Layer Potential Techniques for the Narrow Escape Problem

Habib Ammari*  Kostis Kalimeris†  Hyeonbae Kang‡  Hyundae Lee‡

Abstract

The narrow escape problem consists of deriving the asymptotic expansion of the solution of a drift-diffusion equation with the Dirichlet boundary condition on a small absorbing part of the boundary and the Neumann boundary condition on the remaining reflecting boundaries. Using layer potential techniques, we rigorously find high-order asymptotic expansions of such solutions. We explicitly show the nonlinear interaction of many small absorbing targets. Based on the asymptotic theory for eigenvalue problems developed in [2], we also construct high-order asymptotic formulas for eigenvalues of the Laplace and the drifted Laplace operators for mixed boundary conditions on large and small pieces of the boundary.

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1 Introduction

An interesting problem in cellular and molecular biology is to estimate the mean sojourn time, also called the narrow escape time, of a Brownian particle in a bounded domain Ω before it escapes through a small absorbing arc ∂Ωa on its boundary ∂Ω. The remaining part of the boundary ∂Ωr = ∂Ω \ ∂Ωa is assumed reflecting for the particle. The small arc often represents a small target on a cellular membrane. Its physiological role is to regulate flux, which carries a physiological signal [11], [4].

The narrow escape problem is connected to that of calculating the solution \( u_\varepsilon \) of a mixed Dirichlet-Neumann boundary value problem in Ω, whose Dirichlet boundary is only ∂Ωa on the otherwise Neumann boundary. The escape time can be estimated asymptotically in the limit \( \varepsilon := |\partial \Omega_a|/|\partial \Omega| \to 0 \) by computing the asymptotic expansion of \( u_\varepsilon \) as \( \varepsilon \to 0 \).

In this paper, we first consider a purely diffuse model. We provide mathematically rigorous derivations of the first- and second-order terms in the asymptotic expansion of the solution \( u_\varepsilon \) as \( \varepsilon \to 0 \) in the presence of a single or many small targets. When two
or more Dirichlet targets cluster together they interact nonlinearly. The clustering may affect significantly the asymptotics. Then we study the problem of eigenvalue changes due to the small targets. Finally, accounting for a drift term, we generalize our results to a mixed Dirichlet-Neumann boundary value problem for the drift-diffusion equation.

The narrow escape problem of a free Brownian particle (without drift) through a small target was discussed in [5], [14], [15], and [16]. The method of [14], [15], [16] was generalized in [12] to obtain the leading-order term of the solution to the corresponding mixed boundary value problem for the drift-diffusion equation. Matched asymptotics [17], [18], [19], [8] yield the expansion of the principle eigenvalue of the Laplace operator for mixed boundary conditions on large and small pieces of the boundary. The effect of clustering of the Dirichlet targets on the first eigenvalue was analyzed in [6]. The second-order term in its asymptotic expansion as the size of the target goes to zero was provided in [13] by determining the structure of the boundary singularity of the Neumann function for the Laplacian in a bounded smooth domain. In all of these papers, the derivations are quite formal. It is the purpose of this work to provide a rigorous framework for systematically deriving high-order asymptotic formulas for the solutions of the diffusion and drift-diffusion equations with mixed boundary conditions on large and small pieces of the boundary and the eigenvalues of the corresponding operators. Our derivations are based on layer potential techniques and the asymptotic theory for eigenvalue problems developed in [2].

While this work was in progress, we found out the existence of the work [9]. Some of results of this paper, especially those in Section 3.1, were also obtained in [9]. However, the method of derivation of asymptotics in [9] is formal and uses the method of matched asymptotics.

This paper is organized as follows. In Section 2, we formulate the problem with and without the drift term, and review some facts on relevant Neumann functions and layer potentials on arcs. Section 3 is to derive (higher-order) asymptotic formula for the solutions to the narrow escape problem with a single or multiple (well-separated and closely located) absorbing regions. Section 4 is to deal with the eigenvalue perturbation problem in the presence of small absorbing region using the same layer potential techniques. Section 5 is for the narrow escape problem and the eigenvalue perturbation problem in the presence of the force field. The paper ends with a short discussion.

2 Formulation of the problem

2.1 Physical background

Let Ω be a bounded simply connected domain in \( \mathbb{R}^2 \) with \( C^2 \)-smooth boundary. Suppose that \( \partial \Omega \) has two disjoint parts, the reflecting part \( \partial \Omega_r \) and the absorbing part \( \partial \Omega_a \), satisfying \( \partial \Omega = \partial \Omega_a \cup \partial \Omega_r \). Both \( \partial \Omega_a \) and \( \partial \Omega_r \) consist of finite number of open arcs. We assume that \( \varepsilon := |\partial \Omega_a|/2 \) is much smaller than 1 while \( |\partial \Omega| \) is of order 1. Here and throughout this paper \( |\partial \Omega| \) denotes the arc-length of \( \partial \Omega \).

Suppose that a Brownian particle is confined to Ω. The probability density function \( p_\varepsilon(x, t) \) of finding the Brownian particle at location \( x \) at time \( t \) (prior to its escape) satisfies

\[
p_\varepsilon(x, t) = \frac{1}{\sqrt{4\pi D \varepsilon}} \exp\left(-\frac{|x|^2}{4D \varepsilon}\right)
\]

where \( D \) is the diffusion coefficient.
the Fokker-Planck equation
\[
\frac{\partial p_{\epsilon}(x,t)}{\partial t} = \Delta p_{\epsilon}(x,t) - \nabla \cdot (F(x)p_{\epsilon}(x,t))
\]
with the initial condition
\[p_{\epsilon}(x,0) = \rho(x),\]
and the mixed boundary conditions for \( t > 0 \)
\[
\begin{cases}
p_{\epsilon} = 0 & \text{on } \partial \Omega_a, \\
\frac{\partial p_{\epsilon}}{\partial \nu} - p_{\epsilon} F \cdot \nu = 0 & \text{on } \partial \Omega_r.
\end{cases}
\]
The force field \( F \) is given by \( F(x) = \nabla \phi(x) \) for a smooth potential \( \phi \). The function \( \rho(x) \) is the initial probability density function; e.g., \( \rho(x) = 1/|\Omega| \) for a uniform distribution or \( \rho(x) = \delta_y \), a Dirac mass at \( y \), when the particle is initially located at \( y \).

The function \( v_{\epsilon}(x) := \int_0^{+\infty} p_{\epsilon}(x,t) \, dt \), which is the mean time the particle spends at \( x \) before it escapes through \( \partial \Omega_a \), is the solution to
\[
\begin{cases}
\Delta v_{\epsilon} - \nabla \cdot (Fv_{\epsilon}) = -\rho & \text{in } \Omega, \\
v_{\epsilon} = 0 & \text{on } \partial \Omega_a, \\
\frac{\partial v_{\epsilon}}{\partial \nu} - v_{\epsilon} F \cdot \nu = 0 & \text{on } \partial \Omega_r.
\end{cases}
\]
The function \( w_{\epsilon}(x) := v_{\epsilon}(x)e^{-\phi(x)} \) is the solution of the adjoint problem
\[
\begin{cases}
\Delta w_{\epsilon} + F \cdot \nabla w_{\epsilon} = -\rho e^{-\phi} & \text{in } \Omega, \\
w_{\epsilon} = 0 & \text{on } \partial \Omega_a, \\
\frac{\partial w_{\epsilon}}{\partial \nu} = 0 & \text{on } \partial \Omega_r.
\end{cases}
\]
Suppose that \( \rho(x) = \delta_y \). The function
\[
u_{\epsilon}(x) = \frac{\int_{\Omega} w_{\epsilon}(x,y) \, dy}{\int_{\Omega} e^{-\phi(y)} \, dy}
\]
is the solution to the mixed boundary value problem
\[
\begin{cases}
\Delta u_{\epsilon} + F \cdot \nabla u_{\epsilon} = -1 & \text{in } \Omega, \\
u_{\epsilon} = 0 & \text{on } \partial \Omega_a, \\
\frac{\partial u_{\epsilon}}{\partial \nu} = 0 & \text{on } \partial \Omega_r.
\end{cases}
\]
(2.1)

The narrow escape problem consists of deriving the asymptotic expansion of \( u_{\epsilon} \) as \( \epsilon \to 0 \), from which one can estimate the sojourn time of the Brownian particle. A closely related problem is to construct the asymptotic of the principal eigenvalue of \( -\Delta - F \cdot \nabla \) in \( \Omega \) with the mixed Dirichlet-Neumann boundary conditions for small \( \epsilon \).

The goal of this paper is threefold: (i) To find the asymptotic behavior of the solution \( u_{\epsilon} \) to (2.1) with \( F = 0 \) (free Brownian motion) in the presence of one or multiple small
targets as $\varepsilon \to 0$; (ii) To find the asymptotics of the eigenvalues of $-\Delta$ in $\Omega$ with the Dirichlet boundary condition on $\partial \Omega$ and the Neumann boundary condition on $\partial \Omega_r$; (iii) To generalize these asymptotics to the case with nonzero drift ($F \neq 0$).

We note that if the force field $F = 0$, then by integrating (2.1) over $\Omega$ the following compatibility condition holds:

$$\int_{\partial \Omega} \frac{\partial u_\varepsilon}{\partial \nu} \, d\sigma = -|\Omega|. \quad (2.2)$$

2.2 Neumann functions

Let $N(x, z)$ be the Neumann function for $-\Delta$ in $\Omega$ corresponding to a Dirac mass at $z \in \Omega$. That is, $N$ is the solution to

$$\begin{cases}
\Delta z N(x, z) = -\delta_x, & x, z \in \Omega, \\
\left. \frac{\partial N}{\partial \nu} \right|_{z \in \partial \Omega} = -\frac{1}{|\partial \Omega|}, \\
\int_{\partial \Omega} N(x, z) d\sigma(z) = 0, & x \in \Omega.
\end{cases} \quad (2.3)$$

The Neumann function is symmetric in its arguments; $N(x, z) = N(z, x)$ for $x \neq z \in \Omega$. It furthermore takes the form

$$N(x, z) = -\frac{1}{2\pi} \ln |x - z| + R_\Omega(x, z), \quad x, z \in \Omega, \quad (2.4)$$

where $R_\Omega(\cdot, z)$ belongs to $H^{3/2}(\Omega)$, the standard Sobolev space of order $3/2$, for any $z \in \Omega$ and solves

$$\begin{cases}
\Delta_x R_\Omega(x, z) = 0, & x \in \Omega, \\
\left. \frac{\partial R_\Omega}{\partial \nu_x} \right|_{x \in \partial \Omega} = -\frac{1}{|\partial \Omega|} + \frac{1}{2\pi} \frac{\langle x - z, \nu_x \rangle}{|x - z|^2}, & x \in \partial \Omega.
\end{cases} \quad (2.5)$$

Suppose $z \in \partial \Omega$. Since $\partial \Omega$ is smooth, only a half of any sufficiently small disk about $z$ is contained in $\Omega$, and hence the singularity of $N(x, z)$ is $-(1/\pi) \ln |x - z|$. Therefore, $N(x, z)$ for $z \in \partial \Omega$, which we denote by $N_{\partial \Omega}$, can be written as

$$N_{\partial \Omega}(x, z) = -\frac{1}{\pi} \ln |x - z| + R_{\partial \Omega}(x, z), \quad x \in \Omega, \quad z \in \partial \Omega, \quad (2.6)$$

where $R_{\partial \Omega}(\cdot, z)$ solves the problem

$$\begin{cases}
\Delta_x R_{\partial \Omega}(x, z) = 0, & x \in \Omega, \\
\left. \frac{\partial R_{\partial \Omega}}{\partial \nu_x} \right|_{x \in \partial \Omega} = -\frac{1}{|\partial \Omega|} + \frac{1}{\pi} \frac{\langle x - z, \nu_x \rangle}{|x - z|^2}, & x \in \partial \Omega.
\end{cases} \quad (2.7)$$

Note that the Neumann data in (2.7) is bounded on $\partial \Omega$ uniformly in $z \in \partial \Omega$ since $\partial \Omega$ is $C^2$-smooth, and hence $R_{\partial \Omega}(\cdot, z)$ belongs to $H^{3/2}(\Omega)$ uniformly in $z \in \partial \Omega$.

Let

$$g(x) := \int_\Omega N(x, z) dz, \quad x \in \Omega. \quad (2.8)$$
Then \( \Delta g = -1 \) in \( \Omega \) and \( \frac{\partial g}{\partial \nu} = -|\Omega|/|\partial \Omega| \) on \( \partial \Omega \). Therefore one may use Green’s formula and the mixed boundary conditions in (2.1) with \( F = 0 \) to have

\[
u \varepsilon(x) = g(x) + \int_{\partial \Omega} N_{\partial \Omega}(x,z) \frac{\partial u_{\varepsilon}(z)}{\partial \nu} d\sigma(z) + C_{\varepsilon}, \quad x \in \Omega, \tag{2.9}\]

where

\[
C_{\varepsilon} = \frac{1}{|\partial \Omega|} \int_{\partial \Omega} u_{\varepsilon}(z) d\sigma(z). \tag{2.10}\]

In view of (2.6), (2.9) becomes

\[
u \varepsilon(x) = g(x) - \frac{1}{\pi} \int_{\partial \Omega} \ln |x-z| \frac{\partial u_{\varepsilon}(z)}{\partial \nu} d\sigma(z) + \int_{\partial \Omega} R_{\partial \Omega}(x,z) \frac{\partial u_{\varepsilon}(z)}{\partial \nu} d\sigma(z) + C_{\varepsilon}. \tag{2.11}\]

If \( \Omega \) is the unit disk, then

\[
\langle x-z, \nu_x \rangle = \frac{|x-z|^2 - 2x \cdot z}{|x|^2 + |z|^2} = \frac{1}{2} - \frac{x \cdot z}{|x|^2} = \frac{1}{2}
\]

for \( x \in \partial \Omega \). Therefore, \( \frac{\partial R_{\varepsilon}}{\partial \nu_{\varepsilon}}|_{x \in \partial \Omega} = 0 \), and hence \( R_{\partial \Omega}(x,z) = \text{const.} \). Because of the condition \( \int_{\partial \Omega} N(x,z) d\sigma(z) = 0 \), we have \( R_{\partial \Omega}(x,z) = 0 \) for all \( x \in \Omega \) and \( z \in \partial \Omega \), and hence

\[
N_{\partial \Omega}(x,z) = -\frac{1}{\pi} \ln |x-z|, \quad x \in \Omega, \quad z \in \partial \Omega. \tag{2.12}\]

We also have

\[
g(x) = \int_{\Omega} N(x,z) dz = \frac{1}{4} (1 - |x|), \tag{2.13}\]

which can be seen using Green’s formula. In fact, we have

\[
\int_{\Omega} N(x,z) \Delta |z|^2 - \Delta z N(x,z)|z|^2 dz = \int_{\partial \Omega} \left( 2N(x,z)|z| - \frac{\partial N(x,z)}{\partial \nu_z} |z|^2 \right) d\sigma(z).
\]

### 2.3 Single-layer potential on an arc

Our aim is to derive an asymptotic expansion of \( \partial u_{\varepsilon}(z)/\partial \nu \) on \( \partial \Omega_{\varepsilon} \) for \( \varepsilon \) small enough.

We write this density function as the solution of an integral equation (2.11) on \( \partial \Omega_{\varepsilon} \). For doing so, we introduce two Hilbert spaces: For \( \varepsilon > 0 \), define

\[
X_{\varepsilon} := \{ \varphi : \int_{-\varepsilon}^{\varepsilon} \sqrt{\varepsilon^2 - x^2} |\varphi(x)|^2 dx < +\infty \}
\]

with the norm

\[
\| \varphi \|_{X_{\varepsilon}} = \left( \int_{-\varepsilon}^{\varepsilon} \sqrt{\varepsilon^2 - x^2} |\varphi(x)|^2 dx \right)^{1/2}.
\]

Then one immediately has

\[
\| \varphi \|_{X_{\varepsilon}} = \| \tilde{\varphi} \|_{X_{1}}, \tag{2.15}\]

where \( \tilde{\varphi}(x) := \varepsilon \varphi(\varepsilon x) \). We also define

\[
Y_{\varepsilon} := \{ \psi \in C^0([-\varepsilon, \varepsilon]) : \psi' \in X_{\varepsilon} \}, \tag{2.16}\]

and
with the norm
\[ \|\psi\|_{Y_\varepsilon} = (\|\psi\|_{X_\varepsilon}^2 + \|\psi'\|^2_{X_\varepsilon})^{1/2}, \]
where \(\psi'\) is the derivative of \(\psi\) in the sense of distribution.

We also recall the following lemma; see [2, Chapter 5] and the references therein.

**Lemma 2.1.** The integral operator \(L : X_1 \mapsto Y_1\) defined by
\[
L[\varphi](x) = \int_{-1}^{1} \ln |x - y| \varphi(y) dy
\]
is invertible. For a given function \(\psi \in Y_1\), \(L^{-1}[\psi] \in X_1\) is given by
\[
L^{-1}[\psi](x) = -\frac{1}{\pi^2 \sqrt{1 - x^2}} \int_{-1}^{1} \frac{\sqrt{1 - y^2} \psi'(y)}{x - y} dy - \frac{a(\psi)}{\pi (\ln 2) \sqrt{1 - x^2}}
\]
for \(x \in (-1, 1)\), where the constant \(a(\psi)\) is defined by
\[
a(\psi) = \psi(x) + L \left[ \frac{1}{\pi^2 \sqrt{1 - y^2}} \int_{-1}^{1} \frac{\sqrt{1 - z^2} \psi'(z)}{y - z} dz \right](x).
\]
\[
L^{-1}[1](x) = -\frac{1}{\pi (\ln 2) \sqrt{1 - x^2}}.
\]

**3 Narrow escape of a free Brownian particle**

We now consider the narrow escape problem in the presence of a single or multiple (clustered or well-separated) absorbing arcs.

### 3.1 A single small target

Let \(x(t) : [-\varepsilon, \varepsilon] \to \mathbb{R}^2\) be the arclength parametrization of \(\partial \Omega_a\); namely, \(x\) is a \(C^2\)-function satisfying \(|x'(t)| = 1\) for all \(t \in [-\varepsilon, \varepsilon]\) and
\[
\partial \Omega_a = \{x(t) : t \in [-\varepsilon, \varepsilon]\}.
\]

Since \(u_\varepsilon = 0\) on \(\partial \Omega_a\), it follows from (2.11) that
\[
f(t) - \frac{1}{\pi} \int_{-\varepsilon}^{\varepsilon} \ln |x(t) - x(s)| \varphi_\varepsilon(s) ds + \int_{-\varepsilon}^{\varepsilon} r(t, s) \varphi_\varepsilon(s) ds + C_\varepsilon = 0,
\]
where
\[
f(t) := g(x(t)), \quad \varphi_\varepsilon(t) := \frac{\partial u_\varepsilon(x(t))}{\partial \nu}, \quad r(t, s) := R_{\partial \Omega}(x(t), x(s)).
\]

By a simple change of variables, we obtain that
\[
\frac{1}{\pi} \int_{-1}^{1} \ln |x(\varepsilon t) - x(\varepsilon s)| \tilde{\varphi}_\varepsilon(s) ds - \int_{-1}^{1} r(\varepsilon t, \varepsilon s) \tilde{\varphi}_\varepsilon(s) ds = f(\varepsilon t) + C_\varepsilon,
\]
where
with \( \tilde{\varphi}_\varepsilon(t) = \varepsilon \varphi_\varepsilon(\varepsilon t) \).

Since the compatibility condition (2.2) reads
\[
\int_{-1}^{1} \tilde{\varphi}_\varepsilon(s) ds = -|\Omega|, \tag{3.4}
\]
we may rewrite (3.3) as
\[
\frac{1}{\varepsilon} (\mathcal{L} + \varepsilon \mathcal{L}_1) [\tilde{\varphi}_\varepsilon](t) - \left| \frac{|\Omega| \ln \varepsilon}{\pi} + r(0,0) |\Omega| \right| + f(0) = f(\varepsilon t) + C_\varepsilon, \tag{3.5}
\]
where the operator \( \mathcal{L}_1 \) is given by
\[
\mathcal{L}_1[\varphi] := \frac{1}{\varepsilon} \int_{-1}^{1} \ln \frac{|x(\varepsilon t) - x(\varepsilon s)|}{|t-s|} + \pi r(0,0) - \pi r(\varepsilon t, \varepsilon s) \right) \varphi(s) ds.
\]
Moreover, since
\[
|x(\varepsilon t) - x(\varepsilon s)| = \varepsilon |t-s|(1 + O(\varepsilon)),
\]
one can see that \( \mathcal{L}_1 : X_1 \to Y_1 \) is bounded independently of \( \varepsilon \). Note that \( f \) is \( C^1 \) on \( \overline{\Omega} \) and hence we may rewrite (3.5) as
\[
(\mathcal{L} + \varepsilon \mathcal{L}_1) [\tilde{\varphi}_\varepsilon](t) = |\Omega| \ln \varepsilon - \pi r(0,0) |\Omega| + \pi C_\varepsilon + \pi f(0) + O(\varepsilon). \tag{3.6}
\]
It then follows from (2.20) that
\[
\tilde{\varphi}_\varepsilon(t) = \left[ \left| \frac{|\Omega| \ln \varepsilon}{\pi} - r(0,0) |\Omega| + C_\varepsilon + f(0) \right| \right] \frac{1}{\left( \ln \frac{1}{2} \right) \sqrt{1-t^2}} + O(\varepsilon), \tag{3.7}
\]
where the remainder \( O(\varepsilon) \) is with respect to the norm \( \| \cdot \|_{X_1} \).

Now, plugging (3.7) into (3.4) we arrive at
\[
C_\varepsilon = \left| \frac{|\Omega|}{\pi} \ln \frac{2}{\varepsilon} + r(0,0) |\Omega| - f(0) \right| + O(\varepsilon) \tag{3.8}
\]
and, consequently,
\[
\tilde{\varphi}_\varepsilon(t) = - \left| \frac{|\Omega|}{\pi} \frac{1}{\sqrt{1-t^2}} \right| + O(\varepsilon), \tag{3.9}
\]
where \( O(\varepsilon) \) is with respect to \( \| \cdot \|_{X_1} \). Therefore, we obtain
\[
\varphi_\varepsilon(t) = - \left| \frac{|\Omega|}{\pi} \frac{1}{\sqrt{\varepsilon^2 - t^2}} \right| + O(\varepsilon), \tag{3.10}
\]
where \( O(\varepsilon) \) is with respect to \( \| \cdot \|_{X_\varepsilon} \).

Finally, combining (2.11) and (3.10) yields
\[
u_\varepsilon(x) = g(x) - \left| \frac{|\Omega|}{\pi} \int_{-\varepsilon}^{\varepsilon} \frac{N_{\partial \Omega}(x, x(t))}{\sqrt{\varepsilon^2 - t^2}} dt \right| + C_\varepsilon + O(\varepsilon)
\]
\[
= g(x) - \left| \frac{|\Omega|}{\pi} N_{\partial \Omega}(x, x^*) \right| + C_\varepsilon + O(\varepsilon)
\]
for \( x \in \Omega \) provided that \( \text{dist}(x, \partial \Omega_a) \geq c \) for some constant \( c > 0 \). Thus we have the following theorem.
Theorem 3.1. Suppose that $\partial \Omega_a$ is an arc of center $x^*$ and length $2\varepsilon$. Then the following asymptotic expansion of $u_\varepsilon$ holds
\[
 u_\varepsilon(x) = \frac{|\Omega|}{\pi} \ln \frac{2}{\varepsilon} + \Phi_\Omega(x, x^*) + O(\varepsilon), \tag{3.11}
\]
where
\[
 \Phi_\Omega(x, x^*) = \int_\Omega N(x, z)dz - |\Omega|N_{\partial \Omega}(x, x^*) - \int_\Omega N(x^*, z)dz + |\Omega|R_{\partial \Omega}(x^*, x^*). \tag{3.12}
\]

The remainder $O(\varepsilon)$ is uniform in $x \in \Omega$ satisfying $\text{dist}(x, \partial \Omega_a) \geq c$ for some constant $c > 0$. Moreover, if $x(t), -\varepsilon < t < \varepsilon$, is the arclength parametrization of $\partial \Omega_a$, then
\[
 \frac{\partial u_\varepsilon}{\partial \nu}(x(t)) = -\frac{|\Omega|}{\pi} \frac{1}{\sqrt{\varepsilon^2 - t^2}} + O(\varepsilon),
\]
where $O(\varepsilon)$ is with respect to $\| \cdot \|_{X_\varepsilon}$.

The leading order term in (3.11) was obtained in [14] and the formula (3.11) was obtained in [9] in a formal way (see (2.14) in that paper). We emphasize that the derivation of this paper is mathematically rigorous. Moreover, the method of derivation is quite different from that of [9]; It is based on the layer potential technique. We note that the corrector function $\Phi_\Omega(x, x^*)$ solves the following problem
\[
 \begin{cases}
 \Delta_x \Phi_\Omega(x, x^*) = -1 & \text{in } \Omega, \\
 \frac{\partial \Phi_\Omega}{\partial \nu_x} = -|\Omega|\delta_{x^*} & \text{on } \partial \Omega,
\end{cases} \tag{3.13}
\]
which shows that (3.11) is nothing else than a dipole-type approximation.

If $\Omega$ is the unit disk centered at 0, one can easily see from (2.12) and (2.13) that
\[
 \Phi_\Omega(x, x^*) = \ln |x - x^*| + \frac{1}{4}(1 - |x|^2).
\]
Thus we have the following corollary.

Corollary 3.2. Suppose that $\Omega$ is the unit disk and $\partial \Omega_a$ is an arc of center $x^*$ and length $2\varepsilon$. Then the solution $u_\varepsilon$ is given asymptotically as
\[
 u_\varepsilon(x) = \ln \frac{2}{\varepsilon} + \frac{1}{4}(1 - |x|^2) + \ln |x - x^*| + O(\varepsilon), \tag{3.14}
\]
where the remainder $O(\varepsilon)$ is uniform in $x \in \Omega$ satisfying $\text{dist}(x, \partial \Omega_a) \geq c$ for some constant $c$.

The formula (3.14) was also obtained in [15] (and [9]).
3.2 Two well-separated small targets

Let \( \Omega \) be a unit disk centered at 0. Suppose that \( \partial \Omega_a \) consists of two parts, say \( \partial \Omega_1 \) and \( \partial \Omega_2 \), satisfying \( \partial \Omega_1 \cap \partial \Omega_2 = \emptyset \).

Let \( u \) be the solution to (2.1) with \( F = 0 \). In view of (2.11), \( u \) is now written as

\[
u(x) = g(x) + \int_{\partial \Omega_1} N_{\partial \Omega}(x, z) \frac{\partial u(z)}{\partial \nu} \, d\sigma(z) + \int_{\partial \Omega_2} N_{\partial \Omega}(x, z) \frac{\partial u(z)}{\partial \nu} \, d\sigma(z) + C_\varepsilon. \tag{3.15}\]

Suppose that \( \partial \Omega_i, i = 1, 2 \), are parameterized by

\[
x(t) = (\cos t, \sin t), \quad t \in (s_i - \varepsilon_i, s_i + \varepsilon_i).
\]

Then the (Dirichlet) boundary conditions on \( \partial \Omega_a \) yield the following integral equations

\[
0 = \int_{s_1 - \varepsilon_1}^{s_1 + \varepsilon_1} N_{\partial \Omega}(x(s), x(t)) \phi(t) \, dt + \int_{s_2 - \varepsilon_2}^{s_2 + \varepsilon_2} N_{\partial \Omega}(x(s), x(t)) \phi(t) \, dt + C_\varepsilon \tag{3.16}
\]

for \( s \in (s_i - \varepsilon_i, s_i + \varepsilon_i) \) where \( \phi(t) := \frac{\partial u}{\partial \nu}(x(t)) \).

Recall that \( N_{\partial \Omega}(x, z) = -\frac{1}{\pi} \ln |x - z| \), \( x \in \partial \Omega \), \( z \in \partial \Omega \) for the unit disk. If we make changes of variables \( s \mapsto s_i + \varepsilon_i t \) and \( t \mapsto s_i + \varepsilon_i t \) for \( s, t \in (s_i - \varepsilon_i, s_i + \varepsilon_i) \), the integral equations (3.16) become

\[
\begin{aligned}
&\int_{-1}^{1} \ln |x(s_1 + \varepsilon_1 s) - x(s_1 + \varepsilon_1 t)| \varphi_1(t) \, dt \\
&\quad + \int_{-1}^{1} \ln |x(s_1 + \varepsilon_1 s) - x(s_2 + \varepsilon_2 t)| \varphi_2(t) \, dt = \pi C_\varepsilon, \\
&\int_{-1}^{1} \ln |x(s_2 + \varepsilon_2 s) - x(s_1 + \varepsilon_1 t)| \varphi_1(t) \, dt \\
&\quad + \int_{-1}^{1} \ln |x(s_2 + \varepsilon_2 s) - x(s_2 + \varepsilon_2 t)| \varphi_2(t) \, dt = \pi C_\varepsilon,
\end{aligned} \tag{3.17}
\]

where \( \varphi_i(t) = \varepsilon_i \phi(s_i + \varepsilon_i t) \) for \( t \in (-1, 1) \).

Let

\[
C_i := \frac{1}{\pi} \int_{-1}^{1} \varphi_i(t) \, dt, \quad i = 1, 2.
\tag{3.18}
\]

Because of (2.2), we have

\[
C_1 + C_2 = -1. \tag{3.19}
\]

One can easily check that

\[
\begin{aligned}
&\ln |x(s_i + \varepsilon_i s) - x(s_i + \varepsilon_i t)| = \ln \varepsilon_i |s - t| + O(\varepsilon_i), \\
&\ln |x(s_i + \varepsilon_i s) - x(s_j + \varepsilon_j t)| = \ln |x(s_1) - x(s_2)| + O\left( \frac{\sqrt{\varepsilon_1^2 + \varepsilon_2^2}}{|x(s_1) - x(s_2)|} \right), \quad i \neq j.
\end{aligned} \tag{3.20}
\]

Therefore, the system (3.17) can be written in the form

\[
(A + B) \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \pi \begin{pmatrix} C_\varepsilon - C_1 \ln \varepsilon_1 - C_2 \ln |x(s_1) - x(s_2)| \\ C_\varepsilon - C_1 \ln |x(s_1) - x(s_2)| - C_2 \ln \varepsilon_2 \end{pmatrix}, \tag{3.21}
\]

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where the operator $\mathcal{A} : \mathbf{X}_1 \times \mathbf{X}_1 \mapsto \mathbf{Y}_1 \times \mathbf{Y}_1$ is given by

$$\mathcal{A} := \begin{pmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{L} \end{pmatrix},$$

and $\mathcal{B} : \mathbf{X} \times \mathbf{X} \mapsto \mathbf{Y} \times \mathbf{Y}$ satisfies

$$\mathcal{B} = \begin{pmatrix} O(\varepsilon_1) & O \left( \frac{\sqrt{\varepsilon_1^2 + \varepsilon_2^2}}{|x(s_1) - x(s_2)|} \right) \\ O \left( \frac{\sqrt{\varepsilon_1^2 + \varepsilon_2^2}}{|x(s_1) - x(s_2)|} \right) & O(\varepsilon_2) \end{pmatrix}$$

in the operator norm. Here $\mathcal{L}$ is given by (2.17).

Let us now assume that $|x(s_1) - x(s_2)| \geq c$ for some $c > 0$, in other words, the two targets are well-separated. It then follows from (2.20) that

$$\varphi_1(s) = \frac{C_\varepsilon - C_1 \ln \varepsilon_1 - C_2 \ln |x(s_1) - x(s_2)|}{\ln \frac{1}{2}} \left( \frac{1}{\sqrt{1 - s^2}} + O(\sqrt{\varepsilon_1^2 + \varepsilon_2^2}) \right),$$

$$\varphi_2(s) = \frac{C_\varepsilon - C_1 \ln |x(s_1) - x(s_2)| - C_2 \ln \varepsilon_2}{\ln \frac{1}{2}} \left( \frac{1}{\sqrt{1 - s^2}} + O(\sqrt{\varepsilon_1^2 + \varepsilon_2^2}) \right),$$

where $O(\sqrt{\varepsilon_1^2 + \varepsilon_2^2})$ is with respect to $\| \cdot \|_{X_1}$.

Integrating (3.22) over $(-1, 1)$, we obtain a system of two equations for $C_1, C_2$ and $C_\varepsilon$

$$\begin{cases} C_1 \ln \frac{\varepsilon_1}{2} + C_2 \ln |x(s_1) - x(s_2)| - C_\varepsilon = O(\sqrt{\varepsilon_1^2 + \varepsilon_2^2}), \\
C_1 \ln |x(s_1) - x(s_2)| + C_2 \ln \frac{\varepsilon_2}{2} - C_\varepsilon = O(\sqrt{\varepsilon_1^2 + \varepsilon_2^2}), \end{cases}$$

which together with (3.19) yields

$$\begin{align*}
C_1 &= -\frac{\ln \frac{\varepsilon_2}{2|x(s_1) - x(s_2)|^2}}{\ln \frac{\varepsilon_1^2}{4|x(s_1) - x(s_2)|^2}} + O(\sqrt{\varepsilon_1^2 + \varepsilon_2^2}), \\
C_2 &= -\frac{\ln \frac{\varepsilon_1}{2|x(s_1) - x(s_2)|^2}}{\ln \frac{\varepsilon_1^2}{4|x(s_1) - x(s_2)|^2}} + O(\sqrt{\varepsilon_1^2 + \varepsilon_2^2}), \\
C_\varepsilon &= \frac{(\ln |x(s_1) - x(s_2)|^2 - \ln \frac{\varepsilon_1}{2} \ln \frac{\varepsilon_2}{2})}{\ln \frac{\varepsilon_1^2}{4|x(s_1) - x(s_2)|^2}} + O(\sqrt{\varepsilon_1^2 + \varepsilon_2^2}).
\end{align*}$$

Thus we have

$$\begin{align*}
\varphi_1(s) &= -\frac{\ln \frac{\varepsilon_2}{2|x(s_1) - x(s_2)|^2}}{\ln \frac{\varepsilon_1^2}{4|x(s_1) - x(s_2)|^2}} \frac{1}{\sqrt{1 - s^2}} + O(\sqrt{\varepsilon_1^2 + \varepsilon_2^2}), \\
\varphi_2(s) &= -\frac{\ln \frac{\varepsilon_1}{2|x(s_1) - x(s_2)|^2}}{\ln \frac{\varepsilon_1^2}{4|x(s_1) - x(s_2)|^2}} \frac{1}{\sqrt{1 - s^2}} + O(\sqrt{\varepsilon_1^2 + \varepsilon_2^2}),
\end{align*}$$

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where \( O(\sqrt{\varepsilon_1^2 + \varepsilon_2^2}) \) is with respect to \( \| \cdot \|_1 \). By scaling back, the following expansion of \( \phi(s) \) holds as \( \varepsilon \to 0 \):

\[
\phi(s) = \begin{cases}
- \ln \frac{2|s_1 - x(s_1)|}{4|x(s_1) - x(s_2)|^2} & \frac{1}{\sqrt{\varepsilon_1^2 - (s - s_1)^2}} + O(\sqrt{\varepsilon_1^2 + \varepsilon_2^2}), \quad s \in (s_1 - \varepsilon_1, s_1 + \varepsilon_1), \\
- \ln \frac{2|s_2 - x(s_2)|}{4|x(s_1) - x(s_2)|^2} & \frac{1}{\sqrt{\varepsilon_2^2 - (s - s_2)^2}} + O(\sqrt{\varepsilon_1^2 + \varepsilon_2^2}), \quad s \in (s_2 - \varepsilon_2, s_2 + \varepsilon_2).
\end{cases}
\]

By substituting this formula to (3.15), we obtain the following theorem.

**Theorem 3.3.** Let \( \partial \Omega_i, \ i = 1, 2 \) be parameterized by

\[
x(t) = (\cos t, \sin t), \quad t \in (s_i - \varepsilon_i, s_i + \varepsilon_i).
\]

Assume that there is a constant \( c_0 > 0 \) such that \(|x(s_1) - x(s_2)| \geq c_0\). Then the solution \( u \) to (2.1) is given asymptotically by

\[
u(x) = \left( \frac{\ln |x(s_1) - x(s_2)|^2 - \ln \varepsilon_1^2 \ln \varepsilon_2^2}{\ln \frac{\varepsilon_1^{s_2}}{4|x(s_1) - x(s_2)|^2}} \right) + \frac{1}{4} (1 - |x|^2) + \ln \left| \frac{2|x(s_1) - x(s_2)|}{\varepsilon_1^{s_2}} \right| \ln |x - x(s_1)| + \ln \left| \frac{2|x(s_1) - x(s_2)|}{\varepsilon_1^{s_2}} \right| \ln |x - x(s_2)| + O(\sqrt{\varepsilon_1^2 + \varepsilon_2^2}),
\]

where \( O(\sqrt{\varepsilon_1^2 + \varepsilon_2^2}) \) is uniform in \( x \in \Omega \) satisfying \( \text{dist}(x, \partial \Omega_1 \cup \partial \Omega_2) \geq c \) for some constant \( c > 0 \).

If \( \varepsilon := \varepsilon_1 = \varepsilon_2 \), then (3.24) takes the following simpler form:

\[
u(x) = \frac{1}{2} \ln \frac{2}{\varepsilon} + \frac{1}{4} (1 - |x|^2) + \frac{1}{2} \ln \frac{|x - x(s_1)||x - x(s_2)|}{|x(s_1) - x(s_2)|} + O(\varepsilon).
\]

Formula (3.25) shows that the leading order term of the escape time when there are two targets of the same size is reduced to half of that when there is a single target, which is quite natural. The second-order term (dipole type) shows that the escape time is reduced to half of the sum of two dipoles increased by the relative position of the targets.

### 3.3 Two clustered small targets on the unit circle

We now consider the case when there are two clustered targets. We suppose that

- \( \varepsilon_1 = \varepsilon_2 (=: \varepsilon) \),
- \( s_2 - s_1 = d\varepsilon, \ d > 2 \).

We still assume that \( \Omega \) is a unit disk centered at 0.

Let

\[
\mathcal{L}_1[\phi](s) := \int_{-1}^{1} \ln |d + s - t| \phi(t) \, dt,
\]

\[
\mathcal{L}_2[\phi](s) := \int_{-1}^{1} \ln |d - s + t| \phi(t) \, dt.
\]
The system of equations (3.17) now takes the form

\[ (A + B) \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \pi \begin{pmatrix} C_\varepsilon - C_1 \ln \varepsilon - C_2 \ln \varepsilon \\ C_\varepsilon - C_1 \ln \varepsilon - C_2 \ln \varepsilon \end{pmatrix}, \]

(3.26)

where \( A, B : X \times X \mapsto Y \times Y \) are given by

\[ A := \begin{pmatrix} \mathcal{L} & \mathcal{L}_2 \\ \mathcal{L}_1 & \mathcal{L} \end{pmatrix} \]

and \( B = O(\varepsilon) \) in the operator norm.

The following invertibility result follows from [10].

**Lemma 3.4.** The operator \( A : X \times X \mapsto Y \times Y \) is invertible.

It follows from Lemma 3.4 and (3.19) that

\[ \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \pi (C_\varepsilon + \ln \varepsilon) \left( A^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + O(\varepsilon) \right). \]

(3.27)

Define \( \alpha_j = \alpha_j(d), j = 1, 2 \), by

\[ \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} := \int_{-1}^1 A^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} (t) dt. \]

(3.28)

Then, we have \( \alpha_1 = \alpha_2 (= \alpha) \). In fact, if \( (\varphi_1, \varphi_2)^T \), where \( T \) denotes the transpose, is the solution of

\[ A \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \]

(3.29)

so is \( (\varphi_2(-t), \varphi_1(-t))^T \), and hence we have

\[ \varphi_1(t) = \varphi_2(-t). \]

(3.30)

Integrating (3.27) over \( \partial \Omega_a \) yields

\[ C_\varepsilon = - \ln \varepsilon - \frac{1}{2\alpha(d)} + O(\varepsilon), \]

which combined with (3.15) and (3.27), gives the following result.

**Theorem 3.5.** Let \( \partial \Omega_i, i = 1, 2 \) be parameterized by

\[ x(t) = (\cos t, \sin t), \quad t \in (s_i - \varepsilon, s_i + \varepsilon) \]

with \( |s_1 - s_2| = d \varepsilon \). Then the solution \( u \) to (2.1) is given asymptotically by

\[ u(x) = \ln \frac{1}{\varepsilon} - \frac{1}{2\alpha(d)} + \frac{1}{4} (1 - |x|^2) + \frac{1}{2} \ln |x - x(s_1)| + \frac{1}{2} \ln |x - x(s_2)| + O(\varepsilon), \]

(3.31)

where \( O(\varepsilon) \) is uniform in \( x \in \Omega \) satisfying \( \text{dist}(x, \partial \Omega_1 \cup \partial \Omega_2) \geq c \) for some \( c > 0 \).
In view of (3.30), it follows from (3.29) that
\[ \mathcal{L}[\varphi_1](s) + \mathcal{L}_2[\varphi_1(-\cdot)](s) = 1, \]
or
\[ \int_{-1}^{1} [\ln|s-t| - \ln|d-s-t|] \varphi_1(t) \, dt = 1. \]

By taking \( s = 1 \) one can see that \( \alpha(d) \to \infty \) as \( d \to 2^+ \), and hence the solution converges to the one with an absorbing arc of length \( 4\varepsilon \) (modulo \( O(\varepsilon) \)).

It is interesting to compare the formula (3.31) with (3.25). The nonlinear interaction between the two small targets is described by the term \( \ln \frac{1}{\varepsilon} - \frac{1}{2\alpha(d)} \) in (3.31) while it is described by the term \( -\frac{1}{2} \ln |x(s_1) - x(s_2)| \) in (3.25).

### 3.4 Multiple small targets on the unit circle

We now extend the above analysis to larger clusters. We consider clusters \( \partial \Omega_i, i = 1, \ldots, n \), parameterized by
\[ x(t) = (\cos t, \sin t), \quad t \in (s_i - \varepsilon, s_i + \varepsilon). \]

We assume that each \( \partial \Omega_i \) is separated from its neighbors by a distance comparable to \( \varepsilon \).

Let \( d_{ij} := \frac{s_i - s_j}{\varepsilon} \). Define the operators \( \mathcal{L}_{ij} \) and \( \mathcal{A} \) by

\[ \mathcal{L}_{ij}[\phi](s) := \int_{-1}^{1} \ln |d_{ij} + s - t| \phi(t) \, dt \]
and

\[ \mathcal{A} := (\mathcal{L}_{ij})_{n \times n}. \]

Let
\[ \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \int_{-1}^{1} \mathcal{A}^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \, dt. \]

Following the same lines as in the proof of Theorem 3.5, we can easily prove the following result.

**Theorem 3.6.** Let \( \partial \Omega_i, i = 1, \ldots, n \) be parameterized by
\[ x(t) = (\cos t, \sin t), \quad t \in (s_i - \varepsilon, s_i + \varepsilon) \]
with \( s_i - s_j = d_{ij}\varepsilon \). Then the solution to (2.1) is given asymptotically by
\[ u(x) = -\ln \varepsilon - \frac{1}{\alpha_1 + \cdots + \alpha_n} + \frac{1}{n} \sum_{i=1}^{n} \Phi_{\Omega}(x, x(s_i)) + O(\varepsilon), \]
where the remainder \( O(\varepsilon) \) is uniform in \( x \in \Omega \) with \( \text{dist}(x, \cup_{i=1}^{n} \partial \Omega_i) \geq c \) for some constant \( c > 0 \).
Note that
\[
\frac{1}{n} \sum_{i=1}^{n} \Phi_{\Omega}(x, x(s_i)) \to v_\Omega(x) := \frac{1}{|\partial \Omega|} \int_{\partial \Omega} \Phi_{\Omega}(x, z) \, d\sigma(z) \quad \text{as } n = O(1) \to +\infty,
\]
and the corrector \(v_\Omega\) satisfies the equation
\[
\Delta v_\Omega = -1 \quad \text{in } \Omega,
\]
with, as already known in the homogenization theory [3], the effective Neumann boundary condition
\[
\frac{\partial v_\Omega}{\partial \nu} = -\frac{|\Omega|}{|\partial \Omega|} \quad \text{on } \partial \Omega.
\]

4 Asymptotics of the eigenvalues

Let \(0 = \lambda_0^{(1)} < \lambda_0^{(2)} \leq \lambda_0^{(3)} \leq \ldots\) be the eigenvalues of \(-\Delta\) on \(\Omega\) with Neumann conditions on \(\partial \Omega\). Let \(u_0^{(j)}\) denote the normalized eigenfunction associated with \(\lambda_0^{(j)}\), that is, it satisfies \(\int_{\Omega} |u_0^{(j)}|^2 = 1\). Let \(\omega \notin \{\sqrt{\lambda_0^{(j)}}\}_{j \geq 1}\). Introduce \(N^\omega(x, z)\) as the Neumann function for \(\Delta + \omega^2\) in \(\Omega\) corresponding to a Dirac mass at \(z\). That is, \(N^\omega\) is the unique solution to
\[
\begin{cases}
\langle \Delta_x + \omega^2 \rangle N^\omega(x, z) = -\delta_z & \text{in } \Omega, \\
\frac{\partial N^\omega}{\partial \nu} = 0 & \text{on } \partial \Omega.
\end{cases}
\]
(4.1)

We recall two useful facts on the Neumann function [2, (2.28) & Lemma 5.1].

Lemma 4.1. The following spectral decomposition holds pointwise:
\[
N^\omega(x, z) = \sum_{j=1}^{+\infty} \frac{u_0^{(j)}(x)u_0^{(j)}(z)}{\lambda_0^{(j)} - \omega^2}, \quad x \neq z \in \Omega.
\]
(4.2)

Moreover, if \(\lambda_0^{(j_0)}\) is simple and \(V\) is a complex neighborhood of \(\sqrt{\lambda_0^{(j_0)}}\) which contains no \(\sqrt{\lambda_0^{(j)}}\) other than \(\sqrt{\lambda_0^{(j_0)}}\), then for \(\omega \in V\) the Neumann function \(N^\omega\) on the boundary \(\partial \Omega\), which we denote by \(N^\omega_{\partial \Omega}\), has the form
\[
N^\omega_{\partial \Omega}(x, z) = -\frac{1}{\pi} \ln |x - z| + \frac{u_0^{(j_0)}(x)u_0^{(j_0)}(z)}{\lambda_0^{(j_0)} - \omega^2} + R^\omega_{\partial \Omega}(x, z) \quad \text{for } x \in \Omega, z \in \partial \Omega,
\]
(4.3)
where \(R^\omega_{\partial \Omega}(\cdot, z)\) belongs to \(H^{3/2}(\Omega)\) for any \(z \in \partial \Omega\) and \(\omega \mapsto R^\omega_{\partial \Omega}\) is holomorphic in a complex neighborhood of \(\lambda_0^{(j_0)}\).

Suppose that \(\lambda_0^{(j_0)}\) is simple. Then there exists a simple eigenvalue \(\lambda_\varepsilon^{(j_0)}\) near \(\lambda_0^{(j_0)}\) associated to the normalized eigenfunction \(u_\varepsilon^{(j_0)}\) \((\int_{\Omega} |u_\varepsilon^{(j_0)}|^2 = 1)\) satisfying the eigenvalue problem
\[
\begin{cases}
-\Delta u_\varepsilon^{(j_0)} = \lambda_\varepsilon^{(j_0)} u_\varepsilon^{(j_0)} & \text{in } \Omega, \\
u_\varepsilon^{(j_0)} = 0 & \text{on } \partial \Omega_a, \\
\frac{\partial u_\varepsilon^{(j_0)}}{\partial \nu} = 0 & \text{on } \partial \varepsilon.
\end{cases}
\]
(4.4)
Our aim in this section is to derive a high-order asymptotic expansion of $\lambda_\varepsilon^{(j_0)} - \lambda_0^{(j_0)}$ as $\varepsilon \to 0$.

Set $\varphi_\varepsilon = \partial u_\varepsilon^{(j_0)}/\partial \nu$ on $\partial \Omega_\varepsilon$. By Green’s formula, one can easily see that

$$u_\varepsilon(x) = (\lambda_\varepsilon^{(j_0)} - \omega^2) \int_\Omega u_\varepsilon^{(j_0)}(z) N^\omega(x, z) \, dz + \int_{\partial \Omega_\varepsilon} \varphi_\varepsilon(z) N^\omega(x, z) \, d\sigma(z),$$

which shows that

$$\int_{\partial \Omega_\varepsilon} \varphi_\varepsilon(z) N^\lambda_\varepsilon^{(j_0)}(x, z) \, d\sigma(z) = 0 \text{ on } \partial \Omega_\varepsilon.$$

Therefore, the eigenvalue problem (4.4) reduces to the study of the characteristic values of the operator-valued function $\lambda \mapsto A_\varepsilon(\lambda)$ given by

$$A_\varepsilon(\lambda)[\varphi](x(t)) = \int_{-\varepsilon}^{\varepsilon} \varphi(s) N^{\sqrt{\lambda}}(x(t), x(s)) \, ds,$$

where $x(t)$ is defined by (3.1).

Let $V$ be a neighborhood of $\lambda_0^{(j_0)}$ in the complex plane such that $\lambda_\varepsilon^{(j_0)}$ is the only eigenvalue of $-\Delta$ in $\Omega$ with Neumann boundary condition. The following pole-pencil decomposition of $A_\varepsilon(\lambda) : X_\varepsilon \to Y_\varepsilon$ holds for any $\lambda \in V$:

$$A_\varepsilon(\lambda) = -\frac{1}{\pi} L_\varepsilon + \frac{K_\varepsilon}{\lambda^{(j_0)}_0 - \lambda} + R_\varepsilon(\lambda),$$

where

$$L_\varepsilon[\varphi](x(t)) = \int_{-\varepsilon}^{\varepsilon} \ln |s - t| \varphi(s) \, ds,$$

$K_\varepsilon$ is the one-dimensional operator given by

$$K_\varepsilon[\varphi](x(t)) = \left( \int_{-\varepsilon}^{\varepsilon} u_0^{(j_0)}(x(s)) \varphi(x(s)) \, ds \right) u_0^{(j_0)}(x(t)),$$

and

$$R_\varepsilon(\lambda)[\varphi](x(t)) = \int_{-\varepsilon}^{\varepsilon} R^{\sqrt{\lambda}}(x(t), x(s)) \varphi(s) \, ds.$$

An application of the generalized Rouché theorem [2] shows that the operator-valued function $A_\varepsilon(\lambda)$ has exactly one characteristic value in $V$. Theorem 5.7 in [2] gives its asymptotic expansion for $\varepsilon$ small enough. The following theorem holds.

**Theorem 4.2.** We have

$$\lambda_\varepsilon^{(j_0)} - \lambda_0^{(j_0)} \approx -\pi \int_{-\varepsilon}^{\varepsilon} L_\varepsilon^{-1}[u_0^{(j_0)}](x(t)) u_0^{(j_0)}(x(t)) \, dt + \pi^2 \int_{-\varepsilon}^{\varepsilon} L_\varepsilon^{-1} R_\varepsilon(\lambda_\varepsilon^{(j_0)}) L_\varepsilon^{-1}[u_0^{(j_0)}](x(t)) u_0^{(j_0)}(x(t)) \, dt.$$  \hspace{1cm} (4.5)

In particular, the leading-order term in the expansion of $\lambda_\varepsilon^{(j_0)} - \lambda_0^{(j_0)}$ is given by

$$\lambda_\varepsilon^{(j_0)} - \lambda_0^{(j_0)} \approx -\frac{\pi}{\ln \varepsilon} |u_0^{(j_0)}(x^*)|^2,$$  \hspace{1cm} (4.6)

where $x^*$ is the center of $\partial \Omega_\varepsilon$.

Note that if $\lambda_0^{(j_0)}$ is a multiple eigenvalue of $-\Delta$ in $\Omega$ with Neumann boundary condition, then in exactly the same way as Theorem 3.24 in [2], we can address the splitting problem in the evaluation of eigenvalues of (4.4).
5 Narrow escape problem in a field of force

We now deal with the case when the force field $F = \nabla \phi$ is not zero. As before, we consider the problem of estimating the escape time and that of deriving asymptotics of the perturbed eigenvalue.

5.1 Asymptotics of the solution

Let the Neumann function $N^F(x, z)$ be defined by

$$
\begin{align*}
\Delta_x N^F(x, z) - \nabla_z \cdot (F(z)N^F(x, z)) &= -\delta_x, \quad x, z \in \Omega, \\
\frac{\partial N^F}{\partial \nu_z} - F \cdot \nu_z N^F \bigg|_{z \in \partial \Omega} &= -\frac{1}{|\partial \Omega|}.
\end{align*}
$$

The (boundary) Neumann function, denoted by $N^F_{\partial \Omega}(x, z)$, has the form

$$
N^F_{\partial \Omega}(x, z) = -\frac{1}{\pi} \ln |x - z| + R^F_{\partial \Omega}(x, z), \quad x \in \Omega, \ z \in \partial \Omega,
$$

where $R^F_{\partial \Omega}(\cdot, z)$ belongs to $H^{3/2}(\Omega)$ for any $z \in \partial \Omega$.

The solution $u_\varepsilon$ to (2.1) admits the following integral representation:

$$
u(\nabla \phi (x) u_\varepsilon) = -\varepsilon \phi \quad \text{in } \Omega,
$$

and hence $\varphi_\varepsilon$ satisfies the compatibility condition

$$
\int_{\partial \Omega} e^{\phi(z)} \varphi_\varepsilon(z) d\sigma(z) = -\int_{\Omega} e^{\phi(z)} dz.
$$

In exactly the same way as Theorem 3.1, we can prove that

$$
\varphi_\varepsilon(x(t)) = \left[ e^{-\phi(x^*)} \int_{\Omega} e^{\phi(z)} dz \left( \frac{\ln \varepsilon}{\pi} - R^F_{\partial \Omega}(x^*, x^*) \right) + C_\varepsilon + \int_{\Omega} N^F(x^*, z) dz \right] \left[ \frac{1}{(\ln \frac{1}{2}) \sqrt{1 - t^2}} + O(\varepsilon),
$$

where $x(t)$ is defined by (3.1) and $x^*$ is the center of $\partial \Omega_a$. Consequently, we have

$$
C_\varepsilon = \left( \int_{\Omega} e^{\phi(z) - \phi(x^*)} dz \right) \left( \frac{1}{\pi} \frac{2}{\varepsilon} + R^F_{\partial \Omega}(x^*, x^*) \right) - \int_{\Omega} N^F(x^*, z) dz + O(\varepsilon),
$$

which yields the following result.
Theorem 5.1. The following asymptotic formula
\[ u_{\varepsilon}(x) = \frac{1}{\pi} \left( \int_{\Omega} e^{\phi(z) - \phi(x^*)} \, dz \ln \frac{2}{\varepsilon} + \Phi^F_{\Omega}(x, x^*) + O(\varepsilon), \right) \]
holds uniformly for \( x \in \Omega \) satisfying \( \text{dist}(x, \partial \Omega_a) \geq c \) for some constant \( c > 0 \). Here \( \Phi^F_{\Omega} \) is given by
\[ \Phi^F_{\Omega}(x, x^*) = \int_{\Omega} N^F(x, z) \, dz - \int_{\Omega} N^F(x^*, z) \, dz + \left( \int_{\Omega} e^{\phi(z) - \phi(x^*)} \, dz \right) \left( R^F_{\partial \Omega}(x^*, x^*) - N^F_{\partial \Omega}(x, x^*) \right). \] (5.3)

5.2 Asymptotics of the eigenvalues

Consider the eigenvalue problem
\[ \begin{cases} \Delta u_{\varepsilon, F} + F \cdot \nabla u_{\varepsilon, F} = -\lambda_{\varepsilon, F} u_{\varepsilon, F}, & \text{in } \Omega, \\ u_{\varepsilon, F} = 0 & \text{on } \partial \Omega_a, \\ \frac{\partial u_{\varepsilon, F}}{\partial \nu} = 0 & \text{on } \partial \Omega_r. \end{cases} \] (5.4)

Since \( F = \nabla \phi \) for a smooth potential function \( \phi \), we get
\[ e^{-\phi} \nabla \cdot (e^{\phi} \nabla u) = \Delta u + F \cdot \nabla u, \]
and hence we can realize the Neumann Laplacian with a drift \( F \) in \( \Omega \) as a non-negative, self-adjoint operator on \( L^2(\Omega, e^{-\phi} \, dx) \) via the closure of the Friedrichs extension of the nonnegative quadratic form
\[ (u, v) \mapsto \int_{\Omega} \nabla u \cdot \nabla ve^{\phi} \, dx. \]

Let \( 0 = \lambda^{(1)}_{0, F} < \lambda^{(2)}_{0, F} \leq \lambda^{(3)}_{0, F} \leq \ldots \) be the eigenvalues of \(-\Delta - F \cdot \nabla\) on \( \Omega \) with Neumann conditions on \( \partial \Omega \) and let \( u^{(1)}_{0, F}, u^{(2)}_{0, F}, u^{(3)}_{0, F}, \ldots \) be the corresponding eigenfunctions with
\[ \int_{\Omega} |u^{(j)}_{0, F}|^2 e^{\phi} \, dx = 1. \]
Define for \( \omega \notin \{ \sqrt{\lambda^{(j)}_0} \}_{j \geq 1} \) the Neumann function by
\[ \begin{cases} \Delta z N^F_{\omega}(x, z) - \nabla_z \cdot (F(z) N^F_{\omega}(x, z)) + \omega^2 N^F_{\omega}(x, z) = -\delta_z, & x, z \in \Omega, \\ \frac{\partial N^F_{\omega}}{\partial z} - F \cdot \nu_z N^F_{\omega} \bigg|_{z \in \partial \Omega} = 0. \end{cases} \] (5.5)

Set
\[ \varphi_{\varepsilon, F} := \frac{\partial u_{\varepsilon, F}}{\partial \nu} \text{ on } \partial \Omega_a. \]

From
\[ \int_{\partial \Omega_a} \varphi_{\varepsilon, F}(z) N^F_{\sqrt{\lambda_{\varepsilon, F}}(x, z)} \, d\sigma(z) = 0 \text{ on } \partial \Omega_a, \]
it follows that the eigenvalue problem (5.4) reduces to the study of the characteristic values of the operator-valued function $\lambda \mapsto \mathcal{A}_{\varepsilon,F}(\lambda)$ given by

$$\mathcal{A}_{\varepsilon,F}(\lambda)[\varphi](x(t)) = \int_{-\varepsilon}^{\varepsilon} \varphi(s) F_N F^{1/2}(x(t), x(s)) \, ds.$$ 

Following the same arguments as for the case without drift, we can prove the following result.

**Theorem 5.2.** If $\lambda^{(j_0)}_{0,F}$ is simple then there exists a simple eigenvalue $\lambda^{(j_0)}_{\varepsilon,F}$ of (5.4) converging to $\lambda^{(j_0)}_{0,F}$ as $\varepsilon$ goes to zero and the following formula holds:

$$\lambda^{(j_0)}_{\varepsilon,F} - \lambda^{(j_0)}_{0,F} \approx -\frac{\pi}{\ln \varepsilon} |u^{(j_0)}_{0,F}(x^*)|^2 e^{\phi(x^*)}. \quad (5.6)$$

6 Conclusion

In this paper, we have provided mathematically rigorous derivations of the first- and second-order terms in the asymptotic expansion of the solutions of diffusion equations in the presence of a single or many small targets. We have shown the nonlinear interaction between many targets. We have also studied the problem of eigenvalue changes due to the small targets using the approach developed in [2]. The generalization of the results of this work to nonsmooth domains containing corners and cusps will be the subject of a forthcoming paper.

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