Self-force regularization in the Schwarzschild spacetime

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Abstract
We discuss the gravitational self-force on a particle in a black-hole spacetime. For a point particle, the full (bare) self-force diverges. The metric perturbation induced by a particle can be divided into two parts, the direct part (or the S part) and the tail part (or the R part), in the Lorenz gauge, and the regularized self-force is derived from the R part which is regular and satisfies the source-free perturbed Einstein equations. But this formulation is abstract, so when we apply it to black-hole-particle systems, there are many problems to be overcome in order to derive a concrete self-force. These problems are roughly divided into two parts. They are the problem of regularizing the divergent self-force, i.e., ‘subtraction problem’ and the problem of the singularity in gauge transformation, i.e., ‘gauge problem’. In this paper, we discuss these problems in the Schwarzschild background and report some recent progress.

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1. Introduction

Thanks to recent advances in technology, an era of gravitational wave astronomy has almost arrived. There are already several large-scale laser interferometric gravitational wave detectors that are in operation in the world. Among them are TAMA300 [1], LIGO [2], GEO-600 [3], and VIRGO [4] is expected to start its operation soon. The primary targets of these ground-based detectors are inspiralling compact binaries, which are expected to be detected in the near future.

There are also space-based interferometric detector projects. LISA is now on its R&D stage [5], and there is a future plan called DECIGO/BBO [6, 7]. These space-based detectors can detect gravitational waves from solar-mass compact objects orbiting supermassive black
holes. To extract physical information from such binary systems, it is essential to know the theoretical gravitational waveforms with sufficient accuracy. The black-hole perturbation approach is most suited for this purpose. In this approach, one considers gravitational waves emitted by a point particle that represents a compact object orbiting a black hole, assuming the mass of the particle ($\mu$) to be much less than that of the black hole ($M$); $\mu \ll M$.

In the lowest order in the mass ratio ($\mu/M$)$^0$, the orbit of the particle is a geodesic on the background geometry of a black hole. Already in this lowest order, by combining with the assumption of adiabatic orbital evolution, this approach is proved to be very powerful for evaluating general relativistic corrections to the gravitational waveforms, even for neutron star–neutron star (NS–NS) binaries where the assumption of this approach is maximally violated [8].

In the next order, the orbit deviates from the geodesic because the spacetime is perturbed by the particle. We can interpret this deviation as the effect of the self-force of the particle on itself. Since it is essential to take account of this deviation to predict the orbital evolution accurately, we have to derive the equation of motion that includes the self-force. The gravitational self-force is, however, not easily obtainable. There are two main obstacles.

First, the full (bare) metric perturbation diverges at the location of the particle, which is point-like, hence so does the self-force. The full metric perturbation can be formally divided into two parts; the direct part that comes from the perturbation propagating along the light cone of the background spacetime directly from the source particle, and the tail part that arises from the curvature scattering of the perturbation, which exists within the light cone. The self-force is given by the tail part of the metric perturbation which is regular at the location of the particle [9, 10]. Recently, this has been reformulated in a more elaborated way by Detweiler and Whiting [11]. In this new formulation, the direct part is replaced by the S part and the tail part is replaced by the R part. The R part is equivalent to the tail part as far as the self-force calculation is concerned, but has a much better property that it is a solution of source-free linearized Einstein equations. Thus, one has to identify the R part of the metric perturbation to obtain a meaningful self-force. However, by construction, the R part cannot be determined locally but depends on the whole history of the particle. Therefore, one usually identifies the direct part (or the S part) which can be evaluated locally in the vicinity of the particle to a necessary order, i.e., by local analysis, and subtracts it from the full metric perturbation. This identification of the S part is the subtraction problem.

Second, the regularized self-force is formally defined only in the Lorenz gauge, in which linearized Einstein equations are hyperbolic, which enables us to define the S part and the R part uniquely. On the other hand, the metric perturbation of a black-hole geometry can be calculated only in the radiation gauge in the Kerr background [12], or in the Regge–Wheeler (–Zerilli) gauge in the Schwarzschild background [13, 14]. Hence, one has to find a gauge transformation to express the full metric perturbation and the S part in the same gauge, to identify the R part in that gauge correctly. This is the gauge problem.

As a method to solve the subtraction problem, extending earlier work [15, 16], Mino, Nakano and Sasaki formulated two types of regularization methods, namely, the ‘power expansion regularization’ and the ‘mode-decomposition regularization’ [17, 18]. In particular, the mode-decomposition regularization was found to be quite powerful. Meanwhile, Barack and Ori independently developed the ‘mode-sum regularization’ [19–21]. Although the mode-decomposition regularization and the mode-sum regularization were quite different in their

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3 This gauge is often called the harmonic gauge in the literature, including our previous papers. But because the meaning of the term ‘harmonic gauge’ is different in the standard post-Newtonian approach, we use the term ‘Lorenz gauge’ throughout this paper in order to avoid possible confusion.
formulations, they are essentially the same in the sense that the spherical harmonic expansion is used to regularize the divergent metric. In fact, it was shown that the two methods give the same result for the S part in the Lorenz gauge, in the form of a (divergent) spherical harmonic series, for an arbitrary orbit in the Schwarzschild background [22].

However, it seems extremely difficult to solve the metric perturbation under the Lorenz gauge because the metric components couple to each other in a complicated way [23]. This is one of the reasons why the gauge problem is difficult to solve. Recently, Barack and Ori [24] gave a useful insight into the gauge problem. They proposed an intermediate gauge approach in which only the direct part of the metric in the Lorenz gauge is subtracted from the full metric perturbation in the Regge–Wheeler (RW) gauge. They then argued that the gauge dependence of the self-force is unimportant when averaged over a sufficiently long lapse of time. Using this approach, the gravitational self-force for an orbit plunging into a Schwarzschild black hole was calculated by Barack and Lousto [25].

In the case of the Kerr background, which is of our ultimate interest, there is additionally a much more intricate technical issue of how to treat the spheroidal harmonics: neither the eigenfunction nor the eigenvalue has simple analytical expressions, and they are entangled with the frequency eigenvalues of Fourier modes. Further, the only known gauge in which the metric perturbation can be evaluated is the radiation gauge formulated by Chrzanowski [12]. However, the Chrzanowski construction of the metric perturbation becomes ill-defined in the neighbourhood of the particle, i.e., the Einstein equations are not satisfied there [24]. Some progress was made by Ori [26] to obtain the correct, full metric perturbation in the Kerr background. The regularization parameters in the mode-sum regularization for the Kerr case are calculated by Barack and Ori [27]. Although our ultimate goal is to overcome these difficulties altogether, since each one is sufficiently involved, we choose to proceed step-by-step, and focus on the Schwarzschild background.

In this paper, in order to obtain the gravitational self-force, we consider a particle orbiting a Schwarzschild black hole, and propose a method to calculate the regularized self-force by solving the subtraction and gauge problems simultaneously. Namely, we develop a method to regularize the self-force in the Regge–Wheeler gauge. The regularization is done by the mode-decomposition regularization, and we introduce the concept of a ‘finite gauge transformation’ of the S part [28, 29]. It is noted that, although our approach is philosophically quite different from the intermediate gauge approach by Barack and Ori [24], practically both approaches turn out to give the same result as far as the calculation of the gauge transformation is concerned.

The paper is organized as follows. In section 2, we review the mode-decomposition regularization. We first derive the direct part of the self-force (which is equivalent to the S part in the coincidence limit of a field point with a point on the particle trajectory) by local analysis. Then we transform it to an infinite series of spherical harmonic modes and derive the regularization counter terms for a general orbit. In section 3, further focusing on a circular orbit, we perform a gauge transformation from the Lorenz gauge to the RW gauge, and present the first post-Newtonian order self-force. One of the reasons why we had to focus on circular orbits is that it is formidable to obtain a closed analytic expression of the full metric perturbation for a general orbit even under the post-Newtonian expansion. In section 4, turning back to a simpler case of the self-force of a scalar-charged particle, we present a new regularization method that can apply to an arbitrary orbit. This new method is a variation (or an improved version) of the mode-decomposition regularization. The essential point is to divide the full field in the frequency domain into two parts, the $\hat{S}$ part and $\hat{R}$ part, in such a way that the divergent part is totally contained in the $\hat{S}$ part whose frequency integral can be analytically performed.
2. Mode-decomposition regularization

In this section, we review the mode-decomposition regularization method, and derive the regularization counter terms for a general orbit in the Schwarzschild background. For simplicity, we consider a scalar-charged particle which sources a scalar field $\phi$. The extension to the gravitational case is straightforward. The only complication is the presence of additional tensor indices associated with the metric perturbation.

Since the tail part depends non-locally on the geometry of a background spacetime, it is almost impossible to calculate it directly. However, for a certain class of spacetimes such as Schwarzschild/Kerr geometries, there is a way to calculate the full field generated by a point source. Considering a field point slightly off the particle trajectory, it is then possible to obtain the tail part by subtracting the locally given divergent direct part from the full field.

Let $x = z(\tau)$ be a trajectory of the particle with $\tau$ being the proper time along the trajectory. Then the field $\phi(x)$ generated by the particle is given by

$$\phi^{\text{full}}(x) = -q \int \, d\tau \, G^{\text{full}}(x, z(\tau)), \quad (1)$$

where $G^{\text{full}}(x, x')$ is the retarded Green function satisfying the Klein–Gordon equation,

$$\nabla^a \nabla_a G^{\text{full}}(x, x') = -\frac{\delta^{(4)}(x - x')}{\sqrt{-g}}. \quad (2)$$

The self-force at $x = z(\tau)$ is schematically given by

$$F_\alpha(\tau_0) = \lim_{x \to z(\tau_0)} F_\alpha[\phi^{\text{tail}}](x), \quad (3)$$

where

$$F_\alpha[\phi^{\text{tail}}](x) = F_\alpha[\phi^{\text{full}}](x) - F_\alpha[\phi^{\text{dir}}](x), \quad (x \neq z(\tau)).$$

The symbols $\phi^{\text{full}}$, $\phi^{\text{dir}}$ and $\phi^{\text{tail}}$ stand for the full field, the direct part and the tail part, respectively. Both $\phi^{\text{full}}$ and $\phi^{\text{dir}}$ diverge in the coincidence limit $x \to z(\tau)$, while $\phi^{\text{tail}}$ is finite in the coincidence limit. $F_\alpha[\cdots]$ is a tensor operator on the field, and is defined as

$$F_\alpha[\phi] = q \, P_\alpha^\beta \nabla_\beta \phi, \quad (4)$$

where $P_\alpha^\beta = \delta_\alpha^\beta + V_\alpha V^\beta$ is the projection tensor with $V^\alpha$ being an appropriate extension of the 4-velocity $v^\alpha(\tau_0)$ off the orbital point. Explicitly, we specify an extension of the 4-velocity off the orbit later by equation (16).

In what follows, we use the usual Schwarzschild coordinates for the Schwarzschild background,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) \, dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} \, dr^2 + r^2(\, d\theta^2 + \sin^2 \theta \, d\phi^2), \quad (5)$$

and denote a field point by $x = \{t, r, \theta, \phi\}$ and a point on the orbit by $z(\tau_0) = z_0 = \{t_0, r_0, \theta_0, \phi_0\}$.

2.1. Basis

Even in the Newtonian limit, the integration of an orbit involves an elliptic function. Thus, numerical computations will be necessary at some stage of deriving the self-force. However, it will not be easy to perform the regularization of the divergence numerically. The idea to

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4 In this section, we consider the direct part instead of the S part, because the direct part has a slightly simpler form for local analysis.
overcome this difficulty is to replace the divergence by an infinite series, each term of which is finite and analytically calculable.

The derivation of the direct part $\varphi^{\text{dir}}(x)$ is one of the main issues in the regularization calculation. The direct part of the scalar field is obtained by integrating the direct part of the retarded Green function with the scalar charge. The direct part of the retarded Green function $G^{\text{dir}}$ is given in a covariant manner as

$$G^{\text{dir}}(x, x') = -\frac{1}{4\pi} \theta[\Sigma(x), x'] \sqrt{\Delta(x, x')} \delta(\sigma(x, x')), \quad (6)$$

where $\sigma(x, x')$ is the biscalar of half the squared geodesic distance, $\Delta(x, x')$ is the generalized van Vleck–Morette determinant, $\Sigma(x)$ is an arbitrary spacelike hypersurface containing $x$, and $\theta[\Sigma(x), x'] = 1 - \theta[x', \Sigma(x)]$ is equal to 1 when $x'$ lies in the past of $\Sigma(x)$ and vanishes when $x'$ lies in the future. The physical meaning of the direct part is understood by the factor $\theta[\Sigma(x), x'] \delta(\sigma(x, x'))$ in equation (6). Since $\sigma(x, x')$ describes the geodesic distance between $x$ and $x'$, the direct part of the Green function becomes non-zero only when $x'$ lies on the past light cone of $x$. Hence the direct part describes the effect of the waves propagated directly from $x'$ to $x$ without being scattered by the background curvature.

For the actual evaluation of the direct part, a couple of methods have been proposed. In [31, 32], the direct part of the field is calculated by picking up a limiting contribution in the full Green function from the light cone as

$$\varphi^{\text{dir}}(x) = \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} dr \ G^{\text{full}}(x, z(\tau)) S(\tau), \quad (7)$$

where $G^{\text{full}}$ is the retarded Green function, $S(\tau)$ is the scalar charge density, and $\tau_{\text{ret}}(x)$ is the retarded time defined by the past light cone condition of the field point $x$ as

$$\theta[\Sigma(x), z(\tau_{\text{ret}})] \delta(\sigma(x, z(\tau_{\text{ret}}))) = 0. \quad (8)$$

However, the calculation seems rather cumbersome when we apply this method to a general orbit.

In [16], the direct part was evaluated using the local bitensor expansion technique. Using the bitensor, the direct part is expanded around the particle location as

$$\varphi^{\text{dir}}(x) = q \left[ \frac{1}{\sigma(x, z(\tau_{\text{ret}})) v(\tau_{\text{ret}})} \right] + O(y^2), \quad (9)$$

where the letters $\mu, \nu, \ldots$ are used for the indices of the field point $x$, $\alpha, \beta, \ldots$ for the indices of the orbital point $z$, and $v^\mu(\tau)$ is the orbital 4-velocity at $z(\tau)$. The order of the local expansion is represented by the powers of $y$, where $y$ is the coordinate difference between the field point $x$ and the orbital point $z_0$; $y^\mu = x^\mu - z_0^\mu$. Here, from the local analysis around the particle location, we find that the full force which includes the direct part is quadratically divergent. Because of this divergence, we must carry out the local bitensor expansion of the full field up through $O(y)$. By evaluating the local coordinate values of the relevant bitensors, we obtain the local expansion of the full force in a given coordinate system. As described in [16], this may be done in a systematic manner, and it is possible to obtain the explicit form of the divergence for a general orbit. However, the problem is how to decompose it into an appropriate infinite series.

### 2.2. Mode-decomposition regularization

The mode-decomposition regularization method is based on the spherical harmonic series expansion [18]. There is a delicate problem in this approach. The exact decomposition
calculation usually needs the global analytic structure of the field so that we can uniquely
define each term in the infinite series. On the other hand, the divergent direct part to be
subtracted is given only by the local analysis of the field \([9, 10, 30, 33]\). Thus the direct part is
defined (and known) only in a local neighbourhood of the particle. Because of this, to obtain
a harmonic series expression of the direct part, we need a global extension of the direct part.
But this extension cannot be done uniquely in practice. Nevertheless, the final result of the
self-force should be unique. Although we have no explicit proof for the uniqueness of the
regularization counterterms for the self-force, the uniqueness is intuitively apparent because
the extension is performed in such a way that the local behaviour is retained with sufficient
accuracy. In addition, it is reassuring to find that the resulting counterterms, presented at the
end of this section, agree completely with those obtained in the mode-sum regularization by
Barack and Ori \([19–22]\).

The harmonic decomposition is defined on a 2-sphere. However, both the direct
field and the full field have divergence on the sphere containing the particle location, the
mode decomposition is ill-defined on that sphere. Therefore, we perform the harmonic
decomposition of the direct and full fields on a sphere which does not include, but is sufficiently
close to, the orbit. The steps in the mode-decomposition regularization are as follows:

1. We evaluate both the full field and the direct field at

\[ x = \{t, r, \theta, \phi\} \]

(10)
where we do not take the coincidence limit of either \(t\) or \(r\).

2. We decompose the full force and direct force into infinite harmonic series as

\[
F\alpha[\varphi_{\text{full}}](x) = \sum_{\ell m} F_{\ell m}^{\alpha}[\varphi_{\text{full}}](x),
\]

(11)

\[
F\alpha[\varphi_{\text{dir}}](x) = \sum_{\ell m} F_{\ell m}^{\alpha}[\varphi_{\text{dir}}](x),
\]

(12)
where \(F_{\ell m}^{\alpha}[\varphi_{\text{full/dir}}](x)\) are expanded in terms of the spherical harmonics \(Y_{\ell m}(\theta, \phi)\) with the
coefficients dependent on \(t\) and \(r\). For the direct part, the harmonic expansion is done by
extending the locally defined direct force over the whole 2-sphere in a way that correctly
reproduces the divergent behaviour around the orbital point \(z_0\) up to the finite term.

3. We subtract the direct part from the full part in each \(\ell, m\) mode to obtain

\[
F_{\ell m}^{\alpha}[\varphi_{\text{tail}}] = (F_{\ell m}^{\alpha}[\varphi_{\text{full}}] - F_{\ell m}^{\alpha}[\varphi_{\text{dir}}]).
\]

(13)
Then, we take the coincidence limit \(x \to z_0\). Here we note that one can exchange the
order of the procedure, i.e., first take the coincidence limit and then subtract, provided the
mode coefficients are finite in the coincidence limit.

4. Finally, by taking the sum over the modes, we obtain the self-force,

\[
F_{\alpha}(\tau_0) = \sum_{\ell m} F_{\ell m}^{\alpha}[\varphi_{\text{tail}}](z_0),
\]

(14)

2.3. Decomposition of the direct part

The advantage of using expression (9) for the direct part is that we have a systematic method
for evaluating it. Here we describe a method to evaluate the bitensors in a general regular
coordinate system.
Before we consider the local expansion in a given coordinate system, we calculate the derivative of equation (9), and derive the direct part of the force with the local bitensor expansion using the equal-time condition,

$$0 = \left[ \frac{d}{d\tau} \sigma(x, z(\tau)) \right]_{\tau = \tau_{eq}(x)}. \quad (15)$$

We define an extension of the 4-velocity off the orbit by

$$V^\alpha(x) := \bar{g}_{\alpha\beta}(x, \bar{z}_{eq}) v^\beta_{eq}; \quad \bar{z}_{eq} = z(\tau_{eq}(x)), \quad v^\alpha_{eq} = \frac{dz^\beta_{eq}}{d\tau} \bigg|_{\tau = \tau_{eq}(x)}, \quad (16)$$

where $\bar{g}_{\alpha\beta}$ is the parallel displacement bivector.

Using the formulae in [9, 30], we have

$$F_\alpha[\psi^{dir}](x) = q \bar{g}^\alpha_{\beta}(x, \bar{z}_{eq}) \frac{1}{\epsilon \kappa} \left( \sigma_{\beta}(x, \bar{z}_{eq}) + \frac{1}{3} \epsilon^2 R_{\alpha\beta\rho\delta}(z_{eq}) v^\beta_{eq} \sigma^\rho(x, z_{eq}) v^\delta_{eq} \right) + O(y), \quad (17)$$

$$\epsilon = \sqrt{2 \sigma(x, z_{eq})}, \quad (18)$$

$$\kappa = -\sigma_{\alpha\beta}(x, z_{eq}) v^\alpha_{eq} v^\beta_{eq} = 1 + \frac{1}{6} R_{\alpha\beta\rho\delta}(z_{eq}) v^\beta_{eq} \sigma^\rho(x, z_{eq}) v^\delta_{eq} + O(y^3). \quad (19)$$

The bitensors necessary for the evaluation of the direct force (17) are $\sigma(x, \bar{x})$ and $\bar{g}_{\alpha\beta}(x, \bar{x})$, for which we consider the local coordinate expansion of these bitensors around the coincidence limit $x \to \bar{x}$. In a general regular coordinate system, $\sigma(x, \bar{x})$ and $\bar{g}_{\alpha\beta}(x, \bar{x})$ can be expanded as

$$\sigma(x, \bar{x}) = \frac{1}{2} \bar{g}_{\alpha\beta}(\bar{x}) y^\alpha_{\beta} + \sum_{n=3,4,\ldots} \frac{1}{n!} A_{\alpha_1^\alpha_2^\ldots^\alpha_n}(\bar{x}) y^{\alpha_1^\ldots^\alpha_n}, \quad (20)$$

$$\bar{g}_{\alpha\beta}(x, \bar{x}) = g_{\alpha\beta}(\bar{x}) + \sum_{n=1,2,\ldots} \frac{1}{n!} R_{\alpha_1 \alpha_2^\ldots^\alpha_n}(\bar{x}) y^{\alpha_1^\ldots^\alpha_n}, \quad (21)$$

$$y^{\alpha_1^\ldots^\alpha_n} = \left( x^{\alpha_1} - \bar{x}^{\alpha_1} \right) \left( x^{\alpha_2} - \bar{x}^{\alpha_2} \right) \ldots. \quad (22)$$

To calculate the reaction force to a particle, it is enough to know the expansion coefficients of $n = 3, 4$ of equation (20) and $n = 1, 2$ of equation (21). These are shown in appendix A of [18]. The local expansion of the force (17) on the Schwarzschild coordinates is quite tedious, though systematic. However, most of the terms give no contribution to the harmonic coefficients in the coincidence limit $x \to z_0$. Below, we shall focus on the terms that are non-vanishing in the coincidence limit.

Without loss of generality, we may assume that the particle is located at $t_0 = \pi/2, \phi_0 = 0$ at time $t_0$. Since the full force is calculated in the form of the Fourier-harmonic expansion and the Fourier modes are independent of the spherical harmonics, we may take the field point to lie on the hypersurface $t = t_0$ in the full force. Hence we may take $t = t_0$ before we perform the local coordinate expansion of the direct force. That is, we consider the local coordinate expansion of the direct force at a point $\{ t_0, r, \theta, \phi \}$ near the particle location $\{ t_0, r_0, \pi/2, 0 \}$. 


The local expansion of the direct force in the Schwarzschild coordinates can be done in such a way that it consists of terms of the form:

\[ R^{\alpha_1 \phi_1 ; \phi_1} = \xi^{2n_1+1} . \]  

\[ (\pi/2) \]  

\[ (23) \]

\[ \xi := \sqrt{2} r_0 \left( a - \cos \theta + \frac{b}{2} (\phi - \phi')^2 \right)^{1/2}, \]

\[ R := r - r_0, \quad \theta := \theta - \frac{\pi}{2}, \]  

\[ (24) \]

\[ (25) \]

where \( n_1, n_2, n_3, n_4 \) are non-negative integers, and \( a, b, \) and \( \phi' \) are defined by:

\[ a := 1 + \frac{1}{2} \frac{r_0^2}{r_0^2 + L^2 (r_0 - 2M)^2} \xi^2 R^2, \]

\[ b := \frac{L^2}{r_0}, \]

\[ \phi' := -r_0^2 \xi^3 \]

\[ (27) \]

\[ (26) \]

where \( \xi := -g_{tt} \frac{dr}{d\tau} , L := g_{\phi \phi} \frac{d\phi}{d\tau} , \) and \( u_r := g_{rr} \frac{dr}{d\tau} , \) and \( \hat{\theta} \) is the relative angle between \( (\theta, \phi) \) and \( (\pi/2, \phi') \) and 

There are two apparently different terms in the covariant form of the direct force given by equation (17); the first term in the curly braces exhibiting the quadratic divergence, and the second term proportional to the curvature tensor that appears to be finite in the coincidence limit. In the local coordinate expansion, the second term will give terms of the form \( R/\xi \) or \( \phi/\xi \). As discussed in section 2.4, the harmonic coefficients of \( R/\xi \) vanish in the coincidence limit, while those of \( \phi/\xi \) are finite but they give no contribution to the final result when the infinite harmonic modes are summed up after the coincidence limit is taken. Hence we may focus on the first term in the curly braces of equation (17).

Since the orbit always remains in the equatorial plane, the force is symmetric under the transformation \( \theta \rightarrow \pi - \theta \), which implies that there is no term proportional to odd powers of \( \Theta \). Hence we only need to consider the case of \( n_2 \) being an even number in the general form given by equation (23). Then the factor \( \Theta^{n_2} \) may be eliminated by expressing \( \Theta^3 \) in terms of \( \xi, R \) and \( \phi \), and we are left with terms of the form

\[ R^{\alpha_1 \phi_1} = \xi^{2n_1+1} . \]  

Explicitly, we find:

\[ F^\text{dir}_\tau = q \left( \frac{E u_r + E \xi^2 \phi}{\xi^{3/2}} + \frac{1}{2} \frac{(r_0 - 2M) E u_r}{r_0^2} \frac{1}{\xi^{1/2}} + \frac{2(r_0 - 2M) \xi^2 u_r}{r_0^2} \frac{\phi^2}{\xi^{3/2}} \right) + \frac{3}{2} \frac{(r_0 - 2M) \xi^2 u_r}{r_0^2} \frac{\phi^4}{\xi^{5/2}} \right), \]  

\[ F^\text{dir}_r = q \left( \frac{E u_r + E \xi^2 \phi}{\xi^{3/2}} + \frac{1}{2} \frac{(r_0 - 2M) E u_r}{r_0^2} \frac{1}{\xi^{1/2}} + \frac{2(r_0 - 2M) \xi^2 u_r}{r_0^2} \frac{\phi^2}{\xi^{3/2}} \right) + \frac{3}{2} \frac{(r_0 - 2M) \xi^2 u_r}{r_0^2} \frac{\phi^4}{\xi^{5/2}} \right), \]  

\[ (31) \]
Self-force regularization

\[ F^\text{dir}_a = 0, \]

\[ F^\text{dir}_\phi = q \left( -L u_r \frac{R}{\xi^3} - (r_0^2 + L^2) \frac{\phi}{\xi^3} + \frac{1}{2} \frac{(r_0 - 2M)}{r_0^2} \frac{1}{\xi} \right. \]

\[ \left. - \frac{1}{2} \frac{(r_0 - 2M)(r_0^2 + 4L^2)}{r_0^2} \frac{\phi^2}{\xi^3} + \frac{3}{2} \frac{(r_0 - 2M)(r_0^2 + L^2)}{r_0^2} \frac{\phi^4}{\xi^5} \right). \]

where \( F^\text{dir}_a = F_a[\psi^\text{dir}] \). The absence of \( F^\text{dir}_a \) is because of the symmetry of the background.

2.4. Regularization counterterms

What we have to do now is to perform the harmonic decomposition of the components of the direct force given above. To do so, we note the following important fact. Apart from the trivial multiplicative factor of \( \mathcal{R}^n \) which is independent of the spherical coordinates, the terms to be expanded in the spherical harmonics are of the form \( \phi^n / \xi^{2n+1} \), or \( (\phi - \phi')^n / \xi^{2n+1} \). To the order of accuracy we need, the factor \( (\phi - \phi')^n \) may be eliminated by replacing it with an equivalent \( \phi \)-derivative operator of degree \( n_3 \) acting on \( \xi^{2n_3 - 2n_4 - 1} \), which is further converted to a polynomial in \( m \) after the harmonic expansion of \( \xi^{2n_4 - 1} \). Thus the only basic formula we need is the harmonic expansion of \( \xi^{2p-1} \) where \( p \) is an integer. Note that, apart from the term \( b(\phi - \phi')^2 / 2 \) in \( \xi \) with respect to which we expand \( \xi \) in a convergent infinite series, \( \xi^{2p-1} \) is defined over the whole sphere to allow the straightforward harmonic decomposition. The result to the leading order in the coincidence limit \( a \to 1 + 0 \) is

\[ \left( \frac{\xi}{\sqrt{2r_0}} \right)^{2p-1} = \left( a - \cos \theta + \frac{b}{2}(\phi - \phi')^2 \right)^{p-1/2} \]

\[ = 2\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} D^{p-1/2}_{\ell m}(a) Y_{\ell m}(\theta, \phi) Y_\ell^*(\theta', \phi'), \]

\[ D^{p-1/2}_{\ell m}(a) \rightarrow \begin{cases} 
\frac{1}{\sqrt{1+b}} - p - 1/2 (a - 1)^{p+1/2}, & \text{for } p + \frac{1}{2} < 0, \\
(-1)^{\ell(p+1/2)} \frac{1}{\sqrt{1+b}} \sum_{n=0}^{\infty} K_n, & \text{for } p + \frac{1}{2} > 0; 
\end{cases} \]

\[ K_n = \frac{\Gamma(p + 1/2)\Gamma(p + n + 1/2)}{\Gamma(p + n - \ell + 1/2)\Gamma(p + n + \ell + 3/2) n!} \left( \frac{-m^2b}{1+b} \right)^n. \]

We note that, although what we need here is only the case of an integer \( p \), the above formula is valid for any \( p \) (except for the case \( p = -1/2 \)).

After the decomposition, we can take the radial coincidence limit \( r \to r_0 \) (followed by the angular coincidence limit if desired). Here we briefly explain the reason why the terms proportional to \( R/\xi \) and \( \phi/\xi \) give no contribution to the final result. The term \( R/\xi \) corresponds to \( R \) times the case of \( p = 0 \), for which \( D^{p-1/2}_{\ell m} \) is finite in the limit \( a \to 1 \) (i.e., \( R \to 0 \)). Hence all the coefficients vanish in the radial coincidence limit. As for \( \phi/\xi \), it can be replaced by \( (\phi - \phi')/\xi \) which is equivalent to \( \partial_\theta \xi \) in the coincidence limit. This corresponds to the case of \( p = 1 \) multiplied by \( m \). Hence all the harmonic coefficients become odd functions of \( m \), and their sum over \( m \) for each \( \ell \) vanishes in the angular coincidence limit. As a result, the non-vanishing contribution comes only from the terms \( R/\xi^3, \phi/\xi^3, 1/\xi, \phi^2/\xi^3 \) and \( \phi^4/\xi^5 \).
Barack and Ori define the regularization counterterms as
\[
\lim_{x \to z_0} F_{\text{dir}}^\alpha l = A_\alpha L + B_\alpha + C_\alpha/L + O(L^{-2}).
\]
(36)
\[
D_\alpha = \sum_{l=0}^\infty \left[ \lim_{x \to z_0} F_{\text{dir}}^\alpha l - A_\alpha L - B_\alpha - C_\alpha/L \right]
\]
(37)
where \( F_{\text{dir}}^\alpha l \) is the multipole \( l \) mode of \( F_{\text{dir}}^\alpha \), \( L = \ell + 1/2 \), and \( A_\alpha \), \( B_\alpha \) and \( C_\alpha \) are independent of \( L \). The \( A_\alpha \) term is to subtract the quadratic divergence, the \( B_\alpha \) term is the linear divergence, and the \( C_\alpha \) term is the logarithmic divergence. The \( D_\alpha \) term is the remaining finite contribution of the direct force to be subtracted. We find \( C_\alpha = D_\alpha = 0 \) in agreement with Barack and Ori. We also find the complete agreement of \( A_\alpha \) and \( B_\alpha \) terms with their results for a general geodesic orbit as given below. We call these features of regularization counterterms the standard form.

The \( A \) term describes the quadratic divergent terms of the direct force. It comes from the terms \( R/\xi^3 \) in equations (30)–(33). The important fact is that it is odd in \( R \). This leads to the harmonic coefficients proportional to \( \text{sign}(R) \).

\[
A_t = \text{sign}(R) \frac{q^2 r_0 - 2M}{r_0^3} \frac{u_r}{1 + L^2/r_0^2},
\]
(38)
\[
A_r = -\text{sign}(R) \frac{q^2 r_0}{r_0^3} \frac{e}{1 + L^2/r_0^2},
\]
(39)
\[
A_\phi = 0.
\]
(40)
These \( A \) terms vanish when averaged over both limits \( R \to \pm 0 \).

The \( B \) term describes linearly divergent terms, which are of the form \( \phi^{2n}/\xi^{2n+1} \) in equations (30)–(33). We find the \( B \) term in terms of the hypergeometric functions as

\[
B_t = -\frac{(r_0 - 2M)e u_r}{2r_0^3} \left( \frac{\text{F}}{\frac{3}{2}, \frac{3}{2}; 1; -\frac{L^2}{r_0^2}} \right),
\]
(41)
\[
B_r = \frac{(r_0 - 2M)u_r^2}{2r_0^3} \left( \frac{\text{F}}{\frac{3}{2}, \frac{3}{2}; 1; -\frac{L^2}{r_0^2}} - \frac{1}{2r_0^3} \left( \text{F} \left( \frac{1}{2}, \frac{1}{2}; \frac{1}{2}; -\frac{L^2}{r_0^2} \right) + \frac{L^2}{2r_0^2} \text{F} \left( \frac{3}{2}, \frac{3}{2}; 2; -\frac{L^2}{r_0^2} \right) \right) \right),
\]
(42)
\[
B_\phi = \frac{(r_0 - 2M)u_r}{16r_0^3} \left( 8 \text{F} \left( \frac{3}{2}, \frac{3}{2}; 1; -\frac{L^2}{r_0^2} \right) - 4 \text{F} \left( \frac{3}{2}, \frac{3}{2}; 2; -\frac{L^2}{r_0^2} \right) \right) + \frac{9L^2}{r_0^5} \text{F} \left( \frac{5}{2}, \frac{5}{2}; 3; -\frac{L^2}{r_0^2} \right).
\]
(43)
As mentioned at the beginning of this section, the above results for the \( A \) and \( B \) terms perfectly agree with the results obtained by Barack and Ori in a quite different fashion [20–22].

3. Gravitational self-force in the Regge–Wheeler gauge

Recently, Detweiler and Whiting found a slight but important modification of the direct and tail parts of the metric perturbation [11]. The full metric perturbation in the Lorenz gauge is
now decomposed as
\[ h^{\text{full},L}_{\mu\nu} = h^{S,L}_{\mu\nu} + h^{R,L}_{\mu\nu}, \]  
(44)
where the superscript ‘L’ stands for the Lorenz gauge, and the S part, \( h^{S,L}_{\mu\nu} \), satisfies the same linearized Einstein equations with source under the Lorenz gauge as the full metric perturbation, \( \Box h^{S,L}_{\mu\nu} + 2 R^\alpha_{\mu \nu} \bar{h}^{S,L}_{\alpha\beta} = -16\pi T_{\mu\nu} \),
(45)
where \( \bar{h}^{S,L}_{\alpha\beta} = h^{S,L}_{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} h^{\text{full},L}_{\mu\mu} \). The new tail part, called the R part, \( h^{R,L}_{\mu\nu} \), is then a homogeneous solution\(^5\). Detweiler and Whiting showed that the properly regularized self-force is given by the R part of the metric perturbation. Namely, we have
\[ \frac{d^2 z^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu} \frac{dz^\mu}{d\tau} \frac{dz^\nu}{d\tau} = \frac{1}{\mu^2} F^\alpha_{\mu}[h^{R,L}], \]  
(46)
where \( z^\alpha(\tau) \) is an orbit of the particle parametrized by the background proper time (i.e., \( g_{\mu\nu} \frac{dz^\mu}{d\tau} \frac{dz^\nu}{d\tau} = -1 \)), and
\[ F^\alpha_{\mu} = -\mu P^\beta_{\mu,\beta} \left( h_{\gamma\delta,\epsilon,\beta}^{R,L} - \frac{1}{2} \bar{h}^{\gamma\delta,\epsilon,\beta} + \frac{1}{2} \bar{h}^{\gamma\delta,\epsilon,\beta} \right) u^\mu u^\beta, \]  
(47)
where \( P^\beta_{\mu,\beta} = \delta^\beta_{\alpha} + u^\alpha u^\beta \) and \( u^\alpha = \frac{dz^\alpha}{d\tau} \).

Since the R part is a solution of the Einstein equations, it is perfectly legitimate to consider a gauge transformation of it to another gauge. Therefore we can define the R part in an arbitrary gauge by a gauge transformation of this part from the Lorenz gauge to that gauge. We thus have an unambiguous definition of the self-force in an arbitrary gauge. In particular, we may consider the self-force in the Regge–Wheeler (RW) gauge, in which we may be able to obtain the full metric perturbation.

In this section, we formulate a method to obtain the self-force in the RW gauge by applying the mode-decomposition regularization. Then, as a simple but non-trivial example, we consider a circular orbit and derive the regularized self-force in the RW gauge to 1PN order\(^2\).\(^9\)

### 3.1. Gauge transformation to the RW gauge

The gravitational self-force acting on the particle is given by the R part in the Lorenz gauge, formally obtained as
\[ F^L_{\alpha}(\tau) = \lim_{x \to z(\tau)} F^\alpha_{\mu}[h^{R,L}_{\mu\nu}] (x) \]
\[ = \lim_{x \to z(\tau)} F^\alpha_{\mu}[h^{\text{full},L}_{\mu\nu} - h^{S,L}_{\mu\nu}] (x) \]
\[ = \lim_{x \to z(\tau)} (F^\alpha_{\mu}[h^{\text{full},L}_{\mu\nu}] (x) - F^\alpha_{\mu}[h^{S,L}_{\mu\nu}] (x)). \]
(48)
The S part can be calculated by the method discussed in the previous section. It is important to note that the S and R parts independently satisfy the Lorenz gauge condition.

We consider the gauge transformation of the R part of the metric perturbation from the Lorenz gauge to the RW gauge,
\[ x^\mu_{\mu} \to x^\mu_{\mu}^{\text{RW}} = x^\mu_{\mu} + \xi^\mu_{\mu} \to ^{\text{RW}}[h^{R,L}_{\mu\nu}], \]
(49)
\[ h^{R,L}_{\mu\nu} \to h^{R,\text{RW}}_{\mu\nu} = h^{R,L}_{\mu\nu} - 2 \nabla^\mu \xi_{\mu,\nu}^{\text{RW}}[h^{R,L}_{\mu\nu}], \]
(50)
where \( \xi^\mu_{\mu} \to ^{\text{RW}} \) is the generator of the gauge transformation.

\(^5\) When we consider the R part of the metric perturbation by using the tensor-harmonic expansion mentioned below, the \( \ell = 0 \) and 1 modes are pure gauge modes because the R part is a homogeneous solution.
It is noted that Barack and Ori [24] proposed the use of an intermediate gauge in which the singular part of the metric takes the same form as that in the Lorenz gauge. Namely, the singular part to be subtracted from the full force is always the one in the Lorenz gauge. In the same paper, they considered the self-force in the RW gauge as an example. In the case of a point particle moving along a strictly radial free-fall orbit, they successfully obtained the regularized self-force [24, 25]. But, in the case of a circular orbit, they found that the gauge vector for the full metric perturbation is discontinuous at the location of the particle, and concluded that the self-force is ill-defined in the RW gauge.

Contrary to this argument by Barack and Ori that the self-force in the RW gauge is ill-defined, we do not have this problem in our case because what we consider is the gauge transformation of the R part of the metric perturbation, which is a regular, homogeneous solution of the perturbed Einstein equations. Hence the self-force under the RW gauge is well defined. As we shall see immediately below, in the language of the intermediate gauge approach of Barack and Ori [24], this is technically equivalent to allowing a singular gauge transformation. In this sense, performing a gauge transformation of the R part may be regarded as a method for defining a transformation of the singular part uniquely, up to non-singular, possible residual gauge degrees of freedom.

From the gauge transformation in equation (50), the self-force in the RW gauge is given by

\[
F_{\alpha}^{\text{RW}}(\tau) = \lim_{x \to z(\tau)} F_{\alpha}[h^{\text{R,L}}[h^{\text{R,L}}]](x) = \lim_{x \to z(\tau)} F_{\alpha}[h^{\text{full,L}}[h^{\text{full,L}} - 2\nabla \xi^{\text{L,RW}}[h^{\text{full,L}} - h^{\text{S,L}}]](x) = \lim_{x \to z(\tau)} (F_{\alpha}[h^{\text{full,RW}}[h^{\text{S,L}} - 2\nabla \xi^{\text{L,RW}}[h^{\text{S,L}}]](x) - F_{\alpha}[h^{\text{S,L}} - 2\nabla \xi^{\text{L,RW}}[h^{\text{S,L}}]](x)),
\]

where we have omitted the spacetime indices of \( h_{\mu\nu} \) and \( \nabla_{(\mu} \xi_{\nu)} \) for notational simplicity, and we assume that any singular quantity appearing in the above manipulation is properly regularized by a certain method (the mode-by-mode decomposition in our case). The full metric perturbation \( h^{\text{full,RW}} \) can be calculated by using the Regge–Wheeler–Zerilli formalism, while the S part \( h^{\text{S,L}} \) can be obtained with sufficient accuracy by the local analysis near the particle location. Thus provided that the gauge transformation of the S part can be unambiguously defined, we will be able to obtain the regularized self-force in the RW gauge. Namely, the following gauge transformation should be uniquely determined:

\[
\xi^{\text{S,L,RW}}_a \equiv \xi^{\text{L,RW}}_a [h^{\text{S,L}}_a],
\]

Provided that this is the case, the singular part of the full metric perturbation in the RW gauge will be unambiguously subtracted and we obtain a well-defined self-force in the RW gauge.

We note that the self-force (51) is almost identical to the expression obtained in the intermediate gauge approach [24], if we replace the S and R parts by the direct and tail parts, respectively. The only formal difference is that the S and R parts are now solutions of the inhomogeneous and homogeneous Einstein equations, respectively. But this formal difference makes a big difference because we can now consider the gauge transformation for the R part of the metric perturbation. For our method to be consistent, technically, the problem is whether the S part under the RW gauge is uniquely determined. As will be shown later in equations (80), this turns out to be indeed the case. Therefore one may identify the
self-force (51) to be actually the one evaluated in the RW gauge [28], not in some intermediate gauge.

It is also noted that the RW gauge condition uniquely determines only the \( \ell \geq 2 \) modes of the metric perturbation. Thus there remain gauge degrees of freedom in the \( \ell = 0 \) and 1 modes. On the other hand, because of the fact that the regularized self-force is derived from the \( R \) part of the metric perturbation, the \( \ell = 0 \) and 1 modes of the regularized self-force must be pure gauge modes. This latter fact seems to imply that it is not necessary to consider the \( \ell = 0 \) and 1 modes. However, it is not quite so because we need the \( R \) part to satisfy the asymptotic flatness at infinity. In particular, this implies that we should recover the standard Newton force at the Newtonian order [37]. The issues on the \( \ell = 0 \) and 1 modes will be discussed in detail in section 3.5.

3.2. Full force for circular orbits

We consider the full metric perturbation and its self-force in the case of a circular orbit. Hereafter, we will need to consider the case of general, non-circular orbits. However, because of several technical problems, which are not particular to the gravitational self-force but common to all types of the self-force, we leave it for future work. A promising method to deal with general, non-circular orbits will be presented later in section 4, in which the case of scalar self-force is discussed for simplicity. An extension to the gravitational case is under progress [45].

First, the metric perturbation is calculated by the Regge–Wheeler–Zerilli formalism in which the metric is expanded in terms of Fourier series in frequency \( \omega \) and tensor spherical harmonics with eigenvalue indices \((\ell, m)\) by using the symmetry of the background spacetime. Then, we derive the Fourier-harmonic components of the self-force. As noted earlier, although the full self-force diverges on the orbit of the particle, each Fourier-harmonic component is finite. We may even sum over \( \omega \) and \( m \) and the result is still finite. Thus we formally express the full self-force in the coincidence limit \( x \to z(\tau) \) as

\[
F^\alpha[\h_{\text{full}}] = \lim_{x \to z(\tau)} \sum_{\ell=0}^{\infty} F^\alpha_{\ell}(r_0),
\]

where \( r = r_0 \) is the radius of a circular orbit.

On the Schwarzschild background, the metric perturbation \( h_{\mu\nu} \) can be expanded in terms of tensor harmonics as

\[
\begin{align*}
\mathbf{h} = \sum_{\ell m} & \left[ f(r) H_{00\ell m}(t, r) \mathbf{a}_{\ell m}^{(0)} - i \sqrt{2} H_{11\ell m}(t, r) \mathbf{a}_{\ell m}^{(1)} + \frac{1}{f(r)} H_{22\ell m}(t, r) \mathbf{a}_{\ell m} \right] \\
& - \frac{i}{r} \sqrt{2} (\ell + 1) H_{00\ell m}(t, r) \mathbf{b}_{\ell m}^{(0)} + \frac{1}{r} \sqrt{2} (\ell + 1) H_{11\ell m}(t, r) \mathbf{b}_{\ell m}^{(1)} \\
& + \frac{1}{2} \ell (\ell + 1) (\ell - 1) (\ell + 2) G_{\ell m}(t, r) \mathbf{f}_{\ell m} \\
& + \left( \sqrt{2} K_{\ell m}(t, r) - \frac{\ell (\ell + 1)}{\sqrt{2}} G_{\ell m}(t, r) \right) \mathbf{g}_{\ell m} \\
& - \frac{\sqrt{2} (\ell + 1)}{r} H_{00\ell m}(t, r) \mathbf{c}_{\ell m}^{(0)} + \frac{i \sqrt{2} (\ell + 1)}{r} H_{11\ell m}(t, r) \mathbf{c}_{\ell m}^{(1)} \\
& + \frac{\sqrt{2} (\ell + 1) (\ell - 1) (\ell + 2)}{2r^2} H_{22\ell m}(t, r) \mathbf{d}_{\ell m} \right],
\end{align*}
\]
The new radial function $R^{(od)}(r)$ is given by

$$h_{1}\ell m_{\omega} = \frac{r^2}{r - 2M} R^{(od)}_{\ell m}\omega.$$

The new radial function $R^{(od)}_{\ell m}\omega(r)$ satisfies the Regge–Wheeler equation,

$$\frac{d^2}{dr^2} R^{(od)}_{\ell m}\omega + [\omega - V_{\ell}(r)] R^{(od)}_{\ell m}\omega = \frac{8\pi i}{(r - 2M)^2} \left[ \frac{\ell(\ell + 1)(\ell - 1)(\ell + 2)}{2} \right]^{1/2} \frac{r - 2M}{r^2}$$

$$\times \left( -r^2 \frac{d}{dr} \left[ 1 - \frac{2M}{r} \right] D_{\ell m}\omega \right) + (r - 2M)[(\ell - 1)(\ell + 2)]^{1/2} Q_{\ell m}\omega(r),$$

where $r^* = r + 2M \log(r/2M - 1)$, and the potential $V_{\ell}(r)$ is given by

$$V_{\ell}(r) = \left( 1 - \frac{2M}{r} \right) \left( \frac{\ell(\ell + 1)}{r^2} - \frac{6M}{r^3} \right).$$

The source term $Q_{\ell m}\omega(r)$ vanishes in the case of a circular orbit and

$$D_{\ell m}\omega(r) = \left( \frac{\ell(\ell + 1)(\ell - 1)(\ell + 2)}{2} \right)^{-1/2} \mu u^\prime \delta(r - r_0) m \partial_{\theta} Y_{\ell m}(\theta_0, \phi_0),$$

where the orbit is given by

$$z^\theta(\tau) = \left\{ u^\prime \tau, r_0, \frac{\pi}{2}, u^\prime \theta \right\}, \quad u^\prime = \sqrt{\frac{r_0}{r_0 - 3M}}, \quad u^\theta = \frac{1}{r_0} \sqrt{\frac{M}{r_0 - 3M}},$$

where $\Omega = u^\theta/u^\prime = \sqrt{M/r_0}$ which is the orbital frequency. The orbit is assumed to be on the equatorial plane without loss of generality.

For even parity modes with the parity $(-1)^l$, we introduce a new radial function $R^{(o)}_{\ell m}(r)$ in terms of which the four radial functions of the metric perturbation are expressed as (see also [29] for $h_{0\ell m o}$, $H_{1\ell m o}^{RW}$, $H_{0\ell m o}^{RW}$)

$$K_{\ell m}\omega = \frac{r^2}{r^2 + 3M} \frac{3M r + 6M^2}{r^2(r + 3M)} R_{\ell m}\omega + \frac{r - 2M}{r} \frac{d}{dr} R_{\ell m}\omega$$

$$= \frac{r(r - 2M)}{r^2 + 3M} \frac{i(r - 2M)^2}{r^2 + 3M} \frac{r^2}{r^2 + 3M},$$

where $a_{\ell m 0, a_{\ell m} \ldots}$ are the tensor harmonics introduced by Zerilli [14]. The energy–momentum tensor of a point particle takes the form

$$T^{\mu\nu} = \mu \int_{-\infty}^{\infty} \delta^{(4)}(x - z(\tau)) \frac{d\mu}{d\tau} \frac{d\nu}{d\tau} d\tau$$

$$= \mu \frac{1}{\mu^r} u^\mu u^\nu \delta(r - r_0(\tau)) \delta(\Omega - \Omega_0(\tau)).$$

The RW gauge is defined by the conditions on the metric perturbation as

$$h_{2}^{RW} = h_{1}^{RW} = h_{1}^{GRW} = G^{RW} = 0.$$
where \( \lambda = (\ell - 1)(\ell + 2)/2 \) and the local source terms are given by

\[
\hat{B}_{\ell m\omega} = \frac{8\pi r^2 (r - 2M)}{\lambda r + 3M} \left\{ A_{\ell m\omega} + \left[ \frac{1}{2} \ell (\ell + 1) \right]^{-1/2} B_{\ell m\omega} \right\} - \frac{4\pi \sqrt{2}}{\lambda r + 3M} \frac{Mr}{\omega} A_{(1)\ell m\omega} \\
\hat{C}_{1\ell m\omega} = \frac{8\pi}{\sqrt{2} \omega} A_{(1)\ell m\omega} + \frac{1}{r} \frac{B_{\ell m\omega}}{r - 2M} - 16\pi r \left[ \frac{1}{2} \ell (\ell + 1)(\ell - 1)(\ell + 2) \right]^{-1/2} F_{\ell m\omega}, \\
\hat{C}_{2\ell m\omega} = -\frac{8\pi r^2 \left[ \frac{1}{2} (\ell + 1) \right]^{-1/2}}{\omega} B^{(0)}_{\ell m\omega} - \frac{ir}{r - 2M} \frac{B_{\ell m\omega}}{r - 2M} \\
+ \frac{16\pi ir^3}{r - 2M} \left[ \frac{1}{2} \ell (\ell + 1)(\ell - 1)(\ell + 2) \right]^{-1/2} F_{\ell m\omega}. 
\] (63)

Here the harmonic coefficients of the source terms \( A_{\ell m\omega}, A_{(1)\ell m\omega} \) and \( B_{\ell m\omega} \) vanish in the circular case and

\[
B^{(0)}_{\ell m\omega} = \left[ \frac{\ell (\ell + 1)}{2} \right]^{-1/2} \mu u^p \left( 1 - \frac{2M}{r} \right) \frac{1}{r} \delta(r - r_0) m Y^*_{\ell m}(\theta_0, \phi_0), \\
F_{\ell m\omega} = \frac{1}{2} \left[ \frac{\ell (\ell + 1)(\ell - 1)(\ell + 2)}{2} \right]^{-1/2} \mu \left( \frac{u^p}{u^l} \right)^2 \frac{1}{r} \delta(r - r_0) \lambda (\ell + 1) - 2m^2) Y^*_{\ell m}(\theta_0, \phi_0). 
\] (64)

The new radial function \( R^{(Z)\ell}_{\ell m\omega}(r) \) obeys the Zerilli equation,

\[
\frac{d^2}{dr^2} R^{(Z)\ell}_{\ell m\omega} + \left[ \omega^2 - V^{(Z)}_{\ell}(r) \right] R^{(Z)\ell}_{\ell m\omega} = S^{(Z)\ell}_{\ell m\omega}, 
\] (65)

where

\[
V^{(Z)}_{\ell}(r) = \left( 1 - \frac{2M}{r} \right) \frac{2\ell (\ell + 1)r^3 + 6\lambda^2 M^2 r^2 + 18\lambda M^2 r + 18M^3}{r^3(\lambda r + 3M)^2}, 
\] (66)

and

\[
S^{(Z)\ell}_{\ell m\omega} = -i \frac{r - 2M}{r} \frac{d}{dr} \left[ \frac{r - 2M}{r(\lambda r + 3M)} \left( \frac{ir^2}{r - 2M} \hat{C}_{1\ell m\omega} + \hat{C}_{2\ell m\omega} \right) \right] + \frac{(r - 2M)^2}{r^2(\lambda r + 3M)^2} \frac{r}{(r - 2M)} \hat{C}_{1\ell m\omega} \\
\times \left[ \lambda (\ell + 1)^2 + 3\lambda M r + 6M^2 \hat{C}_{2\ell m\omega} + \frac{\lambda r^2 - 3\lambda M r - 3M^2}{(r - 2M)} \hat{C}_{1\ell m\omega} \right]. 
\] (67)

The Zerilli equation can be transformed to the Regge–Wheeler equation by the Chandrasekhar transformation [35] if desired.

The homogeneous solutions of the Regge–Wheeler equation are discussed in detail by Mano et al [36]. By constructing the retarded Green function from the homogeneous solutions with appropriate boundary conditions, namely, the two independent solutions with the ingoing and up-going wave boundary conditions, we can solve the Regge–Wheeler and Zerilli equations to obtain the full metric perturbation in the RW gauge. Here, we consider the radial functions up to the first post-Newtonian (1PN) order.

The radial function for the odd part of the metric perturbation is obtained for \( r < r_0 \) as

\[
R^{(odd)\ell}_{\ell m\omega}(r) = \frac{16i\pi \mu \Omega^2 m r}{(2\ell + 1)\ell (\ell + 1)(\ell + 2)} \left( \frac{r}{r_0} \right) \omega Y^*_{\ell m}(\theta_0, \phi_0). 
\] (68)
where $\Omega = u^\phi /u^r$ (see [29] for $r > r_0$). For the even part, the radial function is obtained for $r < r_0$ (see also [29] for $r > r_0$) as

$$R_{\ell m}^{(Z)} = \frac{8\Omega m\pi u^\mu u_\mu}{(2\ell + 1)(\ell + 2)(\ell + 1)\omega} \left( \frac{r}{r_0} + 2 \frac{(\ell^2 - 2\ell - 1)Mr}{(\ell - 1)r_0^2} \right) \left( \frac{r}{r_0} \right) \frac{r^3}{(2\ell + 3)r_0} + \frac{r}{(2\ell - 1)(\ell - 1)} \omega^2 \right) \left( \frac{r}{r_0} \right)^{\ell} \frac{2(\ell^3 + 6\ell^2 - 4\ell - 4)M}{\ell(\ell - 1)(\ell + 2)r_0} Y_{\ell m}^*(\theta_0, \phi_0). \quad (69)$$

The metric perturbation in the RW gauge is obtained from equations (57) and (62).

Next, we calculate the self-force. We apply the ‘mode-decomposition regularization’ method, in which the force is decomposed into harmonic modes and subtract the harmonic-decomposed S part mode-by-mode before the coincidence limit $x \rightarrow z(\tau)$ is taken. Since the orbit under consideration is circular, the source term contains the factor $\delta(\omega - m\Omega)$, and the frequency integral can be trivially performed. Hence we can calculate the harmonic coefficients of the full metric perturbation in the time domain. This is a great advantage of the circular orbit case, since the S part can be given only in the time domain. We also note that the $\theta$-component of the force vanishes because of the symmetry, and $F^\theta = [(r_0 - 2M)/(r_0^2\Omega)] F^\theta$ for a circular orbit.

Performing the summation over $m$, we find in the end

$$F_{\ell}^{\text{full,RW}} = \left. F_{\ell}^{\text{full,RW}} \right| = \left. F_{\ell}^{\text{full,RW}} \right| = 0,$$

$$F_{\ell}^{(\pm)} = - \left( \frac{\ell + 1}{\ell} \right) \mu^2 \left( 12\ell^2 + 25\ell^2 \right) + \frac{1}{2} r_0^2 (2\ell + 3)(2\ell - 1) M,$$

$$F_{\ell}^{(\pm)} = \left. \frac{\ell + 1}{\ell} \mu^2 \left( 12\ell^2 + 11\ell^2 - 10\ell + 12 \right) M. \quad (70)$$

We see that the only non-vanishing component is the radial component as expected because there is no radiation reaction effect at 1PN order. In the above, the index $(\pm)$ denotes that the coincidence limit is taken from outside ($r > r_0$) of the orbit, and $(\pm)$ from inside ($r < r_0$) of the orbit, and the vertical bar with $\ell, \ldots | \ell$, denotes the coefficient of the $\ell$ mode in the coincidence limit. We note that the above result is valid for $\ell \geqslant 2$. There are some complications with $\ell = 0$ and 1 modes, and they need to be treated separately. The details are discussed in [29, 37]. Here we focus on the modes $\ell \geqslant 2$.

### 3.3. The S part in the Lorenz gauge

In this subsection, we present the S part of the metric perturbation and its self-force by using the local coordinate expansion. It may be noted that if we simply apply the scalar harmonic expansion to each component of the self-force, we obtain exactly the same regularization counterterms as in the case of a scalar charge discussed in section 2. However, here we consider the tensor-harmonic expansion. So, each $\ell$-component may be different from the scalar case.

The S part of the metric perturbation in the Lorenz gauge is given covariantly as

$$\tilde{h}_{\mu \nu}^{\text{S,\ell}}(x) = 4\mu \left[ \tilde{h}_{\mu \nu}(x, z_{\text{ret}}) \tilde{h}_{\rho \sigma}(x, z_{\text{ret}}) u^\rho (\tau_{\text{ret}}) u^\sigma (\tau_{\text{ret}}) \right] \delta_{\nu}^{\text{ret}} (\pi(x, z_{\text{ret}})) + 2\mu (\tau_{\text{ret}} - \tau_{\text{ret}}) \tilde{h}_{\mu \nu}(x, z_{\text{ret}}) \tilde{h}_{\rho \sigma}(x, z_{\text{ret}}) R_{\nu \mu \rho \sigma}(z_{\text{ret}}) u^\rho (\tau_{\text{ret}}) u^\sigma (\tau_{\text{ret}}) + O(\gamma^2), \quad (71)$$
where $\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h_{\alpha\beta}$ is the trace-reversed metric perturbation, $z_{ret} = z(\tau_{ret})$, $\tau_{ret}$ and $\tau_{adv}$ are the retarded proper time and advanced proper time, respectively, defined by the cross section of the past and future light cones of the field point $x$ with the orbit. As in section 2, $\tilde{g}_{\mu\nu}$ is the parallel displacement bivector, and $y$ is the difference of the coordinates between $x$ and $z_0$, $y^\mu = x^\mu - z_0^\mu$. Details of the local expansion are given in [18]. The difference between the S part and the direct part appears in the terms of $O(y)$, i.e., the second term on the right-hand side of equation (71). It turns out, however, that this term does not contribute to the self-force.

The local coordinate expansion of the S part is then performed in exactly the same manner as in section 2, with a suitable extension of local quantities over the whole sphere. It is, in principle, possible to calculate the harmonic coefficients of the extended S part exactly. However, it is neither necessary nor meaningful because the extended S part is only approximate. In fact, corresponding to the fact that all the terms in positive powers of $y$ vanish in the coincidence limit, it is known that all the terms of $O(1/L^2)$ or higher, where $L = \ell + 1/2$, vanish when summed over $\ell$ [18] in the Lorenz gauge. It should be noted, however, that this result is obtained by expanding the force in the scalar spherical harmonics.

In our present analysis, we employ the tensor spherical harmonic expansion. So, the meaning of the index $\ell$ is different. Nevertheless, the same is found to be true. Namely, by expanding the S part of the metric perturbation in the tensor spherical harmonics, in the Lorenz gauge the S force is found to have the form

$$F^{(\pm)}_{\ell}(\tau) = \pm A^\ell L + B^\ell + D^\ell_\ell,$$

(72)

where $A^\ell$ and $B^\ell$ are independent of $\ell$, and the $\pm$ denotes that the limit to $r_0$ is taken from the greater or smaller value of $r$, and

$$D^\ell_\ell = \frac{q^\ell}{L^2 - 1} + \frac{e^\ell}{(L^2 - 1)(L^2 - 4)} + \frac{f^\ell}{(L^2 - 1)(L^2 - 4)(L^2 - 9)} + \cdots.$$

(73)

Then, the summation of $D^\ell_\ell$ over $\ell$ (from $\ell = 0$ to $\infty$) vanishes. For convenience, let us call this the standard form. As we shall see later, the standard form of the S force is found to persist also in the RW gauge, at least in the present case of a circular orbit to 1PN order.

For the moment, let us assume the standard form of the S force both in the Lorenz gauge and the RW gauge. Then, we may focus our discussion on the divergent terms. When we calculate the S force in the RW gauge, we first transform the metric perturbation from the Lorenz gauge to the RW gauge, and then take appropriate linear combinations of their first derivatives. We then find that the harmonic coefficients $h_{2\ell m}^{S.L}$, $h_{\ell 0 m}^{e(S.L)}$ and $h_{\ell 0 m}^{(e)S.L}$ are differentiated two times, and $G_{\ell m}(S.L)$ is differentiated three times, while the rest are differentiated once, to obtain the S force. So, it is necessary and sufficient to perform the Taylor expansion of the harmonic coefficients up to $O(X^2)$ for $h_{2\ell m}^{S.L}$, $h_{\ell 0 m}^{e(S.L)}$ and $h_{\ell 0 m}^{(e)S.L}$, and up to $O(X^3)$ for $G_{\ell m}^{S.L}$, and the rest up to $O(X^4)$, where $X = T - t_0$ or $R - r_0$. To the accuracy mentioned above, the harmonic coefficients of the S part are found in the form, for example,

$$h_{00 m}^{S.L}(r, r) = \frac{2}{L^2} \tau \mu \left[ \frac{4i T m r_0 (L^2 - 2)(\mu^2)}{L^2} + \cdots \right] \partial_0 r^{*} \ell m (\theta_0, \phi_0),$$

(74)

where we have defined

$$L^{(2)} = \ell (\ell + 1) = \left( L^2 - \frac{1}{4} \right).$$

(75)

The above example is the coefficient obtained by approaching the orbit from inside ($r < r_0$). The other coefficients, as well as the coefficients obtained in the case of approaching the orbit from outside ($r > r_0$), can be found on the webpage: http://www2.yukawa.kyoto-u.ac.jp/misao/BHPC/.
Now we consider the S force in the Lorenz gauge. It is noted that the $t$-, $\theta$- and $\phi$- components of the S force vanish after summing over $m$ modes. Using the formulae for summation over $m$

\[
\sum_{m} \frac{2\pi}{L} m^2 |Y_{lm}(\pi/2, 0)|^2 = \frac{L}{2}, \quad \sum_{m} \frac{2\pi}{L} |\partial_\theta Y_{lm}(\pi/2, 0)|^2 = \frac{L}{2}
\]

the $r$-component of the S force is derived as

\[
F_{S, L}^r \bigg|_{\ell} \bigg|_{L} = F_{S, L}^\theta \bigg|_{\ell} \bigg|_{L} = F_{S, L}^\phi \bigg|_{\ell} \bigg|_{L} = 0,
\]

\[
F_{S, L}^{(\pm)} \bigg|_{\ell} = \pm \frac{1}{2} \frac{\mu^2 (2r_0 - 3M)}{r_0^3} L - \frac{1}{8} \frac{\mu^2 (4r_0 - 7M)}{r_0^3} + \frac{\mu^2 M (172L^4 - 14784L^2 + 299)}{128r_0^3 (L^2 - 1)(L^2 - 4)(L^2 - 9)} \]

\[
= \pm \frac{1}{2} \frac{\mu^2 (2r_0 - 3M)}{r_0^3} L - \frac{1}{8} \frac{\mu^2 (4r_0 - 7M)}{r_0^3} + O \left( \frac{1}{L^2} \right).
\]

(77)

This is indeed of the standard form. It should be noted that the factor $L$ which is present in the denominators before summing over $m$ is cancelled by the same factor that arises from summation over $m$. If it were present in the final result, we would not be able to conclude that the summation of $D_{\mu}^{(\pm)}$ over $\ell$ vanishes. We note that, apart from the fact that the denominator of the $D_{\mu}^{(\pm)}$ term takes the standard form, the numerical coefficients appearing in the numerator should not be taken rigorously. This is because our calculation is accurate only to $O(y^0)$ of the S force, while the numerical coefficients depend on the $O(y)$ behaviour of it. It is also noted that the $O(1/L)$ terms are absent in the S force, implying the absence of logarithmic divergence.

### 3.4. The S part in the Regge–Wheeler gauge

Now, we transform the S part of the metric perturbation from the Lorenz gauge to the RW gauge. The gauge transformation functions are given in the tensor-harmonic expansion form as

\[
\xi_{\mu}^{(\text{odd})} = \sum_{\ell m} N_{\ell m}^{S, L \rightarrow RW} (t, r) \left\{ 0, 0, -\frac{1}{\sin \theta} \partial_\phi Y_{lm}(\theta, \phi), \sin \theta \partial_\theta Y_{lm}(\theta, \phi) \right\},
\]

\[
\xi_{\mu}^{(\text{even})} = \sum_{\ell m} \left\{ M_{0\ell m}^{S, L \rightarrow RW} (t, r) Y_{lm}(\theta, \phi), M_{1\ell m}^{S, L \rightarrow RW} (t, r) Y_{lm}(\theta, \phi),
\right.
\]

\[
M_{2\ell m}^{S, L \rightarrow RW} (t, r) \partial_\theta Y_{lm}(\theta, \phi), M_{2\ell m}^{S, L \rightarrow RW} (t, r) \partial_\phi Y_{lm}(\theta, \phi) \right\}.
\]

(78)

There is one degree of gauge freedom for the odd part and three for the even part. To satisfy the RW gauge condition (56), we obtain the equations for the gauge functions that are found to be rather simple, and then we find

\[
N_{\ell m}^{S, L \rightarrow RW} (t, r) = \frac{i}{2} h_{2\ell m}^{S, L} (t, r),
\]

\[
M_{2\ell m}^{S, L \rightarrow RW} (t, r) = -\frac{r^2}{2} \partial_\theta h_{2\ell m}^{S, L} (t, r),
\]

\[
M_{0\ell m}^{S, L \rightarrow RW} (t, r) = -h_{0\ell m}^{S, L} (t, r) - \partial_\theta M_{2\ell m}^{S, L \rightarrow RW} (t, r),
\]

\[
M_{1\ell m}^{S, L \rightarrow RW} (t, r) = -h_{1\ell m}^{S, L} (t, r) - r^2 \partial_\theta \left( \frac{M_{2\ell m}^{S, L \rightarrow RW} (t, r)}{r^2} \right).
\]

(80)
We note that it is not necessary to calculate any integration with respect to \( t \) or \( r \). It is also noted that the gauge functions are determined uniquely. This is because the RW gauge is a gauge in which there is no residual gauge freedom (for \( \ell \geq 2 \)).

Then, the S part of the metric perturbation in the RW gauge is expressed in terms of those in the Lorenz gauge as follows. The odd parity components are found as

\[
h^{S,RW}_{00m}(t, r) = h^{S,L}_{00m}(t, r) + \partial_t \Lambda^{S,L\rightarrow RW}_{1m}(t, r),
\]

\[
h^{S,RW}_{11m}(t, r) = h^{S,L}_{11m}(t, r) + r^2 \partial_r \left( \Lambda^{S,L\rightarrow RW}_{1m}(t, r) \right),
\]

and the even parity components are found as

\[
H^{S,RW}_{00m}(t, r) = H^{S,L}_{00m}(t, r) + \frac{2r}{r - 2M} \left[ \partial_t M^{S,L\rightarrow RW}_{00m}(t, r) - \frac{M(r - 2M)}{r^3} M^{S,L\rightarrow RW}_{11m}(t, r) \right],
\]

\[
H^{S,RW}_{11m}(t, r) = H^{S,L}_{11m}(t, r) + \left[ \partial_t M^{S,L\rightarrow RW}_{11m}(t, r) + \frac{2r}{r - 2M} M^{S,L\rightarrow RW}_{00m}(t, r) \right],
\]

\[
H^{S,RW}_{22m}(t, r) = H^{S,L}_{22m}(t, r) + \frac{2(r - 2M)}{r} \left[ \partial_r M^{S,L\rightarrow RW}_{11m}(t, r) + \frac{M}{r(r - 2M)} M^{S,L\rightarrow RW}_{11m}(t, r) \right],
\]

\[
K^{S,RW}_{\ell m}(t, r) = K^{S,L}_{\ell m}(t, r) + \frac{2(r - 2M)}{r^2} M^{S,L\rightarrow RW}_{11m}(t, r),
\]

where the gauge functions are given by equations (79) and (80).

Inserting the S part of the metric perturbation obtained as described in the previous subsection to equations (79) and (80), we obtain the gauge functions that transform the S part from the Lorenz gauge to the RW gauge. It may be noted that the gauge functions do not contribute to the metric at the Newtonian order. In other words, both the RW gauge and the Lorenz gauge reduce to the same (Newtonian) gauge in the Newtonian limit. The S part of the metric perturbation in the RW gauge is now found in the form, for example,

\[
h^{S,RW}_{00m}(t, r) = \frac{2}{r^3} \pi \mu \left[ \frac{4i \pi t r_0 (L^2 - 2)(u^5)^2}{L^2 - 1} + \cdots \right] \partial_\theta Y^{*}_{\ell m}(\theta_0, \phi_0).
\]

Next we calculate the S part of the self-force. Of course, it diverges in the coincidence limit. However, as we noted several times, in the mode-decomposition regularization in which the regularization is done for each harmonic mode (\( \ell \)-mode), the harmonic coefficients of the S part are finite. The calculation is straightforward. We find that the \( t, \theta \)- and \( \phi \)-components of the S force vanish after summing over \( m \). Summing the above over \( m \), the \( r \)-component of the S force is derived as

\[
F^r_{S,\ell m} = F^\theta_{S,\ell m} = F^\phi_{S,\ell m} = 0, \quad F^{(\pm)}_{S,\ell m} = F^{(\pm)}_{S,L \ell m} = 0.
\]

We now see that the S force in the RW gauge has the standard form as in the case of the Lorenz gauge and there is no \( O(1/L) \) term. This is because the difference of the S force between the Lorenz and RW gauge arises from 2PN order as

\[
\delta F^r_{S,L\rightarrow RW}_{\ell m} = \sum_m \left[ -\frac{3M(r_0 - 2M)^2 h^{(e)S,L}_{11m}(0, r_0)}{r_0^2 (r_0 - 3M)} + \frac{3M(r_0 - 2M)^2 \partial_r G^{S,L}_{1m}(0, r_0)}{r_0^2 (r_0 - 3M)} \right] Y_{\ell m}(\theta_0, \phi_0).
\]
In the previous two subsections, we calculated the full and S parts of the self-force in the RW gauge for \( \ell \geq 2 \) modes. Now we are ready to evaluate the regularized self-force. But there is one more issue to be discussed, namely, the treatment of the \( \ell = 0 \) and 1 modes.

The full metric perturbation and its self-force are derived by the Regge–Wheeler–Zerilli formalism. This means that they contain only the harmonic modes with \( \ell \geq 2 \). The knowledge of the modes \( \ell \geq 2 \) would be sufficient to derive the regular, R part of the self-force, because the R part of the metric perturbation is known to satisfy the homogeneous Einstein equations [11], and because there are no non-trivial homogeneous solutions in the \( \ell = 0 \) and 1 modes. To be more precise, apart from pure gauge modes that are always present, the \( \ell = 0 \) solution corresponds to a shift of the black-hole mass and the \( \ell = 1 \) odd parity to adding a small angular momentum to the black hole, both of which should be put to zero in the absence of an orbiting particle. As for the \( \ell = 1 \) even mode, however, it turns out that it is important to take it into account properly because this mode corresponds to a dipolar shift of the coordinates. Namely, the asymptotic flatness condition on the R part of the metric perturbation will not be satisfied unless a gauge is properly chosen. This means that the \( \ell = 1 \) even mode arises already at the Newtonian level, as the Newtonian term of the self-force. In other words, apart from this \( \ell = 1 \) even mode contribution, the \( \ell = 0 \) and 1 modes of the full force should be exactly cancelled by those of the S part.

First, we consider the contributions of \( \ell \geq 2 \) to the self-force. As noted before, for the 1PN calculation, the only \( r \)-component of the full and S part of the self-force is non-zero. The \( \ell \) mode coefficients are derived as

\[
F_{\text{RW}}^{r}(\ell \geq 2) = -\frac{3\mu^2 M}{4r_0^2}.
\]  

(87)

Next, we consider the \( \ell = 0 \) and 1 modes. It is noted that the \( \ell = 0 \) and \( \ell = 1 \) odd modes are easily determinable in the Lorenz gauge, with the retarded boundary condition. On the other hand, we were unable to solve for the \( \ell = 1 \) even mode in the Lorenz gauge. Since it is locally a gauge mode describing a shift of the centre of mass coordinates, this gives rise to an ambiguity in the final result of the self-force. Nevertheless, we were able to resolve this ambiguity at Newtonian order, and hence to obtain an unambiguous interpretation of the resulting self-force. The corrections to the regularized self-force that arise from the \( \ell = 0 \) and \( \ell = 1 \) modes were calculated by Detweiler and Poisson [37]\(^6\). The result is

\[
\delta F_{\text{GW}}^{r}(\ell = 0, 1) = \frac{2\mu^2}{r_0^2} - \frac{57\mu^2 M}{4r_0^4}.
\]  

(88)

Finally, adding equations (87) and (88), we obtain the regularized gravitational self-force to the 1PN order as

\[
F_{\text{RW}}^{r} = \frac{2\mu^2}{r_0^2} - \frac{15\mu^2 M}{4r_0^4}.
\]  

(89)

Since there will be no effect of the gravitational radiation at the 1PN order, i.e., the \( t \)- and \( \phi \)-components are zero, the above force describes the correction to the radius of the orbit

\(^6\) The calculation of the \( \ell = 0 \) mode given in [29] failed to be regular on the event horizon.
that deviates from the geodesic on the unperturbed background. It is noted that the first term proportional to $\mu^2$ is just the correction to the total mass of the system at the Newtonian order, where $r_0$ is interpreted as the distance from the centre of mass of the system to the particle. Although this is a bit of a cumbersome way to obtain the Newtonian force, it is nevertheless an important result because it was derived from the relativistic self-force for the first time.

4. New regularization

As mentioned in the previous section, the frequency spectrum becomes monochromatic in the circular case so that the frequency integral can be trivially performed to give the harmonic coefficients of the full metric perturbation in the time domain. But when we consider the general orbit, the frequency spectrum is highly non-trivial and the frequency integral becomes formidable to perform analytically, making the $S$ part subtraction difficult to carry out. In this section, we present a new analytical method that can circumvent this difficulty [38].

Here, again for simplicity, we go back to the case of a particle with a scalar charge $q$, and consider the self-force given by $F_{\alpha}[\phi]$ defined in equation (4). We first describe a new decomposition, which we call the $\tilde{S}$–$\tilde{R}$ decomposition, of the Green function in the Fourier-harmonic domain. Next we show that only the $\tilde{S}$ part needs to be regularized, but it can be easily transformed to an expression in the time domain if we employ the post-Newtonian expansion. We then calculate the ($\tilde{S}$–$S$) part, which is finite and regular, for general orbit to 6PN order and for circular orbit to 18PN order up to now, although we present the explicit result only to 4PN order in this paper. Finally, specializing to the case of circular orbits, for which we can calculate the $\tilde{R}$ part analytically as well, we evaluate the self-force for several radii. In particular, we compare our result with the result numerically obtained by Detweiler, Messaritaki, Whiting and Diaz-Rivera [39, 40].

4.1. The $\tilde{S}$–$\tilde{R}$ decomposition

The full scalar field induced by a scalar-charged particle is given by equation (1) in terms of the retarded Green function $G_{\text{full}}^{\text{fall}}(x, x')$. The retarded Green function is represented in terms of the Fourier-harmonic decomposition as

$$G_{\text{full}}^{\text{fall}}(x, x') = \int \frac{d\omega}{2\pi} e^{i \omega(t - t')} \sum_{\ell m} g_{\ell m}(r, r') Y_{\ell m}(\theta, \phi) Y_{\ell m}^*(\theta', \phi'). \quad (90)$$

Then, the Klein–Gordon equation (2) reduces to an ordinary differential equation for the radial Green function as

$$\left[ \left( 1 - \frac{2M}{r} \right) \frac{d^2}{dr^2} + \frac{2(r - M)}{r^2} \frac{d}{dr} + \left( \frac{\ell(\ell + 1)}{r^2} - \frac{\ell \omega^2}{r^2} \right) \right] g_{\ell m}(r, r') = - \frac{1}{\omega^2} \delta(r - r'). \quad (91)$$

The radial part of the full Green function can be expressed in terms of homogeneous solutions of equation (91), which can be obtained using a systematic analytic method developed in [36]. We have

$$g_{\ell m}(r, r') = - \frac{1}{W_{\ell m}(\phi_{\text{in}}^v, \phi_{\text{up}}^v)} \left( \phi_{\text{in}}^v(r) \phi_{\text{up}}^v(r') \delta(r - r') + \phi_{\text{up}}^v(r) \phi_{\text{in}}^v(r') \delta(r - r') \right),$$

$$W_{\ell m}(\phi_{\text{in}}^v, \phi_{\text{up}}^v) = r^2 \left( 1 - \frac{2M}{r} \right) \left[ \left( \frac{d}{dr} \phi_{\text{up}}^v(r) \right) \phi_{\text{in}}^v(r) - \left( \frac{d}{dr} \phi_{\text{in}}^v(r) \right) \phi_{\text{up}}^v(r) \right]. \quad (92)$$
Here, the in-going and up-going homogeneous solutions are denoted, respectively, by \( \psi^v_{\text{in}} \) and \( \psi^v_{\text{up}} \), and \( v \) is called the ‘renormalized angular momentum’ \([36, 41, 42]\), which is equal to \( \ell \) in the limit \( M \omega \to 0 \).

We express the homogeneous solutions \( \psi^v_{\text{in}} \) and \( \psi^v_{\text{up}} \) in terms of the Coulomb wavefunctions \( \psi^v_z \) and \( \psi^v_{-z} \) \([41–43]\):

\[
\psi^v_{\text{in}} = \psi^v_z + \tilde{\beta}_v \psi^v_{-z}, \quad \psi^v_{\text{up}} = \tilde{\gamma}_v \psi^v_z + \psi^v_{-z}.
\]

The properties and the relations of the coefficients \( \tilde{\beta}_v, \tilde{\gamma}_v \) are studied in \([41]\) and \([42]\). An essential point to be noted here is that when we consider the post-Newtonian expansion, i.e., when \( \psi^v_z \) is expanded in terms of \( z := \omega r \) and \( \epsilon := 2M\omega \), \( \Phi^\nu := (2z)^{-\nu} \psi^v_z \) contains only terms that are integer powers of \( z \) and \( \epsilon \). Furthermore, this PN expansion turns out to be a double Taylor series expansion in \( z^2 \) and \( \epsilon/z \), i.e., there appear only positive powers of \( \omega^2 \).

We now divide the Green function into two parts, as

\[
g^\text{full}_{\ell m\omega}(r, r') = g^S_{\ell m\omega}(r, r') + g^R_{\ell m\omega}(r, r'),
\]

where

\[
g^S_{\ell m\omega}(r, r') = \frac{-1}{W_{\ell m\omega}(\psi^v_z, \psi^v_{-z})} \left[ \psi^v_z(r)\psi^v_{-z}(r')\theta(r' - r) + \psi^v_{-z}(r)\psi^v_z(r')\theta(r - r') \right],
\]

\[
g^R_{\ell m\omega}(r, r') = \frac{-1}{(1 - \tilde{\beta}_v \tilde{\gamma}_v)W_{\ell m\omega}(\psi^v_z, \psi^v_{-z})} \left[ \tilde{\beta}_v \tilde{\gamma}_v \left( \psi^v_z(r)\psi^v_{-z}(r') + \psi^v_{-z}(r)\psi^v_z(r') \right) + \tilde{\gamma}_v \psi^v_z(r)\psi^v_z(r') + \tilde{\beta}_v \psi^v_{-z}(r)\psi^v_{-z}(r') \right].
\]

Using results derived in \([41]\), we obtain the behaviour of the coefficients \( \{\tilde{\beta}_v, \tilde{\gamma}_v\} \) in the PN expansion as

\[
\tilde{\beta}_v = O(v^{\ell+2}), \quad \tilde{\gamma}_v = O(v^{-3}),
\]

where we have set \( z = O(v) \) and \( \epsilon = O(v^3) \) with \( v \) being the characteristic orbital velocity. The functions \( \psi^v_z \) and \( \psi^v_{-z} \) are, respectively, of \( O(v^{\ell}) \) and \( O(v^{\ell-1}) \) (except at \( \ell = 0 \)). Therefore, the three terms in the \( R \) part of the Green function become, respectively, \( O(v^{\ell+4}), O(v^{\ell+2}) \) and \( O(v^{\ell+2}) \) relative to the \( S \) part.

As should be clear from equations \((95)\) and \((96)\), the \( R \) part is regular at \( r = r' \), hence the part that needs to be regularized is the \( S \) part. Furthermore, if we truncate the PN expansion at a finite order, the \( R \) part terminates at finite \( \ell \). Therefore the expression for the \( R \) part of the self-force in the new decomposition takes the following form:

\[
F^R_a = F^\text{full}_a - F^S_a = (F^S_a - F^S_a) + \sum_{\ell=0}^{\ell_{\text{max}}} F^R_a|_{\ell}.
\]

4.2. Computation of the \( \tilde{S} \) part

We now compute the force due to the \( \tilde{S} \) part for a general orbit. The difference between the \( S \) force and the \( \tilde{S} \) force should be finite, because the \( R \) force is finite. Thus, in general, when expanded in terms of the spherical harmonics, the \( S \) force must take the same form as the \( S \) force

\[
F^S_a|_{\ell} = A_a L + B_a + \tilde{D}_{a,\ell}.
\]

Below we confirm explicitly that both \( A_a \) and \( B_a \) for the \( S \) force coincide with those of equations \((40)\) and \((43)\). Therefore, the \( \tilde{S} \) force minus the \( S \) force, which is finite, is given by

\[
F^{\tilde{S} - S}_a = \sum_{\ell=0}^{\infty} (F^S_a|_{\ell} - F^S_a|_{\ell}) = \sum_{\ell=0}^{\infty} \tilde{D}_{a,\ell}.
\]
To obtain an expression for the \( \tilde{S} \) force in the form of equation (99), it is necessary to perform the \( \omega \) integration explicitly. Here, the key fact is that there appears no fractional power of \( \omega \) in the \( \tilde{S} \) part. This is because we have chosen \( \phi \nu \) and \( \phi - \nu - 1 \) as the two independent basis functions. As noted above, except for the overall fractional powers \( z^\nu \) and \( z^{-\nu-1} \), they contain only the terms with positive integer powers of \( \omega^2 \). When we consider a product of these two functions, \( \omega \) contained in the overall factors \( z^\nu \) and \( z^{-\nu-1} \) just produces \( \omega^{-1} \), which is cancelled by \( \omega \) from the inverse of the Wronskian. Thus, \( g_{\ell m \omega}(r, r') \) is expanded as

\[
g_{\ell m \omega}(r, r') = \sum_{k=0}^{\infty} \omega^{2k} G_{\ell m k}(r, r').
\]  

(101)

The Fourier transform of \( \omega^{2n} \) simply produces

\[
\int \omega^{2n} e^{-i\omega(t'-t)} = 2\pi (-1)^n \partial_t^n \delta(t - t').
\]

(101)

Differentiation of the \( \delta \)-function in the expression above can be integrated by parts to act on the source term. Thus, the integration over \( \omega \) can be performed easily, and we can express the \( \tilde{S} \) force in the time domain as

\[
F_{\tilde{S}}|_{\ell} = q^2 \lim_{x \to z(t)} \left[ \partial_t \partial_\phi \sum_{m,k} \left[ (-1)^k (-1)^j \sum_{\ell} (\delta_\ell) Y_{\ell m}(\theta, \phi) Y_{\ell m}^*(\theta', \phi') \right] \right].
\]

(102)

The transformation of the \( \tilde{S} \) part into the time domain makes it possible to subtract the divergent \( S \) part analytically. If we were to perform this subtraction numerically, the fraction to be subtracted would become closer and closer to unity as \( \ell \) increases. Apparently, this would mean a stringent requirement on numerical accuracy. In this sense, we anticipate a clear advantage in the analytical subtraction.

4.3. The (\( \tilde{S} \)-\( S \)) force

The result for the \( \tilde{S} \) force is

\[
F_{\tilde{S}}|_{\ell} = \frac{q^2 u'}{4\pi r_0^2} \sum_{n=0}^{\infty} K_{t,\ell}^{(n)}
\]

\[
F_{\tilde{S}}|_{\phi} = 0,
\]

\[
F_{\tilde{S}}|_{t} = \frac{q^2 u' L}{4\pi r_0^2} \sum_{n=0}^{\infty} K_{\phi,\ell}^{(n)},
\]

(103)

and

\[
F_{\tilde{S}}|_{t} = -\frac{\mathcal{E}}{u' (1 - 2M/r_0)} F_{\tilde{S}}|_{\ell} - \frac{L}{u' r_0^2} F_{\tilde{S}}|_{\phi},
\]

(104)

where the coefficients \( K_{\alpha,\ell}^{(n)} \), whose superscript \( (n) \) represents the PN order, are formally given by

\[
K_{t,\ell}^{(n)} = \sum_{i+j+k=n} d_{t,\ell}^{(i,j,k)} (\delta_\ell)^j \left( \frac{L^2}{r_0^2} \right)^k U^k,
\]

(105)

\[
K_{\phi,\ell}^{(n)} = \sum_{i+j+k=n} d_{\phi,\ell}^{(i,j,k)} (\delta_\ell)^j \left( \frac{L^2}{r_0^2} \right)^k U^k.
\]

Here, the quantities \( d_{\alpha,\ell}^{(i,j,k)} \) are some functions of \( \ell, r_0 \equiv z'(t_0), \delta_\ell \equiv 1 - \frac{1}{r_0}, U \equiv \frac{M}{r_0} \), and \( \mathcal{E} \) and \( \mathcal{L} \) are, respectively, the energy and the angular momentum of the particle. To obtain these expressions, we have used the first integrals of the geodesic motion and we have also reduced higher order derivatives with respect to \( r \) by using the equations of motion.
To summarize, the (S–S) force is given by

\[
F_{\alpha}^{S-S} = \sum_{\ell=0}^{\infty} \left( F_{\alpha}^{S} \bigg|_{\ell} - A_{\alpha} \left( \ell + \frac{1}{2} \right) - B_{\alpha} \right).
\]

The final result is

\[
F_{t}^{S-S} = \frac{q^{2}u'}{4\pi r_{0}} \sum_{n=0}^{\infty} C_{t}^{S-S(n)}, \quad F_{\theta}^{S-S} = 0, \quad F_{\phi}^{S-S} = \frac{q^{2}u'c}{4\pi r_{0}} \sum_{n=0}^{\infty} C_{\phi}^{S-S(n)},
\]

and

\[
F_{r}^{S-S} = -\frac{\epsilon}{u'(1-2M/r_{0})} F_{t}^{S-S} - \frac{\epsilon}{u'\epsilon r_{0}} F_{\phi}^{S-S}.
\]

with the coefficients \(C_{a}^{S-S(n)}\), whose upper index, \(n\), represents the PN order. The calculation of \(C_{a}^{S-S(n)}\) to a given order of post-Newtonian expansion is straightforward. Most importantly, once we compute them, we can just keep the results and apply them to any general orbit at any time. Up to now, we have computed the (S–S) force to 6 PN order. As an example, we list some low PN order terms:

\[
\begin{align*}
C_{t}^{S-S(0)} &= \frac{73}{133}, \\
C_{t}^{S-S(1)} &= -\frac{610}{31521} \delta_{\epsilon} + \frac{282 c^{2}}{1501 \epsilon}, \\
C_{t}^{S-S(2)} &= -\frac{2296958}{8878415} \delta_{\epsilon} + \left[ \frac{14127898 c^{2}}{8878415 \epsilon} - \frac{20571064 c}{26635245} \right] \delta_{\epsilon} - \frac{5579893 c^{4}}{1775683 \epsilon}, \\
C_{t}^{S-S(3)} &= -\frac{115291414894}{269415503175} \delta_{\epsilon}^{3} + \left[ \frac{43471970326 c^{2}}{17961033545 \epsilon} - \frac{48448379368 c}{89805167725} \right] \delta_{\epsilon}^{2}, \\
&\quad + \left[ -\frac{22584903396 c^{4}}{2565861935 \epsilon} - \frac{508295808 c^{2} U}{2565861935 \epsilon} \right] \delta_{\epsilon}, \\
&\quad - \left[ \frac{244692415685084 c}{8336485426815} + \frac{21 c^{2}}{32} \right] U \delta_{\epsilon}, \\
&\quad + \left[ \frac{16156048 c^{4}}{1301367 \epsilon} - \frac{32479265 c}{32479265 \epsilon} \right] U^{2} \\
&\quad + \left[ \frac{122513312775814 c^{2}}{1667297085363} + \frac{105 c^{2}}{64} \right] U^{2} \\
&\quad - \left[ \frac{170795214491529 c}{25009456280445} + \frac{7 c^{2}}{4} \right] U^{3},
\end{align*}
\]

\[
\begin{align*}
C_{\phi}^{S-S(0)} &= \frac{960}{10507}, \\
C_{\phi}^{S-S(1)} &= \frac{23145656 c}{26635245} \delta_{\epsilon} + \frac{33594 c^{2}}{1775683 \epsilon} + \frac{2403368 c}{761007} U.
\end{align*}
\]
\[ C_{\phi}^{\tilde{S}-S(2)} = \begin{vmatrix} 46189&522292 \\ 53883&100635 \end{vmatrix} \delta \varepsilon + \begin{vmatrix} -94186639&\mathcal{L}^2 \\ 2565861935&r_0^2 \end{vmatrix} + \begin{vmatrix} 1182191716 \\ 2565861935 \end{vmatrix} U \delta \varepsilon \]

\[ -86499760\mathcal{L}^4 + 965295376\mathcal{L}^2U \]

\[ + \begin{vmatrix} -231720301397292 \\ 8336485426815 \end{vmatrix} \delta E - \frac{21}{8} \pi^2 \]

Once we obtain the general expression for the \((\tilde{S}-S)\) part of the force, computation of the remaining \(R\) part is rather easy, because only terms up to a finite value of \(\ell\) contribute to the force for a given PN order. Then, the \(R\) force, which is what we want in the end, is given by the form (98).

We note that the \((\tilde{S}-S)\) force contains only the conservative part of the self-force. To show this, we use the fact that the equations of motion take the form of \(\ddot{u}^\mu = \text{even in } u^\mu\) and \(\dot{E} = \mathcal{L} = 0\) at leading order in \(\mu\). Here an overdot represents the differentiation with respect to \(t\). Recalling the general expression for the \(\tilde{S}\) force in equation (102), we see that the \(t\)-component contains an odd number of time derivatives \(\partial_{2k+1}/\partial t^{2k+1}\), the \(r\)-component an even number of time derivatives plus one radial derivative \(\partial_{2k+1}/(\partial t^2 \partial r)\), and the \(\varphi\)-component an even number of time derivatives plus one \(\varphi\)-derivative \(\partial_{2k+1}/(\partial t^2 \partial \varphi)\). Now, using the equations of motion, the \(t\)-derivative may be replaced by

\[ \frac{\partial}{\partial t} = \dot{z}_r(t) \frac{\partial}{\partial z_r} + \ddot{z}_r(t) \frac{\partial}{\partial \dot{z}_r} - i m \dot{z}_\varphi(t), \]

and the \(\varphi\)-derivative by

\[ \frac{\partial}{\partial \varphi} = +im. \]

Note that one differentiation with respect to \(t\) or \(\varphi\) changes the total power of \(u^\mu\) and \(m\) by an odd number, while a differentiation with respect to \(r\) does not. Note also that only the terms even in \(m\) remain after summation over \(m\). Therefore, the \(\ell\) mode of the \(\tilde{S}\) force takes the form

\[ F^{\tilde{S}}_t |_{\ell} = \mathcal{F}_{t,\ell}(r, E, \mathcal{L}) u^\mu, \quad F^{\tilde{S}}_r |_{\ell} = \mathcal{F}_{r,\ell}(r, E, \mathcal{L}), \quad F^{\tilde{S}}_\varphi |_{\ell} = \mathcal{F}_{\varphi,\ell}(r, E, \mathcal{L}) u^\mu. \] (109)

The above equation can be directly found from equations (103). The \(S\) part of the force is known to have exactly the same form. This implies that the \((\tilde{S}-S)\) force also takes the same form. Thus, after summing over \(\ell\), we conclude that the final form of the \((\tilde{S}-S)\) force is

\[ F^{{\tilde{S}-S}}_t = \mathcal{F}_t(r, E, \mathcal{L}) u^\mu, \quad F^{{\tilde{S}-S}}_r = \mathcal{F}_r(r, E, \mathcal{L}), \quad F^{{\tilde{S}-S}}_\varphi = \mathcal{F}_{\varphi}(r, E, \mathcal{L}) u^\mu. \] (110)

Next, we can explicitly show that the above form of the force implies the absence of a dissipative reaction effect. In other words, the force is conservative. The equations of motion to \(O(\mu^2)\) are given by

\[ \frac{\mu}{d\tau} \ddot{u}^\mu = F^\mu, \]

where \(\ddot{u}^\mu\) is the perturbed 4-velocity and \(D/d\tau\) is the covariant derivative. Then, we obtain the evolution equation for the perturbed energy \(\dot{E} := -\mu \dot{t}_\mu \ddot{u}^\mu\) as

\[ \frac{dE}{d\tau} = -\mu \frac{d}{d\tau} (\dot{t}_\mu \ddot{u}^\mu) = -\dot{t}_\mu F^\mu = -\mathcal{F}_t(r) \frac{dr}{d\tau}, \] (112)
where $\hat{\tau}^\mu = (\partial_r)^\mu$ is the timelike Killing vector. This equation is integrated to give
\[
\hat{\mathcal{E}} = \mathcal{E} - \int F_r(r) \, dr.
\] (113)
Here, $\mathcal{E}$ is an integration constant, which we can interpret as the unperturbed energy. In the same manner, for the perturbed angular momentum $\mathcal{L}$, we obtain
\[
\hat{\mathcal{L}} = \mathcal{L} + \int F_\phi(r) \, dr.
\] (114)
Thus, for an orbit with bounded radial motion, i.e., a periodic orbit, we find that there is no cumulative effect on the evolution of the energy and angular momentum of the particle. In other words, a force of the form (110) guarantees the presence of the constants of motion $E$ and $L$.

### 4.4. Testing the efficiency for circular orbits

To examine the efficiency of this new regularization method, we revisit the problem of the self-force for circular orbits. For this purpose, we first calculate the $S$ force for a circular orbit to 18PN order. Then, combining with the calculation of the $R$ part, which can be done with sufficient accuracy in order not to spoil the 18PN order accuracy of the $(S-S)$ part, the regularized scalar self-force is evaluated and is compared with the result obtained by Detweiler, Messaritaki and Whiting \[39\], and very recently by Diaz-Rivera et al \[40\].

The components of the $(S-S)$ self-force, $F_{S-S}^\alpha = F_S^\alpha - F_S^\alpha$, have been given for general orbits in equation (107). Since the $(S-S)$ force is expressed in terms of local quantities of the particle, i.e., its position and velocity, what we have to do is just to specify the orbit. In the present case, we consider a circular orbit, given by equation (61). Then we find only the $r$-component is non-vanishing for the $(S-S)$ force because the $t$- and $\phi$-components are directly related to rates of change of the energy and angular momentum, and are purely dissipative for circular orbits. To 4PN order, it is given explicitly by
\[
F_{S-S}^r = \frac{q^2}{4\pi r_0^3} \left[ \frac{73}{133} + \frac{16151}{21014} V^2 + \frac{395567}{106808} V^4 + \left( \frac{1107284037660637}{400151300487120} + \frac{7}{64\pi^2} \right) V^6 \right. \\
+ \left. \left( \frac{182118981911377689978271}{8548630707351386171520} + \frac{29\pi^2}{1024} \right) V^8 \right],
\] (116)
where $V = \sqrt{M/r_0} = r_0 \Omega$.

The components of the $R$ force are formally given by
\[
F_t^R = -\frac{iq^2}{\mu^t} \sum_{lm} m g_{lm,m\Omega}^{\delta \ell_m, m\Omega}(r_0, r_0) \left| Y_{lm} \left( \frac{\pi}{2}, 0 \right) \right|^2, \\
F_r^R = \frac{q^2}{\mu^t} \sum_{lm} \delta_r S_{lm,m\Omega}(r, r_0) \left| Y_{lm} \left( \frac{\pi}{2}, 0 \right) \right|^2, \\
F_\theta^R = 0, \quad F_\phi^R = -\frac{1}{\Omega} F_t^R.
\] (117)
The $R$ force in this case can be completely obtained analytically because the integration with respect to $\omega$ is done by substituting $m\Omega$ for $\omega$. Also note that if we need the precision up to
$n$-PN order inclusive, it is sufficient for us to calculate the modes up to $\ell \leq n + 1$. The 4PN results, after summation over $\ell$ modes, are

$$F^{R}_{r} = \frac{q^{2}}{4\pi r_{0}^{2}} \left[ \frac{73}{133} - \frac{16151}{21014} V^{2} - \frac{395567}{106808} V^{4} \right. \\
+ \left( \frac{4}{3} \gamma - \frac{4}{3} \frac{\ln(2V)}{400151300487120} \right) V^{6} \\
+ \left( \frac{59372}{120} 592 232 147 984 979 - \frac{14}{3} \ln V - \frac{66}{5} \ln(2) - \frac{14\gamma}{3} \right) V^{8} \bigg],$$

$$F^{R}_{t} = \frac{q^{2}V}{4\pi r_{0}^{2}} \left[ \frac{1}{3} V^{3} - \frac{1}{6} V^{5} + \frac{2\pi}{3} V^{6} - \frac{77}{24} V^{7} + \frac{9\pi}{5} V^{8} \bigg].$$

(118)

Here $\gamma$ is Euler’s constant, $\gamma = 0.57 \ldots$. As expected, the $t$-component which represents the energy loss rate starts at 1.5 PN order, corresponding to dipole radiation.

An important property of the $\tilde{R}$ force is that each $\ell$ mode of it may be evaluated without performing the post-Newtonian expansion. In other words, we do not have to expand the $\tilde{R}$ part of the Green function in powers of $\omega$. Instead, we can easily compute each $\ell$ mode with sufficient accuracy for any given radius $r_{0}$. Thus, as far as the $\tilde{R}$ part is concerned, what we actually employ is not the standard post-Newtonian expansion, but the truncation of the series expansion in $\ell$ at a given $\ell_{\text{max}}$ compatible with the PN order of our interest.

The total self-force is obtained by summing the $(\tilde{S} - S)$ force and the $\tilde{R}$ force. We have computed the force accurate to 18PN order so far. But here, we present the result explicitly only to 4PN order,

$$F^{R}_{r} = \frac{q^{2}}{4\pi r_{0}^{2}} \left[ \left( \frac{4}{3} \gamma - \frac{7}{64} \pi^{2} - \frac{4}{3} \frac{\ln(2V)}{400151300487120} \right) V^{6} \\
+ \left( \frac{604}{45} + \frac{29\pi^{2}}{1024} - \frac{66}{5} \ln(2) - \frac{14}{5} \ln V - \frac{14\gamma}{5} \right) V^{8} \bigg],$$

(119)

$$F^{R}_{t} = F^{R}_{r}.$$  

(120)

In the $r$-component of the scalar self-force there is a significant cancellation between the $(\tilde{S} - S)$ part and the $\tilde{R}$ part, and the total force begins at 3PN order.

The accuracy is actually limited by the $(\tilde{S} - S)$ part. In figure 1, we show the convergence of the $r$-component of the $(\tilde{S} - S)$ force as a function of the PN order for several representative orbital radii $r_{0}$, so the accuracy of the full regularized force can be read from this figure. Here an estimator of convergence of the PN expansion is defined by

$$\Delta_{a}^{S-S}(n) := \left| \frac{F_{a}^{S\text{-}S}(n) - F_{a}^{S\text{-}S}(n-1)}{F_{a}} \right|,$$

(121)

where $F_{a}^{S\text{-}S}(n)$ denotes the $(\tilde{S} - S)$ part of the force truncated at $n$PN order (inclusive), and the denominator $F_{a}$ denotes the exact (fully relativistic) self-force including the $\tilde{R}$ part. In practice, since it is impossible to know the exact value of it, we use the most accurate result in our calculation. It is found that the convergence of the PN expansion is steady even near ISCO, although it slows down there. The convergence improves slightly by using the Padé approximation near ISCO. Here, in the Padé approximation, we have chosen the denominator to be quadratic in $V^{2}$.  

Figure 1. The relative error of the post-Newtonian formulae in the $r$-component of the $(\tilde{S}-S)$ force for the radii $r_0 = 6M, 10M, 20M$ and $50M$. The horizontal axis is the order of post-Newtonian expansion. The top figure shows the convergence in the Taylor expansion and the bottom figure is the one obtained by using the Padé approximation.

Now let us compare our result with that obtained by Detweiler and his collaborators [39, 40]. They calculated the radial component of the self-force for various radii in units of $M = 1$ and $q^2 = 4\pi$. In our case, we employed a Padé approximation for the $(\tilde{S}-S)$ force accurate to 18PN order and used the most accurate $R$ force in our calculation including the terms up to $\ell = 19$. Their results are compared with ours for $r_0 = 6M$ (ISCO), $10M$ and $20M$ in table 1. As is clear from it, the agreement is impressive for $r_0 = 10M$ and $20M$, while there is a relative error of $\sim 10^{-4}$ for $r_0 = 6M$. This is consistent with the error estimate given in figure 1. This error may seem large, but if we use our result as a template for a space gravitational wave detector such as LISA, it turns out that the error is small enough [44]. Thus, we conclude that our new method is capable of computing the self-force with sufficient accuracy, and the result obtained to 18PN order seems accurate enough even in the limit of ISCO.
Table 1. The $r$-component of the self-force in units of $M = 1$ and $q^2 = 4\pi$ obtained by our method (top row) and by Detweiler et al [39, 40] (bottom row). The number for $r_0 = 10M$ is taken from [39] in which they estimate the error by a Monte Carlo simulation, and the numbers for $r_0 = 6M$ and $20M$ are taken from table I of [40].

| $r_0$   | 6M      | 10M     | 20M     |
|---------|---------|---------|---------|
| $F_R^r(r_0)$ | 1.676 820 878 $\times 10^{-4}$ | 1.378 448 171 $\times 10^{-5}$ | 4.937 905 866 $\times 10^{-7}$ |
| Detweiler et al | 1.677 2834 $\times 10^{-4}$ | 1.378 448 28(2) $\times 10^{-5}$ | 4.937 906 $\times 10^{-7}$ |

5. Conclusion

In this paper, we have reviewed recent progress in the regularization and computation of the self-force acting on a particle orbiting a black hole.

We have presented a method to regularize the self-force analytically, employing the post-Newtonian expansion. If we were to perform regularization numerically, the fraction to be subtracted would become closer and closer to unity as $\ell$ increases. Apparently, this would mean a stringent requirement on numerical accuracy. This is a clear advantage in the analytical approach over a numerical one. Thus, as far as the Schwarzschild background is concerned, we are almost ready to calculate the gravitational self-force for general orbits with a sufficiently high accuracy in post-Newtonian expansion. Research along this direction is now in progress [45].

At the same time, however, we should mention that our analytic method will become ineffective in the very high-frequency regime. Therefore, it seems important to develop an alternative numerical method that may play a complementary role.

Furthermore, although these recent developments are substantial, they are not quite good enough because our final goal is to compute the regularized self-force in the Kerr background.

As mentioned in the introduction, there is some progress in the case of the Kerr background [26, 27]. However, we should admit that we are still at a primitive stage in the case of the Kerr background.

Recently, extending the adiabatic orbital evolution in a more systematic manner, Mino proposed an alternative, possibly much more powerful method to deal with the gravitational reaction force to the orbit of a particle [46]. Perhaps we should test his idea by applying it to orbits in the Schwarzschild background and comparing the orbital evolution with that obtained from the self-force regularization method. Together with computation of the gravitational self-force for general orbits in the Schwarzschild background, we hope to come back to this issue in the near future.

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