CLUSTER TILTING MODULES FOR MESH ALGEBRAS

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Abstract. We study cluster tilting modules in mesh algebras of Dynkin type. We show that these are precisely the maximal rigid modules, and in all but one case we show they are equivariant for a certain automorphism. We also study their mutation, providing an explicit example of mutation in an abelian category which is not stably 2-Calabi-Yau.

1. Introduction

In recent years, cluster tilting theory has gained traction in the study of representation theory of finite dimensional algebras, and in algebraic Lie theory. On one hand, it is a tool to study combinatorial phenomena arising in cluster theory, in the context of additive categorifications of cluster algebras. On the other hand, it generalizes classical tilting theory, which is crucial in the understanding of derived equivalences.

In this article, we study 2-cluster tilting modules (cluster tilting modules for short, cf. Definition 2.1) for finite dimensional self-injective algebras. The existence of a cluster tilting module for such an algebra $A$ has powerful implications. It was shown by Erdmann and Holm in [6] that if $A$ has cluster tilting modules, all modules must have complexity at most 1, that is, the terms in a minimal projective resolution have bounded dimensions. Furthermore, the representation dimension of $A$ must be at most 3. The notion of representation dimension was introduced by Auslander [2], who also showed that the representation dimension of an algebra is at most 2 if and only if it is of finite type. In this sense, the existence of a cluster tilting module shows that the algebra $A$ is not too far from being representation finite.

Cluster tilting modules for self-injective algebras are notoriously elusive. In [6], the only examples found were for certain algebras of finite type; for group algebras of infinite type it is still open as to whether they have cluster tilting modules. However, there are cluster tilting modules for an important class of finite dimensional self-injective algebras: In a series of papers (including [8], [9], [10]), Geiß, Leclerc and Schröer showed the existence of cluster tilting modules for preprojective algebras of simply laced Dynkin type, and studied the cluster, as well as representation, theoretic implications.

In this article, our main objects of interest are mesh algebras of Dynkin type, which were studied by Erdmann and Skowroński in [7]. Each mesh algebra is associated to a generalized Cartan matrix, with type either an arbitrary Dynkin type, or a type called $L_n$ (for $n \geq 1$). They are a natural generalization of preprojective algebras, which are precisely the mesh algebras of simply laced Dynkin type. Therefore, it is a natural question to ask whether mesh algebras have cluster tilting modules. Our first main result shows that all mesh algebras (except those of type $L_n$) indeed have cluster tilting modules (cf. Theorem 3.4). The proof uses a result by Darpó and Iyama [4] (which they also use to find cluster tilting modules), and exploits the work of Geiß, Leclerc and Schröer [9], [10].

Hence, mesh algebras of (non-simply laced) Dynkin type provide a new example of a naturally occurring class of finite dimensional self-injective algebras that have cluster tilting modules. In particular, this yields that these algebras are 2-representation finite, in the sense of Iyama, see [13]. We also show that the cluster tilting modules coincide with the maximal rigid modules (cf. Theorem 7.1).

Our second main result describes mutation of cluster tilting modules (i.e. maximal rigid modules) for a mesh algebra $\Lambda$ of Dynkin type other than $G_2$. Mutation replaces a summand of a cluster tilting module by a unique other module to obtain what is again a cluster tilting module, and this mutation is encoded in certain short exact sequences, called exchange sequences. Classically, in a (stably) 2-Calabi Yau setting, the summands we replace are indecomposable. In our case, however, they need not be. Instead, they are what we call minimal $\gamma$-equivariant (cf. Definition 5.5) under a certain automorphism $\gamma$. The number of non-isomorphic minimal $\gamma$-equivariant summands in a (reachable) cluster tilting module is exactly the number of positive roots in the Dynkin diagram associated to $\Lambda$ (cf. Proposition 7.5).
We are able to exploit a symmetry of Ext$^1$-spaces for modules which are equivariant under $\gamma$ (cf. Proposition 4.4), to describe mutation of cluster tilting modules of $\Lambda$. The key observation is that, if the automorphism $\gamma$ has order $\leq 2$, which is the case for all mesh algebras of Dynkin type except $G_2$, then every basic maximal rigid module is $\gamma$-equivariant. More generally, our mutation theory can be applied to any finite dimensional self-injective algebra for which a suitable automorphism $\gamma$ exists, as long as we restrict ourselves to cluster tilting modules that are $\gamma$-equivariant (provided those exist, as they do for example for $P(G_2)$).

On the stable level, our mutation agrees with the mutation of 2-cluster tilting subcategories with respect to almost complete 2-cluster tilting subcategories as described by Iyama and Yoshino in [14]. For an approach to combinatorially similar mutations, with different starting conditions, see Demonet’s work [5].

The paper is structured as follows. In Section 2 we recall some preliminaries on Galois covers, mesh algebras and cluster tilting modules. In Section 3 we show the existence of cluster tilting modules for mesh algebras and cluster tilting modules. In Section 4 we discuss certain useful automorphisms of our algebras, and show how to exploit a symmetry of Ext$^1$-spaces for modules which are equivariant under a certain automorphism. In Section 5 we introduce the notion of mutation of maximal rigid modules for a specific class of self-injective algebras, which includes the mesh algebras $P(\mathbb{B}_k)$ for $k \geq 2$, $P(\mathbb{C}_n)$ for $n \geq 3$ and $P(\mathbb{F}_4)$. In Section 6 we study the endomorphism algebras of basic maximal rigid modules for such an algebra $\Lambda$. Finally, in Section 7 we show that when $\Lambda$ is a mesh algebra of Dynkin type, the global dimension of these endomorphism rings is 3, and hence the representation dimension of $\Lambda$ is at most 3.

2. Preliminaries

Throughout this article, we work over an algebraically closed field $\mathbb{K}$.

2.1. Mesh categories. Let $\Delta$ be an orientation of a simply laced Dynkin diagram, and let $\mathcal{C} = \mathbb{K}(\mathbb{Z}\Delta)$ be the mesh category of the translation quiver $(\mathbb{Z}\Delta, \tau)$ (see Happel’s work [11] Chapter I, 5.6] for a reminder on this construction). It is well-known and easy to see that the mesh category $\mathcal{C}$ does not depend on the orientation of $\Delta$. The vertices in $\mathbb{Z}\Delta$ are labelled by $\mathbb{Z} \times \Delta_0$, where $\Delta_0$ denotes the vertex set of $\Delta$, and arrows are given as follows: For every arrow $\alpha$ in $\Delta$ starting in $v \in \Delta_0$ and ending in $w \in \Delta_0$ and for every $i \in \mathbb{Z}$ we get a pair of arrows

- $\alpha_i$ starting in $(i, v)$ and ending in $(i, w)$ and
- $\alpha'_i$ starting in $(i, w)$ and ending in $(i - 1, v)$.

Following the convention adopted in [8] Section 9 we denote the vertex in $\mathbb{Z}\Delta$ labelled by $(i, q)$ by $q_i$. With this notation, the translation on $\mathbb{Z}\Delta$ is given by

$$\tau(q_i) = q_{i+1}.$$ 

In Figure 1 we give an illustration of the translation quiver $(\mathbb{Z}\Delta, \tau)$ for $\Delta$ of type $A_{2k-1}$, and with labelling induced by the labelling of the orientation of $A_{2k-1}$ as given in Figure 2.

2.2. Orbit categories of mesh categories. The mesh category $\mathcal{C}$ is a locally bounded $\mathbb{K}$-linear category such that its additive closure $\text{add}(\mathcal{C})$ is Krull-Schmidt, and we can apply the results of Darpö and Iyama given in [4]. Here, locally bounded means that for all objects $x$ in $\mathcal{C}$ we have

$$\sum_{y \in \mathcal{C}} (\dim_{\mathbb{K}} \text{Hom}_{\mathcal{C}}(x, y) + \dim_{\mathbb{K}} \text{Hom}_{\mathcal{C}}(y, x)) < \infty.$$ 

Assume $G$ is a group of $\mathbb{K}$-linear automorphisms of the category $\mathcal{C}$. Following [4], the action of $G$ is admissible if $g(x) \neq x$ for every object $x$ in $\mathcal{C}$ and every $1 \neq g \in G$. Assuming this, the orbit category $\mathcal{C}/G$ is again a locally bounded category such that its additive closure $\text{add}(\mathcal{C}/G)$ is Krull-Schmidt.

We focus on the case where $G$ is induced by graph automorphisms of $\mathbb{Z}\Delta$ which have finitely many orbits on vertices of $\mathbb{Z}\Delta$. In this case the action is admissible if any $g \in G$ with $g \neq 1$ acts freely on $\mathbb{Z}\Delta$. Assuming this, the orbit category $\mathcal{C}/G$ is the $\mathbb{K}$-category of a finite-dimensional $\mathbb{K}$-algebra $\mathbb{K}(\mathbb{Z}\Delta/G)$, whose quiver has vertices and arrows labelled by the orbits of $G$, and with relations induced by the mesh relations of $\mathcal{C}$.

The algebra obtained in this way is called the mesh algebra of $\mathbb{Z}\Delta/G$, denoted by $\mathbb{K}(\mathbb{Z}\Delta/G)$. We will then, by common convention, identify the $\mathbb{K}$-algebra $\mathbb{K}(\mathbb{Z}\Delta/G)$ with its $\mathbb{K}$-category $\mathcal{C}/G$. 

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2.3. **Mesh algebras.** We consider the following automorphisms of \( \mathcal{C} \) induced by graph automorphisms of \( \mathbb{Z}\Delta \).

(i) First, the translation \( \tau \) of \( \mathcal{C} \), which is admissible.

(ii) Second, suppose \( \sigma \) is a graph automorphism of the underlying Dynkin diagram of \( \Delta \), and suppose that the orientation of \( \Delta \) is invariant under \( \sigma \). Then \( \sigma \) induces the following graph automorphism of \( \mathbb{Z}\Delta \) of finite order:

\[
q_i \mapsto \sigma(q)_i, \quad a_i \mapsto \sigma(a)_i,
\]

for all \( i \in \mathbb{Z} \), and every vertex \( q \) and arrow \( a \) of \( \Delta \). By abuse of notation, we denote this graph automorphism of \( \mathbb{Z}\Delta \), and the induced automorphism of \( \mathcal{C} \), also by \( \sigma \). The automorphism \( \sigma \) commutes with \( \tau \), and we consider the admissible automorphism \( \sigma \tau \).

This leads us to consider the following cases. We refer the reader to [7] for a detailed description of the respective mesh algebras.

1. Let \( G = \langle \tau \rangle \). Then \( \mathcal{C}/G \cong P(\Delta) \), the preprojective algebra of type \( \Delta \). Recall that \( P(\Delta) \) is defined as the algebra \( K\overline{\Delta}/\langle I_\rho \rangle \), where \( \overline{\Delta} \) is the double quiver of \( \Delta \) obtained from \( \Delta \) by adding an arrow \( a^* \) starting in vertex \( w \) and ending in vertex \( v \) for each arrow \( a \) in \( \Delta \) starting in \( v \) and ending in \( w \), and where \( \langle I_\rho \rangle \) is the ideal generated by the element

\[
I_\rho = \sum_{a \in Q} (a^*a - aa^*).
\]
Note that $P(\Delta)$ only depends on the underlying Dynkin diagram of $\Delta$, and not on its orientation.

(2) For $k \geq 2$ let $\Delta$ be of type $A_{2k-1}$, and consider the automorphism $\sigma$ of $C$ induced by the graph automorphism of order 2 of $A_{2k-1}$. Assume the orientation of $\Delta$ is invariant under $\sigma$, e.g. take the orientation of $A_{2k-1}$ from Figure 2. Then $C/\sigma$ is isomorphic to the mesh algebra $P(\mathbb{B}_k)$.

(3) For $n \geq 3$, let $\Delta$ be of type $D_{n+1}$, and consider the automorphism $\sigma$ of $C$ induced by the graph automorphism of order 2 (or, if $n = 3$, some choice $\sigma$ of automorphism of order 2) of $D_{n+1}$. Assume the orientation of $\Delta$ is invariant under $\sigma$, e.g. take the orientation of $D_{n+1}$ from Figure 2. Then $C/\sigma$ is isomorphic to the mesh algebra $P(\mathbb{G}_n)$.

(4) Let $\Delta$ be of type $E_6$ and consider the automorphism $\sigma$ of $C$ induced by the graph automorphism of order 2 of $E_6$. Assume the orientation of $\Delta$ is invariant under $\sigma$, e.g. take the orientation of $E_6$ from Figure 2. Then $C/\sigma$ is isomorphic to the mesh algebra $P(\mathbb{F}_4)$.

(5) Let $\Delta$ be of type $D_4$, and consider the automorphism $\sigma$ of $C$ induced by the graph automorphism of order 3 of $D_4$. Assume the orientation of $\Delta$ is invariant under $\sigma$, e.g. take the orientation of $D_4$ from Figure 2 (setting $n = 3$). Then $C/\sigma$ is isomorphic to the mesh algebra $P(\mathbb{G}_2)$.

2.4. Modules of $C$. In line with [8], we work with left modules throughout. For a $\mathbb{K}$-linear category $B$ a left $B$-module is a $\mathbb{K}$-functor from $B$ to the category of $\mathbb{K}$-vector spaces. We denote by $\text{Mod}B$ the category of left $B$-modules, and by $\text{mod}B$ the category of finitely presented left $B$-modules.

Assume that $G$ is a group of $\mathbb{K}$-linear automorphisms of $B$, acting from the left on $B$. Then $G$ acts naturally on $\text{Mod}B$: If $g \in G$ and $M$ is a left $B$-module, we write $g_*(M) := M(g^{-1}(-))$ (denoted by $M^g$ in [8]). Here we use the notation from [4], to facilitate comparing the relevant results we are referring to from this paper, even though, following [8], we do work with left modules throughout. If $U$ is a full subcategory of $B$ we write $g_*(U)$ for the full subcategory of objects $g_*(M)$ for $M$ in $U$. We say that a full subcategory $U$ of $\text{Mod}B$ is $G$-equivariant if $g_*(U) = U$ for all $g \in G$.

2.5. The “start module”. Let $\Delta$ be an orientation of a simply laced Dynkin diagram. Following the approach in [10] Section 2.4] we define the Auslander category $\Gamma_\Delta$ of $\Delta$ to be the full subcategory of $C$ whose objects are the vertices in the Auslander-Reiten quiver $A_\Delta$ of $\mathbb{K}\Delta^\text{opp}$, viewed as a subquiver of $\mathbb{Z}\Delta$. Note that $\text{mod}\Gamma_\Delta$ can be identified with a full subcategory of $\text{mod}C$.

For a group $G$ of $\mathbb{K}$-linear automorphisms of $C$, the covering functor $F_C : C \to C/G$ induces the pull-up

$$F^* : \text{Mod}C/G \to \text{Mod}C$$

and the push-down (denoted by $F_\lambda$ by Bongartz and Gabriel in [3])

$$F_* : \text{Mod}C \to \text{Mod}C/G.$$

Both of these functors are exact and $(F_*, F^*)$ is an adjoint pair. Note that restriction of $F_*$ to the subcategory of finitely presented modules induces a functor $F_* : \text{mod}C \to \text{mod}C/G$.

In [10] the main object of study is the $P(\Delta)$-module $I_\Delta$, where, as in Section 2.3 $P(\Delta)$ denotes the preprojective algebra of type $\Delta$. Recall that $P(\Delta) \cong C/\tau$, where $C = \mathbb{K}(\mathbb{Z}\Delta)$ with translation $\tau$ on $\mathbb{Z}\Delta$, and set $F \gamma : C \to C/\tau$ to be the covering functor. For each vertex $x$ in the quiver $A_\Delta$ (we write $x \in A_\Delta$ for brevity), we denote by $S(x)$ the simple at $x$ and by $I(x)$ the injective $\Gamma_\Delta$-module with socle $S(x)$. Observe that $I(x)$ is spanned by all paths in $A_\Delta$ ending at vertex $x$. Then the module $I_\Delta$ is the push-down

$$I_\Delta = F_*(\bigoplus_{x \in A_\Delta} I(x)),$$

of the $C$-module $\bigoplus_{x \in A_\Delta} I(x)$ which belongs to $\text{mod}\Gamma_\Delta$, i.e. which is supported on $A_\Delta$, to the preprojective algebra $P(\Delta)$. As in [10], we will refer to $I_\Delta$ as the start module of $P(\Delta)$ with respect to $\Delta$. Note that while $P(\Delta)$ does not depend on the orientation of $\Delta$, the start module $I_\Delta$ does.
2.6. Cluster tilting modules. Let $\mathcal{T}$ be a triangulated or abelian category. If $U$ is an object in $\mathcal{T}$, we write $\text{add}(U)$ for its additive hull. An object $U$ in $\mathcal{T}$ is called rigid, if $\text{Ext}^1_\mathcal{T}(U,U) = 0$. It is called maximal rigid, if whenever $X \oplus U$ is rigid, then $X \in \text{add}(U)$.

Definition 2.1. Let $\mathcal{T}$ be a triangulated or abelian category and let $\mathcal{U}$ be a full subcategory of $\mathcal{T}$ that is closed under isomorphisms, finite direct sums and direct summands. Then $\mathcal{U}$ is a 2-cluster tilting subcategory of $\mathcal{T}$, or cluster tilting subcategory for short if it is functorially finite and it satisfies

$$\mathcal{U} = \{ T \in \mathcal{T} \mid \text{Ext}^1(\mathcal{U}, T) = 0 \text{ for all } U \in \mathcal{U} \}$$

If $\mathcal{U} = \text{add}(U)$ for an object $U \in \mathcal{T}$, then we call $U$ a 2-cluster tilting object of $\mathcal{T}$, and if $\mathcal{T}$ is a category of modules, then we call $U$ a 2-cluster tilting module, or cluster tilting module for short.

Note that any 2-cluster tilting object is maximal rigid, but the converse is not true in general. The next theorem provides a concrete example of a cluster tilting module, which will be essential for us.

Theorem 2.2 ([11] Theorem 1, [9] Theorem 2.2). The start module $I_\Delta$ of $P(\Delta)$ with respect to $\Delta$ is a cluster tilting module of $P(\Delta)$.

The following crucial result due to Darpö and Iyama allows us to exploit Theorem 2.2 to show existence of cluster tilting modules for mesh algebras. While the result in [4] is stated more generally for $d$-cluster tilting subcategories and $d$-cluster tilting modules, we only use the case $d=2$. Recall that our field $\mathbb{K}$ is algebraically closed, and note that a $\mathbb{K}$-linear category is Morita equivalent to its additive closure.

Theorem 2.3 ([4] Corollary 2.14]). Let $\mathcal{T}$ be a locally bounded $\mathbb{K}$-linear category such that $\text{add}(\mathcal{C})$ is Krull-Schmidt and let $G$ be a finitely generated free abelian group, acting admissibly on $\mathcal{T}$. The push-down functor $F_* : \text{mod}\mathcal{T} \rightarrow \text{mod}\mathcal{T}/G$ induces a bijection between

(a) the set of locally bounded $G$-equivariant cluster tilting subcategories of $\text{mod}\mathcal{T}$;
(b) the set of locally bounded cluster tilting subcategories of $\text{mod}\mathcal{T}/G$.

Remark 2.4. In particular, it follows that in the set-up as in Theorem 2.3 the push-down $F_*$ induces a bijection between

(a) the set of locally bounded $G$-equivariant cluster tilting subcategories of $\text{mod}\mathcal{T}$ that have finitely many $G$-orbits of indecomposable objects;
(b) the set of basic cluster tilting modules in $\text{mod}\mathcal{T}/G$.

3. Existence of cluster tilting modules for mesh algebras

In this section we show that all algebras from the list in Section 2, have cluster tilting modules. By Theorem 2.2 this is true for preprojective algebras of type $\Delta$, and in Theorem 3.3 we show that it also holds for mesh algebras of non-simply laced Dynkin type.

3.1. Invariance under twists. Let $\Delta$ be an orientation of a simply laced Dynkin diagram, and consider a group $G$ of automorphisms acting admissibly on the mesh category $\mathcal{C} = \mathbb{K}(\mathbb{Z}\Delta)$. Let $A_\Delta$ be the quiver of $\Gamma_\Delta$ as in Section 2.3. For $g \in G$ denote by $g(\Gamma_\Delta)$ the full subcategory of $\mathcal{C}$ with indecomposable objects

$$\{ g(x) \mid x \text{ is indecomposable in } \Gamma_\Delta \}.$$ 

Its isomorphism classes of indecomposable objects are the vertices in the full subquiver $g(A_\Delta)$ of $\mathbb{Z}\Delta$ with vertices $\{ g(x) \mid x \in A_\Delta \}$, where $g$ is considered as the induced graph automorphism of $\mathbb{Z}\Delta$. Recall from Section 2.3 that $I(x)$ denotes the $\mathcal{C}$-module with support on $A_\Delta$, which, as a $\Gamma_\Delta$-module, is injective with socle $S(x)$ for $x$ a vertex in $A_\Delta$.

Lemma 3.1. Let $g \in G$. The module $g_*I(x)$ is the $\mathcal{C}$-module supported on $g(A_\Delta)$ which is injective as a module for $g(\Gamma_\Delta)$ with socle $S(g(x))$.

Proof. If $M$ is a $\mathcal{C}$-module supported on $A_\Delta$ then $g_*(M) = M(g^{-1}(\cdot))$ is supported on $g(A_\Delta)$. (This can be thought of as $M$ “shifted to $g(A_\Delta)^\circ$.) If $M$ has a simple socle $S(x)$ then $g_*(M)$ has simple socle $S(g(x))$. If $M$ is injective as a module for $\Gamma_\Delta$ then the shift of $M$ by $g$ is injective as a module for $g(\Gamma_\Delta)$. \qed
Definition 3.2. Let $\mathcal{B}$ be a $\mathbb{K}$-linear category, and let $g$ be an automorphism of $\mathcal{B}$. A module $M$ in $\text{mod}\mathcal{B}$ is called $g$-equivariant, if $g_*(M) \cong M$.

Corollary 3.3. Assume $\sigma$ is an automorphism of $\mathcal{C}$ induced by a graph automorphism of the underlying diagram of $\Delta$ leaving the orientation of $\Delta$ invariant. Then the module $\bigoplus_{x \in A_{\Delta}} I(x)$ is $\sigma$-equivariant.

3.2. Existence of cluster tilting modules.

Theorem 3.4. Let $\Lambda$ be a mesh algebra of non-simply laced Dynkin type, i.e. one of the algebras

1. $P(B_k)$ for $k \geq 2$,
2. $P(C_n)$ for $n \geq 3$,
3. $P(G_2)$,
4. $P(F_4)$.

Then $\Lambda$ has cluster tilting modules.

Proof. In each of the cases respectively, let $\Delta$ be the quiver and $\sigma$ the automorphism of $\mathcal{C}$ induced by the graph automorphism, also denoted by $\sigma$, on the underlying diagram of $\Delta$ listed below:

1. For $k \geq 2$ and $\Delta = P(B_k)$ let $\Delta$ be the orientation of $A_{2k-1}$ from Figure 2 and let $\sigma$ be the graph automorphism of $A_{2k-1}$ of order two, i.e. the automorphism given by the permutation of vertices $(1,2)(3,4)\cdots(2k-3,2k-2)$.
2. For $n \geq 3$ and $\Delta = P(C_n)$ let $\Delta$ be the orientation of $D_{n+1}$ from Figure 2 and let $\sigma$ be the graph automorphism of $D_{n+1}$ of order two, i.e. the automorphism given by the permutation of vertices $(0,1)$.
3. For $\Lambda = P(G_2)$ let $\Delta$ be the orientation of $D_4$ from Figure 2 (setting $n = 3$) and let $\sigma$ be the graph automorphism of $D_4$ of order three, i.e. the automorphism given by the permutation of vertices $(0,1,3)$.
4. For $\Lambda = P(F_4)$ let $\Delta$ be the orientation of $E_6$ from Figure 2 and let $\sigma$ be the graph automorphism of $E_6$ of order two, i.e. the automorphism given by the permutation of vertices $(1,2)(3,4)$.

Consider now in each case the preprojective algebra $P(\Delta)$. By Theorem 2.2 the start module $I_\Delta$ of $P(\Delta)$ with respect to $\Delta$ is cluster tilting. Consider the push-down

$$F_\cdot : \text{mod}\mathcal{C} \to \text{mod}P(\Delta)$$

of the covering functor. By Corollary 2.3 and Remark 2.4 the subcategory $F_\cdot^{-1}(I_\Delta)$ is cluster tilting in $\text{mod}\mathcal{C}$, it is $\langle \tau \rangle$-equivariant and has finitely many $\langle \tau \rangle$-orbits of indecomposable objects. Since $\Delta$ is invariant under $\sigma$, by Corollary 3.3 the $\mathcal{C}$-module $\bigoplus_{x \in A_{\Delta}} I(x)$ is $\sigma$-equivariant and hence so is $F_\cdot^{-1}(I_\Delta) = F_\cdot^{-1}(F_\cdot(\bigoplus_{x \in A_{\Delta}} I(x)))$.

Therefore, the subcategory $F_\cdot^{-1}(I_\Delta)$ is $\langle \sigma \tau \rangle$-equivariant, and there are finitely many $\langle \sigma \tau \rangle$-orbits of indecomposable objects. Consider now the push-down

$$\tilde{F}_\cdot : \text{mod}\mathcal{C} \to \text{mod}\mathcal{C}/\langle \sigma \tau \rangle \cong \text{mod}\Lambda.$$ 

Again by Theorem 2.3 and Remark 2.4 the push-down $\tilde{F}_\cdot(\bigoplus_{x \in A_{\Delta}} I(x))$ is a cluster tilting module of $\Lambda$, which proves the claim.

\[
\square
\]

4. Automorphisms and $\text{Ext}^1$-symmetry for mesh algebras

Throughout this section we denote by $\Lambda$ a mesh algebra of non-simply laced Dynkin type, i.e. as in Theorem 3.4. Further, let $\Delta$ be the respective Dynkin quiver and $\sigma$ the respective automorphism of $\mathcal{C} = \mathbb{K}(\mathbb{Z}\Delta)$, such that $\Lambda = \mathcal{C}/\langle \sigma \tau \rangle$, as outlined in the list from Section 2.3.

We recall and further investigate certain automorphisms of $\Lambda$, and show that for modules which are equivariant under a specific automorphism, we obtain a useful $\text{Ext}^1$-symmetry. We first start with some important facts about general self-injective algebras.
4.1. Automorphisms of a self-injective algebra. Let $A$ be a finite dimensional self-injective algebra. For an algebra automorphism $\varphi$ and a module $M$ in $\text{mod}A$, we denote from now on by $\varphi M$, instead of by $\varphi_* (M)$, the twist of $M$ by $\varphi$; since we now take the point of view of finite dimensional algebras and their naturally arising automorphisms, we switch to this more algebraic notation. Let $N$ be the Nakayama functor on $\text{mod}A$, that is

$$N = D\text{Hom}_{A}(-, A),$$

where $D(-) = \text{Hom}_{K}(-, K)$. It is well known that $D(A)$ is isomorphic to the twisted module

$$D(A) \cong \eta^{-1} A_{id},$$

where $\eta$ is an algebra automorphism, which is constructed by Yamagata in [16] using a non-degenerate associative bilinear form. The morphism $\eta$ is called a Nakayama automorphism of $A$. Note that for any module $Y$ in $\text{mod}A$ we have

$$NY = D\text{Hom}_{A} (Y, A) \cong DA \otimes_{A} Y \cong \eta^{-1} A \otimes_{A} Y \cong \eta^{-1} Y.$$

We denote by $A^e$ the enveloping algebra of $A$ and by $\Omega$ the first syzygy in a bi-module resolution; note that left $A^e$ modules are just $A$-$A$-bimodules. Putting the next lemma into a wider context, note that if some syzygy of $A$ as an $A^e$-module is isomorphic to a twist of $A$ as a bimodule, then it follows that all left $A$-modules have complexity $\leq 1$: The terms of a minimal bimodule resolution of $A$ have bounded dimension, and tensoring this with a left module $Y$ yields a projective resolution of $Y$, where the terms still have bounded dimension. Therefore, the condition in [6] for the existence of cluster tilting modules is satisfied, and potentially $A$ might have cluster tilting modules. Here, we focus on algebras where $\Omega^3_A (A)$ is a twist of $A$ as a bimodule.

Lemma 4.1. Assume $A$ is a finite dimensional self-injective algebra such that

$$\Omega^3_A (A) \cong \mu A_{id}$$

for some automorphism $\mu$ of $A$ and let $\eta$ be a Nakayama automorphism of $A$. Consider the automorphism $\gamma = \eta^{-1} \circ \mu$ on $A$, and let $X$ and $Y$ be modules in $\text{mod}A$. Then

$$D\text{Ext}^1_A (X, Y) \cong \text{Ext}^1_A (X, \gamma Y).$$

Proof. The first assumption implies that for any left module $M$ of $A$ we have $\Omega^3 (M) \cong \mu M$. We underline whenever we work in the stable category $\text{mod}A$. Recall that suspension in $\text{mod}A$ is given by $\Omega^{-1}$ and that we have the Serre functor $\Omega N$. Since $N$ is a self-Morita equivalence for $A$ (cf. for example Zimmermann’s book [17, Lemma 4.5.6]), it commutes with $\Omega$. So we obtain

$$D\text{Ext}^1_A (X, Y) \cong D\text{Hom} (Y, \Omega^{-1} X) \cong \text{Hom} (\Omega^{-1} X, \Omega Y) \cong \text{Hom} (\Omega X, \Omega^3 Y) \cong \text{Hom} (\Omega X, \Omega^3 (\eta Y)) \cong \text{Hom} (X, \Omega^{-1} (\eta^{-1} \mu Y) ) \cong \text{Ext}^1_A (X, \gamma Y).$$

Consider now $\Lambda$, our mesh algebra of Dynkin type. We know that $\Omega^3_A (A) \cong \mu A_{id}$ for some automorphism $\mu$ (cf. Section 4.2), so we will be able to apply Lemma 4.1 to $\Lambda$. In [1], Andreu Juan and Saorín explicitly describe the following automorphisms of $\Lambda$.

- The Nakayama automorphism $\eta$, which is induced by an automorphism $\theta$ of $C$.
- An algebra automorphism $\mu$ of $\Lambda$ such that $\Omega^3_A (\Lambda) \cong \mu A_{1}$. In particular they determine the period of $\Lambda$ as a bimodule, and hence the order of $\mu$ in the factor group modulo inner automorphisms.

We fix $\eta$ and $\mu$ to be these automorphisms for $\Lambda$, and, for convenience of notation, discuss the latter in slightly more detail in Section 4.2.

4.2. The third syzygy. Consider the automorphism $\vartheta$ on $C$ which fixes each vertex of $\mathbb{Z} \Delta$ and acts on arrows by

$$\vartheta(a) = (-1)^{s(a)} (s^{-1}(a)) + s(a) a,$$

where $s$ denotes the signature map described in [1 Proposition 3.3]. To be slightly more precise, first, [1] define a set $X$ of $(\sigma \tau)$-orbits on arrows of $C$ so that every mesh contains precisely one arrow in $X$, and moreover for each arrow $a$ of $C$ precisely one of $a$ and $\tau^{-1}(a)$ belongs to $X$. Then the signature map $s$ is defined from the set of arrows of $C$ to $\{1, 0\}$ by $s(a) = 1$ for $a \in X$ and $s(a) = 0$ otherwise.
If $\Delta$ is of type other than $A_{2k-1}$, then $\vartheta$ acts as the identity (cf. [1, Remark 5.4]). The ungraded version of [1, Corollary 5.5] states the following:

**Proposition 4.2** ([1, Corollary 5.5]). The automorphism $\mu$ of $\Lambda$, such that $\Omega^3\Lambda_\varphi(\Lambda) \cong \mu_{\text{id}}$, is induced by the automorphism

1. $\mu = \eta^{-1} \circ \mu = \eta^{-1} \circ \tau^{-1} = \tau^{-1} = \vartheta$, if $\Delta$ is of type $A_{2k-1}$, and
2. $\mu = \eta^{-1} \circ \mu = \eta^{-1} \circ \tau^{-1}$ otherwise,

of $\mathcal{C}$, where $\vartheta$ is the automorphism of $\mathcal{C}$ as in [1, Corollary 5.5] inducing the Nakayama automorphism $\eta$ of $\Lambda$.

### 4.3. The automorphism $\gamma$

Consider our mesh algebra $\Lambda$ of Dynkin type with automorphisms $\mu$ as discussed in Section 4.2 and Nakayama automorphism $\eta$, and as in Lemma 4.1 set

$$\gamma = \eta^{-1} \circ \mu.$$ 

Recall that $\Lambda = C/G$ where $\mathcal{C} = K(Z\Delta)$ is the mesh category and $G = \langle \sigma \rangle$, for $\sigma$ and $\Delta$ as in the list from Section 2.3. The automorphisms $\sigma$ and $\tau$ commute with each element of $G$ and hence induce the automorphisms $\sigma$ and $\tau$ of $\Lambda$. Moreover, since $\sigma \tau$ is the identity for $C/G$ it follows that $\sigma = \tau^{-1}$.

**Lemma 4.3.** The automorphism $\gamma$ of $\Lambda$ is equal to $\tilde{\sigma}$, up to inner automorphism.

**Proof.** Recall that $\mu$ is induced by the automorphism of $\mathcal{C}$ from Proposition 4.2. In case (2) of Proposition 4.2 when $\Delta$ is not of type $A_{2k-1}$, we obtain

$$\gamma = \eta^{-1} \circ \mu = \eta^{-1} \circ \eta \circ \tau^{-1} = \tau^{-1} = \tilde{\sigma}.$$ 

Now consider case (1), where $\sigma$ has order 2. Recall the definition of $\vartheta$ from the beginning of Section 4.2 via the signature map $s$. We have that $\tau \vartheta(a) = -\tau(a)$ and $\vartheta(\tau(a)) = -\tau(a)$ for an arrow $a$. So $\tau$ and $\vartheta$ commute.

Now by definition, the signature map $s$ is constant on $G$-orbits and commutes with $\sigma \tau$. Also, as maps on arrows, $s \circ (\tau^{-1} + 1)$ and $\sigma \tau$ commute. Hence the automorphism $\vartheta$ of $\mathcal{C}$ commutes with $\sigma \tau$. Then we get an induced automorphism $\tilde{\vartheta}$ of $\Lambda$. It follows that

$$\mu = \eta \circ \tau^{-1} \circ \tilde{\vartheta}$$

and we have

$$\gamma = \eta^{-1} \circ \mu = \eta^{-1} \circ \eta \circ \tilde{\vartheta} = \tilde{\sigma} \circ \tilde{\vartheta},$$

and the automorphism $\tilde{\vartheta}$ of $\Lambda$ is inner. \hfill \square

### 4.4. Ext$_1$-symmetry

In [9, Section 5] mutation of rigid modules in preprojective algebras of simply laced Dynkin type is studied. The results presented there heavily rely on the fact that the category of finitely generated modules over a preprojective algebra is stably 2-Calabi-Yau, which affords a symmetry of Ext$_1$-spaces that we do not in general find in our $\text{mod}\Lambda$. We can however still exploit some of the methods from [9, Section 5] in our situation, relying on the following symmetry, which holds in particular when $\Lambda = A$, one of our mesh algebras.

**Proposition 4.4.** Let $A$ be as in Lemma 4.1, and let $X$ and $Y$ be modules in $\text{mod}A$. Assume that $Y$ is $\gamma$-equivariant. Then we have

$$\text{DExt}_1^\Lambda(Y, X) \cong \text{Ext}_A^1(X, Y).$$

**Proof.** This follows directly from Lemma 4.1. \hfill \square

### 5. Mutation of Rigid Modules in Mesh Algebras

Throughout this section, let $\Lambda$ be a finite dimensional self-injective algebra with automorphism $\mu$ such that $\Omega^3\Lambda_\varphi(\Lambda) \cong \mu_{\text{id}}$ and Nakayama automorphism $\eta$. As before, we set $\gamma = \eta^{-1} \circ \mu$. Assume further that any basic maximal rigid module in $\text{mod}A$ is $\gamma$-equivariant. In particular, we will show in Theorem 5.3 that we can choose $\Lambda$ to be any mesh algebra from the following list:

1. $P(B_k)$ for $k \geq 2$,
2. $P(C_n)$ for $n \geq 3$,
Corollary 5.1. Let $T$ be a basic maximal rigid module in $\text{mod}\Lambda$. Then for every module $X$ in $\text{mod}\Lambda$ we have
\[
D\text{Ext}^1_{\Lambda}(T, X) \cong \text{Ext}^1_{\Lambda}(X, T).
\]

5.1. Equivariance of maximal rigid modules for mesh algebras. Asking that $\gamma T \cong T$ for any basic maximal rigid $\Lambda$-module $T$ is not some elusive condition on the algebra $\Lambda$: In fact, this holds for the majority of mesh algebras, as we show below.

Lemma 5.2. Assume that $\gamma^2 = \text{id}$. Then every basic maximal rigid module $T$ in $\text{mod}\Lambda$ is $\gamma$-equivariant.

Proof. Assume as a contradiction that $T$ is not $\gamma$-equivariant, and write $T = \bigoplus_{j \in I} T_j$ with $T_j$ indecomposable. Then there is some $i \in I$ such that $\gamma T_i \not\cong T_j$ for all $j \in I$. By Lemma 4.1 we have
\[
\text{Ext}^1_{\Lambda}(T, \gamma T_i) \cong D\text{Ext}^1_{\Lambda}(T_i, T) = 0.
\]

Furthermore, since $\gamma^2 = \text{id}$ we have
\[
\text{Ext}^1_{\Lambda}(\gamma T_i, T) \cong \text{Ext}^1_{\Lambda}(T_i, \gamma T) \cong D\text{Ext}^1_{\Lambda}(T_i, T) = 0.
\]

Clearly, we have that $\gamma T_i$ is rigid, and it follows that $T \oplus \gamma T_i$ is rigid; a contradiction to the assumption that $T$ is maximal rigid. 

\[\square\]

Theorem 5.3. Let $A$ be a mesh algebra from the following list:
\begin{enumerate}
\item[(1)] $P(B_k)$ for $k \geq 2$,
\item[(2)] $P(C_n)$ for $n \geq 3$,
\item[(3)] $P(F_4)$.
\end{enumerate}

Then every basic maximal rigid module $T$ in $\text{mod}\Lambda$ is $\gamma$-equivariant.

Proof. By Lemma 4.3 the automorphism $\gamma$ has order 2. The claim follows from Lemma 5.2. 

\[\square\]

We can thus choose our algebra $\Lambda$ with the desired conditions listed at the start of Section 5, to be one of the mesh algebras from Theorem 5.3. Note that the only mesh algebra of non-simply laced Dynkin type that is not included in the list, is $P(G_2)$. There, the automorphism $\gamma$ has order 3, and we cannot apply Lemma 5.2.

5.2. Minimal $\gamma$-equivariant modules. Mutation of maximal rigid modules of preprojective algebras as studied in [9] replaces an indecomposable summand of the maximal rigid module: [9, Proposition 6.7] states that if $T \oplus X$ is a basic maximal rigid module of a preprojective algebra $P(\Delta)$ of Dynkin type, with $X$ indecomposable, then there exists a unique indecomposable module $Y \not\cong X$ of $P(\Delta)$ such that $T \oplus Y$ is again a basic maximal rigid module. The following observation makes clear why we cannot expect to be able to mutate at a single indecomposable summand in our algebra $\Lambda$, unless this summand is $\gamma$-equivariant.

Lemma 5.4. Assume $T \oplus X$ is a basic maximal rigid module in $\text{mod}\Lambda$, such that $X$ is indecomposable and not $\gamma$-equivariant. Then there cannot exist a module $Y \not\cong X$ such that $T \oplus Y$ is also basic maximal rigid.

Proof. Assume as a contradiction there exists a module $Y \not\cong X$, such that $T \oplus Y$ is basic maximal rigid. Then we have non-trivial extensions between $X$ and $Y$ by maximal rigidity of $T \oplus X$, and thus also $\text{Ext}^1_{\Lambda}(\gamma X, \gamma Y) \neq 0$ or $\text{Ext}^1_{\Lambda}(\gamma Y, \gamma X) \neq 0$. By our assumptions on $\Lambda$, any basic maximal rigid module is $\gamma$-equivariant. So, since $T \oplus X$ is basic, and $X$ is indecomposable and not $\gamma$-equivariant, we must have that $\gamma X$ is a summand of $T$. However, by $\gamma$-equivariance of $T \oplus Y$, the module $\gamma Y$ must also be a summand of $T \oplus Y$, contradicting rigidity of $T \oplus Y$.

\[\square\]

Instead, in our algebra $\Lambda$ we want to mutate with respect to minimal $\gamma$-equivariant summands. We introduce this concept more generally. Let $A$ be a finite dimensional self-injective algebra.

Notation 1. Let $X$ be a module in $\text{mod}\Lambda$. Then we write
\[
|X| = \text{number of indecomposable summands of } X \text{ up to isomorphism}.
\]
Definition 5.5. Let $X$ be a module in $\text{mod}A$ and let $\varphi$ be an automorphism of $A$. We say that $X$ is minimal $\varphi$-equivariant, if it is basic and the indecomposable summands of $X$ form a $\varphi$-orbit, i.e. setting $n = |X| - 1$ we have

$$X \cong \bigoplus_{i=0}^{n} \varphi^i \tilde{X},$$

where $\tilde{X}$ is an indecomposable $A$-module with $\varphi^{n+1} \tilde{X} \cong \tilde{X}$.

Remark 5.6. In the following, we will consider minimal $\gamma$-equivariant modules of $\Lambda$. Note that if $\Lambda$ is a mesh algebra as in Theorem 5.3, then since $\gamma^2 = \text{id}$ by Lemma 4.3, any minimal $\gamma$-equivariant module in $\text{mod}\Lambda$ has one or two indecomposable summands.

5.3. Mutation of maximal rigid modules. Our strategy to construct mutations is based upon modifying [9, Section 5] by using our results on equivariance. For a brief reminder on approximations, we refer the reader to the preliminaries in [9, Section 3.1].

Lemma 5.7. Let $T$ and $X$ be rigid $\Lambda$-modules with $\gamma T \cong T$. If $0 \to X \xrightarrow{f} T' \xrightarrow{g} Y \to 0$ is a short exact sequence with $f$ a left $\text{add}(T)$-approximation, then $T \oplus Y$ is rigid.

Proof. This follows analogously to the proof of [9, Lemma 5.1] and exploiting the Ext$^1$-symmetry from Proposition 4.4. \qed

Corollary 5.8. Let $T$ and $X$ be rigid $\Lambda$-modules. If $T$ is basic maximal rigid then there exists a short exact sequence

$$0 \to X \to T' \to T'' \to 0$$

with $T'$ and $T''$ in $\text{add}(T)$.

Proof. This follows analogously to the proof of [9, Corollary 5.2], where we use Lemma 5.7 and Corollary 5.1. \qed

Corollary 5.9. Let $T$ and $X$ be rigid $\Lambda$-modules and set $E = \text{End}_\Lambda(T)$. If $T$ is basic maximal rigid, then $\text{proj.dim}_E(\text{Hom}_\Lambda(X, T)) \leq 1$.

Proof. This follows analogously to the proof of [9, Corollary 5.3]: Applying $\text{Hom}_\Lambda(-, T)$ to the short exact sequence from Corollary 5.8 yields the projective resolution

$$0 \to \text{Hom}_\Lambda(T'', T) \to \text{Hom}_\Lambda(T', T) \to \text{Hom}_\Lambda(X, T) \to 0.$$ \qed

Theorem 5.10. Let $T_1$ and $T_2$ be basic maximal rigid $\Lambda$-modules. For $i = 1, 2$ set $E_i = \text{End}_\Lambda(T_i)$. Then $T = \text{Hom}_\Lambda(T_2, T_1)$ is tilting over $E_1$ and $\text{End}_{E_i}(T) \cong E_i^{op}$. In particular, the endomorphism algebras $\text{End}_I(T_1)$ and $\text{End}_I(T_2)$ are derived equivalent.

Proof. Use Corollaries 5.8 and 5.9 and follow the proof of [13, Theorem 5.3.2] (cf. also the comments following Theorem 5.4 in [9]). \qed

Proposition 5.11. Let $T \oplus X$ be a basic $\gamma$-equivariant rigid module in $\text{mod}\Lambda$ such that

- $X$ is minimal $\gamma$-equivariant;
- $\Lambda \in \text{add}(T)$.

Then for $0 \leq i \leq n = |X| - 1$ there exists a short exact sequence

$$S_i = 0 \to \gamma_i \tilde{X} \xrightarrow{\gamma_i f} \gamma_i \tilde{T}' \xrightarrow{\gamma_i g} \gamma_i \tilde{Y} \to 0$$

such that taking the direct sum yields an exact sequence

$$\bigoplus_{i=0}^{n} S_i = 0 \to X \xrightarrow{f} T' \xrightarrow{g} Y \to 0$$

with the following properties:
(a) $f = \bigoplus_{i=0}^{n} \gamma_i \tilde{f}$ is a minimal left add($T$)-approximation and $g = \bigoplus_{i=0}^{n} \gamma_i \tilde{g}$ is a minimal right add($T$)-approximation.

(b) $Y = \bigoplus_{i=0}^{n} \gamma_i \tilde{Y}$ is minimal $\gamma$-equivariant, and $X \not\cong Y$.

(c) $T \oplus Y$ is basic $\gamma$-equivariant rigid.

Proof. It follows analogously to the proof of [3] Proposition 5.6] that we have a short exact sequence

$$0 \longrightarrow \tilde{X} \xrightarrow{\tilde{f}} \tilde{T} \xrightarrow{\tilde{g}} \tilde{Y} \longrightarrow 0$$

such that

- $\tilde{f}$ is a minimal left add($T$)-approximation and $\tilde{g}$ is a minimal right add($T$)-approximation.
- $\tilde{Y}$ is indecomposable and $\tilde{X} \not\cong \tilde{Y}$.
- $T \oplus \tilde{Y}$ is basic rigid.

Consider now the basic module $X \cong \bigoplus_{i=0}^{n} \gamma_i \tilde{X}$.

(a) Since $\tilde{f}$ is a minimal left add($T$)-approximation of $\tilde{X}$, we get that $\gamma_i \tilde{f}$ is a minimal left add($T$)-approximation of $\gamma_i \tilde{X}$ for $0 \leq i \leq n$, since by assumption add($T$) is $\gamma$-equivariant. Since minimal approximations commute with direct sums, we obtain that $f = \bigoplus_{i=0}^{n} \gamma_i \tilde{f}$ is a minimal left add($T$)-approximation. Dually, the map $g = \bigoplus_{i=0}^{n} \gamma_i \tilde{g}$ is a minimal right add($T$)-approximation.

(b) Since $X \not\in$ add($T$), sequence (1) does not split, and since $X$ is rigid, it follows that $X \not\cong Y$. Furthermore, we have $\gamma_i \tilde{Y} \not\cong \gamma_i \tilde{X}$ for any $0 \leq i < j \leq n$, or equivalently $\tilde{Y} \not\cong \gamma_k \tilde{Y}$ for any $1 \leq k \leq n$. Else, by uniqueness (up to isomorphism) of the minimal right add($T$)-approximation $\tilde{g}$ we would have $T' \cong \gamma_k T'$, and thus isomorphic kernels $\tilde{X} \cong \gamma_k \tilde{X}$; a contradiction to the assumption. By an analogous argument, we must have $\gamma_i \tilde{X} \cong \gamma_i \tilde{Y}$. Therefore, $Y = \bigoplus_{i=0}^{n} \gamma_i \tilde{Y}$ is minimal $\gamma$-equivariant.

(c) By Lemma [5.7] the module $T \oplus Y = T \oplus \bigoplus_{i=0}^{n} \gamma_i \tilde{Y}$ is rigid. Furthermore, it is basic: We know that $T \oplus \gamma_i \tilde{Y}$ is basic for any $0 \leq i \leq n$, and we have $\gamma_i \tilde{Y} \not\cong \gamma_i \tilde{X}$ for any $0 \leq i < j \leq n$. Since both $T$ and $Y$ are $\gamma$-equivariant, so is $T \oplus Y$.

5.4. Exchange pairs and exchange sequences. The following definition is analogous to the definitions of exchange pair, exchange sequence, etc. in [3] Section 5. Note that we keep track of the order of the two arguments $X$ and $Y$, whereas this is not needed in the classical definition.

Definition 5.12. In the situation of Proposition [5.17] we call $(X = \bigoplus_{i=0}^{n} \gamma_i \tilde{X}, Y = \bigoplus_{i=0}^{n} \gamma_i \tilde{Y})$ a pointed exchange pair with base $(\tilde{X}, \tilde{Y})$ associated to $T$, and we call the sequence

$$0 \longrightarrow X \xrightarrow{f} T' \xrightarrow{g} Y \longrightarrow 0$$

the exchange sequence starting in $X$ and ending in $Y$. The module $T \oplus Y$ is called the mutation of $T \oplus X$ in direction $X$ and we write

$$\mu_X(T \oplus X) = T \oplus Y.$$

Proposition 5.13. Let $X = \bigoplus_{i=0}^{n} \gamma_i \tilde{X}$ and $Y = \bigoplus_{i=0}^{n} \gamma_i \tilde{Y}$ be basic rigid minimal $\gamma$-equivariant modules in mod$\Lambda$ with $n = |X| - 1 = |Y| - 1$. Assume that for $0 \leq i \leq n$ we have

$$\dim(\text{Ext}^1_{\Lambda}(\tilde{Y}, \gamma_i \tilde{X})) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$

and let

$$0 \longrightarrow \tilde{X} \xrightarrow{f} \tilde{M} \xrightarrow{\bar{g}} \tilde{Y} \longrightarrow 0$$

be a non-split exact sequence. Set

$$0 \longrightarrow X \xrightarrow{f} M \xrightarrow{g} Y \longrightarrow 0 \cong \bigoplus_{i=0}^{n} \left(0 \longrightarrow \gamma_i \tilde{X} \xrightarrow{\gamma_i \tilde{f}} \gamma_i \tilde{M} \xrightarrow{\gamma_i \tilde{g}} \gamma_i \tilde{Y} \longrightarrow 0 \right).$$
Then $M \oplus X$ and $M \oplus Y$ are rigid, and $X, Y \notin \text{add}(M)$. If additionally there exists a module $T$ in $\text{mod}\Lambda$ such that $T \oplus X$ and $T \oplus Y$ are basic maximal rigid, then $f$ is a minimal left $\text{add}(T)$-approximation and $g$ is a minimal right $\text{add}(T)$-approximation.

**Proof.** For this part, we use a similar argument to the proof of [9, Proposition 3.4]. Assume $X \in \text{add}(M)$. Since $X$ is basic, we get $M \cong X \oplus M'$ for some $\Lambda$-module $M'$ and Sequence 3 reads as

$$0 \to X \to X \oplus M' \to Y \to 0.$$  

By Riedtmann [15, Proposition 4.3] $M'$ degenerates to $Y$. Since $Y$ is rigid, this implies $M' = Y$. Thus the above sequence splits; a contradiction. Dually one shows that $Y \notin \text{add}(M)$.

**Corollary 5.14.** Let $(X, Y)$ be a pointed exchange pair with base $(\tilde{X}, \tilde{Y})$ associated to a basic rigid module $T$, such that $T \oplus X$ and $T \oplus Y$ are maximal rigid. Assume further that for $0 \leq i \leq n$

$$\dim(\text{Ext}_A^i(\tilde{Y}, \gamma_i \tilde{X})) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{if } i \neq 0. \end{cases}$$

Then we have

$$\mu_Y(\mu_X(T \oplus X)) = T \oplus X.$$  

**Proof.** Let

$$0 \to X \to T' \to Y \to 0 \cong \bigoplus_{i=0}^n \left( 0 \to \gamma_i \tilde{X} \to \gamma_i \tilde{T}' \to \gamma_i \tilde{Y} \to 0 \right)$$

be the short exact sequence from Proposition 5.11 so we have $\mu_X(T \oplus X) = T \oplus Y$. Note that by Lemma 4.1, we have

$$\text{DExt}_A^1(\tilde{X}, \gamma_i \tilde{Y}) \cong \text{Ext}_A^1(\gamma_i \tilde{Y}, \gamma_i \tilde{X}) \cong \text{Ext}_A^1(\tilde{Y}, \gamma_1 \ldots \gamma_i \tilde{X}).$$
Therefore, by assumption, for $0 \leq i \leq n$ we have
\[
\dim \text{Ext}^1(\tilde{X}, \gamma_0 Y) = \begin{cases} 
1 & \text{if } i = 1 \\
0 & \text{if } i \neq 1.
\end{cases}
\]

Further, since $T \oplus X$ and $T \oplus Y$ are basic maximal rigid, Proposition 5.13 yields a non-split short exact sequence
\[
0 \rightarrow Y \xrightarrow{h} M \rightarrow X \rightarrow 0 \cong \bigoplus_{i=0}^{n} \left(0 \rightarrow \gamma_{i+1} \tilde{Y} \rightarrow \gamma_i \tilde{M} \rightarrow \gamma_i \tilde{X} \rightarrow 0\right)
\]
where $h$ is a minimal left $\text{add}(T)$-approximation. Thus $\mu_Y(T \oplus Y) = T \oplus X$. \hfill \Box

**Remark 5.15.** It follows from the Proof of 5.14 that if $(X, Y)$ is a pointed exchange pair with base $(\tilde{X}, \tilde{Y})$ associated to $T$ such that $T \oplus X$ and $T \oplus Y$ are basic maximal rigid, then $(Y, X)$ is a pointed exchange pair with base $(\gamma \tilde{Y}, \tilde{X})$ associated to $T$. Namely, when we exploit Ext$^1$-symmetry for $\gamma$-equivariant modules, we have to be mindful of what happens to their summands. Thus, for a pointed exchange pair $(X, Y)$ with base $(\tilde{X}, \tilde{Y})$ as in Corollary 5.14 where we have that $\mu_Y(\mu_X(T \oplus X)) = T \oplus X$, the associated exchange sequences decompose in a different manner: We have
\[
0 \rightarrow X \rightarrow T' \rightarrow Y \rightarrow 0 \cong \bigoplus_{i=0}^{n} \left(0 \rightarrow \gamma_i \tilde{X} \rightarrow \gamma_i \tilde{T'} \rightarrow \gamma_i \tilde{Y} \rightarrow 0\right),
\]
for the exchange sequence starting in $X$ and ending in $Y$ yet the backwards mutation sees a twist by $\gamma$; we have
\[
0 \rightarrow Y \rightarrow M \rightarrow X \rightarrow 0 \cong \bigoplus_{i=0}^{n} \left(0 \rightarrow \gamma_{i+1} \tilde{Y} \rightarrow \gamma_i \tilde{M} \rightarrow \gamma_i \tilde{X} \rightarrow 0\right)
\]
for the exchange sequence starting in $Y$ and ending in $X$.

**Remark 5.16.** Note that in the stable module category, our mutations induce mutations in the sense of Iyama and Yoshino [14, Section 5]. Indeed, in $\text{mod} \Lambda$ we have Serre functor $S = \Omega \gamma$. The functor denoted by $S_n$ in [14], which is of interest in our case for $n = 2$, takes on the form
\[
S_2 = N \gamma^2.
\]
Thus for any $M \in \text{mod} \Lambda$ we have
\[
S_2 M \cong \gamma M.
\]
Note that, in addition to being defined already on the abelian category of modules, our mutation theory for our specific algebras is more explicit, in that it describes precisely which summands can be exchanged, and how mutation induces a twist on the indecomposable summands.

The following shows that one can mutate a cluster tilting module at any minimal $\gamma$-equivariant non-projective summand.

**Proposition 5.17.** Let $T$ be a basic maximal rigid module in $\text{mod} \Lambda$, and let $X$ be a minimal $\gamma$-equivariant direct summand of $T$. If $X$ is non-projective, then up to isomorphism there exists exactly one minimal $\gamma$-equivariant $Y$ in $\text{mod} \Lambda$ such that $X \not\cong Y$ and $Y \oplus T/X$ is maximal rigid.

**Proof.** This follows directly from Remark 5.16 together with [14, Proposition 5.3]. \hfill \Box

**Remark 5.18.** More generally, throughout this section we could replace $\Lambda$ by any finite dimensional self-injective algebra $A$ with automorphism $\mu$ such that $\Omega^2_A(A) \cong \mu \text{Id}$. While we do not know if every basic maximal rigid module is $\gamma$-equivariant, all of our previous results starting from Section 5.3 still apply if, whenever the term “maximal rigid module” appears, we replace it by “$\gamma$-equivariant maximal rigid module”, and rely on Proposition 5.14 rather than Corollary 5.1.

In particular, we could pick $\Lambda$ to be the remaining mesh algebra of Dynkin type, namely $P(\mathbb{G}_2)$. It does have a $\gamma$-equivariant maximal rigid module, namely the cluster tilting module constructed in Theorem 5.4.
6. Endomorphism algebras of maximal rigid modules

Throughout this section, let $\Lambda$ be a finite dimensional self-injective algebra with automorphism $\mu$ such that $\Omega^3_{\Lambda}(A) \cong \mu A_{ad}$ and with Nakayama automorphism $\eta$. As before, we set $\gamma = \eta^{-1} \circ \mu$. In line with Remark 5.13, we do not need to know whether every basic maximal rigid module is $\gamma$-equivariant.

In the following, we study endomorphism algebras and their quivers. The strategy of this section, and the beginning of the next, is analogous to Section [9, Section 6]. If $M$ is a module in $\text{mod}\Lambda$, then the quiver of $\text{End}_\Lambda(M)$ has vertices labelled by the indecomposable modules in $\text{add}(M)$. By abuse of notation, we will denote the vertex associated to an indecomposable $M$ in $\text{add}(M)$ by $M$ as well.

Lemma 6.1. Let $(X = \bigoplus_{i=0}^n \gamma_i X, Y = \bigoplus_{i=0}^n \gamma_i Y)$ be a pointed exchange pair with base $(\tilde{X}, \tilde{Y})$ associated to a basic rigid module $T$ in $\text{mod}\Lambda$. The following are equivalent:

1. The quiver of $\text{End}_\Lambda(T \oplus X)$ has no arrows from $\tilde{X}$ to $\gamma_i \tilde{X}$ for all $i \in \mathbb{Z}$.
2. Every radical map $X \to X$ factors through $\text{add}(T)$.
3. For $0 \leq i \leq n$ we have
   $$\dim \text{Ext}^3_\Lambda(\gamma_i X, \gamma_j X) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{else.} \end{cases}$$

Proof. We first show that (1) is equivalent to (2), and then that (2) is equivalent to (3).

1) $\Rightarrow$ (2): Assume that $f: X \to X$ is a radical map which does not factor through $\text{add}(T)$. Then, in the quiver of $\text{End}_\Lambda(T \oplus X)$, this yields an arrow from $\gamma_i \tilde{X}$ to $\gamma_j \tilde{X}$ for some $i, j \in \mathbb{Z}$, and thus an arrow from $\tilde{X}$ to $\gamma_j \tilde{X}$.

2) $\Rightarrow$ (1): If, in the quiver of $\text{End}_\Lambda(T \oplus X)$, we have an arrow from $\tilde{X}$ to $\gamma_j \tilde{X}$, it is in the radical of $\text{End}_\Lambda(T \oplus X)$. Furthermore, it is not in $\text{rad}^2(\text{End}_\Lambda(T \oplus X))$, so it cannot factor through $\text{add}(T)$.

2) $\Rightarrow$ (3): Since $(X, Y)$ is a pointed exchange pair with base $(\tilde{X}, \tilde{Y})$, there is a short exact sequence

$$0 \longrightarrow \tilde{X} \xrightarrow{\tilde{f}} \tilde{T} \longrightarrow \tilde{Y} \longrightarrow 0$$

with $\tilde{T} \in \text{add}(T)$. Applying $\text{Hom}_\Lambda(-, X)$ yields the exact sequence

$$0 \longrightarrow \text{Hom}_\Lambda(\tilde{Y}, X) \longrightarrow \text{Hom}_\Lambda(\tilde{T}, X) \longrightarrow \text{Hom}_\Lambda(\tilde{X}, X) \longrightarrow \text{Ext}^1_\Lambda(\tilde{Y}, X) \longrightarrow 0$$

By assumption any radical map $h: \tilde{X} \to X$ must factor through $\text{add}(T)$, so we must have $h = \mu \circ \theta$ for some $\theta: \tilde{X} \to T'$ and $\mu: T' \to X$ with $T' \in \text{add}(T)$. Now, since $\tilde{f}$ is a left $\text{add}(T)$-approximation, we have $\theta = \psi \circ \tilde{f}$ for some $\psi: \tilde{T}' \to T'$, and we obtain a factorization

$$h = \mu \circ \psi \circ \tilde{f}.$$ 

This means that $\text{coker}(\text{Hom}_\Lambda(\tilde{f}, X))$ is 1-dimensional; it is spanned by the coset of the inclusion of $\tilde{X}$ into $X$. Therefore, we have

$$\dim \text{Ext}^1_\Lambda(\tilde{Y}, X) = 1.$$

Now, by existence of the short exact sequence (4), we know that $\text{Ext}^1_\Lambda(\tilde{Y}, \tilde{X}) \neq 0$ and the claim follows.

3) $\Rightarrow$ (2): Consider the sequence (5). By assumption, $\dim \text{Ext}^1_\Lambda(\tilde{Y}, X) = 1$. The inclusion $\iota: \tilde{X} \to X$ does not factor through $\tilde{f}$, since the short exact sequence (4) does not split. So $\text{coker}(\text{Hom}_\Lambda(\tilde{f}, X))$ is spanned by the coset of $\iota$. The claim follows. \qed

Let now $T$ be a basic $\gamma$-equivariant maximal rigid module in $\text{mod}\Lambda$. Note that this is not the same $T$ as previously, which was just assumed to be basic rigid. If $\Lambda$ is a mesh algebra of Dynkin type other than $P(\mathbb{G}_2)$, or any $\Lambda$ where $\gamma$ has order at most 2, we can choose $T$ to be any basic maximal rigid module by Lemma 6.2. Consider the quiver of $E = \text{End}_\Lambda(T)$. We denote by $S_{\tilde{X}}$ the simple $E$-module associated to an indecomposable $\tilde{X} \in \text{add}(T)$.

Proposition 6.2. If the quiver of $E$ has no arrows from $\tilde{X}$ to $\gamma_i \tilde{X}$ for all indecomposable modules $\tilde{X} \in \text{add}(T)$ and all $i \in \mathbb{Z}$, then

$$\text{gl.dim}(E) = 3.$$
Proof. Consider an indecomposable module $\hat{X}$ in $\text{add}(T)$. Assume first that $\hat{X}$ is non-projective. Let $X$ be the minimal $\gamma$-equivariant summand of $T$ having $\hat{X}$ as a summand:

$$X = \bigoplus_{i=0}^{\frac{|X|-1}} \gamma^i \hat{X}.$$ 

Consider the exchange pair $(X, Y)$ with base $(\hat{X}, \tilde{Y})$ associated to $T/X$. Since $\hat{X}$ is non-projective, the module $X$ has no projective summands, and since $T$ is maximal rigid, we have $\Lambda \in \text{add}(T/X)$.

By the discussions in Remark 5.13 based on Propositions 5.11 and 5.13 we have short exact sequences

$$0 \to \tilde{X} \xrightarrow{\tilde{f}} T' \xrightarrow{\tilde{Y}} 0$$

and

$$0 \to \tilde{Y} \xrightarrow{\tilde{Y}} T'' \xrightarrow{\gamma^{-1} \hat{X}} 0$$

with $T'$ and $T''$ in $\text{add}(T/X)$. Applying $\text{Hom}_\Lambda(-, T)$ we obtain sequences

$$0 \to \text{Hom}_\Lambda(\tilde{Y}, T) \to \text{Hom}_\Lambda(T', T) \xrightarrow{\text{Hom}_\Lambda(\tilde{f}, T)} \text{Hom}_\Lambda(\tilde{X}, T) \xrightarrow{\text{Ext}^1(\tilde{Y}, T)} 0,$$

where $\text{Ext}^1(\tilde{Y}, T) \cong \text{Ext}^1(\tilde{f}, \tilde{X})$, and

$$0 \to \text{Hom}_\Lambda(\gamma^{-1} \hat{X}, T) \to \text{Hom}_\Lambda(T'', T) \to \text{Hom}_\Lambda(T', T) \to \text{Hom}_\Lambda(\tilde{X}, T) \to \text{Ext}_\Lambda(\gamma^{-1} \hat{X}, T) = 0.$$ 

The cokernel of $\text{Hom}_\Lambda(\tilde{f}, T)$, i.e. $\text{Ext}^1(\tilde{Y}, \tilde{X})$, is one-dimensional by Lemma 6.1. Therefore it is isomorphic to $S_{\hat{X}}$, as $\text{Hom}_\Lambda(\tilde{X}, T)$ is the projective of $E$ associated to $\hat{X}$. Combining the two sequences yields an exact sequence

$$0 \to \text{Hom}_\Lambda(\gamma^{-1} \hat{X}, T) \to \text{Hom}_\Lambda(T'', T) \to \text{Hom}_\Lambda(T', T) \to \text{Hom}_\Lambda(\tilde{X}, T) \to S_{\hat{X}} \to 0.$$ 

This is a projective resolution of $S_{\hat{X}}$. Applying $\text{Hom}_E(-, S_{\gamma^{-1} \hat{X}})$ yields

$$\text{Ext}^1_E(S_{\hat{X}}, S_{\gamma^{-1} \hat{X}}) \cong \text{Hom}_E(\text{Hom}_\Lambda(\gamma^{-1} \hat{X}, T), S_{\gamma^{-1} \hat{X}}) \neq 0,$$

and hence $\text{pdim} S_{\hat{X}} = 3$.

Next, assume that $\hat{X}$ is projective. Let $Z \cong \hat{X}/S$, where $S$ is the (simple) socle of $\hat{X}$. We claim that $Z$ is rigid. Indeed we have

$$\text{Ext}_{\Lambda}(Z, Z) \cong \text{Ext}_1(\Omega^{-1}(Z), \Omega^{-1}(Z)) \cong \text{Ext}_1(S, S) = 0,$$

since the quiver of $\Lambda$ has no loops.

As before, let $X$ be the minimal $\gamma$-equivariant summand of $T$ with summand $\hat{X}$. Let $f: Z \to T'$ be the minimal left $\text{add}(T/X)$-approximation of $Z$. This is injective: We have

$$\text{soc}_\Lambda(Z) = \bigoplus_{L \to S \text{ in the quiver of } X} L.$$

For each $L$ occurring as a summand of $\text{soc}_\Lambda(Z)$, we have that $L \not\cong \gamma_i S$ for all $i \in \mathbb{Z}$ by the assumption. Hence the injective hull of $Z$ is in $\text{add}(T/X)$ which implies that $f$ is injective. Thus we get a short exact sequence

$$0 \to Z \xrightarrow{f} T' \to \overline{Z} \to 0.$$ 

By Lemma 5.7 we have that $\overline{Z} \oplus T/X$ is rigid. Since $\overline{Z}$ is not projective we have $\overline{Z} \not\in \text{add}(X)$. However, the module $T$ is maximal rigid, so we must have $\overline{Z} \in \text{add}(T)$.

Consider now the projection $\pi: \hat{X} \to Z \cong \hat{X}/S$. We have an exact sequence

$$\hat{X} \xrightarrow{h=\pi f} T' \xrightarrow{Z} 0$$

and applying $\text{Hom}_\Lambda(-, T)$ yields an exact sequence

$$0 \to \text{Hom}_\Lambda(\overline{Z}, T) \to \text{Hom}_\Lambda(T', T) \xrightarrow{\text{Hom}_\Lambda(h, T)} \text{Hom}_\Lambda(\hat{X}, T) \to W \to 0.$$

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where \( W = \text{coker}(\text{Hom}_\Lambda(h, T)) \). We want to show that \( \dim W = 1 \). We have

\[
\text{Hom}_\Lambda(\tilde{X}, T) = \text{Hom}_\Lambda(\tilde{X}, T/X) \oplus \text{Hom}_\Lambda(\tilde{X}, X).
\]

For \( g \in \text{Hom}_\Lambda(\tilde{X}, T/X) \) we have \( g(S) = 0 \), since \( g \) is not a monomorphism, as \( \tilde{X} \) is not a summand of \( T/X \) and \( \Lambda \) is self-injective. This gives a factorization \( g = g' \circ \pi \) for some \( g' : Z \to T/X \). Since \( f : Z \to T' \) is an \( \text{add}(T/X) \)-approximation, we get \( g'' : T' \to T/X \) with \( g'' = g'' \circ f \). So the following diagram commutes:

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{g}} & T/X \\
\downarrow \pi & & \downarrow \tilde{g}' \\
Z & \xrightarrow{f} & T'
\end{array}
\]

We have \( g = g'' \circ f \circ \pi = g'' \circ h \). Now, since in the quiver of \( E \) there are no arrows from \( \tilde{X} \) to \( \gamma_i \tilde{X} \) for all \( i \geq 0 \), by Lemma 6.1 all radical maps from \( \tilde{X} \) to \( X \) factor through \( T/X \). By the same argument, they factor through \( h \). Therefore \( \dim W = 1 \) and \( W \cong S_{\tilde{X}} \) which thus has projective dimension \( \leq 2 \).

\[
\square
\]

7. Endomorphism algebras of cluster tilting modules for mesh algebras

We now return our focus on mesh algebras. Throughout this section let \( \Lambda \) denote a mesh algebra of non-simply laced Dynkin type, and as before let \( \gamma \) be the automorphism of \( \Lambda \) described in Section 4.3. Note that in all cases but \( P(\mathbb{G}_2) \), any basic maximal rigid module is automatically \( \gamma \)-equivariant, and we could remove this assumption from the statements of the results in these cases.

**Theorem 7.1.** Let \( T \) be a basic \( \gamma \)-equivariant maximal rigid module in \( \text{mod}\Lambda \), and set \( E = \text{End}_\Lambda(T) \). Then the following hold:

1. The quiver of \( E \) has no arrows from \( \tilde{X} \) to \( \gamma_i \tilde{X} \) for any indecomposable \( \tilde{X} \in \text{add}(T) \).
2. We have \( \text{gl.dim}(E) = 3 \).
3. The module \( T \) is cluster tilting.
4. We have \( \text{dom.dim}(E) = 3 \).

Before we provide the Proof of Theorem 7.1 let us recall some facts from Sections 2 and 3. In the proof of Theorem 7.1 we will consider the module

\[
I_\Lambda = \hat{F}_*(I \bigoplus_{x \in \Delta_\Lambda} I(x))
\]

in \( \text{mod}\Lambda \) from the proof of Theorem 3.4 where \( \hat{F}_* : \text{mod}\mathcal{C} \to \text{mod}\Lambda \) as usual denotes the push-down of the covering functor. Recall that \( I(x) \) are the indecomposable injective \( \Gamma_\Delta \)-modules, viewed as \( \mathcal{C} \)-modules, and that \( I_\Lambda \) is the push-down of the pull-up of the start module \( I_\Delta \) in \( \text{P}(\Delta) \). It is a cluster tilting module, and we think of it as our “start module” in \( \text{mod}\Lambda \). Note that, in particular, our start module \( I_\Lambda \) in \( \text{mod}\Lambda \) is basic \( \gamma \)-equivariant maximal rigid.

**Proof.** By Theorem 5.10 the endomorphism algebra \( \text{End}_\Lambda(T) \) is derived equivalent to \( \text{End}_\Lambda(I_\Lambda) \), the endomorphism algebra of our start module \( I_\Lambda \) in \( \text{mod}\Lambda \) (for \( P(\mathbb{G}_2) \)) keep in mind Remark 5.18. Now, since \( I_\Lambda \) is cluster tilting we have by [13 Theorem 0.2] that

\[
\text{gl.dim} \text{End}_\Lambda(I_\Lambda) \leq 3 < \infty.
\]

This implies that \( \text{End}_\Lambda(T) \) has finite global dimension. Therefore, by Igusa’s work on the automorphism conjecture [12 Theorem 3.2], the quiver of \( \text{End}_\Lambda(T) \) has no arrows from \( \tilde{X} \) to \( \gamma_i \tilde{X} \) for every indecomposable module \( \tilde{X} \in \text{add}(T) \) and all \( i \in \mathbb{Z} \). Thus, by Proposition 6.2 we have \( \text{gl.dim}(\text{End}_\Lambda(T)) = 3 \). So \( T \) is cluster tilting by [13 Theorem 5.1(3)]. Again by [13 Theorem 0.2] we get that \( \text{dom.dim}(\text{End}_\Lambda(T)) = 3 \), since \( \text{gl.dim} \Lambda = 3 \) also implies \( \text{dom.dim}(A) \leq 3 \).

**Corollary 7.2.** The algebra \( \Lambda \) has representation dimension \( \leq 3 \). In particular, if it is of infinite type, it has representation dimension 3.
Corollary 7.3. Assume $\Lambda$ is not of type $P(G_2)$. Then a basic module $T$ in $\mod\Lambda$ is cluster tilting if and only if it is maximal rigid.

**Definition 7.4.** Let $T$ and $T'$ be cluster tilting modules in $\mod\Lambda$. We say that $T'$ is reachable from $T$ if there exists an $l \geq 0$ and a sequence $T_0, T_1, \ldots, T_l$ of cluster tilting modules with $T_0 = T$ and $T_l = T'$ and with a minimal $\gamma$-equivariant summand $X_i$ of $T_i$ for $0 \leq i \leq l-1$ such that $T_{i+1}$ is the mutation of $T_i$ in direction $X_i$.

**Proposition 7.5.** Let $T$ be a basic cluster tilting module of $\Lambda$ reachable from the start module $I_\Delta$. Then the number of minimal $\gamma$-equivariant summands of $T$ is the number of positive roots of the corresponding root system, cf. Table 1.

**Proof.** We show, case by case, that the claim is true for $T = I_\Lambda$, our start module. It then follows that it is true for any cluster tilting module that is reachable from $I_\Delta$, since the number of minimal $\gamma$-equivariant summands is invariant under mutation. Observe that $\gamma$ acts as $\sigma$ on the summands of our start module (cf. Lemma 4.3) so what we want to count is the number of $\sigma$-orbits on $A_\Delta$.

1. Assume $\Lambda = P(B_k)$ for $k \geq 2$. Then we start with the quiver $\Delta$ of type $A_{2k-1}$ from Figure 2 and form $A_\Delta$ which consists of $k$ copies of $\Delta$ suitably connected. The automorphism $\sigma$ fixes each copy of $\Delta$; on it fixes the central vertex and has $k-1$ orbits of length 2. In total, the automorphism $\sigma$ has $k \cdot (1 + (k-1)) = k^2$ orbits.

2. Assume $\Lambda = P(C_n)$ for $n \geq 3$. Start with the quiver $\Delta$ of type $D_{n+1}$ from Figure 2. Then $A_\Delta$ consists of $n$ copies of $\Delta$ suitably connected. In each copy of $\Delta$, the automorphism $\sigma$ fixes $n-1$ vertices and has one orbit of length 2. In total, it has $(1 + (n-1)) \cdot n = n^2$ orbits.

3. Assume $\Lambda = P(F_4)$. Start with the quiver $\Delta$ of type $E_5$ from Figure 2. Then $A_\Delta$ consists of six copies of $\Delta$ suitably connected. The automorphism $\sigma$ fixes each of these copies. On it, there are two fixed points and two orbits of length 2, i.e. 4 orbits. In total, there are $4 \cdot 6 = 24$ orbits.

4. Assume $\Lambda = P(G_2)$. Start with the quiver $\Delta$ of type $D_4$ from Figure 2 (setting $n = 3$). Then $A_\Delta$ consists of 3 copies of $\Delta$ suitably connected. On each copy, the automorphism $\sigma$ has one orbit of length 3 and one fixed point. In total, there are $3 \cdot (1 + 1) = 6$ orbits.

As for preprojective algebras (cf. [3] Conjecture 6.10]), it is an open question whether any cluster tilting module of a mesh algebra is reachable from a fixed cluster tilting module. If this is the case, all our results on mutation in Section 5 go through unchanged for $P(G_2)$, since $\gamma$-equivariance is preserved under mutation.

Finally, for the last remaining class of mesh algebras, it can be seen with a direct calculation that $P(L_4)$ (with two vertices) has no cluster tilting modules, but it is still open if and when $P(L_n)$ for $n \geq 2$ has a cluster tilting module.

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