A systematic route to subcritical dynamo branches

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We demonstrate that the nonlinear optimisation of a finite-amplitude disturbance over a freely evolving and possibly even turbulent flow, can successfully identify subcritical dynamo branches as well as the structure and amplitude of their critical perturbations. As this approach does not require prior knowledge of the magnetic field amplification mechanisms, it opens a new avenue for systematically probing subcritical dynamo flows.

Long-lived astrophysical magnetic fields display a remarkable diversity of spatial scales, structures and intensities - with detected fields of order \( \sim 1 \mu G \) in galaxies [1], to \( \sim 10^{14} G \) in magnetars [2]. A century ago, Joseph Larmor argued that intense magnetic fields can be born out of a dynamo instability [3], whereby favorable flow motions amplify a magnetic seed field in the electrically conducting fluid layers of celestial bodies. Ever since, the origin of astrophysical magnetic fields has continued to raise many fundamental questions.

Indeed, exhibiting a flow capable of amplifying and maintaining a magnetic field by dynamo action has proved a challenging task, both from a theoretical and experimental point of view [4]: firstly, fluid motions need to be vigorous enough, and the fluid resistivity sufficiently low, for induction to overcome the destructive effect of ohmic dissipation. As a result, igniting a dynamo instability out of a weak magnetic seed, often translates into intractable parameter regimes for global numerical simulations or unsustainable energy costs for laboratory devices, so that experimental dynamos are very scarce, whether in liquid metals [5–7] or plasma flows [8, 9]. Secondly, flows too simple - in the sense that they present too many symmetries, such as (typically) Keplerian flows - are linearly stable to dynamo action [10].

In some situations however, a dynamo can be triggered by finite-amplitude disturbances even in a highly symmetric or (comparatively) poorly conducting flow. This springs from the nonlinear nature of the magnetohydrodynamics (MHD) equations, and can occur whenever the feedback of the magnetic field on the flow, sustains its own amplification by subcritical dynamo instability. Natural systems where subcritical dynamos are assumed to operate abound, with important examples including the Earth’s core [11], but also accretion disks and radiative stars [12, 13], where magnetogenesis is poorly understood despite its suspected role in explaining the observed anomalous transport of angular momentum [14–15].

Due to the lack of a systematic nonlinear stability method however, the identification and modeling of subcritical dynamos so far involves a substantial amount of luck. Successful attempts have relied on imposing (and subsequently removing) a specifically tailored electromotive force or magnetic field [12–16]. This however, requires some prior knowledge of the field’s structure and the intensity required to sustain a dynamo in a given flow. Another way is identifying a dynamo branch as it linearly bifurcates from the base state, and then following the branch backwards in parameter space, below its linear instability threshold - a method however doomed to fail when the threshold parameter regime is intractable, or when the base state is always linearly stable to dynamo instability.

In this Letter, we show how subcritical dynamo branches can be systematically explored in a given MHD base flow. Furthermore, we aim at determining the minimal dynamo seed - that is, the spatial structure of the smallest-amplitude (and often low dimensional) magnetic disturbance, capable of non-linearly triggering a self-sustaining dynamo in a given system. Our approach builds on the mathematical tools of optimal control and nonmodal stability analysis [17], which have recently been used in the context of subcritical transition to hydrodynamic turbulence in shear flows [18–20]. The robustness of the proposed method is demonstrated by considering two contrasting reference flows - namely, a local model of laminar quasi-Keplerian Couette flow, and a turbulent Taylor-Green flow.

To that end, we consider the MHD system:

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nu \Delta \mathbf{u} + \nabla P - (\nabla \times \mathbf{b}) \times \mathbf{b} = 0, \tag{1}
\]

\[
\frac{\partial \mathbf{b}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{b}) - \eta \Delta \mathbf{b} = 0, \tag{2}
\]

\[
\nabla \cdot \mathbf{u} = 0, \tag{3}
\]

\[
\nabla \cdot \mathbf{b} = 0, \tag{4}
\]

where \( \mathbf{u}(x,t), \mathbf{b}(x,t) \) and \( P(x,t) \) are respectively the velocity, magnetic and pressure fields; \( \nu \) is the kinematic viscosity and \( \eta \) the magnetic diffusivity. Here the magnetic field is rescaled by \( 1/\sqrt{\mu_0 \rho} \) as an Alfvén velocity, where \( \rho \) is the (constant) fluid density and \( \mu_0 \) the void magnetic permeability.

At long times, any initial condition \( \mathbf{b}_0 = \mathbf{b}(x,0) \) capable of non-linearly exciting a dynamo instability, will yield strong amplification of the magnetic energy. We thus consider the following question: given a reference flow \( \mathbf{u}_0 = \mathbf{u}(x,0) \), and an initial magnetic energy density budget \( M_0 = \langle |\mathbf{b}_0|^2 \rangle \) (where \( \langle f \rangle = \frac{1}{V} \int_V f \, dV \) denotes spatial average), what initial condition \( \mathbf{b}_0 \) maximises the objective function \( J(\mathbf{b}_0) = \int_0^T \langle |\mathbf{b}|^2 \rangle \, dt \) by some target time \( T \)? Maximisation of the magnetic energy has been previously used to optimize a frozen velocity field in the (linear) kinematic dynamo problem [21–24]. Here we address the optimisation question by extremising the
following functional:

\[
\mathcal{L} = \int_0^T \left( \langle |\mathbf{b}|^2 \rangle dt - \int_0^T (\mathbf{b} \cdot \left[ \text{IND} \right]) dt - \int_0^T \langle \nabla \cdot (\nabla \cdot \mathbf{b}) \rangle dt \right) \delta \mathbf{b} \bigg|_{t=0} - \int_0^T (\Delta \mathbf{b}) \cdot \mathbf{u} \, dt - \int_0^T \langle P \nabla \cdot \mathbf{u} \rangle \, dt,
\]

where \( \mathbf{u}(x,t), \mathbf{b}(x,t), P(x,t), \Pi(x,t) \) are Lagrange multipliers used to enforce the nonlinear constraints \([1-4]\) in the whole domain for \( t \in [0,T] \), and "NS", "IND" stand for the left-hand side (LHS) of \([1] \) and \([2] \), respectively. Maximisation with respect to \( \mathbf{b}_0 \) requires estimation of the variational derivative \( \delta \mathcal{L} / \delta \mathbf{b}_0 \), which has to vanish when an optimum is reached. This estimate is obtained at affordable computational cost using a direct-adjoint looping method. First, canceling the variational derivatives of \( \mathcal{L} \) provides (i) compatibility conditions that relate the physical ("direct") variables \( \mathbf{u}, \mathbf{b} \) to the Lagrange multipliers (or "adjoint variables") \( \tilde{\mathbf{u}}, \tilde{\mathbf{b}} \) at time \( T \); (ii) backward evolution equations and boundary conditions for the adjoint variables, and (iii) an optimality condition relating the desired information \( \delta \mathcal{L} / \delta \mathbf{b}_0 \) to the adjoint fields at \( t = 0 \), which in the present case reduces to

\[
\delta \mathcal{L} / \delta \mathbf{b}_0 = \tilde{\mathbf{b}}(x,0) + \nabla \Pi. \tag{6}
\]

Then for a given reference flow \( \mathbf{u}_0 \), the looping procedure uses (i)-(iii) as follows: starting from a (random) first guess \( \mathbf{b}_0 \) with energy \( M_0 \), we evolve \( \mathbf{u}, \mathbf{b} \) from \( t = 0 \rightarrow T \) according to \([1-4]\). We then use condition (i), here that \( \tilde{\mathbf{u}}(T) = \tilde{\mathbf{b}}(T) = 0 \), to evolve \( \tilde{\mathbf{u}}, \tilde{\mathbf{b}} \) back from \( t = T \rightarrow 0 \) according to the adjoint equations (ii):

\[
\frac{\partial \tilde{\mathbf{u}}}{\partial t} = N(\tilde{\mathbf{u}}, \mathbf{u}) + \mathbf{b} \times (\nabla \times \tilde{\mathbf{b}}) + \nabla \Delta \tilde{\mathbf{u}} + \nabla P,
\]

\[
\frac{\partial \tilde{\mathbf{b}}}{\partial t} = -N(\tilde{\mathbf{u}}, \mathbf{b}) - \mathbf{u} \times (\nabla \times \tilde{\mathbf{b}}) + \eta \Delta \tilde{\mathbf{b}} + \nabla \Pi + 2 \mathbf{b},
\]

where \( N(\tilde{\mathbf{u}}, \mathbf{u}) \equiv (\nabla \times \mathbf{u}) \times \tilde{\mathbf{u}} + \nabla \times (\tilde{\mathbf{u}} \times \mathbf{u}) \), and \( \tilde{\mathbf{u}} \) and \( \tilde{\mathbf{b}} \) are subject to divergence-free conditions \([25]\) and the same boundary conditions as \( \mathbf{u}, \mathbf{b} \), respectively. This yields the desired gradient information \([6]\) at \( t = 0 \), which we use to update \( \mathbf{b}_0 \) by means of a classical descent method. The latter is combined with the rotation method of \([26, 27]\) to enforce the condition that \( \mathbf{b}_0 \) be of energy \( M_0 \). The whole process is then repeated until the system converges to an optimum. The resulting optimal seed \( \mathbf{b}_0 \) exhibits whatever spatial structure yields largest amplification of the magnetic energy by target time \( T \). In order to rule out transient states, or even self-killing dynamo processes \([28, 29]\), the optimal seed is then used as an initial condition for integrating the governing equations \([1-4]\) (direct numerical simulation (DNS)) over longer timescales \( t \gg T \), which finally confirms whether a self-sustained dynamo develops or not.

We first consider a laminar incompressible flow, sheared between two infinite parallel rigid plates, in a rotating frame of reference (rotating plane Couette flow). This highly symmetric flow is linearly stable to dynamo action by virtue of Zel dovich’s antidynamo theorem \([31]\). The existence of subcritical dynamo solutions in quasi-Keplerian Couette flow was numerically demonstrated in \([12]\): using Newton iteration they construct steady dynamo states, which we find to be stable however, only with certain symmetries enforced. Their approach relies on prescribing an artificial electromotive force, carefully chosen so as to replenish a large-scale field prone to (linear) destabilization via the magnetorotational instability (MRI). Then by progressively removing it until nonlinear interactions between MRI modes can take over in replenishing the large-scale field, they close the dynamo loop. Because the obtained solutions are time-independent, their approach is inherently restricted to low Reynolds numbers.

Here we choose the same geometry and boundary conditions as described in \([12]\). The domain aspect ratios are
The control parameters are $Pm$ and the Grashof number $Gr = f l^3/v^2 = 1875$, where $l = L/(2\pi)$ is the unit length.
FIG. 3. From Left to Right: Snapshots of (A) the minimal seed for the Taylor-Green dynamo at \( \{Gr = 1875; Pm = 0.44\} \) (magnetic lines in blue, current lines in red), (B) magnetic field during the algebraic growth (rescaled and time-averaged over the growth period); (C) growing magnetic field during the exponential growth (rescaled and time-averaged), (D) saturated state (recovers \[36\]). The volume rendering shows the density of magnetic energy. The magnetic energy at saturation displays cigar-shaped structures aligned in two parallel planes.

and \( f \) the intensity of the Taylor-Green forcing. (A diagnostic Reynolds number built on the root-mean square velocity is computed as in \[38\], with \( Re_{rms} = 194.7 \).) For each value of \( Pm \) considered here, we use the target time \( T = 10\sqrt{l/f} \), and apply the procedure described above for some (large) energy budget \( M_0 \), accompanied by DNS for \( \sim 100T \). We typically use a numerical resolution of \( 64^3 \) gridpoints.

The black squares in the bifurcation diagram of Fig. 2 (Left) denote the dynamo branch previously identified by \[36\], while \( Pm_c \sim 0.54 \) denotes the linear dynamo instability threshold, above which infinitesimal magnetic fields are amplified up to their saturation energy \( M \). Restarting DNS from the saturated state above \( Pm_c \), while gradually decreasing \( Pm \) (dashed arrows), allows the dynamo branch to be tracked below its linear onset. The full triangles denote new results: they indicate the smallest energy \( M_0 \), of the optimal \( b_0 \), found to trigger a self-excited dynamo. Time-series of magnetic and kinetic energies are shown in Fig. 2 (Right) for \( Pm = 0.44 \). Here again, strong algebraic growth occurs first, followed by a short plateau. Exponential energy growth finally kicks in, as the optimal seed spontaneously evolves toward the saturated state corresponding to the (previously known) subcritical dynamo branch, driving larger, but slower, fluctuations in the kinetic energy. Note that the magnetic energy undergoes considerable amplification (\( \sim 10^4 \) times) during this process. On the other hand, the empty triangles in Fig. 2 (a) correspond to failed dynamos, suggesting that the frontier between the two (dynamo and non-dynamo) basins of attraction lies somewhere between the full and empty triangles lines.

We re-emphasize that no assumptions were ever made about the structure or amplitude of the final state when searching for \( b_0 \); furthermore, the structure of the minimal seed field (illustrated in Fig. 3 (A)) bears no topological resemblance with the saturated state (Fig. 3 (D)). This implies that identifying the former by means of DNS and continuation from the latter seems a hopeless endeavor. Instead, the minimal seed structure is found to be both very simple and localized in space, as well as being robust to changes in the initialization of the optimisation procedure. Although the nonconvexity of the optimisation problem considered, makes it impossible to guarantee that the optimal seed identified corresponds to a global extremum, repeated optimisations have reassuringly identified such magnetic loops. Remarkably, these seeds are located near one of the flow’s stagnation points: indeed in such regions, the intense stretching of magnetic field lines ensures efficient growth of the magnetic energy \[10\], while localization ensures optimal expenditure of the energy budget \( M_0 \) \[20\].

Finally, let us note that while the minimal energy \( M_0 \) required to trigger the subcritical dynamo seems to decrease slightly as \( T \) increases, or that it changes slightly when maximizing \( \langle |b(x,T)|^2 \rangle \) in place of its time-integrated quantity, the seed structure remains consistent in both cases. Due to the deterioration of the gradient estimates with larger optimisation times, or alternative objective functions in highly fluctuating regimes, these issues were not further investigated in the present study. Since we anticipate the minimal dynamo seed to be independent of the target time or the objective function if the former is sufficiently large, future improvement of the numerical methods used to estimate the gradient will be interesting for the purpose of accurate nonlinear stability analysis.

In this Letter we have shown how nonlinear optimisation, previously used to identify minimal disturbances in shear flows \[18–20\], is a numerically viable and flexible approach to systematically probe subcritical dynamo action in electrically conducting flows. The significance of this approach lies in its numerical feasibility, given the existence of highly parallel open source codes \[32\]. It also lies in its here demonstrated ability to identify at the same time stable dynamo branches and their critical perturbations, without imposing symmetries, restricting time-dependence or making prior assumptions on the physical processes involved. Although we restricted our attention here to purely magnetic seeds, the same approach can readily identify optimal velocity, magnetic disturbances or a combination of both, with only minor modifications in
the optimisation procedure. We thus argue that this approach represents a promising way of numerically obtaining elusive dynamo models, such as the strong branch of the geodynamo or the dynamo of Keplerian flows. Moreover, the possibility of placing additional constraints on the structure of the optimized disturbance can be exploited for experimental purposes, to design a way of kickstarting self-excited dynamos at sustainable energy cost in the laboratory.

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