BV QUANTIZATION OF COVARIANT (POLYSYMPLECTIC) HAMILTONIAN FIELD THEORY

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Covariant (polysymplectic) Hamiltonian field theory is the Hamiltonian counterpart of classical Lagrangian field theory. They are quasi-equivalent in the case of almost-regular Lagrangians. This work addresses BV quantization of polysymplectic Hamiltonian field theory. We compare BV quantizations of associated Lagrangian and polysymplectic Hamiltonian field systems in the case of almost-regular quadratic Lagrangians.

1 Introduction

The Hamiltonian counterpart of classical first-order Lagrangian field theory is covariant Hamiltonian formalism which is developed in the polysymplectic, multisymplectic and Hamilton – De Donder variants (see [1, 2, 3, 4, 5] and references therein). In order to quantize covariant Hamiltonian field theory, one usually attempts to construct the multisymplectic generalization of a Poisson bracket [6, 7, 8]. We provide its BV quantization.

Let us consider a field system represented by sections of a fiber bundle \( \pi : Y \to X \) coordinated by \((x^\lambda, y^i)\). Its configuration space is the first-order jet manifold \( J^1Y \) of \( Y \) equipped with the adapted coordinates \((x^\mu, y^i, y^i_\mu)\), compatible with the composite fibration

\[
J^1Y \overset{\pi_1}{\longrightarrow} Y \longrightarrow X.
\]

A first-order Lagrangian of fields is defined as a horizontal density

\[
L = \mathcal{L}\omega : J^1Y \to \wedge^n T^*X, \quad \omega = dx^1 \wedge \cdots dx^n, \quad n = \dim X, \quad (1)
\]
on the jet manifold $J^1Y$. The corresponding Euler–Lagrange equations are given by the subset

$$\delta L_i = (\partial_i - d_\lambda \partial^\lambda)L = 0, \quad d_\lambda = \partial_\lambda + y^i_i \partial_i + y^i_i \partial^\mu,$$

of the second-order jet manifold $J^2Y$ of $Y$ coordinated by $(x^\mu, y^i, y^i_\lambda, y^i_\lambda\mu)$.

The polysymplectic phase space of a field system on a fiber bundle $Y \to X$ is the Legendre bundle

$$\Pi = \overset{n}{\wedge} T^*X \otimes V^*Y \otimes T^*Y = V^*Y \wedge (\overset{n-1}{\wedge} T^*X)$$

equipped with the holonomic bundle coordinates $(x^\lambda, y^i, p^\mu_i)$ [1, 2]. They are compatible with the composite fibration

$$\Pi \xrightarrow{\pi_Y} Y \to X.$$

A covariant Hamiltonian $\mathcal{H}$ on $\Pi$ (3) is defined as a section $p = -\mathcal{H}$ of the trivial one-dimensional fiber bundle

$$Z_Y = T^*Y \wedge (\overset{n-1}{\wedge} T^*X) \to \Pi,$$

coordinated by $(x^\lambda, y^i, p^\mu_i, p)$. This fiber bundle is provided with the canonical multisymplectic Liouville form

$$\Xi = p\omega + p^\lambda_i dy^i \wedge \omega_\lambda, \quad \omega_\lambda = \partial_\lambda \omega.$$

The pull-back of $\Xi$ onto $\Pi$ by a Hamiltonian $\mathcal{H}$ is a Hamiltonian form

$$H = \mathcal{H}^*\Xi_Y = p^\lambda_i dy^i \wedge \omega_\lambda - \mathcal{H}\omega$$

on $\Pi$. The corresponding covariant Hamilton equations on $\Pi$ are given by the closed submanifold

$$y^i_\lambda = \partial^i_\lambda \mathcal{H}, \quad p^\lambda_i = -\partial_i \mathcal{H}$$

of the jet manifold $J^1\Pi$ of $\Pi \to X$. A covariant Hamiltonian system on $\Pi$ is equivalent to a particular Lagrangian system on $\Pi$ as follows [2].

**Proposition 1.** The covariant Hamilton equations (6) are equivalent to the Euler–Lagrange equations for the first-order Lagrangian

$$L_\mathcal{H} = h_0(H) = \mathcal{L}_\mathcal{H}\omega = (p^\lambda_i y^i_\lambda - \mathcal{H})\omega$$

(7)
on $J^1\Pi$, where $h_0$ is the so called horizontal projector sending exterior forms on $\Pi$ onto horizontal exterior forms on $J^1\Pi \to X$ by the rule
\[
h_0(dy^\lambda) = y^\lambda dx, \quad h_0(dp^\mu_i) = p^\mu_i dx^\lambda.
\]

This fact motivates us to quantize covariant Hamiltonian field theory with a Hamiltonian $H$ on $\Pi$ as a Lagrangian system with the Lagrangian $L_H$ (7). This Lagrangian system can be quantized in the framework of familiar perturbation quantum field theory. If there is no constraint and the matrix
\[
\frac{\partial^2 H}{\partial p^\mu_i \partial p^\nu_j} = -\frac{\partial^2 L}{\partial p^\mu_i \partial p^\nu_j}
\]
is positive-definite and nondegenerate on an Euclidean space-time, this quantization is given by the generating functional
\[
Z = \mathcal{N}^{-1} \int \exp\left\{ \int (L_H + \Lambda + iJ_i y^i + iJ_\mu p^\mu_i) \omega \right\} \prod_x [dp(x)][dy(x)] \quad (8)
\]
of Euclidean Green functions [9], where $\Lambda$ comes from the normalization condition
\[
\int \exp\left\{ \frac{1}{2} \frac{\partial^\mu}{\partial^\nu} L_H p^\mu_i p^\nu_j + \Lambda \right\} dx \prod_x [dp(x)] = 1.
\]

A constrained covariant Hamiltonian system can be quantized as follows.

Let $i_N : N \to \Pi$ be a closed imbedded subbundle of the Legendre bundle $\Pi \to Y$ which is regarded as a constraint space of a covariant Hamiltonian field system with a Hamiltonian $H$. Let $H_N = i_N^* H$ be the pull-back of the Hamiltonian form $H$ (5) onto $N$. This form defines the constrained Lagrangian
\[
L_N = h_0(H_N) = (J^1 i_N)^* L_H \quad (9)
\]
on the jet manifold $J^1 N_L$ of the fiber bundle $N_L \to X$. The Euler–Lagrange equations for this Lagrangian are called the constrained Hamilton equations.

Remark 1. In fact, the Lagrangian $L_H$ (7) is the pull-back onto $J^1\Pi$ of the horizontal form $L_H$ on the bundle product $\Pi \times J^1 Y$ over $Y$ by the canonical map $J^1\Pi \to \Pi \times J^1 Y$. Therefore, the constrained Lagrangian $L_N$ (9) is simply the restriction of $L_H$ to $N \times J^1 Y$. 

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Let us refer to the following results [2].

Proposition 2. A section $r$ of $\Pi \to X$ is a solution of the covariant Hamilton equations (6) iff it satisfies the condition $r^*(u_\Pi]dH) = 0$ for any vertical vector field $u_\Pi$ on $\Pi \to X$.

Proposition 3. A section $r$ of the fiber bundle $N \to X$ is a solution of constrained Hamilton equations iff it satisfies the condition $r^*(u_N]dH) = 0$ for any vertical vector field $u_N$ on $N \to X$.

Propositions 2 and 3 result in the following.

Proposition 4. Any solution of the covariant Hamilton equations (6) which lives in the constraint manifold $N$ is also a solution of the constrained Hamilton equations on $N$.

This fact motivates us to quantize covariant Hamiltonian field theory on a constraint manifold $N$ as a Lagrangian system with the pull-back Lagrangian $L_N$ (9). Furthermore, a closed imbedded constraint submanifold $N$ of $\Pi$ admits an open neighbourhood $U$ which is a fibered manifold $U \to N$. If $\Pi$ is a fibered manifold $\pi_N : \Pi \to N$ over $N$, it is often convenient to quantize a Lagrangian system on $\Pi$ with the pull-back Lagrangian $L_\Pi = \pi_N^* L_N$. Since this Lagrangian possesses gauge symmetries, Batalin–Vilkoviski (henceforth BV) quantization should be called into play.

Recall that BV quantization [10, 11] provides the universal scheme of quantization of gauge-invariant Lagrangian field systems. Given a classical Lagrangian, it enables one to obtain a gauged-fixed BRST invariant Lagrangian in the generating functional of perturbation quantum field theory. However, the BV quantization scheme does not automatically provide the path-integral measure.

Here, we apply BV quantization to covariant Hamiltonian systems associated to Lagrangian field systems with quadratic Lagrangians

$$L = \frac{1}{2} a_{ij}^{\lambda \mu} y^i_\lambda y^j_\mu + b^i_\lambda y^i_\lambda + c, \quad (10)$$

where $a$, $b$ and $c$ are functions on $Y$. If the Lagrangian (10) is hyperregular (i.e., the matrix function $a$ is nondegenerate), there exists a unique associated Hamiltonian system whose Hamiltonian $H$ is quadratic in momenta $p_i^\mu$, and so is the Lagrangian $L_H$ (7). If the matrix function $a$ is positive-definite on an Euclidean space-time, the generating functional (8) is a Gaussian integral of momenta $p_i^\mu(x)$. Integrating $Z$ with respect to $p_i^\mu(x)$, one
restarts the generating functional of quantum field theory with the original Lagrangian $\mathcal{L}$ (10). Using the BV quantization procedure, we generalize this result to field theories with almost-regular Lagrangians $\mathcal{L}$ (10), e.g., Yang–Mills gauge theory. The key point is that such a Lagrangian $\mathcal{L}$ yields constraints and admits different associated Hamiltonians $\mathcal{H}$, but all the Lagrangians $\mathcal{L}_\mathcal{H}$ coincide on the constraint manifold and, therefore, we have a unique constrained Hamiltonian system which is quasi-equivalent to the original Lagrangian one.

2 Associated Lagrangian and Hamiltonian systems

In order to relate classical Lagrangian and covariant Hamiltonian field theories, let us recall that, besides the Euler–Lagrange equations, a Lagrangian $\mathcal{L}$ (1) also yields the Cartan equations which are given by the subset

$$
(\overline{y}_\mu^j - y_\mu^i) \partial_i^\lambda \partial_j^\mu \mathcal{L} = 0, \quad \partial_i \mathcal{L} - \overline{d}_\lambda \partial_i^\lambda \mathcal{L} + (\overline{y}_\mu^j - y_\mu^i) \partial_i \partial_j^\mu \mathcal{L} = 0,
$$

(11)

of the repeated jet manifold $J^1 J^1 Y$ coordinated by $(x^\mu, y^i, y^i_\lambda, \overline{y}_\lambda^i, \overline{y}_\lambda^i_\mu)$. The jet prolongation $J^1 s$ of any solution $s$ of the Euler–Lagrange equations (2) is a solution of the Cartan equations (11). If $\mathfrak{s}$ is a solution of the Cartan equations and $\mathfrak{s} = J^1 s$, then $s$ is a solution of the Euler–Lagrange equations. If a Lagrangian $\mathcal{L}$ is regular, the equations (2) and (11) are equivalent.

Any Lagrangian $\mathcal{L}$ (1) yields the Legendre map

$$
\hat{\mathcal{L}} : J^1 Y \longrightarrow \Pi, \quad \hat{p}_i^\lambda \circ \hat{\mathcal{L}} = \partial_i^\lambda \mathcal{L},
$$

(12)

over $\text{Id} Y$ whose image $N_L = \hat{\mathcal{L}}(J^1 Y)$ is called the Lagrangian constraint space. A Lagrangian $\mathcal{L}$ is said to be hyperregular if the Legendre map (12) is a diffeomorphism. A Lagrangian $\mathcal{L}$ is called almost-regular if the Lagrangian constraint space is a closed imbedded subbundle $i_N : N_L \longrightarrow \Pi$ of the Legendre bundle $\Pi \longrightarrow Y$ and the surjection $\hat{\mathcal{L}} : J^1 Y \rightarrow N_L$ is a fibered manifold possessing connected fibers. Conversely, any Hamiltonian $\mathcal{H}$ yields the Hamiltonian map

$$
\hat{\mathcal{H}} : \Pi \longrightarrow J^1 Y, \quad \hat{y}_\lambda^i \circ \hat{\mathcal{H}} = \partial_\lambda^i \mathcal{H}.
$$

(13)

A Hamiltonian $\mathcal{H}$ on $\Pi$ is said to be associated to a Lagrangian $\mathcal{L}$ on $J^1 Y$ if $\mathcal{H}$ satisfies the relations

$$
\hat{\mathcal{L}} \circ \hat{\mathcal{H}} \circ \hat{\mathcal{L}} = \hat{\mathcal{L}}, \quad \hat{p}_i^\mu = \partial_i^\mu \mathcal{L}(x^\mu, y^i, \partial_\lambda^i \mathcal{H}), \quad (x^\mu, y^i, \hat{p}_i^\mu) \in N_L,
$$

(14)
\[
\tilde{H}^* L_H = \tilde{H}^* \mathcal{L}, \quad p^\mu_\nu \partial^\nu_\mu \mathcal{H} - \mathcal{H} = \mathcal{L}(x^\mu, y^j, \partial^\lambda_\lambda),
\]  
(15)

where \( L_H \) is the Lagrangian (7) on \( \Pi \times J^1 Y \) (see Remark 1). If an associated Hamiltonian \( \mathcal{H} \) exists, the Lagrangian constraint space \( N_L \) is given by the coordinate relations (14) and \( \hat{L} \circ \tilde{H} \) is a projector of \( \Pi \) onto \( N_L \).

For instance, any hyperregular Lagrangian \( L \) admits a unique associated Hamiltonian \( \mathcal{H} \) such that

\[
\tilde{H} = \hat{L}^{-1}, \quad \mathcal{H} = p^\mu_\nu \hat{L}^{-1\nu}_\mu - \mathcal{L}(x^\lambda, y^j, \hat{L}^{-1} \lambda).
\]

In this case, any solution \( s \) of the Euler–Lagrange equations (2) defines the solution \( r = \hat{L} \circ J^1 s \) of the covariant Hamilton equations (6). Conversely, any solution \( r \) of these Hamilton equations yields the solution \( s = \pi_Y \circ r \) of the Euler–Lagrange equations (2).

A degenerate Lagrangian need not admit an associated Hamiltonian. If such a Hamiltonian exists, it is not necessarily unique. Let us restrict our consideration to almost-regular Lagrangians. From the physical viewpoint, the most of Lagrangian field theories is of this type. From the mathematical one, this notion of degeneracy is particularly appropriate for the study of relations between Lagrangian and covariant Hamiltonian formalisms as follows.

**Theorem 5.** Let \( L \) be an almost-regular Lagrangian and \( \mathcal{H} \) an associated Hamiltonian. Let a section \( r \) of \( \Pi \rightarrow X \) be a solution of the covariant Hamilton equations (6) for \( \mathcal{H} \). If \( r \) lives in the Lagrangian constraint manifold \( N_L \), then \( s = \pi_Y \circ r \) satisfies the Euler–Lagrange equations (2) for \( L \), while \( \pi = \tilde{H} \circ r \) obeys the Cartan equations (11). Conversely, let \( s \) be a solution of the Cartan equations (11) for \( L \). If \( \mathcal{H} \) satisfies the relation

\[
\tilde{H} \circ \hat{L} \circ s = J^1 (\pi_0^1 \circ s),
\]

the section \( r = \hat{L} \circ s \) of the Legendre bundle \( \Pi \rightarrow X \) is a solution of the covariant Hamilton equations (6) for \( \mathcal{H} \). If \( s = J^1 s \), we obtain the relation between solutions the Euler–Lagrange and Cartan ones.

By virtue of Theorem 5, one need a set of different associated Hamiltonians in order to recover all solutions of the Euler–Lagrange and Cartan equations for an almost-regular Lagrangian \( L \). This ambiguity can be overcome as follows.

**Proposition 6.** Let \( \mathcal{H}, \mathcal{H}' \) be two different Hamiltonians associated to an almost-regular Lagrangian \( L \). Let \( H, H' \) be the corresponding Hamiltonian forms (5). Their pull-backs \( i_N^* H \) and \( i_N^* H' \) onto the Lagrangian constraint manifold \( N_L \) coincide.
It follows that, if an almost-regular Lagrangian admits associated Hamiltonians $\mathcal{H}$, it defines a unique constrained Hamiltonian form $H_N = i_{N}^{*}H$ on the Lagrangian constraint manifold $N_L$ and a unique constrained Lagrangian $L_N = h_0(H_N)$ on the jet manifold $J^1 N_L$ of the fiber bundle $N_L \to X$. Basing on Proposition 4 and Theorem 5, one can prove the following.

**Theorem 7.** Let an almost-regular Lagrangian $L$ admit associated Hamiltonians. A section $\mathfrak{s}$ of the jet bundle $J^1 Y \to X$ is a solution of the Cartan equations for $L$ iff $\hat{L} \circ \mathfrak{s}$ is a solution of the constrained Hamilton equations. In particular, any solution $r$ of the constrained Hamilton equations provides the solution $\mathfrak{s} = \hat{H} \circ r$ of the Cartan equations.

Theorem 7 shows that the constrained Hamilton equations and the Cartan equations are quasi-equivalent. Thus, one can associate to an almost-regular Lagrangian $L$ a unique constrained Lagrangian system on the constraint Lagrangian manifold $N_L$ (14). Let us compare quantizations of these Lagrangian systems on $Y$ and $N_L \subset \Pi$ in the case of an almost-regular quadratic Lagrangian $L$.

### 3 Quadratic degenerate systems

Given a fiber bundle $Y \to X$, let us consider a quadratic Lagrangian $L$ (10), where $a$, $b$ and $c$ are local functions on $Y$. This property is coordinate-independent since $J^1 Y \to Y$ is an affine bundle modeled over the vector bundle $T^*X \otimes V Y$. The associated Legendre map (12) reads

$$ p_i^\lambda \circ \hat{L} = a_{ij}^\lambda y_j^\mu + b_i^\lambda. \quad (16) $$

Let a Lagrangian $L$ (10) be almost-regular, i.e., the matrix function $a$ is a linear bundle morphism

$$ a : T^*X \otimes V Y \to \Pi, \quad p_i^\lambda = a_{ij}^\nu \Gamma_\mu^j, \quad (17) $$
of constant rank, where $(x^\lambda, y^i, \bar{y}_\lambda)$ are bundle coordinates on $T^*X \otimes V Y$. Then the Lagrangian constraint space $N_L$ (16) is an affine subbundle of the Legendre bundle $\Pi \to Y$. Hence, $N_L \to Y$ has a global section. For the sake of simplicity, let us assume that it is the canonical zero section $\hat{0}(Y)$ of $\Pi \to Y$. The kernel of the Legendre map (16) is also an affine subbundle of the affine jet bundle $J^1 Y \to Y$. Therefore, it admits a global section

$$ \Gamma : Y \to \text{Ker} \hat{L} \subset J^1 Y, \quad a_{ij}^\nu \Gamma_\mu^j + b_i^\lambda = 0, \quad (18) $$
which is a connection on $Y \to X$. With such a connection, the Lagrangian (10) is brought into the form
\[
\mathcal{L} = \frac{1}{2} \Omega^{\lambda \mu}_{ij}(y^j_\lambda - \Gamma^j_\lambda)(y^i_\mu - \Gamma^i_\mu) + c'.
\]  

(19)

Let us refer to the following theorems [1, 2].

**Theorem 8.** There exists a linear bundle morphism
\[
\sigma : \Pi \to T^*Y \otimes VY, \quad \bar{y}^j_\lambda \circ \sigma = \sigma^{ij}_{\lambda \mu} p^\mu_j,
\]

(20)
such that
\[
a \circ \sigma \circ a = a, \quad a^{\lambda \mu}_{ij} \sigma^{\alpha \nu}_{\mu a} a^{\alpha \nu}_{kb} = a^{\lambda \nu}_{ib}.
\]

(21)

The equalities (18) and (21) give the relation $(a \circ \sigma_0)^{ij}_{\lambda \mu} b^\mu_j = b^j_\lambda$. Note that the morphism $\sigma$ (20) is not unique, but it falls into the sum $\sigma = \sigma_0 + \sigma_1$ such that
\[
\sigma_0 \circ a \circ \sigma_0 = \sigma_0, \quad a \circ \sigma_1 = \sigma_1 \circ a = 0,
\]

(22)

where $\sigma_0$ is uniquely defined.

**Theorem 9.** There are the splittings
\[
J^1Y = \text{Ker} \hat{L} \oplus \text{Im}(\sigma_0 \circ \hat{L}),
\]

(23)

\[
y^i_\lambda = S^i_\lambda + F^i_\lambda = [y^i_\lambda - \sigma_0^{ik \alpha} (a^{\alpha \mu}_{kj} y^j_\mu + b^\mu_k)] + [\sigma_0^{ik \alpha} (a^{\alpha \mu}_{kj} y^j_\mu + b^\mu_k)],
\]

(24)

\[
\Pi = \text{Ker} \sigma_0 \oplus N_L,
\]

(25)

\[
p^\lambda_i = R^\lambda_i + P^\lambda_i = [p^\lambda_i - a^{\lambda \mu}_{ij} \sigma_0^{\alpha \nu}_{j \mu a} p^\alpha_k] + [a^{\lambda \mu}_{ij} \sigma_0^{\alpha \nu}_{j \mu a} p^\alpha_k].
\]

(26)

The relations (22) lead to the equalities
\[
\sigma_0^{jk \alpha} R^\alpha_k = 0, \quad \sigma_1^{jk \alpha} P^\alpha_k = 0, \quad R^\lambda_i F^i_\lambda = 0.
\]

(27)

By virtue of the equalities (22) and the relation
\[
F^i_\mu = (\sigma_0 \circ a)^{ij \alpha}_{\mu j} (y^j_\lambda - \Gamma^j_\lambda),
\]

(28)
the Lagrangian (10) takes the form
\[ L = L_\omega, \quad L = \frac{1}{2} a_{ij}^{\lambda \mu} F^{i}_{\lambda} F^{j}_{\mu} + c'. \] (29)

One can show that this Lagrangian admits a set of associated Hamiltonians
\[ \mathcal{H}_\Gamma = (\mathcal{R}_{i}^{\lambda} + \mathcal{P}_{i}^{\lambda}) \Gamma^{i}_{\lambda} + \frac{1}{2} \sigma_{0 \lambda \mu}^{ij} \mathcal{P}_{i}^{\lambda} \mathcal{P}_{j}^{\mu} + \frac{1}{2} \sigma_{1 \lambda \mu}^{ij} \mathcal{R}_{i}^{\lambda} \mathcal{R}_{j}^{\mu} - c' \] (30)
indexed by connections \( \Gamma \) (18). Accordingly, the Lagrangian constraint manifold (16) is characterized by the equalities
\[ \mathcal{R}_{i}^{\lambda} = p_{i}^{\lambda} - a_{ij}^{\lambda \mu} p_{k}^{\mu} = 0. \] (31)

Given a Hamiltonian \( \mathcal{H}_\Gamma \), the corresponding Lagrangian (7) on \( \Pi \times \mathcal{J}^{1}Y \) (see Remark 1) reads
\[ \mathcal{L}_{\mathcal{H}_\Gamma} = \mathcal{R}_{i}^{\lambda} (S_{\lambda}^{i} - \Gamma^{i}_{\lambda}) + \mathcal{P}_{i}^{\lambda} F^{i}_{\lambda} - \frac{1}{2} \sigma_{0 \lambda \mu}^{ij} \mathcal{P}_{i}^{\lambda} \mathcal{P}_{j}^{\mu} - \frac{1}{2} \sigma_{1 \lambda \mu}^{ij} \mathcal{R}_{i}^{\lambda} \mathcal{R}_{j}^{\mu} + c'. \] (32)

Its restriction (9) to the constraint manifold \( N_{L} \times \mathcal{J}^{1}Y \) is
\[ L_{N} = L_{N}\omega, \quad L_{N} = \mathcal{P}_{i}^{\lambda} F^{i}_{\lambda} - \frac{1}{2} \sigma_{0 \lambda \mu}^{ij} \mathcal{P}_{i}^{\lambda} \mathcal{P}_{j}^{\mu} + c'. \] (33)

It is independent of the choice of a Hamiltonian (30).

The Hamiltonian \( \mathcal{H}_\Gamma \) yields the Hamiltonian map \( \widehat{H}_\Gamma \) (13) and the projector
\[ T = \widehat{L} \circ \widehat{H}_\Gamma, \quad p_{i}^{\lambda} \circ T = T_{i\mu}^{\lambda} p_{\mu}^{j} = a_{ik}^{\lambda \mu} \sigma_{0 \mu \nu} p_{j}^{\nu} = \mathcal{P}_{i}^{\lambda}, \]
of \( \Pi \) onto its summand \( N_{L} \) in the decomposition (25). It is a linear morphism over \( \text{Id}Y \).

Therefore, \( T : \Pi \to N_{L} \) is a vector bundle. We aim to quantize the pull-back
\[ L_{\Pi} = T^{*} L_{N} = L_{\Pi}\omega, \quad L_{\Pi} = \mathcal{P}_{i}^{\lambda} F^{i}_{\lambda} - \frac{1}{2} \sigma_{0 \lambda \mu}^{ij} \mathcal{P}_{i}^{\lambda} \mathcal{P}_{j}^{\mu} + c', \] (34)
of the constrained Lagrangian \( L_{N} \) (33) onto \( \Pi \times \mathcal{J}^{1}Y \).

Note that the splittings (23) and (25) result from the splitting of the vector bundle
\[ T^{*}X \otimes VY = \text{Ker} a \oplus E, \]
which can be provided with the adapted coordinates \( (\overline{y}^{i}, \overline{y}^{A}) \) such that \( a \) (17) is brought into a diagonal matrix with nonvanishing components \( a_{AA} \). Then the Legendre bundle \( \Pi \to Y \).
(3) is endowed with the dual (nonholonomic) coordinates \((p_a, p_A)\) where \(p_A\) are coordinates on the Lagrangian constraint manifold \(N_L\), given by the equalities \(p_a = 0\). Written relative to these coordinates, \(\sigma_0\) becomes the diagonal matrix

\[
\sigma_0^{AA} = (a_{AA})^{-1}, \quad \sigma_0^{aa} = 0,
\]

while \(\sigma_1^{Aa} = \sigma_1^{AB} = 0\). Let us write

\[
p_a = M_a^i \rho_i, \quad p_A = M_A^i \rho_i,
\]

where \(M\) are the matrix functions on \(Y\) obeying the relations

\[
M_a^i a_{ij} = 0, \quad (M^{-1})_i^j \sigma_{0,\mu} = 0, \quad M_A^j (a \circ \sigma_0)_{i\mu} = M_A^i j, \quad (M^{-1})_A^j A_{\mu} = a^{\mu}\sigma_{0,\nu}.
\]

4 Gauge symmetries

The Lagrangian \(L_\Pi\) (34) possesses gauge symmetries. By gauge transformations are meant automorphisms \(\Phi\) of the composite fiber bundle \(\Pi \to Y \to X\) over bundle automorphisms \(\phi\) of \(Y \to X\) over \(\text{Id} X\). Such an automorphism \(\Phi\) gives rise to the automorphism \((\Phi, J_\Pi^1)\) of the composite fiber bundle

\[
\Pi \times J^1 Y \to Y \to X.
\]

An automorphism \(\Phi\) is said to be a gauge symmetry of the Lagrangian \(L_\Pi\) if \((\Phi, J_\Pi^1)\) \(\cdot L_\Pi = L_\Pi\). If the Lagrangian (10) is degenerate, the group \(G\) of gauge symmetries of the Lagrangian \(L_\Pi\) (34) is never trivial. Indeed, any vertical automorphism of the vector bundle \(\text{Ker} \sigma_0 \to Y\) in the decomposition (25) is obviously a gauge symmetry of the Lagrangian \(L_\Pi\) (34). The gauge group \(G\) acts on the space \(\Pi(X)\) of sections of the Legendre bundle \(\Pi \to X\). For the purpose of quantization, it suffices to consider a subgroup \(G\) of \(G\) which acts freely on \(\Pi(X)\) and satisfies the relation \(\Pi(X)/G = \Pi(X)/G\). Moreover, we need one-parameter subgroups of \(G\). Their infinitesimal generators are represented by projectable vector fields

\[
u_\Pi = u^i(x^\mu, y^j) \partial_i + u_A^\lambda (x^\mu, y^j, p^\mu_j) \partial_\lambda
\]

on the Legendre bundle \(\Pi \to Y\) which give rise to the vector fields

\[
\pi = u^i \partial_i + u_A^\lambda \partial_\lambda + u^i \lambda \partial_i, \quad d_\lambda = \partial_\lambda + y^j_\lambda \partial_i,
\]
on $\Pi \times J^1Y$. A Lagrangian $L_{\Pi}$ is invariant under a one-parameter group of gauge transformations iff its Lie derivative

$$L_{\Pi L_{\Pi}} = \pi(L_{\Pi})\omega$$

along the infinitesimal generator $\pi$ (39) of this group vanishes.

Since linear and affine spaces of fields are only quantized, let us assume that $Y \to X$ is an affine bundle modeled over a vector bundle $\overline{Y} \to X$ (e.g., $\overline{Y} = Y$ if $Y$ is a vector bundle). In this case, the vertical tangent and cotangent bundles $VY$ and $V^*Y$ of $Y \to X$ are canonically isomorphic to the products

$$VY = Y \times \overline{Y}, \quad V^*Y = Y \times \overline{Y}^*.$$  \hfill (40)

Accordingly, the Legendre bundle $\Pi \to X$ (3) is isomorphic to the product

$$\Pi = Y \times (\overline{Y}^* \otimes \overline{Y}^* \otimes T^*X \otimes TX)$$

such that transition functions of coordinates $p_\lambda^i$ are independent of $y$. Then the splitting (25) takes the form

$$\Pi = Y \times (\text{Ker} \sigma_0 \oplus \overline{N}_L), \quad \text{where} \quad \text{Ker} \sigma_0 \text{ and } \overline{N}_L \text{ are fiber bundles over } X \text{ such that } \text{Ker} \sigma_0 = \pi^*\text{Ker} \sigma_0 \text{ and } N_L = \pi^*\overline{N}_L \text{ are their pull-backs onto } Y. \quad \text{The splitting (41) keeps the coordinate form (26). Due to the splittings (40) and (41), one can choose the coordinates } p_a \text{ on } \text{Ker} \sigma_0 \to Y \text{ such that } \sigma_1 \text{ becomes a diagonal matrix with nonzero positive entities } \sigma_{i}^{ab} = \delta^{ab}\nu^{-1}, \text{ where } \nu \omega \text{ is a volume form on } X. \text{ In this case, the matrix functions } (M^{-1})^{a\mu}_\lambda \text{ (36) are independent of } y^i.

The splittings (23) and (41) lead to the decomposition

$$\Pi \times J^1Y = (\text{Ker} \sigma_0 \oplus \overline{N}_L) \times (\text{Ker} \mathcal{L} \oplus \text{Im} (\sigma_0 \circ \mathcal{L})). \quad \text{(42)}$$

In view of this decomposition, let us define the gauge group $G$ as a direct product of the additive group $G_{\Pi}$ of sections of the vector bundle $\text{Ker} \sigma_0 \to X$ and some group $G_Y$ of gauge symmetries of the Lagrangian $L$ (10). The infinitesimal generators of the group $G_{\Pi}$ are vector fields

$$u_{\Pi} = \xi_a (M^{-1})^{a\lambda}_i \partial^i_{\lambda}, \quad \text{(43)}$$
parameterized by components $\xi_a$ of sections of $\text{Ker} \sigma_0 \to X$ with respect to the coordinates $p_a$ (36). The vector fields (43) mutually commute. Their lift (39) onto $\Pi^1 Y$ keeps the coordinate form

$$\mathbf{u}_\Pi = \xi_a (M^{-1})^{\alpha i} \partial_\alpha i$$

(44). By virtue of the relations (37), the Lie derivatives of $L_\Pi$ along vector fields (44) vanish.

Though it may happen that the Lagrangian $L$ (10) does not possess gauge symmetries, let us assume that it is invariant under some group $G_Y$ of vertical automorphisms of the fiber bundle $Y \to X$. The infinitesimal generators of one-parameter subgroups of $G_Y$ are represented by vertical vector fields $u = u^i \partial_i$ on $Y \to X$ which give rise to the vector fields

$$J^1 u = u^i \partial_i + d_\lambda u^i \partial_\lambda,$$

(45) on $J^1 Y$. The Lie derivatives $L_{\mu} J^1 u$ of the Lagrangian $L$ (29) along the vector fields (39) vanish, i.e.,

$$(u^i \partial_i + d_\lambda u^i \partial_\lambda) L = 0.$$

(46)

In order to study the invariance condition (46), let us consider the Lagrangian (10) written in the form (19). Since

$$J^1 u (y^i_\lambda - \Gamma^i_\lambda) = \partial_k u^i (y^k_\lambda - \Gamma^k_\lambda),$$

(47) one easily obtains from the equality (46) that

$$u^k \partial_k a^j_{ij} + \partial_i u^k a^j_{kj} + a^\lambda k \partial_j u^k = 0.$$

(48) It follows that the summands of the Lagrangian $L$ (19) and, consequently, the summands of the Lagrangian (29) are separately gauge-invariant, i.e.,

$$J^1 u (a^j_{ij} \mathcal{F}_j^i) = 0, \quad J^1 u (c') = u^k \partial_k c' = 0.$$

(49) The equalities (28), (47) and (48) give the transformation law

$$J^1 u (a^\lambda j_{ij} \mathcal{F}_j^\mu) = - \partial_i u^k a^\lambda k_{ij} \mathcal{F}_j^\mu.$$

(50) The relations (22) and (48) lead to the equality

$$a^\lambda j_{ij} \mathcal{F}^i_j [u^k \partial_k \sigma_0^{jn} - \partial_k u^j \sigma_0^{kn} - \sigma_0^{nk} \partial_k u^i] a^{\alpha \nu}_{ab} = 0.$$

(51)
Given the Legendre map $\hat{L}$ (16) and the tangent morphism

$$T\hat{L} : T^1Y \to TN_L,$$

let us consider the map

$$T\hat{L} \circ J^1u : J^1Y \ni (x^\lambda, y^i, \lambda^i) \mapsto u^i \partial_i + (u^k \partial_k + \partial_u u^k \partial_k^i)(M^i_{\lambda \mu} F^j_\mu) \partial^A =$$

$$u^i \partial_i + [u^k \partial_k (M^i_{\lambda \mu}) a^\lambda_{ij} F^j_\mu + M^i_{\lambda \mu} J^1u (a^\lambda_{ij} F^j_\mu)] \partial^A =$$

$$u^i \partial_i + [u^k \partial_k (M^i_{\lambda \mu}) a^\lambda_{ij} F^j_\mu - M^i_{\lambda \mu} \partial_i u^k a^\lambda_{ij} F^j_\mu] \partial^A =$$

$$u^i \partial_i + [u^k \partial_k (a \circ \sigma_0)^\mu_i a^\lambda_{ij} F^j_\mu - (a \circ \sigma_0)^\mu_i \partial_i u^k P^\lambda P^i_k] \partial^j \in TN_L,$$

where the relations (37) and (50) have been used. Let us assign to a point $(x^\lambda, y^i, P^\lambda_i) \in N_L$ some point

$$(x^\lambda, y^i, \lambda^i) \in \hat{L}^{-1}(x^\lambda, y^i, P^\lambda_i)$$

and then the image of the point (53) under the morphism (52). We obtain the map

$$v_N : (x^\lambda, y^i, P^\lambda_i) \mapsto u^i \partial_i + [u^k \partial_k (a \circ \sigma_0)^\mu_i P^\lambda_i - (a \circ \sigma_0)^\mu_i \partial_i u^k P^\lambda P^i_k] \partial^j$$

which is independent of the choice of a point (53). Therefore, it is a vector field on the Lagrangian constraint manifold $N_L$. This vector field gives rise to the vector field

$$\nu_N = u^i \partial_i + [u^k \partial_k (a \circ \sigma_0)^\mu_i P^\lambda_i - (a \circ \sigma_0)^\mu_i \partial_i u^k P^\lambda P^i_k] \partial^j + d_{\lambda} u^i \partial_i$$

on $N_L \times J^1Y$.

Proposition 10. The Lie derivative $\mathbf{L}_{\pi_N} L_N$ of the Lagrangian $L_N$ (33) along the vector field $\nu_N$ (55) vanishes.

Proof. One can show that

$$v_N(P^\lambda_i) = -\partial_i u^k P^\lambda_k$$

on the constraint manifold $\mathcal{R}_i^\lambda = 0$. Then the invariance condition $\nu_N(L_N) = 0$ falls into the three equalities

$$\nu_N(\sigma_0)_{ij} P^\lambda_i P^\mu_j = 0, \quad \nu_N(P^\lambda_i F^j_\lambda) = 0, \quad \nu_N(c') = 0.$$

(57)
The latter is exactly the second equality (49). The first equality (57) is satisfied due to the relations (51) and (56). The second one takes the form

$$v_N(P_i^\lambda(y_i^\lambda - \Gamma_i^\lambda)) = 0.$$

(58)

It holds owing to the relations (47) and (56).

Thus, the gauge invariance of the Lagrangian $L$ (29) implies that of the Lagrangian $L_N$ (33). Turn now to gauge symmetries of the pull-back $L_\Pi$ (34) of the Lagrangian $L_N$ onto $\Pi \times J^1Y$.

Due to the splitting (41), the vector field $v_N$ (54) on $N_L$ can be extended onto $\Pi$ by putting it zero on $\text{Ker} \sigma_0$. It keeps the coordinate form

$$v_\Pi = u^i \partial_i + [u^k \partial_k (a \circ \sigma_0)^{ij}_{\mu} P_i^\lambda - (a \circ \sigma_0)^{ij}_{\mu} \partial_i u^k P_k^\lambda] \partial_j.$$

(59)

but the transformation law (56) is modified as

$$v_\Pi(P_i^\lambda) = u^k \partial_k (a \circ \sigma_0)^\lambda_{\mu j} P^\mu_j - \partial_i u^k P_k^\lambda.$$

(60)

As a consequence, the invariance condition (58) does not hold, and the Lagrangian $L_\Pi$ fails to be gauge-invariant in general.

In order to overcome this difficulty, let us assume that the gauge group $G_Y$ preserves the splitting (23), i.e., its infinitesimal generators $u$ obey the condition

$$u^k \partial_k (\sigma_0^{im}_{\lambda \nu} a^{\nu \mu}_{mj}) + \sigma_0^{im}_{\lambda \nu} a^{\nu \mu}_{mk} \partial_j u^k - \partial_k u^i \sigma_0^{km}_{\lambda \nu} a^{\nu \mu}_{mj} = 0.$$

(61)

The relations (47) and (61) lead to the transformation law

$$J^1 u(\mathcal{F}_\mu^i) = \partial_j u^i \mathcal{F}_j^\mu.$$

(62)

Since $S_i^\lambda = y_i^\lambda - \mathcal{F}_i^\lambda$, we also obtain

$$J^1 u(S_\mu^i) = d_\mu u^i - \partial_j u^i \mathcal{F}_\mu^j = d_\mu u^i - \partial_j u^i (y_\mu^j - S_\mu^j) = \partial_\mu u^i + \partial_j u^i S_\mu^j.$$

(63)

A glance at this relation shows that the gauge group $G_Y$ acts freely on the space of sections $S(x)$ of the fiber bundle $\text{Ker} \hat{L} \to Y$ in the splitting (23). Then some combinations $b^{\mu \nu}_i(x) S_\mu^i$ of $S_\mu^i$ can be used as the gauge-fixing condition

$$b^{\mu \nu}_i S_\mu^i(x) = \alpha^\tau(x).$$

(64)
similar to the generalized Lorentz gauge in Yang–Mills gauge theory.

By virtue of the condition (61), the vector field $v_{\Pi}$ (59) takes the form

$$v_{\Pi} = u^i \partial_i - \partial_j u^i p^j_\lambda \partial_\lambda. \quad \text{(65)}$$

However, the transformation law (60) holds and the Lagrangian $L_{\Pi}$ fails to be gauge-invariant. Therefore, let us consider a different vector field on $\Pi$ projected onto the vector field $v_N$ (54) on $N_L$.

Any vertical vector field $u$ on $Y \to X$ gives rise to the vector field

$$u_{\Pi} = u^i \partial_i - \partial_j u^i p^j_\lambda \partial_\lambda \quad \text{(66)}$$
on the Legendre bundle $\Pi$ and to the vector field

$$\overline{u}_{\Pi} = u^i \partial_i - \partial_j u^i p^j_\lambda \partial_\lambda + d_\lambda u^i \partial_\lambda \quad \text{(67)}$$
on $\Pi \times \tilde{J}^1 Y$. The vector field (66) as like as the vector field (65) is projected onto the vector field $v_N$ (54). We have

$$u_{\Pi} - v_{\Pi} = -\partial_j u^i \overline{R}_i^\lambda \partial_\lambda.$$

Proposition 11. If the condition (61) holds, the vector field $u_{\Pi}$ (66) is an infinitesimal gauge symmetry of the Lagrangian $L_{\Pi}$ (34) iff $u$ is an infinitesimal gauge symmetry of the Lagrangian $L$ (29).

Proof. Due to the condition (61), the infinitesimal gauge symmetry $u_{\Pi}$ (66) preserves the splitting (25), i.e.,

$$\overline{u}_{\Pi}(\sigma^i_{\lambda\mu}) = -\partial_i u^k p^\lambda_k, \quad \overline{u}(R^\lambda_i) = -\partial_i u^k R^\lambda_k. \quad \text{(68)}$$

We have the gauge invariance condition

$$(u^i \partial_i - \partial_j u^i p^j_\lambda \partial_\lambda + d_\lambda u^i \partial_\lambda) L_{\Pi} = 0. \quad \text{(69)}$$

It is readily observed that the first and third terms of the Lagrangian $L_{\Pi}$ are separately gauge-invariant due to the relations (49) and (62). Its second term is gauge-invariant owing to the equality (51). Conversely, let the invariance condition (69) holds. It falls into the independent equalities

$$\overline{u}_{\Pi}(\sigma_\mu^i p^\lambda_i p^\mu_j) = 0, \quad \overline{u}_{\Pi}(p^\lambda_i \mathcal{F}^j_\lambda) = 0, \quad u^i \partial_i c' = 0. \quad \text{(70)}$$
i.e., the Lagrangian \( L_\Pi \) is gauge-invariant iff its three summands are separately gauge-invariant. One obtains at once from the second condition (70) that the quantity \( F \) is transformed as the dual of momenta \( p \). Then the first condition (70) shows that the quantity \( \sigma_0 p \) is transformed by the same law as \( F \). It follows that the term \( a F F \) in the Lagrangian \( L \) (29) is transformed as \( a(\sigma_0 p)(\sigma_0 p) = \sigma_0 pp \), i.e., is gauge-invariant. Then this Lagrangian is gauge-invariant due to the third equality (70).

Though vector fields \( u_\Pi \) (66) are infinitesimal generators of gauge symmetries of the Lagrangian \( L_\Pi \) (34) in accordance with Proposition 11, they are not infinitesimal generators of the gauge group \( G_Y \) because they fail to commute with the vector fields (43). Moreover, the system of vector fields (43) and (66) is not closed with respect to the Lie bracket, and this fact makes the BV quantization procedure rather complicated.

Therefore, let us return to the vector fields \( v_\Pi \) (65), but require that

\[
u^k \partial_k a_{\mu
u} = 0.
\]

(71)

For instance, this condition is always satisfied if the matrix function \( a \) in the Lagrangian (10) is independent of \( y^i \). This is the standard case of perturbation quantum field theory, e.g., quantum gauge theory.

If \( u \) obeys the condition (71), the transformation law (60) comes to

\[
u_\Pi(\mathcal{P}^\lambda_i) = -\partial_i u^k \mathcal{P}^\lambda_k.
\]

(72)

Due to this transformation law, the vector fields (65) become infinitesimal generators of gauge symmetries of the Lagrangian \( L_\Pi \) (34). Moreover, if the condition (71) holds, vector field (65) commute with the infinitesimal generators (43) of the gauge group \( G_\Pi \). Therefore, they are infinitesimal generators of the gauge group \( G_Y \) acting on \( \Pi \times J^1 Y \). Thus, the vector fields \( u_\Pi \) (43) and \( v_\Pi \) (65) are infinitesimal generators of one-parameter groups of gauge symmetries of the Lagrangian \( L_\Pi \) (34) which we aim to quantize.

Let us further assume that the gauge group \( G_Y \) of the Lagrangian \( L \) (10) is indexed by \( m \) parameter functions \( \xi^r(x) \) such that the components \( u^i(x^\lambda, y^j, \xi^r) \) of its infinitesimal generators \( u \) are linear first order differential operators

\[
u^i(x^\lambda, y^j, \xi^r) = u^i_\nu(x^\lambda, y^j) \xi^r + u^i_\mu(x^\lambda, y^j) \partial_\mu \xi^r
\]

(73)
on the space of parameters $\xi^r(x)$ and the vector fields $u(\xi^r)$ (38) satisfy the commutation relations
\[ [u(\xi^q), u(\xi^p)] = u(c^r_{pq} \xi^p \xi^q), \]
where $c^r_{pq}$ are structure constants.

5 BV quantization

Turn now to the BV quantization of the Lagrangian $\mathcal{L}_\Pi$ (34) on $\Pi \times J^1Y$ whose infinitesimal gauge symmetries are represented by vector field
\[ \pi = u^i \partial_i + (\xi_a (M^{-1})^a_{\lambda i} - \partial_i u^k \mathcal{P}^\lambda_k) \partial^i_{\lambda} + d u^i \partial^i_{\lambda}, \] (74)
where $u$ are vector fields (73) depending on parameters $\xi^r$.

We follow the quantization procedure in [10, 11] reformulated in the jet terms [12, 13]. Note that odd fields $C^r$ can be introduced as the basis for a graded manifold determined by the dual $E^*$ of a vector space $E \to X$ coordinated by $(x^\lambda, e^r)$. Then the $k$-order jets $C^r_{\lambda_k \ldots \lambda_1}$ are defined as the basis for a graded manifold determined by the dual of the $k$-order jet bundle $J^kE \to X$, which is a vector bundle [14]. The BV quantization procedure falls into the two steps. At first, one obtains a proper solution of the classical master equation and, afterwards, the gauge-fixed BRST invariant Lagrangian is constructed.

Let the number $m$ of parameters of the gauge group $G$ do not exceed the fiber dimension of $\text{Ker} \hat{L} \to Y$. Then we can follow the standard BV quantization procedure for irreducible gauge theories in [11].

Firstly, one should introduce odd ghosts $C^r$, $C_a$ of ghost number 1 together with odd antifields $y^i$, $p^i_{\lambda} \xi^r$ and even antifields $C^*_r$, $C^*_a$ of ghost number $-2$. Then a proper solution of the classical master equation reads
\[ \mathcal{L}_{PS} = \mathcal{L}_\Pi + y^i u^i_C + p^i_{\lambda} u^i_C - \frac{1}{2} c^r_{pq} C^*_r C^p C^q, \] (75)
where $u_C$ is the vector field
\[ u^i_C = u^i_r C^r + u^i_r C^a, \quad u^i_{C_i} = C_a (M^{-1})^a_{\lambda i} - \partial_i u^k \mathcal{P}^\lambda_k \] (76)
obtained from the vector field (74) by replacement of parameter functions $\xi_a, \xi^r$ and their derivatives $\partial^i_{\mu} \xi^r$ with the ghosts $C_a, C^r$ and their jets $C^r_{\mu}$. 

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Secondly, one introduces the gauge-fixing density depending on fields \( y^i, p_1^\lambda \) ghosts \( C^r, C_a \) and additional auxiliary fields, which are odd fields \( \overline{C}_p, \overline{C}_a \) of ghost number \(-1\) and even fields \( B_r, B^a \) of zero ghost number. Passing to the Euclidean space-time, this gauge-fixing density reads

\[
\Psi = \overline{C}_p \left( \frac{i}{2} h^{pr} B_r + b^{\mu}_{i} S^i_{\mu} \right) + \overline{C}_a \left( \frac{i}{2} (\sigma_1^{-1})_{ab} B^b + M_a i p^i \right),
\]

where \( h^{pr}(x) \) is a non-degenerate positive-definite matrix function on \( X \) and \( b^{\mu}_{i} S^i_{\mu} \) are the gauge-fixing combinations \( (64) \).

Thirdly, the desired gauge-fixing Lagrangian \( \mathcal{L}_{GF} \) is derived from the extended Lagrangian

\[
\mathcal{L}'_{PS} = \mathcal{L}_{PS} + i \overline{C}_p B_p + i \overline{C}_a B^a,
\]

where \( \overline{C}_p, \overline{C}_a \) are antifields of auxiliary fields \( C_p, C_a \) by replacement of antifields with the variational derivatives

\[
y_i^* = \frac{\delta \Psi}{\delta y^i}, \quad C_p^* = \frac{\delta \Psi}{\delta C_p} = 0, \quad \overline{C}_p^* = \frac{\delta \Psi}{\delta \overline{C}_p} = \frac{i}{2} h^{pr} B_r + b^{\mu}_{i} S^i_{\mu},
\]

\[
p_\lambda^i = \frac{\delta \Psi}{\delta p_\lambda^i} = \overline{C} a M_a i, \quad \overline{C} a^* = \frac{\delta \Psi}{\delta \overline{C}_a} = \frac{i}{2} (\sigma_1^{-1})_{ab} B^b + M_a i p^i.
\]

We obtain

\[
\mathcal{L}_{GF} = \mathcal{L}_{\Pi} + (\delta_i \Psi u_C^i + \delta_\lambda^i \Psi u_C^\lambda) - B_p \left( \frac{i}{2} h^{pr} B_r - i b^{\mu}_{i} S^i_{\mu} \right) - B^a \left( \frac{i}{2} (\sigma_1^{-1})_{ab} B^b - i M_a i p^i \right).
\]

Let us bring its second term into the form

\[
(\partial_i \Psi - d_\lambda \partial_\lambda^i \Psi) u_C^i + \partial_\lambda^i \Psi u_C^\lambda = \partial_i \Psi u_C^i + \partial_\lambda^i \Psi d_\lambda (u_C^i) - d_\lambda (\partial_\lambda^i \Psi u_C^\lambda) + \partial_\lambda^i \Psi u_C^\lambda =
\]

\[
\pi_C(\Psi) - d_\lambda (\partial_\lambda^i \Psi u_C^i),
\]

where

\[
\pi_C = u_C^i \partial_i + u_C^\lambda \partial_\lambda^i + d_\lambda u_C^i \partial_\lambda^i, \quad d_\lambda = \partial_\lambda + y_\lambda^i \partial_i + C_\lambda \frac{\partial}{\partial \overline{C}_r}
\]

is the jet prolongation of the vector field \( u_C \) \( (76) \). In view of the transformation law \( (63) \), we have

\[
\pi_C(\Psi) = -C_p b^{\lambda i}_i J^1 u_C^\lambda (S_\lambda^i) - C_a u_C (M a i p^i) = -C_p b^{\lambda i}_i [\partial_\lambda u_C^\lambda C^r + u_C^i C_\lambda^r + \partial_\lambda u_{i}^{\mu} C^r_{\mu} + u_{i}^{\mu} C_{\lambda^r \mu} +
\]

\[
(\partial_j u_C^r C^r + \partial_j u_{i}^{\mu} C_{\mu} C_i^r S_\lambda^i) - C_a M a i ((M^{-1})_{i}^{b \lambda} C_b - \partial_i u^k p_{\lambda}^{k i} = -C_p M_a C^r - C_a
\]

\[\text{(81)}\]
where $\mathcal{M}^\mu C^\nu$ is a second order differential operator on ghosts $C^\nu$. Then the gauge-fixing Lagrangian (79) up to a divergence term takes the form

$$\mathcal{L}_{GF} = \mathcal{L}_\Pi - \overline{C}_P \mathcal{M}^\mu P^\nu C^\nu - \overline{C}^\alpha C_a - \frac{1}{2} h^{pr} B_p B_r + i B_p b^{\mu\nu} S^i_\mu - \frac{1}{2} B^a (\sigma_1^{-1})_{ab} B^b + i B^a M_a i p_1^\lambda.$$

Finally, one can write the generating functional

$$Z = N^{-1} \int \exp\{ \int (\mathcal{L}_\Pi - \overline{C}_P \mathcal{M}^\mu P^\nu C^\nu - \overline{C}^\alpha C_a - \frac{1}{2} h^{pr} B_p B_r + i B_p b^{\mu\nu} S^i_\mu - \frac{1}{2} B^a (\sigma_1^{-1})_{ab} B^b + i B^a M_a i p_1^\lambda + i J_\lambda y^k + i J_\lambda^1 p_1^\lambda) \omega \}
\prod_x [dB_p] [dB^a] [dC]\overline{C} [dC^\nu] [dC^\mu] [dc_a][dy(x)][dp(x)].$$

Of course, the Lagrangian

$$\mathcal{L}_\Pi - \overline{C}_P \mathcal{M}^\mu P^\nu C^\nu - \overline{C}^\alpha C_a - \frac{1}{2} h^{pr} b^{\mu\nu} b^{\nu\mu} S^i_\mu S^i_\nu - \frac{1}{2} \sigma_1^{ab} M_a i M_b j^\mu p_1^\lambda p_2^\mu$$

in the generating functional (83) is not gauge-invariant, but it is invariant under the BRST transformation

$$\vartheta = u^\lambda C_\lambda + u^\lambda \partial_\lambda + d_\lambda y^i \partial_\lambda + v^a \frac{\partial}{\partial C_a} + v_a \frac{\partial}{\partial C_\lambda} +$$

$$\overline{v}^\mu \frac{\partial}{\partial \overline{C}^\mu} + v^\nu \frac{\partial}{\partial \overline{C}^\nu} + d_\lambda v^\nu \frac{\partial}{\partial \overline{C}^\nu} + d_\lambda v^\nu \frac{\partial}{\partial \overline{C}^\nu}_\lambda +$$

$$d_\lambda \frac{\partial}{\partial C^\lambda} + y^i_\lambda \partial_\lambda + y^i_\lambda^\mu \partial_\mu + C^\nu_\lambda \frac{\partial}{\partial \overline{C}^\nu} + C^\nu_\lambda^\mu \frac{\partial}{\partial \overline{C}^\mu},$$

whose components $v$ are given by the antibrackets

$$v_a = (C_a, \mathcal{L}_{PS}) = \frac{\delta \mathcal{L}_{PS}'}{\delta C^*_a} = 0, \quad v^i = (\overline{C}^i, \mathcal{L}_{PS}) = \frac{\delta \mathcal{L}_{PS}'}{\delta C^*_a} = i B^a$$

$$v^r = (C^r, \mathcal{L}_{PS}) = \frac{\delta \mathcal{L}_{PS}'}{\delta C^*_r} = -\frac{1}{2} c^r_{pq} C^p C^q, \quad \overline{v}^r = (\overline{C}^r, \mathcal{L}_{PS}) = \frac{\delta \mathcal{L}_{PS}'}{\delta \overline{C}^r} = i B_r.$$
restricted to the shell (78) and to the solutions

\[ B_r = i h^{-1}_{rp} b^p \xi_\mu^i, \quad B^a = i \sigma^i_{1\lambda} M_{ab} \rho^\lambda \]

of the Euler–Lagrange equations \( \delta \mathcal{L}_{GF} / \delta B_r = 0 \) and \( \delta \mathcal{L}_{GF} / \delta B^a = 0 \).

Integration of the generating functional \( Z (83) \) as a Gaussian integral with respect to the variables \( C^i, C_a \) results in the desired BV quantization

\[ Z = N' \int \exp \left\{ \int (\mathcal{L}_\Pi - \mathcal{C} \mathcal{M} \mathcal{C} - \frac{1}{2} h^{-1}_{pr} b^{\rho_i} b^{r} \xi_\mu^i \xi_\nu^j - \frac{1}{2} \sigma_{ij} \rho^\mu \xi_\mu^i \xi_\nu^j + i J_k y^k + i J_i \rho^\lambda \omega) \prod_x [dC_r][dC^r][dy(x)][dp(x)] \right\} \]

of the Lagrangian \( L_\Pi (34) \). Integrating the generating functional \( Z (86) \) as a Gaussian integral with respect to the momenta \( p(x) \) under the condition \( J_\lambda = 0 \), one restarts the generating functional

\[ Z = N' \int \exp \left\{ \int (\mathcal{L} - \mathcal{C} \mathcal{M} \mathcal{C} - \frac{1}{2} h^{-1}_{pr} b^{\rho_i} b^{r} \xi_\mu^i \xi_\nu^j + i J_k y^k) \omega \prod_x [dC_r][dC^r][dy(x)] \right\} \]

of the BV quantization of the original quadratic Lagrangian \( L (10) \).

6 Hamiltonian gauge theory

For example, let us consider gauge theory of principal connections on a principal bundle \( P \to X \) with a structure Lie group \( G \). Principal connections on \( P \to X \) are represented by sections of the affine bundle

\[ C = J^1 P / G \to X, \]

modelled over the vector bundle \( T^* X \otimes V_G P \) [1]. Here, \( V_G P = VP / G \) is the fiber bundle in Lie algebras \( \mathfrak{g} \) of the group \( G \). Given the basis \( \{ \varepsilon_r \} \) for \( \mathfrak{g} \), we obtain the local fiber bases \( \{ e_r \} \) for \( V_G P \). The connection bundle \( C (87) \) is coordinated by \( (x^\mu, a^\mu_r) \) such that, written relative to these coordinates, sections \( A = A^\mu_r dx^\mu \otimes e_r \) of \( C \to X \) are the familiar local connection one-forms, regarded as gauge potentials.

There is one-to-one correspondence between the sections \( \xi = \xi^r e_r \) of \( V_G P \to X \) and the vector fields on \( P \) which are infinitesimal generators of one-parameter groups of vertical
automorphisms (gauge transformations) of $P$. Any section $\xi$ of $V_G P \to X$ yields the vector field

$$u(\xi) = u^r_\mu \frac{\partial}{\partial a^r_\mu} = \left( c^r_{pq} a^p_\mu \xi^q + \partial_\mu \xi^r \right) \frac{\partial}{\partial a^r_\mu}$$

on $C$, where $c^r_{pq}$ are the structure constants of the Lie algebra $g$.

The configuration space of gauge theory is the jet manifold $J^1 C$ equipped with the coordinates $(x^\lambda, a^r_\lambda, a^r_{\mu \lambda})$. It admits the canonical splitting (23) given by the coordinate expression

$$a^r_{\mu \lambda} = S^r_{\mu \lambda} + \mathcal{F}^r_{\mu \lambda} = \frac{1}{2} (a^r_{\mu \lambda} + a^r_{\lambda \mu} - c^r_{pq} a^p_\mu a^q_\lambda) + \frac{1}{2} (a^r_{\mu \lambda} - a^r_{\lambda \mu} + c^r_{pq} a^p_\mu a^q_\lambda),$$

where $\mathcal{F}$ is the strength of gauge fields up to the factor $1/2$. The Yang–Mills Lagrangian on the configuration space $J^1 C$ reads

$$L_{\text{YM}} = a_G^{\mu \nu} g^{\alpha \beta} \mathcal{F}^p_{\alpha \beta} \mathcal{F}^q_{\mu \nu} \sqrt{|g|} \omega, \quad g = \det(g_{\mu \nu}),$$

where $a^G$ is a non-degenerate $G$-invariant metric in the dual of the Lie algebra of $g$ and $g$ is a nondegenerate metric on $X$.

The phase space $\Pi$ (3) of the gauge theory is endowed with the canonical coordinates $(x^\lambda, a^p_\lambda, p^{\mu \lambda})$. It admits the canonical splitting (25) given by the coordinate expression

$$p^{\mu \lambda}_m = R^{\mu \lambda}_m + \mathcal{P}^{\mu \lambda}_m = p^{(\mu \lambda)}_m + p^{[\mu \lambda]}_m = \frac{1}{2} (p^{(\mu \lambda)}_m + p^{(\lambda \mu)}_m) + \frac{1}{2} (p^{(\mu \lambda)}_m - p^{(\lambda \mu)}_m).$$

With respect to this splitting, the Legendre map induced by the Lagrangian (90) takes the form

$$p^{(\mu \lambda)}_m \circ \hat{L}_{\text{YM}} = 0,$$

$$p^{[\mu \lambda]}_m \circ \hat{L}_{\text{YM}} = 4 a_G^{\mu \alpha} g^{\alpha \beta} \mathcal{F}^n_{\alpha \beta} \sqrt{|g|}.$$

The equalities (92) define the Lagrangian constraint space $N_L$ of Hamiltonian gauge theory. Obviously, it is an imbedded submanifold of $\Pi$, and the Lagrangian $L_{\text{YM}}$ is almost-regular.

In order to construct an associated Hamiltonian, let us consider a connection $\Gamma$ (18) on the fiber bundle $C \to X$ which take their values into $\text{Ker} \hat{L}$, i.e.,

$$\Gamma^r_{\lambda \mu} - \Gamma^r_{\mu \lambda} + c^r_{pq} a^p_\lambda a^q_\mu = 0.$$
Given a symmetric linear connection $K$ on $X$ and a principal connection $B$ on $P \to X$, this connection reads

$$\Gamma_{\lambda \mu}^r = \frac{1}{2} \left[ \partial_\mu B_\lambda^r + \partial_\lambda B_\mu^r - c_{pq}^r a_\lambda^p a_\mu^q + c_{pq}^r (a_\lambda^p B_\mu^q + a_\mu^p B_\lambda^q) \right] - K_\lambda^\beta (a_\beta^r - B_\beta^r).$$

The corresponding Hamiltonian (30) associated to $L_{YM}$ is

$$\mathcal{H}_\Gamma = \rho^{\lambda \mu} \Gamma_{\lambda \mu}^r + a^{mn}_G g_{\mu \nu} g_{\lambda \beta} p^{|\mu \lambda|}_m p^{|\nu \beta|}_n \sqrt{|g|}. $$

Then we obtain the Lagrangian

$$\mathcal{L}_N = \rho^{\lambda \mu} F_{\lambda \mu}^r - a^{mn}_G g_{\mu \nu} g_{\lambda \beta} p^{|\mu \lambda|}_m p^{|\nu \beta|}_n \sqrt{|g|}$$

(33) on $N_L \times J^1 Y$ and its pull-back

$$L_{\Pi} = \mathcal{L}_{\Pi \omega}, \quad \mathcal{L}_{\Pi} = \rho^{\lambda \mu} F_{\lambda \mu}^r - a^{mn}_G g_{\mu \nu} g_{\lambda \beta} p^{|\mu \lambda|}_m p^{|\nu \beta|}_n \sqrt{|g|},$$

(94) onto $\Pi \times J^1 Y$.

Both the Lagrangian $L_{YM}$ (90) and the Lagrangian $L_{\Pi}$ (94) are invariant under gauge transformations whose infinitesimal generators are the lifts

$$J^1 u(\xi) = (c_{pq}^r a_\mu^p a_\xi^q + \partial_\mu \xi^r) \frac{\partial}{\partial a_\mu^p} + \left( c_{pq}^r (a_\lambda^p \xi^q + a_\mu^p \partial_\lambda \xi^r) + \partial_\lambda \partial_\mu \xi^r \right) \frac{\partial}{\partial a_\mu^p},$$

$$\Pi(\xi) = J^1 u(\xi) - c_{pq}^{\lambda \mu} a_\lambda^p \xi^q \frac{\partial}{\partial p^{\lambda \mu}}$$

of the vector fields (88) onto $J^1 C$ and $\Pi \times J^1 C$, respectively. We have the transformation laws

$$J^1 u(\xi)(F_{\lambda \mu}^r) = c_{pq}^r F_{\lambda \mu}^p \xi^q, \quad J^1 u(\xi)(S_{\lambda \mu}^r) = c_{pq}^r S_{\lambda \mu}^p \xi^q + c_{pq}^r a_\mu^p \partial_\lambda \xi^q + \partial_\lambda \partial_\mu \xi^r.$$ 

Therefore, one can choose the gauge conditions

$$g^{\lambda \mu} S_{\lambda \mu}^r(x) - a_\lambda^r(x) = \frac{1}{2} g^{\lambda \mu} (\partial_\lambda a_\mu^r(x) + \partial_\mu a_\lambda^r(x)) - a_\lambda^r(x) = 0,$$

which are the familiar generalized Lorentz gauge. The corresponding second-order differential operator (81) reads

$$\mathcal{M}_s^r \xi^s = g^{\lambda \mu} \left( \frac{1}{2} c_{pq}^r (\partial_\lambda a_\mu^p + \partial_\mu a_\lambda^p) \xi^q + c_{pq}^r a_\mu^p \partial_\lambda \xi^q + \partial_\lambda \partial_\mu \xi^r \right).$$
Passing to the Euclidean space and repeating the quantization procedure in previous Section, we come to the generating functional

\[ Z = N^{-1} \int \exp \left\{ \int (p^r_\mu F^r_{\lambda \mu} - a^m_G g_{\lambda \beta} p^p_m p^p_\mu \sqrt{|g|} - \frac{1}{8} a^C_{rs} g^{\lambda \mu} (\partial_\alpha a^r_\alpha + \partial_{\alpha} a^r_\mu + \partial_{\mu} a^s_\alpha) - g^{\lambda \mu} C_r \left( \frac{1}{2} e^r_q (\partial_\lambda a^p_\mu + \partial_\mu a^p_\lambda) C^q + c^s_{pq} a^p_\mu C^q + C^r_{\lambda \mu} \right) + iJ_r^p a^r_\mu + iJ^r_{\mu \lambda} p^r_\lambda \right\} \prod_x [dC][dP][dp(x)][da(x)]. \]

Equation (86) of BV quantization of Hamiltonian gauge theory. Its integration with respect to momenta restarts the familiar BV quantization of gauge theory.

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