the NLS\(^+\) spectral problem
\[
\psi_x = U(\lambda, q, q^\ast)\psi
\]
(4.11)
with \( q \) defined by \( q \) is given by
\[
\psi = \psi_0 - \frac{f\sigma(f, \psi_0)}{c(t) + \sigma(f, f)} = \left( \frac{\rho e^{i\alpha t}}{(\kappa + \lambda) e^{-i\beta t}} \right) e^{\kappa(x+2i\lambda t)} - \left( \frac{\sqrt{\kappa_1 - i\ell} e^{i\alpha t}}{\sqrt{\kappa_1 + i\ell} e^{-i\beta t}} \right) \frac{\kappa_1 e^{2\xi \kappa(x+2i\lambda t)} e^{\lambda(x+2i\lambda t)}}{\rho(\lambda - i\ell)(1 + e^{2\xi})} \times
\]
\[
\sqrt{\kappa_1 - i\ell} \left| \frac{\rho}{\sqrt{\kappa_1 + i\ell}} \right| \sqrt{\kappa_1 + i\ell} \kappa + \lambda
\]
\[
\times \left( \frac{\rho^2 + i\ell \lambda - \kappa_1 \kappa \kappa_1 e^{2\xi t} - \left( (\kappa + \lambda)(\kappa_1 + i\ell) e^{-i\beta t} \right) e^{\kappa(x+2i\lambda t)}. \right)
\]
(4.12)
Based on formulas \( q \), we can analyze the asymptotic features of the dark one-soliton solution. For fixed \( t \), we have
\[
q = \begin{cases} 
\rho e^{2i\beta t}[1 + o(1)], & x \to -\infty, \\
\rho e^{i(\pi - 4\beta)} e^{2i\beta t}[1 + o(1)], & x \to +\infty,
\end{cases}
\]
(4.13)
\[
w_1 \to 0, \quad x \to \pm\infty.
\]
(4.14)
It is easy to see that \( q \) belongs to the class of potentials satisfying the finite density boundary condition \( q \)
\[
q(x, t) = \rho e^{i\alpha \pm(t)}[1 + o(1)], \quad x \to \pm\infty,
\]
(4.15)
where \( \alpha_\pm(t) \) are real functions and \( \beta \equiv \frac{1}{2}(\alpha_+(t) - \alpha_-(t)) \) is a real constant independent of \( t \). We now define the scattering data for this class of potentials in a similar way in \( q \).

First, we define \( u = q e^{-i\alpha_-(t)} \), then \( u \) satisfies the standard finite density boundary condition
\[
u(x, t) = \begin{cases} 
\rho[1 + o(1)], & x \to -\infty, \\
\rho e^{2i\beta}[1 + o(1)], & x \to +\infty.
\end{cases}
\]
(4.16)
Next, we define transmission and reflection coefficients for the NLS\(^+\) spectral system
\[
\phi_x = \left( \begin{array}{c}
-\lambda \\
u \\
\lambda
\end{array} \right) \phi.
\]
(4.17)
For \( u \equiv \rho \), the system \( q \) has two linearly independent solutions
\[
\left( \frac{\rho}{\kappa + \lambda} \right) e^{\kappa x}, \quad \left( -\frac{1}{\kappa + \lambda} \right) e^{-\kappa x},
\]
while for \( u \equiv \rho e^{2i\beta} \), the system \( q \) has two linearly independent solutions
\[
Q(\beta) \left( \frac{\rho}{\kappa + \lambda} \right) e^{\kappa x}, \quad Q(\beta) \left( -\frac{1}{\kappa + \lambda} \right) e^{-\kappa x},
\]
where \( Q(\beta) = \text{diag}(e^{i\beta}, e^{-i\beta}) \). We fix a Jost solution \( \phi \) of the system \( q \) by imposing the asymptotic property
\[
\phi = \left( \frac{\rho}{\kappa + \lambda} \right) e^{\kappa x}[1 + o(1)], \quad x \to -\infty,
\]
(4.18)
while the transmission and reflection coefficients \( a(\lambda, t) \) and \( b(\lambda, t) \) are determined by the asymptotic estimate

\[
\phi = a(\lambda, t)Q(\beta) \left( \frac{1}{\kappa + \lambda} \right) e^{\kappa x} + b(\lambda, t)Q(\beta) \left( \frac{-1}{\kappa + \lambda} \right) e^{-\kappa x}, \quad x \to +\infty. \tag{4.19}
\]

We can now calculate the scattering data for the dark one-soliton solution. In this case, we have \( u = qe^{-i\rho^2 t} \) and \( \beta = \pi/2 - 2\theta \). Formula \[18\] implies that the function \( \psi \) has the asymptotic behaviors

\[
\psi = \frac{\rho e^{i\rho^2 t}}{(\kappa + \lambda)e^{-i\rho^2 t}} e^{\kappa(x+2i\lambda t)}[1 + o(1)], \quad x \to -\infty, \tag{4.20}
\]

\[
\psi = \frac{\rho^2 + i\ell\lambda - \kappa_1\kappa}{i\rho(\lambda - i\ell)} \left( \frac{\rho(\kappa_1 - i\ell)e^{i\rho^2 t}}{-(\kappa + \lambda)(\kappa_1 + i\ell)e^{-i\rho^2 t}} \right) e^{\kappa(x+2i\lambda t)}[1 + o(1)], \quad x \to +\infty, \tag{4.21}
\]

We now take the Jost solution

\[
\phi = Q(-\rho^2 t)(\kappa + \lambda)^{-1} e^{-2i\kappa\lambda t} \psi, \tag{4.22}
\]

then we have

\[
\phi = \frac{\rho^2 + i\ell\lambda - \kappa_1\kappa}{i\rho(\lambda - i\ell)} Q(\pi/2 - 2\theta) \left( \frac{1}{\kappa + \lambda} \right) e^{\kappa x}[1 + o(1)], \quad x \to +\infty, \tag{4.23}
\]

which implies that

\[
a(\lambda, t) = \frac{\rho^2 + i\ell\lambda - \kappa_1\kappa}{i\rho(\lambda - i\ell)}, \quad b(\lambda, t) = 0. \tag{4.24}
\]

The dark one-soliton solution is a reflectionless potential.

(2) One-positon solution and super-reflectionless property

We take \( \rho < |\ell| \) and choose \( \kappa = (\text{sign } \lambda f) i\sqrt{-\lambda^2 - \rho^2} \), then \( \kappa \) is analytic at \( \lambda = i\ell \), and \( \kappa(i\ell) = ik_1 \), where \( k_1 = (\text{sign } \ell) \sqrt{\ell^2 - \rho^2} \) is a real constant. Choose a periodic solution of the system \[3\] with \( \lambda = i\ell \) as

\[
f = \begin{bmatrix} \Psi \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{bmatrix} \bigg|_{\lambda=i\ell} = \begin{pmatrix} \rho e^{\kappa(x+2i\lambda t)} + i\rho^2 t - (\kappa + \lambda)e^{-\kappa(x+2i\lambda t)} + i\rho^2 t \\ (\kappa + \lambda)e^{\kappa(x+2i\lambda t)} - i\rho^2 t + \rho e^{-\kappa(x+2i\lambda t)} - i\rho^2 t \end{pmatrix} \bigg|_{\lambda=i\ell} \]

\[
= \begin{pmatrix} \rho e^{i(\Theta + \rho^2 t)} - i(k_1 + \ell)e^{-i(\Theta - \rho^2 t)} \\ i(k_1 + \ell)e^{i(\Theta - \rho^2 t)} + \rho e^{-i(\Theta + \rho^2 t)} \end{pmatrix}, \tag{4.25}
\]

where \( \Theta = k_1(x - 2\ell t) \). One finds \( f^{(2)} = f^{(1)*} \), and

\[
(f^{(1)})^2 = 2\ell(k_1 + \ell)[-k_1 \ell^{-1} \cos 2\Theta + i(\sin 2\Theta - \rho \ell^{-1})] e^{2i\rho^2 t},
\]

\[
\sigma(f, f) = 2\ell(k_1 + \ell)[x - 2(k_1 \ell^{-1} + \ell) t + \rho(2k_1 \ell)^{-1} \cos 2\Theta].
\]

Choose \( c(t) = 2\ell(k_1 + \ell)(at + b) \) with \( a \in \mathbb{R}^+, b \in \mathbb{R} \) being constants, which implies that \( c(t) \geq 0 \). Then formulas \[4\] give a one-positon solution

\[
q = pe^{2i\rho^2 t} - \frac{(f^{(1)})^2}{c(t) + \sigma(f, f)} = \left[ \frac{\rho + k_1 \ell^{-1} \cos 2\Theta - i(\sin 2\Theta - \rho \ell^{-1})}{\gamma + \rho(2k_1 \ell)^{-1} \cos 2\Theta} \right] e^{2i\rho^2 t}, \tag{4.26a}
\]

\[
w_1 = \sqrt{c(t) f^{(1)}} = \sqrt{\frac{a}{2} \cdot \frac{\sqrt{1 - k_1 \ell^{-1}} e^{i\Theta} - i\sqrt{1 + k_1 \ell^{-1}} e^{-i\Theta}}{\gamma + (2k_1 \ell)^{-1} \rho \cos 2\Theta}} e^{i\rho^2 t}, \tag{4.26b}
\]

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where

\[ \gamma = x + [a - 2(\ell + k_1^2 \ell^{-1})]t + b. \]

Formula (4.17) implies

\[
|q|^2 = \rho^2 - \partial_x^2 \log[\gamma + (2k_1 \ell)^{-1} \rho \cos 2\Theta] = \rho^2 + \frac{1 + \rho^2 \ell^{-2} + 2\rho \ell^{-1}(k_1 \gamma \cos 2\Theta - \sin 2\Theta)}{[\gamma + \rho(2k_1 \ell)^{-1} \cos 2\Theta]^2}. \tag{4.27}
\]

When \( \rho = a = 0 \), we have \( k_1 = \ell \) and \( w_1 \equiv 0 \), and formulas (4.26) degenerate to a solution of the NLS* equation (4.11) with the potential \( \gamma = 0 \), which was given in [25].

A solution of the NLS* spectral problem (4.11) with the potential \( \gamma \) defined by (4.26) is

\[
\psi = \psi_0 - \frac{f \sigma(f, \psi_0)}{e(t) + \sigma(f, f)} = \left( \rho e^{iy^2 t} \right) e^{\kappa(x + 2i\lambda t)} - \left( \frac{\rho e^{i\Theta} - i(k_1 + \ell)e^{-i\Theta} e^{iy^2 t}}{[i(k_1 + \ell) e^{i\Theta} + \rho e^{-i\Theta}]} e^{iy^2 t} \right) \times
\]

\[
\frac{e^{\kappa(x + 2i\lambda t)}}{4(\lambda - i\ell)(k_1 + \ell)[\gamma + (2k_1 \ell)^{-1} \rho \cos 2\Theta]}
\left| \begin{array}{cc}
\rho e^{i\Theta} - i(k_1 + \ell) e^{-i\Theta} & \rho \\
[i(k_1 + \ell) e^{i\Theta} + \rho e^{-i\Theta}] & \kappa + \lambda
\end{array} \right|
\tag{4.29}
\]

Based on formulas (4.26) and (4.29), we can analyze the basic features of the one-positon solution. Formulas (4.26) imply that for fixed \( t \) and \( x \to \pm \infty \), we have the asymptotic estimate

\[
q e^{-2iy^2 t} = \rho + [k_1 \ell^{-1} \cos 2\Theta - i(\sin 2\Theta - \rho \ell^{-1})]x^{-1}[1 + O(x^{-1})],
\tag{4.30}
\]

\[
w_1 e^{-iy^2 t} = \sqrt{a/2} (\sqrt{1 - k_1 \ell^{-1}} e^{i\Theta} - i \sqrt{1 + k_1 \ell^{-1}} e^{-i\Theta}) x^{-1}[1 + O(x^{-1})]
\tag{4.31}
\]

for all \( \rho \in \mathbb{R}_+ \). However, the asymptotic behavior of \( |q|^2 \) for \( \rho = 0 \) is different with that for \( \rho > 0 \). Actually, for \( \rho = 0 \), we have

\[
|q|^2 = x^{-2}[1 + O(x^{-1})],
\]

while for \( \rho > 0 \), we have

\[
|q|^2 = \rho^2 + 2k_1 \rho \ell^{-1} x^{-1} \cos 2\Theta[1 + O(x^{-1})].
\tag{4.32}
\]

Compared to the dark one-soliton solution, the one-positon solution converges to its background slowly.

As a function of \( x \), the potential \( q \) and the source \( w_1 \) share the same first-order pole \( x = x_0(t) \), which is implicitly determined by the equation

\[
2k_1 [x_0 + (a - 2\ell - 2k_1^2 \ell^{-1}]t + b] = \rho \cos(2k_1 x_0 - 4k_1 \ell t).
\]

The uniqueness of the solution \( x_0 \) can be easily proved. Let \( \gamma_0(t) = x_0(t) + (a - 2\ell - 2k_1^2 \ell^{-1})t + b \), then \( \gamma_0 \) satisfies

\[
2k_1 \ell \gamma_0 = \rho \cos(2k_1 [\gamma_0 - (a - 2k_1^2 \ell^{-1})t - b]).
\]

This equation implies that \( \gamma_0(t) \) is a periodic function of \( t \) with period \( \ell \pi/(2k_1^3) \). We define the velocity of a positon as the velocity of its pole. From this definition, the velocity of the positon is

\[
v(t) = v(t + T) = \dot{x}_0(t) = [2\ell + 2k_1^2 \ell^{-1} - a + \dot{\gamma}_0(t)],
\]
Figure 1: The one-positon solution of the NLS$^+$ESCS with $\ell = 5$. The data is $\rho = 3$, $a = 2$ and $b = 1$. The plots are taken at $t = 0$. The two upper graphs show the real and imaginary parts of $q$ respectively while the two lower graphs show the modulus of $q$ and the real part of $w_1$ respectively.

where $T = \pi/(2k_1^2)$, and the average speed of the positon is

$$\frac{1}{T} \int_0^T v(t) dt = (2\ell + 2k_1^2\ell^{-1} - a).$$

In Figure 1, we plot a one-positon solution of the NLS$^+$ESCS. In this case, $u = qe^{-i\rho^2t}$ and $\beta = 0$. Formula (4.29) implies the asymptotic behavior of the function $\psi$

$$\psi = \left( \frac{\rho e^{i\rho^2 t}}{(\kappa + \lambda)e^{-i\rho^2 t}} \right) e^{\kappa(x + 2i\lambda t)}[1 + o(1)], \quad x \to \pm \infty.$$

We take the Jost solution as

$$\phi = Q(-\rho^2 t)(\kappa + \lambda)^{-1} e^{-2i\kappa \lambda t} \psi,$$

then we have

$$\phi \to \left( \frac{\rho}{\kappa + \lambda} \right) e^{\kappa x}, \quad x \to \pm \infty, \quad a(\lambda, t) = 1, \quad b(\lambda, t) = 0.$$

Potentials with reflection coefficient $b = 0$ and transmission coefficient $a = 1$ are called superreflectionless or supertransparent potentials. By this definition, the one-positon solution is superreflectionless.

In [23], positons are defined as long-range analogous of solitons and slowly decreasing, oscillating solutions of nonlinear integrable equations. If we stick to the property of slowly decreasing, the potential $q$ defined by (4.26) should not be called a one-positon solution unless $\rho = 0$. However, we see that other properties such as long-range analogous of a soliton, super-reflectionless property are still valid. Thus it is reasonable to extend the definition of positons as: long-range analogous of solitons, slowly converging, oscillating solutions of nonlinear integrable equations. According to this extended definition, the solution (4.26) is a positon solution.
4.1.2 Solutions of the NLS$^+$ equation with sources with $m = 0$ and $n = 2$.

The NLS$^+$ equation with sources with $m = 0$ and $n = 2$ reads

\[ w_{1,x} = i\ell_1 w_1 + qw_1^*, \quad w_{2,x} = i\ell_2 w_2 + qw_2^*, \quad (4.33a) \]

\[ q_t = i(2|q|^2q - q_{xx}) + w_1^2 + w_2^2, \quad (4.33b) \]

where $\ell_1$ and $\ell_2$ are two distinct real constants. For $j = 1, 2$, let $f_j$ be a solution of the system with $\lambda = i\ell_j$ and satisfy $f_j^{(1)} = f_j^{(2)*}$, and let $c_j(t)$ be an arbitrary function with $c_j(t) \geq 0$. Then by Proposition 3.3, a solution of the Eqs. is given by

\[ q = pe^{2i\rho^2 t} + \frac{2\sigma(f_1, f_2) f_1^{(1)} f_2^{(1)} - (c_1(t) + \sigma(f_2, f_2))(f_1^{(1)})^2 - (c_2(t) + \sigma(f_1, f_1))(f_2^{(1)})^2}{(c_1(t) + \sigma(f_1, f_1))(c_2(t) + \sigma(f_2, f_2)) - \sigma(f_1, f_2)^2}, \quad (4.34a) \]

\[ w_1 = \frac{\sqrt{c_1(t)} [(c_2(t) + \sigma(f_2, f_2)) f_1^{(1)} - \sigma(f_1, f_2) f_2^{(1)}]}{(c_1(t) + \sigma(f_1, f_1))(c_2(t) + \sigma(f_2, f_2)) - \sigma(f_1, f_2)^2}, \quad (4.34b) \]

\[ w_2 = \frac{\sqrt{c_2(t)} [(c_1(t) + \sigma(f_1, f_1)) f_2^{(1)} - \sigma(f_1, f_2) f_1^{(1)}]}{(c_1(t) + \sigma(f_1, f_1))(c_2(t) + \sigma(f_2, f_2)) - \sigma(f_1, f_2)^2}. \quad (4.34c) \]

Moreover, we have

\[ |q|^2 = \rho^2 - \frac{\partial_t^2}{\rho^2 - \ell_j^2}. \quad (4.35) \]

For simplicity, we assume $|\ell_1| > |\ell_2|$. According to the three cases for $\rho$: (i) $\rho > |\ell_j|$, $j = 1, 2$, (ii) $\rho < |\ell_j|$, $j = 1, 2$, and (iii) $|\ell_1| > \rho > |\ell_2|$, formulas 4.34 will give three classes of solutions respectively: dark two-soliton solution, two-positon solution and one-soliton-one-positon solution.

(1) Dark two-soliton solution.

For $j = 1, 2$, we take $\rho > |\ell_j|$, and choose

\[ f_j = \left[ \begin{array}{c} \sqrt{\kappa - \lambda / \rho} \\ 0 \end{array} \right]_{\lambda = i\ell_j} = \frac{\sqrt{\kappa_j - i\ell_j} e^{\kappa_j(x-2\ell_j t) + i\rho^2 t}}{\sqrt{\kappa_j + i\ell_j} e^{\kappa_j(x-2\ell_j t) - i\rho^2 t}}, \]

where $\kappa = \sqrt{\lambda^2 + \rho^2}$ and $\kappa_j = \sqrt{\rho^2 - \ell_j^2}$. Let

\[ c_j(t) = \frac{\rho}{2\kappa_j} e^{2\kappa_j(x-2\ell_j t)}; \quad \theta_j = \frac{1}{2} \arcsin \frac{\ell_j}{\rho}, \quad j = 1, 2, \]

where $\alpha_j \in \mathbb{R}_+$, $b_j \in \mathbb{R}$ are constants, then one finds

\[ \sigma(f_j, f_j) = \frac{\rho}{2\kappa_j} e^{2\kappa_j(x-2\ell_j t)}, \quad j = 1, 2, \quad \sigma(f_1, f_2) = \frac{\rho \sin(\theta_1 - \theta_2)}{\ell_1 - \ell_2} e^{\kappa_1(x-2\ell_1 t) + \kappa_2(x-2\ell_2 t)}. \]

Formulas 4.34 yield a dark two-soliton solution

\[ q = \frac{1}{\Delta} \begin{vmatrix} \rho^2(1 + e^{2\xi_1}) & \rho \sin(\theta_1 - \theta_2) e^{\xi_1 + \xi_2} \\ \rho \sin(\theta_1 - \theta_2) e^{\xi_1 + \xi_2} & \rho^2(1 + e^{2\xi_2}) \end{vmatrix} \begin{vmatrix} \rho^2 e^{\xi_1 + i(\rho^2 t - \theta_1)} \\ \rho^2 e^{\xi_2 + i(\rho^2 t - \theta_2)} \end{vmatrix} \begin{vmatrix} \rho^2 e^{2i\rho^2 t} \\ \rho^2 e^{2i\rho^2 t} \end{vmatrix}, \quad (4.36a) \]
\[ w_1 = \frac{\sqrt{a_1 \rho e^{ip^2 t}}}{\Delta} \left| \frac{\rho}{2k_2}(1 + e^{2\xi_2}) \frac{e^{i(\theta_1 - \theta_2)_{\xi_2}}}{e^{i\xi_2}} \right|, \quad (4.36b) \]

\[ w_2 = \frac{\sqrt{a_2 \rho e^{ip^2 t}}}{\Delta} \left| \frac{\rho}{2k_1}(1 + e^{2\xi_1}) \frac{e^{i(\theta_1 - \theta_2)_{\xi_1}}}{e^{i\xi_1}} \right|, \quad (4.36c) \]

where \( \xi_j = \kappa_j[x - (2\ell_j + a_j)t - b_j], \) \( j = 1, 2, \) and

\[ \Delta = \left| \frac{\rho}{2k_1}(1 + e^{2\xi_1}) \frac{e^{i(\theta_1 - \theta_2)_{\xi_1}}}{e^{i\xi_1}} \right| - \left| \frac{\rho}{2k_2}(1 + e^{2\xi_2}) \right|. \]

(2) Two-positon solutions and positon-positon interaction

For \( j = 1, 2, \) we let \( \rho < |\ell_j|, \) and choose

\[ f_j = \left[ \Psi \left( \frac{1}{-1} \right) \right]_{\lambda=\ell_j} = \left( \left[ \rho e^{i\Theta_j} - i(k_j + \ell_j) e^{-i\Theta_j} e^{ip^2 t} \right] \frac{[i(k_j + \ell_j) e^{i\Theta_j} + \rho e^{-i\Theta_j}] e^{-ip^2 t}}{\rho e^{i\Theta_j} - i(k_j + \ell_j) e^{-i\Theta_j} e^{ip^2 t}} \right), \]

where \( \kappa = (\text{sign } \lambda_j) i \sqrt{-\lambda^2 - \rho^2}, \) and \( \Theta_j = k_j(x - 2\ell_j + a_j t - b_j), \) \( k_j = (\text{sign } \ell_j) \sqrt{\ell_j^2 - \rho^2}. \) Let

\[ c_j(t) = 2\ell_j(k_j + \ell_j)(a_j t + b_j), \quad \gamma_j = x + [a_j - 2(\ell_j + k_j^2 \ell_j^{-1})] t + b_j, \quad j = 1, 2, \]

where \( a_j \in \mathbb{R}_+, \) \( b_j \in \mathbb{R} \) are constants. Then one finds

\[ c_j(t) + \sigma(f_j, f_j) = 2\ell_j(k_j + \ell_j)[\gamma_j + \rho(2k_j \ell_j)^{-1} \cos 2\Theta_j], \quad j = 1, 2, \]

\[ \sigma(f_1, f_2) = \rho(1 + \frac{k_1^2 - k_2^2}{\ell_1 + \ell_2}) \cos(\Theta_1 + \Theta_2) - [\rho^2 - (k_1 + \ell_1)(k_2 + \ell_2)] \frac{\sin(\Theta_1 - \Theta_2)}{\ell_1 - \ell_2} \]

Formulas \( 4.35 \) give a two-positon solution

\[ q = \rho e^{2ip^2 t} + \frac{2f_1^{(1)} f_2^{(1)} \sigma(f_1, f_2) - 2\ell_2(k_2 + \ell_2) \Gamma_2 f_1^{(1)} - 2\ell_1(k_1 + \ell_1) \Gamma_1 f_2^{(1)}}{4\ell_1 \ell_2(k_1 + \ell_1)(k_2 + \ell_2) \Gamma_1 \Gamma_2 - \sigma(f_1, f_2)^2}, \quad (4.37a) \]

\[ w_1 = \sqrt{2a_1 \ell_1(k_1 + \ell_1)} \frac{2\ell_2(k_2 + \ell_2) \Gamma_2 f_1^{(1)} - \sigma(f_1, f_2) f_2^{(1)}}{4(k_1 + \ell_1)(k_2 + \ell_2) \Gamma_1 \Gamma_2 - \sigma(f_1, f_2)^2}, \quad (4.37b) \]

\[ w_2 = \sqrt{2a_2 \ell_2(k_2 + \ell_2)} \frac{2\ell_1(k_1 + \ell_1) \Gamma_1 f_2^{(1)} - \sigma(f_1, f_2) f_1^{(1)}}{4(k_1 + \ell_1)(k_2 + \ell_2) \Gamma_1 \Gamma_2 - \sigma(f_1, f_2)^2}, \quad (4.37c) \]

where

\[ \Gamma_j = \gamma_j + \rho(2k_j \ell_j)^{-1} \cos 2\Theta_j, \quad j = 1, 2. \]

Assume that \( 2\ell_1 + 2k_1^2 \ell_1^{-1} - a_1 \neq 2\ell_2^2 + 2k_2^2 \ell_2^{-1} - a_2. \) Fixing \( \gamma_1 \) and letting \( t \to \infty \) (which implies \( \gamma_2 \to \infty \), we obtain the asymptotic estimate

\[ q = \rho e^{2ip^2 t} - \frac{k_1 \ell_1^{-1} \cos 2\Theta_1 - i(\sin 2\Theta_1 - \rho \ell_1^{-1})}{\gamma_1 + (2k_1 \ell_1)^{-1} \rho \cos 2\Theta_1} e^{2ip^2 t} [1 + O(t^{-1})], \quad (4.38a) \]

\[ w_1 = \sqrt{a_1} \left\{ \frac{1 - k_1 \ell_1^{-1} e^{i\Theta_1} - i \sqrt{1 + k_1 \ell_1^{-1} e^{-i\Theta_1}}}{\gamma_1 + (2k_1 \ell_1)^{-1} \rho \cos 2\Theta_1} e^{ip^2 t} [1 + O(t^{-1})], \quad w_2 = O(t^{-1}). \quad (4.38b) \]
Conversely, if we fix $\gamma_2$ and let $t \to \infty$, then we obtain

$$q = pe^{2i\epsilon_2 t} - \frac{k_2 \ell_2^{-1} \cos 2\Theta_2 - i(i \sin 2\Theta_2 - \rho k_2^{-1})e^{2i\epsilon_2 t}[1 + O(t^{-1})]}{\gamma_2 + (2k_2 \ell_2)^{-1} \rho \cos 2\Theta_2},$$

(4.39a)

$$w_1 = O(t^{-1}), \quad w_2 = \sqrt{\frac{\gamma_2}{2}} \cdot \frac{1 - k_2 \ell_2^{-1} e^{i\epsilon_2} - i1 + k_2 \ell_2^{-1} e^{-i\epsilon_2}}{\gamma_2 + (2k_2 \ell_2)^{-1} \rho \cos 2\Theta_2} e^{i\epsilon_2 t}[1 + O(t^{-1})].$$

(4.39b)

Thus we have proved that the two-positon solution decays into two positons asymptotically as $t \to \infty$, and the collision of the two positons are completely insensitive. Even the additional phase shifts in the collision of two dark solitons are absent here.

(3) One-soliton-one-positon solution and soliton-positon interaction.

We let $\rho$ satisfy $|\ell_1| < \rho < |\ell_2|$, and choose

$$f_1 = \left[ \Psi \left( \begin{array}{c} \sqrt{\kappa - \lambda / \rho} \\ 0 \end{array} \right) \right]_{\lambda = i\ell_1} = \left( \sqrt{\kappa_1 - i\ell_1} e^{\kappa_1(x - 2\ell_1 t) + i\epsilon_2 t} \right) \left( \sqrt{\kappa_1 + i\ell_1} e^{\kappa_1(x - 2\ell_1 t) - i\epsilon_2 t} \right) = \sqrt{\rho} e^{\kappa_1(x - 2\ell_1 t)} \left( e^{i(\rho^2 t - \theta_1)} \right),$$

where

$$\kappa = \sqrt{\lambda^2 + \rho^2}, \quad \kappa_1 = \sqrt{\rho^2 - \ell_1^2}, \quad \theta_1 = \frac{1}{2} \arcsin \frac{\ell_1}{\rho},$$

and choose

$$f_2 = \left[ \Psi \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \right]_{\lambda = i\ell_2} = \left( \left[ pe_{i\epsilon_2} - i(k_2 + \ell_2) e^{-i\epsilon_2} \right] e^{i\epsilon_2 t} \right) \left( \left[ i(k_2 + \ell_2) e^{i\epsilon_2} + pe^{-i\epsilon_2} \right] e^{-i\epsilon_2 t} \right),$$

where

$$\kappa = (\text{sign Im} \lambda) i \sqrt{-\lambda^2 - \rho^2}, \quad \Theta_2 = k_2(x - 2\ell_2 t), \quad k_2 = (\text{sign } \ell_2) \sqrt{\ell_2^2 - \rho^2}.$$

Let

$$c_1(t) = \frac{\rho}{2\kappa_1} e^{2\kappa_1(a_1t + b_1)}, \quad \xi_1 = \kappa_1[x - (2\ell_1 + a_1)t - b_1], \quad c_2(t) = 2\ell_2(k_2 + \ell_2)(a_2 t + b_2), \quad \xi_2 = x + [a_2 - 2(\ell_2 + k_2^2 \ell_2^{-1})]t + b_2,$$

where $a_j \in \mathbb{R}_+, b_j \in \mathbb{R}, j = 1, 2,$ are constants. Then one finds

$$c_1(t) + \sigma(f_1, f_1) = \frac{\rho}{2\kappa_1} e^{2\kappa_1(a_1t + b_1)}(1 + e^{2\xi_1}), \quad c_2(t) + \sigma(f_2, f_2) = 2\ell_2(k_2 + \ell_2)[\gamma_2 + (2k_2 \ell_2)^{-1} \rho \cos 2\Theta_2],$$

$$\sigma(f_1, f_2) = \sqrt{\rho} e^{\kappa_1(x - 2\ell_1 t)} \frac{\ell_2 - \ell_1}{\ell_2} \left[ (k_2 + \ell_2) \cos (\theta_1 - \Theta_2) - \rho \sin (\theta_1 + \Theta_2) \right].$$

Formulas (4.33a) give a one-soliton-one-positon solution

$$q = e^{2i\epsilon_2 t} \left[ \frac{\rho + 2\sqrt{\rho} e^{2\xi_1 - i\delta_1} A B - 2\ell_2(k_2 + \ell_2)\rho e^{2(\xi_1 - i\delta_1)} \Gamma_2 - \rho(2\kappa_1)^{-1}(1 + e^{2\xi_1})A^2}{\rho\kappa_1^{-1} \ell_2(k_2 + \ell_2)(1 + e^{2\xi_1}) \Gamma_2 - e^{2\xi_1}B^2} \right]$$

(4.40a)

$$w_1 = \sqrt{\rho\kappa_1} \left[ 2\ell_2(k_2 + \ell_2)\sqrt{\rho} e^{\xi_1 - i\delta_1} \Gamma_2 - e^{\xi_1} A B] e^{i\epsilon_2 t} \right] \frac{1}{\rho\kappa_1^{-1} \ell_2(k_2 + \ell_2)(1 + e^{2\xi_1}) \Gamma_2 - e^{2\xi_1}B^2}$$

(4.40b)

$$w_2 = \sqrt{2a_2 \ell_2(k_2 + \ell_2)} \left[ \rho(2\kappa_1)^{-1}(1 + e^{2\xi_1})A - \sqrt{\rho} e^{2\xi_1 - i\delta_1} B \right] e^{i\epsilon_2 t} \frac{1}{\rho\kappa_1^{-1} \ell_2(k_2 + \ell_2)(1 + e^{2\xi_1}) \Gamma_2 - e^{2\xi_1}B^2}$$

(4.40c)
\[ \Gamma_2 = \gamma_2 + \rho(2k_2 \ell_2)^{-1} \cos 2\Theta_2, \quad A = \rho e^{i\Theta_2} - i(k_2 + \ell_2)e^{-i\Theta_2}, \]

\[ B = \sqrt{\frac{\rho}{\ell_2 - \ell_1}} [(k_2 + \ell_2) \cos(\theta_1 - \Theta_2) - \rho \sin(\theta_1 + \Theta_2)]. \]

Formula (4.35) implies that

\[ |q|^2 = \rho^2 - \partial_x^2 \log[\rho \kappa_1^{-1} \ell_2 (k_2 + \ell_2) (1 + e^{2\xi_1}) \Gamma_2 - e^{2\xi_1} B^2]. \quad (4.41) \]

It is easy to see that \( \kappa_1^{-1} \xi_1 - \gamma_2 = [2(\ell_2 + k_2 \ell_2^{-1} - \ell_1) - a_1 - a_2]t - b_1 - b_2. \) Assume \( 2(\ell_2 + k_2 \ell_2^{-1} - \ell_1) - a_1 - a_2 > 0. \)

We now fix \( \gamma_2, \) and let \( t \to -\infty \) (which implies \( \xi_1 \to -\infty \)), then we obtain the estimate

\[ q = \rho e^{2\nu^2 t} + \frac{k_2 \ell_2^{-1} \cos 2\Theta_2 - i(\sin 2\Theta_2 - \rho \ell^{-1}}{\gamma_2 + (2k_2 \ell_2)^{-1} \rho \cos 2\Theta_2} e^{2\nu^2 t}[1 + O(e^{-2[\xi_1]}], \quad (4.42a) \]

\[ w_1 = O(e^{\xi_1}), \quad w_2 = \sqrt{\frac{a_2}{2}} \sqrt{\frac{1 - k_2 \ell_2^{-1}}{\gamma_2 + (2k_2 \ell_2)^{-1} \rho \cos 2\Theta_2} e^{i\Theta_2} - \frac{i}{\rho} \frac{1 + k_2 \ell_2^{-1}}{\gamma_2 + (2k_2 \ell_2)^{-1} \rho \cos 2\Theta_2} e^{-i\Theta_2} e^{i\nu^2 t}[1 + O(e^{-2[\xi_1]}], \quad (4.42b) \]

and

\[ |q|^2 = \rho^2 - \partial_x^2 \log[\gamma_2 + \rho(2k_2 \ell_2)^{-1} \cos 2\Theta_2][1 + O(e^{-2[\xi_1]}]. \quad (4.42c) \]

Let \( t \to +\infty, \) then we obtain the estimate (for simplicity, we only give the estimate for \( |q|^2 \))

\[ |q|^2 = \rho^2 - \partial_x^2 \log[\gamma_2 + \delta_1 + \rho(2k_2 \ell_2)^{-1} \cos 2(\Theta_2 + \delta_2)][1 + O(e^{-2[\xi_1]}], \]

where

\[ \delta_1 = -\frac{\kappa_1}{\ell_2(\ell_2 - \ell_1)}, \quad \delta_2 = \frac{1}{2} \arcsin \frac{2 \kappa_1 k_2(\ell_1 \ell_2 - \rho^2)}{\rho^2(\ell_2 - \ell_1)^2}. \]

If we fix \( \xi_1 \) and let \( t \to \pm \infty \) (which implies \( \gamma_2 \to \pm \infty \)), then we have the asymptotic estimate

\[ q = \rho e^{2\nu^2 t} - \frac{1 + e^{-4i\Theta_1}}{1 + e^{2\xi_1}} \frac{e^{2\xi_1 + 2\nu^2 t}[1 + O(t^{-1})], \quad (4.43a) \]

\[ w_1 = \frac{2\sqrt{\nu_1} \kappa_1 e^{\xi_1 - i\Theta_1}}{1 + e^{2\xi_1}} e^{i\nu^2 t}[1 + O(t^{-1})], \quad w_2 = O(t^{-1}). \quad (4.43b) \]

Thus we have proved that the one-soliton-one-positon solution decays asymptotically into a dark soliton and a positon for large \( t. \) The dark soliton recovers completely after the collision with a positon, in other words, a positon is totally transparent to a dark soliton. However, the positon gains phase shifts when colliding with the dark soliton.

In Figure 2, we plot the one-soliton-one-positon solution.

### 4.1.3 Solutions of the NLS\(^+\)ESCS with \( m = 0 \) and \( n = N. \)

The NLS\(^+\)ESCS with \( m = 0 \) and \( n = N \) reads

\[ w_{j,x} = i\ell_j w_j + qw_j, \quad j = 1, \ldots, N, \quad (4.44a) \]

\[ q_t = i|q|^2 q - q_{xx} + \sum_{j=1}^{N} w_j^2, \quad (4.44b) \]
For simplicity, we assume

Moreover, we have

where \( \ell_j \neq 0, j = 1, \ldots, N \), are \( N \) distinct real constants. For \( j = 1, \ldots, N \), let \( f_j \) be a solution of the system \((1.44)\) with \( \lambda = i\ell_j \) and satisfy \( f_j^{(1)} = f_j^{(2)*} \), and let \( c_j(t) \) be an arbitrary real function satisfying \( c_j(t) \geq 0 \). Then by Proposition 3.3, a solution of the equations \((4.44)\) is given by

\[
q = \rho e^{2i\rho^2 t} + \frac{\Delta_2}{\Delta_0}, \quad w_j = \frac{\sqrt{c_j(t)} \Delta_{1j}}{\Delta_0}, \quad j = 1, \ldots, N, \quad (4.45)
\]

where

\[
\Delta_0 = W_0(\{c_1, f_1\}, \ldots, \{c_N, f_N\}), \quad \Delta_2 = W_2^{(1)}(\{c_1, f_1\}, \ldots, \{c_N, f_N\}; 0),
\]

\[
\Delta_{1j} = W_1^{(1)}(\{c_1, f_1\}, \ldots, \{c_{j-1}, f_{j-1}\}, \{c_{j+1}, f_{j+1}\}, \ldots, \{c_N, f_N\}; f_j).
\]

Moreover, we have

\[
|q|^2 = \rho^2 - \Delta_x^2 \log \Delta_0. \quad (4.46)
\]

For simplicity, we assume \(|f_1| > \ldots > |f_N|\). Then according to the different choice of \( \rho \), we can obtain different classes of solutions.

(1) **Multi-soliton solutions.**

We take \( \rho > |\ell_j|, j = 1, \ldots, N \), and choose

\[
c_j(t) = \frac{\rho}{2\kappa_j} e^{2\kappa_j(a_j t + b_j)}, \quad f_j = \begin{bmatrix} \Psi \left( \frac{\sqrt{\kappa - \lambda}}{\rho} \right) \\ 0 \end{bmatrix}_{\lambda = i\ell_j} = \begin{bmatrix} \sqrt{\kappa_j - i\ell_j} e^{\kappa_j(x - 2\ell_j t) + i\rho^2 t} \\ \sqrt{\kappa_j + i\ell_j} e^{\kappa_j(x - 2\ell_j t) - i\rho^2 t} \end{bmatrix},
\]

where \( \kappa = \sqrt{\lambda^2 + \rho^2}, \kappa_j = \sqrt{\rho^2 - \ell_j^2}, a_j \in \mathbb{R}_+ \) and \( b_j \in \mathbb{R} \), then formulas \((4.45)\) give the dark \( N \)-soliton solution.

(2) **Multi-positon solutions.**

We take \( \rho < |\ell_j|, j = 1, \ldots, N \), and choose

\[
c_j(t) = 2\ell_j(k_j + \ell_j)(a_j t + b_j), \quad f_j = \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{\lambda = i\ell_j} = \begin{bmatrix} [\rho e^{i\Theta_j} - i(k_j + \ell_j) e^{-i\Theta_j}] e^{i\rho^2 t} \\ [i(k_j + \ell_j) e^{i\Theta_j} + \rho e^{-i\Theta_j}] e^{-i\rho^2 t} \end{bmatrix},
\]

where \( \kappa = (\text{sign Im} \lambda) i\sqrt{-\lambda^2 - \rho^2}, k_j = (\text{sign } \ell_j) \sqrt{\ell_j^2 - \rho^2}, a_j \in \mathbb{R}_+ \) and \( b_j \in \mathbb{R} \), then formulas \((4.45)\) give the \( N \)-positon solution.

(3) **Multi-soliton-multi-positon solutions.**
We let $\rho$ satisfy $|\ell_{N_1}| > \rho > |\ell_{N_1+1}|$, where $1 \leq N_1 \leq N$, and choose

$$c_j(t) = \frac{\rho}{2\kappa_j} e^{2\kappa_j(a_j t + b_j)}, \quad f_j = \left[ \Psi \left( \frac{\sqrt{\kappa - \lambda/\rho}}{0} \right) \right]_{\lambda = \ell_j}, \quad j = 1, \ldots, N_1,$$

where $\kappa = \sqrt{\lambda^2 + \rho^2}$ and $\kappa_j = \sqrt{\rho^2 - \ell_j^2}$, and

$$c_j(t) = 2\ell_j(k_j + \ell_j)(a_j t + b_j), \quad f_j = \left[ \Psi \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \right]_{\lambda = \ell_j}, \quad j = N_1 + 1, \ldots, N,$$

where $\kappa = (\text{sign } \text{Im } \lambda) i \sqrt{\lambda^2 - \rho^2}$ and $k_j = (\text{sign } \ell_j) \sqrt{\ell_j^2 - \rho^2}$. Here $a_j \in \mathbb{R}^+$ and $b_j \in \mathbb{R}$ for $j = 1, \ldots, N$. Then formulas (4.45) give the $N_1$-soliton-$N_2$-positon solution ($N_2 = N - N_1$).

### 4.2 Solutions of the NLS^−ESCS

We start from the NLS^− equation without sources

$$q_t = -i(2|q|^2 q + q_{xx}),$$

(4.47)

and its solution

$$q = \rho e^{-2i\rho^2 t}.$$  

(4.48)

We need to solve the linear system

$$\psi_x = U(\lambda, \rho e^{-2i\rho^2 t}, -\rho e^{2i\rho^2 t})\psi, \quad \psi_t = V(\lambda, \rho e^{-2i\rho^2 t}, -\rho e^{2i\rho^2 t})\psi.$$  

(4.49)

The fundamental solution matrix for the linear system (4.49) is

$$\Phi = \begin{pmatrix} (\kappa + \lambda)e^{\kappa(x+2i\lambda t) - i\rho^2 t} & -\rho e^{\kappa(x+2i\lambda t) - i\rho^2 t} \\ -\rho e^{\kappa(x+2i\lambda t) + i\rho^2 t} & (\kappa + \lambda)e^{-\kappa(x+2i\lambda t) + i\rho^2 t} \end{pmatrix},$$

(4.50)

where $\kappa = \kappa(\lambda)$ satisfies $\kappa^2 = \lambda^2 - \rho^2$.

#### 4.2.1 Solutions of the NLS^−ESCS with $n = 1$.

The NLS^− ESCS with $n = 1$ reads

$$\varphi_{1,x} = U(\lambda_1, q, -q^*)\varphi_1,$$

(4.51a)

$$q_t = -i(2|q|^2 q + q_{xx}) + (\varphi_1^{(1)})^2 - (\varphi_1^{(2)*})^2,$$

(4.51b)

where $\lambda_1 = \lambda_{1R} + i\lambda_{1I}$ is a complex constant with $\lambda_{1R} > 0$, $\lambda_{1I} \neq 0$. Let $f$ be a solution of the system (4.49) with $\lambda = \lambda_1$, $c(t)$ be an arbitrary complex function, then by Proposition 3.3 a solution of the equations (4.51) is given by

$$q = \rho e^{-2i\rho^2 t} + \frac{\Delta_2}{\Delta_0}, \quad \varphi_1 = \frac{\sqrt{c(t)}}{\Delta_0} \left( \frac{\Delta_1^{(1)}}{\Delta_1^{(2)}} \right),$$

(4.52)

where

$$\Delta_0 = \left| \begin{array}{cc} c(t) + \sigma(f, f) & -|f^{(1)}|^2 + |f^{(2)}|^2 \frac{\lambda_{1R}}{4\lambda_{1R}} \\ -|f^{(1)}|^2 + |f^{(2)}|^2 \frac{\lambda_{1R}}{4\lambda_{1R}} & -c(t)^* - \sigma(f, f)^* \end{array} \right| = -|c(t) + \sigma(f, f)|^2 - \left( \frac{|f^{(1)}|^2 + |f^{(2)}|^2}{4\lambda_{1R}} \right)^2,$$
\[ \Delta_1^{(1)} = \begin{vmatrix} -c(t)^* - \sigma(f, f)^* & -f(2)^* \\ \frac{|f(1)|^2 + |f(2)|^2}{4 \lambda_1 R} & \frac{4 \lambda_1 R}{f(1)} \end{vmatrix}, \quad \Delta_1^{(2)} = \begin{vmatrix} -c(t)^* - \sigma(f, f)^* & -f(2)^* \\ \frac{|f(1)|^2 + |f(2)|^2}{4 \lambda_1 R} & \frac{4 \lambda_1 R}{f(2)} \end{vmatrix}, \]

\[ \Delta_2 = \begin{vmatrix} c(t) + \sigma(f, f) & f(1)^* \\ \frac{|f(1)|^2 + |f(2)|^2}{4 \lambda_1 R} & -c(t)^* - \sigma(f, f)^* \end{vmatrix} \]

Moreover, we have

\[ |q|^2 = \rho^2 + \partial_x^2 \log \Delta_0 \quad (4.53) \]

**Topological deformation of the bright one-soliton.**

We choose \( f \) as

\[ f = \begin{cases} \Phi \\ 1 \\ 0 \end{cases} \lambda = \lambda_1 \]

where \( \kappa_1 = \kappa(\lambda_1) \). Here, we choose \( \kappa = \kappa(\lambda) = (\text{sign} \lambda_1) \sqrt{\lambda^2 - \rho^2} \) for \( \Phi \) defined by \( \Phi \), then \( \kappa \) is analytic at \( \lambda = \lambda_1 \). Furthermore, under this choice of \( \kappa \), we have \( \lim_{\rho \to 0} \kappa = \lambda \). Calculation yields

\[ \sigma(f, f) = \frac{\rho(\kappa_1 + \lambda_1)}{2\kappa_1} e^{2\kappa_1(x + 2i\lambda_1 t)}, \quad |f(1)|^2 = |\kappa_1 + \lambda_1|^2 e^{2(\kappa_1 R x - 2\lambda_1 t)}, \quad |f(2)|^2 = \rho^2 e^{2(\kappa_1 R x - 2\lambda_1 t)}. \]

We choose \( c(t) = (2\kappa_1)^{-1}(\kappa_1 + \lambda_1) e^{2(\alpha t + b)} \), where \( a \) and \( b \) are two arbitrary complex numbers, then formulas \( \Phi \) give the topological deformation of bright one-soliton solution

\[ q = \left[ \begin{array}{c} \rho e^{2(\kappa_1 + \lambda_1) \xi} e^{-2i\eta} - |\kappa_1 + \lambda_1|^2 (\kappa_1 + \lambda_1) e^{2i\eta} + \frac{\rho(\kappa_1 + \lambda_1)}{2 \kappa_1} e^{2i\eta} + \frac{|\kappa_1 + \lambda_1|^2}{4 \lambda_1 R} e^{2i\eta} + \frac{|\kappa_1 + \lambda_1|^2}{4 \lambda_1 R} e^{2i\eta} \end{array} \right] e^{-2i\eta^2 t} \quad (4.54a) \]

\[ \varphi_1^{(1)} = \sqrt{\frac{\kappa_1}{\kappa_1}} \left( \frac{(\kappa_1 + \lambda_1)^2}{2 \kappa_1} e^{-\xi + i\eta} + \rho e^{-\xi - i\eta} \frac{|\kappa_1 + \lambda_1|^2}{4 \lambda_1 R} e^{\xi - i\eta} \right) e^{-ip^2 t}, \quad (4.54b) \]

\[ \varphi_1^{(2)} = \sqrt{\frac{\kappa_1}{\kappa_1}} \left( -\frac{\kappa_1 + \lambda_1}{2 \kappa_1} e^{-\xi + i\eta} + \rho e^{-\xi - i\eta} \frac{|\kappa_1 + \lambda_1|^2}{4 \lambda_1 R} e^{\xi - i\eta} \right) e^{ip^2 t}, \quad (4.54c) \]

where

\[ \xi = \kappa_1 R x - (2\lambda_1 t + a R) t - b_R, \quad \eta = \kappa_1 x + (2\lambda_1 R - a I) t - b_I. \]

Formula \( \Phi \) implies that

\[ |q|^2 = \rho^2 + \partial_x^2 \log \left[ 4 \lambda_1^2 R \kappa_1 + |\lambda_1|^2 (e^{-2\xi} + 2\rho \cos 2\eta + \rho^2 e^{2\xi}) + |\kappa_1|^2 (|\kappa_1 + \lambda_1|^2 + \rho^2) e^{2\xi} \right]. \]

When \( \rho = 0 \), we have \( \kappa_1 = \lambda_1 \) and the solution given by \( \Phi \) corresponds to the bright one-soliton solution

\[ q = \frac{2\lambda_1 R e^{2i\eta_0}}{\cosh 2\xi_0}, \quad \varphi_1 = \frac{\sqrt{2\lambda_1 R}}{\cosh 2\xi_0} \left( e^{-\xi_0 + i\eta_0} \right) \left( -e^{-\xi_0 - i\eta_0} \right), \]

where

\[ \xi_0 = \lambda_1 R x - (2\lambda_1 t + a R) t - b_R + \log(|\lambda_1|/\sqrt{\lambda_1 R}), \quad \eta_0 = \lambda_1 t + (2\lambda_1 R - a I) t - b_I + \arg \lambda_1. \]

The topological deformation of bright one-soliton solution for the NLS\(^-\) equation was already known. Here, we have given its correspondence for the NLS\(^-\)-ESCS.

In Figure 3, we plot the topological deformation of bright one-soliton solution.
Figure 3: The topological deformation of bright one-soliton solution of the NLS−ESCS with $\lambda_1 = 2 + i$. The data is $\rho = \sqrt{7}$ and $a_1 = a_2 = b_1 = b_2 = 1$. The two graphs show the modulus of $q$ (the left) and the real part of $\phi_1^{(1)}$ (the right) at $t = 0$.

4.2.2 Solutions of the NLS−ESCS with $n = N$.

The NLS−ESCS with $n = N$ reads

$$\varphi_{j,x} = U(\lambda_j, q, -q^*) \varphi_j, \quad j = 1, \ldots, N,$$

and

$$q_t = -i(2|q|^2 q + q_{xx}) + (\varphi_1^{(1)})^2 - (\varphi_1^{(2)*})^2,$$

where $\lambda_j = \lambda_{jR} + i\lambda_{jI}$ are distinct complex constants with $\lambda_{jR} > 0$, $\lambda_{jI} \neq 0$. For $j = 1, \ldots, N$, let

$$F_j = \{c_j, f_j\}, \quad F_j^* = \{-c_j^*, S_f f_j\}, \quad c_j(t) = \frac{\kappa_j + \lambda_j}{2\kappa_j} e^{a_j t + b_j}, \quad f_j = \left[ \begin{array}{c} \Phi(1) \\ 0 \end{array} \right]_{\lambda = \lambda_j}, \quad \kappa_j = \text{sign} \lambda_{jI} \sqrt{\lambda_j^2 - \rho^2},$$

then the topological deformation of bright $N$-soliton solution of Eqs. (4.55) is given by

$$q = \rho e^{-2i\rho^2 t} + \frac{\Delta_2}{\Delta_0}, \quad \varphi_j = \frac{\sqrt{c_j(t)}}{\Delta_0} \left( \frac{\Delta_1(j)}{\Delta_2(j)} \right), \quad j = 1, \ldots, N,$$

where

$$\Delta_0 = W_0(F_1, F_1', \ldots, F_N, F_N'), \quad \Delta_2 = W_2^{(0)}(F_1, F_1', \ldots, F_N, F_N'; 0),$$

$$\Delta_1^{(l)} = W_1^{(l)}(F_1, F_1', \ldots, F_{j-1}, F_{j-1}', F_j, F_{j+1}, F_{j+1}', \ldots, F_N, F_N'; f_j), \quad l = 1, 2, \quad j = 1, \ldots, N.$$

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