Quantum antibrackets: polarization and parametrization

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Abstract

By proceeding from a simple non-polarized formalism, we consider in detail the polarization procedure as applied to the generating equations of the quantum antibracket algebra, in terms of the parametrized generating operator.

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1 Introduction

Given a fermionic nilpotent operator $Q$, one can define, for any two operators $X$ and $Y$, their quantum 2-antibracket \[ (X,Y)_Q := \frac{1}{2}([X,[Q,Y]]-[Y,[Q,X]](-1)^{(\epsilon_X+1)(\epsilon_Y+1)}), \] (1.1) or the derived bracket in mathematical terminology \[3, 4, 5\]. These objects have very nice algebraic properties such as the anti-symmetry, the Jacobi relations, and the Leibnitz rule. It appears remarkably that the quantum 2-antibracket (1.1) arises naturally when studying BRST - invariant constraint algebra \[6, 7\] as well as in formulating quantum dynamical equations \[8\] and in the general representation of the gauge fields \[9\].

The quantum 2-antibracket (1.1) generalizes the usual antibracket of the field-antifield BV formalism \[10, 11\] (see also \[12, 13, 14\]). Indeed, when being operators $X$ and $Y$ in (1.1) functions of the field-antifield variables only, one identifies $Q = -\Delta$, where $\Delta$ is the odd Laplacian, and then uses $[\Delta,Y] = (\Delta Y) + \text{ad}(Y)(-1)^{\epsilon_Y}$, to get $(X,Y)_Q = (X,Y)^{\text{BV}}$.

In the present paper, we proceed from the non-polarized version of the (1.1) as defined at $X=Y=B$, with any bosonic operator $B$,

\[ (B,B)_Q := -[B,[B,Q]]. \] (1.2)

So, we call the (1.1) as the polarized quantum 2-antibracket. In fact, that is just being $X \neq Y$ we mean as the polarization, in the general sense. Given the (1.2), one can polarize that by considering formally $B = \alpha X + \beta Y$, with $\alpha$ and $\beta$ being the respective parameters, and then taking the $\alpha\beta$ derivative of the (1.2),

\[ (X,Y)_Q = \partial_\alpha \partial_\beta \frac{1}{2}(B,B)_Q(-1)^{\epsilon_Y}. \] (1.3)

It appears that the basic definitions and the general analysis look simpler essentially in the non-polarized formalism. On the other hand, the generating equations of the quantum antibracket algebra can be formulated naturally in terms of the parametrized generating operator,

\[ \tilde{Q} =: \exp\{\lambda^a f_a\}Q \exp\{-\lambda^a f_a\}, \] (1.4)

being $\{f_a\}$ a chain of operators, and being $\{\lambda^a\}$ the respective parameters. In this way, the generating equations of the quantum antibracket algebra do acquire their geometrically-covariant status. We call the representation (1.4) as the parametrization. All higher quantum antibrackets are defined in terms of derivatives of the generating operator (1.4) with respect to the parameters. Complete set of the structure relations of the quantum antibracket algebra is generated by the nilpotence of the operator (1.4).
2 Basics on quantum antibrackets

Let $Q$ be a fermionic nilpotent operator,

$$Q : \varepsilon(Q) = 1, \quad Q^2 = 0, \quad (2.1)$$

and let $B$ be an arbitrary bosonic operator,

$$B : \varepsilon(B) = 0. \quad (2.2)$$

Introduce a quantum 2-antibracket,

$$(B, B)_Q =: -[B, [B, Q]], \quad (2.3)$$

then we have the main property

$$[Q, (B, B)_Q] = [[Q, B], [Q, B]]. \quad (2.4)$$

Let $A$ be the associator multiplied with a parameter $\beta$,

$$A =: \beta(B, (B, B)_Q). \quad (2.5)$$

It follows then

$$[Q, A] = \beta[[Q, B], [[Q, B], [Q, B]]] = 0 \implies A = [A, Q]. \quad (2.6)$$

By choosing the operator $\mathcal{A}$ as

$$\mathcal{A} = [B, (B, B)_Q] = (B, B)_{Q} =: -[B, [B, Q]], \quad (2.7)$$

we arrive at the non-polarized form of the Jacobi relation

$$6(B, (B, B)_Q) = [(B, B, B)_Q, Q], \quad (2.8)$$

as an identity with respect to $B$. Indeed, denote the operators

$$X =: [B, [[B, Q], [B, Q]]], \quad Y =: [[B, Q], [B, [Q, B]]], \quad (2.9)$$

such that

$$X = -2Y. \quad (2.10)$$

Then, we have

$$(B, (B, B)_Q)_Q = \frac{1}{2}(X + Y) = -\frac{1}{2}Y, \quad (2.11)$$

$$[(B, B, B)_Q, Q] = X - Y = -3Y, \quad (2.12)$$

which is equivalent to (2.8).
3 Polarization

In the definition (2.3), consider the Boson $B$ of the form

$$B = \alpha X + \beta Y + \gamma Z,$$

(3.1)

with $\alpha, \beta$ and $\gamma$ being parameters. It follows then the polarized quantum 2-antibracket

$$\partial_\alpha \partial_\beta \frac{1}{2} (B, B)_Q (-1)^{\varepsilon_Y} = (X, Y)_Q =: \frac{1}{2} ([X, [Q, Y]] - [Y, [Q, X]] (-1)^{(\varepsilon_X+1)(\varepsilon_Y+1)}).$$

(3.2)

In analogy with the main property (2.4) we have its counterpart for polarized quantum 2-antibracket (3.2)

$$[Q, (X, Y)_Q] = [[Q, X], [Q, Y]].$$

(3.3)

In turn, by using (3.2), we have the polarized version of the Jacobi relation (2.8)

$$(X, (Y, Z)_Q)_Q =: \frac{1}{6} ([B, B, B)_Q(-1)^{(\varepsilon_X+1)(\varepsilon_Z+1)} + \text{cyclic permutations } (X, Y, Z)).$$

(3.4)

Thus, we identify the polarized quantum 3-antibracket,

$$(X, Y, Z)_Q =: \partial_\alpha \partial_\beta \partial_\gamma \frac{1}{6} (B, B, B)_Q (-1)^{\varepsilon_Y} =$$

$$= -(-1)^{(\varepsilon_X+1)(\varepsilon_Z+1)} \frac{1}{3} ([X, (Y, Z)_Q](-1)^{\varepsilon_X(\varepsilon_Z+1)+\varepsilon_Y} + \text{cyclic permutations } (X, Y, Z)).$$

(3.5)

The modified Leibnitz rule for quantum 2-antibracket (3.2) reads

$$(XY, Z)_Q - X(Y, Z)_Q - (X, Z)_Q Y(-1)^{\varepsilon_Y(\varepsilon_Z+1)} =$$

$$= \frac{1}{2} ([X, Z][Y, Q](-1)^{\varepsilon_Z(\varepsilon_Y+1)} + [X, Q][Y, Z](-1)^{\varepsilon_Y}).$$

(3.6)

4 Parametrization

Here we include in short the generating equations for the quantum antibracket algebra [2].

Let us introduce an operator valued exponential

$$U = \exp \{\lambda^a f_a\}, \quad U|_{\lambda=0} = 1,$$

(4.1)

where $\{f_a, a = 1, 2, \ldots\}$, is a chain of operators, $\varepsilon(f_a) = \varepsilon_a$, and $\lambda^a$ are parameters, $\varepsilon(\lambda^a) = \varepsilon_a$.

Introduce the $U$-transformed $Q$-operator,

$$\tilde{Q} = UQU^{-1}, \quad \tilde{Q}^2 = 0.$$
The latter equation $\text{(4.2)}$ does generate the complete set of the higher Jacobi relations following $\text{(3.5)}$.

Further, we have the generating equations

$$\partial_a \tilde{Q} = [R_a, \tilde{Q}], \quad R_a = (\partial_a U) U^{-1}, \quad \partial_a = \frac{\partial}{\partial \lambda_a}; \quad \tilde{Q}|_{\lambda=0} = Q,$$

$$\partial_a R_b - \partial_b R_a (-1)^{\varepsilon_a \varepsilon_b} = [R_a, R_b].$$

It follows from $\text{(4.4)}$ that the equation holds with the Euler operator, $N =: \lambda^a \partial_a$,

$$(N + 1) R_b = \partial_b \Psi - [R_b, \Psi],$$

where

$$\Psi = \lambda^a R_a,$$

is an operator describing the arbitrariness in a choice of $\lambda$ parametrization.

By making the rescaling

$$\lambda^a \rightarrow t \lambda^a,$$

$$R_a \rightarrow \tilde{R}_a =: t R_a (t \lambda),$$

$$\Psi \rightarrow \tilde{\Psi} =: \lambda^a R_a (t \lambda),$$

we convert the equation $\text{(4.5)}$ to the form

$$\frac{\partial \tilde{R}_b}{\partial t} = \partial_b \tilde{\Psi} - [\tilde{R}_b, \tilde{\Psi}],$$

with the boundary condition

$$\tilde{R}_b|_{t=0} = 0.$$

The Cauchy problem $\text{(4.10)}, \text{(4.11)}$ resolves in the form

$$\tilde{R}_b(t) = \int_0^t dt' U(t, t') (\partial_b \tilde{\Psi}(t')) U^{-1}(t, t'),$$

where $U(t, t')$ resolves the Cauchy problem

$$\frac{\partial U(t, t')}{\partial t} = \tilde{\Psi}(t) U(t, t'),$$

$$U(t, t')|_{t=t'} = 1.$$
In parallel to the above geometric formulae (4.10) - (4.14), we suggest a simpler derivation for the "current" $R_a$ (4.3),

$$R_a = \int_0^1 dt \exp\{t\psi\} (\partial_a \psi) \exp\{-t\psi\} = \int_0^1 dt \exp\{t \text{ad}(\psi)\}(\partial_a \psi) =$$

$$= \frac{\exp\{\text{ad}(\psi)\} - 1}{\text{ad}(\psi)} (\partial_a \psi), \quad (4.15)$$

where

$$\psi = \lambda^a f_a, \quad (4.16)$$

and we have used the identity

$$[A, \exp{B}] = \int_0^1 dt \exp\{tB\}[A, B]\exp\{(1 - t)B\}, \quad (4.17)$$

for $A = \partial_a$, $B = \psi$.

Notice that the following relation holds between the $\Psi$, (4.6), and the $\psi$, (4.16),

$$\Psi = \exp\{\text{ad}(\psi)\} - 1 \text{ad}(\psi) (N\psi). \quad (4.18)$$

By multiplying the equation (4.10) with $\lambda^b$ from the left we get an identity, as expected, which implies that the $\tilde{\Psi}$, (4.9), is an arbitrary operator. Then, by choosing in (4.12) - (4.14) $\tilde{\Psi} = \psi$, with $\psi$ being given in (4.16), we arrive at the representation (4.15). Indeed, it follows from (4.10) that the equation

$$\partial_t \tilde{\Psi} = N\tilde{\Psi} \quad (4.19)$$

holds, which implies in turn

$$t \partial_t \tilde{\Psi} = \lambda^a NR_a(t\lambda). \quad (4.20)$$

That is just an identity expected. Due to (4.9), the left-hand side of (4.20) rewrites in the form

$$\lambda^a t \lambda^b \frac{\partial}{\partial(t^b)} R_a(t\lambda) = \lambda^a NR_a(t\lambda). \quad (4.21)$$

Thus, the $\tilde{\Psi}$, (4.9), remains arbitrary.

The Lie equation (4.3) and the Maurer-Cartan equation (4.4) do serve as the generating equations for quantum antibrackets. Here we present explicitly only the case of quantum 2-antibracket. It follows from (4.3) by $\lambda$ differentiating, that

$$-\partial_a \partial_b \tilde{Q}(-1)^{\varepsilon_b} + \frac{1}{2}[(\partial_a R_b + \partial_b R_a(-1)^{\varepsilon_a \varepsilon_b})(-1)^{\varepsilon_b}, \tilde{Q}] =$$

$$= \frac{1}{2} \left( [R_a, [\tilde{Q}, R_b]] - (a \leftrightarrow b)(-1)^{(\varepsilon_a+1)(\varepsilon_b+1)}\right) = (R_a, R_b)\tilde{Q}. \quad (4.22)$$
In turn, it follows from (3.5) and (4.22) that the next equation holds,
\[-\frac{1}{3}(-1)^{(\varepsilon_a+1)(\varepsilon_c+1)} \left( [R_a, \Delta_{bc}\tilde{Q}](-1)^{(\varepsilon_a(\varepsilon_c+1)+\varepsilon_b)} + \text{cyclic permutations } (a, b, c) \right) = (R_a, R_b, R_c)\tilde{Q}, \tag{4.23} \]
where we have denoted
\[\Delta_{ab} := -\partial_a \partial_b (-1)^{\varepsilon_b} + [Y_{ab}, (\cdot)], \tag{4.24} \]
\[Y_{ab} := \frac{1}{2} \left( \partial_a R_b + \partial_b R_a (-1)^{\varepsilon_a \varepsilon_b} \right) (-1)^{\varepsilon_b}. \tag{4.25} \]

In the latter notation, the equation (4.22) takes the form
\[\Delta_{ab}\tilde{Q} = (R_a, R_b)\tilde{Q}. \tag{4.26} \]

It follows from (4.22) at \(\lambda = 0\),
\[-(-\partial_a \partial_b \tilde{Q})(-1)^{\varepsilon_b}|_{\lambda=0} = (f_a, f_b)Q, \tag{4.27} \]
where we have used
\[(\partial_a R_b)|_{\lambda=0} = \frac{1}{2}[f_a, f_b]. \tag{4.28} \]

The next equation (4.23) takes the form
\[\Delta_{abc}\tilde{Q} = (R_a, R_b, R_c)\tilde{Q}, \tag{4.29} \]
where
\[\Delta_{abc} = -\partial_a \partial_b \partial_c (-1)^{\varepsilon_b} + \text{ expression vanishing at } \lambda = 0. \tag{4.30} \]

In its more explicit form, the second term in right-hand side in (4.30) reads
\[
\text{expression } =: \frac{1}{3}(-1)^{(\varepsilon_a+1)(\varepsilon_c+1)} \left\{ \left[ \partial_b [\partial_c R_a, (\cdot)](-1)^{\varepsilon_a (\varepsilon_c+1)} + [\partial_b R_a, \partial_c (\cdot)](-1)^{\varepsilon_a \varepsilon_c} + \right. \right. \\
+ \left[ R_a, \frac{1}{2} \left( \partial_b R_c + \partial_c R_b (-1)^{\varepsilon_b \varepsilon_c} \right), (\cdot) \right] \left[ (-1)^{(\varepsilon_a+1)(\varepsilon_c+1)+\varepsilon_b} + \right. \right. \\
\left. \left. \text{cyclic permutations } (a, b, c) \right\} \right\}. \tag{4.31} \]

It follows in a similar way that higher \(\lambda\) derivatives of \(\tilde{Q}\) do yield all higher quantum antibrackets,
\[-(\partial_{a_1} \cdots \partial_{a_n} \tilde{Q})(-1)^{E_n}|_{\lambda=0} = (f_{a_1}, \ldots, f_{a_n})Q = -\text{Sym}([f_{a_1}, \ldots, [f_{a_n}, Q], \ldots])(-1)^{E_n}, \tag{4.32} \]
where we have denoted
\[E_n = \sum_{k=1}^{[n/2]} \varepsilon_{a_{2k}}, \tag{4.33} \]
\[\text{Sym}(X_{a_1 \ldots a_n}) = S_{b_1 \ldots b_n}^{a_1 \ldots a_n} X_{b_1 \ldots b_n}, \quad n!S_{a_1 \ldots a_n}^{b_1 \ldots b_n} = \partial_{a_1} \ldots \partial_{a_n} \lambda^{b_1} \ldots \lambda^{b_n}. \tag{4.34} \]

It has also been shown in [2, 15], how these equations enable one to derive the modified Jacobi relations for subsequent higher quantum antibrackets.
5 Summary

In the present article we have considered a simple non-polarized form of the quantum antibracket algebra (Section 2), and then derived its polarized form (Section 3). Also, we have introduced a natural parametrization (4.2), and then derived the respective generating equations (4.26), (4.29) (Section 4). In an obvious way the construction can be extended to cover all higher quantum antibrackets.

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